Cohomology of the structure sheaf of Deligne–Lusztig varieties for GL_n

A thesis submitted for the degree of
Doctor of Philosophy
at the Bergische Universität Wuppertal

by

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2022

The PhD thesis can be quoted as follows:

urn:nbn:de:hbz:468-20220413-162926-4
[http://nbn-resolving.de/urn/resolver.pl?urn=urn%3Anbn%3Ade%3Ahbz%3A468-20220413-162926-4]

DOI: 10.25926/kx3b-y913
[https://doi.org/10.25926/kx3b-y913]

COHOMOLOGY OF THE STRUCTURE SHEAF OF DELIGNE–LUSZTIG VARIETIES FOR GL_n

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ABSTRACT. In this paper, we give a description of the cohomology groups of the structure sheaf on smooth compactifications $\overline{X}(w)$ of Deligne–Lusztig varieties $X(w)$ for GL_n, for all elements w in the Weyl group. To this end we adapt the double induction on the presentation and length of w from [Orl18] for ℓ-adic cohomology. Then we use the Artin–Schreier–Witt sequence to obtain the mod p^m and integral p-adic étale cohomology of $\overline{X}(w)$. Moreover, using our result for $\overline{X}(w)$ and a spectral sequence associated to a stratification of $\overline{X}(w)$, we deduce the mod p^m and integral p-adic étale cohomology with compact support of $X(w)$.

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Date: March 1, 2022.

Introduction

Deligne–Lusztig varieties were introduced in [DL76] for studying irreducible representations of finite groups of Lie type. Fix an algebraic closure $\overline{\mathbb{F}}_p$ of $\mathbb{F}_p$ and let $\mathbb{F}_q$ be the subfield of $\overline{\mathbb{F}}_p$ with q elements, where q is a power of p. Let G_0 be a connected reductive group over $\mathbb{F}_q$, and let G be the base change of G_0 to $\overline{\mathbb{F}}_p$. Fix a maximal torus and a Borel subgroup $T^* \subseteq B^*$ of G. For elements w of the Weyl group $W = N(T^*)/T^*$, the Deligne–Lusztig varieties $X(w)$ associated to G are smooth quasi-projective $\overline{\mathbb{F}}_p$-schemes of finite type. Considered as subschemes of G/B^*, they are stable under the action of $G_0(\mathbb{F}_q)$. Deligne and Lusztig considered the virtual representations arising from the ℓ-adic cohomology with compact support of $X(w)$ and their coverings for $\ell \neq p$. Furthermore, they constructed smooth compactifications $\overline{X}(w)$ of $X(w)$, for each reduced expression of w, inspired by the Demazure–Hansen desingularization of Schubert varieties.

In this paper, for $G = \mathrm{GL}_n$, we give a description of the coherent cohomology of the structure sheaf on $\overline{X}(w)$ as $\overline{\mathbb{F}}_p$-vector spaces and as $\mathrm{GL}_n(\mathbb{F}_q)$-representations. In particular, the higher cohomology groups vanish. Since taking global sections of the structure sheaf detects the number of irreducible components of $\overline{X}(w)$, and $\mathrm{GL}_n(\mathbb{F}_q)$ acts on the set of irreducible components of $\overline{X}(w)$ transitively, the global sections carry a structure of induced representations. Moreover, as a consequence of the Artin–Schreier–Witt sequence, we describe the $\mathbb{Z}/p^m\mathbb{Z}$- and $\mathbb{Z}_p$-cohomology of $\overline{X}(w)_{\text{ét}}$. On the other hand, $\overline{X}(w)$ has a stratification, in which the complement of $X(w)$ is the union of suitable $\overline{X}(v)$ of lower dimension. Using a spectral sequence akin to the Mayer–Vietoris sequence with respect to this stratification, we obtain the $\mathbb{Z}/p^m\mathbb{Z}$- and $\mathbb{Z}_p$-cohomology with compact support of $X(w)_{\text{ét}}$ as $\mathrm{GL}_n(\mathbb{F}_q)$-modules. Furthermore, for each w, these cohomology groups vanish except at the degree equal to the Bruhat length of w, and the non-vanishing term affords an induced representation of the Steinberg module of a Levi subgroup of $\mathrm{GL}_n(\mathbb{F}_q)$.

A prominent example of Deligne–Lusztig varieties in the case of $G = \mathrm{GL}_n$ is the Drinfeld half space. It is isomorphic to $X(\mathbf{w})$ ($\mathrm{GL}_n(\mathbb{F}_q)$-equivariantly), where $\mathbf{w} = s_1 \cdots s_{n-1}$ is the standard Coxeter element (under suitable choice of $T^* \subseteq B^*$), where s_i corresponds to the reflection $(i, i+1)$ in the symmetric group $S_n \cong W$. This construction originated from the work of Drinfeld in 1973 on the study of the discrete series representations of $\mathrm{SL}_2(\mathbb{F}_\mathrm{q})$. The Drinfeld half space $X(\mathbf{w})$ can be embedded into the projective space G/P^*, where $P^* \supset B^*$ is the parabolic subgroup corresponding to $s_2, \ldots, s_{n-1}$. In fact, this gives an isomorphism $X(w) \cong \mathbb{P}^{n-1}_{\overline{\mathbb{F}}_p} \setminus \mathcal{H}$, where $\mathcal{H}$ is the union of all $\mathbb{F}_q$-rational hyperplanes in

$\mathbb{P}^{n-1}_{\overline{\mathbb{F}}_p}$. The cohomology of the structure sheaf for the smooth compactification $\overline{X}(s_1 \cdots s_{n-1})$ of $X(\mathbf{w})$ is known [GK05, §2].

Theorem ([GK05]). *Let $G = \mathrm{GL}_n$ and $\mathbf{w} = s_1 \cdots s_{n-1}$ the standard Coxeter element.*

$$H^k\left(\overline{X}(\mathbf{w}), \mathcal{O}_{\overline{X}(\mathbf{w})}\right) = \begin{cases} \overline{\mathbb{F}}_p, & k = 0, \\ 0, & k > 0. \end{cases}$$

The Drinfeld half space also has analogues when considered over finite extensions of $\mathbb{Q}_p$ cf. [SS91].

In [Orl18], Orlik provided a strategy for computing the ℓ-adic cohomology groups with compact support of Deligne–Lusztig varieties for $G = \mathrm{GL_n}$ and their realizations as ℓ-adic representations of $\mathrm{GL_n}(\mathbb{F}_\mathrm{q})$, which we recall here. Let F^+ be the free monoid generated by a fixed set of standard generators of W. The generalized Deligne–Lusztig variety attached to $w \in F^+$ were introduced in [DMR07]. The idea of Orlik's strategy is to study the relation between cohomology groups of $\overline{X}(w)$ and $\overline{X}(w')$, where $w, w' \in F^+$ and the set of the standard generators of W showing up in the presentation of w coincides with that of w'. More specifically, w and w' may differ in length or differ by a relation in the group presentation of W. One may now establish a double induction procedure with respect to the length of w and the number of relations applied. The base case is the cohomology of $\overline{X}(w)$ with $w \in F^+$ having no repeating terms in its expression.

We adapt Orlik's method of double induction to the cohomology of the structure sheaf on $\overline{X}(w), w \in F^+$, and establish the base case using Grosse-Klönne's result as stated above.

After preparatory work from Section 1 to 6, we obtain our main theorem:

Theorem (6.1). *Let $G = \mathrm{GL}_n$ and $w \in F^+$ with $w = s_{i_1} \cdots s_{i_r}$. Let $I = \{s_{i_1} \dots s_{i_r}\}$ and $P_I \supseteq B^*$ be the standard parabolic subgroup associated to I, then*

$$H^k\left(\overline{X}(w), \mathcal{O}_{\overline{X}(w)}\right) = \begin{cases} \mathrm{ind}_{P_I(\mathbb{F}_q)}^{\mathrm{GL}_d(\mathbb{F}_q)} \mathbb{1}_{\overline{\mathbb{F}}_p}, & k = 0, \\ 0, & k > 0, \end{cases}$$

where $\mathbb{1}_{\overline{\mathbb{F}}_p}$ is the trivial $P_I(\mathbb{F}_q)$-representation with coefficients in $\overline{\mathbb{F}}_p$.

Moreover, if we consider the sheaf of Witt vectors $W_m\left(\mathcal{O}_{\overline{X}(w)}\right)$ of length m on $\overline{X}(w)$, the theorem above implies that

$$H^0\left(\overline{X}(w), W_m\left(\mathcal{O}_{\overline{X}(w)}\right)\right) = \mathrm{ind}_{P_I(\mathbb{F}_q)}^{\mathrm{GL}_d(\mathbb{F}_q)} W_m\left(\overline{\mathbb{F}}_p\right).$$

Then we can use the Artin–Schreier–Witt sequence to compute the cohomology of the constant sheaves $\mathbb{Z}/p^m\mathbb{Z}$ and $\mathbb{Z}_p$ on $\overline{X}(w)_{\text{ét}}$ with $w \in F^+$:

Corollary (6.6, 6.7). *Let $G = \mathrm{GL}_n$, and $w \in F^+$ with $w = s_{i_1} \cdots s_{i_r}$. Let R be $\mathbb{Z}/p^m\mathbb{Z}, m \geq 1$, or $\mathbb{Z}_p$.*

$$H^k_{\text{ét}}\left(\overline{X}(w), R\right) = \begin{cases} \mathrm{ind}_{P_I(\mathbb{F}_q)}^{\mathrm{GL}_n(\mathbb{F}_q)} \mathbb{1}_R, & k = 0, \\ 0, & k > 0, \end{cases}$$

where $\mathbb{1}_R$ is the 1-dimensional trivial $P_I(\mathbb{F}_q)$-module with coefficients in R.

In Section 7, we analyze a spectral sequence associated to a stratification of $\overline{X}(w)$:

$$E_1^{i,j} = \bigoplus_{\substack{u \preceq w \\ \ell(u) = \ell(w) - i}} H^j_{\text{ét}}\left(\overline{X}(u), \mathbb{Z}/p^m\mathbb{Z}\right) \Rightarrow H^{i+j}_{\text{ét},c}\left(X(w), \mathbb{Z}/p^m\mathbb{Z}\right).$$

The corollary above implies that this spectral sequence degenerates at the E_2-page. In particular, it can be computed by the Solomon-Tits complex, which is a simplicial complex constructed from the group $\mathrm{GL}_n(\mathbb{F}_q)$. This method has been used to study the compactly supported ℓ-adic cohomology of $X(w)$ [Orl18, §5, §7], c.f. [DOR10, Ch VII].

Corollary (7.5, 7.6)**.** *Let $G = \mathrm{GL}_n$ and $w \in F^+$. Let $L_I \supseteq T^*$ be the standard Levi subgroup of GL_n such that $P_I = U_I \rtimes L_I$, where U_I is the unipotent radical of P_I. Let R be $\mathbb{Z}/p^m\mathbb{Z}, m \geq 1$ or $\mathbb{Z}_p$. Then*

$$H^k_{\text{ét},c}\left(X(w), R\right) = \begin{cases} 0, & k \neq \ell(w), \\ \mathrm{ind}_{P_I(\mathbb{F}_q)}^{\mathrm{GL}_n(\mathbb{F}_q)} \mathrm{St}_{L_I}, & k = \ell(w), \end{cases}$$

where St_{L_I} is the Steinberg module for L_I with coefficients in R. In particular, when $I = S$, we have $H^{\ell(w)}_{\text{ét},c}\left(X(w), R\right) = \mathrm{St}_{\mathrm{GL}_n}$.

Finally, it is natural to ask whether the double induction method applies to the case for $X(w)$ instead of the smooth compactification. One possibility is to adapt the steps directly to $H^k\left(X(w), \mathcal{O}\right)$. As the base case, Appendix A contains a description of $H^0\left(X(w), \mathcal{O}\right)$ for $w \leq \mathbf{w}$, analogous to Kuschkowitz's result for the Drinfeld half space [Kus16, Theorem 2.1.2.1], which gives a filtration on $H^0\left(X(\mathbf{w}), \mathcal{O}\right)$ such that each subquotient fits into an extension of $\mathrm{GL}_n(\mathbb{F}_q)$-representations. Another possibility is to adapt the steps of the double induction to the local cohomology groups $H^k_{\overline{X}(v)}\left(\overline{X}(w), \mathcal{O}\right)$. If all such local cohomology groups are known for all $\overline{X}(v)$ in $\overline{X}(w)\backslash X(w)$, then one could construct a spectral sequence for these local cohomology groups and examine whether they could help us to understand $H^k\left(X(w), \mathcal{O}\right)$. We will give some examples for these local cohomology of $\overline{X}(w)$ with support in $\overline{X}(v)$ in Appendix B.

Acknowledgements. I am grateful to my advisor, Sascha Orlik, for suggesting many very interesting problems and projects on the topic of Deligne–Lusztig varieties for GL_n, and for his guidance on my research. Many thanks to Thomas Hudson and Georg Linden for giving detailed comments and constructive advice on this thesis. Furthermore, thanks to Kay Rülling for explaining [CR11] to me and suggesting that the Artin–Schreier–Witt sequence can be used to describe the mod p^m-cohomology. I want to also thank Stefan Schröer for helpful discussions. Moreover, I wish to thank Pierre Deligne for answering my questions about constructions used in [DL76].

This project is supported by the research training group *GRK 2240: Algebro-Geometric Methods in Algebra, Arithmetic and Topology*, which is funded by the Deutsche Forschungsgemeinschaft (DFG).

1. Deligne–Lusztig varieties for $\mathrm{GL_n}$

In this section we first fix the notations and conventions for Deligne–Lusztig varieties in general, and then specifically for the case of $G = \mathrm{GL}_n$ in Section 1.7. Finally, we will conclude this section with some examples of Deligne–Lusztig varieties for GL_n.

1.1. **Notations.** Let p be a prime number, and $q = p^r, r \geq 1$. Fix an algebraic closure $\overline{\mathbb{F}}_p$ of the finite field $\mathbb{F}_p$ that contains the finite field $\mathbb{F}_q$. Here we recall some basic notions from the theory of reductive groups. The standard references we have used are [DG70], [Jan03], [Hum75] and [SGA3-I].

Let G be a *reductive algebraic $\overline{\mathbb{F}}_p$-group* defined via base change by a connected reductive $\mathbb{F}_q$-group G_0. Let F denote the *Frobenius endomorphism* on G, obtained by extension from the Frobenius endomorphism of G_0. Denote with G^F the fixed points of G by the Frobenius.

Note that the datum of maximal torus, Borel subgroup, and the Weyl group is unique up to unique isomorphisms (cf. [DL76, §1.1]). Fix a *F-stable Borel subgroup* B^* of G and a *F-stable maximal torus* T^* such that $T^* \subseteq B^*$.

Let $W := N(T^*)/T^*$ be the *Weyl group*, where $N(T^*)$ is the normalizer of T^* in G. At the same time, W is the Weyl group of the root system of T, which contains a set S of simple roots that are generators of W. In the literature, elements of S are sometimes called elementary reflections or simple reflections. We denote by $\ell(w)$ the *Bruhat length* of $w \in W$. It is the minimal number r such that w can be written as the product $w = s_{i_1} \cdots s_{i_r}$, where

$s_{i_j} \in S$, $j = 1, ..., r$. Here we call $s_{i_1} \cdots s_{i_r}$ a *reduced expression* of w. The *Bruhat order* $\leq$ on W is defined by: $w \leq v$ whenever $w, v \in W$ have reduced expressions $w = s_{i_1} \cdots s_{i_r}$ and $v = t_1 \cdots t_k$, $t_1, \ldots, t_k \in S$ such that $1 \leq i_1 \leq \cdots \leq i_r \leq k$, and $s_{i_j} = t_{i_j}$ for all $j = 1, ..., r$ cf. [GP00, §1.2.4].

Denote the *opposite Borel subgroup* by B^+, recall that we have decompositions

$$B^* = U^* T^* = U^* \rtimes T^* \qquad B^+ = U^+ T^* = U^+ \rtimes T^*,$$

where U^* and U^+ are the unipotent radicals of B^* and B^+ respectively.

Since by definition G and B^* are $\overline{\mathbb{F}}_p$-schemes, the quotient G/B^* exists in the category of $\overline{\mathbb{F}}_p$-schemes [DG70, III §3.5.4]. In particular, G/B^* is integral, projective and smooth [Jan03, §II.13.3]. Since $\overline{\mathbb{F}}_p$ is algebraically closed, then the $\overline{\mathbb{F}}_p$-rational points on G/B^* correspond bijectively to the elements in $G\left(\overline{\mathbb{F}}_p\right)/B^*\left(\overline{\mathbb{F}}_p\right)$. Thus by Borel fixed point theorem, we know that the Borel subgroups of G correspond to the $\overline{\mathbb{F}}_p$-rational points on G/B^* and they are all conjugate to B^*. Throughout this text, let X be the set of all Borel subgroups of G on which G acts by conjugation. In particular, via the set theoretic identification between G/B^* and X given by $gB^* \mapsto gB^*g^{-1}$, X obtains a structure of a $\overline{\mathbb{F}}_p$-scheme with an G-action such that the identification $G/B^* \cong X$ is G-equivariant. By abuse of notation, I write $gB^* \in G/B^*$, $B \in X$ or $x \in X$ for $\overline{\mathbb{F}}_p$-rational points on G/B^* and X respectively, when there is no ambiguity.

1.2. **Basic Constructions.** There is a left G-action on the product $X \times X$ given by the diagonal action:

$$\begin{array}{ccc} G \times (X \times X) & \longrightarrow & X \times X \\ (g, (B_1, B_2)) & \longmapsto & \left(gB_1g^{-1}, gB_2g^{-1}\right). \end{array}$$

The quotient $G \backslash (X \times X)$ is in bijection with the Weyl group W as a result of the Bruhat decomposition [BT65, §2.11]. For each $w \in W$, the orbit $O(w)$ is of the form $G.\left(B^*, \dot{w}B^*\dot{w}^{-1}\right)$, where $\dot{w} \in N(T^*)$ is a representative of w. In particular, the orbit corresponding to the identity $e \in W$ is $O(e) \cong X$. We refer to [DL76, §1] for basic properties of the orbits $O(w)$. Since $F : G \to G$ induces an automorphism on $X = G/B$, let $\Gamma_F \subseteq X \times X$ be the graph of F.

Definition 1.1. The *Deligne–Lusztig variety* $X(w)$ for G corresponding to $w \in W$ is defined as the intersection in $X \times X$ of $O(w)$ and the graph of F.

$$X(w) := O(w) \times_{(X \times X)} \Gamma.$$

Remark 1.2. Note that this intersection is transverse. Moreover, the set of (rational) points of $X(w)$ corresponds to the subset of Borel subgroups B in X such that B and $F(B)$ are in relative position w.

$$X(w) = \{B \in X | (B, F(B)) \in O(w)\}.$$

Additionally, $X(w)$ is a subscheme of X of dimension $\ell(w)$ that is locally closed and smooth. Considered as a subscheme of X, the Deligne–Lusztig variety $X(w)$ is stable under the G^F-action. Thus we have a G^F-action on $X(w)$.

Deligne–Lusztig varieties may alternatively be defined as follows cf. [DMR07, §2.3].

Definition 1.3. Let $w = s_{i_1} \cdots s_{i_r}$ with $s_{i_j} \in S$ be a reduced expression. The *Deligne–Lusztig variety associated to the reduced expression* $s_{i_1} \cdots s_{i_r}$ is defined as:

$$X(s_{i_1}, ..., s_{i_r}) := \left\{(B_0, ..., B_r) \in X^{r+1} | (B_{j-1}, B_j) \in O(s_{i_j}), j = 1, ..., r, FB_0 = B_r\right\}.$$

Remark 1.4. Let $w \in W$. For any reduced decomposition $s_{i_1} \cdots s_{i_r}$ of w with $s_{i_j} \in S$, there is an isomorphism $X(w) \xrightarrow{\sim} X(s_{i_1}, ..., s_{i_r})$. Thus this definition is independent of the reduced expression of w up to (canonical) isomorphisms.

1.3. **Smooth compactifications.** In general, the Zariski closure $\overline{X(w)}$ of $X(w)$ in X is not smooth. For each reduced decomposition $w = s_{i_1} \cdots s_{i_r}$ of $w \in W$ with $s_{i_j} \in S$, $j = 1, ..., r$, we have a smooth compactification $\overline{X}(w)$ of $X(w)$ with a normal crossing divisor at infinity [DL76, §9.10] defined as follows:

Definition 1.5. Let $w \in W$, and let $w = s_{i_1} \cdots s_{i_r}$ be a reduced expression with $s_{i_j} \in S$, $j = 1, ..., r$. We define

$$\overline{O}(s_{i_1}, ..., s_{i_r}) := \left\{ (B_0, ..., B_r) \in X^{r+1} | (B_{j-1}, B_j) \in \overline{O(s_{i_j})}, j = 1, ..., r \right\},$$

where $\overline{O(s_{i_j})} = O(s_{i_j}) \dot{\bigcup} O(e)$, and

$$\overline{X}(s_{i_1}, ..., s_{i_r}) := \overline{O}(s_{i_1}, ..., s_{i_r}) \times_{(X \times X)} \Gamma,$$

where $\Gamma \subseteq X \times X$ is the graph of the Frobenius and $\overline{O}(s_{i_1}, ..., s_{i_r}) \to X \times X$ is the projection map $(B_0, ..., B_r) \mapsto (B_0, B_r)$.

We also use the notation $\overline{X}(w)$ for $\overline{X}(s_{i_1}, ..., s_{i_r})$ when the reduced expression $w = s_{i_1} \cdots s_{i_r}$ is specified.

Remark 1.6. In view of [DL76, Lemma 9.11], for each reduced expression $s_{i_1} \cdots s_{i_r}$ of w, the scheme $\overline{X}(s_{i_1}, ..., s_{i_r})$ gives a smooth compactification of $X(w)$ with a normal crossing divisor $\overline{X}(s_{i_1}, ..., s_{i_r}) \backslash X(w)$ at infinity. We may also write

$$\overline{X}(s_{i_1}, ..., s_{i_r}) = \left\{ (B_0, ..., B_r) \in X^{r+1} | (B_{j-1}, B_j) \in \overline{O(s_{i_j})}, j = 1, ..., r, FB_0 = B_r \right\}.$$

1.4. **Affiness and Irreducibility.**

Affineness. (i) Let $G/\overline{\mathbb{F}}_p$ be any connected reductive group that is defined by $G_0/\mathbb{F}_q$. Let h be the Coxeter number of G, which is the Bruhat length of the Coxeter element W. If $q > h$, then $X(w)$ is affine for any $w \in W$ [DL76, Theorem 9.7].

(ii) If $w \in W$ is a Coxeter element, then $X(w)$ is affine [Lus77, Corollary 2.8].

Irreducibility. Whether a Deligne–Lusztig variety is irreducible is completely dependent on the support of the corresponding Weyl group element.

(i) When $w \in W$ is a Coxeter element, then $X(w)$ is irreducible [Lus77, Proposition 4.8].

(ii) Let $w \in W$. If

$$\mathrm{supp}(w) := \{s \in S | s \leq w\} = S,$$

where $\leq$ is the Bruhat order on W, then $X(w)$ is irreducible. In addition, $X(w)$ has the same number of irreducible components with its smooth compactification $\overline{X}(w)$ (when one fixes a reduced expression of w). There exists many proofs for this statement. Here we refer to [BR06] and [Gör09].

(iii) Let $v, w \in W$ such that

$$\mathrm{supp}(v) = \mathrm{supp}(w),$$

then $X(w)$ and $X(v)$ have the same number of irreducible components. By [DMR07, Proposition 2.3.8], one may write down the irreducible components, as well as the number of irreducible components. In Section 5.3, we will see this in more detail for $G = \mathrm{GL}_n$.

1.5. **Fibrations over X.** We recall basic constructions from [DL76, 1.2(b)]. We use the notations from Section 1.1. We say a morphism of $\overline{\mathbb{F}}_p$-schemes $f : Y_1 \to Y_2$ is a bundle with fiber E, if Y_2 admits an open covering with respect to a topology τ (e.g. Zariski open covering, fppf open covering) such that all (closed) fibers of f are isomorphic to E and f is locally trivial with respect to this covering.

Let $s \in S$ be a simple reflection. Let $\pi : G \to G/B^*$ be the canonical projection map. The Zariski open covering $\{\dot{w}U^+B^*\}_{w \in W}$ of G gives a Zariski open covering $\{\pi(\dot{w}U^+B^*)\}_{w \in W}$ of G/B cf. [Jan03, II §1.10].

Lemma 1.7. *The projection map $pr_1 : O(s) \to X$ is an $\mathbb{A}^1$-bundle with respect to the Zariski open covering $\{\pi(\dot{w}U^+B^*)\}_{w \in W}$ of X.*

Proof. We recall a construction from [Jan03, F.23]. The product $B^* \times B^*$ acts on $G \times G$ from the right via $(g_1, g_2).(b_1, b_2) \mapsto (g_1 b_1, b_1^{-1} g_2 b_2)$. One may take the quotient and get $G \times^{B^*} G/B^*$. There is also an isomorphism

$$\varphi : G \times^{B^*} G/B^* \xrightarrow{\sim} G/B^* \times G/B^*$$

$$(g_1, g_2 B^*)B^* \longmapsto (g_1 B^*, g_1 g_2 B^*)$$

of $\overline{\mathbb{F}}_p$-schemes. We may take the embedding of $O(s)$ into $G \times^{B^*} G/B^*$ via the inverse of φ. The image of $O(s)$ is isomorphic to $G \times^{B^*} B^* s B^*/B^*$. The projection map to the first factor remains the same for $G \times^{B^*} G/B^*$ by construction. We may easily see that

$$\mathrm{pr}_1^{-1}(\pi(\dot{w}U^+B^*)) \xrightarrow{\sim} \dot{w}U^+B^* \times^{B^*} B^* s B^*/B^*$$

Since $\pi : G \to G/B^*$ is Zariski locally trivial with respect to the open covering $\{\pi(\dot{w}U^+B^*)\}_{w\in W}$, the projection to the first factor $\mathrm{pr}_1 : G \times^{B^*} B^* s B^*/B^* \to G/B^*$ is Zariski locally trivial with respect to the same open covering $\{\pi(\dot{w}U^+B^*)\}_{w\in W}$ [Jan03, I §5.16]. In other words, we have an isomorphism

$$\dot{w}U^+B^* \times^{B^*} B^* s B^*/B^* = \pi^{-1}\left(\pi(\dot{w}U^+B^*)\right) \times^{B^*} B^* s B^*/B^* \xrightarrow{\sim} \pi(\dot{w}U^+B^*) \times B^* s B^*/B^*.$$

Since $B^* s B^*/B^* \cong \mathbb{A}^1$, this is an $\mathbb{A}^1$-bundle. □

Lemma 1.8. *The projection map $pr_2 : O(s) \to X$ is an $\mathbb{A}^1$-bundle with respect to the Zariski open covering $\{\pi(\dot{w}U^+B^*)\}_{w\in W}$ of X.*

Proof. Via the isomorphism $G/B^* \times G/B^* \to G/B^* \times G/B^*$ defined by $(g_1 B^*, g_2 B^*) \mapsto (g_2 B^*, g_1 B^*)$, the result follows from Lemma 1.7. □

Lemma 1.9. *Let $pr_i : O(s) \to X, i = 1, 2,$ be the projection maps and let $\alpha : T^* \to \mathbb{G}_m$ be the simple root associated s. Then $O(s)$ is isomorphic to a homogeneous G-space, with fibres isomorphic to $B^*/(B^* \cap \dot{s}B^*\dot{s})$. Moreover, the B^*-action on the fiber $B^*/(B^* \cap \dot{s}B^*\dot{s})$ is given by $b.y = \alpha(b)y + \mu(b)$, $y \in B^*/(B^* \cap \dot{s}B^*\dot{s})$, where $\mu : B^* \to \mathbb{G}_a$ is defined by $\mu(ut) = u_{j,j+1}$.*

Proof. Recall that $O(s)$ is a G-orbit coming from the diagonal G-action on $X \times X$. Thus G acts transitively on $O(s)$ cf. [Spr09, §2.3]. If we choose a lift $\dot{s} \in G$, then $X \to O(s)$ defined by $B \mapsto (B, \dot{s}B\dot{s})$ gives a section of pr_1. (Respectively, $B \mapsto (\dot{s}B\dot{s}, B)$ gives a section of pr_2.) Thus $O(s)$ is a homogeneous G-space.

Note that $\mathrm{Stab}_G(B^*) = B^*$ and $\mathrm{Stab}_G(B^*, \dot{s}B^*\dot{s}) = B^* \cap \dot{s}B^*\dot{s}$. There is a G-equivariant bijection

$$\iota : O(s) \longrightarrow G/(B^* \cap \dot{s}B^*\dot{s}),$$

which is G-equivariant with respect to the diagonal G-action on $O(s)$ and the left G-action on $G/(B^* \cap \dot{s}B^*\dot{s})$.

In fact, this is also an isomorphism of $\overline{\mathbb{F}}_p$-schemes cf. [Jan03, §I.5.6 (8)], and the canonical projection maps

$$\pi : G \xrightarrow{\pi_2} G/(B^* \cap \dot{s}B^*\dot{s}) \xrightarrow{\pi_1} G/B^*.$$

are Zariski locally trivial with respect to the covering $\{\pi(\dot{w}U^+B^*)\}_{w\in W}$ of G/B^* [Jan03, §II 1.10]. In particular, $G/(B^* \cap \dot{s}B^*\dot{s})$ is a homogeneous G-space over G/B^* via π_1, and the fibers are isomorphic to $B^*/(B^* \cap \dot{s}B^*\dot{s})$.

The G-action that preserves the fiber at B^* comes from $\mathrm{Stab}_G(B^*) = B^*$. Note that $B^*/(B^* \cap \dot{s}B^*\dot{s})$ is isomorphic to the unipotent subgroup $U_{-\alpha}$ cf. [Jan03, §II. 1. 8, II 13.1], and $U_{-\alpha} \cong \mathbb{A}^1$.

The B^*-action on $U_{-\alpha}$ is given by the following: Let $ut \in B^*$, where $u \in U^*$ and $t \in T^*$, and $u_0 \in U_{-\alpha}$. Then $utu_0 = u(tu_0t^{-1})t$. Conjugation of the matrix u_0 by t is multiplication on the nonzero entry of u_0 by an element of $\mathbb{G}_m(\overline{\mathbb{F}}_p)$. For example, if s acts on T^* by permuting the $(j, j+1)$-th entries, then tu_0t^{-1} is the unipotent matrix of $U_{-\alpha}$ with its nonzero entry on the upper triangular part multiplied by $t_j t_{j+1}^{-1}$. Observe that conjugation by $t \in T^*(\overline{\mathbb{F}}_p)$ gives a group action on $U_{-\alpha}$ whose character coincides with the simple root α associated to s. The action of $u \in U^*$ on tu_0t^{-1} is simply adding the nonzero entry of tu_0t^{-1} on the upper triangular part by the $(j, j+1)$-th entry of u, which is an element of $\mathbb{G}_a(\overline{\mathbb{F}}_p)$. □

Recall that $\overline{O(s)}$ is the Zariski closure of $O(s)$ in $X \times X$, and $\overline{O(s)}$ is the disjoint union of $O(s)$ and $O(e)$:

$$\overline{O(s)} = \{(B_0, B_1) \in X \times X | (B_0, B_1) \in O(s), \text{ or } (B_0, B_1) \in O(e)\}.$$

Note that we have an identity section of the projection maps $\mathrm{pr}_i : \overline{O(s)} \to X, i = 1, 2$, given by $X \cong O(e) \to \overline{O(s)}$. The complement of this section gives us $\mathrm{pr}_i : O(s) \to X$.

Let $B \subseteq G$ be a Borel subgroup. Consider the parabolic subgroup $P := B \cup BsB$. Denote the unipotent radical of P by U_P and denote $L_P := P/U_P$. The quotient group $\overline{L_P} := L_P/Z(L_P)$ is semisimple of rank 1. Then $\overline{L_P}$ is isomorphic to PGL_2 [Mil17, Theorem 20.16].

Similar to $X = G/B^*$, we may construct a homogeneous space $X_{\overline{L_P}}$ associated to the reductive group $\overline{L}$. In particular, $X_{\overline{L_P}}$ is a smooth projective scheme of dimension 1 with a nontrivial action of $\overline{L_P}$. Note that it is isomorphic to $\mathbb{P}^1_{\overline{\mathbb{F}}_p}$.

Lemma 1.10. *The projection map $pr_i : \overline{O(s)} \to X, i = 1, 2,$ is a $\mathbb{P}^1$-bundle and it is locally trivial with respect to the Zariski open covering $\{\pi(\dot{w}U^+B^*)\}_{w \in W}$ of X.*

Proof. The proof is the same as Lemma 1.7. Let $i = 1$. The isomorphism of $\overline{\mathbb{F}}_p$-schemes

$$\varphi : G \times^{B^*} G/B^* \xrightarrow{\sim} G/B^* \times G/B^*$$

$$(g_1, g_2B^*)B^* \longmapsto (g_1B^*, g_1g_2B^*)$$

gives an isomorphism between $\overline{O(s)}$ and $G \times^{B^*} \overline{B^*sB^*}/B^*$. We have $\overline{B^*sB^*}/B^* \cong \mathbb{P}^1$. For any $\pi(\dot{w}U^+B^*)$ in the open covering for X, we have an isomorphism

$$\mathrm{pr}_1^{-1}(\pi(\dot{w}U^+B^*)) \xrightarrow{\sim} \dot{w}U^+B^* \times^{B^*} \overline{B^*sB^*}/B^*.$$

Since $\pi : G \to G/B^*$ is locally trivial with respect to the open covering $\{\pi(\dot{w}U^+B^*)\}_{w \in W}$, the projection to the first factor $\mathrm{pr}_1 : G \times^{B^*} \overline{B^*sB^*}/B^* \to G/B^*$ is Zariski locally trivial with respect to the same open covering $\{\pi(\dot{w}U^+B^*)\}_{w \in W}$ [Jan03, I §5.16]. In other words, we have an isomorphism

$$\dot{w}U^+B^* \times^{B^*} \overline{B^*sB^*}/B^* \xrightarrow{\sim} \pi(\dot{w}U^+B^*) \times \overline{B^*sB^*}/B^*$$

The case for $i = 2$ is symmetric. □

Remark 1.11. Let $P^* := B^*sB^* \cup B^*$. Then $\mathrm{pr}_i : \overline{O(s)} \to G/B^*$ is the base change of the homogeneous G-space $G/B^* \to G/P^*$ via the projection map $\pi_P : G/B^* \to G/P^*$.

Indeed, let $(g_1B^*, g_2B^*) \in \overline{O(s)}$, then we have $g_1^{-1}g_2 \in P^*$. Thus we have $g_1P^* = g_2P^*$ in G/P^*. We have the canonical projection map $\pi_P : G/B^* \to G/P^*$. Observe that $(g_1B^*, g_2B^*) \in G/B^* \times G/B^*$ is in $\overline{O(s)}$ if and only if $\pi_P(g_1B^*) = \pi_P(g_2B^*)$. Thus $\overline{O(s)}$ can be identified with $G/B^* \times_{G/P^*} G/B^*$. In other words, we have a cartesian diagram of $\overline{\mathbb{F}}_p$-schemes:

$$\begin{tikzcd} \overline{O(s)} \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"'] & G/B^* \arrow[d, "\pi_P"] \\ G/B^* \arrow[r, "\pi_P"] & G/P^*, \end{tikzcd} \tag{1.1}$$

where all maps are G-equivariant.

Let $\pi' : G \to G/P^*$ be the canonical projection map, and since π' is locally trivial with respect to the the (Zariski) open cover $\{\pi'(\dot{w}U^+B^*)\}_{w \in W}$ of G/P^* [Jan03, §II.1.10 (5)], we see that π_P is also locally trivial with respect to this cover. As a result, $\mathrm{pr}_i, i = 1, 2$, is locally trivial with respect to the Zariski open covering $\{\pi(\dot{w}U^+B^*)\}_{w\in W}$ of G/B^*.

In particular, the fibers of pr_i are isomorphic to P^*/B^*. The fiber at B^* is precisely P^*/B^* and has a P^*-action. Note that $P^*/B^* \cong \mathbb{P}^1$ and the P^*-action on P^*/B^* induces the natural PGL_2-action on $\mathbb{P}^1$ cf. [Dem74, §2.5 Lemma 3].

1.6. **The induced G^F-action on the cohomology groups.** Let G be as in Section 1.1, and let Y be a $\overline{\mathbb{F}}_p$-scheme with G^F-action. Recall that $G^F = G_0(\mathbb{F}_q)$, so it is in particular a finite group. Following [MFK94, Definition 1.6], we explain how G^F acts on the cohomology groups of G^F-equivariant $\mathcal{O}_Y$-modules.

Definition 1.12. Let Y be a $\overline{\mathbb{F}}_p$-scheme with G^F-action $\sigma : G^F \times Y \to Y$, and let $\mathcal{V}$ be an invertible sheaf of $\mathcal{O}_Y$-modules. Denote by $\mu : G^F \times G^F \to G^F$ the multiplication. A G^F-*linearization* of $\mathcal{V}$ constists of the datum of an isomorphism of sheaves of $\mathcal{O}_{G^F\times Y}$-modules,

$$\phi : \sigma^*\mathcal{V} \xrightarrow{\sim} \mathrm{pr}_2^*\mathcal{V},$$

such that $\phi|_{\{1\}\times Y}$ is the identity and the cocycle condition on $G^F \times G^F \times Y$

$$\left(\mathrm{pr}_{2,3}^*\phi\right) \circ ((\mathrm{id}_{G^F} \times \sigma)^*\phi) = (\mu \times \mathrm{id}_Y)^*\phi$$

is satisfied.

We say that $\mathcal{V}$ is G^F-equivariant if it possesses a G^F-linearization.

Example 1.13. (i) For $\mathcal{V} = \mathcal{O}_Y$, we naturally have $\sigma^*\mathcal{O}_Y \cong \mathcal{O}_{G^F\times Y}$ and $\mathrm{pr}_2^*\mathcal{O}_Y \cong \mathcal{O}_{G^F\times Y}$. Thus ϕ is given by the composition of these two isomorphisms. The cocycle condition follows because the pullback of the structure sheaf is the structure sheaf. Thus $\mathcal{O}_Y$ is G^F-equivariant.

(ii) There is a natural morphisms of sheaves of $\mathcal{O}_{G^F\times Y}$-modules:

$$\sigma^*\Omega^1_Y \longrightarrow \Omega^1_{G^F\times Y}.$$

Note that the projection map induces a projection map of sheaves of $\mathcal{O}_{G^F\times Y}$-modules:

$$\Omega^1_{G^F\times Y} \longrightarrow \mathrm{pr}_2^*\Omega^1_Y.$$

The composition yields a morphism of $\mathcal{O}_{G^F\times Y}$-modules $\phi : \sigma^*\Omega^1_Y \to \mathrm{pr}_2^*\Omega^1_Y$. By checking on the level of stalk, one sees that ϕ is an isomorphism and the cocycle condition is satisfied.

(iii) Let $\dim_{\overline{\mathbb{F}}_p} Y = r$. Let Ω^r_Y be the canonical sheaf of Y. One shows that Ω^r_Y is G^F-equivariant by following the steps above and taking r-th exterior powers. Note that exterior powers commute with taking the inverse image.

Lemma 1.14. *Let Y be a $\overline{\mathbb{F}}_p$-scheme with G^F-action, and let $\mathcal{V}$ be a G^F-equivariant sheaf of $\mathcal{O}_Y$-modules. Then the cohomology groups*

$$H^k(Y, \mathcal{V})$$

are G^F-modules.

Proof. For each $g \in G^F$, the map $g : Y \to Y$ induces isomorphisms of

$$H^k(Y, g^*\mathcal{V}) \xrightarrow{\sim} H^k(Y, \mathcal{V})$$

by functoriality. For any $g, g' \in G^F$, the isomorphism ϕ induces an isomorphism $\phi_g : g^*\mathcal{V} \xrightarrow{\sim} \mathcal{V}$ such that the cocycle condition implies that $\phi_{gg'} = \phi_{g'} \circ g'^*(\phi_g)$. Thus $H^k(Y, \mathcal{V})$ are G^F-modules. □

Lemma 1.15. *Let Y be a $\overline{\mathbb{F}}_p$-scheme together with a G^F-action. Suppose $H^0(Y, \mathcal{O}_Y) = \overline{\mathbb{F}}_p$. Then $H^0(Y, \mathcal{O}_Y)$ is the trivial G^F-representation.*

Proof. Let $g \in G^F$. We have an isomorphism $\mathcal{O}_Y \xrightarrow{\sim} g_*\mathcal{O}_Y$. Let $\varphi \in \mathcal{O}_Y(Y)$, for all $y \in Y$,

$$g.\varphi(y) = \varphi(g^{-1}.y).$$

As we have $\mathcal{O}_Y(Y) = \overline{\mathbb{F}}_p$, any $\varphi \in \mathcal{O}_Y(Y)$ is constant, and so $g.\varphi = \varphi$. Therefore G^F acts on $H^0(Y, \mathcal{O}_Y)$ trivially.

□

1.7. **Conventions for $G = \mathrm{GL}_n$.** In the following, we will specifically consider the case $G = \mathrm{GL_n}$. In this case, we have $G^F(\overline{\mathbb{F}}_p) = \mathrm{GL_n}(\mathbb{F}_q)$. Fix a maximal torus $T^* \subseteq \mathrm{GL}_n$ such that $T^*(\overline{\mathbb{F}}_p)$ corresponds to the diagonal matrices in $\mathrm{GL}_n(\overline{\mathbb{F}}_p)$. We also fix a Borel subgroup $B^* \subseteq \mathrm{GL}_n$ whose $\overline{\mathbb{F}}_p$-rational points correspond to the upper triangular matrices in $\mathrm{GL}_n(\overline{\mathbb{F}}_p)$ cf. [Jan03, §II.1.8].

There is a canonical isomorphism between the Weyl group W associated to $T^* \subseteq B^*$ as above and the symmetric group S_n, in the sense that the elements of W acts on $T^*(\overline{\mathbb{F}}_p)$ by permuting the diagonal entries. We fix a set of generators $S := \{s_1, ..., s_{n-1}\}$ of W such that s_i acts on the elements of $T^*(\overline{\mathbb{F}}_p)$ by permuting the i-th and $(i+1)$-th entries. Any product of the s_i's in W such that each s_i shows up exactly once is called a Coxeter element. In particular, we call the product $s_1 \cdots s_{n-1}$ the standard Coxeter element, which we denote by $\mathbf{w}$. Let $\{\alpha_1, ..., \alpha_{n-1}\}$ be the simple roots with each α_i corresponding to s_i. The the unipotent subgroup $U_{-\alpha_i}$ has $\overline{\mathbb{F}}_p$-rational points consisting of matrices whose only nonzero entries lie on the diagonal and the $(i, i+1)$-th entry, with the entries on the diagonal all 1's. Moreover, each $U_{-\alpha_i}$ is isomorphic to the additive group $\mathbb{G}_a$.

For any subset $I \subseteq S$, denote W_I the subgroup of W generated by I. We define the associated standard parabolic subgroup cf. [Hum75, Theorem 29.2]:

$$P_I = B^* W_I B^* := \bigcup_{w \in W_I} B^* \dot{w} B^*.$$

Let L_I be the standard Levi subgroup containing T^* such that we have a Levi decomposition

$$P_I \xrightarrow{\sim} U_I \rtimes L_I,$$

where U_I is the unipotent radical of P_I.

Finally, recall that the Frobenius endomorphism F acts as the identity on W in this case cf.[DM20, Example 2.3.13].

1.8. **Examples of Deligne–Lusztig varieties for GL_n.** Let $G = \mathrm{GL}_n$. Note that the standard Coxeter element $\mathbf{w}$ corresponds to the n-cycle $(1, ..., n)$ in the symmetric group on n-elements. Recall from [DL76, §2] that $X(\mathbf{w})$ can be identified with the following subspace of of the complete flag variety X:

$$\left\{ D_\bullet \,\middle|\, \dim_{\overline{\mathbb{F}}_p} D_i = i,\ D_0 = \{0\},\ D_i = \bigoplus_{j=1}^{i} F^{j-1} D_1,\ i = 1, ..., n-1,\ D_n = \overline{\mathbb{F}}_p^n \right\}.$$

Via the projection $D_\bullet \mapsto D_1$, one obtains an embedding of $X(\mathbf{w})$ into $\mathbb{P}^{n-1}_{\overline{\mathbb{F}}_p}$. In particular, $X(\mathbf{w})$ is isomorphic to the *Drinfeld half space* of dimension $n-1$, which is the complement of all $\mathbb{F}_q$-rational hyperplanes in $\mathbb{P}^{n-1}_{\overline{\mathbb{F}}_p}$. In other words, there is an $\mathrm{GL}_n(\mathbb{F}_q)$-equivariant isomorphism

$$X(\mathbf{w}) = X(s_1 \cdots s_{n-1}) \xrightarrow{\sim} \mathbb{P}^{n-1}_{\overline{\mathbb{F}}_p} - \mathcal{H},$$

where $\mathcal{H}$ is the union of all $\mathbb{F}_q$-rational hyperplanes in $\mathbb{P}^{n-1}_{\overline{\mathbb{F}}_p}$.

The smooth compactification $\overline{X}(\mathbf{w})$ associated to the expression $\mathbf{w} = s_1 \cdots s_{n-1}$ is isomorphic to the successive blow up $\widetilde{Y}$ of $\mathbb{P}^{n-1}_{\overline{\mathbb{F}}_p}$ along all $\mathbb{F}_q$-rational linear subvarieties [Ito05, §4.1], cf. [GK05, §1], and [Lin18, §2.5].

$$\widetilde{Y} = Y_{n-1} \longrightarrow Y_{n-2} \longrightarrow \cdots \longrightarrow Y_{-1} = \mathbb{P}^{n-1}_{\overline{\mathbb{F}}_p}$$

where $Y_i \to Y_{i-1}$ is the blow up of Y_{i-1} along the strict transform of all $\mathbb{F}_q$-rational linear subvarieties $H \subseteq \mathbb{P}^{n-1}_{\overline{\mathbb{F}}_p}$ with $\dim H = i$. The maps $Y_i \to Y_{i-1}$ are $\mathrm{GL}_n(\mathbb{F}_q)$-equivariant, so the map $\widetilde{Y} \to \mathbb{P}^{n-1}_{\overline{\mathbb{F}}_p}$ is equivariant under $\mathrm{GL}_n(\mathbb{F}_q)$-action.

Let $w \in W$ such that $w \leq \mathbf{w}$. Then $X_{\mathrm{GL}_n}(w)$ is isomorphic to a disjoint union of products of Drinfeld half spaces. This isomorphism extends to the corresponding smooth compactifications. In Section 5.2 and 5.3, we will discuss this example in more detail.

2. The Weyl group of GL_n and generalized Deligne–Lusztig varieties

The goal of this section is to recall some constructions related to the Weyl group W of GL_n, and to give the definition of generalized Deligne–Lusztig varieties associated to an element of the free monoid F^+ (resp. $\hat{F}^+$).

2.1. **Conjugacy classes and cyclic shifting.** We would like to review some definitions and theorems from [GP00, §3], specifically in the case when W is the Weyl group of GL_n. The conventions of Section 1.7 applies.

Definition 2.1. We say $w, w' \in W$ are *conjugated by cyclic shifts* when there exists elements $v_0, ..., v_m \in W$ such that $v_0 = w, v_m = w'$ and for all $i = 1, ..., m$, we have $x_i, y_i \in W$ such that $v_{i-1} = x_i y_i$, $v_i = y_i x_i$, and $\ell(v_{i-1}) = \ell(x_i) + \ell(y_i) = \ell(v_i)$.

Remark 2.2. Note that if $w, w' \in W$ are conjugate by cyclic shift, then $\ell(w) = \ell(w')$.

The following theorem is from [GP00, Theorem 3.1.4].

Theorem 2.3. *Any two Coxeter elements of W are conjugated by cyclic shifts.*

Let C be a conjugacy class of W. We write $C_{\min}$ for the subset of C that consists of elements with the shortest Bruhat length:

$$C_{\min} := \{v \in C | \ell(v) \leq \ell(w) \text{ for all } w \in C\}.$$

Definition 2.4. Let $w, v \in W$. We write $w \to v$ if and only if there exists elements $w = w_0, w_1, w_2, ..., w_m = v \in W$, such that $w_i = s_i w_{i-1} s_i$ and $\ell(w_i) \geq \ell(w_{i-1})$ for $i = 1, ..., m$, where $s_i \in W$ is an elementary reflection.

Now we may present a special case of the theorem from [GP00, Theorem 3.2.9], with wording adapted to our situation.

Theorem 2.5 (Geck-Pfeifer)**.** *(i) Let $w \in W$, and let C be a conjugacy class of W containing w. Then there exists $w' \in C_{\min}$ such that $w \to w'$.*

(ii) Let $w_1, w_2 \in W$ be two Coxeter elements, then $w_1 \to w_2$ and $w_2 \to w_1$.

Corollary 2.6. *Let $w \in W$, and let $C \subseteq W$ be the conjugacy class of w. Then either $w \in C_{\min}$ or there exists $s \in S, v \in W$ such that $sws = v$ and $\ell(v) + 2 = \ell(w)$.*

Proof. By the above theorem [GP00, Theorem 3.2.9], we know that $w \to w'$ for some $w' \in C_{\min}$. If $\ell(w) = \ell(w')$, then by definition of the minimal elements in a conjugacy class, we know that $w \in C_{\min}$. If $w \notin C_{\min}$, then there exists some $s \in S, v \in W$ such that $sws = v$ and $\ell(v) < \ell(w)$. It is easy to see that $\ell(v) = \ell(w) - 1$ is impossible. Thus we have $\ell(v) + 2 = \ell(w)$. ☐

2.2. **Height.** Corollary 2.6 allows us to define the height of any $w \in W$. This definition was first presented in [Orl18, Definition 5.1].

Definition 2.7. Let W be the Weyl group (for GL_n). Let $w \in W$ and denote the conjugacy class of w in W by C. We define the *height* for w inductively as follows:

i. If $w \in C_{\min}$, then we define $\mathrm{ht}(w) = 0$,
ii. If $w \notin C_{\min}$, then there exists $v \in W$ and $s \in S$ such that $w = svs$ and $\ell(w) = \ell(v)+2$. We define the height $\mathrm{ht}(w) = \mathrm{ht}(v) + 1$.

Example 2.8. Let $G = \mathrm{GL}_n$, W be the Weyl group, S a set of generators for W. Take $w \in W$ and denote the conjugacy class of w in W by C.

i. By [GP00, Example 3.1.16], $w \in C_{\min}$ if and only if w is the Coxeter element for some $W_J \subseteq W$. Thus any height 0 element in W is a Coxeter element for some parabolic subgroup W_J of W.
ii. Let $w' = s_1 \cdots s_{i-1} s_{i+1} \cdots s_{n-1}$ and $w := s_i w' s_i$. As w' is the Coxeter element of the subgroup of W generated by the set $\{s_1, ..., s_{i-1}, s_{i+1}, ..., s_{n-1}\}$, we know that $w' \in C_{\min}$, where C is the conjugacy class in W containing w'. Thus $\mathrm{ht}(w') = 0$ and $\mathrm{ht}(w) = 1$.

2.3. **Support.**

Definition 2.9. Let $w \in W$. The *support* of w is

$$\mathrm{supp}(w) := \{s \in S | s \leq w\}.$$

Note that $|\mathrm{supp}(w)| \leq \ell(w)$ and that the equality holds when w has a reduced expression $w = s_{i_1} \cdots s_{i_r}$ with $s_{i_j} \in S$ all distinct.

Let $G = \mathrm{GL}_n$ and $w \in W$. Set $I := \mathrm{supp}(w)$, and let C be the conjugacy class of w in W. Let $W_I \subseteq W$ be the subgroup generated by I. Then there exists $w' \in C_{\min}$ such that $w \to w'$ and w' is a Coxeter element in W_I.

2.4. **The free monoid associated to the Weyl group.** Let us introduce the free monoid F^+, cf. [DMR07] and [Orl18, §2].

Definition 2.10. We define F^+ as the free monoid generated by $s_1, \ldots, s_n$:

$$F^+ := \langle s_1, ..., s_{n-1} \rangle$$

Remark 2.11. (i) There is a natural surjective morphism of monoids:

$$\alpha : F^+ \longrightarrow W$$

with kernel generated by the relations in the group presentation of W.

(ii) There is a partial order $\preccurlyeq$ on F^+ defined by: $w' \preccurlyeq w$ whenever $w' = s_{i_1} \cdots s_{i_r}$ and $w = t_1 \cdots t_k$, $t_1, \ldots, t_k \in S$ such that $1 \leq i_1 \leq \cdots \leq i_r \leq k$, and $s_{i_j} = t_{i_j}$ for all $j = 1, ..., r$. We call this the Bruhat order on F^+. Note that this is not entirely compatible with the Bruhat order on W.

(iii) There is a Bruhat length function on F^+, not compatible with the Bruhat length on W. When $w = s_{i_1} \cdots s_{i_r} \in F^+$, we have $\ell(w) = r$.

(iv) For $w, v \in F^+$, we always have $\ell(wv) = \ell(w) + \ell(v)$.

There is also a variant of F^+ defined in [Orl18, p. 22].

Definition 2.12. Let $\widehat{W}$ be a copy of W. Define $\hat{F}^+$ as the free monoid generated by $S \dot{\cup} T'$, where

$$T' := \left\{ \widehat{sts} \in \widehat{W} \middle| \, st \neq ts \text{ in } W, s, t \in S \right\}.$$

Remark 2.13. Note that T' and S are forced to be disjoint in $\hat{F}^+$.

We define the Bruhat length function on $\hat{F}^+$ as the function counting the number of elements of S and $\widehat{S}$ showing up in the expression, where $\widehat{S}$ is the set of generators in $\widehat{W}$ corresponding to S, instead of counting the number of generators.

2.5. **Constructing generalized Deligne–Lusztig varieties.** As in Section 1, we may define generalized Deligne–Lusztig varieties for elements of the free monoids F^+ and $\hat{F}^+$. We only recall the definitions and properties here. Refer to [DMR07, §2.2.11] and [Orl18, end of §3] for more detailed discussions.

Definition 2.14. Let $w = s_{i_1} \cdots s_{i_r} \in F^+$. We define the *Deligne–Lusztig variety corresponding to* $s_{i_1} \cdots s_{i_r}$ as follows:

$$X(s_{i_1}, ..., s_{i_r}) := \{(B_0, ..., B_r) \in X^{r+1} | (B_{j-1}, B_j) \in O(s_{i_j}), \forall j = 1, ..., r, B_r = F(B_0)\}.$$

This variety is a smooth $\overline{\mathbb{F}}_p$-variety with a G^F-action.

Remark 2.15. For $s_{i_j} \in F^+$ with $\alpha(s_{i_j}) \in S$. We set $O(s_{i_j}) := O(\alpha(s_{i_j}))$. However, note that for all $a, b \in F^+$, we have $O(a) \times_X O(b) \xrightarrow{\sim} O(a, b)$, reflecting the structure of F^+ as a free monoid.

Let $w = s_{i_1} \cdots s_{i_r} \in F^+$. If $\ell(w) = \ell(\alpha(w))$, then there is a G^F-equivariant isomorphism

$$\begin{array}{ccc} X(s_{i_1}, ..., s_{i_r}) & \xrightarrow{\sim} & X(\alpha(w)) \\ (B_0, ..., B_r) & \longmapsto & B_0 \end{array}$$

of $\overline{\mathbb{F}}_p$-schemes. Note that if we consider $w \in W$, then for each reduced expression $s_{i_1} \cdots s_{i_r}$ of w with $s_{i_j} \in S$, we get an element of F^+ and a corresponding (generalized) Deligne–Lusztig variety.

Similarly, we define the (generalized) Deligne–Lusztig variety corresponding to elements in $\hat{F}^+$. After post composing with the isomorphism $W \cong \hat{W}$, we may extend the surjective map α to $\hat{\alpha} : \hat{F}^+ \to W$.

Definition 2.16. Let $w = t_1 \cdots t_r \in \hat{F}^+$. We define the *Deligne–Lusztig variety corresponding to w* as follows:

$$X(t_1, ..., t_r) := \left\{ (B_0, ..., B_r) \in X^{r+1} \middle| (B_{j-1}, B_j) \in O(t_j), \forall j = 1, ..., r, B_r = F(B_0) \right\}.$$

Remark 2.17. This variety is a smooth $\overline{\mathbb{F}}_p$-variety with a G^F-action. Note that there could be some t_j in the expression of w such that $t_j = \widehat{sts}$ for some $s, t \in S, st \neq ts$. For $\widehat{sts} \in \hat{F}^+$, we set $O(\widehat{sts}) := O(\hat{\alpha}(\widehat{sts}))$. Similarly, we have for all $a, b \in \hat{F}^+$, we have $O(a) \times_X O(b) \xrightarrow{\sim} O(a, b)$. Thus $X(t_1, \ldots, t_r)$ has dimension $\ell(w) \geq r$.

2.6. **Smooth compactification of generalized Deligne–Lusztig varieties.** We write down the smooth compactifications for Deligne–Lusztig varieties corresponding to $w \in F^+$ and $w \in \hat{F}^+$. They are the same as the construction in [DL76, §9] when $w \in F^+$ or $w \in \hat{F}^+$ is the reduced expression for some $w' \in W$.

Definition 2.18. Let $w = s_{i_1} \cdots s_{i_r} \in F^+$. The *smooth compactification of* $X(s_{i_1}, ..., s_{i_r})$ is defined as:

$$\overline{X}(s_{i_1}, ..., s_{i_r}) := \left\{ (B_0, ..., B_r) \in X^{r+1} \middle| (B_{j-1}, B_j) \in \overline{O(s_{i_j})}, \forall j = 1, ..., r, B_r = F(B_0) \right\}.$$

Remark 2.19. Let $w \in W$, and let $w = s_{i_1} \cdots s_{i_r}$ be a reduced expression with $s_{i_j} \in S$. Then $\overline{X}(s_{i_1}, ..., s_{i_r})$ can be identified with $\overline{X}(w)$, corresponding to the reduced expression $w = s_{i_1} \cdots s_{i_r}$, constructed in Definition 1.5. Moreover, the smooth compactification $\overline{X}(s_{i_1}, ..., s_{i_r})$ is a smooth projective $\overline{\mathbb{F}}_p$-variety with G^F-action cf. [DL76, §9], [Orl18, Proposition 3.17].

Similarly, we have the following definition for $w \in \hat{F}^+$.

Definition 2.20. Let $w = t_1 \cdots t_r \in \hat{F}^+$. We define the following $\overline{\mathbb{F}}_p$-scheme containing $X(t_1, ..., t_r)$:

$$\overline{X}(t_1, ..., t_r) := \left\{ (B_0, ..., B_r) \in X^{r+1} \middle| (B_{j-1}, B_j) \in \overline{O(t_j)}, \forall j = 1, ..., r, B_r = F(B_0) \right\}.$$

Lemma 2.21. *Let $w = t_1 \cdots t_r \in \hat{F}^+$, then $\overline{X}(t_1, ..., t_r)$ is projective and smooth.*

Proof. Since $\overline{O(t_j)}$ is projective for all $j = 1, ..., r$, it follows from [DMR07, Proposition 2.3.6 (iv)] that $\overline{X}(t_1, ..., t_r)$ is projective.

Note that $\overline{O(t)}$ are smooth for all $t \in S$. We may have $t_j, j = 1, ..., r$ to be of the form $t_j = \widehat{sts}$, where $st \neq ts, s, t, \in W$. Since $st \neq ts$, we see that s and t do not correspond to non-adjacent simple reflections in S_n. Now let $I = \{s, t\}$, the parabolic subgroup W_I of the Weyl group W is thus isomorphic to the symmetric group S_3. In particular, sts is a reduced expression of the longest element in W_I. By [DMR07, Corollary 2.2.10], we see that $\overline{O(sts)}$ is a smooth $\overline{\mathbb{F}}_p$-scheme.

Hence for all $j = 1, ..., r$, $\overline{O(t_j)}$ is smooth. By [DMR07, Proposition 2.3.5], we may conclude that $\overline{X}(t_1, ..., t_r)$ is smooth. $\square$

2.7. **Stratifications of $\overline{X(w)}$ and $\overline{X}(w)$.** Let $w_1, w_2 \in W$ such that $\ell(w_1w_2) = \ell(w_1) + \ell(w_2)$, recall that we have an isomorphism of schemes $O(w_1) \times_X O(w_2) \xrightarrow{\sim} O(w_1w_2)$. Hence for $w, w' \in W$, $w' \leq w$ if and only if $\overline{O(w')} \subseteq \overline{O(w)}$. This implies that we have a stratification as follows for any $w \in W$, cf. [DL76, §1.2], [Jan03, §II.13.7],

$$\overline{O(w)} = \bigcup_{w' \leq w} O(w').$$

By intersecting with the graph of the Frobenius Γ transversally in $X \times X$, we get a similar stratification for any $w \in W$,

$$\overline{X(w)} = \bigcup_{w' \leq w} X(w').$$

Let $w \in F^+$ or $\hat{F}^+$, with the expression $s_{i_1} \cdots s_{i_r}$, then for all $w' \preceq w$ a subword, we know that $X(w')$ is isomorphic to a locally closed subscheme of $\overline{X}(w)$, thus we have the following stratification, cf. [Orl18, §3],

$$\overline{X}(w) = \bigcup_{w' \preceq w} X(w').$$

Example 2.22. Recall that we assume $G = \mathrm{GL_n}$. Let $m \leq n-1$ be a positive integer and $w \in W$ with a reduced expression $s_{i_1} \cdots s_{i_m}$ such that all s_{i_j}, $j = 1, ..., m$, are distinct. Recall from [DL76, Lemma 9.11] that

$$D = \bigcup_{w' \prec w} X(w'),$$

with each $w' \prec w$ considered as a subword of $s_{i_1} \cdots s_{i_m} \in F^+$, is the normal crossing divisor of $\overline{X}(w)$ at infinity.

Note that the projection map

$$X^m \longrightarrow X$$
$$(B_0, ..., B_{m-1}) \longmapsto B_0$$

induces an isomorphism on the open subschemes

$$\overline{X}(w) \backslash D \longrightarrow X(w).$$

This map extends to the Zariski closure of $X(w)$ in X, and so we have a surjective morphism

$$\overline{X}(w) \longrightarrow \overline{X(w)}$$

Since $w = s_{i_1} \cdots s_{i_m}$ with all s_{i_j} distinct, $\overline{X}(w)$ and $\overline{X(w)}$ have stratifications indexed by the same set, and each corresponding strata is isomorphic. In fact, they are isomorphic as $\overline{\mathbb{F}}_p$-schemes [Han99, Lemma 1.9].

3. Geometry of Deligne–Lusztig varieties via $\mathbb{P}^1$-bundles

From now on let $G = \mathrm{GL}_n$. In this section, we consider $\mathbb{P}^1$-bunldes $\pi_1 : \overline{X}(sws) \to \overline{X}(ws)$ and $\pi_2 : \overline{X}(sws) \to \overline{X}(sw)$ constructed from the morphism $\overline{O(s)} \to X$ from Section 1.5.

3.1. **The structure of certain morphisms as $\mathbb{P}^1$-bundles.** Let $w = t_1 \cdots t_r \in \hat{F}^+$ and $s \in S$. Consider the following smooth compactification of the Deligne–Lusztig varieties $X(sws)$, $X(ws)$:

$$\overline{X}(sws) = \left\{ (B_0, ..., B_{r+2}) \in X^{r+3} \;\middle|\; \begin{array}{l} (B_j, B_{j+1}) \in \overline{O(t_j)}, j = 1, ..., r, \\ (B_0, B_1) \in \overline{O(s)}, (B_{r+1}, B_{r+2}) \in \overline{O(s)}, B_{r+2} = FB_0 \end{array} \right\},$$

$$\overline{X}(ws) = \left\{ (B'_0, ..., B'_{r+1}) \in X^{r+2} \;\middle|\; \begin{array}{l} (B'_{j-1}, B'_j) \in \overline{O(t_j)}, j = 1, ..., r, \\ (B'_r, B'_{r+1}) \in \overline{O(s)}, B'_{r+1} = FB'_0 \end{array} \right\}.$$

Lemma 3.1. *The map $\pi_1 : \overline{X}(sws) \to \overline{X}(ws)$ defined by*

$$(B_0, B_1, ..., B_{r+1}, FB_0) \mapsto (B_1, B_2, ..., B_{r+1}, FB_1)$$

is a $\mathbb{P}^1$-bundle over $\overline{X}(ws)$ locally trivial with respect to a Zariski covering of $\overline{X}(ws)$. Furthermore, π_1 has a section $\sigma : \overline{X}(ws) \to \overline{X}(sws)$ defined by

$$(B'_0, ..., B'_r, FB'_0) \mapsto (B'_0, B'_0, ..., B'_r, FB'_0)$$

with $\pi_1 \circ \sigma = \mathrm{id}_{\overline{X}(ws)}$.

Proof. Let us first check the well-definedness of π_1. Take $(B_0, B_1, ..., B_{r+1}, FB_0) \in \overline{X}(sws)$. Since $(B_0, B_1) \in \overline{O(s)}$, and F fixes any $s \in S$, we know that $(FB_0, FB_1) \in \overline{O(s)}$. As we also have $(B_{r+1}, FB_0) \in \overline{O(s)}$, by [DL76, §1.2 (b1)], we have $(B_{r+1}, FB_1) \in \overline{O(s)}$. Thus $(B_1, B_2, ..., B_{r+1}, FB_1) \in \overline{X}(ws)$.

To see that π_1 gives a $\mathbb{P}^1$-bundle, take any $(B'_0, ..., B'_r, FB'_0) \in \overline{X}(ws)$, and take any $B_0 \in X$ such that $(B_0, B'_0) \in \overline{O(s)}$. Since F fixes s, we have $(FB_0, FB'_0) \in \overline{O(s)}$. As $(B'_r, FB'_0) \in \overline{O(s)}$, we know by [DL76, §1.2 (b1)] that $(FB_0, B'_r) \in \overline{O(s)}$. Thus $(B_0, B'_0, ..., B'_r, FB_0) \in \overline{X}(sws)$, and the preimage of $(B'_0, ..., B'_r, FB'_0)$ under π_1 is

$$\left\{ (B_0, B'_0, ..., B'_r, FB_0) \in \overline{X}(sws) \,\middle|\, (B_0, B'_0) \in \overline{O(s)} \right\}.$$

Thus the fibre of π_1 at any $(B'_0, ..., B'_r, FB'_0)$ is isomorphic to the fibre of $\overline{O(s)} \to X$ at B'_0.

Let $\mathrm{pr}_{2,...,r+3} : \overline{O(s)} \times X^{r+1} \to X^{r+2}$ be the projection map to the 2nd to $(r+3)$-th component. If we take the embedding of $\overline{X}(ws)$ into X^{r+2}, we find that $\overline{X}(sws)$ is isomorphic to the fibre product of $\overline{X}(ws) \hookrightarrow X^{r+2}$ and $\mathrm{pr}_{2,...,r+3}$.

$$\begin{array}{ccc} \overline{X}(sws) & \longrightarrow & \overline{O(s)} \times X^{r+1} \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \mathrm{pr}_{2,...,r+3}} \\ \overline{X}(ws) & \hookrightarrow & X^{r+2}. \end{array}$$

Let $\pi_G : G \to G/B^*$ be the canonical projection map for $X = G/B^*$. By Lemma 1.8, we know that $\mathrm{pr}_2 : \overline{O(s)} \to X$ is locally trivial with respect to the Zariski covering $\{\pi_G(\dot{w}U^+B^*)\}_{w\in W}$ of X. Thus $\mathrm{pr}_{2,...,r+3}$ is locally trivial with respect to the Zariski covering $\{\pi_G(\dot{w}U^+B^*) \times X^{r+1}\}_{w\in W}$ of X^{r+2}. Via embedding $\overline{X}(ws)$ into X^{r+2}, we see that π_1 is locally trivial with respect to a Zariski open covering of $X(ws)$. Therefore $\overline{X}(sws)$ is a $\mathbb{P}^1$-bundle over $\overline{X}(ws)$.

Finally, the statement $\pi_1 \circ \sigma = \mathrm{id}_{\overline{X}(ws)}$ can be easily verified. ☐

Remark 3.2. Note that our construction of π_1 is dependent on the fact that $G = \mathrm{GL}_n$ is a split group and we have fixed suitable $T^* \subseteq B^*$ such that every $w \in W$ is fixed by the Frobenius endomorphism F.

We also have a smooth compactification of the Deligne–Lusztig variety $X(sw)$:

$$\overline{X}(sw) = \left\{ (B'_0, ..., B'_{r+1}) \in X^{r+2} \,\middle|\, \begin{array}{l} (B'_j, B'_{j+1}) \in \overline{O(t_j)}, j = 1, ..., r, \\ (B'_0, B'_1) \in \overline{O(s)}, B'_{r+1} = FB'_0 \end{array} \right\}.$$

Lemma 3.3. *The map $\pi_2 : \overline{X}(sws) \to \overline{X}(sw)$ defined by*

$$(B_0, B_1, ..., B_{r+1}, FB_0) \mapsto (B_{r+1}, FB_1, ..., FB_{r+1})$$

is a $\mathbb{P}^1$-bundle over $\overline{X}(sw)$ locally trivial with respect to an fppf-covering of $\overline{X}(sw)$.

Proof. We first check the well-definedness of π_2. Take $(B_0, B_1, ..., B_{r+1}, FB_0) \in \overline{X}(sws)$. Since $(B_{r+1}, FB_0) \in \overline{O(s)}$ and $(FB_0, FB_1) \in \overline{O(s)}$, we know that $(B_{r+1}, FB_1) \in \overline{O(s)}$. Thus $(B_{r+1}, FB_1, ..., FB_{r+1}) \in \overline{X}(sw)$.

Note that $F : \overline{X}(sw) \to \overline{X}(sw)$ is a flat and finite morphism. Via flat base change by $F : \overline{X}(sw) \to \overline{X}(sw)$, $\overline{X}(sws)$ becomes a fppf $\mathbb{P}^1$-bundle over $\overline{X}(sw)$. More precisely, we have a cartesian square:

$$\begin{tikzcd} Y \arrow[r, "\mathrm{pr}_1'"] \arrow[d, "\pi_2'"'] & \overline{X}(sws) \arrow[d, "\pi_2"] \\ \overline{X}(sw) \arrow[r, "F"] & \overline{X}(sw), \end{tikzcd} \tag{3.1}$$

where

$$Y := \{(B, B') \in \overline{X}(sws) \times \overline{X}(sw) | FB' = \pi_2(B)\},$$

and π_2' is projection to the second component cf. [DL76, Theorem 1.6]. Moreover, π_2' fits into another cartesian square:

$$\begin{tikzcd} Y \arrow[r, "\iota"] \arrow[d, "\pi_2'"'] & \overline{O(s)} \arrow[d, "\mathrm{pr}_2"] \\ \overline{X}(sw) \arrow[r, "\mathrm{pr}_0"] & X. \end{tikzcd}$$

Let $B = (B_0, ..., B_{r+1}, FB_0)$ and $B' = (B_0', B_1', ..., B_r', FB_0')$ such that $(B, B') \in Y$. The map pr_0 is given by

$$(B_0', B_1', ..., B_r', FB_0') \mapsto B_0'.$$

Also note that ι is the map defined by

$$((B_0, ..., B_{r+1}, FB_0), (B_0', B_1', ..., B_r', FB_0')) \mapsto (B_0, B_0').$$

Since $F(s) = s$, we know that (B_0, B_0') and (FB_0, FB_0') must belong to the same orbit in $X \times X$. The condition $FB_0' = B_{r+1}$ implies that $(FB_0, FB_0') \in \overline{O(s)}$, so $(B_0, B_0') \in \overline{O(s)}$.

For all $B' = (B_0', B_1', ..., B_r', FB_0') \in \overline{X}(sw)$, we have by the condition $FB' = \pi_2(B)$ that $\pi_2'^{-1}(B') = (B, B')$ where $B = (B_0, ..., B_{r+1}, FB_0)$ such that $FB_0' = B_{r+1}$. In particular, $(B_0, B_1') \in \overline{O(s)}$. Conversely, for any $B_0 \in X$ such that $(B_0, B_0') \in \overline{O(s)}$, we have $(FB_0, FB_0') \in \overline{O(s)}$ and $(B_0, B_1') \in \overline{O(s)}$. Thus $(B_0, B_1', ..., B_r', FB_0', FB_0) \in \overline{X}(sws)$. In particular,

$$((B_0, B_1', ..., B_r', FB_0', FB_0), (B_0', B_1', ..., B_r', FB_0')) \in \pi_2'^{-1}(B_0', B_1', ..., B_r', FB_0')$$

Hence via this cartesian square, we know that π_2' is a $\mathbb{P}^1$-bundle over $\overline{X}(sw)$. The argument that π_2' is locally trivial with respect to the Zariski topology is analogous to the one used in Lemma 3.1.

Finally, we return to the cartesian diagram (3.1). Since $\overline{X}(sw)$ is a $\overline{\mathbb{F}}_p$-scheme of finite type, we know that the Frobenius endomorphism $F : \overline{X}(sw) \to \overline{X}(sw)$ is flat, and is a universal homeomorphism. Thus the base change pr_1' of F is also flat, and is a universal homeomorphism. Since π_2' gives a Zariski locally trivial $\mathbb{P}^1$-bundle, and the fppf topology is finer than the Zariski topology, there exists a fppf open covering $\mathcal{U} := \{f_i : U_i \to \overline{X}(sw)\}_i$ such that π_2' is locally trivial with respect to π_2'. We get a composition of cartesian diagrams for each i:

$$\begin{tikzcd} U_i \times \mathbb{P}^1 \arrow[r] \arrow[d] & Y \arrow[r, "\mathrm{pr}_1'"] \arrow[d, "\pi_2'"'] & \overline{X}(sws) \arrow[d, "\pi_2"] \\ U_i \arrow[r, "f_i"] & \overline{X}(sw) \arrow[r, "F"] & \overline{X}(sw). \end{tikzcd}$$

Since F is flat and surjective, we see that the composition morphism $F \circ f_i$ is flat, locally of finite presentation for all i and

$$\bigcup_i (F \circ f_i(U_i)) = \overline{X}(sws).$$

Thus $\mathcal{U}' := \{F \circ f_i : U_i \to \overline{X}(sw)\}$ gives a fppf covering for $\overline{X}(sw)$ such that π_2 is locally trivial with respect to $\mathcal{U}'$. $\square$

Remark 3.4. Recall that proper morphisms are preserved under fpqc (hence fppf) base change and descent [SGA1, Exposé VIII, Corollary 4.8]. We see that the map π_1 and π_2 in Lemma 3.1 and 3.3 are also proper morphisms of $\overline{\mathbb{F}}_p$-schemes.

Lemma 3.5. *The maps $\pi_1 : \overline{X}(sws) \to \overline{X}(ws)$ and $\pi_2 : \overline{X}(sws) \to \overline{X}(sw)$ defined above are $\mathrm{GL}_n(\mathbb{F}_q)$-equivariant.*

Proof. Take $(B_0, B_1, ..., B_{r+1}, FB_0) \in \overline{X}(sws)$. Recall that $\mathrm{GL}_n(\mathbb{F}_q)$ acts on $\overline{X}(sws)$ via conjugation in each component. Let $g \in \mathrm{GL}_n(\mathbb{F}_q)$.

$$\pi_1\left(gB_0g^{-1}, ..., gB_{r+1}g^{-1}, g(FB_0)g^{-1}\right) = \left(gB_1g^{-1}, ..., gB_{r+1}g^{-1}, F(gB_1g^{-1})\right)$$

Since any $g \in \mathrm{GL}_n(\mathbb{F}_q)$ is fixed by F, we have $F(gB_1g^{-1}) = g(FB_1)g^{-1}$ and thus

$$\pi_1\left(gB_0g^{-1}, ..., gB_{r+1}g^{-1}, g(FB_0)g^{-1}\right) = g.\pi_1\left(B_0, B_1, ..., B_{r+1}, FB_0\right).$$

For π_2, let $(B_0, B_1, ..., B_{r+1}, FB_0) \in \overline{X}(sws)$, and $g \in \mathrm{GL}_n(\mathbb{F}_q)$. Consider the following:

$$\pi_2\left(gB_0g^{-1}, gB_1g^{-1}, ..., g(FB_0)g^{-1}\right) = \left(gB_{r+1}g^{-1}, F(gB_1g^{-1}), ..., F(gB_{r+1}g^{-1})\right).$$

Since F fixes g and g^{-1}, we have $F(g^{-1}B_ig) = B_i$ for all i. Thus

$$\pi_2\left(gB_0g^{-1}, gB_1g^{-1}, ..., g(FB_0)g^{-1}\right) = g.\pi_2\left(B_0, B_1, ..., B_{r+1}, FB_0\right).$$

□

3.2. **Cohomology of the structure sheaf of the $\mathbb{P}^1$-bundles.** Recall that the smooth compactifications of Deligne–Lusztig varieties are smooth, separated scheme of finite type over $\overline{\mathbb{F}}_p$.

Proposition 3.6. *Let $w \in \hat{F}^+$, $s \in S$. Then for all $k \geq 0$, there are isomorphisms of $\overline{\mathbb{F}}_p$-vector spaces:*

$$H^k\left(\overline{X}(ws), \mathcal{O}_{\overline{X}(ws)}\right) \xrightarrow{\sim} H^k\left(\overline{X}(sws), \mathcal{O}_{\overline{X}(sws)}\right)$$

and

$$H^k\left(\overline{X}(sw), \mathcal{O}_{\overline{X}(sw)}\right) \xrightarrow{\sim} H^k\left(\overline{X}(sws), \mathcal{O}_{\overline{X}(sws)}\right).$$

Proof. To simplify notation, we use $X := \overline{X}(sws), Y := \overline{X}(ws)$ (resp. $Y := \overline{X}(sw)$). Let π be as in Lemma 3.1 (resp. Lemma 3.3), and consider the Leray spectral sequence for $\mathcal{O}_X$:

$$E_2^{p,q} = H^p\left(Y, R^q\pi_*\mathcal{O}_X\right) \implies H^{p+q}\left(X, \mathcal{O}_X\right). \tag{3.2}$$

Note that X, Y are smooth schemes of finite type over $\overline{\mathbb{F}}_p$. Since the (closed) fibres of π are equidimentional and $\dim_{\overline{\mathbb{F}}_p} X = \dim_{\overline{\mathbb{F}}_p} Y + \dim_{\overline{\mathbb{F}}_p} \mathbb{P}^1_{\overline{\mathbb{F}}_p}$, by miracle flatness [Mat89, Theorem 23.1], we know that $\pi : X \to Y$ is a flat morphism. Since the fibres are isomorphic to $\mathbb{P}^1_{\overline{\mathbb{F}}_p}$, they are geometrically connected and geometrically reduced. By [Fan+05, §9.3.11], we know that $\pi_*\mathcal{O}_X \cong \mathcal{O}_Y$.

Note that X is geometrically reduced and π has connected fibres. By [EGAIII-1, Prop 1.4.10], we know that for $q > 0$, $R^q\pi_*(\mathcal{O}_X)$ are coherent $\mathcal{O}_Y$-modules. For $q \geq 0$, let $\mathcal{F}^q_{\mathrm{fppf}}$ be the presheaf on Y_{fppf} defined by

$$(\varphi_V : V \to Y) \mapsto \Gamma(V, \varphi_V^* R^q\pi_*(\mathcal{O}_X)), \tag{3.3}$$

s where $\varphi_V : V \to Y$ is any flat morphism of finite presentation. Let $\mathcal{U} := \{\varphi_{U_i} : U_i \to Y\}_i$ be a fppf open covering for Y such that for each i, we have a cartesian diagram

$$\begin{tikzcd} X \times_Y U_i \arrow[r, "\varphi_{i,X}"] \arrow[d, "\pi_{U_i}"'] & X \arrow[d, "\pi"] \\ U_i \arrow[r, "\varphi_{U_i}"] & Y \end{tikzcd},$$

with $X \times_Y U_i \xrightarrow{\sim} \mathbb{P}^1_{U_i}$. By [SGA7-II, Exposé XI, Theorem 1.1], we have $R^q\pi_{U_i,*}(\mathcal{O}_{X\times_Y U_i}) = 0$ for all $q > 0$. Since $\mathcal{O}_{X\times_Y U_i} \cong \varphi^*_{i,X}\mathcal{O}_X$, we see by [Har77, Proposition III.9.3] that $\varphi^*_{U_i}R^q\pi_*(\mathcal{O}_X) = 0$ for all $q > 0$. Thus for all $q > 0$, we have $\mathcal{F}^q_{\mathrm{fppf}} = 0$.

Since quasi-coherent sheaves satisfy fpqc descent [SGA1, Exposé VIII, Corollary 1.3] and thus fppf descent, we know that for $q \geq 0$, there is an isomorphism

$$H^p\big(Y, \mathcal{F}^q_{\mathrm{fppf}}\big) \xrightarrow{\sim} H^p\left(Y, R^q\pi_*(\mathcal{O}_X)\right).$$

Therefore the Leray spectral sequence (3.2) degenerates and we have

$$H^p(Y, \mathcal{O}_Y) \xrightarrow{\sim} H^p(X, \mathcal{O}_X).$$

□

Remark 3.7. Since π is a $\mathrm{GL}_n(\mathbb{F}_q)$-equivariant morphism, the isomorphisms of cohomology groups

$$H^k(Y, \mathcal{O}_Y) \xrightarrow{\sim} H^k(X, \mathcal{O}_X)$$

are also equivariant under the $\mathrm{GL}_n(\mathbb{F}_q)$-action for all $k \geq 0$.

4. Towards induction steps

We now set the stage for the (double) induction. Our goal is to reduce the problem of computing the cohomology groups of coherent sheaves on the smooth compactifications of Deligne–Lusztig varieties to those corresponding to a Coxeter element of W or a Coxeter element corresponding to a parabolic subgroup $P \subseteq \mathrm{GL}_n$.

In loose terms, we may describe the strategy as follows: Let w be an element of the free monoid F^+ or $\hat{F}^+$. Its expression may contain a repeating $s \in S$. We introduce operations C, K, R on F^+ and $\hat{F}^+$ so that after applying finitely many such operations on w, we may obtain a word of the form $sw's$. The operations C, K, R preserve the length of w, so we will still have $\ell(w) = \ell(sw's) = \ell(w') + 2$. Then Section 3 helps us to reduce this to the case of sw' and thus removing one of the repeating s. This process has finitely many steps and we will eventually reduce it to the case of $v \in W$ being a product of non-repeating simple reflections with $\ell(v) = |\mathrm{supp}(w)|$.

We will introduce each of the operations C, K, R and discuss how they affect the cohomology groups of the corresponding smooth compactifications of Deligne–Lusztig varieties.

One can find the original definitions of these operations and the double induction strategy in [Orl18, before Proposition 7.9] for the case of ℓ-adic cohomology with compact support.

4.1. The Cyclic shifting operation.

The elements $sw', w's \in W$ are conjugated by $s \in S$. Recall from Definition 2.1 and 2.4 that this can be generalized to elements of W being conjugated by cyclic shifts. The following operator is constructed to impose the concept of elements being conjugated by cyclic shifts on F^+ and $\hat{F}^+$.

Definition 4.1. Let $w \in F^+$ $\big(\text{resp. } \hat{F}^+\big)$. If $w = sw'$, where $w' \in F^+$ $\big(\text{resp. } \hat{F}^+\big)$ and $s \in S$, we define the operator C on F^+ $\big(\text{resp. } \hat{F}^+\big)$ by $C(w) := w's$.

Proposition 4.2. *Let $w \in \hat{F}^+$, such that $w = sw'$ with $s \in S$. Then we have isomorphisms of $\overline{\mathbb{F}}_p$-vector spaces for all $k \geq 0$:*

$$H^k\left(\overline{X}(w), \mathcal{O}_{\overline{X}(w)}\right) \xrightarrow{\sim} H^k\left(\overline{X}(C(w)), \mathcal{O}_{\overline{X}(C(w))}\right).$$

Furthermore, the isomorphism is $\mathrm{GL}_n(\mathbb{F}_q)$-equivariant.

Proof. Let $w \in \hat{F}^+$, such that $w = sw'$ with $s \in S$ and $w' \in \hat{F}^+$. Consider the product $ws = sw's$ in $\hat{F}^+$. We consider from Section 3.1 the surjective morphisms $\pi_1 : \overline{X}(sw's) \to \overline{X}(w's)$ and $\pi_2 : \overline{X}(sw's) \to \overline{X}(sw')$ that make $\overline{X}(sw's)$ a fppf $\mathbb{P}^1$-bundle over $\overline{X}(w's)$ and $\overline{X}(sw')$ respectively.

By Proposition 3.6, we have $\mathrm{GL}_n(\mathbb{F}_q)$-equivariant isomorphisms for all $k \geq 0$,

$$H^k\left(\overline{X}(sw'), \mathcal{O}_{\overline{X}(sw')}\right) \cong H^k\left(\overline{X}(sw's), \mathcal{O}_{\overline{X}(sw's)}\right) \cong H^k\left(\overline{X}(w's), \mathcal{O}_{\overline{X}(w's)}\right).$$

Thus

$$H^k\left(\overline{X}(w), \mathcal{O}_{\overline{X}(w)}\right) \cong H^k\left(\overline{X}(C(w)), \mathcal{O}_{\overline{X}(C(w))}\right).$$

$\square$

4.2. **Operations corresponding to relations.** Recall from Definition 2.12 that $\hat{F}^+$ is generated by $S \amalg T'$.

Definition 4.3. Let $w \in \hat{F}^+$, such that $w = w_1 st w_2$ with $w_1, w_2 \in \hat{F}^+$, $s, t \in S$ and $st = ts$ in W. Define the operator K on $\hat{F}^+$ by $K(w) := w_1 ts w_2$.

Let $w \in \hat{F}^+$, such that $w = w_1 sts w_2$ with $w_1, w_2 \in \hat{F}^+$, $s, t \in S$ and $sts = tst$ in W. Define the operator R on $\hat{F}^+$ by $R(w) := w_1 tst w_2$.

Remark 4.4. We clearly have $K(w), R(w) \in \hat{F}^+$. Also observe that the operators K and R are analogous to two of the relations in the presentation of the symmetric group S_n.

Proposition 4.5. *(i) Let $w = w_1 st w_2$ such that $w_1, w_2 \in \hat{F}^+$ and $s, t \in S$ with $st = ts$ in W. Then for all $i \geq 0$, we have isomorphisms of $\overline{\mathbb{F}}_p$-vector spaces:*

$$H^i\left(\overline{X}(w), \mathcal{O}_{\overline{X}(w)}\right) \xrightarrow{\sim} H^i\left(\overline{X}(K(w)), \mathcal{O}_{\overline{X}(K(w))}\right).$$

(ii) Let $w = w_1 sts w_2$ such that $w_1, w_2 \in \hat{F}^+$ and $s, t \in S$ with $sts = tst$ in W. For all $i \geq 0$, we have isomorphisms of $\overline{\mathbb{F}}_p$-vector spaces:

$$H^i\left(\overline{X}(w), \mathcal{O}_{\overline{X}(w)}\right) \xrightarrow{\sim} H^i\left(\overline{X}(R(w)), \mathcal{O}_{\overline{X}(R(w))}\right).$$

Proof. Let w be as in the assumption of (ii). Recall that $\overline{X}(w)$ is smooth and projective over $\overline{\mathbb{F}}_p$. Recall from Definition 2.12 that we have $\widehat{sts} \in T'$ because $st \neq ts$. The $\overline{\mathbb{F}}_p$-scheme $\overline{X}(w_1 \widehat{sts} w_2)$ is projective and smooth by Lemma 2.21. In particular, elements of $\overline{X}(w_1 \widehat{sts} w_2)$ are of the form $(B'_0, ..., B'_j, B'_{j+1}, ..., FB'_0)$, where $(B'_j, B'_{j+1}) \in \overline{O(sts)}$ for some j. Now we have a cartesian square:

$$\begin{array}{ccc} \overline{X}(w) & \longrightarrow & \overline{O}(s,t,s) \\ {\scriptstyle f}\big\downarrow & & \big\downarrow \\ \overline{X}(w_1 \widehat{sts} w_2) & \longrightarrow & \overline{O(sts)}, \end{array}$$

where the horizontal maps are projections, and the vertical map on the right is the resolution of singularities from [DL76, §9.1]. Thus the projection map f is proper.

Observe that the open subscheme $X(w)$ of $\overline{X}(w)$ is also contained in $\overline{X}(w_1 \widehat{sts} w_2)$ such that the restriction of f to $X(w)$ is the identity. Hence f is a birational morphism.

By [CR11, Theorem 3.2.8], we have for all $i \geq 0$, an isomorphism of $\overline{\mathbb{F}}_p$-vector spaces:

$$H^i\left(\overline{X}(w), \mathcal{O}_{\overline{X}(w)}\right) \xrightarrow{\sim} H^i\left(\overline{X}(w_1 \widehat{sts} w_2), \mathcal{O}_{\overline{X}(w_1 \widehat{sts} w_2)}\right).$$

For $\widehat{tst} \in T'$, we use the same argument as above to construct a proper birational morphism $f' : \overline{X}(R(w)) \to \overline{X}(w_1 \widehat{sts} w_2)$. The key observation is that as $sts = tst$ in W, $st \neq ts$, the scheme $\overline{O}(t,s,t)$ gives a smooth compactification of $\overline{O(sts)}$.

By [CR11, Theorem 3.2.8], we have for all $i \geq 0$, an isomorphism of $\overline{\mathbb{F}}_p$-vector spaces:

$$H^i\left(\overline{X}(w_1 \widehat{sts} w_2), \mathcal{O}_{\overline{X}(w_1 \widehat{sts} w_2)}\right) \xrightarrow{\sim} H^i\left(\overline{X}(R(w)), \mathcal{O}_{\overline{X}(R(w))}\right).$$

This concludes (ii).

Let w be as in the assumption of (i), and set $\ell(w_1) = r_1$ and $\ell(w_2) = r_2$. For this proof, we define the following projective $\overline{\mathbb{F}}_p$-scheme:

$$Y := \Big\{ (B_0', ..., B_{r_1}', B_{r_1+1}', ..., B_{r_1+r_2}', FB_0)' | (B_{r_1}', B_{r_1+1}') \in \overline{O(st)}, \\ (B_0', ..., B_{r_1}') \in \overline{O}(w_1), (B_{r_1+1}', ..., B_{r_1+r_2}', FB_0) \in \overline{O}(w_2), \Big\}.$$

Since $st = ts$, we see that they are associated to non-adjacent simple reflections. By [DMR07, Proposition 2.2.16 (iii), (iv)], there exists an isomorphism $\overline{O}(s,t) \to \overline{O(st)}$ such that the restriction to the open subschemes $O(s,t) \to O(st)$ remains an isomorphism. Then $\overline{O(st)}$ is smooth and thus Y is smooth.

We have an cartesian square:

$$\begin{tikzcd} \overline{X}(w) \arrow[r] \arrow[d, "f"'] & \overline{O}(s,t) \arrow[d] \\ Y \arrow[r] & \overline{O(st)}. \end{tikzcd}$$

Thus the projection map f on the left is proper. As the restriction of f to $X(w)$ is the identity morphism, we see that f is birational. By [CR11, Theorem 3.2.8], for all $i \geq 0$, there is an isomorphism of $\overline{\mathbb{F}}_p$-vector spaces:

$$H^i\left(\overline{X}(w), \mathcal{O}_{\overline{X}(w)}\right) \xrightarrow{\sim} H^i\left(Y, \mathcal{O}_Y\right).$$

On the other hand, since $st = ts$, we have $O(st) = O(ts)$ and thus $\overline{O(st)} = \overline{O(ts)}$. As in [DL76, §9.1], there is a resolution of singularity $\overline{O}(t,s) \to \overline{O(ts)}$. Hence we have a proper birational morphism $\overline{O}(t,s) \to \overline{O(st)}$. Thus we have a cartesian square with horizontal maps being projections:

$$\begin{tikzcd} \overline{X}(K(w)) \arrow[r] \arrow[d, "f'"'] & \overline{O}(t,s) \arrow[d] \\ Y \arrow[r] & \overline{O(st)}. \end{tikzcd}$$

Thus f' is proper. Note that the restriction of f' to $X(w_1tsw_2)$ gives an isomorphism between the respective open subschemes $X(w_1tsw_2)$ and $X(w_1stw_2)$ of $\overline{X}(K(w))$ and Y. Thus f' is birational. By [CR11, Theorem 3.2.8], for all $i \geq 0$, there is an isomorphism of $\overline{\mathbb{F}}_p$-vector spaces:

$$H^i\left(Y, \mathcal{O}_Y\right) \xrightarrow{\sim} H^i\left(\overline{X}(K(w)), \mathcal{O}_{\overline{X}(K(w))}\right).$$

This concludes the proof of (i).

□

5. The base case

Let $G = \mathrm{GL}_n$. Sections 3 and 4 have reduced the study of $H^k\big(\overline{X}(w), \mathcal{O}_{\overline{X}(w)}\big)$ to the case in which w is a Coxeter element or a Coxeter element corresponding to a parabolic subgroup $P \subseteq \mathrm{GL}_n$. We shall now treat the cases introduced in Section 1.8.

5.1. **Cohomology of $\overline{X}(w)$ for w a Coxeter element.** We use the notations of Section 1.7 and 1.8. Recall that $\mathbf{w}$ denotes the standard Coxeter element $s_1 \cdots s_{n-1}$.

Proposition 5.1. *For $k > 0$, we have $H^k\big(\overline{X}(\mathbf{w}), \mathcal{O}_{\overline{X}(\mathbf{w})}\big) = 0$. Then the space of global section*

$$H^0\left(\overline{X}(\mathbf{w}), \mathcal{O}_{\overline{X}(\mathbf{w})}\right) = \overline{\mathbb{F}}_p$$

is the trivial $\mathrm{GL}_n(\mathbb{F}_q)$-representation.

Proof. Recall that $\overline{X}(\mathbf{w})$ is isomorphic to the successive blow up $\tilde{Y}$ of $\mathbb{P}^{n-1}_{\overline{\mathbb{F}}_p}$ along all $\mathbb{F}_q$-rational linear subvarieties [Ito05, §4.1]. Thus there exists a birational morphism of $\overline{\mathbb{F}}_p$-schemes $\overline{X}(\mathbf{w}) \to \mathbb{P}^{n-1}_{\overline{\mathbb{F}}_p}$. Thus by [CR11, Theorem 3.2.8], for all $k \geq 0$, there is an isomorphism of $\overline{\mathbb{F}}_p$-vector spaces:

$$H^k\left(\overline{X}(\mathbf{w}), \mathcal{O}_{\overline{X}(\mathbf{w})}\right) \xrightarrow{\sim} H^k\left(\mathbb{P}^{n-1}_{\overline{\mathbb{F}}_p}, \mathcal{O}_{\overline{X}(\mathbf{w})}\right).$$

Thus we have

$$H^k\left(\overline{X}(\mathbf{w}), \mathcal{O}_{\overline{X}(\mathbf{w})}\right) = \begin{cases} \overline{\mathbb{F}}_p, & k = 0 \\ 0, & k > 0. \end{cases}$$

Finally, it follows from Lemma 1.15 that $\mathrm{GL}_n(\mathbb{F}_q)$ acts on $H^0\big(\overline{X}(\mathbf{w}), \mathcal{O}_{\overline{X}(\mathbf{w})}\big)$ trivially. □

Remark 5.2. The above Proposition also follows from [GK05, Theorem 2.3].

Corollary 5.3. *Let $w \in W$ be an arbitrary Coxeter element. Then the cohomology of $\overline{X}(w)$ is as follows:*

$$H^k\left(\overline{X}(w), \mathcal{O}_{\overline{X}(w)}\right) = \begin{cases} \overline{\mathbb{F}}_p, & k = 0 \\ 0, & k > 0. \end{cases}$$

In particular, $H^0\big(\overline{X}(w), \mathcal{O}_{\overline{X}(w)}\big)$ is the trivial $\mathrm{GL}_n(\mathbb{F}_q)$-representation.

Proof. After fixing a reduced expression of w, we have a corresponding element w' of $\hat{F}^+$ with $\hat{\alpha}(w') = w$ and $\overline{X}(w') \cong \overline{X}(w)$. By [GP00, Theorem 3.1.4], any two Coxeter elements in W are conjugate through a cyclic shift. Thus we have $C^k(w') = \mathbf{w}$ for some integer $k \geq 0$. We may apply Proposition 4.2 to reduce to the case when w is a standard Coxeter element. Use Proposition 5.1 to get the result on the cohomology groups, and it follows from Lemma 1.15 that $H^0\big(\overline{X}(w), \mathcal{O}_{\overline{X}(w)}\big)$ is a trivial $\mathrm{GL}_n(\mathbb{F}_q)$-representation. □

5.2. Cohomology of $\overline{X}_{L_I}(w)$ for $w \leq \mathbf{w}$ and $L_I \subseteq \mathrm{GL}_n$ a standard Levi subgroup.

Lemma 5.4. *Let $w \in W$ such that $w := s_{i_1} \cdots s_{i_m}$, $s_{i_j} \in S$, and let $I = \mathrm{supp}(w)$. Then we have an isomorphism of $\overline{\mathbb{F}}_p$-schemes compatible with $L_I(\mathbb{F}_q)$-action:*

$$X_{L_I}(w) \xrightarrow{\sim} X_{\mathrm{GL}_{\mathrm{n}_1}}(w_1) \times \cdots \times X_{\mathrm{GL}_{\mathrm{n}_r}}(w_r),$$

where w_a is an element of the Weyl group of GL_{n_a}, $a = 1, ..., r$ and $n_1 + \cdots + n_r = n$.

When the s_{i_j}'s do not repeat, w_a is a Coxeter element in the Weyl group of GL_{n_a} for all a. In particular, when $w \leq \mathbf{w}$, $w_1 \cdots w_r = w$.

Proof. For $a = 1, ..., r$, denote the Weyl group of $\mathrm{GL}_{\mathrm{n_a}}$ by W_a. The intersection $L_I \cap B^*$ is a Borel subgroup of L_I, and it is a product $B_1 \times \cdots \times B_r$, where B_a is a Borel subgroup of $\mathrm{GL}_{\mathrm{n_a}}$. Then we have the homogeneous spaces $X_a := \mathrm{GL}_{n_a}/B_a$, and the orbit of $v \in W_a$ in $X_a \times X_a$ is $O_a(v)$.

Note that the Weyl group W_I of L_I is isomorphic to the product of symmetric groups $S_{n_1} \times \cdots \times S_{n_r}$. Hence the Bruhat decomposition of L_I is compatible with the Bruhat decomposition of each $\mathrm{GL}_{\mathrm{n_a}}$, and thus the orbit $O_{L_I}(w)$ is the product $O_1(w_1) \times \cdots \times O_r(w_r)$ over Spec $\overline{\mathbb{F}}_p$. Thus by construction, when s_{i_j} are all distinct, they will each show up exactly once in w_a for exactly one a. When $w \preceq \mathbf{w}$, we have $w_1 \cdots w_r = w$.

The restriction of the Frobenius endomorphism from GL_n to L_I respects the product as well. To finish the proof, it suffices to go through the definitions of Deligne–Lusztig varieties with respect to products over Spec $\overline{\mathbb{F}}_p$. □

The same applies to the corresponding smooth compactifications with respect to the expression $w = s_{i_1} \cdots s_{i_m}$.

Lemma 5.5. *Using the same notations as in Lemma 5.4, we have an isomorphism of $\overline{\mathbb{F}}_p$-schemes equivariant under $L_I(\mathbb{F}_q)$-action.*

$$\overline{X}_{L_I}(w) \xrightarrow{\sim} \overline{X}_{\mathrm{GL}_{n_1}}(w_1) \times \cdots \times \overline{X}_{\mathrm{GL}_{n_r}}(w_r)$$

Proof. It suffices check the definitions of smooth compactifications for Deligne–Lusztig varieties with respect product of reductive groups. □

Remark 5.6. If $\mathrm{supp}(w)$ is a proper subset of S, then sometimes we could have $\mathrm{GL}_{n_i} = \mathrm{GL}_1$ for some i, thus $\overline{X}_{\mathrm{GL}_{n_i}}(e)$ is the point corresponding to the only Borel subgroup $1 \in \mathrm{GL}_1(\mathbb{F}_q) = \mathbb{F}_q^\times$.

Proposition 5.7. *Let $w \in W$ such that $w \leq \mathbf{w}$. Let $I = \mathrm{supp}(w)$. Then*

$$H^k\left(\overline{X}_{L_I}(w), \mathcal{O}_{\overline{X}_{L_I}(w)}\right) = 0,$$

for $k > 0$, and

$$H^0\left(\overline{X}_{L_I}(w), \mathcal{O}_{\overline{X}_{L_I}(w)}\right) = \overline{\mathbb{F}}_p$$

is the trivial $L_I(\mathbb{F}_q)$-representation.

Proof. By Lemma 5.5, we may compute the cohomology for $\overline{X}_{\mathrm{GL}_{n_1}}(w_1) \times \cdots \times \overline{X}_{\mathrm{GL}_{n_r}}(w_r)$, with notations as before. To simplify the notation, set

$$V_j := \overline{X}_{\mathrm{GL}_{n_1}}(w_1) \times \cdots \times \overline{X}_{\mathrm{GL}_{n_j}}(w_j).$$

By applying induction on the Künneth formula for coherent sheaves [EGAIII-2, Theorem 6.7.8], we have

$$H^k\left(V_j, \mathcal{O}_{V_j}\right) = \bigoplus_{p+q=k} H^p\left(V_{j-1}, \mathcal{O}_{V_{j-1}}\right) \otimes_{\overline{\mathbb{F}}_p} H^q\left(\overline{X}_{\mathrm{GL}_{n_j}}(w_j), \mathcal{O}_{\overline{X}_{\mathrm{GL}_{n_j}}(w_j)}\right).$$

By Proposition 5.1, we know that for all $j = 1, ..., r$,

$$H^k\left(\overline{X}_{\mathrm{GL}_{n_j}}(w_j), \mathcal{O}_{\overline{X}_{\mathrm{GL}_{n_j}}(w_j)}\right) = \begin{cases} \overline{\mathbb{F}}_p, & k = 0 \\ 0, & k > 0. \end{cases}$$

Thus,

$$H^k\left(V_j, \mathcal{O}_{V_j}\right) = H^k\left(V_{j-1}, \mathcal{O}_{V_{j-1}}\right) \otimes_{\overline{\mathbb{F}}_p} \overline{\mathbb{F}}_p.$$

Hence by induction on j we know that for any j,

$$H^k\left(V_j, \mathcal{O}_{V_j}\right) = \begin{cases} \overline{\mathbb{F}}_p, & k = 0 \\ 0, & k > 0. \end{cases}$$

The case $j = r$ yields the desired result. It follows from

$$H^0\left(\overline{X}_{L_I}(w), \mathcal{O}_{\overline{X}_{L_I}(w)}\right) = \overline{\mathbb{F}}_p,$$

that $H^0(\overline{X}_{L_I}(w), \mathcal{O}_{\overline{X}_{L_I}(w)})$ is the trivial $L_I(\mathbb{F}_q)$-representation by Lemma 1.15. □

Corollary 5.8. *Let $w \in W$, $w = s_{i_1} \cdots s_{i_m}$ and $I = \mathrm{supp}(w)$, such that the s_{i_j}'s are all distinct. Then*

$$H^k\left(\overline{X}_{L_I}(w), \mathcal{O}_{\overline{X}_{L_I}(w)}\right) = \begin{cases} \overline{\mathbb{F}}_p, & k = 0 \\ 0, & k > 0. \end{cases}$$

Furthermore, $H^0\left(\overline{X}_{L_I}(w), \mathcal{O}_{\overline{X}_{L_I}(w)}\right)$ is the trivial $L_I(\mathbb{F}_q)$-representation.

Proof. Repeat the proof of Proposition 5.7 and use Corollary 5.3. □

5.3. **Construction of $\overline{X}_{\mathrm{GL}_n}(w)$ for w a Coxeter element in a Levi subgroup W_I of W.** The constructions in this section apply to general G as in Section 1.1, but here we treat $G = \mathrm{GL}_n$ as in Section 1.7.

Let $I \subseteq S$, $t_1, ..., t_k \in W_I$. Recall that we have the standard parabolic subgroup $P_I \cong U_I \rtimes L_I$. Consider the product schemes

$$G^F/U_I^F \times_{\mathrm{Spec}\, \overline{\mathbb{F}}_p} X_{L_I}(t_1, ..., t_k)$$

and

$$G^F/U_I^F \times_{\mathrm{Spec}\, \overline{\mathbb{F}}_p} \overline{X}_{L_I}(t_1, ..., t_k).$$

When there is no ambiguity, a product of $\overline{\mathbb{F}}_p$-schemes will be assumed to be taken over Spec $\overline{\mathbb{F}}_p$.

There are natural L_I^F-actions on G^F/U_I^F. Here we consider a right L_I^F-action. Since $P_I \cong U_I \rtimes L_I$, we know that U_I is a normal subgroup of P_I, and so $lU_I = U_I l$ for any $l \in L_I$. Similarly, $P_I^F \cong U_I^F \rtimes L_I^F$ yields: for any $l \in L_I^F$,

$$lU_I^F = U_I^F l.$$

For any $xU_I^F \in G/U_I$ and $l \in L_I^F$, we consider the following right action

$$l \cdot xU_I^F = xl^{-1}U_i^F$$

Note that this action is free.

By the definition of Deligne–Lusztig varieties, there is a left L_I^F-action on $X_{L_I}(t_1, ..., t_k)$ (resp. $\overline{X}_{L_I}(t_1, ..., t_k)$). For any $l \in L_I^F$ and $(B_0, ..., B_k) \in X_{L_I}(t_1, ..., t_k)$, the action is

$$l \cdot (B_0, ..., B_k) = \left(lB_0l^{-1}, ..., lB_kl^{-1}\right).$$

We now describe a L_I^F-action on the product of G^F/U_I^F and $X_{L_I}(t_1, ..., t_k)$ (resp. $\overline{X}_{L_I}(t_1, ..., t_k)$). For any $\left(xU_I^F, (B_0, ..., B_k)\right) \in G^F/U_I^F \times X_{L_I}(t_1, ..., t_k)$ and $l \in L_I^F$, we have

$$l \cdot \left(xU_I^F, (B_0, ..., B_k)\right) = \left(xl^{-1}U_I^F, \left(lB_0l^{-1}, ..., lB_kl^{-1}\right)\right).$$

Since L_I^F is a finite group, and $G^F/U_I^F \times X_{L_I}(t_1, ..., t_k)$ (resp. $G^F/U_I^F \times \overline{X}_{L_I}(t_1, ..., t_k)$) is a $\overline{\mathbb{F}}_p$-scheme, the quotient of the L_I^F-action exists in the category of $\overline{\mathbb{F}}_p$-schemes. We recall the standard proof for any quasi-projective schemes with a finite group action.

Lemma 5.9. *Any orbit of the L_I^F-action on $G^F/U_I^F \times X_{L_I}(t_1, ..., t_k)$ (resp. $G^F/U_I^F \times \overline{X}_{L_I}(t_1, ..., t_k)$) is contained in some affine open subset of $G^F/U_I^F \times X_{L_I}(t_1, ..., t_k)$ (resp. $G^F/U_I^F \times \overline{X}_{L_I}(t_1, ..., t_k)$).*

Proof. This lemma is standard for any quasi-projective scheme with a suitable finite group action. See [Sta20, Tag 09NV] for a proof. ☐

Proposition 5.10. *The quotients*

$$\left(G^F/U_I^F \times X_{L_I}(t_1, ..., t_k)\right)/L_I^F$$

and

$$\left(G^F/U_I^F \times \overline{X}_{L_I}(t_1, ..., t_k)\right)/L_I^F$$

exists in the category of $\overline{\mathbb{F}}_p$-schemes.

Proof. This follows from [SGA1, Exposé V, Proposition 1.8] and Lemma 5.9. ☐

Remark 5.11. We will write

$$G^F/U_I^F \times^{L_I^F} X_{L_I}(t_1, ..., t_k) \quad \text{and} \quad G^F/U_I^F \times^{L_I^F} \overline{X}_{L_I}(t_1, ..., t_k)$$

for the quotients

$$\left(G^F/U_I^F \times X_{L_I}(t_1, ..., t_k)\right)/L_I^F \quad \text{and} \quad \left(G^F/U_I^F \times \overline{X}_{L_I}(t_1, ..., t_k)\right)/L_I^F.$$

There is an associated fibration of the quotient scheme $G^F/U_I^F \times^{L_I^F} X_{L_I}(t_1,...,t_k)$ (resp. $G^F/U_I^F \times^{L_I^F} \overline{X}_{L_I}(t_1,...,t_k)$), and we will now discuss its construction following the steps of [Jan03, §I.5.14]. Once again we only state this for $G^F/U_I^F \times^{L_I^F} \overline{X}_{L_I}(t_1,...,t_k)$ because the case for $G^F/U_I^F \times^{L_I^F} X_{L_I}(t_1,...,t_k)$ is identical.

Let

$$\pi : G^F/U_I^F \longrightarrow G^F/P_I^F$$

be the projection map, which is also a quotient of G^F/U_I^F by L_I^F. Now by composing with the projection to the fist factor, we obtain the following morphism of $\overline{\mathbb{F}}_p$-schemes.

$$\begin{aligned} f : G^F/U_I^F \times \overline{X}_{L_I}(t_1,...,t_k) &\longrightarrow G^F/P_I^F \\ \big(xU_I^F, (B_0,...,B_k)\big) &\longmapsto \pi(xU_I^F) \end{aligned}$$

Observe that f is invariant under L_I^F-action. In other words, for $(xU_I^F, y) \in G^F/U_I^F \times \overline{X}_{L_I}(t_1,...,t_k)$, and any $l \in L_I(\mathbb{F}_q)$,

$$f(xl^{-1}U_I^F, l \cdot y) = \pi(xl^{-1}U_I^F) = \pi(xU_I^F),$$

because $l^{-1} \in P_I(\mathbb{F}_q)$. Now we have an induced morphism

$$\begin{aligned} \pi_I : G^F/U_I^F \times^{L_I^F} \overline{X}_{L_I}(t_1,...,t_k) &\longrightarrow G^F/P_I^F \\ \big(xU_I^F, (B_0,...,B_k)\big) &\longmapsto \pi(xU_I^F), \end{aligned} \tag{5.1}$$

which is equivariant under the G^F-action and constant under the P_I^F-action. We know that π_I is surjective. For any $x \in G^F/U_I^F$, we have

$$\pi_I^{-1}(\pi(x)) \xrightarrow{\sim} \overline{X}_{L_I}(t_1,...,t_k).$$

Hence $G^F/U_I^F \times^{L_I^F} \overline{X}_{L_I}(t_1,...,t_k)$ together with π_I is a fibration over G^F/P_I^F, with fibres $\overline{X}_{L_I}(t_1,...,t_k)$.

Lemma 5.12. *Let $w \in W$ such that $w := s_{i_1} \cdots s_{i_k}$, and $I = \mathrm{supp}(w)$. Then the irreducible components of $X(w)$ (resp. $\overline{X}(w)$) are $|G(\mathbb{F}_q)/P_I(\mathbb{F}_q)|$ isomorphic copies of $X_{L_I}(s_{i_1},...,s_{i_k})$ (resp. $\overline{X}_{L_I}(s_{i_1},...,s_{i_k})$).*

Proof. By [DMR07, Proposition 2.3.8] and [DMR07, §2.3.4], we have isomorphisms of $\overline{\mathbb{F}}_p$-schemes

$$G^F/U_I^F \times^{L_I^F} X_{L_I}(s_{i_1},...,s_{i_k}) \xrightarrow{\sim} X_G(s_{i_1},...,s_{i_k})$$

and

$$G^F/U_I^F \times^{L_I^F} \overline{X}_{L_I}(s_{i_1},...,s_{i_k}) \xrightarrow{\sim} \overline{X}_G(s_{i_1},...,s_{i_k}),$$

which are equivariant under the action of G^F. The fibres of the maps cf. (5.1)

$$\pi_I' : G^F/U_I^F \times^{L_I^F} X_{L_I}(s_{i_1},...,s_{i_k}) \longrightarrow G^F/P_I^F$$

and

$$\pi_I : G^F/U_I^F \times^{L_I^F} \overline{X}_{L_I}(s_{i_1},...,s_{i_k}) \longrightarrow G^F/P_I^F$$

are irreducible as $\mathrm{supp}(w)$ generates the Weyl group W_I of L_I cf. [BR06] and [Gör09]. The proof is concluded by the discussion about the morphism π_I above. $\square$

Remark 5.13. In the above lemma, for any $w \in W$, we can see that the number of irreducible components $X(w)$ and $\overline{X}(w)$ (if one fix a reduced expression of w) depends only on $\mathrm{supp}(w)$.

5.4. **Cohomology of $\overline{X}_{\mathrm{GL}_n}(w)$ for w a Coxeter element in a Levi subgroup W_I of W.** We will now compute the cohomology with respect to the structure sheaf in this case for $G = \mathrm{GL}_n$.

Proposition 5.14. *Let $w \in W$ with $w = s_{i_1} \cdots s_{i_m}$ such that $s_{i_j} \in S$ are all distinct. Then for $I = \mathrm{supp}(w)$, one has*

$$H^0\left(\overline{X}(w), \mathcal{O}_{\overline{X}(w)}\right) = \mathrm{ind}_{P_I(\mathbb{F}_q)}^{\mathrm{GL}_n(\mathbb{F}_q)} \mathbb{1}_{\overline{\mathbb{F}}_p},$$

where $\mathbb{1}_{\overline{\mathbb{F}}_p}$ is the trivial $P_I(\mathbb{F}_q)$-representation. For all $k > 0$,

$$H^k\left(\overline{X}(w), \mathcal{O}_{\overline{X}(w)}\right) = 0.$$

Proof. By [DMR07, Proposition 2.3.8], we have an $\mathrm{GL}_n(\mathbb{F}_q)$-equivariant isomorphism of $\overline{\mathbb{F}}_p$-schemes:

$$\mathrm{GL}_n^F/U_I^F \times^{L_I^F} \overline{X}_{L_I}(s_{i_1}, ..., s_{i_m}) \xrightarrow{\sim} \overline{X}(s_{i_1}, ..., s_{i_m}).$$

Observe that the P_I^F-action on $\mathrm{GL}_n^F/U_I^F \times^{L_I^F} \overline{X}_{L_I}(s_{i_1}, ..., s_{i_m})$ amounts to an U_I^F-action, but U_I^F acts trivially on this scheme by construction. Hence P_I^F acts trivially on $\mathrm{GL}_n^F/U_I^F \times^{L_I^F} \overline{X}_{L_I}(s_{i_1}, ..., s_{i_m})$.

On the other hand, since the morphism

$$\pi_I : \mathrm{GL}_n^F/U_I^F \times^{L_I^F} \overline{X}_{L_I}(s_{i_1}, ..., s_{i_m}) \longrightarrow \mathrm{GL}_n^F/P_I^F$$

as defined in (5.1) is $\mathrm{GL}_n(\mathbb{F}_q)$-equivariant, we see that $\mathrm{GL}_n(\mathbb{F}_q)$ acts on the set of fibers of π_I transitively, and that the stablizer of each fibre corresponds to a conjugate of P_I^F in GL_n^F.

The induced $\mathrm{GL}_n(\mathbb{F}_q)$-action on the global sections gives us

$$H^0\left(\overline{X}(w), \mathcal{O}_{\overline{X}(w)}\right) = \mathrm{ind}_{P_I(\mathbb{F}_q)}^{\mathrm{GL}_d(\mathbb{F}_q)} H^0\left(\overline{X}_{L_I}(w), \mathcal{O}_{\overline{X}_{L_I}(w)}\right).$$

By Lemma 5.5, we know that

$$\overline{X}_{L_I}(w) \xrightarrow{\sim} \overline{X}_{\mathrm{GL}_{\mathrm{n}_1}}(w_1) \times \cdots \times \overline{X}_{\mathrm{GL}_{\mathrm{n}_\mathrm{r}}}(w_r),$$

is an isomorphism of $\mathbb{F}_p$-schemes equivariant under $\mathrm{GL}_n(\mathbb{F}_q)$-action, where $n_1 + \cdots + n_r = n$ and w_a is an element in the Weyl group of GL_{n_a}, $a = 1, ..., r$. This gives an isomorphism of $\mathrm{GL}_n(\mathbb{F}_q)$-modules:

$$H^0\left(\overline{X}_{L_I}(w), \mathcal{O}_{\overline{X}_{L_I}(w)}\right) \xrightarrow{\sim} H^0\left(\overline{X}_{\mathrm{GL}_{\mathrm{n}_1}}(w_1), \mathcal{O}_{\overline{X}_{\mathrm{GL}_{n_1}}(w_1)}\right) \otimes \cdots \otimes H^0\left(\overline{X}_{\mathrm{GL}_{\mathrm{n}_\mathrm{r}}}(w_r), \mathcal{O}_{\overline{X}_{\mathrm{GL}_{n_r}}(w_r)}\right).$$

Since w has full support in W_I, we know that w_a has full support in W_a for all $a = 1, ..., r$, where W_a is the Weyl group of GL_{n_a}. Since we have already shown in Corollary 5.3 that

$$H^0\left(\overline{X}_{\mathrm{GL}_{\mathrm{n_a}}}(w_a), \mathcal{O}_{\overline{X}_{\mathrm{GL}_{n_a}}(w_a)}\right) = \overline{\mathbb{F}}_p$$

is the trivial representation for $\mathrm{GL}_{n_a}(\mathbb{F}_q)$, we know that

$$H^0\left(\overline{X}_{L_I}(w), \mathcal{O}_{\overline{X}_{L_I}(w)}\right) = \overline{\mathbb{F}}_p$$

gives the trivial $L_I(\mathbb{F}_q)$-representation. Therefore

$$H^0\left(\overline{X}(w), \mathcal{O}_{\overline{X}(w)}\right) = \mathrm{ind}_{P_I(\mathbb{F}_q)}^{\mathrm{GL}_d(\mathbb{F}_q)} \mathbb{1}_{\overline{\mathbb{F}}_p},$$

where $\mathbb{1}_{\overline{\mathbb{F}}_p}$ is the trivial $P_I(\mathbb{F}_q)$-representation with coefficients in $\overline{\mathbb{F}}_p$.

□

Remark 5.15. The scheme $\overline{X}(s_{i_1}, ..., s_{i_m})$ as in the proof above may also be written as a disjoint union cf. [Lus77, (1.17)]:

$$\overline{X}(s_{i_1}, ..., s_{i_m}) \xrightarrow{\sim} \dot{\bigcup_{gP_I^F \in \mathrm{GL}_n^F/P_I^F}} gP_I^F.\overline{X}_{L_I}(s_{i_1}, ..., s_{i_m}).$$

6. The main Theorem

6.1. Cohomology of the structure sheaf on $\overline{X}(w)$.

Theorem 6.1. *Let $G = \mathrm{GL}_n$ and $w \in F^+$. Let $I = \mathrm{supp}(w)$ and $P_I = B^* W_I B^*$, then*

$$H^k\left(\overline{X}(w), \mathcal{O}_{\overline{X}(w)}\right) = \begin{cases} \mathrm{ind}_{P_I(\mathbb{F}_q)}^{\mathrm{GL}_n(\mathbb{F}_q)} \mathbb{1}_{\overline{\mathbb{F}}_p}, & k = 0 \\ 0, & k > 0, \end{cases}$$

where $\mathbb{1}_{\overline{\mathbb{F}}_p}$ is the trivial $P_I(\mathbb{F}_q)$-representation with coefficients in $\overline{\mathbb{F}}_p$.

Proof. Denote $w = s_{i_1} \cdots s_{i_m} \in F^+$. First, suppose that $\mathrm{supp}(w) = S$. If w is a Coxeter element, we apply Proposition 5.1 and Corollary 5.3.

If the s_{i_j}'s are not all distinct, we apply the three operators C, K, R and apply Proposition 4.2 and Proposition 4.5 to transform w into the shape $sw's$ with $s \in S$, so that we may apply Proposition 3.6. After repeating this procedure finitely many times, we have for all $k \geq 0$, an isomorphism of $\overline{\mathbb{F}}_p$-vector spaces:

$$H^k\left(\overline{X}(w), \mathcal{O}_{\overline{X}(w)}\right) \xrightarrow{\sim} H^k\left(\overline{X}(v), \mathcal{O}_{\overline{X}(v)}\right),$$

where v is a Coxeter element such that v has no repeating $s_{i_j} \in S$ in its expression and $\mathrm{supp}(v) = \mathrm{supp}(w)$. By Corollary 5.3, we know that for $k > 0$, $H^k\big(\overline{X}(w), \mathcal{O}_{\overline{X}(w)}\big)$ vanish, and

$$H^0\left(\overline{X}(w), \mathcal{O}_{\overline{X}(w)}\right) = \overline{\mathbb{F}}_p.$$

Thus by Lemma 1.15, we know that $H^0\big(\overline{X}(w), \mathcal{O}_{\overline{X}(w)}\big)$ is the trivial $\mathrm{GL}_n(\mathbb{F}_q)$-representation.

Next let w to not have full support and we denote $I = \mathrm{supp}(w)$. If the s_{i_j}'s are all distinct, then we apply Proposition 5.14.

If s_{i_j}'s are not all distinct, we again apply C, K, R and Proposition 4.2 and 4.5 to transform w into the shape $sw's$ with $s \in S$, so that we may apply Proposition 3.6. After repeating this procedure finitely many times, we have for all $k \geq 0$, an isomorphism of $\overline{\mathbb{F}}_p$-vector spaces:

$$H^k\left(\overline{X}(w), \mathcal{O}_{\overline{X}(w)}\right) \xrightarrow{\sim} H^k\left(\overline{X}(v), \mathcal{O}_{\overline{X}(v)}\right),$$

where v has no repeating s_{i_j}'s in its expression, and $\mathrm{supp}(v) = \mathrm{supp}(w)$. By Proposition 5.14, we have the vanishing

$$H^k\left(\overline{X}(w), \mathcal{O}_{\overline{X}(w)}\right) = 0,$$

for all $k > 0$.

In order to analyze the global section as a $\mathrm{GL}_n(\mathbb{F}_q)$-representation, observe that the construction in the proof of Proposition 5.14 carries over. Note that the result of Proposition 5.14 does not directly apply as we only have isomorphism of $\overline{\mathbb{F}}_p$-vector spaces $H^k\big(\overline{X}(w), \mathcal{O}_{\overline{X}(w)}\big) \cong H^k\big(\overline{X}(v), \mathcal{O}_{\overline{X}(v)}\big)$ from above.

We have a $\mathrm{GL}_n(\mathbb{F}_q)$-equivariant morphism of $\overline{\mathbb{F}}_p$-schemes:

$$\mathrm{GL}_n^F / U_I^F \times^{L_I^F} \overline{X}_{L_I}(s_{i_1}, ..., s_{i_m}) \longrightarrow \mathrm{GL}_n^F / P_I^F,$$

such that $\mathrm{GL}_n(\mathbb{F}_q)$ acts on the set of fibres of this morphism transitively and the stablizer of each fibre corresponds to a conjugate of $P_I(\mathbb{F}_q)$ in $\mathrm{GL}_n(\mathbb{F}_q)$. By [DMR07, Proposition 2.3.8], we have an $\mathrm{GL}_n(\mathbb{F}_q)$-equivariant isomorphism of $\overline{\mathbb{F}}_p$-schemes:

$$\mathrm{GL}_n^F / U_I^F \times^{L_I^F} \overline{X}_{L_I}(s_{i_1}, ..., s_{i_m}) \xrightarrow{\sim} \overline{X}(s_{i_1}, ..., s_{i_m})$$

The induced $\mathrm{GL}_n(\mathbb{F}_q)$-action on the global sections thus gives us

$$H^0\left(\overline{X}(w), \mathcal{O}_{\overline{X}(w)}\right) = \mathrm{ind}_{P_I(\mathbb{F}_q)}^{\mathrm{GL}_n(\mathbb{F}_q)} H^0\left(\overline{X}_{L_I}(w), \mathcal{O}_{\overline{X}_{L_I}(w)}\right).$$

By Lemma 5.5, we know that

$$\overline{X}_{L_I}(w) \xrightarrow{\sim} \overline{X}_{\mathrm{GL}_{n_1}}(w_1) \times \cdots \times \overline{X}_{\mathrm{GL}_{n_r}}(w_r),$$

is an isomorphism of $\mathbb{F}_p$-schemes equivariant under $L_I(\mathbb{F}_q)$-action, where $n_1 + \cdots + n_r = n$ and w_a is an element in the Weyl group of GL_{n_a}, $a = 1, ..., r$. This gives an isomorphism of $\mathrm{GL}_n(\mathbb{F}_q)$-modules:

$$H^0\left(\overline{X}_{L_I}(w), \mathcal{O}_{\overline{X}_{L_I}(w)}\right) \xrightarrow{\sim} H^0\left(\overline{X}_{\mathrm{GL_{n_1}}}(w_1), \mathcal{O}_{\overline{X}_{\mathrm{GL}_{n_1}}(w_1)}\right) \otimes \cdots \otimes H^0\left(\overline{X}_{\mathrm{GL_{n_r}}}(w_r), \mathcal{O}_{\overline{X}_{\mathrm{GL}_{n_r}}(w_r)}\right).$$

Since w has full support in W_I, we know that w_a has full support in W_a for all $a = 1, ..., r$, where W_a is the Weyl group of GL_{n_a}. Since we have already shown above that

$$H^0\left(\overline{X}_{\mathrm{GL_{n_a}}}(w_a), \mathcal{O}_{\overline{X}_{\mathrm{GL}_{n_a}}(w_a)}\right) = \overline{\mathbb{F}}_p$$

is the trivial representation for $\mathrm{GL}_{n_a}(\mathbb{F}_q)$, we know that

$$H^0\left(\overline{X}_{L_I}(w), \mathcal{O}_{\overline{X}_{L_I}(w)}\right) = \overline{\mathbb{F}}_p$$

gives the trivial $L_I(\mathbb{F}_q)$-representation. Therefore

$$H^0\left(\overline{X}(w), \mathcal{O}_{\overline{X}(w)}\right) = \mathrm{ind}_{P_I(\mathbb{F}_q)}^{\mathrm{GL}_n(\mathbb{F}_q)} \mathbb{1}_{\overline{\mathbb{F}}_p},$$

where $\mathbb{1}_{\overline{\mathbb{F}}_p}$ is the trivial $P_I(\mathbb{F}_q)$-representation with coefficients in $\overline{\mathbb{F}}_p$. $\square$

Remark 6.2. For $w \in W$, if we fix a reduced expression $w = s_{i_1} \cdots s_{i_m}$ with $s_{i_j} \in S$, then $s_{i_1} \cdots s_{i_m}$ can be considered as an element of F^+ and $\overline{X}(s_{i_1}, ..., s_{i_m}) \cong \overline{X}(w)$. Thus Theorem 6.1 can be applied to $\overline{X}(w)$.

6.2. **The** mod p^m **and** $\mathbb{Z}_p$ **étale cohomology of** $\overline{X}(w)$**.** For the rest of this paper, for any ring R and any finite group H, we fix the notation $\mathbb{1}_R$ for the free 1-dimensional trivial H-module with coefficients in R.

Let X be a k-scheme with k being a field of characteristic $p > 0$. We have the constant sheaf $\mathbb{Z}/p\mathbb{Z}$ on $X_{\text{ét}}$. Note that the associated presheaf of the group scheme $\mathbb{G}_a$ is a sheaf on both $X_{\text{ét}}$ and X_{Zar} [Mil80, p. 52]. In particular, it gives the structure sheaf on X_{Zar}. Recall the Artin–Schreier sequence [Mil80, p. 67].

Lemma/Definition 6.3. Let X be a k-scheme with k being a field of characteristic $p > 0$. Let F_p be the p-Frobenius on $\mathcal{O}_X$ sending $x \mapsto x^p$. There exists a short exact sequence of sheaves on $X_{\text{ét}}$:

$$0 \to \mathbb{Z}/p\mathbb{Z} \longrightarrow \mathbb{G}_a \xrightarrow{F_p - 1} \mathbb{G}_a \to 0.$$

We call this the *Artin–Schreier sequence*.

Note that by [Mil80, p. 114], the cohomology of $\mathbb{G}_a$ on $X_{\text{ét}}$ and X_{Zar} are isomorphic.

Remark 6.4. Recall that for $G = \mathrm{GL}_n$, we have

$$H^0\left(\overline{X}(w), \mathcal{O}_{\overline{X}(w)}\right) = \mathrm{ind}_{P_I(\mathbb{F}_q)}^{\mathrm{GL}_n(\mathbb{F}_q)} \overline{\mathbb{F}}_p,$$

where $I = \mathrm{supp}(w)$ and $P_I = B^* W_I B^*$. Note that p-Frobenius F_p on $\overline{X}(w)$ is given by the identity on the topological space and p-power map on sections of $\mathcal{O}_{\overline{X}(w)}$. Thus the p-power map given by F_p on the global section $H^0\big(\overline{X}(w), \mathcal{O}_{\overline{X}(w)}\big)$ on the left hand side is the same as the p-power map on $\overline{\mathbb{F}}_p$ on the right hand side.

Proposition 6.5. *Let $G = \mathrm{GL}_n$ and $w \in F^+$. Consider the constant sheaf $\mathbb{Z}/p\mathbb{Z}$ on $\overline{X}(w)_{\acute{e}t}$. Let $I = \mathrm{supp}(w)$ and $P_I = B^* W_I B^*$ then we have*

$$H^k_{\acute{e}t}\Big(\overline{X}(w), \mathbb{Z}/p\mathbb{Z}\Big) = \begin{cases} \mathrm{ind}_{P_I(\mathbb{F}_q)}^{\mathrm{GL}_n(\mathbb{F}_q)} \mathbb{1}_{\mathbb{Z}/p\mathbb{Z}}, & k = 0, \\ 0, & k > 0. \end{cases}$$

Proof. Consider the long exact sequence associated to the Artin–Schreier sequence on $\overline{X}(w)$.

$$\begin{aligned}0 \to H^0_{\text{ét}}\big(\overline{X}(w), \mathbb{Z}/p\mathbb{Z}\big) \to H^0\big(\overline{X}(w), \mathcal{O}_{\overline{X}(w)}\big) \xrightarrow{F_p-1} H^0\big(\overline{X}(w), \mathcal{O}_{\overline{X}(w)}\big) \to \\ \to H^1_{\text{ét}}\big(\overline{X}(w), \mathbb{Z}/p\mathbb{Z}\big) \to H^1\big(\overline{X}(w), \mathcal{O}_{\overline{X}(w)}\big) \xrightarrow{F_p-1} H^1\big(\overline{X}(w), \mathcal{O}_{\overline{X}(w)}\big) \to \cdots\end{aligned}$$

By Theorem 6.1, we know that

$$H^k\big(\overline{X}(w), \mathcal{O}_{\overline{X}(w)}\big) = 0,$$

for all $k > 0$, and

$$H^0\big(\overline{X}(w), \mathcal{O}_{\overline{X}(w)}\big) = \operatorname{ind}_{P_I(\mathbb{F}_q)}^{\mathrm{GL}_n(\mathbb{F}_q)} \overline{\mathbb{F}}_p.$$

As a consequence, the long exact sequence above becomes

$$0 \to H^0_{\text{ét}}\big(\overline{X}(w), \mathbb{Z}/p\mathbb{Z}\big) \to H^0\big(\overline{X}(w), \mathcal{O}_{\overline{X}(w)}\big) \xrightarrow{F_p-1} H^0\big(\overline{X}(w), \mathcal{O}_{\overline{X}(w)}\big) \to H^1_{\text{ét}}\big(\overline{X}(w), \mathbb{Z}/p\mathbb{Z}\big) \to 0.$$

Since $\overline{\mathbb{F}}_p$ is algebraically closed, the polynomial $x^p - x \in \overline{\mathbb{F}}_p[x]$ always splits. Note that $\operatorname{ind}_{P_I(\mathbb{F}_q)}^{\mathrm{GL}_n(\mathbb{F}_q)} \overline{\mathbb{F}}_p$ is a finite dimensional $\overline{\mathbb{F}}_p$-vector space, so $F_p - 1$ is the map $x \mapsto x^p - x$ on each coordinate. Hence $F_p - 1$ is surjective. Therefore $H^1_{\text{ét}}\big(\overline{X}(w), \mathbb{Z}/p\mathbb{Z}\big) = 0$.

By Fermat's little theorem, we have that $\ker(F_p - 1) = \mathbb{F}_p$ when $\operatorname{supp}(w) = S$. Since $H^0\big(\overline{X}(w), \mathcal{O}_{\overline{X}(w)}\big)$ is the trivial $\mathrm{GL}_n(\mathbb{F}_q)$-representation, its subring $\ker(F_p-1)$ automatically inherit a trivial $\mathrm{GL}_n(\mathbb{F}_q)$-module structure. Thus we have

$$H^0_{\text{ét}}\Big(\overline{X}(w), \mathbb{Z}/p\mathbb{Z}\Big) = \mathbb{1}_{\mathbb{Z}/p\mathbb{Z}},$$

when $\operatorname{supp}(w) = S$. When $w \in W$ is arbitrary, let $I = \operatorname{supp}(w)$. Recall from Section 5.3 and [DMR07, Proposition 2.3.8] that we have $\mathrm{GL}_n(\mathbb{F}_q)$-equivariant surjective morphism

$$\overline{X}(w) \longrightarrow \mathrm{GL}_n^F / P_I^F,$$

whose fibers are all isomorphic to $\overline{X}_{L_I}(w)$. In particular, $\mathrm{GL}_n(\mathbb{F}_q)$ acts on the set of fibres transitively and the stablizer of each fibre corresponds to a conjugate of $P_I(\mathbb{F}_q)$ in $\mathrm{GL}_n(\mathbb{F}_q)$. Then the long exact sequence associated to the Artin–Schreier sequence gives us:

$$\begin{aligned}0 \to \operatorname{ind}_{P_I(\mathbb{F}_q)}^{\mathrm{GL}_n(\mathbb{F}_q)} H^0_{\text{ét}}\big(\overline{X}_{L_I}(w), \mathbb{Z}/p\mathbb{Z}\big) \to \operatorname{ind}_{P_I(\mathbb{F}_q)}^{\mathrm{GL}_n(\mathbb{F}_q)} H^0\big(\overline{X}_{L_I}(w), \mathcal{O}_{\overline{X}(w)}\big) \xrightarrow{F_p-1} \\ \longrightarrow \operatorname{ind}_{P_I(\mathbb{F}_q)}^{\mathrm{GL}_n(\mathbb{F}_q)} H^0\big(\overline{X}_{L_I}(w), \mathcal{O}_{\overline{X}(w)}\big) \to 0.\end{aligned}$$

We see in the proof of Theorem 6.1 that

$$H^0\big(\overline{X}_{L_I}(w), \mathcal{O}_{\overline{X}(w)}\big) = \overline{\mathbb{F}}_p.$$

Then $H^0\big(\overline{X}_{L_I}(w), \mathcal{O}_{\overline{X}(w)}\big)$ is a trivial $P_I(\mathbb{F}_q)$-representation and so $H^0_{\text{ét}}\big(\overline{X}_{L_I}(w), \mathbb{Z}/p\mathbb{Z}\big)$ is a trivial $P_I(\mathbb{F}_q)$-representation. Hence we have $\ker(F_p - 1) = \operatorname{ind}_{P_I(\mathbb{F}_q)}^{\mathrm{GL}_n(\mathbb{F}_q)} \mathbb{F}_p$. Therefore

$$H^0_{\text{ét}}\Big(\overline{X}(w), \mathbb{Z}/p\mathbb{Z}\Big) = \operatorname{ind}_{P_I(\mathbb{F}_q)}^{\mathrm{GL}_n(\mathbb{F}_q)} \mathbb{1}_{\mathbb{F}_p}$$

□

Corollary 6.6. *Let $G = \mathrm{GL}_n$, and $w \in F^+$ with $I = \operatorname{supp}(w)$ and $P_I = B^* W_I B^*$. For any integer $m > 0$, we have*

$$H^k_{\text{ét}}\left(\overline{X}(w), \mathbb{Z}/p^m\mathbb{Z}\right) = \begin{cases} \operatorname{ind}_{P_I(\mathbb{F}_q)}^{\mathrm{GL}_n(\mathbb{F}_q)} \mathbb{1}_{\mathbb{Z}/p^m\mathbb{Z}}, & k = 0, \\ 0, & k > 0. \end{cases}$$

Proof. For every integer $m \geq 2$, we have the short exact sequence

$$0 \to \mathbb{Z}/p^{m-1}\mathbb{Z} \xrightarrow{\alpha} \mathbb{Z}/p^m\mathbb{Z} \xrightarrow{\beta} \mathbb{Z}/p\mathbb{Z} \to 0,$$

where

$$\alpha : a \bmod p^{m-1} \longmapsto pa \bmod p^m$$

$$\beta : b \bmod p^m \longmapsto b \bmod p.$$

There are also constant sheaves $\mathbb{Z}/p^m\mathbb{Z}$ on $\overline{X}(w)_{\text{ét}}$. Thus we get an induced long exact sequence for every integer $m \geq 2$,

$$0 \to H^0_{\text{ét}}\big(\overline{X}(w), \mathbb{Z}/p^{m-1}\mathbb{Z}\big) \to H^0_{\text{ét}}\big(\overline{X}(w), \mathbb{Z}/p^m\mathbb{Z}\big) \to H^0_{\text{ét}}\big(\overline{X}(w), \mathbb{Z}/p\mathbb{Z}\big) \to \cdots$$

$$\cdots \to H^k_{\text{ét}}\big(\overline{X}(w), \mathbb{Z}/p^{m-1}\mathbb{Z}\big) \to H^k_{\text{ét}}\big(\overline{X}(w), \mathbb{Z}/p^m\mathbb{Z}\big) \to H^k_{\text{ét}}\big(\overline{X}(w), \mathbb{Z}/p\mathbb{Z}\big) \to \cdots.$$

By Proposition 6.5, we know that $H^k_{\text{ét}}\big(\overline{X}(w), \mathbb{Z}/p\mathbb{Z}\big) = 0$ for all $k > 0$. By induction on m, assume that $H^k_{\text{ét}}\big(\overline{X}(w), \mathbb{Z}/p^{m-1}\mathbb{Z}\big) = 0$ for all $k > 0$, so the long exact sequence gives that $H^k_{\text{ét}}\big(\overline{X}(w), \mathbb{Z}/p^m\mathbb{Z}\big) = 0$ for all $k > 0$. Therefore for any integer $m > 0$, we have

$$H^k_{\text{ét}}\big(\overline{X}(w), \mathbb{Z}/p^m\mathbb{Z}\big) = 0$$

for all $k > 0$.

For any commutative ring A of characteristic p, denote $W_j(A)$ to be the ring of Witt vectors of length j with coefficients in A. Recall that $W_j(A)$ is set-theoretically in bijection with the product A^j, but the bijection is not an isomorphism of rings when $j > 1$. However, the p-Frobenius F_p on $W_j(A)$ is compatible with the p-Frobenius on A, in the sense that F_p on $W_j(A)$ is the map $x \mapsto x^p$ on each coordinate. See [Ill79, §0.1] for an introduction on the ring of Witt vectors.

Let $W_m\left(\mathcal{O}_{\overline{X}(w)}\right)$ be the sheaf of Witt vectors of length m on $\overline{X}(w)$ [Ill79, §0.1.5]. The stalk of $W_m\left(\mathcal{O}_{\overline{X}(w)}\right)$ at a point $x \in X$ is $W_m\left(\mathcal{O}_{\overline{X}(w),x}\right)$ [Ill79, (01.5.6)]. Note that $W_m\left(\mathcal{O}_{\overline{X}(w)}\right)$ are coherent sheaves on $\overline{X}(w)$ [Ser58, §2]. Similar to the ring of Witt vectors, the sections of the coherent sheaf $W_m\left(\mathcal{O}_{\overline{X}(w)}\right)$ are set-theoretically in bijection with the corresponding sections of $\mathcal{O}^m_{\overline{X}(w)}$, but the bijection is not an isomorphism of rings when $m > 1$. Again, the p-Frobenius F_p on $W_m\left(\mathcal{O}_{\overline{X}(w)}\right)$ is compatible with the p-Frobenius on $\mathcal{O}_{\overline{X}(w)}$, in the sense that it is $x \mapsto x^p$ on each coordinate.

On $\overline{X}(w)_{\text{ét}}$, we have the *Artin–Schreier–Witt exact sequence*, cf. [Ill79, Proposition 3.28],

$$0 \longrightarrow \mathbb{Z}/p^m\mathbb{Z} \to W_m\left(\mathcal{O}_{\overline{X}(w)}\right) \xrightarrow{F_p - 1} W_m\left(\mathcal{O}_{\overline{X}(w)}\right) \to 0.$$

One attains the long exact sequence

$$0 \longrightarrow H^0\left(\overline{X}(w), \mathbb{Z}/p^m\mathbb{Z}\right) \to H^0\left(\overline{X}(w), W_m\left(\mathcal{O}_{\overline{X}(w)}\right)\right) \xrightarrow{F_p - 1} H^0\left(\overline{X}(w), W_m\left(\mathcal{O}_{\overline{X}(w)}\right)\right) \to \cdots.$$

We see that $H^0\left(\overline{X}(w), \mathbb{Z}/p^j\mathbb{Z}\right) = \ker(F_p - 1)$.

First, let $\operatorname{supp}(w) = S$. Since $\overline{X}(w)$ is smooth projective over $\overline{\mathbb{F}}_p$ and

$$H^0\left(\overline{X}(w), \mathcal{O}_{\overline{X}(w)}\right) = \overline{\mathbb{F}}_p,$$

we know that $H^0\left(\overline{X}(w), W_m\left(\mathcal{O}_{\overline{X}(w)}\right)\right) = W_m\left(\overline{\mathbb{F}}_p\right)$. Note that the p-Frobenius F_p on $W_m\left(\mathcal{O}_{\overline{X}(w)}\right)$ is compatible with taking sections, so it is compatible with the Frobenius on $W_m\left(\overline{\mathbb{F}}_p\right)$. Thus we have

$$\ker(F_p - 1) = W_m(\mathbb{F}_p)$$

for all $j > 1$. We know that $W_m(\mathbb{F}_p) = \mathbb{Z}/p^m\mathbb{Z}$ for all $m > 1$. Again, since $\mathrm{GL}_n(\mathbb{F}_q)$ acts trivially on $H^0\left(\overline{X}(w), \mathcal{O}_{\overline{X}(w)}\right) = \overline{\mathbb{F}}_p$, $W_m\left(\overline{\mathbb{F}}_p\right)$ inherits a trivial $\mathrm{GL}_n(\mathbb{F}_q)$-action because $\mathrm{GL}_n(\mathbb{F}_q)$-acts trivially on each of the coordinate of $W_m\left(\overline{\mathbb{F}}_p\right)$.

Now let $w \in W$ be an arbitrary element and set $I = \mathrm{supp}(w)$. Via Section 5.3 and [DMR07, Proposition 2.3.8] we have a $\mathrm{GL}_n(\mathbb{F}_q)$-equivariant surjective morphism

$$\overline{X}(w) \longrightarrow \mathrm{GL}_n^F/P_I^F$$

such that the fibres are $\overline{X}_{L_I}(w)$. In particular, $\mathrm{GL}_n(\mathbb{F}_q)$ acts on the set of fibres transitively and the stablizer of each fibre corresponds to a conjugate of $P_I(\mathbb{F}_q)$ in $\mathrm{GL}_n(\mathbb{F}_q)$.

By definition one has

$$W_m\left(\mathcal{O}_{\overline{X}(w)}\right)\left(\overline{X}(w)\right) = W_m\left(\mathcal{O}_{\overline{X}(w)}\left(\overline{X}(w)\right)\right),$$

and since the functor W_m is a finite limit, we have

$$W_m\left(\mathcal{O}_{\overline{X}(w)}\left(\overline{X}(w)\right)\right) = \mathrm{ind}_{P_I(\mathbb{F}_q)}^{\mathrm{GL}_n(\mathbb{F}_q)} W_m\left(\mathcal{O}_{\overline{X}_{L_I}(w)}\left(\overline{X}_{L_I}(w)\right)\right).$$

It follows from the proof of Theorem 6.1 that $\mathcal{O}_{\overline{X}_{L_I}(w)}\left(\overline{X}_{L_I}(w)\right) = \overline{\mathbb{F}}_p$, so $\mathcal{O}_{\overline{X}_{L_I}(w)}\left(\overline{X}_{L_I}(w)\right)$ is the trivial $P_I(\mathbb{F}_q)$-representation. As before, this makes $W_m\left(\mathcal{O}_{\overline{X}_{L_I}(w)}\left(\overline{X}_{L_I}(w)\right)\right)$ the 1-dimensional trivial $P_I(\mathbb{F}_q)$-module. Thus we have

$$\ker(F_p - 1) = \mathrm{ind}_{P_I(\mathbb{F}_q)}^{\mathrm{GL}_n(\mathbb{F}_q)} W_m(\mathbb{F}_p) = \mathrm{ind}_{P_I(\mathbb{F}_q)}^{\mathrm{GL}_n(\mathbb{F}_q)} \mathbb{1}_{\mathbb{Z}/p^m\mathbb{Z}}$$

where $\mathbb{1}_{\mathbb{Z}/p^m\mathbb{Z}}$ is the trivial $P_I(\mathbb{F}_q)$-representation with coefficients in $\mathbb{Z}/p^m\mathbb{Z}$. This finishes the proof. □

Corollary 6.7. *Let $G = \mathrm{GL}_n$, and $w \in F^+$ with $I = \mathrm{supp}(w)$. Let $P_I = B^*W_IB^*$. Then one has*

$$H^k_{\text{ét}}\left(\overline{X}(w), \mathbb{Z}_p\right) = \begin{cases} \mathrm{ind}_{P_I(\mathbb{F}_q)}^{\mathrm{GL}_n(\mathbb{F}_q)} \mathbb{1}_{\mathbb{Z}_p}, & k = 0, \\ 0, & k > 0. \end{cases}$$

Proof. By Corollary 6.6, since for all $k > 0$ and $m > 0$, we have $H^k_{\text{ét}}\big(\overline{X}(w), \mathbb{Z}/p^m\mathbb{Z}\big) = 0$, the tower $\left\{H^k_{\text{ét}}\left(\overline{X}(w), \mathbb{Z}/p^m\mathbb{Z}\right)\right\}_m$ of abelian groups satisfy the Mittag–Leffler condition trivially. Thus for all $k > 0$, we have

$$H^k_{\text{ét}}\big(\overline{X}(w), \mathbb{Z}_p\big) = \varprojlim_m H^k_{\text{ét}}\big(\overline{X}(w), \mathbb{Z}/p^m\mathbb{Z}\big) = 0.$$

On the other hand, the higher vanishing implies that whenever we have $m > l$ and a mod p^l map

$$\begin{aligned} \mathbb{Z}/p^m\mathbb{Z} &\longrightarrow \mathbb{Z}/p^l\mathbb{Z} \\ b \bmod p^m &\longmapsto b \bmod p^l, \end{aligned}$$

Take the induced short exact sequence of sheaves on $\overline{X}(w)$:

$$0 \to \mathbb{Z}/p^{m-l}\mathbb{Z} \to \mathbb{Z}/p^m\mathbb{Z} \to \mathbb{Z}/p^l\mathbb{Z} \to 0.$$

Take the associated long exact sequence of cohomology groups

$$\cdots \to H^0_{\text{ét}}\big(\overline{X}(w), \mathbb{Z}/p^m\mathbb{Z}\big) \to H^0_{\text{ét}}\big(\overline{X}(w), \mathbb{Z}/p^l\mathbb{Z}\big) \to H^1_{\text{ét}}\big(\overline{X}(w), \mathbb{Z}/p^{m-l}\mathbb{Z}\big) \to \cdots.$$

By the higher vanishing from Corollary 6.6, we see that the morphism

$$H^0_{\text{ét}}\big(\overline{X}(w), \mathbb{Z}/p^m\mathbb{Z}\big) \longrightarrow H^0_{\text{ét}}\big(\overline{X}(w), \mathbb{Z}/p^l\mathbb{Z}\big)$$

is surjective. Therefore the tower $\{H^0_{\text{ét}}\left(\overline{X}(w), \mathbb{Z}/p^m\mathbb{Z}\right)\}_m$ of abelian groups satisfies the Mittag–Leffler condition. Thus we have

$$H^0_{\text{ét}}\left(\overline{X}(w), \mathbb{Z}_p\right) = \varprojlim_m H^0_{\text{ét}}\left(\overline{X}(w), \mathbb{Z}/p^m\mathbb{Z}\right),$$

and the identification

$$H^0_{\text{ét}}\left(\overline{X}(w), \mathbb{Z}_p\right) = \mathrm{ind}_{P_I(\mathbb{F}_q)}^{\mathrm{GL}_n(\mathbb{F}_q)} \mathbb{Z}_p.$$

$\square$

6.3. **Cohomology of $\Omega^{\ell(w)}$ on $\overline{X}(w)$.** Let $G = \mathrm{GL}_n$ and $w \in W$. Recall that $\overline{X}(w)$ is smooth projective of dimension $\ell(w)$. Let Ω be the sheaf of differentials on $\overline{X}(w)$ and write $\Omega^p = \wedge^p\Omega$ for the sheaf of differential p-forms. In particular, $\omega := \wedge^{\ell(w)}\Omega = \Omega^{\ell(w)}$ is the dualizing sheaf on $\overline{X}(w)$.

Proposition 6.8. *Let $G = \mathrm{GL}_n, w \in F^+$, $I = \mathrm{supp}(w)$ and $P_I = B^*W_IB^*$, then*

$$H^k\left(\overline{X}(w), \Omega^{\ell(w)}\right) = \begin{cases} \mathrm{ind}_{P_I(\mathbb{F}_q)}^{\mathrm{GL}_d(\mathbb{F}_q)} \mathbb{1}_{\overline{\mathbb{F}}_p}, & k = \ell(w) \\ 0, & k \neq \ell(w). \end{cases}$$

Proof. Since $\overline{X}(w)$ is a smooth projective $\overline{\mathbb{F}}_p$-scheme of dimension $\ell(w)$, Serre duality implies that there is an isomorphism of $\mathbb{F}_p$-schemes

$$H^q\big(\overline{X}(w), \Omega^p\big) \xrightarrow{\sim} H^{\ell(w)-q}\big(\overline{X}(w), \Omega^{\ell(w)-p}\big)^\vee$$

for all $p, q \geq 0$. Fix $p = 0$ and we may apply Theorem 6.1 to get

$$H^k\big(\overline{X}(w), \Omega^{\ell(w)}\big) = 0,$$

when $k \neq \ell(w)$. Setting $p = 0$ and $q = 0$, we get by Theorem 6.1 that

$$H^0\big(\overline{X}(w), \mathcal{O}_{\overline{X}(w)}\big) \xrightarrow{\sim} H^{\ell(w)}\big(\overline{X}(w), \Omega^{\ell(w)}\big)^\vee.$$

This isomorphism is equivariant under $\mathrm{GL}_n(\mathbb{F}_q)$-action by [Has09, Theorem 29.5]. Finally, recall that since $\mathrm{GL}_n(\mathbb{F}_q)$ is a finite group, the induction functor commutes with taking the dual of a representation. $\square$

7. The compactly supported mod p^m and $\mathbb{Z}_p$ étale cohomology of $X(w)$

Throughout this section, let $G = \mathrm{GL}_n$ and $w \in W$. Fix a reduced expression $w = t_1 \cdots t_r$, $t_j \in S$. This reduced expression determines a smooth compactification $\overline{X}(w)$ for $X(w)$. We also have an isomorphism $X(t_1, \ldots, t_r) \xrightarrow{\sim} X(w)$ cf. Remark 2.15. By [DMR07, Proposition 3.2.2], we have the following disjoint union:

$$\overline{X}(w) := \overline{X}(t_1, \ldots, t_r) = X(t_1, \ldots, t_r) \dot{\bigcup} \left(\bigcup_{\substack{v \prec w \\ \ell(v) = \ell(w)-1}} \overline{X}(v) \right),$$

where $\prec$ is the Bruhat order on F^+. Let us denote $Y := \overline{X}(w) \big\backslash X(t_1, \ldots, t_r)$ in this section. We want to make use of this stratification to compute the $\mathbb{Z}/p^m\mathbb{Z}$-cohomology of $X(w)$.

Denote $j : X(t_1, \ldots, t_r) \hookrightarrow \overline{X}(w)$ the obvious open immersion and $i : Y \hookrightarrow \overline{X}(w)$ the closed immersion. For $m \geq 1$, let $\mathbb{Z}/p^m\mathbb{Z}$ be the constant sheaf on $\overline{X}(w)_{\text{ét}}$ defined by the ring $\mathbb{Z}/p^m\mathbb{Z}$.

Recall that by the definition of the cohomology with compact support, we have

$$H^k_{\text{ét},c}\left(X(t_1, \ldots, t_r), \mathbb{Z}/p^m\mathbb{Z}\right) \xrightarrow{\sim} H^k\left(\overline{X}(w), j_!\mathbb{Z}/p^m\mathbb{Z}\right)$$

for all k. Furthermore, since $X(w)$ is stable under the $\mathrm{GL}_n(\mathbb{F}_q)$-action, the identification of the cohomology groups above is equivariant under the $\mathrm{GL}_n(\mathbb{F}_q)$-action. Since i_* is exact, for all $k > 0$, we have $R^k i_*\mathbb{Z}/p^m\mathbb{Z} = 0$. As a consequence, the Leray spectral sequence

$$E_1^{r,s} = H^r\left(\overline{X}(w)_{\text{ét}}, R^s i_*\mathbb{Z}/p^m\mathbb{Z}\right) \implies H^{r+s}\left(Y_{\text{ét}}, \mathbb{Z}/p^m\mathbb{Z}\right)$$

collapses and we have $H^r_{\text{ét}}\left(Y, \mathbb{Z}/p^m\mathbb{Z}\right) = H^r_{\text{ét}}\left(\overline{X}(w), i_*\mathbb{Z}/p^m\mathbb{Z}\right)$ for all r. Note that this identification is also equivariant under the $\mathrm{GL}_n(\mathbb{F}_q)$-action.

Note that these sheaves fit into a short exact sequence on $\overline{X}(w)_{\text{ét}}$:

$$0 \to j_!\mathbb{Z}/p^m\mathbb{Z} \longrightarrow \mathbb{Z}/p^m\mathbb{Z} \longrightarrow i_*\mathbb{Z}/p^m\mathbb{Z} \to 0$$

which has the following associated long exact sequence

$$\begin{aligned}0 \to H^0_{\text{ét,c}}\left(X(w), \mathbb{Z}/p^m\mathbb{Z}\right) &\longrightarrow H^0_{\text{ét}}\left(\overline{X}(w), \mathbb{Z}/p^m\mathbb{Z}\right) \longrightarrow H^0_{\text{ét}}\left(Y, \mathbb{Z}/p^m\mathbb{Z}\right) \to \cdots \\ \cdots \to H^k_{\text{ét,c}}\left(X(w), \mathbb{Z}/p^m\mathbb{Z}\right) &\longrightarrow H^k_{\text{ét}}\left(\overline{X}(w), \mathbb{Z}/p^m\mathbb{Z}\right) \longrightarrow H^k_{\text{ét}}\left(Y, \mathbb{Z}/p^m\mathbb{Z}\right) \to \cdots .\end{aligned}$$

We already know the $\mathbb{Z}/p^m\mathbb{Z}$-cohomology for $\overline{X}(w)$, so this gives us some information about the cohomology groups $H^k_{\text{ét},c}\left(X(w), \mathbb{Z}/p^m\mathbb{Z}\right)$. In particular, we already have for $k > 0$,

$$H^0_{\text{ét},c}\left(X(w), \mathbb{Z}/p^m\mathbb{Z}\right) = 0 \quad\text{and}\quad H^k_{\text{ét},c}\left(X(w), \mathbb{Z}/p^m\mathbb{Z}\right) \xrightarrow{\sim} H^{k-1}_{\text{ét}}\left(Y, \mathbb{Z}/p^m\mathbb{Z}\right).$$

Furthermore, in order to compute exactly $H^k_{\text{ét},c}\left(X(w), \mathbb{Z}/p^m\mathbb{Z}\right)$, we may construct an exact sequence similar to Mayer–Vietoris spectral sequence with respect to the stratification of Y. The method has been used to compute the compactly supported ℓ-adic cohomology groups of $X(w)$ in [Orl18, §5, §7]. We shall adapt this for our case.

7.1. **An acyclic resolution for the Steinberg module for a Levi subgroup of** GL_n**.** Let $w \in W$ such that $w = t_1 \cdots t_r$ with $t_j \in S$ are all distinct from one another. We have the associated parabolic subgroup $P_I = B^* W_I B^*$, where $I = \mathrm{supp}(w)$. Set $I_u = \mathrm{supp}(u)$ for $u \preceq w$. Consider the following sequence:

$$\begin{aligned}&\mathrm{ind}_{P_I(\mathbb{F}_q)}^{\mathrm{GL}_n(\mathbb{F}_q)} \mathbb{1}_{\mathbb{Z}/p^m\mathbb{Z}} \xrightarrow{d_0} \bigoplus_{\substack{u \prec w \\ \ell(u)=\ell(w)-1}} \mathrm{ind}_{P_{I_u}(\mathbb{F}_q)}^{\mathrm{GL}_n(\mathbb{F}_q)} \mathbb{1}_{\mathbb{Z}/p^m\mathbb{Z}} \to \cdots \\ &\bigoplus_{\substack{u \prec w \\ \ell(u)=\ell(w)-i+1}} \mathrm{ind}_{P_{I_u}(\mathbb{F}_q)}^{\mathrm{GL}_n(\mathbb{F}_q)} \mathbb{1}_{\mathbb{Z}/p^m\mathbb{Z}} \xrightarrow{d_{i-1}} \bigoplus_{\substack{u \prec w \\ \ell(u)=\ell(w)-i}} \mathrm{ind}_{P_{I_u}(\mathbb{F}_q)}^{\mathrm{GL}_n(\mathbb{F}_q)} \mathbb{1}_{\mathbb{Z}/p^m\mathbb{Z}} \xrightarrow{d_i} \bigoplus_{\substack{u \prec w \\ \ell(u)=\ell(w)-i-1}} \mathrm{ind}_{P_{I_u}(\mathbb{F}_q)}^{\mathrm{GL}_n(\mathbb{F}_q)} \mathbb{1}_{\mathbb{Z}/p^m\mathbb{Z}} \\ &\qquad\cdots \to \bigoplus_{\substack{u \prec w \\ \ell(u)=1}} \mathrm{ind}_{P_{I_u}(\mathbb{F}_q)}^{\mathrm{GL}_n(\mathbb{F}_q)} \mathbb{1}_{\mathbb{Z}/p^m\mathbb{Z}} \xrightarrow{d_{\ell(w)-1}} \mathrm{ind}_{B(\mathbb{F}_q)}^{\mathrm{GL}_n(\mathbb{F}_q)} \mathbb{1}_{\mathbb{Z}/p^m\mathbb{Z}}.\end{aligned} \tag{7.1}$$

For all $u_{i+1} \preceq u_i \preceq w$ with $\ell(u_{i+1}) = \ell(u_i) - 1$, let

$$\iota_{u_i}^{u_{i+1}} : \mathrm{ind}_{P_{u_i}(\mathbb{F}_q)}^{\mathrm{GL}_n(\mathbb{F}_q)} \mathbb{1}_{\mathbb{Z}/p^m\mathbb{Z}} \to \mathrm{ind}_{P_{u_{i+1}}(\mathbb{F}_q)}^{\mathrm{GL}_n(\mathbb{F}_q)} \mathbb{1}_{\mathbb{Z}/p^m\mathbb{Z}}$$

be the inclusion map, where $P_{u_i} := P_{\mathrm{supp}(u_i)}$. Then the map d_i is defined by

$$(f_{u_i})_{u_i} \mapsto \left(\sum_{u_i} (-1)^{\alpha(u_i \to u_{i+1})} \iota_{u_i}^{u_{i+1}}(f_{u_i})\right)_{u_{i+1}},$$

where $\alpha(u_i \to u_{i+1})$ is a map defined as follows: if u_{i+1} is obtained from u_i by deleting the r-th $s \in S$ in its product expression, then $\alpha(u_i \to u_{i+1}) = r$.

We will see in the following Proposition that this sequence is an acyclic complex. In particular, if $w \in W$ is a Coxeter element, then (up to augmentation) the complex (7.1) gives a resolution for the Steinberg module:

$$\mathrm{St}_{\mathrm{GL}_n} := \mathrm{ind}_{B(\mathbb{F}_q)}^{\mathrm{GL}_n(\mathbb{F}_q)} \mathbb{1}_{\mathbb{Z}/p^m\mathbb{Z}} \Big/ \sum_{P \supsetneq B} \mathrm{ind}_{P(\mathbb{F}_q)}^{\mathrm{GL}_n(\mathbb{F}_q)} \mathbb{1}_{\mathbb{Z}/p^m\mathbb{Z}}.$$

Proposition 7.1. *Let $w = t_1 \cdots t_r$ such that t_j are all distinct. Let $P_I = B^* W_I B^*$, where $I = \mathrm{supp}(w)$. Set $I_u = \mathrm{supp}(u)$ for $u \preceq w$. Then the sequence (7.1) is an acyclic complex. Furthermore, d_0 is injective and the cokernel of $d_{\ell(w)-1}$ is*

$$\mathrm{ind}_{B(\mathbb{F}_q)}^{\mathrm{GL}_n(\mathbb{F}_q)} \mathbb{1}_{\mathbb{Z}/p^m\mathbb{Z}} \Big/ \sum_{\substack{u \prec w \\ \ell(u)=1}} \mathrm{ind}_{P_{I_u}(\mathbb{F}_q)}^{\mathrm{GL}_n(\mathbb{F}_q)} \mathbb{1}_{\mathbb{Z}/p^m\mathbb{Z}}.$$

Proof. We denote $\mathrm{ind}_{H(\mathbb{F}_q)}^{G(\mathbb{F}_q)} \mathbb{1}_{\mathbb{Z}/p^m\mathbb{Z}}$ by i_H^G for any subgroup H of a group G when there is no ambiguity. Recall that we have $P_I = L_I \ltimes U_I$, where L_I is a Levi subgroup of GL_n. In particular, L_I is a reductive algebraic group over $\overline{\mathbb{F}}_p$ defined over $\mathbb{F}_q$. Note that the Weyl group of L_I is exactly W_I. Since L_I is reductive, by [Sol69, Theorem 1] cf. [CLT80, §7], the following sequence

$$\bigoplus_{\substack{u\prec w\\ \ell(u)=\ell(w)-1}} i^{L_I}_{P_{I_u}\cap L_I} \to \cdots \to \bigoplus_{\substack{u\prec w\\ \ell(u)=1}} i^{L_I}_{P_{I_u}\cap L_I} \xrightarrow{d_{\ell(w)-1}} i^{L_I}_{B\cap L_I} \tag{7.2}$$

identifies with the combinatorial Tits complex Δ of L_I tensored with $\mathbb{Z}/p^m\mathbb{Z}$. In particular,

$$H_0(\Delta,\mathbb{Z})=\mathbb{Z} \qquad\text{and}\qquad H_{\ell(w)}(\Delta,\mathbb{Z})=\mathbb{Z}^{|U_{L_I}(\mathbb{F}_q)|},$$

where U_{L_I} is a maximal $\mathbb{F}_q$-split unipotent subgroup of L_I, and $H_j(\Delta,\mathbb{Z})=0$ otherwise. By the Universal Coefficients Theorem, we know that the complex (7.2) extends to the following acyclic complex:

$$0\to i^{L_I}_{L_I}\to \bigoplus_{\substack{u\prec w\\ \ell(u)=\ell(w)-1}} i^{L_I}_{P_{I_u}\cap L_I} \to \cdots \to \bigoplus_{\substack{u\prec w\\ \ell(u)=1}} i^{L_I}_{P_{I_u}\cap L_I} \xrightarrow{d_{\ell(w)-1}} i^{L_I}_{B\cap L_I} \to \mathrm{St}_{L_I}\to 0, \tag{7.3}$$

where

$$\mathrm{St}_{L_I} := \mathrm{ind}^{L_I(\mathbb{F}_q)}_{(B\cap L_I)(\mathbb{F}_q)} \mathbb{1}_{\mathbb{Z}/p^m\mathbb{Z}} \Big/ \sum_{(P\cap L_I)\supsetneq (B\cap L_I)} \mathrm{ind}^{L_I(\mathbb{F}_q)}_{(P\cap L_I)(\mathbb{F}_q)} \mathbb{1}_{\mathbb{Z}/p^m\mathbb{Z}}.$$

For any parabolic subgroup $P\subseteq \mathrm{GL}_n$ with $P=U_P\rtimes L_P$, where L_P is the Levi subgroup corresponding to P, we have the identification

$$\mathrm{ind}^{P(\mathbb{F}_q)}_{H(\mathbb{F}_q)} \mathbb{1}_{\mathbb{Z}/p^m\mathbb{Z}} = \mathrm{ind}^{L_P(\mathbb{F}_q)}_{(H\cap L_P)(\mathbb{F}_q)} \mathbb{1}_{\mathbb{Z}/p^m\mathbb{Z}}$$

for all subgroups $H\subseteq P$. Hence may rewrite the acyclic complex (7.3) as follows:

$$0\to i^{P_I}_{P_I} \longrightarrow \bigoplus_{\substack{u\prec w\\ \ell(u)=\ell(w)-1}} i^{P_I}_{P_{I_u}} \to\cdots\to i^{P_I}_{B} \to \mathrm{ind}^{P_I(\mathbb{F}_q)}_{B(\mathbb{F}_q)} \mathbb{1}_{\mathbb{Z}/p^m\mathbb{Z}} \Big/ \sum_{P\supsetneq B} \mathrm{ind}^{P_I(\mathbb{F}_q)}_{P(\mathbb{F}_q)} \mathbb{1}_{\mathbb{Z}/p^m\mathbb{Z}} \to 0. \tag{7.4}$$

Recall that since $P_I(\mathbb{F}_q)$ is a finite subgroup of $\mathrm{GL}_n(\mathbb{F}_q)$, the functor $\mathrm{ind}^{\mathrm{GL}_n(\mathbb{F}_q)}_{P_I(\mathbb{F}_q)}$ is exact. Thus we may apply the functor $\mathrm{ind}^{\mathrm{GL}_n(\mathbb{F}_q)}_{P_I(\mathbb{F}_q)}$ to the acyclic complex (7.4) and obtain the complex (7.1). Therefore the complex (7.1) is acyclic and $d_{\ell(w)-1}$ has the cokernel as desired. $\square$

We can also prove Proposition 7.1 algebraically.

Alternate proof. Denote representation with coefficients in $\mathbb{Q}$ by $i^G_H(\mathbb{Q})$. Set $A_k=i^{L_I}_{P_{u_k}\cap L_I}(\mathbb{Q})$ with $k=1,...,\ell(w)$ such that $u_k\prec w, \ell(u_k)=\ell(w)-1$ and $P_{u_k}=P_{\mathrm{supp}(u_k)}$. It is verified in [DOR10, Theorem 3.2.5] that for any subsets $I,J\subseteq\{1,...,\ell(w)\}$,

$$\left(\sum_{i\in I} A_i\right)\cap\left(\bigcap_{j\in J} A_j\right) = \sum_{i\in I}\left(A_i\cap\left(\bigcap_{j\in J}A_j\right)\right). \tag{7.5}$$

Note that loc.cit. was proved for generalized Steinberg representations, but it does apply for the scenario of the Steinberg representation itself. By [SS91, Proposition 2.6], one obtains an acyclic complex:

$$0\to i^{L_I}_{L_I}(\mathbb{Q})\to \bigoplus_{\substack{u\prec w\\ \ell(u)=\ell(w)-1}} i^{L_I}_{P_{I_u}\cap L_I}(\mathbb{Q}) \to\cdots\to i^{L_I}_{B\cap L_I}(\mathbb{Q})\to \mathrm{St}_{L_I}(\mathbb{Q})\to 0, \tag{7.6}$$

where $\mathrm{St}_{L_I}(\mathbb{Q})$ is the Steinberg representation with coefficients in $\mathbb{Q}$. The complex (7.6) gives a basis for $i^{L_I}_{B\cap L_I}(\mathbb{Q})$ such that for all $P\supset B$, the subrepresentation $i^{L_I}_{P\cap L_I}(\mathbb{Q})$ is generated by a subset of this basis. More precisely, one start with fixing a basis for $i^{L_I}_{L_I}(\mathbb{Q})$ and inductively fix bases for each constituent of the next term in the complex. This ensures that the intersections would still be free modules generated by a basis element.

Taking the $\mathbb{Z}$-lattice with respect to this basis yields the sub-representations $i_{P\cap L_I}^{L_I}(\mathbb{Z})$. In particular, the equality (7.5) holds after intersecting with the $\mathbb{Z}$-lattice and the intersections are free $\mathbb{Z}$-modules generated by basis elements. Thus we can tensor this $\mathbb{Z}$-lattice with $\mathbb{Z}/p^m\mathbb{Z}$ to obtain $B_k := i_{P_{u_k}\cap L_I}^{L_I}(\mathbb{Z}/p^m\mathbb{Z})$, such that B_k's satisfy the condition (7.5). Thus by [SS91, Proposition 2.6], we obtain the acyclic complex as desired.

□

7.2. **A spectral sequence associated to the stratification.** Denote the category of sheaves on $\overline{X}(w)_{\text{ét}}$ by $\mathrm{Sh}(\overline{X}(w)_{\text{ét}})$. For any closed subscheme Z of $\overline{X}(w)$ with the inclusion map $\iota : Z \to \overline{X}(w)$, we denote $\iota_*\mathbb{Z}/p^m\mathbb{Z}$ by $(\mathbb{Z}/p^m\mathbb{Z})_Z$. Again, since ι_* is exact, we have an isomorphism of $\mathbb{Z}/p^m\mathbb{Z}$-modules:

$$H^r_{\text{ét}}\left(Z, \mathbb{Z}/p^m\mathbb{Z}\right) \xrightarrow{\sim} H^r_{\text{ét}}\left(\overline{X}(w), (\mathbb{Z}/p^m\mathbb{Z})_Z\right).$$

In addition, if we assume that Z is stable under $\mathrm{GL}_n(\mathbb{F}_q)$-action, then the above isomorphism is $\mathrm{GL}_n(\mathbb{F}_q)$-equivariant.

We have a sequence of constant sheaves on $\overline{X}(w)_{\text{ét}}$, where $R = \mathbb{Z}/p^m\mathbb{Z}$, for any $m > 0$. For $u \preceq w$, we denote the constant sheaf R on $\overline{X}(u)$ by $R_{\overline{X}(u)}$.

$$\begin{aligned} R_{\overline{X}(w)} \to \bigoplus_{\substack{u\prec w\\ \ell(u)=\ell(w)-1}} R_{\overline{X}(u)} \to \bigoplus_{\substack{u\prec w\\ \ell(u)=\ell(w)-2}} R_{\overline{X}(u)} \to \cdots \\ \cdots \to \bigoplus_{\substack{u\prec w\\ \ell(u)=\ell(w)-i}} R_{\overline{X}(u)} \to \cdots \to \bigoplus_{\substack{u\prec w\\ \ell(u)=1}} R_{\overline{X}(u)} \to R_{X(e)} \end{aligned} \tag{7.7}$$

Let $\{U_\alpha \to \overline{X}(w)\}_\alpha$ be an étale cover of $\overline{X}(w)$, and consider the $i-1, i, i+1$-th terms of this complex

$$\bigoplus_{\substack{u\prec w\\ \ell(u)=\ell(w)-i+1}} R_{\overline{X}(u)}(U_\alpha) \xrightarrow{d_{i-1}} \bigoplus_{\substack{u\prec w\\ \ell(u)=\ell(w)-i}} R_{\overline{X}(u)}(U_\alpha) \xrightarrow{d_i} \bigoplus_{\substack{u\prec w\\ \ell(u)=\ell(w)-i-1}} R_{\overline{X}(u)}(U_\alpha).$$

We now describe the maps d_i. Label each summand of the i-th term in the complex by u_i, by abuse of notation. Let

$$(f_{u_i})_{u_i} \in \bigoplus_{\substack{u\prec w\\ \ell(u)=\ell(w)-i}} R_{\overline{X}(u)}(U_\alpha)$$

be a section, then we have

$$d_i\left((f_{u_i})_{u_i}\right) = \left(\sum_{u_i} (-1)^{\alpha(u_i\to u_{i+1})} f_{u_i}|_{\overline{X}(u_{i+1})}\right)_{u_{i+1}}.$$

Here we define α as follows: when $\overline{X}(u_i)$ contains $\overline{X}(u_{i+1})$, if u_{i+1} is obtained from u_i by deleting the r-th term in the product expression of u_i, then $\alpha(u_i \to u_{i+1}) = r$, otherwise α takes value in 0. Here the restriction of f_{u_i} to $\overline{X}(u_{i+1})$ can be nonzero if and only if $\overline{X}(u_{i+1}) \subseteq \overline{X}(u_i)$.

When we fix $u_{i+1} \preceq u_{i-1} \preceq w$, as $\ell(u_{i-1}) = \ell(u_{i+1}) - 2$, there are only two ways to take restrictions from $\overline{X}(u_{i-1})$ to $\overline{X}(u_{i+1})$ via $\overline{X}(u_i')$ for some $u_{i-1} \preceq u_i' \preceq u_{i+1}$. Thus by the definition of the function α, we may conclude that $d_i \circ d_{i-1} = 0$. Therefore (7.7) is a complex.

Lemma 7.2. *Let $w = t_1 \cdots t_r$ such that t_j are all distinct. Then the complex (7.7) of sheaves on $\overline{X}(w)_{\text{ét}}$ is acyclic.*

Proof. It suffices to check the acyclicity of the complex (7.7) on the stalks. Let $x \in \overline{X}(w)$, then the complex would simplify depending on which closed subvariety x lives in.

$$\bigoplus_{\substack{u \prec w \\ \ell(u)=\ell(w)-i+1}} R_{\overline{X}(u),x} \xrightarrow{d_{i-1}} \bigoplus_{\substack{u \prec w \\ \ell(u)=\ell(w)-i}} R_{\overline{X}(u),x} \xrightarrow{d_i} \bigoplus_{\substack{u \prec w \\ \ell(u)=\ell(w)-i-1}} R_{\overline{X}(u),x} \tag{7.8}$$

If we have $x \in X(u_{i-1})$ for some $u_{i-1} \prec w$, $\ell(u_{i-1}) = \ell(w) - i + 1$. Then we know that $x \notin \overline{X}(u)$ for all $u \preceq w$ with $\ell(u) \leq i$ and thus $R_{\overline{X}(u),x} = 0$. Then the complex 7.8 is trivially exact at the i-th term.

If we have $x \in X(u_i)$ for some $u_i \prec w$, $\ell(u_i) = \ell(w) - i$. Then for all $R_{\overline{X}(u),x} = 0$ for $u \prec w$, $\ell(u) \leq i - 1$. The complex 7.8 becomes

$$\bigoplus_{\substack{u_i \prec u \prec w \\ \ell(u)=\ell(w)-i+1}} R_{\overline{X}(u),x} \xrightarrow{d_{i-1}} R_{\overline{X}(u_i),x} \longrightarrow 0.$$

Since d_{i-1} is obviously surjective, this complex is exact at the i-th term in this case.

If $x \in \overline{X}(u_{i+1})$, then for any $f \in \mathrm{Ker}(d_i)$, all the summand of $d_i(f)$ are 0. Now if $f = (f_t)_t$, then for each summand of $d_i(f)$, there exists an even number of nonzero f_t's that maps to it. Each such pair of f_t, f'_t would have the property $f_t = -f'_t$ or $f_t = f'_t$. This is because all summands of $d_i(f)$ are 0. Now by the definition of d_{i-1}, we may build an element of the $i-1$-th term of below using the nonzero terms of f_t.

$$\bigoplus_{\substack{u \prec w \\ \ell(u)=\ell(w)-i+1}} R_{\overline{X}(u),x} \xrightarrow{d_{i-1}} \bigoplus_{\substack{u \prec w \\ \ell(u)=\ell(w)-i}} R_{\overline{X}(u),x} \xrightarrow{d_i} \bigoplus_{\substack{u \prec w \\ \ell(u)=\ell(w)-i-1}} R_{\overline{X}(u),x}$$

□

Corollary 7.3. *Let $j : X(w) \hookrightarrow \overline{X}(w)$ be the open immersion. Then the complex (7.7) gives a resolution for $j_!\mathbb{Z}/p^m\mathbb{Z}$.*

Proof. We need to verify that the following complex is exact at R:

$$0 \to j_!R \xrightarrow{d_{-1}} R \xrightarrow{d_0} \bigoplus_{\substack{u \prec w \\ \ell(u)=\ell(w)-1}} R_{\overline{X}(u)}.$$

We verify on the stalks. When $x \in X(w)$, we have $j_!R_x \cong R_x$. When $x \in \overline{X}(w)\backslash X(w)$, then

$$R_x \xrightarrow{d_0} \bigoplus_{\substack{u \prec w \\ \ell(u)=\ell(w)-1}} R_{\overline{X}(u),x}$$

is injective. □

Since the category $\mathrm{Sh}_{\text{ét}}(\overline{X}(w))$ has enough injective objects, by Corollary 7.3, the complex (7.7) is quasi-isomorphic to an injective resolution of $j_!\mathbb{Z}/p^m\mathbb{Z}$. Take the spectral sequence associated to the complex (7.7), we have

$$E_1^{i,j} = \bigoplus_{\substack{u \prec w \\ \ell(u)=\ell(w)-i}} H^j_{\text{ét}}\left(\overline{X}(u), \mathbb{Z}/p^m\mathbb{Z}\right) \Rightarrow H^{i+j}_{\text{ét},c}\left(X(w), \mathbb{Z}/p^m\mathbb{Z}\right). \tag{7.9}$$

Proposition 7.4. *The spectral sequence 7.9 degenerates at the E_2-page.*

Proof. By Corollary 6.6, we have for all $j > 0$ and $u \preceq w$,

$$H^j\left(\overline{X}(u), \mathbb{Z}/p^m\mathbb{Z}\right) = 0.$$

Hence there is no nonzero terms at $E_1^{i,j}$ when $j \neq 0$. By Proposition 7.1, we know that $E_2 = E_\infty$.

□

7.3. **The étale cohomology with compact support for $X(w)$ with coefficients in $\mathbb{Z}/p^m\mathbb{Z}$ and $\mathbb{Z}_p$.**

Theorem 7.5. *Let $G = \mathrm{GL}_n$, and $w = t_1 \cdots t_r \in W$ such that $t_j \in S$ are distinct from one another. Set $I = \mathrm{supp}(w), I_u = \mathrm{supp}(u)$ and $P_I = B^* W_I B^* = U_I \rtimes L_I$. Then for $k \neq \ell(w), m > 0$,*

$$H^k_{\text{ét},c}(X(w), \mathbb{Z}/p^m\mathbb{Z}) = 0,$$

and

$$H^{\ell(w)}_{\text{ét},c}(X(w), \mathbb{Z}/p^m\mathbb{Z}) \cong \mathrm{ind}_{B(\mathbb{F}_q)}^{\mathrm{GL}_n(\mathbb{F}_q)} \mathbb{1}_{\mathbb{Z}/p^m\mathbb{Z}} \Big/ \sum_{\substack{u \prec w \\ \ell(u)=1}} \mathrm{ind}_{P_{I_u}(\mathbb{F}_q)}^{\mathrm{GL}_n(\mathbb{F}_q)} \mathbb{1}_{\mathbb{Z}/p^m\mathbb{Z}},$$

In particular,

$$H^{\ell(w)}_{\text{ét},c}(X(w), \mathbb{Z}/p^m\mathbb{Z}) \cong \mathrm{ind}_{P_I(\mathbb{F}_q)}^{\mathrm{GL}_n(\mathbb{F}_q)} \mathrm{St}_{L_I},$$

where St_{L_I} is the Steinberg module for $L_I(\mathbb{F}_q)$ with coefficients in $\mathbb{Z}/p^m\mathbb{Z}$.

Proof. By Proposition 7.4, we have for all $i \geq 0$,

$$E_2^{i,0} \cong H^i_{\text{ét, c}}(X(w), \mathbb{Z}/p^m\mathbb{Z}).$$

It follows from Proposition 7.1 that

$$E_2^{\ell(w),0} \cong \mathrm{ind}_{B(\mathbb{F}_q)}^{\mathrm{GL}_n(\mathbb{F}_q)} \mathbb{1}_{\mathbb{Z}/p^m\mathbb{Z}} \Big/ \sum_{\substack{u \prec w \\ \ell(u)=1}} \mathrm{ind}_{P_{I_u}(\mathbb{F}_q)}^{\mathrm{GL}_n(\mathbb{F}_q)} \mathbb{1}_{\mathbb{Z}/p^m\mathbb{Z}},$$

and $E_2^{i,j} = 0$ otherwise. □

Corollary 7.6. *Let $G = \mathrm{GL}_n$, and $w = t_1 \cdots t_r \in W$ such that the t_j's are distinct from one another. Denote $I = \mathrm{supp}(w), I_u = \mathrm{supp}(u), u \preceq w$ and $P_I = B^* W_I B^*$. Then for $k \neq \ell(w), m > 0$,*

$$H^k_{\text{ét},c}(X(w), \mathbb{Z}_p) = 0,$$

and

$$H^{\ell(w)}_{\text{ét},c}(X(w), \mathbb{Z}_p) \cong \mathrm{ind}_{B(\mathbb{F}_q)}^{\mathrm{GL}_n(\mathbb{F}_q)} \mathbb{1}_{\mathbb{Z}_p} \Big/ \sum_{\substack{u \prec w \\ \ell(u)=1}} \mathrm{ind}_{P_{I_u}(\mathbb{F}_q)}^{\mathrm{GL}_n(\mathbb{F}_q)} \mathbb{1}_{\mathbb{Z}_p}.$$

In particular,

$$H^{\ell(w)}_{\text{ét},c}(X(w), \mathbb{Z}_p) \cong \mathrm{ind}_{P_I(\mathbb{F}_q)}^{\mathrm{GL}_n(\mathbb{F}_q)} \mathrm{St}_{L_I},$$

where St_{L_I} is the Steinberg module for $L_I(\mathbb{F}_q)$ with coefficients in $\mathbb{Z}_p$.

Proof. By Theorem 7.5, we have for all $k \neq \ell(w)$ and $m > 0$, $H^k_{\text{ét},c}(X(w), \mathbb{Z}/p^m\mathbb{Z}) = 0$. Thus the tower $\left\{H^k_{\text{ét},c}(X(w), \mathbb{Z}/p^m\mathbb{Z})\right\}_m$ of abelian groups satisfy the Mittag–Leffler condition. Thus for all $k \neq \ell(w)$, we have

$$H^k_{\text{ét},c}(X(w), \mathbb{Z}_p) = \varprojlim_m H^k_{\text{ét},c}(X(w), \mathbb{Z}/p^m\mathbb{Z}) = 0.$$

On the other hand, whenever we have $m > l$ and a mod p^l map

$$\begin{aligned} \mathbb{Z}/p^m\mathbb{Z} &\longrightarrow \mathbb{Z}/p^l\mathbb{Z} \\ b \bmod p^m &\longmapsto b \bmod p^l, \end{aligned}$$

there is a short exact sequence of sheaves on $\overline{X}(w)_{\text{ét}}$:

$$0 \to \mathbb{Z}/p^{m-l}\mathbb{Z} \to \mathbb{Z}/p^m\mathbb{Z} \to \mathbb{Z}/p^l\mathbb{Z} \to 0.$$

By [Mil13, Corollary 8.14], since $j : X(w) \to \overline{X}(w)$ is an open immersion, we know that $j_!$ is an exact functor, so there is a short exact sequence

$$0 \to j_!\mathbb{Z}/p^{m-l}\mathbb{Z} \to j_!\mathbb{Z}/p^m\mathbb{Z} \to j_!\mathbb{Z}/p^l\mathbb{Z} \to 0.$$

Taking the associated long exact sequence yields

$$\cdots \to H^{\ell(w)}_{\text{ét}}\left(\overline{X}(w), j_!\mathbb{Z}/p^m\mathbb{Z}\right) \to H^{\ell(w)}_{\text{ét}}\left(\overline{X}(w), j_!\mathbb{Z}/p^l\mathbb{Z}\right) \to H^{\ell(w)+1}_{\text{ét}}\left(\overline{X}(w), j_!\mathbb{Z}/p^{m-l}\mathbb{Z}\right) \to \cdots.$$

The vanishing result for $k \neq \ell(w)$ from Theorem 7.5 implies that

$$H^{\ell(w)}_{\text{ét},c}\left(X(w), \mathbb{Z}/p^m\mathbb{Z}\right) \to H^{\ell(w)}_{\text{ét},c}\left(X(w), \mathbb{Z}/p^l\mathbb{Z}\right)$$

is surjective. Hence the tower of abelian groups $\left\{H^{\ell(w)}_{\text{ét},c}\left(X(w), \mathbb{Z}/p^m\mathbb{Z}\right)\right\}_m$ satisfies the Mittag–Leffler condition. Therefore

$$H^{\ell(w)}_{\text{ét},c}\left(X(w), \mathbb{Z}_p\right) = \varprojlim_m H^{\ell(w)}_{\text{ét},c}\left(X(w), \mathbb{Z}/p^m\mathbb{Z}\right),$$

and

$$H^{\ell(w)}_{\text{ét},c}\left(X(w), \mathbb{Z}_p\right) \cong \mathrm{ind}^{\mathrm{GL}_n(\mathbb{F}_q)}_{B(\mathbb{F}_q)} \mathbb{1}_{\mathbb{Z}_p} \Big/ \sum_{\substack{u \prec w \\ \ell(u)=1}} \mathrm{ind}^{\mathrm{GL}_n(\mathbb{F}_q)}_{P_{I_u}(\mathbb{F}_q)} \mathbb{1}_{\mathbb{Z}_p}.$$

□

Corollary 7.7. *Let* $G = \mathrm{GL}_n$ *and* $w \in F^+$. *Let* $v \in F^+$ *such that* $\mathrm{supp}(v) = \mathrm{supp}(w)$ *and* $v = s_{\alpha_1} \cdots s_{\alpha_r} \in W$ *with* s_{α_t} *all distinct. Let* $R = \mathbb{Z}/p^m\mathbb{Z}$ *or* $\mathbb{Z}_p$, $m > 0$. *Set* $I = \mathrm{supp}(w), I_u = \mathrm{supp}(u)$, *and* $P_I = B^* W_I B^*$. *Then for* $k \neq \ell(w)$,

$$H^k_{\text{ét},c}\left(X(w), R\right) = 0$$

and

$$H^{\ell(w)}_{\text{ét},c}\left(X(w), R\right) \cong H^{\ell(v)}_{\text{ét},c}\left(X(v), R\right) \cong \mathrm{ind}^{\mathrm{GL}_n(\mathbb{F}_q)}_{B(\mathbb{F}_q)} \mathbb{1}_R \Big/ \sum_{\substack{u \prec v \\ \ell(u)=1}} \mathrm{ind}^{\mathrm{GL}_n(\mathbb{F}_q)}_{P_{I_u}(\mathbb{F}_q)} \mathbb{1}_R.$$

In particular,

$$H^{\ell(w)}_{\text{ét},c}\left(X(w), R\right) \cong \mathrm{ind}^{\mathrm{GL}_n(\mathbb{F}_q)}_{P_I(\mathbb{F}_q)} \mathrm{St}_{L_I},$$

where St_{L_I} *is the Steinberg module for* $L_I(\mathbb{F}_q)$ *with coefficients in* R.

Proof. Analogous to [Orl18, Proposition 2.11], by induction on $\ell(w) - \ell(v)$, the complex (7.1) we get for $w \in F^+$ is homotopic to the complex (7.1) for any $v \in F^+$ such that $\mathrm{supp}(v) = \mathrm{supp}(w)$. The rest follows from Theorem 7.5 and Corollary 7.6. □

Appendix A. Filtrations on the global section of $\mathcal{O}_{X(w)}$

We are interested in the cohomology groups $H^k\left(X(w), \mathcal{O}_{X(w)}\right), k \geq 0$ for $w \in W$ in the case of $G = \mathrm{GL}_n$. We would also like to understand their structure as $\mathrm{GL}_n(\mathbb{F}_q)$-representations. In view of the double induction procedure in [Orl18], one would like to study the relation between the cohomology groups of Deligne–Lusztig varieties corresponding to Weyl group elements with the same support. For example, between $X(sws)$ and $X(ws)$, or between $X(w)$ and $X(K(w))$. If one could deduce all the relevant relations, then the determination of the cohomology groups $H^k\left(X(w), \mathcal{O}_{X(w)}\right)$ for arbitrary $w \in W$ depends on the case for the Coxeter elements (of a parabolic subgroup $W_I \subseteq W$).

As a first step in the base case of the strategy above, we consider in this section the cohomology of $\mathrm{GL}_n(\mathbb{F}_q)$-equivariant vector bundles $\mathcal{E}$ on $X(w)$ when $w \leq \mathbf{w}$. Since the Deligne–Lusztig varieties $X(w)$ are all affine $\overline{\mathbb{F}}_p$-schemes in this case, the only non-vanishing cohomology group is at degree 0. Moreover, we will consider filtrations on $\mathcal{E}(X(w))$.

A.1. **Background for the case of the Drinfeld half space.** In [Orl08, Theorem 2.2.8], Orlik has given a filtration for $H^0\left(\mathcal{X}_K^{n-1}, \mathcal{E}\right)$, where K is a finite extension of $\mathbb{Q}_p$, and $\mathcal{X}_K^{n-1}$ is the Drinfeld half space and $\mathcal{E}$ is a GL_n-equivariant homogeneous vector bundle on $\mathcal{X}_K^{n-1}$. Kuschkowitz has adapted Orlik's result for the Drinfeld half space defined over a finite field. We recall the result of [Kus16, Theorem 2.1.2.1] after base change to the algebraic closure $\overline{\mathbb{F}}_p$.

Theorem A.1 (Kuschkowitz '16)**.** Let $\mathcal{E}$ be a $\mathrm{GL}_n(\mathbb{F}_q)$-equivariant homogeneous vector bundle on $\mathbb{P}^{n-1}_{\overline{\mathbb{F}}_p}$. Then there is a filtration on $H^0(X(\mathbf{w}), \mathcal{E}) =: \mathcal{E}(X(\mathbf{w}))^0$:

$$\mathcal{E}(X(\mathbf{w}))^0 \supset \mathcal{E}(X(\mathbf{w}))^1 \supset \cdots \supset \mathcal{E}(X(\mathbf{w}))^{n-2} \supset \mathcal{E}(X(\mathbf{w}))^{n-1} = H^0\left(\mathbb{P}^{n-1}_{\overline{\mathbb{F}}_p}, \mathcal{E}\right)$$

such that each subquotient of this filtration fits into an extension of $\mathrm{GL}_n(\mathbb{F}_q)$-representations

$$\begin{aligned} 0 \to \operatorname{ind}^{\mathrm{GL}_n(\mathbb{F}_q)}_{P_{(j+1,n-1-j)}(\mathbb{F}_q)} \left(\tilde{H}^{n-1-j}_{\mathbb{P}^j_{\overline{\mathbb{F}}_p}}\left(\mathbb{P}^{n-1}_{\overline{\mathbb{F}}_p}, \mathcal{E}\right) \otimes \mathrm{St}_{n-j}\right) \to \mathcal{E}(X(\mathbf{w}))^j / \mathcal{E}(X(\mathbf{w}))^{j+1} \to \\ \to v^{\mathrm{GL}_n(\mathbb{F}_q)}_{P_{(j+1,1,\ldots,1)}(\mathbb{F}_q)} \otimes H^{n-1-j}\left(\mathbb{P}^{n-1}_{\overline{\mathbb{F}}_p}, \mathcal{E}\right) \to 0 \quad \text{(A.1)}\end{aligned}$$

for $j = 0, ..., n-2$, where P_d, for a partition d of n, is the standard parabolic subgroup of GL_n corresponding to d in the obvious way, and $v^{\mathrm{GL}_n(\mathbb{F}_q)}_{P_{(j+1,1,\ldots,1)}(\mathbb{F}_q)}$ is a generalized Steinberg representation. Also,

$$\tilde{H}^{n-1-j}_{\mathbb{P}^j_{\overline{\mathbb{F}}_p}}\left(\mathbb{P}^{n-1}_{\overline{\mathbb{F}}_p}, \mathcal{E}\right) := \ker\left(H^{n-1-j}_{\mathbb{P}^j_{\overline{\mathbb{F}}_p}}\left(\mathbb{P}^{n-1}_{\overline{\mathbb{F}}_p}, \mathcal{E}\right) \to H^{n-1-j}\left(\mathbb{P}^{n-1}_{\overline{\mathbb{F}}_p}, \mathcal{E}\right)\right).$$

The representations arising from Kuschkowitz' theorem are studied in the preprint [Orl21].

A.2. **Notations and constructions.** We fix some notations for the rest of this section. Let $w \leq \mathbf{w}$. Denote $I = \mathrm{supp}(w)$, $P_I = B^* W_I B^*$, where $W_I \subseteq W$ is the subgroup generated by I. We have a decomposition $P_I = U_I \rtimes L_I$, where U_I is the unipotent radical of P_I and $L_I \supseteq T^*$ is the standard Levi subgroup of GL_n associated to P_I. Recall that

$$L_I \xrightarrow{\sim} \mathrm{GL}_{n_1} \times \cdots \times \mathrm{GL}_{n_r},$$

where $n_1 + \cdots + n_r = n$ and w_i is a Coxeter element for GL_{n_i} such that $w = w_1 \cdots w_r$. Note that the homogeneous space for GL_1 is just a point $\mathrm{Spec}\, \overline{\mathbb{F}}_p$ and the Deligne–Lusztig variety for GL_1 is $\mathrm{Spec}\, \overline{\mathbb{F}}_p$.

For GL_m, the projective space $\mathbb{P}^{m-1}_{\overline{\mathbb{F}}_p}$ is isomorphic to the quotient GL_m/P_J, where P_J is the parabolic subgroup generated by $J = \{s_2, ..., s_{m-1}\}$. Now for any $P_J(\mathbb{F}_q)$-module M that is also a $\overline{\mathbb{F}}_p$-vector space, there is an associated vector bundle

$$E := \mathrm{GL}_m \times^{P_J} M$$

over GL_m/P_J. Note that we have the identification $(gp, m) \sim (g, p.m)$. The map $\pi : E \to \mathrm{GL}_m/P_J$ is locally trivial with respect to a Zariski cover cf. [Jan03, §I 5, §II 1]. We may take the associated sheaf of $\mathcal{O}_{\mathrm{GL}_m/P_J}$-module $\mathcal{E}$ defined by

$$\mathcal{E}(U) := \Gamma(U, E) = \{s : U \to E | \pi \circ s = \mathrm{id}_U\},$$

for all $U \subseteq \mathrm{GL}_m/P_J$ open. As we have identified GL_m/P_J with $\mathbb{P}^{m-1}_{\overline{\mathbb{F}}_p}$, we call such $\mathcal{E}$ homogeneous vector bundles on $\mathbb{P}^{m-1}_{\overline{\mathbb{F}}_p}$. In particular, $\mathcal{E}$ is $\mathrm{GL}_m(\mathbb{F}_q)$-equivariant by construction.

Let Y_1, Y_2 be two Drinfeld half spaces over $\overline{\mathbb{F}}_p$ defined over $\mathbb{F}_q$, of dimension $m_1 - 1$ and $m_2 - 1$ respectively. Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be homogeneous vector bundles on $\mathbb{P}^{m_1-1}_{\overline{\mathbb{F}}_p}$ and $\mathbb{P}^{m_2-1}_{\overline{\mathbb{F}}_p}$ respectively. Denote their restrictions to Y_1 and Y_2 also by $\mathcal{E}_1$ and $\mathcal{E}_2$. Then $\mathrm{pr}_1^*\mathcal{E}_i$, $i = 1, 2$, is a $\mathrm{GL}_n(\mathbb{F}_q)$-equivariant locally free $\mathcal{O}_{Y_1 \times Y_2}$-module on $Y_1 \times Y_2$. Thus $\mathrm{pr}_1^*\mathcal{E}_1 \otimes \mathrm{pr}_2^*\mathcal{E}_2$ gives a $\mathrm{GL}_n(\mathbb{F}_q)$-equivariant locally free $\mathcal{O}_{Y_1 \times Y_2}$-module on $Y_1 \times Y_2$, which we denote by $\mathcal{E}$. Now we have an isomorphism of $\overline{\mathbb{F}}_p$-vector spaces:

$$\mathcal{E}(Y_1 \times Y_2) \xrightarrow{\sim} \mathcal{E}_1(Y_1) \otimes \mathcal{E}_2(Y_2). \quad \text{(A.2)}$$

Hence the filtrations for $\mathcal{E}_1(Y_1)$ and $\mathcal{E}_2(Y_2)$ would carry over to a filtration for $\mathcal{E}(Y_1 \times Y_2)$. The natural $\mathrm{GL}_{m_1}(\mathbb{F}_q)$-action (resp. $\mathrm{GL}_{m_2}(\mathbb{F}_q)$-action) on Y_1 (resp. Y_2) yields a $\mathrm{GL}_{m_1} \times \mathrm{GL}_{m_2}(\mathbb{F}_q)$-action on the product scheme $Y_1 \times Y_2$. Observe that the isomorphism (A.2) is in fact also $\mathrm{GL}_{m_1} \times \mathrm{GL}_{m_2}(\mathbb{F}_q)$-equivariant.

A.3. **Filtrations of vector bundles on $X(w)$ with $w \leq \mathbf{w}$.**

Lemma A.2. *Let $G = \mathrm{GL}_n$ and $w \leq \mathbf{w}$. The Deligne–Lusztig variety $X_{\mathrm{GL}_n}(w)$ is affine.*

Proof. Using the notations defined above, we have an $L_I(\mathbb{F}_q)$-equivariant isomorphism of $\overline{\mathbb{F}}_p$-schemes:

$$X_{L_I}(w) \xrightarrow{\sim} X_{\mathrm{GL}_{n_1}}(w_1) \times \cdots \times X_{\mathrm{GL}_{n_r}}(w_r).$$

By construction, we know that $X_{\mathrm{GL}_{n_i}}(w_i)$ is the Drinfeld half space of dimension $\ell(w_i)$ for each $i = 1, .., r$. Each $X_{\mathrm{GL}_{n_i}}(w_i)$ is affine by [Lus77, Corollary 2.8]. Thus $X_{L_I}(w)$ is affine.

Recall that

$$X_{\mathrm{GL}_n}(w) = \mathrm{GL}_n^F / U_I^F \times^{L_I^F} X_{L_I}(w)$$

is a fibration over GL_n^F / P_I^F with fibres $X_{L_I}(w)$. Thus $X_{\mathrm{GL}_n}(w)$ is affine. □

Lemma A.3. *Let $\mathcal{E}_i$ be $\mathrm{GL}_{n_i}(\mathbb{F}_q)$-equivariant locally free $\mathcal{O}$-modules on $X_{\mathrm{GL}_{n_i}}(w_i)$ each coming from a homogeneous vector bundle on $\mathbb{P}^{n_i-1}_{\overline{\mathbb{F}}_p}$. Let*

$$\mathcal{E}_I := \bigotimes_{i=1}^{r} pr_i^* \mathcal{E}_i \qquad \text{and} \qquad \mathcal{E} := \bigoplus_{gP_I^F \in \mathrm{GL}_n^F / P_I^F} \iota_*^{gP_I^F} \mathcal{E}_I,$$

where $\iota^{gP_I^F} : \{gP_I^F\} \times X_{L_I}(w) \to X_{\mathrm{GL}_n}(w)$ is the inclusion for each irreducible component of $X_{\mathrm{GL}_n}(w)$. Then there is a $\mathrm{GL}_n(\mathbb{F}_q)$-equivariant isomorphism of cohomology groups:

$$H^0\left(X_{\mathrm{GL}_n}(w), \mathcal{E}\right) \xrightarrow{\sim} \mathrm{ind}_{P_I^F}^{\mathrm{GL}_n^F} H^0\left(X_{L_I}(w), \mathcal{E}_I\right).$$

Proof. This follows directly from the construction mentioned in the previous lemma. As justified in Section A.2, we know that $\mathcal{E}_I$ is a $L_I(\mathbb{F}_q)$-equivariant $\mathcal{O}_{X_{L_I}(w)}$-module. The construction of $\mathcal{E}$ follow from the construction of sheaves on disjoint unions of schemes. In particular, $\mathcal{E}$ is a $\mathrm{GL}_n(\mathbb{F}_q)$-equivariant $\mathcal{O}_{X(w)}$-module. □

Lemma A.4. *We have a filtration*

$$H^0\left(X_{L_I}(w), \mathcal{E}_I\right) =: M_{L_I}^0 \supset M_{L_I}^1 \supset \cdots \supset M_{L_I}^{\sum_{i=1}^r (n_i - 1)} = \bigotimes_{i=1}^{r} H^0\left(\mathbb{P}^{n_i-1}_{\overline{\mathbb{F}}_p}, \mathcal{E}_i\right)$$

such that for $\left(\sum_{i=1}^{t-1}(n_i - 1)\right) \leq j \leq \left(\sum_{i=1}^{t}(n_i - 1)\right)$, $t = 1, ..., r$,

$$M_{L_I}^j = \left(\bigotimes_{i=1}^{t-1} H^0\left(\mathbb{P}^{n_i-1}_{\overline{\mathbb{F}}_p}, \mathcal{E}_i\right)\right) \otimes N^{j'} \otimes \left(\bigotimes_{i=t+1}^{r} H^0\left(X_{\mathrm{GL}_{n_i}}(w_i), \mathcal{E}_i\right)\right)$$

where $N^{j'}$ is the j'-th term in Kuschkowitz' filtration for $H^0\big(X_{\mathrm{GL}_{n_t}}(w_t), \mathcal{E}_t\big)$ with $j' = j - \big(\sum_{i=1}^{t-1} n_i - 1\big)$.

Furthermore, when $\left(\sum_{i=1}^{t-1}(n_i - 1)\right) \leq j \leq \left(\sum_{i=1}^{t}(n_i - 1)\right)$, for $t = 1, ..., r$, there is an extension of $L_I(\mathbb{F}_q)$-representations:

$$0 \to A^t \otimes \mathrm{ind}_{P^F_{(j'+1, n_t-1-j')}}^{\mathrm{GL}_{n_t}^F} \left(\tilde{H}^{n_t-1-j'}_{\mathbb{P}^{j'}}\left(\mathbb{P}^{n_t-1}_{\overline{\mathbb{F}}_p}, \mathcal{E}_t\right) \otimes \mathrm{St}_{n_t-j'}\right) \otimes B^t \to M_{L_I}^j / M_{L_I}^{j+1} \to$$
$$\to A^t \otimes v_{P^F_{(j'+1,1,\ldots,1)}}^{\mathrm{GL}_{n_t}^F} \otimes H^{n_t-1-j'}\left(\mathbb{P}^{n_t-1}_{\overline{\mathbb{F}}_p}, \mathcal{E}_t\right) \otimes B^t \to 0, \quad \text{(A.3)}$$

where $j' = j - \left(\sum_{i=1}^{t-1} n_i - 1\right)$, *and*

$$A^t := \bigotimes_{i=1}^{t-1} H^0\left(\mathbb{P}^{n_i-1}_{\overline{\mathbb{F}}_p}, \mathcal{E}_i\right) \qquad B^t := \bigotimes_{i=t+1}^{r} H^0\left(X_{\mathrm{GL}_{n_i}}(w_i), \mathcal{E}_i\right).$$

Proof. This lemma follows from [Kus16, Theorem 2.1.2.1] and the following $L_I(\mathbb{F}_q)$-equivariant isomorphism of $\overline{\mathbb{F}}_p$-schemes:

$$X_{L_I}(w) \xrightarrow{\sim} X_{\mathrm{GL}_{n_1}}(w_1) \times \cdots \times X_{\mathrm{GL}_{n_r}}(w_r).$$

Since A^t and B^t are $\overline{\mathbb{F}}_p$-vector spaces for all $t = 1, \ldots, r$, taking tensor products with A^t and B^t is exact. Thus extensions in (A.1) for $N^{j'}/N^{j'+1}$ yields the extensions in (A.3). ☐

Proposition A.5. *Let* $w \le \mathbf{w}$, *such that* $w = w_1 \cdots w_r$, *with* w_i *being the standard Coxeter elment for* GL_{n_i}, $i = 1, ..., r$, $n_1 + \cdots n_r = n$. *Denote* $I = \mathrm{supp}(w)$. *For each* i, *let* $\mathcal{E}_i$ *be a* $\mathrm{GL}_{n_i}(\mathbb{F}_q)$*-equivariant locally free* $\mathcal{O}$*-modules on* $X_{\mathrm{GL}_{n_i}}(w_i)$ *coming from a homogeneous vector bundle on* $\mathbb{P}^{n_i-1}_{\overline{\mathbb{F}}_p}$. *Let*

$$\mathcal{E}_I := \bigotimes_{i=1}^{r} pr_i^* \mathcal{E}_i \qquad \text{and} \qquad \mathcal{E} := \bigoplus_{gP_I^F \in \mathrm{GL}_n^F/P_I^F} \iota_*^{gP_I^F} \mathcal{E}_I,$$

where $\iota^{gP_I^F} : \left\{gP_I^F\right\} \times X_{L_I}(w) \to X_{\mathrm{GL}_n}(w)$ *is the inclusion for each irreducible component of* $X_{\mathrm{GL}_n}(w)$. *Then*

$$H^0\left(X_{\mathrm{GL}_n}(w), \mathcal{E}\right) = \mathrm{ind}_{P_I^F}^{\mathrm{GL}_n^F} H^0\left(X_{L_I}(w), \mathcal{E}\right)$$

has a filtration

$$H^0\left(X_{\mathrm{GL}_n}(w), \mathcal{E}\right) =: M^0 \supset M^1 \supset \cdots \supset M^{\sum_{i=1}^{r}(n_i-1)} = \mathrm{ind}_{P_I^F}^{\mathrm{GL}_n^F}\left(\bigotimes_{i=1}^{r} H^0\left(\mathbb{P}^{n_i-1}_{\overline{\mathbb{F}}_p}, \mathcal{E}_i\right)\right)$$

such that when $\left(\sum_{i=1}^{t-1}(n_i - 1)\right) \le j \le \left(\sum_{i=1}^{t}(n_i - 1)\right)$, *for* $t = 1, ..., r$, *there is an extension* $\mathrm{GL}_n(\mathbb{F}_q)$*-representations:*

$$\begin{aligned} 0 \to \mathrm{ind}_{P_I^F}^{\mathrm{GL}_n^F}\left(A^t \otimes \mathrm{ind}_{P^F_{(j'+1, n_t-1-j')}}^{\mathrm{GL}_{n_t}^F}\left(\tilde{H}^{n_t-1-j'}_{\mathbb{P}^{j'}}\left(\mathbb{P}^{n_t-1}_{\overline{\mathbb{F}}_p}, \mathcal{E}_t\right) \otimes \mathrm{St}_{n_t-j'}\right) \otimes B^t\right) \to M^j/M^{j+1} \to \\ \to \mathrm{ind}_{P_I^F}^{\mathrm{GL}_n^F}\left(A^t \otimes v_{P^F_{(j'+1,1,\ldots,1)}}^{\mathrm{GL}_{n_t}^F} \otimes H^{n_t-1-j'}\left(\mathbb{P}^{n_t-1}_{\overline{\mathbb{F}}_p}, \mathcal{E}_t\right) \otimes B^t\right) \to 0, \quad \text{(A.4)} \end{aligned}$$

where $j' = j - \left(\sum_{i=1}^{t-1} n_i - 1\right)$ *and*

$$A^t := \bigotimes_{i=1}^{t-1} H^0\left(\mathbb{P}^{n_i-1}_{\overline{\mathbb{F}}_p}, \mathcal{E}_i\right) \qquad B^t := \bigotimes_{i=t+1}^{r} H^0\left(X_{\mathrm{GL}_{n_i}}(w_i), \mathcal{E}_i\right).$$

In particular, when $\mathcal{E}_i$ *are the structure sheaf for all* i, *then the sequence (A.4) is also exact on the left.*

Proof. Since $\mathrm{GL}_n^F = \mathrm{GL}_n(\mathbb{F}_q)$ and $P_I^F = P_I(\mathbb{F}_q)$ are finite groups, $\mathrm{ind}_{P_I^F}^{\mathrm{GL}_n^F}$ is an exact functor. Thus $\mathrm{ind}_{P_I^F}^{\mathrm{GL}_n^F}$ preserves injective maps as well as short exact sequences. ☐

It is also possible to rewrite this filtration in a different way to resemble the Hodge filtration.

Proposition A.6. *Let* $w \le \mathbf{w}$, *such that* $w = w_1 \cdots w_r$, *with* w_i *being the standard Coxeter elment for* GL_{n_i}, $i = 1, ..., r$. *Denote* $I = \mathrm{supp}(w)$. *For each* i, *let* $\mathcal{E}_i$ *be a* $\mathrm{GL}_{n_i}(\mathbb{F}_q)$*-equivariant locally free* $\mathcal{O}$*-modules on* $X_{\mathrm{GL}_{n_i}}(w_i)$ *coming from a homogeneous vector bundle on* $\mathbb{P}^{n_i-1}_{\overline{\mathbb{F}}_p}$. *Let*

$$\mathcal{E}_I := \bigotimes_{i=1}^{r} pr_i^* \mathcal{E}_i \qquad \text{and} \qquad \mathcal{E} := \bigoplus_{gP_I^F \in \mathrm{GL}_n^F/P_I^F} \iota_*^{gP_I^F} \mathcal{E}_I,$$

where $\iota^{gP_I^F}: \{gP_I^F\} \times X_{L_I}(w) \to X_{\mathrm{GL}_n}(w)$ *is the inclusion for each irreducible component of* $X_{\mathrm{GL}_n}(w)$. *Then*

$$H^0\left(X_{\mathrm{GL}_n}(w), \mathcal{E}\right) = \mathrm{ind}_{P_I^F}^{\mathrm{GL}_n^F} H^0\left(X_{L_I}(w), \mathcal{E}\right)$$

has a filtration

$$H^0\left(X_{\mathrm{GL}_n}(w), \mathcal{E}\right) =: M^0 \supset M^1 \supset \cdots \supset M^{\sum_{i=1}^r (n_i-1)} = \mathrm{ind}_{P_I^F}^{\mathrm{GL}_n^F} \left(\bigotimes_{i=1}^r H^0\left(\mathbb{P}_{\overline{\mathbb{F}}_p}^{n_i-1}, \mathcal{E}_i\right)\right)$$

with graded terms being $\mathrm{GL}_n(\mathbb{F}_q)$*-representations:*

$$M^k/M^{k+1} = \bigoplus_{\alpha_1+\cdots+\alpha_r=k} \left((N_{n_1}^{\alpha_1}/N_{n_1}^{\alpha_1+1}) \otimes \cdots \otimes (N_{n_r}^{\alpha_r}/N_{n_r}^{\alpha_r+1})\right),$$

where $N_{n_t}^{\alpha_t}$ *is the* α_t*-th term in the filtration of* $H^0\left(X_{\mathrm{GL}_{n_t}}(w_t), \mathcal{E}_t\right)$.

Proof. When $r = 2$, this follows from the observation that the graded terms of a tensor product come from the direct sum of tensor products of the graded terms of the two filtrations, similar to Hodge filtrations. For $r > 2$ it follows by induction. $\square$

A.4. **Examples.** (a) Let $w \leq \mathbf{w}$ such that $w = s_1 \cdots s_{n-2}$. Let $I = \mathrm{supp}(w)$, so in this case we have $L_I \cong \mathrm{GL}_{n-1} \times \mathrm{GL}_1$. Thus we have

$$H^0\left(X_{\mathrm{GL}_n}(w), \mathcal{O}_{X(w)}\right) = \mathrm{ind}_{P_I^F}^{\mathrm{GL}_n^F} H^0\left(X_{\mathrm{GL}_{n-1}}(w), \mathcal{O}_{X_{\mathrm{GL}_{n-1}}(w)}\right).$$

If $N^0 \supseteq \cdots \supseteq N^{n-2}$ is the GL_{n-1}^F-equivariant filtration for $H^0\big(X_{\mathrm{GL}_{n-1}}(w), \mathcal{O}\big)$ as in [Kus16, Theorem 2.1.2.1], then

$$\mathrm{ind}_{P_I^F}^{\mathrm{GL}_n^F} N^0 \supseteq \cdots \supseteq \mathrm{ind}_{P_I^F}^{\mathrm{GL}_n^F} N^{n-2} = \mathrm{ind}_{P_I^F}^{\mathrm{GL}_n^F} H^0\left(\mathbb{P}_{\overline{\mathbb{F}}_p}^{n-2}, \mathcal{O}_{\mathbb{P}_{\overline{\mathbb{F}}_p}^{n-2}}\right)$$

gives a GL_n^F-equivariant filtration for $H^0\big(X_{\mathrm{GL}_n}(w), \mathcal{O}\big)$.

(b) Let $w \leq \mathbf{w}$ such that $w = s_1 \cdots s_{f-1} s_{f+1} \cdots s_{n-1}$, $1 < f < n-1$. In this case, we have

$$X_{L_I}(w) \xrightarrow{\sim} X_{\mathrm{GL}_{n_1}}(w_1) \times X_{\mathrm{GL}_{n_2}}(w_2),$$

where $w_1 = s_1 \cdots s_{f-1}$, $w_2 = s_{f+1} \cdots s_{n-1}$. Hence

$$H^0\left(X_{\mathrm{GL}_n}(w), \mathcal{O}_{X(w)}\right) = \mathrm{ind}_{P_I^F}^{\mathrm{GL}_n^F} \left(H^0\left(X_{\mathrm{GL}_{n_1}}(w_1), \mathcal{O}_{X_{\mathrm{GL}_{n_1}}(w_1)}\right) \otimes H^0\left(X_{\mathrm{GL}_{n_2}}(w_2), \mathcal{O}_{X_{\mathrm{GL}_{n_2}}(w_2)}\right)\right).$$

If $N_{n_1}^0 \supseteq \cdots \supseteq N_{n_1}^{n_1-1}$ (resp. $N_{n_2}^0 \supseteq \cdots \supseteq N_{n_2}^{n_2-1}$) is the $\mathrm{GL}_{n_1}^F$-equivariant (resp.GL_{n_2}-equivariant) filtration for $H^0\big(X_{\mathrm{GL}_{n_1}}(w_1), \mathcal{O}_{X_{\mathrm{GL}_{n_1}}(w_1)}\big)$ (resp. $H^0\big(X_{\mathrm{GL}_{n_2}}(w_2), \mathcal{O}_{X_{\mathrm{GL}_{n_2}}(w_2)}\big)$) as in [Kus16, Theorem 2.1.2.1], then

$$\begin{aligned}\mathrm{ind}_{P_I^F}^{\mathrm{GL}_n^F}\left(N_{n_1}^0 \otimes N_{n_2}^0\right) \supset \mathrm{ind}_{P_I^F}^{\mathrm{GL}_n^F}\left(N_{n_1}^1 \otimes N_{n_2}^0\right) \supset \cdots \supset \mathrm{ind}_{P_I^F}^{\mathrm{GL}_n^F}\left(N_{n_1}^{n_1-1} \otimes N_{n_2}^0\right) \supset \\ \supset \mathrm{ind}_{P_I^F}^{\mathrm{GL}_n^F}\left(N_{n_1}^{n_1-1} \otimes N_{n_2}^1\right) \supset \cdots \supset \mathrm{ind}_{P_I^F}^{\mathrm{GL}_n^F}\left(N_{n_1}^{n_1-1} \otimes N_{n_2}^{n_2-1}\right)\end{aligned}$$

gives a filtration for $H^0\big(X_{\mathrm{GL}_n}(w), \mathcal{O}_{X(w)}\big)$. Note that

$$\mathrm{ind}_{P_I^F}^{\mathrm{GL}_n^F}\left(N_{n_1}^{n_1-1} \otimes N_{n_2}^{n_2-1}\right) = \mathrm{ind}_{P_I^F}^{\mathrm{GL}_n^F}\left(H^0\left(\mathbb{P}_{\overline{\mathbb{F}}_p}^{n_1-1}, \mathcal{O}_{\mathbb{P}_{\overline{\mathbb{F}}_p}^{n_1-1}}\right) \otimes H^0\left(\mathbb{P}_{\overline{\mathbb{F}}_p}^{n_2-1}, \mathcal{O}_{\mathbb{P}_{\overline{\mathbb{F}}_p}^{n_2-1}}\right)\right).$$

We have extensions of GL_n^F-representations

$$\begin{aligned}0 \to \mathrm{ind}_{P_I^F}^{\mathrm{GL}_n^F}\left(\mathrm{ind}_{P_{(j'+1,n_1-1-j')}^F}^{\mathrm{GL}_{n_1}^F}\left(\tilde{H}_{\mathbb{P}^{j'}}^{n_t-1-j'}\left(\mathbb{P}_{\overline{\mathbb{F}}_p}^{n_1-1}, \mathcal{O}_{\mathbb{P}_{\overline{\mathbb{F}}_p}^{n_1-1}}\right) \otimes \mathrm{St}_{n_1-j'}\right) \otimes H^0(X_{\mathrm{GL}_{n_2}}(w_2), \mathcal{O}_{X_{\mathrm{GL}_{n_2}}(w_2)})\right) \to \\ \to \mathrm{ind}_{P_I^F}^{\mathrm{GL}_n^F}\left(N_{n_1}^{j'} \otimes N_{n_2}^0\right) / \mathrm{ind}_{P_I^F}^{\mathrm{GL}_n^F}\left(N_{n_1}^{j'+1} \otimes N_{n_2}^0\right) \to \\ \to \mathrm{ind}_{P_I^F}^{\mathrm{GL}_n^F}\left(v_{P_{(j'+1,1,\ldots,1)}^F}^{\mathrm{GL}_{n_1}^F} \otimes H^{n_1-1-j'}\left(\mathbb{P}_{\overline{\mathbb{F}}_p}^{n_1-1}, \mathcal{O}_{\mathbb{P}_{\overline{\mathbb{F}}_p}^{n_1-1}}\right) \otimes H^0(X_{\mathrm{GL}_{n_2}}(w_2), \mathcal{O}_{X_{\mathrm{GL}_{n_2}}(w_2)})\right) \to 0\end{aligned}$$

for $j' = 0, ..., n_1 - 1$ and

$$0 \to \mathrm{ind}_{P_I^F}^{\mathrm{GL}_n^F} \left(H^0 \left(\mathbb{P}_{\overline{\mathbb{F}}_p}^{n_1}, \mathcal{O}_{\mathbb{P}_{\overline{\mathbb{F}}_p}^{n_1}} \right) \otimes \mathrm{ind}_{P^F_{(j'+1, n_2-1-j')}}^{\mathrm{GL}_{n_2}^F} \left(\tilde{H}_{\mathbb{P}^{j'}}^{n_2-1-j'} \left(\mathbb{P}_{\overline{\mathbb{F}}_p}^{n_2-1}, \mathcal{O}_{\mathbb{P}_{\overline{\mathbb{F}}_p}^{n_2-1}} \right) \otimes \mathrm{St}_{n_2-j'} \right) \right) \to$$
$$\to \mathrm{ind}_{P_I^F}^{\mathrm{GL}_n^F} \left(N_{n_1}^{n_1-1} \otimes N_{n_2}^{j'} \right) / \mathrm{ind}_{P_I^F}^{\mathrm{GL}_n^F} \left(N_{n_1}^{n_1-1} \otimes N_{n_2}^{j'+1} \right) \to$$
$$\to \mathrm{ind}_{P_I^F}^{\mathrm{GL}_n^F} \left(H^0 \left(\mathbb{P}_{\overline{\mathbb{F}}_p}^{n_1}, \mathcal{O}_{\mathbb{P}_{\overline{\mathbb{F}}_p}^{n_1}} \right) \otimes v_{P^F_{(j'+1,1,\ldots,1)}}^{\mathrm{GL}_{n_2}^F} \otimes H^{n_2-1-j'} \left(\mathbb{P}_{\overline{\mathbb{F}}_p}^{n_2-1}, \mathcal{O}_{\mathbb{P}_{\overline{\mathbb{F}}_p}^{n_2-1}} \right) \right) \to 0$$

for $j' = 0, ..., n_2 - 1$.

Appendix B. Examples of local cohomology of $\overline{X}(w)$ with support in $\overline{X}(v)$

As mentioned in Appendix A, we are interested in $H^k(X(w), \mathcal{O}_{X(w)}), k \geq 0$. Another possible approach would be via the computation of local cohomology groups on $\overline{X}(w)$. In particular, one could study the terms in the spectral sequence

$$E_1^{i,j} = \bigoplus_{\substack{v \prec w \\ \ell(v) = \ell(w) - i}} H^j_{\overline{X}(v)} \left(\overline{X}(w), \mathcal{O}_{\overline{X}(w)} \right).$$

In this case, we are interested in how the local cohomology groups are related when we replace the locally closed subscheme $\overline{X}(v)$ with another locally closed subscheme $\overline{X}(v')$ with $\mathrm{supp}(v) = \mathrm{supp}(v')$.

We will examine some examples of local cohomology of $\overline{X}(w)$ in this section.

B.1. **Examples of computations of local cohomology of $\overline{X}(w)$ with support in $\overline{X}(v)$.** We recall some definitions and lemmas from [SGA2, Exposé I].

Denote $X := \overline{X}(w)$, $U := X(w)$, $Z := \overline{X}(w) \backslash X(w)$. By [SGA2, Exposé I, Theorem 2.8], we have the following long exact sequence

$$0 \to \Gamma_{\overline{X}(w) \backslash X(w)} \left(\overline{X}(w), \mathcal{O} \right) \longrightarrow \Gamma \left(\overline{X}(w), \mathcal{O} \right) \longrightarrow \Gamma \left(X(w), \mathcal{O} \right) \longrightarrow H^1_{\overline{X}(w) \backslash X(w)} \left(\overline{X}(w), \mathcal{O} \right) \to \cdots$$
$$\cdots \to H^k_{\overline{X}(w) \backslash X(w)} \left(\overline{X}(w), \mathcal{O} \right) \longrightarrow H^k \left(\overline{X}(w), \mathcal{O} \right) \longrightarrow H^k \left(X(w), \mathcal{O} \right) \to \cdots.$$

By Theorem 6.1, we know $H^k\big(\overline{X}(w), \mathcal{O}_{\overline{X}(w)}\big)$ for all $k \geq 0$. If we also know the local cohomology groups $H^k_{\overline{X}(w) \backslash X(w)}\big(\overline{X}(w), \mathcal{O}_{\overline{X}(w)}\big)$, then we will be able to deduce $H^k\big(X(w), \mathcal{O}_{X(w)}\big)$ from the long exact sequence.

Example B.1. Let $G = \mathrm{GL}_n, w \in W$. Fix a reduced expression $w = s_{i_1} \cdots s_{i_r}$. Denote $\overline{X}(w) := \overline{X}(s_{i_1}, ..., s_{i_r})$. We want to study $H^k_{X(v)}\big(\overline{X}(w), \mathcal{O}_{\overline{X}(w)}\big)$ and $H^k_{\overline{X}(v)}\big(\overline{X}(w), \mathcal{O}_{\overline{X}(w)}\big)$, for $v \preceq w$ a subword of $s_{i_1} \cdots s_{i_r} \in F^+$.

i. Let $v = e$ be the identity element in F^+, then $X(e) = (\mathrm{GL}_n / B^*)(\mathbb{F}_q)$. For $X(e) \subseteq \overline{X}(w)$, we have for all $k \geq 0$,

$$H^k_{X(e)} \left(\overline{X}(w), \mathcal{O}_{\overline{X}(w)} \right) = \bigoplus_{x \in (\mathrm{GL}_n / B^*)(\mathbb{F}_q)} H^k_x \left(\overline{X}(w), \mathcal{O}_{\overline{X}(w)} \right) = \mathrm{Ind}_{B^*(\mathbb{F}_q)}^{\mathrm{GL}_n(\mathbb{F}_q)} H^k_x \left(\overline{X}(w), \mathcal{O}_{\overline{X}(w)} \right),$$

for any fixed $x \in (\mathrm{GL_n} / B^*)(\mathbb{F}_q)$.

ii. Note that $X(e)$ is a finite disjoint union of the $\mathbb{F}_q$-rational points of GL_n / B^*, so $\overline{X}(e) = X(e)$. Then

$$H^k_{\overline{X}(e)} \left(\overline{X}(w), \mathcal{O}_{\overline{X}(w)} \right) = \bigoplus_{x \in (\mathrm{GL}_n / B^*)(\mathbb{F}_q)} H^k_x \left(\overline{X}(w), \mathcal{O}_{\overline{X}(w)} \right) = \mathrm{Ind}_{B^*(\mathbb{F}_q)}^{\mathrm{GL}_n(\mathbb{F}_q)} H^k_x \left(\overline{X}(w), \mathcal{O}_{\overline{X}(w)} \right).$$

iii. Now if $n=2$, $w=s$, then we have

$$\overline{X}(s) = X(s) \dot{\cup} \overline{X}(e).$$

Computing as above and we get

$$H^k_{\overline{X}(e)}\left(\overline{X}(s), \mathcal{O}_{\overline{X}(s)}\right) = \bigoplus_{x\in(\mathrm{GL}_2/B^*)(\mathbb{F}_q)} H^k_x\left(\overline{X}(s), \mathcal{O}_{\overline{X}(s)}\right) = \mathrm{Ind}_{B^*(\mathbb{F}_q)}^{\mathrm{GL}_2(\mathbb{F}_q)} H^k_x\left(\overline{X}(s), \mathcal{O}_{\overline{X}(s)}\right).$$

The computation of the local cohomology with support in $\overline{X}(e)$ is now reduced to the computation of $H^k_x\big(\overline{X}(s), \mathcal{O}_{\overline{X}(s)}\big)$ for $x \in (\mathrm{GL}_2/B^*)(\mathbb{F}_q)$.

B.2. **Local cohomology and some $\mathbb{P}^1$-bundles.** Let $G = \mathrm{GL}_n, w \in W$.

Lemma B.2. *Let $v, w, s \in W$ and let v be a subword of w in F^+. Then $\pi_1 : \overline{X}(sws) \to \overline{X}(ws)$ restricts to $\pi_1' : \overline{X}(svs) \to \overline{X}(vs)$, and $\pi_2 : \overline{X}(sws) \to \overline{X}(sw)$ restricts to $\pi_2' : \overline{X}(svs) \to \overline{X}(sv)$.*

Proof. Let $\ell(w) = r$. Recall that $\leq$ is the Bruhat order on W. We have $\overline{X}(svs)$ and $\overline{X}(sv)$, together with a morphism $\pi_1' : \overline{X}(svs) \to \overline{X}(vs)$ defined by $(B_0, ..., B_{r+1}, FB_0) \mapsto (B_1, ..., B_{r+1}, FB_1)$. Thus it agrees with π_1 and identifies with its restriction to $\overline{X}(svs)$.

Similarly, define $\pi_2' : \overline{X}(svs) \to \overline{X}(sv)$ by $(B_0, ..., B_{r+1}, FB_0) \mapsto (B_{r+1}, FB_1, ..., FB_{r+1})$. This also agrees with π_2 and identifies with its restriction to $\overline{X}(svs)$. □

Lemma B.3. *There are $\mathrm{GL}_n(\mathbb{F}_q)$-equivariant isomorphisms:*

$$H^k_{\overline{X}(vs)}\left(\overline{X}(ws), \mathcal{O}_{\overline{X}(ws)}\right) \cong H^k_{\overline{X}(vs)}\left(\overline{X}(sws), \mathcal{O}_{\overline{X}(sws)}\right)$$
$$H^k_{\overline{X}(sv)}\left(\overline{X}(sw), \mathcal{O}_{\overline{X}(sw)}\right) \cong H^k_{\overline{X}(sv)}\left(\overline{X}(sws), \mathcal{O}_{\overline{X}(sws)}\right).$$

Proof. Consider the Grothendieck spectral sequence for the functors $\Gamma_{\overline{X}(sv)}$ and $\pi_{2,*}$ for the sheaf $\mathcal{O}_{\overline{X}(sws)}$:

$$E_2^{p,q} = \left(R^p\Gamma_{\overline{X}(sv)} \circ R^q\pi_{2,*}\right)\big(\mathcal{O}_{\overline{X}(sws)}\big) \implies R^{p+q}\left(\Gamma_{\overline{X}(sv)} \circ \pi_{2,*}\right)\big(\mathcal{O}_{\overline{X}(sws)}\big).$$

By Lemma B.2, the canonical restriction of π_2 to $\overline{X}(svs)$ is π_2'. Hence $\Gamma_{\overline{X}(sv)}(\overline{X}(sw), -) \circ \pi_{2,*}$ identifies with the functor $\Gamma_{\overline{X}(sv)}(\overline{X}(sws), -)$. Thus the spectral sequence becomes

$$E_2^{p,q} = \left(R^p\Gamma_{\overline{X}(sv)} \circ R^q\pi_{2,*}\right)\big(\mathcal{O}_{\overline{X}(sws)}\big) \implies R^{p+q}\Gamma_{\overline{X}(sv)}\big(\mathcal{O}_{\overline{X}(sws)}\big).$$

As in Proposition 3.6, we have $\pi_{2,*}\mathcal{O}_{\overline{X}(sws)} = \mathcal{O}_{\overline{X}(sw)}$ and $R^q\pi_{2,*}\mathcal{O}_{\overline{X}(sws)} = 0$ for $q > 0$. Thus we have

$$R^p\Gamma_{\overline{X}(sv)}\big(\pi_{2,*}\mathcal{O}_{\overline{X}(sws)}\big) \xrightarrow{\sim} R^p\Gamma_{\overline{X}(sv)}\big(\mathcal{O}_{\overline{X}(sws)}\big).$$

and

$$H^k_{\overline{X}(sv)}\left(\overline{X}(sw), \mathcal{O}_{\overline{X}(sw)}\right) \cong H^k_{\overline{X}(sv)}\left(\overline{X}(sws), \mathcal{O}_{\overline{X}(sws)}\right).$$

The case of $\Gamma_{\overline{X}(vs)}$ uses the same argument. □

Remark B.4. Let $w \in \hat{F}^+$. Recall that $\overline{X}(w)$ and $\overline{X}(R_i(w))$, $i = 1, 2$, are birationally equivalent. We would like to know whether local cohomology groups for the structure sheaf are birational invariants. One would then require a generalization of [CR11, Theorem 3.2.8] for local cohomology groups.

Finally, if one can apply the operators C, K, R to local cohomology functors and obtain a formula for local cohomology groups of $\overline{X}(w)$ for arbitrary $w \in W$ with a reduced expression $t_1 \cdots t_r$, $t_j \in S$, then it could be possible to use the stratification

$$\overline{X}(w) = X(t_1, \ldots, t_r) \dot{\bigcup} \left(\bigcup_{\substack{v \prec w \\ \ell(v)=\ell(w)-1}} \overline{X}(v) \right).$$

We may construct a sequence of local cohomology sheaves on $\overline{X}(w)$:

$$0 \to \mathcal{O}_{\overline{X}(w)} \to \bigoplus_{\substack{v \prec w \\ \ell(v)=\ell(w)-1}} \underline{\Gamma}_{\overline{X}(v)}\left(\mathcal{O}_{\overline{X}(w)}\right) \to \cdots$$
$$\cdots \to \bigoplus_{\substack{v \prec w \\ \ell(v)=\ell(w)-i}} \underline{\Gamma}_{\overline{X}(v)}\left(\mathcal{O}_{\overline{X}(w)}\right) \to \cdots \to \underline{\Gamma}_{\overline{X}(e)}\left(\mathcal{O}_{\overline{X}(w)}\right) \to 0. \quad \text{(B.1)}$$

The associated spectral sequence would be

$$E_1^{i,j} = \bigoplus_{\substack{v \prec w \\ \ell(v)=\ell(w)-i}} H^j_{\overline{X}(v)}\left(\overline{X}(w), \mathcal{O}_{\overline{X}(w)}\right). \quad \text{(B.2)}$$

If we denote

$$Y := \bigcup_{\substack{v \prec w \\ \ell(v)=\ell(w)-1}} \overline{X}(v),$$

then there is a canonical immersion

$$\underline{\Gamma}_Y\left(\mathcal{O}_{\overline{X}(w)}\right) \hookrightarrow \mathcal{O}_{\overline{X}(w)}.$$

Thus if the sequence (B.1) is an acyclic complex, then the spectral sequence (B.2) would compute the local cohomology with support in Y, and therefore would give a description for the cohomology groups $H^k(X(w), \mathcal{O}_{X(w)})$. This strategy is used in [Orl08] for Drinfeld half space over a finite extension K of $\mathbb{Q}_p$.

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