Early History of Symplectic Geometry

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by

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Introduction

Symplectic Geometry

Currently, symplectic geometry refers to the study of symplectic manifolds. A symplectic manifold is an even dimensional manifold endowed with a closed non-degenerated 2-form. This 2-form is called a symplectic form or structure. Some examples of symplectic manifolds are given in chapter 5.2.

In euclidean geometry, for example, the concepts of distance can be obtained from the inner product, which is a symmetric bilinear form. In symplectic geometry, an even-dimensional euclidean space is endowed with an antisymmetric bilinear form. $\omega_0 = (v, w) = v^t J w$ where $v,\ w \in \mathbb{R}^{2n}$ and

$$J = \begin{pmatrix} 0 & Id \\ -Id & 0 \end{pmatrix}.$$

If the dimension is equal 2, i.e. a two dimensional euclidean plane, the antisymmetric bilinear form is

$$\omega_0(v, w) = v^t J w = v_1 w_2 - v_2 w_2$$

where $v, w \in \mathbb{R}^2$ and

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

This antisymmetric bilinear form is the symplectic form on the plane, and it gives the area of the parallelogram formed by the vectors v and w, therefore, the symplectic form on the plane represents an area. In symplectic geometry the length of every vector in the plane is zero, and every vector is orthogonal to itself.

In symplectic geometry there are no local invariant results in Darboux's theorem, which states that every symplectic structure on a manifold is locally diffeomorphic to the standard symplectic structure

$$\omega_0 = \sum_{j=1}^{n} dx_j \wedge dy_j \quad \text{on} \quad \mathbb{R}^{2n}.$$

The diffeomorphisms that preserve the symplectic form are called symplectomorphisms. The linear symplectomorphisms of a symplectic vector space form a group called the symplectic group of the symplectic vector space.

On the plane a symplectomorphism sends any region of finite volume into one of the same volume.

The symplectic group of degree $2n$ over a field F is denoted by $Sp(2n,(F)$, it consists of all $2n \times 2n$ matrices

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

with A, B, C and D $n \times n$-matrices, which satisfy $M^t J M = J$ i.e.

$$A^t C = C^t A, \quad B^t D = D^t B, \quad A^t D - C^t B = Id.$$

The set of symplectomorphisms of the plane is the symplectic group $Sp(2, \mathbb{R})$.

Before 1938, the symplectic group was known as the complex group or Abelian linear group. In 1938, Hermann Weyl proposed to change the name and named it symplectic to avoid confusion of the complex group with complex numbers. The adjective "symplectic" is the Greek adjective for complex.

Historically another type of geometry has been called "symplectic geometry" and I will be deal twith in this tesis in Siegel's work on (even dimensional) hyperbolic spaces on chapter 2.

Nowadays, symplectic geometry is related to classical mechanics. This relation comes because the structure of Hamilton's equations can be described through the symplectic form. The matrices that map a Hamiltonian system into a Hamiltonian system are symplectic, and, in Hamilton mechanics, if the configuration space is an n-dimensional manifold, the momentum phase space is a symplectic manifold of dimension $2n$.[1]

State of the Art

At the end of the 1960s, the study of non-degenerate 2-differential forms over a differential manifold attracted the interest of many mathematicians and physicists, including Jean-Marie Souriau, Ralph H. Abraham, Jerrold E. Marsden, Alan Weinstein and Vladimir Igorevich Arnold. This interest arose from the fact that symplectic geometry can be seen as a mathematical and geometrical formulation of classical mechanics. Currently, some of the textbooks about symplectic geometry and topology start with classical mechanics as a motivation.[2]

Patrick Iglesias, Souriau's student, wrote in 2002,

La géométrie symplectique est devenue le cadre par excellence

[1] See (Abraham & Marsden 1978, p.178)

[2] See (Berndt 1998, McDuff & Salamon 1995).

> *de la mécanique à tel point que l'on peut dire aujourd'hui que ces théories se confondent.*[3]
>
> (Iglesias 2002, p.2)

The Hamilton's equations can be described through a symplectic form, therefore, many mathematicians date the origin of symplectic structures to the work "Analytical Mechanics" of Lagrange (Lagrange 1811). One of these mathematicians was Souriau, who published an article entitled *La structure symplectique de la mécanique décrite par Lagrange en 1811* in 1986.[4] The same claim was made later by Iglesias.[5] It must be said that Souriau and Iglesias formulated their claim carefully, and they did not say that this was the genesis of symplectic geometry, but it is difficult to support their hypothesis that the symplectic structure can be found in the work of Lagrange. The symplectic structure is a 2-form which is closed, and it was not already defined in Lagrange's work of 1811. The calculus of differential forms was developed by Élie Cartan at the end of the 19th century and at the beginning of the 20th century.[6]

The transformation of the Euler-Lagrange equations to the Hamiltonian equations can be derived through a Legendre transformation. This was known in the 19th century, and, therefore, one might think that the symplectic form can be found in the work of Lagrange and in the classical mechanics of the 19th century.

Something similar happens with symplectic manifolds. Nowadays, if you look in some popular science articles or in Wikipedia, there they claim, "sym-

[3] Symplectic geometry has become the frame par excellence of mechanics, to a point that it can be said that these theories have been mixed.

[4] The symplectic structure in mechanics described by Lagrange in 1811. (Souriau 1986).

[5] See (Iglesias 2002, Iglesias 1995*b*).

[6] See (Katz 1981, Katz 1985).

plectic manifolds arise from classical mechanics, in particular they are generalizations of the phase space of a closed system."[7] This claim arises because, as Iglesias said, "symplectic geometry has become the frame par excellence of mechanics" and mathematicians know the use of symplectic geometry in classical mechanics; moreover, they see it as part of the language of mechanics. The claim that symplectic manifolds arose from classical mechanics is not at all correct because the first manifolds endowed with a symplectic structure were the Kähler manifolds. Kähler manifolds are complex manifolds endowed with a closed 2-form which is known as a Kähler form, and they were defined by Erich Kähler in 1933. But, even though all Kähler manifolds are symplectic manifolds, the symplectic manifolds were not defined until 1950 by Charles Ehresmann.[8] Ehresmann's motivation to define the notion of a symplectic manifold was to find out whether any $2n$ real dimensional manifolds admit a complex structure, but then he found out that not all even real dimensional spheres admit a complex structure.

Concerning Darboux's theorem, it is said that the theorem for symplectic manifolds was given by Darboux in 1882, but again, the symplectic manifolds were not defined until the middle of the 20th century.

These claims and assertions about the origin of symplectic structure and Darboux's theorem, and the fact that symplectic geometry can be seen as a mathematical formulation of classical mechanics, means that many mathematicians think that the origins and the development of symplectic geometry took place in the 19th century.

But if it was not during the 19th century:

[7] See Wikipedia, *Symplectic manifold*, [on-line], available from: https://en.wikipedia.org/wiki/Symplectic_ manifold, 29.01.2019

[8] See chapter 5.

- Where is the origin of the symplectic form and symplectic manifolds?
- When was Darboux's theorem understood as stating that locally two symplectic manifolds of the same dimension are isomorphic?
- When did the symplectic group become associated with describing the diffeomorphisms over a symplectic manifold?
- When did symplectic geometry become the frame of classical mechanics?
- Where, how and why all this development took place?

The combination "symplectic geometry" appeared for the first time in history in 1943. The first time that the adjective "symplectic" was used to denoted a geometry was in 1943 by Carl Ludwig Siegel. In 1943, Siegel published his article "Symplectic Geometry" (Siegel 1943*b*). In his article he generalized the theory of automorphic functions to the case of m complex variables, investigated the invariant geometric properties of a simple domain called E, identified the discontinuous subgroups operating on E and constructed their fundamental domains. An example of a simple domain is the Siegel half space. Siegel's half space $\mathcal{H}_n = \{Z \in M_n(\mathbb{C}) | Z = Z^t, Im\, Z > 0\}$ where $Z = X + iY$ is a complex symplectic matrix with $Y > 0$ and Z is a symmetric matrix $Z = (z_{kl})$. The group of all analytic transformations that map a simple domain E with dimension m into itself is a group of symplectic tranformations.[9] This is the reason why Siegel used *symplectic* to characterize this geometry. Siegel did not endow Siegel's half space with a symplectic structure, but Siegel's half space is a symplectic space. For Siegel, symplectic

[9](Siegel 1943*b*, p. 3).

geometry is a generalization of hyperbolic geometry to $\frac{1}{2}n(n+1)$ complex dimensions.

The symplectic group is used in celestial mechanics, and Siegel use explicitly the symplectic group to describe the transformations of a Hamiltonian system into a Hamiltonian system.[10] Celestial mechanics is a field that Siegel used to teach since he was appointed as a professor in Frankfurt am Main in 1922 (Siegel 1956). Siegel's publications in this field are (Siegel 1941) and (Siegel 1956). The latter one are his lecture notes on celestial mechanics for a course he gave in Göttingen during the winter-semester 1951-1952. Jürgen K. Moser compiled the notes for the book four years later (Siegel 1956, p. i).

Three years before the publication of Siegel's lectures notes, Jean-Marie Souriau gave a conference in Strasbourg at the Colloques Internationaux du Centre National de la Recherche Scientifique with the title "Géométrie symplectique différentielle" (Souriau 1953). It seems that after these two events, the publication of Souriau's presentation and Siegel's lecture, symplectic geometry became fully linked to the study of classical mechanics and the calculus of variations. Fields, which have been of interest to many mathematicians for a long time.

It is to be noted that Siegel's article (Siegel 1943*b*) is quoted by some textbooks on symplectic geometry and mathematical methods of mechanics. [11] However, between these publications and Siegel's article, beside the symplectic group and Siegel's half space, what is understood as symplectic geometry seems not to be the same field.

But as Rolf Berndt said:

> The study of the geometry of these manifolds [Siegel's half

[10] See chapter 8.

[11] See (Abraham & Marsden 1967, Arnold 1989, Berndt 1998, Da Silva 2000).

> space] and the holomorphic, as well as the meromorphic, functions on $\mathcal{H}_n$ [Siegel's half space] with known invariant or covariant properties under the operation of the group $Sp(2n, \mathbb{Z})$ or its subgroups was [sic] initiated by Siegel; for some time it was exactly these topics introduced by Siegel that formed the subject of symplectic geometry. [...] Nowadays, however, symplectic geometry refers to a much broader range of topics.[...]
>
> (Berndt 2001, p.34)

The topics that Berndt refers to are the study of symplectic manifolds which are spaces endowed with a symplectic form, and symplectic geometry as the geometrical interpretation of classical mechanics.

I will argue that the field of symplectic geometry emerged after some mathematical objects of symplectic geometry were defined and developed in an explicit way, as the symplectic group, symplectic structure, symplectic manifold and Darboux's theorem.

As well, I will state that Siegel and Souriau are the authors who introduced the concept of symplectic structures into Hamilton mechanics, and it seems possible that they applied it independently of each other. After the introduction of the concept of symplectic structures into Hamilton mechanics, symplectic geometry is linked to Hamilton mechanics and to the study of even dimensional manifolds endowed with a symplectic structure.

To prove these hypothesis I will follow the development of the above mentioned mathematical objects in the field of symplectic geometry during the 1930s to the 1950s. I will show the context in which these objects were developed, and not only the mathematical context but as well the social and biographical context of the mathematicians who developed these ideas. For the second assumption, I will present the work that Siegel and Souriau did when they exposed symplectic geometry as a field of study.

A short summary of the history of symplectic geometry was given by Yvette Kosmann-Schwarzbach in her book *"Simeon-Denis Poisson : Les mathématiques au service de la science"* published in 2013.

About this work

This work has ten chapters. They are set in an almost chronological order. First I give a short chapter on the history of the symplectic group and a short biography of Marius Souphous Lie and Hermann Weyl. Nor Lie or Weyl were direct actor to the development of symplectic geometry but they describe the symplectic group and its properties. Even though Weyl was not a direct actor in the field of symplectic geometry, his personal network provided the connections between the different mathematicians and institutions being involved in the development of symplectic geometry.

The objective of the second chapter is to describe the achievements of Carl Ludwig Siegel during the middle of the 1930s and the beginning of the 1940s by generalizing the theory of automorphic functions to the case of an arbitrary number of variables.

The chapter starts with some biographical notes on Siegel to provide the historical context.

Then there is a short description of Siegel's number theoretical work, on which Siegel extended the theorem of Hasse-Minkowski for quadratic forms and gave an analytical interpretation in 1935. This interpretation led him to develop the theory of modular functions of degree n.

Through this result Siegel was able to made another generalization about hyperbolic geometry on the $\frac{1}{2}n(n+1)$ dimensional space to the space of symmetric complex matrices. This generalization was published in 1943 in the American Journal of Mathematics under the name *"Symplectic Geometry"*. It is the first time that the words "symplectic" and "geometry" appeared together to characterize a mathematical field.

Siegel generalized the half plane to the half space of dimension $\frac{1}{2}n(n+1)$. Nowadays, this half space is known as Siegel's half space. The group that acts biholomorphically on Siegel's half space is the symplectic group, and, therefore, the generalization of hyperbolic geometry was named *"Symplectic Geometry"* by Siegel. Siegel proved that on the upper half space, the group that acts biholomorphically is the modular group, $SL(2,\mathbb{R})$, which is isomorphic to the symplectic group $Sp(2,\mathbb{R})$.

Siegel's half space is as well a symplectic vector space, but Siegel did not define the symplectic vector space, and, therefore, the Siegel half space is not defined as a symplectic vector space.

At the end of the second chapter a parallel work is described. This work was done in 1943 by Hua Loo-Keng (*1910, †1985), a Chinese mathematician, who at that time lived in China but studied in England with Hardy from 1936 to 1938. In 1943 Hua worked at the National Tsing Hua University of China, at the Institute of Mathematics at the Academia Sinica. In 1943 Weyl qualified Hua's work as a double of Siegel's article *Symplectic Geometry*. In this chapter there is as well a short biography of Hua and the story of this doubling.

In Chapter three, a short passage of how the theory of exterior differential forms was developed by Élie Cartan is presented. The development of the theory of exterior differential forms was done by Cartan in his work and research on the Pfaffoan Problem. He had developed the theory of the exterior calculus of differential forms in (Cartan 1896), (Cartan 1899) and (Cartan 1901). Élie Cartan, like Weyl, had an important personal network. Even more, many of Cartan's students developed what is nowadays understood as symplectic geometry.

Chapter 4 deals with the genesis of the studies on Kähler manifolds and the biography of Erich Kähler. Kähler manifolds were defined by Erich Kähler in 1933 before symplectic manifolds were defined. Today, it is known that Kähler manifold are symplectic manifolds. Therefore, the genesis of Kähler manifolds can be considered as part of the development of symplectic manifolds and as part of the early history of symplectic geometry.

Kähler manifolds are n-dimensional complex manifolds with an Hermitian metric endowed with a Kähler form, which is a closed and non degenerated 2-form. Kähler noticed that there is a relationship between the Hermitian metric and its associated differential form if the 2-form is closed and non degenerated. If a differential 2-form is closed and non degenerated over an even dimensional manifold, then it is a symplectic manifold. This differential form is a symplectic form and in the case of complex manifolds it is known as "Kähler form".

The last part of the chapter deals with the reception of Kähler manifolds in the work of Eckmann and Guggenheimer at the end of the 1940s. At that time the name "Kähler manifold" was established.

Chapter 5 describes the development provided by Charles Ehresmann, who gave the first definition of the notion "symplectic manifolds" in 1950. The context of this definition is the development of fibre bundles, fields to which Ehresmann and Jacques Feldbau contributed at the beginning of the 1940s.

Ehresmann showed that on a real even dimensional differential manifold the existence of an almost complex structure is equivalent to the existence of a differential 2-form of rank $2n$ in all points over the manifold. Ehresmann proved that the four-dimensional real sphere does not admit a complex structure, and later he asked himself: "Do all real even dimensional differential manifolds admit a complex structure?"[12] The question was rhetorical because he knew at that time that not all real even differential manifolds admit a complex structure. The actual question was: Which real even-dimensional differential manifolds admit a complex structure? This led to the definition of symplectic manifolds because the real even dimensional manifold, which admits a complex structure, admits a 2-form in all points over the manifold, and if the form is closed and non-degenerated, it is a symplectic form. In this context the symplectic structure arose as a necessary condition for the existence of a complex structure.

After Ehresmann's definition of symplectic manifolds, other mathematicians such as Guggenheimer and Eckmann used it for their studies on Kähler manifolds. At that time, all symplectic manifolds were Kähler manifolds, and it was like this until 1976 when William Thurston (*1946, †2012) showed that there exist symplectic manifolds which are not Kähler manifolds.

Chapter 6 is about the development of Darboux's theorem. In symplectic geometry the Darboux theorem states that every symplectic structure on a manifold is locally diffeomorphic to the canonical symplectic structure. First, I look at the development done by Darboux when he solved Pfaff's problem at the end of the 19th century. At the end of the 19th century to solve Pfaff's problem was understood as to find a suitable change of variables for

[12](Ehresmann 1950*a*, p. 412)

the differential equation

$$\omega = \sum_{i=1}^{n} a_i(x)dx_i = 0, \quad (x = (x_1, ..., x_n)),$$

such that it could be expressed through a minimal number of variables, i.e. to find the canonical form of the differential equation. The solution that Darboux gave in 1882 was named Darboux's theorem by Shalomo Sternberg in 1964 in his book *Lectures on Differential Geometry*. Sternberg discussed the problem of finding the canonical form for a Pfaffian differential form, i.e. a 1-form, which is the translation of Darboux's theorem of differential equations to the theory of differential form. Darboux theorem in Sternberg's book states how to find the canonical form of a 1-form on a differential manifold and by using an exterior differentiation he obtained the canonical form for a closed 2-form with rank p everywhere on an n-dimensional manifold. For the case the manifold is even dimensional it becomes the theorem of Darboux of symplectic geometry.

This last result was given by Paulette Libermann at the end of the 1940s and published on her PhD thesis. Paulette Libermann was a student of Ehresmann. In her PhD thesis she studied the equivalent problem of Cartan. The equivalent problem of Cartan was expressed in 1953 as the problem of local equivalence between infinitesimal structures, i.e. between differential forms. Libermann formulated this in relation to fibre spaces. In her thesis she stated Darboux's theorem for symplectic geometry, which shows that all symplectic manifolds with the same dimension are local isomorphic between them, and they are local isomorphic to a symplectic vector space of the same dimension. Therefore, in symplectic geometry they are not local invariants, and all points on the symplectic manifold are equivalent.

Chapter 7 is an excursus because it explores the work done by the Chinese mathematician Lee Hwa-Chung during the 1940s. Lee's work is not well known within the scientific community. Lee Hwa-Chung worked in China on differential geometry, and some of the work he did in the 1940s was the differential geometry over even dimensional spaces endowed with a closed 2-form. In chapter 7 Lee's work on even dimensional geometry is presented, which is the name he gave to symplectic geometry. In this work he defined an even dimensional flat space, which is the actual definition of symplectic manifolds, he found the automorphisms over symplectic manifolds, and stated the diffeomorphisms between even dimensional flat manifolds, i.e. between symplectic manifolds. Lee also used his even dimensional geometry on classical mechanics.

Although Lee endowed an even dimensional manifold with a closed 2-form before Ehresmann did, but Lee's work and Ehresmann's work were independent of each other. Ehresmann knew about Lee's first article (Lee 1943) in 1950 but it is clear that Ehresmann did not take the ideas from Lee's article, and when Ehresmann defined a symplectic manifold it was because he was interested in finding the conditions when a space and a manifold accept a complex structure.

The work of Lee was and is not well known by the mathematical community, even though he published in the American Journal of Mathematics. Not only is his work not known, Lee's life is also not known, and it was only possible to track the places where he worked through his articles he published in western journals.

The moment when symplectic geometry can be seen as a mathematical and geometrical formulation of classical mechanics is described in chapter 8. The chapter starts with short introduction to classical mechanics and Hamiltonian mechanicsan exposition. It continues with the work of Aurel Wintner in 1941 about classical mechanics and the use of symplectic matrices. Wintner noticed that the matrix of a linear canonical transformation between Hamiltonian functions belongs to the symplectic group. But it was not until 1956 in Siegel's lecture notes that the canonical transformation was explicitly identified as belonging to the symplectic group. In 1952 Georges Reeb identified the connection between the phase space and symplectic manifolds.

The definitions of Lagrangian submanifolds and symplectomorphisms will be presented in chapter 9. This development was done by Souriau at the beginning of the 1950s. In Souriau's work the Lagrangian submanifolds were called isotropic stature manifolds. The name Lagrangian submanifold was given by Vladimir Arnold in 1967. The first time when Souriau presented the Lagrangian submanifolds was in 1953 at the COLLOQUE INTERNATIONAL DE GÉOMÉTRIE DIFFÉRENTIELLE, which took place at the University of Strasbourg. Souriau's presentation had the title *Géométrie symplectique differentielle-Applications.*

In his presentation Souriau defined the symplectic vector space, developed what nowadays is known as Lagrangian submanifolds, and gave some applications to classical mechanics.

To close this work about the early history of symplectic geometry, the moment of symplectic geometry took off is introduced. It took place with the publication of Abraham and Marsden's book *Foundation of Mechanics* in 1967. In this publication Abraham and Marsden called explicitly the field symplectic geometry the study of symplectic manifolds, and they linked it almost inseparably to the study of symplectic manifolds with classical mechanics. In 1973 and 1974, two international conference about symplectic geometry took place. The first one was in Rome, where not so many mathematicians and physicists participated, and one year later near Marseille, where Souriau had a position, a bigger conference about symplectic geometry took place.

After these conferences the number of publications about symplectic geometry increased and it can be said that it became established as a field at that moment.

Chapter 1

Symplectic group

Introduction

In this chapter a brief introduction to the development of the sympletic group will be given. The study of symplectic groups goes back to the work of Souphous Lie, Felix Klein and Wilhelm Karl Killing about 1870s and 1880s. Élie Cartan called it complex groups and finally Hermann Weyl change the terminology to symplectic groups.

At first I define the symplectic group, and then there is a short passage on the historical development of it. A biography of the mathematicians who acted in this development is also given. The biography of Élie Cartan is given in chapter 3.

1.1 Symplectic group

The symplectic group is defined as the set of linear transformations of a $2n$-dimensional vector space over a field $\mathbb{F}$, which is denote by $Sp(2n, \mathbb{F})$. The symplectic group over the a field $\mathbb{F}$ consists of all matrices

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{1.1}$$

with A, B, C and D square matrices of order n, so that $M^t J M = J$ with

$$J = \begin{pmatrix} 0 & Id \\ -Id & 0 \end{pmatrix}.$$

This is true if

$$A^t C = C^t A, \quad B^t D = D^t B, \quad A^t D - C^t B = Id. \tag{1.2}$$

1.1.1 Marius Souphous Lie

Marius Souphous Lie was born in December 17th, 1842 in Norway and died in February 18, 1899. He studied science at the University of Christiana (Oslo) and ended this studies in 1965. He was thinking to become an observational astronomer and he gave lectures on astronomy targeted at a wide audience. After graduating, Lie began teaching natural science at a school and also gave private lessons in mathematics to support himself financially. It might be possible that the private instruction gave him the impulse to study the geometrical work of Poncelet and Plücker and finally to gain interest in mathematics and to develop his own theories in the area of line geometry.[1] Early in 1869 he wrote the article *"Repräsentation der Imaginären der Plangeometrie"*[2], which is concerned about the real representation of imaginary

[1] See (Fritzsche et al. 1999) and (Hawkins 2012).

[2] Representation of Imaginaries in Plane Geometry.

quantities in projective geometry being published in the *Journal für die reine und angewandte Mathematik*.[3] This publication caught the attention of some mathematicians and therefore Lie got a fellowship to study at the university of Berlin for one year (1869-1870). In Berlin he participated in the seminar of Ernst Kummer (*1810, †1893) and became a friend of Felix Klein (*1849, †1925). With Klein, who was student of Plücker, Lie discussed about line geometry and both started to apply the concept of a group.[4] Later Klein developed what is known as the *Erlange Programm*, where the concept of group applied in a geometric context, derived in the classification of different geometries via their respective transformation groups.[5]

1.1.2 Emergence of Symplectic Groups

The symplectic group was known as the *"complex"* group, since Lie described this group from the transformations that occurred within line geometry in the so-called line complexes.

Following (Hawkins 2012), line geometry uses the straight lines as the basic elements to research geometrical objects. Therefore, the study of geometry consists of studying the configuration composed by lines. This configuration of lines where called as "line complex" by Plücker.[6] A description of a line complex on a 3 dimensional complex projective spaces $\mathbb{P}^3(\mathbb{C})$ is the following: Let l be a line in $\mathbb{P}^3(\mathbb{C})$ defined by two points in homogeneous

[3]See (Hawkins 2012, p. 2).
[4]See (Fritzsche et al. 1999, p. 3).
[5]See (Rowe 1989).
[6]See (Hawkins 2012, p. 4).

coordinates (x_1, x_2, x_3, x_4) and (y_1, y_2, y_3, y_4).[7] For $i \neq j$, let

$$p_{ij} = \begin{vmatrix} x_i & x_j \\ y_i & y_j \end{vmatrix} = x_i y_j - x_j y_i. \tag{1.3}$$

So a complex line is defined by a linear relation

$$\Omega = p_{12}p_{24} + p_{13}p_{42} + p_{14}p_{23} = 0. \; ^{8} \tag{1.4}$$

The group of symplectic transformations appears through Lie's discovery of a correspondence between the geometry of line complex and the geometry of spheres in a 3 dimensional space. Lie establishes a correspondence between the group of projective transformations that leave a complex line invariant defined by (1.4) and the 10-dimensional conformal group of a 3-dimensional space.[9] So, there is a relation between a projective transformation which leaves the line complexes invariant and a contact transformation[10] which leaves the canonical symmetric bilinear form invariant of (1.3).[11] The group to which the protective transformations belongs is the protective version of the symplectic group $Sp(2, \mathbb{C})$.[12] Around 1884, Lie studied contact transformations of dimension n, as he was interested in studying the solutions for differential equations. He found that the simple groups of dimension $n(2n+1)$

[7]This homogeneous coordinates can be called as "Plücker" coordinates, see (Kosmann-Schwarzbach 2013, p. 135).

[8]See (Hawkins 2012, p. 4).

[9]See (Kosmann-Schwarzbach 2013, p.135).

[10]The term "contact transformation" was first used for transformations which maps a contact element in $\mathbb{R}^3$ in to a contact element onto another contact element. A contact element was defined as a point in $\mathbb{R}^3$ together with a plane passing through that point. The general theory of contact transformations was introduced by Lie in his studies of the reduction of Pfaffian forms." See (Aa.Vv. 1991*a*, p. 372).

[11]See (Kosmann-Schwarzbach 2013, p. 135) and (Hawkins 2012, p. 30 and pp. 96-99).

[12]See (Kosmann-Schwarzbach 2013, p. 135).

which are symplectic groups of $\mathbb{C}^{2n}$, for $n > 2$ are different from the orthogonal special group of the same dimension $SO(2n+1,\mathbb{C})$. [13] Wilhelm Karl Killing (*1847, †1923) classified the different simple groups into types. For example, type C_n for the nowadays called symplectic groups $Sp(2n,\mathbb{C})$. A_n is the type for the linear special group family $SL(n+1,\mathbb{C})$ and B_n is the type for the orthogonal special group family $S0(2n+1,\mathbb{C})$. This classification allowed Lie to realize that the contact transformations of dimension n are the groups of the C_n family. The name *"complex"* groups was given by Élie Cartan (*1869, †1951) because the symplectic groups has its origins by the study of the line complex.[14]

The symplectic group was also called as Abelian linear group. This terminology goes back to Camille Jordan, who, "introduced the analogue of the symplectic groups for the coefficient field of the integers modulo a prime p" in 1869.[15] The symplectic group did not have a name at that moment. Jordan noticed that the symplectic groups are subgroups of the group of all transformations $T \in GL(2n,\mathbb{F})$ leaving a line complex invariant (1.4).[16]

1.2 Hermann Weyl and the symplectic group

Before 1938, the symplectic group was known as complex or abelian linear group. In 1938, Hermann Weyl proposed to change the name and named it symplectic.

Hermann Weyl was born on November 9, 1885 in Elmshorn, Germany, and died on December 8, 1955 in Zürich, Switzerland. In 1904, Weyl started

[13]See (Kosmann-Schwarzbach 2013, p. 135).
[14]See (Kosmann-Schwarzbach 2013, pp. 135-136).
[15](Hawkins 2012, p. 98).
[16]See (Hawkins 2012, p. 99).

to study mathematics and physics at the University of Göttingen. At the time Weyl was studying, Felix Klein (*1849, †1925), Hermann Minkowski (*1864, †1909) and David Hilbert (*1862, †1943) were teaching at the Mathematical Institute of Göttingen. David Hilbert was Weyl's doctoral advisor. Weyl received his mathematical doctoral degree in 1908.

Three years later in 1910 Weyl became lecturer at the Mathematical Institute in Göttingen, where he taught for the next three years. In 1913, he left Göttingen for a position as a professor of mathematics at the *Eidgenössische Technische Hochschule Zürich*[17] (ETH Zürich).

When Hilbert retired in 1930, Weyl went back to Göttingen to become his successor. Weyl taught in Göttingen for the next three years until 1933 when the Nazis seized power in Germany. Weyl and his wife Helene Weyl[18], who was born into a Jewish family, decided to emigrate to the United States. They made this decision before the laws restricting professions for Jewish people were enacted and before the persecution of families with Jewish members began. Their early decision made their emigration relatively easy. Weyl accepted a position at the Institute for Advanced Study in Princeton[19] (IAS) where he stayed until his retirement in 1951 when he moved to Zürich.[20]

The different places where Hermann Weyl taught and the academical recognition he had in the international mathematical and physicists community allowed him to build a network of contacts.

During his time in Zürich, Weyl has been the colleague of Albert Einstein

[17] Swiss Federal Institute of Technology in Zürich.

[18] Helene Weyl was born in 1893 as Helene Joseph and died in 1948 in Princeton. She was a German writer and translated the philosophical work of José Ortega y Gasset into German.

[19] The Institute was founded in 1930.

[20] Helene died before Hermann Weyl's retirement and their children stayed in the United States.

(*1879, †1955) and Erwin Schrödinger (*1887, †1961), and he corresponded with Élie Cartan about infinitesimal geometry.[21] Later, during his second period in Göttingen, some mathematicians, for example Charles Ehresmann, went to study with him.[22]

Hermann Weyl was an important connection between mathematicians in the United States because he invited many mathematicians to spend research periods at the IAS from all over the world.

During the Nazi persecution against the Jewish and opponents of the regime, Hermann Weyl used and activated his personal network to help a lot of the persecuted mathematicians to emigrate to the United States.[23] Furthermore, he helped to organise funding from the Rockefeller Foundation with the help of Richard Courant.[24]

Weyl's scientific contributions are related to mathematics, physics, philosophy and the foundations of mathematics. In mathematics he contributed to different fields such as real and complex analysis, geometry and topology, number theory and mathematical physics.

Weyl's contributions to group theory are collected in his book *The classical groups: their invariants and representations* (Weyl 1939), which contains his work about representation theory of Lie groups from the 1920s, and his later lectures given in the IAS Princeton in the 1930s.[25] Striktly speaking, Weyl's contribution to symplectic geometry "was only" the denotation of it as "symplectic" because his contributions had no direct impact on the genesis

[21]See (Scholz 2011).

[22]Charles Ehresmann was in Göttingen between 1930 and 1931.

[23]See (Siegmund-Schultze 2009, Siegmund-Schultze 2012).

[24]Richard Courant was born in Germany in 1888 and died in New York in 1972. He was a Jewish mathematician who left Göttingen at the same time as Weyl. Later, he lived in New York City and from there helped others to emigrate to the United States.

[25]See (Eckes 2011).

and the development of the field of symplectic geometry.

The word "symplectic" appears for the first time in a mathematical context in Weyl's book *The classical groups*, which was published in 1939. Hermann Weyl proposed a change of name for the *complex* group to the *symplectic* group. The word *"symplectic"* is the Greek word for *"complex"* and Weyl considered that the word *"complex"* led to confusion with the complex numbers. The arguments for this change were given by Weyl in the footnote as transcribed as follows:

> The name "complex group" formerly advocated by me in allusion to line complexes, as these are defined by the vanishing of antisymmetric bilinear forms, has become more and more embarrassing through collision with the word "complex" in the connotation of complex number. I therefore propose to replace it by the corresponding Greek adjective "symplectic." Dickson calls the group the "Abelian linear group" in homage to Abel who first studied it.
>
> (Weyl 1939, p. 165)

The term "symplectic" to describe this group was accepted and used by the mathematical community. In 1941, Aurel Wintner (*1903, †1958) used in his work *The Analytical Foundation of Celestial Mechanics* on brackets the term symplectic as an alternative denotation for the complex group.[26] In 1943 Carl Ludwig Siegel (*1896, †1981) used it for the group that operates on the Siegel half space and in his article *"Symplectic Geometry"* (Siegel 1943*b*).[27] Later from 1950 to 1953, Charles Ehresmann and Libermann used the adjective symplectic for a manifold that is endowed with a skewsymmetric

[26] See chapter 8.

[27] See chapter 2.

non-degenerate 2-differential form, i.e. a symplectic form.[28] In 1953, Jean-Marie Souriau (*1922, †2012) called the vector space endowed with a 2-differential form, which is non-degenerate, a symplectic vector space.[29]

[28]The terms 2-differential form, 2-form and two form are equivalent.

[29]See chapter 9.

Chapter 2

Symplectic Geometry by Carl Ludwig Siegel

Introduction

The focus of this chapter is to describe the achievements of Carl Ludwig Siegel (*1896 ,†1981) during the middle of the 1930s and the beginning of the 1940s in generalizing the theory of automorphic functions to the case of an arbitrary number of variables, which is the generalization of hyperbolic geometry on the $\frac{1}{2}n(n+1)$ complex dimensional space, named by Siegel as "symplectic geometry".

At first, some biographical notes on Siegel will be given to provide the historical context.

In 1935 Siegel extended the theorem of Hasse-Minkowski for quadratic forms. For the extension, Siegel gave an analytically interpretation which led him to developing the theory of modular functions of degree n. With these results, Siegel achieved the generalization of hyperbolic geometry on the $\frac{1}{2}n(n+1)$ dimensional space. The generalization of hyperbolic geometry was published in 1943 in the American Journal of Mathematics under the name *"Symplectic Geometry"*.[1]

Siegel generalized the half plane to the half space of dimension $\frac{1}{2}n(n+1)$, which is a bounded simple domain. The half space is nowadays known as Siegel's half space. The group that acts biholomorphically on Siegel's half space is the symplectic group, and, therefore, the generalization of hyperbolic geometry was named *"Symplectic Geometry"* by Siegel.

In 1943 after Siegel's publication, a similar work was done by Hua Loo-Keng (*1910, †1985). Hua Loo-Keng was a Chinese mathematician who worked in the National Tsing Hua University of China in 1943, at the institute of mathematics Academia Sinica. A short biography of Hua will be given and, as well, the story of this doubling.

2.1 Carl Ludwig Siegel

Carl Ludwig Siegel was born in Berlin on December 31 in 1896. He died in Göttingen on April 4 in 1981. Siegel was a specialist in number theory. However, he also made contributions to other mathematical fields such as complex analysis and celestial mechanics.

Although he wanted to study mathematics at first, in 1915 Siegel started his studies in astronomy. This decision resulted from the fact that in 1914 the first World War began, which was a threat to Siegel, who was at that time 17 years old. He was a pacifist and against the recruitment of people for the war. Because of his rejection of the war, he decided to dissociate himself from it as much as possible, not only physically but also professionally. Therefore, he

[1] During 1941 to 1945 Hermann Weyl was one of the editors of the American Journal of Mathematics.

decided to study astronomy, which moreover complied with his mathematical interests.[2]

Thus, in 1915 Siegel started his studies in astronomy at the university of Berlin, but the lectures on astronomy were delayed by two weeks. Therefore, Siegel decided to attend the lecture on number theory given by Frobenius.[3] Frobenius' lecture had an important influence on Siegel's career. Because of this lecture, Siegel decided to study mathematics, although the risk being recruited was high, and developed his interest in number theory, for which he became an expert.

But although he changed his studies to mathematics, he never fully abandoned astronomy. During his life, Siegel published some articles about celestial mechanics and gave lectures on celestial mechanics at the universities where he was active.

In 1918 Siegel's fears to be recruited became true and he had to interrupt his studies in order to serve in the German army in Strasbourg. Neither the fact that he was studying mathematics, nor the effort of some professors who were impressed by Siegel's skills in mathematics, were able to release him from military service.[4]

In Strasbourg, Egon Schaffeld was in charge of organizing the papers of the new recruits. Before Egon Schaffeld[5] was recruited and sent to Strasbourg, he also studied mathematics. When he read Siegel's documents he was impressed by the professors' references and decided to help Siegel in

[2](Frobenius 1968, iv-vi).

[3](Frobenius 1968, iv-vi).

[4]It is not clear who wrote the letters requesting the release of Siegel from military service. It is an assertion made by Hel Braun in her memories when she wrote about Siegel and Siegel's friend Schaffeld. (Braun 1990, p.20).

[5]Hel Braun mentioned in her memories (Braun 1990, p.20) that Siegel always said that Schaffeld saved his life.

Strasbourg so Siegel could have some privileges during his military service. Siegel even asked Schaffeld if it was possible to leave his military service, but this proved impossible. For Siegel the army was terrible, and after six weeks in Strasbourg, Siegel had a psychological break down. Siegel was interned in the psychiatric hospital in Strasbourg, and his father asked his neighbor Dr. Leopold Landau for medical help. Leopold Landau treated Siegel in Strasbourg.

Leopold Landau was the father of Edmund Landau,[6] who was the professor of mathematics at the University of Göttingen and worked in the field of number theory and complex analysis. Through Leopold Landau, Siegel came into contact with Edmund Landau, who invited him to continue his studies in Göttingen.[7]

In 1919 Siegel left Berlin and enrolled at the Georg-August University of Göttingen. In Göttingen he achieved his doctoral degree in 1920 with his thesis on algebraic number theory, *"Approximation algebraischer Zahlen"*[8] under the supervision of Edmund Landau.

After his doctoral degree, Siegel continued his academic life in Göttingen by giving lectures and working as Richard Courant's[9] research assistant from 1920 to 1922.[10] During this period, Siegel started his habilitation in which

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[6]Edmund Landau was born on February 14, 1877 and died on February 19, 1938. He studied mathematics in Berlin and received his Ph.D. In 1899, he wrote his thesis on number theory under the supervision of Frobenius and Fuchs.

[7]See (Braun 1990, p. 20).

[8]"Approximation of algebraic numbers"

[9]Richard Courant was born on January 8, 1888 and died on January 27, 1972. He was a German mathematician mainly dealing with differential calculus and calculus of variations.

[10]Siegel's friendship with Courant helped him to obtain a scholarship for the Institute of Advanced Studies (IAS) in Princeton in 1935 and in 1940. See (Brühne 2003).

he used analytical methods developed by Hardy (*1877, †1947), Littelwood (*1885, †1977) and Ramanujan (*1887, †1920)[11] and developed an interest in analytical number theory and in complex analysis. Complex analysis served as a tool for solving number theoretical problems. This interest was expressed in his habilitation request on August of 1921:

> *Zuerst interessierte mich mehr die algebraische Richtung der Zahlentheorie, sowie der Gruppentheorie. Als ich dann durch eingehende Beschäftigung mit Funktionentheorie die mächtigen Hilfsmittel kennen gelernt hatte, mit denen sie insbesondere die Theorie der algebraischen Zahlkörper fördert, wandte ich mich mehr der analytischen Zahlentheorie zu.*[12]
>
> Siegel 1921 quotation by (Schneider 1983, pp.151-152)

Siegel achieved his "habilitation" at the end of 1921 with the "Habilitationschrift: *Zur additiven Theorie der Zahlkörper*".[13]

In 1922 Siegel obtained his habilitation qualification and afterwards, in the winter semester 1922, Siegel went to the Johann-Wolfgang Goethe University of Frankfurt to work as a professor as Arthur Moritz Schönflies[14] successor.[15]

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[11] (Schneider 1983, p.152).

[12] First, I was more interested in the algebraic branch of number theory, as well as on group theory. I started to work with complex analysis, and I realized the powerful tool it was for the theory of algebraic number fields, so I started to work more in analytic number theory.

[13] "Additive theory of number fields."

[14] Arthur Moritz Schönflies was a German mathematician born on April 17, 1853 and died on May 5, 1928. He made contributions to the application of group theory to crystallography, and he worked on topology as well.

[15] See (Brühne 2003).

In Frankfurt, Siegel attended the history of mathematics seminar with Ernst Hellinger (*1883, †1950), Max Dehn (*1878, †1952), Paul Epstein (*1871, †1939), and Otto Szász (*1884, †1952), with whom he built a strong friendship. Siegel recalled the period between 1922 and 1934 in Frankfurt as the most beautiful years in his life.[16]

With the seizure of power by the Nazi Party and its allies in 1933, Siegel became worried about the danger of the Nazi-regime for the peace in Europe and the danger for his Jewish colleagues and friends in the mathematical seminar. Then, in 1933 Otto Szász was expelled from Frankfurt University; in 1934, Epstein was expelled as well, and, one year later in 1935, Dehn and Hellinger were prematurely retired, all of them because of their Jewish roots.[17]

In 1935 Siegel went for a sabbatical year to the Institute of Advanced Study (IAS) in Princeton in the United States of America.[18]

It was during Siegel's Sabbatical year that he published two works about analytical number theory (Siegel 1935*b*) and (Siegel 1936) in which he presented the modular functions of degree n. The first one (Siegel 1935*b*) was finished before he went on his sabbatical year in 1935 as he sent it in November of 1934. Siegel dedicated it to Hellinger. The second one (Siegel 1936) was written during his sabbatical year.

In January 1938 Siegel went to Göttingen for a professorship with the

[16]See (Siegel 1965).

[17]See (Schwarz, Wolfgang and Wolfart, Jürgen 1988), and, for more information about the Nazi persecution of Jewish German mathematician see (Szabó 2000), (Bergmann, Epple & Ungar 2012).

[18]During this time, Weyl and Courant asked Siegel to stay but he refused and gave some excuses about not understanding the moral of US society, and that he wanted to go back because of his father and because he wanted to support his friend Hellinger. Instead, he asked for a Scholarship for Max Dehn.

aspiration for an academic exchange with Helmut Hasse (*1898, †1979), but Siegel was disappointed because Hasse's political position, and his pride of having served in the Navy during the First World War were intolerable for Siegel, as Hel Braun described.[19]

In 1940 Siegel went into exile to the United States. He stayed there from 1940 to 1951. Siegel went into exile without being persecuted like his Jewish colleagues. He did it because of his rejection of the Nazi government. He made clear his opposition in his actions and comments. For example, after Hellinger and Dehn were expelled from the *Mathematische Seminar* in Frankfurt, Siegel wanted them to attend at least the mathematical colloquium.

> *Es gehörte immer mehr Mut dazu Kontakte mit jüdischen Kollegen aufrecht zu erhalten.*[20]
>
> (Braun 1990, p. 42)

An incident occurred when Siegel went to Paris to give some lectures for a couple of weeks at the Sorbonne University in Paris in 1937. Siegel, as a German professor, was requested to visit the "Deutscher Akademischer Austauschdienst" (DAAD, German Academic Exchange Service) and the representative of the "Nationalsozialistische Deutsche Arbeiterpartei" (NSDAP, National Socialist German Workers' Party) in Paris. Siegel visited the DAAD but he did not visit the representatives of the NSDAP. He argued:

> *[...] konnte ich zu meinem größten Bedauern den Pariser Leiter der Auslandsstelle der NSDAP [...] nicht erreichen, da die Sprechstunden dieses Herrn in die Zeit meiner Vorlesungen*

[19] (Braun 1990, p. 46).

[20] It took courage to keep contact with the Jewish colleagues.

> *und wissenschaftlichen Besprechungen fielen.*[21]
>
> (Schwarz, Wolfgang and Wolfart, Jürgen 1988, p. 87)

In 1940 Siegel did not want to live in Germany under the control of the Nazis anymore, and therefore, he went into exile as his personal opposition against the Nazis regime.

With the help of Weyl and Courant, Siegel gained a scholarship for the first five years at the Institute of Advanced Study (IAS) in Princeton, New Jersey. Later, in the middle of the 1940s, Siegel obtained his U.S. nationality and a chair at the IAS.

Because of Jewish persecution in Germany, the IAS started to be an important center for the development of mathematics and physic research. A lot of Jewish mathematicians and physicists who were persecuted by the fascists in Europe went into exile to the United States, and some of them entered the IAS in Princeton. Therefore, there were some of Siegel's European colleagues at the IAS.

Siegel returned to Göttingen to give lectures as an invited professor in the winter semester of 1946/1947, and in 1951 he returned there for good until his retirement.

2.2 Symplectic geometry by Siegel

In 1943 Carl Ludwig Siegel published his article *"Symplectic Geometry"* in the American Journal of Mathematics.[22] There, he continued with the generalization of the theory of automorphic functions of degree n started by Élie

[21] [...] to my great regret, I could not meet the director of the representation of the NSDAP in Paris [...], because the office hours were at the same time as my lectures and my academic activity.

[22] See (Siegel 1943*b*).

Cartan (Cartan 1936) and the work done by himself during last half of the 1930s.[23]

The focus of this work is absolutely geometrical, as the name of the article suggests. In the article Siegel studied a bounded simple domain E on $\mathbb{C}^n$ which are homogeneous symmetric spaces of the group Ω of analytical transformations, its discret subgroups $\triangle$ of Ω and the fundamental domains, with the objective to studied the automorphic functions over $E/\triangle$.[24]

In Siegel's article (Siegel 1943*b*) the real symplectic group $Sp(2n, \mathbb{R})$ is the generalization of the special linear group $SL(2, \mathbb{R})$ and he showed that $Sp(2n, \mathbb{R})$ operated on the upper half space

$$\mathcal{H}_n = \{Z \in M_n(\mathbb{C}) | Z = Z^t, Im\, Z > 0\}. \tag{2.1}$$

This domain is nowadays known as Siegel's half space and as it can be seen, is a generalization of the Poincaré half plane to n- dimensinal case. Therefore, following Klein's classification of geometries as given in the *"Erlangen Programm"* Siegel called this geometry "symplectic geometry". Siegel generalized, as well, the unit disk. This generalization consist of all complex symmetric matrices W of $n \times n$, for which the Hermitian Matrix $Id - \overline{W}$ is positive definite, i.e.

$$\mathcal{E}_n = \{W \in M_{n\times n}(\mathbb{C}) \,|\, W^t = W,\ Id - W\overline{W} > 0\}. \tag{2.2}$$

Analog to the way is done between the Poincaré half plane $\mathcal{H}_n$ and unit disk $\mathcal{E}_1 = \{z \in \mathbb{C} |\, |z| < 1\}$ and Siegel used a Cayley transformation he maps this unit ball 2.2 onto the Siegel's half space.

[23] See (Siegel 1935*b*), (Siegel 1935*a*), (Siegel 1936), (Siegel 1937*c*), (Siegel 1937*a*), (Siegel 1937*b*), (Siegel 1939).

[24] (Kosmann-Schwarzbach 2013, p. 137)

In his article of symplectic geometry, he generalized some properties of the Poincaré model of non-euclidean geometry and find the fundamental domains of the modular group of degree n. Therefore, Siegel's symplectic geometry is a generalization of hyperbolic geometry to the complex space of dimension $\frac{1}{2}n(n+1)$.

This development was the consequence of the work done by Siegel about number theory in the 1930s, where Siegel found a generalization of the theory of modular forms, functions, automorphic functions. and he constructed the fundamental domain of the modular group of degree n. The step from a generalized hyperbolic geometry to $\frac{1}{2}n(n+1)$ was not far from the development that Siegel achieved concerning modular functions of degree n in the 1930s. Siegels works is similar to Poincarés works on Fuchsian functions in the 1880s.[25]

In the next sections I will show Siegel's work about modular forms in the 1930s, which contributed to the developed of Siegel's symplectic geometry.

2.2.1 Modular forms of degree n

The findings of Siegel on modular functions of degree n theory helps for the development of the theory of automorphic functions of several variables, because the theory of modular functions of several variables is a special case of the theory of automorphic function of several variables. One result was that Siegel was able to give the generalization of hyperbolic geometry.

In 1934 Siegel was working on analytical number theory. As a result of this research, he published the article *"Über die analytische Theorie der quadratischen Formen"*[26] in which he presented a connection between the

[25] See (Gray 2000).

[26] "About the analytic theory of the quadratic forms", See (Siegel 1935*b*).

analytical theory of quadratic forms and automorphic functions in several complex variables. The automorphic functions that Siegel discovered are the modular functions which are known as the Siegel modular functions.[27]

Siegel discovered that definite quadratic forms lead to a construction of modular forms of degree n through the generalization of the Eisenstein series, and the Theta series is a special case of the Siegel modular form.[28] This is similar to Poincarés series of works on Fuchsian functions in the 1880.[29] Siegel find as well the fundamental domain for the modular group of degree n.[30]

A modular form of weight k ($k \in \mathbb{N}$) for the modular group $PSL(2,\mathbb{Z})$ is a complex valued function f on the upper half plane which satisfies that f is a holomorphic function on the upper half plane, for any z in the upper half plane, and for any matrix in $PSL(2,\mathbb{Z})$ the equation $f(\gamma(z)) = (cz+d)^k f(z)$ holds with $\gamma \in PSL(2,\mathbb{Z})$, i.e. $\gamma(z) = \frac{az+b}{cz+d}$ with $a, b, c, d \in \mathbb{Z}$ and f is required to be holomorphic as $z \to i\infty$.

A modular form of degree n and weight k for the symplectic group $Sp(2n,\mathbb{Z})$ is a complex valued function f defined on Siegel's half space. The Siegel half space of degree n consists of all n-rowed complex symmetric matrices Z, with positive definite imaginery parts.[31] The function f satisfies that is holomophic, and the equation $f(\gamma(Z)) = (CZ+D)^k f(Z)$ holds for all γ's in $Sp(2n,\mathbb{Z})$, i.e. $\gamma(Z) = (AZ+B)(CZ+D)^{-1}$ with A, B, C, D satisfying (1.2), for any matrix Z in Siegel's half space, and f is bounded on a Siegel

[27]A modular function f of degree n is a meromorphic function on the "Siegel" half space of dimension n which is invariant under the action of the modular group $Sp(2n,\mathbb{Z})$.(Klingen 1990, p.130).

[28](Klingen 1983, p.161).

[29]See (Gray 2000).

[30]See (Siegel 1935*a*).

[31]See equation (2.13).

fundamental domain.[32] The group $Sp(2n, \mathbb{Z})/\{\pm Id\}$ is currently known as Siegel modular group. A fundamental domain of it, is called Siegel fundamental domain, is a closed subset of Siegel half space bounded by finitely many algebraic surfaces.[33]

In 1936 Siegel presented his work on modular functions at the INTERNATIONAL CONGRESS OF MATHEMATICS in Oslo.[34] Oslo was one of the few congresses where Siegel participated and gave a talk.[35]

In 1939 Siegel published *"Einführung in die Theorie der Modulfunktionen n-ten Grades"*[36] which is a special case of the study on automorphic functions in a domain with a discontinuous group.

The developed on the theory of the modular functions of degree n is a result of Siegel's work about the analytic theory of quadratic forms, where he find a connection to the modular functions. This work was published in three parts over three years.[37]

Siegel's main result on the analytic theory of quadratic forms can be interpreted in analytically through the Eisenstein series and he find out that the modular forms of degree n can be constructed through the Eisenstein series.[38] This turned Siegel's attention to complex analysis in several variables.

In the next section I will give an overview about Siegel's work on the analytic theory of quadratic forms.

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[32] (Klingen 1990, p.43).
[33] (Klingen 1990, p. 31).
[34] See (Siegel 1937*a*).
[35] See (Klingen, Rüssmann & Schneider 1983).
[36] Introduction to the modular functions of degree n
[37] See (Siegel 1935*b*, Siegel 1936, Siegel 1937*c*).
[38] See (Siegel 1935*b*)

2.2.2 Analytic theory of quadratic forms

Siegel's research on automorphic functions of several complex variables started when he extended the Hasse theorem, which he called Legendre-Hasse theorem.[39]

Helmut Hasse[40] gave the generalization of the Legendre theorem (Hasse 1923), and the Legendre theorem states that a quadratic equation

$$ax^2 + bxy + cy^2 = d \quad a, b, c, d \in \mathbb{Z} \tag{2.3}$$

can be solved with rational solutions if and only if the congruence

$$ax^2 + bxy + cy^2 \equiv d \mod q \tag{2.4}$$

has a rational solution for every module q, $q \in \mathbb{N}$.[41]

The Hasse theorem solves the problem how to represent a quadratic form R with n variable by a quadratic form Q of m variables. So, if S is the symmetric matrix that represents the quadratic form Q with m variables, T the symmetric matrix for the quadratic form R with n variables and X the matrix of the linear transformation, the equation

$$X^t S X = T. \tag{2.5}$$

[39]It was not possible to find an actual mathematical publication where the name Legendre-Hasse theorem is used. Instead for the same theorem the name Hasse-Minkowski is used. It seems that only Siegel's students called this theorem as Siegel named it. (See (Klingen et al. 1983)). Here the name Hasse Theorem is going to be used for the theorem.

[40]Helmut Hasse was born on August 25, 1898 and died on December 26, 1979. He worked on and developed the field number theory. Hasse and Siegel were together, as students, form 1919 to 1920. After 1920, Hasse went to Magdeburg to finish his doctoral degree. Hasse went back to Göttingen when he replaced Weyl in 1934, after Weyl's emigration to the USA (Princeton).

[41]Two rational numbers are congruent module q to each other if the difference $a - b$ is an integer number divisible by q. For example: $\frac{16}{5} \equiv \frac{1}{5} \bmod 3$.

is the generalization of (2.3). Hasse compared the last equation with the rational solvability of the congruence

$$X^t S X \equiv T \mod q \tag{2.6}$$

for each module q and proved that from the rational solvability of the congruence (2.6) for each module q follows the rational solvability of the equation (2.5).[42]

The Legendre theorem fulfills the case if the quadratic form Q has $m = 2$ variables and the quadratic form R has $n = 1$ variables. Then Q has the form

$$au_1^2 + 2bu_1u_2 + cu_2^2,$$

and the quadratic form is represented by the symmetric matrix S, which has the form

$$\begin{pmatrix} a & b \\ b & c \end{pmatrix}.$$

The quadratic from R has the form dv_1^2 and is represented by the matrix T of the form (d). The homogeneous linear substitution is $u_1 = xv_1$ and $u_2 = yv_1$ and the substitution matrix is $X = \begin{pmatrix} x \\ y \end{pmatrix}$.

The equation (2.5) is

$$(x, y)\begin{pmatrix} a & b \\ b & c \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = (d). \tag{2.7}$$

By multiplying the left side of equation (2.7), equation (2.3) is obtained, which is the case of the Legendre theorem. The case $m = n$ was considered by Minkowski.[43]

[42] Two matrices A and B are congruent module q, i.e. $A \equiv B \bmod q$, if all elements of the matrix $A - B$ are divisible by q.

[43] (Siegel 1937*b*, p.334).

The Hasse theorem gives a qualitative result, i.e. existence of the rational solutions of the equation. Siegel generalized it to be a quantitative result, finding the numbers of solutions for (2.5). He mentioned that if the equation (2.5) has rational solutions, there is an infinite number of solutions. To obtain a finite number of solutions, Siegel considered only the cases in which the solutions are integers, i.e. that all the elements x_{ij} of the matrix X are integers and so are the elements of the matrices S and T.[44] The number of integer solutions of the equation (2.5) is denoted by $A(S,T)$. The number of integer solutions of the congruence (2.6) is denoted by $A_q(S,T)$. This means that the number of solutions X for the congruence (2.6) are integer and not congruent with respect to module q. For the quadratic transformation itself, the number of solutions is denoted by $A(S,S) = E(S)$.

Siegel showed the relationship between the number of solutions of the equation (2.5) and the number of solutions of the congruence (2.6). To show this relationship, Siegel defined the notion of class and genus.[45]

Two quadratic forms Q and Q_1 belong to the same class, i.e. are equivalent if Q_1 can be represented by Q and vice versa. For example, if S is a representation matrix of the quadratic form Q and S_1 is a representation matrix of Q_1, then the two equations

$$X^t S X = S_1$$

and

$$X_1^t S_1 X_1 = S$$

can be solved for integer matrices X and X_1. $A(S,T)$ is invariant if one substitutes the quadratic form Q with matrix S by the quadratic form Q_1 with matrix S_1. It is said the quadratic forms are equivalent.

[44] (Siegel 1937*b*, p. 334).

[45] (Klingen et al. 1983, p.159).

Two quadratic forms belong to the same genus if the congruence relations

$$X^t S X \equiv S_1 \bmod q$$

and

$$X_1^t S_1 X_1 \equiv S \bmod q$$

can be solved for integer matrices X and X_1 for each q, and the numbers $A_q(S,T)$ and $A_q(S_1,T)$ are identical.

Siegel gave a simple example of two quadratic forms

$$Q = x^2 + 55y^2$$

and

$$Q_1 = 5x^2 + 11y^2.$$

belonging to the same genus, but not to the same class. Both have the same number of solutions for the congruence

$$x^2 + 55y^2 \equiv 1 \text{ mod } q$$

and

$$5x^2 + 11y^2 \equiv 1 \text{ mod } q,$$

but $x^2 + 55y^2 = 1$ has integer solutions and $5x^2 + 11y^2 = 1$ does not, i.e. Q and Q_1 are not equivalent.[46]

Siegel used a theorem stated by Charles Hermite (*1822, †1901), which says that to each genus there only exist a finite number of classes. [47]

For the relationship between $A_q(S,T)$ and $A(S,T)$, Siegel defined, what he called, the average value of $A_q(S,T)$ and $A(S,T)$. For the average value of $A_q(S,T)$, Siegel took the number of integer solutions X of the congruence

$$X^t S X \equiv T \bmod q$$

[46]See (Siegel 1935*b*, p.528).
[47](Siegel 1935*b*, p.554).

that are not congruent module q. Then, for each element x_{kl} of X with value $1, ..., q$, all matrices X can be obtained that are incongruent with module q. Because X is an integer matrix with m rows and n columns, the matrix has mn elements and there are q possibilities for each element of X. Then, X runs through all the module q incongruent matrices and not only through the solutions of the congruency. Therefore, there exist q^{mn} matrices X.

For all these matrices X, the matrix $X^tSX = Y$ will be an integer symmetric matrix of $n \times n$. Y can only have $\frac{1}{2}n(n+1)$ independent elements. This means that there can only be $q^{\frac{1}{2}n(n+1)}$ possible integer matrices for Y, which are incongruent module q; therefore,

$$\sum_{Y \bmod q} A_q(S, Y) = q^{mn}$$

and

$$\sum_{Y \bmod q} 1 = q^{\frac{1}{2}n(n+1)}$$

while Y runs through all the incongruent matrices with module q. The number $q^{mn-\frac{1}{2}n(n+1)}$ can be named as the average value of the number $A_q(S, T)$.

For the average value of $A(S, T)$, Siegel considered the independent elements of the symmetric matrices T, which can be interpreted as Cartesian coordinates of a point in the $\frac{1}{2}n(n+1)$ dimensional space. Then, he lets y be a domain in the space, which includes the point T and which has a volume. He looked for all real matrices X which satisfy the equation $X^tSX = Y$, where Y is any point of the domain y. If Y runs through all the points of y, the real elements of the matrix X run through the mn Cartesian coordinates of points of the domain x, which is a mn-dimensional space.

Siegel calculated the volume for the domains y and x and built the quotient of these volumes. The average value of $A(S, T)$ is the limit of the quotient of the volumes when the domain y converges to Y, where Y is any point of the domain y, i.e. the limit of the volume quotient

$$\lim_{y\to Y}\frac{\int_x dX}{\int_y dY} = A_\infty(S,Y)$$

exists if the domain y converges to any point Y of y. Then, he concluded the relationship

$$\int_y A_\infty(S,Y)dY = \int_x dX. \tag{2.8}$$

Moreover, if the matrices X run through the integer matrices located in the domain x and Y are the integer matrices located in the domain y, the relation

$$\sum_{Y\in y} A(S,Y) = \sum_{X\in x} 1, \tag{2.9}$$

is true because the number $A(S,Y)$ of integer matrices X corresponds to the same matrices $Y = X^tSX$.

Therefore, the average value of $A(S,T)$ is the volume quotient

$$A_\infty(S,T) = \lim_{y\to T}\frac{\int_x dX}{\int_y dY}.\text{[48]} \tag{2.10}$$

After he defined the classes, the genus, and average values, Siegel was able to give the fundamental theorem of his work which consists of the formula

$$\frac{\frac{A(S_1,T)}{E(S_1)} + ... + \frac{A(S_h,T)}{E(S_h)}}{\frac{A_\infty(S_1,T)}{E(S_1)} + ... + \frac{A_\infty(S_h,T)}{E(S_h)}} = \lim_{q\to\infty}\frac{A_q(S,T)}{q^{mn-\frac{1}{2}n(n+1)}} \quad (m > n+1) \tag{2.11}$$

where q tends suitably to infinity, for example, q can be the sequence of the factorial numbers, as Siegel mentioned.[49]

For the case $m \leq n+1$, Siegel added to the right side of the equation the factor 2^{-1}. If $m = n$, he added the factor $2^{\omega(q)}$ to the denominator of the right side, where $\omega(q)$ are the numbers of the prime divisors of q. Both sides

[48] (Siegel 1937*b*, p.338).

[49] (Siegel 1937*b*, p.338).

of the equation (2.11) are divided by their average values.[50] The matrices $S_1, S_2, ..., S_h$ are the representatives of the different classes of the genus of S.[51]

Siegel rewrote the right side of the equation (2.11) as an infinite product and through this change he could give an analytical interpretation of the main theorem. For this Siegel found out that if q and r are relative primes, then

$$A_{qr}(S,T) = A_q(S,T)\,A_r(S,T);$$

moreover, for a power $q = p^a$ with p being prime and for a being sufficiently large, the quotient

$$\frac{A_q(S,T)}{q^{mn-\frac{1}{2}n(n+1)}} = \alpha_p(S,T)$$

is constant with S and T fixed. Siegel identified $\alpha_p(S,T)$ as the density of the rational solution of $X^tSX = T$ in the p-adic field numbers.[52] The expression can be written as an infinite product of the densities of the rational solutions over all prime numbers, i.e.

$$\prod_p \alpha_p(S,T) = \lim_{q\to\infty} \frac{A_q(S,T)}{q^{mn-\frac{n(n+1)}{2}}}.^{53}$$

Siegel rewrote the equation (2.11) as follows

$$\frac{\frac{A(S_1,T)}{E(S_1)} + ... + \frac{A(S_h,T)}{E(S_h)}}{\frac{1}{E(S_1)} + ... + \frac{1}{E(S_h)}} = A_\infty(S,T)\prod_p \alpha_p(S,T), \tag{2.12}$$

using the fact that the genus is invariant, i.e.

$$A_\infty(S_1,T) = ... = A_\infty(S_h,T) = A_\infty(S,T).$$

[50] (Siegel 1935*b*, p.529).
[51] (Siegel 1937*a*, p.107).
[52] (Siegel 1935*b*, p.552).
[53] (Siegel 1935*b*, p.555).

2.2.3 Analytical interpretation

For the analytical interpretation of the main theorem (2.12), Siegel used the relation that exists between some analytic functions of $\frac{n}{2}(n+1)$ variables and the $2n$ periodic meromorphic functions, which is the same relation as the one between the modular function and the elliptic functions.[54] An elliptic function is a meromorphic function that is periodic in two directions and every modular function can be expressed as a rational function of an elliptic function.[55]

Siegel noticed that there is a relationship between the Riemann Theta function[56] and the generalization of the Eisenstein series.[57]

First, Siegel established a relationship with the number of solutions $A(S, T)$ and gave an interpretation for the left side of equation (2.12). For this, Siegel constructed the Theta series for a real positive definite square matrix S of order m and a symmetric matrix Z of order n with complex elements and the imaginary part of Z being positive definite, i.e. $\mathrm{Im}Z > 1$. Siegel let all integer matrices G with m rows and n columns run through to get an infinite

[54](Siegel 1937*b*, p.109).

[55](Apostol 2012, p.40).

[56]A Theta series is a series of functions used in the representation of automophic forms and functions (E.D. Solomentsev 2017). The Theta series of a lattice is the generating function for the number of vectors with norm n in the lattice (Weisstein 2017). The Riemann Theta function is defined in the Siegel half space (see 2.13) for Z and τ in the complex n dimensional space and the function is as the following:

$$F(\tau, Z) = \sum_{m \in \mathbb{Z}^n} \exp\left(2\pi i(\frac{1}{2}m^t Z m + m^t \tau)\right)$$

[57](Siegel 1935*b*, p.601).

series

$$f(S,Z) = \sum_{G \in M_{m\times n}(\mathbb{Z})} e^{\pi i \operatorname{tr}(G^t SGZ)}.$$

This series is the Theta series. All the matrices $M = G^t SGZ$ are located in a domain

$$\mathcal{H}_n = \{Z \in M_n(\mathbb{C}) | Z = Z^t, Im\, Z > 0\}. \tag{2.13}$$

of the space of $\frac{1}{2}n(n+1)$ complex variables z_{kl}. This domain is the half space, which was defined by Siegel in (Siegel 1935*b*, p.572), and is nowadays known as Siegel's half space.

Siegel proved that the series $f(S,Z)$ is uniformly convergent in all finite parts of the domain $\mathcal{H}_n$.[58] Therefore, it is holomorphic in the domain $\mathcal{H}_n$.[59] The number of solutions of $G^t SG = T$, where T is a symmetric matrix, is then $A(S,T)$. Siegel obtained the relation

$$f(S,Z) = \sum_T A(S,T) e^{\pi i \operatorname{tr}(TZ)} \tag{2.14}$$

through the Fourier development.

The left side of the equation (2.12) can be written for all T by the genus invariant as

$$F(S,Z) = \frac{\frac{f(S_1,T)}{E(S_1)} + ... + \frac{f(S_h,T)}{E(S_h)}}{\frac{1}{E(S_1)} + ... + \frac{1}{E(S_h)}}$$

where $S_1, ..., S_2$ represent all the classes of the genus of S and the analytical function $F(S,Z)$ is determined by the genus of S, which Siegel called the analytical invariant of genus and with (2.14), $F(S,Z)$ can be rewritten as

$$F(S,Z) = \sum_T \frac{\frac{A(S_1,T)}{E(S_1)} + ... + \frac{A(S_h,T)}{E(S_h)}}{\frac{1}{E(S_1)} + ... + \frac{1}{E(S_h)}}. \tag{2.15}$$ [60]

[58] Hel Braun exposed this in details in her PhD thesis (Braun 1938).

[59] See (Siegel 1937*b*).

[60] (Siegel 1937*c*, p. 345).

Through some identities of the Fourier series and through the analytical invariant genus, Siegel transformed the right side of (2.15) into a series of simple fractions, i.e.

$$F(S,Z) = \sum_{C,D} \gamma(S,C,D)\det(CZ+D)^{-\frac{m}{2}}, \tag{2.16}$$

$m > 2(n+1)$, while C and D run through a system of $n \times n$ integer matrices as follows: C and D are symmetric to each other. They are primitive, i.e. if for any matrix M the matrices MC and MD are integer then M is itself integer. C and D are not associated, i.e. for any two pairs C_1, D_1 and C_2, D_2 of the system C, D, $C_1 D_2^t \neq D_1 C_2^t$.

The coefficient $\gamma(S,C,D)$ does not depend on the variable Z, it only depends on C and D. If the determinant of S is equal 1, has a even number of elements in its diagonal and the elements of S are integers numbers, the coefficients $\gamma(S,C,D) = 1$. and so

$$F(S,Z) = \sum_{C,D} \det(CZ+D)^{-\frac{m}{2}}.$$

Siegel did not mention, in 1937, when the simplification (2.16) occurs, but he gave two examples for it that may be an indication that he knew that the simplification of equation only occurs if the grade m if the quadratic form holds $m \equiv 0 \mod 8$. Nowadays this is know.[61] Through this generalization and simplification, Siegel developed the modular forms of degree n.

The classical Eisenstein series are defined by

$$G_{2k}(z) = \sum_{m,n \in \mathbb{Z}-\{0\}} \frac{1}{(m+nz)^{2k}} \tag{2.17}$$

where $z \in \mathcal{H}_1$ and $2k$ is the weight, $k \geq 2$. The Eisenstein series are absolutely convergent to a holomorphic function of z in the upper half plane,

[61] See (Klingen 1983).

and are holomorphic as $z \to i\infty$. For any modular matrix $M \in SL(2, \mathbb{Z})$ the equation

$$G_{2k}\left(\frac{az+b}{cz+d}\right) = (cz+d)^{2k} G_{2k}(z)$$

is satisfy. [62]

For the case $n = 1$, the modular form of weight k, with $k \in \mathbb{N}_0$ and k being a pair, is a complex value function f on the upper half plane

$$\mathcal{H}_1 = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}. \tag{2.18}$$

The function f is an holomorphic function on $\mathcal{H}_1$. It satisfies for any $z \in \mathcal{H}_1$ and any matrix in the modular group, i.e. $A \in PSL(2, \mathbb{Z})$ the equation

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z), \tag{2.19}$$

and f is holomorphic as $z \to i\infty$. So, an example for $n = 1$ of modular forms are the Eisenstein series.

The generalization of the Eisenstein series is

$$F(S, Z) = \sum_{C,D} \det(CZ + D)^{-\frac{m}{2}}.$$

Siegel's two examples of the generalization of the Eisenstein series are. for $m = 8$ and the symmetric matrix that represents the quadratic form is the identity matrix $S = E_8$.[63]

The first example that Siegel gave of the generalization of the Eisenstein series is for $m = 8$ and $n = 1$. The resulting matrix $S = E_8$ is symmetric, and, therefore, the classes of the genus equal 1. Z is a complex number z. The Theta series is

$$F(E_8, z) = f(E_8, z) = \sum_{\kappa_1,\ldots,\kappa_2=+\infty}^{-\infty} e^{\pi i z(\kappa_1^2+\ldots+\kappa_8^2)} = \left(\sum_{\kappa=-\infty}^{+\infty} e^{\pi i z \kappa^2 z}\right)^8.$$

[62](Apostol 2012, p.113).

[63]See (Siegel 1937*b*).

Siegel calculated $\gamma(S, C, D)$, and ascertained that it is equal 1 or equal 0.

Therefore,

$$\left(\sum_{\kappa=-\infty}^{+\infty} e^{\pi i z \kappa^2 z}\right)^8 = \sum_{c,d} (cz+d)^{-4} \tag{2.20}$$

and c, d runs through all integer numbers with the properties that they are relative primes, cd being a pair number; $c > 0$ or $c = 0$, $d > 0$. So, in the left side of the equation (2.20), a Jacobi theta function to power eight is equal to an Eisenstein series. Therefore, equation (2.20) shows the relation between the Theta functions and the Eisenstein series.[64]

The second example is for $m = 8$, $n = 2$ and

$$Z = \begin{pmatrix} x & y \\ y & x \end{pmatrix}.$$

The second example is analogue to the first one.

$$F(E_8, Z) = \left(\sum_{a,b=-\infty}^{+\infty} e^{\pi i (xa^2 + 2yab + zb^2)}\right)^8$$

and

$$\left(\sum_{a,b=-\infty}^{+\infty} e^{\pi i (xa^2 + 2yab + zb^2)}\right)^8 = \sum_{C,D} \det(CZ+D)^{-4}, \tag{2.21}$$

where C, D have the properties described above.

The left side of the equation (2.21) is a Riemann Theta function of power eight. Siegel noticed that the Riemann Theta function corresponds to an algebraic curve of genus two.[65] The second member of equation (2.21) is a generalization of the Eisenstein series, which is related to the theory of automorphic functions of several variables.

With the last example, Siegel linked the result of his fundamental theorem on quadratic forms to the theory of algebraic curves.[66]

[64](Siegel 1937*b*, p.346).
[65]See (Siegel 1937*b*).
[66]See (Siegel 1937*b*).

Siegel used the fact that the Riemann Theta function corresponds to an algebraic curve and an algebraic curve is a compact Riemann surface to obtain the modular functions of degree n.

2.2.4 Siegel modular functions of degree n

For the construction of the modular functions of degree n, Siegel searched first for the modular group. After finding the modular group which left invariant the Siegel upper half space $\mathcal{H}_n$. He find the fundamental domain of the modular group on $\mathcal{H}_n$ and so he was able to constructed the modular functions of degree n as the quotient of modular forms, with the modular forms being holomorphic in $\mathcal{H}_n$.

Modular group

To gain the modular functions of degree n, Siegel started with a compact Riemann surface of genus n. On the Riemann surface, Siegel fixed a set of $2n$ simple closed curves inducing a canonical homology base. A canonical homology base is a cut-class system $\{\alpha_i, \beta_i\}_{i=1}^{n}$.[67] Siegel called this *"ein canonisches Schnittsystem von* $2n$ *Schnitten"*.[68] The simple closed curves are cuts on the Riemann surface, and a $2n$ canonical homology base is a cut-class system. In the cut-class system $\{\alpha_i, \beta_i\}_{i=1}^{n}$, the intersections are between the path α_i and β_i, so that

$$\alpha_i \cdot \alpha_j = 0; \quad \beta_i \cdot \beta_j = 0; \quad \alpha_i \cdot \beta_j = \delta_{i,j} = -\beta_i \cdot \alpha_j \tag{2.22}$$

$i = 1, ..., n$; $j = 1, ..., n$.

In the case that α_i cuts in the same direction as β_i, the intersection number is 1, otherwise it would be -1. These relations produce the antisymmetric

[67]See (Siegel 1935*b*, p. 595).

[68]A canonical cut-class system of $2n$ cuts. See (Siegel 1935*b*).

matrix J,

$$\begin{pmatrix} \alpha_1 \\ . \\ . \\ \alpha_n \\ \beta_1 \\ . \\ . \\ \beta_n \end{pmatrix} (\alpha_1, ..., \alpha_n, \beta_1, ..., \beta_n) = \begin{pmatrix} 0 & Id \\ -Id & 0 \end{pmatrix} = J.$$

Siegel did not mention that the compact Riemann surface most be oriented, but is needed to be oriented so it is possible to give the cuts a directions.

On a Riemann surface of genus n, Siegel defined a system of independent abelian differentials of the first kind. An abelian differential is a meromorphic 1-form on an open subset of a Riemann surface. If the abelian differential is holomorphic everywhere on the Riemann surface, it is of the first kind.[69]

A period of an integral of an abelian differential of the first kind belongs to a simple closed curve on a Riemann surface, i.e.

$$\int_{\alpha_i} dw_j = p_{ij} \quad \int_{\beta_i} dw_j = q_{ij},$$

where dw_j is an independent system of differentials of first kind and $(p_{ij}) = P$, $(q_{ij}) = Q$ are matrices. If $Q \neq 0$ then $PQ^{-1} = Z = X + iY$ is a symmetric matrix with a positive definite imaginary part, i.e. $Y > 0$.[70]

Siegel showed how to change the canonical homology base $\{\alpha_i, \beta_i\}_{i=1}^n$ to a different canonical homology base using a linear matrix $M = GL(2n, \mathbb{Z})$

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

[69] See (Forster 1999).

[70] (Siegel 1935*b*, p.602).

where A, B, C, D are symmetric matrices of $n \times n$ each.

The change of the canonical homology base is then

$$\begin{pmatrix} \alpha'_1 \\ . \\ . \\ \alpha'_n \\ \beta'_1 \\ . \\ . \\ \beta'_n \end{pmatrix} = M \begin{pmatrix} \alpha_1 \\ . \\ . \\ \alpha_n \\ \beta_1 \\ . \\ . \\ \beta_n \end{pmatrix}.$$

The basis is exactly canonical if, and only if,

$$\begin{pmatrix} \alpha'_1 \\ . \\ . \\ \alpha'_n \\ \beta'_1 \\ . \\ . \\ \beta'_n \end{pmatrix} (\alpha'_1, ..., \alpha'_n, \beta'_1, ..., \beta'_n) = J,$$

i.e. $M^t J M = J$. Siegel called the matrix M the canonical matrix because $M^t J M = J$, which it means that M is a symplectic matrix with integer entries, i.e. $M \in Sp(2n, \mathbb{Z})$. So, the matrix J is defined using the homology basis $\{\alpha'_1, ..., \alpha'_n, \beta'_1, ..., \beta n'\}$ and in this way the compact Riemann surface get the structure of a symplectic module, then Siegel is working over $\mathbb{Z}$. Siegel introduced the terminology *canonical* group then in 1935 the symplectic group did not have that name at that time.[71]

[71]See chapter 1.

With this linear transformation, Siegel was able to map the Z matrix onto another matrix Z' with positive a definite imaginary part. He defined the Siegel half space $\mathcal{H}_n$,

> *Bedeutet H das Gebiet aller symmetrischen [Matrizen] $\mathfrak{X}$ mit positivem Imaginärteil.*[72]
>
> (Siegel 1935*b*, p. 597)

Currenly is describe as:

$$\mathcal{H}_n = \{Z \in M_n(\mathbb{C}) | Z = Z^t, Im\, Z > 0\}. \tag{2.23}$$

So, he showed that the symplectic group operates on the half space $\mathcal{H}_n$.[73] This means that the symplectic matrix M operates on the Siegel half space, i.e. a symmetric matrix $Z = (z_{kl})$, where $Z = X + iY$ with $Y > 0$ is mapped into another symmetric matrix $Z' = X' + iY'$ that is as well positive definite. In a current terminology it means that the transformation maps the half space onto itself, and the automorphic group of $\mathcal{H}_n$ is described by the operation:

$$Sp(2n, \mathbb{Z}) \times \mathcal{H}_n \longrightarrow \mathcal{H}_n, \quad (\mu, Z) \mapsto Z' = \mu\langle Z\rangle := (AZ + B)(CZ + D)^{-1} \tag{2.24}$$

In the case that the matrix M is symplectic, or in Siegel's notation, the matrix is canonical, the matrices M and $-M$ provide the same substitution and the Modular group of degree n arises as a factor group of the group of all canonical matrices, i.e. $Sp(2n, \mathbb{Z})/\{\pm Id\}$. We used the current notation for the modular group of degree n, i.e.

$$\Gamma_n = Sp(2n, \mathbb{Z})/\{\pm Id\}. \tag{2.25}$$

---

[72] Means H the area of all symmetric [matrices] $\mathfrak{X}$ with positive imaginary part.

[73] This half space, as is mention in sec 2.2, is currently known as the Siegel half space.

If $n = 1$, the symplectic group is the modular group, i.e.

$$Sp(2, \mathbb{Z})/\{\pm Id\} \cong PSL(2, \mathbb{Z}).$$

As mentioned before, the modular group of degree $n > 1$ is today known as the Siegel modular group.[74]

Fundamental domain of the Modular Group of degree n

Siegel constructed the fundamental domain in $\mathcal{H}_n$ for the modular group of degree n operating on it.[75]

A fundamental domain for a discontinuous group is a domain $\mathcal{F}_n$ in $\mathcal{H}_n$, if the images of $\mathcal{F}_n$ under the discontinuous group cover $\mathcal{H}_n$ without gaps and overlaps. A group is discontinuous on $\mathcal{H}_n$ if no set of equivalent points has a limit point in $\mathcal{H}_n$. The discontinuous group is the modular group of degree n.

Siegel proved that a subset $\mathcal{F}_n$ of $\mathcal{H}_n$ which fulfills the conditions

1) $\det(CZ + D) \geq 1$ for all $M \in \Gamma_n$.

2) Y fulfill the Minkowski reduction.

3) $|x_{ij}| \leq \frac{1}{2}$ for $i \leq j$ and $i, j \in \{1, ..., n\}$.

is a fundamental domain of the modular group Γ_n.

Modular Functions of degree n

After giving the fundamental regions, Siegel had the tools to give the modular functions of degree n.[76] The modular functions of degree n are meromorphic

[74] Currently some mathematicians called the modular group of degree n as $\Gamma_n = Sp(2n, \mathbb{Z})$. See (Klingen 1990).

[75] (Siegel 1935*b*, p.598 ff) and (Siegel 1939, p.625 ff).

[76] See (Siegel 1935*b*, Siegel 1939).

functions on the half space $\mathcal{H}_n$ and are invariant under the action of the modular group of degree n, i.e.

$$f(M(Z)) = f(Z) \text{ for all } Z \in \mathcal{H}_n \text{ and for all } M \in Sp(2n, \mathbb{Z}).^{77} \qquad (2.26)$$

Siegel constructed the modular function as the quotient of modular forms, with the modular forms being holomorphic in $\mathcal{H}_n$. The modular function is

$$f_{rs}(Z) = \psi_r^s(Z)\psi_s^{-r}(Z)$$

where the modular forms are ψ_r^s and ψ_s^{-r}. A modular form of degree n and weight k is a complex valued function ψ_r defined on the half space and satisfies

- ψ_r is holomorphic,
- ψ_r is bounded on the fundamental domain of the modular group,
- $\psi_r(M\langle Z\rangle) = \det(CZ + D)^r \psi_r$ for all $M \in Sp(2n, \mathbb{Z})$, $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ satisfying $A^tC = C^tA, \;\; B^tD = D^tB, \;\; A^tD - C^tB = Id.$

Siegel used the generalization of the Eisenstein series (2.16) for the construction of the modular form:

$$\psi_r(Z) = \sum_{C,D} \det(CZ + D)^{-r}$$

where $r = \frac{m}{2}$ $(r = 2, 4, 6, ...)$ and $m > 2(n + 1)$. The quotient function

$$f_{rs}(Z) = \psi_r^s(Z)\psi_s^{-r}(Z)$$

is a modular function of degree n, and all the modular functions are expressed as a ratio of a generalization of Eisenstein series.[78]

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[77] See (2.24).

[78] See (Siegel 1935*b*).

2.2.5 Symplectic geometry 1943

As mentioned above, Siegel discovers the modular forms of degree n and thus obtains the objects, such as the modular group and its fundamental regions and the Siegel half space, to carry out a generalization of the hyperbolic geometry in $\frac{n}{2}(n+1)$-dimensions which he had done in 1943.

Siegel exposed this generalization, as is said at the beginning of this chapter the article "Symplectic Geometry" , which is linked to the generalization of the theory of automorphic functions of degree n. As the title of the article already suggests, the focus of it is absolutely geometrical.

For the generalization of the automorphic functions of degree n, Siegel mentions the steps that are needed in his article "Symplectic Geometry" of 1943.:

> A generalization of the theory of automorphic functions to the case of an arbitrary number of variables requires the following three steps:
>
> 1. To determine all bounded simple domains E in the space of m complex variables, which are symmetric spaces with respect to a group Ω of analytic mappings.
> 2. To investigate the invariant geometric properties of E, to find the discontinuous subgroups Δ of Ω and construct their fundamental domains.
> 3. To study the field of automophic functions in E with the group Δ.
>
> (Siegel 1943*b*, p. 1)

In the quotation, Siegel used the name *symmetric space* which nowadays would be called a bounded homogeneous symmetric domain. A bounded domain in the complex space can be Siegel's half space (2.13). A bounded domain is called symmetric if to each point there exists an involution in the group of biholomorphic automorphisms with the given point as a single fixed point. A bounded domain is called homogeneous if the group of biholomorphic automorphisms acts transitively.[79] A group action $G \times X \to X$ is transitive if for every pair of elements x and y, there is $g \in G$ such that $gx = y$. The space X, with a transitive group operation, is called a homogeneous space if the group is a Lie group.[80]

Siegel considered that the first step was already done by Élie Cartan.[81] In 1936 Cartan obtained six irreducible bounded symmetric domains and noticed that all other bounded symmetric domains can be derived by biholomorphic transformations.

If a domain is bounded and symmetric, the existence of other biholomorphical automorphisms that operate on the domain besides the identity is garantueed.

That the domain is irreducible means that the domain cannot be decomposed into the products of two domains of the same kind.[82]

As Siegel said, and as it is shown in the last sections, the third step for the generalization of the theory of automophic functions had been done by Siegel in 1939 for the special case of the modular group of degree n.[83]

In 1943 Siegel worked out the second step, he found the invariant properties of a bounded symmetric domain, as the metric, the volume. Siegel found the discontinuous groups and constructs their fundamental domains.[84] Siegel

[79](Klingen 1990, p.4).
[80]See (Rowland n.d.).
[81]See (Cartan 1936).
[82](Klingen 1990, p.4).
[83]See (Siegel 1939).
[84]See (Siegel 1943*b*).

reorganized the results he gave in the articles about the modular group and showed that Siegel's half space is one of the six Cartan bounded symmetric domains, because through a Calyey transformation, which Siegel had done, the Siegel half space can be map to the generalization of the unit disk, a unit ball.[85] So, Siegel restricted his work to one bounded symmetric domain, the unit ball of degree $n > 1$, which is a generalization of the unit disk.

The generalization of the unit disk for $n > 1$ consists of all complex symmetric matrices W of $n \times n$, for which the Hermitian matrix $Id - W\overline{W}$ is positive definite, i.e.

$$\mathcal{E}_n = \{W \in M_{n\times n}(\mathbb{C}) \,|\, W^t = W,\ Id - W\bar{W} > 0\}. \tag{2.27}$$

The unit ball can be mapped onto Siegel's half space (2.13) through a Cayley transformation $K : \mathcal{E}_n \longrightarrow \mathcal{H}_n$,

$$W \mapsto Z := K\langle W\rangle = i(Id + W)(Id - W)^{-1}. \tag{2.28}$$

Siegel did this and gave the inverse transformation

$$Z \mapsto W := L\langle Z\rangle = (Z - iId)(Z + iId)^{-1}. \tag{2.29}$$

The Cayley transformation and the inverse transformation are analogue to the case $n = 1$, i.e. the Cayley transformation that maps the unit disk on the upper half plane, $K : \mathcal{E}_1 \longrightarrow \mathcal{H}_1$. On the unit ball and Siegel's half space, Siegel found the metric, the volume, the discontinuous subgroups that are operate on them, and their fundamental domains. It is to be remembered that $Z \in \mathcal{H}_n$ is a Hermitian matrix, i.e. $Z = X + iY$ where X is the real part of Z and Y the imaginary part of Z.

[85](Siegel 1943*b*, p.10).

Siegel generalized the transformation from the upper half to the upper half plane, which belongs to the group of biholomorphic automorphisms:

$$\mu(z) = \frac{az + b}{cz + d}$$

with $a, b, c, d \in \mathbb{R}$ and $ac - bd = 1$.

The generalization of the transformation from Siegel's half space to itself

$$\mu : Sp(2n, \mathbb{R}) \times \mathcal{H}_n \to \mathcal{H}_n, \tag{2.30}$$

is then

$$(\mu, Z) \mapsto \mu\langle Z\rangle := (AZ + B)(CZ + D)^{-1}. \tag{2.31}$$

For the generalization Siegel introduced the symplectic group of all real matrices, i.e. $Sp(2n, \mathbb{R})$. The transformations μ form the symplectic group obtained by identifying the symplectic matrices M with $-M$.[86] The symplectic group $Sp(2n, \mathbb{R})$ operates on $\mathcal{H}_n$ as a group of automorphisms. This is the reason Siegel called this geometry Symplectic geometry because then the group of automorphisms operating on Siegel's half space is the symplectic group.

To find the metric on $\mathcal{H}_n$, Siegel generalized the way it can be found on Poincarés model of the hyperbolic geometry, through the cross-ration.[87]

So, Siegel first generalized the theorem which says that there exists a Möbius transformation $\mu \in PSL(2, \mathbb{R})$ mapping two given points z, z_1 on the upper half plane $\mathcal{H}_1$ onto two other given points w, w_1 on the upper half plane for an arbitrary n, if, and only if, the cross-ratio of z, $z_1 \in \mathcal{H}$

$$R(z, z_1) = \frac{z - z_1}{z - \overline{z}_1} \frac{\overline{z} - \overline{z}_1}{\overline{z} - z_1}$$

is equal to the cross-ratio of w, w_1, i.e. $R(z, z_1) = R(w, w_1)$.

[86]For the case $n = 1$ then $PSL(2, \mathbb{R}) \cong Sp_1(\mathbb{R})$.

[87]See (Gray 2000).

For the generalization Siegel considered the cross-ratio for $Z, Z_1 \in \mathcal{H}_n$ which are any two "points" on Siegel's half space:

$$R(Z, Z_1) = (Z - Z_1)(Z - \overline{Z}_1)^{-1}(\overline{Z} - \overline{Z}_1)(\overline{Z} - Z_1)^{-1}. \tag{2.32}$$

The generalization of the cross-ratio is a matrix. So, the generalization of the theorem for hyperbolic geometry is:

> **Theorem 2.** There exists a symplectic transformation mapping a given pair Z, Z_1 of $\mathcal{H}_n$ into [sic] another given pair W, W_1 of $\mathcal{H}_n$, if, and only if, the two matrices $R(Z, Z_1)$ and $R(W, W_1)$ have the same characteristic roots.
>
> (Siegel 1943*b*, p. 3)

The trace of the cross-ratio $tr(R(Z, Z_1))$ is invariant under any symplectic transformation of the points Z and Z_1, for any pair of points in Siegel's half spaces.

Siegel derived R twice at the point $Z = Z_1$ and obtained the value

$$d^2R = 2dZ(Z - \overline{Z})^{-1}d\overline{Z}(\overline{Z} - Z)^{-1} = \frac{1}{2}dZY^{-1}d\overline{Z}Y^{-1}. \tag{2.33}$$

where Y denotes the imaginary part of $Z = X + iY$

So $tr(d^2R) = d^2(tr(R))$ is invariant with respect to the operation of the symplectic group $Sp(2n, \mathbb{R})$. He defines a quadratic differential form

$$ds^2 = \mathrm{tr}(Y^{-1}dZY^{-1}d\overline{Z}). \tag{2.34}$$

which is as well invariant with respect to the operation of $Sp(2n, \mathbb{R})$. By introducing $X = (x_{kl})$ and $Y = (y_{kl})$ he obtain

$$ds^2 = tr(Y^{-1}dX\,Y^{-1}\,dX + Y^{-1}dY\,Y^{-1}\,dY) \tag{2.35}$$

and for $Z = i\,Id$,

$$ds^2 = \sum_{k=1}^{n}(dx_{kk}^2 + dy_{kk}^2) + 2\sum_{k<l}(dx_{kl}^2 + dy_{kl}^2). \tag{2.36}$$

The quadratic differential form (2.34) was called by Siegel as an *hermitian differential form* and he proved that it defines a Riemannian metric on $\mathcal{H}_n$. He showed that it is positive definite everywhere on $\mathcal{H}_n$ using that $Sp(2n, \mathbb{R})$ is transitive in $\mathcal{H}_n$.[88]

Through this result Siegel was able to find the shortest arc connecting two arbitrary points Z and Z_1 of Siegel's half space $\mathcal{H}_n$, i.e. he found the geodesics for the symplectic metric (2.34) so that there exists exactly one geodesic arc connecting two arbitrary points[89].

Using the results for the geodesic and the cross-ratio, Siegel was able to give the length on Siegel's half space

$$\rho^2 = \sigma\left(\log^2\frac{1+R^{\frac{1}{2}}}{1-R^{\frac{1}{2}}}\right) \tag{2.37}$$

with R being the cross-ratio between Z and Z_1 and

$$\log^2\frac{1+R^{\frac{1}{2}}}{1-R^{\frac{1}{2}}} = 4R\left(\sum_{k=0}^{\infty}\frac{R^k}{2k+1}\right)^2.$$ [90]

The volume element for the symplectic metric is

$$2^{\frac{1}{2}n(n-1)}dv. \tag{2.38}$$

Siegel showed that the Gauss curvature for Siegel's half space is less or equal zero. The Gauss curvature is characterized by Siegel through the equation

$$K = -\frac{1}{4}tr(Y^{-1}F\,Y^{-1}(\overline{F}^t)^{-1}) \leq 0 \tag{2.39}$$

88See (Siegel 1943*b*, p. 17). Henri Cartan showed in 1957 that this Hermitian differential form is a Kähler form, this will be shown in chapter 5.2.5.

89(Siegel 1943*b*, pp.20-21).

90(Siegel 1943*b*, p. 20).

where

$$F = \delta_1 Z\, Y^{-1} \delta_2 \overline{Z} - \delta_2 Z\, Y^{-1} \delta_1 \overline{Z}$$

is a matrix and $\delta_1 Z$, $\delta_2 Z$ are two arbitrary different directions at the point $Z \in \mathcal{H}_n$. $K = 0$ for $F = 0$ and the curvature is negative for

$$\delta_1 Z\, Y^{-1} \delta_c \overline{Z} \neq \delta_2 Z\, Y^{-1} \delta_1 \overline{Z},$$

and 0 otherwise.[91]

Siegel also generalized the Euler characteristic

> **Theorem 5.** The Euler characteristic of a closed manifold F with the metric (2.34) is
>
> $$\chi = c_n (-\pi)^{\frac{1}{2} n(n+1)} \int_F dv$$
>
> where c_n denotes a positive rational number depending only upon n; in particular, $c_1 = \frac{1}{2}$, $c_2 = \frac{3}{8}$, $c_3 = \frac{45}{256}$.
>
> (Siegel 1943*b*, p. 4)

After having generalized the geometrical properties of Siegel's half space $\mathcal{H}_n$, Siegel continued with the generalization of the Fuchsian groups and their fundamental domains.

Part of this work was done before, in the case of the modular group of degree n which is a discontinuous subgroup first kind of the symplectic group,

$$Sp(2n, \mathbb{Z}) < Sp(2n, \mathbb{R})$$

A discontinuous group of first kind is defined as:

> A discontinuous group Δ is called of first kind, if there exists a normal fundamental domain $\mathcal{F}_n$ having the following three properties:

[91] See (Siegel 1943*b*, p. 23).

1. Every compact domain $\mathcal{H}_n$ is covered by a finite number of images of $\mathcal{F}_n$;
2. only a finite number of images of $\mathcal{F}_n$ are neighbours of $\mathcal{F}_n$;
3. the integral
$$V(\Delta) = \int_{\mathcal{F}_n} dv$$
converges.

(Siegel 1943*b*, p.4)

These properties were used for the Minkowski reduction to find the fundamental domain of the modular group of degree n in his previous work.

2.2.6 Reception of Siegel's paper 1943

In 1943, Siegel reorganized his previous results from the 1930s and generalized them for other discontinuous groups. As a result of this reorganization, Siegel gave the generalization of hyperbolic geometry, which is called symplectic geometry by Siegle, because the group that operates on Siegel's half space is a symplectic group.

Siegel's article from 1943 is quoted by authors of textbooks about symplectic geometry (See e.g. (Da Silva 2000) or (Berndt 1998)). These textbooks deal with the study of symplectic manifolds and not with the generalization of hyperbolic geometry. Siegel's article is quoted as well in books about classical mechanics (See e.g. (Abraham & Marsden 1978) or (Arnold 1989)). The relationship between classical mechanics and the study of symplectic manifolds is that the phase space has a structure of a symplectic manifold.

Today, there exist mathematicians working in the field of modular forms recognize Siegel's geometrical investigations of the upper half space, and the field of research is called the field of Siegel modular forms as Klingen or Van Der Geer,.[92]

Furthermore, the research that Siegel started about discontinuous groups that operate on the Siegel half spaces has continued. Siegel himself published in May of 1943 an article with the title *"Discontinuous groups"*.[93]

It seems that there was interest in Siegel's article *"Symplectic Geometry"* because the American Journal of Mathematics reprinted it as a book in 1964. Siegel wrote in the preface:

> There still seems to be considerable interest in my paper on Symplectic Geometry which appeared 21 years ago in the *American Journal of Mathematics.* Since copies are no longer available. I am grateful to the editors of Academic Press for this new publication.
>
> (Siegel 1964)

In the 1950s mathematicians started to call the modular group of degree n Siegel modular form, and the $\frac{1}{2}n(n+1)$ dimensional half space the Siegel half space. This can be seen in the work of Ichiro Satake[94], who published his article *"On the compactification of the Siegel space"*, i.e. the Siegel half space.[95]

In the 1950s, the mathematician Henri Cartan[96] continued with the geometric work of Siegel. In 1957, Henri Cartan published his work *"Ouverts*

[92] See (Klingen 1990), (Van Der Geer 2008).

[93] See (Siegel 1943*a*).

[94] Ichifo Satake was born in Japan in 1927 and died on October 10, 2014. He was mostly active as a mathematician in the United States at the IAS and Berkeley

[95] See (Satake 1956).

[96] Henri Cartan was the son of Élie Cartan. He was born on July 8, 1904 and died on August 13, 2008.

fondamentaux pour le groupe modulaire"[97] in which he found a fundamental open domain for the modular group of degree n and he found that Siegel's half space is a Kähler manifold, therefore Siegel's half space is a symplectic manifold.[98] This development will be discussed in chapter 5.2.4. In this work Cartan used the denotation *espace de Siegel* for $\mathcal{H}_n$.

In the 1950s mathematicians used Siegel's name to refer to the half space and to the modular forms of degree n, but they did not use the name symplectic geometry to refer to the field that they were investigating.

Nowadays, symplectic geometry refers to another field of study which is not a generalization of hyperbolic geometry to the Siegel's half space.

Nevertheless, some textbooks about symplectic geometry and about classical mechanics have a bibliographical reference to Siegel's article *"Symplectic Geometry"* even though they did not address the generalization Siegel made. However, they use the Siegel half space, because is a symplectic manifold, as an example of a symplectic space, where the symplectic group acts as a group of automorphisms.[99] In this case, the symplectic group is the group of the symplectomophism of the Siegel half space into itself.[100]

Siegel's students seems to be the only ones that used the name "Symplectic Geometry" to refer to this generalization.[101]

2.3 Siegel's and Hua's parallel works

In the history of science it often happens that more than one person or research group find or discover similar results or theories. This seems to be

[97] Foundamental open regions for the modular group.
[98] See (Cartan 1957).
[99] See (McDuff & Salamon 1995) or (Berndt 1998).
[100] For the definition of symplectomophism see section 9.1.4.
[101] See (Klingen et al. 1983).

the case for the generalization of hyperbolic geometry given by Siegel, because on the other side of the world, in Kunming in the province of Yunna,[102] China, the mathematician Hua Loo-Keng was working in the same field and found similar results as Siegel gave in his article *"Symplectic Geometry"*. The results were so similar that Hermann Weyl, who read Hua's draft, qualified it as a double of Siegel's article.[103] This can be read in his letter that he sent to some professors in March 1943:

> Professor Alexander, Einstein, Morse, Veblen
>
> March 24, 1943
>
> In my opinion, the two outstanding Chinese mathematicians are Chern and Hua Loo-Keng (National Tsing Hua University, Kunming). The latter has made a number of profound contributions to the Hardy–Littlewood–Vinogradoff line of analytic number theory, and in a manuscript which he recently sent me duplicated a considerable part of Siegel's results in his big paper on symplectic geometry. It would be of the greatest value to him to get into closer contact with Siegel; [...]
>
> Hermann Weyl
>
> Hermann Weyl in (Richard & Serme 2013, p. 75)

It seems unrealistic that two mathematicians that work far away from each other during a period of war come to the same results, but it is important to consider the context and the involved researchers.

[102]Yunna is located in the south west of the country and Kunming is the capital of the province. The province borders Myanmar, Laos, and Vietnam.

[103]Weyl and Hua had an exchange of correspondence during the 1940s; this story was written by Richard and Serme in 2013. See (Richard & Serme 2013).

Hua, indeed, claimed that the works were absolutely different.[104] However, Hua was trained in mathematics in China and England. Moreover, the exchange of letters with Hermann Weyl during the 1940s show that Hua was not isolated at all, and he used to have contact with mathematicians outside China during politically difficult times.

China in the context of the 20$^{\text{th}}$ century

At the beginning of the 20th century the exchange between Chinese and Western mathematicians had increased.

In 1928 the Academia Sinica was founded in Mainland China. It was not only a mathematical institute but also an academy of science. It shows the interest of the Chinese in developing and promoting the exchange of scientific work between European and North American scientists.

During the 1920s and the 1930s China invited some Western mathematicians to give lessons at the arithmetical institutes, which were later renamed mathematical institutes.[105] Through this contact Chinese students were invited as well to study in the USA or in Europe, and in the 1920s several Chinese mathematicians started to publish in Western Journals.[106] This was the case for Hua Loo-Keng and Shiing-Shen Chern (*1911, †2004), two of the mathematicians that Weyl mentioned in his letter.

2.3.1 Hua Loo-Keng

Hua Loo-Keng was born on November 12, 1910 in the town Jintan in the province of Jiangsu, China.[107] Several articles written about his live claim

[104] (Richard & Serme 2013, p.76).
[105] See (Salaff 1972).
[106] (Salaff 1972, p.145).
[107] Jintan is a town 200 km west from Shanghai.

that Hua was the founder of modern mathematics in China.[108]

The biography of Hua provides an example of a person who managed to work in mathematics and made contributions to mathematical research despite disadvantaged conditions. Hua did not achieve an academic degree, and he only finished junior middle school because his family was not able to afford education for their children. Therefore, Hua could not attend senior middle school after finishing middle school.

In 1924 Hua gained admission to the Shanghai Chung-Hua Vocational School. In Shanghai Hua showed some mathematical skills and won the national abacus competition. Unfortunately, he was forced to leave six months before he finished his education because Hua had to return to his family to work with his father in their store. During his free time Hua dedicated himself to reading and learning modern mathematics.[109]

In December of 1929 Hua published his first article *"Some Researches on the Theorem of Sturm"*[110] in the Shanghai Journal *Science* .

A year later Hua made another contribution in the same Journal by handing in a commentary in which he showed that a paper published in 1926, which claimed to have solved the quintic, was wrong.

Through this commentary Hua attracted the attention of professor Hsiung Qin-Lai, who was the chairman of the department of Arithmetic at the Tsinghua University in Beijing. In 1930 the department for mathematics at the Tsinghua University in Beijing was called department of Arithmetic.[111]

[108]See (Salaff 1972).

[109]See (Salaff 1972, p.144).

[110]Sturm's theorem expresses the number of distinct real roots of a polynomial in an interval in terms of the number of changes of signs of Sturm's sequence of a polynomial. Sturm's sequence is a sequence of polynomials associated to the polynomial and its derivative by a variant of Euclidean algorithm for polynomial.

[111]See (Chern 2001).

Hsiung wanted to contact Hua, and he thought that Hua was a student or a graduate student. However, neither the assistants nor the professors in the department of Arithmetics in Beijing had ever heard about a student named Hua Loo-Keng. Eventually a teacher called Tang, who was born in Jintan, informed Hsiung that Hua Loo-Keng was born in his home town, and that he had not even finished senior middle school.[112]

Hsiung contacted Hua and offered him a job in the department of Arithmetics in Beijing. Hua accepted and went to Beijing in the summer of 1931. He was an employee at the library. During this time Hua attended some lectures in the department. Later, this gave Hua the opportunity to start an assistantship in the department.[113]

It was here that he met Chern for the first time because the two men had to share a desk. Hua and Chern did not only share desk, they also became friends.[114]

In 1934 Hua was promoted to the rank of lecturer, despite his lack of an academic degree. Hua's initial research was in the field of number theory on the Waring problem. The Waring problem is to find for a given natural number k the least positive integer s, so that every natural number n is the sum of at most s k^{th} power of natural numbers, i.e. the equation

$$n = x_1^k + x_2^k + ... + x_s^k$$

is solvable for every natural number n.

The theme was encouraged by Hsiung, who had done his PhD thesis in additive number theory at the University of Chicago.[115]

---

[112] See (Gong 2001).
[113] (Halberstam 1988, p.99).
[114] See (Chern 2001).
[115] (Salaff 1972, p.145).

During 1935 and 1936 Jacques Hadamard[116] and Norbert Wiener[117] visited the Arithmetic department at Beijing University and gave lectures there, which Hua attended.

Wiener's visit was important for the development of Hua's mathematical career because after his stay at Beijing University, Wiener went to Cambridge, England. There, he recommended Hua to Godfrey Harold Hardy.[118]

In 1936, Hua received and accepted an invitation and a scholarship to go to Cambridge.

2.3.2 Hua's time in Europe 1936-1938

In the summer of 1936 Hua made his way to Cambridge with the Trans-Siberian Railway from Beijing to Berlin to meet Chern. They went together to see the Olympic Games, which took place in the first two weeks of August. Chern was at that time at the University of Hamburg. After the games, Hua and Chern went to Cambridge, England together, where Hua spent two years working and attending lessons with Hardy.[119]

Hardy offered Hua to do his PhD in Cambridge, but this was not possible for Hua because, again, he could not afford the registration fee, and so he declined the offer.

[116] Jacques Salomon Hadamard was born on December 8, 1865, in France and died on October 17, 1963. His scientific contributions are in number theory, complex function theory, differential geometry, and partial differential equations.

[117] Norbert Wiener was born on November 26, 1894, in the United States and died on March 18, 1964. At the time, Wiener was a professor of mathematics at Massachusetts Institute of Technology (MIT).

[118] Godfrey Harold Hardy (born in England in 1877 and died in 1947) was the mentor of the Indian mathematician Srinivasa Ramanujan, who also could not follow the usual path for an academic career.

[119] See (Chern 2001).

In Cambridge, Hua had the possibility to come into contact with other mathematicians, and he became friends with Harold Davenport and Hans Heilbronn.[120]

Hua spent two years in Cambridge. During this time, he published eighteen papers in number theory.[121]

Hua's return to China was earlier than expected. Hua returned to China in 1938, as in 1937 Japan invaded China, and Hua wanted to see if he could help.[122] One may think that Hua did not have enough money to stay in England, but it seems that throughout this live, Hua always wished to return to China when there was a historical turning point. Another instance that leads to this impression was in 1950, two years after getting a chair at the University of Illinois. Hua preferred to return to China because he wanted to participate in the revolutionary activities of the Chinese Communist Revolution.[123]

2.3.3 Remarks on Siegel's and Hua's parallel work

During Hua's time in England he continued to work on the Waring problem and, in addition, worked on Goldbach's conjecture. The Goldbach conjecture says that every even integer greater than 2 can be expressed as the sum of two primes.

In Cambridge Hua learned from Hardy the analytical techniques for num-

[120]Harold Davenport was born on October 30, 1907 in England and died on June 9, 1969. His field of studies was number theory and he worked, as well, on Waring's problem. Heilbronn was the last assistant Landau's in Göttingen. See (Rogers 1971, Halberstam 2002, Salaff 1972).

[121]See (Halberstam 1988).

[122]See (Schweigman & Zhang 1994).

[123](Salaff 1972, p. 149).

ber theory and used them to try the Waring problem.[124]

In 1938 Hua returned to China. He went back as a professor at the South West Associated University in the city of Kunming.[125] The South West Associated University was founded by the Tsing Hua University of Beijing and the University of Nanka. In 1938 both universities went into exile during the Japanese invasion.[126] Hua and his family stayed in Kunming until 1945.

In Kunming, Hua finished his manuscript *"Additive Prime Number Theory"* in 1941 in which he unified the results of previous papers about the Waring problem and Goldbach's conjecture and improved them.[127]

Hua claimed that in the same year, he started to work on the theory of several complex variables, automorphic functions, and the geometry of matrices. This field is the same field that Siegel presented in *Symplectic Geometry.*

In 1943 Hua sent the draft of his paper *On the Theory of Automorphic Functions of a Matrix Variable, I - Geometrical Basis*[128] to Weyl for revision, who classified it as a double of Siegel's paper, but Hua denied being influenced by Siegel in his work[129].

One year later, Hua's article was published in the American Journal of Mathematics. The article was split into two parts. The first part (Hua 1944*a*) gave results, which doubled Siegel's results, and in the second part (Hua 1944*b*) he presented some new results.

This fact is mentioned by Hua in the introduction:

> The present paper is a revised form of another manuscript

[124] See (Salaff 1972, p.146).

[125] Kunming is the capital of the Yunnan Province in southwest China

[126] (Fairbank & Feuerwerker 1986, p. 564).

[127] See (Salaff 1972).

[128] See (Hua 1944*a*).

[129] See (Richard & Serme 2013).

> which the author had previously submitted for publication. The revision was necessary because the original manuscript contained some results (found independently by the author in some research begun in 1941) that have been recently published in Prof. C. L. Siegel's paper on Symplectic Geometry. It is the aim of this paper to give a brief account of those results which are interfluent with Siegel's contributions. The remaining part of the author's research will be given later separately.
>
> (Hua 1944*a*, p.470)

Hua added thanks to Weyl, who was his interlocutor for sending a copy of Siegel's paper *Symplectic Geometry*, and remarked on the fact that he did not know about Siegel's publication.

It shows that Hua had been working on the same generalization as Siegel before the publication of Siegel's article (Siegel 1943*b*).

Another remark was made by the editor, who wrote how difficult the communication had been between China and the USA, and, therefore, the paper was corrected by Hua's friend Dr. Hsio-Fu Tuan and by Siegel:

> Because of the poor mail service between the U. S. and China, a number of minor changes in this paper have been made here, with the consent of the editors, by Prof. Hua's friend Dr. Hsio-Fu Tuan and Prof. C. L. Siegel.
>
> (Hua 1944*a*)

The question is if Hua had had some access to Siegel's work between 1935 and 1938 or, afterwards, between 1938 and 1943.

It is to discard that Hua attended the conference about the analytic theory of the quadratic forms given by Siegel at the International Mathematical Congress in Oslo in 1936. This is because Hua arrived in Europe with the Trans Siberia Railway from Beijing to Berlin in the first two weeks of August, 1936 to see the Olympic Games. The International Congress took place from 13-18 July. However, maybe Hua read Congress Reports later.

If Hua had knew Siegel's work on modular forms of the 1930s, it is possible that Hua could have reorganized Siegel's results to research the geometric properties, as well.

Hua knew Siegel's work when he was in Cambridge. In Hua's work (Hua 1938) giving some results on additive prime number theory, he quoted Siegel's from 1935. Later, he published a note about quadratic forms.[130]

In 1943, Weyl invited Hua for a stay at the IAS in Princeton, which Hua rejected. Hua argued that he wanted to work on his ideas by himself, and he did not want to be influenced by Siegel.[131] On 15 March 1943 Hua changed his mind and expressed his interest to go to the IAS and study with Weyl and Siegel.[132]

When the second world war ended, Hua received an invitation to the USSR to visit the Soviet Academy of Science by Ivan Vinogradov[133], which he accepted, and, after that he went to Princeton to the IAS.

In Princeton Hua continued to work about automorphic functions and developed the theory of the Geometry of Matrices.[134] In an article from 1947

[130] See (Hua 1941).

[131] See(Halberstam 2002, p.144).

[132] (Richard & Serme 2013, p. 74).

[133] Ivan Matveevich Vinogradov (*1891, †1983) was a Soviet mathematician. He worked on analytic number theory, and he provided a partial solution to the Goldbach conjecture (of Encyclopædia Britannica 2017). See (Salaff 1972).

[134] Geometry of Matrices is the geometry of rectangular alternate, symmetric and hermi-

Hua presented his results about this topic.[135] In this work Hua presented the automorphism of the generalization of the elliptic space and called this generalization Elliptic Geometry.[136] He again included some results that Siegel gave in 1943 and remarked on the importance of Siegel's *Symplectic Geometry*, as can be read in the next quotation:

> [...] with a result due to the author (Hua 1946), we solved also the corresponding problem for the group of automorphism of the elliptic space. It should be remarked that the corresponding problem for hyperbolic space was solved by C. L. Siegel in a recent important paper.
>
> (Hua 1947, p. 229)

Hua did not only find the same results as Siegel and continued to develop the field. The letters and Hua's own assertion show that he did not know Siegel's work (Siegel 1943*b*). Unfortunately, no evidence was found to ensure that Hua was aware of Siegel's work.

Remarks about Hua

Hua did not have fortunate conditions in his early life, but he seized the opportunities that occurred. Hua had to take the hard way to work on mathematics, without all the opportunities that other mathematicians had at that time. Nevertheless without any academical degree in the 1940s, Hua gained the recognizion of the international mathematical society as a great mathematician, as can be read in Weyl's letters.[137]

tian matrices over a division ring or over a field.

[135] See (Hua 1947).

[136] (Hua 1947, p. 250).

[137] See (Richard & Serme 2013).

During the wars in China in the 1930s and 1940s, i.e. Japan's invasion of China, and the second world war, mathematical exchange was difficult for Hua, but it was not impossible. Hua was able to exchange letters with Hermann Weyl, which enabled him to continue his mathematical research.

In 1950 Hua returned to China to work on the reform of the educational system in mathematics for the Peoples Republic of China. In 1952 he became the first director of the Mathematical Institute of the Academia Sinica.[138]

During the cultural revolution from 1966 to 1976 Hua Loo-keng had a hard time, and he was not able to work and publish on mathematics or have any influence on the educational system.[139]

After the cultural revolution, Hua became vice-president of the Academia Sinica and science advisor for the Chinese government, and in 1980 he obtained an honorary doctorate from the University of Nancy.[140]

Hua died of a heart attack at the end of a lecture he gave in Tokyo on June 12, 1985.

Finally, the recognition of Hua's mathematical skills are shown in the remarks that Derrick Lehme made about Hua in 1970:

> Something that was seen in Hua as Hua had the uncanny ability of taking the best work of others and finding the exact points where their results could be sharpened. He had many tricks of his own, too. He read widely and commanded an overview of all of twentieth-century number theory. His chief interest was to improve upon the whole field; he would have, if left to himself, tried to generalize every result he came upon. His work was in some respects like that of [I] Schur, or even [Norbert] Wiener,

[138] See (Salaff 1972).
[139] See (Gong 2001).
[140] See (Gong 2001).

both of whom made deep contributions to number theory, but also branched off to other fields.

(Derrick Lehme in (Salaff 1972, p.151))

Chapter 3

Differential forms

Introduction

Élie Cartan developed the theory of differential forms, and, like Weyl, he also had an important personal network. Even more important, many of his students developed what is nowadays understood as symplectic geometry. It can be said that Élie Cartan established the basis for the development of symplectic geometry with his theory of integral invariant.

A symplectic form on a finite dimensional vector space V over a field $\mathbb{R}$ is a non-degenerate two form ω on V. A 2-form over V is a bilinear form on V. A symplectic form on a smooth manifold is a closed non-degenerate differential 2-form.

The first part of the chapter provides the definition of a symplectic form and a symplectic vector space. Then a short passage on how the theory of exterior differential forms was developed by Élie Cartan is presented and it contains a short biography of Élie Cartan.

3.1 Symplectic form and symplectic space

A symplectic form on a finite dimensional vector space V over a field $\mathbb{R}$ is a non-degenerate two form ω on V. A 2-form over V is a bilinear form on V, i.e.

$$\omega : V \times V \to \mathbb{R},$$

which is antisymmetric, so that

$$\omega(v, v) = 0 \quad \text{for all } v \in V$$

holds. Non-degenerate means that

$$\omega(v, w) = 0 \quad \text{for all } v \in V \quad \text{implies } w = 0.$$

A vector space provided with a symplectic form is called a symplectic vector space.[1] Therefore, if V is a symplectic vector space with dimension $r = 2n$, i.e. $V \simeq \mathbb{R}^{2n}$, the standard symplectic form is defined by

$$\omega(v, w) = v^t J w = \sum_{k=0}^{n} v_k w_{n+k} - v_{n+k} w_k, \quad v, w \in V \tag{3.1}$$

with

$$J = \begin{pmatrix} 0 & Id \\ -Id & 0 \end{pmatrix}.$$

J belongs to the symplectic group, i.e. $J \in Sp(2n, \mathbb{R})$.

A symplectic manifold (M, ω) is a $2n$-dimensional differentiable manifold M with ω as symplectic form.[2]

A symplectic form on a $2n$-dimensional differentiable manifold M is a closed and non-degenerate differential 2-form. ω is called closed if the exterior

[1] The vector space could be over any commutative field $\mathbb{K}$ with characteristic zero.
[2] (Abraham & Marsden 1978, p. 176).

differential is equal to zero. ω is called non-degenerate if on each tangent space T_pM, with $p \in M$, it holds that

$$\omega(\xi, \eta) = 0 \quad \text{for all } \xi \in T_pM \quad \text{implies } \eta = 0.$$

Locally the symplectic manifold can be seen as symplectic vector space.

3.2 Differential forms

Because a symplectic form is a differential 2-form, and in the case of a symplectic manifold the 2-form is closed, the next section will track the developments on differential forms made by Élie Cartan. Before dealing with the development of differentiable forms by Élie Cartan a brief biography of Cartan is given below.

3.2.1 Élie Cartan

Élie Cartan was born in Dolomic, France, on April 9, 1869 and he died in Paris, on May 6, 1951. Some of his fundamental contributions were on the theory of Lie groups, Lie algebras, differential equations, differential geometry and topology.

In 1888 he started his studies in Paris at the *École Normal Supérieure* and obtained his doctor degree in 1894 under the supervision of Jean-Gaston Darboux (*1842, †1917). After having graduated, Cartan gave lectures at the University of Montpellier from 1894 to 1896. Later on, during 1896 to 1903, he also worked as a lecturer in Faculty of Sciences at the University of Lyon. In 1903 he obtained a position as a Professor in the Faculty of Sciences at the University of Nancy. He left Nancy in 1909 and moved to Paris to give lectures at the Sorbonne in Paris. In 1912, with the support of Henri

Poincaré (*1854, †1912), he became a Professor at the Sorbonne.[3]

In his doctoral thesis Cartan gave a complete treatment of the classification of finite-dimensional, semi-simple and complex Lie algebras, which at the time were called the infinitesimal groups. [4] After having finished his doctoral thesis, Cartan worked on the theory of partial differential equations (Cartan 1896), in which he exposed the systems of partial differential equations whose solutions only depend on arbitrary constants.[5]

In 1899 he formalized the notion of a differential form and developed the theory of exterior differential forms. Before Cartan's formalization, the differential forms were seen as "the things under integral signs".[6] In 1922 Cartan published his lecture notes about this theory, and in 1936 Cartan gave a lecture about the application of differential forms to geometry, which was published in 1945.[7]

The genesis of the symplectic form correlates with the development of the theory on differential forms and their applications in geometry.[8]

Élie Cartan had an active influence on the development as he was the person who fostered the communication between the mathematicians. Among his students were André Lichnerowicz (*1915, †1998), André Weil (*1906, †1998) and Charles Ehresmann (*1905, †1979).[9]

He participated in many International Congresses of Mathematics be-

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[3](Akivis & Rosenfeld 2011, p. 7-9).

[4]A Lie algebra is a vector space $\mathfrak{g}$ over a field F together with a binary operation $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ called the Lie brackets, which is bilinear, alternating on $\mathfrak{g}$ and satisfy the Jacobi identity.

[5]See (Cogliati 2011).

[6](Katz 1985, p. 322).

[7]See (Cartan 1945).

[8]For a more detailed historical description of the development of the differential form the reader may consult (Katz 1981), (Katz 1985) and (Hawkins 2005).

[9]See (Akivis & Rosenfeld 2011).

tween 1920 and 1936.[10] During his life Élie Cartan had many international students, such as the German mathematician Ernst August Weisse (*1900, †1942), who spend two semesters with Cartan in Paris; or as Shiing-Shen Chern, who went to Paris to study with Cartan in 1936.[11] Another international student of Cartan was Mohand Hashtroudi (*1908, †1976) from Iran, who finished his PhD in 1936.

Cartan maintained intense relations with Soviet mathematicians such as Serge P. Finikov (*1883, †1964), who attended Cartan's lectures in Paris from 1926 to 1927 and later founded a Soviet differential geometric school.[12] In 1934 Cartan went to the International Conference on Tensor Differential Geometry in Moscow. At this conference he came into in contact and became friends with Erich Kähler (*1906, †2000). Kähler had come to Moscow with the research group of Wilhelm Blaschke (*1885, †1962).[13]

Later in 1931 Cartan was elected to the French Academy of Sciences. In 1945 he became vice-president of the Academy and in 1946 its president.[14] There, Cartan presented reports of other mathematicians work in progress in the fields differential geometry and topology. These reports were short contributions of two pages. They gave the possibility to exchange ideas and to see what other mathematicians were researching. These publications were important because they provided a medium through which mathematicians could communicate their new results; they also provide historians with the possibility to track the developments that were taking place.

[10] See (Akivis & Rosenfeld 2011) and (Gispert & Leloup 2009).
[11] See (Akivis & Rosenfeld 2011, p. 28) and appendix A.
[12] (Akivis & Rosenfeld 2011, p.30).
[13] See (Berndt & Riemenschneider 2003).
[14] See (Chern, Chevalley et al. 1952) and (Gispert & Leloup 2009).

3.2.2 Cartan differential form

Cartan was primarily interested in the problem of equivalences between systems of differential equations and looked for conditions under which these can be equivalent. One of his first publications about this problem was published in 1899 when he worked on the problem of Pfaff and developed the theory of exterior calculus of differential forms.[15] He introduced the idea of a differential 1-form and its multiplication rules. Later, Cartan wrote another publication about the Pfaff problem (Cartan 1901), in which he introduced differential forms of higher degrees.

Pfaff's problem is a problem in the field of differential equations and has its roots in the theory of first order partial differential equations in the work of Euler (*1707- †1783) and Monge (*1746- †1818).[16] Currently, in the theory of differential forms, Pfaff's problem is concerned with the conditions under which the number of variables in a 1-form can be reduced by a change of variables.[17]

During the 19th century some mathematicians such as Lagrange (*1736, †1813), Pfaff (*1765, †1825), Jacobi (*1804, †1851), Clebsch (*1837, †1872), Grassmann (*1809, †1877), Frobenius (*1849, †1917) and Darboux (*1841, †1917) worked on the problem.

In the 18th century and at the beginning of the 19th century the problem was the following: has the differential equation

$$\omega = a_1(x)dx_1 + a_2(x)dx_2 + ... + a_n(x)dx_n = 0 \quad (x = x_1, ..., x_n),^{18} \tag{3.2}$$

a meaning, when there does not exist any function f so that $df = \omega$? Euler

[15] See (Cartan 1899).
[16] See (Katz 1985).
[17] (Katz 1981, p. 185).
[18] The equation (3.2) was later known as the Pfaffian equation.

argued that such a differential equation was meaningless, but Monge noticed that it can be a system of integral equations which form an integral equivalent to the equation (3.2).[19]

In 1815 Johann Friedrich Pfaff[20] made the assertion that the integral equivalent of (3.2) can be satisfied by a system of $\frac{n}{2}$ integral equations in the case that n is even and a system of $\frac{n+1}{2}$ integral equations, if n is odd.[21]

To find this result Pfaff extended some results of Lagrange on the theory of differential equations. Lagrange found out how to obtain a general solution for a first order differential equation with any number of variables m.

Lagrange was able to integrate any first order partial differential equation for two independent variables $m = 2$. In 1815, Pfaff was able to integrate non-linear partial differential equations of $m > 2$ variables.[22]

In modern notation a first order partial differential equation can be expressed as

$$F\left(x_1, ..., x_m, u, \frac{\partial u}{\partial x_1}, ..., \frac{\partial u}{\partial x_m}\right) = 0, \tag{3.3}$$

with solution $u = u(x_1, ..., x_m, C_1, ..., C_m)$, where C_i are arbitrary constants.[23]

Pfaff found out that any system of first order partial differential equations with m variables can be reduced to one system of differential equations with $n = 2m$ variables of the form (3.2).[24], and that this reduction led to a solution of equation (3.2).[25]

[19](Katz 1985, p. 324).

[20]Johann Friedrich Pfaff who worked on differential equations was born on December 22, 1765 in Stuttgart, Germany and died on April 21, 1825 in Halle, Germany.

[21](Katz 1985, p.324).

[22](Hawkins 2005, p. 387).

[23](Hawkins 2005, p.386).

[24](Hawkins 2005, p.387). For more information about the Pfaffian problem see chapter 6.2.1

[25]In modern terms the solution can be characterized by a submanifold of co-dimension m in $\mathbb{R}$, i.e. the “integral variety”.

At the end of the 19th century mathematicians identified that the problem of Pfaff was to find a suitable change of variables such that the Pfaffian equation ω could be expressed through a minimal number of variables, i.e. to find a canonical form of the Pfaffian form ω.

This problem was solved by Frobenius in (Frobenius 1877) and later by Darboux (Darboux 1882). Darboux's solution for Pfaff's problem is known as Darboux's theorem and also became an important theorem in symplectic geometry. This will be discussed in chapter 6.

In 1899 Cartan gave an exposition of the Pfaffian problem in his work (Cartan 1899). He defined a differential "expression" ω with n variables as a homogeneous expression formed by a finite number of additions and alternating multiplications of the n differentials $dx_1, ..., dx_n$.[26]

A differential *expression* was called a differential form after the publication of the lecture notes of Weyl about Lie groups in 1935.[27]

A Pfaffian expression was defined as a differential expression of degree one, i.e.

$$\omega = \sum_{i=1}^{n} a_i dx_i. \tag{3.4}$$

In 1901 Cartan introduced the exterior differential forms of higher degree, i.e.

$$\omega = \sum a_{ij...k} dx_i \wedge dx_j \wedge ... \wedge dx_k. \tag{3.5}$$

Cartan did not introduce the actual notation of the wedge product $\wedge$, his notation for the differential expression was

$$\omega = \sum a_{ij...k} dx_i dx_j ... dx_k.$$

In 1899, Cartan had introduced the idea of the value of a form. For the value of the differential "expression" ω of degree h, Cartan considered

[26](Cartan 1899, p.7), (Katz 1985, p.322).

[27](Katz 1985, p.333).

$x_1, ..., x_n$ to be functions of h indeterminate parameters $\alpha_1, ..., \alpha_h$. He assumed that the parameters have a rank in a certain order which he called the *natural order*. The total value of the differential form ω of degree h is defined by Cartan as the sum over all $h!$ permutations of arbitrary parameters $\alpha_1, ..., \alpha_h$ of the variables $x_1, ..., x_n$. The value that corresponds to a permutation $\beta_1, ..., \beta_h$ is the value that ω takes if the differentials dx of each term of ω are replaced on the i^{th} position by the corresponding derivative of $x_1, ..., x_n$.[28]

Cartan gave an example for the value of the differential form in the case

$$a_1 dx_2 \wedge dx_1 + a_2 dx_3 \wedge dx_2$$

The value is

$$a_1 \frac{\partial x_2}{\partial \alpha_1} \frac{\partial x_1}{\partial \alpha_2} + a_2 \frac{\partial x_3}{\partial \alpha_1} \frac{\partial x_2}{\partial \alpha_2} - a_1 \frac{\partial x_2}{\partial \alpha_2} \frac{\partial x_1}{\partial \alpha_1} - a_2 \frac{\partial x_3}{\partial \alpha_2} \frac{\partial x_2}{\partial \alpha_1}.$$

If two or more differential forms of degree h have the same value, independent of the choice of the parameters $\alpha_1, ..., \alpha_2$, Cartan defined them as equivalent.[29]

With the value and the equivalence of the differential forms, Cartan was able to establish Grassmann's multiplication rules because there is no distinction between differential forms with the same value.[30]

Grassmann's multiplication rule for two Pfaffian *expressions* or, in modern terms, of two differential 1-forms is

$$dx_i \wedge dx_j = -dx_j \wedge dx_i \text{ or } dx_i \wedge dx_i = 0,$$

which is a differential expression of degree 2, i.e. a 2-form.[31]

[28] See (Cartan 1899, p.246) and (Katz 1985, p. 323).

[29] (Cartan 1899, p. 246).

[30] (Katz 1985, p.323).

[31] Nowadays, the set of all differential 1-forms on a manifold is a vector space denoted

Cartan introduced the exterior derivative for differential forms such that for a Pfaffian form

$$a_1 dx + ... + a_n dx_n$$

the derived expression of degree two is

$$d\omega = \sum_{i=1}^{n} da_i \wedge dx_i$$

with

$$da_i = \sum \left(\frac{\partial a_i}{\partial x_j} \right) dx_j.$$

It should be noted that Cartan used for the exterior derivative the notation ω' and not the current notation $d\omega$.[32]

Cartan noticed that $d\omega = 0$ if, and only if, $\omega = df$, when the function f is well defined in a simple-connected domain. The fundamental property of the operation of the exterior derivative is that it is invariant with respect to any change of variables.

3.2.3 Closed differential forms

The symplectic form is a closed form, so a differential form is called closed if its exterior derivative is equal to zero. This terminology was introduced by De Rham[33] in his dissertation (De Rham 1931) on cohomology, in which he used Cartan's theory of differential forms. Cartan was the president in the jury of the defense of de Rham's PhD thesis.[34]

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by Ω^1, for the k-forms the vector space is denoted by Ω^k. So, the exterior multiplication of a k-form with an l-form belongs to the vector space Ω^{k+l}.

[32]The current notation was introduced by Erich Kähler in (Kähler 1934).

[33]Georges de Rham was born on September 10, 1903 and died on October 9, 1990. He was a Swiss mathematician who contributed to the field of differential topology. De Rham developed in his dissertation the theory of cohomology. For further information about this development the reader can consult (Katz 1985) and (Dieudonné 1989).

[34](de Rham 1980, p. 25).

In his dissertation De Rham completed the generalization of Poincaré's Lemma, which is known as De Rham's theorem.[35] In this generalization, De Rham considered the relation of differential forms to algebraic topology.[36] De Rham defined a closed form as:

> *Une forme régulière de degré p, ω, sera dite fermée, si sa dérivée extérieure est nulle: $\omega' = 0$. On dira encore que ω est homologue à zéro, $\omega \sim 0$, s'il existe une forme régulière $\tilde{\omega}$ telle que $\tilde{\omega}' = \omega$.*[37]
>
> (De Rham 1931, p.176)

De Rham defined the objects over which differential q-forms are to be integrated in a variety V as the the i-chain. An i-chain which is the homeomorphic image in the variety of an i-dimensional hypertetrahedron is an elementary i-chain and an arbitrary i-chain is the linear combination of elementary chains. An i-dimensional hypertetrahedron is as well known as i-simplex.[38]

The term"closed" came about when De Rham made an explicit analogy between forms and chains in homology theory using the generalization of Stoke's theorem

$$\int_{\partial C} \omega = \int_C d\omega. \tag{3.6}$$

[35] Poincaré lemma: For an arbitrary manifold M, every point has an open neighbourhood U in which for every k-form ω and $d\omega = 0$, there exists a $k-1$-form α with $d\alpha = \omega|U$. (Janich & Kay 2001, p.204).

[36] (Katz 1985, p.332).

[37] "ω is called closed, if the exterior derivative is zero: $\omega' = 0$. It is also said that ω is homolog to zero, $\omega \sim 0$ if there exists a closed regular form $\tilde{\omega}$ so that $\tilde{\omega}' = \omega$."

[38] (Katz 1985, p.332).

The analogy was made by introducing the definition of the forms such a way that they fit to the definition of the chains.

A closed chain is homologous to zero, so a chain c is closed if the boundary is zero while a differential form is closed if its exterior derivative $d\omega = 0$.

> *La formule de Stokes montre que l'intégrale d'une forme homologue a zéro étendue à un champ fermè est nulle, et que l'intégrale d'une forme fermée étendue à un champ homologue à zèro est nulle.*[39]
>
> (De Rham 1931, p. 176)

3.3 Early applications of differential forms

Cartan used the theory of differential forms in the three body problem. This was done in Cartan's book *Leçons sur les invariants intégraux* in 1922. This book contains lecture notes about theory of exterior calculus of differential forms.[40] In this lectures Cartan used exterior algebra and added the time to the invariants integral of Poincaré.[41] So, Cartan introduced a Pfaffian form known currently as Cartan integral invariant

$$\omega = \sum_{i=1}^{n} p_i dq_i - H dt,$$

[39] "Stokes's formula shows that the integral of a form homologous to zero, extended to a closed domain is zero and that the integral of a closed form extended to a domain is homologous to zero."

[40] See (Cartan 1922).

[41] Poincaré introduced the concept of invariant integrals in 1886. An invariant integral of a system of differential equations is an expression which maintains a constant value at all times. This development is linked to the three body problem in celestial mechanics. For the history of Poincaré and the three body problem, the reader may consult the book by (Barrow-Green 1997).

where q_i are the coordinates on $\mathbb{R}^{2n}$, t is the time, p_i are the momenta of the system and H is the energy.[42]

Erich Kähler applied exterior differentials in his paper *"Einführung in die Theorie der Systeme von Differentialgleichungen"* in 1934 in which the theory of exterior differential forms is applied to the problem of solving systems of partial differential equations.[43]

In Cartan's lectures of 1936, which were published in 1945, he gave some application of the differential forms to differential geometry. He introduced the algebra of the differential forms over a manifold.[44]

During the 1940s the theory of differential forms was used for the theory of fibre bundle by Charles Ehresmann, and this lead to the definition of the symplectic manifold discussed in chapter 5.

[42](Libermann 2005, p.194).
[43]See (Kähler 1934).
[44]See (Cartan 1945).

Chapter 4

Kähler form - Symplectic form

Introduction

This chapter deals with the genesis of the studies on Kähler manifolds. Kähler manifolds were defined by Erich Kähler in 1933 before symplectic manifolds were defined. Today, we know that the Kähler manifold are symplectic manifolds. Therefore, the genesis of Kähler manifolds must be considered as a part of the development of symplectic manifolds and as a part of the early history of symplectic geometry.

The genesis of symplectic forms took place when mathematicians, such as Erich Kähler, applied the theory of differential forms to complex manifolds. Kähler used this to find new invariants on complex manifolds.

This chapter deals with the reception of Kähler manifolds in the work of Eckmann and Guggenheimer at the end of the 1940s, and how the name Kähler manifold was established.

4.1 Erich Kähler

Erich Kähler was born on January 16, 1906 in Leipzig and died on May 31, 2000. Since his early school days, Erich Kähler was interested in science, in particular in astronomy and mathematics. During his time at the *Gymnasium*[1], Kähler read some lecture notes provided of Karl Weierstrass (*1815, †1897) by the school principal, who had realized Kähler's mathematical skills.[2]

Weierstrass's notes encouraged Kähler to write a "doctoral thesis" on mathematics. Kähler's so called first "doctoral thesis" was about fractional differentiation. He submitted his "doctoral thesis" to professor Otto Hölder at the University of Leipzig expecting to gain his PhD degree.[3] Hölder replied to Kähler that the requirements for obtaining a doctoral title include studies of at least six semester at the University.[4] Consequently, after Kähler finished his education at the *Gymnasium*, Kähler enrolled at the University of Leipzig in 1924 to study mathematics.

Four years later in 1928 Kähler obtained his doctoral degree under the supervision of Leon Lichtenstein.[5] His dissertation was titled *"Über die Existenz von Gleichgewichtsfiguren rotierender Flüssigkeiten, die sich aus gewis-*

[1]High school. The *Gymnasium*, is a secondary school in German speaking countries. It prepares the student for higher education.

[2]The lecture notes were about elliptic functions, Abelian function and Gauß surveying. (Berndt 2000, p. 179).

[3]Otto Hölder (*1857, †1937) was one of the developers of group theory and Galois theory. (See (Nicholson 1993).

[4](Berndt 2000, p. 179)

[5]Leon Lichtenstein (*1878, †1933) researched on differential equations, conformal mapping, and potential theory. He was also interested in theoretical physics, publishing research in hydrodynamics and astronomy. See (Anonymous 1934).

sen Lösungen des n-Körperproblems ableiten".[6]

Kähler went to Hamburg in 1929 to have an interview with Emil Artin looking for a position at the University of Hamburg. Artin was, as well, a former student of Lichtenstein.[7] After the interview, Artin proposed to Wilhem Blaschke, who also worked at the *Mathematische Seminar* in Hamburg, to hire Kähler as Blaschke's scientific assistant.[8] Blaschke hired Kähler in 1929, and in 1930 Kähler finished his *Habilitation* about integrals of algebraic differential equations.[9]

Although Blaschke recommended Kähler for a full professorship in Rostock in 1931, Kähler declined Blaschke's support and stayed in Hamburg. Kähler considered Hamburg as a good place for mathematical exchange.[10]

In the same year Kähler received a Rockefeller scholarship to spend a year in Italy, where he studied differential and algebraic geometry with Enriques, Castelnuovo, Levi-Civita, Serveri and Beniamino Segre.[11] In 1932 he published *"Forme differenziali et funzioni algebriche"* in which he applied the theory of differential forms to

> *[...], den eleganten Kalkül der symbolischen Differentialformen.*[12]
>
> (Kähler 1933, 173)

[6]See (Kähler 1928). The existence of equilibrium figures of rotate fluids that are derived from certain solutions of the n-body problem.

[7]Emil Artin (*1898, †1962) was an Austrian mathematician, who worked in number theory.

[8]Wilhem Blaschke (*1885, †1962) worked at that time on differential geometry.

[9](Berndt 2000, p. 179).

[10](Berndt 2000, p. 179).

[11]For more information about the Italian school of algebraic geometry the reader can consult (Guerraggio & Nastasi 2005).

[12][...], the elegant calculation of symbolic differential forms.

In the same year Kähler finished the article in which he introduced the notion of the Kähler metrics, which led to the notion of Kähler manifold: *"Über eine bemerkenswerte Hermitesche Metrik"*.[13] The article was published in 1933 by the University of Hamburg.[14]

In 1935 Erich Kähler went to the University of Königsberg for a professorship. The focus of his research was on mathematical physics. During his stay in Königsberg, he rewrote the Maxwell equations with the help of differential forms.[15]

In 1939, when the Second World War started, Kähler enrolled in the German navy. During the war, Kähler did not conduct mathematical research. He started again when he was a prisoner of war in France from 1945 to 1947 because as a naval officer he did not have to work. Therefore, Kähler resumed his work in mathematics using his contacts with French mathematicians such as Élie Cartan and his son Henri Cartan.[16] Kähler had met Élie Cartan in Moscow in 1934 during a conference about systems of differential equations.[17]

After two years in prison, Kähler went to Leipzig for a professorship in 1947. Later, he moved to Berlin which was a stopover before he returned to the University of Hamburg, where he stayed until his retirement.

After the war, Kähler's work focused on mathematical physics and no longer on differential geometry. Therefore, his contributions to differential geometry can be referred to the 1930s. As a consequence, some mathematicians who were working on differentials and Kähler geometry at the end of the 20th century, did not know that Erich Kähler was still alive in the 1980s as Jean-Pierre Bourgouignon mentioned:

[13] "On a remarkable Hermitian metric."
[14] See (Kähler 1933).
[15] (Berndt 2000, p. 180).
[16] (Berndt 2000, p. 180).
[17] (Berndt 2000, p. 179).

> Many of the specialist in the field of Kählerian Geometry living in the last third of the 20th century did not even imagine that Erich Kähler was still alive in 1980s, [...] the author of this note is one of them.
>
> (Bourguignon 2003, p.745)

Bourguignon mentioned, as well, that some of the mathematicians considered the Kähler geometry as a "classical" geometry, i.e. as the Riemannian Geometry and they thought that Kähler made his developments in geometry at the beginning of the 20th century and not in the 1930s.

4.2 Kähler metric over a Hermitian manifold

Kähler wrote the article *"Über eine bemerkenswerte Hermitesche Metrik"* during the time he was in Italy studying differential and algebraic geometry with the Italian mathematicians (Kähler 1933). In this article Kähler introduced the Kähler metric, Kähler potential and Kähler manifolds.

A Hermitian manifold is a complex manifold M with a Hermitian metric,

$$ds^2 = \sum h_{j,\bar{k}} dz_j d\bar{z}_k \tag{4.1}$$

where $H = (h_{j,\bar{k}})$ is a positive definite Hermitian matrix.

A Hermitian matrix is a complex matrix so that $H = \overline{H}^t$. It is positive definite, if and only if $\sum_{i,\bar{k}} h_{j,\bar{k}} dz_j d\bar{z}_k \geq 0$ and it is equal zero only for $z = 0$. The Hermitian matrix can be split into $H = A + iB$, A being a real symmetric matrix and B a real antisymmetric matrix. In the case that the manifold is a $\mathbb{C}$-vector space W, the Hermitian metric is a Hermitian bilinear form $h : W \times W \to \mathbb{C}$ where $h(v, w) = g(v, w) - i\omega(v, w)$ with $g : W \times W \to \mathbb{R}$

is a symmetric bilinear form and $\omega : W \times W \to \mathbb{R}$ the antisymmetric form. The real part g of the Hermitian metric h induce a Rimannian metric.

An almost complex structure J on a complex manifold M is a linear map

$$J : TM \to TM \tag{4.2}$$

such that $J^2 = -Id$. For all vector fields X, Y and an almost complex structure J on a complex manifold M, a Hermitian form defines on M a differential 2-form

$$\omega(X, Y) = g(X, JY) \tag{4.3}$$

which is called the canonical 2-form of the Hermitian manifold.

A Riemannian metric $g(\cdot, \cdot)$ is compatible with a corresponding complex structure J over the complex manifold M, if

$$g(JX, JY) = g(X, Y) \tag{4.4}$$

for all vector fields X, Y on M.

It can be said that a Hermitian manifold is a complex manifold M together with a compatible Riemannian metric $g = g(\cdot, \cdot)$.

The first and main result of Kähler's article (Kähler 1933) is the relation between a Hermitian metric and a 2-form. If the canonical 2-form is closed, then the Hermitian metric can be derived from the local existence of a smooth real function called Kähler potential. Nowadays, a closed 2-form over a complex manifold is called the Kähler form.

This relation was important for the studies of Hermitian manifolds, as Kähler noticed, and, therefore, he used the adjective *"bemerkenswert"* which means remarkable in the title of the article (Kähler 1933).

Nowadays, the fact that a manifold has a closed non-degenerate differential 2-form means that the manifold is a symplectic manifold. Kähler did not define the symplectic manifold, but he defined what later would be known as Kähler manifolds.

This does not mean that Kähler started the research on symplectic geometry; the results that Kähler presented were part of the development of differential geometry on complex manifolds. Today, the study of complex manifolds with a Hermitian metric and an associated closed 2-form, i.e. that the complex manifold has a Kähler metric, is part of the field named Kählerian geometry.[18]

In 1933 Kähler considered a complex manifold M with $2n$ real dimension. The complex manifold is covered by a family of systems of local coordinates $z_1, ..., z_n$. The relation between two such complex coordinate systems in the intersection of their existence domains is given by holomorphic functions.[19] Nowadays, the system of local coordinates are known as an Atlas.

In addition to the Hermitian metric, Kähler found its associated antisymmetric quadratic differential form

$$\omega = \sum h_{j,\bar{k}} dz_j \wedge d\bar{z}_k. \tag{4.5}$$

If the exterior derivative of a differential quadratic form is closed, it is equivalent to the local existence of a smooth real function U in a neighborhood of a point p_0 in the Hermitian manifold M. Therefore, the Hermitian matrix of (4.5) is

$$h_{j,\bar{k}} = \frac{\partial^2 U}{\partial z_j \partial \bar{z}_k}. \tag{4.6}$$

So, the Hermitian metric will be

$$ds^2 = \sum \frac{\partial^2 U}{\partial z_j \partial \bar{z}_k} dz_j d\bar{z}_k, \tag{4.7}$$

[18](Bourguignon 2003, p. 739)

[19]See (Kähler 1933)

and the differential form will be

$$\omega = \sum \frac{\partial^2 U}{\partial z_j \partial \bar{z}_k} dz_j \wedge d\bar{z}_k. \tag{4.8}$$

The function U is called the potential by Kähler. This potential is nowadays called Kähler potential.

Currently, if a quadratic differential form is closed and non-degenerate over a Hermitian manifold, the Hermitian metric is called Kähler metric, and the manifold is called Kähler manifold.

Every Kähler manifold is a symplectic manifold, but not every symplectic manifold is a Kähler manifold. This result was proved by William Paul Thurston (*1946, †2012) in 1976.[20]

Examples of a Kähler metric over a Hermitian manifold

Kähler gave two examples of a Kähler metric and remarked that this ariese in the theory of automorphic functions.[21] The first example is the kähler metric with potential

$$U = \sum_{\nu=1}^{n} k \log(1 - \sum z_\nu \bar{z}_\nu) \ \ (k \text{ constant}), \tag{4.9}$$

which is invariant under a discontinous group of transformations

$$z_i' = \frac{\alpha_{i0} + \alpha_{i1} z_1 + \ldots + \alpha_{in} z_n}{\alpha_{00} + \alpha_{01} z_1 + \ldots + \alpha_{0n} z_n}, \tag{4.10}$$

which transform the unit hypersphere

$$1 - \sum_{i=1}^{n} z_i \bar{z}_i = 0 \tag{4.11}$$

[20] See (Thurston 1976)

[21] See (Kähler 1933).

into itself.[22] Kähler called the transformations (4.10) as the “hyperfuchsian” transformations using the name that Picard gave to it.[23]

The second example is analogue to the first one. The metric satisfies the condition $d\omega = 0$ and has the potential

$$U = \sum_{i=1}^{n} k_i log(1 - z_i \bar{z}_i), \quad k_i \text{ constant.} \tag{4.12}$$

which is defined in the *”Einheitskreise”*

$$1 - z_i \bar{z}_i = 0. \tag{4.13}$$

The group of transformations that leaves the metric invariant is the automorphic group of the form

$$z_i' = \frac{\alpha_i z_i + \beta_i}{\gamma_i z_i + \delta_i}, \quad i = 1, ..., n, \tag{4.14}$$

which Kähler named the group of *hyperabelian* transformations.[24]

Kähler mentioned that the combination of the two metrics given in his examples may be used for the algebraic studies on automorphic functions.

Kähler found other properties if the differential form of the complex manifold is closed, for example that the metric (4.7) is an Einstein metric but these properties are not embedded in the development of symplectic geometry, but they are important for the study of Kähler geometry.[25] Currently, Kähler geometry is considered to be “at the crossroads between Riemannian, symplectic and complex geometry.”[26]

[22]See (Kähler 1933).
[23]See (Goldstein 2018, p. 15).
[24]See (Kähler 1933).
[25]See (Kähler 1933, p. 175).
[26](Bourguignon 2003, p. 740)

4.3 Reception of Kähler manifolds

After the publication of Kähler's article (Kähler 1933), the research on complex manifolds continued, as can be seen in the work of Élie Cartan (Cartan 1936). The interest in this closed 2-form and in Kähler manifolds was not immediate. In the 1940s the use of the theory of differential forms was combined with other mathematical tools such as the homology group and the de Rham cohomology.

The notion of homology had been introduced by Poincaré in 1895. Georges De Rham (*1903, †1990) treated the analogies between chains and forms in his dissertation (De Rham 1931), and the notion of cohomology without this name was introduced by Kähler in his publications (Kähler 1932, Kähler 1934).[27]

Also in the 1940s Kähler manifolds were researched and developed, by among others, William V. D. Hodge[28], Shiing-Shen Chern, André Weil[29], Beno Eckmann and Heinrich Guggenheimer.

At the end of the 1940s some mathematicians such as André Weil and Beno Eckmann started to call the Hermitian metric associated with a closed differential form a Kähler metric.[30]

During the 1930s and 1940s the research on complex manifolds continued

[27]See (Katz 1985, p.333)

[28]William Vallance Douglas Hodge was born on June 17, 1903 in Edinburgh and died on July 7, 1975. He was a Scottish mathematician and researched in the field of algebraic geometry. Hodge held the Lowndean Professorship of Astronomy and Geometry at Cambridge until his retirement in 1970.

[29]André Weil was born in Paris on May 6, 1906 and died on August 6, 1998. His work was fundamental in number theory and algebraic geometry. He was, as well, a student and friend of Carl Ludwig Siegel. He lived in the United States from 1941 to 1945 because of the persecution against the Jews in Europe.

[30]See (Weil 1947, Eckmann & Guggenheimer 1949*c*).

and Chern was one of the mathematicians who developed this research with Weil. Chern was a student of Kähler, and he used the Hermitian manifolds to prove the generalization of the Gauss-Bonnet Theorem.[31]

4.3.1 Eckmann and Guggenheimer

Beno Eckmann, with his student Heinrich Guggenheimer, contributed to the study of Kähler manifolds at the end of 1940s. He was born on March 31, 1917 in Bern and died on November 25, 2008 in Zürich. He attended the *Eidgenössische Technische Hochschule Zürich* (ETH) from 1935 to 1939, where he studied mathematics and obtained his Dr. Sc. math. in 1941 under the supervision of Professor Heinz Hopf.[32] He received his habilitation qualification in 1942 at the ETH. Eckmann's interest was in topology and its relation to algebra. In his article (Eckmann 1941), he worked on the homotopy theory of fibre spaces and in his following work (Eckmann 1942).

In 1942 he worked as a lecturer at the University of Lausanne, where Georges de Rham had been working as a professor since 1931, as well. Eckmann and de Rham exchanged mathematical ideas about homology theory in Lausanne.[33] In 1943 he received a professorship in Lausanne, and in May, 1943 he received a full professorship at the ETH Zürich. In his inaugural lecture (Eckmann 1944), he discussed the relation between topology and algebra. Later in 1947, Eckmann went to the IAS in Princeton for a research

[31] See (Wu 2008).

[32] Heinz Hopf was born on November 19, 1894, in Gäbschen (At that time Germany and nowadays Poland) and died on June 3, 1971, in Zillikon, Switzerland. He worked on topology. In his doctoral thesis he classified simply connected Riemannian 3-manifolds of constant curvature. In 1931 he went to ETH Zürich to take up the chair of Hermann Weyl.

[33] See (Eckmann 1992).

year. [34]He returned to Zürich in 1948 and started to work on what later would be known as Kähler manifolds with his student Heinrich Guggenheimer.

Heinrich Guggenheimer was born on July 21, 1924 in Nuremberg, Germany. Guggenheimer immigrated to Jerusalem in 1954, there he taught at the Hebrew University from 1954 to 1956, and later from 1956 to 1959, he taught at the Bar-Ilan University in Tel Aviv.[35] In 1959 he immigrated to the United States, where he taught at different universities, and continued his work on differential geometry, topology, and algebraic geometry.

Currently, Guggenheimer is Professor Emeritus of the Polytechnic Institute of Brooklyn.

4.3.2 Eckmann and Guggenheimer's work on Kähler manifolds

In 1949 Eckmann and Guggenheimer were working on Kähler manifolds.[36] The first reviews about this research were published in *Comptes Rendus de l'Académie de Sciences.* These reviews were Eckmann's work in progress (Eckmann 1952), and Guggenheimer's dissertation *"Über komplex-analytische Mannigfaltigkeiten mit Kählerscher Metrik"*, (Guggenheimer 1951*c*).[37]

Eckmann's and Guggenheimer's first two reports, are classified within the field of differential geometry because they dealt with local properties.[38] The last note (Eckmann & Guggenheimer 1949*c*) was classified within the field

[34]See (Hopf 2001).

[35]Heinrich Guggenheimer is Jewish and in 1992 he published with his wife the Dictionary, Jewish Family Names and Their Origins: An Etymological Dictionary.

[36]See (Eckmann & Guggenheimer 1949*a*, Eckmann & Guggenheimer 1949*b*, Eckmann & Guggenheimer 1949*c*).

[37]"On complex analytic manifolds with Kähler metric."

[38]See (Eckmann & Guggenheimer 1949*a*, Eckmann & Guggenheimer 1949*b*).

of topology because they researched the global properties.

> *Dans cette Note nous appliquerons à des variétés complexes closes les considérations purement locales de deux Note antérieures sur la structure complexe et la métrique hermitienne, et nous en déduirons des propiétés globales.*[39]
>
> (Eckmann & Guggenheimer 1949*c*, p.503)

Eckmann's and Guggenheimer's research during 1949 was on Hodge's theory about homology structures over algebraic manifolds.

In 1932 Hodge started to developed his theory about harmonic integrals. This work was published in 1941.[40] Hodge showed that in the Kähler manifolds the complex harmonic differential forms have special properties.[41]

Eckmann's and Guggenheimer's research simplified Hodge's theory, as mentioned in the introduction to their first note in 1949:

> *Dans cette Note et une Note suivante nous indiquerons une série de formules et de relations de caractère purement local concernant les formes différentielles dans un espace à metrique hermitienne sans torsion (1). Ces résultats seront utilisés dans des*

[39] In these notes, we apply to the closed complex manifolds the purely local consideration of the last two notes on the complex structure and the Hermitian metric, and we deduce the global properties.

[40] See (Hodge 1941). The book of Hodge was last re-issued in 1989 and the digital re-issue of it in 2008.

[41] These properties are not addressed in this work. The reader can consult Dieudonne for the development of Hodge's theory (Dieudonné 1989)

> *Notes ultérieures pour établir des propriétés topologiques qu'entraîne l'existence et simplifierons ainsi la théorie de Hodge (2) sur la structure homologique des variétés algébriques. Les détails des énoncés et des démonstrations seront publiés dans un Mémoire en préparation.*[42]
>
> (Eckmann & Guggenheimer 1949*a*, p.464)

A Hermitian metric without torsion is a Kähler metric. Eckmann and Guggenheimer used this term because a Hermitian manifold is a Kähler manifold if and only if the canonical Hermitian connection has no torsion. Later they defined what is a Hermitian metric without torsion as a Kähler metric. In the footnote (1) of (Eckmann & Guggenheimer 1949*a*), they specified that Hermitian metrics without torsion had been examined by E. Kähler.[43] In the footnote (2) they quoted the work of Hodge (Hodge 1941). The "future" notes mentioned in the quotation are (Eckmann & Guggenheimer 1949*b*, Eckmann & Guggenheimer 1949*c*), and the "memoir in preparation" is Guggenheimer's thesis (Guggenheimer 1951*c*), and Eckmann's publication (Eckmann 1952).

In (Eckmann & Guggenheimer 1949*a*), Eckmann and Guggenheimer gave the definitions of Hodge theory and its operators on the Euclidean space $\mathbb{R}^{2n}$ with $2n$ real variables, which is considered as the E^n space of n complex variables. They considered the Hermitian metric on the $\mathbb{R}^{2n}$ space with

[42] In this note, and in a following note, we indicate a series of formulas and relations with a purely local character concerning the differential forms in a space with Hermitian metric without torsion (1). These results will be used in future notes to establish the topological properties that involve the existence and will simplify Hodge's theory (2) about the homology structure of algebraic manifolds. The details of the theorems and the proofs will be published in a memoir in preparation.

[43] (Eckmann & Guggenheimer 1949*a*, p.464).

complex structure of which the associated 2-form to the Hermitian metric is a Kähler form. They defined the linear vector space of all differential p-forms.

The note (Eckmann & Guggenheimer 1949*c*), was the last note that Beno Eckmann and Heinrich Guggenheimer published together in the *Comptes Rendus de l'Académie de Sciences* in 1949. It is classified within the field of topology, as they announced in their first report. In this note they had applied to a complex manifolds local considerations of two previous notes and they have deduced the global priorities. So, on a complex manifold V of dimension $2n$, they showed that all the notions and results persented in the two previous notes, which are related to complex structure, have a global meaning, if they are applied to defined differential forms over the whole manifold V.

One of the results shown was the existence of an isomorphism between the linear space of all the harmonic forms of degree p with the cohomology group of a complex manifold V. As well as that the range of the linear space of all the harmonic forms of degree p is equal to the p-number of Betti of that complex manifold.

In this note they explicitely called the complex $2n$-dimensional oriented manifolds endowed with a Kähler metric as *variétés kählérienne*.[44]

Eckmann presented the results of his research published in 1949 with Guggenheimer in 1950, at the International Mathematical Congress (IMA) in Cambridge, Massachusetts, USA.[45] In the proceerdings of the congress of the IMA in 1952, Eckmann noticed that some of his results, such as the existence problem of a complex structure on a given manifold M, and the properties of M that are implied by a complex structure, overlap

[44](Eckmann & Guggenheimer 1949*c*, p. 504)

[45]The date of the publication of the IMA proceerdings was in 1952. See (Eckmann 1952).

with the presentation Charles Ehresmann gave at the same congress.[46]

In November 1950, Heinrich Guggenheimer delivered his dissertation about complex analytical manifolds with a Kähler metric, and in January 1951 he gave a talk in the topology colloquium directed by Ehresmann in Strasbourg with the title *"Varietes symplectiques"*. There he discussed his results on "Hodge theory". At the time Guggenheimer participated in Ehresmann's colloquium, Ehresmann had already defined symplectic manifolds. This development will be discussed in chapter 5.

[46]See (Eckmann 1952)

Chapter 5

Symplectic Manifolds

Introduction

The first definition of symplectic manifolds was given by Charles Ehresmann in 1950. It was given within the context of fibre bundles, which Ehresmann's and Jacques Feldbau's contribution to the development at the beginning of the 1940s.

First, Ehresmann showed that on an even-dimensional real differential manifold the existence of an almost complex structure is equivalent to the existence of a differential 2-form of rank $2n$ in all points over the manifold. Ehresmann showed that the four dimensional real sphere does not admit an almost complex structure. Therefore, he asked himself: which even-dimensional real differential manifolds admits an almost complex structure? This led to the definition of symplectic manifolds because the even-dimensional real manifold, which admits an almost complex structure, admits a 2-form in all points over the manifold, and if the form is closed and non-degenerated, it is a symplectic form. In this context the symplectic structure arose as a necessary and sufficient condition for the existence of an almost complex structure.

After Ehresmann's definition of symplectic manifolds, other mathematicians, such as Guggenheimer, noticed that Kähler manifolds are symplectic manifolds.[1]

5.1 Charles Ehresmann and Jacques Feldbau

Charles Ehresmann was born on April 19, 1905 in Strasbourg, France[2] and died on September 22, 1979.

In 1924 Ehresmann entered the *École Normale Supérieure.* After his graduation in 1928, he worked for a year as a teacher at a French Lycée in Rabat, Morocco.

From 1930 to 1931 he went to Göttingen to study with Hermann Weyl, and from 1932 to 1934 he went to the IAS in Princeton. In Princeton, he worked on his doctoral thesis on the topology of homogeneous spaces, and although Élie Cartan was in Paris, Cartan was his doctoral advisor.[3] Most probably Ehresmann exchanged ideas with Weyl.

After his graduation in 1934, Ehresmann worked in the *Centre Nationale de la Recherché Scientifique.*[4] During this period he made his first contributions to the topological properties of differentiable manifolds. He described the homology of classical types of homogeneous manifolds.

In July, 1935 Ehresmann was invited to participate in the Bourbaki

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[1] See (Guggenheimer 1951*a*).

[2] In 1905 Strasbourg was part of Germany, and after the first war the city returned to France.

[3] See (Dieudonné 1984, p.xxi).

[4] See (Ehresmann 1984, p.xix).

group.[5] In 1939 he became professor at the University of Strasbourg and stayed there until 1955 when he was appointed as professor for topology at the University of Paris. The chair for topology was created for Ehresmann.[6]

During Ehresmann's first year in Strasbourg, the Second World War broke out, and in September 1939 the city of Strasbourg was evacuated. The University of Strasbourg and its employees were relocated to Clermont-Ferrand, where Ehresmann continued his mathematical work.

His first doctoral student, out of 76 he supervised in total, was Jacques Feldbau in Strasbourg in 1939 and during this time the University of Strasbourg moved to Clermont-Ferrand.

From 1939 to 1943 Ehresmann and Feldbau developed the theory of fibre spaces.[7] The theory of fibre spaces was the context in which symplectic manifolds were defined.

5.1.1 Jacques Feldbau

Jacques Feldbau played an important role in the development of fibre spaces. He was born into a Jewish family in Strasbourg on October 22, 1914.[8] In 1943 Feldbau was arrested by the Gestapo and was sent to Auschwitz and died on April 22, 1945, in the Ganacker concentration camp a few days before it was liberated by the red army.[9]

As mentioned above, Feldbau was Ehresmann's first doctoral student and worked on the theory of fibre spaces, but he could not finish his doctoral

[5]See (Mashaal 2002, p. 6).

[6]See (Dieudonné 1984, p.xxi).

[7](Zisman 1999), (Audin 2010) and (Kosmann-Schwarzbach 2013).

[8]For a more detailed biography of Feldbau, the reader may consult the book "*Jacques Feldbau, Das Schicksal eines jüischen Mathematikers (1914-1945)*" (Audin 2010).

[9]See (Audin 2010).

thesis because he was taken as a prisoner by the Vichy regime[10] in 1943. Audin claimed it was possible that Feldbau had finished his doctoral thesis at that time, but he was not able to present his work. During his time in the Auschwitz concentration camp, he was not able to do mathematics, although he was in contact with Henri Cartan.[11]

During his life Feldbau published three notes in the *Comptes Rendus de l'Académie des Sciences* (Feldbau 1939, Ehresmann & Feldbau 1941, Ehresmann 1941), two on the *Bulletin de la Société Mathématique de France* (Laboureur 1941, Laboureur 1943), and there is a posthumous publication on the *"Séminaire Ehresmann, Topologie et géométrie différentielle"* (Feldbau 1958-60). The third publication in the *Comptes Rendus de l'Académie des Sciences* is only signed by Charles Ehresmann. In Michèle Audin's biography of Feldbau, she reproduces the manuscript of the note, on which the authors of the note were scratched-out and the name Charles Ehresmann was written on the edge of the page with pencil.[12] The scratched-out names were Charles Ehresmann and Jacques Feldbau. This was a consequence of the anti-semitic legislation by the Vichy regime, and the *Académie de Sciences* could not tolerate publishing something signed by a Jewish person during this time. Therefore, Feldbau adopted the pseudonymous Laboureur to be able to publish his results during the time of occupation.[13]

His last publication of 1958 was a posthumous publication, which is a

[10]The Vichy regime governed unoccupied France from 1940 to 1944. Its official name was French State, and it was an authoritarian state, which promulgated laws against Jews. It collaborated with Nazi-Germany and helped in the detention and deportation of many Jews and political opponents to concentration camps. For further information about the Vichy regime see (Jackson 2003).

[11]See (Audin 2010).

[12]See (Audin 2010, p.8).

[13]See (Audin 2010).

tribute to Feldbau by Ehresmann. Ehresmann printed the article in his seminar publication although many of the results contained in it were not new any more. In the introduction Ehresmann wrote a small review of Feldbau's contributions to the theory of fibre spaces and mentioned that the article, which was written by Feldbau in 1941, was recovered in 1945.[14]

5.2 Symplectic Manifolds

A symplectic manifold is an even-dimensional differentiable manifold M endowed with a closed non-degenerate 2-form. Let $2n$ be the dimension of M endowed with a symplectic form ω, then the top exterior power $\omega^n = \omega\wedge...\wedge\omega$ is a volume form. The volume form does not vanish because ω is non-degenerate. A volume form, is equivalent to ω not been degenerate. That any symplectic manifold has a volume form, implies that any symplectic manifold is an orientable manifold.[15]

Some examples of a symplectic manifold are:

- The vector space $\mathbb{R}^{2n}$, with coordinates $(x_1, ..., x_n, y_1, ..., y_n)$, endowed with the 2-form $\omega_0 = \sum dx_i \wedge dy_i$.
- The Kähler manifolds.
- The 2-sphere S^2 with its standard area form.
- Any oriented Riemann surface with its area form.
- The cotangent bundle over a Riemannian manifold with the Liouville form.

[14] See (Audin 2010, p. 12).
[15] See (Abraham & Marsden 1978, p. 165).

A Kähler manifold is a complex n-dimensional manifold endowed with a Kähler metric. A Kähler metric is a hermitian metric, and its associated 2-form is closed. Therefore, the 2-form is symplectic.

The 2-sphere embedded in $\mathbb{R}^3$ has a symplectic form defined by

$$\omega_p(u, v) := \langle p, u \times v \rangle \tag{5.1}$$

where $p \in S^2$, $u, v \in T_pS^2$ and "$\times$" is the vector product in $\mathbb{R}^3$. This is the standard area form of S^2 with total area 4π.

The last example is applied in classical mechanics.[16] The cotangent bundle is the phase space of a Hamiltonian dynamical system. The Riemannian manifold is the configuration space. The symplectic form is $\omega = -d\alpha$, where α is the canonical 1-form defined by

$$\sum_{i=1}^{n} \xi dx_i \tag{5.2}$$

and where the local cotangent bundle coordinates are (x, ξ). The (5.2) is called the Liouville form.

5.2.1 Fibre Spaces, Fibre Bundle

Modernly speaking, a fibre bundle is a structure (E, p, B, F) where E, B, F are topological spaces, and p is a continuous surjective map. E is called the total space, B the base space, F the (generic) fibre, p is the projection $p : E \to B$, and for any point $x \in B$, $p^{-1}(x)$ is called the fibre over x. The projection p must be a locally trivial map that means that for every point $x \in B$ there exists an open neighborhood $U \subset B$ such that there is a homeomorphism .

$$\varphi : p^{-1}(U) \longrightarrow U \times F$$

[16] See chapter 8.

so that the diagram

$$\begin{array}{ccc} \varphi : p^{-1}(U) & \longrightarrow & U \times F \\ p \searrow & & \swarrow \ proj_1 \\ & U & \end{array}$$

commutes. The $proj_1 : U \times F \to U$ is the natural projection onto the first component.

Fibre spaces were proposed and considered by Hopf, Herbert Karl Johannes Seifert[17] and Hassler Whitney[18] in their articles.[19]

In 1934 Whitney wrote the article "Sphere space" (Whitney 1934), in which he researched a sphere bundle which is a fibre bundle whose fibre is a n-sphere. There he introduced the terminology of base space, total space and the coordinate system for the fibre space, and he used, as well, the terminology tangent space of a differentiable manifold and the normal space of an embedded manifold.[20]

A year after Whitney's publication in 1935, Hopf published "*Über die Abbildungen von Sphären auf Sphären niedriger Dimension*" (Hopf 1935).[21] He gave a description of the families of fibrations where the fibres are spheres. The total space is a sphere, and the base spaces are projective spaces.[22]

From 1939 to 1943 Feldbau and Ehresmann were working on Whitney's theory of fibre bundle on spheres.

[17]Herbert Karl Johannes Seifert was born on May 27, 1907, in Bernstadt, Germany and died on October 1, 1996, in Heidelberg. He worked on topology.

[18]Hassler Whitney was born on March 23, 1907, in New York City and died on May 10, 1989, in Switzerland. He was a topologist and participated in the development of differential topology. See (Chern 1994).

[19]See (Seifert 1933, Whitney 1934, Hopf 1935, Whitney 1937).

[20](Zisman 1999, p.609).

[21]About transformations between spheres into spheres with low dimension.

[22]See (James 1999).

In Feldbau's work (Feldbau 1939), he extended the theory of fibrations, so that the total space and the base space are manifolds, and the fibres are compact manifolds. Feldbau introduced a family of homeomorphisms from the fibre over x in the base into the generic fibre, i.e.

$$H : F_x \to F.$$ [23]

In (Ehresmann & Feldbau 1941, Ehresmann 1941), Ehresmann and Feldbau gave a method to construct a fibre bundle. A bundle is constructed with a given base space B, a generic fibre F, the automorphismus of F are given by G of the generic fibre, and an open covering of the base spaces.

To defined an automorphisms of generic fibre F it is needed an open covering $U = (U_i)_{i\in I}$ of the base space B and homeomorphisms

$$\varphi_i : p^{-1}(U_i) \to U_i \times F. \tag{5.3}$$

For any given any $x \in U_i \cap U_j$ the equation

$$g_{ij}(x)(y) = \varphi_j \circ \varphi_i^{-1}, (x, y) \quad y \in F \tag{5.4}$$

defines an automorphism $g_{ij}(x)$ of F satisfying the relation

$$g_{jk} = g_{jk}(x) \circ g_{ij}(x) \ \text{ for } \ x \in U_i \cap U_j \cap U_k. \tag{5.5}$$

The group of automorphism G of F is called a structure group of the fibre bundle if G is a topological group and $g_{ij} \in G$ maps $g_{ij} : U_i \cap U_j \to G$ continuously. So, the collection (E, p, B, F, G) is called a fibre bundle with structure group G.[24]

Ehresmann discussed how to reduce the structure group G of the fibre bundle, when the reduction is a fibre bundle with a structure group G',

[23](Zisman 1999, p.611).

[24](Zisman 1999, p.605).

which is a subgroup of the original structure group G.[25] This problem can be reduced to the existence of a section in the associated fibre of typ G/G', which gave Ehresmann the necessary and sufficient conditions when an even dimensional manifold accepts an almost complex structure.[26]

Later in 1943 he gave the tangent bundle of a differentiable manifold.

> *A toute variété differentiable V^n on peut associer un espace fibré appelé variété des vecteurs tangents à V^n.*[27]
>
> (Ehresmann 1943, p. 628)

V^n is Ehresmann's notation for an n-dimensional manifold, and the fibre space is what is currently called tangent bundle. Ehresmann defined the differentiable manifolds using local charts and atlases. So, if the transition maps are differentiable, then the manifold is called differentiable; and he mentioned that if the base space of the tangent bundle is an n-dimensional a differentiable manifold M, the total space TM is a $2n$-dimensional manifold.

Ehresmann showed that the structure group of the tangent bundle is the linear group $GL(n, \mathbb{R})$. Therefore, the tangent bundle of a differentiable manifold is a collection $(TM, p, M, \mathbb{R}, GL(n, \mathbb{R}))$.

After giving the tangent bundle, he asked himself if the structural group of the tangent bundle has a reduction:

> *La structure $\breve{V}^n(V^n, \mathbb{R}^n, L, H)$ contient-elle des structures plus précises $\breve{V}^n(V^n, \mathbb{R}^n, L', H')$ où L' est un sous-groupe de L?*[28]

[25] See (Ehresmann 1942).

[26] See section (5.2.2).

[27] All differentiable manifolds V^n can be associated to a fibre space called the manifold of tangent vectors of V^n.

[28] Does the structure $\breve{V}^n(V^n, \mathbb{R}^n, L, H)$ have a more precise structure $\breve{V}^n(V^n, \mathbb{R}^n, L', H')$?

(Ehresmann 1943, p. 629)

Ehresmann used $\breve{V}$ to denote the tangent bundle, V^n the n-dimensional manifold, L the linear group $GL(n, \mathbb{R})$, and H the homeomorphism between the fibre $\mathbb{R}^n_x$ over x and the generic fiber $\mathbb{R}^n$. The reduction is a subgroup of $GL(n, \mathbb{R})$.

Ehresmann proved that the reduction of $GL(n, \mathbb{R})$ to the orthogonal group $O(n, re)$ is always possible and, consequently, the existence of a *"forme différentielle quadratique définie positive"* (2-form) over the differentiable manifold.[29] He showed that the reduction for non-definite signatures is not always possible but depends on the topology of the base space.[30]

5.2.2 Almost Complex Structure on $2n$ Dimensional Manifolds

In the middle of the 1940s Ehresmann continued his research on the tangent bundle of a differentiable manifold and the reduction of its structure group, which led him to research the almost complex structures over $2n$-dimensional orientable real differentiable manifolds.

In current terminology, an almost complex structure on a $2n$-dimensional real manifold M is a complex structure J on the tangent bundle TM, i.e. an almost complex structure on M is a section J of the bundle such that $J_x^2 = -Id_{T_xM}$ for every point x in the manifold M.[31]

Ehresmann found out that there exist orientable $2n$ real differentiable

[29](Ehresmann 1943, p. 629).

[30]The signature of a non-degenerate quadratic form $Q(x) = x_1^2 + ... + x_p^2 - x_{p+1}^2 - ... - x_n^2$ of rank n is the ordered pair $(p, q) = (p, n - p)$ of the numbers of positive or negative, respectively squared terms in its reduced form.

[31](Audin & Lafontaine 1994, p.42).

manifolds which do no admit an almost complex structure, for example the 2n dimensional sphere for $n \neq 1$ and $n \neq 3$.[32] Ehresmann proved this assertion for the 4 dimensional sphere in (Ehresmann 1947, sec. 10).

Ehresmann's first publication about the conditions for an even-dimensional real manifold to admit complex structure was (Ehresmann 1947). This publication is based on a conference at the SÉMINAIRE N. BOURBAKI in Paris at the beginning of 1947.[33]

He researched the conditions over a $2n$ real dimensional manifold M for the existence of a 2-form of rank $2n$ on every point of the manifold.

> *L'existence d'une telle forme différentielle est équivalente à l'existence, dans l'espace vectoriel tangent à V_{2n} au point x, d'une structure d'espace vectoriel complexe de dimension n dépendant d'une façon continue de x. Nous dirons que V_{2n} est muni d'une structure presque complexe lorsqu'on a défini, d'une façon continue par rapport à x, une structure d'espace vectoriel complexe dans l'espace tangent en x. L'existence sur V_{2n} d'une structure presque complexe est nécessaire, mais probablement non suffisante, pour qu'on puisse définir sur V_{2n} une structure analytique complexe, subordonnée à la structure différentiable réelle.*[34]
>
> (Ehresmann 1947, p. 133)

[32]See (Audin & Lafontaine 1994, McDuff & Salamon 1995).

[33]See the footnote (1) in (Ehresmann 1947).

[34]The existence of such a differential form is equivalent to the existence, in the tangent space of V_{2n} at a point x, of a structure of the n-dimensional complex vector space, which is dependent of a continuous function of x. We say that V_{2n} is equipped with an almost complex structure when one has defined, in a continuous way with respect to x, a structure of a complex vector space into the tangent space over x. For the existence of an almost complex structure over V_{2n} it is necessary, but may be it is not sufficient, to define over V_{2n} an analytical complex structure, subordinated to the differential real structure.

Ehresmann provided to the fibre $\mathbb{R}^{2n}$ a complex structure through a linear transformation $I : x \mapsto ix$ for x in $\mathbb{R}^{2n}$ so that $i(ix) = -x$. With a complex structure, the even-dimensional real space can be identified with the complex space $\mathbb{C}^n$.[35] On the complex space $\mathbb{C}^n$, the complex linear group $GL(n, \mathbb{C})$ is a subgroup of the linear group $GL(2n, \mathbb{R})$. The space of complex structures on a tangent space is

$$GL(2n, \mathbb{R})/GL(n, \mathbb{C}). \tag{5.6}$$

A section in the reduced bundle corresponds to an almost complex structure on the base manifold.[36]

He used a previous result that says if the reduced structure group is connected it determines an orientation of the manifold. Because $GL(n, \mathbb{C})$ is connected it determines an orientation of the manifold M.[37]

Ehresmann showed that the complex structures of $GL(2n, \mathbb{R})/GL(n, \mathbb{C})$ leaves the quadratic form

$$F(x, x) = x_1^2 + x_2^2 + ... + x_{2n}^2 \quad (x \in \mathbb{R}^{2n}) \tag{5.7}$$

invariant on a fibre $\mathbb{R}^{2n}$, which is equivalent to $F(x, Jx) = 0$ where J is an almost complex structure of M. As well, the complex structures leaves the Hermitian form

$$\Phi(z, z) = z_1\bar{z}_1 + z_2\bar{z}_2 + ... + z_n\bar{z}_n, \quad z \in \mathbb{C}^n, \tag{5.8}$$

invariant.[38]

[35](Ehresmann 1947, p.140).
[36](Ehresmann 1947, p.140).
[37](Ehresmann 1947, p.140).
[38](Ehresmann 1947, p. 141).

Ehresmann considers at first a fibre and comperes the reductions, i.e.

$$GL(2n, \mathbb{R}) \longrightarrow GL(2n, \mathbb{R})/GL(n, \mathbb{C})$$

and

$$O(2n, \mathbb{R}) \longrightarrow O(2n, \mathbb{R})/U(n, \mathbb{C})$$

with $U(n, \mathbb{C})$ the group of unitary trasnformations leaving (5.8) invariant.

He argues that the connected complement of J for $O(2n, \mathbb{R})/U(n, \mathbb{C})$ and $GL(2n, \mathbb{R})/GL(n, \mathbb{C})$ are "isomorphe", i.e. topologically equivalent. Therefore the reduction problems are equivalent.

As was mentioned before in (5.7), Ehresmann noticed that an almost complex structure leaves the quadratic form invariant because it is equivalent to

$$g(x, Jx) = 0$$

and the symmetric bilinear form g is associated to a skew-symmetric form ω of rank $2n$ if

$$\omega(x, y) = g(Jx, y).$$

The hermitian form is positive definite with respect to the complex structure defined by J.[39]

He showed that the existence of an almost complex structure on a $2n$-dimensional differentiable manifold M is equivalent to the existence of a differential 2-form of rank $2n$ in all points over the manifold M.

> *Pour qu'il existe sur V_{2n} une structure presque complexe ou presque hermitienne, il faut et il suffit qu'il existe sur V_{2n} une*

[39]See (Ehresmann 1950*a*, p.414).

forme différentielle extérieure de degré 2 *et en tout point de rang* $2n$.[40]

(Ehresmann 1947, p.141)

To prove that the 4-sphere S^4 does not admit an almost complex structure, Ehresmann started with a connected subgroup of $GL(2n, \mathbb{R})$, which left invariant the quadratic form called by him as Ω and the subgroup of $GL(n, \mathbb{C})$ that left the Hermitian form invariant, called by him as Ω'. He noticed that to determine the associated fibre boundle of a fiber bundle $E(V_{2n}, \mathbb{R}^{2n}, \Omega, H_1)$, which associated fiber boundle is $E(V_{2n}, \mathbb{C}^n, \Omega', H')$, is equivalent to determine the sections of the associated fiber bundle $E(V_{2n}, \Gamma_{(n)}, \Omega_\Gamma, H_\Gamma)$ where $\Gamma_{(n)}$ is the quotient space Ω/Ω'.[41]

Ehresmann proved that for the existence of an almost complex structure or a 2-form of rank 2 on a S^{2n} sphere, it is necessary and sufficient that a continuous map $\sigma' : S^{2n-1} \longrightarrow \Gamma_{(n)}$, in this case $\Gamma_{(n)} \approx S^3 \longrightarrow S^3 \times P^3\mathbb{R}$, has non-vanishing homotopy invariants. Therefore, the 4-sphere does not admit an almost complex structure.[42]

Currently, it is known that the only spheres that admit an almost complex structure are the 2-sphere and the 6-sphere.[43] This result was proved by Armand Borel (*1923, †2003) and Jean-Pierre Serre (*1926) in 1953.[44]

---

[40] For the existence on V_{2n} of an almost complex structure or an almost Hermitian, it is necessary and sufficient that there exists on V_{2n} an exterior differential form of degree 2 and in any point with rank $2n$.

[41] Ehresmann notation (Ehresmann 1947, p. 141).

[42] (Ehresmann 1947, p.143).

[43] For the proof see (Banyaga 1994).

[44] See (Borel & Serre 1953).

5.2.3 Ehresmann: Symplectic Manifolds

The current definition of a symplectic manifold was given in March 1950 at the SÉMINAIRE N. BOURBAKI at Ehresmann's conference *"Sur les variétés presque complexes"*.[45] He presented again the same work at the ICM in Cambridge, United States, in 1950.

Ehresmann started the report of his conference with some questions:

> *Etant donnée una variété topologique V_{2n}, de dimension $2n$, existe-t-il sur V_{2n} une structure analytique complexe? Plus abordable paraît la question suivante:*
> *Etant donnée une variété différentiable V_{2n}, existe-t-il sur V_{2n} une structure analytique complexe subordonnée à sa structure différentiable?*[46]
>
> (Ehresmann 1950*a*, p.412)

These questions were results from his previous work (Ehresmann 1947).

Ehresmann noticed that a consequence of his last work is the existence of an almost complex structure on an oriented manifold, which is equivalent to the existence of a differential exterior quadratic form ω of rank $2n$ over all points of the manifold M. This manifold is orientable if $\omega^n \neq 0$, where the orientation is defined by ω^n, ω^n is nowhere zero is equivalent to the non-degenerateness of the 2-form.

[45]See (Ehresmann 1948-1950*b*).

[46]Given a topological manifold V_{2n} of dimension $2n$, does there exist a complex analytical structure on V_2n? More proper seems the question: given a differentiable manifold V_{2n}, does a complex analytical structure subordinated to its differantiable structure exist on V_{2n}?

He defined an almost symplectic structure as a 2-form over an even dimensional differentiable manifold which is non-degenerate but not closed.

Ehresmann proved that an even-dimensional differentiable manifold M admits an almost complex structure if and only if M admits an almost symplectic structure. After this Ehresmann gave the definition of a symplectic manifold as when the manifold is endowed with a 2-form which is non-degenerate and closed:

> *Appelons variété symplectique une variété V_{2n} muni d'une forme fermée Ω telle que $\Omega^n \neq 0$ en chaque point.*[47]
>
> (Ehresmann 1950*a*, p. 415)

As mentioned at the beginning of section (5.2), $\omega^n \neq 0$ is equivalent to ω not being degenerat. Ehresmann never stated that the 2-form has to be closed in this article. But at that time, Paulette Libermann was his student, and she researched on the equivalence of structures over manifolds. The reason for preferring a closed 2-form is because the symplectic manifold is locally isomorphic to a symplectic vector space. Moreover, all symplectic manifolds with the same dimension are locally isomorphic to a symplectic vector space with the same dimension. This result is known as Darboux's theorem, which was stated by Libermann during her research for her PhD thesis.[48]

Ehresmann stated that if a symplectic manifold is compact i.e. without boundaries and with a finite number of neighborhood that covers all the manifold, then its even dimensional Betti numbers are different of zero.[49] As a remark, Ehresmann previous result about the 4-sphere which does not

[47] We call a symplectic manifold a manifold V_{2n} with a closed form Ω so that $\Omega^n \neq 0$ in all points.

[48] This development will be discussed in chapter 6.

[49] (Ehresmann 1950*a*, p. 150).

admit an almost complex structure implies, as well, that it does not admit a symplectic structure. In other words Ehresman stated that $2n$-dimensional compact manifolds with a trivial even cohomology group $H^{2k}(M;R)$, (with $k = 0, 1, ..., n$) such as spheres S^{2n} with $n > 1$, can never be symplectic.[50] The S^2 is the only sphere that admits a symplectic structure.[51]

5.2.4 The first reception and the relation between symplectic manifolds and Kähler manifolds

Henrick Guggenheimer immediately started to call an even-dimensional differential manifold endowed with a closed non-degenerate differential 2-form a symplectic manifold.

In January 1951, Guggenheimer presented his results about Hodge's theory over closed symplectic manifolds at Ehresmann's seminar in Strasbourg, the Colloque de Topologie de Strasbourg. As mentioned in section 4.3.2, the talk was called *"Varietes symplectiques"*.[52]

In April of the same year, Guggenheimer published a note in the *Comptes Rendus de l'Académie de Sciences* with the title *"Sur les variété qui possèdent une forme extérieure quadratique fermée"*.[53] There Guggenheimer claimed that a closed symplectic manifold has the property that its odd dimensional Betti numbers are even, as is the case for closed Kähler manifolds. But in 1956 Libermann noticed that the proof was incomplete.[54] It was not until 1976 that William Paul Thurston[55] denied Gugggenheimer's assertion and

[50] (Da Silva 2000, p. 7).

[51] See example (5.1).

[52] See (Guggenheimer 1951*b*).

[53] About the manifolds which have a exterior quadratic closed form. See (Guggenheimer 1951*a*).

[54] See (Libermann 1956).

[55] William Paul Thurston was born on October 30, 1946 in Washington, D.C and died

constructed some counterexamples in his publication *"Some Simple Examples of Symplectic Manifolds"*.[56] On the basis of these examples, Thurston proved, as well, that not every closed symplectic manifold has also a Kähler structure, and therefore, not all symplectic manifolds are Kähler manifolds too. A method to find symplectic manifolds that are not Kähler was developed by Robert Gompf[57] in 1995.[58]

5.2.5 Siegel's Half Space is a Symplectic Manifold

In 1957, Henri Cartan[59] (*1904, †2008) published his article entitled *"Ouvert Fondamentaux par le groupe modulaire"*[60] (Cartan 1957) where one of the results is that a Siegel's half-space is a Kähler manifold and therefore symplectic manifold.

As part of the context it should be mentioned that in the 1930s Cartan began working on the theory of functions in several variables, which coincided with Siegel's research interests. This can also be seen in the bibliography of (Cartan 1957). The bibliography of this article consists of three papers, the first one is written by his father Élie Cartan. In this case Henri Cartan cited his father's complete works, but the work that he used was (Cartan 1936),

on August 21, 2012 in Rochester, New York. In 1982 he received the Fields Medal for his research and contributions to the study of 3-manifolds.

[56]See (Thurston 1976).

[57]Robert Ernest Gompf was born in 1957 in the United States. He works at the University of Texas at Austin.

[58]See (Gompf 1995).

[59]Henri Cartan was the son of Élie Cartan. He was a founding member of the Nicolas Bourbaki collective (*1934) along with André Weil (*1906, †1998), Claude Chevalley (*1909, †1984), Jean Delsarte (*1903, †1992), Jean Dieudonné (*1906, †1992) and René de Possel (*1905, †1974).

[60]'Open Fundamentals by the modular group."

as mentioned earlier in chapter 2, in this work Élie Cartan obtained six irreducible bounded symmetric domains. The other two articles cited are the works of Carl Ludwig Siegel about modular functions of several variables (Siegel 1939) and the article "Symplectic Geometry" (Siegel 1943*b*).

In the article (Cartan 1957) the research on the generalization of hyperbolic geometry given by Siegel in (Siegel 1943*b*) is extended where one of the results is that the Siegel half-space is a Kähler manifold. The first thing Cartan did in his article (Cartan 1957) is very similar to the work of Siegel (Siegel 1943*b*). Cartan proved that the symplectic group $Sp(2n, \mathbb{R})$ operates on the Siegel's half-space. Then he searched for differential invariant. For this, as Siegel, he found the hermitian differential form over Siegel's half-space

$$ds^2 = tr(Y^{-1}dZ\, Y^{-1}\, d\overline{Z}). \tag{5.9}$$

To prove that it is a Kähler metric, Cartan considered the exterior quadratic form associated with the metric ds^2, i.e.

$$\omega = \frac{1}{2i} tr(Y^{-1}dZ \wedge Y^{-1}\, d\overline{Z}) = tr(Y^{-1}\, dY\, Y^{-1} \wedge dX) \tag{5.10}$$

where $Y^{-1}\, dX\, Y^{-1} = -dY^{-1}$ and so

$$\omega = tr(dX \wedge dY^{-1}) \tag{5.11}$$

and $d\omega = 0$.[61] Thus he proved that the metric is a Kähler metric. So Siegel's half-space is a Kähler manifold and therefore is a symplectic manifold.

Henri Cartan later dealt with other results that are part of Siegel's generalization of hyperbolic geometry.

The result that Siegel's half-space is a symplectic manifold gives the link between these two geometries that bear the same name but are not the same

[61] See (Cartan 1957, p.5-6).

field. In any case it can be said that Siegel's half space is a symplectic manifold and therefore can be considered as an object to be studied by the currently discipline understood as symplectic geometry.

Chapter 6

Darboux's theorem

Introduction

This chapter is about the development of Darboux's theorem in the context of the Pfaff problem at the end of the 19th century. In 1882 Darboux developed a theorem to reduce a system of linear differential equations. The current formulation is stated in terms of differential forms which is known as Darboux's theorem on Pfaffian forms or on 1-forms. In 1953, Paulette Libermann. developed the version of Darboux's theorem over an even-dimensional manifold, which shows that all symplectic manifolds with the same dimension are locally isomorphic to a symplectic vector space of the same dimension.

In this chapter Darboux's theorem on 1-forms will be referred as the *"classical Darboux theorem"* its historical formulation as Darboux (1882) and *"Darboux's theorem"* as used in symplectic geometry as Darboux's theorem.

6.1 Two versions of Darboux's theorem

6.1.1 The form used in symplectic geometry

An important theorem for symplectic geometry is Darboux's theorem, which is explained in the following.

For any point x of a symplectic manifold M, the tangent space T_xM is provided with a symplectic bilinear form, i.e. the tangent space T_xM is a symplectic vector space.

Darboux's theorem states that for every point x on the symplectic manifold M, there exists an open neighborhood U of x in M so that the symplectic form ω can be transformed into the canonical form, i.e.

$$\omega_0 = \sum_{i=1}^{n} dx_i \wedge dy_i.$$

Theorem 6.1.1. *For each point m of the symplectic manifold (M, ω) with dimension $2n$, there exists an open neighborhood U of m and a smooth transformation*

$$F : U \to \mathbb{R}^{2n}$$

with

$$F^*\omega_0 = \omega|_U$$

where ω_0 is the canonical symplectic form on $\mathbb{R}^{2n}$.

Therefore, there are no local invariants and all points on the symplectic manifold are equivalent in symplectic geometry. This differs from Riemannian geometry, where the metric, which is a bilinear positive definite symmetric form, can always be brought to the canonical form for any given point x of a Riemannian manifold. However, it is not always possible in a neighborhood U around the point x.

6.1.2 The classical form

Before classical Darboux's theorem, it is necessary to give some definitions and results at first.

A Pfaffian form is one differential form

$$\alpha = a_1(x)dx_1 + ... + a_n(x)dx_n, \tag{6.1}$$

defined on an open subset U of a manifold M.[1]

A Pfaffian equation has the form

$$\alpha \equiv a_1(x)dx_1 + ... + a_n(x)dx_n = 0, \tag{6.2}$$

where x is an element of a domain D which is a subset of $\mathbb{R}^{2n}$, α is a Pfaffian form and the functions $a_i(x)$ $i = 1, ..., n$ are real-value functions.[2]

An integral manifold of the Pfaffian equation is a k-dimensional manifold $M \subset \mathbb{R}^{2n}$, where $k \geq 1$ and the manifold is of class C^1, if $\alpha \equiv 0$ on the manifold M. If there is one and only one integral manifold of maximum possible dimension $n-1$ through each point of the domain, then the Pfaffian equation is said to be completely integrable. The necessary and sufficient condition for the Pfaffian equation to be completely integrable is that

$$d\alpha \wedge \alpha \equiv 0. \tag{6.3}$$

This is known as Frobenius' theorem.[3]

The next result was proved by Darboux in 1882. This is discussed in the section 6.2.3. The following theorem is what we call the classical Darboux theorem and is currently it is called Darboux's theorem on Pfaffian equation and it states in modernized notation:

[1](Aa.Vv. 1991*b*, p. 147).
[2](Aa.Vv. 1991*b*, p. 145).
[3](Aa.Vv. 1991*b*, p. 145).

Theorem 6.1.2. *Let α be a 1-form on an n-dimensional manifold M, so that $d\alpha$ has rank p everywhere. If $\alpha \neq 0$ and $\alpha \wedge (d\alpha)^p \equiv 0$, about every point it can be found coordinates $x_1, ..., x_{n-p}, y_1, ..., y_p$ such that*

$$\alpha = x_1 dy_1 + ... + x_p dy_p. \tag{6.4}$$

If $\alpha \wedge (d\alpha)^p \neq 0$ everywhere, about every point it can be introduce coordinates $x_1, ..., x_{n-p}, y_1, ..., y_p$ so that

$$\alpha = x_1 dy_1 + ... + x_p dy_p + dx_{p+1}.\text{[4]} \tag{6.5}$$

This last theorem is the one that Darboux stated in 1882 in the context of systems of linear differential equation in his work about the Pfaffian problem. and which we will discuss in the next section.

6.2 A look back to the late 19th century

6.2.1 Pfaffian's Problem

As mentioned in chapter 3, the Pfaffian problem is a problem which has its origins in the theory of partial differential equations. Because the Pfaffian problem was the basis for Darboux to state the "classical" Darboux's theorem, this section will provide a short review of Pfaffian's problem.

At the end of the 18th century Lagrange provided a method of integrating a general first order partial differential equation

$$F\left(x_1, ..., x_n, u, \frac{\partial u}{\partial x_1}, ..., \frac{\partial u}{\partial x_n}\right) = 0 \tag{6.6}$$

$u = \varphi(x_1, ..., x_n, C_1, ..., C_n)$ where C_i are arbitrary constants in two independent variables.[5] But Lagrange did not succeed in the case when there are

[4]See (Sternberg 1964).

[5](Hawkins 2005, p.386).

more than two independent variables.[6] In 1815 Pfaff solved the problem and even more he solved how to integrate a non-linear partial differential equation with $n > 2$ variables.[7] To integrate a non-linear partial differential equation with $n > 2$, Pfaff worked on a problem that had been discussed by Euler and Monge[8] at the end of the 18th century.[9] Euler and Monge discussed if there exists an integral function φ such that $d\varphi = \alpha$, where $\alpha = 0$ is a Pfaffian equation (6.2).[10] Monge found out that there is no single integral equation $\varphi = C$ equivalent to the given differential equation, but it could be two or more integral equations and together they form an integral equivalent to the differential equation.[11]

Pfaff considered Monge's statement that two or more simultaneous equations could be an integral of the Pfaffian equation (6.2).[12] Pfaff stated that an equation,

$$\alpha \equiv a_1(x)dx_1 + ... + a_n(x)dx_n = 0,$$

in any number of variables $x = (x_1, ..., x_n)$ is possible to integrate through a change of variables $x_i = \psi_i(y_1, ..., y_n)$, $i = 1, ..., n$ such that

$$\alpha = b_1(y)dy_1 + ... + b_p(y)dy_p, \tag{6.7}$$

where $p = \frac{n}{2}$ if n is even and $p = \frac{n+1}{2}$ if n is odd.[13] Pfaff's result provides the existence of solutions to an equation (6.2).[14] At the beginning of the 19th

[6] (Hawkins 1991, p. 203).

[7] (Hawkins 2005, p. 387).

[8] Gaspard Monge (*1746, †1818) was a French mathematician. He invented the descriptive geometry and founder of the École Polytechnique.

[9] (Katz 1985, p.324).

[10] The Pfaffian equation was called as a total integral equation. See (Hawkins 1991, Hawkins 2005).

[11] (Katz 1985, p.324).

[12] (Hawkins 1991, p. 203).

[13] (Hawkins 1991, p. 203).

[14] (Hawkins 1991, p. 204).

century this was known as the Pfaffian problem. A solution to a Pfaffian equation (6.2) is established if $y_i = \varphi_i(x_1, ..., x_n)$, $i = 1, ..., n$, and $C_1, ..., C_p$ are constants, so the solution are the p equations

$$\varphi_j(x_1, ..., x_n) = C_j, \quad j = 1, ..., p,$$

which implies $dy_j = 0$ for all $j = 1, ..., p$.[15] Pfaff was not able to completely prove his result. It was Jacobi who proved it in 1827.[16]

At the end of the 19th century mathematicians understood the problem of Pfaff as challenge to find a suitable change of variables so that the Pfaffian equation (6.1) could be expressed through a minimal number of variables, i.e. to find a canonical form of the Pfaffian form α.[17] To solve this problem mathematicians tried to find the canonical expression for a partial differential equation (6.1) by trying to find integrals. To find integrals of the Pfaffian equation means to find a finite number of relation $\varphi_j(x_1, ..., x_n) = 0$ with $j = 1, ..., p$, between the independent variables $x_1, ..., x_n$ so that the Pfaffian equation vanishes as a consequence of the $2p$ equations

$$\varphi_j(x_1, ..., x_n) = 0 \quad \text{and} \quad d\varphi_j = \sum_{k=1}^{n} \frac{\partial \varphi_j}{\partial x_k} dx_k = 0 \quad j = 1, ..., p. \tag{6.8}$$

Geometrically, $\varphi_j(x_1, ..., x_n) = 0$ can be interpreted as a condition for a hypersurface. In this way the integrals that are found define, in modern terminology, an integral manifold of dimension $n-p$, given by the intersection of the p hypersurfaces.[18]

The Pfaffian problem was treated by Frobenius and Gaston Darboux in the second half of the 19th century. As mentioned in chapter 3.2.2, the

[15] (Hawkins 1991, p. 204).
[16] (Katz 1985, p. 324).
[17] See (Cogliati 2011, Hawkins 2005, Katz 1981).
[18] (Cogliati 2011, p.399, 400).

Pfaffian problem was, as well, treated by Élie Cartan in 1899 and this lead to the development of the theory of differential forms. [19]

Darboux's work of 1882 led to the theorem in terms of systems of linear differential equations.

6.2.2 Gaston Darboux

Jean Gaston Darboux was born on August 14, 1842 in Nimes, France and died in Paris, France on February 23, 1917. Darboux took the entrance examination for the *École Polytechnique* and for the *École Normale Supérieure*. He decided to begin his studies at the *École Normale Supérieure*, which at that time started to gain prestige and to offer high teaching standard under the direction of Louis Pasteur.

He obtained his doctorate in 1866 with his thesis *"Sur les surfaces orthogonales"*.

After his graduation, Darboux worked as a teacher in Paris at the *Lycée Louis-le-Grand* until 1872.[20]

From 1872 to 1881 he taught at the *École Normale Supérieure* and from 1872 to 1878 he substituted Liouville's (*1809, †1882) post at the Sorbonne.[21]

In 1878 Darboux replaced his doctoral advisor Michel Chasles (*1793, †1880) at the Sorbonne. Chasles held the chair of higher geometry at that time. After Chasles's death in 1880, Darboux succeeded him at the Sorbonne. Darboux held this chair until his death in 1917.[22]

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[19] See chaper 3.

[20] The *Lycée Louis-le-Grand* has been a prestigious secondary school since its foundation in 1550. See (Dupont-Ferrier 1925).

[21] (Lützen 1990, p. 248).

[22] See (Picard 1917, Croizat 2016).

Darboux's contributions were in geometry of surfaces and infinitesimal geometry. He wrote four volumes about infinitesimal geometry *"Leçons sur la théorie général des surfaces et les applications géométriques du calcul infinitésimal"* which were published between 1887 and 1896.[23]

6.2.3 The classical theorem of Darboux

In 1882 Darboux published his solution for Pfaff's Problem. Earlier in 1877, Frobenius published a solution for the problem of Pfaff. Although Frobenius had solved the problem of Pfaff, six years later Darboux published his own solution.[24] Darboux knew about Frobenius' publication in 1876. This can be read in a footnote on Darboux's publication:

> *La première Partie de ce travail a été écrite en 1876 et communiquée à M. Bertrand, qui enseignait alors au Collège de France la théorie des équations aux dérivées partielles. M. Bertrand a bien voulu exposer la méthode que je lui avais soumise, dans sa première leçon de janvier 1877.*
>
> *Quelque temps après paraissait dans le* Journal de Borchardt *un beau Mémoire de M. Forbenius qui porte d'ailleurs une date antérieure á celle de janvier 1877 (septembre 1876) et où ce savant géomètre suit une marche assez analogue á celle que j'ai communiquée à M. Bertrand, en ce sens qu'elle repose sur l'emploi de invariants et du covariant bilinéaire de M. Lipschitz. En revenant dans ces derniers temps sur mon travail, il m'a semblé que mon exposition était plus affranchie de calcul et que, par suite de l'importance que la méthode de Pfaff est appelée à prendre, il*

[23]See (Darboux 1887, Darboux 1889, Darboux 1894, Darboux 1896).

[24]See (Frobenius 1877).

y avaint intérèt à la faire connaître.[25]

(Darboux 1882, p.15)

With this footnote, Darboux justified the publication of his solution in 1882. He expressed that his solution of Pfaff's Problem does not use the theory of partial differential equations. Instead, he used the properties of invariants.

Darboux's solution of Pfaff's Problem starts with the following

$$\Theta_d = a_1(x)dx_1 + ... + a_n(x)dx_n. \tag{6.9}$$

which is called *expression differentielle* by Darboux. Currently, the *expression differentielle* may be reinterpreted as a differential form, but as it was mentioned before the theory of differential forms was developed, Darboux stated that his solution was for Pfaff's problem. The differential expression (6.9) has a "direction" expressed by $dx_1, ..., dx_n$. Darboux constructed a pfaffian system. For this he needed a differential in another "direction" δ so he used the differential expression

$$\Theta_\delta = a_1(x)\delta x_1 + ... + a_n(x)\delta x_n. \tag{6.10}$$

Darboux differentiated the differential expressions crossing the differential d

[25]The first part of this work was written in 1876 and communicated to Mr. Bertrand, who taught at the *Collège de France* the theory of partial differential equations. Mr. Bertrand provided the method, that I submitted, in his first lecture on January 1877. Some time later, in the *Journal de Borchardt*, a beautiful memoir by Mr. Forbenius appeared, which showed an earlier date than that of January 1877 (September 1876), and in which the learned geometer followed a manner, which fits quite well to the work I communicated to Mr. Bertrand, in the sense that this article responded to the application of invariants and bilinear covariants of M. Lipschitz. Upon returning recently to my work, it seemed to me that my exposition was more calculation-free and, in view of the importance, that the method of Pfaff has assumed that it would be of interest to make it known.

and δ and subtracted them, leading to the next expression

$$\delta\Theta_d - d\Theta_\delta = \sum_{i<k} a_{ik}(dx_i\delta x_k - dx_k\delta x_i), \tag{6.11}$$

where

$$a_{ik} = \frac{\partial a_i}{\partial x_k} - \frac{\partial a_k}{\partial x_i},$$

$a_{ik} + a_{ki} = 0$, and $a_{ii} = 0$. The expression (6.11) is a system of $\frac{n(n-1)}{2}$ terms.

He made a change of variables in the expression (6.9), with $x_i = \psi(y_1, ..., y_n)$ given by

$$dx_i = \sum_k \frac{\partial \psi_i}{\partial y_k} dy_k$$

and the expression Θ_d has the form

$$\Theta_d = \sum_{i=1}^{n} b_i(y)\, dy_i. \tag{6.12}$$

Hence,

$$\delta\Theta_d - d\Theta_\delta = \sum_{i<k} \; b_{ik}\; dy_i \;\delta y_k, \tag{6.13}$$

$$b_{ik} = \frac{\partial b_i}{\partial y_k} - \frac{\partial b_k}{\partial y_i},$$

where $b_{ik} + b_{ki} = 0$, and $b_{ii} = 0$, he proved that expressions (6.11) and (6.13) are equal, i.e.

$$\sum a_{ik}\, dx_i\, \delta x_k = \sum b_{ik}\, dy_i\, \delta y_k, \tag{6.14}$$

because

$$b_{ik} = \sum_{l<m} a_{lm} \left(\frac{\partial \psi_l}{\partial y_i}\frac{\partial \psi_m}{\partial y_k} - \frac{\partial \psi_l}{\partial y_k}\frac{\partial \psi_m}{\partial y_i} \right)$$

Through this equation Darboux showed the invariant properties by a change of variables. Using this property Darboux showed the equivalents of two systems of linear differential equations. He started with the system associated to the differential expression (6.9)

$$\begin{aligned} a_{11}(x)dx_1 + ... + a_{1n}(x)dx_n &= \lambda a_1(x)dt \\ \vdots \qquad\qquad & \qquad \vdots \\ a_{n1}(x)dx_1 + ... + a_{nn}(x)dx_n &= \lambda a_n(x)dt \end{aligned} \tag{6.15}$$

where λ "would be a quantity, which could arbitrarily chosen 0, a constant or a function of t depending on the case".[26] The system (6.15) is equivalent to

$$\begin{aligned} b_{11}(y)dy_1 + ... + b_{1n}(y)dy_n &= \lambda b_1(y)dt \\ \vdots \qquad\qquad & \qquad \vdots \\ b_{n1}(y)dy_1 + ... + b_{nn}(y)dy_n &= \lambda b_n(y)dt. \end{aligned} \tag{6.16}$$

Darboux replaced the system (6.15) with the equation

$$\delta\Theta_d - d\Theta_\delta = \lambda\Theta_\delta dt. \tag{6.17}$$

To find the canonical form for the Pfaffian system, i.e. to reduce the number of variables of the system so the new system is equivalent to the original one, Darboux established the minimal number of differential terms into which the differential expression Θ_d can be transformed. He did this by looking at the rank of the system and by changing the variables x to y and z. Thereby, Darboux stated and proved the theorem

Theorem 6.2.1. *Une forme Θ_d peut toujours être ramenée à l'un des deux types suivants:*

$$dy - z_1dy_1 - z_2dy_2 - ... - z_pdy_p \tag{6.18}$$

$$z_1dy_1 + z_2dy_2 + ... + z_pdy_p \tag{6.19}$$

où les fonctions $y, y_1, ..., z_k$ constituent un système de variables indépendantes,

[26](Darboux 1882, p.19).

c'est-à-dire sont des fonctions indépendantes de toutes les variables qui entrent dans la forme Θ_d.[27]

(Darboux 1882, p.26)

The next problem solved by Darboux was the value of p. To solve this Darboux used the last theorem, knowing that the differential expression Θ_d is invariant under a change of variables y, z. First, he assumed that the differential expression can be reduced to the case (6.18) of the theorem (6.2.2), i.e. $\Theta_d = dy - z_1dy_1 - z_2dy_2 - ... - z_pdy_p$. So, the equation (6.17) becomes

$$\delta\Theta_d - d\Theta_\delta = dz_1\delta y_1 - dy_1\delta z_1 + ... + dz_p\delta y_p - dy_p\delta z_p = \lambda\Theta_\delta dt. \qquad (6.20)$$

where

$$\delta\Theta_d = \delta dy - \delta z_1dy_1 - \delta z_2dy_2 - ... - \delta z_pdy_p, \qquad (6.21)$$

$$d\Theta_\delta = d\delta y - dz_1\delta y_1 - dz_2\delta y_2 - ... - dz_p\delta y_p \qquad (6.22)$$

and

$$\lambda\Theta_\delta dt = \lambda(\delta y - z_1\delta y_1 - z_2\delta y_2 ... - z_p\delta y_p)dt. \qquad (6.23)$$

Because

$$\delta\Theta_d - d\Theta_\delta = dz_1\delta y_1 - dy_1\delta z_1 + ... + dz_p\delta y_p - dy_p\delta z_p,$$

and

$$\lambda\Theta_\delta dt = \lambda\delta ydt - \lambda z_1\delta y_1dt - \lambda z_2\delta y_2dt - ... - \lambda z_p\delta y_pdt,$$

[27] One form Θ_d can always be reduced to one equation of the following ones:

$$dy - z_1dy_1 - z_2dy_2 - ... - z_pdy_p$$

$$z_1dy_1 + z_2dy_2 + ... + z_pdy_p$$

where the variables $y, y_1, ..., z_k$ constitute a system of independent variables, i.e. they are functions independent of all variables that enter into the form Θ_d.

the system (6.15) associated to the differential expression becomes

$$\begin{aligned} dy_1 = 0,\ dz_1dt &= -\lambda z_1 dt, \\ \vdots \qquad & \qquad \vdots \\ dy_p = 0,\ dz_pdt &= -\lambda z_p dt, \\ 0 &= \lambda dt. \end{aligned} \tag{6.24}$$

This new system has only solutions for $\lambda = 0$, and the system is reduced to a system of $2p$ equations (6.24), which is completely integrable.[28] Here the change of variables can be given by $y_i = \phi_i(x)$, and $z_i = \psi_i(x)$, when the solutions are $\phi_i = C_i$ and $\psi_i = D_i$.[29]

If the differential expression Θ_d can be reduced to the case (6.19), i.e.

$$\Theta_d = z_1dy_1 + ... + z_pdy_p, \tag{6.25}$$

then

$$\delta\Theta_d = \delta z_1dy_1 + ... + \delta z_pdy_p \tag{6.26}$$

and

$$d\Theta_\delta = dz_1\delta y_1 + ... + dz_p\delta y_p. \tag{6.27}$$

Because

$$\delta\Theta_d - d\Theta_\delta = \delta z_1dy_1 - dz_1\delta y_1 + ... + \delta z_pdy_p - dz_p\delta y_p,$$

and

$$\lambda\Theta_\delta dt = \lambda z_1\delta y_1dt + ... + \lambda z_p\delta y_pdt \tag{6.28}$$

the system (6.15) is equivalent to the equations

$$\begin{aligned} dy_1 = 0,\ dz_1(y)dt &= \lambda z_1 dt, \\ \vdots \qquad & \qquad \vdots \\ dy_p = 0,\ dz_p(y)dt &= \lambda z_p dt, \end{aligned} \tag{6.29}$$

[28](Darboux 1882, p. 28).

[29](Hawkins 2005, p.422).

In this case λ is not zero, and it is a constant.[30] The equations admit $2p-1$ independent integrals of t, and the solutions are $y_i = C_i$ $(i = 1, ..., p)$ and $z_i/z_1 = C'_i$ $(i = 2, ..., p)$ upon elimination of t.[31] So, Darboux was able to state (Darboux 1882, p.27-28) the theorem known as Darboux's theorem (classical Darboux's theorem), which can be rewritten as:

Theorem 6.2.2 (Darboux (1882))**.** *If the system of differential equations [6.15] associated to a differential expression Θ_d has solutions only when $\lambda = 0$, a variable change $x \to y, z$ is possible so that*

$$\Theta_d = dy - z_1 dy_1 - ... - z_p dy_p \tag{6.30}$$

and hence the number of distinct equations to which the system [6.15] is reduced if $\lambda = 0$ is $2p$ and the number is $2p+1$ for $\lambda = 1$.
If the system [6.15] has solutions for $\lambda \neq 0$, the differential expression Θ_d may be put in the form

$$\Theta_d = z_1 dy_1 + ... + z_p dy_p \tag{6.31}$$

and the number of equations to which system [6.15] reduces is $2p$.[32]

Darboux's solution deduced the canonical form of a Pfaffian equation. This later helped Élie Cartan to develop the theory of differential forms in 1899 (See chapter 3). After Cartan's theory about differential forms, the differential expression can be seen as a differential form.

Darboux's theorem of 1882 is currently formulated in terms of differential forms over a n-dimensional manifold, see Darboux's theorem (6.1.2). It leads to the same reduced forms as given by Darboux, although the distinguishing criteria are stated in differential forms. A special case is the following:

[30](Darboux 1882, p.28).
[31](Darboux 1882, p.28).
[32](Hawkins 2005, p.422).

If ω is a closed non-degenerate 2-form on an n-dimensional manifold M, with n even, then in a neighborhood of each point of the manifold M, it is possible to find a 1-form α so that $d\alpha = \omega$.[33] If α satisfies that $d\alpha$ has exactly rank p everywhere on the manifold M and $2p = n$, then coordinates $x_1, ..., x_{n-p}, y_1, ..., y_p$ can be locally introduced so that

$$\omega = dx_1 \wedge dy_1 + ... + dx_p \wedge dy_p.^{34} \tag{6.32}$$

The coordinates are canonical coordinates for the symplectic form. Darboux's theorem in symplectic geometry was first stated in 1953 by Paulette Libermann. At the time she was working on her PhD thesis on the local equivalence structures over manifolds.[35]

6.3 The perspective of symplectic manifolds

6.3.1 Paulette Libermann

Paulette Libermann was a student of Élie Cartan and wrote her thesis under the supervision of Charles Ehresmann.

Paulette Libermann was born on November 14, 1919 in Paris and died on July 10, 2007. She was born into a Jewish family with Russian and Ukranian roots.

In 1939 Paulette Libermann entered the *École Normale Supérieure de Jeunes Filles de Sèvres.* At the time Paulette Libermann entered this school, it was run by Eugénie Cotton, a physicist and communist militant, who wanted to raise the level of the school to that of the *École Normale Supérieure of*

[33] This is possible by Poincaré lemma, see (Sternberg 1964, p. 121).
[34] (Sternberg 1964, p.140).
[35] See (Libermann 1953*b*).

Ulm.[36] For this reason, Eugénie Cotton invited Élie Cartan, Andre Lichnerowicz and Jacqueline Ferrand (*1918, †2014) to teach at the *École Normale Supérieure de Jeunes Filles de Sèrves*.

At the end of 1940 Paulette Libermann was studying for the *agrégation*.[37] Libermann was not able to take the examination because the Vichy Regime, which controlled the country during the German occupation of France, established laws that forbade Jewish persons to practice certain professions, one of which was teaching. This meant that Paulette Libermann could not take the *agrégation*. Nevertheless, Eugénie Cotton managed that her Jewish students received a scholarship for one year in the school. During this year Paulette Libermann started her research on mathematics under the supervision of Élie Cartan, and she obtained the *"diplôme d'etudes supérieures de mathématiques"* in 1942.

In the same year the Jewish persecution was intensified in France, and Paulette Libermann's family fled to Lyon.[38] During that period Libermann's family had to live clandestine, using fake names. To survive, Paulette Libermann gave private lessons.

Paulette Libermann survived the Jewish persecution and France was liberated in 1944. She was able to return to Paris to the *École Normale Supérieure de Jeunes Filles de Sèrves* where she finally took the *agrégation* examination.

Paulette Libermann became a secondary school teacher. First, she was sent to the south of Lille, but she went to Strasbourg to teach at a secondary

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[36] In 1987, the *École Normale Supérieure de Jeunes Filles de Sèrves* and the *École Normale Supérieure of Ulm* were combined.

[37] In France, the *agrégation* is a competitive examination for some positions in the public education system.

[38] See (Kosmann-Schwarzbach 2013).

school in 1945.

In Strasbourg Paulette Libermann established contact with Charles Ehresmann, as suggested by Élie Cartan, and in 1947 Paulette Libermann became Ehresmann's student.

Parallel to her work as a secondary teacher, she started to work on her thesis on Cartan's equivalence problem, and she finished it in 1953.[39] Cartan's equivalence problem is to determine whether two geometrical structures are equal.

6.3.2 Equivalence between differential forms

In 1948 Libermann exposed on a $2n$-dimensional manifold M endowed with a 2-form ω the conditions to reduce this 2-form ω to its equivalent canonical form. This first result appeared in the *Comptes Rendus de l'Académie de Sciences* in 1948, which her advisor Ehresmann coauthored.[40] They studied the 2-forms with rank $2n$ on a manifold that is $2n$-dimensional and $n-1$-times differentiable. They wrote the 2-form as

$$\omega = \alpha_1 \wedge \overline{\alpha_1} + ... + \alpha_n \wedge \overline{\alpha_n} \tag{6.33}$$

where α_i, $\overline{\alpha_i}$ are independent $2n$ Pfaffian forms. They argued the 2-forms are completely integrable if, and only if, $d\omega = \alpha \wedge \omega$, and therefore, the exterior derivative is $d\alpha \wedge \omega = 0$ implying that $d\alpha = 0$.[41]

One of the results of the article was a theorem which stats the local equivalence of a complete integrable 2-form on a $2n$-dimensional manifold.

[39] See (Audin 2008).
[40] See (Ehresmann & Libermann 1948).
[41] (Ehresmann & Libermann 1948, p. 420).

Theorem 6.3.1. *[Soit V_{2n} une varietété n fois différentiable de dimension $2n$. Sur V_{2n} soit Ω une forme différentiable extérieure de degree 2.] Pour $n > 2$, toute forme Ω complètement intégrable, de rank $2n$ et $n-1$ fois différentiable, est localement équivalente à l'une des formes suivantes:*

a. $dx_1 \wedge dy_1 + ... + dx_n \wedge dy_n$ *si* $D\Omega = 0$ *en tout point.*

b. $y_1(dx_1 \wedge dy_1 + ... + dx_n \wedge dy_n)$ *au voisinage d'un point oú* $D\Omega \neq 0$.[42]

(Ehresmann & Libermann 1948, p.421)

Here, it should be remarked that Libermann and Ehresmann used the notation $D\Omega$ as the exterior derivation of the differential form. Item a. of the quotation can be reformulated as Darboux's theorem on symplectic geometry. This was not possible at that time because Ehresmann had not yet defined what a symplectic manifold is.[43]

On Mai 21, 1953, Paulette Libermann submitted her doctoral thesis in which she used the current Darboux's theorem for symplectic manifolds to prove a theorem about 2-forms admitting an integral factor.[44] Here is the statement she made given as a theorem:

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[42][Let V_{2n} be a manifold n times differentiable with dimension $2n$ and $n-1$ times differentiable. Over V_{2n} let Ω be a exterior differential of degree 2.] For $n > 2$, every differential form Ω of range 2 is completely integrable, and $n-1$ times differentiable is locally equivalent to one of the next two forms:
a. $dx_1 \wedge dy_1 + ... + dx_n \wedge dy_n$ if the exterior derivative of the differential form is equal to zero in every point.
b. $y_1(dx_1 \wedge dy_1 + ... + dx_n \wedge dy_n)$ if on a point, the exterior derivative is not equal to zero.

[43]See Chapter 5.

[44]See (Libermann 1953*b*, p. 50).

Theorem 6.3.2. *Sur une variété V_{2n} si la forme différentielle extérieure quadratique Ω de rang $2n$ en tout point $x \in V_{2n}$ est fermée, il existe dans le voisinage U de tout point de V_{2n} des systèmes de coordonnées locales*

$$x^1, ..., x^n, y^1, ..., y^n$$

tels que cette forme puisse s'escrire:

$$\Omega = dx^1 \wedge dy^1 + ... + dx^n \wedge dy^n. \tag{6.34}$$ [45]

(Libermann 1953*b*, p.51)

V_{2n} is a symplectic manifold. The 2-form Ω is closed. Therefore, it can be written as $\Omega = d\alpha$ on a neighborhood U where α is a 1-form and, as shown in the last part of section (6.2.3), the "classical" Darboux theorem of 1882 can be applied. To prove the last statement, Libermann did not quote Darboux's article of 1882 but, instead, she quoted the book by Joseph Miller Thomas[46] from 1937 about differential systems. Thomas stated and proved how to reduce a Pfaffian form to its canonical form, but he did not quote Darboux's publication of 1882 either.[47] This is probably because the use of the results of the classical Darboux theorem was a well-known result. This can be seen in the publication of (Goursat 1922).

[45] On a manifold V_{2n} if the form Ω is closed, there exists on an open neighborhood U for every point of V_{2n} a system of local coordinates $x^1, ..., x^n, y^1, ..., y^n$ so that the form can be written:

$$\Omega = dx^1 \wedge dy^1 + ... + dx^n \wedge dy^n.$$

[46] Joseph Miller Thomas was born in 1898 in the United States of America and died in 1979. He was a mathematician and became professor at Duke University in North Carolina in 1941.

[47] (Thomas 1937, p. 44).

Later in 1953 Libermann published an article called *Forme canonique d'une forme différentielle extérieure quadratique fermée*, in which she proved the result of item *a.* of theorem (6.3.1), i.e. Darboux's Theorem.[48]

It was the third publication in which she addressed the problem of finding the conditions that are needed to reduce a closed 2-form on a neighborhood of an even-dimensional manifold to its canonical form.

In 1955 Libermann published work on almost complex structures and other infinitesimal regular structures. She restricted herself to Riemannian structures and Kähler structures. In this work Darboux's Theorem was again presented because it is essential for finding the canonical form on a neighborhood of a closed 2-form over an even-dimensional manifold.[49]

At first look, the link between Darboux's classical theorem and Darboux's theorem is that both problems concern to reduce a differential forms to its canonical expression. However, the two theorems have a stronger bound, since a symplectic manifold has a closed 2-form ω, i.e. $d\omega = 0$. This 2-form can be locally integrated and get $\omega = d\alpha$ with α a differential 1-form. On α it can be applied the classical Darboux theorem. Then under the conditions mentioned above α can be written as $\alpha = \sum_i^p dy_i$. Then $\omega = d\alpha = \sum_i^p dx_i dy_i$.

6.3.3 About the name of Darboux's Theorem

Between 1953 and 1955 Libermann proved and used Darboux's theorem in her publications, but she never called it like this.

During the 1950s mathematicians did not use the name Darboux in connection with finding the canonical form of a closed differential 2-form on a neighborhood over an even-dimensional manifold or in the case of finding the

[48]See (Libermann 1953*a*).

[49]See (Libermann 1955).

canonical form of a Pfaffian form.

A publication in which the name of Darboux's Theorem appeared can be found in in the text book by Shlomo Sternberg[50] *Lectures on Differential Geometry* in 1964. Sternberg addressed a section of his textbook on Differential Geometry to Darboux's theorem in the chapter of *Integral Calculus on Manifolds*, in which he discussed the problem of finding the canonical form for linear differential forms.[51] Sternberg called the canonical form *the normal* form. This is a translation of Darboux's theorem of 1882 from differential equations into the theory of differential forms. Before he stated Darboux's Theorem, he stated an analogous theorem that Libermann and Ehresmann gave in their notes (Ehresmann & Libermann 1948). For Sternberg, Darboux's Theorem was about how to reduce a Pfaffian form to its canonical form.

In the first edition of *The Foundations of Mechanics* written by Ralph H. Abraham[52] and Jerrold E. Marsden[53], the theorem called Darboux's Theorem referred to the conditions for the reduction of a closed 2-form on a neighborhood over a even dimensional manifold to its canonical form.[54] In this work it appeared in the explicit context of symplectic geometry and classical mechanics.

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[50]Shlomo Sternberg was born in 1936 in the United States of America and is a professor of mathematics at Harvard University. He was student of Aurel Wintner. For Aurel Wintner see Chapter (8.2).

[51](Sternberg 1964, p.137).

[52]Ralph H. Abraham was born on July 4, 1936. He is a professor of mathematics at the University of California Santa Cruz.

[53]Jerrold E. Marsden was born on August 17, 1942 and died on September 21, 2010. He was a professor of Control and Dynamical Systems at the California Institute of Technology.

[54]See (Abraham & Marsden 1967).

Abraham and Marsden referred to Sternberg's Darboux's theorem.[55] After Abraham's and Marsden's publication of 1967, the name of Darboux's theorem started to be in use in the field of symplectic geometry.

[55] In the edition of 1964 of the *Foundations of Mechanics*, there is a small bibliographic mistake. It quoted the article written by Sternberg in 1961 for Darboux's theorem, but it actually refers to (Sternberg 1964).

Chapter 7

Lee's work on flat manifolds in the 1940s.

Introduction

This chapter presents the work done in China during the 1940's by the mathematician Lee Hwa-Chung. The work presented here is restricted to even dimensional spaces endowed with a closed 2-form. These even dimensional spaces and even dimensional manifolds were called later by Ehersmann symplectic spaces and symplectic manifolds, as shown in chapter (5.1).

Even though Lee had published on the *American Journal of Mathematics* during the 1940s, his work was and is not well known by the mathematical community. Not only is his work not well known, but Lee's live is also not well known, although it was possible to track the places where he worked through the articles he published in western journals.

The first part of this chapter will try to reconstruct where Lee worked during the 1930's to the 1940's.

The second part will present Lee's work about even dimensional geometry, which is the name he gave to symplectic geometry. In his work he defined an even dimensional flat space, which is the actual definition of symplectic manifolds, found the automorphisms over symplectic manifolds and stated Darboux's theorem as well. Lee noticed that the theory about "even dimensional geometry" can be applied to classical mechanics.

7.1 About Lee Hwa-Chung Live

Lee Hwa-Chung was or is a Chinese mathematician, who published in 1943 his work: *"A kind of even-dimensional geometry and its applications to exterior calculus"*.[1] In it Lee defined what would later be called by Ehresmann symplectic manifolds.

It was difficult to track Lee's biography as if there exists more information about Lee's biography, then maybe it is written in Chinese, and not in English, German, French or Spanish. Nevertheless, it is possible to track the institutions where Lee worked during the late 1930s, in the 1940s and later until the 1960s through the publications he made in the western journals.

Lee must have lived for a few years in Europe because he obtained his PhD at the University of Edinburgh, as it is mention in his publication of 1945.[2]

In 1938 Lee published an article in the *"Comptes Rendus de l'Académie de Sciences"*. The article is presented by Élie Cartan in the field of differential geometry, and the title *"Sur les transformations des congruences hamiltoniennes"*[3]. In 1939 Lee published his article *"On the projective the-*

[1] See (Lee 1943).
[2] See (Lee 1945).
[3] On the Hamiltonian congruences transformations. See (Lee 1938).

ory of spinors"[4] in the Dutch journal *Composition Mathematica*, which has signed in Paris.[5]

Therefore, Lee spent time in Paris in 1938, but it is not clear with whom he worked there with. He used to quote the work of the French mathematician Edouard Goursat (*1858, †1936) *Leçons sur le probleme de Pfaff*[6] in his work about even dimensional geometry, and also Élie Cartan's publications, but this does not give a clear hint with whom he was working with in Paris. or if he was in Paris working at all. In 1939, when the article (Lee 1938) was published, Goursat was already dead, but it is possible that Lee worked with Élie Cartan, but this is only speculation.

During the 1940s Lee was back in China. In 1941 Lee was working at the National Sywchwan University when he submitted the draft of his work *"A kind of even-dimensional geometry and its applications to exterior calculus"* to the *American Journal of Mathematics* (Lee 1943).[7]

Lee published two more publications in the *American Journal of Mathematics* in the 1940s. The first one in 1945, *"On Even-Dimensional Skew-Metric Spaces and Their Groups of Transformations"*[8]. It was signed at the Academia Sinica, which at that time was part of the National Tsing Hua University of China.[9] The second publication in 1947 was signed as well at

[4] See (Lee 1939).

[5] The journal *Composition Mathematica* was founded in the 1930s by the Dutch mathematician Luitzen Egbertus Jan Brouwer (*1881, †1966).

[6] See (Goursat 1922).

[7] It was signed in the National Szechwan University, China. The name Szechwan can as well be written as Sichuan which is a province in the south-west of China.

[8] See (Lee 1945).

[9] During the Second Sino-Japanese War (1937-1945), many Chinese universities moved to the west of China, leaving Japanese occupied territory. See (Fairbank & Feuerwerker 1986, p. 564).

the Academia Sinica.[10]

The last publication during the 1940s was in 1947 at the *Proceedings of the Royal Society of Edinburgh.* Section A. Mathematical and Physical Sciences with the title *"The Universal Integral Invariants of Hamiltonian Systems and Application to the Theory of Canonical Transformations"*. The article was received by the editors in October 1945 and Lee signed it from Wuhan University in China.[11]

After 1960s it was possible to track other publications in the field of mathematical physics. The last article in Western Journals was published in 1978. This publication was received in 1977, and Lee signed from the Chungshan University of Taiwan.[12] Through this information we know that Lee moved to Taiwan.

7.2 Even dimensional geometry (Symplectic geometry)

The first publication in the field of even dimensional differential geometry was tracked through a footnote in Ehresmann and Libermann's publication in 1949.In this publication Libermann and Ehresmann were working on the problem of equivalence between quadratic exterior differential forms on a even dimensional differential manifold. They wrote in the footnote:

Hwa-Chung Lee, A kind of even-dimensional geometry and its

[10]See (Lee 1947*a*).

[11]The city of Wuhan is the capital of Hubei province, People's Republic of China, and nowadays it has about ten million inhabitants and is located in the Central China region. It lies in the eastern Jianghan Plain at the intersection of the middle reaches of the Yangtze and Han rivers.

[12]See (Wu Yong-Shi & Ting-Chang 1978).

> applications to exterior calculus (Am. Journal of Math., 1943, p. 433-438). *Cet article, qui nous a été signalé par M. Yen Chin-Ta après la parution de notre Note (1), contient déjà sous une autre forme plusieurs résultats énoncés par nous.*[13]
>
> (Ehresmann & Libermann 1949, p. 698)

Lee's article was not mentioned or quoted in any other publication that contributed to the development of symplectic geometry during the late 1940s and 1950's. Some references were found during this research to this article, but they were after the 1970s. One of them was given in the textbook about symplectic geometry written by Libermann and Marle in 1987.[14]

A more resent reference can be found in a publication written by Mark J. Gotay and James A. Isenberg. There, they consider Lee's work as not well known. They wrote:

> But symplectic geometry, as a distinct mathematical discipline, did not really appear until 1940s with the (little-known) work of Hwa-Chung Lee in China.
>
> (Gotay & Isenberg 1992, p. 16)

Lee's article (Lee 1943) was published in the *American Journal of Mathematics* during the Second World War. At that time China was resisting the Japanese invasion, east China was occupied by Japan and the United States were helping the Republic of China in the war.[15] So, there was communication between China and the United States, but maybe the communication

[13][...]. This article, was reported to us by M. Yen Chin-Ta after the publication of our note (1), [the article of Lee] contains another possible form of the results exposed by us.

[14](Libermann & Marle 1987).

[15]See (Fairbank & Feuerwerker 1986).

between China and France was not so good. One has to remember that between 1940 and 1942 part of France was occupied by the Germans and the "independent part" was a fascist regime known as the Vichy regime.[16] In 1943, all France was occupied. However, this contradicts the communication that existed at the time between Élie Cartan and Chern. [17]

7.2.1 Even dimensional manifolds endowed with a nonsingular closed 2-form (Symplectic manifold)

In 1943 Lee published the first part of his work on even dimensional geometry, *"A kind of even-dimensional geometry and its applications to exterior calculus"*. It is the first of three articles on *Even-Dimensional Geometry* and *Skew-Metric Spaces* published in the *American Journal of Mathematics*. The manuscript was send to the *American Journal* in 1941.

Lee studied differential geometry on an even dimensional space. The even dimensional spaces are even dimensional manifolds. Lee endowed the even dimensional manifolds with a "nonsingular skewsymmetric matrix" as a fundamental tensor.

Using Lee's notation, an even dimensional manifold endowed with a fundamental tensor is denoted by Lee as L_{2n}. Each point of the manifold is described by a system of coordinates denoted as x^α, where $(\alpha, \beta, \gamma, \rho, \sigma, \tau) = (1, ..., 2n)$. The covariant tensor or fundamental tensor is denoted as $a_{\alpha\beta}$. The fundamental tensor is defined as a nonsingular skewsymmetric covariant tensor, i.e. non degenerate antisymmetric matrix. The components of the fundamental tensor are analytical function of the x's. He defined a "curva-

[16] See (Jackson 2003).
[17] See appendix A.

ture" tensor of the manifold as the differential form

$$K_{\alpha\beta\gamma} = \frac{\partial}{\partial x_\alpha} a_{\beta\gamma} + \frac{\partial}{\partial x_\beta} a_{\gamma\alpha} + \frac{\partial}{\partial x_\gamma} a_{\alpha\beta} \tag{7.1}$$

which satisfies the identity

$$\frac{\partial}{\partial x_\rho} K_{\alpha\beta\gamma} - \frac{\partial}{\partial x_\alpha} K_{\rho\beta\gamma} - \frac{\partial}{\partial x_\beta} K_{\alpha\rho\gamma} - \frac{\partial}{\partial x_\gamma} K_{\alpha\beta\rho} = 0 \tag{7.2}$$

"which is a tensorial equation that holds for all coördinates [sic] systems".[18]

Lee's "fundamental tensor" $a_{\alpha\beta}$ is associated with 2-form

$$\Omega = \sum a_{\alpha\beta} dx^\alpha \wedge dx^\beta, \tag{7.3}$$

which Lee calls the *fundamental form* of the even dimensional manifold L_{2n}, and because the *fundamental tensor* is non degenerate, then the 2-form is non degenerate too. The exterior derivative of Ω is

$$d\Omega = \frac{1}{3} \sum K_{\alpha\beta\gamma} dx^\alpha \wedge dx^\beta \tag{7.4}$$

and Lee mentioned that the identity (7.2) is equivalent to $dd\Omega = 0$.

Lee called the equation (7.1) as "curvature tensor". When the "curvature tensor vanishes", i.e. $K_{\alpha\beta\gamma} = 0$, the even dimensional manifold is flat. If the exterior derivative of (7.4) is zero because $K_{\alpha\beta\gamma} = 0$, then the 2-form associated to the fundamental tensor $a_{\alpha\beta}$ is closed, i.e. $d\Omega = 0$.[19]

Lee's definition of a flat manifold L_{2n} is what would later be the definition for symplectic manifolds. A flat manifold is an even dimensional manifold endowed with a closed 2-form which is non degenerate. Today, this even dimensional manifold would be called a symplectic manifold.

Lee wrote the 2-form in canonical coordinates

$$\Omega = \sum_{i=1}^{n} dx_i \wedge dx_{n+i}, \tag{7.5}$$

[18](Lee 1943, p.433).

[19](Lee 1943, pp. 433-434).

and therefore, in the new coordinate system the *fundamental tensor* has the form

$$a_{\alpha\beta} = \begin{pmatrix} 0 & Id \\ -Id & 0, \end{pmatrix}, \tag{7.6}$$

so he stated a theorem for "flat manifolds":

> **Theorem**: An L^{2n} is flat if, and only if, there exist a coördinate [sic] system for which the components of the fundamental tensor $a_{\alpha\beta}$ are constants.
>
> (Lee 1943, p.434)

Lee's definition of a flat $2n$ dimensional manifold raises the question, as to whether Ehresmann used Lee's definition of symplectic manifolds, or if Ehresmann was inspired by Lee's work when he defined the symplectic manifolds.

The answer to this question is that Ehresmann did not use Lee's definition or was inspired by Lee's work, then Ehresmann constructed the symplectic manifolds in another way.[20] Ehresmann's definition arose from the question of which even dimensional manifold admits an almost complex structure as shown in chapter 5.

7.2.2 Conformal flat manifolds

Lee defined that L_{2n} and L'_{2n}, are conformal to each other if the *fundamental tensors* $a_{\alpha\beta}$ of L_{2n} and $a'_{\alpha\beta}$ of L'_{2n}, which refer to the same coordinate system, are connected by the relation

$$a'_{\alpha\beta} = \phi a_{\alpha\beta}$$

[20]See chapter 5.

where ϕ is a scalar function of the coordinates. He remarked that the manifold L_{2n} is conformal to itself. In this context, Lee proposed that all flat manifolds of the same dimension have locally the same skew symmetric form. Currently known as a local symplectomorphism. A symplectomorphisms is a diffeomorphism between symplectic manifolds, which preserves the symplectic structure. In the case that one of the two manifolds is $(\mathbb{R}^{2n}, \omega_0)$, then Darboux's theorem is obtained.

7.2.3 Automorphisms

In 1945 Lee published the second part of his work on *even dimensional geometry*. Lee continued working on even dimensional flat manifolds endowed with a nonsingular skewsymmetric tensor; i.e. symplectic manifolds; and he described the "analytic point-transformation" group and their subgroups on a symplectic manifold, as the subgroup of conformal transformation which leaved the "fundamental tensor invariant save for a non-vanishing factor."[21] Lee defined "for an arbitrary chosen coordinate system a *conformal point transformation* $x \mapsto y$ as"[22]:

$$\psi a_{\alpha\beta}(x) = a_{\rho\sigma}(y)\frac{\partial y^\rho}{\partial x^\alpha}\frac{\partial y^\sigma}{\partial x^\beta} \quad (\psi \neq 0), \tag{7.7}$$

where ψ is a factor. The conformal point transformation forms a group. When ψ is a constant factor, the transformation belongs to a subgroup of the *conformal point transformations* which is the group of special conformal transformations, and if $\psi = 1$ then it is a subgroup of the special conformal transformation. This subgroup Lee called the group of *automorphisms*.[23] This group is the symplectic group $Sp(2n, \mathbb{R})$. This can be seen,

[21] (Lee 1945, p.321).
[22] (Lee 1945, p.321).
[23] (Lee 1945, p.321).

since Lee shows that for $n = 1$ the group of *automorphisms* is the *unimodular* group which is the group $SL(2, \mathbb{R})$ which is equal to the symplectic group $Sp(2n, \mathbb{R})$.[24] For a general n it is *"the group of canonical transformations in* $2n$ *varialbles"* is the symplectic group $Sp(2n, \mathbb{R})$.[25] Currently, the canonical transformations are known as symplectomorphisms.

7.2.4 Poisson bracket

Lee introduced the Poisson bracket. For two differentiable functions of position f and g on a manifold, he formed the expression

$$\{f, g\} = a^{\alpha\beta} \frac{\partial f}{\partial x^\alpha} \frac{\partial g}{\partial x^\beta}. \tag{7.8}$$

where $a^{\alpha\beta}$ is like above..

Lee shows that for a symplectic manifold L_{2n} there exists a coordinate system where the components of the fundamental tensor are constant. This system was called by him as *preferred*.[26] He called a *preferred* coordinate system canonical when

$$a^{\alpha\beta} = \begin{pmatrix} 0 & -Id \\ Id & 0 \end{pmatrix},$$

$$a_{\alpha\beta} = \begin{pmatrix} 0 & Id \\ -Id & 0 \end{pmatrix}.$$

The equation (7.8) is the same the equation (7.7). The brackets (7.8) are antisymmetric, i.e.

$$\{f, g\} = -\{g, f\}.$$

[24]See (Lee 1945, p. 322).
[25]See (Lee 1945, p. 325).
[26](Lee 1945, p. 324).

Lee noticed that if the $K^{\alpha,\beta,\gamma}$ is equal zero in the identity

$$\{f,\{g,h\}\}+\{g,\{h,f\}\}+\{h,\{f,g\}\}=K^{\alpha,\beta,\gamma}\frac{\partial f}{\partial x^{\alpha}}\frac{\partial g}{\partial x^{\beta}}\frac{\partial h}{\partial x^{\gamma}}, \tag{7.9}$$

then it is the Jacobi identity.[27]

Lee was able to introduce the Poisson bracket on the symplectic manifolds

$$\{f,g\}:=\sum_{i=1}^{n}\frac{\partial f}{\partial x^{n-i}}\frac{\partial g}{\partial x^{i}}-\frac{\partial f}{\partial x^{i}}\frac{\partial g}{\partial x^{n+i}}.^{28} \tag{7.10}$$

A manifold with a Poisson bracket is nowadays known as a Poisson manifold, this manifold is a generalization of a symplectic manifold.[29] Lichnerowicz defined Poisson manifolds in 1973.[30]

7.2.5 Hamilton's equations

In 1945 Lee considered a system of curve in the manifold L_{2n}. The system of curve is defined by a system of ordinary differential equations of the form

$$\frac{dx^{\alpha}}{dt}+a^{\alpha\beta}(x)\frac{\partial H(x,t)}{\partial x^{\beta}}=0. \tag{7.11}$$

Lee mentioned that H is a Hamiltonian function, and the system of curve is a *Hamiltonian congruence*.[31] If the manifold L_{2n} is a symplectic manifold, then the equation (7.11) can be rewritten as a Hamiltonian equation

$$\frac{dx^{i}}{dt}=\frac{\partial H}{\partial p_i},\qquad \frac{dp_i}{dt}=-\frac{\partial H}{\partial x^{i}}\qquad (i=1,...,n). \tag{7.12}$$

The group of conformal transformation on a manifold L_{2n} is the special group of conformal transformations, which consist of the "Hamiltonian

[27] Lee called it a contravariant curvature tensor of (7.1).
[28] (Lee 1945, p. 325).
[29] (Libermann & Marle 1987, p. 105).
[30] (Kosmann-Schwarzbach 2013, p. 160).
[31] For the definition of Hamiltonian function see chapter 8.

transformations"[32], i.e. the transformations that transforms every *Hamiltonian congruence* into a *Hamiltonian congruence.* The later terminology for *Hamiltonian congruence* is *Hamiltonian flow.* Lee make clear that the group of *Hamiltonian transformation* is larger than $Sp(2n, \mathbb{R})$.

By rewriting the ordinary differential equations of the form (7.11) of a system of curves on a symplectic manifold as Hamilton's equations (7.12), Lee finds the relationship between symplectic geometry and classical mechanics.

The article (Lee 1945) is linked to his work in the 1930's. Lee published a note (Lee 1938) on the weekly review of the sessions of the French Academy of Science (*Comptes rendus hebdomadoires des séances de l'Academie des Sciences*) in 1938. In the publication Lee presented his work in progress. The title of the note was *"Sur les transformations des congruences halmiltoniennes."*[33]

7.2.6 Equivalence of flat spaces

In the third article about even dimensional spaces, Lee noticed that two symplectic manifolds are equivalent if the 2-form of one can be transformed into the 2-form of the other by a change of coordinates.

> **Theorem.** Two flat spaces L_m are equivalent if and only if they are of the same rank.
>
> (Lee 1947*a*, p. 795)

The dimension of L_m must be even. In this theorem, Lee presents the diffiomorphisms between two symplectic manifolds. Which as mentioned above

[32]See (Lee 1945, p. 326).

[33]About the Hamiltonian congruence transformation.

are symplectomorphisms and in the case that these are local symplectomorphisms between a symplectic manifold and the symplectic manifold $(\mathbb{R}^{2n}, \omega_0)$ then it is the theorem of Darboux.

7.2.7 Lee Hwa Chung Theorem

There is currently a symplectic geometry theorem named after Lee. This theorem was stated by Lee in his paper (Lee 1947*b*). In its current form it statement is as follows:

> **Theorem Lee Hwa Chung:** *Let M be a symplectic manifold with symplectic form ω. Let α be a differential k-form on M which is invariant for all Hamiltonian vector fields. Then:*
>
> - *If k is odd, $\alpha = 0$.*
> - *If k is even, $\alpha = c \times \omega^{\wedge \frac{k}{2}}$, where $c \in \mathbb{R}$.*

It appears in the context of dynamical systems, for example, in publication like (Gomis, Llosa & Roman 1984), (Llosa & Roy 1988), and (Kozlov 1995). The theorem appears, as well, in textbooks of analytical mechanics, for example, in (Gantmacher 1970), or later in the textbook (Tiwari & Thakur 2007).

The context in which Lee formulates the theorem is in the context of Hamiltonian systems. Here one can see the current link between symplectic geometry and classical mechanics. Which Lee had done before, as seen above, but in his work (Lee 1947*b*) this link is not explicit, since he worked on Hamiltonian systems only and did not use the symplectic manifolds as in his paper (Lee 1945).

Lee stated the theorem for the relative integral invariant:

Theorem 5. *There is no universal absolute integral invariant of any odd order, and apart form an arbitrary constant factor there is only one universal absolute integral invariant of every even order* $2s$*, namely*

$$\int \sum \delta p_{i_1} \delta q_{i_1} \dots \delta p_{i_s} \delta q_{i_s}.$$

(Lee 1947*b*, p.241)

An relative integral invariant is a differential form of degree k whose exterior differential is an absolute integral invariant of degree $k+1$.[34]

7.3 The reception of Lee's work

At the beginning of this research on the early history of symplectic geometry, it was amazing to find Chinese mathematicians developing symplectic geometry, but after reading them and tracking part of their lives, it is clear that these Chinese mathematicians were in close contact with mathematicians in Europe and in the United States. For example, in the case of Hua, it was not so difficult to find information about his live and the contacts he had, but in the case of Lee, even though he had published in the *American Journal of Mathematics*, his live and his contact with the mathematical community in the West are still not clear. A few footnotes in the articles of Lee hint with whom he was in contact:

[34] D.V. Anosov (originator), Encyclopaedia of Mathematics

> The author [Lee] wishes to express his thanks to Professor J. M. Thomas for reading the manuscript and giving valuable criticisms.
>
> (Lee 1943, p.433)

Joseph Miller Thomas is the mathematician that Libermann quoted in her PhD Thesis.[35] He published during 1933 and 1934 two articles about Pfaffian systems.[36] But the fact that Lee referenced J. M. Thomas does not mean that he had contact with Thomas. It might have been a sporadic contact because Thomas corrected Lee's article. Therefore, it is difficult to make the assertion that Lee was in direct contact with western mathematicians. But that he knew and worked with some scientists outside China can be seen through his publication of 1947, in which Lee coauthor an article on physics named *"The Theory of Wedge Indentation of Ductile Materials"*, published in the *Proceedings of the Royal Society* in 1947.[37] The article was presented by Neville Francis Mott (*1905 - †1996), who was an English physicist and the coauthors were R. Hill and S. J. Tupper, who were working on plastic deformations and problems of elasticity. But the coauthors were physicists and not mathematicians.

Although Lee spent some years in Europe, his whole work on even dimensional spaces (Lee 1943, Lee 1945, Lee 1947*a*) was not well known in France, where important development in the study of symplectic manifolds took place. This could be because Lee published in the *American Journal of Mathematics* during the Second World War. Therefore, it might have been difficult to get all the journals during and short time after the Second World

[35] See section 6.3.2.
[36] See (Thomas 1933, Thomas 1934).
[37] See (Hill 1947).

War in France, and maybe this was one of the reasons why nobody knew Lee in France in the 1940s.

At least it is clear that Ehresmann and Libermann knew Lee's first work, in which he defined "Flat spaces", but they did not use Lee's work to develop the definition of symplectic manifolds or Darboux's theorem. Ehresmann and Libermann's footnote remarked that "[the article of Lee, (Lee 1943)] contains another possible form of the results exposed."[38] By reading both Lee's article and the note of Ehresmann and Libermann's note it is clear that they developed this independently of each other.

Regarding the three articles "Even dimensional manifolds", it most be noted that in them Lee defined symplectic manifolds, defined the symplectomorphisms on symplectic manifolds and linked the studied of symplectic manifolds with the study of Hamiltonian systems and, therefore, with classical mechanics. Lee presented what nowadays is known as the field of symplectic geometry but the reception did not take place because nobody used his results to develop the field.

[38](Ehresmann & Libermann 1949, p. 698).

Chapter 8

Symplectic Geometry and Classical Mechanics

Introduction

This chapter expose the explicit use of symplectic matrices in classical mechanics for canonical transformations, which map a Hamiltonian system into another one. Not only did symplectic matrices start to be used in classical mechanics, but it was also found that symplectic manifolds could represent a phase space of a configuration space. These results were stated by Lee in the 1940s, but as already, his work was not well known.

The first part of the chapter provides a short introduction to classical mechanics and Hamiltonian mechanics. It continues with Aurel Wintner's work on classical mechanics and the use of symplectic matrices. Wintner noticed that the matrix of linear canonical transformations between Hamiltonian functions belongs to the symplectic group.[1]

In 1951 Siegel gave a course about celestial mechanics in Göttingen and

[1](Wintner 1941).

his notes were published in 1956. He explicitly mention the canonical transformation between Hamilton systems to be a transformation which belong to the symplectic group.

The last part of the chapter deals with Georges Reeb's work identifying the connection between the phase space and symplectic manifolds in 1952.

8.1 Classical Mechanics

Nowadays, symplectic geometry is related to classical mechanics. Indeed, some of the textbooks on symplectic geometry and symplectic topology start with classical mechanics as a motivation (for example in (Berndt 1998, McDuff & Salamon 1995)). The structure of Hamilton's equations, which are a system of ordinary differential equations, can be described through the symplectic form.

8.1.1 Lagrangian and Hamiltonian Mechanics

There are two equivalent formulation for classical mechanics: Lagrangian mechanics and Hamiltonian mechanics. Lagrangian mechanics was introduced by Joseph-Louis Lagrange (*1736, †1813) at the beginning of the 19$^{\text{th}}$ century in his work on calculus of variation, which was published in the second edition of his book *Mécanique analytique*.[2]

Hamiltonian mechanics was developed by the Irish mathematician and physicist William Rowan Hamilton (*1805, †1865) in 1834.[3] For this historical development, please refer to June Barrow-Green's book, "Poincaré and the Three Body Problem".[4]

[2]See (Lagrange 1811).
[3]See (Hamilton 1834).
[4](Barrow-Green 1997).

Lagrangian

The variational problem arises from the n-body problem in celestial mechanics where a number of celestial bodies in a mechanical system move in space under their mutual gravitational attraction. If the initial conditions are given, their subsequent motions have to be determined. The solution of the n-body problem can be solved for $n = 1, 2$ with elementary functions, but for $n \geq 3$ it is a non-linear problem and can not easily be solved.[5]

In Lagrangian mechanics, the development of a mechanical system is described by the solutions of the Euler-Lagrange equation, which is a motion equation. The Euler-Lagrange equation is a second-order partial differential equation and is derived from the variational principle of least action.

The motion of an object can be described by a curve $\gamma(t) = P$ on a configuration space Q, with t as time. The configuration space is an n-dimensional real manifold and the point P is fixed through local coordinates $q_1, ..., q_n$. The local coordinates are named as position variables.

The principle of least action allows to give the curve as a solution to a differential equation if it is assumed that the system has a Lagrange Function. The Lagrange Function is also called Lagrangian. The Lagrangian has the form

$$L = L(q, \dot{q}, t). \tag{8.1}$$

L is a twice continuously differentiable function of $2n + 1$ variables, q is the position variable, $\dot{q}$ the velocity and t the time. The Lagrangian function L can be defined by

$$L = T - V. \tag{8.2}$$

[5] In 1890 Henri Poincaré (*1854, †1912) tried to find a solution for the three body problem using the theory of asymptotic solutions, *Les méthodes nouvelles de la mécanique céleste* in three volumes. See (Barrow-Green 1997).

where T is the total kinetic energy and V is the potential energy of the system.

The principle of least action states that the change in the mechanical system proceeds in such a way that the curve γ, which describes that change, minimizes the path integral

$$\Phi(q) = \int_{t_0}^{t_1} L(q, \dot{q}, t)dt. \tag{8.3}$$

The variational principle states that for the minimal curve γ the mechanical system satisfies the Euler-Lagrange equations

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0. \tag{8.4}$$

The system of Euler-Lagrange equations is as Lagrangian system, which is a system of second-order differential equations on the $2n$-dimensional tangent bundle over the configuration space, i.e. TQ.[6]

Hamiltonian

In Hamiltonian mechanics, the function which corresponds to the total energy of a closed mechanical system is

$$H(p, q, t) = T + V \tag{8.5}$$

and is called the Hamiltonian function, where T is kinetic energy and V is potential energy. Potential energy is only a function of the position variable q, and kinetic energy is a function of the momentum variable p.[7] A Lagrangian system can be transformed into a Hamiltonian system. A Hamiltonian system is a $2n$ first order differential equation on the cotangent bundle T^*Q.

[6] (Berndt 1998, p. 2)

[7] In a single particle, its momentum is the product of mass and velocity, i.e. $p = mv$.

The transformation of an Euler-Lagrangian system into a Hamiltonian system can be achieved through the Legendre transformation, which maps

$$TQ \to T^*Q,$$

$$(q, \dot{q}) \mapsto (q, p)$$

and establishes an equivalence of Euler-Lagrange and Hamilton equations.

Applying the Legendre transformation, the Hamiltonian function is

$$H(p, q, t) = p\dot{q} - L(q, \dot{q}, t), \tag{8.6}$$

and Hamilton's equations are

$$\dot{q} = \frac{\partial H}{\partial p}, \quad \dot{p} = -\frac{\partial H}{\partial q}. \tag{8.7}$$

A Hamiltonian system of equations is also called a canonical system.

8.1.2 Symplectic Group and Classical Mechanics

The relation between the symplectic group and classical mechanics is that canonical transformation, which map each Hamiltonian system of differential equations into another or into itself, belong to the symplectic group, and therefore, differential equations of mechanics have invariance properties relative to the symplectic group.

In 1967 Ralph H. Abraham and Jerrold E. Marsden introduced a mathematical model for mechanics. This model consists of a differentiable manifold provided with a symplectic form, i.e. a symplectic manifold, "together with a Hamiltonian vector-field or a global system of first order differential equations preserving the symplectic structure."[8] If the configuration space is an

[8](Abraham & Marsden 1967, p. 2).

n-dimensional manifold, the momentum phase space is its cotangent bundle with a symplectic form, and therefore, the phase space is a symplectic manifold.[9] The configuration space is a differentiable manifold Q, and the cotangent bundle T^*Q is the phase space of the configuration, which is a $2n$ dimensional manifold.

To define a 1-form on the cotangent bundle T^*Q, let $(q_1, ..., q_n, p_1, ..., p_n)$ be the system of local coordinate for an open neighborhood U of a point $x \in T^*Q$. A 1-form α_Q is defined on the tangent bundle by

$$\alpha_Q = \sum_{i=0}^{n} p_i dx_i. \tag{8.8}$$

This form is known as the Liouville form.[10]

The symplectic form on the cotangent bundle is then

$$d\alpha_Q = \sum_{i=0}^{n} dp_i \wedge dx_i; \tag{8.9}$$

therefore, the cotangent bundle T^*Q is a symplectic manifold. The system $(q_1, ..., q_n, p_1, ..., p_2)$ is a system of canonical coordinates for the symplectic manifold $(T^*Q, d\alpha_Q)$.

A Hamiltonian vector-field X_H on a symplectic manifold is a vector-field defined by an energy function, which is the Hamiltonian function. A Hamiltonian function H on a symplectic manifold (M, ω), i.e. on a phase space, is a smooth function $H : \mathbb{R} \times M \to \mathbb{R}$. A Hamiltonian vector field X_H on M in the canonical coordinates (q, p) is defined by

$$X_H = \left(\frac{\partial H}{\partial p_i}, -\frac{\partial H}{\partial q_i} \right) = J \cdot dH, \tag{8.10}$$

where $J = \begin{pmatrix} 0 & Id \\ -Id & 0 \end{pmatrix}$ and $dH = \begin{pmatrix} \frac{\partial H}{\partial p_i} \\ \frac{\partial H}{\partial q_i} \end{pmatrix}$. The vector field X_H is

[9]See (Abraham & Marsden 1978, p.178).

[10](Berndt 1998, p. 45).

determined by the condition

$$\omega(X_H, \cdot) = dH(\cdot) \tag{8.11}$$

that is, $\iota_{X_H}\omega = dH$, which means that an inner product ι of the Hamiltonian vector field X_H and the symplectic form ω on the manifold M is equivalent to the Hamilton equations (8.7).[11]

The tupel (M, ω, X_H) is called a Hamiltonian system. In a Hamiltonian system, a curve $\gamma(t) = (q(t), p(t))$ is an integral curve for the Hamiltonian vector field, if, and only if, the Hamilton equation (8.7) holds.[12]

Poisson Bracket and Hamiltonian Mechanics

The Poisson bracket on a manifold is a bilinear operation of two smooth functions, i.e. $\{\cdot,\cdot\} : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$, $(f, g) \mapsto \{f, g\}$. To be a Poisson bracket the bilinear operation must be skew symmetry, i.e.

$$\{f, g\} = -\{g, f\}, \quad \{f, g\} = -\{g, f\},$$

must satisfy the Jacobi identity, i.e.

$$\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0,$$

and Leibniz's Rule, i.e.

$$\{fg, h\} = f\{g, h\} + g\{f, h\}.^{13}$$

On any pair of differential functions f, g on the phase space T^*Q the Poisson bracket { , } is defined by the equation

[11] See (Berndt 1998, Abraham & Marsden 1978).

[12] (Abraham & Marsden 1978, p. 187).

[13] The pair $(M, \{\cdot,\cdot\})$ is called a Poisson manifold. For the historical development of the Poisson bracket and Poisson geometry, the reader can consult the work (Kosmann-Schwarzbach 2013).

$$\{f, g\} := \sum_{i=1}^{n} \left(\frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} \right), ^{14} \tag{8.12}$$

which is called the canonical Poisson bracket on $\mathbb{R}^{2n}$ and is completely characterized by its values on the coordinates functions $\{q_i, q_j\} = \{p_i, p_j\} = 0$ and $\{p_i, q_j\} = \delta_{ij}$.[15]

Hamilton's equation can be written with the help of the Poisson bracket. Let (q, p) be the canonical coordinates for the symplectic manifold T^*Q and H a Hamiltonian function, then the Poisson bracket of q and H, and, p and H are

$$\{q, H\} = \sum_{i=1}^{n} \frac{\partial q}{\partial q_i} \frac{\partial H}{\partial p_i} - \frac{\partial q}{\partial p_i} \frac{\partial H}{\partial q_i}, \tag{8.13}$$

$$\{p, H\} = \sum_{i=1}^{n} \frac{\partial p}{\partial q_i} \frac{\partial H}{\partial p_i} - \frac{\partial p}{\partial p_i} \frac{\partial H}{\partial q_i}. \tag{8.14}$$

Therefore, Hamilton's equation can be written as

$$\dot{q} = \{q, H\}, \qquad \dot{p} = \{p, H\}.^{16} \tag{8.15}$$

Canonical Transformations

In celestial mechanics and classical mechanics, astronomers, physicist and mathematicians use canonical transformations which result in a change of coordinates (p, q, t) to (p', q', t) and convert each Hamiltonian system of differential equations into another Hamiltonian system or into itself. The problem is to find the canonical transformation and the conditions needed for this transformation to be canonical. For example, on a mechanical system

[14] (Berndt 2001, p. 6).
[15] (Fernandes & Marcut 2014, p. 3).
[16] See (Berndt 2001, p. 6).

where the phase space is $\mathbb{R}^{2n}$, the linear canonical transformation is a linear mapping of $\mathbb{R}^{2n}$ to $\mathbb{R}^{2n}$, so that $(p,q) \mapsto (p',q')$ and the coordinates p' and q' satisfy Hamilton's equations (8.7). Let

$$x = (p,q), \qquad \nabla_x = \left(\frac{\partial}{\partial p}, \frac{\partial}{\partial q}\right), \qquad \dot{x} = \left(\frac{dp}{dt}, \frac{dq}{dt}\right)$$

and

$$J = \begin{pmatrix} 0 & Id \\ -Id & 0 \end{pmatrix}.$$

the skew-symmetric matrix. Hamilton's equations will be

$$\dot{x} = J\nabla_x H.$$

Consider a linear transformation $x' = Ax$. Then,

$$\dot{x}' = AJA^t\nabla_x H$$

is Hamilton's equation if, and only if, $A \in Sp(2n, \mathbb{R})$.[17] Therefore, the next theorem holds:

Theorem 8.1.1. $Sp(2n,\mathbb{R})$ *is the group of linear canonical transformations.*

8.2 Aurel Wintner and Canonical Transformations

The use of a symplectic matrix to transform a Hamiltonian system into another can be found in the work done by Aurel Wintner (Wintner 1941) *"The Analytical Foundations of Celestial Mechanics."*

[17] See (Brieskom 1983, p.412. ff).

8.2.1 Aurel Wintner

Aurel Wintner was born on April 8, 1903 in Budapest, Hungary and died on January 15, 1958 in Baltimore, the United States of America. He started to study at the University of Budapest in 1920. In 1927 he went to the University of Leipzig to do his doctoral studies under the supervision of Leon Lichtenstein (*1878, †1933). In Leipzig he worked as an editor of the *Mathematische Zeitschrift* and of the *Jahrbuch über die Forschritte der Mathematik.*

In 1929 Wintner received his Doctorate with the dissertation *"Über die Konvergenzfragen der Mondtheorie"*.

In 1930 Wintner emigrated to the United States of America and joined the mathematics faculty of the Johns Hopkins University in Baltimore, Maryland. He worked in Baltimore until his death.

In the United States Wintner became the editor of the *American Journal of Mathematics* in 1944. He published articles in the field of analysis, differential equations, probability, and number theory.[18]

8.2.2 Canonical Transformations

The first publication, in which Wintner dealt with the theory of canonical transformations that transform a Hamiltonian system into another or into itself, was in 1934. In this publication, he worked on the linear case.[19]

In 1941 he published the book *The Analytical Foundations of Celestial Mechanics*, in which he gave a mathematical formalization for celestial mechanics.[20] Part of this formalization is the theory of canonical transformations. He also presented the theory for the general case followed by the linear

[18]See (Hartman 1962).
[19]See (Wintner 1934).
[20]See (Wintner 1941).

case.

Before he addressed the general case of the theory of canonical transformation, Wintner defined the skew symmetric matrix J, and in the footnote he wrote:

> This skew-symmetric matrix, which will play a fundamental rôle [sic] in what follows, is known to represent the normal form of an arbitrary non-singular bilinear form; in the sense that there exists for every non-singular skew-symmetric matrix S a non-singular matrix T such that $T^tST = I$.[21]
>
> (Wintner 1941, p. 17)

Wintner figured out the relation between the skew-symmetric matrix and the bilinear form.

He showed that a transformation between two Hamiltonian systems is canonical if, and only if, the matrix relation

$$MJM^t = \lambda J \tag{8.16}$$

is fulfilled. In equation (8.16), $\lambda \neq 0$ is a constant scalar and M is a $2n$ dimensional Jacobian matrix. In this case, the canonical transformation maps a phase space onto another phase space. This transformation between two symplectic spaces is currently known as symplectomorphism.[22] Wintner remarked that the set of the canonical transformation forms a group which generalizes the symplectic group.[23]

[21]Wintner denoted the skew-symmetric matrix with I, but in this work it is denoted as J.

[22]See chapter 9, subsection 9.1.4

[23]See section 7.2.3.

Equation (8.16) is the general problem. Furthermore, he showed that it is sufficient for the canonical transformation, or, as Wintner called it "completely canonical transformation," if the matrix relation

$$MJM^t = J \tag{8.17}$$

is fulfilled. Then the set of Jacobian matrices of the transformation M forms a subgroup of the last group of transformations with the condition (8.16).

At the end of the book, Wintner included a section with historical notes and references:

> The linear canonical transformations as derived by A. Wintner (Ann. di Mat. (4) *13* (1934), 105-112) may also be described as forming the real subgroup of the "complex" (or "symplectic") group.
>
> (Wintner 1941, p. 415)

This shows that Wintner knew Weyl's new name for the complex group, but he did not use it in this book.

Nowadays, Wintner's book is still recognized, and the last reprint was 2014.

8.3 *Vorlesungen über Himmelsmechanik* by Carl Ludwig Siegel

In 1956, fifteen years after Aurel Wintner's *The Analytical Foundations of Celestial Mechanics*, Carl Ludwig Siegel published his lecture on celestial mechanics, that he gave in Göttingen from 1951 to 1952. These lecture notes were published in German under the name *Vorlesungen über Himmelsmechanik* and in them the name "symplectic group" was explicitly used to denote the group of canonical transformation which maps a Hamiltonian system into such a system.

Siegel started his studies in astronomy in Berlin in 1915, but during the first weeks of his studies he changed to mathematics.[24] Even though his mathematical interest was mainly in number theory, Siegel never left the field of astronomy behind. He wrote twelve papers on celestial mechanics in addition to his lecture notes, and he taught lectures on celestial mechanics during his whole life at universities in Germany and in the United States.[25]

> *Über die im folgenden behandelten Fragen der Himmelsmechanik habe ich in Frankfurt am Main und Baltimore sowie wiederholt in Göttingen und Princeton gelesen, am ausführlichsten in einem vierstündigen Göttinger Kolleg des Wintersemesters 1951/52. Herr Dr. J. Moser, jetzt in New York, hat damals eine sorgfältige Nachschrift angefertigt, welche dieser Veröffentlichung zugrunde liegt.*[26]
>
> (Siegel 1956, p.i)

The notes for the book were taken and worked on by Jürgen Kurt Moser[27]

[24] See chapter 2.1.

[25] (Rüssmann 1983, p.176).

[26] I have lectured on the questions in celestial mechanics treated in this work at Frankfurt on Main and Baltimore as well as again at Göttingen and Princeton, most fully in a lecture series during the winter semester of 1951/52 at Göttingen. At the time Dr. J. Moser, now in New York, prepared a careful set of notes on which this publication is based.(Translation second edition (Siegel & Moser 1971)).

[27] Jürgen Kurt Moser was born on July 4, 1928 in Königsberg, Germany and died on

in 1955. Moser was a doctoral student of Franz Rellich[28] at the university of Göttingen. He participated in Siegel's lecture during the winter semester of 1951/1952. The *Lecture Notes on Celestial Mechanics* became a standard reference after its first publication in German.

> *Die* "Lectures on Celestial Mechanics" *von Siegel und Moser sind zu eine Standardwerk in der Himmelsmechanik geworden.*[29]
>
> (Rüssmann 1983, p.190)

In 1971 the second edition was published in English with Moser as the co-author.

Siegel lecture notes related the symplectic group explicitly to the canonical transformation in mechanics, i.e. the elements of the symplectic group map a Hamiltonian system into such a system.[30] As mentioned in section (8.2.2), Wintner noticed before that the elements of the group are the canonical transformations, but Wintner only mentioned that the canonical transformations belong to the symplectic group in a note.

In the first chapter Siegel's lectures deal with *Das Dreikörperproblem.*[31]. Siegel developed the transformation theory of the Euler-Lagrange equation and Hamilton's equations. He proved that a Lagrangian system can be transformed into a Hamiltonian system through a Legendre transformation.

---

17 December, 1999 in Zürich. He did his doctorate in 1952 in Göttingen on differential equations.

[28]Franz Rellich (*1906, †1955) worked on the field of mathematical physics, quantum mechanics, and the theory of partial differential equations.

[29]The Lecture on Celestial Mechanics (the book) of Siegel and Moser became a standard reference on Celestial Mechanics.

[30]See (Siegel 1956, p.10).

[31]The Three Body Problem.

The second section of the first chapter is about *Kanonische Transformation*[32]. This has been standard knowledge in classical mechanics for the last century.

Siegel showed that the transformation $z = z(\zeta, t)$, which maps a Hamiltonian system into such a system, is a canonical transformation "if, and only if, the Jacobian matrix $z_\zeta = \mathfrak{M}$ is symplectic identically in ζ, t."[33] In Siegel's notation z denotes the $2n$ coordinates of the Hamiltonian function and ζ are the coordinates of the other $2n$ Hamiltonian function. $\mathfrak{M}$ is a symplectic matrix of the form:

$$\mathfrak{M} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

with A, B, C and $D \in \mathfrak{M}_n(\mathbb{R})$, so that $\mathfrak{M}^t J \mathfrak{M} = J$ with

$$J = \begin{pmatrix} 0 & Id \\ -Id & 0 \end{pmatrix}.$$

where

$$A^t C = C^t A, \quad B^t D = D^t B, \quad A^t D - C^t B = Id.$$

Siegel exposed the same results as Wintner, namely that the set of canonical transformations are the symplectic group.[34] But even though Siegel exposed the same results as Wintner, he introduced explicitly the symplectic group to characterize the canonical transformations in classical mechanics. It may be that the relation between canonical and symplectic transformations has been generally known at least in the German-speaking mathematical community. This can be assumed after having read the works of Wintner

[32] Canonical Transformations

[33] The original quote is in German (Siegel 1956, pp. 9,10).

[34] (Siegel 1956, p. 10).

and Siegel in classical mechanics. This work remains pending as it would involve consulting unpublished material such as correspondence, lecture notes, etc., but conducting this research in this paper is beyond the borders of the present work.

Siegel's lectures are an example of how to organize classical mechanics lectures. Many mathematicians followed his example not only in classical mechanics textbooks, but also in textbooks about symplectic geometry and symplectic topology. In some textbooks about symplectic geometry and symplectic topology, an introduction to classical mechanics is given in order to show where symplectic geometry is applied.[35]

8.4 Georges Reeb *Systémes Dynamiques*

A few months after Siegel gave his lectures on Celestial Mechanics in Göttingen in 1952, Georges Reeb wrote a small note in the weekly review of the sessions of the French Academy of Science (*Comptes rendus hebdomadaires des séances de l'Académie des Sciences*) about dynamical systems, where the phase space is a $2n$-dimensional manifold equipped with a differential structure.

8.4.1 George Reeb

George Reeb was born on November 12, 1920 and died on November 6, 1993 in Strasbourg. He was the first graduate student of Charles Ehresmann and obtained his doctoral degree in 1943, when the University of Strasbourg was located in Clermont-Ferrand. Later on, he taught at the University of Greno-

[35]See for classical mechanics (Arnold 1989) and for symplectic geometry and topology (McDuff & Salamon 1995, Berndt 1998).

ble and Strasbourg. In 1953 Reeb participated in the COLLOQUE INTERNATIONAL DE GÉOMÉTRIE DIFFÉRENTIELLE organized by Ehresmann and Lichnerowicz in Strasbourg, and in 1954 he spent a year at IAS Princeton.

Reeb's main topics were differential geometry and topology, and dynamical system theory.

8.4.2 *Systèmes Dynamiques*

In 1949 Reeb published three notes in the *Comptes rendus hebdomadaires des séances de l'Academie des Sciences.* In the first note (Reeb 1949*b*) he deals with closed trajectories of a differential system. Reeb used the theory of fibre bundles.[36]

First, he considered that a vector field defined over an n-dimensional smooth manifold M. The differential system over a manifold is

$$dx = Y(x)dt, \tag{8.18}$$

where x is a point in the manifold, $t \in \mathbb{R}$ and $Y(x)$ is a vector of the vector-field. Reeb studied the case of periodic orbits. In this case, the trajectories of the differential system (8.18) over a vector field are the fibres of the bundle of the manifold on a one-dimensional circle. The basespace of the fibre bundle is a $(n-1)$-dimensional manifold.

In the case that the vector field has singular points, Reeb showed that there exists a finite number of closed trajectories. The objective of the note was to apply this insight to the qualitative study of dynamical systems.

[36] As mentioned in chapter 5, the theory of fibre bundle was developed by Ehresmann and Feldbau at the beginning of the 1940s.

In his second note Reeb worked with canonical transformations of Hamiltonian functions, where the vector fields over a $2n$-dimensional manifold are associated with a Hamiltonian function $H(p,q)$, with p and q being local coordinates in the $2n$-dimensional manifold. This was standard knowledge in the field of celestial mechanics, but Reeb used the theory of fibre bundles here, where the base space of the fibre bundle of the manifold is a $(2n-1)$-dimensional space.[37]

Three years later in 1952 Reeb published another note in which he explicitly used Ehresmann's work of 1950 on almost complex structures, and he applied the results and ideas to the problems of dynamical systems.[38]

First, Reeb defined an almost complex manifold, and, because the existence of an almost complex structure is equivalent to the existence of a 2-form of rank $2n$, he defined a symplectic manifold.[39]

He added to a symplectic manifold $\tilde{N}$, with the dimension $2n$, a dimension which represents the time, i.e. $\tilde{N} \times \mathbb{R} = N$, and therefore, the dimension of N is $2n+1$.

The manifold N endowed with a differential form of rank $2n$ is called a dynamical system by Reeb:

[37] See (Reeb 1949*a*).
[38] See (Reeb 1952).
[39] See chapter 5.2.3.

> DÉFINITION 2. *On appelle S.D. [systéme dynamique] le couple* $(V_{2n+1}, \bigwedge)$*, d'une variété* V_{2n+1} *et d'une forme differentielle extérieure* $\bigwedge$*, de degré 2 et de rang* $2n$*, définie sur* V_{2n+1}*. Un S.D. vérifiant la condition supplémentaire* $d\bigwedge = 0$ *est appelé un S.D.I.*[40]
>
> (Reeb 1952, p.776)

The differential form that Reeb used for the *S.D.I* is the 2-form

$$\omega_N = \sum_{i=1}^{n} dp_i \wedge dq_i - dH \wedge dt \tag{8.19}$$

where $H = H(p, q, t)$ is a Hamiltonian function and $d\omega = 0$. It should be mentioned that Reeb did not explain what the I of *S.D.I.* means, but it could mean *invariant* because if the 2-form (8.19) is closed, it is called by him *l'invariant intègral absolu.* The $2n+1$-dimensional manifold N is called *l'espace phases-temps* by Reeb.

One of the properties that Reeb discovered was that on the tangent bundle of an n-dimensional manifold, it is possible that the cotangent bundle has the structure of a symplectic manifold.[41]

Reeb's work referred to dynamical systems. Therefore, a cotangent bundle of an n-dimensional manifold is a phase space. The n-dimensional manifold in a dynamical system is the configuration space. This is one of the first exposition of Hamiltonian systems on symplectic manifolds after the work done by Lee. As already mentioned, Lee's work on symplectic varieties was little known, however it may be that Reeb knew about it, although I have

[40] We called D.S. [Dynamical system] the pair $(V_{2n+1}, \bigwedge)$, of a manifold V_{2n+1} and a differential exterior form $\bigwedge$, of degree 2, and rank $2n$, defined on V_{2n+1}. A dynamical system satisfying the extra condition that $d\bigwedge = 0$ is called *S.D.I.*

[41] See (Reeb 1952)

not found elements to assure that Reeb knew about Lee's work. Although from the notation and terminology it could be intuited that the works are independent of each other.

Chapter 9

Géométrie Symplectique Différentielle

Introduction

The first Colloque International de Géométrie Différentielle took place at the University of Strasbourg in 1953. One of the presentations given by Jean Marie Souriau had the title *Géométrie symplectique differentielle-Applications*. In his presentation Souriau defined the symplectic vector space, developed what nowadays is known as Lagrangian submanifolds, defined explicitly symplectomorphisms, and gave some applications to classical mechanics.

The chapter will begin with a brief presentation of basic mathematical concepts such as subspaces and subvarieties of symplectic vector space, symplectic subvarieties and symplectomorphisms. Afterwards, Souriau's development will be presented.

9.1 Basic mathematical concepts

9.1.1 Subspaces of a Symplectic Vector Space

Let W be a linear dimensional subspace of dimension k of a symplectic vector space (V, ω). A 2-form on W is induced by symplectic form on V, and is denoted by $\omega|_W$. The ω-orthogonal space $W^\perp$ is defined as

$$W^\perp = \{v \in V \mid \omega(v, w) = 0, \forall w \in W\} \tag{9.1}$$

where $\dim W + \dim W^\perp = \dim V$.[1] The linear subspace W is said to be:

- **Isotropic** if $W \subseteq W^\perp$.
- **Coisotropic** if $W^\perp \subseteq W$.
- **Symplectic** if $W^\perp \cap W = 0$.
- **Lagrangian** if $W = W^\perp$.[2]

A Lagrangian vector space is a subspace W of a symplectic vector space (V, ω), which is ω-orthogonal and equal to its ω-orthogonal space. Lagrangian vector spaces are the maximal isotropic subspaces of a symplectic vector space, and therefore, the dimension of the Lagrangian subspace is half the dimension of the symplectic vector space.

If two Lagrangian subspaces L_n and L'_n of the symplectic vector space V are disjoint, then they are called transversal and then $V = L_n \oplus L'_n$.[3]

[1]The ω-orthogonal space $W^\perp$ of the linear subspace W of a symplectic vector space (V, ω) is as well called the skew-orthogonal space, orthogonal space or symplectic complement of W.

[2](Weinstein 1977).

[3](Banyaga 1994, p.20).

Let $(\mathbb{R}^{2n}, \omega_0)$ be a symplectic vector spaces, $\{q_1, ..., q_n, p_1, ..., p_n\}$ a canonical basis and ω_0 the canonical symplectic form. Let L_k and M_k be two subspaces of $(\mathbb{R}^{2n}, \omega_0)$ so that they are disjoint to each other. So, let the subspace L_k be spanned by $\{q_1, ..., q_k\}$ and the subspace M_k by $\{p_1, ..., p_k\}$ with $1 \leq k \leq n$, then they are isotropic subspaces. In the case $k = n$ the subspace L_n spanned by $\{q_1, ..., q_n\}$ and the subspace M_n spanned by $\{p_1, ..., p_n\}$ are Lagrangian subspaces.[4]

9.1.2 Submanifolds of a Symplectic Manifold

Let N be a differentiable manifold and M a symplectic manifold. A differentiable map $\iota : N \to M$ is called an isotropic (coisotropic, Lagrangian or symplectic) immersion at $x \in N$ if $T_x\iota : T_xN \to T_{\iota(x)}M$ is injective, and $T_x\iota(T_xN)$ is an isotropic (coisotropic, Lagrangian or symplectic) subspace of a symplectic vector space $(T_{\iota(x)}M, \omega_{\iota(x)})$.[5]

The pair (N, ι) is called isotropic (coisotropic, Lagrangian or symplectic) immersion if the map ι is an isotropic (coisotropic, Lagrangian or symplectic) immersion at every point $x \in N$.[6]

For the case that N is a submanifold of M, N is called isotropic (coisotropic, Lagrangian or symplectic) at a point $x \in N$ if T_xN is an isotropic (coisotropic, Lagrangian or symplectic) subspace of (T_xM, ω_x). N is called an isotropic (coisotropic, Lagrangian or symplectic) submanifold of M if it is so at every point.[7]

Of the last defined submanifolds, the Lagrangian submanifolds are important in classical mechanics because the behavior of a mechanical system

[4](Banyaga 1994, p.20).
[5](Libermann & Marle 1987, p. 92).
[6](Libermann & Marle 1987, p. 92).
[7](Libermann & Marle 1987, p. 92).

can be described in terms of Lagrangian submanifolds. Through the Lagrangian submanifolds, functions for the symplectic maps can be generated which trivialize a Hamiltonian system. Through Lagrangian submanifolds it is possible to describe the behavior of the physical system which has an associated symplectic manifold.[8]

An example of a Lagrangian immersion is the following: Let N be a differentiable manifold and T^*N be a 1-form β on a differentiable manifold N which is its cotangent bundle endowed with a symplectic form $\omega = d\alpha$, $\beta : N \to T^*N$, is a Lagrangian immersion if, and only if, β is closed.[9] Here α is a 1-form on the cotangent bundle of a manifold M, i.e.

$$\alpha : M \to T^*M,$$

called the Liouville form, which canonical coordinates is given by

$$\alpha = \sum_{i=1}^{n} p_i dq_i.$$

An example of a Lagrangian submanifold of the symplectic manifold (T^*N, ω) is the image $\beta(N)$ of the closed 1-form β on N. This Lagrangian submanifold is also called as well the graph of β.[10]

9.1.3 About the Name: Lagrangian Subspaces

In 1967 Vladimir Igorevich Arnold (*1937, †2010) published the article "Characteristic class entering in quantization conditions" in the journal *Functional Analysis and its application.*[11] He renamed the isotropic saturated spaces as

[8]See (Abraham & Marsden 1978, sec. 5.3).
[9](Libermann & Marle 1987, p. 92).
[10](Libermann & Marle 1987, p. 93).
[11]See (Arnold 1967).

Lagrangian subspaces, which are ω-orthogonal subspaces. Lagrangian submanifolds were firstly defined under the name *Variétés isotropes saturées*[12] by Jean-Marie Souriau (*1922, †2012). Souriau named the Lagrangian vector spaces as isotropic saturated vector spaces because they are the maximal isotropic subspaces of a symplectic vector spaces. Arnold gave the name Lagrangian subspaces in the following way:

> We consider an n-dimensional plane $\mathbb{R}^n \subset \mathbb{R}^{2n}$. It is called Lagrangian if the skew-scalar product of any two vectors of $\mathbb{R}^n$ equals zero. For example, the planes $p = 0$ and $q = 0$ are Lagrangian. The name comes from the "Lagrange bracket" in classical mechanics.[13]
>
> (Arnold 1967, p.1)

Arnold defined the Lagrangian manifolds as followed:

> Let M be an n-dimensional submanifold of the phase space $\mathbb{R}^{2n}$. The manifold M is called Lagrangian if its tangent plane at each point is Lagrangian. For example, in the case $n = 1$ every curve M on the phase plane $\mathbb{R}^2$ is Lagrangian.
>
> (Arnold 1967, p.3)

9.1.4 Symplectomorphisms

A symplectomorphism of a symplectic vector space (V, ω) is a vector space isomorphism $f : V \to V$ which preserves the symplectic structure

$$f^*\omega = \omega,$$

[12] Isotropic saturated manifold

[13] The Lagrange bracket is the scalar function $[X, Y] = \omega(X, Y)$ where X, Y are vector fields over a symplectic manifold. See (Abraham & Marsden 1978).

where $f^*\omega(v,w) = \omega(fv, fw)$, and $v, w \in V$. The symplectomorphisms of the symplectic vector space (V, ω) form a group called the symplectic group of (V, ω) denoted by $Sp(V)$. On an Euclidean space endowed with the canonical symplectic structure, $(\mathbb{R}^{2n}, \omega_0)$ the group of symplectomorphisms is denoted by $Sp(2n, \mathbb{R})$.

More generally, a symplectomorphism is a diffeomorphism between two symplectic manifolds of the same dimension:

Let (M_1, ω_1) and (M_2, ω_2) be two symplectic manifolds of the same dimension. The diffeomorphism $f : M_1 \to M_2$ is called a symplectomorphism if

$$f^*\omega_2 = \omega_1. \tag{9.2}$$

The canonical transformations of classical mechanics are symplectomorphisms.

A local symplectomorphism of a symplectic manifold M_1 into a symplectic manifold M_2, with the same dimension is a map $f : M_1 \to M_2$, so that every point of M_1 has an open neighborhood U_1 which satisfies that $U_2 = f(U_1)$ is open in M_2, and $f|_{U_1}$ is a symplectomorphism of $(U_1, \omega_1|_{U_1})$ onto $(U_1, \omega_2|_{U_2})$.[14] Any symplectic manifold (M, ω) of dimension $2n$ is locally symplectomorphic to $(\mathbb{R}^{2n}, \omega_0)$. This is just another way of stating Darboux's theorem.

9.2 Strasbourg 1953

From May 26 to June 1th 1953, the Colloque International de Géométrie Différentielle took place at the University of Strasbourg. It was organized by Charles Ehresmann and André Lichnerowicz[15].

[14] (Libermann & Marle 1987, p.94).

[15] Lichnerowicz (*1915, †1998) was a French mathematician. He contributed to differential geometry and mathematical physics, and he was the second doctoral advisor of Jean-Marie Souriau. See (Aa.Vv. 1976).

The *Colloque* of 1953 was conceived as an opportunity to exchange new ideas and developments in differential geometry. The objective of the *Colloque* was to create a forum for mathematicians, so that they could present their results and, after the war, to get in contact with other mathematicians around the world. This can be read in the introduction of the conference proceedings:

> *Nous nous sommes surtout efforcés de mettre en évidence certaines des voies nouvelles où s'engage notre science. Nous avons voulu aussi que de jeunes mathematiciens puissent mettre en pleine lumière leurs réflexions et leurs résultats.*[16]
>
> (Ehresmann & Lichneerowicz 1953, p. 10)

Some of the participants were De Rahm, Reeb, Libermann, Guggenheimer, Souriau and Eckmann.[17]

Eckmann, for example, gave a talk about complex structures and almost complex structures (*Sur les structures complexes et presque complexes*). Ehresmann talked about infinitesimal structures and Lie pseudo groups (*Introduction à la théorie des structures infinitésimales et des pseudo-groupes de Lie*). Lichnerowicz' contribution was about Kähler spaces (*Espaces homogènes kähleriénns*). During the conference, Libermann presented some results of her doctoral thesis, which at that time had not been defended yet (*Sur certaines structures infinitésimales régulières*).

Most significant for the development of symplectic geometry was Souriau's

[16]We have tried to bring to light some of the new pathways which our current science has taken. We also wanted that young mathematicians can show their reflections and their results in full light.

[17]The list of all participants in the *colloque* can be consulted in appendix C.

presentation with the title *"Géométrie symplectique differentielle - Applications"*.[18]

9.3 Géométrie Symplectique Différentielle - Applications

9.3.1 Jean-Marie Souriau

Jean-Marie Souriau was born on June 3, 1922 in Paris and died on March 15, 2012 in Aix-en-Provence. In 1942 Souriau started his studies at the *École Normale Supérieure* in Paris, and in contrast to the persecution that Jewish mathematicians in France suffered, such as Feldbaum or Libermann[19], his life as a student during the war in Paris was relatively normal.[20]

At the *École Normale Supérieure*, Souriau took lectures with Henri Cartan and with the physicist Yves Rocard (*1903, †1992).

After the *Libération* in 1944, he enlisted in the Army and interrupted his studies. Souriau continued them in 1945 at the end of the war. In 1945 Souriau took the *agrégation*[21] examination.[22]

In 1948 Souriau started to work on applied mathematics at the *Office National d'Etudes et de Recherches Aérospatiales* (ONERA).[23] There he wrote his dissertation on the stability of airplanes, *"Sur la stabilité des avions"*.

During this time Souriau gave a free public lecture in Paris, with the title

[18]Differential Symplectic Geometry- applications

[19]See chapter (Audin 2010) and sections 5.1 and 6.3.1

[20]See (Iglesias 1995*a*).

[21]In France, the *agrégation* is a competitive examination for some positions in the public education system.

[22]See (Iglesias 1995*a*).

[23]National Aerospace research office. French aerospace research centre.

"Méthodes nouvelles de la Physique mathématique" where he taught calculus of variations and matrix calculus to the general public.[24]

In 1952 Souriau worked as a professor at the *Institut des Hautes Études de Tunis* in Tunisia. In Tunesia, Souriau works on classical mechanics and develops the idea of "symplectic differential geometry". This is clear from his presentation at the colloquium in Strasbourg, but he also mentions it in an interview he gave to Iglesias in 1995.[25]

9.3.2 Differential Symplectic Geometry and applications

Souriau exposed in Strasbourg at the *Colloque* of 1953, his work about symplectic geometry and gave some of the mathematical methods for classical mechanics where the symplectic manifold is an important element.

In 1954 he summarized all the details of his presentation given in Strasbourg at the *Colloque* in an article named *"Equations canoniques et géométrie symplectique"*.[26]

The results that Souriau presented, both in (Souriau 1953) and in (Souriau 1954) did not deal with symplectic manifolds in the sense that Ehresmann did. He studied the submanifolds of a symplectic space.[27]

Souriau's presentation (Souriau 1953) in Strasbourg was divided into three parts. In the first part Souriau defined a vector space with a symmetric bilinear form and the vector space with a skew-symmetric bilinear form. The focus of his work was on the vector space with a skew-symmetric bilinear form. If the vector space is equipped with a skew-symmetric bilinear

[24] See (Vallée 2012).
[25] See (Iglesias 1995*a*, p.164).
[26] See (Souriau 1954).
[27] This can be seen below and was also said in (Kosmann-Schwarzbach 2013, p.143).

form it is a symplectic vector space. On the symplectic vector space Souriau classified the subspaces isotropic, coisotropic, symplectic and lagrangian in which the complement is an orthogonal vector space.[28]

In the second part, Souriau presented what he called differential symplectic geometry. Here Souriau defined the symplectic gradient of a function.[29] He studied the Lagrangian submanifolds contained in the hypersurfaces of a symplectic space. At the end of the second part he deduced a new method for solving first-order partial differential equations generalizing Jacobi's method.[30]

The third part deals with the applications of symplectic geometry to classical mechanics. He applied the new method for solving first order partial differential equations to the calculus of variations, which gives a practical method for finding extremes.[31]

9.3.3 Vector Space and Symplectic Vector Space

Souriau's presentation started with the real n-dimensional vector space and its corresponding dual space. The set of linear maps that defined the dual space were called *covectors* by Souriau, and through them he defined the scalar product on the real n-dimensional vector space. He defined the scalar product over the vector space, and in the case that the scalar product was antisymmetric then it was defined as a symplectic vector space.

[28] See section 9.1.1.

[29] In case the function is the Hamiltonian function then the symplectic gradient will be the vector field X_H.

[30] See (Kosmann-Schwarzbach 2013, p. 143) and (Souriau 1953, p. 58).

[31] See (Kosmann-Schwarzbach 2013, p. 143) and (Souriau 1953, p. 58-59).

> *Le cas du produit scalaire antisymétrique,[...], correspond à la géométrie SYMPLECTIQUE.*[32]
>
> (Souriau 1953, p.55)

After defining the symplectic vector space, Souriau defined its subspaces and showed that the dimension of Lagrangian subspaces are half the dimension of its symplectic vector space.

> *Le nombre de dimensions d'un sous-espace isotrope saturé est constant et égal à la moitié du nombre de dimension de l'espace.*[33]
>
> (Souriau 1954, p.246)

This implies that symplectic vector spaces are even dimensional spaces.

9.3.4 Phase space

In 1954 Souriau added a section *Espace de phase* to his Strasbourg presentation. This additional section pointed to the applications of symplectic geometry in classical mechanics, which, as already seen in chapter 7, Lee had already done in the 1940s.

Souriau constructed the phase space, a $2n$-dimensional vector space through an n-dimensional vector space V_0. Using Souriau's notation of 1954, the phase is denoted by $z = \{p, q\}$ with the following conventions:

$$z + z' = \{p + p', q + q'\}$$

[32] The case where the scalar product is anti-symmetric,[...], corresponds to SYMPLECTIC geometry.

[33] "The number of the dimesion of a Isotropic saturated [Lagrangian] subspace is constant and equal to the half of dimension of the space."

$$g(z, z') = pq' - p'q = \sum_{i=1}^{N} p_i q'_i - p'_i q_i$$

where $g(z, z')$ is the skew-symmetric form and p_i are the covariants coordinates of p and q'_i are the contravariant coordinates of q of the vector space V_0. The space form by the phase is a $2n$-dimensional vector space known as the phase space associated to an n-dimensional vector space V_0 and is called the configuration space. He remarked that all bases of V_0 correspond to the canonical basis of its phase spaces.

9.3.5 Géométrie Symplectique Différentielle

Souriau added the adjective *différentielle* because he wanted to obtain a *"totale différentielle"*[34] over a symplectic vector space V to obtain differentiable manifolds, which in this case are submanifolds of a symplectic space. The submanifolds that he was looking for were Lagrangian manifolds.

Souriau called these submanifolds *Variétés isotropes saturées (V.I.S.)*, and in 1954 he defined Lagrangian submanifolds as:

> *On appelle V.I.S. d'une espace symplectique une variété différentiable dont l'espace vectoriel tangent est en tout point isotrope saturé.*[35]
>
> (Souriau 1954, p.250)

Souriau noticed that a Lagrangian submanifold can be constructed by choosing an arbitrary function:

> *[...], on vérifie que tout V.I.S. s'obtient en choisissant une fonction arbitraire α des q_i et en écrivant la relation*
>
> $$d\alpha = \sum p_i \, dq_i$$

[34] total differential

[35] V.I.S. (Lagrangian) of a symplectic space is called a differentiable manifold where the vectorial tangent space is at every point isotropic and saturated (Lagrangian vector space).

> *les q_i étant éventuellement liés par un nombre arbitraire de relations.*[36]
>
> (Souriau 1953, p.56)

q_i is a canonical coordinate of a vector z, and z is a vector on a symplectic vector space with canonical coordinates (p_i, q_i). This idea of the construction of a Lagrangian submanifold is the same idea that was used in the example of section 9.1.2, which is an example after Souriau's publication.

Souriau started with an n-dimensional vector space V_0. On this vector space Souriau chose an arbitrary point q. The point q is an element of a k-dimensional differentiable manifold M_0, so that $0 \leq k \leq n$, and M_0 is immersed in the vector space V_0. The vector space V_0 was endowed with a scalar function α.

Souriau constructed the dual space of the tangent vector space of M_0 at the point q with the linear map $\frac{\partial\alpha}{\partial q}$. The covectors of the dual tangent vector space are represented by $p = \frac{\partial\alpha}{\partial q}$ and the differential 1-form

$$d\alpha = p\,dq \tag{9.3}$$

depends on $n - k$ parameters. A n-dimensional differentiable manifold M is created by Souriau through the phase space $z = \{p, q\}$, which is bound by the relations $q \in M_0$ and $d\alpha = p\,dq$.[37] The tangent vectors of M form a maximally isotropic subspace of a symplectic vector space with the dimension n, and therefore, the subspace is a Lagrangian subspace. So, every tangent vector of M is perpendicular to each other, and therefore, the manifold M is a Lagrangian submanifold.

---

[36] It is verifed that all V.I.S. can be obtained by choosing an arbitrary function α of q_i and by writing the relation $d\alpha = \sum p_i\,dq_i$.

[37] (Souriau 1954, p.249).

The importance of the Lagrangian subvarieties for Souriau is that by means of these it is possible to give a generalization of Jacobi's method for solving first order partial differential equations in the calculus of variations.[38]

9.3.6 Les Transformations Canoniques - Sympleto-morphisms

In 1954, as mentioned above, Souriau wrote a paper on the results he presented at the Strassbourg colloquium in 1953. In it, he proves and extends the results and in particular shows that

> *Tous les espaces symplectiques à $2N$ dimensions sont isomorphes.*[39]
>
> (Souriau 1954, p.246)

That is, symplectic spaces of dimension $2n$ are isomorphic by means of canonical transformations, which leave the symplectic form invariant. This result was exposed among symplectic manifold by Lee in 1947.[40] These canonical transformations are named in 1970 by Souriau symplectomorphisms. In the case that the symplectomophism is local between a symplectic manifold and the manifold $(\mathbb{R}^{2n}, \omega_0)$ then it is Darboux's theorem, as mentioned above. To obtain the result, Souriau gave the canonical basis of a symplectic vector space using a canonical projection and transforming the canonical basis into another canonical basis. This gave the isomorphisms between two symplectic vector spaces with the same dimension. Souriau showed that a canonical transformation is a diffeomorphism between two symplectic manifolds. The

[38] See (Souriau 1953, p. 57).
[39] All the symplectic spaces of $2N$ [$2n$] dimension are isomorphic.
[40] See chapter 7.2.6.

use of canonical transformations in classical mechanics at that time was in everyday use and therefore it can be seen that the link between symplectic geometry and classical mechanics became stronger.

9.3.7 Applications

The last part of each of his articles (Souriau 1953, Souriau 1954) is dedicated to the applications of symplectic geometry to classical mechanics problems. Some of the current applications were mentioned in section 9.3.4. Symplectomorphisms are the canonical transformation between two Hamiltonian systems.

One goal of the application of differential symplectic geometry was to provide the general solution for a differential equation of first order.

Souriau solved the Cauchy problem. The Cauchy problem asks for solutions of a partial differential equation that satisfies certain conditions which are given on a hypersurface. The hypersurface is a $2n-1$ dimensional surface on the $2n$ dimensional symplectic vector space.[41]

[41] (Souriau 1954, p.252).

Chapter 10

A brief outline of the take-off of symplectic geometry

10.1 Symplectic geometry takes off

The mathematical objects that are part of the field symplectic geometry were developed mainly during the decades 1930s to 1950s and some of the objects were explicitly named with the adjective symplectic.

The acceptance of symplectic geometry, like many of the new ideas in mathematics, took time until 1967. Between 1954 and 1967, the mathematical community did not consider symplectic geometry as an independent field. They studied the symplectic manifolds as a part of other fields like Kähler manifolds.

It was in the late 1960s that the study of symplectic manifolds attracted the interest of many mathematicians and physicists, at least in the Soviet Union, France, and the United States, among them Vladimir Igorevich Arnold, Ralph H. Abraham, Jerrold E. Marsden and Alan Weinstein, and they used the name symplectic geometry to define this field of studies.

10.1.1 Foundations of mechanics

In 1967 Ralph H. Abraham and Jerrold E. Marsden published their book "Foundations of mechanics". Abraham and Marsden developed a mathematical model for mechanics, in which they translated the structure of classical Hamiltonian systems into the language of calculus on manifolds and in which"symplectic manifolds are the setting for Hamiltonian mechanics".[1] It is to be remarked that they did not quote Souriau's work (Souriau 1953, Souriau 1954), but they referenced Siegel's book (Siegel 1956) in the bibliography, and other works by Siegel on celestial mechanics.

First, they exposed concepts of a symplectic vector space and the linear algebra on them, as locally a symplectic manifold is a symplectic vector space. They studied symplectic manifolds as "the globalization of the symplectic algebra"[2], and this globalization was called symplectic geometry. In particular, they proved Darboux's theorem.

After this publication, the mathematical community explicitly considered the study of symplectic geometry to be the study of symplectic manifolds, and symplectic geometry was linked to the study of classical mechanics.

In 1969, Alan Weinstein who is a professor at the University of California in Berkeley and had been a student of Chern, published a note about the symplectic structure on Banach spaces.[3] This article is the first of many other articles which deals with the symplectic structure in infinite dimensions.

Two year later in 1971, Weinstein gave a generalization of Darboux's theorem.[4] A year after Weinstein's publication, Marsden published a paper on symplectic structure in Banach spaces. In this paper, he defines what is

[1](Abraham & Marsden 1967, p.84).
[2](Abraham & Marsden 1967, p.92).
[3]See (Weinstein 1969).
[4]See (Weinstein 1971).

a weak bilinear structure and shows that *"Darboux's theorem fails for weak symplectic forms"*.[5]

So, the research and publications on symplectic geometry with the explicit name symplectic geometry started to become more frequent, as well as the use of symplectic geometry as a formulation for classical mechanics. This fact led some mathematicians such as Souria, Weinstein, Marsden and Iglesias, to think that symplectic geometry had its origins almost only in classical mechanics. All of them place the origin of symplectic geometry in Lagrange's work (Lagrange 1811).[6] In Lagrange's work, from a modern point of view, one can think that symplectic geometry originated there, but many of the tools necessary to say that with Lagrange symplectic geometry was born had not yet been developed. However, it is very likely, and not only that, it is certain that Sourian, Marsden, Weinstein and Iglesias were aware, in the late 1960s and early 1970s, of previous work on symplectic manifolds by Eheresmann and Liebermann in the 1950s. Libermann was academically very active in the 1960s. Weinstein was a student of Chern and Chern participated in the Strassbourg colloquium in 1953.[7] Iglesias was a student of Sourian in the 1980s and Sourian was a contemporary of Liebermann and Ehresmann.

10.1.2 Symplectic geometry conferences in 1973 and 1974

At the end of the 1960s and the beginning of the 1970s, more mathematicians were working on problems of symplectic geometry and classical mechanics, and this provided the incentive to hold a conference about symplectic ge-

[5]See (Marsden 1972).
[6]See (Iglesias 1995*b*, Souriau 1986, Marsden & Weinstein 2001).
[7]See Appendix C.

ometry and classical mechanics. This conference took place in Rome at the *Istituto Nazionale di Alta Matematicia*(INDAM) in January 18 to 23, 1973 under the name *Geometria simplettica e fisica matematica.* It was the first conference whose topic was symplectic geometry. Some of the significant speakers were Souriau, Marsden and Weinstein.[8]

A year later in 1974, Souriau, who was already at the university of Marseilles, organized another conference with the title *"Géométrie symplectique et physique mathématique"*[9] in Aix-en-Provence, France. The conference has been well attended.[10] Other conferences on symplectic geometry follwed in the 1980s.[11]

10.1.3 Textbooks and journals about symplectic geometry

After Abraham and Marsden's book "Foundations of Mechanics" in the 1970s there were more publications about dynamical systems, which dealt with symplectic geometry as the mathematical and geometrical interpretation of mechanics, such as Souriau's book in 1970.[12] In 1974, Arnold published *Mathemacheskie metody klassischeskoi mekhaniki*[13] in Russia. The second edition of the book was translated into English in 1978 by Karen Vogtmann and Weinstein, and a second edition of the translation was printed in 1989.

In 1977, Weinstein published his lecture notes on symplectic manifolds, and this is one of the first textbook about symplectic geometry.[14]

[8]For the complete list of speakers see appendix D.

[9]Symplectic geometry and mathematical physics. See (Aa.Vv. 1975).

[10]See appendix E.

[11](Kosmann-Schwarzbach 2013, p.145).

[12]See (Souriau 1970).

[13]Mathematical methods of classical mechanics

[14]See (Weinstein 1977).

In the 1980s another textbook about symplectic geometry was published by Libermann and Charles-Michel Marle in 1987 with the title "Symplectic Geometry and Analytical Mechanics".[15]

In all these books symplectic geometry is linked to classical mechanics, and this link gives the one-sided impression that the exclusive origin of symplectic geometry is classical mechanics.

A Journal about symplectic geometry was founded in 2001 and some of its co-editors at the beginning were Marsden and Weinstein.

[15] See (Libermann & Marle 1987).

Conclusion

In the 1970s, Arnold worked on symplectic geometry and played a key role together with Abraham, Marsden and Souriau, in the acceptance of the field of symplectic geometry. All of them used symplectic geometry as a method for classical mechanics.[16]

Arnold pointed out that there is a close relation between mathematics and physics. Therefore, his work on symplectic geometry was linked to physics, and classical mechanics. This can be read in the obituary of Arnold written by Gusein-Zade and Varchenko, and remembering that Arnold used to claim:

> Mathematics is a part of physics. Physics is an experimental science, a part of natural sciences. Mathematics is the part of physics where experiments are cheap.
>
> (S. M. Gusein-Zade 2010, p.28)

The origin of the field of symplectic geometry is not exclusively and even not primarily classical mechanics. The objects of symplectic geometry were developed during the 1930s. First by Kähler, who defined the Kählerian manifolds. Then in 1950, Ehresmann defined the symplectic manifolds, as a tool for the study of the introduction of a complex structure on even-dimensional differentiable manifolds. Libermann states Darboux's theorem

[16]See (Abraham & Marsden 1967, Souriau 1970, Arnold 1978).

of symplectic geometry.

The formulation of classical mechanics in terms of symplectic geometry is first given by Lee in the 1940s, although this work was not transcendent since it was well not known. At that time and during the 1950s, Wintner and Siegel introduced the symplectic group to characterize the canonical transformations of classical mechanics. Reeb contributed by applying Ehresmann's results on symplectic varieties to dynamics problems. Souriau's contribution to this formulation is the Lagrangian submanifolds that make it possible to generalize Jacobi's method to find solutions of first order partial equations in the calculus of variations.

Siegel's work (Siegel 1943*b*) did not contribute to the current development of symplectic geometry despite the name "Symplectic geometry". As shown in Chapter 2, Siegel generalized hyperbolic geometry to an m-dimensional complex space. This was later known as "Siegel's half-space". In 1957, Henri Cartan shows that the Siegel half-space is a Kähler manifold and therefore symplectic, but this result remains as one more result in the work of Henri Cartan in the study of complex functions of several variables. This result also shows that Siegel's half-space is an object of study of the current symplectic geometry.

The acceptance of the field symplectic geometry, as it is known nowadays and its link with classical mechanics, took place in the late 1960s and in the 1970s with the contributions of Abraham, Marsden, Souriau, Weinstein and many other mathematicians.

Finally, I would like to mention that I am aware of the existence of the paper *"Differential geometry in symplectic space"* by Chern and his student Hsien-Chung Wang in 1947, which was published in 1947 in the journal of

Tsing Hua University.[17] Kosmann-Schwarzbach mentions that in this article the term “symplectic” was used only in the study of the group of linear transformations of a vector space preserving a nondegenerate antisymmetric bilinear form, the space being necessarily of even dimension. Chern and Wang call a vector space of this type ”symplectic space”.[18] Therefore, it was not used in this work. However, the possibility of analyzing it and commenting on the development Chern and Wang carried out remains open.

[17](Chern & Wang 1947).

[18]See (Kosmann-Schwarzbach 2013, p. 137).

Appendices

Appendix A

Shiing-Shen Chern

This is a very short excursion on Chern's biography and his relationship with Kähler and Cartan. Ending with Chern's context when he proves the generalization of the Gauss-Bonnet theorem.

Chern was born on October 28, 1911 in Jiaxing[1] China and died in December 3, 2004. At the age of fifteen he entered the Nankai University to study mathematics. He was a student of Lifu Jiang[2]. In 1930 Chern went to Tsinghua University in Beijing for his graduated studies. In 1932 Blaschke went to Beijing to gave some lectures on geometry. Chern went to Blaschke's lectures and so he decided that he would like to go to Hamburg for his PhD under Blaschke direction.[3]

Hamburg started to be a important centre for the mathematicians although University of Hamburg was founded in 1919. As is mention in chapter

[1]Jiaxing is on the west of China one hundred kilometres south-west form Shanghai.

[2]Lifu Jiang was one of the mathematicians that went to Harvard in USA to achieve his PhD.

[3]At that time China had a exchange program with the USA and not with Germany and Chern awarded a fellowships for the USA. Chern requested to use his fellowships for his studies in Germany.

4 this was a reason why Kähler did not accept the professorship in Rostock. In 1934 Chern went to Hamburg to achieved his doctor degree under the supervision of Blaschke. In Hamburg Chern work in the Invariant theory on a webbof a n-dimensional manifold in a $2n$-dimensional space.

A.1 Chern and Erich Kähler

The relationship between Kähler and Chern started when Chern went to Hamburg to write his thesis. In 1934, Chern attended Kähler's seminar and he mentioned:

> *"[The seminar of Kähler in Hamburg 1934] looked like a kind of celebration. The classroom was filled, and the book (Kähler 1934) had just come out. Kähler came in with a pile of the books and gave everybody a copy. But the subject was difficult, so after a number of times, people didn't come any more. I think I was essentially the only one who stayed till the end because I followed the subject."*
>
> (Jackson & Kotschick 1998, p.860)

Chern developed a good relationship with Kähler that last their hold live. An example of this friendship is when Kähler was a war prisoner in France, during the years 1945 and 1947, he wrote to Chern who was at that time in the Academia Sinica in China, asking for books and some tea.[4]

[4](Yau 2011, p.1238).

A.2 Chern and Élie Cartan

Chern went to Paris in 1936 to study with Élie Cartan where he spend a year. Chern remember these meetings with Cartan in the interview he gave in 1998:

> Usually the day after [meeting with Cartan] I would get a letter form him. He would say, "After you left, I thought more about your questions [...] I saw him [Élie Cartan] about once every two weeks,[...]
>
> (Jackson & Kotschick 1998, p.861)

Although Chern did not understand much French and Élie Cartan spoke only French, both manage to communicated and to work together during that year.[5]

In 1937 he went back to China to the South-west Associated University[6] as professor of mathematics.

A.3 Gauss-Bonnet Theorem

An example of the used of the Kähler manifolds is the proof that Chern made of generalization of the Gauss-Bonnet Theorem.

In China, even though the hart times that the war brought Chern continued reading and researching about some papers that Élie Cartan had send

[5]Sometime is amazing how people can communicated although the barrier of the language

[6]During the Japanese invasion the Tsing Hua University of Beijing and the University of Nanka were evacuated to Kunming and there the China government formed the South-West Associated University

to him. He continued publishing. His publication gave him the possibility of being recognized by the mathematician in the international circles. In 1943 he went to the IAS in Princeton for two years, which implied at war time a very difficult trip. Chern took seven days to arrive to the United States.

In Princeton he met André Weil. In 1942 Weil published a paper with Carl Barnett Allendoerfer [7] (Allendoerfer & Weil 1943) where they presented a proof of the generalization of Gauss-Bonnet theorem. Weil mention that for the proof they followed the steps of Weyl.[8] The proof is based on the fact that a Riemann manifold can be locally isometrically embedded in a Euclidean space.[9]

Weil talk with Chern about this proof and Chern proposed an other proof which is an intrinsic proof of the generalized Gauss-Bonnet theorem. The generalization states that the Euler characteristic $\chi(M)$ of a closed Riemann manifold M of arbitrary dimension can be express as an integral of the Gauss curvature over a manifold. Chern proof the generalization for hermitian manifolds in which are inserted the Kähler manifolds but not for Riemannian manifolds. For further details about how Chern proof the generalization and the mathematical impact of this proof the reader can consult (Palais & Terng 1992) and (Wu 2008).

[7] Carl Barnett Allendoerfer was an US-American mathematician, he was born on April 4, 1911 in Kansas City and died on September 29, 1974. His field of work was the topology and the geometry. He work as well on mathematical education and produced some mathematical films.

[8] See (Wu 2008)

[9] (Jackson & Kotschick 1998, p.7).

Appendix B

Vladimir Igorevich Arnold and Arnold's conjecture

Vladimir Igorevich Arnold was born on June 12, 1937 in Odessa, Soviet Union and died on June 3, 2010 in Paris. In 1957, he was a student of Andrey Kolmogorov (*1903, †1987) and he graduated at the Moscow State University in 1959. Arnold obtained a chair at the Moscow State University in 1965, where he worked until 1986. From 1965 to 1966, Arnold went to Paris for a research year. After 1986, he worked at the Steklov Mathematical Institute in Moscow and in 1993 he obtained a position at the Paris Dauphine University. He work in both places until his death.

One of Arnold's fields of studies was on dynamical systems and in 1963 Arnold presented his development which is known as Kolmogorov-Arnold-Moser theory in a lecture. There he gave an explanation for the stability of the solar system.[1]

[1] See (Khesin & Tabachinikov 2012).

B.1 Arnold's conjecture

In 1965, Arnold stated a conjecture which is the generalization of what is called by him as the geometric theorem of Poincaré, which stats:

> An area-preserving diffeomorphism of an annulus that moves the two bounding circles in opposite directions has no fewer than two fixed points.
>
> (Arnold 1986, p. 3)

In 1972 and 1976 he formulated a conjecture which is a multidimensional generalization of Poincaré's theorem what is known as the Arnold conjecture.[2] The conjecture states:

> A symplectomorphism of a compact manifold, homologous to the identity transformation (Join by a one-parameter family of symplectomorphisms with single-valued, but time dependent, Hamiltonians.), has at least as many fixed points as a smooth function on the manifold has critical points.
>
> (Arnold 1986, p. 4)

The attempts to prove the Arnold conjecture contributed to the development symplectic topology.[3]

[2] See (Arnold 1986).

[3] See (Arnold 1986, Audin 2014, Khesin & Tabachinikov 2012, McDuff & Salamon 1995).

B.2 Arnold's conjecture and symplectic topology

This section is a short excursion through the history of symplectic topology. Michèle Audin published part of the history of symplectic topology in 2014.[4] As Ian Stewart defined in a popular sciences publication, symplectic topology is the study of symplectic mappings of symplectic manifolds, and symplectic mapping of the plane is thus any transformation that preserves area.[5]

Arnold contributed to the development not only of symplectic geometry, but also contributed to the development of the field symplectic topology. The attempts to prove Arnold's conjecture lead to the development of symplectic topology. In 1986 Arnold attributed in his article with title "First steps in symplectic topology", proof of the conjecture to Mikhael Gromov.[6] Gromov gave a proof for a special case in 1985 (Gromov 1985), but it was not a general proof.[7]

In 1986 Arnold was interested is not only finding in symplectic topology, but he wanted what he called a symplectization of the whole theory.[8] As an example of this, Arnold asked if a symplectic camel can go through the eye of a needle? The symplectic mapping for an $2n$ dimensional space preserves volume. In the case of a camel, it can be stretched out until it is so thin that it can go through the needle, but in the case of the symplectic camel, this is not possible, this was prove by Gromov.[9]

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[4] See (Audin 2014).
[5] (Stewart 1987, p.17,18).
[6] See (Arnold 1986).
[7] (Audin 2014, p.17).
[8] (Arnold 1986).
[9] (Stewart 1987).

Appendix C

Liste of participants at the *Colloque International de Géométrie Différentielle, Strasbourg*

Organizers

- Charles Ehresmann, University of Strassbourg, France.
- André Lichnerowicz, Collège de France.

Participants

- E. Bampiani, Roma, Italy.
- S. S. Chern, Chicago, United States of America.
- E. T. Davies, Southampton, England.

- P. Dedecker, Brussels, Belgium.
- B. Eckmann, Zürich, Switzerland.
- E. Heinz, Göttingen, Germany.
- N. H. Kuiper, Wageningen, Netherlands.
- H. Rund, Bonn, Germany.
- M. Villa, Bologne, Italy.
- T. J. Willmore, Durham, England.
- J. L. Koszul, Strassbourg, France.
- Paulette Libermann, Strassbourg, France.
- G. Reeb, Grenoble, France.
- L. Schwartz, Paris, France.
- J. M. Souriau Tunis, France.
- R. Thom C.N.R.S., France.
- A. Weil Chicago, United States of America.
- A. Aragnal, Paris, France.
- M. Berger, Paris, France.
- D Bernard, Paris, France.
- C. de Carvalho, Rio de Janeiro, Brazil.
- C. Chabauty, Strassbourg, France.

- G. Cerf, Strassbourg, France.
- R. Debever, Brussels, Belgium.
- M. Decuyper, Lille, France.
- J. Deny, Strassbourg, France.
- A. Frölicher, Zürich, Switzerland.
- F. Gallissot, Grenoble, France.
- L. Godeaux, Liège, Belgium.
- H. Guggenheimer, Bôle, Switzerland.
- R. Guy, Neuchâtel, Switzerland.
- M. Heins, Brown University, United States of America.
- R. Hermann, Amsterdam, Netherlands.
- H. Hopf, Zürich, Switzerland.
- M. Iss, Strassbourg, France.
- F. Jongmans, Liège, Belgium.
- G. Legrand, Paris, France.
- S. Lemoine, Paris, France.
- J. Loiseau, Paris, France.
- M. Lyra, Sao Paolo, Brazil.
- B. Malgrange, Paris, France.

- J. Milnor, Zürich, Switzerland.
- R. Piedvache, Poitiers, France.
- G. de Rham, Lausanne, Switzerland.
- M. H. Schwartz, Paris, France.
- H. B. Shutrick, Liverpool, England.
- E. H. Spanier, Chicago, United States of America.
- W. Süss, Freiburg, Germany.
- Y. Thiry, Tunis, France.[1]

[1](Aa.Vv. 1953, pp.8-9)

Appendix D

"Geometria simplettica e fisica matematica", Rome, 1973

List of speakers

- Cushman, R.
- Díaz Miranda, A.
- Elhadad, J.
- García, P. L.
- Klein, J.
- Konstant, B.
- Kumpera, A.
- Künzle, H. P
- Leray, J.
- Lichnerowicz, A.
- Marsden, J. E.
- Ouzilou, R.
- Pérez-Rendón, A.
- Segal, I.
- Souriau, J. M.
- Streater, R. F.
- Tulczyjew, W. M.
- Weinstein, A.[1]

[1] See (Aa.Vv. 1974).

Appendix E

"Géométrie Symplectique et Physique Mathématique", Aix-en-Provence, France, 1974

List of speakers

- J.W Robbin, University of Wisconsin.
- Carl P. Simon, University of Michigan.
- Charles J. Titus, University of Michigan.
- André Lichnerowicz, Collège de France, Paris.
- Shlomo Sternberg, Harvard University.
- Jean-Marie Souriau, Université de Provence.
- Andrei Iacob, Institut de Mathématique de Bucarest.
- H.P. Kunzle, University of Alberta.

- Anthony Leung, University of Cincinnati.
- Kenneth Meyer, University of Cincinnati.
- Clark Robinson, University of Warwick.
- Enrico Onofri, Università di Parma.
- David Simms, Trinity College, Dublin.
- Robert J. Blattner, University of Massachusetts.
- Bertram Kostant, Massachusetts Institute of Technology.
- K. Gawedzki, Université de Varsovie.
- Victor W. Guillemin, Massachusetts Institute of Technology.
- Jean Leray, Collège de France, Paris.
- A. Voros, Centre d'Etudes Nucléaires de Saclay.
- Alan Weinstein, University of California, Berkeley.
- F.J. Bloore, University of Liverpool
- Wilhelm Klingenberg, Université de Bonn.
- B.R. Chernoff, University of California, Berkeley.
- Jerrold E. Marsden, University of California, Berkeley.
- Arthur E. Fischer, University of California, Berkeley.
- Jerzy Kijowski, Université de Varsovie.
- Wiktor Szczyrba, Université de Varsovie.

- Dominique-Paul Chevallier, Ecole Nationale des Ponts et Chaussées, Paris
- Andrés J. Kalnay, Centro de Física, IVIC, Caracas.
- Jean-Pierre Vigier, Institut Henri Poincaré, Paris.[1]

[1]See (Aa.Vv. 1975).

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