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Linear port-Hamiltonian Systems on Multidimensional Spatial Domains

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Nathanael Skrepek

betreut durch

Prof. Birgit Jacob

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Contents

Symbols

Symbol	**Meaning**	**Page**
$\mathbb{K}$	either $\mathbb{R}$ or $\mathbb{C}$	
$B_r(\zeta_0)$	$\{\zeta \in X \mid \|\zeta - \zeta_0\|_X < r\}$ ball with radius r and center ζ_0 in a normed space X	
M^{H}	complex conjugated transposed (Hermitian transposed) of a matrix M	
$\mathcal{D}(\Omega)$	set of $C^\infty(\Omega)$ functions with compact support	1
$\mathcal{D}'(\Omega)$	(anti)dual space of $\mathcal{D}(\Omega)$	2
$\mathcal{D}(\mathbb{R}^n)\big\vert_\Omega$	$\{f\big\vert_\Omega \mid f \in \mathcal{D}(\mathbb{R}^n)\}$	42
$\mathrm{H_f}$	Heaviside function	2
ν	outward pointing normed normal vector on $\partial\Omega$	5
γ_0	$H^1(\Omega, X) \to \mathsf{L}^2(\partial\Omega, X)$; extension of $f \mapsto f\big\vert_{\partial\Omega}$	42
$\langle\cdot,\cdot\rangle_A$	$\langle\cdot,\cdot\rangle + \langle A\cdot, A\cdot\rangle$ graph inner product of the operator A	24
$\\|\cdot\\|_A$	graph norm of the operator A; corresponding norm to the graph inner product	25
$\langle\!\langle\cdot,\cdot\rangle\!\rangle$	Stoke-Dirac product	12
L_∂	$\sum_{i=1}^n \partial_i L_i$ a differential operator from $\mathsf{L}^2(\Omega)^{m_2}$ to $\mathsf{L}^2(\Omega)^{m_1}$	42
L_∂^{H}	$\sum_{i=1}^n \partial_i L_i^{\mathsf{H}}$	42
$\mathsf{H}(L_\partial, \Omega)$	$\{f \in \mathsf{L}^2(\Omega)^{m_2} \mid L_\partial f \in \mathsf{L}^2(\Omega)^{m_1}\}$ maximal domain of L_∂	43
$\mathsf{H}_0(L_\partial, \Omega)$	The closure of $\mathcal{D}(\Omega)^{m_2}$ in $\mathsf{H}(L_\partial, \Omega)$	43
$\mathsf{H}_{\Gamma_0}(L_\partial^{\mathsf{H}}, \Omega)$	$\ker \bar{\pi}_L^{\Gamma_0}$	82
L_ν	$\sum_{i=1}^n \nu_i L_i \colon \mathsf{L}^2(\partial\Omega)^{m_2} \to \mathsf{L}^2(\partial\Omega)^{m_1}$	43
$\bar{L}_\nu^\Gamma$	$\mathsf{H}(L_\partial, \Omega) \to \mathcal{V}'_{L,\Gamma}$; extension of $\mathbb{1}_\Gamma L_\nu \gamma_0$ on $\mathsf{H}(L_\partial, \Omega)$	84

Symbol	Meaning	Page
$\bar{L}_\nu$	$\bar{L}_\nu^{\partial\Omega} \colon \mathsf{H}(L_\partial, \Omega) \to \mathcal{V}_L'$; extension of $L_\nu \gamma_0$ on $\mathsf{H}(L_\partial, \Omega)$	85
$\mathsf{L}^2_\pi(\Gamma)$	$\overline{\operatorname{ran} \mathbb{1}_\Gamma L_\nu \gamma_0} \subseteq \mathsf{L}^2(\Gamma)^{m_1}$	80
π_L^Γ	$H^1(\Omega)^{m_1} \to \mathsf{L}^2_\pi(\Gamma)$; projection on $\mathsf{L}^2_\pi(\Gamma)$ composed with γ_0	80
π_L	$\pi_L^{\partial\Omega} \colon H^1(\Omega)^{m_1} \to \mathsf{L}^2(\partial\Omega)^{m_1}$	80
$\bar{\pi}_L^\Gamma$	$\mathsf{H}(L_\partial^{\mathsf{H}}, \Omega) \to \mathcal{V}_{L,\Gamma}$; extension of π_L^Γ on $\mathsf{H}(L_\partial^{\mathsf{H}}, \Omega)$	81
$\bar{\pi}_L$	$\bar{\pi}_L^{\partial\Omega} \colon \mathsf{H}(L_\partial^{\mathsf{H}}, \Omega) \to \mathcal{V}_L$	81
M_Γ	$\operatorname{ran} \pi_L^\Gamma$ as subset of $\mathsf{L}^2_\pi(\Gamma)$	81
$\mathcal{V}_{L,\Gamma_1}$	$\operatorname{ran} \bar{\pi}_L\big\vert_{\mathsf{H}_{\Gamma_0}(L_\partial^{\mathsf{H}}, \Omega)}$	82
$\mathcal{V}_L$	$\mathcal{V}_{L,\partial\Omega}$	82
γ_ν	extension of $f \mapsto \nu \cdot f\big\vert_{\partial\Omega}$	103,124
γ_τ	extension of $f \mapsto (\nu \times f\big\vert_{\partial\Omega}) \times \nu$	105,123
$\gamma_{\tau_\times}$	extension of $f \mapsto \nu \times f\big\vert_{\partial\Omega}$	105,123
$\mathsf{H}_{\Gamma_0,0}(\operatorname{rot}, \Omega)$	$\mathsf{H}_{\Gamma_0}(\operatorname{rot}, \Omega) \cap \ker \operatorname{rot}$	123
$\mathsf{H}_{\Gamma_1,0}(\operatorname{div}, \Omega)$	$\mathsf{H}_{\Gamma_1}(\operatorname{div}, \Omega) \cap \ker \operatorname{div}$	123
$\widehat{\mathsf{H}}_{\Gamma_0}(\operatorname{rot}, \Omega)$	$\{E \in \mathsf{L}^2(\Omega) \mid \operatorname{rot} E \in \mathsf{L}^2(\Omega), \gamma_\tau^{\Gamma_0} E \in \mathsf{L}^2(\Gamma_0)\}$	123
$\widehat{\mathsf{H}}_{\Gamma_1}(\operatorname{div}, \Omega)$	$\{E \in \mathsf{L}^2(\Omega) \mid \operatorname{div} E \in \mathsf{L}^2(\Omega), \gamma_\nu^{\Gamma_1} E \in \mathsf{L}^2(\Gamma_1)\}$	123
$\mathcal{H}_{\Gamma_0,\Gamma_1,\varepsilon}(\Omega)$	$\mathsf{H}_{\Gamma_0,0}(\operatorname{rot}, \Omega) \cap \varepsilon^{-1} \mathsf{H}_{\Gamma_1,0}(\operatorname{div}, \Omega)$	123
$\mathcal{H}$	Hamiltonian density	51
$\mathcal{X}_\mathcal{H}$	$\mathsf{L}^2(\Omega)^m$ equipped with $\langle \mathcal{H}\cdot, \cdot \rangle_{\mathsf{L}^2(\Omega)^m}$; the state space	51
$\begin{bmatrix} (\mathcal{H}x)_{L^{\mathsf{H}}} \\ (\mathcal{H}x)_L \end{bmatrix}$	splitting of $\mathcal{H}x$ w.r.t. the dimensions of L	91
$\mathcal{X}_0$	Hilbert space; pivot space of a quasi Gelfand triple	59
$\tilde{D}_+$	dense subspace of $\mathcal{X}_0$ with an alternative inner product	59
D_-	$\left\{ g \in \mathcal{X}_0 \,\middle\vert\, \sup_{g \in \tilde{D}_+ \setminus \{0\}} \frac{\lvert \langle g, f \rangle_{\mathcal{X}_0} \rvert}{\lVert f \rVert_{\mathcal{X}_+}} < +\infty \right\}$	60

Introduction

In this thesis we develop a framework for linear *port-Hamiltonian systems* (PHS) on multidimensional spatial domains that justifies existence and uniqueness of solutions. The inner dynamic of those systems can be described by the following equations

$$\begin{aligned} \frac{\partial}{\partial t} x(t,\zeta) &= \sum_{i=1}^{n} \frac{\partial}{\partial \zeta_i} P_i \mathcal{H}(\zeta) x(t,\zeta) + P_0 \mathcal{H}(\zeta) x(t,\zeta), && \zeta \in \Omega, t \geq 0, \\ x(0,\zeta) &= x_0(\zeta), && \zeta \in \Omega, \end{aligned} \tag{1}$$

where x is the state, P_i and P_0 are matrices, $\mathcal{H}$ is the Hamiltonian density, and Ω is an open subset of $\mathbb{R}^n$ with bounded Lipschitz boundary. We will restrict ourselves to the case, where the matrices P_i have the block shape $\left[\begin{smallmatrix} 0 & L_i \\ L_i^{\mathsf{H}} & 0 \end{smallmatrix}\right]$ for $i \in \{1, \ldots, n\}$.

We also introduce "natural" boundary controls and observations, the so called ports, which make the system a scattering passive (energy preserving) or impedance passive (energy preserving) boundary control system

$$\begin{aligned} u(t,\zeta) &= \mathcal{B}(\zeta)\mathcal{H}(\zeta)x(t,\zeta), && \zeta \in \partial\Omega, t \geq 0, \\ y(t,\zeta) &= \mathcal{C}(\zeta)\mathcal{H}(\zeta)x(t,\zeta), && \zeta \in \partial\Omega, t \geq 0. \end{aligned}$$

Moreover, we are interested in stability/stabilizability of such systems. As showcase we regard the wave equation. However, we conclude what has to be done for general port-Hamiltonian systems. Amongst others a compact embedding of the domain of the differential operator is necessary. We will also show that the Maxwell operator with mixed non-homogeneous boundary conditions satisfies this.

The partial differential equation (PDE) in (1) perfectly matches the description of port-Hamiltonian systems in one spatial dimension in the book of Jacob and Zwart [25], if we set $n = 1$. The additional restriction $P_1 = \left[\begin{smallmatrix} 0 & L_1 \\ L_1^{\mathsf{H}} & 0 \end{smallmatrix}\right]$ is not needed in [25], since the boundary of a line automatically satisfies certain symmetry properties. We decided to not demand an analogous symmetry from Ω in the multidimensional case, because it did not seem very restrictive to ask for $P_i = \left[\begin{smallmatrix} 0 & L_i \\ L_i^{\mathsf{H}} & 0 \end{smallmatrix}\right]$ as all the examples satisfy this anyway. However, it is probably possible to drop this restriction and ask instead for a certain symmetry of the boundary. The references [25, 61] treat the existence and uniqueness of solutions for these port-Hamiltonian systems with a one-dimensional spatial domain.

The port-Hamiltonian formulation has proven to be a powerful tool for the modeling and control of complex multiphysics systems. Port-Hamiltonian systems encode the underlying physical principles such as conservation laws directly into the structure of the system. An introductory overview can be found in [59]. This theory originates from Bernhard M. Maschke and Arjan van der Schaft [36]. For finite-dimensional systems there is by now a well-established theory [58, 14, 13]. The port-Hamiltonian approach has been further extended to the infinite-dimensional situation, see e.g. [60, 30, 32, 26, 67, 61, 25, 28]. In [28] the authors showed that the port-Hamiltonian formulation of the wave equation in n spatial dimensions possesses unique mild and classical solutions.

Chapter 8 of the Ph.D. thesis [61] also regards such port-Hamiltonian systems that have multidimensional spatial domains, but the results demand very strong assumptions on the boundary operators (they have to map into $\mathsf{H}^{1/2}(\partial\Omega)^k$ and its dual respectively), which are not satisfied in case of Maxwell's equations and the Mindlin plate model, as Example 5.1.8 shows for Maxwell's equations. With the following approach we will overcome these limits.

The strategy is to find a boundary triple associated to the differential operator. The multidimensional integration by parts formula

$$\begin{aligned}\int_\Omega \Big\langle \sum_{i=1}^n \frac{\partial}{\partial \zeta_i} L_i x(\zeta), y(\zeta) \Big\rangle \,\mathrm{d}\zeta + \int_\Omega \Big\langle x(\zeta), \sum_{i=1}^n \frac{\partial}{\partial \zeta_i} L_i^{\mathsf{H}} y(\zeta) \Big\rangle \,\mathrm{d}\zeta \\ = \int_{\partial\Omega} \Big\langle \sum_{i=1}^n \nu_i(\zeta) L_i x(\zeta), y(\zeta) \Big\rangle,\end{aligned}$$

where ν_i is the i-th component of the normal vector on the boundary of Ω, already suggests possible operators for a boundary triple (we will show this integrations by parts formula in Lemma 3.1.8). Unfortunately these operators cannot be extended to the entire domain of the differential operator. Hence, we need to adapt the codomain of these boundary operators, which will lead to the construction of suitable boundary spaces for this problem. These boundary spaces behave like a Gelfand triple with the original codomain as pivot space, but lack of a chain inclusion.

To the author's best knowledge there is no earlier theory about this setting. So we will develop the notion of *quasi Gelfand triples* in Chapter 4, which equips us with the tools to state the boundary condition in terms of the pivot space instead of the artificially constructed boundary spaces (Theorem 4.4.6).

One can think of using a quasi boundary triple $(\mathcal{G}, \Gamma_0, \Gamma_1)$ (see [7]) to overcome the extension problem of the boundary mappings, but unfortunately the condition $\ker \Gamma_0$ is self-adjoint (or in this setting skew-adjoint) is in general not satisfied for our class of systems.

The approach to the wave equation in [28] perfectly fits the framework presented in this thesis. In fact, many ideas from [28] are generalized in this work. Also Maxwell's equations can be formulated as such a port-Hamiltonian system and the results in [64] can also be derived with the tools of this framework. Moreover, this theory can be applied on the model of the Mindlin plate in [8, 33]. In Chapter 6 we give examples of how this framework can be applied to these three PDEs.

The core of this thesis has been published in form of an article, see [54]. However, in this thesis we have enough space to deepen some aspects and give extra information. Chapter 7 and Chapter 8 are the result of the papers [24] and [46], respectively.

We start this thesis with some preliminaries. In particular we give a short introduction to distributions and Lipschitz boundaries to be self-contained (up to a certain point) and precise. Then we will introduce a (maybe not entirely standard) concept of dualities of Banach spaces and adjoint operators, that covers both Banach spaces and Hilbert spaces at once. Moreover, this enables us to easily switch between a Banach space adjoint and a Hilbert space adjoint. In this work we work with linear operators from the point of view of linear relations, which is a generalization of linear operators. They can be seen as multi-valued linear operators. This concept is presented in Chapter 2.

Finally, in Chapter 3 we define port-Hamiltonian systems and the corresponding differential operators. We take care of all the technical details of these differential operators. Furthermore, we give relevant examples of this class of PDEs.

In order to develop a suitable solutions theory for these systems we create the concept of *quasi Gelfand triple* in Chapter 4. These triples behave essentially like Gelfand triples, but lack of a chain inclusion. Afterwards in Chapter 5 we construct suitable boundary spaces for our port-Hamiltonian differential operator that establish a quasi Gelfand triple with $\mathsf{L}^2(\partial\Omega)$ as pivot space. This enables us to formulate boundary conditions that admit existence and uniqueness of solutions. Thus, at this point we reach one goal of the thesis.

In Chapter 6 we regard the port-Hamiltonian system with an input and an output function as a *boundary control and observation* system. We will see that certain choices of these inputs and outputs result in well-posed boundary control and observation systems.

We will apply a scattering passive feedback to the wave equation in Chapter 7 and show that this stabilizes the wave equation in a semi-uniform way. This also implies strong stability of the closed loop system. As a by-product we show that the differential operator of this system possesses a compact resolvent.

Finally in Chapter 8 we show a compact embedding of the domain of Maxwell's equations with mixed and non-homogeneous boundary conditions. This can be used to show that the "pure dynamic" (displacement from an equilibrium) of Maxwell's equation is described by an operator with compact resolvent.

Important for Cherry Picker

Here are you few things that might help readers that are only interested in selected chapters.

- We use linear relations instead of operators, i.e. every linear operator $A\colon X \to Y$ will be also treated as linear relation. This means that you will often find something like $\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] \in A$, which just means $Ax = y$. This nuance is sometimes helpful.
- We will always regard the antidual space instead of dual space. This prevents unnecessary inconveniences when switching between a dual pairing and an inner product.
- We define the adjoint with respect to a dual pair. This allows us to treat Banach space adjoints and Hilbert space adjoints with the same framework.

Figure 1 illustrates the dependencies of the chapters. A dashed arrow indicates that it is also understandable without the previous chapters, but some results of the previous chapters might be used.

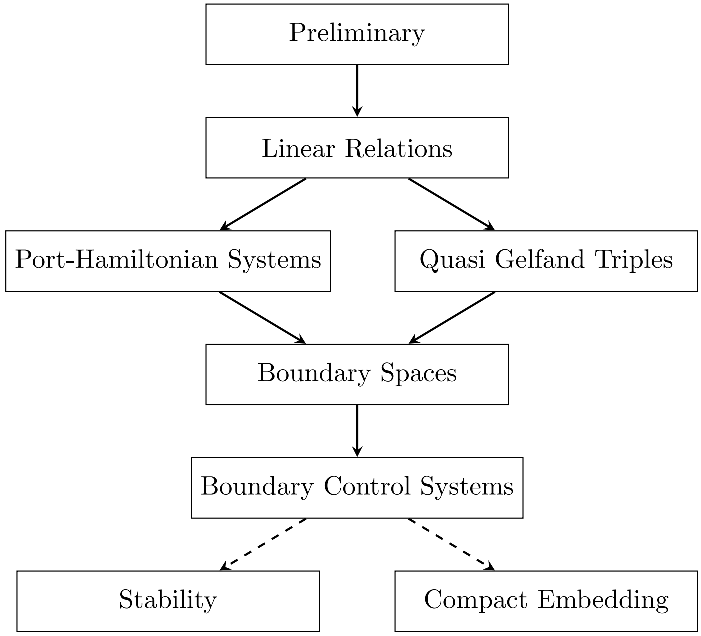

Figure 1: Dependencies

Chapter 1

Preliminaries

Sometimes it can be confusing to pay attention to the antilinear structure of an inner product of a Hilbert space, when switching between the inner product and the dual pairing. Thus, for the sake of clarity we will always consider the antidual space instead of the dual space, which is the space of all continuous antilinear mappings from the topological vector space into its scalar field. Hence, both the inner product and the (anti)dual pairing is linear in one component and antilinear in the other.

1.1 Distributions

In this section we want to recall the definition of the space of distributions and the most important results. For detailed information see [66, ch. I, sec. 8] or [22, ch. II].

Let Ω be an open subset of $\mathbb{R}^n$. Before we can introduce distributions we have to introduce the space of test functions on Ω. We define

$$\mathcal{D}(\Omega) \coloneqq \mathsf{C}_{\mathrm{c}}^{\infty}(\Omega) \coloneqq \{\phi \in \mathsf{C}^{\infty}(\Omega) \,|\, \operatorname{supp}\phi \text{ is compact in } \Omega\}.$$

We use the notation $\mathcal{D}(\Omega)$ instead of $\mathsf{C}_{\mathrm{c}}^{\infty}(\Omega)$, because we will endow this space with a special topology. Note that this space is dense in $\mathsf{L}^p(\Omega)$ for every $p \in [1, \infty)$.

For a multi-index $\alpha \in \mathbb{N}_0^n$ we define $|\alpha| \coloneqq \sum_{i=1}^n \alpha_i$ and

$$\mathrm{D}^{\alpha}\phi \coloneqq \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}}\phi.$$

We want to give an idea of the topology, but for a precise discussion we refer to [22, 66]. For $\Omega \subseteq \mathbb{R}^n$ open and $K \subseteq \Omega$ we define

$$\mathcal{D}_K(\Omega) \coloneqq \{\phi \in \mathcal{D}(\Omega) \,|\, \operatorname{supp}\phi \subseteq K\}.$$

On $\mathcal{D}_K(\Omega)$ we can define the semi norms

$$p_m(\phi) := \sup_{|\alpha|\leq m} \sup_{x\in K} |\mathrm{D}^\alpha \phi(x)|, \quad m \in \mathbb{N}_0.$$

These semi norms establish a topology on $\mathcal{D}_K(\Omega)$, with which $\mathcal{D}_K(\Omega)$ is a locally convex topological vector space. We have

$$\mathcal{D}(\Omega) = \bigcup_{K\subseteq\Omega \text{ compact}} \mathcal{D}_K(\Omega).$$

and we will endow $\mathcal{D}(\Omega)$ with the finest locally convex topology such that all inclusion mappings $\iota_K : \mathcal{D}_K(\Omega) \to \mathcal{D}(\Omega), f \mapsto f$ are continuous.

We can characterize convergence in $\mathcal{D}(\Omega)$ by $\phi_n \to \phi$, if and only if

- there exists a compact $K \subseteq \Omega$ such that $\operatorname{supp}(\phi_n - \phi) \subseteq K$ for all $n \in \mathbb{N}$
- and $\sup_{x\in K} |\mathrm{D}^\alpha(\phi_n - \phi)(x)| \to 0$ for all $\alpha \in \mathbb{N}_0^n$.

Definition 1.1.1. We define the space of distributions $\mathcal{D}'(\Omega)$ as the (anti)dual space of test functions $\mathcal{D}(\Omega)$. For $\Lambda \in \mathcal{D}'(\Omega)$ and $\phi \in \mathcal{D}(\Omega)$ we define

$$\langle \Lambda, \phi \rangle_{\mathcal{D}'(\Omega),\mathcal{D}(\Omega)} := \Lambda(\phi).$$

We will write just $\langle \Lambda, \phi \rangle_{\mathcal{D}',\mathcal{D}}$, if Ω is clear and $\langle \Lambda, \phi \rangle$, if it can't be confused with another dual pairing.

Remark 1.1.2. Every $f \in \mathsf{L}^1_{\mathrm{loc}}(\Omega)$ can be regarded as a distribution by

$$\langle f, \phi \rangle_{\mathcal{D}'(\Omega),\mathcal{D}(\Omega)} = \int_\Omega f \overline{\phi} \, \mathrm{d}\lambda,$$

where λ denotes the Lebesgue measure. Since $\operatorname{supp} \phi \subseteq \Omega$, we can replace the integral over Ω by an integral over $\mathbb{R}^n$, if we extend f outside of Ω with 0. A distribution that can be represented by an $f \in \mathsf{L}^1_{\mathrm{loc}}(\Omega)$ via the previous integral is called *regular*.

Inspired by the integration by parts formula we define $\mathrm{D}^\alpha \Lambda$ for a distribution.

Definition 1.1.3. Let $\Lambda \in \mathcal{D}'(\Omega)$ we define the *distributional derivative* $\mathrm{D}^\alpha \Lambda$ pointwise for every $\phi \in \mathcal{D}(\Omega)$ by

$$\langle \mathrm{D}^\alpha \Lambda, \phi \rangle_{\mathcal{D}'(\Omega),\mathcal{D}(\Omega)} = (-1)^{|\alpha|} \langle \Lambda, \mathrm{D}^\alpha \phi \rangle_{\mathcal{D}'(\Omega),\mathcal{D}(\Omega)}.$$

Note that a distribution is arbitrarily often differentiable (in the distributional sense).

Example 1.1.4. We define the *Heaviside function* $\mathrm{H_f} : \mathbb{R} \to \mathbb{R}$ by

$$\mathrm{H_f}(x) = \begin{cases} 0, & x \leq 0, \\ 1, & x > 0. \end{cases}$$

Clearly, $\mathrm{H_f} = \mathbb{1}_{(0,+\infty)}$. Its distributional derivative can be calculated by

$$\langle \mathrm{H}_\mathrm{f}', \phi\rangle_{\mathcal{D}'(\mathbb{R}),\mathcal{D}(\mathbb{R})} = -\int_{\mathbb{R}} \mathrm{H_f}\overline{\phi'}\,\mathrm{d}\lambda = -\int_0^{+\infty} \overline{\phi'(x)}\,\mathrm{d}x = -\overline{\phi(x)}\Big|_0^{+\infty} = \overline{\phi(0)}$$

where $\phi \in \mathcal{D}(\mathbb{R})$. Note that $\overline{\delta}_0\colon \phi \mapsto \overline{\phi(0)}$ is continuous and antilinear, and therefore an element of $\mathcal{D}'(\mathbb{R})$.

Lemma 1.1.5. *Let $(f_m)_{m\in\mathbb{N}}$ be a sequence in $\mathsf{L}^1_{\mathrm{loc}}(\Omega)$ that converges pointwise to $f \in \mathsf{L}^1_{\mathrm{loc}}(\Omega)$ such that for every compact $K \subseteq \Omega$ there exists an integrable function g_K such that $|f_m(x)| \leq g_K(x)$ for a.e. $x \in K$. Then f_m converges to f in $\mathcal{D}'(\Omega)$, i.e.*

$$\lim_{m\in\mathbb{N}} \langle f_m, \phi\rangle = \langle f, \phi\rangle \quad \textit{for all} \quad \phi \in \mathcal{D}(\Omega).$$

Proof. Let $\phi \in \mathcal{D}(\Omega)$ be arbitrary. Then $\operatorname{supp}\phi$ is compact and therefore there exists an integrable function $g_{\operatorname{supp}\phi}$ such that $|f_m(x)| \leq g_{\operatorname{supp}\phi}(x)$ for a.e. $x \in \operatorname{supp}\phi$. Hence, by Lebesgue's dominated convergence theorem, we have

$$\begin{aligned}\lim_{m\to\infty} |\langle f_m - f, \phi\rangle| &= \lim_{m\to\infty}\left|\int_\Omega (f_m - f)\overline{\phi}\,\mathrm{d}\lambda\right| \\ &\leq \|\phi\|_\infty \lim_{m\to\infty}\int_{\operatorname{supp}\phi} |f_m - f|\,\mathrm{d}\lambda = 0.\end{aligned}$$ ❑

Lemma 1.1.6. *Let $(\Lambda_m)_{m\in\mathbb{N}}$ be a sequence in $\mathcal{D}'(\Omega)$ that converges to $\Lambda \in \mathcal{D}'(\Omega)$ in $\mathcal{D}'(\Omega)$. Then $\mathrm{D}^\alpha\Lambda_m$ converges to $\mathrm{D}^\alpha\Lambda$ in $\mathcal{D}'(\Omega)$ for every $\alpha \in \mathbb{N}_0^n$.*

Proof. Let $\phi \in \mathcal{D}(\Omega)$. Note that also $\mathrm{D}^\alpha\phi \in \mathcal{D}(\Omega)$. Hence,

$$\langle \mathrm{D}^\alpha(\Lambda_m - \Lambda), \phi\rangle = (-1)^{|\alpha|}\langle \Lambda_m - \Lambda, \mathrm{D}^\alpha\phi\rangle \to 0.$$ ❑

Note that every Lipschitz continuous function $f\colon \Omega \to \mathbb{R}$ possesses an almost everywhere defined derivative by Rademacher's theorem, see [1, th. 2.6]. Moreover, if we restrict it on a line in Ω, then this function is absolutely continuous. Hence, by the fundamental theorem of calculus for absolutely continuous function we can derive an integration by parts formula for every $\phi \in \mathcal{D}(\Omega)$ by integrating

$$\frac{\partial}{\partial x_i}(f\phi) = \Big(\frac{\partial}{\partial x_i} f\Big)\phi + f\Big(\frac{\partial}{\partial x_i}\phi\Big)$$

over $\mathbb{R}$. Consequently the distributional derivative $\mathrm{D}^\alpha f$ coincides with $\frac{\partial}{\partial x_i} f$ as distribution for $\alpha = e_i$, where e_i is the n-tuple that with 1 in the i-th entry and 0 else.

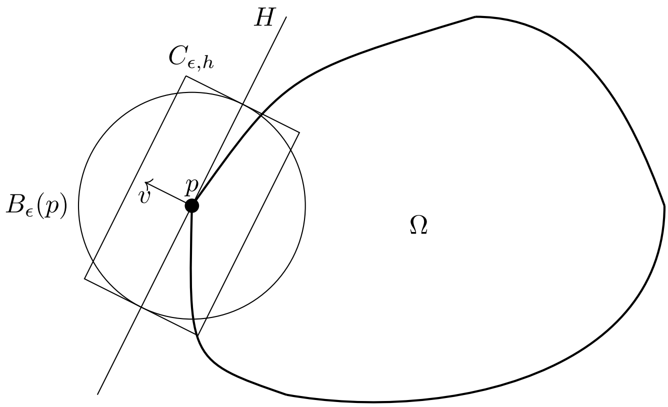

Figure 1.1: Lipschitz boundary

1.2 Lipschitz Boundary

In this thesis we will only deal with *strong Lipschitz boundaries*, hence we will not mention weak Lipschitz boundaries and we will just use the term Lipschitz boundary for strong Lipschitz boundary. Sets with Lipschitz boundaries are nice enough to allow to define an outer normal vector, which will be important to define boundary operators. More details can be found in [21].

Definition 1.2.1. A set $\Omega \subseteq \mathbb{R}^n$ is said to have a *Lipschitz boundary*, if for every $p \in \partial\Omega$ there exists an ϵ, $h > 0$, a hyper plane H in p and a Lipschitz continuous function $g\colon H \cap B_\epsilon(p) \to \mathbb{R}$ such that

$$\begin{aligned} \partial\Omega \cap C_{\epsilon,h} &= \{x + g(x)v \mid x \in H \cap B_\epsilon(p)\}, \\ \Omega \cap C_{\epsilon,h} &= \{x + yv \mid x \in H \cap B_\epsilon(p), -h < y < g(x)\}, \end{aligned}$$

where v is the normal vector on H and $C_{\epsilon,h}$ is the cylinder $\{x + \eta v \mid x \in H \cap B_\epsilon(p), \eta \in (-h, h)\}$.

Figure 1.1 illustrates this definition. Locally the boundary of Ω is the graph of a Lipschitz function. Alternatively, you can regard the hyper plane and its normal vector as an orthogonal basis (a different coordinate system). Sometimes this is used to define Lipschitz boundaries. The condition on $\Omega \cap C_{\epsilon,h}$ makes sure that the surface of Ω is orientable.

Locally we can define an embedding on a Lipschitz boundary by

$$\phi\colon \left\{ \begin{array}{rcl} B_\epsilon(p) \cap H & \to & \partial\Omega, \\ x & \mapsto & x + g(x)v. \end{array} \right.$$

Clearly, since $B_\epsilon(p) \cap H$ is isomorphic to a ball $B_\epsilon(0)$ in $\mathbb{R}^{n-1}$, we can also define a Lipschitz continuous embedding $\tilde{\phi}$, whose domain is $B_\epsilon(0) \subseteq \mathbb{R}^{n-1}$.

Note that for $x \in B_\epsilon(p)$ the vector $x - \langle x, v\rangle v \in H$ and we can characterize the boundary locally at p by the zeros of

$$F\colon \begin{cases} B_\epsilon(p) & \to & \mathbb{R}, \\ x & \mapsto & \langle x-p, v\rangle - g\big(x - \langle x-p, v\rangle v\big). \end{cases}$$

In the coordinate system given by H and v (origin in p) this function can be expressed by

$$\tilde{F}\colon \begin{cases} B_\epsilon(p) & \to & \mathbb{R}, \\ \begin{bmatrix} \zeta \\ \xi \end{bmatrix} & \mapsto & \xi - \tilde{g}(\zeta), \end{cases}$$

where $\tilde{g}$ is an appropriate modification of g. Since g is Lipschitz continuous, also F is Lipschitz continuous.

By Rademacher's theorem [1, th. 2.6] every Lipschitz function is almost everywhere differentiable. Hence, we can define tangential space on almost every point of $\partial\Omega$ and an outward pointing normal vector. Let $(b_i)_{i=1}^{n-1}$ be a basis of H. Then the tangential space of $\partial\Omega$ in almost every q is given by

$$\operatorname{span}\left\{\frac{\partial}{\partial b_1}\phi\big(\phi^{-1}(q)\big), \dots, \frac{\partial}{\partial b_{n-1}}\phi\big(\phi^{-1}(q)\big)\right\}, \tag{1.1}$$

or

$$\operatorname{span}\left\{\frac{\partial}{\partial e_1}\tilde{\phi}\big(\tilde{\phi}^{-1}(q)\big), \dots, \frac{\partial}{\partial e_{n-1}}\tilde{\phi}\big(\tilde{\phi}^{-1}(q)\big)\right\},$$

where $(e_i)_{i=1}^{n}$ is the standard basis of $\mathbb{R}^n$. It can be shown that the tangential space does not depend on the embedding ϕ, i.e. if ψ is another embedding such that q is in the image of ψ, then the corresponding tangential space w.r.t. ψ is the same.

Lemma 1.2.2. *The normal vector is given (almost everywhere) by*

$$\nu\colon \begin{cases} \partial\Omega \cap B_\epsilon(p) & \to & \mathbb{R}^n, \\ q & \mapsto & \frac{(\mathrm{d}F(q))^\mathsf{T}}{\|\mathrm{d}F(q)\|}. \end{cases}$$

Proof. Let $q \in \partial\Omega \cap B_\epsilon(p)$ such that the tangential space exists. By (1.1) $\left\{\frac{\partial}{\partial b_1}\phi\big(\phi^{-1}(q)\big), \dots, \frac{\partial}{\partial b_{n-1}}\phi\big(\phi^{-1}(q)\big)\right\}$ is a basis of the tangential space. Hence, we only have to show that $\nu(q)$ is orthogonal on each basis vector. Let $s = \phi^{-1}(q)$. Then

$$\begin{aligned}\left\langle \mathrm{d}F(q)^\mathsf{T}, \frac{\partial\phi}{\partial b_i}(\phi^{-1}(q))\right\rangle &= \mathrm{d}F(q)\frac{\partial\phi}{\partial b_i}(\phi^{-1}(q)) = \mathrm{d}F(\phi(s))\frac{\partial\phi}{\partial b_i}(s) \\ &= \frac{\partial}{\partial b_i}(F\circ\phi)(s) = 0,\end{aligned}$$

which proves the claim. ❑

By working in the coordinate system centered in p given by an orthogonal basis of H and v we can assume

$$H = \operatorname{span}\{e_1, \dots, e_{n-1}\} \quad \text{and} \quad v = e_n. \tag{1.2}$$

Moreover, we will identify H with $\mathbb{R}^{n-1}$, which allows us to write the embedding ϕ as

$$\phi(x) = \begin{bmatrix} x \\ g(x) \end{bmatrix}.$$

Note that in this new coordinate system $p = 0$. Let $q \in \partial\Omega \cap B_\epsilon(p)$, π denotes the orthogonal projection on H, which is given by $x \mapsto \begin{bmatrix} x_1 & x_2 & \cdots & x_{n-1} \end{bmatrix}^\mathsf{T}$ and $s = \pi(q)$. Then the tangential space in q can by written as

$$\operatorname{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ \frac{\partial g(s)}{\partial e_1} \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \\ \frac{\partial g(s)}{\partial e_2} \end{bmatrix}, \dots, \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \frac{\partial g(s)}{\partial e_{n-1}} \end{bmatrix} \right\}$$

or in a matrix form

$$\operatorname{ran} \begin{bmatrix} E_{n-1} \\ \mathrm{d}g(s) \end{bmatrix},$$

where E_{n-1} denotes the identity matrix in $\mathbb{R}^{n-1}$. Hence, we can easily see that the tangential space exists in every point where $g(s)$ is differentiable. The normal vector in Lemma 1.2.2 is then given by

$$\nu(q) = \frac{1}{\sqrt{1 + \|\nabla g(s)\|^2}} \begin{bmatrix} -\nabla g(s) \\ 1 \end{bmatrix}$$

Theorem 1.2.3. *Let ν be the function given in Lemma 1.2.2. Then $\nu(q)$ points outward Ω for almost every $q \in \partial\Omega \cap C_{\epsilon,h}$.*

Figure 1.2 illustrates the proof.

Proof. Let $q \in \partial\Omega \cap C_{\epsilon,h}$ such that the tangential space exists and $s = \pi(q)$. For an $x \in H \cap B_\epsilon(p)$ we can express the corresponding point on surface of Ω by

$$\begin{bmatrix} x \\ g(x) \end{bmatrix}.$$

Since g is differentiable in s, we have

$$\begin{bmatrix} s - \mu\nabla g(s) \\ g(s - \mu\nabla g(s)) \end{bmatrix} = \begin{bmatrix} s \\ g(s) \end{bmatrix} - \mu \begin{bmatrix} \nabla g(s) \\ \|\nabla g(s)\|^2 \end{bmatrix} + \begin{bmatrix} 0 \\ o(\mu) \end{bmatrix}.$$

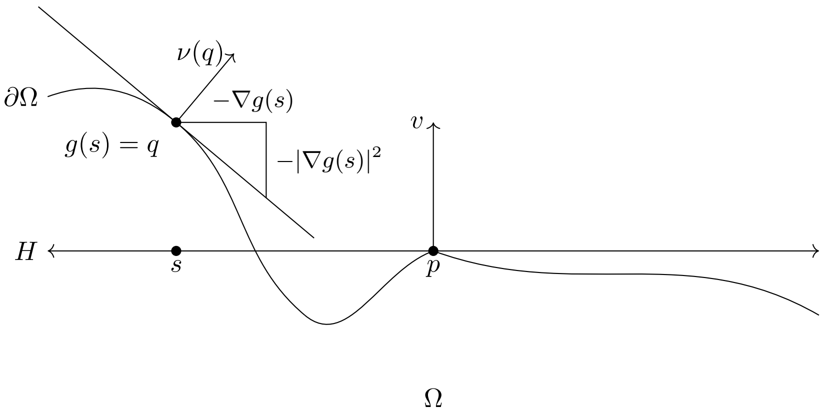

Figure 1.2: Outer normal vector

Hence, for $\mu > 0$ sufficiently small we have $g(s) \geq g(s - \mu\nabla g(s))$, which implies $\begin{bmatrix} s-\mu\nabla g(s) \\ g(s) \end{bmatrix} \notin \Omega$. Consequently,

$$q + \mu\nu(q) = \begin{bmatrix} s \\ g(s) \end{bmatrix} + \mu \begin{bmatrix} -\nabla g(s) \\ 1 \end{bmatrix} \notin \Omega.$$

Therefore, $\nu(q)$ points outward Ω. ❑

Hence, there is a function $\nu \colon \partial\Omega \to \mathbb{R}^n$ that is defined almost everywhere (w.r.t. surface measure of $\partial\Omega$) such that ν is an outward pointing normal vector.

We mentioned the surface measure of $\partial\Omega$ a few times without really saying what it is. Hence, we catch up on this. The set of Borel sets on $\partial\Omega$ can be described by

$$\mathcal{B}(\partial\Omega) = \mathcal{B}(\mathbb{R}^n) \cap \Omega = \{A \cap \Omega \mid A \in \mathcal{B}(\mathbb{R}^n)\},$$

where $\mathcal{B}(\mathbb{R}^n)$ are the Borel sets on $\mathbb{R}^n$. We still assume (1.2). For simplicity we will only define the surface measure on $\partial\Omega \cap C_{\epsilon,h}$, but this can easily be extended on $\partial\Omega$ by a covering consisting of sets $C_{\epsilon,h}$ centered in different points.

Definition 1.2.4. For $A \in \mathcal{B}(\partial\Omega) \cap C_{\epsilon,h}$ we define the *surface measure* of $\partial\Omega$ by

$$\mu(A) := \int_{g^{-1}(\pi_n(A))} \sqrt{1 + \|\nabla g\|^2} \, \mathrm{d}\lambda_{n-1}.$$

where λ_{n-1} denotes the Lebesgue measure in $\mathbb{R}^{n-1}$ and π_n denotes the projection on the n-th coordinate.

Note that $g^{-1}(\pi_n(A)) = \phi^{-1}(A)$. For a measurable function f we can calculate the surface integral by

$$\int_{\partial\Omega \cap C_{\epsilon,h}} f \, \mathrm{d}\mu = \int_{B_\epsilon(0)} f\left(\begin{bmatrix} x \\ g(x) \end{bmatrix}\right) \sqrt{1 + \|\nabla g(x)\|^2} \, \mathrm{d}\lambda_{n-1}(x).$$

In order to prove the *divergence theorem* or *Gauß's theorem*,

$$\int_{\Omega} \operatorname{div} f \,\mathrm{d}\lambda = \int_{\partial\Omega} \nu \cdot f \,\mathrm{d}\mu,$$

we will prove locally $\int_{\Omega} \partial_i \psi \,\mathrm{d}\lambda = \int_{\partial\Omega} \nu_i \psi \,\mathrm{d}\mu$ and then obtain the global result by a partition of unity. Finally, the divergence theorem/Gauß's theorem is just an easy consequence.

Theorem 1.2.5. *Let $\psi \in \mathcal{D}(C_{\epsilon,h})$. Then*

$$\int_{\Omega \cap C_{\epsilon,h}} \partial_i \psi \,\mathrm{d}\lambda = \int_{\partial\Omega \cap C_{\epsilon,h}} \nu_i \psi \,\mathrm{d}\mu$$

for every $i \in \{1, \dots, n\}$.

We will again (without loss of generality) assume (1.2).

Proof. Let $h \in \mathsf{C}^{\infty}(\mathbb{R})$ be such that

$$h(\zeta) \in \begin{cases} 0, & \zeta \in (-\infty, 0), \\ [0,1], & \zeta \in [0,1], \\ 1, & \zeta \in (1, \infty). \end{cases}$$

Figure 1.3 illustrates the function h. We define $h_m(x) := h(mx)$, which converges

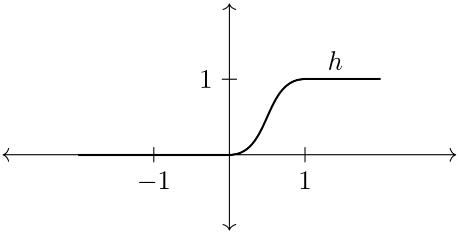

Figure 1.3: The function h

pointwise to the heaviside function $\mathrm{H_f} = \mathbb{1}_{(0,+\infty)}$. By the second condition of Lipschitz boundaries, we have $x \in \Omega \cap C_{\epsilon,h}$ if and only if $x_n < g(\tilde{x})$, where $\tilde{x} = \pi(x)$, the projection of x on the first $n-1$ coordinates. Therefore, we can write $\mathbb{1}_\Omega$ for $x \in C_{\epsilon,h}$ as a pointwise limit

$$\mathbb{1}_\Omega(x) = \lim_{m\to\infty} h\big(m(g(\tilde{x}) - x_n)\big) = \lim_{m\to\infty} h_m(g(\tilde{x}) - x_n) = \lim_{m\to\infty} \tilde{h}_m(x),$$

where $\tilde{h}_m := h_m(g(\tilde{x}) - x_n)$. Hence $\mathbb{1}_\Omega$ regarded as distribution, i.e. as element of $\mathcal{D}'(C_{\epsilon,h})$, is also the limit of $\tilde{h}_m$ (Lemma 1.1.5). The distributional derivatives of $\mathbb{1}_\Omega$ can be written as (Lemma 1.1.6)

$$\frac{\partial}{\partial x_i} \mathbb{1}_\Omega = \lim_{m\to\infty} m w_i \tilde{h}'_m,$$

where $w(\tilde{x}) = \begin{bmatrix} \nabla g(\tilde{x}) \\ -1 \end{bmatrix} = -\sqrt{1+\|\nabla g(\tilde{x})\|^2}\nu\left(\left[\begin{smallmatrix} \tilde{x} \\ g(\tilde{x}) \end{smallmatrix}\right]\right)$ (for a.e. x). For $\psi \in \mathcal{D}(C_{\epsilon,h})$ we have

$$\begin{aligned}
\int_{\Omega\cap C_{\epsilon,h}} \partial_i\psi \,\mathrm{d}\lambda &= -\left\langle \frac{\partial}{\partial x_i}\mathbb{1}_\Omega, \overline{\psi}\right\rangle_{\mathcal{D}'(C_{\epsilon,h}),\mathcal{D}(C_{\epsilon,h})} \\
&= -\lim_{m\to\infty}\int_{\mathbb{R}^{n-1}}\int_{\mathbb{R}} m w_i(\tilde{x})h'(m(g(\tilde{x})-x_n))\psi(\left[\begin{smallmatrix} \tilde{x} \\ x_n \end{smallmatrix}\right])\,\mathrm{d}x_n\,\mathrm{d}\tilde{x} \\
&= \int_{\mathbb{R}^{n-1}} w_i(\tilde{x})\int_{\mathbb{R}}\lim_{m\to\infty}\left[\frac{\partial}{\partial x_n}h(mx_n)\right]\psi\left(\left[\begin{smallmatrix} \tilde{x} \\ g(\tilde{x})-x_n \end{smallmatrix}\right]\right)\mathrm{d}x_n\,\mathrm{d}\tilde{x} \\
&= \int_{\mathbb{R}^{n-1}} w_i(\tilde{x})\left\langle \mathrm{H}_\mathrm{f}', \overline{\psi}\left(\left[\begin{smallmatrix} \tilde{x} \\ g(\tilde{x})-\cdot \end{smallmatrix}\right]\right)\right\rangle_{\mathcal{D}'(\mathbb{R}),\mathcal{D}(\mathbb{R})}\mathrm{d}\tilde{x} \\
&= \int_{\mathbb{R}^{n-1}} \nu_i\left(\left[\begin{smallmatrix} \tilde{x} \\ g(\tilde{x}) \end{smallmatrix}\right]\right)\psi\left(\left[\begin{smallmatrix} \tilde{x} \\ g(\tilde{x}) \end{smallmatrix}\right]\right)\sqrt{1+\|\nabla g(\tilde{x})\|^2}\,\mathrm{d}\tilde{x} \\
&= \int_{\partial\Omega\cap C_{\epsilon,h}} \nu_i\psi\,\mathrm{d}\mu.
\end{aligned}$$
❑

Lemma 1.2.6. *Let $K \subseteq \mathbb{R}^n$ be a compact set and $\Omega \subseteq \mathbb{R}^n$ open with Lipschitz boundary. Then there exists an open covering $(O_j)_{j=0}^k$ of $\overline{\Omega}\cap K$, such that O_j for $j \geq 1$ are cylinders $C_{\epsilon_j,h_j}(p_j)$ that fulfill the conditions in the definition of a Lipschitz boundary (Definition 1.2.1) and $O_0 \subseteq \Omega$.*

Proof. By definition of a Lipschitz boundary for each $p \in \partial\Omega$ there exists a cylinder $C_{\epsilon,h}(p)$ (ϵ and h depend on p). Clearly, $\bigcup_{p\in\partial\Omega\cap K} C_{\epsilon,h}(p)$ is an open covering of the compact set $\partial\Omega\cap K$. Therefore, there exist finitely many p_j such that

$$\bigcup_{j=1}^k C_{\epsilon_j,h_j}(p_j) \supseteq \partial\Omega\cap K.$$

We define O_j as $C_{\epsilon_j,h_j}(p_j)$ for every $j \in \{1,\dots,k\}$. Since the distance δ of $(\Omega\cap K)\setminus\bigcup_{j=1}^k O_j$ to $\partial\Omega$ is positive, the $\frac{\delta}{2}$ neighborhood $B_{\frac{\delta}{2}}((\Omega\cap K)\setminus\bigcup_{j=1}^k O_j)$ of $(\Omega\cap K)\setminus\bigcup_{j=1}^k O_j$ is contained in Ω. We define O_0 as $B_{\frac{\delta}{2}}((\Omega\cap K)\setminus\bigcup_{j=1}^k O_j)$. Then $(O_j)_{j=0}^k$ is the desired open covering of $\overline{\Omega}\cap K$. ❑

Theorem 1.2.7. *Let $\psi \in \mathcal{D}(\mathbb{R})$ and $\Omega \subseteq \mathbb{R}^n$ be open with Lipschitz boundary. Then*

$$\int_\Omega \partial_i\psi\,\mathrm{d}\lambda = \int_{\partial\Omega}\nu_i\psi\,\mathrm{d}\mu$$

for every $i \in \{1,\dots,n\}$.

Proof. We apply Lemma 1.2.6 on $K = \operatorname{supp}\psi$. Then we have an open covering $\overline{\Omega}\cap K$ consisting of $O_0 \subseteq \Omega$ and cylinder $O_j = C_{\epsilon_j,h_j}(p_j)$ for $j \in \{1,\dots,k\}$. We employ a partition of unity and obtain $(\alpha_j)_{j=0}^k$ subordinate to this covering, i.e.

$$\alpha_i \in \mathcal{D}(O_j), \quad \alpha_j(x) \in [0,1], \quad \text{and} \quad \sum_{j=0}^k \alpha_j(x) = 1 \quad \text{for} \quad \zeta \in \overline{\Omega}\cap K.$$

We define $\psi_j = \alpha_j \psi \in \mathcal{D}(O_j)$. Hence, we have $\psi = \sum_{j=0}^k \psi_j$ and

$$\int_\Omega \partial_i \psi \,\mathrm{d}\lambda = \int_{\Omega\cap K} \partial_i \sum_{j=0}^k \psi_j \,\mathrm{d}\lambda = \sum_{j=0}^k \int_{\Omega\cap O_j} \partial_i \psi_j \,\mathrm{d}\lambda.$$

Note that $\int_{\Omega\cap O_0} \partial_i \psi_0 \,\mathrm{d}\lambda = \int_{O_0} \partial_i \psi_0 \,\mathrm{d}\lambda = \int_{\mathbb{R}^n} \partial_i \psi_0 \,\mathrm{d}\lambda = 0$. Therefore, by Theorem 1.2.5 we have

$$\sum_{j=0}^k \int_{\Omega\cap O_j} \partial_i \psi_j \,\mathrm{d}\lambda = \sum_{j=1}^k \int_{\Omega\cap O_j} \partial_i \psi_j \,\mathrm{d}\lambda = \sum_{j=1}^k \int_{\partial\Omega\cap O_j} \nu_i \psi_j \,\mathrm{d}\mu = \int_{\partial\Omega} \nu_i \psi \,\mathrm{d}\mu,$$

which proves the claim. ❑

Theorem 1.2.8 (Gauß's theorem). *Let $\Omega \subseteq \mathbb{R}^n$ be open with Lipschitz boundary and $f \in \mathcal{D}(\mathbb{R}^n)^n$. Then*

$$\int_\Omega \operatorname{div} f \,\mathrm{d}\lambda = \int_{\partial\Omega} \nu \cdot f \,\mathrm{d}\mu.$$

Proof. Note that $f_i \in \mathcal{D}(\mathbb{R}^n)$. Hence, by Theorem 1.2.7

$$\int_\Omega \operatorname{div} f \,\mathrm{d}\lambda = \sum_{i=1}^n \int_\Omega \partial_i f_i \,\mathrm{d}\lambda = \sum_{i=1}^n \int_{\partial\Omega} \nu_i f_i \,\mathrm{d}\mu = \int_{\partial\Omega} \nu \cdot f \,\mathrm{d}\mu.$$ ❑

This result can be extended to a more general class of functions by continuity, e.g. $\mathsf{H}^1(\Omega)^n$, if Ω is bounded. Note that for an unbounded Ω this formula cannot be extended to $\mathsf{H}^1(\Omega)^n$ as shown in [57, Re. 13.7.4]. We will later use this result to introduce an integration by parts formula for a certain class of $\mathsf{L}^2(\Omega)$ functions.

1.3 Dual Pairs

In this section we will introduce the notion of dual pairs, that allows us to treat dualities for Hilbert spaces and for Banach spaces in the same framework.

Definition 1.3.1. Let X and Y be Banach spaces and let $\langle\cdot,\cdot\rangle_{Y,X} : Y \times X \to \mathbb{C}$ be continuous and sesquilinear (linear in the first argument and antilinear in the second argument). We define

$$\Phi\colon \begin{cases} Y & \to & X', \\ y & \mapsto & \langle y, \cdot\rangle_{Y,X}, \end{cases} \quad \text{and} \quad \Psi\colon \begin{cases} X & \to & Y', \\ x & \mapsto & \overline{\langle \cdot, x\rangle_{Y,X}}. \end{cases}$$

If Φ is isometric and bijective, then we say that (X,Y) is a *(anti)dual pair* and $\langle\cdot,\cdot\rangle_{Y,X}$ is its *(anti)dual pairing*.

We define

$$\langle x, y\rangle_{X,Y} \coloneqq \overline{\langle y, x\rangle_{Y,X}},$$

which is again a sesquilinear form.

If also Ψ is isometric and bijective, then we say that (X, Y) is a *complete (anti)dual pair*.

Remark 1.3.2. Since Φ is isometric, the duality mapping of a dual pair (X, Y) satisfies

$$|\langle y, x\rangle_{Y,X}| \leq \|y\|_Y \|x\|_X.$$

Example 1.3.3. There are some "natural" and well-known dual pairs.

- Let X be a Banach space, then (X, X') is a dual pair by the dual pairing

  $$\langle y, x\rangle_{X',X} \coloneqq y(x).$$

  If X is additionally reflexive, then (X, X') is even a complete dual pair.

- Let H be a Hilbert space, then (H, H) is a complete dual pair by its inner product

  $$\langle y, x\rangle_{H,H} \coloneqq \langle y, x\rangle_H.$$

- Let $p \in [1, +\infty)$, $X = \mathsf{L}^p(\Omega)$ and $Y = \mathsf{L}^q(\Omega)$, where $\frac{1}{p} + \frac{1}{q} = 1$. Then (X, Y) is a dual pair with the dual pairing

  $$\langle y, x\rangle_{Y,X} \coloneqq \int_\Omega y\overline{x} \,\mathrm{d}\lambda.$$

  For $p \neq 1$ it is even a complete dual pair. Note that this is not a special case of the first example as $\mathsf{L}^q(\Omega)$ is "only" isomorphic to the dual space of $\mathsf{L}^p(\Omega)$.

Clearly, every dual pair (X, Y) can be identified with (X, X'). However, sometimes such identifications can make things a little bit confusing. Especially for Hilbert spaces H it is (in most cases) more "natural" to regard the dual pair (H, H) instead of (H, H'). Nevertheless, sometimes also for Hilbert spaces it is more convenient to regard another dual pair, e.g. for the Sobolev space $\mathsf{H}^1(\mathbb{R})$ it is more handy to work with $\mathsf{H}^{-1}(\mathbb{R})$. Furthermore, if you deal with both the dual pairs (H, H) and (H, H') simultaneously, then it is less confusing, if you are able to properly distinguish between them, even if the difference is only an isomorphism.

Unfortunately, building this theory is a little bit harder than doing duality theory only for Hilbert spaces, but on the plus side it gives a framework in which the duality of Banach spaces and Hilbert spaces is the same. This is especially an advantage, when it comes to adjoint mappings.

If we do not explicitly choose a dual pair, we work with (X, X'), if X is a Banach space and (H, H) if H is a Hilbert space.

Remark 1.3.4. If (X,Y) is a complete dual pair, then (Y,X) is also a complete dual pair. A Banach space X is reflexive, if and only if there exists a Banach space Y such that (X,Y) is a complete dual pair.

Definition 1.3.5. For a dual pair (X,Y) we define a sesquilinear form on $X \times Y$ by

$$\left\langle\!\!\left\langle \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \right\rangle\!\!\right\rangle_{X\times Y} := \langle x_2, y_1\rangle_{Y,X} + \langle x_1, y_2\rangle_{X,Y}$$

We call this sesquilinear form the *Stokes-Dirac product.* We will just write $\langle\!\langle \cdot,\cdot \rangle\!\rangle$, if the space is clear. If we regard canonical dual pairs like (X,X') or (H,H) for a Hilbert space H we will sometime just write $\langle\!\langle \cdot,\cdot \rangle\!\rangle_X$ and $\langle\!\langle \cdot,\cdot \rangle\!\rangle_H$, respectively. We say similar to inner products $\left[\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right] \perp_{\langle\!\langle,\rangle\!\rangle} \left[\begin{smallmatrix} y_1 \\ y_2 \end{smallmatrix}\right]$, if $\langle\!\langle \left[\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right], \left[\begin{smallmatrix} y_1 \\ y_2 \end{smallmatrix}\right] \rangle\!\rangle = 0$ and correspondingly $\left[\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right] \perp_{\langle\!\langle,\rangle\!\rangle} M$ and $N \perp_{\langle\!\langle,\rangle\!\rangle} M$ for sets $M, N \subseteq X \times X'$.

Lemma 1.3.6. *Let (X,Y) be a dual pair. Then $\langle\!\langle \cdot,\cdot \rangle\!\rangle$ is a non-degenerated sesquilinear form, i.e. $(X\times Y)^{\perp_{\langle\!\langle,\rangle\!\rangle}} = \{0\}$.*

Proof. Let $\left[\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right] \in X \times Y$ be such that $\left[\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right] \perp_{\langle\!\langle,\rangle\!\rangle} X \times Y$. Then, in particular, $\left[\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right] \perp_{\langle\!\langle,\rangle\!\rangle} \left[\begin{smallmatrix} 0 \\ y \end{smallmatrix}\right]$ for all $y \in Y$, which means $0 = \langle\!\langle \left[\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right], \left[\begin{smallmatrix} 0 \\ y \end{smallmatrix}\right] \rangle\!\rangle = \langle x_1, y\rangle_{X,Y}$. This implies $x_1 = 0$, since (X,Y) is a dual pair. Analogously, we can show that $x_2 = 0$. ❑

Clearly, for a complete dual pair (X,Y) it is easy to show that $(X\times Y, Y\times X)$ is a complete dual pair. However, the next lemma shows that there is another complete dual pairing for $X \times Y$, which comes from an indefinite inner product. We endow $X \times Y$ with $\|\left[\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right]\|_{X\times Y} := \sqrt{\|x_1\|_X^2 + \|x_2\|_Y^2}$.

Lemma 1.3.7. *Let (X,Y) be a complete dual pair. Then $(X\times Y, X\times Y)$ with $\langle\!\langle \cdot,\cdot \rangle\!\rangle_{X\times Y}$ is a complete dual pair.*

Proof. Since this is a duality between the space $X \times Y$ and itself, it is enough to show that

$$\Phi\colon \begin{cases} X\times Y & \to & (X\times Y)', \\ \left[\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right] & \mapsto & \langle\!\langle \left[\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right], \cdot \rangle\!\rangle_{X\times Y} \end{cases}$$

is isometric and bijective.

Let $\left[\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right], \left[\begin{smallmatrix} y_1 \\ y_2 \end{smallmatrix}\right] \in X\times Y$. Then by the triangle inequality and Cauchy Schwarz's inequality

$$\begin{aligned} \left|\left\langle\!\!\left\langle \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \right\rangle\!\!\right\rangle\right| &= \left|\langle x_2, y_1\rangle_{Y,X} + \langle x_1, y_2\rangle_{X,Y}\right| \le \|x_2\|_Y \|y_1\|_Y + \|x_1\|_X \|y_2\|_Y \\ &\le \sqrt{\|x_2\|_Y^2 + \|x_1\|_X^2} \cdot \sqrt{\|y_1\|_X^2 + \|y_2\|_Y^2} \\ &= \left\| \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right\|_{X\times Y} \left\| \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \right\|_{X\times Y}. \end{aligned}$$

On the other hand for $x_1 \in X$ there exits a $y_2 \in Y$ with $\|y_2\| = \|x_1\|$ such that $\langle x_1, y_2\rangle = \|x_1\|^2$ and for $x_2 \in Y$ there exists a $y_1 \in Y$ with $\|y_1\| = \|x_2\|$ such that $\langle x_2, y_1\rangle = \|x_2\|^2$ (this is a consequence of the Hahn Banach theorem and the fact that (X, Y) is a complete dual pair). Hence,

$$\left|\left\langle\!\!\left\langle \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \right\rangle\!\!\right\rangle\right| = \|x_1\|_X^2 + \|x_2\|_Y^2 = \left\| \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right\|_{X\times Y}^2 = \left\| \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right\|_{X\times Y} \left\| \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \right\|_{X\times Y},$$

which implies $\|\Phi[\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}]\|_{(X\times Y)'} = \|[\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}]\|_{X\times Y}$.

To show surjectivity let $f \in (X \times Y)'$ be arbitrary. Then

$$f([\begin{smallmatrix} y_1 \\ y_2 \end{smallmatrix}]) = f([\begin{smallmatrix} y_1 \\ 0 \end{smallmatrix}]) + f([\begin{smallmatrix} 0 \\ y_2 \end{smallmatrix}]).$$

Since both parts can be seen as elements of X' and Y', we find $x_2 \in Y$ and $x_1 \in X$ such that

$$f([\begin{smallmatrix} y_1 \\ y_2 \end{smallmatrix}]) = \langle x_2, y_1\rangle_{Y,X} + \langle x_1, y_2\rangle_{X,Y} = \langle\!\langle [\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}], [\begin{smallmatrix} y_1 \\ y_2 \end{smallmatrix}] \rangle\!\rangle.$$

Consequently, $\Phi[\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}] = f$. ❑

Definition 1.3.8. Let (X, Y) be a dual pair, $M \subseteq X$ and $N \subseteq Y$. Then we define the *annihilators* of M and N by

$$\begin{aligned} M^{\perp_Y} &:= \{y \in Y \mid \langle y, x\rangle_{Y,X} = 0 \ \forall x \in M\}, \\ {}^{\perp_X}N &:= \{x \in X \mid \langle y, x\rangle_{Y,X} = 0 \ \forall y \in N\}. \end{aligned}$$

We will just write $M^\perp$ and ${}^\perp N$, if the dual pair is clear. If (X, Y) is a complete dual pair we also write $N^\perp$ for ${}^\perp N$.

Note that for a complete dual pair (X, Y) also (Y, X) is a complete dual pair. Hence, the notation $N^{\perp_X}$ for ${}^{\perp_X}N$ is justified for $N \subseteq Y$ as both describe the same set.

The next theorem is a translation of [53, Theorem 4.7] in our notation. Nevertheless we will present a proof.

Proposition 1.3.9. *Let (X, Y) be a complete dual pair, and $M \subseteq X$ and $N \subseteq Y$ be subspaces. Then*

$$M^{\perp\perp} = \overline{M} \quad \textit{and} \quad N^{\perp\perp} = \overline{N}.$$

Proof. It is sufficient to show $M^{\perp\perp} = \overline{M}$ as the second assertion follows from the first applied to the complete dual pair (Y, X).

It is obvious that $M \subseteq M^{\perp\perp}$ and since $M^{\perp\perp}$ is closed, we conclude $\overline{M} \subseteq M^{\perp\perp}$. On the other hand, if $x \notin \overline{M}$, then we can separate x and M by a Hahn-Banach theorem ([53, Theorem 3.5]) with a functional ψ such that $\psi(x) = 1$ and $\psi(M) = 0$. This Ψ can be represented by $\langle y, \cdot\rangle_{Y,X}$ for a $y \in Y$. This y satisfies $y \in M^\perp$ and $\langle y, x\rangle_{Y,X} \neq 0$, therefore $x \notin M^{\perp\perp}$. ❑

Chapter 2

Linear Relations

In this chapter we will introduce linear relations, which can be seen as a generalization of linear operators or as multi-valued linear operators. Although it may be possible to completely avoid this concept, it is worth to use it, as otherwise proofs can become cumbersome and some interesting links will stay hidden. We will introduce well-known concepts like adjoints, skew-symmetry, dissipativity and the Cayley transform for linear relations. Then we will present the most important results on boundary triples (for our purposes). Finally, we recall strongly continuous semigroups and in particular contraction semigroups.

2.1 Basics

Definition 2.1.1. Let X, Y be two vector spaces over the same scalar field. Then we will call a subspace T of $X \times Y$ a *linear relation* between X and Y. A linear relation between X and X will be called a linear relation on X.

Remark 2.1.2. Every linear operator $T\colon X \to Y$ can be identified by a linear relation by considering the graph of T. In fact, if we consider mappings from X to Y as subsets of $X \times Y$ then T is already a linear relation. On the other hand not every linear relation comes from an operator, as $\{0\} \times Y$ demonstrates the most degenerate example.

The statement $\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] \in T$ can be interpreted as Tx"="y, if T comes from a linear operator, this is also its literal meaning. However, for a general linear relation y is not uniquely determined by x. So from a multi-valued operator perspective this can be interpreted as $y \in Tx$.

Definition 2.1.3. For a linear relation T between the vector spaces X and Y we define

- $\operatorname{dom} T := \{x \in X \mid \exists\, y \in Y \text{ such that } \left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] \in T\}$ the *domain* of T,
- $\operatorname{ran} T := \{y \in Y \mid \exists\, x \in X \text{ such that } \left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] \in T\}$ the *range* of T,

- $\ker T := \{x \in X \mid \left[\begin{smallmatrix} x \\ 0 \end{smallmatrix}\right] \in T\}$ the *kernel* of T,
- $\operatorname{mul} T := \{y \in Y \mid \left[\begin{smallmatrix} 0 \\ y \end{smallmatrix}\right] \in T\}$ the *multi-value-part* of T.

We say T is *single-valued* or a *linear operator*, if $\operatorname{mul} T = \{0\}$.

Remark 2.1.4. Every linear relation T which satisfies $\operatorname{mul} T = \{0\}$ can be regarded as a linear mapping T defined on $\operatorname{dom} T$, where $Tx = y$ is well defined by $\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] \in T$.

Definition 2.1.5. Let X, Y, Z be vector spaces and S, T be linear relations between X and Y, and R a linear relation between Y and Z.

- $S + T := \{\left[\begin{smallmatrix} x \\ y_1 + y_2 \end{smallmatrix}\right] \in X \times Y \mid \left[\begin{smallmatrix} x \\ y_1 \end{smallmatrix}\right] \in S \text{ and } \left[\begin{smallmatrix} x \\ y_2 \end{smallmatrix}\right] \in T\}$,
- $\lambda T := \{\left[\begin{smallmatrix} x \\ \lambda y \end{smallmatrix}\right] \in X \times Y \mid \left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] \in T\}$,
- $T^{-1} := \{\left[\begin{smallmatrix} y \\ x \end{smallmatrix}\right] \in Y \times X \mid \left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] \in T\}$,
- $RS := \{\left[\begin{smallmatrix} x \\ z \end{smallmatrix}\right] \in X \times Z \mid \exists\, y \in Y \text{ such that } \left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] \in S \text{ and } \left[\begin{smallmatrix} y \\ z \end{smallmatrix}\right] \in R\}$.

It is easy to check that the sets defined in the previous definition are also linear relations. Furthermore, if S, T and R are linear operators, then the previous definition coincide with the usual definition of addition, scalar multiplication, inverse and composition.

Definition 2.1.6. For a Banach space $(X, \|\cdot\|)$ and a linear relation A on X, we define

- $\rho(A) := \{\lambda \in \mathbb{C} \cup \{\infty\} \mid (A - \lambda)^{-1} \in \mathcal{L}_\mathrm{b}(X)\}$ as the *resolvent set*,
- $\sigma(A) := (\mathbb{C} \cup \{\infty\}) \setminus \rho(A)$ as the *spectrum*,
- $\sigma_\mathrm{p}(A) := \{\lambda \in \mathbb{C} \cup \{\infty\} \mid \ker(A - \lambda)^{-1} \neq \{0\}\}$ as *point spectrum*, and
- $r(A) := \{\lambda \in \mathbb{C} \cup \{\infty\} \mid (A - \lambda)^{-1} \in \mathcal{L}_\mathrm{b}(\operatorname{ran}(A - \lambda), X)\}$ as the *points of regular type*,

where $\operatorname{ran}(A - \lambda)$ is endowed with the norm of X and we set $(T - \infty)^{-1} := T$ and $\operatorname{ran}(T - \infty) := \operatorname{dom} T$.

Note that definition of $(A - \infty)^{-1}$ is just to ensure that $\infty \in \sigma(A)$, if A is unbounded.

Definition 2.1.7. Let X be a vector space over $\mathbb{C}$ and $M = \left[\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right] \in \mathbb{C}^{2\times 2}$, then we define the mapping $\tau_M \colon X \times X \to X \times X$ by

$$\tau_M \begin{bmatrix} x \\ y \end{bmatrix} := \begin{bmatrix} \delta I & \gamma I \\ \beta I & \alpha I \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \delta x + \gamma y \\ \beta x + \alpha y \end{bmatrix}.$$

Lemma 2.1.8. *For $M, N \in \mathbb{C}^{2\times 2}$ we have $\tau_M \tau_N = \tau_{MN}$ and therefore, for invertible M also $\tau_{M^{-1}} = {\tau_M}^{-1}$.*

Proof. Note that

$$\tau_M \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} M \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

Hence,

$$\tau_M \tau_N = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} M \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} N \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} MN \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \tau_{MN}.$$

From the already shown, we can immediately conclude that $\tau_{M^{-1}} = \tau_M^{-1}$. ❑

Lemma 2.1.9. *Let A be a linear relation on a vector space X and $M = \left[\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right] \in \mathbb{C}^{2\times 2}$. If $\operatorname{mul} A = \{0\}$, then*

$$\tau_M(A) = (\alpha A + \beta I)(\gamma A + \delta I)^{-1}.$$

Proof. Let $\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right] \in \tau_M(A)$. Then there exists a $\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] \in A$ such that $\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right] = \left[\begin{smallmatrix} \delta x + \gamma y \\ \beta x + \alpha y \end{smallmatrix}\right]$. By the definition of the addition and multiplication by a scalar for linear relations we have $\left[\begin{smallmatrix} x \\ \alpha y + \beta x \end{smallmatrix}\right] \in (\alpha A + \beta I)$, $\left[\begin{smallmatrix} x \\ \gamma y + \delta x \end{smallmatrix}\right] \in (\gamma A + \delta I)$ and therefore $\left[\begin{smallmatrix} \gamma y + \delta x \\ x \end{smallmatrix}\right] \in (\gamma A + \delta I)^{-1}$. Consequently $\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right] \in (\alpha A + \beta I)(\gamma A + \delta I)^{-1}$.

On the other hand let $\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right] \in (\alpha A + \beta I)(\gamma A + \delta I)^{-1}$. Then there exists a $x \in \operatorname{dom} A$ such that $\left[\begin{smallmatrix} a \\ x \end{smallmatrix}\right] \in (\gamma A + \delta I)^{-1}$ and $\left[\begin{smallmatrix} x \\ b \end{smallmatrix}\right] \in (\alpha A + \beta I)$. Since $\operatorname{mul} A = \{0\}$, there exists a unique $y \in X$ such that $\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] \in A$. Hence, $a = \gamma y + \delta x$ and $b = \alpha y + \beta x$ and consequently $\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right] \in \tau_M(A)$. ❑

Remark 2.1.10. For $M = \left[\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right] \in \mathbb{C}^{2\times 2}$ with $\det M \neq 0$ we have the Möbius transformation

$$\phi_M(z) = \frac{\alpha z + \beta}{\gamma z + \delta} = (\alpha z + \beta)(\gamma z + \delta)^{-1}.$$

By Lemma 2.1.9, we can see that $\phi_M(A) := (\alpha A + \beta)(\gamma A + \delta)^{-1} = \tau_M(A)$ for any linear relation A with $\operatorname{mul} A = \{0\}$.

Hence, the previous definition of $\tau_M(A)$ can be seen as the Möbius transformation of a linear relation.

2.2 Adjoint Linear Relations

We will introduce a slightly more general approach to the adjoint of a linear relation (or operator). This is again a nuance coming from a proper distinction of identifications of dual spaces. Clearly, all adjoints for different (isomorphic) dual spaces are isomorphic in some sense, nevertheless this differentiation can sometimes reveal details, that are otherwise hard to spot.

Definition 2.2.1. Let (X_1, Y_1), (X_2, Y_2) be dual pairs and A a linear relation between X_1 and X_2. Then we define the *adjoint linear relation* by

$$A^{*_{Y_2 \times Y_1}} := \left\{ \begin{bmatrix} y_2 \\ y_1 \end{bmatrix} \in Y_2 \times Y_1 \,\middle|\, \langle y_2, x_2 \rangle_{Y_2,X_2} = \langle y_1, x_1 \rangle_{Y_1,X_1} \text{ for all } \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in A \right\}.$$

We will just write A^*, if the dual pairs are clear.

For a Banach space X, we will regard the dual pair (X, X') for the adjoint, if no other dual pair is given. Similar, for a Hilbert space H we will regard the dual pair (H, H), if no other dual pair is given.

Remark 2.2.2. Let H_1, H_2 be Hilbert spaces and let us regard the natural complete dual pairs (H_1, H_1) and (H_2, H_2). Then the adjoint of a densely defined linear operator $A\colon H_1 \to H_2$ can be characterized by

$$\begin{bmatrix} y_2 \\ y_1 \end{bmatrix} \in A^* \quad \Leftrightarrow \quad \langle y_2, Ax \rangle_{H_2} = \langle y_1, x \rangle_{H_1} \quad \text{for all} \quad x \in \operatorname{dom} A.$$

This matches the usual definition of a Hilbert space adjoint, if we regard y_1 as $A^* y_2$

$$\langle y_2, Ax \rangle_{H_2} = \langle A^* y_2, x \rangle_{H_1}.$$

In fact we will later see that for a densely defined linear relation its adjoint is an operator.

In the operator case the next lemma is sometimes used as the definition of $\operatorname{dom} A^*$.

Lemma 2.2.3. *Let A be an operator ($\operatorname{mul} A = \{0\}$). Then we can characterize the domain of A^* by*

$$x \in \operatorname{dom} A^* \quad \Leftrightarrow \quad \operatorname{dom} A \ni u \mapsto \langle x, Au \rangle_{Y_2,X_2} \textit{ is continuous w.r.t. } \|\cdot\|_{X_1}.$$

Proof. If $x \in \operatorname{dom} A^*$, then there exists (at least one) $y \in Y_1$ such that

$$\langle x, Au \rangle_{Y_2,X_2} = \langle y, u \rangle_{Y_1,X_1} \quad \text{for all} \quad u \in \operatorname{dom} A.$$

Hence, $u \mapsto \langle x, Au \rangle_{Y_2,X_2}$ is bounded by $\|y\|_{Y_1}$ and therefore continuous.

If $\phi\colon \operatorname{dom} A \to \mathbb{C}, u \mapsto \langle x, Au \rangle_{Y_2,X_2}$ is continuous, then we can extend this mapping by continuity on $\overline{\operatorname{dom} A}$. By Hahn-Banach we can further continuously extend this on X_1 (not necessarily uniquely), denoted by $\hat{\phi}$. Since (X_1, Y_1) is a dual pair, there exists a $y \in Y_1$ such that $\hat{\phi}(\cdot) = \langle y, \cdot \rangle_{Y_1,X_1}$. Hence,

$$\langle x, Au \rangle_{Y_2,X_2} = \hat{\phi}(u) = \langle y, u \rangle_{Y_1,X_1}$$

which implies $\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] \in A^*$ and $x \in \operatorname{dom} A^*$. ❑

Lemma 2.2.4. *Let (X_1, Y_1), (X_1, Z_1), (X_2, Y_2) and (X_2, Z_2) be dual pairs and $\Psi_1 \colon Y_1 \to Z_1$ and $\Psi_2 \colon Y_2 \to Z_2$ be the isomorphisms between Y_1 and Z_1, and Y_2 and Z_2, respectively. Then we have for a linear relation between X_1 and X_2*

$$A^{*_{Z_2 \times Z_1}} = \Psi_1 A^{*_{Y_2 \times Y_1}} \Psi_2^{-1} = \begin{bmatrix} \Psi_2 & 0 \\ 0 & \Psi_1 \end{bmatrix} A^{*_{Y_2 \times Y_1}}.$$

Proof. By the definition of the adjoint relation we have

$$\begin{aligned}
\begin{bmatrix} x \\ y \end{bmatrix} \in A^{*Y_2\times Y_1} \quad &\Leftrightarrow \quad \langle x, v\rangle_{Y_2,X_2} = \langle y, u\rangle_{Y_1,X_1} \quad \forall \begin{bmatrix} u \\ v \end{bmatrix} \in A \\
&\Leftrightarrow \quad \langle \Psi_2 x, v\rangle_{Z_2,X_2} = \langle \Psi_1 y, u\rangle_{Z_1,X_1} \quad \forall \begin{bmatrix} u \\ v \end{bmatrix} \in A \\
&\Leftrightarrow \quad \begin{bmatrix} \Psi_2 x \\ \Psi_1 y \end{bmatrix} \in A^{*Z_2\times Z_1}.
\end{aligned}$$

This implies the claim. ❑

Remark 2.2.5. Let (X, Y) be a complete dual pair. For a linear relation A between X and Y we use the dual pairs (X, Y) and (Y, X) such that the adjoint relation $A^{*X\times Y}$ is also between X and Y. In this case we can characterize the adjoint relation by

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \in -A^* \quad \Leftrightarrow \quad \left\langle\!\!\left\langle \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \right\rangle\!\!\right\rangle_{X\times Y} = 0 \quad \text{for all} \quad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in A,$$

or shorter by

$$-A^* = A^{\perp\langle\!\langle,\rangle\!\rangle}.$$

Lemma 2.2.6. *Let (X_1, Y_1), (X_2, Y_2) be dual pairs and let A be a linear relation between X_1 and X_2. Then*

- $(-A)^{-1} = -A^{-1}$,
- $(-A)^{\perp_{Y_1\times Y_2}} = -A^{\perp_{Y_1\times Y_2}}$ *and*
- $(A^{-1})^{\perp_{Y_2\times Y_1}} = (A^{\perp_{Y_1\times Y_2}})^{-1}$.

Proof. We show $(-A)^{-1} = -A^{-1}$ by

$$\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] \in (-A)^{-1} \Leftrightarrow \left[\begin{smallmatrix} y \\ -x \end{smallmatrix}\right] \in A \Leftrightarrow \left[\begin{smallmatrix} -y \\ x \end{smallmatrix}\right] \in A \Leftrightarrow \left[\begin{smallmatrix} x \\ -y \end{smallmatrix}\right] \in A^{-1} \Leftrightarrow \left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] \in -A^{-1}.$$

The second assertion $(-A)^\perp = -A^\perp$ follows from

$$\begin{aligned}
\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] \in (-A)^\perp &\Leftrightarrow \langle \left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right], \left[\begin{smallmatrix} u \\ -v \end{smallmatrix}\right]\rangle_{Y_1\times Y_2, X_1\times X_2} = 0 \quad \forall \left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right] \in A \\
&\Leftrightarrow \langle \left[\begin{smallmatrix} x \\ -y \end{smallmatrix}\right], \left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right]\rangle_{Y_1\times Y_2, X_1\times X_2} = 0 \quad \forall \left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right] \in A \\
&\Leftrightarrow \left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] \in -(A^\perp).
\end{aligned}$$

Finally, $(A^{-1})^\perp = (A^\perp)^{-1}$ can be seen by

$$\begin{aligned}
\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] \in (A^{-1})^\perp &\Leftrightarrow \langle \left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right], \left[\begin{smallmatrix} v \\ u \end{smallmatrix}\right]\rangle_{Y_1\times Y_2, X_1\times X_2} = 0 \quad \forall \left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right] \in A \\
&\Leftrightarrow \langle \left[\begin{smallmatrix} y \\ x \end{smallmatrix}\right], \left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right]\rangle_{Y_2\times Y_1, X_2\times X_1} = 0 \quad \forall \left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right] \in A \\
&\Leftrightarrow \left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] \in (A^\perp)^{-1}.
\end{aligned}$$

❑

Proposition 2.2.7. *Let (X_1, Y_1), (X_2, Y_2) be dual pairs and A a linear relation between X_1 and X_2. Then we have the following identities*

$$A^* = ((-A)^{-1})^\perp = -(A^{-1})^\perp = -(A^\perp)^{-1}.$$

Moreover, A^* is closed.

Proof. Note that

$$\langle y, u\rangle_{Y_1,X_1} + \langle x, v\rangle_{Y_2,X_2} = \langle [\begin{smallmatrix} y \\ x \end{smallmatrix}], [\begin{smallmatrix} u \\ v \end{smallmatrix}]\rangle_{Y_1\times Y_2, X_1\times X_2}$$

Therefore we can reformulate the condition in the definition of A^*

$$A^* = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in Y_2 \times Y_1 \,\middle|\, \left\langle \begin{bmatrix} -y \\ x \end{bmatrix}, \begin{bmatrix} u \\ v \end{bmatrix} \right\rangle = 0 \text{ for all } \begin{bmatrix} u \\ v \end{bmatrix} \in A \right\} = (-A^\perp)^{-1}.$$

The other characterizations follow from Lemma 2.2.6. The closedness follows from the closedness of the annihilator. ❑

Lemma 2.2.8. *Let (X_1, Y_1), (X_2, Y_2) and (X_3, Y_3) be dual pairs and A a linear relation between X_1 and X_2. Then*

(i) $\operatorname{mul} A^* = (\operatorname{dom} A)^\perp$, $\ker A^* = (\operatorname{ran} A)^\perp$,

(ii) $(BA)^* \supseteq A^*B^*$ *for all linear relations B between X_2 and X_3,*

(iii) $(BA)^* = A^*B^*$ *for all operators $B \in \mathcal{L}_\mathrm{b}(X_2, X_3)$,*

Proof.

(i) By the definition of A^*, we have

$$\begin{aligned} \operatorname{mul} A^* &= \left\{ y \in Y_2 \,\middle|\, [\begin{smallmatrix} 0 \\ y \end{smallmatrix}] \in A^* \right\} = \left\{ y \in Y_2 \,\middle|\, \overbrace{\langle 0, v\rangle}^{=0} = \langle y, u\rangle \text{ for all } [\begin{smallmatrix} u \\ v \end{smallmatrix}] \in A \right\} \\ &= (\operatorname{dom} A)^\perp, \end{aligned}$$

$$\begin{aligned} \ker A^* &= \left\{ x \in Y_1 \,\middle|\, [\begin{smallmatrix} x \\ 0 \end{smallmatrix}] \in A^* \right\} = \left\{ x \in Y_1 \,\middle|\, \langle x, v\rangle = \overbrace{\langle 0, u\rangle}^{=0} \text{ for all } [\begin{smallmatrix} u \\ v \end{smallmatrix}] \in A \right\} \\ &= (\operatorname{ran} A)^\perp. \end{aligned}$$

(ii) If $[\begin{smallmatrix} x \\ y \end{smallmatrix}] \in A^*B^*$, then there exist a $z \in Y_2$ such that $[\begin{smallmatrix} x \\ z \end{smallmatrix}] \in B^*$ and $[\begin{smallmatrix} z \\ y \end{smallmatrix}] \in A^*$. Moreover,

$$\begin{aligned} \langle x, w\rangle_{Y_3,X_3} &= \langle z, v\rangle_{Y_2,X_2} \quad \text{for all} \quad [\begin{smallmatrix} v \\ w \end{smallmatrix}] \in B, \\ \langle z, v\rangle_{Y_2,X_2} &= \langle y, u\rangle_{Y_1,X_1} \quad \text{for all} \quad [\begin{smallmatrix} u \\ v \end{smallmatrix}] \in A. \end{aligned}$$

Hence, $\langle x, w\rangle_{Y_3,X_3} = \langle y, u\rangle_{Y_1,X_1}$ for all $[\begin{smallmatrix} u \\ w \end{smallmatrix}] \in BA$ and consequently $[\begin{smallmatrix} x \\ y \end{smallmatrix}] \in (BA)^*$.

(iii) Since B is an everywhere defined operator, we can write $BA = \{[\begin{smallmatrix} u \\ Bv \end{smallmatrix}] \mid [\begin{smallmatrix} u \\ v \end{smallmatrix}] \in A\}$. Therefore,

$$(BA)^* = \left\{[\begin{smallmatrix} x \\ y \end{smallmatrix}] \in Y_3 \times Y_1 \,\middle|\, \langle x, Bv\rangle_{Y_3,X_3} = \langle y, u\rangle_{Y_1,X_1} \text{ for all } [\begin{smallmatrix} u \\ v \end{smallmatrix}] \in A\right\}.$$

If $[\begin{smallmatrix} x \\ y \end{smallmatrix}] \in (BA)^*$, then

$$\langle x, Bv\rangle_{\mathcal{K}_3} = \langle B^*x, v\rangle_{\mathcal{K}_2} = \langle y, u\rangle_{\mathcal{K}_1} \quad \text{for all} \quad [\begin{smallmatrix} u \\ v \end{smallmatrix}] \in A,$$

and in turn $\left[\begin{smallmatrix} B^*x \\ y \end{smallmatrix}\right] \in A^*$. Clearly, we also have $[\begin{smallmatrix} x \\ B^*x \end{smallmatrix}] \in B^*$. Hence $[\begin{smallmatrix} x \\ y \end{smallmatrix}] \in A^*B^*$.

❑

For complete dual pairs (X_1, Y_1) and (X_2, Y_2) the adjoint of a linear relation A between X_1 and X_2 is defined by $A^* = A^{*Y_2\times Y_1}$. However, we can also define the adjoint of a linear relation B between Y_2 and Y_1 by $B^* = B^{*X_1\times X_2}$ as (Y_2, X_2) and (Y_1, X_1) are also dual pairs. Therefore, we can take the double adjoint of A which is

$$A^{**} = (A^{*Y_2\times Y_1})^{*X_1\times X_2}.$$

The next lemma will show that this is just the closure of A in $X_1 \times X_2$.

Lemma 2.2.9. *Let (X_1, Y_1), (X_2, Y_2) be complete dual pairs and A a linear relation between X_1 and X_2. Then*

$$A^{**} = \overline{A}.$$

Proof. By the identities in Proposition 2.2.7 and Proposition 1.3.9 we have

$$A^{**} = \big(-(A^\perp)^{-1}\big)^* = \Big(\big((A^\perp)^{-1}\big)^{-1}\Big)^\perp = A^{\perp\perp} = \overline{A}.$$ ❑

In some sense a linear relation on a complete dual pair (X, Y), i.e. A is a linear relation between X and Y, is the closest thing to a linear relation on a Hilbert space. Note that A^* is again a linear relation between X and Y. Hence, we can define things like symmetry.

Definition 2.2.10. Let (X, Y) be a complete dual pair and A a linear relation between X and Y. We call A

- *symmetric*, if $A \subseteq A^*$ and *self-adjoint*, if $A = A^*$.
- *skew-symmetric*, if $A \subseteq -A^*$ and *skew-adjoint*, if $A = -A^*$.
- *dissipative*, if $\operatorname{Re}\langle y, x\rangle_{Y,X} \leq 0$ for all $[\begin{smallmatrix} x \\ y \end{smallmatrix}] \in A$ and *maximal dissipative*, if A is dissipative and there is no proper dissipative extension of A.
- *accretive*, if $-A$ is dissipative and *maximal accretive*, if $-A$ is maximal dissipative.

Note that, if H is a Hilbert space and we regard the dual pair (H,H) and a linear relation A between H and H, then the previous definition coincides with the standard definition in the literature. However, for Banach spaces dissipativity and accretivity are usually defined differently.

Remark 2.2.11. If A is symmetric/self-adjoint, then $\mathrm{i}A$ is skew-symmetric/skew-adjoint. Conversely, if A is skew-symmetric/skew-adjoint, then $\mathrm{i}A$ is symmetric/self-adjoint.

Lemma 2.2.12. *A self-adjoint operator A, i.e. $A^* = A$ and $\operatorname{mul} A = \{0\}$, is densely defined. A skew-adjoint operator B is densely defined.*

Proof. By Lemma 2.2.8 we have

$$\overline{\operatorname{dom} A} = (\operatorname{mul} A)^\perp = \{0\}^\perp = X,$$

which proves the claim.

Clearly, this already implies the result for skew-adjoint operators, as $\mathrm{i}B$ is self-adjoint. ❑

Lemma 2.2.13. *A linear relation A is skew-symmetric, if and only if A is dissipative and accretive, i.e.* $\operatorname{Re}\langle x_2, x_1\rangle_{Y,X} = 0$ *for all* $\left[\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right] \in A$.

For operators the condition can be read as $\operatorname{Re}\langle Ax, x\rangle_{Y,X} = 0$. In other words $\langle Ax, x\rangle_{Y,X} \in \mathrm{i}\mathbb{R}$.

Proof. Let $\operatorname{Re}\langle x_2, x_1\rangle_{Y,X} = 0$. Note that $\langle\!\langle \left[\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right], \left[\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right] \rangle\!\rangle = 2\operatorname{Re}\langle x_2, x_1\rangle_{Y,X}$. By the polarization identity (Lemma A.3.1), we have

$$\left\langle\!\!\left\langle \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \right\rangle\!\!\right\rangle = 0 \quad \forall \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \in A \quad \Leftrightarrow \quad \left\langle\!\!\left\langle \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right\rangle\!\!\right\rangle = 0 \quad \forall \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in A.$$

The right hand side of the equivalence is satisfied by assumption and the left hand side implies by Remark 2.2.5 that $-A \subseteq A^*$.

If A is skew-symmetric, then $\left[\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right] \in A$ implies $\left[\begin{smallmatrix} x_1 \\ -x_2 \end{smallmatrix}\right] \in A^*$. Hence, by the definition of the adjoint

$$\langle x_2, x_1\rangle_{Y,X} = \langle x_1, -x_2\rangle_{X,Y}$$

and consequently $\langle x_2, x_1\rangle_{Y,X} + \overline{\langle x_2, x_1\rangle_{Y,X}} = 0$, where the left hand side equals the real part of $\langle x_2, x_1\rangle_{Y,X}$. ❑

Remark 2.2.14. We can characterize skew-symmetry, dissipativity and accretivity in the following way

$$\begin{aligned} A \text{ skew-symmetric} &\Leftrightarrow \operatorname{Re}\langle y, x\rangle = 0 \quad \forall \left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] \in A \Leftrightarrow \langle\!\langle \left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right], \left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] \rangle\!\rangle = 0 \quad \forall \left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] \in A, \\ A \text{ dissipative} &\Leftrightarrow \operatorname{Re}\langle y, x\rangle \leq 0 \quad \forall \left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] \in A \Leftrightarrow \langle\!\langle \left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right], \left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] \rangle\!\rangle \leq 0 \quad \forall \left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] \in A, \\ A \text{ accretive} &\Leftrightarrow \operatorname{Re}\langle y, x\rangle \geq 0 \quad \forall \left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] \in A \Leftrightarrow \langle\!\langle \left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right], \left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] \rangle\!\rangle \geq 0 \quad \forall \left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] \in A. \end{aligned}$$

2.3 Skew-symmetry and Dissipativity on Hilbert Spaces

In the following we will regard linear relations on Hilbert spaces. Similar to defect indices for symmetric operators we want to introduce the analogon for skew-symmetric operators. We will discuss the spaces $\operatorname{ran}(A-\lambda)$ and $\operatorname{ran}(A+\overline{\lambda})$, where $\operatorname{Re}\lambda \neq 0$. Note that contrary to the concept for symmetric operators we regard a pair of complex numbers λ, $-\overline{\lambda}$ mirrored along the imaginary axis, instead of the real axis. This is not surprising as the point spectrum of a skew-symmetric operator is on the imaginary axis.

Lemma 2.3.1. *Let A be a closed dissipative linear relation on a Hilbert space X and $\lambda \in \mathbb{C}$ such that $\operatorname{Re}\lambda > 0$. Then $\operatorname{ran}(A-\lambda)$ is closed in X.*

Proof. Let $\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] \in A$ and $\left[\begin{smallmatrix} x \\ z \end{smallmatrix}\right] \in (A-\lambda)$ such that $z = y - \lambda x$. Note that $(A - \mathrm{i}\operatorname{Im}\lambda)$ is also a dissipative linear relation and therefore $\operatorname{Re}\langle y - \mathrm{i}\operatorname{Im}\lambda x, x\rangle \leq 0$. Then we have the following inequality

$$\begin{aligned} \|z\|_X^2 &= \|y-\lambda x\|_X^2 = \|y - \mathrm{i}\operatorname{Im}\lambda x - \operatorname{Re}\lambda x\|_X^2 \\ &= \|y - \mathrm{i}\operatorname{Im}\lambda x\|_X^2 \underbrace{-2\operatorname{Re}\lambda \overbrace{\operatorname{Re}\langle y - \mathrm{i}\operatorname{Im}\lambda x, x\rangle_X}^{\leq 0}}_{\geq 0} + |\operatorname{Re}\lambda|^2 \|x\|_X^2 \\ &\geq |\operatorname{Re}\lambda|^2\|x\|_X^2. \end{aligned}$$

Let $(\left[\begin{smallmatrix} x_n \\ z_n \end{smallmatrix}\right])_{n\in\mathbb{N}}$ be a sequence in $(A-\lambda)$ such that $(z_n)_{n\in\mathbb{N}}$ converges to $z \in X$. Then the previous inequality implies that also $(x_n)_{n\in\mathbb{N}}$ converges to a limit $x \in X$. Since A is closed (and therefore also $(A-\lambda)$), we conclude that $\left[\begin{smallmatrix} x \\ z \end{smallmatrix}\right] \in (A-\lambda)$ and consequently that $\operatorname{ran}(A-\lambda)$ is closed. ❑

Corollary 2.3.2. *Let A be a closed skew-symmetric operator on a Hilbert space X and $\lambda \in \mathbb{C}$ such that $\operatorname{Re}\lambda \neq 0$. Then $\operatorname{ran}(A-\lambda)$ is closed in X.*

Proof. Note that A and $-A$ are dissipative. Hence, for $\operatorname{Re}\lambda > 0$ we apply Lemma 2.3.1 on A and for $\operatorname{Re}\lambda < 0$ we apply Lemma 2.3.1 on $-A$. ❑

Lemma 2.3.3. *If A is a maximal dissipative linear relation on a Hilbert space X, then $(A-1)$ is surjective, i.e.* $\operatorname{ran}(A-1) = X$.

Proof. Note that $\operatorname{ran}(A-1)$ is closed. Assume that $(A-1)$ is not surjective. Then there is a non zero $z \in X$ that is orthogonal on $\operatorname{ran}(A-1)$, i.e.

$$0 = \langle y - x, z\rangle = \langle y, z\rangle - \langle x, z\rangle \quad \text{for all} \quad \left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] \in A. \tag{2.1}$$

If $z \in \operatorname{dom} A$, then, by the previous equation and the dissipativity of A, we have for all $\left[\begin{smallmatrix} z \\ w \end{smallmatrix}\right] \in A$

$$\operatorname{Re}\|z\|^2 = \operatorname{Re}\langle w, z\rangle \leq 0.$$

Therefore, $z = 0$, which contradicts our assumption $z \neq 0$. On the other hand, if $z \notin \operatorname{dom} A$, then we extend A to $B := \operatorname{span}(A \cup \{[\begin{smallmatrix} z \\ -z \end{smallmatrix}]\})$, which is again dissipative. This can be seen by using (2.1)

$$\operatorname{Re}\langle \alpha z + x, -\alpha z + y\rangle = -|\alpha|^2\|z\|^2 + \underbrace{\operatorname{Re}(\langle \alpha z, y\rangle - \langle x, \alpha z\rangle)}_{=0} + \operatorname{Re}\langle x, y\rangle \leq 0$$

for $[\begin{smallmatrix} x \\ y \end{smallmatrix}] \in A$. However, this contradicts the maximal dissipativity of A. Hence, such a z cannot exist. ❑

Lemma 2.3.4. *Let A be a closed skew-symmetric operator on a Hilbert space and $\lambda \in \mathbb{C}$ such that $\operatorname{Re}\lambda \neq 0$. Then*

$$\operatorname{dom} A^* = \operatorname{dom} A \dotplus \operatorname{ran}(A - \lambda)^\perp \dotplus \operatorname{ran}(A + \overline{\lambda})^\perp. \tag{2.2}$$

Proof. We start showing that this sum is indeed direct. Let

$$f \in \operatorname{dom} A, \quad g \in \underbrace{\operatorname{ran}(A - \lambda)^\perp}_{=\ker(A^* - \overline{\lambda})} \quad \text{and} \quad h \in \underbrace{\operatorname{ran}(A + \overline{\lambda})^\perp}_{=\ker(A^* + \lambda)}$$

be such that

$$f + g + h = 0. \tag{2.3}$$

Applying $A^* - \overline{\lambda}$ on this equations yields ($A^* f = -Af$ and $A^* h = -\lambda h$)

$$-(A + \overline{\lambda})f - \underbrace{(\lambda + \overline{\lambda})}_{=2\operatorname{Re}\lambda} h = 0$$

By assumption the summands are orthogonal and $\operatorname{Re}\lambda \neq 0$, therefore

$$(A + \overline{\lambda})f = 0 \quad \text{and} \quad h = 0.$$

Since A is skew-symmetric, only a pure imaginary number can be an eigenvalue and consequently $f = 0$. Because of (2.3), we also see $g = 0$.

Finally, we show that there is equality in (2.2). Let $u \in \operatorname{dom} A^*$. Since $\operatorname{ran}(A - \lambda)$ is closed by Corollary 2.3.2, we have $X = \operatorname{ran}(A - \lambda) \oplus (\operatorname{ran}(A - \lambda))^\perp$. Therefore, we can decompose

$$(A^* + \lambda)u = u_1 + u_2, \quad \text{where} \quad u_1 \in \operatorname{ran}(A - \lambda), u_2 \in (\operatorname{ran}(A - \lambda))^\perp.$$

We can write $u_1 = (A - \lambda)f$ for an $f \in \operatorname{dom} A$ and $u_2 = (\overline{\lambda} + \lambda)g$, where $g = \frac{1}{\overline{\lambda} + \lambda} u_2 \in \ker(A^* - \overline{\lambda})$.

$$(A^* + \lambda)u = (A - \lambda)f + (\overline{\lambda} + \lambda)g = -(A^* + \lambda)f + (A^* + \lambda)g.$$

Therefore, $h := u + f - g \in \ker(A^* + \lambda) = (\operatorname{ran}(A + \overline{\lambda}))^\perp$ and $u = -f + g + h$. ❑

Definition 2.3.5. Let X and Y be Hilbert spaces and $A\colon X \to Y$ be a linear operator. Then we define the *graph inner product* by

$$\langle f, g\rangle_A := \langle f, g\rangle_X + \langle Af, Ag\rangle_Y.$$

The corresponding *graph norm* is given by

$$\|f\|_A = \sqrt{\|f\|_X^2 + \|Af\|_Y^2}.$$

It is easy to see that $\operatorname{dom} A$ is a Hilbert space with the graph inner product, if A is a closed operator (if A is closed in $X \times Y$).

Lemma 2.3.6. *Let A be a closed skew-symmetric operator on an Hilbert space X. If we regard $\operatorname{dom} A^*$ with the graph inner product of A^*, then we have the following orthogonal decomposition*

$$\operatorname{dom} A^* = \operatorname{dom} A \oplus_{A^*} \operatorname{ran}(A-1)^\perp \oplus_{A^*} \operatorname{ran}(A+1)^\perp,$$

where $\operatorname{ran}(A-1)^\perp$ is still the orthogonal complement of $\operatorname{ran}(A-1)$ in X.

Proof. By Lemma 2.3.4 for $\lambda = 1$ we know that the sum on the right hand side spans all of $\operatorname{dom} A^*$. Hence, it is left to show that is an orthogonal sum.

Let $f \in \operatorname{dom} A$ and $g \in \operatorname{ran}(A-\lambda)^\perp = \ker(A^* - \overline{\lambda})$. Then by assumption $A^* f = -Af$ and $A^* g = \overline{\lambda} g$, therefore

$$\begin{aligned} \langle f, g\rangle_{A^*} &= \langle f, g\rangle_X + \langle A^* f, A^* g\rangle_X = \langle f, \overline{\lambda}^{-1} A^* g\rangle_X - \langle Af, \overline{\lambda} g\rangle_X \\ &= \left\langle f, \left(\overline{\lambda}^{-1} - \overline{\lambda}\right) A^* g\right\rangle_X. \end{aligned}$$

Hence, if $\overline{\lambda} = \pm 1$, then we have orthogonality.

Let $g \in \ker(A-1)$ and $h \in \ker(A+1)$. Then

$$\langle g, h\rangle_{A^*} = \langle g, h\rangle_X + \langle A^* g, A^* h\rangle_X = \langle g, h\rangle_X + \langle g, -h\rangle_X = 0,$$

which finishes the proof. ❑

Remark 2.3.7. Since $\operatorname{dom} A^*$ endowed with the graph inner product of A^* is isomorphic to A^* as subspace of $X \times X$. Hence, by Lemma 2.3.6 we can also decompose A^* into

$$A^* = A \oplus A\big|_{\operatorname{ran}(A-1)^\perp} \oplus A\big|_{\operatorname{ran}(A+1)^\perp},$$

where the orthogonal sum is in $X \times X$.

Lemma 2.3.8. *Let A be a densely defined, closed skew-symmetric operator on a Hilbert space X. Then*

$$\langle f, g\rangle_{A^*} = \pm \left\langle\!\!\left\langle \begin{bmatrix} f \\ A^* f \end{bmatrix}, \begin{bmatrix} g \\ A^* g \end{bmatrix} \right\rangle\!\!\right\rangle \quad \textit{for} \quad f \in \operatorname{dom} A^*, g \in \operatorname{ran}(A - (\pm 1))^\perp$$

Proof. Let $f \in \operatorname{dom} A^*$ and $g \in \operatorname{ran}(A - (\pm 1))^\perp$. Note that $\operatorname{ran}(A - (\pm 1))^\perp = \ker(A^* - (\pm 1))$. Therefore, we have $g = \pm A^* g$ and

$$\begin{aligned} \langle f, g\rangle_{A^*} &= \langle f, g\rangle_X + \langle A^* f, A^* g\rangle_X = \pm\langle f, A^* g\rangle_X \pm \langle A^* f, g\rangle_X \\ &= \pm \left\langle\!\!\left\langle \begin{bmatrix} f \\ A^* f \end{bmatrix}, \begin{bmatrix} g \\ A^* g \end{bmatrix} \right\rangle\!\!\right\rangle. \end{aligned}$$

❑

Definition 2.3.9. Let A be a linear relation on a Hilbert space. Then we define the *Cayley transform* of A by

$$\mathcal{C}(A) := \tau_M(A) = \left\{ \begin{bmatrix} -x+y \\ x+y \end{bmatrix} \,\middle|\, \begin{bmatrix} x \\ y \end{bmatrix} \in A \right\}, \quad \text{where} \quad M = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}.$$

Remark 2.3.10.

- Note that the factor $\frac{1}{\sqrt{2}}$ is only for cosmetic reason as $\tau_{\alpha M} = \tau_M$ for $\alpha \neq 0$.
- Note that the Cayley transform comes from the Möbius tranformation

  $$\phi_M(z) = \frac{z+1}{z-1},$$

  which maps the imaginary axis on the unit circle ring[1] and the left half plane on the interior of the unit circle[2]. Therefore, it will not come as a surprise that the Cayley transform of a skew-adjoint operator is a unitary operator and the Cayley transform of a dissipative linear relation is a contractive operator.
- The inverse Cayley transform is given by

  $$\mathcal{C}^{-1}(A) = \mathcal{C}(A).$$

  This can be easily seen by $M^2 = \mathrm{I}$ and Lemma 2.1.8.
- If A is an operator then we can write the Cayley transform as

  $$\mathcal{C}(A) = (A+1)(A-1)^{-1}.$$

  If additionally $1 \notin \sigma_{\mathrm{p}}(A)$, then $\mathcal{C}(A)$ is an operator.

Definition 2.3.11. We say a linear relation A between two Banach spaces X and Y is *contractive*, if

$$\begin{bmatrix} x \\ y \end{bmatrix} \in A \quad \Rightarrow \quad \|y\|_Y \leq \|x\|_X.$$

Note that every contractive linear relation A satisfies $\operatorname{mul} A = \{0\}$, or in other words A is an operator. This can be easily seen by

$$\begin{bmatrix} 0 \\ y \end{bmatrix} \in A \quad \Rightarrow \quad \|y\| \leq 0.$$

Lemma 2.3.12. *Let A be a dissipative linear relation on a Hilbert space. Then its Cayley transform is a contractive operator. Conversely, the Cayley transform of a contractive operator K is a dissipative linear relation.*

[1] often denoted by $\mathbb{T}$
[2] often denoted by $\mathbb{D}$

Note that a contractive operator does not have to be everywhere defined.

Proof. By definition, for $\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right] \in \mathcal{C}(A)$ there exits $\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] \in A$ such that

$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} -x+y \\ x+y \end{bmatrix}$$

So we have

$$\|u\|^2 = \|x\|^2 + \|y\|^2 - 2\operatorname{Re}\langle x, y\rangle$$
$$\|v\|^2 = \|x\|^2 + \|y\|^2 + 2\operatorname{Re}\langle x, y\rangle.$$

Since A is dissipative, this implies $\|v\| \leq \|u\|$. Consequently $\mathcal{C}(A)$ is a contractive linear relation, which is automatically an operator.

By definition,

$$\mathcal{C}(K) = \left\{ \begin{bmatrix} -x+y \\ x+y \end{bmatrix} \middle| \begin{bmatrix} x \\ y \end{bmatrix} \in K \right\} = \left\{ \begin{bmatrix} -x+Kx \\ x+Kx \end{bmatrix} \middle| x \in \operatorname{dom} K \right\}$$

Hence, for $\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right] \in \mathcal{C}(K)$ we have

$$\operatorname{Re}\langle v, u\rangle = \operatorname{Re}\langle x+Kx, -x+Kx\rangle = -\|x\|^2 + \|Kx\|^2 + \underbrace{\operatorname{Re}(\langle x, Kx\rangle - \langle Kx, x\rangle)}_{=0} \leq 0,$$

because K is contractive. This gives the dissipativity of $\mathcal{C}(K)$. ❑

Lemma 2.3.13. *Let K be a linear relation on a Hilbert space with* $\operatorname{mul} K = \{0\}$ *(single-valued). Then*

$$\ker \begin{bmatrix} 1+K & 1-K \end{bmatrix} = \mathcal{C}(K).$$

Proof. Let $\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right] \in \ker \begin{bmatrix} 1+K & 1-K \end{bmatrix}$. Then

$$(K+1)a = (K-1)b.$$

Subtracting Kb and b on both sides gives

$$K(a-b) + (a-b) = -2b,$$

which implies $\left[\begin{smallmatrix} a-b \\ -2b \end{smallmatrix}\right] \in (K+1)$. Moreover

$$(K-1)(a-b) = Ka - a - \underbrace{(K-1)b}_{=(K+1)a} = -2a,$$

which implies $\left[\begin{smallmatrix} -2a \\ (a-b) \end{smallmatrix}\right] \in (K-1)^{-1}$. Hence, $\left[\begin{smallmatrix} -2a \\ -2b \end{smallmatrix}\right] \in (K+1)(K-1)^{-1}$ and by linearity $\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right] \in (K+1)(K-1)^{-1} = \mathcal{C}(K)$.

For the reverse inclusion note

$$\mathcal{C}(K) = \left\{ \begin{bmatrix} -x+y \\ x+y \end{bmatrix} \middle| \begin{bmatrix} x \\ y \end{bmatrix} \in K \right\} = \left\{ \begin{bmatrix} -(1-K) \\ (1+K) \end{bmatrix} x \middle| x \in \operatorname{dom} K \right\}.$$

Therefore, if $\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right] \in \mathcal{C}(K)$, then

$$\begin{bmatrix} 1-K & 1+K \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1-K & 1+K \end{bmatrix} \begin{bmatrix} -(1+K) \\ (1-K) \end{bmatrix} x = 0,$$

which proves the assertion. ❑

Theorem 2.3.14. *The Cayley transform of a maximal dissipative linear relation is an everywhere defined contractive operator and conversely the Cayley transform of an everywhere defined contractive operator is a maximal dissipative linear relation.*

We could also say "maximal contractive operator" instead of "everywhere defined". In this notation the Cayley transform preserves maximality.

Proof. We only have to check whether the Cayley transform of these linear relations are everywhere defined and maximal, respectively, as we have already shown in Lemma 2.3.12, that the Cayley transformation maps dissipative linear relations on contractive linear relations and vice versa.

If A is a maximal dissipative linear relation, then by Lemma 2.3.3 $\operatorname{ran}(A - 1) = X$ and therefore $\operatorname{dom} \mathcal{C}(A) = X$.

If K is a everywhere defined contractive operator, then $\mathcal{C}(K)$ is dissipative. If $\mathcal{C}(K)$ would not be maximal dissipative, then it would have a proper dissipative extension A. The Cayley transform of A would be a proper contractive extension of K, which is impossible as K is already everywhere defined. ❑

Corollary 2.3.15. *A linear relation A on a Hilbert space X is maximal dissipative, if and only if there exists an everywhere defined K contractive operator on X ($\operatorname{dom} K = X$) such that*

$$A = \ker \begin{bmatrix} \mathrm{I}+K & \mathrm{I}-K \end{bmatrix}. \tag{2.4}$$

We can even say $\mathcal{C}(A) = K$ and $\mathcal{C}(K) = A$.

Proof. This immediately follows from Theorem 2.3.14 and Lemma 2.3.13. ❑

Remark 2.3.16. There is also a slightly different characterization of maximal dissipative operators, given by

$$A = \ker \begin{bmatrix} \mathrm{I}-K & \mathrm{I}+K \end{bmatrix}.$$

This gives the inverse linear relation of the linear relation in (2.4) (the inverse of a dissipative linear relation is again dissipative). This becomes even more obvious, if we notice that $-K$ is also contractive.

2.4 Boundary Triples

Boundary triples were investigated primarily to determine self-adjoint extensions of symmetric operators. We will use them to find dissipative extensions of skew-symmetric operators. We will make a slight modification on the standard definition, which allows us to work with a complete dual pair as boundary space instead of a Hilbert space.

Boundary triples are studied extensively in [18].

Definition 2.4.1. Let A_0 be a densely defined, skew-symmetric, and closed operator on a Hilbert space X. By a *boundary triple* for A_0^* we mean a triple $((\mathcal{B}_+, \mathcal{B}_-), B_1, B_2)$ consisting of a complete dual pair $(\mathcal{B}_+, \mathcal{B}_-)$, and two linear operators $B_1\colon \operatorname{dom} A_0^* \to \mathcal{B}_+$ and $B_2\colon \operatorname{dom} A_0^* \to \mathcal{B}_-$ such that

(i) the mapping $B = \left[\begin{smallmatrix} B_1 \\ B_2 \end{smallmatrix}\right]\colon \operatorname{dom} A_0^* \to \mathcal{B}_+ \times \mathcal{B}_-$, $x \mapsto \left[\begin{smallmatrix} B_1 x \\ B_2 x \end{smallmatrix}\right]$ is surjective, and

(ii) for $x, y \in \operatorname{dom} A_0^*$ there holds

$$\langle A_0^* x, y\rangle_X + \langle x, A_0^* y\rangle_X = \langle B_1 x, B_2 y\rangle_{\mathcal{B}_+,\mathcal{B}_-} + \langle B_2 x, B_1 y\rangle_{\mathcal{B}_-,\mathcal{B}_+}. \tag{2.5}$$

In order to avoid too much notation we will assume that the boundary space is a reflexive Banach space and we regard the complete dual pair $(\mathcal{B}, \mathcal{B}')$ instead of $(\mathcal{B}_+, \mathcal{B}_-)$ and denote the boundary triple as $(\mathcal{B}, B_1, B_2)$ instead of $((\mathcal{B}, \mathcal{B}'), B_1, B_2)$, which represents the setting of complete dual pairs.

When we say $(\mathcal{B}, B_1, B_2)$ is a boundary triple for A_0^*, we implicitly assume that A_0 is densely defined, closed and skew-symmetric. Clearly, if A_0 is densely defined and skew-symmetric, we can always regard $\overline{A_0}$ instead to have a closed operator.

Note that if $\mathcal{B}$ is a Hilbert space and B_2 maps also into $\mathcal{B}$, then we can replace the dual pairing in (2.5) by the inner product in $\mathcal{B}$. If $\mathcal{B}$ is a Hilbert space, this can always be forced, since we can replace B_2 by ΨB_2, where Ψ is the natural isomorphism between $\mathcal{B}'$ and $\mathcal{B}$. Nevertheless, we will allow this nuance.

Alternatively we can write (2.5) as

$$\left\langle\!\!\left\langle \begin{bmatrix} x \\ A_0^* x \end{bmatrix}, \begin{bmatrix} y \\ A_0^* y \end{bmatrix} \right\rangle\!\!\right\rangle_{X \times X} = \left\langle\!\!\left\langle \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} x, \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} y \right\rangle\!\!\right\rangle_{\mathcal{B} \times \mathcal{B}'}.$$

Since $\operatorname{dom} A_0^*$ equipped with the graph inner product of A_0^* is isomorphic to A_0^* as subspace of $X \times X$, we can also regard B_1 and B_2 as mapping defined on A_0^* instead of $\operatorname{dom} A_0^*$. In fact, by this approach we could generalize boundary triples for linear relations, as done in [7]. However, also for our usage this can sometimes simplify some arguments. In [7] they present an even weaker concept of boundary triples: so called *quasi boundary triple*.

Definition 2.4.2. Let $(\mathcal{B}, B_1, B_2)$ be a boundary triple for A_0^*. Then we define

$$\tilde{B}_1 \colon \begin{cases} A_0^* & \to & \mathcal{B}, \\ \left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] & \mapsto & B_1 x, \end{cases} \qquad \tilde{B}_2 \colon \begin{cases} A_0^* & \to & \mathcal{B}', \\ \left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] & \mapsto & B_2 x, \end{cases} \qquad \tilde{B} \colon \begin{cases} A_0^* & \to & \mathcal{B} \times \mathcal{B}', \\ \left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] & \mapsto & \left[\begin{smallmatrix} B_1 \\ B_2 \end{smallmatrix}\right] x. \end{cases}$$

The only difference between $\tilde{B}_1$ and B_1 is that $\tilde{B}_1$ is defined on the operator A_0^*, where we regard A_0^* as subspace of $X \times X$, and B_1 is defined on $\operatorname{dom} A_0^*$. If we introduce $\pi_1 \colon X \times X \to X$, $\left[\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right] \mapsto x_1$ the projection on the first component, then we have the relation

$$\tilde{B}_1 \left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] = B_1 \pi_1 \left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right].$$

On the other hand, since A_0^* is an operator (well-defined), we can also find an embedding ι from $\operatorname{dom} A_0^*$ onto A_0^*. Therefore, we have $B_1 x = \tilde{B}_1 \iota x$. Accordingly, we have the same for $\tilde{B}_2$ and $\tilde{B}$.

If we would fully commit to linear relations, we could write (2.5) even more compactly, as

$$\langle\!\langle f, g \rangle\!\rangle_{X \times X} = \langle\!\langle \tilde{B} f, \tilde{B} g \rangle\!\rangle_{\mathcal{B} \times \mathcal{B}'} \quad \text{for all} \quad f, g \in A_0^*.$$

In [7] this notation is used.

Remark 2.4.3. Let A be a densely defined, closed, and symmetric operator. Then $\mathrm{i}A$ is a densely defined, closed, and skew-symmetric operator. If there is a boundary triple $(\mathcal{B}, B_1, B_2)$ for $(\mathrm{i}A)^*$, then we have the following adaption of (2.5)

$$\langle A^* x, y \rangle - \langle x, A^* y \rangle = \langle \mathrm{i} B_1 x, B_2 y \rangle - \langle B_2 x, \mathrm{i} B_1 y \rangle.$$

Hence, $(\mathcal{B}, \mathrm{i}B_1, B_2)$ is a boundary triple for A^* in the notion of symmetric operators.

One could wonder why we introduced a boundary triple for the adjoint of a skew-symmetric operator instead of replacing A_0^* just with any operator A in Definition 2.4.1. One could think that these properties already imply that A^* is given by the restriction of $-A$ to $\ker B_1 \cap \ker B_2$ and A is the adjoint of a skew-symmetric operator anyway, but this is not necessarily true as we will see later in Example 2.4.6.

Example 2.4.4. Let $X = \mathsf{L}^2(0,1)^2$ and

$$A_0 = -\begin{bmatrix} 0 & \frac{\mathrm{d}}{\mathrm{d}\xi} \\ \frac{\mathrm{d}}{\mathrm{d}\xi} & 0 \end{bmatrix} \quad \text{with} \quad \operatorname{dom} A_0 = \mathsf{H}_0^1(0,1) \times \mathsf{H}_0^1(0,1).$$

Then the adjoint of this operator is given by

$$A_0^* = \begin{bmatrix} 0 & \frac{\mathrm{d}}{\mathrm{d}\xi} \\ \frac{\mathrm{d}}{\mathrm{d}\xi} & 0 \end{bmatrix} \quad \text{with} \quad \operatorname{dom} A_0^* = \mathsf{H}^1(0,1) \times \mathsf{H}^1(0,1).$$

For smooth functions we have by the integration by parts formula

$$\begin{aligned}
\langle A_0^* f, g\rangle + \langle f, A_0^* g\rangle &= \int_0^1 \left\langle \begin{bmatrix} f_2' \\ f_1' \end{bmatrix}, \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} \right\rangle \mathrm{d}\xi + \int_0^1 \left\langle \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}, \begin{bmatrix} g_2' \\ g_1' \end{bmatrix} \right\rangle \mathrm{d}\xi \\
&= \int_0^1 (f_2' g_1 + f_1' g_2 + f_1 g_2' + f_2 g_1') \,\mathrm{d}\xi = f_2 g_1 \Big|_0^1 + f_1 g_2 \Big|_0^1 \\
&= f_2(1) g_1(1) - f_2(0) g_1(0) + f_1(1) g_2(1) - f_1(0) g_2(0) \\
&= \Bigg\langle \underbrace{\begin{bmatrix} f_2(1) \\ -f_2(0) \end{bmatrix}}_{B_2 f}, \underbrace{\begin{bmatrix} g_1(1) \\ g_1(0) \end{bmatrix}}_{B_1 g} \Bigg\rangle + \Bigg\langle \underbrace{\begin{bmatrix} f_1(1) \\ f_1(0) \end{bmatrix}}_{B_1 f}, \underbrace{\begin{bmatrix} g_2(1) \\ -g_2(0) \end{bmatrix}}_{B_2 g} \Bigg\rangle.
\end{aligned}$$

Defining $B_1 f := \left[\begin{smallmatrix} f_1(1) \\ f_1(0) \end{smallmatrix}\right]$ and $B_2 f := \left[\begin{smallmatrix} f_2(1) \\ -f_2(0) \end{smallmatrix}\right]$ yields (by continuous extension)

$$\langle A_0^* f, g\rangle + \langle f, A_0^* g\rangle = \langle B_1 f, B_2 g\rangle + \langle B_2 f, B_1 g\rangle. \tag{2.6}$$

The mapping $\left[\begin{smallmatrix} B_1 \\ B_2 \end{smallmatrix}\right] : \operatorname{dom} A_0^* \to \mathbb{K}^2 \times \mathbb{K}^2$ is surjective (this can be seen by choosing f_1 and f_2 to be linear interpolations). Hence, $(\mathbb{K}^2, B_1, B_2)$ is a boundary triple for A_0^*.

The operator A_0 can be recovered by restricting $-A_0^*$ to $\ker B_1 \cap \ker B_2$ as the next lemma will show. However, if A_0^* satisfied item (i) and item (ii) but wasn't the adjoint of a skew-symmetric operator, then the next lemma would not hold as Example 2.4.6 demonstrates.

Lemma 2.4.5. *Let A_0 be a densely defined, skew-symmetric, and closed operator on a Hilbert space X and $(\mathcal{B}, B_1, B_2)$ be a boundary triple for A_0^*. Then*

$$A_0 = -A_0^*\big|_{\ker B_1 \cap \ker B_2} = -A_0^*\big|_{\ker B}.$$

In other words $\operatorname{dom} A_0 = \ker B_1 \cap \ker B_2 = \ker B$.

Proof. Let $x \in \ker B_1 \cap \ker B_2$ and $y \in \operatorname{dom} A_0^*$. Then the right-hand-side of (2.5) is 0. Hence,

$$\langle x, A_0^* y\rangle_X = \langle -A_0^* x, y\rangle_X \quad \text{for all} \quad y \in \operatorname{dom} A_0^*.$$

This yields $(x, -A_0^* x) \in A_0^{**} = A_0$. Hence, $-A_0^*\big|_{\ker B_1 \cap \ker B_2} \subseteq A_0$.

On the other hand if $x \in \operatorname{dom} A_0$, then $A_0^* x = -A_0 x$ and consequently

$$\langle \underbrace{A_0^* x}_{=-A_0 x}, y\rangle_X + \langle x, A_0^* y\rangle_X = \langle -x, A_0^* y\rangle_X + \langle x, A_0^* y\rangle_X = 0.$$

Therefore, the right-hand-side of (2.5) is 0 which can be written as

$$\left\langle\!\!\left\langle \left[\begin{smallmatrix} B_1 \\ B_2 \end{smallmatrix}\right] x, \left[\begin{smallmatrix} B_1 \\ B_2 \end{smallmatrix}\right] y \right\rangle\!\!\right\rangle_{\mathcal{B} \times \mathcal{B}'} = 0 \quad \text{for all} \quad y \in \operatorname{dom} A_0^*.$$

Since $\left[\begin{smallmatrix} B_1 \\ B_2 \end{smallmatrix}\right]$ is surjective on $\mathcal{B} \times \mathcal{B}'$, we have

$$\begin{bmatrix} B_1 x \\ B_2 x \end{bmatrix} \perp_{\langle\!\langle , \rangle\!\rangle} \mathcal{B} \times \mathcal{B}',$$

which implies $x \in \ker B_1 \cap \ker B_2 = \ker B$ (because $\langle\!\langle \cdot, \cdot \rangle\!\rangle$ is non-degenerated). ❑

The next example shows that it is possible to have item (i) and item (ii) of a "boundary triple" for an operator A (replacing A_0^* with A in Definition 2.4.1) without A being the adjoint of a skew-symmetric operator. Moreover, it shows that in this situation Lemma 2.4.5 does not hold. This demonstrates the importance of A being the adjoint of a skew-symmetric operator in the definition.

Example 2.4.6. Let A_0 be the operator on $\mathsf{L}^2(0,1)^2$ from Example 2.4.4. Then we have a boundary triple $(\mathbb{K}^2, B_1, B_2)$ for A_0^*, where $B_1 f := \left[\begin{smallmatrix} f_1(1) \\ f_1(0) \end{smallmatrix}\right]$ and $B_2 f := \left[\begin{smallmatrix} f_2(1) \\ -f_2(0) \end{smallmatrix}\right]$.

We define A as the restriction of A_0^* on $\mathsf{H}^1_{\{1\}=0}(0,1) \times \mathsf{H}^1_{\{0\}=\{1\}}(0,1)$, where

$$\begin{aligned} \mathsf{H}^1_{\{1\}=0}(0,1) &:= \{f \in \mathsf{H}^1(0,1) \mid f(1) = 0\}, \\ \text{and} \quad \mathsf{H}^1_{\{0\}=\{1\}}(0,1) &:= \{f \in \mathsf{H}^1(0,1) \mid f(0) = f(1)\}. \end{aligned}$$

Therefore, we can reformulate (2.6) for $f, g \in \operatorname{dom} A$:

$$\langle Af, g\rangle + \langle f, Ag\rangle = \langle B_1 f, B_2 g\rangle + \langle B_2 f, B_1 g\rangle = -f_1(0)g_2(0) + f_2(0)(-g_1(0)).$$

By defining $F_1 f := -f_1(0)$ and $F_2 f := f_2(0)$ we again have that $\left[\begin{smallmatrix} F_1 \\ F_2 \end{smallmatrix}\right] \colon \operatorname{dom} A \to \mathbb{K} \times \mathbb{K}$ is surjective. However A is not the adjoint of a skew-symmetric operator. If it were, then $(\mathbb{K}, F_1, F_2)$ would be a boundary triple for A and

$$A^* = -A|_{\ker F_1 \cap \ker F_2} = -A_0^*|_{\mathsf{H}^1_0(0,1)^2} = A_0.$$

This would imply $\overline{A} = A^{**} = A_0^*$, which is certainly not true.

In fact, with the boundary triple for A_0^* we can apply Corollary 2.4.11, which will give us that the adjoint of A is $-A_0^*|_{\mathsf{H}^1_{\{0\}=\{1\}}(0,1) \times \mathsf{H}^1_{\{0\}=0}(0,1)}$.

Lemma 2.4.7. *Let $(\mathcal{B}, B_1, B_2)$ be a boundary triple for A_0^*. We endow* $\operatorname{dom} A_0^*$ *with the graph inner product of A_0^*. Then the following statements are true*

(i) $\tilde{B} \colon A_0^* \to \mathcal{B} \times \mathcal{B}'$ *is bounded.*

(ii) $B \colon \operatorname{dom} A_0^* \to \mathcal{B} \times \mathcal{B}'$ *is bounded (w.r.t. to the graph norm).*

(iii) $\tilde{B}$ *restricted to* $A_0^*|_{(\operatorname{ran}(A-1))^\perp \oplus_{A_0^*} (\operatorname{ran}(A+1))^\perp}$ *is bijective, bounded and boundedly invertible.*

(iv) *B restricted to* $(\operatorname{ran}(A-1))^{\perp} \oplus_{A_0^*} (\operatorname{ran}(A+1))^{\perp}$ *is bijective, bounded and boundedly invertible.*

Clearly, this also implies that $\tilde{B}_1$, $\tilde{B}_2$ and B_1, B_2 are bounded.

Proof. Recall that we can decompose Lemma 2.3.6 $\operatorname{dom} A_0^*$ into

$$\operatorname{dom} A_0^* = \operatorname{dom} A_0 \oplus_{A^*} \operatorname{ran}(A-1)^{\perp} \oplus_{A^*} \operatorname{ran}(A+1)^{\perp}.$$

By Lemma 2.4.5 $\operatorname{dom} A_0 = \ker B$, which implies that B restricted to $\operatorname{ran}(A-1)^{\perp} \oplus_{A^*} \operatorname{ran}(A+1)^{\perp}$ is bijective. Hence, it is enough to show item (iv).

We will show that $B\big|^{-1}_{\operatorname{ran}(A_0-1)^{\perp}\oplus\operatorname{ran}(A_0+1)^{\perp}}$ is bounded. By the open mapping theorem this also implies that $B\big|_{\operatorname{ran}(A_0-1)^{\perp}\oplus\operatorname{ran}(A_0+1)^{\perp}}$ is bounded. For notational simplicity we will replace $B\big|_{\operatorname{ran}(A_0-1)^{\perp}\oplus\operatorname{ran}(A_0+1)^{\perp}}$ by B.

Let $(\left[\begin{smallmatrix} u_n \\ v_n \end{smallmatrix}\right])_{n\in\mathbb{N}}$ be a sequence in $\operatorname{ran} B\big|_{\operatorname{ran}(A-1)^{\perp}} \subseteq \mathcal{B} \times \mathcal{B}'$ that converges to $\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right]$ in $\mathcal{B} \times \mathcal{B}'$. Then we define $x_n := B^{-1}\left[\begin{smallmatrix} u_n \\ v_n \end{smallmatrix}\right]$ and $x := B^{-1}\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right]$, which gives the sequence $(x_n)_{n\in\mathbb{N}}$ in $\operatorname{ran}(A_0-1)^{\perp}$. We can uniquely and orthogonally decompose x into $x = x^+ + x^-$, where $x^{\pm} \in \operatorname{ran}(A_0 - (\pm 1))^{\perp}$. By Lemma 2.3.8

$$\begin{aligned}
&\|x_n - x\|^2_{A_0^*} \\
&\quad= \langle x_n - x, x_n - x^+\rangle_{A_0^*} - \langle x_n - x, x^-\rangle_{A_0^*} \\
&\quad= \left\langle\!\!\left\langle \begin{bmatrix} x_n - x \\ A_0^*(x_n - x)\end{bmatrix}, \begin{bmatrix} x_n - x^+ \\ A_0^*(x_n - x^+)\end{bmatrix} \right\rangle\!\!\right\rangle_{X\times X} + \left\langle\!\!\left\langle \begin{bmatrix} x_n - x \\ A_0^*(x_n - x)\end{bmatrix}, \begin{bmatrix} x^- \\ A_0^* x^-\end{bmatrix} \right\rangle\!\!\right\rangle_{X\times X} \\
&\quad= \left\langle\!\!\left\langle \begin{bmatrix} B_1(x_n - x) \\ B_2(x_n - x)\end{bmatrix}, \begin{bmatrix} B_1(x_n - x^+) \\ B_2(x_n - x^+)\end{bmatrix} \right\rangle\!\!\right\rangle_{\mathcal{B}\times\mathcal{B}'} + \left\langle\!\!\left\langle \begin{bmatrix} B_1(x_n - x) \\ B_2(x_n - x)\end{bmatrix}, \begin{bmatrix} B_1 x^- \\ B_2 x^-\end{bmatrix} \right\rangle\!\!\right\rangle_{\mathcal{B}\times\mathcal{B}'} \\
&\quad= \left\langle\!\!\left\langle \begin{bmatrix} u_n - u \\ v_n - v\end{bmatrix}, \begin{bmatrix} u_n - B_1 x^+ \\ v_n - B_2 x^+\end{bmatrix} \right\rangle\!\!\right\rangle_{\mathcal{B}\times\mathcal{B}'} + \left\langle\!\!\left\langle \begin{bmatrix} u_n - u \\ v_n - v\end{bmatrix}, \begin{bmatrix} B_1 x^- \\ B_2 x^-\end{bmatrix} \right\rangle\!\!\right\rangle_{\mathcal{B}\times\mathcal{B}'} \to 0,
\end{aligned} \tag{2.7}$$

as $\left[\begin{smallmatrix} u_n \\ v_n \end{smallmatrix}\right]$ is bounded and converges to $\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right]$. Hence, x_n converges to x (w.r.t. $\|\cdot\|_{A_0^*}$) and $x \in \operatorname{ran}(A_0-1)^{\perp}$, by the closedness of $\operatorname{ran}(A_0-1)^{\perp}$. Moreover, $Bx = \left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right] \in \operatorname{ran} B\big|_{\operatorname{ran}(A_0-1)^{\perp}}$, which implies that $\operatorname{ran} B\big|_{\operatorname{ran}(A_0-1)^{\perp}}$ is closed. Equation (2.7) also implies the continuity of $B\big|^{-1}_{\operatorname{ran}(A_0-1)^{\perp}}$. Analogously, we can show that $B\big|^{-1}_{\operatorname{ran}(A_0+1)^{\perp}}$ is continuous. Since B is bijective from $\operatorname{ran}(A_0-1)^{\perp} \oplus_{A_0^*} \operatorname{ran}(A_0+1)^{\perp}$ to $\mathcal{B} \times \mathcal{B}'$, we have $\mathcal{B} \times \mathcal{B}' = \operatorname{ran} B\big|_{\operatorname{ran}(A_0-1)^{\perp}} \dot{+} \operatorname{ran} B\big|_{\operatorname{ran}(A_0+1)^{\perp}}$, which is a decomposition of closed subspaces. Hence, the continuity of $B\big|^{-1}_{\operatorname{ran}(A_0-1)^{\perp}}$ and $B\big|^{-1}_{\operatorname{ran}(A_0+1)^{\perp}}$ implies the continuity of B. ❑

Remark 2.4.8. $\mathcal{B} \times \mathcal{B}'$ with $\langle\!\langle\cdot,\cdot\rangle\!\rangle_{\mathcal{B}\times\mathcal{B}'}$ is a Krein space. Its fundamental decomposition is given by $B\operatorname{ran}(A_0-1)^{\perp}$ and $B\operatorname{ran}(A_0+1)^{\perp}$. The Krein space topology and the norm topology of $\mathcal{B}\times\mathcal{B}'$ coincide. Hence, we can endow $\mathcal{B}\times\mathcal{B}'$

with an inner product such that it is a Hilbert space. Consequently, there is an inner product on $\mathcal{B}$ whose induced norm is equivalent to its original norm. Therefore, we do not restrict ourselves, if we ask $\mathcal{B}$ to be a Hilbert space in Definition 2.4.1 from the beginning.

Definition 2.4.9. Let A_0 be a skew-symmetric operator and $(\mathcal{B}, B_1, B_2)$ a boundary triple for A_0^*. Then for a linear relation Θ between $\mathcal{B}$ and $\mathcal{B}'$ ($\Theta \subseteq \mathcal{B} \times \mathcal{B}'$) we define

$$A_\Theta := \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in A_0^* \,\middle|\, \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} x \in \Theta \right\} = \tilde{B}^{-1}\Theta.$$

So A_Θ is the restriction of A_0^* to $\operatorname{dom} A_\Theta = \left\{ x \in \operatorname{dom} A_0^* \,\middle|\, \left[\begin{smallmatrix} B_1 \\ B_2 \end{smallmatrix}\right] x \in \Theta \right\}$. If Θ is even an operator on $\mathcal{B}$, then $\operatorname{dom} A_\Theta = \ker(B_2 - \Theta B_1)$. If Θ^{-1} is an operator, then $\operatorname{dom} A_\Theta = \ker(B_1 - \Theta^{-1} B_2)$.

On the other hand if we have $-A_0 \subseteq A \subseteq A_0^*$, then we can construct a linear relation $\Theta(A)$ such that $A_{\Theta(A)} = A$, by

$$\Theta(A) := \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \operatorname{dom} A = \begin{bmatrix} \tilde{B}_1 \\ \tilde{B}_2 \end{bmatrix} A = \tilde{B} A.$$

Hence, every operator A that satisfies $-A_0 \subseteq A \subseteq A_0^*$ is given by A_Θ for some linear relation Θ (for $\Theta = \tilde{B} A$).

Proposition 2.4.10. *Let A_0 be a closed and skew-symmetric operator on a Hilbert space X, $(\mathcal{B}, B_1, B_2)$ a boundary triple for A_0^* and Θ a linear relation between $\mathcal{B}$ and $\mathcal{B}'$. Then*

(i) $\overline{A_\Theta} = A_{\overline{\Theta}}$,

(ii) $A_\Theta^* = -A_{-\Theta^*}$,

(iii) *A_Θ is (maximal) dissipative/accretive, if and only if Θ is (maximal) dissipative/accretive.*

In particular, A_Θ is skew-adjoint, if and only if, Θ is skew-adjoint.

Proof.

(i) Since $\tilde{B}$ is bounded and boundedly invertible, we have

$$\overline{A_\Theta} = \overline{\tilde{B}^{-1}\Theta} = \tilde{B}^{-1}\overline{\Theta} = A_{\overline{\Theta}}.$$

(ii) Note that by $-A_0 \subseteq A_\Theta \subseteq A_0^*$, the adjoint of A_Θ is contained in $-A_0^*$. Moreover, by assumption we have

$$\left\langle\!\!\left\langle \begin{bmatrix} x \\ y \end{bmatrix}, \begin{bmatrix} u \\ v \end{bmatrix} \right\rangle\!\!\right\rangle_{X \times X} = \left\langle\!\!\left\langle \tilde{B} \begin{bmatrix} x \\ y \end{bmatrix}, \tilde{B} \begin{bmatrix} u \\ v \end{bmatrix} \right\rangle\!\!\right\rangle_{\mathcal{B} \times \mathcal{B}'} \tag{2.8}$$

for all $\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right], \left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right] \in A_0^*$. Hence, by Remark 2.2.5

$$\begin{bmatrix} u \\ v \end{bmatrix} \in -A_\Theta^* \quad \Leftrightarrow \quad \tilde{B}\begin{bmatrix} u \\ v \end{bmatrix} \in -\Theta^* \quad \Leftrightarrow \quad \begin{bmatrix} u \\ v \end{bmatrix} \in \underbrace{\tilde{B}^{-1}(-\Theta^*)}_{=A_{-\Theta^*}}.$$

(iii) Note that dissipativity of a linear relation R between Y and Y' can be characterized by $\langle\!\langle \left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right], \left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] \rangle\!\rangle_{Y \times Y'} \le 0$ for all $\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] \in R$. Hence, again by (2.8) we conclude the assertion. ❑

The next corollary is same result in a different notation, as it is presented in [28].

Corollary 2.4.11. *Let A_0 be a skew-symmetric operator and $(\mathcal{B}, B_1, B_2)$ be a boundary triple for A_0^*. Consider the restriction A of A_0^* to a subspace $\mathcal{D}$ containing* $\ker B_1 \cap \ker B_2$. *Define a subspace of $\mathcal{B} \times \mathcal{B}'$ by $\Theta := \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \mathcal{D}$. Then the following claims are true:*

(i) *The domain of A can be written as*

$$\operatorname{dom} A = \mathcal{D} = \left\{ d \in \operatorname{dom} A_0^* \;\middle|\; \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} d \in \Theta \right\}.$$

(ii) *The operator closure of A is A_0^* restricted to*

$$\tilde{\mathcal{D}} := \left\{ d \in \operatorname{dom} A_0^* \;\middle|\; \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} d \in \overline{\Theta} \right\},$$

where $\overline{\Theta}$ is the closure in $\mathcal{B}^2$. Therefore, A is closed, if and only if Θ is closed.

(iii) *The adjoint A^* is the restriction of $-A_0^*$ to $\mathcal{D}'$, where*

$$\mathcal{D}' := \left\{ d' \in \operatorname{dom} A_0^* \;\middle|\; \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} d' \in \underbrace{\begin{bmatrix} 0 & \mathrm{I} \\ \mathrm{I} & 0 \end{bmatrix} \Theta^\perp}_{=-\Theta^*} \right\}.$$

(iv) *The operator A is (maximal) dissipative if and only if Θ is a (maximal) dissipative relation. It also holds that A is (maximal) accretive, if and only if Θ is (maximal) accretive.*

2.5 Strongly Continuous Semigroups

We will be very minimalistic in this section and only introduce really necessary results. However, there is a lot more to say about strongly continuous semigroups. We refer to [15] for a detailed introduction.

Let A be a $n \times n$ matrix and $x_0 \in \mathbb{C}^n$ any initial vector. Then we regard the following differential equation (Cauchy problem)

$$\begin{aligned} \dot{x}(t) &= Ax(t), \quad t \in [0, +\infty) \\ x(0) &= x_0. \end{aligned}$$

The solution of this equation is given by $x(t) = \mathrm{e}^{tA}x_0$. The exponential function is not only defined for matrices, but also for bounded linear mappings on a Banach space. Hence, this approach to solve differential equations can easily extended to so called abstract Cauchy problems: Let X be a Banach space, A be a bounded linear mapping and $x_0 \in X$. Find a function $x\colon [0, +\infty)$ such that

$$\begin{aligned} \dot{x}(t) &= Ax(t), \quad t \in [0, +\infty) \\ x(0) &= x_0. \end{aligned} \tag{2.9}$$

Again the solution is given by $x(t) = \mathrm{e}^{tA}x_0$.

However, we want to go even further and want to solve this abstract Cauchy problem for unbounded operators. For unbounded operators the exponential function is harder to define or not even possible, but if A satisfies a few conditions we can find something that carries the essence to solve the abstract Cauchy problem.

Definition 2.5.1. Let X be a Banach space and $T\colon [0, +\infty) \to \mathcal{L}_\mathrm{b}(X)$. We say T is a *strongly continuous semigroup* or C_0-*semigroup*, if

- $T(0) = \mathrm{I}$,
- $T(t + s) = T(t)T(s)$ for all $t, s \in [0, +\infty)$,
- and $t \mapsto T(t)x$ is continuous for every $x \in X$, i.e. T is strongly continuous.

Note that it is actually enough to ask for T is strongly continuous in 0, as $T(t + s) = T(t)T(s)$ then already implies that T is strongly continuous in every $t \in [0, +\infty)$.

By the properties of the exponential function we can see that $T(t) := \mathrm{e}^{tA}$, for $A \in \mathcal{L}_\mathrm{b}(X)$, is a C_0-semigroup.

Definition 2.5.2. Let T be a strongly continuous semigroup on a Banach space X. We define its *infinitesimal generator* by

$$A := \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in X \times X \;\middle|\; y = \lim_{t \to 0} \frac{T(t)x - x}{t} \right\}.$$

Note that the infinitesimal generator A is an operator ($\operatorname{mul} A = \{0\}$), since limits in Hausdorff spaces are unique. So for $x \in \operatorname{dom} A$ we can also write

$$Ax = \lim_{t \to 0} \frac{T(t)x - x}{t}.$$

Lemma 2.5.3. *Let T be a strongly continuous semigroup. Then there exists an $M \geq 1$ and an $\omega \in \mathbb{R}$ such that*

$$\|T(t)\| \leq M\mathrm{e}^{\omega t} \quad \textit{for all} \quad t \in [0, +\infty).$$

Proof. First we will show that there is an $\epsilon > 0$ such that $\|T(t)\|$ is uniformly bounded for $t \in [0, \epsilon]$:

Let us assume that this is not true. Then for each $n \in \mathbb{N}$ we there exists a $t_n \in [0, \frac{1}{n}]$ such that

$$\|T(t_n)\| \geq n. \tag{2.10}$$

Since $(t_n)_{n\in\mathbb{N}}$ converges to 0 and T is strongly continuous we have $T(t_n)x \to x$ for all $x \in X$. Consequently, the set $\{T(t_n)x \,|\, n \in \mathbb{N}\}$ is bounded in X for every $x \in X$. The principle of uniform boundedness implies that the set $\{T(t_n) \,|\, n \in \mathbb{N}\}$ is bounded in $\mathcal{L}_\mathrm{b}(X)$, which contraticts (2.10). Thus there exists an $\epsilon > 0$ such that $\|T(t)\| \leq M$ on $[0, \epsilon]$.

We can write every $t = n\epsilon + \delta$, where $\delta < \epsilon$ and $n \in \mathbb{N}$ ($n = \lfloor \frac{t}{\epsilon} \rfloor$). This leads to

$$\|T(t)\| = \|T(n\epsilon + \delta)\| = \|T(\epsilon)^n T(\delta)\| \leq M^n M \leq M M^{\frac{t}{\epsilon}} = M\mathrm{e}^{\frac{1}{\epsilon}\ln(M)t}.$$

Defining ω as $\frac{1}{\epsilon}\ln(M)$ finishes the proof. ❑

Lemma 2.5.4. *Let T be a strongly continuous semigroup, A its infinitesimal generator and $x \in \operatorname{dom} A$. Then*

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(T(t)x\big) = T(t)Ax = AT(t)x.$$

Proof. Note that for fixed t the operator $T(t)$ is continuous. Therefore,

$$\begin{aligned}\lim_{s\to 0+} \frac{T(t+s)x - T(t)x}{s} &= \lim_{s\to 0+} \frac{T(t)T(s)x - T(t)x}{s} \\ &= T(t) \lim_{s\to 0+} \frac{T(s)x - x}{s} = T(t)Ax.\end{aligned}$$

On the other hand, we have to check the limit from the left hand side, which we can rewrite as a right hand side limit

$$\lim_{s\to 0-} \frac{T(t+s)x - T(t)x}{s} = \lim_{s\to 0+} \frac{T(t)x - T(t-s)x}{s}.$$

Hence, we have to check whether the limit agrees with $T(t)Ax$. Note that

$T(t) \leq M\mathrm{e}^{\omega t}$ (by Lemma 2.5.3) and that T is strongly continuous:

$$\begin{aligned}
&\left\| T(t-s)\frac{T(s)x-x}{s} - T(t)Ax \right\| \\
&\qquad \leq \left\| T(t-s)\frac{T(s)x-x}{s} - T(t-s)Ax \right\| + \|T(t-s)Ax - T(t)Ax\| \\
&\qquad \leq M\mathrm{e}^{\omega(t-s)} \underbrace{\left\| \frac{T(s)x-x}{s} - Ax \right\|}_{\to 0} + \underbrace{\|T(t-s)Ax - T(t)Ax\|}_{\to 0}.
\end{aligned}$$

Hence, $\frac{\mathrm{d}}{\mathrm{d}t}T(t)x = T(t)Ax$ and $T(t)x \in \operatorname{dom} A$. Therefore,

$$\lim_{s\to 0+} \frac{T(t+s)x - T(t)x}{s} = \lim_{s\to 0+} \frac{T(s)T(t)x - T(t)x}{s} = AT(t)x.$$

For the limit from left hand side we obtain the same since we have already shown that the limits agree. ❑

Now let T be a strongly continuous semigroup, A its infinitesimal generator and $x_0 \in \operatorname{dom} A$. Then the abstract Cauchy problem

$$\begin{aligned}
\dot{x}(t) &= Ax(t), \quad t \in [0, +\infty), \\
x(0) &= x_0,
\end{aligned}$$

is solved by $x(t) := T(t)x_0$, as

$$\dot{x}(t) = \frac{\mathrm{d}}{\mathrm{d}t}\big(T(t)x_0\big) = AT(t)x_0 = Ax(t)$$

and $x(0) = T(0)x_0 = x_0$. It can even be shown that this is the only solution, see [15, ch. II prop. 6.2].

Therefore, it is natural to ask when a linear operator A is an infinitesimal generator of a strongly continuous semigroup.

We can even extend the solution term for initial conditions that are not in $\operatorname{dom} A$.

Definition 2.5.5. We say a function $x\colon [0, +\infty) \to X$ is a *mild solution* of an abstract Cauchy problem (2.9), where A is an unbounded operator on X, if

$$x(t) - x(0) = A\int_0^t x(s)\,\mathrm{d}s,$$

where we implicitly demand that $\int_0^t x(s)\,\mathrm{d}s \in \operatorname{dom} A$.

Every mild solution is given by $x(\cdot) = T(\cdot)x_0$ for $x_0 \in X$, if A is the infinitesimal generator of $T(\cdot)$.

Since we are only interested in solutions that respect certain physical conservation laws, we restrict ourselves to semigroups that produce non-increasing solutions w.r.t. the norm (in our applications the norm will represent the energy).

Definition 2.5.6. We say a strongly continuous semigroup T is a *contraction semigroup*, if $\|T(t)\| \leq 1$ for all $t \in [0, +\infty)$.

Note that our definition of dissipativity only matches the standard definition in literature for Hilbert spaces.

Theorem 2.5.7 (Lumer-Phillips Theorem)**.** *Let A be a linear operator on a Hilbert space X. Then A is the infinitesimal generator of a contraction semigroup T, if and only if A is dissipative and* $\operatorname{ran}(A - \mathrm{I}) = X$*.*

Note that A is dissipative and $\operatorname{ran}(A - \mathrm{I}) = X$ is equivalent to A is maximal dissipative (in the Hilbert space case).

Proof. For the proof we refer to [15, ch. II, th. 3.15] ❑

Corollary 2.5.8. *Let A be a closed and densely defined linear operator on a Hilbert space X. Then A is the infinitesimal generator of a contraction semigroup T, if and only if A and A^* are dissipative.*

Proof. For the proof we refer to [15, ch. II, cor. 3.17]. ❑

Chapter 3

Port-Hamiltonian Systems

The port-Hamiltonian formulation has proven to be a powerful tool for the modeling and control of complex multiphysics systems. Port-Hamiltonian systems encode the underlying physical principles such as conservation laws directly into the structure of the system structure. An introductory overview can be found in [59]. This theory originates from B. M. Maschke and A. van der Schaft [36]. For finite-dimensional systems there is by now a well-established theory [58, 14, 13]. The port-Hamiltonian approach has been further extended to the infinite-dimensional situation, see e.g. [60, 30, 32, 26, 67, 61, 25, 28]. In [28] the authors showed that the port-Hamiltonian formulation of the wave equation in n spatial dimensions possess unique mild and classical solutions. We want to develop a port-Hamiltonian framework in n spatial dimension that provides existence and uniqueness of solutions.

In this chapter we will give a precise definition of what we understand under a linear first order port-Hamiltonian system. We aim to lift the theory of infinite dimensional port-Hamiltonian systems in one spatial variable, that is presented in the book of Jacob and Zwart [25] and Ph.D. thesis of Villegas [61], to n spatial variables—at least in some aspects. Although *Dirac structures* play an important role in most of the previous references, we choose a semigroup approach as in [25]. In the Ph.D. thesis [61] there is even one chapter dedicated to port-Hamiltonian systems in n spatial variables. We will adopt the system equation:

$$\begin{aligned} \frac{\partial}{\partial t} x(t,\zeta) &= \sum_{i=1}^{n} \frac{\partial}{\partial \zeta_i} P_i \mathcal{H}(\zeta) x(t,\zeta) + P_0 \mathcal{H}(\zeta) x(t,\zeta), & \zeta \in \Omega, t \geq 0, \\ x(0,\zeta) &= x_0(\zeta), & \zeta \in \Omega, \end{aligned}$$

where P_i are symmetric matrices, P_0 is a skew-symmetric matrix and $\mathcal{H}$ is the matrix-valued Hamiltonian density. The details are given in Definition 3.2.1. However, the theory in [61] is limited, e.g. it can not handle Maxwell's equations as it requires that the boundary operators establish a Gelfand triples. We will see that the boundary operators of Maxwell's equations cannot be extended to operators that map into $\mathsf{L}^2(\partial\Omega)$ in Example 5.1.8. We will overcome these limits by, among others, introducing quasi Gelfand triple in Chapter 4.

We will associate "natural" boundary operators to this PDE, which can be used to control and observe the system

$$\begin{aligned} u(t,\zeta) &= \mathcal{B}(\zeta)\mathcal{H}(\zeta)x(t,\zeta), && \zeta \in \partial\Omega, t \geq 0, \\ y(t,\zeta) &= \mathcal{C}(\zeta)\mathcal{H}(\zeta)x(t,\zeta), && \zeta \in \partial\Omega, t \geq 0. \end{aligned}$$

However, for now we restrict ourselves to the case of no input ($u = 0$), which essentially gives a boundary condition. We will later see that answering the question of existence and uniqueness of solutions for no input will be crucial also for non-zero input. We will also ignore the output function y for now as we focus on existence and uniqueness of solutions of the inner dynamic. We will regard the entire system (with input and output) in Chapter 6.

3.1 Differential Operators

Before we start analyzing port-Hamiltonian systems we will make some observations about the differential operators that will appear in the PDE. In this section we take care of all the technical details of these differential operators. Since it doesn't really make a difference whether we use the scalar field $\mathbb{R}$ or $\mathbb{C}$ we will use $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ for the scalar field. The following assumption will be made for the rest of this chapter.

Assumption 3.1.1. Let $m_1, m_2, n \in \mathbb{N}$, $\Omega \subseteq \mathbb{R}^n$ be open with a bounded Lipschitz boundary, and $L = (L_i)_{i=1}^n$ such that $L_i \in \mathbb{K}^{m_1 \times m_2}$ for all $i \in \{1, \ldots, n\}$. Corresponding to L we also have $L^{\mathsf{H}} := (L_i^{\mathsf{H}})_{i=1}^n$, where L_i^{H} denotes the complex conjugated transposed (Hermitian transposed) matrix.

By bounded Lipschitz boundary we mean that the surface measure of the boundary is finite. Hence, we can also regard the exterior of a domain. Moreover, $\Omega = \mathbb{R}^n$ is also allowed as the boundary of $\mathbb{R}^n$ is empty.

Moreover, we will write $\mathcal{D}(\mathbb{R}^n)\big|_\Omega$ for $\{f\big|_\Omega \,|\, f \in \mathcal{D}(\mathbb{R}^n)\}$. We will use ∂_i as a short notation for $\frac{\partial}{\partial\zeta_i}$. We denote the boundary trace operator by $\gamma_0 \colon \mathsf{H}^1(\Omega, X) \to \mathsf{L}^2(\partial\Omega, X)$ for a Banach space X.

Definition 3.1.2. Let L be as in Assumption 3.1.1. Then we define

$$L_\partial := \sum_{i=1}^n \partial_i L_i \quad \text{and} \quad L_\partial^{\mathsf{H}} := (L^{\mathsf{H}})_\partial = \sum_{i=1}^n \partial_i L_i^{\mathsf{H}}$$

as operators from $\mathcal{D}'(\Omega)^{m_2}$ to $\mathcal{D}'(\Omega)^{m_1}$ and from $\mathcal{D}'(\Omega)^{m_1}$ to $\mathcal{D}'(\Omega)^{m_2}$, respectively. Furthermore, we define the space

$$\mathsf{H}(L_\partial, \Omega) := \left\{ f \in \mathsf{L}^2(\Omega, \mathbb{K}^{m_2}) \,\middle|\, L_\partial f \in \mathsf{L}^2(\Omega, \mathbb{K}^{m_1}) \right\}.$$

This space is endowed with the inner product

$$\langle f, g\rangle_{\mathsf{H}(L_\partial,\Omega)} := \langle f, g\rangle_{\mathsf{L}^2(\Omega,\mathbb{K}^{m_2})} + \langle L_\partial f, L_\partial g\rangle_{\mathsf{L}^2(\Omega,\mathbb{K}^{m_1})}.$$

The space $\mathsf{H}_0(L_\partial, \Omega)$ is defined as $\overline{\mathcal{D}(\Omega)^{m_2}}^{\|\cdot\|_{\mathsf{H}(L_\partial,\Omega)}}$, the closure of $\mathcal{D}(\Omega)^{m_2}$ in $\mathsf{H}(L_\partial, \Omega)$. We denote the outward pointing normalized normal vector on $\partial\Omega$ by ν and its i-th component by ν_i. Moreover, we define

$$L_\nu := \sum_{i=1}^{n} \nu_i L_i \colon \begin{cases} \mathsf{L}^2(\partial\Omega, \mathbb{K}^{m_2}) & \to \quad \mathsf{L}^2(\partial\Omega, \mathbb{K}^{m_1}), \\ \qquad\qquad f & \mapsto \quad \sum_{i=1}^{n} \nu_i L_i f, \end{cases}$$

and $L_\nu^{\mathsf{H}} := (L^{\mathsf{H}})_\nu$.

The operator L_∂ can also be regarded as a linear unbounded operator from $\mathsf{L}^2(\Omega, \mathbb{K}^{m_2})$ to $\mathsf{L}^2(\Omega, \mathbb{K}^{m_1})$ with domain $\mathsf{H}(L_\partial, \Omega)$. In fact this is what we will do most of the time. The same goes for L_∂^{H} with domain $\mathsf{H}(L_\partial^{\mathsf{H}}, \Omega)$. Since $\nu \in \mathsf{L}^\infty(\partial\Omega, \mathbb{R}^n)$ the mappings L_ν and L_ν^{H} are well-defined and bounded.

For convenience we will write $\mathsf{H}^1(\Omega)^k$ instead of $\mathsf{H}^1(\Omega, \mathbb{K}^k)$ and $\mathsf{L}^2(\Omega)^k$ instead of $\mathsf{L}^2(\Omega, \mathbb{K}^k)$ for $k \in \mathbb{N}$.

Clearly, $\mathcal{D}(\mathbb{R}^n)^{m_2}\big|_\Omega \subseteq \mathsf{H}^1(\Omega)^{m_2} \subseteq \mathsf{H}(L_\partial, \Omega)$ and $\mathcal{D}(\mathbb{R}^n)^{m_1}\big|_\Omega \subseteq \mathsf{H}^1(\Omega)^{m_1} \subseteq \mathsf{H}(L_\partial^{\mathsf{H}}, \Omega)$.

Example 3.1.3. Let us regard the following matrices

$$L_1 = \begin{bmatrix}1 & 0 & 0\end{bmatrix}, \quad L_2 = \begin{bmatrix}0 & 1 & 0\end{bmatrix}, \quad \text{and} \quad L_3 = \begin{bmatrix}0 & 0 & 1\end{bmatrix}.$$

Then we obtain the corresponding differential operators

$$L_\partial = \begin{bmatrix}\partial_1 & \partial_2 & \partial_3\end{bmatrix} = \operatorname{div} \quad \text{and} \quad L_\partial^{\mathsf{H}} = \begin{bmatrix}\partial_1 \\ \partial_2 \\ \partial_3\end{bmatrix} = \operatorname{grad}.$$

The corresponding operator L_ν that acts on $\mathsf{L}^2(\partial\Omega)$ can be written as an inner product

$$L_\nu f = \begin{bmatrix}\nu_1 & \nu_2 & \nu_3\end{bmatrix} \begin{bmatrix}f_1 \\ f_2 \\ f_3\end{bmatrix} = \nu \cdot f.$$

Clearly, the previous example can be extended to any finite dimension.

Example 3.1.4. The following matrices will construct the rotation operator.

$$L_1 = \begin{bmatrix}0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0\end{bmatrix}, \quad L_2 = \begin{bmatrix}0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0\end{bmatrix}, \quad \text{and} \quad L_3 = \begin{bmatrix}0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0\end{bmatrix}.$$

In this example we have $L_i^{\mathsf{H}} = -L_i$. Furthermore, the corresponding differential operator is

$$L_\partial = \begin{bmatrix} 0 & -\partial_3 & \partial_2 \\ \partial_3 & 0 & -\partial_1 \\ -\partial_2 & \partial_1 & 0 \end{bmatrix} = \mathrm{rot} = -L_\partial^{\mathsf{H}}.$$

The corresponding operator L_ν that acts on $\mathsf{L}^2(\partial\Omega)$ can be written as a vector cross product

$$L_\nu f = \begin{bmatrix} 0 & -\nu_3 & \nu_2 \\ \nu_3 & 0 & -\nu_1 \\ -\nu_2 & \nu_1 & 0 \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} = \nu \times f.$$

The previous two examples give us a definition of the spaces $\mathsf{H}(\mathrm{rot}, \Omega)$, $\mathsf{H}(\mathrm{div}, \Omega)$ and $\mathsf{H}(\mathrm{grad}, \Omega)$ by the corresponding L and $\mathsf{H}(L_\partial, \Omega)$. These definition matches the standard definition in literature. It is easy to see that $\mathsf{H}(\mathrm{grad}, \Omega) = \mathsf{H}^1(\Omega)$.

Lemma 3.1.5. *The operator L_∂ with $\operatorname{dom} L_\partial = \mathsf{H}(L_\partial, \Omega)$ is a closed operator from $\mathsf{L}^2(\Omega)^{m_2}$ to $\mathsf{L}^2(\Omega)^{m_1}$ and $\mathsf{H}(L_\partial, \Omega)$ endowed with the inner product $\langle \cdot, \cdot \rangle_{\mathsf{H}(L_\partial, \Omega)}$ is a Hilbert space.*

Note that for $f \in \mathcal{D}'(\Omega)^{m_2}$ and $\phi \in \mathcal{D}(\Omega)^{m_1}$ we have

$$\begin{aligned} \langle L_\partial f, \phi \rangle_{\mathcal{D}'(\Omega)^{m_1}, \mathcal{D}(\Omega)^{m_1}} &= \sum_{i=1}^{n} \langle \partial_i L_i f, \phi \rangle_{\mathcal{D}'(\Omega)^{m_1}, \mathcal{D}(\Omega)^{m_1}} \\ &= \sum_{i=1}^{n} \langle f, -\partial_i L_i^{\mathsf{H}} \phi \rangle_{\mathcal{D}'(\Omega)^{m_2}, \mathcal{D}(\Omega)^{m_2}} = \langle f, -L_\partial^{\mathsf{H}} \phi \rangle_{\mathcal{D}'(\Omega)^{m_2}, \mathcal{D}(\Omega)^{m_2}}. \end{aligned}$$

Proof. Let $\left(\begin{bmatrix} f_k \\ L_\partial f_k \end{bmatrix}\right)_{k \in \mathbb{N}}$ be a sequence in L_∂ that converges to a point $\begin{bmatrix} f \\ g \end{bmatrix} \in \mathsf{L}^2(\Omega)^{m_2} \times \mathsf{L}^2(\Omega)^{m_1}$. For an arbitrary $\phi \in \mathcal{D}(\Omega)^{m_1}$ we have

$$\begin{aligned} \langle g, \phi \rangle_{\mathcal{D}'(\Omega)^{m_1}, \mathcal{D}(\Omega)^{m_1}} &= \lim_{k \to \infty} \langle L_\partial f_k, \phi \rangle_{\mathcal{D}'(\Omega)^{m_1}, \mathcal{D}(\Omega)^{m_1}} \\ &= \lim_{k \to \infty} \langle f_k, -L_\partial^{\mathsf{H}} \phi \rangle_{\mathcal{D}'(\Omega)^{m_1}, \mathcal{D}(\Omega)^{m_1}} \\ &= \langle f, -L_\partial^{\mathsf{H}} \phi \rangle_{\mathcal{D}'(\Omega)^{m_2}, \mathcal{D}(\Omega)^{m_2}} \\ &= \langle L_\partial f, \phi \rangle_{\mathcal{D}'(\Omega)^{m_1}, \mathcal{D}(\Omega)^{m_1}}, \end{aligned}$$

which implies $g = L_\partial f$. Since g is also in $\mathsf{L}^2(\Omega)^{m_1}$, we conclude that L_∂ is closed. Hence, $\operatorname{dom} L_\partial = \mathsf{H}(L_\partial, \Omega)$ endowed with the graph norm of L_∂, which is induced by $\langle \cdot, \cdot \rangle_{\mathsf{H}(L_\partial, \Omega)}$, is a Hilbert space. ❑

Lemma 3.1.6. *The adjoint of L_∂ with $\operatorname{dom} L_\partial = \mathsf{H}(L_\partial, \Omega)$ (as an unbounded operator/linear relation from $\mathsf{L}^2(\Omega)^{m_2}$ to $\mathsf{L}^2(\Omega)^{m_1}$) is given by $L_\partial^* g = -L_\partial^{\mathsf{H}} g$ for $g \in \operatorname{dom} L_\partial^* \subseteq \mathsf{H}(L_\partial^{\mathsf{H}}, \Omega)$, i.e. $L_\partial^* \subseteq -L_\partial^{\mathsf{H}}$.*

Proof. For an arbitrary $g \in \operatorname{dom} L_\partial^*$ and an arbitrary $\phi \in \mathcal{D}(\Omega)^{m_2}$ we have

$$\langle L_\partial^* g, \phi\rangle_{\mathcal{D}',\mathcal{D}} = \langle L_\partial^* g, \phi\rangle_{\mathsf{L}^2} = \langle g, L_\partial \phi\rangle_{\mathsf{L}^2} = \langle g, L_\partial \phi\rangle_{\mathcal{D}',\mathcal{D}} = \langle -L_\partial^{\mathsf{H}} g, \phi\rangle_{\mathcal{D}',\mathcal{D}}.$$

Therefore, $L_\partial^* g = -L_\partial^{\mathsf{H}} g$ and $L_\partial^* g \in \mathsf{L}^2(\Omega)^{m_2}$ implies $L_\partial^{\mathsf{H}} g \in \mathsf{L}^2(\Omega)^{m_2}$. Consequently, $\operatorname{dom} L_\partial^* \subseteq \mathsf{H}(L_\partial^{\mathsf{H}}, \Omega)$. ❑

Remark 3.1.7. If L contains only Hermitian matrices ($L_i^{\mathsf{H}} = L_i$), then $L_\partial^{\mathsf{H}} = L_\partial$ and L_∂^* is skew-symmetric by the previous lemma.

The next result is an integration by parts version for L_∂. This will be helpful to construct a boundary triple for the differential operator in the port-Hamiltonian PDE.

Lemma 3.1.8. *Let $f \in \mathsf{H}^1(\Omega)^{m_2}$ and $g \in \mathsf{H}^1(\Omega)^{m_1}$. Then we have*

$$\begin{aligned}\langle L_\partial f, g\rangle_{\mathsf{L}^2(\Omega)^{m_1}} + \langle f, L_\partial^{\mathsf{H}} g\rangle_{\mathsf{L}^2(\Omega)^{m_2}} &= \langle L_\nu \gamma_0 f, \gamma_0 g\rangle_{\mathsf{L}^2(\partial\Omega)^{m_1}} \\ &= \langle \gamma_0 f, L_\nu^{\mathsf{H}} \gamma_0 g\rangle_{\mathsf{L}^2(\partial\Omega)^{m_2}}.\end{aligned} \tag{3.1}$$

Proof. Let $f \in \mathcal{D}(\mathbb{R}^n)^{m_2}\big|_\Omega$ and $g \in \mathcal{D}(\mathbb{R}^n)^{m_1}\big|_\Omega$. By the definition of L_∂ and L_∂^{H}, and the linearity of the scalar product we can write the left-hand-side of (3.1) as

$$\int_\Omega \sum_{i=1}^n \langle \partial_i L_i f, g\rangle + \langle f, \partial_i L_i^{\mathsf{H}} g\rangle \,\mathrm{d}\lambda = \int_\Omega \sum_{i=1}^n \langle \partial_i L_i f, g\rangle + \langle L_i f, \partial_i g\rangle \,\mathrm{d}\lambda,$$

where λ denotes the Lebesgue measure. By the product rule for derivatives and Gauß's theorem (divergence theorem, Theorem 1.2.8) this is equal to

$$\int_\Omega \sum_{i=1}^n \partial_i \langle L_i f, g\rangle \,\mathrm{d}\lambda = \int_{\partial\Omega} \sum_{i=1}^n \nu_i \gamma_0 \langle L_i f, g\rangle \,\mathrm{d}\mu = \int_{\partial\Omega} \langle L_\nu \gamma_0 f, \gamma_0 g\rangle \,\mathrm{d}\mu,$$

where μ denotes the surface measure of $\partial\Omega$. By density we can extend this equality for $f \in \mathsf{H}^1(\Omega)^{m_2}$ and $g \in \mathsf{H}^1(\Omega)^{m_1}$. ❑

Note that Gauß's theorem (Theorem 1.2.8) cannot be extended to $\mathsf{H}^1(\Omega)^n$ for an unbounded Ω as we have already remarked (see [57, Re. 13.7.4]). However, in (3.1) the dependencies on f and g are continuous w.r.t. the norm of $\mathsf{H}^1(\Omega)$.

Corollary 3.1.9. *Let $f \in \mathsf{H}^1(\Omega)^{m_2}$ and $g \in \mathsf{H}^1(\Omega)^{m_1}$. Then we have*

$$\left|\langle L_\nu \gamma_0 f, \gamma_0 g\rangle_{\mathsf{L}^2(\partial\Omega)^{m_1}}\right| \le \|f\|_{\mathsf{H}(L_\partial,\Omega)} \|g\|_{\mathsf{H}(L_\partial^{\mathsf{H}},\Omega)}.$$

Proof. Lemma 3.1.8, the triangle inequality and the Cauchy Schwarz inequality yield

$$\begin{aligned}\left|\langle L_\nu \gamma_0 f, \gamma_0 g\rangle_{\mathsf{L}^2(\partial\Omega)^{m_1}}\right| &\le \left|\langle L_\partial f, g\rangle_{\mathsf{L}^2(\Omega)^{m_1}}\right| + \left|\langle f, L_\partial^{\mathsf{H}} g\rangle_{\mathsf{L}^2(\Omega)^{m_2}}\right| \\ &\le \|L_\partial f\|_{\mathsf{L}^2(\Omega)^{m_1}} \|g\|_{\mathsf{L}^2(\Omega)^{m_1}} + \|f\|_{\mathsf{L}^2(\Omega)^{m_2}} \|L_\partial^{\mathsf{H}} g\|_{\mathsf{L}^2(\Omega)^{m_2}} \\ &\le \sqrt{\|L_\partial f\|_{\mathsf{L}^2}^2 + \|f\|_{\mathsf{L}^2}^2} \sqrt{\|g\|_{\mathsf{L}^2}^2 + \|L_\partial^{\mathsf{H}} g\|_{\mathsf{L}^2}^2} \\ &= \|f\|_{\mathsf{H}(L_\partial,\Omega)} \|g\|_{\mathsf{H}(L_\partial^{\mathsf{H}},\Omega)}.\end{aligned}$$

❑

Note that $\Omega = \mathbb{R}^n$ satisfies the assumptions in Assumption 3.1.1. Hence, all the previous results hold true for $\Omega = \mathbb{R}^n$ (and also the following).

Our next goal is to show that $\mathcal{D}(\mathbb{R}^n)^{m_2}\big|_\Omega$ is dense in $\mathsf{H}(L_\partial, \Omega)$; see Theorem 3.1.18. In order to archive this we will present some regularization and continuity results. In particular the density is needed to extend the integration by parts formula (Lemma 3.1.8) for $f \in \mathsf{H}(L_\partial, \Omega)$ and $g \in \mathsf{H}(L_\partial^{\mathsf{H}}, \Omega)$.

Lemma 3.1.10. *The mapping* $\iota\colon \mathsf{H}(L_\partial, \mathbb{R}^n) \to \mathsf{H}(L_\partial, \Omega), f \mapsto f\big|_\Omega$ *is well-defined and continuous for any open set* $\Omega \subseteq \mathbb{R}^n$*. In particular,* $L_\partial(f\big|_\Omega) = (L_\partial f)\big|_\Omega$*. Moreover, if* $f_k \to f$ *in* $\mathsf{H}(L_\partial, \mathbb{R}^n)$*, then* $f_k \to f$ *in* $\mathsf{H}(L_\partial, \Omega)$*.*

Hence, we can always regard an $f \in \mathsf{H}(L_\partial, \mathbb{R}^n)$ as an element of $\mathsf{H}(L_\partial, \Omega)$, especially when $\operatorname{supp} f \subseteq \overline{\Omega}$ – then it is also possible to recover f from $f\big|_\Omega$.

Proof. If $f \in \mathsf{H}(L_\partial, \mathbb{R}^n)$, then $f \in \mathsf{L}^2(\mathbb{R}^n)^{m_2}$ and $L_\partial f \in \mathsf{L}^2(\mathbb{R}^n)^{m_1}$. Hence, it is easy to see that $\|f\big|_\Omega\|_{\mathsf{L}^2(\Omega)} \leq \|f\|_{\mathsf{L}^2(\mathbb{R}^n)}$ and $\|(L_\partial f)\big|_\Omega\|_{\mathsf{L}^2(\Omega)} \leq \|L_\partial f\|_{\mathsf{L}^2(\mathbb{R}^n)}$. Note that $\mathcal{D}(\Omega) \subseteq \mathcal{D}(\mathbb{R}^n)$, and that for $g \in \mathsf{L}^2(\mathbb{R}^n)$ and $\phi \in \mathcal{D}(\Omega)$

$$\langle g, \phi \rangle_{\mathcal{D}'(\mathbb{R}^n), \mathcal{D}(\mathbb{R}^n)} = \langle g, \phi \rangle_{\mathsf{L}^2(\mathbb{R}^n)} = \big\langle g\big|_\Omega, \phi \big\rangle_{\mathsf{L}^2(\Omega)} = \big\langle g\big|_\Omega, \phi \big\rangle_{\mathcal{D}'(\Omega), \mathcal{D}(\Omega)}.$$

Hence, for $f \in \mathsf{H}(L_\partial, \mathbb{R}^n)$ and $\phi \in \mathcal{D}(\Omega)$ we have

$$\begin{aligned}
\big\langle L_\partial(f\big|_\Omega), \phi \big\rangle_{\mathcal{D}'(\Omega), \mathcal{D}(\Omega)} &= \big\langle f\big|_\Omega, -L_\partial^{\mathsf{H}} \phi \big\rangle_{\mathcal{D}'(\Omega), \mathcal{D}(\Omega)} = \langle f, -L_\partial^{\mathsf{H}} \phi \rangle_{\mathcal{D}'(\mathbb{R}^n), \mathcal{D}(\mathbb{R}^n)} \\
&= \langle L_\partial f, \phi \rangle_{\mathcal{D}'(\mathbb{R}^n), \mathcal{D}(\mathbb{R}^n)} \\
&= \big\langle (L_\partial f)\big|_\Omega, \phi \big\rangle_{\mathcal{D}'(\Omega), \mathcal{D}(\Omega)},
\end{aligned}$$

which implies $L_\partial(f\big|_\Omega) = (L_\partial f)\big|_\Omega$ in $\mathcal{D}'(\Omega)$. Since the latter is in $\mathsf{L}^2(\Omega)$, we conclude $f\big|_\Omega \in \mathsf{H}(L_\partial, \Omega)$. Consequently, ι is well-defined and $\|\iota f\|_{\mathsf{H}(L_\partial, \Omega)} \leq \|f\|_{\mathsf{H}(L_\partial, \mathbb{R}^n)}$ by the norm estimates from the beginning. Since ι is linear this implies the continuity of ι and in turn the last assertion of the lemma. ❑

Lemma 3.1.11. *Let* $D_\eta\colon \mathsf{L}^2(\mathbb{R}^n)^k \to \mathsf{L}^2(\mathbb{R}^n)^k$ *be the mapping defined by*

$$(D_\eta f)(\zeta) := f(\eta\zeta),$$

where $\eta \in (0, +\infty)$ *and* $k \in \mathbb{N}$*. Then* D_η *converges in the strong operator topology to* I *for* $\eta \to 1$*.*

Proof. For $\phi \in \mathcal{D}(\mathbb{R}^n)^k$ we will show that $\eta \mapsto D_\eta \phi$ from $(0, +\infty)$ to $\mathsf{L}^2(\mathbb{R}^n)^k$ is continuous:

$$\begin{aligned}
\|D_{\eta_1}\phi - D_{\eta_2}\phi\|_{\mathsf{L}^2}^2 &= \int_{\mathbb{R}^n} \|\phi(\eta_1\zeta) - \phi(\eta_2\zeta)\|_{\mathbb{K}^k}^2 \,\mathrm{d}\lambda(\zeta) \\
&= \frac{1}{\eta_2^{2n}} \int_{\mathbb{R}^n} \Big\|\phi\Big(\frac{\eta_1}{\eta_2}\zeta\Big) - \phi(\zeta)\Big\|_{\mathbb{K}^k}^2 \,\mathrm{d}\lambda(\zeta) \to 0 \quad \text{for} \quad \eta_2 \to \eta_1
\end{aligned}$$

by Lebesgue's dominated convergence theorem, where λ denotes the Lebesgue measure. For $f \in \mathsf{L}^2(\mathbb{R}^n)^k$ there exists a sequence $(\phi_m)_{m\in\mathbb{N}}$ of $\mathcal{D}(\mathbb{R}^n)^k$ functions that converges to f (w.r.t. $\|\cdot\|_{\mathsf{L}^2}$). Hence,

$$\|D_\eta\phi_m - D_\eta f\|_{\mathsf{L}^2} = \frac{1}{\eta^n}\|\phi_m - f\|_{\mathsf{L}^2}$$

and $D_\eta\phi_m$ converges uniformly in $\eta \in (\epsilon, +\infty), \epsilon > 0$ to $D_\eta f$ for $m \to \infty$. Consequently $\eta \mapsto D_\eta f$ is also continuous from $(\epsilon, +\infty)$ to $\mathsf{L}^2(\Omega)^k$ and in particular $D_\eta f \to f$ for $\eta \to 1$. ❑

Definition 3.1.12. A set $O \subseteq \mathbb{R}^n$ is *strongly star-shaped with respect to* ζ_0, if for every $\zeta \in \overline{O}$ the half-open line segment $\{\theta(\zeta - \zeta_0) + \zeta_0 \,|\, \theta \in [0,1)\}$ is contained in O. We call O *strongly star-shaped*, if there is a ζ_0 such that O is strongly star-shaped with respect to ζ_0.

Note that this is equivalent to

$$\theta(\overline{O} - \zeta_0) + \zeta_0 \subseteq O \quad \text{for all} \quad \theta \in [0,1).$$

Lemma 3.1.13. *Let $f \in \mathsf{H}(L_\partial, \mathbb{R}^n)$ and $\zeta_0 \in \mathbb{R}^n$. Furthermore, let $f_\theta(\zeta) := f(\frac{1}{\theta}(\zeta - \zeta_0) + \zeta_0)$ for $\theta \in (0,1)$ and a.e. $\zeta \in \mathbb{R}^n$. Then $f_\theta \in \mathsf{H}(L_\partial, \mathbb{R}^n)$ and $f_\theta \to f$ in $\mathsf{H}(L_\partial, \mathbb{R}^n)$ as $\theta \to 1$. If there exists a strongly star-shaped set O with respect to the previous ζ_0 such that* $\operatorname{supp} f \subseteq \overline{O}$, *then* $\operatorname{supp} f_\theta \subseteq O$ *for* $\theta \in (0,1)$.

Proof. Let $f \in \mathsf{H}(L_\partial, \mathbb{R}^n)$ and $\alpha(\zeta) := \frac{1}{\theta}(\zeta - \zeta_0) + \zeta_0$. Then it is easy to see that $f_\theta = f \circ \alpha$ and $f_\theta \in \mathsf{L}^2(\mathbb{R}^n)^{m_2}$. By change of variables we have

$$\begin{aligned}
&\langle L_\partial(f \circ \alpha), \phi\rangle_{\mathcal{D}'(\mathbb{R}^n),\mathcal{D}(\mathbb{R}^n)} \\
&\quad= \langle f, -(L_\partial^{\mathsf{H}}\phi) \circ \alpha^{-1}\theta^n\rangle_{\mathsf{L}^2(\mathbb{R}^n)} = \Big\langle f, -\sum_{i=1}^n L_i^{\mathsf{H}}\partial_i\Big(\phi \circ \alpha^{-1}\frac{1}{\theta}\Big)\theta^n\Big\rangle_{\mathsf{L}^2(\mathbb{R}^n)} \\
&\quad= \Big\langle f, -L_\partial^{\mathsf{H}}\Big(\frac{1}{\theta}\phi \circ \alpha^{-1}\Big)\theta^n\Big\rangle_{\mathsf{L}^2(\mathbb{R}^n)} = \Big\langle \frac{1}{\theta}(L_\partial f) \circ \alpha, \phi\Big\rangle_{\mathsf{L}^2(\mathbb{R}^n)} \\
&\quad= \Big\langle \frac{1}{\theta}(L_\partial f) \circ \alpha, \phi\Big\rangle_{\mathcal{D}'(\mathbb{R}^n),\mathcal{D}(\mathbb{R}^n)}.
\end{aligned}$$

Therefore, $L_\partial f_\theta = \frac{1}{\theta}(L_\partial f)_\theta$ and $f_\theta \in \mathsf{H}(L_\partial, \mathbb{R}^n)$. We can also write f_θ as $T_{\zeta_0}D_{\frac{1}{\theta}}T_{-\zeta_0}f$, where $T_\xi\colon \mathsf{L}^2(\mathbb{R}^n)^{m_2} \to \mathsf{L}^2(\mathbb{R}^n)^{m_2}$ is the translation mapping $f \mapsto f(\cdot + \xi)$ and $D_\eta\colon \mathsf{L}^2(\mathbb{R}^n)^{m_2} \to \mathsf{L}^2(\mathbb{R}^n)^{m_2}$ is the mapping from Lemma 3.1.11. Since T_ξ is bounded and D_η converges strongly to I as $\eta \to 1$, we conclude $f_\theta \to f$ in $\mathsf{L}^2(\mathbb{R}^n)^{m_2}$ as $\theta \to 1$ and $L_\partial f_\theta = \frac{1}{\theta}(L_\partial f)_\theta \to L_\partial f$ in $\mathsf{L}^2(\mathbb{R}^n)^{m_1}$ as $\theta \to 1$. Hence, $f_\theta \to f$ in $\mathsf{H}(L_\partial, \mathbb{R}^n)$.

Let O be strongly star-shaped with respect to ζ_0 and $\operatorname{supp} f \subseteq \overline{O}$. Then for $\theta \in (0,1)$

$$\operatorname{supp} f_\theta = \theta(\operatorname{supp} f - \zeta_0) + \zeta_0 \subseteq \theta(\overline{O} - \zeta_0) + \zeta_0 \subseteq O.$$ ❑

Lemma 3.1.14. *If* $f \in \mathsf{H}(L_\partial, \Omega)$ *and* $\psi \in \mathcal{D}(\mathbb{R}^n)\big|_\Omega$, *then also* $\psi f \in \mathsf{H}(L_\partial, \Omega)$ *and*

$$L_\partial(\psi f) = \psi L_\partial f + \sum_{i=1}^n (\partial_i \psi) L_i f.$$

Proof. Note that $\langle \psi f, \phi \rangle_{\mathcal{D}',\mathcal{D}} = \langle f, \overline{\psi}\phi \rangle_{\mathcal{D}',\mathcal{D}}$ and by the product rule

$$\overline{\psi} L_i^{\mathsf{H}} \partial_i \phi = \partial_i(\overline{\psi} L_i^{\mathsf{H}} \phi) - (\partial_i \overline{\psi}) L_i^{\mathsf{H}} \phi.$$

Hence,

$$\begin{aligned}
\langle L_\partial(\psi f), \phi \rangle_{\mathcal{D}',\mathcal{D}} &= -\big\langle \psi f, L_\partial^{\mathsf{H}} \phi \big\rangle_{\mathcal{D}',\mathcal{D}} = -\sum_{i=1}^n \big\langle f, \overline{\psi} L_i \partial_i^{\mathsf{H}} \phi \big\rangle_{\mathcal{D}',\mathcal{D}} \\
&= -\sum_{i=1}^n \Big\langle f, \partial_i(\overline{\psi} L_i^{\mathsf{H}} \phi) - (\partial_i \overline{\psi}) L_i^{\mathsf{H}} \phi \Big\rangle_{\mathcal{D}',\mathcal{D}} \\
&= \sum_{i=1}^n -\big\langle f, \partial_i(L_i^{\mathsf{H}} \overline{\psi} \phi) \big\rangle_{\mathcal{D}',\mathcal{D}} + \big\langle f, (\partial_i \overline{\psi}) L_i^{\mathsf{H}} \phi \big\rangle_{\mathcal{D}',\mathcal{D}} \\
&= \sum_{i=1}^n \langle \psi L_i \partial_i f, \phi \rangle_{\mathcal{D}',\mathcal{D}} + \langle (\partial_i \psi) L_i f, \phi \rangle_{\mathcal{D}',\mathcal{D}} \\
&= \langle \psi L_\partial f, \phi \rangle_{\mathcal{D}',\mathcal{D}} + \Big\langle \sum_{i=1}^n (\partial_i \psi) L_i f, \phi \Big\rangle_{\mathcal{D}',\mathcal{D}} \\
&= \Big\langle \psi L_\partial f + \sum_{i=1}^n (\partial_i \psi) L_i f, \phi \Big\rangle_{\mathcal{D}',\mathcal{D}}.
\end{aligned}$$

Thus, $L_\partial(\psi f) = \psi L_\partial f + \sum_{i=1}^n (\partial_i \psi) L_i f$. ❑

Lemma 3.1.15. *For every* $f \in \mathsf{H}(L_\partial, \mathbb{R}^n)$ *exists a sequence* $(f_k)_{k\in\mathbb{N}}$ *in* $\mathsf{H}(L_\partial, \mathbb{R}^n)$, *whose terms have compact support* $\operatorname{supp} f_k \subseteq \operatorname{supp} f$, *that converges to* f *in* $\mathsf{H}(L_\partial, \mathbb{R}^n)$.

Proof. Let $\psi \in \mathsf{C}^\infty(\mathbb{R}^n, \mathbb{R})$ be such that

$$\psi(\zeta) \in \begin{cases} \{1\}, & \text{if } \|\zeta\| \leq 1, \\ [0,1], & \text{if } 1 < \|\zeta\| < 2, \\ \{0\}, & \text{if } \|\zeta\| \geq 2. \end{cases}$$

Then $f_k := \psi(\frac{\cdot}{k}) f \in \mathsf{L}^2(\mathbb{R}^n)^{m_2}$ and $f_k \to f$ in L^2. By Lemma 3.1.14 we have $L_\partial f_k = \psi(\frac{\cdot}{k}) L_\partial f + \frac{1}{k}\sum_{i=1}^n (\partial_i \psi)(\frac{\cdot}{k}) L_i f$ and therefore $f_k \in \mathsf{H}(L_\partial, \mathbb{R}^n)$. Since $\|\partial_i \psi\|_\infty < \infty$ and $\|L_i f\|_{\mathsf{L}^2} \leq \|L_i\| \|f\|_{\mathsf{L}^2} < \infty$, we have $L_\partial f_k \to L_\partial f$ as $\psi(\frac{\cdot}{k}) L_\partial f \to L_\partial f$ in $\mathsf{L}^2(\mathbb{R}^n)^{m_2}$ and consequently $f_k \to f$ in $\mathsf{H}(L_\partial, \mathbb{R}^n)$. ❑

The next result is essentially [9, Proposition 2.5.4, page 69], except that we allow Ω to be unbounded.

Lemma 3.1.16. *For $\Omega \subseteq \mathbb{R}^n$ (open with bounded Lipschitz boundary) there exists an open covering $(O_i)_{i=0}^k$ of $\overline{\Omega}$ such that $O_i \cap \Omega$ is bounded and strongly star-shaped for $i \in \{1, \ldots, k\}$ and $\overline{O_0} \subseteq \Omega$.*

Proof. Since Ω has a bounded Lipschitz boundary, there is an open ball $B_r(0)$ such that $\partial\Omega \subseteq B_r(0)$. Hence, $B_r(0) \cap \Omega$ is bounded and open with bounded Lipschitz boundary and we can apply [9, Proposition 2.5.4, page 69]. This gives an open covering $(O_i)_{i=1}^k$ of $B_r(0) \cap \Omega$ and in particular of $\partial\Omega$ such that $O_i \cap \Omega$ is strongly star-shaped. We define O_0 as $B_\epsilon(\Omega \setminus \bigcup_{i=1}^k O_i)$, where $\epsilon > 0$ is small enough such that $\overline{O_0} \subseteq \Omega$. ❑

The next lemma is similar to [12, Lemma 1, page 206], which proves the result for $L_\partial = \operatorname{rot}$. The main idea of the proof can be adopted.

Lemma 3.1.17. *If $f \in \mathsf{H}(L_\partial, \Omega)$ is such that*

$$\langle L_\partial f, \phi \rangle_{\mathsf{L}^2(\Omega)} + \langle f, L_\partial^{\mathsf{H}} \phi \rangle_{\mathsf{L}^2(\Omega)} = 0 \quad \text{for all} \quad \phi \in \mathcal{D}(\mathbb{R}^n)^{m_1}, \tag{3.2}$$

then $f \in \mathsf{H}_0(L_\partial, \Omega)$.

Recall the definition of a positive mollifier: Let $\rho \in \mathcal{D}(\mathbb{R}^n)$. Then we define ρ_ϵ by $\rho_\epsilon(\zeta) = \epsilon^{-n} \rho(\frac{\zeta}{\epsilon})$. We say that ρ_ϵ is a positive mollifier, if $\rho(\zeta) \geq 0$, $\int_{\mathbb{R}^n} \rho(\zeta) \,\mathrm{d}\zeta = 1$ and $\lim_{\epsilon \to 0} \rho_\epsilon = \delta_0$ in the sense of distributions, where δ_0 is the Dirac delta function, i.e. $\langle \delta_0, \phi \rangle_{\mathcal{D}', \mathcal{D}} = \phi(0)$.

In particular, for every $f \in \mathsf{L}^2(\mathbb{R}^n)$ holds

$$\rho_\epsilon * f := \int_{\mathbb{R}^n} \rho_\epsilon(\zeta) f(\cdot - \zeta) \,\mathrm{d}\zeta \xrightarrow{\epsilon \to 0} f \quad \text{in} \quad \mathsf{L}^2(\mathbb{R}^n).$$

Proof. Let $f \in \mathsf{H}(L_\partial, \Omega)$ satisfy (3.2). Then we have to find a sequence $(f_n)_{n \in \mathbb{N}}$ in $\mathcal{D}(\Omega)^{m_2}$ that converges to f with respect to $\|\cdot\|_{\mathsf{H}(L_\partial, \Omega)}$.

We define $\tilde{f}$ and $\widetilde{L_\partial f}$ as the extension of f and $L_\partial f$ respectively on $\mathbb{R}^n$ such that these functions are 0 outside of Ω. By

$$\begin{aligned}
\langle \widetilde{L_\partial f}, \phi \rangle_{\mathcal{D}'(\mathbb{R}^n), \mathcal{D}(\mathbb{R}^n)} &= \langle \widetilde{L_\partial f}, \phi \rangle_{\mathsf{L}^2(\mathbb{R}^n)} = \langle L_\partial f, \phi \rangle_{\mathsf{L}^2(\Omega)} \overset{(3.2)}{=} \langle f, -L_\partial^{\mathsf{H}} \phi \rangle_{\mathsf{L}^2(\Omega)} \\
&= \langle \tilde{f}, -L_\partial^{\mathsf{H}} \phi \rangle_{\mathsf{L}^2(\mathbb{R}^n)} = \langle \tilde{f}, -L_\partial^{\mathsf{H}} \phi \rangle_{\mathcal{D}'(\mathbb{R}^n), \mathcal{D}(\mathbb{R}^n)} \\
&= \langle L_\partial \tilde{f}, \phi \rangle_{\mathcal{D}'(\mathbb{R}^n), \mathcal{D}(\mathbb{R}^n)}
\end{aligned}$$

for $\phi \in \mathcal{D}(\mathbb{R}^n)^{m_1}$, we see that $\widetilde{L_\partial f} = L_\partial \tilde{f}$ and $\tilde{f} \in \mathsf{H}(L_\partial, \mathbb{R}^n)$ with $\operatorname{supp} \tilde{f} \subseteq \overline{\Omega}$.

By Lemma 3.1.16 there is a finite open covering $(O_i)_{i=0}^k$ of $\overline{\Omega}$ such that $O_i \cap \overline{\Omega}$ is bounded and strongly star-shaped for $i \in \{1, \ldots, k\}$ and $\overline{O_0} \subseteq \Omega$. We employ a partition of unity and obtain $(\alpha_i)_{i=0}^k$, subordinate to this covering, that is

$$\alpha_i \in C^\infty(\mathbb{R}^n), \quad \operatorname{supp} \alpha_i \subseteq O_i, \quad \alpha_i(\zeta) \in [0, 1], \quad \text{and} \quad \sum_{i=0}^k \alpha_i(\zeta) = 1 \quad \text{for} \quad \zeta \in \Omega.$$

Hence, $\tilde{f} = \sum_{i=0}^k \alpha_i \tilde{f}$ and we define $f_i := \alpha_i \tilde{f}$. By construction $f_i \in \mathsf{H}(L_\partial, \mathbb{R}^n)$ and $\operatorname{supp} f_i \subseteq \overline{O_i \cap \Omega}$. For $i \neq 0$ the set $\overline{O_i \cap \Omega}$ is compact.

- For $i \in \{1, \dots, k\}$ we have $O_i \cap \Omega$ is strongly star-shaped. Lemma 3.1.13 ensures that $\operatorname{supp}(f_i)_\theta \subseteq O_i \cap \Omega$ for $\theta \in (0,1)$ and $(f_i)_\theta \to f_i$ in $\mathsf{H}(L_\partial, \mathbb{R}^n)$ for $\theta \to 1$.

  Let ρ_ϵ be a positive mollifier. Then $\rho_\epsilon * g \to g$ in $\mathsf{L}^2(\mathbb{R}^n)$ for an arbitrary $g \in \mathsf{L}^2(\mathbb{R}^n)$. Since $L_\partial(\rho_\epsilon * h) = \rho_\epsilon * L_\partial h$, we also have $\rho_\epsilon * h \to h$ in $\mathsf{H}(L_\partial, \mathbb{R}^n)$ for $h \in \mathsf{H}(L_\partial, \mathbb{R}^n)$ and since $\rho_\epsilon \in \mathsf{C}^\infty(\mathbb{R}^n)$ we have $\rho_\epsilon * h \in \mathsf{C}^\infty(\mathbb{R}^n)^{m_2}$.

  For fixed $\theta \in (0,1)$ and ϵ sufficiently small, we can say $\operatorname{supp} \rho_\epsilon * (f_i)_\theta \subseteq O_i \cap \Omega$. Hence, by a diagonalization argument we find a sequence $(\rho_{\epsilon_j} * (f_i)_{\theta_j})_{j\in\mathbb{N}}$ in $\mathcal{D}(\Omega)^{m_2}$ converging to f_i in $\mathsf{H}(L_\partial, \mathbb{R}^n)$. Doing this for every $i \in \{1, \dots, k\}$ yields sequences $(f_{i,j})_{j\in\mathbb{N}}$ in $\mathcal{D}(\Omega)^{m_2}$ converging to f_i in $\mathsf{H}(L_\partial, \mathbb{R}^n)$.

- For f_0 we have $\operatorname{supp} f_0 \subseteq \overline{O_0} \subseteq \Omega$ and by Lemma 3.1.15 there exists a sequence $(g_l)_{l\in\mathbb{N}}$ in $\mathsf{H}(L_\partial, \mathbb{R}^n)$ that converges to f_0 in $\mathsf{H}(L_\partial, \mathbb{R}^n)$ such that every g_l has compact support in Ω. Every g_l can be approximated by $\rho_\epsilon * g_l$ for $\epsilon \to 0$ in $\mathsf{H}(L_\partial, \mathbb{R}^n)$ and if ϵ is sufficiently small $\operatorname{supp} \rho_\epsilon * g_l \subseteq \Omega$. Hence, $\rho_\epsilon * g_l \in \mathcal{D}(\Omega)^{m_2}$. A diagonalization argument establishes a sequence $(f_{0,j})_{j\in\mathbb{N}}$ in $\mathcal{D}(\Omega)^{m_2}$ that converges to f_0 in $\mathsf{H}(L_\partial, \mathbb{R}^n)$.

Consequently, $\left(\sum_{i=0}^k f_{i,j}\right)_{j\in\mathbb{N}}$ is a sequence in $\mathcal{D}(\Omega)^{m_2}$ that converges to $\tilde{f}$ in $\mathsf{H}(L_\partial, \mathbb{R}^n)$ and by Lemma 3.1.10 also in $\mathsf{H}(L_\partial, \Omega)$. ❑

Theorem 3.1.18. *$\mathcal{D}(\mathbb{R}^n)^{m_2}\big|_\Omega$ is dense in $\mathsf{H}(L_\partial, \Omega)$.*

Proof. Suppose $\mathcal{D}(\mathbb{R}^n)^{m_2}\big|_\Omega$ is not dense in $\mathsf{H}(L_\partial, \Omega)$. Then there exists a non zero $f \in \mathsf{H}(L_\partial, \Omega)$ such that

$$\langle f, g\rangle_{\mathsf{H}(L_\partial,\Omega)} = \langle f, g\rangle_{\mathsf{L}^2} + \langle L_\partial f, L_\partial g\rangle_{\mathsf{L}^2} = 0 \quad \text{for all} \quad g \in \mathcal{D}(\mathbb{R}^n)^{m_2}\big|_\Omega. \tag{3.3}$$

In particular, for an arbitrary $h \in \mathcal{D}(\Omega)^{m_2}$ we have

$$\langle f, h\rangle_{\mathcal{D}',\mathcal{D}} = \langle f, h\rangle_{\mathsf{L}^2} = -\langle L_\partial f, L_\partial h\rangle_{\mathsf{L}^2} = -\langle L_\partial f, L_\partial h\rangle_{\mathcal{D}',\mathcal{D}} = \langle L_\partial^{\mathsf{H}} L_\partial f, h\rangle_{\mathcal{D}',\mathcal{D}},$$

which implies that $f = L_\partial^{\mathsf{H}} L_\partial f \in \mathsf{L}^2(\Omega)^{m_2}$ and $f_0 := L_\partial f \in \mathsf{H}(L_\partial^{\mathsf{H}}, \Omega)$. Hence we can rewrite (3.3) as

$$\langle L_\partial^{\mathsf{H}} \underbrace{L_\partial f}_{=f_0}, g\rangle_{\mathsf{L}^2(\Omega)} + \langle \underbrace{L_\partial f}_{=f_0}, L_\partial g\rangle_{\mathsf{L}^2(\Omega)} = 0 \quad \text{for all} \quad g \in \mathcal{D}(\mathbb{R}^n)^{m_2}\big|_\Omega.$$

By Lemma 3.1.17 (switching the roles of L_∂ and L_∂^{H}) we have $f_0 \in \mathsf{H}_0(L_\partial^{\mathsf{H}}, \Omega)$. Since $\mathcal{D}(\Omega)^{m_1}$ is dense in $\mathsf{H}_0(L_\partial^{\mathsf{H}}, \Omega)$, there is a sequence $(f_n)_{n\in\mathbb{N}}$ in $\mathcal{D}(\Omega)^{m_1}$ converging to f_0 with respect to $\|\cdot\|_{\mathsf{H}(L_\partial^{\mathsf{H}},\Omega)}$. The fact $f = L_\partial^{\mathsf{H}} L_\partial f = L_\partial^{\mathsf{H}} f_0$ implies

$$\begin{aligned}\langle f_0, f_n\rangle_{\mathsf{H}(L_\partial^{\mathsf{H}},\Omega)} &= \langle f_0, f_n\rangle_{\mathsf{L}^2} + \langle L_\partial^{\mathsf{H}} f_0, L_\partial^{\mathsf{H}} f_n\rangle_{\mathsf{L}^2} = \langle L_\partial f, f_n\rangle_{\mathsf{L}^2} + \langle f, L_\partial^{\mathsf{H}} f_n\rangle_{\mathsf{L}^2} \\ &= \langle L_\partial f, f_n\rangle_{\mathcal{D}',\mathcal{D}} - \langle L_\partial f, f_n\rangle_{\mathcal{D}',\mathcal{D}} \\ &= 0.\end{aligned}$$

Since $\|f_0\|^2_{\mathsf{H}(L_\partial^{\mathsf{H}},\Omega)} = \lim_{n\to\infty} \langle f_0, f_n\rangle_{\mathsf{H}(L_\partial^{\mathsf{H}},\Omega)} = 0$, we have $f_0 = 0$, which implies $f = L_\partial^{\mathsf{H}} f_0 = 0$. Hence, $\mathcal{D}(\mathbb{R}^n)^{m_2}\big|_\Omega$ is dense in $\mathsf{H}(L_\partial, \Omega)$. ❑

3.2 Port-Hamiltonian Systems

In this section we will introduce linear first order port-Hamiltonian systems on multidimensional spatial domains and illustrate the difficulties we want to overcome.

Definition 3.2.1. Let $m \in \mathbb{N}$ and $P = (P_i)_{i=1}^n$, where P_i is a Hermitian $m \times m$ matrix. Moreover, let $\mathcal{H}\colon \Omega \to \mathbb{K}^{m\times m}$ be measurable such that $\mathcal{H}(\zeta)^{\mathsf{H}} = \mathcal{H}(\zeta)$ and $c\mathrm{I} \leq \mathcal{H}(\zeta) \leq C\mathrm{I}$ for a.e. $\zeta \in \Omega$ and some constants $c, C > 0$ independent of ζ. Then we endow the space $\mathcal{X}_{\mathcal{H}} \coloneqq \mathsf{L}^2(\Omega)^m$ with the scalar product

$$\langle f, g\rangle_{\mathcal{X}_{\mathcal{H}}} \coloneqq \langle \mathcal{H}f, g\rangle_{\mathsf{L}^2(\Omega)^m} = \int_\Omega \langle \mathcal{H}(\zeta)f(\zeta), g(\zeta)\rangle_{\mathbb{K}^m} \,\mathrm{d}\lambda(\zeta).$$

We will refer to $\mathcal{X}_{\mathcal{H}}$ as the *state space* and to its elements as *state variables* or *states*. Furthermore, let $P_0 \in \mathbb{K}^{m\times m}$ be such that $P_0^{\mathsf{H}} = -P_0$. Then we will call the differential equation

$$\begin{aligned} \frac{\partial}{\partial t}x(t,\zeta) &= \sum_{i=1}^n \frac{\partial}{\partial \zeta_i} P_i\big(\mathcal{H}(\zeta)x(t,\zeta)\big) + P_0\big(\mathcal{H}(\zeta)x(t,\zeta)\big), && t \in \mathbb{R}_+, \zeta \in \Omega, \\ x(0,\zeta) &= x_0(\zeta), && \zeta \in \Omega \end{aligned} \tag{3.4}$$

a *linear, first order port-Hamiltonian system*, where $x_0 \in \mathsf{L}^2(\Omega)^m$ is the initial state. The associated *Hamiltonian* $H\colon \mathcal{X}_{\mathcal{H}} \to \mathbb{R}_+ \cup \{0\}$ is defined by

$$H(x) \coloneqq \frac{1}{2}\langle x, x\rangle_{\mathcal{X}_{\mathcal{H}}} = \frac{1}{2}\int_\Omega \langle \mathcal{H}(\zeta)x(\zeta), x(\zeta)\rangle_{\mathbb{K}^m} \,\mathrm{d}\lambda(\zeta),$$

where $\mathcal{H}$ is called the *Hamiltonian density.*

In most applications the Hamiltonian describes the energy in the state space. It may seem more natural to define the inner product of $\mathcal{X}_{\mathcal{H}}$ as

$$\frac{1}{2}\langle \mathcal{H}f, g\rangle_{\mathsf{L}^2(\Omega)},$$

because then the Hamiltonian is just $\|x\|_{\mathcal{X}_{\mathcal{H}}}^2$ and the name energy norm is accurate. However, then we also have to pay attention to the factor $\frac{1}{2}$, when we switch between the inner products. Therefore, for convenience we leave out this factor.

By the convention of regarding a function $x\colon \mathbb{R}_+ \times \Omega \to \mathbb{K}^m$ as $x\colon \mathbb{R}_+ \to \mathsf{L}^2(\Omega;\mathbb{K}^m)$ by setting $x(t) = x(t,\cdot)$, we can rewrite the PDE (3.4) as

$$\dot{x}(t) = \Big(\sum_{i=1}^n \partial_i P_i + P_0\Big)\mathcal{H}x(t) = (P_\partial + P_0)\mathcal{H}x(t), \quad x(0) = x_0,$$

where P_∂ is defined by Definition 3.1.2 replacing L with P. This is an abstract Cauchy problem. Hence, we are interested whether $(P_\partial + P_0)\mathcal{H}$ is a generator of a contraction semigroup.

We want to add the following assumption on P.

Assumption 3.2.2. Let $m, m_1, m_2 \in \mathbb{N}$ such that $m = m_1 + m_2$ and let $L = (L_i)_{i=1}^n$ such that $L_i \in \mathbb{K}^{m_1 \times m_2}$. Then we assume that $P = (P_i)_{i=1}^n$ has the block structure

$$P_i = \begin{bmatrix} 0 & L_i \\ L_i^{\mathsf{H}} & 0 \end{bmatrix}.$$

Assumption 3.2.2 implies that P contains only Hermitian matrices. According to the block structure we split $x \in \mathbb{K}^m$ into $\left[\begin{smallmatrix} x_{L^{\mathsf{H}}} \\ x_L \end{smallmatrix}\right]$, where $x_{L^{\mathsf{H}}} = (x_i)_{i=1}^{m_1}$ and $x_L = (x_i)_{i=m_1+1}^m$.

We have introduced differential operators L_∂ for a family of matrices L. Clearly, we can do the same with the family P. Because of the block structure of P we can immediately derive the following identities: $\mathsf{H}(P_\partial, \Omega) = \mathsf{H}(L_\partial^{\mathsf{H}}, \Omega) \times \mathsf{H}(L_\partial, \Omega)$,

$$P_\partial = \begin{bmatrix} 0 & L_\partial \\ L_\partial^{\mathsf{H}} & 0 \end{bmatrix} \quad \text{and} \quad P_\nu = \begin{bmatrix} 0 & L_\nu \\ L_\nu^{\mathsf{H}} & 0 \end{bmatrix}.$$

By Lemma 3.1.8 we have for $x, y \in \mathsf{H}^1(\Omega)^m$

$$\begin{aligned} \langle P_\partial x, y\rangle_{\mathsf{L}^2(\Omega)} &+ \langle x, P_\partial y\rangle_{\mathsf{L}^2(\Omega)} \\ &= \langle P_\nu \gamma_0 x, \gamma_0 y\rangle_{\mathsf{L}^2(\partial\Omega)} \\ &= \left\langle \begin{bmatrix} 0 & L_\nu \\ L_\nu^{\mathsf{H}} & 0 \end{bmatrix} \gamma_0 \begin{bmatrix} x_{L^{\mathsf{H}}} \\ x_L \end{bmatrix}, \gamma_0 \begin{bmatrix} y_{L^{\mathsf{H}}} \\ y_L \end{bmatrix} \right\rangle_{\mathsf{L}^2(\partial\Omega)} \\ &= \langle L_\nu \gamma_0 x_L, \gamma_0 y_{L^{\mathsf{H}}}\rangle_{\mathsf{L}^2(\partial\Omega)} + \langle L_\nu^{\mathsf{H}} \gamma_0 x_{L^{\mathsf{H}}}, \gamma_0 y_L\rangle_{\mathsf{L}^2(\partial\Omega)} \\ &= \langle L_\nu \gamma_0 x_L, \gamma_0 y_{L^{\mathsf{H}}}\rangle_{\mathsf{L}^2(\partial\Omega)} + \langle \gamma_0 x_{L^{\mathsf{H}}}, L_\nu \gamma_0 y_L\rangle_{\mathsf{L}^2(\partial\Omega)}. \end{aligned} \tag{3.5}$$

Hence, $\mathcal{B} = \mathsf{L}^2(\partial\Omega)^{m_1}$, $B_1 x = L_\nu \gamma_0 x_L$ and $B_2 x = \gamma_0 x_{L^{\mathsf{H}}}$ is reminiscent of a boundary triple for $A_0^* = P_\partial$ ($A_0 = P_\partial^*$ is skew-symmetric by Remark 3.1.7). However, we need to extend (3.5) for $x, y \in \mathsf{H}(P_\partial, \Omega)$. In order to do this we have to introduce a new norm on $\mathsf{L}^2(\partial\Omega)^{m_1}$, which will lead to the notion of quasi Gelfand triples.

If we are a little bit sloppy about the details (just for now), then we can easily calculate the change of the Hamiltonian (energy) along a solution x of the port-Hamiltonian system:

$$\begin{aligned} 2\frac{\mathrm{d}}{\mathrm{d}t} H(x(t)) &= \frac{\mathrm{d}}{\mathrm{d}t}\langle x(t), x(t)\rangle_{\mathcal{X}_\mathcal{H}} = \langle \dot{x}(t), x(t)\rangle_{\mathcal{X}_\mathcal{H}} + \langle x(t), \dot{x}(t)\rangle_{\mathcal{X}_\mathcal{H}} \\ &= \big\langle (P_\partial + P_0)\mathcal{H}x(t), x(t)\big\rangle_{\mathcal{X}_\mathcal{H}} + \big\langle x(t), (P_\partial + P_0)\mathcal{H}x(t)\big\rangle_{\mathcal{X}_\mathcal{H}} \\ &= \big\langle (P_\partial + P_0)\mathcal{H}x(t), \mathcal{H}x(t)\big\rangle_{\mathsf{L}^2(\Omega)} + \big\langle \mathcal{H}x(t), (P_\partial + P_0)\mathcal{H}x(t)\big\rangle_{\mathsf{L}^2(\Omega)} \end{aligned}$$

(since P_0 is skew-adjoint, we can eliminate P_0)

$$= \big\langle P_\partial \mathcal{H}x(t), \mathcal{H}x(t)\big\rangle_{\mathsf{L}^2(\Omega)} + \big\langle \mathcal{H}x(t), P_\partial \mathcal{H}x(t)\big\rangle_{\mathsf{L}^2(\Omega)}$$

(we can use (3.5) and $\mathcal{H}x = \left[\begin{smallmatrix} (\mathcal{H}x)_{L^\mathsf{H}} \\ (\mathcal{H}x)_L \end{smallmatrix}\right]$)

$$= \langle L_\nu \gamma_0 (\mathcal{H}x)_L, \gamma_0(\mathcal{H}x)_{L^\mathsf{H}} \rangle_{\mathsf{L}^2(\partial\Omega)} + \langle \gamma_0(\mathcal{H}x)_{L^\mathsf{H}}, L_\nu \gamma_0 (\mathcal{H}x)_L \rangle_{\mathsf{L}^2(\partial\Omega)}.$$

Thus, the change of the Hamiltonian (energy) only occurs on the boundary. Moreover, we can see that the change of the Hamiltonian is connected to Stokes-Dirac product of $\mathsf{L}^2(\partial\Omega)$:

$$2\frac{\mathrm{d}}{\mathrm{d}t} H(x(t)) = \left\langle\!\!\left\langle \left[\begin{smallmatrix} L_\nu \gamma_0(\mathcal{H}x)_L \\ \gamma_0(\mathcal{H}x)_{L^\mathsf{H}} \end{smallmatrix}\right], \left[\begin{smallmatrix} L_\nu \gamma_0(\mathcal{H}x)_L \\ \gamma_0(\mathcal{H}x)_{L^\mathsf{H}} \end{smallmatrix}\right] \right\rangle\!\!\right\rangle_{\mathsf{L}^2(\partial\Omega)}.$$

3.3 The Wave Equation as port-Hamiltonian System

In [28] the wave equation in n-D is investigated as a port-Hamiltonian system. We follow their reformulation of the wave equation such that it fits the port-Hamiltonian framework.

The classical formulation of the wave equation without boundary conditions is given by

$$\begin{aligned} \frac{\partial^2}{\partial t^2} w(t,\zeta) &= \frac{1}{\rho(\zeta)} \operatorname{div} \Big(T(\zeta) \operatorname{grad} w(t,\zeta) \Big), && t \in \mathbb{R}_+, \zeta \in \Omega, \\ w(0,\zeta) &= w_0(\zeta), && \zeta \in \Omega, \\ \frac{\partial}{\partial t} w(0,\zeta) &= w_1(\zeta), && \zeta \in \Omega, \end{aligned}$$

where $\Omega \subseteq \mathbb{R}^n$, ρ is the mass density, T is Young's modulus and w_0, w_1 are the initial conditions. Furthermore, $T(\zeta)$ is symmetric for a.e. $\zeta \in \Omega$,

$$\rho, \frac{1}{\rho} \in \mathsf{L}^\infty(\Omega) \quad \text{and} \quad T, T(\cdot)^{-1} \in \mathsf{L}^\infty(\Omega)^{n\times n}.$$

We can reformulate the wave equation in a port-Hamiltonian fashion, by introducing the state variable

$$x(t,\zeta) := \begin{bmatrix} x_1(t,\zeta) \\ x_2(t,\zeta) \end{bmatrix} := \begin{bmatrix} \rho(\zeta) \frac{\partial}{\partial t} w(t,\zeta) \\ \operatorname{grad} w(t,\zeta) \end{bmatrix}.$$

Hence, if w is a solution of the wave equation, then

$$\begin{aligned} \frac{\partial}{\partial t} x(t,\zeta) &= \begin{bmatrix} \rho(\zeta) \frac{\partial^2}{\partial t^2} w(t,\zeta) \\ \operatorname{grad} \frac{\partial}{\partial t} w(t,\zeta) \end{bmatrix} = \begin{bmatrix} \operatorname{div} T(\zeta) \operatorname{grad} w(t,\zeta) \\ \operatorname{grad} \frac{1}{\rho(\zeta)} \rho(\zeta) \frac{\partial}{\partial t} w(t,\zeta) \end{bmatrix} \\ &= \begin{bmatrix} 0 & \operatorname{div} \\ \operatorname{grad} & 0 \end{bmatrix} \underbrace{\begin{bmatrix} \frac{1}{\rho(\zeta)} & 0 \\ 0 & T(\zeta) \end{bmatrix}}_{=:\mathcal{H}(\zeta)} \begin{bmatrix} x_1(t,\zeta) \\ x_2(t,\zeta) \end{bmatrix}. \end{aligned} \tag{3.6}$$

Note that div and grad can be written as L_∂ and L_∂^{H} for L given by the n-dimensional analogon of Example 3.1.3. Therefore we have

$$\begin{bmatrix} 0 & \operatorname{div} \\ \operatorname{grad} & 0 \end{bmatrix} = \sum_{i=1}^n \frac{\partial}{\partial \zeta_i} \begin{bmatrix} 0 & L_i \\ L_i^{\mathsf{H}} & 0 \end{bmatrix}$$

and we can write (3.6) as

$$\begin{aligned} \frac{\partial}{\partial t} x(t,\zeta) &= \sum_{i=1}^n \frac{\partial}{\partial \zeta_i} \begin{bmatrix} 0 & L_i \\ L_i^{\mathsf{H}} & 0 \end{bmatrix} \mathcal{H}(\zeta) x(t,\zeta), \\ x(0,\zeta) &= \begin{bmatrix} \rho(\zeta) w_1(\zeta) \\ \operatorname{grad} w_0(\zeta) \end{bmatrix}. \end{aligned} \tag{3.7}$$

It is easy to see that (3.7) fits the definition a port-Hamiltonian system. Hence, we will regard this system as abstract Cauchy problem on $\mathsf{L}^2(\Omega)^{n+1}$:

$$\dot{x}(t) = \begin{bmatrix} 0 & \operatorname{div} \\ \operatorname{grad} & 0 \end{bmatrix} \mathcal{H} x(t),$$

where $x(t) = x(t,\cdot)$ ($x(t) \in \mathsf{L}^2(\Omega)^{n+1}$ for all $t \in \mathbb{R}_+$). However, we will later see that for stability analysis the state space $\mathsf{L}^2(\Omega)^{n+1}$ is too large, as it allows solutions that are unrelated to the original wave equation. Nevertheless for well-posedness we can even work with this larger space.

For uniqueness of solutions we need boundary conditions, like Dirichlet boundary conditions

$$w(t,\zeta) = h(\zeta), \qquad \zeta \in \partial\Omega,$$

or Neumann boundary conditions

$$\frac{\partial}{\partial \nu} T(\zeta) w(t,\zeta) = g(\zeta), \qquad \zeta \in \partial\Omega.$$

We can reduce ourselves to homogeneous boundary condition, by subtracting a solution of the time invariant system

$$\begin{aligned} \operatorname{div} T(\zeta) \operatorname{grad} w(\zeta) &= 0, & \zeta &\in \Omega, \\ w(\zeta) &= h(\zeta), & \zeta &\in \partial\Omega. \end{aligned}$$

Accordingly, for Neumann boundary conditions. Note that $w(t,\zeta) = h(\zeta)$ for $\zeta \in \partial\Omega$ can be translated to $\frac{\partial}{\partial t} w(t,\zeta) = 0$, if the initial condition w_0 satisfies the boundary condition. Therefore, we can translate the boundary conditions in the port-Hamiltonian formulation to

$$\gamma_0 \frac{1}{\rho} x_1(t) = 0 \quad \text{or} \quad \nu \cdot \gamma_0 T x_2(t) = 0,$$

where $\nu \cdot \gamma_0 y = L_\nu \gamma_0 y$ is the "natural" boundary operator corresponding to L given by $L_\partial = \operatorname{div}$.

3.4 Maxwell's Equations

Let $\Omega \subseteq \mathbb{R}^3$. We will see that Maxwell's equations in a non-conducting medium

$$\frac{\partial}{\partial t}\mathbf{D}(t,\zeta) = \operatorname{rot}\mathbf{H}(t,\zeta), \qquad \frac{\partial}{\partial t}\mathbf{B}(t,\zeta) = -\operatorname{rot}\mathbf{E}(t,\zeta), \quad t \in \mathbb{R}_+, \zeta \in \Omega, \quad (3.8)$$

$$\operatorname{div}\mathbf{D}(t,\zeta) = \rho(\zeta), \qquad \operatorname{div}\mathbf{B}(t,\zeta) = 0, \quad t \in \mathbb{R}_+, \zeta \in \Omega, \quad (3.9)$$

$$\mathbf{D}(t,\zeta) = \epsilon(\zeta)\mathbf{E}(t,\zeta), \qquad \mathbf{B}(t,\zeta) = \mu(\zeta)\mathbf{H}(t,\zeta), \quad t \in \mathbb{R}_+, \zeta \in \Omega, \quad (3.10)$$

$$\mathbf{D}(0,\zeta) = \mathbf{D}_0(\zeta), \qquad \mathbf{B}(0,\zeta) = \mathbf{B}_0(\zeta), \quad \zeta \in \Omega, \quad (3.11)$$

where $\epsilon, \mu, \frac{1}{\epsilon}, \frac{1}{\mu} \in \mathsf{L}^\infty(\Omega)$ and $\rho \in \mathsf{L}^2(\Omega)$, fit the port-Hamiltonian structure.

We choose the state variable

$$x(t,\zeta) = \begin{bmatrix} \mathbf{D}(t,\zeta) \\ \mathbf{B}(t,\zeta) \end{bmatrix},$$

so that

$$\frac{\partial}{\partial t}x(t,\zeta) = \begin{bmatrix} 0 & \operatorname{rot} \\ -\operatorname{rot} & 0 \end{bmatrix} \underbrace{\begin{bmatrix} \frac{1}{\epsilon(\zeta)} & 0 \\ 0 & \frac{1}{\mu(\zeta)} \end{bmatrix}}_{=:\mathcal{H}(\zeta)} x(t,\zeta). \quad (3.12)$$

Note that by Example 3.1.4 there is an L such that $L_\partial = \operatorname{rot}$ and $L_\partial^{\mathsf{H}} = -\operatorname{rot}$. Hence, (3.12) fits the definition of a port-Hamiltonian system Definition 3.2.1. We will see that (3.9) is automatically fulfilled, if the initial condition satisfies (3.9).

The "natural" boundary operator L_ν is given by $f \mapsto \nu \times f$. We will see that $\nu \times \gamma_0$ cannot be continuously extended on $\mathsf{H}(\operatorname{rot},\Omega)$ such that its codomain is still $\mathsf{L}^2(\partial\Omega)$, see Example 5.1.8. Hence, we have to find another way to get a boundary triple for the Maxwell differential operator.

3.5 Mindlin Plate Model

The Mindlin plate model was formulated in a port-Hamiltonian fashion in [33, 8]. We just want to show the equations without going into its physical background.

Let $\Omega \subseteq \mathbb{R}^2$ be as in Assumption 3.1.1. Let us consider the differential operator P_∂ and the skew-symmetric matrix P_0 given by

$$P_\partial := \left[\begin{array}{ccc|ccccc} 0 & 0 & 0 & 0 & 0 & 0 & \partial_1 & \partial_2 \\ 0 & 0 & 0 & \partial_1 & 0 & \partial_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & \partial_2 & \partial_1 & 0 & 0 \\ \hline 0 & \partial_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \partial_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & \partial_2 & \partial_1 & 0 & 0 & 0 & 0 & 0 \\ \partial_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \partial_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right], \quad P_0 := \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The corresponding $L = (L_i)_{i=1}^2$ is given by

$$L_1 = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} \quad \text{and} \quad L_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix},$$

and therefore $P = (P_i)_{i=1}^2$ is given by

$$P_i = \begin{bmatrix} 0 & L_i \\ L_i^{\mathsf{H}} & 0 \end{bmatrix}.$$

We define a Hamiltonian density by

$$\mathcal{H} = \left[\begin{array}{ccc|ccc|cc} \frac{1}{\rho h} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{12}{\rho h^3} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{12}{\rho h^3} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & & & & 0 & 0 \\ 0 & 0 & 0 & & \boldsymbol{D}_b & & 0 & 0 \\ 0 & 0 & 0 & & & & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \multicolumn{2}{c}{\boldsymbol{D}_s} \\ 0 & 0 & 0 & 0 & 0 & 0 & & \end{array}\right],$$

where ρ, h are strictly positive functions, $\boldsymbol{D}_b(\zeta)$ is a coercive 3×3 matrix and $\boldsymbol{D}_s(\zeta)$ is a coercive 2×2 matrix, such that all conditions on $\mathcal{H}$ in Definition 3.2.1 are satisfied. We write the state variable x as

$$\boldsymbol{\alpha} := \begin{bmatrix} \rho h v & \rho\frac{h^3}{12} w_1 & \rho\frac{h^3}{12} w_2 & \kappa_{1,1} & \kappa_{2,2} & \kappa_{1,2} & \gamma_{1,3} & \gamma_{2,3} \end{bmatrix}^{\mathsf{T}},$$

where we stick to the notation in [8] except that we renamed the coordinates x, y and z as 1, 2 and 3. Furthermore, we have

$$\mathbf{e} := \mathcal{H}\boldsymbol{\alpha} = \begin{bmatrix} v & w_1 & w_2 & M_{1,1} & M_{2,2} & M_{1,2} & Q_1 & Q_2 \end{bmatrix}^{\mathsf{T}}.$$

We don't want to go into details about the physical meaning of these state variables. We just want to make it easier to translate the results into the notation of [33, 8]. So the port-Hamiltonian PDE

$$\frac{\partial}{\partial t} x = (P_\partial + P_0)\mathcal{H}x \quad \text{looks like} \quad \frac{\partial}{\partial t}\boldsymbol{\alpha} = (P_\partial + P_0)\mathbf{e},$$

which is the formulation in [33, 8]. The corresponding boundary operator is

$$L_\nu f = \begin{bmatrix} 0 & 0 & 0 & \nu_1 & \nu_2 \\ \nu_1 & 0 & \nu_2 & 0 & 0 \\ 0 & \nu_2 & \nu_1 & 0 & 0 \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \end{bmatrix} = \begin{bmatrix} \nu \cdot \begin{bmatrix} f_4 \\ f_5 \end{bmatrix} \\ \nu \cdot \begin{bmatrix} f_1 \\ f_3 \end{bmatrix} \\ \nu \cdot \begin{bmatrix} f_3 \\ f_2 \end{bmatrix} \end{bmatrix}.$$

Since $\|\nu(\zeta)\| = 1$, at least $\nu_1(\zeta) \neq 0$ or $\nu_2(\zeta) \neq 0$. This can be used to show that $\operatorname{ran} L_\nu = \mathsf{L}^2(\partial\Omega)^3$.

Since there is no direct physical meaning to the boundary variables we will later apply a unitary transformation on them, to obtain the boundary variables of [33, 8].

Chapter 4

Quasi Gelfand Triples

Normally when we talk about Gelfand triples we have a Hilbert space $\mathcal{X}_0$ and a reflexive Banach space $\mathcal{X}_+$ that can be continuously and densely embedded into $\mathcal{X}_0$. The third space $\mathcal{X}_-$ is given by the completion of $\mathcal{X}_0$ with respect to

$$\|g\|_{\mathcal{X}_-} := \sup_{f \in \mathcal{X}_+ \setminus \{0\}} \frac{|\langle g, f\rangle_{\mathcal{X}_0}|}{\|f\|_{\mathcal{X}_+}}.$$

The duality between $\mathcal{X}_+$ and $\mathcal{X}_-$ is given by

$$\langle g, f\rangle_{\mathcal{X}_-,\mathcal{X}_+} = \lim_{k\to\infty} \langle g_k, f\rangle_{\mathcal{X}_0},$$

where $(g_k)_{k\in\mathbb{N}}$ is a sequence in $\mathcal{X}_0$ that converges to g in $\mathcal{X}_-$. Details for "ordinary" Gelfand triple can be found in [18, ch. 2.1, p. 54] or in [57, ch. 2.9, p. 56]. We want to weaken the assumptions such that the norm of $\mathcal{X}_+$ is not necessarily related to the norm of $\mathcal{X}_0$. This is in particular necessary for Maxwell's equations.

4.1 Motivation

In Section 6.4 we point out that is not possible to associate an "ordinary" Gelfand triple to the spatial differential operator of Maxwell's equations.

We will have the following setting: Let $\mathcal{X}_0$ be a Hilbert space with the inner product $\langle\cdot,\cdot\rangle_{\mathcal{X}_0}$ and $\langle\cdot,\cdot\rangle_{\mathcal{X}_+}$ be another inner product on $\mathcal{X}_0$ (not necessarily related to $\langle\cdot,\cdot\rangle_{\mathcal{X}_0}$), which is defined on a dense (w.r.t. $\|\cdot\|_{\mathcal{X}_0}$) subspace $\tilde{D}_+$ of $\mathcal{X}_0$. We denote the completion of $\tilde{D}_+$ w.r.t. $\|\cdot\|_{\mathcal{X}_+}$ ($\|f\|_{\mathcal{X}_+} := \sqrt{\langle f, f\rangle_{\mathcal{X}_+}}$) by $\mathcal{X}_+$. This completion is, by construction a Hilbert space with the extension of $\langle\cdot,\cdot\rangle_{\mathcal{X}_+}$, for which we use the same symbol. Now we have $\tilde{D}_+$ is dense in $\mathcal{X}_0$ w.r.t. $\|\cdot\|_{\mathcal{X}_0}$ and dense in $\mathcal{X}_+$ w.r.t. $\|\cdot\|_{\mathcal{X}_+}$. Figure 4.1 illustrates this setting.

Note that $\mathcal{X}_+$, as a Hilbert space, is automatically reflexive. For the further construction the crucial property of $\mathcal{X}_+$ is its reflexivity. Hence, we will weaken the previous setting such that $\mathcal{X}_+$ is only a reflexive Banach space:

- $\mathcal{X}_0$ Hilbert space endowed with $\langle\cdot,\cdot\rangle_{\mathcal{X}_0}$.
- $\tilde{D}_+$ dense subspace of $\mathcal{X}_0$ (w.r.t. $\|\cdot\|_{\mathcal{X}_0}$).
- $\|\cdot\|_{\mathcal{X}_+}$ another norm defined on $\tilde{D}_+$.
- $\mathcal{X}_+$ completion of $\tilde{D}_+$ with respect to $\|\cdot\|_{\mathcal{X}_+}$ is reflexive.

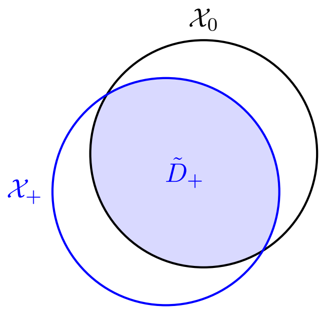

Figure 4.1: Setting of $\mathcal{X}_0$, $\tilde{D}_+$ and $\mathcal{X}_+$.

Example 4.1.1. Let $\mathcal{X}_0 = \ell^2(\mathbb{Z}\setminus\{0\})$ with the standard inner product $\langle x,y\rangle_{\mathcal{X}_0} = \sum_{n=1}^\infty x_n\overline{y_n} + x_{-n}\overline{y_{-n}}$. We define the inner product

$$\langle x,y\rangle_{\mathcal{X}_+} := \sum_{n=1}^\infty n^2 x_n\overline{y_n} + \frac{1}{n^2}x_{-n}\overline{y_{-n}}$$

and the set $\tilde{D}_+ := \{f \in \mathcal{X}_0 \,|\, \|f\|_{\mathcal{X}_+} < +\infty\}$. Clearly, this inner product is well-defined on $\tilde{D}_+$. Let e_i denote the sequence which is 1 on the i-th position and 0 elsewhere. Since $\{e_i \,|\, i \in \mathbb{Z}\setminus\{0\}\}$ is a orthonormal basis of $\mathcal{X}_0$ and contained in $\tilde{D}_+$, $\tilde{D}_+$ is dense in $\mathcal{X}_0$. The sequence $\left(\sum_{i=1}^n e_{-i}\right)_{n\in\mathbb{N}}$ is a Cauchy sequence with respect to $\|\cdot\|_{\mathcal{X}_+}$, but not with respect to $\|\cdot\|_{\mathcal{X}_0}$.

Definition 4.1.2. We define

$$\|g\|_{\mathcal{X}_-} := \sup_{f\in\tilde{D}_+\setminus\{0\}} \frac{|\langle g,f\rangle_{\mathcal{X}_0}|}{\|f\|_{\mathcal{X}_+}} \text{ for } g\in\mathcal{X}_0 \text{ and } D_- := \left\{g\in\mathcal{X}_0 \,\middle|\, \|g\|_{\mathcal{X}_-} < +\infty\right\}.$$

We denote the completion of D_- w.r.t. $\|\cdot\|_{\mathcal{X}_-}$ by $\mathcal{X}_-$. We will also denote the extension of $\|\cdot\|_{\mathcal{X}_-}$ to $\mathcal{X}_-$ by $\|\cdot\|_{\mathcal{X}_-}$.

Remark 4.1.3. By definition of D_- we can identify every $g \in D_-$ with an element of $\mathcal{X}_+'$ by the continuous extension of

$$\psi_g\colon \left\{\begin{array}{ccc} D_+ & \to & \mathbb{C}, \\ f & \mapsto & \langle g,f\rangle_{\mathcal{X}_0}, \end{array}\right.$$

on $\mathcal{X}_+$. We denote this extension again by ψ_g. By definition of D_- we have $\|\psi_g\|_{\mathcal{X}_+'} = \|g\|_{\mathcal{X}_-}$ for $g \in D_-$. Hence, we can extend the isometry

$$\Psi\colon \begin{cases} D_- & \to & \mathcal{X}_+', \\ g & \mapsto & \psi_g, \end{cases}$$

by continuity on $\mathcal{X}_-$. So $\mathcal{X}_-$ can be seen as the closure of D_- in $\mathcal{X}_+'$.

We can define a dual pairing of between $\mathcal{X}_+$ and $\mathcal{X}_-$ by

$$\langle g, f\rangle_{\mathcal{X}_-,\mathcal{X}_+} := \langle \Psi g, f\rangle_{\mathcal{X}_+',\mathcal{X}_+}.$$

However, this does not necessarily make $(\mathcal{X}_+, \mathcal{X}_-)$ a dual pair in the sense of Definition 1.3.1, because we do not know whether Ψ is surjective.

Lemma 4.1.4. *D_- is complete with respect to* $\|g\|_{\mathcal{X}_-\cap\mathcal{X}_0} := \sqrt{\|g\|_{\mathcal{X}_0}^2 + \|g\|_{\mathcal{X}_-}^2}$.

Proof. Let $(g_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in D_- with respect to $\|\cdot\|_{\mathcal{X}_-\cap\mathcal{X}_0}$. Then $(g_n)_{n\in\mathbb{N}}$ is a convergent sequence in $\mathcal{X}_0$ (w.r.t. $\|\cdot\|_{\mathcal{X}_0}$) and a Cauchy sequence in D_- (w.r.t. $\|\cdot\|_{\mathcal{X}_-}$). We denote the limit in $\mathcal{X}_0$ by g_0. By definition of $\|\cdot\|_{\mathcal{X}_-}$ we obtain for $f \in \tilde{D}_+$

$$|\langle g_0, f\rangle_{\mathcal{X}_0}| = \lim_{n\to\infty} |\langle g_n, f\rangle_{\mathcal{X}_0}| \le \lim_{n\to\infty} \|g_n\|_{\mathcal{X}_-}\|f\|_{\mathcal{X}_+} \le C\|f\|_{\mathcal{X}_+}$$

and consequently $g_0 \in D_-$.

Let $\epsilon > 0$ be arbitrary. Since $(g_n)_{n\in\mathbb{N}}$ is a Cauchy sequence with respect to $\|\cdot\|_{\mathcal{X}_-}$, there is an $n_0 \in \mathbb{N}$ such that for all $f \in \tilde{D}_+$ with $\|f\|_{\mathcal{X}_+} = 1$

$$|\langle g_n - g_m, f\rangle_{\mathcal{X}_0}| \le \frac{\epsilon}{2}, \quad \text{if} \quad n, m \ge n_0$$

holds true. Furthermore, for every $f \in \tilde{D}_+$ there exists an $m_f \ge n_0$ such that $|\langle g_0 - g_{m_f}, f\rangle_{\mathcal{X}_0}| \le \frac{\epsilon\|f\|_{\mathcal{X}_+}}{2}$, because $g_m \to g_0$ w.r.t. $\|\cdot\|_{\mathcal{X}_0}$. This yields

$$\frac{|\langle g_0 - g_n, f\rangle_{\mathcal{X}_0}|}{\|f\|_{\mathcal{X}_+}} \le \frac{|\langle g_0 - g_{m_f}, f\rangle_{\mathcal{X}_0}|}{\|f\|_{\mathcal{X}_+}} + \frac{|\langle g_{m_f} - g_n, f\rangle_{\mathcal{X}_0}|}{\|f\|_{\mathcal{X}_+}} \le \epsilon, \quad \text{if} \quad n \ge n_0.$$

Since the right-hand-side is independent of f, we obtain

$$\|g_0 - g_n\|_{\mathcal{X}_-} = \sup_{f\in\tilde{D}_+\setminus\{0\}} \frac{|\langle g_0 - g_n, f\rangle_{\mathcal{X}_0}|}{\|f\|_{\mathcal{X}_+}} \le \epsilon, \quad \text{if} \quad n \ge n_0.$$

Hence, g_0 is also the limit of $(g_n)_{n\in\mathbb{N}}$ with respect to $\|\cdot\|_{\mathcal{X}_-}$ and consequently the limit of $(g_n)_{n\in\mathbb{N}}$ with respect to $\|\cdot\|_{\mathcal{X}_-\cap\mathcal{X}_0}$. ❑

Strictly speaking $\tilde{D}_+$ and D_- are subsets of $\mathcal{X}_0$, but most of the time we rather want to regard them as subsets of $\mathcal{X}_+$ and $\mathcal{X}_-$, respectively. Hence, introduce the following embedding mappings

$$\tilde{\iota}_+\colon \begin{cases} \tilde{D}_+ \subseteq \mathcal{X}_+ & \to & \mathcal{X}_0, \\ f & \mapsto & f, \end{cases} \quad \text{and} \quad \iota_-\colon \begin{cases} D_- \subseteq \mathcal{X}_- & \to & \mathcal{X}_0, \\ g & \mapsto & g. \end{cases}$$

This allows us to distinguish between $f \in \tilde{D}_+$ as element of $\mathcal{X}_+$ and $\tilde{\iota}_+(f)$ as element of $\mathcal{X}_0$, if necessary. Clearly, the same for $g \in D_-$.

Lemma 4.1.5. *The embedding $\tilde{\iota}_+$ is a densely defined operator with $\operatorname{ran}\tilde{\iota}_+$ is dense in $\mathcal{X}_0$ and $\ker\tilde{\iota}_+ = \{0\}$. Furthermore, the embedding ι_- is closed and $\ker\iota_- = \{0\}$.*

Proof. By assumption on $\tilde{D}_+$ and definition of $\mathcal{X}_+$ the embedding $\tilde{\iota}_+$ is densely defined and has a dense range. Clearly, $\ker\tilde{\iota}_+ = \{0\}$ and $\ker\iota_- = \{0\}$. By Lemma 4.1.4 ι_- is closed. ❑

Lemma 4.1.6. *Let $\tilde{\iota}_+^* = \tilde{\iota}_+^{*\mathcal{X}_+\times\mathcal{X}_+'}$ denote the adjoint relation (w.r.t. the dualities $(\mathcal{X}_0, \mathcal{X}_0)$ and $(\mathcal{X}_+, \mathcal{X}_+')$) of $\tilde{\iota}_+$. Then $\tilde{\iota}_+^*$ is an operator (single-valued, i.e. $\operatorname{mul}\tilde{\iota}_+^* = \{0\}$) and $\ker\tilde{\iota}_+^* = \{0\}$. Its domain coincides with D_- and $\tilde{\iota}_+^*\iota_-\colon D_- \subseteq \mathcal{X}_- \to \mathcal{X}_+'$ is isometric.*

If $\ker\overline{\tilde{\iota}_+} = \{0\}$, then $\operatorname{ran}\tilde{\iota}_+^*$ is dense in $\mathcal{X}_+'$.

Proof. The density of the domain of $\tilde{\iota}_+$ yields $\operatorname{mul}\tilde{\iota}_+^* = (\operatorname{dom}\tilde{\iota}_+)^\perp = \{0\}$, and $\overline{\operatorname{ran}\tilde{\iota}_+}^{\mathcal{X}_0} = \mathcal{X}_0$ yields $\ker\tilde{\iota}_+^* = \{0\}$. The following equivalences show $\operatorname{dom}\tilde{\iota}_+^* = D_-$:

$$
\begin{aligned}
g \in \operatorname{dom}\tilde{\iota}_+^* &\Leftrightarrow \langle g, \tilde{\iota}_+ f\rangle_{\mathcal{X}_0} \text{ is continuous in } f \in \tilde{D}_+ \text{ w.r.t. } \|\cdot\|_{\mathcal{X}_+} \\
&\Leftrightarrow \sup_{f\in\tilde{D}_+\setminus\{0\}} \frac{|\langle g, f\rangle_{\mathcal{X}_0}|}{\|f\|_{\mathcal{X}_+}} < +\infty \\
&\Leftrightarrow g \in D_-.
\end{aligned}
$$

For $g \in D_- \subseteq \mathcal{X}_-$ we have

$$
\|g\|_{\mathcal{X}_-} = \sup_{f\in\tilde{D}_+\setminus\{0\}} \frac{|\langle \iota_- g, f\rangle_{\mathcal{X}_0}|}{\|f\|_{\mathcal{X}_+}} = \sup_{f\in\tilde{D}_+\setminus\{0\}} \frac{|\langle \tilde{\iota}_+^*\iota_- g, f\rangle_{\mathcal{X}_+',\mathcal{X}_+}|}{\|f\|_{\mathcal{X}_+}} = \|\tilde{\iota}_+^*\iota_- g\|_{\mathcal{X}_+'},
$$

which proves that $\tilde{\iota}_+^*\iota_-$ is isometric.

Note that the reflexivity of $\mathcal{X}_+$ implies $\overline{\tilde{\iota}_+} = \tilde{\iota}_+^{**}$. If $\ker\overline{\tilde{\iota}_+} = \{0\}$, then the following equation implies the density of $\operatorname{ran}\tilde{\iota}_+^*$ in $\mathcal{X}_+$

$$
\{0\} = \ker\overline{\tilde{\iota}_+} = \ker\tilde{\iota}_+^{**} = (\operatorname{ran}\tilde{\iota}_+^*)^\perp. \qquad ❑
$$

Remark 4.1.7. As mentioned in Remark 4.1.3 every $g \in D_-$ can be regarded as an element of $\mathcal{X}_+'$ by ψ_g. Let $g \in D_-$, $f \in \mathcal{X}_+$ and $(f_n)_{n\in\mathbb{N}}$ in D_+ converging to f in $\mathcal{X}_+$ (w.r.t. $\|\cdot\|_{\mathcal{X}_+}$). Since $D_- = \operatorname{dom}\tilde{\iota}_+^*$, we have

$$
\langle \psi_g, f\rangle_{\mathcal{X}_+',\mathcal{X}_+} = \lim_{n\to\infty} \langle \iota_- g, \tilde{\iota}_+ f_n\rangle_{\mathcal{X}_0} = \langle \tilde{\iota}_+^*\iota_- g, f\rangle_{\mathcal{X}_+',\mathcal{X}_+}
$$

and consequently $\psi_g = \tilde{\iota}_+^*\iota_- g$. Hence, $\Psi D_- = \tilde{\iota}_+^*\iota_- D_-$.

Proposition 4.1.8. *The following assertions are equivalent.*

(i) *There is a Hausdorff topological vector space* $(Z, \mathcal{T})$ *and two continuous embeddings* $\phi_{\mathcal{X}_+}: \mathcal{X}_+ \to Z$ *and* $\phi_{\mathcal{X}_0}: \mathcal{X}_0 \to Z$ *such that the diagram*

$$\begin{tikzcd} & \mathcal{X}_+ \arrow[dr, "\phi_{\mathcal{X}_+}"] & \\ \tilde{D}_+ \arrow[ur, "\mathrm{id}"] \arrow[dr, "\tilde{\iota}_+"'] & & Z \\ & \mathcal{X}_0 \arrow[ur, "\phi_{\mathcal{X}_0}"'] & \end{tikzcd}$$

commutes.

(ii) *If* $\tilde{D}_+ \ni f_n \to 0$ *w.r.t.* $\|\cdot\|_{\mathcal{X}_+}$ *and* $\lim_{n\to\infty} f_n$ *exists w.r.t.* $\|\cdot\|_{\mathcal{X}_0}$*, then this limit is also* 0 *and if* $\tilde{D}_+ \ni f_n \to 0$ *w.r.t.* $\|\cdot\|_{\mathcal{X}_0}$ *and* $\lim_{n\to\infty} f_n$ *exists w.r.t.* $\|\cdot\|_{\mathcal{X}_+}$*, then this limit is also* 0*.*

(iii) $\tilde{\iota}_+: \tilde{D}_+ \subseteq \mathcal{X}_+ \to \mathcal{X}_0, f \mapsto f$ *is closable (as an operator) and its closure is injective.*

(iv) D_- *is dense in* $\mathcal{X}_0$ *and dense in* $\mathcal{X}'_+$*, i.e.* ΨD_- *is dense in* $\mathcal{X}'_+$*.*

Proof. (i) $\Rightarrow$ (ii): Let $(f_n)_{n\in\mathbb{N}}$ be a sequence in $\tilde{D}_+$ such that $f_n \to \hat{f}$ w.r.t. $\mathcal{X}_+$ and $f_n \to f$ w.r.t. $\mathcal{X}_0$. Since $\mathcal{T}$ is coarser than both of the topologies induced by these norms, we also have

$$\begin{tikzcd}[row sep=small] & \hat{f} \\ f_n \arrow[ur, "\mathcal{T}"] \arrow[dr, "\mathcal{T}"'] & \\ & f \end{tikzcd} \quad \text{in } Z.$$

Since $\mathcal{T}$ is Hausdorff, we conclude $f = \hat{f}$. Hence, if either $\hat{f}$ or f is 0, then also the other is 0.

(ii) $\Rightarrow$ (iii): If $(f_n, f_n)_{n\in\mathbb{N}}$ is a sequence in $\tilde{\iota}_+$ that converges to $(0, f) \in \mathcal{X}_+ \times \mathcal{X}_0$, then $f = 0$ by (ii). Hence, $\operatorname{mul} \overline{\tilde{\iota}_+} = \{0\}$ and consequently $\tilde{\iota}_+$ is closable. On the other hand, if $(f_n, f_n)_{n\in\mathbb{N}}$ is a sequence in $\tilde{\iota}_+$ that converges to $(f, 0)$, then $f = 0$ by (ii). Consequently, $\ker \overline{\tilde{\iota}_+} = \{0\}$ and $\overline{\tilde{\iota}_+}$ is injective.

(iii) $\Rightarrow$ (iv): We have $(\operatorname{dom} \tilde{\iota}_+^*)^\perp = \operatorname{mul} \tilde{\iota}_+^{**} = \operatorname{mul} \overline{\tilde{\iota}_+}$. Since $\tilde{\iota}_+$ is closable, we have $\operatorname{mul} \overline{\tilde{\iota}_+} = \{0\}$, which yields that $\operatorname{dom} \tilde{\iota}_+^*$ is dense in $\mathcal{X}_0$. By Lemma 4.1.6 $\operatorname{dom} \tilde{\iota}_+^*$ coincides with D_-.

The second assertion of Lemma 4.1.6 yields that $\operatorname{ran} \tilde{\iota}_+^*$ is dense in $\mathcal{X}'_+$. By Remark 4.1.7 we have $\operatorname{ran} \tilde{\iota}_+^* = \Psi D_-$.

(iv) $\Rightarrow$ (i): Let $Y := D_-$ be equipped with

$$\|g\|_Y := \|g\|_{\mathcal{X}_- \cap \mathcal{X}_0} = \sqrt{\|g\|_{\mathcal{X}_-}^2 + \|g\|_{\mathcal{X}_0}^2}.$$

We define $Z := Y'$ as the (anti)dual space of Y. Then we have

$$\begin{aligned} &|\langle f, g\rangle_{\mathcal{X}_0}| \leq \|f\|_{\mathcal{X}_0}\|g\|_{\mathcal{X}_0} \leq \|f\|_{\mathcal{X}_0}\|g\|_Y && \text{for} \quad f \in \mathcal{X}_0, g \in Y \\ \text{and} \quad &|\langle f, \tilde{\iota}_+^* g\rangle_{\mathcal{X}_+,\mathcal{X}_+'}| \leq \|f\|_{\mathcal{X}_+} \underbrace{\|\tilde{\iota}_+^* g\|_{\mathcal{X}_+'}}_{=\|g\|_{\mathcal{X}_-}} \leq \|f\|_{\mathcal{X}_+}\|g\|_Y && \text{for} \quad f \in \mathcal{X}_+, g \in Y. \end{aligned}$$

Hence, $\phi_{\mathcal{X}_0} : f \mapsto \langle f, \cdot\rangle_{\mathcal{X}_0}$ and $\phi_{\mathcal{X}_+} : f \mapsto \langle f, \tilde{\iota}_+^* \cdot\rangle_{\mathcal{X}_+,\mathcal{X}_+'}$ are continuous mappings from $\mathcal{X}_0$ and $\mathcal{X}_+$, respectively, into Z. The injectivity of these mappings follows from the density of D_- in $\mathcal{X}_0$ and D_- in $\mathcal{X}_+'$ ($\tilde{\iota}_+^* D_-$ dense in $\mathcal{X}_+'$), respectively. For $f \in \tilde{D}_+$ we have

$$\phi_{\mathcal{X}_+} f = \langle f, \tilde{\iota}_+^* \cdot\rangle_{\mathcal{X}_+,\mathcal{X}_+'} = \langle \tilde{\iota}_+ f, \cdot\rangle_{\mathcal{X}_0} = \phi_{\mathcal{X}_0} \circ \tilde{\iota}_+ f$$

and consequently the diagram in (i) commutes. ❑

If one and therefore all assertions in Proposition 4.1.8 are satisfied, then $\mathcal{X}_+ \cap \mathcal{X}_0$ is defined as the intersection in Z and complete with the norm $\|\cdot\|_{\mathcal{X}_+\cap\mathcal{X}_0} := \sqrt{\|\cdot\|_{\mathcal{X}_+}^2 + \|\cdot\|_{\mathcal{X}_0}^2}$. Moreover, we define D_+ as the closure of $\tilde{D}_+$ in $\mathcal{X}_+ \cap \mathcal{X}_0$ (w.r.t. $\|\cdot\|_{\mathcal{X}_+\cap\mathcal{X}_0}$). Note that although $\mathcal{X}_+ \cap \mathcal{X}_0$ may depend on Z, D_+ is independent of Z. We will denote the extension of $\tilde{\iota}_+$ to D_+ by ι_+, which can be expressed by $\iota_+ = \overline{\tilde{\iota}_+}$. The adjoint ι_+^* coincides with $\tilde{\iota}_+^*$. Also D_- does not change, if we replace $\tilde{D}_+$ by D_+ in Definition 4.1.2 and all previous results in this section also hold for D_+ and ι_+ instead of $\tilde{D}_+$ and $\tilde{\iota}_+$, respectively. If $\tilde{\iota}_+$ is already closed, then $D_+ = \tilde{D}_+$.

Lemma 4.1.9. *Let one assertion in Proposition 4.1.8 be satisfied. Let $Z = Y'$, where $Y = D_-$ endowed with $\|g\|_Y := \|g\|_{\mathcal{X}_-\cap\mathcal{X}_0} = \sqrt{\|g\|_{\mathcal{X}_-}^2 + \|g\|_{\mathcal{X}_0}^2}$ (from Proposition 4.1.8* (iv) $\Rightarrow$ (i)*). Then we have the following characterization for D_+:*

- $D_+ = \operatorname{dom} \iota_-^*$,
- $D_+ = \mathcal{X}_+ \cap \mathcal{X}_0$ *in* Y'.

Proof. Note that for $g \in D_-$ we have $g = (\iota_+^*)^{-1}\iota_+^* g$ and that $\iota_+^*\iota_-$ is isometric from $D_- = \operatorname{dom}\iota_- \subseteq \mathcal{X}_-$ onto $\operatorname{ran}\iota_+^* = \operatorname{dom}(\iota_+^*)^{-1} \subseteq \mathcal{X}_+'$. The following equivalences show the first assertion:

$$\begin{aligned} f \in \operatorname{dom}\iota_-^* &\Leftrightarrow D_- \ni g \mapsto \langle f, \iota_- g\rangle_{\mathcal{X}_0} \text{ is continuous w.r.t. } \|\cdot\|_{\mathcal{X}_-} \\ &\Leftrightarrow D_- \ni g \mapsto \langle f, (\iota_+^*)^{-1}\iota_+^*\iota_- g\rangle_{\mathcal{X}_0} \text{ is continuous w.r.t. } \|\cdot\|_{\mathcal{X}_-} \\ &\Leftrightarrow \operatorname{dom}(\iota_+^*)^{-1} \ni h \mapsto \langle f, (\iota_+^*)^{-1} h\rangle_{\mathcal{X}_0} \text{ is continuous w.r.t. } \|\cdot\|_{\mathcal{X}_+'} \\ &\Leftrightarrow f \in \operatorname{dom}\left((\iota_+^*)^{-1}\right)^* = \operatorname{dom}\iota_+^{-1} = \operatorname{ran}\iota_+ = D_+. \end{aligned}$$

For the second characterization we define $P_+ := \mathcal{X}_+ \cap \mathcal{X}_0$ and we define P_- analogously to D_- in Definition 4.1.2:

$$\|g\|_{P_-} := \sup_{f \in P_+\setminus\{0\}} \frac{|\langle g, f\rangle_{\mathcal{X}_0}|}{\|f\|_{\mathcal{X}_+}} \quad \text{and} \quad P_- := \{g \in \mathcal{X}_0 \mid \|g\|_{P_-} < +\infty\}.$$

Clearly, $\|g\|_{\mathcal{X}_-} \leq \|g\|_{P_-}$ for $g \in P_-$ and consequently $P_- \subseteq D_-$. Furthermore, we can define $\iota_{P_+}: P_+ \subseteq \mathcal{X}_+ \to \mathcal{X}_0, f \mapsto f$ analogously to $\tilde{\iota}_+$. Note that ι_{P_+} is closed due the completeness of $(\mathcal{X}_+ \cap \mathcal{X}_0, \|\cdot\|_{\mathcal{X}_+ \cap \mathcal{X}_0})$. Then we have $\operatorname{dom} \iota_{P_+}^* = P_-$ and $\tilde{\iota}_+ \subseteq \iota_{P_+}$ and therefore $\iota_{P_+}^* \subseteq \tilde{\iota}_+^*$. For $g \in D_-$ and $f \in P_+$ we have, by definition of $P_+ = \mathcal{X}_+ \cap \mathcal{X}_0$ in Z,

$$|\langle g, f \rangle_{\mathcal{X}_0}| = |\langle \tilde{\iota}_+^* g, f \rangle_{\mathcal{X}_+', \mathcal{X}_+}| \leq \|\tilde{\iota}_+^* g\|_{\mathcal{X}_+'} \|f\|_{\mathcal{X}_+} = \|g\|_{\mathcal{X}_-} \|f\|_{\mathcal{X}_+},$$

which yields $\|g\|_{P_-} \leq \|g\|_{\mathcal{X}_-}$. Hence, $P_- = D_-$, $\iota_{P_+}^* = \tilde{\iota}_+^*$ and $\iota_{P_+} = \overline{\tilde{\iota}_+}$, which is equivalent to $P_+ = \mathcal{X}_+ \cap \mathcal{X}_0 = \overline{\tilde{D}_+}^{\mathcal{X}_+ \cap \mathcal{X}_0} = D_+$. ❑

Theorem 4.1.10. *Let one assertion in Proposition 4.1.8 be satisfied. Then the continuous extension of $\iota_+^* \iota_-$ denoted by $\overline{\iota_+^* \iota_-}$ equals Ψ. Moreover, Ψ is surjective and $(\mathcal{X}_+, \mathcal{X}_-)$ is a complete dual pair with*

$$\langle g, f \rangle_{\mathcal{X}_-, \mathcal{X}_+} \coloneqq \langle \Psi g, f \rangle_{\mathcal{X}_+', \mathcal{X}_+}.$$

Proof. We have already shown, that $\iota_+^* \iota_- g = \Psi g$ for $g \in D_-$. Since D_- is dense in $\mathcal{X}_-$, we also have $\overline{\iota_+^* \iota_-} g = \Psi g$ for $g \in \mathcal{X}_-$.

If one assertion in Proposition 4.1.8 is true, then all of them are true. Hence, ΨD_- is dense in $\mathcal{X}_+'$ and because Ψ is isometric $\operatorname{ran} \Psi$ is closed and therefore $\operatorname{ran} \Psi = \mathcal{X}_+'$.

Since Ψ is an isomorphism between $\mathcal{X}_-$ and $\mathcal{X}_+'$, it immediately follows that $(\mathcal{X}_+, \mathcal{X}_-)$ is a complete dual pair with the dual pairing $\langle \cdot, \cdot \rangle_{\mathcal{X}_-, \mathcal{X}_+}$. ❑

Remark 4.1.11. For $f \in D_+$ and $g \in D_-$ we have

$$\langle g, f \rangle_{\mathcal{X}_-, \mathcal{X}_+} = \langle \Psi g, f \rangle_{\mathcal{X}_+', \mathcal{X}_+} = \langle \iota_+^* \iota_- g, f \rangle_{\mathcal{X}_+', \mathcal{X}_+} = \langle \iota_- g, \iota_+ f \rangle_{\mathcal{X}_0} = \langle g, f \rangle_{\mathcal{X}_0}.$$

Since these two sets are dense in $\mathcal{X}_+$ and $\mathcal{X}_-$ respectively, we have for $f \in \mathcal{X}_+$ and $g \in \mathcal{X}_-$

$$\langle g, f \rangle_{\mathcal{X}_-, \mathcal{X}_+} = \lim_{(n,m) \to (\infty, \infty)} \langle g_n, f_m \rangle_{\mathcal{X}_0},$$

where $(f_m)_{m \in \mathbb{N}}$ is a sequence in D_+ that converges to f in $\mathcal{X}_+$ and $(g_n)_{n \in \mathbb{N}}$ is a sequence in D_- that converges to g in $\mathcal{X}_-$.

4.2 Definition and Results

The previous section leads to the following definition.

Definition 4.2.1. Let $(\mathcal{X}_+, \mathcal{X}_-)$ be a complete dual pair and $\mathcal{X}_0$ be a Hilbert space. Furthermore, let $\iota_+ \colon \operatorname{dom} \iota_+ \subseteq \mathcal{X}_+ \to \mathcal{X}_0$ and $\iota_- \colon \operatorname{dom} \iota_- \subseteq \mathcal{X}_- \to \mathcal{X}_0$ be densely defined, closed, and injective linear mappings with dense range. We call $(\mathcal{X}_+, \mathcal{X}_0, \mathcal{X}_-)$ a *quasi Gelfand triple*, if

$$\langle g, f \rangle_{\mathcal{X}_-, \mathcal{X}_+} = \langle \iota_- g, \iota_+ f \rangle_{\mathcal{X}_0} \tag{4.1}$$

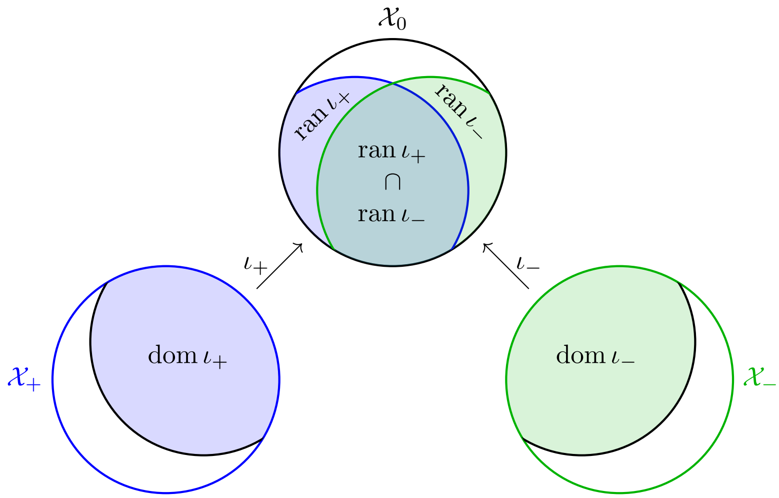

Figure 4.2: Illustration of a quasi Gelfand triple

for all $f \in \operatorname{dom}\iota_+$ and $g \in \operatorname{dom}\iota_-$, and $\operatorname{dom}\iota_+^* = \operatorname{ran}\iota_-$. The space $\mathcal{X}_0$ will be referred as *pivot space*.

Figure 4.2 illustrates the setting of a quasi Gelfand triple. Contrary to the previous section we will regard the adjoint of ι_+ and ι_- with respect to the complete dual pairs $(\mathcal{X}_+, \mathcal{X}_-)$ and $(\mathcal{X}_0, \mathcal{X}_0)$. Therefore, ι_+^* is a densely defined operator from $\mathcal{X}_0$ to $\mathcal{X}_-$ and ι_-^* is a densely defined operator from $\mathcal{X}_0$ to $\mathcal{X}_+$. We could not do this before, because we did not know from the beginning that $(\mathcal{X}_+, \mathcal{X}_-)$ is a complete dual pair.

Example 4.2.2. Let $\mathcal{X}_+ = \mathsf{L}^p(\mathbb{R})$, $\mathcal{X}_- = \mathsf{L}^q(\mathbb{R})$ and $\mathcal{X}_0 = \mathsf{L}^2(\mathbb{R})$, where $p \in (1, +\infty)$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then $(\mathcal{X}_+, \mathcal{X}_-)$ is a complete dual pair. Note that $\mathsf{L}^p(\mathbb{R}) \cap \mathsf{L}^2(\mathbb{R})$ is already well-defined. We can define

$$\iota_+ : \begin{cases} \mathsf{L}^p(\mathbb{R}) \cap \mathsf{L}^2(\mathbb{R}) \subseteq \mathsf{L}^p(\mathbb{R}) & \to & \mathsf{L}^2(\mathbb{R}), \\ f & \mapsto & f, \end{cases}$$

$$\text{and} \quad \iota_- : \begin{cases} \mathsf{L}^q(\mathbb{R}) \cap \mathsf{L}^2(\mathbb{R}) \subseteq \mathsf{L}^q(\mathbb{R}) & \to & \mathsf{L}^2(\mathbb{R}), \\ g & \mapsto & g. \end{cases}$$

These mapping are densely defined, injective and closed with dense range. By definition of the dual pairing of $(\mathsf{L}^p(\mathbb{R}), \mathsf{L}^q(\mathbb{R}))$ we have

$$\langle g, f \rangle_{\mathsf{L}^q(\mathbb{R}), \mathsf{L}^p(\mathbb{R})} = \int_{\mathbb{R}} g\overline{f} \,\mathrm{d}\lambda = \langle g, f \rangle_{\mathcal{X}_0} = \langle \iota_- g, \iota_+ f \rangle_{\mathcal{X}_0}$$

for $g \in \mathsf{L}^q(\mathbb{R}) \cap \mathsf{L}^2(\mathbb{R})$ and $f \in \mathsf{L}^p(\mathbb{R}) \cap \mathsf{L}^2(\mathbb{R})$. Hence, $(\mathsf{L}^p(\mathbb{R}), \mathsf{L}^2(\mathbb{R}), \mathsf{L}^q(\mathbb{R}))$ is a quasi Gelfand triple.

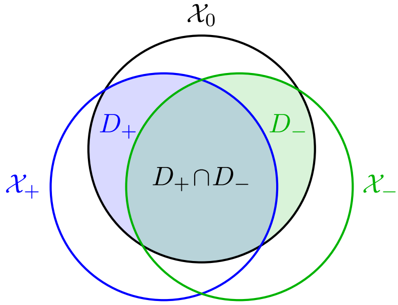

Figure 4.3: Illustration of a quasi Gelfand triple, where $D_+ = \operatorname{ran} \iota_+$ and $D_- = \operatorname{ran} \iota_-$.

Note that the mapping ι_+ gives us an identification of $\operatorname{dom} \iota_+$ and $\operatorname{ran} \iota_+$. Hence, we can introduce the norm of $\mathcal{X}_+$ on $\operatorname{ran} \iota_+$ by $\|f\|_{\mathcal{X}_+} = \|\iota_+^{-1} f\|_{\mathcal{X}_+}$ for $f \in \operatorname{ran} \iota_+$. Then the completion of $\operatorname{ran} \iota_+$ with respect to $\|\cdot\|_{\mathcal{X}_+}$ is isometrically isomorphic to $\mathcal{X}_+$. Accordingly, we can do the same for $\mathcal{X}_-$. This justifies the following definition and Figure 4.3

Definition 4.2.3. For a quasi Gelfand triple $(\mathcal{X}_+, \mathcal{X}_0, \mathcal{X}_-)$ we define

$$\mathcal{X}_+ \cap \mathcal{X}_0 \coloneqq \operatorname{ran} \iota_+ \quad \text{and} \quad \mathcal{X}_- \cap \mathcal{X}_0 \coloneqq \operatorname{ran} \iota_-.$$

If either ι_+ or ι_- is continuous, then a quasi Gelfand triple is an "ordinary" Gelfand triple. Clearly, every "ordinary" Gelfand triple is also a quasi Gelfand triple.

The condition $\operatorname{dom} \iota_+^* = \operatorname{ran} \iota_-$ is not really necessary as this condition can always be forced as we will see later. Actually, I believe that this condition is automatically implied by all the others, but I could not find a proof. Moreover, the next lemma shows that we can also ask for the converse condition $\operatorname{dom} \iota_-^* = \operatorname{ran} \iota_+$ instead. Note that from (4.1) we can immediately see that $\operatorname{dom} \iota_+^* \supseteq \operatorname{ran} \iota_-$ and $\operatorname{dom} \iota_-^* \supseteq \operatorname{ran} \iota_+$. Hence, for $f \in \operatorname{dom} \iota_+$ and $g \in \operatorname{dom} \iota_-$ we have

$$\langle g, f \rangle_{\mathcal{X}_-, \mathcal{X}_+} = \langle \iota_- g, \iota_+ f \rangle_{\mathcal{X}_0} = \begin{cases} \langle \iota_+^* \iota_- g, f \rangle_{\mathcal{X}_-, \mathcal{X}_+} \\ \langle g, \iota_-^* \iota_+ f \rangle_{\mathcal{X}_-, \mathcal{X}_+}, \end{cases} \tag{4.2}$$

which implies $\iota_+^* \iota_- g = g$ and $\iota_-^* \iota_+ f = f$.

Lemma 4.2.4. *Let $(\mathcal{X}_+, \mathcal{X}_0, \mathcal{X}_-)$ with ι_+ and ι_- satisfy all conditions of Definition 4.2.1 except* $\operatorname{dom} \iota_+^* = \operatorname{ran} \iota_-$. *Then*

$$\operatorname{dom} \iota_+^* = \operatorname{ran} \iota_- \quad \Leftrightarrow \quad \operatorname{dom} \iota_-^* = \operatorname{ran} \iota_+.$$

In particular, if $(\mathcal{X}_+, \mathcal{X}_0, \mathcal{X}_-)$ is a quasi Gelfand triple, then also $\operatorname{dom} \iota_-^* = \operatorname{ran} \iota_+$ *holds true.*

The proof of this is basically the first part of the proof of Lemma 4.1.9.

Proof. Let $\operatorname{dom}\iota_+^* = \operatorname{ran}\iota_-$. The following equivalences

$$\begin{aligned} f \in \operatorname{dom}\iota_-^* &\Leftrightarrow \operatorname{dom}\iota_- \ni g \mapsto \langle f, \iota_- g\rangle_{\mathcal{X}_0} \text{ is continuous w.r.t. } \|\cdot\|_{\mathcal{X}_-} \\ &\Leftrightarrow \operatorname{dom}\iota_- \ni g \mapsto \langle f, (\iota_+^*)^{-1} \underbrace{\iota_+^*\iota_- g}_{=g}\rangle_{\mathcal{X}_0} \text{ is continuous w.r.t. } \|\cdot\|_{\mathcal{X}_-} \\ &\Leftrightarrow f \in \operatorname{dom}\left((\iota_+^*)^{-1}\right)^* = \operatorname{dom}\iota_+^{-1} = \operatorname{ran}\iota_+ \end{aligned}$$

imply $\operatorname{dom}\iota_-^* = \operatorname{ran}\iota_+$.

The other implication follows analogously. ❑

In contrast to “ordinary” Gelfand triple, the setting for quasi Gelfand triple is somehow “symmetric”, i.e. the roles of $\mathcal{X}_+$ and $\mathcal{X}_-$ are interchangeable, since neither of the embeddings ι_+ and ι_- has to be continuous, as indicated in the beginning of this section.

Lemma 4.2.5. *Let $(\mathcal{X}_+, \mathcal{X}_0, \mathcal{X}_-)$ with ι_+ and ι_- satisfy all conditions of Definition 4.2.1 except* $\operatorname{dom}\iota_+^* = \operatorname{ran}\iota_-$. *Then there exists an extension $\hat\iota_-$ of ι_- that respects* (4.1) *and satisfies* $\operatorname{dom}\iota_+^* = \operatorname{ran}\hat\iota_-$. *In particular, $(\mathcal{X}_+, \mathcal{X}_0, \mathcal{X}_-)$ with ι_+ and $\hat\iota_-$ forms a quasi Gelfand triple.*

Proof. Note that $\iota_+^*\iota_- g = g$. Hence, $\iota_+^* \supseteq \iota_-^{-1}$ and $(\iota_+^*)^{-1} \supseteq \iota_-$. We define $\hat\iota_-$ as $(\iota_+^*)^{-1}$. Then clearly $\operatorname{ran}\hat\iota_- = \operatorname{dom}\iota_+^*$. For $f \in \operatorname{dom}\iota_+$ and $g \in \operatorname{dom}\hat\iota_-$ we have

$$\langle \hat\iota_- g, \iota_+ f\rangle_{\mathcal{X}_0} = \langle \iota_+^*\hat\iota_- g, f\rangle_{\mathcal{X}_-,\mathcal{X}_+} = \langle g, f\rangle_{\mathcal{X}_-,\mathcal{X}_+}.$$ ❑

Remark 4.2.6. If $(\mathcal{X}_+, \mathcal{X}_0, \mathcal{X}_-)$ is a quasi Gelfand triple and $(\mathcal{X}_+, \widetilde{\mathcal{X}_-})$ is another dual pair for $\mathcal{X}_+$, then also $(\mathcal{X}_+, \mathcal{X}_0, \widetilde{\mathcal{X}_-})$ is a quasi Gelfand triple.

Lemma 4.2.7. *Let $(\mathcal{X}_+, \mathcal{X}_0, \mathcal{X}_-)$ be a quasi Gelfand triple. Then*

$$\iota_+^* = \iota_-^{-1} \quad \textit{and} \quad \iota_-^* = \iota_+^{-1}.$$

Proof. By (4.2) we have $\iota_+^*\iota_- g = g$ for all $g \in \operatorname{dom}\iota_+$. Since $\operatorname{ran}\iota_- = \operatorname{dom}\iota_+^*$ (by assumption), we conclude that $\iota_+^* = \iota_-^{-1}$.

Analogously, the second equality can be shown. ❑

Theorem 4.2.8. *Let $\mathcal{X}_+$ be a reflexive Banach space and $\mathcal{X}_0$ be a Hilbert space and $\iota_+ \colon \operatorname{dom}\iota_+ \subseteq \mathcal{X}_+ \to \mathcal{X}_0$ be a densely defined, closed, and injective linear mapping with dense range. Then there exists a Banach space $\mathcal{X}_-$ and a mapping ι_- such that $(\mathcal{X}_+, \mathcal{X}_0, \mathcal{X}_-)$ is a quasi Gelfand triple.*

In particular, $\mathcal{X}_-$ is given by Definition 4.1.2, where $D_+ = \operatorname{ran}\iota_+$.

Proof. We will identify $\operatorname{dom}\iota_+$ with $\operatorname{ran}\iota_+$ and denote it by D_+. Then item (iii) of Proposition 4.1.8 is satisfied. Hence, the corresponding D_- (Definition 4.1.2) is dense in $\mathcal{X}_0$ and its completion $\mathcal{X}_-$ (w.r.t. to $\|\cdot\|_{\mathcal{X}_-}$) establishes the complete dual pair $(\mathcal{X}_+, \mathcal{X}_-)$, by Theorem 4.1.10. The mapping

$$\iota_- : \begin{cases} D_- \subseteq \mathcal{X}_- & \to & \mathcal{X}_0, \\ g & \mapsto & g, \end{cases}$$

is densely defined and injective by construction. By the already shown $\operatorname{ran}\iota_- = D_-$ is dense in $\mathcal{X}_0$. Finally, by Lemma 4.1.5 ι_- is closed and by Lemma 4.1.6 $\operatorname{dom}\iota_+^* = D_- = \operatorname{ran}\iota_-$. ❑

Remark 4.2.9. By Theorem 4.2.8 the setting in the beginning of this chapter establishes a quasi Gelfand triple, if one assertion of Proposition 4.1.8 is satisfied.

Until the end of this chapter we will assume that $(\mathcal{X}_+, \mathcal{X}_0, \mathcal{X}_-)$ is a quasi Gelfand triple and we will identify $\operatorname{dom}\iota_+$ with $\operatorname{ran}\iota_+$ and denote it by D_+. The set D_- is defined by Definition 4.1.2 for D_+. This set coincides with $\operatorname{ran}\iota_-$, which we will identify with $\operatorname{dom}\iota_-$.

Proposition 4.2.10. *The space $D_+ \cap D_-$ is complete with respect to*

$$\|\cdot\|_{\mathcal{X}_+\cap\mathcal{X}_-} \coloneqq \sqrt{\|\cdot\|_{\mathcal{X}_+}^2 + \|\cdot\|_{\mathcal{X}_-}^2}.$$

Proof. For $f \in D_+ \cap D_-$ we have

$$\|f\|_{\mathcal{X}_0}^2 = |\langle f, f\rangle_{\mathcal{X}_0}| = |\langle f, f\rangle_{\mathcal{X}_-,\mathcal{X}_+}| \leq \|f\|_{\mathcal{X}_-}\|f\|_{\mathcal{X}_+} \leq \|f\|_{\mathcal{X}_+\cap\mathcal{X}_-}^2.$$

Hence, every Cauchy sequence in $D_+ \cap D_-$ with respect to $\|\cdot\|_{\mathcal{X}_+\cap\mathcal{X}_-}$ is also a Cauchy sequence with respect to $\|\cdot\|_{\mathcal{X}_0}$, $\|\cdot\|_{\mathcal{X}_+}$ and $\|\cdot\|_{\mathcal{X}_-}$.

Let $(f_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $D_+ \cap D_-$ with respect to $\|\cdot\|_{\mathcal{X}_+\cap\mathcal{X}_-}$. By the closedness of ι_+ the limit with respect to $\|\cdot\|_{\mathcal{X}_0}$ and the limit with respect to $\|\cdot\|_{\mathcal{X}_+}$ coincide. The same argument for ι_- yields that the limit with respect to $\|\cdot\|_{\mathcal{X}_0}$ and the limit with respect $\|\cdot\|_{\mathcal{X}_-}$ also coincide. Therefore, all these limits have to coincide and $(f_n)_{n\in\mathbb{N}}$ converges to that limit in $D_+ \cap D_-$ w.r.t. $\|\cdot\|_{\mathcal{X}_+\cap\mathcal{X}_-}$. ❑

Lemma 4.2.11. *The operator*

$$\begin{bmatrix}\iota_+ & \iota_-\end{bmatrix} : \begin{cases} D_+ \times D_- \subseteq \mathcal{X}_+ \times \mathcal{X}_- & \to & \mathcal{X}_0, \\ \begin{bmatrix} f \\ g \end{bmatrix} & \mapsto & f + g, \end{cases}$$

is closed.

Proof. Let $\left(\left(\left[\begin{smallmatrix} f_n \\ g_n \end{smallmatrix}\right], z_n\right)\right)_{n\in\mathbb{N}}$ be a sequence in $\begin{bmatrix}\iota_+ & \iota_-\end{bmatrix}$ that converges to $\left(\left[\begin{smallmatrix} f \\ g \end{smallmatrix}\right], z\right)$ in $\mathcal{X}_+ \times \mathcal{X}_- \times \mathcal{X}_0$. Then we have

$$\|z\|_{\mathcal{X}_0}^2 = \lim_{n\to\infty} \|f_n + g_n\|_{\mathcal{X}_0}^2 = \lim_{n\to\infty} \left(\|f_n\|_{\mathcal{X}_0}^2 + \|g_n\|_{\mathcal{X}_0}^2 + 2\operatorname{Re}\langle f_n, g_n\rangle_{\mathcal{X}_0}\right).$$

Since $2\,\mathrm{Re}\langle f_n, g_n\rangle_{\mathcal{X}_0}$ converges to $2\,\mathrm{Re}\langle f, g\rangle_{\mathcal{X}_+,\mathcal{X}_-}$, we conclude that $\|f_n\|_{\mathcal{X}_0}$ and $\|g_n\|_{\mathcal{X}_0}$ are bounded. Hence, there exists a subsequence of $(f_n)_{n\in\mathbb{N}}$ that converges weakly to an $\tilde{f} \in \mathcal{X}_0$. Moreover, by Lemma A.3.3 we can pass on to a further subsequence $(f_{n(k)})_{k\in\mathbb{N}}$ such that $\left(\frac{1}{j}\sum_{k=1}^{j} f_{n(k)}\right)_{j\in\mathbb{N}}$ converges to $\tilde{f}$ strongly (w.r.t. $\|\cdot\|_{\mathcal{X}_0}$). The sequence $\left(\frac{1}{j}\sum_{k=1}^{j} f_{n(k)}\right)_{j\in\mathbb{N}}$ has still the limit f in $\mathcal{X}_+$ (w.r.t. $\|\cdot\|_{\mathcal{X}_+}$) and because ι_+ is closed we conclude that $f = \tilde{f} \in D_+$. By linearity we also have $\frac{1}{j}\sum_{k=1}^{j} g_{n(k)} \to z - f$ in $\mathcal{X}_0$ for the same subsequence. Since $\frac{1}{j}\sum_{k=1}^{j} g_{n(k)}$ is a Cauchy sequence in both $\mathcal{X}_-$ and $\mathcal{X}_0$, the closedness of ι_- gives that $g = z - f \in D_-$. Hence, $z = \begin{bmatrix}\iota_+ & \iota_-\end{bmatrix}\begin{bmatrix} f \\ g\end{bmatrix}$ and the operator $\begin{bmatrix}\iota_+ & \iota_-\end{bmatrix}$ is closed. ❑

Proposition 4.2.12. *$D_+ \cap D_-$ is dense in $\mathcal{X}_0$ with respect to $\|\cdot\|_{\mathcal{X}_0}$.*

Proof. By $\operatorname{dom}\iota_\pm^* = \operatorname{ran}\iota_\mp = D_\mp$ (Lemma 4.2.4) we have

$$\mathcal{X}_0 = \left(\operatorname{mul}\begin{bmatrix}\iota_+ & \iota_-\end{bmatrix}\right)^\perp = \overline{\operatorname{dom}\begin{bmatrix}\iota_+ & \iota_-\end{bmatrix}^*} = \overline{\operatorname{dom}\iota_+^* \cap \operatorname{dom}\iota_-^*} = \overline{D_- \cap D_+}.$$ ❑

4.3 Quasi Gelfand Triple with Hilbert Spaces

In this section we will regard a quasi Gelfand triple $(\mathcal{X}_+, \mathcal{X}_0, \mathcal{X}_-)$, where $\mathcal{X}_+$ and $\mathcal{X}_-$ (and of course $\mathcal{X}_0$) are Hilbert spaces. Maybe also these results can be proven for general quasi Gelfand triple, but I could not find a substitute for Theorem 4.3.1.

For a quasi Gelfand triple $(\mathcal{X}_+, \mathcal{X}_0, \mathcal{X}_-)$ consisting of Hilbert spaces, there exists a unitary mapping Ψ from $\mathcal{X}_-$ to $\mathcal{X}_+$ satisfying

$$\langle g, f\rangle_{\mathcal{X}_-,\mathcal{X}_+} = \langle \Psi g, f\rangle_{\mathcal{X}_+} \quad \text{and} \quad \langle f, g\rangle_{\mathcal{X}_+,\mathcal{X}_-} = \langle \Psi^{-1} f, g\rangle_{\mathcal{X}_-}.$$

We will refer to this mapping as the *duality map* of the quasi Gelfand triple.

Theorem 4.3.1 (J. von Neumann)**.** *Let T be a closed linear operator from the Hilbert space X to the Hilbert space Y. Then T^*T and TT^* are self-adjoint, and $(\mathrm{I}_X + T^*T)$ and $(\mathrm{I}_Y + TT^*)$ are boundedly invertible.*

Note that here the adjoint T^* is calculated with respect to the "natural" dual pairs (X, X) and (Y, Y), i.e. $T^* = T^{*_{Y\times X}}$.

Proof. Since $T^* = \begin{bmatrix} 0 & \mathrm{I}_Y \\ -\mathrm{I}_X & 0\end{bmatrix} T^\perp$, we have $T \oplus \begin{bmatrix} 0 & -\mathrm{I}_X \\ \mathrm{I}_Y & 0\end{bmatrix} T^* = X \times Y$. Hence, for $\begin{bmatrix} h \\ 0\end{bmatrix} \in X \times Y$ there are unique $x \in \operatorname{dom} T$ and $y \in \operatorname{dom} T^*$ such that

$$\begin{bmatrix} h \\ 0\end{bmatrix} = \begin{bmatrix} x \\ Tx\end{bmatrix} + \begin{bmatrix} -T^*y \\ y\end{bmatrix}. \tag{4.3}$$

Consequently, $h = x - T^*y$ and $y = -Tx$, which implies $x \in \operatorname{dom} T^*T$ and

$$h = x + T^*Tx.$$

Because of the uniqueness of the decomposition in (4.3), $x \in \operatorname{dom} T^*T$ is uniquely determined by $h \in X$. Therefore, $(\mathrm{I}_X + T^*T)^{-1}$ is a well-defined and everywhere defined operator.

For $h_1, h_2 \in X$, we define $x_1 := (\mathrm{I}_X + T^*T)^{-1} h_1$ and $x_2 := (\mathrm{I}_X + T^*T)^{-1} h_2$. Then $x_1, x_2 \in \operatorname{dom} T^*T$ and, by the closedness of T, $T^{**} = T$. Hence,

$$\begin{aligned}\langle h_1, (\mathrm{I}_X + T^*T)^{-1} h_2 \rangle &= \langle (\mathrm{I}_X + T^*T) x_1, x_2 \rangle = \langle x_1, x_2 \rangle + \langle T^*T x_1, x_2 \rangle \\ &= \langle x_1, x_2 \rangle + \langle T x_1, T x_2 \rangle = \langle x_1, x_2 \rangle + \langle x_1, T^*T x_2 \rangle \\ &= \langle x_1, (\mathrm{I}_X + T^*T) x_2 \rangle = \langle (\mathrm{I}_X + T^*T)^{-1} h_1, h_2 \rangle,\end{aligned}$$

which yields that $(\mathrm{I}_X + T^*T)^{-1}$ is self-adjoint. Therefore $(\mathrm{I}_X + T^*T)$ and T^*T are also self-adjoint. Moreover, $(\mathrm{I}_X + T^*T)^{-1}$ is bounded as a closed and everywhere defined operator.

By $TT^* = (T^*)^*(T^*)$ the other statements follow by the already shown. ❑

Applying this theorem to $S = \lambda T$ implies that $\mathbb{R}_-$ is contained in the resolvent set of T^*T.

Note that we previously regarded the adjoint of ι_+ with respect to the dual pairs $(\mathcal{X}_0, \mathcal{X}_0)$ and $(\mathcal{X}_+, \mathcal{X}_-)$. However, in order to apply Theorem 4.3.1 we have to regard the adjoint with respect to $(\mathcal{X}_0, \mathcal{X}_0)$ and $(\mathcal{X}_+, \mathcal{X}_+)$. Hence, we will emphazise this difference by the notation $\iota_+^{*\mathcal{X}_0 \times \mathcal{X}_+}$, which was introduced in Definition 2.2.1.

Corollary 4.3.2. *The set $D_+ \cap D_-$ is dense in $\mathcal{X}_+$ and $\mathcal{X}_-$ with respect to their corresponding norms.*

Proof. Applying Theorem 4.3.1 to ι_+ yields $\iota_+^{*\mathcal{X}_0 \times \mathcal{X}_+} \iota_+$ is self-adjoint. Note that by Lemma 2.2.4 we have $\iota_+^{*\mathcal{X}_0 \times \mathcal{X}_+} = \Psi \iota_+^{*\mathcal{X}_0 \times \mathcal{X}_-} = \Psi \iota_+^*$, where Ψ is the duality map introduced in the beginning of this section. Hence, $\operatorname{dom} \iota_+^{*\mathcal{X}_0 \times \mathcal{X}_+} \iota_+ = \operatorname{dom} \iota_+^* \iota_+$ is dense in $\mathcal{X}_+$. By Lemma 4.2.4 $\operatorname{dom} \iota_+^* = D_-$, consequently

$$\operatorname{dom} \iota_+^* \iota_+ = D_+ \cap D_-. \tag{4.4}$$

An analogous argument for ι_- yields $D_+ \cap D_-$ is dense in $\mathcal{X}_-$. ❑

Corollary 4.3.3. $D_+ + D_- = \mathcal{X}_0$.

Proof. Applying Theorem 4.3.1 to ι_+ gives that $(\mathrm{I}_{\mathcal{X}_0} + \iota_+ \iota_+^{*\mathcal{X}_0 \times \mathcal{X}_+})$ is onto. Hence, for every $x \in \mathcal{X}_0$ there exists a $g_x \in \operatorname{dom} \iota_+ \iota_+^{*\mathcal{X}_0 \times \mathcal{X}_+} \subseteq D_-$ such that

$$x = \underbrace{g_x}_{\in D_-} + \underbrace{\iota_+ \iota_+^{*\mathcal{X}_0 \times \mathcal{X}_+} g_x}_{\in D_+}.$$

Since $g_x \in \operatorname{dom} \iota_+ \iota_+^{*\mathcal{X}_0 \times \mathcal{X}_+}$, we have $\iota_+^{*\mathcal{X}_0 \times \mathcal{X}_+} g_x \in D_+$ and consequently $x \in D_+ + D_-$. ❑

Note that $D_+ \cap D_-$ with $\|\cdot\|_{\mathcal{X}_+\cap\mathcal{X}_-}$ is complete and therefore a Banach space. Since $\mathcal{X}_+$ and $\mathcal{X}_0$ are Hilbert spaces (in this section) we can define the inner product

$$\langle g, f\rangle_{\mathcal{X}_+\cap\mathcal{X}_-} \coloneqq \langle g, f\rangle_{\mathcal{X}_+} + \langle g, f\rangle_{\mathcal{X}_-}$$

on $D_+ \cap D_-$. This inner product induces the previous norm $\|\cdot\|_{\mathcal{X}_+\cap\mathcal{X}_-}$. Consequently $D_+ \cap D_-$ is a Hilbert space with $\langle\cdot,\cdot\rangle_{\mathcal{X}_+\cap\mathcal{X}_-}$. For shorter notation we denote $D_+ \cap D_-$ as $\mathcal{Z}_+$. Note that $\mathcal{Z}_+$ is dense in $\mathcal{X}_+$, $\mathcal{X}_0$ and $\mathcal{X}_-$ with respect to their norms (up to embeddings). Hence, we can continuously embed all of theses spaces into $\mathcal{Z}_+'$. For notational harmony we will denote $\mathcal{Z}_+'$ as $\mathcal{Z}_-$. Clearly $(\mathcal{Z}_+, \mathcal{Z}_-)$ is a complete dual pair. Moreover, by Theorem 4.2.8 and Remark 4.2.6 $(\mathcal{Z}_+, \mathcal{X}_0, \mathcal{Z}_-)$ is a quasi Gelfand triple. Actually, it is even a Gelfand triple, as the embedding of $\mathcal{Z}_+$ into $\mathcal{X}_0$ is continuous. Figure 4.4 illustrates this scenario.

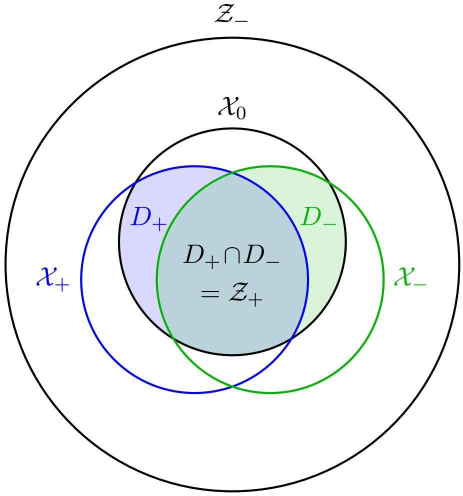

Figure 4.4: quasi Gelfand triple embedded in $\mathcal{Z}_-$

Lemma 4.3.4. *$\mathcal{Z}_- = \mathcal{X}_+ + \mathcal{X}_-$ and*

$$\|h\|_{\mathcal{Z}_-} = \inf_{f+g=h} \sqrt{\|f\|_{\mathcal{X}_+}^2 + \|g\|_{\mathcal{X}_-}^2}.$$

Proof. Note that $\mathcal{Z}_+$ is a Hilbert space with $\langle z_1, z_2\rangle_{\mathcal{Z}_+} = \langle z_1, z_2\rangle_{\mathcal{X}_+} + \langle z_1, z_2\rangle_{\mathcal{X}_-}$. Hence, there is a duality map Φ from $\mathcal{Z}_-$ to $\mathcal{Z}_+$ and we can write

$$\langle h, z\rangle_{\mathcal{Z}_-,\mathcal{Z}_+} = \langle \Phi h, z\rangle_{\mathcal{Z}_+} = \langle \Phi h, z\rangle_{\mathcal{X}_+} + \langle \Phi h, z\rangle_{\mathcal{X}_-}.$$

Furthermore, with the duality map Ψ from $\mathcal{X}_-$ to $\mathcal{X}_+$ we have

$$\langle h, z\rangle_{\mathcal{Z}_-,\mathcal{Z}_+} = \langle \Psi^{-1}\Phi h, z\rangle_{\mathcal{X}_-,\mathcal{X}_+} + \langle \Psi\Phi h, z\rangle_{\mathcal{X}_+,\mathcal{X}_-}$$

and $h = \Psi^{-1}\Phi h + \Psi\Phi h$ in $\mathcal{Z}_-$, where $\Psi^{-1}\Phi h \in \mathcal{X}_-$ and $\Psi\Phi h \in \mathcal{X}_+$.

Let $h \in \mathcal{Z}_-$. Then for every $f \in \mathcal{X}_+$, $g \in \mathcal{X}_-$ that satisfy $h = f + g$ in $\mathcal{Z}_-$ we have

$$\begin{aligned}
|\langle h, z\rangle_{\mathcal{Z}_-,\mathcal{Z}_+}| &= |\langle f, z\rangle_{\mathcal{X}_+,\mathcal{X}_-} + \langle g, z\rangle_{\mathcal{X}_-,\mathcal{X}_+}| \leq |\langle f, z\rangle_{\mathcal{X}_+,\mathcal{X}_-}| + |\langle g, z\rangle_{\mathcal{X}_-,\mathcal{X}_+}| \\
&\leq \|f\|_{\mathcal{X}_+}\|z\|_{\mathcal{X}_-} + \|g\|_{\mathcal{X}_-}\|z\|_{\mathcal{X}_+} \\
&\leq \sqrt{\|f\|_{\mathcal{X}_+}^2 + \|g\|_{\mathcal{X}_-}^2}\sqrt{\|z\|_{\mathcal{X}_-}^2 + \|z\|_{\mathcal{X}_+}^2} \\
&= \sqrt{\|f\|_{\mathcal{X}_+}^2 + \|g\|_{\mathcal{X}_-}^2}\,\|z\|_{\mathcal{Z}_+},
\end{aligned}$$

which implies $\|h\|_{\mathcal{Z}_-} \leq \inf_{h=f+g}\sqrt{\|f\|_{\mathcal{X}_+}^2 + \|g\|_{\mathcal{X}_-}^2}$. On the other hand

$$\|h\|_{\mathcal{Z}_-}^2 = \|\Phi h\|_{\mathcal{Z}_+}^2 = \|\Phi h\|_{\mathcal{X}_+}^2 + \|\Phi h\|_{\mathcal{X}_-}^2 = \|\Psi^{-1}\Phi h\|_{\mathcal{X}_-}^2 + \|\Psi\Phi h\|_{\mathcal{X}_+}^2$$

finishes the proof. ❑

Theorem 4.3.5. *The intersection $\mathcal{X}_+ \cap \mathcal{X}_-$ in $\mathcal{Z}_-$ is $D_+ \cap D_-$.*

This means that area of $\mathcal{X}_+ \cap \mathcal{X}_-$ in Figure 4.4 outside of $\mathcal{X}_0$ is actually empty.

Proof. Let $h \in \mathcal{X}_+ \cap \mathcal{X}_- \subseteq \mathcal{Z}_-$, i.e. it exists an $f \in \mathcal{X}_+$ and a $g \in \mathcal{X}_-$ such that

$$\langle h, z\rangle_{\mathcal{Z}_-,\mathcal{Z}_+} = \langle f, \iota_-^{-1}z\rangle_{\mathcal{X}_+,\mathcal{X}_-} = \langle g, \iota_+^{-1}z\rangle_{\mathcal{X}_-,\mathcal{X}_+} \quad \text{for all} \quad z \in D_+ \cap D_-.$$

We define $x := \iota_+^{-1}z$. Since $z \in \operatorname{dom}\iota_-^{-1}$, we have $x \in \operatorname{dom}\iota_-^{-1}\iota_+$. Note that $\iota_-^{-1} = \iota_+^*$ and $\iota_+^{-1}\mathcal{Z}_+ = \operatorname{dom}\iota_+^*\iota_+$ (see (4.4)). Hence,

$$\langle f, \iota_+^*\iota_+ x\rangle_{\mathcal{X}_+,\mathcal{X}_-} = \langle g, x\rangle_{\mathcal{X}_-,\mathcal{X}_+} \quad \text{for all} \quad x \in \operatorname{dom}\iota_+^*\iota_+,$$

which implies $(\iota_+^*\iota_+)^* f = g$ and $f \in \operatorname{dom}(\iota_+^*\iota_+)^* = \operatorname{dom}\iota_+^*\iota_+$. In particular, $\iota_+ f \in D_+ \cap D_-$. Note that again by $\iota_-^{-1} = \iota_+^*$ we have $\iota_-^{-1}\iota_+ f = g$. Therefore, $g \in \operatorname{dom}\iota_-$ and $\iota_+ f = \iota_- g$. This gives

$$\langle h, z\rangle_{\mathcal{Z}_-,\mathcal{Z}_+} = \langle \iota_+ f, z\rangle_{\mathcal{X}_0} = \langle \iota_- g, z\rangle_{\mathcal{X}_0}.$$

Therefore, $h = f = g = \iota_+ f = \iota_- g$ in $\mathcal{Z}_-$. ❑

4.4 Quasi Gelfand Triples and Boundary Triples

By Remark 2.4.8 the boundary spaces of a boundary triple are always Hilbert spaces. Hence, without loss of generality we will again assume that $(\mathcal{X}_+, \mathcal{X}_0, \mathcal{X}_-)$ is a quasi Gelfand triple, where $\mathcal{X}_+$ and $\mathcal{X}_-$ are Hilbert spaces.

Proposition 4.4.1. *Let T be a bounded and boundedly invertible mapping from $\mathcal{X}_0$ to another Hilbert space $\mathcal{Y}_0$. Then $P_+ := TD_+$ equipped with $\|f\|_{\mathcal{Y}_+} := \|T^{-1}f\|_{\mathcal{X}_+}$ establishes a quasi Gelfand triple $(\mathcal{Y}_+, \mathcal{Y}_0, \mathcal{Y}_-)$, where $\mathcal{Y}_+$ is the completion of P_+ and $\mathcal{Y}_-$ is the completion of P_- defined as in Definition 4.1.2, where D_+ is replaced by P_+. Moreover, $P_- = (T^*)^{-1}D_-$ and $\|g\|_{\mathcal{Y}_-} = \|T^*g\|_{\mathcal{X}_-}$ for $g \in P_-$.*

Proof. The mapping $T\big|_{D_+} : D_+ \to P_+$ is isometric and surjective, if we equip its domain with $\|\cdot\|_{\mathcal{X}_+}$ and its codomain with $\|\cdot\|_{\mathcal{Y}_+}$. So the linear (single-valued) relation $\left[\begin{smallmatrix} T & 0 \\ 0 & T \end{smallmatrix}\right]\iota_+ = \left\{ \left[\begin{smallmatrix} Tf \\ Tg \end{smallmatrix}\right] \,\middle|\, \left[\begin{smallmatrix} f \\ g \end{smallmatrix}\right] \in \iota_+ \right\} \subseteq \mathcal{Y}_+ \times \mathcal{Y}_0$ is closed. Since this linear relation coincides with the embedding $\iota_{P_+} : P_+ \subseteq \mathcal{Y}_+ \to \mathcal{Y}_0, f \mapsto f$, Theorem 4.2.8 yields that $(\mathcal{Y}_+, \mathcal{Y}_0, \mathcal{Y}_-)$ is a quasi Gelfand triple.

For $g \in P_-$ we have

$$\|g\|_{\mathcal{Y}_-} = \sup_{h \in P_+ \setminus \{0\}} \frac{|\langle g, h \rangle_{\mathcal{Y}_0}|}{\|h\|_{\mathcal{Y}_+}} = \sup_{f \in D_+ \setminus \{0\}} \frac{|\langle g, Tf \rangle_{\mathcal{Y}_0}|}{\|Tf\|_{\mathcal{Y}_+}}$$
$$= \sup_{f \in D_+ \setminus \{0\}} \frac{|\langle T^*g, f \rangle_{\mathcal{X}_0}|}{\|f\|_{\mathcal{X}_+}} = \|T^*g\|_{\mathcal{X}_-}$$

and consequently $P_- = (T^*)^{-1}D_-$. ❑

Corollary 4.4.2. *With the assumption from Proposition 4.4.1 the operators $T\big|_{D_+}$ and $(T^*)^{-1}\big|_{D_-}$ can be continuously extended to unitary operators from $\mathcal{X}_+$ and $\mathcal{X}_-$ to $\mathcal{Y}_+$ and $\mathcal{Y}_-$ respectively. These extension will be denoted by T_+ and $(T^*)^{-1}_-$.*

Proof. Since $T\big|_{D_+}$ is isometric from D_+ onto P_+, we can extend this mapping by continuity. This extension T_+ is again isometric and since $P_+ \subseteq \operatorname{ran} T_+$ is dense, T_+ has to be surjective. Analogously, we can show the same for $(T^*)^{-1}$. ❑

Note that we regard the dual pairs $(\mathcal{X}_+, \mathcal{X}_-)$ and $(\mathcal{Y}_+, \mathcal{Y}_-)$ and therefore the adjoint of T_+ is not its inverse. However, the adjoint with respect to $(\mathcal{X}_+, \mathcal{X}_+)$ and $(\mathcal{Y}_+, \mathcal{Y}_+)$ denoted by $T_+^{*\mathcal{Y}_+ \times \mathcal{X}_+}$ is the inverse of T_+. Clearly, the same goes for $(T^*)^{-1}$. In fact we have another identity for the adjoint of T_+.

Corollary 4.4.3. *Let us continue with the assumptions of Proposition 4.4.1 and Corollary 4.4.2. Then $(T_+^*)^{-1} = (T^*)^{-1}_-$ and*

$$\left\langle (T_+^*)^{-1} g, T_+ f \right\rangle_{\mathcal{Y}_-, \mathcal{Y}_+} = \langle g, f \rangle_{\mathcal{X}_-, \mathcal{X}_+}$$

for $g \in \mathcal{X}_-$ and $f \in \mathcal{X}_+$.

Proof. Note that for $f \in D_+$ and $g \in D_-$ we have

$$\langle g, f \rangle_{\mathcal{X}_-, \mathcal{X}_+} = \langle g, f \rangle_{\mathcal{X}_0} = \left\langle (T^*)^{-1} g, Tf \right\rangle_{\mathcal{Y}_0} = \left\langle (T^*)^{-1} g, Tf \right\rangle_{\mathcal{Y}_-, \mathcal{Y}_+}.$$

Hence, we can extend this by continuity for $f \in \mathcal{X}_+$ and $g \in \mathcal{X}_-$. ❑

Corollary 4.4.4. *Let S, T be a bounded and boundedly invertible mappings on $\mathcal{X}_0$. Then $\left[\, ST\big|_{D_+} \;\; S(T^*)^{-1}\big|_{D_-} \,\right]$ is a densely defined closed surjective linear operator from $\mathcal{X}_+ \times \mathcal{X}_-$ to $\mathcal{X}_0$. In particular* $\operatorname{ran}\left[\, ST\big|_{D_+} \;\; S(T^*)^{-1}\big|_{D_-} \,\right] = \mathcal{X}_0$.

Proof. Let $P_+ = TD_+$. Then by Proposition 4.4.1 the corresponding P_- can be obtained by $(T^*)^{-1}D_-$. The mapping

$$\Xi\colon \begin{cases} \mathcal{X}_+ \times \mathcal{X}_- \times \mathcal{X}_0 & \to \quad \mathcal{Y}_+ \times \mathcal{Y}_- \times \mathcal{X}_0, \\ \qquad\qquad \begin{bmatrix} f \\ g \\ z \end{bmatrix} & \mapsto \quad \begin{bmatrix} T_+ & 0 & 0 \\ 0 & (T^*)^{-1}_- & 0 \\ 0 & 0 & S^{-1} \end{bmatrix} \begin{bmatrix} f \\ g \\ z \end{bmatrix} \end{cases}$$

is linear bounded and boundedly invertible, where $\mathcal{Y}_\pm$ is the completion of $P_\pm$ as in Proposition 4.4.1. Since $(\mathcal{Y}_+, \mathcal{X}_0, \mathcal{Y}_-)$ is a quasi Gelfand triple,

$$\begin{bmatrix} \iota_{P_+} & \iota_{P_-} \end{bmatrix} = \left\{ \begin{bmatrix} Tf \\ (T^*)^{-1}g \\ Tf + (T^*)^{-1}g \end{bmatrix} : f \in D_+, g \in D_- \right\}$$

is closed in $\mathcal{Y}_+ \times \mathcal{Y}_- \times \mathcal{X}_0$ (Lemma 4.2.11) and therefore also its pre-image under Ξ

$$\Xi^{-1}\left(\begin{bmatrix} \iota_{P_+} & \iota_{P_-} \end{bmatrix}\right) = \begin{bmatrix} T^{-1} & 0 & 0 \\ 0 & T^* & 0 \\ 0 & 0 & S \end{bmatrix} \begin{bmatrix} \iota_{P_+} & \iota_{P_-} \end{bmatrix} = \begin{bmatrix} ST\iota_+ & S(T^*)^{-1}\iota_- \end{bmatrix}$$

is closed in $\mathcal{X}_+ \times \mathcal{X}_- \times \mathcal{X}_0$. Furthermore, by Corollary 4.3.3

$$\operatorname{ran}\left[ST\big|_{D_+} \quad S(T^*)^{-1}\big|_{D_-} \right] = S \operatorname{ran}\begin{bmatrix} \iota_{P_+} & \iota_{P_-} \end{bmatrix} = S\mathcal{X}_0 = \mathcal{X}_0. \qquad ❑$$

Lemma 4.4.5. *Let A_0 be a densely defined, closed, skew-symmetric operator on $\mathcal{X}_0$, $\mathcal{Y}_0$ be a Hilbert space, and let $T\colon \mathcal{X}_0 \to \mathcal{Y}_0$ be a bounded and boundedly invertible. Let $(\mathcal{X}_+, \mathcal{X}_0, \mathcal{X}_-)$ be a quasi Gelfand triple such that $((\mathcal{X}_+, \mathcal{X}_-), B_1, B_2)$ is a boundary triple for A_0^*. Furthermore, let $\mathcal{Y}_+$ and $\mathcal{Y}_-$ be as defined in Proposition 4.4.1. Then $(\mathcal{Y}_+, \mathcal{Y}_0, \mathcal{Y}_-)$ is also a quasi Gelfand triple such that $((\mathcal{Y}_+, \mathcal{Y}_-), T_+B_1, (T_+^*)^{-1}B_2)$ is a boundary triple for A_0^*.*

Proof. By Proposition 4.4.1 $(\mathcal{Y}_+, \mathcal{Y}_0, \mathcal{Y}_-)$ is a quasi Gelfand triple. For $x, y \in \operatorname{dom} A_0^*$ we have, by Corollary 4.4.3,

$$2\operatorname{Re}\langle A_0^* x, y\rangle = 2\operatorname{Re}\langle B_1 x, B_2 y\rangle_{\mathcal{X}_+,\mathcal{X}_-} = 2\operatorname{Re}\langle T_+ B_1 x, (T_+^*)^{-1} B_2 y\rangle_{\mathcal{Y}_+,\mathcal{Y}_-}.$$

Since $T_+\colon \mathcal{X}_+ \to \mathcal{Y}_+$ and $(T_+^*)^{-1}\colon \mathcal{X}_- \to \mathcal{Y}_-$ are surjective, the surjectivity of $\begin{bmatrix} T_+B_1 \\ (T_+^*)^{-1}B_2 \end{bmatrix} = \begin{bmatrix} T_+ & 0 \\ 0 & (T_+^*)^{-1} \end{bmatrix} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$ follows from the surjectivity of $\begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$. ❑

The following result is a generalization of [28, Theorem 2.6] for quasi Gelfand triple and also fixes some minor issues, like the closedness of $\left[\, V_1\big|_{\mathcal{B}_+\cap\mathcal{B}_0} \;\; V_2\big|_{\mathcal{B}_-\cap\mathcal{B}_0} \,\right]$ as an operator from $\mathcal{B}_+ \times \mathcal{B}_-$ to $\mathcal{K}$. This theorem is the main tool to justify existence and uniqueness of solutions for port-Hamiltonian systems.

Theorem 4.4.6. *Let $(\mathcal{B}_+, \mathcal{B}_0, \mathcal{B}_-)$ be a quasi Gelfand triple, A_0 be a closed skew-symmetric operator and $((\mathcal{B}_+, \mathcal{B}_-), B_1, B_2)$ be a boundary triple for A_0^*, where Ψ is the duality map of the quasi Gelfand triple. For $V_1, V_2 \in \mathcal{L}_\mathrm{b}(\mathcal{B}_0, \mathcal{K})$ we define*

$$D := \left\{ a \in \operatorname{dom} A_0^* \,\middle|\, B_1 a, B_2 a \in \mathcal{B}_0 \ \text{ and } \ \begin{bmatrix} V_1 & V_2 \end{bmatrix} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} a = 0 \right\}$$

and the operator $A := A_0^*\big|_D$. If

(i) $\begin{bmatrix} V_1\big|_{\mathcal{B}_0 \cap \mathcal{B}_+} & V_2\big|_{\mathcal{B}_0 \cap \mathcal{B}_-} \end{bmatrix}$ *is closed as an operator from $\mathcal{B}_+ \times \mathcal{B}_-$ to $\mathcal{K}$,*

(ii) $\ker \begin{bmatrix} V_1 & V_2 \end{bmatrix}$ *is dissipative as linear relation on $\mathcal{B}_0$,*

(iii) $V_1 V_2^* + V_2 V_1^* \geq 0$ *as operator on $\mathcal{K}$,*

then A is a generator of a contraction semigroup.

Proof. It is sufficient to show that A is closed, and A and A^* are dissipative.

Step 1. Showing that A is closed and dissipative. We have

$$\begin{aligned} a \in D \Leftrightarrow \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} a &\in (\mathcal{B}_0 \times \mathcal{B}_0) \cap \ker \begin{bmatrix} V_1 & V_2 \end{bmatrix} \\ \Leftrightarrow \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} a &\in \underbrace{\ker \begin{bmatrix} V_1\big|_{\mathcal{B}_0 \cap \mathcal{B}_+} & V_2\big|_{\mathcal{B}_0 \cap \mathcal{B}_-} \end{bmatrix}}_{=:\Theta}. \end{aligned}$$

We can write

$$\Theta = \left\{ \begin{bmatrix} p \\ q \end{bmatrix} \in \mathcal{B}_+ \times \mathcal{B}_- \,\middle|\, p \in \mathcal{B}_0, q \in \mathcal{B}_0 \text{ and } V_1 q + V_2 p = 0 \right\}.$$

Since $(\mathcal{B}_+, \mathcal{B}_0, \mathcal{B}_-)$ is a quasi Gelfand triple we have for $\left[\begin{smallmatrix} p \\ q \end{smallmatrix}\right] \in \Theta$

$$\operatorname{Re}\langle q, p \rangle_{\mathcal{B}_-, \mathcal{B}_+} = \operatorname{Re}\langle q, p \rangle_{\mathcal{B}_0} \leq 0,$$

which implies the dissipativity of A by Corollary 2.4.11. Assumption (i) implies that Θ is closed in $\mathcal{B}_+ \times \mathcal{B}_-$, which implies the closedness of A by Corollary 2.4.11.

Step 2. Showing that A^* is dissipative. By Corollary 2.4.11 we can characterize the domain of A^* by

$$\begin{aligned} d \in \operatorname{dom} A^* \Leftrightarrow \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} d &\in \begin{bmatrix} 0 & \mathrm{I} \\ \mathrm{I} & 0 \end{bmatrix} \Theta^{\perp_{\mathcal{B}_- \times \mathcal{B}_+}} \\ \Leftrightarrow \begin{bmatrix} B_2 \\ B_1 \end{bmatrix} d &\in \overline{\operatorname{ran} \begin{bmatrix} \left(V_1\big|_{\mathcal{B}_0 \cap \mathcal{B}_+}\right)^{*_{\mathcal{K} \times \mathcal{B}_-}} \\ \left(V_2\big|_{\mathcal{B}_0 \cap \mathcal{B}_-}\right)^{*_{\mathcal{K} \times \mathcal{B}_+}} \end{bmatrix}}^{\mathcal{B}_- \times \mathcal{B}_+}. \end{aligned}$$

The second equivalence needed the closedness in assumption (i), since $(\ker T)^\perp = \overline{\operatorname{ran} T^*}$ for a linear relation (or even unbounded operator) T is not true in general.

Hence, by Lemma 4.2.7

$$\left(V_1\big|_{\mathcal{B}_0\cap\mathcal{B}_+}\right)^{*_{\mathcal{K}\times\mathcal{B}_-}} = (V_1\iota_+)^* = \iota_+^* V_1^* = \iota_-^{-1} V_1^* = V_1^*\big|_{V_1^{*-1}(\mathcal{B}_0\cap\mathcal{B}_-)},$$

where $\iota_+ : \mathcal{B}_+ \cap \mathcal{B}_0 \subseteq \mathcal{B}_+ \to \mathcal{B}_0$ and $\iota_- : \mathcal{B}_- \cap \mathcal{B}_0 \subseteq \mathcal{B}_- \to \mathcal{B}_0$ are the embeddings of the quasi Gelfand triple. Analogously, we have

$$\left(V_2\big|_{\mathcal{B}_0\cap\mathcal{B}_-}\right)^{*_{\mathcal{K}\times\mathcal{B}_+}} = (V_2\iota_-)^* = \iota_-^* V_2^* = \iota_+^{-1} V_2^* = V_2\big|_{V_2^{*-1}(\mathcal{B}_0\cap\mathcal{B}_+)}.$$

Hence, for

$$\begin{bmatrix} x \\ y \end{bmatrix} \in \operatorname{ran} \begin{bmatrix} \left(V_1\big|_{\mathcal{B}_0\cap\mathcal{B}_+}\right)^{*_{\mathcal{K}\times\mathcal{B}_-}} \\ \left(V_2\big|_{\mathcal{B}_0\cap\mathcal{B}_-}\right)^{*_{\mathcal{K}\times\mathcal{B}_+}} \end{bmatrix}$$
$$= \left\{ \begin{bmatrix} V_1^* \\ V_2^* \end{bmatrix} k \,\middle|\, k \in V_1^{*-1}(\mathcal{B}_0 \cap \mathcal{B}_-) \cap V_2^{*-1}(\mathcal{B}_0 \cap \mathcal{B}_+) \right\},$$

we have

$$\begin{aligned} \operatorname{Re}\langle x, y\rangle_{\mathcal{B}_-,\mathcal{B}_+} &= \operatorname{Re}\langle V_1^* k, V_2^* k\rangle_{\mathcal{B}_-,\mathcal{B}_+} = \operatorname{Re}\langle V_1^* k, V_2^* k\rangle_{\mathcal{B}_0} \\ &= \operatorname{Re}\langle V_2 V_1^* k, k\rangle_{\mathcal{K}} \geq 0. \end{aligned}$$

Therefore, $\Theta^\perp$ is accretive and by Corollary 2.4.11 also $A_0\big|_{\operatorname{dom} A^*}$ is accretive, which yields $A^* = -A_0\big|_{\operatorname{dom} A^*}$ is dissipative. ❑

Remark 4.4.7. If we are already satisfied with the operator closure $\overline{A}$ is a generator (instead of A) in the previous theorem, then we can replace condition (i) by

$$\ker \overline{\begin{bmatrix} V_1\big|_{\mathcal{B}_0\cap\mathcal{B}_+} & V_2\big|_{\mathcal{B}_0\cap\mathcal{B}_-} \end{bmatrix}} \subseteq \overline{\ker \begin{bmatrix} V_1\big|_{\mathcal{B}_0\cap\mathcal{B}_+} & V_2\big|_{\mathcal{B}_0\cap\mathcal{B}_-} \end{bmatrix}}^{\mathcal{B}_+\times\mathcal{B}_-}, \tag{4.5}$$

where $\overline{\begin{bmatrix} V_1|_{\mathcal{B}_0\cap\mathcal{B}_+} & V_2|_{\mathcal{B}_0\cap\mathcal{B}_-} \end{bmatrix}}$ is the closure as linear relation (possibly multi-valued). Clearly, if (4.5) holds, then there is already equality.

Example 4.4.8. Let $(\mathcal{B}_+, \mathcal{B}_0, \mathcal{B}_-)$ be a quasi Gelfand triple that satisfies all conditions of Theorem 4.4.6 and let $M \in \mathcal{L}_{\mathrm{b}}(\mathcal{B}_0)$ be coercive (i.e. $M \geq c\mathrm{I}$, $c > 0$). Then $V_1 := \mathrm{I}$, $V_2 := M$ fulfill all conditions of Theorem 4.4.6:

(i) Setting $S = M^{\frac{1}{2}}$ and $T = M^{-\frac{1}{2}}$ in Corollary 4.4.4 implies the closedness of $\begin{bmatrix} \mathrm{I}\big|_{\mathcal{B}_0\cap\mathcal{B}_+} & M\big|_{\mathcal{B}_0\cap\mathcal{B}_-} \end{bmatrix}$.

(ii) For $\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] \in \ker \begin{bmatrix} V_1 & V_2 \end{bmatrix}$ we have $x = -My$. Since M is positive this yields

$$\operatorname{Re}\langle x, y\rangle_{\mathcal{B}_0} = \operatorname{Re}\langle -My, y\rangle_{\mathcal{B}_0} = -\langle My, y\rangle_{\mathcal{B}_0} \leq 0.$$

(iii) $V_1 V_2^* + V_2 V_1^* = M^* + M = 2\operatorname{Re} M \geq 0$.

Moreover, Corollary 4.4.4 also implies the surjectivity of $\left[\mathrm{I}\big|_{\mathcal{B}_0\cap\mathcal{B}_+} \; M\big|_{\mathcal{B}_0\cap\mathcal{B}_-}\right]$.

Clearly, also $V_1 := M$, $V_2 := \mathrm{I}$ fulfill all conditions.

Lemma 4.4.9. *Let the assumptions of Theorem 4.4.6 be true. Additionally, let V_1, V_2 be boundedly invertible and we replace condition* (iii) *by the stricter condition $V_1 V_2^*$ is coercive, i.e. $V_1 V_2^* \geq c\mathrm{I}$, for $c > 0$. The adjoint of A is then given by $A^* = -A_0^*$ restricted to*

$$\operatorname{dom} A^* = \{x \in \operatorname{dom} A_0 \mid B_1 x, B_2 x \in \mathcal{B}_0, (V_2^*)^{-1} B_1 x - (V_1^*)^{-1} B_2 x = 0\}.$$

Proof. In the proof of Theorem 4.4.6 we characterized the domain of A^* by

$$\begin{aligned} &a \in \operatorname{dom} A^* \\ &\Leftrightarrow \begin{bmatrix} B_1 a \\ B_2 a \end{bmatrix} \in \overline{\left\{ \begin{bmatrix} V_2^* k \\ V_1^* k \end{bmatrix} \,\middle|\, k \in {V_1^*}^{-1}(\mathcal{B}_- \cap \mathcal{B}_0) \cap {V_2^*}^{-1}(\mathcal{B}_+ \cap \mathcal{B}_0) \right\}}^{\mathcal{B}_+ \times \mathcal{B}_-}. \end{aligned}$$

First we show that the set on the right-hand-side is already closed: Let $(k_n)_{n\in\mathbb{N}}$ be a sequence in ${V_1^*}^{-1}(\mathcal{B}_+ \cap \mathcal{B}_0) \cap {V_2^*}^{-1}(\mathcal{B}_- \cap \mathcal{B}_0)$ such that

$$\begin{bmatrix} V_2^* k_n \\ V_1^* k_n \end{bmatrix} \to \begin{bmatrix} f \\ g \end{bmatrix} \quad \text{in} \quad \mathcal{B}_+ \times \mathcal{B}_-.$$

Then we have

$$\begin{aligned} \|k_n\|_{\mathcal{K}}^2 \leq \frac{1}{c^2} \langle k_n, V_1 V_2^* k_n \rangle_{\mathcal{K}} &= \langle V_1^* k_n, V_2^* k_n \rangle_{\mathcal{B}_0} \\ &= \langle V_1^* k_n, V_2^* k_n \rangle_{\mathcal{B}_-,\mathcal{B}_+} \to \langle g, f \rangle_{\mathcal{B}_-,\mathcal{B}_+}, \end{aligned}$$

which implies that $(k_n)_{n\in\mathbb{N}}$ is bounded in $\mathcal{K}$. Therefore, there exists a weakly convergent subsequence with limit $k \in \mathcal{K}$ and by Lemma A.3.3 there is even a further subsequence such that k is the strong limit of $\tilde{k}_N := \frac{1}{N}\sum_{j=1}^N k_{n(j)}$. Hence, $V_1^* \tilde{k}_n \to V_1^* k$ and $V_2^* \tilde{k}_n \to V_2^* k$ w.r.t. $\|\cdot\|_{\mathcal{B}_0}$. Clearly, the limit of $V_1^* \tilde{k}_n$ in $\mathcal{B}_-$ is still g and the same goes for $V_2^* \tilde{k}_n$. By the closedness of the embeddings ι_+ and ι_- of a quasi Gelfand triple, we conclude that $g = V_1^* k$ and $f = V_2^* k$ and

$$a \in \operatorname{dom} A^* \Leftrightarrow \begin{bmatrix} B_1 a \\ B_2 a \end{bmatrix} \in \left\{ \begin{bmatrix} V_2^* k \\ V_1^* k \end{bmatrix} \,\middle|\, k \in {V_1^*}^{-1}(\mathcal{B}_- \cap \mathcal{B}_0) \cap {V_2^*}^{-1}(\mathcal{B}_+ \cap \mathcal{B}_0) \right\}.$$

Hence, $a \in \operatorname{dom} A^*$ is equivalent to there exists a $k \in {V_1^*}^{-1}(\mathcal{B}_- \cap \mathcal{B}_0) \cap {V_2^*}^{-1}(\mathcal{B}_+ \cap \mathcal{B}_0)$ such that

$$(V_2^*)^{-1} B_1 a = k \quad \text{and} \quad (V_1^*)^{-1} B_2 a = k,$$

which is equivalent to

$$(V_2^*)^{-1} B_1 a - (V_1^*)^{-1} B_2 a = 0. \qquad \square$$

Chapter 5

Boundary Spaces

In this chapter we come back to the port-Hamiltonian PDE and combine the previous theory to justify well-posedness of the port-Hamiltonian PDE. We had boundary operators that gave us almost a boundary triple in (3.5). We will construct suitable boundary spaces to extend those operators such that we obtain a boundary triple. Hence, we can apply boundary triple theory to characterize boundary conditions such that the systems has for every initial condition a unique solution that does not grow in the Hamiltonian. Furthermore, we will see that our boundary spaces establish a quasi Gelfand triple with a subspace of $\mathsf{L}^2(\partial\Omega)$ as pivot space. Hence, we can also apply Theorem 4.4.6 to obtain suitable boundary conditions. This enables us to formulate the boundary conditions in the pivot space.

5.1 Boundary Spaces for L_∂

In this section we will construct a suitable boundary space $\mathcal{V}_L$ (Definition 5.1.6), such that we can extend the integration by parts formula for L_∂ (Lemma 3.1.8). We will formulate the boundary conditions in this space in Section 5.3. This space will provide a quasi Gelfand triple with a subspace of $\mathsf{L}^2(\partial\Omega)$ as pivot space. In order to impose different boundary conditions on different parts of the boundary we introduce boundary operators that only act on a part of the boundary and their boundary spaces $\mathcal{V}_{L,\Gamma_1}$.

Definition 5.1.1. We say $(\Gamma_j)_{j=1}^k$, where $\Gamma_j \subseteq \partial\Omega$, is a *splitting with thin boundaries* of $\partial\Omega$, if

(i) $\bigcup_{j=1}^k \overline{\Gamma_j} = \partial\Omega$,

(ii) the sets Γ_j are pairwise disjoint,

(iii) the sets Γ_j are relatively open in $\partial\Omega$,

(iv) the boundaries of Γ_j have zero measure w.r.t. the surface measure of $\partial\Omega$.

For $\Gamma \subseteq \partial\Omega$ we will denote by P_Γ the orthogonal projection from $\mathsf{L}^2(\partial\Omega)^{m_1}$ on $\mathsf{L}^2_\pi(\Gamma) := \overline{\operatorname{ran} \mathbb{1}_\Gamma L_\nu} \subseteq \mathsf{L}^2(\Gamma)^{m_1}$, where $\mathbb{1}_M$ denotes the indicator function for a set M. We endow $\mathsf{L}^2_\pi(\Gamma)$ with the inner product of $\mathsf{L}^2(\partial\Omega)^{m_1}$. Therefore, we can adapt (3.1) to obtain

$$\langle L_\partial f, g\rangle_{\mathsf{L}^2(\Omega)^{m_1}} + \langle f, L_\partial^{\mathsf{H}} g\rangle_{\mathsf{L}^2(\Omega)^{m_2}} = \langle L_\nu \gamma_0 f, \underbrace{P_{\partial\Omega}\gamma_0 g}_{\pi_L g}\rangle_{\mathsf{L}^2(\partial\Omega)^{m_1}}. \tag{5.1}$$

We define $\pi_L^\Gamma \colon \mathsf{H}^1(\Omega)^{m_1} \to \mathsf{L}^2_\pi(\Gamma)$ by $\pi_L^\Gamma := P_\Gamma \gamma_0$ and $\pi_L := \pi_L^{\partial\Omega}$. Since both P_Γ and γ_0 are continuous, the mapping π_L^Γ is also continuous. Therefore, $\ker \pi_L^\Gamma$ is closed. Note that $P_\Gamma = \mathbb{1}_\Gamma P_{\partial\Omega}$ and consequently $\pi_L^\Gamma = \mathbb{1}_\Gamma \pi_L$, and $\mathbb{1}_\Gamma L_\nu = L_\nu \mathbb{1}_\Gamma$.

Example 5.1.2. Let L be as in Example 3.1.3 ($L_\partial = \operatorname{div}$). Then $L_\nu f = \nu \cdot f$ and L_ν is certainly surjective. Therefore, $\mathsf{L}^2_\pi(\partial\Omega) = \mathsf{L}^2(\partial\Omega)$, $\pi_L = \gamma_0$ and $\pi_L^\Gamma = \mathbb{1}_\Gamma\gamma_0$. Since $L_\partial^{\mathsf{H}} = \operatorname{grad}$, we have $\mathsf{H}(L_\partial^{\mathsf{H}}, \Omega) = \mathsf{H}^1(\Omega)$.

Example 5.1.3. Let L be as in Example 3.1.4 ($L_\partial = \operatorname{rot}$). Then $L_\nu f = \nu \times f$. Note that for every $w \in \mathbb{K}^3$ and every $u \in \mathbb{R}^3$ with $\|u\| = 1$ we have $w = (u \times w) \times u + (u \cdot w) \cdot u$. It is not hard to conclude $P_{\partial\Omega} f = (\nu \times f) \times \nu$. Hence, $\pi_L = (\nu \times \gamma_0) \times \nu$.

Lemma 5.1.4. *Let $\Gamma \subseteq \partial\Omega$ be relatively open and let the boundary of Γ have zero measure (w.r.t. the surface measure of $\partial\Omega$). Then $\ker \pi_L^\Gamma$ is closed as subspace of $\mathsf{H}^1(\Omega)^{m_1}$ endowed with the trace topology of $\|\cdot\|_{\mathsf{H}(L_\partial^{\mathsf{H}},\Omega)}$, i.e.*

$$\overline{\ker \pi_L^\Gamma}^{\|\cdot\|_{\mathsf{H}(L_\partial^{\mathsf{H}},\Omega)}} \cap \mathsf{H}^1(\Omega)^{m_1} = \ker \pi_L^\Gamma.$$

Proof. Clearly, $\overline{\ker \pi_L^\Gamma}^{\|\cdot\|_{\mathsf{H}(L_\partial^{\mathsf{H}},\Omega)}} \cap \mathsf{H}^1(\Omega)^{m_1} \supseteq \ker \pi_L^\Gamma$. So we will show the other inclusion. Note that for $\Upsilon \subseteq \partial\Omega$ we have

$$\mathsf{H}^1_\Upsilon(\Omega)^{m_2} := \left\{ f \in \mathsf{H}^1(\Omega)^{m_2} \,\middle|\, \mathbb{1}_\Upsilon \gamma_0 f = 0 \in \mathsf{L}^2(\partial\Omega)^{m_2} \right\}.$$

Hence, $\mathsf{H}^1_{\partial\Omega\setminus\Gamma}(\Omega)^{m_2} = \mathsf{H}^1_{\partial\Omega\setminus\overline{\Gamma}}(\Omega)^{m_2}$, since the boundary of Γ has zero measure. Let $(g_n)_{n\in\mathbb{N}}$ be a sequence in $\ker \pi_L^\Gamma$ which converges to $g \in \mathsf{H}^1(\Omega)^{m_1}$ with respect to $\|\cdot\|_{\mathsf{H}(L_\partial^{\mathsf{H}},\Omega)}$. By Corollary 3.1.9 we have for an arbitrary $f \in \mathsf{H}^1_{\partial\Omega\setminus\Gamma}(\Omega)^{m_2}$

$$|\langle L_\nu\gamma_0 f, \pi_L^\Gamma(g - g_n)\rangle_{\mathsf{L}^2}| = |\langle L_\nu\gamma_0 f, \pi_L(g - g_n)\rangle_{\mathsf{L}^2}| \leq \|f\|_{\mathsf{H}(L_\partial,\Omega)} \|g - g_n\|_{\mathsf{H}(L_\partial^{\mathsf{H}},\Omega)}.$$

Since $\pi_L^\Gamma(g - g_n) = \pi_L^\Gamma g$ and the right-hand-side converges to 0, we can see that $\pi_L^\Gamma g \perp L_\nu\gamma_0 \mathsf{H}^1_{\partial\Omega\setminus\Gamma}(\Omega)^{m_2}$. By [57, Th. 13.6.10, Re. 13.6.12] $\gamma_0\mathsf{H}^1_{\partial\Omega\setminus\Gamma}(\Omega)^{m_2}$ is dense in $\mathsf{L}^2(\Gamma)^{m_2}$, which implies $\pi_L^\Gamma g \perp \operatorname{ran} \mathbb{1}_\Gamma L_\nu$. By definition $\pi_L^\Gamma g$ is also in $\overline{\operatorname{ran} \mathbb{1}_\Gamma L_\nu}$, which leads to $\pi_L^\Gamma g = 0$. Hence, $\ker \pi_L^\Gamma$ is closed in $\mathsf{H}^1(\Omega)^{m_1}$ with respect to $\|\cdot\|_{\mathsf{H}(L_\partial^{\mathsf{H}},\Omega)}$. ❑

By the previous lemma

$$\|\phi\|_{M_\Gamma} := \inf\left\{ \|g\|_{\mathsf{H}(L_\partial^{\mathsf{H}},\Omega)} \,\middle|\, \pi_L^\Gamma g = \phi \right\}$$

is a norm on $M_\Gamma := \operatorname{ran} \pi_L^\Gamma$. The next lemma will show that this norm is induced by an inner product.

Lemma 5.1.5. *Let $\Gamma \subseteq \partial\Omega$ be relatively open and let the boundary of Γ have zero measure (w.r.t. the surface measure of $\partial\Omega$). Then the space $(M_\Gamma, \|\cdot\|_{M_\Gamma})$ is a pre-Hilbert space. Furthermore, its completion denoted by $(\overline{M_\Gamma}, \|\cdot\|_{\overline{M_\Gamma}})$ is isomorphic to the Hilbert space $\mathsf{H}(L_\partial^{\mathsf{H}}, \Omega) \big/ \overline{\ker \pi_L^\Gamma}^{\mathsf{H}(L_\partial^{\mathsf{H}},\Omega)}$.*

The mapping $\pi_L^\Gamma \colon H^1(\Omega)^{m_1} \to M_\Gamma$ can be continuously extended to a surjective contraction $\bar{\pi}_L^\Gamma \colon \mathsf{H}(L_\partial^{\mathsf{H}}, \Omega) \to \overline{M_\Gamma}$. The kernel of $\bar{\pi}_L^\Gamma$ satisfies $\ker \bar{\pi}_L^\Gamma = \overline{\ker \pi_L^\Gamma}^{\mathsf{H}(L_\partial^{\mathsf{H}},\Omega)}$.

Instead of $\bar{\pi}_L^{\partial\Omega}$ we will just write $\bar{\pi}_L$.

Proof. By Lemma 5.1.4 $\ker \pi_L^\Gamma$ is closed in $\mathsf{H}^1(\Omega)^{m_1}$ with respect to trace topology of $\|\cdot\|_{\mathsf{H}(L_\partial^{\mathsf{H}},\Omega)}$, which implies that $\left(\mathsf{H}^1(\Omega)^{m_1} \big/ \ker \pi_L^\Gamma, \|\cdot\|_{\mathsf{H}(L_\partial^{\mathsf{H}},\Omega) / \ker \pi_L^\Gamma}\right)$ is a normed space (normed space factorized by a closed subspace is again a normed space). Since

$$\left\|[g]_\sim\right\|_{\mathsf{H}(L_\partial^{\mathsf{H}},\Omega) / \ker \pi_L^\Gamma} = \left\|\pi_L^\Gamma g\right\|_{M_\Gamma},$$

it is straight forward that $[g]_\sim \mapsto \pi_L^\Gamma g$ is an isometry from $\Big(\mathsf{H}^1(\Omega)^{m_1} \big/ \ker \pi_L^\Gamma,$ $\|\cdot\|_{\mathsf{H}(L_\partial^{\mathsf{H}},\Omega) / \ker \pi_L^\Gamma}\Big)$ onto $(M_\Gamma, \|\cdot\|_{M_\Gamma})$.

Clearly, $(M_\Gamma, \|\cdot\|_{M_\Gamma})$ has a completion $(\overline{M_\Gamma}, \|\cdot\|_{\overline{M_\Gamma}})$. By definition of the norm $\|\cdot\|_{M_\Gamma}$ we have for every $g \in \mathsf{H}^1(\Omega)^{m_1}$

$$\|\pi_L^\Gamma g\|_{\overline{M_\Gamma}} = \|\pi_L^\Gamma g\|_{M_\Gamma} \leq \|g\|_{\mathsf{H}(L_\partial^{\mathsf{H}},\Omega)}.$$

Therefore, we can extend π_L^Γ by continuity on $\mathsf{H}(L_\partial^{\mathsf{H}}, \Omega)$. This extension is denoted by $\bar{\pi}_L^\Gamma$ and is a contraction by the previous equation.

Let $g \in \mathsf{H}(L_\partial^{\mathsf{H}}, \Omega)$. Then by Theorem 3.1.18 there exists a sequence $(g_n)_{n\in\mathbb{N}}$ in $\mathsf{H}^1(\Omega)^{m_1}$, which converges to g. Therefore, we have

$$\|\bar{\pi}_L^\Gamma g\|_{\overline{M_\Gamma}} = \lim_{n\to\infty} \|\pi_L^\Gamma g_n\|_{M_\Gamma} = \lim_{n\to\infty} \inf_{k \in \ker \pi_L^\Gamma} \|g_n + k\|_{\mathsf{H}(L_\partial^{\mathsf{H}},\Omega)}.$$

The triangular inequality yields

$$\inf_{k \in \ker \pi_L^\Gamma} \|g + k\| - \|g_n - g\| \leq \inf_{k \in \ker \pi_L^\Gamma} \|g_n + k\| \leq \inf_{k \in \ker \pi_L^\Gamma} \|g + k\| + \|g_n - g\|.$$

Hence, we have

$$\|\bar{\pi}_L^\Gamma g\|_{\overline{M_\Gamma}} = \inf_{k \in \ker \pi_L^\Gamma} \|g + k\|_{\mathsf{H}(L_\partial^{\mathsf{H}},\Omega)} = \inf_{k \in \overline{\ker \pi_L^\Gamma}} \|g + k\|_{\mathsf{H}(L_\partial^{\mathsf{H}},\Omega)} \tag{5.2}$$

and consequently $\mathsf{H}(L_\partial^{\mathsf{H}},\Omega)\big/\overline{\ker\pi_L^\Gamma}$ is isomorphic to $\operatorname{ran}\bar\pi_L^\Gamma$. Since the quotient space $\mathsf{H}(L_\partial^{\mathsf{H}},\Omega)\big/\overline{\ker\pi_L^\Gamma}$ is a Hilbert space, in particular complete, and $M_\Gamma \subseteq \operatorname{ran}\bar\pi_L^\Gamma \subseteq \overline{M_\Gamma}$, we have $\overline{M_\Gamma} = \operatorname{ran}\bar\pi_L^\Gamma$. This makes $\overline{M_\Gamma}$ also a Hilbert space and M_Γ a pre-Hilbert space.

Finally, equation (5.2) implies $\ker\bar\pi_L^\Gamma = \overline{\ker\pi_L^\Gamma}$. ❑

Now we are able to define a complete subspace of $\mathsf{H}(L_\partial^{\mathsf{H}},\Omega)$ that is in some sense 0 at one part of the boundary and the corresponding boundary space for the other part of the boundary.

Definition 5.1.6. Let $\Gamma_0,\Gamma_1 \subseteq \partial\Omega$ be a splitting with thin boundaries and $\bar\pi_L$ the extension of π_L introduced in Lemma 5.1.5. Then we define

$$\mathsf{H}_{\Gamma_0}(L_\partial^{\mathsf{H}},\Omega) := \ker\bar\pi_L^{\Gamma_0} \quad\text{and}\quad \mathcal{V}_{L,\Gamma_1} := \operatorname{ran}\bar\pi_L\big|_{\mathsf{H}_{\Gamma_0}(L_\partial^{\mathsf{H}},\Omega)},$$

where we endow $\mathsf{H}_{\Gamma_0}(L_\partial^{\mathsf{H}},\Omega)$ with $\|\cdot\|_{\mathsf{H}(L_\partial^{\mathsf{H}},\Omega)}$ and $\mathcal{V}_{L,\Gamma_1}$ with $\|\cdot\|_{\mathcal{V}_{L,\Gamma_1}} := \|\cdot\|_{\overline{M_{\partial\Omega}}}$. Instead of $\mathcal{V}_{L,\partial\Omega} = \operatorname{ran}\bar\pi_L = \overline{M_{\partial\Omega}}$ we just write $\mathcal{V}_L$.

From now on until the end of this section we will assume that $\Gamma_0,\Gamma_1 \subseteq \partial\Omega$ is a splitting with thin boundaries. By Lemma 5.1.5 $\mathcal{V}_L$ is a Hilbert space.

Note that $\mathcal{V}_{L,\Gamma_1}$ and $\overline{M_{\Gamma_1}}$ are not necessarily the same space. Although, we have $\bar\pi_L^{\Gamma_1} g = \bar\pi_L g$ (in $\mathsf{L}^2(\partial\Omega)^{m_1}$) for $g \in \mathsf{H}^1(\Omega)^{m_1} \cap \mathsf{H}_{\Gamma_0}(L_\partial^{\mathsf{H}},\Omega)$, but we can only say $\|\bar\pi_L^{\Gamma_1} g\|_{\overline{M_{\Gamma_1}}} \le \|\bar\pi_L g\|_{\mathcal{V}_{L,\Gamma_1}}$.

Example 5.1.7. Continuing Example 5.1.2 yields $\mathsf{H}_{\Gamma_0}(L_\partial^{\mathsf{H}},\Omega) = \mathsf{H}^1_{\Gamma_0}(\Omega)^{m_1} = \{f \in \mathsf{H}^1(\Omega)^{m_1} \mid \mathbb{1}_{\Gamma_1}\gamma_0 f = 0\}$ which already appeared in the proof of Lemma 5.1.4. Moreover, we have $\bar\pi_L = \gamma_0$, $\bar\pi_L^{\Gamma_1} = \mathbb{1}_{\Gamma_1}\gamma_0$, $\mathcal{V}_L = \mathsf{H}^{1/2}(\partial\Omega)$, and $\mathcal{V}_{L,\Gamma_1} = \{f \in \mathsf{H}^{1/2}(\partial\Omega) \mid f\big|_{\Gamma_0} = 0\}$.

The next example shows that for L from Example 5.1.3 ($L_\partial = \operatorname{rot}$) neither of the “natural” boundary operators π_L and L_ν can be continuously extended to $\mathsf{H}(\operatorname{rot},\Omega)$ ($= \mathsf{H}(L_\partial,\Omega)$) such that the codomain is contained in $\mathsf{L}^2(\partial\Omega)$. Note that $L_\nu\phi = \nu\times\phi$ for $\phi \in \mathsf{L}^2(\partial\Omega)$ and $\pi_L f = (\nu\times\gamma_0 f)\times\nu$ for $f \in \mathsf{H}^1(\Omega)$.

Example 5.1.8. Let $\Omega = (0,1)^3$ and $F\colon \Omega \to \mathbb{R}$ be defined by

$$F(x) = \frac{1}{\|x\|_2^{2/5}} = (x_1^2 + x_2^2 + x_3^2)^{-2/10}.$$

Then we define $f = \operatorname{grad} F$, which is

$$f(x) = \begin{bmatrix} -\frac{4}{10}x_1(x_1^2+x_2^2+x_3^2)^{-6/5} \\ -\frac{4}{10}x_2(x_1^2+x_2^2+x_3^2)^{-6/5} \\ -\frac{4}{10}x_3(x_1^2+x_2^2+x_3^2)^{-6/5} \end{bmatrix}.$$

Hence, $\operatorname{rot} f = \operatorname{rot}\operatorname{grad} F = 0$. We will show that f is in $\mathsf{L}^2(\Omega)^3$:

$$\begin{aligned}
\int_\Omega \|f(x)\|_2^2 \,\mathrm{d}x &= \int_\Omega \sum_{i=1}^{3} \frac{16}{100} x_i^2 (x_1^2 + x_2^2 + x_3^2)^{-12/5} \,\mathrm{d}x \\
&= \frac{16}{100} \int_\Omega (x_1^2 + x_2^2 + x_3^2)^{-7/5} \,\mathrm{d}x \\
&\leq \int_{B_{\sqrt{3}}(0)} (x_1^2 + x_2^2 + x_3^2)^{-7/5} \,\mathrm{d}x = 2\pi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{\sqrt{3}} r^{-14/5} r^2 \cos\theta \,\mathrm{d}r \,\mathrm{d}\theta \\
&= 4\pi \int_0^{\sqrt{3}} r^{-4/5} \,\mathrm{d}r = 4\pi 5 r^{1/5} \Big|_0^{\sqrt{3}} < +\infty.
\end{aligned}$$

Therefore, f is even in $\mathsf{H}(\operatorname{rot}, \Omega)$. Let ν denote the normal vector on $\partial\Omega$. Then we show that $\nu \times f\big|_{\partial\Omega}$ is not in $\mathsf{L}^2(\partial\Omega)^3$: Note that $\nu(\zeta) = \left[\begin{smallmatrix} 0 \\ 0 \\ -1 \end{smallmatrix}\right]$ on $[0,1] \times [0,1] \times \{0\}$. Therefore,

$$\nu(\zeta) \times f(\zeta) = \begin{bmatrix} -\frac{4}{10}\zeta_2(\zeta_1^2 + \zeta_2^2)^{-6/5} \\ \frac{4}{10}\zeta_1(\zeta_1^2 + \zeta_2^2)^{-6/5} \\ 0 \end{bmatrix} \quad \text{for} \quad \zeta \in [0,1] \times [0,1] \times \{0\}$$

and consequently

$$\begin{aligned}
\int_{\partial\Omega} \|\nu(\zeta) \times f(\zeta)\|_2^2 \,\mathrm{d}\zeta &\geq \int_{[0,1]\times[0,1]\times\{0\}} \|\nu(\zeta) \times f(\zeta)\|_2^2 \,\mathrm{d}\zeta \\
&= \frac{16}{100} \int_{[0,1]\times[0,1]} (\xi_1^2 + \xi_2^2)^{-7/5} \,\mathrm{d}\xi.
\end{aligned}$$

Since $[0,1] \times [0,1]$ contains the circular sector with arc $\frac{\pi}{2}$ and radius 1, we further have (by applying polar coordinates)

$$\begin{aligned}
&\geq \frac{16}{100} \frac{\pi}{2} \int_0^1 r^{-14/5} r \,\mathrm{d}r = \frac{16}{100} \frac{\pi}{2} \int_0^1 r^{-9/5} \,\mathrm{d}r \\
&= -\frac{16}{100} \frac{\pi}{2} \frac{5}{4} r^{-4/5} \Big|_0^1 = +\infty.
\end{aligned}$$

Hence, $f \in \mathsf{H}(\operatorname{rot}, \Omega)$, but $\nu \times f\big|_{\partial\Omega} \notin \mathsf{L}^2(\partial\Omega)^3$. Since

$$(\nu(\zeta) \times f(\zeta)) \times \nu(\zeta) = \begin{bmatrix} -\frac{4}{10}\zeta_1(\zeta_1^2 + \zeta_2^2)^{-6/5} \\ -\frac{4}{10}\zeta_2(\zeta_1^2 + \zeta_2^2)^{-6/5} \\ 0 \end{bmatrix} \quad \text{for} \quad \zeta \in [0,1] \times [0,1] \times \{0\},$$

we also have $\big(\nu \times f\big|_{\partial\Omega}\big) \times \nu \notin \mathsf{L}^2(\partial\Omega)^3$.

Lemma 5.1.9. *The space $\mathsf{H}_{\Gamma_0}(L_\partial^{\mathsf{H}}, \Omega)$ equipped with $\langle \cdot, \cdot \rangle_{\mathsf{H}(L_\partial^{\mathsf{H}}, \Omega)}$ is a Hilbert space and $\mathsf{H}^1(\Omega)^{m_1} \cap \mathsf{H}_{\Gamma_0}(L_\partial^{\mathsf{H}}, \Omega)$ is dense in $\mathsf{H}_{\Gamma_0}(L_\partial^{\mathsf{H}}, \Omega)$. Moreover, $\mathcal{V}_{L,\Gamma_1}$ is a closed subspace of $\mathcal{V}_L$ and therefore also a Hilbert space.*

Proof. By definition of $\mathsf{H}_{\Gamma_0}(L_\partial^{\mathsf{H}},\Omega)$ and Lemma 5.1.5 we have

$$\mathsf{H}_{\Gamma_0}(L_\partial^{\mathsf{H}},\Omega) = \ker\bar{\pi}_L^{\Gamma_0} = \overline{\ker\pi_L^{\Gamma_0}} = \overline{\mathsf{H}^1(\Omega)^{m_1}\cap\mathsf{H}_{\Gamma_0}(L_\partial^{\mathsf{H}},\Omega)}.$$

Note that $\ker\pi_L \subseteq \ker\pi_L^{\Gamma_0}$, since $\pi_L^{\Gamma_0} = \mathbb{1}_{\Gamma_0}\pi_L$. Again by Lemma 5.1.5, we have

$$\ker\bar{\pi}_L = \overline{\ker\pi_L} \subseteq \overline{\ker\pi_L^{\Gamma_0}} = \ker\bar{\pi}_L^{\Gamma_0}.$$

Therefore, $\bar{\pi}_L^{\Gamma_0}\circ\bar{\pi}_L^{-1} : \mathcal{V}_L \to \overline{M_{\Gamma_0}}$ is single-valued (well-defined). For arbitrary $\phi\in\mathcal{V}_L$ and $g\in\bar{\pi}_L^{-1}\phi$ we have

$$\|\bar{\pi}_L^{\Gamma_0}\circ\bar{\pi}_L^{-1}\phi\|_{\overline{M_{\Gamma_0}}} = \inf_{k\in\ker\bar{\pi}_L^{\Gamma_0}}\|g+k\|_{\mathsf{H}(L_\partial^{\mathsf{H}},\Omega)} \le \inf_{k\in\ker\bar{\pi}_L}\|g+k\|_{\mathsf{H}(L_\partial^{\mathsf{H}},\Omega)} = \|\phi\|_{\mathcal{V}_L}.$$

Hence, $\bar{\pi}_L^{\Gamma_0}\circ\bar{\pi}_L^{-1}$ is continuous and $\ker\bar{\pi}_L^{\Gamma_0}\circ\bar{\pi}_L^{-1}$ is closed in $\mathcal{V}_L$ and therefore also a Hilbert space endowed with $\langle\cdot,\cdot\rangle_{\mathcal{V}_L}$. The equivalences

$$\phi\in\ker\bar{\pi}_L^{\Gamma_0}\circ\bar{\pi}_L^{-1} \quad\Leftrightarrow\quad \bar{\pi}_L^{-1}\phi\subseteq\ker\bar{\pi}_L^{\Gamma_0} \quad\Leftrightarrow\quad \phi\in\underbrace{\operatorname{ran}\bar{\pi}_L\big|_{\ker\bar{\pi}_L^{\Gamma_0}}}_{=\mathcal{V}_{L,\Gamma_1}}$$

imply that $\mathcal{V}_{L,\Gamma_1}$ is closed and therefore a Hilbert space. ❑

Proposition 5.1.10. *The mapping* $\mathbb{1}_{\Gamma_1}L_\nu\gamma_0\colon \mathsf{H}^1(\Omega)^{m_2}\to\mathsf{L}^2_\pi(\Gamma_1)$ *can be extended to a linear continuous mapping*

$$\bar{L}_\nu^{\Gamma_1}\colon \mathsf{H}(L_\partial,\Omega)\to\mathcal{V}'_{L,\Gamma_1},$$

such that $\|\bar{L}_\nu^{\Gamma_1}f\|_{\mathcal{V}'_{L,\Gamma_1}} \le \|f\|_{\mathsf{H}(L_\partial,\Omega)}$.

Proof. Let $f\in\mathsf{H}^1(\Omega)^{m_2}$. For $g\in\mathsf{H}^1(\Omega)^{m_1}\cap\mathsf{H}_{\Gamma_0}(L_\partial^{\mathsf{H}},\Omega)$ we have by Corollary 3.1.9

$$\left|\langle\mathbb{1}_{\Gamma_1}L_\nu\gamma_0 f,\bar{\pi}_L g\rangle_{\mathsf{L}^2(\Gamma_1)^{m_1}}\right| = \left|\langle L_\nu\gamma_0 f,\bar{\pi}_L g\rangle_{\mathsf{L}^2(\partial\Omega)^{m_1}}\right| \le \|f\|_{\mathsf{H}(L_\partial,\Omega)}\|g\|_{\mathsf{H}(L_\partial^{\mathsf{H}},\Omega)}.$$

By Lemma 5.1.9 the subspace $M := \operatorname{ran}\bar{\pi}_L\big|_{\mathsf{H}^1(\Omega)^{m_1}\cap\mathsf{H}_{\Gamma_0}(L_\partial^{\mathsf{H}},\Omega)} \subseteq \mathsf{L}^2_\pi(\Gamma_1)^{m_1}$ of $\mathcal{V}_{L,\Gamma_1}$ is dense in $\mathcal{V}_{L,\Gamma_1}$. For $\phi\in M$ there exists at least one $g\in\mathsf{H}^1(\Omega)^{m_1}\cap\mathsf{H}_{\Gamma_0}(L_\partial^{\mathsf{H}},\Omega)$ such that $\pi_L g=\phi$. Hence, we can rewrite the inequality as

$$\begin{aligned}\left|\langle\mathbb{1}_{\Gamma_1}L_\nu\gamma_0 f,\phi\rangle_{\mathsf{L}^2(\Gamma_1)^{m_1}}\right| &\le \|f\|_{\mathsf{H}(L_\partial,\Omega)} \inf_{\substack{g\in\mathsf{H}^1(\Omega)^{m_1}\cap\mathsf{H}_{\Gamma_0}(L_\partial^{\mathsf{H}},\Omega)\\ \bar{\pi}_L g=\phi}}\|g\|_{\mathsf{H}(L_\partial^{\mathsf{H}},\Omega)}\\ &= \|f\|_{\mathsf{H}(L_\partial,\Omega)}\|\phi\|_{\mathcal{V}_{L,\Gamma_1}}.\end{aligned}$$

We extend the mapping $\phi\mapsto\langle\mathbb{1}_{\Gamma_1}L_\nu\gamma_0 f,\phi\rangle_{\mathsf{L}^2(\Gamma_1)^{m_1}}$ by continuity on $\mathcal{V}_{L,\Gamma_1}$ and denote this extension by Ξ_f. Therefore, we have

$$|\Xi_f(\phi)| \le \|f\|_{\mathsf{H}(L_\partial,\Omega)}\|\phi\|_{\mathcal{V}_{L,\Gamma_1}}.$$

This means that the mapping $f\mapsto\Xi_f$ from $\mathsf{H}^1(\Omega)^{m_2}$ to $\mathcal{V}'_{L,\Gamma_1}$ is continuous, if we endow $\mathsf{H}^1(\Omega)^{m_2}$ with $\|\cdot\|_{\mathsf{H}(L_\partial,\Omega)}$. Once again, we will extend this mapping by continuity on $\mathsf{H}(L_\partial,\Omega)$ and denote it by $\bar{L}_\nu^{\Gamma_1}$. ❑

Instead of writing $\bar{L}_\nu^{\partial\Omega}$ we will just write $\bar{L}_\nu$.

Remark 5.1.11. Since $\mathcal{V}_{L,\Gamma_1}$ is a subspace of $\mathcal{V}_{L,\partial\Omega} = \mathcal{V}_L$ every element of $\mathcal{V}_L'$ can also be treated as an element of $\mathcal{V}_{L,\Gamma_1}'$. By definition of $\bar{L}_\nu^{\Gamma_1}$ and $\bar{L}_\nu$ it is easy to see that $\bar{L}_\nu^{\Gamma_1} f = \bar{L}_\nu f\big|_{\mathcal{V}_{L,\Gamma_1}}$ or equivalently $\bar{L}_\nu^{\Gamma_1} f$ and $\bar{L}_\nu f$ coincide as elements of $\mathcal{V}_{L,\Gamma_1}'$ for $f \in \mathsf{H}(L_\partial, \Omega)$. Hence, we can say $\mathcal{V}_L'\big|_{\mathcal{V}_{L,\Gamma_1}} \subseteq \mathcal{V}_{L,\Gamma_1}'$. Since Hahn-Banach gives the reverse inclusion we can even say $\mathcal{V}_L'\big|_{\mathcal{V}_{L,\Gamma_1}} = \mathcal{V}_{L,\Gamma_1}'$.

The reason for even defining $\bar{L}_\nu^{\Gamma_1}$ instead of just using $\bar{L}_\nu$ is that the range of its restriction to $\mathsf{H}^1(\Omega)^{m_2}$ is also contained in $\mathsf{L}^2_\pi(\Gamma_1)$, which will be important for getting a quasi Gelfand triple.

Corollary 5.1.12. *For $f \in \mathsf{H}(L_\partial, \Omega)$ and $g \in \mathsf{H}_{\Gamma_0}(L_\partial^{\mathsf{H}}, \Omega)$ we have*

$$\langle L_\partial f, g\rangle_{\mathsf{L}^2(\Omega)^{m_1}} + \langle f, L_\partial^{\mathsf{H}} g\rangle_{\mathsf{L}^2(\Omega)^{m_2}} = \langle \bar{L}_\nu f, \bar{\pi}_L g\rangle_{\mathcal{V}_{L,\Gamma_1}', \mathcal{V}_{L,\Gamma_1}}.$$

For $f \in \mathsf{H}(L_\partial, \Omega)$ and $g \in \mathsf{H}(L_\partial^{\mathsf{H}}, \Omega)$ we have

$$\begin{aligned}\langle L_\partial f, g\rangle_{\mathsf{L}^2(\Omega)^{m_1}} + \langle f, L_\partial^{\mathsf{H}} g\rangle_{\mathsf{L}^2(\Omega)^{m_2}} &= \langle \bar{L}_\nu f, \bar{\pi}_L g\rangle_{\mathcal{V}_L', \mathcal{V}_L} \\ &= \langle \bar{\pi}_{L^{\mathsf{H}}} f, \bar{L}_\nu^{\mathsf{H}} g\rangle_{\mathcal{V}_{L^{\mathsf{H}}}, \mathcal{V}_{L^{\mathsf{H}}}'}.\end{aligned}$$

Proof. Since $\mathsf{H}^1(\Omega)^{m_2}$ is dense in $\mathsf{H}(L_\partial, \Omega)$ and $\mathsf{H}^1(\Omega)^{m_1} \cap \mathsf{H}_{\Gamma_0}(L_\partial^{\mathsf{H}}, \Omega)$ is dense in $\mathsf{H}_{\Gamma_0}(L_\partial^{\mathsf{H}}, \Omega)$, the first equation follows from (5.1) by continuity. The second equation is just the special case $\Gamma_0 = \emptyset$ and switching the roles of L_∂ and L_∂^{H} yields the last equation. ❑

Theorem 5.1.13. *The mapping $\bar{L}_\nu \colon \mathsf{H}(L_\partial, \Omega) \to \mathcal{V}_L'$ is linear, bounded and onto.*

Proof. By Proposition 5.1.10 we already know that $\bar{L}_\nu$ is linear and bounded from $\mathsf{H}(L_\partial, \Omega)$ to $\mathcal{V}_L'$.

Let $\mu \in \mathcal{V}_L'$ be arbitrary. Since $\bar{\pi}_L$ is continuous from $\mathsf{H}(L_\partial^{\mathsf{H}}, \Omega)$ to $\mathcal{V}_L$, the mapping $g \mapsto \langle \mu, \bar{\pi}_L g\rangle_{\mathcal{V}_L', \mathcal{V}_L}$ is continuous from $\mathsf{H}(L_\partial^{\mathsf{H}}, \Omega)$ to $\mathbb{C}$. Consequently, there exists an $h \in \mathsf{H}(L_\partial^{\mathsf{H}}, \Omega)$ such that

$$\langle h, g\rangle_{\mathsf{H}(L_\partial^{\mathsf{H}}, \Omega)} = \langle \mu, \pi_L g\rangle_{\mathcal{V}_L', \mathcal{V}_L} \quad \text{for all} \quad g \in \mathsf{H}(L_\partial^{\mathsf{H}}, \Omega).$$

For a test function $v \in \mathcal{D}(\Omega)^{m_1}$ we have

$$\begin{aligned}0 = \langle \mu, \pi_L v\rangle_{\mathcal{V}_L', \mathcal{V}_L} &= \langle h, v\rangle_{\mathsf{H}(L_\partial^{\mathsf{H}}, \Omega)} = \langle h, v\rangle_{\mathsf{L}^2(\Omega)^{m_1}} + \langle L_\partial^{\mathsf{H}} h, L_\partial^{\mathsf{H}} v\rangle_{\mathsf{L}^2(\Omega)^{m_2}} \\ &= \langle h, v\rangle_{\mathcal{D}'(\Omega)^{m_1}, \mathcal{D}(\Omega)^{m_1}} + \big\langle L_\partial^{\mathsf{H}} h, L_\partial^{\mathsf{H}} v\big\rangle_{\mathcal{D}'(\Omega)^{m_2}, \mathcal{D}(\Omega)^{m_2}} \\ &= \big\langle (\mathrm{I} - L_\partial L_\partial^{\mathsf{H}}) h, v\big\rangle_{\mathcal{D}'(\Omega)^{m_1}, \mathcal{D}(\Omega)^{m_1}}.\end{aligned}$$

This means $L_\partial L_\partial^{\mathsf{H}} h = h$ in the sense of distributions. However, $h \in \mathsf{H}(L_\partial^{\mathsf{H}}, \Omega)$ implies $h \in \mathsf{L}^2(\Omega)$, which in turn gives $L_\partial L_\partial^{\mathsf{H}} h \in \mathsf{L}^2(\Omega)^{m_1}$, and $L_\partial^{\mathsf{H}} h \in \mathsf{L}^2(\Omega)^{m_2}$.

Therefore, $f := L_\partial h \in \mathsf{H}(L_\partial, \Omega)$. By Corollary 5.1.12 for $f = L_\partial^{\mathsf{H}} h \in \mathsf{H}(L_\partial, \Omega)$ and $g \in \mathsf{H}(L_\partial^{\mathsf{H}}, \Omega)$ we have

$$\begin{aligned}\langle \mu, \pi_L g\rangle_{\mathcal{V}_L', \mathcal{V}_L} &= \langle h, g\rangle_{\mathsf{H}(L_\partial^{\mathsf{H}}, \Omega)} = \langle h, g\rangle_{\mathsf{L}^2(\Omega)^{m_1}} + \langle L_\partial^{\mathsf{H}} h, L_\partial^{\mathsf{H}} g\rangle_{\mathsf{L}^2(\Omega)^{m_2}} \\ &= \langle (\mathrm{I} - L_\partial L_\partial^{\mathsf{H}}) h, g\rangle_{\mathsf{L}^2(\Omega)^{m_1}} + \langle \bar{L}_\nu L_\partial^{\mathsf{H}} h, \pi_L g\rangle_{\mathcal{V}_L', \mathcal{V}_L} \\ &= \big\langle \bar{L}_\nu \underbrace{(L_\partial^{\mathsf{H}} h)}_{=f}, \pi_L g\big\rangle_{\mathcal{V}_L', \mathcal{V}_L}.\end{aligned}$$

Hence, $\bar{L}_\nu f = \mu$ and $\bar{L}_\nu$ is onto. ❑

Corollary 5.1.14. *The mapping $\bar{L}_\nu^{\Gamma_1} \colon \mathsf{H}(L_\partial, \Omega) \to \mathcal{V}_{L,\Gamma_1}'$ is linear, bounded and onto.*

Proof. By Proposition 5.1.10 we already know that $\bar{L}_\nu^{\Gamma_1}$ is linear and bounded form $\mathsf{H}(L_\partial, \Omega)$ to $\mathcal{V}_L'$. Remark 5.1.11 gives $\bar{L}_\nu f\big|_{\mathcal{V}_{L,\Gamma_1}} = \bar{L}_\nu^{\Gamma_1} f$ for $f \in \mathsf{H}(L_\partial, \Omega)$ and $\mathcal{V}_{L,\Gamma_1}' = \mathcal{V}_L'\big|_{\mathcal{V}_{L,\Gamma_1}}$, which completes the proof. ❑

Theorem 5.1.15. $(\mathcal{V}_{L,\Gamma_1}, \mathsf{L}^2_\pi(\Gamma_1), \mathcal{V}_{L,\Gamma_1}')$ *is a quasi Gelfand triple.*

Proof. Let $\tilde{D}_+ := \operatorname{ran} \pi_L\big|_{\mathsf{H}^1_{\Gamma_0}(\Omega)^{m_1}}$ equipped with $\|\cdot\|_{\mathcal{X}_+} = \|\cdot\|_{\mathcal{V}_{L,\Gamma_1}}$ and let D_- denote the corresponding set from Definition 4.1.2 with $\mathcal{X}_0 = \mathsf{L}^2_\pi(\Gamma_1)$. Then by Remark 4.1.3 $\|g\|_{\mathcal{X}_-} = \|g\|_{\mathcal{V}_{L,\Gamma_1}'}$ for $g \in D_-$ and $\operatorname{ran} \mathbb{1}_{\Gamma_1} L_\nu \gamma_0 \subseteq D_-$ (by Proposition 5.1.10). By definition $\operatorname{ran} \mathbb{1}_{\Gamma_1} L_\nu \gamma_0$ is dense in $\mathsf{L}^2_\pi(\Gamma_1)$ and by Proposition 5.1.10 and Corollary 5.1.14 also dense in $\mathcal{V}_{L,\Gamma_1}'$. Consequently, also D_- is dense in both $\mathsf{L}^2_\pi(\Gamma_1)$ and $\mathcal{V}_{L,\Gamma_1}'$. Hence, assertion (iv) of Proposition 4.1.8 is satisfied, and by Remark 4.2.9 the completions of $\tilde{D}_+$ and D_- form a quasi Gelfand triple with pivot space $\mathsf{L}^2_\pi(\Gamma_1)$. By construction the completion of $\tilde{D}_+$ is $\mathcal{V}_{L,\Gamma_1}$. By the density of D_- in $\mathcal{V}_{L,\Gamma_1}'$ and $\|g\|_{\mathcal{X}_-} = \|g\|_{\mathcal{V}_{L,\Gamma_1}'}$ for $g \in D_-$ the completion of D_- is $\mathcal{V}_{L,\Gamma_1}'$. ❑

Corollary 5.1.16. $\mathsf{H}_0(L_\partial^{\mathsf{H}}, \Omega) = \mathsf{H}_{\partial\Omega}(L_\partial^{\mathsf{H}}, \Omega) = \ker \bar{\pi}_L = \ker \bar{L}_\nu^{\mathsf{H}}$ *and* $\mathsf{H}_0(L_\partial, \Omega) = \mathsf{H}_{\partial\Omega}(L_\partial, \Omega) = \ker \bar{\pi}_{L^{\mathsf{H}}} = \ker \bar{L}_\nu$.

Proof. For $g \in \mathsf{H}_0(L_\partial^{\mathsf{H}}, \Omega)$ there is a sequence $(g_n)_{n\in\mathbb{N}}$ in $\mathcal{D}(\Omega)$ converging to g, which implies $\bar{\pi}_L g = \lim_{n\to\infty} \bar{\pi}_L g_n = 0$. Therefore, $\mathsf{H}_0(L_\partial^{\mathsf{H}}, \Omega) \subseteq \ker \bar{\pi}_L = \mathsf{H}_{\partial\Omega}(L_\partial^{\mathsf{H}}, \Omega)$. On the other hand, if $g \in \mathsf{H}_{\partial\Omega}(L_\partial^{\mathsf{H}}, \Omega)$, then

$$\langle L_\partial f, g\rangle_{\mathsf{L}^2(\Omega)^{m_1}} + \langle f, L_\partial^{\mathsf{H}} g\rangle_{\mathsf{L}^2(\Omega)^{m_2}} = \langle \bar{L}_\nu f, \bar{\pi}_L g\rangle_{\mathcal{V}_L', \mathcal{V}_L} = 0$$

for all $f \in \mathsf{H}(L_\partial, \Omega)$. Hence, by Lemma 3.1.17 $g \in \mathsf{H}_0(L_\partial^{\mathsf{H}}, \Omega)$. Consequently, $\mathsf{H}_0(L_\partial^{\mathsf{H}}, \Omega) = \mathsf{H}_{\partial\Omega}(L_\partial^{\mathsf{H}}, \Omega)$. The second equality of the statement holds by

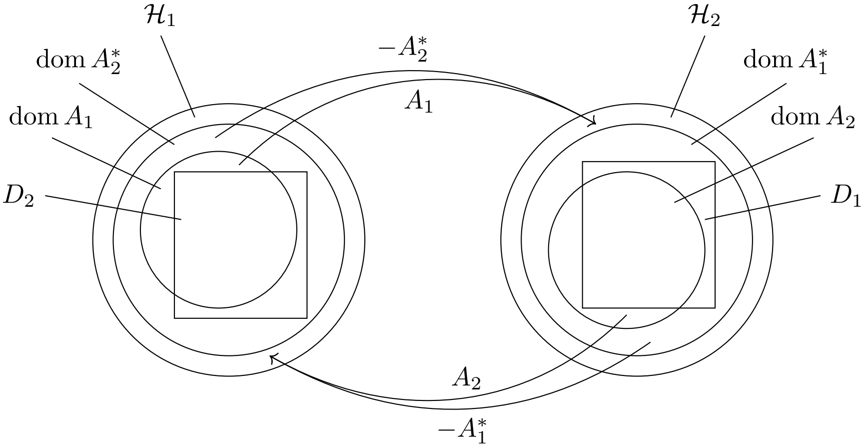

Figure 5.1: Setting of Section 5.2

definition and the third will be proven by the following equivalences

$$\begin{aligned} g \in \ker \pi_L &\Leftrightarrow \langle \bar\pi_L g, \psi\rangle_{\mathcal{V}_L', \mathcal{V}_L} = 0 \quad \text{for all} \quad \psi \in \mathcal{V}_L' \\ &\Leftrightarrow \langle \bar\pi_L g, \bar L_\nu f\rangle_{\mathcal{V}_L', \mathcal{V}_L} = 0 \quad \text{for all} \quad f \in \mathsf{H}(L_\partial, \Omega) \\ &\overset{C.5.1.12}{\Leftrightarrow} \langle \bar L_\nu^{\mathsf{H}} g, \bar\pi_{L^{\mathsf{H}}} f\rangle_{\mathcal{V}_{L^{\mathsf{H}}}', \mathcal{V}_{L^{\mathsf{H}}}} = 0 \quad \text{for all} \quad f \in \mathsf{H}(L_\partial, \Omega) \\ &\Leftrightarrow \langle \bar L_\nu^{\mathsf{H}} g, \phi\rangle_{\mathcal{V}_{L^{\mathsf{H}}}', \mathcal{V}_{L^{\mathsf{H}}}} = 0 \quad \text{for all} \quad \phi \in \mathcal{V}_{L^{\mathsf{H}}} \\ &\Leftrightarrow g \in \ker \bar L_\nu^{\mathsf{H}}. \end{aligned}$$

Switching L with L^{H} yields $\mathsf{H}_0(L_\partial, \Omega) = \mathsf{H}_{\partial\Omega}(L_\partial, \Omega) = \ker \bar\pi_{L^{\mathsf{H}}} = \ker \bar L_\nu$. ❑

5.2 Abstract Approach

This section we extract the essence of the previous and present an abstract approach to boundary spaces with differential operators of arbitrary order in mind.

Let $\mathcal{H}_1, \mathcal{H}_2$ be Hilbert spaces, $A_1\colon \operatorname{dom} A_1 \subseteq \mathcal{H}_1 \to \mathcal{H}_2$ and $A_2\colon \operatorname{dom} A_2 \subseteq \mathcal{H}_2 \to H_1$ be a densely defined and closed operators, such that $A_1 \subseteq -A_2^*$. Moreover, let D_1, D_2 be dense subspaces of $\mathcal{H}_2$ and $\mathcal{H}_1$, respectively, such that D_i is also dense in $\operatorname{dom} A_i^*$ for $i \in \{1,2\}$ with respect to the graph norm. Furthermore, let $\mathcal{X}_0$ be another Hilbert space and $B_1\colon D_1 \to \mathcal{X}_0$, $B_2\colon D_2 \to \mathcal{X}_0$ are linear with dense range.

In this section we will show the following theorem

Theorem 5.2.1. *Let A_1, A_2, B_1 and B_2 fulfill the previous assumptions and* *an abstract integration by parts formula:*

$$\langle A_1^* f, g\rangle_{\mathcal{H}_1} + \langle f, A_2^* g\rangle_{\mathcal{H}_2} = \langle B_1 f, B_2 g\rangle_{\mathcal{X}_0} \tag{5.3}$$

for $f \in D_1$ *and* $g \in D_2$. *Then we can construct a boundary triple* $(\mathcal{X}, \hat{B}_1, \hat{B}_2)$ *for* $\left[\begin{smallmatrix} 0 & A_1^* \\ A_2^* & 0 \end{smallmatrix}\right]$ *such that* $(\mathcal{X}, \mathcal{X}_0, \mathcal{X}')$ *is a quasi Gelfand triple.*

Clearly, the previous setting with $A_1 = L_\partial$, $A_2 = L_\partial^{\mathsf{H}}$, $B_1 = L_\nu$ and $B_2 = \pi_L$ is an example of this setting.

Lemma 5.2.2. *For* $f \in D_1$ *and* $g \in D_2$ *we have*

$$|\langle B_1 f, B_2 g\rangle_{\mathcal{X}_0}| \leq \|f\|_{A_1^*}\|g\|_{A_2^*}.$$

Proof. By (5.3), the triangular inequality and Cauchy-Schwarz's inequality we have

$$\begin{aligned}
\left|\langle B_1 f, B_2 g\rangle_{\mathcal{X}_0}\right| &= \left|\langle A_1^* f, g\rangle_{\mathcal{H}_1} + \langle f, A_2^* g\rangle_{\mathcal{H}_2}\right| \\
&\leq \|A_1^* f\|_{\mathcal{H}_1}\|g\|_{\mathcal{H}_1} + \|f\|_{\mathcal{H}_2}\|A_2^* g\|_{\mathcal{H}_2} \\
&\leq \sqrt{\|A_1^* f\|_{\mathcal{H}_1}^2 + \|f\|_{\mathcal{H}_2}^2}\sqrt{\|g\|_{\mathcal{H}_1}^2 + \|A_2^* g\|_{\mathcal{H}_2}^2} \\
&= \|f\|_{A_1^*}\|g\|_{A_2^*}.
\end{aligned}$$

❑

Lemma 5.2.3. $\ker B_2$ *is closed in* D_2 *with respect to* $\|\cdot\|_{A_2^*}$.

Proof. Let $(g_n)_{n\in\mathbb{N}}$ be a sequence in $\ker B_2$ that converges to $g \in D_2$ with respect to $\|\cdot\|_{A_2^*}$. Then for every $f \in D_1$

$$|\langle B_1 f, B_2 g\rangle_{\mathcal{X}_0}| = |\langle B_1 f, B_2(g - g_n)\rangle_{\mathcal{X}_0}| \leq \|f\|_{A_1^*}\|g - g_n\|_{A_2^*} \to 0$$

Since $\operatorname{ran} B_1$ is dense in $\mathcal{X}_0$, we have $B_2 g \perp \mathcal{X}_0$ and consequently $g \in \ker B_2$. ❑

Therefore, $\operatorname{ran} B_2$ equipped with

$$\|\phi\|_{\operatorname{ran} B_2} := \inf\left\{\|g\|_{A_2^*} \,\middle|\, B_2 g = \phi\right\} \tag{5.4}$$

is a normed space. Its completion is a Hilbert space as the next lemma will show.

Lemma 5.2.4. *the completion of* $\operatorname{ran} B_2$ *with respect to the norm* (5.4) *is isometric isomorphic to* $\operatorname{dom} A_2^* \big/ \overline{\ker B_2}^{\|\cdot\|_{A_2^*}}$ *equipped with the factor norm* $\|[f]_\sim\| := \inf\{\|g\|_{A_2^*} \mid g \sim f\}$.

Proof. By Lemma 5.2.3 that $\ker B_2$ is closed in D_2 w.r.t. $\|\cdot\|_{A_2^*}$. Hence, $D_2 \big/ \ker B_2$ is a normed space. Moreover, we have

$$\|[g]_\sim\|_{D_2/\ker B_2} := \inf_{k\in\ker B_2}\|g + k\|_{A_2^*} = \|B_2 g\|_{\operatorname{ran} B_2}$$

and therefore it is straight forward that $[g]_\sim \mapsto B_2 g$ is an isometry from $D_2\big/\ker B_2$ onto $\operatorname{ran} B_2$.

Consequently every completion of $D_2\big/\ker B_2$ is also a completion of $\operatorname{ran} B_2$ with $\|\cdot\|_{\operatorname{ran} B_2}$. It is not hard to see that $\operatorname{dom} A_2^*\Big/\overline{\ker B_2}^{\|\cdot\|_{A_2^*}}$ is a completion of $D_2\big/\ker B_2$. ❑

We will denote the completion of $\operatorname{ran} B_2$ w.r.t. $\|\cdot\|_{\operatorname{ran} B_2}$ by $\mathcal{X}$. We have that B_2 as a mapping from D_2 equipped with $\|\cdot\|_{A_2^*}$ onto $\operatorname{ran} B_2$ equipped with $\|\cdot\|_{\operatorname{ran} B_2}$ is a contraction ($\|B_2\| \leq 1$).

Lemma 5.2.5. *We can continuously extend the mapping B_2 to a surjective mapping $\bar{B}_2\colon \operatorname{dom} A_2^* \to \mathcal{X}$, where $\operatorname{dom} A_2^*$ is equipped with the graph norm. Moreover, $\|\bar{B}_2\| \leq 1$ and $\ker \bar{B}_2 = \overline{\ker B_2}^{\|\cdot\|_{A_2^*}}$.*

Proof. Since $\operatorname{dom} B_2 = D_2$ is dense in $\operatorname{dom} A_2^*$ (w.r.t. $\|\cdot\|_{A_2^*}$), the continuous extension of B_1 is defined on $\operatorname{dom} A_2^*$ and still satisfies $\|\bar{B}_2\| \leq 1$. Clearly, $\overline{\ker B_2}^{\|\cdot\|_{A_2^*}} \subseteq \ker \bar{B}_2$ and on the other hand for $g \in \ker \bar{B}_2$ there exists a convergent (w.r.t. $\|\cdot\|_{A_2^*}$) sequence $(g_n)_{n\in\mathbb{N}}$ in D_2 such that $\lim B_2 g_n = 0$. By the triangular inequality we have

$$\inf_{k\in\ker B_2} \|g+k\|_{A_2^*} - \|g_n - g\|_{A_2^*} \leq \inf_{k\in\ker B_2} \|g_n+k\|_{A_2^*} \leq \inf_{k\in\ker B_2} \|g+k\|_{A_2^*} + \|g_n-g\|_{A_2^*}$$

Hence,

$$0 = \|\bar{B}_2 g\|_{\mathcal{X}} = \lim_{n\in\mathbb{N}} \|B_2 g_n\|_{\operatorname{ran} B_2} = \lim_{n\in\mathbb{N}} \inf_{k\in\ker B_2} \|g_n + k\|_{A_2^*} = \inf_{k\in\ker B_2} \|g + k\|_{A_2^*}$$

which implies that $f \in \overline{\ker B_2}^{\|\cdot\|_{A_2^*}}$. ❑

Lemma 5.2.6. *The mapping $B_1\colon D_1 \to \mathcal{X}_0$ can be extended to a mapping $\bar{B}_1\colon \operatorname{dom} A_1^* \to \mathcal{X}'$, such that for every $f \in D_1$*

$$\bar{B}_1 f(\phi) = \langle B_1 f, \phi\rangle_{\mathcal{X}_0} \quad \textit{for all} \quad \phi \in \operatorname{ran} B_2$$

Proof. For a fixed $f \in D_1$ and $\phi \in \operatorname{ran} B_2$ we have $|\langle B_1 f, \phi\rangle_{\mathcal{X}_0}| \leq \|f\|_{A_1^*}\|\phi\|_{\mathcal{X}}$ by Lemma 5.2.2. Hence, we can extend the mapping

$$\Xi_f\colon \phi \in D_2 \mapsto \langle B_1 f, \phi\rangle_{\mathcal{X}_0}$$

by continuity on $\mathcal{X}$. So $\Xi_f \in \mathcal{X}'$ and $f \mapsto \Xi_f$ is linear and bounded by $\|\cdot\|_{A_1^*}$. Therefore, we can also extend this mapping on $\operatorname{dom} A_1^*$, we will denote this extension by $\bar{B}_1$. ❑

Lemma 5.2.7. *For $f \in \operatorname{dom} A_1^*$ and $g \in \operatorname{dom} A_2^*$ we have*

$$\langle A_1^* f, g\rangle_{\mathcal{H}_1} + \langle f, A_2^* g\rangle_{\mathcal{H}_2} = \langle \bar{B}_1 f, \bar{B}_2 g\rangle_{\mathcal{X}',\mathcal{X}}.$$

Proof. By assumption D_i is dense in $\operatorname{dom} A_i^*$ w.r.t. the graph norm of A_i^*. Moreover, $\bar{B}_i$ are continuous extensions of B_i (w.r.t. the graph norm of A_i^*). Hence, the assertion follows by continuity from (5.3). ❑

Clearly this also implies $A_1 = -A_2^*\big|_{\ker \bar{B}_2}$ and consequently $\operatorname{dom} A_1 = \ker \bar{B}_2 = \overline{\ker B_2}^{\|\cdot\|_{A_2^*}}$

Theorem 5.2.8. *$\bar{B}_1$ is surjective.*

Proof. Let $\mu \in \mathcal{X}'$ be arbitrary. Then $\phi\colon g \in \operatorname{dom} A_2^* \mapsto \langle \mu, \bar{B}_2 g\rangle_{\mathcal{X}',\mathcal{X}}$ is an element in $(\operatorname{dom} A_2^*)'$ and therefore there exists an $h \in \operatorname{dom} A_2^*$ such that $\langle \mu, \bar{B}_2 g\rangle_{\mathcal{X}',\mathcal{X}} = \langle h, g\rangle_{A_2}$. For $g \in \operatorname{dom} A_1 = \ker \bar{B}_2$ we have

$$\begin{aligned}
0 = \langle \mu, \bar{B}_2 g\rangle_{\mathcal{X}',hs} &= \langle h, g\rangle_{A_2^*} = \langle h, g\rangle_{H_1} + \langle A_2^* h, A_2^* g\rangle_{H_2} \\
&= \langle h, g\rangle_{H_1} - \langle A_2^* h, A_1 g\rangle_{H_2},
\end{aligned}$$

which implies that $A_2^* h \in \operatorname{dom} A_1^*$ and $h = A_1^* A_2^* h$. Hence, we have for a $g \in \operatorname{dom} A_2^*$

$$\begin{aligned}
\langle \mu, \bar{B}_2 g\rangle_{\mathcal{X}',\mathcal{X}} &= \langle h, g\rangle_{H_1} + \langle A_2^* h, A_2^* g\rangle_{H_2} \\
&= \langle h, g\rangle_{H_1} - \langle A_1^* A_2^* h, g\rangle_{H_1} + \langle \bar{B}_1 A_2^* h, \bar{B}_2 g\rangle_{\mathcal{X}',\mathcal{X}} \\
&= \langle \underbrace{h - A_1^* A_2^* h}_{=0}, g\rangle_{H_1} + \langle \bar{B}_1 A_2^* h, \bar{B}_2 g\rangle_{\mathcal{X}',\mathcal{X}}.
\end{aligned}$$

Consequently, $\mu = \bar{B}_1 A_2^* h$ and $\bar{B}_1$ is surjective. ❑

Proof of Theorem 5.2.1. We define $\tilde{D}_+ = \operatorname{ran} B_1$ which is by assumption dense in $\mathcal{X}_0$. Moreover, $\operatorname{ran} B_2$ is also dense in $\mathcal{X}_0$ and since its extension $\bar{B}_2$ maps into $\mathcal{X}'$, we conclude that $\operatorname{ran} B_2 \subseteq D_-$ (where D_- is the corresponding set to $\tilde{D}_+$ given by Definition 4.1.2). By construction of $\bar{B}_2$, $\operatorname{ran} B_2$ is also dense in $\mathcal{X}'$. Hence, assertion (iv) of Proposition 4.1.8 is satisfied and by Remark 4.2.9 $(\mathcal{X}, \mathcal{X}_0, \mathcal{X}')$ is a quasi Gelfand triple. ❑

5.3 Boundary Triple for a port-Hamiltonian System

In this section we will show that there is a boundary triple associated to the port-Hamiltonian differential operator $(P_\partial + P_0)\mathcal{H}$, which enables us to formulate boundary conditions that admit existence and uniqueness of solutions. In particular, we can parameterize all boundary conditions that provide unique solutions that are non-increasing in the Hamiltonian.

Recall the setting in Section 3.2. We had the following PDE:

$$\begin{aligned}
\frac{\partial}{\partial t} x(t,\zeta) &= \sum_{i=1}^{n} \frac{\partial}{\partial \zeta_i} P_i \mathcal{H}(\zeta) x(t,\zeta) + P_0 \mathcal{H}(\zeta) x(t,\zeta), \qquad && t \in \mathbb{R}_+, \zeta \in \Omega, \\
x(0,\zeta) &= x_0(\zeta), && \zeta \in \Omega,
\end{aligned}$$

where we assume that $P_i = \left[\begin{smallmatrix} 0 & L_i \\ L_i^{\mathsf{H}} & 0 \end{smallmatrix}\right]$ for $i \in \{1, \ldots, n\}$, see Assumption 3.2.2. We wrote this system as an abstract Cauchy problem:

$$\dot{x}(t) = (P_\partial + P_0)\mathcal{H}x(t), \quad x(0) = x_0,$$

so that we can use semigroup theory to characterize solvability and uniqueness of solutions.

In (3.5) we have already found an almost boundary triple for $P_\partial + P_0$ (note that P_0 is skew-adjoint, therefore $\langle P_0 x, y\rangle + \langle x, P_0 y\rangle = 0$). We had

$$\langle (P_\partial + P_0)x, y\rangle + \langle x, (P_\partial + P_0)y\rangle = \langle L_\nu \gamma_0 x_L, \gamma_0 y_{L^{\mathsf{H}}}\rangle + \langle \gamma_0 x_{L^{\mathsf{H}}}, L_\nu \gamma_0 y_L\rangle.$$

We will see that we can extend this to a boundary triple such that the boundary spaces establish a quasi Gelfand triple with $\mathsf{L}^2(\partial\Omega)$ as pivot space.

Lemma 5.3.1. *Let P and L be as in Assumption 3.2.2. Then*

$$\mathcal{V}_P = \mathcal{V}_L \times \mathcal{V}_{L^{\mathsf{H}}} \quad \textit{and} \quad \mathcal{V}_P' = \mathcal{V}_L' \times \mathcal{V}_{L^{\mathsf{H}}}'.$$

Moreover,

$$\bar{\pi}_P = \begin{bmatrix} \bar{\pi}_L & 0 \\ 0 & \bar{\pi}_{L^{\mathsf{H}}} \end{bmatrix} \quad \textit{and} \quad \bar{P}_\nu = \begin{bmatrix} 0 & \bar{L}_\nu \\ \bar{L}_\nu^{\mathsf{H}} & 0 \end{bmatrix}.$$

Proof. Note that π_P is defined as the orthogonal projection on $\overline{\operatorname{ran} P_\nu}$. Since $P_\nu = \left[\begin{smallmatrix} 0 & L_\nu \\ L_\nu^{\mathsf{H}} & 0 \end{smallmatrix}\right]$, we can easily derive $\pi_P = \left[\begin{smallmatrix} \pi_L & 0 \\ 0 & \pi_{L^{\mathsf{H}}} \end{smallmatrix}\right]$. By definition $\mathcal{V}_P$ is the completion of $\operatorname{ran} \pi_P$ with respect to the range norm. We will denote the completion of a normed space S by $\tilde{S}$. Thus,

$$\mathcal{V}_P = \widetilde{\operatorname{ran} \pi_P} = \widetilde{\operatorname{ran} \pi_L \times \operatorname{ran} \pi_{L^{\mathsf{H}}}} = \widetilde{\operatorname{ran} \pi_L} \times \widetilde{\operatorname{ran} \pi_{L^{\mathsf{H}}}} = \mathcal{V}_L \times \mathcal{V}_{L^{\mathsf{H}}}.$$

Clearly, this implies $\mathcal{V}_P' = \mathcal{V}_L' \times \mathcal{V}_{L^{\mathsf{H}}}'$ and $\bar{\pi}_P$ as the continuous extension of $\pi_P = \left[\begin{smallmatrix} \pi_L & 0 \\ 0 & \pi_{L^{\mathsf{H}}} \end{smallmatrix}\right]$ equals $\left[\begin{smallmatrix} \bar{\pi}_L & 0 \\ 0 & \bar{\pi}_{L^{\mathsf{H}}} \end{smallmatrix}\right]$. Finally, the continuous extension $\bar{P}_\nu$ of $P_\nu = \left[\begin{smallmatrix} 0 & L_\nu \\ L_\nu^{\mathsf{H}} & 0 \end{smallmatrix}\right]$ equals $\left[\begin{smallmatrix} 0 & \bar{L}_\nu \\ \bar{L}_\nu^{\mathsf{H}} & 0 \end{smallmatrix}\right]$. ❑

Recall the splitting $x = \left[\begin{smallmatrix} x_{L^{\mathsf{H}}} \\ x_L \end{smallmatrix}\right]$. Accordingly, we introduce $\mathcal{H}x = \left[\begin{smallmatrix} (\mathcal{H}x)_{L^{\mathsf{H}}} \\ (\mathcal{H}x)_L \end{smallmatrix}\right]$ for $x \in \mathcal{H}^{-1}(\mathsf{H}(P_\partial, \Omega))$, so that

$$P_\partial \mathcal{H}x = \begin{bmatrix} L_\partial (\mathcal{H}x)_L \\ L_\partial^{\mathsf{H}} (\mathcal{H}x)_{L^{\mathsf{H}}} \end{bmatrix}, \quad \begin{bmatrix} 0 & \bar{L}_\nu \end{bmatrix} \mathcal{H}x = \bar{L}_\nu (\mathcal{H}x)_L, \quad \begin{bmatrix} \bar{\pi}_L & 0 \end{bmatrix} \mathcal{H}x = \bar{\pi}_L (\mathcal{H}x)_{L^{\mathsf{H}}}.$$

The next theorem gives us a boundary triple for the port-Hamiltonian differential operator, such that the boundary spaces establish a quasi Gelfand space with $\mathsf{L}^2(\partial\Omega)$ as pivot space. Recall that $\mathcal{X}_\mathcal{H}$ is $\mathsf{L}^2(\Omega)^m$ equipped with

$$\langle x, y\rangle_{\mathcal{X}_\mathcal{H}} = \langle x, \mathcal{H}y\rangle_{\mathsf{L}^2(\Omega)}.$$

Theorem 5.3.2. *The operator*

$$A_0 := -(P_\partial + P_0)\mathcal{H}, \quad \operatorname{dom} A_0 := \mathcal{H}^{-1}(\ker \bar{P}_\nu)$$

is closed, skew-symmetric, and densely defined on $\mathcal{X}_\mathcal{H}$. Its adjoint is

$$A_0^* = (P_\partial + P_0)\mathcal{H}, \quad \operatorname{dom} A_0^* = \mathcal{H}^{-1}(\mathsf{H}(P_\partial, \Omega)).$$

Let $B_1 = \begin{bmatrix}\bar{\pi}_L & 0\end{bmatrix}\mathcal{H}$, $B_2 = \begin{bmatrix}0 & \bar{L}_\nu\end{bmatrix}\mathcal{H}$. Then $(\mathcal{V}_L, B_1, B_2)$ is a boundary triple for A_0^*.

Proof. We define $\tilde{A}$ as $(P_\partial + P_0)\mathcal{H}$ with $\operatorname{dom}\tilde{A} = \mathcal{H}^{-1}(\mathsf{H}(P_\partial,\Omega))$ on $\mathcal{X}_\mathcal{H}$. By Lemma 3.1.5 $P_\partial \colon \mathsf{H}(P_\partial,\Omega) \subseteq \mathsf{L}^2(\Omega)^m \to \mathsf{L}^2(\Omega)^m$ is a closed operator. Since $\mathcal{H}$ is a bounded operator on $\mathsf{L}^2(\Omega)^m$, and $\mathcal{X}_\mathcal{H}$ and $\mathsf{L}^2(\Omega)^m$ have equivalent norms, it is easy to see that $\tilde{A}\colon \mathcal{H}^{-1}(\mathsf{H}(P_\partial,\Omega)) \subseteq \mathcal{X}_\mathcal{H} \to \mathcal{X}_\mathcal{H}$ is closed. The adjoint of $\tilde{A}$ can be calculated by

$$\tilde{A}^* = \big((P_\partial + P_0)\mathcal{H}\big)^{*_{\mathcal{X}_\mathcal{H}\times\mathcal{X}_\mathcal{H}}} = \mathcal{H}^{-1}\big((P_\partial + P_0)\mathcal{H}\big)^{*_{\mathsf{L}^2\times\mathsf{L}^2}}\mathcal{H} = (P_\partial^{*_{\mathsf{L}^2\times\mathsf{L}^2}} + P_0^{*_{\mathsf{L}^2\times\mathsf{L}^2}})\mathcal{H}$$

and according to Remark 3.1.7 we have $P_\partial^{*_{\mathsf{L}^2\times\mathsf{L}^2}} = -P_\partial\big|_{\operatorname{dom} P_\partial^{*_{\mathsf{L}^2\times\mathsf{L}^2}}}$, where $\operatorname{dom} P_\partial^{*_{\mathsf{L}^2\times\mathsf{L}^2}} \subseteq \mathsf{H}(P_\partial,\Omega)$. Hence,

$$\tilde{A}^* = -(P_\partial + P_0)\mathcal{H}\big|_{\mathcal{H}^{-1}(\operatorname{dom} P_\partial^{*_{\mathsf{L}^2\times\mathsf{L}^2}})} = -\tilde{A}\big|_{\mathcal{H}^{-1}(\operatorname{dom} P_\partial^{*_{\mathsf{L}^2\times\mathsf{L}^2}})} \subseteq -\tilde{A}.$$

Since $\tilde{A}$ is closed, we have $\tilde{A}^{**} = \tilde{A}$. Consequently, $\tilde{A}^*$ is skew-symmetric on $\mathcal{X}_\mathcal{H}$.

Now we know that $\tilde{A}$ is the adjoint of a skew-symmetric operator. So we can talk about boundary triples for $\tilde{A}$. First we note that

$$\operatorname{ran}\begin{bmatrix}B_1\\B_2\end{bmatrix} = \operatorname{ran}\bar{\pi}_L \times \operatorname{ran}\bar{L}_\nu = \mathcal{V}_L \times \mathcal{V}_L'.$$

Since $\mathcal{H}$ is self-adjoint and P_0 is skew-adjoint, we have for $x, y \in \operatorname{dom}\tilde{A}$

$$\begin{aligned}\langle \tilde{A}x, y\rangle_{\mathcal{X}_\mathcal{H}} &+ \langle x, \tilde{A}y\rangle_{\mathcal{X}_\mathcal{H}} \\ &= \langle P_\partial \mathcal{H}x, \mathcal{H}y\rangle_{\mathsf{L}^2} + \langle \mathcal{H}x, P_\partial\mathcal{H}y\rangle_{\mathsf{L}^2}\end{aligned}$$

by the the identity $P_\partial = \begin{bmatrix}0 & L_\partial\\ L_\partial^{\mathsf{H}} & 0\end{bmatrix}$ and Corollary 5.1.12 we further have

$$\begin{aligned}
&= \left\langle \begin{bmatrix} L_\partial(\mathcal{H}x)_L \\ L_\partial^{\mathsf{H}}(\mathcal{H}x)_{L^{\mathsf{H}}}\end{bmatrix}, \begin{bmatrix}(\mathcal{H}y)_{L^{\mathsf{H}}} \\ (\mathcal{H}y)_L\end{bmatrix}\right\rangle_{\mathsf{L}^2} + \left\langle \begin{bmatrix}(\mathcal{H}x)_{L^{\mathsf{H}}} \\ (\mathcal{H}x)_L\end{bmatrix}, \begin{bmatrix} L_\partial(\mathcal{H}y)_L \\ L_\partial^{\mathsf{H}}(\mathcal{H}y)_{L^{\mathsf{H}}}\end{bmatrix}\right\rangle_{\mathsf{L}^2} \\
&= \langle L_\partial(\mathcal{H}x)_L, (\mathcal{H}y)_{L^{\mathsf{H}}}\rangle_{\mathsf{L}^2} + \big\langle (\mathcal{H}x)_L, L_\partial^{\mathsf{H}}(\mathcal{H}y)_{L^{\mathsf{H}}}\big\rangle_{\mathsf{L}^2} \\
&\qquad + \big\langle L_\partial^{\mathsf{H}}(\mathcal{H}x)_{L^{\mathsf{H}}}, (\mathcal{H}y)_L\big\rangle_{\mathsf{L}^2} + \big\langle (\mathcal{H}x)_{L^{\mathsf{H}}}, L_\partial^{\mathsf{H}}(\mathcal{H}y)_L\big\rangle_{\mathsf{L}^2} \\
&= \langle \bar{L}_\nu(\mathcal{H}x)_L, \bar{\pi}_L(\mathcal{H}y)_{L^{\mathsf{H}}}\rangle_{\mathcal{V}_L', \mathcal{V}_L} + \langle \bar{\pi}_L(\mathcal{H}x)_{L^{\mathsf{H}}}, \bar{L}_\nu(\mathcal{H}y)_L\rangle_{\mathcal{V}_L, \mathcal{V}_L'} \\
&= \langle B_2 x, B_1 y\rangle_{\mathcal{V}_L', \mathcal{V}_L} + \langle B_1 x, B_2 y\rangle_{\mathcal{V}_L, \mathcal{V}_L'}.
\end{aligned}$$

Therefore, $(\mathcal{V}_L, B_1, B_2)$ is a boundary triple for $\tilde{A}$.

By Lemma 2.4.5 $\operatorname{dom} \tilde{A}^* = \ker B_1 \cap \ker B_2$, which is equal to

$$\ker B_1 \cap \ker B_2 = \mathcal{H}^{-1}\big(\ker \begin{bmatrix}\bar{\pi}_L & 0\end{bmatrix} \cap \ker \begin{bmatrix}0 & \bar{L}_\nu\end{bmatrix}\big) = \mathcal{H}^{-1}\big(\ker \bar{\pi}_L \times \ker \bar{L}_\nu\big).$$

By Corollary 5.1.16 this is equal to $\mathcal{H}^{-1}(\ker \bar{L}_\nu^{\mathsf{H}} \times \ker \bar{L}_\nu) = \mathcal{H}^{-1}(\ker \bar{P}_\nu)$. Hence, $\tilde{A}^* = A_0$ and $A_0^* = \tilde{A}$. ❑

Theorem 5.3.3. *Let A_0^* be the operator from the previous theorem. Then $(\mathcal{V}_{L,\Gamma_1}, \begin{bmatrix}\bar{\pi}_L & 0\end{bmatrix}\mathcal{H}, \begin{bmatrix}0 & \bar{L}_\nu^{\Gamma_1}\end{bmatrix}\mathcal{H})$ is a boundary triple for*

$$A := A_0^*\big|_{\mathcal{H}^{-1}\left(\mathsf{H}_{\Gamma_0}(L_\partial^{\mathsf{H}},\Omega)\times \mathsf{H}(L_\partial,\Omega)\right)}.$$

Proof. Since we already have a boundary triple for A_0^*, we can show that A is the adjoint of a skew-symmetric operator by Corollary 2.4.11 (iii). Hence, we have to check, whether $\left[\begin{smallmatrix}0 & \mathrm{I}\\ \mathrm{I} & 0\end{smallmatrix}\right]\mathcal{C}^\perp \subseteq \mathcal{C}$ in $\mathcal{V}_L \times \mathcal{V}_L'$, where $\mathcal{C}$ is the corresponding relation to the domain of A according to Corollary 2.4.11. For B_1, B_2 being the mappings from the previous theorem we have (Note that $\mathcal{V}_{L,\Gamma_1}$ is a subspace of $\mathcal{V}_L$; Lemma 5.1.9)

$$\mathcal{C} = \begin{bmatrix}B_1\\ B_2\end{bmatrix}\operatorname{dom} A = \mathcal{V}_{L,\Gamma_1} \times \mathcal{V}_L'$$
$$\begin{bmatrix}0 & \mathrm{I}\\ \mathrm{I} & 0\end{bmatrix}\mathcal{C}^\perp = \{0\} \times \mathcal{V}_{L,\Gamma_1}^\perp \subseteq \mathcal{V}_{L,\Gamma_1} \times \mathcal{V}_L' = \mathcal{C}.$$

For $x, y \in \operatorname{dom} A$ we have, using Remark 5.1.11,

$$\begin{aligned}
\langle B_1 x, B_2 y\rangle_{\mathcal{V}_L,\mathcal{V}_L'} &= \big\langle \bar{\pi}_L(\mathcal{H}x)_{L^{\mathsf{H}}}, \bar{L}_\nu(\mathcal{H}y)_L\big\rangle_{\mathcal{V}_L,\mathcal{V}_L'}\\
&= \big\langle \bar{\pi}_L(\mathcal{H}x)_{L^{\mathsf{H}}}, \bar{L}_\nu^{\Gamma_1}(\mathcal{H}y)_L\big\rangle_{\mathcal{V}_{L,\Gamma_1},\mathcal{V}_{L,\Gamma_1}'}\\
&= \big\langle \begin{bmatrix}\bar{\pi}_L & 0\end{bmatrix}\mathcal{H}x, \begin{bmatrix}0 & \bar{L}_\nu^{\Gamma_1}\end{bmatrix}\mathcal{H}y\big\rangle_{\mathcal{V}_{L,\Gamma_1},\mathcal{V}_{L,\Gamma_1}'},
\end{aligned}$$

which yields item (ii) in Definition 2.4.1. By $\operatorname{ran}\left[\begin{smallmatrix}\bar{\pi}_L & 0\\ 0 & \bar{L}_\nu^{\Gamma_1}\end{smallmatrix}\right]\Big|_{\mathsf{H}_{\Gamma_0}(L_\partial^{\mathsf{H}},\Omega)\times\mathsf{H}(L_\partial,\Omega)} = \mathcal{V}_{L,\Gamma_1} \times \mathcal{V}_{L,\Gamma_1}'$, the remaining item (i) is fulfilled. ❑

With the next theorem from [28, Theorem 2.5] we can characterize boundary conditions such that an operator A that possesses a boundary triple generates a contraction semigroup.

Theorem 5.3.4. *Let A_0 be a skew-symmetric operator on a Hilbert space X and $(\mathcal{B}, B_1, B_2)$ be a boundary triple for A_0^*. Furthermore let $\mathcal{K}$ be a Hilbert space, $W_B = \begin{bmatrix}W_1 & W_2\end{bmatrix}$, where $W_1, W_2 \in \mathcal{L}_{\mathrm{b}}(\mathcal{B}, \mathcal{K})$, and $A := A_0^*\big|_{\operatorname{dom} A}$, where $\operatorname{dom} A = \ker W_B \begin{bmatrix}B_1\\ B_2\end{bmatrix}$. If $\operatorname{ran} W_1 - W_2 \subseteq \operatorname{ran} W_1 + W_2$ then the following assertions are equivalent.*

(i) *The operator A generates a contraction semigroup on X.*

(ii) *The operator A is dissipative.*

(iii) *The operator $W_1 + W_2$ is injective and the following operator inequality holds*

$$W_1 W_2^* + W_2 W_1^* \geq 0.$$

We will reformulate this theorem to fit our situation.

Corollary 5.3.5. *Let $\mathcal{K}$ be some Hilbert space and $W = \begin{bmatrix} W_1 & W_2 \end{bmatrix} : \mathcal{V}_{L,\Gamma_1} \times \mathcal{V}_{L,\Gamma_1} \to \mathcal{K}$ a bounded linear mapping such that* $\operatorname{ran} W_1 - W_2 \subseteq \operatorname{ran} W_1 + W_2$. *Let*

$$D := \{x \in \mathcal{H}^{-1}(\mathsf{H}_{\Gamma_0}(L_\partial^{\mathsf{H}}, \Omega) \times \mathsf{H}(L_\partial, \Omega)) \\ \,|\, W_1 \begin{bmatrix} \bar{\pi}_L & 0 \end{bmatrix} \mathcal{H}x + W_2 \Psi \begin{bmatrix} 0 & \bar{L}_\nu^{\Gamma_1} \end{bmatrix} \mathcal{H}x = 0\},$$

where $\Psi \colon \mathcal{V}'_{L,\Gamma_1} \to \mathcal{V}_{L,\Gamma_1}$ is the duality mapping corresponding to the quasi Gelfand triple. Then the following assertions are equivalent.

(i) *$(P_\partial + P_0)\mathcal{H}\big|_D$ generates a contraction semigroup.*

(ii) *$(P_\partial + P_0)\mathcal{H}\big|_D$ is dissipative.*

(iii) *The operator $W_1 + W_2$ is injective and the following operator inequality holds*

$$W_1 W_2^* + W_2 W_1^* \geq 0.$$

Note that by Theorem 5.3.3 the port-Hamiltonian differential operator $P_\partial \mathcal{H}$ has the boundary triple $(\mathcal{V}_{L,\Gamma_1}, [\,\bar{\pi}_L \ 0\,]\mathcal{H}, [\,0 \ \bar{L}_\nu^{\Gamma_1}\,]\mathcal{H})$. Additionally, $(\mathcal{V}_{L,\Gamma_1}, \mathsf{L}^2_\pi(\Gamma_1), \mathcal{V}'_{L,\Gamma_1})$ is a quasi Gelfand triple. The previous results characterize dissipative boundary conditions (and therefore existence and uniqueness of solutions) in terms of the boundary spaces $\mathcal{V}_{L,\Gamma_1}$, which posses a slightly unhandy inner product/norm. This makes it sometimes impracticable to check the conditions of the previous results. Fortunately, in Theorem 4.4.6 we have already shown that we can formulate the boundary conditions also in terms of the pivot space of the quasi Gelfand triple, thus in $\mathsf{L}^2_\pi(\Gamma_1)$. Moreover, in Example 4.4.8 we have given concrete boundary operators that fulfill all conditions of Theorem 4.4.6.

Theorem 5.3.6. *Let M be a linear positive operator on $\mathsf{L}^2_\pi(\Gamma_1)$. Then $A = (P_\partial + P_0)\mathcal{H}$ with domain*

$$\operatorname{dom} A = \{a \in \mathcal{H}^{-1}(\mathsf{H}_{\Gamma_0}(L_\partial^{\mathsf{H}}, \Omega) \times \mathsf{H}(L_\partial, \Omega)) \,|\, \bar{\pi}_L(\mathcal{H}x)_{L^{\mathsf{H}}} + M\bar{L}_\nu(\mathcal{H}x)_L = 0\}$$

generates a contraction semigroup. Its adjoint is given by $-(P_\partial + P_0)\mathcal{H}$ restricted to

$$\operatorname{dom} A^* = \{a \in \mathcal{H}^{-1}(\mathsf{H}_{\Gamma_0}(L_\partial^{\mathsf{H}}, \Omega) \times \mathsf{H}(L_\partial, \Omega)) \,|\, \bar{\pi}_L(\mathcal{H}x)_{L^{\mathsf{H}}} - M\bar{L}_\nu(\mathcal{H}x)_L = 0\}.$$

Note that $\bar{\pi}_L(\mathcal{H}x)_{L^{\mathsf{H}}} + M\bar{L}_\nu(\mathcal{H}x)_L = 0$ and $\bar{\pi}_L(\mathcal{H}x)_{L^{\mathsf{H}}} - M\bar{L}_\nu(\mathcal{H}x)_L = 0$ implicitly imply that each summand is in the pivot space $\mathsf{L}^2_\pi(\Gamma_1)$ (Theorem 4.3.5).

Proof. We want to apply Theorem 4.4.6. Hence, we need a boundary triple, which dual pair comes from a quasi Gelfand triple. By Theorem 5.3.3

$$(\mathcal{V}_{L,\Gamma_1}, \begin{bmatrix}\bar{\pi}_L & 0\end{bmatrix}, \begin{bmatrix}0 & \bar{L}_\nu^{\Gamma_1}\end{bmatrix})$$

is a boundary triple for $(P_\partial + P_0)\mathcal{H}$ with domain $\mathcal{H}^{-1}(\mathsf{H}_{\Gamma_0}(L_\partial^{\mathsf{H}}, \Omega) \times \mathsf{H}(L_\partial, \Omega))$. Moreover, $(\mathcal{V}_{L,\Gamma_1}, \mathsf{L}^2(\Gamma_1), \mathcal{V}'_{L,\Gamma_1})$ is a quasi Gelfand triple. In Example 4.4.8 we checked that the boundary operators $V_1 = \mathrm{I}$ and $V_2 = M$ satisfy the conditions of Theorem 4.4.6. Hence, by Theorem 4.4.6 the operator A generates a contraction semigroup.

By Lemma 4.4.9 the adjoint of A is given by $-(P_\partial + P_0)\mathcal{H}$ restricted to

$$\left\{x \in \mathcal{H}^{-1}(\mathsf{H}_{\Gamma_1}(L_\partial^{\mathsf{H}}, \Omega) \times \mathsf{H}(L_\partial, \Omega)) \,\middle|\, M^{-1}\bar{\pi}_L(\mathcal{H}x)_{L^{\mathsf{H}}} - \bar{L}_\nu(\mathcal{H}x)_L = 0\right\}.$$

Applying the operator M on the boundary condition yields the claim. ❑

Corollary 5.3.7. *The port-Hamiltonian system with boundary condition*

$$\bar{\pi}_L(\mathcal{H}x)_{L^{\mathsf{H}}} + M(\mathcal{H}x)_L = 0$$

possesses a unique mild solution for every initial condition in $\mathsf{L}^2(\Omega)^m$.

Proof. This is just an easy consequence of Theorem 5.3.6. ❑

5.4 Conclusion

For the port-Hamiltonian operator $(P_\partial + P_0)\mathcal{H}$, there exists a boundary triple $(\mathcal{V}_L, [\,\bar{\pi}_L\ 0\,], [\,0\ \bar{L}_\nu\,])$ such that $(\mathcal{V}_L, \mathsf{L}^2_\pi(\partial\Omega), \mathcal{V}'_L)$ is a quasi Gelfand triple. Hence, we can characterize every boundary conditions such that $(P_\partial + P_0)\mathcal{H}$ generates a contraction semigroup in terms of the boundary space $\mathcal{V}_L$. However, we can also characterize boundary conditions in the pivot space $\mathsf{L}^2_\pi(\partial\Omega)$ such that $(P_\partial + P_0)\mathcal{H}$ generates a contraction semigroup by Theorem 4.4.6. In any case we have existence and uniqueness of solutions. Moreover, the Hamiltonian along solutions is non-increasing. This can be seen by

$$H(x(t)) = \frac{1}{2}\|x(t)\|^2_{\mathcal{X}_\mathcal{H}} = \frac{1}{2}\|T(t-s)x(s)\|^2_{\mathcal{X}_\mathcal{H}} \leq \frac{1}{2}\|x(s)\|^2_{\mathcal{X}_\mathcal{H}} = H(x(s)), \quad s \leq t,$$

as T (the semigroup generated by $(P_\partial + P_0)\mathcal{H}$) is a contraction semigroup.

Instead of using a semigroup approach to show existence and uniqueness of solutions we could have used the tools of [37], which provide a more general approach. The crucial property is that $P_\partial\mathcal{H}$ (with adequate boundary conditions) is a maximal dissipative operator.

Chapter 6

Boundary Control and Observation Systems

We will recall the notion of boundary control systems, scattering passive and impedance passive in the manner of [35]. We will show that a port-Hamiltonian system can be described as such a system. This concept already provides solution theory (see i.e. [34, Lemma 2.6]). It is well known that every scattering passive boundary control system induces a scattering passive well-posed linear system.

Finally, we will show that port-Hamiltonian systems, with the input and output function that were indicated at the beginning, can be described as such boundary control and observation systems, either scattering passive or impedance passive.

6.1 Basics

Definition 6.1.1. A *colligation* $\Xi := \left(\begin{bmatrix} G \\ L \\ K \end{bmatrix}; \begin{bmatrix} \mathcal{U} \\ \mathcal{X} \\ \mathcal{Y} \end{bmatrix} \right)$ consists of the three Hilbert spaces $\mathcal{U}$, $\mathcal{X}$, and $\mathcal{Y}$, and the three linear maps G, L, and K, with the same domain $\mathcal{Z} \subseteq \mathcal{X}$ and with values in $\mathcal{U}$, $\mathcal{X}$, and $\mathcal{Y}$, respectively.

Definition 6.1.2. A colligation $\Xi := \left(\begin{bmatrix} G \\ L \\ K \end{bmatrix}; \begin{bmatrix} \mathcal{U} \\ \mathcal{X} \\ \mathcal{Y} \end{bmatrix} \right)$ is an *(internally well-posed) boundary control and observation system*, if

(i) the operator $\begin{bmatrix} G \\ L \\ K \end{bmatrix}$ is closed from $\mathcal{X}$ to $\begin{bmatrix} \mathcal{U} \\ \mathcal{X} \\ \mathcal{Y} \end{bmatrix}$,

(ii) the operator G is surjective, and

(iii) the operator $A := L\big|_{\ker G}$ generates a contraction semigroup on $\mathcal{X}$.

We will sometimes use *boundary control system* as an abbreviation for *boundary control and observation system*

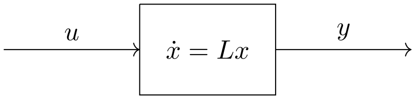

Figure 6.1: Boundary control and observation system

In literature you will also find the term *boundary node* for what we have defined as *boundary control and observation system.*

We think of the operators in this definition as determining a system via

$$\begin{aligned} u(t) &= Gx(t), \\ \dot{x}(t) &= Lx(t), \quad x(0) = x_0, \\ y(t) &= Kx(t). \end{aligned} \tag{6.1}$$

Figure 6.1 illustrates this system. We call $\mathcal{U}$ the *input space*, $\mathcal{X}$ the *state space*, $\mathcal{Y}$ the *output space* and $\mathcal{Z}$ the *solution space.* Normally, the input space $\mathcal{U}$ and the output space $\mathcal{Y}$ are boundary spaces.

Definition 6.1.3. Let $\Xi = \left(\left[\begin{smallmatrix} G \\ L \\ K \end{smallmatrix}\right]; \left[\begin{smallmatrix} \mathcal{U} \\ \mathcal{X} \\ \mathcal{Y} \end{smallmatrix}\right]\right)$ be a colligation. If Ξ is a boundary control and observation system such that

$$2\,\mathrm{Re}\langle Lx, x\rangle_{\mathcal{X}} + \|Kx\|_{\mathcal{Y}}^2 \leq \|Gx\|_{\mathcal{U}}^2 \quad \text{for} \quad x \in \mathcal{Z}, \tag{6.2}$$

then it is *scattering passive* and it is *scattering energy preserving*, if we have equality in (6.2).

We say Ξ is *impedance passive* (*energy preserving*), if $\mathcal{Y} = \mathcal{U}'$, $\Psi : \mathcal{U}' \to \mathcal{U}$ is the unitary identification mapping and $\tilde{\Xi} := \left(\left[\begin{smallmatrix} \frac{1}{\sqrt{2}}(G+\Psi K) \\ Ln \\ \frac{1}{\sqrt{2}}(G-\Psi K) \end{smallmatrix}\right]; \left[\begin{smallmatrix} \mathcal{U} \\ \mathcal{X} \\ \mathcal{U} \end{smallmatrix}\right]\right)$ is scattering passive (energy preserving).

Note that an impedance passive (energy preserving) colligation Ξ does not need to be a boundary control and observation system. If $\mathcal{U} = \mathcal{Y}$, then Ψ is the identity mapping.

We defined impedance passive (energy preserving) for a colligation not directly, but by its external Cayley transform. This prevents difficulties with boundary control and observation systems as already remarked. Normally we would ask for

$$\mathrm{Re}\langle Lx, x\rangle_{\mathcal{X}} \leq \mathrm{Re}\langle Gx, Kx\rangle_{\mathcal{U},\mathcal{Y}},$$

where $(\mathcal{U}, \mathcal{Y})$ is a complete dual pair. This would also allow $\mathcal{U}$ and $\mathcal{Y}$ to be Banach spaces.

6.2 Port-Hamiltonian System as Boundary Control and Observation System

Corresponding to a port-Hamiltonian system we want to introduce the following operators

$$\begin{aligned} G_{\mathrm{p}} &:= S_+ \begin{bmatrix} \bar{\pi}_L & 0 \end{bmatrix} \mathcal{H}\colon && \mathcal{H}^{-1}(\mathsf{H}(P_\partial,\Omega)) \subseteq \mathcal{X}_{\mathcal{H}} \to S\mathcal{V}_L, \\ L_{\mathrm{p}} &:= (P_\partial + P_0)\mathcal{H}\colon && \mathcal{H}^{-1}(\mathsf{H}(P_\partial,\Omega)) \subseteq \mathcal{X}_{\mathcal{H}} \to \mathcal{X}_{\mathcal{H}}, \\ K_{\mathrm{p}} &:= (S^*)_-^{-1} \begin{bmatrix} 0 & \bar{L}_\nu \end{bmatrix} \mathcal{H}\colon && \mathcal{H}^{-1}(\mathsf{H}(P_\partial,\Omega)) \subseteq \mathcal{X}_{\mathcal{H}} \to (S\mathcal{V}_L)', \end{aligned}$$

where $S \in \mathcal{L}_{\mathrm{b}}(\mathsf{L}^2(\partial\Omega)^{m_1})$ is boundedly invertible, and S_+ and $(S^*)_-^{-1}$ denote their extension on $\mathcal{V}_L$ and $\mathcal{V}_L'$ respectively (see Corollary 4.4.2). By Lemma 4.4.5 also G_{p} and K_{p} establish a boundary triple for L_{p} restricted to $\mathsf{H}_{\Gamma_0}(L_\partial^{\mathsf{H}},\Omega) \times \mathsf{H}(L_\partial,\Omega)$ and $(S_+\mathcal{V}_{L,\Gamma_1}, S\mathsf{L}^2_\pi(\Gamma_1), (S_+\mathcal{V}_{L,\Gamma_1})')$ is a quasi Gelfand triple For simplification S can be imagined to be the identity mapping. We still have Γ_0, Γ_1 as a splitting with thin boundaries of $\partial\Omega$.

Definition 6.2.1. We say the colligation

$$\Xi = \left(\begin{bmatrix} G\left[\begin{smallmatrix} G_{\mathrm{p}} \\ K_{\mathrm{p}} \end{smallmatrix}\right] \\ L_{\mathrm{p}} \\ K\left[\begin{smallmatrix} G_{\mathrm{p}} \\ K_{\mathrm{p}} \end{smallmatrix}\right] \end{bmatrix}; \begin{bmatrix} \mathcal{U} \\ \mathcal{X}_{\mathcal{H}} \\ \mathcal{Y} \end{bmatrix} \right)$$

is a *port-Hamiltonian boundary control and observation system*, where G and K are linear mappings from $\mathcal{V}_L \times \mathcal{V}_L'$ to Hilbert spaces $\mathcal{U}$ and $\mathcal{Y}$, repectively.

In particular we will regard

$$G\left[\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right] = x_1 \quad \text{and} \quad K\left[\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right] = x_2,$$

and

$$G\left[\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right] = \frac{1}{\sqrt{2}}(x_1 + x_2) \quad \text{and} \quad K\left[\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right] = \frac{1}{\sqrt{2}}(x_1 - x_2),$$

where in the second case we have to specify the solution space such that $x_1 + x_2$ and $x_1 - x_2$ is defined.

Corollary 6.2.2. *The colligation* $\Xi = \left(\left[\begin{smallmatrix} G_{\mathrm{p}} \\ L_{\mathrm{p}} \\ K_{\mathrm{p}} \end{smallmatrix}\right]; \left[\begin{smallmatrix} S_+\mathcal{V}_{L,\Gamma_1} \\ \mathcal{X}_{\mathcal{H}} \\ (S_+\mathcal{V}_{L,\Gamma_1})' \end{smallmatrix}\right] \right)$ *with solution space*

$$\mathcal{Z} = \mathcal{H}^{-1}\big(\mathsf{H}_{\Gamma_0}(L_\partial^{\mathsf{H}},\Omega) \times \mathsf{H}(L_\partial,\Omega)\big)$$

is a boundary control and observation system. Moreover, it is impedance energy preserving.

Proof. Since L_{p} is closed on $\mathcal{X}_{\mathcal{H}}$ with domain $\mathcal{Z}$, and G_{p} and K_{p} are continuous with the graph norm of L_{p}, we have $\left[\begin{smallmatrix} G_{\mathrm{p}} & L_{\mathrm{p}} & K_{\mathrm{p}} \end{smallmatrix}\right]^{\mathsf{T}}$ is closed. By construction G_{p}

with domain $\mathcal{Z}$ maps onto $S_+\mathcal{V}_{L,\Gamma_0}$. Since G_p is one operator of a boundary triple for L_p, the restriction $L_\mathrm{p}\big|_{\ker G_\mathrm{p}}$ is skew-adjoint and therefore a generator of a contraction semigroup.

We denote $\mathcal{U} = S_+\mathcal{V}_{L,\Gamma_1}$ and $\mathcal{Y} = \mathcal{U}' = (S_+\mathcal{V}_L\Gamma_1)'$ and we write Ψ for the duality map between $\mathcal{U}$ and $\mathcal{Y}$. Note that $(\mathcal{U}, G_\mathrm{p}, K_\mathrm{p})$ is a quasi Gelfand triple for L_p. Therefore,

$$\begin{aligned}
&2\,\mathrm{Re}\langle L_\mathrm{p}x, x\rangle_{\mathcal{X}_\mathcal{H}} \\
&\quad = 2\,\mathrm{Re}\langle G_\mathrm{p}x, K_\mathrm{p}x\rangle_{\mathcal{V}_L,\mathcal{V}_L'} = 2\,\mathrm{Re}\langle G_\mathrm{p}x, \Psi K_\mathrm{p}x\rangle_{\mathcal{V}_L} \\
&\quad = \frac{1}{2}\big(\langle G_\mathrm{p}x, G_\mathrm{p}x\rangle_\mathcal{U} + 2\,\mathrm{Re}\langle G_\mathrm{p}x, \Psi K_\mathrm{p}x\rangle_\mathcal{U} + \langle \Psi K_\mathrm{p}x, \Psi K_\mathrm{p}x\rangle_\mathcal{U}\big) \\
&\qquad - \frac{1}{2}\big(\langle G_\mathrm{p}x, G_\mathrm{p}x\rangle_\mathcal{U} - 2\,\mathrm{Re}\langle G_\mathrm{p}x, \Psi K_\mathrm{p}x\rangle_\mathcal{U} + \langle \Psi K_\mathrm{p}x, \Psi K_\mathrm{p}x\rangle_\mathcal{U}\big) \\
&\quad = \big\|\tfrac{1}{\sqrt{2}}(G_\mathrm{p} + \Psi K_\mathrm{p})x\big\|_\mathcal{U}^2 - \big\|\tfrac{1}{\sqrt{2}}(G_\mathrm{p} - \Psi K_\mathrm{p})x\big\|_\mathcal{U}^2,
\end{aligned}$$

which makes $\tilde{\Xi} = \left(\begin{bmatrix} \frac{1}{\sqrt{2}}(G_\mathrm{p}+\Psi K_\mathrm{p}) \\ L_\mathrm{p} \\ \frac{1}{\sqrt{2}}(G_\mathrm{p}-\Psi K_\mathrm{p}) \end{bmatrix}\right.$ a scattering energy preserving colligation. Thus, Ξ is impedance energy preserving ❑

Proposition 6.2.3. *Let $R \in \mathcal{L}_\mathrm{b}(S\mathsf{L}^2_\pi(\Gamma_1))$ be coercive. Then the colligation $\Xi = \left(\begin{bmatrix} \frac{1}{\sqrt{2}}(G_\mathrm{p}+RK_\mathrm{p}) \\ L_\mathrm{p} \\ \frac{1}{\sqrt{2}}(G_\mathrm{p}-RK_\mathrm{p}) \end{bmatrix}; \begin{bmatrix} \mathcal{U} \\ \mathcal{X}_\mathcal{H} \\ \mathcal{Y} \end{bmatrix}\right)$ with $\mathcal{U} = \mathcal{Y} = S\mathsf{L}^2_\pi(\Gamma_1)$ endowed with $\|f\|_\mathcal{U} = \|f\|_\mathcal{Y} = \|R^{-1/2}f\|_{\mathsf{L}^2}$ and solution space*

$$\mathcal{Z} = \{x \in \mathcal{H}^{-1}(\mathsf{H}_{\Gamma_0}(L_\partial^\mathsf{H}, \Omega) \times \mathsf{H}(L_\partial, \Omega)) \,|\, G_\mathrm{p}x, K_\mathrm{p}x \in S\mathsf{L}^2_\pi(\Gamma_1)\}.$$

is a scattering energy preserving boundary control and observation system

Proof. Let $(x_n, [G_\mathrm{p}x_n \quad L_\mathrm{p}x_n \quad K_\mathrm{p}x_n]^\mathsf{T})_{n\in\mathbb{N}}$ be a sequence in $[G_\mathrm{p} \quad L_\mathrm{p} \quad K_\mathrm{p}]^\mathsf{T}$ (restricted to $\mathcal{Z}$) that converges to $(x, [f \quad y \quad g]^\mathsf{T}) \in \mathcal{X}_\mathcal{H} \times \mathcal{U} \times \mathcal{X}_\mathcal{H} \times \mathcal{U}$. Since L_p with domain $\mathsf{H}(P_\partial, \Omega)$ is a closed operator and $\mathsf{H}_{\Gamma_0}(L_\partial^\mathsf{H}, \Omega) \times \mathsf{H}(L_\partial, \Omega)$ is closed in $\mathsf{H}(P_\partial, \Omega)$, we conclude that $x \in \mathcal{H}^{-1}(\mathsf{H}_{\Gamma_0}(L_\partial^\mathsf{H}, \Omega) \times \mathsf{H}(L_\partial, \Omega))$ and $y = L_\mathrm{p}x$. Hence, $G_\mathrm{p}x_n$ converges in $S_+\mathcal{V}_{L,\Gamma_1}$ to $G_\mathrm{p}x$ and in $S\mathsf{L}^2_\pi(\Gamma_1)$ to f. Since $(S_+\mathcal{V}_{L,\Gamma_1}, S\mathsf{L}^2_\pi(\Gamma_1), (S_+\mathcal{V}_{L,\Gamma_1})')$ is a quasi Gelfand triple, we have $G_\mathrm{p}x = f$. Analogously, we conclude $K_\mathrm{p}x = g$. Therefore, $x \in \mathcal{Z}$ and $[G_\mathrm{p} \quad L_\mathrm{p} \quad K_\mathrm{p}]^\mathsf{T}$ is closed, which implies that also $\big[\frac{1}{\sqrt{2}}(G_\mathrm{p} + RK_\mathrm{p}) \quad L_\mathrm{p} \quad \frac{1}{\sqrt{2}}(G_\mathrm{p} - RK_\mathrm{p})\big]^\mathsf{T}$ is closed.

By Example 4.4.8 and Theorem 4.4.6 $L_\mathrm{p}\big|_{\ker \frac{1}{\sqrt{2}}(G_\mathrm{p}+RK_\mathrm{p})}$ generates a contraction semigroup.

The surjectivity of $\left[\begin{smallmatrix} G_\mathrm{p} \\ K_\mathrm{p} \end{smallmatrix}\right]$ and Example 4.4.8 gives the surjectivity of $\frac{1}{\sqrt{2}}(G_\mathrm{p} + RK_\mathrm{p})$.

Since $(\mathcal{V}_L, G_\mathrm{p}, K_\mathrm{p})$ is a boundary triple for L_p, we have

$$\begin{aligned}
&2\operatorname{Re}\langle L_\mathrm{p}x, x\rangle_{\mathcal{X}_\mathcal{H}} \\
&\quad= 2\operatorname{Re}\langle G_\mathrm{p}x, K_\mathrm{p}x\rangle_{\mathcal{V}_L,\mathcal{V}_L'} = 2\operatorname{Re}\langle G_\mathrm{p}x, K_\mathrm{p}x\rangle_{\mathsf{L}^2_\pi(\Gamma_1)} \\
&\quad= \frac{1}{2}\big(\langle R^{-1}G_\mathrm{p}x, G_\mathrm{p}x\rangle_{\mathsf{L}^2} + 2\operatorname{Re}\langle G_\mathrm{p}x, K_\mathrm{p}x\rangle_{\mathsf{L}^2} + \langle RK_\mathrm{p}x, K_\mathrm{p}x\rangle_{\mathsf{L}^2}\big) \\
&\qquad - \frac{1}{2}\big(\langle R^{-1}G_\mathrm{p}x, G_\mathrm{p}x\rangle_{\mathsf{L}^2} - 2\operatorname{Re}\langle G_\mathrm{p}x, K_\mathrm{p}x\rangle_{\mathsf{L}^2} + \langle RK_\mathrm{p}x, K_\mathrm{p}x\rangle_{\mathsf{L}^2}\big) \\
&\quad= \big\|\tfrac{1}{\sqrt{2}}(G_\mathrm{p} + RK_\mathrm{p})x\big\|_{\mathcal{U}}^2 - \big\|\tfrac{1}{\sqrt{2}}(G_\mathrm{p} - RK_\mathrm{p})x\big\|_{\mathcal{Y}}^2,
\end{aligned}$$

which makes Ξ scattering energy preserving. ❑

Remark 6.2.4. Clearly, the previous proposition holds also true for the operator triple $\begin{bmatrix}\frac{1}{\sqrt{2}}(RK_\mathrm{p}+G_\mathrm{p}) & L_\mathrm{p} & \frac{1}{\sqrt{2}}(RK_\mathrm{p}-G_\mathrm{p})\end{bmatrix}^\mathsf{T}$ and for G_p and K_p being swapped. Moreover, replacing L_p by $L_\mathrm{p} + J$, where $J \in \mathcal{L}_\mathrm{b}(\mathcal{X}_\mathcal{H})$ is dissipative, yields a scattering passive system.

Hence, the port-Hamiltonian system with input u and output y described by the equations

$$\begin{aligned}
\sqrt{2}u(t,\zeta) &= \pi_L\big(\mathcal{H}(\zeta)x(t,\zeta)\big)_{L^\mathsf{H}} + RL_\nu\big(\mathcal{H}(\zeta)x(t,\zeta)\big)_L, && t\in\mathbb{R}_+, \zeta\in\Gamma_1,\\
\frac{\partial}{\partial t}x(t,\zeta) &= \sum_{i=1}^n \frac{\partial}{\partial\zeta_i}P_i\big(\mathcal{H}(\zeta)x(t,\zeta)\big) + P_0\big(\mathcal{H}(\zeta)x(t,\zeta)\big), && t\in\mathbb{R}_+, \zeta\in\Omega,\\
\sqrt{2}y(t,\zeta) &= \pi_L\big(\mathcal{H}(\zeta)x(t,\zeta)\big)_{L^\mathsf{H}} - RL_\nu\big(\mathcal{H}(\zeta)x(t,\zeta)\big)_L, && t\in\mathbb{R}_+, \zeta\in\Gamma_1,\\
0 &= \pi_L\big(\mathcal{H}(\zeta)x(t,\zeta)\big)_{L^\mathsf{H}}, && t\in\mathbb{R}_+, \zeta\in\Gamma_0,\\
x(0,\zeta) &= x_0(\zeta), && \zeta\in\Omega,
\end{aligned} \tag{6.3}$$

is scattering passive and in particular well-posed, as the following corollary will clarify. The mappings π_L and L_ν are used a little bit sloppy. There is always a pointwise a.e. description for these mappings, but due to compact notation we use π_L and L_ν.

Corollary 6.2.5. *The system* (6.4) *can be interpreted as the scattering energy preserving boundary control and observation system*

$$\left(\begin{bmatrix}\frac{1}{\sqrt{2}}(G_\mathrm{p}+RK_\mathrm{p})\\ L_\mathrm{p}\\ \frac{1}{\sqrt{2}}(G_\mathrm{p}-RK_\mathrm{p})\end{bmatrix}; \begin{bmatrix}\mathcal{U}\\ \mathcal{X}_\mathcal{H}\\ \mathcal{Y}\end{bmatrix}\right),$$

with the assumptions of Proposition 6.2.3 and $S = \mathrm{I}$*. Replacing* L_p *with* $L_\mathrm{p} + J$ *for a dissipative* $J \in \mathcal{L}_\mathrm{b}(\mathcal{X}_\mathcal{H})$ *yields a scattering passive boundary control and observation system.*

Corollary 6.2.6. *With the setting of Proposition 6.2.3 the colligation*

$$\left(\begin{bmatrix}G_\mathrm{p}\\ L_\mathrm{p}\\ K_\mathrm{p}\end{bmatrix}; \begin{bmatrix}S\mathsf{L}^2_\pi(\Gamma_1)\\ \mathcal{X}_\mathcal{H}\\ S\mathsf{L}^2_\pi(\Gamma_1)\end{bmatrix}\right)$$

with solution space

$$\mathcal{Z} = \{x \in \mathcal{H}^{-1}(\mathsf{H}_{\Gamma_0}(L_\partial^{\mathsf{H}}, \Omega) \times \mathsf{H}(L_\partial, \Omega)) \mid G_{\mathrm{p}}x, K_{\mathrm{p}}x \in S\mathsf{L}^2_\pi(\Gamma_1)\}$$

is impedance energy preserving.

Proof. This is a direct consequence of Proposition 6.2.3 for $R = \mathrm{I}$. ❑

Note that the colligations in Corollary 6.2.2 and Corollary 6.2.6 are the same but the solution spaces are slightly different. The colligation in Corollary 6.2.6 is in general not necessarily a boundary control and observation system.

Hence, the port-Hamiltonian system with input u and output y described by the equations

$$\begin{aligned}
u(t,\zeta) &= RL_\nu\big(\mathcal{H}(\zeta)x(t,\zeta)\big)_L, && t \in \mathbb{R}_+, \zeta \in \Gamma_1,\\
\frac{\partial}{\partial t}x(t,\zeta) &= \sum_{i=1}^n \frac{\partial}{\partial \zeta_i} P_i\big(\mathcal{H}(\zeta)x(t,\zeta)\big) + P_0\big(\mathcal{H}(\zeta)x(t,\zeta)\big), && t \in \mathbb{R}_+, \zeta \in \Omega,\\
y(t,\zeta) &= \pi_L\big(\mathcal{H}(\zeta)x(t,\zeta)\big)_{L^{\mathsf{H}}}, && t \in \mathbb{R}_+, \zeta \in \Gamma_1,\\
0 &= \pi_L\big(\mathcal{H}(\zeta)x(t,\zeta)\big)_{L^{\mathsf{H}}}, && t \in \mathbb{R}_+, \zeta \in \Gamma_0,\\
x(0,\zeta) &= x_0(\zeta), && \zeta \in \Omega,
\end{aligned} \tag{6.4}$$

is impedance energy preserving.

6.3 Wave Equation

This section can be seen as a continuation of Section 3.3. For convenience we recap the assumptions. Let $\Omega \subseteq \mathbb{R}^n$ be as in Assumption 3.1.1 and Γ_0, Γ_1 is a splitting with thin boundary of $\partial\Omega$ (Definition 5.1.1). Let $\rho \in \mathsf{L}^\infty(\Omega)$ be the mass density and $T \in \mathsf{L}^\infty(\Omega)^{n\times n}$ be the Young modulus, such that $\frac{1}{\rho} \in \mathsf{L}^\infty(\Omega)$, $T(\zeta)^{\mathsf{H}} = T(\zeta)$ and $T(\zeta) \geq \delta\mathrm{I}$ for a $\delta > 0$ and almost every $\zeta \in \Omega$.

In Section 3.3 we have already seen, that we can rewrite the wave equation as a port-Hamiltonian system. The wave equation

$$\frac{\partial^2}{\partial t^2} w(t,\xi) = \frac{1}{\rho(\xi)} \operatorname{div}\big(T(\xi)\operatorname{grad} w(t,\xi)\big),$$

can be formulated as a port-Hamiltonian system by choosing the state variable $x(t,\zeta) = \begin{bmatrix} x_1(t,\zeta) \\ x_2(t,\zeta)\end{bmatrix} = \begin{bmatrix} \rho\frac{\partial}{\partial t}w(t,\zeta) \\ \operatorname{grad} w(t,\zeta)\end{bmatrix}$. Then the PDE looks like

$$\dot{x} = \underbrace{\begin{bmatrix} 0 & \operatorname{div} \\ \operatorname{grad} & 0 \end{bmatrix}}_{=P_\partial} \underbrace{\begin{bmatrix} \frac{1}{\rho} & 0 \\ 0 & T \end{bmatrix}}_{=\mathcal{H}} x.$$

This is exactly the port-Hamiltonian system we get from choosing L as in Example 3.1.3. From Example 5.1.2 and Example 5.1.7 we know that the boundary operators are γ_0 and the extension $\bar{L}_\nu$ of $L_\nu\gamma_0 = \nu\cdot\gamma_0$. We will denote the normal trace $\bar{L}_\nu$ by γ_ν. Therefore,

$$\begin{aligned}
\sqrt{2}u(t,\zeta) &= \nu(\zeta)\cdot\big(T(\zeta)\operatorname{grad} w(t,\zeta)\big) + \frac{\partial}{\partial t}w(t,\zeta), && t\in\mathbb{R}_+, \zeta\in\Gamma_1,\\
\frac{\partial^2}{\partial t^2}w(t,\xi) &= \frac{1}{\rho(\xi)}\operatorname{div}\big(T(\xi)\operatorname{grad} w(t,\xi)\big), && t\in\mathbb{R}_+, \zeta\in\Omega,\\
\sqrt{2}y(t,\zeta) &= \nu(\zeta)\cdot\big(T(\zeta)\operatorname{grad} w(t,\zeta)\big) - \frac{\partial}{\partial t}w(t,\zeta), && t\in\mathbb{R}_+, \zeta\in\Gamma_1,\\
0 &= \frac{\partial}{\partial t}w(t,\zeta), && t\in\mathbb{R}_+, \zeta\in\Gamma_0,\\
w(0,\zeta) &= w_0(\zeta), && \zeta\in\Omega,\\
\frac{\partial}{\partial t}w(0,\zeta) &= w_1(\zeta), && \zeta\in\Omega,
\end{aligned}$$

can be modeled by a scattering passive and well-posed boundary control system, by Corollary 6.2.5. In the port-Hamiltonian formulation this system is described by

$$\begin{aligned}
u(t) &= \frac{1}{\sqrt{2}}\Big(\gamma_\nu T x_2(t) + \gamma_0 \tfrac{1}{\rho} x_1(t)\Big), && t\in\mathbb{R}_+,\\
\dot{x}(t) &= \begin{bmatrix} 0 & \operatorname{div}\\ \operatorname{grad} & 0\end{bmatrix}\mathcal{H}x(t), && t\in\mathbb{R}_+,\\
y(t) &= \frac{1}{\sqrt{2}}\Big(\gamma_\nu T x_2(t) + \gamma_0 \tfrac{1}{\rho} x_1(t)\Big), && t\in\mathbb{R}_+,\\
x(0) &= x_0,
\end{aligned}$$

where we choose the solution space as

$$\mathcal{Z} = \{x\in\mathcal{H}^{-1}(\mathsf{H}^1_{\Gamma_0}(\Omega)\times\mathsf{H}(\operatorname{div},\Omega))\,|\,\gamma_0 x_1, \gamma_\nu T x_2 \in \mathsf{L}^2(\Gamma_1)\}. \tag{6.5}$$

Moreover, with different input and output operators we have by Corollary 6.2.6 the impedance passive boundary control system

$$\begin{aligned}
u(t,\zeta) &= \nu(\zeta)\cdot\big(T(\zeta)\operatorname{grad} w(t,\zeta)\big), && t\in\mathbb{R}_+, \zeta\in\Gamma_1,\\
\frac{\partial^2}{\partial t^2}w(t,\xi) &= \frac{1}{\rho(\xi)}\operatorname{div}\big(T(\xi)\operatorname{grad} w(t,\xi)\big), && t\in\mathbb{R}_+, \zeta\in\Omega,\\
y(t,\zeta) &= \frac{\partial}{\partial t}w(t,\zeta), && t\in\mathbb{R}_+, \zeta\in\Gamma_1,\\
0 &= \frac{\partial}{\partial t}w(t,\zeta), && t\in\mathbb{R}_+, \zeta\in\Gamma_0.
\end{aligned}$$

Again in the port-Hamiltonian formulation this system is described by

$$\begin{aligned} u(t) &= \gamma_\nu T x_2(t), && t \in \mathbb{R}_+, \\ \dot{x}(t) &= \begin{bmatrix} 0 & \operatorname{div} \\ \operatorname{grad} & 0 \end{bmatrix} \mathcal{H} x(t), && t \in \mathbb{R}_+, \\ y(t) &= \gamma_0 \tfrac{1}{\rho} x_1(t), && t \in \mathbb{R}_+, \\ x(0) &= x_0 \end{aligned}$$

with solution space either (6.5) or

$$\mathcal{Z} = \mathcal{H}^{-1}(\mathsf{H}^1_{\Gamma_0}(\Omega) \times \mathsf{H}(\operatorname{div}, \Omega))$$

depending of whether we use Corollary 6.2.6 or Corollary 6.2.2. Thus there are two ways of describing the wave equation as impedance port-Hamiltonian boundary control and observation system.

In any case (scattering energy preserving or impedance energy preserving) $\left[\begin{smallmatrix} 0 & \operatorname{div} \\ \operatorname{grad} & 0 \end{smallmatrix}\right]\mathcal{H}$ with boundary condition $\gamma_0 \frac{1}{\rho} x_1 + \gamma_\nu T x_2 = 0$, $\gamma_\nu T x_2$ or $\gamma_0 \frac{1}{\rho} x_1$ generates a contraction semigroup.

6.4 Maxwell's Equations

This section is continuation of Section 3.4. However, we will recall the most important things. Let $\Omega \subseteq \mathbb{R}^3$ be as in Assumption 3.1.1 and $L = (L_i)_{i=1}^3$ be as in Example 3.1.4. In this example we have already showed $L_\partial = \operatorname{rot}$ and $L_\nu f = \nu \times f$. The corresponding differential operator for the port-Hamiltonian PDE is

$$P_\partial = \begin{bmatrix} 0 & L_\partial \\ L_\partial^{\mathsf{H}} & 0 \end{bmatrix} = \begin{bmatrix} 0 & \operatorname{rot} \\ -\operatorname{rot} & 0 \end{bmatrix}.$$

We write the state as $x = \left[\begin{smallmatrix} \mathbf{D} \\ \mathbf{B} \end{smallmatrix}\right]$, where $\mathbf{D}, \mathbf{B} \in \mathbb{K}^3$. We also want to introduce the positive scalar functions ϵ, μ, g and r such that

$$\epsilon, \frac{1}{\epsilon}, \mu, \frac{1}{\mu}, g \in \mathsf{L}^\infty(\Omega) \quad \text{and} \quad r, \frac{1}{r} \in \mathsf{L}^\infty(\Gamma_1).$$

Furthermore, we define the Hamiltonian density by $\mathcal{H}(\zeta) := \begin{bmatrix} \frac{1}{\epsilon(\zeta)} & 0 \\ 0 & \frac{1}{\mu(\zeta)} \end{bmatrix}$, where each block is a 3×3 matrix. At last we define $\left[\begin{smallmatrix} \mathbf{E} \\ \mathbf{H} \end{smallmatrix}\right] := \mathcal{H}\left[\begin{smallmatrix} \mathbf{D} \\ \mathbf{B} \end{smallmatrix}\right]$, so that we have the same notation as in [64].

The projection on $\overline{\operatorname{ran} L_\nu}$ is given by $g \mapsto (\nu \times g) \times \nu$, therefore $\bar{\pi}_L$ is the extension of $g \mapsto (\nu \times \gamma_0 g) \times \nu$ to $\mathsf{H}(L_\partial^{\mathsf{H}}, \Omega)$. The mapping π_τ from [64] can be compared with $\bar{\pi}_L$ but is not exactly the same, since they have different domains and codomains. We have $\pi_\tau \colon H^1(\Omega)^3 \to V_\tau \subseteq \mathsf{L}^2(\partial\Omega)^3$ and $\bar{\pi}_L \colon H(\operatorname{rot}, \Omega) \to \mathcal{V}_L$ is its extension, if we change the norms in the domain and codomain of π_τ. However, $\mathcal{V}_L$ cannot be embedded into $\mathsf{L}^2(\partial\Omega)^3$.

For this particular L we denote $\bar{L}_\nu$ by $\gamma_{\tau_\times}$ and $\bar{\pi}_L$ by γ_τ.

Note that by Example 5.1.8 neither γ_τ nor $\gamma_{\tau_\times}^{\Gamma_1}$ map even into $\mathsf{L}^2_\pi(\Gamma_1)$, therefore it is really necessary to use a quasi Gelfand triple instead of an "ordinary" Gelfand triple.

The corresponding boundary control system is a model for Maxwell's equations in the following form

$$\begin{aligned}
\sqrt{2}u(t,\zeta) &= r(\zeta)\nu(\zeta)\times\mathbf{H}(t,\zeta) + (\nu(\zeta)\times\mathbf{E}(t,\zeta))\times\nu(\zeta), && t\in\mathbb{R}_+,\zeta\in\Gamma_1,\\
\frac{\partial}{\partial t}\mathbf{D}(t,\zeta) &= \operatorname{rot}\mathbf{H}(t,\zeta) - g(\zeta)\mathbf{E}(t,\zeta), && t\in\mathbb{R}_+,\zeta\in\Omega,\\
\frac{\partial}{\partial t}\mathbf{B}(t,\zeta) &= -\operatorname{rot}\mathbf{E}(t,\zeta), && t\in\mathbb{R}_+,\zeta\in\Omega,\\
\sqrt{2}y(t,\zeta) &= r(\zeta)\nu(\zeta)\times\mathbf{H}(t,\zeta) - (\nu(\zeta)\times\mathbf{E}(t,\zeta))\times\nu(\zeta), && t\in\mathbb{R}_+,\zeta\in\Gamma_1,\\
0 &= (\nu(\zeta)\times\mathbf{E}(t,\zeta))\times\nu(\zeta), && t\in\mathbb{R}_+,\zeta\in\Gamma_0,\\
\mathbf{D}(0,\zeta) &= \mathbf{D}_0(\zeta) && \zeta\in\Omega,\\
\mathbf{B}(0,\zeta) &= \mathbf{B}_0(\zeta) && \zeta\in\Omega,
\end{aligned}$$

and is scattering passive by Corollary 6.2.5, where we set $J = \left[\begin{smallmatrix}-g & 0\\ 0 & 0\end{smallmatrix}\right]\mathcal{H}$.

Note that, following the trick in [64, Proposition 6.1], Gauß's law $\operatorname{div}\mathbf{D} = \rho$ is satisfied by simply defining ρ by this formula and Gauß's law for magnetism $\operatorname{div}\mathbf{B} = 0$ is automatically satisfied, if the initial condition satisfies it. This can be seen, if we apply div on both sides of $\frac{\partial}{\partial t}\mu\mathbf{H} = -\operatorname{rot}\mathbf{E}$ and noting that $\operatorname{div}\mu\mathbf{H} = \operatorname{div}\mathbf{B}$ is constant in time ($\operatorname{div}\operatorname{rot} = 0$). This has to be understood in the sense of distributions. However, for classical solutions this can also be understood in the classical sense. We will explain this in more detail in Section 8.5.3, where we separate the static solutions from the dynamic solutions.

In the port-Hamiltonian formulation this system looks like

$$\begin{aligned}
u(t) &= \tfrac{1}{\sqrt{2}}\big(r\gamma_{\tau_\times}\tfrac{1}{\epsilon}\mathbf{B} + \gamma_\tau\tfrac{1}{\mu}\mathbf{D}\big),\\
\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix}\mathbf{D}(t)\\ \mathbf{B}(t)\end{bmatrix} &= \begin{bmatrix}0 & \operatorname{rot}\\ -\operatorname{rot} & 0\end{bmatrix}\begin{bmatrix}\frac{1}{\epsilon} & 0\\ 0 & \frac{1}{\mu}\end{bmatrix}\begin{bmatrix}\mathbf{D}(t)\\ \mathbf{B}(t)\end{bmatrix} + \begin{bmatrix}-g & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}\mathbf{D}(t)\\ \mathbf{B}(t)\end{bmatrix},\\
y(t) &= \tfrac{1}{\sqrt{2}}\big(r\gamma_{\tau_\times}\tfrac{1}{\epsilon}\mathbf{B} - \gamma_\tau\tfrac{1}{\mu}\mathbf{D}\big),\\
\begin{bmatrix}\mathbf{D}(0)\\ \mathbf{B}(0)\end{bmatrix} &= \begin{bmatrix}\mathbf{D}_0\\ \mathbf{B}_0\end{bmatrix},
\end{aligned}$$

with solution space

$$\mathcal{Z} = \left\{\begin{bmatrix}\mathbf{D}(t)\\ \mathbf{B}(t)\end{bmatrix} \in \epsilon\mathsf{H}_{\Gamma_0}(\operatorname{rot},\Omega)\times\mu\mathsf{H}(\operatorname{rot},\Omega)\,\middle|\,\gamma_{\tau_\times}\tfrac{1}{\epsilon}\mathbf{B}, \gamma_\tau\tfrac{1}{\mu}\mathbf{D}\in\mathsf{L}^2(\Gamma_1)\right\}.$$

6.5 Mindlin Plate Model

This section is a continuation of Section 3.5. Nevertheless, we will recall the setting. Let $\Omega \subseteq \mathbb{R}^2$ be as in Assumption 3.1.1. Let us consider the differential operator P_∂ and the skew-symmetric matrix P_0 given by

$$P_\partial := \left[\begin{array}{ccc|ccccc} 0 & 0 & 0 & 0 & 0 & 0 & \partial_1 & \partial_2 \\ 0 & 0 & 0 & \partial_1 & 0 & \partial_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & \partial_2 & \partial_1 & 0 & 0 \\ \hline 0 & \partial_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \partial_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & \partial_2 & \partial_1 & 0 & 0 & 0 & 0 & 0 \\ \partial_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \partial_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right], P_0 := \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

It is easy to derive the corresponding $P = (P_i)_{i=1}^2$ and $L = (L_i)_{i=1}^2$. We define a Hamiltonian density by

$$\mathcal{H} = \begin{bmatrix} \frac{1}{\rho h} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{12}{\rho h^3} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{12}{\rho h^3} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & & & & 0 & 0 \\ 0 & 0 & 0 & & \boldsymbol{D}_b & & 0 & 0 \\ 0 & 0 & 0 & & & & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \multicolumn{2}{c}{\multirow{2}{*}{$\boldsymbol{D}_s$}} \\ 0 & 0 & 0 & 0 & 0 & 0 & & \end{bmatrix},$$

where ρ, h are strictly positive functions, $\boldsymbol{D}_b(\zeta)$ is a coercive 3×3 matrix and $\boldsymbol{D}_s(\zeta)$ is a coercive 2×2 matrix, such that all conditions on $\mathcal{H}$ in Definition 3.2.1 are satisfied. We have written (in Section 3.5) the state variable x as

$$\boldsymbol{\alpha} := \begin{bmatrix} \rho h v & \rho \frac{h^3}{12} w_1 & \rho \frac{h^3}{12} w_2 & \kappa_{1,1} & \kappa_{2,2} & \kappa_{1,2} & \gamma_{1,3} & \gamma_{2,3} \end{bmatrix}^\mathsf{T}$$

and defined

$$\mathbf{e} := \mathcal{H}\boldsymbol{\alpha} = \begin{bmatrix} v & w_1 & w_2 & M_{1,1} & M_{2,2} & M_{1,2} & Q_1 & Q_2 \end{bmatrix}^\mathsf{T}.$$

Thus, we can write the port-Hamiltonian PDE

$$\frac{\partial}{\partial t} x = (P_\partial + P_0)\mathcal{H}x \quad \text{as} \quad \frac{\partial}{\partial t}\boldsymbol{\alpha} = (P_\partial + P_0)\mathbf{e}.$$

The corresponding boundary operator is

$$L_\nu f = \begin{bmatrix} 0 & 0 & 0 & \nu_1 & \nu_2 \\ \nu_1 & 0 & \nu_2 & 0 & 0 \\ 0 & \nu_2 & \nu_1 & 0 & 0 \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \end{bmatrix} = \begin{bmatrix} \nu \cdot \begin{bmatrix} f_4 \\ f_5 \end{bmatrix} \\ \nu \cdot \begin{bmatrix} f_1 \\ f_3 \end{bmatrix} \\ \nu \cdot \begin{bmatrix} f_3 \\ f_2 \end{bmatrix} \end{bmatrix}.$$

Since $\|\nu(\zeta)\| = 1$, at least $\nu_1(\zeta) \neq 0$ or $\nu_2(\zeta) \neq 0$. This can be used to show that $\operatorname{ran} L_\nu = \mathsf{L}^2(\partial\Omega)^3$. Therefore, $\bar{\pi}_L$ is the extension of the boundary trace operator γ_0 to $\mathsf{H}(L_\partial^{\mathsf{H}}, \Omega)$.

Since there is no direct physical meaning to the boundary variables

$$\begin{bmatrix} 0 & L_\nu \end{bmatrix} \mathbf{e} = \begin{bmatrix} \nu \cdot \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} \\ \nu \cdot \begin{bmatrix} M_{1,1} \\ M_{1,2} \end{bmatrix} \\ \nu \cdot \begin{bmatrix} M_{1,2} \\ M_{2,2} \end{bmatrix} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \pi_L & 0 \end{bmatrix} \mathbf{e} = \begin{bmatrix} v \\ w_1 \\ w_2 \end{bmatrix},$$

we define $\eta := \left[\begin{smallmatrix} -\nu_2 \\ \nu_1 \end{smallmatrix}\right]$ and apply the unitary transformation $S = \left[\begin{smallmatrix} 1 & 0 & 0 \\ 0 & \nu_1 & \nu_2 \\ 0 & -\nu_2 & \nu_1 \end{smallmatrix}\right]$ to obtain

$$\begin{bmatrix} Q_\nu \\ M_{\nu,\nu} \\ M_{\nu,\eta} \end{bmatrix} := S \begin{bmatrix} \nu \cdot \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} \\ \nu \cdot \begin{bmatrix} M_{1,1} \\ M_{1,2} \end{bmatrix} \\ \nu \cdot \begin{bmatrix} M_{1,2} \\ M_{2,2} \end{bmatrix} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} v \\ w_\nu \\ w_\eta \end{bmatrix} := \underbrace{(S^*)^{-1}}_{=S} \begin{bmatrix} v \\ w_1 \\ w_2 \end{bmatrix},$$

which have a physical interpretation; see [8]. Hence, by Corollary 6.2.6 the system

$$\begin{aligned} u &= \begin{bmatrix} Q_\nu & M_{\nu,\nu} & M_{\nu,\eta} \end{bmatrix}^{\mathsf{T}}, && \text{on } \mathbb{R}_+ \times \Gamma_1, \\ \frac{\partial}{\partial t}\boldsymbol{\alpha} &= (P_\partial + P_0)\mathbf{e}, && \text{on } \mathbb{R}_+ \times \Omega, \\ y &= \begin{bmatrix} v & w_\nu & w_\eta \end{bmatrix}^{\mathsf{T}}, && \text{on } \mathbb{R}_+ \times \Gamma_1, \\ 0 &= \begin{bmatrix} v & w_\nu & w_\eta \end{bmatrix}^{\mathsf{T}}, && \text{on } \mathbb{R}_+ \times \Gamma_0, \end{aligned}$$

for the Mindlin plate is impedance energy preserving, which is exactly the system in [8].

Chapter 7

Stabilization of the Wave Equation

We investigate the stability of the wave equation with spatial dependent coefficients on a bounded and connected multidimensional domain. The system is stabilized via a scattering passive feedback law. We show that the system is semi-uniform stable, which is a stability concept between exponential stability and strong stability. Hence, this also implies strong stability of the system. In particular, classical solutions are uniformly stable. This will be achieved by showing that the spectrum of the port-Hamiltonian operator is contained in the left half plane $\mathbb{C}_-$ and the port-Hamiltonian operator generates a contraction semigroup. Moreover, we show that the spectrum consists of eigenvalues only and the port-Hamiltonian operator has a compact resolvent.

This chapter is the result of joint work with Birgit Jacob [24].

7.1 Introduction

Recall the setting of the wave equation in Section 3.3. We had the Young's elasticity modulus $T\colon \Omega \to \mathbb{C}^{n\times n}$, which is a Lipschitz continuous matrix-valued function such that $T(\zeta)$ is a positive and invertible matrix (a.e.) and $T, T^{-1} \in \mathsf{L}^\infty(\Omega)^{n\times n}$. Moreover, we had the Lipschitz continuous mass density $\rho\colon \Omega \to \mathbb{R}_+$, that satisfies $\rho, \rho^{-1} \in \mathsf{L}^\infty(\Omega)$. In this chapter we investigate stability of following boundary control system

$$
\begin{aligned}
u(t,\zeta) &= \frac{\partial w}{\partial T\nu}(t,\zeta), && t \geq 0, \zeta \in \Gamma_1, \\
\frac{\partial^2 w}{\partial t^2}(t,\zeta) &= \frac{1}{\rho(\zeta)} \operatorname{div}\left(T(\zeta)\operatorname{grad} w(t,\zeta)\right), && t \geq 0, \zeta \in \Omega, \\
w(t,\zeta) &= h(\zeta), && t \geq 0, \zeta \in \Gamma_0, \\
w(0,\zeta) &= w_0(\zeta), && \zeta \in \Omega, \\
\frac{\partial w}{\partial t}(0,\zeta) &= w_1(\zeta), && \zeta \in \Omega, \\
y(t,\zeta) &= \frac{\partial w}{\partial t}(t,\zeta), && t \geq 0, \zeta \in \Gamma_1,
\end{aligned}
\tag{7.1a}
$$

with feedback law

$$
u(t,\zeta) = -k(\zeta) y(t,\zeta), \qquad t \geq 0, \zeta \in \Gamma_1, \tag{7.1b}
$$

where u and y are the boundary control and observation, respectively and $\Omega \subseteq \mathbb{R}^n$ is a bounded and connected domain with Lipschitz boundary $\partial\Omega = \overline{\Gamma_0} \cup \overline{\Gamma_1}$ with $\Gamma_0 \cap \Gamma_1 = \emptyset$, Γ_0 and Γ_1 are open in the relative topology of $\partial\Omega$ and the boundaries of Γ_0 and Γ_1 have surface measure zero. Note, that Γ_0 and Γ_1 do not have to be connected. Furthermore, $w(\zeta, t)$ is the deflection at point $\zeta \in \Omega$ and $t \geq 0$, and a profile h is given on Γ_0, where the deflection is fixed. The vector ν denotes the outward normal at the boundary and $\frac{\partial}{\partial T\nu} w(t,\zeta) = T(\zeta)\nu(\zeta) \cdot \operatorname{grad} w(t,\zeta) = \nu(\zeta) \cdot T(\zeta) \operatorname{grad} w(t,\zeta)$ is the conormal derivative. Further, $k \colon \Gamma_1 \to \mathbb{R}$ is a measurable positive and bounded function such that also its pointwise inverse $k^{-1} = \frac{1}{k}$ is bounded. Finally, w_0 and w_1 are the initial conditions.

Stability of (7.1) has been studied in the literature by several authors, see e.g. [3, 23, 29, 52]. Strong stability has been investigated in [52]. Further, exponential stability of the wave equation with constant T and ρ has been shown in [29] using multiplier methods. For smooth domains, in [3] the equivalence of exponential stability and the so-called *geometric control condition* was shown by methods from micro-local analysis. In [23] this system also appears in port-Hamiltonian formulation, but with constant T and ρ and C^2 boundary. Under these restrictions it could be shown that this system is even exponential stable. However, semi-uniform stability, a notion which is stronger than strong stability and weaker than exponential stability, of the multidimensional wave equation with spatial dependent functions ρ and T on quite general domains has not been studied in the literature.

We aim to show semi-uniform stability of the multidimensional wave equation (7.1) using a port-Hamiltonian formulation. Semi-uniform stability implies strong stability, and thus we extend the results obtained in [52]. To prove our main result we use the fact that semi-uniform stability is satisfied if the port-Hamiltonian operator generates a contraction semigroup and possesses no spectrum in the closed right half plane.

We proceed as follows. In Section 7.2 we model the multidimensional wave equation as a port-Hamiltonian system with a suitable state space. The main results concerning stability are then obtained in Section 7.3, where we analyze the spectrum of the differential operator of the port-Hamiltonian formulation. We will see that finding points in the resolvent set is linked to solvability of lossy Helmholtz equations. We will show that our operator has a compact resolvent and its resolvent set contains the imaginary axis. At that point we can apply existing theory to justify semi-uniform stability. Finally, used notations and results on Sobolev spaces and Gårdings inequalities are presented in the Appendix.

7.2 Port-Hamiltonian Formulation of the System

In order to find a port-Hamiltonian formulation of our system, that is suitable for our purpose, we split the system (7.1) into a time independent system for the equilibrium and a dynamical system with homogeneous boundary conditions. The time static system for the equilibrium is given by

$$\begin{aligned} \operatorname{div} T(\zeta) \operatorname{grad} w_{\mathrm{e}}(\zeta) &= 0, & \zeta &\in \Omega, \\ w_{\mathrm{e}}(\zeta) &= h(\zeta), & \zeta &\in \Gamma_0, \\ \frac{\partial w_{\mathrm{e}}}{\partial T\nu}(\zeta) &= 0, & \zeta &\in \Gamma_1, \end{aligned} \tag{7.2}$$

and a dynamical system with homogeneous Dirichlet boundary conditions on Γ_0 is given by

$$\begin{aligned} \frac{\partial^2 w_{\mathrm{d}}}{\partial t^2}(t,\zeta) &= \frac{1}{\rho(\zeta)} \operatorname{div}(T(\zeta) \operatorname{grad} w_{\mathrm{d}}(t,\zeta)), & t &\geq 0, \zeta \in \Omega, \\ w_{\mathrm{d}}(t,\zeta) &= 0, & t &\geq 0, \zeta \in \Gamma_0, \\ w_{\mathrm{d}}(0,\zeta) &= w_0(\zeta) - w_{\mathrm{e}}(\zeta), & \zeta &\in \Omega, \\ \frac{\partial w_{\mathrm{d}}}{\partial t}(0,\zeta) &= w_1(\zeta), & \zeta &\in \Omega, \\ \frac{\partial w_{\mathrm{d}}}{\partial T\nu}(t,\zeta) &= -k\frac{\partial w_{\mathrm{d}}}{\partial t}(t,\zeta), & t &\geq 0, \zeta \in \Gamma_1. \end{aligned} \tag{7.3}$$

The original system is solved by $w(t,\zeta) = w_{\mathrm{e}}(t,\zeta) + w_{\mathrm{d}}(\zeta)$. As in Section 3.3 (and in [28]) the system in (7.3) can be described in a port-Hamiltonian manner by choosing the state $x(t,\zeta) = \begin{bmatrix} \rho(\zeta)\frac{\partial}{\partial t} w_{\mathrm{d}}(t,\zeta) \\ \operatorname{grad} w_{\mathrm{d}}(t,\zeta) \end{bmatrix}$. By using the convention $x(t) :=$

$x(t,\cdot)$ we can write the system (7.3) as

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}x(t) &= \begin{bmatrix} 0 & \operatorname{div} \\ \operatorname{grad} & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\rho} & 0 \\ 0 & T \end{bmatrix} x(t), \\
x(0) &= \begin{bmatrix} \rho w_1 \\ \operatorname{grad}(w_0 - w_{\mathrm{e}}) \end{bmatrix}, \\
\gamma_0 \tfrac{1}{\rho} x_1(t)\big|_{\Gamma_0} &= 0, \\
\gamma_\nu T x_2(t)\big|_{\Gamma_1} &= -k\gamma_0 \tfrac{1}{\rho} x_1(t)\big|_{\Gamma_1},
\end{aligned}$$

where γ_0 is the boundary trace and γ_ν is the normal trace (the extension of $f \mapsto \nu \cdot \gamma_0 f$). In Section 3.3 we chose the state space $\mathsf{L}^2(\Omega)^{n+1}$ equipped with the energy inner product

$$\langle x, y \rangle := \left\langle x, \left[\begin{smallmatrix} \frac{1}{\rho} & 0 \\ 0 & T \end{smallmatrix}\right] y \right\rangle_{\mathsf{L}^2(\Omega)^{n+1}},$$

which is equivalent to the standard inner product of $\mathsf{L}^2(\Omega)^{n+1}$ thanks to the assumptions on T and ρ. For well-posedness this is a suitable state space, but when it comes to stability this state space is too large as it does not reflect the fact that the second component of the state variable x_2 is of the form $\operatorname{grad} v$, for some function v in the Sobolev space $\mathsf{H}^1_{\Gamma_0}(\Omega)$. Thus, we choose the state space $\mathcal{X}_{\mathcal{H}}$ as $\mathsf{L}^2(\Omega) \times \operatorname{grad} \mathsf{H}^1_{\Gamma_0}(\Omega)$, instead of $\mathsf{L}^2(\Omega)^{n+1}$. Note that $\operatorname{grad} \mathsf{H}^1_{\Gamma_0}(\Omega)$ is closed in $\mathsf{L}^2(\Omega)^n$ by Poincaré's inequality. Hence, $\mathcal{X}_{\mathcal{H}}$ is also a Hilbert space with the L^2-inner product. Nevertheless, we also use the energy inner product on $\mathcal{X}_{\mathcal{H}}$, that is

$$\langle x, y \rangle_{\mathcal{X}_{\mathcal{H}}} := \left\langle x, \left[\begin{smallmatrix} \frac{1}{\rho} & 0 \\ 0 & T \end{smallmatrix}\right] y \right\rangle_{\mathsf{L}^2(\Omega)^{n+1}}.$$

Furthermore, we define

$$\begin{aligned}
\mathfrak{A} &:= \left[\begin{smallmatrix} 0 & \operatorname{div} \\ \operatorname{grad} & 0 \end{smallmatrix}\right] \left[\begin{smallmatrix} \frac{1}{\rho} & 0 \\ 0 & T \end{smallmatrix}\right] \\
\text{with} \quad \operatorname{dom}(\mathfrak{A}) &:= \left[\begin{smallmatrix} \frac{1}{\rho} & 0 \\ 0 & T \end{smallmatrix}\right]^{-1} \big(\mathsf{H}^1_{\Gamma_0}(\Omega) \times \mathsf{H}(\operatorname{div}, \Omega)\big)
\end{aligned}$$

as densely defined operator on $\mathsf{L}^2(\Omega)^{n+1}$. The definition of $\mathsf{H}(\operatorname{div}, \Omega)\big)$ is given in the appendix. Note that we have already packed the boundary condition $\gamma_0 \frac{1}{\rho} x_1 = 0$ on Γ_0 into the domain of $\mathfrak{A}$. Moreover, by construction $\operatorname{ran} \mathfrak{A} = \mathcal{X}_{\mathcal{H}}$. Taking the state space and the remaining boundary condition (feedback) into account gives

$$\begin{aligned}
A &:= \mathfrak{A}\big|_{\operatorname{dom}(A)}, \\
\text{where} \quad \operatorname{dom}(A) &:= \left\{ x \in \operatorname{dom}(\mathfrak{A}) \;\middle|\; \gamma_\nu T x_2\big|_{\Gamma_1} = -k\gamma_0 \tfrac{1}{\rho} x_1\big|_{\Gamma_1} \right\} \cap \mathcal{X}_{\mathcal{H}}
\end{aligned} \tag{7.4}$$

as an operator on $\mathcal{X}_{\mathcal{H}}$. Note that $\operatorname{ran} A \subseteq \operatorname{ran} \mathfrak{A} = \mathcal{X}_{\mathcal{H}}$. Therefore the operator A indeed maps into $\mathcal{X}_{\mathcal{H}}$.

The corresponding operator on $\mathsf{L}^2(\Omega)^{n+1}$ would be

$$\begin{aligned} A_0 &:= \mathfrak{A}\big|_{\mathrm{dom}(A_0)}, \\ \text{where} \quad \mathrm{dom}(A_0) &:= \left\{ x \in \mathrm{dom}(\mathfrak{A}) \;\middle|\; \gamma_\nu T x_2\big|_{\Gamma_1} = -k\gamma_0 \tfrac{1}{\rho} x_1\big|_{\Gamma_1} \right\}. \end{aligned} \tag{7.5}$$

By Section 6.3, A_0 generates a contraction semigroup on $\mathsf{L}^2(\Omega)^{n+1}$ endowed with $\langle x, y\rangle := \left\langle x, \left[\begin{smallmatrix} \frac{1}{\rho} & 0 \\ 0 & T \end{smallmatrix}\right] y \right\rangle_{\mathsf{L}^2}$. Note that this operator allows elements in its domain which do not respect that the second component is a gradient field. This can lead to solutions that are not related to the original problem anymore, as by construction of the state $x(t,\zeta)$ the second component is $\operatorname{grad} w_{\mathrm{d}}(t,\zeta)$ and therefore a gradient field. Lemma 7.3.15 shows that this is problematic for stability.

We do not need to rebuild the semigroup theory in [28] for the "new" state space $\mathcal{X}_{\mathcal{H}}$. We will see that A inherits most of the properties of A_0 as $A = A_0\big|_{\mathcal{X}_{\mathcal{H}}}$.

Lemma 7.2.1. *Let $(T(t))_{t\geq 0}$ be a strongly continuous semigroup on a Hilbert space X and $\tilde{A}$ its generator. Then every subspace $V \supseteq \operatorname{ran} \tilde{A}$ is invariant under $(T(t))_{t\geq 0}$.*

Moreover, $\tilde{A}\big|_V$ generates the strongly continuous semigroup

$$(T_V(t))_{t\geq 0} := (T(t)\big|_V)_{t\geq 0},$$

if V is additionally closed.

Proof. Let $t \geq 0$ and $x \in V$. Then it is well-known that

$$\underbrace{\tilde{A}\int_0^t T(s)x \,\mathrm{d}s}_{\in \operatorname{ran} \tilde{A} \subseteq V} = T(t)x - \underbrace{x}_{\in V}.$$

Hence, $T(t)x \in V$, because the left-hand-side is in $\operatorname{ran} \tilde{A} \subseteq V$ and V is a subspace. The remaining assertion follows from [15, ch. II sec. 2.3]. ❑

Remark 7.2.2. If the strongly continuous semigroup $(T(t))_{t\geq 0}$ is even a contraction semigroup, then also $(T_V(t))_{t\geq 0}$ is a contraction semigroup.

Proposition 7.2.3. *The operator A given by* (7.4) *is a generator of contraction semigroup.*

Proof. By [28], A_0 (defined in (7.5)) is a generator of a contraction semigroup $(T_0(t))_{t\geq 0}$. Because of $\operatorname{ran} A_0 \subseteq \operatorname{ran} \mathfrak{A} = \mathcal{X}_{\mathcal{H}}$ and Lemma 8.5.5 $A = A_0\big|_{\mathcal{X}_{\mathcal{H}}}$ generates the contraction semigroup $(T(t))_{t\geq 0} := (T_0(t)\big|_{\mathcal{X}_{\mathcal{H}}})_{t\geq 0}$. ❑

The following lemma is the boundary triple property for the port-Hamiltonian system given by the wave equation.

Lemma 7.2.4. *Let A be given by* (7.4) *and $x, y \in \operatorname{dom}(A)$. Then*

$$\langle Ax, y\rangle_{\mathcal{X}_{\mathcal{H}}} + \langle x, Ay\rangle_{\mathcal{X}_{\mathcal{H}}} = \langle \gamma_\nu T x_2, \gamma_0 \tfrac{1}{\rho} y_1\rangle_{\mathsf{L}^2(\Gamma_1)} + \langle \gamma_0 \tfrac{1}{\rho} x_1, \gamma_\nu T y_2\rangle_{\mathsf{L}^2(\Gamma_1)}.$$

And in particular

$$\operatorname{Re}\langle Ax, x\rangle_{\mathcal{X}_{\mathcal{H}}} = \operatorname{Re}\langle \gamma_\nu T x_2, \gamma_0 \tfrac{1}{\rho} x_1\rangle_{\mathsf{L}^2(\Gamma_1)}.$$

7.3 Stability Results

In this section we prove semi-uniform stability of the multidimensional wave equation (7.1). We start with the definition of semi-uniform stability and strong stability.

Definition 7.3.1. We say a strongly continuous semigroup $(T(t))_{t\geq 0}$ on a Hilbert space X is *strongly stable*, if for every $x \in X$

$$\lim_{t\to\infty} \|T(t)x\|_X = 0.$$

We say a continuous semigroup $(T(t))_{t\geq 0}$ on a Hilbert space X is *semi-uniform stable*, if there exists a continuous monotone decreasing function $f\colon [0,\infty) \to [0,\infty)$ with $\lim_{t\to\infty} f(t) = 0$ and

$$\|T(t)x\|_X \leq f(t)\|x\|_A$$

for every $x \in \operatorname{dom}(A)$.

Remark 7.3.2. Note that in [10, sec. 3] semi-uniform stability is defined by $\|T(t)A^{-1}\| \to 0$, where A is the generator of $(T(t))_{t\geq 0}$. It can be easily seen that this is equivalent to our definition.

Moreover, in [10, sec. 3] it is explained that semi-uniform stability is a concept between exponential stability and strong stability. In particular, semi-uniform stability implies strong stability.

The already mentioned article [10] is an overview article on semi-uniform stability. We remark that this notion is sometimes called differently, e.g. in [55] it is called uniform stability for smooth data (USSD).

In the following we denote by A the operator given by (7.4) which is associated to the port-Hamiltonian formulation of (7.1).

Our main result is the following theorem.

Theorem 7.3.3. *The semigroup generated by A is semi-uniform stable.*

The proof of Theorem 7.3.3 is given at the end of the section.

Remark 7.3.4. For the original system (7.1) strong stability of A translates to: There is a $w_{\mathrm{e}} \in \mathsf{H}^1(\Omega)$ such that for every initial value $w_0 \in \mathsf{H}^1(\Omega)$, $w_1 \in \mathsf{L}^2(\Omega)$ the corresponding solution w satisfies

$$\lim_{t\to\infty} \|w(t,\cdot) - w_{\mathrm{e}}(\cdot)\|_{\mathsf{H}^1(\Omega)} = 0.$$

We will make use of a characterization of semi-uniform stability in [10, Theorem 3.4] to show that A, given by (7.4), generates a semi-uniform stable semigroup. As A generates a bounded strongly continuous semigroup, by this theorem a sufficient condition for semi-uniform stability is given by $\sigma(A) \cap i\mathbb{R} = \emptyset$. Here $\sigma(A)$ denotes the spectrum of the operator A. Hence, it suggests itself to analyse the spectrum of A or its complement in $\mathbb{C}$, the resolvent set.

We will show that calculating the resolvent set $\rho(A)$ is related to a lossy Helmholtz problem: Find a function $u\colon \Omega \to \mathbb{C}$ that satisfies

$$\begin{aligned} \operatorname{div} T \operatorname{grad} u - \lambda^2 \rho u &= f \quad \text{in} \quad \Omega, \\ \tfrac{\partial}{\partial T\nu} u + \lambda k u &= g \quad \text{on} \quad \Gamma_1, \end{aligned} \tag{7.6}$$

where $\lambda \in \mathbb{C} \setminus \{0\}$, $f \in \mathsf{L}^2(\Omega)$, $g \in \mathsf{L}^2(\Gamma_1)$, and k, ρ and T are the functions from the beginning. A weak formulation of this problem can be derived by taking the inner product with $v \in \mathsf{H}^1_{\Gamma_0}(\Omega)$, apply an integration by parts formula for div-grad and taking the boundary conditions into account:

$$\begin{aligned} \langle T \operatorname{grad} u, \operatorname{grad} v\rangle_{\mathsf{L}^2(\Omega)} + \lambda^2 \langle \rho u, v\rangle_{\mathsf{L}^2(\Omega)} + \lambda \langle k\gamma_0 u, \gamma_0 v\rangle_{\mathsf{L}^2(\Gamma_1)} \\ = \langle -f, v\rangle_{\mathsf{L}^2(\Omega)} + \langle g, \gamma_0 v\rangle_{\mathsf{L}^2(\Gamma_1)}. \end{aligned} \tag{7.7}$$

We define

$$\begin{aligned} b(u,v) &:= \langle T \operatorname{grad} u, \operatorname{grad} v\rangle_{\mathsf{L}^2(\Omega)} + \lambda^2 \langle \rho u, v\rangle_{\mathsf{L}^2(\Omega)} + \lambda \langle k\gamma_0 u, \gamma_0 v\rangle_{\mathsf{L}^2(\Gamma_1)} \\ F(v) &:= \langle -f, v\rangle_{\mathsf{L}^2(\Omega)} + \langle g, \gamma_0 v\rangle_{\mathsf{L}^2(\Gamma_1)}, \end{aligned}$$

so that we can write (7.7) as

$$b(u,v) = F(v). \tag{7.8}$$

A *weak solution* of (7.6) is a function $u \in \mathsf{H}^1_{\Gamma_0}(\Omega)$ that satisfies (7.8) for every $v \in \mathsf{H}^1_{\Gamma_0}(\Omega)$.

Lemma 7.3.5. *Let u be a weak solution of the Helmholtz problem* (7.6). *Then $u \in \mathsf{H}^1_{\Gamma_0}(\Omega)$, $T \operatorname{grad} u \in \mathsf{H}(\operatorname{div}, \Omega)$ and in particular,*

$$\begin{aligned} \operatorname{div} T \operatorname{grad} u - \lambda^2 \rho u &= f \quad \textit{in} \quad \mathsf{L}^2(\Omega), \\ \gamma_\nu T \operatorname{grad} u + \lambda k \gamma_0 u &= g \quad \textit{in} \quad \mathsf{L}^2(\Gamma_1). \end{aligned}$$

Proof. A weak solution u is by definition in $\mathsf{H}^1_{\Gamma_0}(\Omega)$ and satisfies $b(u,v) = F(v)$ for all $v \in \mathsf{H}^1_{\Gamma_0}(\Omega)$. If we choose $v \in \mathsf{C}^\infty_c(\Omega)$, then all boundary integrals vanish. Hence,

$$\langle T \operatorname{grad} u, \operatorname{grad} v\rangle_{\mathsf{L}^2(\Omega)} = \langle -f, v\rangle_{\mathsf{L}^2(\Omega)} - \lambda^2 \langle \rho u, v\rangle_{\mathsf{L}^2(\Omega)},$$

which implies that $T \operatorname{grad} u \in \mathsf{H}(\operatorname{div}, \Omega)$ and $\operatorname{div} T \operatorname{grad} u = f + \lambda^2 \rho u$. Using this and choosing again $v \in \mathsf{H}^1_{\Gamma_0}(\Omega)$ in the weak formulation gives

$$\langle \gamma_\nu T u, \gamma_0 v\rangle_{\mathsf{H}^{-1/2}(\Gamma_1), \mathsf{H}^{1/2}(\Gamma_1)} + \lambda \langle k\gamma_0 u, \gamma_0 v\rangle_{\mathsf{L}^2(\Gamma_1)} = \langle g, \gamma_0 v\rangle_{\mathsf{L}^2(\Gamma_1)}.$$

Therefore, $\gamma_\nu T u$ has an $\mathsf{L}^2(\Gamma_1)$ representative and $\gamma_\nu T u + \lambda k \gamma_0 u = g$. ❑

Note that for $y = \left[\begin{smallmatrix} y_1 \\ y_2 \end{smallmatrix}\right] \in \mathcal{X}_{\mathcal{H}}$ there exists a $\phi \in \mathsf{H}^1_{\Gamma_0}(\Omega)$ such that $y_2 = \operatorname{grad}\phi$. This ϕ continuously depends on y_2 by Poincaré's inequality. If $\Gamma_0 = \emptyset$, then we choose $\phi \in \mathsf{H}^1(\Omega)/_{\mathbb{R}}$ ($\phi \in \mathsf{H}^1(\Omega)$ and $\int_\Omega \phi \,\mathrm{d}\lambda = 0$) for uniqueness and continuity.

Lemma 7.3.6. *Let A be the operator defined in* (7.4)*. Then $\lambda \in \rho(A) \setminus \{0\}$ is equivalent to: The system*

$$\begin{aligned} \operatorname{div} T \operatorname{grad} u - \lambda^2 \rho u &= \lambda y_1 + \lambda^2 \rho\phi \quad &&\text{in} \quad \Omega, \\ \tfrac{\partial}{\partial T\nu} u + \lambda k u &= -\lambda k \phi &&\text{on} \quad \Gamma_1, \end{aligned} \tag{7.9}$$

is weakly solvable for every $y = \left[\begin{smallmatrix} y_1 \\ y_2 \end{smallmatrix}\right] \in \mathcal{X}_{\mathcal{H}}$, where ϕ is defined by $\operatorname{grad}\phi = y_2$ as described above.

Proof. For $\lambda \in \rho(A) \setminus \{0\}$ and $y \in \mathcal{X}_{\mathcal{H}}$ there exists an $x \in \operatorname{dom}(A)$ such that $(A - \lambda)x = y$. Hence,

$$\begin{aligned} \operatorname{div} T x_2 - \lambda x_1 &= y_1 \\ \operatorname{grad} \tfrac{1}{\rho} x_1 - \lambda x_2 &= \operatorname{grad}\phi \quad \Rightarrow \quad x_2 = \tfrac{1}{\lambda} \operatorname{grad}(\tfrac{1}{\rho} x_1 - \phi). \end{aligned}$$

Substituting x_2 in the first equation, multiplying by λ and adding $\lambda^2\rho\phi$ on both sides yields

$$\operatorname{div} T \operatorname{grad}(\tfrac{1}{\rho} x_1 - \phi) - \lambda^2 \rho (\tfrac{1}{\rho} x_1 - \phi) = \lambda y_1 + \lambda^2 \rho \phi.$$

Since $x \in \operatorname{dom}(A)$ we have $k\gamma_\nu T x_2 + \gamma_0 \frac{1}{\rho} x_1 = 0$ which becomes

$$\gamma_\nu T \operatorname{grad}(\tfrac{1}{\rho} x_1 - \phi) + \lambda k \gamma_0 (\tfrac{1}{\rho} x_1 - \phi) = -\lambda k \gamma_0 \phi.$$

Hence, $u := (\frac{1}{\rho} x_1 - \phi)$ is a weak solution of the system (7.9). On the other hand if u is a weak solution of (7.9), then $x := \left[\begin{smallmatrix} \rho(u+\phi) \\ \frac{1}{\lambda} \operatorname{grad} u \end{smallmatrix}\right] \in \operatorname{dom}(A)$ and $(A - \lambda)x = y$ by Lemma 7.3.5. ❑

Theorem 7.3.7. *For every $\lambda \in \mathrm{i}\mathbb{R} \setminus \{0\}$ the system* (7.9) *is weakly solvable.*

Proof. We set $\lambda = \mathrm{i}\eta$, where $\eta \in \mathbb{R} \setminus \{0\}$.

Note that by

$$\operatorname{Re} b(u,u) = \|T^{1/2} \operatorname{grad} u\|^2_{\mathsf{L}^2(\Omega)} - \eta^2 \|\rho^{1/2} u\|^2_{\mathsf{L}^2(\Omega)},$$

$b(\cdot,\cdot)$ satisfies a Gårding inequality (see Definition A.1.1).

By Gårding's inequality it is sufficient to show that $b(\cdot,\cdot)$ is a non-degenerated sesquilinear form, (see e.g. Theorem A.1.2). Suppose there is a $u \in \mathsf{H}^1_{\Gamma_0}(\Omega)$ such that $b(u,v) = 0$ for all $v \in \mathsf{H}^1_{\Gamma_0}(\Omega)$. Then $b(u,u) = 0$ and by separating the imaginary part we have

$$\mathrm{i}\eta \langle k\gamma_0 u, \gamma_0 u \rangle_{\mathsf{L}^2(\Gamma_1)} = 0.$$

Hence, $u \in \mathsf{H}_0^1(\Omega)$. Moreover, u is a weak solution of the corresponding system to $b(u, v) = \tilde{F}(v)$, where $\tilde{F}(v) := 0$. By Lemma 7.3.5, $\operatorname{div} T \operatorname{grad} u + \eta^2 \rho u = 0$ in $\mathsf{L}^2(\Omega)$ and $\gamma_\nu T u = 0$ in $\mathsf{L}^2(\Gamma_1)$. Summed up u satisfies

$$\begin{aligned} \operatorname{div} T \operatorname{grad} u + \eta^2 \rho u &= 0, \\ \gamma_0 u &= 0, \\ \gamma_\nu T u\big|_{\Gamma_1} &= 0. \end{aligned}$$

By the unique continuation principle (see e.g. [56, Theorem 1.7, Remark 1.8]), u has to be 0 and consequently $b(\cdot, \cdot)$ is non-degenerated. ❑

Remark 7.3.8. The system (7.9) is also solvable for $\lambda \in \mathbb{C}_+$, but we already knew from the dissipativity of A that $\mathbb{C}_+ \subseteq \rho(A)$.

Corollary 7.3.9. $\mathrm{i}\mathbb{R} \setminus \{0\} \cup \mathbb{C}_+ \subseteq \rho(A)$.

Proof. This is a direct consequence of Lemma 7.3.6 and Theorem 7.3.7. ❑

Lemma 7.3.10. *If $\lambda \in \mathrm{i}\mathbb{R}$ is an eigenvalue of A, then a corresponding eigenvector x satisfies $\gamma_\nu T x_2\big|_{\Gamma_1} = \gamma_0 \frac{1}{\rho} x_1\big|_{\Gamma_1} = 0$.*

Proof. By Lemma 7.2.4 we have

$$\begin{aligned} \operatorname{Re}\langle (A - \lambda)x, x\rangle_{\mathcal{X}_\mathcal{H}} &= \operatorname{Re}\langle Ax, x\rangle_{\mathcal{X}_\mathcal{H}} - \operatorname{Re} \lambda \langle x, x\rangle_{\mathcal{X}_\mathcal{H}} \\ &= \operatorname{Re}\langle \gamma_\nu T x_2, \gamma_0 \tfrac{1}{\rho} x_1\rangle_{\mathsf{L}^2(\Gamma_1)} - \operatorname{Re} \lambda \|x\|_{\mathcal{X}_\mathcal{H}}^2 \\ &= -\operatorname{Re}\langle k \gamma_0 \tfrac{1}{\rho} x_1, \gamma_0 \tfrac{1}{\rho} x_1\rangle_{\mathsf{L}^2(\Gamma_1)} - \operatorname{Re} \lambda \|x\|_{\mathcal{X}_\mathcal{H}}^2 \\ &= -\|k^{1/2} \gamma_0 \tfrac{1}{\rho} x_1\|_{\mathsf{L}^2(\Gamma_1)}^2 - \operatorname{Re} \lambda \|x\|_{\mathcal{X}_\mathcal{H}}^2. \end{aligned}$$

If x is an eigenvector of $\lambda \in \mathrm{i}\mathbb{R}$, then this equation becomes

$$0 = -\|k^{1/2} \gamma_0 \tfrac{1}{\rho} x_1\|_{\mathsf{L}^2(\Gamma_1)}^2,$$

which implies $\gamma_0 \frac{1}{\rho} x_1\big|_{\Gamma_1} = 0$ and $\gamma_\nu T x_2\big|_{\Gamma_1} = 0$ by the boundary condition. ❑

Lemma 7.3.11. *Let $A\colon \operatorname{dom}(A) \subseteq \mathcal{X}_\mathcal{H} \to \mathcal{X}_\mathcal{H}$ be the operator from the beginning. Then 0 is not an eigenvalue of A.*

Proof. Let us assume that 0 is an eigenvalue of A and x be an eigenvector. Then $\operatorname{div} T x_2 = 0$ and $\operatorname{grad} \frac{1}{\rho} x_1 = 0$ and by Lemma 7.3.10 x satisfies $\gamma_\nu T x_2\big|_{\Gamma_1} = 0 = \gamma_0 \frac{1}{\rho} x_1\big|_{\Gamma_1}$. Hence, for arbitrary $f \in \mathsf{H}_{\Gamma_0}^1(\Omega)$ we have

$$0 = \langle \operatorname{div} T x_2, f\rangle_{\mathsf{L}^2} = -\langle T x_2, \operatorname{grad} f\rangle_{\mathsf{L}^2},$$

which implies $T x_2 \perp \operatorname{grad} \mathsf{H}_{\Gamma_0}^1(\Omega)$. Since by assumption $T x_2 \in \operatorname{grad} \mathsf{H}_{\Gamma_0}^1(\Omega)$ we conclude $x_2 = 0$. Finally, $x_1 = 0$ by Poincaré's inequality. Therefore, 0 cannot be an eigenvalue. ❑

Theorem 7.3.12. *Let*

$$X \coloneqq \operatorname{grad} \mathsf{H}^1_{\Gamma_0}(\Omega) \cap \{f \in \mathsf{H}(\operatorname{div}, \Omega) \mid \gamma_\nu f\big|_{\Gamma_1} \in \mathsf{L}^2(\Gamma_1)\}$$

with $$\|f\|_X \coloneqq \sqrt{\|f\|^2_{\mathsf{L}^2(\Omega)^n} + \|\operatorname{div} f\|^2_{\mathsf{L}^2(\Omega)} + \|\gamma_\nu f\|^2_{\mathsf{L}^2(\Gamma_1)}}.$$

Then X can be compactly embedded into $\mathsf{L}^2(\Omega)^n$.

Proof. Let $(f_n)_{n\in\mathbb{N}}$ be a bounded sequence in X, i.e. $\sup_{n\in\mathbb{N}}\|f_n\|_X \leq K \in \mathbb{R}$. By assumption there exists a $\phi_n \in \mathsf{H}^1_{\Gamma_0}(\Omega)$ such that $f_n = \operatorname{grad}\phi_n$ for every $n \in \mathbb{N}$. By Poincaré's inequality we have

$$\|\phi_n\|_{\mathsf{H}^1(\Omega)} \leq C\|\operatorname{grad}\phi_n\|_{\mathsf{L}^2(\Omega)} \leq C\|f_n\|_X.$$

Hence, $(\phi_n)_{n\in\mathbb{N}}$ is a bounded sequence in $\mathsf{H}^1(\Omega)$. Moreover, $(\gamma_0\phi_n)_{n\in\mathbb{N}}$ is a bounded sequence in $\mathsf{H}^{1/2}(\partial\Omega)$. By the compact embedding of $\mathsf{H}^1(\Omega)$ into $\mathsf{L}^2(\Omega)$ and $\mathsf{H}^{1/2}(\partial\Omega)$ into $\mathsf{L}^2(\partial\Omega)$, there exists a subsequence $(\phi_{n(k)})_{k\in\mathbb{N}}$ that converges in $\mathsf{L}^2(\Omega)$ such that also $(\gamma_0\phi_{n(k)})_{k\in\mathbb{N}}$ converges in $\mathsf{L}^2(\partial\Omega)$. W.l.o.g. we assume that this is already true for the original sequence. By

$$\begin{aligned}
&\|f_n - f_m\|^2_{\mathsf{L}^2(\Omega)} \\
&\quad = \langle f_n - f_m, \operatorname{grad}(\phi_n - \phi_m)\rangle_{\mathsf{L}^2(\Omega)} \\
&\quad = -\langle \operatorname{div}(f_n - f_m), \phi_n - \phi_m\rangle_{\mathsf{L}^2(\Omega)} \\
&\qquad\qquad + \underbrace{\langle \gamma_\nu(f_n - f_m), \gamma_0(\phi_n - \phi_m)\rangle_{\mathsf{H}^{-1/2}(\Gamma_1),\mathsf{H}^{1/2}(\Gamma_1)}}_{\langle \gamma_\nu(f_n - f_m), \gamma_0(\phi_n - \phi_m)\rangle_{\mathsf{L}^2(\Gamma_1)}} \\
&\quad \leq 2K\|\phi_n - \phi_m\|_{\mathsf{L}^2(\Omega)} + 2K\|\gamma_0\phi_n - \gamma_0\phi_m\|_{\mathsf{L}^2(\Gamma_1)} \\
&\quad \to 0,
\end{aligned}$$

we have that $(f_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathsf{L}^2(\Omega)^n$ and therefore convergent. ❑

Theorem 7.3.13. $\operatorname{dom}(A)$ *can be compactly embedded into $\mathcal{X}_\mathcal{H}$.*

Proof. Note that $\operatorname{dom}(A) \subseteq \mathcal{X}_\mathcal{H}$ and that $\|\cdot\|_{\mathcal{X}_\mathcal{H}}$ is equivalent to $\|\cdot\|_{\mathsf{L}^2(\Omega)^{n+1}}$. We regard $\operatorname{dom}(A)$ with $\langle x, y\rangle_A = \langle x, y\rangle_{\mathcal{X}_\mathcal{H}} + \langle Ax, Ay\rangle_{\mathcal{X}_\mathcal{H}}$ as inner product. Note that $\operatorname{dom}(A)$ is a Hilbert space with the previous inner product. The induced norm can be written as

$$\|x\|_A = \sqrt{\|x\|^2_{\mathcal{X}_\mathcal{H}} + \|T\operatorname{grad}\tfrac{1}{\rho}x_1\|^2_{\mathsf{L}^2} + \|\tfrac{1}{\rho}\operatorname{div} Tx_2\|^2_{\mathsf{L}^2}}.$$

Note that $\|\gamma_\nu Tx_2\|_{\mathsf{L}^2(\Gamma_1)}$ is automatically bounded by $C\|x\|_A$ for some $C > 0$, since $\|\gamma_0\frac{1}{\rho}x_1\|_{\mathsf{H}^{1/2}(\partial\Omega)}$ is bounded by $C\|x\|_A$ for some $C > 0$ and $\gamma_\nu Tx_2\big|_{\Gamma_1} = -k\gamma_0\frac{1}{\rho}x_1\big|_{\Gamma_1}$. Let X be the space from Theorem 7.3.12. Then

$$\Phi\colon \begin{cases} \operatorname{dom}(A) & \to & \mathsf{H}^1_{\Gamma_0}(\Omega) \times X, \\ x & \mapsto & \begin{bmatrix} \frac{1}{\rho} & 0 \\ 0 & T \end{bmatrix} x, \end{cases}$$

is continuous. Moreover, both $\mathsf{H}^1_{\Gamma_0}(\Omega)$ and X can be compactly embedded into $\mathsf{L}^2(\Omega)$ and $\mathsf{L}^2(\Omega)^n$, respectively. We denote this combined compact embeddeding by $\iota\colon \mathsf{H}^1_{\Gamma_0}(\Omega)\times X \to \mathsf{L}^2(\Omega)^{n+1}$. Hence, also $\operatorname{dom}(A)$ can be compactly embedded into $\mathcal{X}_{\mathcal{H}}$ by $\Phi^{-1}\iota\Phi$. ❑

Corollary 7.3.14. *The resolvent operators of A are compact, the spectrum of A contains only eigenvalues and* $\mathrm{i}\mathbb{R}\cup\mathbb{C}_+\subseteq\rho(A)$.

Proof. By Theorem 7.3.13, $\operatorname{dom}(A)$ can be compactly embedded into $\mathcal{X}_{\mathcal{H}}$, which implies that every resolvent operator is compact. Hence, the spectrum of A contains only eigenvalues. Since 0 is not an eigenvalue by Lemma 7.3.11, we conclude that $0\in\rho(A)$. Moreover, by Corollary 7.3.9 also every other point on $\mathrm{i}\mathbb{R}$ is in $\rho(A)$. ❑

Finally we will prove Theorem 7.3.3.

Proof of Theorem 7.3.3. By Corollary 7.3.14 we have $\sigma(A)\cap\mathrm{i}\mathbb{R}=\emptyset$. Therefore, as announced in the beginning, [10, Theorem 3.4] implies the semi-uniform stability of the semigroup generated by A. ❑

We conclude this section with an investigation of the strong stability of the operator A_0 given by (7.5), which is an extension of A and generates a strongly continuous semigroup on $\mathsf{L}^2(\Omega)^{n+1}$.

Lemma 7.3.15. *Let $\Omega\subseteq\mathbb{R}^n$ be bounded and open with Lipschitz boundary, $n\geq 2$. Then the operator A_0 (defined in* (7.5)*) has $\lambda=0$ as an eigenvalue and thus, does not generate a strongly stable semigroup.*

Proof. Choose the components of $x=\left[\begin{smallmatrix}x_1\\x_2\end{smallmatrix}\right]$ as

$$x_1=0\quad\text{and}\quad x_2=T^{-1}\begin{bmatrix}\partial_2\phi\\-\partial_1\phi\\0\\\vdots\\0\end{bmatrix},$$

where ϕ is any non zero $\mathsf{C}^\infty_{\mathrm{c}}(\Omega)$ function. Then $x_2\neq 0$ and $\operatorname{div}Tx_2=\partial_1\partial_2\phi-\partial_2\partial_1\phi=0$. Since ϕ has compact support, x satisfies the boundary conditions. Thus A_0 cannot generate a strongly stable semigroup, since the eigenvector x to $\lambda=0$ is a constant solution of the Cauchy problem. ❑

7.4 Conclusion

In this paper we showed semi-uniform stability of the multidimensional wave equation equipped with a scattering passive feedback law. Further, we proved that the corresponding port-Hamiltonian operator has a compact resolvent.

To get compact embeddings for the port-Hamiltonian operator of the wave equation it is necessary to choose an adequate state space. This is a new aspect that arises for spatial multidimensional port-Hamiltonian systems as in the one-dimensional spatial setting the compact embedding is always given. It is likely that most of the techniques presented in this chapter will translate for general linear port-Hamiltonian systems on multidimensional spatial domains like Maxwell's equations and the Mindlin plate model. Probably the crucial tool will be a unique continuation principle.

Moreover, there is an interesting link between the resolvent set of the port-Hamiltonian operator of the wave equation and solvability of lossy Helmholtz equations. Since in the theory of Helmholtz equations (especially in view of finite element methods) a uniform bound of the solution operator is of interest, it might be possible to use results from that theory to give explicit decay rates for the semi-uniform stability or even obtain exponential stability under certain assumptions. For constant coefficients we can find such estimates in [38, 39, 17]. There are some recent works on these estimates with non constant coefficients [19, 20].

Chapter 8

Compact Embedding for div-rot Systems

We show the following *compactness theorem*: Any L^2-bounded sequence of vector fields with L^2-bounded rotations and L^2-bounded divergences as well as L^2-bounded tangential traces on one part of the boundary and L^2-bounded normal traces on the other part of the boundary, contains a strongly L^2-convergent subsequence. This generalises recent results for homogeneous mixed boundary conditions in [4, 6]. As applications we present a related *Friedrichs/Poincaré type estimate*, a *div-curl lemma*, and show that the Maxwell operator with mixed tangential and impedance boundary conditions (Robin type boundary conditions) has *compact resolvents.*

This chapter is the result of a joint work with Dirk Pauly [46].

8.1 Introduction

Let $\Omega \subset \mathbb{R}^3$ be open with boundary Γ, composed of the boundary parts Γ_0 (tangential) and Γ_1 (normal). In [4, Theorem 4.7] the following version of Weck's selection theorem has been shown. In fact they showed the theorem for weak Lipschitz boundaries, but we will stick to strong Lipschitz boundaries.

Theorem 8.1.1 (compact embedding for vector fields with homogeneous mixed boundary conditions)**.** *Let Ω be a bounded strong Lipschitz domain and Γ_0, Γ_1 a splitting with thin boundaries (see Definition 5.1.1). Furthermore, let ε be admissible. Then*

$$\mathsf{H}_{\Gamma_0}(\mathrm{rot}, \Omega) \cap \varepsilon^{-1}\mathsf{H}_{\Gamma_1}(\mathrm{div}, \Omega) \overset{\mathsf{cpt}}{\hookrightarrow} \mathsf{L}^2(\Omega).$$

Here, $\overset{\mathsf{cpt}}{\hookrightarrow}$ denotes a compact embedding, and – in classical terms and in the smooth case – we have for a vector field E (ν denotes the exterior unit normal at Γ)

$$\begin{aligned} E \in \mathsf{H}_{\Gamma_0}(\mathrm{rot},\Omega) \quad &\Leftrightarrow \quad E \in \mathsf{L}^2(\Omega), \quad \mathrm{rot}\, E \in \mathsf{L}^2(\Omega), \quad \nu \times E|_{\Gamma_0} = 0, \\ E \in \varepsilon^{-1}\mathsf{H}_{\Gamma_1}(\mathrm{div},\Omega) \quad &\Leftrightarrow \quad \varepsilon E \in \mathsf{L}^2(\Omega), \quad \mathrm{div}\, \varepsilon E \in \mathsf{L}^2(\Omega), \quad \nu \cdot \varepsilon E|_{\Gamma_1} = 0. \end{aligned}$$

Note that Theorem 8.1.1 even holds for bounded weak Lipschitz pairs (Ω, Γ_0). For exact definitions and notations see Section 8.2, and for a history of related compact embedding results see, e.g., [63, 49, 62, 11, 65, 27, 51] and [31]. The general importance of compact embeddings in a functional analytical setting (FA-ToolBox) for Hilbert complexes (such as de Rham, elasticity, biharmonic) is described, e.g., in [42, 44, 45, 43] and [47, 48, 2].

In this chapter, we shall generalise Theorem 8.1.1 to the case of inhomogeneous boundary conditions, i.e., we will show that the compact embedding in Theorem 8.1.1 still holds if the space

$$\mathsf{H}_{\Gamma_0}(\mathrm{rot},\Omega) \cap \varepsilon^{-1}\mathsf{H}_{\Gamma_1}(\mathrm{div},\Omega)$$

is replaced by

$$\widehat{\mathsf{H}}_{\Gamma_0}(\mathrm{rot},\Omega) \cap \varepsilon^{-1}\widehat{\mathsf{H}}_{\Gamma_1}(\mathrm{div},\Omega),$$

where in classical terms and in the smooth case

$$\begin{aligned} E \in \widehat{\mathsf{H}}_{\Gamma_0}(\mathrm{rot},\Omega) \quad &\Leftrightarrow \quad E \in \mathsf{L}^2(\Omega), \quad \mathrm{rot}\, E \in \mathsf{L}^2(\Omega), \quad \nu \times E|_{\Gamma_0} \in \mathsf{L}^2(\Gamma_0), \\ E \in \varepsilon^{-1}\widehat{\mathsf{H}}_{\Gamma_1}(\mathrm{div},\Omega) \quad &\Leftrightarrow \quad \varepsilon E \in \mathsf{L}^2(\Omega), \quad \mathrm{div}\, \varepsilon E \in \mathsf{L}^2(\Omega), \quad \nu \cdot \varepsilon E|_{\Gamma_1} \in \mathsf{L}^2(\Gamma_1). \end{aligned}$$

The main result (compact embedding) is formulated in Theorem 8.4.1. As applications we show in Theorem 8.5.1 that the compact embedding implies a related Friedrichs/Poincaré type estimate, showing well-posedness of related systems of partial differential equations. Moreover, in Theorem 8.5.3 we prove that Theorem 8.4.1 yields a div-curl lemma. Note that corresponding results for exterior domains are straight forward using weighted Sobolev spaces, see [40, 41]. Another application is presented in Section 8.5.3 where we show that our compact embedding result implies compact resolvents of the Maxwell operator with inhomogeneous mixed boundary conditions, even of impedance type.

8.2 Notations

Throughout this chapter, let $\Omega \subset \mathbb{R}^3$ be an open and bounded strong Lipschitz domain, and let ε be an *admissible* tensor (matrix) field, i.e., a symmetric, L^∞-bounded, and uniformly positive definite tensor field $\varepsilon\colon \Omega \to \mathbb{R}^{3\times 3}$. Moreover, let the boundary Γ of Ω be decomposed into two relatively open and strong Lipschitz subsets Γ_0 and $\Gamma_1 := \Gamma \setminus \overline{\Gamma_0}$ forming the interface $\overline{\Gamma_0} \cap \overline{\Gamma_1}$ for the mixed boundary conditions. See [4, 5, 6] for exact definitions. We call (Ω, Γ_0) a bounded strong Lipschitz pair.

The usual Lebesgue and Sobolev Hilbert spaces (of scalar or vector valued fields) are denoted by $\mathsf{L}^2(\Omega)$, $\mathsf{H}^1(\Omega)$, $\mathsf{H}(\mathrm{rot},\Omega)$, $\mathsf{H}(\mathrm{div},\Omega)$, and spaces with vanishing rot and div are denoted by

$$\mathsf{H}_{\Gamma_0,0}(\mathrm{rot},\Omega) := \mathsf{H}_{\Gamma_0}(\mathrm{rot},\Omega) \cap \ker \mathrm{rot} \quad \text{and}$$
$$\mathsf{H}_{\Gamma_0,0}(\mathrm{div},\Omega) := \mathsf{H}_{\Gamma_0}(\mathrm{div},\Omega) \cap \ker \mathrm{div},$$

respectively. Moreover, we introduce the cohomology space of Dirichlet or Neumann fields (generalised harmonic fields)

$$\mathcal{H}_{\Gamma_0,\Gamma_1,\varepsilon}(\Omega) := \mathsf{H}_{\Gamma_0,0}(\mathrm{rot},\Omega) \cap \varepsilon^{-1}\mathsf{H}_{\Gamma_1,0}(\mathrm{div},\Omega).$$

The $\mathsf{L}^2(\Omega)$-inner product and norm (of scalar or vector valued $\mathsf{L}^2(\Omega)$-spaces) will be denoted by $\langle\,\cdot\,,\,\cdot\,\rangle_{\mathsf{L}^2(\Omega)}$ and $\|\cdot\|_{\mathsf{L}^2(\Omega)}$, respectively, and the weighted Lebesgue space $\mathsf{L}^2_\varepsilon(\Omega)$ is defined as $\mathsf{L}^2(\Omega)$ (of vector fields) but being equipped with the weighted $\mathsf{L}^2(\Omega)$-inner product and norm $\langle\,\cdot\,,\,\cdot\,\rangle_{\mathsf{L}^2_\varepsilon(\Omega)} := \langle\varepsilon\,\cdot\,,\,\cdot\,\rangle_{\mathsf{L}^2(\Omega)}$ and $\|\cdot\|_{\mathsf{L}^2_\varepsilon(\Omega)}$, respectively. The norms in, e.g., $\mathsf{H}^1(\Omega)$ and $\mathsf{H}(\mathrm{rot},\Omega)$ are denoted by $\|\cdot\|_{\mathsf{H}^1(\Omega)}$ and $\|\cdot\|_{\mathsf{H}(\mathrm{rot},\Omega)}$, respectively. Orthogonality and orthogonal sum in $\mathsf{L}^2(\Omega)$ and $\mathsf{L}^2_\varepsilon(\Omega)$ are indicated by $\perp_{\mathsf{L}^2(\Omega)}$, $\perp_{\mathsf{L}^2_\varepsilon(\Omega)}$, and $\oplus_{\mathsf{L}^2(\Omega)}$, $\oplus_{\mathsf{L}^2_\varepsilon(\Omega)}$, respectively.

Finally, we introduce inhomogeneous tangential and normal L^2-boundary conditions in

$$\widehat{\mathsf{H}}_{\Gamma_0}(\mathrm{rot},\Omega) := \big\{E \in \mathsf{H}(\mathrm{rot},\Omega) \,\big|\, \gamma_\tau^{\Gamma_0} E \in \mathsf{L}^2(\Gamma_0)\big\},$$
$$\widehat{\mathsf{H}}_{\Gamma_1}(\mathrm{div},\Omega) := \big\{E \in \mathsf{H}(\mathrm{div},\Omega) \,\big|\, \gamma_\nu^{\Gamma_1} E \in \mathsf{L}^2(\Gamma_1)\big\}$$

with norms given by, e.g., $\|E\|^2_{\widehat{\mathsf{H}}_{\Gamma_0}(\mathrm{rot},\Omega)} := \|E\|^2_{\mathsf{H}(\mathrm{rot},\Omega)} + \|\gamma_\tau^{\Gamma_0} E\|^2_{\mathsf{L}^2(\Gamma_0)}$. The definitions of the latter Hilbert spaces need some explanations:

Definition 8.2.1. (L^2-traces)

(i) The tangential trace of a vector field $E \in \mathsf{H}(\mathrm{rot},\Omega)$ is a well-defined tangential vector field $\gamma_\tau^\Gamma E \in \mathsf{H}^{-1/2}(\Gamma)$ generalising the classical tangential trace $\gamma_\tau^\Gamma \widetilde{E} = -\nu \times \nu \times \widetilde{E}|_\Gamma$ for smooth vector fields $\widetilde{E}$. By the notation $\gamma_\tau^{\Gamma_0} E \in \mathsf{L}^2(\Gamma_0)$ we mean, that there exists a tangential vector field $E_{\Gamma_0} \in \mathsf{L}^2(\Gamma_0)$, such that for all vector fields $\Phi \in \mathsf{H}^1_{\Gamma_1}(\Omega)$ it holds

$$\langle \mathrm{rot}\,\Phi, E\rangle_{\mathsf{L}^2(\Omega)} - \langle \Phi, \mathrm{rot}\,E\rangle_{\mathsf{L}^2(\Omega)} = \langle \gamma_{\tau_\times}^{\Gamma_0}\Phi, E_{\Gamma_0}\rangle_{\mathsf{L}^2(\Gamma_0)}.$$

Then we set $\gamma_\tau^{\Gamma_0} E := E_{\Gamma_0} \in \mathsf{L}^2(\Gamma_0)$. Here and in the following, the twisted tangential trace of the smooth vector field Φ is given by the tangential vector field $\gamma_{\tau_\times}^\Gamma \Phi = \nu \times \Phi|_\Gamma \in \mathsf{L}^2(\Gamma)$ with $\gamma_{\tau_\times}^{\Gamma_1}\Phi = \gamma_{\tau_\times}^\Gamma \Phi|_{\Gamma_1} = 0$ and $\gamma_{\tau_\times}^{\Gamma_0}\Phi = \gamma_{\tau_\times}^\Gamma \Phi|_{\Gamma_0} \in \mathsf{L}^2(\Gamma_0)$. Note that $\gamma_\tau^{\Gamma_0} E$ is well defined as $\gamma_{\tau_\times}^{\Gamma_0}\mathsf{H}^1_{\Gamma_1}(\Omega)$ is dense in $\mathsf{L}^2_t(\Gamma_0) = \{w \in \mathsf{L}^2(\Gamma_0) \,|\, \nu \cdot w = 0\}$.

(ii) Analogously, the normal trace of a vector field $E \in \mathsf{H}(\mathrm{div}, \Omega)$ is a well-defined function $\gamma_\nu^\Gamma E \in \mathsf{H}^{-1/2}(\Gamma)$ generalising the classical normal trace $\gamma_\nu^\Gamma \widetilde{E} = \nu \cdot \widetilde{E}|_\Gamma$ for smooth vector fields $\widetilde{E}$. Again, by the notation $\gamma_\nu^{\Gamma_1} E \in \mathsf{L}^2(\Gamma_1)$ we mean, that for all functions $\phi \in \mathsf{H}^1_{\Gamma_0}(\Omega)$ it holds

$$\langle \mathrm{grad}\, \phi, E \rangle_{\mathsf{L}^2(\Omega)} + \langle \phi, \mathrm{div}\, E \rangle_{\mathsf{L}^2(\Omega)} = \langle \gamma_0^{\Gamma_1} \phi, \gamma_\nu^{\Gamma_1} E \rangle_{\mathsf{L}^2(\Gamma_1)}.$$

Here, the well-known scalar trace of the smooth function ϕ is given by $\gamma_0^\Gamma \phi = \phi|_\Gamma \in \mathsf{L}^2(\Gamma)$ with $\gamma_0^{\Gamma_0} \phi = \gamma_0^\Gamma \phi|_{\Gamma_0} = 0$ and $\gamma_0^{\Gamma_1} \phi = \gamma_0^\Gamma \phi|_{\Gamma_1} \in \mathsf{L}^2(\Gamma_1)$. Note that $\gamma_\nu^{\Gamma_1} E$ is well defined as $\gamma_0^{\Gamma_1} \mathsf{H}^1_{\Gamma_0}(\Omega)$ is dense in $\mathsf{L}^2(\Gamma_1)$.

Remark 8.2.2. Analogously to Definition 8.2.1 (i) and as

$$\gamma_{\tau_\times}^{\Gamma_0} \widetilde{E} \cdot \gamma_\tau^{\Gamma_0} \widetilde{H} = (\nu \times \widetilde{E}) \cdot (-\nu \times \nu \times \widetilde{H}) = (\nu \times \nu \times \widetilde{E}) \cdot (\nu \times \widetilde{H}) = -\gamma_\tau^{\Gamma_0} \widetilde{E} \cdot \gamma_{\tau_\times}^{\Gamma_0} \widetilde{H}$$

holds on Γ_0 for smooth vector fields $\widetilde{E}$, $\widetilde{H}$, we can define the twisted tangential trace $\gamma_{\tau_\times}^{\Gamma_0} E \in \mathsf{L}^2(\Gamma_0)$ of a vector field $E \in \mathsf{H}(\mathrm{rot}, \Omega)$ as well by

$$\langle \mathrm{rot}\, \Phi, E \rangle_{\mathsf{L}^2(\Omega)} - \langle \Phi, \mathrm{rot}\, E \rangle_{\mathsf{L}^2(\Omega)} = -\langle \gamma_\tau^{\Gamma_0} \Phi, \gamma_{\tau_\times}^{\Gamma_0} E \rangle_{\mathsf{L}^2(\Gamma_0)}$$

for all vector fields $\Phi \in \mathsf{H}^1_{\Gamma_1}(\Omega)$.

8.3 Preliminaries

In [6, Theorem 5.5], see [5, Theorem 7.4] for more details and compare to [4], the following theorem about the existence of regular potentials for the rotation with homogeneous mixed boundary conditions has been shown.

Theorem 8.3.1 (regular potential for rot with homogeneous mixed boundary conditions)**.**

$$\mathsf{H}_{\Gamma_1,0}(\mathrm{div}, \Omega) \cap \mathcal{H}_{\Gamma_0,\Gamma_1}(\Omega)^{\perp_{\mathsf{L}^2(\Omega)}} = \mathrm{rot}\, \mathsf{H}_{\Gamma_1}(\mathrm{rot}, \Omega) = \mathrm{rot}\, \mathsf{H}^1_{\Gamma_1}(\Omega)$$

holds together with a regular potential operator mapping $\mathrm{rot}\, \mathsf{H}_{\Gamma_1}(\mathrm{rot}, \Omega)$ *to* $\mathsf{H}^1_{\Gamma_1}(\Omega)$ *continuously. In particular, the latter ranges are closed subspaces of* $\mathsf{L}^2(\Omega)$.

Moreover, we need [6, Theorem 5.2]:

Theorem 8.3.2 (Helmholtz decompositions with homogeneous mixed boundary conditions)**.** *The ranges* $\mathrm{grad}\, \mathsf{H}^1_{\Gamma_0}(\Omega)$ *and* $\mathrm{rot}\, \mathsf{H}_{\Gamma_1}(\mathrm{rot}, \Omega)$ *are closed subspaces of* $\mathsf{L}^2(\Omega)$*, and the* $\mathsf{L}^2_\varepsilon(\Omega)$*-orthogonal Helmholtz decompositions*

$$\begin{aligned}
\mathsf{L}^2_\varepsilon(\Omega) &= \mathrm{grad}\, \mathsf{H}^1_{\Gamma_0}(\Omega) \oplus_{\mathsf{L}^2_\varepsilon(\Omega)} \varepsilon^{-1} \mathsf{H}_{\Gamma_1,0}(\mathrm{div}, \Omega) \\
&= \mathsf{H}_{\Gamma_0,0}(\mathrm{rot}, \Omega) \oplus_{\mathsf{L}^2_\varepsilon(\Omega)} \varepsilon^{-1}\, \mathrm{rot}\, \mathsf{H}_{\Gamma_1}(\mathrm{rot}, \Omega) \\
&= \mathrm{grad}\, \mathsf{H}^1_{\Gamma_0}(\Omega) \oplus_{\mathsf{L}^2_\varepsilon(\Omega)} \mathcal{H}_{\Gamma_0,\Gamma_1,\varepsilon}(\Omega) \oplus_{\mathsf{L}^2_\varepsilon(\Omega)} \varepsilon^{-1}\, \mathrm{rot}\, \mathsf{H}_{\Gamma_1}(\mathrm{rot}, \Omega)
\end{aligned}$$

hold (with continuous potential operators). Moreover, $\mathcal{H}_{\Gamma_0,\Gamma_1,\varepsilon}(\Omega)$ *has finite dimension.*

Combining Theorem 8.3.1 and Theorem 8.3.2 shows immediately the following.

Corollary 8.3.3 (regular Helmholtz decomposition with homogeneous mixed boundary conditions)**.** *The* $\mathsf{L}^2_\varepsilon(\Omega)$*-orthogonal regular Helmholtz decomposition*

$$\mathsf{L}^2_\varepsilon(\Omega) = \operatorname{grad} \mathsf{H}^1_{\Gamma_0}(\Omega) \oplus_{\mathsf{L}^2_\varepsilon(\Omega)} \mathcal{H}_{\Gamma_0,\Gamma_1,\varepsilon}(\Omega) \oplus_{\mathsf{L}^2_\varepsilon(\Omega)} \varepsilon^{-1} \operatorname{rot} \mathsf{H}^1_{\Gamma_1}(\Omega)$$

holds (with continuous potential operators) and $\mathcal{H}_{\Gamma_0,\Gamma_1,\varepsilon}(\Omega)$ *has finite dimension. More precisely, any* $E \in \mathsf{L}^2_\varepsilon(\Omega)$ *may be* $\mathsf{L}^2_\varepsilon(\Omega)$*-orthogonally (and regularly) decomposed into*

$$E = \operatorname{grad} u_{\mathrm{grad}} + E_{\mathcal{H}} + \varepsilon^{-1} \operatorname{rot} E_{\mathrm{rot}}$$

with $u_{\mathrm{grad}} \in \mathsf{H}^1_{\Gamma_0}(\Omega)$*,* $E_{\mathrm{rot}} \in \mathsf{H}^1_{\Gamma_1}(\Omega)$*, and* $E_{\mathcal{H}} \in \mathcal{H}_{\Gamma_0,\Gamma_1,\varepsilon}(\Omega)$*, and there exists a constant* $c > 0$*, independent of* $E, u_{\mathrm{grad}}, E_{\mathcal{H}}, E_{\mathrm{rot}}$*, such that*

$$\begin{aligned} \|E_{\mathcal{H}}\|_{\mathsf{L}^2_\varepsilon(\Omega)} &\le \|E\|_{\mathsf{L}^2_\varepsilon(\Omega)}, \\ c\|u_{\mathrm{grad}}\|_{\mathsf{H}^1_{\Gamma_0}(\Omega)} \le \|\operatorname{grad} u_{\mathrm{grad}}\|_{\mathsf{L}^2_\varepsilon(\Omega)} &\le \|E\|_{\mathsf{L}^2_\varepsilon(\Omega)}, \\ c\|E_{\mathrm{rot}}\|_{\mathsf{H}^1_{\Gamma_1}(\Omega)} \le \|\varepsilon^{-1} \operatorname{rot} E_{\mathrm{rot}}\|_{\mathsf{L}^2_\varepsilon(\Omega)} &\le \|E\|_{\mathsf{L}^2_\varepsilon(\Omega)}. \end{aligned}$$

8.4 Compact Embeddings

Our main result reads as follows:

Theorem 8.4.1 (compact embedding for vector fields with inhomogeneous mixed boundary conditions)**.**

$$\widehat{\mathsf{H}}_{\Gamma_0}(\mathrm{rot}, \Omega) \cap \varepsilon^{-1} \widehat{\mathsf{H}}_{\Gamma_1}(\mathrm{div}, \Omega) \overset{\mathsf{cpt}}{\hookrightarrow} \mathsf{L}^2(\Omega).$$

Proof. Let (E_ℓ) be a bounded sequence in $\widehat{\mathsf{H}}_{\Gamma_0}(\mathrm{rot}, \Omega) \cap \varepsilon^{-1} \widehat{\mathsf{H}}_{\Gamma_1}(\mathrm{div}, \Omega)$. By the Helmholtz decomposition in Corollary 8.3.3 we $\mathsf{L}^2_\varepsilon(\Omega)$-orthogonally and regularly decompose

$$E_\ell = \operatorname{grad} u_{\mathrm{grad},\ell} + E_{\mathcal{H},\ell} + \varepsilon^{-1} \operatorname{rot} E_{\mathrm{rot},\ell}$$

with $u_{\mathrm{grad},\ell} \in \mathsf{H}^1_{\Gamma_0}(\Omega)$, $E_{\mathrm{rot},\ell} \in \mathsf{H}^1_{\Gamma_1}(\Omega)$, and $E_{\mathcal{H},\ell} \in \mathcal{H}_{\Gamma_0,\Gamma_1,\varepsilon}(\Omega)$, and there exists a constant $c > 0$ such that independent of $E_{\dots}$ and for all ℓ

$$\|u_{\mathrm{grad},\ell}\|_{\mathsf{H}^1_{\Gamma_0}(\Omega)} + \|E_{\mathcal{H},\ell}\|_{\mathsf{L}^2_\varepsilon(\Omega)} + \|E_{\mathrm{rot},\ell}\|_{\mathsf{H}^1_{\Gamma_1}(\Omega)} \le c\|E_\ell\|_{\mathsf{L}^2_\varepsilon(\Omega)}.$$

As $\mathcal{H}_{\Gamma_0,\Gamma_1,\varepsilon}(\Omega)$ is finite dimensional we may assume (after extracting a subsequence) that $E_{\mathcal{H},\ell}$ converges strongly in $\mathsf{L}^2_\varepsilon(\Omega)$. Since $\mathsf{H}^1(\Omega) \overset{\mathsf{cpt}}{\hookrightarrow} \mathsf{L}^2(\Omega)$ by Rellich's selection theorem, we may assume that also the regular potentials $u_{\mathrm{grad},\ell}$ and $E_{\mathrm{rot},\ell}$ converge strongly in $\mathsf{L}^2(\Omega)$. Moreover, $u_{\mathrm{grad},\ell}|_\Gamma$ and $E_{\mathrm{rot},\ell}|_\Gamma$ are bounded in $\mathsf{H}^{1/2}(\Gamma)$ by the (scalar) trace theorem, and thus we may assume by the compact embedding $\mathsf{H}^{1/2}(\Gamma) \overset{\mathsf{cpt}}{\hookrightarrow} \mathsf{L}^2(\Gamma)$ that $u_{\mathrm{grad},\ell}|_\Gamma$ and $E_{\mathrm{rot},\ell}|_\Gamma$ converge strongly in $\mathsf{L}^2(\Gamma)$. In particular, $u_{\mathrm{grad},\ell}|_{\Gamma_1}$ and $E_{\mathrm{rot},\ell}|_{\Gamma_0}$ converge strongly in

$\mathsf{L}^2(\Gamma_1)$ and $\mathsf{L}^2(\Gamma_0)$, respectively. After all this successively taking subsequences we obtain (using $\mathsf{L}^2_\varepsilon(\Omega)$-orthogonality and the definition of the $\mathsf{L}^2(\Gamma_1)$-traces of $\gamma_\nu^{\Gamma_1}\varepsilon E_\ell$ and the $\mathsf{L}^2(\Gamma_0)$-traces of $\gamma_\tau^{\Gamma_0}E_\ell$ from Definition 8.2.1)

$$\begin{aligned}
&\big\|\operatorname{grad}(u_{\mathrm{grad},\ell}-u_{\mathrm{grad},k})\big\|^2_{\mathsf{L}^2_\varepsilon(\Omega)}\\
&\qquad= \big\langle \operatorname{grad}(u_{\mathrm{grad},\ell}-u_{\mathrm{grad},k}), E_\ell - E_k\big\rangle_{\mathsf{L}^2_\varepsilon(\Omega)}\\
&\qquad= -\big\langle u_{\mathrm{grad},\ell}-u_{\mathrm{grad},k}, \operatorname{div}\varepsilon(E_\ell - E_k)\big\rangle_{\mathsf{L}^2(\Omega)}\\
&\qquad\quad +\big\langle \gamma_0^{\Gamma_1}(u_{\mathrm{grad},\ell}-u_{\mathrm{grad},k}), \gamma_\nu^{\Gamma_1}\varepsilon(E_\ell - E_k)\big\rangle_{\mathsf{L}^2(\Gamma_1)}\\
&\qquad\le c\|u_{\mathrm{grad},\ell}-u_{\mathrm{grad},k}\|_{\mathsf{L}^2(\Omega)} + c\big\|(u_{\mathrm{grad},\ell}-u_{\mathrm{grad},k})|_{\Gamma_1}\big\|_{\mathsf{L}^2(\Gamma_1)} \to 0
\end{aligned}$$

and

$$\begin{aligned}
&\big\|\varepsilon^{-1}\operatorname{rot}(E_{\mathrm{rot},\ell}-E_{\mathrm{rot},k})\big\|^2_{\mathsf{L}^2_\varepsilon(\Omega)}\\
&\qquad= \big\langle \varepsilon^{-1}\operatorname{rot}(E_{\mathrm{rot},\ell}-E_{\mathrm{rot},k}), E_\ell - E_k\big\rangle_{\mathsf{L}^2_\varepsilon(\Omega)}\\
&\qquad= \big\langle E_{\mathrm{rot},\ell}-E_{\mathrm{rot},k}, \operatorname{rot}(E_\ell - E_k)\big\rangle_{\mathsf{L}^2(\Omega)}\\
&\qquad\quad + \big\langle \gamma_{\tau_\times}^{\Gamma_0}(E_{\mathrm{rot},\ell}-E_{\mathrm{rot},k}), \gamma_\tau^{\Gamma_0}(E_\ell - E_k)\big\rangle_{\mathsf{L}^2(\Gamma_0)}\\
&\qquad\le c\|E_{\mathrm{rot},\ell}-E_{\mathrm{rot},k}\|_{\mathsf{L}^2(\Omega)} + c\big\|(E_{\mathrm{rot},\ell}-E_{\mathrm{rot},k})|_{\Gamma_0}\big\|_{\mathsf{L}^2(\Gamma_0)} \to 0.
\end{aligned}$$

Hence, (E_ℓ) contains a strongly $\mathsf{L}^2_\varepsilon(\Omega)$-convergent (and thus strongly $\mathsf{L}^2(\Omega)$-convergent) subsequence. ❑

Remark 8.4.2 (compact embedding for vector fields with inhomogeneous mixed boundary conditions). Theorem 8.4.1 even holds for weaker boundary data. For this, let $0 \le s < 1/2$. Taking into account the compact embedding $\mathsf{H}^{1/2}(\Gamma) \overset{\mathsf{cpt}}{\hookrightarrow} \mathsf{H}^s(\Gamma)$ and looking at the latter proof, we see that

$$\begin{aligned}
&\{E \in \mathsf{H}(\operatorname{rot},\Omega) \,|\, \gamma_\tau^{\Gamma_0}E \in \mathsf{H}^{-s}(\Gamma_0)\}\\
&\qquad\qquad \cap \{E \in \varepsilon^{-1}\mathsf{H}(\operatorname{div},\Omega) \,|\, \gamma_\nu^{\Gamma_1}\varepsilon E \in \mathsf{H}^{-s}(\Gamma_1)\} \overset{\mathsf{cpt}}{\hookrightarrow} \mathsf{L}^2(\Omega).
\end{aligned}$$

8.5 Applications

8.5.1 Friedrichs/Poincaré Type Estimates

A first application is the following estimate:

Theorem 8.5.1 (Friedrichs/Poincaré type estimate for vector fields with inhomogeneous mixed boundary conditions)**.** *There exists a positive constant c such that for all vector fields E in $\widehat{\mathsf{H}}_{\Gamma_0}(\operatorname{rot},\Omega) \cap \varepsilon^{-1}\widehat{\mathsf{H}}_{\Gamma_1}(\operatorname{div},\Omega) \cap \mathcal{H}_{\Gamma_0,\Gamma_1,\varepsilon}(\Omega)^{\perp_{\mathsf{L}^2_\varepsilon(\Omega)}}$ it holds*

$$c\|E\|_{\mathsf{L}^2_\varepsilon(\Omega)} \le \|\operatorname{rot} E\|_{\mathsf{L}^2(\Omega)} + \|\operatorname{div}\varepsilon E\|_{\mathsf{L}^2(\Omega)} + \|\gamma_\tau^{\Gamma_0}E\|_{\mathsf{L}^2(\Gamma_0)} + \|\gamma_\nu^{\Gamma_1}\varepsilon E\|_{\mathsf{L}^2(\Gamma_1)}.$$

Proof. For a proof we use a standard compactness argument using Theorem 8.4.1. If the estimate was wrong, then there exists a sequence $(E_\ell) \in \widehat{\mathsf{H}}_{\Gamma_0}(\mathrm{rot},\Omega) \cap \varepsilon^{-1}\widehat{\mathsf{H}}_{\Gamma_1}(\mathrm{div},\Omega) \cap \mathcal{H}_{\Gamma_0,\Gamma_1,\varepsilon}(\Omega)^{\perp_{\mathsf{L}^2_\varepsilon(\Omega)}}$ with $\|E_\ell\|_{\mathsf{L}^2_\varepsilon(\Omega)} = 1$ and

$$\|\mathrm{rot}\, E_\ell\|_{\mathsf{L}^2(\Omega)} + \|\mathrm{div}\,\varepsilon E_\ell\|_{\mathsf{L}^2(\Omega)} + \|\gamma_\tau^{\Gamma_0} E_\ell\|_{\mathsf{L}^2(\Gamma_0)} + \|\gamma_\nu^{\Gamma_1}\varepsilon E_\ell\|_{\mathsf{L}^2(\Gamma_1)} \to 0.$$

Thus, by Theorem 8.4.1 (after extracting a subsequence)

$$E_\ell \to E \quad \text{in} \quad \widehat{\mathsf{H}}_{\Gamma_0}(\mathrm{rot},\Omega) \cap \varepsilon^{-1}\widehat{\mathsf{H}}_{\Gamma_1}(\mathrm{div},\Omega) \cap \mathcal{H}_{\Gamma_0,\Gamma_1,\varepsilon}(\Omega)^{\perp_{\mathsf{L}^2_\varepsilon(\Omega)}} \qquad \text{(strongly)}$$

and $\mathrm{rot}\, E = 0$ and $\mathrm{div}\,\varepsilon E = 0$ (by testing). Moreover, for all $\Phi \in \mathsf{C}^\infty_{\Gamma_1}(\Omega)$ and for all $\phi \in \mathsf{C}^\infty_{\Gamma_0}(\Omega)$

$$\langle \mathrm{rot}\,\Phi, E_\ell\rangle_{\mathsf{L}^2(\Omega)} - \langle \Phi, \mathrm{rot}\, E_\ell\rangle_{\mathsf{L}^2(\Omega)} = \langle \gamma_{\tau_\times}^{\Gamma_0}\Phi, \gamma_\tau^{\Gamma_0} E_\ell\rangle_{\mathsf{L}^2(\Gamma_0)} \leq c\|\gamma_\tau^{\Gamma_0} E_\ell\|_{\mathsf{L}^2(\Gamma_0)} \to 0,$$

and

$$\begin{aligned}\langle \mathrm{grad}\,\phi, \varepsilon E_\ell\rangle_{\mathsf{L}^2(\Omega)} &+ \langle \phi, \mathrm{div}\,\varepsilon E_\ell\rangle_{\mathsf{L}^2(\Omega)} \\ &= \langle \gamma_0^{\Gamma_1}\phi, \gamma_\nu^{\Gamma_1}\varepsilon E_\ell\rangle_{\mathsf{L}^2(\Gamma_1)} \leq c\|\gamma_\nu^{\Gamma_1}\varepsilon E_\ell\|_{\mathsf{L}^2(\Gamma_1)} \to 0,\end{aligned}$$

cf. Definition 8.2.1, implying

$$\langle \mathrm{rot}\,\Phi, E\rangle_{\mathsf{L}^2(\Omega)} = 0 \quad \text{and} \quad \langle \mathrm{grad}\,\phi, \varepsilon E\rangle_{\mathsf{L}^2(\Omega)} = 0.$$

Hence, $E \in \mathsf{H}_{\Gamma_0,0}(\mathrm{rot},\Omega) \cap \varepsilon^{-1}\mathsf{H}_{\Gamma_1,0}(\mathrm{div},\Omega) = \mathcal{H}_{\Gamma_0,\Gamma_1,\varepsilon}(\Omega)$ by [6, Theorem 4.7] (weak and strong homogeneous boundary conditions coincide). This shows $E = 0$ as $E \perp_{\mathsf{L}^2_\varepsilon(\Omega)} \mathcal{H}_{\Gamma_0,\Gamma_1,\varepsilon}(\Omega)$, in contradiction to $1 = \|E_\ell\|_{\mathsf{L}^2_\varepsilon(\Omega)} \to \|E\|_{\mathsf{L}^2_\varepsilon(\Omega)} = 0$. ❑

Remark 8.5.2 (Friedrichs/Poincaré type estimate for vector fields with inhomogeneous mixed boundary conditions)*.* As in Remark 8.4.2 there are corresponding generalised Friedrichs/Poincaré type estimates for weaker boundary data, where the $\mathsf{L}^2(\Gamma_{0/1})$-spaces and norms are replaced by $\mathsf{H}^{-s}(\Gamma_{0/1})$-spaces and norms.

8.5.2 A div-curl Lemma

Another immediate consequence is a div-curl-lemma.

Theorem 8.5.3 (div-curl lemma for vector fields with inhomogeneous mixed boundary conditions)**.** *Let (E_n) and (H_n) be bounded sequences in $\widehat{\mathsf{H}}_{\Gamma_0}(\mathrm{rot},\Omega)$ and $\widehat{\mathsf{H}}_{\Gamma_1}(\mathrm{div},\Omega)$, respectively. Then there exist $E \in \widehat{\mathsf{H}}_{\Gamma_0}(\mathrm{rot},\Omega)$ and $H \in \widehat{\mathsf{H}}_{\Gamma_1}(\mathrm{div},\Omega)$ as well as subsequences, again denoted by (E_n) and (H_n), such that $E_n \rightharpoonup E$ in $\widehat{\mathsf{H}}_{\Gamma_0}(\mathrm{rot},\Omega)$ and $H_n \rightharpoonup H$ in $\widehat{\mathsf{H}}_{\Gamma_1}(\mathrm{div},\Omega)$ as well as*

$$\langle E_n, H_n\rangle_{\mathsf{L}^2(\Omega)} \to \langle E, H\rangle_{\mathsf{L}^2(\Omega)}.$$

Proof. We follow in closed lines the proof of [43, Theorem 3.1]. Let (E_n) and (H_n) be as stated. First, we pick subsequences, again denoted by (E_n) and (H_n), and E and H, such that $E_n \rightharpoonup E$ in $\widehat{\mathsf{H}}_{\Gamma_0}(\mathrm{rot},\Omega)$ and $H_n \rightharpoonup H$ in $\widehat{\mathsf{H}}_{\Gamma_1}(\mathrm{div},\Omega)$. In particular,

$$\gamma_\nu^{\Gamma_1} H_n \rightharpoonup \gamma_\nu^{\Gamma_1} H \quad \text{in} \quad \mathsf{L}^2(\Gamma_1). \tag{8.1}$$

To see (8.1), let $\gamma_\nu^{\Gamma_1} H_n \rightharpoonup H_{\Gamma_1}$ in $\mathsf{L}^2(\Gamma_1)$. Since for all $\phi \in \mathsf{H}^1_{\Gamma_0}(\Omega)$

$$\begin{aligned}\langle \gamma_0^{\Gamma_1}\phi, H_{\Gamma_1}\rangle_{\mathsf{L}^2(\Gamma_1)} \leftarrow \langle \gamma_0^{\Gamma_1}\phi, \gamma_\nu^{\Gamma_1} H_n\rangle_{\mathsf{L}^2(\Gamma_1)} &= \langle \operatorname{grad}\phi, H_n\rangle_{\mathsf{L}^2(\Omega)} + \langle \phi, \operatorname{div} H_n\rangle_{\mathsf{L}^2(\Omega)}\\ &\to \langle \operatorname{grad}\phi, H\rangle_{\mathsf{L}^2(\Omega)} + \langle \phi, \operatorname{div} H\rangle_{\mathsf{L}^2(\Omega)},\end{aligned}$$

we get $H \in \widehat{\mathsf{H}}_{\Gamma_1}(\mathrm{div},\Omega)$ and $\gamma_\nu^{\Gamma_1} H = H_{\Gamma_1}$. Moreover, $\langle \gamma_0^{\Gamma_1}\phi, \gamma_\nu^{\Gamma_1} H_n\rangle_{\mathsf{L}^2(\Gamma_1)} \to \langle \gamma_0^{\Gamma_1}\phi, \gamma_\nu^{\Gamma_1} H\rangle_{\mathsf{L}^2(\Gamma_1)}$. As $\gamma_0^{\Gamma_1}\mathsf{H}^1_{\Gamma_0}(\Omega)$ is dense in $\mathsf{L}^2(\Gamma_1)$ and $\big(\langle\,\cdot\,, \gamma_\nu^{\Gamma_1} H_n\rangle_{\mathsf{L}^2(\Gamma_1)}\big)$ is uniformly bounded with respect to n we obtain (8.1).

By Theorem 8.3.2 we have the orthogonal Helmholtz decomposition

$$\widehat{\mathsf{H}}_{\Gamma_0}(\mathrm{rot},\Omega) \ni E_n = \operatorname{grad} u_n + \widetilde{E}_n$$

with $u_n \in \mathsf{H}^1_{\Gamma_0}(\Omega)$ and $\widetilde{E}_n \in \widehat{\mathsf{H}}_{\Gamma_0}(\mathrm{rot},\Omega) \cap \mathsf{H}_{\Gamma_1,0}(\mathrm{div},\Omega)$ as $\operatorname{grad}\mathsf{H}^1_{\Gamma_0}(\Omega) \subset \mathsf{H}_{\Gamma_0,0}(\mathrm{rot},\Omega) \subset \widehat{\mathsf{H}}_{\Gamma_0}(\mathrm{rot},\Omega)$. By orthogonality and the Friedrichs/Poincaré estimate, (u_n) is bounded in $\mathsf{H}^1_{\Gamma_0}(\Omega)$ and hence contains a strongly $\mathsf{L}^2(\Omega)$-convergent subsequence, again denoted by (u_n). (For $\Gamma_0 = \emptyset$ we may have to add a constant to each u_n.) Moreover, as $(u_n|_\Gamma)$ is bounded in $\mathsf{H}^{1/2}(\Gamma) \overset{\mathsf{cpt}}{\hookrightarrow} \mathsf{L}^2(\Gamma)$ we may assume that $(u_n|_\Gamma)$ converges strongly in $\mathsf{L}^2(\Gamma)$. In particular, $(\gamma_0^{\Gamma_1} u_n) = (u_n|_{\Gamma_1})$ converges strongly in $\mathsf{L}^2(\Gamma_1)$. The sequence $(\widetilde{E}_n)$ is bounded in $\widehat{\mathsf{H}}_{\Gamma_0}(\mathrm{rot},\Omega) \cap \mathsf{H}_{\Gamma_1,0}(\mathrm{div},\Omega)$ by orthogonality and since $\operatorname{rot}\widetilde{E}_n = \operatorname{rot} E_n$ and $\gamma_\tau^{\Gamma_0}\widetilde{E}_n = \gamma_\tau^{\Gamma_0} E_n$. Theorem 8.4.1 yields a strongly $\mathsf{L}^2(\Omega)$-convergent subsequence, again denoted by $(\widetilde{E}_n)$. Hence, there exist $u \in \mathsf{H}^1_{\Gamma_0}(\Omega)$ and $\widetilde{E} \in \widehat{\mathsf{H}}_{\Gamma_0}(\mathrm{rot},\Omega) \cap \mathsf{H}_{\Gamma_1,0}(\mathrm{div},\Omega)$ such that $u_n \rightharpoonup u$ in $\mathsf{H}^1_{\Gamma_0}(\Omega)$ and $u_n \to u$ in $\mathsf{L}^2(\Omega)$ and $\gamma_0^{\Gamma_1} u_n \to \gamma_0^{\Gamma_1} u$ in $\mathsf{L}^2(\Gamma_1)$ as well as $\widetilde{E}_n \rightharpoonup \widetilde{E}$ in $\widehat{\mathsf{H}}_{\Gamma_0}(\mathrm{rot},\Omega) \cap \mathsf{H}_{\Gamma_1,0}(\mathrm{div},\Omega)$ and $\widetilde{E}_n \to \widetilde{E}$ in $\mathsf{L}^2(\Omega)$. Finally, we compute

$$\begin{aligned}\langle E_n, H_n\rangle_{\mathsf{L}^2(\Omega)} &= \langle \operatorname{grad} u_n, H_n\rangle_{\mathsf{L}^2(\Omega)} + \langle \widetilde{E}_n, H_n\rangle_{\mathsf{L}^2(\Omega)}\\ &= -\langle u_n, \operatorname{div} H_n\rangle_{\mathsf{L}^2(\Omega)} + \langle \gamma_0^{\Gamma_1} u_n, \gamma_\nu^{\Gamma_1} H_n\rangle_{\mathsf{L}^2(\Gamma_1)} + \langle \widetilde{E}_n, H_n\rangle_{\mathsf{L}^2(\Omega)}\\ &\to -\langle u, \operatorname{div} H\rangle_{\mathsf{L}^2(\Omega)} + \langle \gamma_0^{\Gamma_1} u, \gamma_\nu^{\Gamma_1} H\rangle_{\mathsf{L}^2(\Gamma_1)} + \langle \widetilde{E}, H\rangle_{\mathsf{L}^2(\Omega)}\\ &= \langle \operatorname{grad} u, \operatorname{div} H\rangle_{\mathsf{L}^2(\Omega)} + \langle \widetilde{E}, H\rangle_{\mathsf{L}^2(\Omega)} = \langle E, H\rangle_{\mathsf{L}^2(\Omega)},\end{aligned}$$

since indeed $E = \operatorname{grad} u + \widetilde{E}$ holds by the weak convergence. ❑

Remark 8.5.4 (div-curl lemma for vector fields with inhomogeneous mixed boundary conditions). As in Remark 8.4.2 and Remark 8.5.2 there are corresponding generalised div-curl lemmas for weaker boundary data, where the $\mathsf{L}^2(\Gamma_{0/1})$-spaces and norms are replaced by $\mathsf{H}^{-s}(\Gamma_{0/1})$-spaces and norms.

8.5.3 Maxwell's Equations with Mixed Impedance Type Boundary Conditions

Let ε, μ be admissible and time-independent matrix fields, and let $T, k \in \mathbb{R}_+$. We consider Maxwell's equations with mixed tangential and impedance boundary conditions

$$\begin{aligned}
\tfrac{\partial}{\partial t} E - \varepsilon^{-1} \operatorname{rot} H &= F && \text{(Ampère/Maxwell law)} && (8.2a)\\
\tfrac{\partial}{\partial t} H + \mu^{-1} \operatorname{rot} E &= G && \text{(Faraday/Maxwell law)} && (8.2b)\\
\operatorname{div} \varepsilon E &= \rho, && \text{(Gauß law)} && (8.2c)\\
\operatorname{div} \mu H &= 0, && \text{(Gauß law for magnetism)} && (8.2d)\\
\gamma_\tau^{\Gamma_0} E &= 0, && \text{(perfect conductor bc)} && (8.2e)\\
\gamma_\nu^{\Gamma_0} H &= f, && \text{(normal trace bc)} && (8.2f)\\
\gamma_\tau^{\Gamma_1} E + k\gamma_{\tau\times}^{\Gamma_1} H &= 0, && \text{(impedance bc)} && (8.2g)\\
E(0) &= E_0 && \text{(electric initial value)} && (8.2h)\\
H(0) &= H_0 && \text{(magnetic initial value)} && (8.2i)
\end{aligned}$$

Note that the impedance boundary condition, also called Leontovich boundary condition, is of Robin type and that the impedance is given by $\lambda = 1/k = \sqrt{\varepsilon/\mu}$, if $\varepsilon, \mu \in \mathbb{R}_+$.

Despite of other recent and very powerful approaches such as the concept of "evolutionary equations", see the pioneering work of Rainer Picard, e.g., [50, 37], one can use classical semigroup theory for solving the Maxwell system (8.2).

We will split the system (8.2) into two static systems and a dynamic system. For simplicity we set $\varepsilon = \mu = 1$ and $F = G = 0$. The static systems are

$$\begin{aligned}
\operatorname{rot} E &= 0, & \operatorname{rot} H &= 0, && (8.3a)\\
\operatorname{div} E &= \rho, & \operatorname{div} H &= 0, && (8.3b)\\
\gamma_\tau^{\Gamma_0} E &= 0, & \gamma_\nu^{\Gamma_0} H &= f, && (8.3c)\\
\gamma_\tau^{\Gamma_1} E &= -kg, & \gamma_{\tau\times}^{\Gamma_1} H &= g, && (8.3d)
\end{aligned}$$

where g is any suitable tangential vector field in $\mathsf{L}^2(\Gamma_1)$. For simplicity we put $g = 0$, then these two systems are solvable by [4, Theorem 5.6]. However, the same result also gives conditions for which $g \neq 0$ this system is solvable. The dynamic system is

$$\begin{aligned}
\tfrac{\partial}{\partial t} E &= \operatorname{rot} H, && \text{(8.4a)}\\
\tfrac{\partial}{\partial t} H &= -\operatorname{rot} E, && \text{(8.4b)}\\
\operatorname{div} E &= 0, && \text{(8.4c)}\\
\operatorname{div} H &= 0, && \text{(8.4d)}\\
\gamma_{\nu}^{\Gamma_0} H &= 0, && \text{(8.4e)}\\
\gamma_{\tau}^{\Gamma_0} E &= 0, && \text{(8.4f)}\\
\gamma_{\tau}^{\Gamma_1} E + k\gamma_{\tau_\times}^{\Gamma_1} H &= 0. && \text{(8.4g)}
\end{aligned}$$

The initial conditions for the dynamic system are $E(0) = E_0 - E_{\text{stat}}$ and $H(0) = H_0 - H_{\text{stat}}$, where E_{stat} and H_{stat} are the solutions of the two static systems (8.3). We can write (8.4a) and (8.4b) as

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} E \\ H \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & \operatorname{rot} \\ -\operatorname{rot} & 0 \end{bmatrix}}_{=:A_0} \begin{bmatrix} E \\ H \end{bmatrix},$$

and the boundary conditions (8.4f) and (8.4g) shall be covered by the domain of A_0:

$$\operatorname{dom} A_0 := \Big\{ (E,H) \in \widehat{\mathsf{H}}_{\Gamma}(\operatorname{rot},\Omega) \times \widehat{\mathsf{H}}_{\Gamma_1}(\operatorname{rot},\Omega) \,\Big|\, \gamma_{\tau}^{\Gamma_0} E = 0,\ \gamma_{\tau}^{\Gamma_1} E + k\gamma_{\tau_\times}^{\Gamma_1} H = 0 \Big\}.$$

Here, we did ignore the equations $\operatorname{div} E = 0$, $\operatorname{div} H = 0$ and $\gamma_{\nu}^{\Gamma_0} H = 0$. However, A_0 is a generator of a C_0-semigroup, by Section 6.4. The next lemma provides a tool to respect the remaining conditions of (8.4) as well.

Lemma 8.5.5. *Let $T(\cdot)$ be a C_0-semigroup on a Banach space X, and let A be its generator. Then every subspace $V \supseteq \operatorname{ran} A$ is invariant under $T(\cdot)$. Moreover, $A\big|_V$ generates the strongly continuous semigroup $T_V(\cdot) := T(\cdot)\big|_V$, if V is additionally closed in X.*

Proof. Let $t \geq 0$ and let $x \in V$. Then $\operatorname{ran} A \ni A\int_0^t T(s)x \,\mathrm{d}s = T(t)x - x$ and hence $T(t)x \in V$. The remaining assertion follows from [15, Chapter II, Section 2.3]. ❑

Therefore, it is left to show that the remaining conditions establish a closed and invariant subspace under the semigroup T_0 generated by A_0 or contains $\operatorname{ran} A_0$. Note that by Theorem 8.3.1

$$\begin{aligned}
S &:= \big\{ (E,H) \,\big|\, \operatorname{div} E = 0,\ \operatorname{div} H = 0,\ \gamma_{\nu}^{\Gamma_0} H = 0 \big\}\\
&= \mathsf{H}_0(\operatorname{div},\Omega) \times \mathsf{H}_{\Gamma_0,0}(\operatorname{div},\Omega)\\
&= \big(\operatorname{rot} \mathsf{H}(\operatorname{rot},\Omega) \times \operatorname{rot} \mathsf{H}_{\Gamma_0}(\operatorname{rot},\Omega) \big) \oplus \big(\mathcal{H}_{\Gamma,\emptyset}(\Omega) \times \mathcal{H}_{\Gamma_1,\Gamma_0}(\Omega) \big).
\end{aligned}$$

This space is closed as the intersection of kernels of closed operators. Clearly, $\mathcal{H}_{\Gamma,\emptyset}(\Omega) \times \mathcal{H}_{\Gamma_1,\Gamma_0}(\Omega)$ is invariant under T_0, since every $(E,H) \in \mathcal{H}_{\Gamma,\emptyset}(\Omega) \times \mathcal{H}_{\Gamma_1,\Gamma_0}(\Omega)$ is a constant in time solution of the system (8.4), i.e.,

$$T_0(t)\begin{bmatrix} E \\ H \end{bmatrix} = \begin{bmatrix} E \\ H \end{bmatrix}.$$

By

$$\operatorname{rot} \mathsf{H}(\operatorname{rot},\Omega) \times \operatorname{rot} \mathsf{H}_{\Gamma_0}(\operatorname{rot},\Omega) = \begin{bmatrix} 0 & \operatorname{rot} \\ -\operatorname{rot} & 0 \end{bmatrix} \big(\mathsf{H}_{\Gamma_0}(\operatorname{rot},\Omega) \times \mathsf{H}(\operatorname{rot},\Omega)\big) \supseteq \operatorname{ran} A_0$$

and Lemma 8.5.5 we have that also $\operatorname{rot} \mathsf{H}(\operatorname{rot},\Omega) \times \operatorname{rot} \mathsf{H}_{\Gamma_0}(\operatorname{rot},\Omega)$ is invariant under T_0. Hence, Lemma 8.5.5 and Theorem 8.4.1 imply the next theorem.

Theorem 8.5.6. $A \coloneqq A_0\big|_S$ *is a generator of a* C_0*-semigroup and*

$$\operatorname{dom} A \subseteq \big(\widehat{\mathsf{H}}_{\Gamma}(\operatorname{rot},\Omega) \cap \mathsf{H}(\operatorname{div},\Omega)\big) \times \big(\widehat{\mathsf{H}}_{\Gamma_1}(\operatorname{rot},\Omega) \cap \mathsf{H}_{\Gamma_0}(\operatorname{div},\Omega)\big) \overset{\mathsf{cpt}}{\hookrightarrow} \mathsf{L}^2(\Omega).$$

Consequently, every resolvent operator of A *is compact.*

If $\mathcal{H}_{\Gamma,\emptyset}(\Omega) = \{0\}$ and $\mathcal{H}_{\Gamma_1,\Gamma_0}(\Omega) = \{0\}$, then 0 is in the resolvent set of A and A^{-1} is compact. Alternatively, we can further restrict A to $\mathcal{H}_{\Gamma,\emptyset}(\Omega)^{\perp_{\mathsf{L}^2(\Omega)}} \times \mathcal{H}_{\Gamma_1,\Gamma_0}(\Omega)^{\perp_{\mathsf{L}^2(\Omega)}}$. This would also match our separation of static solutions and dynamic solutions, since solutions with initial condition in $\mathcal{H}_{\Gamma,\emptyset}(\Omega) \times \mathcal{H}_{\Gamma_1,\Gamma_0}(\Omega)$ are constant in time.

Appendix A

Appendix

A.1 Gårding Inequalities

In this section we want to show that there is a Fredholm alternative for sesquilinear forms that are non-coercive, but satisfy a *Gårding inequality*. In [66] this concept is presented in a less abstract way for differential operators.

Definition A.1.1. Let X_0 and X_1 be Hilbert spaces and $K\colon X_1 \to X_0$ be a compact linear operator. A sesquilinear form $b\colon X_1 \times X_1 \to \mathbb{C}$ satisfies a *Gårding inequality*, if

$$\operatorname{Re} b(u,u) \geq C_1 \|u\|_{X_1}^2 - C_2 \|Ku\|_{X_0}^2 \quad \text{for all} \quad u \in X_1.$$

In most applications K is a compact embedding, e.g. the embedding of $\mathsf{H}^1(\Omega)$ into $\mathsf{L}^2(\Omega)$. Note that (by Lax-Milgram, e.g. [16]) for every bounded sesquilinear form $b(\cdot,\cdot)$ on a Hilbert space there exists a bounded operator $B\colon X_1 \to X_1$ such that

$$b(u,v) = \langle Bu, v\rangle_{X_1} \quad \text{for all} \quad u, v \in X_1.$$

The operator B is injective if and only if $b(\cdot,\cdot)$ is non-degenerated.

Theorem A.1.2 (Fredholm alternative)**.** *Let $b(\cdot,\cdot)$ be a bounded sesquilinear form on X_1 that satisfies a Gårding inequality. If the corresponding operator B is injective ($b(\cdot,\cdot)$ is non-degenerated), then B is bijective.*

Proof. The sesquilinear form b satisfies the Gårding inequality

$$\operatorname{Re} b(u,u) \geq C_1 \|u\|_{X_1}^2 - C_2 \|Ku\|_{X_0}^2 \quad \text{for all} \quad u \in X_1.$$

Hence, $\tilde{b}(u,v) := b(u,v) + C_2 \langle Ku, Kv\rangle_{X_0}$ is coercive. The corresponding operator $\tilde{B}$ is given by $B + C_2 K^* K$. By the Lax-Milgram theorem $\tilde{B}$ is bijective. Note that

$$B = \tilde{B} - C_2 K^* K = \tilde{B}(\mathrm{I} - \tilde{B}^{-1} C_2 K^* K).$$

The injectivity of B implies that 1 is not an eigenvalue of $\tilde{B}^{-1} C_2 K^* K$ and since $\tilde{B}^{-1} C_2 K^* K$ is compact, it is surjective. Consequently B is also surjective. ❑

A.2 Solution Theory for the Wave Equation

In this section we will discuss a suitable solution concept for (7.1). We will regard a solution $w(\cdot,\cdot)$ as a function in time mapping into spatial function space.

An integrated version of the PDE is

$$\rho(\zeta)\frac{\partial}{\partial t}w(t,\zeta) - \rho(\zeta)w_1(\zeta) = \int_0^t \operatorname{div} T(\zeta) \operatorname{grad} w(s,\zeta)\,\mathrm{d}s.$$

We will demand that a solution will satisfy this integrated version of the PDE.

If we assume that both

$$\int_0^t \operatorname{div} T(\zeta) \operatorname{grad} w(s,\zeta)\,\mathrm{d}s \quad \text{and} \quad \operatorname{div} T(\zeta) \int_0^t \operatorname{grad} w(s,\zeta)\,\mathrm{d}s$$

exist, then they coincide and

$$\rho(\zeta)\frac{\partial}{\partial t}w(t,\zeta) - \rho(\zeta)w_1(\zeta) = \operatorname{div} T(\zeta) \int_0^t \operatorname{grad} w(s,\zeta)\,\mathrm{d}s.$$

This is a consequence of the closedness of div. For a classical solution ($w \in \mathsf{C}^2(\mathbb{R}_+ \times \Omega) \cap \mathsf{C}^1(\mathbb{R}_+ \times \overline{\Omega})$) these integrals coincide.

We will also regard an integrated version of the boundary conditions:

$$\int_0^t \frac{\mathrm{d}}{\mathrm{d}s}w(s,\zeta)\,\mathrm{d}s = -k\int_0^t \nu \cdot T \operatorname{grad} w(s,\zeta)\,\mathrm{d}s$$

for all $\zeta \in \Gamma_1$. Again for classical solutions this can be manipulated to

$$w(t,\zeta) - w(0,\zeta) = -k\nu \cdot T \int_0^t \operatorname{grad} w(s,\zeta)\,\mathrm{d}s \quad \text{for all} \quad \zeta \in \Gamma_1,$$

$$\gamma_0 w(t,\cdot)\big|_{\Gamma_1} - \gamma_0 w(0,\cdot)\big|_{\Gamma_1} = -k\gamma_\nu\Big(T\int_0^t \operatorname{grad} w(s,\cdot)\,\mathrm{d}s\Big)\Big|_{\Gamma_1}.$$

Definition A.2.1. Let $w_0 \in \mathsf{H}^1(\Omega)$ and $w_1 \in \mathsf{L}^2(\Omega)$. Then we say that $w(\cdot,\cdot)$ is a *solution* of (7.1), if $t \mapsto w(t,\cdot)$ is $\mathsf{C}^1(\mathbb{R}_+;\mathsf{L}^2(\Omega)) \cap \mathsf{C}^0(\mathbb{R}_+;\mathsf{H}^1(\Omega))$, and

$$\begin{aligned}
\rho\frac{\mathrm{d}}{\mathrm{d}t}w(t,\cdot) - \rho w_1 &= \operatorname{div} T \int_0^t \operatorname{grad} w(s,\cdot)\,\mathrm{d}s,\\
w(0,\cdot) &= w_0,\\
\frac{\mathrm{d}}{\mathrm{d}t}w(t,\cdot)\Big|_{t=0} &= w_1,\\
\gamma_0 w(t,\cdot)\big|_{\Gamma_0} &= h,\\
\gamma_0 w(t,\cdot)\big|_{\Gamma_1} - \gamma_0 w_0\big|_{\Gamma_1} &= -k\gamma_\nu\Big(T\int_0^t \operatorname{grad} w(s,\cdot)\,\mathrm{d}s\Big)\Big|_{\Gamma_1},
\end{aligned}$$

for all $t \in \mathbb{R}_+$.

Proposition A.2.2. *Let w is a solution of* (7.1) *in the sense of Definition A.2.1 and w_{e} the solution of the equilibrium system* (7.2)*. Then*

$$\begin{bmatrix} \rho\frac{\partial}{\partial t}w(t,\cdot) \\ \operatorname{grad} w(t,\cdot) - \operatorname{grad} w_{\mathrm{e}} \end{bmatrix} \quad \text{and} \quad T(t)\begin{bmatrix} \rho w_1 \\ \operatorname{grad} w_0 - \operatorname{grad} w_{\mathrm{e}} \end{bmatrix}$$

coincide, where T is the semigroup generated by A.

On the other hand, let x_1 denote the first component of the solution given by the semigroup. Then

$$w(t,\cdot) := \int_0^t \frac{1}{\rho} x_1(s)\,\mathrm{d}s + w_0 + w_{\mathrm{e}}$$

is a solution of (7.1) *in the sense of Definition A.2.1.*

Remark A.2.3. If we regard the semigroup T_0 generated by A_0, we can even cancel out $\operatorname{grad} w_{\mathrm{e}}$ and obtain

$$\begin{bmatrix} \rho\frac{\partial}{\partial t}w(t,\cdot) \\ \operatorname{grad} w(t,\cdot) \end{bmatrix} = T_0(t)\begin{bmatrix} \rho w_1 \\ \operatorname{grad} w_0 \end{bmatrix}$$

Theorem A.2.4. *The system* (7.2) *is solvable for $h \in \mathsf{H}^{1/2}(\Gamma_0)$.*

Proof. Let $H \in \mathsf{H}^1(\Omega)$ such that $h = \gamma_0 H\big|_{\Gamma_0}$. The weak formulation of (7.2) is: find a $\tilde{w} \in \mathsf{H}^1_{\Gamma_0}(\Omega)$ such that

$$\langle \operatorname{grad} \tilde{w}, \operatorname{grad} v\rangle_{\mathsf{L}^2(\Omega)} = -\langle \operatorname{grad} H, \operatorname{grad} v\rangle_{\mathsf{L}^2(\Omega)}$$

for all $v \in \mathsf{H}^1_{\Gamma_0}(\Omega)$. Then $w_{\mathrm{e}} = \tilde{w} + H$. By the Lax-Milgram theorem this is solvable. ❑

A.3 Uncategorized

Lemma A.3.1 (Polarization identity)**.** *Let X be a vector space and $b\colon X \times X \to \mathbb{C}$ be a sesquilinear form. Then*

$$4b(x,y) = b(x+y,x+y) - b(x-y,x-y) + \mathrm{i}b(x+\mathrm{i}y,x+\mathrm{i}y) - \mathrm{i}b(x-\mathrm{i}y,x-\mathrm{i}y).$$

Proof. Note that

$$\begin{aligned} b(x+y,x+y) &= b(x,x) + b(x,y) + b(y,x) + b(y,y), \\ -b(x-y,x-y) &= -b(x,x) + b(x,y) + b(y,x) - b(y,y), \\ \mathrm{i}b(x+\mathrm{i}y,x+\mathrm{i}y) &= \mathrm{i}b(x,x) + b(x,y) - b(y,x) + \mathrm{i}b(y,y), \\ -\mathrm{i}b(x-\mathrm{i}y,x-\mathrm{i}y) &= -\mathrm{i}b(x,x) + b(x,y) - b(y,x) - \mathrm{i}b(y,y). \end{aligned}$$

Summing this four equations yields the statement. ❑

Lemma A.3.2. *Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in a normed vector space X that converges w.r.t. the weak topology to an $x_0 \in X$. Then $(x_n)_{n\in\mathbb{N}}$ is bounded i.e. $\sup_{n\in\mathbb{N}}\|x_n\|_X < +\infty$.*

Proof. Let ι denote the canonical embedding from X into X'' that maps x to $\langle x, \cdot\rangle_{X,X'}$. Then, by assumption, for every fixed $\phi \in X'$ $(\iota x_n)(\phi) \to (\iota x_0)(\phi)$, in particular $\sup_{n\in\mathbb{N}}|(\iota x_n)(\phi)| < \infty$. The principle of uniform boundedness yields $\sup_{n\in\mathbb{N}}\|\iota x_n\|_{X''} < +\infty$. Since $\|\iota x\|_{X''} = \|x\|_X$ for every $x \in X$, this proves the assertion. ❑

Lemma A.3.3. *Let $(x_n)_{n\in\mathbb{N}}$ be a weak convergent sequence in a Hilbert space H with limit x. Then there exists a subsequence $(x_{n(k)})_{k\in\mathbb{N}}$ such that*

$$\left\|\frac{1}{N}\sum_{k=1}^{N} x_{n(k)} - x\right\| \to 0.$$

Proof. We assume that $x = 0$. For the general result we just need to replace x_n by $x_n - x$.

We define the subsequence inductively: $n(1) = 1$ and for $k > 1$ we choose $n(k)$ such that

$$|\langle x_{n(k)}, x_{n(j)}\rangle| \leq \frac{1}{k} \quad \text{for all} \quad j < k.$$

This is possible, because $(x_n)_{n\in\mathbb{N}}$ converges weakly to 0. Hence, by Lemma A.3.2 $\sup_{n\in\mathbb{N}}\|x_n\| \leq C$. This yields

$$\begin{aligned}
\left\|\frac{1}{N}\sum_{k=1}^{N} x_{n(k)}\right\|^2 &= \frac{1}{N^2}\sum_{k=1}^{N}\sum_{j=1}^{N}\langle x_{n(k)}, x_{n(j)}\rangle \\
&= \frac{1}{N^2}\sum_{k=1}^{N}\|x_{n(k)}\|^2 + \frac{1}{N^2}\sum_{j=1}^{N}\sum_{k=j+1}^{N} 2\,\mathrm{Re}\langle x_{n(k)}, x_{n(j)}\rangle \\
&\leq \frac{1}{N}C^2 + \frac{2}{N^2}\sum_{j=1}^{N}\sum_{k=j+1}^{N}\frac{1}{k} \leq \frac{C^2}{N} + \frac{1}{N}\ln(N) \to 0.
\end{aligned}$$

❑

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