Tree structures in the geometric representation theory of quivers

Habilitationsschrift

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Contents

1 Introduction

One of the major goals in the representation theory of quivers is the classification of isomorphism classes of representations of arbitrary quivers and the homomorphism spaces between them. Except for representations of quivers of Dynkin type, which were first classified by Gabriel in [24], and those of extended Dynkin type, which were independently classified by Nazarova in [48] as well as by Donovan and Freislich in [16], the classification problem of the remaining so-called wild quivers is generally regarded as hopeless. Nevertheless, in [34], Kac could show that the dimension vectors which appear as those of indecomposable representations are precisely the positive roots of the associated Kac-Moody Lie algebra. In [35], he was even able to express the number of parameters which parametrize the isomorphism classes of indecomposable representations in terms of the Euler form of the root. For a fixed root of the quiver, he also showed that the number of absolutely indecomposable representations over a finite field is given by a polynomial with integer coefficients. By now, Hausel, Letellier and Rodriguez-Villegas were able to prove in [31] that all these integers are positive which generalizes a former result of Crawley-Boevey and van den Bergh for coprime dimension vectors, see [15]. The main ingredient in proving this former conjecture of Kac is to write the counting polynomial as the Poincaré polynomial (resp. a related polynomial) of a moduli space associated with the fixed root. This emphasizes the importance of moduli spaces of stable quiver representations as treated in King's work [37] and results concerning their cohomology as obtained by Reineke [52].

There are also several results which deal with the classification problem itself and which help to understand the nature of indecomposable representations at least partially. A powerful tool is the reflection functor of Bernstein, Gelfand and Ponomarev introduced in [3] which can be used to restrict the classification problem to a smaller number of roots. Apart from this functor, there are plenty of others which can be applied to construct indecomposable representations recursively. In [62], Schofield established an induction which can be used to describe the perpendicular subcategories coming along with a fixed exceptional sequence as the representation category of a smaller quiver. But also the reflection functor introduced by Ringel in [57], which can especially be used to construct non-Schurian indecomposable representations by extending a given representation by a certain number of copies of an exceptional representation, is a famous example. As we see, exceptional representations play an important role in this field. This underlines the significance of both Ringel's theorem [58], which says that every exceptional representation is a tree module, and Schofield induction, which can be used to construct them explicitly.

Apart from the question of the classification of indecomposable representations, another problem has been of growing interest throughout the last years. For a fixed representation of a quiver, one can ask for the subset of subrepresentations of a fixed dimension which is in fact a subvariety of a certain product of Grassmannians. These so-called quiver Grassmannians are insofar particularly interesting as their cohomology was discovered to be strongly linked to the cluster variables of the cluster algebra associated with the quiver, see [6]. But quiver Grassmannians are clearly interesting in their own right. For instance, it is remarkable that the mentioned finite-tame-wild trichotomy seems to extend to quiver Grassmannians. More precisely, quiver Grassmannians attached to indecomposable representations of Dynkin quivers are smooth, those attached to extended Dynkin quivers seem to admit a cell decomposition into affine spaces while those attached to wild quivers are generally neither smooth nor admit a cell decomposition into affine spaces. We should point out that the case of quiver Grassmannians of quivers of type $\tilde{E}$ has not been investigated in detail until now.

Therefore, despite this disillusioning prediction of being hopeless, lots of statements concerning the representation theory of wild quivers can be made. One of the main goals of this habilitation is to make a contribution to the representation theory of wild quivers focusing on the mentioned issues. In order to achieve this, we mainly try to give a geometric description of the objects under consideration, or we investigate geometric objects which naturally appear in the representation theory of quivers. Thereby, tree structures happen to play an important role, for instance as tree modules, which are hoped to serve as a starting point for a normal form of certain isomorphism classes of indecomposable representations, but also as torus fixed points of moduli spaces. But also the Euler characteristic of quiver Grassmannians of representations of extended Dynkin quivers of type $\tilde{A}_n$ and $\tilde{D}_n$ turns out to be determined by the number of certain subgraphs of the coefficient quiver of the representation which is a tree in this case. So we take this as an opportunity to analyze tree structures in the representation theory of quivers in greater detail. Since these tree structures can be described very explicitly, it turns out that it is often very promising trying to understand certain tree structures in a first step and to generalize the obtained results in a second one.

In the second chapter, we introduce notation and recall several results, which we use frequently throughout the paper. Afterwards, we focus on three aspects. The first one contributes to the mentioned classification problem in such a way that we introduce methods which can be used to construct isomorphism classes of indecomposable representations which can be described by as many parameters as predicted by Kac's Theorem. Often tree modules serve as a skeleton in this construction and the investigations raise hope that, at least for a certain class of roots, further considerations can lead to a normal form for indecomposable quiver representations. Thereby, tree modules should play the role of nilpotent endomorphisms in the theory of Jordan normal forms for endomorphisms of vector spaces. Another advantage of tree modules is that they are given very explicitly in terms of their coefficient quiver. Furthermore, it is straightforward to write down a basis of the groups of extensions of two tree modules.

Since we are interested in the recursive construction of indecomposable representations, it is of advantage to know the groups of extensions explicitly. The basic idea is to fix a sequence of representations of a fixed quiver and to consider the quiver whose vertices correspond to the representations and whose arrows correspond to the bases of the groups of extensions. Now representations of the new quiver give rise to those of the original one. This can be described very explicitly in terms of a functor whose properties we study in the third chapter. More precisely, we investigate under which conditions it is fully faithful or at least preserves indecomposability. This setup admits lots of applications: for instance, we can generalize Schofield induction and Ringel's reflection functor. This recursive construction is described in detail in [71]. We also use the introduced language in order to describe the main results of [68] implying the existence of indecomposable tree modules for every imaginary Schur root.

But we also focus on new aspects as we use the results of [71] in the case of Schur roots in order to construct as many isomorphism classes of indecomposable representations as predicted by Kac's Theorem. Furthermore, we see that it is possible to show that the number of indecomposable tree modules grows exponentially in this case. In turn, we can use this fact to show that the value of the Kac polynomial at one, which is attached to a root of the Kronecker quiver, grows exponentially with the dimension vector. To obtain this, it is fundamental to establish a connection between (cover-thin) tree modules and torus fixed points of moduli spaces of stable representations of the corresponding preprojective algebra.

Another approach to tackle the classification problem is to combine arguments from intersection theory with this recursive construction as done in [23]. In the case of non-Schurian roots of quivers with three vertices, this turns out to be a very powerful tool. In fact, the points of the intersection of two subvarieties of a Grassmannian, which are prescribed by a fixed root, are in correspondence to those representations which can be constructed by Ringel's reflection functor. Thereby, it is substantial that the dimension of each irreducible component of this intersection is exactly as predicted by Kac's Theorem as soon as it contains a Schur representation. It also turns out that, under certain extra conditions, we can glue Schur representations which are contained in two different intersections in order to obtain Schur representations of a third one.

The Kac polynomial and geometric considerations are also the starting point of the fourth chapter and build the bridge to the second, cohomological, aspect of our investigation. A generalization of a conjecture of Douglas to the case of the Kac polynomial at one serves as an additional motivation. The main focus of our considerations is on the (singular) cohomology of various moduli spaces of quiver representations. As far as the Euler characteristic is concerned, tree structures again play a very important role. Actually, (iterated) torus fixed points are representations of the universal covering quiver which is a tree. In turn, stable tree modules are already torus fixed points. Since the odd cohomology of the moduli spaces under investigation vanishes, counting them yields at least a lower bound for the Euler characteristic, but also exact formulae in many cases.

Maybe the main knowledge gained in this theory throughout the last years is that the Euler characteristic can be obtained in a purely combinatorial way when counting certain trees. This observation is obtained when combining the MPS degeneration formula of [45] and the localization theorem of [69]. More precisely, the first result reduces the determination of the Poincaré polynomial to the case of dimension vectors of type one. Now we can use the second result in order to show that there are only finitely many torus fixed points in this case. In [70] and [55], this is used to show that the Euler characteristic of Kronecker moduli spaces grows exponentially with the dimension vector and, moreover, to obtain certain exact formulae which include several cases where the Euler characteristic vanishes. We recall some of these formulae, but also focus on a new instance of vanishing Euler characteristics which seems to be very useful for future considerations.

But this is not the end of the story as there is an equivalent formula in the theory of Gromov-Witten invariants, see [29]. In order to derive this equivalence which is established in [54], the so-called refined GW/Kronecker correspondence of [55] is fundamental as it discloses an identity between Euler characteristics of quiver moduli spaces of refined Kronecker quivers and Gromov-Witten invariants counting rational curves on weighted projective planes. In a sense, it turns out that in the theory of the afore mentioned Gromov-Witten invariants certain tropical curves, whose underlying graphs are also trees, play the role of torus fixed points. This clearly raises the question if there is a direct link between these two kinds of objects. Actually, it can be answered positively in several cases. In this work, we outline the main results of [55, 54], but also give an alternative proof of the equivalence where the notion of scattering diagrams is not needed.

In the last chapter, we focus on the third aspect and investigate quiver Grassmannians attached to representations of extended Dynkin quivers of type $\tilde{D}_n$. This is done by giving an overview of the main results of [43, 44] extended by an analysis of torus actions on quiver Grassmannians. In a first step, we show that every quiver Grassmannian admits a cell decomposition into affine spaces. This is not only a contribution to the above conjectured trichotomy as it turns out that the cells are in one-to-one correspondence to so-called non-contradictory subquivers of certain coefficient quivers, which happen to be trees in the case of real roots. This means that the Euler characteristic is already given by their number. The upshot is that we obtain an explicit description of the generating functions of the Euler characteristics. These generating functions are then again important in the theory of cluster algebras as they can be used to determine the corresponding cluster variables.

Apart from the applications to generalized Kronecker quivers, quivers of type $\tilde{D}_4$, imaginary Schur roots and especially Theorem 3.1.7, the results of Section 3.1 already appeared in [71, 68]. Except for Theorems 3.2.2 and 3.2.3 and the counterexample in Section 3.2.3, which appeared in the articles indicated at the respective places of this paper, as far as I know, the results of Section 3.2 did not appear in literature before. The results of Section 3.3 are collected from [23].

Section 4.1 is supposed to motivate the subsequent considerations by recalling and generalizing a conjecture of Michael Douglas. Section 4.2 can be understood as a revised version of [54, Section 5]. In Section 4.3, the results of [70] are summarized while, in Section 4.4, an overview of the formulae obtained in [55, 70] is given. But also a new formula is derived in Theorem 4.4.12. Finally, in Section 4.5, some of the main results of [54] and [55] are reviewed including an alternative proof of [54, Proposition

4.3].

The results of Sections 5.1 and 5.2 already appeared in the preprints [43, 44] while Section 5.3 gives a new perspective on torus actions on quiver Grassmannians.

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2 Basic definitions and results

In this chapter, we fix most of the notation and, furthermore, important results which are used frequently. Throughout this work, we fix an algebraically closed ground field k which is of characteristic zero unless otherwise stated. In the last two chapters, we even assume that k is the field of complex numbers. Actually, several results, especially many of those of the third chapter, remain valid when passing to arbitrary algebraically closed fields. We refer to the particular references for more details.

2.1 Quiver representations

For a detailed introduction to the representation theory of quivers we refer to [1]. A quiver Q consists of vertices Q_0 and arrows Q_1 which we denote by $\rho : p \to q$. For an arrow $\rho : p \to q$, let $s(\rho) = p$ be its head and $t(\rho) = q$ its tail. Moreover, let $a(p, q)$ be the number of arrows from p to q and let N_q be the set of neighbors of q. We denote by Q^{op} the quiver obtained from Q when turning around all arrows.

Let $\mathrm{Rep}(Q)$ be the category of finite-dimensional k-representations of Q whose objects are tuples

$$M = ((M_q)_{q\in Q_0}, (M_\rho : M_{s(\rho)} \to M_{t(\rho)})_{\rho\in Q_1})$$

consisting of finite-dimensional k-vectors spaces M_q and k-linear maps M_ρ. Taking dual vector spaces and adjoint linear maps, we obtain the dual representation $M^* \in \mathrm{Rep}(Q^{\mathrm{op}})$ of a representation $M \in \mathrm{Rep}(Q)$. The dimension vector $\underline{\dim}\, M \in \mathbb{N}Q_0$ of a representation M is given by $\underline{\dim}\, M = \sum_{q\in Q_0} \dim M_q \cdot q$. Sometimes it is more convenient to use the notation $\underline{\dim}\, M = (\dim M_q)_{q\in Q_0}$. We say that a representation M is of type one if $\dim M_q \in \{0, 1\}$ for every $q \in Q_0$. Let $R_\alpha(Q)$ denote the affine space of representations of dimension α.

On $\mathbb{Z}Q_0$ we have a non-symmetric bilinear form, the Euler form, which is defined by

$$\langle \alpha, \beta\rangle = \sum_{q\in Q_0} \alpha_q\beta_q - \sum_{\rho\in Q_1} \alpha_{s(\rho)}\beta_{t(\rho)}$$

for $\alpha, \beta \in \mathbb{Z}Q_0$. The antisymmetrization of the Euler form is defined by $\{\alpha, \beta\} = \langle\alpha, \beta\rangle - \langle\beta, \alpha\rangle$.

Let M and N be two representations of a quiver Q. We consider the linear map

$$d_{M,N} : \bigoplus_{q\in Q_0} \mathrm{Hom}_k(M_q, N_q) \to \bigoplus_{\rho\in Q_1} \mathrm{Hom}_k(M_{s(\rho)}, N_{t(\rho)})$$

defined by $d_{M,N}((f_q)_{q\in Q_0}) = (N_\rho f_{s(\rho)} - f_{t(\rho)}M_\rho)_{\rho\in Q_1}$. Then we have $\ker(d_{M,N}) = \mathrm{Hom}(M,N)$ and $\mathrm{coker}(d_{M,N}) = \mathrm{Ext}(M,N)$, see [56, Section 2.1]. If Q has no oriented cycles, following [56, Section 2.2], we have

$$\langle \underline{\dim}\, M, \underline{\dim}\, N\rangle = \dim_k \mathrm{Hom}(M,N) - \dim_k \mathrm{Ext}(M,N)$$

and $\mathrm{Ext}^i(M,N) = 0$ for $i \geq 2$ and any two representations M and N of Q.

Recall that every morphism $g \in \bigoplus_{\rho\in Q_1} \mathrm{Hom}_k(M_{s(\rho)}, N_{t(\rho)})$ defines a short exact sequence $E(g) \in \mathrm{Ext}(M,N)$ by

$$0 \to N \to ((N_q \oplus M_q)_{q\in Q_0}, (\begin{pmatrix} N_\rho & g_\rho \\ 0 & M_\rho \end{pmatrix})_{\rho\in Q_1}) \to M \to 0$$

with the canonical inclusion on the left hand side and the canonical projection on the right hand side. Then it is straightforward to check that two short exact sequences $E(g)$ and $E(h)$ are equivalent if and only if $g - h \in \mathrm{Im}(d_{M,N})$.

A dimension vector α is called a root if there exists an indecomposable representation of this dimension. We distinguish between real, isotropic and imaginary roots corresponding to the cases $\langle \alpha, \alpha\rangle = 1$, $\langle \alpha, \alpha\rangle = 0$ and $\langle \alpha, \alpha\rangle \leq 0$. A root is called Schur root if there exists a representation of dimension α with trivial endomorphism ring which we call Schur representation. An indecomposable representation M is called exceptional if we have $\mathrm{Ext}(M,M) = 0$. A representation M is exceptional if and only if $\underline{\dim}\, M$ is a real Schur root which we also call exceptional. If α is a real root, it follows from [35] that there only exists one indecomposable representation M of dimension α up to isomorphism which we denote by M_α. Let S_q be the simple representation corresponding to the vertex q and s_q its dimension vector.

If some property is independent of the point chosen in some open subset U of $R_\alpha(Q)$, following [61], we say that this property is true for a general representation of dimension vector α. Since the function $\lambda : R_\alpha(Q) \times R_\beta(Q) \to \mathbb{N}$, $(M,N) \mapsto \dim \mathrm{Hom}(M,N)$, is upper semi-continuous, see for instance [61, Section 1], we can define $\hom(\alpha,\beta)$ to be the minimal, and therefore general, value of this function. In particular, if α is a Schur root of a quiver, it follows that a general representation of dimension α is Schurian. Finally, we define $\mathrm{ext}(\alpha,\beta) := \hom(\alpha,\beta) - \langle \alpha, \beta\rangle$.

2.2 Subcategories of representation categories and functors between them

There are plenty of subcategories of representations of quivers which play an important role throughout this work. They can often be identified with the representation category of other quivers.

A sequence $E = (E_1, \ldots, E_r)$ of representations of Q is called exceptional if every representation E_i is exceptional and, moreover, $\mathrm{Hom}(E_i, E_j) = \mathrm{Ext}(E_i, E_j) = 0$ if $i < j$. If, in addition, $\mathrm{Hom}(E_j, E_i) = 0$ if $i < j$, we call such a sequence reduced. For two roots α and β we denote by $\beta \in \alpha^\perp$ if $\hom(\alpha,\beta) = \mathrm{ext}(\alpha,\beta) = 0$. In this way, we can also refer to exceptional sequences of roots.

For a set $M = \{M_1, \ldots, M_r\}$ of representations of Q, we define its perpendicular categories

$$^\perp M = \{X \in \mathrm{Rep}(Q) \mid \mathrm{Hom}(X, M_j) = \mathrm{Ext}(X, M_j) = 0 \text{ for } j = 1, \ldots, r\},$$

$$M^\perp = \{X \in \mathrm{Rep}(Q) \mid \mathrm{Hom}(M_j, X) = \mathrm{Ext}(M_j, X) = 0 \text{ for } j = 1, \ldots, r\}.$$

It is straightforward to check that these categories are closed under direct sums, direct summands, extensions, images, kernels and cokernels.

Theorem 2.2.1 ([62, Theorems 2.3 and 2.4])**.** *Let Q be a quiver with n vertices and $E = (E_1, \ldots, E_r)$ be an exceptional sequence.*

i) The categories $^\perp E$ and $E^\perp$ are equivalent to the categories of representations of quivers $Q(^\perp E)$ and $Q(E^\perp)$ respectively such that these quivers have $n-r$ vertices and no oriented cycles.

ii) There is an isometry with respect to the Euler form between the dimension vectors of $Q(^\perp E)$ (resp. $Q(E^\perp)$) and the dimension vectors of $^\perp E$ (resp. $E^\perp$) given by $\Phi((d_1, \ldots, d_{n-r})) = \sum_{i=1}^{n-r} d_i\alpha_i$ where $\alpha_1, \ldots, \alpha_{n-r}$ are the dimension vectors of the simple representations of the perpendicular categories.

For an exceptional sequence $E = (E_1, \ldots, E_r)$ with $\underline{\dim}\, E_i = \alpha_i$, let $\mathcal{C}(E_1, \ldots, E_r)$ be the full subcategory of $\mathrm{Rep}(Q)$ which contains $E_1, \ldots, E_r$ and which is closed under extensions, kernels of epimorphisms and cokernels of monomorphisms. Suppose that $E = (E_1, \ldots, E_r)$ is a reduced exceptional sequence. Then by [19, Lemma 2.35], we have that $E_1, \ldots, E_r$ are the simple objects of $\mathcal{C}(E_1, \ldots, E_r)$. Moreover, by Theorem 2.2.1, it follows that the category $\mathcal{C}(E_1, \ldots, E_r)$ is equivalent to the category of representations of the quiver $Q(E)$ which has vertices $Q(E)_0 = \{q_1, \ldots, q_r\}$ and $n_{ij} := \dim \mathrm{Ext}(E_i, E_j)$ arrows from q_i to q_j if $i \neq j$. Thus, an immediate consequence of Theorem 2.2.1 is the following, see also [62, Section 2] and [19, Theorem 2.38]:

Corollary 2.2.2. *Let $E = (E_1, \ldots, E_r)$ with $\underline{\dim}\, E_i = \alpha_i$ be a reduced exceptional sequence. Then $\alpha = \sum_{i=1}^{r} d_i\alpha_i$ is a root of Q if and only if $(d_1, \ldots, d_r)$ is a root of $Q(E)$.*

BGP-reflection functor

Since we frequently use the reflection functor introduced by Bernstein, Gelfand and Ponomarev in [3], we shortly review its definition. For a quiver Q, consider the matrix $A = (a_{p,q})_{p,q \in Q_0}$ with $a_{p,p} = 2$ and $a_{p,q} = a_{q,p}$ for $p \neq q$, in which $a_{p,q} = a(p,q) + a(q,p)$. For a fixed $q \in Q_0$, we define $\sigma_q : \mathbb{Z}Q_0 \to \mathbb{Z}Q_0$ as

$$\sigma_q(p) = p - a_{p,q}q.$$

Let $q \in Q_0$ be a sink (resp. a source). Then by $\sigma_q Q$ we denote the quiver which is obtained from Q by turning around all arrows with head (resp. tail) q. If M is a representation of Q and q is a sink (resp. a source), we consider the linear maps

$$\phi_q^M : \bigoplus_{\rho:p\to q} M_p \xrightarrow{M_\rho} M_q \text{ (resp. } \quad \phi_q^M : M_q \xrightarrow{M_\rho} \bigoplus_{\rho:p\to q} M_p).$$

Now in both cases, we define $(\sigma_q M)_p = M_p$ if $p \neq q$. Moreover, we define $(\sigma_q M)_q = \ker(\phi_q^M)$ (resp. $(\sigma_q M)_q = \operatorname{coker}(\phi_q^M)$). Moreover, the linear maps $(\sigma_q M)_{\rho^*}$ for $\rho : p \to q$ (resp. $(\sigma_q M)_{\rho^*}$ for $\rho : q \to p$) are the natural ones while the maps $M_{\rho'}$ for the remaining arrows $\rho' \in Q_1$ do not change. In both cases, we obtain an additive functor $\sigma_q : \operatorname{Rep}(Q) \to \operatorname{Rep}(\sigma_q Q)$ which is called (BGP-)reflection functor. It has the following properties:

i) If $M \cong S_q$, then $\sigma_q(S_q) = 0$.

ii) If $M \ncong S_q$ is indecomposable, then $\sigma_q(M)$ is indecomposable such that $\sigma_q^2(M) \cong M$ and $\underline{\dim}\, \sigma_q(M) = \sigma_q(\underline{\dim}\, M)$.

Ringel's reflection functor

We review some of the results of [57, Section 1]. For a fixed exceptional representation S and a full subcategory $\mathcal{C}$ of $\operatorname{Rep}(Q)$, we denote by $\mathcal{C}/S$ the category which has the same objects as $\mathcal{C}$ and the same morphisms modulo those factorizing through $\bigoplus_{i=1}^n S$ for some $n \in \mathbb{N}$. We define the following full subcategories of $\operatorname{Rep}(Q)$:

i) $\mathcal{M}^{-S} = \{M \in \operatorname{Rep}(Q) \mid \operatorname{Hom}(M, S) = 0\}$

ii) $\mathcal{M}_{-S} = \{M \in \operatorname{Rep}(Q) \mid \operatorname{Hom}(S, M) = 0\}$.

iii) $\mathcal{M}^S$ as the category of representations $M \in \operatorname{Rep}(Q)$ with $\operatorname{Ext}(S, M) = 0$ such that, moreover, there does not exist a direct summand of M which can be embedded into a direct sum of copies of S.

iv) $\mathcal{M}_S$ as the category of representations $M \in \operatorname{Rep}(Q)$ with $\operatorname{Ext}(M, S) = 0$ such that, moreover, no direct summand of M is a quotient of a direct sum of copies of S.

Let $M \in \mathcal{M}^S$ and $\mathcal{B} := \{\varphi_1, \ldots, \varphi_n\}$ be a basis of $\operatorname{Hom}(M, S)$. Following [57, Lemma 2], there exists a short exact sequence

$$0 \to M^{-S} \to M \to \bigoplus_{i=1}^n S \to 0$$

induced by $\mathcal{B}$ such that the induced sequences $e_1, \ldots, e_n$ form a basis of $\operatorname{Ext}(S, M^{-S})$. Moreover, we have $M^{-S} \in \mathcal{M}^{-S}$. The other way around, if $N \in \mathcal{M}^{-S}$ and $\{e_1, \ldots, e_n\}$

is a basis of $\mathrm{Ext}(S,N)$, we have an induced short exact sequence

$$0 \to N \to N^S \to \bigoplus_{i=1}^{n} S \to 0$$

such that $N^S \in \mathcal{M}^S$. We can proceed similarly for $M \in \mathcal{M}_S$ and $N \in \mathcal{M}_{-S}$. Then we have the following theorem summarizing the results of [57, Section 1]:

Theorem 2.2.3.

i) There exists an equivalence of categories given by the functor $F : \mathcal{M}^S/S \to \mathcal{M}^{-S}$, $M \mapsto M^{-S}$.

ii) There exists an equivalence of categories given by the functor $G : \mathcal{M}_S/S \to \mathcal{M}_{-S}$, $M \mapsto M_{-S}$.

iii) There exist equivalences $\Psi : \mathcal{M}^S_{-S} \to \mathcal{M}^{-S}_S$ and $\Phi : \mathcal{M}^S_S/S \to \mathcal{M}^{-S}_{-S}$ induced by composing the functors from above.

2.3 Coefficient quivers and tree modules

We introduce coefficient quivers and tree modules following [58]. Let M with $\underline{\dim}\, M = \alpha$ be a representation of Q. A basis of M is a subset $\mathcal{B}$ of $\bigoplus_{q\in Q_0} M_q$ such that

$$\mathcal{B}_q := \mathcal{B} \cap M_q$$

is a basis of M_q for every vertex $q \in Q_0$. For every arrow $\rho : p \to q$, we may write M_ρ as a $(\alpha_q \times \alpha_p)$-matrix $M_{\rho,\mathcal{B}}$ with coefficients in k such that the rows and columns are indexed by $\mathcal{B}_q$ and $\mathcal{B}_p$ respectively.

Definition 2.3.1. The coefficient quiver $\Gamma(M,\mathcal{B})$ of a representation M with a fixed basis $\mathcal{B}$ has vertex set $\mathcal{B}$ and arrows between vertices are defined by the following condition: if $(M_{\rho,\mathcal{B}})_{b',b} \neq 0$, there exists an arrow $(\rho, b, b') : b \to b'$ where $b \in \mathcal{B}_p$, $b' \in \mathcal{B}_q$ and $\rho : p \to q$.

A representation M is called a tree module if there exists a basis $\mathcal{B}$ for M such that the corresponding coefficient quiver is a tree.

There exists a natural map $\Gamma(M,\mathcal{B}) \to Q$ which we denote by $F_{\Gamma(M,\mathcal{B})}$. In Chapter 5, we assume the bases $\mathcal{B}_q$ for every $q \in Q_0$ to be linearly ordered. In this case, we call $\mathcal{B}$ ordered. In order to shorten notation, we mostly denote the arrows (ρ, b, b') by ρ. In the following considerations, all matrix representations of linear maps are with respect to $\mathcal{B}$. Let $M_{m,n}(k)$ be the set of $(m \times n)$-matrices with coefficients in k. For $A \in M_{m,n}(k)$, let $A_{i,j}$ be the (i,j)-entry. We denote by $E(s,t) \in M_{m,n}(k)$ the matrix with $E(s,t)_{i,j} = \delta_{i,s}\delta_{j,t}$. If M, N are representations of Q, we call a basis $\{E(f_1), \ldots, E(f_n)\}$ of $\mathrm{Ext}(M,N)$ tree-shaped if, for all $i = 1, \ldots, n$, there exists s,t such that we have $(f_i)_\rho = E(s,t)$ for exactly one $\rho \in Q_1$ and $(f_i)_{\rho'} = 0$ if $\rho' \neq \rho$. Since we can clearly choose a tree-shaped basis $\mathcal{C}$ of $\bigoplus_{\rho\in Q_1} \mathrm{Hom}_k(M_{s(\rho)}, N_{t(\rho)})$, we can choose a tree-shaped basis of $\mathrm{Ext}(M, N)$ consisting of elements of the form $b+\mathrm{Im}(d_{M,N})$ with $b \in \mathcal{C}$, see also the proof of [72, Lemma 3.16].

For a quiver Q, we denote by $\tilde{Q}$ its universal covering quiver given by the vertex set

$$\tilde{Q}_0 = \{(q, w) \mid q \in Q_0, w \in W(Q)\}$$

and the arrow set

$$\tilde{Q}_1 = \{\alpha_{(p,w)} : (p, w) \to (q, w\rho) \mid \rho : p \to q \in Q_1, w \in W(Q)\}.$$

Here $W(Q)$ denotes the set of words of Q where we refer to [69, Section 3.4] for a precise definition.

Remark 2.3.2. Since tree modules are already representations of $\tilde{Q}$, it is often convenient to work with the universal covering quiver from the beginning when investigating tree modules. Recall that, the other way around, every representation of $\tilde{Q}$ naturally defines a representation of the original quiver in such a way that indecomposability is preserved, see [25, Lemma 3.5].

A special feature of tree modules (or more generally of representations with a coefficient quiver which is almost a tree) is that one can read off certain sub- and factor representations. This gives the motivation to call a full subquiver Q' of a quiver Q of sink-type if we have $t(\rho) \in Q'_0$ for all arrows $\rho \in Q_1$ with $s(\rho) \in Q'_0$. Analogously, we use the term source-type. As far as coefficient quivers are concerned, every subquiver of sink-type (resp. source-type) defines a subrepresentation (resp. factor representation) in the natural way.

2.4 Moduli spaces of stable representations and their cohomology

In the last three sections of this chapter, we assume that $k = \mathbb{C}$. We choose a level $l : Q_0 \to \mathbb{N}^+$ on the set of vertices and $\Theta \in \mathbb{Z}Q_0$. Define two linear forms $\Theta, \kappa \in \mathrm{Hom}(\mathbb{Z}Q_0, \mathbb{Z})$ by $\Theta(\alpha) = \sum_{q\in Q_0} \Theta_q\alpha_q$, $\kappa(\alpha) = \sum_{q\in Q_0} l(q)\alpha_q$ and a slope function $\mu : \mathbb{N}Q_0\backslash\{0\} \to \mathbb{Q}$ by

$$\mu(\alpha) = \frac{\Theta(\alpha)}{\kappa(\alpha)}.$$

For $\mu \in \mathbb{Q}$, we denote by ${}'\Lambda^+_\mu \subset \mathbb{N}Q_0$ the set of dimension vectors of slope μ and define $\Lambda^+_\mu = {}'\Lambda^+_\mu \cup \{0\}$. This is a subsemigroup of $\mathbb{N}Q_0$.

For a representation M of the quiver Q, we define $\mu(M) := \mu(\underline{\dim}\, M)$. The representation M is called (semi-)stable if the slope (weakly) decreases on proper non-zero subrepresentations. For a fixed slope function as above, we denote by $R^{\Theta-\mathrm{sst}}_\alpha(Q)$ the set of semistable points and by $R^{\Theta-\mathrm{st}}_\alpha(Q)$ the set of stable points in $R_\alpha(Q)$. Following [37], there exist moduli spaces $M^{\Theta-\mathrm{st}}_\alpha(Q)$ (resp. $M^{\Theta-\mathrm{sst}}_\alpha(Q)$) of stable (resp. semistable) representations parametrizing isomorphism classes of stable (resp. polystable) representations. If Q is acyclic and $M_\alpha^{\Theta-\mathrm{st}}(Q)$ is non-empty, it is a smooth irreducible variety of dimension $1-\langle\alpha,\alpha\rangle$. Moreover, it is projective if semistability and stability coincide. Recall that this is the case if α is Θ-coprime, i.e. if we have $\mu(\beta)\neq\mu(\alpha)$ for all dimension vectors $0\neq\beta<\alpha$. In the remaining part of this section, we assume that $k=\mathbb{C}$. The following result of Reineke enables us to compute the Poincaré polynomial (in singular cohomology) of these moduli spaces:

Theorem 2.4.1 ([52, Corollary 6.8])**.** *For Θ-coprime α, we have*

$$\sum_i \dim H^i(M_\alpha^{\Theta-\mathrm{st}}(Q))q^{i/2}=(q-1)\sum_{\alpha^*}(-1)^{s-1}q^{-\sum_{k\leq l}\langle\alpha^l,\alpha^k\rangle}\prod_{k=1}^{s}\prod_{i\in Q_0}\prod_{j=1}^{\alpha_i^k}(1-q^{-j})^{-1},$$

where the sum ranges over all decompositions $\alpha=\alpha^1+\ldots+\alpha^s$ of α such that all α^k are non-zero, and $\mu(\alpha^1+\ldots+\alpha^k)>\mu(\alpha)$ for all $k<s$.

We also recall the notion of moduli spaces of framed representations called smooth models in [21]. We choose complex vector spaces V_q of dimension n_q where $0\neq n\in\mathbb{N}Q_0$. Then there exists a moduli space $M_{\alpha,n}^{\Theta-\mathrm{st}}(Q)$ which parametrizes equivalence classes of tuples (M,f) consisting of a Θ-semistable representation M of dimension α and a tuple $f=(f_q:V_q\to M_q)_{q\in Q_0}$ of linear maps such that for all subrepresentations $U\subset M$ with $f(V_q)\subset U_q$ we have $\mu(U)<\mu(M)$. Here we call (M,f) and (M',f') equivalent if there exists an isomorphism $\varphi:M\to M'$ such that $f'=\varphi\circ f$. If it is not empty, $M_{\alpha,n}^{\Theta-\mathrm{st}}(Q)$ is a smooth irreducible variety of dimension $n\cdot\alpha-\langle\alpha,\,\alpha\rangle$, see [21, Proposition 3.6].

Later the generating function of the Euler characteristics

$$Q_\mu^{(n)}(x)=\sum_{\alpha\in\Lambda_\mu^+}\chi(M_{\alpha,n}^{\Theta-\mathrm{st}}(Q))x^\alpha\in\mathbb{Z}[\![\Lambda_\mu^+]\!]$$

will be important for us. Moreover, for $\eta\in(\mathbb{Q}Q_0)^*$, let $Q_\mu^\eta(x)=\prod_{q\in Q_0}Q_\mu^{(q)}(x)^{\eta(q)}$. Then it can be shown that $Q_\mu^{n\cdot}(x)=Q_\mu^{(n)}(x)$, see [51, Theorem 3.4].

In the case when Q is acyclic, we can order the vertices $\{q_1,\ldots,q_r\}$ of Q in such a way that $k>l$ provided there exists an arrow $q_k\to q_l$. Following [53], we define a Poisson algebra $B(Q)=\mathbb{Q}[\![x_q\mid q\in Q_0]\!]$ with Poisson bracket $\{x^\alpha,x^\beta\}=\{\alpha,\beta\}x^{\alpha+\beta}$. For a vertex $q\in Q_0$, we can define a Poisson automorphism $T_q\in\mathrm{Aut}(B(Q))$ by $T_q(x^\alpha)=x^\alpha(1+x_q)^{\{q,\alpha\}}$. Then we have:

Theorem 2.4.2 ([53, Theorem 2.1])**.** *In $\mathrm{Aut}(B(Q))$, there exists the following factorization:*

$$T_{q_1}\circ\ldots\circ T_{q_r}=\overleftarrow{\prod_{\mu\in\mathbb{Q}}}T_\mu,$$

where

$$T_\mu(x^\alpha)=x^\alpha\prod_{q\in Q_0}Q_\mu^{(q)}(x)^{\{q,\alpha\}}.$$

Moreover, we have:

Lemma 2.4.3 ([53, Lemma 3.6]). *The series $Q_\mu^{\Theta-\mu\dim}$ equals* 1.

Finally, the main result of [51] is important for us:

Theorem 2.4.4 ([51, Theorem 3.4]). *The series $Q_\mu^{(n)}(x)$ is given by*

$$Q_\mu^{(n)}(x) = \prod_{\alpha\in'\Lambda_\mu^+} R^\alpha(x)^{\chi(M_\alpha^{\Theta-\mathrm{st}}(Q))(n\cdot\alpha)},$$

where the series $R^\alpha(x) \in \mathbb{Z}[\![\Lambda_\mu^+]\!]$, $\alpha \in {}'\Lambda_\mu^+$, is uniquely determined by the system of functional equations defined by

$$R^\alpha(x) = \left(1 - x^\alpha \cdot \prod_{\beta\in'\Lambda_\mu^+} R^\beta(x)^{-\chi(M_\beta^{\Theta-\mathrm{st}}(Q))\langle\alpha,\beta\rangle}\right)^{-1}$$

for all $\alpha \in {}'\Lambda_\mu^+$.

We obtain the double $\overline{Q}$ of a quiver Q by adding an arrow $\rho^* : q \to p$ for every arrow $\rho : p \to q \in Q_1$. For $\lambda \in \mathbb{C}Q_0$, we consider the deformed preprojective algebra

$$\Pi^\lambda(Q) = \mathbb{C}\overline{Q} / \sum_{\rho\in Q_1} [\rho, \rho^*] - \sum_{q\in Q_0} \lambda_q e_q$$

and the closed subset $\mathrm{Rep}(\Pi^\lambda(Q), \alpha)$ of $\mathrm{Rep}(\overline{Q}, \alpha)$. For a fixed dimension vector $\alpha \in \mathbb{N}Q_0$, we fix a linear form $\Theta : \mathbb{Z}Q_0 \to \mathbb{Z}$. If α is coprime, we can choose Θ such that $\Theta(\alpha) = 0$ and $\Theta(\beta) \neq 0$ for all dimension vectors $0 < \beta < \alpha$. Also with this setup, we obtain moduli spaces $M_\alpha^{0,\Theta-\mathrm{st}}(Q)$ of stable representations of $\Pi^0(Q)$ of dimension α where we set $\kappa = \dim$.

For a dimension vector α, we denote by $a_\alpha(q)$ the number of absolutely indecomposable representations of Q of dimension α over $\mathbb{F}_q$. It was proved by Kac in [36] that $a_\alpha(q)$ is a polynomial in q with integral coefficients. As conjectured by Kac, it was proved in [15] for coprime dimension vectors and in [31] in full generality that a_α actually has positive coefficients. If α is coprime, by [15], we have

$$a_\alpha(q) = \sum_{i=0}^{d} \dim H^{2d-2i}(M_\alpha^{0,\Theta-\mathrm{st}}(Q), \mathbb{C}) q^i$$

where we consider singular cohomology and where $d = \frac{1}{2}\dim M_\alpha^{0,\Theta-\mathrm{st}}(Q)$.

2.5 The tropical vertex

We briefly review the definition of the tropical vertex group following [29, Section 0]. It is insofar interesting as it reveals a connection between two seemingly unrelated invariants which is the Euler characteristic of quiver moduli spaces and Gromov-Witten invariants counting rational curves on weighted projective planes.

We fix non-negative integers $l_1, l_2 \geq 1$ and define R as the formal power series ring $R = \mathbb{Q}[\![s_1, \ldots, s_{l_1}, t_1, \ldots, t_{l_2}]\!]$ with maximal ideal $\mathfrak{m}$. Let B be the R-algebra

$$B = \mathbb{Q}[x^{\pm 1}, y^{\pm 1}][\![s_1, \ldots, s_{l_1}, t_1, \ldots, t_{l_2}]\!] = \mathbb{Q}[x^{\pm 1}, y^{\pm 1}]\widehat{\otimes} R$$

(i.e. a suitable completion of the tensor product). For $(d, e) \in \mathbb{Z}^2$ and a series

$$f \in 1 + x^d y^e \mathbb{Q}[x^d y^e]\widehat{\otimes}\mathfrak{m},$$

we consider the R-linear automorphism of B defined by

$$\theta_{(d,e),f} : \begin{cases} x & \mapsto & xf^{-e} \\ y & \mapsto & yf^{d}. \end{cases}$$

Note that these automorphisms respect the symplectic form $\frac{dx}{x} \wedge \frac{dy}{y}$.

Definition 2.5.1. The tropical vertex group $\mathbb{V}_R \subset \mathrm{Aut}_R(B)$ is defined as the completion with respect to $\mathfrak{m}$ of the subgroup of $\mathrm{Aut}_R(B)$ generated by all elements $\theta_{(d,e),f}$ as above.

We recall that, by [40] (see also [29, Section 1.3]), there exists a unique infinite ordered product factorization in $\mathbb{V}_R$ which is of the form

$$\theta_{(1,0),\prod_k(1+s_k x)}\theta_{(0,1),\prod_l(1+t_l y)} = \prod_{e/d \text{ decreasing}} \theta_{(d,e),f_{(d,e)}},$$

the product ranging over all coprime pairs $(d, e) \in \mathbb{N}^2$. One of the main issues of [29] is to describe the series $f_{(d,e)}$.

We introduce Gromov-Witten invariants on projective planes following [29, Section 0.4]. Following the main result of [29], we will see later that the series $f_{(d,e)}$ can be expressed in terms of these Gromov-Witten invariants. Denote by $\Sigma \subset \mathbb{Z}^2$ the fan with rays generated by $-(1, 0), -(0, 1)$ and (d, e). Let $X_{d,e}$ be the toric surface over $\mathbb{C}$ associated to Σ with corresponding toric divisors $D_1, D_2, D_{\mathrm{out}}$. It is isomorphic to the weighted projective plane $(\mathbb{C}^3 \setminus \{0\})/\mathbb{C}^*$ for the action $t(x, y, z) = (t^d x, t^e y, tz)$. Let $X^o_{d,e} \subset X_{d,e}$ be the open surface obtained by removing the torus fixed points and let $D^o_1, D^o_2, D^o_{\mathrm{out}}$ be the restrictions of the toric divisors to $X^o_{d,e}$.

We consider pairs $(\mathbf{P}_1, \mathbf{P}_2)$ of ordered partitions of length l_1 and l_2 respectively which we write as

$$\mathbf{P}_1 = p_{1,1} + \ldots + p_{1,l_1}, \quad \mathbf{P}_2 = p_{2,1} + \ldots + p_{2,l_2}$$

where we allow parts to be zero. Assume that $|\mathbf{P}_1| = \sum_{l=1}^{l_1} p_{1,l} = kd$ and $|\mathbf{P}_2| = \sum_{l=1}^{l_2} p_{2,l} = ke$ for some $k \geq 1$. Let $\nu : X_{d,e}[(\mathbf{P}_1, \mathbf{P}_2)] \to X_{d,e}$ be the blow-up of $X_{d,e}$ along l_1 (resp. l_2) points of D_1^o (resp. D_2^o) and define $X_{d,e}^o[(\mathbf{P}_1, \mathbf{P}_2)] = \nu^{-1}(X_{d,e}^o)$. Let $\beta_k \in H^2(X_{d,e}, \mathbb{Z})$ be the unique cohomology class with intersection numbers

$$\beta_1 \cdot D_1 = kd,\ \beta_2 \cdot D_2 = ke,\ \beta_k \cdot D_{\text{out}} = k.$$

Define a cohomology class $\beta_k[(\mathbf{P}_1, \mathbf{P}_2)] \in H^2(X_{d,e}[(\mathbf{P}_1, \mathbf{P}_2)], \mathbb{Z})$ by

$$\beta_k[(\mathbf{P}_1, \mathbf{P}_2)] = \nu^*(\beta_k) - \sum_{j=1}^{l_1} p_{1,j}[E_{1,k}] - \sum_{l=1}^{l_2} p_{2,l}[E_{2,l}],$$

where $E_{i,j}$ for $j = 1, \ldots, l_i$ denotes the j-th exceptional divisor over D_i^o for $i = 1, 2$. The moduli space $\overline{\mathfrak{M}}(X_{d,e}^o[(\mathbf{P}_1, \mathbf{P}_2)]/D_{\text{out}}^o)$ of genus 0 maps to $X_{d,e}^o[(\mathbf{P}_1, \mathbf{P}_2)]$ in class $\beta_k[(\mathbf{P}_1, \mathbf{P}_2)]$ with full contact order k at an unspecified point of D_{out}^o is proper and of virtual dimension 0. This means that a corresponding Gromov-Witten invariant $N_{(d,e)}[(\mathbf{P}_1, \mathbf{P}_2)] \in \mathbb{Q}$ is well-defined, see [29, Section 4]. We refer to [29, Section 6.4] for examples.

Theorem 2.5.2 ([29, Theorem 5.4])**.** *For all coprime (d, e), we have*

$$\log f_{(d,e)} = \sum_{k \geq 1} \sum_{\substack{|\mathbf{P}_1| = kd, \\ |\mathbf{P}_2| = ke}} k N_{(d,e)}[(\mathbf{P}_1, \mathbf{P}_2)] s^{\mathbf{P}_1} t^{\mathbf{P}_2} (x^d y^e)^k.$$

2.6 Quiver Grassmannians and F-polynomials

For a representation M with $\alpha = \underline{\dim}\, M$, the quiver Grassmannian $\mathrm{Gr}_e(M)$ is the set of subrepresentations U of M with $\underline{\dim}\, U = e$. It is a closed subvariety of the product $\prod_{q \in Q_0} \mathrm{Gr}_{e_q}(\alpha_q)$ of the usual Grassmannians $\mathrm{Gr}_{e_q}(\alpha_q)$.

Let $\mathbb{Q}[x_q^{\pm 1} \mid q \in Q_0]$ be the $\mathbb{Q}$-algebra of Laurent polynomials in the variables x_q. Denoting by χ the Euler characteristic in singular cohomology, as in [5], we set

$$X_M = \sum_{e \in \mathbb{N}Q_0} \chi(\mathrm{Gr}_e(M)) \prod_{q \in Q_0} x_q^{-\langle e,\, s_q \rangle - \langle s_q,\, \alpha - e \rangle}.$$

With Q we can associate a cluster algebra $\mathcal{A}(Q)$, which were introduced by [22], and its cluster category $\mathcal{C}_Q$ introduced in [4].

Theorem 2.6.1 ([6, Theorem 4])**.** *The correspondence $M \mapsto X_M$ provides a bijection between the set of indecomposable objects of $\mathcal{C}_Q$ without self-extensions and the set of cluster variables of $\mathcal{A}(Q)$.*

Actually, this bijection restricts to a bijection between exceptional representations of Q and cluster variables of $\mathcal{A}(Q)$ excluding the initial variables. In [7, Theorem 2], which generalizes [5, Proposition 3.10], the following multiplication formula is shown:

Theorem 2.6.2. *Let M and N be two indecomposable objects of $\mathcal{C}_Q$ such that we have $\dim \operatorname{Ext}_{\mathcal{C}_Q}(M,N) = 1$. Then we have*

$$X_M X_N = X_B + X'_B$$

where B and B' are up to isomorphism the unique middle terms of the non-split triangles

$$N \to B \to M \to SN, \quad M \to B' \to N \to SM.$$

Note that we have

$$\dim \operatorname{Ext}_{\mathcal{C}_Q}(M,N) = \dim \operatorname{Ext}(M,N) + \dim \operatorname{Ext}(N,M),$$

see [4]. Moreover, if $\operatorname{Ext}(M,N) = k$, the middle term B is the one induced by the non-splitting sequence in the module category. But since $\operatorname{Ext}(N,M) = 0$ in this case, using the terminology of [6], the middle term B' is just an object of $\mathcal{C}_Q$. But it actually has a corresponding representation in the module category which can be determined explicitly.

In this work, we are interested in the generating function F_M of the Euler characteristics of the corresponding quiver Grassmannians of M, which we also call F-polynomial due to its name in the theory of cluster algebras. It is defined by

$$F_M(x) = \sum_{e \in \mathbb{N}Q_0} \chi(\operatorname{Gr}_e(X)) x^e.$$

It is closely related to the cluster variables X_M. Indeed, setting

$$\alpha'_q = \sum_{p \in Q_0} a(q,p)\alpha_p - \alpha_q$$

and considering the variable transformation $x_q \mapsto x'_q$ defined by

$$x'_q = \prod_{p \in Q_0} x_p^{a(q,p)-a(p,q)},$$

it is straightforward to check that we have

$$X_M = x^{\alpha'} F_M(x').$$

3 Indecomposable representations and tree modules

As already mentioned in the introduction, a major goal in the representation theory of quivers is the classification of indecomposable representations and of the homomorphisms between them. This problem is far from being solved in full generality. But there are plenty of tuples of quivers and roots for which this problem can be tackled. In this chapter, we consider approaches yielding a recursive construction of indecomposable representations coming along with certain decompositions of a fixed root. For roots which allow such a decomposition, it is mostly possible to construct sets of isomorphism classes of indecomposable representations which can be described by as many parameters as predicted by Kac's Theorem. In all cases, tree modules can be constructed by imposing conditions on the glued representations and on the bases of the corresponding groups of extensions.

Tree modules are therefore very interesting as there is hope that they play an important role as far as the question of normal forms is concerned (under the hypothesis such a normal form exists for a fixed root). The idea is that they can be seen as the skeleton for it. The most famous example is probably the case of the quiver with one vertex and one loop where the nilpotent matrices consisting of only one Jordan block are exactly the indecomposable tree modules. As is known, the remaining indecomposable representations can be obtained by adding a multiple of the identity matrix.

In the first section of this chapter, we introduce a functor which builds the basis for the recursive construction investigated later on where we follow the presentation of [71]. We recall the generalizations of Schofield induction and Ringel's reflection functor established there. But we also obtain a new result showing that we can construct a $(1 - \langle \alpha, \alpha \rangle)$-parameter family of indecomposable representations with the introduced methods and for every imaginary Schur root α. Finally, we consider several conditions for decompositions of roots which are sufficient to construct indecomposable representations.

In the second section, we concentrate on tree modules. We actually deal with three basic questions concerning them. The first was stated by Ringel in [59, Problem 9]: Does there exist an indecomposable tree module for every wild hereditary quiver and every root? In particular, Ringel conjectured that there should be more than one isomorphism class for imaginary roots. We use the language of the first section in order to re-obtain the main result of [68] which shows that there exists an indecomposable tree module for every imaginary Schur root. But we also concentrate on a second question which is the one for the number of indecomposable tree modules and which was somewhat neglected in research before. In the case of imaginary Schur roots, it turns out that it grows exponentially with the dimension vector. In the case of the generalized Kronecker quivers, this has a consequence for the Kac polynomial at one as it can be shown that it also grows exponentially.

If there is a positive answer to the first question, a natural third one is how the existence of tree modules can be used to construct families of indecomposable representations. Actually, investigation in this direction is in early stages and was only carried out in special cases.

In the third section, we follow [23] and focus on Ringel's reflection functor in the case of quivers with three vertices. For non-Schurian roots, the canonical decomposition suggests that there exist indecomposable representations which can be constructed in this way. We tackle this problem geometrically in terms of subvarieties of Grassmannians. More precisely, we investigate a subvariety whose points correspond to representations which can be constructed by Ringel's functor. Finally, we combine this construction with the recursive construction of the first section.

3.1 Recursive construction of indecomposable representations

A naive approach to construct an indecomposable (tree) module is to write down a coefficient quiver (which is a tree) and to determine the endomorphism ring. This is only promising in cases of very small roots. A more encouraging approach is a recursive construction which means to find a decomposition of a fixed root α into smaller roots $\alpha_1, \ldots, \alpha_n$ in such a way that every tuple of representations $M \in \prod_{i=1}^{n} \mathrm{Rep}_{\alpha_i}(Q)$ gives rise to representations of dimension α when glueing them. In order to make this approach successful, it is necessary to impose conditions on the decomposition. We investigate several conditions which can be used to construct indecomposable representations. Under some extra conditions, it turns out that the resulting representations are even tree modules.

3.1.1 A functor between categories of quiver representations

Following [71, Section 2.2], it is convenient to describe this glueing in terms of a faithful functor between categories of quiver representations. Actually, we can define the functor for every fixed sequence of representations of a fixed quiver Q. It turns out that this functor does not have any nice properties in general as it does not preserve indecomposability or homomorphism spaces. Nevertheless, if the fixed sequence of representations does satisfy certain extra conditions, it has these nice properties or it at least preserves indecomposability. Thus, the main aim is to work out these conditions.

Let Q be a quiver without oriented cycles and fix a sequence $M = (M_1, \ldots, M_r)$ of representations of Q. Then we consider the quiver $Q(M)$ which has vertices $Q(M)_0 = \{m_1, \ldots, m_r\}$ and $n_{ij} := \dim \mathrm{Ext}(M_i, M_j)$ arrows from m_i to m_j if $i \neq j$ and no loops.

For each pair i, j with $i \neq j$, we also fix a basis

$$\mathcal{B}_{ij} = \{\chi_1^{ij}, \ldots, \chi_{n_{ij}}^{ij}\} \subseteq \bigoplus_{\rho \in Q_1} \mathrm{Hom}_k((M_i)_{s(\rho)}, (M_j)_{t(\rho)})$$

such that the corresponding residue classes are a basis of $\mathrm{Ext}(M_i, M_j)$. Since the arrows of $Q(M)$ are in correspondence with these basis elements, we also denote the arrows of $Q(M)$ by χ_l^{ij}. Finally, for a representation X of $Q(M)$ define $\tilde{X}_{i,q} := (M_i)_q \otimes_k X_{m_i}$ where $q \in Q_0$ and $i \in \{1, \ldots, r\}$.

This gives rise to a functor $F_M : \mathrm{Rep}(Q(M)) \to \mathrm{Rep}(Q)$: we define a representation $F_M X$ of Q by the vector spaces

$$(F_M X)_q = \bigoplus_{i=1}^{r} \tilde{X}_{i,q} \text{ for all } q \in Q_0$$

and for $\rho : p \to q$ we define linear maps $(F_M X)_\rho = \bigoplus_{i=1}^r \tilde{X}_{i,p} \to \bigoplus_{i=1}^r \tilde{X}_{i,q}$ by

$$((F_M X)_\rho)_{i,i} = (M_i)_\rho \otimes_k \mathrm{id}_{X_{m_i}} : \tilde{X}_{i,p} \to \tilde{X}_{i,q}$$

and

$$((F_M X)_\rho)_{i,j} = \sum_{l=1}^{n_{ji}} (\chi_l^{ji})_\rho \otimes_k X_{\chi_l^{ji}} : \tilde{X}_{j,p} \to \tilde{X}_{i,q}$$

for $i \neq j$.

Let $f = (f_{m_i})_{i=1,\ldots,r} : X \to X'$ be a morphism. Then we define $F_M f : F_M X \to F_M X'$ by

$$((F_M f)_q)_{i,j} = \begin{cases} \mathrm{id}_{(M_j)_q} \otimes_k f_{m_j} : \tilde{X}_{j,q} \to \tilde{X}'_{i,q} \text{ if } i = j \\ 0 : \tilde{X}_{j,q} \to \tilde{X}'_{i,q} \text{ if } i \neq j \end{cases}.$$

In abuse of notation, we will often skip the M in F_M. Note that F indeed defines a functor because, for a morphism $f : X \to X'$, we have that

$$\begin{aligned}
((FX')_\rho \circ (Ff)_p)_{i,j} &= \sum_{l=1}^{r} ((FX')_\rho)_{i,l} \circ ((Ff)_p)_{l,j} = ((FX')_\rho)_{i,j} \circ ((Ff)_p)_{j,j} \\
&= \left(\sum_{l=1}^{n_{ji}} (\chi_l^{ji})_\rho \otimes_k X'_{\chi_l^{ji}} \right) \circ \mathrm{id}_{(M_j)_p} \otimes_k f_{m_j} \\
&= \mathrm{id}_{(M_i)_q} \otimes_k f_{m_i} \circ \left(\sum_{l=1}^{n_{ji}} (\chi_l^{ji})_\rho \otimes_k X_{\chi_l^{ji}} \right) \\
&= ((Ff)_q \circ (FX)_\rho)_{i,j}
\end{aligned}$$

for $i \neq j$. Since this identity is also true if $i = j$, it follows that $(FX')_\rho \circ (Ff)_p = (Ff)_q \circ (FX)_\rho$ for all $\rho : p \to q$. In summary, every vertex of $Q(M)$ corresponds to a representation of M and every arrow of $Q(M)$ corresponds to a basis element of the group of extensions of the representations corresponding to the tail and the head of the arrow. The representation of $Q(M)$ then describes how to glue the original representations. We should point out that $Q(M)$ is allowed to have oriented cycles.

Remark 3.1.1. In order to construct tree modules, it is necessary that all representations M_i are tree modules and, moreover, to choose tree-shaped Ext-bases as defined in Section 2.3. In this case, every tree module of $Q(M)$ gives rise to a tree module of Q. The construction can be made very explicit in these cases because glueing translates to drawing extra arrows between the corresponding coefficient quivers of the different tree modules.

3.1.2 Schofield induction and generalizations

The most encouraging instance of this functor so far is a generalization of Schofield induction [62]. As a starting point, we fix a sequence of Schur representations $M = (M_1, \ldots, M_r)$ with $\mathrm{Hom}(M_i, M_j) = 0$ for $i \neq j$. In [71], this is called an elementary sequence (of Schur representations). Keeping in mind the notion of general representations introduced by Schofield, we also speak about elementary sequences of Schur roots.

In comparison, Schofield induction deals with exceptional sequences which can be without loss of generality assumed to be reduced, which means that they satisfy $\mathrm{Hom}(M_i, M_j) = 0$ for $i > j$. Thus, the main generalization is that we allow extensions in both directions and, moreover, Schur representations rather than exceptional ones. The first main result of [71] is the following:

Theorem 3.1.2 ([71, Theorem 3.3]). *Let $M = (M_1, \ldots, M_r)$ be an elementary sequence of Schur representations. Then F_M is a fully faithful embedding. In particular, $F_M X$ is indecomposable if and only if X is indecomposable.*

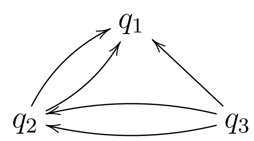
This immediately raises the following question:

Question 3.1.3. *For which roots α of a quiver Q does there exist a non-trivial decomposition into Schur roots $\alpha = \bigoplus_{i=1}^{n} \alpha_i^n$ such that $\hom(\alpha_i, \alpha_j) = 0$ for $i \neq j$?*

By non-trivial, we mean that the decomposition neither consists of the root itself nor consists only of simple roots. There are plenty of examples where this is the case. But unfortunately, a structured approach which tries to answer this question in full generality is missing. Actually, it would be interesting to tackle this problem because the answer to it would be very helpful when dealing with the question of the existence of tree modules, but also for the construction of indecomposable representations in general.

Even for non-Schurian roots of the quiver with three vertices, it is not clear how to approach this problem. In comparison to the canonical decomposition of a root, such a decomposition is also not unique as the following example shows.

Example 3.1.4. Consider the quiver



and the root $\alpha = (1,4,1)$. Then it admits the following decompositions into exceptional roots

$$\alpha = (0,1,0)^4 \oplus (1,0,1), \quad \alpha = (1,2,0) \oplus (0,2,1)$$

yielding quivers (and dimension vectors)

Thus, the functor can be used to construct representations of dimensions $(0,1,0)^m + (1,0,1)^n$ and $(1,2,0)^m + (0,2,1)^n$.

Let us investigate two more cases in which such a decomposition exists.

Generalized Kronecker quivers

We denote the generalized Kronecker quivers by $K(m)$. Moreover, we denote their vertices by $K(m)_0 = \{q_0, q_1\}$ and their arrows by $K(m)_1 = \{\rho_i : q_0 \to q_1 \mid i = 1, \dots, m\}$. Let $(d,e) \in \mathbb{N}K(m)_0$ be a root.

Following the considerations of [23, Section 2.3] which are based on [69, Section 4], every imaginary Schur root decomposes into roots $(d,e) = (d_s, e_s) + l(d', e')$ such that $\hom((d',e'),(d_s + kd', e_s + ke')) = \hom((d_s + kd', e_s + ke'),(d',e')) = 0$ for all $k \geq 0$. Thus, in the respective representation spaces exist open subsets of Schur representations such that we have $\mathrm{Hom}(M,N) = \mathrm{Hom}(N,M) = 0$ for every pair (M,N) of dimensions (d',e') and $(d_s,e_s) + k(d',e')$ respectively such that M and N are contained in these open subsets. By Theorem 3.1.2, this ensures that $F_{(M,N)} : \mathrm{Rep}(Q(M,N)) \to \mathrm{Rep}(Q)$ is a fully faithful embedding. In particular, every indecomposable representation of dimension (a,b) of $Q(M,N)$ gives rise to an indecomposable representation of $K(m)$ of dimension $a(d',e') + b((d_s,e_s) + k(d',e'))$.

We can without loss of generality assume that $k = 0$. In the special case $(a,b) = (1,1)$, we can construct a $(1 - \langle (d,e),\, (d,e) \rangle)$-parameter family of isomorphism classes of indecomposable representations with this method. This is actually true because (d',e') and (d_s,e_s) are Schur roots satisfying $\hom((d',e'),(d_s,e_s)) = \hom((d_s,e_s),(d',e')) = 0$. Thus, there exist open subsets U_1 of $R_{(d',e')}(K(m))$ of dimension $md'e'$ and U_2 of $R_{(d_s,e_s)}(K(m))$ of dimension md_se_s respectively such that $\mathrm{Hom}(M,N) = \mathrm{Hom}(N,M) = 0$ for all pairs $(M,N) \in U_1 \times U_2$. Moreover, for every such pair there exist open subsets V_1 of $\bigoplus_{i=1}^m \mathrm{Hom}(M_{q_0}, N_{q_1})$ and V_2 of $\bigoplus_{i=1}^m \mathrm{Hom}(N_{q_0}, M_{q_1})$ such that every pair $(e_1,e_2) \in V_1 \times V_2$ gives rise to a pair of short exact sequences $(\overline{e_1}, \overline{e_2}) \in \mathrm{Ext}(M,N) \times \mathrm{Ext}(N,M)$ and thus to a representation X of $Q(M,N)$ such that $F_{(M,N)}X$ is Schurian. This is because $(1,1)$ is a Schur root of $Q(M,N)$. Thus, without taking care of isomorphisms between the constructed representations, we have constructed a mde-parameter family of Schurian (!) representations contained in $R_{(d,e)}(K(m))$. Indeed, we have $mde = md'e' + md_se_s + md'e_s + md_se'$. In turn, this yields a $(1 - \langle (d,e),\, (d,e) \rangle)$-parameter family of isomorphism classes of Schur representations of dimension (d,e). Note that we have

$$1 - \langle (d,e),\, (d,e) \rangle = 1 - \langle (d_s,e_s),\, (d_s,e_s) \rangle + 1 - \langle (d',e'),\, (d',e') \rangle + (1 - \langle (1,1),\, (1,1) \rangle_{Q(M)}.$$

Let $m \geq 3$. Then the easiest example is the case of the dimension vectors $(d_s, e_s) = (1,2)$ and $(d', e') = (1,1)$. Then we have

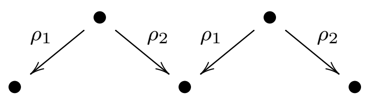
$$\mathrm{ext}((1,1),(1,2)) = 2m-3, \quad \mathrm{ext}((1,2),(1,1)) = m-3.$$

All indecomposable tree modules are of the form

If the three arrows m_{i_1}, m_{i_2} and m_j correspond to three different arrows of $K(m)$, it is ensured that the homomorphism spaces vanish. In this case, a tree-shaped Ext-basis of $\mathrm{Ext}(M_{(1,2)}, M_{(1,1)})$ is induced by the arrows $K(m)_1 \backslash \{m_{i_1}, m_{i_2}, m_j\}$. A tree-shaped basis of $\mathrm{Ext}(M_{(1,1)}, M_{(1,2)})$ is given by the maps connecting the source on the right hand side to one of the two sinks on the left hand side via an arrow $K(m)\backslash\{m_j\}$ additionally excluding one of the maps corresponding to m_{i_k}.

Extended Dynkin quivers

The case of $\tilde{A}_n$ is very similar to the case of the quiver $K(2)$. In this case, the real roots and the corresponding representations can be explicitly determined in terms of coefficient quivers. For instance, the real root representation of dimension $(2,3)$ is given by



All other real root representations of dimension $(d, d+1)$ are of the same shape, those of dimension $(d+1, d)$ are obtained by turning around all arrows in the coefficient quiver. The classification of the indecomposable representations of the imaginary roots (d,d) can be obtained analogously to the classification of endomorphisms of k^d or of indecomposable representations of the one-loop quiver.

Let us concentrate on $\tilde{D}_n$ where the unique imaginary Schur root is denoted by δ. For the defect of preprojective roots α, we have $\delta(\alpha) := \langle \delta, \alpha \rangle \in \{-1,-2\}$. For simplicity, we assume that $n = 4$ and that $\tilde{D}_n$ is in subspace orientation with vertices $q_0, \ldots, q_4$. By slight modifications, one can straightforwardly generalize the following observations to general n. Up to permutation, the roots with $\delta(\alpha) = -1$ are of the form $\alpha = (2n+1, n+1, n, n, n)$ and those with $\delta(\alpha) = -2$ are $(2n+1, n, n, n, n)$. Thus, roots with $\delta(\alpha) = -1$ admit a decomposition of the form $\alpha = \beta^{n+1} + \gamma^n$ where $\beta = (1,1,0,0,0)$, $\gamma = (1,0,1,1,1)$, $\mathrm{ext}(\gamma, \beta) = 2$ and $\mathrm{ext}(\beta, \gamma) = 0$. Furthermore, it is easy to check that the roots with $\delta(\alpha) = -2$ admit a decomposition into two smaller preprojective roots β and $\tau^{-1}\beta$ because the corresponding representation appears as the single middle term of the respective Auslander-Reiten sequence. In particular, we have $\tau^{-1}\beta \in \beta^{\perp}$ and $\mathrm{ext}(\tau^{-1}\beta, \beta) = 1$.

Also the indecomposable representations in the tubes can be obtained using the functor. For two quasi-simple representations M and N from one exceptional tube, say of dimensions $(1,1,1,0,0)$ and $(1,0,0,1,1)$, we have $\mathrm{Ext}(M,N) = \mathrm{Ext}(N,M) = k$ and $\mathrm{Hom}(M,N) = \mathrm{Hom}(N,M) = 0$. It is easy to see that every indecomposable representation in the tube containing M and N can be obtained by glueing M and N. But it is also straightforward to check that we obtain all representations of the homogeneous tubes by glueing two quasi-simples of this shape.

Thus, since the preinjective and the preprojective representations are dual to each other, it turns out that representations of $\tilde{D}_n$ are easy to describe using this method. Moreover, this strategy is indeed very useful when constructing explicit coefficient quivers as needed in Chapter 5.

Even if the case of quivers of extended Dynkin type $\tilde{E}_n$ for $n = 6, 7, 8$ was not studied in detail, it seems likely that every root admits a decomposition into exceptional roots such that the corresponding sequence of roots is either elementary or even reduced.

Imaginary Schur roots

The case of imaginary Schur roots is studied in [68] using a slightly different language. But also the construction introduced there fits into this setup. Actually, the main aim of [68] was to construct indecomposable tree modules for every imaginary Schur root. A particular investigation shows that we can actually construct a $(1 - \langle\alpha, \alpha\rangle)$-parameter family of isomorphism classes of indecomposable representations using this method. This is insofar very interesting as this is, by Kac's Theorem [35, Theorem C], the number of parameters describing the isomorphism classes of indecomposable representations.

The key observation is that there is a positive answer to Question 3.1.3 in the case of imaginary Schur roots. This is actually the case because the canonical decomposition of Schur roots only consists of the root itself. More precisely, based on the algorithm of Derksen and Weyman established in [18, Section 4], we obtain the following:

Proposition 3.1.5 ([68, Proposition 3.15]). *Let α be a Schur root. Then at least one the following cases holds:*

i) There exist a real Schur root β and a real or isotropic Schur root γ and $d, e \in \mathbb{N}_+$ such that $\alpha = \beta^d + \gamma^e$. Moreover, we have $\beta \in \gamma^\perp$ and $\mathrm{hom}(\beta,\gamma) = 0$ or $\beta \in {}^\perp\gamma$ and $\mathrm{hom}(\gamma,\beta) = 0$ and (d,e) is a root of $K(\mathrm{ext}(\beta,\gamma))$ or $K(\mathrm{ext}(\gamma,\beta))$.

ii) There exist a real Schur root β, an imaginary Schur root γ and $t \leq \mathrm{ext}(\beta,\gamma) + \mathrm{ext}(\gamma,\beta)$ such that $\alpha = \beta + t\gamma$. Moreover, we have $\beta \in \gamma^\perp$ and $\mathrm{hom}(\beta,\gamma) = 0$ or $\beta \in {}^\perp\gamma$ and $\mathrm{hom}(\gamma,\beta) = 0$.

iii) There exist two imaginary Schur roots β and γ such that $\alpha = \beta + \gamma$. Moreover, we have $\gamma \in \beta^\perp$ and $\mathrm{hom}(\gamma,\beta) = 0$.

In the next section, we will use this proposition to construct indecomposable tree modules for every Schur root. While doing the respective considerations to construct the mentioned $(1 - \langle \alpha, \alpha \rangle)$-parameter family of indecomposable representations, we should keep in mind the snake lemma. More detailed, we should keep in mind the following lemma induced by it:

Lemma 3.1.6. *Assume that* $0 \to L \to M \to N \to 0$ *and* $0 \to L' \to M' \to N' \to 0$ *are two short exact sequences with* $\underline{\dim}\, L = \underline{\dim}\, L'$ *and* $\underline{\dim}\, N = \underline{\dim}\, N'$. *Then* M *and* M' *are isomorphic if and only if both* L *and* L', *and* N *and* N' *are isomorphic.*

Now we are ready to prove the following:

Theorem 3.1.7. *For every Schur root* α, *there exists a* $(1 - \langle \alpha, \alpha \rangle)$*-parameter family* I_α *of isomorphism classes of indecomposable representations of dimension* α *and a non-trivial decomposition* $\alpha = \beta^d + \gamma^e$ *into Schur roots such that*

- $\gamma \in \beta^\perp$ *and* $\hom(\gamma, \beta) = 0$;
- (d, e) *is a root of* $K(\mathrm{ext}(\gamma, \beta))$;
- *For every representation* $M \in I_\alpha$, *there exist two Schur representations* $L \in R_\beta(Q)$, $N \in R_\gamma(Q) \cap L^\perp$ *and a representation* $X \in R_{(d,e)}(Q(L, N))$ *such that* $M \cong F_{(L,N)} X$.

Proof. First note that we have $Q(L, N) = K(\mathrm{ext}(\gamma, \beta))$. We consider the three different cases of Proposition 3.1.5. In the first case we have

$$\langle \alpha, \alpha \rangle = \langle \beta^d + \gamma^e, \beta^d + \gamma^e \rangle = d^2 + e^2 - \mathrm{ext}(\gamma, \beta) de = \langle (d, e), (d, e) \rangle$$

if β and γ are real. Thereby, we have to keep in mind that the Euler form depends on the quiver. Since F is a fully faithful embedding in this case, we can construct a $(1 - \langle (d, e), (d, e) \rangle)$-parameter family of indecomposable representations. If one of the roots β and γ is an isotropic Schur root, we can combine these observations with those we make in the following.

In the case $\alpha = \beta + \gamma^l$ decomposes into an imaginary Schur root and a multiple of an exceptional root, every tuple consisting of a Schur representation of $K(\mathrm{ext}(\gamma, \beta))$ of dimension $(1, l)$ and a Schur representation $L \in {}^\perp\gamma$ gives rise to a Schur representation of dimension α. Since there exists a $(1 - \langle \beta, \beta \rangle)$-parameter family of Schur representations in ${}^\perp\gamma$, keeping in mind Lemma 3.1.6, we can construct a family which can be described by

$$1 - \langle \beta, \beta \rangle + 1 - \langle (1, l), (1, l) \rangle = 1 - \langle \alpha, \alpha \rangle$$

parameters with these methods.

In the case α decomposes into two imaginary Schur roots of dimension β and γ, we have to keep in mind that $\gamma \in \beta^\perp$ means that there exist open subsets I_β and I_γ of Schur representations in $R_\beta(Q)$ and $R_\gamma(Q)$ respectively such that $\hom(\gamma, \beta) = 0$. Thus, every triple consisting of an indecomposable representation of $K(\mathrm{ext}(\gamma, \beta))$ and Schur representations contained in I_β and I_γ gives rise to a Schur representation of dimension α. Therefore, the claim follows analogously to the last one because

$$\langle \alpha, \alpha \rangle = \langle \beta, \beta \rangle + \langle \gamma, \gamma \rangle + \langle \gamma, \beta \rangle$$

and $\mathrm{ext}(\gamma, \beta) = -\langle \gamma, \beta \rangle$. □

3.1.3 Ringel's reflection functor and generalizations

We can also translate the first two instances of Ringel's reflection functor into the introduced language. Following [71, Section 3.2], we fix a sequence $M = (M_1, \dots, M_r)$ of indecomposable representations such that the following conditions are satisfied:

i) The representations M_j, $j \geq 2$, are Schurian;

ii) We have $\mathrm{Hom}(M_i, M_j) = \mathrm{Ext}(M_i, M_j) = 0$ for $i < j$;

iii) If $i, j \neq 1$, we have $\mathrm{Hom}(M_j, M_i) = 0$ for $i < j$.

This means that we allow homomorphisms $M_j \to M_1$ for $j > 1$. In comparison to the last section, the second condition assures that $Q(M)$ has no oriented cycles. For a representation X of $Q(M)$, we denote by X_2 the corresponding representation of the full subquiver $Q(M)\backslash\{m_1\}$. Moreover, we denote by S_iX the semisimple representation

$$(S_iX)_{m_j} = \begin{cases} X_{m_i} \text{ if } j = i \\ 0 \text{ otherwise} \end{cases}.$$

Then we have $FS_iX = M_i^{\dim X_{m_i}}$. Let

$$\Theta^{ji}_{X,X'} : \bigoplus_{l=1}^{n_{ji}} \mathrm{Hom}_k(S_jX, S_iX') \cong \mathrm{Ext}(M_j^{\dim X_{m_j}}, M_i^{\dim X'_{m_i}})$$

denote the natural isomorphisms induced by $(\phi_l)_l \mapsto (\sum_{l=1}^{n_{ji}} (\chi_l^{ji})_\rho \otimes_k \phi_l)_{\rho \in Q_1} = F((\phi_l)_l)$ if $i \neq j$. It is straightforward to check that Θ^{ji} is functorial in X and X'. Define $\Theta^1_X := (\Theta^{i,1}_{X,X})_{i=2,\dots,r}$. The following statement is the second main result of [71]:

Theorem 3.1.8 ([71, Theorem 3.8]). *Let M be a sequence of indecomposable representations satisfying conditions i)-iii). If X is a representation of $Q(M)$ such that $\dim X_{m_1} = 1$ and such that Θ^1_X induces an isomorphism*

$$\bigoplus_{i=2}^{r} \mathrm{Ext}(M_i^{\dim X_{m_i}}, M_1) \cong \mathrm{Ext}(FX_2, M_1),$$

we have that F_MX is indecomposable whenever X is indecomposable.

The rather technical condition that Θ^1_X induces the mentioned isomorphism is very important. This is because otherwise we could have

$$\dim \mathrm{Ext}(FX_2, M_1) < \dim \mathrm{Ext}(\bigoplus_{i=2}^{r} M_i^{\dim X_{m_i}}, M_1)$$

which means that FX might be decomposable. This condition is for instance satisfied if $\mathrm{Hom}(M_i, M_1) = 0$ for all $i = 2, \dots, r$ or if we have a decomposition of $\{2, \dots, r\}$ into disjoint sets I_1 and I_2 such that $\mathrm{Hom}(M_i, M_1) = 0$ for $i \in I_2$ and $\mathrm{Ext}(M_i, M_j) = 0$ for $i \in I_1$ or $j \in I_1$, $i \neq j$ and $i, j \geq 2$. This is clearly satisfied if $r = 2$ or if $\mathrm{Ext}(M_i, M_j) = 0$ for $i \neq j$ and $i, j \geq 2$.

Remark 3.1.9. Note that there is a dual statement when considering sequences $M = (M_1, \dots, M_r)$ of indecomposable representations satisfying the following conditions:

i) The representations M_j, $j \leq r-1$, are Schurian;

ii) We have $\mathrm{Hom}(M_i, M_j) = \mathrm{Ext}(M_i, M_j) = 0$ for $i < j$;

iii) If $i, j \neq r$, we have $\mathrm{Hom}(M_j, M_i) = 0$ for $i < j$.

Thus, we allow homomorphisms $M_r \to M_i$ for $i \in \{1, \dots, r-1\}$. The proof in this case is obtained by a slight modification of the arguments or when considering the opposite quiver obtained when turning around all arrows.

Since this construction includes the first two instances of Ringel's reflection functor, this also serves as our first example. For a fixed representation S, every representation $M \in \mathcal{M}^{-S}$ (resp. $M \in \mathcal{M}_{-S}$) gives rise to a sequence of representations (M, S) (resp. (S, M)) satisfying the three required conditions. Thus, every representation X of $K(\dim \mathrm{Ext}(S, M))$ of dimension $(1, l)$ gives rise to a representation FX of the original quiver of dimension $\underline{\dim}\, M + l \cdot \underline{\dim}\, S$. In this special case, the representation FX can be obtained as the middle term of a short exact sequence

$$0 \to M \to FX \to S^l \to 0.$$

If $l = \dim \mathrm{Ext}(S, M)$ and if X is the respective real root representation, using the notation of Section 2.2, we have $FX \cong M^S$. If $l \leq \dim \mathrm{Ext}(S, M)$, we obtain another proof of the following result:

Lemma 3.1.10 ([68, Lemma 3.12])**.** *Let M and S be indecomposable representations such that* $\mathrm{Ext}(S, S) = 0$*. Let $e_1, \dots, e_n$ be a basis of* $\mathrm{Ext}(S, M)$*. Consider the exact sequence induced by this basis:*

$$0 \to M \to M^S \to \bigoplus_{i=1}^{n} S \to 0.$$

Moreover, consider

$$0 \to M \to N^S \to \bigoplus_{i=1}^{l} S \to 0$$

induced by $e_1, \dots, e_l$. If M^S is indecomposable, then N^S is indecomposable.

The dual statement of this lemma also holds.

In terms of the quiver $K(\dim \mathrm{Ext}(S, M))$, the representation N^S just corresponds to an indecomposable factor representation of dimension $(1, l)$ of the real root representation.

Also the combination of both, the functor $\Phi^{-1} : \mathcal{M}^{-S}_{-S} \to \mathcal{M}^S_S / S$, can be obtained in this way. To see this, let M be a representation of Q and S an exceptional representation such that $\mathrm{Hom}(M,S) = \mathrm{Hom}(S,M) = 0$, $\dim \mathrm{Ext}(M,S) = n_1$ and $\dim \mathrm{Ext}(S,M) = n_2$. Then every indecomposable representation of

$$n_2 \underset{\rho_{n_2}}{\overset{\rho_1}{\underset{\vdots}{\rightrightarrows}}} 1$$

with dimension vector as indicated yields an indecomposable representation of dimension $\underline{\dim}\, M + n_2 \cdot \underline{\dim}\, S$. In particular, we can construct the indecomposable representation obtained in Theorem 2.2.3 in this way, i.e. the middle term of the short exact sequence

$$0 \to M \to M^S \to \bigoplus_{i=1}^{n_2} S \to 0.$$

Now we have $\mathrm{Ext}(S, M^S) = \mathrm{Hom}(S, M^S) = 0$ and since we have $\mathrm{Hom}(M,S) \cong \mathrm{Ext}(M^S, S)$, every indecomposable representation of the quiver

$$1 \underset{\rho_{n_1}}{\overset{\rho_1}{\underset{\vdots}{\rightrightarrows}}} n_1$$

gives rise to an indecomposable representation of Q of dimension $\underline{\dim}\, M^S + n_1 \cdot \underline{\dim}\, S$. In particular, we can construct the representation M_S^S from Theorem 2.2.3.

3.2 Existence and number of indecomposable tree modules

In this section, we concentrate on indecomposable tree modules. To do so, we consider the methods introduced in the last section in this special case. More precisely, we fix a sequence $M = (M_1, \ldots, M_r)$ of tree modules of a fixed quiver Q with coefficient quivers Γ_{M_i}. Moreover, for every pair (M_i, M_j) with $i \neq j$, we fix a tree-shaped basis of $\mathrm{Ext}(M_i, M_j)$. This means that we assume that every basis element corresponds to a triple (p, q, ρ) where $p \in (\Gamma_{M_i})_0$, $q \in (\Gamma_{M_j})_0$ and $\rho \in Q_1$. Then the following statement is straightforward:

Lemma 3.2.1. *If $X \in \mathrm{Rep}(Q(M))$ is a tree module, $F_M X$ is also a tree module.*

This suggests to start a recursion on quivers which can be used to answer the question of the existence of tree modules. Even if not all roots are covered by it, a rather huge class can. We use the following two results in order to get the recursion started:

Theorem 3.2.2 ([58],[67, Theorem 3.9])**.** *We have:*

i) Every exceptional representation is a tree module.

ii) For every root of the Kronecker quiver, there exists an indecomposable tree module.

Thus, additionally using Proposition 3.1.5 and the considerations from Section 3.1.1, we obtain the main result of [68]:

Theorem 3.2.3 ([68, Theorem 3.18])**.** *Let α be an imaginary Schur root. Then there exists more than one isomorphism class of indecomposable tree modules of dimension α.*

It is worth mentioning that the introduced methods give a very explicit recursive construction of tree modules. In particular, having constructed the exceptional tree modules of the generalized Kronecker quivers explicitly, they can be used to construct the exceptional representations of arbitrary quivers.

It is natural to ask for the number of indecomposable tree modules. In the case of imaginary Schur roots, one can show that the number of indecomposable tree modules grows exponentially with the dimension vector using the introduced methods. This is our next topic.

3.2.1 Exponential growth of the number of tree modules

In this section, we show that the number of indecomposable tree modules for imaginary Schur roots grows exponentially with the dimension vector. This can be used to show that the Kac polynomial at one grows exponentially in the case of the Kronecker quiver.

Recall that a tree module of Q is already a representation of the universal covering quiver $\tilde{Q}$ of Q. Following [60], we call a tree module of Q cover-thin if its dimension vector is of type one as a representation of $\tilde{Q}$.

Lemma 3.2.4. *Every indecomposable tree module M which is cover-thin is exceptional as a representation of the universal covering quiver. In particular, we have that $\sigma_q M$ is also an indecomposable tree module for any sink (resp. source) $q \in Q_0$.*

Proof. The first part follows because M is Schurian and because of $\langle \underline{\dim} M, \underline{\dim} M \rangle = \dim_k \operatorname{Hom}(M, M) - \dim_k \operatorname{Ext}(M, M) = 1$. In particular, $\sigma_q M$ is also exceptional as a representation of the universal covering quiver and thus a tree module by the second part of Theorem 3.2.2. □

Let $Q = K(m)$. Note that, for roots (d, e) with $d \leq e$, there only exist cover-thin tree modules if $e \leq (m-1)d + 1$. The next aim is to count them. This can be done exactly as, up to translation, the indecomposable cover-thin tree modules are exactly the connected subtrees of $\tilde{Q}$ with d sources and e sinks. Or equivalently, they can be described as certain colored spanning trees of the full bipartite graph with d sources and e sinks. Recall that spanning trees are by definition connected. We make use of the multivariate Lagrangian Inversion formula, see [26] for a combinatorial proof and a collection of references:

Theorem 3.2.5. *Let $h, g_1, \ldots, g_d \in k[\![x_1, \ldots, x_d]\!]$ such that every g_i has a non-zero constant term. Then there exist unique formal power series $f_i \in k[\![x_1, \ldots, x_d]\!]$ such that*

$$f_i(\mathbf{x}) = x_i g_i(\mathbf{f})$$

for $i = 1, \ldots, d$. We have

$$[\mathbf{x^n}]h(\mathbf{f}) = [\mathbf{t^n}]h(\mathbf{t})\mathbf{g}(\mathbf{t})^{\mathbf{n}} \det\left(\delta_{i,j} - \frac{t_i}{g_i(\mathbf{t})}\frac{\partial g_i(\mathbf{t})}{\partial t_j}\right).$$

Now we use this result to prove the following:

Proposition 3.2.6. *Let $n := m - 1$. The number of indecomposable cover-thin tree modules of $K(m)$ which are of dimension (d, e) is*

$$\frac{1}{d}\sum_{i=1}^{m}\binom{m}{i}\binom{ne}{d-1}\binom{n(d-1)}{e-i}\frac{i}{e}.$$

Proof. Let $t_{d,e}$ be the number of indecomposable cover-thin tree modules of dimension (d, e) and let

$$T(x_1, x_2) = \sum_{(d,e)} t_{(d,e)} x_1^d x_2^e$$

be the generating function. All connected subtrees with d sources and e sinks of $\tilde{Q}_0$ can be obtained recursively by glueing subquivers of type $(1, i)$ for some $i = 1, \ldots, m$ which are called simple (of type i) in what follows. In turn, every such subtree has d simple subtrees. Thus, we have d choices to fix one of them. Now the whole tree module is obtained recursively by glueing up to n simple subquivers to each sink of a simple subquiver. In turn, the number of possibilities to glue k_i simple subquivers of type i for $i = 1, \ldots, m$ to a fixed sink is

$$z_{\underline{k}} := \prod_{i=1}^{m}\binom{n - \sum_{j=1}^{i-1} k_j}{k_i}\prod_{i=1}^{m}\binom{n}{i-1}^{k_i}.$$

In terms of generating functions, the simple subquivers of type i correspond to the monomials $x_1 x_2^i$. The following identities hold

$$\sum_{\substack{\sum k_i = a\\ \sum (i-1)k_i = b}} z_{\underline{k}} = \binom{n}{a}\binom{na}{b} = [x_1^a x_2^b](1 + x_1(1 + x_2)^n)^n$$

where the first one can be seen as follows: in order to glue a subquiver with a sources and b sinks to a fixed sink, we glue n subquivers of type $(1, n)$ to this fixed sink in a first step. Taking colorings into account, we have exactly one possibility to do this. In a second step, we obtain a subquiver with a sources and b sinks when choosing a out of the n glued subquivers and afterwards b out of the remaining na sinks. The second identity follows from the subsequent considerations.

Let $z(x_1, x_2) = 1+x_1(1+x_2)^n$, $g_1(x_1, x_2) = 1$, $g_2(x_1, x_2) = z(x_1, x_2)^n$ and $h_i(x_1, x_2) = x_2^i$. Let f_2 be the generating function counting the number of subtrees which can be glued to a fixed sink. Then f_2 satisfies the function equation

$$f_2(x_1, x_2) = x_2 g_2(x_1, f_2(x_1, x_2)).$$

Note that $f_2^i(x_1, x_2)$ counts the number of subtrees which can be glued to i fixed sinks. We have

$$\begin{aligned}
[x_1^a x_2^b] z(x_1, x_2)^{ne} &= [x_1^a x_2^b] \sum_{i=0}^{ne} \binom{ne}{i} (x_1(1+x_2)^n)^i \\
&= [x_1^a x_2^b] \sum_{i=0}^{ne} \binom{ne}{i} x_1^i \sum_{j=0}^{ni} \binom{ni}{j} x_2^j \\
&= \binom{ne}{a}\binom{na}{b}.
\end{aligned}$$

Using

$$(1+x_2)^{-1} = \sum_{i=0}^{\infty} (-1)^i x_2^i,$$

we can similarly show that

$$[x_1^a x_2^b] z(x_1, x_2)^{ne} (1+x_2)^{-1} = \binom{ne}{a} \sum_{j=0}^{e} (-1)^j \binom{na}{e-j}.$$

Thus, by Theorem 3.2.5, we have

$$\begin{aligned}
[t_1^d t_2^e] f_2^i(t_1, t_2) &= [x_1^d x_2^e] x_2^i g_2^e \left(1 - \frac{x_2}{g_2} \frac{\partial g_2(\mathbf{x})}{\partial x_2}\right) \\
&= [x_1^d x_2^e] x_2^i z^{ne} - [x_1^d x_2^e] x_1 x_2^{i+1} n^2 (1+x_2)^{n-1} z^{ne-1} \\
&= [x_1^d x_2^{e-i}] z^{ne} - [x_1^{d-1} x_2^{e-(i+1)}] n^2 (1+x_2)^{n-1} z^{ne-1} \\
&= \binom{ne}{d}\binom{nd}{e-i} - n^2 \binom{ne-1}{d-1} \sum_{j=0}^{e-(i+1)} (-1)^j \binom{nd}{e-(i+1)-j} \\
&= \binom{ne}{d}\binom{nd}{e-i} - n^2 \binom{ne-1}{d-1}\binom{nd-1}{e-(i+1)} \\
&= \frac{(ne)!(nd)! - n^2 (ne-1)!(nd-1)! d(e-i)}{(ne-d)!(nd-(e-i))! d! (e-i)!} \\
&= \frac{n^2 (ne-1)!(nd-1)! id}{(ne-d)!(nd-(e-i))! d! (e-i)!} \\
&= \binom{ne}{d}\binom{nd}{e-i} \frac{i}{e}.
\end{aligned}$$

Taking into account that there are $\binom{m}{i}$ simple subtrees of type i and that every subtree of dimension (d, e) has d sources, we have

$$t_{(d,e)} = \frac{1}{d} \sum_{i=1}^{m} \binom{m}{i} [x_1^{d-1} x_2^e] f_2^i(x_1, x_2)$$

$$= \frac{1}{d}\sum_{i=1}^{m}\binom{m}{i}\binom{ne}{d-1}\binom{n(d-1)}{e-i}\frac{i}{e}.$$

□

Applying Lemma 3.2.4, we thus obtain:

Corollary 3.2.7. *The number of indecomposable tree modules of $K(m)$ grows (at least) exponentially with the dimension vector, i.e. for every imaginary Schur root (d,e) of $K(m)$ there exists a real number $K_{d,e} > 1$ such that $t_{n(d,e)} > K_{d,e}^n$.*

For a fixed dimension vector α of any quiver Q, we denote by T_α the number of indecomposable tree modules of dimension α. Now we are ready to prove the following result:

Theorem 3.2.8. *If α is an imaginary Schur root, then the number of indecomposable tree modules grows exponentially with the dimension vector. More precisely, for every imaginary Schur root α, there exists a real number $K_\alpha > 1$ such that $T_{n\cdot\alpha} > K_\alpha^n$.*

Proof. Since α is an imaginary Schur root, by Proposition 3.1.5 and up to ordering, we have that it decomposes in one of the following ways:

i) $\alpha = \beta^d + \gamma^e$ and β, γ are exceptional or isotropic Schur roots such that $\gamma \in \beta^\perp$ and (d,e) is an imaginary Schur root of $K(\mathrm{ext}(\gamma,\beta))$;

ii) $\alpha = \beta + \gamma^e$, β is an imaginary Schur root and γ exceptional. Moreover, we have $\beta \in {}^\perp\gamma$;

iii) $\alpha = \beta + \gamma$ and β, γ are imaginary Schur roots such that $\gamma \in \beta^\perp$.

The first case is a consequence of Corollary 3.2.7. In the second case, we can actually construct tree modules in two different ways. Firstly, every indecomposable tree module M of dimension $n\beta$ such that $M \in {}^\perp\gamma$ gives rise to short exact sequences of the form $0 \to M \to N \to M_\gamma^{ne} \to 0$ where $\underline{\dim}\, N = n\alpha$. Now we can argue by induction that the number of indecomposable tree modules of dimension $n\beta$ grows exponentially with n. Thus, the same is true for the number of indecomposable tree modules of dimension $n\alpha$. Alternatively, we can fix an indecomposable tree module M of dimension β. Then we can consider representations of the form $N = F_{(M,M_\gamma)}(X)$ where $X \in R_{(n,nl)}(K(\mathrm{ext}(\gamma,\beta)))$. Clearly, N is a tree module if X is a tree module. Now we can again use Corollary 3.2.7.

In the last case, the claim follows because we already know that $T_{n\beta}$ and $T_{n\gamma}$ grow exponentially. □

Example 3.2.9. If $m = 3$ and $(d,e) = (d, d+1)$, it is straightforward to check that we have

$$t_{(d,d+1)} = \frac{3}{(d+2)(d+3)}\binom{2d}{d}\binom{2(d+1)}{d+1}.$$

The respective sequence of natural numbers appears as sequence A186266 in [49]. It seems that there was no combinatorial interpretation of this sequence before.

3.2.2 Kac polynomial and tree modules

The considerations of Section 3.2.1 have a consequence for the Kac polynomials evaluated at one which we investigate in the case of the Kronecker quiver. As already mentioned, if α is coprime, by [15], we have

$$a_\alpha(q) = \sum_{i=0}^{d} \dim H^{2d-2i}(M_\alpha^{0,\Theta-\mathrm{st}}(Q), \mathbb{C}) q^i$$

where we consider singular cohomology and where $d = \frac{1}{2}\dim M_\alpha^{0,\Theta-\mathrm{st}}(Q)$. Since $M_\alpha^{0,\Theta-\mathrm{st}}(Q)$ is cohomological pure, the existence of a polynomial with integer coefficients which counts the rational points yields that the odd cohomology vanishes, see [15, Appendix A]. In particular, we obtain $a_\alpha(1) = \chi(M_\alpha^{0,\Theta-\mathrm{st}}(Q))$. By a well-known result, we have $\chi(X) = \chi(X^T)$ for any complex variety with a torus action, see for instance [9, Section 2.5]. Here X^T denotes the fixed point set. It is straightforward to transfer the results of [69] to the case of the moduli spaces $M_\alpha^{0,\Theta-\mathrm{st}}(Q)$. This enables us to understand the corresponding fixed point components as moduli spaces attached to the universal abelian covering of Q. More precisely, let $T := (\mathbb{C}^*)^{|Q_1|}$ act on $\mathrm{Rep}(\Pi^0(Q), \alpha)$ by

$$(t_\rho)_\rho * (M_\rho, M_{\rho^*})_{\rho\in Q_1} = (t_\rho M_\rho, t_\rho^{-1} M_{\rho^*}).$$

This indeed defines an action on $\mathrm{Rep}(\Pi^0(Q), \alpha)$ which commutes with the usual base change action of $\prod_{q\in Q_0} \mathrm{Gl}_{\alpha_q}(\mathbb{C})$. Now the same proofs as those of [69, Section 3] apply to show the following:

Theorem 3.2.10. *The set of torus fixed points $M_\alpha^{0,\Theta-\mathrm{st}}(Q)^T$ is isomorphic to the disjoint union of moduli spaces*

$$\bigcup_{\hat\alpha} M_{\hat\alpha}^{0,\hat\Theta-\mathrm{st}}(\hat Q)$$

where $\hat\alpha$ ranges over all equivalence classes of dimension vectors compatible with α.

Here $\hat Q$ denotes the universal abelian covering quiver as defined in [69, Section 3.1]. Note that the moduli spaces $M_{\hat\alpha}^{0,\hat\Theta-\mathrm{st}}(\hat Q)$ consist of representations of $\Pi^0(\hat Q)$ which are stable with respect to the slope function induced by $\hat\Theta$ where $\hat\Theta_{(q,\chi)} = \Theta_q$ for all $q \in Q_0$, $\chi \in \mathbb{Z}^{|Q_0|}$.

Analogously to the case of acyclic quivers and their moduli spaces, the fixed points remaining after iterated localization are stable representations of $\Pi^0(\tilde Q)$. The main advantage is that the connected components of $\tilde Q$ are trees which makes the counting of fixed points very easy in many cases. This has the following consequence for the Euler characteristic in singular cohomology and the Kac polynomial at one:

Corollary 3.2.11. *For a coprime dimension vector α, we have*

$$a_\alpha(1) = \chi(M_\alpha^{0,\Theta-\mathrm{st}}(Q)) = \chi(M_\alpha^{0,\Theta-\mathrm{st}}(Q)^T) = \sum_{\tilde\alpha} \chi(M_{\tilde\alpha}^{0,\tilde\Theta-\mathrm{st}}(\tilde Q)) = \sum_{\tilde\alpha} a_{\tilde\alpha}(1)$$

where $\tilde\alpha$ ranges over all equivalence classes which are compatible with α.

Remark 3.2.12. In an unpublished note, Ryan Kinser sketched a proof, which he worked out with Harm Derksen, showing that Corollary 3.2.11 is true for arbitrary dimension vectors. Actually, their proof uses completely different methods as preprojective algebras and the corresponding moduli spaces do not play any role.

This natural generalization of considering torus fixed of the moduli spaces $M_{\alpha}^{0,\Theta-\mathrm{st}}(Q)$ has plenty of interesting consequences which are closely related to several results of this paper. Some of them are due to the fact that we have $\chi(M_{\alpha}^{0,\Theta-\mathrm{st}}(Q)) = 1$ if α is a real root or if the moduli space is a point.

Corollary 3.2.13. *Let α be a coprime dimension vector such that all equivalence classes of compatible dimension vectors consist of exceptional roots. Then the number of indecomposable tree modules is equal to the Kac polynomial at one.*

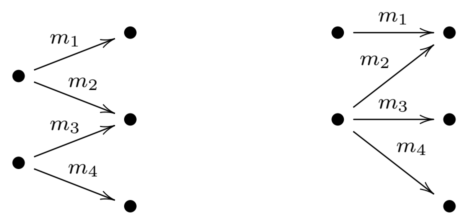
Proof. By the main result of [58], every exceptional representation is an indecomposable tree module and thus a representation of the universal covering quiver, say of dimension $\tilde{\alpha}$. For the Kac polynomial of a real root $\tilde{\alpha}$, we have $a_{\tilde{\alpha}}(q) = a_{\tilde{\alpha}}(1) = 1$.

The other way around, every stable representation of $M_{\tilde{\alpha}}^{0,\tilde{\Theta}-\mathrm{st}}(\tilde{Q})$ yields a dimension vector $\tilde{\alpha}$ of $\tilde{Q}$ which is compatible with α. Since $\tilde{\alpha}$ is exceptional by assumption, it yields a tree module of dimension α. □

Example 3.2.14. Consider the generalized Kronecker quiver $K(3)$ and the dimension vector $(2,3)$. By use of Hua's formula, we obtain

$$a_{(2,3)} = q^6 + q^5 + 3q^4 + 4q^3 + 5q^2 + 3q + 2$$

and thus $a_{(2,3)}(1) = 19$, see [33, Section 5]. There are 18 cover-thin tree modules of $K(3)$ of dimension $(2,3)$ which are given by



Here the arrows $m_i \in \{\rho_1, \rho_2, \rho_3\}$ satisfy the conditions $m_1 \neq m_2 \neq m_3 \neq m_4$ in the first case and the conditions $m_1 \neq m_2$ and m_2, m_3, m_4 pairwise disjoint in the second case. Finally, there is one tree module which is not cover-thin and whose coefficient quiver is the one on the left hand side in the case when $m_2 = m_3$ and m_1, m_2, m_4 are pairwise disjoint. Note that, as a representation of $\widetilde{K(3)}$, this is just the real root representation of D_4 of dimension $(2,1,1,1)$. Thus, the number of indecomposable tree modules is 19.

We also re-obtain the following result:

Corollary 3.2.15 ([47, Corollary 4.4]). *If α is a dimension vector of Q such that $\alpha_q = 1$ for all $q \in Q_0$, the Kac polynomial at one is equal to the number of spanning trees of Q.*

Corollary 3.2.7 immediately yields the following:

Corollary 3.2.16. *Let (d, e) be a coprime root of the generalized Kronecker quiver $K(m)$. Then the Kac polynomial at one grows exponentially with the dimension vector, i.e. there exists a real number $K_{(d,e)} > 0$ such that*

$$\lim_{n\to\infty} a_{(d_s,e_s)+n(d,e)}(1) > K^n_{(d,e)}.$$

Here (d_s, e_s) is the root obtained by the considerations of Section 3.1.2.

We conclude this section with several open questions. Our observations raise the following natural question, which was for instance posed in [38, Question 7], but also posed to the author by W. Crawley-Boevey and A. Hubery:

Question 3.2.17. *Do we always have $T_\alpha \geq a_\alpha(1)$?*

The considerations of this section yield that this needs to be checked only for quivers which are trees. But actually, similar to the question of the existence of tree modules, it seems that this does not make things much easier. Many examples which can be found in the literature suggest that this true. In the case of extended Dynkin quivers of type $\tilde{D}_n$, this can be checked by hand. We also conjecture that equality holds if and only if the assumptions of Corollary 3.2.13 hold.

Question 3.2.18. *Does Corollary 3.2.11 also hold for quivers with loops?*

In this case, the results of [15] do not apply. But examples suggest that this is true. For instance, for the quiver with only one vertex q and g loops, one can check by hand that this is true for $\alpha_q \leq 6$. The only non-trivial moduli space appears for $\alpha_q = 6$. There we need to consider the moduli space $M_\delta^{0,\Theta-\mathrm{st}}(\tilde{D}_4)$ for the imaginary Schur root $\delta = (2, 1, 1, 1, 1)$ of $\tilde{D}_4$ (for generic Θ). Note that we have $a_\delta(q) = q + 4$. Moreover, there exist six indecomposable tree modules of dimension δ. Taking into account the different possibilities of coloring the arrows of $\tilde{D}_4$ with the colors $\{1, \ldots, g\}$, one checks that this indeed fills the gap between $a_6(1)$, see [32, Section 1], and the number of indecomposable tree modules of dimension $\alpha = 6$, see [38, Section 4.1].

Question 3.2.19. *What can we say if α is not coprime?*

Also in this case, the considerations of Crawley-Boevey and van den Bergh do not apply. For non-coprime roots, the Kac polynomial is not described by the cohomology of the moduli space introduced in this section. Actually, one needs to deal with the setup of [31]. There it is shown that the coefficients of the Kac polynomial are given by the dimensions of the sign-isotypical components of the cohomology groups of certain moduli spaces and not by the dimensions of the whole group. Thus, one of the first things to study would be how the torus action agrees with the decomposition into isotypical components.

3.2.3 Tree modules as skeleton

As already mentioned, the hope is that indecomposable tree modules serve as skeleton for a certain normal form. A step into this direction could be that tree modules give rise to torus fixed points of the torus action introduced in Section 3.2.2. But there are several problems to deal with. Actually, one would like to consider a Bialynicki-Birula decomposition, see [2], coming along with the torus action. But because of the definition of the torus action the attracting sets are not well-defined and the limits $\lim_{t\to 0} t * M$ might not exist respectively. Actually, there is hope that further investigation in this direction yields interesting results.

Another direction which could be promising is to take self-extensions of the indecomposable tree modules into account. As the next example shows, this is not as straightforward as it might seem on first sight. But there is the feeling that one needs to consider only a certain subspace of the space of self-extensions instead of the whole space. This would also fit better to results coming along with Bialynicki-Birula decompositions.

Investigations in this direction are only just beginning. There are some promising examples as the one at the end of this section, but also some counterexamples as the following one which was first considered in [71, Section 5].

As far as the functor from Section 3.1.1 is concerned, taking self-extensions into account, should translate to adding n loops to a vertex which corresponds to a Schur representation M with $\dim \operatorname{Ext}(M, M) = n$. Even if there is a natural generalization of the functor to this case, it can be shown that it is not full. Let $L(n)$ be the quiver having one vertex denoted by m and n loops. Let M be a representation of Q with $\operatorname{End}(M) = k$ and $\dim \operatorname{Ext}(M, M) = n$. Fix a tree-shaped basis $\{b_1, \ldots, b_n\}$ of $\operatorname{Ext}(M, M)$. Let $X = (X_m, (X_l)_{l=1,\ldots,n})$ be a representation of $L(n)$ and define $\tilde{X}_q := M_q \otimes_k X_m$ for $q \in Q_0$. Then we can define a representation FX of Q by the vector spaces

$$(FX)_q = \tilde{X}_q \text{ for all } q \in Q_0$$

and by the linear maps $(FX)_\rho = \tilde{X}_p \to \tilde{X}_q$ where

$$(FX)_\rho = M_\rho \otimes_k \mathrm{id}_{X_m} + \sum_{l=1}^{n} (b_l)_\rho \otimes_k X_l : \tilde{X}_p \to \tilde{X}_q$$

for every $\rho : p \to q$.

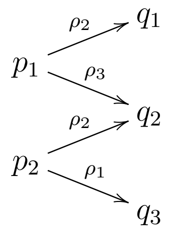
Let $f : X \to X'$ be a morphism. Then we define $Ff : FX \to FX'$ by

$$(Ff)_q := \mathrm{id}_{M_q} \otimes_k f_m : \tilde{X}_p \to \tilde{X}'_q.$$

It is checked in [71, Section 5] that F is a faithful functor. But the following example given there also shows that it is not full, even for Schur representation (resp. for stable representations). Let M be the representation of dimension $(2, 3)$ of $K(3)$ defined by the matrices

$$M_{\rho_1} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}, M_{\rho_2} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}, M_{\rho_3} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

The corresponding coefficient quiver is given by



Now it is easy to check that M is even stable with respect to the standard stability. Every arrow of the set $\{\rho : p \to q \mid \rho \in K(3)_1,\, p \in \{p_1, p_2\},\, q \in \{q_1, q_2, q_3\}\}$ defines an element of $\bigoplus_{\rho \in K(3)_1} \mathrm{Hom}_k(M_{s(\rho)}, M_{t(\rho)})$ in the natural way. Using this notation, a tree-shaped bases of $\mathrm{Ext}(M, M)$ is given by the arrows

$$p_1 \xrightarrow{\rho_1} q_1,\ p_1 \xrightarrow{\rho_1} q_2,\ p_2 \xrightarrow{\rho_2} q_1,\ p_2 \xrightarrow{\rho_2} q_3,\ p_2 \xrightarrow{\rho_3} q_1,\ p_2 \xrightarrow{\rho_3} q_3.$$

Thus, the representation M' defined by the matrices

$$M'_{\rho_1} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix},\ M'_{\rho_2} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{pmatrix},\ M'_{\rho_3} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$$

can be constructed using the functor from above. But since

$$g_p = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix},\ g_q = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

is a non-trivial idempotent of $\mathrm{End}(M')$, the representation M' is not indecomposable.

Example 3.2.20. Let us consider the n-subspace quiver $S(n)$ with sink q_0, sources $q_1, \ldots, q_n$ and arrows $\rho_1, \ldots, \rho_n$. Let α be the root with $\alpha_{q_0} = 2$ and $\alpha_{q_i} = 1$ for $i \geq 1$ and let M be the tree module defined by

$$M_{\rho_1} = e_1 + e_2,\ M_{\rho_2} = e_2 \text{ and } M_{\rho_i} = e_1 \text{ for } i \geq 3.$$

Now it is easy to write down the coefficient quiver whose sources we denote by the original sources and whose sinks we denote by q_0^1 and q_0^2. We have $\dim \mathrm{Ext}(M, M) = n - 3$ and a tree-shaped basis is induced by the arrows $q_i \xrightarrow{\rho_i} q_0^2$ for $i \geq 4$. Now it is straightforward that every representation X of dimension one of the $(n-3)$-loop quiver gives rise to an indecomposable (!) representation FX of $S(n)$ of the same dimension. The indecomposability simply follows because the first three subspaces are pairwise disjoint for all representations FX.

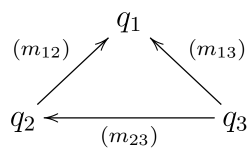
Finally, note that in this manner we can construct a $(1 - \langle \alpha, \alpha \rangle)$-parameter family of isomorphism classes of indecomposable representations. Moreover, it is straightforward that we can construct all indecomposable representations of dimension α starting with an appropriate tree module.

3.3 Non-Schurian indecomposables via intersection theory

In this section, we review the main constructions and results of [23] where representations of quivers with three vertices are considered which have non-Schurian roots as dimension vector. The canonical decomposition of such roots consists of an imaginary root and a multiple of a real Schur root. This setup suggests that there exist indecomposable representations of this dimension which can be constructed using Ringel's reflection functor. In general, it is not clear if such representations exist and how to characterize them, see [73] and [68, Section 4] for counterexamples. In the case of an acyclic quiver with three vertices, this problem can be approached geometrically.

3.3.1 Intersection of subvarieties of Grassmannians

For a fixed vector $\underline{m} := (m_{12}, m_{13}, m_{23}) \in \mathbb{N}^3$, we denote by $Q(\underline{m})$ the quiver



where m_{ij} in brackets indicates the number of arrows between the corresponding vertices. We denote the arrows by $\rho_1^i : q_2 \to q_1$ for $i = 1, \ldots, m_{12}$, $\rho_2^i : q_3 \to q_1$ for $i = 1, \ldots, m_{13}$ and $\rho_3^i : q_3 \to q_2$ for $i = 1, \ldots, m_{23}$. If α is a non-Schurian root of $Q(\underline{m})$, we can without loss of generality assume that the canonical decomposition of α is $\alpha = \alpha_1^{d_1} \oplus \hat{\alpha}$ where $\hat{\alpha}$ is an imaginary root and α_1 is a real Schur root, see also [18, Section 6]. Note that $\hat{\alpha}$ is Schurian if it is not isotropic. Then we have $\hat{\alpha} = \alpha_2^{d_2} + \alpha_3^{d_3}$ where α_2 and α_3 are the two simple roots in $\alpha_1^{\perp}$ which are also exceptional. In particular, $\hat{\alpha}$ corresponds to a root of the generalized Kronecker quiver $K(\mathrm{ext}(\alpha_3, \alpha_2))$. As in [23], we call the unique decomposition of α into exceptional roots $\alpha = \alpha_1^{d_1} + \alpha_2^{d_2} + \alpha_3^{d_3}$, which is obtained in this way, the canonical exceptional decomposition of α. Define

$$l := \mathrm{hom}(\alpha_2, \alpha_1),\ m := \mathrm{ext}(\alpha_3, \alpha_1),\ n := \mathrm{ext}(\alpha_3, \alpha_2)$$

where it can be checked that in this setup $l > 0$ always holds if α is a root. We recall the following lemma:

Lemma 3.3.1 ([23, Lemmas 3.3, 3.4])**.**

i) We have $\mathrm{hom}(\alpha_3, \alpha_2) = \mathrm{hom}(\alpha_3, \alpha_1) = \mathrm{ext}(\alpha_2, \alpha_1) = 0$.

ii) For an exceptional root α*, we have* $\mathrm{Gr}_{s\alpha}(M_\alpha^r) \cong \mathrm{Gr}_s(r)$ *for* $0 \leq s \leq r$.

The basic question of [23] is if and how it is possible to classify and construct indecomposable representations $M_{\hat{\alpha}}$ of dimension $\hat{\alpha}$ such that $M_{\hat{\alpha}} \in M_{\alpha_1}^{\perp}$ and such that $\dim \mathrm{Hom}(M_{\hat{\alpha}}, M_{\alpha_1}) \geq \langle \hat{\alpha}, \alpha_1 \rangle + d_1$. Then we have $\dim \mathrm{Ext}(M_{\hat{\alpha}}, M_{\alpha_1}) \geq d_1$ and, by

Lemma 3.1.10, every such representation gives rise to short exact sequences of the form

$$e : 0 \to M_{\alpha_1}^{d_1} \to M \to M_{\hat{\alpha}} \to 0$$

in such a way that the middle term is indecomposable if and only if e corresponds to an indecomposable representation of $K(\dim \mathrm{Ext}(M_{\hat{\alpha}}, M_{\alpha_1}))$ of dimension $(1, d_1)$.

The most important things when constructing such representations can be found in commutative diagrams of the form

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M_{\alpha_2}^{r} & \xrightarrow{i_1} & M_{\delta}^{d_3} & \xrightarrow{\pi_1} & M_{\hat{\alpha}} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f_1} & \searrow{\scriptstyle f_p} & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_3} & & \\ 0 & \longrightarrow & M_{\alpha_1}^{t} & \xrightarrow{i_2} & M_{\alpha_1}^{s} & \xrightarrow{\pi_2} & M_{\alpha_1}^{s-t} & \longrightarrow & 0 \end{array}$$

where $r = nd_3 - d_2$ and $s = \dim \mathrm{Hom}(M_{\delta}^{d_3}, M_{\alpha_1})$. Here M_δ and M_{α_2} are the indecomposable projective representations in the category $M_{\alpha_1}^{\perp}$ which means that the upper row is a minimal projective resolution of $M_{\hat{\alpha}}$. In the case when $d_3 = 1$, the representation $M_{\hat{\alpha}}$ is automatically indecomposable. In this regard, the case $d_3 \geq 2$ turns out to be more difficult.

Now the main question is what kind of conditions ensure that such a diagram exists and that the morphisms f_i are of maximal rank. In particular, f_3 would be of maximal rank which would ensure that $\dim \mathrm{Hom}(M_{\hat{\alpha}}, M_{\alpha_1}) \geq s - t$. Thus, if $s - t \geq \langle \hat{\alpha}, \alpha_1 \rangle + d_1$, we would have constructed a representation $M_{\hat{\alpha}}$ of dimension $\hat{\alpha}$ with $\dim \mathrm{Ext}(M_{\hat{\alpha}}, M_{\alpha_1}) \geq d_1$.

We follow the considerations of [23] and answer this question geometrically. For simplicity, we assume that $s - t = \langle \hat{\alpha}, \alpha_1 \rangle + d_1$. This can be obtained if we set $r = nd_3 - d_2$ and $t = (nl - m)d_3 - d_1 - \langle \hat{\alpha}, \alpha_1 \rangle$ in the commutative diagram under consideration. Then we additionally have that $M_{\hat{\alpha}}$ is of dimension $\hat{\alpha}$.

Moreover, we only consider non-Schurian roots α of type one, i.e. we assume that $r \leq lt$ and $r \leq nd_3 \leq ls$. Special cases where at least one of these conditions is violated can be found in [23, Section 4]. Even if a detailed analysis of roots which violate one of these inequalities is missing, it seems that this is very rarely the case.

It turns out that our considerations ensure that we get "the maximal rank"-property for free. Thus, we have to deal with the question in which case a fixed morphism of maximal rank $f_p \in \mathrm{Hom}(M_{\alpha_2}^{r}, M_{\alpha_1}^{s})$ factors through both $M_{\delta}^{d_3}$ and $M_{\alpha_1}^{t}$. This can be formulated in terms of two subvarieties of $\mathrm{Gr}_r(V)$ where $V := \mathrm{Hom}(M_{\alpha_2}, M_{\alpha_2}^{ls})$ and the question whether they intersect or not. We recall their construction from [23, Section 3.2].

For a fixed k-vector space V of dimension n and natural numbers $1 \leq d_1 < \ldots < d_k \leq n$ with $k \leq n$, we denote the corresponding (partial) flag variety by

$$\mathrm{Fl}_{(d_1,\ldots,d_k)}(V) := \{U_1 \subset \ldots \subset U_k \subset V \mid \dim U_i = d_i\}.$$

Every r-dimensional subspace of V as defined above, i.e. every point of the usual Grassmannian $\mathrm{Gr}_r(V)$, defines an injection $M_{\alpha_2}^{r} \hookrightarrow M_{\alpha_2}^{ls}$. Moreover, every basis $(\phi_1, \ldots, \phi_l)$

of $\mathrm{Hom}(M_{\alpha_2}, M_{\alpha_1})$ induces a morphism $\phi : M_{\alpha_2}^l \to M_{\alpha_1}$. In turn, we get an injective map

$$\mathrm{Hom}(M_{\alpha_1}^r, M_{\alpha_1}^t) \hookrightarrow \mathrm{Hom}((M_{\alpha_2}^l)^r, M_{\alpha_1}^t) \cong \mathrm{Hom}((M_{\alpha_2}^l)^r, (M_{\alpha_2}^l)^t).$$

This embedding induces a closed embedding $\Delta : \mathrm{Gr}_t(V_0) \to \mathrm{Gr}_{lt}(V)$, $U \mapsto U^l$, where $V_0 = \mathrm{Hom}(M_{\alpha_1}, M_{\alpha_1}^s) \cong \mathrm{Hom}(M_{\alpha_2}, M_{\alpha_2}^s)$. By Lemma 3.3.1, this also yields an inclusion

$$\mathrm{Gr}_t(k^s) \cong \mathrm{Gr}_{t\alpha_1}(M_{\alpha_1}^s) \subset \mathrm{Gr}_{t(l\alpha_2)}((M_{\alpha_2}^l)^s)$$

induced by commutative diagrams of the form

$$\begin{array}{ccc} (M_{\alpha_2}^l)^t & \hookrightarrow & (M_{\alpha_2}^l)^s \\ \downarrow \phi^t & & \downarrow \phi^s \\ M_{\alpha_1}^t & \hookrightarrow & M_{\alpha_1}^s \end{array}$$

Thus, morphisms $f_p : M_{\alpha_2}^r \to M_{\alpha_1}^s$ of maximal possible rank factoring through $M_{\alpha_1}^t$ are in one-to-one correspondence to certain morphisms $\hat{f}_p : M_{\alpha_2}^r \to (M_{\alpha_2}^l)^s$ factoring through $(M_{\alpha_2}^l)^t$. But these morphisms can be described in terms of subvarieties of the Grassmannian $\mathrm{Gr}_r(V)$. The flag variety $\mathrm{Fl}_{(r,lt)}(V)$ comes along with projections

$$\mathrm{Gr}_r(V) \xleftarrow{\psi_1} \mathrm{Fl}_{(r,lt)}(V) \xrightarrow{\psi_2} \mathrm{Gr}_{lt}(V).$$

Every point in the image of ψ_1 defines a morphism $M_{\alpha_2}^r \to (M_{\alpha_2}^l)^s$ factoring through $(M_{\alpha_2}^l)^t$. But since we are only interested in morphisms $h : M_{\alpha_2}^r \to M_{\alpha_1}^s$ factoring through $M_{\alpha_1}^t$, we need to restrict ψ_1 to the subvariety

$$Y := \psi_2^{-1}\Delta(\mathrm{Gr}_t(V_0)) = \{(U_1, U_2) \in \mathrm{Fl}_{(r,lt)}(V) \mid U_2 \in \Delta(\mathrm{Gr}_t(V_0))\}.$$

We denote the subvariety $\psi_1(Y)$ of $\mathrm{Gr}_r(V)$ by X_1^α. The considerations of [23, Section 3.3] yield that

$$\dim X_1^\alpha = \dim \mathrm{Gr}_r(k^{ls}) + \dim \mathrm{Gr}_t(k^s) = r(ls - r) + t(s - t).$$

In order to define the second subvariety, we first note that, since $\hom(\hat{\alpha}, \alpha_1) \geq 0$ and since (d_2, d_3) is a root of $K(n)$, we have $m \leq n \cdot l$ and thus $\mathrm{ext}(\delta, \alpha_1) = 0$. Thus, (α_1, δ) is again an exceptional sequence, and we get a morphism $M_\delta^{d_3} \to M_{\alpha_1}^s$ induced by a basis of $\mathrm{Hom}(M_\delta^{d_3}, M_{\alpha_1})$. This induces a linear map

$$\mathrm{Hom}(M_{\alpha_2}, M_\delta^{d_3}) \to \mathrm{Hom}(M_{\alpha_2}, M_{\alpha_1}^s) \cong \mathrm{Hom}(M_{\alpha_2}, (M_{\alpha_2}^l)^s) = V.$$

This already means that every r-dimensional subspace which is contained in $W := \mathrm{Hom}(M_{\alpha_2}, M_\delta^{d_3})$ defines a point of the Grassmannian $\mathrm{Gr}_r(V)$ which corresponds to an injection of $M_{\alpha_2}^r \to (M_{\alpha_2}^l)^s$ factoring through $M_\delta^{d_3}$. In turn, such morphisms correspond to points of the subvariety

$$X_2^\alpha = \{U \in \mathrm{Gr}_r(V) \mid U \subset W\} \cong \mathrm{Gr}_r(W).$$

Clearly, we have $\dim X_2^\alpha = r(\dim W - r)$.

In summary, fixing a non-Schurian root α of type one we get two subvarieties X_i^α of $\mathrm{Gr}_r(V)$ such that every point in their intersection I^α defines a morphism of maximal rank $M_{\alpha_2}^r \hookrightarrow M_{\alpha_1}^s$ which factors through $M_\delta^{d_3}$ and $M_{\alpha_1}^t$. A priori, it is not clear that X_1^α and X_2^α intersect, even if the dimensions of X_i^α at least sum up to the dimension of $\mathrm{Gr}_r(V)$. But it can be shown using intersection theory that this particular subvarieties do always intersect. More precisely we have:

Theorem 3.3.2 ([23, Theorems 3.6, 3.15, Corollary 3.21])**.** *Let α be a non-Schurian root of $Q(\underline{m})$ which is of type one. Then we have:*

i) Every morphism f_p induced by a point $p \in I^\alpha$ gives rise to a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M_{\alpha_2}^r & \xrightarrow{i_1} & M_\delta^{d_3} & \xrightarrow{\pi_1} & M_{\hat\alpha} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f_1} & \searrow^{f_p} & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_3} & & \\ 0 & \longrightarrow & M_{\alpha_1}^t & \xrightarrow{i_2} & M_{\alpha_1}^s & \xrightarrow{\pi_2} & M_{\alpha_1}^{s-t} & \longrightarrow & 0 \end{array}$$

such that f_3 is of maximal rank. Moreover, if $d_3 = 1$ the cokernel of π_1 is indecomposable.

ii) The subvarieties X_1^α and X_2^α always intersect in such a way that every irreducible component of $X_1^\alpha \cap X_2^\alpha$ has at least dimension $d_3^2 - \langle \alpha, \alpha \rangle$.

Remark 3.3.3. We have that $P := M_\delta \oplus M_{\alpha_2}$ is a partial tilting module. Moreover, $\mathrm{End}(P)$ is isomorphic to the path algebra of $K(\hom(\alpha_2, \delta))$ where $\hom(\alpha_2, \delta) = \mathrm{ext}(\alpha_3, \alpha_2)$. This implies that the representations $M_{\hat\alpha}$ obtained as the cokernel of an exact sequence of the form

$$0 \to M_{\alpha_2}^r \to M_\delta^{d_3} \to M_{\hat\alpha} \to 0$$

are in one-to-one correspondence to representations X of $K(n)$ of dimension $d := (r, d_3)$ such that $\mathrm{Hom}(X, S_{q_1}) = 0$. Here S_{q_1} denotes the simple representation corresponding to $q_1 \in K(n)_0$. As already mentioned, M_{α_2} and M_δ are the indecomposable projective representations in $M_{\alpha_1}^\perp$ which means that the exact sequence yields a minimal projective resolution of $M_{\hat\alpha}$ in $M_{\alpha_1}^\perp$. Now the natural group action of $\mathrm{Gl}_r(k) \times \mathrm{Gl}_{d_3}(k)$ on $R_d(K(n))$ corresponds to diagrams

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M_{\alpha_2}^r & \xrightarrow{i_1} & M_\delta^{d_3} & \xrightarrow{\pi_1} & M_{\hat\alpha} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle g_1} & & \downarrow{\scriptstyle g_2} & & \downarrow{\scriptstyle g_3} & & \\ 0 & \longrightarrow & M_{\alpha_2}^r & \xrightarrow{i_2} & M_\delta^{d_3} & \xrightarrow{\pi_2} & M'_{\hat\alpha} & \longrightarrow & 0 \end{array}$$

where the maps g_i are isomorphisms. It is straightforward that, on the Grassmannian side, the $\mathrm{Gl}_r(k)$-action corresponds to the usual base change action. Thus, if we want to classify representations in I^α up to isomorphism, we only need to consider the $\mathrm{Gl}_{d_3}(k)$-action.

Remark 3.3.4. Since we have $\dim I^\alpha \geq d_3^2 - \langle \alpha, \alpha \rangle$, taking into account the $\mathrm{Gl}_{d_3}(k)$-action, there exists at least a $(1 - \langle \alpha, \alpha \rangle)$-parameter family of isomorphism classes of representations in I^α. By Kac's Theorem [35, Theorem C], this is also an upper bound if all representations in I^α are indecomposable. The same is true for an irreducible component U of I_α if one representation in U, and thus an open subset of representations in U, is Schurian. Here we have to keep in mind that there is a trivial k^*-action on $R_{(d_2,d_3)}(K(n))$.

The following corollary establishes the connection to Ringel's reflection functor:

Corollary 3.3.5 ([23, Corollary 3.10])**.** *Let α be of type one and let t and r be defined as above. Then the points of I^α correspond to those representations $M_{\hat{\alpha}}$ which can be written as the cokernel of short exact sequences*

$$0 \to M_{\alpha_1}^{d_1} \to M_\alpha \to M_{\hat{\alpha}} \to 0$$

such that M_α is of dimension α with $\mathrm{Hom}(M_{\alpha_1}, M_\alpha) = d_1$ *and such that M_α has no direct summand which is isomorphic to M_{α_1} or M_{α_2}.*

Proof. We include the proof for the convenience of the reader. By construction, for every representation $M_{\hat{\alpha}}$ corresponding to a point of I^α, we have $M_{\hat{\alpha}} \in M_{\alpha_1}^{\perp}$ and $\dim \mathrm{Ext}(M_{\hat{\alpha}}, M_{\alpha_1}) \geq d_1$. Moreover, $M_{\hat{\alpha}}$ has no direct summand which is isomorphic to M_{α_1} or M_{α_2}. Thus, there exist short exact sequences $0 \to M_{\alpha_1}^{d_1} \to M_\alpha \to M_{\hat{\alpha}} \to 0$ such that the middle terms satisfy the claimed properties.

The other way around let M_α be of dimension α such that $\dim \mathrm{Hom}(M_{\alpha_1}, M_\alpha) = d_1$ and such that M_{α_1} and M_{α_2} are no direct summands of M_α. Then we have $\mathrm{Ext}(M_{\alpha_1}, M_\alpha) = 0$ and thus there exists a short exact sequence $0 \to M_{\alpha_1}^{d_1} \to M_\alpha \to M_{\hat{\alpha}} \to 0$ such that $M_{\hat{\alpha}} \in M_{\alpha_1}^{\perp}$ and $\dim \mathrm{Ext}(M_{\hat{\alpha}}, M_{\alpha_1}) \geq d_1$. It follows that we have $\dim \mathrm{Hom}(M_{\hat{\alpha}}, M_{\alpha_1}) \geq \langle \hat{\alpha}, \alpha_1 \rangle + d_1$. Since M_α has no direct summand isomorphic to M_{α_2}, the same is true for $M_{\hat{\alpha}}$ because $\mathrm{Ext}(M_{\alpha_2}, M_{\alpha_1}) = 0$. In particular, $M_{\hat{\alpha}}$ fits into a commutative diagram as in Theorem 3.3.2. □

3.3.2 Glueing representations

The next step is to investigate which points in I_α correspond to indecomposable representations. We have already seen that all representations in I_α are indecomposable if $d_3 = 1$. In general, this is not easy to decide and needs further investigation. But in many cases, we are at least able to give a recipe how to construct $(1 - \langle \alpha, \alpha \rangle)$-parameter families of isomorphism classes of indecomposables in I_α recursively. It is very interesting that the decomposition of roots of the generalized Kronecker quiver obtained in Section 3.1.2 again plays an important role. In the following, we recall the basic ideas of [23, Section 3.5].

With every non-Schurian root $\alpha = \alpha_1^{d_1} + \hat{\alpha}$, we can associate an imaginary Schur $\hat{\alpha}$ which, in turn, defines a root $(d, e) := (d_2, d_3)$ of the generalized Kronecker quiver $K(\mathrm{ext}(\alpha_3, \alpha_2))$. Initially, we assume that (d, e) is coprime because the general case only needs a slight generalization. Thus, we can decompose (d, e) into smaller coprime roots $(d, e) = (d_s, e_s) + b(d', e')$ for some $b \geq 1$ yielding two Schur roots $\hat{\beta}$ and $\hat{\gamma}$ of $Q(\underline{m})$ which correspond to the Schur roots $(d_s, e_s)+(b-1)(d', e')$ and (d', e'). The basic question is whether there exists a decomposition $d_1 = c_s + c$ such that $\alpha_1^{c_s} + \hat{\beta}$ and $\alpha_1^c + \hat{\gamma}$ are again roots of $Q(\underline{m})$. If this is the case, we know that the varieties I_β and I_γ are not empty and we can ask for a method of glueing representations of dimension $\hat{\beta}$ and $\hat{\gamma}$ such that the glued representation are indecomposable representations which are contained in I_α. Since we have $\hom(\hat{\beta}, \hat{\gamma}) = \hom(\hat{\gamma}, \hat{\beta}) = 0$, this suggests to apply Theorem 3.1.2. But there are two things we have to bear in mind. On the one hand, we have to ensure that the glueing is additive on Hom- and Ext-spaces, i.e. for fixed $M_{\hat{\beta}} \in I_\beta$ and $M_{\hat{\gamma}} \in I_\gamma$, we would like to have

$$\begin{aligned} \dim \operatorname{Hom}(M_{\hat{\alpha}}, M_{\alpha_1}) &= \dim \operatorname{Hom}(M_{\hat{\beta}}, M_{\alpha_1}) + \dim \operatorname{Hom}(M_{\hat{\gamma}}, M_{\alpha_1}), \\ \dim \operatorname{Ext}(M_{\hat{\alpha}}, M_{\alpha_1}) &= \dim \operatorname{Ext}(M_{\hat{\beta}}, M_{\alpha_1}) + \dim \operatorname{Ext}(M_{\hat{\gamma}}, M_{\alpha_1}). \end{aligned}$$

Here we have $M_{\hat{\alpha}} = F_{(M_{\hat{\beta}}, M_{\hat{\gamma}})}(X)$ for a representation X of $Q(M_{\hat{\beta}}, M_{\hat{\gamma}})$ of dimension $(1, 1)$. On the other hand, we need two Schur representations $M_{\hat{\beta}}$ and $M_{\hat{\gamma}}$ satisfying $\operatorname{Hom}(M_{\hat{\beta}}, M_{\hat{\gamma}}) = \operatorname{Hom}(M_{\hat{\gamma}}, M_{\hat{\beta}}) = 0$.

Let $h_\beta := \dim \operatorname{Hom}(M_{\hat{\beta}}, M_{\alpha_1})$. Every $M_{\hat{\beta}} \in I_\beta$ induces a natural projection $M_{\hat{\beta}} \to M_{\alpha_1}^{h_\beta}$. The analogous statement is clearly true for $M_{\hat{\gamma}} \in I_\gamma$. Now the additivity is ensured when restricting to exact sequences which are contained in the kernels of the natural surjections $\operatorname{Ext}(M_{\hat{\beta}}, M_{\hat{\gamma}}) \to \operatorname{Ext}(M_{\hat{\beta}}, M_{\alpha_1}^{h_\gamma})$ and $\operatorname{Ext}(M_{\hat{\gamma}}, M_{\hat{\beta}}) \to \operatorname{Ext}(M_{\hat{\gamma}}, M_{\alpha_1}^{h_\beta})$ respectively, keeping in mind the following easy lemma:

Lemma 3.3.6 ([23, Lemma 3.22]). *Let M, N be two representations such that M is exceptional with coefficient quiver Γ_M. Then we have:*

i) If M^n is a subrepresentation of N, there exists a coefficient quiver Γ_N of N such that Γ_N has n subquivers Γ_M^i of sink-type where $\Gamma_M^i = \Gamma_M$ for all $i = 1, \ldots, n$.

ii) If N has a coefficient quiver Γ_N which has n subquivers Γ_M^i of sink-type such that $\Gamma_M^i = \Gamma_M$ for all $i = 1, \ldots, n$ and such that $\Gamma_M^i \cap \Gamma_M^j = \emptyset$ for $i \neq j$, we have $\dim \operatorname{Hom}(M, N) \geq n$.

The analogous statements hold if M^n is a factor of N and for quivers of source-type respectively.

As far as the existence of Schur representations with vanishing homomorphism spaces is concerned, it is very useful that the semi-continuity of $\dim \operatorname{Hom}_k$ on $R_{\hat{\beta}}(Q(\underline{m})) \times R_{\hat{\gamma}}(Q(\underline{m}))$, see [61, Section 1], transfers when considering $\dim \operatorname{Hom}_k$ on $I_\beta \times I_\gamma$. On the one hand, this ensures that the existence of one Schur representation in I_β (resp. I_γ) implies the existence of an open subset of Schur representations. On the other hand, this means that the existence of a tuple of Schur representations $(M_{\hat{\beta}}, M_{\hat{\gamma}})$ with $\operatorname{Hom}(M_{\hat{\beta}}, M_{\hat{\gamma}}) = 0$ already ensures that there exist non-empty open subsets of Schur representations in I_β and I_γ respectively satisfying this property. Note that these subsets are not necessarily dense because I_β and I_γ might be reducible.

For the non-coprime case, we need a slight generalization. Assume that we have $(d_2, d_3) = n(\hat{d}_2, \hat{d}_3)$ with $n = \gcd(d_2, d_3)$. Then we can decompose $(\hat{d}_2, \hat{d}_3)$ as done before. Moreover, a general representation of dimension $(\hat{d}_2, \hat{d}_3)$ is Schurian. Since $(\hat{d}_2, \hat{d}_3)$ is also imaginary, by [61, Theorem 3.5], we have $\hom((\hat{d}_2, \hat{d}_3), (\hat{d}_2, \hat{d}_3)) = 0$. Therefore, we obtain $\hom(k(\hat{d}_2, \hat{d}_3), l(\hat{d}_2, \hat{d}_3)) = 0$ for $k, l \geq 1$. Taking into account the considerations of Section 3.1.2, we obtain the following result:

Theorem 3.3.7 ([23, Theorem 3.29])**.** *Let α be a non-Schurian root of type one. Then we have:*

i) Every pair consisting of a representation $M_{\hat{\alpha}}$ corresponding to a point of I^α and a point $e \in \mathrm{Gr}_{d_1}(\mathrm{Ext}(M_{\hat{\alpha}}, M_{\alpha_1}))$ gives rise to a short exact sequence $0 \to M_{\alpha_1}^{d_1} \to M_\alpha \to M_{\hat{\alpha}} \to 0$ such that M_α is indecomposable if and only if $M_{\hat{\alpha}}$ is indecomposable.

ii) For two pairs $(p, e), (p', e') \in I^\alpha \times \mathrm{Gr}_{d_1}(\mathrm{Ext}(M_{\hat{\alpha}}, M_{\alpha_1}))$ and their corresponding short exact sequences $0 \to M_{\alpha_1}^{d_1} \to M_\alpha \to M_{\hat{\alpha}} \to 0$ and $0 \to M_{\alpha_1}^{d_1} \to M'_\alpha \to M'_{\hat{\alpha}} \to 0$, the following are equivalent:

- *M_α and M'_α are isomorphic;*
- *$M_{\hat{\alpha}}$ and $M'_{\hat{\alpha}}$ are isomorphic and e and e' are equivalent;*
- *p and p' lie in the same $\mathrm{Gl}_{d_3}(k)$-orbit and e and e' are equivalent.*

iii) The closed subset I^α is non-empty and each of its irreducible components has dimension at least $d_3^2 - \langle \alpha, \alpha \rangle$.

iv) If I^α contains one Schur representation $M_{\hat{\alpha}}$, then the corresponding irreducible component is of dimension $d_3^2 - \langle \alpha, \alpha \rangle$ and contains an open subset of Schur representations. In this case, there exists a $(1 - \langle \alpha, \alpha \rangle)$-parameter family of isomorphism classes of Schur representations of dimension $\hat{\alpha}$ with $M_{\hat{\alpha}} \in M_{\alpha_1}^\perp$ and $\dim \mathrm{Ext}(M_{\hat{\alpha}}, M_{\alpha_1}) \geq d_1$.

v) If there exists a decomposition into roots $\alpha = \beta + \gamma$, which is induced by the glueing conditions, and such that I^β and I^γ both contain one Schur representation such that the corresponding homomorphism spaces vanish, then there exists an open subset of Schur representations in I^α which can be constructed by glueing Schur representations of dimension β and γ respectively.

4 On the Euler characteristic of moduli spaces of stable representations

Tree quivers, tree modules and more generally tree structures also play an important role in geometric representation theory and related topics. For instance, torus fixed points of moduli spaces of stable representations are representations of the universal covering quiver which is a tree. In turn, stable tree modules are torus fixed point as they are representations of the universal covering quiver. More general, indecomposable tree modules give rise to torus fixed points of the moduli spaces obtained when passing to the preprojective algebra. Thus, we can make use of this to make statements concerning the Euler characteristic of the respective moduli spaces.

But there is also a seemingly unrelated topic which is connected to all this which is the Gromov-Witten theory of rational curves on weighted projective planes. Here Gromov-Witten invariants play the role of Euler characteristics and tropical curves play the one of torus fixed points. Also here tree structures show up as the underlying graphs of rational tropical curves are trees.

The main knowledge gained in this theory throughout the last years is that the corresponding invariants, i.e. Euler characteristics and Gromov-Witten invariants, can be obtained in a purely combinatorial way when counting certain trees. On the quiver side, this is encoded when combining the degeneration formula of Manschot, Pioline and Sen [45] with the localization theorem of [69]. On the Gromov-Witten side, the respective results were obtained by Gross, Pandharipande and Siebert [28, 29] (including an analogous degeneration formula). The upshot is that, via the refined GW/Kronecker correspondence developed in [54, 55], it can be shown that the two degeneration formulae are equivalent.

In the first section of this chapter, we focus on new aspects and generalize a conjecture of Douglas concerning the Euler characteristic of moduli spaces which leads to an analogous version for the Kac polynomial at one. In the case of generalized Kronecker quivers, the number of cover-thin tree modules as obtained in Proposition 3.2.6 gives a lower bound for the function induced by the conjecture. This section is also meant to serve as a motivation for further investigations of the presented material in the case of moduli spaces coming along with preprojective algebras.

In the second section, we recall how the MPS degeneration formula and the localization theorem can be combined. This is also described in [54]. As already mentioned, this makes the determination of the Euler characteristics to a purely combinatorial problem of counting trees.

In the third section, we recall the main results of [70] and apply the introduced methods to the case of generalized Kronecker quivers which yields an upper bound for the Euler characteristic of Kronecker moduli spaces. In particular, we can confirm that it grows exponentially as implied by the conjecture of Douglas.

As a further example of the combination of the MPS degeneration formula and the localization theorem, we derive several formulae for Euler characteristics in the fourth section which are, apart from one new formula, derived in [70, 55]. In particular, those cases where it vanishes are very helpful when determining the corresponding Gromov-Witten invariants. Furthermore, since it is in general easier to compute Euler characteristics than Gromov-Witten invariants, the explicit formulae can be used to give formulae for Gromov-Witten invariants which were unknown before.

Finally, the refined GW/Kronecker correspondence established in [55] is reviewed in the fifth section. In there, we also consider the Gromov-Witten analogue of the MPS degeneration formula and recall that the two formulae are indeed equivalent where we mainly follow [54]. As already mentioned, the Gromov-Witten analogue of torus fixed points turns out to be certain tropical curves. This motivates the attempt of giving a direct correspondence between them. Actually, we deal with examples where such a correspondence can be written down explicitly. In general, it is not clear how such a correspondence can be obtained and further investigation is needed.

4.1 (Generalized) Douglas conjecture on the Euler characteristic of moduli spaces

Throughout this chapter, we only consider quivers without oriented cycles, and we fix the field of complex numbers as our ground field. Moreover, we fix a level $l : Q_0 \to \mathbb{N}^+$ and a stability induced by a fixed linear form $\Theta \in \mathrm{Hom}(\mathbb{Z}Q_0, \mathbb{Z})$.

We begin this chapter with motivating considerations concerning the Euler characteristic of moduli spaces and the Kac polynomial at one. Assume that $\kappa = \dim$ which means that we have $l(q) = 1$ for all vertices $q \in Q_0$. We start with the following question, which is based on a conjecture of Michael Douglas, concerning moduli spaces of stable representations, see [69, Section 6.1]. Initially, only the case of stable representations of the Kronecker quiver was considered, but actually there is a straightforward generalization:

Conjecture 4.1.1. *Assume that Θ is chosen in such a way that every coprime dimension vector is already Θ-coprime. Then there exists a continuous function $f : \mathbb{R}Q_0 \to \mathbb{R}$ such that*

$$f(\alpha) = \lim_{n\to\infty} \frac{\ln(\chi(M^{\Theta-\mathrm{st}}_{\alpha_s+n\alpha}(Q)))}{n}$$

for all coprime roots $\alpha \in \mathbb{N}Q_0$. Here α_s is a fixed root chosen in such a way that $\alpha_s + n\alpha$ is a coprime root for all $n \geq 1$.

But this does not seem to be the end of the story because Proposition 3.2.6 suggests that we can extend this result to more general moduli spaces and thus to the Kac polynomial at one.

Conjecture 4.1.2. *There exists a continuous function $f : \mathbb{R}Q_0 \to \mathbb{R}$ such that*

$$f(\alpha) = \lim_{n\to\infty} \frac{\ln(a_{\alpha_s+n\alpha}(1))}{n}$$

for all coprime roots $\alpha \in \mathbb{N}Q_0$. Here α_s is a fixed root chosen in such a way that $\alpha_s + n\alpha$ is a coprime root for all $n \geq 1$.

Note that, for a fixed coprime dimension vector α, the existence of a coprime dimension vector α_s such that $\alpha_s + n\alpha$ is coprime for all $n \geq 0$ can be proved by induction on the number of vertices. For $n = 2$, we are faced with the Kronecker quiver. In particular, the dimension vector α_s is obtained as in Section 3.1.2.

If a function as predicted in Conjecture 4.1.2 exists, Proposition 3.2.6 immediately yields a lower bound in the case of the Kronecker quiver and for coprime dimension vectors (d, e) such that $d \leq e \leq (m-1)d + 1$.

Lemma 4.1.3. *Let (d, e) be a coprime root of the Kronecker quiver such that $d \leq e \leq (m-1)d + 1$ and define $k := e/d$ and $n := m - 1$. Then we have*

$$\begin{aligned} \lim_{d\to\infty} \frac{a_{(d_s,e_s)+(d,kd)}(1)}{d} &\geq \lim_{d\to\infty} \frac{t_{(d_s,e_s)+(d,kd)}}{d} \\ &= n(k+1)\ln n + k(n-1)\ln k - (nk-1)\ln(nk-1) \\ &\quad -(n-k)\ln(n-k). \end{aligned}$$

Proof. Obviously, we only have to consider one of the m summands of the formula obtained in Proposition 3.2.6. Then the claim follows straightforwardly when applying the Stirling formula. □

Note that the numbers $t_{(d,e)}$ are not invariant under the reflection functor. Nevertheless, the reflection functor can clearly be used to obtain a lower bound for $a_{(d,e)}(1)$ for every coprime dimension vector (d, e). Furthermore, it would be interesting to know if there are tuples (d, e) for which equality holds or to know more about the contribution of cover-thin tree modules to the Kac polynomial at one.

This considerations motivate asking for techniques which can be used to determine the Euler characteristic of moduli spaces or the Kac polynomial at one. Actually, in the first case much more is known and possible generalizations to the Kac polynomial need to be investigated in greater detail. Throughout the next sections, we mostly restrict to the case of Euler characteristics.

4.2 Combining the MPS degeneration formula with the localization theorem

The good news is that, in the case of Θ-coprime dimension vectors, the Euler characteristic of the moduli spaces $M_\alpha^{\Theta-\mathrm{st}}(Q)$ can be obtained in a purely combinatorial way by counting certain trees. Indeed, we can combine the MPS degeneration formula of [45] with the localization theorem of [69]. We shortly recall this fact which has plenty of applications as we will see.

4.2.1 Euler characteristic of moduli spaces via counting trees

For a vertex $r \in Q_0$, we denote by $A_r \subseteq Q_1$ the set of arrows ρ such that r is the head or tail of ρ. Moreover, let $Q(r)$ be the quiver which has vertices

$$Q(r)_0 = Q_0 \backslash \{r\} \cup \{r_{l,m} \mid (l,m) \in \mathbb{N}_+^2\}$$

and arrows

$$\begin{aligned} Q(r)_1 &= Q_1 \backslash A_r \cup \{\alpha_1, \ldots, \alpha_l : p \to r_{l,m} \mid \alpha : p \to r,\, m \in \mathbb{N}_+\} \\ &\quad \cup \{\alpha_1, \ldots, \alpha_l : r_{l,m} \to q \mid \alpha : r \to q,\, m \in \mathbb{N}_+\}. \end{aligned}$$

The level on Q induces a level $l : Q(r)_0 \to \mathbb{N}$ by

$$l(q) = \begin{cases} l \text{ if } q = r_{l,m} \\ l(q) \text{ if } q \in Q_0 \backslash \{r\} \end{cases} .$$

If we fix a dimension vector $\alpha \in \mathbb{N}Q_0$ and a weighted partition $\alpha_r = \sum_{l=1}^{t} lk_l$, this induces a dimension vector $\bar{\alpha}$ of $Q(r)$ in the following way: we set $\bar{\alpha}_q = \alpha_q$ for all $q \neq r$ and $\bar{\alpha}_{r_{l,m}} = 1$ for $1 \leq l \leq t$ and $1 \leq m \leq k_l$ and $\bar{\alpha}_{r_{l,m}} = 0$ otherwise. If we think of a dimension vector of $Q(r)$, it is convenient to think of a tuple $\alpha(k_*) := (\alpha, k_*)$ where $k_* \vdash \alpha_r$ is a weighted partition of α_r. We call $Q(r)$ the MPS-quiver of Q with respect to r. Clearly, we can inductively apply this construction to all vertices of Q and obtain the full MPS-quiver of Q which we denote by $Q_{\mathcal{F}}$.

We define a linear form $\Theta^r \in \mathrm{Hom}(\mathbb{Z}Q(r)_0, \mathbb{Z})$ by $(\Theta^r)_q = \Theta_q$ for all $q \neq r$ and $(\Theta^r)_{r_{l,m}} = l\Theta_r$ for all $l, m \geq 1$. We denote the corresponding linear form on $Q_{\mathcal{F}}$ by $\Theta_{\mathcal{F}}$ or just by Θ in order to shorten notation.

Now we can formulate the following result concerning the Poincaré polynomial and the Euler characteristic of moduli spaces of stable representations, see [45, Appendix D] and also [54, Sections 3.2, 3.3] for a more general setting:

Theorem 4.2.1. *If α is Θ-coprime, we have*

$$t^{\alpha_r(\alpha_r-1)} P(M_\alpha^{\Theta-\mathrm{st}}(Q), t) = \sum_{k_* \vdash \alpha_r} \prod_{l \geq 1} \frac{1}{k_l!} \left(\frac{(-1)^{l-1}}{l[l]_{t^2}} \right)^{k_l} P(M_{\alpha(k_*)}^{\Theta^r-\mathrm{st}}(Q(r)), t)$$

and

$$\chi(M_\alpha^{\Theta-\mathrm{st}}(Q)) = \sum_{k_* \vdash \alpha_r} \prod_{l \geq 1} \frac{1}{k_l!} \left(\frac{(-1)^{l-1}}{l^2} \right)^{k_l} \chi(M_{\alpha(k_*)}^{\Theta^r-\mathrm{st}}(Q(r)))$$

where the sums are over all weighted partitions of α_r and where $[l]_{t^2} = (t^{2l}-1)/(t-1)$.

As far as the full MPS-quiver is concerned, every tuple $k_* := (k_*^q)_q$ of weighted partitions of α, denoted by $k_* \vdash \alpha$ defines a dimension vector $\alpha(k_*)$ of the full MPS-quiver $Q_{\mathcal{F}}$. Moreover, every such tuple corresponds to a summand

$$\prod_{q\in Q_0}\prod_{l\geq 1}\frac{1}{k_l^q!}\left(\frac{(-1)^{l-1}}{l^2}\right)^{k_l^q}\chi(M_{\alpha(k_*)}^{\Theta_{\mathcal{F}}-\mathrm{st}}(Q_{\mathcal{F}})) =: \kappa(k_*)\chi(M_{\alpha(k_*)}^{\Theta_{\mathcal{F}}-\mathrm{st}}(Q_{\mathcal{F}})).$$

in the formula for the Euler characteristic. The aim is to combine the second part of Theorem 4.2.1 with the localization theorem which is particularly powerful in the case of dimension vectors of type one. We recall the localization theorem:

Theorem 4.2.2 ([69, Corollary 3.14])**.** *We have*

$$\chi(M_{\alpha}^{\Theta-\mathrm{st}}(Q)) = \sum_{\tilde{\alpha}}\chi(M_{\tilde{\alpha}}^{\tilde{\Theta}-\mathrm{st}}(\tilde{Q})),$$

where $\tilde{\alpha}$ ranges over all equivalence classes being compatible with α, and where the stability considered on $\tilde{Q}$ is the one naturally induced by the stability fixed on Q.

We call a tuple $(\mathbf{Q},\tilde{\alpha})$ consisting of a finite subquiver $\mathbf{Q}$ of $\tilde{Q}$ and a dimension vector $\tilde{\alpha}\in\mathbb{N}\mathbf{Q}_0$ a localization datum if $M_{\tilde{\alpha}}^{\tilde{\Theta}-\mathrm{st}}(\mathbf{Q})\neq\emptyset$.

Remark 4.2.3. Each localization datum comes along with two maps $c_1 : \mathbf{Q}_1 \to Q_1$ and $c_2 : \mathbf{Q}_0 \to Q_0$ which we will call coloring and labelling in what follows. In order to determine the Euler characteristic of certain moduli spaces explicitly with the localization theorem, it is often convenient to consider so-called uncolored or even unlabelled localization data and to count the number of possible colorings in a second step. Note that every coloring already induces a labelling. In other words, we consider tuples $(\mathbf{Q},\tilde{\alpha})$ of quivers and dimension vectors such that $\mathbf{Q}$ can be embedded into $\tilde{Q}$ and, moreover, $\tilde{\alpha}$ is compatible with α under this embedding. Then the number of embeddings is the same as the number of colorings. In order to be clear, when forgetting the coloring, we call a localization datum uncolored; when additionally forgetting the labelling, we call it unlabelled. Clearly, all embeddings into $\tilde{Q}$ are considered up to translation.

It is very important for us that, in the case of dimension vectors of type one, every compatible dimension vector is the original dimension vector itself. Thus, every localization datum defines a spanning tree of Q and, therefore, the corresponding moduli space is a point. We call a spanning tree stable if the corresponding moduli space is not empty. We denote the set of stable spanning trees with respect to the chosen stability by $N^{\Theta}(Q)$.

Corollary 4.2.4. *Let Q be quiver and $\alpha\in\mathbb{N}Q_0$ defined by $\alpha_q = 1$ for all $q\in Q_0$. Then we have*

$$\chi(M_{\alpha}^{\Theta-\mathrm{st}}(Q)) = |N^{\Theta}(Q)|.$$

For $\alpha \in \mathbb{N}Q_0$, let $\mathrm{supp}(\alpha)$ be its support, i.e. the full subquiver of Q which has vertices $\{q \in Q_0 \mid \alpha_q \neq 0\}$. Applying the MPS degeneration formula to all vertices and the localization theorem afterwards, we end up with the following formula:

Theorem 4.2.5 ([54, Corollary 5.3]). *Let α be a Θ-coprime dimension vector of an acyclic quiver Q. Then we have*

$$\chi(M_\alpha^{\Theta-\mathrm{st}}(Q)) = \sum_{k_* \vdash \alpha} \kappa(k_*) |N^\Theta(\mathrm{supp}(\alpha(k_*)))|.$$

4.2.2 Application to bipartite quivers

We mainly restrict to bipartite quivers as the generalized Kronecker quivers $K(m)$, the full bipartite quivers $K(l_1, l_2)$ with l_1 sources and l_2 sinks and, finally, the quiver $\mathcal{N}(m)$ which is defined by the vertices

$$\mathcal{N}(m)_0 = \{i_{(l,k)} \mid (l,k) \in \mathbb{N}_+^2\} \cup \{j_{(l,k)} \mid (l,k) \in \mathbb{N}_+^2\}$$

and the arrows

$$\mathcal{N}(m)_1 = \{\rho_1, \ldots, \rho_{l \cdot l' \cdot m} : i_{(l,k)} \to j_{(l',k')} \mid l, l', k, k' \in \mathbb{N}_+\}.$$

Note that $\mathcal{N}(m)$ is the full MPS-quiver of $K(m)$ and $\mathcal{N}(1)$ is (isomorphic to) the full MPS-quiver of $K(l_1, l_2)$. We consider the usual stability on $K(m)$ and $K(l_1, l_2)$ respectively, i.e. we set $\Theta_q = 1$ if q is a source, $\Theta_q = 0$ if q is a sink while we consider the level defined by $l(q) = 1$ for all $q \in Q_0$. Hence, the level on $\mathcal{N}(m)$ is given by $l(q_{(l,k)}) = l$ for $q \in \{i, j\}, k \in \mathbb{N}_+$ and the respective stability is obtained as described in Section 4.2.

Every pair of ordered partitions

$$(\mathbf{P}_1, \mathbf{P}_2) = \left(\sum_{i=1}^{l_1} p_{1i}, \sum_{j=1}^{l_2} p_{2j} \right)$$

defines a dimension vector of $K(l_1, l_2)$. In turn, every refinement of $(k^1, k^2) \vdash (\mathbf{P}_1, \mathbf{P}_2)$, i.e. every pair of sets of integers

$$(k^1, k^2) = (\{k^1_{wi}\}, \{k^2_{wj}\})$$

such that, for $i = 1, \ldots, l_1$ and $j = 1, \ldots, l_2$, we have

$$p_{1i} = \sum_w w k^1_{wi},\ p_{2j} = \sum_w w k^2_{wj},$$

defines a dimension vector of $\mathcal{N}(1)$. For $p = 1, 2$, the number of entries of weight w in k^p is defined by

$$m_w(k^p) = \sum_{j=1}^{l_p} k^p_{wj}.$$

Now a fixed refinement (k^1, k^2) induces a dimension vector α of $\mathcal{N}(1)$ by setting

$$\alpha_{q_{(w,m)}} = \begin{cases} 1 \text{ for } m = 1, \dots, m_w(k^p), \\ 0 \text{ for } m > m_w(k^p), \end{cases}$$

for $(q,p) \in \{(i,1),(j,2)\}$. With this notation in place, the MPS degeneration formula at the level of Euler characteristics can be expressed by

$$\chi(M^{\Theta-\mathrm{st}}_{(\mathbf{P}_1,\mathbf{P}_2)}(K(l_1,l_2))) = \sum_{(k^1,k^2)\vdash(\mathbf{P}_1,\mathbf{P}_2)} \chi(M^{\Theta-\mathrm{st}}_{(k^1,k^2)}(\mathcal{N}(1))) \prod_{i=1}^{2}\prod_{j=1}^{l_i}\prod_{w} \frac{(-1)^{k^i_{w,j}(w-1)}}{k^i_{w,j}!w^{2k^i_{w,j}}}. \tag{4.2.1}$$

4.3 An upper bound for the Euler characteristic of Kronecker moduli spaces

In the case of the Kronecker quiver, the choice of the linear form $\Theta = (1,0)$ implies that Θ-coprime dimension vectors are precisely the coprime dimension vectors. If the conjecture of Douglas is true, it implies an exceptional growth of the Euler characteristic. In [69], it could be shown that it grows at least exponentially by constructing a lower bound for the Euler characteristic. Together with the results of [70], which yield an upper bound and which we review in this section, this shows that the Euler characteristic of Kronecker moduli spaces indeed grows exponentially.

We use the abbreviation $M^m_{d,e} := M^{\Theta-\mathrm{st}}_{(d,e)}(K(m))$. The BGP-reflection functor and taking transpose representations respectively yield isomorphisms

$$M^m_{d,e} \cong M^m_{d,md-e} \text{ and } M^m_{d,e} \cong M^m_{e,d}$$

which we use frequently. For a fixed weighted partition $\sum l d_l = d$ also denoted by $d_* \vdash d$, we define $\hat{d} = \sum_l d_l$ and $\tilde{d} = d - \hat{d}$. The considerations of the last two subsections yield

$$\chi(M^m_{d,e}) = \sum_{(d_*,e_*)\vdash(d,e)} \chi(M^{\Theta_{\mathcal{F}}-\mathrm{st}}_{d_*,e_*}(\mathcal{N}(m))) \prod_l \frac{(-1)^{(d_l+e_l)(l-1)}}{d_l!e_l!l^{2(d_l+e_l)}}.$$

Thus, it remains to determine the number of stable spanning trees $N^\Theta(\mathrm{supp}(d_*,e_*))$ for every weighted partition of (d,e). For a quiver Q, let $\widehat{Q}$ be the quiver obtained from Q when replacing multiple arrows between any two vertices by single arrows. Then, as a quiver, $\widehat{\mathcal{N}(m)}$ is just the full bipartite quiver with infinitely many sources and sinks. But we have to keep in mind that the level function is not constant on $\widehat{\mathcal{N}(m)}_0$.

Instead of counting the number of stable spanning trees of $\mathrm{supp}(d_*,e_*) \subset \mathcal{N}(m)$ for $(d_*,e_*) \vdash (d,e)$, we can also proceed as pointed out in Remark 4.2.3. This means that we count the number of stable spanning trees of $\widehat{\mathrm{supp}(d_*,e_*)} \subset \widehat{\mathcal{N}(m)}$ in a first step.

In a second step, we determine the number of colorings of each stable spanning tree. Every spanning tree of $\widehat{\mathrm{supp}(d_*, e_*)}$ has $\hat{d} + \hat{e} - 1$ arrows. Thus, taking into account that there are mll' arrows between $i_{l,k}$ and $j_{l',k'}$, for a fixed stable spanning tree $\mathcal{T}$ of $\widehat{\mathrm{supp}(d_*, e_*)}$, there are

$$m^{\hat{d}+\hat{e}-1} \prod_{\rho \in \mathcal{T}_1} l(s(\rho))l(t(\rho))$$

possibilities to color the arrows. We define

$$v(\mathcal{T}) := \prod_{\rho \in \mathcal{T}_1} l(s(\rho))l(t(\rho)).$$

In summary, we obtain

$$\chi(M^{\Theta-\mathrm{st}}_{(d_*,e_*)}(\mathcal{N}(m))) = m^{\hat{d}+\hat{e}-1} \sum_{\mathcal{T} \in N^{\Theta}(\widehat{\mathrm{supp}(d_*,e_*)})} v(\mathcal{T}).$$

As usual, the degree of a vertex is simply the number of its incident edges. There are several results concerning the number τ of spanning trees of (multi)graphs. We make use of the following two, see [63] and [66]:

Theorem 4.3.1. *We have:*

i) Let $K_{d,e} := (I + J, E)$ be the complete bipartite graph with $|I| = d$ and $|J| = e$. Then for the number of spanning trees, we have $\tau(K_{d,e}) = d^{e-1}e^{d-1}$.

ii) For the number of spanning trees of a multigraph G with vertices $q_1, \dots, q_n$ of degrees $d_1, \dots, d_n$ we have $\tau(G) \leq d_1 \dots d_{n-1}$.

Clearly, we have $K_{d,e} = \widehat{\mathrm{supp}(1 \cdot d, 1 \cdot e)} \subset \widehat{\mathcal{N}(m)}$. Since also $|N^{\Theta}(\widehat{\mathrm{supp}(d_*, e_*)})| \leq \tau(K_{d,e})$ is satisfied, the following result is immediate:

Corollary 4.3.2 ([70, Corollary 3.6])**.** *Let $(d_*, e_*) = (\sum_l ld_l, \sum_l le_l)$ be a pair of weighted partitions. Then we have*

$$\chi(M^{\Theta-\mathrm{st}}_{(1\cdot d,1\cdot e)}(\mathcal{N}(m))) \leq m^{d+e-1}d^{e-1}e^{d-1}$$

and

$$\chi(M^{\Theta-\mathrm{st}}_{(d_*,e_*)}(\mathcal{N}(m))) \leq m^{\hat{d}+\hat{e}-1}e^{\hat{d}}d^{\hat{e}} \prod_l l^{d_l+e_l}.$$

As far as an upper bound is concerned, we are left with the coefficients $\kappa(d_*, e_*)$ appearing in the MPS degeneration formula. We recall the following lemma:

Lemma 4.3.3 ([70, Lemma 3.9])**.** *Let $\frac{1}{m}d \leq e \leq md$ for d, $e \in \mathbb{N}$. Then we have*

$$\frac{1}{\hat{d}!\hat{e}!\prod_l l^{d_l+e_l}} \binom{d}{\tilde{d}} \binom{e}{\tilde{e}} \leq 2^{d+e}(de)^{\frac{1}{2}} \frac{1}{d!e!} e^{\tilde{d}} d^{\tilde{e}} m^{\tilde{d}+\tilde{e}}.$$

Now we can use that the number of partitions of a natural number n is bounded by $\exp(\pi\sqrt{2n/3})$, see [39, Section 6], in order to prove the main results of [70]:

Theorem 4.3.4 ([70, Theorem 3.12, Corollary 3.14])**.** *Let (d, e) be a coprime dimension vector of $K(m)$ such that $e \approx kd$.*

i) We have

$$\chi(M_{d,e}^m) \leq \frac{1}{d!e!} 2^{d+e} m^{d+e-1} \exp(\pi\sqrt{\frac{2}{3}}(\sqrt{d}+\sqrt{e})) e^{d+1/2} d^{e+1/2}.$$

ii) We have

$$\lim_{d\to\infty} \frac{1}{d} \ln \chi(M_{d,e}^m) \leq (k+1)(\ln(m)+\ln 2+1) - (k-1)\ln k.$$

In particular, the Euler characteristic of Kronecker moduli spaces grows exponentially with the dimension vector.

4.4 Exact formulae for the Euler characteristic of moduli spaces

Sometimes it is possible to determine the Euler characteristic of certain moduli spaces of stable representations exactly. In most of these cases, the idea of the derivation of a formula is similar. Initially, one has to identify the stable spanning trees. If this turns out to follow a certain recursion, there is hope to find a generating function for these trees whose coefficients can be determined explicitly. As already noticed in Remark 4.2.3, it is often convenient to forget the coloring or even the labelling of the spanning trees, i.e. to count only the possible underlying graphs up to automorphism.

As far as the determination of the coefficients is concerned, one can often proceed analogously to Section 3.2.1. In particular, the number of trees we are looking for is given by the respective coefficient of the generating function which can be obtained by the Lagrangian Inversion Theorem. In a possible further step, one has to count the number of colorings in order to obtain a formula for the Euler characteristic which is of the following shape:

$$\chi(M_{(d_*,e_*)}^{\Theta-\mathrm{st}}(\mathrm{supp}(d_*,e_*))) = \sum_{\mathcal{T}\in N^\Theta(\widehat{\mathrm{supp}(d_*,e_*)})} \frac{c(\mathcal{T})}{|\mathrm{Aut}(\mathcal{T})|} \qquad (4.4.1)$$

where $c(\mathcal{T})$ denotes the number of colorings of a fixed spanning tree.

4.4.1 The trivial partition of $\mathcal{N}(m)$

If we consider the quiver $\mathcal{N}(m)$ with the trivial partition $(1\cdot d, 1\cdot e) \vdash (d,e)$, by Theorem 4.2.5, we have

$$\chi(M_{(1\cdot d,1\cdot e)}^{\Theta-\mathrm{st}}(\mathcal{N}(m))) = |N^\Theta(\mathrm{supp}(1\cdot d, 1\cdot e))|.$$

Thus, we are left with the problem of counting stable spanning trees of $\operatorname{supp}(1 \cdot d, 1 \cdot e)$. In some particular cases, one even gets a formula for this counting problem. Let us consider the case where $e = kd + 1$ for some $k \in \mathbb{N}_+$. We make use of the following lemma and also include the proof in order to give an idea for the derivation of certain formulae for Euler characteristics:

Lemma 4.4.1 ([70, Lemma 5.1])**.** *Every source of a stable spanning tree $\mathcal{T}$ of* $\operatorname{supp}(1 \cdot d, 1 \cdot (kd+1)) \subset \mathcal{N}(m)$ *has exactly $k+1$ neighbors.*

Proof. The stability condition yields that every source $i \in \mathcal{T}_0$ has at least $k+1$ neighbors. Now we can be proceed by induction on the number of sources d. Since $\mathcal{T}$ is a tree, it has a subquiver $(i, j_1, \ldots, j_n)$ such that $N_{j_s} = \{i\}$ for all but one $s \in \{1, \ldots, n\}$. If we had $n \geq k+2$, the remaining part of the induced localization datum would contradict the stability condition because

$$k(d+1) + 1 - (k+1) = kd < \frac{k(d+1)+1}{d+1} d = kd + \frac{d}{d+1}.$$

This means that $n = k+1$. Deleting this subquiver except the sink j_s with $|N_{j_s}| > 1$, it is straightforward to check that we obtain a stable spanning tree of $\operatorname{supp}(1 \cdot d, 1 \cdot (kd+1))$ because

$$\frac{k(d+1)+1}{d+1} d' < e' \Leftrightarrow \frac{kd+1}{d} d' < e'$$

for all $d' < d$ and $e' \in \mathbb{N}$. Thus, the claim follows by the induction hypothesis. □

For the proof to work, it is very important that we get a stable spanning tree of the same shape after we have removed a subquiver. This does not hold for general dimension vectors. Since all sinks and sources have level one, the number of colorings is constant along the unlabelled stable spanning trees. More precisely, we have

$$c(\mathcal{T}) = d!(kd+1)! m^{(k+1)d}$$

because we have d sources, $kd+1$ sinks and thus $(k+1)d$ arrows where we have m possible colors for each of them. Thus, we are left with the task of counting the number of stable spanning trees of $\operatorname{supp}(1 \cdot d, 1 \cdot (kd+1))$ taking into account its automorphisms. Therefore, keeping in mind (4.4.1), we need to determine

$$\mathcal{S}(d, kd+1) := \sum_{\mathcal{T}} \frac{1}{\operatorname{Aut}(\mathcal{T})}$$

where the sum is taken over all unlabelled stable spanning trees of $\operatorname{supp}(1 \cdot d, 1 \cdot (kd+1))$.

Proposition 4.4.2 ([70, Proposition 5.2])**.** *We have*

$$\mathcal{S}(d, kd+1) = \frac{1}{(kd+1)^2} \frac{1}{d!} \left(\frac{(kd+1)}{k!} \right)^d.$$

Proof. We only sketch the proof because it is similar to the proof of Proposition 3.2.6. Let $y(x)$ be the generating function of rooted (!) unlabelled spanning trees of dimension $(1 \cdot d, 1 \cdot (kd+1))$ taking into account quiver automorphism. Then $y(x)$ satisfies the functional equation $y(x) = x\Phi(y(x))$ where

$$\Phi(x) := \exp\left(\frac{x^k}{k!}\right) = \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{x^k}{k!}\right)^n.$$

By the usual Lagrangian inversion formula, see for instance [64, Section 5.4], we thus have:

$$[x^t]y(x) = \frac{1}{t}[u^{t-1}]\Phi(u)^t = \frac{1}{t}[u^{t-1}]\sum_{n=0}^{\infty}\frac{1}{n!}\left(t\frac{u^k}{k!}\right)^n = \frac{1}{t}\begin{cases} \frac{1}{(\frac{t-1}{k})!}\left(\frac{t}{k!}\right)^{\frac{t-1}{k}} & \text{if } k|(t-1) \\ 0 & \text{otherwise} \end{cases}.$$

In particular, we have

$$[x^{kd+1}]y(x) = \frac{1}{(kd+1)}\frac{1}{d!}\left(\frac{(kd+1)}{k!}\right)^d.$$

Since every graph has $kd+1$ sinks which can be assigned to be the root, the result follows. □

Thus, we obtain the following result:

Theorem 4.4.3 ([70, Theorem 5.4, Corollary 5.5])**.** *We have:*

i)

$$\chi(M^{\Theta-\mathrm{st}}_{(1\cdot d,1\cdot(kd+1))}(\mathcal{N}(m))) = m^{(k+1)d}\frac{(kd)!}{kd+1}\left(\frac{kd+1}{k!}\right)^d.$$

ii)

$$\lim_{d\to\infty}\frac{1}{d}\ln\left(\frac{\chi(M^{\Theta-\mathrm{st}}_{(1\cdot d,1\cdot(kd+1))}(\mathcal{N}(m)))}{d!(kd+1)!}\right) = \ln(m)(k+1) + 1 - \ln((k-1)!).$$

By [69, Theorem 6.6], we have

$$\lim_{d\to\infty}\frac{\ln(\chi(M^m_{d,d+1}))}{d} = (m-1)^2\ln(m-1)^2 - (m^2-2m)\ln(m^2-2m).$$

Comparing this to Theorem 4.4.3, we see that we have

$$\lim_{d\to\infty}\frac{1}{d}\ln\left(\frac{\chi(M^{\Theta-\mathrm{st}}_{(1\cdot d,1\cdot(d+1))}(\mathcal{N}(m)))}{d!(d+1)!}\right) - \lim_{d\to\infty}\frac{\ln(\chi(M^m_{d,d+1}))}{d} > 0$$

already for small m. As far as the MPS degeneration formula is concerned, this means that it does not suffice to consider the summand corresponding to the trivial partition in order to investigate the asymptotic behavior of the Euler characteristic.

4.4.2 Formulae for the Euler characteristic of moduli spaces of $K(l_1, l_2)$

In [55, Section 15], several formulae for Euler characteristics of moduli spaces associated with the quiver $K(l_1, l_2)$ are derived. They are particularly interesting as they can be used to determine certain Gromov-Witten invariants. Notice that in almost all cases it is easier to determine Euler characteristics than Gromov-Witten invariants. Thus, this formulae turn out to be very helpful in Gromov-Witten theory as we will see later in this work.

We denote the sources of $K(l_1, l_2)$ by I and its sinks by J. Every pair of partitions $(\mathbf{P}_1, \mathbf{P}_2)$ of length l_1 and l_2 respectively defines a dimension vector of $K(l_1, l_2)$. We call $(d, e) := (|\mathbf{P}_1|, |\mathbf{P}_2|)$ the Kronecker type of the partition. Again the case where $|\mathbf{P}_2| = (l_2 - 1)|\mathbf{P}_1| + 1$ is special because all localization data are known in this case and, moreover, the corresponding moduli spaces are points. Analogous to Lemma 4.4.1, one can show the following:

Lemma 4.4.4. *Let $(\mathbf{Q}, (\tilde{\mathbf{P}}_1, \tilde{\mathbf{P}}_2))$ be a localization datum of $K(l_1, l_2)$ with sources $\mathbf{I}$ and sinks $\mathbf{J}$ such that $|\tilde{\mathbf{P}}_1| = d$ and $|\tilde{\mathbf{P}}_2| = (l_2 - 1)d + 1$. Then the partitions $\tilde{\mathbf{P}}_1$ and $\tilde{\mathbf{P}}_2$ are trivial and for every $i \in \mathbf{I}$ we have $|N_i| = l_2$. The other way around, every subtree of $\widetilde{K(l_1, l_2)}$ such that every source has l_2 neighbors defines a localization datum.*

Proceeding similar to Theorem 4.4.3, we obtain:

Theorem 4.4.5 ([55, Theorem 15.3])**.** *We have*

$$\sum_{|\mathbf{P}_1|=d, |\mathbf{P}_2|=(l_2-1)d+1} \chi(M^{\Theta-\mathrm{st}}_{(\mathbf{P}_1,\mathbf{P}_2)}(K(l_1, l_2)) = \frac{l_1 l_2}{d((l_2 - 1)d + 1)} \binom{(l_1 - 1)(l_2 - 1)d + l_1 - 1}{d - 1}.$$

Reflecting at each sink, this covers also the case $|\mathbf{P}_1| = d - 1$, $|\mathbf{P}_2| = d$ of $K(l_2, l_1)$. Actually, this formula has the same shape as the one obtained in [69, Theorem 6.6] for $\chi(M^m_{d,d+1})$. This is no coincidence and the reason for this lies in the fact that unlabelled localization data of $K(l_1, l_2)$ induce unlabelled localization data of the Kronecker quiver. In terms of quiver homomorphisms this just means that we get an embedding $\widetilde{K(l_1, l_2)} \hookrightarrow \widetilde{K(m)}$ where $m = \max\{l_1, l_2\}$.

These ideas can also be used to prove the next statement for which we need some more notation. We assume that $l_2 \geq l_1$. Denote the set of uncolored localization data of dimension $(\mathbf{P}_1, \mathbf{P}_2)$ by $\mathcal{L}_{\mathbf{P}_1,\mathbf{P}_2}(K(l_1, l_2))$ and define

$$\mathcal{L}_{(d,e)}(K(l_1, l_2)) := \bigcup_{|\mathbf{P}_1|=d, |\mathbf{P}_2|=e} \mathcal{L}_{\mathbf{P}_1,\mathbf{P}_2}(K(l_1, l_2)).$$

Since there only exists at most one arrow between any two vertices, this coincides with the set of localization data of dimension type $(\mathbf{P}_1, \mathbf{P}_2)$. Moreover, for the Kronecker quiver $K(l_2)$ we denote by $\mathcal{L}^{l_1}_{(d,e)}(K(l_2))$ those uncolored localization data $(\mathbf{Q}, \tilde{\alpha})$ such that $|N_j| \leq l_1$ for all sinks $j \in \mathbf{Q}_0$. Keeping in mind Remark 4.2.3, the following lemma can be proved:

Lemma 4.4.6 ([55, Lemma 15.1]). *There exists a one-to-one correspondence between the set of localization data* $\mathcal{L}_{(d,e)}(K(l_1,l_2))$ *and tuples* $(\mathbf{Q},\tilde{\alpha},c:\mathbf{Q}_1\to K(l_1,l_2)_1)$ *where* $(\mathbf{Q},\tilde{\alpha})\in\mathcal{L}^{l_1}_{(d,e)}(K(l_2))$ *and* $c:\mathbf{Q}_1\to K(l_1,l_2)_1$ *is a coloring such that arrows which have the same sink or source are colored differently.*

Using this lemma and, moreover, an induction on the sources, the following can be shown:

Theorem 4.4.7 ([55, Theorem 15.2]). *Fix a Kronecker type* (d,e) *of* $K(l_1,l_2)$*. Then we have*

$$\sum_{|\mathbf{P}_1|=d,|\mathbf{P}_2|=e}\chi(M^{\Theta-\mathrm{st}}_{(\mathbf{P}_1,\mathbf{P}_2)}(K(l_1,l_2)))=\sum_{(\mathbf{Q},\tilde{\alpha})\in\mathcal{L}^{l_1}_{(d,e)}(K(l_2))}\mathcal{C}_{K(l_1,l_2)}(\mathbf{Q})\chi(M^{\Theta-\mathrm{st}}_{\tilde{\alpha}}(\mathbf{Q}))$$

where $\mathcal{C}_{K(l_1,l_2)}(\mathbf{Q})$ *is the number of colorings* $c:\mathbf{Q}_1\to K(l_1,l_2)_1$*. If* $m:=l_1=l_2$*, we thus have* $\mathcal{C}_{K(m,m)}(\mathbf{Q})=m\mathcal{C}_{K(m)}(\mathbf{Q})$ *for all uncolored localization data* $(\mathbf{Q},\tilde{\alpha})$*. In particular, we have*

$$\sum_{|\mathbf{P}_1|=d,|\mathbf{P}_2|=e}\chi(M^{\Theta-\mathrm{st}}_{(\mathbf{P}_1,\mathbf{P}_2)}(K(m,m)))=m\chi(M^m_{d,e}).$$

4.4.3 Vanishing of the Euler characteristic

To prove that the Euler characteristic of a certain moduli space vanishes, is one of the major applications of iterated localization. The key observation is that the quivers belonging to localization data of a certain dimension type are forced to have cycles which they are not allowed to have (respectively which do not remain after iterated localization). It turns out that this can be used to prove certain conjectures in Gromov-Witten theory via the refined GW/Kronecker correspondence which is reviewed in Section 4.5. Most of the material presented in this section can be found in [55, Section 16].

Let Q be a bipartite quiver with vertices $I\cup J$, $a(i,j)$ arrows from i to j and a level $l:Q_0\to\mathbb{N}_+$. Then we consider $\Theta,\kappa\in\mathrm{Hom}(\mathbb{Z}Q_0,\mathbb{Z})$ defined by

$$\Theta(\alpha)=\sum_{i\in I}l(i)\alpha_i,\quad \kappa(\alpha)=\sum_{q\in Q_0}l(q)\alpha_q$$

and the corresponding slope denoted by μ. If I and J are finite, we denote the vertices by $I=\{i_1,\ldots,i_m\}$ and $J=\{j_1,\ldots,j_n\}$. It can be checked that a representation $M\in\mathrm{Rep}(Q)$ is stable if and only if

$$\sum_{j\in J}l(j)\beta_j>\frac{\sum_{j\in J}l(j)\alpha_j}{\sum_{i\in I}l(i)\alpha_i}\sum_{i\in I}l(i)\beta_i \tag{4.4.2}$$

for all $0,\alpha\neq\beta\in\mathbb{N}Q_0$ where $\mathrm{Gr}_\beta(M)\neq\emptyset$.

Let $K \in \mathbb{Q}$ such that α satisfies

$$K \sum_{i \in I} l(i)\alpha_i = \sum_{j \in J} l(j)\alpha_j$$

and let $(\mathbf{Q}, \tilde{\alpha})$ be a localization datum with sources $\mathbf{I}$ and sinks $\mathbf{J}$. Every source $i \in \mathbf{I}$ gives rise to a subrepresentation of dimension $i + \sum_{j \in N_i} j$. Similarly, every sink $j \in \mathbf{J}$ gives rise to a factor representation of dimension $j + \sum_{i \in N_j} i$. Using inequality (4.4.2), this yields conditions for the level of i and j respectively, see also [55, Proposition 16.2], which are

$$Kl(i) < \sum_{j \in N_i} l(j), \quad l(j) < K \sum_{i \in N_j} l(i).$$

Here we assume that the above subrepresentations are proper subrepresentations. If this is the case and if the level is constant on the sources and sinks, say that we have $l(i) = \lambda_1$ and $l(j) = \lambda_2$ for all $i \in I$ and $j \in J$, and if $K = \frac{\lambda_2}{\lambda_1}$, these inequalities become

$$\frac{\lambda_2}{\lambda_1}\lambda_1 < |N_i|\lambda_2, \quad \lambda_2 < \frac{\lambda_2}{\lambda_1}|N_j|\lambda_1.$$

This simply means that every source and every sink has at least two neighbors. This observation can be used to obtain the first instances of vanishing Euler characteristics:

Theorem 4.4.8 ([55, Theorem 16.3, Theorem 16.5])**.** *Let $\alpha \in \mathbb{N}Q_0$ be a dimension vector such that $\sum_{i \in I} \alpha_i \neq 1$ and $\sum_{j \in J} \alpha_j \neq 1$. Moreover, let*

$$K \sum_{i \in I} l(i)\alpha_i = \sum_{j \in J} l(j)\alpha_j$$

i) If $l(i) = \lambda_1$ for all $i \in I$, $l(j) = \lambda_2$ for all $j \in J$ and $K = \frac{\lambda_2}{\lambda_1}$, there does not exist any localization datum of this dimension type.

ii) If there exists a sink $j \in J$ such that

$$l(j) \geq K \sum_{i \in N_j} a(i,j) l(i)$$

or if there exists a source $i \in I$ such that

$$l(i) \geq \frac{1}{K} \sum_{j \in N_i} a(i,j) l(j),$$

there does not exist any localization datum of this dimension type.

In particular, the Euler characteristic of the corresponding moduli space vanishes in both cases.

The most important application for us is the following generalization of [69, Corollary 6.14]:

Corollary 4.4.9 ([55, Corollary 16.4]). *Let $l(q) = 1$ for all $q \in Q_0$, $K = 1$ and let $\alpha \in \mathbb{N}Q_0$ be a dimension vector such that*

$$\sum_{i \in I} \alpha_i = \sum_{j \in J} \alpha_j.$$

If $\sum_{i \in I} \alpha_i = 1$, there exists exactly one $i \in I$ and $j \in J$ such that $\alpha_i = \alpha_j = 1$ and we have $\chi(M_\alpha^{\Theta-\mathrm{st}}(Q)) = a(i,j)$. If $\sum_{i \in I} \alpha_i \neq 1$, we have $\chi(M_\alpha^{\Theta-\mathrm{st}}(Q)) = 0$.

Let us for completeness point out that there is another generalization of [69, Corollary 6.14]. Actually, this is the most important special case of [55, Theorem 16.7]:

Theorem 4.4.10 ([55, Corollary 16.8]). *Assume that $l(q) = 1$ for all vertices of the quiver Q and assume that*

$$K \sum_{i \in I} \alpha_i = \sum_{j \in J} \alpha_j$$

where $\sum_{i \in I} \alpha_i \neq 1$ and $K \in \mathbb{N}_+$. If we have $\sum_{j \in J} a(i,j) \leq K + 1$ for all $i \in I$, there exists no localization data of dimension type α. In particular, the Euler characteristic of the corresponding moduli space vanishes.

If we have $I = \{i\}$ and $J = \{j\}$, the quiver Q is a generalized Kronecker quiver $K(m)$ and the condition we obtain simplifies to $K \geq m-1$. In this case, the statement is only non-trivial for $K = m - 1$, and it can also be obtained from Corollary 4.4.9 when applying BGP-reflections.

Remark 4.4.11.

i) In [55, Section 16], the no-peak condition plays an important role. Since it is rather straightforward to check that it is automatically satisfied in the cases we treat here, we do not need to take care of it.

ii) The most famous example for a non-empty moduli space with vanishing Euler characteristic is probably the case of the dimension vectors (d, d) of the generalized Kronecker quiver $K(m)$ for $d \geq 2$. Since this is true for $d \geq 2$ and general m, the question for a reason and generalizations arises. One explanation might be that (d, d) is an isotropic root for $m = 2$, i.e. $\langle (d, d), (d, d) \rangle = 0$. Thus, we have $M_{d,d}^2 = \emptyset$ for $d \geq 2$. This is actually not true for $m \geq 3$ where the Euler characteristic vanishes even if the moduli space is not empty. In all cases, there do not exist stable representations of the universal covering of dimension type (d, d), but only semi-stable ones. Further investigation in this direction could be very interesting.

Let us consider a final case which could also be a starting point for generalizations:

Theorem 4.4.12. *Assume that the level $l : Q_0 \to \mathbb{N}_+$ satisfies $l(j) = 1$ for all $j \in J$ and that $\alpha \in \mathbb{N}Q_0$ satisfies $\alpha_i = 1$ for all $i \in I$. If, moreover,*

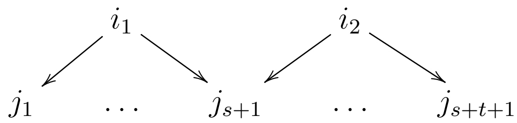
$$K \sum_{i \in I} l(i)\alpha_i = \sum_{j \in J} l(j)\alpha_j$$

for some $K \in \mathbb{N}_+$, we have $\chi(M_\alpha^{\Theta-\mathrm{st}}(Q)) = 0$ if $|I| \geq 2$. If $I = \{i\}$ and $\mathrm{supp}(\alpha) = Q_0$*, we have $\chi(M_\alpha^{\Theta-\mathrm{st}}(Q)) = \prod_{j \in J} a(i,j)$ if, in addition, $\alpha_j = 1$ for all $j \in J$ and $\chi(M_\alpha^{\Theta-\mathrm{st}}(Q)) = 0$ otherwise.*

Proof. Let $I = \{i_1, \ldots, i_m\}$. If $|I| = 1$, α is a root if and only if $\alpha_j \in \{0, 1\}$ for all $j \in J$. Thus, if $\mathrm{supp}(\alpha) = Q_0$, there is actually exactly one uncolored localization datum which is given by (Q, α). In particular, we have

$$\chi(M_\alpha^{\Theta-\mathrm{st}}(Q)) = \prod_{j \in J} a(i_1, j).$$

We proceed by induction on $|I|$. If $|I| = 2$, since $\alpha_i = 1$ and since there need to exist stable representations, an uncolored localization datum $(\mathbf{Q}, \tilde{\alpha})$ would need to have the form



with $\tilde{\alpha}_{i_l} = \tilde{\alpha}_{j_l} = 1$. By assumption, we have $K(l(i_1) + l(i_2)) = s + t + 1$. But the stability condition yields

$$s + 1 > Kl(i_1), \quad t + 1 > Kl(i_2),$$

and thus it also yields a contradiction.

The induction step follows a common argument. More precisely, we consider a subquiver S of a localization datum $(\mathbf{Q}, \tilde{\alpha})$ which has only one common sink with the remainder of the quiver. Let S have vertices $\{i_1, j_1 \ldots, j_n\}$ and let j_n be the common vertex. Since $(\mathbf{Q}, \tilde{\alpha})$ is forced to admit stable representations, we have $\tilde{\alpha}_{j_l} = 1$ for $l = 1, \ldots, n-1$ and thus $\sum_{l=1}^{n-1} \tilde{\alpha}_{j_l} = n - 1$. Moreover, we have $Kl(i_1) < n$. We claim that $Kl(i_1) = n - 1$. Indeed, if $Kl(i_1) < n - 1$, we had

$$K \sum_{i \in I \setminus \{i_1\}} l(i) = \sum_{j \in J} \tilde{\alpha}_j - Kl(i_1) > \sum_{j \in J \setminus \{j_1, \ldots, j_{n-1}\}} \tilde{\alpha}_j$$

which yields a contradiction to the stability condition. This means that, after removing the subquiver with vertices $\{i_1, j_1, \ldots, j_{n-1}\}$, we obtain a localization datum with

$$K \sum_{i \in I \setminus \{i_1\}} l(i) = \sum_{j \in J \setminus \{j_1, \ldots, j_{n-1}\}} \tilde{\alpha}_j.$$

Since there does not exist such a localization datum, the claim follows.

□

Let us conclude this section with the following remark:

Remark 4.4.13.

i) The former result could be indeed very helpful for future considerations, particularly if there were a certain generalization of the MPS degeneration formula to non-coprime dimension vectors. As an example, consider the quiver $K(m)(q_0)$, i.e. the MPS-quiver of $K(m)$ with respect to the source of $K(m)$. In this case, all localization data $(\mathbf{Q}, \alpha)$ would satisfy $\alpha_i = 1$ and $l(j) = 1$ for all $i \in \mathbf{I}$ and $j \in \mathbf{J}$. The dimension vectors considered in the preceding theorem would then correspond to the dimension vector (d, Kd) of the Kronecker quiver where K is a natural number. Thus, there were hope to make statements concerning the Euler characteristics of the corresponding moduli spaces. Note that they do not vanish in general. An example for this is $\chi(M^4_{2,4}) = -1$. Also note that here the no-peak condition comes into play.

ii) In [12], a (non-explicit) formula for the Euler characteristic of the Kronecker moduli spaces $M^m_{d,Kd+1}$ is obtained when applying the MPS degeneration formula only to the source of $K(m)$. In there, after applying more involved methods coming from theoretical physics, one is again left with counting certain trees. Without checking the details, it is likely that this trees precisely correspond to the respective torus fixed points (obtained after applying the MPS degeneration formula to the source). Moreover, it seems that in this case there are only finitely many torus fixed points.

4.5 Refined GW/Kronecker correspondence

It was noted in [29] and [53] that there is a numerical correspondence between the Euler characteristic of moduli spaces of stable quiver representations and Gromov-Witten invariants coming along with weighted projective planes. This was explained in detail in [28] leading to the following result:

Theorem 4.5.1 ([28, Corollary 3])**.** *For every natural number $m > 0$ and coprime $(d, e) \in \mathbb{N}^2_+$, we have*

$$\exp\left(\sum_{k=1}^{\infty} \sum_{|\mathbf{P}_1|=kd, |\mathbf{P}_2|=ke} kN_{(d,e)}[(\mathbf{P}_1, \mathbf{P}_2)](tx)^{kd}(ty)^{ke}\right)$$

$$= \left(\sum_{k=0}^{\infty} \chi(M^{\Theta-\mathrm{st}}_{(kd,ke),(1,0)}(K(m)))(tx)^{kd}(ty)^{ke}\right)^{m/d}$$

$$= \left(\sum_{k=0}^{\infty} \chi(M^{\Theta-\mathrm{st}}_{(kd,ke),(0,1)}(K(m)))(tx)^{kd}(ty)^{ke}\right)^{m/e}$$

where the second sum in the first line is over all ordered partitions of length m permitting zero entries.

Note that the moduli spaces which appear in the formulae are framed moduli spaces as defined in Section 2.4. Apparently, the partitions of kd and ke play only a role on the Gromov-Witten side of the formula. One major motivation in [55] was to find a correspondence which takes care of the partitions on the quiver side as well. This can be achieved when replacing the generalized Kronecker quivers by the quivers $K(l_1, l_2)$ and leads to the so-called refined GW/Kronecker correspondence which we investigate in the following.

4.5.1 Gromov-Witten invariants vs. Euler characteristics

The main observation is that both the generating function of Gromov-Witten invariants and the one of the Euler characteristic of moduli spaces are linked to the series $f_{(d,e)}$ appearing in the tropical vertex via Theorems 2.5.2 and 2.4.2. This refined correspondence enables us to determine several Gromov-Witten invariants explicitly which were unknown before. This is due to the already mentioned fact that Euler characteristics are often easier to compute than Gromov-Witten invariants. In turn, we can prove several conjectures of [28, 29] concerning the invariants

$$N_{(d,e)}[k] := \sum_{|\mathbf{P}_1|=kd, |\mathbf{P}_2|=ke} N_{(d,e)}[(\mathbf{P}_1, \mathbf{P}_2)]$$

with the help of the invariants

$$\chi_{(d,e)}(k) := \sum_{|\mathbf{P}_1|=kd, |\mathbf{P}_2|=ke} \chi(M^{\Theta-\mathrm{st}}_{(\mathbf{P}_1,\mathbf{P}_2)}(K(l_1, l_2))).$$

More detailed, using the factorization of Theorem 2.4.2, the following is obtained in [55]:

Theorem 4.5.2 ([55, Theorem 6.1])**.** *For all coprime (d, e) we have*

$$\begin{aligned} f_{(d,e)} &= \left(\sum_{k\geq 0} \sum_{|\mathbf{P}_1|=kd, |\mathbf{P}_2|=ke} \chi(M^{\Theta-\mathrm{st}}_{(\mathbf{P}_1,\mathbf{P}_2),f}(K(l_1, l_2)))(tx)^{kd}(ty)^{ke}\right)^{1/e} \\ &= \left(\sum_{k\geq 0} \sum_{|\mathbf{P}_1|=kd, |\mathbf{P}_2|=ke} \chi(M^{\Theta-\mathrm{st}}_{(\mathbf{P}_1,\mathbf{P}_2),b}(K(l_1, l_2)))(tx)^{kd}(ty)^{ke}\right)^{1/d} \end{aligned}$$

where the functionals b and f are defined by $b(i) = 1$, $b(j) = 0$, $f(i) = 0$ and $f(j) = 1$ for all sources $i \in K(l_1, l_2)_0$ and all sinks $j \in K(l_1, l_2)_0$.

This immediately yields the following refined version of Corollary 4.5.1:

Corollary 4.5.3 ([55, Corollary 6.2]). *For all coprime (d,e), we have*

$$\exp\left(\sum_{k=1}^{\infty}\sum_{|\mathbf{P}_1|=kd,|\mathbf{P}_2|=ke} kN_{(d,e)}[(\mathbf{P}_1,\mathbf{P}_2)]s^{\mathbf{P}_1}t^{\mathbf{P}_2}x^{kd}y^{ke}\right)$$

$$=\left(\sum_{k=0}^{\infty}\sum_{|\mathbf{P}_1|=kd,|\mathbf{P}_2|=ke}\chi(M^{\Theta-\mathrm{st}}_{(\mathbf{P}_1,\mathbf{P}_2),f}(K(l_1,l_2)))s^{\mathbf{P}_1}t^{\mathbf{P}_2}x^{kd}y^{ke}\right)^{1/e}$$

$$=\left(\sum_{k=0}^{\infty}\sum_{|\mathbf{P}_1|=kd,|\mathbf{P}_2|=ke}\chi(M^{\Theta-\mathrm{st}}_{(\mathbf{P}_1,\mathbf{P}_2),b}(K(l_1,l_2)))s^{\mathbf{P}_1}t^{\mathbf{P}_2}x^{kd}y^{ke}\right)^{1/d}$$

where the sums are over all ordered partitions of kd and ke of length l_1 and l_2 respectively.

In a next step, we can apply Theorem 2.4.4 in order to determine both the series $f_{(d,e)}$ and its specialization $f_{(d,e)}(t)$ which shows up when specializing all variables s_k, t_l to one variable t. This is done in [55, Sections 7, 8] and leads to the following result:

Theorem 4.5.4 ([55, Theorem 7.1, Theorem 8.1, Corollary 8.2]). *For coprime (d,e), the series $f_{(d,e)}$ is given by*

$$f_{(d,e)}=\prod_{k\geq 1}\prod_{\substack{|\mathbf{P}_1|=kd\\|\mathbf{P}_2|=ke}}(R^{\mathbf{P}_1,\mathbf{P}_2})^{k\chi(M^{\Theta-\mathrm{st}}_{(\mathbf{P}_1,\mathbf{P}_2)}(K(l_1,l_2)))},$$

where the series $R^{\mathbf{P}_1,\mathbf{P}_2}\in B$ are determined by the following system of functional equations:
For all pairs of ordered partitions $(\mathbf{P}_1,\mathbf{P}_2)$ as above, we have

$$R^{\mathbf{P}_1,\mathbf{P}_2}=\left(1-s^{\mathbf{P}_1}t^{\mathbf{P}_2}(x^dy^e)^k\prod_{k'\geq 1}\prod_{\substack{|\mathbf{P}'_1|=k'd\\|\mathbf{P}'_2|=k'e}}(R^{\mathbf{P}'_1,\mathbf{P}'_2})^{-\langle(\mathbf{P}_1,\mathbf{P}_2),(\mathbf{P}'_1,\mathbf{P}'_2)\rangle\chi(M^{\Theta-\mathrm{st}}_{(\mathbf{P}'_1,\mathbf{P}'_2)}(K(l_1,l_2)))}\right)^{-1}.$$

Specializing, the series $f_{(d,e)}(t)$ is determined by the single functional equation

$$f_{(d,e)}(t)=\prod_{k\geq 1}\left(1-((tx)^d(ty)^ef_{(d,e)}(t)^E)^k\right)^{-k\chi_{(d,e)}(k)}$$

where

$$E=\frac{l_1l_2de-l_2d^2-l_1e^2}{l_1l_2}\in\mathbb{Q}$$

which, in turn, yields

$$N_{(d,e)}[k]=\frac{1}{Ek^2}\sum_{r\vdash k}\prod_l\binom{Ekl\chi_{(d,e)}(l)+r_l-1}{r_l},$$

where the sum runs over all ordered partitions $\sum_l lr_l=k$.

The results of this section also yield the following corollary:

Corollary 4.5.5 ([55, Corollary 9.1]). *For coprime (d,e) and a pair of ordered partitions $(\mathbf{P}_1,\mathbf{P}_2)$ such that $|\mathbf{P}_1|=d$ and $|\mathbf{P}_2|=e$, we have*

$$\begin{aligned} N_{(d,e)}[(\mathbf{P}_1,\mathbf{P}_2)] &= \frac{1}{d}\cdot\chi(M^{\Theta-\mathrm{st}}_{(\mathbf{P}_1,\mathbf{P}_2),b}(K(l_1,l_2)))=\frac{1}{e}\cdot\chi(M^{\Theta-\mathrm{st}}_{(\mathbf{P}_1,\mathbf{P}_2),f}(K(l_1,l_2)))\\ &= \chi(M^{\Theta-\mathrm{st}}_{(\mathbf{P}_1,\mathbf{P}_2)}(K(l_1,l_2))). \end{aligned}$$

Example 4.5.6. We can use Corollary 4.5.5 to determine several Gromov-Witten invariants explicitly. For instance, when applying Theorem 4.4.5, we obtain the explicit formula

$$N_{(d-1,d)}[1]=\chi_{(d-1,d)}(1)=\frac{l_1l_2}{d((l_1-1)d+1)}\binom{(l_1-1)(l_2-1)d+l_2-1}{d-1}.$$

One can also work out that $N_{(3,5)}[1]=204$ if $l_1=l_2=3$. This is done in [55, Section 15].

Obviously, the functional equations of Theorem 4.5.4 simplify in the case when certain Euler characteristics vanish. This is the case when $d=e=1$ and $k\geq 2$. If $k=1$, the appearing moduli spaces are points and we obtain $\chi_{(d,d)}[k]=l_1l_2$. Since we have

$$E=\frac{l_1l_2-l_1-l_2}{l_1l_2}$$

in this case, we obtain

$$f_{(1,1)}=\prod_{k=1}^{l_1}\prod_{l=1}^{l_2}R^{k,l},$$

where the series $R^{k,l}$ are determined by the system of functional equations

$$R^{k,l}=1+s_kt_lxy\prod_{k'\neq k}\prod_{l'\neq l}R^{k',l'}.$$

This system can be solved using multivariate Lagrange inversion, but this is not very explicit. When specializing the variables s_k,t_l to one variable t, the system of functional equations simplifies further so that only one equation remains. More detailed, we obtain

$$f_{(1,1)}(t)=H(t)^{l_1l_2}$$

where the series H is determined by the single functional equation

$$H(t)=(1-t^2xyH(t)^{l_1l_2-l_1-l_2})^{-1}.$$

But then it follows immediately from [50, Theorem 1.4] that:

Corollary 4.5.7 ([55, Corollary 11.2]). *We have*

$$f_{(1,1)}(t)=\left(\sum_{k\geq 0}\frac{1}{(l_1l_2-l_1-l_2)k+1}\binom{(l_1-1)(l_2-1)k}{k}(t^2xy)^k\right)^{l_1l_2},$$

confirming [28, Conjecture 1.4].

We can also apply the last formula of Theorem 4.5.4 and obtain:

Corollary 4.5.8 ([55, Corollary 11.3]). *We have*

$$N_{(1,1)}[k]=\frac{l_1l_2}{k^2}\binom{(l_1-1)(l_2-1)k-1}{k-1}.$$

Finally, we apply Theorem 4.4.7 to the specialized case under the additional assumption that $l_1=l_2=m$. Actually, we can confirm a variant of [29, Conjecture 6.2] in the balanced case and for specialized variables as already indicated in [28, Section 6]:

Corollary 4.5.9 ([55, Corollary 12.2]). *If $l_1=m=l_2$, every specialized series $f_{(d,e)}(t)$ admits a product factorization*

$$f_{(d,e)}(t)=\prod_{k\geq 1}\left(1-((-1)^{mde-d^2-e^2}t)^k\right)^{-k\mathbf{d}(d,e,k)}$$

for integral $\mathbf{d}(d,e,k)$.

Proof. Applying Theorem 4.5.4 and using Theorem 4.4.7, the series $f_{(d,e)}(t)^{1/m}$ is determined by the functional equation

$$f_{(d,e)}(t)^{1/m}=\prod_{k\geq 1}\left(1-\left((tx)^d(ty)^e(f_{(d,e)}^{1/m})^{mde-d^2-e^2}\right)^k\right)^{-k\chi(M^{\Theta-\mathrm{st}}_{(kd,ke)}(K(l_1,l_2)))}.$$

Applying [51, Theorem 4.9], the statement follows. □

4.5.2 Comparison of degeneration formulae: GPS vs. MPS

We again restrict to dimension vectors of the quiver $K(l_1,l_2)$ which are of coprime Kronecker type. We already treated the MPS degeneration formula in Section 4.2 which expresses the Euler characteristic of moduli spaces as an alternating sum of Euler characteristics of the full MPS-quiver. Due to Gross, Pandharipande and Siebert, there is a comparable formula on the Gromov-Witten side. In [54, Section 4], it is shown that these formulae are equivalent using the notion of scattering diagrams. In this work, we reprove this result inductively starting with the identity established in Corollary 4.5.5. In order to understand this equivalence in more detail, we need the basic notion of tropical curves as introduced in [29, Section 2.1].

Let $\overline{\Gamma}$ be a weighted, connected tree with only 1-valent and 3-valent vertices. When embedding $\overline{\Gamma}$ into $\mathbb{R}^2$, we can think of it as a compact topological space. The graph Γ is obtained from $\overline{\Gamma}$ when removing the 1-valent vertices. The non-compact edges are called unbounded edges. We denote the weight function on the edges of Γ by w_Γ.

Definition 4.5.10.

i) A parametrized rational tropical curve in $\mathbb{R}^2$ is a proper map $h : \Gamma \to \mathbb{R}^2$ such that:

- the restriction of h to an edge is an embedding whose image is contained in an affine line of rational slope;
- the following balancing condition holds at the vertices: denoting by m_i the primitive integral vector emanating from the image of a vertex $h(V)$ in the direction of an edge $h(E_i)$, we have

$$\sum_{i=1}^{3} w_\Gamma(E_i)m_i = 0,$$

where the sum runs over all edges which are adjacent to V.

ii) A (rational) tropical curve is the equivalence class of a rational parametrized tropical curve under reparametrizations which respect w_Γ.

iii) The multiplicity at a vertex V is defined as

$$\text{mult}_V(h) = w_\Gamma(E_1)w_\Gamma(E_2)|m_1 \wedge m_2|.$$

iv) The total multiplicity of h is defined as

$$\text{mult}(h) = \prod_V \text{mult}_V(h).$$

Note that, due to the balancing condition, in iii) we can choose E_1, E_2 to be any two out of the three edges adjacent to V. We define a weight vector $\mathbf{w}_i$ as a sequence of non-negative integers $(w_{i1}, \ldots, w_{it_i})$ satisfying $w_{ij} \leq w_{ij+1}$. For a fixed partition $\mathbf{P}_i$ together with a refinement k^i, we can define a weight vector $\mathbf{w}(k^i) = (w_{i1}, \ldots, w_{it_i})$ of length $t_i = \sum_w m_w(k^i)$ by

$$w_{ij} = w \text{ for all } j = \sum_{r=1}^{w-1} m_r(k^i) + 1, \ldots, \sum_{r=1}^{w} m_r(k^i).$$

Of course the weight vector $\mathbf{w}(k^i)$ only depends on k^i through $\{m_w(k^i)\}_w$. With every pair of weight vectors $(\mathbf{w}_1, \mathbf{w}_2)$, we can associate a tropical invariant which counts rational tropical curves h satisfying the following conditions:

- the unbounded edges of Γ are E_{ij} for $1 \leq i \leq 2, 1 \leq j \leq t_i$, and a single "outgoing" edge E_{out};
- the image $h(E_{ij})$ is contained in a line $e_{ij} + \mathbb{R}e_i$ for some fixed vectors e_{ij}, and its unbounded direction is $-e_i$;

- $w_\Gamma(E_{ij}) = w_{ij}$.

Note that the balancing condition implies that $h(E_{\mathrm{out}})$ lies on an affine line with direction $(|\mathbf{w}_1|, |\mathbf{w}_2|)$. The set of such tropical tropical curves h is finite, and it follows from the general theory that, when we count curves h taking into account the multiplicity $\mathrm{mult}(h)$, we get an integer $N^{\mathrm{trop}}[(\mathbf{w}_1, \mathbf{w}_2)]$ which is independent of the (generic) choice of the vectors e_{ij}. Here refer to [27] and [46] for the general theory on tropical counts.

Using the degeneration formula for Gromov-Witten invariants [29, Proposition 5.3], [29, Theorem 3.4, Theorem 4.4] together with [54, Lemma 4.2], we obtain the following variant of the degeneration formula for the Gromov-Witten invariant $N_{(d,e)}[\mathbf{P}_1, \mathbf{P}_2]$ in terms of tropical invariants:

Theorem 4.5.11. *We have*

$$N_{(d,e)}[(\mathbf{P}_1, \mathbf{P}_2)] \quad = \sum_{(k_1,k_2)\vdash(\mathbf{P}_1,\mathbf{P}_2)} N^{\mathrm{trop}}[(\mathbf{w}(k^1), \mathbf{w}(k^2))] \prod_{i=1}^{2}\prod_{j=1}^{l_i}\prod_{w} \frac{(-1)^{k^i_{w,j}(w-1)}}{k^i_{w,j}!w^{2k^i_{w,j}}}.$$

Now we have the following result:

Proposition 4.5.12 ([54, Proposition 4.3])**.** *We have an equality of Euler characteristics and tropical counts*

$$N^{\mathrm{trop}}[(\mathbf{w}(k^1), \mathbf{w}(k^2))] = \chi(M^{\Theta-\mathrm{st}}_{(k^1,k^2)}(\mathcal{N}(1))). \tag{4.5.1}$$

Proof. A direct proof using scattering diagrams is given in [54, Proposition 4.3]. But, as already indicated in [54, Remark 4.4], it is also possible to prove the result by induction on $|p_{ij}|$ using the refined GW/Kronecker correspondence $N_{(d,e)}[(\mathbf{P}_1, \mathbf{P}_2)] = \chi(M^{\Theta-\mathrm{st}}_{(\mathbf{P}_1,\mathbf{P}_2)}(K(l_1, l_2)))$ established in Corollary 4.5.5. More precisely, we fix a coprime tuple (d, e). Then we start our induction with the trivial partition $(\mathbf{P}_1, \mathbf{P}_2)$. In this case, there exists only one refinement (k^1, k^2) which is the partition itself, and we have

$$N^{\mathrm{trop}}[(\mathbf{w}(k^1), \mathbf{w}(k^2))] = N_{(d,e)}[(\mathbf{P}_1, \mathbf{P}_2)] = \chi(M^{\Theta-\mathrm{st}}_{(\mathbf{P}_1,\mathbf{P}_2)}(K(l_1, l_2))) = \chi(M^{\Theta-\mathrm{st}}_{(k^1,k^2)}(\mathcal{N}(1))).$$

In order to get the induction started, we have to keep in mind that there exists a canonical refinement (c^1, c^2) of every partition $(\mathbf{P}_1, \mathbf{P}_2)$ which is given by $p_{1i} = p_{1i} \cdot 1$ and $p_{2j} = p_{2j} \cdot 1$ for every i, j. Let

$$C_{k_*} := \prod_{i=1}^{2}\prod_{j=1}^{l_i}\prod_{w} \frac{(-1)^{k^i_{w,j}(w-1)}}{k^i_{w,j}!w^{2k^i_{w,j}}}.$$

By induction hypothesis together with Corollary 4.5.5, we obtain

$$\begin{aligned} N^{\mathrm{trop}}[(\mathbf{w}(c^1), \mathbf{w}(c^2))] &= N_{(d,e)}[(\mathbf{P}_1, \mathbf{P}_2)] - \sum_{\substack{(k^1,k^2)\vdash(\mathbf{P}_1,\mathbf{P}_2)\\(k^1,k^2)\neq(c^1,c^2)}} C_{k_*} N^{\mathrm{trop}}[(\mathbf{w}(k^1), \mathbf{w}(k^2))] \\ &= \chi(M^{\Theta-\mathrm{st}}_{(\mathbf{P}_1,\mathbf{P}_2)}(K(l_1, l_2))) - \sum_{\substack{(k^1,k^2)\vdash(\mathbf{P}_1,\mathbf{P}_2)\\(k^1,k^2)\neq(c^1,c^2)}} C_{k_*} \chi(M^{\Theta-\mathrm{st}}_{(k^1,k^2)}(\mathcal{N}(1))) \\ &= \chi(M^{\Theta-\mathrm{st}}_{(c^1,c^2)}(\mathcal{N}(1))). \end{aligned}$$

Indeed, every refinement $(k^1, k^2) \neq (c^1, c^2)$ already appears as a refinement of a partition covered by the induction hypothesis. $\square$

This proves the following result:

Theorem 4.5.13 ([54, Theorem 4.1]). *Let $\mathbf{P}_1, \mathbf{P}_2$ be coprime. Then the MPS degeneration formula for Euler characteristics of moduli spaces of quiver representations established in (4.2.1) and the degeneration formula for Gromov-Witten invariants (Theorem 4.5.11) are equivalent.*

4.5.3 Tropical curves vs. localization data

In [28], [53], Gross and Pandharipande and Reineke respectively posed the question if there is a more or less direct correspondence between rational (tropical) curves and quiver representations. Also Proposition 4.5.12, Theorem 4.5.1 and Corollary 4.5.5 suggest such a relationship. Motivated by Theorem 4.5.1, in [65] Stoppa tried to construct a tropical curve from a stable representation of $\widetilde{K(m)}$. It turned out that this is unrewarding and that it is more promising to consider stable representations of the refined Kronecker quivers $K(l_1, l_2)$. This is also suggested by the identity $N^{\text{trop}}[(\mathbf{w}(k^1), \mathbf{w}(k^2))] = \chi(M^{\Theta-\text{st}}_{(k^1,k^2)}(\mathcal{N}(1)))$. Indeed, on the one hand, the Gromov-Witten invariant $N^{\text{trop}}[(\mathbf{w}(k^1), \mathbf{w}(k^2))]$ counts rational tropical curves, and, on the other hand, also the Euler characteristic counts representations because there exist only finitely many torus fixed points in $M^{\Theta-\text{st}}_{(k^1,k^2)}(\mathcal{N}(1))$. Even if there is no general recipe which is applicable for all partitions (resp. refinements), in [54, Section 6], examples were worked out where an explicit correspondence exists. Besides these examples, we review two similar recursive constructions which can be used to construct tropical curves and torus fixed points respectively.

Recursive construction of curves

Throughout this section, we fix a coprime Kronecker type (d, e) and a weight vector $(\mathbf{w}_1, \mathbf{w}_2)$ of length (t_1, t_2) such that $|\mathbf{w}_1| = d$ and $|\mathbf{w}_2| = e$. The key observation which is needed for this recursive construction is that we can choose the lines $h(E_{ij})$ in such a (still generic) way that, after removing the last edge E_{2t_2}, the tropical curve decomposes into smaller ones such their corresponding slopes define a w_{t_2}-admissible decomposition as defined below. The other way around, we can glue n tropical curves, whose slopes define a w_{t_2}-admissible decomposition, obtaining a curve which is compatible with the fixed weight vector.

We shortly recall the construction of [54, Section 6.1]. Let $\mathbf{P}_i$ be a partition of length l_i and $\mathbf{w}_i$ be a weight vector such that $|\mathbf{w}_i| = |\mathbf{P}_i|$. A set partition $I_\bullet$ of $\mathbf{w}_i$ is a decomposition of the index set

$$I_1 \cup \cdots \cup I_{l_i} = \{1, \ldots, t_i\}$$

into l_i disjoint, possibly empty parts. We say that $I_\bullet$ is compatible with $\mathbf{P}_i$ if we have

$$p_{ij} = \sum_{r \in I_j} w_{ir}$$

for all j. We say that it is proper if all parts are not empty.

We denote a vector $x \in \mathbb{R}^2$ by (x^1, x^2). Let $h : \Gamma \to \mathbb{R}^2$ be a connected parametrized rational tropical curve where the unbounded edges E_{ij} have weight w_{ij}. Moreover, we assume that $h(E_{ij}) \subset e_{ij} + \mathbb{R}e_i$. Let $V(E)$ be the unique vertex which is adjacent to an unbounded edge E, and $V_1(E)$, $V_2(E)$ those vertices adjacent to a compact edge E where we choose the numbering such that $h(V_1(E))^1 < h(V_2(E))^1$. Using general methods of tropical geometry as treated for instance in [27] and [29, Proposition 2.7], it can be seen that the invariant $N^{\text{trop}}[(\mathbf{w}_1, \mathbf{w}_2)]$ does not depend on the generic choice of the vectors e_{ij}. In particular, we can assume that $e^1_{2j} > e^1_{2j-1}$ and $e^2_{1j} > e^2_{1j-1}$. With every compact edge E, we can associate a slope

$$\mu(E) = \frac{w_\Gamma(E)m^2}{w_\Gamma(E)m^1}$$

of $h(E)$ where $m = (m^1, m^2)$ denotes the primitive vector emanating from $h(V_1(E))$ in the direction of $h(V_2(E))$. It is straightforward that the balancing condition implies $m^1, m^2 > 0$.

Definition 4.5.14. Let F_k for $k = 1, \ldots, n$ be those compact edges which satisfy $h(V_1(F_k))^1 < e^1_{2t_2} \leq h(V_2(F_k))^1$. We call a rational tropical curve slope ordered if we can order the F_k in such a way that

$$\mu(F_k) \leq \mu(F_l) \text{ and } h(V_2(F_k))^i < h(V_2(F_l))^i$$

for $i = 1, 2$ whenever $k < l$. Moreover, we call the edges F_k glueing edges.

Lemma 4.5.15 ([54, Lemmas 6.1 and 6.2])**.**

i) We can choose the lines $h(E_{ij})$ in such a way that every tropical curve is slope ordered.

ii) Let $h : \Gamma \to \mathbb{R}^2$ be a slope ordered tropical curve with glueing edges $F_1, \ldots F_n$ where $\mu(F_k) = e_k/d_k$. Then there exist edges $G_0, G_1, \ldots, G_n$ and vertices $V_1, \ldots, V_n$ such that V_k is adjacent to F_k, G_{k-1} and G_k and such that

$$\mu(G_k) = \frac{w_{2,t_2} + \sum_{i=1}^k e_i}{\sum_{i=1}^k d_i} > \frac{e_{k+1}}{d_{k+1}}$$

for $k = 1, \ldots, n-1$. Moreover, we have $G_0 = E_{2t_2}$ and $G_n = E_{out}$.

Note that these conditions are part of the glueing conditions of [69, Section 4.3]. The Lemma shows that every slope ordered tropical curve is glued by n connected tropical curves with outgoing arrows F_k of increasing slopes e_k/d_k such that

$$w_{2t_2} + \sum_{i=1}^{n} e_i =: e,\ \sum_{i=1}^{n} d_i =: d,\ \frac{e_i}{d_i} \leq \frac{e_{i+1}}{d_{i+1}} \text{ and } \frac{w_{2t_2} + \sum_{i=1}^{k} e_i}{\sum_{i=1}^{k} d_i} > \frac{e_{k+1}}{d_{k+1}}. \tag{4.5.2}$$

We call such a decomposition a w_{2t_2}-admissible decomposition of (d, e).

The other way around, in [54, Section 6.1] it is shown that n tropical curves $h_i : \Gamma_i \to \mathbb{R}^2$, such that their slopes $\mu(E_{i,\text{out}}) = e_i/d_i$ define a w_{2,t_2}-admissible decomposition of (d, e), can be glued in the natural way such that a connected tropical curve $h : \Gamma \to \mathbb{R}^2$ of slope e/d is obtained. In order to compute the tropical invariants, we have to take the number of possibilities for choosing the unbounded edges into account. More detailed, we fix a weight vector $(\mathbf{w}_1, \mathbf{w}_2)$ with $d = \sum_{j=1}^{t_1} w_{1j}$ and $e = \sum_{j=1}^{t_2} w_{2j}$ satisfying $w_{ij} \leq w_{i(j+1)}$. Every w_{2,t_2}-admissible decomposition of (d, e) defines two ordered partitions of d and $e - w_{2,t_2}$ respectively. Then every tuple of set partitions of $\{1, \ldots, t_1\}$ and $\{1, \ldots, t_2 - 1\}$ respectively which is compatible with $(d_i, e_i)_{i=1,\ldots,n}$ defines n tuples of weight vectors $(\mathbf{w}(i)_1, \mathbf{w}(i)_2)$ with $d_i = \sum_j \mathbf{w}(i)_{1j}$ and $e_i = \sum_j \mathbf{w}(i)_{2j}$. This means that every compatible set partition can be thought of as a way of assigning the unbound edges of the tropical curves h_i to those of the curve h.

Considering all w_{2t_2}-admissible decompositions and all tuples of tropical curves (embedded into the chosen line arrangement) as above at once, we can assume that all tropical curves under consideration are slope ordered and obtained by glueing smaller ones. This leads to the following result:

Theorem 4.5.16 ([54, Theorem 6.4]). *We have*

$$N^{\text{trop}}[(\mathbf{w}_1, \mathbf{w}_2)] = \sum_{(d_i,e_i)_i} \sum_{I_\bullet} \prod_{i=1}^{n} N^{\text{trop}}[(\mathbf{w}(i)_1, \mathbf{w}(i)_2)] \left| \prod_{k=1}^{n} \left(e_k \sum_{i=1}^{k-1} d_i - d_k (\sum_{i=1}^{k-1} e_i + w_{2,t_2}) \right) \right|$$

where we first sum over all w_{t_2}-admissible decompositions of (d, e) and then over all proper set partitions $I_\bullet$ which are compatible with the partitions $(d, e - w_{2,t_2}) = (\sum_{i=1}^{n} d_i, \sum_{i=1}^{n} e_i)$.

Proof. We just need to determine the multiplicities of the vertices where the original curves are glued. For their multiplicities, we get

$$\text{Mult}_{V_k}(h) = \left| \begin{pmatrix} \sum_{i=1}^{k-1} d_i & d_k \\ \sum_{i=1}^{k-1} e_i + w_{2,t_2} & e_k \end{pmatrix} \right|$$

for $k = 1, \ldots, n$. □

This formula gives also a recursive formula for the Euler characteristic of moduli spaces. Indeed, even for w-admissible decompositions of (d, e) involving non-coprime vectors (d_i, e_i), one ends up with coprime vectors after finitely many steps.

Recursive construction of localization data

We have a similar construction on the quiver side. We again fix a coprime tuple (d, e) and a w-admissible decomposition $(d_i, e_i)_{i=1,\dots,n}$ of (d, e). Furthermore, we fix tuples $(\mathbf{Q}_1, \alpha_1), \dots, (\mathbf{Q}_n, \alpha_n)$ consisting of disjoint subquivers $\mathbf{Q}_i \subset \mathcal{N}(1)$ and dimension vectors α_i of type one such that $M_{\alpha_i}^{\Theta-\mathrm{sst}}(\mathbf{Q}_i) \neq \emptyset$ and such that

$$\sum_{q \in \mathbf{Q}_i(I)} l(q) = d_i \text{ and } \sum_{q \in \mathbf{Q}_i(J)} l(q) = e_i.$$

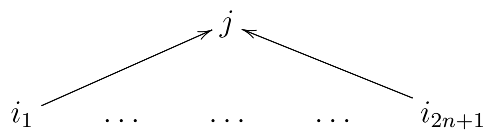
Here $\mathbf{Q}_i(I)$ denotes the set of sources of $\mathbf{Q}_i$ and $\mathbf{Q}_i(J)$ the set of sinks. Consider the tuple $(\mathbf{Q}, \alpha)$ consisting of the quiver $\mathbf{Q}$ defined by the vertices $\mathbf{Q}_0 = \bigcup_{i=1}^n (\mathbf{Q}_i)_0 \cup \{q\}$ with $l(q) = w$ and the arrows $\mathbf{Q}_1 = \bigcup_{i=1}^n (\mathbf{Q}_i)_1 \cup \{\rho : p \to q \mid p \in \mathbf{Q}_i(I), i = 1, \dots, n\}$ and the dimension vector α obtained by setting $\alpha_q = 1$.

Theorem 4.5.17 ([54, Theorem 6.6])**.** *We have $M_\alpha^{\Theta-\mathrm{st}}(\mathbf{Q}) \neq \emptyset$. In particular, every torus fixed point of $M_\alpha^{\Theta-\mathrm{st}}(\mathbf{Q})$ defines a torus fixed point of $M_\alpha^{\Theta-\mathrm{st}}(\mathcal{N}(1))$ of type (d, e).*

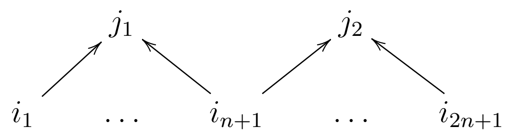
There exists a dual version of this theorem which is obtained when turning around all arrows. In this case, the quivers are glued in a source. Note that, in comparison to the tropical analogue, it is an open question if all localization data of type (d, e) can be obtained by Theorem 4.5.17. Moreover, there is no notion of multiplicity for a localization datum. This leads to the task of associating m localization data with a tropical curve of multiplicity m. Up to now, no general statement in this direction is known, but promising examples are. In this section, we recall the example in which $(d, e) = (2, 2n + 1)$. This case also illustrates the introduced recursions. Another example is given in [54, Section 6.3.2] where the case $(d, d + 1)$ is treated.

The case $(2, 2n + 1)$

Analogously to [54, Section 6.3.1], we consider the example of $2n + 1$ points in the projective plane, i.e. $(\mathbf{P}_1, \mathbf{P}_2) = (1^{2n+1}, 2)$. There exist two refined partitions which are the partition itself and $(k^1, k^2) = (1^{2n+1}, 1 + 1)$. In the first case, the only tree to consider is



with $l(j) = 2$ and $l(i_k) = 1$. In the second case, the only tree to consider is



Now it is easy to check that we have 2^{2n+1} different colorings in the first case and

$$\binom{2n+1}{n}\binom{n+1}{n}$$

different colorings in the second case.

Thus, by MPS degeneration formula 4.2.1, we get

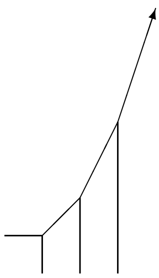
$$\chi(M^{\Theta-\mathrm{st}}_{(1^{2n+1},2)}(K(2n+1,1))) = \frac{1}{2}\binom{2n+1}{n}\binom{n+1}{n} - \frac{1}{4}2^{2n+1}. \qquad (4.5.3)$$

Since there is also exactly one tropical curve of weight 2^{2n+1} for the weight vector $(1^{2n+1},2)$, we are left with constructing a correspondence in the second case. On the curve side, we consider a line arrangement with $2n+1$ vertical legs of the following shape:

In order to apply the construction of the last section, we have to decompose the vector $(2n+1,2)$ into a 1-admissible tuple (d_i, e_i). It is easy to see that the only two possibilities are

$$(d_1, e_1) = (d_2, e_2) = (n, 1) \text{ and } (d_1, e_1) = (2n, 2).$$

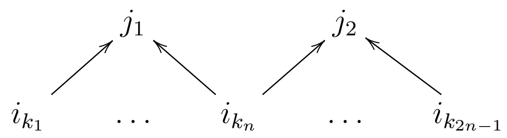
In turn, the only 1-admissible decomposition of $(2n, 2)$ is $(d_1, e_1) = (2n-1, 2)$. The first case corresponds to picking the two corresponding localization data of type $(n, 1)$ and glue them in a source i_{2n+1}. It can be checked straightforwardly that there are $\binom{2n}{n}$ localization data of this form. In order to determine the corresponding tropical curves, we first consider the tropical curve of weight one for the partition $(1^n, 1)$ (here for $n = 3$), i.e.:



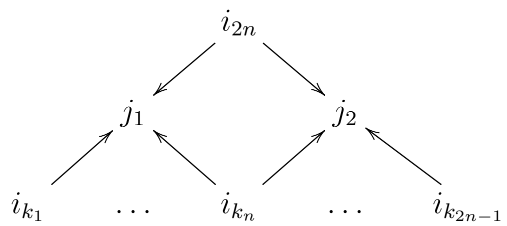
For general n, we have $\binom{2n}{n}$ possibilities to embed the curves corresponding to the partition $(1^n, 1)$ into the upper row of the line arrangement above and another curve of the same shape into the lower row. Then we can glue them as described above. Note that all curves obtained in this way have multiplicity one.

In the second case, assume that we have already constructed the curves corresponding to $(2n-1,2)$. The only way to obtain a curve corresponding to $(2n+1,2)$ from such a curve is to glue twice a curve of slope $(1,0)$ to it. If m is the multiplicity of the tropical curve of slope $(2n-1,2)$, the multiplicity of the resulting curve is $4m$.

On the quiver side this means that we have to construct four localization data of type $(2n+1,2)$ from every localization datum of type $(2n-1,2)$. Consider the uncolored localization datum



with $1 \leq k_j \leq 2n-1$. Thus, we can construct the following stable tuple of type $(2n,2)$ starting with this one



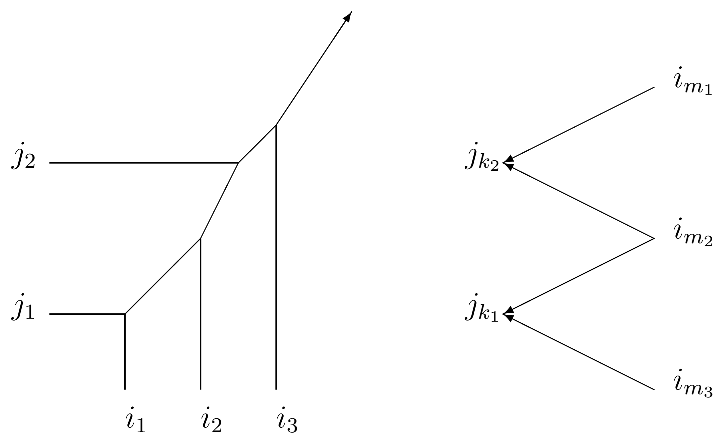
This leads to four semistable tuples by deleting one of the four arrows with source i_{k_n} or i_{2n}. But now it is easy to check that we have exactly one possibility to obtain a localization datum from this by glueing the source i_{2n+1}.

Since in the case of the partition itself we get one curve of weight 2^{2n+1}, in total we get

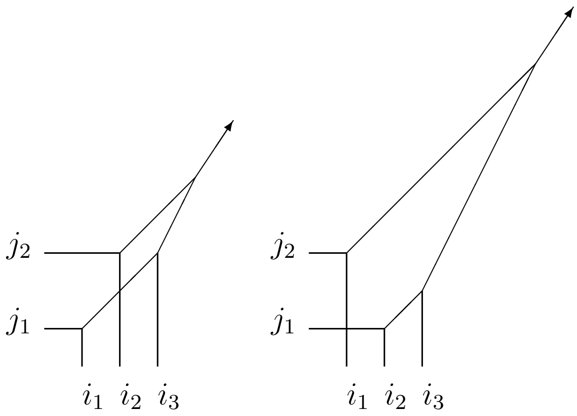
$$N_{(2n+1,2)}[(1^{2n+1},2)] = \frac{1}{2}(\binom{2n}{n} + 4\binom{n}{n-1}\binom{2n-1}{n-1}) - \frac{1}{4}2^{2n+1}$$

which is easily seen to be the same as the expression (4.5.3).

For $n=1$, we get the following localization data and the following curves of multiplicity four, one and one respectively:

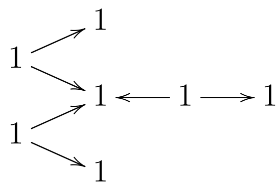


with $m_2 \in \{1,2\}$ and $k_1, k_2 \in \{1,2\}$ and $m_1, m_3 \in \{1,2,3\}$ which are four localization data. For the curves

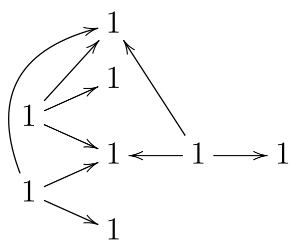


we get the same quiver colored by $k_1 = 1$, $k_2 = 2$, $m_1 = 2$, $m_2 = 3$, $m_3 = 1$ and $k_1 = 1$, $k_2 = 2$, $m_1 = 1$, $m_2 = 3$, $m_3 = 2$ respectively.

Let us close this section with a negative example showing that our construction does not always give a canonical correspondence. Consider the data



with a fixed coloring. Glueing an additional sink, we get the data



which has six subdata defining localization data, obtained by deleting two arrows in an appropriate way. But on the tropical curve side, we only get three new curves in this way (taking multiplicities into account) because we glue a curve of slope $(3,4)$ and one of slope $(0,1)$.

5 Quiver Grassmannians and F-polynomials of representations of quivers of extended Dynkin type $\tilde{D}_n$

Quiver Grassmannians are certainly interesting in their own right as they parametrize subrepresentations of a fixed dimension of a fixed quiver representations. Around 25 years ago, they first appeared in Schofield's paper [61] where he used them to make statements concerning subrepresentations of general representations of quivers and the canonical decomposition of roots. Nowadays they, additionally, attract attention because of their relevance for cluster algebras. As already indicated in Section 2.6, the generating function of the Euler characteristics of quiver Grassmannians of a fixed representation determines a cluster variable and vice versa.

In the case of quivers of extended Dynkin type $\tilde{A}_n$ and $\tilde{D}_n$, a key observation is that every real root representation is a tree module. It turns out that an appropriate coefficient quiver can be chosen such that the Euler characteristic of their quiver Grassmannians is given by the number of certain subgraphs of the respective dimension type. Hence, it is given in a purely combinatorial way. For type $\tilde{A}_n$, the Euler characteristics were determined in [10] and [30]. In this chapter, we review the main results of [43, 44] and try to motivate possible future considerations which are based on the introduced or related methods. In there, the case of quivers of extended Dynkin type $\tilde{D}_n$ is treated which turns out to be by far more considerable than the case of quivers of type $\tilde{A}_n$. To deal with it, the notion of Schubert decompositions introduced in [41, 42] is needed which happens to be a cell decomposition into affine spaces for a certain basis.

The first step is to show this for indecomposable representations of small defect. The second one is to generalize methods of Caldero and Chapoton [5] to the present case. This can be used to obtain a cell decomposition of quiver Grassmannians of representations of large defect. This is treated and reviewed in the first section of this chapter.

In the second section, we derive an explicit description of the F-polynomials of representations of quivers of extended Dynkin type $\tilde{D}_n$. We present several methods which can be used to restrict our observations to subspace orientation and to $n \leq 6$ respectively. For subspace orientation and indecomposable preprojective representations of small defect, the Euler characteristic is given by the number of so-called admissible subsets of the respective coefficient quiver. The upshot is that the corresponding generating function can be determined explicitly. Moreover, we can use a formula similar to the multiplication formula of Theorem 2.6.2 in order to obtain the generating functions for representations of large defect. The remaining cases, i.e. all representations lying in the tubes, are also covered by one of the methods pointed out.

In the final section, we concentrate on an aspect which was not treated in [43, 44] and consider torus actions on quiver Grassmannians. It turns out that, for representations which are not cover-thin, there rarely exists a well-defined and non-trivial action. This is also a reason why the methods used in [10, 30] cannot be generalized to type $\tilde{D}_n$. But actually, in many cases there exists a non-trivial torus action on the particular Schubert cells. We investigate it for $\tilde{D}_4$ in subspace orientation and end up with the formula for the Euler characteristic also obtained with the methods of the preceding sections. This section should also be understood as a motivation for future considerations.

We should note that it is likely, but still not investigated in detail, that there are similar results for quivers of type $\tilde{E}$.

5.1 Decomposition into affine spaces

Throughout this chapter, we fix the field of complex numbers. Moreover, we fix a quiver of extended Dynkin type $\tilde{D}_n$ unless otherwise stated.

In this section, we construct cell decompositions into affine spaces of quiver Grassmannians of representations of quivers of extended Dynkin type $\tilde{D}_n$. We first introduce the notion of Schubert decompositions and describe the coefficient quivers which need to be considered for our purposes. This turns out to be a cell decomposition for representations of small defect. This is also the starting point for constructing cell decompositions for representations of large defect as considered in the last subsection.

5.1.1 Schubert decomposition of quiver Grassmannians

We review some of the results of [41] and [42] which can be used to transfer the Schubert decomposition of the usual Grassmannians to a decomposition of quiver Grassmannians.

A point of the Grassmannian $\mathrm{Gr}_e(d)$ is an e-dimensional subspace V of $\mathbb{C}^d$. If V is spanned by vectors $w_1, \ldots, w_e \in \mathbb{C}^d$, we may write $w = (w_{i,j})_{i=1\ldots d, j=1\ldots,e}$ for the matrix of all coordinates of $w_1, \ldots, w_e$. For a subset $\beta \subset \{1, \ldots, d\}$ of cardinality e, the Plücker coordinates

$$\Delta_\beta(V) = \det(w_{i,j})_{i\in\beta, j=1\ldots,e}$$

define a point $(\Delta_\beta(V))_\beta$ in $\mathbb{P}(\Lambda^e\mathbb{C}^d)$. For two ordered subsets $\beta = \{i_1, \ldots, i_e\}$ and $\beta' = \{j_1, \ldots, j_e\}$ of $\{1, \ldots, d\}$, we define $\beta \leq \beta'$ if $i_l \leq j_l$ for all $l = 1\ldots, e$. The Schubert cell $C_\beta(d)$ of $\mathrm{Gr}_e(d)$ is defined as the locally closed subvariety of all subspaces V such that $\Delta_\beta(V) \neq 0$ and $\Delta_{\beta'}(V) = 0$ for all $\beta' > \beta$.

For a complex representation M with a fixed basis $\mathcal{B}$, this motivates to define the Schubert cell

$$C_\beta^M = C_\beta(d) \cap \mathrm{Gr}_e(M) \subset \prod_{q\in Q_0} \mathrm{Gr}_{e_q}(\dim M_q) \subset \mathrm{Gr}_e(d)$$

where $d = \sum_{q \in Q_0} \underline{\dim}\, M_q$, $e \in \mathbb{N}Q_0$ and β is of cardinality $e = \sum_{q \in Q_0} e_q$. This yields the so-called Schubert decomposition of the quiver Grassmannian

$$\mathrm{Gr}_e(M) \quad = \quad \coprod_{\substack{\beta \subset \mathcal{B} \\ \text{of type } e}} C_\beta^M .$$

Note that the cells are in general no affine spaces. The next step is to describe the Schubert cells by explicit equations. For a subset β of $\mathcal{B}$, let N be a point of C_β^M. For every $p \in Q_0$, the vector space N_p has a basis $(w_j)_{j \in \beta_p}$ where $w_j = (w_{i,j})_{i \in \mathcal{B}_p}$ are column vectors in M_p with respect to the coordinates given by $\mathcal{B}_p$. If we define $w_{i,j} = 0$ for $i, j \in \mathcal{B}$ whenever $j \notin \beta$, or $i \in \mathcal{B}_p$ and $j \in \mathcal{B}_q$ with $p \neq q$, then we obtain a matrix $w = (w_{i,j})_{i,j \in \mathcal{B}}$. We call such a matrix w a matrix representation of N. A matrix $w \in \mathrm{Mat}_{\mathcal{B} \times \mathcal{B}}$ is said to be in β-normal form if it satisfies

i) $w_{i,i} = 1$ for all $i \in \beta$;

ii) $w_{i,j} = 0$ for all $i, j \in \beta$ with $j \neq i$;

iii) $w_{i,j} = 0$ for all $i \in \mathcal{B}$ and $j \in \beta$ with $j < i$;

iv) $w_{i,j} = 0$ for all $i \in \mathcal{B}$ and $j \in \mathcal{B} - \beta$;

v) $w_{i,j} = 0$ for all $i \in \mathcal{B}_p$ and $j \in \beta_q$ with $p \neq q$.

We say that $w_{i,j}$ is a constant coefficient (with respect to β) if it appears in i)-v); otherwise we say that $w_{i,j}$ is a free coefficient (with respect to β), which is the case if and only if there is a $p \in Q_0$ such that $i \in \mathcal{B}_p - \beta_p$, $j \in \beta_p$ and $i < j$. The following lemma is crucial for our studies:

Lemma 5.1.1 ([42, Lemmas 2.1, 2.2], [41, Section 4.1]). *Let M be a complex representation with coefficient quiver $\Gamma(M, \mathcal{B})$ with matrix coefficients denoted by $\mu_{\rho,s,t}$.*

i) Every $N \in C_\beta^M$ has a unique matrix representation $w = (w_{i,j})_{i,j \in \mathcal{B}}$ in β-normal form.

ii) There is a natural embedding $\iota_\beta : C_\beta^M \to \mathrm{Mat}_{\mathcal{B} \times \mathcal{B}}$ such that its image is the intersection of the solution set of i)-v) and the vanishing set of the polynomials

$$E(\rho, t, s) \quad = \quad \sum_{(\rho, s', t') \in \Gamma_1} \mu_{\rho, s', t'} w_{t,t'} w_{s',s} \quad - \quad \sum_{(\rho, s', t) \in \Gamma_1} \mu_{\rho, s', t} w_{s',s} \qquad (5.1.1)$$

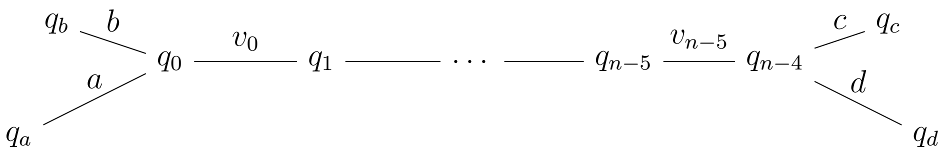
for all arrows $\rho : p \to q$ in Q_1 and all vertices $s \in F^{-1}(p)$ and $t \in F^{-1}(q)$. Here $F : \Gamma(M, \mathcal{B}) \to Q$ is the natural morphism.

Our major question is whether the Schubert decomposition is a decomposition into affine spaces. If this is the case, the closures of the non-empty Schubert cells form an additive basis for the singular cohomology ring of $\mathrm{Gr}_e(M)$, and they show that the cohomology is concentrated in even degree. Therefore, we can compute the Euler characteristic of $\mathrm{Gr}_e(M)$ as

$$\chi(\mathrm{Gr}_e(M)) \quad = \quad \#\{\beta \subset (\Gamma_M)_0 \text{ of type } e \text{ such that } C_\beta^M \text{ is not empty}\}.$$

5.1.2 Representations of small defect

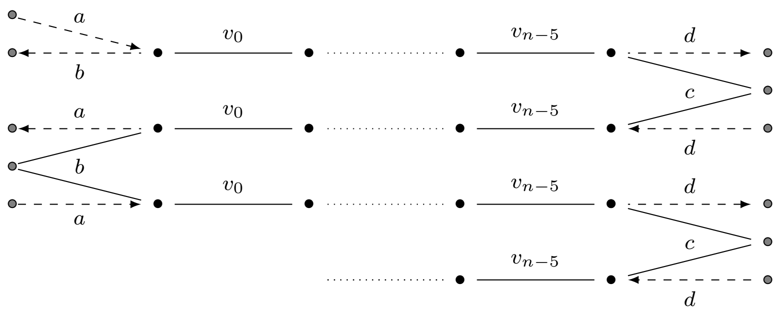
For a quiver of extended Dynkin type with imaginary Schur root δ, let $\delta(M) := \langle \delta, \underline{\dim}\, M\rangle$ be the defect of M. We have $|\delta(M)| \leq 2$ if M is indecomposable, see for instance [13, Sections 7-9]. We say that an indecomposable representation M is of small defect if it lies not in an homogeneous tube and if it satisfies $|\delta(M)| \leq 1$. If Q is of type $\tilde{D}_n$, the underlying graph is given by



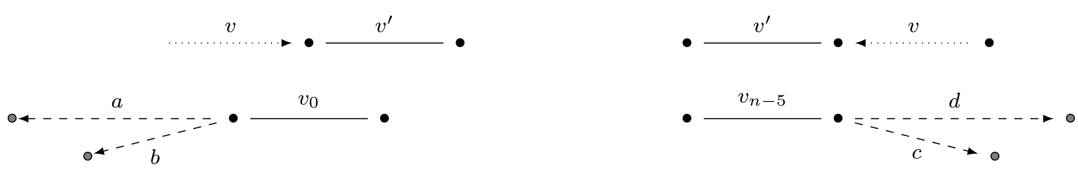
where we use the indicated notation. In order to shorten notation, we write P_a instead of P_{q_a} etc. for the indecomposable projective representations. For our investigations, it is crucial that the coefficient quivers of representations of small defect are of a special shape. To do so, we consider preprojective representations rather than preinjective ones. Since the cases are dual, it is straightforward to transfer our methods to the case of preinjective representations.

In the following diagrams, the dashed arrows are only contained in Γ if the orientation of the corresponding arrow of Q is as indicated. Moreover, a dotted arrow is not contained in the coefficient quiver, and it is only displayed to indicate the orientation of the corresponding arrow of Q. Moreover, we assume without loss of generality that the coefficients corresponding to the arrows of a tree-shaped coefficient quiver of a tree module are one.

Theorem 5.1.2 ([43, Appendix B]). *Up to automorphism of the underlying Dynkin diagram of the quiver $\widetilde{D}_n$, an indecomposable preprojective representation M with $\delta(M) = -1$ has an ordered basis such that the coefficient quiver $\Gamma = \Gamma(M, \mathcal{B})$ is given by*

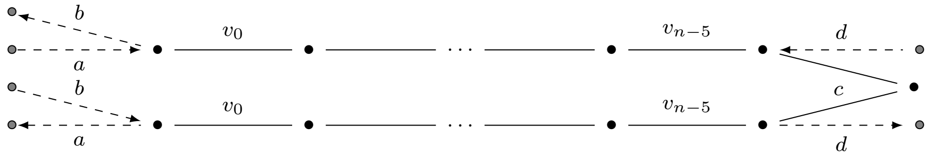


where the bottom row ends in one of the following situations

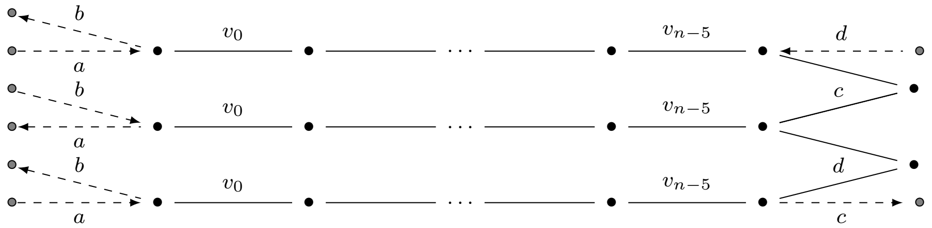


where $v \in \{v_0, \ldots, v_{n-5}\}$ and $v' \in \{b, v_0, \ldots, v_{n-5}, c\}$.

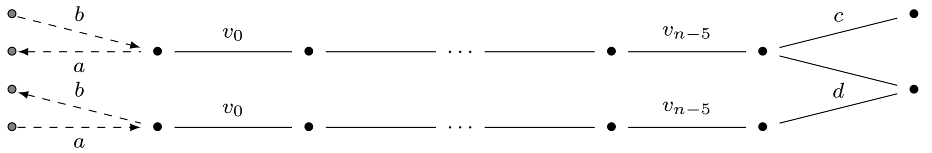
Up to an automorphism of $\tilde{D}_n$, the coefficient quivers of indecomposable representations in the tubes of rank 2 are recursively given by



and

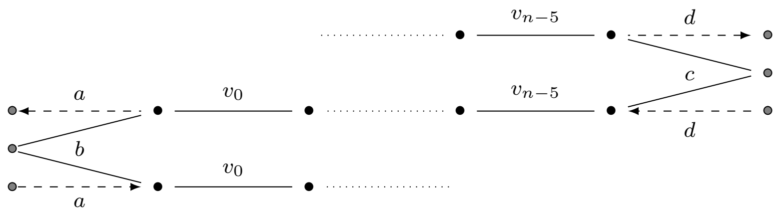


when glueing an appropriate number of copies of the coefficient quiver



between the first and the second row of one of the above quivers.

Up to an automorphism of $\tilde{D}_n$, the coefficient quivers of indecomposable representations in the tubes of rank $n-2$ are recursively given by



where the top row ends in one of the following situations

and the bottom row ends in one of the following situations

where $v \in \{v_0, \ldots, v_{n-5}\}$ *and* $v' \in \{b, v_0, \ldots, v_{n-5}, c\}$.

The ordering on the considered basis $\mathcal{B}$ can be deduced from the following rule: if a vertex i is drawn on top of another vertex j, then we have $i < j$. Since the Schubert decomposition $\mathrm{Gr}_e(M) = \coprod C_\beta^M$ depends only on the ordering on each fibre of $F_\Gamma : \Gamma \to Q$, we can extend this partial order arbitrarily to a linear order of $\mathcal{B}$.

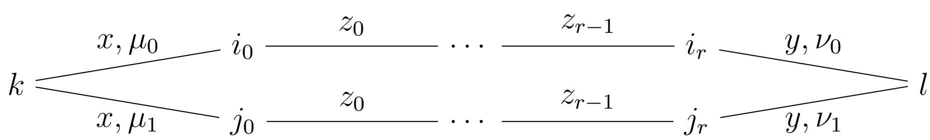
For coefficient quivers of this shape, it turns out that the system of equations given by (5.1.1) can be solved recursively. Moreover, it can be shown that the solution set is either an empty set or isomorphic to an affine space. This depends on the subgraph described by $\beta \subset \mathcal{B}$. Actually, there are two kinds of subsets β which lead to empty vanishing sets and which are called contradictory. The first one is rather easy to identify because we only need the notion of extremal arrows and extremal successor closed subsets β. The second one depends on the orientation of the arrows a, b, c, d and is slightly more complicated. But it is still possible to identify these subsets straightforwardly because the conditions for being contradictory are local conditions involving only a small number of arrows and vertices.

Definition 5.1.3. An arrow $(v, s, t) \in \Gamma_1$ is called extremal if for all arrows $(v, s', t') \in \Gamma_1$ either $s < s'$ or $t' < t$ holds. A subset β of $\mathcal{B} = \Gamma_0$ is called extremal successor closed if for all extremal arrows $(v, s, t) \in \Gamma_1$, $s \in \beta$ implies $t \in \beta$. Let M be a representation of Q with ordered basis $\mathcal{B}$. We say that β is contradictory of the first kind if it is not extremal successor closed.

A subset β of $\mathcal{B}$ is contradictory of the second kind if it satisfies the following conditions:

i) β is not contradictory of the first kind;

ii) There is a subgraph Γ' of Γ of the form



where $i_e < j_e$ for $e = 0, \ldots, r$, the arrows $x, y, z_0, \ldots, z_{r-1} \in Q_1$ are pairwise distinct and of arbitrary orientation, and one of the weights $\mu_0, \mu_1, \nu_0, \nu_1 \in \mathbb{C}$ is allowed to be zero, which means that the corresponding arrow is not part of Γ';

iii) If both k and l are sinks or both are sources of Γ', then $\mu_0\nu_1 \neq \mu_1\nu_0$. If one of k and l is a sink and the other vertex is a source, then $\mu_0\nu_0 \neq -\mu_1\nu_1$;

iv) $i_0, \ldots, i_r \notin \beta$, $j_0, \ldots, j_r \in \beta$; we have $k \in \beta$ if and only if k is a source in Γ'; we have $l \in \beta$ if and only if l is a source in Γ';

v) If (v, s, t) is an arrow of Γ that is not contained in Γ' with $v \in F_\Gamma(\Gamma')_1$, $F_\Gamma(s) = F_\Gamma(j)$ and $F_\Gamma(t) = F_\Gamma(i)$ where $j \in \{k, j_0, \ldots, j_r, l\}$ and $i \in \{k, i_0, \ldots, i_r, l\}$, then $s < j$ or $i < t$.

In general, it is convenient to use the following lemma in order to pass to a much simpler system of equations (called reduced Schubert system in [43]):

Lemma 5.1.4 ([43, Lemma 2.8]). *Let M be a representation of Q. If β is not extremal successor closed, the Schubert cell C_β^M is empty.*

This means that we can restrict to subsets $\beta \subset \mathcal{B}$ which are extremal successor closed. In particular, we can substitute w_{ii} by one or zero. Moreover, it is straightforward to check that the equations $E(v,t,s)$ are trivial if $t \in \beta$ or $s \notin \beta$ which means that we only need to consider the equations $E(v,t,s)$ in which $t \notin \beta$ and $s \in \beta$, see also [42, Lemma 2.2]. Under the assumption that M is a tree module with coefficient quiver Γ, as already mentioned, the coefficients $\mu_{v,s,t}$ can be assumed to be one or zero. The reduced equations have the following shape:

i) If (v,s,t) is an extremal arrow in Γ_M and β is extremal successor closed, the equation $E(v,t,s)$ is constant zero;

ii) If $t \notin \beta$ and $s \in \beta$, we obtain the reduced form

$$\overline{E}(v,t,s) = \sum_{\substack{(v,s,t')\in\Gamma_1\\ t<t'}} w_{tt'} + \sum_{\substack{(v,s',t')\in\Gamma_1\\ t<t',s'<s}} w_{s's}w_{tt'} - \sum_{\substack{(v,s',t)\in\Gamma_1\\ s'<s}} w_{s's} - \mu_{v,s,t}$$

where $\mu_{v,s,t} = 1$ if (v,s,t) is an arrow of Γ and $\mu_{v,s,t} = 0$ otherwise.

This simplification is crucial in the case of representations treated in Theorem 5.1.2 as this makes it possible to use the theory of Schubert systems. In a first step, the following can be shown:

Proposition 5.1.5 ([43, Proposition 4.2]). *If β is contradictory of the first or of the second kind, the Schubert cell C_β^M is empty.*

Now in a second step, this result can be used to show that every cell C_β^M is isomorphic to an affine space if the corresponding subset β is neither contradictory of the first nor of the second kind. In [43], this is obtained by solving the equations, which are not trivial, recursively. One major reason for this to be possible is the orientation of the dotted arrows of the coefficient quivers under consideration. Roughly speaking, this ensures that there are no additional equations induced by the tail of the dotted arrow. The other reason is the recursive construction of the coefficient quivers. In fact, for a fixed coefficient quiver as above, it is possible to glue a coefficient quiver, whose corresponding representation has the imaginary root as dimension vector, in order to obtain a new coefficient quiver whose top and end row remain the same.

Theorem 5.1.6 ([43, Theorem 4.4]). *Let e be a dimension vector of Q, let M be one of the indecomposable representations considered in Theorem 5.1.2 and let $\Gamma(M,\mathcal{B})$ be the respective coefficient quiver. Then the Schubert decomposition*

$$\mathrm{Gr}_e(M) = \coprod_{\substack{\beta\subset\mathcal{B}\\ \text{of type } e}} C_\beta^M$$

with respect to $\mathcal{B}$ is a decomposition into affine spaces. A Schubert cell C_β^M is empty if and only if β is contradictory of the first or of the second kind.

Note that the dimension of the affine cells is also explicitly given by [43, Theorem 4.4].

Corollary 5.1.7 ([43, Corollary 4.5])**.** *The Euler characteristic of* $\mathrm{Gr}_e(M)$ *is*

$$\chi(\mathrm{Gr}_e(M)) \;=\; \#\left\{ \beta\subset\mathcal{B} \;\middle|\; \begin{array}{c}\beta \textit{ of type } e \textit{ and not contradictory} \\ \textit{of the first or of the second kind}\end{array} \right\}.$$

In the case of representations of large defect, this approach fails (at least for those coefficient quivers we considered during our investigation) because the arrows, which correspond to the dotted arrows in the present situation, are partially oriented the other way around. This leads to additional equations while the number of variables remain the same. But still the corresponding quiver Grassmannians do have a cell decomposition into affine spaces as we will see in the next section.

5.1.3 Representations of large defect

For two representations M, N, we define $[M,N] := \dim \mathrm{Hom}(M,N)$. For a fixed short exact sequence

$$0 \to M \xrightarrow{i} B \xrightarrow{\pi} N \to 0,$$

we consider the following morphism of algebraic varieties

$$\Psi_e : \mathrm{Gr}_e(B) \to \coprod_{f+g=e} \mathrm{Gr}_f(M) \times \mathrm{Gr}_g(N),\, U \mapsto (i^{-1}(U), \pi(U))$$

as introduced in [5, Section 3]. Note that we have $i^{-1}(U) \cong U \cap M$ and $\pi(U) \cong (U+M)/M \cong U/U\cap M$.

It is crucial for us that the proof of [5, Lemma 3.11] applies in a more general setup and can be used to show that the fibres of the morphism are either empty or affine spaces:

Lemma 5.1.8. *If $\Psi_e^{-1}(A,V)$ is not empty, we have $\Psi_e^{-1}(A,V) = \mathbb{A}^{[V,M/A]}$.*

In the case of almost split sequences, [5, Lemma 3.11] and [8, Proposition 2] respectively, show that the fibre of $(A,V) \in \mathrm{Gr}_f(M) \times \mathrm{Gr}_g(N)$ is only empty for $(A,V) = (0,N)$ and only depends on the dimension vectors $\underline{\dim}\, A$ and $\underline{\dim}\, V$ otherwise. This can be used to show that cell decompositions into affine spaces are preserved by almost split sequences. In our situation, we are faced with two major problems. In general, preprojective representations of $\tilde{D}_n$ with $n \geq 6$ cannot be written as the middle term of an almost split sequence. Furthermore, it is by far more difficult to identify empty fibres and their dimensions respectively in the case of general short exact sequences. This also means that a cell decomposition might not be preserved.

Our first step is to write a preprojective representation of large defect as the middle term of a short exact sequence between representations of small defect which is closed to almost split. In a second step, it is possible to combine Theorem 5.1.6 and generalizations of the methods of Caldero and Chapoton, which are developed in [5], in order to show that every quiver Grassmannian of type $\tilde{D}_n$ has a decomposition into affine spaces.

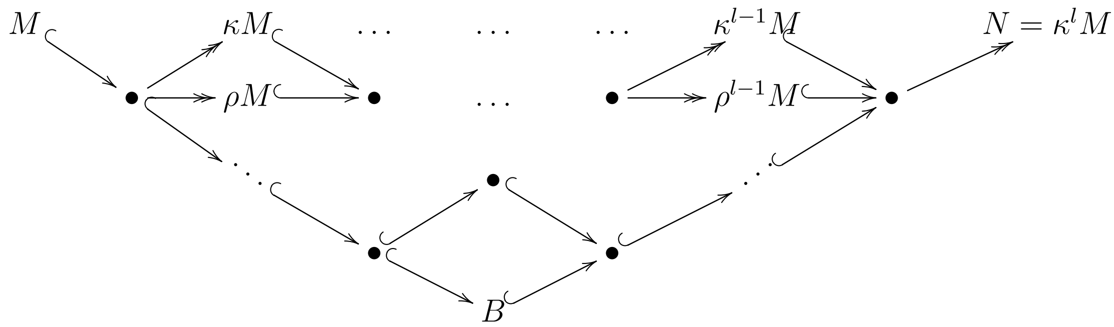
With every pair of preprojective representations $(\tau^{-m}P_a, \tau^{-m}P_b)$, we can recursively associate the tuple $(\rho^l\tau^{-m}P_a, \rho^l\tau^{-m}P_b)$ when defining

$$(\rho\tau^{-m}P_a, \rho\tau^{-m}P_b) := (\tau^{-(m+1)}P_b, \tau^{-(m+1)}P_a).$$

Moreover, we define

$$(\kappa^l\tau^{-m}P_a, \kappa^l\tau^{-m}P_b) := \begin{cases} (\tau^{-(m+l)}P_a, \tau^{-(m+l)}P_b) \text{ if } l \text{ is odd} \\ (\tau^{-(m+l)}P_b, \tau^{-(m+l)}P_a) \text{ if } l \text{ is even} \end{cases}.$$

As already mentioned, a representation of large defect cannot be written as the middle term of an almost split sequence. But it turns out that, for an indecomposable preprojective representation B with $\delta(B) = -2$, the following subquiver of the Auslander-Reiten quiver induces a substitute for it:



Here M and N are two indecomposable preprojective representations of $\tilde{D}_n$ of defect -1. More precisely, we have:

Lemma 5.1.9 ([44, Lemma 1.10]). *Every indecomposable preprojective representation B with $\delta(B) = -2$ is obtained as the middle term of a short exact sequence*

$$0 \to M \to B \to N \to 0$$

such that $N = \kappa^l M \in M^\perp$ with $l \leq n-3$, $\mathrm{Ext}(N, M) = k$ *and* $\mathrm{Hom}(N, M) = 0$.

In the following, we refer to the indecomposable representations lying properly in the above triangle or corresponding to a point $T \neq B$ on the path from B to N as (M,N)-inner representation. The remaining ones, i.e. those which are outside the triangle or on the path from M to B, as (M,N)-outer representations. We drop (M,N) if it is clear which representations are considered. Finally, if a (M,N)-inner (resp. (M,N)-outer) representation is also a subrepresentation of N, we call it inner (resp. outer) subrepresentation of N if we fixed a triangle before. In order to investigate such a triangle, the Auslander-Reiten formulae assure that we can mostly without loss of generality assume that $M = P_a$ and $N = \kappa^l M$.

The next step is supposed to investigate the morphism Ψ_e in the present situation. In particular, we need to study which fibres are empty, and we have to determine the dimensions of the non-empty fibres. This is done in [44, Section 1.6]. A key step is the following lemma:

Lemma 5.1.10 ([44, Lemma 1.12])**.**

i) The inner subrepresentations C of N are precisely the representations $C = \kappa^i M$ for $i = 1, \dots, l$.

ii) If A is a non-zero subrepresentation of M and V is any subrepresentation of N, we have $\mathrm{Ext}(V, M/A) = 0$.

iii) If $0 \neq V \subseteq N$, the corresponding injection either factors through B or $V \cong C \oplus L$ where C is an inner subrepresentation of N with $\mathrm{Hom}(C, B) = \mathrm{Hom}(C, M) = 0$.

iv) If $C \oplus L \subseteq N$ such that C is an inner subrepresentation, we have that $L \subseteq N/C$ is preprojective, and, moreover, we have that

$$\{V' \in \mathrm{Gr}_{\underline{\dim} C + \underline{\dim} L}(N) \mid V' \cong C \oplus L',\ L' \text{ prep.}\} \cong \{L' \in \mathrm{Gr}_{\underline{\dim} L}(N/C) \mid L' \text{ prep.}\}$$

is a union of cells C^N_β of the cell decomposition of $\mathrm{Gr}_{\underline{\dim} C + \underline{\dim} L}(N)$ into affine spaces obtained in Theorem 5.1.6.

Combining the methods of [5, Lemma 3.11] with Lemmas 5.1.8 and 5.1.10, it is possible to prove the following essential proposition:

Proposition 5.1.11 ([44, Proposition 1.14])**.**

i) The fibre $\Psi_e^{-1}(A, V)$ is empty if and only if $A = 0$ and $V \cong C \oplus L$ where C is an inner subrepresentation and $L = 0$ or $L \subset N/C$ is preprojective.

ii) If $\Psi_e^{-1}(A, V)$ is not empty, we have $\Psi_e^{-1}(A, V) = \mathbb{A}^{[V, M/A]}$.

iii) For all subrepresentations $V \subseteq N$ and $A \subseteq M$ of a fixed dimension, we have $[V, M/A] = \langle \underline{\dim} V, \underline{\dim} M/A \rangle$. In particular, the dimensions of the non-empty fibres of Ψ_e only depend on the dimension vectors of V and A respectively.

The considerations of this section together with [43, Theorem 4.4] now yield that there exists a cell decomposition for every quiver Grassmannian attached to preprojective representations (resp. preinjective representations).

Theorem 5.1.12 ([44, Theorem 1.15])**.** *Let $B \in \mathrm{Rep}(Q)$ be an indecomposable preprojective representation with $\delta(B) = -2$. Then there exist two preprojective representations M and $N = \kappa^l M$ with $\delta(M) = \delta(N) = -1$ and a short exact sequence*

$$0 \to M \to B \to N \to 0$$

such that we have

$$\Psi_e^{-1}(A,V) = \begin{cases} \emptyset \text{ if } A=0,\ V\cong C\oplus L,\ C \text{ an } (M,N)\text{-inner subrepresentation} \\ \mathbb{A}^{\langle \underline{\dim}\, V, \underline{\dim}\, M/A\rangle} \text{ otherwise} \end{cases}.$$

For the bases fixed in Theorem 5.1.2, we have that $\Psi_e^{-1}(A,V)$ *is constant on* $C^M_\beta \times C^N_{\beta'} \subseteq \mathrm{Gr}_{\underline{\dim}\, A}(M)\times \mathrm{Gr}_{\underline{\dim}\, V}(N)$ *for each pair* (β,β') *of type* $(\underline{\dim}\, A, \underline{\dim}\, V)$*. In particular,* $\mathrm{Gr}_e(B)$ *has a cell decomposition into affine spaces.*

The results of this section can now be used to obtain the F-polynomials of indecomposable representations of large defect. It is straightforward to check that, in terms of cluster variables, this corresponds to the multiplication formula of Theorem 2.6.2:

Theorem 5.1.13 ([44, Theorem 1.17])**.** *Let* B *be an indecomposable representation with* $\delta(B)=-2$*. If* $0\to M\to B\to N\to 0$ *is a short exact sequence as in Theorem 5.1.12, we have*

$$F_B = F_N F_M - x^{\underline{\dim}\,\tau^{-1}M} F_{N/\tau^{-1}M}.$$

Proof. We include the proof for the convenience of the reader. It is straightforward to check that $\mathrm{Gr}_{\underline{\dim}\, C}(N)=\{\mathrm{pt}\}$ for every inner subrepresentation of N. Every regular subrepresentation V of N/C gives rise to an inner subrepresentation C' of N obtained as the middle term of the unique exact sequence between C and V. Also the fibre of $(0,C')$ is empty. Moreover, every preprojective subrepresentation V of N/C gives rise to a subrepresentation $C\oplus V$ of N such that the fibre of $(0, C\oplus V)$ is empty. We can also combine both cases in the natural way. Choosing $C=\tau^{-1}M$, these observations can be summarized to

$$\chi(\mathrm{Gr}_e(B)) = \sum_{f+g=e} \chi(\mathrm{Gr}_f(M))\chi(\mathrm{Gr}_g(N)) - \sum_{f=e-\underline{\dim}\,\tau^{-1}M} \chi(\mathrm{Gr}_f(N/\tau^{-1}M)).$$

Now it is straightforward that, in terms of F-polynomials, this translates to the claim. □

5.1.4 Representations of the homogeneous tubes

Quiver Grassmannians of representations of dimension $r\delta$ of the homogeneous tubes are independent of the chosen tube. This was already shown in [20, Lemma 5.3], but follows also from the independence for $r=1$ together with the observations of [44, Section 1.7] which we review in what follows. These investigations additionally show that each quiver Grassmannian associated with a representation of a homogeneous tube has a cell decomposition into affine spaces.

Let us fix a homogeneous tube and let $M_{r\delta}$ be the representation of dimension $r\delta$ in this tube. Then there are non-splitting sequences of the form

$$0\to M_{(r-1)\delta} \xrightarrow{i_r} M_{r\delta} \xrightarrow{\pi_r} M_\delta \to 0$$

where i_r is irreducible. We again consider the morphism

$$\Psi_e^r : \mathrm{Gr}_e(M_{r\delta}) \to \coprod_{f+g=e} \mathrm{Gr}_f(M_{(r-1)\delta}) \times \mathrm{Gr}_g(M_\delta),\ U \mapsto (i_r^{-1}(U), \pi_r(U)).$$

Similar to the case of preprojective representations of large defect, the idea is to identify the empty fibres in a first step and to show that the non-empty fibres only depend on the dimension vectors of the subrepresentations in a second one. Actually, it turns out that the empty fibres can be described recursively.

Lemma 5.1.14 ([44, Lemmas 1.20, 1.21])**.**

i) The fibre $(\Psi_e^r)^{-1}(A, V)$ is empty if and only if $V = M_\delta$ and $i_{r-1}^{-1}(A) \cong A$, i.e. A is already a subrepresentation of $M_{(r-2)\delta}$.

ii) If $(\Psi_e^r)^{-1}(A, V) \neq \emptyset$, we have $(\Psi_e^r)^{-1}(A, V) = \mathbb{A}^{\langle \underline{\dim} V, \underline{\dim} M_{(r-1)\delta}/A \rangle}$.

This enables us to prove the main result of [44, Section 1.7].

Theorem 5.1.15 ([44, Theorem 1.22])**.** *Every quiver Grassmannian $\mathrm{Gr}_e(M_{r\delta})$ has a cell decomposition into affine spaces. Moreover, this decomposition is compatible with the decomposition*

$$\mathrm{Gr}_e(M_{r\delta}) = \{U \in \mathrm{Gr}_e(M_{r\delta}) \mid \pi_r(U) = 0\} \cup \{U \in \mathrm{Gr}_e(M_{r\delta}) \mid \pi_r(U) \neq 0\}. \qquad (5.1.2)$$

Proof. We include the proof for the convenience of the reader. We proceed by induction on r. If $r = 1$, the claim follows for instance by [44, Theorem 4.4], but can also be checked by hand. Since we clearly have $\pi_1(U) \neq 0$ for every subrepresentation $U \subset M_\delta$, also the compatibility follows.

Thus, let $r \geq 2$. By Lemma 5.1.14, the fibre of (A, V) is empty if and only if A is a subrepresentation of $M_{(r-2)\delta}$ and $V = M_\delta$. Since $\mathrm{Gr}_f(M_{(r-1)\delta})$ and $\mathrm{Gr}_g(M_\delta)$ have cell decompositions, by Lemma 5.1.14, it follows that

$$(\Psi_e^r)^{-1}(\mathrm{Gr}_f(M_{(r-1)\delta}) \times \mathrm{Gr}_g(M_\delta))$$

has a cell decomposition if $g \neq \delta$. If $g = \delta$, the fibre is empty if $\pi_{r-1}(A) = 0$. Since the cell decompositions of the quiver Grassmannians $\mathrm{Gr}_f(M_{(r-1)\delta})$ are compatible with the decomposition (5.1.2) by induction hypothesis, the claim follows in this case in the same way.

Since we have $\pi_r((\Psi_e^r)^{-1}(A, V)) = 0$ if and only if $V = 0$, it follows that

$$\{U \in \mathrm{Gr}_e(M_{r\delta}) \mid \pi_r(U) = 0\} = (\Psi_e^r)^{-1}(\mathrm{Gr}_e(M_{(r-1)\delta}) \times \{0\})$$

and

$$\{U \in \mathrm{Gr}_e(M_{r\delta}) \mid \pi_r(U) \neq 0\} = (\Psi_e^r)^{-1}(\coprod_{\substack{f+g=e \\ g \neq 0}} \mathrm{Gr}_f(M_{(r-1)\delta}) \times \mathrm{Gr}_g(M_\delta)).$$

This already shows that the cell decompositions of the quiver Grassmannians $\mathrm{Gr}_e(M_{r\delta})$ are also compatible with decomposition (5.1.2). □

We define $F_{r\delta} := F_{M_{r\delta}}$. Now the following corollary is straightforward:

Corollary 5.1.16 ([44, Corollary 1.23])**.** *We have*

$$F_{r\delta} = F_\delta F_{(r-1)\delta} - x^\delta F_{(r-2)\delta}$$

for $r \geq 1$ *where* $F_0 = 1$ *and* $F_{-\delta} := 0$.

5.2 F-polynomials

The determination of the generating functions (resp. F-polynomials) relies on two ingredients. Firstly, we need the connections between quiver Grassmannians and F-polynomials of a representation M and those of its BGP-reflections σM as studied in [17, Section 5] and [74, Section 5]. This can be used to restrict our considerations to $\tilde{D}_n$ in subspace orientation. Secondly, we can use that most of the linear maps of indecomposable representations of $\tilde{D}_n$ are isomorphisms. In turn, this can be used to restrict our considerations to the quivers $\tilde{D}_4, \tilde{D}_5$ and $\tilde{D}_6$.

5.2.1 F-polynomials and BGP-reflections

We review some of the results of [74] and [17] where in the latter paper the more general case of mutations is treated. For a fixed representation M of dimension α and a dimension vector e, we consider the subvarieties

$$\mathrm{Gr}_e(M, q^r) = \{U \in \mathrm{Gr}_e(M) \mid \dim \mathrm{Hom}(U, S_q) = r\}$$

and

$$\mathrm{Gr}_e(q^r, M) = \{U \in \mathrm{Gr}_e(M) \mid \dim \mathrm{Hom}(S_q, U) = r\}.$$

If q is a sink, we consider the map

$$\pi_q^r : \mathrm{Gr}_e(M, q^r) \to \mathrm{Gr}_{e-rs_q}(M, q^0)$$

where π_q^r is defined by $\pi_q^r(U)_q = \mathrm{Im}\phi_q^U$ and $\pi_q^r(U)_p = U_p$ if $p \neq q$. Here ϕ_q^U is the linear map which is used to define the BGP-reflection functor in Section 2.2. Note that $\pi_q^r(U)$ is indeed a subrepresentation of dimension $e - rs_q$ of M such that $\mathrm{Hom}(U, S_q) = 0$.

Theorem 5.2.1 ([74, Theorem 5.11])**.** *The morphism* π_q^r *is surjective with fibres isomorphic to* $\mathrm{Gr}_r(\alpha_q - e_q + r)$. *Moreover, there exists an isomorphism of varieties*

$$\sigma_q : \mathrm{Gr}_e(M, q^0) \to \mathrm{Gr}_{\sigma_q e}(q^0, \sigma_q M),\ U \mapsto \sigma_q U.$$

The analogous statement holds if q *is a source.*

Now it can be checked that there exists an $e_q \in \mathbb{N}$ such that $\mathrm{Gr}_e(M) = \mathrm{Gr}_e(M, q^0)$. If we fix such an e_q, together with Theorem 5.2.1, [44, Proposition 3.2] shows that we have

$$\chi(\mathrm{Gr}_{e+ms_q}(M, q^0)) = \sum_{i=0}^{m} (-1)^{m-i} \binom{\alpha_q - e_q - i}{m-i} \chi(\mathrm{Gr}_{e+is_q}(M)).$$

Using the identities of (sums of) binomial coefficients of [44, Lemma 3.3], it is more or less straightforward to prove the following:

Theorem 5.2.2 ([44, Theorem 3.4])**.** *Let M be a representation of dimension α. Let q be a sink and $e \in \mathbb{N}Q_0$ such that $\mathrm{Gr}_e(M) = \mathrm{Gr}_e(M, q^0)$. Let $n := (\sigma_q e)_q$ and $t := \alpha_q - e_q$. Then we have*

$$\chi(\mathrm{Gr}_{\sigma_q e - ms_q}(\sigma_q M)) = \sum_{j=0}^{m} \chi(\mathrm{Gr}_{e+js_q}(M)) \binom{n-t}{m-j}.$$

In terms of F-polynomials, this can be used to re-obtain [17, Lemma 5.2] in the special case of BGP-reflections as done in [44, Theorem 3.5]. We again consider the case where $\mathrm{Gr}_e(M) = \mathrm{Gr}_e(M, q^0)$. By Theorem 5.2.2, we know that

$$\chi(\mathrm{Gr}_{e+rs_q}(M)) \binom{n-t}{i}$$

contributes to the coefficient of $x^{\sigma_q(e)-(r+i)s_q}$ for $r = 0, \ldots, t$ and $i = 0, \ldots n-t$. In other words, for the coefficient of $x^{\sigma_q(e)-(r+i)s_q}$ in $F_{\sigma_q M}(x)$, we get

$$\sum_{i=0}^{n-t} \binom{n-t}{i} \chi(\mathrm{Gr}_{e+rs_q}(M)) x^{\sigma_q(e)-(r+i)s_q} = x^{\sigma_q(e+rs_q)} \chi(\mathrm{Gr}_{e+rs_q}(M))(1 + x_q^{-1})^{n-t}.$$

Define $\sigma_q(x^e) := x^{\sigma_q(e)}(1+x_q^{-1})^{\sigma_q(e)_q + e_q}$. Now we can formulate the following statement:

Theorem 5.2.3 ([17, Lemma 5.2])**.** *Let M be a representation of dimension α.*

i) Let q be a sink. Then we have

$$F_{\sigma_q M}(x) = (1 + x_q^{-1})^{-\dim M_q} \sum_{e \in \mathbb{N}Q_0} \chi(\mathrm{Gr}_e(M)) \sigma_q(x^e) = (1 + x_q^{-1})^{-\dim M_q} F_M(x')$$

where

$$x'_i = \begin{cases} x_i^{-1} \text{ if } i = q \\ x_i x_q^{a(i,q)} (1 + x_q^{-1})^{a(i,q)} \text{ if } i \neq q \end{cases}.$$

ii) Let q be a source. Then we have

$$F_{\sigma_q M}(x) = (1 + x_q^{-1})^{(\sigma_q \underline{\dim} M)_q} F_M(x')$$

where

$$x'_i = \begin{cases} x_i^{-1} \text{ if } i = q \\ x_i x_q^{a(q,i)} (1 + x_q)^{-a(q,i)} \text{ if } i \neq q \end{cases}.$$

5.2.2 Reduction of type one

We shortly recall the basic results of [44, Section 2.1] which lead to the so-called reductions of quiver Grassmannians of type one. Actually, this kind of reduction works for any quiver Q which has a full subquiver of the form

$$q_2 \xrightarrow{\rho_2} q_1 \xrightarrow{\rho_1} q_0$$

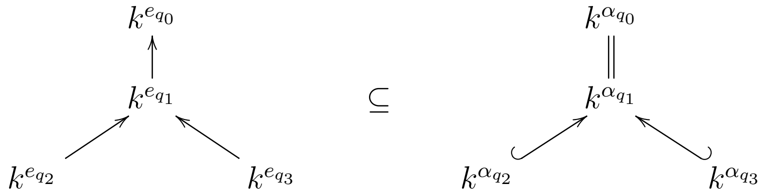
and for any representation M of dimension α such that the linear maps M_{ρ_i} are of maximal rank. By [72], this is for instance true for real root representations.

If we additionally have that $\alpha_{q_2} \leq \alpha_{q_1} = \alpha_{q_0}$, we have $\mathrm{Gr}_e(M) \neq \emptyset$ only in the case if $e_{q_2} \leq e_{q_1} \leq e_{q_0}$. Then we can consider the quiver $Q(q_1)$ obtained when deleting the vertex q_1 and the two corresponding arrows while adding an extra arrow $q_2 \to q_0$. Let $\hat{M}$ and $\hat{e}$ be the induced representation and the induced dimension vector. The maximal rank property now yields that

$$\mathrm{Gr}_e(M) \cong \mathrm{Gr}_{e_{q_1}-e_{q_2}}(e_{q_0} - e_{q_2}) \times \mathrm{Gr}_{\hat{e}}(\hat{M}).$$

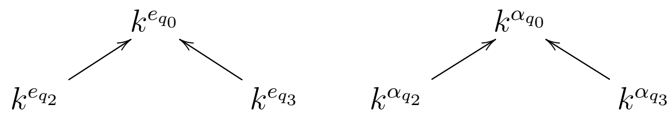
Clearly, we obtain a dual version of this statement when turning around the arrows.

We want to consider a similar case and assume that we are faced with the following situation



(with injective linear maps). Moreover, we assume that $\alpha_{q_2} + \alpha_{q_3} \leq \alpha_{q_1} = \alpha_{q_0}$, $e \leq \alpha$ and $k^{\alpha_{q_2}} \cap k^{\alpha_{q_3}} = \{0\}$. Here we think of these two vector spaces as subspaces of $k^{\alpha_{q_1}}$.

Since all maps in the diagram are injective, similar to the preceding case, we can consider the reduction



and obtain

$$\mathrm{Gr}_e(M) \cong \mathrm{Gr}_{\hat{e}}(\hat{M}) \times \mathrm{Gr}_{e_{q_1}-e_{q_2}-e_{q_3}}(e_{q_0} - e_{q_2} - e_{q_3}).$$

Note that there is again a dual case obtained when turning around all arrows. In terms of F-polynomials, this leads to the following result:

Lemma 5.2.4 ([44, Lemma 4.4])**.** *Let M be a representation of Q which can be reduced to a representation $\hat{M}$ by the first instance of reduction of type one. Then we have*

$$F_M(x) = \sum_{\hat{e} \in \mathbb{N}Q_0} \chi(\mathrm{Gr}_{\hat{e}}(\hat{M})) x^{\hat{e}} x_{q_1}^{\hat{e}_{q_2}} (1 + x_{q_1})^{\hat{e}_{q_0} - \hat{e}_{q_2}}.$$

In other words, considering the variable transformation $x_q \mapsto x'_q$ *where*

$$x'_{q_0} := x_{q_0}(1 + x_{q_1}),\ x'_{q_2} := x_{q_1}x_{q_2}(1 + x_{q_1})^{-1},\ x'_q = x_q \text{ for all } q' \notin \{q_0, q_2\},$$

we have $F_M(x) = F_{\hat{M}}(x')$.

Moreover, we obtain an analogous statement for the second instance of reduction of type one.

5.2.3 F-polynomials of representations of the homogeneous tubes

Later it turns out that the F-polynomials of representations of the homogeneous tubes play an important role in the description of F-polynomials of indecomposable representations of $\tilde{D}_n$. We make use of the recursion obtained in Corollary 5.1.16 in order to develop an explicit formula for these F-polynomials. To do so, we use methods which are similar to those used when solving the recursion of the Fibonacci numbers and which are recalled in detail in [44, Section 4.2]. More detailed, we consider a recursion of the following shape where f_0, $f_1 \in k[x_i \mid i \in I]$ and where f_j for $j \geq 2$ is recursively defined by

$$\begin{pmatrix} f_{2n} \\ f_{2n+1} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} f_{2n-2} \\ f_{2n-1} \end{pmatrix}$$

for some $a, b, c, d \in k[x_i \mid i \in I]$ and where $n \geq 1$. Then the following holds:

$$f_{2n} = \frac{1}{2z}((a(\lambda_+^n - \lambda_-^n) - (ad - bc)(\lambda_+^{n-1} - \lambda_-^{n-1}))f_0 - b(\lambda_-^n - \lambda_+^n)f_1), \tag{5.2.1}$$

$$f_{2n+1} = \frac{-1}{2bz}((a - \lambda_+)(a - \lambda_-)(\lambda_+^n - \lambda_-^n)f_0 + b(\lambda_+^n(a - \lambda_+) - \lambda_-^n(a - \lambda_-))f_1) \tag{5.2.2}$$

where

$$\lambda_\pm = \frac{a + d}{2} \pm \sqrt{\frac{(a + d)^2}{4} - ad + bc}.$$

Now the recursion of Corollary 5.1.16 can be rewritten as

$$\begin{pmatrix} F_{r\delta} \\ F_{(r+1)\delta} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -x^\delta & F_\delta \end{pmatrix}^{r+1} \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Defining

$$z = \frac{1}{2}\sqrt{F_\delta^2 - 4x^\delta}, \quad \lambda_\pm = \frac{F_\delta}{2} \pm z,$$

the following explicit formula can be obtained:

Corollary 5.2.5 ([44, Corollary 4.12])**.** *We have*

$$F_{r\delta} = \frac{1}{2z}(\lambda_+^{r+1} - \lambda_-^{r+1}).$$

Remark 5.2.6. For $r \geq 1$, we can also consider the almost split sequences

$$0 \to M_{r\delta} \to M_{(r-1)\delta} \oplus M_{(r+1)\delta} \to M_{r\delta} \to 0$$

where $M_{0\delta} = M_0 := 0$ and $F_{M_0} = 1$. By [5, Lemma 3.11], we thus obtain the recursive description

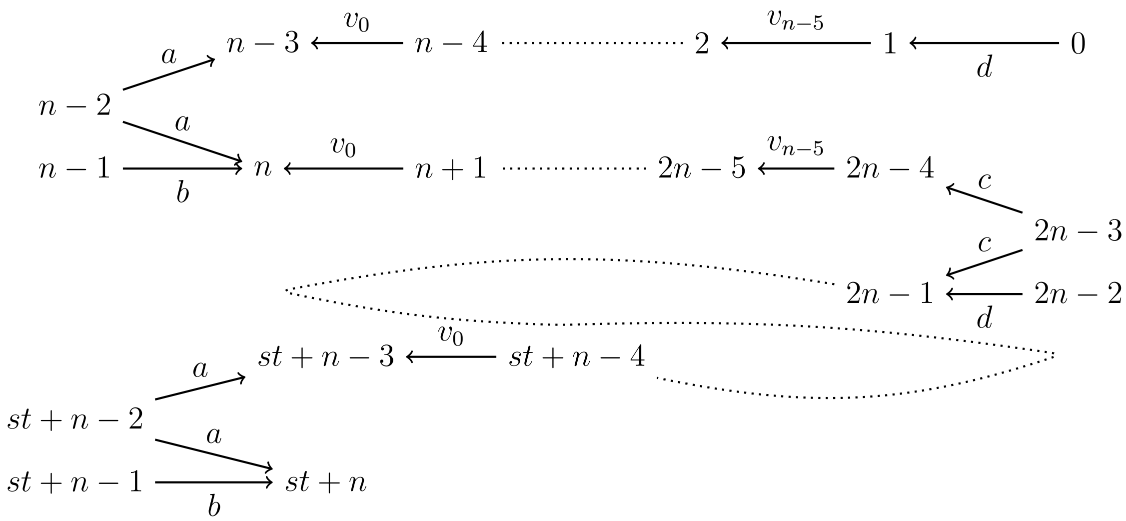
$$F_{(r+1)\delta} F_{(r-1)\delta} = F_{r\delta}^2 - x^{r\delta}$$

for $r \geq 1$.

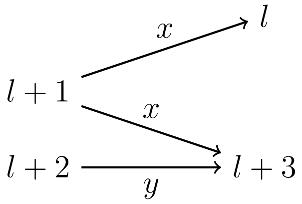
5.2.4 Admissible subsets

In order to determine the Euler characteristics of quiver Grassmannians $\mathrm{Gr}_e(M)$, it suffices to count the number of subsets $\beta \subset \mathcal{B}$ of type e which are not contradictory of the first and second kind. It turns out that, even if Definition 5.1.3 might look complicated, for a fixed orientation it is actually easy to check if a subset is contradictory of the first or second kind. In the following, we restrict to subspace orientation which is sufficient for the determination of F-polynomials.

The involved combinatorics can be described in terms of so-called admissible subsets of the following snake-shaped coefficient quiver which defines a preprojective representation. The F-polynomials of representations in the tubes can be obtained by slight modifications as we will see in Section 5.2.5. Let $\mathcal{Q}(s, n)$ (where $t := 2n - 2$) be the following coefficient quiver where we omit the vertices and only state the numbering:



We denote the corresponding preprojective representation by $M(s, n)$. By a ramification subgraph, we mean a subgraph of the form



Note that in our case we have $x \in \{a, c\}$ and $y \in \{b, d\}$. Moreover, the extremal arrows of $\mathcal{Q}(s, n)$ are all arrows but those of the form $l+1 \xrightarrow{x} l$ contained in the ramification subgraphs.

Definition 5.2.7. We call a subset G_0 of $\mathcal{Q}(s, n)_0$ admissible if the following holds:

i) G_0 is extremal successor closed, i.e. if the tail of an extremal arrow is contained in G_0, the head is also contained in G_0;

ii) For all ramification subgraphs, we have: if $l+1$, $l+2 \in G_0$, it follows that $l \in G_0$.

Note that we automatically have $l+3 \in G_0$ if G is extremal successor closed and if $l+1 \in G_0$ or $l+2 \in G_0$. Every such subset G_0 induces a dimension vector $e \in \mathbb{N}Q_0$, called the type of G_0 in what follows. The next step is to determine the number of admissible subsets of $\mathcal{Q}(s, n)_0$ of a fixed type e because it is precisely the Euler characteristic of the corresponding quiver Grassmannian. More precisely, using Theorem 5.1.6, we obtain:

Theorem 5.2.8 ([44, Theorem 4.8])**.** *Let $e \in \mathbb{N}Q_0$. Then $\chi(\mathrm{Gr}_e(M(s, n)))$ coincides with the number of admissible subsets of $\mathcal{Q}(s, n)_0$ of type e.*

Consider $I := \{0, \ldots, 2s+1\}$ and $J := \{0, \ldots, n-4\}$. If we delete the sources of $\mathcal{Q}(s, n)$ corresponding to the ramification subgraphs, we can think of the remaining graph as a matrix having entries which are vertices, i.e. with every index (i, j) we associate the vertex in the ith row and jth column of the remaining graph. Note that we start the indexing by $(0, 0)$.

For $(i, j) \in I \times J$, let $\mathcal{G}(i, j)$ be the full (connected) subgraph of $\mathcal{Q}(s, n)$ which has vertices $\{(0, n-4), (0, n-5), ..., (i, j)\}$ and where we add the subgraph $1 \leftarrow 0$ and also all sources of ramification subgraphs whose remaining vertices are all contained in $\{(0, n-4), (0, n-5), ..., (i, j)\}$. Let

$$\mathcal{F}_i^j = \sum_{e \in \mathbb{N}Q_0} \chi(i, j, e) x^e$$

be the generating function counting the number $\chi(i, j, e)$ of admissible subsets of $\mathcal{G}(i, j)_0$ of type e. We define $\mathcal{F}_{-1}^{n-4} := 1$ and $\mathcal{F}_0^{n-4} := 1 + x_{n-4} + x_{n-4}x_d$. The following lemma can be checked straightforwardly:

Lemma 5.2.9 ([44, Lemma 4.9])**.** *We have the following recursive relations:*

i) For all $m \geq 0$, $j = n-5, \ldots, 0$, we have $\mathcal{F}_{2m}^j = x_j \mathcal{F}_{2m}^{j+1} + \mathcal{F}_{2m-1}^{n-4}$.

ii) For all $m \geq 0$, we have

$$\mathcal{F}_{2m+1}^0 = (1 + x_0 + x_0x_a + x_0x_b + x_0x_ax_b)\mathcal{F}_{2m}^0 - x_0x_ax_b\mathcal{F}_{2m-1}^{n-4}.$$

iii) For all $m \geq 0$, we have $\mathcal{F}_{2m+1}^1 = (x_1 + 1)\mathcal{F}_{2m+1}^0 - x_1\mathcal{F}_{2m}^0$.

iv) For all $m \geq 0$, $j = 2, \ldots, n-4$, we have $\mathcal{F}_{2m+1}^j = (x_j + 1)\mathcal{F}_{2m+1}^{j-1} - x_j\mathcal{F}_{2m}^{j-2}$.

v) For all $m \geq 1$, we have

$$\mathcal{F}^{n-4}_{2m} = (1 + x_{n-4} + x_{n-4}x_c + x_{n-4}x_d + x_{n-4}x_cx_d)\mathcal{F}^{n-4}_{2m-1} - x_{n-4}x_cx_d\mathcal{F}^{n-5}_{2m-1}.$$

Let $H(x, y, z) = 1 + x + xy + xz + xyz$. Together with the formulae obtained in [44, Lemma 4.6], one immediately obtains the following recursion:

Corollary 5.2.10 ([44, Corollary 4.10]). *We have*

$$\begin{aligned}\begin{pmatrix}\mathcal{F}^{n-4}_{2m+1}\\ \mathcal{F}^{n-4}_{2m+2}\end{pmatrix} &= \begin{pmatrix}0 & 1\\ -x_{n-4}x_cx_d & H(x_{n-4}, x_c, x_d)\end{pmatrix}\begin{pmatrix}-F_{1,n-5}+1 & F_{1,n-5}\\ -F_{1,n-4}+1 & F_{1,n-4}\end{pmatrix}\\ &\quad \begin{pmatrix}0 & 1\\ -x_0x_ax_b & H(x_0, x_a, x_b)\end{pmatrix}\begin{pmatrix}1 & 0\\ F_{n-6} & \prod_{i=0}^{n-5} x_i\end{pmatrix}\begin{pmatrix}\mathcal{F}^{n-4}_{2m-1}\\ \mathcal{F}^{n-4}_{2m}\end{pmatrix}\end{aligned}$$

where $F_m := \sum_{i=-1}^{m}\prod_{j=0}^{i} x_j$ and $F_{1,m} := \sum_{i=0}^{m}\prod_{j=1}^{i} x_j$. Thus, for $n = 4$, we get

$$\begin{pmatrix}\mathcal{F}^{0}_{2m+1}\\ \mathcal{F}^{0}_{2m+2}\end{pmatrix} = \begin{pmatrix}0 & 1\\ -x_0x_cx_d & H(x_0, x_c, x_d)\end{pmatrix}\begin{pmatrix}0 & 1\\ -x_0x_ax_b & H(x_0, x_a, x_b)\end{pmatrix}\begin{pmatrix}\mathcal{F}^{0}_{2m-1}\\ \mathcal{F}^{0}_{2m}\end{pmatrix}$$

This turns out to be the essential recursion which can actually be solved as we will see in the subsequent section.

5.2.5 The main theorem concerning F-polynomials

We introduce some more notation. For a real root representation M_α, we denote by F_α the corresponding F-polynomial. In a tube of rank t, there exist t chains of irreducible morphisms

$$M_{0,1} \hookrightarrow M_{0,2} \hookrightarrow \ldots \hookrightarrow M_{0,t-1} \hookrightarrow M_{1,0} := M_\delta \hookrightarrow M_{1,1} \hookrightarrow \ldots$$

where the $m_l(r) := \underline{\dim}\, M_{r,l}$ are real roots and the imaginary root representations $M_{r,0} := M_{r\delta}$ are uniquely determined by this chain. Furthermore, for every real root α in the tube of rank t, there exists an exceptional root $m_l(0)$ such that $\alpha = r\delta + m_l(0)$. Under the convention that $F_\alpha = 0$ if $\alpha \in \mathbb{Z}Q_0$ has at least one negative component, setting $m_t(0) := \delta$, we can summarize the main results of [44, Section 4] as follows:

Theorem 5.2.11 ([44, Theorems 1.17, 4.14, 4.18 and 4.25]).

i) For the representations $M_{m_l(r)}$ lying in an exceptional tube of rank t and $l = 0, \ldots, t-1$, we have

$$F_{m_l(r)} = F_{m_l(0)}F_{r\delta} + x^{m_{l+1}(0)}F_{m_{t-1}(0)-m_{l+1}(0)}F_{(r-1)\delta}.$$

ii) Let M be preprojective of defect -1 such that $t_M := \underline{\dim}\, M - r\delta \leq \delta$. If $\delta - t_M$ is an injective root, we have

$$F_M = F_{t_M}F_{r\delta} - x^\delta F_{(r-1)\delta}.$$

If $\delta - t_M$ is no injective root, we have

$$F_M = F_{t_M}F_{r\delta} - x^{\tau^{-1}t_M}F_{\delta-\tau^{-1}t_M}F_{(r-1)\delta}.$$

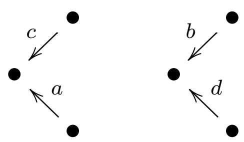
iii) Let B be an indecomposable representation of defect -2. Then there exist indecomposable representations M and N of defect -1 such that

$$F_B = F_N F_M - x^{\underline{\dim}\,\tau^{-1}M} F_{N/\tau^{-1}M}.$$

iv) Passing to the dual, we obtain analogous formulae for indecomposable representations of positive defect.

The third result was already obtained in Theorem 5.1.13. As already mentioned, the idea to prove the first two results is the same. Firstly, one proves the identities for $\tilde{D}_4$ in subspace orientation using the notion of admissible subsets and by solving the recursion obtained in Corollary 5.2.10. Using (inverse) reduction of type one, we can show the identities for $\tilde{D}_n$ in subspace orientation for arbitrary n. Finally, one can show that the identities are preserved under BGP-reflections. To give an idea, we review the proof in the case of representations lying in the homogeneous tubes of rank two. For more details, we refer to [44, Section 4]. We should point out that in the case of preprojective representations of small defect, the only difference is that we start our glueing process with the coefficient quiver $\bullet \xleftarrow{d} \bullet$ while, in the present situation, we start with the empty coefficient quiver, cf. Section 5.2.4. This leads to the fact that combinatorics are slightly easier.

Similar to the case of representations of defect -1, we obtain all coefficient quivers of representations lying in the exceptional tube of rank two by glueing the coefficient quivers



in turns. We denote the representation on the left hand side by T_1 and the representation on the right hand side by T_2. The generating functions are

$$F_{T_1} = 1 + x_0 + x_0x_a + x_0x_c + x_0x_ax_c, \quad F_{T_2} = 1 + x_0 + x_0x_b + x_0x_d + x_0x_bx_d.$$

Without loss of generality, we can assume that we start our glueing process with the coefficient quiver of T_1. We set $f_{-1} = 0$, $f_0 = 1$ and

$$f_{2r+1} = F_{T_1} f_{2r} - x_0x_ax_c f_{2r-1}, \quad f_{2r+2} = F_{T_2} f_{2r+1} - x_0x_bx_d f_{2r}.$$

Then f_{2r+1} is the generating function of the unique indecomposable of dimension $t(r) := \underline{\dim}\, T_1 + r\cdot\delta$ and f_{2r+2} is the generating function of the unique indecomposable $M^1_{(r+1)\delta}$ of dimension $(r+1)\cdot\delta$ such that $T_1 \subset M^1_{(r+1)\delta}$. Note that if $n = 4$, all tubes have rank two so that this construction applies for all exceptional tubes in this case. Now using the methods of Section 5.2.3, we obtain the following statement:

Proposition 5.2.12 ([44, Propostition 4.13])**.** *For $r \geq 0$, we have*

$$F_{t(r)} = f_{2r+1} = F_{T_1} F_{r\delta} = F_{t(0)} F_{r\delta},$$

$$F_{M^1_{(r+1)\delta}} = f_{2r+2} = F_{(r+1)\delta} + x_0x_ax_c F_{r\delta}.$$

Proof. We include the proof for the convenience of the reader. Using the notation from Section 5.2.3, we have

$$a = -x_0 x_a x_c, \quad b = F_{T_1}, \quad c = -x_0 x_a x_c F_{T_2}, \quad d = -x_0 x_b x_d + F_{T_1} F_{T_2}.$$

Then it is easy to check that we have

$$a + d = F_\delta, \quad z = \frac{1}{2}\sqrt{F_\delta^2 - 4x^\delta}, \quad ad - bc = \lambda_+ \lambda_- = x^\delta.$$

Moreover, we get

$$\lambda_+ = \frac{1}{2}\left(F_\delta + \sqrt{F_\delta^2 - 4x^\delta}\right), \quad \lambda_- = \frac{1}{2}\left(F_\delta - \sqrt{F_\delta^2 - 4x^\delta}\right).$$

Since $f_{-1} = 0$ and $f_0 = 1$, Equation (5.2.1) yields

$$f_{2r+1} = \frac{1}{2z}(F_{T_1}(\lambda_+^{r+1} - \lambda_-^{r+1})).$$

Thus, it remains to show that

$$F_{r\delta} = \frac{1}{2z}(\lambda_+^{r+1} - \lambda_-^{r+1}). \tag{5.2.3}$$

For $r = 0$, this is clearly true. Since $a + d = F_\delta$, this is also true for $r = 1$. By Remark 5.2.6, it suffices to show that

$$\left(\frac{\lambda_+^{r+1} - \lambda_-^{r+1}}{2z}\right)\left(\frac{\lambda_+^{r-1} - \lambda_-^{r-1}}{2z}\right) = \left(\frac{\lambda_+^{r} - \lambda_-^{r}}{2z}\right)^2 - x^{(r-1)\delta}$$

for $r \geq 1$, what follows from

$$2z = \lambda_+ - \lambda_-, \quad \lambda_+\lambda_- = x^\delta.$$

Using Equation (5.2.2), we have

$$\begin{aligned} f_{2r+2} &= \frac{1}{2z}(\lambda_-^{r+1}(-x_0x_ax_c - \lambda_-) - (\lambda_+^{r+1}(-x_0x_ax_c - \lambda_+))) \\ &= \frac{1}{2z}(\lambda_+^{r+2} - \lambda_-^{r+2}) + \frac{1}{2z}x_0x_ax_c(\lambda_+^{r+1} - \lambda_-^{r+1}) \\ &= F_{(r+1)\delta} + x_0x_ax_cF_{r\delta}, \end{aligned}$$

which completes the proof of the proposition. □

Let us consider the tubes of rank two for general n with arbitrary orientation. For a fixed tube, we denote by $t_1(0)$ and $t_2(0)$ the quasi-simple roots. The real roots in this tube are given by $t_i(r) = t_i(0) + r\delta$. Finally, we denote the representation of dimension $r\delta$ with subrepresentation $M_{t_i(0)}$ by $M^i_{r\delta}$.

Theorem 5.2.13 ([44, Theorem 4.14]). *For the indecomposable representations $M_{t_i(r)}$ and $M^i_{r\delta}$ lying in one of the exceptional tubes of rank two of $\tilde{D}_n$, we have:*

i) $F_{t_i(r)} = F_{t_i(0)} F_{r\delta}$.

ii) $F_{M^i_{r\delta}} = F_{r\delta} + x^{t_i(0)} F_{(r-1)\delta}$.

Proof. We include the proof for the convenience of the reader. Under consideration of Lemma 5.2.4, it is straightforward to generalize Proposition 5.2.12 to arbitrary $\tilde{D}_n$ in subspace orientation.

Assume that M with $\underline{\dim}\, M = t_i(r) + r\delta$ lies in one of the exceptional tubes of rank two of $\tilde{D}_n$ (with arbitrary orientation) and satisfies $F_M = F_{t_i(0)} F_{r\delta}$. Applying Theorem 5.2.3, we have

$$F_{\sigma_q M} = F_{\sigma_q t_i(0)} F_{r\delta}.$$

Thus, the first statement follows by induction.

For a fixed sink q of $\tilde{D}_n$ (with arbitrary orientation), it is straightforward to check that

$$\sum_{p \in Q_0} a(p,q) t_i(0)_p = \delta_q.$$

Indeed, if $q \in \{q_a, q_b, q_c, q_d\}$, both sides are one. Otherwise both sides are two. Assume that $F_{M^i_{r\delta}} = F_{r\delta} + x^{t_i(0)} F_{(r-1)\delta}$. Then, again by Theorem 5.2.3, we have

$$F_{\sigma_q M^i_{r\delta}} = F_{r\delta} + x^{\sigma_q t_i(0)} (1 + x_q^{-1})^{\sum_{p \in Q_0} a(p,q) t_i(0)_p} (1 + x_q^{-1})^{-\delta_q} F_{(r-1)\delta} = F_{r\delta} + x^{\sigma_q t_i(0)} F_{(r-1)\delta}.$$

Thus, the second statement also follows by induction. □

5.3 Torus actions on quiver Grassmannians

In this section, we consider torus actions on quiver Grassmannians. As already investigated in this work, a torus action can be used to reduce the determination of the Euler characteristic of a complex variety to the one of the fixed point set. This approach was already investigated by Cerulli Irelli and Esposito in [10] and [11] for representations of the Kronecker quiver and so-called string modules. The fixed point sets of the torus action considered there are finite. Thus, if the quiver Grassmannian is smooth, the corresponding Bialynicki-Birula decomposition is a decomposition into affine spaces. In general, such a nice torus action does not exist and, in addition to it, it is not straightforward to find any non-trivial torus action on the whole Grassmannian. This leads to the idea to define a torus action only on the cells C^M_β and to study in which cases it is not trivial. With the help of these methods, it is possible to give a short proof of Corollary 5.1.7 for preprojective representations of small defect of $\tilde{D}_4$ in subspace orientation.

5.3.1 A condition for torus actions on Schubert cells

Let $T = \mathbb{C}^*$ be the one-dimensional torus and M be a representation of an arbitrary acyclic quiver Q of dimension α. We fix a basis $\mathcal{B}_q = (b_i^q)_{i=1,\ldots,\alpha_q}$ for every $q \in Q_0$ and a map $d : \bigcup_{q\in Q_0} \mathcal{B}_q \to \mathbb{Z}$ called degree in what follows. We define $\lambda.b_i^q = \lambda^{d(b_i^q)} b_i^q$ for every $\lambda \in T$ and extend this definition to M_q taking linearity into account. This induces an action of T on $\bigoplus_{q\in Q_0} M_q$. This raises the question under which conditions such an action defines a (non-trivial) action on a quiver Grassmannian $\mathrm{Gr}_e(M)$ for a fixed dimension vector $e \in \mathbb{N}^{Q_0}$. If $U \in \mathrm{Gr}_e(M)$, this yields the condition $\lambda.U \in \mathrm{Gr}_e(M)$ for every $\lambda \in T$ which means

$$M_\rho(\lambda.u) \in \lambda.U_{t(\rho)} \tag{5.3.1}$$

for every $u \in U_{s(\rho)}$, $\lambda \in T$, $\rho \in Q_1$. Under no further conditions, this is rarely the case. Thus, assume that M is a tree module with respect to the fixed basis $\mathcal{B}$ and let Γ_M be the corresponding ordered coefficient quiver. In particular, we have an injection $\mathcal{B} \hookrightarrow \mathbb{N}$. Even in this case it seems that we only get an induced torus action on the whole quiver Grassmannian if M is cover-thin.

But actually, we may consider the Schubert decomposition $\mathrm{Gr}_e(M) = \coprod_{\beta\subset\mathcal{B}} C_\beta^M$ and investigate under which conditions there exists a torus action on some fixed locally closed subset C_β^M. Let $w \in C_\beta^M$ be in β-normal form. Thus, the equations $E(\rho, t, s)$ are satisfied for $\rho : s \to t \in (\Gamma_M)_1$. The equations $E(\rho, t, s)$ can be reduced to

$$\sum_{\substack{s'\in\mathcal{B}\backslash\beta\cup\{s\}\\ s'\le s,\, s'\xrightarrow{\rho}t}} w_{s's} = \sum_{\substack{s'\in\mathcal{B}\backslash\beta\cup\{s\}\\ t'\in\beta,\, s'\xrightarrow{\rho}t'\\ s'\le s,\, t<t'}} w_{s's} w_{tt'}$$

where we can assume that $t \notin \beta$ and $s \in \beta$, see also [42, Section 2].

Now for w in β-normal form and $i, j \in F^{-1}(q)$, we have that $(\lambda.w)_{ij} = \lambda^{d(b_i^q)} w_{ij}$ is, in general, not in β-normal form. Recall that for $i \neq j$ the inequality $w_{ij} \neq 0$ can only hold if $i \notin \beta$ and $j \in \beta$. Now we can easily obtain the corresponding β-normal form of $\lambda.w$ when multiplying the j^{th} column of w with $\lambda^{-d(b_j^q)}$ for each $q \in Q_0$ and $j \in \beta$.

This means that Equation (5.3.1) translates to the condition

$$\sum_{\substack{s'\in\mathcal{B}\backslash\beta\cup\{s\}\\ s'\le s,\, s'\xrightarrow{\rho}t}} \lambda^{d(s')-d(s)} w_{s's} = \sum_{\substack{s'\in\mathcal{B}\backslash\beta\cup\{s\}\\ t'\in\beta,\, s'\xrightarrow{\rho}t'\\ s'\le s,\, t<t'}} \lambda^{d(s')-d(s)} w_{s's} \lambda^{d(t)-d(t')} w_{tt'}$$

which is equivalent to

$$\sum_{\substack{s'\in\mathcal{B}\backslash\beta\cup\{s\}\\ s'\le s,\, s'\xrightarrow{\rho}t}} \lambda^{d(s')-d(t)} w_{s's} = \sum_{\substack{s'\in\mathcal{B}\backslash\beta\cup\{s\}\\ t'\in\beta,\, s'\xrightarrow{\rho}t'\\ s'\le s,\, t<t'}} \lambda^{d(s')-d(t')} w_{s's} w_{tt'}.$$

We refer to this equation as $\lambda.E(\rho, t, s)$. This leads to the following observation:

Lemma 5.3.1. *The degree d defines a torus action on C_β^M if we have*

$$d(\rho, s, t) := d(s') - d(t) = d(s') - d(t')$$

for all $s' \xrightarrow{\rho} t$ with $s' \leq s$, $s' \in \mathcal{B} \backslash \beta \cup \{s\}$ and for all $s' \xrightarrow{\rho} t'$ with $t' \in \beta$, $s' \in \mathcal{B} \backslash \beta \cup \{s\}$ and $s' \leq s$, $t < t'$ where $t \notin \beta$, $s \in \beta$.

This raises the question under which conditions the torus action is non-trivial, by what we mean that not all points are fixed points. The torus action is easily seen to be trivial if we have $d(b_i^q) = d_q$ for all $q \in Q_0$ and some fixed $d_q \in \mathbb{Z}$. But, as we will see, there are cases with a non-trivial torus action.

Remark 5.3.2. Roughly speaking, the idea is the following: assume that the degree d induces a torus action. If $q' \leq q$ are vertices of the same color, then T can be seen to act on $w_{q'q}$ via scalar multiplication by $\lambda^{d(q')-d(q)}$. By the considerations from above, we see the following: if w was in β-normal form, $\lambda.w$ is also in β-normal form. If $d(q) \neq d(q')$, by the uniqueness of the β-normal form, we are forced to have $w_{q'q} = 0$ for a torus fixed point under the assumption that the degree is chosen generically.

Since Γ_M is a tree, there exists a unique path $p(q, q')$ between each two vertices q and q' of Γ_M. In order to shorten notation, let $F = F_{\Gamma_M}$.

Definition 5.3.3. Let $s, s', s'', t, t', t'', q, q' \in (\Gamma_M)_0$ such that $F(s) = F(s') = F(s'')$, $F(t) = F(t') = F(t'')$ and $F(q) = F(q')$.

i) We say that an arrow $\rho : s' \to t'$ is β-fixed if $w_{s's}w_{tt'}$ appears in a non-trivial equation $E(\rho, t, s)$ where $t \notin \beta, s \in \beta$. Otherwise, we say that ρ is β-free.

ii) We say that $w_{s's''}$ is β-free if there is at least one arrow $\rho : s \to t$ in $p(s', s'')$ which is β-free.

iii) We say that $w_{t't''}$ is β-free if there is at least one arrow $\rho : s \to t$ in $p(t', t'')$ which is β-free.

iv) If $w_{q,q'}$ is not β-free, we say that it is β-fixed.

Note that if $w_{q,q'}$ is β-fixed, q and q' are forced to have the same degree. Later it turns out that torus fixed points have vanishing β-free variables.

Lemma 5.3.4. *Let $\rho : s' \to t'$ be an arrow of the coefficient quiver Γ_M.*

i) If $s' \in \beta$, $t' \in \beta$, then $\rho : s' \to t'$ is β-fixed if and only if there exists a vertex $t \in \mathcal{B} \backslash \beta$ with $F(t) = F(t')$ such that $t < t'$.

ii) If $s' \notin \beta$, $t' \notin \beta$, then $\rho : s' \to t'$ is β-fixed if and only if there exists a vertex $s \in \beta$ with $F(s) = F(s')$ such that $s' < s$.

iii) If $s' \notin \beta$, $t' \in \beta$, then $\rho : s' \to t'$ is β-fixed if and only if there exists a vertex $s \in \beta$ with $F(s) = F(s')$ and a vertex $t \in \mathcal{B} \backslash \beta$ with $F(t) = F(t')$ such that $s' < s$ and $t < t'$.

iv) If $s' \in \beta$, $t' \notin \beta$, then $\rho : s' \to t'$ is β-fixed.

Proof. In the first case, we have that $\lambda^{d(s')-d(t')} w_{s's'} w_{tt'}$ appears in $\lambda.E(\rho, t, s')$. In the second case, $\lambda^{d(s')-d(t')} w_{s's}$ appears in $\lambda.E(\rho, t', s)$. In the third case, $\lambda^{d(s')-d(t')} w_{s's} w_{t't}$ appears in $\lambda.E(\rho, t, s)$. Finally, in the last case, $\lambda^{d(s')-d(t')} w_{s's'}$ appears in $\lambda.E(\rho, t', s')$. □

We can reformulate the preceding lemma in terms of β-free variables and arrows:

Lemma 5.3.5.

i) Let $s' \notin \beta$. If we have $s \notin \beta$ for all $s \geq s'$ with $F(s') = F(s)$, then all arrows $\rho : s' \to t'$ are β-free.

ii) Let $t' \in \beta$. If we have $t \in \beta$ for all $t \leq t'$ with $F(t) = F(t')$, then all arrows $\rho : s' \to t'$ are β-free.

Fix integers $d(\rho)$ for all $\rho \in Q_0$. Then we set $d(\rho, s, t) = d(\rho)$ for every β-non-trivial equation $E(\rho, t, s)$. We introduce a sequence of subquivers $\Gamma_1, \ldots, \Gamma_n$ of the coefficient quiver by saying that two β-fixed arrows ρ, ρ' (including head and tail) belong to the same subquiver if and only if there is a path of β-fixed arrows in Γ_M to which ρ and ρ' belong. In each subquiver Γ_i, we fix a vertex $q_i \in (\Gamma_i)_0$ and degrees $d(q_i)$. This recursively determines the degrees $d(q')$ of all vertices q' in Γ_i by the formula $d(s) + d(\rho) = d(t)$ if $\rho : s \to t \in (\Gamma_i)_1$. Clearly, we can assume that $d(q) \neq d(q')$ for all $q \in (\Gamma_i)_0$, $q' \in (\Gamma_j)_0$ with $i \neq j$. Let $(\Gamma_0)_0$ be the set of vertices q with $q \notin (\Gamma_i)_0$ for all $i = 1, \ldots, n$. We also fix a degree for every vertex q in this set where we assume that $d(q) \neq d(q')$ whenever $q' \neq q$. In the following, we refer to such a degree as β-compatible. Then we have the following statement:

Theorem 5.3.6. *Let M be a tree module with coefficient quiver Γ_M. Let $\beta \in (\Gamma_M)_0$ be a subset of type e and d a degree which is β-compatible. Then d defines a torus action on $C^M_\beta \subset \mathrm{Gr}_e(M)$ in such a way that $w \in (C^M_\beta)^T$ if and only if*

$$w_{qq'} = 0 \text{ for all pairs of vertices } (q, q') \text{ with } q \in (\Gamma_k)_0, q' \in (\Gamma_l)_0$$

and $k \neq l$ or $q \neq q'$ if $k = l = 0$.

Proof. By construction, d defines a torus action on C^M_β. As already mentioned in Remark 5.3.2, the variable $w_{q,q'}$ is multiplied by $\lambda^{d(q)-d(q')}$. Since q and q' have different degrees, by the uniqueness of the β-normal form, we have $w_{q,q'} = 0$ if w is a torus fixed point. □

5.3.2 Application to preprojective representations of $\tilde{D}_4$ of small defect

Let $Q = \tilde{D}_4$ be in subspace orientation where we use the notation from Section 5.1.2. If Γ_M is a coefficient quiver of a representation M and $q \in Q_0$, define $(\Gamma_M)_0(q) = \{q' \in (\Gamma_M)_0 \mid F(q') = q\}$.

Moreover, we consider the representation M_n given by the coefficient quiver Γ_n with vertices

$$(\Gamma_n)_0(q_0) = \{3k+1 \mid k = 0, \ldots, 2n\},\ (\Gamma_n)_0(q_d) = \{6k+2 \mid k = 0, \ldots, n\},$$
$$(\Gamma_n)_0(q_b) = \{6k+3 \mid k = 0, \ldots, n-1\},\ (\Gamma_n)_0(q_a) = \{6k+5 \mid k = 0, \ldots, n-1\},$$
$$(\Gamma_n)_0(q_c) = \{6k \mid k = 1, \ldots, n\}$$

and arrows

$$(\Gamma_n)_1 = \{6k+2 \xrightarrow{d} 6k+1,\ 6k+3 \xrightarrow{b} 6k+1,\ 6k+3 \xrightarrow{b} 6k+4,\ 6k+5 \xrightarrow{a} 6k+4,$$
$$6k \xrightarrow{c} 6k+1,\ 6k \xrightarrow{c} 6k-2\},$$

which is up to automorphism and a slight reordering the same as the one considered in Section 5.2.4. The root corresponding to it is $(2n+1, n, n, n, n+1)$. A coefficient quiver for the indecomposable representation of dimension $(2n+2, n+1, n, n+1, n+1)$, which we denote by S_{n+1}, is obtained by glueing the subquiver N defined by

$$6n+3 \xrightarrow{b} 6n+4 \xleftarrow{a} 6n+5.$$

In turn, we obtain a coefficient quiver for M_{n+1} by glueing the subquiver

$$6(n+1) \xrightarrow{c} 6(n+1)+1 \xleftarrow{d} 6(n+1)+2.$$

Here we add extra arrows $6n+3 \xrightarrow{b} 6n+1$ and $6(n+1) \xrightarrow{c} 6n+4$ respectively. Note that all preprojective representations can be obtained in this way. We use the definition of admissible subsets introduced in Definition 5.2.7. The aim of this section is to determine the Euler characteristic, partially with the help of the introduced torus action, but also using a recursion which relies on the recursive construction of the coefficient quivers for this fixed orientation. More precisely, we give a short proof of the following statement which clearly also follows from Theorem 5.1.6:

Theorem 5.3.7. *Let M be an indecomposable preprojective representation of $\tilde{D}_4$ in subspace orientation and consider its coefficient quiver Γ_M as defined above. Then $\chi(\mathrm{Gr}_e(M))$ coincides with the number of admissible subsets of Γ_M of type e.*

Proof. We proceed by induction and prove that

$$\chi(C_\beta^M) = \begin{cases} 1 \text{ if } \beta \text{ is admissible} \\ 0 \text{ otherwise} \end{cases}.$$

The claim is easily checked to be true for M_0 and N respectively. Now consider the representation S_{n+1} and let $\beta_1 \cup \beta_2 = \beta \subset (S_{n+1})_0$ with $\beta_1 \subseteq \{6n+3, 6n+4, 6n+5\}$ and $\beta_2 \subseteq (\Gamma_{M_n})_0$. We proceed by a case-by-case analysis considering the different possibilities for β_1.

First assume that $6n+3 \xrightarrow{b} 6n+1$ is β-free, which is by Lemma 5.3.5 the cases if $6n+3 \notin \beta$. Applying Theorem 5.3.6, we can define a torus action such that for $w \in (C_\beta^{S_{n+1}})^T$ we have $w_{qq'} = 0$ whenever $q' \in \{6n+3, 6n+4, 6n+5\}$ and $q \in \beta_2$. In particular, the fixed points decompose into $w_1 + w_2$ with $w_1 \in (C_{\beta_1}^N)^T$ and $w_2 \in (C_{\beta_2}^{M_n})^T$. This implies that

$$(C_\beta^{S_{n+1}})^T = (C_{\beta_2}^{M_n})^T \times (C_{\beta_1}^N)^T.$$

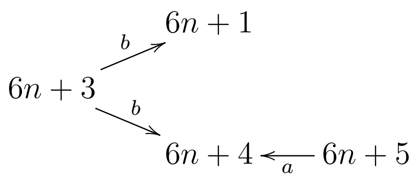
Thus, it follows that

$$\chi(C_\beta^{S_{n+1}}) = \chi((C_\beta^{S_{n+1}})^T) = \chi((C_{\beta_2}^{M_n})^T) \cdot \chi((C_{\beta_1}^N)^T) = \chi(C_{\beta_2}^{M_n}) \cdot \chi(C_{\beta_1}^N).$$

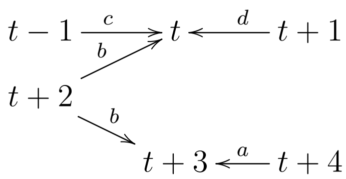
By induction hypothesis, it follows that $\chi(C_\beta^{S_{n+1}}) = 1$ if both β_1 and β_2 are admissible. But since $6n+3 \notin \beta$, this is equivalent to the requirement that β is admissible.

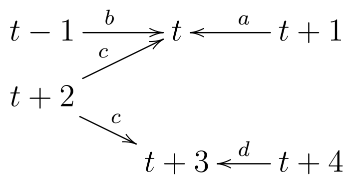
Moreover, if $6n+3 \in \beta$ and $6n+4 \notin \beta$, we have that β is not extremal successor closed which means that $E(b, 6n+4, 6n+3)$ already yields the contradiction $1 = 0$. It follows that $\chi(C_\beta^{S_{n+1}}) = 0$.

Thus, it remains to consider the cases in which $\{6n+3, 6n+4\} \subseteq \beta_1$ what we assume from now on. The strategy is to consider and investigate the additional (non-trivial) equations and variables arising when glueing the subquiver N in the different cases. It turns out that the solution set of the original equations is not affected by the new equations and variables. Moreover, the solution set of the new equations can be seen to be an affine space when fixing the variables appearing in the original equations. Consider the subquiver



First assume that $6n+5 \notin \beta$. Then we get additional equations $E(b, t, 6n+3)$ for $t \notin \beta$ and $F(t) = q_0$. Moreover, we get additional variables $w_{s,6n+3}$ for $s \notin \beta$ with $F(s) = q_b$ and $w_{t,6n+4}$ for $t \notin \beta$ with $F(t) = q_0$. More detailed, we consider the following subquivers





with $t \notin \beta$. Then we have $t-1 \notin \beta$ because β is forced to be extremal successor closed. In the case of the subquiver on the right hand side, $E(b, t, 6n+3)$ becomes

$$w_{t-1,6n+3} = w_{t,6n+4} + w_{t,6n+1} + \sum_{\substack{s' \notin \beta,\, t' \in \beta \\ s' < s,\, t < t' \\ s' \xrightarrow{b} t'}} w_{s',6n+3} w_{tt'}.$$

Note that since $t \notin \beta$, we have that $t < 6n+4$. Moreover, $w_{t,6n+1} = 0$ is possible. Nevertheless, we can solve this equation for $w_{t,6n+4}$. In the case of the second subquiver, $E(b,t,6n+3)$ becomes

$$w_{t+2,6n} = w_{t,6n+4} + w_{t,6n+1} + \sum_{\substack{s' \notin \beta,\, t' \in \beta \\ s' < s,\, t < t' \\ s' \xrightarrow{b} t'}} w_{s',6n+3} w_{tt'}$$

which can again be solved for $w_{t,6n+4}$. Since the equations and variables corresponding to the subquiver M_n do not change, we obtain that

$$C_{\beta}^{S_{n+1}} = C_{\beta_2}^{M_n} \times \mathbb{A}^r$$

for some $r \geq 0$. Thus, the claim follows also for this choice of β.

Secondly, assume that $6n+5 \in \beta$. Then, in addition to the equations $E(b,t,6n+3)$ from above, we get the equations $E(a,t,6n+5)$ for $t \notin \beta$ with $F(t) = q_0$. Then it is straightforward that we have $6n+1 \in \beta$. Indeed, otherwise $E(a,6n+1,6n+5)$ would yield $w_{6n+1,6n+4} = 0$ while $E(b,6n+1,6n+3)$ would yield $w_{6n+1,6n+4} = 1$.

In addition to the variables $w_{s,6n+3}$ for $s \notin \beta$ with $F(s) = q_b$ and $w_{t,6n+4}$ for $t \notin \beta$ with $F(t) = q_0$ we get the variables $w_{s,6n+5}$ for $s \notin \beta$ and $F(s) = q_a$. We again consider the two subquivers from above. In the case of the subquiver on the right hand side, $E(a,t,6n+5)$ becomes

$$w_{t+1,6n+5} = w_{t,6n+4} + \sum_{\substack{s' \notin \beta,\, t' \in \beta \\ s' < s,\, t < t' \\ s' \xrightarrow{a} t'}} w_{s',6n+5} w_{tt'}.$$

Now we can solve this equation for $w_{t+1,6n+5}$ and, as before, we can solve $E(b,t,6n+3)$ for $w_{t,6n+4}$. Note that this is possible because these two variables do not appear in the sum which consists of the quadratic terms.

The instance of the subquiver on the left hand side happens to be the most complicated one: In this case, $E(a,t,6n+5)$ becomes

$$0 = w_{t,6n+4} + \sum_{\substack{s' \notin \beta,\, t' \in \beta \\ s' < s,\, t < t' \\ s' \xrightarrow{a} t'}} w_{s',6n+5} w_{tt'}.$$

Moreover, for $E(b,t,6n+3)$, we obtain

$$w_{t+2,6n+3} = w_{t,6n+4} + w_{t,6n+1} + \sum_{\substack{s' \notin \beta,\, t' \in \beta \\ s' < s,\, t < t' \\ s' \xrightarrow{b} t'}} w_{s',6n+3} w_{tt'}.$$

We need a further case-by-case analysis:

i) Let $t+2 \in \beta$. In this case, $E(b,t,t+2)$ becomes

$$w_{t+2,t+2} = w_{t+2,t+2}w_{t,t+3} + w_{t+2,t+2}w_{t,t} = w_{t+2,t+2}w_{t,t+3}.$$

Since $t+2 \in \beta$, we have $t+3 \in \beta$. In particular, we have $w_{t,t+3} = 1$. Moreover, we have $t+4 \notin \beta$ because $t \notin \beta$ which follows by the same argument as above. Thus, for $E(a,t,6n+5)$, we have

$$0 = w_{t,6n+4} + w_{t+4,6n+5} + \sum_{\substack{s' \notin \beta,\, t' \in \beta \\ s' < s,\, t+3 < t' \\ s' \xrightarrow{a} t'}} w_{s',6n+5}w_{tt'}.$$

Moreover, for $E(b,t,6n+3)$, we have

$$0 = w_{t,6n+4} + w_{t,6n+1} + \sum_{\substack{s' \notin \beta,\, t' \in \beta \\ s' < s,\, t < t' \\ s' \xrightarrow{b} t'}} w_{s',6n+3}w_{tt'}.$$

In particular, we can solve the first equation for $w_{t+4,6n+5}$ and the second for $w_{t,6n+4}$.

ii) $t+2 \notin \beta$, $t+3 \notin \beta$. Then we can solve $E(b,t,6n+3)$ which becomes

$$w_{t+2,6n+3} = w_{t,6n+4} + w_{t,6n+1} + \sum_{\substack{s' \notin \beta,\, t' \in \beta \\ s' < s,\, t < t' \\ s' \xrightarrow{b} t'}} w_{s',6n+3}w_{tt'}$$

for $w_{t+2,6n+3}$ because $w_{t,t+3} = 0$. This means that $w_{t+2,6n+3}$ does not appear on the right hand side. Moreover, we can solve $E(a,t,6n+5)$ for $w_{t,6n+4}$.

iii) $t+2 \notin \beta$, $t+3 \in \beta$, $t+4 \in \beta$. Then we have $w_{t+4,q} = 0$ for every $q \neq t+4$ and $w_{t,t+3} \neq 0$ so that we cannot simply solve $E(b,t,6n+3)$ for $w_{t+2,6n+3}$. In this case, we make a variable transformation:

$$w_{t,6n+4} \mapsto \tilde{w}_{t,6n+4} := w_{t,6n+4} + w_{t,t+3}w_{t+2,6n+3}.$$

Thus, $E(b,t,6n+3)$ becomes

$$w_{t+2,6n+3} = \tilde{w}_{t,6n+4} - w_{t,t+3}w_{t+2,6n+3} + w_{t,6n+1} + \sum_{\substack{s' \notin \beta,\, t' \in \beta \\ s' < s,\, t < t' \\ s' \xrightarrow{b} t'}} w_{s',6n+3}w_{tt'}$$

which means that $-w_{t,t+3}w_{t+2,6n+3}$ cancels with $w_{t,t+3}w_{t+2,6n+3}$ as it also appears in the sum of quadratic terms. Finally, $E(a,t,6n+5)$ becomes

$$0 = \tilde{w}_{t,6n+4} - w_{t,t+3}w_{t+2,6n+3} + \sum_{\substack{s' \notin \beta,\, t' \in \beta \\ s' < s,\, t < t' \\ s' \xrightarrow{a} t'}} w_{s',6n+5}w_{tt'}.$$

In particular, we can solve the first equation for $w_{t+2,6n+3}$ and the second for $\tilde{w}_{t,6n+4}$.

iv) $t+2 \notin \beta$, $t+3 \in \beta$, $t+4 \notin \beta$. Then the only difference to the preceding case is that $w_{t+4,6n+5}$ is a free variable which appears in a quadratic term of $E(a,t,6n+5)$. But this does not bother us as we can make the same variable transformation as before. We can again solve the first equation for $w_{t+2,6n+3}$ and the second for $\tilde{w}_{t,6n+4}$.

Summarizing, also in all cases coming along with the condition $6n+5 \in \beta$, we obtain

$$C_{\beta}^{S_{n+1}} = C_{\beta_2}^{M_n} \times \mathbb{A}^r$$

and the claim follows.

□

Remark 5.3.8. It does not seem to be clear (or at least a further check is needed) if the proof can be used to prove that the Schubert cells C_{β}^{M} are affine spaces (what they are as we know). One problem is that the classical theorem of Bialynicki-Birula from [2] does not apply. Indeed, we a priori do not know if the Schubert cells are smooth and, moreover, they are definitely not projective. In particular, if we know that $(C_{\beta}^{M})^T$ is an affine space, we still do not know if C_{β}^{M} is an affine space.

But it seems that the consideration of a torus action on the Schubert cells can often be used to simplify the determination of the Euler characteristic; especially if there is no torus action on the whole quiver Grassmannian.

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