Character Triple Conjecture

Towards the inductive condition for Dade's Conjecture in nondefining characteristic

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To my mother Carla

Ma veda, il problema secondo me è un altro. Perchè lei vuole fare teatro? Per arricchire? Per diventare famoso? Perchè le interessa questo mezzo espressivo! Comunque lei farà teatro, perchè quello che si vuol fare a vent'anni lo si fa nella vita.

Nanni Moretti, *Io sono un autarchico*

Contents

Declaration

I hereby declare that this thesis has been composed by myself, that the work presented herein is the result of my own original research except where explicitly stated otherwise and that all external sources of information have been acknowledged as references. I also declare that this thesis was never previously submitted for an academic degree at the Bergische Universität Wuppertal or any other university.

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1

Introduction

The main object of investigation of this thesis is Dade's Projective Conjecture [Dad94, Conjecture 15.5]. This is part of a series of conjectures in representation theory of finite groups aimed at explaining the so called Global-Local principle, according to which the representation theory of a finite group should be determined by the representation theory of its p-local subgroups, for p a varying prime number (see Section 2.4 for further details). The statement of Dade's Projective Conjecture is somewhat involved and we refer the reader to Section 2.5 for a detailed introduction. The importance of Dade's conjecture lies in the fact that it implies many of the other Global-Local conjectures (such as the McKay, Alperin–McKay, Alperin Weight and Brauer's Height Zero conjectures) and unifies them in a single statement. A direct proof of this conjecture seems to be completely out of reach at the present time and the only hope we have to solve this problem is by invoking the Classification of Finite Simple Groups. As is well known, the Classification states that every nonabelian finite simple group falls in one of the following families:

(i) Alternating groups of degree at least 5;

(ii) Finite groups of Lie type;

(iii) Sporadic groups.

The finite groups of Lie type, also known as finite reductive groups, include most of the finite simple groups and for this reason play a very important role in group theory and in representation theory of finite groups.

Since every finite group can be constructed by gluing together simple groups, it is often possible to reduce a group theoretic problem to a question on simple groups. Following this idea, Dade's Projective Conjecture has been reduced by Späth to a question on (quasi)simple groups [Spä17]. Unfortunately, this reduction theorem tell us that, in order to obtain Dade's Projective Conjecture for arbitrary finite groups, we need to prove a much stronger result for simple groups. This new statement, called Character Triple Conjecture, will be introduced in Section 3.5. In simple terms, this is a version of Dade's conjecture that is compatible with Clifford theory and with the action of automorphisms. Using the Character Triple Conjecture, one can formulate the *inductive condition* *for Dade's Conjecture* (see Definition 9.1.3) which, if true for every simple group, implies Dade's Projective Conjecture for every finite group (see Theorem 9.1.4).

Although, the Character Triple Conjecture was formulated for simple groups, its statement makes sense in a general context and is believed to hold for every finite group. Moreover, we suspect that the Character Triple Conjecture is actually the correct statement to reduce to simple groups. By this we mean that, supposedly, the Character Triple Conjecture will hold for every finite group if proved for quasisimple groups. Our first main result provides evidences in this direction. In Chapter 4 we initiate an analysis of a minimal counterexample to the Character Triple Conjecture, a fundamental step towards a possible reduction, and obtain as a consequence a proof for p-solvable groups. This result can be found in [Ros21].

Theorem 1.1. *Let G be a finite group and p a prime number. If G is p-solvable, then the Character Triple Conjecture holds for G with respect to p.*

The next step of the reduction process would require extending some advanced techniques of Külshammer and Puig on nilpotent blocks to include a compatibility with isomorphisms of character triples. Due to these obstructions, the above result is the best we can achieve at the moment.

As mentioned at the beginning of this introduction, Dade's Projective Conjecture implies many of the other Global-Local conjectures. In particular it has been shown in [Dad94, Theorem 18.14] that Dade's Projective Conjecture implies the Alperin–McKay Conjecture. More recently, Navarro has shown that the nonblockwise version of Dade's Ordinary Conjecture implies the McKay Conjecture, while Kessar and Linckelmann proved that Dade's Ordinary Conjecture implies the Alperin–McKay conjecture. It is then natural to ask whether similar implications hold between the inductive conditions for these conjectures. In Chapter 5, we state a general form of the inductive Alperin–McKay condition for arbitrary finite groups (see Conjecture 5.1.1) and show that this statement follows from the Character Triple Conjecture.

Theorem 1.2. *Let p be a prime number. If the Character Triple Conjecture holds for every p-block of every finite group, then the inductive Alperin–McKay condition (see Conjecture 5.1.1) holds for every p-block of every finite group.*

As a consequence of Theorem 1.1 and Theorem 1.2, it follows that Conjecture 5.1.1 holds for p-solvable groups. This result can also be deduced by the main theorem of [NS14b].

As mentioned above, in order to obtain Dade's Projective Conjecture via Späth's reduction theorem, we need to prove the inductive condition for Dade's Conjecture for simple groups. In the second part of this thesis we consider this problem for simple groups of Lie type in the nondefining characteristic. From now on, let $\mathbf{G}$ be a connected reductive group, $F : \mathbf{G} \to \mathbf{G}$ a Frobenius endomorphism associated to an $\mathbb{F}_q$-structure, with q a prime power, and denote by $\mathbf{G}^F$ the set of points of $\mathbf{G}$ fixed under the action of F. Let ℓ be a prime not dividing q and e the multiplicative order of q modulo ℓ (or q modulo 4 if $\ell = 2$). In the second part of the thesis blocks will always be considered with respect to the prime ℓ.

For our purpose, we first need to extend some results on generalized e-Harish-Chandra theory. This is a powerful tool to deal with modular representation theoretic problems for finite groups of Lie type in nondefining characteristic. In [CE99], blocks of finite groups of Lie type have been classified in terms of e-cuspidal pairs. According to [CE99, Theorem 4.1] (see Theorem 6.2.19), for $\ell \geq 7$ the ℓ-blocks of $\mathbf{G}^F$ are in bijection with the conjugacy classes of e-cuspidal pairs $(\mathbf{L}, \lambda)$, where λ is a so-called ℓ'-character of $\mathbf{L}^F$. Moreover, if the block B correspond to the pair $(\mathbf{L}, \lambda)$, then we can recover the set of ℓ'-characters belonging to B by the knowledge of Deligne–Lusztig induction from $(\mathbf{L}, \lambda)$. However, this result does not tell us how to obtain all characters belonging to B. In order to fix this problem, in Chapter 7, we extend Cabanes–Enguehard's result to characters lying in rational Lusztig series associated to ℓ-singular semisimple elements and show that the set of all characters belonging to a block can be recovered, by using Deligne–Lusztig induction, from a unique set of e-cuspidal pairs (up to conjugation).

Theorem 1.3. *Assume Hypothesis 7.2.7 and let B be a block of $\mathbf{G}^F$. Then there exist unique (up to conjugation) e-cuspidal pairs $(\mathbf{L}_1, \lambda_1), \dots, (\mathbf{L}_n, \lambda_n)$ such that*

$$\mathrm{Irr}(B) = \coprod_{i=1}^{n} \mathcal{E}\left(\mathbf{G}^F, (\mathbf{L}_1, \lambda_1)\right),$$

where $\mathcal{E}(\mathbf{G}^F, (\mathbf{L}_i, \lambda_i))$ denotes the e-Harish-Chandra series of $\mathbf{G}^F$ associated with $(\mathbf{L}_i, \lambda_i)$.

We refer the reader to Chapter 7 for further details. Moreover, we remark that Hypothesis 7.2.7 is satisfied in most of the cases we are interested in (see Remark 7.2.8).

Using the above result on e-Harish-Chandra theory, we can then proceed towards our main problem: proving the inductive condition for Dade's Conjecture. In a first step, we give a reformulation of the Character Triple Conjecture tailored to finite groups of Lie type (see Proposition 9.2.10). This extends work of Broué, Fong and Srinivasan on Dade's Projective Conjecture for unipotent blocks [BFS14] which provides a link between ℓ-elementary abelian subgroups and e-split Levi subgroups. Again inspired by the work of Broué, Fong and Srinivasan, we then show how the new reformulation reduces to proving the existence of certain bijections predicted by e-Harish-Chandra theory.

Condition 1.4. *Let $(\mathbf{L}, \lambda)$ be an e-cuspidal pair of $\mathbf{G}$ and denote by $\mathrm{Aut}(\mathbf{G}^F)_{(\mathbf{L},\lambda)}$ its stabilizer. Then there exists a defect preserving $\mathrm{Aut}(\mathbf{G}^F)_{(\mathbf{L},\lambda)}$-equivariant bijection*

$$\Omega^{\mathbf{G}}_{(\mathbf{L},\lambda)} : \mathcal{E}\left(\mathbf{G}^F, (\mathbf{L}, \lambda)\right) \to \mathrm{Irr}\left(\mathbf{N}_{\mathbf{G}}(\mathbf{L})^F \mid \lambda\right)$$

that preserves $\mathbf{G}^F$-block isomorphisms of character triples (see Definition 3.3.6).

The precise statement can be found in Condition 9.1. Notice that, for unipotent 1-cuspidal pairs, bijections similar to the one required in Condition 1.4 can be deduced by [BMM93, Theorem 3.2] together with [Lus84, Theorem 8.6] and [Gec93, Corollary 2]. Having introduced Condition 1.4, we can now state the main result of Chapter 9 as follows (see Theorem 9.2).

Theorem 1.5. *Assume Hypothesis 9.2.11 and suppose that Condition 1.4 holds for every irreducible rational component of every e-split Levi subgroup of $\mathbf{G}$. If $\mathbf{G}^F/\mathbf{Z}(\mathbf{G}^F)$ is a nonabelian simple group with universal covering group $\mathbf{G}^F$, then the inductive condition for Dade's Conjecture holds for $\mathbf{G}^F/\mathbf{Z}(\mathbf{G}^F)$ and the prime ℓ.*

The importance of Theorem 1.5 lies in the fact that the bijections required by Condition 1.4 are closely related to the bijections considered in the proof of the inductive condition for the McKay, the Alperin–McKay and the Alperin weight conjectures. Reducing the inductive condition for Dade's Conjecture to the existence of such bijections allows us to use the techniques developed to deal with these more established inductive conditions. See, for instance, [Spä12], [CS13], [CS15], [CS17a], [CS17b], [CS19], [BS20b], [CSFS] and [BS20a].

Thanks to Theorem 1.5, we are now left with the problem of checking Condition 1.4. We consider this problem in Chapter 10. To start, we prove a criterion for Condition 1.4 similar to the ones proved in [Spä12] for the inductive McKay condition, in [CS15] and [BS20b] for the inductive Alperin–McKay condition and in [BS20a] for the inductive Alperin weight condition. This criterion (see Theorem 10.1.8) is roughly divided into two parts: the first part requires the existence of a character bijection with good properties (for groups with connected center), the second part requires some conditions on the extendibility of characters of e-split Levi subgroups. In Chapter 8, we deal with the first part of the criterion and show that the needed bijections can be constructed by assuming some further conditions on character extendibility (see Corollary 8.2). Combining Theorem 10.1.8 with Corollary 8.2 we obtain a final reduction of Condition 1.4 to questions on character extendibility (see Theorem 10.2). In doing so, we exhaust the theoretic machinery available at the present time and we are left with certain technical extendibility requirements for characters of e-split Levi subgroups. We also mention that these requirements are analogous to certain conditions needed for the proof of the inductive condition for the McKay, the Alperin–McKay and the Alperin weight conjectures. This last remaining problem is part of an important ongoing project in representation theory of finite groups of Lie type and has been checked in some partial cases (see [BS20b] and [Bro]). Using these results, we finally obtain Condition 1.4 for some cases in types $\mathbf{A}_n$ and $\mathbf{C}_n$ (see Corollary 10.3 and Corollary 10.4).

First, by applying the main results of [BS20b] we obtain the following corollary.

Corollary 1.6. *Let ℓ be a prime, q a prime power and $\epsilon \in \{\pm 1\}$ such that $\ell \nmid 3q(q-\epsilon)$. Set $\mathbf{G} := \mathrm{SL}_n(\overline{\mathbb{F}_q})$, $G := \mathrm{SL}_n(\epsilon q)$ and assume that G is the universal covering group of $\mathrm{PSL}_n(\epsilon q)$. Let B be an ℓ-block of G such that, either*

(i) $\mathrm{Out}(G)_{\mathcal{B}}$ is abelian, where $\mathcal{B}$ is the $\mathrm{GL}_n(\epsilon q)$-orbit of B; or

(ii) B is unipotent; or

(iii) B has maximal defect.

Then Condition 1.4 holds for G with respect to every e-cuspidal pair $(\mathbf{L}, \lambda)$ of $(\mathbf{G}, F)$, with λ an ℓ'-character, such that $\mathrm{bl}(\lambda)^{\mathbf{G}^F} = B$ via Brauer's induction, where $\mathrm{bl}(\lambda)$ is the ℓ-block of $\mathbf{L}^F$ to which λ belongs.

Since the outer automorphism group of any simple simply connected group of Lie type $\mathbf{C}_n$ is always abelian, we obtain the following corollary by applying the main results of [Bro].

Corollary 1.7. *Let ℓ be a prime and q a prime power such that $\ell \nmid 6q$. Set $\mathbf{G} := \mathrm{Sp}_{2n}(\overline{\mathbb{F}_q})$, $G := \mathrm{Sp}_{2n}(q)$ and assume that G is the universal covering group of $\mathrm{PSp}_{2n}(q)$. Then Condition 9.1 holds for G with respect to every (e, ℓ')-cuspidal pair $(\mathbf{L}, \lambda)$ of $\mathbf{G}$.*

Although partial, these results show that the path initiated in this thesis towards a proof of the inductive condition for Dade's Conjecture is promising and might eventually lead to a proof of Dade's Projective Conjecture. Nonetheless, still a lot of work remains to be done and this will be the focus of the author for years to come.

In addition to the above mentioned work towards the inductive condition for Dade's Conjecture, many of the obtained results can also be used to deduce the nonblockwise version of Dade's Projective Conjecture for groups of Lie type from certain conditions on character extendibility. Clearly, the conditions required in this case are much simpler then the ones needed for the inductive condition for Dade's Conjecture. Analogous results are obtained in this case: In Chapter 9 we deduce the nonblockwise version of Dade's Projective Conjecture from a simplified version of Condition 1.4 (see Theorem 9.2.23 and Condition 9.2.22), then in Chapter 10 we give a criterion for Condition 9.2.22 (see Theorem 10.1.3) and show how the nonblockwise version of Dade's Projective Conjecture reduces to some extendibility conditions (Theorem 10.1).

2

Preliminaries

In this chapter, attempting to make this thesis as self-contained as possible, we introduce some notation and background terminology that will help the reader following the subsequent chapters. On the other hand, most of the material presented here can be found in standard textbooks such as [Isa76], [NT89], [Nav98]. We assume the reader has some familiarity with these basic notions.

2.1 Notation

In this thesis $\mathbb{N} := \{0, 1, 2, 3, \dots\}$ denotes the set of natural numbers including 0. We denote by $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{C}$ the ring of integers, the field of rational numbers and the field of complex numbers. For a prime number p, the field of p-adic numbers is denoted by $\mathbb{Q}_p$. An algebraic closure of a field $\mathbb{K}$ is denoted by $\overline{\mathbb{K}}$. The multiplicative group of the field $\mathbb{K}$ is denoted by $\mathbb{K}^\times$ and the additive group by $\mathbb{K}^+$. Moreover, for every integral domain R, we denote by $\mathrm{Frac}(R)$ the field of fractions of R. If q is a power of a prime number, then $\mathbb{F}_q$ denotes the field with q elements. For $n, m \in \mathbb{Z}$, the greatest common divisor is denoted by $\gcd(m, n) := (m, n)$ while the lowest common multiple by $\mathrm{lcm}(m, n) := [m, n]$. Moreover, if π is a set of primes and $n \in \mathbb{N}$, then n_π denotes the largest divisor of n whose prime divisors are contained in π. If $n = n_\pi$, then n is called a π-number. Recall that π' denotes the set of all prime numbers that are not contained in π. For a matrix M, we denote by $\mathrm{Tr}(M)$ the trace of M. If X and Y are two sets, then we denote by $X \coprod Y$ the disjoint union of X and Y. Moreover, if $f : X \to Y$ is a map and X' is a subset of X, then $f_{X'} : X' \to Y$ denotes the restriction of f to X'.

Standard group theoretic notation is considered. For a group X and a subgroup Y of X we write $Y \leq X$. If Y is normal or characteristic in X, then we write $Y \trianglelefteq X$ and $Y \leq_{\mathrm{ch}} X$ respectively. The centralizer and the normalizer of Y in X are denoted by $\mathbf{C}_X(Y)$ and $\mathbf{N}_X(Y)$ respectively. Moreover $\mathbf{Z}(X) := \mathbf{C}_X(X)$ is the center of X. Similarly, for an algebra A over a field $\mathbb{F}$, we denote its center by $\mathbf{Z}(A)$. If $x \in X$, then the inner automorphism of X induced by x is denoted by σ_x. Notice that for $x, y \in X$, we define $x^y := y^{-1}xy$ and ${}^y x := x^{y^{-1}} := yxy^{-1}$. The conjugacy class of x is denoted by $\mathfrak{Cl}_X(x) := \{x^y \mid y \in X\}$. As is well known, the set of inner automorphisms

$\mathrm{Inn}(X)$ is a normal subgroup of the automorphism group $\mathrm{Aut}(X)$. The outer automorphism group is the quotient $\mathrm{Out}(X) := \mathrm{Aut}(X)/\mathrm{Inn}(X)$.

If X is a finite set, we denote the size of X by $|X|$. When X is a group and $Y \leq X$, then $|X : Y|$ is the index of Y in X. For a set of prime numbers π, we define $\mathbf{O}_\pi(X)$ as the largest normal π-subgroup of X. If $x \in X$, then we denote by $o(x)$ the order of x. This coincides with the size of the cyclic group $\langle x \rangle$ generated by x. Recall that there exist unique elements $x_\pi, x_{\pi'} \in \langle x \rangle$ such that $x = x_\pi x_{\pi'}$ and $o(x_\pi)$, $o(x_{\pi'})$ are a π-number and a π'-number respectively. With this notation, an element $x \in X$ is called π-**regular** (resp. π-**singular**) if $x_\pi = 1$ (resp. $x_\pi \neq 1$).

2.2 Representations and characters

A complex **representation** of a finite group G is a group homomorphism

$$\mathfrak{X} : G \to \mathrm{GL}_n(\mathbb{C}),$$

for some positive integer n called the **degree** of the representation. Taking the trace function of a representation, we obtain a **character**

$$\begin{aligned} \chi : G &\to \mathbb{C} \\ g &\mapsto \mathrm{Tr}\left(\mathfrak{X}(g)\right). \end{aligned}$$

In this case, we say that χ is afforded by the representation $\mathfrak{X}$. If χ is afforded by two representations $\mathfrak{X}$ and $\mathfrak{Y}$, then the representations $\mathfrak{X}$ and $\mathfrak{Y}$ are **similar**, i.e. there exists $P \in \mathrm{GL}_n(\mathbb{C})$ such that $\mathfrak{X}(g) = P\mathfrak{Y}(g)P^{-1}$ for every $g \in G$. It follows by the definition that characters are **class functions**: these are those maps $G \to \mathbb{C}$ which are constant on G-conjugacy classes. The evaluation of χ at the neutral element gives the degree of the character $\chi(1) = n$. A character is called **irreducible** if it cannot be expressed as a sum of two characters. An example of irreducible characters are the **linear** characters, that is those characters of degree one. The easiest example of a (linear) character is the **trivial character** which we denote by 1_G. The set of irreducible characters of G is denoted by $\mathrm{Irr}(G)$ and forms a basis for the complex vector space of class functions. For two class functions φ, ψ of G, the usual inner product is denoted by $[\varphi, \psi]$.

Every character is a nonzero $\mathbb{N}$-linear combination of irreducible characters, while an integer linear combination of irreducible characters is called a **generalized character**. The set of generalized characters is therefore denoted by $\mathbb{Z}\mathrm{Irr}(G)$. Let $\chi \in \mathbb{Z}\mathrm{Irr}(G)$ and write $\chi = \sum_i \alpha_i \vartheta_i$, where $\vartheta_i \in \mathrm{Irr}(G)$ and $\alpha_i \in \mathbb{Z}$. The irreducible **constituents** of χ are those irreducible characters $\vartheta_i \in \mathrm{Irr}(G)$ for which $\alpha_i \neq 0$. Notice that $\alpha_i = [\chi, \vartheta_i]$ by the first orthogonality relation. The set of irreducible constituents of χ is denoted by $\mathrm{Irr}(\chi)$.

Let $H \leq G$. For every $\chi \in \mathrm{Irr}(G)$, the **restriction** of χ to H is a character denoted by χ_H. On the other hand, if $\psi \in \mathrm{Irr}(H)$, then **induction** yields a character ψ^G of G. In the latter case $\mathrm{Irr}(G \mid \psi)$ denotes the set of irreducible constituents of ψ^G. Of particular interest is the case of normal subgroups $N \trianglelefteq G$. In this situation the group G acts by conjugation on $\mathrm{Irr}(N)$: set $\vartheta^g(x) := \vartheta({}^g x)$ for every $\vartheta \in \mathrm{Irr}(N)$, $g \in G$ and $x \in N$. We denote by G_ϑ the stabilizer of $\vartheta \in \mathrm{Irr}(N)$ in G and by $\mathrm{Irr}_G(N)$ the set of G-**invariant** irreducible characters of N. If $\vartheta \in \mathrm{Irr}(N)$

and $\chi \in \mathrm{Irr}(G \mid \vartheta)$, then we say that χ **lies above** ϑ and that ϑ **lies below** χ. According to the **Clifford correspondence** there exists a bijection

$$\mathrm{Irr}(G_\vartheta \mid \vartheta) \to \mathrm{Irr}(G \mid \vartheta)$$

given by induction of characters. If $\chi \in \mathrm{Irr}(G \mid \vartheta)$, then the unique character $\psi \in \mathrm{Irr}(G_\vartheta \mid \vartheta)$ such that $\psi^G = \chi$ is called the Clifford correspondent of χ above ϑ.

Let χ be a character of G. We define the **kernel** of χ as $\mathrm{Ker}(\chi) \coloneqq \{g \in G \mid \chi(g) = \chi(1)\}$. If χ is afforded by the representation $\mathfrak{X}$, then $\mathrm{Ker}(\chi) = \mathrm{Ker}(\mathfrak{X})$ and therefore is a normal subgroup of G. Let $N \trianglelefteq G$ such that $N \leq \mathrm{Ker}(\chi)$. Setting $\overline{\chi}(Ng) \coloneqq \chi(g)$ for every $g \in G$, we obtain a well defined character of G/N. Conversely, every character $\overline{\chi}$ of G/N defines a character χ of G with $N \leq \mathrm{Ker}(\chi)$. We will usually identify the characters of G whose kernels contain N with the characters of G/N. This process is often referred to as **deflation** and **inflation** of characters.

Let $\mathbb{C}G$ be the **group algebra** of G over $\mathbb{C}$, that is the set of formal sums $\sum_g \alpha_g g$, where g runs over the elements of G and $\alpha_g \in \mathbb{C}$. Notice that, over the complex numbers, the study of characters is tantamount to that of representations or equivalently of $\mathbb{C}G$-modules. This is due to Maschke's theorem (see [Isa76, Theorem 1.9]). For every subset $S \subseteq G$, we denote by S^+ the sum of its elements in the group algebra $\mathbb{C}G$. Then, the set of elements K^+, where K runs over the conjugacy classes of G, forms a basis for $\mathbf{Z}(\mathbb{C}G)$. If $\chi \in \mathrm{Irr}(G)$, we can define an algebra homomorphism $\mathbf{Z}(\mathbb{C}G) \to \mathbb{C}$ by setting

$$\omega_\chi(\mathfrak{Cl}_G(x)^+) \coloneqq \frac{|G : \mathbf{C}_G(x)|\chi(x)}{\chi(1)}$$

for every conjugacy class $\mathfrak{Cl}_K(x)$ of G, and then extending it by linearity. Moreover, all the algebra homomorphism $\mathbf{Z}(\mathbb{C}G) \to \mathbb{C}$ are of this form and $\omega_\chi = \omega_\psi$ if and only if $\chi = \psi$. These morphisms are called **central characters** and can be used to define blocks.

The action of G on $\mathrm{Irr}(N)$ is a special case of a more general construction. For this, suppose that $H \leq G$ and let $\alpha \in \mathrm{Aut}(G)$. If $\psi \in \mathrm{Irr}(H)$, then $\psi^\alpha \in \mathrm{Irr}(H^\alpha)$ is the character obtained by setting $\psi^\alpha(\alpha(h)) \coloneqq \psi(h)$ for every $h \in H$.

We conclude by recalling the **Glauberman–Isaacs correspondence**. Let A be a finite group acting via automorphisms on G and such that $(|G|, |A|) = 1$. If $\mathrm{Irr}_A(G)$ denotes the set of A-invariant characters of G, then there exists a canonical bijection

$$f_A : \mathrm{Irr}_A(G) \to \mathrm{Irr}(\mathbf{C}_G(A)).$$

If $\chi \in \mathrm{Irr}_A(G)$, then $f_A(\chi)$ is called the Glauberman–Isaacs correspondent of χ over A. Notice that, by Feit–Thompson's theorem [FT63], either G or A must be solvable. The bijection was proved by Glauberman when A is solvable [Gla68] and by Isaacs when G is solvable in his pioneering work [Isa73]. Whenever A is solvable, we will often refer to this bijection simply as the Glauberman correspondence.

2.3 Blocks

Let $\mathcal{R}$ be the ring of algebraic integers in $\mathbb{C}$, p a prime and fix a maximal ideal $\mathcal{M}$ of $\mathcal{R}$ containing $p\mathbb{Z}$. The quotient $\mathbb{F} = \mathcal{R}/\mathcal{M}$ is a field of prime characteristic equal to p and we denote by $^* : \mathcal{R} \to \mathbb{F}$ the canonical projection.

The group algebra $\mathbb{F}G$ admits a decomposition

$$\mathbb{F}G = B_1 \oplus \cdots \oplus B_n \tag{2.3.1}$$

into twosided indecomposable ideals called the p-**blocks** of G. We will often omit the prime p when no confusion arises. This corresponds to a decomposition of the identity element

$$1_{\mathbb{F}G} = e_{B_1} + \cdots + e_{B_n}.$$

The e_{B_i}'s are the central primitive idempotents of $\mathbb{F}G$ and are also often referred to as the blocks of G: in fact the blocks can be recovered from the central primitive idempotents as $B_i = e_{B_i}\mathbb{F}G$. We denote the set of blocks of G by $\mathrm{Bl}(G) = \{B_1, \dots, B_n\}$.

Let $\chi \in \mathrm{Irr}(G)$ and consider the associated central character ω_χ. By [Isa76, Theorem 3.7], we know that $\omega_\chi(\mathfrak{Cl}_G(x)^+) \in \mathcal{R}$ for every $x \in G$. Then, we can define a morphism of $\mathbb{F}$-algebras

$$\lambda_\chi : \mathbf{Z}(\mathbb{F}G) \to \mathbb{F}$$

by setting $\lambda_\chi(\mathfrak{Cl}_G(x)^+) := \omega_\chi(\mathfrak{Cl}_G(x)^+)^*$ for every conjugacy class $\mathfrak{Cl}_G(x)$ of G. This induces an equivalence relation on $\mathrm{Irr}(G)$ defined, for every $\chi, \psi \in \mathrm{Irr}(G)$, by $\chi \sim \psi$ if and only if $\lambda_\chi = \lambda_\psi$. Notice that, for $\chi, \psi \in \mathrm{Irr}(G)$, we have $\lambda_\chi = \lambda_\psi$ if and only if $\lambda_\chi(\mathfrak{Cl}_G(x)^+) = \lambda_\psi(\mathfrak{Cl}_G(x)^+)$ for every p-regular element $x \in G$ (see the argument of [NT89, Theorem 3.6.24 (i)]).

It turns out that for every equivalence class $[\chi]_\sim$ there exists a unique block $B \in \mathrm{Bl}(G)$ such that $\lambda_\psi(e_B) \neq 0$ for every $\psi \in [\chi]_\sim$. In this case we say that the character ψ **belongs** to the block B. Conversely, if $\chi, \psi \in \mathrm{Irr}(G)$ belong to the same block B, then $\lambda_\chi = \lambda_\psi$. Therefore, if we denote by $\mathrm{Irr}(B)$ the set of irreducible characters belonging to B, then $\mathrm{Irr}(B)$ coincides with an equivalence class $[\chi]_\sim$. Then we write $\lambda_B := \lambda_\psi$ for any $\psi \in [\chi]_\sim$. Every algebra homomorphism $\mathbf{Z}(\mathbb{F}G) \to \mathbb{F}$ is of the form λ_B for some block $B \in \mathrm{Bl}(G)$. It follows from the above discussion that there exists a partition

$$\mathrm{Irr}(G) = \coprod_{B \in \mathrm{Bl}(G)} \mathrm{Irr}(B).$$

For $\chi \in \mathrm{Irr}(G)$, we denote by $\mathrm{bl}(\chi)$ the unique block of G to which χ belongs. The block $B_0 := \mathrm{bl}(1_G)$ is called the **principal block** of G.

Associated to every $B \in \mathrm{Bl}(G)$, there is a G-conjugacy class $\delta(B)$ of p-subgroups D called the **defect groups** of B. The p-**defect** of B is the nonnegative integer $d(B)$ defined by $|D| = p^{d(B)}$. For a fixed p-subgroup $P \leq G$, we denote by $\mathrm{Bl}(G \mid P)$ the set of blocks $B \in \mathrm{Bl}(G)$ such that $P \in \delta(B)$. Next, recalling that $\chi(1)$ divides $|G|$ for every $\chi \in \mathrm{Irr}(G)$, we define the p-**defect** of χ to be the nonnegative integer $d(\chi)$ defined by $p^{d(\chi)} := |G|_p/\chi(1)_p$. Similarly, the p-**residue** of χ is the nonnegative integer $r(\chi)$ defined by $r(\chi) = |G|_{p'}/\chi(1)_{p'}$. It can be shown that

$d(B) = \max\{d(\chi) \mid \chi \in \mathrm{Irr}(B)\}$ and the nonnegative integer $\mathrm{ht}(\chi) := d(B) - d(\chi)$ is called the p-**height** of χ. For a fixed $d \geq 0$ we denote by $\mathrm{Irr}^d(B)$ the set of irreducible character $\chi \in \mathrm{Irr}(B)$ with defect $d(\chi) = d$. Moreover, $\mathrm{Irr}_0(B)$ denotes the set of irreducible characters belonging to B with height zero.

Let $H \leq G$ and consider a block $b \in \mathrm{Bl}(H)$. We define a map $\lambda_b^G : \mathbf{Z}(\mathbb{F}G) \to \mathbb{F}$ via $\lambda_b^G(\mathfrak{Cl}_G(x)^+) := \lambda_b((\mathfrak{Cl}_G(x) \cap H)^+)$ for every conjugacy class $\mathfrak{Cl}_G(x)$ of G. If λ_b^G is an algebra homomorphism, then there exists a unique block of G, called the **induced block**, denoted by b^G such that $\lambda_b^G = \lambda_{b^G}$. Block induction from H to G is always defined, for instance, when there exists a p-subgroup $D \leq G$ such that $D\mathbf{C}_G(D) \leq H \leq \mathbf{N}_G(D)$ (see [Nav98, Theorem 4.14]). The First Main Theorem of Brauer (see [Nav98, Theorem 4.12]) shows that there exists a bijection

$$\mathrm{Bl}(\mathbf{N}_G(D) \mid D) \to \mathrm{Bl}(G \mid D)$$

given by block induction. This is also known as the **Brauer correspondence**.

Consider now $N \trianglelefteq G$, $b \in \mathrm{Bl}(N)$ and $B \in \mathrm{Bl}(G)$. We say that B **covers** b (and b is **covered** by B) if there exists $\chi \in \mathrm{Irr}(B)$ and $\vartheta \in \mathrm{Irr}(b)$ such that χ lies above ϑ. Then, we denote by $\mathrm{Bl}(G \mid b)$ the set of blocks of G covering b. As for characters, the group G acts by conjugation on $\mathrm{Bl}(N)$ and we denote by G_b the stabilizer of $b \in \mathrm{Bl}(N)$ under this action. Notice that, if $\vartheta \in \mathrm{Irr}(b)$ and $g \in G$, then $\vartheta^g \in \mathrm{Irr}(b^g)$ and hence $G_\vartheta \leq G_b$. The blockwise analogue to the Clifford correspondence is the **Fong–Reynolds correspondence** according to which we have a bijection

$$\mathrm{Bl}(G_b \mid b) \to \mathrm{Bl}(G \mid b)$$

given by induction of blocks. If $B \in \mathrm{Bl}(G \mid b)$, then the unique block $C \in \mathrm{Bl}(G_b \mid b)$ such that $C^G = B$ is called the Fong–Reynolds correspondent of B over b.

The compatibility between blocks covering and the Brauer correspondence was proved by Harris and Knörr in [HK85]. According to their theorem, if $b \in \mathrm{Bl}(N \mid D)$ has Brauer correspondent b', then block induction gives a bijection

$$\mathrm{Bl}(\mathbf{N}_G(D) \mid b') \to \mathrm{Bl}(G \mid b)$$

such that $\delta(B') \subseteq \delta(B'^G)$ for every $B' \in \mathrm{Bl}(\mathbf{N}_G(D) \mid b')$.

2.3.1 A consequence of the Harris–Knörr theorem

Here, we collect some consequences of the Harris–Knörr theorem that will be used in the sequel.

Lemma 2.3.1. *Let $N \trianglelefteq G$ and P be a p-subgroup of N. Consider a block $b \in \mathrm{Bl}(N \mid P)$ and its Brauer first main correspondent $b' \in \mathrm{Bl}(\mathbf{N}_N(P) \mid P)$. Let $B' \in \mathrm{Bl}(\mathbf{N}_G(P))$ and set $B := (B')^G$. Then B' covers b' if and only if B covers b.*

Proof. The result follows from the proof of the Harris–Knörr theorem [HK85]. □

Next, we apply the above lemma in a particular case given by the Glauberman correspondence.

Corollary 2.3.2. *Let N be a normal p'-subgroup of G, P be a p-subgroup of G and $\mathbf{C}_G(P) \leq H \leq \mathbf{N}_G(P)$. Consider $\mu \in \mathrm{Irr}_P(N)$ and set $\mu' := f_P(\mu) \in \mathrm{Irr}(\mathbf{C}_N(P))$. If $B' \in \mathrm{Bl}(H)$, then B' covers $\mathrm{bl}(\mu')$ if and only if $(B')^{NH}$ covers $\mathrm{bl}(\mu)$. Moreover, if μ is G-invariant, then B' covers $\mathrm{bl}(\mu')$ if and only if $(B')^G$ covers $\mathrm{bl}(\mu)$.*

Proof. Let b' be the unique block of $\mathbf{N}_{NP}(P)$ that covers $\mathrm{bl}(\mu')$, b the unique block of NP that covers $\mathrm{bl}(\mu)$ (see [Nav98, Corollary 9.6]) and notice that b and b' are Brauer first main correspondents over P. Now, $\mathrm{Bl}(NH \mid \mathrm{bl}(\mu)) = \mathrm{Bl}(NH \mid b)$ and $\mathrm{Bl}(H \mid \mathrm{bl}(\mu')) = \mathrm{Bl}(H \mid b')$ and, since $H = \mathbf{N}_{NH}(P)$, Lemma 2.3.1 implies that B' covers $\mathrm{bl}(\mu')$ if and only if $(B')^{NH}$ covers $\mathrm{bl}(\mu)$. Moreover, if μ is G-invariant, then $\mathrm{bl}(\mu)$ is covered by $(B')^{NH}$ if and only if it is covered by $(B')^G$. □

2.4 Global-Local Counting Conjectures

Consider a finite group G and fix a prime number p. The p-structure of G gives rise to a collection of p-local subgroups: these include the nontrivial p-subgroups of G together with their normalizers and centralizers as well as their intersections and products. Opposed to the p-local subgroups, the group G plays the role of a global object. Hinted by many known and conjectural results, the Global-Local principle in representation theory of finite groups has quickly become the leading object of investigation in the field comprehending a network of deep and interconnected statements. In an extremely naive and vague way, according to this principle, the p-representation theory of G is determined by the p-representation theory of its p-local subgroups. Here, the term p-representation theory has to be interpreted as any representation theoretic invariant affected by the choice of the prime p and therefore includes both concepts from p-modular and ordinary representation theory. For more details on this topic we refer the reader to [Nav18, Chapter 9] and [Cra19, Chapter 4]. From the group theoretic point of view, the interplay between the group structure of G and that of its p-local subgroups has been widely investigated and exploited for a much longer time. As is well known, this was the key to one of the premier achievements of twentieth century mathematics: the classification of finite simple groups.

In this thesis we restrict our attention to the so called Global-Local counting conjectures. The first of these statements was proposed by McKay in [McK72]. In this paper, he observed that the number of irreducible characters of odd degree of certain simple groups coincided with the number of irreducible characters of odd degree of the normalizers of Sylow 2-subgroups. After that, Isaacs proved in [Isa73] that the same observation was true for groups of odd order and with respect to every prime. This paper suggested that, perhaps, neither the simplicity of the group nor the restrictions on the prime were necessary and that a general statement would hold for every finite group. Recall that, for a finite group X, we denote by $\mathrm{Irr}_{p'}(X)$ the set of irreducible characters of X whose degree is not divisible by p. Then, the McKay Conjecture can be stated as follows.

Conjecture 2.4.1 (McKay Conjecture)**.** *Let G be a finite group and p a prime number. Then*

$$|\mathrm{Irr}_{p'}(G)| = |\mathrm{Irr}_{p'}(\mathbf{N}_G(P))|,$$

where P is a Sylow p-subgroup of G.

A few years later, Alperin introduced a generalization of the McKay Conjecture which involves p-blocks. For this, notice that a character $\chi \in \mathrm{Irr}(G)$ has p'-degree if and only if it belongs to a block B of maximal defect, i.e. $|G|_p = p^{d(B)}$, and $\mathrm{ht}(\chi) = 0$. This observation was used by Alperin to formulate the Alperin–McKay Conjecture in [Alp76].

Conjecture 2.4.2 (Alperin–McKay Conjecture)**.** *Let G be a finite group, p a prime number and consider a p-block B of G with defect group D. Then*

$$|\mathrm{Irr}_0(B)| = |\mathrm{Irr}_0(b)|,$$

where $b \in \mathrm{Bl}(\mathbf{N}_G(D))$ is the Brauer correspondent of B.

As mentioned before, the McKay Conjecture follows from the Alperin–McKay Conjecture by considering blocks of maximal defect. The Alperin-McKay Conjecture was proved by Olsson for symmetric groups [Ols76] and by Okuyama–Wajima [OW80] and Dade [Dad80] for p-solvable groups. However, it appears that the proofs in Dade's paper contain some gaps.

Alperin also proposed another conjecture of a slightly different flavour. For this, let B be a block and denote by $\ell(B)$ the number of irreducible $\mathbb{F}G$-modules, up to isomorphism, belonging to B. Next, define a p-**weight** of G to be a pair (Q, μ) with Q a p-subgroup of G and $\mu \in \mathrm{Irr}^0(\mathbf{N}_G(Q)/Q)$. We denote by $\mathcal{W}_p(G)$ the set of p-weights of G. Observe that, for every weight (Q, μ), the induced block $\mathrm{bl}(\mu)^G$ is defined and we say that (Q, μ) is a p-weight of B if $\mathrm{bl}(\mu)^G = B$. The set of p-weights of B is denoted by $\mathcal{W}_p(B)$. The conjugacy action of G induces an action on the set of weights and, since B is G-invariant, we also obtain an action of G on the set of B-weights. With this in mind, the Alperin Weight Conjecture can be stated as follows (see [Alp87]).

Conjecture 2.4.3 (Alperin Weight Conjecture)**.** *Let G be a finite group, p a prime number and consider a p-block B of G. Then*

$$\ell(B) = |\mathcal{W}_p(B)/G|,$$

where $\mathcal{W}_p(B)/G$ is the set of G-orbits on $\mathcal{W}_p(B)$.

The Alperin Weight Conjecture was proved by Cabanes for groups of Lie type in the defining characteristic [Cab88] (notice that this was proved before the conjecture was even published) and by Alperin and Fong for symmetric and general linear groups [AF90]. For p-solvable groups the result is attributed to Okuyama, although his proof seems to have some gaps (see the discussion in [Bar97, p.134]). The first published proof of this result was given by Isaacs and Navarro in [IN95].

It should now be clear why the above conjectures are called Global-Local counting conjectures: in all these statements, a certain global numerical invariant is determined by other local numerical invariants. Although, as we have seen, these conjectures hold in many cases, a general argument for all finite groups seems to be out of reach at the present time. Nonetheless, as it has been nicely expressed by Alperin in his review of [Dad94], all these statements are believed to be true:

> *"Proofs of all these results elude us still but the evidence for them is overwhelming and includes proofs of special cases and examples, derivation of known results from the*

conjectures as well as connections between all the conjectures. If the subject were physics and not mathematics all these special conjectures would be accepted truths."

After the proof of the classification of finite simple groups was completed, it seemed clear that a reduction to simple groups could provide a way of proving these conjectures. This was, for instance, proposed by Feit in [Fei80] for the (Alperin–)McKay Conjecture. Almost thirty years after Feit's comment, a reduction theorem for the McKay Conjecture was proved by Isaacs, Malle and Navarro [IMN07]. In their paper, the McKay Conjecture is reduced to a problem on (quasi-)simple groups. This requires proving a strong version of the McKay conjecture for universal covering groups of nonabelian finite simple groups called the *inductive McKay condition* (see [Spä18] and [Nav18, Chapter 10] for more details). This major breakthrough in the field was followed by other reduction theorems: for the nonblockwise Alperin Weight Conjecture by Navarro and Tiep [NT11], for the Alperin–McKay Conjecture by Späth [Spä13a], and for the blockwise Alperin Weight Conjecture by Späth [Spä13b].

Since then, the conditions on quasi-simple groups required by these reduction theorems have been checked for many families of simple groups. As a proof of the validity of this program, using the reduction theorem in [IMN07], Malle and Späth proved in [MS16] that the McKay Conjecture holds for $p = 2$. The inductive McKay condition has then been checked in many other cases, see [Mal08], [Spä12], [CS13], [CS17a], [CS17b] and [CS19]. At the present time, it only remains to prove the case of groups of Lie type $\mathbf{D}_n$ for p odd and different from the defining characteristic.

Fewer results have been proved for the inductive Alperin–McKay condition. The condition has been verified for groups of Lie type when $p \geq 5$ coincides with the defining characteristic [Spä13a, Theorem 8.4], for alternating groups and $p = 2$ [Den14], for groups of Lie type $\mathbf{A}_n$ and blocks of maximal defect (or unipotent) in the nondefining characteristic in [CS15] and for blocks of quasi-simple groups with cyclic defect in [KS16a] and [KS16b]. More recently, the inductive Alperin–McKay condition has been verified for some blocks of groups of Lie type $\mathbf{A}_n$ in nondefining characteristic in [BS20b] and, providing a reduction to quasi-isolated blocks, for all blocks in nondefining characteristic $p \geq 5$ in [Ruh21b]. The techniques used in the latter paper might lead to a proof of the Alperin–McKay Conjecture for $p = 2$, this is work in progress by Brough and Ruhstorfer. We also mention that most of the results of [BS20b] have been extended to groups of Lie type $\mathbf{C}_n$ in the upcoming paper [Bro]. These results can be used to verify the inductive Alperin–McKay condition for simple groups of Lie type $\mathbf{C}_n$.

We conclude this section by mentioning some known results on the inductive Alperin Weight condition. This condition has been checked for groups of Lie type in the defining characteristic [Spä13b], for blocks of quasi-simple groups with cyclic defect [KS16a] and [KS16b], for alternating, Suzuki and Ree groups [Mal14] and for groups of Lie type $\mathbf{C}_n(q)$ with q odd and $p = 2$ [FM]. Many preprints on this topic have recently been uploaded: we mention [AHL21] in which the condition is checked for groups of Lie type $\mathbf{F}_4$ and p odd not equal to the defining characteristic and a series of papers by Z. Feng, C. Li, Z. Li and J. Zhang trying to adapt the results of [CS15], [BS20b] and [Ruh21b] to the inductive Alperin Weight condition (among others, we mention [FLZ20] and [FLZ21]).

2.5 Dade's Projective Conjecture

Although at first sight the Alperin Weight Conjecture and the (Alperin–)McKay Conjecture seem unrelated, this is not the case thanks to work of Knörr and Robinson [KR89] where the Alperin Weight Conjecture was reformulated. To understand this equivalent statement we need to introduce some more notation. Let $\mathfrak{P}(G)$ be the set of p-**chains** of G: these are the chains $\mathbb{D} = \{D_0 < D_1 < \cdots < D_n\}$ where each D_i is a p-subgroup of G. The number $|\mathbb{D}| := n$ is called the **length** of the chain $\mathbb{D}$. Observe that the group G acts by conjugation on $\mathfrak{P}(G)$ and that the stabilizer $G_\mathbb{D}$ of a p-chain $\mathbb{D} \in \mathfrak{P}(G)$ is equal to $G_\mathbb{D} = \bigcap_i \mathbf{N}_G(D_i)$. According to [KR89, Lemma 3.2], whenever $\mathbb{D} \in \mathfrak{P}(G)$ and $b \in \mathrm{Bl}(G_\mathbb{D})$, the induced block b^G is defined and we can define the set

$$\mathrm{Irr}(B_\mathbb{D}) := \left\{\psi \in \mathrm{Irr}(G_\mathbb{D}) \;\middle|\; \mathrm{bl}(\psi)^G = B\right\}.$$

Now, Knörr–Robinson's reformulation can be stated as follows (see [KR89, Theorem 4.6]).

Theorem 2.5.1 (Knörr–Robinson)**.** *The Alperin Weight Conjecture holds for the prime p if and only if for every finite group G we have*

$$\sum_{\mathbb{D}\in\mathfrak{P}(G)/\sim_G} (-1)^{|\mathbb{D}|}|\mathrm{Irr}(B_\mathbb{D})| = 0$$

for every p-block B of G with $d(B) > 0$ and where $\mathfrak{P}(G)/\sim_G$ denotes a G-transversal in $\mathfrak{P}(G)$.

Using this reformulation, we can establish a connection between the Alperin–McKay Conjecture and the Alperin Weight Conjecture. In fact, by using [KM13, Theorem 1.1], it can be shown that these two statements are equivalent when considering blocks with abelian defect groups (see [KR89, Proposition 5.6]). A more far-reaching consequence of the Knörr–Robinson reformulation is the introduction of a unifying conjecture by Dade. First, define $\mathrm{Irr}^d(B_\mathbb{D}) := \mathrm{Irr}(B_\mathbb{D}) \cap \mathrm{Irr}^d(G_\mathbb{D})$ for every $\mathbb{D} \in \mathfrak{P}(G)$, $B \in \mathrm{Bl}(G)$ and $d \geq 0$. Extending Knörr and Robinson's idea, Dade introduced the following statement (see [Dad92, Conjecture 6.3]).

Conjecture 2.5.2 (Dade's Ordinary Conjecture)**.** *Let G be a finite group, p a prime number and B a block of G with $d(B) > 0$. If $\mathbf{O}_p(G) = 1$, then*

$$\sum_{\mathbb{D}\in\mathfrak{P}(G)/\sim_G} (-1)^{|\mathbb{D}|}|\mathrm{Irr}^d(B_\mathbb{D})| = 0$$

for every $d \geq 0$ and where $\mathfrak{P}(G)/\sim_G$ denotes a G-transversal in $\mathfrak{P}(G)$.

The importance of Dade's conjecture is that it implies both the (Alperin–)McKay Conjecture (see [KL19]) and the Alperin Weight Conjecture (see [Dad92, Theorem 8.3]). Moreover, long before the reduction theorem of Isaacs, Malle and Navarro, Dade tried to reduce his conjecture to a statement on quasisimple groups. In order to do so, in [Dad92], [Dad94] and [Dad97], he proposed a series of increasingly deeper statements with the aim of finding a version of his conjecture strong enough to hold for every finite group if proved for quasisimple groups. The candidate for this role was claimed to be Dade's Inductive Conjecture [Dad97, 5.8], in Dade's words:

> *"With a great amount of work it can be shown to hold for all finite groups if it holds whenever G is a nonabelian finite simple group."*

Although his reduction theorem has never been published, Dade deserves credit for foreseeing the possibility of proving such a result and for all the crucial work he made towards this achievement. Nowadays, we know that Dade's claim was in all likelihood correct. In fact, a reduction theorem for Dade's conjecture was proved by Späth [Spä17] using a strong form of his conjecture called the Character Triple Conjecture . This new conjecture is also believed to imply Dade's Inductive Conjecture (see Section 3.5 for further details).

For later purpose, we now introduce the most notorious of Dade's conjectures called Dade's Projective Conjecture. Perhaps, this is also the statement which got the most attention amongst those proposed by Dade. Once more, we need to introduce some notation. Let G be a finite group, p a prime and consider a normal p-subgroup $N \trianglelefteq G$. We denote by $\mathfrak{P}(G,N)$ the subset of $\mathfrak{P}(G)$ consisting of those p-chains $\mathbb{D}$ whose first term coincide with N. Since N is normal in G, the action of G on $\mathfrak{P}(G)$ restricts to $\mathfrak{P}(G,N)$. Then, we denote by $\mathfrak{P}(G,N)/\sim_G$ a G-transversal in $\mathfrak{P}(G,N)$. Consider now $\mathbb{D} \in \mathfrak{P}(G)$ and suppose that $N \leq G_{\mathbb{D}}$. If $\lambda \in \mathrm{Irr}(N)$, then we define $\mathrm{Irr}^d(B_{\mathbb{D}} \mid \lambda) := \mathrm{Irr}(G_{\mathbb{D}} \mid \lambda) \cap \mathrm{Irr}^d(B_{\mathbb{D}})$ for every $B \in \mathrm{Bl}(G)$ and $d \geq 0$. Then, Dade's Projective Conjecture can be stated as follows.

Conjecture 2.5.3 (Dade's Projective Conjecture)**.** *Let G be a finite group, p a prime and consider $Z \leq \mathbf{Z}(G)$ and $\lambda \in \mathrm{Irr}(Z)$. Set $Z_p := \mathbf{O}_p(Z)$ and consider a block $B \in \mathrm{Bl}(G)$ with defect groups larger than Z_p. Then*

$$\sum_{\mathbb{D} \in \mathfrak{P}(G,Z_p)/\sim_G} (-1)^{|\mathbb{D}|} |\mathrm{Irr}^d(B_{\mathbb{D}} \mid \lambda)| = 0$$

for every $d \geq 0$.

Dade's Projective Conjecture is known for groups of Lie type and unipotent blocks with abelian defect groups in the nondefining characteristic [BMM93], for tame blocks [Uno94], for blocks of cyclic defect [Dad96], for symmetric groups [OU95] and [An98], for $\mathrm{GL}_n(q)$ and $\mathrm{SL}_n(q)$ in the defining characteristic [OU96] and [Suk99] and for p-solvable groups [Rob00]. The latter paper, together with [Rob02] and [ER02] provide results on the structure of a minimal counterexample. Späth's reduction theorem heavily depends on these fundamental results. Moreover, the conjecture has been checked for many sporadic groups by An, Dade, O' Brien and many others.

By considering all blocks at once we obtain the following nonblockwise version of Dade's Projective Conjecture (see also [Nav18, Conjecture 9.25]).

Conjecture 2.5.4 (Nonblockwise Dade's Projective Conjecture)**.** *Let G be a finite group, p a prime and consider $Z \leq \mathbf{Z}(G)$ and $\lambda \in \mathrm{Irr}(Z)$. Set $Z_p := \mathbf{O}_p(Z)$ and consider a positive integer $d > 0$. Then*

$$\sum_{\mathbb{D} \in \mathfrak{P}(G,Z_p)/\sim_G} (-1)^{|\mathbb{D}|} |\mathrm{Irr}^d(G_{\mathbb{D}} \mid \lambda)| = 0.$$

We conclude by introducing some important sets of chains. Let $\mathbb{D} = \{D_0 < D_1 < \cdots < D_n\}$ be a p-chain of G. We say that $\mathbb{D}$ is a **normal p-chain** if $D_i \trianglelefteq D_n$ for every $i \leq n$. The set of normal p-chains is denoted by $\mathfrak{N}(G)$. As before, if N is a normal p-subgroup of G, then we define $\mathfrak{N}(G, N) := \mathfrak{N}(G) \cap \mathfrak{P}(G, N)$. Next, recall that a p-subgroup P of G is called p**-radical** if $P = \mathbf{O}_p(\mathbf{N}_G(P))$. Then, we say that $\mathbb{D}$ is a p**-radical chain** if $D_0 = \mathbf{O}_p(G)$ and D_i is p-radical in $G_{\mathbb{D}_i}$, where $\mathbb{D}_i$ is the subchain of $\mathbb{D}$ given by $\{D_0 < D_1 < \cdots < D_i\}$. The set of p-radical chains of G is denoted by $\mathfrak{R}(G)$. Finally, we say that $\mathbb{D}$ is a p**-elementary abelian chain** if $\mathbb{D}$ is a normal p-chain and D_i/D_0 is a p-elementary abelian group for every $i \geq 0$. The set of p-elementary abelian chains is denoted by $\mathfrak{E}(G)$. As before we define $\mathfrak{E}(G, N) := \mathfrak{E}(G) \cap \mathfrak{P}(G, N)$ for any normal p-subgroup N of G. Different types of p-chains are suited to different families of groups. For instance, we will see in the subsequent chapters that working with normal p-chains is convenient when dealing with p-solvable groups while p-elementary abelian chains are more beneficial when considering groups of Lie type in the nondefining characteristic. As another example, p-radical chains are a good choice to tackle groups of Lie type in the defining characteristic (see [OU96]). Thankfully, all these different kinds of p-chains can be replaced with one another when proving Dade's conjectures. This result is due to Knörr–Robinson [KR89, Proposition 3.3 and Corollary 3.4] and goes back to work of Bouc, Brown, Quillen and Thévenaz on simplicial complexes.

3

Character Triples

Let G be a finite group and consider $N \trianglelefteq G$. If $\vartheta \in \mathrm{Irr}(N)$ is G-invariant, then we say that (G, N, ϑ) is a **character triple**. This representation theoretic notion plays an important role in the reduction theorems of the counting Global-Local conjectures. For this purpose, various relations on character triples have been introduced. Each of these relations has specific features tailored on the problems arising from a Global-Local conjecture. In this chapter we will introduce some of these relations on character triples and show their main properties. In order to do so we first need to introduce the notion of projective representation. For this, we follow [Isa76], [Nav18] and [Spä18]. The results presented here are essential for understanding the arguments that will be given in the following chapters.

3.1 Character Triple Isomorphisms

Character triples can be used to control the Clifford theory of characters. One of the principal examples of this fact can be found in the reduction theorems of the Global-Local conjectures. In Section 3.3 we will see how character triples can be compared. First, in this section, we introduce isomorphisms of character triples and show some of their properties. The results of this section can be found in [Isa76, Chapter 11].

Definition 3.1.1. Let (G, N, ϑ) and (H, M, φ) be character triples. An **isomorphism** between these two character triples is the datum of a group isomorphism $\iota : G/N \to H/M$ and a map

$$\sigma_J : \mathrm{Char}(J \mid \vartheta) \to \mathrm{Char}(J^\iota \mid \varphi)$$

for every $N \leq J \leq G$ and where J/N and $J^\iota/M := \iota(J/M)$ are isomorphic via ι, such that the following properties hold for all $N \leq K \leq J \leq G$, $\chi, \psi \in \mathrm{Char}(J \mid \vartheta)$ and $\nu \in \mathrm{Irr}(J)$ with $N \leq \mathrm{Ker}(\nu)$.

(i) $\sigma_J(\chi + \psi) = \sigma_J(\chi) + \sigma_J(\psi)$;

(ii) $[\chi, \psi] = [\sigma_J(\chi), \sigma_J(\psi)]$;

(iii) $\sigma_K(\chi_K) = [\sigma_J(\chi)]_{K^\iota}$;

(iv) $\sigma_J(\chi\nu) = \sigma_J(\chi)\nu^\iota$, where ν^ι is the character of J^ι corresponding to ν via the isomorphism ι.

The isomorphism $(\iota, \sigma_\bullet)$ is **strong** if it satisfies the following additional property:

(v) $[\sigma_J(\chi)]^{\iota(\overline{g})} = \sigma_{J^g}\left(\chi^{\overline{g}}\right)$, where $\chi^{\overline{g}} \in \mathrm{Char}(J^g \mid \vartheta)$ is defined by $\chi^{\overline{g}}(x^g) = \chi(x)$ for every $\overline{g} = Ng \in G/N$ and $x \in J$.

For more information about isomorphisms of character triples we refer to [Isa76, Chapter 11]. For our purpose we only need some basic properties which we collect in the following lemma.

Lemma 3.1.2. *Let $(\iota, \sigma_\bullet)$ be an isomorphism between (G, N, ϑ) and (H, M, φ). Then the following properties hold for every $N \le J \le G$.*

(i) $\chi(1)/\vartheta(1) = \sigma_J(\chi)(1)/\varphi(1)$ *for every* $\chi \in \mathrm{Char}(J \mid \vartheta)$.

(ii) $\sigma_J : \mathrm{Char}(J \mid \vartheta) \to \mathrm{Char}(J^\iota \mid \varphi)$ *is a bijection. Moreover, it restricts to* $\sigma_J : \mathrm{Irr}(J \mid \vartheta) \to \mathrm{Irr}(J^\iota \mid \varphi)$.

(iii) ϑ *extends to* J *if and only if* φ *extends to* J^ι.

(iv) If $N \le K \le J$ *and* $\psi \in \mathrm{Char}(K \mid \vartheta)$, *then* $\sigma_J(\psi^J) = [\sigma_K(\psi)]^{J^\iota}$.

Proof. Set $e(\chi) := \chi(1)/\vartheta(1)$ and $e(\eta) := \eta(1)/\varphi(1)$ for every $\chi \in \mathrm{Char}(J \mid \vartheta)$ and $\eta \in \mathrm{Char}(J^\iota \mid \varphi)$. Since $N^\iota = M$ and $\sigma_N(\vartheta) = \varphi$ we have $e(\sigma_J(\chi))\varphi = [\sigma_J(\chi)]_M = \sigma_N(\chi_N) = \sigma_N(e(\chi)\vartheta) = e(\chi)\varphi$. It follows that

$$\chi(1)/\vartheta(1) = e(\chi) = e(\sigma_J(\chi)) = \sigma_J(\chi)(1)/\varphi(1).$$

Next, by using property (i) of Definition 3.1.1, observe that σ_J is completely determined by its image on $\mathrm{Irr}(J \mid \vartheta)$. Then, using (ii), we deduce that the map is injective. In order to prove surjectivity, it is enough to show that $\mathrm{Irr}(J^\iota \mid \varphi)$ is contained in the image of $\mathrm{Irr}(J \mid \vartheta)$. Consider the characters

$$\vartheta^J = \sum_{\chi \in \mathrm{Irr}(J|\vartheta)} e(\chi)\chi, \qquad \varphi^{J^\iota} = \sum_{\eta \in \mathrm{Irr}(J^\iota|\varphi)} e(\eta)\eta$$

and notice, by evaluating the degrees, that $\sum_\chi e(\chi)^2 = |J : N| = |J^\iota : M| = \sum_\eta e(\eta)^2$. Since σ_J is injective, we deduce that $\sum_\chi e(\chi)^2 = \sum_\chi e(\sigma_J(\chi))^2 \le \sum_\eta e(\eta)^2$ and therefore $\mathrm{Irr}(J^\iota \mid \varphi)$ must be contained in the image of σ_J. This proves (ii). The claim on extendibility follows directly by (iii) of Definition 3.1.1.

Finally, let $N \le K \le J$ and $\psi \in \mathrm{Irr}(K \mid \vartheta)$. In order to prove (iv), it's enough to show that $[\sigma_J(\psi^J), \chi] = [\sigma_K(\psi)^{J^\iota}, \chi]$ for every $\chi \in \mathrm{Irr}(J^\iota \mid \varphi)$. By (ii) above, we can write $\chi = \sigma_J(\xi)$, for some $\xi \in \mathrm{Irr}(J \mid \vartheta)$. Then $[\sigma_J(\psi^J), \chi] = [\sigma_J(\psi^J), \sigma_J(\xi)] = [\psi^J, \xi] = [\psi, \xi_K] = [\sigma_K(\psi), \sigma_K(\xi_K)] = [\sigma_K(\psi), \sigma_J(\xi)_{K^\iota}] = [\sigma_K(\psi)^{J^\iota}, \sigma_J(\xi)] = [\sigma_K(\psi)^{J^\iota}, \chi]$. This concludes the proof. □

3.2 Projective Representations

Let G be a group. A **complex projective representation** of G is a map

$$\mathcal{P} : G \to \mathrm{GL}_n(\mathbb{C})$$

such that

$$\mathcal{P}(x)\mathcal{P}(y) = \alpha(x,y)\mathcal{P}(xy)$$

for every $x, y \in G$ and some $\alpha : G \times G \to \mathbb{C}^\times$. The scalar map α satisfies

$$\alpha(x,y)\alpha(x,yz) = \alpha(x,yz)\alpha(y,z)$$

for every $x, y, z \in G$ and is called a **factor set** of G. In fact this is nothing but a 2-cocyle $\alpha \in \mathbf{Z}^2(G, \mathbb{C}^\times)$. We denote by $\mathrm{Proj}(G)$ the set of projective representations of G. Moreover, for a fixed factor set α, we denote by $\mathrm{Proj}(G \mid \alpha)$ the set of projective representations of G whose factor set coincides with α.

Consider two projective representations $\mathcal{P}$ and $\mathcal{Q}$ of G. We say that $\mathcal{P}$ is **similar** to $\mathcal{Q}$ if there exists $P \in \mathrm{GL}_n(\mathbb{C})$ such that $\mathcal{P}(x) = P^{-1}\mathcal{Q}(x)P$ for every $x \in G$. Then $\mathcal{P}$ is irreducible if it's not similar to any projective representation in proper block form. We say that $\mathcal{P}$ and $\mathcal{Q}$ are **equivalent** if there exists $\mu : G \to \mathbb{C}^\times$ such that $\mathcal{P} = \mu\mathcal{Q}$, where $\mu\mathcal{Q}$ is the projective representation defined by $\mu\mathcal{Q}(x) := \mu(x)\mathcal{Q}(x)$ for every $x \in G$.

Projective representations appear often when studying problems in representation theory. Classical applications of projective representations are in relation with extendibility of characters. Consider $N \trianglelefteq G$ and a G-invariant character $\vartheta \in \mathrm{Irr}(N)$. Although it might not be true that ϑ extends to G, we can always find a projective representation $\mathcal{P}$ of G such that ϑ is afforded by $\mathcal{P}_N$. This leads us to the next definition.

Definition 3.2.1. Let (G, N, ϑ) be a character triple. A projective representation $\mathcal{P}$ of G is **associated** with (G, N, ϑ) if:

(i) $\mathcal{P}_N$ is a representation of N affording ϑ, and

(ii) $\mathcal{P}(gn) = \mathcal{P}(g)\mathcal{P}(n)$ and $\mathcal{P}(ng) = \mathcal{P}(n)\mathcal{P}(g)$ for every $n \in N$ and $g \in G$.

If no confusion arises, then we simply say that $\mathcal{P}$ is associated with ϑ.

In order to prove the existence of projective representations associated with a character triple, we need to recall a result on extendibility in cyclic factors.

Theorem 3.2.2. *Let (G, N, ϑ) be a character triple with G/N cyclic and let $\mathfrak{X}$ be a representation affording ϑ. Then there exists an irreducible representation $\tilde{\mathfrak{X}}$ of G such that $\tilde{\mathfrak{X}}_N = \mathfrak{X}$. In particular ϑ extends to G.*

Proof. See [Nav18, Theorem 5.1]. □

The next Lemma can be found in [Nav18, Lemma 5.4].

Lemma 3.2.3. *Let (G,N,ϑ) be a character triple and consider a representation $\mathfrak{X}$ affording ϑ. For $x,y\in G$, let $\mathfrak{X}_x$, $\mathfrak{X}_y$ and $\mathfrak{X}_{xy}$ be extensions of $\mathfrak{X}$ to $\langle N,x\rangle$, $\langle N,y\rangle$ and $\langle N,xy\rangle$ respectively. Then there exists $\alpha(x,y)\in\mathbb{C}^\times$ such that $\mathfrak{X}_x(x)\mathfrak{X}_y(y)=\alpha(x,y)\mathfrak{X}_{xy}(xy)$.*

Proof. First, observe that $\mathfrak{X}_x$, $\mathfrak{X}_y$ and $\mathfrak{X}_{xy}$ exist by Theorem 3.2.2. Let $n\in N$ and notice that

$$\mathfrak{X}\left(yny^{-1}\right)=\mathfrak{X}_y\left(yny^{-1}\right)=\mathfrak{X}_y(y)\mathfrak{X}_y(n)\mathfrak{X}_y\left(y^{-1}\right)=\mathfrak{X}_y(y)\mathfrak{X}(n)\mathfrak{X}_y(y)^{-1} \tag{3.2.1}$$

and similarly

$$\mathfrak{X}(xyny^{-1}x^{-1})=\mathfrak{X}_{xy}(xy)\mathfrak{X}(n)\mathfrak{X}_{xy}(xy)^{-1}. \tag{3.2.2}$$

If we conjugate (3.2.1) by $\mathfrak{X}_x(x)^{-1}$, then

$$\mathfrak{X}_x(x)\mathfrak{X}_y(y)\mathfrak{X}(n)(\mathfrak{X}_x(x)\mathfrak{X}_y(y))^{-1}=\mathfrak{X}_x(x)\mathfrak{X}(yny^{-1})\mathfrak{X}_x(x)^{-1}=\mathfrak{X}(xyny^{-1}x^{-1})$$

and using (3.2.2) we obtain

$$\mathfrak{X}_x(x)\mathfrak{X}_y(y)\mathfrak{X}(n)(\mathfrak{X}_x(x)\mathfrak{X}_y(y))^{-1}=\mathfrak{X}_{xy}(xy)\mathfrak{X}(n)\mathfrak{X}_{xy}(xy)^{-1}.$$

This shows that $(\mathfrak{X}_x(x)\mathfrak{X}_y(y))^{-1}\mathfrak{X}_{xy}(xy)$ commutes with $\mathfrak{X}(n)$ for every $n\in N$. Using Schur's lemma [Isa76, Lemma 2.25], we conclude that $(\mathfrak{X}_x(x)\mathfrak{X}_y(y))^{-1}\mathfrak{X}_{xy}(xy)$ is a scalar matrix. □

We can now show the existence of projective representations associated with character triples (see [Nav18, Theorem 5.5]).

Theorem 3.2.4. *Let (G,N,ϑ) be a character triple and fix a representation $\mathfrak{X}$ of N affording ϑ. Then there exists a projective representation $\mathcal{P}\in\mathrm{Proj}(G)$ associated with (G,N,ϑ) and with factor set α such that*

(i) $\mathcal{P}_N=\mathfrak{X}$, *and*

(ii) $\alpha(g,h)^{|G:N|\vartheta(1)}=1$ *for every* $g,h\in G$.

Proof. For $\overline{g}\in\overline{G}:=G/N$, using Theorem 3.2.2, fix an extension $\mathfrak{X}_{\overline{g}}$ of $\mathfrak{X}$ to $\langle N,g\rangle$. Set $\mathcal{P}(g):=\mathfrak{X}_{\overline{g}}(g)$, for every $g\in G$, and observe that $\mathcal{P}$ is a projective representation of G by Lemma 3.2.3. It's immediate to show $\mathcal{P}_N=\mathfrak{X}$. Moreover, since $\overline{g}=\overline{gn}$, we deduce that $\mathfrak{X}_{\overline{g}}=\mathfrak{X}_{\overline{gn}}$ and therefore

$$\mathcal{P}(g)\mathcal{P}(n)=\mathfrak{X}_{\overline{g}}(g)\mathfrak{X}(n)=\mathfrak{X}_{\overline{g}}(gn)=\mathfrak{X}_{\overline{gn}}(gn)=\mathcal{P}(gn)$$

for every $g\in G$, $n\in N$. Likewise $\mathcal{P}(n)\mathcal{P}(g)=\mathcal{P}(ng)$. For $g\in G$ and $m\in\mathbb{N}$, if $g^m\in N$, then

$$\mathcal{P}(g^m)=\mathfrak{X}(g^m)=\mathfrak{X}_{\overline{g}}(g)^m=\mathcal{P}(g)^m.$$

Therefore $\mathcal{P}(g)^{|G:N|}=\mathcal{P}(1)=I_{\vartheta(1)}$ and $\det(\mathcal{P}(g))^{|G:N|}=\det(\mathcal{P}(g)^{|G:N|})=1$ for every $g\in G$, and it follows that

$$1=\det\left(\mathcal{P}(g)\mathcal{P}(h)\mathcal{P}(gh)^{-1}\right)^{|G:N|}=\det\left(\alpha(g,h)I_{\vartheta(1)}\right)^{|G:N|}=\alpha(g,h)^{|G:N|\vartheta(1)}$$

for every $g,h\in G$. This completes the proof. □

Next, we describe some properties of the projective representations associated with character triples (see [Nav18, Lemma 5.3]).

Proposition 3.2.5. *Let $\mathcal{P} \in \mathrm{Proj}(G)$ be a projective representation associated with the character triple (G, N, ϑ) and with factor set α. Then:*

(i) $\alpha(g_1 n_1, g_2 n_2) = \alpha(g_1, g_2) = \alpha(n_1 g_1, n_2 g_2)$ *for every $n_i \in N$ and $g_i \in G$.*

(ii) $1 = \alpha(1,1) = \alpha(g,n) = \alpha(n,g)$ *for every $n \in N$ and $g \in G$.*

(iii) $\mathcal{P}(n^g) = \mathcal{P}(g)^{-1}\mathcal{P}(n)\mathcal{P}(g)$ *for every $n \in N$ and $g \in G$.*

(iv) $\mathcal{P}(x)$ *is a scalar matrix for every $x \in \mathbf{C}_G(N)$.*

(v) $\alpha(g,z) = \alpha(z,g)$ *for every $z \in \mathbf{Z}(G)$ and $g \in G$.*

Proof. Let $g_1, g_2 \in G$ and $n_1, n_2 \in N$. Using the fact that $\mathcal{P}(gn) = \mathcal{P}(g)\mathcal{P}(n)$, for every $g \in G$ and $n \in N$, we deduce that $\mathcal{P}(g_1 g_2)\mathcal{P}(n_1^{g_2} n_2) = \mathcal{P}(g_1 g_2 n_1^{g_2} n_2) = \mathcal{P}(g_1 n_1 g_2 n_2)$. Then we obtain

$$\alpha(g_1, g_2)\mathcal{P}(g_1 n_1 g_2 n_2) = \mathcal{P}(g_1)\mathcal{P}(g_2)\mathcal{P}(n_1^{g_2} n_2) = \mathcal{P}(g_1)\mathcal{P}(g_2 n_1^{g_2} n_2) = \mathcal{P}(g_1 n_1)\mathcal{P}(g_2 n_2)$$

and therefore $\alpha(g_1 n_1, g_2 n_2) = \alpha(g_1, g_2)$. This proves (i) and (ii) follows easily. To prove (iii), just notice that $\mathcal{P}(n)\mathcal{P}(g) = \mathcal{P}(ng) = \mathcal{P}(gn^g) = \mathcal{P}(g)\mathcal{P}(n^g)$. Now, take $x \in \mathbf{C}_G(N)$ and recall that $\mathfrak{X} := \mathcal{P}_N$ is an irreducible representation of N. Then, for every $n \in N$, it follows from (iii) that

$$[\mathcal{P}(x), \mathfrak{X}(n)] = \mathcal{P}(x)^{-1}\mathfrak{X}(n)^{-1}\mathcal{P}(x)\mathfrak{X}(n) = \mathfrak{X}\left(\left(n^{-1}\right)^x\right)\mathfrak{X}(n) = I_{\vartheta(1)}.$$

By Schur's lemma [Isa76, Lemma 2.25], we obtain (iv). Finally, consider $g \in G$ and $z \in \mathbf{Z}(G)$. Since $z \in \mathbf{C}_G(N)$, we deduce from (iv) that $\mathcal{P}(g)\mathcal{P}(z) = \mathcal{P}(z)\mathcal{P}(g)$ and thus

$$\alpha(g,z) = \mathcal{P}(g)\mathcal{P}(z)\mathcal{P}(gz)^{-1} = \mathcal{P}(z)\mathcal{P}(g)\mathcal{P}(zg)^{-1} = \alpha(z,g).$$

Now the proof is complete. □

By the above result, if $\mathcal{P}$ is a projective representation associated with a character triple (G, N, ϑ) and with factor set α, we obtain a well defined map $\overline{\alpha} : G/N \times G/N \to \mathbb{C}$ given by $\overline{\alpha}(Ng_1, Ng_2) = \alpha(g_1, g_2)$ for every $g_1, g_2 \in G$.

We consider another important feature of projective representations. Let (G, N, ϑ) be a character triple and choose a projective representation $\mathcal{P}$ associated with it. This choice allows us to construct a central extension $\widehat{G}$ of G together with a character ϑ_0 of a subgroup of $\widehat{G}$, that may be identified with ϑ, that extends to $\widehat{G}$. This process is often useful to reduce to the case where ϑ extends to G. The next result can be found in [Nav18, Theorem 5.6].

Theorem 3.2.6. *Let (G, N, ϑ) be a character triple and $\mathcal{P} \in \mathrm{Proj}(G)$ a projective representation of G with factor set α associated with ϑ as in Theorem 3.2.4. Set $S := \langle \alpha(g,h) \mid g, h \in G \rangle$ the finite subgroup of $\mathbb{C}^\times$ generated by the values of α. Consider the set $\hat{G} := G \times S$ endowed with the group multiplication*

$$(x,s) \cdot (y,t) := (xy, \alpha(x,y)st)$$

for every $(x,s), (y,t) \in \widehat{G}$. Then the following holds.

(i) *$\widehat{G}$ is a central extension of G with projection $\epsilon : \widehat{G} \to G$, $(x,s) \mapsto x$, with kernel $S_0 := \{(1,s) \mid s \in S\} \simeq S$.*

(ii) *For every $X \leq G$, set $\widehat{X} := \epsilon^{-1}(X)$. Moreover define $Y_0 := \{(y,1) \mid y \in Y\}$ for every $Y \leq N$. Then $\widehat{Y} = Y_0 \times S_0$ (as a group) and Y_0 is isomorphic to Y via ϵ.*

(iii) *If $Y \leq N$ with $Y \trianglelefteq G$, then $Y_0, \widehat{Y} \trianglelefteq \widehat{G}$ and $\widehat{Y}/Y_0 \leq \mathbf{Z}(\widehat{G}/Y_0)$.*

(iv) *The irreducible representation*

$$\begin{aligned}\widehat{\mathcal{P}} : \widehat{G} &\to \mathrm{GL}_{\vartheta(1)}(\mathbb{C})\\ (g,z) &\mapsto z\mathcal{P}(g)\end{aligned}$$

affords an extension τ of ϑ_0, where ϑ_0 is the character of N_0 corresponding to ϑ via the isomorphism $\epsilon : N_0 \to N$, i.e. $\vartheta_0(n,1) := \vartheta(n)$ for every $n \in N$.

(v) *Let $\widehat{\vartheta} := \vartheta_0 \times 1_{S_0}$. There exists a bijection $\mathrm{Irr}\left(\widehat{G} \mid \widehat{\vartheta}\right) \to \mathrm{Irr}\left(G \mid \vartheta\right)$, $\widehat{\chi} \mapsto \chi$. This bijection preserves the decomposition into blocks: for every $\chi, \psi \in \mathrm{Irr}(G \mid \vartheta)$, we have $\mathrm{bl}(\chi) = \mathrm{bl}(\psi)$ if and only if $\mathrm{bl}(\widehat{\chi}) = \mathrm{bl}(\widehat{\psi})$.*

Proof. Since α is a factor set, the operation defined on $\widehat{G}$ is a group multiplication. By using the properties described in Proposition 3.2.5, we deduce that $(1,1)$ is the identity of $\widehat{G}$ and observe that $(x,s)^{-1} = (x^{-1}, \alpha(x,x^{-1})s^{-1})$ for every $(x,s) \in \widehat{G}$. Straightforward computations show that (i), (ii) and (iii) hold (see [Nav18, Theorem 5.6] for more details). The fact that $\widehat{\mathcal{P}}$ is an ordinary representation of $\widehat{G}$ follows from the fact that $\mathcal{P}$ has factor set α and by the definition of the multiplication on $\widehat{G}$. Furthermore, as $\mathcal{P}_N$ affords ϑ, we deduce that $\tau_N = \vartheta_0$. Finally, for every $\chi \in \mathrm{Irr}(G)$, let $\widehat{\chi}$ be the inflation to $\widehat{G}$ of the character of $\widehat{G}/S_0$ corresponding to χ via the isomorphism $\epsilon : \widehat{G}/S_0 \to G$, i.e. $\widehat{\chi}(g,s) := \chi(g)$ for every $(g,s) \in \widehat{G}$. Then we have a bijection between the set of characters $\chi \in \mathrm{Irr}(G)$ and those $\widehat{\chi}$ of $\widehat{G}$ with $S_0 \leq \mathrm{Ker}(\widehat{\chi})$. To conclude notice that χ lies above ϑ if and only if $\widehat{\chi}$ lies above $\widehat{\vartheta}$. The claim on blocks follows by [NT89, Theorem 5.8.8 and Theorem 5.8.11] recalling that S_0 is central in $\widehat{G}$. □

We will refer to the group $\widehat{G}$ constructed above as the central extension of G **induced** (or **defined**) by $\mathcal{P}$. Next, we go one step further and define a central extension of G/N. This construction was used by Fong in his fundamental paper [Fon61]. Later we will see some additional properties of the following bijection (see Section 4.2).

Theorem 3.2.7. *Let (G,N,ϑ) be a character triple and $\mathcal{P} \in \mathrm{Proj}(G)$ a projective representation of G associated with ϑ and with factor set α. Consider the central extension $\widehat{G}$ of G defined by $\mathcal{P}$ and set $\widetilde{X} := \widehat{X}N_0/N_0$ for every $X \leq G$ and where $N_0 := \{(n,1) \mid n \in N\}$.*

(i) *$\widetilde{G}$ is a central extension of G/N with projection $\widetilde{G} \to G/N$, $N_0(x,s) \mapsto Nx$, with kernel $\widehat{N}/N_0$.*

(ii) *Let $\widehat{\lambda}$ be the character of $\widehat{N}$ defined by $\widehat{\lambda}(n,s) := s^{-1}$, so that τ extends $\widehat{\vartheta}\widehat{\lambda}^{-1}$. Observe that $N_0 \leq \mathrm{Ker}(\widehat{\lambda})$ and denote by $\widetilde{\vartheta}$ the character of $\widetilde{N}$ corresponding to $\widehat{\lambda}$ via inflation: that is $\widetilde{\vartheta}(N_0(n,s)) := s^{-1}$. Then there exists a bijection $\mathrm{Irr}\left(G \mid \vartheta\right) \to \mathrm{Irr}\left(\widetilde{G} \mid \widetilde{\vartheta}\right)$.*

Proof. By Theorem 3.2.6, we know that $\widetilde{N} \leq \mathbf{Z}(\widetilde{G})$, while it is clear that $\widetilde{G}/\widetilde{N}$ is isomorphic to G/N. To prove the second part, let $\chi \in \mathrm{Irr}(G \mid \vartheta)$. By Theorem 3.2.6 the character χ corresponds to a unique $\widehat{\chi} \in \mathrm{Irr}(\widehat{G} \mid \widehat{\vartheta})$. By Gallagher's theorem [Isa76, Corollary 6.17] there exists a unique character $\widehat{\chi}' \in \mathrm{Irr}(\widehat{G})$, with $N_0 \leq \mathrm{Ker}(\widehat{\chi}')$, such that $\widehat{\chi} = \tau\widehat{\chi}'$. Since τ lies over $\widehat{\vartheta}\widehat{\lambda}^{-1}$, we deduce that $\widehat{\chi}'$ lies above $\widehat{\lambda}$. Denote by $\widetilde{\chi}$ the deflation of $\widehat{\chi}'$ to $\widetilde{G}$ and observe that $\widetilde{\chi}$ lies over $\widetilde{\vartheta}$. Then we define the required bijection by $\chi \mapsto \widetilde{\chi}$. □

The central extensions constructed in Theorem 3.2.6 and Theorem 3.2.7 yield isomorphisms of character triples (see also [Isa76, Theorem 11.28] and [Nav18, Corollary 5.9]).

Corollary 3.2.8. *Let $\mathcal{P}$ be a projective representation associated with (G, N, ϑ). Consider the central extensions $\widehat{G}$ of G and $\widetilde{G}$ of G/N defined by $\mathcal{P}$ and let $\widehat{\vartheta} \in \mathrm{Irr}(\widehat{N})$ and $\widetilde{\vartheta} \in \mathrm{Irr}(\widetilde{N})$ as defined in Theorem 3.2.6 and Theorem 3.2.7 respectively. Then the character triples (G, N, ϑ), $(\widehat{G}, \widehat{N}, \widehat{\vartheta})$ and $(\widetilde{G}, \widetilde{N}, \widetilde{\vartheta})$ are strongly isomorphic.*

Proof. By [Isa76, Lemma 11.26] we obtain an isomorphism between the character triples (G, N, ϑ) and $(\widehat{G}, \widehat{N}, \widehat{\vartheta})$. Since $\tau_{\widehat{N}} = \widehat{\vartheta}\widehat{\lambda}^{-1}$, [Isa76, Lemma 11.27] implies that $(\widehat{G}, \widehat{N}, \widehat{\vartheta})$ is isomorphic to $(\widehat{G}, \widehat{N}, \widehat{\lambda})$. Applying once again [Isa76, Lemma 11.26], we conclude that $(\widehat{G}, \widehat{N}, \widehat{\lambda})$ is isomorphic to $(\widetilde{G}, \widetilde{N}, \widetilde{\vartheta})$. Since isomorphism of character triples is an equivalence relation, it follows that (G, N, ϑ), $(\widehat{G}, \widehat{N}, \widehat{\vartheta})$ and $(\widetilde{G}, \widetilde{N}, \widetilde{\vartheta})$ are isomorphic. Finally, it follows by straightforward computations that all above mentioned isomorphisms are strong. □

We end this section by recalling the following result on Clifford theory for projective representations. See [Nav98, Theorem 8.16 and Theorem 8.18] for a proof.

Theorem 3.2.9. *Let (G, N, ϑ) be a character triple with associated projective representation $\mathcal{P} \in \mathrm{Proj}(G)$ with factor set α. Then:*

(i) $\mathrm{Proj}(G/N \mid \alpha^{-1}) \to \mathrm{Rep}(G \mid \vartheta), \mathcal{Q} \mapsto \mathcal{Q} \otimes \mathcal{P}$ *is injective, where* $\mathrm{Rep}(G \mid \vartheta)$ *is the set of representations of G whose characters lies over ϑ.*

(ii) For every $\chi \in \mathrm{Char}(G \mid \vartheta)$ there exists $\mathcal{Q} \in \mathrm{Proj}(G/N \mid \alpha^{-1})$ such that χ is afforded by $\mathcal{Q} \otimes \mathcal{P}$.

(iii) If $\mathcal{Q} \in \mathrm{Proj}(G/N \mid \alpha^{-1})$, then $\mathcal{Q}$ is irreducible if and only if $\mathcal{Q} \otimes \mathcal{P}$ is irreducible.

(iv) If $\mathcal{Q}, \mathcal{Q}' \in \mathrm{Proj}(G/N \mid \alpha^{-1})$, then $\mathcal{Q}$ is similar to $\mathcal{Q}'$ if and only if $\mathcal{Q} \otimes \mathcal{P}$ is similar to $\mathcal{Q}' \otimes \mathcal{P}$.

(v) Let $N \leq J \leq G$ and $\psi \in \mathrm{Irr}(J \mid \vartheta)$. Set $H := \mathbf{N}_G(J)_\psi$ and consider $\mathcal{D} \in \mathrm{Proj}(H \mid \beta)$ associated with (H, J, ψ) such that

$$\mathcal{D}_J = \mathcal{Q} \otimes \mathcal{P}_J$$

for some $\mathcal{Q} \in \mathrm{Proj}(J/N \mid \alpha^{-1}_{J\times J})$. Then there exists $\widehat{\mathcal{Q}} \in \mathrm{Proj}(H/N \mid \beta\alpha^{-1}_{H\times H})$ such that

$$\mathcal{D} = \widehat{\mathcal{Q}} \otimes \mathcal{P}_H$$

and $\widehat{\mathcal{Q}}_J = \mathcal{Q}$.

Proof. The proof of (i)-(iv) can be found in [Nav98, Theorem 8.16 and Theorem 8.18] (see also [Nav18, Theorem 10.11]). We now consider (v). Since $\mathcal{P}_J$ is associated with (J, N, ϑ), by (ii) there exists $\mathcal{Q} \in \mathrm{Proj}(J/N \mid \alpha^{-1}_{J\times J})$ such that $\mathcal{Q} \otimes \mathcal{P}_J$ affords ψ. Using Theorem 3.2.4 we can find $\mathcal{D}$ associated with (H, J, ψ) such that $\mathcal{D}_J = \mathcal{Q} \otimes \mathcal{P}_J$. Then, using the proof of [Nav98, Theorem 8.16] (see also the argument used in [NS14b, Proposition 3.9 (b)]), we can construct $\widehat{\mathcal{Q}}$. □

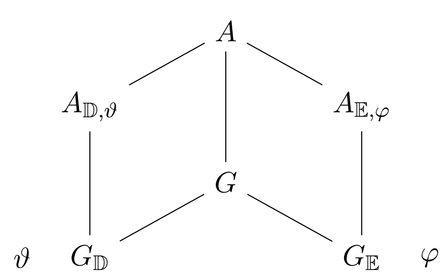
3.3 Relations on Character Triples

As we already mentioned earlier, character triples can be used as a tool to control Clifford theory. With this goal in mind, it is useful to introduce partial relations on the set of character triples in order to be able to compare them. Depending on the problem that we are dealing with, there could be different aspects of representation theory that we would like to control and this leads to a variety of relations on character triples (see [Spä18] and [Nav18, Chapter 10]). For instance, in the reduction theorem of the McKay Conjecture it is used a relation that allows us to control the restriction of characters to the center (this is due to the fact that a relative version of the conjecture needs to be addressed). On the other hand, and not surprisingly, in the reduction theorem of the Alperin–McKay Conjecture it is used a relation that provides control on the block theoretic aspects of the problem. In this thesis we consider relations that are tailored to deal with Dade's Conjecture. As we will see, these relations offer a general setting that allows us to recover, as special cases, many of the previously introduced relations. All the results presented in this section can be found in [NS14b], [Spä17], [Nav18, Chapter 10] or [Spä18] with the exception of Proposition 3.4.1 (i), Proposition 3.4.3 and Proposition 3.4.4.

3.3.1 N-central isomorphism

We start by introducing a simpler relation called N-**central isomorphism**. Then, we will introduce additional block theoretic requirements and define the N-**block isomorphism**. In this way, we will obtain a relation that incorporates all aspects of representation theory needed to deal with Dade's conjecture.

Before giving the first definition, we try to motivate what we are doing. Fix a prime p and let $\mathbb{D}$ and $\mathbb{E}$ be two p-chains of the group G. To deal with Dade's conjecture we consider $\vartheta \in \mathrm{Irr}(G_{\mathbb{D}})$ and $\varphi \in \mathrm{Irr}(G_{\mathbb{E}})$. If $G \trianglelefteq A$, then we wish to compare the character triples $(A_{\mathbb{D},\vartheta}, G_{\mathbb{D}}, \vartheta)$ and $(A_{\mathbb{E},\varphi}, G_{\mathbb{E}}, \varphi)$



The first natural properties that one might ask is to have $GA_{\mathbb{D},\vartheta} = GA_{\mathbb{E},\varphi}$ so that $A_{\mathbb{D},\vartheta}/G_{\mathbb{D}}$ is naturally isomorphic to $A_{\mathbb{E},\varphi}/G_{\mathbb{E}}$. Next, we would like to have an isomorphism between the two character triples $(A_{\mathbb{D},\vartheta}, G_{\mathbb{D}}, \vartheta)$ and $(A_{\mathbb{E},\varphi}, G_{\mathbb{E}}, \varphi)$. To do so, consider projective representations $\mathcal{P} \in \operatorname{Proj}(A_{\mathbb{D},\vartheta})$ with factor set α and $\mathcal{Q} \in \operatorname{Proj}(A_{\mathbb{E},\varphi})$ with factor set β associated with ϑ and φ respectively. Notice that, by Proposition 3.2.5 the factor sets α and β can be seen as factor sets of $A_{\mathbb{D},\vartheta}/G_{\mathbb{D}}$ and $A_{\mathbb{E},\varphi}/G_{\mathbb{E}}$. If α and β coincide via the isomorphism $A_{\mathbb{D},\vartheta}/G_{\mathbb{D}} \simeq A_{\mathbb{E},\varphi}/G_{\mathbb{E}}$, then we can produce an isomorphism of character triples by applying Theorem 3.2.9. The following result is [Spä17, Theorem 3.3].

Proposition 3.3.1. *Let $N \trianglelefteq G$ and consider two character triples (H_1, M_1, ϑ_1) and (H_2, M_2, ϑ_2) such that $G = NH_1 = NH_2$, $M_1 = N \cap H_1$ and $M_2 = N \cap H_2$. Denote by $\iota : H_1/M_1 \to H_2/M_2$ the canonical isomorphism and assume there exist projective representations $\mathcal{P}_i \in \operatorname{Proj}(H_i)$ associated with ϑ_i and with factor set α_i such that*

$$\overline{\alpha}_1(x, y) = \overline{\alpha}_2(\iota(x), \iota(y))$$

for every $x, y \in H_1/M_1$ and where $\overline{\alpha}_i$ is the factor set of H_i/M_i corresponding to α_i. Then there exists a strong isomorphism of character triples

$$(\iota, \sigma_\bullet) : (H_1, M_1, \vartheta_1) \to (H_2, M_2, \vartheta_2)$$

such that, for every $N \leq J \leq G$,

$$\begin{aligned} \sigma_{J_1} : \operatorname{Irr}(J_1 \mid \vartheta_1) &\to \operatorname{Irr}(J_2 \mid \vartheta_2) \\ \operatorname{Tr}\left(\mathcal{Q}_{J_1} \otimes \mathcal{P}_{1,J_1}\right) &\mapsto \operatorname{Tr}\left(\mathcal{Q}_{J_2} \otimes \mathcal{P}_{2,J_2}\right) \end{aligned}$$

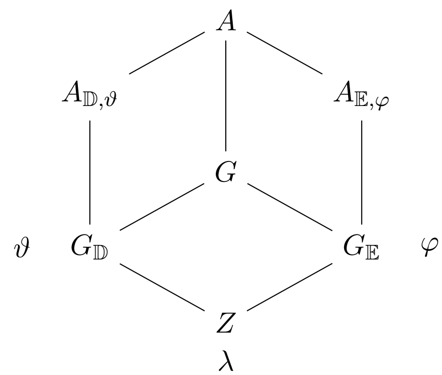
where $J_i := J \cap H_i$ and $\mathcal{Q} \in \operatorname{Proj}(J/N)$ is given by Theorem 3.2.9.

Proof. Recall that by the definition of isomorphism of character triples it is enough to define the map σ_{J_1} on the set of irreducible characters. Fix $\psi_i \in \operatorname{Irr}(J_1 \mid \vartheta_1)$ and consider $\mathcal{Q}_i \in \operatorname{Proj}(J_1/M_1 \mid \alpha^{-1}_{1,J_1\times J_1})$ such that ψ_1 is afforded by $\mathcal{Q}_1 \otimes \mathcal{P}_{1,J_1}$ as in Theorem 3.2.9. As usual, we identify a (projective) representation of a quotient group with its lift. Via the isomorphism $H_1/M_1 \simeq G/N$ we can define a projective representation $\mathcal{Q} \in \operatorname{Proj}(J/N)$ such that $\mathcal{Q}_i = \mathcal{Q}_{J_1}$. Since α_1 and α_2 coincides via ι, we deduce that $\mathcal{Q}_{J_2} \in \operatorname{Proj}(J_2/M_2)$ has factor set $\alpha^{-1}_{2,J_2\times J_2}$. It follows that $\sigma_{J_1}(\psi_1) := \operatorname{Tr}(\mathcal{Q}_{J_2} \otimes \mathcal{P}_{2,J_2})$ is a character in $\operatorname{Irr}(J_2 \mid \vartheta_2)$. Using Theorem 3.2.9 we conclude that σ_{J_1} is a well defined bijection and by standard computations it can be seen that the requirements of Definition 3.1.1 are satisfied. □

Definition 3.3.2. Consider $N \trianglelefteq G$ and two character triples (H_1, M_1, ϑ_1) and (H_2, M_2, ϑ_2) satisfying the assumptions of Proposition 3.3.1 with respect to the pair $(\mathcal{P}_1, \mathcal{P}_2)$. Then we say that the isomorphism $(\iota, \sigma_\bullet)$ from Proposition 3.3.1 is an N-**isomorphism** between (H_1, M_1, ϑ_1) and (H_2, M_2, ϑ_2) . Moreover we say that (H_1, M_1, ϑ_1) and (H_2, M_2, ϑ_2) are N-**isomorphic**. If we want to specify a choice of projective representations, then we say that the N-isomorphism is given by $(\mathcal{P}_1, \mathcal{P}_2)$ or that the character triples are N-isomorphic via $(\mathcal{P}_1, \mathcal{P}_2)$.

We now go back to our discussion on Dade's Conjecture. Consider the two character triples $(A_{\mathbb{D},\vartheta}, G_{\mathbb{D}}, \vartheta)$ and $(A_{\mathbb{E},\varphi}, G_{\mathbb{E}}, \varphi)$. In the statement of Dade's Projective Conjecture, a subgroup

$Z \leq \mathbf{Z}(G)$ is given together with a character $\lambda \in \mathrm{Irr}(Z)$. Then, noticing that $Z \leq G_{\mathbb{D}} \cap G_{\mathbb{E}}$, we wish to control those characters $\vartheta \in \mathrm{Irr}(G_{\mathbb{D}})$ and $\varphi \in \mathrm{Irr}(G_{\mathbb{E}})$ that lie above the fixed character λ. In particular, we would like to say that ϑ lies above λ if and only if so does φ.



Consider a pair of projective representations $(\mathcal{P}_1, \mathcal{P}_2)$ giving an N-isomorphism between (H_1, M_1, ϑ_1) and (H_2, M_2, ϑ_2). For every $x \in \mathbf{C}_{H_i}(M_i)$, we know by Proposition 3.2.5 that the matrix $\mathcal{P}_i(x)$ is of the form $\zeta_i(x) I_{\vartheta_i(1)}$ for some $\zeta_i(x) \in \mathbb{C}^\times$. This defines a map

$$\zeta_i : \mathbf{C}_{H_i}(M_i) \to \mathbb{C}^\times.$$

We refer to this map as the **scalar function** of $\mathcal{P}_i$. In particular, if $G := H_i N$ and $\mathbf{C}_G(N) \leq H_i$ for $i = 1, 2$, then $\mathbf{C}_G(N) \leq \mathbf{C}_{H_i}(M_i)$ and we can compare the two scalar functions ζ_1 and ζ_2 on $\mathbf{C}_G(N)$. The next result can be found in [Spä17, Lemma 3.4] and should be compared with [NS14b, Lemma 3.3].

Lemma 3.3.3. *Let (H_1, M_1, ϑ_1) and (H_2, M_2, ϑ_2) be N-isomorphic via $(\mathcal{P}_1, \mathcal{P}_2)$ and consider the isomorphism $(\iota, \sigma_\bullet)$ from Proposition 3.3.1. If $\mathbf{C}_G(N) \leq H_1 \cap H_2$ for $G := NH_i$, then the following are equivalent:*

(i) the scalar functions of $\mathcal{P}_1$ and $\mathcal{P}_2$ coincide on $\mathbf{C}_G(N)$;

(ii) for every $N \leq J \leq G$ and $\psi \in \mathrm{Irr}(J_1 \mid \vartheta_1)$, we have

$$\mathrm{Irr}\left(\psi_{\mathbf{C}_J(N)}\right) = \mathrm{Irr}\left(\sigma_{J_1}(\psi)_{\mathbf{C}_J(N)}\right),$$

where $J_i := J \cap H_i$.

Proof. Consider the scalar function ζ_i of $\mathcal{P}_i$, for $i = 1, 2$, and assume first that $\zeta_{1,\mathbf{C}_G(N)} = \zeta_{2,\mathbf{C}_G(N)}$. Fix $N \leq J \leq G$ and set $J_i := J \cap H_i$ and $C := \mathbf{C}_J(N)$. Let $\psi_1 \in \mathrm{Irr}(J_1 \mid \vartheta_1)$ and write $\psi_2 := \sigma_{J_1}(\psi_1)$. Then, there exists $\mathcal{Q} \in \mathrm{Proj}(J/N)$ such that ψ_i is afforded by $\mathcal{Q}_{J_i} \otimes \mathcal{P}_{i,J_i}$. Now, for $i = 1, 2$,

$$\psi_{i,C} = \vartheta_i(1)\zeta_{i,C} \cdot \mathrm{Tr}\left(\mathcal{Q}_C\right),$$

and therefore $\vartheta_2(1)\psi_{1,C} = \vartheta_1(1)\psi_{2,C}$. We deduce that $\mathrm{Irr}(\psi_{1,C}) = \mathrm{Irr}(\psi_{2,C})$.

Assume now that (ii) holds. Let $c \in \mathbf{C}_G(N)$ and set $J := \langle N, c\rangle$. Notice that $C := \mathbf{C}_J(N) = \mathbf{Z}(J)$ is abelian. Let $\psi_1 \in \mathrm{Irr}(J_1 \mid \vartheta_1)$, set $\psi_2 := \sigma_{J_1}(\psi_1)$ and consider $Q \in \mathrm{Proj}(J/N)$ such that

$Q_{J_i} \otimes \mathcal{P}_{i,J_i}$ affords ψ_i. Then, for $i = 1, 2$, we have $\psi_i(c) = \vartheta_i(1)\zeta_i(c) \cdot \mathrm{Tr}(\mathcal{Q}(c))$. However, from the assumption and since C is abelian, we can find a linear character $\lambda \in \mathrm{Irr}(C)$ such that $\psi_i(c) = \psi_i(1)\lambda(c)$. It follows that $\mathrm{Tr}(\mathcal{Q}(c)) \neq 0$ and therefore

$$\zeta_i(c) = \frac{\lambda(c)}{\mathrm{Tr}\left(\mathcal{Q}(c)\right)} \frac{\psi_i(1)}{\vartheta_i(1)}.$$

By Lemma 3.1.2, we obtain $\psi_1(1)/\vartheta_1(1) = \psi_2(1)/\vartheta_2(1)$ and we conclude that $\zeta_1(c) = \zeta_2(c)$. □

Definition 3.3.4. For $N \trianglelefteq G$, we say that two character triples (H_1, M_1, ϑ_1) and (H_2, M_2, ϑ_2) are N-**central isomorphic**, and write

$$(H_1, M_1, \vartheta_1) \sim_N^c (H_2, M_2, \vartheta_2),$$

if $G = NH_1 = NH_2$, $\mathbf{C}_G(N) \leq H_1 \cap H_2$ and there exists a pair of projective representations $(\mathcal{P}_1, \mathcal{P}_2)$ giving an N-isomorphism $(\iota, \sigma_\bullet)$ and satisfying the equivalent conditions of Lemma 3.3.3. In this case $(\iota, \sigma_\bullet)$ is called an N-**central isomorphism**. As in Definition 3.3.2, if we want to specify a choice of projective representations, then we say that the N-central isomorphism is given by $(\mathcal{P}_1, \mathcal{P}_2)$ or that the character triples are N-central isomorphic via $(\mathcal{P}_1, \mathcal{P}_2)$.

3.3.2 N-block isomorphism

We now add one further requirement to the definition of N-central isomorphism. To motivate this requirement we go back to Dade's Conjecture. Consider the two character triples $(A_{\mathbb{D},\vartheta}, G_{\mathbb{D}}, \vartheta)$ and $(A_{\mathbb{E},\varphi}, G_{\mathbb{E}}, \varphi)$. For a fixed block B of G, we want to consider only those $\vartheta \in \mathrm{Irr}(G_{\mathbb{D}})$ and $\varphi \in \mathrm{Irr}(G_{\mathbb{E}})$ such that $\mathrm{bl}(\vartheta)^G = B = \mathrm{bl}(\varphi)^G$. In this case, we would like to say that, if the two character triples are isomorphic, then the bijection $\sigma : \mathrm{Irr}(J_{\mathbb{D},\vartheta} \mid \vartheta) \to \mathrm{Irr}(J_{\mathbb{E},\varphi} \mid \varphi)$ is compatible with block induction, whenever $G \leq J \leq A$. This requirement is included in our next definition.

Lemma 3.3.5. *Let $M \trianglelefteq J_0 \leq J$ and $\vartheta \in \mathrm{Irr}(M)$. Set $b := \mathrm{bl}(\vartheta)$ and suppose there exists a defect group D of b such that $\mathbf{C}_J(D) \leq J_0$. Then the induced block c^J is defined for every block c of J_0 that covers b.*

Proof. By [Nav98, Theorem 9.26], there exists a defect group D of c such that $D = Q \cap N$. It follows that $\mathbf{C}_J(Q) \leq \mathbf{C}_J(D) \leq J_0$ and therefore c is admissible with respect to G (see [Nav98, p. 213]). By the argument preceding [Nav98, Theorem 9.24] we conclude that c^J is defined. □

Suppose that (H_1, M_1, ϑ_1) and (H_2, M_2, ϑ_2) are two N-isomorphic character triples and let $N \leq J \leq G$ and $J_i := J \cap H_i$. If there exists a defect group D_i of the block $\mathrm{bl}(\vartheta_i)$ such that $\mathbf{C}_G(D_i) \leq H_i$, then it follows by Lemma 3.3.5 that the induced block $\mathrm{bl}(\psi_i)^J$ is defined for every $\psi_i \in \mathrm{Irr}(J_i \mid \vartheta_i)$. Notice that in this case $\mathbf{C}_G(N) \leq \mathbf{C}_G(D_i) \leq H_i$. The next definition can be found in [Spä17, Definition 6.3].

Definition 3.3.6. For $N \trianglelefteq G$, we say that two character triples (H_1, M_1, ϑ_1) and (H_2, M_2, ϑ_2) are N-**block isomorphic**, and write

$$(H_1, M_1, \vartheta_1) \sim_N (H_2, M_2, \vartheta_2),$$

if $G = NH_1 = NH_2$ and, for $i = 1,2$, there exists a defect group D_i of $\mathrm{bl}(\vartheta_i)$ such that $\mathbf{C}_G(D_i) \leq H_i$, and if there exists a pair of projective representations $(\mathcal{P}_1, \mathcal{P}_2)$ giving an N-central isomorphism $(\iota, \sigma_\bullet)$ such that

$$\mathrm{bl}\left(\psi\right)^J = \mathrm{bl}\left(\sigma_{J_1}(\psi)\right)^J$$

for every $N \leq J \leq G$ and $\psi \in \mathrm{Irr}(J_1 \mid \vartheta_1)$ where $J_1 := J \cap H_1$. In this case $(\iota, \sigma_\bullet)$ is called an N-**block isomorphism**. If we want to specify a choice of projective representations, then we say that the N-block isomorphism is given by $(\mathcal{P}_1, \mathcal{P}_2)$ or that the character triples are N-block isomorphic via $(\mathcal{P}_1, \mathcal{P}_2)$.

Before proceeding further, we give a more explicit list of all the properties that need to be checked in order to have an N-block isomorphism.

Remark 3.3.7. For $N \trianglelefteq G$ and $H_1, H_2 \leq G$, two character triples (H_1, M_1, ϑ_1) and (H_2, M_2, ϑ_2) are N-block isomorphic if the following conditions are satisfied:

(i) $N \trianglelefteq NH_1 = NH_2 =: G$, $M_1 = H_1 \cap N$ and $M_2 = H_2 \cap N$. We denote the canonical isomorphisms by $l_i : H_i/M_i \to G/N$ and by $\iota := l_2^{-1} \circ l_1 : H_1/M_1 \to H_2/M_2$;

(ii) for $i = 1,2$, there exists a defect group $D_i \in \delta(\mathrm{bl}(\vartheta_i))$ such that $\mathbf{C}_G(D_i) \leq H_i$. In particular $\mathbf{C}_G(N) \leq H_1 \cap H_2$;

(iii) For $i = 1,2$, there exists a projective representation $\mathcal{P}_i \in \mathrm{Proj}(H_i)$ associated with (H_i, M_i, ϑ_i) and with factor set α_i such that

$$\overline{\alpha}_1(x,y) = \overline{\alpha}_2(\iota(x), \iota(y))$$

for every $x, y \in H_1/M_1$ and where $\overline{\alpha}_i$ is the factor set of H_i/M_i corresponding to α_i;

(iv) the scalar functions ζ_1 of $\mathcal{P}_1$ and ζ_2 of $\mathcal{P}_2$ satisfy

$$\zeta_{1,\mathbf{C}_G(N)} = \zeta_{2,\mathbf{C}_G(N)};$$

(v) if $(\iota, \sigma_\bullet)$ is the N-isomorphism given by $(\mathcal{P}_1, \mathcal{P}_2)$, then

$$\mathrm{bl}(\psi)^J = \mathrm{bl}(\sigma_{J_1}(\psi))^J$$

for every $N \leq J \leq G$ and $\psi \in \mathrm{Irr}(J_1 \mid \vartheta_1)$ with $J_1 := J \cap H_1$.

If we exclude the last condition and we replace (ii) with $\mathbf{C}_G(N) \leq H_1 \cap H_2$, then we have an N-central isomorphism between the character triples, as defined in 3.3.4.

As we mentioned previously, other relations on character triples can be recovered as special cases of $\sim_N^c$ and $\sim_N$. For instance consider the relations $\geq$, $\geq_c$ and $\geq_b$ introduced in [Spä18, Definition 2.1, 2.7 and 4.2]. Then, it is immediate to see that $(G, N, \vartheta) \geq (H, M, \varphi)$ if and only if (G, N, ϑ) and (H, M, φ) are N-isomorphic, that $(G, N, \vartheta) \geq_c (H, M, \varphi)$ if and only if $(G, N, \vartheta) \sim_N^c (H, M, \varphi)$ and that $(G, N, \vartheta) \geq_b (H, M, \varphi)$ if and only if $(G, N, \vartheta) \sim_N (H, M, \varphi)$. Moreover, observe that $\sim_N^c$ and $\sim_N$ are equivalence relations. We collect this and other basic properties in the next lemma (see [Spä17, Lemma 3.8]).

Lemma 3.3.8. *Let N be a finite group.*

(i) If $(H_1, M_1, \vartheta_1) \sim_N (H_2, M_2, \vartheta_2)$ and $(H_2, M_2, \vartheta_2) \sim_N (H_3, M_3, \vartheta_3)$, then $(H_1, M_1, \vartheta_1) \sim_N (H_3, M_3, \vartheta_3)$. An analogue statement holds for $\sim_N^c$.

(ii) Suppose that $(H_1, M_1, \vartheta_1) \sim_N (H_2, M_2, \vartheta_2)$ and consider $N \leq J \leq G := NH_i$. Then $(J_1, M_1, \vartheta_1) \sim_N (J_2, M_2, \vartheta_2)$ where $J_i := J \cap H_i$. A similar statement holds for $\sim_N^c$.

(iii) Suppose that $(H_1, M_1, \vartheta_1) \sim_N (H_2, M_2, \vartheta_2)$ and let γ be an automorphism of $G = H_i N$. Then $(H_1^\gamma, M_1^\gamma, \vartheta_1^\gamma) \sim_{N^\gamma} (H_2^\gamma, M_2^\gamma, \vartheta_2^\gamma)$. A similar statement holds for $\sim_N^c$.

Proof. All the claims follows directly from the definition of $\sim_N$ and $\sim_N^c$. □

In some special cases, the conditions listed in Remark 3.3.7 can be simplified. We consider the case where the characters ϑ_1 and ϑ_2 extend to H_1 and H_2 respectively. Notice that, in this case, a projective representation $\mathcal{P}_i$ associated with ϑ_i is just a representation of H_i affording an extension of ϑ_i. The next result can be found in [Spä17, Lemma 3.10] and applies with minor changes if we replace N-block with N-central isomorphism.

Lemma 3.3.9. *Let $N \trianglelefteq G$ and consider two character triples (H_1, M_1, ϑ_1) and (H_2, M_2, ϑ_2) with $H_1, H_2 \leq G$. For $i = 1, 2$, let $\widetilde{\vartheta}_i$ be an extension of ϑ_i to H_i. Suppose that the following conditions are satisfied:*

(i) $G = NH_1 = NH_2$, $M_1 = H_1 \cap N$ and $M_2 = H_2 \cap N$. Moreover, there exists a defect group $D_i \in \delta(\mathrm{bl}(\vartheta_i))$ such that $\mathbf{C}_G(D_i) \leq H_i$;

(ii) $\mathrm{Irr}\left(\widetilde{\vartheta}_{1,\mathbf{C}_G(N)}\right) = \mathrm{Irr}\left(\widetilde{\vartheta}_{2,\mathbf{C}_G(N)}\right)$; and

(iii) $\mathrm{bl}\left(\widetilde{\vartheta}_{1,J_1}\right)^J = \mathrm{bl}\left(\widetilde{\vartheta}_{2,J_2}\right)^J$ for every $N \leq J \leq G$ and where $J_i := J \cap H_i$.

Then

$$(H_1, M_1, \vartheta_1) \sim_N (H_2, M_2, \vartheta_2),$$

via any pair of representations affording $\widetilde{\vartheta}_1$ and $\widetilde{\vartheta}_2$.

Proof. We check the requirements of Remark 3.3.7. By assumption we already have Remark 3.3.7 (i) and (ii). Let $\mathcal{R}_i$ be a representation affording $\widetilde{\vartheta}_i$, for $i = 1, 2$, and notice that the factor sets of $\mathcal{R}_1$ and $\mathcal{R}_2$ are both trivial. In particular Remark 3.3.7 (iii) is satisfied. Observe that $\mathbf{C}_G(N) \leq \mathbf{C}_G(D_i) \leq H_i$. By [Isa76, Lemma 2.27] it follows that $\widetilde{\vartheta}_{i,\mathbf{C}_G(N)}$ has a unique linear constituent, say ν_i. Moreover, by assumption $\nu_1 = \nu_2 =: \nu$. An easy computation shows that, if ζ_i is the scalar function of $\mathcal{R}_i$, then $\zeta_{i,\mathbf{C}_G(N)} = \nu$. Now Remark 3.3.7 (iv) follows. Finally, consider the N-isomorphism $(\iota, \sigma_\bullet)$ given by $(\mathcal{R}_1, \mathcal{R}_2)$. Let $N \leq J \leq G$ and set $J_i := J \cap H_i$. If $\psi_1 \in \mathrm{Irr}(J_1 \mid \vartheta_1)$, then by Gallagher's lemma we can find $\eta \in \mathrm{Irr}(J/N)$ such that $\psi_1 = \widetilde{\vartheta}_{1,J_1}\eta_{J_1}$. Then $\psi_2 := \sigma_{J_1}(\psi_1) = \widetilde{\vartheta}_{2,J_2}\eta_{J_2}$ and, using the hypothesis and [Spä17, Proposition 2.3], we conclude that $\mathrm{bl}(\psi_1)^J = \mathrm{bl}(\psi_2)^J$. □

We end this subsection with an elementary but useful observation. Suppose given N-block isomorphic character triples and consider $N \leq \widehat{N}$. Under certain conditions, it is possible to deduce that those character triples are in fact $\widehat{N}$-block isomorphic. A similar result can be stated for N-central isomorphisms.

Lemma 3.3.10. *Let $(H_1, M_1, \vartheta_1) \sim_N (H_2, M_2, \vartheta_2)$ with $H_i N = G$. Suppose that $G \leq \widehat{G}$ and let $N \leq \widehat{N} \trianglelefteq \widehat{G}$ with $\widehat{G} = G\widehat{N}$ and $N = G \cap \widehat{N}$. If $\mathbf{C}_{\widehat{G}}(D_i) \leq G$ for some $D_i \in \delta(\mathrm{bl}(\vartheta_i))$, then $(H_1, M_1, \vartheta_1) \sim_{\widehat{N}} (H_2, M_2, \vartheta_2)$.*

Proof. This follows directly from Definition 3.3.6. □

3.4 Construction of N-central and N-block isomorphism

In this section we introduce some rather technical methods that can be used to construct N-central and N-block isomorphisms of character triples. The results presented here will be used in subsequent chapters in order to prove our main theorems. For simplicity, we will only prove these results for the N-block isomorphisms. However, the reader should observe that all these results hold, with minor changes, when replacing N-block isomorphisms with N-central isomorphisms.

Let $(\iota, \sigma_\bullet)$ be an N-block isomorphism between (H_1, M_1, ϑ_1) and (H_2, M_2, ϑ_2). Set $G := H_1 N = H_2 N$. As shown in the previous section (see Proposition 3.3.1), for every $N \leq J \leq G$, there exists a bijection

$$\sigma_{J_1} : \mathrm{Irr}(J_1 \mid \vartheta_1) \to \mathrm{Irr}(J_2 \mid \vartheta_2),$$

where $J_i := J \cap H_i$. As we've already seen this bijections have many nice features. In the next proposition, we show that this bijections are compatible with N-block isomorphisms (this should be compared to [NS14b, Proposition 3.9] and [Spä17, Proposition 3.9]).

Proposition 3.4.1. *Let $(H_1, M_1, \vartheta_1) \sim_N (H_2, M_2, \vartheta_2)$ be given by $(\mathcal{P}_1, \mathcal{P}_2)$ and set $G := H_i N$. Consider $N \leq J \leq G$ and define $J_i := J \cap H_i$. Let $\psi_1 \in \mathrm{Irr}(J_1 \mid \vartheta_1)$ and set $\psi_2 := \sigma_{J_1}(\psi_1)$, where $(\iota, \sigma_\bullet)$ is the N-block isomorphism given by $(\mathcal{P}_1, \mathcal{P}_2)$. Then:*

(i) $(\mathbf{N}_{H_1}(J)_{\psi_1}, J_1, \psi_1) \sim_J (\mathbf{N}_{H_2}(J)_{\psi_2}, J_2, \psi_2)$;

(ii) $d(\psi_1) - d(\psi_2) = d(\vartheta_1) - d(\vartheta_2)$.

Proof. We first prove (i). As $J\mathbf{N}_{H_1}(J) = \mathbf{N}_G(J) = J\mathbf{N}_{H_2}(J)$, we may assume $J \trianglelefteq G$. Moreover, since $(\iota, \sigma_\bullet)$ is a strong isomorphism of character triples, we know that $\sigma_{J_1}(\psi_1)^{x_2} = \sigma_{J_1}(\psi_1^{x_1})$ for every $x_1 \in H_1$ and $x_2 \in H_2$ such that $\iota(M_1 x_1) = M_2 x_2$. In particular $\iota(H_{1,\psi_1}/M_1) = H_{2,\psi_2}/M_2$ and so $JH_{1,\psi_1} = JH_{2,\psi_2}$. Without loss of generality, we may assume $H_i = H_{i,\psi_i}$.

Using Theorem 3.2.9 (v) together with the isomorphisms $H_1/J_1 \simeq G/J \simeq H_2/J_2$, we can find a projective representation $\mathcal{Q} \in \mathrm{Proj}(G/J)$ such that $\mathcal{D}_i := \mathcal{Q}_{H_i} \otimes \mathcal{P}_i$ is associated with ψ_i, for $i = 1, 2$. We claim that the pair $(\mathcal{D}_1, \mathcal{D}_2)$ gives

$$(H_1, J_1, \psi_1) \sim_J (H_2, J_2, \psi_2).$$

Remark 3.3.7 (i) is clearly satisfied. Let D_i be a defect group of $\mathrm{bl}(\vartheta_i)$ such that $\mathbf{C}_G(D_i) \leq H_i$. Since $\mathrm{bl}(\psi_i)$ covers $\mathrm{bl}(\vartheta_i)$, we can find a defect group Q_i of $\mathrm{bl}(\psi_i)$ such that $D_i \leq Q_i$. Then

$\mathbf{C}_G(Q_i) \leq \mathbf{C}_G(D_i) \leq H_i$ and Remark 3.3.7 (ii) holds. In order to prove Remark 3.3.7 (iii), let β be the factor set of $\mathcal{Q}$ and set $\beta_i := \beta_{H_i \times H_i}$. Then $\mathcal{D}_i$ has factor set $\gamma_i = \alpha_i \beta_i$. Since α_1 and α_2 coincide via ι, then γ_1 and γ_2 coincide via the canonical isomorphism $\kappa : H_1/J_1 \to H_2/J_2$. Moreover, since $\mathbf{C}_G(J) \leq \mathbf{C}_G(N)$, if $x \in \mathbf{C}_G(J)$, then both $\mathcal{P}_i(x)$ and $\mathcal{D}_i(x)$ are scalar matrices. It follows that $\mathcal{Q}(x)$ is a scalar matrix and therefore, since the scalar functions of $\mathcal{P}_1$ and $\mathcal{P}_2$ coincide on $\mathbf{C}_G(N)$, we conclude that Remark 3.3.7 holds for $(\mathcal{D}_1, \mathcal{D}_2)$. Now, consider the J-isomorphism (κ, ρ) given by $(\mathcal{D}_1, \mathcal{D}_2)$ according to Proposition 3.3.1. Let $J \leq K \leq G$ and set $K_i := K \cap H_i$. For $i = 1, 2$, observe that $\mathrm{Irr}(K_i \mid \psi_i)$ is contained in $\mathrm{Irr}(K_i \mid \vartheta_i)$. We claim that σ_{K_1} and ρ_{K_1} coincide on $\mathrm{Irr}(K_1 \mid \psi_1)$. Fix $\chi_1 \in \mathrm{Irr}(K_1 \mid \psi_1)$ and let $\mathcal{R} \in \mathrm{Proj}(K/J)$ such that χ_1 is afforded by $\mathcal{R}_{K_1} \otimes \mathcal{D}_{1,K_1}$. Then

$$\begin{aligned}
\rho_{K_1}(\chi_1) &= \mathrm{Tr}(\mathcal{R}_{K_2} \otimes \mathcal{D}_{2,K_2}) \\
&= \mathrm{Tr}(\mathcal{R}_{K_2} \otimes \mathcal{Q}_{K_2} \otimes \mathcal{P}_{2,K_2}) \\
&= \sigma_{K_1}(\mathrm{Tr}(\mathcal{R}_{K_1} \otimes \mathcal{Q}_{K_1} \otimes \mathcal{P}_{1,K_1})) \\
&= \sigma_{K_1}(\mathrm{Tr}(\mathcal{R}_{K_1} \otimes \mathcal{D}_{1,K_1})) \\
&= \sigma_{K_1}(\chi_1)
\end{aligned}$$

and this proves our claim. It follows that

$$\mathrm{bl}(\chi_1)^K = \mathrm{bl}(\sigma_{K_1}(\chi_1))^K = \mathrm{bl}(\rho_{K_1}(\chi_1))^K.$$

This proves Remark 3.3.7 (v) and therefore the first half of this proposition.

To prove (ii) notice that, since $\psi_1(1)/\vartheta_1(1) = \psi_2(1)/\vartheta_2(1)$ by Lemma 3.1.2 and $|J : J_i| = |N : M_i|$, it follows that

$$p^{d(\psi_1) - d(\psi_2)} = \frac{|J_1|_p \psi_2(1)_p}{|J_2|_p \psi_1(1)_p} = \frac{|M_1|_p \vartheta_2(1)_p}{|M_2|_p \vartheta_1(1)_p} = p^{d(\vartheta_1) - d(\vartheta_2)}.$$

This completes the proof. □

The next proposition can be used to obtain new N-block isomorphic character triples involving irreducibly induced characters. This is the case, for instance, when we apply the Fong–Reynolds correspondence or the Clifford correspondence. Before proving this result, we need an easy lemma.

Lemma 3.4.2. *Let $N \trianglelefteq G$ and $\vartheta \in \mathrm{Irr}(N)$. If $\vartheta^G \in \mathrm{Irr}(G)$, then $\mathbf{C}_G(N) \leq N$.*

Proof. Set $H := N\mathbf{C}_G(N)$ and observe that $\psi := \vartheta^H \in \mathrm{Irr}(H)$. Since ϑ is H-invariant we have $\psi_N = e\vartheta$ with $e = |H : N|$. However $e = [\psi_N, \vartheta] = [\psi, \psi] = 1$ and therefore $\mathbf{C}_G(N) \leq N$. □

The next result should be compared to [NS14b, Theorem 3.14].

Proposition 3.4.3. *Let $N \trianglelefteq G$ and $G_0 \leq G$. For $i = 1, 2$, consider $H_i \leq G$ such that $G = NH_i$ and set $M_i := N \cap H_i$, $H_{0,i} := G_0 \cap H_i$, $M_{0,i} := G_0 \cap M_i$ and $N_0 := G_0 \cap N \trianglelefteq G_0$. Suppose that $G = G_0 N$, that $H_i = H_{0,i} M_i$ and that $\varphi_i := (\varphi_{0,i})^{M_i} \in \mathrm{Irr}(M_i)$, for some $\varphi_{0,i} \in \mathrm{Irr}(M_{0,i})$. If*

(i) $(H_{0,1}, M_{0,1}, \varphi_{0,1}) \sim_{N_0} (H_{0,2}, M_{0,2}, \varphi_{0,2})$;

(ii) *there exists a defect group* $D_i \in \delta(\mathrm{bl}(\varphi_i))$ *such that* $\mathbf{C}_G(D_i) \leq H_i$ *and*

(iii) *induction* $\mathrm{Ind}_{J_{0,i}}^{J_i} : \mathrm{Irr}(J_{0,i} \mid \varphi_{0,i}) \to \mathrm{Irr}(J_i \mid \varphi_i)$ *defines a bijection for every* $N \leq J \leq G$, *where* $J_i := J \cap H_i$ *and* $J_{0,i} := J \cap H_{0,i}$,

then $(H_1, M_1, \varphi_1) \sim_N (H_2, M_2, \varphi_2)$.

Proof. Assume $(H_{0,1}, M_{0,1}, \varphi_{0,1}) \sim_{N_0} (H_{0,2}, M_{0,2}, \varphi_{0,2})$ via $(\mathcal{P}_{0,1}, \mathcal{P}_{0,2})$ and let $\alpha_{0,i}$ be the factor set of $\mathcal{P}_{0,i}$. Consider the canonical isomorphisms $l_{0,i} : H_{0,i}/M_{0,i} \to G_0/N_0$ and $l_i : H_i/M_i \to G/N$ and set $i_0 := l_{0,2}^{-1} \circ l_{0,1}$ and $i = l_2^{-1} \circ l_1$. If $j : G/N \to G_0/N_0$ and $j_i : H_i/M_i \to H_{0,i}/M_{0,i}$ are the canonical isomorphisms, then we have a commutative diagram

$$\begin{array}{ccccc} & & \overset{i}{\frown\!\!\!\longrightarrow} & & \\ H_1/M_1 & \xrightarrow{l_1} & G/N & \xleftarrow{l_2} & H_2/M_2 \\ {\scriptstyle j_1}\downarrow & & \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle j_2} \\ H_{0,1}/M_{0,1} & \xrightarrow[l_{0,1}]{} & G_0/N_0 & \xleftarrow[l_{0,2}]{} & H_{0,2}/M_{0,2} \\ & & \underset{i_0}{\smile\!\!\!\longrightarrow} & & \end{array}$$

As in [NS14b, Theorem 3.14], consider the projective representation $\mathcal{P}_i := (\mathcal{P}_{0,i})^{H_i} \in \mathrm{Proj}(H_i)$ with factor set α_i defined as follows: let $\{t_{i,1}, \ldots, t_{i,n}\}$ be an H_i-transversal for $H_{0,i}$ contained in M_i, where $n := |G : G_0| = |H_i : H_{0,i}|$. For every $x \in H_i$, let

$$\mathcal{P}_{i,j,k}(x) := \begin{cases} \mathcal{P}_{0,i}(t_{i,j}^{-1} x t_{i,k}), & \text{if } t_{i,j}^{-1} x t_{i,k} \in H_{0,i} \\ 0, & \text{otherwise} \end{cases}$$

and define

$$\mathcal{P}_i(x) := \begin{pmatrix} \mathcal{P}_{i,1,1}(x) & \ldots & \mathcal{P}_{i,1,n}(x) \\ \vdots & & \vdots \\ \mathcal{P}_{i,n,1}(x) & \ldots & \mathcal{P}_{i,n,n}(x) \end{pmatrix}.$$

Then, $\mathcal{P}_i$ is a projective representation of H_i associated with $\varphi_i = \varphi_{0,i}^{M_i}$ with factor set α_i satisfying $\alpha_i(x, y) = \alpha_{0,i}(j_i(x), j_i(y))$ for every $x, y \in H_i/M_i$. Since

$$\alpha_{0,1}(j_1(x), j_1(y)) = \alpha_{0,2}(i_0(j_1(x)), i_0(j_1(y))),$$

we conclude that $\alpha_1(x, y) = \alpha_2(i(x), i(y))$, for all $x, y \in H_1/M_1$.

We claim that $\mathbf{C}_{H_i}(M_i) \leq G_0$. In this case, since $\mathbf{C}_G(N) \leq \mathbf{C}_G(D_i) \leq H_i$, we deduce $\mathbf{C}_G(N) \leq \mathbf{C}_{G_0}(N_0)$. To prove the claim, fix $x \in \mathbf{C}_{H_i}(M_i)$, set $J_i := \langle M_i, x \rangle$ and $J_{0,i} := G_0 \cap J_i$ and let $\varphi_{i,x}$ be an extension of φ_i to J_i. Since $\mathrm{Ind}_{J_{0,i}}^{J_i} : \mathrm{Irr}(J_{0,i} \mid \varphi_{0,i}) \to \mathrm{Irr}(J_i \mid \varphi_i)$ is a bijection, we can find an irreducible character $\varphi_{0,i,x} \in \mathrm{Irr}(J_{0,i} \mid \varphi_{0,i})$ such that $\varphi_{0,i,x}^{J_i} = \varphi_{i,x}$. By Lemma 3.4.2 we conclude that $x \in \mathbf{C}_{J_i}(J_{0,i}) \leq J_{0,i} \leq G_0$. This proves the claim and hence $\mathbf{C}_G(N) \leq \mathbf{C}_{G_0}(N_0)$. Now, since the scalar functions of $\mathcal{P}_{0,1}$ and $\mathcal{P}_{0,2}$ coincide on $\mathbf{C}_{G_0}(N_0)$ and $[t_{i,j}, \mathbf{C}_G(N)] = 1$, for every $i = 1, 2$ and $j = 1, \ldots, n$, then the scalar functions of $\mathcal{P}_1$ and $\mathcal{P}_2$ coincide on $\mathbf{C}_G(N)$.

To conclude, fix $N \leq J \leq G$, set $J_0 := J \cap G_0$, $J_i := J \cap H_i$ and $J_{0,i} := J \cap H_{0,i}$, and consider the bijections given by the character triple isomorphisms induced by $(\mathcal{P}_{0,1}, \mathcal{P}_{0,2})$ and $(\mathcal{P}_1, \mathcal{P}_2)$:

$$\begin{aligned} \sigma_{0,J_{0,1}} : \operatorname{Irr}(J_{0,1} \mid \varphi_{0,1}) &\to \operatorname{Irr}(J_{0,2} \mid \varphi_{0,2}) \\ \operatorname{Tr}(\mathcal{Q}_{0,J_{0,1}} \otimes \mathcal{P}_{0,1,J_{0,1}}) &\mapsto \operatorname{Tr}(\mathcal{Q}_{0,J_{0,2}} \otimes \mathcal{P}_{0,2,J_{0,2}}) \end{aligned}$$

where $\mathcal{Q}_0 \in \operatorname{Proj}(J_0/N_0)$, and

$$\begin{aligned} \sigma_{J_1} : \operatorname{Irr}(J_1 \mid \varphi_1) &\to \operatorname{Irr}(J_2 \mid \varphi_2) \\ \operatorname{Tr}(\mathcal{Q}_{J_1} \otimes \mathcal{P}_{1,J_1}) &\mapsto \operatorname{Tr}(\mathcal{Q}_{J_2} \otimes \mathcal{P}_{2,J_2}) \end{aligned}$$

where $Q \in \operatorname{Proj}(J/N)$. Observe that $\sigma_{J_1}\left(\psi_0^{J_1}\right) = \left(\sigma_{0,J_{0,1}}(\psi_0)\right)^{J_2}$ for every $\psi_0 \in \operatorname{Irr}(J_{0,1} \mid \varphi_{0,1})$. Let $\psi \in \operatorname{Irr}(J_1 \mid \varphi_1)$ and write $\psi = \psi_0^{J_1}$, for some $\psi_0 \in \operatorname{Irr}(J_{0,1} \mid \varphi_{0,1})$. Since by hypothesis $\operatorname{bl}(\psi_0)^{J_0} = \operatorname{bl}(\sigma_{0,J_{0,1}}(\psi_0))^{J_0}$, it follows that $\operatorname{bl}(\psi)^J = \operatorname{bl}(\sigma_{J_1}(\psi))^J$. □

As a consequence of Proposition 3.4.1 and Proposition 3.4.3, we obtain one of the most powerful tools that can be deduced in the presence of N-block isomorphisms. Let $N \trianglelefteq G$ and $H \leq G$ with $G = HN$. Set $M := H \cap N$ and suppose that there exists a bijection between the character sets $\mathcal{S} \subseteq \operatorname{Irr}(N)$ and $\mathcal{S}' \subseteq \operatorname{Irr}(M)$. If $(G, N, \vartheta) \sim_N (H, M, \varphi)$, for every $\vartheta \in \mathcal{S}$ corresponding to $\varphi \in \mathcal{S}'$, then we can construct a bijection between the set of characters of G lying above some character of $\mathcal{S}$ and the set of characters of H lying above some character of $\mathcal{S}'$. Moreover this bijection can be shown to be compatible with N-block isomorphisms. This result will have a fundamental impact in Chapter 9 (this can be compared to [NS14b, Proposition 4.7 (b)]; see also Proposition 9.1.5).

Proposition 3.4.4. *Let $K \trianglelefteq A$, $A_0 \leq A$ with $A = KA_0$ and, for every subgroup $X \leq A$, set $X_0 := X \cap A_0$. Consider A_0-stable subsets of characters $\mathcal{S} \subseteq \operatorname{Irr}(K)$ and $\mathcal{S}_0 \subseteq \operatorname{Irr}(K_0)$. Assume there exists an A_0-equivariant bijection*

$$\Psi : \mathcal{S} \to \mathcal{S}_0$$

such that

$$(A_\vartheta, K, \vartheta) \sim_K \left(A_{0,\vartheta}, K_0, \Psi(\vartheta)\right)$$

and

$$\mathbf{C}_A(D) \leq A_0$$

for every $\vartheta \in \mathcal{S}$ and some defect group D of $\operatorname{bl}(\Psi(\vartheta))$. Then, for every $K \leq J \leq A$, there exists an $A_{0,J}$-equivariant bijection

$$\Phi_J : \operatorname{Irr}(J \mid \mathcal{S}) \to \operatorname{Irr}(J_0 \mid \mathcal{S}_0)$$

such that

$$(A_{J,\chi}, J, \chi) \sim_J (A_{0,J,\chi}, J_0, \Phi_J(\chi))$$

and

$$\mathbf{C}_A(Q) \leq A_0$$

for every $\chi \in \operatorname{Irr}(J \mid \mathcal{S})$ and some defect group Q of $\operatorname{bl}(\Phi_J(\chi))$. Moreover Ψ preserves the defect of characters if and only if so does Φ_J.

Proof. Consider an $\mathbf{N}_{A_0}(J)$-transversal $\mathbb{S}$ in $\mathcal{S}$ and define $\mathbb{S}_0 := \{\Psi(\vartheta) \mid \vartheta \in \mathbb{S}\}$. Since Ψ is A_0-equivariant, it follows that $\mathbb{S}_0$ is an $\mathbf{N}_{A_0}(J)$-transversal in $\mathcal{S}_0$. For every $\vartheta \in \mathbb{S}$, with $\vartheta_0 := \Psi(\vartheta) \in \mathbb{S}_0$, we fix a pair of projective representations $(\mathcal{P}^{(\vartheta)}, \mathcal{P}_0^{(\vartheta_0)})$ giving $(A_\vartheta, K, \vartheta) \sim_K (A_{0,\vartheta}, K_0, \vartheta_0)$. Now, let $\mathbb{T}$ be an $\mathbf{N}_{A_0}(J)$-transversal in $\mathrm{Irr}(J \mid \mathcal{S})$ such that every character $\chi \in \mathbb{T}$ lies above a character $\vartheta \in \mathbb{S}$. Moreover, as $A = KA_0$, we have $J = KJ_0$ and therefore every $\chi \in \mathbb{T}$ lies over a unique $\vartheta \in \mathbb{S}$ by Clifford's theorem.

For $\chi \in \mathbb{T}$ lying over $\vartheta \in \mathbb{S}$, let $\varphi \in \mathrm{Irr}(J_\vartheta \mid \vartheta)$ be the Clifford correspondent of χ over ϑ. Set $\vartheta_0 := \Psi(\vartheta) \in \mathbb{S}_0$ and consider the $\mathbf{N}_{A_0}(J)_\vartheta$-equivariant bijection $\sigma_{J_\vartheta} : \mathrm{Irr}(J_\vartheta \mid \vartheta) \to \mathrm{Irr}(J_{0,\vartheta} \mid \vartheta_0)$ induced by our choice of projective representations $(\mathcal{P}^{(\vartheta)}, \mathcal{P}_0^{(\vartheta_0)})$. Let $\varphi_0 := \sigma_{J_\vartheta}(\varphi)$. Since Ψ is A_0-equivariant, we deduce that $J_{0,\vartheta} = J_{0,\vartheta_0}$ and therefore $\Phi_J(\chi) := \varphi^{J_0}$ is irreducible by the Clifford correspondence. Then, we define

$$\Phi_J(\chi^x) := \Phi_J(\chi)^x$$

for every $\chi \in \mathbb{T}$ and $x \in \mathbf{N}_{A_0}(J)$. This defines an $\mathbf{N}_{A_0}(J)$-equivariant bijection $\Psi : \mathrm{Irr}(J \mid \mathcal{S}) \to \mathrm{Irr}(J_0 \mid \mathcal{S}_0)$. Furthermore, using Proposition 3.4.1 it's clear that Ψ preserves the defect of characters if and only if so does Φ_J.

Next, using the fact that $(A_\vartheta, K, \vartheta) \sim_K \left(A_{0,\vartheta}, K_0, \vartheta_0\right)$ together with Proposition 3.4.1, we have

$$\left(A_{\vartheta,J_\vartheta,\psi}, J_\vartheta, \psi\right) \sim_{J_\vartheta} \left(A_{0,\vartheta,J_\vartheta,\psi}, J_{0,\vartheta}, \psi_0\right)$$

and, because $A_{\vartheta,J} \leq A_{\vartheta,J_\vartheta}$, it follows from Lemma 3.3.8 that

$$\left(A_{\vartheta,J,\psi}, J_\vartheta, \psi\right) \sim_{J_\vartheta} \left(A_{0,\vartheta,J,\psi}, J_{0,\vartheta}, \psi_0\right). \tag{3.4.1}$$

By hypothesis there exists a defect group D of $\mathrm{bl}(\vartheta_0)$ such that $\mathbf{C}_A(D) \leq A_0$. Since $\mathrm{bl}(\chi_0)$ covers $\mathrm{bl}(\vartheta_0)$ we can find a defect group Q of $\mathrm{bl}(\chi_0)$ such that $D \leq Q$. It follows that $\mathbf{C}_A(Q) \leq \mathbf{C}_A(D) \leq A_0$. Finally, we obtain

$$(A_{J,\chi}, J, \chi) \sim_J (A_{0,J,\chi}, J_0, \Phi_J(\chi))$$

by applying Proposition 3.4.3 together with (3.4.1). □

Next, we study the behaviour of N-block isomorphisms with respect to quotients. On one hand, N-block isomorphisms in a quotient can always be lifted. Although the converse doesn't hold in general, some partial results can be shown under additional assumptions. First, we recall a lemma on block induction in quotient groups.

Lemma 3.4.5. *Let $K \leq H \leq G$ with $K \trianglelefteq G$. Set $\overline{G} := G/K$ and $\overline{H} := H/K$ and consider $\overline{B} \in \mathrm{Bl}(\overline{G})$ dominated by $B \in \mathrm{Bl}(G)$ and $\overline{b} \in \mathrm{Bl}(\overline{H})$ dominated by $b \in \mathrm{Bl}(H)$.*

(i) If $\overline{b}^{\overline{G}} = \overline{B}$, then b^G is defined and coincides with B.

(ii) Assume that K is a p'-group or $K \leq \mathbf{Z}(G)$ and that $\overline{b}^{\overline{G}}$ is defined. If $b^G = B$, then $\overline{b}^{\overline{G}} = \overline{B}$.

Proof. This is [NS14b, Proposition 2.4]. For the second part of the statement notice that, when $K \leq \mathbf{Z}(G)$, we can write $K = \mathbf{O}_p(K) \times \mathbf{O}_{p'}(K)$. Then, applying first [NS14b, Proposition 2.4 (b)] with $Z := \mathbf{O}_{p'}(K)$ and then [NS14b, Proposition 2.4 (c)] with $Z := \mathbf{O}_p(G)$ we obtain our result. □

Using the above lemma, we can show that N-block isomorphisms can always be lifted from a quotient group. The next lemma might be compared to [Spä17, Corollary 4.4].

Lemma 3.4.6. *Let $K \trianglelefteq G$, $N \trianglelefteq G$ and, for $i = 1, 2$, consider $H_i \leq G$ such that $G = NH_i$ and $K \leq M_i := N \cap H_i$. Let $\vartheta_i \in \operatorname{Irr}(M_i)$ be H_i-invariant with $K \leq \operatorname{Ker}(\vartheta_i)$ and suppose that*

$$\left(\overline{H}_1, \overline{M}_1, \overline{\vartheta}_1\right) \sim_{\overline{N}} \left(\overline{H}_2, \overline{M}_2, \overline{\vartheta}_2\right),$$

where $\overline{X} := XK/K$, for every $X \leq G$, and $\overline{\vartheta}_i$ corresponds to ϑ_i via inflation of characters. Then $(H_1, M_1, \vartheta_1) \sim_N (H_2, M_2, \vartheta_2)$.

Proof. We check the requirements of Remark 3.3.7. The group theoretic conditions are satisfied. By hypothesis, there exists a defect group $\overline{D}_i$ of $\operatorname{bl}(\overline{\vartheta}_i)$ such that $\mathbf{C}_{\overline{G}}(\overline{D}_i) \leq \overline{H}_i$. Using [Nav98, Theorem 9.9], we can find a defect group Q_i of $\operatorname{bl}(\vartheta_i)$ such that $\overline{D}_i \leq \overline{Q}_i$. Then

$$\overline{\mathbf{C}_G(Q_i)} \leq \mathbf{C}_{\overline{G}}\left(\overline{Q}_i\right) \leq \mathbf{C}_{\overline{G}}\left(\overline{D}_i\right) \leq \overline{H}_i$$

and therefore $\mathbf{C}_G(Q_i) \leq H_i$. Let $(\overline{\mathcal{P}}_1, \overline{\mathcal{P}}_2)$ be a pair of projective representations giving the above $\overline{N}$-block isomorphism. For $i = 1, 2$, define the map $\mathcal{P}_i(x) := \overline{\mathcal{P}}_i(Kx)$ for every $x \in H_i$. Then $\mathcal{P}_i$ is a projective representation associated with (H_i, M_i, ϑ_i). By assumptions the factor sets of $\overline{\mathcal{P}}_1$ and $\overline{\mathcal{P}}_2$ coincide via the canonical isomorphism $\overline{\iota} : \overline{H}_1/\overline{M}_1 \to \overline{H}_2/\overline{M}_2$. By definition and using the third isomorphism theorem, we deduce that the factor sets of $\mathcal{P}_1$ and $\mathcal{P}_2$ coincide via $\iota : H_1/M_1 \to H_2/M_2$. Moreover, recalling that $\overline{\mathbf{C}_G(N)} \leq \mathbf{C}_{\overline{G}}(\overline{N})$, using the fact that the scalar functions of $\overline{\mathcal{P}}_1$ and $\overline{\mathcal{P}}_2$ coincide on $\mathbf{C}_{\overline{G}}(\overline{N})$ it follows that the scalar functions of $\mathcal{P}_1$ and $\mathcal{P}_2$ agree on $\mathbf{C}_G(N)$. To finish the proof, we need to check Remark 3.3.7 (v). Let $N \leq J \leq G$ and consider $\psi_1 \in \operatorname{Irr}(J_1 \mid \vartheta_1)$, where $J_1 := H_1 \cap J$. Observe that $K \leq \operatorname{Ker}(\vartheta_1) \leq \operatorname{Ker}(\psi_1)$ and denote by $\overline{\psi}_1 \in \operatorname{Irr}(\overline{J}_1 \mid \overline{\vartheta}_1)$ the character corresponding to ψ_1 via inflation. Let $(\iota, \sigma_\bullet)$ be the N-isomorphism given by $(\mathcal{P}_1, \mathcal{P}_2)$ and $(\overline{\iota}, \overline{\sigma}_\bullet)$ be the $\overline{N}$-isomorphism given by $(\overline{\mathcal{P}}_1, \overline{\mathcal{P}}_2)$. By definition, notice that $\overline{\sigma}_{\overline{J}_1}(\overline{\psi}_1)$ coincides with the character $\overline{\sigma_{J_1}(\psi)}$ whose inflation is $\sigma_{J_1}(\psi_1)$. By hypothesis we have

$$\operatorname{bl}\left(\overline{\psi}_1\right)^{\overline{J}} = \operatorname{bl}\left(\overline{\sigma}_{\overline{J}_1}\left(\overline{\psi}_1\right)\right)^{\overline{J}} = \left(\overline{\sigma_{J_1}(\psi_1)}\right)^{\overline{J}}$$

and it follows by Lemma 3.4.5 (i) that $\operatorname{bl}(\psi_1)^J = \operatorname{bl}(\sigma_{J_1}(\psi_1)^J)$. Now the proof is complete. □

As mentioned before, under additional requirements, we can show that some partial converse to the above statement also holds. The next result should be compared to [Spä17, Corollary 4.5].

Lemma 3.4.7. *Let $(H_1, M_1, \vartheta_1) \sim_N (H_2, M_2, \vartheta_2)$ and consider $K \trianglelefteq G = NH_i$ with $K \leq \operatorname{Ker}(\vartheta_1) \cap \operatorname{Ker}(\vartheta_2)$. Set $\overline{X} := XK/K$, for every $X \leq G := NH_i$, and denote by $\overline{\vartheta}_i$ the character of $\overline{M}_i$ corresponding to ϑ_i via inflation. If $\overline{\mathbf{C}_G(N)} = \mathbf{C}_{\overline{G}}(\overline{N})$ and there exist defect groups $\overline{D}_i$*

of $\mathrm{bl}(\overline{\vartheta}_i)$ *such that* $\mathbf{C}_{\overline{G}}(\overline{D}_i) \leq \overline{H}_i$*, then*

$$\left(\overline{H}_1, \overline{M}_1, \overline{\vartheta}_1\right) \sim_{\overline{N}} \left(\overline{H}_2, \overline{M}_2, \overline{\vartheta}_2\right)$$

provided that K *is a* p'*-group or that* $K \leq \mathbf{Z}(G)$*.*

Proof. By assumptions we already have conditions (i) and (ii) of Remark 3.3.7. Let $(\mathcal{P}_1, \mathcal{P}_2)$ be a pair of projective representations giving $(H_1, M_1, \vartheta_1) \sim_N (H_2, M_2, \vartheta_2)$. By Proposition 3.2.5 (ii) and recalling that $K \leq \mathrm{Ker}(\vartheta_i)$, we deduce that $\mathcal{P}_i$ is constant on K-cosets. Hence we can define a projective representation $\overline{\mathcal{P}}_i$ associated with $(\overline{H}_i, \overline{M}_i \overline{\vartheta}_i)$. By definition the factor sets of $\overline{\mathcal{P}}_1$ and $\overline{\mathcal{P}}_2$ coincide via the natural isomorphism. Moreover, since the scalar functions of $\mathcal{P}_1$ and $\mathcal{P}_2$ coincide on $\mathbf{C}_G(N)$ and using the assumption $\overline{\mathbf{C}_G(N)} = \mathbf{C}_{\overline{G}}(\overline{N})$, we deduce that the scalar functions of $\mathcal{P}_1$ and $\mathcal{P}_2$ coincide on $\mathbf{C}_{\overline{G}}(\overline{N})$. Finally, consider $N \leq J \leq G$ and $\psi \in \mathrm{Irr}(J_1 \mid \vartheta_1)$, where $J_1 := H_1 \cap N$. Denote by $\overline{\psi}_1 \in \mathrm{Irr}(\overline{J}_1 \mid \overline{\vartheta}_1)$ the character corresponding to ψ_1 via inflation. If (ι, σ) is the N-isomorphism given by $(\mathcal{P}_1, \mathcal{P}_2)$ and $(\overline{\iota}, \overline{\sigma})$ is the $\overline{N}$-isomorphism given by $(\overline{\mathcal{P}}_1, \overline{\mathcal{P}}_2)$, then $\overline{\sigma}_{\overline{J}_1}(\overline{\psi}_1)$ coincides with $\overline{\sigma_{J_1}(\psi_1)}$. Since $\mathrm{bl}(\psi_1)^J = \mathrm{bl}(\sigma_{J_1}(\psi_1))^J$, it follows by Lemma 3.4.5 that $\mathrm{bl}(\overline{\psi}_1)^{\overline{J}} = \mathrm{bl}(\overline{\sigma}_{\overline{J}_1}(\overline{\psi}_1))^{\overline{J}}$. □

In the next result we present a slight variation of the previous lemma. Notice that the group $S \leq \mathbf{Z}(\widehat{G})$ introduced in Theorem 3.2.6 satisfies the requirements of the following statement.

Lemma 3.4.8. *Let* $(H_1, M_1, \vartheta_1) \sim_N (H_2, M_2, \vartheta_2)$ *and consider* $K \leq H_1 \cap H_2$ *with* $K \trianglelefteq G$ *and* $N \cap K = 1$*. Set* $\overline{X} := XK/K$*, for every* $X \leq G := NH_i$*, and consider the character* $\overline{\vartheta}_i$ *corresponding to* ϑ_i *via the isomorphism* $\overline{M}_i \simeq M_i$*. Then*

$$\left(\overline{H}_1, \overline{M}_1, \overline{\vartheta}_1\right) \sim_{\overline{N}} \left(\overline{H}_2, \overline{M}_2, \overline{\vartheta}_2\right)$$

provided that K *is a* p'*-group or that* $K \leq \mathbf{Z}(G)$*.*

Proof. This is [Spä17, Proposition 3.13]. □

The next lemma shows that N-block isomorphisms of character triples are compatible with direct products.

Lemma 3.4.9. *For* $j = 1, 2$ *let* $(H_{j,1}, M_{j,1}, \vartheta_{j,1}) \sim_{N_j} (H_{j,2}, M_{j,2}, \vartheta_{j,2})$*. Then*

$$(H_1, M_1, \vartheta_1) \sim_N (H_2, M_2, \vartheta_2),$$

where $N := N_1 \times N_2$, $H_i := H_{1,i} \times H_{2,i}$, $M_i := M_{1,i} \times M_{2,i}$ *and* $\vartheta_i := \vartheta_{1,i} \times \vartheta_{2,i} \in \mathrm{Irr}(M_i)$*.*

Proof. This is [Spä17, Theorem 5.1]. □

In addition, N-block isomorphisms of character triples are compatible with wreath products.

Lemma 3.4.10. *Let* r *be a positive integer and consider* $(H_1, M_1, \vartheta_1) \sim_N (H_2, M_2, \vartheta_2)$*. Then*

$$\left(H_1 \wr S_r, M_1^r, \vartheta_1^r\right) \sim_{N^r} \left(H_2 \wr S_r, M_2^r, \vartheta_2^r\right).$$

Proof. This is [Spä17, Theorem 5.2]. □

As we have seen in Theorem 3.2.6, we can associate a central extension to any choice of projective representation associated with a character triple. In this way, it is sometime possible to reduce a problem on character triples to the case where the character extends. In the next theorem, we show that this construction is compatible with N-block isomorphisms. Then, whenever we have an N-isomorphism between the character triples (H_1, M_1, ϑ_1) and (H_2, M_2, ϑ_2), in order to check if this is an N-block isomorphism we can assume that ϑ_1 extends to H_1 and ϑ_2 extends to H_2. In this situation we can apply Lemma 3.3.9. The next result in [Spä17, Theorem 4.1].

Theorem 3.4.11. *Let (ι, σ) be an N-isomorphism between the character triples (H_1, M_1, ϑ_1) and (H_2, M_2, ϑ_2) given by $(\mathcal{P}_1, \mathcal{P}_2)$. For $i = 1, 2$, let α_i be the factor set of $\mathcal{P}_i$ and recall that $G = NH_i$. Since α_1 and α_2 coincide via ι, we can define a factor set α of G/N such that $\alpha_i = \alpha_{H_i \times H_i}$. As in Theorem 3.2.6, we can define a group multiplication on the set $\widehat{G} := G \times S$ given by*

$$(x, s) \cdot (y, t) := (xy, st\alpha(x, y))$$

for every $(x, s), (y, t) \in \widehat{G}$ and where S is the group generated by the values of α. Then $\widehat{G}$ is a central extension of G with projection $\epsilon : \widehat{G} \to G$, $(x, s) \mapsto x$ with kernel $S_0 := \{(1, s) \mid s \in S\} \simeq S$. Set $\widehat{X} := \epsilon^{-1}(X)$ and $Y_0 := \{(y, 1) \mid y \in Y\}$ for every $X \leq G$ and $Y \leq N$. Then $\widehat{Y} = Y_0 \times S_0$ (as a group) and Y_0 is isomorphic to Y via ϵ.

(i) For $i = 1, 2$, the group $\widehat{H}_i$ is the central extension of H_i induced by $\mathcal{P}_i$ (see Theorem 3.2.6).

(ii) If $N \leq J \leq G$ and $\mathbf{C}_G(J) \leq H_i$, then $\mathbf{C}_{\widehat{G}}(\widehat{J}) = \widehat{\mathbf{C}_G(J)}$.

(iii) There is an N_0-isomorphism $(\widehat{\iota}, \widehat{\sigma})$ between $(\widehat{H}_1, M_{1,0}, \vartheta_{1,0})$ and $(\widehat{H}_2, M_{2,0}, \vartheta_{2,0})$, where $\vartheta_{i,0}$ is the character of $M_{i,0} := (M_i)_0$ corresponding to ϑ_i via the isomorphism $\epsilon : M_{i,0} \to M_i$.

(iv) If $(\widehat{\iota}, \widehat{\sigma})$ is an N_0-block isomorphism, then (ι, σ) is an N-block isomorphism.

Proof. The first part of the statement follows by Theorem 3.2.6. Let $N \leq J \leq G$ and notice that $\mathbf{C}_{\widehat{G}}(\widehat{C}) \leq \widehat{\mathbf{C}_G(J)}$. On the other hand, if $\mathbf{C}_G(J) \leq H_1$ and $x \in \mathbf{C}_G(J) \leq \mathbf{C}_{H_1}(M_1)$, then $\mathcal{P}_1(c)$ is a scalar matrix. Let $(c, s) \in \widehat{\mathbf{C}_G(J)}$, with $c \in \mathbf{C}_G(J)$, and consider $(y, t) \in \widehat{J}$, with $y \in J$. We want to show that (c, s) commutes with (y, t). For this, it is enough to prove that $\alpha(c, y) = \alpha(y, c)$. If $J_1 := J \cap H_1$, then $J = NJ_1$ and we can write $y = ny_1$, with $n \in N$ and $y_1 \in J_1$. Since α is constant on N-cosets, it is enough to show that $\alpha(c, y_1) = \alpha(y_1, c)$ or equivalently that $\alpha_1(c, y_1) = \alpha(y_1, c)$. This last equality follows immediately from the fact that $\mathcal{P}_1(c)$ is a scalar matrix. We conclude that $\widehat{\mathbf{C}_G(J)} \leq \mathbf{C}_{\widehat{G}}(\widehat{J})$.

As in Theorem 3.2.6 consider the representation $\widehat{\mathcal{P}}_i$ of $\widehat{H}_i$ given by $\widehat{\mathcal{P}}_i(x, s) := s\mathcal{P}_i(x)$ for every $(x, s) \in \widehat{H}_i$. The factor sets of $\widehat{\mathcal{P}}_1$ and $\widehat{\mathcal{P}}_2$ are trivial and therefore coincide via the canonical isomorphism $\widehat{\iota} : \widehat{H}_1/M_{1,0} \to \widehat{H}_2/M_{2,0}$. In particular the character triples $(\widehat{H}_1, M_{1,0}, \vartheta_{1,0})$ and $(\widehat{H}_2, M_{2,0}, \vartheta_{2,0})$ are N_0-isomorphic via $(\widehat{\mathcal{P}}_1, \widehat{\mathcal{P}}_2)$. Let $(\widehat{\iota}, \widehat{\sigma})$ be the associated N_0-isomorphism. If $(\widehat{\iota}, \widehat{\sigma})$ is an N_0-block isomorphism, then (ι, σ) is an N-block isomorphism by Lemma 3.4.8. □

We conclude this section with a fundamental result. Consider a group N and two character triples (H_1, M_1, ϑ_1) and (H_2, M_2, ϑ_2) satisfying Remark 3.3.7 (i). What is the role of $G = NH_1 = NH_2$

when considering N-block isomorphism? According to the Butterfly theorem, what really matters is not the group G itself but the automorphism group induced by G on N.

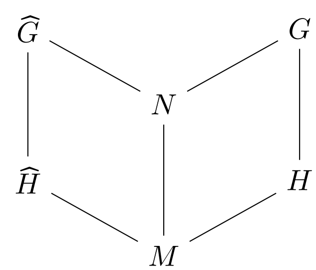
Theorem 3.4.12 (Butterfly Theorem)**.** *Let* $(H_1, M_1, \vartheta_1) \sim_N (H_2, M_2, \vartheta_2)$ *with* $G = NH_1 = NH_2$*. Let* $N \trianglelefteq \widehat{G}$ *and consider the canonical maps* $\epsilon : G \to \mathrm{Aut}(N)$ *and* $\widehat{\epsilon} : \widehat{G} \to \mathrm{Aut}(N)$*. If* $\epsilon(G) = \widehat{\epsilon}(\widehat{G})$*, then*

$$\left(\widehat{H}_1, M_1, \vartheta_1\right) \sim_N \left(\widehat{H}_2, M_2, \vartheta_2\right),$$

where $\widehat{H}_i := \widehat{\epsilon}^{-1}(\epsilon(H_i))$*, for* $i = 1, 2$*.*

Proof. The proof of this result is rather technical and can be found in [Spä17, Theorem 5.3] □

The reader might wonder about the origin of the name of the previous result. In order to explain it, we need to consider a simplified situation: let $(G, N, \vartheta) := (H_1, M_1, \vartheta_1)$ and $(H, M, \varphi) := (H_2, M_2, \vartheta_2)$. Then the setting of Theorem 3.4.12 can be described by the following diagram which resembles a butterfly



here $\widehat{H} = \widehat{\epsilon}^{-1}(\epsilon(H))$.

3.5 The Character Triple Conjecture

Having introduced N-block isomorphisms of character triples we can now introduce Späth's Character Triple Conjecture. This is a version of Dade's Projective Conjecture (see Conjecture 2.5.3) adapted in order to include the possibility of controlling Clifford theory via N-block isomorphisms of character triples. As we will see, the Character Triple Conjecture works as an inductive condition for Dade's Projective Conjecture: namely, it can be used to reduce Dade's conjecture to a problem on quasisimple groups. This reduction theorem was proved by Späth in [Spä17] which is the main reference for the results presented in this section.

Let G be a finite group, Z a central subgroup of G and $\lambda \in \mathrm{Irr}(Z)$. Set $Z_p := \mathbf{O}_p(Z)$ and consider a block $B \in \mathrm{Bl}(G)$ whose defect groups strictly contain Z_p. Then, recall that Dade's Projective Conjecture (see Section 2.5) posits that

$$\sum_{\mathbb{D} \in \mathfrak{P}(G, Z_p)/\sim_G} (-1)^{|\mathbb{D}|} \left|\mathrm{Irr}^d\left(B_{\mathbb{D}} \mid \lambda\right)\right| = 0, \tag{3.5.1}$$

where the sum runs over the set of p-chains of G, with first term equal to Z_p, up to G-conjugation (see Section 2.5 for further details). We want to restate this equality in a more suitable way. To do so, let $\epsilon \in \{+,-\}$ and consider the subset $\mathfrak{P}(G, Z_p)_\epsilon$ of $\mathfrak{P}(G, Z_p)$ consisting of those p-chains $\mathbb{D}$ that satisfy $(-1)^{|\mathbb{D}|} = \epsilon 1$. Then $\mathfrak{P}(G, Z_p)_+$ is the set of chains of even length and $\mathfrak{P}(G, Z_p)_-$ is the set of chains of odd length. Now, (3.5.1) is equivalent to

$$\sum_{\mathbb{D} \in \mathfrak{P}(G,Z_p)_+/\sim_G} (-1)^{|\mathbb{D}|} \left|\mathrm{Irr}^d\left(B_{\mathbb{D}} \mid \lambda\right)\right| = \sum_{\mathbb{D} \in \mathfrak{P}(G,Z_p)_-/\sim_G} (-1)^{|\mathbb{D}|} \left|\mathrm{Irr}^d\left(B_{\mathbb{D}} \mid \lambda\right)\right|. \tag{3.5.2}$$

Next, we restate the equality (3.5.2) as the existence of a bijection between two sets of finite order. To see this, for $\epsilon = \pm$, define

$$\mathcal{C}^d(B, Z_p, \lambda)_\epsilon := \left\{(\mathbb{D}, \vartheta) \;\middle|\; \mathbb{D} \in \mathfrak{P}(G, Z_p)_\epsilon, \vartheta \in \mathrm{Irr}^d\left(B_{\mathbb{D}} \mid \lambda\right)\right\}$$

and denote by $\mathcal{C}^d(B, Z_p, \lambda)_\epsilon/G$ the corresponding set of G-orbits. Then Dade's Projective Conjecture is equivalent to saying that there exists a bijection

$$\Omega : \mathcal{C}^d(B, Z_p, \lambda)_+/G \to \mathcal{C}^d(B, Z_p, \lambda)_-/G. \tag{3.5.3}$$

Finally, we strengthen Dade's Projective Conjecture by requiring additional properties on the bijection (3.5.3). First, if $G \trianglelefteq A$, then we ask that the bijection Ω is $A_{B,Z,\lambda}$-equivariant. Moreover, we want to control Clifford theory via G-block isomorphisms of character triples. For this purpose, denote by $\overline{(\mathbb{D}, \vartheta)}$ the G-orbit of any $(\mathbb{D}, \vartheta) \in \mathcal{C}^d(B, Z_p, \lambda)$ and notice that we can associate the character triple $(A_{\mathbb{D},\vartheta}, G_{\mathbb{D}}, \vartheta)$ to the pair $(\mathbb{D}, \vartheta)$. By Lemma 3.3.8, the equivalence class of the character triple $(A_{\mathbb{D},\vartheta}, G_{\mathbb{D}}, \vartheta)$ under $\sim_G$ does not depend on the representative of $\overline{(\mathbb{D}, \vartheta)}$ and therefore we can require that the bijection Ω satisfies

$$\left(A_{\mathbb{D},\vartheta}, G_{\mathbb{D}}, \vartheta\right) \sim_G \left(A_{\mathbb{E},\chi}, G_{\mathbb{E}}, \chi\right) \tag{3.5.4}$$

for every $(\mathbb{D}, \vartheta) \in \mathcal{C}^d(B, Z_p, \lambda)$ and $(\mathbb{E}, \chi) \in \Omega(\overline{(\mathbb{D}, \vartheta)})$. As a last step we make a minor simplification. Define

$$\mathcal{C}^d(B, Z_p) := \coprod_{\lambda \in \mathrm{Irr}(Z)} C^d(B, Z_p, \lambda).$$

Then, by using Lemma 3.3.3, a bijection $\Omega : \mathcal{C}^d(B, Z_p)_+/G \to \mathcal{C}^d(B, Z_p)_-/G$ that satisfies (3.5.4) restricts to $\Omega_\lambda : \mathcal{C}^d(B, Z_p, \lambda) \to \mathcal{C}^d(B, Z_p, \lambda)$ for every $\lambda \in \mathrm{Irr}(Z)$. We are now ready to state the Character Triple Conjecture (see [Spä17, Conjecture 6.3]).

Conjecture 3.5.1 (Character Triple Conjecture). *Let G be a finite group, $Z \leq \mathbf{Z}(G)$ be a p-subgroup and consider $B \in \mathrm{Bl}(G)$ with defect groups strictly larger than Z. Suppose that $G \trianglelefteq A$. Then, for every $d \geq 0$, there exists an $A_{B,Z}$-equivariant bijection*

$$\Omega : \mathcal{C}^d(B, Z)_+/G \to \mathcal{C}^d(B, Z)_-/G$$

such that

$$\left(A_{\mathbb{D},\vartheta}, G_{\mathbb{D}}, \vartheta\right) \sim_G \left(A_{\mathbb{E},\chi}, G_{\mathbb{E}}, \chi\right)$$

for every $(\mathbb{D}, \vartheta) \in \mathcal{C}^d(B, Z)_+$ and $(\mathbb{E}, \chi) \in \Omega(\overline{(\mathbb{D}, \vartheta)})$.

We point out that, by using Theorem 3.4.12, we could have equivalently stated Conjecture 3.5.1 by considering the group $A := G \rtimes \mathrm{Aut}(G)$.

By the above argument, the Character Triple Conjecture implies Dade's Projective Conjecture. But even more is true, in fact in [Spä17, Proposition 6.4] it was shown that Conjecture 3.5.1 implies Dade's Extended Projective Conjecture (see [Dad97, 4.10]). It is also suspected that Conjecture 3.5.1 also implies the final version of Dade's conjecture, i.e. Dade's Inductive Conjecture (see [Dad97, 5.8]).

As mentioned previously, the Character Triple Conjecture can be used to reduce Dade's Projective Conjecture to a question on quasisimple groups. In fact, in [Spä17] it is shown that if Conjecture 3.5.1 holds for every quasisimple group, then Dade's Projective Conjecture holds for every finite group. Recall that a group X is **involved** in a group G if there exists $H \leq G$ and $N \trianglelefteq H$ such that $H/N \simeq X$.

Theorem 3.5.2. *Let G be a finite group and suppose that every covering group X of a nonabelian simple group involved in G satisfies Conjecture 3.5.1 with respect to $X \trianglelefteq X \rtimes \mathrm{Aut}(X)$. Then Dade's Projective Conjecture holds for G.*

Proof. This is [Spä17, Theorem 1.3]. □

In the rest of this section we make some helpful comments and simplifications of Conjecture 3.5.1. We start with a fundamental remark regarding the type of p-chains used in Conjecture 3.5.1. As we have seen in Section 2.5, Dade's Projective Conjecture can be equivalently stated by using the set $\mathfrak{P}(G)$ of all p-chains of G, the set $\mathfrak{N}(G)$ of normal p-chains, the set $\mathfrak{E}(G)$ of elementary abelian p-chains or the set $\mathfrak{R}(G)$ of radical p-chains. This is an important feature since it allows to work with different sets of p-chains with specific properties depending on the groups that we are dealing with. For instance, as we will see later on, the set of normal p-chains is a good choice when working with p-solvable groups while in the case of groups of Lie type in nondefining characteristic it is preferable to work with the set of elementary abelian chains. An analogous property is shared by the Character Triple Conjecture. These results can be found in [Dad94, Proposition 2.10] and [Spä17, Proposition 6.10]. Let Z be a central p-subgroup of G, consider a block B of G and a nonnegative integer d. For $\epsilon = \pm$ and $\kappa \in \{\mathrm{norm}, \mathrm{elem}, \mathrm{rad}\}$, define $\mathcal{C}^d_\kappa(B,Z)_\epsilon$ to be the subset of $\mathcal{C}^d(B,Z)_\epsilon$ consisting of those pairs $(\mathbb{D},\vartheta)$ that satisfy $\mathbb{D} \in \mathfrak{N}(G,Z)$, $\mathbb{D} \in \mathfrak{E}(G,Z)$ or $\mathbb{D} \in \mathfrak{R}(G,Z)$ respectively.

Lemma 3.5.3. *Let G be a finite group, $Z \leq \mathbf{Z}(G)$ be a p-subgroup and consider $B \in \mathrm{Bl}(G)$ with defect groups strictly larger than Z. Suppose that $G \trianglelefteq A$ and fix $\kappa \in \{\mathrm{norm}, \mathrm{elem}, \mathrm{rad}\}$. Then Conjecture 3.5.1 holds if and only if, for every $d \geq 0$, there exists an $A_{B,Z}$-equivariant bijection*

$$\Omega : \mathcal{C}^d_\kappa(B,Z)_+/G \to \mathcal{C}^d_\kappa(B,Z)_-/G$$

such that

$$\left(A_{\mathbb{D},\vartheta}, G_{\mathbb{D}}, \vartheta\right) \sim_G \left(A_{\mathbb{E},\chi}, G_{\mathbb{E}}, \chi\right)$$

for every $(\mathbb{D},\vartheta) \in \mathcal{C}^d_\kappa(B,Z)_+$ and $(\mathbb{E},\chi) \in \Omega(\overline{(\mathbb{D},\vartheta)})$.

Proof. This is [Spä17, Proposition 6.10]. □

The next result tell us that in Conjecture 3.5.1 it is no loss of generality to assume $\mathbf{O}_p(G) = Z \leq \mathbf{Z}(G)$. For simplicity we will usually denote $\mathcal{C}^d(B, \mathbf{O}_p(\mathbf{G}))_\epsilon$ simply by $\mathcal{C}^d(B)_\epsilon$.

Lemma 3.5.4. *Conjecture 3.5.1 holds whenever $Z < \mathbf{O}_p(G)$.*

Proof. Consider $\mathbb{D} \in \mathfrak{N}(G, Z)$ with $\mathbb{D} = \{D_0 < D_1 \cdots < D_n\}$. If $\mathbf{O}_p(G) \nleq D_n$, then define $\mathbb{D}^*$ to be the p-chain obtained by adding $\mathbf{O}_p(G)D_n$ to $\mathbb{D}$. Assume $\mathbf{O}_p(G) \leq D_n$ and let k be the unique nonnegative integer such that $\mathbf{O}_p(G) \leq D_k$ and $\mathbf{O}_p(G) \nleq D_{k-1}$. If $\mathbf{O}_p(G)D_{k-1} = D_k$, then we define $\mathbb{D}^*$ by deleting the term D_k from $\mathbb{D}$. If $\mathbf{O}_p(G)D_{k-1} < D_k$, then we define $\mathbb{D}^*$ by adding the term $\mathbf{O}_p(G)D_{k-1}$ to $\mathbb{D}$. This defines a self-inverse $\mathbf{N}_A(Z)$-equivariant bijection $^* : \overline{\mathfrak{N}(G, Z)} \to \overline{\mathfrak{N}(G, Z)}$ such that $|\mathbb{D}| = |\mathbb{D}^*| \pm 1$. In particular $G_{\mathbb{D}} = G_{\mathbb{D}^*}$ and we define $\Omega(\overline{(\mathbb{D}, \vartheta)}) := \overline{(\mathbb{D}^*, \vartheta)}$ for every $(\mathbb{D}, \vartheta) \in \mathcal{C}^d(B, Z)_+$. □

3.5.1 The nonblockwise Character Triple Conjecture

In the above section we have shown how the Character Triple Conjecture implies Dade's Projective Conjecture. Similarly, it can be seen that the following nonblockwise version of the Character Triple Conjecture implies the nonblockwise version of Dade's Projective Conjecture (see Conjecture 2.5.4).

For a finite group G, a p-subgroup $Z \leq \mathbf{Z}(G)$, a nonnegative integer d and $\epsilon \in \{+, -\}$ we define

$$\mathcal{C}^d(G, Z)_\epsilon := \bigcup_{B \in \mathrm{Bl}(G)} \mathcal{C}^d(B, Z)_\epsilon.$$

As usual G acts on $\mathcal{C}^d(G, Z)_\epsilon$ and we denote by $\mathcal{C}^d(G, Z)_\epsilon / G$ the corresponding set of G-orbits.

Conjecture 3.5.5 (Nonblockwise Character Triple Conjecture). *Let G be a finite group, $Z \leq \mathbf{Z}(G)$ be a p-subgroup and suppose that $G \trianglelefteq A$. Then, for every positive integer $d > 0$, there exists an $A_{B,Z}$-equivariant bijection*

$$\Omega : \mathcal{C}^d(G, Z)_+ / G \to \mathcal{C}^d(G, Z)_- / G$$

such that

$$\left(A_{\mathbb{D},\vartheta}, G_{\mathbb{D}}, \vartheta\right) \sim^c_G \left(A_{\mathbb{E},\chi}, G_{\mathbb{E}}, \chi\right)$$

for every $(\mathbb{D}, \vartheta) \in \mathcal{C}^d(G, Z)_+$ and $(\mathbb{E}, \chi) \in \Omega(\overline{(\mathbb{D}, \vartheta)})$.

Notice that in the above statement we consider G-central isomorphisms of character triples and not G-block isomorphisms of character triples. As we will see in Chapter 10 this is a much easier condition to check.

The argument used in Lemma 3.5.4 applies also in this case and shows that Conjecture 3.5.5 always holds if $Z < \mathbf{O}_p(G)$. For this reason we will often assume $Z = \mathbf{O}_p(G) \leq \mathbf{Z}(G)$. Moreover, in this case we denote $\mathcal{C}^d(G, Z)_\epsilon$ simply by $\mathcal{C}^d(G)_\epsilon$.

We mention that it is natural to expect that Conjecture 3.5.5 could be used as an inductive condition for Conjecture 2.5.4 in order to obtain a nonblockwise version of Theorem 3.5.2.

4

Character Triple Conjecture for p-Solvable Groups

As mentioned in Chapter 2, Global-Local counting conjectures are amongst the deepest problems in representation theory of finite groups. At the moment of writing, no conceptual explanation for any of these conjectures can be provided and the only way of proving them is by brute force, namely by using the classification of finite simple groups. In his papers [Dad92], [Dad94] and [Dad97], Dade introduced a series of increasingly deeper statements with the aim of reducing his conjecture to finite (quasi)simple groups. In order to do so, he had to find a version of his conjecture strong enough to hold for every finite group if proved for all (quasi)simple groups. Such a statement should incorporate aspects of Clifford theory that could be compatibly clued together when assuming the result for chief factors of an abmient group. The candidate for this purpose was found in Dade's Inductive Conjecture [Dad97, 5.8]. In Dade's words (see [Dad97])

"With a great amount of work it can be shown to hold for all finite groups if it holds whenever G is a nonabelian simple group"

However such a result has never been published. Ten years after Dade's claim, a fundamental step towards the solution of the McKay Conjecture has been achieved by Isaacs, Malle and Navarro in [IMN07]. In their paper, the McKay Conjecture is reduced to a stronger statement for (quasi)simple groups. Inspired by this result, other reduction theorems have been proved by Navarro and Tiep [NT11], by Navarro and Späth [NS14b], by Späth [Spä13a], [Spä13b], [Spä17] and by Navarro, Späth and Vallejo [NSV20]. However, contrary to Dade's philosophy, all the reduction theorems mentioned above reduce a certain statement for arbitrary finite groups to a much stronger statement for (quasi)simple groups.

Although these stronger statements, known as inductive conditions, have been originally formulated for (quasi)simple groups, they can be stated for all finite groups. Going back to Dade's philosophy, it should be possible to obtain, not only the original conjecture, but even the inductive condition itself, for every finite group, by proving the inductive condition for (quasi)simple groups. Therefore we now have the need of stronger reduction theorems that might be referred to as second generation reductions. In the case of the Alperin–McKay Conjecture this was achieved in [NS14b]. For the McKay Conjecture see [Ros]. The reader should observe that these stronger inductive conditions are not just mere strengthenings of the Global-Local conjectures. In fact, very deep consequences can be deduced from them: for instance, as shown in [NS14b], Brauer's Height Zero Conjecture follows from the inductive condition for the Alperin–McKay Conjecture.

We now consider the case of Dade's Projective Conjecture. As we know, the Character Triple Conjecture plays the role of an inductive condition for Dade's Projective Conjecture (see [Spä17, Theorem 1.3]). Following [NS14b] and [Ros], we would like to show that the Character Triple Conjecture holds for every finite group if it holds for all quasisimple groups. This would also complete Dade's plan by replacing Dade's Inductive Conjecture with the Character Triple Conjecture. To prove such a reduction theorem, it is necessary to study the structure of a minimal counterexample to the Character Triple Conjecture. As for all the above mentioned reductions, the first step in this direction is to show that such a counterexample cannot be p-solvable. This will be the main result of the present chapter. The results of this chapter can be found in the preprint [Ros21].

Theorem 4.1. *Let G be a finite p-solvable group with $\mathbf{O}_p(G) \leq \mathbf{Z}(G)$ and consider a p-block B of G with noncentral defect groups. Suppose that $G \trianglelefteq A$. Then, for every $d \geq 0$, there exists an A_B-equivariant bijection*

$$\Omega : \mathcal{C}^d(B)_+/G \to \mathcal{C}^d(B)_-/G$$

such that

$$\left(A_{\mathbb{D},\vartheta}, G_{\mathbb{D}}, \vartheta\right) \sim_G \left(A_{\mathbb{E},\chi}, G_{\mathbb{E}}, \chi\right)$$

for every $(\mathbb{D},\vartheta) \in \mathcal{C}^d(B)_+$ and $(\mathbb{E},\chi) \in \Omega(\overline{(\mathbb{D},\vartheta)})$.

Next, recall that for $\chi \in \mathrm{Irr}(G)$, the p-residue of χ is the nonnegative integer $r(\chi) := |G|_{p'}/\chi(1)_{p'}$. Following ideas of Isaacs and Navarro [IN02], we include the p-**residue** of characters into the picture.

Theorem 4.2. *There exists a bijection Ω satisfying the conditions of Theorem 4.1 and such that*

$$r(\vartheta) \equiv \pm r(\chi) \pmod p$$

for every $(\mathbb{D},\vartheta) \in \mathcal{C}^d(B)_+$ and some $(\mathbb{E},\chi) \in \Omega(\overline{(\mathbb{D},\vartheta)})$.

As a corollary to our results, we show that Dade's Extended Projective Conjecture [Dad97, 4.10], with the Isaacs-Navarro refinement, holds for every p-solvable group.

Corollary 4.3. *Dade's Extended Projective Conjecture with the Isaacs-Navarro refinement holds for every p-solvable group.*

Proof. This follows from Theorem 4.2 and [Spä17, Proposition 6.4]. □

According to Lemma 3.5.3 the Character Triple Conjecture can be equivalently stated considering various sets of p-chains. It appears that normal p-chains are more suitable when dealing with p-solvable groups. Therefore, every p-chain $\mathbb{D}$ considered in this chapter will be a normal p-chain, that is a chain $\mathbb{D} = \{P_0 < \cdots < P_n\}$ with $P_i \trianglelefteq P_n$ for every $i = 0, \ldots, n$.

4.1 N-block isomorphic character triples and Glauberman correspondence

The aim of this section is to prove Theorem 4.1.8 which will be one of the main ingredients in the proof of Theorem 4.1. To prove this result, we need to extend the bijection given in [NS14b, Theorem 5.13] to characters of positive height. This is done in Proposition 4.1.7 for the case where the D-correspondence (see [NS14b, Definition 5.3]) coincides with the Glauberman correspondence. Moreover, in this situation, we obtain a canonical bijection.

Recall that, for every $\chi \in \mathrm{Irr}(G)$, it is defined a linear character $\det(\chi)$ of G: let $\mathfrak{X}$ be a representation affording χ and set

$$\det(\chi)(g) := \det(\mathfrak{X}(g)).$$

Observe that this definition does not depend on the choice of $\mathfrak{X}$. The character $\det(\chi)$ is called the **determinant** of χ. Then, we can define the **determinantal order** of χ as

$$o(\chi) := |G : \mathrm{Ker}(\det(\chi))|.$$

Let $N \trianglelefteq G$ and $\vartheta \in \mathrm{Irr}(N)$ be a G-invariant character such that $(o(\vartheta)\vartheta(1), |G : N|) = 1$. By [Isa76, Corollary 8.16], there exists a unique extension $\vartheta^\diamond$ of ϑ to G such that $(o(\vartheta^\diamond), |G : N|) = 1$. The character $\vartheta^\diamond$ is called the **canonical extension** of ϑ to G. For instance, this happens if $(|G : N|, |N|) = 1$. In order to prove Proposition 4.1.7, we need some results on the extendibility of the canonical extension.

Lemma 4.1.1. *Let $H \leq G$ and $\chi \in \mathrm{Irr}(G)$ such that $\chi_H \in \mathrm{Irr}(H)$. Then $o(\chi_H)$ divides $o(\chi)$.*

Proof. Set $\psi := \chi_H$ and observe that $\det(\psi) = \det(\chi)_H$. If $K := \mathrm{Ker}(\det(\chi))$, then $K \cap H = \mathrm{Ker}(\det(\psi))$ and it follows that $o(\psi) = |H : K \cap H|$ divides $|G : K| = o(\chi)$. □

Corollary 4.1.2. *Let $N, K \trianglelefteq G$ with $N \leq K$ and $(|K : N|, |N|) = 1$. Let $\mu \in \mathrm{Irr}_G(N)$ and consider its canonical extension $\mu^\diamond \in \mathrm{Irr}_G(K)$. Then μ extends to G if and only if $\mu^\diamond$ extends to G.*

Proof. Notice that $\mu^\diamond$ is G-invariant since μ is G-invariant and $\mu^\diamond$ is uniquely determined by μ. Clearly, if $\mu^\diamond$ extends to G, then so does μ. Conversely, assume that μ has an extension $\chi \in \mathrm{Irr}(G)$. By [Isa76, Corollary 11.31], in order to show that $\mu^\diamond$ extends to G, it is enough to show that $\mu^\diamond$ extends to H for every $H/K \in \mathrm{Syl}_p(G/K)$ and every prime p. If p does not divide $|N|$, then μ has a canonical extension to H, which is also an extension of $\mu^\diamond$ by Lemma 4.1.1.

Assume that p divides $|N|$. By [Isa76, Corollary 6.17] there exists a linear character $\lambda \in \mathrm{Irr}(K/N)$ such that $\mu^\diamond = \lambda\chi_K$. Notice that, as $\mu^\diamond$ and χ_K are G-invariant, the character λ is G-invariant. Since $|K : N|$ and $|H : K|$ are coprime, we deduce that λ has a canonical extension $\lambda^\diamond$ to H. Then $\lambda^\diamond\chi_H$ is an extension of $\mu^\diamond$ to H. This concludes the proof. □

If P is a finite group acting via automorphisms on a finite group N with $(|N|, |P|) = 1$, then as in Section 2.2 we denote by

$$f_P : \mathrm{Irr}_P(N) \to \mathrm{Irr}(\mathbf{C}_N(P))$$

the Glauberman correspondence (see [Isa76, Chapter 13] and [Nav18, Section 2.3]). For the remainder of this section we consider the following setting. As mentioned already, when necessary, we denote the normalizer $\mathbf{N}_X(Y)$ simply by X_Y.

Hypothesis 4.1.3. Let N be a normal p'-subgroup of A and P be a p-subgroup of A such that $K := NP \trianglelefteq A$. Consider $\mu \in \mathrm{Irr}_A(N)$ and its Glauberman correspondent $f_P(\mu) \in \mathrm{Irr}_{A_P}(N_P)$. Let $\mu^\diamond \in \mathrm{Irr}_A(K)$ and $f_P(\mu)^\diamond \in \mathrm{Irr}_{A_P}(K_P)$ be the canonical extensions of μ and $f_P(\mu)$ respectively.

Lemma 4.1.4. *Assume Hypothesis 4.1.3 and let C be an abelian normal subgroup of A with $C \leq \mathbf{C}_A(K)$. Suppose that $\mu^\diamond$ has an extension $\widetilde{\mu}$ to A. Then there exists an extension $\widetilde{f_P(\mu)}$ of $f_P(\mu)^\diamond$ to A_P such that*

$$\mathrm{Irr}\left(\widetilde{\mu}_C\right) = \mathrm{Irr}\left(\widetilde{f_P(\mu)}_C\right).$$

Proof. Write $C_p := \mathbf{O}_p(C)$ and $C_{p'} := \mathbf{O}_{p'}(C)$ and set $\kappa := \widetilde{\mu}_{NC_{p'}}$. Let $\kappa^\diamond$ be the canonical extension of κ to KC. Since κ extends to A, there exists an extension $\widetilde{\kappa}$ of $\kappa^\diamond$ to A by Corollary 4.1.2. Using Lemma 4.1.1, observe that $\kappa^\diamond$ extends $\mu^\diamond$ and so does $\widetilde{\kappa}$. Now, by [Isa76, Corollary 6.17], there exists a linear character $\eta \in \mathrm{Irr}(A/K)$ such that $\widetilde{\mu} = \widetilde{\kappa}\eta$. Let λ and λ_1 be the unique irreducible constituent of $\widetilde{\mu}_C$ and of $\widetilde{\kappa}_C$ respectively. Then $\lambda = \lambda_1\eta_C$. Next, consider the Glauberman correspondent $f_P(\kappa) \in \mathrm{Irr}((NC_{p'})_P)$ of κ and let $f_P(\kappa)^\diamond$ be its canonical extension to $(KC)_P$. Using [Tur08, Theorem 6.5] and [Tur09, Theorem 7.12], as κ extends to A, we conclude that $f_P(\kappa)$ extends to A_P. By Corollary 4.1.2 there exists an extension $\widetilde{f_P(\kappa)}$ of $f_P(\kappa)^\diamond$ to A_P. As before, using Lemma 4.1.1, notice that $\widetilde{f_P(\kappa)}$ is an extension of $f_P(\mu)^\diamond$. Define $\widetilde{f_P(\mu)} := \widetilde{f_P(\kappa)}\eta_{A_P}$. Since $K_P \leq \mathrm{Ker}(\eta_{A_P})$, it follows that $\widetilde{f_P(\mu)}$ is an extension of $f_P(\mu)^\diamond$. If λ' and λ'_1 are the unique irreducible constituents of $\widetilde{f_P(\mu)}_C$ and $\widetilde{f_P(\kappa)}_C$ respectively, then $\lambda' = \lambda'_1\eta_C$. Therefore, in order to conclude, it is enough to show that $\lambda_1 = \lambda'_1$. Write $\lambda_1 = \lambda_{1,p} \times \lambda_{1,p'}$ and $\lambda'_1 = \lambda'_{1,p} \times \lambda'_{1,p'}$, with $\lambda_{1,p}, \lambda'_{1,p} \in \mathrm{Irr}(C_p)$ and $\lambda_{1,p'}, \lambda'_{1,p'} \in \mathrm{Irr}(C_{p'})$. First, because $f_P(\kappa)$ is an irreducible constituent of $\kappa_{NC_{p'}}$ and $C_{p'} \leq \mathbf{Z}(NC_{p'})$, it follows that

$$\mathrm{Irr}\left(\widetilde{\kappa}_{C_{p'}}\right) = \mathrm{Irr}\left(\kappa_{C'_p}\right) = \mathrm{Irr}\left(f_P(\kappa)_{C_{p'}}\right) = \mathrm{Irr}\left(\widetilde{f_P(\kappa)}_{C_{p'}}\right)$$

and therefore $\lambda_{1,p'} = \lambda'_{1,p'}$. Observe that $\widetilde{\kappa}_{N\times C_p} = (\kappa^\diamond)_{N\times C_p} = \mu \times \lambda_{1,p}$. Since p does not divide $o(\kappa^\diamond)$, Lemma 4.1.1 implies that p does not divide $o(\mu \times \lambda_p)$. In particular $(p, o(\lambda_p)) = 1$ and therefore $\lambda_{1,p} = 1_{C_p}$. By the same argument, we obtain $\lambda'_{1,p} = 1_{C_p}$. This shows that $\lambda_1 = \lambda'_1$ and the proof is complete. □

Next, we extend Lemma 4.1.4 to the case where C is not necessarily abelian.

Corollary 4.1.5. *Assume Hypothesis 4.1.3 and suppose that $\mu^\diamond$ has an extension $\widetilde{\mu}$ to A. Then there exists an extension $\widetilde{f_P(\mu)}$ of $f_P(\mu)^\diamond$ to A_P such that*

$$\mathrm{Irr}\left(\widetilde{\mu}_{\mathbf{C}_A(K)}\right) = \mathrm{Irr}\left(\widetilde{f_P(\mu)}_{\mathbf{C}_A(K)}\right).$$

Proof. Set $C := \mathbf{C}_A(K)$, $C' := [C, C]$ and $\overline{A} := A/C'$. Since $\widetilde{\mu}_K$ is irreducible, as recalled before Lemma 3.3.3, we have $C \leq \mathbf{Z}(\widetilde{\mu})$ and [Isa76, Lemma 2.27] implies that $\widetilde{\mu}_C = \mu(1)\lambda$, for some linear character $\lambda \in \mathrm{Irr}(C)$. In particular $C' \leq \mathrm{Ker}(\lambda) \leq \mathrm{Ker}(\widetilde{\mu})$. It follows that $C' \cap K$ is contained in $\mathrm{Ker}(\mu^\diamond)$ and $\mathrm{Ker}(f_P(\mu)^\diamond)$ while $C' \cap N$ is contained in $\mathrm{Ker}(\mu)$ and $\mathrm{Ker}(f_P(\mu))$. Via the canonical isomorphism $\overline{N} \simeq N/C' \cap N$, we can identify μ with a character $\overline{\mu}$ of $\overline{N}$. Similarly we can consider $\overline{\mu^\diamond}$ as a character of $\overline{K}$, $\overline{f_P(\mu)}$ as a character of $\overline{N_P}$ and $\overline{f_P(\mu)^\diamond}$ as a character of $\overline{K_P}$. Notice that $\overline{A_P} = \overline{A}_{\overline{P}}$, $\overline{K_P} = \overline{K}_{\overline{P}}$ and $\overline{N_P} = \overline{N}_{\overline{P}}$. By [NS14b, Lemma 5.10] the character $\overline{f_P(\mu)}$ coincides with the Glauberman correspondent $f_{\overline{P}}(\overline{\mu})$ of $\overline{\mu}$. Moreover $\overline{\mu^\diamond}$ and $\overline{f_P(\mu)^\diamond}$ are the canonical extensions of $\overline{\mu}$ and of $\overline{f_P(\mu)}$. Applying Lemma 4.1.4, we find an extension ψ of $\overline{f_P(\mu)^\diamond}$ to $\overline{A}_{\overline{P}}$ such that $\mathrm{Irr}\left(\widetilde{\overline{\mu}}_{\overline{C}}\right) = \mathrm{Irr}\left(\psi_{\overline{C}}\right)$, where $\widetilde{\overline{\mu}}$ is the character of $\overline{A}$ corresponding to $\widetilde{\mu}$ via inflation. Now the inflation $\widetilde{f_P(\mu)} \in \mathrm{Irr}(A_P)$ of ψ satisfies the required hypothesis. □

Recall that, if $\mathbf{R}$ is the ring of algebraic integers and $\mathbf{S}$ is the localization of $\mathbf{R}$ at some maximal ideal containing $p\mathbf{R}$, then $^* : \mathbf{S} \to \mathbb{F}$ denotes the canonical epimorphism, where $\mathbb{F}$ is the residue field of characteristic p (see [Nav98, Chapter 2] for details).

Lemma 4.1.6. *Assume Hypothesis 4.1.3. If $\mu^\diamond$ extends to $\widetilde{\mu} \in \mathrm{Irr}(A)$, then there exists an extension $\widetilde{f_P(\mu)}$ of $f_P(\mu)^\diamond$ to A_P such that*

$$\mathrm{Irr}\left(\widetilde{\mu}_{\mathbf{C}_A(K)}\right) = \mathrm{Irr}\left(\widetilde{f_P(\mu)}_{\mathbf{C}_A(K)}\right)$$

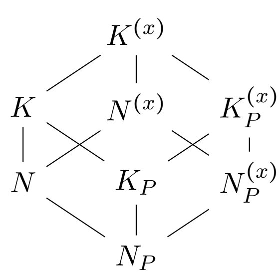
and

$$\widetilde{\mu}(x)^* = e\widetilde{f_P(\mu)}(x)^*$$

for every p-regular $x \in A$ with $P \in \mathrm{Syl}_p(\mathbf{C}_K(x))$, where $e := [\mu_{N_P}, f_P(\mu)]$.

Proof. By Corollary 4.1.5 there exists an extension χ of $f_P(\mu)^\diamond$ that satisfies the first condition. In order to conclude, it is enough to find a linear character $\widetilde{\xi} \in \mathrm{Irr}(A_P/\mathbf{C}_A(K)K_P)$ such that $\widetilde{f_P(\mu)} := \widetilde{\xi} \cdot \chi$ satisfies the second condition.

First, we construct the linear character $\widetilde{\xi}$. Let x be a p-regular element of $\mathbf{C}_A(P)K_P$, set $N^{(x)} := N\langle x\rangle$, $K^{(x)} := K\langle x\rangle$ and observe that $(N^{(x)})_P = (N_p)^{(x)} := K_P\langle x\rangle$ and $(K^{(x)})_P = (K_p)^{(x)} := K_P\langle x\rangle$.



Since x is p-regular, the subgroup $N^{(x)}$ has order coprime to p and we can consider the Glauberman correspondent $f_P(\widetilde{\mu}_{N^{(x)}})$ of $\widetilde{\mu}_{N^{(x)}}$. Moreover $f_P(\widetilde{\mu}_{N^{(x)}})_{N_P} = f_P(\mu)$ by [IN91, Theorem A]. Now, if $f_P(\widetilde{\mu}_{N^{(x)}})^\diamond$ is the canonical extension of $f_P(\widetilde{\mu}_{N^{(x)}})$ to $K_P^{(x)}$, then Lemma 4.1.1 implies that $(f_P(\widetilde{\mu}_{N^{(x)}})^\diamond)_{K_P} = f_P(\mu)^\diamond$. Since $\chi_{K_P^{(x)}}$ is another extension of $f_P(\mu)^\diamond$ to $K_P^{(x)}$, it follows that there exists a unique linear character $\xi^{(x)} \in \mathrm{Irr}(K_P^{(x)}/K_P)$ such that $\xi^{(x)}\chi_{K_P^{(x)}} = f_P(\widetilde{\mu}_{N^{(x)}})^\diamond$. We define the map

$$\begin{aligned}\xi : \mathbf{C}_A(P)K_P &\to \mathbb{C}\\ x &\mapsto \xi^{(x_{p'})}(x_{p'}).\end{aligned}$$

We claim that ξ is a linear character of $\mathbf{C}_A(P)K_P$ with an extension $\widetilde{\xi}$ to A_P. To show that ξ is an irreducible character we apply [Isa76, Corollary 8.12]. Clearly $\xi(1) = 1$. Next, in order to show that ξ is a class function we check that $\xi^{(x^n)} = (\xi^{(x)})^n$ for every $n \in A_P$ and every p-regular $x \in \mathbf{C}_A(P)K_P$. If this is the case, then

$$\xi(x^n) = \xi^{((x^n)_{p'})}((x^n)_{p'}) = \xi^{((x_{p'})^n)}((x_{p'})^n) = (\xi^{(x_{p'})})^n((x_{p'})^n) = \xi^{(x_{p'})}(x_{p'}) = \xi(x)$$

for every $x \in \mathbf{C}_A(P)K_P$ and $n \in A_P$. In particular ξ is a class function. To prove the claim, just notice that $(\chi_{K_P^{(x)}})^n = \chi_{K_P^{(x^n)}}$ and that $(f_P(\widetilde{\mu}_{N^{(x)}})^\diamond)^n$ is the canonical extension of $f_P(\widetilde{\mu}_{N^{(x)}})^n = f_P(\widetilde{\mu}_{N^{(x^n)}})$ for every $n \in A_P$ and every p-regular $x \in \mathbf{C}_A(P)K_P$. Next, since $\xi^{(x)} = \xi^{(x^{-1})}$ for every p-regular $x \in \mathbf{C}_A(P)K_P$, we deduce that $\xi(x^{-1}) = \xi^{-1}(x)$ for every $x \in \mathbf{C}_A(P)K_P$ and therefore $[\xi, \xi] = 1$. Finally, fix $S \times T \leq \mathbf{C}_A(P)K_P$ with S a p-group and T a p'-group. Observe that $\xi_S = 1_S$. On the other hand χ_{K_PT} and $f_P(\widetilde{\mu}_{NT})^\diamond$ are both extensions of $f_P(\mu)^\diamond$ and we can find a linear character $\lambda \in \mathrm{Irr}(K_PT/K_P)$ such that $\lambda\chi_{K_PT} = f_P(\widetilde{\mu}_{NT})^\diamond$. Moreover, for every $x \in T$, we have $(f_P(\widetilde{\mu}_{NT})^\diamond)_{K_P^{(x)}} = f_P(\widetilde{\mu}_{N^{(x)}})^\diamond$ and therefore $\xi_T = \lambda_T$. It follows that $\xi_{S\times T} \in \mathbb{Z}\mathrm{Irr}(S \times T)$ and hence ξ is a linear character by [Isa76, Corollary 8.12].

Next, we show that ξ extends to A_P. To do so, we use [Isa76, Theorem 6.26]. Let q be a prime dividing $o(\xi)$ and consider $S_q/\mathbf{C}_A(P)K_P \in \mathrm{Syl}_q(A_P/\mathbf{C}_A(P)K_P)$. Noticing that every p-element x of $\mathbf{C}_A(P)K_P$ is contained in $\mathrm{Ker}(\xi)$, we deduce that p does not divide $|\mathbf{C}_A(P)K_P : \mathrm{Ker}(\xi)|$ and hence $q \neq p$. Let $Q \in \mathrm{Syl}_q(A_P/N_P)$ such that $S_q = \mathbf{C}_A(P)K_PQ$ and define $Q_1 := Q \cap \mathbf{C}_A(P)K_P$ and $\xi_1 := \xi_{Q_1}$. By [Spä10, Lemma 4.1], we deduce that ξ extends to A_P if and only if ξ_{Q_1} extends to Q. We are going to check the latter condition. Because $Q_1 \leq A_P$ we deduce that NQ_1 is a P-invariant p'-group and that $(NQ_1)_P = N_PQ_1 = Q_1$. We also have $KQ_1 = (NQ_1) \rtimes P$ and $(KQ_1)_P = K_PQ_1 = Q_1P$. Now we can consider the Glauberman correspondent $f_P(\widetilde{\mu}_{NQ_1})$ and its canonical extension $f_P(\widetilde{\mu}_{NQ_1})^\diamond$ to Q_1P. By [IN91, Theorem A] we have $f_P(\widetilde{\mu}_{NQ_1})_{N_P} = f_P(\mu)$ and so $(f_P(\widetilde{\mu}_{NQ_1})^\diamond)_{K_P} = f_P(\mu)^\diamond$ by Lemma 4.1.1. Using Corollary 4.1.2, we obtain an extension ψ of $f_P(\widetilde{\mu}_{NQ_1})^\diamond$ to $(KQ)_P = K_PQ$. By Gallagher's theorem there exists a unique linear character $\nu \in \mathrm{Irr}(K_PQ/K_P)$ such that $\chi_{K_PQ} \cdot \nu = \psi$. Finally, for every $x \in Q_1$, we have

$$\begin{aligned}\xi^{(x)}\chi_{K_P^{(x)}} &= f_P(\widetilde{\mu}_{N^{(x)}})^\diamond = (f_P(\widetilde{\mu}_{NQ_1})_{N_P^{(x)}})^\diamond\\ &= (f_P(\widetilde{\mu}_{NQ_1})^\diamond)_{K_P^{(x)}} = (\psi_{PQ_1})_{K_P^{(x)}}\\ &= \psi_{K_P^{(x)}} = \chi_{K_P^{(x)}}\nu_{K_P^{(x)}}\end{aligned}$$

and it follows that $\nu_{Q_1} = \xi_1$. This shows that ν_Q is an extension of ξ_1 to Q and therefore ξ extends to S_q. We conclude that ξ has an extension $\widetilde{\xi}$ to A_P.

Define $\widetilde{f_P(\mu)} := \widetilde{\xi}\chi$. By [NS14a, Theorem 2.6] we deduce that

$$\widetilde{\mu}(x)^* = \widetilde{\mu}_{N(x)}(x)^* = ef_P(\widetilde{\mu}_{N(x)})(x)^* = e(\xi(x)\chi(x))^* = e\widetilde{f_P(\mu)}(x)^*$$

for every p-regular $x \in E$ such that $P \in \mathrm{Syl}_p(\mathbf{C}_K(x))$.

It remains to show that $C := \mathbf{C}_A(K)$ is contained in the kernel of ξ. First, observe that $\mathrm{Ker}(\xi)$ contains $C' := [C, C] \leq \mathrm{Ker}(\xi_C)$. Moreover, $\mathrm{Ker}(\xi)$ contains every p-element of C. Since C/C' is abelian it's enough to show that every p-regular element x of C lies in $\mathrm{Ker}(\xi)$. By the Alperin argument we know that $B := \mathrm{bl}(\widetilde{\mu}_{K(x)})$ and $B' := \mathrm{bl}(f_P(\widetilde{\mu}_{N(x)})^\diamond)$ are Brauer correspondents with B covering $b := \mathrm{bl}(\widetilde{\mu}_{N(x)}) = \{\widetilde{\mu}_{N(x)}\}$ and B' covering $b' := \mathrm{bl}(f_P(\widetilde{\mu}_{N(x)})) = \{f_P(\widetilde{\mu}_{N(x)})\}$. According to [Nav98, Theorem 4.14] it follows that $\lambda_B = \lambda_{B'} \circ \mathrm{Br}_P$. Since $x \in \mathbf{C}_A(K)$, we have $x \leq \mathbf{Z}(K^{(x)})$ and hence

$$\lambda_B(x) = \lambda_B\left(\left(x^{K(x)}\right)^+\right) = \lambda_{B'}\left(\left(x^{K(x)} \cap \mathbf{C}_{K(x)}(P)\right)^+\right) = \lambda_{B'}(x).$$

By [Nav98, Theorem 9.5], we conclude that

$$\left(\frac{\widetilde{\mu}(x)}{\widetilde{\mu}(1)}\right)^* = \lambda_B(x) = \lambda_{B'}(x) = \left(\frac{f_P(\widetilde{\mu}_{N(x)})(x)}{f_P(\widetilde{\mu}_{N(x)})(1)}\right)^* = \left(\frac{\widetilde{f_P(\mu)}(x)}{\widetilde{f_P(\mu)}(1)}\right)^*.$$

As $\mathrm{Irr}(\widetilde{\mu}_C) = \mathrm{Irr}(\chi_C)$ and x is p-regular, we obtain

$$\frac{\chi(x)}{\chi(1)} = \frac{\widetilde{\mu}(x)}{\widetilde{\mu}(1)} = \frac{\widetilde{f_P(\mu)}(x)}{\widetilde{f_P(\mu)}(1)}$$

and, in particular, $\xi(x) = 1$. This concludes the proof. □

Using the above result we are able to extend the bijection given in [NS14b, Theorem 5.13] to characters of positive height. This is done in the particular case where the group K from [NS14b, Hypothesis 5.1] has order not divisible by p. In this particular situation we obtain a canonical bijection.

Proposition 4.1.7. *Assume Hypothesis 4.1.3. Then there exists a canonical defect preserving A_P-equivariant bijection*

$$\begin{aligned}\Psi_{\mu,P} : \mathrm{Irr}\left(K \mid \mu\right) &\to \mathrm{Irr}\left(K_P \mid f_P(\mu)\right)\\ \mu^\diamond\nu &\mapsto f_P(\mu)^\diamond\nu_{K_P}\end{aligned}$$

for every $\nu \in \mathrm{Irr}(K/N)$. Moreover

$$(A_\vartheta, K, \vartheta) \sim_K (A_{P,\vartheta}, K_P, \Psi_{\mu,P}(\vartheta))$$

and

$$\mathbf{C}_A(D) \leq A_P$$

for every $\vartheta \in \mathrm{Irr}(K \mid \mu)$ and some defect group D of $\mathrm{bl}(\Psi_{\mu,P}(\vartheta))$.

Proof. Since $K = N \rtimes P$ and $K_P = N_P \times P$ are p-nilpotent groups with Sylow p-subgroup P, μ is K-invariant and $f_P(\mu)$ is K_P-invariant, we deduce that

$$\mathrm{Irr}(K \mid \mu) = \mathrm{Irr}(\mathrm{bl}(\mu^\diamond)) = \{\mu^\diamond \nu \mid \nu \in \mathrm{Irr}(K/N)\}$$

and that

$$\mathrm{Irr}(K_P \mid f_P(\mu)) = \mathrm{Irr}(\mathrm{bl}(f_P(\mu)^\diamond)) = \{f_P(\mu)^\diamond \nu \mid \nu \in \mathrm{Irr}(K_P/N_P)\}.$$

Then, we obtain an A_P-quivariant defect preserving bijection by setting

$$\Psi_{\mu,P}(\mu^\diamond \cdot \nu) := f_P(\mu)^\diamond \cdot \nu_{K_P}$$

for every $\nu \in \mathrm{Irr}(K/N)$. Furthermore, as P is a common defect group of the two blocks $\mathrm{bl}(\vartheta)$ and $\mathrm{bl}(\Psi_{\mu,P}(\vartheta))$ for every $\vartheta \in \mathrm{Irr}(K \mid \mu)$, the condition on defect groups is clearly satisfied.

Let $\mathcal{P} \in \mathrm{Proj}(A)$ be a projective representation associated with $\mu^\diamond$ with factor set α and observe that $\mathcal{P}$ is also associated with μ. Consider the central extension $\widehat{A}$ of A defined by $\mathcal{P}$ and $\epsilon : \widehat{A} \to A$ the map given by $\epsilon(x,s) := x$, for every $x \in A$ and $s \in S$, with kernel $S := \langle \alpha(x,y) \mid x, y \in A\rangle$. For $H \leq A$, set $\widehat{H} := \epsilon^{-1}(H)$. By Theorem 3.2.6 the set $H_0 := \{(h,1) \mid h \in H\}$ is a subgroup of $\widehat{A}$, whenever $H \leq K$. In this case let $\vartheta_0 \in \mathrm{Irr}(H_0)$ be the character corresponding to $\vartheta \in \mathrm{Irr}(H)$ via the isomorphism $\epsilon_{H_0} : H_0 \to H$. Moreover $\widehat{H} = H_0 \times S$ and we define $\widehat{\vartheta} := \vartheta_0 \times 1_S \in \mathrm{Irr}(\widehat{H})$. Notice that $(\mu^\diamond)_0 \in \mathrm{Irr}(K_0)$ is the canonical extension of μ_0 and that $\widehat{\mu^\diamond} \in \mathrm{Irr}(\widehat{K})$ is the canonical extension of $\widehat{\mu}$. Furthermore $f_P(\mu)_0 = f_{P_0}(\mu_0)$ and $(f_P(\mu)^\diamond)_0$ is its canonical extension. As no confusion can arise, we just write $\mu_0^\diamond$ (resp. $f_P(\mu)_0^\diamond$) instead of $(\mu^\diamond)_0 = (\mu_0)^\diamond$ (resp. $(f_P(\mu)^\diamond)_0 = f_{P_0}(\mu_0)^\diamond$).

Recall that the map defined by $\widehat{\mathcal{P}}(x,s) := s\mathcal{P}(x)$, for every $(x,s) \in \widehat{A}$, is an irreducible representation of $\widehat{A}$ affording an extension τ of $\mu_0^\diamond$. Set $S_p := \mathbf{O}_p(S)$, $S_{p'} := \mathbf{O}_{p'}(S)$, $M := N_0 \times S_{p'}$, $Q := P_0 \times S_p$ and notice that $\widehat{K} = M \rtimes Q$, $M_Q = (N_P)_0 \times S_{p'}$ and $\widehat{K}_Q = \widehat{K_P}$. Let $\varphi := \tau_M \in \mathrm{Irr}_{\widehat{A}}(M)$ and consider its canonical extension $\varphi^\diamond \in \mathrm{Irr}(\widehat{K})$. By Corollary 4.1.2, there exists an extension $\widetilde{\varphi}$ of $\varphi^\diamond$ to $\widehat{A}$. Lemma 4.1.1 implies that $\widetilde{\varphi}_{K_0} = \mu_0^\diamond$. Now, if $\widehat{\mathcal{R}}$ is an irreducible representation of $\widehat{A}$ affording $\widetilde{\varphi}$, then $\mathcal{R}(x) := \widehat{\mathcal{R}}(x,1)$ defines a projective representation of A associated with $\mu^\diamond$. Replacing $\mathcal{P}$ with $\mathcal{R}$, it is no loss of generality to assume that τ extends $\varphi^\diamond$.

Now, Lemma 4.1.6 yields an extension $\widetilde{f_Q(\varphi)}$ of $f_Q(\varphi)^\diamond$ to $\widehat{A}_Q = \widehat{A_P}$ such that

$$\mathrm{Irr}\left(\widetilde{\varphi}_{\mathbf{C}_{\widehat{A}}(\widehat{K})}\right) = \mathrm{Irr}\left(\widetilde{f_Q(\varphi)}_{\mathbf{C}_{\widehat{A}}(\widehat{K})}\right) \tag{4.1.1}$$

and

$$\widetilde{\varphi}(x)^* = e\widetilde{f_Q(\varphi)}(x)^* \tag{4.1.2}$$

for every p-regular $x \in \widehat{A}$ such that $Q \in \mathrm{Syl}_p(\mathbf{C}_{\widehat{K}}(x))$, where $e := [\varphi_{M_Q}, f_Q(\varphi)]$. Observe that, by [IN91, Theorem A] and using the fact that $S_p \leq \mathbf{Z}(\widehat{A})$, we have $\widetilde{f_Q(\varphi)}_{(N_P)_0} = f_{P_0}(\mu_0)$ and $\widetilde{f_Q(\varphi)}_{(K_P)_0} = f_{P_0}(\mu_0)^\diamond$.

Let $\widehat{\mathcal{P}}'$ be an irreducible representation of $\widehat{A_P}$ affording $\widetilde{f_Q(\varphi)}$ and consider the projective representation $\mathcal{P}'$ of A_P defined by $\mathcal{P}'(x) := \widehat{\mathcal{P}}'(x,1)$ for every $x \in A_P$. Notice that $\mathcal{P}'$ is associated with $f_P(\mu)^\diamond$ and that its factor set coincides with $\alpha_{A_P\times A_P}$. Furthermore, as $\mathbf{C}_{\widehat{A}}(\widehat{K}) = \widehat{\mathbf{C}_A(K)}$ (see Theorem 3.4.11 (ii)) and by (4.1.1), we deduce that $\mathcal{P}_{\mathbf{C}_A(K)}$ and $\mathcal{P}'_{\mathbf{C}_A(K)}$ are associated with the same scalar function. This shows that

$$(A, K, \mu^\diamond) \sim_K^c (A_P, K_P, f_P(\mu)^\diamond)\,.$$

Next, let $\vartheta = \mu^\diamond\nu \in \mathrm{Irr}(K \mid \mu)$, with $\nu \in \mathrm{Irr}(K/N)$, and observe that $A_\vartheta = A_\nu$. Let $\mathcal{Q}$ be a projective representation of A_ν associated with ν and notice that $\mathcal{Q}_{A_{P,\nu}}$ is a projective representation of $A_{P,\nu}$ associated with ν_{K_P}. Now $\mathcal{S} := \mathcal{P}_{A_\nu} \otimes \mathcal{Q}$ is a projective representation of A_ν associated with ϑ, while $\mathcal{S}' := \mathcal{P}'_{A_{P,\nu}} \otimes \mathcal{Q}_{A_{P,\nu}}$ is a projective representation of $A_{P,\nu}$ associated with $\Psi_{\mu,P}(\vartheta) = f_P(\mu)^\diamond\nu_{K_P}$. We claim that $(A_\vartheta, K, \vartheta) \sim_K (A_{P,\vartheta}, K_P, \Psi_{\mu,P}(\vartheta))$ via $(\mathcal{S}, \mathcal{S}')$. By the previous paragraph, one can easily check that conditions (i)-(iv) of Remark 3.3.7 are satisfied. To conclude, it remains to check Remark 3.3.7 (v). By the proof of [NS14b, Theorem 4.4] it's enough to show that

$$\left(\frac{|K|_{p'}\mathrm{Tr}(\mathcal{S}(x))}{p^{\mathrm{ht}(\vartheta)}\vartheta(1)_{p'}}\right)^* = \left(\frac{|K_P|_{p'}\mathrm{Tr}(\mathcal{S}'(x))}{p^{\mathrm{ht}(\Psi_{\mu,P}(\vartheta))}\Psi_{\mu,P}(\vartheta)(1)_{p'}}\right)^*$$

for every p-regular $x \in A_{P,\vartheta}$ such that $P \in \mathrm{Syl}_p(\mathbf{C}_K(x))$. Fix a p-regular element $x \in A_{P,\vartheta}$ with $P \in \mathrm{Syl}_p(\mathbf{C}_K(x))$. Then $Q \in \mathrm{Syl}_p(\mathbf{C}_{\widehat{K}}(x,1))$ and (4.1.2) implies

$$\mathrm{Tr}\left(\mathcal{S}(x)\right)^* = \widetilde{\varphi}(x,1)^*\mathrm{Tr}(\mathcal{Q}(x))^* = \left(e\widetilde{f_Q(\varphi)}(x,1)\right)^* \mathrm{Tr}(\mathcal{Q}(x))^* = e^*\mathrm{Tr}\left(\mathcal{S}'(x)\right)^*.$$

As $e = [\mu_{N_P}, f_P(\mu)]$ and by [NS14b, Theorem 5.2 (b)], we obtain

$$\vartheta(1)_{p'} = \mu(1) \equiv [\mu_{N_P}, f_P(\mu)]|N : N_P| f_P(\mu)(1) \equiv e\frac{|K|_{p'}}{|K_P|_{p'}}\Psi_{\mu,P}(\vartheta)(1)_{p'} \pmod p$$

and therefore

$$\left(\frac{|K|_{p'}\mathrm{Tr}(\mathcal{S}(x))}{p^{\mathrm{ht}(\vartheta)}\vartheta(1)_{p'}}\right)^* = \left(\frac{e|K|_{p'}\mathrm{Tr}(\mathcal{S}'(x))}{\nu(1)\vartheta(1)_{p'}}\right)^* = \left(\frac{|K_P|_{p'}\mathrm{Tr}(\mathcal{S}'(x))}{p^{\mathrm{ht}(\Psi_{\mu,P}(\vartheta))}\Psi_{\mu,P}(\vartheta)(1)_{p'}}\right)^*.$$

Now the proof is complete. □

As a consequence, applying Proposition 3.4.4 and Proposition 4.1.7, for every $N \leq J \leq A$ we obtain an $A_{P,J}$-equivariant defect preserving bijection

$$\Phi : \mathrm{Irr}(J \mid \mu) \to \mathrm{Irr}(J_P \mid f_P(\mu))$$

such that

$$(A_{J,\chi}, J, \chi) \sim_J (A_{J,P,\chi}, J_P, \Phi(\chi))$$

for every $\chi \in \mathrm{Irr}(J \mid \mu)$. Finally, we obtain the main result of this section by considering a normal p-chain $\mathbb{D}$ with last term P and $J = NG_{\mathbb{D}}$.

Theorem 4.1.8. *Let $N \leq G \trianglelefteq A$, with $N \trianglelefteq A$ a p'-subgroup, and consider a normal p-chain $\mathbb{D}$ of G with final term P. Let $\mu \in \mathrm{Irr}_A(N)$ and $f_P(\mu) \in \mathrm{Irr}(N_P)$ be its Glauberman correspondent. Then there exists a defect preserving $A_{\mathbb{D}}$-equivariant bijection*

$$\Phi_{\mu,\mathbb{D}} : \mathrm{Irr}(NG_{\mathbb{D}} \mid \mu) \to \mathrm{Irr}(G_{\mathbb{D}} \mid f_P(\mu))$$

such that

$$\left(NA_{\mathbb{D},\chi}, NG_{\mathbb{D}}, \chi\right) \sim_G \left(A_{\mathbb{D},\chi}, G_{\mathbb{D}}, \Phi_{\mu,\mathbb{D}}(\chi)\right)$$

for every $\chi \in \mathrm{Irr}(NG_{\mathbb{D}} \mid \mu)$.

Proof. Let $K := NP$ and observe that, without loss of generality, we may assume $K \trianglelefteq A$. Set $\mathcal{S} := \mathrm{Irr}(K \mid \mu)$ and $\mathcal{S}' := \mathrm{Irr}(K_P \mid f_P(\mu))$. By Proposition 4.1.7 there exists an A_P-equivariant defect preserving bijection

$$\Psi_{\mu,P} : \mathcal{S} \to \mathcal{S}'$$

such that

$$(A_\vartheta, K, \vartheta) \sim_K (A_{P,\vartheta}, K_P, \Psi_{\mu,P}(\vartheta))$$

and

$$\mathbf{C}_A(D) \leq A_P$$

for every $\vartheta \in \mathrm{Irr}(K \mid \mu)$ and some defect group D of $\mathrm{bl}(\Psi_{\mu,P}(\vartheta))$. Let $J := NG_{\mathbb{D}}$ and notice that, since N is a p'-group, we have $N_P = \mathbf{C}_N(P) \leq G_{\mathbb{D}}$ and therefore $(NG_{\mathbb{D}})_P = G_{\mathbb{D}}$. Moreover, observe that $\mathrm{Irr}(J \mid \mathcal{S}) = \mathrm{Irr}(NG_{\mathbb{D}} \mid \mu)$ and $\mathrm{Irr}(J_P \mid \mathcal{S}') = \mathrm{Irr}(G_{\mathbb{D}} \mid f_P(\mu))$. Now, as $A_{\mathbb{D}} \leq A_{P,J}$, Proposition 3.4.4 yields an $A_{\mathbb{D}}$-equivariant defect preserving bijection

$$\Phi_{\mu,\mathbb{D}} : \mathrm{Irr}(NG_{\mathbb{D}} \mid \mu) \to \mathrm{Irr}(G_{\mathbb{D}} \mid f_P(\mu))$$

such that

$$(A_{J,\chi}, J, \chi) \sim_J \left(A_{P,J,\chi}, J_P, \Phi_{\mu,\mathbb{D}}(\chi)\right)$$

for every $\chi \in \mathrm{Irr}(J \mid \mu)$. By Lemma 3.3.8 it follows that

$$\left(NA_{\mathbb{D},\chi}, J, \chi\right) \sim_J \left(A_{\mathbb{D},\chi}, J_P, \Phi_{\mu,\mathbb{D}}(\chi)\right)$$

and then, by using Lemma 3.3.10, we obtain

$$\left(NA_{\mathbb{D},\chi}, NG_{\mathbb{D}}, \chi\right) \sim_G \left(A_{\mathbb{D},\chi}, G_{\mathbb{D}}, \Phi_{\mu,\mathbb{D}}(\chi)\right).$$

Observe that, in order to apply Lemma 3.3.10, we need to check that $\mathbf{C}_{GA_{\mathbb{D},\chi}}(D) \leq NA_{\mathbb{D},\chi}$, for some $D \in \delta(\mathrm{bl}(\Phi_{\mu,\mathbb{D}}(\chi)))$. To see this, observe that $P \leq \mathbf{O}_p(G_{\mathbb{D}})$ because $\mathbb{D}$ is a normal p-chain and hence $P \leq D$. In particular $\mathbf{C}_{GA_{\mathbb{D},\chi}}(D) \leq A_{\mathbb{D}} \cap GA_{\mathbb{D},\chi} = A_{\mathbb{D},\chi} \leq NA_{\mathbb{D},\chi}$. □

4.2 N-block isomorphic character triples and Fong correspondence

In this section, we show that the Fong correspondence [Fon61] can be used to construct N-block isomorphic character triples. For completeness, we state the Fong correspondence in the form we need.

Hypothesis 4.2.1. Let N be a normal p'-subgroup of A and consider $\mu \in \mathrm{Irr}_A(N)$. Let $\mathcal{P} \in \mathrm{Proj}(A)$ be a projective representation associated with (A, N, μ) and with factor set α such that $\alpha(x,y)^{|N|^2} = 1$, for every $x, y \in A$ (see [NT89, Theorem 3.5.7]), and denote by $\widehat{A}$ the p'-central extension of A by $S := \langle \alpha(x,y) \mid x, y \in A\rangle$ defined by $\mathcal{P}$ as in Theorem 3.2.6. Let $\epsilon : \widehat{A} \to A$ be the epimorphism given by $\epsilon(x,s) := x$, for every $x \in A$ and $s \in S$, and consider $N_0 := \{(n,1) \mid n \in N\} \trianglelefteq \widehat{A}$. For every $X \leq A$, set $\widehat{X} := \epsilon^{-1}(X)$ and $\widetilde{X} := \widehat{X}N_0/N_0$. Consider the irreducible representation $\widehat{\mathcal{P}}$ of $\widehat{A}$ defined by $\widehat{\mathcal{P}}(x,s) := s\mathcal{P}(x)$, for every $x \in A$ and $s \in S$, and denote its character by τ. Let $\widehat{\lambda} \in \mathrm{Irr}(\widehat{N})$ be the linear character defined by $\widehat{\lambda}(n,s) := s^{-1}$, for every $n \in N$ and $s \in S$, and set $\widehat{\mu} := \mu_0 \times 1_S \in \mathrm{Irr}(\widehat{N})$, where μ_0 correspond to μ via the isomorphism $N \simeq N_0$. Notice that τ extends $\widehat{\mu}\widehat{\lambda}^{-1}$. Finally, denote by $\widetilde{\mu}$ the character $\widehat{\lambda}$ viewed as a character of $\widetilde{N} = \widehat{N}/N_0$, that is $\widetilde{\mu}(N_0(n,s)) = s^{-1}$ for every $n \in N$ and $s \in S$ (see Theorem 3.2.7).

The following result shows that the bijection given in Theorem 3.2.7 is compatible with block decomposition.

Theorem 4.2.2 (Fong)**.** *Assume Hypothesis 4.2.1. If $N \leq H \leq A$, then:*

(i) $\widetilde{H}$ is a p'-central extension of H/N by the central p'-subgroup $\widetilde{N} \simeq S$;

(ii) There exists a bijection

$$\begin{aligned}\mathrm{Bl}(H \mid \mathrm{bl}(\mu)) &\to \mathrm{Bl}(\widetilde{H} \mid \mathrm{bl}(\widetilde{\mu}))\\ B &\mapsto \widetilde{B}\end{aligned}$$

(iii) Let $D \in \delta(B)$ and consider $Q \in \mathrm{Syl}_p(\widehat{D})$ so that $\widehat{D} = Q \times S$. Then $QN_0/N_0 \in \delta(\widetilde{B})$. In particular B and $\widetilde{B}$ have isomorphic defect groups;

(iv) For every $B \in \mathrm{Bl}(H \mid \mathrm{bl}(\mu))$ corresponding to $\widetilde{B} \in \mathrm{Bl}(\widetilde{H} \mid \mathrm{bl}(\widetilde{\mu}))$ via the bijection in (ii), there exists a defect preserving bijection

$$\begin{aligned}\mathrm{Irr}(B) &\to \mathrm{Irr}(\widetilde{B})\\ \psi &\mapsto \widetilde{\psi}\end{aligned}$$

such that, if $\widehat{\psi}$ is the inflation to $\widehat{H}$ of the character of $\widehat{H}/S \simeq H$ corresponding to ψ and $\widehat{\psi}'$ is the inflation to $\widehat{H}$ of $\widetilde{\psi}$, then $\widehat{\psi} = \tau_{\widehat{H}}\widehat{\psi}'$;

(v) For $\widehat{x} \in \widehat{A}$ set $x := \epsilon(\widehat{x})$ and $\widetilde{x} := N_0\widehat{x}$. Then $\widetilde{\psi^x} = \left(\widetilde{\psi}\right)^{\widetilde{x}}$ and $\widetilde{B^x} = \left(\widetilde{B}\right)^{\widetilde{x}}$ for every $B \in \mathrm{Bl}(G \mid \mathrm{bl}(\mu))$ and $\psi \in \mathrm{Irr}(B)$.

Proof. Consider $\psi \in \mathrm{Irr}(H \mid \mu)$ afforded by $\mathfrak{X}$. We just show how to construct $\widetilde{\psi}$. By [Nav18, Theorem 10.11], there exists an irreducible projective representation $\mathcal{Q} \in \mathrm{Proj}(H/N \mid \alpha^{-1}_{H\times H})$ such that $\mathfrak{X} = \mathcal{Q} \otimes \mathcal{P}_H$ unique up to similarity. Now, $\widehat{\mathcal{Q}}(x,s) := \mathcal{Q}(x)s^{-1}$, for every $x \in H$ and $s \in S$, defines an irreducible linear representation of $\widehat{H}$ with $N_0 \leq \mathrm{Ker}(\widehat{\mathcal{Q}})$ and whose character lies over $\widehat{\lambda}$. If we consider the inflation $\widehat{\mathfrak{X}}$ to $\widehat{H}$ of the representation of $\widehat{H}/S \simeq H$ corresponding to $\mathfrak{X}$, that is $\widehat{\mathfrak{X}}(\widehat{x}) := \mathfrak{X}(\epsilon(\widehat{x}))$ for every $\widehat{x} \in \widehat{H}$, then $\widehat{\mathfrak{X}} = \widehat{\mathcal{Q}} \otimes \widehat{\mathcal{P}}_{\widehat{H}}$. Define $\widetilde{\mathfrak{X}}$ to be the irreducible representation of $\widetilde{H} = \widehat{H}/N_0$ whose inflation is $\widehat{\mathcal{Q}}$, and let $\widetilde{\psi}$ be the character afforded by $\widetilde{\mathfrak{X}}$. Then $\widetilde{\psi} \in \mathrm{Irr}(\widetilde{A} \mid \widetilde{\mu})$ and, if $\widehat{\psi}$ is the inflation to $\widehat{H}$ of the character of $\widehat{H}/S \simeq H$ corresponding to ψ and $\widetilde{\psi}'$ is the inflation of $\widetilde{\psi}$ to $\widehat{H}$, then $\widehat{\psi} = \tau_{\widehat{H}}\widetilde{\psi}'$. The result follows from [Fon61]. The description of defect groups is a consequence of the proof of [Fon61, 2C]. To conclude, for $\widehat{x} \in \widehat{A}$, set $x := \epsilon(\widehat{x})$ and $\widetilde{x} := N_0\widehat{x}$. Then $\widehat{\psi^x} = (\widehat{\psi})^{\widehat{x}} = (\widetilde{\psi}'\tau_{\widehat{H}})^{\widehat{x}} = (\widetilde{\psi}')^{\widehat{x}}\tau_{\widehat{H}} = (\widetilde{\psi}^{\widetilde{x}})'\tau_{\widehat{H}}$, where $\widehat{\psi^x}$ is the inflation to $\widehat{H}$ of the character of $\widehat{H}/S \simeq H$ corresponding to ψ^x and $(\widetilde{\psi}^{\widetilde{x}})'$ is the inflation of $\widetilde{\psi}^{\widetilde{x}}$ to $\widehat{H}$. Thus $(\widetilde{\psi})^{\widetilde{x}}$ coincides with $\widetilde{\psi^x}$ the Fong correspondent of ψ^x. In particular, since $\widetilde{\psi^x} \in \mathrm{Irr}(\widetilde{B^x})$ and $(\widetilde{\psi})^{\widetilde{x}} \in \mathrm{Irr}(\widetilde{B}^{\widetilde{x}})$, we conclude that $\widetilde{B^x} = \widetilde{B}^{\widetilde{x}}$. □

In the situation of Theorem 4.2.2, we refer to $\widetilde{B}$ as the **Fong correspondent** of B and to $\widetilde{\psi}$ as the **Fong correspondent** of ψ. An important feature of the Fong correspondence is that it is compatible with block induction.

Proposition 4.2.3. *Assume Hypothesis 4.2.1 and let $N \leq X \leq Y \leq A$. Let $b \in \mathrm{Bl}(X \mid \mathrm{bl}(\mu))$ with Fong correspondent $\widetilde{b} \in \mathrm{Bl}(\widetilde{X} \mid \mathrm{bl}(\widetilde{\mu}))$ and suppose that the induced blocks b^Y and $(\widetilde{b})^{\widetilde{Y}}$ are defined. Then $\widetilde{b^Y} = (\widetilde{b})^{\widetilde{Y}}$.*

Proof. This result has been shown in [Rob00]. It can also be deduced from [Dad94, Theorem 14.3]. □

Now, we prove a rather technical result that shows that Fong's reduction is compatible with N-block isomorphism of character triples. More precisely, we have the following.

Theorem 4.2.4. *Assume Hypothesis 4.2.1. For $i = 1, 2$, consider $N \leq L_i \trianglelefteq H_i \leq A$ and a H_i-invariant $\psi_i \in \mathrm{Irr}(L_i \mid \mu)$. Notice that $\widetilde{L}_i \trianglelefteq \widetilde{H}_i$ and that the Fong correspondent $\widetilde{\psi}_i \in \mathrm{Irr}(\widetilde{L}_i \mid \widetilde{\mu})$ is $\widetilde{H}_i$-invariant. Let $L_i \leq G \trianglelefteq A$ and assume*

$$\left(\widetilde{H}_1, \widetilde{L}_1, \widetilde{\psi}_1\right) \sim_{\widetilde{G}} \left(\widetilde{H}_2, \widetilde{L}_2, \widetilde{\psi}_2\right).$$

Then

$$\left(H_1, L_1, \psi_1\right) \sim_{G} \left(H_2, L_2, \psi_2\right).$$

Proof. The group theoretical conditions are clearly satisfied and without loss of generality we may assume $A = GH_i$, $\widehat{A} = \widehat{G}\widehat{H}_i$ and $\widetilde{A} = \widetilde{G}\widetilde{H}_i$. Consider $B_i := \mathrm{bl}(\psi_i)$ and its Fong correspondent $\widetilde{B}_i = \mathrm{bl}(\widetilde{\psi}_i)$. By hypothesis, there exists a defect group $D_i \in \delta(\widetilde{B}_i)$ such that $\mathbf{C}_{\widetilde{A}}(D_i) \leq \widetilde{H}_i$. Furthermore, by Theorem 4.2.2 (iii) we can find a defect group $P_i \in \delta(B_i)$ such that, if $Q_i \in \mathrm{Syl}_p(\widehat{P}_i)$, then $D_i = Q_iN_0/N_0$. In particular

$$\mathbf{C}_{\widehat{A}}(Q_i) \leq \widehat{H}_i \tag{4.2.1}$$

and, noticing that

$$\epsilon\left(\mathbf{C}_{\widehat{A}}(Q_i)\right) = \mathbf{C}_A(P_i),$$

we obtain $\mathbf{C}_A(P_i) \leq H_i$. Fix a pair of projective representations $(\widetilde{\mathcal{R}}_1, \widetilde{\mathcal{R}}_2)$ associated with $\left(\widetilde{H}_1, \widetilde{L}_1, \widetilde{\psi}_1\right) \sim_{\widetilde{G}} \left(\widetilde{H}_2, \widetilde{L}_2, \widetilde{\psi}_2\right)$ and let $\widetilde{\alpha}_i$ be the factor set of $\widetilde{\mathcal{R}}_i$. Consider a projective representation $\mathcal{R}_i \in \mathrm{Proj}(H_i)$ with factor set α_i associated with ψ_i and define the projective representation $\widehat{\mathcal{R}}_i \in \mathrm{Proj}(\widehat{H}_i)$ given by

$$\widehat{\mathcal{R}}_i(h) \coloneqq \mathcal{R}_i(\epsilon(h))$$

for every $h \in \widehat{H}_i$. Notice that the factor set $\widehat{\alpha}_i$ of $\widehat{\mathcal{R}}_i$ satisfies $\widehat{\alpha}_i(h,k) = \alpha_i(\epsilon(h), \epsilon(k))$, for every $h, k \in \widehat{H}_i$, and that $\widehat{\mathcal{R}}_i$ is associated with $\widehat{\psi}_i$. Let $\widetilde{\mathcal{R}}'_i \in \mathrm{Proj}(\widehat{H}_i)$ be the projective representation defined by

$$\widetilde{\mathcal{R}}'_i(h) \coloneqq \widetilde{\mathcal{R}}_i(N_0 h)$$

for every $h \in \widehat{H}_i$. The factor set $\widetilde{\alpha}'_i$ of $\widetilde{\mathcal{R}}'_i$ satisfies $\widetilde{\alpha}'_i(h,k) = \widetilde{\alpha}_i(N_0 h, N_0 k)$, for every $h, k \in \widehat{H}_i$, and $\widetilde{\mathcal{R}}'_i$ is associated with $\widetilde{\psi}'_i$. As $\widehat{\mathcal{R}}_i$ and $\widehat{\mathcal{P}}_{\widehat{H}_i} \otimes \widetilde{\mathcal{R}}'_i$ are a projective representations of $\widehat{H}_i$ associated with $\tau_{\widehat{L}_i} \widetilde{\psi}'_i = \widehat{\psi}_i$, by [Nav18, Lemma 10.10 (b)] there exists a map $\widehat{\xi}_i : \widehat{H}_i/\widehat{L}_i \to \mathbb{C}^\times$ such that $\widehat{\xi}_i \widehat{\mathcal{R}}_i = \widehat{\mathcal{P}}_{\widehat{H}_i} \otimes \widetilde{\mathcal{R}}'_i$. Let $\xi_i : H_i/L_i \to \mathbb{C}^\times$ corresponds to $\widehat{\xi}_i$ via the isomorphism $H_i/L_i \simeq \widehat{H}_i/\widehat{L}_i$. Replacing $\mathcal{R}_i$ with $\xi_i \mathcal{R}_i$, we may assume

$$\widehat{\mathcal{R}}_i = \widehat{\mathcal{P}}_{\widehat{H}_i} \otimes \widetilde{\mathcal{R}}'_i. \tag{4.2.2}$$

Now, as the factor sets $\widetilde{\alpha}_1$ and $\widetilde{\alpha}_2$ coincide under the isomorphism $\widetilde{H}_1/\widetilde{L}_1 \simeq \widetilde{H}_2/\widetilde{L}_2$, we deduce that α_1 and α_2 coincide under the isomorphism $H_1/L_1 \simeq H_2/L_2$. By hypothesis $\widetilde{\mathcal{R}}_1$ and $\widetilde{\mathcal{R}}_2$ define the same scalar function on $\mathbf{C}_{\widetilde{A}}(\widetilde{G})$. As $\mathbf{C}_{\widehat{A}}(\widehat{G})N_0/N_0 \leq \mathbf{C}_{\widetilde{A}}(\widetilde{G})$ and $\mathbf{C}_{\widehat{A}}(\widehat{G}) \leq \widehat{H}_1 \cap \widehat{H}_2$ by (4.2.1), the scalar functions defined by $\widetilde{\mathcal{R}}'_1$ and $\widetilde{\mathcal{R}}'_2$ on $\mathbf{C}_{\widehat{A}}(\widehat{G})$ coincide. Now $\widehat{\mathcal{R}}_{1,\mathbf{C}_{\widehat{A}}(\widehat{G})}$ and $\widehat{\mathcal{R}}_{2,\mathbf{C}_{\widehat{A}}(\widehat{G})}$ are associated with the same scalar function and, since $\epsilon(\mathbf{C}_{\widehat{A}}(\widehat{G})) = \mathbf{C}_A(G)$ (see Theorem 3.4.11 (ii)), the same is true for $\mathcal{R}_{1,\mathbf{C}_A(G)}$ and $\mathcal{R}_{2,\mathbf{C}_A(G)}$.

Next, consider $G \leq J \leq A$ and set $J_i \coloneqq J \cap H_i$. Notice that, if $\chi \in \mathrm{Irr}(J_1 \mid \psi_1)$, then Theorem 4.2.2 (iv) implies that $\widetilde{\chi} \in \mathrm{Irr}(\widetilde{J}_1 \mid \widetilde{\psi}_1)$. Write $\chi = \mathrm{Tr}(\mathcal{Q}_{J_1} \otimes \mathcal{R}_{1,J_1})$, for some $\mathcal{Q} \in \mathrm{Proj}(J/G)$. If we set $\widehat{Q}(x) \coloneqq \mathcal{Q}(\epsilon(x))$ for every $x \in \widehat{J}$, then (4.2.2) implies

$$\begin{aligned}
\widehat{\chi}_1 &= \mathrm{Tr}\left(\widehat{\mathcal{Q}}_{\widehat{J}_1} \otimes \widehat{\mathcal{R}}_{1,\widehat{J}_1}\right) \\
&= \mathrm{Tr}\left(\widehat{\mathcal{Q}}_{\widehat{J}_1} \otimes \widetilde{\mathcal{R}}'_{1,\widehat{J}_1} \otimes \widehat{\mathcal{P}}_{\widehat{J}_1}\right)
\end{aligned}$$

and therefore $\widetilde{\chi}' = \mathrm{Tr}(\widehat{\mathcal{Q}}_{\widehat{J}_1} \otimes \widetilde{\mathcal{R}}'_{1,\widehat{J}_1})$. Now, let $\widetilde{\mathcal{Q}} \in \mathrm{Proj}(\widetilde{J}/\widetilde{G})$ correspond to $\widehat{\mathcal{Q}}$ via the isomorphism $\widetilde{J}/\widetilde{G} \simeq \widehat{J}/\widehat{G}$ and observe that the Fong correspondent of χ can be written as $\widetilde{\chi} = \mathrm{Tr}(\widetilde{\mathcal{Q}}_{\widetilde{J}_1} \otimes \widetilde{\mathcal{R}}_{1,\widetilde{J}_1})$. By definition $\widetilde{\sigma}_{\widetilde{J}_1}(\widetilde{\chi}) = \mathrm{Tr}(\widetilde{\mathcal{Q}}_{\widetilde{J}_2} \otimes \widetilde{\mathcal{R}}_{2,\widetilde{J}_2})$ so that its inflation $\widetilde{\sigma}_{\widetilde{J}_1}(\widetilde{\chi})' = \mathrm{Tr}(\widehat{\mathcal{Q}}_{\widehat{J}_2} \otimes \widetilde{\mathcal{R}}'_{2,\widehat{J}_2})$. By Theorem 4.2.2 (iv) and (4.2.2) we obtain

$$\begin{aligned}
\tau_{\widehat{J}_2} \widetilde{\sigma}_{\widetilde{J}_1}(\widetilde{\chi})' &= \mathrm{Tr}\left(\widehat{\mathcal{P}}_{\widehat{J}_2} \otimes \widehat{\mathcal{Q}}_{\widehat{J}_2} \otimes \widetilde{\mathcal{R}}'_{2,\widehat{J}_2}\right) \\
&= \mathrm{Tr}\left(\widehat{\mathcal{Q}}_{\widehat{J}_2} \otimes \widehat{\mathcal{R}}_{2,\widehat{J}_2}\right) \\
&= \widehat{\sigma_{J_1}(\chi)} \\
&= \tau_{\widehat{J}_2} \widetilde{\sigma_{J_1}(\chi)}'
\end{aligned}$$

and therefore

$$\widetilde{\sigma_{J_1}(\chi)} = \widetilde{\sigma}_{\widetilde{J}_1}(\widetilde{\chi}).$$

Since by hypothesis $\mathrm{bl}(\widetilde{\sigma_{J_1}(\chi)})^{\widetilde{J}} = \mathrm{bl}(\widetilde{\chi})^{\widetilde{J}}$, we conclude from Proposition 4.2.3 that

$$\mathrm{bl}(\sigma_{J_1}(\chi))^J = \mathrm{bl}(\chi)^J.$$

This completes the proof. □

From now on we consider $N \leq G \trianglelefteq A$. Since $\widetilde{N}$ is a central p'-subgroup of $\widetilde{G}$, for every p-subgroup P of G we have a decomposition $\widetilde{P} = \widetilde{N} \times \mathbf{O}_p(\widetilde{P})$. We write $\widetilde{P}_p := \mathbf{O}_p(\widetilde{P})$. Mapping P to $\widetilde{P}_p$ induces a length preserving bijection

$$\begin{aligned} \mathfrak{N}(G,Z)/G &\to \mathfrak{N}(\widetilde{G},\widetilde{Z}_p)/\widetilde{G} \\ \mathbb{D} &\mapsto \widetilde{\mathbb{D}} \end{aligned} \tag{4.2.3}$$

which commutes with the action of A and $\widetilde{A}$. In particular, observe that $\widetilde{NG_{\mathbb{D}}} = \widetilde{G}_{\widetilde{\mathbb{D}}}$. Using Theorem 4.1.8, Theorem 4.2.2 and Theorem 4.2.4 we obtain the following corollaries.

Corollary 4.2.5. *Assume Hypothesis 4.2.1 and let $N \leq G \trianglelefteq A$. Consider a normal p-chain $\mathbb{D}$ of G with final term P and let $f_P(\mu) \in \mathrm{Irr}(N_P)$ be the Glauberman correspondent of μ. Then there exists a defect preserving bijection*

$$\Gamma_{\mu,\mathbb{D}} : \mathrm{Irr}\left(G_{\mathbb{D}} \mid f_P(\mu)\right) \to \mathrm{Irr}\left(\widetilde{G}_{\widetilde{\mathbb{D}}} \mid \widetilde{\mu}\right)$$

commuting with the action of A and $\widetilde{A}$.

Proof. This follows immediately by Theorem 4.1.8 and Theorem 4.2.2. □

The bijections described in the previous corollary are compatible with N-block isomorphisms of character triples in the following sense.

Corollary 4.2.6. *Assume Hypothesis 4.2.1 and let $N \leq G \trianglelefteq A$. Consider normal p-chains $\mathbb{D}$ and $\mathbb{E}$ of G with final term P and Q respectively and let $\Gamma_{\mu,\mathbb{D}}$ and $\Gamma_{\mu,\mathbb{E}}$ be the corresponding bijections given by Corollary 4.2.5. Let $\vartheta \in \mathrm{Irr}(G_{\mathbb{D}} \mid f_P(\mu))$ and $\chi \in \mathrm{Irr}(G_{\mathbb{E}} \mid f_Q(\mu))$ and suppose that*

$$\left(\widetilde{A}_{\widetilde{\mathbb{D}},\Gamma_{\mu,\mathbb{D}}(\vartheta)}, \widetilde{G}_{\widetilde{\mathbb{D}}}, \Gamma_{\mu,\mathbb{D}}(\vartheta)\right) \sim_{\widetilde{G}} \left(\widetilde{A}_{\widetilde{\mathbb{E}},\Gamma_{\mu,\mathbb{E}}(\chi)}, \widetilde{G}_{\widetilde{\mathbb{E}}}, \Gamma_{\mu,\mathbb{E}}(\chi)\right).$$

Then

$$\left(A_{\mathbb{D},\vartheta}, G_{\mathbb{D}}, \vartheta\right) \sim_G \left(A_{\mathbb{E},\chi}, G_{\mathbb{E}}, \chi\right).$$

Proof. This is a consequence of Theorem 4.1.8, Theorem 4.2.4 and Corollary 4.2.5 together with the fact that $\sim_G$ is an equivalence relation (see Lemma 3.3.8). □

Finally, by putting together all the results obtained so far in this chapter, we obtain the following result which will play a fundamental role in the proof of Theorem 4.1.

Corollary 4.2.7. *Assume Hypothesis 4.2.1 and let $N \leq G \trianglelefteq A$. Let Z be a central p-subgroup of G and consider a block $B \in \mathrm{Bl}(G \mid \mathrm{bl}(\mu))$ whose defect groups are larger than Z. Then the Fong correspondent $\widetilde{B} \in \mathrm{Bl}(\widetilde{G})$ has defect groups larger than $\widetilde{Z}_p$ and there exists a bijection*

$$\Delta : \mathcal{C}^d(B, Z)/G \to \mathcal{C}^d(\widetilde{B}, \widetilde{Z}_p)/\widetilde{G}$$

that preserves the length of the p-chains, commutes with the ation of A and $\widetilde{A}$ and such that, if

$$\left(\widetilde{A}_{(\widetilde{\mathbb{D}},\widetilde{\vartheta})}, \widetilde{G}_{\widetilde{\mathbb{D}}}, \widetilde{\vartheta}\right) \sim_{\widetilde{G}} \left(\widetilde{A}_{(\widetilde{\mathbb{E}},\widetilde{\chi})}, \widetilde{G}_{\widetilde{\mathbb{D}}}, \widetilde{\chi}\right),$$

then

$$\left(A_{(\mathbb{D},\vartheta)}, G_{\mathbb{D}}, \vartheta\right) \sim_G \left(A_{(\mathbb{E},\chi)}, G_{\mathbb{E}}, \chi\right)$$

for every $(\mathbb{D}, \vartheta), (\mathbb{E}, \chi) \in \mathcal{C}^d(B, Z)$, $(\widetilde{\mathbb{D}}, \widetilde{\vartheta}) \in \Delta(\overline{(\mathbb{D}, \vartheta)})$ and $(\widetilde{\mathbb{E}}, \widetilde{\chi}) \in \Delta(\overline{(\mathbb{E}, \chi)})$.

Proof. Let $\mathbb{D} \in \mathfrak{N}(G, Z)$ with last term P and consider $\widetilde{\mathbb{D}} \in \mathfrak{N}(\widetilde{G}, \widetilde{Z}_p)$. If $\vartheta \in \mathrm{Irr}(G_{\mathbb{D}})$ and $\mathrm{bl}(\vartheta)^G = B$, then ϑ lies over $f_P(\mu)$ by Corollary 2.3.2. Now, there exists a unique $\psi \in \mathrm{Irr}(NG_{\mathbb{D}} \mid \mu)$ such that $\vartheta = \Phi_{\mu,\mathbb{D}}(\psi)$ and $\Gamma_{\mu,\mathbb{D}}(\vartheta) = \widetilde{\psi}$ is the Fong correspondent of ψ. By Theorem 4.1.8, we know that $\mathrm{bl}(\vartheta)^{NG_{\mathbb{D}}} = \mathrm{bl}(\psi)$, hence $\mathrm{bl}(\vartheta)^G = B$ if and only if $\mathrm{bl}(\psi)^G = B$. Furthermore, by Proposition 4.2.3 it follows that $\mathrm{bl}(\psi)^G = B$ if and only if $\mathrm{bl}(\widetilde{\psi})^{\widetilde{G}} = \widetilde{B}$. This shows that the set of characters of $G_{\mathbb{D}}$ whose block induces to B is mapped via $\Gamma_{\mu,\mathbb{D}}$ to the set of characters of $\widetilde{G}_{\widetilde{\mathbb{D}}}$ whose block induces to $\widetilde{B}$. We define

$$\Delta\left(\overline{(\mathbb{D}, \vartheta)}\right) := \overline{\left(\widetilde{\mathbb{D}}, \Gamma_{\mu,\mathbb{D}}(\vartheta)\right)}$$

for every $(\mathbb{D}, \vartheta) \in \mathcal{C}^d(B, Z)$. By (4.2.3), Corollary 4.2.5 and Corollary 4.2.6, we conclude that Δ is a bijection with the required properties. □

4.3 Structure of a minimal counterexample

In this section, we finally prove the Character Triple Conjecture for p-solvable groups. Our proof is inspired by the argument developed in [Rob00]. As in Robinson's work, what we are actually going to show is that a minimal counterexample G to Conjecture 3.5.1 satisfies $\mathbf{O}_p(G)\mathbf{O}_{p'}(G) \leq \mathbf{Z}(G)$ (see Theorem 4.3.2). Since the conjecture trivially holds for abelian groups, Theorem 4.1 will then follow as a corollary of (the proof of) Theorem 4.3.2. In the follwing proof, we consider subpairs in the sense of [Ols82], i.e. pairs (P, b_P), where P is a p-subgroup of G and $b_P \in \mathrm{Bl}(P\mathbf{C}_G(P))$.

Proposition 4.3.1. *Assume that $G \trianglelefteq A$ is a minimal counterexample to Conjecture 3.5.1 with respect to $|G : \mathbf{Z}(G)|$ first and then to $|A|$ and consider $Z \leq \mathbf{Z}(G)$, $B \in \mathrm{Bl}(G)$ and $d \geq 0$ for which the conjecture fails to hold. Then every block $b \in \mathrm{Bl}(\mathbf{O}_{p'}(G))$ covered by B is A-invariant.*

Proof. Set $N := \mathbf{O}_{p'}(G)$ and fix a block $b \in \mathrm{Bl}(N)$ covered by B. Let μ be the unique irreducible ordinary character of b. For every subgroup $H \leq A$, set $H^\vee := H_\mu$ and notice that $H^\vee = H_b$. Let $B^\vee \in \mathrm{Bl}(G^\vee \mid \mathrm{bl}(\mu))$ be the Fong–Reynolds correspondent of B over $\mathrm{bl}(\mu)$. Since B and $B^\vee$ have a common defect group $D \leq G^\vee$, by using [Ols82, Theorem 2.1] we can find a $B^\vee$-Sylow subpair $(D, b_D^\vee)$ such that (D, b_D) is a B-Sylow subpair, where $b_D := (b_D^\vee)^{D\mathbf{C}_G(D)}$. We claim that, for every B-subpair $(Q, b_Q) \leq (D, b_D)$, the block b_Q covers $\mathrm{bl}(f_Q(\mu))$. To see this, notice that $Q\mathbf{C}_{G^\vee}(Q)N \leq G^\vee$ and, as N is a p'-subgroup of $G^\vee$, that $Q\mathbf{C}_{G^\vee}(Q) = (Q\mathbf{C}_{G^\vee}(Q)N)_Q$. By [Ols82, Theorem 1.7], there exists a unique block $b_Q^\vee$ of $Q\mathbf{C}_{G^\vee}(Q)$ such that $(Q, b_Q^\vee) \leq (D, b_D^\vee)$. Moreover $(b_Q^\vee)^{Q\mathbf{C}_G(Q)} = b_Q$. Since $(b_Q^\vee)^{G^\vee} = B^\vee$ covers the $G^\vee$-invariant block b, it follows that $(b_Q^\vee)^{Q\mathbf{C}_{G^\vee}(Q)N}$ covers b. Now, applying Corollary 2.3.2, we obtain that $b_Q^\vee$ covers $\mathrm{bl}(f_Q(\mu))$ and therefore so does $b_Q = (b_Q^\vee)^{Q\mathbf{C}_G(Q)}$. This proves the claim.

Using our claim, we can construct an A-transversal $\mathbb{T}$ in $\mathcal{C}^d(B, Z)$ such that $P \leq G^\vee$ and $\vartheta \in \mathrm{Irr}(G_\mathbb{D} \mid f_P(\mu))$ for every $(\mathbb{D}, \vartheta) \in \mathbb{T}$ with P the last term of $\mathbb{D}$. In fact, let $(\mathbb{D}, \vartheta) \in \mathcal{C}^d(B, Z)$ and P the last term of $\mathbb{D}$. Consider a block b_P of $P\mathbf{C}_G(P)$ covered by $\mathrm{bl}(\vartheta)$. By [Nav98, Corollary 9.21] it follows that $(b_P)^{G_\mathbb{D}} = \mathrm{bl}(\vartheta)$ and, since $\mathrm{bl}(\vartheta)^G = B$, we deduce that (P, b_P) is a B-subpair. By [Ols82, Theorem 2.2] there exists $g \in G$ such that $(P, b_P)^g \leq (D, b_D)$. Now, by the previous paragraph we conclude that $(\mathbb{D}^g, \vartheta^g)$ satisfies $P^g \leq G^\vee$ and $\mathrm{bl}(\vartheta^g)$ covers $\mathrm{bl}(f_{P^g}(\mu))$. This shows that every $(\mathbb{D}, \vartheta) \in \mathcal{C}^d(B, Z)$ is G-conjugate to a pair with required properties. In particular we can find an A-transversal $\mathbb{T}$ as above.

Now, consider $(\mathbb{D}, \vartheta) \in \mathbb{T}$ and let P be the last term of $\mathbb{D}$. Notice that $G_\mathbb{D}^\vee = G_{\mathbb{D}, f_P(\mu)}$ and let $\vartheta^\vee \in \mathrm{Irr}(G_\mathbb{D}^\vee \mid f_P(\mu))$ be the Clifford correspondent of ϑ over $f_P(\mu)$. As $A = GA^\vee$ (recall that B is A-invariant), we obtain an $A^\vee$-equivariant bijection

$$\Upsilon : \mathcal{C}^d(B, Z)/G \to \mathcal{C}^d(B^\vee, Z)/G^\vee$$

by defining $\Upsilon(\overline{(\mathbb{D}, \vartheta)}^y) := \overline{(\mathbb{D}, \vartheta^\vee)}^y$ for every $(\mathbb{D}, \vartheta) \in \mathbb{T}$ and $y \in A^\vee$. Since $|G^\vee : \mathbf{Z}(G^\vee)| \leq |G : \mathbf{Z}(G)|$, if μ is not A-invariant, then there exists an $A^\vee$-equivariant bijection

$$\Omega^\vee : \mathcal{C}^d(B^\vee, Z)_+/G^\vee \to \mathcal{C}^d(B^\vee, Z)_-/G^\vee$$

such that

$$\left(A^\vee_{(\mathbb{D}, \vartheta^\vee)}, G^\vee_\mathbb{D}, \vartheta^\vee\right) \sim_{G^\vee} \left(A^\vee_{(\mathbb{E}, \chi^\vee)}, G^\vee_\mathbb{E}, \chi^\vee\right)$$

for every $(\mathbb{D}, \vartheta^\vee) \in \mathcal{C}^d(B^\vee, Z)_+$ and $(\mathbb{E}, \chi^\vee) \in \Omega^\vee(\overline{(\mathbb{D}, \vartheta^\vee)})$. Combining $\Omega^\vee$ with Υ and applying Proposition 3.4.3, we obtain an A-equivariant bijection

$$\Omega : \mathcal{C}^d(B, Z)_+/G \to \mathcal{C}^d(B, Z)_-/G$$

such that

$$\left(A_{(\mathbb{D}, \vartheta)}, G_\mathbb{D}, \vartheta\right) \sim_G \left(A_{(\mathbb{E}, \chi)}, G_\mathbb{E}, \chi\right)$$

for every $(\mathbb{D}, \vartheta) \in \mathcal{C}^d(B, Z)_+$ and $(\mathbb{E}, \chi) \in \Omega(\overline{(\mathbb{D}, \vartheta)})$. This is a contradiction and therefore μ must be A-invariant. □

Theorem 4.3.2. *Assume that $G \trianglelefteq A$ is a minimal counterexample to Conjecture 3.5.1 with respect to $|G : \mathbf{Z}(G)|$ first and then to $|A|$ and consider $Z \leq \mathbf{Z}(G)$, $B \in \mathrm{Bl}(G)$ and $d \geq 0$ for which the conjecture fails to hold. Then $\mathbf{O}_p(G)\mathbf{O}_{p'}(G) \leq \mathbf{Z}(G)$.*

Proof. Set $N := \mathbf{O}_{p'}(G)$ and fix a block $\mathrm{bl}(\mu) \in \mathrm{Bl}(N)$ covered by B. Notice that by Lemma 3.5.4 we must have $Z = \mathbf{O}_p(G)$. Thus it's enough to show that N is contained in the center. By Proposition 4.3.1, we know that μ is A-invariant and therefore we can apply the results obtained in Section 4.1 and 4.2. Consider the setting of Hypothesis 4.2.1 corresponding to a choice of a projective representation $\mathcal{P}$ associated with (A, N, μ). Let $\widetilde{B} \in \mathrm{Bl}(\widetilde{G})$ be the Fong correspondent of B (see Theorem 4.2.2). Since $\widetilde{N} \leq \mathbf{Z}(\widetilde{G})$, if $N \nleq \mathbf{Z}(G)$, then $|\widetilde{G} : \mathbf{Z}(\widetilde{G})| \leq |\widetilde{G} : \widetilde{N\mathbf{Z}(G)}| = |G : N\mathbf{Z}(G)| < |G : \mathbf{Z}(G)|$ and we obtain an $\widetilde{A}$-equivariant bijection

$$\widetilde{\Omega} : \mathcal{C}^d(\widetilde{B}, \widetilde{Z}_p)_+/\widetilde{G} \to \mathcal{C}^d(\widetilde{B}, \widetilde{Z}_p)_-/\widetilde{G}$$

such that

$$\left(\widetilde{A}_{(\widetilde{\mathbb{D}},\widetilde{\vartheta})}, \widetilde{G}_{\widetilde{\mathbb{D}}}, \widetilde{\vartheta}\right) \sim_{\widetilde{G}} \left(\widetilde{A}_{(\widetilde{\mathbb{E}},\widetilde{\chi})}, \widetilde{G}_{\widetilde{\mathbb{D}}}, \widetilde{\chi}\right)$$

for every $(\widetilde{\mathbb{D}}, \widetilde{\vartheta}) \in \mathcal{C}^d(\widetilde{B}, \widetilde{Z}_p)_+$ and $(\widetilde{\mathbb{E}}, \widetilde{\chi}) \in \widetilde{\Omega}(\overline{(\widetilde{\mathbb{D}}, \widetilde{\vartheta})})$ and where $\widetilde{Z}_p$ is defined by (4.2.3). Combining $\widetilde{\Omega}$ with the bijection Δ given by Corollary 4.2.7, we obtain an A-equivariant bijection

$$\Omega : \mathcal{C}^d(B, Z)_+/G \to \mathcal{C}^d(B, Z)_-/G$$

such that

$$\left(A_{(\mathbb{D},\vartheta)}, G_{\mathbb{D}}, \vartheta\right) \sim_G \left(A_{(\mathbb{E},\chi)}, G_{\mathbb{E}}, \chi\right)$$

for every $(\mathbb{D}, \vartheta) \in \mathcal{C}^d(B, Z)_+$ and $(\mathbb{E}, \chi) \in \Omega(\overline{(\mathbb{D}, \vartheta)})$. This contradiction shows that N must be contained in the center of G. □

Next, we consider the p-residue of characters. We are going to obtain Theorem 4.2 as a consequence of an analogous study of a minimal counterexample. Namely, Theorem 4.2 will follow from (the proof of) Theorem 4.3.5. We start by comparing the residues of characters that correspond under the bijection from Corollary 4.2.5.

Lemma 4.3.3. *Let $\Gamma_{\mu,\mathbb{D}}$ be the bijection of Corollary 4.2.5. Then*

$$r\left(\Gamma_{\mu,\mathbb{D}}(\vartheta)\right)|N| \equiv \pm\mu(1)r(\vartheta)|\widetilde{N}| \pmod p$$

for every $\vartheta \in \mathrm{Irr}(G_{\mathbb{D}} \mid f_P(\mu))$.

Proof. Let P be the last term of the p-chain $\mathbb{D}$ and fix $\vartheta \in \mathrm{Irr}(G_{\mathbb{D}} \mid f_P(\mu))$. Let χ be the unique character in $\mathrm{Irr}(NG_{\mathbb{D}} \mid \mu)$ such that $\Phi_{\mu,\mathbb{D}}(\chi) = \vartheta$ (see Theorem 4.1.8). Then $\Gamma_{\mu,\mathbb{D}}(\vartheta)$ coincides with the Fong correspondent $\widetilde{\chi} \in \mathrm{Irr}(\widetilde{G}_{\widetilde{\mathbb{D}}} \mid \widetilde{\mu})$ of χ (see Theorem 4.2.2 and Corollary 4.2.5). First, we show that

$$r(\vartheta) \equiv \pm r(\chi) \pmod p. \tag{4.3.1}$$

Let ϑ_0 be the Clifford correspondent of ϑ over $f_P(\mu)$ and notice that $G_{\mathbb{D},f_P(\mu)} = G_{\mathbb{D},\mu}$. Similarly, let χ_0 be the Clifford correspondent of χ over μ. Since induction of characters preserves the p-residue, we deduce that $r(\vartheta) = r(\vartheta_0)$ and $r(\chi) = r(\chi_0)$ and it's enough to show that $r(\vartheta_0) \equiv \pm r(\chi_0) \pmod p$. By the definition of $\Phi_{\mu,\mathbb{D}}$ (see the proof of Theorem 4.1.8 and of Proposition 3.4.4) and using Lemma 3.1.2, it follows that $\vartheta_0(1) = \chi_0(1)f_P(\mu)(1)/\mu(1)$. Then

$$r(\vartheta_0) = \frac{|G_{\mathbb{D},\mu}|_{p'}}{\vartheta_0(1)_{p'}} = \frac{|G_{\mathbb{D},\mu}|_{p'}\mu(1)}{\chi_0(1)_{p'}f_P(\mu)(1)}.$$

By [NS14b, Theorem 5.2 (b)], we have $\mu(1) \equiv \pm|N : N_P| f_P(\mu)(1) \pmod p$ and so

$$\frac{|G_{\mathbb{D},\mu}|_{p'}\mu(1)}{\chi_0(1)_{p'} f_P(\mu)(1)} \equiv \pm \frac{|G_{\mathbb{D},\mu}|_{p'}|N|}{\chi_0(1)_{p'}|N_P|} \pmod p.$$

Finally, since N is a p'-subgroup, we have $G_{\mathbb{D},\mu} \cap N = \mathbf{C}_N(P) = N_P$ and therefore

$$\frac{|G_{\mathbb{D},\mu}|_{p'}|N|}{\chi_0(1)_{p'}|N_P|} = \frac{|NG_{\mathbb{D},\mu}|_{p'}}{\chi_0(1)_{p'}} = r(\chi_0).$$

This proves (4.3.1). Next, noticing that $\chi(1) = \mu(1)\widetilde{\chi}(1)$ and that $|NG_{\mathbb{D}} : N| = |\widetilde{G}_{\widetilde{\mathbb{D}}} : \widetilde{N}|$, we obtain

$$r(\chi) = r(\widetilde{\chi}) \frac{|N|}{|\widetilde{N}|\mu(1)}. \tag{4.3.2}$$

Now the result follows by combining (4.3.1) and (4.3.2). □

For completeness, we state the Character Triple Conjecture with the Isaacs-Navarro refinement.

Conjecture 4.3.4 (Isaacs-Navarro refinement of the Character Triple Conjecture)**.** *There exists a bijection Ω as in Conjecture 3.5.1 such that*

$$r(\vartheta) \equiv \pm r(\chi) \pmod p$$

for every $(\mathbb{D}, \vartheta) \in \mathcal{C}^d(B, Z)_+$ and $(\mathbb{E}, \chi) \in \Omega(\overline{(\mathbb{D}, \vartheta)})$.

Finally, using the proof of Theorem 4.3.2 and Lemma 4.3.3 we obtain a similar structure theorem for a minimal counterexample of Conjecture 4.3.4.

Theorem 4.3.5. *Assume that $G \trianglelefteq A$ is a minimal counterexample to Conjecture 4.3.4 with respect to $|G : \mathbf{Z}(G)|$ first and then to $|A|$ and consider $Z \leq \mathbf{Z}(G)$, $B \in \mathrm{Bl}(G)$ and $d \geq 0$ for which the conjecture fails to hold. Then $\mathbf{O}_p(G)\mathbf{O}_{p'}(G) \leq \mathbf{Z}(G)$.*

Proof. Set $N := \mathbf{O}_{p'}(G)$ and fix a block $\mathrm{bl}(\mu) \in \mathrm{Bl}(N)$ covered by B. By the proof of Lemma 2.3.1 we know that $Z = \mathbf{O}_p(G)$ and it's enough to show that $N \leq \mathbf{Z}(G)$. Proceeding as in the proof of Proposition 4.3.1 and noticing that induction of characters preserves the residue of characters, we deduce that μ must be A-invariant. Then, using Lemma 4.3.3 and adapting the the proof of Theorem 4.3.2, we obtain $N \leq \mathbf{Z}(G)$. □

5

On the Inductive Alperin–McKay Condition

It was shown by Dade in [Dad94] that the projective form of his conjecture (see Conjecture 2.5.3) implies the Alperin–McKay Conjecture (see Conjecture 2.4.2). Later, Navarro [Nav18, Theorem 9.27] proved that the nonblockwise version of Dade's Ordinary Conjecture (see Conjecture 2.5.2) implies the McKay Conjecture (see Conjecture 2.4.1), while Kessar and Linckelmann [KL19] extended these results by proving that Dade's Ordinary Conjecture implies the Alperin–McKay Conjecture.

By work of Späth [Spä13a], the Alperin-McKay Conjecture has been reduced to the inductive Alperin–McKay condition for quasisimple groups. Nonetheless, the inductive Alperin–McKay condition can be formulated for every finite group (see Conjecturr 5.1.1). In this chapter, we provide further evidence for the validity of the Character Triple Conjecture by showing that it implies the inductive Alperin–McKay condition for every finite group. Let p be a prime number. Every block in this chapter will be considered with respect to the prime p.

Theorem 5.1. *If the Character Triple Conjecture holds for every p-block of every finite group, then the inductive Alperin–McKay condition (see Conjecture 5.1.1) holds for every p-block of every finite group.*

By work of Navarro and Späth, we know that Conjecture 5.1.1 holds for every p-solvable group (this follows from [NS14b, Theorem 7.1]). We obtain another proof of this fact by using the proof of Theorem 5.1 together with the fact that the Character Triple Conjecture holds for p-solvable groups (see Theorem 4.1).

5.1 The inductive Alperin–McKay condition

Here, we state the inductive Alperin–McKay condition in a general form adapted to arbitrary finite groups.

Conjecture 5.1.1 (Inductive Alperin-McKay condition). *Let G be a finite group with $G \trianglelefteq A$. Let $B \in \mathrm{Bl}(G \mid D)$ and consider its Brauer's first main correspondent $b \in \mathrm{Bl}(\mathbf{N}_G(D) \mid D)$. Then there exists a $\mathbf{N}_A(D)_B$-equivariant bijection*

$$\Theta : \mathrm{Irr}_0(B) \to \mathrm{Irr}_0(b)$$

such that

$$(A_\vartheta, G, \chi) \sim_G (\mathbf{N}_A(D)_\vartheta, \mathbf{N}_G(D), \Theta(\vartheta)),$$

for every $\vartheta \in \mathrm{Irr}_0(B)$.

We point out that, arguing as in the proof of [Spä17, Proposition 6.8], it follows that the inductive Alperin–McKay condition from [Spä18, Definition 4.12] holds for every universal covering group if and only if Conjecture 5.1.1 holds for every quasisimple group X with respect to $X \trianglelefteq X \rtimes \mathrm{Aut}(X)$. We include here a proof of this fact.

Lemma 5.1.2. *Let S be a nonabelian simple group with universal covering group X and consider $B \in \mathrm{Bl}(X)$. Then B is AM-good (in the sense of [Spä18, Definition 4.12]) if and only if Conjecture 5.1.1 holds for every $\overline{B} \in \mathrm{Bl}(\overline{X})$ with respect to $\overline{X} \rtimes \mathrm{Aut}(\overline{X})$, where $\overline{X}$ is a quotient of X by a central subgroup and $\overline{B}$ is dominated by B.*

Proof. Suppose that Conjecture 5.1.1 holds for every $\overline{B} \in \mathrm{Bl}(\overline{X})$ with respect to $\overline{X} \rtimes \mathrm{Aut}(\overline{X})$, where $\overline{X}$ is a quotient of X by a central subgroup and $\overline{B}$ is dominated by B. Let $b \in \mathrm{Bl}(\mathbf{N}_X(D))$ be the Brauer correspondent of B, where D is a defect group of B. We construct a bijection $\Lambda : \mathrm{Irr}_0(B) \to \mathrm{Irr}_0(b)$ satisfying the requirements of [Spä18, Definition 4.12]. The set $\mathrm{Irr}_0(B)$ can be partitioned into sets of the form $\mathrm{Irr}_0(B \mid 1_Z)$, where $Z \leq \mathbf{Z}(X)$. Then, the set $\mathrm{Irr}_0(B \mid 1_Z)$ can be identified via inflation with the set $\mathrm{Irr}_0(\overline{B})$, where $\overline{B}$ is the block of $\overline{X} := X/Z$ dominated by B (see [NT89, Theorem 5.8.8 and Theorem 5.8.11]). Similarly the set $\mathrm{Irr}_0(b)$ can be identified with the set $\mathrm{Irr}_0(\overline{b})$, where $\overline{b}$ is the block of $\overline{\mathbf{N}_X(D)} := \mathbf{N}_X(D)/Z$ dominated by b. By [NT89, Theorem 5.8.8 and Theorem 5.8.11] we know that $\overline{B}$ has defect group $\overline{D} := DZ/Z$ and, noticing that $\overline{\mathbf{N}_X(D)} = \mathbf{N}_{\overline{X}}(\overline{D})$, we deduce that $\overline{b}$ is the Brauer correspondent of $\overline{B}$. By assumption there exists a bijection $\Theta_Z : \mathrm{Irr}_0(\overline{B}) \to \mathrm{Irr}_0(\overline{b})$ satisfying the properties of Conjecture 5.1.1. Then, combining the bijections Θ_Z, where Z runs over the subgroup of $\mathbf{Z}(X)$, we obtain a bijection $\Lambda : \mathrm{Irr}_0(B) \to \mathrm{Irr}_0(b)$ satisfying the requirements of [Spä18, Definition 4.12]. The other implication follows by a similar argument. □

Now, the reduction theorem for the Alperin–McKay Conjecture can be stated as follows.

Theorem 5.1.3. *Let G be a finite group and suppose that every covering group X of a nonabelian simple group involved in G satisfies Conjecture 5.1.1 with respect to $X \trianglelefteq X \rtimes \mathrm{Aut}(X)$. Then the Alperin–McKay Conjecture holds for G.*

Proof. This is [Spä13a, Theorem C]. □

A much stronger result has been proved in [NS14b] where the authors proved a reduction of Conjecture 5.1.1 to quasisimple groups.

Theorem 5.1.4. *Let G be a finite group and suppose that every covering group X of a nonabelian simple group involved in G satisfies Conjecture 5.1.1 with respect to $X \trianglelefteq X \rtimes \mathrm{Aut}(X)$. Then Conjecture 5.1.1 holds for G.*

Proof. This is [NS14b, Theorem 7.1]. □

5.2 Proof of Theorem 5.1

We now start working towards a proof of Theorem 5.1. In order to do so, we need to understand the structure of a minimal counterexample to Conjecture 5.1.1. Results in this direction can be found in [NS14b]. Here we remark that, although in [NS14b, Section 7] the inductive Alperin-McKay condition is assumed for quasisimple groups in order to prove [NS14b, Theorem 7.1], this hypothesis is only used in [NS14b, Proposition 7.7]. In particular the following result can be deduced by the proof of [NS14b, Proposition 7.4].

Proposition 5.2.1. *Let $G \trianglelefteq A$ be a minimal counterexample to Conjecture 5.1.1 with respect to $|G : \mathbf{Z}(G)|$. Then $\mathbf{O}_p(G) \leq \mathbf{Z}(G)$.*

Proof. This follows immediately from [NS14b, Proposition 7.4]. □

Let $G \trianglelefteq A$ be a minimal counterexample as in Proposition 5.2.1 and consider a block $B \in \mathrm{Bl}(G)$ for which Conjecture 5.1.1 fails to hold. Clearly, the defect groups of B are not contained in the center of G and therefore they properly contain $\mathbf{O}_p(G)$. For $d \geq 0$, we define

$$\mathcal{C}_0^d(B) := \left\{(\{\mathbf{O}_p(G)\}, \vartheta) \in \mathcal{C}^d(B)_+\right\}$$

and

$$\mathcal{C}_1^d(B) := \left\{(\{\mathbf{O}_p(G) < D\}, \vartheta) \in \mathcal{C}^d(B)_- \mid D \in \delta(B)\right\}.$$

Moreover, set $\mathcal{G}_+^d := \mathcal{C}^d(B)_+ \smallsetminus \mathcal{C}_0^d(B)$ and $\mathcal{G}_-^d := \mathcal{C}^d(B)_- \smallsetminus \mathcal{C}_1^d(B)$. Notice that G acts via conjugation on $\mathcal{G}_\pm^d$ and let $\mathcal{G}_\pm^d/G$ denote the corresponding set of G-orbits. For any element $(\mathbb{D}, \vartheta) \in \mathcal{G}_\pm^d$ let $\overline{(\mathbb{D}, \vartheta)}$ denote its G-orbit.

Corollary 5.2.2. *Let $G \trianglelefteq A$ be a minimal counterexample to Conjecture 5.1.1 with respect to $|G : \mathbf{Z}(G)|$ and let $B \in \mathrm{Bl}(G \mid D)$ be a block for which the result fails to hold. If $d := d(B)$, then there exists an $\mathbf{N}_A(D)_B$-equivariant bijection*

$$\Pi : \mathcal{G}_+^d/G \to \mathcal{G}_-^d/G$$

such that

$$\left(A_{\mathbb{D},\vartheta}, G_{\mathbb{D}}, \vartheta\right) \sim_G \left(A_{\mathbb{E},\chi}, G_{\mathbb{E}}, \chi\right),$$

for every $(\mathbb{D}, \vartheta) \in \mathcal{G}_+^d$ and $(\mathbb{E}, \chi) \in \Pi(\overline{(\mathbb{D}, \vartheta)})$.

Proof. For $\epsilon \in \{+,-\}$, consider the set $\widehat{\mathcal{G}}^d_\epsilon$ of p-chains $\mathbb{D}$ for which there exists a character $\vartheta \in \mathrm{Irr}(G_\mathbb{D})$ such that $(\mathbb{D},\vartheta) \in \mathcal{G}^d_\epsilon$. Let $\widehat{\mathcal{G}}^d_\epsilon/G$ be the set of G-orbits on $\widehat{\mathcal{G}}^d_\epsilon$ and denote by $\overline{\mathbb{D}}$ the G-orbit of $\mathbb{D} \in \widehat{\mathcal{G}}^d_\epsilon$. Notice that, if $\mathbb{D} \in \widehat{\mathcal{G}}^d_\epsilon$ has final term D_n, then there exists $g \in G$ such that

$$D_n \leq D^g \leq G_\mathbb{D}$$

and D^g is a defect group of some block of $G_\mathbb{D}$. In fact, if $(\mathbb{D},\vartheta) \in \mathcal{G}^d_\epsilon$ and Q is a defect group of $\mathrm{bl}(\vartheta)$, then $D_n \leq \mathbf{O}_p(G_\mathbb{D}) \leq Q$ and there exists $g \in G$ such that $Q \leq D^g$. Moreover, if $d_0 := d(\mathrm{bl}(\vartheta))$, then $d = d(\vartheta) \leq d_0 \leq d(\mathrm{bl}(\vartheta)^G) = d(B) =: d$ and therefore $D^g = Q \leq G_\mathbb{D}$.

Next, we define an $\mathbf{N}_A(D)_B$-equivariant bijection

$$\widehat{\Pi} : \widehat{\mathcal{G}}^d_+/G \to \widehat{\mathcal{G}}^d_-/G$$

by setting $\widehat{\Pi}(\overline{\mathbb{D}}) := \overline{\mathbb{D} \smallsetminus \{D_n\}}$, if the last term D_n of $\mathbb{D}$ is a defect group of B and where $\mathbb{D} \smallsetminus \{D_n\}$ is defined to be the p-chains obtained by removing D_n from $\mathbb{D}$, and $\widehat{\Pi}(\overline{\mathbb{D}}) := \overline{\mathbb{D} \cup \{D^g\}}$, if the last term D_n of $\mathbb{D}$ is properly contained in some defect group of B and $g \in G$ is an element such that D^g is a defect group of some block of $G_\mathbb{D}$, here $\mathbb{D} \cup \{D^g\}$ denotes the p-chain obtained by adding D^g to $\mathbb{D}$. Notice that the above definition does not depend on the choice of D^g, but only on its $G_\mathbb{D}$-conjugacy class, nor on the representative $\mathbb{D}$ in $\overline{\mathbb{D}}$. Furthermore, as $D^g \leq G_\mathbb{D}$ we deduce that the map sends normal chains to normal chains. To conclude that $\widehat{\Pi}$ is well defined we need to check that, for every $\mathbb{E} \in \widehat{\Pi}(\overline{\mathbb{D}})$, there exists $\chi \in G_\mathbb{E}$ such that $(\mathbb{E},\chi) \in \mathcal{G}^d_-$. Without loss of generality, assume that $\mathbb{E}$ is the chain obtain from $\mathbb{D}$ by adding D as a final term. Notice that $G_\mathbb{D} < G$ since the last term of $\mathbb{D}$ properly contains $\mathbf{O}_p(G)$. Then $|G_\mathbb{D} : \mathbf{Z}(G_\mathbb{D})| < |G : \mathbf{Z}(G)|$ and $G_\mathbb{D}$ satisfies Conjecture 5.1.1. Therefore, there exists an $A_\mathbb{E}$-equivariant bijection

$$\Pi_\mathbb{D} : \mathrm{Irr}^d(G_\mathbb{D} \mid D) \to \mathrm{Irr}^d(G_\mathbb{E} \mid D)$$

such that

$$\left(A_{\mathbb{D},\vartheta}, G_\mathbb{D}, \vartheta\right) \sim_{G_\mathbb{D}} \left(A_{\mathbb{E},\vartheta}, G_\mathbb{E}, \Pi_\mathbb{D}(\vartheta)\right),$$

for every $\vartheta \in \mathrm{Irr}^d(G_\mathbb{D} \mid D)$. Noticing that $\mathbf{C}_{A_{\mathbb{D},\vartheta}\cdot G}(D) \leq A_{\mathbb{D},\vartheta}$ and applying Lemma 3.3.10, we obtain

$$\left(A_{\mathbb{D},\vartheta}, G_\mathbb{D}, \vartheta\right) \sim_{G} \left(A_{\mathbb{E},\vartheta}, G_\mathbb{E}, \Pi_\mathbb{D}(\vartheta)\right),$$

for every $\vartheta \in \mathrm{Irr}^d(G_\mathbb{D} \mid D)$. In particular, $(\mathbb{D},\vartheta) \in \mathcal{G}^d_+$ if and only if $(\mathbb{E},\Pi_\mathbb{D}(\vartheta)) \in \mathcal{G}^d_-$ and so $\widehat{\Pi}$ is well defined. Moreover, it's clear that $\widehat{\Pi}$ is a bijection (the inverse map is defined analogously) and that it is $\mathbf{N}_A(D)_B$-equivariant.

To conclude, using the character bijections obtained in the previous paragraph, we immediately obtain a bijection Π with the required properties by defining $\Pi(\overline{(\mathbb{D},\vartheta)}) := \overline{(\mathbb{E},\Pi_\mathbb{D}(\vartheta))}$, for every $(\mathbb{D},\vartheta) \in \mathcal{G}^d_+$ and where $\overline{\mathbb{E}} = \widehat{\Pi}(\overline{\mathbb{D}})$. □

We can now prove Theorem 5.1.

Theorem 5.2.3. *Let $G \trianglelefteq A$ be finite groups and consider $B \in \mathrm{Bl}(G \mid D)$ with Brauer correspondent $b \in \mathrm{Bl}(\mathbf{N}_G(D) \mid D)$. If Conjecture 3.5.1 holds for every p-block of every finite group, then there exists an $\mathbf{N}_A(D)_B$-equivariant bijection*

$$\Theta : \mathrm{Irr}_0(B) \to \mathrm{Irr}_0(b)$$

such that

$$(A_\vartheta, G, \vartheta) \sim_G (\mathbf{N}_A(D)_\vartheta, \mathbf{N}_G(D), \Theta(\vartheta)),$$

for every $\vartheta \in \mathrm{Irr}_0(B)$.

Proof. Suppose the result is false and let G be a minimal counterexample with respect to $|G : \mathbf{Z}(G)|$. Consider $Z := \mathbf{O}_p(G)$ and $d := d(B)$. By Proposition 5.2.1 we have $Z \leq \mathbf{Z}(G)$. Clearly $Z < D$, as otherwise D would be central. Now, since by assumption Conjecture 3.5.1 holds for G, we can find an A_B-equivariant bijection

$$\Omega : \mathcal{C}^d(B)_+/G \to \mathcal{C}^d(B)_-/G,$$

with the required properties. Consider the sets $\mathcal{C}_0^d(B)$ and $\mathcal{C}_1^d(B)$ defined above and notice that

$$\mathcal{C}_0^d(B)/G = \left\{\overline{(\{\mathbf{O}_p(G)\}, \vartheta)} \;\middle|\; \vartheta \in \mathrm{Irr}_0(B)\right\}$$

and that

$$\mathcal{C}_1^d(B)/G = \left\{\overline{(\{\mathbf{O}_p(G) < D\}, \chi)} \;\middle|\; \chi \in \mathrm{Irr}_0(b)\right\}.$$

If $\Omega(\mathcal{C}_0^d(B)/G) = \mathcal{C}_1^d(B)/G$, then we obtain a bijection $\Theta : \mathrm{Irr}_0(B) \to \mathrm{Irr}_0(b)$ satisfying the required properties by defining $\Theta(\chi) := \vartheta$ whenever $\Omega(\overline{(\{\mathbf{O}_p(G)\}, \vartheta)}) = \overline{(\{\mathbf{O}_p(G) < D\}, \chi)}$. This would contradict the choice of our counterexample.

Let $\Pi : \mathcal{G}_+^d/G \to \mathcal{G}_-^d/G$ be the bijection given by Proposition 5.2.1 and observe that

$$|\mathcal{C}_0^d(B)/G| = |\mathcal{C}^d(B)_+/G| - |\mathcal{G}_+^d/G| = |\mathcal{C}^d(B)_-/G| - |\mathcal{G}_-^d/G| = |\mathcal{C}_1^d(B)/G|.$$

Since $\Omega(\mathcal{C}_0^d(B)/G) \neq \mathcal{C}_1^d(B)/G$, there exists $(\mathbb{D}_0, \vartheta_0) \in \mathcal{C}_0^d(B)$ such that $\Omega(\overline{(\mathbb{D}_0, \vartheta_0)}) \notin \mathcal{C}_1^d(B)/G$. We now proceed as follows: as $\Omega(\overline{(\mathbb{D}_0, \vartheta_0)}) \in \mathcal{G}_-^d/G$, we can define

$$\overline{(\mathbb{D}_1, \vartheta_1)} := \Pi\left(\Omega\left(\overline{\mathbb{D}_0, \vartheta_0}\right)\right).$$

If $\Omega(\overline{(\mathbb{D}_1, \vartheta_1)}) \in \mathcal{C}_1^d(B)/G$, then we stop. Otherwise we define

$$\overline{(\mathbb{D}_2, \vartheta_2)} := \Pi\left(\Omega\left(\overline{(\mathbb{D}_1, \vartheta_1)}\right)\right).$$

Continuing this way, for $i \geq 1$, we define

$$\overline{(\mathbb{D}_i, \vartheta_i)} := \Pi\left(\Omega\left(\overline{(\mathbb{D}_{i-1}, \vartheta_{i-1})}\right)\right),$$

if $\Omega(\overline{(\mathbb{D}_{i-1}, \vartheta_{i-1})}) \notin \mathcal{C}_1^d(B)/G$. It is important to observe that, for every $i \geq 1$, the pair $(\mathbb{D}_i, \vartheta_i)$ does not lie in $\mathcal{C}_0^d(B)$ and satisfies

$$(A_{\mathbb{D}_0}, G_{\mathbb{D}_0}, \vartheta_0) \sim_G (A_{\mathbb{D}_i}, G_{\mathbb{D}_i}, \vartheta_i). \tag{5.2.1}$$

We claim that there exists some $n \geq 1$ such that $\Omega(\overline{(\mathbb{D}_n, \vartheta_n)}) \in \mathcal{C}_1^d(B)/G$. If this is not the case, then the set

$$\mathcal{S} := \left\{(\Pi \circ \Omega)^i \left(\overline{(\mathbb{D}_0, \vartheta_0)}\right) \;\middle|\; i \geq 0\right\} \subseteq \mathcal{C}^d(B)_+/G$$

is well defined, $\Omega(\mathcal{S}) \subseteq \mathcal{G}^d_-/G$ and, as $\mathcal{S}$ is finite, it satisfies

$$\Pi \circ \Omega(\mathcal{S}) = \mathcal{S}.$$

Since $\overline{(\mathbb{D}_0, \vartheta_0)} \in \mathcal{S} \cap \mathcal{C}^d_0(B)/G$, we obtain

$$\begin{aligned} |\mathcal{S}| &= |\Omega(\mathcal{S})| \\ &= |\Omega(\mathcal{S}) \cap \mathcal{G}^d_-/G| \\ &= |\Pi\left(\Omega(\mathcal{S}) \cap \mathcal{G}^d_-/G\right)| \\ &= |\Pi(\Omega(\mathcal{S})) \cap \Pi(\mathcal{G}^d_-/G)| \\ &= |\mathcal{S} \cap \mathcal{G}^d_+/G| \\ &\leq |\mathcal{S}| - 1. \end{aligned}$$

This contradiction proves our claim. Now, since $\mathcal{C}^d_1(B)$ is $\mathbf{N}_A(D)_B$-stable and Ω and Π are $\mathbf{N}_A(D)_B$-equivariant, the pairs $(\mathbb{D}_0, \vartheta_0)$ and $(\mathbb{D}_n, \vartheta_n)$ are not $\mathbf{N}_A(D)_B$-conjugate. Then, we can find a $\mathbf{N}_A(D)_B$-transversal $\mathcal{T}$ in $\mathcal{C}^d(B)_+/G$ containing $\overline{(\mathbb{D}_0, \vartheta_0)}$ and $\overline{(\mathbb{D}_n, \vartheta_n)}$. We define a new $\mathbf{N}_A(D)_B$-equivariant bijection $\Omega' : \mathcal{C}^d(B)_+/G \to \mathcal{C}^d(B)_-/G$ by setting

$$\Omega'\left(\overline{(\mathbb{D}, \vartheta)}^x\right) := \begin{cases} \Omega\left(\overline{(\mathbb{D}, \vartheta)}^x\right), & \text{if } \overline{(\mathbb{D}, \vartheta)} \in \mathcal{T} \smallsetminus \left\{\overline{(\mathbb{D}_0, \vartheta_0)}, \overline{(\mathbb{D}_n, \vartheta_n)}\right\} \\ \Omega\left(\overline{(\mathbb{D}_n, \vartheta_n)}^x\right), & \text{if } \overline{(\mathbb{D}, \vartheta)} = \overline{(\mathbb{D}_0, \vartheta_0)} \\ \Omega\left(\overline{(\mathbb{D}_0, \vartheta_0)}^x\right), & \text{if } \overline{(\mathbb{D}, \vartheta)} = \overline{(\mathbb{D}_n, \vartheta_n)} \end{cases},$$

for every $\overline{(\mathbb{D}, \vartheta)} \in \mathcal{T}$ and $x \in \mathbf{N}_A(D)_B$. Using (5.2.1), we deduce that Ω' satisfies the conditions of Conjecture 3.5.1. Noticing that $(\mathbb{D}_n, \vartheta_n) \notin \mathcal{C}^d_0(B)$, we conclude that a multiple application of the previous argument yields a bijection Ω'' satisfying the conditions of Conjecture 3.5.1 and such that $\Omega''(\mathcal{C}^d_0(B)/G) = \mathcal{C}^d_1(B)/G$. As remarked at the beginning of the proof this is a contradiction. □

6

Representation Theory of Finite Groups of Lie Type

In this chapter we introduce some preliminary results on the representation theory of finite groups of Lie type. Our presentation follows [DM91], [CE04] and [GM20]. In order to make this thesis self-contained we also give a brief introduction to the main definitions and results in the structure theory of linear algebraic groups and finite groups of Lie type. For this group theoretic part we follow [MT11].

6.1 Finite groups of Lie type

6.1.1 Linear algebraic groups

Let p be a prime, q a power of p and set $\mathbb{F} := \overline{\mathbb{F}_q}$ where $\mathbb{F}_q$ is a finite field with q elements. Recall that, for an ideal I of the polynomial ring $\mathbb{F}[x_1, \dots, x_n]$, the set $\mathcal{V}(I) := \{x \in \mathbb{F}^n \mid f(x) = 0 \text{ for every } f \in I\}$ is called an **algebraic set**. Taking algebraic sets as closed subsets defines a topology on the set $\mathbb{F}^n$ called the **Zariski topology**. An **affine algebraic variety** (defined over $\mathbb{F}$) is an algebraic set with the induced Zariski topology.

A **linear algebraic group** is an affine variety $\mathbf{G}$ endowed with a group structure in such a way that multiplication and inversion are morphisms of varieties. The simplest examples of algebraic groups are $\mathbb{G}_{\mathrm{a}}$ and $\mathbb{G}_{\mathrm{m}}$: these are defined as the additive group $\mathbb{F}$ and the multiplicative group $\mathbb{F}^\times$ respectively, endowed with the Zariski topology. It can be shown that linear algebraic groups are exactly the (Zariski) closed subgroups of $\mathrm{GL}_n(\mathbb{F})$. For an algebraic group $\mathbf{G}$, we denote by $\mathbf{G}^\circ$ the connected component containing the identity element. In the following, we will often write $\mathbf{Z}^\circ(\mathbf{G}) := \mathbf{Z}(\mathbf{G})^\circ$ and $\mathbf{C}^\circ_{\mathbf{G}}(x) := \mathbf{C}_{\mathbf{G}}(x)^\circ$ for any $x \in \mathbf{G}$.

An element $g \in \mathbf{G}$ is called **unipotent** (resp. **semisimple**) if, given any embedding of algebraic groups $\rho : \mathbf{G} \to \mathrm{GL}_n(\mathbb{F})$, the matrix $\rho(g)$ is unipotent (resp. semisimple). Then, we have a **Jordan**

decomposition of elements as $g = g_u g_s = g_s g_u$, where g_u is unipotent and g_s is semisimple. This does not depend on the choice of the embedding. A group $\mathbf{G}$ is called **unipotent** if all of its elements are unipotent. The set of semisimple elements of $\mathbf{G}$ is denoted by $\mathbf{G}_{\mathrm{ss}}$. A closed subgroup $\mathbf{T}$ of $\mathbf{G}$ isomorphic to the direct product of a finite number of copies of $\mathbb{G}_{\mathrm{m}}$ is called a **torus**. All maximal tori are $\mathbf{G}$-conjugate and any semisimple element is contained in a maximal torus.

An algebraic group $\mathbf{G}$ is (algebraically) **simple** if it has no proper nontrivial closed connected normal subgroup. We define the **unipotent radical** $\mathbf{R}_u(\mathbf{G})$ of $\mathbf{G}$ to be the largest closed connected normal unipotent subgroup of $\mathbf{G}$. Then $\mathbf{G}$ is **reductive** if $\mathbf{R}_u(\mathbf{G}) = 1$. If $\mathbf{G}$ is connected and reductive, then $\mathbf{G} = \mathbf{Z}^\circ(\mathbf{G})[\mathbf{G},\mathbf{G}]$ and $[\mathbf{G},\mathbf{G}] = \mathbf{G}_1 \dots \mathbf{G}_n$ for some simple algebraic groups $\mathbf{G}_i$ (see [MT11, Theorem 8.21 and Corollary 8.22]). Notice that $\mathbf{Z}^\circ(\mathbf{G})$ is a torus. A connected reductive group is called **semisimple** if $\mathbf{Z}^\circ(\mathbf{G}) = 1$.

Semisimple algebraic groups are classified in terms of root data. A **root system** of a finite dimensional real vector space E endowed with the standard scalar product $(-,-)$ is a finite subset $\Phi \subseteq E$ such that $0 \notin \Phi$, E is generated by Φ and the following conditions are satisfied:

(R1) if $\alpha, c\cdot\alpha \in \Phi$ with $c \in \mathbb{R}$, then $c = \pm 1$;

(R2) for every $\alpha \in \Phi$ there exists a reflection $s_\alpha \in \mathrm{GL}(E)$ along α that stabilizes Φ;

(R3) for $\alpha, \beta \in \Phi$, the element $s_\alpha(\beta) - \beta$ is an integral multiple of α.

Then we can define a **root datum** as a quadruple $(X, \Phi, Y, \Phi^\vee)$ where:

(RD1) X and Y are free abelian groups of the same rank with a perfect pairing $\langle -,- \rangle : X \times Y \to \mathbb{Z}$ inducing isomorphisms $Y \simeq \mathrm{Hom}(X,\mathbb{Z})$ and $X \simeq \mathrm{Hom}(Y,\mathbb{Z})$;

(RD2) $\Phi \subseteq X$ and $\Phi^\vee \subseteq Y$ are root systems of $\mathbb{Z}\Phi \otimes_{\mathbb{Z}} \mathbb{R}$ and $\mathbb{Z}\Phi^\vee \otimes_{\mathbb{Z}} \mathbb{R}$ respectively;

(RD3) there exists a bijection $\Phi \to \Phi^\vee$, $\alpha \mapsto \alpha^\vee$ such that $\langle \alpha, \alpha^\vee \rangle = 2$;

(RD4) for every $\alpha \in \Phi$ and $\alpha^\vee \in \Phi$ we have $s_\alpha(x) = x - \langle x, \alpha^\vee \rangle \alpha$ and $s_{\alpha^\vee}(y) = y - \langle \alpha, y \rangle \alpha^\vee$ for every $x \in X$ and $y \in Y$.

Given a connected reductive group $\mathbf{G}$ with maximal torus $\mathbf{T}$, we obtain a root datum as follows: let $X := X(\mathbf{T}) := \mathrm{Hom}(\mathbf{T}, \mathbb{G}_{\mathrm{m}})$ be the group of characters of $\mathbf{T}$, $Y := Y(\mathbf{T}) := \mathrm{Hom}(\mathbb{G}_{\mathrm{m}}, \mathbf{T})$ the group of cocarachters of $\mathbf{T}$, $\Phi := \Phi(\mathbf{G},\mathbf{T})$ the set of roots arising from the action of $\mathbf{T}$ on the Lie algebra of $\mathbf{G}$ and $\Phi^\vee := \Phi(\mathbf{G},\mathbf{T})^\vee$ the corresponding set of coroots (see [MT11, Proposition 9.11] for a more detailed description). Then semisimple algebraic groups are classified by the following theorem of Chevalley

Theorem 6.1.1 (Chevalley)**.** *Two connected algebraic groups are isomorphic if and only if they have isomorphic root data. Moreover, for every root datum there exists a connected reductive group with such a root datum.*

Proof. See [Spr09, Theorem 9.6.2 and Theorem 10.1.1]. □

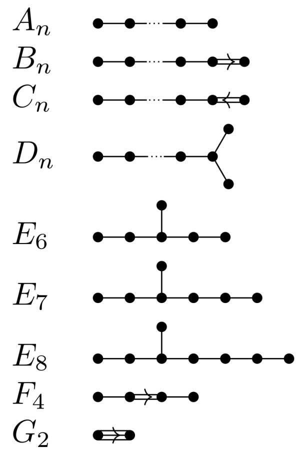


Figure 6.1: Dynkin diagrams of indecomposable root systems

To describe all the possible root data, we need to introduce one further object. Set $\Omega := \mathrm{Hom}(\mathbb{Z}\Phi^\vee, \mathbb{Z})$ and notice that restriction to $\mathbb{Z}\Phi^\vee \leq \mathbb{Z}Y$ induces an injection

$$X \simeq \mathrm{Hom}(Y, \mathbb{Z}) \hookrightarrow \mathrm{Hom}(\mathbb{Z}\Phi^\vee, \mathbb{Z}) =: \Omega.$$

Then, we obtain an inclusion of groups $\mathbb{Z}\Phi \leq X \leq \Omega$. The quotient $\Omega/\mathbb{Z}\Phi$ is a finite group called the **fundamental group** of the root system Φ. The root data with fixed root system Φ are determined by the subgroups of $\Omega/\mathbb{Z}\Phi$. Moreover, the possible indecomposable root systems are classified by the Dynkin diagrams listed in Figure 6.1.

If $\mathbf{G}$ is a semisimple algebraic group with root datum $(X, \Phi, Y, \Phi^\vee)$ corresponding to the choice of a maximal torus $\mathbf{T}$, then we say that $\mathbf{G}$ is **simply connected** if $\Omega = X$ and that it is **adjoint** if $X = \mathbb{Z}\Phi$. Given a semisimple algebraic group $\mathbf{G}$ with root system Φ, we denote by $\mathbf{G}_{\mathrm{sc}}$ (resp. $\mathbf{G}_{\mathrm{ad}}$) a simply connected (resp. adjoint) group with the same root system. Notice that $\mathbf{Z}(\mathbf{G}_{\mathrm{ad}}) = 1$ (see [GM20, Example 1.5.3 (a)]). If $\mathbf{G}$ is connected reductive, then we say that $\mathbf{G}$ is simply connected (resp. adjoint) if the semisimple group $[\mathbf{G}, \mathbf{G}]$ is simply connected (resp. adjoint).

A **Borel subgroup** of $\mathbf{G}$ is any maximal closed connected solvable subgroup. If $\mathbf{T}$ is a maximal torus contained in a Borel subgroup $\mathbf{B}$, then $\mathbf{B} = \mathbf{R}_u(\mathbf{B}) \rtimes \mathbf{T}$. A **parabolic subgroup** of $\mathbf{G}$ is any closed subgroup $\mathbf{P} \leq \mathbf{G}$ containing a Borel subgroup. For any parabolic subgroup $\mathbf{P}$ of $\mathbf{G}$ there exists a closed subgroup $\mathbf{L} \leq \mathbf{P}$ such that $\mathbf{P} = \mathbf{R}_u(\mathbf{P}) \rtimes \mathbf{L}$. The group $\mathbf{L}$ is called a **Levi complement** of $\mathbf{P}$. It can be shown that the centralizer $\mathbf{C}_{\mathbf{G}}(\mathbf{S})$ of a torus $\mathbf{S}$ is a Levi subgroup of $\mathbf{G}$ (see [DM91, Proposition 1.22]). On the other hand, if $\mathbf{L}$ is a Levi subgroup of $\mathbf{G}$, then $\mathbf{L} = \mathbf{C}_{\mathbf{G}}(\mathbf{Z}^\circ(\mathbf{L}))$ (see [DM91, Proposition 1.21]). Recalling that $\mathbf{Z}^\circ(\mathbf{L})$ is a torus, it follows that the Levi subgroups of $\mathbf{G}$ are exactly the centralizers of tori. As a consequence, [DM91, Proposition 0.32 (ii)] implies that every Levi subgroup of a connected reductive group is connected and reductive. Notice that maximal tori, being self-centralizing, are Levi subgroups (see [DM91, Proposition 0.32 (iii)]).

6.1.2 Finite groups of Lie type

Let q be a power of p, consider a finite field $\mathbb{F}_q$ with q elements and set $\mathbb{F} := \overline{\mathbb{F}_q}$. An affine variety V has an $\mathbb{F}_q$-**structure** if there exists an isomorphism of varieties $\iota : V \to V'$ where $V' \subseteq \mathbb{F}^n$ is a Zariski closed subset stable under the **standard Frobenius map**

$$F_q : \mathbb{F}^n \to \mathbb{F}^n, (\xi_1, \dots, \xi_n) \mapsto (\xi_1^q, \dots, \xi_n^q).$$

In this case there exists a unique endomorphism $F : V \to V$ such that $\iota \circ F = F_q \circ \iota$. The morphism F is often referred to as the **Frobenius endomorphism** defining an $\mathbb{F}_q$-structure on V. Notice that F_q is a bijective morphism with fixed points set $\mathbb{F}_q^n$. In particular the set

$$V^F := \{v \in V \mid F(v) = v\}$$

is finite.

Let $\mathbf{G}$ be a linear algebraic group. An endomorphism $F : \mathbf{G} \to \mathbf{G}$ is called a **Steinberg endomorphism** if there exists a nonnegative integer m such that $F^m : \mathbf{G} \to \mathbf{G}$ is the Frobenius endomorphism corresponding to an $\mathbb{F}_q$-structure on $\mathbf{G}$. The set of fixed points $\mathbf{G}^F$ is a finite group. If $\mathbf{G}$ is connected and reductive, then we say that $\mathbf{G}^F$ is a **finite group of Lie type** or a **finite reductive group**.

Let $\mathbf{G}$ be a connected linear algebraic group. By the Lang–Steinberg theorem, the Lang map

$$\begin{aligned} \mathbf{G} &\to \mathbf{G} \\ g &\mapsto g^{-1}F(g) \end{aligned}$$

is surjective. One of the main consequences of this result is the existence of a maximal torus and a Borel subgroup $\mathbf{T} \leq \mathbf{B}$ that are stable under the action of F. Such a maximal torus is called 1-**split** or **maximally split**. Although not all F-stable maximal tori are $\mathbf{G}^F$-conjugate, it can be shown that 1-split tori are all conjugate under the action of $\mathbf{G}^F$. Similarly, one can show that there exists an F-stable Levi subgroup in any F-stable parabolic subgroup. These are called 1-**split Levi subgroups**. Another important consequence of the Lang–Steinberg theorem is the following. Suppose that $\mathbf{H}$ is a closed F-stable subgroup of $\mathbf{G}$. If $\mathbf{H}$ is connected, then $(\mathbf{G}/\mathbf{H})^F = \mathbf{G}^F/\mathbf{H}^F$. To conclude this subsection, observe that $\mathbf{C}_{\mathbf{G}}(\mathbf{G}^F) = \mathbf{Z}(\mathbf{G})$ (see [DM20, Proposition 12.2.17]) and hence $\mathbf{Z}(\mathbf{G}^F) = \mathbf{Z}(\mathbf{G})^F$. This fact will often be used in the sequel without further reference.

6.1.3 Duality

Let $\mathbf{G}$ be a connected reductive group with root datum $(X(\mathbf{T}), \Phi(\mathbf{G}, \mathbf{T}), Y(\mathbf{T}), \Phi(\mathbf{G}, \mathbf{T})^\vee)$ with respect to a maximal torus $\mathbf{T}$. By replacing roots with coroots, we obtain another root datum $(Y(\mathbf{T}), \Phi(\mathbf{G}, \mathbf{T})^\vee, X(\mathbf{T}), \Phi(\mathbf{G}, \mathbf{T}))$. A connected reductive group $\mathbf{G}^*$ is in **duality** with $\mathbf{G}$ if the root datum $(X(\mathbf{T}^*), \Phi(\mathbf{G}^*, \mathbf{T}^*), Y(\mathbf{T}^*), \Phi(\mathbf{G}^*, \mathbf{T}^*)^\vee)$ is isomorphic to the root datum $(Y(\mathbf{T}), \Phi(\mathbf{G}, \mathbf{T})^\vee, X(\mathbf{T}), \Phi(\mathbf{G}, \mathbf{T}))$ for some maximal torus $\mathbf{T}^*$ of $\mathbf{G}^*$. More precisely, if there exists an isomorphism $\delta : X(\mathbf{T}) \to Y(\mathbf{T}^*)$ such that $\delta(\Phi(\mathbf{G}, \mathbf{T})) = \Phi(\mathbf{G}^*, \mathbf{T}^*)^\vee$ and

$$\langle \lambda, \alpha^\vee \rangle = \langle \delta(\alpha)^\vee, \delta(\lambda) \rangle$$

for every $\lambda \in X(\mathbf{T})$ and $\alpha \in \Phi(\mathbf{G}, \mathbf{T})$. If we need to specify the choice of maximal tori then we say that $(\mathbf{G}, \mathbf{T})$ is dual to $(\mathbf{G}^*, \mathbf{T}^*)$. Observe that $\mathbf{G}$ can be identified with the dual of $\mathbf{G}^*$. It is also worth noting that if $\mathbf{G}$ is semisimple then $\mathbf{G}^*$ is semisimple. Furthermore if $\mathbf{G}$ is semisimple and simply connected (resp. adjoint) then $\mathbf{G}^*$ is adjoint (resp. simply connected) (see the comment following [DM20, Example 11.1.13]).

Let $F : \mathbf{G} \to \mathbf{G}$ be a Steinberg endomorphism and consider an F-stable torus $\mathbf{T}$. A pair $(\mathbf{G}^*, F^*)$, with $\mathbf{G}^*$ a connected reductive group and F^* a Steinberg endomorphism of $\mathbf{G}^*$, is **dual** to $(\mathbf{G}, F)$ if there exists an F^*-stable maximal torus $\mathbf{T}^*$ of $\mathbf{G}^*$ such that $(\mathbf{G}, \mathbf{T})$ is dual to $(\mathbf{G}^*, \mathbf{T}^*)$ and

$$\delta\left(\lambda \circ F\,|_{\mathbf{T}}\right) = F^*\,|_{\mathbf{T}^*} \circ \delta(\lambda)$$

for every $\lambda \in X(\mathbf{T})$. If $(\mathbf{G}, \mathbf{T}, F)$ is dual to $(\mathbf{G}^*, \mathbf{T}^*, F^*)$, then there exists a bijection

$$\mathbf{L} \mapsto \mathbf{L}^* \tag{6.1.1}$$

between the set of Levi subgroups of $\mathbf{G}$ containing $\mathbf{T}$ and the set of Levi subgroups of $\mathbf{G}^*$ containing $\mathbf{T}^*$ (see [CE04, p.123]). This bijection induces a correspondence between the set of F-stable Levi subgroups of $\mathbf{G}$ and the set of F^*-stable Levi subgroups of $\mathbf{G}^*$. Moreover, it is compatible with the action of $\mathbf{G}^F$ and $\mathbf{G}^{*F^*}$. This bijection can be described as follows (see [CS13, Section 2.3]): the Levi subgroups $\mathbf{L}$ and $\mathbf{L}^*$ correspond via (6.1.1) if $\Phi(\mathbf{L}, \mathbf{T})$ corresponds to $\Phi(\mathbf{L}^*, \mathbf{T}^*)^\vee$ via the isomorphism $\delta : X(\mathbf{T}) \to Y(\mathbf{T}^*)$. Furthermore, in this case $(\mathbf{L}, \mathbf{T}, F)$ is dual to $(\mathbf{L}^*, \mathbf{T}^*, F^*)$.

6.1.4 Regular embeddings

Let $\mathbf{G}$, $\widetilde{\mathbf{G}}$ be connected reductive groups with Steinberg endomorphisms $F : \mathbf{G} \to \mathbf{G}$ and $\widetilde{F} : \widetilde{\mathbf{G}} \to \widetilde{\mathbf{G}}$. A morphism of algebraic groups $i : \mathbf{G} \to \widetilde{\mathbf{G}}$ is a **regular embedding** if $\widetilde{F} \circ i = i \circ F$ and i induces an isomorphism of $\mathbf{G}$ with a closed subgroup $i(\mathbf{G})$ of $\widetilde{\mathbf{G}}$, the center $\mathbf{Z}(\widetilde{\mathbf{G}})$ of $\widetilde{\mathbf{G}}$ is connected and $[i(\mathbf{G}), i(\mathbf{G})] = [\widetilde{\mathbf{G}}, \widetilde{\mathbf{G}}]$. In this case we can identify $\mathbf{G}$ with its image $i(\mathbf{G})$ and $\widetilde{F}$ with an extension of F to $\widetilde{\mathbf{G}}$ which, by abuse of notation, we denote again by F.

Since $[\widetilde{\mathbf{G}}, \widetilde{\mathbf{G}}]$ is contained in $\mathbf{G}$, we deduce that $\mathbf{G}$ is normal in $\widetilde{\mathbf{G}}$ and that $\widetilde{\mathbf{G}}/\mathbf{G}$ is abelian. Moreover, since $\widetilde{\mathbf{G}}$ is connected and reductive, we have $\widetilde{\mathbf{G}} = \mathbf{Z}(\widetilde{\mathbf{G}})[\widetilde{\mathbf{G}}, \widetilde{\mathbf{G}}] = \mathbf{Z}(\widetilde{\mathbf{G}})\mathbf{G}$. In particular, it follows that $\mathbf{Z}(\mathbf{G}) = \mathbf{Z}(\widetilde{\mathbf{G}}) \cap \mathbf{G}$. Similarly, $[\widetilde{\mathbf{G}}^F, \widetilde{\mathbf{G}}^F] \leq \mathbf{G}^F$ and hence $\mathbf{G}^F$ is a normal subgroup of $\widetilde{\mathbf{G}}^F$ with abelian quotient $\widetilde{\mathbf{G}}^F/\mathbf{G}^F$. Notice, however, that $\widetilde{\mathbf{G}}^F$ might be larger than $\mathbf{Z}(\widetilde{\mathbf{G}}^F)\mathbf{G}^F$. We point out that, when $\mathbf{G}$ is simple of simply connected type different from $\mathbf{D}_n$, then one can construct a regular embedding such that $\widetilde{\mathbf{G}}^F/\mathbf{G}^F$ is cyclic (see [GM20, Proposition 1.7.5]).

By the description given in the previous paragraph it follows that, when dealing with the representation theory of the groups $\mathbf{G}^F$ and $\widetilde{\mathbf{G}}^F$, we can apply Clifford theory for abelian quotients (see [Isa76, Problem 6.2]). Another fundamental ingredient to understand the representation theory with respect to $\mathbf{G}^F \trianglelefteq \widetilde{\mathbf{G}}^F$ is the fact that restriction from $\widetilde{\mathbf{G}}^F$ to $\mathbf{G}^F$ is multiplicity free. This results was first stated by Lusztig [Lus88] while the details of the proof were provided by Cabanes–Enguehard (see [CE04, Chapter 16]). We state a slightly more general result.

Theorem 6.1.2. *Let $\mathbf{G}$ be a connected reductive group with a Frobenius endomorphism $F : \mathbf{G} \to \mathbf{G}$. Then, for every $\chi \in \mathrm{Irr}(\mathbf{G}^F)$ the restriction of χ to $[\mathbf{G}^F, \mathbf{G}^F]$ is a sum of distinct irreducible characters.*

Proof. This is [CE04, Theorem 15.11]. □

Let now $\mathbf{L}$ be an F-stable Levi subgroup of $\mathbf{G}$. Then, the group $\widetilde{\mathbf{L}} := \mathbf{Z}(\widetilde{\mathbf{G}})\mathbf{L}$ is an F-stable Levi subgroup of $\widetilde{\mathbf{G}}$. In fact, if $\mathbf{L} = \mathbf{C_G}(\mathbf{S})$ with $\mathbf{S} := \mathbf{Z}^\circ(\mathbf{L})$, then $\widetilde{\mathbf{L}} := \mathbf{L}\mathbf{Z}(\widetilde{\mathbf{G}}) = \mathbf{C_G}(\mathbf{S})\mathbf{Z}(\widetilde{\mathbf{G}}) \leq \mathbf{C}_{\widetilde{\mathbf{G}}}(\mathbf{S}) = \mathbf{C}_{\mathbf{GZ}(\widetilde{\mathbf{G}})}(\mathbf{S}) \leq \mathbf{C_G}(\mathbf{S})\mathbf{Z}(\widetilde{\mathbf{G}}) = \mathbf{L}\mathbf{Z}(\widetilde{\mathbf{G}}) = \widetilde{\mathbf{L}}$. Then, it is clear that $\mathbf{L} = \widetilde{\mathbf{L}} \cap \mathbf{G}$ and therefore $\mathbf{N_G}(\mathbf{L}) = \mathbf{N_G}(\mathbf{S})$ and $\mathbf{N}_{\widetilde{\mathbf{G}}}(\widetilde{\mathbf{L}}) = \mathbf{N}_{\widetilde{\mathbf{G}}}(\mathbf{L}) = \mathbf{N}_{\widetilde{\mathbf{G}}}(\mathbf{S})$. In addition, as $\mathbf{Z}(\widetilde{\mathbf{G}})$ is contained in $\widetilde{\mathbf{L}}$, observe that $\widetilde{\mathbf{G}} = \widetilde{\mathbf{L}}\mathbf{G}$ which implies $\widetilde{\mathbf{G}}/\mathbf{G} \simeq \widetilde{\mathbf{L}}/\mathbf{L}$. Similarly, we have $\widetilde{\mathbf{G}}^F = \widetilde{\mathbf{L}}^F\mathbf{G}^F$ and $\widetilde{\mathbf{G}}^F/\mathbf{G}^F \simeq \mathbf{N}_{\widetilde{\mathbf{G}}}(\mathbf{L})^F/\mathbf{N_G}(\mathbf{L})^F \simeq \widetilde{\mathbf{L}}^F/\mathbf{L}^F$. Observe that, since $\widetilde{\mathbf{L}}$ has connected center by [DM91, Lemma 13.14] and $[\widetilde{\mathbf{L}}, \widetilde{\mathbf{L}}] = [\mathbf{L}\mathbf{Z}(\widetilde{\mathbf{G}}), \mathbf{L}\mathbf{Z}(\widetilde{\mathbf{G}})] = [\mathbf{L}, \mathbf{L}]$, the map $i\,|_{\mathbf{L}} \colon \mathbf{L} \to \widetilde{\mathbf{L}}$ is a regular embedding.

Next, consider pairs $(\mathbf{G}^*, F^*)$ and $(\widetilde{\mathbf{G}}^*, F^*)$ dual to $(\mathbf{G}, F)$ and $(\widetilde{\mathbf{G}}, F)$ respectively. The map $i : \mathbf{G} \to \widetilde{\mathbf{G}}$ induces a surjective morphism $i^* : \widetilde{\mathbf{G}}^* \to \mathbf{G}^*$ such that $\mathrm{Ker}(i^*)$ is a connected subgroup of $\mathbf{Z}(\widetilde{\mathbf{G}}^*)$ (see [CE04, Section 15.1]). When $\mathbf{G}$ is simply connected, we have $\mathrm{Ker}(i^*) = \mathbf{Z}(\widetilde{\mathbf{G}}^*)$: in fact, the center $\mathbf{Z}(\mathbf{G}^*)$ is trivial since $\mathbf{G}^*$ is adjoint and therefore, using the isomorphism $\widetilde{\mathbf{G}}^*/\mathrm{Ker}(i^*) \simeq \mathbf{G}^*$, we deduce that $\mathbf{Z}(\widetilde{\mathbf{G}}^*) \leq \mathrm{Ker}(i^*)$. As shown in [CE04, (15.2)], there exists an isomorphism

$$\begin{aligned} \mathrm{Ker}(i^*)^F &\to \mathrm{Irr}\left(\widetilde{\mathbf{G}}^F/\mathbf{G}^F\right) \\ z &\mapsto \widehat{z}_{\widetilde{\mathbf{G}}^F} \end{aligned} \tag{6.1.2}$$

If $\mathbf{L}$ is an F-stable Levi subgroup of $\mathbf{G}$, noticing that $\mathrm{Ker}(i^*) \leq \mathbf{Z}(\widetilde{\mathbf{G}}^*) \leq \widetilde{\mathbf{L}}^*$, it follows that $\mathrm{Ker}(i^*) = \mathrm{Ker}(i^*\,|_{\widetilde{\mathbf{L}}^*})$. As before we obtain a map $\mathrm{Ker}(i^*\,|_{\widetilde{\mathbf{L}}^*})^F \to \mathrm{Irr}(\widetilde{\mathbf{L}}^F/\mathbf{L}^F)$, $z \mapsto \widehat{z}_{\widetilde{\mathbf{L}}^F}$ which coincides with the restriction of the map defined above, i.e. $\widehat{z}_{\widetilde{\mathbf{L}}^F} = (\widehat{z}_{\widetilde{\mathbf{G}}^F})_{\widetilde{\mathbf{L}}^F}$. If no confusion arises, we will denote $\mathcal{K} := \mathrm{Ker}(i^*)^F = \mathrm{Ker}(i^*\,|_{\widetilde{\mathbf{L}}^*})^F$ and obtain bijections

$$\begin{aligned} \mathcal{K} &\to \mathrm{Irr}\left(\widetilde{\mathbf{L}}^F/\mathbf{L}^F\right) \\ z &\mapsto \widehat{z}_{\widetilde{\mathbf{L}}^F} \end{aligned}$$

for every F-stable Levi subgroup $\mathbf{L} \leq \mathbf{G}$.

We summarize the above discussion in the following lemma.

Lemma 6.1.3. *Let $i : \mathbf{G} \to \widetilde{\mathbf{G}}$ be a regular embedding and consider a Levi subgroup $\mathbf{L} = \mathbf{C_G}(\mathbf{S})$ of $\mathbf{G}$, where $\mathbf{S} = \mathbf{Z}^\circ(\mathbf{L})$. Set $\widetilde{\mathbf{L}} := \mathbf{L} \cdot \mathbf{Z}(\widetilde{\mathbf{G}})$. Then the following statements hold:*

(i) $\widetilde{\mathbf{L}} = \mathbf{C}_{\widetilde{\mathbf{G}}}(\mathbf{S})$ *is a Levi subgroup of* $\widetilde{\mathbf{G}}$*;*

(ii) $\mathbf{N_G}(\mathbf{L}) = \mathbf{N_G}(\mathbf{S})$ *and* $\mathbf{N}_{\widetilde{\mathbf{G}}}(\widetilde{\mathbf{L}}) = \mathbf{N}_{\widetilde{\mathbf{G}}}(\mathbf{S}) = \mathbf{N}_{\widetilde{\mathbf{G}}}(\mathbf{L})$*;*

(iii) if $\mathbf{L}$ *is* e*-split then so is* $\widetilde{\mathbf{L}}$*;*

(iv) $i|_{\mathbf{L}} : \mathbf{L} \to \widetilde{\mathbf{L}}$ *is a regular embedding;*

(v) *Let* $(\mathbf{G}^*, F^*)$ *be in duality with* $(\mathbf{G}, F)$*, consider the morphism* $i^* : \widetilde{\mathbf{G}}^* \to \mathbf{G}^*$ *given in [CE04, Section 15.1] and set* $\mathcal{K} := \mathrm{Ker}(i^*)^F$. *There are canonical isomorphisms* $\mathcal{K} \simeq \widetilde{\mathbf{G}}^F/\mathbf{G}^F \simeq \widetilde{\mathbf{L}}^F/\mathbf{L}^F$. *Moreover* $\mathrm{Ker}(i^*) \leq \mathbf{Z}(\widetilde{\mathbf{G}}^*) \leq \widetilde{\mathbf{L}}^*$*, so that* $\mathcal{K} = \mathrm{Ker}(i^*|_{\widetilde{\mathbf{L}}^*})^F$.

6.1.5 Automorphisms

Let $\mathbf{G}$ be a connected reductive group with a Frobenius endomorphism F defining an $\mathbb{F}_q$-structure on $\mathbf{G}$. If $\sigma : \mathbf{G} \to \mathbf{G}$ is a bijective morphism of algebraic groups satisfying $\sigma \circ F = F \circ \sigma$, then the restriction of σ to $\mathbf{G}^F$, which by abuse of notation we denote again by σ, is an automorphism of the finite group $\mathbf{G}^F$. We denote by $\mathrm{Aut}_{\mathbb{F}}(\mathbf{G}^F)$ the set of those automorphisms of $\mathbf{G}^F$ obtained in this way. As mentioned in [CS13, Section 2.4], a morphism $\sigma \in \mathrm{Aut}_{\mathbb{F}}(\mathbf{G}^F)$ is determined by its restriction to $\mathbf{G}^F$ up to a power of F. It follows that $\mathrm{Aut}_{\mathbb{F}}(\mathbf{G}^F)$ acts on the set of F-stable closed connected subgroup $\mathbf{H}$ of $\mathbf{G}$. In particular, for any F-stable closed connected subgroup $\mathbf{H}$ of $\mathbf{G}$, there is a well defined set $\mathrm{Aut}_{\mathbb{F}}(\mathbf{G}^F)_{\mathbf{H}}$ whose elements are the restrictions to $\mathbf{G}^F$ of those morphisms σ as above that stabilize $\mathbf{H}$. When $\mathbf{G}$ is a simple algebraic group of simply connected type such that $\mathbf{G}^F/\mathbf{Z}(\mathbf{G}^F)$ is a nonabelian simple group, then we have $\mathrm{Aut}_{\mathbb{F}}(\mathbf{G}^F) = \mathrm{Aut}(\mathbf{G}^F)$ (see [GLS98, Section 1.15]).

Now, we want to describe the connection between automorphisms of a connected reductive group and its dual. Here, we follow [Tay18, Section 4 and Section 5]. This is done by studying isogenies. If $\mathcal{R} = (X, \Phi, Y, \Phi^\vee)$ and $\mathcal{R}' = (X', \Phi', Y', \Phi'^\vee)$ are root data, then a group homomorphism $\varphi : X' \to X$ is a p-**isogeny** if the following two conditions are satisfied (see, for instance, [GM20, Definition 1.2.9]):

(i) φ and $\varphi^\vee$ are injective, where $\varphi^\vee : Y \to Y'$ is the **dual** of φ, i.e. $\varphi^\vee$ is the unique element of $\mathrm{Hom}(Y, Y')$ such that $\langle \varphi(x'), y\rangle = \langle x', \varphi^\vee(y)\rangle$ for every $x' \in X'$ and $y \in Y$;

(ii) there exists a bijection $\Phi \to \Phi'$, $\alpha \mapsto \alpha^\dagger$ and a map $q : \Phi \to \{p^n \mid n \in \mathbb{Z}_{\geq 0}\}$ such that $\varphi(\alpha^\dagger) = q(\alpha)\alpha$ and $\varphi^\vee(\alpha^\vee) = q(\alpha)(\alpha^\dagger)^\vee$ for every $\alpha \in \Phi$.

We denote by $\mathrm{Iso}_p(\mathcal{R}, \mathcal{R}')$ the set of all p-isogenies from $\mathcal{R}$ to $\mathcal{R}'$. Next, recall that a morphism of connected reductive groups $\sigma : \mathbf{G} \to \mathbf{G}'$ is an **isogeny** if it is surjective and has a finite kernel. In this case, notice that $\mathrm{Ker}(\sigma) \leq \mathbf{Z}(\mathbf{G})$. Consider the pairs $\mathcal{G} = (\mathbf{G}, \mathbf{T})$ and $\mathcal{G}' = (\mathbf{G}', \mathbf{T}')$, where $\mathbf{T}$ and $\mathbf{T}'$ are maximal tori of $\mathbf{G}$ and $\mathbf{G}'$ respectively. Following [Tay18, Section 4.5], we denote by $\mathrm{Iso}(\mathcal{G}, \mathcal{G}')$ the set of isogenies $\sigma : \mathbf{G} \to \mathbf{G}'$ such that $\sigma(\mathbf{T}) = \mathbf{T}'$. Observe that the torus $\mathbf{T}$ acts on $\mathrm{Iso}(\mathcal{G}, \mathcal{G}')$ via $\sigma \cdot t := \sigma \circ \sigma_t$, where for any group X and $x \in X$ we denote by $\sigma_x : X \to X$ the homomorphism given by $y \mapsto x^{-1}yx$. Then, if $\sigma \in \mathrm{Iso}(\mathcal{G}, \mathcal{G}')$, we obtain a map $X(\sigma) : X(\mathbf{T}') \to \mathbf{X}(\mathbf{T})$ given by $X(\sigma)(x') := x' \circ \sigma$. Notice that $X(\sigma)$ is a p-isogeny of the corresponding root data $\mathcal{R} = (X(\mathbf{T}), \Phi(\mathbf{G}, \mathbf{T}), Y(\mathbf{T}), \Phi(\mathbf{G}, \mathbf{T})^\vee)$ and $\mathcal{R}' = (X(\mathbf{T}'), \Phi(\mathbf{G}', \mathbf{T}'), Y(\mathbf{T}'), \Phi(\mathbf{G}', \mathbf{T}')^\vee)$ and that the map $\mathrm{Iso}(\mathcal{G}, \mathcal{G}') \to \mathrm{Iso}_p(\mathcal{R}, \mathcal{R}')$ is constant on $\mathbf{T}$-orbits. By [GM20, Theorem 1.3.12] this induces a bijection

$$\mathrm{Iso}(\mathcal{G}, \mathcal{G}')/\mathbf{T} \to \mathrm{Iso}_p(\mathcal{R}', \mathcal{R}). \tag{6.1.3}$$

Consider now pairs $(\mathbf{G}^*, \mathbf{T}^*)$ and $(\mathbf{G}'^*, \mathbf{T}'^*)$ dual to $(\mathbf{G}, \mathbf{T})$ and $(\mathbf{G}', \mathbf{T}')$ respectively and let $\delta : X(\mathbf{T}) \to Y(\mathbf{T}^*)$ and $\delta' : X(\mathbf{T}') \to Y(\mathbf{T}'^*)$ be the isomorphisms introduced in Section 6.1.3.

By [Tay18, Lemma 5.2] the map

$$^* : \mathrm{Hom}(X(\mathbf{T}'), X(\mathbf{T})) \to \mathrm{Hom}(X(\mathbf{T}^*), X(\mathbf{T}'^*))$$

defined by $(\varphi^*)^\vee := \delta' \circ \varphi^{-1} \circ \delta^{-1}$ induces a bijection

$$\mathrm{Iso}_p(\mathcal{R}', \mathcal{R}) \to \mathrm{Iso}_p(\mathcal{R}^*, \mathcal{R}'^*), \tag{6.1.4}$$

where we define the root data $\mathcal{R}^* := (Y(\mathbf{T}), \Phi(\mathbf{G},\mathbf{T})^\vee, X(\mathbf{T}), \Phi(\mathbf{G},\mathbf{T}))$ corresponding to $\mathcal{G}^* := (\mathbf{G}^*, \mathbf{T}^*)$ and $\mathcal{R}'^* := (Y(\mathbf{T}'), \Phi(\mathbf{G}',\mathbf{T}')^\vee, X(\mathbf{T}'), \Phi(\mathbf{G}',\mathbf{T}'))$ corresponding to $\mathcal{G}'^* := (\mathbf{G}'^*, \mathbf{T}'^*)$. Now, (6.1.3) and (6.1.4) implies that there exists a bijection

$$\mathrm{Iso}(\mathcal{G}, \mathcal{G}')/\mathbf{T} \to \mathrm{Iso}(\mathcal{G}'^*, \mathcal{G}^*)/\mathbf{T}'^* \tag{6.1.5}$$

sending the $\mathbf{T}$-orbit of σ to the $\mathbf{T}'^*$-orbit of a corresponding element $\sigma^* \in \mathrm{Iso}(\mathcal{G}'^*, \mathcal{G}^*)$.

Let $F : \mathbf{G} \to \mathbf{G}$ be a Frobenius endomorphism and consider a pair $(\mathbf{G}^*, F^*)$ dual to $(\mathbf{G}, F)$. According to [CS13, Section 2.4], there exists an isomorphism

$$\mathrm{Aut}_{\mathbb{F}}\left(\mathbf{G}^F\right)/\mathrm{Inn}\left(\mathbf{G}^F_{\mathrm{ad}}\right) \simeq \mathrm{Aut}_{\mathbb{F}}\left(\mathbf{G}^{*F^*}\right)/\mathrm{Inn}\left(\mathbf{G}^{*F^*}_{\mathrm{ad}}\right).$$

If the coset of σ corresponds to the coset of σ^* via the above isomorphism, then we write $\sigma \sim \sigma^*$ (see [CS13, Definition 2.1]). In the following remark, we point out the relation between this definition and the similar situation given by the bijection (6.1.5).

Remark 6.1.4. Let $\sigma \in \mathrm{Aut}_{\mathbb{F}}(\mathbf{G}^F)$ and $\sigma^* \in \mathrm{Aut}_{\mathbb{F}}(\mathbf{G}^{*F^*})$ and, by abuse of notation, denote extensions of these morphisms to the algebraic groups again by $\sigma : \mathbf{G} \to \mathbf{G}$ and $\sigma^* : \mathbf{G}^* \to \mathbf{G}^*$. Then $\sigma \sim \sigma^*$ if and only if the $\mathbf{T}$-orbit of σ corresponds to the $\mathbf{T}^*$-orbit of σ^{*-1} via the bijection (6.1.5).

The above remark allows us to compare the results of [CS13] with the ones of [Tay18].

Lemma 6.1.5. *Let $\mathbf{L} \leq \mathbf{K}$ be F-stable Levi subgroups of $\mathbf{G}$ in duality with the Levi subgroups $\mathbf{L}^* \leq \mathbf{K}^*$ of $\mathbf{G}^*$. Then, for every $\sigma \in \mathrm{Aut}_{\mathbb{F}}(\mathbf{G}^F)_{\mathbf{L},\mathbf{K}}$ there exists $\sigma^* \in \mathrm{Aut}_{\mathbb{F}}(\mathbf{G}^{*F^*})_{\mathbf{L}^*,\mathbf{K}^*}$ such that $\sigma \sim \sigma^*$.*

Proof. Notice that by the comment at the beginning of this section, the groups $\mathrm{Aut}_{\mathbb{F}}(\mathbf{G}^F)_{\mathbf{L},\mathbf{K}} := \mathrm{Aut}_{\mathbb{F}}(\mathbf{G}^F)_{\mathbf{L}} \cap \mathrm{Aut}_{\mathbb{F}}(\mathbf{G}^F)_{\mathbf{K}}$ and $\mathrm{Aut}_{\mathbb{F}}(\mathbf{G}^{*F^*})_{\mathbf{L}^*,\mathbf{K}^*} := \mathrm{Aut}_{\mathbb{F}}(\mathbf{G}^{*F^*})_{\mathbf{L}^*} \cap \mathrm{Aut}_{\mathbb{F}}(\mathbf{G}^{*F^*})_{\mathbf{K}^*}$ are well defined. If $\mathbf{L} = \mathbf{K}$, then this is [CS13, Proposition 2.2] while a similar argument applies in the general case. □

Assume now that $\mathbf{G}$ is simple of simply connected type. Fix a maximally split torus $\mathbf{T}_0$ contained in an F-stable Borel subgroup $\mathbf{B}_0$ of $\mathbf{G}$. This choice corresponds to a set of simple roots $\Delta \subseteq \Phi := \Phi(\mathbf{G}, \mathbf{T}_0)$. For every $\alpha \in \Phi$ consider a one-parameter subgroup $x_\alpha : \mathbb{G}_{\mathrm{a}} \to \mathbf{G}$. Then $\mathbf{G}$ is generated by the elements $x_\alpha(t)$, where $t \in \mathbb{G}_{\mathrm{a}}$ and $\alpha \in \pm\Delta$. Consider the **field endomorphism** $F_0 : \mathbf{G} \to \mathbf{G}$ given by $F_0(x_\alpha(t)) := x_\alpha(t^p)$ for every $t \in \mathbb{G}_{\mathrm{a}}$ and $\alpha \in \Phi$. Moreover, for every symmetry γ of the Dynkin diagram of Δ, we have a **graph automorphism** $\gamma : \mathbf{G} \to \mathbf{G}$ given by $\gamma(x_\alpha(t)) := x_{\gamma(\alpha)}(t)$ for every $t \in \mathbb{G}_{\mathrm{a}}$ and $\alpha \in \pm\Delta$. Then, up to inner automorphisms of $\mathbf{G}$, any

Frobenius endomorphism F defining an $\mathbb{F}_q$-structure on $\mathbf{G}$ can be written as $F = F_0^m\gamma$, for some symmetry γ and $m \in \mathbb{Z}$ with $q = p^m$ (see [MT11, Theorem 22.5]). We say that F is **untwisted** if γ is the identity and **twisted** otherwise. In this case one can construct a regular embedding $\mathbf{G} \leq \widetilde{\mathbf{G}}$ in such a way that the Frobenius endomorphism F_0 extends to an algebraic group endomorphism $F_0 : \widetilde{\mathbf{G}} \to \widetilde{\mathbf{G}}$ defining an $\mathbb{F}_p$-structure on $\widetilde{\mathbf{G}}$. Moreover, every graph automorphism γ can be extended to an algebraic group automorphism of $\widetilde{\mathbf{G}}$ commuting with F_0 (see [MS16, Section 2B]). If we denote by $\mathcal{A}$ the group generated by γ and F_0, then we can construct the semidirect product $\widetilde{\mathbf{G}}^F \rtimes \mathcal{A}$. Finally, we define the set of **diagonal automorphisms** of $\mathbf{G}^F$ to be the set of those automorphisms induced by the action of $\widetilde{\mathbf{G}}^F$ on $\mathbf{G}^F$. If $\mathbf{G}^F/\mathbf{Z}(\mathbf{G}^F)$ is a nonabelian simple group with universal covering group $\mathbf{G}^F$, then the group $\widetilde{\mathbf{G}}^F \rtimes \mathcal{A}$ acts on $\mathbf{G}^F$ and induces all the automorphisms of $\mathbf{G}^F$ (see, for instance, the proof of [Spä12, Proposition 3.4] and of [CS19, Theorem 2.4]).

We conclude this section, by recalling an important property that will be needed in subsequent chapters.

Lemma 6.1.6. *Let $\mathbf{G}$, $\widetilde{\mathbf{G}}$, F and $\mathcal{A}$ as in the above paragraph and suppose that $\mathbf{G}^F$ is the universal covering group of $\mathbf{G}^F/\mathbf{Z}(\mathbf{G}^F)$. Let $Z \leq \mathbf{Z}(\mathbf{G}^F)$ and denote by $(\widetilde{\mathbf{G}}^F\mathcal{A})_Z$ the normalizer of Z in $\widetilde{\mathbf{G}}^F\mathcal{A}$. Then*

$$\mathbf{C}_{(\widetilde{\mathbf{G}}^F\mathcal{A})_Z/Z}\left(\mathbf{G}^F/Z\right) = \mathbf{Z}\left(\widetilde{\mathbf{G}}^F\right)/Z$$

and the canonical map $(\widetilde{\mathbf{G}}^F\mathcal{A})_Z \to \mathrm{Aut}(\mathbf{G}^F/Z)$ induces an isomorphism

$$\left(\widetilde{\mathbf{G}}^F\mathcal{A}\right)_Z/\mathbf{Z}\left(\widetilde{\mathbf{G}}^F\right) \simeq \mathrm{Aut}\left(\mathbf{G}^F/Z\right).$$

Proof. By the above paragraph, we know that $\widetilde{\mathbf{G}}^F\mathcal{A}/\mathbf{C}_{\widetilde{\mathbf{G}}^F\mathcal{A}}(\mathbf{G}^F) \simeq \mathrm{Aut}(\mathbf{G}^F)$ and therefore, using the fact that $\mathbf{C}_{\widetilde{\mathbf{G}}^F\mathcal{A}}(\mathbf{G}^F) = \mathbf{Z}(\widetilde{\mathbf{G}}^F)$ (for this fact see the argument used in [Spä12, Proposition 3.4 (a)], [CS19, Theorem 2.4] and ultimately [GLS98, Theorem 2.5.1]), we obtain $(\widetilde{\mathbf{G}}^F\mathcal{A})_Z/\mathbf{Z}(\widetilde{\mathbf{G}}^F) \simeq \mathrm{Aut}(\mathbf{G}^F)_Z$. Then, by [GLS98, Corollary 5.1.4 (b)], it follows that

$$\left(\widetilde{\mathbf{G}}^F\mathcal{A}\right)_Z/\mathbf{Z}\left(\widetilde{\mathbf{G}}^F\right) \simeq \mathrm{Aut}\left(\mathbf{G}^F\right)_Z \simeq \mathrm{Aut}\left(\mathbf{G}^F/Z\right).$$

On the other hand, since

$$\mathrm{Aut}\left(\mathbf{G}^F/Z\right) \simeq \frac{\left(\widetilde{\mathbf{G}}^F\mathcal{A}\right)_Z/Z}{\mathbf{C}_{\left(\widetilde{\mathbf{G}}^F\mathcal{A}\right)_Z/Z}\left(\mathbf{G}^F/Z\right)},$$

the third isomorphism theorem yields the desired isomorphism. □

6.1.6 Polynomial orders and E-split Levi subgroups

It is a well known fact that, given a connected reductive group $\mathbf{G}$ with a Frobenius endomorphism F defining an $\mathbb{F}_q$-structure on $\mathbf{G}$, the order of the corresponding finite group $\mathbf{G}^F$ is given by the evaluation at q of a polynomial with coefficients in $\mathbb{Z}$. More precisely, there exists a polynomial $P_{\mathbf{G},F}(x) \in \mathbb{Z}[x]$, called the **polynomial order** of $\mathbf{G}^F$, and a positive integer a such that

$$\left|\mathbf{G}^{F^m}\right| = P_{\mathbf{G},F}(q^m)$$

for every $m \equiv 1 \pmod{a}$. By the comment at the beginning of [MT11, Section 25.1], if $\Phi_e(x)$ denotes the e-th cyclotomic polynomial for a positive integer e, then there exist nonnegative integers N, n_e such that

$$P_{\mathbf{G},F}(x) = x^N \prod_{e\geq 1} \Phi_e(x)^{n_e}.$$

The theory of polynomial orders was developed in [BM92]. In this paper it was shown that the cyclotomic polynomials $\Phi_e(x)$ should play the role of "generic prime numbers". Moreover the authors showed that an analogue of the Sylow theorem holds in this context. In the sequel we follow the presentation given in [CE04, Chapter 13]. For further details we refer to [BM92], [MT11, Chapter 25] and [GM20, Section 3.5].

It can be shown (see [CE04, Proposition 13.2 (ii)]) that, if $\mathbf{H}$ is an F-stable closed connected reductive subgroup of $\mathbf{G}$, then $P_{\mathbf{H},F}(x)$ divides $P_{\mathbf{G},F}(x)$. Now, given a set of positive integers E, we define a Φ_E**-torus** of $\mathbf{G}$ to be any F-stable torus $\mathbf{T}$ such that $P_{\mathbf{T},F}(x) = \prod_{e\in E} \Phi_e(x)^{n_e}$, for some nonnegative integers n_e. The centralizer in $\mathbf{G}$ of a Φ_E-torus is called an E**-split Levi subgroup** (or Φ_E**-split Levi subgroup**) of $\mathbf{G}$. When $E = \{e\}$, then we write Φ_e-torus and e-split Levi subgroup instead of $\Phi_{\{e\}}$-torus and $\{e\}$-split Levi subgroup. By [GM20, Example 3.5.2] a Levi subgroup is 1-split if and only if it is contained in an F-stable parabolic subgroup. This terminology agrees with the definition given in Section 6.1.2.

As mentioned before, if we replace prime numbers with cyclotomic polynomials, than an analogue of the Sylow theorem holds in this situation.

Theorem 6.1.7. *Let $\mathbf{G}$ be a connected reductive group with Frobenius endomorphism $F : \mathbf{G} \to \mathbf{G}$ defining an $\mathbb{F}_q$-structure on $\mathbf{G}$. Let e be a positive integer and $\Phi_e(x)^{n_e}$ be the largest power of $\Phi_e(x)$ dividing $P_{\mathbf{G},F}(x)$.*

*(i) There exists an F-stable torus $\mathbf{S}$ of $\mathbf{G}$ such that $P_{\mathbf{S},F}(x) = \Phi_e(x)^{n_e}$. All such tori are $\mathbf{G}^F$-conjugate and are called **Sylow** Φ_e**-tori**.*

(ii) For every Φ_e-torus $\mathbf{T}$ of $\mathbf{G}$ there exists a Sylow Φ_e-torus $\mathbf{S}$ such that $\mathbf{T} \leq \mathbf{S}$.

Proof. This is [BM92, Theorem 3.4]. See also [CE04, Theorem 13.18]. □

It follows by the above theorem that, if $\mathbf{G}$ is abelian, then there exists a unique Sylow Φ_e-torus which we denote by $\mathbf{G}_{\Phi_e}$. The same is true if we replace the singleton $\{e\}$ with any set of positive integers E (see [CE04, Proposition 13.5]). We can now prove the following result on the intersection of E-split Levi subgroups.

Lemma 6.1.8. *Consider a set of positive integers E. Let $\mathbf{L}_1$ and $\mathbf{L}_2$ be two E-split Levi subgroups of $\mathbf{G}$ containing a common F-stable maximal torus $\mathbf{T}$. Then $\mathbf{L}_1 \cap \mathbf{L}_2$ is an E-split Levi subgroup of $\mathbf{G}$.*

Proof. For $i = 1, 2$, let $\mathbf{S}_i$ be a Φ_E-torus of $\mathbf{G}$ such that $\mathbf{L}_i = \mathbf{C}_{\mathbf{G}}(\mathbf{S}_i)$. Notice that $\mathbf{S}_i \leq \mathbf{Z}^\circ(\mathbf{L}_i) \leq \mathbf{T}$. Then $\mathbf{S}_i \leq \mathbf{T}_{\Phi_E}$. Moreover, as $\mathbf{T}$ is abelian, we deduce that $\mathbf{S} := \mathbf{S}_1\mathbf{S}_2$ is a subgroup of $\mathbf{T}$. Since $\mathbf{S}$ is connected it follows that $\mathbf{S}$ is a torus contained in $\mathbf{T}$. By [CE04, Proposition 13.2] it follows that $\mathbf{S}$ is a Φ_E-torus and therefore $\mathbf{L} := \mathbf{C}_{\mathbf{G}}(\mathbf{S})$ is an E-split Levi subgroup of $\mathbf{G}$. To conclude, observe that $\mathbf{L} = \mathbf{L}_1 \cap \mathbf{L}_2$. □

We conclude by studying the behaviour of E-split Levi subgroups with respect to duality and regular embeddings.

Lemma 6.1.9. *Let* $(\mathbf{G}^*, F^*)$ *be a pair dual to* $(\mathbf{G}, F)$*. Then the bijection* (6.1.1) *restricts to a bijection*

$$\mathbf{L} \mapsto \mathbf{L}^*$$

between E*-split Levi subgroups of* $\mathbf{G}$ *and* E*-split Levi subgroups of* $\mathbf{G}^*$*.*

Proof. This is [CE04, Proposition 13.9]. □

Lemma 6.1.10. *Let* $i : \mathbf{G} \to \widetilde{\mathbf{G}}$ *be a regular embedding. Consider a Levi subgroup* $\mathbf{L}$ *of* $\mathbf{G}$ *and set* $\widetilde{\mathbf{L}} := \mathbf{L}\mathbf{Z}(\widetilde{\mathbf{G}})$*. If* $\mathbf{L}$ *is an* E*-split Levi subgroup of* $\mathbf{G}$ *then* $\widetilde{\mathbf{L}}$ *is an* E*-split Levi subgroup of* $\widetilde{\mathbf{G}}$*.*

Proof. Since $\mathbf{L}$ is an E-split Levi subgroup of $\mathbf{G}$, we can find a Φ_E-torus $\mathbf{T} \leq \mathbf{G}$ such that $\mathbf{L} = \mathbf{C}_{\mathbf{G}}(\mathbf{T})$. Recalling that $\widetilde{\mathbf{G}} = \mathbf{Z}(\widetilde{\mathbf{G}})\mathbf{G}$, it follows that $\mathbf{C}_{\widetilde{\mathbf{G}}}(\mathbf{T}) = \mathbf{C}_{\mathbf{Z}(\widetilde{\mathbf{G}})\mathbf{G}}(\mathbf{T}) = \mathbf{Z}(\widetilde{\mathbf{G}})\mathbf{C}_{\mathbf{G}}(\mathbf{T})$. Therefore $\widetilde{\mathbf{L}} = \mathbf{Z}(\widetilde{\mathbf{G}})\mathbf{L} = \mathbf{C}_{\widetilde{\mathbf{G}}}(\mathbf{T})$ and, because $\mathbf{T}$ is a Φ_E-torus of $\widetilde{\mathbf{G}}$, we conclude that $\widetilde{\mathbf{L}}$ is an E-split Levi subgroup of $\widetilde{\mathbf{G}}$. □

6.2 Representation theory of finite groups of Lie type

Let $\mathbf{G}$ be a connected reductive group with Frobenius endomorphism $F : \mathbf{G} \to \mathbf{G}$. For every F-stable Levi subgroup $\mathbf{L}$ of a (not necessarily F-stable) parabolic subgroup $\mathbf{P}$ of $\mathbf{G}$, Deligne–Lusztig and Lusztig associated two maps

$$\mathbf{R}^{\mathbf{G}}_{\mathbf{L} \leq \mathbf{P}} : \mathbb{Z}\mathrm{Irr}\left(\mathbf{L}^F\right) \to \mathbb{Z}\mathrm{Irr}\left(\mathbf{G}^F\right)$$

and

$$^*\mathbf{R}^{\mathbf{G}}_{\mathbf{L} \leq \mathbf{P}} : \mathbb{Z}\mathrm{Irr}\left(\mathbf{G}^F\right) \to \mathbb{Z}\mathrm{Irr}\left(\mathbf{L}^F\right)$$

adjoint to each other with respect to the usual scalar product on class functions. The representation theory of finite groups of Lie type relies heavily on the work of Deligne and Lusztig and it is the aim of this section to recall some of the main ideas and definitions of this fascinating field.

6.2.1 Deligne–Lusztig induction and restriction

Fix a prime ℓ different from p and let $\overline{\mathbb{Q}_\ell}$ be an algebraic closure of the field $\mathbb{Q}_\ell$ of ℓ-adic numbers. So far, we have only considered affine algebraic varieties. In this section we will encounter a larger family of varieties. However, whenever we say variety we mean quasi-projective variety, i.e. a locally closed subvariety of a projective variety (see [CE04, Appendix 2] for more details). One can associate to every variety $\mathbf{X}$ a family of finite dimensional $\overline{\mathbb{Q}_\ell}$-vector spaces $H^i_{\mathrm{c}}(\mathbf{X}, \overline{\mathbb{Q}_\ell})$, $i \in \mathbb{Z}$, called the ℓ-adic cohomology groups with compact support (see [CE04, Appendix 3]). This cohomology theory is functorial and, for every morphism $f : \mathbf{X} \to \mathbf{X}'$, there is an induced linear map $f^* : H^i_{\mathrm{c}}(\mathbf{X}', \overline{\mathbb{Q}_\ell}) \to H^i_{\mathrm{c}}(\mathbf{X}, \overline{\mathbb{Q}_\ell})$. Then, if G is a finite group acting via algebraic automorphisms on $\mathbf{X}$, then the vector space $H^i_{\mathrm{c}}(\mathbf{X}, \overline{\mathbb{Q}_\ell})$ has a structure of $\overline{\mathbb{Q}_\ell}G$-module given by

$g \cdot v := (g^*)^{-1}(v)$ for every $g \in G$ and $v \in H^i_c(\mathbf{X}, \overline{\mathbb{Q}_\ell})$. We then define the **Lefschetz number** of g on $\mathbf{X}$ to be

$$\mathfrak{L}(g, \mathbf{X}) := \sum_i (-1)^i \mathrm{Tr}\left((g^*)^{-1}, H^i_c(\mathbf{X}, \overline{\mathbb{Q}_\ell})\right),$$

where $\mathrm{Tr}((g^*)^{-1}, H^i_c(\mathbf{X}, \overline{\mathbb{Q}_\ell}))$ denotes the trace of the linear map $(g^*)^{-1}$ on $H^i_c(\mathbf{X}, \overline{\mathbb{Q}_\ell})$. Notice that, since the vector spaces $H^i_c(\mathbf{X}, \overline{\mathbb{Q}_\ell})$ are finite dimensional and zero whenever $i < 0$ or $i > 2\dim(\mathbf{X})$, it follows that the above sum is well defined (see [DM91, Proposition 10.1]). It is also worth noting that $\mathfrak{L}(g, \mathbf{X})$ does not depend on ℓ (see [DM91, Corollary 10.6]).

Let $\mathbf{G}$ be a connected reductive group with a Frobenius endomorphism F associated to an $\mathbb{F}_q$-structure on $\mathbf{G}$ for some power q of p. For every parabolic subgroup $\mathbf{P}$ of $\mathbf{G}$ with Levi decomposition $\mathbf{P} = \mathbf{L} \ltimes \mathbf{U}$ such that $\mathbf{L}$ is F-stable, we can define the variety

$$\mathbf{Y}_\mathbf{U} := \left\{ g\mathbf{U} \in \mathbf{G}/\mathbf{U} \;\middle|\; g^{-1}F(g) \in \mathbf{U}F(\mathbf{U}) \right\}.$$

Since the finite groups $\mathbf{G}^F$ and $\mathbf{L}^F$ acts on $\mathbf{Y}_\mathbf{U}$ by left and right multiplication respectively, it follows that the vector spaces $H^i_c(\mathbf{Y}_\mathbf{U}, \overline{\mathbb{Q}_\ell})$ are $(\mathbf{G}^F, \mathbf{L}^F)$-bimodules. Then, we define **Deligne–Lusztig induction** as the map

$$\mathbf{R}^\mathbf{G}_{\mathbf{L} \leq \mathbf{P}} : \mathbb{Z}\mathrm{Irr}\left(\mathbf{L}^F\right) \to \mathbb{Z}\mathrm{Irr}\left(\mathbf{G}^F\right)$$

defined by

$$\begin{aligned} \mathbf{R}^\mathbf{G}_{\mathbf{L} \leq \mathbf{P}}(\lambda)(g) &:= \sum_i (-1)^i \mathrm{Tr}\left((g^*)^{-1}, H^i_c(\mathbf{Y}_\mathbf{U}, \overline{\mathbb{Q}_\ell}) \otimes_{\mathbb{C}\mathbf{L}^F} \Lambda\right) \\ &= |\mathbf{L}^F|^{-1} \sum_{l \in \mathbf{L}^F} \mathfrak{L}\left((g, l), \mathbf{Y}_\mathbf{U}\right) \lambda(l), \end{aligned}$$

where Λ is a $\mathbb{C}\mathbf{L}^F$-module affording the character λ and we consider $H^i_c(\mathbf{Y}_\mathbf{U}, \overline{\mathbb{Q}_\ell})$ as a module over $\mathbb{C}$ (see [GM20, Remark 2.1.5]). The map

$${}^*\mathbf{R}^\mathbf{G}_{\mathbf{L} \leq \mathbf{P}} : \mathbb{Z}\mathrm{Irr}\left(\mathbf{G}^F\right) \to \mathbb{Z}\mathrm{Irr}\left(\mathbf{L}^F\right)$$

dual to $\mathbf{R}^\mathbf{G}_{\mathbf{L} \leq \mathbf{P}}$ with respect to the usual scalar product is called **Deligne–Lusztig restriction**. This means that

$$\left[\mathbf{R}^\mathbf{G}_{\mathbf{L} \leq \mathbf{P}}(\lambda), \chi\right] = \left[\lambda, {}^*\mathbf{R}^\mathbf{G}_{\mathbf{L} \leq \mathbf{P}}(\chi)\right]$$

for every $\lambda \in \mathrm{Irr}(\mathbf{L}^F)$ and $\chi \in \mathrm{Irr}(\mathbf{G}^F)$. For more details on this topic and for the main properties of Deligne–Lusztig induction, we refer the reader to [DM91, Chapter 10 and 11], [CE04, Section 8.3] and [GM20, Section 3.3]. The idea of using ℓ-adic cohomology to obtain representations of finite groups was introduced by Deligne–Lusztig [DL76] and Lusztig [Lus76]. The constructions given above where introduced in [DL76] only for the case where $\mathbf{L}$ is a maximal torus and then generalized to arbitrary F-stable Levi subgroups in [Lus76]. For this reason, some authors use the term Deligne–Lusztig induction (resp. restriction) only when $\mathbf{L}$ is a maximal torus and use the term Lusztig induction (resp. restriction) for the general case. Another term used in the literature is twisted induction (resp. restriction). To avoid confusion, in this thesis we will refer to these maps as Deligne–Lusztig induction (resp. restriction) in every case. The generalized characters of the form $\mathbf{R}^{\mathbf{G}}_{\mathbf{T}\leq\mathbf{B}}(\vartheta)$, where $\mathbf{T}$ is an F-stable maximal torus contained in a Borel subgroup $\mathbf{B}$ of $\mathbf{G}$ and $\vartheta \in \mathrm{Irr}(\mathbf{T}^F)$, are called **Deligne–Lusztig characters**.

It is conjectured that Deligne–Lusztig induction and restriction do not depend on the choice of a parabolic subgroup. This fact can be derived as a consequence of the so called **Mackey formula**. Let $\mathbf{L}$ and $\mathbf{M}$ be F-stable Levi complements of the parabolic subgroups $\mathbf{P}$ and $\mathbf{Q}$ of $\mathbf{G}$ respectively. Then the Mackey formula asserts that

$$ {}^*\mathbf{R}^{\mathbf{G}}_{\mathbf{L}\leq\mathbf{P}} \circ \mathbf{R}^{\mathbf{G}}_{\mathbf{M}\leq\mathbf{Q}} = \sum_{g \in \mathbf{L}^F \backslash \mathcal{S}_{\mathbf{G}}(\mathbf{L},\mathbf{M})^F/\mathbf{M}^F} \mathbf{R}^{\mathbf{L}}_{\mathbf{L}\cap {}^g\mathbf{M}\leq \mathbf{L}\cap {}^g\mathbf{Q}} \circ {}^*\mathbf{R}^{{}^g\mathbf{M}}_{\mathbf{L}\cap {}^g\mathbf{M}\leq \mathbf{P}\cap {}^g\mathbf{M}} \circ (\mathrm{ad}\, g)_{\mathbf{M}^F}, \tag{6.2.1} $$

where the sum runs over a set of representatives for the $(\mathbf{L}^F, \mathbf{M}^F)$-double cosets in

$$ \mathcal{S}_{\mathbf{G}}(\mathbf{L},\mathbf{M})^F := \left\{ g \in \mathbf{G}^F \;\middle|\; \mathbf{L} \cap^g \mathbf{M} \text{ contains a maximal torus of } \mathbf{G} \right\} $$

and $(\mathrm{ad}\, g)_{\mathbf{M}^F} : \mathbb{Z}\mathrm{Irr}(\mathbf{M}^F) \to \mathbb{Z}\mathrm{Irr}({}^g\mathbf{M}^F)$ is the map defined by $(\mathrm{ad}\, g)_{\mathbf{M}^F}(\psi)({}^g x) := \psi(x)$ for every $x \in \mathbf{M}^F$. We will say that the Mackey formula holds for a connected reductive group $\mathbf{G}$ if it holds with respect to every parabolic and Levi subgroups of $\mathbf{G}$. As mentioned above, assuming the Mackey formula, one can show the independence of Deligne–Lusztig induction and restriction from the choice of a parabolic subgroup (see [GM20, Theorem 3.3.8]).

Lemma 6.2.1. *Assume that the Mackey formula holds for a connected reductive group $\mathbf{G}$ with Frobenius endomorphism $F : \mathbf{G} \to \mathbf{G}$. Then $\mathbf{R}^{\mathbf{G}}_{\mathbf{L}\leq\mathbf{P}}$ and ${}^*\mathbf{R}^{\mathbf{G}}_{\mathbf{L}\leq\mathbf{P}}$ are independent of the parabolic subgroup $\mathbf{P}$.*

It is conjectured that the Mackey formula always holds. Unfortunately, at the time of writing, this has not yet been proved in full generality. Initially the formula was shown in the case where both parabolic subgroups are F-stable (see [LS79, Lemma 2.5]) and in the case where one of the two Levi subgroups is a maximal torus (see [DL83] and [DM91, Theorem 11.13]), while the case where both Levi subgroups are maximal tori was already dealt with in [DL76]. The best known result in this direction shows that the formula holds for every connected reductive group $\mathbf{G}$ endowed with an $\mathbb{F}_q$-structure induced by a Frobenius endomorphism $F : \mathbf{G} \to \mathbf{G}$, unless $q = 2$ and $\mathbf{G}^F$ has a quasisimple component of type ${}^2\mathbf{E}_6$, $\mathbf{E}_7$ or $\mathbf{E}_8$ (see [BM11]). Other evidences appeared in [Tay18].

Theorem 6.2.2. *Let $\mathbf{G}$ be a connected reductive group with a Frobenius endomorphism $F : \mathbf{G} \to \mathbf{G}$ associated to an $\mathbb{F}_q$-structure on $\mathbf{G}$. Then* (6.2.1) *holds whenever one of the following conditions is met:*

(i) both $\mathbf{P}$ and $\mathbf{Q}$ are F-stable;

(ii) either $\mathbf{L}$ or $\mathbf{M}$ is a maximal torus;

(iii) $q \neq 2$;

(iv) $\mathbf{G}^F$ does not contain a quasisimple component of type ${}^2\mathbf{E}_6$, $\mathbf{E}_7$ or $\mathbf{E}_8$.

As a final remark, we warn the reader that from now on we will always write $\mathbf{R}^{\mathbf{G}}_{\mathbf{L}}$ (resp. ${}^*\mathbf{R}^{\mathbf{G}}_{\mathbf{L}}$) instead of $\mathbf{R}^{\mathbf{G}}_{\mathbf{L}\leq\mathbf{P}}$ (resp. ${}^*\mathbf{R}^{\mathbf{G}}_{\mathbf{L}\leq\mathbf{P}}$) whenever the result does not depend on the choice of a parabolic subgroup.

6.2.2 Rational Lusztig series

We now introduce a fundamental partition of the characters of a finite group of Lie type. We follow the description given in [Bon06, Section 9, Section 11]. Other references are [DM91, Chapter 13], [CE04, Section 8.4] and [GM20, Section 2.6].

Let $\mathbf{G}$ be a connected reductive group with a Frobenius endomorphism $F : \mathbf{G} \to \mathbf{G}$ defining an $\mathbb{F}_q$-structure on $\mathbf{G}$. Denote by $\nabla(\mathbf{G}, F)$ the set of pairs $(\mathbf{T}, \vartheta)$ where $\mathbf{T}$ is an F-stable maximal torus of $\mathbf{G}$ and $\vartheta \in \mathrm{Irr}(\mathbf{T}^F)$. The finite group $\mathbf{G}^F$ acts by conjugation on $\nabla(\mathbf{G}, F)$ and we denote by $\nabla(\mathbf{G}, F)/\mathbf{G}^F$ the set of $\mathbf{G}^F$-orbits on $\nabla(\mathbf{G}, F)$. Let $(\mathbf{G}^*, F^*)$ be a pair dual to $(\mathbf{G}, F)$ and consider the set $\nabla^*(\mathbf{G}, F)$ consisting of pairs $(\mathbf{T}^*, s)$ where $\mathbf{T}^*$ is an F^*-stable maximal torus of $\mathbf{G}^*$ and $s \in \mathbf{T}^{*F^*}_{\mathrm{ss}}$. As before, the group $\mathbf{G}^{*F^*}$ acts by conjugation on the set $\nabla^*(\mathbf{G}, F)$ and we denote by $\nabla^*(\mathbf{G}, F)/\mathbf{G}^{*F^*}$ the set of $\mathbf{G}^{*F^*}$-orbits on $\nabla^*(\mathbf{G}, F)$. By [DM91, Proposition 13.13] there exists a bijection

$$\nabla(\mathbf{G}, F)/\mathbf{G}^F \to \nabla^*(\mathbf{G}, F)/\mathbf{G}^{*F^*}. \tag{6.2.2}$$

Since, for every $(\mathbf{T}_1, \vartheta_1), (\mathbf{T}_2, \vartheta_2) \in \nabla(\mathbf{G}, F)$, we have $\mathbf{R}^{\mathbf{G}}_{\mathbf{T}_1}(\vartheta_1) = \mathbf{R}^{\mathbf{G}}_{\mathbf{T}_2}(\vartheta_2)$ whenever $(\mathbf{T}_1, \vartheta_1)$ and $(\mathbf{T}_2, \vartheta_2)$ are $\mathbf{G}^F$-conjugate (see [GM20, Corollary 2.2.10]), using the bijection (6.2.2) we can define

$$\mathbf{R}^{\mathbf{G}}_{\mathbf{T}^*}(s) := \mathbf{R}^{\mathbf{G}}_{\mathbf{T}}(\vartheta)$$

for every $(\mathbf{T}, \vartheta) \in \nabla(\mathbf{G}, F)$ and $(\mathbf{T}^*, s) \in \nabla^*(\mathbf{G}, F)$ whose orbits correspond via (6.2.2).

We now define the **(rational) Lusztig series** associated to the $\mathbf{G}^{*F^*}$-conjugacy class of a semisimple element $s \in \mathbf{G}^{*F^*}_{\mathrm{ss}}$ to be the set $\mathcal{E}(\mathbf{G}^F, [s])$ of irreducible constituents of some $\mathbf{R}^{\mathbf{G}}_{\mathbf{T}^*}(s)$, where $\mathbf{T}^*$ is an F^*-stable maximal torus of $\mathbf{G}^*$ containing s, that is

$$\mathcal{E}\left(\mathbf{G}^F, [s]\right) := \left\{\chi \in \mathrm{Irr}\left(\mathbf{G}^F\right) \;\middle|\; \left[\chi, \mathbf{R}^{\mathbf{G}}_{\mathbf{T}^*}(s)\right] \neq 0, \text{ for some } (\mathbf{T}^*, s) \in \nabla^*(\mathbf{G}, F)\right\}.$$

The elements of $\mathcal{E}(\mathbf{G}^F, [1])$ are called **unipotent characters**. The importance of Lusztig series lies in the following result of Lusztig (see [Lus77, 7.6]). This is the first step towards a Jordan decomposition for characters.

Theorem 6.2.3. *Lusztig series give a partition of the irreducible characters of $\mathbf{G}^F$ as*

$$\mathrm{Irr}\left(\mathbf{G}^F\right) = \coprod_s \mathcal{E}\left(\mathbf{G}^F, [s]\right),$$

where s runs over a set of representatives for the $\mathbf{G}^{*F^*}$-conjugacy classes of semisimple elements of $\mathbf{G}^{*F^*}$.

We remark that there is another type of series which is often considered in the literature. These are the so-called **geometric Lusztig series**. One can show that geometric Lusztig series are unions of rational Lusztig series and that the two notions coincide when $\mathbf{G}$ has connected center. However, in this thesis we will only consider rational Lusztig series and we will refer to them simply as Lusztig series.

We will now state some properties that will be often used in the sequel. First, consider an F-stable Levi subgroup $\mathbf{L}$ of $\mathbf{G}$ and let $\mathbf{P}$ be a parabolic subgroup of $\mathbf{G}$ of which $\mathbf{L}$ is a Levi complement. It is important to know how the Deligne–Lusztig induction map $\mathbf{R}^{\mathbf{G}}_{\mathbf{L} \leq \mathbf{P}} : \mathbb{Z}\mathrm{Irr}(\mathbf{L}^F) \to \mathbb{Z}\mathrm{Irr}(\mathbf{G}^F)$ behaves with respect to the Lusztig series of $\mathbf{L}^F$ and of $\mathbf{G}^F$.

Lemma 6.2.4. *Let* $\mathbf{P}$ *be a parabolic subgroup with* F*-stable Levi complement* $\mathbf{L}$ *and consider an* F^**-stable Levi subgroup* $\mathbf{L}^*$ *of* $\mathbf{G}^*$ *in duality with* $\mathbf{L}$*. Let* $t \in \mathbf{L}_{\mathrm{ss}}^{*F^*}$*,* $s \in \mathbf{G}_{\mathrm{ss}}^{*F^*}$ *and consider* $\lambda \in \mathcal{E}(\mathbf{L}^F, [t])$ *and* $\chi \in \mathcal{E}(\mathbf{G}^F, [s])$*. Then*

$$\mathbf{R}_{\mathbf{L}\leq\mathbf{P}}^{\mathbf{G}}(\lambda) \in \mathbb{Z}\mathcal{E}\left(\mathbf{G}^F, [t]\right)$$

and, if λ *is an irreducible constituent of* ${}^*\mathbf{R}_{\mathbf{L}\leq\mathbf{P}}^{\mathbf{G}}(\chi)$*, then* s *and* t *are* $\mathbf{G}^{*F^*}$*-conjugate.*

Proof. This is [CE04, Proposition 15.7]. ☐

Next, we consider Lusztig series under regular embeddings.

Lemma 6.2.5. *Let* $i : \mathbf{G} \to \widetilde{\mathbf{G}}$ *be a regular embedding and let* $i^* : \widetilde{\mathbf{G}}^* \to \mathbf{G}^*$ *be the dual morphism. Let* $\widetilde{s}$ *be a semisimple element of* $\widetilde{\mathbf{G}}^{*F^*}$ *and consider its image* $s := i^*(\widetilde{s})$*. Then*

$$\mathcal{E}\left(\mathbf{G}^F, [s]\right) = \left\{\chi \in \mathrm{Irr}\left(\mathbf{G}^F\right) \;\middle|\; [\chi, \widetilde{\chi}_{\mathbf{G}^F}] \neq 0, \text{ for some } \widetilde{\chi} \in \mathcal{E}\left(\widetilde{\mathbf{G}}^F, [\widetilde{s}]\right)\right\}.$$

Proof. This is [CE04, Proposition 15.6]. ☐

6.2.3 Jordan decomposition of characters

In the previous section we have seen how rational Lusztig series provide a partition of the irreducible characters of a finite reductive group. The next step towards a "Jordan decomposition" for characters was proved by Lusztig in [Lus84] (for groups with connected center) and [Lus88] (for groups with disconnected center under some mild restrictions). In order to state this result, we need to introduce some notation. Let $s \in \mathbf{G}_{\mathrm{ss}}^{*F^*}$, since the centralizer $\mathbf{C}_{\mathbf{G}^*}(s)$ might not be connected, we denote by $\mathcal{E}(\mathbf{C}_{\mathbf{G}^*}(s)^{F^*}, [1])$ the set of irreducible constituents of those characters of $\mathbf{C}_{\mathbf{G}^*}(s)^{F^*}$ induced from an element of $\mathcal{E}(\mathbf{C}_{\mathbf{G}^*}^{\circ}(s)^{F^*}, [1])$. Notice that, if $\mathbf{Z}(\mathbf{G})$ is connected, then $\mathbf{C}_{\mathbf{G}^*}(s)$ is connected (see [DM91, Remark 13.15 (ii)]) and the above definition coincides with the notion of Lusztig series.

Theorem 6.2.6. *Let* $\mathbf{G}$ *be a connected reductive group with Frobenius endomorphism* F*. Consider a pair* $(\mathbf{G}^*, F^*)$ *dual to* $(\mathbf{G}, F)$ *and a semisimple element* $s \in \mathbf{G}_{\mathrm{ss}}^{*F^*}$*. Then there exists a bijection*

$$J_{\mathbf{G},s} : \mathcal{E}\left(\mathbf{G}^F, [s]\right) \to \mathcal{E}\left(\mathbf{C}_{\mathbf{G}^*}(s)^{F^*}, [1]\right).$$

Moreover

$$\chi(1) = \left|\mathbf{G}^{*F^*} : \mathbf{C}_{\mathbf{G}^*}(s)^{F^*}\right|_{p'} J_{\mathbf{G},s}(\chi)(1)$$

for every $\chi \in \mathcal{E}(\mathbf{G}^F, [s])$*.*

Proof. See [DM20, Theorem 11.5.1 and Proposition 11.5.6] and [GM20, Theorem 2.6.22 and Remark 2.6.26]. ☐

As mentioned before Theorem 6.2.6 was mainly proven by Lusztig in [Lus88] generalizing the connected center case which was already shown in Lusztig's book [Lus84]. The argument of [Lus88] needs to be complemented by some multiplicity one statements (see [CE04, Chapter 16], [DM20, Section 11.5] and [Lus08]). In this way, every irreducible character χ of $\mathbf{G}^F$ can be parametrized by a rational conjugacy class $[s]$ of semisimple elements of $\mathbf{G}^{*F^*}$ and a unipotent character of $\mathbf{C}_{\mathbf{G}^*}(s)^{F^*}$. This provides a Jordan decomposition of the characters of $\mathbf{G}^F$.

When the centralizer $\mathbf{C}_{\mathbf{G}^*}(s)$ is a Levi subgroup, then Jordan decomposition can be explicitly described via Deligne–Lusztig induction. This follows by the next two results.

Proposition 6.2.7. *Let $s \in \mathbf{G}^{*F^*}$ be a semisimple element and consider $z \in \mathbf{Z}(\mathbf{G}^*)^{F^*}$. Then there exists a linear character $\widehat{z}_{\mathbf{G}^F} \in \mathcal{E}(\mathbf{G}^F, [z])$ and a bijection*

$$\mathcal{E}\left(\mathbf{G}^F, [s]\right) \to \mathcal{E}\left(\mathbf{G}^F, [sz]\right)$$

given by multiplication by $\widehat{z}_{\mathbf{G}^F}$.

Proof. This is [CE04, Proposition 8.26]. □

For the next statement, we need to define the sign $\epsilon_{\mathbf{G}} := (-1)^{\sigma(\mathbf{G})}$ for every linear algebraic group $\mathbf{G}$, where $\sigma(\mathbf{G})$ is the $\mathbb{F}_q$-rank of $\mathbf{G}$ as in [DM91, Definition 8.3].

Proposition 6.2.8. *Let $\mathbf{L}$ be a Levi subgroup of $\mathbf{G}$ corresponding to the Levi subgroup $\mathbf{L}^*$ of $\mathbf{G}^*$ via duality. Suppose that $s \in \mathbf{G}^{*F^*}$ is a semisimple element such that $\mathbf{C}^\circ_{\mathbf{G}^*}(s)\mathbf{C}_{\mathbf{G}^*}(s)^{F^*} \leq \mathbf{L}^*$. Then there exists a bijection*

$$\epsilon_{\mathbf{L}}\epsilon_{\mathbf{G}}\mathbf{R}^{\mathbf{G}}_{\mathbf{L}\leq\mathbf{P}} : \mathcal{E}\left(\mathbf{L}^F, [s]\right) \to \mathcal{E}\left(\mathbf{G}^F, [s]\right)$$

given by Deligne–Lusztig induction, where $\mathbf{P}$ is a parabolic subgroup of $\mathbf{G}$ having $\mathbf{L}$ as Levi complement.

Proof. See [CE04, Theorem 8.27] and [DM91, Theorem 13.25]. □

To conclude this section, we consider an important property of Jordan decomposition. Namely, we ask whether a Jordan decomposition can always be chosen in such a way that it commutes with Deligne–Lusztig induction (resp. restriction). Although this property still needs to be proved in full generality, some partial results have been shown for groups with connected center. For classical groups this was first proved by Fong and Srinivasan by using results of Shoji and Asai [FS89, Appendix A]

Theorem 6.2.9. *Let $\mathbf{G}$ be a connected reductive group with connected center and with components only of classical type $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ or $\mathbf{D}$. Let F be a Frobenius endomorphism of $\mathbf{G}$ and suppose that F does not induce the triality automorphism on components of type $\mathbf{D}_4$. Let $(\mathbf{G}^*, F^*)$ be a pair dual to $(\mathbf{G}, F)$ and consider F-stable Levi subgroups $\mathbf{L} \leq \mathbf{M} \leq \mathbf{G}$ corresponding to $\mathbf{L}^* \leq \mathbf{M}^* \leq \mathbf{G}^*$ via duality. If $s \in \mathbf{L}^{*F^*}_{\mathrm{ss}}$, then the diagram*

$$\begin{array}{ccc} \mathbb{Z}\mathcal{E}\left(\mathbf{M}^F,[s]\right) & \xrightarrow{J_{\mathbf{M},s}} & \mathbb{Z}\mathcal{E}\left(\mathbf{C}_{\mathbf{M}^*}(s)^{F^*},[1]\right) \\ {\scriptstyle \mathbf{R}_{\mathbf{L}}^{\mathbf{M}}}\uparrow & & \uparrow{\scriptstyle \mathbf{R}_{\mathbf{C}_{\mathbf{L}^*}(s)}^{\mathbf{C}_{\mathbf{M}^*}(s)}} \\ \mathbb{Z}\mathcal{E}\left(\mathbf{L}^F,[s]\right) & \xrightarrow[J_{\mathbf{L},s}]{} & \mathbb{Z}\mathcal{E}\left(\mathbf{C}_{\mathbf{L}^*}(s)^{F^*},[1]\right) \end{array}$$

commutes, where $J_{\bullet,s}$ is a Jordan decomposition as in Theorem 6.2.6.

Proof. This is [GM20, Theorem 4.7.2]. □

For groups of exceptional type we have the following result.

Theorem 6.2.10. *Let $\mathbf{G}$ be a simple algebraic group with connected center, $F : \mathbf{G} \to \mathbf{G}$ a Steinberg endomorphism and suppose that the Mackey formula holds for $\mathbf{G}^F$. Let $(\mathbf{G}^*, F^*)$ be a pair dual to $(\mathbf{G}, F)$ and consider F-stable Levi subgroups $\mathbf{L} \leq \mathbf{M} \leq \mathbf{G}$ corresponding to $\mathbf{L}^* \leq \mathbf{M}^* \leq \mathbf{G}^*$ via duality. If $s \in \mathbf{L}^{*F^*}_{\mathrm{ss}}$, then the diagram*

$$\begin{array}{ccc} \mathbb{Z}\mathcal{E}\left(\mathbf{M}^F,[s]\right) & \xrightarrow{J_{\mathbf{M},s}} & \mathbb{Z}\mathcal{E}\left(\mathbf{C}_{\mathbf{M}^*}(s)^{F^*},[1]\right) \\ {\scriptstyle \mathbf{R}_{\mathbf{L}}^{\mathbf{M}}}\uparrow & & \uparrow{\scriptstyle \mathbf{R}_{\mathbf{C}_{\mathbf{L}^*}(s)}^{\mathbf{C}_{\mathbf{M}^*}(s)}} \\ \mathbb{Z}\mathcal{E}\left(\mathbf{L}^F,[s]\right) & \xrightarrow[J_{\mathbf{L},s}]{} & \mathbb{Z}\mathcal{E}\left(\mathbf{C}_{\mathbf{L}^*}(s)^{F^*},[1]\right) \end{array}$$

commutes, where $J_{\bullet,s}$ is a Jordan decomposition as in Theorem 6.2.6, unless possibly when $\mathbf{G} = \mathbf{M}$ is of type $\mathbf{E}_8$.

Proof. This is [GM20, Theorem 4.7.5]. □

6.2.4 Generalized e-Harish-Chandra theories

We now introduce the main results in e-Harish-Chandra theories. The classical ordinary Harish-Chandra theory provides an inductive way of classifying the irreducible characters of finite groups of Lie type. More generally this theory can be developed for groups with a BN-pair (see [GM20, Section 3.1]). Ordinary Harish-Chandra theory was first introduced by Harish-Chandra in [HC70] and then developed further by Howelett and Lehrer in [HL80]. This theory provides a partition of the irreducible characters of a finite group of Lie type into, so-called, Harish-Chandra series determined via Harish-Chandra induction from cuspidal characters of 1-split Levi subgroups. By replacing Harish-Chandra induction (resp. restriction) with Deligne–Lusztig induction (resp. restriction) and by considering e-split Levi subgroups instead of 1-split Levi subgroups, we then obtain the, so-called, e-Harish-Chandra theories first introduced by Fong and Srinivasan in [FS86] and then fully developed in [BMM93] in the unipotent case. Then, ordinary Harish-Chandra series can be recovered as 1-Harish-Chandra theory. Here, we will present the main features of these theories following [GM20, Chapter 3]. In the next section we will see how e-Harish-Chandra theories can be applied in order to obtain a deep insight into the modular representation theory of finite groups of Lie type in nondefining characteristic.

Let $\mathbf{G}$ be a connected reductive group with a Frobenius endomorphism F defining an $\mathbb{F}_q$-structure on $\mathbf{G}$. For the rest of this section we fix a positive integer e. An irreducible character $\chi \in \operatorname{Irr}(\mathbf{G}^F)$ is e-**cuspidal** if ${}^*\mathbf{R}^{\mathbf{G}}_{\mathbf{L}\leq\mathbf{P}}(\chi) = 0$ for every e-split Levi subgroup $\mathbf{L} < \mathbf{G}$ which is the complement of a parabolic subgroup $\mathbf{P}$. An e-**cuspidal pair** of $(\mathbf{G}, F)$ (or simply of $\mathbf{G}$ if no confusion arises) is any pair $(\mathbf{L}, \lambda)$, where $\mathbf{L}$ is an e-split Levi subgroup of $\mathbf{G}$ and λ is an e-cuspidal character of $\mathbf{L}^F$ (see also Definition 7.2.1). Next, we relate the notion of e-cuspidality with Jordan decomposition.

Proposition 6.2.11. *Let $\chi \in \mathcal{E}(\mathbf{G}^F, [s])$ be e-cuspidal with $s \in \mathbf{G}^{*F^*}_{\mathrm{ss}}$. If $\psi \in \mathcal{E}(\mathbf{C}^\circ_{\mathbf{G}^*}(s)^F, [1])$ lies in the $\mathbf{C}_{\mathbf{G}^*}(s)^{F^*}$-orbit of unipotent characters lying below $J_{\mathbf{G},s}(\chi)$, then:*

(i) ψ is e-cuspidal; and

(ii) $\mathbf{Z}^\circ(\mathbf{G}^*)_{\Phi_e} = \mathbf{Z}^\circ(\mathbf{C}^\circ_{\mathbf{G}^*}(s))_{\Phi_e}$.

Proof. This is [CE99, Proposition 1.10]. □

It is conjectured that the above property of χ, called e-**Jordan cuspidality** in [KM15, Definition 2.1], is equivalent to the notion of e-cuspidality. In particular, Proposition 6.2.11 implies that, if $\chi \in \mathcal{E}(\mathbf{G}^F, [s])$ is e-cuspidal, then $\mathbf{G}^*$ is the only e-split Levi subgroup of $\mathbf{G}^*$ containing $\mathbf{C}^\circ_{\mathbf{G}^*}(s)$.

For every e-cuspidal pair $(\mathbf{L}, \lambda)$, define the associated e-**Harish-Chandra series** $\mathcal{E}(\mathbf{G}^F, (\mathbf{L}, \lambda))$ to be the set of $\chi \in \operatorname{Irr}(\mathbf{G}^F)$ such that $[\chi, \mathbf{R}^{\mathbf{G}}_{\mathbf{L}\leq\mathbf{P}}(\lambda)] \neq 0$, for some parabolic subgroup $\mathbf{P}$ of $\mathbf{G}$ having $\mathbf{L}$ has Levi complement. By Lemma 6.2.4, it follows that e-Harish-Chandra series are contained in rational Lusztig series. More precisely, if $\lambda \in \mathcal{E}(\mathbf{L}^F, [t])$ with $t \in \mathbf{L}^{*F^*}_{\mathrm{ss}}$, then $\mathcal{E}(\mathbf{G}^F, (\mathbf{L}, \lambda)) \subseteq \mathcal{E}(\mathbf{G}^F, [t])$. It is expected that rational Lusztig series are unions of e-Harish-Chandra series and therefore that e-Harish-Chandra series partition the set of irreducible characters of $\mathbf{G}^F$. This has been shown for ordinary Harish-Chandra theory and for e-Harish-Chandra series associated with unipotent e-cuspidal pairs.

Theorem 6.2.12. *Let $\mathbf{G}$ be a connected reductive group with Frobenius endomorphism F and consider a positive integer e.*

(i) The set of 1-Harish-Chandra series partition $\operatorname{Irr}(\mathbf{G}^F)$. More precisely

$$\operatorname{Irr}\left(\mathbf{G}^F\right) = \coprod_{(\mathbf{L},\lambda)} \mathcal{E}\left(\mathbf{G}^F, (\mathbf{L}, \lambda)\right),$$

where the union runs over 1-cuspidal pairs $(\mathbf{L}, \lambda)$ up to $\mathbf{G}^F$-conjugation.

(ii) The set of e-Harish-Chandra series corresponding to unipotent e-cuspidal pairs partition the set of unipotent characters of $\mathbf{G}^F$. More precisely

$$\mathcal{E}\left(\mathbf{G}^F, [1]\right) = \coprod_{(\mathbf{L},\lambda)} \mathcal{E}\left(\mathbf{G}^F, (\mathbf{L}, \lambda)\right),$$

where the union runs over the set of unipotent e-cuspidal pairs $(\mathbf{L}, \lambda)$ up to $\mathbf{G}^F$-conjugation.

Proof. See [DM91, Theorem 6.4] and [BMM93, Theorem 3.2 (1)]. Other references are [GM20, Corollary 3.1.17 and Theorem 4.6.20]. □

In the next chapter we will show that a similar result holds for arbitrary e-Harish-Chandra series under some suitable conditions. In fact we will see that e-Harish-Chandra series can be used to describe Brauer–Lusztig blocks (see Definition 7.3.1) in characteristic ℓ, where e is the multiplicative order of q modulo ℓ.

Given a partition as the ones in Theorem 6.2.12, it is natural to look for a description of every single e-Harish-Chandra series. For every e-cuspidal pair $(\mathbf{L},\lambda)$ of $\mathbf{G}$ set

$$\mathbf{N}_{\mathbf{G}}(\mathbf{L},\lambda)^F := (\mathbf{N}_{\mathbf{G}}(\mathbf{L})^F)_\lambda$$

and define the **relative Weyl group** of $(\mathbf{L},\lambda)$ in $\mathbf{G}$ to be the quotient group

$$\mathbf{W}_{\mathbf{G}}(\mathbf{L},\lambda)^F := \mathbf{N}_{\mathbf{G}}(\mathbf{L},\lambda)^F/\mathbf{L}^F,$$

where $\mathbf{N}_{\mathbf{G}}(\mathbf{L})^F_\lambda$ is the stabilizer of λ in $\mathbf{N}_{\mathbf{G}}(\mathbf{L})^F$. Then, the e-Harish-Chandra series corresponding to the e-cuspidal pair $(\mathbf{L},\lambda)$ can be described in terms of the relative Weyl group $\mathbf{W}_{\mathbf{G}}(\mathbf{L},\lambda)^F$. For ordinary Harish-Chandra theory this result is due to Howlett and Lehrer [HL83].

Theorem 6.2.13. *For every* 1*-cuspidal pair* $(\mathbf{L},\lambda)$ *of* $\mathbf{G}^F$ *and any* 1*-split Levi subgroup* $\mathbf{M}$ *with* $\mathbf{L}\leq\mathbf{M}\leq\mathbf{G}$ *there exists a bijection*

$$I^{\mathbf{M}}_{(\mathbf{L},\lambda)}:\mathrm{Irr}\left(\mathbf{W}_{\mathbf{M}}(\mathbf{L},\lambda)^F\right)\to\mathcal{E}\left(\mathbf{M}^F,(\mathbf{L},\lambda)\right).$$

These bijections can be chosen in such a way that, if extended $\mathbb{Z}$*-linearly, then the following diagram commutes*

$$\begin{array}{ccc}
\mathbb{Z}\mathrm{Irr}\left(\mathbf{W}_{\mathbf{G}}(\mathbf{L},\lambda)^{\mathbf{F}}\right) & \xrightarrow{I^{\mathbf{G}}_{(\mathbf{L},\lambda)}} & \mathbb{Z}\mathcal{E}\left(\mathbf{G}^F,(\mathbf{L},\lambda)\right)\\
\mathrm{Ind}^{\mathbf{G}^F}_{\mathbf{M}^F}\big\uparrow & & \big\uparrow \mathbf{R}^{\mathbf{G}}_{\mathbf{M}}\\
\mathbb{Z}\mathrm{Irr}\left(\mathbf{W}_{\mathbf{M}}(\mathbf{L},\lambda)^{\mathbf{F}}\right) & \xrightarrow[I^{\mathbf{M}}_{(\mathbf{L},\lambda)}]{} & \mathbb{Z}\mathcal{E}\left(\mathbf{M}^F,(\mathbf{L},\lambda)\right)
\end{array}$$

where $\mathrm{Ind}^{\mathbf{G}^\mathrm{F}}_{\mathbf{M}^\mathrm{F}}$ *denotes the induction of characters from* $\mathbf{M}^F$ *to* $\mathbf{G}^F$*.*

Proof. See [HL83] and [GM20, Theorem 3.2.7]. □

In the case of e-Harish-Chandra series corresponding to unipotent e-cuspidal pairs this is due to Broué, Malle and Michel [BMM93].

Theorem 6.2.14. *For every unipotent* e*-cuspidal pair* $(\mathbf{L},\lambda)$ *of* $\mathbf{G}^F$ *and any* e*-split Levi subgroup* $\mathbf{L}\leq\mathbf{M}\leq\mathbf{G}$ *there exist an isometry*

$$I^{\mathbf{M}}_{(\mathbf{L},\lambda)}:\mathbb{Z}\mathrm{Irr}\left(\mathbf{W}_{\mathbf{M}}(\mathbf{L},\lambda)^F\right)\to\mathbb{Z}\mathcal{E}\left(\mathbf{M}^F,(\mathbf{L},\lambda)\right).$$

These isometries can be chosen in such a way that the following diagram commutes

$$\begin{array}{ccc}
\mathbb{Z}\mathrm{Irr}\left(\mathbf{W}_{\mathbf{G}}(\mathbf{L},\lambda)^{\mathbf{F}}\right) & \xrightarrow{I^{\mathbf{G}}_{(\mathbf{L},\lambda)}} & \mathbb{Z}\mathcal{E}\left(\mathbf{G}^F,(\mathbf{L},\lambda)\right)\\
\mathrm{Ind}^{\mathbf{G}^F}_{\mathbf{M}^F}\big\uparrow & & \big\uparrow \mathbf{R}^{\mathbf{G}}_{\mathbf{M}}\\
\mathbb{Z}\mathrm{Irr}\left(\mathbf{W}_{\mathbf{M}}(\mathbf{L},\lambda)^{\mathbf{F}}\right) & \xrightarrow[I^{\mathbf{G}}_{(\mathbf{L},\lambda)}]{} & \mathbb{Z}\mathcal{E}\left(\mathbf{M}^F,(\mathbf{L},\lambda)\right)
\end{array}$$

Proof. See [BMM93, Theorem 3.2 (2)] and [GM20, Theorem 4.6.21]. □

In Chapter 10 we will obtain similar bijections for any e-cuspidal pair when $\mathbf{G}$ has connected center. This is done by applying Theorem 6.2.9 and Theorem 6.2.10 and will therefore require some restrictions on the type of $\mathbf{G}$. It is expected that the bijections from Theorem 6.2.13 and Theorem 6.2.14, and more generally similar bijections for arbitrary e-cuspidal pairs $(\mathbf{L}, \lambda)$, can be chosen to be equivarinat with respect to those automorphism of $\mathbf{G}^F$ stabilizing $(\mathbf{L}, \lambda)$ (see [MS16, Theorem 5.2] and [CS13, Theorem 3.4] for some special cases).

6.2.5 Blocks in nondefining characteristic

In this section we consider Brauer blocks of finite groups of Lie type in nondefining characteristic. All blocks will be considered with respect to the prime ℓ. Let q be a prime power such that $\ell \nmid q$ and let e be the order of q modulo ℓ. A strong connection between the block structure of classical groups of Lie type and the decomposition of Deligne–Lusztig induction has been established by Fong and Srinivasan in [FS86]. These results show that generalized e-Harish-Chandra theory provides a very effective tool to study and classify the blocks of finite groups of Lie type in nondefining characteristic. The work of Fong and Srinivasan on classical groups has been extended to unipotent blocks (i.e. blocks containing a unipotent character) by Broué, Malle and Michel [BMM93] (for large primes ℓ) and by Cabanes and Enguehard [CE94], while the case of arbitrary blocks, for primes $\ell \geq 7$, has been described by Cabanes and Enguehard in [CE99]. In a recent paper by Kessar and Malle [KM15], all of the previous results have been unified and extended to the highest possible generality. Our aim is to introduce the reader to these and other results on ℓ-modular representation theory of finite groups of Lie type that are used in the subsequent chapters.

Let $\mathbf{G}$ be a connected reductive group with a Frobenius $F : \mathbf{G} \to \mathbf{G}$ defining an $\mathbb{F}_q$-structure on $\mathbf{G}$. Let $(\mathbf{G}^*, F^*)$ be a pair dual to $(\mathbf{G}, F)$ and consider a semisimple ℓ-regular element $s \in \mathbf{G}^{*F^*}$. We define the union of rational Lustig series

$$\mathcal{E}_\ell\left(\mathbf{G}^F, [s]\right) := \bigcup_t \mathcal{E}\left(\mathbf{G}^F, [st]\right),$$

where the union runs over the ℓ-elements $t \in \mathbf{C}_{\mathbf{G}^*}(s)^{F^*}$. The next result, due to Broué and Michel, shows that to every ℓ-block B of $\mathbf{G}^F$ is associated a unique $\mathbf{G}^{*F^*}$-conjugacy class of semisimple ℓ-regular elements(see [BM89, Theorem 2.2]).

Theorem 6.2.15. *Let s be a semisimple ℓ-regular element of $\mathbf{G}^{*F^*}$. Then $\mathcal{E}_\ell(\mathbf{G}^F, [s])$ is a union of (characters of) blocks of $\mathbf{G}^F$.*

Next, define the union of rational Lusztig series associated to semisimple ℓ-regular elements of $\mathbf{G}^{*F^*}$

$$\mathcal{E}\left(\mathbf{G}^F, \ell'\right) := \bigcup_{s \in \mathbf{G}^{*F^*}_{\mathrm{ss},\ell'}} \mathcal{E}\left(\mathbf{G}^F, [s]\right).$$

Let s be a semisimple ℓ-regular element of $\mathbf{G}^{*F^*}$ and consider a block B of $\mathbf{G}^F$ such that $\mathrm{Irr}(B) \subseteq \mathcal{E}_\ell(\mathbf{G}^F, [s])$. It was shown by Hiss in his habilitation [Hiß90] (see also [CE04, Theorem 9.12 (ii)]) that $\mathrm{Irr}(B) \cap \mathcal{E}(\mathbf{G}^F, [s]) \neq \emptyset$. Notice also that $\mathrm{Irr}(B) \cap \mathcal{E}(\mathbf{G}^F, [s]) = \mathrm{Irr}(B) \cap \mathcal{E}(\mathbf{G}^F, \ell')$.

Theorem 6.2.16. *Let s be a semisimple ℓ-regular element of $\mathbf{G}^{*F^*}$ and consider a block B of $\mathbf{G}^F$ such that $\mathrm{Irr}(B) \subseteq \mathcal{E}_\ell(\mathbf{G}^F, [s])$. Then $\mathrm{Irr}(B) \cap \mathcal{E}(\mathbf{G}^F, [s]) \neq \emptyset$.*

Suppose now that, for a semisimple ℓ-regular element s of $\mathbf{G}^{*F^*}$, there exists an F-stable Levi subgroup $\mathbf{L}$ of $\mathbf{G}$ satisfying $\mathbf{C}^\circ_{\mathbf{G}^*}(s)\mathbf{C}_{\mathbf{G}^*}(s)^{F^*} \leq \mathbf{L}^*$, where $\mathbf{L}^*$ is the Levi subgroup of $\mathbf{G}^*$ corresponding to $\mathbf{L}$ via duality. By Proposition 6.2.8, and using the fact that $\mathbf{C}_{\mathbf{G}^*}(st) \leq \mathbf{C}_{\mathbf{G}^*}(s)$ whenever t is an ℓ-element of $\mathbf{C}_{\mathbf{G}^*}(s)^{F^*} \leq \mathbf{L}^{*F^*}$, we deduce that Deligne–Lusztig induction yields a bijection

$$\epsilon_{\mathbf{L}}\epsilon_{\mathbf{G}}\mathbf{R}^{\mathbf{G}}_{\mathbf{L}} : \mathcal{E}_\ell\left(\mathbf{L}^F, [s]\right) \to \mathcal{E}_\ell\left(\mathbf{G}^F, [s]\right).$$

In [Bro90], Broué showed that the above bijection preserves the partition of characters into blocks. In fact Broué showed that the above bijection is a perfect isometry [Bro90, Theorem 2.3]. We refere the reader to [Bro90] and [Sam20] for more details on perfect isometries. It was conjectured by Broué that the above mentioned perfect isometry should be a consequence of a Morita equivalence. This conjecture has been proved by Bonnafé and Rouqier [BR03] and by Bonnafé, Dat and Rouquier [BDR17].

Theorem 6.2.17. *Let $\mathbf{L}$ be an F-stable Levi subgroup of $\mathbf{G}$. Let s be a semisimple ℓ-regular element of $\mathbf{L}^{*F^*}$ such that $\mathbf{C}^\circ_{\mathbf{G}^*}(s)\mathbf{C}_{\mathbf{G}^*}(s)^F \leq \mathbf{L}^*$. Then the bijection*

$$\epsilon_{\mathbf{G}}\epsilon_{\mathbf{L}}\mathbf{R}^{\mathbf{G}}_{\mathbf{L}} : \mathcal{E}_\ell\left(\mathbf{L}^F, [s]\right) \to \mathcal{E}_\ell\left(\mathbf{G}^F, [s]\right)$$

satisfies

$$\mathrm{bl}(\lambda_1) = \mathrm{bl}(\lambda_2) \Leftrightarrow \mathrm{bl}\left(\epsilon_{\mathbf{L}}\epsilon_{\mathbf{G}}\mathbf{R}^{\mathbf{G}}_{\mathbf{L}}(\lambda_1)\right) = \mathrm{bl}\left(\epsilon_{\mathbf{L}}\epsilon_{\mathbf{G}}\mathbf{R}^{\mathbf{G}}_{\mathbf{L}}(\lambda_2)\right)$$

for every $\lambda_1, \lambda_2 \in \mathcal{E}_\ell(\mathbf{L}^F, [s])$.

Proof. This is [Bro90, Theorem 2.3]. □

The next statement, which is one of the main ingredients for the parametrization of blocks of groups of Lie type given by Cabanes and Enguehard, shows how Deligne–Lusztig induction can be used to define a twisted induction for blocks.

Theorem 6.2.18. *Assume $\ell \geq 7$ if $\mathbf{G}$ has a component of type $\mathbf{E}_8$ and $\ell \geq 5$ otherwise. Let $\mathbf{L}$ be an e-split Levi subgroup of $\mathbf{G}$ and consider a block b of $\mathbf{L}^F$. Then there exists a block B of $\mathbf{G}^F$ such that*

$$\mathbf{R}^{\mathbf{G}}_{\mathbf{L}\leq\mathbf{P}}(\lambda) \in \mathbb{Z}\mathrm{Irr}(B)$$

for every $\lambda \in \mathrm{Irr}(b) \cap \mathcal{E}(\mathbf{L}^F, \ell')$ and any parabolic subgroup $\mathbf{P}$ of $\mathbf{G}$ of which $\mathbf{L}$ is a Levi complement. Moreover, if $\mathbf{L} = \mathbf{C}^\circ_{\mathbf{G}}(\mathbf{Z}(\mathbf{L})^F_\ell)$, then B coincides with the block $b^{\mathbf{G}^F}$ obtained via Brauer's induction.

Proof. This is [CE99, Theorem 2.5] □

Using the above result, Cabanes and Enguehard have showed in [CE99] that the blocks of finite groups of Lie type, for $\ell \geq 7$, can be parametrized by $\mathbf{G}^F$-conjugacy classes of e-cuspidal pairs $(\mathbf{L}, \lambda)$ where $\lambda \in \mathcal{E}(\mathbf{L}^F, \ell')$. The case of unipotent blocks can be found in [CE94] while a generalization to arbitrary primes and blocks is the main result of [KM15].

Theorem 6.2.19. *Assume $\ell \geq 7$ if $\mathbf{G}$ has a component of type $\mathbf{E}_8$ and $\ell \geq 5$ otherwise. Let e be the order of q modulo ℓ. Then there exists a bijection*

$$(\mathbf{L}, \lambda) \mapsto b_{\mathbf{G}^F}(\mathbf{L}, \lambda)$$

between the set of $\mathbf{G}^F$-conjugacy classes of e-cuspidal pairs $(\mathbf{L}, \lambda)$ with $\lambda \in \mathcal{E}(\mathbf{L}^F, \ell')$ and the set of blocks of $\mathbf{G}^F$. Moreover, we have

$$\operatorname{Irr}\left(b_{\mathbf{G}^F}(\mathbf{L}, \lambda)\right) \cap \mathcal{E}\left(\mathbf{G}^F, \ell'\right) = \left\{\chi \in \operatorname{Irr}\left(\mathbf{G}^F\right) \;\middle|\; (\mathbf{L}, \lambda) \ll_e (\mathbf{G}, \chi)\right\}$$

where $\ll_e$ is the order relation introduced after Definition 7.2.1.

Proof. This is [CE99, Theorem 4.1]. □

7

Brauer–Lusztig Blocks and e-Harish-Chandra Series

Let $\mathbf{G}$ be a connected reductive group and $F : \mathbf{G} \to \mathbf{G}$ a Frobenius endomorphism endowing $\mathbf{G}$ with an $\mathbb{F}_q$-structure for some prime power q. Let ℓ be a prime number not dividing q and denote by e the multiplicative order of q modulo ℓ (modulo 4 if $\ell = 2$). Let $(\mathbf{G}^*, F^*)$ be a dual pair to $(\mathbf{G}, F)$. As we have seen in Section 6.2.4, blocks of finite groups of Lie type have been parametrized by work of Fong–Srinivasan, Cabanes–Enguehard and Kessar–Malle. We recall briefly how this parametrization works. For simplicity assume $\ell \geq 7$. By Theorem 6.2.15, to every ℓ-block B of $\mathbf{G}^F$ is associated a unique rational conjugacy class $[s]$ of semisimple ℓ-regular elements of $\mathbf{G}^{*F^*}$ such that

$$\mathrm{Irr}(B) \subseteq \mathcal{E}_\ell(\mathbf{G}^F, [s]).$$

Then, according to Theorem 6.2.19, there exists a unique $\mathbf{G}^F$-conjugacy class of e-cuspidal pairs $(\mathbf{L}, \lambda)$ such that $\lambda \in \mathcal{E}(\mathbf{L}^F, [s'])$ for some $\mathbf{G}^{*F^*}$-conjugate s' of s and every irreducible constituent of $\mathbf{R}^{\mathbf{G}}_{\mathbf{L} \leq \mathbf{P}}(\lambda)$ is contained in $\mathrm{Irr}(B)$ for every parabolic subgroup $\mathbf{P}$ having $\mathbf{L}$ as Levi complement. In this situation we write

$$B = b_{\mathbf{G}^F}(\mathbf{L}, \lambda).$$

In this case, we also have a characterization of the ℓ'-characters in the block B as

$$\mathcal{E}\left(\mathbf{G}^F, [s]\right) \cap \mathrm{Irr}(B) = \left\{\chi \in \mathrm{Irr}\left(\mathbf{G}^F\right) \;\middle|\; (\mathbf{L}, \lambda) \ll_e (\mathbf{G}, \chi)\right\}, \tag{7.0.1}$$

where $\ll_e$ is the transitive closure of a relation $\leq_e$ defined on the set of e-pairs, i.e. pairs $(\mathbf{M}, \mu)$ with $\mathbf{M}$ an e-split Levi subgroup of $\mathbf{G}$ and $\mu \in \mathrm{Irr}(\mathbf{M}^F)$ (see the discussion following Definition 7.2.1 for more details). The next step is to obtain information on all irreducible characters contained in the block B.

Combining Brauer ℓ-blocks and Lusztig series, Broué, Fong and Srinivasan introduced the so-called **Brauer–Lusztig blocks** of $\mathbf{G}^F$: these are defined to be those nonempty sets of the form

$$\mathcal{E}\left(\mathbf{G}^F, C, [x]\right) := \mathcal{E}\left(\mathbf{G}^F, [x]\right) \cap \mathrm{Irr}\left(C\right),$$

where C is an ℓ-block of $\mathbf{G}^F$ and x is any semisimple element of $\mathbf{G}^{*F^*}$ (see Definition 7.3.1). Using once again Theorem 6.2.15, observe that

$$\mathrm{Irr}(B) = \coprod_{t \in \mathbf{C}_{\mathbf{G}^*}(s)_\ell^{F^*}} \mathcal{E}\left(\mathbf{G}^F, B, [st]\right),$$

where s lies in the rational conjugacy class of semisimple ℓ-regular elements of $\mathbf{G}^{*F^*}$ determined by $\mathrm{Irr}(B) \subseteq \mathcal{E}_\ell(\mathbf{G}^F, [s])$. In particular, in order to obtain all the characters in $\mathrm{Irr}(B)$, we have to describe the Brauer–Lusztig blocks $\mathcal{E}(\mathbf{G}^F, B, [st])$. In [BMM93] Broué, Malle and Michel proved that, on the set of unipotent e-pairs, the relation $\leq_e$ is transitive and therefore coincides with $\ll_e$. This fact was conjectured in full generality in [CE99, 1.11] (see Conjecture 7.2.2). Under this assumption, (7.0.1) can be restated by saying that, when s is ℓ-regular, the Brauer Lusztig block $\mathcal{E}(\mathbf{G}^F, B, [s])$ coincides with the e-Harish-Chandra series $\mathcal{E}(\mathbf{G}^F, (\mathbf{L}, \lambda))$.

In this chapter, we generalize ideas of Broué, Fong and Srinivasan on unipotent blocks and we extend the results of Cabanes and Enguehard. Namely, we remove the condition on the semisimple element s and we show that Brauer–Lusztig blocks are disjoint unions of e-Harish-Chandra series. In particular, this gives a parametrization of all the characters in a block B in terms of e-cuspidal pairs and shows that e-Harish-Chandra series partition $\mathrm{Irr}(\mathbf{G}^F)$.

Theorem 7.1. *Assume Hypothesis 7.2.7. Then, for every Brauer–Lusztig block $\mathcal{E}(\mathbf{G}^F, B, [s])$, there exist e-cuspidal pairs $(\mathbf{L}_i, \lambda_i)$, for $i = 1, \ldots, n$, such that*

$$\mathcal{E}\left(\mathbf{G}^F, B, [s]\right) = \coprod_{i=1}^{n} \mathcal{E}\left(\mathbf{G}^F, (\mathbf{L}_i, \lambda_i)\right).$$

Moreover the $(\mathbf{L}_i, \lambda_i)$ are unique up to $\mathbf{G}^F$-conjugation and $B = \mathrm{bl}(\lambda_i)^{\mathbf{G}^F}$ via Brauer induction.

We believe that the integer n in Theorem 7.1 is always 1 and therefore that Brauer–Lusztig blocks and e-Harish-Chandra series coincide. This issue will be the subject of future investigations and it can be reduced to showing that the reverse implication of Corollary 7.2.12 holds. As an immediate consequence, we obtain a description of all the characters in a block.

Corollary 7.2. *Assume Hypothesis 7.2.7 and let B be a block of $\mathbf{G}^F$. Then*

$$\mathrm{Irr}(B) = \coprod_{(\mathbf{L},\lambda)} \mathcal{E}\left(\mathbf{G}^F, (\mathbf{L}, \lambda)\right),$$

where the union runs over a $\mathbf{G}^F$-transversal in the set of e-cuspidal pairs $(\mathbf{L}, \lambda)$ of $\mathbf{G}$ such that $\mathrm{bl}(\lambda)^{\mathbf{G}^F} = B$.

Notice that Hypothesis 7.2.7 is satisfied whenever $\mathbf{G}$ is simple simply connected such that $\mathbf{G}^F \neq {}^2\mathbf{E}_6(2), \mathbf{E}_7(2), \mathbf{E}_8(2)$ and considering $\ell \in \Gamma(\mathbf{G}, F)$ with $\ell \geq 5$ (see Remark 7.2.8).

As we have mentioned before, Cabanes–Enguehard results have been generalized to bad primes by Kessar and Malle in [KM15] and the reader might wonder why we are not considering this more general situation. Unfortunately, many of the techniques used in this chapter fail for bad primes and a different proof needs to be found in this case.

7.1 Good primes and e-split Levi subgroups

For the rest of this chapter we will consider the following setting.

Notation 7.1.1. Let $\mathbf{G}$ be a connected reductive linear algebraic group defined over an algebraic closure $\mathbb{F}$ of a finite field of characteristic p and $F : \mathbf{G} \to \mathbf{G}$ a Frobenius endomorphism defining an $\mathbb{F}_q$-structure on $\mathbf{G}$, for a power q of p. Consider a prime ℓ different from p and denote by e the multiplicative order of q modulo ℓ (modulo 4 if $\ell = 2$). All blocks are considered with respect to the prime ℓ.

In what follows we will consider some restrictions on the prime ℓ. First, recall that ℓ is a **good prime** for $\mathbf{G}$ if it is good for each simple factor of $\mathbf{G}$, while the conditions for the simple factors are

$$\begin{aligned} \mathbf{A}_n &: \text{every prime is good} \\ \mathbf{B}_n, \mathbf{C}_n, \mathbf{D}_n &: \ell \neq 2 \\ \mathbf{G}_2, \mathbf{F}_4, \mathbf{E}_6, \mathbf{E}_7 &: \ell \neq 2, 3 \\ \mathbf{E}_8 &: \ell \neq 2, 3, 5. \end{aligned}$$

We say that ℓ is a **bad prime** for $\mathbf{G}$ if it is not a good prime. Next, we introduce the set of primes $\Gamma(\mathbf{G}, F)$ that is of fundamental importance in the rest of this thesis (see [CE94, Notation 1.1]).

Definition 7.1.2. We denote by $\gamma(\mathbf{G}, F)$ the set of primes ℓ such that: ℓ is odd, $\ell \neq p$, ℓ is good for $\mathbf{G}$ and ℓ doesn't divide $|\mathbf{Z}(\mathbf{G})^F : \mathbf{Z}^\circ(\mathbf{G})^F|$. Let $(\mathbf{G}^*, F^*)$ be in duality with $(\mathbf{G}, F)$ and set $\Gamma(\mathbf{G}, F) := (\gamma(\mathbf{G}, F) \cap \gamma(\mathbf{G}^*, F^*)) \setminus \{3\}$ if $\mathbf{G}_{\mathrm{ad}}^F$ has a component of type ${}^3\mathbf{D}_4(q^m)$ and $\Gamma(\mathbf{G}, F) := \gamma(\mathbf{G}, F) \cap \gamma(\mathbf{G}^*, F^*)$ otherwise.

Remark 7.1.3. Notice that, if $\ell \in \Gamma(\mathbf{G}, F)$, then $\ell \in \Gamma(\mathbf{G}^*, F^*)$ and $\ell \in \Gamma(\mathbf{H}, F)$, where $\mathbf{H}$ is any F-stable connected reductive subgroup of $\mathbf{G}$ containing an F-stable maximal torus of $\mathbf{G}$ (see [CE04, Proposition 13.12]). In particular, if $\ell \in \Gamma(\mathbf{G}, F)$ and $\mathbf{L}$ is an F-stable Levi subgroup of $\mathbf{G}$, then $\ell \in \Gamma(\mathbf{L}, F)$.

If $\mathbf{G}$ is simple of simply connected type with Frobenius endomorphism F defining an $\mathbb{F}_q$-structure on $\mathbf{G}$, then the primes $\ell \in \Gamma(\mathbf{G}, F)$ are as follows (see [CE04, Table 13.11])

$$\begin{aligned} \mathbf{A}_n(q) &: \ell \nmid 2q(n+1, q-1), \\ {}^2\mathbf{A}_n(q) &: \ell \nmid 2q(n+1, q+1), \\ \mathbf{B}_n(q), \mathbf{C}_n(q), \mathbf{D}_n(q), {}^2\mathbf{D}_n(q) &: \ell \neq 2, p \\ {}^3\mathbf{D}_4(q), \mathbf{G}_2(q), \mathbf{F}_4(q), \mathbf{E}_6(q), {}^2\mathbf{E}_6(q), \mathbf{E}_7(q) &: \ell \neq 2, 3, p \\ \mathbf{E}_8(q) &: \ell \neq 2, 3, 5, p. \end{aligned}$$

As a consequence, if a connected reductive group $\mathbf{G}$ has no simple components of type $\mathbf{A}$, then $\ell \in \Gamma(\mathbf{G}, F)$ if and only if ℓ is good for $\mathbf{G}$ and $\ell \neq p$.

Lemma 7.1.4. *Let $\mathbf{G}$ be a connected reductive group with Frobenius endomorphism F. Let ℓ be a good prime for $\mathbf{G}$. If $\mathbf{G}$ has no simple component of type $\mathbf{A}$, then ℓ does not divide $|\mathbf{Z}(\mathbf{G})^F : \mathbf{Z}^\circ(\mathbf{G})^F|$ nor $|\mathbf{Z}(\mathbf{G}^*)^{F^*} : \mathbf{Z}^\circ(\mathbf{G}^*)^{F^*}|$.*

Proof. This is [CE04, Proposition 13.12]. □

When $\ell \in \Gamma(\mathbf{G}, F)$ some really nice consequences on the structure of e-split Levi subgroups can be drawn. For instance, as pointed out by Broué, Fong and Srinivasan [BFS14], under this assumption one can establish a link between ℓ-elementary abelian chains in $\mathbf{G}^F$ and descending chains of e-split Levi subgroup of $\mathbf{G}$ (see Section 9.2.1). This will be one of the main ingredients used in Chapter 9 to tackle the Character Triple Conjecture for finite groups of Lie type (see Section 9.2.1).

Lemma 7.1.5. *Let $\mathbf{L}$ be an F-stable Levi subgroup of $\mathbf{G}$.*

(i) Let E be a set of positive integers. Then $\mathbf{L}$ is E-split if and only if $\mathbf{L} = \mathbf{C}_{\mathbf{G}}(\mathbf{Z}^\circ(\mathbf{L})_{\Phi_E})$.

(ii) Set $E_{q,\ell} := \{e \cdot \ell^m \mid m \in \mathbb{N}\}$. If $\mathbf{L} = \mathbf{C}^\circ_{\mathbf{G}}\left(\mathbf{Z}^\circ(\mathbf{L})^F_\ell\right)$, then $\mathbf{L}$ is $E_{q,\ell}$-split. The converse holds if $\ell \in \Gamma(\mathbf{G}, F)$.

Proof. The first statement follows directly from the definition. In fact, since $\mathbf{Z}^\circ(\mathbf{L})$ is a torus, we deduce that $\mathbf{Z}^\circ(\mathbf{L})_{\Phi_E}$ is a Φ_E-torus and therefore $\mathbf{C}_{\mathbf{G}}(\mathbf{Z}^\circ(\mathbf{L})_{\Phi_E})$ is E-split. Conversely, assume that $\mathbf{L}$ is E-split. Then there exists a Φ_E-torus $\mathbf{T}$ such that $\mathbf{L} = \mathbf{C}_{\mathbf{G}}(\mathbf{T})$. Since $\mathbf{T}$ is abelian, we deduce that $\mathbf{T} \leq \mathbf{Z}(\mathbf{L})$. Then, as $\mathbf{T}$ is connected, we have $\mathbf{T} \leq \mathbf{Z}^\circ(\mathbf{L})$ and therefore $\mathbf{T} \leq \mathbf{Z}^\circ(\mathbf{L})_{\Phi_E}$ because $\mathbf{T} = \mathbf{T}_{\Phi_E}$. By [DM91, Proposition 1.21], we conclude that $\mathbf{L} = \mathbf{C}_{\mathbf{G}}(\mathbf{Z}^\circ(\mathbf{L})) \leq \mathbf{C}_{\mathbf{G}}(\mathbf{Z}^\circ(\mathbf{L})_{\Phi_E}) \leq \mathbf{C}_{\mathbf{G}}(\mathbf{T}) = \mathbf{L}$. For the second statement see [CE04, Proposition 13.19]. □

Before stating the next proposition, recall that for any finite ℓ-group X and positive integer n we can define the subgroup
$$\Omega_n(X) := \langle x \in X \mid x^{\ell^n} = 1\rangle.$$
In particular, when X is abelian, $\Omega_1(X)$ is the largest ℓ-elementary abelian subgroup of X.

Proposition 7.1.6. *Let Y be an ℓ-subgroup of $\mathbf{G}^F$.*

(i) If ℓ is good for $\mathbf{G}$ and Y is abelian, then $\mathbf{C}^\circ_{\mathbf{G}}(Y)$ is a Levi subgroup.

(ii) If $\ell \in \Gamma(\mathbf{G}, F)$, then:

- *(a) $\mathbf{C}_{\mathbf{G}}(Y)^F = \mathbf{C}^\circ_{\mathbf{G}}(Y)^F$;*
- *(b) if Y is abelian, then $Y \leq \mathbf{Z}^\circ(\mathbf{C}^\circ_{\mathbf{G}}(Y))$;*
- *(c) if Y is abelian and either $\mathbf{Z}(\mathbf{G}_{\mathrm{sc}})^F_\ell = 1$ or $\ell \in \Gamma(\mathbf{G}_{\mathrm{ad}}, F)$, then $\mathbf{C}^\circ_{\mathbf{G}}(Y)$ is an e-split Levi subgroup of $\mathbf{G}$;*
- *(d) if $\mathbf{S}$ is any Φ_e-torus of $\mathbf{G}$, then $\mathbf{S} \leq \mathbf{Z}(\mathbf{G})$ if and only if $\mathbf{S}^F_\ell \leq \mathbf{Z}(\mathbf{G}^F)$ if and only if $\Omega_1(\mathbf{S}^F_\ell) \leq \mathbf{Z}(\mathbf{G}^F)$. Moreover $\mathbf{C}_{\mathbf{G}}(\mathbf{S}) = \mathbf{C}^\circ_{\mathbf{G}}(\mathbf{S}^F_\ell) = \mathbf{C}^\circ_{\mathbf{G}}(\Omega_1(\mathbf{S}^F_\ell))$;*
- *(e) let $\mathbf{L}$ be an e-split Levi subgroup of $\mathbf{G}$ and define $X := \Omega_1\left(\mathbf{Z}^\circ(\mathbf{L})^F_\ell\right)$. Then $\mathbf{L} = \mathbf{C}^\circ_{\mathbf{G}}(\mathbf{Z}(\mathbf{L})^F_\ell) = \mathbf{C}^\circ_{\mathbf{G}}(X)$.*

Proof. The first statement is [CE04, Proposition 13.16 (ii)] while (ii.a) is [CE94, Proposition 2.1 (iii)] (see also [CE04, Proposition 13.16 (i)]). To prove (ii.b) notice that, since Y is abelian and using (ii.a), $Y \leq \mathbf{C}_{\mathbf{G}}(Y)^F = \mathbf{C}^\circ_{\mathbf{G}}(Y)^F$. Then $Y \leq \mathbf{Z}(\mathbf{C}^\circ_{\mathbf{G}}(Y))$. By (i) we know that $\mathbf{C}^\circ_{\mathbf{G}}(Y)$ is a Levi subgroup of $\mathbf{G}$ and hence $\ell \in \Gamma(\mathbf{C}^\circ_{\mathbf{G}}(Y), F)$ by Remark 7.1.3. In particular ℓ does not divide $|\mathbf{Z}(\mathbf{C}^\circ_{\mathbf{G}}(Y))^F : \mathbf{Z}^\circ(\mathbf{C}^\circ_{\mathbf{G}}(Y))^F|$ and so $Y \leq \mathbf{Z}^\circ(\mathbf{C}^\circ_{\mathbf{G}}(Y))$.

Next, consider (ii.c). Set $\mathbf{L} := \mathbf{C}^\circ_{\mathbf{G}}(Y)$ and notice that, using (i) and (ii.b), $\mathbf{L}$ is a Levi subgroup with $Y \leq \mathbf{Z}^\circ(\mathbf{L})$. By [DM91, Proposition 1.21] it follows that $\mathbf{L} = \mathbf{C}^\circ_{\mathbf{G}}(\mathbf{Z}^\circ(\mathbf{L})^F_\ell)$ and Lemma 7.1.5 implies that $\mathbf{L} = \mathbf{C}_{\mathbf{G}}(\mathbf{Z}^\circ(\mathbf{L})_{\Phi_{E_{q,\ell}}})$. Now $\mathbf{L} \leq \mathbf{C}_{\mathbf{G}}(\mathbf{Z}^\circ(\mathbf{L})_{\Phi_e}) =: \mathbf{M}$ and $\mathbf{Z}^\circ(\mathbf{L})_{\Phi_e} \leq \mathbf{Z}^\circ(\mathbf{M})$. Using [CE04, Lemma 22.3 (ii)] (if $\mathbf{Z}(\mathbf{G}_{\mathrm{sc}})^F_\ell = 1$) or [CE94, Proposition 1.6] (if $\ell \in \Gamma(\mathbf{G}_{\mathrm{ad}}, F)$) we conclude that $\mathbf{Z}^\circ(\mathbf{L})_{\Phi_{E_{q,\ell}}} \leq \mathbf{Z}^\circ(\mathbf{M})$ and therefore that $\mathbf{M} = \mathbf{C}_{\mathbf{G}}(\mathbf{Z}^\circ(\mathbf{M})) \leq \mathbf{C}_{\mathbf{G}}\left(\mathbf{Z}^\circ(\mathbf{L})_{\Phi_{E_{q,\ell}}}\right) = \mathbf{L}$. This shows that $\mathbf{L} = \mathbf{M}$ is an e-split Levi subgroup.

We now prove (ii.d). This follows from an adaptation of the proof of [CE04, Proposition 13.17 (ii)]. In order to prove the first part, it is enough to show that, if $\mathbf{S} \nleq \mathbf{Z}(\mathbf{G})$, then $\Omega_1(\mathbf{S}^F_\ell) \nleq \mathbf{Z}(\mathbf{G}^F)$. So assume that $\mathbf{S} \nleq \mathbf{Z}(\mathbf{G})$ and consider the canonical morphism $\pi : \mathbf{G} \to \mathbf{G}/\mathbf{Z}(\mathbf{G})$. Observe, by the proof of [CE04, Proposition 13.7], that $\pi(\mathbf{S}) \neq 1$ is a Φ_e-torus. Moreover, notice that ℓ divides $\Phi_e(q)$ (see [Mal07, Lemma 5.2 (a)]) and that, if ℓ^a is the largest power of ℓ dividing $\Phi_e(q)$, then $\mathbf{T}^F_\ell$ is the direct product of copies of C_{ℓ^a} for every Φ_e-torus $\mathbf{T}$ (see [BM92, Proposition 3.3]). Let $y \in \pi(\mathbf{S})^F_\ell$ be an element of order ℓ^a. Since $\pi(\mathbf{S})^F_\ell = \pi(\mathbf{S}^F_\ell)$ by [CE04, Lemma 13.17 (i)], it follows that there exists $x \in \mathbf{S}^F_\ell$ such that $\pi(x) = y$. Moreover, notice that the order of $y = \pi(x)$ divides the order of x. On the other hand, since $\mathbf{S}$ is a Φ_e-torus, the above discussion implies that the order of x divides ℓ^a. We conclude that x has order ℓ^a. Then $s := x^{\ell^{a-1}} \in \Omega_1(\mathbf{S}^F_\ell)$ and $\pi(s) = y^{\ell^{a-1}} \neq 1$. This shows that $\Omega_1(\mathbf{S}^F_\ell) \nleq \mathbf{Z}(\mathbf{G}^F)$. To prove the second part of (ii.d), we proceed by induction on the dimension of $\mathbf{G}$. Notice that $\mathbf{C}_{\mathbf{G}}(\mathbf{S}) \leq \mathbf{C}^\circ_{\mathbf{G}}(\mathbf{S}^F_\ell) \leq \mathbf{C}^\circ_{\mathbf{G}}(\Omega_1(\mathbf{S}^F_\ell))$ and it's enough to show that $\mathbf{L} := \mathbf{C}^\circ_{\mathbf{G}}(\Omega_1(\mathbf{S}^F_\ell)) \leq \mathbf{C}_{\mathbf{G}}(\mathbf{S})$. Observe that $\mathbf{L}$ is a Levi subgroup by (i) above. If $\mathbf{S} \leq \mathbf{Z}(\mathbf{G})$, then $\mathbf{L} = \mathbf{G} = \mathbf{C}_{\mathbf{G}}(\mathbf{S})$. Therefore, we can assume $\mathbf{S} \nleq \mathbf{Z}(\mathbf{G})$. By the above argument, we know that $\Omega_1(\mathbf{S}^F_\ell) \nleq \mathbf{Z}(\mathbf{G})$ and therefore $\dim(\mathbf{L}) < \dim(\mathbf{G})$. As by Remark 7.1.3 we have $\ell \in \Gamma(\mathbf{L}, F)$, applying the inductive hypothesis we conclude that $\mathbf{C}_{\mathbf{L}}(\mathbf{S}) = \mathbf{C}^\circ_{\mathbf{L}}(\Omega_1(\mathbf{S}^F_\ell))$. Then the result follows by noticing that $\mathbf{C}_{\mathbf{L}}(\mathbf{S}) = \mathbf{C}_{\mathbf{G}}(\mathbf{S})$ and $\mathbf{C}_{\mathbf{L}}(\Omega_1(\mathbf{S}^F_\ell)) = \mathbf{C}^\circ_{\mathbf{G}}(\Omega_1(\mathbf{S}^F_\ell))$.

Now (ii.e) follows from (ii.d). In fact, let $\mathbf{L}$ be an e-split Levi and suppose that $\mathbf{L} = \mathbf{C}_{\mathbf{G}}(\mathbf{S})$ for a Φ_e-torus $\mathbf{S}$. We need to show that $\mathbf{L} = \mathbf{C}^\circ_{\mathbf{G}}(X)$, where $X := \Omega_1(\mathbf{Z}^\circ(\mathbf{L})^F_\ell)$. As $\mathbf{S} \leq \mathbf{Z}(\mathbf{L})$, using (ii.d) we obtain $\Omega_1(\mathbf{S}^F_\ell) \leq \mathbf{Z}(\mathbf{G}^F)$ and $\mathbf{C}^\circ_{\mathbf{G}}(\mathbf{S}) = \mathbf{C}^\circ_{\mathbf{G}}(\Omega_1(\mathbf{S})^F_\ell)$. In particular $\Omega_1(\mathbf{S}^F_\ell) \leq X$. It follows that $\mathbf{L} = \mathbf{C}_{\mathbf{G}}(\mathbf{Z}^\circ(\mathbf{L})) \leq \mathbf{C}^\circ_{\mathbf{G}}(X) \leq \mathbf{C}^\circ_{\mathbf{G}}(\Omega_1(\mathbf{S}^F_\ell)) = \mathbf{C}_{\mathbf{G}}(\mathbf{S}) = \mathbf{L}$. □

7.2 e-Harish-Chandra series and ℓ-blocks

Consider $\mathbf{G}$, F, ℓ and e as in Notation 7.1.1.

Definition 7.2.1. An e-pair of $(\mathbf{G}, F)$ (or simply of $\mathbf{G}$ when no confusion arises) is a pair $(\mathbf{L}, \lambda)$ where $\mathbf{L}$ is an e-split Levi subgroup of $\mathbf{G}$ and $\lambda \in \mathrm{Irr}(\mathbf{L}^F)$. For any semisimple element $s \in \mathbf{G}^{*F^*}$, we say that an e-pair $(\mathbf{L}, \lambda)$ is an (e, s)**-pair** if $\lambda \in \mathcal{E}(\mathbf{L}^F, [s'])$ for some $s' \in \mathbf{L}^{*F^*}$ that is $\mathbf{G}^{*F^*}$-conjugate to s. Finally, we say that $(\mathbf{L}, \lambda)$ is an (e, ℓ')**-pair** if it is an (e, s)-pair for some ℓ-regular semisimple element $s \in \mathbf{G}^{*F^*}$.

In [CE99, Notation 1.11] a binary relation, denoted by $\leq_e$, was defined on the set of e-pairs. Namely, write $(\mathbf{L}, \lambda) \leq_e (\mathbf{K}, \kappa)$ provided that $\mathbf{L} \leq \mathbf{K}$ are e-split Levi subgroups of $\mathbf{G}$ and there exists a parabolic subgroup $\mathbf{P}$ of $\mathbf{K}$ containing $\mathbf{L}$ as a Levi complement such that κ is an irreducible constituent of the virtual character $\mathbf{R}^{\mathbf{K}}_{\mathbf{L}\leq\mathbf{P}}(\lambda)$. Since Deligne–Lusztig induction and restriction send characters to generalized characters, the relation $\leq_e$ might not be transitive. In fact, suppose that $(\mathbf{L}, \lambda) \leq_e (\mathbf{M}, \mu) \leq_e (\mathbf{K}, \kappa)$. Assume for simplicity that Deligne–Lusztig induction does not depend on the choice of parabolic subgroups. By assumption we know that μ is an irreducible constituent of $\mathbf{R}^{\mathbf{M}}_{\mathbf{L}}(\lambda)$ and that κ is an irreducible constituent of $\mathbf{R}^{\mathbf{K}}_{\mathbf{M}}(\mu)$. In particular, we can write $\mathbf{R}^{\mathbf{M}}_{\mathbf{L}}(\lambda) = \Delta + a\mu$, where $0 \neq a \in \mathbb{Z}$ and $[\Delta, \mu] = 0$. Now, by the transitivity of Deligne–Lusztig induction, we deduce that

$$\mathbf{R}^{\mathbf{K}}_{\mathbf{L}}(\lambda) = \mathbf{R}^{\mathbf{K}}_{\mathbf{M}}(\Delta) + a\mathbf{R}^{\mathbf{K}}_{\mathbf{M}}(\mu).$$

Although κ is an irreducible constituent of $\mathbf{R}^{\mathbf{K}}_{\mathbf{M}}(\mu)$, since $\mathbf{R}^{\mathbf{K}}_{\mathbf{M}}(\Delta)$ is a generalized character, it might happen that $[\mathbf{R}^{\mathbf{K}}_{\mathbf{M}}(\Delta), \kappa] = -a[\mathbf{R}^{\mathbf{K}}_{\mathbf{M}}(\mu), \kappa]$ and hence that $[\mathbf{R}^{\mathbf{K}}_{\mathbf{L}}(\lambda), \kappa] = 0$.

In order to overcome this problem, we consider the transitive closure $\ll_e$ of $\leq_e$. Since the set of e-pairs of $(\mathbf{G}, F)$ is finite, we deduce that, for two e-pairs $(\mathbf{L}, \lambda)$ and $(\mathbf{K}, \kappa)$, we have $(\mathbf{L}, \lambda) \ll_e (\mathbf{K}, \kappa)$ if and only if there exists a finite number of e-pairs $(\mathbf{L}_i, \lambda_i)$, with $i = 1, \dots, n$, such that

$$(\mathbf{L}, \lambda) \leq_e (\mathbf{L}_1, \lambda_1) \leq_e \cdots \leq_e (\mathbf{L}_n, \lambda_n) \leq_e (\mathbf{K}, \kappa).$$

Observe that a pair $(\mathbf{L}, \lambda)$ is e-cuspidal (see the discussion preceding Proposition 6.2.11) if and only if it is minimal with respect to $\ll_e$. Moreover, by using Lemma 6.2.4, the relations $\leq_e$ and $\ll_e$ restrict to the set of (e, s)-pairs for every $s \in \mathbf{G}^{*F^*}_{\mathrm{ss}}$. A minimal element in the induced poset of (e, s)-cuspidal pairs is called (e, s)-**cuspidal**.

The following conjecture was made in [CE99, 1.11].

Conjecture 7.2.2 (Cabanes–Enguehard Conjecture)**.** *The relation $\leq_e$ is transitive and therefore coincides with $\ll_e$.*

We point out an important consequence of Conjecture 7.2.2. Let $(\mathbf{L}, \lambda)$ be an e-pair of $\mathbf{G}$. If Conjecture 7.2.2 holds, then

$$\left\{\chi \in \mathrm{Irr}\left(\mathbf{G}^F\right) \;\middle|\; (\mathbf{L}, \lambda) \ll_e (\mathbf{G}, \chi)\right\} = \mathcal{E}\left(\mathbf{G}^F, (\mathbf{L}, \lambda)\right),$$

where $\mathcal{E}(\mathbf{G}^F, (\mathbf{L}, \lambda))$ is the e-Harish-Chandra series determined by $(\mathbf{L}, \lambda)$, that is the set of irreducible constituents of $\mathbf{R}^{\mathbf{G}}_{\mathbf{L}\leq\mathbf{P}}(\lambda)$ for every parabolic subgroup $\mathbf{P}$ of $\mathbf{G}$ having $\mathbf{L}$ as a Levi complement. In addition, if Deligne–Lusztig induction does not depend on the choice of a parabolic subgroup, then

$$\left\{\chi \in \mathrm{Irr}\left(\mathbf{G}^F\right) \;\middle|\; (\mathbf{L}, \lambda) \ll_e (\mathbf{G}, \chi)\right\} = \mathrm{Irr}\left(\mathbf{R}^{\mathbf{G}}_{\mathbf{L}}(\lambda)\right),$$

where we recall that, for any finite group X and $\chi \in \mathbb{Z}\mathrm{Irr}(X)$, we denote by $\mathrm{Irr}(\chi)$ the set of irreducible constituent of χ. Since this remark will be used many times in the sequel, we introduce the following condition.

Condition 7.2.3. Consider $\mathbf{G}$, F, ℓ and e as in Notation 7.1.1 and assume that Deligne–Lusztig induction does not depend on the choice of parabolic subgroups and

$$\left\{\kappa \in \mathrm{Irr}\left(\mathbf{K}^F\right) \;\middle|\; (\mathbf{L},\lambda) \ll_e (\mathbf{K},\kappa)\right\} = \mathrm{Irr}\left(\mathbf{R}_{\mathbf{L}}^{\mathbf{K}}(\lambda)\right)$$

for every e-split Levi subgroup $\mathbf{K}$ of $\mathbf{G}$ and every (e,ℓ')-cuspidal pair $(\mathbf{L},\lambda)$ of $\mathbf{K}$.

Observe that Conjecture 7.2.2 is known for $(e,1)$-pairs by [BMM93, 3.11] while Condition 7.2.3 has been proved for $\mathbf{G}$ simple of exceptional simply connected type and good primes in [Hol22, Theorem 1.1]. Exceptional simple groups and bad primes have been considered in [KM13, Theorem 1.4]. Moreover Condition 7.2.3 is known to hold for groups with connected center and good primes $\ell \geq 5$ by [Eng13, Proposition 2.2.4]. We extend these results and show that Condition 7.2.3 holds for every simply connected reductive group and good primes $\ell \geq 5$. Notice that our proof does not depend on [Eng13] in any way.

The following result is well known to the experts.

Lemma 7.2.4. *Let $\mathbf{L}$ be an e-split Levi subgroup of a connected reductive group $\mathbf{G}$ and consider $\mathbf{G}_0 := [\mathbf{G},\mathbf{G}]$ and $\mathbf{L}_0 := \mathbf{L} \cap \mathbf{G}_0$.*

(i) Let $\lambda \in \mathrm{Irr}(\mathbf{L}_0^F)$ and $\chi_0 \in \mathrm{Irr}(\mathbf{G}_0^F)$. If $(\mathbf{L}_0,\lambda_0) \leq_e (\mathbf{G}_0,\chi_0)$ and $\chi \in \mathrm{Irr}(\mathbf{G}^F \mid \chi_0)$, then there exists $\lambda \in \mathrm{Irr}(\mathbf{L}^F \mid \lambda_0)$ such that $(\mathbf{L},\lambda) \leq_e (\mathbf{G},\chi)$.

(ii) Let $\lambda \in \mathrm{Irr}(\mathbf{L}^F)$ and $\chi \in \mathrm{Irr}(\mathbf{G}^F)$. If $(\mathbf{L},\lambda) \leq_e (\mathbf{G},\chi)$ and $\lambda_0 \in \mathrm{Irr}(\lambda_{\mathbf{L}_0^F})$, then there exists $\chi_0 \in \mathrm{Irr}(\chi_{\mathbf{G}_0^F})$ such that $(\mathbf{L}_0,\lambda_0) \leq_e (\mathbf{G}_0,\chi_0)$.

Proof. First observe that $\mathbf{L}_0$ is an e-split Levi subgroup of $\mathbf{G}_0$. By [GM20, Proposition 3.3.24] (see also the proof of [GM20, Corollary 3.3.25]) and since $\mathbf{G} = \mathbf{Z}^\circ(\mathbf{G})\mathbf{G}_0$, it follows that

$$\mathbf{R}_{\mathbf{L}}^{\mathbf{G}} \circ \mathrm{Ind}_{\mathbf{L}_0^F}^{\mathbf{L}^F} = \mathrm{Ind}_{\mathbf{G}_0^F}^{\mathbf{G}^F} \circ \mathbf{R}_{\mathbf{L}_0}^{\mathbf{G}_0} \tag{7.2.1}$$

and

$$^*\mathbf{R}_{\mathbf{L}_0}^{\mathbf{G}_0} \circ \mathrm{Res}_{\mathbf{G}_0^F}^{\mathbf{G}^F} = \mathrm{Res}_{\mathbf{L}_0^F}^{\mathbf{L}^F} \circ {}^*\mathbf{R}_{\mathbf{L}}^{\mathbf{G}} \tag{7.2.2}$$

Suppose first that $(\mathbf{L}_0,\lambda_0) \leq_e (\mathbf{G}_0,\chi_0)$ and consider $\chi \in \mathrm{Irr}(\mathbf{G}_0^F \mid \chi_0)$. Then χ is an irreducible constituent of $\mathrm{Ind}_{\mathbf{G}_0^F}^{\mathbf{G}^F}(\mathbf{R}_{\mathbf{L}_0}^{\mathbf{G}_0}(\lambda_0))$ and by (7.2.1) we can find $\lambda \in \mathrm{Irr}(\mathbf{L}^F \mid \lambda_0)$ such that $(\mathbf{L},\lambda) \leq_e (\mathbf{G},\chi)$.

Suppose now that $(\mathbf{L},\lambda) \leq_e (\mathbf{G},\chi)$ and let λ_0 be an irreducible constituent of $\lambda_{\mathbf{L}_0^F}$. Since Deligne–Lusztig induction and restriction are adjoint with respect to the usual scalar product, we deduce that λ_0 is an irreducible constituent of $\mathrm{Res}_{\mathbf{L}_0^F}^{\mathbf{L}^F}({}^*\mathbf{R}_{\mathbf{L}}^{\mathbf{G}}(\chi))$. By (7.2.2) there exists $\chi_0 \in \mathrm{Irr}(\chi_{\mathbf{G}_0^F})$ such that λ_0 is a constituent of ${}^*\mathbf{R}_{\mathbf{L}_0}^{\mathbf{G}_0}(\chi_0)$ and therefore $(\mathbf{L}_0,\lambda_0) \leq_e (\mathbf{G}_0,\chi_0)$. □

The following result shows that Condition 7.2.3 holds when $\mathbf{G}$ has only components of classical types or when $\mathbf{G}$ is simple, $\mathbf{K} = \mathbf{G}$ and λ lies in a rational Lusztig series associated with a quasi-isolated element. Recall that a semisimple element s of a reductive group $\mathbf{G}$ is called **quasi-isolated** if $\mathbf{C}_{\mathbf{G}}(s)$ is not contained in any proper Levi subgroup of $\mathbf{G}$.

Lemma 7.2.5. *Let $\mathbf{G}$ be connected reductive, $\chi \in \mathrm{Irr}(\mathbf{G}^F)$ and consider an e-cuspidal pair $(\mathbf{L},\lambda) \ll_e (\mathbf{G},\chi)$, where $\lambda \in \mathcal{E}(\mathbf{L}^F,[s])$ for some $s \in \mathbf{L}^{*F^*}_{\mathrm{ss},\ell'}$. Suppose that $\ell \geq 5$ is good for $\mathbf{G}$ and that the Mackey formula holds for $(\mathbf{G},F)$. If either $\mathbf{G}$ has only components of classical types and F does not induce the triality automorphism on components of type $\mathbf{D}_4$ or $\mathbf{G}$ is simple and s is quasi-isolated in $\mathbf{G}^*$, then $(\mathbf{L},\lambda) \leq_e (\mathbf{G},\chi)$.*

Proof. Consider a regular embedding $i : \mathbf{G} \to \widetilde{\mathbf{G}}$. By applying Theorem 6.2.9 and [GM20, Corollary 4.7.8] to $\widetilde{\mathbf{G}}$, it follows that Conjecture 7.2.2 holds in $\widetilde{\mathbf{G}}$ unless s is quasi-isolated in $\mathbf{G}$ and $\mathbf{G}$ is simple of simply connected type $\mathbf{E}_6$ or $\mathbf{E}_7$ or $\mathbf{G}^F = {}^3\mathbf{D}_4(q)$. However, in these excluded cases the result holds by [Hol22, Theorem 1.1] and we can therefore assume that Conjecture 7.2.2 holds in $\widetilde{\mathbf{G}}$. Now, [CE99, Proposition 5.2] shows that

$$\left\{\psi \in \mathrm{Irr}(\mathbf{G}^F) \;\middle|\; (\mathbf{L},\lambda) \leq_e (\mathbf{G},\psi)\right\} = \mathrm{Irr}\left(b_{\mathbf{G}^F}(\mathbf{L},\lambda)\right) \cap \mathcal{E}\left(\mathbf{G}^F,\ell'\right) \tag{7.2.3}$$

while

$$\left\{\psi \in \mathrm{Irr}(\mathbf{G}^F) \;\middle|\; (\mathbf{L},\lambda) \ll_e (\mathbf{G},\psi)\right\} = \mathrm{Irr}\left(b_{\mathbf{G}^F}(\mathbf{L},\lambda)\right) \cap \mathcal{E}\left(\mathbf{G}^F,\ell'\right) \tag{7.2.4}$$

according to Theorem 6.2.19. Combining (7.2.3) and (7.2.4) the result follows. □

We can now prove our claimed result. For a connected reductive group $\mathbf{G}$, we say that $\mathbf{G}$ is simply connected (resp. adjoint) if the semisimple group $[\mathbf{G},\mathbf{G}]$ is simply connected (resp. adjoint).

Proposition 7.2.6. *Let $\mathbf{G}$ be a simply connected reductive group, $\chi \in \mathrm{Irr}(\mathbf{G}^F)$ and consider an (e,ℓ')-cuspidal pair $(\mathbf{L},\lambda) \ll_e (\mathbf{G},\chi)$. If $\ell \geq 5$ is good for $\mathbf{G}$ and the Mackey formula holds for $(\mathbf{G},F)$, then $(\mathbf{L},\lambda) \leq_e (\mathbf{G},\chi)$.*

Proof. Let $(\mathbf{G}^*,F^*)$ be dual to $(\mathbf{G},F)$ and let $\mathbf{L}^*$ be the e-split Levi subgroup of $\mathbf{G}^*$ corresponding to $\mathbf{L}$. Consider $s \in \mathbf{L}^{*F^*}_{\mathrm{ss},\ell'}$ such that $\lambda \in \mathcal{E}(\mathbf{L}^F,[s])$ and notice that $\chi \in \mathcal{E}(\mathbf{G}^F,[s])$ because $(\mathbf{L},\lambda) \ll_e (\mathbf{G},\chi)$ (see Lemma 6.2.4). By induction on $\dim(\mathbf{G})$, we claim that s is quasi-isolated in $\mathbf{G}^*$. Suppose that $\mathbf{G}_1$ is a proper F-stable Levi subgroup of $\mathbf{G}$ such that $\mathbf{C}_{\mathbf{G}^*}(s) \leq \mathbf{G}_1^*$. Observe that $\mathbf{G}_1$ is simply connected by [MT11, Proposition 12.14]. Set $\mathbf{L}_1^* := \mathbf{C}_{\mathbf{G}_1^*}(\mathbf{Z}^\circ(\mathbf{L}^*)_{\Phi_e}) = \mathbf{L}^* \cap \mathbf{G}_1^*$ and notice that its dual $\mathbf{L}_1 \leq \mathbf{L}$ is an e-split Levi subgroup of $\mathbf{G}_1$ and that $\mathbf{C}_{\mathbf{L}^*}(s) \leq \mathbf{L}^* \cap \mathbf{G}_1^* = \mathbf{L}_1^*$. By Proposition 6.2.8 there exist unique $\lambda_1 \in \mathcal{E}(\mathbf{L}_1^F,[s])$ and $\chi_1 \in \mathcal{E}(\mathbf{G}_1^F,[s])$ such that $\lambda = \pm\mathbf{R}^{\mathbf{L}}_{\mathbf{L}_1}(\lambda_1)$ and $\chi = \pm\mathbf{R}^{\mathbf{G}}_{\mathbf{G}_1}(\chi_1)$. Since $(\mathbf{L},\lambda) \ll_e (\mathbf{G},\chi)$, it follows by the transitivity of Deligne–Lusztig induction that $(\mathbf{L}_1,\lambda_1) \ll_e (\mathbf{G}_1,\chi_1)$. A similar argument also shows that λ_1 is e-cuspidal. Since $\dim(\mathbf{G}_1) < \dim(\mathbf{G})$, we obtain $(\mathbf{L}_1,\lambda_1) \leq_e (\mathbf{G}_1,\chi_1)$. This shows that χ_1 is an irreducible constituent of $\mathbf{R}^{\mathbf{G}_1}_{\mathbf{L}_1}(\lambda_1)$ and, because all constituents of $\mathbf{R}^{\mathbf{G}_1}_{\mathbf{L}_1}(\lambda_1)$ are contained in $\mathcal{E}(\mathbf{G}_1^F,[s])$ and $\mathbf{R}^{\mathbf{G}}_{\mathbf{G}_1}$ induces a bijection between $\mathcal{E}(\mathbf{G}_1^F,[s])$ and $\mathcal{E}(\mathbf{G}^F,[s])$, we conclude that χ is an irreducible constituent of $\pm\mathbf{R}^{\mathbf{G}}_{\mathbf{G}_1}(\mathbf{R}^{\mathbf{G}_1}_{\mathbf{L}_1}(\lambda_1)) = \pm\mathbf{R}^{\mathbf{G}}_{\mathbf{L}}(\lambda)$. Hence $(\mathbf{L},\lambda) \leq_e (\mathbf{G},\chi)$ and we may assume that s is quasi-isolated in $\mathbf{G}^*$.

Let $\mathbf{G}_0 := [\mathbf{G},\mathbf{G}]$ and $\mathbf{L}_0 := \mathbf{L} \cap \mathbf{G}_0$. By assumption, there exist e-split Levi subgroups $\mathbf{L}_i$ of $\mathbf{G}$ containing $\mathbf{L}$ and characters $\lambda_i \in \mathrm{Irr}(\mathbf{L}_i^F)$ such that $(\mathbf{L},\lambda) \leq_e (\mathbf{L}_1,\lambda_1) \leq_e \cdots \leq_e (\mathbf{G},\chi)$. If we define $\mathbf{L}_{i,0} := \mathbf{L}_i \cap \mathbf{G}_0$, then a repeated application of Lemma 7.2.4 yields characters $\lambda_0 \in \mathrm{Irr}(\lambda_{\mathbf{L}_0^F})$, $\lambda_{i,0} \in \mathrm{Irr}(\lambda_{i,\mathbf{L}_{i,0}^F})$ and $\chi_0 \in \mathrm{Irr}(\chi_{\mathbf{G}_0^F})$ such that $(\mathbf{L}_0,\lambda_0) \leq_e (\mathbf{L}_{1,0},\lambda_{1,0}) \leq_e \cdots \leq_e (\mathbf{G}_0,\chi_0)$. Then $(\mathbf{L}_0,\lambda_0) \ll_e (\mathbf{G}_0,\chi_0)$ with $(\mathbf{L}_0,\lambda_0)$ an (e,ℓ')-cuspidal pair. Moreover, if the result is true for

$\mathbf{G}_0$, then $(\mathbf{L}_0, \lambda_0) \leq_e (\mathbf{G}_0, \chi_0)$ and using Lemma 7.2.4 we find $\lambda' \in \mathrm{Irr}(\mathbf{L}^F \mid \lambda_0)$ such that $(\mathbf{L}, \lambda') \leq_e (\mathbf{G}, \chi)$. Then Theorem 6.2.19 shows that $\lambda'^g = \lambda$, for some $g \in \mathbf{N}_\mathbf{G}(\mathbf{L})^F$, and hence $(\mathbf{L}, \lambda) = (\mathbf{L}, \lambda')^g \leq_e (\mathbf{G}, \chi)^g = (\mathbf{G}, \chi)$. Notice that the inclusion $\mathbf{G}_0 \to \mathbf{G}$ induces a dual morphism $\mathbf{G}^* \to \mathbf{G}_0^*$ and that, if $s \in \mathbf{G}_{\mathrm{ss}}^{*F^*}$ is quasi-isolated, then the corresponding element $s_0 \in \mathbf{G}_{0,\mathrm{ss}}^{*F^*}$ is quasi-isolated by [Bon05, Proposition 2.3]. Without loss of generality we can assume $\mathbf{G} = [\mathbf{G}, \mathbf{G}]$.

Now, $\mathbf{G}$ is a direct product of simple algebraic groups $\mathbf{H}_1, \ldots, \mathbf{H}_n$ (see [Mar91, Proposition 1.4.10]). The action of F induces a permutation on the set of simple components $\mathbf{H}_i$. For every orbit of F we denote by $\mathbf{G}_j$, $j = 1, \ldots, t$, the direct product of simple components in such orbit. Then $\mathbf{G}_j$ is F-stable and

$$\mathbf{G}^F = \mathbf{G}_1^F \times \cdots \times \mathbf{G}_t^F.$$

If $\mathbf{H}_{i_j}$ is a simple component of $\mathbf{G}_j$ and n_j is the size of the F-orbit of $\mathbf{H}_{i_j}$, then we have an isomorphism

$$\mathbf{G}_j^F \simeq \mathbf{H}_{i_j}^{F^{n_j}}. \tag{7.2.5}$$

Define $\mathbf{L}_j := \mathbf{L} \cap \mathbf{G}_j$ and observe that $\mathbf{L}_j$ is an e-split Levi subgroup of $\mathbf{G}_j$ and that

$$\mathbf{L}^F = \mathbf{L}_1^F \times \cdots \times \mathbf{L}_j^F.$$

Then we can write $\chi = \chi_1 \times \cdots \times \chi_t$ and $\lambda = \lambda_1 \times \cdots \times \lambda_t$, with $\chi_j \in \mathrm{Irr}(\mathbf{G}_j^F)$ and $\lambda_j \in \mathrm{Irr}(\mathbf{L}_j^F)$. Since $\mathbf{R}_\mathbf{L}^\mathbf{G} = \mathbf{R}_{\mathbf{L}_1}^{\mathbf{G}_1} \times \cdots \times \mathbf{R}_{\mathbf{L}_t}^{\mathbf{G}_t}$ (see [DM91, Proposition 10.9 (ii)]), eventually considering intermediate e-split Levi subgroups, the fact that $(\mathbf{L}, \lambda) \ll_e (\mathbf{G}, \chi)$ implies that $(\mathbf{L}_j, \lambda_j) \ll_e (\mathbf{G}_j, \chi_j)$ for every j. Noticing that $\mathbf{G}^{*F^*} = \mathbf{G}_1^{*F^*} \times \cdots \times \mathbf{G}_t^{*F^*}$, we can write $s = s_1 \times \cdots \times s_t$ for some ℓ-regular semisimple elements $s_j \in \mathbf{G}_j^{*F^*}$. Moreover, since s is quasi-isolated in $\mathbf{G}^*$, it follows that s_j is quasi-isolated in $\mathbf{G}_j^*$. Finally, by (7.2.5) and Lemma 7.2.5, it follows that the result holds in $\mathbf{G}_j$ and so $(\mathbf{L}_j, \lambda_j) \leq_e (\mathbf{G}_j, \chi_j)$. From this, we conclude that $(\mathbf{L}, \lambda) \leq_e (\mathbf{G}, \chi)$. □

Since the hypotheses of the above proposition are inherited by Levi subgroups, it follows that Condition 7.2.3 holds whenever $\mathbf{G}$ is a simply connected reductive group.

In the sequel we will assume the following conditions.

Hypothesis 7.2.7. Let $\mathbf{G}$, $F : \mathbf{G} \to \mathbf{G}$, ℓ and e be as in Notation 7.1.1. Assume that:

(i) $\ell \in \Gamma(\mathbf{G}, F)$ with $\ell \geq 5$ and the Mackey formula hold for $(\mathbf{G}, F)$;

(ii) either $\mathbf{Z}((\mathbf{G}^*)_{\mathrm{sc}}^{F^*})_\ell = 1$ or $\ell \in \Gamma((\mathbf{G}^*)_{\mathrm{ad}}, F)$; and

(iii) Condition 7.2.3 holds for $(\mathbf{G}, F)$.

Notice that under Hypothesis 7.2.7 (i), Deligne–Lusztig induction and restriction do not depend on the choice of parabolic subgroups (see the comment following [DM91, Theorem 11.13]). For this reason, in what follows we will usually write $\mathbf{R}_\mathbf{L}^\mathbf{G}$ (resp. ${}^*\mathbf{R}_\mathbf{L}^\mathbf{G}$) instead of $\mathbf{R}_{\mathbf{L} \leq \mathbf{P}}^\mathbf{G}$ (resp. ${}^*\mathbf{R}_{\mathbf{L} \leq \mathbf{P}}^\mathbf{G}$). Regarding the validity of the Mackey formula we refer the reader to Theorem 6.2.2 (see also [BM11] and [Tay18]).

In the following remark we show that Hypothesis 7.2.7 is satisfied in most of the cases we are interested in.

Remark 7.2.8. Let $\mathbf{G}$ be simple of simply connected type and consider a Forbenius endomorphism F of $\mathbf{G}$ associated with an $\mathbb{F}_q$-structure. Suppose that $\mathbf{G}^F \neq {}^2\mathbf{E}_6(2), \mathbf{E}_7(2), \mathbf{E}_8(2)$ and that $\ell \in \Gamma(\mathbf{G}, F)$ with $\ell \geq 5$. Then Hypothesis 7.2.7 is satisfied. In fact, under our assumption, the Mackey formula holds by [BM11] and [Tay18] while Condition 7.2.3 holds by Proposition 7.2.6. This shows that Hypothesis 7.2.7 (i) and (iii) are satisfied. Moreover, since $\mathbf{G}$ is simple and simply connected, our assumption on ℓ shows that $\ell \in \Gamma((\mathbf{G}^*)_{\mathrm{ad}}, F)$ (see [CE04, Table 13.11]). Notice that in this case we also have $\ell \in \Gamma(\mathbf{G}_{\mathrm{ad}}, F)$.

We now start working towards a proof of our main result. The next result shows how to associate to every (e, s)-pair an $(e, s_{\ell'})$-pair via Jordan decomposition. This will be used to extend some of the results of [CE99] from (e, ℓ')-pairs to arbitrary e-pairs.

Lemma 7.2.9. *Assume Hypothesis 7.2.7 (i)-(ii).*

(i) If $(\mathbf{L}, \lambda)$ is an (e, s)-pair of $\mathbf{G}$ with $s \in \mathbf{L}^*$, then there exists an $(e, s_{\ell'})$-pair $(\mathbf{L}(s_\ell), \lambda(s_\ell))$ and a linear character $\widehat{s_\ell}$ of $\mathbf{L}(s_\ell)^F$ such that $\lambda = \epsilon_{\mathbf{L}}\epsilon_{\mathbf{L}(s_\ell)}\mathbf{R}^{\mathbf{L}}_{\mathbf{L}(s_\ell)}(\lambda(s_\ell) \cdot \widehat{s_\ell})$.

(ii) If $(\mathbf{L}, \lambda)$ is (e, s)-cuspidal, then $(\mathbf{L}(s_\ell), \lambda(s_\ell))$ is $(e, s_{\ell'})$-cuspidal. In this case $\mathbf{L} = \mathbf{L}(s_\ell)$.

Proof. Under our assumptions, Proposition 7.1.6 implies that $\mathbf{C}^\circ_{\mathbf{G}^*}(s_\ell)$ is an e-split Levi subgroup of $\mathbf{G}^*$. If $\mathbf{T}^*$ is an F^*-stable maximal torus of $\mathbf{L}^*$ such that $s_\ell \in \mathbf{T}^*$, then $\mathbf{T}^*$ is a maximal torus of $\mathbf{C}^\circ_{\mathbf{G}^*}(s_\ell)$ and Lemma 6.1.8 implies that $\mathbf{C}^\circ_{\mathbf{L}^*}(s_\ell) = \mathbf{C}^\circ_{\mathbf{G}^*}(s_\ell) \cap \mathbf{L}^*$ is an e-split Levi subgroup of $\mathbf{G}^*$. As $\ell \in \Gamma(\mathbf{G}, F)$, Remark 7.1.3 implies that $\ell \in \Gamma(\mathbf{L}^*, F^*)$ and therefore $\mathbf{C}^\circ_{\mathbf{L}^*}(s_\ell)^F = \mathbf{C}_{\mathbf{L}^*}(s_\ell)^F$ by Proposition 7.1.6 (ii.a). Recalling that $\langle s_\ell \rangle \leq \langle s \rangle$, it follows that $\mathbf{C}^\circ_{\mathbf{L}^*}(s)\mathbf{C}_{\mathbf{L}^*}(s)^F \subseteq \mathbf{C}^\circ_{\mathbf{L}^*}(s_\ell)\mathbf{C}_{\mathbf{L}^*}(s_\ell)^F = \mathbf{C}^\circ_{\mathbf{L}^*}(s_\ell)$. Let $\mathbf{L}(s_\ell)$ be an e-split Levi subgroup of $\mathbf{G}$ in duality with $\mathbf{C}^\circ_{\mathbf{L}^*}(s_\ell)$. By Proposition 6.2.7 and Proposition 6.2.8 there exists a unique character $\lambda(s_\ell) \in \mathcal{E}(\mathbf{L}(s_\ell)^F, [s_{\ell'}])$ such that

$$\lambda = \epsilon_{\mathbf{L}}\epsilon_{\mathbf{L}(s_\ell)}\mathbf{R}^{\mathbf{L}}_{\mathbf{L}(s_\ell)}\left(\widehat{s_\ell} \cdot \lambda(s_\ell)\right),$$

where $\widehat{s_\ell}$ is the linear character corresponding to $s_\ell \in \mathbf{Z}(\mathbf{C}_{\mathbf{L}^*}(s_\ell)^{F^*})$ (see Proposition 6.2.7). This proves (i). Assume now that $(\mathbf{L}, \lambda)$ is (e, s)-cuspidal. Then Proposition 6.2.11 implies that $\mathbf{L}^*$ is the largest e-split Levi subgroup containing $\mathbf{C}^\circ_{\mathbf{G}^*}(s)$. Therefore $\mathbf{L}(s_\ell) = \mathbf{L}$ and $\lambda = \widehat{s_\ell} \cdot \lambda(s_\ell)$. It follows that $(\mathbf{L}(s_\ell), \lambda(s_\ell))$ is $(e, s_{\ell'})$-cuspidal. □

Next, we show that the relation $\ll_e$ is preserved under the construction of Lemma 7.2.9.

Lemma 7.2.10. *Assume Hypothesis 7.2.7 (i)-(ii). Let $(\mathbf{L}, \lambda)$ and $(\mathbf{K}, \kappa)$ be two (e, s)-cuspidal pairs and consider the corresponding $(e, s_{\ell'})$-cuspidal pairs $(\mathbf{L}(s_\ell), \lambda(s_\ell))$ and $(\mathbf{K}(s_\ell), \kappa(s_\ell))$ given by Lemma 7.2.9. If $(\mathbf{L}, \lambda) \ll_e (\mathbf{K}, \kappa)$, then $(\mathbf{L}(s_\ell), \lambda(s_\ell)) \ll_e (\mathbf{K}(s_\ell), \kappa(s_\ell))$.*

Proof. Without loss of generality we may assume $s \in \mathbf{L}^*$. Since $(\mathbf{L}, \lambda) \ll_e (\mathbf{K}, \kappa)$, there exist (e, s)-pairs $(\mathbf{L}_i, \lambda_i)$, for $i = 1, \dots, n$, such that

$$(\mathbf{L}, \lambda) = (\mathbf{L}_1, \lambda_1) \leq_e \cdots \leq_e (\mathbf{L}_n, \lambda_n) = (\mathbf{K}, \kappa).$$

For $i = 1, \dots, n$, consider the $(e, s_{\ell'})$-pair $(\mathbf{L}_i(s_\ell), \lambda_i(s_\ell))$ given by Lemma 7.2.9. If we show that $(\mathbf{L}_i(s_\ell), \lambda_i(s_\ell)) \leq_e (\mathbf{L}_{i+1}(s_\ell), \lambda_{i+1}(s_\ell))$, then we obtain $(\mathbf{L}(s_\ell), \lambda(s_\ell)) \ll_e (\mathbf{K}(s_\ell), \kappa(s_\ell))$.

Since $(\mathbf{L}_i, \lambda_i) \leq_e (\mathbf{L}_{i+1}, \lambda_{i+1})$, we know that λ_{i+1} is an irreducible constituent of $\mathbf{R}_{\mathbf{L}_i}^{\mathbf{L}_{i+1}}(\lambda_i)$. By the transitivity of Deligne–Lusztig induction (see [GM20, Theorem 3.3.6]), we have

$$\mathbf{R}_{\mathbf{L}_i}^{\mathbf{L}_{i+1}}(\lambda_i) = \epsilon_{\mathbf{L}_i}\epsilon_{\mathbf{L}_i(s_\ell)}\mathbf{R}_{\mathbf{L}_i(s_\ell)}^{\mathbf{L}_{i+1}}\left(\widehat{s_\ell}\cdot\lambda_i(s_\ell)\right) = \epsilon_{\mathbf{L}}\epsilon_{\mathbf{L}_i(s_\ell)}\mathbf{R}_{\mathbf{L}_{i+1}(s_\ell)}^{\mathbf{L}_{i+1}}\left(\mathbf{R}_{\mathbf{L}_i(s_\ell)}^{\mathbf{L}_{i+1}(s_\ell)}\left(\widehat{s_\ell}\cdot\lambda_i(s_\ell)\right)\right).$$

Moreover, by Lemma 6.2.4, every irreducible constituent of $\mathbf{R}_{\mathbf{L}_i(s_\ell)}^{\mathbf{L}_{i+1}(s_\ell)}(\widehat{s_\ell}\cdot\lambda_i(s_\ell))$ is contained in $\mathcal{E}(\mathbf{L}_{i+1}(s_\ell)^F, [s])$. Then, since

$$\epsilon_{\mathbf{L}_{i+1}}\epsilon_{\mathbf{L}_{i+1}(s_\ell)}\mathbf{R}_{\mathbf{L}_{i+1}(s_\ell)}^{\mathbf{L}_{i+1}} : \mathcal{E}(\mathbf{L}_{i+1}(s_\ell)^F, [s]) \to \mathcal{E}(\mathbf{L}_{i+1}^F, [s])$$

is a bijection, we deduce that $\widehat{s_\ell}\cdot\lambda_{i+1}(s_\ell)$ is an irreducible constituent of $\mathbf{R}_{\mathbf{L}_i(s_\ell)}^{\mathbf{L}_{i+1}(s_\ell)}(\widehat{s_\ell}\cdot\lambda_i(s_\ell))$. It follows that $\lambda_{i+1}(s_\ell)$ is an irreducible constituent of $\mathbf{R}_{\mathbf{L}_i(s_\ell)}^{\mathbf{L}_{i+1}(s_\ell)}(\lambda_i(s_\ell))$ (see [Bon06, 10.2]) and this completes the proof. □

The following lemma is a fundamental ingredient to understand the distribution of characters into blocks. This idea was first used in [CE94] in order to deal with unipotent blocks.

Lemma 7.2.11. *Assume Hypothesis 7.2.7 (i)-(ii). Let $(\mathbf{K}, \kappa)$ be an (e, s)-pair of $\mathbf{G}$ and consider the $(e, s_{\ell'})$-pair $(\mathbf{K}(s_\ell), \kappa(s_\ell))$ given by Lemma 7.2.9. Consider an $(e, s_{\ell'})$-cuspidal pair $(\mathbf{L}, \lambda)$ of $\mathbf{K}(s_\ell)$ such that $\mathrm{bl}(\kappa(s_\ell)) = b_{\mathbf{K}(s_\ell)^F}(\mathbf{L}, \lambda)$ (see Theorem 6.2.19). Then $\mathrm{bl}(\kappa) = b_{\mathbf{K}^F}(\mathbf{L}, \lambda)$.*

Proof. Using Theorem 6.2.18, observe that all irreducible constituents of $\mathbf{R}_{\mathbf{K}(s_\ell)}^{\mathbf{K}}(\kappa(s_\ell))$ are contained in a unique block b of $\mathbf{K}^F$. Moreover, under our assumptions, Proposition 7.1.6 (ii.e) implies that $\mathbf{K} = \mathbf{C}_{\mathbf{G}}^\circ(\mathbf{Z}(\mathbf{K})_\ell^F)$ and therefore $b = \mathrm{bl}(\kappa(s_\ell))^{\mathbf{K}^F}$. Similarly $b_{\mathbf{K}(s_\ell)^F}(\mathbf{L}, \lambda) = \mathrm{bl}(\lambda)^{\mathbf{K}(s_\ell)^F}$ and $b_{\mathbf{K}^F}(\mathbf{L}, \lambda) = \mathrm{bl}(\lambda)^{\mathbf{K}^F}$. Then, by the transitivity of block induction, we deduce that

$$b = \mathrm{bl}(\kappa(s_\ell))^{\mathbf{K}^F} = \left(\mathrm{bl}(\lambda)^{\mathbf{K}(s_\ell)^F}\right)^{\mathbf{K}^F} = \mathrm{bl}(\lambda)^{\mathbf{K}^F} = b_{\mathbf{K}^F}(\mathbf{L}, \lambda)$$

and it is enough to show that $b = \mathrm{bl}(\kappa)$. In order to do so, we apply Brauer's second Main Theorem (see [CE04, Theorem 5.8]). Then, it suffices to show that $d^1(\mathbf{R}_{\mathbf{K}(s_\ell)}^{\mathbf{K}}(\kappa(s_\ell)))$ has an irreducible constituent in $\mathrm{bl}(\kappa)$. By [CE04, Proposition 21.4] and since $\mathbf{R}_{\mathbf{K}(s_\ell)}^{\mathbf{K}}$ and ${}^*\mathbf{R}_{\mathbf{K}(s_\ell)}^{\mathbf{K}}$ are adjoint, it follows that

$$\begin{aligned} d^1\left(\mathbf{R}_{\mathbf{K}(s_\ell)}^{\mathbf{K}}(\kappa(s_\ell))\right) &= \mathbf{R}_{\mathbf{K}(s_\ell)}^{\mathbf{K}}\left(d^1(\kappa(s_\ell))\right) \\ &= \mathbf{R}_{\mathbf{K}(s_\ell)}^{\mathbf{K}}\left(d^1(\widehat{s_\ell}\cdot\kappa(s_\ell))\right) \\ &= \epsilon_{\mathbf{K}}\epsilon_{\mathbf{K}(s_\ell)}d^1(\kappa). \end{aligned}$$

Since by Brauer's second Main Theorem $d^1(\kappa) \in \mathbb{N}\mathrm{Irr}(\mathrm{bl}(\kappa))$, the proof is now complete. □

As a corollary we deduce that the construction given in Lemma 7.2.9 preserves the decomposition of characters into blocks.

Corollary 7.2.12. *Assume Hypothesis 7.2.7 (i)-(ii). Let* $\mathbf{L}$ *be an* e*-split Levi subgroup of* $\mathbf{G}$ *and consider* $s \in \mathbf{L}_{ss}^{*F^*}$. *For* $i = 1, 2$, *let* $\lambda_i \in \mathcal{E}(\mathbf{L}^F, [s])$ *and consider* $\lambda_i(s_\ell) \in \mathcal{E}(\mathbf{L}(s_\ell)^F, [s_{\ell'}])$ *given by Lemma 7.2.9. If* $\lambda_1(s_\ell)$ *and* $\lambda_2(s_\ell)$ *are in the same block of* $\mathbf{L}(s_\ell)^F$, *then* λ_1 *and* λ_2 *are in the same block of* $\mathbf{L}^F$.

Proof. Let c be the block of $\mathbf{L}(s_\ell)$ containing $\lambda_1(s_\ell)$ and $\lambda_2(s_\ell)$ and consider an e-cuspidal pair $(\mathbf{M}, \mu)$ such that $c = b_{\mathbf{L}(s_\ell)}(\mathbf{M}, \mu)$. Then, Lemma 7.2.11 implies that $\mathrm{bl}(\lambda_1) = b_{\mathbf{L}^F}(\mathbf{M}, \mu) = \mathrm{bl}(\lambda_2)$. □

Remark 7.2.13. We believe that the reverse implication of Corollary 7.2.12 also holds. Namely we believe that, if λ_1 and λ_2 are in the same block, then $\lambda_1(s_\ell)$ and $\lambda_2(s_\ell)$ are in the same block. We point out that this is true when s is ℓ-regular, and more generally when $\mathbf{C}^\circ_{\mathbf{G}^*}(s_{\ell'})\mathbf{C}_{\mathbf{G}^*}(s_{\ell'})^{F^*} \leq \mathbf{L}(s_\ell)^*$, by results of Broué on perfect isometries (see [Bro90, Theorem 2.3]).

The next result can be seen as an extension of Theorem 6.2.18 to (e, s)-pairs with s not necessarily ℓ-regular. Notice that, if $\ell \in \Gamma(\mathbf{G}, F)$ and $\mathbf{L}$ is an e-split Levi subgroup of $\mathbf{G}$, then $\mathbf{L}^F = \mathbf{C}_{\mathbf{G}^F}(X)$ for some abelian ℓ-subgroup $X \leq \mathbf{G}^F$ by Proposition 7.1.6. Therefore, block induction from $\mathbf{L}^F$ to $\mathbf{G}^F$ is defined by [Nav98, Theorem 4.14].

Proposition 7.2.14. *Assume Hypothesis 7.2.7 (i)-(ii). Let* $\mathbf{K}$ *be an* e*-split Levi subgroup of* $\mathbf{G}$ *and* $(\mathbf{L}, \lambda)$ *an* (e, s)*-pair of* $\mathbf{K}$. *Then there exists a block* b *of* $\mathbf{K}^F$ *such that* $\mathbf{R}_{\mathbf{L}}^{\mathbf{K}}(\lambda) \in \mathbb{Z}\mathrm{Irr}(b)$. *Moreover* $b = \mathrm{bl}(\lambda)^{\mathbf{K}^F}$.

Proof. Without loss of generality we may assume $s \in \mathbf{L}^*$. Consider the $(e, s_{\ell'})$-pairs $(\mathbf{L}(s_\ell), \lambda(s_\ell))$ given by Lemma 7.2.9. By Theorem 6.2.18, there exists a block $b(s_\ell)$ of $\mathbf{K}(s_\ell)$ such that $\mathbf{R}_{\mathbf{L}(s_\ell)}^{\mathbf{K}(s_\ell)}(\lambda(s_\ell)) \in \mathbb{Z}\mathrm{Irr}(b(s_\ell))$. Furthermore $b(s_\ell) = \mathrm{bl}(\lambda(s_\ell))^{\mathbf{K}(s_\ell)^F}$ by Proposition 7.1.6 (e). If we denote by $\widehat{s_\ell} \cdot b(s_\ell)$ the block of $\mathbf{K}(s_\ell)$ consisting of those characters of the form $\widehat{s_\ell} \cdot \xi$, for $\xi \in \mathrm{Irr}(b(s_\ell))$, then

$$\mathbf{R}_{\mathbf{L}(s_\ell)}^{\mathbf{K}(s_\ell)}\left(\widehat{s_\ell} \cdot \lambda(s_\ell)\right) = \widehat{s_\ell} \cdot \mathbf{R}_{\mathbf{L}(s_\ell)}^{\mathbf{K}(s_\ell)}(\lambda(s_\ell)) \in \mathbb{Z}\mathrm{Irr}\left(\widehat{s_\ell} \cdot b(s_\ell)\right). \tag{7.2.6}$$

By Corollary 7.2.12 and (7.2.6) it follows that there exists a unique block b of $\mathbf{K}^F$ such that

$$\mathbf{R}_{\mathbf{L}}^{\mathbf{K}}(\lambda) = \mathbf{R}_{\mathbf{L}}^{\mathbf{K}}\left(\mathbf{R}_{\mathbf{L}(s_\ell)}^{\mathbf{L}}\left(\widehat{s_\ell} \cdot \lambda(s_\ell)\right)\right) = \mathbf{R}_{\mathbf{L}(s_\ell)}^{\mathbf{L}}\left(\mathbf{R}_{\mathbf{L}(s_\ell)}^{\mathbf{K}(s_\ell)}\left(\widehat{s_\ell} \cdot \lambda(s_\ell)\right)\right) \in \mathbb{Z}\mathrm{Irr}(b).$$

Next, set $c := \mathrm{bl}(\lambda(s_\ell))$. Consider an (e, ℓ')-cuspidal pair $(\mathbf{M}, \mu)$ such that $c = b_{\mathbf{L}(s_\ell)}(\mathbf{M}, \mu)$. Since $c = \mathrm{bl}(\mu)^{\mathbf{L}(s_\ell)^F}$ and $b(s_\ell) = c^{\mathbf{K}(s_\ell)^F}$ it follows that

$$b(s_\ell) = c^{\mathbf{K}(s_\ell)^F} = \mathrm{bl}(\mu)^{\mathbf{K}(s_\ell)^F} = b_{\mathbf{K}(s_\ell)}(\mathbf{M}, \mu).$$

Now, Lemma 7.2.11 implies that $\mathrm{bl}(\lambda) = b_{\mathbf{L}^F}(\mathbf{M}, \mu)$ and that $b = b_{\mathbf{K}^F}(\mathbf{M}, \mu)$. We conclude that $b = \mathrm{bl}(\mu)^{\mathbf{K}^F} = (\mathrm{bl}(\mu)^{\mathbf{L}^F})^{\mathbf{K}^F} = \mathrm{bl}(\lambda)^{\mathbf{K}^F}$ and this completes the proof. □

Finally, we show that for every e-pair $(\mathbf{K}, \kappa)$ there exists a unique (up to $\mathbf{K}^F$-conjugation) e-cuspidal pair $(\mathbf{L}, \lambda) \leq_e (\mathbf{K}, \kappa)$.

Proposition 7.2.15. *Assume Hypothesis 7.2.7. Let* $(\mathbf{L},\lambda) \ll_e (\mathbf{K},\kappa)$ *be* (e,s)*-pairs such that* $(\mathbf{L},\lambda)$ *is* (e,s)*-cuspidal. Then* $(\mathbf{L},\lambda) \leq_e (\mathbf{K},\kappa)$*. Moreover, if* $(\mathbf{L}',\lambda')$ *is another* (e,s)*-cuspidal pair satisfying* $(\mathbf{L}',\lambda') \ll_e (\mathbf{K},\kappa)$*, then* $(\mathbf{L},\lambda)$ *and* $(\mathbf{L}',\lambda')$ *are* $\mathbf{K}^F$*-conjugate.*

Proof. By Lemma 6.2.4 we may assume $s \in \mathbf{L}^* \leq \mathbf{K}^*$. Consider the $(e,s_{\ell'})$-pairs $(\mathbf{L}(s_\ell),\lambda(s_\ell))$ and $(\mathbf{K}(s_\ell),\kappa(s_\ell))$ given by Lemma 7.2.9 and notice that $(\mathbf{L}(s_\ell),\lambda(s_\ell))$ is e-cuspidal. By Lemma 7.2.10 it follows that $(\mathbf{L}(s_\ell),\lambda(s_\ell)) \ll_e (\mathbf{K}(s_\ell),\kappa(s_\ell))$ and Condition 7.2.3 shows that $\kappa(s_\ell)$ is an irreducible constituent of $\mathbf{R}_{\mathbf{L}(s_\ell)}^{\mathbf{K}(s_\ell)}(\lambda(s_\ell))$. Then $\widehat{s_\ell}\cdot\kappa(s_\ell)$ is an irreducible constituent of $\widehat{s_\ell}\cdot\mathbf{R}_{\mathbf{L}(s_\ell)}^{\mathbf{K}(s_\ell)}(\lambda(s_\ell))$. Since by Lemma 6.2.4 we have

$$\widehat{s_\ell}\cdot\mathbf{R}_{\mathbf{L}(s_\ell)}^{\mathbf{K}(s_\ell)}(\lambda(s_\ell)) = \mathbf{R}_{\mathbf{L}(s_\ell)}^{\mathbf{K}(s_\ell)}(\widehat{s_\ell}\cdot\lambda(s_\ell)) \in \mathbb{Z}\mathcal{E}(\mathbf{K}(s_\ell),[s]),$$

we deduce form Proposition 6.2.8 that $\kappa = \epsilon_{\mathbf{K}}\epsilon_{\mathbf{K}(s_\ell)}\mathbf{R}_{\mathbf{K}(s_\ell)}^{\mathbf{K}}(\widehat{s_\ell}\cdot\kappa(s_\ell))$ is an irreducible constituent of $\mathbf{R}_{\mathbf{K}(s_\ell)}^{\mathbf{K}}(\mathbf{R}_{\mathbf{L}(s_\ell)}^{\mathbf{K}(s_\ell)}(\widehat{s_\ell}\cdot\lambda(s_\ell))) = \mathbf{R}_{\mathbf{L}}^{\mathbf{K}}(\lambda)$. This shows that $(\mathbf{L},\lambda) \leq_e (\mathbf{K},\kappa)$.

Next consider another (e,s)-cuspidal pair $(\mathbf{L}',\lambda') \ll_e (\mathbf{K},\kappa)$. Let $\lambda' \in \mathcal{E}(\mathbf{L}',[s'])$ and notice that s and s' are $\mathbf{K}^{*F^*}$-conjugate by Lemma 6.2.4. Replacing $(\mathbf{L}',\lambda')$ with a $\mathbf{K}^F$-conjugate we may assume that $s = s'$. As before consider the $(e,s_{\ell'})$-cuspidal pair $(\mathbf{L}'(s_\ell),\lambda'(s_\ell))$ and observe that $(\mathbf{L}'(s_\ell),\lambda'(s_\ell)) \ll_e (\mathbf{K}(s_\ell),\kappa(s_\ell))$. By Theorem 6.2.19 it follows that $(\mathbf{L}(s_\ell),\lambda(s_\ell))$ and $(\mathbf{L}'(s_\ell),\lambda'(s_\ell))$ are $\mathbf{K}(s_\ell)$-conjugate. Since $\widehat{s_\ell}$ is $\mathbf{K}(s_\ell)^F$-invariant, we deduce that $(\mathbf{L}(s_\ell),\widehat{s_\ell}\cdot\lambda(s_\ell))$ and $(\mathbf{L}'(s_\ell),\widehat{s_\ell}\cdot\lambda'(s_\ell))$ are $\mathbf{K}(s_\ell)^F$-conjugate. Recalling that $\mathbf{L} = \mathbf{L}(s_\ell)$ and $\mathbf{L}' = \mathbf{L}'(s_\ell)$, we obtain that $(\mathbf{L},\lambda)$ and $(\mathbf{L}',\lambda')$ are $\mathbf{K}^F$-conjugate. □

As an immediate consequence of Proposition 7.2.15 we deduce that the set $\mathrm{Irr}(\mathbf{K}^F)$ is a disjoint union of e-Harish-Chandra series. This should be compared with the classical Harish-Chandra theory (see [GM20, Corollary 3.1.17]) and with the analogous result for unipotent characters [GM20, Theorem 4.6.20]. These two results can be recovered by considering $(1,s)$-pairs and $(e,1)$-pairs respectively.

Corollary 7.2.16. *Assume Hypothesis 7.2.7. If* $\mathbf{K}$ *is an* e*-split Levi subgroup of* $\mathbf{G}$*, then*

$$\mathrm{Irr}(\mathbf{K}^F) = \coprod_{(\mathbf{L},\lambda)} \mathcal{E}\left(\mathbf{K}^F,(\mathbf{L},\lambda)\right),$$

where the union runs over a $\mathbf{K}^F$*-transversal in the set of* e*-cuspidal pairs of* $\mathbf{K}$*.*

Combining Corollary 7.2.16 and Proposition 7.2.14 we can describe all the characters in the blocks of $\mathbf{K}^F$ in terms of e-Harish-Chandra series.

Theorem 7.2.17. *Assume Hypothesis 7.2.7. Let* $\mathbf{K}$ *be an* e*-split Levi subgroup of* $\mathbf{G}$ *and* b *a block of* $\mathbf{K}^F$*. Then*

$$\mathrm{Irr}(b) = \coprod_{(\mathbf{L},\lambda)} \mathcal{E}\left(\mathbf{K}^F,(\mathbf{L},\lambda)\right),$$

where the union runs over a $\mathbf{K}^F$*-transversal in the set of* e*-cuspidal pairs* $(\mathbf{L},\lambda)$ *of* $\mathbf{K}$ *such that* $\mathrm{bl}(\lambda)^{\mathbf{K}^F} = b$*.*

Proof. First, for every e-cuspidal pair $(\mathbf{L},\lambda)$ such that $\mathrm{bl}(\lambda)^{\mathbf{K}^F}=b$, Proposition 7.2.14 shows that $\mathcal{E}(\mathbf{K}^F,(\mathbf{L},\lambda))\subseteq\mathrm{Irr}(b)$. On the other hand, if $k\in\mathrm{Irr}(b)$, then by Corollary 7.2.16 there exists an e-cuspidal pair $(\mathbf{L},\lambda)$ of $\mathbf{K}$ such that $k\in\mathcal{E}(\mathbf{K}^F,(\mathbf{L},\lambda))$. Moreover, applying Proposition 7.2.14 once more, it follows that $b=\mathrm{bl}(\kappa)=\mathrm{bl}(\lambda)^{\mathbf{K}^F}$. Finally, the union is disjoint by Proposition 7.2.15. □

Corollary 7.2 is now an immediate consequence of Theorem 7.2.17.

7.3 Brauer–Lusztig blocks

We now extend Theorem 7.2.17 in order to obtain Theorem 7.1. To start, following Broué, Fong and Srinivasan, we define the Brauer–Lusztig blocks of $\mathbf{G}^F$.

Definition 7.3.1. A **Brauer–Lusztig block** of $\mathbf{G}^F$ is any nonempty set of the form

$$\mathcal{E}\left(\mathbf{G}^F,B,[s]\right):=\mathcal{E}\left(\mathbf{G}^F,[s]\right)\cap\mathrm{Irr}(B),$$

where B is an ℓ-block of $\mathbf{G}^F$ and s is a semisimple element of $\mathbf{G}^{*F^*}$. In this case, we say that $(\mathbf{G},B,[s])$ is the associated **Brauer–Lusztig triple** of $\mathbf{G}^F$. Moreover, we denote by $\mathcal{BL}(\mathbf{G},F)$ the set of all Brauer–Lusztig triples of $\mathbf{G}^F$. We also define the set

$$\mathcal{BL}^*(\mathbf{G},F):=\coprod_{\mathbf{L}\leq\mathbf{G}}\mathcal{BL}(\mathbf{L},F),$$

where $\mathbf{L}$ runs over all e-split Levi subgroups of $\mathbf{G}$.

Next, assume $\ell\in\Gamma(\mathbf{G},F)$. If $\mathbf{L}$ is an e-split Levi subgroup of $\mathbf{G}$, then $\mathbf{L}^F=\mathbf{C}_{\mathbf{G}^F}(A)$ for some abelian ℓ-subgroup $A\leq\mathbf{G}^F$ by Proposition 7.1.6. Therefore, for $b\in\mathrm{Bl}(\mathbf{L}^F)$, the Brauer induced block $b^{\mathbf{G}^F}$ is defined (see [Nav98, Theorem 4.14]). Then, we can introduce a partial order relation on $\mathcal{BL}^*(\mathbf{G},F)$ by defining

$$(\mathbf{L},b,[s])\leq(\mathbf{K},c,[t])$$

if $\mathbf{L}\leq\mathbf{K}$, $b^{\mathbf{K}^F}=c$ and the semisimple elements s and t are conjugate by an element of $\mathbf{K}^{*F^*}$. If $(\mathbf{L},b,[s])$ is a minimal element of the poset $(\mathcal{BL}^*(\mathbf{G},F),\leq)$, then we say that $(\mathbf{L},b,[s])$ is a **cuspidal** Brauer–Lusztig triple.

In the next lemma we compare the relation $\leq$ on Brauer–Lusztig triples with the relation $\leq_e$ on e-pairs.

Lemma 7.3.2. *Assume Hypothesis 7.2.7. Let $\mathbf{L}$ and $\mathbf{K}$ be e-split Levi subgroups of $\mathbf{G}$ and consider semisimple elements $s\in\mathbf{L}^{*F^*}$ and $t\in\mathbf{K}^{*F^*}$.*

(i) Let $\lambda\in\mathcal{E}(\mathbf{L}^F,b,[s])$ and $\kappa\in\mathcal{E}(\mathbf{K}^F,c,[t])$. If $(\mathbf{L},\lambda)\ll_e(\mathbf{K},\kappa)$, then $(\mathbf{L},b,[s])\leq(\mathbf{K},c,[t])$.

(ii) Let $\lambda\in\mathcal{E}(\mathbf{L}^F,b,[s])$. If $(\mathbf{L},b,[s])$ is cuspidal, then $(\mathbf{L},\lambda)$ is e-cuspidal.

(iii) If $(\mathbf{L},b,[s])\leq(\mathbf{K},c,[t])$, then for every $\lambda\in\mathcal{E}(\mathbf{L}^F,b,[s])$ there exists $\kappa\in\mathcal{E}(\mathbf{K}^F,c,[t])$ such that $(\mathbf{L},\lambda)\leq_e(\mathbf{K},\kappa)$.

Proof. We start by proving (i). Let $(\mathbf{L},\lambda) \ll_e (\mathbf{K},\kappa)$. By Lemma 6.2.4, we may assume $s = t$ and it is enough to show that $\mathrm{bl}(\lambda)^{\mathbf{K}^F} = \mathrm{bl}(\kappa)$. To see this, choose an e-cuspidal pair $(\mathbf{M},\mu) \ll_e (\mathbf{L},\lambda)$ and notice that $(\mathbf{M},\mu) \ll_e (\mathbf{K},\kappa)$. By Proposition 7.2.15 we deduce that $(\mathbf{M},\mu) \leq_e (\mathbf{L},\lambda)$ and $(\mathbf{M},\mu) \leq_e (\mathbf{K},\kappa)$. Then, Proposition 7.2.14 implies that $\mathrm{bl}(\lambda) = \mathrm{bl}(\mu)^{\mathbf{L}^F}$ and $\mathrm{bl}(\kappa) = \mathrm{bl}(\mu)^{\mathbf{K}^F}$. By the transitivity of block induction, we conclude that $\mathrm{bl}(\kappa) = \mathrm{bl}(\lambda)^{\mathbf{K}^F}$. This proves (i) and (ii) is an immediate consequence. In fact, if $(\mathbf{L}, b, [s])$ is a cuspidal Brauer–Lusztig triple and we consider an e-cuspidal $(\mathbf{M},\mu) \ll_e (\mathbf{L},\lambda)$, then (i) shows that $(\mathbf{M}, \mathrm{bl}(\mu), [r]) \leq (\mathbf{L}, b, [s])$, where $\mu \in \mathcal{E}(\mathbf{M}^F, [r])$. It follows that $\mathbf{L} = \mathbf{M}$ and that $(\mathbf{L},\lambda) = (\mathbf{M},\mu)$ is e-cuspidal.

Finally, let $(\mathbf{L}, b, [s]) \leq (\mathbf{K}, c, [t])$ and consider $\lambda \in \mathcal{E}(\mathbf{L}^F, b, [s])$. Let κ be an irreducible constituent of $\mathbf{R}_{\mathbf{L}}^{\mathbf{K}}(\lambda)$ so that $(\mathbf{L},\lambda) \leq_e (\mathbf{K},\kappa)$. We need to show that $\kappa \in \mathcal{E}(\mathbf{K}^F, c, [t])$. By Lemma 6.2.4 we have $\kappa \in \mathcal{E}(\mathbf{K}^F, [s]) = \mathcal{E}(\mathbf{K}^F, [t])$. Moreover, applying Proposition 7.2.14, we obtain $\mathrm{bl}(\kappa) = \mathrm{bl}(\lambda)^{\mathbf{K}^F} = b^{\mathbf{K}^F} = c$. We conclude that $\kappa \in \mathcal{E}(\mathbf{K}^F, c, [t])$. □

Finally, we prove the following strong form of Theorem 7.1.

Theorem 7.3.3. *Assume Hypothesis 7.2.7. Let* $(\mathbf{K}, c, [t]) \in \mathcal{BL}^*(\mathbf{G}, F)$*. Then*

$$\mathcal{E}\left(\mathbf{K}^F, c, [t]\right) = \coprod_{(\mathbf{L},\lambda)} \mathcal{E}\left(\mathbf{K}^F, (\mathbf{L},\lambda)\right), \tag{7.3.1}$$

where the union runs over a $\mathbf{K}^F$*-transversal in the set of* (e,t)*-cuspidal pairs* $(\mathbf{L},\lambda)$ *of* $\mathbf{K}$ *with* $\lambda \in \mathcal{E}(\mathbf{L}^F, [s_\lambda])$ *such that* $(\mathbf{L}, \mathrm{bl}(\lambda), [s_\lambda]) \leq (\mathbf{K}, c, [t])$*.*

Proof. Consider an e-cuspidal pair $(\mathbf{L},\lambda)$ such that $(\mathbf{L}, \mathrm{bl}(\lambda), [s]) \leq (\mathbf{K}, c, [t])$, where $s \in \mathbf{L}^{*F^*}_{\mathrm{ss}}$ and $\lambda \in \mathcal{E}(\mathbf{L}^F, [s])$. Since s and t are $\mathbf{K}^{*F^*}$-conjugate, Lemma 6.2.4 implies that $\mathcal{E}(\mathbf{K}^F, (\mathbf{L},\lambda)) \subseteq \mathcal{E}(\mathbf{K}^F, [t])$. Moreover, using the fact that $c = \mathrm{bl}(\lambda)^{\mathbf{K}^F}$, Proposition 7.2.14 shows that the e-Harish-Chandra series $\mathcal{E}(\mathbf{K}^F, (\mathbf{L},\lambda))$ is contained in $\mathrm{Irr}(c)$. This shows that the union on the right hand side of (7.3.1) is contained in the Brauer–Lusztig block $\mathcal{E}(\mathbf{K}^F, c, [t])$. Moreover the union is disjoint by Proposition 7.2.15. To conclude, let $\kappa \in \mathcal{E}(\mathbf{K}, c, [t])$ and notice that there exists an e-cuspidal pair $(\mathbf{L},\lambda)$ of $\mathbf{K}$ such that $\kappa \in \mathrm{Irr}(\mathbf{R}_{\mathbf{L}}^{\mathbf{K}}(\lambda))$ by Proposition 7.2.15. If $\lambda \in \mathcal{E}(\mathbf{L}^F, [s])$, then s and t are $\mathbf{K}^{*F^*}$-conjugate by Lemma 6.2.4. Moreover, $c = \mathrm{bl}(\kappa) = \mathrm{bl}(\lambda)^{\mathbf{K}^F}$ by Proposition 7.2.14. It follows that $(\mathbf{L}, \mathrm{bl}(\lambda), [s]) \leq (\mathbf{K}, c, [t])$. □

We conclude this section with a remark concerning Theorem 7.3.3. Here we have shown that Brauer–Lusztig blocks are disjoint unions of e-Harish-Chandra series. However, we believe that there exists a unique (up to $\mathbf{K}^F$-conjugation) e-cuspidal pair $(\mathbf{L},\lambda)$ such that $(\mathbf{L}, \mathrm{bl}(\lambda), [s_\lambda]) \leq (\mathbf{K}, c, [t])$. By Theorem 6.2.19 this happens when t is an ℓ'-element. In particular this would show that the concepts of Brauer–Lusztig blocks and e-Harish-Chandra series coincide, at least under the above restrictions on primes. It can be seen that to prove such a statement it is enough to show that the reverse implication of Corollary 7.2.12 would hold (see Remark 7.2.13). We will see this condition again in Theorem 10.2.

7.4 Cuspidal Brauer–Lusztig triples have central defect

Consider $\mathbf{G}$, F, ℓ and e as in Notation 7.1.1. In this section we show that under suitable assumptions, if $(\mathbf{L},\lambda)$ is an e-cuspidal pair of $\mathbf{G}$, then λ has ℓ-defect zero. This result will be used in the main theorem of Chapter 9 (see Theorem 9.2.21).

Proposition 7.4.1. *Let $\ell \in \Gamma(\mathbf{G},F) \cap \Gamma((\mathbf{G}^*)_{\mathrm{ad}},F^*)$. If $(\mathbf{L},\lambda)$ is an e-cuspidal pair of $\mathbf{G}$ such that $\mathbf{Z}(\mathbf{L}^*)_\ell^{F^*} = 1$, then λ has ℓ-defect zero, i.e. $d(\lambda) = 0$.*

Proof. Let $s \in \mathbf{L}^{*F^*}$ such that $\lambda \in \mathcal{E}(\mathbf{L}^F,[s])$. By Jordan decomposition (see Theorem 6.2.6), λ corresponds to a unique $\lambda(s) \in \mathcal{E}(\mathbf{C}_{\mathbf{L}^*}(s)^{F^*},1)$ lying over some unipotent character $\lambda^\circ(s) \in \mathcal{E}(\mathbf{C}^\circ_{\mathbf{L}^*}(s)^{F^*},1)$. Notice that

$$\lambda(1)_\ell = \frac{|\mathbf{L}^F|_\ell}{|\mathbf{C}_{\mathbf{L}^*}(s)^{F^*}|_\ell}\lambda(s)(1)_\ell.$$

Since $\ell \in \Gamma(\mathbf{G},F)$, we obtain $\ell \in \Gamma(\mathbf{L},F)$ by Remark 7.1.3. Now, since $|\mathbf{C}_{\mathbf{L}^*}(s)^F : \mathbf{C}^\circ_{\mathbf{L}^*}(s)^F|$ divides $|\mathbf{Z}(\mathbf{L}^*)^{F^*} : \mathbf{Z}^\circ(\mathbf{L}^*)^{F^*}|$ by [DM20, Lemma 11.2.1 (iii)], Clifford's theorem implies

$$\lambda(1)_\ell = \frac{|\mathbf{L}^F|_\ell}{|\mathbf{C}^\circ_{\mathbf{L}^*}(s)^{F^*}|_\ell}\lambda^\circ(s)(1)_\ell. \tag{7.4.1}$$

Set $\mathbf{H} := \mathbf{C}^\circ_{\mathbf{L}^*}(s)$ and notice that, by [CE94, Theorem (ii)], the block $\mathrm{bl}(\lambda^\circ(s))$ has defect group $D \in \mathrm{Syl}_\ell(\mathbf{C}^\circ_{\mathbf{H}}([\mathbf{H},\mathbf{H}])^{F^*})$. Since $\mathbf{H} = \mathbf{Z}^\circ(\mathbf{H})[\mathbf{H},\mathbf{H}]$, it follows that $\mathbf{C}^\circ_{\mathbf{H}}([\mathbf{H},\mathbf{H}]) = \mathbf{Z}^\circ(\mathbf{H})$. Thus $D \leq \mathbf{Z}(\mathbf{H})^{F^*} \leq \mathbf{Z}(\mathbf{H}^{F^*})$ and, using [Nav98, Theorem 9.12], we obtain

$$\lambda^\circ(s)(1)_\ell = |\mathbf{H}^{F^*} : D|_\ell.$$

This implies

$$\lambda^\circ(s)(1)_\ell = |\mathbf{H}^{F^*} : \mathbf{Z}^\circ(\mathbf{H})^{F^*}|_\ell. \tag{7.4.2}$$

Combining (7.4.1) and (7.4.2) we see that it is enough to show that $Z := \mathbf{Z}^\circ(\mathbf{H})_\ell^{F^*} = 1$. To do so, observe that $\mathbf{Z}^\circ(\mathbf{L}^*)_{\Phi_e} = \mathbf{Z}^\circ(\mathbf{H})_{\Phi_e}$ by Proposition 6.2.11. In particular, for every e-split Levi subgroup $\mathbf{K}^*$ of $\mathbf{L}^*$ containing $\mathbf{H}$, we have $\mathbf{K}^* = \mathbf{L}^*$. Notice that $\mathbf{H} \leq \mathbf{C}^\circ_{\mathbf{L}^*}(Z)$ and that $\mathbf{C}^\circ_{\mathbf{L}^*}(Z)$ is an e-split Levi subgroup of $\mathbf{L}^*$ by Proposition 7.1.6 (ii.c). Therefore $\mathbf{C}^\circ_{\mathbf{L}^*}(Z) = \mathbf{L}^*$ and $Z \leq \mathbf{Z}(\mathbf{L}^*)_\ell^{F^*} = 1$. □

8

Bijections for Groups with Connected Center

The main goal of this chapter is to construct certain bijections (see Corollary 8.2) which will be used in Chapter 10 to obtain results towards a proof of the inductive condition for Dade's Conjecture for groups of Lie type. More precisely, let $\mathbf{G}$, $F : \mathbf{G} \to \mathbf{G}$, ℓ and e be as in Notation 7.1.1 and recall that, if $\mathbf{L} \leq \mathbf{K}$ are F-stable Levi subgroups of $\mathbf{G}$ and $\lambda \in \mathrm{Irr}(\mathbf{L}^F)$, then $W_{\mathbf{K}^F}(\mathbf{L}, \lambda) := \mathbf{N}_{\mathbf{K}}(\mathbf{L}, \lambda)^F / \mathbf{L}^F$ denotes the relative Weyl group of $(\mathbf{L}, \lambda)$ in $\mathbf{K}$ (see Section 6.2.4). By Theorem 6.2.14 there exists a collection of isometries

$$I^{\mathbf{K}}_{(\mathbf{L},\lambda)} : \mathbb{Z}\mathrm{Irr}(W_{\mathbf{K}}(\mathbf{L}, \lambda)^F) \to \mathbb{Z}\mathcal{E}(\mathbf{K}^F, (\mathbf{L}, \lambda))$$

for every e-split Levi subgroup $\mathbf{K}$ of $\mathbf{G}$ and every unipotent e-cuspidal pair $(\mathbf{L}, \lambda)$ of $\mathbf{K}$. This gives rise to bijections with signs between the sets $\mathrm{Irr}(W_{\mathbf{K}}(\mathbf{L}, \lambda)^F)$ and $\mathcal{E}(\mathbf{K}^F, (\mathbf{L}, \lambda))$. By eventually changing the signs, we can then define bijections

$$I^{\mathbf{K}}_{(\mathbf{L},\lambda)} : \mathrm{Irr}(W_{\mathbf{K}}(\mathbf{L}, \lambda)^F) \to \mathcal{E}(\mathbf{K}^F, (\mathbf{L}, \lambda)) \tag{8.0.1}$$

which by abuse of notation are denoted by the same symbol. Moreover, if λ has an extension $\widehat{\lambda}$ to $\mathbf{N}_{\mathbf{K}}(\mathbf{L}, \lambda)^F$, then by Gallagher theorem and the Clifford correspondence we obtain a bijection

$$\begin{aligned} \mathcal{E}\left(\mathbf{K}^F, (\mathbf{L}, \lambda)\right) &\to \mathrm{Irr}\left(\mathbf{N}_{\mathbf{K}}(\mathbf{L})^F \;\middle|\; \lambda\right) \\ I^{\mathbf{K}}_{(\mathbf{L},\lambda)}(\eta) &\mapsto \left(\widehat{\lambda}\eta\right)^{\mathbf{N}_{\mathbf{K}}(\mathbf{L})^F} \end{aligned} \tag{8.0.2}$$

for every $\eta \in \mathrm{Irr}(W_{\mathbf{K}}(\mathbf{L}, \lambda)^F)$. Similar bijections will be considered in the following chapters to prove some results on the inductive condition for Dade's Conjecture (see Condition 9.1)

As a first step, we construct bijections as in (8.0.1) for nonunipotent e-cuspidal pairs of groups with connected center. For this, consider a regular embedding $i : \mathbf{G} \to \widetilde{\mathbf{G}}$ compatible with F and set $\widetilde{\mathbf{L}} := \mathbf{L}\mathbf{Z}(\widetilde{\mathbf{G}})$ for every Levi subgroup $\mathbf{L}$ of $\mathbf{G}$. Consider the subset $\mathrm{Aut}_{\mathbb{F}}(\mathbf{G}^F)$ of automorphisms of $\mathbf{G}^F$ defined in Section 6.1.5. Then, for every F-stable subgroup $\mathbf{H}$ of $\mathbf{G}$, the stabilizer of $\mathbf{H}$ in $\mathrm{Aut}_{\mathbb{F}}(\mathbf{G}^F)$ is well defined and is denoted by $\mathrm{Aut}_{\mathbb{F}}(\mathbf{G}^F)_{\mathbf{H}}$. The same observation applies to $\widetilde{\mathbf{G}}$.

Theorem 8.1. *Let $\mathbf{G}$, $F:\mathbf{G}\to\mathbf{G}$, ℓ and e be as in Notation 7.1.1 and suppose that $\mathbf{G}$ is simple not of type $\mathbf{E}_6$, $\mathbf{E}_7$ or $\mathbf{E}_8$. Then there exist a collection of bijections*

$$I^{\widetilde{\mathbf{K}}}_{(\widetilde{\mathbf{L}},\widetilde{\lambda})}:\mathrm{Irr}\left(W_{\widetilde{\mathbf{K}}}\left(\widetilde{\mathbf{L}},\widetilde{\lambda}\right)^F\right)\to\mathcal{E}\left(\widetilde{\mathbf{K}}^F,\left(\widetilde{\mathbf{L}},\widetilde{\lambda}\right)\right)$$

where $\widetilde{\mathbf{K}}$ runs over the set of e-split Levi subgroup of $\widetilde{\mathbf{G}}$ and $(\widetilde{\mathbf{L}},\widetilde{\lambda})$ runs over the set of e-cuspidal pairs of $\widetilde{\mathbf{K}}$, such that:

(i) $I^{\widetilde{\mathbf{K}}}_{(\widetilde{\mathbf{L}},\widetilde{\lambda})}$ *is* $\mathrm{Aut}_{\mathbb{F}}(\widetilde{\mathbf{G}}^F)_{\widetilde{\mathbf{K}},(\widetilde{\mathbf{L}},\widetilde{\lambda})}$*-equivariant;*

(ii) $I^{\widetilde{\mathbf{K}}}_{(\widetilde{\mathbf{L}},\widetilde{\lambda})}(\widetilde{\eta})(1)_\ell=\left|\widetilde{\mathbf{K}}^F:\mathbf{N}_{\widetilde{\mathbf{K}}}(\widetilde{\mathbf{L}},\widetilde{\lambda})^F\right|_\ell\cdot\widetilde{\lambda}(1)_\ell\cdot\widetilde{\eta}(1)_\ell$*; and*

(iii) If $z\in\mathbf{Z}(\widetilde{\mathbf{K}}^{*F^*})$ corresponds to characters $\widehat{z}_{\widetilde{\mathbf{L}}}\in\mathrm{Irr}(\widetilde{\mathbf{L}}^F/\mathbf{L}^F)$ and $\widehat{z}_{\widetilde{\mathbf{K}}}\in\mathrm{Irr}(\widetilde{\mathbf{K}}^F/\mathbf{K}^F)$ (see (6.1.2)*), then $\widetilde{\lambda}\cdot\widehat{z}_{\widetilde{\mathbf{L}}^F}$ is e-cuspidal, $W_{\widetilde{\mathbf{K}}}(\widetilde{\mathbf{L}},\widetilde{\lambda})^F=W_{\widetilde{\mathbf{K}}}(\widetilde{\mathbf{L}},\widetilde{\lambda}\cdot\widehat{z}_{\widetilde{\mathbf{L}}})^F$ and*

$$I^{\widetilde{\mathbf{K}}}_{(\widetilde{\mathbf{L}},\widetilde{\lambda})}(\widetilde{\eta})\cdot\widehat{z}_{\widetilde{\mathbf{K}}}=I^{\widetilde{\mathbf{K}}}_{(\widetilde{\mathbf{L}},\widetilde{\lambda}\cdot\widehat{z}_{\widetilde{\mathbf{L}}})}(\widetilde{\eta})$$

for every $\widetilde{\eta}\in\mathrm{Irr}(W_{\widetilde{\mathbf{K}}}(\widetilde{\mathbf{L}},\widetilde{\lambda}))^F$.

Notice that the restrictions on the type of $\mathbf{G}$ are mainly due to the fact that the Mackey formula is not known to hold in full generality. In addition for types $\mathbf{E}_6$, $\mathbf{E}_7$ and $\mathbf{E}_8$ it is not known whether there exists a Jordan decomposition map which commutes with Deligne–Lusztig induction.

Next, using Theorem 8.1 together with the results obtained in Chapter 7, assuming the existence of an equivariant extension map for e-split Levi subgroups (see Definition 8.2.1) we obtain the following result which will be used in Chapter 10 (see Assumption 10.1.1 and Assumption 10.1.4). For any connected reductive group $\mathbf{H}$ with Frobenius endomorphism F, we denote by $\mathrm{Cusp}_e(\mathbf{H}^F)$ the set of (irreducible) e-cuspidal character of $\mathbf{H}^F$.

Corollary 8.2. *Suppose that $\mathbf{G}$ is simple, simply connected and not of type $\mathbf{E}_6$, $\mathbf{E}_7$ or $\mathbf{E}_8$. Consider $\ell\in\Gamma(\mathbf{G},F)$ with $\ell\geq 5$ and let $\mathbf{K}$ be an e-split Levi subgroup of $\mathbf{G}$ and $(\mathbf{L},\lambda)$ be an e-cuspidal pair of $\mathbf{K}$. Set $A:=\mathrm{Aut}_{\mathbb{F}}(\mathbf{G}^F)_{\mathbf{K},\mathbf{L}}\ltimes\mathrm{Irr}(\widetilde{\mathbf{G}}^F/\mathbf{G}^F)$ and assume there exists an A-equivariant extension map for $\mathrm{Cusp}_e(\widetilde{\mathbf{L}}^F)$ with respect to $\widetilde{\mathbf{L}}^F\trianglelefteq\mathbf{N}_{\widetilde{\mathbf{K}}}(\mathbf{L})^F$ as in Definition 8.2.1. Then there exists an $A_{(\mathbf{L},\lambda)}$-equivariant bijection*

$$\widetilde{\Omega}^{\mathbf{K}}_{(\mathbf{L},\lambda)}:\mathrm{Irr}\left(\widetilde{\mathbf{K}}^F\ \middle|\ \mathcal{E}\left(\mathbf{K}^F,(\mathbf{L},\lambda)\right)\right)\to\mathrm{Irr}\left(\mathbf{N}_{\widetilde{\mathbf{K}}}(\mathbf{L})^F\ \middle|\ \lambda\right)$$

that preserves the ℓ-defect of characters and such that

$$\mathrm{Irr}\left(\widetilde{\chi}_{\mathbf{Z}(\widetilde{\mathbf{K}}^F)}\right)=\mathrm{Irr}\left(\widetilde{\Omega}^{\mathbf{K}}_{(\mathbf{L},\lambda)}(\widetilde{\chi})_{\mathbf{Z}(\widetilde{\mathbf{K}}^F)}\right)$$

and

$$\mathrm{bl}(\widetilde{\chi})=\mathrm{bl}\left(\widetilde{\Omega}^{\mathbf{K}}_{(\mathbf{L},\lambda)}(\widetilde{\chi})\right)^{\widetilde{\mathbf{K}}^F}$$

for every $\widetilde{\chi}\in\mathrm{Irr}(\widetilde{\mathbf{K}}^F\mid\mathcal{E}(\mathbf{K}^F,(\mathbf{L},\lambda))$.

As a by-product, we obtain the following consequence of Corollary 8.2 that is of independent interest. This provides a way to obtain bijections as in (8.0.2) for nonunipotent e-cuspidal pairs of connected reductive groups with disconnected center.

Corollary 8.3. *Suppose that $\mathbf{G}$ is simple, simply connected and not of type $\mathbf{E}_6$, $\mathbf{E}_7$ or $\mathbf{E}_8$. Consider $\ell \in \Gamma(\mathbf{G}, F)$ with $\ell \geq 5$ and let $\mathbf{K}$ be an e-split Levi subgroup of $\mathbf{G}$ and $(\mathbf{L}, \lambda)$ be an e-cuspidal pair of $\mathbf{K}$. Set $A := \mathrm{Aut}_{\mathbb{F}}(\mathbf{G}^F)_{\mathbf{K},\mathbf{L}} \ltimes \mathrm{Irr}(\widetilde{\mathbf{G}}^F/\mathbf{G}^F)$ and assume there exists an A-equivariant extension map for $\mathrm{Cusp}_e(\widetilde{\mathbf{L}}^F)$ with respect to $\widetilde{\mathbf{L}}^F \trianglelefteq \mathbf{N}_{\widetilde{\mathbf{K}}}(\mathbf{L})^F$ as in Definition 8.2.1. Then, there exists a defect preserving $\mathrm{Aut}_{\mathbb{F}}(\mathbf{G}^F)_{\mathbf{K},(\mathbf{L},\lambda)}$-equivariant bijection*

$$\Omega^{\mathbf{K}}_{(\mathbf{L},\lambda)} : \mathcal{E}\left(\mathbf{K}^F, (\mathbf{L}, \lambda)\right) \to \mathrm{Irr}\left(\mathbf{N}_{\mathbf{K}}(\mathbf{L})^F \;\middle|\; \lambda\right).$$

8.1 Generalized e-Harish-Chandra theory for groups with connected center

In this section, we construct bijections as in (8.0.1) for nonunipotent e-cuspidal pairs of reductive groups with connected center. This will prove Theorem 8.1.

Let $\mathbf{G}$, $F : \mathbf{G} \to \mathbf{G}$, ℓ and e be as in Notation 7.1.1. To start, we define an action of the group $\mathcal{K}$ introduced in Lemma 6.1.3 (v) on the set of irreducible characters.

Definition 8.1.1. Let $\mathcal{K}$ be as in Lemma 6.1.3 (v). For $z \in \mathcal{K}$ and $\chi \in \mathrm{Irr}(\widetilde{\mathbf{G}}^F)$, let

$$\chi^z := \chi \cdot \widehat{z}_{\widetilde{\mathbf{G}}},$$

where $\widehat{z}_{\widetilde{\mathbf{G}}} \in \mathrm{Irr}(\widetilde{\mathbf{G}}^F/\mathbf{G}^F)$ corresponds to z via the isomorphism (6.1.2). Similarly, for a Levi subgroup $\mathbf{L}$ of $\mathbf{G}$, the group $\mathcal{K}$ acts on $\mathrm{Irr}(\widetilde{\mathbf{L}}^F)$. Moreover, noticing that $\widetilde{\mathbf{G}}^F/\mathbf{G}^F \simeq \mathbf{N}_{\widetilde{\mathbf{G}}}(\mathbf{L})^F/\mathbf{N}_{\mathbf{G}}(\mathbf{L})^F$, we deduce that $z \in \mathcal{K}$ also acts on the characters $\psi \in \mathrm{Irr}(\mathbf{N}_{\widetilde{\mathbf{G}}}(\mathbf{L})^F)$ via

$$\psi^z := \psi \cdot \widehat{z}_{\mathbf{N}_{\widetilde{\mathbf{G}}}(\mathbf{L})}.$$

In the same way, an action of $\mathcal{K}$ on $\mathrm{Irr}(\widetilde{\mathbf{K}}^F)$ and on $\mathrm{Irr}(\mathbf{N}_{\widetilde{\mathbf{K}}}(\mathbf{L})^F)$ can be defined for all F-stable Levi subgroups $\mathbf{L}$ and $\mathbf{K}$ of $\mathbf{G}$ satisfying $\mathbf{L} \leq \mathbf{K}$.

In what follows we will make use of the fact that, under suitable hypotheses, there exists a Jordan decomposition for $\widetilde{\mathbf{G}}$ which commutes with Deligne–Lusztig induction (see Theorem 6.2.9 and Theorem 6.2.10). In order to be able to apply this result, from now on we will make the following assumption.

Hypothesis 8.1.2. Let $\mathbf{G}$, $F : \mathbf{G} \to \mathbf{G}$, ℓ and e be as in Notation 7.1.1 and suppose that $\mathbf{G}$ is a simple algebraic group not of type $\mathbf{E}_6$, $\mathbf{E}_7$ or $\mathbf{E}_8$.

Theorem 8.1.3. *Assume Hypothesis 8.1.2. For every F-stable Levi subgroup $\widetilde{\mathbf{L}}$ of $\widetilde{\mathbf{G}}$ and every semisimple element $s \in \widetilde{\mathbf{L}}^{*F^*}$, there exists a bijection*

$$J_{\widetilde{\mathbf{L}},s} : \mathcal{E}\left(\widetilde{\mathbf{L}}^F, [s]\right) \to \mathcal{E}\left(\mathbf{C}_{\widetilde{\mathbf{L}}^*}(s)^{F^*}, [1]\right)$$

satisfying the following properties:

(i) $J_{\widetilde{\mathbf{L}},s}\left(\widetilde{\lambda}\right)^{\sigma^*} = J_{\widetilde{\mathbf{L}},\sigma^*(s)}\left(\widetilde{\lambda}^{\sigma}\right)$ *for every* $\sigma \in \mathrm{Aut}_{\mathbb{F}}(\widetilde{\mathbf{G}}^F)_{\widetilde{\mathbf{L}}}$ *and* $\sigma^* \in \mathrm{Aut}_{\mathbb{F}}(\widetilde{\mathbf{G}}^{*F^*})_{\widetilde{\mathbf{L}}^*}$ *with* $\sigma \sim \sigma^*$ *(see Lemma 6.1.5 and [CS13, Proposition 2.2]);*

(ii) $J_{\widetilde{\mathbf{K}},s} \circ \mathbf{R}_{\widetilde{\mathbf{L}}}^{\widetilde{\mathbf{K}}} = \mathbf{R}_{\mathbf{C}_{\widetilde{\mathbf{L}}^*}(s)}^{\mathbf{C}_{\widetilde{\mathbf{K}}^*}(s)} \circ J_{\widetilde{\mathbf{L}},s}$ *for every F-stable Levi subgroup* $\widetilde{\mathbf{K}}$ *of* $\widetilde{\mathbf{G}}$ *containing* $\widetilde{\mathbf{L}}$*;*

(iii) $\widetilde{\lambda}(1) = \left|\widetilde{\mathbf{L}}^{*F^*} : \mathbf{C}_{\widetilde{\mathbf{L}}^*}(s)^{F^*}\right|_{p'} \cdot J_{\widetilde{\mathbf{L}},s}(\widetilde{\lambda})(1)$ *for every* $\widetilde{\lambda} \in \mathcal{E}(\widetilde{\mathbf{L}}^F,[s])$*;*

(iv) If $z \in \mathbf{Z}(\widetilde{\mathbf{L}}^{*F^*})$ *corresponds to the character* $\widehat{z}_{\widetilde{\mathbf{L}}} \in \mathrm{Irr}(\widetilde{\mathbf{L}}^F/\mathbf{L}^F)$ *via* (6.1.2)*, then*

$$J_{\widetilde{\mathbf{L}},s}\left(\widetilde{\lambda}\right) = J_{\widetilde{\mathbf{L}},sz}\left(\widetilde{\lambda}\cdot\widehat{z}_{\widetilde{\mathbf{L}}}\right)$$

for every $\widetilde{\lambda} \in \mathcal{E}(\widetilde{\mathbf{L}}^F,[s])$*, or equivalently*

$$J^{-1}_{\widetilde{\mathbf{L}},s}(\widetilde{\nu})\cdot\widehat{z}_{\widetilde{\mathbf{L}}} = J^{-1}_{\widetilde{\mathbf{L}},sz}(\widetilde{\nu})$$

for every $\widetilde{\nu} \in \mathcal{E}(\mathbf{C}_{\widetilde{\mathbf{L}}^*}(s)^{F^*},[1]) = \mathcal{E}(\mathbf{C}_{\widetilde{\mathbf{L}}^*}(sz)^{F^*},[1])$

Proof. This follows from [DM90, Theorem 7.1], [CS13, Theorem 3.1], Theorem 6.2.9 and Theorem 6.2.10 together with the fact that the Mackey Formula holds under our assumptions (see Theorem 6.2.2). □

As a consequence of the equivariance of the above Jordan decomposition, we obtain an isomorphism of relative Weyl groups.

Corollary 8.1.4. *Assume Hypothesis 8.1.2, let* $\widetilde{\mathbf{L}} \leq \widetilde{\mathbf{K}}$ *be F-stable Levi subgroups of* $\widetilde{\mathbf{G}}$ *and* $\widetilde{\lambda} \in \mathcal{E}(\widetilde{\mathbf{L}}^F,[s])$*. Then, there exists an isomorphism*

$$i^{\widetilde{\mathbf{K}}}_{\widetilde{\mathbf{L}},\widetilde{\lambda}} : W_{\widetilde{\mathbf{K}}}\left(\widetilde{\mathbf{L}},\widetilde{\lambda}\right)^F \to W_{\mathbf{C}_{\widetilde{\mathbf{K}}^*}(s)}\left(\mathbf{C}_{\widetilde{\mathbf{L}}^*}(s), J_{\widetilde{\mathbf{L}},s}\left(\widetilde{\lambda}\right)\right)^{F^*}$$

such that

$$\sigma^*\circ i^{\widetilde{\mathbf{K}}}_{\widetilde{\mathbf{L}},\widetilde{\lambda}} = i^{\widetilde{\mathbf{K}}}_{\widetilde{\mathbf{L}},\widetilde{\lambda}^{\sigma}}\circ\sigma$$

for every $\sigma \in \mathrm{Aut}_{\mathbb{F}}(\widetilde{\mathbf{G}}^F)_{\widetilde{\mathbf{K}},\widetilde{\mathbf{L}}}$ *and* $\sigma^* \in \mathrm{Aut}_{\mathbb{F}}(\widetilde{\mathbf{G}}^{*F^*})_{\widetilde{\mathbf{K}}^*,\widetilde{\mathbf{L}}^*}$ *with* $\sigma\sim\sigma^*$ *(see Lemma 6.1.5 and [CS13, Proposition 2.2]). Moreover, if* $z \in \mathbf{Z}(\widetilde{\mathbf{K}}^{*F^*})$ *corresponds to the character* $\widehat{z}_{\widetilde{\mathbf{L}}} \in \mathrm{Irr}(\widetilde{\mathbf{L}}^F/\mathbf{L}^F)$ *via* (6.1.2)*, then*

$$W_{\widetilde{\mathbf{K}}}\left(\widetilde{\mathbf{L}},\widetilde{\lambda}\right)^F = W_{\widetilde{\mathbf{K}}}\left(\widetilde{\mathbf{L}},\widetilde{\lambda}\cdot\widehat{z}_{\widetilde{\mathbf{L}}}\right)^F,$$

$$W_{\mathbf{C}_{\widetilde{\mathbf{K}}^*}(s)}\left(\mathbf{C}_{\widetilde{\mathbf{L}}^*}(s), J_{\widetilde{\mathbf{L}},s}\left(\widetilde{\lambda}\right)\right)^{F^*} = W_{\mathbf{C}_{\widetilde{\mathbf{K}}^*}(sz)}\left(\mathbf{C}_{\widetilde{\mathbf{L}}^*}(sz), J_{\widetilde{\mathbf{L}},sz}\left(\widetilde{\lambda}\cdot\widehat{z}_{\widetilde{\mathbf{L}}}\right)\right)^{F^*}$$

and

$$i^{\widetilde{\mathbf{K}}}_{\widetilde{\mathbf{L}},\widetilde{\lambda}} = i^{\widetilde{\mathbf{K}}}_{\widetilde{\mathbf{L}},\widetilde{\lambda}\cdot\widehat{z}_{\widetilde{\mathbf{L}}}}$$

Proof. The first statement follows from the proof of [CS13, Theorem 3.3]. The second statement follows from 8.1.3 (iv). □

Before proving Theorem 8.1, we state an equivariant version of Theorem 6.2.14 that has been proved in [CS13].

Theorem 8.1.5. *Let $\mathbf{H}$ be a connected reductive group with a Frobenius endomorphism $F:\mathbf{H}\to\mathbf{H}$ defining an $\mathbb{F}_q$-structure on $\mathbf{H}$, ℓ a prime not dividing q and e the order of q modulo ℓ (modulo 4 if $\ell=2$). For any e-split Levi subgroup $\mathbf{M}$ of $\mathbf{H}$ and $\mu\in\mathcal{E}(\mathbf{M}^F,[1])$ with $(\mathbf{M},\mu)$ a unipotent e-cuspidal pair, there exists an $\mathrm{Aut}_{\mathbb{F}}(\mathbf{H}^F)_{(\mathbf{M},\mu)}$-equivariant bijection*

$$I^{\mathbf{H}}_{(\mathbf{M},\mu)}:\mathrm{Irr}\left(W_{\mathbf{H}}(\mathbf{M},\mu)^F\right)\to\mathcal{E}\left(\mathbf{H}^F,(\mathbf{M},\mu)\right)$$

such that

$$I^{\mathbf{H}}_{(\mathbf{M},\mu)}(\eta)(1)_\ell=\left|\mathbf{H}^F:\mathbf{N}_{\mathbf{H}}(\mathbf{M},\mu)^F\right|_\ell\cdot\mu(1)_\ell\cdot\eta(1)_\ell$$

for every $\eta\in\mathrm{Irr}(W_{\mathbf{H}}(\mathbf{M},\mu)^F)$.

Proof. This follows from the proof of [CS13, Theorem 3.4] applied to arbitrary e-split Levi subgroups (see the comment in the proof of [BS20b, Proposition 5.5]). Regarding the statement on character degrees, see [Mal07, Theorem 4.2] and the argument used to prove [BS20b, Lemma 5.3]. □

We now extend Theorem 8.1.5 to nonunipotent e-cuspidal pairs in the case that $\mathbf{H}$ has a connected center. Let $\widetilde{\mathbf{L}}$ and $\widetilde{\mathbf{K}}$ be e-split Levi subgroups of $\widetilde{\mathbf{G}}$ with $\widetilde{\mathbf{L}}\leq\widetilde{\mathbf{K}}$ and consider $\widetilde{\lambda}\in\mathcal{E}(\widetilde{\mathbf{L}}^F,[s])$ such that $(\widetilde{\mathbf{L}},\widetilde{\lambda})$ is an e-cuspidal pair. Notice that, by Proposition 6.2.11, the unipotent character $J_{\widetilde{\mathbf{L}},s}(\widetilde{\lambda})$ is e-cuspidal. Moreover, using the fact that $\widetilde{\mathbf{L}}$ is an e-split Levi subgroup of $\widetilde{\mathbf{K}}$, we conclude that $\mathbf{C}_{\widetilde{\mathbf{L}}^*}(s)$ is an e-split Levi subgroup of $\mathbf{C}_{\widetilde{\mathbf{K}}^*}(s)$. This shows that $(\mathbf{C}_{\widetilde{\mathbf{L}}^*}(s),J_{\widetilde{\mathbf{L}},s}(\widetilde{\lambda}))$ is a unipotent e-cuspidal pair of $\mathbf{C}_{\widetilde{\mathbf{K}}^*}(s)$. Now, we can define the map

$$I^{\widetilde{\mathbf{K}}}_{(\widetilde{\mathbf{L}},\widetilde{\lambda})}:\mathrm{Irr}\left(W_{\widetilde{\mathbf{K}}}\left(\widetilde{\mathbf{L}},\widetilde{\lambda}\right)^F\right)\to\mathcal{E}\left(\widetilde{\mathbf{K}}^F,\left(\widetilde{\mathbf{L}},\widetilde{\lambda}\right)\right)\tag{8.1.1}$$

given by

$$I^{\widetilde{\mathbf{K}}}_{(\widetilde{\mathbf{L}},\widetilde{\lambda})}\left(\widetilde{\eta}\right):=J^{-1}_{\widetilde{\mathbf{K}},s}\left(I^{\mathbf{C}_{\widetilde{\mathbf{K}}^*}(s)}_{(\mathbf{C}_{\widetilde{\mathbf{L}}^*}(s),J_{\widetilde{\mathbf{L}},s}(\widetilde{\lambda}))}\left(\left(\widetilde{\eta}\right)^{i^{\widetilde{\mathbf{K}}}_{\widetilde{\mathbf{L}},\widetilde{\lambda}}}\right)\right)$$

for every $\widetilde{\eta}\in\mathrm{Irr}(W_{\widetilde{\mathbf{K}}}(\widetilde{\mathbf{L}},\widetilde{\lambda})^F)$ and where $(\widetilde{\eta})^{i^{\widetilde{\mathbf{K}}}_{\widetilde{\mathbf{L}},\widetilde{\lambda}}}\in\mathrm{Irr}(W_{\mathbf{C}_{\widetilde{\mathbf{K}}^*}(s)}(\mathbf{C}_{\widetilde{\mathbf{L}}^*}(s),J_{\widetilde{\mathbf{L}},s}(\widetilde{\lambda}))^{F^*})$ corresponds to $\widetilde{\eta}$ via the isomorphism $i^{\widetilde{\mathbf{K}}}_{\widetilde{\mathbf{L}},\widetilde{\lambda}}$ of Corollary 8.1.4.

Lemma 8.1.6. *Assume Hypothesis 8.1.2. Then the map $I^{\widetilde{\mathbf{K}}}_{(\widetilde{\mathbf{L}},\widetilde{\lambda})}$ is an $\mathrm{Aut}_{\mathbb{F}}(\widetilde{\mathbf{G}}^F)_{\widetilde{\mathbf{K}},(\widetilde{\mathbf{L}},\widetilde{\lambda})}$-equivariant bijection.*

Proof. First, we observe that the map $I^{\widetilde{\mathbf{K}}}_{(\widetilde{\mathbf{L}},\widetilde{\lambda})}$ is a bijection because of Theorem 8.1.3 (ii), in fact

$$\begin{aligned}I^{\widetilde{\mathbf{K}}}_{(\widetilde{\mathbf{L}},\widetilde{\lambda})}\left(\mathrm{Irr}\left(W_{\widetilde{\mathbf{K}}}\left(\widetilde{\mathbf{L}},\widetilde{\lambda}\right)^F\right)\right)&=J^{-1}_{\widetilde{\mathbf{K}},s}\left(\mathrm{Irr}\left(\mathbf{R}^{\mathbf{C}_{\widetilde{\mathbf{K}}^*}(s)}_{\mathbf{C}_{\widetilde{\mathbf{L}}^*}(s)}\left(J_{\widetilde{\mathbf{L}},s}\left(\widetilde{\lambda}\right)\right)\right)\right)\\&=\mathrm{Irr}\left(J^{-1}_{\widetilde{\mathbf{K}},s}\circ\mathbf{R}^{\mathbf{C}_{\widetilde{\mathbf{K}}^*}(s)}_{\mathbf{C}_{\widetilde{\mathbf{L}}^*}(s)}\circ J_{\widetilde{\mathbf{L}},s}\left(\widetilde{\lambda}\right)\right)\\&=\mathrm{Irr}\left(\mathbf{R}^{\widetilde{\mathbf{K}}}_{\widetilde{\mathbf{L}}}\left(\widetilde{\lambda}\right)\right).\end{aligned}$$

Next, to show that the bijection is equivariant, let $\sigma\in\mathrm{Aut}_{\mathbb{F}}(\widetilde{\mathbf{G}}^F)_{\widetilde{\mathbf{K}},\widetilde{\mathbf{L}}}$ and consider $\sigma^*\in\mathrm{Aut}_{\mathbb{F}}(\widetilde{\mathbf{G}}^{*F^*})_{\widetilde{\mathbf{K}}^*,\widetilde{\mathbf{L}}^*}$ with $\sigma\sim\sigma^*$ (see Lemma 6.1.5). If $\sigma\in\mathrm{Aut}_{\mathbb{F}}(\widetilde{\mathbf{G}}^F)_{\widetilde{\mathbf{K}},(\widetilde{\mathbf{L}},\widetilde{\lambda})}$, then σ^* stabilizes the

$\widetilde{\mathbf{L}}^{*F^*}$-orbit of s. Without loss of generality, we may assume that $\sigma^*(s) = s$. Then Theorem 8.1.3 (i) implies that σ^* stabilizes $J_{\widetilde{\mathbf{L}},s}(\widetilde{\lambda})$. Applying Theorem 8.1.3 (i) and the equivariance properties of Corollary 8.1.4 and Theorem 8.1.5, we conclude that

$$\begin{aligned}
I^{\widetilde{\mathbf{K}}}_{(\widetilde{\mathbf{L}},\widetilde{\lambda})}(\widetilde{\eta})^\sigma &:= J^{-1}_{\widetilde{\mathbf{K}},s}\left(I^{\mathbf{C}_{\widetilde{\mathbf{K}}^*}(s)}_{(\mathbf{C}_{\widetilde{\mathbf{L}}^*}(s),J_{\widetilde{\mathbf{L}},s}(\widetilde{\lambda}))}\left(\widetilde{\eta}^{\,i^{\widetilde{\mathbf{K}}}_{\widetilde{\mathbf{L}},\widetilde{\lambda}}}\right)\right)^\sigma \\
&= J^{-1}_{\widetilde{\mathbf{K}},s}\left(I^{\mathbf{C}_{\widetilde{\mathbf{K}}^*}(s)}_{(\mathbf{C}_{\widetilde{\mathbf{L}}^*}(s),J_{\widetilde{\mathbf{L}},s}(\widetilde{\lambda}))}\left(\left(\widetilde{\eta}^{\,i^{\widetilde{\mathbf{K}}}_{\widetilde{\mathbf{L}},\widetilde{\lambda}}}\right)^{\sigma^*}\right)\right) \\
&= J^{-1}_{\widetilde{\mathbf{K}},s}\left(I^{\mathbf{C}_{\widetilde{\mathbf{K}}^*}(s)}_{(\mathbf{C}_{\widetilde{\mathbf{L}}^*}(s),J_{\widetilde{\mathbf{L}},s}(\widetilde{\lambda}))}\left((\widetilde{\eta}^\sigma)^{i^{\widetilde{\mathbf{K}}}_{\widetilde{\mathbf{L}},\widetilde{\lambda}}}\right)\right) \\
&=: I^{\widetilde{\mathbf{K}}}_{(\widetilde{\mathbf{L}},\widetilde{\lambda})}(\widetilde{\eta}^\sigma)
\end{aligned}$$

for every $\widetilde{\eta} \in \mathrm{Irr}(W_{\widetilde{\mathbf{K}}}(\widetilde{\mathbf{L}},\widetilde{\lambda})^F)$. □

Lemma 8.1.7. *Assume Hypothesis 8.1.2. Then* $I^{\widetilde{\mathbf{K}}}_{(\widetilde{\mathbf{L}},\widetilde{\lambda})}(\widetilde{\eta})(1)_\ell = \left|\widetilde{\mathbf{K}}^F : \mathbf{N}_{\widetilde{\mathbf{K}}}(\widetilde{\mathbf{L}},\widetilde{\lambda})^F\right|_\ell \cdot \widetilde{\lambda}(1)_\ell \cdot \widetilde{\eta}(1)_\ell$ *for every* $\widetilde{\eta} \in \mathrm{Irr}(W_{\widetilde{\mathbf{K}}}(\widetilde{\mathbf{L}},\widetilde{\lambda})^F)$.

Proof. By the condition on character degrees given in Theorem 8.1.5 together with Theorem 8.1.3 (iii), we deduce that

$$\begin{aligned}
\widetilde{\eta}(1)_\ell = (\widetilde{\eta})^{i^{\widetilde{\mathbf{K}}}_{\widetilde{\mathbf{L}},\widetilde{\lambda}}}(1)_\ell &= \frac{I^{\mathbf{C}_{\widetilde{\mathbf{K}}^*}(s)}_{(\mathbf{C}_{\widetilde{\mathbf{L}}^*}(s),J_{\widetilde{\mathbf{L}},s}(\widetilde{\lambda}))}\left((\widetilde{\eta})^{i^{\widetilde{\mathbf{K}}}_{\widetilde{\mathbf{L}},\widetilde{\lambda}}}\right)(1)_\ell}{J_{\widetilde{\mathbf{L}},s}\left(\widetilde{\lambda}\right)(1)_\ell \cdot \left|\mathbf{C}_{\widetilde{\mathbf{K}}^*}(s)^{F^*} : \mathbf{N}_{\mathbf{C}_{\widetilde{\mathbf{K}}^*}(s)}(\mathbf{C}_{\widetilde{\mathbf{L}}^*}(s), J_{\widetilde{\mathbf{L}},s}\left(\widetilde{\lambda}\right))^{F^*}\right|_\ell} \\
&= \frac{I^{\widetilde{\mathbf{K}}}_{(\widetilde{\mathbf{L}},\widetilde{\lambda})}(\widetilde{\eta})(1)_\ell \cdot \left|\mathbf{C}_{\widetilde{\mathbf{K}}^*}(s)^{F^*}\right|_\ell \cdot \left|\widetilde{\mathbf{L}}^F\right|_\ell}{\widetilde{\lambda}(1)_\ell \cdot |\mathbf{C}_{\widetilde{\mathbf{L}}^*}(s)^{F^*}|_\ell \cdot |\widetilde{\mathbf{K}}^F|_\ell \cdot \left|\mathbf{C}_{\widetilde{\mathbf{K}}^*}(s)^{F^*} : \mathbf{N}_{\mathbf{C}_{\widetilde{\mathbf{K}}^*}(s)}(\mathbf{C}_{\widetilde{\mathbf{L}}^*}(s), J_{\widetilde{\mathbf{L}},s}\left(\widetilde{\lambda}\right))^{F^*}\right|_\ell} \\
&= \frac{I^{\widetilde{\mathbf{K}}}_{(\widetilde{\mathbf{L}},\widetilde{\lambda})}(\widetilde{\eta})(1)_\ell}{\widetilde{\lambda}(1)_\ell \cdot \left|\widetilde{\mathbf{K}}^F : \mathbf{N}_{\widetilde{\mathbf{K}}}(\widetilde{\mathbf{L}},\widetilde{\lambda})^F\right|_\ell}
\end{aligned}$$

for every $\widetilde{\eta} \in \mathrm{Irr}(W_{\widetilde{\mathbf{K}}}(\widetilde{\mathbf{L}},\widetilde{\lambda})^F)$. □

Lemma 8.1.8. *Assume Hypothesis 8.1.2. If* $z \in \mathbf{Z}(\widetilde{\mathbf{K}}^{*F^*})$ *corresponds to the characters* $\widehat{z}_{\widetilde{\mathbf{L}}} \in \mathrm{Irr}(\widetilde{\mathbf{L}}^F/\mathbf{L}^F)$ *and* $\widehat{z}_{\widetilde{\mathbf{K}}} \in \mathrm{Irr}(\widetilde{\mathbf{K}}^F/\mathbf{K}^F)$*, then* $\widetilde{\lambda}\cdot\widehat{z}_{\widetilde{\mathbf{L}}}$ *is* e*-cuspidal,* $W_{\widetilde{\mathbf{K}}}(\widetilde{\mathbf{L}},\widetilde{\lambda})^F = W_{\widetilde{\mathbf{K}}}(\widetilde{\mathbf{L}},\widetilde{\lambda}\cdot\widehat{z}_{\widetilde{\mathbf{L}}})^F$ *and*

$$I^{\widetilde{\mathbf{K}}}_{(\widetilde{\mathbf{L}},\widetilde{\lambda})}(\widetilde{\eta})\cdot\widehat{z}_{\widetilde{\mathbf{K}}} = I^{\widetilde{\mathbf{K}}}_{(\widetilde{\mathbf{L}},\widetilde{\lambda}\cdot\widehat{z}_{\widetilde{\mathbf{L}}})}(\widetilde{\eta})$$

for every $\widetilde{\eta} \in \mathrm{Irr}(W_{\widetilde{\mathbf{K}}}(\widetilde{\mathbf{L}},\widetilde{\lambda})^F)$.

Proof. We start by noticing that, by [Bon06, Proposition 12.1], the character $\widetilde{\lambda}\cdot\widehat{z}_{\widetilde{\mathbf{L}}}$ is e-cuspidal, while Corollary 8.1.4 shows that $W_{\widetilde{\mathbf{K}}}(\widetilde{\mathbf{L}},\widetilde{\lambda})^F = W_{\widetilde{\mathbf{K}}}(\widetilde{\mathbf{L}},\widetilde{\lambda}\cdot\widehat{z}_{\widetilde{\mathbf{L}}})^F$ and that $i^{\widetilde{\mathbf{K}}}_{\widetilde{\mathbf{L}},\widetilde{\lambda}} = i^{\widetilde{\mathbf{K}}}_{\widetilde{\mathbf{L}},\widetilde{\lambda}\cdot\widehat{z}_{\widetilde{\mathbf{L}}}}$. Using Theorem 8.1.3 (iv) we obtain

$$I^{\mathbf{C}_{\widetilde{\mathbf{K}}^*}(s)}_{(\mathbf{C}_{\widetilde{\mathbf{L}}^*}(s),J_{\widetilde{\mathbf{L}},s}(\widetilde{\lambda}))} = I^{\mathbf{C}_{\widetilde{\mathbf{K}}^*}(sz)}_{(\mathbf{C}_{\widetilde{\mathbf{L}}^*}(sz),J_{\widetilde{\mathbf{L}},sz}(\widetilde{\lambda}\cdot\widehat{z}_{\widetilde{\mathbf{L}}}))}$$

and

$$\begin{aligned}I^{\widetilde{\mathbf{K}}}_{(\widetilde{\mathbf{L}},\widetilde{\lambda})}(\widetilde{\eta})\cdot\widehat{z}_{\widetilde{\mathbf{K}}} &:= J^{-1}_{\widetilde{\mathbf{K}},s}\left(I^{\mathbf{C}_{\widetilde{\mathbf{K}}^*}(s)}_{(\mathbf{C}_{\widetilde{\mathbf{L}}^*}(s),J_{\widetilde{\mathbf{L}},s}(\widetilde{\lambda}))}\left((\widetilde{\eta})^{i^{\widetilde{\mathbf{K}}}_{\widetilde{\mathbf{L}},\widetilde{\lambda}}}\right)\right)\cdot\widehat{z}_{\widetilde{\mathbf{K}}}\\
&= J^{-1}_{\widetilde{\mathbf{K}},s}\left(I^{\mathbf{C}_{\widetilde{\mathbf{K}}^*}(sz)}_{(\mathbf{C}_{\widetilde{\mathbf{L}}^*}(sz),J_{\widetilde{\mathbf{L}},sz}(\widetilde{\lambda}\cdot\widehat{z}_{\widetilde{\mathbf{L}}}))}\left((\widetilde{\eta})^{i^{\widetilde{\mathbf{K}}}_{\widetilde{\mathbf{L}},\widetilde{\lambda}\cdot\widehat{z}_{\widetilde{\mathbf{L}}}}}\right)\right)\cdot\widehat{z}_{\widetilde{\mathbf{K}}}\\
&= J^{-1}_{\widetilde{\mathbf{K}},sz}\left(I^{\mathbf{C}_{\widetilde{\mathbf{K}}^*}(sz)}_{(\mathbf{C}_{\widetilde{\mathbf{L}}^*}(sz),J_{\widetilde{\mathbf{L}},sz}(\widetilde{\lambda}\cdot\widehat{z}_{\widetilde{\mathbf{L}}}))}\left((\widetilde{\eta})^{i^{\widetilde{\mathbf{K}}}_{\widetilde{\mathbf{L}},\widetilde{\lambda}\cdot\widehat{z}_{\widetilde{\mathbf{L}}}}}\right)\right)\\
&= I^{\widetilde{\mathbf{K}}}_{(\widetilde{\mathbf{L}},\widetilde{\lambda}\cdot\widehat{z}_{\widetilde{\mathbf{L}}})}(\widetilde{\eta})\end{aligned}$$

for every $\widetilde{\eta}\in\mathrm{Irr}(W_{\widetilde{\mathbf{K}}}(\widetilde{\mathbf{L}},\widetilde{\lambda})^F)$. □

Now, combining Lemma 8.1.6, Lemma 8.1.7 and Lemma 8.1.8, we obtain Theorem 8.1.

8.2 Equivariant maximal extendibility

We start by recalling the definition of maximal extendibility (see [MS16, Definition 3.5]).

Definition 8.2.1. Let $Y \trianglelefteq X$ be finite groups and consider $\mathcal{Y}\subseteq\mathrm{Irr}(Y)$. Then, we say that **maximal extendibility** holds for $\mathcal{Y}$ with respect to $Y\trianglelefteq X$ if every $\vartheta\in\mathcal{Y}$ extends to X_ϑ. In this case, an **extension map** is any map

$$\Lambda:\mathcal{Y}\to\coprod_{Y\leq X'\leq X}\mathrm{Irr}(X')$$

such that for every $\vartheta\in\mathcal{Y}$, the character $\Lambda(\vartheta)\in\mathrm{Irr}(X_\vartheta)$ is an extension of ϑ. If $\mathcal{Y}=\mathrm{Irr}(Y)$, then we just say that maximal extendibility holds with respect to $Y\trianglelefteq X$.

From now on, suppose that $\mathbf{G}$ is simple of simply connected type. As in Section 6.1.5 fix a maximally split torus $\mathbf{T}_0$ contained in an F-stable Borel subgroup $\mathbf{B}_0$, and consider the corresponding group $\mathcal{A}$ generated by field and graph automorphisms (see the discussion preceding 6.1.6) in such a way that $\mathcal{A}$ acts on $\widetilde{\mathbf{G}}^F$. We then form the semidirect product $\widetilde{\mathbf{G}}^F\rtimes\mathcal{A}$.

Now, let $\widetilde{\mathbf{L}}$ and $\widetilde{\mathbf{K}}$ be e-split Levi subgroups of $\widetilde{\mathbf{G}}$ and consider an extension map $\widetilde{\Lambda}$ with respect to $\widetilde{\mathbf{L}}\trianglelefteq\mathbf{N}_{\widetilde{\mathbf{K}}}(\mathbf{L})^F$. The group $(\widetilde{\mathbf{G}}^F\mathcal{A})_{\mathbf{K},\mathbf{L}}\ltimes\mathcal{K}$ acts on the set $\mathrm{Irr}(\widetilde{\mathbf{L}}^F)$ via

$$\widetilde{\lambda}^{xz}:=\widetilde{\lambda}^x\cdot\widehat{z}_{\widetilde{\mathbf{L}}}$$

for every $\widetilde{\lambda}\in\mathrm{Irr}(\widetilde{\mathbf{L}}^F)$, $x\in(\widetilde{\mathbf{G}}^F\mathcal{A})_{\mathbf{K},\mathbf{L}}$ and $z\in\mathcal{K}$. In this case notice that

$$\widetilde{\Lambda}(\widetilde{\lambda})^{xz}:=\widetilde{\Lambda}(\widetilde{\lambda})^x\cdot\widehat{z}_{\mathbf{N}_{\widetilde{\mathbf{K}}}(\widetilde{\mathbf{L}},\widetilde{\lambda}^x)^F}$$

is an extension of $\widetilde{\lambda}^{xz}$ to $\mathbf{N}_{\widetilde{\mathbf{K}}}(\widetilde{\mathbf{L}},\widetilde{\lambda}^{xz})^F=\mathbf{N}_{\widetilde{\mathbf{K}}}(\widetilde{\mathbf{L}},\widetilde{\lambda}^x)^F$. We say that the an extension map $\widetilde{\Lambda}$ with respect to $\widetilde{\mathbf{L}}^F\trianglelefteq\mathbf{N}_{\widetilde{\mathbf{K}}}(\mathbf{L})^F$ is $(\widetilde{\mathbf{G}}^F\mathcal{A})_{\mathbf{K},\mathbf{L}}\ltimes\mathcal{K}$-equivariant if $\widetilde{\Lambda}(\widetilde{\lambda}^{xz})=\widetilde{\Lambda}(\widetilde{\lambda})^{xz}$ for every $\widetilde{\lambda}\in\mathrm{Irr}(\widetilde{\mathbf{L}}^F)$, $x\in(\widetilde{\mathbf{G}}^F\mathcal{A})_{\mathbf{K},\mathbf{L}}$ and $z\in\mathcal{K}$. Moreover, if $\mathrm{Cusp}_e(\widetilde{\mathbf{L}}^F)$ denotes the set of (irreducible)

e-cuspidal characters of $\mathbf{L}^F$, then $(\widetilde{\mathbf{G}}^F\mathcal{A})_{\mathbf{K},\mathbf{L}} \ltimes \mathcal{K}$ acts on $\mathrm{Cusp}_e(\widetilde{\mathbf{L}}^F)$ (see [Bon06, Proposition 12.1]) and therefore we can also ask for a $(\widetilde{\mathbf{G}}^F\mathcal{A})_{\mathbf{K},\mathbf{L}} \ltimes \mathcal{K}$-equivariant extension map $\widetilde{\Lambda}$ for $\mathrm{Cusp}_e(\widetilde{\mathbf{L}}^F)$ with respect to $\widetilde{\mathbf{L}}^F \trianglelefteq \mathbf{N}_{\widetilde{\mathbf{K}}}(\mathbf{L})^F$.

Now, let $\widetilde{\mathbf{K}}$ be an e-split Levi subgroup of $\widetilde{\mathbf{G}}$ and consider an e-cuspidal pair $(\widetilde{\mathbf{L}},\widetilde{\lambda})$ of $\widetilde{\mathbf{K}}$. Using the bijection $I^{\widetilde{\mathbf{K}}}_{(\widetilde{\mathbf{L}},\widetilde{\lambda})}$ from (8.1.1) and assuming the existence of an extension map $\widetilde{\Lambda}$ for $\mathrm{Cusp}_e(\widetilde{\mathbf{L}}^F)$ with respect to $\widetilde{\mathbf{L}}^F \trianglelefteq \mathbf{N}_{\widetilde{\mathbf{K}}}(\mathbf{L})^F$, we can define the map

$$\Upsilon : \mathcal{E}\left(\widetilde{\mathbf{K}}^F, \left(\widetilde{\mathbf{L}},\widetilde{\lambda}\right)\right) \to \mathrm{Irr}\left(\mathbf{N}_{\widetilde{\mathbf{K}}}(\mathbf{L})^F \;\middle|\; \widetilde{\lambda}\right) \qquad (8.2.1)$$
$$I^{\widetilde{\mathbf{K}}}_{(\widetilde{\mathbf{L}},\widetilde{\lambda})}(\widetilde{\eta}) \mapsto \left(\widetilde{\Lambda}\left(\widetilde{\lambda}\right)\cdot\widetilde{\eta}\right)^{\mathbf{N}_{\widetilde{\mathbf{K}}}(\mathbf{L})^F}$$

for every $\widetilde{\eta} \in \mathrm{Irr}\left(W_{\widetilde{\mathbf{K}}}\left(\widetilde{\mathbf{L}},\widetilde{\lambda}\right)^F\right)$. Notice that Υ is a bijection by the Clifford correspondence and Gallagher's theorem (see [Isa76, Theorem 6.11 and Corollary 6.17]).

First we show that the bijection Υ from (8.2.1) preserves the ℓ-defect of characters.

Lemma 8.2.2. *Assume Hypothesis 8.1.2 and suppose there exists an extension map $\widetilde{\Lambda}$ for $\mathrm{Cusp}_e(\widetilde{\mathbf{L}}^F)$ with respect to $\widetilde{\mathbf{L}}^F \trianglelefteq \mathbf{N}_{\widetilde{\mathbf{K}}}(\mathbf{L})^F$. For every $\widetilde{\eta} \in \mathrm{Irr}\left(W_{\widetilde{\mathbf{K}}}\left(\widetilde{\mathbf{L}},\widetilde{\lambda}\right)^F\right)$ we have*

$$d\left(I^{\widetilde{\mathbf{K}}}_{(\widetilde{\mathbf{L}},\widetilde{\lambda})}(\widetilde{\eta})\right) = d\left(\left(\widetilde{\Lambda}\left(\widetilde{\lambda}\right)\cdot\widetilde{\eta}\right)^{\mathbf{N}_{\widetilde{\mathbf{K}}}(\mathbf{L})^F}\right).$$

Proof. This follows immediately from Lemma 8.1.7 after noticing that induction of characters preserves the defect (this follows from the degree formula for induced characters). □

The bijection Υ from (8.2.1) also preserves central characters.

Lemma 8.2.3. *Assume Hypothesis 8.1.2 and suppose there exists an extension map $\widetilde{\Lambda}$ for $\mathrm{Cusp}_e(\widetilde{\mathbf{L}}^F)$ with respect to $\widetilde{\mathbf{L}}^F \trianglelefteq \mathbf{N}_{\widetilde{\mathbf{K}}}(\mathbf{L})^F$. For every $\widetilde{\eta} \in \mathrm{Irr}\left(W_{\widetilde{\mathbf{K}}}\left(\widetilde{\mathbf{L}},\widetilde{\lambda}\right)^F\right)$ we have*

$$\mathrm{Irr}\left(I^{\widetilde{\mathbf{K}}}_{(\widetilde{\mathbf{L}},\widetilde{\lambda})}(\widetilde{\eta})_{\mathbf{Z}(\widetilde{\mathbf{K}}^F)}\right) = \mathrm{Irr}\left(\left(\left(\widetilde{\Lambda}\left(\widetilde{\lambda}\right)\cdot\widetilde{\eta}\right)^{\mathbf{N}_{\widetilde{\mathbf{K}}}(\mathbf{L})^F}\right)_{\mathbf{Z}(\widetilde{\mathbf{K}}^F)}\right).$$

Proof. First, by Clifford theory we deduce that

$$\mathrm{Irr}\left(\left(\left(\widetilde{\Lambda}\left(\widetilde{\lambda}\right)\cdot\widetilde{\eta}\right)^{\mathbf{N}_{\widetilde{\mathbf{K}}}(\mathbf{L})^F}\right)_{\mathbf{Z}(\widetilde{\mathbf{G}}^F)}\right) = \mathrm{Irr}\left(\widetilde{\lambda}_{\mathbf{Z}(\widetilde{\mathbf{G}}^F)}\right). \qquad (8.2.2)$$

On the other hand, by using the character formula [DM91, Proposition 12.2 (i)], we obtain

$$\mathbf{R}^{\widetilde{\mathbf{K}}}_{\widetilde{\mathbf{L}}}(\widetilde{\lambda})_{\mathbf{Z}(\widetilde{\mathbf{K}}^F)} = \mathbf{R}^{\widetilde{\mathbf{K}}}_{\widetilde{\mathbf{L}}}(\widetilde{\lambda})(1)\cdot\widetilde{\lambda}_{\mathbf{Z}(\widetilde{\mathbf{K}}^F)}$$

and hence

$$\mathrm{Irr}\left(I^{\widetilde{\mathbf{K}}}_{(\widetilde{\mathbf{L}},\widetilde{\lambda})}(\widetilde{\eta})_{\mathbf{Z}(\widetilde{\mathbf{K}}^F)}\right) = \mathrm{Irr}\left(\widetilde{\lambda}_{\mathbf{Z}(\widetilde{\mathbf{K}}^F)}\right). \qquad (8.2.3)$$

Now the result follows from (8.2.2) together with (8.2.3). □

Using the results obtain in Chapter 7, we show that the bijection Υ from (8.2.1) is compatible with block induction.

Lemma 8.2.4. *Assume Hypothesis 7.2.7, Hypothesis 8.1.2 and suppose there exists an extension map* $\widetilde{\Lambda}$ *for* $\mathrm{Cusp}_e(\widetilde{\mathbf{L}}^F)$ *with respect to* $\widetilde{\mathbf{L}}^F \trianglelefteq \mathbf{N}_{\widetilde{\mathbf{K}}}(\mathbf{L})^F$*. Then* $\mathrm{bl}(\widetilde{\chi}) = \mathrm{bl}(\Upsilon(\widetilde{\chi}))^{\widetilde{\mathbf{K}}^F}$ *for every* $\widetilde{\chi} \in \mathcal{E}(\widetilde{\mathbf{K}}^F, (\widetilde{\mathbf{L}}, \widetilde{\lambda}))$.

Proof. Since $\mathrm{bl}(\widetilde{\lambda})^{\widetilde{\mathbf{K}}^F} = \mathrm{bl}(\widetilde{\chi})$ by Proposition 7.2.14 and $\mathrm{bl}(\Upsilon(\widetilde{\chi})) = \mathrm{bl}(\widetilde{\lambda})^{\widetilde{\mathbf{K}}^F_{\mathbf{L}}}$ by Lemma 9.2.5, the result follows by the transitivity of block induction. □

Finally, we show that the bijection Υ from (8.2.1) is equivariant.

Lemma 8.2.5. *Assume Hypothesis 8.1.2 and suppose there exists a* $(\widetilde{\mathbf{G}}^F\mathcal{A})_{\mathbf{K},\mathbf{L}} \ltimes \mathcal{K}$*-equivariant extension map* $\widetilde{\Lambda}$ *for* $\mathrm{Cusp}_e(\widetilde{\mathbf{L}}^F)$ *with respect to* $\widetilde{\mathbf{L}}^F \trianglelefteq \mathbf{N}_{\widetilde{\mathbf{K}}}(\mathbf{L})^F$*. Then* Υ *is* $\left((\widetilde{\mathbf{G}}^F\mathcal{A})_{\mathbf{K},\mathbf{L}} \ltimes \mathcal{K}\right)_{\widetilde{\lambda}}$*-equivariant.*

Proof. Let $(x,z) \in \left((\widetilde{\mathbf{G}}^F\mathcal{A})_{\mathbf{K},\mathbf{L}} \ltimes \mathcal{K}\right)_{\widetilde{\lambda}}$ and notice that, as $\widetilde{\lambda} = \widetilde{\lambda}^x \cdot \widehat{z}_{\widetilde{\mathbf{L}}}$, we have $\mathbf{N}_{\widetilde{\mathbf{K}}}(\mathbf{L}, \widetilde{\lambda})^F = \mathbf{N}_{\widetilde{\mathbf{K}}}(\mathbf{L}, \widetilde{\lambda}^x \cdot \widehat{z}_{\widetilde{\mathbf{L}}})^F = \mathbf{N}_{\widetilde{\mathbf{K}}}(\mathbf{L}, \widetilde{\lambda}^x)^F$. By using the equivariance properties of $\widetilde{\Lambda}$, we obtain

$$\begin{aligned}
\left(\left(\widetilde{\Lambda}\left(\widetilde{\lambda}\right) \cdot \widetilde{\eta}\right)^{\mathbf{N}_{\widetilde{\mathbf{K}}}(\mathbf{L})^F}\right)^{(x,z)} &= \left(\left(\widetilde{\Lambda}\left(\widetilde{\lambda}\right) \cdot \widetilde{\eta}\right)^x\right)^{\mathbf{N}_{\widetilde{\mathbf{K}}}(\mathbf{L})^F} \cdot \widehat{z}_{\mathbf{N}_{\widetilde{\mathbf{K}}}(\mathbf{L})} \\
&= \left(\left(\widetilde{\Lambda}\left(\widetilde{\lambda}\right) \cdot \widetilde{\eta}\right)^x \cdot \widehat{z}_{\mathbf{N}_{\widetilde{\mathbf{K}}}(\mathbf{L},\widetilde{\lambda})^F}\right)^{\mathbf{N}_{\widetilde{\mathbf{K}}}(\mathbf{L})^F} \\
&= \left(\widetilde{\Lambda}\left(\widetilde{\lambda}^x \cdot \widehat{z}_{\widetilde{\mathbf{L}}}\right) \cdot \widetilde{\eta}^x\right)^{\mathbf{N}_{\widetilde{\mathbf{K}}}(\mathbf{L})^F} \\
&= \left(\widetilde{\Lambda}\left(\widetilde{\lambda}\right) \cdot \widetilde{\eta}^x\right)^{\mathbf{N}_{\widetilde{\mathbf{K}}}(\mathbf{L})^F}.
\end{aligned} \tag{8.2.4}$$

On the other hand Lemma 8.1.8 implies

$$\begin{aligned}
I^{\widetilde{\mathbf{K}}}_{(\widetilde{\mathbf{L}},\widetilde{\lambda})}\left(\widetilde{\eta}\right)^{(x,z)} &= I^{\widetilde{\mathbf{K}}}_{(\widetilde{\mathbf{L}},\widetilde{\lambda})}\left(\widetilde{\eta}\right)^x \cdot \widehat{z}_{\widetilde{\mathbf{K}}} \\
&= I^{\widetilde{\mathbf{K}}}_{(\widetilde{\mathbf{L}},\widetilde{\lambda}^x \cdot \widehat{z}_{\widetilde{\mathbf{L}}})}\left(\widetilde{\eta}^x\right) \\
&= I^{\widetilde{\mathbf{K}}}_{(\widetilde{\mathbf{L}},\widetilde{\lambda})}\left(\widetilde{\eta}^x\right).
\end{aligned} \tag{8.2.5}$$

Now, the result follows immediately from (8.2.4) and (8.2.5). □

8.3 e-Harish-Chandra series and regular embeddings

In this section, we combine the bijections given in (8.2.1) in order to obtain Corollary 8.2. To do so, we study the behaviour of e-Harish-Chandra series with respect to regular embeddings. We use the notation introduced in the previous sections. Fix an e-split Levi subgroup $\mathbf{K}$ of $\mathbf{G}$ and an e-cuspidal pair $(\mathbf{L}, \lambda)$ of $\mathbf{K}$.

Definition 8.3.1. Let $\mathcal{HC}(\widetilde{\mathbf{K}}^F, (\mathbf{L}, \lambda))$ be the set consisting of those e-Harish-Chandra series $\mathcal{E}(\widetilde{\mathbf{K}}^F, (\widetilde{\mathbf{L}}, \widetilde{\lambda}))$ with $\widetilde{\lambda} \in \mathrm{Irr}(\widetilde{\mathbf{L}}^F \mid \lambda)$. The group $\mathcal{K}$ from Lemma 6.1.3 acts on the set $\mathcal{HC}(\widetilde{\mathbf{K}}^F, (\mathbf{L}, \lambda))$ via

$$\mathcal{E}\left(\widetilde{\mathbf{K}}^F, \left(\widetilde{\mathbf{L}}, \widetilde{\lambda}\right)\right)^z := \mathcal{E}\left(\widetilde{\mathbf{K}}^F, \left(\widetilde{\mathbf{L}}, \widetilde{\lambda} \cdot \widehat{z}_{\widetilde{\mathbf{L}}}\right)\right)$$

for every $\mathcal{E}(\widetilde{\mathbf{K}}^F, (\widetilde{\mathbf{L}}, \widetilde{\lambda})) \in \mathcal{HC}(\widetilde{\mathbf{K}}^F, (\mathbf{L}, \lambda))$ and $z \in \mathcal{K}$, where $\widehat{z}_{\widetilde{\mathbf{L}}}$ corresponds to z via (6.1.2). Here notice that, as λ is e-cuspidal, then so are $\widetilde{\lambda}$ and $\widetilde{\lambda} \cdot \widehat{z}_{\widetilde{\mathbf{L}}}$ (see [Bon06, Proposition 10.10 and Proposition 10.11]). Moreover, if we define $\mathcal{E}(\widetilde{\mathbf{K}}^F, (\widetilde{\mathbf{L}}, \widetilde{\lambda})) \cdot \widehat{z}_{\widetilde{\mathbf{K}}}$ to be the set of characters $\widetilde{\chi} \cdot \widehat{z}_{\widetilde{\mathbf{K}}}$ for $\widetilde{\chi} \in \mathcal{E}(\widetilde{\mathbf{K}}^F, (\widetilde{\mathbf{L}}, \widetilde{\lambda}))$, then

$$\mathcal{E}(\widetilde{\mathbf{K}}^F, (\widetilde{\mathbf{L}}, \widetilde{\lambda}))^z = \mathcal{E}(\widetilde{\mathbf{K}}^F, (\widetilde{\mathbf{L}}, \widetilde{\lambda})) \cdot \widehat{z}_{\widetilde{\mathbf{K}}}$$

by [Bon06, Proposition 10.11].

We want to compare the action of $\mathcal{K}$ on $\mathcal{HC}(\widetilde{\mathbf{K}}^F, (\mathbf{L}, \lambda))$ with the action of $\mathcal{K}$ on the set of characters $\mathrm{Irr}(\widetilde{\mathbf{L}}^F \mid \lambda)$. To start, notice that both these two actions of $\mathcal{K}$ are transitive by [Isa76, Problem 6.2].

Lemma 8.3.2. *Assume Hypothesis 7.2.7 and let* $\widetilde{\lambda}_i \in \mathrm{Irr}(\widetilde{\mathbf{L}}^F \mid \lambda)$ *for* $i = 1, 2$*. Let* $z \in \mathcal{K}$*, then*

$$\mathcal{E}\left(\widetilde{\mathbf{K}}^F, \left(\widetilde{\mathbf{L}}, \widetilde{\lambda}_1\right)\right) = \mathcal{E}\left(\widetilde{\mathbf{K}}^F, \left(\widetilde{\mathbf{L}}, \widetilde{\lambda}_2\right)\right)^z$$

if and only if

$$\widetilde{\lambda}_1 = \widetilde{\lambda}_2^x \cdot \widehat{z}_{\widetilde{\mathbf{L}}}$$

for some $x \in \mathbf{N}_{\widetilde{\mathbf{K}}}(\mathbf{L}, \lambda)^F$.

Proof. First, assume $\mathcal{E}(\widetilde{\mathbf{K}}^F, (\widetilde{\mathbf{L}}, \widetilde{\lambda}_1)) = \mathcal{E}(\widetilde{\mathbf{K}}^F, (\widetilde{\mathbf{L}}, \widetilde{\lambda}_2))^z$. By Proposition 7.2.15, there exists $u \in \widetilde{\mathbf{K}}^F$ such that $(\widetilde{\mathbf{L}}, \widetilde{\lambda}_1) = (\widetilde{\mathbf{L}}, \widetilde{\lambda}_2 \cdot \widehat{z}_{\widetilde{\mathbf{L}}})^u$. This implies that $u \in \mathbf{N}_{\widetilde{\mathbf{K}}}(\mathbf{L})^F$ and that $\widetilde{\lambda}_1 = \widetilde{\lambda}_2^u \cdot \widehat{z}_{\widetilde{\mathbf{L}}}$. Moreover, since $\widetilde{\lambda}_1$ lies over both λ and λ^u, it follows by Clifford's theorem that $\lambda = \lambda^{uv}$, for some $v \in \widetilde{\mathbf{L}}^F$. Then $x := uv \in \mathbf{N}_{\widetilde{\mathbf{K}}}(\mathbf{L}, \lambda)^F$ and $\widetilde{\lambda}_1 = \widetilde{\lambda}_2^x \cdot \widehat{z}_{\widetilde{\mathbf{L}}}$. Conversely, if $\widetilde{\lambda}_1 = \widetilde{\lambda}_2^x \cdot \widehat{z}_{\widetilde{\mathbf{L}}}$ for some $x \in \mathbf{N}_{\widetilde{\mathbf{K}}}(\mathbf{L}, \lambda)^F$, then [Bon06, Proposition 10.11] yields the desired equality. □

Next, consider the external semidirect product $(\widetilde{\mathbf{G}}^F \mathcal{A}) \ltimes \mathcal{K}$ where, for $x \in \widetilde{\mathbf{G}}^F \mathcal{A}$ and $z \in \mathcal{K}$, the element z^x is defined as the unique element of $\mathcal{K}$ corresponding to $(\widehat{z}_{\widetilde{\mathbf{G}}})^x \in \mathrm{Irr}(\widetilde{\mathbf{G}}^F / \mathbf{G}^F)$ via (6.1.2). Notice that $(\widetilde{\mathbf{G}}^F \mathcal{A})_{\mathbf{L}} \ltimes \mathcal{K}$ acts on $\mathrm{Irr}(\widetilde{\mathbf{L}}^F)$ by

$$\widetilde{\lambda}^{xz} := \widetilde{\lambda}^x \cdot \widehat{z}_{\widetilde{\mathbf{L}}}$$

for every $x \in (\widetilde{\mathbf{G}}^F \mathcal{A})_{\mathbf{L}}$ and $z \in \mathcal{K}$. We denote by $((\widetilde{\mathbf{G}}^F \mathcal{A})_{\mathbf{L}} \ltimes \mathcal{K})_{\widetilde{\lambda}}$ the stabilizer of $\widetilde{\lambda}$ under this action.

Corollary 8.3.3. *Assume Hypothesis 7.2.7 and consider* $\widetilde{\lambda} \in \mathrm{Irr}(\widetilde{\mathbf{L}}^F \mid \lambda)$*. Then* $\mathcal{K}_{\mathcal{E}(\widetilde{\mathbf{K}}^F, (\widetilde{\mathbf{L}}, \widetilde{\lambda}))} \leq \mathbf{N}_{\widetilde{\mathbf{K}}}(\mathbf{L}, \lambda)^F (\mathbf{N}_{\widetilde{\mathbf{K}}}(\mathbf{L}, \lambda)^F \ltimes \mathcal{K})_{\widetilde{\lambda}}$.

Proof. Let $z \in \mathcal{K}$ stabilize $\mathcal{E}(\widetilde{\mathbf{K}}^F, (\widetilde{\mathbf{L}}, \widetilde{\lambda}))$. By Lemma 8.3.2 there exists $x \in \mathbf{N}_{\widetilde{\mathbf{K}}}(\mathbf{L}, \lambda)^F$ such that $\widetilde{\lambda} = \widetilde{\lambda}^x \cdot \widehat{z}_{\widetilde{\mathbf{L}}}$ and hence $z = x^{-1}xz \in \mathbf{N}_{\widetilde{\mathbf{K}}}(\mathbf{L}, \lambda)^F (\mathbf{N}_{\widetilde{\mathbf{K}}}(\mathbf{L}, \lambda)^F \ltimes \mathcal{K})_{\widetilde{\lambda}}$. □

Our next goal is to show how the set $\widetilde{\mathcal{G}}$ of characters of $\widetilde{\mathbf{K}}^F$ lying over some character of the e-Harish-Chandra series $\mathcal{E}(\mathbf{K}^F,(\mathbf{L},\lambda))$ can be partitioned into e-Harish-Chandra series $\mathcal{E}(\widetilde{\mathbf{G}}^F,(\widetilde{\mathbf{L}},\widetilde{\lambda}))\in\mathcal{HC}(\widetilde{\mathbf{G}}^F,(\mathbf{L},\lambda))$. On the other hand the set $\widetilde{\mathcal{N}}$ of characters of $\mathbf{N}_{\widetilde{\mathbf{K}}}(\mathbf{L})^F$ lying over λ can be partitioned into the sets $\mathrm{Irr}(\mathbf{N}_{\widetilde{\mathbf{K}}}(\mathbf{L})^F\mid\widetilde{\lambda})$, where $\widetilde{\lambda}\in\mathrm{Irr}(\widetilde{\mathbf{L}}^F\mid\lambda)$.

Proposition 8.3.4. *Assume Hypothesis 7.2.7 and let* $\widetilde{\lambda}\in\mathrm{Irr}(\widetilde{\mathbf{L}}^F\mid\lambda)$. *Consider* $\widetilde{\mathcal{G}}:=\mathrm{Irr}(\widetilde{\mathbf{K}}^F\mid\mathcal{E}(\mathbf{K}^F,(\mathbf{L},\lambda)))$ *and* $\widetilde{\mathcal{N}}:=\mathrm{Irr}(\mathbf{N}_{\widetilde{\mathbf{K}}}(\mathbf{L})^F\mid\lambda)$. *If* $\mathcal{T}$ *is a transversal for the stabilizer* $\mathcal{K}_{\mathcal{E}(\widetilde{\mathbf{K}}^F,(\widetilde{\mathbf{L}},\widetilde{\lambda}))}$ *in* $\mathcal{K}$, *then*

$$\widetilde{\mathcal{G}}=\coprod_{z\in\mathcal{T}}\mathcal{E}\left(\widetilde{\mathbf{K}}^F,\left(\widetilde{\mathbf{L}},\widetilde{\lambda}\right)\right)\cdot\widehat{z}_{\widetilde{\mathbf{K}}}\tag{8.3.1}$$

and

$$\widetilde{\mathcal{N}}=\coprod_{z\in\mathcal{T}}\mathrm{Irr}\left(\mathbf{N}_{\widetilde{\mathbf{K}}}(\mathbf{L})^F\ \middle|\ \widetilde{\lambda}\right)\cdot\widehat{z}_{\mathbf{N}_{\widetilde{\mathbf{K}}}(\mathbf{L})},\tag{8.3.2}$$

where $\mathrm{Irr}(\mathbf{N}_{\widetilde{\mathbf{K}}}(\mathbf{L})^F\mid\widetilde{\lambda})\cdot\widehat{z}_{\mathbf{N}_{\widetilde{K}}(\mathbf{L})}$ *is the set of characters* $\widetilde{\psi}\cdot\widehat{z}_{\mathbf{N}_{\widetilde{\mathbf{K}}}(\mathbf{L})}$ *for* $\widetilde{\psi}\in\mathrm{Irr}(\mathbf{N}_{\widetilde{\mathbf{K}}}(\mathbf{L})^F\mid\widetilde{\lambda})$.

Proof. First, we claim that $\widetilde{\mathcal{G}}$ is the union of the e-Harish-Chandra series in the set $\mathcal{HC}(\widetilde{\mathbf{K}}^F,(\mathbf{L},\lambda))$. In fact, if $\widetilde{\chi}\in\widetilde{\mathcal{G}}$, then there exists $\chi\in\mathcal{E}(\mathbf{K}^F,(\mathbf{L},\lambda))$ lying below $\widetilde{\chi}$. By [GM20, Corollary 3.3.25], it follows that $\widetilde{\chi}$ is an irreducible constituent of $\mathbf{R}_{\widetilde{\mathbf{L}}}^{\widetilde{\mathbf{K}}}(\lambda^{\widetilde{\mathbf{L}}^F})$ and therefore there exists $\widetilde{\nu}\in\mathrm{Irr}(\widetilde{\mathbf{L}}^F\mid\lambda)$ such that $\widetilde{\chi}\in\mathcal{E}(\widetilde{\mathbf{K}},(\widetilde{\mathbf{L}},\widetilde{\nu}))$. On the other hand, if $\widetilde{\nu}\in\mathrm{Irr}(\widetilde{\mathbf{L}}^F\mid\lambda)$ and $\widetilde{\chi}\in\mathcal{E}(\widetilde{\mathbf{K}}^F,(\widetilde{\mathbf{L}},\widetilde{\nu}))$, then [GM20, Corollary 3.3.25] implies that $\widetilde{\chi}$ lies over some character $\chi\in\mathcal{E}(\mathbf{K}^F,(\mathbf{L},\lambda))$. Since the action of $\mathcal{K}$ on $\mathcal{HC}(\widetilde{\mathbf{K}}^F,(\mathbf{L},\lambda))$ is transitive, we obtain (8.3.1).

Now we prove (8.3.2). By Clifford theory, we know that every element of $\mathcal{G}$ lies above some character $\widetilde{\nu}\in\mathrm{Irr}(\widetilde{\mathbf{L}}\mid\lambda)$. Since $\mathcal{K}$ is transitive on $\mathrm{Irr}(\widetilde{\mathbf{L}}^F\mid\lambda)$, we deduce that $\widetilde{\mathcal{N}}$ is contained in the union

$$\bigcup_{z\in\mathcal{K}}\mathrm{Irr}\left(\mathbf{N}_{\widetilde{\mathbf{K}}}(\mathbf{L})^F\ \middle|\ \widetilde{\lambda}\right)\cdot\widehat{z}_{\mathbf{N}_{\widetilde{\mathbf{K}}}(\mathbf{L})}.$$

Moreover, we claim that the above union coincides with

$$\bigcup_{z\in\mathcal{T}}\mathrm{Irr}\left(\mathbf{N}_{\widetilde{\mathbf{K}}}(\mathbf{L})^F\ \middle|\ \widetilde{\lambda}\right)\cdot\widehat{z}_{\mathbf{N}_{\widetilde{\mathbf{K}}}(\mathbf{L})}.\tag{8.3.3}$$

To see this, let $z\in\mathcal{K}$ and write $z=z_0t$, for some $z_0\in\mathcal{K}_{\mathcal{E}(\widetilde{\mathbf{K}}^F,(\widetilde{\mathbf{L}},\widetilde{\lambda}))}$ and $t\in\mathcal{T}$. By Corollary 8.3.3 we obtain $z_0\in\mathbf{N}_{\widetilde{\mathbf{K}}}(\mathbf{L},\lambda)^F(\mathbf{N}_{\widetilde{\mathbf{K}}}(\mathbf{L},\lambda)^F\ltimes\mathcal{K})_{\widetilde{\lambda}}$ and therefore

$$\mathrm{Irr}\left(\mathbf{N}_{\widetilde{\mathbf{K}}}(\mathbf{L})^F\ \middle|\ \widetilde{\lambda}\right)\cdot\widehat{z}_{\mathbf{N}_{\widetilde{\mathbf{K}}}(\mathbf{L})}=\mathrm{Irr}\left(\mathbf{N}_{\widetilde{\mathbf{K}}}(\mathbf{L})^F\ \middle|\ \widetilde{\lambda}\right)\cdot\widehat{t}_{\mathbf{N}_{\widetilde{\mathbf{K}}}(\mathbf{L})}.$$

This proves our claim and it remains to show that the union in (8.3.3) is disjoint. Assume that, for some $z\in\mathcal{T}$, there exists a character $\widetilde{\psi}$ inside both $\mathrm{Irr}(\mathbf{N}_{\widetilde{\mathbf{K}}}(\mathbf{L})^F\mid\widetilde{\lambda})$ and $\mathrm{Irr}(\mathbf{N}_{\widetilde{\mathbf{K}}}(\mathbf{L})^F\mid\widetilde{\lambda})\cdot\widehat{z}_{\mathbf{N}_{\widetilde{\mathbf{K}}}(\mathbf{L})}$. By [Isa76, Problem 5.3] we deduce that $\mathrm{Irr}(\mathbf{N}_{\widetilde{\mathbf{K}}}(\mathbf{L})^F\mid\widetilde{\lambda})\cdot\widehat{z}_{\mathbf{N}_{\widetilde{\mathbf{K}}}(\mathbf{L})}=\mathrm{Irr}(\mathbf{N}_{\widetilde{\mathbf{K}}}(\mathbf{L})^F\mid\widetilde{\lambda}\cdot\widehat{z}_{\widetilde{\mathbf{L}}})$ and hence $\widetilde{\psi}$ lies above $\widetilde{\lambda}$ and $\widetilde{\lambda}\cdot\widehat{z}_{\widetilde{\mathbf{L}}}$. By Clifford's theorem $\widetilde{\lambda}=(\widetilde{\lambda}\cdot\widehat{z}_{\widetilde{\mathbf{L}}})^u=\widetilde{\lambda}^u\cdot\widehat{z}_{\widetilde{\mathbf{L}}}$, for some $u\in\mathbf{N}_{\widetilde{\mathbf{K}}}(\mathbf{L})^F$ and now Lemma 8.3.2 implies $\mathcal{E}(\widetilde{\mathbf{K}}^F,(\widetilde{\mathbf{L}},\widetilde{\lambda}))=\mathcal{E}(\widetilde{\mathbf{K}}^F,(\widetilde{\mathbf{L}},\widetilde{\lambda}))^z$. By the definition of $\mathcal{T}$ it follows that the union in (8.3.3) is disjoint. □

As a corollary of Proposition 8.3.4 and using the bijection Υ from (8.2.1), we are finally able to prove Corollary 8.2.

Theorem 8.3.5. *Assume Hypothesis 7.2.7 and Hypothesis 8.1.2. Suppose there exists a* $((\widetilde{\mathbf{G}}^F\mathcal{A})_{\mathbf{K},\mathbf{L}}\ltimes\mathcal{K})$*-equivariant extension map for* $\mathrm{Cusp}_e(\widetilde{\mathbf{L}}^F)$ *with respect to* $\widetilde{\mathbf{L}}^F\trianglelefteq\mathbf{N}_{\widetilde{\mathbf{K}}}(\mathbf{L})^F$*. Then, there exists a defect preserving* $((\widetilde{\mathbf{G}}^F\mathcal{A})_{\mathbf{K},(\mathbf{L},\lambda)}\ltimes\mathcal{K})$*-equivariant bijection*

$$\widetilde{\Omega}^{\mathbf{K}}_{(\mathbf{L},\lambda)}:\mathrm{Irr}\left(\widetilde{\mathbf{K}}^F\;\middle|\;\mathcal{E}\left(\mathbf{K}^F,(\mathbf{L},\lambda)\right)\right)\to\mathrm{Irr}\left(\mathbf{N}_{\widetilde{\mathbf{K}}}(\mathbf{L})^F\;\middle|\;\lambda\right)$$

such that, for every $\widetilde{\chi}\in\mathrm{Irr}(\widetilde{\mathbf{K}}^F\mid\mathcal{E}(\mathbf{K}^F,(\mathbf{L},\lambda)))$*, the following conditions hold:*

(i) $\mathrm{Irr}\left(\widetilde{\chi}_{\mathbf{Z}(\widetilde{\mathbf{K}}^F)}\right)=\mathrm{Irr}\left(\widetilde{\Psi}(\widetilde{\chi})_{\mathbf{Z}(\widetilde{\mathbf{K}}^F)}\right)$;

(ii) $\mathrm{bl}(\widetilde{\chi})=\mathrm{bl}\left(\widetilde{\Psi}(\widetilde{\chi})\right)^{\widetilde{\mathbf{K}}^F}$.

Proof. Set $\widetilde{\mathcal{G}}:=\mathrm{Irr}(\widetilde{\mathbf{K}}^F\mid\mathcal{E}(\mathbf{K}^F,(\mathbf{L},\lambda)))$ and $\widetilde{\mathcal{N}}:=\mathrm{Irr}(\mathbf{N}_{\widetilde{\mathbf{K}}}(\mathbf{L})^F\mid\lambda)$ and fix $\widetilde{\lambda}\in\mathrm{Irr}(\widetilde{\mathbf{L}}^F\mid\lambda)$. Let

$$\Upsilon:\mathcal{E}\left(\widetilde{\mathbf{K}}^F,(\widetilde{\mathbf{L}},\widetilde{\lambda})\right)\to\mathrm{Irr}\left(\mathbf{N}_{\widetilde{\mathbf{K}}}(\mathbf{L})^F\;\middle|\;\widetilde{\lambda}\right)$$

be the bijection constructed in (8.2.1). Let $\widetilde{\mathbb{T}}_{\mathrm{glo}}$ be a $((\widetilde{\mathbf{G}}^F\mathcal{A})_{\mathbf{K},\mathbf{L}}\ltimes\mathcal{K})_{\widetilde{\lambda}}$-transversal in $\mathcal{E}(\widetilde{\mathbf{K}}^F,(\widetilde{\mathbf{L}},\widetilde{\lambda}))$ and observe that, by Lemma 8.2.5, the set $\widetilde{\mathbb{T}}_{\mathrm{loc}}:=\{\Upsilon(\widetilde{\chi})\mid\widetilde{\chi}\in\widetilde{\mathbb{T}}_{\mathrm{glo}}\}$ is a $((\widetilde{\mathbf{G}}^F\mathcal{A})_{\mathbf{K},\mathbf{L}}\ltimes\mathcal{K})_{\widetilde{\lambda}}$-transversal in $\mathrm{Irr}(\mathbf{N}_{\widetilde{\mathbf{K}}}(\mathbf{L})^F\mid\lambda)$.

Next, we fix a transversal $\mathcal{T}$ for $\mathcal{K}_{\mathcal{E}(\widetilde{\mathbf{K}}^F,(\widetilde{\mathbf{L}},\widetilde{\lambda}))}$ in $\mathcal{K}$ and we claim that

$$\widetilde{\mathbf{K}}^F_{\mathbf{L}}\left(\left(\widetilde{\mathbf{G}}^F\mathcal{A}\right)_{\mathbf{K},\mathbf{L}}\ltimes\mathcal{K}\right)_{\widetilde{\lambda}}\cdot\mathcal{T}=\widetilde{\mathbf{K}}^F_{\mathbf{L}}\left(\widetilde{\mathbf{G}}^F\mathcal{A}\right)_{\mathbf{K},(\mathbf{L},\lambda)}\ltimes\mathcal{K}.\tag{8.3.4}$$

To prove this equality, consider $xz\in(\widetilde{\mathbf{G}}^F\mathcal{A})_{\mathbf{K},(\mathbf{L},\lambda)}\ltimes\mathcal{K}$. Then both $\widetilde{\lambda}$ and $\widetilde{\lambda}^x\cdot\widehat{z}_{\widetilde{\mathbf{L}}}$ lie over λ and by [Isa76, Problem 6.2] there exists $u\in\mathcal{K}$ such that $\widetilde{\lambda}=\widetilde{\lambda}^x\cdot\widehat{z}_{\widetilde{\mathbf{L}}}\cdot\widehat{u}_{\widetilde{\mathbf{L}}}$. Therefore $xz\in((\widetilde{\mathbf{G}}^F\mathcal{A})_{\mathbf{K},\mathbf{L}}\ltimes\mathcal{K})_{\widetilde{\lambda}}\cdot\mathcal{K}$. On the other hand, applying Corollary 8.3.3, we obtain $\mathcal{K}_{\mathcal{E}(\widetilde{\mathbf{K}}^F,(\widetilde{\mathbf{L}},\widetilde{\lambda}))}\leq\widetilde{\mathbf{K}}^F_{\mathbf{L}}(\widetilde{\mathbf{K}}^F_{\mathbf{L}}\ltimes\mathcal{K})_{\widetilde{\lambda}}$ and by the definition of $\mathcal{T}$, we conclude that

$$\widetilde{\mathbf{K}}^F_{\mathbf{L}}\left(\left(\widetilde{\mathbf{G}}^F\mathcal{A}\right)_{\mathbf{K},\mathbf{L}}\ltimes\mathcal{K}\right)_{\widetilde{\lambda}}\cdot\mathcal{T}\geq\widetilde{\mathbf{K}}^F_{\mathbf{L}}\left(\left(\widetilde{\mathbf{G}}^F\mathcal{A}\right)_{\mathbf{K},(\mathbf{L},\lambda)}\ltimes\mathcal{K}\right).$$

To prove the remaining inclusion it's enough to show that

$$\widetilde{\mathbf{L}}^F\left(\left(\widetilde{\mathbf{G}}^F\mathcal{A}\right)_{\mathbf{K},(\mathbf{L},\lambda)}\ltimes\mathcal{K}\right)_{\widetilde{\lambda}}=\left(\left(\widetilde{\mathbf{G}}^F\mathcal{A}\right)_{\mathbf{K},\mathbf{L}}\ltimes\mathcal{K}\right)_{\widetilde{\lambda}}.$$

Since $\widetilde{\lambda}$ is $\widetilde{\mathbf{L}}^F$-invariant, one inclusion is trivial. So let $xz\in((\widetilde{\mathbf{G}}^F\mathcal{A})_{\mathbf{K},\mathbf{L}}\ltimes\mathcal{K})_{\widetilde{\lambda}}$ and observe that $\widetilde{\lambda}=\widetilde{\lambda}^x\cdot\widehat{z}_{\widetilde{\mathbf{L}}}$ lies both over λ and over λ^x. By Clifford's theorem there exists $y\in\widetilde{\mathbf{L}}^F$ such that $\lambda=\lambda^{xy}$ and hence $xz\in\widetilde{\mathbf{L}}^F((\widetilde{\mathbf{G}}^F\mathcal{A})_{\mathbf{K},(\mathbf{L},\lambda)}\ltimes\mathcal{K})_{\widetilde{\lambda}}$. This proves the claim.

Now, using (8.3.4), we show that $\widetilde{\mathbb{T}}_{\mathrm{glo}}$ is a $((\widetilde{\mathbf{G}}^F\mathcal{A})_{\mathbf{K},(\mathbf{L},\lambda)}\ltimes\mathcal{K})$-transversal in $\widetilde{\mathcal{G}}$. Consider $\widetilde{\chi}\in\widetilde{\mathcal{G}}$. By Proposition 8.3.4 there exist unique $z\in\mathcal{T}$ and $\widetilde{\chi}'_0\in\mathcal{E}(\widetilde{\mathbf{K}}^F,(\widetilde{\mathbf{L}},\widetilde{\lambda}))$ such that $\widetilde{\chi}=\widetilde{\chi}'_0\cdot\widehat{z}_{\widetilde{\mathbf{K}}}$. Let $\widetilde{\chi}_0$ be the unique element in $\widetilde{\mathbb{T}}_{\mathrm{glo}}$ such that $\widetilde{\chi}'_0=\widetilde{\chi}_0^x\cdot\widehat{u}_{\widetilde{\mathbf{K}}}$, for some $xu\in((\widetilde{\mathbf{G}}^F\mathcal{A})_{\mathbf{K},\mathbf{L}}\ltimes\mathcal{K})_{\widetilde{\lambda}}$. Then $\widetilde{\chi}=\widetilde{\chi}_0^x\cdot\widehat{u}_{\widetilde{K}}\cdot\widehat{z}_{\widetilde{K}}$, for $xuz\in((\widetilde{\mathbf{G}}^F\mathcal{A})_{\mathbf{K},\mathbf{L}}\ltimes\mathcal{K})_{\widetilde{\lambda}}\cdot\mathcal{T}$. But using (8.3.4) and since $\widetilde{\chi}$ and $\widetilde{\chi}_0$ are $\mathbf{N}_{\widetilde{\mathbf{K}}}(\mathbf{L})^F$-invariant, we conclude that $\widetilde{\chi}=\widetilde{\chi}_0^y\cdot\widehat{v}_{\widetilde{\mathbf{K}}}$, for some $y\in(\widetilde{\mathbf{G}}^F\mathcal{A})_{\mathbf{K},(\mathbf{L},\lambda)}$ and $v\in\mathcal{K}$. This argument also shows that $\widetilde{\chi}_0$ is the unique element of $\widetilde{\mathbb{T}}_{\mathrm{glo}}$ with this property.

Similarly, using (8.3.4), we deduce that the set $\widetilde{\mathbb{T}}_{\rm loc}$ is a $((\widetilde{\mathbf{G}}^F\mathcal{A})_{\mathbf{K},(\mathbf{L},\lambda)}\ltimes\mathcal{K})$-transversal in $\widetilde{\mathcal{N}}$. Now, the map

$$\widetilde{\Omega}^{\mathbf{K}}_{(\mathbf{L},\lambda)}:\widetilde{\mathcal{G}}\to\widetilde{\mathcal{N}}$$

defined by

$$\widetilde{\Psi}\left(\widetilde{\chi}^x\cdot\widehat{z}_{\widetilde{\mathbf{K}}}\right):=\Upsilon(\widetilde{\chi})^x\cdot\widehat{z}_{\mathbf{N}_{\widetilde{\mathbf{K}}}(\mathbf{L})},$$

for every $\widetilde{\chi}\in\widetilde{\mathbb{T}}_{\rm glo}$, $x\in(\widetilde{\mathbf{G}}^F\mathcal{A})_{\mathbf{K},(\mathbf{L},\lambda)}$ and $z\in\mathcal{K}$, is a $((\widetilde{\mathbf{G}}^F\mathcal{A})_{\mathbf{K},(\mathbf{L},\lambda)}\ltimes\mathcal{K})$-equivariant bijection. The remaining properties follow from Lemma 8.2.2, Lemma 8.2.3 and Lemma 8.2.4 after noticing that $\widehat{z}_{\widetilde{\mathbf{K}}}$ and $\widehat{z}_{\mathbf{N}_{\widetilde{\mathbf{K}}}(\mathbf{L})}$ are linear characters and that

$$\mathrm{bl}\left(\widetilde{\psi}\cdot\widehat{z}_{\mathbf{N}_{\widetilde{\mathbf{K}}}(\mathbf{L})}\right)^{\widetilde{\mathbf{K}}^F}=\mathrm{bl}\left(\widetilde{\psi}\right)^{\widetilde{\mathbf{K}}^F}\cdot\widehat{z}_{\widetilde{\mathbf{K}}}$$

for every $\widetilde{\psi}\in\mathrm{Irr}(\mathbf{N}_{\widetilde{\mathbf{K}}}(\mathbf{L})^F)$ and $z\in\mathcal{K}$. □

As an immediate consequence, we obtain Corollary 8.3.

Corollary 8.3.6. *Assume Hypothesis 7.2.7 and Hypothesis 8.1.2. Suppose there exists a* $((\widetilde{\mathbf{G}}^F\mathcal{A})_{\mathbf{K},\mathbf{L}}\ltimes\mathcal{K})$*-equivariant extension map for* $\mathrm{Cusp}_e(\widetilde{\mathbf{L}}^F)$ *with respect to* $\widetilde{\mathbf{L}}^F\trianglelefteq\mathbf{N}_{\widetilde{\mathbf{K}}}(\mathbf{L})^F$*. Then, there exists a defect preserving* $(\widetilde{\mathbf{G}}^F\mathcal{A})_{\mathbf{K},(\mathbf{L},\lambda)}$*-equivariant bijection*

$$\Omega^{\mathbf{K}}_{(\mathbf{L},\lambda)}:\mathcal{E}\left(\mathbf{K}^F,(\mathbf{L},\lambda)\right)\to\mathrm{Irr}\left(\mathbf{N}_{\mathbf{K}}(\mathbf{L})^F\;\middle|\;\lambda\right).$$

Proof. Fix a $((\widetilde{\mathbf{G}}^F\mathcal{A})_{\mathbf{K},(\mathbf{L},\lambda)}\ltimes\mathcal{K})$-transversal $\widetilde{\mathbb{T}}_{\rm glo}$ in $\mathrm{Irr}(\widetilde{\mathbf{K}}^F\mid\mathcal{E}(\mathbf{K}^F,(\mathbf{L},\lambda)))$. By Theorem 8.3.5 the set $\widetilde{\mathbb{T}}_{\rm loc}:=\{\widetilde{\Omega}^{\mathbf{K}}_{(\mathbf{L},\lambda)}(\widetilde{\chi})\mid\widetilde{\chi}\in\widetilde{\mathbb{T}}_{\rm glo}\}$ is a $((\widetilde{\mathbf{G}}^F\mathcal{A})_{\mathbf{K},(\mathbf{L},\lambda)}\ltimes\mathcal{K})$-transversal in $\mathrm{Irr}(\mathbf{N}_{\widetilde{\mathbf{K}}}(\mathbf{L})^F\mid\lambda)$. For every $\widetilde{\chi}\in\widetilde{\mathbb{T}}_{\rm glo}$ fix an irreducible constituent $\chi\in\mathcal{E}(\mathbf{G}^F,(\mathbf{L},\lambda))$ of $\widetilde{\chi}_{\mathbf{G}^F}$ and define the set $\mathbb{T}_{\rm glo}$ consisting of such characters χ, while $\widetilde{\chi}$ runs over the elements of $\widetilde{\mathbb{T}}_{\rm glo}$. Similarly, for every $\widetilde{\psi}\in\widetilde{\mathbb{T}}_{\rm loc}$, fix an irreducible constituent $\psi\in\mathrm{Irr}(\mathbf{N}_{\mathbf{K}}(\mathbf{L})^F\mid\lambda)$ of $\widetilde{\psi}$ and define the set $\mathbb{T}_{\rm loc}$ consisting of such characters ψ, while $\widetilde{\psi}$ runs over the elements of $\widetilde{\mathbb{T}}_{\rm loc}$. Then $\mathbb{T}_{\rm glo}$ and $\mathbb{T}_{\rm loc}$ are $(\widetilde{\mathbf{G}}^F\mathcal{A})_{\mathbf{K},(\mathbf{L},\lambda)}$-transversals in $\mathcal{E}(\mathbf{G}^F,(\mathbf{L},\lambda))$ and $\mathrm{Irr}(\mathbf{N}_{\mathbf{K}}(\mathbf{L})^F\mid\lambda)$ respectively. Fix $\chi\in\mathbb{T}_{\rm glo}$ and let $\widetilde{\chi}$ be the unique element of $\widetilde{\mathbb{T}}_{\rm glo}$ lying above χ. Let $\widetilde{\psi}:=\widetilde{\Omega}^{\mathbf{K}}_{(\mathbf{L},\lambda)}(\widetilde{\chi})\in\widetilde{\mathbb{T}}_{\rm loc}$ and consider the unique element ψ of $\mathbb{T}_{\rm loc}$ lying below $\widetilde{\psi}$. This defines a bijection

$$\mathbb{T}_{\rm glo}\to\mathbb{T}_{\rm loc}. \tag{8.3.5}$$

Then, defining

$$\Omega^{\mathbf{K}}_{(\mathbf{L},\lambda)}(\chi^x):=\psi^x$$

for every $x\in(\widetilde{\mathbf{G}}^F\mathcal{A})_{\mathbf{K},(\mathbf{L},\lambda)}$ and every $\chi\in\mathbb{T}_{\rm glo}$ corresponding to $\psi\in\mathbb{T}_{\rm loc}$ via (8.3.5) we obtain the wanted bijection. □

9

Towards the Character Triple Conjecture for Groups of Lie Type

As we have seen in Theorem 3.5.2, in order to obtain Dade's Projective Conjecture for all finite groups via Späth's reduction theorem, one needs to prove the Character Triple Conjecture for quasisimple groups. In this chapter we provide a strategy to tackle the Character Triple Conjecture for quasisimple groups of Lie type. This project originates from ideas introduced by Broué, Fong and Srinivasan in an attempt to solve Dade's Projective Conjecture for unipotent blocks. Inspired by this and using the main results of the previous chapter (see Theorem 7.1 and Theorem 7.2.17) we provide a strategy to prove the Character Triple Conjecture tailored to finite groups of Lie type. Namely, we show that the Character Triple Conjecture holds provided that some bijections related to e-Harish-Chandra theory exist (see Condition 9.1 below). In the next chapter we will see that the main obstruction to the construction of the above mentioned bijections is given by some rather technical conditions on extendibility of characters of e-split Levi subgroups. These conditions also appear in the proofs of the inductive conditions for the McKay, the Alperin–McKay and the Alperin Weight conjectures and the checking of these requirements is part of an important ongoing project in representation theory of finite groups of Lie type (see [CS17a], [CS17b], [Tay18], [CS19] and [BS20b]).

More precisely, using the description of characters in blocks given by Theorem 7.2.17, we give a first reformulation of the Character Triple Conjecture (see Conjecture 9.1.1) tailored to groups of Lie type. To do so, we restate this conjecture by replacing chains of ℓ-elementary abelian subgroups with chains of e-split Levi subgroups and related e-cuspidal pairs (see Proposition 9.2.10). This is inspired by a clever argument given by Broué, Fong and Srinivasan for Dade's Projective Conjecture and unipotent blocks. The next step in their plan was to reduce the new reformulation of Dade's Projective Conjecture to the existence of certain bijections associated to e-cuspidal pairs similar to the one given by [BMM93, Theorem 3.2] for the unipotent case. Generalizing this argument, we reduce the Character Triple Conjecture to the existence of analogous bijections satisfying some additional Clifford theoretic requirements.

Condition 9.1. *Let* $\mathbf{G}$, $F : \mathbf{G} \to \mathbf{G}$, ℓ *and* e *be as in Notation 7.1.1 and consider an* e*-cuspidal pair*

$(\mathbf{L},\lambda)$ of $\mathbf{G}$. Then there exists a defect preserving $\mathrm{Aut}_{\mathbb{F}}(\mathbf{G}^F)_{(\mathbf{L},\lambda)}$-equivariant bijection

$$\Omega^{\mathbf{G}}_{(\mathbf{L},\lambda)} : \mathcal{E}\left(\mathbf{G}^F,(\mathbf{L},\lambda)\right) \to \mathrm{Irr}\left(\mathbf{N}_{\mathbf{G}}(\mathbf{L})^F \;\middle|\; \lambda\right)$$

such that

$$\left(X_\vartheta, \mathbf{G}^F, \vartheta\right) \sim_{\mathbf{G}^F} \left(\mathbf{N}_{X_\vartheta}(\mathbf{L}), \mathbf{N}_{\mathbf{G}^F}(\mathbf{L}), \Omega^{\mathbf{G}}_{(\mathbf{L},\lambda)}(\vartheta)\right)$$

for every $\vartheta \in \mathcal{E}\left(\mathbf{G}^F,(\mathbf{L},\lambda)\right)$ and where $X := \mathbf{G}^F \rtimes \mathrm{Aut}_{\mathbb{F}}(\mathbf{G}^F)$.

We will say that Condition 9.1 holds for $(\mathbf{G},F)$ at the prime ℓ if it holds for every e-cuspidal pair $(\mathbf{L},\lambda)$ where e is order of q modulo ℓ and q is the prime power associated to F. As anticipated, we show that using the bijections given by Condition 9.1 we can prove the Character Triple Conjecture (in the form of Conjecture 9.1.1) and therefore the inductive condition for Dade's Conjecture (see Definition 9.1.3).

Theorem 9.2. *Assume that Hypothesis 9.2.11 is satisfied with respect to $(\mathbf{G},F)$ and the prime ℓ and denote by e the order of q modulo ℓ. If Condition 9.1 holds at ℓ for every irreducible rational component $(\mathbf{H},F)$ of every e-split Levi subgroup of $\mathbf{G}$ (see Definition 9.2.13), then Conjecture 9.1.1 holds at ℓ for $\mathbf{G}^F$ with respect to $\mathbf{G}^F \trianglelefteq \mathbf{G}^F \rtimes \mathrm{Aut}_{\mathbb{F}}(\mathbf{G}^F)$. Moreover, if $\mathbf{G}^F/\mathbf{Z}(\mathbf{G}^F)$ is a nonabelian simple group with universal covering group $\mathbf{G}^F$, then the inductive condition for Dade's Conjecture (see Definition 9.1.3) holds for $\mathbf{G}^F/\mathbf{Z}(\mathbf{G}^F)$ at ℓ.*

In Remark 9.2.12 it is shown that Hypothesis 9.2.11 holds in most of the cases we are interested in.

The same argument used to prove Theorem 9.2 can be used to obtain the nonblockwise version of Dade's Projective Conjecture from a weaker version of Condition 9.1 (see Condition 9.2.22 and Theorem 9.2.23). Moreover, as said before, in the next chapter we will show how the checking of Condition 9.1 (and Condition 9.2.22) reduces to proving some technical requirements on extendibility of characters of e-split Levi subgroups.

9.1 Preliminaries

Before proving the main results of this chapter, we introduce some preliminary remarks. First, we introduce a version of the inductive condition for Dade's Conjecture (see [Spä17, Definition 6.7]) better suited to our purpose. Moreover, we consider a variant of Proposition 3.4.4 tailored to finite groups of Lie type.

9.1.1 The inductive condition for Dade's Conjecture

In order to state the inductive condition for Dade's Conjecture, we need to introduce a strong form of the Character Triple Conjecture.

Conjecture 9.1.1. *Let G be a finite group such that $\mathbf{O}_\ell(G) \leq \mathbf{Z}(G)$ and consider a block $B \in \mathrm{Bl}(G)$ with defect groups larger than $\mathbf{O}_\ell(G)$. Suppose that $G \trianglelefteq A$. Then, for every $d \geq 0$, there exists an A_B-equivariant bijection*

$$\Omega : \mathcal{C}^d(B)_+/G \to \mathcal{C}^d(B)_-/G$$

such that $\operatorname{Ker}(\vartheta_{\mathbf{Z}(G)}) = \operatorname{Ker}(\chi_{\mathbf{Z}(G)}) =: Z$ *and*

$$\left(A_{\mathbb{D},\vartheta}/Z, G_{\mathbb{D}}/Z, \overline{\vartheta}\right) \sim_{G/Z} \left(A_{\mathbb{E},\chi}/Z, G_{\mathbb{E}}/Z, \overline{\chi}\right)$$

for every $(\mathbb{D},\vartheta) \in \mathcal{C}^d(B)_+$ *and* $(\mathbb{E},\chi) \in \Omega(\overline{(\mathbb{D},\vartheta)})$.

Notice that the above statement is stronger than the original version of the Character Triple Conjecture (see Conjecture 3.5.1), in fact, block isomorphisms of character triples can be lifted from quotients with respect to central subgroups (see [Spä17, Corollary 4.4]).

Using Conjecture 9.1.1 we can reformulate the inductive condition for Dade's Conjecture as defined in [Spä17, Definition 6.7]. The next result should be compared with Lemma 5.1.2.

Lemma 9.1.2. *Let S be a nonabelian simple group with universal covering group X and consider $B \in \operatorname{Bl}(X)$ with noncentral defect groups. Then the inductive condition for Dade's Conjecture (in the sense of [Spä17, Definition 6.7]) holds for B if and only if Conjecture 9.1.1 holds for B with respect to $X \trianglelefteq X \rtimes \operatorname{Aut}(X)$.*

Proof. This is [Spä17, Proposition 6.8]. □

The above lemma tells us that, in order to prove the inductive condition for Dade's Conjecture (see [Spä17, Definition 6.7]) for a nonabelian simple group S, it is enough to show that Conjecture 9.1.1 holds for all blocks of its universal covering group X with respect to $X \trianglelefteq X \rtimes \operatorname{Aut}(X)$. Due to this fact, we introduce the following reformulation of the inductive condition for Dade's Conjecture.

Definition 9.1.3. Let S be a nonabelian simple group with universal covering group X. We say that the **inductive condition for Dade's Conjecture** holds for S if Conjecture 9.1.1 holds for X with respect to $X \trianglelefteq X \rtimes \operatorname{Aut}(X)$ and every $d \geq 0$ and $B \in \operatorname{Bl}(X)$ with defect groups larger than $\mathbf{O}_\ell(G)$.

Now, the reduction theorem of Dade's Projective Conjecture (see Theorem 3.5.2) can be restated as follows.

Theorem 9.1.4. *Let G be a finite group and suppose that every nonabelian simple group involved in G satisfies the inductive condition for Dade's Conjecture. Then Dade's Projective Conjecture holds for G.*

Noticing that, in the majority of cases, the universal covering group of a finite simple group of Lie type is of the form $\mathbf{G}^F$, where $\mathbf{G}$ is a simple algebraic group of simply connected type with a Frobenius endomorphism F, we now turn our attention to proving Conjecture 9.1.1 for such groups $\mathbf{G}^F$.

9.1.2 Bijections and N-block isomorphic character triples

Next, we prove a technical result involving N-block isomorphic character triples. Using this result we will be able to lift the bijections given by Condition 9.1. This is a version of Proposition 3.4.4 adapted to finite groups of Lie type. Recall that, for $Y \trianglelefteq X$ and $\mathcal{S} \subseteq \operatorname{Irr}(Y)$, we denote by $\mathrm{Irr}(X \mid \mathcal{S})$ the set of irreducible characters of X whose restriction to Y has an irreducible constituent contained in $\mathcal{S}$. Moreover, we define $X_{\mathcal{S}} := \{x \in X \mid \mathcal{S}^x = \mathcal{S}\}$.

Proposition 9.1.5. *Let $K \leq G \leq A$ be finite groups with $G \trianglelefteq A$, consider $A_0 \leq A$. and set $H_0 := H \cap A_0$ for every $H \leq A$. Consider $\mathcal{S} \subseteq \mathrm{Irr}(K)$ and $\mathcal{S}_0 \subseteq \mathrm{Irr}(K_0)$ and suppose there exists $K \leq V \leq X \leq \mathbf{N}_A(K)$ and $U \leq X_0$ such that:*

(i) $V \leq X_{\mathcal{S}}$. Moreover, if $x \in X$ and $\mathcal{S} \cap \mathcal{S}^x \neq \varnothing$, then $x \in V$;

(ii) $U \leq X_{0,\mathcal{S}_0}$. Moreover, if $x \in X_0$ and $\mathcal{S}_0 \cap \mathcal{S}_0^x \neq \varnothing$, then $x \in K_0U$;

(iii) $V = KU$.

Assume there exists a U-equivariant bijection

$$\Psi : \mathcal{S} \to \mathcal{S}_0$$

such that

$$(X_\vartheta, K, \vartheta) \sim_K \left(X_{0,\vartheta}, K_0, \Psi(\vartheta)\right)$$

for every $\vartheta \in \mathcal{S}$. If $K \leq J \leq X \cap G$ and $\mathbf{C}_X(Q) \leq X_0$ for every radical ℓ-subgroup Q of J_0, then there exists an $\mathbf{N}_U(J)$-equivariant bijection

$$\Phi_J : \mathrm{Irr}\left(J \mid \mathcal{S}\right) \to \mathrm{Irr}\left(J_0 \mid \mathcal{S}_0\right)$$

such that

$$(\mathbf{N}_X(J)_\chi, J, \chi) \sim_J (\mathbf{N}_{X_0}(J)_\chi, J_0, \Phi_J(\chi))$$

for every $\chi \in \mathrm{Irr}(J \mid \mathcal{S})$.

Proof. Consider an $\mathbf{N}_U(J)$-transversal $\mathbb{S}$ in $\mathcal{S}$ and define $\mathbb{S}_0 := \{\Psi(\vartheta) \mid \vartheta \in \mathbb{S}\}$. Since Ψ is U-equivariant, it follows that $\mathbb{S}_0$ is an $\mathbf{N}_U(J)$-transversal in $\mathcal{S}_0$. For every $\vartheta \in \mathbb{S}$, with $\vartheta_0 := \Psi(\vartheta) \in \mathbb{S}_0$, we fix a pair of projective representations $(\mathcal{P}^{(\vartheta)}, \mathcal{P}_0^{(\vartheta_0)})$ giving $(X_\vartheta, K, \vartheta) \sim_K (X_{0,\vartheta}, K_0, \vartheta_0)$. Now, let $\mathbb{T}$ be an $\mathbf{N}_U(J)$-transversal in $\mathrm{Irr}(J \mid \mathcal{S})$ such that every character $\chi \in \mathbb{T}$ lies above a character $\vartheta \in \mathbb{S}$ (this can be done by the choice of $\mathbb{S}$). Moreover, using Clifford's theorem together with hypotheses (i) and (iii), it follows that every $\chi \in \mathbb{T}$ lies over a unique $\vartheta \in \mathbb{S}$.

For $\chi \in \mathbb{T}$ lying over $\vartheta \in \mathbb{S}$, let $\psi \in \mathrm{Irr}(J_\vartheta \mid \vartheta)$ be the Clifford correspondent of χ over ϑ. Set $\vartheta_0 := \Psi(\vartheta) \in \mathbb{S}_0$ and consider the $\mathbf{N}_U(J)_\vartheta$-equivariant bijection $\sigma_{J_\vartheta} : \mathrm{Irr}(J_\vartheta \mid \vartheta) \to \mathrm{Irr}(J_{0,\vartheta} \mid \vartheta_0)$ induced by our choice of projective representations $(\mathcal{P}^{(\vartheta)}, \mathcal{P}_0^{(\vartheta_0)})$. Let $\psi_0 := \sigma_{J_\vartheta}(\psi)$. Observe that $J_{0,\vartheta_0} = J_{0,\vartheta}$. To see this, notice that $U_\vartheta = U_{\vartheta_0}$ since Ψ is U-equivariant and that $J_{0,\vartheta_0} \leq K_0U$ by (ii) above. Therefore $J_{0,\vartheta_0} \leq J_{0,\vartheta}$. On the other hand, since $(J \cap U)_\vartheta = (J \cap U)_{\vartheta_0}$ because Ψ is U-equivariant and noticing that $J_{0,\vartheta} \leq J_0 \cap V = K_0(J \cap U)$ by using (iii), it follows that $J_{0,\vartheta} \leq J_{0,\vartheta_0}$. Now $\Phi_J(\chi) := \psi^{J_0}$ is irreducible by the Clifford correspondence. We define

$$\Phi_J\left(\chi^x\right) := \Phi_J(\chi)^x$$

for every $\chi \in \mathbb{T}$ and $x \in \mathbf{N}_U(J)$. This defines an $\mathbf{N}_U(J)$-equivariant bijection $\Psi : \mathrm{Irr}(J \mid \mathcal{S}) \to \mathrm{Irr}(J_0 \mid \mathcal{S}_0)$.

To prove the condition on character triples, consider $\chi \in \mathrm{Irr}(J \mid \mathcal{S})$, $\vartheta \in \mathrm{Irr}(\chi_K) \cap \mathcal{S}$, $\psi \in \mathrm{Irr}(J_\vartheta \mid \vartheta)$ and $\vartheta_0 := \Psi(\vartheta)$, $\psi_0 := \sigma_{J_\vartheta}(\psi)$ and $\chi_0 := \Phi_J(\chi)$ as in the previous paragraph. Since $(X_\vartheta, K, \vartheta) \sim_K (X_{0,\vartheta}, K_0, \vartheta_0)$, Proposition 3.4.1 (i) implies that

$$\left(\mathbf{N}_{X_\vartheta}(J_\vartheta)_\psi, J_\vartheta, \psi\right) \sim_{J_\vartheta} \left(\mathbf{N}_{X_{0,\vartheta}}(J_\vartheta)_\psi, J_{0,\vartheta}, \psi_0\right)$$

and, because $\mathbf{N}_X(J)_\vartheta \leq \mathbf{N}_{X_\vartheta}(J_\vartheta)$, Lemma 3.3.8 implies

$$\left(\mathbf{N}_X(J)_{\vartheta,\psi}, J_\vartheta, \psi\right) \sim_{J_\vartheta} \left(\mathbf{N}_{X_0}(J)_{\vartheta,\psi}, J_{0,\vartheta}, \psi_0\right). \tag{9.1.1}$$

To conclude, observe that by hypothesis we have

$$\mathbf{C}_{\mathbf{N}_X(J)_\chi}(Q) \leq \mathbf{N}_{X_0}(J)_\chi$$

for every $\chi_0 \in \mathrm{Irr}(J_0 \mid \mathcal{S})$ and $Q \in \delta(\mathrm{bl}(\chi_0))$ and therefore we can apply Proposition 3.4.3 which, together with (9.1.1), yields

$$(\mathbf{N}_X(J)_\chi, J, \chi) \sim_J (\mathbf{N}_{X_0}(J)_\chi, J_0, \chi_0).$$

The proof is now complete. □

Remark 9.1.6. Consider the setup of Proposition 9.1.5. Then, the bijection Φ_J is defect preserving if and only if Ψ is defect preserving.

Proof. For $\chi \in \mathrm{Irr}(J \mid \mathcal{S})$, let ψ be the Clifford correspondent of χ over some $\vartheta \in \mathrm{Irr}(\chi_K) \cap \mathcal{S}$ and let $\psi_0 := \sigma_{J_\vartheta}(\psi)$ and $\vartheta_0 := \Psi(\vartheta)$. If $\chi_0 := \Phi_J(\chi) = \psi_0^{J_0}$, then $d(\chi) = d(\psi)$ and $d(\chi_0) = d(\psi_0)$. By Proposition 3.4.1 (ii) we deduce that $d(\psi) - d(\psi_0) = d(\vartheta) - d(\vartheta_0)$. □

9.2 The reformulation

Showing how Conjecture 9.1.1 can be deduced from Condition 9.1, requires two main steps: first, we replace elementary abelian ℓ-subgroups with e-split Levi subgroups, then we use the bijections given by Condition 9.1 to prove Conjecture 9.1.1. Consider $\mathbf{G}$, F, ℓ and e as in Notation 7.1.1.

9.2.1 From ℓ-elementary abelian subgroups to e-split Levi subgroups

We start by replacing chains of ℓ-elementary abelian subgroups with chains of e-split Levi subgroups and related e-cuspidal pairs. This will give us a version of Conjecture 9.1.1 tailored to finite reductive groups. To do so we adapt a clever argument of Broué, Fong and Srinivasan. For this purpose we make the following assumption.

Hypothesis 9.2.1. Let $\mathbf{G}$ be a connected reductive group with Frobenius endomorphism F defining an $\mathbb{F}_q$-structure on $\mathbf{G}$. Let $\ell \in \Gamma(\mathbf{G}, F)$ and suppose that $\mathbf{O}_\ell(\mathbf{G}^F) = 1$ and that either $\mathbf{Z}(\mathbf{G}_{\mathrm{sc}})_\ell^F = 1$ or $\ell \in \Gamma(\mathbf{G}_{\mathrm{ad}}, F)$.

The next definition has been introduced by Broué, Fong and Srinivasan.

Definition 9.2.2. Let E be an ℓ-elementary abelian subgroup of $\mathbf{G}^F$. Then E is said to be **good** if

$$E = \Omega_1\left(\mathbf{O}_\ell\left(\mathbf{Z}^\circ\left(\mathbf{C}^\circ_{\mathbf{G}}(E)\right)^F\right)\right),$$

and **bad** otherwise. An ℓ-elementary abelian chain $\mathbb{E} \in \mathfrak{E}(\mathbf{G}^F, 1)$ starting with $E_0 = 1$ is said to be **good** if E_i is good for every i, while it is **bad** otherwise. The set of good (resp. bad) ℓ-elementary abelian chains of $\mathbf{G}^F$ is denote by $\mathfrak{E}_g(\mathbf{G}^F)$ (resp. $\mathfrak{E}_b(\mathbf{G}^F)$).

Denote by $\mathcal{L}(\mathbf{G}, F)$, or simply by $\mathcal{L}(\mathbf{G})$ when F is clear from the context, the set of decreasing chains $\mathbb{L} = (\mathbf{G} = \mathbf{L}_0 > \cdots > \mathbf{L}_n)$ of e-split Levi subgroups of $\mathbf{G}$. We define the **length** of the chain $\mathbb{L}$ as $|\mathbb{L}| := n$. For $\epsilon \in \{+,-\}$, let $\mathcal{L}(\mathbf{G}, F)_\epsilon$ be the subset of $\mathcal{L}(\mathbf{G}, F)$ consisting of those chains $\mathbb{L}$ such that $(-1)^{|\mathbb{L}|} = \epsilon 1$. We show that there exists an equivariant length preserving bijection between decreasing chains of e-split Levi subgroup of $\mathbf{G}$ and good ℓ-elementary abelian chains of $\mathbf{G}^F$. Recall that every automorphism $\alpha \in \mathrm{Aut}_{\mathbb{F}}(\mathbf{G}^F)$ extends to a bijective endomorphism of $\mathbf{G}$ commuting with F. Then $\mathrm{Aut}_{\mathbb{F}}(\mathbf{G}^F)$ acts on the set of F-stable closed connected subgroups of $\mathbf{G}$ (see Section 6.1.5 and [CS13, Section 2.4] for more details).

Lemma 9.2.3. *Assume Hypothesis 9.2.1. Then the maps*

$$\begin{aligned}\mathcal{L}(\mathbf{G}) &\to \mathfrak{E}_g\left(\mathbf{G}^F\right)\\ \mathbb{L} = (\mathbf{L}_i) &\mapsto \mathbb{E} = \left(\Omega_1\left(\mathbf{O}_\ell\left(\mathbf{Z}^\circ(\mathbf{L}_i)^F\right)\right)\right)\end{aligned}$$

and

$$\begin{aligned}\mathfrak{E}_g(\mathbf{G}^F) &\to \mathcal{L}(\mathbf{G})\\ \mathbb{E} = (E_i) &\mapsto \mathbb{L} := (\mathbf{C}^\circ_{\mathbf{G}}(E_i))\end{aligned}$$

are $\mathrm{Aut}_{\mathbb{F}}(\mathbf{G}^F)$*-equivariant length preserving bijections inverses of each other.*

Proof. First, consider a chain of e-split Levi subgroups $\mathbb{L} = (\mathbf{G} = \mathbf{L}_0 > \cdots > \mathbf{L}_n)$. Under Hypothesis 9.2.1 we can apply Proposition 7.1.6 (ii.e) to deduce that $E_i := \Omega_1(\mathbf{O}_\ell(\mathbf{Z}^\circ(\mathbf{L}_i)^F))$ is a good ℓ-elementary abelian subgroup and that $\mathbf{L}_i = \mathbf{C}^\circ_{\mathbf{G}}(E_i)$. Since $\mathbf{L}_i > \mathbf{L}_{i+1}$, this also shows that $E_i < E_{i+1}$ for every $i = 0, \ldots, n-1$. Moreover, as $\mathbf{O}_\ell(\mathbf{G}^F) = 1$, we deduce that $E_0 = \mathbf{O}_\ell(\mathbf{G}^F)$. On the other hand, if $\mathbb{D} = (\mathbf{O}_\ell(\mathbf{G}^F) = D_0 < \cdots < D_n)$ is a good ℓ-elementary abelian chain, then all terms D_i are elementary abelian (since $\mathbf{O}_\ell(\mathbf{G}^F) = 1$) and by Proposition 7.1.6 (ii.c) we deduce that $\mathbf{K}_i := \mathbf{C}^\circ_{\mathbf{G}}(D_i)$ is an e-split Levi subgroup. Furthermore $D_i = \Omega_1(\mathbf{O}_\ell(\mathbf{Z}^\circ(\mathbf{K}_i)^F))$, because D_i is good in the sense of Definition 9.2.2, and $K_0 = \mathbf{G}$. As a consequence, since $D_i < D_{i+1}$, we obtain that $\mathbf{K}_i > \mathbf{K}_{i+1}$ for every $i = 0, \ldots, n-1$. It follows that the above maps are $\mathrm{Aut}_{\mathbb{F}}(\mathbf{G}^F)$-equivariant, inverses of each other and preserve the length of chains. □

Next, we show that there exists a self inverse $\mathrm{Aut}_{\mathbb{F}}(\mathbf{G}^F)$-equivariant bijection between bad ℓ-elementary abelian chains such that, if $\mathbb{E}$ is mapped to $\mathbb{E}'$, then $|\mathbb{E}| = |\mathbb{E}'| \pm 1$. This allow us to only consider good ℓ-elementary abelian chains and therefore, by Lemma 9.2.3, we can replace ℓ-elementary abelian chains with chains of e-split Levi subgroups.

Lemma 9.2.4. *Assume Hypothesis 9.2.1. Then there exists an* $\mathrm{Aut}_{\mathbb{F}}(\mathbf{G}^F)$*-equivariant bijection*

$$\mathfrak{E}_b(\mathbf{G}^F) \to \mathfrak{E}_b(\mathbf{G}^F)$$

such that, if $\mathbb{E}$ *is mapped to* $\mathbb{E}'$*, then* $|\mathbb{E}| = |\mathbb{E}'| \pm 1$.

Proof. Let $\mathbb{E} = (E_0 < \cdots < E_n) \in \mathfrak{E}_b(\mathbf{G}^F)$ and set $D_i := \Omega_1(\mathbf{O}_\ell(\mathbf{Z}^\circ(\mathbf{C}^\circ_{\mathbf{G}}(E_i))^F))$. Notice that $E_i \leq D_i$ by Proposition 7.1.6 (ii.b) and therefore that $\mathbf{C}^\circ_{\mathbf{G}}(D_i) \leq \mathbf{C}^\circ_{\mathbf{G}}(E_i)$. On the other hand, as $D_i \leq \mathbf{Z}^\circ(\mathbf{C}^\circ_{\mathbf{G}}(E_i))^F$, we have $\mathbf{C}^\circ_{\mathbf{G}}(E_i) \leq \mathbf{C}^\circ_{\mathbf{G}}(D_i)$. Thus $\mathbf{C}^\circ_{\mathbf{G}}(E_i) = \mathbf{C}^\circ_{\mathbf{G}}(D_i)$ and we conclude that D_i is a good ℓ-elementary abelian subgroup. Now, since $\mathbb{E}$ is a bad chain, there exists a maximal index j such that $E_j < D_j$. If $j = n$, then we define $\mathbb{E}'$ by adding D_n to the chain $\mathbb{E}$. Assume $j < n$. In this case we claim that $D_j \leq E_{j+1}$ and we define $\mathbb{E}'$ to be the chain obtained from $\mathbb{E}$ by adding (resp. removing) D_j to $\mathbb{E}$ when $D_j < E_{j+1}$ (resp. $D_j = E_{j+1}$). To prove the claim, notice that $E_{j+1} \leq \mathbf{C}_{\mathbf{G}}(E_{j+1})^F \leq \mathbf{C}_{\mathbf{G}}(E_j)^F = \mathbf{C}^\circ_{\mathbf{G}}(E_j)^F$ by Proposition 7.1.6 (ii.a). As D_j centralizes $\mathbf{C}^\circ_{\mathbf{G}}(E_j)$, we deduce that $D_j \leq \mathbf{C}_{\mathbf{G}}(E_{j+1})^F = \mathbf{C}^\circ_{\mathbf{G}}(E_{j+1})^F$ and that D_j centralizes $\mathbf{C}^\circ_{\mathbf{G}}(E_{j+1})$. Thus $D_j \leq \mathbf{Z}(\mathbf{C}^\circ_{\mathbf{G}}(E_{j+1}))$ and hence $D_j \leq \mathbf{Z}^\circ(\mathbf{C}^\circ_{\mathbf{G}}(E_{j+1}))$ by Proposition 7.1.6 as $\ell \in \Gamma(\mathbf{C}^\circ_{\mathbf{G}}(E_{j+1}), F)$ (see Remark 7.1.3). It follows that $D_j \leq D_{j+1} = E_{j+1}$. □

Before proving the main result of this subsection, we need a lemma. Notice that the group $\mathbf{G}^F$ acts on the set $\mathcal{L}(\mathbf{G}, F)$. For a chain $\mathbb{L} \in \mathcal{L}(\mathbf{G}, F)$, we denote by $\mathbf{G}^F_{\mathbb{L}}$ the stabilizer of $\mathbb{L}$ in $\mathbf{G}^F$. Observe that this stabilizer coincides with the intersection of the normalizers of the individual terms $\mathbf{L}_i$ of the chain $\mathbb{L}$.

Lemma 9.2.5. *Let* $\mathbf{G}^F$ *be a finite reductive group and consider a chain of* e*-split Levi subgroups* $\mathbb{L} \in \mathcal{L}(\mathbf{G})$ *with final term* $\mathbf{L}$*. If* $\ell \in \Gamma(\mathbf{G}, F)$*, then:*

- *(i) Every block of* $\mathbf{G}^F_{\mathbb{L}}$ *is* $\mathbf{L}^F$*-regular (see [Nav98, p.210]). In particular, for* $b \in \mathrm{Bl}(\mathbf{L}^F)$*, the induced block* $b^{\mathbf{G}^F_{\mathbb{L}}}$ *is defined and is the unique block of* $\mathbf{G}^F_{\mathbb{L}}$ *that covers* b*.*
- *(ii) Assume Hypothesis 7.2.7. There is a partition of the irreducible characters of* $\mathbf{G}^F_{\mathbb{L}}$ *given by*

$$\mathrm{Irr}\left(\mathbf{G}^F_{\mathbb{L}}\right) = \coprod_{(\mathbf{M},\mu)/\sim} \mathrm{Irr}\left(\mathbf{G}^F_{\mathbb{L}} \;\middle|\; \mathcal{E}(\mathbf{L}^F, (\mathbf{M}, \mu))\right),$$

where the union is taken over the e*-cuspidal pairs* $(\mathbf{M}, \mu)$ *of* $\mathbf{L}$ *up to* $\mathbf{G}^F_{\mathbb{L}}$*-conjugation.*

Proof. To prove the first statement, let $E := \Omega_1(\mathbf{O}_\ell(\mathbf{Z}^\circ(\mathbf{L})^F))$ and observe that $\mathbf{L}^F = \mathbf{C}^\circ_{\mathbf{G}}(E)^F = \mathbf{C}_{\mathbf{G}}(E)^F$ by Proposition 7.1.6 (ii.a)-(ii.e) and that $E \leq \mathbf{O}_\ell(\mathbf{G}^F_{\mathbb{L}})$. If $B \in \mathrm{Bl}(G_{\mathbb{L}})$ has defect group D, then $E \leq D$ (see [Nav98, Theorem 4.8]) and $\mathbf{C}_{\mathbf{G}^F_{\mathbb{L}}}(D) \leq \mathbf{C}^F_{\mathbf{G}}(E) = \mathbf{L}^F$. This shows that B is $\mathbf{L}^F$-regular. In particular, if the block B covers $b \in \mathrm{Bl}(\mathbf{L}^F)$, then $B = b^{\mathbf{G}^F_{\mathbb{L}}}$ by [Nav98, Theorem 9.19].

We now consider the second statement. As $\mathrm{Irr}(\mathbf{L}^F)$ is the union of the e-Harish-Chandra series $\mathcal{E}(\mathbf{L}^F, (\mathbf{M}, \mu))$ by Corollary 7.2.16, we deduce that every character $\chi \in \mathrm{Irr}(\mathbf{G}^F_{\mathbb{L}})$ lies over some character of an e-Harish-Chandra series $\mathcal{E}(\mathbf{L}^F, (\mathbf{M}, \mu))$, where $(\mathbf{M}, \mu)$ is an e-cuspidal pair of $\mathbf{L}$. To conclude we have to show that, if $(\mathbf{M}', \mu')$ is another e-cuspidal pair of $\mathbf{L}$, then $\mathrm{Irr}(\mathbf{G}^F_{\mathbb{L}} \mid \mathcal{E}(\mathbf{L}^F, (\mathbf{M}, \mu)))$ and $\mathrm{Irr}(\mathbf{G}^F_{\mathbb{L}} \mid \mathcal{E}(\mathbf{L}^F, (\mathbf{M}', \mu')))$ are disjoint unless $(\mathbf{M}, \mu)$

and $(\mathbf{M}',\mu')$ are $\mathbf{G}_{\mathbb{L}}^F$-conjugate. For this, suppose that χ is a character contained in the intersection of $\operatorname{Irr}(\mathbf{G}_{\mathbb{L}}^F \mid \mathcal{E}(\mathbf{L}^F,(\mathbf{M},\mu)))$ and $\operatorname{Irr}(\mathbf{G}_{\mathbb{L}}^F \mid \mathcal{E}(\mathbf{L}^F,(\mathbf{M}',\mu')))$. Let $\psi \in \mathcal{E}(\mathbf{L}^F,(\mathbf{M},\mu))$ and $\psi' \in \mathcal{E}(\mathbf{L}^F,(\mathbf{M}',\mu'))$ lie below χ and consider $g \in \mathbf{G}_{\mathbb{L}}^F$ such that $\psi = \psi'^g$. Then, $\psi \in \mathcal{E}(\mathbf{L}^F,(\mathbf{M},\mu)) \cap \mathcal{E}(\mathbf{L}^F,(\mathbf{M}',\mu')^g)$ and Corollary 7.2.16 implies that $(\mathbf{M},\mu) = (\mathbf{M}',\mu')^{gx}$, for some $x \in \mathbf{L}^F$. Since $gx \in \mathbf{G}_{\mathbb{L}}^F$ the proof is now complete. □

Before proving the main result of this section (see Proposition 9.2.10) we give some definitions.

Definition 9.2.6. Let $\mathbf{M}$ be an e-split Levi subgroup of a connected reductive group $\mathbf{H}$ with Frobenius endomorphism F. For a set $\mathcal{Y} \subseteq \operatorname{Irr}(\mathbf{M}^F)$ of e-cuspidal characters, we define

$$\mathcal{E}\left(\mathbf{H}^F,(\mathbf{M},\mathcal{Y})\right) := \bigcup_{\mu\in\mathcal{Y}} \mathcal{E}\left(\mathbf{H}^F,(\mathbf{M},\mu)\right).$$

Moreover, for a fixed character $\mu \in \operatorname{Irr}(\mathbf{M}^F)$ we define the set

$$\mathcal{Y}(\mu) := \left\{\mu\eta \;\middle|\; \eta \in \operatorname{Irr}\left(\mathbf{M}^F/[\mathbf{M},\mathbf{M}]^F\right)\right\}.$$

By [Bon06, Proposition 12.1], if μ is e-cuspidal, then every character in $\mathcal{Y}(\mu)$ is e-cuspidal.

Definition 9.2.7. For any e-split Levi subgroup $\mathbf{K}$ of $\mathbf{G}$, we denote by $\mathcal{CP}_e(\mathbf{K}^F)$ the set of all e-cuspidal pairs $(\mathbf{L},\lambda)$ of $\mathbf{K}$. Moreover, when $\ell \in \Gamma(\mathbf{G},F)$, for every block b of $\mathbf{K}^F$ we define the subset $\mathcal{CP}_e(b)$ consisting of those e-cuspidal pairs $(\mathbf{L},\lambda)$ of $\mathbf{K}$ such that $\operatorname{bl}(\lambda)^{\mathbf{K}^F} = b$ (see the comment preceding Proposition 7.2.14 concernig block induction).

Next, we introduce the following set which can be thought of as an adaptation to groups of Lie type of the set $\mathcal{C}^d(B)_\epsilon$ from Conjecture 9.1.1.

Definition 9.2.8. Fix a block $B \in \operatorname{Bl}(\mathbf{G}^F)$. For every nonnegative integer d and $\epsilon \in \{+,-\}$ we define

$$\mathcal{L}^d(B)_\epsilon := \left\{(\mathbb{L},\mathbf{M},\mathcal{Y}(\mu),\vartheta) \;\middle|\; \substack{\mathbb{L}\in\mathcal{L}(\mathbf{G})_\epsilon,(\mathbf{M},\mu)\in\mathcal{CP}_e(B) \text{ with } \mathbf{M}\leq\mathbf{L},\\ \vartheta\in\operatorname{Irr}^d\left(\mathbf{G}_{\mathbb{L}}^F \;\mid\; \mathcal{E}\left(\mathbf{L}^F,(\mathbf{M},\mathcal{Y}(\mu))\right)\right) \text{ with } \operatorname{bl}(\vartheta)^{\mathbf{G}^F}=B}\right\},$$

where $\mathbf{L}$ is the final term of the chain $\mathbb{L}$ while $\mathcal{Y}(\mu)$ and $\mathcal{CP}_e(B)$ are as in Definition 9.2.6 and Definition 9.2.7 respectively. Notice that the group $\mathbf{G}^F$ acts by conjugation on $\mathcal{L}^d(B)_\epsilon$ and denote by $\mathcal{L}^d(B)_\epsilon/\mathbf{G}^F$ the corresponding set of $\mathbf{G}^F$-orbits. As usual, for $(\mathbb{L},\mathbf{M},\mathcal{Y}(\mu),\vartheta) \in \mathcal{L}^d(B)_\epsilon$ we denote the corresponding $\mathbf{G}^F$-orbit by $\overline{(\mathbb{L},\mathbf{M},\mathcal{Y}(\mu),\vartheta)}$.

Remark 9.2.9. We remark that, if $(\mathbf{M},\mu) \in \mathcal{CP}_e(B)$ and $\mu' \in \mathcal{Y}(\mu)$ then we have $\mathcal{Y}(\mu) = \mathcal{Y}(\mu')$ although it might happen that $(\mathbf{M},\mu') \notin \mathcal{CP}_e(B)$. On the other hand, let $\mathbb{L} \in \mathcal{L}(\mathbf{G})$ with last term $\mathbf{L}$ and consider an e-cuspidal pair $(\mathbf{M},\mu)$ of $\mathbf{L}$. If $\vartheta \in \operatorname{Irr}(\mathbf{G}_{\mathbb{L}}^F \mid \mathcal{E}(\mathbf{L}^F,(\mathbf{M},\mathcal{Y}(\mu))))$ and $\operatorname{bl}(\vartheta)^{\mathbf{G}^F} = B$, then there exists $\mu' \in \mathcal{Y}(\mu)$, so that $\mathcal{Y}(\mu) = \mathcal{Y}(\mu')$, such that $(\mathbf{M},\mu') \in \mathcal{CP}_e(B)$. In fact, there exists $\mu' \in \mathcal{Y}(\mu)$ such that $\vartheta \in \operatorname{Irr}(\mathbf{G}_{\mathbb{L}}^F \mid \mathcal{E}(\mathbf{L}^F,(\mathbf{M},\mu')))$. By Proposition 7.2.14 every character of $\mathcal{E}(\mathbf{L}^F,(\mathbf{M},\mu'))$ is contained in $\operatorname{bl}(\mu')^{\mathbf{L}^F}$. Then, applying Lemma 9.2.5 (i) and using the transitivity of block induction, it follows that $\operatorname{bl}(\vartheta) = (\operatorname{bl}(\mu')^{\mathbf{L}^F})^{\mathbf{G}_{\mathbb{L}}^F} = \operatorname{bl}(\mu')^{\mathbf{G}_{\mathbb{L}}^F}$. We deduce that $\operatorname{bl}(\mu')^{\mathbf{G}^F} = \operatorname{bl}(\vartheta)^{\mathbf{G}^F} = B$ and hence $(\mathbf{M},\mu') \in \mathcal{CP}_e(B)$. It follows from the above discussion that the set defined in Definition 9.2.8 coincides with

$$\mathcal{L}^d(B)_\epsilon := \left\{(\mathbb{L},\mathbf{M},\mathcal{Y}(\mu),\vartheta) \;\middle|\; \substack{\mathbb{L}\in\mathcal{L}(\mathbf{G})_\epsilon,(\mathbf{M},\mu)\in\mathcal{CP}_e(\mathbf{L}^F),\\ \vartheta\in\operatorname{Irr}^d\left(\mathbf{G}_{\mathbb{L}}^F \;\mid\; \mathcal{E}\left(\mathbf{L}^F,(\mathbf{M},\mathcal{Y}(\mu))\right)\right) \text{ with } \operatorname{bl}(\vartheta)^{\mathbf{G}^F}=B}\right\},$$

where $\mathbf{L}$ is the final term of the chain $\mathbb{L}$.

We are now able to prove a reformulation of Conjecture 9.1.1 tailored to finite groups of Lie type.

Proposition 9.2.10. *Assume Hypothesis 9.2.1 and Hypothesis 7.2.7. Then Conjecture 9.1.1 holds for a block $B \in \mathrm{Bl}(\mathbf{G}^F)$ with nontrivial defect groups and $d \geq 0$ with respect to $\mathbf{G}^F \trianglelefteq \mathbf{G}^F \rtimes \mathrm{Aut}_{\mathbb{F}}(\mathbf{G}^F) =: A$ provided there exists an $\mathrm{Aut}_{\mathbb{F}}(\mathbf{G}^F)_B$-equivariant bijection*

$$\Lambda : \mathcal{L}^d(B)_+/\mathbf{G}^F \to \mathcal{L}^d(B)_-/\mathbf{G}^F$$

such that $\mathrm{Ker}(\vartheta_{\mathbf{Z}(\mathbf{G}^F)}) = \mathrm{Ker}(\chi_{\mathbf{Z}(\mathbf{G}^F)}) =: Z$ *and*

$$\left(A_{\mathbb{L},\vartheta}/Z, \mathbf{G}^F_{\mathbb{L}}/Z, \overline{\vartheta}\right) \sim_{\mathbf{G}^F/Z} \left(A_{\mathbb{K},\chi}/Z, \mathbf{G}^F_{\mathbb{K}}/Z, \overline{\chi}\right)$$

for every $(\mathbb{L}, \mathbf{M}, \mathcal{Y}(\mu), \vartheta) \in \mathcal{L}^d(B)_+$ *and* $(\mathbb{K}, \mathbf{N}, \mathcal{Y}(\nu), \chi) \in \Lambda(\overline{(\mathbb{L}, \mathbf{M}, \mathcal{Y}(\mu), \vartheta)})$.

Proof. Consider $(\mathbb{E}, \vartheta) \in \mathcal{C}^d(B)_+$. By Lemma 3.5.3 we may assume that $\mathbb{E}$ is an ℓ-elementary abelian chain. If $\mathbb{E}$ is a bad ℓ-elementary abelian chain (see Definition 9.2.2), then we define

$$\Omega\left(\overline{(\mathbb{E}, \vartheta)}\right) := \overline{(\mathbb{E}', \vartheta)},$$

where $\mathbb{E}'$ is the chain corresponding to $\mathbb{E}$ via the bijection given by Lemma 9.2.4. Notice in this case that $\mathbf{G}^F_{\mathbb{E}} = \mathbf{G}^F_{\mathbb{E}'}$ and therefore that $(\mathbb{E}', \vartheta) \in \mathcal{C}^d(B)_-$. Then, assume that $\mathbb{E}$ is a good ℓ-elementary abelian chain and consider the corresponding chain of e-split Levi subgroups $\mathbb{L}$ given by Lemma 9.2.3. Notice that $\mathbf{G}^F_{\mathbb{E}} = \mathbf{G}^F_{\mathbb{L}}$ and let $\mathbf{L}$ be the final term of $\mathbb{L}$. By Lemma 9.2.5 (ii), there exists an e-cuspidal pair $(\mathbf{M}, \mu)$ of $\mathbf{L}$, unique up to $\mathbf{G}^F_{\mathbb{L}}$-conjugation, such that $\vartheta \in \mathrm{Irr}^d(\mathbf{G}^F_{\mathbb{L}} \mid \mathcal{E}(\mathbf{L}^F, (\mathbf{M}, \mu))$. We claim that $(\mathbf{M}, \mu) \in \mathcal{CP}_e(B)$. First, observe that every character of $\mathcal{E}(\mathbf{L}^F, (\mathbf{M}, \mu))$ is contained in the block $\mathrm{bl}(\mu)^{\mathbf{L}^F}$ by Proposition 7.2.14. Then, applying Lemma 9.2.5 (i) and using the transitivity of block induction, it follows that $\mathrm{bl}(\vartheta) = (\mathrm{bl}(\mu)^{\mathbf{L}^F})^{\mathbf{G}^F_{\mathbb{L}}} = \mathrm{bl}(\mu)^{\mathbf{G}^F_{\mathbb{L}}}$. Since $(\mathbb{D}, \vartheta) \in \mathcal{C}^d(B)$, we deduce that $\mathrm{bl}(\vartheta)^{\mathbf{G}^F} = B$ and hence $(\mathbf{M}, \mu) \in \mathcal{CP}_e(B)$. This proves the claim. Now $(\mathbb{L}, \mathbf{M}, \mathcal{Y}(\mu), \vartheta) \in \mathcal{L}^d(B)_+$ and we choose $(\mathbb{K}, \mathbf{N}, \mathcal{Y}(\nu), \chi) \in \Lambda(\overline{(\mathbb{L}, \mathbf{M}, \mathcal{Y}(\mu), \vartheta)})$. Let $\mathbb{D}$ be the ℓ-elementary abelian chain corresponding to $\mathbb{K}$ via the bijection given by 9.2.3 and observe that $(\mathbb{D}, \chi) \in \mathcal{C}^d(B)_-$. Finally, we define

$$\Omega\left(\overline{(\mathbb{E}, \vartheta)}\right) := \overline{(\mathbb{D}, \chi)}.$$

Since $(\mathbf{M}, \mu)$ is unique up to $\mathbf{G}^F_{\mathbb{L}}$-conjugation while Λ and the bijections given by Lemma 9.2.4 and Lemma 9.2.3 are equivariant, we conclude that Ω is a well defined A_B-equivariant bijection. Moreover, using the property on character triples of Λ it is immediate to show that Ω satisfies the analogous properties required by Conjecture 9.1.1. This completes the proof. □

9.2.2 From Condition 9.1 to Conjecture 9.1.1

We now come to the proof of Theorem 9.2, namely we show how to deduce Conjecture 9.1.1 from Condition 9.1. Because our final aim is to show the inductive condition for Dade's Projective Conjecture, from now on we restrict our attention to simple algebraic group of simply connected type. Furthermore, in order to be able to use the results from Section 7.2 and Section 9.2.1, we will make the following assumptions.

Hypothesis 9.2.11. Assume Hypothesis 7.2.7 and suppose that $\mathbf{G}$ is simple of simply connected type such that $\mathbf{O}_\ell(\mathbf{G}^F) \leq \mathbf{Z}(\mathbf{G}^F)$.

Remark 9.2.12. Observe that under Hypothesis 9.2.11 the prime ℓ does not divide $|\mathbf{Z}(\mathbf{G})^F|$ and therefore $\mathbf{O}_\ell(\mathbf{G}^F) = 1$. Thus Hypothesis 9.2.1 is satisfied. In addition, the requirements of Proposition 7.4.1 are satisfied with $\mathbf{G} = \mathbf{L}$ (this will be used in the proof of Theorem 9.2.21).

By using Remark 7.2.8 we deduce that Hypothesis 9.2.11 holds whenever $\mathbf{G}$ is simple of simply connected type such that $\mathbf{G}^F \neq {}^2\mathbf{E}_6(2)$, $\mathbf{E}_7(2)$, $\mathbf{E}_8(2)$ such that $\mathbf{G}^F/\mathbf{Z}(\mathbf{G}^F)$ is a nonabelian simple group and $\ell \in \Gamma(\mathbf{G}, F)$ with $\ell \geq 5$.

Before proceeding further we introduce the notion of irreducible rational component (see [CE94, Section 1.1]).

Definition 9.2.13. Let $\mathbf{G}$ be a connected reductive group with Frobenius endomorphism $F : \mathbf{G} \to \mathbf{G}$. Recall that $[\mathbf{G}, \mathbf{G}]$ is the product of simple algebraic groups $\mathbf{G}_1, \ldots, \mathbf{G}_n$ and that F acts on the set $\{\mathbf{G}_1, \ldots, \mathbf{G}_n\}$. For any orbit $\mathcal{O}$ of F, we denote by $\mathbf{G}_\mathcal{O}$ the product of those simple algebraic groups in the orbit $\mathcal{O}$. Notice that $\mathbf{G}_\mathcal{O}$ is F-stable and, by abuse of notation, denote by F the restriction of F to $\mathbf{G}_\mathcal{O}$. Then, we say that $(\mathbf{G}_\mathcal{O}, F)$ is an **irreducible rational component** of $(\mathbf{G}, F)$.

The proof of the next result should be compared to the argument used in [Ruh21a, Proposition 3.8]. Recall that a connected reductive group $\mathbf{G}$ is simply connected if the semisimple algebraic group $[\mathbf{G}, \mathbf{G}]$ is simply connected.

Proposition 9.2.14. *Assume that Hypothesis 7.2.7 holds for $(\mathbf{G}, F)$ and that $\mathbf{G}$ is simply connected. Consider an e-split Levi subgroup $\mathbf{K}$ of $\mathbf{G}$ and suppose that Condition 9.1 holds at the prime ℓ for every irreducible rational component of $(\mathbf{K}, F)$. Let $\mathbf{K}_0 := [\mathbf{K}, \mathbf{K}]$ and consider an e-cuspidal pair $(\mathbf{L}_0, \lambda_0)$ of $\mathbf{K}_0$. Then there exists a defect preserving $\mathrm{Aut}_\mathbb{F}(\mathbf{K}_0^F)_{(\mathbf{L}_0,\lambda_0)}$-equivariant bijection*

$$\Omega^{\mathbf{K}_0}_{(\mathbf{L}_0,\lambda_0)} : \mathcal{E}\left(\mathbf{K}_0^F, (\mathbf{L}_0, \lambda_0)\right) \to \mathrm{Irr}\left(\mathbf{N}_{\mathbf{K}_0}(\mathbf{L}_0)^F \;\middle|\; \lambda_0\right)$$

such that

$$\left(Y_\vartheta, \mathbf{K}_0^F, \vartheta\right) \sim_{\mathbf{K}_0^F} \left(\mathbf{N}_{Y_\vartheta}(\mathbf{L}_0), \mathbf{N}_{\mathbf{K}_0}(\mathbf{L}_0), \Omega^{\mathbf{K}_0}_{(\mathbf{L}_0,\lambda_0)}(\vartheta)\right)$$

for every $\vartheta \in \mathcal{E}(\mathbf{K}_0^F, (\mathbf{L}_0, \lambda_0))$ and where $Y := \mathbf{K}_0^F \rtimes \mathrm{Aut}_\mathbb{F}(\mathbf{K}_0^F)$.

Proof. Since $\mathbf{G}$ is simply connected, we deduce that $\mathbf{K}_0$ is a semisimple group of simply connected type (see [MT11, Proposition 12.14]). By [Mar91, Proposition 1.4.10], $\mathbf{K}_0$ is the direct product of simple algebraic groups $\mathbf{K}_1, \ldots, \mathbf{K}_n$ and the action of F induces a permutation on the set of simple components $\mathbf{K}_i$. For every orbit of F we denote by $\mathbf{H}_j$, $j = 1, \ldots, t$, the direct product of simple components in such orbit. Then $\mathbf{H}_j$ is F-stable and

$$\mathbf{K}_0^F = \mathbf{H}_1^F \times \cdots \times \mathbf{H}_t^F,$$

where by abuse of notation we denote the restriction of F to $\mathbf{H}_j$ again by F. Observe that the $(\mathbf{H}_j, F)$'s are the irreducible rational components of $(\mathbf{K}, F)$. Define $\mathbf{M}_j := \mathbf{L}_0 \cap \mathbf{H}_j$ and observe that $\mathbf{M}_j$ is an e-split Levi subgroup of $\mathbf{H}_j$ and that

$$\mathbf{L}_0^F = \mathbf{M}_1^F \times \cdots \times \mathbf{M}_t^F.$$

Then, we can write $\lambda_0 = \mu_1 \times \cdots \times \mu_t$ with $\mu_j \in \mathrm{Irr}(\mathbf{M}_j^F)$. As $\mathbf{R}_{\mathbf{L}_0}^{\mathbf{K}_0} = \mathbf{R}_{\mathbf{M}_1}^{\mathbf{H}_1} \times \cdots \times \mathbf{R}_{\mathbf{M}_t}^{\mathbf{H}_t}$ (see [DM91, Proposition 10.9 (ii)]), it follows that $(\mathbf{M}_j, \mu_j)$ is an e-cuspidal pair of $\mathbf{H}_j$ for every $j = 1, \ldots, t$ and, using our assumption, there exist bijections

$$\Omega^{\mathbf{H}_j}_{(\mathbf{M}_j,\mu_j)} : \mathcal{E}\left(\mathbf{H}_j^F, (\mathbf{M}_j, \mu_j)\right) \to \mathrm{Irr}\left(\mathbf{N}_{\mathbf{H}_j}(\mathbf{M}_j)^F \;\middle|\; \mu_j\right) \tag{9.2.1}$$

as in Condition 9.1. By using the fact that $\mathbf{R}_{\mathbf{L}_0}^{\mathbf{K}_0} = \mathbf{R}_{\mathbf{M}_1}^{\mathbf{H}_1} \times \cdots \times \mathbf{R}_{\mathbf{M}_t}^{\mathbf{H}_t}$, we deduce that $\mathcal{E}(\mathbf{K}_0^F, (\mathbf{L}_0, \lambda_0))$ coincides with the set of characters of the form $\vartheta_1 \times \cdots \times \vartheta_t$ with $\psi_j \in \mathcal{E}(\mathbf{H}_j^F, (\mathbf{M}_j, \mu_j))$, while it is not hard to see that $\mathrm{Irr}(\mathbf{N}_{\mathbf{K}_0}(\mathbf{L}_0)^F \mid \lambda_0)$ coincides with the set of characters of the form $\xi_1 \times \cdots \times \xi_t$ with $\xi_j \in \mathrm{Irr}(\mathbf{N}_{\mathbf{H}_j}(\mathbf{M}_j)^F \mid \mu_j)$. Hence, we obtain a map

$$\begin{aligned}\Omega^{\mathbf{K}_0}_{(\mathbf{L}_0,\lambda_0)} : \mathcal{E}\left(\mathbf{K}_0^F, (\mathbf{L}_0, \lambda_0)\right) &\to \mathrm{Irr}\left(\mathbf{N}_{\mathbf{K}_0}(\mathbf{L}_0)^F \;\middle|\; \lambda_0\right)\\ \vartheta_1 \times \cdots \times \vartheta_t &\mapsto \Omega^{\mathbf{H}_1}_{(\mathbf{M}_1,\mu_1)}(\vartheta_1) \times \cdots \times \Omega^{\mathbf{H}_t}_{(\mathbf{M}_t,\mu_t)}(\vartheta_t).\end{aligned}$$

We now show that $\Omega^{\mathbf{K}_0}_{(\mathbf{L}_0,\lambda_0)}$ satisfies the required properties.

First, consider the partition $\{1, \ldots, t\} = \coprod_l A_l$ given by $j, k \in A_l$ if there exists a bijective morphism $\varphi : \mathbf{H}_j \to \mathbf{H}_k$ commuting with F such that $\varphi(\mathbf{M}_j, \mu_j) = (\mathbf{M}_k, \mu_k)$. Fix $j_l \in A_l$. By Lemma 3.3.8 (iii), we may assume without loss of generality that

$$\mathbf{K}_0^F = \bigtimes_l \mathbf{H}_{A_l}^F$$

and

$$\Omega^{\mathbf{K}_0}_{(\mathbf{L}_0,\lambda_0)} = \bigtimes_l \Omega^{\mathbf{H}_{A_l}}_{(\mathbf{M}_{A_l},\mu_{A_l})}$$

where $\mathbf{H}_{A_l} := \mathbf{H}_{j_l}^{|A_l|}$, $\mathbf{M}_{A_l} := \mathbf{M}_{j_l}^{|A_l|}$, $\mu_{A_l} = \mu_{j_l}^{\otimes|A_l|}$ and $\Omega^{\mathbf{H}_{A_l}}_{(\mathbf{M}_{A_l},\lambda_{A_l})} := (\Omega^{\mathbf{H}_{j_l}}_{(\mathbf{M}_{j_l},\lambda_{j_l})})^{\otimes|A_l|}$. Fix $\vartheta = \times_l \vartheta_{A_l}$, with $\vartheta_{A_l} \in \mathcal{E}(\mathbf{H}_{A_l}^F, (\mathbf{M}_{A_l}, \mu_{A_l}))$, and write $\xi := \Omega^{\mathbf{K}_0}_{(\mathbf{L}_0,\lambda_0)}(\vartheta) = \times_l \xi_{A_l}$ with $\xi_{A_l} = \Omega^{\mathbf{H}_{A_l}}_{(\mathbf{M}_{A_l},\mu_{A_l})}(\vartheta_{A_l})$. Then, noticing that $\mathrm{Aut}_{\mathbb{F}}(\mathbf{K}_0^F) = \bigtimes_l \mathrm{Aut}_{\mathbb{F}}(\mathbf{H}_{A_l}^F)$, by Lemma 3.4.9 it is enough to check that

$$\left(Y_{A_l,\vartheta_{A_l}}, \mathbf{H}_{A_l}^F, \vartheta_{A_l}\right) \sim_{\mathbf{H}_{A_l}^F} \left(\mathbf{N}_{Y_{A_l,\vartheta_{A_l}}}(\mathbf{M}_{A_l}), \mathbf{N}_{\mathbf{H}_{A_l}}(\mathbf{M}_{A_l})^F, \xi_{A_l}\right) \tag{9.2.2}$$

where $Y_{A_l} := \mathbf{H}_{A_l}^F \rtimes \mathrm{Aut}_{\mathbb{F}}(\mathbf{H}_{A_l}^F)$.

To prove (9.2.2), observe that ϑ_{A_l} is $\mathrm{Aut}_{\mathbb{F}}(\mathbf{H}_{A_l}^F)_{(\mathbf{M}_{A_l},\mu_{A_l})}$-conjugate to a character of the form $\times_u \vartheta_u$ such that for every u, v we have either $\vartheta_u = \vartheta_v$ or ϑ_u and ϑ_v are not $\mathrm{Aut}_{\mathbb{F}}(\mathbf{H}_{A_l}^F)$-conjugate. By Lemma 3.3.8 (iii), we may assume without loss of generality that $\vartheta_{A_l} = \times_u \nu_u^{m_u}$, where for every $u \neq v$ the characters ν_u and ν_v are distinct and not $\mathrm{Aut}_{\mathbb{F}}(\mathbf{H}_{A_l})$-conjugate while m_u are some nonnegative integers. Then

$$\mathrm{Aut}_{\mathbb{F}}(\mathbf{H}_{A_l}^F)_{\vartheta_{A_l}} = \bigtimes_u \left(\mathrm{Aut}_{\mathbb{F}}(\mathbf{H}_{j_l})_{\nu_u} \wr S_{m_u}\right)$$

and hence (9.2.2) follows by the properties of the bijections (9.2.1) by applying Lemma 3.4.10. A similar argument shows that the bijection $\Omega^{\mathbf{K}_0}_{(\mathbf{L}_0,\lambda_0)}$ is $\mathrm{Aut}_{\mathbb{F}}(\mathbf{K}_0^F)_{(\mathbf{L}_0,\lambda_0)}$-equivariant. Moreover $\Omega^{\mathbf{K}_0}_{(\mathbf{L}_0,\lambda_0)}$ preserves the defect of characters by the analogous property of Condition 9.1. □

We now prove an easy lemma which will be used to combine bijections $\Omega^{\mathbf{K}_0}_{(\mathbf{L}_0,\lambda_0)}$ given by Proposition 9.2.14 for various e-cuspidal pairs $(\mathbf{L}_0,\lambda_0)$.

Lemma 9.2.15. *Let $X\leq Y\leq Z$ with $X,Y\trianglelefteq Z$ and Y/X abelian. Consider $\eta\in\mathrm{Irr}(Y)$ and define the set $\mathcal{Y}:=\{\eta\nu\mid\nu\in\mathrm{Irr}(Y/X)\}$. If $z\in Z$ and $\mathcal{Y}^z\cap\mathcal{Y}\neq\varnothing$, then $\mathcal{Y}^z=\mathcal{Y}$.*

Proof. Suppose that $\eta\nu\in\mathcal{Y}^z\cap\mathcal{Y}$, then there exists $\nu_1\in\mathrm{Irr}(Y/X)$ such that $\eta\nu=(\eta\nu_1)^z$. Since Y/X is abelian we deduce that $\eta^z=\eta\nu(\nu_1^z)^{-1}$. Now, if $\eta\nu_2\in\mathcal{Y}$, then $(\eta\nu_2)^z=\eta^z\nu_2^z=\eta\nu(\nu_1^z)^{-1}\nu_2^z$. Noticing that $\nu(\nu_1^z)^{-1}\nu_2^z\in\mathrm{Irr}(Y/X)$, we conclude that $\mathcal{Y}^z\subseteq\mathcal{Y}$ and the result follows. □

Corollary 9.2.16. *Assume that Hypothesis 7.2.7 holds for $(\mathbf{G},F)$ and that $\mathbf{G}$ is simply connected. Consider an e-split Levi subgroup $\mathbf{K}$ of $\mathbf{G}$ and suppose that Condition 9.1 holds at the prime ℓ for every irreducible rational component of $(\mathbf{K},F)$. Let $(\mathbf{L},\lambda)$ be an e-cuspidal pair of $\mathbf{K}$, set $\mathbf{K}_0:=[\mathbf{K},\mathbf{K}]$ and $\mathbf{L}_0:=\mathbf{L}\cap\mathbf{K}_0$ and consider $\lambda_0\in\mathrm{Irr}(\lambda_{\mathbf{L}_0^F})$. Define $\mathcal{Y}_0:=\{\lambda_0\xi\mid\xi\in\mathrm{Irr}(\mathbf{L}_0^F/[\mathbf{L},\mathbf{L}]^F)\}$. Then there exists a defect preserving $\mathrm{Aut}_{\mathbb{F}}(\mathbf{G}^F)_{\mathbf{K},\mathbf{L},\mathcal{Y}(\lambda_0)}$-equivariant bijection*

$$\Psi^{\mathbf{K}_0}_{(\mathbf{L}_0,\lambda_0)}:\mathcal{E}\left(\mathbf{K}_0^F,(\mathbf{L}_0,\mathcal{Y}_0)\right)\to\mathrm{Irr}\left(\mathbf{N}_{\mathbf{K}_0}(\mathbf{L}_0)^F\;\middle|\;\mathcal{Y}_0\right)$$

such that

$$\left(Y_\vartheta,\mathbf{K}_0^F,\vartheta\right)\sim_{\mathbf{K}_0^F}\left(\mathbf{N}_{Y_\vartheta}(\mathbf{L}_0),\mathbf{N}_{\mathbf{K}_0}(\mathbf{L}_0)^F,\Psi^{\mathbf{K}_0}_{(\mathbf{L}_0,\lambda_0)}(\vartheta)\right)$$

for every $\vartheta\in\mathcal{E}(\mathbf{K}_0^F,(\mathbf{L}_0,\mathcal{Y}_0))$ and where $Y:=\mathbf{K}_0^F\rtimes\mathrm{Aut}_{\mathbb{F}}(\mathbf{K}_0^F)$.

Proof. First observe that for every $\lambda_0\xi\in\mathcal{Y}_0$ the pair $(\mathbf{L}_0,\lambda_0\xi)$ is e-cuspidal in $\mathbf{K}_0$ (see [Bon06, Proposition 12.1]). Moreover, notice that $\mathbf{L}=\mathbf{Z}(\mathbf{K})\mathbf{L}_0$ and therefore $\mathbf{N}_{\mathbf{K}}(\mathbf{L}_0)=\mathbf{N}_{\mathbf{K}}(\mathbf{L})$. Let $\mathbb{T}$ be a $\mathbf{N}_{\mathbf{K}_0}(\mathbf{L})^F_{\mathcal{Y}_0}\rtimes\mathrm{Aut}_{\mathbb{F}}(\mathbf{G}^F)_{\mathbf{K},\mathbf{L},\mathcal{Y}_0}$-transversal in $\mathcal{Y}_0$. For each $\lambda_0\xi\in\mathbb{T}$ consider an $\mathrm{Aut}_{\mathbb{F}}(\mathbf{G}^F)_{\mathbf{K},\mathbf{L},\lambda_0\xi}$-transversal $\mathcal{T}_{\lambda_0\xi}$ in $\mathcal{E}(\mathbf{K}_0^F,(\mathbf{L}_0,\lambda_0\xi))$ and define $\mathcal{T}$ as the union of the sets $\mathcal{T}_{\lambda_0\xi}$ with $\lambda_0\xi\in\mathbb{T}$.

We claim that $\mathcal{T}$ is an $\mathrm{Aut}_{\mathbb{F}}(\mathbf{G}^F)_{\mathbf{K},\mathbf{L},\mathcal{Y}_0}$-transversal in

$$\mathcal{E}\left(\mathbf{K}_0^F,(\mathbf{L}_0,\mathcal{Y}_0)\right).$$

First let $\chi\in\mathcal{E}(\mathbf{K}_0^F,(\mathbf{L}_0,\lambda_0\xi))$ with $\xi\in\mathrm{Irr}(\mathbf{L}_0^F/[\mathbf{L},\mathbf{L}]^F)$ and consider the unique $\lambda_0\widehat{\xi}\in\mathbb{T}$ such that $(\lambda_0\xi)^{xy}=\lambda_0\widehat{\xi}$ for some $x\in\mathbf{N}_{\mathbf{K}_0}(\mathbf{L})^F_{\mathcal{Y}_0}$ and $y\in\mathrm{Aut}_{\mathbb{F}}(\mathbf{G}^F)_{\mathbf{K},\mathbf{L},\mathcal{Y}_0}$. Then $\chi^y=\chi^{xy}\in\mathcal{E}(\mathbf{K}_0^F,(\mathbf{L}_0,\lambda_0\widehat{\xi}))$ and there exist a unique $\widehat{\chi}\in\mathcal{T}_{\lambda_0\widehat{\xi}}$ and $z\in\mathrm{Aut}_{\mathbb{F}}(\mathbf{G}^F)_{\mathbf{K},\mathbf{L},\lambda_0\widehat{\xi}}$ such that $\chi^{yz}=\widehat{\chi}$. By Lemma 9.2.15 it follows that $\mathrm{Aut}_{\mathbb{F}}(\mathbf{G}^F)_{\mathbf{K},\mathbf{L},\lambda_0\widehat{\xi}}\leq\mathrm{Aut}_{\mathbb{F}}(\mathbf{G}^F)_{\mathbf{K},\mathbf{L},\mathcal{Y}_0}$ and hence $yz\in\mathrm{Aut}_{\mathbb{F}}(\mathbf{G}^F)_{\mathbf{K},\mathbf{L},\mathcal{Y}_0}$. Next, for $i=1,2$ consider $\chi_i\in\mathcal{T}_{\lambda_0\xi_i}$ with $\lambda\xi_i\in\mathbb{T}$ such that $\chi_1=\chi_2^y$ with $y\in\mathrm{Aut}_{\mathbb{F}}(\mathbf{G}^F)_{\mathbf{K},\mathbf{L},\mathcal{Y}_0}$. In particular $\chi_1\in\mathcal{E}(\mathbf{K}_0^F,(\mathbf{L}_0,\lambda_0\xi_1))\cap\mathcal{E}(\mathbf{K}_0^F,(\mathbf{L}_0,\lambda_0\xi_2)^y)$ and Proposition 7.2.15 implies that $\lambda_0\xi_1=(\lambda_0\xi_2)^{yx}$ for some $x\in\mathbf{N}_{\mathbf{K}_0}(\mathbf{L})^F$. Moreover, Lemma 9.2.15 yields $x\in\mathbf{N}_{\mathbf{K}_0}(\mathbf{L})^F_{\mathcal{Y}_0}$ and by the choice of $\mathbb{T}$ it follows that $\lambda_0\xi_1=\lambda_0\xi_2$. Now $yx\in\mathrm{Aut}_{\mathbb{F}}(\mathbf{G}^F)_{\mathbf{K},\mathbf{L},\lambda_0\xi_1}$ satisfies $\chi_1=\chi_2^{yx}$ and the choice of $\mathcal{T}_{\lambda_0\xi_1}$ implies that $\chi_1=\chi_2$. This proves the claim.

Next, using Proposition 9.2.14, for every $\lambda_0\xi\in\mathbb{T}$, $\chi\in\mathcal{T}_{\lambda_0\xi}$ and $x\in\mathrm{Aut}_{\mathbb{F}}(\mathbf{G}^F)_{\mathbf{K},\mathbf{L},\mathcal{Y}_0}$ we define

$$\Psi^{\mathbf{K}_0}_{(\mathbf{L}_0,\lambda_0)}(\chi^x):=\Omega^{\mathbf{K}_0}_{(\mathbf{L}_0,\lambda_0\xi)}(\chi)^x.$$

Noticing that $\Psi^{\mathbf{K}_0}_{(\mathbf{L}_0,\lambda_0)}(\mathcal{T})$ is an $\mathrm{Aut}_{\mathbb{F}}(\mathbf{G}^F)_{\mathbf{K},\mathbf{L},\mathcal{Y}_0}$-transversal in

$$\mathrm{Irr}\left(\mathbf{N}_{\mathbf{K}_0}(\mathbf{L}_0)^F \;\middle|\; \mathcal{Y}_0\right)$$

we deduce that $\Psi^{\mathbf{K}_0}_{(\mathbf{L}_0,\lambda_0)}$ is an $\mathrm{Aut}_{\mathbb{F}}(\mathbf{G}^F)_{\mathbf{K},\mathbf{L},\mathcal{Y}_0}$-equivariant bijection. The remaining properties follow directly from the corresponding properties of the bijections $\Omega^{\mathbf{K}_0}_{(\mathbf{K}_0,\lambda_0\xi)}$ given by Proposition 9.2.14. □

Using Theorem 3.4.12 we rewrite the relations on character triples given by Corollary 9.2.16 replacing $\mathbf{K}_0 \rtimes \mathrm{Aut}_{\mathbb{F}}(\mathbf{K}_0^F)$ with $(\mathbf{G}^F \rtimes \mathrm{Aut}_{\mathbb{F}}(\mathbf{G}^F))_{\mathbf{K}}$.

Corollary 9.2.17. *Consider the setup of Corollary 9.2.16. Then*

$$\left(X_\vartheta, \mathbf{K}_0^F, \vartheta\right) \sim_{\mathbf{K}_0^F} \left(\mathbf{N}_{X_\vartheta}(\mathbf{L}_0), \mathbf{N}_{\mathbf{K}_0}(\mathbf{L}_0), \Psi^{\mathbf{K}_0}_{(\mathbf{L}_0,\lambda_0)}(\vartheta)\right)$$

for every $\vartheta \in \mathcal{E}(\mathbf{K}_0^F, (\mathbf{L}_0, \mathcal{Y}_0))$ and where $X := (\mathbf{G}^F \rtimes \mathrm{Aut}_{\mathbb{F}}(\mathbf{G}^F))_{\mathbf{K}}$.

Proof. Fix ϑ as in the statement, let $Y := \mathbf{K}_0^F \rtimes \mathrm{Aut}_{\mathbb{F}}(\mathbf{K}_0^F)$ and consider the canonical maps

$$\epsilon : Y_\vartheta \to \mathrm{Aut}(\mathbf{K}_0^F)$$

and

$$\widehat{\epsilon} : X_\vartheta \to \mathrm{Aut}(\mathbf{K}_0^F).$$

Define $U := \widehat{\epsilon}^{-1}(\epsilon(X_\vartheta)) \leq Y_\vartheta$. By Corollary 9.2.16 we know that

$$\left(Y_\vartheta, \mathbf{K}_0^F, \vartheta\right) \sim_{\mathbf{K}_0^F} \left(\mathbf{N}_{Y_\vartheta}(\mathbf{L}_0), \mathbf{N}_{\mathbf{K}_0}(\mathbf{L}_0), \Psi^{\mathbf{K}_0}_{(\mathbf{L}_0,\lambda_0)}(\vartheta)\right)$$

and applying Lemma 3.3.8 (ii) we obtain

$$\left(U, \mathbf{K}_0^F, \vartheta\right) \sim_{\mathbf{K}_0^F} \left(\mathbf{N}_{U}(\mathbf{L}_0), \mathbf{N}_{\mathbf{K}_0}(\mathbf{L}_0), \Psi^{\mathbf{K}_0}_{(\mathbf{L}_0,\lambda_0)}(\vartheta)\right).$$

Now Theorem 3.4.12 implies that

$$\left(X_\vartheta, \mathbf{K}_0^F, \vartheta\right) \sim_{\mathbf{K}_0^F} \left(\mathbf{N}_{X_\vartheta}(\mathbf{L}_0), \mathbf{N}_{\mathbf{K}_0}(\mathbf{L}_0), \Psi^{\mathbf{K}_0}_{(\mathbf{L}_0,\lambda_0)}(\vartheta)\right)$$

and this concludes the proof. □

Our next goal is to lift the bijection $\Psi^{\mathbf{K}_0}_{(\mathbf{L}_0,\lambda_0)}$ to a similar bijection $\Psi^{\mathbf{K}}_{(\mathbf{L},\lambda)}$. To do so we need the following preliminary result.

Lemma 9.2.18. *Consider the setup of Corollary 9.2.16 with $\mathcal{Y}_0 := \{\lambda_0\xi \mid \xi \in \mathrm{Irr}(\mathbf{L}_0^F/[\mathbf{L},\mathbf{L}]^F)\}$ and let $\mathcal{Y}(\lambda) := \{\lambda\eta \mid \eta \in \mathrm{Irr}(\mathbf{L}^F/[\mathbf{L},\mathbf{L}]^F)\}$ (see Definition 9.2.6). Then*

$$\mathrm{Irr}\left(\mathbf{K}^F \;\middle|\; \mathcal{E}(\mathbf{K}_0^F, (\mathbf{L}_0, \mathcal{Y}_0))\right) = \mathcal{E}\left(\mathbf{K}^F, (\mathbf{L}, \mathcal{Y}(\lambda))\right)$$

and

$$\mathrm{Irr}\left(\mathbf{N}_{\mathbf{K}}(\mathbf{L})^F \;\middle|\; \mathcal{Y}_0\right) = \mathrm{Irr}\left(\mathbf{N}_{\mathbf{K}}(\mathbf{L})^F \;\middle|\; \mathcal{Y}(\lambda)\right).$$

Proof. Let $\lambda_0\xi \in \mathcal{Y}_0$ and consider $\chi \in \mathrm{Irr}(\mathbf{K}^F \mid \mathcal{E}(\mathbf{K}_0^F, (\mathbf{L}_0, \lambda_0\xi)))$. Since $\mathbf{L}^F/[\mathbf{L},\mathbf{L}]^F$ is abelian, ξ has an extension $\widehat{\xi} \in \mathrm{Irr}(\mathbf{L}^F/[\mathbf{L},\mathbf{L}]^F)$. By [GM20, Corollary 3.3.25] and [Isa76, Problem 5.3] we obtain

$$\mathrm{Ind}_{\mathbf{K}_0^F}^{\mathbf{K}^F}\left(\mathbf{R}_{\mathbf{L}_0}^{\mathbf{K}_0}(\lambda_0\xi)\right) = \mathbf{R}_{\mathbf{L}}^{\mathbf{K}}\left(\mathrm{Ind}_{\mathbf{L}_0^F}^{\mathbf{L}^F}(\lambda_0\xi)\right) = \mathbf{R}_{\mathbf{L}}^{\mathbf{K}}\left(\mathrm{Ind}_{\mathbf{L}_0^F}^{\mathbf{L}^F}(\lambda_0)\widehat{\xi}\right).$$

Then, by [Isa76, Problem 6.2] there exists $\eta \in \mathrm{Irr}(\mathbf{L}^F/\mathbf{L}_0^F)$ such that $\chi \in \mathcal{E}(\mathbf{K}^F, (\mathbf{L}, \lambda\eta\widehat{\xi}))$ with $\eta\widehat{\xi} \in \mathrm{Irr}(\mathbf{L}^F/[\mathbf{L},\mathbf{L}]^F)$. Assume now that $\chi \in \mathcal{E}(\mathbf{K}^F, (\mathbf{L}, \lambda\eta))$ with $\lambda\eta \in \mathcal{Y}(\lambda)$. Applying [GM20, Corollary 3.3.25], we obtain

$$\mathrm{Res}_{\mathbf{K}_0^F}^{\mathbf{K}^F}\left(\mathbf{R}_{\mathbf{L}}^{\mathbf{K}}(\lambda\eta)\right) = \mathbf{R}_{\mathbf{L}_0}^{\mathbf{K}_0}\left(\mathrm{Res}_{\mathbf{L}_0^F}^{\mathbf{L}^F}(\lambda\eta)\right).$$

By Clifford's theorem we deduce that $\mathrm{Res}_{\mathbf{K}_0^F}^{\mathbf{K}^F}(\chi)$ has an irreducible constituent in $\mathbf{R}_{\mathbf{L}_0}^{\mathbf{K}_0}(\lambda_0^g\xi)$ for some $g \in \mathbf{L}^F$ and $\xi := \eta_{\mathbf{L}_0^F} \in \mathrm{Irr}(\mathbf{L}_0^F/[\mathbf{L},\mathbf{L}]^F)$. This proves the first equality.

Next, consider $\psi \in \mathrm{Irr}(\mathbf{N}_{\mathbf{K}}(\mathbf{L})^F \mid \lambda\eta)$ with $\lambda\eta \in \mathcal{Y}(\lambda)$. Since $\lambda\eta$ lies above $\lambda_0\xi$, with $\xi := \eta_{\mathbf{L}_0^F} \in \mathrm{Irr}(\mathbf{L}_0^F/[\mathbf{L},\mathbf{L}]^F)$, we deduce that $\psi \in \mathrm{Irr}(\mathbf{N}_{\mathbf{K}}(\mathbf{L})^F \mid \mathcal{Y}_0)$. Conversely suppose that $\psi \in \mathrm{Irr}(\mathbf{N}_{\mathbf{K}}(\mathbf{L})^F \mid \lambda_0\xi)$ with $\lambda_0\xi \in \mathcal{Y}_0$ and consider an extension $\eta_1 \in \mathrm{Irr}(\mathbf{L}^F/[\mathbf{L},\mathbf{L}]^F)$ of ξ. By [Isa76, Problem 5.3 and Problem 6.2], we conclude that there exists $\eta_2 \in \mathrm{Irr}(\mathbf{L}^F/\mathbf{L}_0^F)$ such that ψ lies above $\lambda\eta_1\eta_2$. Since $\eta := \eta_1\eta_2 \in \mathrm{Irr}(\mathbf{L}^F/[\mathbf{L},\mathbf{L}]^F)$ the result follows. □

Corollary 9.2.19. *Assume that Hypothesis 7.2.7 holds for $(\mathbf{G}, F)$ and that $\mathbf{G}$ is simply connected, let $\mathbf{K}$ be an e-split Levi subgroup of $\mathbf{G}$ and suppose that Condition 9.1 holds at the prime ℓ for every irreducible rational component of $(\mathbf{K}, F)$. Let $(\mathbf{L}, \lambda)$ be an e-cuspidal pair of $\mathbf{K}$ and consider $\mathcal{Y}(\lambda)$ as in Definition 9.2.6. Then there exists a defect preserving $\mathrm{Aut}_{\mathbb{F}}(\mathbf{G}^F)_{\mathbf{K},\mathbf{L},\mathcal{Y}(\lambda)}$-equivariant bijection*

$$\Psi_{(\mathbf{L},\lambda)}^{\mathbf{K}} : \mathcal{E}\left(\mathbf{K}^F, (\mathbf{L}, \mathcal{Y}(\lambda))\right) \to \mathrm{Irr}\left(\mathbf{N}_{\mathbf{K}}(\mathbf{L})^F \;\middle|\; \mathcal{Y}(\lambda)\right)$$

such that

$$\left(X_\vartheta, \mathbf{K}^F, \vartheta\right) \sim_{\mathbf{K}^F} \left(\mathbf{N}_{X_\vartheta}(\mathbf{L}), \mathbf{N}_{\mathbf{K}}(\mathbf{L})^F, \Psi_{(\mathbf{L},\lambda)}^{\mathbf{K}}(\vartheta)\right)$$

for every $\vartheta \in \mathcal{E}(\mathbf{K}^F, (\mathbf{L}, \mathcal{Y}(\lambda)))$ and where $X := (\mathbf{G}^F \rtimes \mathrm{Aut}_{\mathbb{F}}(\mathbf{G}^F))_{\mathbf{K}}$.

Proof. Define $\mathbf{K}_0 := [\mathbf{K},\mathbf{K}]$, $\mathbf{L}_0 := \mathbf{L} \cap \mathbf{K}_0$, fix an irreducible constituent λ_0 of $\lambda_{\mathbf{L}_0^F}$ and set $\mathcal{Y}_0 := \{\lambda_0\xi \mid \xi \in \mathrm{Irr}(\mathbf{L}_0^F/[\mathbf{L},\mathbf{L}]^F)\}$. We apply Proposition 9.1.5 with $A := \mathbf{G}^F \rtimes \mathrm{Aut}_{\mathbb{F}}(\mathbf{G}^F)$, $A_0 := \mathbf{N}_A(\mathbf{L})$, $K := \mathbf{K}_0^F$, $K_0 = \mathbf{N}_{\mathbf{K}_0}(\mathbf{L})^F = \mathbf{N}_{\mathbf{K}_0}(\mathbf{L}_0)^F$, $G := \mathbf{G}^F$, $X := (\mathbf{G}^F \rtimes \mathrm{Aut}_{\mathbb{F}}(\mathbf{G}^F))_{\mathbf{K}}$, $\mathcal{S} := \mathcal{E}(\mathbf{K}_0^F, (\mathbf{L}_0, \mathcal{Y}_0))$, $\mathcal{S}_0 := \mathrm{Irr}(\mathbf{N}_{\mathbf{K}_0}(\mathbf{L}_0)^F \mid \mathcal{Y}_0)$, $V := (\mathbf{G}^F \rtimes \mathrm{Aut}_{\mathbb{F}}(\mathbf{G}^F))_{\mathbf{K},\mathcal{S}}$ and $U := (\mathbf{G}^F \rtimes \mathrm{Aut}_{\mathbb{F}}(\mathbf{G}^F))_{\mathbf{K},\mathbf{L},\mathcal{Y}_0}$. Observe that properties (ii) and (iii) of Proposition 9.1.5 are satisfied by Proposition 7.2.15 and Lemma 9.2.15. Consider the bijection between $\mathcal{S}$ and $\mathcal{S}_0$ given by Corollary 9.2.16 and Corollary 9.2.17. In order to apply Proposition 9.1.5 with $J := \mathbf{K}^F$ we need to show that $\mathbf{C}_X(Q) \leq X_0$ for every radical ℓ-subgroup Q of $J_0 = \mathbf{N}_{\mathbf{K}}(\mathbf{L})^F$. By Lemma 7.1.5 (ii), we know that $\mathbf{L} = \mathbf{C}_{\mathbf{G}}^\circ(E)$ with $E := \mathbf{Z}^\circ(\mathbf{L})_\ell^F$ and hence $E \leq \mathbf{O}_\ell(\mathbf{N}_{\mathbf{K}}(\mathbf{L})^F)$. Since Q is a radical ℓ-subgroup of J_0, it follows that $E \leq Q$ (see [Dad92, Proposition 1.4]) and therefore $\mathbf{C}_X(Q) \leq \mathbf{C}_X(E) \leq \mathbf{N}_X(E) = \mathbf{N}_X(\mathbf{L}) = X_0$. We can thus apply Proposition 9.1.5 together with Lemma 9.2.18 to obtain an $\mathrm{Aut}_{\mathbb{F}}(\mathbf{G}^F)_{\mathbf{K},\mathbf{L},\mathcal{Y}(\lambda_0)}$-equivariant bijection

$$\Psi_{(\mathbf{L},\lambda)}^{\mathbf{K}} : \mathcal{E}\left(\mathbf{K}^F, (\mathbf{L}, \mathcal{Y}(\lambda))\right) \to \mathrm{Irr}\left(\mathbf{N}_{\mathbf{K}}(\mathbf{L})^F \;\middle|\; \mathcal{Y}(\lambda)\right)$$

such that

$$\left(X_\vartheta, \mathbf{K}^F, \vartheta\right) \sim_{\mathbf{K}^F} \left(\mathbf{N}_{X_\vartheta}(\mathbf{L}), \mathbf{N}_{\mathbf{K}}(\mathbf{L})^F, \Psi^{\mathbf{K}}_{(\mathbf{L},\lambda)}(\vartheta)\right)$$

for every $\vartheta \in \mathcal{E}(\mathbf{K}^F, (\mathbf{L}, \mathcal{Y}(\lambda)))$. Moreover, $\Psi^{\mathbf{K}}_{(\mathbf{L},\lambda)}$ preserves the defect of characters by Remark 9.1.6. To conclude, notice that by a Frattini argument and using Clifford's theorem and Lemma 9.2.15 we have

$$\mathrm{Aut}_{\mathbb{F}}(\mathbf{G}^F)_{\mathbf{K},\mathbf{L},\mathcal{Y}(\lambda)} \leq \mathbf{L}^F \mathrm{Aut}_{\mathbb{F}}(\mathbf{G}^F)_{\mathbf{K},\mathbf{L},\mathcal{Y}(\lambda_0)}$$

and therefore the bijection $\Psi^{\mathbf{K}}_{(\mathbf{L},\lambda)}$ is $\mathrm{Aut}_{\mathbb{F}}(\mathbf{G}^F)_{\mathbf{K},\mathbf{L},\mathcal{Y}(\lambda)}$-equivariant. □

Now, applying Proposition 9.1.5, we show how to lift the bijection given by Corollary 9.2.19 to a bijection

$$\Omega^{\mathbf{K},H}_{(\mathbf{L},\lambda)} : \mathrm{Irr}\left(H \mid \mathcal{E}(\mathbf{K}^F, (\mathbf{L}, \mathcal{Y}(\lambda)))\right) \to \mathrm{Irr}\left(\mathbf{N}_H(\mathbf{L}) \mid \mathcal{Y}(\lambda)\right)$$

for every $\mathbf{K}^F \leq H \leq \mathbf{N}_{\mathbf{G}}(\mathbf{K})^F$. The proof of the next result is similar to the argument used in Corollary 9.2.19.

Proposition 9.2.20. *Consider the setup of Corollary 9.2.19 and let $\mathbf{K}^F \leq H \leq \mathbf{N}_{\mathbf{G}}(\mathbf{K})^F$. Then there exists a defect preserving $\mathrm{Aut}_{\mathbb{F}}(\mathbf{G}^F)_{H,\mathbf{K},(\mathbf{L},\lambda)}$-equivariant bijection*

$$\Omega^{\mathbf{K},H}_{(\mathbf{L},\lambda)} : \mathrm{Irr}\left(H \mid \mathcal{E}\left(\mathbf{K}^F, (\mathbf{L}, \mathcal{Y}(\lambda))\right)\right) \to \mathrm{Irr}\left(\mathbf{N}_H(\mathbf{L}) \mid \mathcal{Y}(\lambda)\right)$$

such that

$$(\mathbf{N}_X(H)_\chi, H, \chi) \sim_H (\mathbf{N}_X(H, \mathbf{L})_\chi, \mathbf{N}_H(\mathbf{L}), \psi)$$

for every $\chi \in \mathrm{Irr}(H \mid \mathcal{E}(\mathbf{K}^F, (\mathbf{L}, \mathcal{Y}(\lambda))))$ and where $X := (\mathbf{G}^F \rtimes \mathrm{Aut}_{\mathbb{F}}(\mathbf{G}^F))_{\mathbf{K}}$.

Proof. We apply Proposition 9.1.5 to the bijection given by Corollary 9.2.19. We consider $A := \mathbf{G}^F \rtimes \mathrm{Aut}_{\mathbb{F}}(\mathbf{G}^F)$, $G := \mathbf{G}^F$, $K := \mathbf{K}^F$, $A_0 := \mathbf{N}_A(\mathbf{L})$, $X := \mathbf{N}_A(\mathbf{K})$, $\mathcal{S} := \mathcal{E}(\mathbf{K}^F, (\mathbf{L}, \mathcal{Y}(\lambda)))$, $\mathcal{S}_0 := \mathrm{Irr}(\mathbf{N}_{\mathbf{K}}(\mathbf{L})^F \mid \mathcal{Y}(\lambda))$, $U := X_{0,\mathcal{Y}(\lambda)}$, $V := X_{\mathcal{S}}$ and $J := H$. By Proposition 7.2.15 and Lemma 9.2.15 we deduce that conditions (ii) and (iii) of Proposition 9.1.5 hold. Next, let Q be a radical ℓ-subgroup of $\mathbf{N}_H(\mathbf{L})$. Set $E := \mathbf{Z}^\circ(\mathbf{L})^F_\ell$ and notice that under our assumptions $\mathbf{L} = \mathbf{C}^\circ_{\mathbf{G}}(E)$ by Lemma 7.1.5. Then $E \leq \mathbf{O}_\ell(\mathbf{N}_H(\mathbf{L})) \leq Q$ because Q is radical and we conclude that $\mathbf{C}_X(Q) \leq \mathbf{C}_X(E) \leq \mathbf{N}_X(\mathbf{L}) = X_0$. We can therefore apply Proposition 9.1.5 to obtain an $\mathrm{Aut}_{\mathbb{F}}(\mathbf{G}^F)_{H,\mathbf{K},\mathbf{L},\mathcal{Y}(\lambda)}$-equivariant bijection $\Omega^{\mathbf{K},H}_{(\mathbf{L},\lambda)}$ as in the statement. Notice that $\Omega^{\mathbf{K},H}_{(\mathbf{L},\lambda)}$ is defect preserving by Remark 9.1.6 while it is $\mathrm{Aut}_{\mathbb{F}}(\mathbf{G}^F)_{H,\mathbf{K},(\mathbf{L},\lambda)}$-equivariant because

$$\mathrm{Aut}_{\mathbb{F}}(\mathbf{G}^F)_{H,\mathbf{K},(\mathbf{L},\lambda)} \leq \mathrm{Aut}_{\mathbb{F}}(\mathbf{G}^F)_{H,\mathbf{K},\mathbf{L},\mathcal{Y}(\lambda)}$$

by Lemma 9.2.15. □

We can finally prove the main result of this section. Theorem 9.2 will be an immediate consequence of the following result. Notice that when $\mathbf{G}$ is simple, simply connected and $\mathbf{G}^F/\mathbf{Z}(\mathbf{G}^F)$ is a nonabelian simple group, then $\mathrm{Aut}_{\mathbb{F}}(\mathbf{G}^F) = \mathrm{Aut}(\mathbf{G}^F)$ (see [GLS98, 1.15] and [CS13, 2.4]).

Theorem 9.2.21. *Assume that Hypothesis 9.2.11 holds for $(\mathbf{G}, F)$ and suppose that Condition 9.1 holds at the prime ℓ for every irreducible rational component of any e-split Levi subgroup of $(\mathbf{G}, F)$. Then Conjecture 9.1.1 holds at ℓ for $\mathbf{G}^F$ with respect to $\mathbf{G}^F \trianglelefteq \mathbf{G}^F \rtimes \mathrm{Aut}_{\mathbb{F}}(\mathbf{G}^F)$.*

Proof. We start by noticing that, under Hypothesis 9.2.11, in order to prove Conjecture 9.1.1 it's enough to check the requirements of Proposition 9.2.10 (see also Remark 9.2.12). Set $A := \mathbf{G}^F \rtimes \mathrm{Aut}_{\mathbb{F}}(\mathbf{G}^F)$ and fix $B \in \mathrm{Bl}(\mathbf{G}^F)$ with nontrivial defect. Let $\mathcal{T}_{1,+}$ be an A_B-transversal in the set

$$\mathcal{S}_{1,+} := \{(\mathbb{L}, \mathbf{M}, \mathcal{Y}(\mu)) \mid \mathbb{L} \in \mathcal{L}(\mathbf{G})_+, (\mathbf{M}, \mu) \in \mathcal{CP}_e(B) \text{ with } \mathbf{M} \leq \mathbf{L}\}$$

where $\mathbf{L}$ is the smallest term of $\mathbb{L}$. For $(\mathbb{L}, \mathbf{M}, \mathcal{Y}(\mu)) \in \mathcal{T}_{1,+}$, let $\mathcal{T}_{2,+}^{(\mathbb{L},\mathbf{M},\mathcal{Y}(\mu))}$ be an $A_{\mathbb{L},(\mathbf{M},\mu)}$-transversal in the set $\{\vartheta \in \mathrm{Irr}^d(\mathbf{G}_{\mathbb{L}}^F \mid \mathcal{E}(\mathbf{L}^F, (\mathbf{M}, \mathcal{Y}(\mu)))) \mid \mathrm{bl}(\vartheta)^{\mathbf{G}^F} = B\}$. Then

$$\mathcal{T}_+ := \left\{\overline{(\mathbb{L}, \mathbf{M}, \mathcal{Y}(\mu), \vartheta)} \;\middle|\; (\mathbb{L}, \mathbf{M}, \mathcal{Y}(\mu)) \in \mathcal{T}_{1,+}, \vartheta \in \mathcal{T}_{2,+}^{(\mathbb{L},\mathbf{M},\mathcal{Y}(\mu))}\right\}$$

is an A_B-transversal in $\mathcal{L}^d(B)_+/G$.

Fix $(\mathbb{L}, \mathbf{M}, \mathcal{Y}(\mu)) \in \mathcal{T}_{1,+}$ and let $\mathbf{L}$ be the smallest term of $\mathbb{L}$. If $\mathbf{L} = \mathbf{M}$, then define $\mathbb{K}$ to be the chain obtained by deleting $\mathbf{L}$ from $\mathbb{L}$ and denote by $\mathbf{K}$ the final term of $\mathbb{K}$. Since B has nontrivial defect, Proposition 7.4.1 implies that $\mathbf{M} < \mathbf{G}$ and hence the chain $\mathbb{K}$ is nonempty. On the other hand if $\mathbf{M} < \mathbf{L}$, then define $\mathbb{K}$ to be the chain obtained by adding $\mathbf{M}$ to $\mathbb{L}$. In this case the last term $\mathbf{K}$ of $\mathbb{K}$ coincides with $\mathbf{M}$. This construction yields an A_B-equivariant bijection

$$\Delta : \mathcal{S}_{1,+} \to \mathcal{S}_{1,-}$$

where

$$\mathcal{S}_{1,-} := \{(\mathbb{K}, \mathbf{N}, \mathcal{Y}(\nu)) \mid \mathbb{K} \in \mathcal{L}(\mathbf{G})_-, (\mathbf{N}, \nu) \in \mathcal{CP}_e(B) \text{ with } \mathbf{N} \leq \mathbf{K}\}$$

with $\mathbf{K}$ the smallest term of $\mathbb{K}$. In particular the image $\mathcal{T}_{1,-}$ of $\mathcal{T}_{1,+}$ under Δ is an A_B-transversal in $\mathcal{S}_{1,-}$. Moreover, notice that if $\Delta((\mathbb{L}, \mathbf{M}, \mathcal{Y}(\mu))) = (\mathbb{K}, \mathbf{M}, \mathcal{Y}(\mu))$, then we have

$$A_{\mathbb{L},(\mathbf{M},\mu)} = A_{\mathbb{K},(\mathbf{M},\mu)}. \tag{9.2.3}$$

Next, consider $(\mathbb{L}, \mathbf{M}, \mathcal{Y}(\mu)) \in \mathcal{T}_{1,+}$ and $(\mathbb{K}, \mathbf{M}, \mathcal{Y}(\mu)) := \Delta((\mathbb{L}, \mathbf{M}, \mathcal{Y}(\mu))) \in \mathcal{T}_{1,-}$ with $(\mathbf{M}, \mu) \in \mathcal{CP}_e(B)$. Assume first that $\mathbf{L} = \mathbf{M}$. By Proposition 9.2.20 applied with $H = \mathbf{G}_{\mathbb{K}}^F$, we obtain a bijection

$$\Omega_{(\mathbf{M},\mu)}^{\mathbf{K},\mathbf{G}_{\mathbb{K}}^F} : \mathrm{Irr}\left(\mathbf{G}_{\mathbb{K}}^F \;\middle|\; \mathcal{E}\left(\mathbf{K}^F, (\mathbf{M}, \mathcal{Y}(\mu))\right)\right) \to \mathrm{Irr}\left(\mathbf{N}_{\mathbf{G}_{\mathbb{K}}^F}(\mathbf{M}) \;\middle|\; \mathcal{Y}(\mu)\right).$$

Since $\mathbf{M} = \mathbf{L}$, notice that $\mathbf{N}_{\mathbf{G}_{\mathbb{K}}^F}(\mathbf{M}) = \mathbf{G}_{\mathbb{L}}^F$ and that $\mathrm{Irr}(\mathbf{G}_{\mathbb{L}}^F \mid \mathcal{E}(\mathbf{L}^F, (\mathbf{M}, \mathcal{Y}(\mu)))) = \mathrm{Irr}(\mathbf{N}_{\mathbf{G}_{\mathbb{K}}^F}(\mathbf{M}) \mid \mathcal{Y}(\mu))$. We define

$$\mathcal{T}_{2,-}^{(\mathbb{K},\mathbf{M},\mathcal{Y}(\mu))} := \left(\Omega_{(\mathbf{M},\mu)}^{\mathbf{K},\mathbf{G}_{\mathbb{K}}^F}\right)^{-1}\left(\mathcal{T}_{2,+}^{(\mathbb{L},\mathbf{M},\mathcal{Y}(\mu))}\right).$$

Similarly, if $\mathbf{M} < \mathbf{L}$, then Proposition 9.2.20 applied with $H = \mathbf{G}_{\mathbb{L}}^F$ yields a bijection

$$\Omega_{(\mathbf{M},\mu)}^{\mathbf{L},\mathbf{G}_{\mathbb{L}}^F} : \mathrm{Irr}\left(\mathbf{G}_{\mathbb{L}}^F \;\middle|\; \mathcal{E}\left(\mathbf{L}^F, (\mathbf{M}, \mathcal{Y}(\mu))\right)\right) \to \mathrm{Irr}\left(\mathbf{N}_{\mathbf{G}_{\mathbb{L}}^F}(\mathbf{M}) \;\middle|\; \mathcal{Y}(\mu)\right).$$

Noticing that $\mathbf{N}_{\mathbf{G}_{\mathbb{L}}^F}(\mathbf{M}) = \mathbf{G}_{\mathbb{K}}^F$ and recalling that the last term $\mathbf{K}$ of $\mathbb{K}$ coincide with $\mathbf{M}$, it follows that $\mathrm{Irr}(\mathbf{N}_{\mathbf{G}_{\mathbb{L}}^F}(\mathbf{M}) \mid \mathcal{Y}(\mu)) = \mathrm{Irr}(\mathbf{G}_{\mathbb{K}}^F \mid \mathcal{E}(\mathbf{K}^F, (\mathbf{M}, \mathcal{Y}(\mu))))$. In this case we define

$$\mathcal{T}_{2,-}^{(\mathbb{K},\mathbf{M},\mathcal{Y}(\mu))} := \Omega_{(\mathbf{M},\mu)}^{\mathbf{L},\mathbf{G}_{\mathbb{L}}^F}\left(\mathcal{T}_{2,+}^{(\mathbb{L},\mathbf{M},\mathcal{Y}(\mu))}\right).$$

Since $\mathcal{T}_{2,+}^{(\mathbb{L},\mathbf{M},\mathcal{Y}(\mu))}$ is an $A_{\mathbb{L},(\mathbf{M},\mu)}$-transversal in the set $\{\vartheta \in \mathrm{Irr}^d(\mathbf{G}_{\mathbb{L}}^F \mid \mathcal{E}(\mathbf{L}^F,(\mathbf{M},\mathcal{Y}(\mu)))) \mid \mathrm{bl}(\vartheta)^{\mathbf{G}^F}\}$, it follows by Proposition 9.2.20 and (9.2.3) that $\mathcal{T}_{2,-}^{(\mathbb{K},\mathbf{M},\mathcal{Y}(\mu))}$ is an $A_{\mathbb{K},(\mathbf{M},\mu)}$-transversal in the set $\{\chi \in \mathrm{Irr}^d(\mathbf{G}_{\mathbb{K}}^F \mid \mathcal{E}(\mathbf{K}^F,(\mathbf{M},\mathcal{Y}(\mu)))) \mid \mathrm{bl}(\chi)^{\mathbf{G}^F} = B\}$. In particular the set

$$\mathcal{T}_- := \left\{\overline{(\mathbb{K},\mathbf{M},\mathcal{Y}(\mu),\chi)} \;\middle|\; (\mathbb{K},\mathbf{M},\mathcal{Y}(\mu)) \in \mathcal{T}_{1,-}, \chi \in \mathcal{T}_{2,-}^{(\mathbb{K},\mathbf{M},\mathcal{Y}(\mu))}\right\}$$

is an A_B-transversal in $\mathcal{L}^d(B)_-/G$ in bijection with $\mathcal{T}_+$. By setting

$$\Lambda\left(\overline{(\mathbb{L},\mathbf{M},\mathcal{Y}(\mu),\vartheta)}^x\right) := \overline{(\mathbb{K},\mathbf{M},\mathcal{Y}(\mu),\chi)}^x,$$

for every $x \in A_B$ and every $(\mathbb{L},\mathbf{M},\mathcal{Y}(\mu),\vartheta) \in \mathcal{T}_+$ corresponding to $(\mathbb{K},\mathbf{M},\mathcal{Y}(\mu),\chi) \in \mathcal{T}_-$, we obtain an A_B-equivariant bijection

$$\Lambda : \mathcal{L}^d(B)_+/\mathbf{G}^F \to \mathcal{L}^d(B)_-/\mathbf{G}^F.$$

It remains to check the condition on character triples. Let $(\mathbb{L},\mathbf{M},\mathcal{Y}(\mu),\vartheta)$ and $(\mathbb{K},\mathbf{M},\mathcal{Y}(\mu),\chi)$ be as above. Without loss of generality we may assume that $\mathbf{M} < \mathbf{L}$ and so $\mathbf{K} = \mathbf{M}$. By the construction given in the previous paragraph and using Proposition 9.2.20 and Lemma 3.3.8 (ii) we know that

$$\left(A_{\mathbb{L},\vartheta},\mathbf{G}_{\mathbb{L}}^F,\vartheta\right) \sim_{\mathbf{G}_{\mathbb{L}}^F} \left(A_{\mathbb{K},\chi},\mathbf{G}_{\mathbb{K}}^F,\chi\right). \tag{9.2.4}$$

First we show that

$$\left(A_{\mathbb{L},\vartheta},\mathbf{G}_{\mathbb{L}}^F,\vartheta\right) \sim_{\mathbf{G}^F} \left(A_{\mathbb{K},\chi},\mathbf{G}_{\mathbb{K}}^F,\chi\right). \tag{9.2.5}$$

To do so, applying Lemma 3.3.10, it is enough to check that

$$\mathbf{C}_{\mathbf{G}^F A_{\mathbb{L},\vartheta}}(D) \leq A_{\mathbb{K},\chi} \tag{9.2.6}$$

for some defect group D of $\mathrm{bl}(\chi)$. By (9.2.4) we already know that $\mathbf{C}_{A_{\mathbb{L},\vartheta}}(D) \leq A_{\mathbb{K},\chi}$ and noticing that $A_{\mathbb{K},\chi} = A_{\mathbb{K},\vartheta}$ it remains to show that $\mathbf{C}_{\mathbf{G}^F A_{\mathbb{L},\vartheta}}(D) \leq A_{\mathbb{L},\vartheta}$. Write $\mathbb{L} = \{\mathbf{G} = \mathbf{L}_0 > \cdots > \mathbf{L}_n = \mathbf{L}\}$ and set $E_i := \mathbf{Z}^\circ(\mathbf{L}_i)_\ell^F$. By the argument used at the end of the proof of Proposition 9.2.20 and noticing that $\mathbf{G}_{\mathbb{K}}^F \leq \mathbf{G}_{\mathbb{L}}^F$, we have $E_i \leq D$ and hence $\mathbf{C}_{\mathbf{G}^F A_{\mathbb{L},\vartheta}}(D) \leq \mathbf{C}_{\mathbf{G}^F A_{\mathbb{L},\vartheta}}(E_i)$ for every $i = 0,\dots,n$. This implies that $\mathbf{C}_{\mathbf{G}^F A_{\mathbb{L},\vartheta}}(D) \leq (\mathbf{G}^F A_{\mathbb{L},\vartheta})_{\mathbb{L}} = A_{\mathbb{L},\vartheta}$ and so we obtain (9.2.6). We can now apply Lemma 3.3.10 to (9.2.4) in order to obtain (9.2.5). Moreover, by Lemma 3.3.3 we deduce that $Z := \mathrm{Ker}(\vartheta_{\mathbf{Z}(\mathbf{G}^F)}) = \mathrm{Ker}(\chi_{\mathbf{Z}(\mathbf{G}^F)})$ and, since under our assumption $\mathbf{Z}(\mathbf{G}^F)$ has order coprime to ℓ, it follows from Lemma 3.4.7 (see also Lemma 6.1.6) that

$$\left(A_{\mathbb{L},\vartheta}/Z,\mathbf{G}_{\mathbb{L}}^F/Z,\overline{\vartheta}\right) \sim_{\mathbf{G}^F/Z} \left(A_{\mathbb{K},\chi}/Z,\mathbf{G}_{\mathbb{K}}^F/Z,\overline{\chi}\right)$$

where $\overline{\vartheta}$ and $\overline{\chi}$ correspond to ϑ and χ via inflation of characters. This shows that all the conditions required by Proposition 9.2.10 are satisfied and hence Conjecture 9.1.1 holds for B with respect to $\mathbf{G}^F \rtimes \mathrm{Aut}_{\mathbb{F}}(\mathbf{G}^F)$. This completes the proof. □

9.2.3 Proving the nonblockwise Character Triple Conjecture

It should be clear to the reader that, with minor changes, all results that are deduced in the presence of N-block isomorphic character triples admit versions which hold when the starting character triples are (only) N-central isomorphic. For instance, with some natural adjustment, Proposition 9.1.5 clearly holds if we replace N-block isomorphic character triples with N-central isomorphic character triples. Similarly, a nonblockwise version of the Character Triple Conjecture can be introduced by requiring the involved character triples to be N-central isomorphic instead of N-block isomorphic (see Conjecture 3.5.5). As mentioned in Section 3.5.1 the nonblockwise Character Triple Conjecture could be used as an inductive condition for the nonblockwise version of Dade's Projective Conjecture (see Conjecture 2.5.4).

By the argument used in the previous section, we can show how to deduce the nonblockwise Character Triple Conjecture (see Conjecture 3.5.5) from the following weaker version of Condition 9.1.

Condition 9.2.22. Let $\mathbf{G}$, $F:\mathbf{G}\to\mathbf{G}$, ℓ and e be as in Notation 7.1.1 and consider an e-cuspidal pair $(\mathbf{L},\lambda)$ of $\mathbf{G}$. Then there exists a defect preserving $\mathrm{Aut}_{\mathbb{F}}(\mathbf{G}^F)_{(\mathbf{L},\lambda)}$-equivariant bijection

$$\Omega^{\mathbf{G}}_{(\mathbf{L},\lambda)}:\mathcal{E}\left(\mathbf{G}^F,(\mathbf{L},\lambda)\right)\to\mathrm{Irr}\left(\mathbf{N}_{\mathbf{G}}(\mathbf{L})^F\;\middle|\;\lambda\right)$$

such that

$$\left(X_\vartheta,\mathbf{G}^F,\vartheta\right)\sim^{c}_{\mathbf{G}^F}\left(\mathbf{N}_{X_\vartheta}(\mathbf{L}),\mathbf{N}_{\mathbf{G}^F}(\mathbf{L}),\Omega^{\mathbf{G}}_{(\mathbf{L},\lambda)}(\vartheta)\right)$$

for every $\vartheta\in\mathcal{E}\left(\mathbf{G}^F,(\mathbf{L},\lambda)\right)$ and where $X:=\mathbf{G}^F\rtimes\mathrm{Aut}_{\mathbb{F}}(\mathbf{G}^F)$.

We say that Condition 9.2.22 holds for $(\mathbf{G},F)$ at the prime ℓ if it holds for every e-cuspidal pair $(\mathbf{L},\lambda)$ where e is the order of q modulo ℓ. Then, proceeding in the exact same way as to prove Theorem 9.2, we obtain the following result.

Theorem 9.2.23. *Let $\mathbf{G}$, $F:\mathbf{G}\to\mathbf{G}$, ℓ and e be as in Notation 7.1.1 and assume that Hypothesis 9.2.11 is satisfied with respect to $(\mathbf{G},F)$. If Condition 9.2.22 holds at the prime ℓ for every irreducible rational component $(\mathbf{H},F)$ of every e-split Levi subgroup of $\mathbf{G}$, then Conjecture 3.5.5 holds at ℓ for $\mathbf{G}^F$ with respect to $\mathbf{G}^F\trianglelefteq\mathbf{G}^F\rtimes\mathrm{Aut}_{\mathbb{F}}(\mathbf{G}^F)$.*

10

Criteria for Condition 9.1 and Condition 9.2.22

In Theorem 9.2 we have shown that, in order to prove the Character Triple Conjecture and hence the inductive condition for Dade's Conjecture (see Definition 9.1.3), it is enough to check Condition 9.1. This significantly simplifies the verification of the inductive condition for Dade's Conjecture for quasisimple groups of Lie type. Similarly, Theorem 9.2.23 shows how Condition 9.2.22, a weak version of Condition 9.1, implies the nonblockwise Character Triple Conjecture (see Conjecture 3.5.5). In this chapter we prove criteria for Condition 9.1 and Condition 9.2.22 (see Theorem 10.1.3 and Theorem 10.1.8). In particular, we show that the main obstruction to the validation of Condition 9.1 and Condition 9.2.22 is given by some technical requirements related to the extendibility of characters of e-split Levi subgroups (see Definition 10.2.1 and [CS19, Definition 2.2]). This approach has already proved effective in dealing with the inductive condition for the McKay conjecture (see [Spä12], [CS13], [MS16], [CS17b], [CS17a], [CS19]) as well as with the inductive conditions for the Alperin–McKay and the Alperin Weight conjectures ([Spä13a], [Mal14], [SF14], [CS15], [KS16a], [KS16b], [BS20b]). It is therefore a natural and necessary step to extend this approach to Dade's Projective Conjecture and its inductive condition.

The requirements for the criteria that we will prove (see Assumption 10.1.1 and Assumption 10.1.4) are roughly divided into two parts: the first part requires the existence of certain bijections with good properties (see Assumption 10.1.1 (ii) and Assumption 10.1.4 (ii)), the second part requires some conditions on extendibility of characters of e-split Levi subgroups (see Assumption 10.1.1 (iii)-(iv) and Assumption 10.1.4 (iii)-(iv)). In Chapter 8, we have shown how to obtain the bijections required by these criteria by assuming the existence of certain extension maps (see Corollary 8.2). Therefore, it only remains to check the requirements on extendibility of characters. This remaining problem is part of an important ongoing project in representation theory of finite groups of Lie type.

Let $\mathbf{G}$, $F : \mathbf{G} \to \mathbf{G}$, ℓ and e be as in Notation 7.1.1 and consider a regular embedding $i : \mathbf{G} \to \widetilde{\mathbf{G}}$ compatible with F. Consider the subset $\mathrm{Aut}_{\mathbb{F}}(\mathbf{G}^F)$ of automorphisms of $\mathbf{G}^F$ defined in Section 6.1.5 and observe that, for every F-stable subgroup $\mathbf{H}$ of $\mathbf{G}$, the stabilizer of $\mathbf{H}$ in $\mathrm{Aut}_{\mathbb{F}}(\mathbf{G}^F)$ is well defined and is denoted by $\mathrm{Aut}_{\mathbb{F}}(\mathbf{G}^F)_{\mathbf{H}}$. Recall that if $\mathbf{L}$ is a Levi subgroup of $\mathbf{G}$, then we set $\widetilde{\mathbf{L}} := \mathbf{L}\mathbf{Z}(\widetilde{\mathbf{G}})$. Moreover, notice that if $\mathbf{G}$ is simple of simply connected type and $\mathbf{G}^F/\mathbf{Z}(\mathbf{G}^F)$ is a finite nonabelian simple group, then $\mathrm{Aut}_{\mathbb{F}}(\mathbf{G}^F) = \mathrm{Aut}(\mathbf{G}^F)$ (see [CS13, 2.4]).

As a consequence of the criteria proved in this chapter (Theorem 10.1.3 and Theorem 10.1.8) and applying Corollary 8.2, we can then show how to obtain Condition 9.2.22 by assuming some conditions on extendibility of characters (see Definition 10.2.1). These conditions should be compared with those introduced in [CS19, Definition 2.2].

Theorem 10.1. *Suppose that $\mathbf{G}$ is simple, simply connected not of type $\mathbf{E}_6$, $\mathbf{E}_7$ or $\mathbf{E}_8$ and consider $\ell \in \Gamma(\mathbf{G}, F)$ with $\ell \geq 5$. Let $\mathbf{L}$ be an e-split Levi subgroup of $\mathbf{G}$ and suppose that the following conditions hold:*

- *(i) maximal extendibility (see Definition 8.2.1) holds with respect to $\mathbf{G}^F \trianglelefteq \widetilde{\mathbf{G}}^F$ and to $\mathbf{N}_{\mathbf{G}}(\mathbf{L})^F \trianglelefteq \mathbf{N}_{\widetilde{\mathbf{G}}}(\mathbf{L})^F$;*
- *(ii) the requirement from Definition 10.2.1 holds for $\mathbf{L} \leq \mathbf{G}$;*
- *(iii) there exists an $(\mathrm{Aut}_{\mathbb{F}}(\mathbf{G}^F)_{\mathbf{L}} \ltimes \mathrm{Irr}(\widetilde{\mathbf{G}}^F/\mathbf{G}^F))$-equivariant extension map for $\mathrm{Cusp}_e(\widetilde{\mathbf{L}}^F)$ with respect to $\widetilde{\mathbf{L}}^F \trianglelefteq \mathbf{N}_{\widetilde{\mathbf{G}}}(\mathbf{L})^F$;*

then Condition 9.2.22 holds for every e-cuspidal pair $(\mathbf{L}, \lambda)$ of $\mathbf{G}^F$.

A similar result can be obtained for Condition 9.1. For this, we need to add some additional block theoretic requirements (see [CS15, Theorem 4.1 (v)] and [BS20b, Theorem 2.4 (v)]). These additional restrictions can be shown to hold for unipotent blocks and blocks with maximal defect and in general for every group not of type $\mathbf{A}$, $\mathbf{D}$ or $\mathbf{E}_6$ (see Remark 10.1.5).

Theorem 10.2. *Suppose that $\mathbf{G}$ is simple, simply connected not of type $\mathbf{E}_6$, $\mathbf{E}_7$ or $\mathbf{E}_8$ and consider $\ell \in \Gamma(\mathbf{G}, F)$ with $\ell \geq 5$. Let $\mathbf{L}$ be an e-split Levi subgroup of $\mathbf{G}$, B an ℓ-block of $\mathbf{G}^F$ and suppose that the following conditions hold:*

- *(i) maximal extendibility holds with respect to $\mathbf{G}^F \trianglelefteq \widetilde{\mathbf{G}}^F$ and to $\mathbf{N}_{\mathbf{G}}(\mathbf{L})^F \trianglelefteq \mathbf{N}_{\widetilde{\mathbf{G}}}(\mathbf{L})^F$;*
- *(ii) the requirement from Definition 10.2.1 holds for $\mathbf{L} \leq \mathbf{G}$;*
- *(iii) there exists an $(\mathrm{Aut}_{\mathbb{F}}(\mathbf{G}^F)_{\mathbf{L}} \ltimes \mathrm{Irr}(\widetilde{\mathbf{G}}^F/\mathbf{G}^F))$-equivariant extension map for $\mathrm{Cusp}_e(\widetilde{\mathbf{L}}^F)$ with respect to $\widetilde{\mathbf{L}}^F \trianglelefteq \mathbf{N}_{\widetilde{\mathbf{G}}}(\mathbf{L})^F$;*
- *(iv) the ℓ-block B satisfies either*
  - *(a) $\mathrm{Out}(\mathbf{G}^F)_{\mathcal{B}}$ is abelian, where $\mathcal{B}$ is the $\widetilde{\mathbf{G}}^F$-orbit of B, or*
  - *(b) for every subgroup $\mathbf{G}^F \leq H \leq \widetilde{\mathbf{G}}^F$, we have that every block C of H covering B is $\widetilde{\mathbf{G}}^F$-invariant.*

Then Condition 9.1 holds for every e-cuspidal pair $(\mathbf{L}, \lambda) \in \mathcal{CP}_e(B)$ such that $\mathcal{E}(\mathbf{G}^F, (\mathbf{L}, \lambda)) = \mathcal{E}(\mathbf{G}^F, B, [s])$, where $s \in \mathbf{L}^{*F^*}_{\mathrm{ss}}$ and $\lambda \in \mathcal{E}(\mathbf{L}^F, [s])$ (see the discussion following Theorem 7.3.3).

As a corollary, by using [BS20b, Theorem 1.2], we obtain Condition 9.1 and Condition 9.2.22 for some cases in type $\mathbf{A}$.

Corollary 10.3. *Let ℓ be a prime, q a prime power and $\epsilon \in \{+1,-1\}$ such that $\ell \nmid 3q(q-\epsilon)$. Set $\mathbf{G} := \mathrm{SL}_n(\overline{\mathbb{F}_q})$, $G := \mathrm{SL}_n(\epsilon \cdot q)$ and assume that G is the universal covering group of $\mathrm{PSL}_n(\epsilon \cdot q)$. If B is a block of G such that, either*

(i) $\mathrm{Out}(G)_{\mathcal{B}}$ *is abelian, where $\mathcal{B}$ is the $\mathrm{GL}_n(\epsilon \cdot q)$-orbit of B; or*

(ii) B *is unipotent; or*

(iii) B *has maximal defect.*

Then Condition 9.2.22 holds for every e-cuspidal pair $(\mathbf{L}, \lambda) \in \mathcal{CP}(B)$ and Condition 9.1 holds for every (e, ℓ')-cuspidal pair $(\mathbf{L}, \lambda) \in \mathcal{CP}_e(B)$.

Similarly, by using the main result of [Bro], we obtain Condition 9.1 and Condition 9.2.22 for some cases in type $\mathbf{C}$.

Corollary 10.4. *Let ℓ be a prime and q a prime power such that $\ell \nmid 6q$. Set $\mathbf{G} := \mathrm{Sp}_{2n}(\overline{\mathbb{F}_q})$, $G := \mathrm{Sp}_{2n}(q)$ and assume that G is the universal covering group of $\mathrm{PSp}_{2n}(q)$, e.g. $n \geq 2$ and q odd. Then Condition 9.2.22 holds for every e-cuspidal pair of $\mathbf{G}$ and Condition 9.1 holds for every (e, ℓ')-cuspidal pair $\mathbf{G}$.*

10.1 The criteria

It will be clear to the experts that the bijections involved in Condition 9.1 and Condition 9.2.22 are closely related to the bijections used to prove the inductive conditions for the McKay Conjecture and the Alperin–McKay Conjecture for simple groups of Lie type. These bijections were introduced, under certain assumptions, by Malle in [Mal07] and [Mal14] and later strenghtened by Cabanes–Späth and Brough–Späth in order to obtain the inductive conditions for the above mentioned conjectures for some cases for groups of Lie type $\mathbf{A}$ (see [CS17a] and [BS20b]). The main idea that allows us to tackle inductive conditions for groups of Lie type comes from a criterion introduced in [Spä12, Theorem 2.12] and in particular in [Spä12, Lemma 2.11] which allows the construction of projective representations.

In this section, we are going to generalize this approach and obtain similar criteria for Condition 9.1 and Condition 9.2.22. Due to some obstructions arising in Clifford theory for blocks, we can only prove a criterion for the stronger Condition 9.1 by adding some additional restrictions on the type of blocks considered which are analogous to the ones introduced in [CS15, Theorem 4.1], [BS20b, Theorem 2.4] and [BS20a, Theorem 4.5]. The criteria proved in this chapter should be compared to [Spä12, Theorem 2.12], [CS15, Theorem 4.1], [BS20a, Theorem 2.4], [Ruh21a, Theorem 2.1] and [Ruh21b, Theorem 9.2].

Throughout this section we will be assuming Hypothesis 9.2.11 (see also Remark 9.2.12).

10.1.1 The criterion for Condition 9.2.22

We start by dealing with Condition 9.2.22. The results obtained in this section will then be used in the next one to prove the criterion for Condition 9.1 under additional restrictions.

As recalled in Section 6.1.4, the group $\mathcal{K}$ from Lemma 6.1.3 (v) (see also (6.1.2)) is isomorphic to the group of linear characters of $\widetilde{\mathbf{G}}^F$ via

$$\begin{aligned}\mathcal{K} &\to \mathrm{Irr}\left(\widetilde{\mathbf{G}}^F/\mathbf{G}^F\right)\\ z &\mapsto \widehat{z}_{\widetilde{\mathbf{G}}}.\end{aligned}$$

Since for every e-split Levi subgroups $\mathbf{L}$ of $\mathbf{G}$ we have $\mathbf{N}_{\widetilde{\mathbf{G}}}(\mathbf{L})^F/\mathbf{N}_{\mathbf{G}}(\mathbf{L})^F \simeq \widetilde{\mathbf{G}}^F/\mathbf{G}^F$, restriction of characters then gives an isomorphism

$$\begin{aligned}\mathcal{K} &\to \mathrm{Irr}\left(\mathbf{N}_{\widetilde{\mathbf{G}}}(\mathbf{L})^F/\mathbf{N}_{\mathbf{G}}(\mathbf{L})^F\right)\\ z &\mapsto \widehat{z}_{\mathbf{N}_{\widetilde{\mathbf{G}}}(\mathbf{L})}.\end{aligned}$$

and hence the group $\mathcal{K}$ acts on the sets of irreducible characters of $\widetilde{\mathbf{G}}^F$ and $\mathbf{N}_{\widetilde{\mathbf{G}}}(\mathbf{L})^F$.

We now introduce the requirements for our first criterion.

Assumption 10.1.1. Let $(\mathbf{L},\lambda)$ be an e-cuspidal pair of $\mathbf{G}$ and consider

$$\mathcal{G} := \mathcal{E}\left(\mathbf{G}^F,(\mathbf{L},\lambda)\right) \quad \text{and} \quad \mathcal{N} := \mathrm{Irr}\left(\mathbf{N}_{\mathbf{G}}(\mathbf{L})^F \mid \lambda\right)$$

and

$$\widetilde{\mathcal{G}} := \mathrm{Irr}\left(\widetilde{\mathbf{G}}^F \mid \mathcal{G}\right) \quad \text{and} \quad \widetilde{\mathcal{N}} := \mathrm{Irr}\left(\mathbf{N}_{\widetilde{\mathbf{G}}}(\mathbf{L})^F \mid \mathcal{N}\right).$$

Assume that:

- (i)
  - (a) There is a semidirect decomposition $\widetilde{\mathbf{G}}^F \rtimes \mathcal{A}$, with $\mathcal{A}$ a finite abelian group, such that $\mathbf{C}_{\widetilde{\mathbf{G}}^F\mathcal{A}}(\mathbf{G}^F) = \mathbf{Z}(\widetilde{\mathbf{G}}^F)$ and $\widetilde{\mathbf{G}}^F\mathcal{A}/\mathbf{Z}(\widetilde{\mathbf{G}}^F) \simeq \mathrm{Aut}(\mathbf{G}^F)$ via the natural map;
  - (b) Maximal extendibility holds with respect to $\mathbf{G}^F \trianglelefteq \widetilde{\mathbf{G}}^F$;
  - (c) Maximal extendibility holds with respect to $\mathbf{N}_{\mathbf{G}}(\mathbf{L})^F \trianglelefteq \mathbf{N}_{\widetilde{\mathbf{G}}}(\mathbf{L})^F$.
- (ii) For $A := (\widetilde{\mathbf{G}}^F\mathcal{A})_{(\mathbf{L},\lambda)}$ there exists a defect preserving $(A \ltimes \mathcal{K})$-equivariant bijection

  $$\widetilde{\Omega}^{\mathbf{G}}_{(\mathbf{L},\lambda)} : \widetilde{\mathcal{G}} \to \widetilde{\mathcal{N}}$$

  such that $\mathrm{Irr}\left(\widetilde{\chi}_{\mathbf{Z}(\widetilde{\mathbf{G}}^F)}\right) = \mathrm{Irr}\left(\widetilde{\Omega}^{\mathbf{G}}_{(\mathbf{L},\lambda)}(\widetilde{\chi})_{\mathbf{Z}(\widetilde{\mathbf{G}}^F)}\right)$ for every $\widetilde{\chi} \in \widetilde{\mathcal{G}}$.
- (iii) For every $\widetilde{\chi} \in \widetilde{\mathcal{G}}$ there exists $\chi \in \mathcal{G} \cap \mathrm{Irr}\left(\widetilde{\chi}_{\mathbf{G}^F}\right)$ such that:
  - (a) $\left(\widetilde{\mathbf{G}}^F\mathcal{A}\right)_\chi = \widetilde{\mathbf{G}}^F_\chi\mathcal{A}_\chi$;
  - (b) χ extends to $\chi' \in \mathrm{Irr}\left(\mathbf{G}^F\mathcal{A}_\chi\right)$.
- (iv) For every $\widetilde{\psi} \in \widetilde{\mathcal{N}}$ there exists $\psi \in \mathcal{N} \cap \mathrm{Irr}\left(\widetilde{\psi}_{\mathbf{N}_{\mathbf{G}}(\mathbf{L})^F}\right)$ such that:

(a) $\left(\widetilde{\mathbf{G}}^F\mathcal{A}\right)_{\mathbf{L},\psi} = \mathbf{N}_{\widetilde{\mathbf{G}}}(\mathbf{L})^F_\psi\left(\mathbf{G}^F\mathcal{A}\right)_{\mathbf{L},\psi}$;

(b) ψ extends to $\psi' \in \mathrm{Irr}\left(\left(\mathbf{G}^F \rtimes \mathcal{A}\right)_{\mathbf{L},\psi}\right)$.

Our aim is to show that Assumption 10.1.1 implies Condition 9.2.22. Before giving a proof of this result, we show that Assumption 10.1.1 (iii.a) and Assumption 10.1.1 (iv.a) are equivalent in the presence of an equivariant bijection $\Omega^{\mathbf{G}}_{(\mathbf{L},\lambda)} : \mathcal{G} \to \mathcal{N}$.

Lemma 10.1.2. *Assume Hypothesis 9.2.11. Let $(\mathbf{L},\lambda)$ be an e-cuspidal pair of $\mathbf{G}$ and suppose that there exists a $(\widetilde{\mathbf{G}}^F\mathcal{A})_{(\mathbf{L},\lambda)}$-equivariant bijection*

$$\Omega^{\mathbf{G}}_{(\mathbf{L},\lambda)} : \mathcal{E}(\mathbf{G}^F,(\mathbf{L},\lambda)) \to \mathrm{Irr}\left(\mathbf{N}_{\mathbf{G}}(\mathbf{L})^F \mid \lambda\right).$$

If $\chi \in \mathcal{E}(\mathbf{G}^F,(\mathbf{L},\lambda))$ and $\psi := \Omega^{\mathbf{G}}_{(\mathbf{L},\lambda)}(\chi)$, then

$$\left(\widetilde{\mathbf{G}}^F\mathcal{A}\right)_\chi = \widetilde{\mathbf{G}}^F_\chi\mathcal{A}_\chi \tag{10.1.1}$$

if and only if

$$\left(\widetilde{\mathbf{G}}^F\mathcal{A}\right)_{\mathbf{L},\psi} = \mathbf{N}_{\widetilde{\mathbf{G}}}(\mathbf{L})^F_\psi\left(\mathbf{G}^F\mathcal{A}\right)_{\mathbf{L},\psi}. \tag{10.1.2}$$

Proof. As the two implications can be shown by similar arguments we will only show that (10.1.1) implies (10.1.2). To start, consider the subgroups

$$T := \mathbf{N}_{\mathbf{G}}(\mathbf{L})^F\left(\widetilde{\mathbf{G}}^F_{(\mathbf{L},\lambda),\chi}\cdot(\mathbf{G}^F\mathcal{A})_{(\mathbf{L},\lambda),\chi}\right) = \mathbf{N}_{\mathbf{G}}(\mathbf{L})^F\left(\widetilde{\mathbf{G}}^F_{(\mathbf{L},\lambda),\psi}\cdot(\mathbf{G}^F\mathcal{A})_{(\mathbf{L},\lambda),\psi}\right)$$

and

$$V := \mathbf{N}_{\mathbf{G}}(\mathbf{L})^F(\widetilde{\mathbf{G}}^F\mathcal{A})_{(\mathbf{L},\lambda),\chi} = \mathbf{N}_{\mathbf{G}}(\mathbf{L})^F(\widetilde{\mathbf{G}}^F\mathcal{A})_{(\mathbf{L},\lambda),\psi},$$

where the equalities follow since $\Omega^{\mathbf{G}}_{(\mathbf{L},\lambda)}$ is equivariant by assumption.

Define $U(\chi) := (\widetilde{\mathbf{G}}^F\mathcal{A})_{\mathbf{L},\chi}$ and $U(\psi) := (\widetilde{\mathbf{G}}^F\mathcal{A})_{\mathbf{L},\psi}$. We claim that $U(\chi) = U(\psi) =: U$. To prove this fact, notice that it is enough to show that $U(\chi)$ and $U(\psi)$ are contained in V, in fact this would imply $U(\chi) = U(\chi) \cap V = (\widetilde{\mathbf{G}}^F\mathcal{A})_{\mathbf{L}} \cap V = U(\psi) \cap V = U(\psi)$. If $x \in U(\chi)$, then $\chi \in \mathcal{E}(\mathbf{G}^F,(\mathbf{L},\lambda)) \cap \mathcal{E}(\mathbf{G}^F,(\mathbf{L},\lambda)^x)$ and, by Proposition 7.2.15, there exists $y \in \mathbf{G}^F$ such that $(\mathbf{L},\lambda) = (\mathbf{L},\lambda)^{xy}$. Notice that $y \in \mathbf{N}_{\mathbf{G}}(\mathbf{L})^F$ and hence $x \in V$. On the other hand, if $x \in U(\psi)$, then ψ lies over λ^x and by Clifford's theorem $\lambda^{xy} = \lambda$, for some $y \in \mathbf{N}_{\mathbf{G}}(\mathbf{L})^F$. Also in this case $x \in V$. Now $U(\chi) = U(\psi)$ and we denote this group by U.

Next, we claim that $T = U$. If this is true, then we deduce that $T \leq \mathbf{N}_{\widetilde{\mathbf{G}}}(\mathbf{L})^F_\psi(\mathbf{G}^F\mathcal{A})_{\mathbf{L},\psi} \leq U = T$ and therefore (10.1.2) holds. First, observe that $T \leq U$. As $T \cap \mathbf{G}^F = \mathbf{N}_{\mathbf{G}}(\mathbf{L})^F = U \cap \mathbf{G}^F$ and $T \leq U \leq (\widetilde{\mathbf{G}}^F\mathcal{A})_\chi$, it is enough to show that $T\mathbf{G}^F = (\widetilde{\mathbf{G}}^F\mathcal{A})_\chi$. First, repeating the same argument as before, a Frattini argument shows that

$$\left(\widetilde{\mathbf{G}}^F\mathcal{A}\right)_\chi = \mathbf{G}^F\left(\widetilde{\mathbf{G}}^F\mathcal{A}\right)_{(\mathbf{L},\lambda),\chi} \tag{10.1.3}$$

and

$$\widetilde{\mathbf{G}}^F_\chi = \mathbf{G}^F\widetilde{\mathbf{G}}^F_{(\mathbf{L},\lambda),\chi}. \tag{10.1.4}$$

Then using the hypothesis we finally obtain

$$\begin{aligned}\left(\widetilde{\mathbf{G}}^F\mathcal{A}\right)_\chi &\overset{(10.1.3)}{=} \mathbf{G}^F\left(\widetilde{\mathbf{G}}^F\mathcal{A}\right)_{(\mathbf{L},\lambda),\chi}\\ &\overset{(10.1.1)}{=} \mathbf{G}^F\left(\widetilde{\mathbf{G}}^F_\chi\left(\mathbf{G}^F\mathcal{A}\right)_\chi\right)_{(\mathbf{L},\lambda)}\\ &\overset{(10.1.4)}{=} \mathbf{G}^F\left(\mathbf{G}^F\widetilde{\mathbf{G}}^F_{(\mathbf{L},\lambda),\chi}\left(\mathbf{G}^F\mathcal{A}\right)_\chi\right)_{(\mathbf{L},\lambda)}\\ &= \mathbf{G}^F\widetilde{\mathbf{G}}^F_{(\mathbf{L},\lambda),\chi}\left(\mathbf{G}^F\mathcal{A}\right)_{(\mathbf{L},\lambda),\chi}\\ &= \mathbf{G}^F T.\end{aligned}$$

This concludes the proof. □

We are now ready to prove the criterion for Condition 9.2.22. It will be clear from the proof of this result that, by using Lemma 10.1.2, only one amongst Assumption 10.1.1 (iii.a) and Assumption 10.1.1 (iv.a) is actually necessary. In fact, the equivariant map required in Lemma 10.1.2 is constructed in the following proof independently form the choices of characters satisfying Assumption 10.1.1 (iii.a) and Assumption 10.1.1 (iv.a).

Theorem 10.1.3. *Assume Hypothesis 9.2.11 and Assumption 10.1.1 with respect to an e-cuspidal pair $(\mathbf{L},\lambda)$ of $\mathbf{G}$. Then Condition 9.2.22 holds for $(\mathbf{L},\lambda)$ and $\mathbf{G}$.*

Proof. We start by fixing an $(A\ltimes\mathcal{K})$-transversal $\widetilde{\mathbb{T}}_{\mathrm{glo}}$ in $\widetilde{\mathcal{G}}$. As $\widetilde{\Omega}^{\mathbf{G}}_{(\mathbf{L},\lambda)}$ is $(A\ltimes\mathcal{K})$-equivariant, we deduce that the set $\widetilde{\mathbb{T}}_{\mathrm{loc}} := \{\widetilde{\Omega}^{\mathbf{G}}_{(\mathbf{L},\lambda)}(\widetilde{\chi}) \mid \widetilde{\chi}\in\widetilde{\mathbb{T}}_{\mathrm{glo}}\}$ is an $(A\ltimes\mathcal{K})$-transversal in $\widetilde{\mathcal{N}}$. For every $\widetilde{\chi}\in\widetilde{\mathbb{T}}_{\mathrm{glo}}$, we chose a character $\chi\in\mathcal{G}\cap\mathrm{Irr}(\widetilde{\chi}_{\mathbf{G}^F})$ satisfying Assumption 10.1.1 (iii). Denote by $\mathbb{T}_{\mathrm{glo}}$ the set of such characters χ, where $\widetilde{\chi}$ runs over $\widetilde{\mathbb{T}}_{\mathrm{glo}}$. Similarly, for every $\widetilde{\psi}\in\widetilde{\mathbb{T}}_{\mathrm{loc}}$, fix a character $\psi\in\mathcal{N}\cap\mathrm{Irr}(\widetilde{\psi}_{\mathbf{N}_{\mathbf{G}}(\mathbf{L})^F})$ satisfying Assumption 10.1.1 (iv) and denote by $\mathbb{T}_{\mathrm{loc}}$ the set of such characters ψ. Observe that $\mathbb{T}_{\mathrm{glo}}$ (resp. $\mathbb{T}_{\mathrm{loc}}$) is an A-transversal in $\mathcal{G}$ (resp. in $\mathcal{N}$). In fact, if $\chi\in\mathcal{G}$, then let $\widetilde{\chi}\in\widetilde{\mathcal{G}}$ lying over χ. Then there exists $\widetilde{\chi}_0\in\widetilde{\mathbb{T}}_{\mathrm{glo}}$ such that $\widetilde{\chi}_0=\widetilde{\chi}^{xz}$, for some $x\in A$ and $z\in\mathcal{K}$. Let $\chi_0\in\mathbb{T}_{\mathrm{glo}}$ correspond to $\widetilde{\chi}_0$ and observe that χ^x and χ_0 lie under $\widetilde{\chi}_0$. By Clifford's theorem there exists $y\in\widetilde{\mathbf{G}}^F$ such that $\chi^{xy}=\chi_0$. Now $\chi\in\mathcal{E}(\mathbf{G}^F,(\mathbf{L},\lambda))\cap\mathcal{E}(\mathbf{G}^F,(\mathbf{L},\lambda)^{xy})$ and, by Proposition 7.2.15, there exists $u\in\mathbf{G}^F$ such that $(\mathbf{L},\lambda)=(\mathbf{L},\lambda)^{xyu}$. Set $v:=xyu$ and notice that $\chi_0=\chi^v$ and that $v\in A$. Moreover, if $\chi_0=\chi^x$ for some $\chi,\chi_0\in\mathbb{T}_{\mathrm{glo}}$ and $x\in A$, then $\widetilde{\chi}_0$ and $\widetilde{\chi}^x$ lie over χ, where $\widetilde{\chi}_0$ (resp. $\widetilde{\chi}$) is the element of $\widetilde{\mathbb{T}}_{\mathrm{glo}}$ corresponding to χ_0 (resp. χ). Therefore $\widetilde{\chi}_0=\widetilde{\chi}^{xz}$, for some $z\in\mathcal{K}$, which implies $\widetilde{\chi}_0=\widetilde{\chi}$ and so $\chi_0=\chi$. This shows that $\mathbb{T}_{\mathrm{glo}}$ is an A-transversal in $\mathcal{G}$. This argument also shows that, for every $\widetilde{\chi}\in\widetilde{\mathbb{T}}_{\mathrm{glo}}$, there exists a unique character $\chi\in\mathbb{T}_{\mathrm{glo}}\cap\mathrm{Irr}(\widetilde{\chi}_{\mathbf{G}^F})$ and that, for every $\chi\in\mathbb{T}_{\mathrm{glo}}$, there exists a unique $\widetilde{\chi}\in\widetilde{\mathbb{T}}_{\mathrm{glo}}\cap\mathrm{Irr}(\chi^{\widetilde{\mathbf{G}}^F})$. A similar argument shows that $\mathbb{T}_{\mathrm{loc}}$ is a transversal in $\mathcal{N}$ and that the correspondence between ψ and $\widetilde{\psi}$ defines a bijection between $\mathbb{T}_{\mathrm{loc}}$ and $\widetilde{\mathbb{T}}_{\mathrm{loc}}$.

Now, setting

$$\Omega^{\mathbf{G}}_{(\mathbf{L},\lambda)}(\chi^x) := \psi^x$$

for every $x\in A$ and $\chi\in\mathbb{T}_{\mathrm{glo}}$, where ψ is the unique character in $\mathbb{T}_{\mathrm{loc}}$ lying below $\widetilde{\psi}:=\widetilde{\Omega}^{\mathbf{G}}_{(\mathbf{L},\lambda)}(\widetilde{\chi})$ and $\widetilde{\chi}$ is the unique character in $\widetilde{\mathbb{T}}_{\mathrm{glo}}$ lying over χ, defines an A-equivariant bijection between $\mathcal{G}$ and $\mathcal{N}$. By Assumption 10.1.1 (i.a) this means that $\Omega^{\mathbf{G}}_{(\mathbf{L},\lambda)}$ is $\mathrm{Aut}_{\mathbb{F}}(\mathbf{G}^F)_{(\mathbf{L},\lambda)}$-equivariant.

To show that $\Omega^{\mathbf{G}}_{(\mathbf{L},\lambda)}$ preserves the defect, we use Assumption 10.1.1 (i.b) and (i.c). Clearly it's enough to show that $d(\chi) = d(\psi)$, for $\chi \in \mathbb{T}_{\mathrm{glo}}$ and $\psi := \Omega^{\mathbf{G}}_{(\mathbf{L},\lambda)}(\chi) \in \mathbb{T}_{\mathrm{loc}}$. Let $\widetilde{\chi}$ (resp. $\widetilde{\psi}$) be the unique element of $\widetilde{\mathbb{T}}_{\mathrm{glo}}$ (resp. $\widetilde{\mathbb{T}}_{\mathrm{loc}}$) lying over χ (resp. ψ). Then $\widetilde{\Omega}^{\mathbf{G}}_{(\mathbf{L},\lambda)}(\widetilde{\chi}) = \widetilde{\psi}$ and $d(\widetilde{\chi}) = d(\widetilde{\psi})$ by Assumption 10.1.1 (ii). Moreover, since $\widetilde{\mathbf{G}}^F/\mathbf{G}^F \simeq \mathbf{N}_{\widetilde{\mathbf{G}}}(\mathbf{L})^F/\mathbf{N}_{\mathbf{G}}(\mathbf{L})^F$ is abelian and using Assumption 10.1.1 (i.b) and (i.c), we deduce that the Clifford correspondent $\widehat{\chi} \in \mathrm{Irr}(\widetilde{\mathbf{G}}^F_\chi)$ of $\widetilde{\chi}$ over χ is an extension of χ and, similarly, that the Clifford correspondent $\widehat{\psi} \in \mathrm{Irr}(\mathbf{N}_{\widetilde{\mathbf{G}}}(\mathbf{L})^F_\psi)$ of $\widetilde{\psi}$ over ψ is an extension of ψ. As a consequence

$$\ell^{d(\chi)} = \ell^{d(\widehat{\chi})} \cdot \left|\widetilde{\mathbf{G}}^F_\chi : \mathbf{G}^F\right|_\ell$$

and

$$\ell^{d(\psi)} = \ell^{d(\widehat{\psi})} \cdot \left|\mathbf{N}_{\widetilde{\mathbf{G}}}(\mathbf{L})^F_\psi : \mathbf{N}_{\mathbf{G}}(\mathbf{L})^F\right|_\ell .$$

Therefore, as the defect is preserved by induction of character, we obtain $d(\widehat{\chi}) = d(\widetilde{\chi}) = d(\widetilde{\psi}) = d(\widehat{\psi})$ and it remains to show that $|\widetilde{\mathbf{G}}^F_\chi : \mathbf{G}^F|_\ell = |\mathbf{N}_{\widetilde{\mathbf{G}}}(\mathbf{L})^F_\psi : \mathbf{N}_{\mathbf{G}}(\mathbf{L})^F|_\ell$. This follows from the proof of Lemma 10.1.2: in fact there it is shown that $\mathbf{N}_{\widetilde{\mathbf{G}}}(\mathbf{L})^F_\psi = \mathbf{N}_{\widetilde{\mathbf{G}}}(\mathbf{L})^F_\chi$ and therefore $\widetilde{\mathbf{G}}^F_\chi/\mathbf{G}^F \simeq \mathbf{N}_{\widetilde{\mathbf{G}}}(\mathbf{L})^F_\chi/\mathbf{N}_{\mathbf{G}}(\mathbf{L})^F = \mathbf{N}_{\widetilde{\mathbf{G}}}(\mathbf{L})^F_\psi/\mathbf{N}_{\mathbf{G}}(\mathbf{L})^F$.

Next, we prove the condition on character triples. Applying a simplified version of [Spä17, Theorem 5.3] adapted to N-central isomorphic character triples (this immediately follows by part of the proof of [Spä17, Theorem 5.3]), it is enough to show that

$$\left((\widetilde{\mathbf{G}}^F\mathcal{A})_\chi, \mathbf{G}^F, \chi\right) \sim^c_{\mathbf{G}^F} \left((\widetilde{\mathbf{G}}^F\mathcal{A})_{\mathbf{L},\psi}, \mathbf{N}_{\mathbf{G}}(\mathbf{L})^F, \Omega^{\mathbf{G}}_{(\mathbf{L},\lambda)}(\chi)\right). \tag{10.1.5}$$

Moreover, as the equivalence relation $\sim^c_{\mathbf{G}^F}$ is compatible with conjugation, it's enough to prove this condition for a fixed $\chi \in \mathbb{T}_{\mathrm{glo}}$ and $\psi := \Omega^{\mathbf{G}}_{(\mathbf{L},\lambda)}(\chi) \in \mathbb{T}_{\mathrm{loc}}$.

First of all, notice that the required group theoretical properties are satisfied by the proof of Lemma 10.1.2. In fact, there we have shown that $(\widetilde{\mathbf{G}}^F\mathcal{A})_{\mathbf{L},\chi} = (\widetilde{\mathbf{G}}^F\mathcal{A})_{\mathbf{L},\psi}$ and that $(\widetilde{\mathbf{G}}^F\mathcal{A})_\chi = \mathbf{G}^F(\widetilde{\mathbf{G}}^F\mathcal{A})_{\mathbf{L},\chi}$, while

$$\mathbf{C}_{(\widetilde{\mathbf{G}}^F\mathcal{A})_\chi}(\mathbf{G}^F) \leq \mathbf{C}_{(\widetilde{\mathbf{G}}^F\mathcal{A})_\chi}(\mathbf{L}^F) \leq \left(\widetilde{\mathbf{G}}^F\mathcal{A}\right)_{\mathbf{L},\chi} = \left(\widetilde{\mathbf{G}}^F\mathcal{A}\right)_{\mathbf{L},\psi}.$$

To construct the relevant projective representations, we make use of [Spä12, Lemma 2.11]. As before, consider the corresponding $\widetilde{\chi} \in \widetilde{\mathbb{T}}_{\mathrm{glo}}$ and $\widetilde{\psi} \in \widetilde{\mathbb{T}}_{\mathrm{loc}}$ with $\widetilde{\Omega}^{\mathbf{G}}_{(\mathbf{L},\lambda)}(\widetilde{\chi}) = \widetilde{\psi}$, $\widetilde{\chi}$ lying over χ and $\widetilde{\psi}$ lying over ψ. Furthermore, consider the Clifford correspondent $\widehat{\chi} \in \mathrm{Irr}(\widetilde{\mathbf{G}}^F_\chi \mid \chi)$ of $\widetilde{\chi}$ and the Clifford correspondent $\widehat{\psi} \in \mathrm{Irr}(\mathbf{N}_{\widetilde{\mathbf{G}}}(\mathbf{L})^F_\psi \mid \psi)$ of $\widetilde{\psi}$. Let $\widehat{\mathcal{D}}_{\mathrm{glo}}$ be a representation affording $\widehat{\chi}$ and notice that, by the choice of χ and using Assumption 10.1.1 (iii.b), there exists a representation $\mathcal{D}'_{\mathrm{glo}}$ affording an extension $\chi' \in \mathrm{Irr}(\mathbf{G}^F\mathcal{A}_\chi)$ of χ. Similarly, let $\widehat{\mathcal{D}}_{\mathrm{loc}}$ be a representation affording $\widehat{\psi}$ and observe that, by the choice of ψ, there is a representation $\mathcal{D}'_{\mathrm{loc}}$ affording an extension $\psi' \in \mathrm{Irr}((\mathbf{G}^F\mathcal{A})_{\mathbf{L},\psi})$ of ψ. Applying [Spä12, Lemma 2.11] with $L := \mathbf{G}^F$, $\widetilde{L} := \widetilde{\mathbf{G}}^F_\chi$, $C := \mathbf{G}^F\mathcal{A}_\chi$, $X := (\widetilde{\mathbf{G}}^F\mathcal{A})_\chi$ and recalling that $X = \widetilde{L}C$ because Assumption 10.1.1 (iii.a) holds for χ, we deduce that the map

$$\mathcal{P}_{\mathrm{glo}} : \left(\widetilde{\mathbf{G}}^F\mathcal{A}\right)_\chi \to \mathrm{GL}_{\chi(1)}(\mathbb{C})$$

given by $\mathcal{P}_{\rm glo}(x_1x_2) \coloneqq \widehat{\mathcal{D}}_{\rm glo}(x_1)\mathcal{D}'_{\rm glo}(x_2)$, for every $x_1 \in \widetilde{\mathbf{G}}^F_\chi$ and $x_2 \in \mathbf{G}^F\mathcal{A}_\chi$, is a projective representation associated with χ whose factor set $\alpha_{\rm glo}$ satisfies

$$\alpha_{\rm glo}(x_1x_2, y_1y_2) = \mu^{\rm glo}_{x_2}(y_1) \tag{10.1.6}$$

for every $x_1, y_1 \in \widetilde{\mathbf{G}}^F_\chi$ and $x_2, y_2 \in \mathbf{G}^F\mathcal{A}_\chi$, where $\mu^{\rm glo}_{x_2} \in {\rm Irr}(\widetilde{\mathbf{G}}^F_\chi/\mathbf{G}^F)$ is determined by the equality $\widehat{\chi} = \mu^{\rm glo}_{x_2}\widehat{\chi}^{x_2}$ via Gallagher's theorem. In a similar way, considering $L \coloneqq \mathbf{N}_\mathbf{G}(\mathbf{L})^F$, $\widetilde{L} \coloneqq \mathbf{N}_{\widetilde{\mathbf{G}}}(\mathbf{L})^F_\chi$, $C \coloneqq (\mathbf{G}^F\mathcal{A})_{\mathbf{L},\chi}$, $X \coloneqq (\widetilde{\mathbf{G}}^F\mathcal{A})_{\mathbf{L},\chi}$ and noticing that $X = \widetilde{L}C$ because Assumption 10.1.1 (iv.a) holds for ψ, we deduce that the map

$$\mathcal{P}_{\rm loc} : \left(\widetilde{\mathbf{G}}^F\mathcal{A}\right)_{\mathbf{L},\chi} \to {\rm GL}_{\psi(1)}(\mathbb{C})$$

given by $\mathcal{P}_{\rm loc}(x_1x_2) \coloneqq \widehat{\mathcal{D}}_{\rm loc}(x_1)\mathcal{D}'_{\rm loc}(x_2)$, for every $x_1 \in \mathbf{N}_{\widetilde{\mathbf{G}}}(\mathbf{L})^F_\chi$ and $x_2 \in (\mathbf{G}^F\mathcal{A})_{\mathbf{L},\chi}$, is a projective representation associated with ψ whose factor set $\alpha_{\rm loc}$ satisfies

$$\alpha_{\rm loc}(x_1x_2, Zy_1y_2) = \mu^{\rm loc}_{x_2}(y_1) \tag{10.1.7}$$

for every $x_1, y_1 \in \mathbf{N}_{\widetilde{\mathbf{G}}}(\mathbf{L})^F_\chi$ and $x_2, y_2 \in (\mathbf{G}^F\mathcal{A})_{\mathbf{L},\chi}$, where $\mu^{\rm loc}_{x_2} \in {\rm Irr}(\mathbf{N}_{\widetilde{\mathbf{G}}}(\mathbf{L})^F_\chi/\mathbf{N}_\mathbf{G}(\mathbf{L})^F)$ is determined by $\widehat{\psi} = \mu^{\rm loc}_{x_2}\widehat{\psi}^{x_2}$. In order to obtain the condition on factor sets required to prove (10.1.5) we have to show that the restriction of $\alpha_{\rm glo}$ to $(\mathbf{G}^F\mathcal{A})_{\mathbf{L},\chi} \times (\mathbf{G}^F\mathcal{A})_{\mathbf{L},\chi}$ coincides with $\alpha_{\rm loc}$. Using (10.1.6) and (10.1.7), it is enough to show that

$$\left(\mu^{\rm glo}_x\right)_{\mathbf{N}_{\widetilde{\mathbf{G}}}(\mathbf{L})^F_\chi} = \mu^{\rm loc}_x$$

for every $x \in (\mathbf{G}^F\mathcal{A})_{\mathbf{L},\chi}$ and where $\widehat{\chi} = \mu^{\rm glo}_x\widehat{\chi}^x$ and $\widehat{\psi} = \mu^{\rm loc}_x\widehat{\psi}^x$. To prove this equality, since $(\mathbf{G}^F\mathcal{A})_{\mathbf{L},\chi} = \mathbf{N}_\mathbf{G}(\mathbf{L})^F A_\chi$ (see the proof of Lemma 10.1.2), we may assume $x \in A_\chi$. Then, we conclude since $\widetilde{\Omega}^\mathbf{G}_{(\mathbf{L},\lambda)}$ is $(A \ltimes \mathcal{K})$-equivariant.

To conclude we need to check one of the equivalent conditions of Lemma 3.3.3. Recalling that $\mathbf{C}_{(\widetilde{\mathbf{G}}^F\mathcal{A})_\chi}(\mathbf{G}^F) = \mathbf{Z}(\widetilde{\mathbf{G}}^F)$ by Assumption 10.1.1 (i.a), if $\zeta_{\rm glo}$ and $\zeta_{\rm loc}$ are the scalar functions of $\mathcal{P}_{\rm glo}$ and $\mathcal{P}_{\rm loc}$ respectively, we have to show that $\zeta_{\rm glo}$ and $\zeta_{\rm loc}$ coincide as characters of $\mathbf{Z}(\widetilde{\mathbf{G}}^F)$. By the definition of $\mathcal{P}_{\rm glo}$, it follows that $\zeta_{\rm glo}$ coincide with the unique irreducible constituent ν of $\widehat{\chi}_{\mathbf{Z}(\widetilde{\mathbf{G}}^F)}$. Moreover, by Clifford theory we know that ν is also the unique irreducible constituent of $\widetilde{\chi}_{\mathbf{Z}(\widetilde{\mathbf{G}}^F)}$. Therefore, we conclude that $\{\zeta_{\rm glo}\} = {\rm Irr}(\widetilde{\chi}_{\mathbf{Z}(\widetilde{\mathbf{G}}^F)})$ and a similar argument shows that $\{\zeta_{\rm loc}\} = {\rm Irr}(\widetilde{\psi}_{\mathbf{Z}(\widetilde{\mathbf{G}}^F)})$. Then, Assumption 10.1.1 (ii) implies that $\zeta_{\rm glo} = \zeta_{\rm loc}$. This completes the proof. □

10.1.2 The criterion for Condition 9.1

Our aim is now to prove a criterion for Condition 9.1. To do so, we will sharpen the argument used in the proof of Theorem 10.1.3. As mentioned at the beginning of Section 10.1, some additional restrictions will be required in order to deal with Clifford theory for blocks.

Assumption 10.1.4. Let $(\mathbf{L}, \lambda)$ be an e-cuspidal pair of $\mathbf{G}$, suppose that $\ell \in \Gamma(\mathbf{G}, F)$ and set $B \coloneqq {\rm bl}(\lambda)^{\mathbf{G}^F}$. Consider

$$\mathcal{G} \coloneqq \mathcal{E}\left(\mathbf{G}^F, (\mathbf{L}, \lambda)\right) \quad \text{and} \quad \mathcal{N} \coloneqq {\rm Irr}\left(\mathbf{N}_\mathbf{G}(\mathbf{L})^F \;\middle|\; \lambda\right)$$

and set

$$\widetilde{\mathcal{G}} := \mathrm{Irr}\left(\widetilde{\mathbf{G}}^F \;\middle|\; \mathcal{G}\right) \quad \text{and} \quad \widetilde{\mathcal{N}} := \mathrm{Irr}\left(\mathbf{N}_{\widetilde{\mathbf{G}}}(\mathbf{L})^F \;\middle|\; \mathcal{N}\right).$$

Assume that:

(i) (a) There is a semidirect decomposition $\widetilde{\mathbf{G}}^F \rtimes \mathcal{A}$, with $\mathcal{A}$ a finite abelian group, such that $\mathbf{C}_{(\widetilde{\mathbf{G}}^F\mathcal{A})_Z/Z}(\mathbf{G}^F/Z) = \mathbf{Z}(\widetilde{\mathbf{G}}^F)/Z$ and $(\widetilde{\mathbf{G}}^F\mathcal{A})_Z/\mathbf{Z}(\widetilde{\mathbf{G}}^F) \simeq \mathrm{Aut}(\mathbf{G}^F/Z)$ via the natural map for every $Z \leq \mathbf{Z}(\mathbf{G}^F)$ (see Lemma 6.1.6);

(b) Maximal extendibility holds with respect to $\mathbf{G}^F \trianglelefteq \widetilde{\mathbf{G}}^F$;

(c) Maximal extendibility holds with respect to $\mathbf{N}_{\mathbf{G}}(\mathbf{L})^F \trianglelefteq \mathbf{N}_{\widetilde{\mathbf{G}}}(\mathbf{L})^F$.

(ii) For $A := (\widetilde{\mathbf{G}}^F\mathcal{A})_{(\mathbf{L},\lambda)}$ there exists a defect preserving $(A \ltimes \mathcal{K})$-equivariant bijection

$$\widetilde{\Omega}^{\mathbf{G}}_{(\mathbf{L},\lambda)} : \widetilde{\mathcal{G}} \to \widetilde{\mathcal{N}}$$

such that, for every $\widetilde{\chi} \in \widetilde{\mathcal{G}}$, the following conditions hold:

(a) $\mathrm{Irr}\left(\widetilde{\chi}_{\mathbf{Z}(\widetilde{\mathbf{G}}^F)}\right) = \mathrm{Irr}\left(\widetilde{\Omega}^{\mathbf{G}}_{(\mathbf{L},\lambda)}(\widetilde{\chi})_{\mathbf{Z}(\widetilde{\mathbf{G}}^F)}\right)$;

(b) $\mathrm{bl}\left(\widetilde{\chi}\right) = \mathrm{bl}\left(\widetilde{\Omega}^{\mathbf{G}}_{(\mathbf{L},\lambda)}\left(\widetilde{\chi}\right)\right)^{\widetilde{\mathbf{G}}^F}$.

(iii) For every $\widetilde{\chi} \in \widetilde{\mathcal{G}}$ there exists $\chi \in \mathrm{Irr}\left(\widetilde{\chi}_{\mathbf{G}^F}\right)$ such that:

(a) $\left(\widetilde{\mathbf{G}}^F\mathcal{A}\right)_\chi = \widetilde{\mathbf{G}}^F_\chi\mathcal{A}_\chi$;

(b) χ extends to $\chi' \in \mathrm{Irr}\left(\mathbf{G}^F\mathcal{A}_\chi\right)$.

(iv) For every $\widetilde{\psi} \in \widetilde{\mathcal{N}}$ there exists $\psi \in \mathcal{N} \cap \mathrm{Irr}\left(\widetilde{\psi}_{\mathbf{N}_{\mathbf{G}}(\mathbf{L})^F}\right)$ such that:

(a) $\left(\widetilde{\mathbf{G}}^F\mathcal{A}\right)_{\mathbf{L},\psi} = \mathbf{N}_{\widetilde{\mathbf{G}}}(\mathbf{L})^F_\psi\left(\mathbf{G}^F\mathcal{A}\right)_{\mathbf{L},\psi}$;

(b) ψ extends to $\psi' \in \mathrm{Irr}\left(\left(\mathbf{G}^F \rtimes \mathcal{A}\right)_{\mathbf{L},\psi}\right)$.

(v) Assume one of the following conditions:

(a) $\mathrm{Out}(\mathbf{G}^F)_{\mathcal{B}}$ is abelian, where $\mathcal{B}$ is the $\widetilde{\mathbf{G}}^F$-orbit of B. In particular (iii) holds for every $\widetilde{\mathbf{G}}^F$-conjugate of χ (see the proof of [BS20a, Lemma 4.7]).

(b) for every subgroup $\mathbf{G}^F \leq H \leq \widetilde{\mathbf{G}}^F$ we have that every block $C \in \mathrm{Bl}(H \mid B)$ is $\widetilde{\mathbf{G}}^F$-invariant.

(vi) If $s \in \mathbf{L}^{*F^*}_{\mathrm{ss}}$ and $\lambda \in \mathcal{E}(\mathbf{L}^F, [s])$, then $\mathcal{G} = \mathcal{E}(\mathbf{G}^F, B, [s])$ (see the discussion following Theorem 7.3.3).

Remark 10.1.5. Here we comment on Assumption 10.1.4. First, observe that (v.a) holds for every block of $\mathbf{G}^F$ whenever $\mathbf{G}$ is a simple algebraic group not of type $\mathbf{A}$, $\mathbf{D}$ or $\mathbf{E}_6$. Next, notice that condition (v.b) holds for blocks of maximal defect (see [CS15, Proposition 5.4] and observe that the proof of this result holds in general in our situation by Lemma 7.1.5(ii)) and for unipotent blocks: if B is a unipotent block of $\mathbf{G}^F$, then there exists a unipotent character $\chi \in \mathrm{Irr}(B)$. By [DM91, Proposition 13.20] we deduce that χ extends to a character $\widetilde{\chi} \in \mathrm{Irr}(\widetilde{\mathbf{G}}^F)$. If $\mathbf{G}^F \leq H \leq \widetilde{\mathbf{G}}^F$ and C is a block of H that covers B, then we can find a character $\psi \in \mathrm{Irr}(C)$ that lies above χ. Since $\widetilde{\chi}_H$ is an irreducible character of H lying above χ, we deduce that $\psi = \widetilde{\chi}_H \widehat{z}_H$ for some $z \in \mathcal{K}$ corresponding to $\widehat{z}_{\widetilde{\mathbf{G}}} \in \mathrm{Irr}(\widetilde{\mathbf{G}}^F/\mathbf{G}^F)$ and where $\widehat{z}_H$ is the restriction of $\widehat{z}_{\widetilde{\mathbf{G}}^F}$ to H. Then ψ is $\widetilde{\mathbf{G}}^F$-invariant and therefore C is $\widetilde{\mathbf{G}}^F$-invariant. This proves that (v.a) holds for unipotent blocks.

Next, we point out that the character χ from Assumption 10.1.4 (iii) is not required to lie in $\mathcal{G}$. In fact, if such a character χ exists, then a character with the same properties and lying in $\mathcal{G}$ can always be found under Assumption 10.1.4 (v)-(vi). To see this, fix $\widetilde{\chi} \in \widetilde{\mathcal{G}}$ and $\chi \in \mathrm{Irr}(\widetilde{\chi}_{\mathbf{G}^F})$ satisfying Assumption 10.1.4 (iii). By the definition of $\widetilde{\mathcal{G}}$ there exists $\chi_0 \in \mathrm{Irr}(\widetilde{\chi}_{\mathbf{G}^F}) \cap \mathcal{G}$. In particular χ and χ_0 are $\widetilde{\mathbf{G}}^F$-conjugate. Now, if (v.a) holds, then all $\widetilde{\mathbf{G}}^F$-conjugates of χ satisfy Assumption 10.1.4 (iii.a) and (iii.b) according to the proof of [BS20a, Lemma 4.7]. Then χ_0 is the character we were looking for. If (v.b) holds, then B is $\widetilde{\mathbf{G}}^F$-invariant and, since $\mathrm{bl}(\chi_0) = B$, we deduce that $\mathrm{bl}(\chi) = B$. On the other hand $\chi \in \mathcal{E}(\mathbf{G}^F, [s])$ by Lemma 6.2.5 and therefore $\chi \in \mathrm{Irr}(B) \cap \mathcal{E}(\mathbf{G}^F, [s])$. By Assumption 10.1.4 (vi) we conclude that $\chi \in \mathcal{G}$.

We now prove the criterion for Condition 9.1. This proof will make large use of the notion of Dade's **ramification group**. For every block b of a normal subgroup N of G, Dade introduced a normal subgroup $G[b]$ of the subgroup G_b such that $G[b] \leq G_\chi$ for every $\chi \in \mathrm{Irr}(b)$. Here we use the following equivalent definition given by Murai in [Mur13] (see also [CS15, Definition 3.1]).

Definition 10.1.6. For every $N \trianglelefteq G$ and $b \in \mathrm{Bl}(G)$ define

$$G[b] := \left\{ g \in G_b \;\middle|\; \lambda_{b^{(g)}}\left(\mathfrak{Cl}_{\langle N, g\rangle}(h)^+\right) \neq 0, \text{ for some } h \in Ng \right\}$$

where $b^{(g)}$ is any block of $\langle N, g\rangle$ covering b (this definition does not depend on the choices of the blocks $b^{(g)}$).

See [Dad73], [Mur13] and [KS15] for further details on ramification groups.

Before proving the criterion for Condition 9.1, we need the following result in which we show how to choose transversals with good properties.

Proposition 10.1.7. *Assume Hypothesis 9.2.11 and Assumption 10.1.4. Let $\widetilde{\mathbb{T}}_{\mathrm{glo}}$ be any $(A \ltimes \mathcal{K})$-transversal in $\widetilde{\mathcal{G}}$ and consider the $(A \ltimes \mathcal{K})$-transversal $\widetilde{\mathbb{T}}_{\mathrm{loc}} := \{\widetilde{\Omega}^{\mathbf{G}}_{(\mathbf{L},\lambda)}(\widetilde{\chi}) \mid \widetilde{\chi} \in \widetilde{\mathbb{T}}_{\mathrm{glo}}\}$ in $\widetilde{\mathcal{N}}$. Then there exist A-transversals $\mathbb{T}_{\mathrm{glo}}$ in $\mathcal{G}$ and $\mathbb{T}_{\mathrm{loc}}$ in $\mathcal{N}$ with the following properties:*

(i) Every $\chi \in \mathbb{T}_{\mathrm{glo}}$ satisfies Assumption 10.1.4 (iii.a) and (iii.b);

(ii) Every $\psi \in \mathbb{T}_{\mathrm{loc}}$ satisfies Assumption 10.1.4 (iv.a) and (iv.b);

(iii) For every $\chi \in \mathbb{T}_{\mathrm{glo}}$ there exists a unique $\widetilde{\chi} \in \widetilde{\mathbb{T}}_{\mathrm{glo}}$ lying over χ. Conversely χ is the only character of $\mathbb{T}_{\mathrm{glo}}$ lying under $\widetilde{\chi}$;

(iv) For every $\psi \in \mathbb{T}_{\mathrm{loc}}$ there exists a unique $\widetilde{\psi} \in \widetilde{\mathbb{T}}_{\mathrm{loc}}$ lying over ψ. Conversely ψ is the only character of $\mathbb{T}_{\mathrm{loc}}$ lying under $\widetilde{\psi}$;

(v) Let $\chi \in \mathbb{T}_{\mathrm{glo}}$ and $\psi \in \mathbb{T}_{\mathrm{loc}}$ such that $\widetilde{\Omega}^{\mathbf{G}}_{(\mathbf{L},\lambda)}(\widetilde{\chi}) = \widetilde{\psi}$, where $\widetilde{\chi}$ is the unique character of $\widetilde{\mathbb{T}}_{\mathrm{glo}}$ lying above χ and $\widetilde{\psi}$ is the unique character ot $\widetilde{\mathbb{T}}_{\mathrm{loc}}$ lying above ψ. Then

$$\mathrm{bl}\left(\widehat{\chi}_J\right) = \mathrm{bl}\left(\widehat{\psi}_{\mathbf{N}_{\widetilde{\mathbf{G}}}(\mathbf{L})^F_\chi \cap J}\right)^J \tag{10.1.8}$$

for every $\mathbf{G}^F \leq J \leq \widetilde{\mathbf{G}}^F$, where $\widehat{\chi} \in \mathrm{Irr}(\widetilde{\mathbf{G}}^F_\chi)$ is the Clifford correspondent of $\widetilde{\chi}$ over χ and $\widehat{\psi} \in \mathrm{Irr}(\mathbf{N}_{\widetilde{\mathbf{G}}}(\mathbf{L})^F_\psi)$ is the CLifford correspondent of $\widetilde{\psi}$ over ψ.

Proof. For every $\widetilde{\psi} \in \widetilde{\mathbb{T}}_{\mathrm{loc}}$ fix a character $\psi \in \mathcal{N} \cap \mathrm{Irr}(\widetilde{\psi}_{\mathbf{N}_\mathbf{G}(\mathbf{L})^F})$ satisfying Assumption 10.1.4 (iv) and denote by $\mathbb{T}_{\mathrm{loc}}$ the set of such characters ψ, while $\widetilde{\psi}$ runs over $\widetilde{\mathbb{T}}_{\mathrm{loc}}$. As proven in Theorem 10.1.3, the set $\mathbb{T}_{\mathrm{loc}}$ is an A-transversal in $\mathcal{N}$ satisfying (iv) above. Next, for every $\widetilde{\chi} \in \widetilde{\mathbb{T}}_{\mathrm{glo}}$, we are going to find a character $\chi \in \mathcal{G} \cap \mathrm{Irr}(\widetilde{\chi}_{\mathbf{G}^F})$ satisfying Assumption 10.1.4 (iii.a) and (iii.b) and such that

$$\mathrm{bl}\left(\widehat{\chi}_J\right) = \mathrm{bl}\left(\widehat{\psi}_{\mathbf{N}_{\widetilde{\mathbf{G}}}(\mathbf{L})^F_\chi \cap J}\right)^J \tag{10.1.9}$$

for every $\mathbf{G}^F \leq J \leq \widetilde{\mathbf{G}}^F_\chi$ and where $\widehat{\chi} \in \mathrm{Irr}(\widetilde{\mathbf{G}}^F_\chi \mid \chi)$ is the Clifford correspondent of $\widetilde{\chi}$ over χ and $\widehat{\psi} \in \mathrm{Irr}(\mathbf{N}_{\widetilde{\mathbf{G}}}(\mathbf{L})^F_\psi \mid \psi)$ is the Clifford correspondent of $\widetilde{\psi}$ over ψ with $\widetilde{\psi} := \widetilde{\Omega}^{\mathbf{G}}_{(\mathbf{L},\lambda)}(\widetilde{\chi})$ and $\psi \in \mathbb{T}_{\mathrm{loc}}$ corresponding to $\widetilde{\psi}$. Then, as shown in the proof of Theorem 10.1.3, the set $\mathbb{T}_{\mathrm{glo}}$ of such characters χ while $\widetilde{\chi}$ runs over $\widetilde{\mathbb{T}}_{\mathrm{glo}}$ will be an A-transversal in $\mathcal{G}$ satisfying (iii) above. Moreover (v) will be satisfied by our choice.

We first prove the claim assuming Assumption 10.1.4 (v.a). We start by showing that, for every $\widetilde{\chi} \in \widetilde{\mathbb{T}}_{\mathrm{glo}}$, there exists a character $\chi \in \mathcal{G} \cap \mathrm{Irr}(\widetilde{\chi}_{\mathbf{G}^F})$ such that

$$\mathrm{bl}\left(\widehat{\chi}_{\widetilde{\mathbf{G}}^F[B]}\right) = \mathrm{bl}\left(\widehat{\psi}_{\mathbf{N}_{\widetilde{\mathbf{G}}}(\mathbf{L})^F[C]}\right)^{\widetilde{\mathbf{G}}^F[B]}, \tag{10.1.10}$$

where $\widehat{\chi} \in \mathrm{Irr}(\widetilde{\mathbf{G}}^F_\chi \mid \chi)$ is the Clifford correspondent of $\widetilde{\chi}$ over χ and $\widehat{\psi} \in \mathrm{Irr}(\mathbf{N}_{\widetilde{\mathbf{G}}}(\mathbf{L})^F_\psi \mid \psi)$ is the Clifford correspondent of $\widetilde{\psi}$ over ψ with $\widetilde{\psi} := \widetilde{\Omega}^{\mathbf{G}}_{(\mathbf{L},\lambda)}(\widetilde{\chi})$ and $\psi \in \mathbb{T}_{\mathrm{loc}}$ corresponding to $\widetilde{\psi}$ and $C := \mathrm{bl}(\psi)$. Notice that, as pointed out in Remark 10.1.5, under Assumption 10.1.4 (v.a) such a character χ will automatically satisfy Assumption 10.1.4 (iii.a) and (iii.b).

Set $b := \mathrm{bl}(\lambda)$ and recall that, as every block of $\mathbf{N}_\mathbf{G}(\mathbf{L})^F$ is $\mathbf{L}^F$-regular (see Lemma 9.2.5), C must coincide with $b^{\mathbf{N}_\mathbf{G}(\mathbf{L})^F}$ and therefore $C^{\mathbf{G}^F} = b^{\mathbf{G}^F} = B$. Moreover, for $E := \mathbf{Z}^\circ(\mathbf{L})^F_\ell$, we have $\mathbf{N}_\mathbf{H}(\mathbf{L}) = \mathbf{N}_\mathbf{H}(E)$ for every F-stable $\mathbf{G} \leq \mathbf{H} \leq \widetilde{\mathbf{G}}$ (see Proposition 7.1.6). Then, for every block $C_1 \in \mathrm{Bl}(\mathbf{N}_\mathbf{H}(\mathbf{L})^F \mid C)$, the induced block $B_1 := C_1^{\mathbf{H}^F}$ is well defined and covers $C^{\mathbf{G}^F} = B$ (see [KS15, Theorem B]): in fact for a defect group $D \in \delta(C)$ we have $E \leq \mathbf{O}_\ell(\mathbf{N}_\mathbf{G}(\mathbf{L})^F) \leq D$ and hence $\mathbf{C}_{\mathbf{H}^F}(D) \leq \mathbf{N}_\mathbf{H}(E)^F = \mathbf{N}_\mathbf{H}(\mathbf{L})^F$.

Consider $\widetilde{C} := \mathrm{bl}(\widetilde{\psi})$, $\widetilde{B} := \mathrm{bl}(\widetilde{\chi})$ and recall that $\widetilde{B} = (\widetilde{C})^{\widetilde{\mathbf{G}}^F}$ by Assumption 10.1.4 (ii.b). Notice that $\widetilde{\mathbf{G}}^F[B] = \mathbf{N}_{\widetilde{\mathbf{G}}}(\mathbf{L})^F[C] \cdot \mathbf{G}^F$ (see [KS15, Lemma 3.2 (c) and Lemma 3.6]) and set $C_1 := \mathrm{bl}(\widehat{\psi}_{\mathbf{N}_{\widetilde{\mathbf{G}}}(\mathbf{L})^F[C]})$ and $B_1 := C_1^{\widetilde{\mathbf{G}}^F[B]}$. By the previous paragraph the block B_1 covers B and the exact same argument can be used to show that $\widetilde{B}$ covers B_1. In particular there exists $\chi_1 \in \mathrm{Irr}(B_1)$ lying under $\widetilde{\chi}$. We claim that $\chi_{1,\mathbf{G}^F}$ is irreducible and lies in $\mathcal{G}$. If χ is an irreducible constituent of $\chi_{1,\mathbf{G}^F}$, then B_1 covers $\mathrm{bl}(\chi)$. As B is $\widetilde{\mathbf{G}}^F[B]$-invariant, we conclude that $\mathrm{bl}(\chi) = B$. Then $\widetilde{\mathbf{G}}^F[B] \leq \widetilde{\mathbf{G}}^F_\chi$ and Assumption 10.1.4 (i.b) implies that $\chi_{1,\mathbf{G}^F} = \chi$. Furthermore, since for every $\mathbf{G}^F \leq J \leq \widetilde{\mathbf{G}}^F_\chi$ there exists a unique irreducible character of J lying over χ and under $\widetilde{\chi}$, we conclude that $\chi_1 = \widehat{\chi}_{\widetilde{\mathbf{G}}^F[B]}$, where $\widehat{\chi} \in \mathrm{Irr}(\widetilde{\mathbf{G}}^F_\chi)$ is the Clifford correspondent of $\widetilde{\chi}$ over χ. To conclude, since $\widetilde{\chi} \in \widetilde{\mathcal{G}}$ covers χ_1 and hence χ, Lemma 6.2.5 implies that $\chi \in \mathcal{E}(\mathbf{G}^F, [s])$, where

$s \in \mathbf{L}_{\mathrm{ss}}^{*F^*}$ and $\lambda \in \mathcal{E}(\mathbf{L}^F, [s])$. By Assumption 10.1.4 (vi) we conclude that $\chi \in \mathcal{G} \cap \mathrm{Irr}(\widetilde{\chi}_{\mathbf{G}^F}) = \mathcal{G}$ and satisfies (10.1.10).

Next, we deduce (10.1.9) from (10.1.10). First, since $\mathrm{bl}(\widehat{\psi}_{\mathbf{N}_{\widetilde{\mathbf{G}}}(\mathbf{L})^F[C]})$ is covered by $\mathrm{bl}(\widehat{\psi})$, by the same argument used before we deduce that $\mathrm{bl}(\widehat{\psi}_{\mathbf{N}_{\widetilde{\mathbf{G}}}(\mathbf{L})^F[C]})^{\widetilde{\mathbf{G}}^F[B]} = \mathrm{bl}(\widehat{\chi}_{\widetilde{\mathbf{G}}^F[B]})$ is covered by $\mathrm{bl}(\widehat{\psi})^{\widetilde{\mathbf{G}}^F_\chi}$. Since $\widetilde{\mathbf{G}}^F_\chi$ has a unique block that covers $\mathrm{bl}(\widehat{\chi}_{\widetilde{\mathbf{G}}^F[B]})$ (see [Mur13, Theorem 3.5]), we conclude that $\mathrm{bl}(\widehat{\psi})^{\widetilde{\mathbf{G}}^F_\chi} = \mathrm{bl}(\widehat{\chi})$. Finally, for $\mathbf{G}^F \leq J \leq \widetilde{\mathbf{G}}^F_\chi$, observe that $\mathrm{bl}(\widehat{\chi}_J)$ is $\widetilde{\mathbf{G}}^F_\chi$-stable and therefore it is the unique block of J covered by $\mathrm{bl}(\widehat{\chi})$. Since, again by using the previous argument, $\mathrm{bl}(\widehat{\psi}_{\mathbf{N}_{\widetilde{\mathbf{G}}}(\mathbf{L})^F_\chi \cap J})^J$ is covered by $\mathrm{bl}(\widehat{\psi})^{\widetilde{\mathbf{G}}^F_\chi} = \mathrm{bl}(\widehat{\chi})$ we conclude that χ is a character of $\mathrm{Irr}(\widetilde{\chi}_{\mathbf{G}^F}) \cap \mathcal{G}$ satisfying Assumption 10.1.4 (iii.a) and (iii.b) and such that (10.1.9) holds. This proves the claim under Assumption 10.1.4 (v.a).

We now prove the claim under Assumption 10.1.4 (v.b). Consider $\chi \in \mathrm{Irr}(\widetilde{\chi}_{\mathbf{G}^F})$ satisfying Assumption 10.1.4 (iii) and notice that, as shown in Remark 10.1.5, under Assumption 10.1.4 (v.b) we automatically have $\chi \in \mathcal{G}$. As shown in the previous part, the block $\widehat{B} := \mathrm{bl}(\widehat{\psi})^{\widetilde{\mathbf{G}}^F_\chi}$ is covered by $\widetilde{B} := \mathrm{bl}(\widetilde{\chi})$ and covers B. Since $\widetilde{B}$ covers $\widehat{B}$, we deduce that $\widehat{B}$ and $\mathrm{bl}(\widehat{\chi})$ are $\widetilde{\mathbf{G}}^F$-conjugate. On the other hand our assumption implies that $\widehat{B}$ is $\widetilde{\mathbf{G}}^F$-stable and therefore coincide with $\mathrm{bl}(\widehat{\chi})$. This shows that $\mathrm{bl}(\widehat{\chi}) = \mathrm{bl}(\widehat{\psi})^{\widetilde{\mathbf{G}}^F_\chi}$ and, arguing as in the final part of the previous paragraph, we conclude that (10.1.9) holds. This completes the proof. □

We can finally prove the criterion for Condition 9.1.

Theorem 10.1.8. *Assume Hypothesis 9.2.11 and Assumption 10.1.4 with respect to the e-cuspidal pair $(\mathbf{L}, \lambda)$. Then Condition 9.1 holds for $(\mathbf{L}, \lambda)$ and $\mathbf{G}$.*

Proof. Choose transversals $\widetilde{\mathbb{T}}_{\mathrm{glo}}$, $\widetilde{\mathbb{T}}_{\mathrm{loc}}$, $\mathbb{T}_{\mathrm{glo}}$ and $\mathbb{T}_{\mathrm{loc}}$ as in Proposition 10.1.7. As in the proof of Theorem 10.1.3, setting

$$\Omega^{\mathbf{G}}_{(\mathbf{L},\lambda)}(\chi^x) := \psi^x$$

for every $x \in A$ and $\chi \in \mathbb{T}_{\mathrm{glo}}$, where ψ is the unique character in $\mathbb{T}_{\mathrm{loc}}$ lying below $\widetilde{\psi} := \widetilde{\Omega}^{\mathbf{G}}_{(\mathbf{L},\lambda)}(\widetilde{\chi})$ and $\widetilde{\chi}$ is the unique character in $\widetilde{\mathbb{T}}_{\mathrm{glo}}$ lying over χ, defines an A-equivariant bijection between $\mathcal{G}$ and $\mathcal{N}$. By Assumption 10.1.4 (i.a) this means that $\Omega^{\mathbf{G}}_{(\mathbf{L},\lambda)}$ is $\mathrm{Aut}(\mathbf{G}^F)_{(\mathbf{L},\lambda)}$-equivariant.

The argument used in the proof of Theorem 10.1.3 shows that $\Omega^{\mathbf{G}}_{(\mathbf{L},\lambda)}$ is defect preserving and that $\mathrm{Ker}(\chi_{\mathbf{Z}(\mathbf{G}^F)}) = \mathrm{Ker}(\Omega^{\mathbf{G}}_{(\mathbf{L},\lambda)}(\chi)_{\mathbf{Z}(\mathbf{G}^F)})$ for every $\chi \in \mathcal{G}$. By [Spä17, Theorem 5.3], we deduce that to conclude the proof it's enough to show that

$$\left((\widetilde{\mathbf{G}}^F\mathcal{A})_\chi/Z, \mathbf{G}^F/Z, \overline{\chi}\right) \sim_{\mathbf{G}^F/Z} \left((\widetilde{\mathbf{G}}^F\mathcal{A})_{\mathbf{L},\chi}/Z, \mathbf{N}_{\mathbf{G}}(\mathbf{L})^F/Z, \overline{\psi}\right), \tag{10.1.11}$$

where $\psi := \Omega^{\mathbf{G}}_{(\mathbf{L},\lambda)}(\chi)$. Moreover, as the equivalence relation $\sim_{\mathbf{G}^F/Z}$ is compatible with conjugation, it is enough to prove (10.1.11) for a fixed $\chi \in \mathbb{T}_{\mathrm{glo}}$ and $\psi := \Omega^{\mathbf{G}}_{(\mathbf{L},\lambda)}(\chi) \in \mathbb{T}_{\mathrm{loc}}$. As before, consider the corresponding $\widetilde{\chi} \in \widetilde{\mathbb{T}}_{\mathrm{glo}}$ and $\widetilde{\psi} \in \widetilde{\mathbb{T}}_{\mathrm{loc}}$ with $\widetilde{\Omega}^{\mathbf{G}}_{(\mathbf{L},\lambda)}(\widetilde{\chi}) = \widetilde{\psi}$, $\widetilde{\chi}$ lying over χ and $\widetilde{\psi}$ lying over ψ. Furthermore, consider the Clifford correspondent $\widehat{\chi} \in \mathrm{Irr}(\widetilde{\mathbf{G}}^F_\chi \mid \chi)$ of $\widetilde{\chi}$ and the Clifford correspondent $\widehat{\psi} \in \mathrm{Irr}(\mathbf{N}_{\widetilde{\mathbf{G}}}(\mathbf{L})^F_\psi \mid \psi)$ of $\widetilde{\psi}$.

Proceeding as in the proof of Theorem 10.1.3, we can construct a projective representation associated with $\overline{\chi}$

$$\overline{\mathcal{P}}_{\mathrm{glo}} : \left(\widetilde{\mathbf{G}}^F\mathcal{A}\right)_\chi/Z \to \mathrm{GL}_{\chi(1)}(\mathbb{C})$$

given by $\overline{\mathcal{P}}_{\mathrm{glo}}(Zx_1x_2) \coloneqq \widehat{\mathcal{D}}_{\mathrm{glo}}(x_1)\mathcal{D}'_{\mathrm{glo}}(x_2)$ for every $x_1 \in \widetilde{\mathbf{G}}^F_\chi$ and $x_2 \in (\mathbf{G}^F\mathcal{A})_\chi$. Similarly, we obtain a projective representation associated with $\overline{\psi}$

$$\overline{\mathcal{P}}_{\mathrm{loc}} : \left(\widetilde{\mathbf{G}}^F\mathcal{A}\right)_{\mathbf{L},\chi}/Z \to \mathrm{GL}_{\psi(1)}(\mathbb{C})$$

given by $\overline{\mathcal{P}}_{\mathrm{loc}}(Zx_1x_2) \coloneqq \widehat{\mathcal{D}}_{\mathrm{loc}}(x_1)\mathcal{D}'_{\mathrm{loc}}(x_2)$ for every $x_1 \in \mathbf{N}_{\widetilde{\mathbf{G}}}(\mathbf{L})^F_\psi$ and $x_2 \in (\mathbf{G}^F\mathcal{A})_{\mathbf{L},\psi}$. Moreover, by the proof of Theorem 10.1.3, we know that

$$\left((\widetilde{\mathbf{G}}^F\mathcal{A})_\chi/Z, \mathbf{G}^F/Z, \overline{\chi}\right) \sim^c_{\mathbf{G}^F/Z} \left((\widetilde{\mathbf{G}}^F\mathcal{A})_{\mathbf{L},\psi}/Z, \mathbf{N}_{\mathbf{G}}(\mathbf{L})^F/Z, \overline{\psi}\right)$$

via the projective representations $(\overline{\mathcal{P}}_{\mathrm{glo}}, \overline{\mathcal{P}}_{\mathrm{loc}})$. Consider the factor sets $\overline{\alpha}_{\mathrm{glo}}$ of $\overline{\mathcal{P}}_{\mathrm{glo}}$ and $\overline{\alpha}_{\mathrm{loc}}$ of $\overline{\mathcal{P}}_{\mathrm{loc}}$. Let S be the group generated by the values of $\overline{\alpha}_{\mathrm{glo}}$ and denote by A_{glo} the central extension of $(\widetilde{\mathbf{G}}^F\mathcal{A})_\chi/Z$ by S induced by $\overline{\alpha}_{\mathrm{glo}}$. Let $\epsilon : A_{\mathrm{glo}} \to (\widetilde{\mathbf{G}}^F\mathcal{A})_\chi/Z$ be the canonical morphism with kernel S. As $\overline{\alpha}_{\mathrm{glo}}$ is trivial on $(\widetilde{\mathbf{G}}^F_\chi/Z) \times (\widetilde{\mathbf{G}}^F_\chi/Z)$, every subgroup $X \leq \widetilde{\mathbf{G}}^F_\chi/Z$ is isomorphic to the subgroup $X_0 \coloneqq \{(x,1) \mid x \in X\}$ of A_{glo} and $\epsilon^{-1}(X) = X_0 \times S$. In particular, we have $H_{\mathrm{glo}} \coloneqq \epsilon^{-1}\left(\widetilde{\mathbf{G}}^F_\chi/Z\right) = (\widetilde{\mathbf{G}}^F_\chi/Z)_0 \times S$. The map given by

$$\mathcal{Q}_{\mathrm{glo}}(x,s) \coloneqq s\overline{\mathcal{P}}_{\mathrm{glo}}(x),$$

for every $s \in S$ and $x \in (\widetilde{\mathbf{G}}^F\mathcal{A})_\chi/Z$, is an irreducible representation of A_{glo} affording an extension χ_1 of the character $\overline{\chi}_0$ of $(\mathbf{G}^F/Z)_0$ corresponding to $\overline{\chi}$. Notice that

$$\chi_{1,H_{\mathrm{glo}}} = \left(\widehat{\overline{\chi}}\right)_0 \times \iota, \tag{10.1.12}$$

where $\iota(s) \coloneqq s$ and $(\widehat{\overline{\chi}})_0$ is the character of $(\widetilde{\mathbf{G}}^F_\chi/Z)_0$ corresponding to $\widehat{\overline{\chi}} \in \mathrm{Irr}(\widetilde{\mathbf{G}}^F_\chi/Z)$. Next, set $A_{\mathrm{loc}} \coloneqq \epsilon^{-1}((\widetilde{\mathbf{G}}^F\mathcal{A})^F_{\mathbf{L},\chi}/Z)$ and notice that, because the factor set $\overline{\alpha}_{\mathrm{loc}}$ of $\overline{\mathcal{P}}_{\mathrm{loc}}$ is the restriction of the factor set $\overline{\alpha}_{\mathrm{glo}}$ of $\overline{\mathcal{P}}_{\mathrm{glo}}$, the map given by

$$\mathcal{Q}_{\mathrm{loc}}(x,s) \coloneqq s\overline{\mathcal{P}}_{\mathrm{loc}}(x),$$

for every $s \in S$ and $x \in (\widetilde{\mathbf{G}}^F\mathcal{A})^F_{\mathbf{L},\chi}/Z$, is an irreducible representation of A_{loc} affording an extension ψ_1 of the character $\overline{\psi}_0$ of $(\mathbf{N}_{\mathbf{G}}(\mathbf{L})^F/Z)_0$ corresponding to $\overline{\psi}$. As before, we have

$$\psi_{1,H_{\mathrm{loc}}} = \left(\widehat{\overline{\psi}}\right)_0 \times \iota, \tag{10.1.13}$$

where $H_{\mathrm{loc}} \coloneqq \epsilon^{-1}(\mathbf{N}_{\widetilde{\mathbf{G}}}(\mathbf{L})^F_\chi/Z) = (\mathbf{N}_{\widetilde{\mathbf{G}}}(\mathbf{L})^F_\chi/Z)_0 \times S$ and $(\widehat{\overline{\psi}})_0$ is the character of $(\mathbf{N}_{\widetilde{\mathbf{G}}}(\mathbf{L})^F_\chi/Z)_0$ corresponding to $\widehat{\overline{\psi}} \in \mathrm{Irr}(\mathbf{N}_{\widetilde{\mathbf{G}}}(\mathbf{L})^F_\chi/Z)$. Now, (10.1.12), (10.1.13) and (10.1.9) imply that

$$\mathrm{bl}\left(\chi_{1,J}\right) = \mathrm{bl}\left(\psi_{1,J\cap H_{\mathrm{glo}}}\right)^J \tag{10.1.14}$$

for every $(\mathbf{G}^F/Z)_0 \leq J \leq H_{\mathrm{glo}}$ (see the argument at the end of the proof of [CS13, proposition 4.2]). By [KS15, Theorem C] there exists $\varphi_1 \in \mathrm{Irr}(A_{\mathrm{glo}}[B_0])$ such that $\varphi_{1,(\mathbf{G}^F/Z)_0}$ is irreducible and lies in the block B_0 and

$$\mathrm{bl}\left(\varphi_{1,J}\right) = \mathrm{bl}\left(\psi_{1,J\cap A_{\mathrm{loc}}}\right)^J \tag{10.1.15}$$

for every $(\mathbf{G}^F/Z)_0 \leq J \leq A_{\mathrm{glo}}[B_0]$. It follows from (10.1.14) and (10.1.15) that

$$\mathrm{bl}\left(\varphi_{1,J}\right) = \mathrm{bl}\left(\psi_{1,J\cap H_{\mathrm{loc}}}\right)^J = \mathrm{bl}\left(\chi_{1,J}\right)$$

for every $(\mathbf{G}^F/Z)_0 \leq J \leq H_{\mathrm{glo}}[B_0] = H_{\mathrm{glo}} \cap A_{\mathrm{glo}}[B_0]$. In particular $B_0 = \mathrm{bl}(\chi_{1,(\mathbf{G}^F/Z)_0}) = \mathrm{bl}(\overline{\chi}_0)$. Therefore the condition of [CS13, Lemma 3.2] are satisfied and we obtain an extension $\chi_2 \in \mathrm{Irr}(A_{\mathrm{glo}})$ of $\chi_{1,H_{\mathrm{glo}}}$ satisfying

$$\mathrm{bl}\left(\varphi_{1,J}\right) = \mathrm{bl}\left(\chi_{2,J}\right) \tag{10.1.16}$$

for every $(\mathbf{G}^F/Z)_0 \leq J \leq A_{\mathrm{glo}}[B_0]$. From (10.1.15) and (10.1.16) we obtain

$$\mathrm{bl}\left(\psi_{1,J\cap A_{\mathrm{loc}}}\right)^J = \mathrm{bl}\left(\chi_{2,J}\right)$$

for every $(\mathbf{G}^F/Z)_0 \leq J \leq A_{\mathrm{glo}}[B_0]$. The latter equation, together with [Mur13, Theorem 3.5], yields

$$\begin{aligned}\mathrm{bl}\left(\psi_{1,J\cap A_{\mathrm{loc}}}\right)^J &= \left(\mathrm{bl}\left(\psi_{1,J\cap A_{\mathrm{loc}}\cap A_{\mathrm{glo}}[B_0]}\right)^{J\cap A_{\mathrm{loc}}}\right)^J \\ &= \left(\mathrm{bl}\left(\chi_{2,J\cap A_{\mathrm{glo}}[B_0]}\right)\right)^J \\ &= \mathrm{bl}\left(\chi_{2,J}\right)\end{aligned} \tag{10.1.17}$$

for every $(\mathbf{G}^F/Z)_0 \leq J \leq A_{\mathrm{glo}}$. Finally, observe that using Assumption 10.1.4 (i.a) and [Spä17, Theorem 4.1 (d)] we obtain

$$\begin{aligned}\mathbf{C}_{A_{\mathrm{glo}}}((\mathbf{G}^F/Z)_0) &= \mathbf{C}_{A_{\mathrm{glo}}}((\mathbf{G}^F/Z)_0 \times S) \\ &\leq \epsilon^{-1}\left(\mathbf{C}_{(\widetilde{\mathbf{G}}^F\mathcal{A})_\chi/Z}\left(\mathbf{G}^F/Z\right)\right) \\ &= \epsilon^{-1}\left(\mathbf{Z}(\widetilde{\mathbf{G}}^F)/Z\right) \\ &= \left(\mathbf{Z}\left(\widetilde{\mathbf{G}}^F\right)/Z\right)_0 \times S.\end{aligned}$$

Recalling that $\mathrm{Irr}(\chi_{\mathbf{Z}(\mathbf{G}^F)}) = \mathrm{Irr}(\psi_{\mathbf{Z}(\mathbf{G}^F)})$, we obtain $\mathrm{Irr}(\widehat{\chi}_{\mathbf{Z}(\widetilde{\mathbf{G}}^F)}) = \mathrm{Irr}(\widehat{\psi}_{\mathbf{Z}(\widetilde{\mathbf{G}}^F)})$ and hence

$$\begin{aligned}\mathrm{Irr}\left(\chi_{2,(\mathbf{Z}(\widetilde{\mathbf{G}}^F)/Z)_0\times S}\right) &= \mathrm{Irr}\left(\chi_{1,(\mathbf{Z}(\widetilde{\mathbf{G}}^F)/Z)_0\times S}\right) \\ &= \mathrm{Irr}\left(\left(\overline{\widehat{\chi}}\right)_{0,(\mathbf{Z}(\widetilde{\mathbf{G}}^F)/Z)_0} \times \iota\right) \\ &= \mathrm{Irr}\left(\left(\overline{\widehat{\psi}}\right)_{0,(\mathbf{Z}(\widetilde{\mathbf{G}}^F)/Z)_0} \times \iota\right) \\ &= \mathrm{Irr}\left(\psi_{1,(\mathbf{Z}(\widetilde{\mathbf{G}}^F)/Z)_0\times S}\right).\end{aligned} \tag{10.1.18}$$

Thanks to (10.1.17) and (10.1.18), we can apply [Spä17, Lemma 3.10] which implies

$$\left(A_{\mathrm{glo}}, \left(\mathbf{G}^F/Z\right)_0, \overline{\chi}_0\right) \sim_{(\mathbf{G}^F/Z)_0} \left(A_{\mathrm{loc}}, \left(\mathbf{N}_{\mathbf{G}}(\mathbf{L})^F/Z\right)_0, \overline{\psi}_0\right).$$

Then (10.1.11) follows by using [Spä17, Theorem 4.1 (i)]. This completes the proof. □

10.2 Proof of Theorem 10.1 and Theorem 10.2

We now come to the proofs of Theorem 10.1 and Theorem 10.2. As said before this reduces the verification of Condition 9.1 and hence of the inductive condition for Dade's Conjecture for finite quasisimple groups of Lie type to problems on character stabilizers and extendibility. Before proceeding further, we give an exact definition of these extendibility conditions. The following should be compared to [CS19, Definition 2.2].

Definition 10.2.1. Let $\mathbf{G}$ be a simple algebraic group of simply connected type and consider F, $\widetilde{\mathbf{G}}$ and $\mathcal{A}$ as in the previous sections. For every e-split Levi subgroup $\mathbf{L}$ of $\mathbf{G}$, we define the following condition.

There exists a $\widetilde{\mathbf{L}}^F$-transversal $\mathcal{T}$ in $\mathrm{Cusp}_e(\mathbf{L}^F)$ such that:

(G) For every $\lambda \in \mathcal{T}$ and every $\chi \in \mathcal{E}(\mathbf{G}^F, (\mathbf{L}, \lambda))$ there exists an $\mathbf{N}_{\widetilde{\mathbf{G}}}(\mathbf{L})^F_\lambda$-conjugate χ_0 of χ such that:

  (i) $\left(\widetilde{\mathbf{G}}^F \mathcal{A}\right)_{\chi_0} = \widetilde{\mathbf{G}}^F_{\chi_0} \mathcal{A}_{\chi_0}$, and

  (ii) χ_0 extends to $\mathbf{G}^F \mathcal{A}_{\chi_0}$.

(L) For every $\lambda \in \mathcal{T}$ and every $\psi \in \mathrm{Irr}(\mathbf{N}_{\mathbf{G}}(\mathbf{L})^F \mid \lambda)$ there exists an $\mathbf{N}_{\widetilde{\mathbf{G}}}(\mathbf{L})^F_\lambda$-conjugate ψ_0 of ψ such that:

  (i) $\left(\widetilde{\mathbf{G}}^F \mathcal{A}\right)_{\mathbf{L}, \psi_0} = \mathbf{N}_{\widetilde{\mathbf{G}}}(\mathbf{L})^F_{\psi_0} \left(\mathbf{G}^F \mathcal{A}\right)_{\mathbf{L}, \psi_0}$, and

  (ii) ψ_0 extends to $\left(\mathbf{G}^F \mathcal{A}\right)_{\mathbf{L}, \psi_0}$.

We now make two remarks on the conditions of Definition 10.2.1. First we consider the local condition for groups of type $\mathbf{A}$.

Remark 10.2.2. Notice that condition (L) from Definition 10.2.1 holds with respect to every e-split Levi subgroup $\mathbf{L}$ in the case that $\mathbf{G}$ is of type $\mathbf{A}_n$. This follows from [BS20b, Section 4].

Proof. To see this observe first that the results obtained in [BS20b, Section 4] (in particular [BS20b, Theorem 4.1 and Corollary 4.7]) rely on the proof of [CS17b, Theorem 4.3] and therefore on the arguments introduced in [CS17a, Section 5]. In particular, consider the argument used in [CS17a, Proposition 5.13]. Consider $\psi \in \mathrm{Irr}(\mathbf{N}_{\mathbf{G}}(\mathbf{L})^F \mid \lambda)$ and notice that λ has an extension $\widehat{\lambda} \in \mathrm{Irr}(\mathbf{N}_{\mathbf{G}}(\mathbf{L})^F_\lambda)$ by [BS20b, Theorem 1.2 (a)]. Using Gallagher's theorem and the Clifford correspondence, we can write $\psi = (\widehat{\lambda}\eta)^{\mathbf{N}_{\mathbf{G}}(\mathbf{L})^F}$ for some $\eta \in \mathrm{Irr}(\mathbf{N}_{\mathbf{G}}(\mathbf{L})^F_\lambda / \mathbf{L}^F)$. By the argument of [CS17a, Proposition 5.13], there exists $\eta_0 \in \mathrm{Irr}(\mathbf{N}_{\mathbf{G}}(\mathbf{L})^F_\lambda / \mathbf{L}^F)$ such that $\psi_0 := (\widehat{\lambda}\eta_0)^{\mathbf{N}_{\mathbf{G}}(\mathbf{L})^F}$ satisfies Definition 10.2.1 (L.i)-(L.ii) and $\psi = \psi_0^x$ for some $x \in \mathbf{N}_{\widetilde{\mathbf{G}}}(\mathbf{L})^F$. By the definition of ψ_0, we deduce that ψ_0 lies above λ and therefore ψ lies above λ and λ^x. By Clifford's theorem, it follows that $\lambda = \lambda^{xy}$ for some $y \in \mathbf{N}_{\mathbf{G}}(\mathbf{L})^F$ and we conclude that $\psi = \psi_0^{xy}$ with $xy \in \mathbf{N}_{\widetilde{\mathbf{G}}}(\mathbf{L})^F_\lambda$. □

Next, we make a comment on the global condition (see also Remark 10.1.5).

Remark 10.2.3. Assume that Hypothesis 7.2.7 holds for $(\mathbf{G}, F)$ and let $(\mathbf{L}, \lambda)$ be an e-cuspidal pair of $\mathbf{G}$. Set $B := \mathrm{bl}(\lambda)^{\mathbf{G}^F}$ (see the discussion following Definition 7.3.1) and suppose that either:

(i) $\mathrm{Out}(\mathbf{G}^F)_{\mathcal{B}}$ is abelian, where $\mathcal{B}$ denotes the $\widetilde{\mathbf{G}}^F$-orbit of B; or

(ii) B is $\widetilde{\mathbf{G}}^F$-invariant and $\mathcal{E}(\mathbf{G}^F, (\mathbf{L}, \lambda)) = \mathcal{E}(\mathbf{G}^F, B, [s])$ where $\lambda \in \mathcal{E}(\mathbf{L}^F, [s])$ with $s \in \mathbf{L}^{*F^*}_{\mathrm{ss}}$.

Then Definition 10.2.1 (G) is equivalent to the following:

(G') For every $\lambda \in \mathcal{T}$ and every $\chi \in \mathcal{E}(\mathbf{G}^F, (\mathbf{L}, \lambda))$ there exists a $\widetilde{\mathbf{G}}^F$-conjugate χ_0 of χ such that:

(i) $\left(\widetilde{\mathbf{G}}^F \mathcal{A}\right)_{\chi_0} = \widetilde{\mathbf{G}}^F_{\chi_0} \mathcal{A}_{\chi_0}$, and

(ii) χ_0 extends to $\mathbf{G}^F \mathcal{A}_{\chi_0}$.

Proof. Clearly Definition 10.2.1 (G) implies (G') above. Conversely let $\chi \in \mathcal{E}(\mathbf{G}^F, (\mathbf{L}, \lambda))$ and consider a $\widetilde{\mathbf{G}}^F$-conjugate χ_1 of χ satisfying the required properties. As explained in Remark 10.1.5, if $\mathrm{Out}(\mathbf{G}^F)_{\mathcal{B}}$ is abelian, then χ also satisfies the required properties (see [BS20a, Lemma 4.7]) and we set $\chi_0 := \chi$. On the other hand by using the argument of Remark 10.1.5, if $\mathcal{E}(\mathbf{G}^F, (\mathbf{L}, \lambda)) = \mathcal{E}(\mathbf{G}^F, B, [s])$ and B is $\widetilde{\mathbf{G}}^F$-invariant, then $\chi_1 \in \mathcal{E}(\mathbf{G}^F, (\mathbf{L}, \lambda))$ and we set $\chi_0 := \chi_1$. This shows that there exists $\chi_0 \in \mathcal{E}(\mathbf{G}^F, (\mathbf{L}, \lambda))$ and $x \in \widetilde{\mathbf{G}}^F$ such that $\chi_0 = \chi^x$ satisfies the required properties. In particular $\chi_0 \in \mathcal{E}(\mathbf{G}^F, (\mathbf{L}, \lambda)) \cap \mathcal{E}(\mathbf{G}^F, (\mathbf{L}, \lambda)^x)$ and Proposition 7.2.15 implies that $(\mathbf{L}, \lambda) = (\mathbf{L}, \lambda)^{xy}$ for some $y \in \mathbf{G}^F$. It follows that $\chi_0 = \chi^{xy}$ with $xy \in \mathbf{N}_{\widetilde{\mathbf{G}}}(\mathbf{L})^F_\lambda$ as required by Definition 10.2.1 (G). □

We can now prove Theorem 10.1.

Theorem 10.2.4. *Assume Hypothesis 9.2.11 and Hypothesis 8.1.2. Let $\mathbf{L}$ be an e-split Levi subgroup of $\mathbf{G}$ and suppose that the following conditions hold:*

(i) maximal extendibility holds with respect to $\mathbf{G}^F \trianglelefteq \widetilde{\mathbf{G}}^F$ and to $\mathbf{N}_{\mathbf{G}}(\mathbf{L})^F \trianglelefteq \mathbf{N}_{\widetilde{\mathbf{G}}}(\mathbf{L})^F$;

(ii) the requirement from Definition 10.2.1 holds for $\mathbf{L} \leq \mathbf{G}$;

(iii) there exists a $(\widetilde{\mathbf{G}}^F \mathcal{A})_{\mathbf{L}} \ltimes \mathcal{K}$-equivariant extension map for $\mathrm{Cusp}_e(\widetilde{\mathbf{L}}^F)$ with respect to $\widetilde{\mathbf{L}}^F \trianglelefteq \mathbf{N}_{\widetilde{\mathbf{G}}}(\mathbf{L})^F$;

then Condition 9.2.22 holds for every e-cuspidal pair $(\mathbf{L}, \lambda)$ of $\mathbf{G}$.

Proof. Fix an e-cuspidal pair $(\mathbf{L}, \lambda)$ of $\mathbf{G}$. We want to find a bijection $\Omega^{\mathbf{G}}_{(\mathbf{L}, \lambda)}$ as in Condition 9.2.22. Let $\mathcal{T}$ be the $\widetilde{\mathbf{L}}^F$-transversal in $\mathrm{Cusp}_e(\mathbf{L}^F)$ given by Definition 10.2.1. Since N-central isomorphisms of character triples are compatible with conjugation, it is no loss of generality to assume $\lambda \in \mathcal{T}$. Now Assumption 10.1.1 (iii) and (iv) hold by Definition 10.2.1 (G) and (L) respectively, while under Hypothesis 8.1.2 the bijection from Assumption 10.1.1 (ii) exists by Theorem 8.3.5. Since we are assuming Hypothesis 9.2.11, we can apply Theorem 10.1.3 to conclude that Condition 9.2.22 holds for $(\mathbf{L}, \lambda)$ and $\mathbf{G}$. □

The same proof can be used to obtain Theorem 10.2.

Theorem 10.2.5. *Assume Hypothesis 9.2.11 and Hypothesis 8.1.2. Let $\mathbf{L}$ be an e-split Levi subgroup of $\mathbf{G}$, $B \in \mathrm{Bl}(\mathbf{G}^F)$ and suppose that the following conditions hold:*

(i) maximal extendibility holds with respect to $\mathbf{G}^F \trianglelefteq \widetilde{\mathbf{G}}^F$ and to $\mathbf{N}_{\mathbf{G}}(\mathbf{L})^F \trianglelefteq \mathbf{N}_{\widetilde{\mathbf{G}}}(\mathbf{L})^F$;

(ii) the requirement from Definition 10.2.1 holds for $\mathbf{L} \leq \mathbf{G}$;

(iii) there exists a $(\widetilde{\mathbf{G}}^F \mathcal{A})_{\mathbf{L}} \ltimes \mathcal{K}$-equivariant extension map for $\mathrm{Cusp}_e(\widetilde{\mathbf{L}}^F)$ with respect to $\widetilde{\mathbf{L}}^F \trianglelefteq \mathbf{N}_{\widetilde{\mathbf{G}}}(\mathbf{L})^F$;

(iv) the block B satisfies either

(a) $\mathrm{Out}(\mathbf{G}^F)_{\mathcal{B}}$ is abelian, where $\mathcal{B}$ is the $\widetilde{\mathbf{G}}^F$-orbit of B, or

(b) for every subgroup $\mathbf{G}^F \leq H \leq \widetilde{\mathbf{G}}^F$, we have that every block C of H covering B is $\widetilde{\mathbf{G}}^F$-invariant;

then Condition 9.1 holds for every e-cuspidal pair $(\mathbf{L}, \lambda) \in \mathcal{CP}_e(B)$ such that $\mathcal{E}(\mathbf{G}^F, (\mathbf{L}, \lambda)) = \mathcal{E}(\mathbf{G}^F, B, [s])$, where $s \in \mathbf{L}^{*F^*}_{\mathrm{ss}}$ and $\lambda \in \mathcal{E}(\mathbf{L}^F, [s])$.

Proof. Consider an e-cuspidal pair $(\mathbf{L}, \lambda)$ of $\mathbf{G}$ as in the statement. Let $\mathcal{T}$ be the $\widetilde{\mathbf{L}}^F$-transversal in $\mathrm{Cusp}_e(\mathbf{L}^F)$ given by Definition 10.2.1. Since N-central isomorphisms of character triples are compatible with conjugation and the assumption (iv) in the statement is preserved by $\widetilde{\mathbf{G}}^F$-conjugation, it is no loss of generality to assume $\lambda \in \mathcal{T}$. Now Assumption 10.1.4 (iii) and (iv) hold by Definition 10.2.1 (G) and (L) respectively, while under Hypothesis 8.1.2 the bijection from Assumption 10.1.4 (ii) exists by Theorem 8.3.5. Finally notice that Assumption 10.1.4 (v) and (vi) hold by our hypothesis. Since we are assuming Hypothesis 9.2.11 we can apply Theorem 10.1.8 to conclude that Condition 9.1 holds for $(\mathbf{L}, \lambda)$ and $\mathbf{G}$. □

By applying the results of [BS20b] and [Bro] we can now prove Corollary 10.3 and Corollary 10.4.

Proof of Corollary 10.3. Let ℓ, q, $\mathbf{G}$, G, B and $(\mathbf{L}, \lambda)$ as in Corollary 10.3 with $(\mathbf{L}, \lambda)$ an (e, ℓ')-cuspidal pair. Let $\widetilde{\mathbf{G}} := \mathrm{GL}_n(\overline{\mathbb{F}_q})$ and $\widetilde{G} = \mathrm{GL}_n(\epsilon \cdot q)$. We show that Condition 9.1 holds for $(\mathbf{L}, \lambda)$ and $\mathbf{G}$ by an application of Theorem 10.2.5. By assumption and using [DM91, Proposition 13.20] and [CS15, Section 5], we deduce that Theorem 10.2.5 (iv) holds for B. Let s be a semsisimple element such that $\lambda \in \mathcal{E}(\mathbf{L}^F, [s])$. Since s has ℓ'-order, it follows by Theorem 6.2.18 and Theorem 6.2.19 that $\mathcal{E}(G, B, [s]) = \mathcal{E}(G, (\mathbf{L}, \lambda))$ (see also Proposition 7.2.6). Next, observe that Theorem 10.2.5 (i) holds because $\widetilde{G}/G$ is cyclic while Theorem 10.2.5 (iii) holds by [BS20b, Corollary 4.7 (b)]. It remains to check the requirements of Definition 10.2.1. First, by [BS20b, Corollary 4.7] together with the argument used in the proof of [CS17a, Proposition 5.13], we deduce that there exists a $\widetilde{\mathbf{L}}^F$-transversal $\mathcal{T}$ in $\mathrm{Cusp}_e(\mathbf{L}^F)$ such that Definition 10.2.1 (L) holds (see Remark 10.2.2). Moreover, by [CS17a, Theorem 4.1] the requirements of Remark 10.2.3 (G') are satisfied and, under our assumption, we deduce that Definition 10.2.1 (G) is satisfied by using Remark 10.2.3. We can now apply Theorem 10.2.5 to conclude that Condition 9.1 holds for $(\mathbf{L}, \lambda)$ and $\mathbf{G}$. A similar argument shows that Condition 9.2.22 holds for $(\mathbf{L}, \lambda)$ and $\mathbf{G}$ by applying Theorem 10.2.4. □

Proof of Corollary 10.4. Let ℓ, q, $\mathbf{G}$, G and $(\mathbf{L}, \lambda)$ as in Corollary 10.4 with $(\mathbf{L}, \lambda)$ an (e, ℓ')-cuspidal pair. Let $\widetilde{\mathbf{G}} := \mathrm{CSp}_{2n}(\overline{\mathbb{F}_q})$ and $\widetilde{G} := \mathrm{CSp}_{2n}(q)$. We show that Condition 9.2.22 holds for $(\mathbf{L}, \lambda)$ and $\mathbf{G}$ by applying Theorem 10.2.5. Under our assumption, observe that Theorem 10.2.5 (iv.a) is always satisfied. Moreover, if $B := \mathrm{bl}(\lambda)^{\mathbf{G}^F}$ (this is defined by Lemma 9.2.5) and s is a semisimple element of ℓ'-order such that $\lambda \in \mathcal{E}(\mathbf{L}^F, [s])$, then $\mathcal{E}(G, B, [s]) = \mathcal{E}(G, (\mathbf{L}, \lambda))$ by Theorem 6.2.18 and Theorem 6.2.19 (see also Proposition 7.2.6). Since $\widetilde{G}/G$ is cyclic we have Theorem 10.2.5 (i), while Theorem 10.2.5 (iii) holds by [Bro]. We now check the requirements of Definition 10.2.1. By [Bro] together with the argument used in the proof of [CS17a, Proposition 5.13], we obtain a $\widetilde{\mathbf{L}}^F$-transversal $\mathcal{T}$ in $\mathrm{Cusp}_e(\mathbf{L}^F)$ satisfying Definition 10.2.1 (L) (this follows by the same argument used in Remark 10.2.2 applied to the results of [Bro]). Furthermore, by [CS17b, Theorem 3.1] the requirements of Remark 10.2.3 (G') are satisfied and, under our hypothesis, we deduce that Definition 10.2.1 (G) holds by Remark 10.2.3. Finally, by Theorem 10.2.5 we conclude that Condition 9.1 holds for $(\mathbf{L}, \lambda)$ and $\mathbf{G}$. □

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Index