Partial Differential Equations and Spatial Structures of Lévy Type

Uncertainty Quantification and Optimization

BERGISCHE
UNIVERSITÄT
WUPPERTAL

Dissertation

zur Erlangung des akademischen Grades eines
Doktors der Naturwissenschaften

an der Fakultät für Mathematik und Naturwissenschaften der
Bergischen Universität Wuppertal

von

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The PhD thesis can be quoted as follows:

urn:nbn:de:hbz:468-urn:nbn:de:hbz:468-20211207-120639-2
[http://nbn-resolving.de/urn/resolver.pl?urn=urn%3Anbn%3Ade%3Ahbz%3A468-20211207-120639-2]

DOI: 10.25926/03pn-nd43
[https://doi.org/10.25926/03pn-nd43]

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Chapter 1

Introduction

Many branches of applied mathematics consider physical, biological, or social systems, governed by highly complex processes, with attributes showing complex patterns of variation in space and time. For example, diffusion processes describe the evolution in time of the density of quantities as heat, fluids, or chemical substances. Fluid dynamics is concerned with the flow of liquids and gases with subdisciplines as aerodynamics or hydrodynamics. In fields like acoustics and electromagnetics, wave equations are used in order to describe light waves or mechanical waves as water waves, sound waves, and seismic waves. These are only a few examples of many more disciplines where the mathematical modeling have to provide a sufficient understanding of processes occuring in the real world. At the same time, as these systems are highly complex, it is often not possible to describe them precisely. In theory, an attribute of a system at a given location, e.g., the conductivity of a porous medium, is deterministic and by measuring it at each location of a system we possibly could describe it deterministically. However, for practical purposes, it would be impractical to do so. When the degree of disorder in a system gets too large, it is more reasonable to approach it in a probabilistic rather than deterministic manner. The inclusion of random fields makes it possible to model complex patterns of variations and spatially correlated values by their covariance structures. A random field allows to capture the essential coefficients of a model and makes them accessible by only a few meaningful parameters. The definition of random fields varies, but is traditionally given by a collection of multidimensional random variables, where the index set is given in time or space. If the index set descibes time, we consider stochastic processes. The time scale is potentially the interval between molecular collsions, as in the study of Brownian motion, or it may in geological units, for example, to describe the variation of properties and thickness of layers of the earth's crust. Spatial random fields are used to e.g., describe subatomic particles in the study of superheated plasma or temperature, density, or chemical composition of matter in interstellar space [112]. As a random field usually cannot cover all effects of a system, the choice of which specific random field should be applied depends on the stochastic nature of the underlying problem.

Generalized random fields are an extension to fields with infinite-dimensional index spaces. Using generalized random fields, we can describe set-valued and distribution-valued random fields as, e.g., point processes or the classical white noise. This thesis presents two applications in which generalized random fields are used to model complex systems of deterministic physical processes in a probabilistic manner. In chapter 4, we extend a well known shape optimization setting into a multi-criteria optimization problem. As an example, we illustrate an application on the optimization process for vanes of gas turbines. Clearly, the reliability of such a component posseses a major role in the optimization process and thus a mathematical language to capture the time of failure must be developed. The event of failure of a vane is associated with the formation of the first crack. Due to the probabilistic nature of crack initiation [60] in time and location, this event is often described using the language of Poisson point processes. This perspective leads to a probability that the time to failure, i.e., the time that passes until the formation of the first crack, lies within a warranty time interval of the lenght t.

The second implementation of generalized random fields is presented in chapter 5 where we investigate the linear stationary diffusion equation. The diffusion equation finds its use in many branches of natural science and engineering. For example, driven by Darcy's law, it describes the flow of a fluid through a porous medium by the equation $-a(x)\nabla u(x)$ on a domain D, where u is the concentration of the diffusing fluid, and a is the conductivity associated to the domain D. In a setting such as the flow of groundwater, the values of the conductivity coefficient are uncertain, as they are derived from sparsed oberservations. The well established approach to deal with these uncertainties includes Gaussian random fields describing the conductivity function a. Gaussian fields have been extensively investigated in the context of uncertainty quantification. However, there are effects which cannot be fully explained by Gaussian random fields, as e.g. the flow diffusion in fractured media or the modeling of heterogeneous materials with two phases. Therefore, an extension to the Gaussian approach is necessary. Chapter 5 uses generalized random fields to describe a possible extension by generalizing the Gaussian coefficients into Lévy type coefficients and gives fundamental results in order to prepare the numerical treatment of diffusion equations with Lévy coefficient functions.

This work is organized as follows: The first two chapters provide introductions to the underlying mathematical language applied in chapter 4 and 5. The second chapter covers partial differential equations (PDEs). As in this thesis we investigate solutions to PDEs in the weak as well as in the classical sense, the second chapter gives a brief overview of definitions and results on Hölder continuity and boundary regularity before it proceeds to the theory of Sobolev spaces. Afterwards, we continue to define and classify systems of partial differential equations in the terms of [4, 5]. For weak and classical solutions, we derive existence results, based on an index theorem for Fredholm operators, along with corresponding Schauder estimates which are applied in chapter 4 in order to prove the compactness of the associated solution spaces. Chapter 2 ends by defining the equation of linear elasticity, a potential flow equation, and the linear diffusion equation as examples for partial differential equations. Chapter 3 introduces generalized random fields to investigate crack initiation on mechanical components and diffusion equation with Lévy coefficient functions in chapter 4 and 5, respectively. Following [42] and [69], it describes Lévy random fields by using Minlos' Theorem based on the concept of multi-Hilbertian spaces. We derive continuity conditions for Lévy noise fields smoothed with smoothing kernels (by a convolution of the random field with the kernel in the distributional sense) from the Matérn class so that we can use them as coefficient function in a differential equation. Chapter 3 ends by providing some examples for Lévy random fields. After the preperation, in chapter 4, we present the first application [59]. Chapter 4 is focused towards multi-criteria shape optimization of mechanical elements. We introduce a multi-criteria shape optimization framework and apply it, as an example, on the optimization of the shape of a turbine vane with respect to lifespan and efficiency. As the integrity of a component is crucial for every optimal design process, the chapter starts with an extensive description of the probabilistic modeling of failure events using Poisson point processes. The last section of this chapter explores scalarization methods and the sensitivity of the sets of optimal shapes in dependency of the scalarization parameters. Chapter 5 deals with random diffusion equation with Lévy diffusion coefficients [42]. We investigate the existence of moments of the H^1-Sobolev norm of the random solution and their approximability by approximating the random diffusion coefficient by a finite modal expansion. This thesis then ends in chapter 6 with its conclusion and an outlook on questions which are worth investigating as part of future work.

The introduction of the generalized random fields in chapter 3 as well as the investigation of random diffusion equation in chapter 5, is based on the joint work [42] with my advisor Hanno Gottschalk from the University of Wuppertal and Oliver Ernst, Thomas Kalmes, and Toni Kowalewitz from the Chemnitz University of Technology. Whereas chapter 3 was mainly written by the collaborators from Chemnitz, most of chapter 5 was the work of myself under the academic supervision of Hanno Gottschalk, including a number of corrections and suggestions by Thomas Kalmes. The shape optimization framework and results of chapter 4 is based on the work [59] of myself under the supervision of Hanno Gottschalk.

Chapter 2

Linear Partial Differential Equations

The first three sections of this chapter provide relevant definitions and results on Hölder continuity, boundary regularity of domains, and Sobolev spaces, which are needed throughout this thesis and are particularly important for the theory of boundary value problems in Section 2.4.

2.1 Hölder Spaces

Let Ω be an open subset of $\mathbb{R}^d$ and $u : \Omega \to \mathbb{R}$ a bounded and continuous function on Ω. For $0 < \alpha \leqslant 1$ let

$$\|u\|_{C^0(\Omega,\mathbb{R})} := \sup_{x\in\Omega}|u| \quad \text{and} \quad [u]_{C^{0,\alpha}(\Omega,\mathbb{R})} := \sup_{\substack{x,y\in\Omega \\ x\neq y}} \left\{ \frac{|u(x)-u(y)|}{|x-y|^\alpha} \right\}.$$

If $[u]_{C^{0,\alpha}(\Omega,\mathbb{R})} < \infty$, then u is Hölder continuous with Hölder coefficient α. The collection of all bounded functions in $C^k(\Omega,\mathbb{R})$, which derivatives up to order $k \in \mathbb{N}_0$ are α-Hölder continuous, will be denoted by

$$C^{k,\alpha}(\Omega,\mathbb{R}).$$

With $\beta \in \mathbb{N}_0^d$ denoting a multi-index, we define the norm on $C^{k,\alpha}(\Omega,\mathbb{R})$

$$\|u\|_{C^{k,\alpha}(\Omega,\mathbb{R})} := \sum_{|\beta|\leqslant k} \|D^\beta u\|_{C^0(\Omega,\mathbb{R})} + \sum_{|\beta|=k} [D^\beta u]_{C^{0,\alpha}(\Omega,\mathbb{R})},$$

where $|\beta| = \sum_{j=1}^d \beta_j$, which makes $C^{k,\alpha}(\Omega,\mathbb{R})$ into a Banach space.

Definition 2.1.1 ($C^{k,\alpha}$-Diffeomorphism)**.** *Consider bounded subsets* Ω, Ω' *of* $\mathbb{R}^d$ *and a bijective map* $f : \Omega \to \Omega'$. *If* $f \in C^{k,\alpha}(\Omega)$ *and* $f^{-1} \in C^{k,\alpha}(\Omega)$, *then we say that* f *is a* $C^{k,\alpha}$*-diffeomorphism. The sets of all* $C^{k,\alpha}$*-diffeomorphism from* Ω *to* Ω' *is denoted with*

$$\mathcal{D}^{k,\alpha}(\Omega, \Omega'),$$

and when $\Omega = \Omega'$, *we also write*

$$\mathcal{D}^{k,\alpha}(\Omega).$$

2.2 Regularity Properties of Domains

Definition 2.2.1 (Cone Condition)**.** *A subset* Ω *of* $\mathbb{R}^d$ *satisfies a uniform (interior) cone condition, based on radius* $r > 0$ *and angle* $\beta \in]0, \frac{\pi}{2}[$, *if, for every* $x \in \partial\Omega$, *there is at least one unit vector* υ *such that the cone* $C_\upsilon := \{y \in \mathbb{R}^d : \langle y, \upsilon \rangle > \|y\| \cos(\beta)\}$ *satisfies*

$$(t + C_\upsilon) \cap B_r(x) \subset \Omega, \quad \textit{for all } t \in \Omega \cap B_r(x),$$

where $B_r(x)$ *is an open ball centered at* x *with radius* r *and* $\|\cdot\|$ *denotes the Euclidean norm. We say a family of subsets* $\mathcal{O}$ *satisfies a uniform cone condition if any subset* $\Omega \in \mathcal{O}$ *fulfills a uniform cone condition based on the same radius* r *and angle* β.

Lemma 2.2.2 ([60, Lemma 5.5])**.** *Let* $\mathcal{O}$ *consist of bounded subsets of* $\mathbb{R}^d$. *If for* $\mathcal{O}$ *a uniform cone condition holds true, then, for every* $\varepsilon > 0$, *there exists a constant* $C_\varepsilon > 0$, *uniform with respect to* $\mathcal{O}$, *such that for any* $u \in C^1(\Omega, \mathbb{R})$

$$\|u\|_{C^0(\Omega,\mathbb{R})} \leqslant \varepsilon \|u\|_{C^1(\Omega,\mathbb{R})} + C_\varepsilon \|u\|_{L^1(\Omega,\mathbb{R})}.$$

Definition 2.2.3 ($C^{k,\alpha}$-boundary)**.** *A subset* Ω *in* $\mathbb{R}^d$ *and its boundary are of class* $C^{k,\alpha}$, *if at each point* $x \in \partial\Omega$ *there is a ball* $B = B_r(x)$ *and a one-to-one mapping* ψ *of* B *onto* $\Omega' \subset \mathbb{R}^d$ *such that*

(i) $\psi(B \cap \Omega) \subset \mathbb{R}^d_{\geqslant 0}$,

(ii) $\psi(B \cap \partial\Omega) \subset \partial\mathbb{R}^d_{\geqslant 0}$,

(iii) $\psi \in C^{k,\alpha}(B, \mathbb{R}^d)$, $\psi^{-1} \in C^{k,\alpha}(\Omega', \mathbb{R}^d)$.

A set Ω *has a boundary portion* $\Gamma \subset \partial\Omega$ *of class* $C^{k,\alpha}$, *if at each point* $x \in \Gamma$ *there is a ball* $B = B_r(x)$ *in which the above conditions are satisfied and such that* $B \cap \partial\Omega \subset \Gamma$.

We shall say that the diffeomorphism ψ straightens the boundary near x. In this work, we shall always assume that $k \in \mathbb{N}_0$ and $0 < \alpha \leqslant 1$ unless we restrict further.

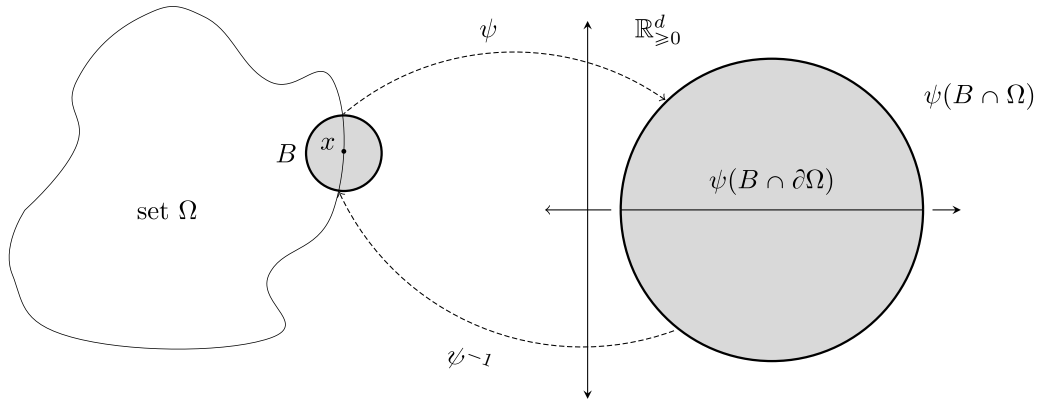

Figure 2-1: Hölder continuous diffeomorphism ψ from B to $\mathbb{R}^d$

Remark 2.2.4. *(i) If only $\psi \in C^k(B, \mathbb{R}^d)$, $\psi^{-1} \in C^k(\Omega, \mathbb{R}^d)$, then we shall say that Ω is of class C^k. In the case of $C^{0,1}$, we say that Ω possesses a Lipschitz boundary.*

(ii) We note that Ω is a $C^{k,\alpha}$-domain if each point of $\partial\Omega$ has a neighborhood in which $\partial\Omega$ is the graph of a $C^{k,\alpha}$ function of $n-1$ of the coordinates $x_1, \ldots, x_n$. The converse is also true if $k \geqslant 1$; see, e.g., [38, Chapter 2, Theorem 5.5].

Definition 2.2.5 (Hemisphere Condition)**.** *Let $\Omega \subset \mathbb{R}^d$ be a domain with regular boundary portion $\Gamma \subset \partial\Omega$. Consider a subdomain $S \subset \Omega$ with $\partial S \cap \partial\Omega \subset \mathring{\Gamma}$. We shall say S satisfies a hemisphere condition if there is a distance $r > 0$ such that every $x \in S$, with $\operatorname{dist}(x, \partial\Omega) \leqslant r$, possesses a neighborhood U_x with*

(i) $\operatorname{cl}(U_x) \cap \partial\Omega \subset \Gamma$,

(ii) $B_{r/2}(x) \subset U_x$,

(iii) $\operatorname{cl}(U_x) \cap \operatorname{cl}(\Omega) = \mathbb{T}_x\left(\Sigma_{R(x)}\right), \quad \operatorname{cl}(U_x) \cap \partial\Omega = \mathbb{T}_x\left(F_{R(x)}\right), \quad 0 < R(x) \leqslant 1.$

Here, $\Sigma_{R(x)} = \{y \in \mathbb{R}^d : \|y\| \leqslant R(x),\ y_d \geqslant 0\}$ is a hemisphere, $F_{R(x)} = \{y \in \mathbb{R}^d : \|y\| \leqslant R(x),\ y_d = 0\}$ is a disk, and $\mathbb{T}_x$ is a one-to-one mapping and called hemisphere transform. If S and $\mathbb{T}_x, \mathbb{T}_x^{-1}$ are of class $C^{k,\alpha}$, we say that S satisfies a $C^{k,\alpha}$-hemisphere condition.

Remark 2.2.6. *A domain Ω which satisfies a hemisphere condition already possesses a boundary of class $C^{k,\alpha}$. However, the converse is only true for bounded domains (see, e.g., [19, Lemma 6.1]), since a hemisphere condition needs a lower bound for the size of the neighborhood of the points near to the boundary, which is not provided in the definition of Hölder continuous boundaries.*

Definition 2.2.7 (Uniform Hemisphere Condition)**.** *Let $\mathcal{O}$ be a family of domains where each domain $\Omega \in \mathcal{O}$ satisfies a hemisphere condition. If the distance of each hemisphere condition as well as the bounds of the corresponding hemisphere transforms are uniform over $\mathcal{O}$, i.e., we have $\|\mathbb{T}_{\Omega,x}\|_{C^{k,\alpha}(\mathbb{R}^d,\mathbb{R}^d)} \leqslant C_{\mathcal{O}}$, for some $C_{\mathcal{O}} > 0$, we shall say that $\mathcal{O}$ satisfies a uniform hemisphere condition.*

Lemma 2.2.8 ([55, Lemma 6.36])**.** *Let Ω be a domain in $\mathbb{R}^d$ with boundary of class $C^{k,\alpha}$, with $k \geqslant 1$, and let S be a bounded set in $C^{k,\alpha}(\mathrm{cl}(\Omega),\mathbb{R})$. Then, S is precompact in $C^{j,\beta}(\mathrm{cl}(\Omega),\mathbb{R})$ if $j+\beta < k+\alpha$.*

Lemma 2.2.9 ([55, Lemma 6.37])**.** *Let Ω be a bounded domain in $\mathbb{R}^d$ with boundary of class $C^{k,\alpha}$, with $k \geqslant 1$, and let Ω' be an open set containing $\mathrm{cl}(\Omega)$. For every $u \in C^{k,\alpha}(\mathrm{cl}(\Omega),\mathbb{R})$ there exists an extension operator $p : C^{k,\alpha}(\Omega,\mathbb{R}) \to C^{k,\alpha}_c(\Omega',\mathbb{R})$ such that $p(u) = u$ in Ω and*

$$\|p(u)\|_{C^{k,\alpha}(\Omega',\mathbb{R})} \leqslant C\|u\|_{C^{k,\alpha}(\Omega,\mathbb{R})}, \tag{2.1}$$

where C depends on Ω, Ω' and k, and where $C^{k,\alpha}_c(\Omega',\mathbb{R})$ denotes the space of all functions in $C^{k,\alpha}(\Omega',\mathbb{R})$ with compact support in Ω'.

Remark 2.2.10. *Let $\mathcal{O}$ be a family of sets in $\mathbb{R}^d$ with boundary of class $C^{k,\alpha}$ ($k \geqslant 1$). If $\mathcal{O}$ possesses a uniform hemisphere condition, then it is shown in [19, Lemma 7.2] that the constant C in Equation (2.1), which depends on the hemisphere transforms to Ω, can be choosen independently with respect to $\mathcal{O}$.*

2.3 Sobolev Spaces

Let Ω be an open subset of $\mathbb{R}^d$. $L^1_{\text{loc}}(\Omega, \mathbb{R})$ denotes the space of all locally integrable functions $u : \Omega \to \mathbb{R}$. Assuming we are given functions $u, v \in L^1_{\text{loc}}(\Omega, \mathbb{R})$ and let $\gamma \in \mathbb{N}_0^d$ be a multi-index. We say that v is the γ^{th}-weak partial derivative of u, written

$$D^\gamma u = v,$$

provided

$$\int_\Omega u D^\gamma \phi \,\mathrm{d}x = (-1)^{|\gamma|} \int_\Omega v\phi \,\mathrm{d}x$$

for all test functions (smooth functions with compact support) $\phi \in C_c^\infty(\Omega, \mathbb{R})$. If u possesses a weak γ^{th}-partial derivative v, then this is uniquely defined up to a set of measure zero.

Definition 2.3.1. *Fix $1 \leqslant p \leqslant \infty$ and let k be a non-negative integer. The space*

$$W^{k,p}(\Omega, \mathbb{R})$$

is called Sobolev space and consists of all locally integrable functions $u : \Omega \to \mathbb{R}$ such that for each multi-index γ with $|\gamma| \leqslant k$, $D^\gamma u$ exists in the weak sense and belongs to $L^p(\Omega)$. We equip this space with the norm

$$\|u\|_{W^{k,p}(\Omega,\mathbb{R})} := \left(\sum_{|\gamma| \leqslant k} \int_\Omega |D^\gamma u(x)|^p \,\mathrm{d}x \right)^{\frac{1}{p}},$$

which makes $W^{k,p}(\Omega, \mathbb{R})$ a Banach space (see [3, Theorem 3.3]*).*

In the case of $p = 2$, the Sobolev space defines a Hilbert space and we shall follow the common convention and write

$$H^k(\Omega, \mathbb{R}) := W^{k,2}(\Omega, \mathbb{R}),$$

and

$$\|u\|_{H^k(\Omega,\mathbb{R})} := \|u\|_{W^{k,2}(\Omega,\mathbb{R})}.$$

The first theorem of this section describes the inclusions between certain Sobolev spaces and possesses a major role for section 2.4 and the regularity of weak and classical solutions of boundary value problems.

Theorem 2.3.2 (The Sobolev Imbedding Theorem, [3, Theorem 4.12])**.** *Let Ω be a domain in $\mathbb{R}^d$ satisfying a cone condition, $j \geqslant 0$, $m \geqslant 1$ be integers, and $1 \leqslant p < \infty$.*

(i) If either $mp > d$ or $m = d$ and $p = 1$, then

$$W^{m,p}(\Omega, \mathbb{R}) \hookrightarrow L^q(\Omega, \mathbb{R}), \quad \text{for } p \leqslant q \leqslant \infty.$$

(ii) If $mp = d$, then

$$W^{m,p}(\Omega, \mathbb{R}) \hookrightarrow L^q(\Omega, \mathbb{R}), \quad \text{for } p \leqslant q < \infty.$$

(iii) If $mp < d$ and either $d - mp < k \leqslant d$ or $p = 1$ and $d - m \leqslant k \leqslant d$, then

$$W^{m,p}(\Omega, \mathbb{R}) \hookrightarrow L^q(\Omega, \mathbb{R}), \quad \text{for } p \leqslant q \leqslant p^* = dp/(d - mp).$$

Given Ω is a domain with Lipschitz boundary, the imbeddings can be further refined as follows:

(iv) If $mp > d > (m-1)p$, then

$$W^{j+m,p}(\Omega, \mathbb{R}) \hookrightarrow C^{j,\alpha}(\mathrm{cl}(\Omega), \mathbb{R}), \quad \text{for } 0 < \alpha \leqslant m - (d/p).$$

(v) If $d = (m-1)p$, then

$$W^{j+m,p}(\Omega, \mathbb{R}) \hookrightarrow C^{j,\alpha}(\mathrm{cl}(\Omega), \mathbb{R}), \quad \text{for } 0 < \alpha < 1.$$

Also if, $d = m - 1$ and $p = 1$, then the last imbedding also holds for $\alpha = 1$.

Remark 2.3.3. *The imbeddings (i) and (ii) also hold for $1 \leqslant q < p$ in addition to the values of q asserted in the statement if and only if the domain Ω has finite volume.*

2.3.1 Sobolev Spaces of Fractional Order

In this subsection, we introduce one possible extension of Sobolev spaces to fractional orders $k \in \mathbb{R}$ using the Bessel potential. For $u \in L^1(\mathbb{R}^d, \mathbb{R})$, we define its Fourier transform by

$$\hat{u}(\xi) := \mathscr{F}(u)(\xi) := \int_{\mathbb{R}^d} u(x) e^{-i\xi x} \, \mathrm{d}x.$$

Let $\mathscr{S} = \mathscr{S}(\mathbb{R}^d, \mathbb{R})$ denote the Schwartz space (see Subsection 3.1.1). For $k \in \mathbb{R}$, we define a continuous linear operator $\mathcal{J}^k : \mathscr{S}(\mathbb{R}^d, \mathbb{R}) \to \mathscr{S}(\mathbb{R}^d, \mathbb{R})$, called the Bessel potential of order k, by

$$\mathcal{J}^k(u)(x) := \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} (1 + |\xi|)^{\frac{k}{2}} \hat{u}(\xi) e^{i\xi x} \, \mathrm{d}\xi, \quad \text{for } x \in \mathbb{R}^d.$$

Since the Bessel potential of some Schwartz function u is defined as the inverse Fourier transform of $(1 + |\xi|)^{\frac{k}{2}} \hat{u}(\xi)$, we have

$$\mathscr{F}\left(\mathcal{J}^k(u)\right)(\xi) = (1 + |\xi|)^{\frac{k}{2}} \hat{u}(\xi). \tag{2.2}$$

For a tempered distribution $u \in \mathscr{S}'$ (the topological dual of $\mathscr{S}$) and a Schwartz function $g \in \mathscr{S}$ we denote by $\langle u, g \rangle := u(g)$. Furthermore, for any fixed $u \in L^1_{\text{loc}}(\mathbb{R}^d, \mathbb{R})$, the integral

$$v \mapsto (u, v) := \int_{\mathbb{R}^d} u(x) v(x) \mathrm{d}x, \quad \text{for all } v \in \mathscr{S},$$

defines a tempered distribution. Since $(u, \cdot)$ is uniquely determined by u (see [83, Theorem 3.7]), this yields an injection of $\mathscr{S}$ into $\mathscr{S}'$. Therefore, for any $u, v \in \mathscr{S}$ we shall write

$$\langle u, v \rangle = (u, v).$$

Conversely, for any $u \in \mathscr{S}'$ we say u lies in $L^1_{\text{loc}}(\mathbb{R}^d, \mathbb{R})$ if there is some $w \in L^1_{\text{loc}}(\mathbb{R}^d, \mathbb{R})$ such that

$$\langle u, v \rangle = (w, v), \quad \text{for all } v \in \mathscr{S}.$$

Following these notations, by using the natural extension $\mathcal{J}^k : \mathscr{S}'(\mathbb{R}^d, \mathbb{R}) \to \mathscr{S}'(\mathbb{R}^d, \mathbb{R})$ on the space of tempered distribution, given by

$$\mathscr{S}' \ni u \mapsto \langle \mathcal{J}^k(u), v \rangle = \langle v, \mathcal{J}^k(u) \rangle, \quad \text{for all } v \in \mathscr{S},$$

we can define the following extension of Sobolev spaces to fractional orders.

Definition 2.3.4 (Sobolev Spaces of Fractional Order)**.** *For any $k \in \mathbb{R}$, we define $W^{k,p}(\mathbb{R}^d, \mathbb{R})$, the Sobolev space of order k on $\mathbb{R}^d$, by*

$$W^{k,p}(\mathbb{R}^d, \mathbb{R}) := \{u \in \mathscr{S}' \,:\, \mathcal{J}^k(u) \in L^p(\mathbb{R}^d, \mathbb{R})\},$$

and equip this space with the norm

$$\|u\|_{W^{k,p}(\mathbb{R}^d,\mathbb{R})} := \|\mathcal{J}^k(u)\|_{L^p(\mathbb{R}^d,\mathbb{R})}, \quad \textit{for } u \in W^{k,p}(\mathbb{R}^d,\mathbb{R}).$$

In [3, 7.6.2] it is shown, that for $k \in \mathbb{N}$ this definition of $W^{k,p}(\mathbb{R}^d,\mathbb{R})$ coincide with the definition above up to equivalence of norms. Moreover, for $k \in \mathbb{R}$ and $p = 2$, it follows immediately that $H^k(\mathbb{R}^d,\mathbb{R}) = W^{k,2}(\mathbb{R}^d,\mathbb{R})$ remains a Hilbert space. By (2.2) and Plancherel's Theorem (see [83, Theorem 3.12]) we also have that

$$\|u\|_{H^k(\Omega,\mathbb{R})} = (2\pi)^{-d}\|(1+|\xi|^2)^{\frac{k}{2}}\hat{u}\|_{L^2(\Omega,\mathbb{R})}. \tag{2.3}$$

Definition 2.3.5. *For a non-empty open subset Ω of $\mathbb{R}^d$ and $k \in \mathbb{R}$ we define the Hilbert space*

$$W^{k,p}(\Omega,\mathbb{R}) := \{u \in \mathscr{S}'(\Omega,\mathbb{R}) : u = U\restriction_\Omega \textit{ for some } U \in W^{k,p}(\mathbb{R}^d,\mathbb{R})\}$$

with the norm

$$\|u\|_{W^{k,p}(\mathbb{R}^d,\mathbb{R})} := \inf_{\substack{U\in W^{k,p}(\Omega,\mathbb{R})\\ U\restriction_\Omega = u}} \|U\|_{W^{k,p}(\mathbb{R}^d,\mathbb{R})}, \quad \textit{for } u \in W^{k,p}(\Omega,\mathbb{R}).$$

In general, for a function $u \in L^p(\Omega,\mathbb{R})$ it is meaningless to speak of the value at $\partial\Omega$ as the d-dimensional Lebesgue measure of the boundary is zero. Fortunately, for Sobolev spaces we can give a reasonable definition of traces assuming Ω is a Lipschitz domain. Let $\mathrm{tr} : C^0(\mathrm{cl}(\Omega),\mathbb{R}) \to C^0(\partial\Omega,\mathbb{R})$ be the trace operator, mapping functions in $C^0(\mathrm{cl}(\Omega),\mathbb{R})$ to their traces on $\partial\Omega$. If Ω is open, bounded, and Lipschitz, we can extend this operator continuously to $W^{1,p}(\Omega,\mathbb{R})$ (see [44, Theorem 4.6]). By an abuse of the notation, we still denote this extension with tr.

Theorem 2.3.6 (Trace Theorem, [41, Theorem B.52 and Corollary B.53])**.** *Let Ω be an open and bounded domain in $\mathbb{R}^d$ with Lipschitz boundary and let $1 \leqslant p, q \leqslant \infty$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Then,*

(i) $\mathrm{tr} : W^{1,p}(\Omega,\mathbb{R}) \to W^{\frac{1}{q},p}(\partial\Omega,\mathbb{R})$ *is surjective.*

(ii) The kernel of the trace operator is given by the closure of $C_c^\infty(\Omega,\mathbb{R})$ in $W^{1,p}(\Omega,\mathbb{R})$, denoted by $W_0^{1,p}(\Omega,\mathbb{R})$.

(iii) There exists a constant $C > 0$ such that for all $u \in W^{1,p}(\Omega, \mathbb{R})$ it holds

$$\|u\|_{W^{1,p}(\Omega,\mathbb{R})} \leqslant \|\mathrm{tr}(u)\|_{W^{\frac{1}{q},p}(\partial\Omega,\mathbb{R})},$$

where the constant C only depends on p and Ω.

Since by (i) we can characterize the trace of a function u in $W^{1,p}(\Omega, \mathbb{R})$ with some function $v \in W^{\frac{1}{q},p}(\partial\Omega, \mathbb{R})$, we denote the trace of u also with u instead of $\mathrm{tr}(u)$.

Due to the trace theorem, the following subspaces of $W^{k,p}(\Omega, \mathbb{R})$ are well-defined.

Definition 2.3.7. *Let Ω be a bounded domain with Lipschitz boundary and Γ_D a boundary portion of $\partial\Omega$. With $k \in \mathbb{R}$ and $1 \leqslant p < \infty$, we define the subspaces*

$$W_D^{k,p}(\Omega, \mathbb{R}) := \{u \in W^{k,p}(\Omega, \mathbb{R}) : u = 0 \textit{ along } \Gamma_D\}.$$

and

$$H_D^k(\Omega, \mathbb{R}) := W_D^{k,2}(\Omega, \mathbb{R}).$$

One can show that $H_D^1(\Omega, \mathbb{R})$ defines a closed subspace of $H^1(\Omega, \mathbb{R})$ (see [32, Theorem 6.3-4]*).*

Lemma 2.3.8 (Poincaré Inequality, [41, Lemma B.61])**.** *Let $1 \leqslant p < \infty$ and let Ω be an open and bounded set with Lipschitz boundary. Then, there exists a constant $C_P > 0$ such that*

$$C_P\|u\|_{L^p(\Omega,\mathbb{R})} \leqslant \|\nabla u\|_{L^p(\Omega,\mathbb{R})}, \quad \textit{for all } u \in W_D^{1,p}(\Omega, \mathbb{R}),$$

where C_P depends on p and Ω.

2.4 Elliptic System of Linear Partial Differential Equation

In this section, we develop the relevant theory of elliptic systems of linear partial differential equations, needed for this thesis. The emphasis here lies on the regularity of solutions to these systems and on corresponding Schauder estimates, which are important, e.g., to, derive uniform bounds of solutions over shape spaces. Our starting point is a more general introduction on the underlying theory, based on the work of

Geymonat [54], and Agmon, Douglis and Nirenberg [4, 5]. Afterwards, we provide some examples of systems of linear elliptic partial differential equations which are considered in this work

Let Ω be a subset of $\mathbb{R}^d$ and $\beta \in \mathbb{N}_0^d$ a multi-index. We consider a polynomial in $\Xi = (\xi_1, \ldots, \xi_d)$ given by

$$p(x, \Xi) := \sum_{|\beta|=0}^{E} p^{(\beta)}(x)\Xi^{\beta} = \sum_{|\beta|=0}^{E} p^{(\beta_1,\ldots,\beta_d)}(x)\xi_1^{\beta_1} \cdots \xi_d^{\beta_d},$$

with coefficient functions $p^{(\beta)}$ on Ω and degree $\deg(p\,(x, \Xi)) = E$. In the case of $\xi_i = \partial/\partial_{x_i}$, we obtain a partial differential operator of order E

$$[pu](x, D) := \sum_{|\beta|=0}^{E} p^{(\beta_1,\ldots,\beta_d)}(x) \left(\frac{\partial}{\partial_{x_1}}\right)^{\beta_1} \ldots \left(\frac{\partial}{\partial_{x_d}}\right)^{\beta_d} u(x) = \sum_{|\beta|=0}^{E} p^{(\beta)} D^{\beta} u(x).$$

A matrix $\mathbf{a}(x, \Xi) := (a_{i,j}(x, \Xi))_{i,j=1,\ldots,N}$, consisting of such polynomials $a_{i,j}(x, \Xi) = \sum_{|\beta|=0}^{N_{i,j}} a_{i,j}^{(\beta)}(x)\Xi^{\beta}$ of degree $E_{i,j} \geqslant 0$, gives, with the above substitution, a system of partial differential equations

$$([\mathbf{a}u](x, D))_i = \sum_{j=1}^{N} a_{i,j}[u_j](x, D) = f_i(x), \quad \text{for } i = 1, \ldots, N. \tag{2.4}$$

The matrix $\mathbf{a}(x, \Xi)$ is called the symbol of the differential operator $\mathbf{a}(x, D)$. We assume that the orders of these operators depend on two system of integer weights – which do not have to be unique – $s_1, \ldots, s_N$ and $t_1, \ldots, t_N$ attached to the i^{th} equation and to the unknown j^{th} dependend variable u_j. The manner of the dependence is expressed by the inequality

$$\deg\left(a_{i,j}(x, \Xi)\right) \leqslant s_i + t_j, \quad \text{for } i, j = 1, \ldots, N, \tag{2.5}$$

where it is to be understood that $a_{i,j} = 0$ if $s_i + t_j < 0$. We can choose the two systems such that $s_i \leqslant 0$ and $0 \leqslant t_j \leqslant t'$, where t' is the maximum of the t_j, and write

$$a_{i,j}(x, \Xi) = \sum_{|\beta|=0}^{s_i+t_j} a_{i,j}^{(\beta)} \Xi^{\beta}.$$

Definition 2.4.1 (Ellipticity)**.** *With* $\mathbf{a}'(x,\Xi) = \left(a'_{i,j}(x,\Xi)\right)_{i,j=1,\dots,N}$ *we denote the principal symbol of* $\mathbf{a}(x,D)$ *that consists of the terms in* $\mathbf{a}(x,\Xi)$ *which are just of the order* $s_i + t_j$*. Then, a system of partial differential equation is*

(i) elliptic, if the corresponding characteristic polynomial is non-zero, i.e.,

$$\mathcal{A}(x,\Xi) := \det\left(\mathbf{a}'(x,\Xi)\right) \neq 0, \quad \textit{for all } \Xi \in \mathbb{R}^d \setminus \{0\}.$$

(ii) uniformly elliptic, if the characteristic polynomial $\mathcal{A}(x,\Xi)$ *is of even degree* $\deg(\mathcal{A}(x,\Xi)) = 2M$ *and there exists a constant* $\Lambda > 0$ *such that*

$$\Lambda^{-1}\|\Xi\|^{2M} \leqslant |\mathcal{A}(x,\Xi)| \leqslant \Lambda\|\Xi\|^{2M}, \quad \textit{for all } \Xi \in \mathbb{R}^d,\ x \in \mathrm{cl}(\Omega),$$

where $\|\cdot\|$ *denotes the Euclidean norm.*

Definition 2.4.2 (Supplementary Condition on $\mathcal{A}$)**.** *The characteristic polynomial* $\mathcal{A}$ *is said to satisfy the supplementary condition, if*

(i) $\mathcal{A}(x,\Xi)$ *is, with respect to* Ξ*, of even degree* $2M$*,*

(ii) for every pair of linearly independent real vectors Ξ, Ξ'*, the polynomial* $\mathcal{A}(x,\Xi+\tau\Xi')$ *in the complex variable* τ *has exactly* M *roots with positive imaginary part.*

Remark 2.4.3. *(i) A characteristic polynomial* $\mathcal{A}$ *satisfies the supplementary condition whenever the corresponding system of partial differential equations* (2.4) *is a system in three or more independent variables* u_j*. A proof to this statement can be found in [4, p. 631-632].*

(ii) The supplementary condition is used only at points x *of the boundary of* Ω *with* Ξ *a tangent, and* Ξ' *the normal to the boundary at* x*.*

From now on, the following assumptions will be used repeatedly and for sake of convenience we summarize them here.

Assumptions 2.4.4. *Consider a partial differential operator* $\mathbf{a}(x,D)$*. We state the assumptions:*

(A1) The systems of weights $s_1,\dots,s_N \leqslant 0$ *and* $0 \leqslant t_1,\dots,t_N \leqslant t'$ *satisfy* (2.5)*.*

(A2) The operator $\mathbf{a}$ *is elliptic.*

(A3) The characteristic polynomial $\mathcal{A}$ fulfills the supplementary condition and the number $M = \frac{1}{2}\deg\left(\mathcal{A}(x,\Xi)\right)$ is positive.

The boundary conditions we consider refer to a regular portion Γ of $\partial\Omega$ and are expressed as

$$([\mathbf{b}u](x,D))_h = \sum_{j=1}^{N} b_{h,j}[u_j](x,D) = g_h(x), \quad h = 1,\ldots,M,\ x \in \Gamma \tag{2.6}$$

in terms of given polynomials in Ξ, $b_{h,j}(x,D)$. The order of the boundary operators, like of the operators in (2.4), depends on two system of integer weights, where the system $t_1,\ldots,t_N$ is already attached to the dependend variables u_j and a new system $r_1,\ldots,r_M$, of which r_h pertains to the h^{th} condition, is introduced. The dependence is equally expressed by the inequality

$$\deg\left(b_{h,j}\right) \leqslant r_h + t_j,$$

where it is understood that $b_{h,j} = 0$ if $r_h + t_j < 0$. For this system, we define $r' := \max\{0, r_1+1,\ldots,r_M+1\}$ and $r'' := \max\{0, r_1,\ldots,r_M\}$. By $\mathbf{b}'(x,\Xi) = \left(b'_{h,j}(x,\Xi)\right)$, with $1 \leqslant h \leqslant M$ and $1 \leqslant j \leqslant N$, we denote the matrix that consists of the terms in $\mathbf{b}(x,\Xi)$ which are just of the order $r_h + t_j$. Further, we can write the polynomials $b_{h,j}(x,\Xi)$ as

$$b_{h,j}(x,\Xi) = \sum_{|\beta|=0}^{r_h+t_j} b_{h,j}^{(\beta)}\Xi^\beta.$$

The resulting boundary value problems (BVPs) are described by

$$\begin{cases} [\mathbf{a}u](x,D) = f(x) & \text{on } \Omega, \\ [\mathbf{b}u](x,D) = g(x) & \text{along } \Gamma, \end{cases} \tag{2.7}$$

where $f = (f_1,\ldots,f_N)$ and $g = (g_1,\ldots,g_M)$. For a well-posed problem, the boundary conditions must "complement" the differential equations. The complementing boundary condition we describe below is an algebraic criterion and involves the principal symbols $\mathbf{a}'(x,\Xi)$ and $\mathbf{b}'(x,\Xi)$.

For a point x on Γ, we denote with $n(x)$ the outward normal at x and $\Xi(x) \neq 0$ any tangent to Γ. We denote by $\tau_h^+(x,\Xi)$, $1 \leqslant h \leqslant M$, the M solutions (in τ) with positive imaginary part of the characteristic equation

$$\mathcal{A}\left(x, \Xi(x) + \tau n(x)\right) = 0,$$

which are assured by the supplementary condition on $\mathcal{A}$. Set

$$M^{+}(x, \Xi, \tau) = \prod_{h=1}^{M} \left(\tau - \tau_h^{+}(x, \Xi)\right),$$

and let $\mathbf{a}'^{*}(x, \Xi + \tau n)$ denote the adjoint matrix of $\mathbf{a}'(x, \Xi + \tau n)$. The criterion regarding the coercivity of the boundary value problems (2.7), we mentioned above, is that the following algebraic condition is satisfied.

Definition 2.4.5 (Complementing Boundary Condition)**.** *The boundary value problem* (2.7) *fulfills the complementing boundary condition if, for any point $x \in \Gamma$ and any real, non-zero vector tangent to Γ at x, the matrix*

$$\mathbf{c}(x, \Xi + \tau n)_{h,k} = \sum_{j=1}^{N} b_{h,j}(x, \Xi + \tau n) {a'_{j,k}}^{*}(x, \Xi + \tau n), \quad h = 1, \ldots M, \; k = 1, \ldots, N,$$

has linear independent rows modulo $M^{+}(x, \Xi, \tau)$. We refer to the assumption, that the complementing boundary condition holds, as (B1)*.*

For problems in which the complementing boundary condition is satisfied, we can find coefficients $c_{h,k}^{(\beta)}(x, \Xi) \in \mathbb{R}$ such that

$$\mathbf{c}(x, \Xi + \tau n)_{h,k} = \sum_{\beta=0}^{M-1} c_{h,k}^{(\beta)}(x, \Xi) \tau^{\beta} \pmod{M^{+}(x, \Xi, \tau)}.$$

We construct the matrix $\mathbf{c} = (c_{h,k}^{\beta})$ having M rows $h = 1, \ldots, M$ and MN columns $\beta = 0, \ldots, M-1$, $k = 1, \ldots, N$. Under the complementing boundary condition, the rank of $\mathbf{c}$ will be M. Hence, if

$$M^{1}(x, \Xi), \ldots, M^{\binom{MN}{M}}(x, \Xi)$$

denote all the m-rowed minors of $\mathbf{c}$, the value

$$\max_{j=1,\ldots,\binom{MN}{M}} |M^{j}(x, \Xi)|$$

will not be zero. Also, if Γ is compact, then the infimum Δ'_Γ of these quantities is, for all $x \in \Gamma$ and all real vectors Ξ which are tangent to Γ at x, non-zero either.

Definition 2.4.6 (Minor Constant)**.** *Assuming that* (B1) *holds, we define the minor constant as follows.*

(i) If Γ is plane, the minor constant is given by $\Delta_\Gamma := \Delta'_\Gamma$.

(ii) If Γ is non-plane and a change of variables $\psi : \Gamma \to \mathbb{R}^{d-1}$ exists that makes Γ plane, we define the minor constant by $\Delta_\Gamma := \Delta'_{\psi(\Gamma)}$.

(iii) If Γ itself is covered by a union of subportions Γ_i, and each Γ_i has a minor constant Δ_{Γ_i}, the corresponding minor for Γ will be defined as $\Delta \equiv \Delta_\Gamma = \inf \Delta_{\Gamma_i}$.

We denote with Δ_x the minor constant for $\mathrm{cl}(U_x) \cap \partial\Omega$ that pertains to the hemisphere transform $\mathbb{T}_x$ (see Definition 2.2.5). Hence, by (iii), $\Delta = \inf_{\{x\,:\,\mathrm{dist(x,\Gamma)\leqslant m}\}} \Delta_x$.

Example 2.4.7 (Pure Traction Problem of Linear Elasticity)**.** *Consider a domain $\Omega \subset \mathbb{R}^3$ and vector fields $f : \Omega \to \mathbb{R}^3$, $u \in C^2(\Omega, \mathbb{R}^3)$. The partial differential equation system of linear elasticity with Dirichlet boundary condition is given by*

$$\begin{cases} -\nabla \cdot \boldsymbol{\sigma}(u) = f & \text{on } \Omega, \\ u = 0 & \text{along } \partial\Omega, \end{cases} \tag{2.8}$$

where $\boldsymbol{\sigma}(u) = \lambda(\nabla \cdot u)\mathbf{I} + \mu(Du + Du^\intercal)$, with Lamé coefficient constants $\lambda, \mu > 0$ and 3×3 identity matrix $\mathbf{I} = \mathbf{I}_3$. Componentwise, this system reads

$$\begin{aligned}
\left[(\lambda + 2\mu)\tfrac{\partial^2}{\partial x_1^2} + \mu\left(\tfrac{\partial^2}{\partial x_2^2} + \tfrac{\partial^2}{\partial x_3^2}\right)\right] u_1 + (\lambda+\mu)\tfrac{\partial^2}{\partial x_1 \partial x_2} u_2 + (\lambda+\mu)\tfrac{\partial^2}{\partial x_1 \partial x_3} u_3 &= f_1, \\
(\lambda+\mu)\tfrac{\partial^2}{\partial x_1 \partial x_2} u_1 + \left[(\lambda + 2\mu)\tfrac{\partial^2}{\partial x_2^2} + \mu\left(\tfrac{\partial^2}{\partial x_1^2} + \tfrac{\partial^2}{\partial x_3^2}\right)\right] u_2 + (\lambda+\mu)\tfrac{\partial^2}{\partial x_2 \partial x_3} u_3 &= f_2, \\
(\lambda+\mu)\tfrac{\partial^2}{\partial x_1 \partial x_3} u_1 + (\lambda+\mu)\tfrac{\partial^2}{\partial x_2 \partial x_3} u_2 + \left[(\lambda + 2\mu)\tfrac{\partial^2}{\partial x_3^2} + \mu\left(\tfrac{\partial^2}{\partial x_1^2} + \tfrac{\partial^2}{\partial x_2^2}\right)\right] u_3 &= f_3,
\end{aligned}$$

with symbols

$$\mathbf{a}(\Xi) = \begin{pmatrix} (\lambda+\mu)\xi_1^2 + \mu\|\Xi\|^2 & (\lambda+\mu)\xi_1\xi_2 & (\lambda+\mu)\xi_1\xi_3 \\ (\lambda+\mu)\xi_1\xi_2 & (\lambda+\mu)\xi_2^2 + \mu\|\Xi\|^2 & (\lambda+\mu)\xi_2\xi_3 \\ (\lambda+\mu)\xi_1\xi_3 & (\lambda+\mu)\xi_2\xi_3 & (\lambda+\mu)\xi_3^2 + \mu\|\Xi\|^2 \end{pmatrix},$$

and $\mathbf{b}(\Xi) = \mathbf{I}$. *As systems of weights, we can choose, e.g.,* $s_1 = s_2 = s_3 = 0$, $t_1 = t_2 = t_3 = 2$ *and* $r_1 = r_2 = r_r = -2$. *The form of the corresponding characteristic polynomial*

$$\mathcal{A}(\Xi) = \mu^2(\lambda + 2\mu)\|\Xi\|^6$$

implies obviously the uniform ellipticity and satisfies the supplementary condition, since (2.8) *is a system in three independent variables* $u_1, \ldots, u_3$.

Theorem 2.4.8 (Index Theorem, [54, Theorem 3.5])**.** *Let*

$$\begin{cases} [\mathbf{a}u](x, D) = f(x) & \text{on } \Omega, \\ [\mathbf{b}u](x, D) = g(x) & \text{along } \Gamma \end{cases} \tag{2.9}$$

be a system of partial differential equations with bounded domain $\Omega \subset \mathbb{R}^d$ *of class* $C^{r'+t'+k}$, $k \in \mathbb{N}_0 \cup \{\infty\}$, *differential operator* $\mathbf{a}(x, D)$ *of order* N, *and boundary operator* $\mathbf{b}(x, D)$ *of order* M *on a regular boundary portion* $\Gamma \subset \partial\Omega$. *We assume that the coefficients of* $\mathbf{a}$ *and* $\mathbf{b}$, *respectively, satisfy*

$$a_{i,j}^{(\beta)} \in \begin{cases} C^{r'-s_i+k}(\mathrm{cl}(\Omega), \mathbb{R}) & \text{if } |\beta| = s_i + t_j, \\ W^{r'-s_i+k,\infty}(\Omega, \mathbb{R}) & \text{if } |\beta| < s_i + t_j, \end{cases}$$

$$b_{h,j}^{(\beta)} \in \begin{cases} C^{r'-r_h+k}(\Gamma, \mathbb{R}) & \text{if } |\beta| = r_h + t_j, \\ W^{r'-r_h+k,\infty}(\Gamma, \mathbb{R}) & \text{if } |\beta| < r_h + t_j. \end{cases}$$

Then, the following two assertions are equivalent:

(i) The system (2.9) *is elliptic* (A2) *and fulfills the supplementary and complementing conditions* (A3) *and* (B1).

(ii) If $1 < p < \infty$ *and* $0 \leqslant l \leqslant k$, *the operator*

$$A_{l,p} : \prod_{j=1}^{N} W^{r'+t_j+l,p}(\Omega, \mathbb{R}) \to \prod_{j=1}^{N} W^{r'-s_i+l,p}(\Omega, \mathbb{R}) \times \prod_{j=1}^{M} W^{r'-r_h+l-\frac{1}{p},p}(\Gamma, \mathbb{R}),$$
$$u \mapsto (\mathbf{a}u, \mathbf{b}u)$$

is linear, continuous, and has finite index

$$\mathrm{ind}(A_{l,p}) := \dim(\ker(A_{l,p})) - \dim(\mathrm{coker}(A_{l,p})) \tag{2.10}$$

that is independent of l *and* p.

Remark 2.4.9. *We can consider the differential and boundary operator of linear boundary value problems* (2.7) *in the context of Fredholm operators. Then, the index* (2.10) *of the Fredholm operator $A_{l,p}$ contains informations about the existence and uniqueness of solutions to boundary value problems, encoded into one number. The dimension of the kernel can be treated as a measure for the injectivity of the operator. If the kernel is given by $\{0\}$, and hence the dimension is 0, the operator is injective and solutions are unique. If we treat the dimension of the kernel as a measure for injectivity, we can accordingly consider the dimension of the cokernel as a measure for surjectivity. If the operator is surjective, the corkernel possesses dimension 0. Consequently, if we are able to show that $\dim(\ker(A_{l,p})) = \dim(\operatorname{coker}(A_{l,p})) = 0$, and thus $\operatorname{ind}(A_{l,p}) = 0$, we see that there exists a unique solution to the boundary value problem* (2.7)*. Now, in this situation, the index theorem states, under the necessary assumptions, that the index is zero for each $1 < p < \infty$ and the unique solution lies in the corresponding space L^p. This trait allows us to, e.g., improve the regularity of a standard H^1 solution to $W^{1,p}$, for arbitrary $1 < p < \infty$.*

Theorem 2.4.10 (Schauder Estimate in Sobolev Spaces, [5, Lemma 10.5])**.** *Consider a boundary value problem* (2.7) *on a bounded domain $\Omega \subset \mathbb{R}^d$ that fulfills assumptions* (A1) - (A3) *and* (B1)*. Let $k \geqslant r'$ be a fixed integer and $1 < p < \infty$. We further assume that the following regularity assumption hold:*

(i) The coefficient functions $a_{i,j}^{(\beta)} \in C^{k-s_i}(\mathrm{cl}(\Omega), \mathbb{R})$, $b_{h,j}^{(\beta)} \in C^{k-r_h}(\partial\Omega, \mathbb{R})$ and the functions $f_i \in W^{k-s_i,p}(\Omega, \mathbb{R})$, $g_h \in W^{k-r_h-\frac{1}{p},p}(\partial\Omega, \mathbb{R})$,

(ii) We assume that the right-hand sides, f and g, and the coefficients $a_{i,j}^{(\beta)}$ and $b_{h,j}^{(\beta)}$, are respectively bounded from above by constants $C_{f,g}$, $C_{a,b} > 0$ in their respective norms.

(iii) Ω possesses a $C^{t'+k}$-hemisphere property such that $\Delta_\Gamma > 0$ and that the hemisphere transforms $\mathbb{T}_x$ have finite $C^{t'+k}$-norms, bounded by some constant $C_\mathbb{T}$ independently of x.

Then, a weak solution $u_j \in W^{r'+t_j,p}(\Omega, \mathbb{R})$ to (2.7) *also lies in $W^{t_j+k,p}(\Omega, \mathbb{R})$, for $j = 1, \dots, N$, and we can estimate*

$$\|u_j\|_{W^{t_j+k,p}(\Omega,\mathbb{R})} \leqslant C \left(\sum_{i=1}^{N} \|f_i\|_{W^{k-s_i,p}(\Omega,\mathbb{R})} + \sum_{h=1}^{M} \|g_h\|_{W^{k-r_h-\frac{1}{p},p}(\partial\Omega,\mathbb{R})} + \sum_{l=1}^{N} \|u_l\|_{C^0(\Omega,\mathbb{R})} \right),$$

with constant $C > 0$ that depends on $C_{a,b}, \Lambda, \Delta_\Gamma, C_\mathbb{T}, r, d, N, M, \sum|r_h|, p$ and k.

Theorem 2.4.11 (Schauder Estimate in Hölder Spaces, [5, Theorem 9.3])**.** *Consider the boundary value problem (2.7) on a non-empty domain $\Omega \subset \mathbb{R}^d$. Suppose that assumptions (A1) - (A3) and (B1) are fulfilled. Let S be a subdomain of Ω with the property that $\partial S \cap \partial\Omega$ lies in the interior of Γ. Let $\alpha \in]0,1[$ be a Hölder exponent and fix an integer $k \geqslant r''$. Further, assume that S satisfies a hemisphere condition with minor constant $\Delta_{\Gamma_s} > 0$ and consider the following regularity assumptions:*

(i) $a_{i,j}^{(\beta)}, f_i$ *are elements of* $C^{k-s_i,\alpha}(\mathrm{cl}(\Omega),\mathbb{R})$ *and* $b_{h,j}^{(\beta)}$, g_h *of* $C^{k-r_h,\alpha}(\Gamma,\mathbb{R})$.

(ii) We assume that the right-hand sides, f and g, and the coefficients, $a_{i,j}^{(\beta)}$ and $b_{h,j}^{(\beta)}$, are respectively upper bounded by constants $C_{f,g}$, $C_{a,b} > 0$ in their respective norms.

(iii) The hemisphere transforms $\mathbb{T}_x$ and their inverse are of class $C^{t''+k,\alpha}$, $t'' = \max\{-s_i, -r_h, t_j\}$, and have fintie $C^{t''+k,\alpha}$-norms, bounded by some constant $C_\mathbb{T}$ independently of x.

Then, a classical solution $u_j \in C^{r''+t_j,\alpha}(\Omega\cup\Gamma,\mathbb{R})$ to (2.7) also lies in $C^{t_j+k,\alpha}(\mathrm{cl}(S),\mathbb{R})$, for $j = 1,\dots,N$, and we can estimate

$$\|u_j\|_{C^{t_j+k,\alpha}(S)} \leqslant C \left(\sum_{i=1}^{N} \|f_i\|_{C^{k-s_i,\alpha}(\Omega,\mathbb{R})} + \sum_{h=1}^{M} \|g_h\|_{C^{k-r_h,\alpha}(\Gamma,\mathbb{R})} + \sum_{l=1}^{N} \|u_l\|_{C^0(\Omega,\mathbb{R})} \right),$$

with constant $C > 0$ that depends on $C_{a,b}, \Lambda, \Delta_{\Gamma_s}, C_\mathbb{T}, m, d, N, \alpha$ and k.

For a bounded domain Ω, we have $S = \Omega$ and $\Gamma = \partial\Omega$, and any solution $u \in C^{r''+t_j,\alpha}(\mathrm{cl}(\Omega),\mathbb{R})$ to (2.7) already belongs to $C^{t_j+k,\alpha}(\mathrm{cl}(\Omega),\mathbb{R})$.

2.4.1 Regularity Theory and Linear Elasticity

Now, we apply the regularity results from above in the context of linear elasticity. Let $\Omega \subset \mathbb{R}^3$ be an open, bounded, and connected domain with divided Lipschitz boundary $\partial\Omega = \Gamma_D \cup \Gamma_N$, with $\Gamma_D \cap \Gamma_N = \emptyset$. With $u : \mathrm{cl}(\Omega) \to \mathbb{R}^3$ we denote the displacement field and with $f : \Omega \to \mathbb{R}^3$, $g : \Gamma_N \to \mathbb{R}^3$ a volume load and a surface load acting on Ω. With Lamé coefficients $\lambda, \mu > 0$, the disjoint-traction problem, or the linear elasticity equation, is given by

$$\begin{cases} -\nabla \cdot \sigma(u) = f & \text{on } \Omega, \\ u = 0 & \text{along } \Gamma_D, \\ n \cdot \boldsymbol{\sigma}(u) = g & \text{along } \Gamma_N, \end{cases} \tag{2.11}$$

where $\boldsymbol{\sigma}(u) = \lambda(\nabla \cdot u)\mathbf{I} + \mu\,(Du + Du^\intercal)$. With $\boldsymbol{\varepsilon}(u) := (Du + Du^\intercal)$ we denote the linearized strain tensor. Approximative numerical solution can be computed by a finite element approach; see [41, 67]. The existence of weak H^1 solutions can be found in, e.g., [32, 41]. In [32], Ciarlet also shows existence theory to the pure traction problem (2.8) in Sobolev spaces of higher order.

Results for classical strong solutions of elliptic systems of partial differential equations are, however, somewhat scattered in the literature. Nonetheless, for linear elasticity, we can derive results on the existence of solutions with corresponding Schauder estimates directly from the results provided above.

Lemma 2.4.12 (Korn's Second Inequality, [24, 3.3])**.** *Let Ω be an open and bounded set in $\mathbb{R}^3$ with Lipschitz boundary $\partial\Omega$. In addition, suppose $\Gamma_D \subset \partial\Omega$ has positive two-dimensional surface measure. Then, there exists a constant $C > 0$ such that*

$$\|\boldsymbol{\varepsilon}(u)\|_{L^2(\Omega,\mathbb{R}^{3\times 3})} = \left(\int_\Omega \boldsymbol{\varepsilon}(u) : \boldsymbol{\varepsilon}(u)\, dx\right)^{\frac{1}{2}} \geqslant C\|u\|_{H^1(\Omega,\mathbb{R}^3)}, \quad \text{for all } u \in H^1_D(\Omega, \mathbb{R}^3),$$

where $\boldsymbol{\varepsilon}(u) : \boldsymbol{\varepsilon}(u)$ denotes the Frobenius scalar product of $\boldsymbol{\varepsilon}(u)$ with itself.

Definition 2.4.13. *The weak formulation of (2.11) is given by*

$$B(u, v) = L(v), \quad \text{for all } v \in H^1_D(\Omega, \mathbb{R}^3), \tag{2.12}$$

with

$$B(u, v) := \int_\Omega \lambda \mathbf{tr}\,(\boldsymbol{\varepsilon}(u))\,\mathbf{tr}\,(\boldsymbol{\varepsilon}(v)) + 2\mu \mathbf{tr}\,(\boldsymbol{\varepsilon}(u)\boldsymbol{\varepsilon}(v))\;\mathrm{d}x = \int_\Omega \boldsymbol{\varepsilon}(u) : \boldsymbol{\sigma}(v)\,\mathrm{d}x,$$

and

$$L(v) := \int_\Omega \langle f, v\rangle\,\mathrm{d}x + \int_{\Gamma_N} \langle g, v\rangle\,\mathrm{d}A,$$

where $\mathbf{tr}$ denotes the trace of a $n \times n$ matrix. By Korn's second inequality, the bilinear form B is strictly coercive and continuous on H^1_D. Since Ω has a Lipschitz boundary, *we can define the trace operator* $\mathrm{tr} : H^1(\Omega, \mathbb{R}^3) \to H^{\frac{1}{2}}(\partial\Omega, \mathbb{R}^3)$ *and apply the Sobolev Imbedding Theorem 2.3.2 to estimate*

$$|L(v)| \leqslant \left(C_1 \|f\|_{L^{\frac{6}{5}}(\Omega,\mathbb{R}^3)} + C_2 \|g\|_{L^{\frac{4}{3}}(\Gamma_N,\mathbb{R}^3)} \right) \|v\|_{H^1(\Omega,\mathbb{R}^3)},$$

with constants $C_1, C_2 > 0$*. This yields the continuity of the linear form* L *and eventually the existence of a unique weak solution to* (2.11)*.*

Theorem 2.4.14 (Regularity of the Weak Solution to the Elasticity Equations, [32, Theorem 6.3-5] and [18, Theorem 2.3.4])**.** *Consider elasticity equation* (2.11) *on a* C^2*-domain* $\Omega \subset \mathbb{R}^3$*, and let* Γ_D *be a measurable subset of* $\partial\Omega$ *with positive surface measure. If* $\Gamma_N = \partial\Omega - \Gamma_D$*, with* $\mathrm{dist}(\Gamma_D, \Gamma_N) > 0$*, and* $f \in L^{\frac{6}{5}}(\Omega, \mathbb{R}^3)$*,* $g \in L^{\frac{4}{3}}(\partial\Omega, \mathbb{R}^3)$*, then there is a unique solution* $u \in H^1_D(\Omega, \mathbb{R}^3)$ *solving* (2.12)*. Furthermore, we have:*

- *(i) If* $f \in L^p(\Omega, \mathbb{R}^3)$ *and* $g \in W^{1-\frac{1}{p},p}(\partial\Omega, \mathbb{R}^3)$*,* $p \geqslant \frac{4}{3}$*, then the solution lies in* $W^{2,p}(\Omega, \mathbb{R}^3)$*.*
- *(ii) Assume that* $k \geqslant 3$ *is an integer and* Ω *is a* C^k*-domain. If for* $p \geqslant \frac{4}{3}$ *we have that* $f \in W^{k-2,p}(\Omega, \mathbb{R}^3)$ *and* $g \in W^{k-1-\frac{1}{p},p}(\partial\Omega, \mathbb{R}^3)$*, then the solution is an element of* $W^{k,p}(\Omega, \mathbb{R}^3)$*.*

For $k \geqslant 2$*, any solution* $u \in W^{k,p}(\Omega, \mathbb{R}^3)$ *satisfies*

$$\|u\|_{W^{k,p}(\Omega,\mathbb{R}^3)} \leqslant C \left(\|f\|_{W^{k-2,p}(\Omega,\mathbb{R}^3)} + \|g\|_{W^{k-1-\frac{1}{p},p}(\partial\Omega,\mathbb{R}^3)} + \|u\|_{C^0(\Omega,\mathbb{R}^3)} \right),$$

with constant $C > 0$*, which depends on* $\Delta_{\partial\Omega}, C_{\mathbb{T}}, \lambda, \mu, p, k$*, and* m*.*

Proof. In order to demonstrate the usage of index Theorem 2.4.8, we briefly sketch the main steps of the proof to Theorem 2.4.14 (i). First, by [32, Theorem 6.3-5] there exists a unique solution $u \in H^1_D(\Omega, \mathbb{R}^3)$ to (2.11). Because the linear elasticity equation satisfies assumptions (A1) - (A3) and (B1), by Theorem 2.4.10, for $p = 2$, we have $u \in W^{2,p}_D(\Omega, \mathbb{R}^3)$. Now, by the Index Theorem 2.4.8, the operator

$$\begin{aligned} A_{0,p} : \prod_{j=1}^{3} W^{2,p}_D(\Omega, \mathbb{R}) &\to \prod_{j=1}^{3} W^{2,p}(\Omega, \mathbb{R}) \times \prod_{j=1}^{3} W^{1-\frac{1}{p},p}(\partial\Omega, \mathbb{R}), \\ u &\mapsto (\mathbf{a}u, \mathbf{b}u) \end{aligned}$$

has an index $\mathrm{ind}(A_{0,p}) = 0$ which is independent of $1 < p < \infty$. In the case of $p = 2$, $A_{0,2}$ is a bijection and $\dim(\ker(A_{0,2})) = \dim(\mathrm{coker}(A_{0,2})) = 0$. Since the boundary of Ω is of class C^2, a cone condition is satisfied and the Imbedding Theorem 2.3.2 gives $W^{2,p}_D(\Omega,\mathbb{R}^3) \hookrightarrow H^1_D(\Omega,\mathbb{R}^3)$, for $p \geqslant 6/5$. Therefore, $\dim(\ker(A_{0,p})) = 0$ for any $4/3 < p < \infty$. It is an immediate consequence that $\dim(\operatorname{coker}(A_{0,p})) = 0$ for any $4/3 < p < \infty$ as well. Hence, $u \in W^{2,p}_D(\Omega,\mathbb{R}^3)$ for $4/3 < p < \infty$. □

Theorem 2.4.15 (Regularity of the Strong Solution to the Elasticity Equations, [18, Theorem 2.3.6])**.** *We consider elasticity equation (2.11) on a bounded domain $\Omega \subset \mathbb{R}^3$ of class $C^{k,\alpha}$, with $k \geqslant 2$ and $\alpha \in]0,1[$, and let Γ_D be a portion of $\partial\Omega$, with positive surface measure. Further, we have a portion $\Gamma_N = \partial\Omega - \Gamma_D$ with $\operatorname{dist}(\Gamma_D,\Gamma_N) > 0$, and assume that $f \in C^{k-2,\alpha}(\operatorname{cl}(\Omega),\mathbb{R}^3)$ and $g \in C^{k-1,\alpha}(\Gamma_N,\mathbb{R}^3)$. Then, there exists a unique solution $u \in C^{k,\alpha}(\operatorname{cl}(\Omega),\mathbb{R}^3)$ of (2.11) satisfying*

$$\|u\|_{C^{k,\alpha}(\Omega,\mathbb{R}^3)} \leqslant C\left(\|f\|_{C^{k-2,\alpha}(\Omega,\mathbb{R}^3)} + \|g\|_{C^{k-1,\alpha}(\Gamma_N,\mathbb{R}^3)} + \|u\|_{C^0(\Omega,\mathbb{R}^3)}\right).$$

The constant $C > 0$ depends on $\Delta_{\partial\Omega}, C_{\mathbb{T}}, \lambda, \mu, \alpha, k,$ and m.

Proof. We only describe the main steps here, since a full proof is already provided in [18, Theorem 2.3.6]. First, we assume that $k \geqslant 3$. With the same procedure as in the proof to Theorem 2.4.14, we get, by using Index Theorem 2.4.8, a unique solution $u \in W^{k,p}(\Omega,\mathbb{R}^3)$ for any $1 < p < \infty$. Then, for $p \geqslant 3$, the Sobolev Imbedding Theorem 2.3.2 states that u also lies in $C^{k-1,\alpha}(\Omega,\mathbb{R}^3)$ for any $\alpha \in]0,1[$. As $k \geqslant 3$, u lies thus in $C^{2,\alpha}(\operatorname{cl}(\Omega),\mathbb{R}^3)$ and therefore, by Theorem 2.4.11, in $C^{k,\alpha}(\operatorname{cl}(\Omega),\mathbb{R}^3)$ as well. In addition, Theorem 2.4.11 provides the desired Schauder estimate.

Now, for $k = 2$, we have $f \in C^{0,\alpha}(\Omega,\mathbb{R}^3)$ and $g \in C^{1,\alpha}(\operatorname{cl}(\Gamma_N),\mathbb{R}^3)$. Considering that $C^{k',\alpha}(S,\mathbb{R}^3)$ is dense in $C^{k,\alpha}(S,\mathbb{R}^3)$ for any bounded domain S in $\mathbb{R}^d$, $\alpha \in]0,1]$, and any $k,k' \in \mathbb{N}_0$, with $k < k'$, we can find sequences $(f_n)_{n\in\mathbb{N}} \subset C^{1,\alpha}(\Omega,\mathbb{R}^3)$ $(g_n)_{n\in\mathbb{N}} \subset C^{2,\alpha}(\operatorname{cl}(\Gamma_N),\mathbb{R}^3)$ which converge in their respective norm to f and g. By above considerations, the elasticity equation (2.11) yields a corresponding sequence of solutions $(u_n)_{n\in\mathbb{N}} \subset C^{2,\alpha}(\operatorname{cl}(\Omega),\mathbb{R}^3)$ which forms a bounded subset of $C^{2,\alpha}(\operatorname{cl}(\Omega_N),\mathbb{R}^3)$. One can show that this implies the convergence of the sequence $(u_n)_{n\in\mathbb{N}}$ to a solution $u \in C^{2,\alpha}(\operatorname{cl}(\Omega),\mathbb{R}^3)$ to (2.11) with volume load f and surface load g. Lastly, Theorem 2.4.11 implies the assertion. □

2.4.2 Potential Flow Equation

Potential flow theory possesses a large role in fluid dynamics and describes approximately many processes occuring in nature. In the following, we show a simple way to apply potential flow theory in order to model the flow of an ideal fluid flowing past a component within a shroud.

As component we consider a connected and compact domain $\Omega \subset \mathbb{R}^3$ with $C^{k,\alpha}$-boundary – where we always assume that $k \geqslant 2$ and $\alpha \in \,]0,1]$ unless we specify further – that is partially contained in some larger, connected, and compact domain $D \subset \mathbb{R}^3$ representing a shroud with $C^{k,\alpha}$-boundary such that $\mathrm{int}(D\backslash\Omega)$ is simply connected and has $C^{k,\alpha}$-boundary as well. The shroud D has an inlet and outlet where the fluid flows in and out, respectively. At the remaining boundary part the fluid cannot leak; see Figure 2-2. This model considers an incompressible and rotation free perfect fluid in a steady state. The assumption of zero shearing stresses in a perfect fluid – or zero viscosity – simplifies the equation of motion so that potential theory can be applied. The resulting solution still provides reasonable approximations to many actual flows. The viscous forces are limited to a thin layer of fluid adjacent to the surface and therefore, in favor of simplicity, we leave these effects out since they have little effect on the general flow pattern[1].

A fundamental condition is that no fluid can be created or destroyed within the shroud D. The equation of continuity expresses this condition. Consider a three-dimensional velocity field v on $D \subset \mathbb{R}^3$, then the continuity equation is given by

$$\nabla \cdot v = 0.$$

If we assume that the velocity field v is rotation free, $\nabla \times v = 0$, then there exists a velocity potential or flow potential ϕ such that

$$v = \nabla\phi.$$

Hence, under the assumption that v is divergence free and rotation free, there is a velocity potential ϕ that satisfies the Laplace equation

$$\Delta\phi = \nabla \cdot \nabla\phi = 0.$$

Let n be the unitary outward normal of the boundary ∂D. By applying suitable Neumann boundary conditions g that correspond to our assumptions for a conserved

[1]unless the local effects make the flow separate from the surface

flow through the inlet and outlet of the shroud, we get the potential flow equation

$$\begin{cases} \Delta\phi = 0 & \text{on } \operatorname{int}(D\backslash\Omega), \\ n\cdot\nabla\phi = g & \text{along } \partial D\backslash\partial(D\cap\Omega), \\ n\cdot\nabla\phi = 0 & \text{along } D\cap\partial\Omega. \end{cases} \tag{2.13}$$

Here, we assume that g is only non-zero on the inlet and outlet regions and is continued to be zero on the upper and lower wall of the shroud. Therefore, no discontinuities occur where $\partial\Omega$ meets ∂D.

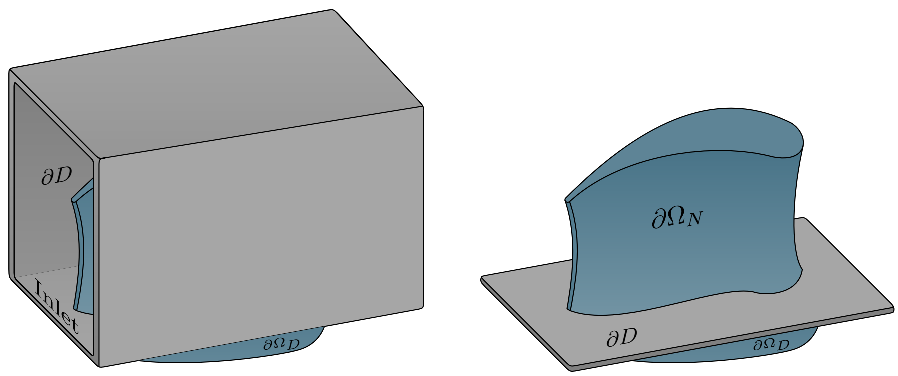

Figure 2-2: A turbine blade Ω within a shroud D. We note that this representation of the domains Ω and D is merely a sketch of the above construction. In particular, every visible edge has to be sufficently smooth in order to ensure the Hölder continuity of the boundaries.

The following theorem ensures the existence of a solution to the potential flow equation and states the corresponding Schauder estimate.

Theorem 2.4.16 (Schauder Estimate for Flow Potentials)**.** *Let us consider the potential flow equation* (2.13)*, where we assume that the boundaries described above are all of class* $C^{k,\alpha}$*, with* $k \geqslant 2$*, and let* $g \in C^{k-1,\alpha}(D,\mathbb{R})$*. If* $\int_{\partial D} g \,\mathrm{d}A = 0$*, then the potential flow problem* (2.13) *possesses at least one solution* $\phi \in C^{2,\alpha}(\operatorname{cl}(D\backslash\Omega),\mathbb{R})$*.*

To obtain uniqueness, we fix $u(x_0) = 0$ *at some point* $x_0 \in \operatorname{int}(D\backslash\Omega)$*. This solution satisfies*

$$\|\phi\|_{C^{2,\alpha}(D\backslash\Omega,\mathbb{R})} \leqslant C\left(\|g\|_{C^{1,\alpha}(\partial D\backslash\partial(D\cap\Omega),\mathbb{R})} + \|\phi\|_{C^{0,\alpha}(D\backslash\Omega,\mathbb{R})}\right), \tag{2.14}$$

where the constant $C > 0$ *depends on the domain* Ω*.*

Proof. Both, the existence of a solution and the Schauder estimate are provided in

[86, Theorem 3.1 and Theorem 4.1]. Alternatively, we can derive this result from Theorem 2.4.11 following the same arguments as in Theorem 2.4.15 □

2.4.3 Diffusion Equations

Diffusion equations help us to understand a rich variety of fundamental processes in physical science. Among these phenoma are thermal processes such as heat conduction in materials, fluid pressure transients in porous media, drying of solids due to moisture depletion, transport of chemicals and pollutants in the environment by gradual reduction in their concentrations, migration of chemicals within concrete and other structural materials, sedimentation and consolidation of geomaterials and in the study of transmission lines. Consider an open, bounded, and connected set $D \subset \mathbb{R}^d$ with Lipschitz boundary ∂D, representing the domain on which the diffusion process takes place. Assuming that there is a measurable partition of $\partial\Omega = \Gamma_D \cup \Gamma_N$ such that Γ_D has positive surface measure, we consider the boundary value problem for the stationary diffusion equation

$$\begin{cases} \nabla \cdot (a\nabla u) = f & \text{on } D, \\ u = g_D & \text{along } \Gamma_D, \\ n \cdot a\nabla u = g_N & \text{along } \Gamma_N, \end{cases} \tag{2.15}$$

where f is a given source term, g_D denotes the Dirichlet boundary data, g_N the given Neumann boundary data, and n denotes as usual the outward normal unit vector field along ∂D. The coefficient function a models the conductivity throughout the domain D.

Theorem 2.4.17. *Given a conductivity $a \in L^\infty(D, \mathbb{R})$ with $\inf a > 0$, $f \in L^2(D, \mathbb{R})$, $g_D \in H^{\frac{1}{2}}(\Gamma_D, \mathbb{R})$, and $g_N \in H^{-\frac{1}{2}}(\Gamma_N, \mathbb{R})$, the diffusion equation (2.15) has a unique solution $u \in H^1(D, \mathbb{R})$. Moreover, there is a constant $C \geqslant 1$ independent of a, f, g_D and g_N such that*

$$\|u\|_{H^1(D,\mathbb{R})} \leqslant C \frac{1 + \|a\|_{C^0(D,\mathbb{R})}}{\inf_{x\in D} a(x)} \left(\|f\|_{L^2(D,\mathbb{R})} + \|g_D\|_{H^{\frac{1}{2}}(\Gamma_D,\mathbb{R})} + \|g_N\|_{H^{-\frac{1}{2}}(\Gamma_N,\mathbb{R})} \right). \tag{2.16}$$

One can choose $C = (1 + C_P^2) \max\{1, 2\|E\|, \|\mathrm{tr}\|\}$, where $C_P > 0$ only depends on D and Γ_D and where $E : H^{\frac{1}{2}}(\Gamma_D, \mathbb{R}) \to H^1(D, \mathbb{R})$ denotes an extension operator and

$\mathrm{tr} : H^1(D,\mathbb{R}) \to H^{\frac{1}{2}}(\Gamma_D,\mathbb{R})$ *the trace operator from Theorem 2.3.6.*

Proof. With tr denoting the trace operator from $H^1(D,\mathbb{R})$ into $H^{\frac{1}{2}}(\Gamma_D,\mathbb{R})$ we seek $u \in H^1(D,\mathbb{R})$ with $\mathrm{tr}(u) = g_D$ on Γ_D and

$$\int_D a\nabla u \cdot \nabla v \,\mathrm{d}x = \int_{\Gamma_N} g_N \,\mathrm{tr}(v)\,\mathrm{d}A + \int_D [fv - a\nabla (E(g_D)) \cdot \nabla v]\,\mathrm{d}x \ \ (=: L(v))$$

for all $v \in H^1_D(D,\mathbb{R})$. By [110, Theorem 6.1.5.4] (and [83, Theorem 3.29, Theorem 3.30]), the left-hand side of (2.16) defines an inner product $(\cdot,\cdot)_a$ on the closed subspace $H^1_D(D,\mathbb{R}) \subset H^1(D,\mathbb{R})$ whose associated norm $\|\cdot\|_a$ satisfies, for some constant $C_P > 0$ originating from the Poincaré inequality 2.3.8,

$$\sqrt{\frac{\inf a}{1+C_P^2}}\|v\|_{H^1(D,\mathbb{R})} \leqslant \|v\|_a \leqslant \|a\|_\infty \|v\|_{H^1(D,\mathbb{R})}, \quad \text{for all } \ v \in H^1_D(D,\mathbb{R}). \tag{2.17}$$

One can show that the linear form L is continuous, hence, by Riesz Representation Theorem, there is a unique $v_l \in H^1_D(D,\mathbb{R})$ satisfying $(v_l, v)_a = L(v)$ and

$$\|v_l\|_{H^1(D,\mathbb{R})} \leqslant \frac{1+C_P^2}{\inf_{x\in D} a(x)} \left(\|f\|_{L^2(D,\mathbb{R})} + \|a\|_\infty \|E\| \|g_D\|_{H^{\frac{1}{2}}(\Gamma_D,\mathbb{R})} + \|g_N\|_{H^{-\frac{1}{2}}(\Gamma_N,\mathbb{R})} \|\mathrm{tr}\| \right).$$

Setting $u := v_l + Eg_D$ gives the unique (weak) solution of (2.15) and the desired inequality follows with $C := (1+C_P^2)\max\{1, 2\|E\|, \|\mathrm{tr}\|\}$. □

Chapter 3

Generalized Random Fields

In this chapter, we introduce generalized random fields indexed by locally convex vector spaces and describe how to obtain smoothed Lèvy noise fields, which we employ in Chapter 5 as random diffusion coefficients in (2.15). We characterize a Lévy noise field as generalized (distribution-valued) random field ”$Z(x)$”, where $Z(f) =$ ”$\int Z(x)f(x)\,\mathrm{d}x$” is only defined as a random variable after "integrating" the distribution-valued random variable ”$Z(x)$” against a test function f. Based on the analysis of Hilbert-Schmidt embeddings, we provide sufficient conditions for Matérn smoothing kernels ”$k(x)$” under which the noise field $Z_k(x) =$ ”$\int k(x-y)Z(y)\,\mathrm{d}y$” has continuous paths. By composing with a continuous positive function T of real arguments, this approach yields Lévy models $a(x) = T(Z_k(x))$ of strictly positive random coefficients so that the boundary value problem (2.15) can be solved strongly, i.e., path-wise for almost all paths $a(x)$.

3.1 Multi-Hilbertian Spaces

Let V be a vector space. A seminorm $p : V \to (0, \infty)$ is called a Hilbertian seminorm, or H-seminorm if p fulfills the polarization property

$$p(f + g)^2 + p(f - g)^2 = 2p(f)^2 + 2p(g)^2$$

for all $f, g \in V$. A vector space V endowed with a H-seminorm is called Hilbertian seminormed space. If (V, p) has a countable dense subspace, then (V, p) (or p) is called separable. Further, orthogonality, orthonormality, and an orthonormal basis (abbr.

ONB) are defined on (V, p) in the same way as in Hilbert spaces. If p is separable, then

$$\tilde{V}_p := V/N_p \quad (N_p := \{f \in V : p(f) = 0\})$$

is a separable pre-Hilbert space with a Hilbertian norm induced by p. We shall denote the quotient norm also with p. The completion of $\tilde{V}_p$ into a separable Hilbert space will be denoted with V_p.

Definition 3.1.1. *Let p and q be separable H-seminorms on a vector space V. We define*

$$(p : q) := \sup_{f \in V}\{p(f) : q(f) \leqslant 1\},$$

$$(p : q)_{\mathrm{HS}} := \left(\sum_n p(e_n)^2\right)^{\frac{1}{2}}, \quad \{e_n\}_{n\in\mathbb{N}} \text{ an ONB on } (V, q)$$

if $(p : q) < \infty$. Otherwise, we set $(p : q)_{\mathrm{HS}} = \infty$. We note that $(p : q)_{\mathrm{HS}}$ is well-defined independently of the choice of $\{e_n\}_{n\in\mathbb{N}}$ (see, e.g., [69, Remark 1.1.2]).

Definition 3.1.2 (Hilbert–Schmidt Operator)**.** *Let V and H be separable Hilbert spaces with respective norms p and q. A linear operator $i : V \to H$ for which*

$$(q \circ i : p)_{\mathrm{HS}} < \infty$$

is called a Hilbert–Schmidt operator.

Definition 3.1.3. *Let p and q be separable H-seminorms on a vector space V. p is said to be bounded by q, written $p \prec q$, if $(p : q) < \infty$. p is said to be Hilbert-Schmidt bounded, or HS bounded by q, written $p \prec_{\mathrm{HS}} q$, if $(p : q)_{\mathrm{HS}} < \infty$.*

Definition 3.1.4 (Multi-Hilbertian Space)**.** *Let V be a vector space and τ a topology on V. τ is called multi-Hilbertian if there exists a family of separable H-seminorms $\mathscr{P}$ such that the sets*

$$\{g \in V : p_i(g - f) < \varepsilon_i,\ i = 1, \ldots, n\}, \quad n \in \mathbb{N}, p_i \in \mathscr{P}, \varepsilon_i > 0$$

form a complete system of $\tau-$neighborhoods of f for every $f \in V$. A vector space with a multi-Hilbertian topology is a special topological vector space and called multi-Hilbertian space. The multi-Hilbertian topology determined by $\mathscr{P}$ is denoted by $\tau(\mathscr{P})$.

Let τ be a multi-Hilbertian topology on a vector space V and let q be a separable H-seminorm on V. If the q topology induced by the semimetric

$$d_q(f,g) := q(f-g), \quad f,g \in V$$

is weaker than τ, we write $q \prec \tau$. Let τ_1 and τ_2 be two multi-Hilbertian topologies on V. We say τ_1 is Hilbert-Schmidt weaker, or HS weaker than τ_2, written $\tau_1 \prec_{\mathrm{HS}} \tau_2$, if for every H-seminorm $p \prec \tau_1$ there exists a H-seminorm $q \prec \tau_2$ such that $p \prec_{\mathrm{HS}} q$.

Definition 3.1.5 (Nuclear Space)**.** *Let τ be a multi-Hilbertian topology on a vector space V. The multi-Hilbertian topology determined by*

$$\{p \,:\, p \prec_{\mathrm{HS}} q \textit{ for some } q \prec \tau\}$$

is called the Kolmogorov-I-topology of τ denoted by $I(\tau)$ [72]. $I(\tau)$ is the strongest of all multi-Hilbertian topologies HS weaker than τ. If τ is determined by a single separable H-seminorm, then $I(\tau)$ is called the Sazonov topology or the Gross topology of τ [36, 96]. If $I(\tau) = \tau$, then τ is called nuclear and (V,τ) nuclear space.

Remark 3.1.6. *An equivalent definition of nuclear spaces would be the following. Consider a multi-Hilbertian topology τ on a vector space V that is determined by a family of continuous H-seminorms $\mathscr{P}$ such that for every $p, q \in \mathscr{P}$ there is a $r \in \mathscr{P}$ with $p, q \prec r$. If for every $p \in \mathscr{P}$ there is some $q \in \mathscr{P}$ with $p \prec q$ and such that the so-called canonical linking map $i_q^p : V_q \to V_p$, i.e., the extension of the inclusion from the pre-Hilbert space $\tilde{V}_q$ into the pre-Hilbert space $\tilde{V}_p$ to their respective completion V_q and V_p, is a Hilbert-Schmidt operator, then τ is called nuclear and (V,τ) nuclear space.*

3.1.1 The Schwartz Space $\mathscr{S}(\mathbb{R}^d, \mathbb{R})$

Let $\mathscr{S} = \mathscr{S}(\mathbb{R}^d, \mathbb{R})$ denote the space of all (real-valued) rapidly decreasing smooth functions on $\mathbb{R}^d$ [84, Example 29.4]. The standard topology τ of $\mathscr{S}$ is induced by the family of norms

$$\|f\|_{\alpha,\beta} := \sup_{x\in\mathbb{R}^d} |x^\alpha D^\beta f(x)|, \quad \alpha, \beta \in \mathbb{N}_0^d,$$

which make $\mathscr{S}$ a separable Fréchet space. Clearly, $\mathscr{S}$ is a subspace of $L^2(\mathbb{R}^d, \mathbb{R})$. These norms are not H-norms, but we can generate the same locally convex topology

τ by a sequence of H-(semi-)norms based on Hermite polynomials as well (see [69, Section I.1.3], [84, Example 29.4], or [92, Appendix to Section V.3] for the case $d = 1$). For $k \in \mathbb{N}_0$ we denote by h_k the k^{th}-Hermite function on $\mathbb{R}$, defined as

$$h_k(x) := \left(2^k k! \sqrt{\pi}\right)^{-\frac{1}{2}} (-1)^k \mathrm{e}^{\frac{x^2}{2}} \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^k \mathrm{e}^{-x^2}, \quad x \in \mathbb{R},$$

and for $\alpha \in \mathbb{N}_0^d$ we denote by $h_\alpha := h_{\alpha_1} \otimes \cdots \otimes h_{\alpha_d}$ the tensorized Hermite function $h_\alpha(x) = \prod_{j=1}^d h_{\alpha_j}(x_j)$ on $\mathbb{R}^d$. It is well known that the family $(h_\alpha)_{\alpha \in \mathbb{N}_0^d}$ forms an orthonormal basis of $L^2(\mathbb{R}^d, \mathbb{R})$. By denoting the inner product on $L^2(\mathbb{R}^d, \mathbb{R})$ with $(\cdot, \cdot)$, we observe that for every $p \in \mathbb{R}$ the set

$$\mathscr{S}_p := \left\{ f \in L^2(\mathbb{R}^d, \mathbb{R}) \, : \, |f|_p^2 := \sum_{\alpha \in \mathbb{N}_0^d} (2|\alpha| + d)^{2p} \, |(f, h_\alpha)|^2 < \infty \right\}$$

is a subspace of $L^2(\mathbb{R}^d, \mathbb{R})$ containing $\mathscr{S}$, $|\cdot|_p$ is a norm on $\mathscr{S}_p$ with associated inner product

$$(f, g)_p := \sum_{\alpha \in \mathbb{N}_0^d} (2|\alpha| + d)^{2p} (f, h_\alpha)(h_\alpha, g), \quad f, q \in \mathscr{S}_p,$$

and $\mathscr{S}_q \subseteq \mathscr{S}_p$ with continuous (and even contractive) inclusion for every $p \leqslant q$. Furthermore, we have that

$$\mathscr{S} = \bigcap_{p \in \mathbb{R}} \mathscr{S}_p = \bigcap_{p \geqslant 0} \mathscr{S}_p = \bigcap_{p \in \mathbb{N}_0} \mathscr{S}_p$$

and $(|\cdot|_p)_{p \in \mathbb{R}}$ is an increasing family of H-norms on $\mathscr{S}$ which generates the standard topology τ and makes $\mathscr{S}$ nuclear as we show below.

Proposition 3.1.7. *For each $p \in \mathbb{R}$ and $\ell > \frac{d}{2}$ the linking map*

$$i_{p+\ell}^p : (\mathscr{S}_{p+\ell}, |\cdot|_{p+\ell}) \to (\mathscr{S}_p, |\cdot|_p) \, ,$$

from the local Hilbert space $\mathscr{S}_{p+\ell}$ into the local Hilbert space $\mathscr{S}_p$, is Hilbert–Schmidt.

Proof. First, for the pre-Hilbert space $(\mathscr{S}, |\cdot|_p)$, whose completion is denoted by $(\mathscr{S}_p, |\cdot|_p)$, we select an orthonormal basis $(h_{p,\alpha})_{\alpha \in \mathbb{N}_0^d}$ defined by

$$h_{p,\alpha} := \prod_{j=1}^d (2\alpha_j + 1)^{-p} h_{\alpha_j}, \quad p \in \mathbb{R}, \, \alpha \in \mathbb{N}_0^d.$$

Because for $\beta \in \mathbb{N}_0^d$, $\ell \geqslant 0$ we have

$$|h_{p+\ell,\beta}|_p^2 = \sum_{\alpha \in \mathbb{N}_0^d} (2|\alpha| + d)^{2p} \, |(h_{p+\ell,\beta}, h_\alpha)|^2 = \frac{(2|\beta| + d)^{2p}}{\prod_{j=1}^d (2\beta_j + 1)^{2(p+\ell)}}$$

and since there is $1 \leqslant j \leqslant d$ with $\beta_j \geqslant |\beta|/d$, we can estimate

$$|h_{p+\ell,\beta}|_p^2 \leqslant \frac{(2|\beta| + d)^{2p}}{(2\frac{|\beta|}{d} + 1)^{2(p+\ell)}} = \frac{d^{2(p+\ell)}}{(2|\beta| + d)^{2\ell}}.$$

Now, by using that for any given $k \in \mathbb{N}_0$ the number of $\beta \in \mathbb{N}_0^d$ with $|\beta| = k$ is equal to $\binom{k+d-1}{k}$, we have that

$$\begin{aligned}
\sum_{\beta \in \mathbb{N}_0^d} |h_{p+\ell,\beta}|_p^2 \leqslant \; & d^{2p+\ell} \sum_{k=0}^{\infty} \binom{k+d-1}{k} \frac{1}{(2k+d)^{2\ell}} \\
= \; & \frac{d^{2(p+\ell)}}{(d-1)!} \sum_{k=0}^{\infty} \frac{(k+d-1)!}{k!(2k+d)^{2\ell}} \\
\leqslant \; & \frac{d^{2(p+\ell)}}{(d-1)!} \sum_{k=0}^{\infty} (2k+d)^{d-1-2\ell}.
\end{aligned}$$

Because we assumed that $\ell > \frac{d}{2}$, this proves the assertion. □

3.2 Generalized Random Fields

Random fields are fundamental to model physical processes where uncertainties, originating from sparse informations, occur. By following [42] closely, we introduce the concept of generalized random fields (in the sense of Minlos; see, e.g., [53, 85]) and analyse them thoroughly. To avoid an excessive introduction, we assume that the reader is already familiar with the basic concepts of probability theory and functional analysis. Generalized random fields are families of random variables indexed by an abstract vector space V. In order to ensure the existence of a non-trivial dual space, we assume that V is a locally convex vector space over the real numbers. Let $L^0(\Omega, \mathfrak{A}, \mathbf{P})$ denote the vector space of Borel measurable random variables on a probability space $(\Omega, \mathfrak{A}, \mathbf{P})$. We treat two random variables from the same equivalence class, resulting from almost sure (a.s.) equality, notationally, as equal. Moreover, we set

$$\|X\|_{L^0} := \mathbf{E}\left[|X| \wedge 1\right] = \int_\Omega (|X| \wedge 1) \, \mathrm{d}\mathbf{P} \quad X \in L^0(\Omega, \mathfrak{A}, \mathbf{P})$$

and

$$d_0(X,Y) := \|X - Y\|_{L^0} \quad X, Y \in L^0(\Omega, \mathfrak{A}, \mathbf{P}),$$

where $X \wedge 1$ is the minimum of X and 1. d_0 is a (translation-invariant) metric on $L^0(\Omega, \mathfrak{A}, \mathbf{P})$ and makes $L^0(\Omega, \mathfrak{A}, \mathbf{P})$ a Hausdorff topological vector space. Moreover, since for any $\varepsilon \in]0,1[$ and $X \in L^0(\Omega, \mathfrak{A}, \mathbf{P})$ we have

$$\varepsilon \mathbf{P}(|X| > \varepsilon) \leqslant \|X\|_{L^0} \leqslant \mathbf{P}(|X| > \varepsilon) + \varepsilon$$

it follows that convergence with respect to the metric d_0 coincides with convergence in probability. It is well known that the metric space $(L^0(\Omega, \mathfrak{A}, \mathbf{P}))$ is complete (see e.g. [70, Lemma 3.6]).

Definition 3.2.1 (Generalized Random Field)**.** *A collection of real-valued random variables $\{Z(f)\}_{f \in V}$, indexed by a locally convex topological space V, on a common probability space $(\Omega, \mathfrak{A}, \mathbf{P})$, is called generalized random field if*

(i) Linearity: $Z(\alpha f + \beta g) = \alpha Z(f) + \beta Z(g)$ a.s. for all $f, g \in V$ and $\alpha, \beta \in \mathbb{R}$.

(ii) Stochastic continuity: $f \to f_0$ in V implies $Z(f) \to Z(f_0)$ in probability.

Thus, a generalized random field on $(\Omega, \mathfrak{A}, \mathbf{P})$ indexed by V is a continuous linear mapping $Z : V \to L^0(\Omega, \mathfrak{A}, \mathbf{P})$, where $L^0(\Omega, \mathfrak{A}, \mathbf{P})$ is endowed with the metric d_0.

Two generalized random fields Z and $\tilde{Z}$ on probability spaces $(\Omega, \mathfrak{A}, \mathbf{P})$ and $(\tilde{\Omega}, \tilde{\mathfrak{A}}, \tilde{\mathbf{P}})$ indexed by V are equivalent (in law) if their finite-dimensional distributions coincide, i.e., if

$$\mathbf{P}\left(Z(f_1) \in A_1 \wedge \cdots \wedge Z(f_n) \in A_n\right) = \tilde{\mathbf{P}}\left(\tilde{Z}(f_1) \in A_1 \wedge \cdots \wedge \tilde{Z}(f_n) \in A_n\right)$$

holds for all $n \in \mathbb{N}$, $f_1, \ldots, f_n \in V$ and $A_1, \ldots, A_n \in \mathfrak{B}(\mathbb{R})$, where $\mathfrak{B}(\mathbb{R})$ denotes the Borel $\sigma-$algebra on $\mathbb{R}$.

Remark 3.2.2. *(i) For the topological dual V' of a locally convex vector space V and measurable $X : (\Omega, \mathfrak{A}, \mathbf{P}) \to (V', \mathfrak{B})$, with $\sigma-$algebra $\mathfrak{B}$ on V' for which the evaluation mapping $V' \to \mathbb{R}$, $u \mapsto u(f)$, $f \in V$, are measurable, the application of Lebesgue's Dominated Convergence Theorem shows that*

$$Z : V \to L^0(\Omega, \mathfrak{A}, \mathbf{P}),\ f \mapsto (\omega \mapsto X(\omega)(f)) =: X(f) =: Z(f, \omega)$$

is a generalized random field. Hence, in this setting, V' valued random variables are generalized random fields. Conversely, for a general (metrizable) locally convex space V we cannot characterize (up to equivalence) every general random field Z, indexed by V, by a V' valued random variable. However, by Minlos Theorem, for nuclear locally convex spaces V, this realization also holds true; see below.

(ii) Let V be a dense subspace of a locally convex space $\tilde{V}$ and $Z : V \to L^0(\Omega, \mathfrak{A}, \mathbf{P})$ a generalized random field. Then, due to the fact that $L^0(\Omega, \mathfrak{A}, \mathbf{P})$ endowed with the topology of convergence in probability is a complete Hausdorff space, there is a unique continuous linear extension $\tilde{Z} : \tilde{V} \to L^0(\Omega, \mathfrak{A}, \mathbf{P})$ of Z (see, e.g., [84, Lemma 22.19]*). In particular, for every generalized random field Z on a locally convex space V, there is a unique extension on the completion of V.*

In finite dimensions, Bochner's Theorem [94, Theorem 1.4.3] establishes a one-to-one correspondence between the distributions of $\mathbb{R}^d$-valued random variables and positive definite functions $\varphi : \mathbb{R}^d \to \mathbb{C}$ with $\varphi(0) = 1$ by way of the Fourier transform as $\mathbf{E}\left[e^{iX(f)}\right] = \varphi(f)$ with $f \in \mathbb{R}^d$ and $X(f) = X \cdot f$ denoting the Euclidean inner product on $\mathbb{R}^d$. As we mentioned above, not every generalized random field indexed by a locally convex space V can be represented by a V' valued random variable. However, the ono-to-one correspondence between generalized random fields (up to equivalence in law) and characteristic functionals on V remains valid in this general setting.

Definition 3.2.3 (Characteristic Functional)**.** *A mapping $\varphi : V \to \mathbb{C}$ on a locally convex space V is called characteristic functional if*

(i) $\varphi(0) = 1$,

(ii) φ is continuous,

(iii) φ is positive definite, i.e., the matrix $[\varphi(f_i - f_j)]_{i,j=1}^n$ is Hermitian and positive semidefinite for all $n \in \mathbb{N}$ and $f_1, \ldots, f_n \in V$.

Note that positive definite functions on a locally convex space are continuous if and only if they are continuous at 0, which holds if and only if they are uniformly continuous, i.e., for every $\varepsilon > 0$ there is a continuous seminorm p on V such that $|\varphi(f) - \varphi(g)| < \varepsilon$ whenever $p(f - g) < 1$ [17].

Theorem 3.2.4 ([69, Theorem 2.4.5])**.** *Let $\varphi : V \to \mathbb{C}$ be a characteristic functional on a locally convex space V. Then, there exist a generalized random field Z indexed by V which is unique (up to equivalence in law) and satisfies $\varphi(f) = \mathbf{E}\left[\mathrm{e}^{iZ(f)}\right]$, $f \in V$. Conversely, for any generalized random field Z indexed by V, its Fourier transform $\varphi(f) := \mathbf{E}\left[\mathrm{e}^{iZ(f)}\right]$, $f \in V$, is a characteristic functional.*

In order to be able to state a generalized random field Z indexed by a locally convex space V as a $V'-$valued random variable, a sufficient condition would be that the characteristic functional of Z is not only continuous with respect to the topology given on V but also in the Kolmogorov-I-topology (see Definition 3.1.5). In general, the Kolmogorov-I-topology is strictly weaker than the original topology on V. A notable exception is the case when the locally convex space V is nuclear. The version of Minlos' Theorem stated below plays an important role for this thesis. In order to formulate Minlos' Theorem, we have to introduce and recall the following notation. Let p be a continuous seminorm and V_p the corresponding local Banach space defined above. By an abuse of notation we also denote with p the quotient norm on V_p. Then we can identify the dual space V_p' of V_p in a canonical way with the subspace $\{\omega \in V' \,:\, \exists\, C > 0 \;\forall f \in V \,:\, |\omega(f)| \leqslant Cp(f)\}$ of V'. We denote by $\mathcal{B}$ the Borel $\sigma-$Algebra on V' that is generated by the weak*-topology $\sigma(V', V)$. Due to the Banach-Alaoglu-Bourbaki Theorem, for every continuous seminorm p on V and every $n \in \mathbb{N}$ the set $\{\omega \in V' \,:\, |\omega(f)| \leqslant np(f) \;\forall f \in V\}$ is $\sigma(V', V)-$compact which implies $V_p' \in \mathcal{B}(V')$. The following version of Minlos' Theorem is a combination of [36, Proof of Theorem III.1.1] and [31, Theorem I.3.4].

Theorem 3.2.5 (Minlos)**.** *Consider a nuclear space V with its topological dual V'. For a functional $\varphi : V \to \mathbb{C}$ the following are equivalent:*

(i) *φ is a characteristic functional.*

(ii) *There is a unique probability measure μ on $(V', \mathcal{B}(V'))$ such that its Fourier transform $\hat{\mu}$ coincides with φ, where*

$$\hat{\mu}(f) := \int_{V'} \mathrm{e}^{i\omega(f)}\, \mu(\mathrm{d}\omega), \quad f \in V.$$

Additionally, if a characteristic functional φ on a nuclear space V is continuous with respect to a continuous H-seminorm p on V, then for the unique probability measure

μ on $(V', \mathcal{B}(V'))$ we have that $\mu(V'_q) = 1$ for every continuous H-seminorm q on V for which the canonical linking map $i^p_q : V_q \to V_p$ is Hilbert–Schmidt.

Remark 3.2.6. *(i) For any locally convex space V and any probability measure μ on $(V', \mathcal{B}(V'))$, the evaluation mapping*

$$(V', \mathcal{B}(V'), \mu) \to \mathbb{R}, \quad \omega \mapsto \omega(f)$$

defines a (scalar) random variable for each $f \in V$. Hence, the mapping

$$Z : V \to L^0(V', \mathcal{B}(V'), \mu), \quad f \mapsto (\omega \mapsto \omega(f))$$

defines a generalized random field indexed by V, which is called the canonical process associated with μ. Canonical processes satisfy a stronger continuity condition than an arbitrary generalized random field since $(Z(f_l))_{l \in I}$ converges also pointwise on V' (in particular μ-almost everywhere) to $Z(f)$ whenever $(f_l)_{l \in I}$ is a net converging to f in V.

(ii) Consider a characteristic functional φ on the nuclear space V which is continuous with respect to the H-seminorm p and let μ be the corresponding probability measure on $(V', \mathcal{B}(V'))$. Further, let $q \succ p$ be a continuous H-seminorm on V such that the canonical linking map i^p_q is Hilbert–Schmidt. One can easily show that the trace σ-algebra $\mathcal{B}(V') \cap V'_q$ coincides with the Borel σ-algebra $\mathcal{B}(V'_q)$ which is generated by the weak-topology $\sigma(V'_q, V_q)$. Therefore, the canonical process*

$$Z : V_q \to L^0(V'_q, \mathcal{B}(V'_q), \mu\restriction_{V'_q})$$

associated with the restriction $\mu\restriction_{V'_q}$, satisfies that for any open $D \subseteq \mathbb{R}^d$, the mapping

$$D \to V_q, \quad x \mapsto f_x$$

is continuous, and that $(Z(f_x))_{x \in D}$ is a random field indexed by D with almost surely continuous paths. Since the characteristic function of the random variable $Z(f_x)$ given by $\varphi(f_x)$, $x \in D$, is by assumption uniformly continuous with respect to p, we can extend it uniquely to a uniformly continuous functional on $V_p \supseteq V_q$.

Below we follow the approach outlined in remark 3.2.6 (ii) for generalized random fields indexed by the space $\mathscr{S} = \mathscr{S}(\mathbb{R}^d, \mathbb{R})$ of Schwartz functions on $\mathbb{R}^d$ whose characteristic functionals are continuous with respect to a specific norm which we define below. In order to obtain random continuous functions on $\mathbb{R}^d$ with known pointwise distributions, these random fields will then be convolved with Matérn kernels.

For $f \in \mathscr{S}$ we denote with $|||f|||$ the continuous norm on $\mathscr{S}$ defined by

$$|||f||| := \left(\|f\|^2_{L^1(\mathbb{R}^d,\mathbb{R})} + \|f\|^2_{L^2(\mathbb{R}^d,\mathbb{R})}\right)^{\frac{1}{2}} = \left(\|f\|^2_{L^1(\mathbb{R}^d,\mathbb{R})} + |f|^2_0\right)^{\frac{1}{2}}$$

For $m \in \mathbb{N}$, $m > \frac{d}{2}$, we set

$$c_m := \int_{\mathbb{R}^d} \frac{\mathrm{d}x}{(1+|x|^2)^m}.$$

With another suitable constant C_m, we see by applying Hölder's and Jensen's inequality that, for any $f \in \mathscr{S}$,

$$\begin{aligned}
|||f|||^2 &\leqslant c_m \int_{\mathbb{R}^d} (1+|x|^2)^m |f(x)|^2 \,\mathrm{d}x + \int_{\mathbb{R}^d} |f(x)|^2 \,\mathrm{d}x \\
&\leqslant (1+c_m)2^{m-1} \int_{\mathbb{R}^d} (1+|x|^{2m})|f(x)|^2 \,\mathrm{d}x \\
&\leqslant (1+c_m)(2d)^{m-1} \int_{\mathbb{R}^d} \left(1 + \sum_{j=1}^d x_j^{2m}\right) |f(x)|^2 \,\mathrm{d}x \qquad (3.1)\\
&= (1+c_m)(2d)^{m-1} \left(|f|^2_0 + \sum_{j=1}^d |x_j^m f|^2_0\right) \\
&\leqslant (1+c_m)(2d)^m C_m |f|^2_{\frac{m}{2}}.
\end{aligned}$$

In the last step we used that for each $m \in \mathbb{N}$ there exists $C_m > 0$ such that we can estimate

$$|x_j^m f|^2_0 \leqslant C_m |f|^2_{\frac{m}{2}}, \qquad \text{for all } 1 \leqslant j \leqslant d \text{ and for all } f \in \mathscr{S},$$

which follows easily by induction from the well-known three-term recurrence relation

$$x_j h_\alpha(x) = \sqrt{\frac{\alpha_j}{2}} h_{\alpha-e_j}(x) + \sqrt{\frac{\alpha_j+1}{2}} h_{\alpha+e_j}(x), \quad 1 \leqslant j \leqslant d,\ \alpha \in \mathbb{N}_0^d,\ x \in \mathbb{R}^d$$

satisfied by the Hermite functions. Here, $e_j = (\delta_{\ell,j})_{1 \leqslant \ell \leqslant d}$ denotes the j^{th} unit coordinate vector in $\mathbb{R}^d$.

Theorem 3.2.7. *Consider a positive definite functional $\varphi : \mathscr{S} \to \mathbb{C}$ which is continuous with respect to the norm $|||\cdot|||$ and for which $\varphi(0) = 1$. Then, there exists a unique probability measure μ on $(\mathscr{S}', \mathcal{B}(\mathscr{S}'))$ such that $\hat{\mu} = \varphi$. Moreover, $\mu(\mathscr{S}'_q) = 1$*

whenever $q > \frac{3d}{4}$.

Proof. With inequality (3.1), we showed that the continuous functional φ is also continuous with respect to $|\cdot|_{\frac{m}{2}}$ for any $m > \frac{d}{2}$. Since by Proposition 3.1.7 the linking map $i^{m/2}_{m/2+\ell}$ is Hilbert–Schmidt for every $\ell > \frac{d}{2}$, the assertion follows from Minlos' Theorem 3.2.5. □

Functionals that lie in the dual space of $\mathscr{S}$ are called tempered distributions. The convolution of a tempered distribution $\omega \in \mathscr{S}'$ with a rapidly decreasing function $f \in \mathscr{S}$

$$\omega * f : \mathbb{R}^d \to \mathbb{R}, \quad y \mapsto \langle \omega, \tau_y(f^\vee) \rangle = \langle \omega_x, f(y-x) \rangle$$

is a smooth function (see, e.g., [46, section 10.3]). We recall that we denote by $u(g) = \langle u, g \rangle$ the application of $u \in \mathscr{S}'$ to $u \in \mathscr{S}$. In addition, $(\tau_y g)(x) := g(x-y)$ denotes the translation of g by $y \in \mathbb{R}^d$, $g^\vee(x) := g(-x)$ states the reflection of g at the origin, and the subscript ω_x indicates that the tempered distribution ω acts on test functions depending on the variable x.

For the dual $\mathscr{S}_q'$ of the local Hilbert spaces $\mathscr{S}_q'$, with $q \in \mathbb{N}_0$, we can similarly conclude that whenever a function $f \in \mathscr{S}_q$ is such that $\tau_y f^\vee \in \mathscr{S}_q$ for every $y \in \mathbb{R}^d$, the convolution with a tempered distribution $\omega \in \mathscr{S}'$

$$\mathbb{R}^d \to \mathbb{R}, \quad y \mapsto \langle \omega_x, \tau_y(f^\vee) \rangle$$

is defined and is clearly continuous if the mapping

$$\mathbb{R}^d \to \mathscr{S}_q(\mathbb{R}^d, \mathbb{R}), \quad y \mapsto \tau_y(f^\vee)$$

is continuous. Therefore, whenever we have a $|||\cdot|||$-continuous characteristic functional φ on $\mathscr{S}$ with associated probability measure μ on $(\mathscr{S}', \mathcal{B}(\mathscr{S}'))$, it follows, from Theorem 3.2.7, that if $q > \frac{3d}{4}$, for each $f \in \mathscr{S}_q$, with $\tau_y(f^\vee) \in \mathscr{S}_q$, the convolution $\omega * f$ is a well-defined function for μ-almost all $\omega \in \mathscr{S}_q'$. This convolution yields a continuous function if

$$\mathbb{R}^d \to \mathscr{S}_q, \quad y \mapsto \tau_y(f^\vee)$$

is continuous. The specific type of Schwartz functions we are particularly interested in, is the class of Matérn kernels.

Definition 3.2.8 (Matérn Kernels)**.** *We the describe the class of Matérn kernels by their Fourier transforms. For $\alpha \in \mathbb{R}$ and $m > 0$ we define by*

$$\hat{k}_{\alpha,m} : \mathbb{R}^d \to \mathbb{R}, \quad \xi \mapsto \frac{1}{(|\xi|^2 + m^2)^\alpha}$$

the Fourier transform of the Matérn kernel

$$k_{\alpha,m} := \mathscr{F}^{-1}(\hat{k}_{\alpha,m})$$

with parameters α and m. Note that $\hat{k}_{\alpha,m}$ is a polynomially bounded smooth function, and thus belongs to $\mathscr{S}'$; hence, its inverse Fourier transform is well-defined.

Lemma 3.2.9 ([42, Appendix A])**.** *Let $q \in \mathbb{N}_0$, $\alpha \in \mathbb{R}$, and $m > 0$. In the following, statement (i) implies (ii), (ii) implies (iii), and (iii) implies (iv).*

(i) $\alpha > \frac{d}{4} + q + \max\{0, \frac{q-3}{2}\}$,

(ii) For every $y \in \mathbb{R}^d$ the translation $\tau_y(k^\vee_{\alpha,m})$ lies in $\mathscr{S}_q$ and the mapping

$$\mathbb{R}^d \to (\mathscr{S}_q, |\cdot|_q), \quad y \mapsto \tau_y(k^\vee_{\alpha,m})$$

is continuous.

(iii) For every $y \in \mathbb{R}^d$ the translation $\tau_y(k^\vee_{\alpha,m})$ lies in $\mathscr{S}_q$.

(iv) $\alpha > \frac{d}{4} + q$.

In particular, if $q \in \{0, 1, 2, 3\}$, then (ii), (iii), and (iv) above are equivalent.

We can now state for a random field with $|||\cdot|||$-continuous characteristic functional conditions on the amount of smoothing required such that it has continuous realization after smoothing by convolution with a Matérn kernel $k_{\alpha,m}$.

Theorem 3.2.10. *Consider a positive definite and $|||\cdot|||$-continuous functional φ on $\mathscr{S}(\mathbb{R}^d, \mathbb{R})$ with $\varphi(0) = 1$. Then, there is a unique probability measure μ on $(\mathscr{S}', \mathcal{B}(\mathscr{S}'))$ satisfying $\hat{\mu} = \varphi$ such that for all $\alpha > d + \max\{0, \frac{3d-12}{8}\}$, for every $m > 0$, the function*

$$\mathbb{R}^d \to \mathbb{R}, \quad y \mapsto \omega * k_{\alpha,m}(y) = \langle \omega, \tau_y(k^\vee_{\alpha,m}) \rangle$$

is defined and continuous for μ-almost all $\omega \in \mathscr{S}'$. Moreover, for fixed $y \in \mathbb{R}^d$ the distribution of the random variable

$$(\mathscr{S}', \mathcal{B}(\mathscr{S}'), \mu) \to (\mathbb{R}, \mathcal{B}(\mathbb{R})), \quad \omega \mapsto \omega * k_{\alpha,m}(y)$$

has the Fourier transform $\varphi\left(\tau_y(k_{\alpha,m}^\vee)\right)$.

Proof. Theorem 3.2.7 provides a unique probability measure μ on $(\mathscr{S}', \mathcal{B}(\mathscr{S}'))$ with $\hat{\mu} = \varphi$ and $\mu(\mathscr{S}'_q) = 1$ whenever $q > \frac{3d}{4}$. Now, for $\alpha > \frac{d}{4} + \frac{3d}{4} + \max\{0, \frac{\frac{3d}{4}-3}{2}\} = d + \max\{0, \frac{3d-12}{8}\}$ there is a $q > \frac{3d}{4}$ such that $\alpha > \frac{d}{4} + q + \max\{0, \frac{q-3}{2}\}$, and that by Lemma 3.2.9 the mapping

$$\mathbb{R}^d \to (\mathscr{S}_q, |\cdot|_q), \quad y \mapsto \tau_y(k_{\alpha,m}^\vee)$$

is well-defined and continuous. As $\mu(\mathscr{S}'_q) = 1$ and hence $\mu(\mathscr{S}' \backslash \mathscr{S}'_q) = 0$, this implies that for μ-almost all $\omega \in \mathscr{S}'$ the mapping

$$\mathbb{R}^d \to \mathbb{C}, \quad y \mapsto \langle \omega, \tau_y(k_{\alpha,m}^\vee)\rangle = \omega * k_{\alpha,m}(y)$$

is continuous, since it is the composition of continuous functions.

Finally, by inequality (3.1) we can follow that φ is $|\cdot|_p$-continuous for any $p > \frac{d}{2}$ and in particular $|\cdot|_q$-continuous for q as above. Because $\tau_y(k_{\alpha,m}^\vee)$ belongs to $\mathscr{S}_q$, the $|\cdot|_q$-completion of $\mathscr{S}$, there exists a sequence $(f_n)_{n\in\mathbb{N}}$ in $\mathscr{S}$ which converges to $\tau_y(k_{\alpha,m}^\vee)$ with respect to $|\cdot|_q$. With Lebesgue's Dominated Convergence Theorem we can conclude

$$\begin{aligned} \varphi\left(\tau_y(k_{\alpha,m}^\vee)\right) &= \lim_{n\to\infty} \varphi(f_n) = \lim_{n\to\infty} \int_{\mathscr{S}'} \mathrm{e}^{i\langle \omega, f_n\rangle}\, \mu(\mathrm{d}\omega) \\ &= \lim_{n\to\infty} \int_{\mathscr{S}'_q} \mathrm{e}^{i\langle \omega, f_n\rangle}\, \mu(\mathrm{d}\omega) = \int_{\mathscr{S}'_q} \mathrm{e}^{i\langle \omega, \tau_y(k_{\alpha,m}^\vee)\rangle}\, \mu(\mathrm{d}\omega), \end{aligned}$$

which proves the theorem. □

3.3 Lévy Random Fields

In this section, we introduce Lévy random fields and apply the generalized random field theory from above in order to construct smoothed Lévy noise fields. In this work, we use these smoothed fields to model uncertainties occurring in various physical state problems.

3.3.1 Classification of Noise Fields

Definition 3.3.1 (Lévy Noise Fields)**.** *Let $b \in \mathbb{R}$, $\sigma^2 \geqslant 0$ and ν be a σ-finite Borel measure on $\mathbb{R}\backslash\{0\}$ satisfying $\int_{\mathbb{R}\backslash\{0\}} \min\{1, s^2\}\, \nu(\mathrm{d}s) < \infty$. Then, the function*

$$\psi : \mathbb{R} \to \mathbb{C}, \quad \psi(t) := ibt - \frac{\sigma^2 t^2}{2} + \int_{\mathbb{R}\backslash\{0\}} \left(\mathrm{e}^{its} - 1 - its\mathbb{1}_{\{|s|\leqslant 1\}}(s)\right)\, \nu(\mathrm{d}s) \tag{3.2}$$

is called the Lévy characteristic with characteristic triplet (b, σ^2, ν).

A generalized random field Z indexed by $\mathscr{S}$ is called a Lévy noise field if there is a characteristic triplet (b, σ^2, ν) such that for the characteristic functional φ of Z there holds

$$\varphi(f) = \exp\left(\int_{\mathbb{R}^d} (\psi \circ f)(x)\, \mathrm{d}x\right), \quad f \in \mathscr{S}, \tag{3.3}$$

where ψ is the Lévy characteristic associated with (b, σ^2, ν). (In particular, this assumes $\psi \circ f \in L^1(\mathbb{R}^d, \mathbb{R})$ for all $f \in \mathscr{S}$.) We then say that Z is associated with the characteristic triplet (b, σ^2, ν). A classical reference for Lévy noise fields is, e.g., [53]; see also [6, 7].

Lemma 3.3.2. *Let Z be a Lévy noise field. Then, we can decompose it in $Z = Z_D + Z_G + Z_J$ with deterministic part Z_D, Gaussian white noise Z_G, and pure jump noise Z_J, each of which are independent random fields with characteristic functionals*

$$\begin{aligned}
\varphi_{Z_D}(f) &= \mathrm{e}^{ib\int_{\mathbb{R}^d} f(x)\, \mathrm{d}x}, \\
\varphi_{Z_G}(f) &= \mathrm{e}^{-\frac{1}{2}\sigma^2 \|f\|^2_{L^2(\mathbb{R}^d,\mathbb{R})}}, \ \ \textit{and} \\
\varphi_{Z_J}(f) &= \mathrm{e}^{\int_{\mathbb{R}^d} \int_{\mathbb{R}\backslash\{0\}} \mathrm{e}^{isf(x)} - 1 - itf(x)\mathbb{1}_{\{|s|\leqslant 1\}}(s)\, \nu(\mathrm{d}s)\mathrm{d}x},
\end{aligned}$$

respectively.

Proof. This factorization is directly implicated by Definition 3.3.1, which suggests the factorization of the characteristic functional $\varphi_Z(f) = \varphi_{Z_D}(f)\varphi_{Z_G}(f)\varphi_{Z_J}(f)$. □

By Lemma 3.3.2, Lévy noises are seen to be a generaliation of Gaussian noises, to which they simplify when the pure jump part Z_J is omitted.

Proposition 3.3.3. *We consider a Lévy characteristic ψ with triplet (b, σ^2, ν) such that the Lévy measure satisfies $\int_{\mathbb{R}\setminus\{0\}} |s| \mathbb{1}_{\{|s|>1\}}(s)\, \nu(\mathrm{d}s) < \infty$. Then,*

$$\varphi : \mathscr{S} \to \mathbb{C}, \quad \varphi(f) := \exp\left(\int_{\mathbb{R}^d} (\psi \circ f)(x)\, \mathrm{d}x \right)$$

is a well-defined characteristic functional which is continuous with respect to the norm $|||\cdot|||$. In particular, there is a Lévy noise field Z (unique up to equivalence in law) associated with (b, σ^2, ν). Further, Z is continuous with respect to $|||\cdot|||$.

Proof. Because any $f \in \mathscr{S}$ also lies in $L^1(\mathbb{R}^d, \mathbb{R}) \cap L^2(\mathbb{R}^d, \mathbb{R})$ we have

$$\begin{aligned}
&\int_{\mathbb{R}^d} \int_{\mathbb{R}\setminus\{0\}} |\mathrm{e}^{isf(x)} - 1 - isf(x) \mathbb{1}_{\{|s|\leqslant 1\}}(s)|\, \nu(\mathrm{d}s)\mathrm{d}x \\
&= \int_{\mathbb{R}^d} \int_{\{0<|s|\leqslant 1\}} |\mathrm{e}^{isf(x)} - 1 - isf(x)|\, \nu(\mathrm{d}s)\mathrm{d}x + \int_{\mathbb{R}^d} \int_{\{|s|>1\}} |\mathrm{e}^{isf(x)} - 1|\, \nu(\mathrm{d}s)\mathrm{d}x \\
&\leqslant \int_{\mathbb{R}^d} \int_{\{0<|s|\leqslant 1\}} \frac{|s|^2 |f(x)|^2}{2}\, \nu(\mathrm{d}s)\mathrm{d}x + \int_{\mathbb{R}^d} \int_{\{|s|>1\}} |s||f(x)|\, \nu(\mathrm{d}s)\mathrm{d}x \\
&\leqslant \frac{1}{2} \int_{\mathbb{R}\setminus\{0\}} \min\{1, s^2\}\, \nu(\mathrm{d}s)\, \|f\|^2_{L^2(\mathbb{R}^d,\mathbb{R})} + \int_{\{|s|>1\}} |s|\, \nu(\mathrm{d}s)\, \|f\|_{L^1(\mathbb{R}^d,\mathbb{R})}
\end{aligned}$$

which yields $\psi \circ f \in L^1(\mathbb{R}^d, \mathbb{R})$ and

$$\begin{aligned}
\|\psi \circ f\|_{L^1(\mathbb{R}^d,\mathbb{R})} \leqslant & \left(|b| + \int_{\{|s|>1\}} |s|\, \nu(\mathrm{d}s) \right) \|f\|_{L^1(\mathbb{R}^d,\mathbb{R})} \\
& + \left(\frac{\sigma^2 + \int_{\mathbb{R}\setminus\{0\}} \min\{1, s^2\}\, \nu(\mathrm{d}s)}{2} \right) \|f\|^2_{L^2(\mathbb{R}^d,\mathbb{R})}.
\end{aligned}$$

Therefore,

$$\varphi : \mathscr{S} \to \mathbb{C}, \quad \varphi(f) := \exp\left(\int_{\mathbb{R}^d} (\psi \circ f)(x)\, \mathrm{d}x \right)$$

is well-defined.

Since $\varphi(0) = 1$, the previous inequality implies that φ is continuous at 0 with respect to the norm $|||\cdot|||$. Since, by [53, Theorem 6, p. 283], the restriction of φ to $\mathscr{D} := C^\infty_{\mathrm{c}}(\mathbb{R}^d, \mathbb{R})$ is positive definite, it follows that the restriction of φ to $\mathscr{D}$ is (uniformly) $|||\cdot|||$-continuous. Further, $\mathscr{D}$ is dense in $\mathscr{S}$ with respect to $|||\cdot|||$ and hence, φ is positive definite and continuous. Therefore, due to Theorem 3.2.4 (and inequality (3.1)), there is a generalized random field Z indexed by $\mathscr{S}$ whose Fourier transform is φ and which, in addition, is continuous with respect to the $|||\cdot|||$-norm. Thus, the proposition is proven. □

Remark 3.3.4. *By convolving a compactly supported continuous function on $\mathbb{R}^d$ with an approximate identity, e.g., with a mollifier, we can show that $\mathscr{D}$ is a dense subspace of the space of compactly supported continuous functions on $\mathbb{R}^d$ with respect to the $|||\cdot|||$ norm. It follows immediately that $\mathscr{S}$ is dense in $L^1(\mathbb{R}^d, \mathbb{R}) \cap L^2(\mathbb{R}^d, \mathbb{R})$ assuming the latter space is equipped with the norm $|||\cdot|||$. As it is noted in remark 3.2.2 (ii), it thus follows that for every $|||\cdot|||$-continuous Lévy noise field Z there is a generalized random field indexed by $L^1(\mathbb{R}^d, \mathbb{R}) \cap L^2(\mathbb{R}^d, \mathbb{R})$ that uniquely extends Z. The extension will also be denoted by Z. Therefore, for a Borel subset Λ of $\mathbb{R}^d$ with finite Lebesgue measure, we are able to define the (non-normalized) Λ-average of the Lévy noise field Z by $Z(\mathbb{1}_\Lambda)$.*

Definition 3.3.5 (Stationary Noise Field)**.** *A generalized random field Z indexed by $\mathscr{S}$ is called*

(i) a noise field if for any choice of index functions $f_1, \ldots, f_n \in \mathscr{S}$ with mutually disjoint supports the random variables $Z(f_1), \ldots, Z(f_n)$ are independent,

(ii) a stationary field if for every $f \in \mathscr{S}$ and each $a \in \mathbb{R}^d$ the random variables $Z(f)$ and $Z(f_a)$ have the same probability distribution, i.e., $Z(f) \sim Z(f_a)$, where $f_a(x) = (\tau_a f)(x) = f(x - a)$,

(iii) a stationary noise field if it is both a noise field and a stationary field.

Since the distributional derivatives of noise fields are noise fields as well, a noise field can be arbitrary singular. There are many situations in which one would like for a bounded and measurable set $A \subseteq \mathbb{R}^d$ take the spatial average $Z(\mathbb{1}_A)$ and ensure that this quantity has finite expectation. However, for such A, the indicator function $\mathbb{1}_A \in V = L^1(\mathbb{R}^d, \mathbb{R}) \cap L^2(\mathbb{R}^d, \mathbb{R})$, but $\mathbb{1}_A \notin \mathscr{S}$ which rules out too singular distributional noises. The next theorem is a characterization of stationarity in the setting outlined above.

Theorem 3.3.6. *We assume for a generalized random field Z on $(\Omega, \mathfrak{A}, \mathbf{P})$ indexed by $\mathscr{S}$ which is $|||\cdot|||$-continuous that its unique $|||\cdot|||$-continuous extension to $L^1(\mathbb{R}^d, \mathbb{R}) \cap L^2(\mathbb{R}^d, \mathbb{R})$ satisfies $Z(f) \in L^1(\Omega, \mathfrak{A}, \mathbf{P})$ for all $f \in L^1(\mathbb{R}^d, \mathbb{R}) \cap L^2(\mathbb{R}^d, \mathbb{R})$. Then, the following two properties are equivalent:*

(i) Z is a Lévy noise field.

(ii) Z is a stationary noise field.

Proof. First, let us assume that Z is a Lévy noise field. Then, by definition there is a characteristic triplet (b, σ^2, ν) with associated Lévy characteristic ψ such that the Fourier transform of Z satisfies $\varphi(f) = \exp\left(\int_{\mathbb{R}^d} (\psi \circ f)\,\mathrm{d}x\right)$, for $f \in \mathscr{S}$. Since the Lebesgue measure is translation invariant, we know that for any $a \in \mathbb{R}^d$ we have $\varphi(f_a) = \varphi(f)$, i.e., the random variables $Z(f_a)$ and $Z(f)$ have the same characteristic function and therefore $Z(f) \sim Z(f_a)$. Hence, Z is a stationary field.

Next, for $f_1, \dots, f_n \in \mathscr{S}$ with disjoint supports it follows for all $(\kappa_1, \dots, \kappa_n) \in \mathbb{R}^n$ and every $x \in \mathbb{R}^d$ that $\psi\left(\sum_{j=1}^n \kappa_j f_j(x)\right) = \sum_{j=1}^n \psi\left(\kappa_j f_j(x)\right)$ because at most one of the summands is different from 0 and $\psi(0) = 0$. Therefore,

$$\begin{aligned}\mathbf{E}\left[\mathrm{e}^{i\sum_{j=1}^n \kappa_j Z(f_j)}\right] &= \mathbf{E}\left[\mathrm{e}^{iZ(\sum_{j=1}^n \kappa_j f_j)}\right] = \exp\left(\int_{\mathbb{R}^d} \psi(\sum_{j=1}^n \kappa_j f_j(x))\,\mathrm{d}x\right)\\ &= \prod_{j=1}^n \exp\left(\int_{\mathbb{R}^d} \psi(\kappa_j f_j(x))\,\mathrm{d}x\right) = \prod_{j=1}^n \mathbf{E}\left[\mathrm{e}^{i\kappa_j Z(f_j)}\right],\end{aligned}$$

i.e., we have that the Fourier transform of the joint distribution of the random variables $Z(f_1), \dots, Z(f_n)$ equals the product of the characteristic functions of the $Z(f_j)$, thus, $Z(f_1), \cdots, Z(f_n)$ are independent, so that Z is stationary.

Now, we show that (ii) implies (i). First, as mentioned in remark 3.3.4, due to the $|||\cdot|||$-continuity of Z, there is a unique extension on $L^1(\mathbb{R}^d, \mathbb{R}) \cap L^2(\mathbb{R}^d, \mathbb{R})$ to Z which we denote with Z as well, where the space $L^1(\mathbb{R}^d, \mathbb{R}) \cap L^2(\mathbb{R}^d, \mathbb{R})$ is eqiupped with the norm $|||\cdot|||$. Since the operator τ_a is linear and continuous on $\mathscr{S}$ with respect to the norm $|||\cdot|||$, it follows by the $|||\cdot|||$-density of $\mathscr{S}$ in $L^1(\mathbb{R}^d, \mathbb{R}) \cap L^2(\mathbb{R}^d, \mathbb{R})$ that $Z(f_a) \sim Z(f)$ for all $f \in L^1(\mathbb{R}^d, \mathbb{R}) \cap L^2(\mathbb{R}^d, \mathbb{R})$.

We define a cube in $\mathbb{R}^d$ as a set of the form $\prod_{j=1}^d [\beta_j, \gamma_j)$ where $\beta_j, \gamma_j \in \mathbb{R}$, $1 \leqslant j \leqslant d$ with $\gamma_j - \beta_j > 0$ independent of j, the so-called (side) length of the cube. We consider a cube Λ with side length $L > 0$ and subdivide Λ into n^d non-intersecting cubes Λ_ℓ, each of side length L/n. Therefore, for each $1 \leqslant \ell, k \leqslant n^d$ we have $Z(\mathbb{1}_{\Lambda_\ell}) \sim Z(\mathbb{1}_{\Lambda_k})$ by the extended stationarity of Z. Due to the construction, we can find $\beta_j, \gamma_j \in \mathbb{R}$, $\gamma_j - \beta_j = L/n$, $1 \leqslant j \leqslant d$ as well as $a^{(1)}, \dots, a^{(n^d)} \in \mathbb{R}^d$ such that

$$\Lambda_\ell = \prod_{j=1}^d [\beta_j + a_j^{(\ell)}, \gamma_j + a_j^{(\ell)}), \quad \ell = 1, \dots, n^d.$$

For $\varepsilon \in]0, L/2n[$ we define

$$\Lambda_\ell^\varepsilon := \prod_{j=1}^{d} [\beta_j + a_j^{(\ell)} + \varepsilon, \gamma_j + a_j^{(\ell)} - \varepsilon), \quad \ell = 1, 2, \ldots, n^d.$$

Further, we consider $\phi \in \mathscr{D}$ with $\operatorname{supp} \phi \subset (-1, 1)$, $\phi \geqslant 0$, and $\int_{\mathbb{R}^d} \phi \, \mathrm{d}x = 1$. For $\varepsilon > 0$ we set $\phi_\varepsilon(x) := \varepsilon^{-d} \phi(x/\varepsilon)$. For $\varepsilon \in]0, L/2n[$, the functions $\phi_\varepsilon * \mathbb{1}_{\Lambda_\ell^\varepsilon}$, $1 \leqslant \ell \leqslant n^d$ belongs to $\mathscr{D}$ and satisfy

(1) $\forall 1 \leqslant \ell \leqslant n^d, \varepsilon \in]0, L/2n[:$ $\operatorname{supp} \phi_\varepsilon * \mathbb{1}_{\Lambda_\ell^\varepsilon} \subseteq \Lambda_\ell$,

(2) $\forall 1 \leqslant \ell \leqslant n^d, \varepsilon \in]0, L/2n[:$ $\sup_{x \in \mathbb{R}^d} |\phi_\varepsilon * \mathbb{1}_{\Lambda_\ell^\varepsilon}| \leqslant 1$.

Due to (1), for fixed $\varepsilon \in]0, L/2n[$ the functions $\phi_\varepsilon * \mathbb{1}_{\Lambda_\ell^\varepsilon}$, $1 \leqslant \ell \leqslant n^d$, have mutually disjoint supports and by Lebesgue's Dominated Convergence Theorem, $\lim_{\varepsilon \to 0} \phi_\varepsilon * \mathbb{1}_{\Lambda_\ell^\varepsilon} = \mathbb{1}_{\Lambda_\ell}$ with respect to $|\!|\!|\cdot|\!|\!|$. As Z is $|\!|\!|\cdot|\!|\!|$-continuous, $Z(\phi_\varepsilon * \mathbb{1}_{\Lambda_\ell^\varepsilon}) \to Z(\mathbb{1}_{\Lambda_\ell})$ in probability when $\varepsilon \searrow 0$; see Remark 3.2.2 (ii). Consequently, the vector $(Z(\phi_\varepsilon * \mathbb{1}_{\Lambda_1^\varepsilon}), \cdots, Z(\phi_\varepsilon * \mathbb{1}_{\Lambda_{n^d}^\varepsilon}))$ converges in probability to $(Z(\mathbb{1}_{\Lambda_1}), \cdots, Z(\mathbb{1}_{\Lambda_{n^d}}))$ as well. Since the random variables $Z(\phi_\varepsilon * \mathbb{1}_{\Lambda_\ell^\varepsilon})$ are independent, their joint characteristic function factors to a product of individual characteristic functions, each converging to the characteristic function of the corresponding $Z(\mathbb{1}_{\Lambda_\ell})$. Hence, the joint characteristic function of $(Z(\mathbb{1}_{\Lambda_1}), \cdots, Z(\mathbb{1}_{\Lambda_{n^d}}))$, which coincides with the limit of the joint characteristic function of $(Z(\phi_\varepsilon * \mathbb{1}_{\Lambda_1^\varepsilon}), \cdots, Z(\phi_\varepsilon * \mathbb{1}_{\Lambda_{n^d}^\varepsilon}))$, factors to a product of the characteristic functions of $Z(\mathbb{1}_{\Lambda_\ell})$. This implies the independence of the random variables $Z(\mathbb{1}_{\Lambda_\ell})$.

With

$$B_{\ell,n} := \left[\beta_1 + (\ell - 1)\frac{L}{n}, \, \beta_1 + \ell\frac{L}{n}\right) \times \prod_{j=2}^{n} [\beta_j, \gamma_j), \quad 1 \leqslant \ell \leqslant n,$$

we define a partition of Λ into n sets where each set is a disjoint union of a mutually disjoint subfamily of the $\Lambda_1, \ldots, \Lambda_{n^d}$ such that the corresponding random variables $Z(\mathbb{1}_{B_{1,n}}), \ldots, Z(\mathbb{1}_{B_{n,n}})$ are independent and identically distributed. Since $n \in \mathbb{N}$ was arbitrarily chosen and $Z(\mathbb{1}_\Lambda) = \sum_{\ell=1}^{n} Z(\mathbb{1}_{B_{\ell,n}})$, $Z(\mathbb{1}_\Lambda)$ has an infinitely divisible probability law. By the Lévy-Khinchine Theorem [95, Theorem 8.1], there is a uniquely determined characteristic triplet $(b_\Lambda, \sigma_\Lambda^2, \nu_\Lambda)$ with associated Lévy characteristic ψ_Λ such that

$$\mathbf{E}\left[\mathrm{e}^{iZ(\mathbb{1}_\Lambda)}\right] = \mathrm{e}^{|\Lambda|\psi_\Lambda(\kappa)} \quad \text{and} \quad \mathbf{E}\left[\mathrm{e}^{i\kappa Z(\mathbb{1}_{\Lambda_\ell})}\right] = \mathrm{e}^{|\Lambda_\ell|\psi_\Lambda(\alpha)},$$

for all $\kappa \in \mathbb{R}$ and $\ell = 1, \ldots, n^d$, where for a Borel set $B \subset \mathbb{R}^d$ we denote by $|B|$ its Lebesgue measure.

Now, we consider another cube Λ' with side length L' such that L/L' is a rational number n/m, with $n, m \in \mathbb{N}$. We subdivide the cube Λ into n^d mutually disjoint cubes Λ_ℓ of side length L/n and Λ' into m^d mutually disjoint cubes Λ'_k of side length L'/m as we did above. Since the extension Z is stationary and $L/n = L'/m$, we have that the random variables $Z(\mathbb{1}_{\Lambda_\ell})$ and $Z(\mathbb{1}_{\Lambda'_k})$ have the same distribution, for $1 \leqslant \ell \leqslant n^d$, $1 \leqslant k \leqslant m^d$. This yields

$$\mathrm{e}^{|\Lambda_\ell|\psi_\Lambda(\kappa)} = \mathrm{e}^{|\Lambda'_k|\psi_{\Lambda'}(\kappa)}, \quad \forall \kappa \in \mathbb{R}, 1 \leqslant \ell \leqslant n^d,\ 1 \leqslant k \leqslant m^d,$$

so that, by $|\Lambda_\ell| = (L/n)^d = (L/m)^d = |\Lambda'_k|$ and the continuity of the Lévy characteristics ψ_Λ and $\psi_{\Lambda'}$, it follows that there is $k \in \mathbb{Z}$ with $\psi_\Lambda(\kappa) = \psi_{\Lambda'}(\kappa) + 2\pi i k$. Since $\psi_\Lambda(0) = 0 = \psi_{\Lambda'}(0)$, we can follow that $\psi_\Lambda = \psi_{\Lambda'}$. Therefore, there is a characteristic triplet (b, σ^2, ν) with associated Lévy characteristic ψ such that for all cubes Λ with rational side length we have $\mathbf{E}\left[\mathrm{e}^{i\kappa Z(\mathbb{1}_\Lambda)}\right] = \mathrm{e}^{|\Lambda|\psi(\kappa)}$, $\kappa \in \mathbb{R}$. From [95, Example 25.12] we can follow, since $Z(\mathbb{1}_\Lambda) \in L^1(\Omega, \mathfrak{A}, \mathbf{P})$, that $\int_{\{|s|>1\}} |s|\, \nu(\mathrm{d}s) < \infty$ which yields by Proposition 3.3.3 that

$$\varphi_\psi : \mathscr{S} \to \mathbb{C}, \quad f \mapsto \exp\left(\int_{\mathbb{R}^d} (\psi \circ f)(x)\,\mathrm{d}x\right)$$

is a well-defined, positive definite functional which is $|||\cdot|||$-continuous and which can be extended uniquely to a $|||\cdot|||$-characteristic functional on $L^1(\mathbb{R}^d, \mathbb{R}) \cap L^2(\mathbb{R}^d, \mathbb{R})$.

Since Z is a noise field, for mutually disjoint cubes $\Lambda^{(1)}, \ldots, \Lambda^{(n)}$ in $\mathbb{R}^d$ of respective side length $L_j \in \mathbb{Q}$ we obtain by the same procedure as above, i.e., via mollification of the indicator functions of suitably shrunk cubes, that the random variables $Z(\mathbb{1}_{\Lambda^{(1)}}), \ldots Z(\mathbb{1}_{\Lambda^{(n)}})$ are independent. For the simple function $f = \sum_{j=1}^n \kappa_j \mathbb{1}_{\Lambda^{(j)}}$ we conclude

$$\begin{aligned}\varphi(f) = \mathbf{E}\left[\mathrm{e}^{iZ(f)}\right] &= \prod_{j=1}^n \mathrm{e}^{|\Lambda^{(j)}|\psi} = \prod_{j=1}^n \mathrm{e}^{\int_{\Lambda^{(j)}} \psi(\kappa_j)\,\mathrm{d}x} = \mathrm{e}^{\int_{\mathbb{R}^d} \sum_{j=1}^n \psi(\kappa_j)\mathbb{1}_{\Lambda^{(j)}}\,\mathrm{d}x} \\ &= \mathrm{e}^{\int_{\mathbb{R}^d} (\psi\circ f)(x)\,\mathrm{d}x} = \varphi_\psi(f),\end{aligned} \tag{3.4}$$

where we have used again that for functions with mutually disjoint (essential) supports $f_1, \ldots, f_n \in L^1(\mathbb{R}^d, \mathbb{R}) \cap L^2(\mathbb{R}^d, \mathbb{R})$ we have $\psi(\sum_j f_j) = \sum_j \psi(f_j)$ due to $\psi(0) = 0$.

Finally, since the simple functions we used above are $|||\cdot|||$-dense in $L^1(\mathbb{R}^d, \mathbb{R}) \cap$

$L^2(\mathbb{R}^d,\mathbb{R})$ and φ as well as φ_ψ are $|||\cdot|||$-continuous, it follows from (3.4) that $\varphi(f) = \varphi_\psi(f)$ for all $f \in L^1(\mathbb{R}^d,\mathbb{R}) \cap L^2(\mathbb{R}^d,\mathbb{R})$. In particular, Z is a Lévy noise field. □

The next proposition is numerically motivated. In order to computationally treat Lévy noise fields in terms of e.g., quadrature or Karhunen-Loève expansion, knowledge about the expectation of polynomial expressions in the random fields is needed. Hence, we provide in the next proposition the moments of Lévy noise fields.

Proposition 3.3.7 ([42, Appendix B]). *Let Z be a $|||\cdot|||$-continuous Lévy noise field with characteristic triplet (b,σ^2,ν). We assume that the Lévy measure ν is such that the following integrals exist and are finite:*

$$b_1 := \int_{\{|s|>1\}} s\,\nu(\mathrm{d}s) \quad \text{and} \quad b_n := \int_{\mathbb{R}\setminus\{0\}} s^n\,\nu(\mathrm{d}s), \quad n \in \mathbb{N},\, n \geqslant 2.$$

Then, $Z(f)$ possesses moments of all orders for every $f \in \mathscr{S}$ and

$$\mathbf{E}\left[\prod_{j=1}^{n} Z(f_j)\right] = \sum_{\substack{I\in\mathscr{P}^{(n)} \\ I=\{I_1,\dots,I_k\}}} \prod_{\ell=1}^{k} c_{|I_\ell|} \int_{\mathbb{R}^d} \prod_{j\in I_\ell} f_j \,\mathrm{d}x.$$

Here, $\mathscr{P}^{(n)}$ is the collection of all partitions of $\{1,\dots,n\}$ into non-intersecting and non-empty sets $\{I_1,\dots,I_k\}$, where k is arbitrary. $|I_\ell|$ denotes the number of elements in I_ℓ and c_n is a sequence of constants defined as

$$c_n = \begin{cases} b + b_1 & : n = 1, \\ \sigma^2 + b_2 & : n = 2, \\ b_n & : n \geqslant 3. \end{cases}$$

3.4 Smoothed Stationary Noise Fields

In this thesis, we apply random functions on (an open, bounded subset of) $\mathbb{R}^d$ to model physical uncertainties in various situations. In order to make use of the so far examined $|||\cdot|||$-continuous stationary noise fields indexed by the Schwartz space $\mathscr{S}(\mathbb{R}^d,\mathbb{R})$, which are uniquely extendable on $L^1(\mathbb{R}^d,\mathbb{R}) \cap L^2(\mathbb{R}^d,\mathbb{R})$, we introduce the following notation.

Definition 3.4.1 (Smoothed Random Fields). *For a $|||\cdot|||$-continuous stationary noise field Z on the probability space $(\Omega, \mathfrak{A}, \mathbf{P})$ and a function $k \in L^1(\mathbb{R}^d, \mathbb{R}) \cap L^2(\mathbb{R}^d, \mathbb{R})$ we define the smoothed random field (with window function, smoothing function, or smoothing kernel k) as the family of random variables*

$$Z_k(x) := Z(k_x) \in L^0(\Omega, \mathfrak{A}, \mathbf{P}), \quad x \in \mathbb{R}^d,$$

where $k_x := \tau_x(k^\vee) = k(x - \cdot)$. More generally, we shall call a bivariate function $k : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ a smoothing function (or window function) if $k(x, \cdot) \in L^1(\mathbb{R}^d, \mathbb{R}) \cap L^2(\mathbb{R}^d, \mathbb{R})$ for every $x \in \mathbb{R}^d$ and $\mathbb{R}^d \to (L^1(\mathbb{R}^d, \mathbb{R}) \cap L^2(\mathbb{R}^d, \mathbb{R}), |||\cdot|||)$, $x \mapsto k(x, \cdot)$ is continuous. For a bivariate smoothing function k we set, by an abuse of notation, $k_x := k(x, \cdot)$ and define the smoothed random field with smoothing function k as the family of random variables $Z_k(x) := Z(k_x) \in L^0(\Omega, \mathfrak{A}, \mathbf{P})$, $x \in \mathbb{R}^d$.

Remark 3.4.2. *(i) By Minlos' Theorem 3.2.5, every generalized random field Z on a probability space $(\Omega, \mathfrak{A}, \mathbf{P})$ indexed by $\mathscr{S}(\mathbb{R}^d, \mathbb{R})$ is given by a $\mathscr{S}'(\mathbb{R}^d, \mathbb{R})$-valued random variable (also denoted by Z). It follows for a window function $k \in \mathscr{S}(\mathbb{R}^d, \mathbb{R})$ that*

$$\mathbb{R}^d \to \mathbb{R}, \quad x \mapsto Z(k_x) = \langle Z, k(x - \cdot)\rangle = (Z * k)(x)$$

is $\mathbf{P}$-almost surely a smooth function as a convolution of a random tempered distribution with a Schwartz function.

(ii) By definition, the random variables of a smoothed random field based on a stationary noise field Z and an arbitrary window function $k \in \mathscr{S}(\mathbb{R}^d, \mathbb{R})$ are identically distributed, i.e., $Z_k(x_1) \sim Z_k(x_2)$ for every $x_1, x_2 \in \mathbb{R}^d$. Further, whenever $x_1, \ldots, x_n \in \mathbb{R}^d$ are such that $\tau_{x_1}(k), \ldots, \tau_{x_n}(k)$ have mutually disjoint supports, then the random variables $Z_k(x_1), \ldots, Z_k(x_n)$ are independent. By the same arguments we already used in the proof of Theorem 3.3.6, it follows that for a stationary noise field the "noise field property" as well as stationarity not only hold for $f_1, \ldots, f_n, f \in \mathscr{S}(\mathbb{R}^d, \mathbb{R})$ but for arbitrary $f_1, \ldots, f_n, f \in L^1(\mathbb{R}^d, \mathbb{R}) \cap L^2(\mathbb{R}^d, \mathbb{R})$.

(iii) By Theorem 3.2.7, we can describe every $|||\cdot|||$-continuous stationary noise field Z by a $\mathscr{S}_q'(\mathbb{R}^d, \mathbb{R})$-valued random variable with arbitrary $q > \frac{3d}{4}$. Thus, if $k \in$

$\mathscr{S}_q(\mathbb{R}^d, \mathbb{R})$ is a function such that

$$\mathbb{R}^d \to \mathscr{S}_q(\mathbb{R}^d, \mathbb{R}), \quad x \mapsto \tau_x k \tag{3.5}$$

is well-defined, we may consider the smoothed random field Z_k with window function k even if $k \notin L^1(\mathbb{R}^d, \mathbb{R}) \cap L^2(\mathbb{R}^d, \mathbb{R})$. If the function in (3.5) is also continuous (when $\mathscr{S}_q(\mathbb{R}^d, \mathbb{R})$ is equipped with the Hilbert space norm $|\cdot|_q$), the resulting smoothed random field is almost surely a continuous function on $\mathbb{R}^d$. For Matérn kernels as window function this smoothing procedure is described in Theorem 3.2.10.

(iv) A smoothed Lévy field $Z_{k_{\alpha,m}}$ with a Matérn kernel $k_{\alpha,m}$ as window function can also be viewed as the distributional solution of the linear Stochastic pseudodifferential equation

$$(-\Delta + m^2)^\alpha Z_{k_{\alpha,m}}(f) = Z(f), \quad \text{for all } f \in \mathscr{S},$$

where Z is a Lévy noise field and Δ denotes the Laplacian; see [6, 7]. In this sense, the approach of [42] directly extends the approach in [77] for sampling Gaussian fields.

3.5 Examples

3.5.1 Gaussian Fields

We obtain Gaussian random fields by Lévy fields with associated characteristic triplet $(b, \sigma^2, 0)$ where we set the Lévy measure $\nu = 0$. The corresponding generalized random field is a stationary noise field denoted by G with characteristic functional $\varphi_G(f) = \exp\left(ib \int_{\mathbb{R}^d} f(y)\,\mathrm{d}y - \frac{\sigma^2}{2}\|f\|^2_{L^2(\mathbb{R}^d,\mathbb{R})}\right)$. Since b corresponds to a deterministic background field, we obtain a classical white noise for $b = 0$. If G is indexed by $f \in L^1(\mathbb{R}^d, \mathbb{R}) \cap L^2(\mathbb{R}^d, \mathbb{R})$, the corresponding random variable $G(f)$ has variance $\sigma^2\|f\|_{L^2(\mathbb{R}^d,\mathbb{R})}$.

If we smooth a Gaussian random field with window functions $k \in L^1(\mathbb{R}^d, \mathbb{R}) \cap L^2(\mathbb{R}^d, \mathbb{R})$, then $G(k_x)$ has a Gaussian distribution with mean $b \int_{\mathbb{R}^d} k(y)\,\mathrm{d}y$ for each

$x \in \mathbb{R}^d$. Further, a straightforward calculation shows that

$$\mathbf{Cov}(G_k(x_1), G_k(x_2)) = \sigma^2 \int_{\mathbb{R}^d} k(x_1 - \tau)k(x_2 - \tau)\,\mathrm{d}\tau = \sigma^2(k^\vee * k)(x_1 - x_2).$$

In particular, setting $k = k_{\alpha,m}$ we obtain $\mathbf{Cov}(G_k(x_1), G_k(x_2)) = \sigma^2 k_{2\alpha,m}(x_1 - x_2)$, which is the usual Matérn covariance function with smoothness parameter 2α.

Remark 3.5.1. *By Kolmogorov's continuity criterion for random fields [73] one can see that the lower bound $\alpha > d$ in $d = 1, 2, 3$ obtained in Theorem 3.2.10 is not optimal for the Gaussian case, where $2\alpha > d$ already yields a continuous modification $Z_{k_{\alpha,m}}(x)$.*

3.5.2 Compound Poisson Random Field

Let ν be a finite Lévy measure on $\mathbb{R}\backslash\{0\}$ and set $b := \int_{\{0<|s|\leqslant 1\}} s\,\nu(\mathrm{d}s)$. A compound Poisson random field is a Lévy noise field P with associated triplet $(b, 0, \nu)$ and corresponding characteristic functional

$$\varphi_P(f) = \exp\left(\int_{\mathbb{R}^d}\int_{\mathbb{R}\backslash\{0\}} \left(\mathrm{e}^{isf(x)} - 1\right)\,\nu(\mathrm{d}s)\mathrm{d}x\right), \quad f \in L^1(\mathbb{R}^d, \mathbb{R}) \cap L^2(\mathbb{R}^d, \mathbb{R}). \tag{3.6}$$

We consider $f \in L^1(\mathbb{R}^d, \mathbb{R}) \cap L^2(\mathbb{R}^d, \mathbb{R})$ with essential support in a region $\Lambda \subset \mathbb{R}^d$ with $|\Lambda| < \infty$. The finite Lévy measure ν yields an intensity parameter $\lambda := \nu(\mathbb{R}\backslash\{0\})$ and a probability measure $\tilde{\nu}$ on $\mathbb{R}$ defined by $\tilde{\nu} := \lambda^{-1}\nu$ with $\tilde{\nu}(\{0\}) = 0$.

Now, let $(\Omega, \mathfrak{A}, \mathbf{P})$ be a probability space, let N_Λ be a Poisson-distributed random variable with intensity $\lambda|\Lambda|$, and let $(X_1, S_1), (X_2, S_2), \ldots$ be a sequence of $\mathbb{R}^d \times \mathbb{R}\backslash\{0\}$-valued random variables which are identically distributed with $(X_1, S_1) \sim \frac{\mathrm{d}x}{|\Lambda|} \otimes \tilde{\nu}$, where $\mathrm{d}x$ is restricted to Λ, and such that $N_\Lambda, (X_1, S_1), (X_2, S_2), \ldots$ are independent. Let $\{\Lambda_j\}_{j\in\mathbb{N}}$ define a partition of $\mathbb{R}^d$ such that any compact set intersects at most finitely many Λ_j and such that the random variables $P_{\Lambda_j} := \sum_{j=1}^{N_{\Lambda_j}} S_j \delta_{X_j}$ are mutually independent. For $f \in \mathscr{S}$ with compact support let $I = \{j \in \mathbb{N} : \Lambda_j \cap \mathrm{supp} f \neq \varnothing\}$.

Then,

$$
\begin{aligned}
\varphi_P(f) = \; \mathbf{E}\left[\mathrm{e}^{iP(f)}\right] = \; \mathbf{E}\left[\mathrm{e}^{i\sum_{j\in I} P_{\Lambda_j}(f)}\right] &= \; \prod_{j\in I}\mathbf{E}\left[\mathrm{e}^{iP_{\Lambda_j}(f)}\right] \\
&= \prod_{j\in I}\sum_{\ell_j=1}^{\infty}\mathbf{P}(N_{\Lambda_j}=\ell_j)\prod_{r_j=1}^{\ell_j}\mathbf{E}\left[\mathrm{e}^{iS^{(j)}_{r_j}f(X^{(j)}_{r_j})}\right] \\
&= \prod_{j\in I}\sum_{\ell_j=1}^{\infty}\mathrm{e}^{-\lambda|\Lambda|}\frac{(\lambda|\Lambda_j|)^{\ell_j}}{\ell_j!}\left(\int_{\Lambda_j}\int_{\mathbb{R}}\mathrm{e}^{isf(x)}\,\tilde{\nu}(\mathrm{d}s)\,\frac{\mathrm{d}x}{|\Lambda_j|}\right)^{\ell_j} \\
&= \prod_{j\in I}\exp\left(\lambda\int_{\Lambda_j}\int_{\mathbb{R}}(\mathrm{e}^{isf(x)}-1)\,\tilde{\nu}(\mathrm{d}s)\mathrm{d}x\right) \\
&= \exp\left(\int_{\mathbb{R}^d}\int_{\mathbb{R}\setminus\{0\}}(\mathrm{e}^{isf(x)}-1)\,\nu(\mathrm{d}s)\mathrm{d}x\right).
\end{aligned} \tag{3.7}
$$

Since the space of smooth functions with compact support is $|||\cdot|||$-dense in $\mathscr{S}$, we can extend $P(f)$ to $\mathscr{S}$. As the characteristic functionals coincide, the constructed random field $P(f)$ coincide with the random field from Theorem 3.2.5 up to equivalence in law. By the Borel-Cantelli Lemma, it follows that the locally finite and discrete signed measure P is actually a tempered distribution $\mathbf{P}$-a.s..

Let k be a continuous, bounded function in $L^1(\mathbb{R}^d,\mathbb{R})$ (and hence also in $L^2(\mathbb{R}^d,\mathbb{R})$), then

$$
P_{\Lambda_j,k}(x) = \sum_{\ell=1}^{N_{\Lambda_j}} S^{(j)}_\ell k(x - X^{(j)}_\ell)
$$

is a continuous functions. For the smoothed compount Poisson noise field we obtain $P_k(x) = \sum_{j=1}^{\infty} P_{\Lambda_j,k}$. Although the continuity of $P_{\Lambda_j,k}$ does not immediately imply that P_k is continuous as well, it seems likely that this is true if $k(x)$ has some global uniform continuity and decays sufficiently fast; see [7, Theorem 3.2] for a related argument.

Remark 3.5.2. *We note that the absolute value of the signed measure $|P|$ is given by $\sum_j |P_{\Lambda_j}|$ and $|P_{\Lambda_j}| = \sum_{\ell=1}^{N_{\Lambda_j}} |S^{(j)}_\ell|\delta_{X^{(j)}_\ell}$ holds $\mathbf{P}$-a.s.. This yields the estimate*

$$
|P_k(x)| \leqslant |P|_{|k|}(x), \quad \textit{for all } x \in \mathbb{R}^d \tag{3.8}
$$

$\mathbf{P}$-*almost surely. Since $|P|$ is also a compound Poisson noise field with characteristic triplet $(b^+, 0, \nu^+)$, where ν^+ is the image measure of ν under the mapping $s \mapsto |s|$ which is supported on $]0,\infty[$, the right-hand side of (3.8), for fixed x, is almost surely finite.*

3.5.3 Poisson Point Process

As Poisson point processes, which we use in chapter 4 to model crack initiation events on mechanical elements, are of importance for this thesis, we present them here as an example, despite the fact that they do not necessarily define a Lévy noise field in terms of Definition 3.3.1. Here, we only construct the process, but provide a more detailed introduction on this construction below. We consider a characteristic triplet $(b, 0, \nu)$ where the Lévy measure ν on $\mathbb{R}$ is given by $\nu = \delta_1$, where δ_x denotes the Dirac measure on $x \in \mathbb{R}$, and $b := \int_{\{0<|s|\leqslant 1\}} s\,\nu(\mathrm{d}s) = 1$. Let $\Omega \subset \mathbb{R}^d$ and $\mathcal{C} = \Omega \times \mathbb{R}_{\geqslant 0}$. Let $(\Omega, \mathfrak{A}, \mathbf{P})$ be a probability space, ρ a compactly supported Radon-Nikodym derivative of some Radon measure describing the intensity of the process, and $\Lambda \subseteq \mathcal{C}$. We consider the random variables N_Λ, $X_1, X_2, \ldots$ on $(\Omega, \mathfrak{A}, \mathbf{P})$, where N_Λ is Poisson distributed with intensity $\rho(\Lambda)$ and $X_j \sim \frac{\rho(\mathrm{d}x)}{\rho^{-1}(\Lambda)}$ is $\mathbb{R}^d$ valued. Assuming that all these random variables are additionally independent, we define the Poisson point process on $\mathcal{C}$ by $\gamma_\Lambda = \sum_{j=1}^{N_\Lambda} \delta_{X_j}$. Similarly to above, we can derive that the characteristic functional of γ is given by

$$\varphi_\gamma(f) = \exp\left(\int_{\mathbb{R}^d} (\mathrm{e}^{isf(x)} - 1)\rho(x)\,\mathrm{d}x\right), \tag{3.9}$$

for all $f \in L^1(\mathbb{R}^d, \mathbb{R}) \cap L^2(\mathbb{R}^d, \mathbb{R})$. If ρ is translation invariant, we can show, by following the same steps as in the proof of Theorem 3.3.6, that the associated Poisson point process is a stationary noise field, and thus by Theorem 3.3.6 a Lévy noise field.

3.5.4 Lévy Noise of Infinite Activity

We consider the case when we only have $\int_{\{|s|\leqslant 1\}} |s|\,\nu(\mathrm{d}s)$. In this case, the deterministic compensator term for the small jumps $\int_{\mathbb{R}\setminus\{0\}} s\mathbb{1}_{\{|s|\leqslant 1\}}(s)\,\nu(\mathrm{d}s)$ can still be subsumed into the constant b and the expression for the characteristic functional (3.6) remains valid. Even though the assumption is quite strong, it includes important examples such as (bi-) gamma distributions with $\nu(\mathrm{d}s) = v\mathbb{1}_{\{s>0\}}(s)\frac{\mathrm{e}^{-ws}}{s}\,\mathrm{d}s$ ($\nu(\mathrm{d}s) = v\frac{\mathrm{e}^{-w|s|}}{|s|}\,\mathrm{d}s$), $v, w > 0$. Since in this case and others the jump measure ν is infinite, the representation given in the compound Poisson case needs to be extended as follows: With $\Theta_0 = \{s \in \mathbb{R} : |s| > 1\}$ and $\Theta_\ell = \{s \in \mathbb{R} : \frac{1}{\ell} \geqslant |s| > \frac{1}{\ell+1}\}$ we consider partitions of $\mathbb{R}\backslash\{0\}$. The Lévy measures on these sets $\nu_\ell(\mathrm{d}s) = \mathbb{1}_{\Theta_\ell}(s)\nu(\mathrm{d}s)$ are all finite and define independent compound Poisson processes P_ℓ. Similarly to (3.7), we can show that (3.6) still holds and the same applies to (3.8), where $|P| = \sum_{\ell=1}^{\infty} |P_\ell|$. Further, $|P|$ is a Lévy noise with characteristic triplet $(b^+, 0, \nu^+)$ and the right-hand side of (3.8) is $\mathbf{P}$-a.s. true for all $x \in \mathbb{R}^d$. Figure 3-1 shows sample paths of Gaussian, Poisson (compound Poisson with $\nu = \delta_1$) and bi-directional gamma noise fields with identical covariance.

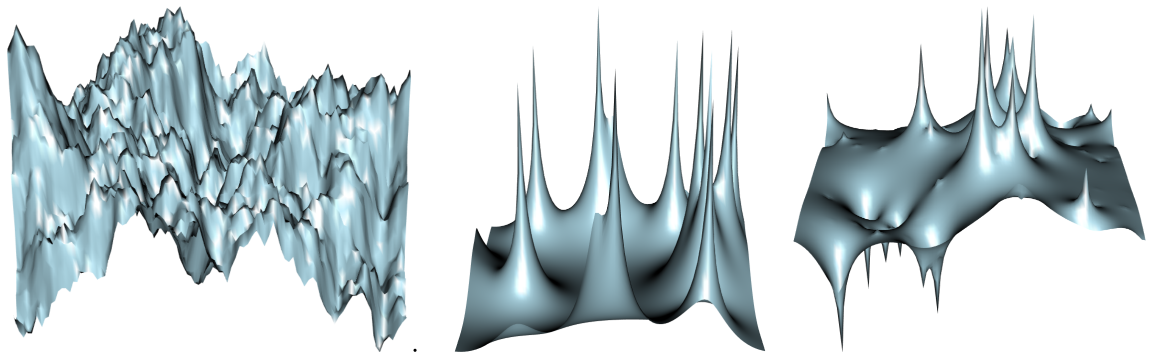

Figure 3-1: Realizations of smoothed noise fields: Gaussian (left), Poisson (middle) and bi-gamma (right) – each with the same Matérn covariance function.

Chapter 4

An Analytical Study in Multi-Physics and Multi-Criteria Shape Optimization

In this chapter, we give our first application of randomness in models expressing real world problems. We present a shape optimization model that includes the randomness of crack initiation processes in the design process of mechanical elements [59]. Component life model from material science are often used to evaluate the integrity of a mechanical component subjected to a certain number of load cycles; see, e.g., [20]. In the literature, there are different approaches described to predict the failure of a component. Many models use deterministic life calculation which predicts failure at the component or component's surface, depending on whether we have a volume or a surface driven damage mechanism. Probabilistic models extend the deterministic life prediction by integrals over local functions of the stress tensor in the component or on the component's surface, respectively; see, e.g., [10, 45, 60, 61, 66, 80, 98, 101]. This approach makes it possible to compute shape derivatives and gradients [18, 25, 58, 104] and therefore places component reliability in the context of shape optimization.

Besides the endurance, the design of a component's shape must often satisfy further primary objectives. For example, the component has to withstand a minimum amount of load cycles before yielding while being as efficient as possible at the same time. This search for an optimal design leads therefore, in the majority of cases, to an at least bi-criteria optimization problem. While several mono-criteria shape optimization frameworks have already been established [8, 27, 38, 49, 65], a general framework for a multi-criteria design process is still missing; see however [28, 39, 65] for numerical studies addressing the topic.

The different types of objectives are typically expressed by objective functionals, which depend on the solutions to boundary value problems. These objective functionals require a certain level of regularity for the solutions and the usual weak theory based on H^1 Sobolev spaces is sometimes not sufficient. This is the case for, e.g., the probability of failure, as remarked in [60], but, as we see here, also for simplistic fluid dynamic models. As in previous works [19, 21, 60], we therefore introduce here a framework based on Hölder continuous classical solution spaces and extend it to multi-criteria optimization. Within this framework, we state conditions which provide the existence of at least one optimal shape in terms of Pareto optimality [40]. Our approach uses the graph compactness property [65, Subsection 2.4] and requires the lower semicontinuity of all objective functionals to provide certain maximality properties of the non-dominated feasible points: Namely that the Pareto front in the set of feasible points coincides with the Pareto front of the closure of the feasible points. Put in other words, each dominated design is also dominated by at least one Pareto optimal design.

In order to illustrate our framework, we describe a simplistic multi-physics system as an example. This mathematical model is motivated from gas turbine engineering and describes a turbine vane lying within a shroud where a fluid is flowing through. We couple a potential flow equation, describing the fluid, with the equation of linear elasticity in order to model the deformations of the component when volume and surface forces are exerted onto it. We identify the objectives by two (rather singular) functionals, namely an aerodynamic loss based on the theory of boundary layers [97] and furthermore, the probability of failure after a certain number of load cycles [60, 80]. For this system, we prove that the assumptions of the general framework are fulfilled and we conclude that a maximal Pareto front exists in this case.

In contrast to solving a multi-criteria optimization problem in the context of Pareto optimality, the traditional approach of scalarization functions reduces the multi-criteria into a mono-criteria shape problem by using a decision maker, which reflects the preferences of the design process. Here, we are interested in continuity properties of Pareto optimal shapes, when the preference is expressed by a parameter in a merit function, which, e.g., could be the weights in a weighted sum approach. The stability of the solution to such scalarization techniques have already been investigated in the literature for finite and infinite-dimensional spaces; see, e.g., [16, 63, 105, 106]. We study our general framework on these known result, i.e., we are interested in how the sets of optimal shapes act, when subjected to small changes in the preference parameter.

The first section of this chapter follows closely [60] and is devoted to the probabilistic modeling of a mechanical components life expectation. Afterwards, we give a general introduction to multi-criteria shape optimization problems governed by PDE constraints. We then present a toy model, where we couple a potential flow equation with the equation of linear elasticity, and which includes the components life model provided in the first section. For this toy model, we prove the existence of an optimal shape and also study scalarization techniques and corresponding continuity properties of the sets of optimals shapes with respect to the weighted sum scalarization and the ε-constraint method.

4.1 Probabilistic Life Prediction

In this section, we introduce Poisson point processes (see Example 3.5.3) from a practical point of view and show how to implement them in crack initiation models by following the construction of [60]. We consider an open and bounded domain $\Omega \subset \mathbb{R}^3$ with Lipschitz boundary $\partial\Omega$ representing the shape of some component in a physical system. In this system, we measure time in load cycles $\mathcal{T} = \mathbb{N}_0$ or in natural time $\mathcal{T} = \text{cl}(\mathbb{R})$, where $\text{cl}(\mathbb{R}) = \mathbb{R} \cup \{-\infty, \infty\}$. The space $\mathcal{C} = \mathcal{T} \times \text{cl}(\Omega)$ endowed with the standard metric topology denotes the configuration space of crack initiation at time $t \in \mathcal{T}$ and location $x \in \text{cl}(\Omega)$. The Borel σ-algebra of the topological space $\mathcal{C}$ is denoted by $\mathfrak{B}(\mathcal{C})$.

Let $\mathcal{R} = \mathcal{R}(\mathcal{C})$ denote the space of all Radon measures on $\mathcal{C}$, i.e., the set of all measure γ on the measurable space $(\mathcal{C}, \mathfrak{B}(\mathcal{C}))$ for which $\gamma(B) < \infty$ for each bounded set $B \in \mathfrak{B}(\mathcal{C})$. The set of all counting measures in $\mathcal{R}$, i.e., the Radon measures $\gamma \in \mathcal{R}$ with $\gamma(B) \in \mathbb{N}_0$ for all bounded $B \in \mathfrak{B}(\mathcal{C})$, is denoted by $\mathcal{R}_c$. In the given context, a counting measure $\gamma \in \mathcal{R}_c$ encodes one particular history of potentially multiple crack initiations on the component Ω. Hence, $\gamma(B)$ gives for some Borel measurable location $B \subseteq \text{cl}(\Omega)$ the number of crack initiations with time-location instances $c = (t, x) \in B$.

We attach the event of failure of the component Ω with the occurrence of the first crack in $\text{cl}(\Omega)$. We define the time of failure $\tau : \mathcal{R}_c \to \mathcal{T}^\bullet = \mathcal{T} \cup \{\infty\}$ regarding some crack initiations history γ by

$$\tau(\gamma) = \inf\{t > 0 \,:\, \gamma(\mathcal{C}_t) > 0\},$$

where $\mathcal{C}_t = \{(\tau, x) \in \mathcal{C} \,:\, \tau \leqslant t\}$. By [71, Theorem 1.6], for each bounded set $B \in \mathfrak{B}(\mathcal{C})$, the restriction of $\gamma \in \mathcal{R}_c$ to B possesses an atomic decomposition

$$\gamma\restriction_B = \sum_{j=1}^{n} b_j \delta_{c_j}, \quad n, b_j \in \mathbb{N},\, c_j \in \mathcal{C},\, c_i \neq c_j \text{ for } i \neq j,$$

where δ_c stands for the Dirac measure in $c \in \mathcal{C}$. This representation is unique up to order of terms. We call the Radon measure γ simple if for all $B \in \mathfrak{B}(\mathcal{C})$ we have $b_j = 1$ for all $j = 1, \ldots, n$. The simplicity of a crack initiation history γ is a natural condition, since it states that two cracks occuring at the same time and at the same place are considered as the same crack. As we cannot predict the precise time and location of the first crack initiation, we have to take into account that crack initiation is a random process. With $\mathcal{N}(\mathcal{R}_c)$ we denote the standard σ-algebra, which is called vague topology, on the space of Radon counting measures generated by the mapping $\gamma \mapsto \int_{\mathcal{C}} f \,\mathrm{d}\gamma$ with $f \in C_c(\mathcal{C}, \mathbb{R})$, the space of compactly supported continuous functions on $\mathcal{C}$. It is easily seen that the time of failure $\tau : (\mathcal{R}_c, \mathcal{N}(\mathcal{R}_c))$ is measurable. The following definition can be found in [71, Chapter 2].

Definition 4.1.1 (Point Process)**.** *Let $(\mathcal{X}, \mathfrak{A}, \mathbf{P})$ be a probability space.*

(i) A point process on $\mathcal{C}$ is a measurable mapping $\gamma : (\mathcal{X}, \mathfrak{A}, \mathbf{P}) \to (\mathcal{R}_c, \mathcal{N}(\mathcal{R}_c))$.

(ii) The point process γ is simple if $\gamma(\cdot, \omega)$ is simple for $\mathbf{P}$-almost all $\omega \in \mathcal{X}$.

(iii) A point process γ is non-atomic if $\mathbf{P}\left(\gamma(\{c\}) > 0\right) = 0$ for all $c \in \mathcal{C}$.

(iv) A point process γ has independent increments if for mutually disjoint $B_1, \ldots, B_n \in \mathfrak{B}(\mathcal{C})$, the random variables $\gamma(B_1), \ldots, \gamma(B_n)$ are independent.

Random crack initiation histories are naturally modeled as simple point processes. The additional assumption that γ does not possess "atoms" states that there is no location $x \in \mathrm{cl}(\Omega)$ where a crack precisely originates with a probability larger than zero.

Definition 4.1.2 (Crack Initiation Process)**.**

(i) A crack initiation process γ is a simple, non-atomic point process on $\mathcal{C}$.

(ii) The time to crack initiation $T : \mathcal{X} \to \mathcal{T}^\bullet$ associated with γ is the random variable $T = \tau(\gamma)$.

It is debatable whether the assumption of independent increments is realistic for random crack initiation. However, since we regard a component Ω as failed after the first crack has occurred, we are only interested in the component's history until the formation of the first crack initiation. A study of a model with interacting crack networks is given in, e.g., [81].

The following proposition is an application of some standard results from the theory of point processes in the given context.

Proposition 4.1.3 (Crack Initiation and PPPs, [60, Proposition 2.3])**.**

(i) Any crack initiation process γ on $\mathcal{C}$ with independent increments is a Poisson point process (PPP), i.e., there exists a unique Radon measure $\rho \in \mathcal{R}$ such that

$$\mathbf{P}\left(\gamma(B) = n\right) = \mathrm{e}^{-\rho(B)} \frac{\rho(B)^n}{n!}, \quad \text{for all bounded } B \in \mathfrak{B}(\mathcal{C}),\ n \in \mathbb{N}_0.$$

ρ is called the intensity measure of γ.

(ii) The distribution function F_T of the time to crack initiation T is given by $F_T = 1 - \mathrm{e}^{-H(t)}$ with cumulative hazard function $H(t) = \rho(\mathcal{C}_t)$.

(iii) If $\rho(\mathcal{C}) = \infty$, then $\mathbf{P}(T = \infty) = 0$ and T can be modified to $T : (\mathcal{X}, \mathfrak{A}) \to (\mathcal{T}, \mathfrak{B}(\mathcal{T}))$.

The reliability of the component Ω at some warranty time t^* or after the passage of a service interval of duration t^* depends on the forces acting on Ω, the material, and the shape Ω itself. In many design applications the loads and material are given and the optimization process to maximize the life span of the component takes place at the choice of the shape. The choice of an optimal design depends crucially on an assignment of failure probabilities to the shape Ω. In the following part, we introduce a reliability optimization model.

Definition 4.1.4 (Crack Initiation Model)**.** *Let $\mathcal{O}$ be a collection of admissible shapes contained in some larger shape $\Omega^{\text{ext}} \subset \mathbb{R}^3$ and let $f, g : \mathcal{C}^{\text{ext}} = \mathcal{T} \times \Omega^{\text{ext}} \to \mathbb{R}^3$ be vector* *fields belonging to some spaces $\mathcal{V}_{\text{vol}}$ and $\mathcal{V}_{\text{sur}}$, respectively. For $\Omega \in \mathcal{O}$ we interpret $f\restriction_\Omega$ as the history – or load collective – of volume force densities on $\partial\Omega$ and $g\restriction_{\partial\Omega}$ as the history of surface force densities.*

A crack initiation model is a mapping γ from $\mathcal{O} \times \mathcal{V}_{\text{vol}} \times \mathcal{V}_{\text{sur}}$ to the space of all crack initiation processes on $\mathcal{T} \times \Omega^{\text{ext}}$ mapping (Ω, f, g) to $\gamma_{\Omega,f,g}$ such that

(i) $\gamma_{\Omega,f,g}(\mathcal{T} \times (\Omega^{\text{ext}} \backslash \text{cl}(\Omega))) = 0$ $\mathbf{P}$*-almost surely,*

(ii) $\gamma_{\Omega,f,g}$ *depends* $\mathbf{P}$*-almost surely only on* $f\restriction_\Omega$ *and* $g\restriction_{\partial\Omega}$*.*

Any crack initiation model induces a mapping of (Ω, f, g) to the crack initiation time random variable $T_{\Omega,f,g}$ associated with $\gamma_{\Omega,f,g}$. Assuming we are given volume and surface loads $f \in \mathcal{V}_{\text{vol}}$ and $g \in \mathcal{V}_{\text{sur}}$, and a fixed warranty time $t^* \in \mathcal{T}$, a crack initiation model leads to the optimal reliability problem.

Definition 4.1.5 (Optimal Reliability Problem)**.** *Given $t^* \in \mathcal{T}$, $f \in \mathcal{V}_{\text{vol}}$, $g \in \mathcal{V}_{\text{sur}}$, and a crack initiation model γ, find $\Omega^* \in \mathcal{O}$ such that*

$$\mathbf{P}(T_{\Omega^*} \leqslant t^*) \leqslant \mathbf{P}(T_\Omega \leqslant t^*), \quad \textit{for all } \Omega \in \mathcal{O}.$$

As the initiation of the first crack interacts with the atomic displacements within the component Ω originating from the exerted volume and surface forces f and g, we construct crack initiation models with independent increments based on the PDE of linear isotropic elasticity (2.11). We restrict ourself to the case where f and g are independent of t such that the model is based on one well-defined load cycle. The time t then counts the number of such load cycles.

Definition 4.1.6 (Local Crack Initiation Model)**.** *Let $\mathcal{O} \times \mathcal{V}_{\text{vol}} \times \mathcal{V}_{\text{sur}}$ be such that for all $\Omega \in \mathcal{O}$, $f \in \mathcal{V}_{\text{vol}}$, and $g \in \mathcal{V}_{\text{sur}}$ there exists a unique (weak) solution u_Ω to* (2.11)*. Let furthermore $\varrho_{\text{vol}} : \mathcal{T} \times \mathbb{R}^d \to \text{cl}(\mathbb{R}_{\geqslant 0})$ and $\varrho_{\text{sur}} : \mathcal{T} \times \mathbb{R}^d \to \text{cl}(\mathbb{R}_{\geqslant 0})$ with $d = 3 + \sum_{j=0}^{r} 3^{j+1} = 3 + \frac{3}{2}(3^{r+1} - 1)$ be measurable, non-negative functions, where $r \in \mathbb{N}$ is the order of the model. Suppose that the k^{th} weak derivatives $\nabla^k u$ of u are measurable functions for $k = 0, \ldots, r$ and that the trace $\nabla^k u\restriction_{\partial\Omega}$ is well defined in the sense of measurable functions. Here, $(\nabla^k u)^j_{j_1,\ldots,j_k}$ stands for $\frac{\partial^k u_j}{\partial x_{j_1} \cdots, \partial x_{j_k}}$. Then, we define the following:*

(i) An r^{th} order local crack initiation model with independent increments and linear elasticity state equation is defined by this data by setting γ_Ω to be the PPP on

$\mathcal{C}^{\mathrm{ext}}$ associated to the intensity measure

$$\rho_\Omega(B) = \int_{B\cap(\mathcal{T}\times\Omega)} \varrho_{\mathrm{vol}}(t,x,u,\nabla u,\ldots,\nabla^r u)\,\mathrm{d}t\mathrm{d}x \\ + \int_{B\cap(\mathcal{T}\times\partial\Omega)} \varrho_{\mathrm{sur}}(t,x,u,\nabla u,\ldots,\nabla^r u)\,\mathrm{d}t\mathrm{d}A, \quad \text{for all } B\in\mathfrak{B}(\mathcal{C}^{\mathrm{ext}}),$$

provided that the resulting measures are Radon measures on $\mathcal{C}^{\mathrm{ext}}$.

(ii) γ is said to be strain driven if ϱ_{vol} and ϱ_{sur} depend only on the elastic strain tensor field $\varepsilon(u)$ from (2.11). As the elastic stress tensor field $\sigma(u)$ can be obtained from $\varepsilon(u)$ and vice versa, strain and stress driven crack initiation models are synonymous.

(iii) If $\varrho_{\mathrm{vol}} = 0$, then γ is surface driven, and if $\varrho_{\mathrm{sur}} = 0$ it is volume driven.

(iv) We say that the r^{th} order crack initiation model has s-regular intensity functions, $s \geqslant 0$, if ϱ_{vol} and ϱ_{sur} are in $C^0(\mathcal{T})\otimes C^s(\mathbb{R}^d)$.

From the previous definition, it follows that the optimal reliability problem from Definition 4.1.5 is, in the case of an r^{th} order local crack initiation model, a PDE constrained shape optimization problem. The following lemma establish the link between optimal reliability and PDE constrained shape optimization.

Lemma 4.1.7 (Optimal Reliability and PDE Constrained Shape Optimization, [60, Lemma 2.7])**.** *For an r^{th} order local crack initiation model with elasticity state equation (2.11), the optimal reliability problem given in Definition 4.1.5 is equivalent to the shape optimization problem given in Definition 4.3.2 below with*

$$J(\Omega,y) = \int_\Omega \mathcal{F}_{\mathrm{vol}}(x,y,\nabla y,\ldots,\nabla^r y)\,\mathrm{d}x \\ + \int_{\partial\Omega} \mathcal{F}_{\mathrm{sur}}(x,y,\nabla y,\ldots,\nabla^r y)\,\mathrm{d}A,$$

with $\mathcal{F}_{\mathrm{vol}}(\cdot) = \int_0^{t*} \varrho_{\mathrm{vol}}(t,\cdot)\,\mathrm{d}t$ and $\mathcal{F}_{\mathrm{sur}}(\cdot) = \int_0^{t*} \varrho_{\mathrm{sur}}(t,\cdot)\,\mathrm{d}t$ and $y:\Omega^{\mathrm{ext}}\to\mathbb{R}^3$ sufficiently regular (continuous differentiable). In particular, for the case of s-regular intensity functions $\mathcal{F}_{\mathrm{vol}},\mathcal{F}_{\mathrm{sur}}\in C^s(\mathbb{R}^d)$.

4.1.1 LCF Driven Crack Initiation Model

We give here an example for crack initiation models based on small deformations which pile up and are the result from cyclic loadings. This degradation is called fatigue and is differentiated into high-cycle fatigue (HCF) and low-cycle fatigue (LCF) [90, 93]. The model we introduce in the following considers repeated small plastic deformations and models LCF crack initiation as a surface and strain driven model with elastic state equation. It is numerically implemented in [99] where it is applied to gas turbines. An extension to thermomechanical equations can be found in [100] and experimental validation is presented in [101].

The use of the equation of isotropic elasticity (2.11) in the context of low-cycle fatigue seems contradictory at first as it models elastic deformations which are completely reversible and therefore does not lead to degradation. In the following, we introduce a method of time-honored elastic-plastic stress conversion which solves this problem.

We define by $\sigma_v = \sqrt{\frac{3}{2}\mathbf{tr}(\boldsymbol{\sigma}'^2)}$ the von Mises stress, where $\boldsymbol{\sigma}' = \boldsymbol{\sigma} - \frac{1}{3}\mathbf{tr}(\boldsymbol{\sigma})\mathbf{I}$ is the trace free part of $\boldsymbol{\sigma}$ capturing non-hydrostatic stresses only. The von Mises stress is often used in order to determine whether an isotropic and ductile metal will yield when subjected complex loading. Next, we define the Ramberg-Osgood relation which is used to locally derive strain levels from scalar comparison stresses σ_v; see [91]. This equation describes stress-strain curves of metals near their yield points.

Definition 4.1.8 (Ramberg-Osgood Relation)**.** *Let $K > 0$ denote the strain hardening coefficient and $n' > 0$ the strain hardening exponent. Then, the Ramberg-Osgood relation between an elastic-plastic comparison strain $\varepsilon_v^{\mathrm{el-pl}} \in \mathrm{cl}(\mathbb{R}_{\geqslant 0})$ and an elastic-plastic comparison stress $\sigma_v^{\mathrm{el-pl}}$ is given by*

$$\varepsilon_v^{\mathrm{el-pl}} = \frac{\sigma_v^{\mathrm{el-pl}}}{E} + \left(\frac{\sigma_v^{\mathrm{el-pl}}}{K}\right)^{\frac{1}{n'}}, \tag{4.1}$$

with Young‘s modulus $E = \frac{\mu(3\lambda+2\mu)}{\lambda+\mu}$. The coefficient $\varepsilon_v^{\mathrm{el-pl}}$ is called the comparison strain and we shall write $\varepsilon_v^{\mathrm{el-pl}} = \mathrm{RO}(\sigma_v^{\mathrm{el-pl}})$.

The problem which arise at this point is that the elastic-plastic comparison stress $\sigma_v^{\mathrm{el-pl}}$ needs to be defined on the basis of the elastic von Mises stress σ_v. We can solve this problem with the method of stress shakedown by Neuber [87, 93].

Definition 4.1.9 (Elastic-Plastic Stress Conversion and Shakedown)**.** *Given* $\sigma_v \in \mathrm{cl}(\mathbb{R}_{\geqslant 0})$ *the associated elastic-plastic comparison stress* $\sigma_v^{\mathrm{el-pl}}$ *is defined as the positive solution to the equation*

$$\frac{\sigma_v^2}{E} = \sigma_v^{\mathrm{el-pl}}\varepsilon_v^{\mathrm{el-pl}} = \frac{(\sigma_v^{\mathrm{el-pl}})^2}{E} + \sigma_v^{\mathrm{el-pl}}\left(\frac{\sigma_v^{\mathrm{el-pl}}}{K}\right)^{\frac{1}{n'}}.$$

Therefore, we can determine the elastic-plastic von Mises stress $\sigma_v^{\mathrm{el-pl}}$ *by using the elastic von Mises stress* σ_v *and, in addition, we are able to obtain* $\varepsilon_v^{\mathrm{el-pl}}$ *from* (4.1)*. We shall write* $\sigma_v^{\mathrm{el-pl}} = \mathrm{SD}(\sigma_v)$*.*

Compared to the static case, with fatigue we describe the damage of material caused by much lower load amplitudes of cyclic loading. As an example, in Figure 4-1 (from [60]) we can see a triangle-shaped uniaxial load-time-curve, where $\sigma_a = [(\sigma_{\max} - \sigma_{\min})/2]_v$ is the elastic von Mises comparison stress amplitude.

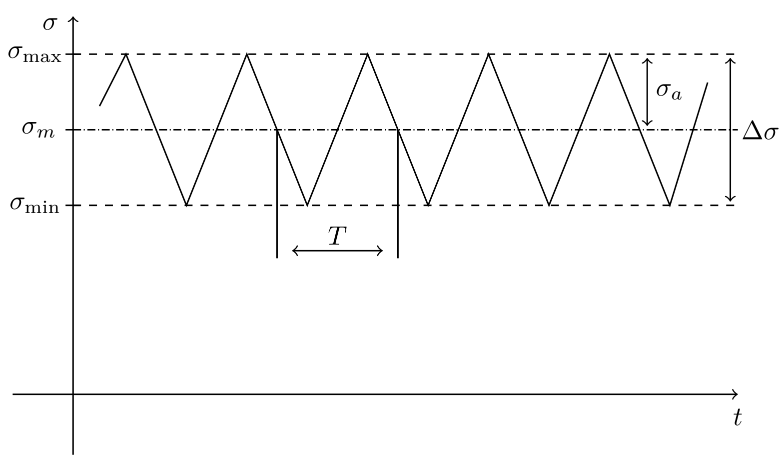

Figure 4-1: Triangle-shaped load-time-curve.

Since in fatigue the number of cycles until failure is determined, so-called E-N diagrams can be drawn if the tests are strain controlled; see Figure 4-2 (from [60]). Figure 4-2 also shows the relation between the strain amplitude $\varepsilon_a^{\mathrm{el-pl}} = \mathrm{RO} \circ \mathrm{SD}(\sigma_a)$ and the life time N_i to crack initiation measured in cycles. The Coffin-Manson-Basquin (CMB), or Wöhler equation, connects both sides of this relationship.

Definition 4.1.10 (CMB Equation)**.** *The Coffin-Manson-Basquin (CMB) equation connects the (deterministic) time to crack initiation* N_i *with the elastic-plastic strain*

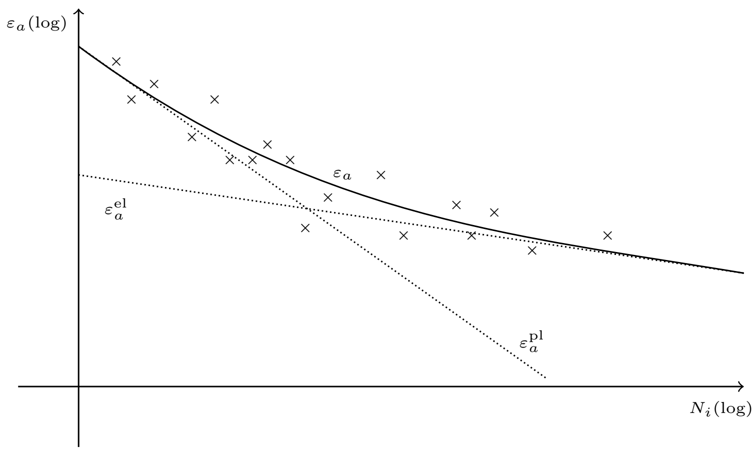

Figure 4-2: E-N diagram of a standardized specimen.

amplitude $\varepsilon_a^{\mathrm{el-pl}}$ *via*

$$\varepsilon_a^{\mathrm{el-pl}} = \mathrm{CMB}(N_i) = \frac{\sigma_f'}{E}(2N_i)^b + \varepsilon_f'(2N_i)^c. \tag{4.2}$$

Here, σ_f' and ε_f' are positive and b and c are negative material parameters. For the sake of simplicity, we assume in the following that the lower edge of the load cycle is stress-free, corresponding to $f_{\min} = 0$ and $g_{\min} = 0$ in (2.11) and thus $\sigma_a = \sigma_v/2$, where we set $\sigma = \sigma_{\max}$, $f = f_{\max}$, and $g = g_{\max}$.

In deterministic design, the lifetime of a component under cyclic loading corresponds to the loading condition at the part's surface position of highest stress. Safety factors are additionally imposed to account for the stochastic nature of LCF and size effects. This method is referred to as the safe-life approach in fatigue design, which is widely used in engineering; see [90, 93, 101, 102].

Lemma 4.1.11 ([60, Lemma 3.1])**.** *The function* $\varphi = \mathrm{CMB}^{-1} \circ \mathrm{RO} \circ \mathrm{SD} : \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$, *which maps the elastic von Mises comparison stress to a predicted life time, satisfies the following:*

(i) φ *is bijective and strictly monotonically decreasing.*

(ii) $\lim_{\sigma_v \to 0} \varphi(\sigma_v) = \infty$.

(iii) φ *lies in* $C^\infty(\mathbb{R}_{\geqslant 0})$.

Definition 4.1.12 (Deterministic LCF-Life at a Surface Point)**.** *Let* $\mathbb{R}^{3\times 3}$ *be the space* *of real* 3×3 *matrices. Define* $N_{\text{det}} : \mathbb{R}^{3\times3} \to \text{cl}(\mathbb{R}_{\geqslant 0})$ *by*

$$N_{\text{det}}(\boldsymbol{M}) = \varphi([\lambda \boldsymbol{tr}(\boldsymbol{M})\boldsymbol{I} + \mu(\boldsymbol{M} + \boldsymbol{M}^{\intercal})]_v)$$

where φ*, as in Lemma 4.1.11, is extended by* $\varphi(0) = \infty$*. Assuming that there is a solution* $u \in H^1(\Omega, \mathbb{R}^3)$ *to* (2.11) *such that the trace* $\nabla u\restriction_{\partial\Omega}$ *can be reasonably defined and can be represented as a continuous function, then* $N_{\text{det}}(\nabla u(x))$ *is the predicted deterministic LCF-life at point* $x \in \partial\Omega$*.*

The usual approach in reliability statistics [114] is to choose the deterministic life prediction as a scale variable of a failure time distribution. Moreover, Weibull distributions are widely used in technical reliability analysis. Recall that the Weibull distribution with scale parameter η and shape parameter m is defined by the cumulative distribution function $F(t) = 1 - \mathrm{e}^{-\left(\frac{t}{\eta}\right)^m}$ for $t > 0$ and zero otherwise.

Definition 4.1.13 (Local Weibull Model for LCF)**.** *For* $m \geqslant 1$*, the strain and surface driven crack initiation model (recall Definition 4.1.6) with independent increments of first order, that is defined by*

$$\varrho_{\text{vol}} = 0, \quad \varrho_{\text{sur}}(t, \nabla u) = \frac{m}{N_{\text{det}}(\nabla u)} \left(\frac{t}{N_{\text{det}}(\nabla u)}\right)^{m-1},$$

is called the local (probabilistic) Weibull model for LCF. The associated optimal reliability problem, as given in Definition 4.1.5 and Lemma 4.1.7, is called the optimal reliability problem for LCF. Here, we employ the convention $\frac{1}{\infty} = 0$*.*

The model could be defined for both $\mathcal{T} = \mathbb{N}_0$ and $\mathcal{T} = \text{cl}(\mathbb{R}_{\geqslant 0})$, but the second option is used more often. In this case, $\mathcal{F}_{\text{sur}}(\nabla u) = (\frac{t}{N_{\text{det}}(\nabla u)})^m$.

Proposition 4.1.14 (Properties of the Local Weibull Model for LCF, [60, Proposition 3.4])**.** *Let the conditions of Definition 4.1.6 be fulfilled such that* $\|\frac{1}{N_{\text{det}}(\nabla u)}\|_{L^m(\partial\Omega)} < \infty$ *for some* $m > 1$ *and for all* $f \in \mathcal{V}_{\text{vol}}$*,* $g \in \mathcal{V}_{\text{sur}}$*, and* $\Omega \in \mathcal{O}$*. Then,*

(i) the local probabilistic Weibull model for LCF actually induces a first order local crack initiation model, i.e., the associated measure ρ_Ω *are Radon measures.*

(ii) the intensities of this model are 0*-regular, i.e., are continuous functions of* ∇u*.*

(iii) the crack initiation time T_Ω *is Weibull distributed with shape parameter* m *and scale parameter* $\eta = \|\frac{1}{N_{\text{det}}(\nabla u)}\|^{-1}_{L^m(\partial\Omega)}$*.*

Notch Support Effect

When loads acting on components with inhomogeneous geometries where, e.g., notches exist, these components exhibit domains of concentrated stress at the respective location. This stress concentration leads to inhomogeneous stress fields. Thus, domains near the surface are quickly plastically strained while domains further inside the body still support the structure as they experience smaller stresses and therefore impede failure. In reality, the predicted crack initiation life of components with inhomogeneous feature are therefore higher than predicted by the Coffin-Manson-Basquin equation (4.2). To approach this phenomenon, known as notch support effect, Siebel et al. [103] proposed to shift LCF life prediction models such as the CMB equation to higher life by using a notch support factor $n_{\mathcal{X}}$. We model $n_{\mathcal{X}}$ as a function of the normalized von Mises stress gradient

$$\mathcal{X}(x) = \frac{1}{\sigma_v(x)} \cdot \nabla\sigma_v(x), \quad \text{for } x \in \partial\Omega$$

at the surface of the component which depends on material dependent notch support parameter A and k. These parameters are simultaneously derived with the CMB parameters from LCF test data as described in Section 1.3 in [79]. By replacing the maximum strain value $\varepsilon_a^{\mathrm{el-pl}}$ in the CMB equation (4.2) by $\varepsilon_a^{\mathrm{el-pl}}/n_{\mathcal{X}}$, we lift the Wöhler curve along the $\varepsilon_a^{\mathrm{el-pl}}$-axis to higher strain values as $n_x \geqslant 1$ for all $\mathcal{X} \geqslant 0$. A numerical validation for this assertion is provided in [79] and [80]. In the extended probabilistic LCF model, all strain values $\varepsilon_a^{\mathrm{el-pl}}$ are divided by the respective local notch support factor $n_{\mathcal{X}}(x)$ at every integration point. Thus, the values of $N_{\mathrm{det}}(\nabla^2 u)$ are obtained by inverting the equation

$$\frac{\varepsilon_a^{\mathrm{el-pl}}}{n_{\mathcal{X}}\left(\mathcal{X}(x)\right)} = \frac{\sigma_f'}{E}(2N_i)^b + \varepsilon_f'(2N_i)^c.$$

This implies that N_{det} depends on the second order derivative of the displacement field u. We also note that in addition to the notch support effect, we also have a statistical size effect which plays an important role in the LCF life of irregularly shaped components, as critical stresses usually occur in confined domains which are small compared to the entire component.

4.2 A Multi-Criteria Shape Optimization Problem

Shape optimization is an indispensable mathematical subject for the design and construction of industrial structures. The optimization of the geometry and topology of structural layout has quite an impact on the performance of structures and the efficient use of materials. For example, mechanical elements of an aircraft and spacecraft have to satisfy, simultaneously, very strict criteria on mechanical performance while weighing as little as possible. The optimization process of these compontents consists of minimizing several loss (or cost) functionals, which represent the demands on the element, while, at the same time, satisfying specific constraint, as, e.g., thickness, strain energy, displacement bounds, or solving boundary value problems standing for physical phenomenas which the component is subjected to. These strict constraints make the search for an optimal shape far from trivial, and often, simplifications of the real world applications are needed in order to be able to solve these problems. For example, we optimize over a set of admissible shapes representing the components. The admissible shapes are given by open and bounded sets whose topology is given, e.g., it may be simply connected or doubly connected. As on these domains boundary value problems take place, the boundary of the domains, which represent the compontent, have to fulfill regularity conditons as, e.g., smoothness or piecewise smoothness. Further, the objective functionals may need to depend continuously on the solutions to the boundary value problems.

In the first section, we give an abstract framework in which our multi-physics shape optimization problem is described, and in which it possesses at least one optimal solution in terms of Pareto optimality. This framework, we define below, is an multi-criteria extension of the uni-criteria setting presented in [65, Chapter 2].

4.2.1 General Definitions

We denote a family of admissible shapes by $\tilde{\mathcal{O}}$ and for every shape $\Omega \in \tilde{\mathcal{O}}$ we denote by $V_1(\Omega), \dots, V_m(\Omega)$, $m \in \mathbb{N}$ state spaces of real-valued functions on Ω. Consider a sequence of shapes $(\Omega_n)_{n\in\mathbb{N}}$ in $\tilde{\mathcal{O}}$, and let $\Omega \in \tilde{\mathcal{O}}$. Assuming a topology on the shape space $\tilde{\mathcal{O}}$ is given, the convergence of Ω_n against Ω is denoted by $\Omega_n \xrightarrow{\tilde{\mathcal{O}}} \Omega$ as $n \to \infty$. For a sequence of functions $(y_n)_{n\in\mathbb{N}}$, with $y_n \in \bigtimes_{i=1}^m V_i(\Omega_n)$ for all $n \in \mathbb{N}$, we denote the convergence against some $y \in \bigtimes_{i=1}^m V_i(\Omega)$ with respect to a given topology on $\bigtimes_{i=1}^m V_i(\Omega)$ with $y_n \rightsquigarrow y$ as $n \to \infty$. We assume that for every $\Omega \in \tilde{\mathcal{O}}$ one can solve uniquely a given set of state problems, e.g., a set of PDEs or variational inequalities. By associating the corresponding unique solutions $v_{i,\Omega} \in V_i(\Omega)$ with $\Omega \in \tilde{\mathcal{O}}$, one obtains the map $v_i : \Omega \mapsto v_{i,\Omega} \in V_i(\Omega)$. Let $\mathcal{O}$ be a subfamily of $\tilde{\mathcal{O}}$, then $\mathcal{G} = \{(\Omega, v_\Omega) : \Omega \in \mathcal{O}\}$ is called the graph of the mapping $v := (v_1, \dots, v_m)$. A cost functional J on $\tilde{\mathcal{O}}$ is given by a map $J : (\Omega, y) \mapsto J(\Omega, y) \in \mathbb{R}$, where $\Omega \in \tilde{\mathcal{O}}$ and $y \in \bigtimes_{i=1}^m V_i(\Omega)$. Then, a vector of l cost functionals is defined by $J := (J_1, \dots, J_l)$, and the image of $\mathcal{O}$ (or $\mathcal{G}$) under J is denoted with $\mathcal{Y} \subset \mathbb{R}^l$. For the sake of convenience, we shall write $J(\Omega, v_\Omega) := (J_1(\Omega, v_\Omega), \dots, J_l(\Omega, v_\Omega))$, and, in addition, we make use of the notation $\nabla v_\Omega := (\nabla v_{1,\Omega}, \dots, \nabla v_{m,\Omega})$.

Definition 4.2.1 (Pareto optimality). *Consider a subfamily $\mathcal{O}$ of $\tilde{\mathcal{O}}$ with corresponding graph $\mathcal{G}$ to given state spaces $V = (V_1, \dots, V_m)$. We call a point $(\Omega^*, v_{\Omega^*}) \in \mathcal{G}$ Pareto optimal with respect to cost functionals $J = (J_1, \dots, J_l)$, if there is no $(\Omega, v_\Omega) \in \mathcal{G}$ such that $J_i(\Omega, v_\Omega) \leqslant J_i(\Omega^*, v_{\Omega^*})$ for all $1 \leqslant i \leqslant l$ and $J_j(\Omega, v_\Omega) < J_j(\Omega^*, v_{\Omega^*})$ for some $j \in \{1, \dots, l\}$. The associated value $J(\Omega^*, v_{\Omega^*})$ is called non-dominated.*

Let $\mathcal{Y} := J(\mathcal{G}) = \{J(\Omega, v_\Omega) : (\Omega, v_\Omega) \in \mathcal{G}\}$ denote the image of the graph $\mathcal{G}$ under the objective functionals mapping J. For a set of Pareto optimal points, we can define $\mathcal{Y}_N := \{J(\Omega, v_\Omega) \in \mathcal{Y} : J(\Omega, v_\Omega) \text{ is non-dominated in } \mathcal{Y}\}$, i.e., the corresponding Pareto front which lies by definition on the boundary of $\mathcal{Y}$.

Definition 4.2.2 (Multi-Criteria Shape Optimization Problem). *Consider a family $\mathcal{O}$ of admissible shapes which is a subspace of a shape space $\tilde{\mathcal{O}}$. For every $\Omega \in \tilde{\mathcal{O}}$, let $v_\Omega = (v_{1,\Omega}, \dots, v_{m,\Omega})$ be the unique solutions to given state problems on Ω, and let $J = (J_1, \dots, J_l)$ be cost functionals on $\tilde{\mathcal{O}}$. We define an optimal shape design problem by*

$$\begin{cases} \text{Find } \Omega^* \in \mathcal{O} \text{ such that} \\ (\Omega^*, v_{\Omega^*}) \text{ is Pareto optimal with respect to } J. \end{cases} \tag{4.3}$$

The next theorem states conditions on the existence of potentially multiple solutions to the optimal shape design problem (4.3). It outlines the succeeding sections, where we define, exemplary, a multi-physics shape optimization problem, and provide existence results for optimal shapes in the given context.

Theorem 4.2.3. *Let $\tilde{\mathcal{O}}$ be a family of shapes with a subfamily of admissible shapes $\mathcal{O}$. Consider cost functionals $J = (J_1, \dots, J_l)$ on $\tilde{\mathcal{O}}$ and assume for each $\Omega \in \tilde{\mathcal{O}}$ we* *have given state problems with state spaces* $V(\Omega) = (V_1(\Omega), \dots, V_m(\Omega))$ *such that each state problem has a unique solution* $v_{i,\Omega} \in V_i(\Omega)$, $1 \leqslant i \leqslant m$. *Let the following two assumptions hold true:*

(i) Compactness of $\mathcal{G} = \{(\Omega, v_\Omega) \,:\, \Omega \in \mathcal{O}\}$*:*
Every sequence $(\Omega_n, v_{\Omega_n})_{n\in\mathbb{N}} \subset \mathcal{G}$ *has a subsequence* $(\Omega_{n_k}, v_{\Omega_{n_k}})_{k\in\mathbb{N}}$ *that satisfies*

$$\Omega_{n_k} \xrightarrow{\tilde{\mathcal{O}}} \Omega, \quad k \to \infty$$
$$v_{\Omega_{n_k}} \rightsquigarrow v_\Omega, \quad k \to \infty,$$

for some $(\Omega, v_\Omega) \in \mathcal{G}$.

(ii) Lower semicontinuity of J_i*:*
Let $(\Omega_n)_{n\in\mathbb{N}}$ *be a sequence of shapes in* $\tilde{\mathcal{O}}$ *and* $(y_n)_{n\in\mathbb{N}}$ *a sequence of functions such that* $y_n \in V(\Omega_n)$ *for all* $n \in \mathbb{N}$. *Consider some elements* Ω, y *in* $\tilde{\mathcal{O}}$ *and* $V(\Omega)$, *respectively. Then,*

$$\left.\begin{array}{r} \Omega_n \xrightarrow{\tilde{\mathcal{O}}} \Omega, \quad n \to \infty \\ y_n \rightsquigarrow y, \quad n \to \infty \end{array}\right\} \Longrightarrow \liminf_{n\to\infty} J_i(\Omega_n, y_n) \geqslant J_i(\Omega, y),$$

for all $1 \leqslant i \leqslant l$.

Then, the multi-criteria shape design problem (4.3) *possesses at least one solution and the Pareto front covers all non-dominated points in* $\mathrm{cl}(\mathcal{Y})$, *i.e.,* $\mathcal{Y}_N = \mathrm{cl}(\mathcal{Y})_N$, *the set of non-dominated points in the closure of* $\mathcal{Y}$.

Proof. First, we prove the existence of an optimal shape. [65, Theorem 2.10] shows that, in this setting, a single lower semicontinuous cost functional possesses at least one minimum. We apply this theorem, without loss of generality, to cost functional J_1 and minimize it on $\mathcal{G}$. Due to the compactness of $\mathcal{G}$ and the lower semicontinuity of J_1, the resulting set of arguments of the minimum $\arg\min_{(\Omega,v_\Omega)\in\mathcal{G}} J_1$ is also compact. Hence, we can again apply [65, Theorem 2.10] to the next cost functional J_2 and minimize it on $\arg\min_{(\Omega,v_\Omega)\in\mathcal{G}} J_1$ as well. We continue this procedure until we minimized each cost functional on its preceding cost functionals set of arguments of the minimum. The last set then contains at least one Pareto optimal point.

For the second assertion, we recall that $\mathcal{Y}_N$ lies on the boundary of $\mathcal{Y}$, and thus it is an immediate consequence that $\mathcal{Y}_N \subseteq \mathrm{cl}(\mathcal{Y})_N$. Conversely, let $J^* \in \mathrm{cl}(\mathcal{Y})_N$.

Consider a sequence $(J(\Omega_n, v_{\Omega_n}))_{n\in\mathbb{N}} \subset \mathcal{Y}$ with $J(\Omega_n, v_{\Omega_n}) \to J^* \in \mathrm{cl}(\mathcal{Y})_N$ as $n \to \infty$. We assume that the corresponding sequence $(\Omega_n, v_{\Omega_n})_{n\in\mathbb{N}} \subset \mathcal{G}$ converges to some $(\Omega, v_\Omega) \in \mathcal{G}$ as well (since $\mathcal{G}$ is compact we can always find such a subsequence). Due to the lower semicontinuity of J, we have

$$J_i(\Omega, v_\Omega) \leqslant \lim_{n\to\infty} J_i(\Omega_n, v_{\Omega_n}) = J_i^*, \quad \text{for all } 1 \leqslant i \leqslant l.$$

The Pareto optimality of $J^* = J(\Omega^*, v_{\Omega^*})$ gives that $J(\Omega, v_\Omega) = J(\Omega^*, v_{\Omega^*})$, and since $J(\Omega, v_\Omega) \in \mathcal{Y}$, it follows that $J^* \in \mathcal{Y}$ and therefore $\mathrm{cl}(\mathcal{Y})_N \subseteq \mathcal{Y}_N$. □

4.3 Multi-Physics Shape Optimization

Shape optimization techniques have to consider the various different physical processes a mechanical component is exposed to. In practice, a component is subjected to more than one simultaneously occurring physical processes as, e.g.,internal and external flows of fluids, centrifugal forces, or thermal effects occurring during service. Thus, a multi-physics design approach is necessary and the different physical fields have to be coupled. In this section, we present an example for an multi-physics shape optimization system, where we model a gas turbine vane lying within a shroud in which a fluid is flowing. Our goal is to maximize its lifespan while it shall be as energy efficient as possible. We present a simple toy model, which describes the material behavior of a vane when it is exposed to the pressure that is inflected by the fluid onto the vane. Therefore, we couple the elasticity equation, presented in Subsection 2.4.1, which describes the deformations of a shape Ω under volume and surface loadings, with the potential flow equation from Subsection 2.4.2, which models the fluid flowing past the vane.

4.3.1 Multi-Physics Equation Coupling

We consider the system of the potential flow equation given in Subsection 2.4.2 and adjust it so that, in this setting, the elasticity equation is on the shape Ω solvable as well. For this purpose, we introduce an open ball $B := B_r$, with radius $r > 0$, where we clutch the component and which is fixed and excluded from the optimization process. Assuming that $\mathrm{cl}(B) \subset \mathrm{int}(\Omega\backslash D)$, we define the domain $\Omega_B = \mathrm{int}(\Omega\backslash B)$. For the boundary conditions, we get as Dirichlet boundary $\Gamma_D = \partial B$ and as Neumann boundary $\Gamma_N = \partial\Omega$.

As we already suggested, the surface force g in equation (2.11) is given by the pressure the fluid exerts onto the component. The potential flow equation (2.13) yields the velocity field v at the part of the boundary of Ω that lies within the shroud D. Assuming that the total energy density, also denoted as stagnation pressure p_{st}, is constant at the inlet, we can derive the static pressure p_{s} from Bernoulli's law

$$p_{\mathrm{st}} = \frac{1}{2}\rho|\nabla\phi|^2 + p_{\mathrm{s}} \Leftrightarrow -p_{\mathrm{s}} = \frac{1}{2}\rho|\nabla\phi|^2 - p_{\mathrm{st}}, \tag{4.4}$$

where ρ is the density of the fluid at all points in the fluid. The surface force g on the component Ω is then described by the directed static pressure $-p_{\mathrm{s}}n$, where n denotes the unitary outward normal on $\partial\Omega$. Therefore, by continuously extending p_{s} to be zero on $\partial\Omega\backslash D$, the surface load g is given by

$$g = g_{\mathrm{s}} = -p_{\mathrm{s}}n = \left(\frac{1}{2}\rho|\nabla\phi|^2 - p_{\mathrm{st}}\right) n.$$

As Ω possesses, by the construction of the potential flow equation, a Hölder continuous boundary of class $C^{k,\alpha}$ (with $k \geqslant 2$), the outward normal n is a function of class $C^{k-1,\alpha}$. Additionally, since we model an incompressible flow, the density ρ of the fluid is constant as well as the stagnation pressure p_{st} by assumption. Therefore, as the flow potential ϕ lies in $C^{2,\alpha}(\mathrm{cl}(D\backslash\Omega), \mathbb{R})$ (see Theorem 2.4.16), the surface load g_{s} is a function in $C^{1,\alpha}(\partial\Omega, \mathbb{R}^3)$. In summary, the elasticity equation coupled with the potential flow equation can be stated as

$$\begin{cases} -\nabla\cdot\sigma(u) = f & \text{on } \Omega_B, \\ u = 0 & \text{along } \partial B, \\ n\cdot\sigma(u) = g_{\mathrm{s}} & \text{along } \partial\Omega. \end{cases} \tag{4.5}$$

By Theorem 2.4.15, for any volume force $f \in C^{k-2,\alpha}(\mathrm{cl}(\Omega_B), \mathbb{R}^3)$, there exists a unique solution $u_\Omega \in C^{2,\alpha}(\mathrm{cl}(\Omega_B), \mathbb{R}^3)$, with corresponding Schauder estimate

$$\|u_\Omega\|_{C^{2,\alpha}(\Omega_B,\mathbb{R}^3)} \leqslant C\left(\|f\|_{C^{0,\alpha}(\Omega_B,\mathbb{R}^3)} + \|g_{\mathrm{s}}\|_{C^{1,\alpha}(\partial\Omega,\mathbb{R}^3)} + \|u_\Omega\|_{C^0(\Omega_B,\mathbb{R}^3)}\right). \tag{4.6}$$

At the same time, due to the regularity assumptions on Ω, the potential flow equation (2.13) possesses, by Theorem 2.4.16, for any Neumann boundary conditon $g \in$

$C^{1,\alpha}(D,\mathbb{R})$ a unique solution $\phi_\Omega \in C^{2,\alpha}(\mathrm{cl}(D\backslash\Omega),\mathbb{R})$ with Schauder estimate

$$\|\phi_\Omega\|_{C^{2,\alpha}(D\backslash\Omega,\mathbb{R})} \leqslant C\left(\|g\|_{C^{1,\alpha}(\partial D\backslash\partial(D\cap\Omega),\mathbb{R})} + \|\phi_\Omega\|_{C^{0,\alpha}(D\backslash\Omega,\mathbb{R})}\right). \tag{4.7}$$

Henceforth, we shall always associate in this chapter a function g with the Neumann boundary conditon of the potential flow equation, and a function g_s with the Neumann boundary conditon, or the surface load, of the elasticity equation.

4.3.2 Admissible Shapes and Criteria

Our problem formulation starts with choosing a space of admissible shapes $\mathcal{O}$ that contains all possible candidates among which an optimal one is sought. In order to satisfy the assumptions of Theorem 4.2.3, the shape space shall lie compact in a larger system $\tilde{\mathcal{O}}$. In addition, the definition of $\mathcal{O}$ has to respect all technical constraint characterizing the problem which means that, in particular, for each shape $\Omega \in \tilde{\mathcal{O}}$ the coupled multi-physics system of Subsection 4.3.1 must possess a unique solution. Hence, it is initially clear that $\mathcal{O}$ is a space consisting of compact domains that possess Hölder continuous boundaries. Within the shape space, we want to be able to freely deform one shape into another. For this purpose we introduce the shapes Ω_0 and Ω^{ext}, which suffice the construction of Subsection 4.3.1. The shape Ω_0 serves as a baseline domain which we deform continuously into various shapes in order to construct a shape space. Ω^{ext} provides an upper bound for the shapes, i.e., any shape in $\mathcal{O}$ is a subset of Ω^{ext}, and therefore $\Omega_0 \subset \Omega^{\mathrm{ext}}$ in particular. Any of these transforms which deform Ω_0 must retain the assumptions of Subsection 4.3.1. All these considerations lead to the following definitions. Let $K > 0$ be a positive constant, then the elements of the set

$$U^{\mathrm{ad}}_{k,\alpha} := U^{\mathrm{ad}}_{k,\alpha}(\Omega^{\mathrm{ext}}) := \Big\{\psi \in \mathcal{D}^{k,\alpha}(\Omega^{\mathrm{ext}}) \;:\; \psi\restriction_{\mathrm{cl}(\Omega\backslash D)} = \mathrm{id},\ \|\psi\|_{C^{k,\alpha}(\Omega^{\mathrm{ext}},\mathbb{R}^3)} \leqslant K, \\ \|\psi^{-1}\|_{C^{k,\alpha}(\Omega^{\mathrm{ext}},\mathbb{R}^3)} \leqslant K\Big\}$$

are called design variables. These design variables induce, in a natural way, the set of admissible shapes

$$\mathcal{O}_{k,\alpha} := \mathcal{O}_{k,\alpha}(\Omega_0,\Omega^{\mathrm{ext}}) := \left\{\psi(\Omega_0) \;:\; \psi \in U^{\mathrm{ad}}_{k,\alpha}(\Omega^{\mathrm{ext}})\right\},$$

assigned to Ω_0. In order to measure the distance between two shapes in $\mathcal{O}_{k,\alpha}$, we need a suited metric. For two non-empty subsets Ω, Ω' of a metric space (M, d) we define their Hausdorff distance by

$$d_H(\Omega, \Omega') := \max\{\sup_{x\in\Omega} \inf_{y\in\Omega'} d(x-y), \sup_{y\in\Omega'} \inf_{x\in\Omega} d(x-y)\}.$$

Since by definition the shapes in $\mathcal{O}_{k,\alpha}$ are compact, the Hausdorff distance defines a metric on $\mathcal{O}_{k,\alpha}$. Furthermore, in Corollary 4.4.4 we see that the metric space $(\mathcal{O}_{k,\alpha}, d_H)$ is also compact.

Lemma 4.3.1. *Let $k \geqslant 1$ and $\alpha \in]0, 1]$, then the shape space $\mathcal{O}_{k,\alpha}$ satisfies a uniform hemisphere condition and a uniform cone condition.*

Proof. First, we show that $\mathcal{O}_{k,\alpha}$ satisfies a uniform hemisphere condition. We consider a set of hemisphere transforms $\mathbb{T}_1, \ldots, \mathbb{T}_m$, with $m \in \mathbb{N}$, straightening piecewise the boundary of the baseline shape Ω_0, and define $C_{\mathbb{T}} := \max_{i=1,\ldots,m} \|\mathbb{T}_i\|_{C^{k,\alpha}(\Omega_0,\mathbb{R}^3)}$. By definition, for each shape $\Omega \in \mathcal{O}_{k,\alpha}$ we have $\Omega = \psi(\Omega_0)$ with some admissible variable $\psi \in U^{\mathrm{ad}}_{k,\alpha}$. Consequently, the compositions $\mathbb{T}_i \circ \psi^{-1}$, $1 \leqslant i \leqslant m$, define a set of hemisphere transforms for Ω. Since $k \geqslant 1$, the functions $\mathbb{T}$ and ψ^{-1} are Lipschitz continuous up to order $k-1$, and thus the hemisphere transforms $\mathbb{T}_i \circ \psi^{-1}$ are a subset of $C^{k,\alpha}(\psi(\Omega_0), \mathbb{R}^3)$. The admissible variables are, by definition, uniformly bounded by some constant $K > 0$ in their respective norm, and hence, it follows directly that $\max_{i=1,\ldots,m} \|\mathbb{T}_i \circ \psi^{-1}\|_{C^{k,\alpha}(\Omega,\mathbb{R}^3)} \leqslant K C_{\mathbb{T}}$ uniformly with respect to $\mathcal{O}_{k,\alpha}$.

Next, as $k \geqslant 1$, the shape Ω_0 is a domain with Lipschitz boundary and therefore fulfills a uniform cone condition. Since, in addition, every transform $\psi \in U^{\mathrm{ad}}_{k,\alpha}(\Omega^{\mathrm{ext}})$ and its inverse is bounded by K, we have

$$\frac{1}{K}|x-y| \leqslant |\psi(x) - \psi(y)| \leqslant K|x-y|, \quad \text{for all } x, y \in \Omega_0. \tag{4.8}$$

Let $C(x)$ be the cone associated with the cone condition satisfied by Ω_0, where $x \in \partial\Omega_0$ denotes the vertex. Further, we denote with $C_K(x)$ the cone where we decreased the radius of C with factor $\frac{1}{K}$. Then, by the lower bound in (4.8), we can always place the shrinked cone C_K within the transformed cone $\psi(C(x))$ at the boundary point $\psi(x) \in \partial\psi(\Omega_0)$ for every $\psi \in U^{\mathrm{ad}}_{k,\alpha}(\Omega^{\mathrm{ext}})$ and $x \in \partial\Omega_0$. Consequently, the cone C_K provides the uniform cone condition for $\mathcal{O}_{k,\alpha}$. □

Now, to specify a shape optimization problem, it only remains to define a class of cost functionals which depend lower semicontinuously on the shapes in $\mathcal{O}_{k,\alpha}$ and respect the demands and constraints of the component.

Definition 4.3.2 (Local Cost Functionals)**.** *Let $\mathcal{O} \subset \mathcal{P}(\mathbb{R}^3)$ denote a shape space with corresponding state spaces $V_1(\Omega), \ldots, V_m(\Omega)$, $\Omega \in \mathcal{O}$ and graph $\mathcal{G} := \{(\Omega, v_\Omega) : \Omega \in \mathcal{O}\}$. Assuming that $V_i(\Omega) \subseteq C^k(\Omega, \mathbb{R}^{d_i})$ for all $1 \leqslant i \leqslant m$, then the local cost functional on $\mathcal{G}$ is given by*

$$\begin{aligned} J(\Omega, v_\Omega) :=& \int_\Omega \mathcal{F}_{vol}(x, v_\Omega, \nabla v_\Omega, \ldots, \nabla^k v_\Omega)\,\mathrm{d}x \\ &+ \int_{\partial\Omega} \mathcal{F}_{sur}(x, v_\Omega, \nabla v_\Omega, \ldots, \nabla^k v_\Omega)\,\mathrm{d}A, \end{aligned} \tag{4.9}$$

where $\mathcal{F}_{vol}$, $\mathcal{F}_{vol} : \mathbb{R}^r \to \mathrm{cl}(\mathbb{R}_{\geqslant 0})$ and $r = 3 + \sum_{i=1}^m \sum_{j=1}^k d_i^j$. We denote the volume integral and surface integral with

$$\begin{aligned} J_{vol}(\Omega, v_\Omega) &:= \int_\Omega \mathcal{F}_{vol}(x, v_\Omega, \nabla v_\Omega, \ldots, \nabla^k v_\Omega)\,\mathrm{d}x, \\ J_{sur}(\Omega, v_\Omega) &:= \int_{\partial\Omega} \mathcal{F}_{sur}(x, v_\Omega, \nabla v_\Omega, \ldots, \nabla^k v_\Omega)\,\mathrm{d}A. \end{aligned} \tag{4.10}$$

Assuming J_{vol}, $J_{\mathrm{sur}} \in C^0(\mathbb{R}^r, \mathbb{R})$ and $v \in C^k(\Omega, \mathbb{R}^d)$, with $\sum_{i=1}^m d_i$, we will show in Lemma 4.4.9 that these cost functionals not only depend lower semicontinuously on the shapes $\Omega \in \mathcal{O}_{k,\alpha}$ but also continuously. In the following, we present two examples of cost functionals connected with the linear elasticity equation and the potential flow equation, which one could possibly implement in this optimization framework.

4.3.3 Examples

Optimal Reliability

Low cycle fatigue (LCF) driven surface crack initiation is particularly important for the reliability of highly loaded engineering parts as turbine components [79, 99]. The design of such mechanical elements therefore requires a model that is capable of accurately quantifying risk levels for LCF crack initiation, crack growth and ultimate failure. Here we refer to the model that we introduced in Subsection 4.1.1 that models the statistical size effect but also includes the notch support factor, by using stress gradients arising from the coupled elasticity equation (4.5):

$$J_R(\Omega, u_\Omega) := \int_{D \cap \partial\Omega} \left(\frac{1}{N_{\text{det}}(\nabla u_\Omega, \nabla^2 u_\Omega(x))} \right)^m \mathrm{d}A. \tag{4.11}$$

Ω represents the shape of the component, u_Ω is the displacement field and the solution to the coupled elasticity equation on Ω_B, N_{det} is the deterministic number of life cycles at each point of the surface of Ω and m is the Weibull shape parameter. The probability of failure (PoF) after t load cycles is then given as $PoF(t) = 1 - \mathrm{e}^{-t^m J_R(\Omega, u_\Omega)}$. Minimizing the probability of failure thus clearly is equivalent to minimizing $J_R(\Omega, u_\Omega)$.

By Lemma 4.1.11, N_{det} is a smooth function on $\mathbb{R}_{\geqslant 0}$. In addition, by Theorem 2.4.15, the solution u_Ω is a function in $C^{2,\alpha}(\mathrm{cl}(\Omega_B), \mathbb{R}^3)$ for any shape $\Omega \in \mathcal{O}_{2,\alpha}$. Thus, as we see later, J_R defines a continuous cost functional on $\mathcal{O}_{2,\alpha}$

Efficiency

The second primary objective of a vane we use as an example is the energy efficiency that is connected with the viscosity of the fluid flowing through the shroud. Viscosity is a measure which describes the internal friction of a moving fluid. In a laminar fluid the effect of viscosity is limited to a thin layer near the surface of the component. The fluid does not slip along the surface, but adheres to it. In the case of potential flow, there is a transition from zero velocity at the surface to the full velocity which is present at a certain distance from the surface. The layer where this transition takes place is called the boundary layer or frictional layer. The thickness of the boundary layer is not constant but (roughly) proportional to the square root of the kinematic viscosity ν and is growing from the leading edge, the location where the fluid first impinge on the surface of the component. Friction of the fluid on the surface leads to energy dissipation. A coefficient for the inflicted local wall shear stress is given by

$$\tau_w(x) = \frac{0.322 \cdot \mu \|v_\Omega\|^{\frac{3}{2}}}{\sqrt{\nu \cdot \mathrm{dist}_{\text{LE}}(x)}}, \tag{4.12}$$

where we denote with $\|\cdot\|$ the Euclidean norm, μ is the viscosity, and $\mathrm{dist}_{\text{LE}}$ the distance to the leading edge along the component's surface $\partial\Omega$. For a detailed introduction to boundary layer theory one can see, e.g., [23, 97]. With this coefficient one can derive an estimate for the loss of power due to friction given by

$$J_E(\Omega, \phi_\Omega) := \int_{D \cap \partial\Omega} \|v_\Omega\| \tau_w \, \mathrm{dA}. \tag{4.13}$$

Again, Ω is a shape in $\mathcal{O}_{k,\alpha}$ and $v_\Omega = \nabla\phi_\Omega$ originates from the potential flow equation (2.13). Even though τ_w does not satisfy the continuity assumptions we can find, since we see later that v_Ω is uniformly bounded on $\mathcal{O}_{k,\alpha}$, an integrable majorant for $|v_\Omega|\tau_w$ for every $\Omega \in \mathcal{O}_{k,\alpha}$ which implies the continuity of J_E on $\mathcal{O}_{k,\alpha}$, by applying Lebesgue's Dominated Convergence Theorem.

4.3.4 Multi-Physics Shape Optimization Problem

At last, we are able to merge all the needed parts and state our multi-physics shape optimization problem. Consider the space $(\mathcal{O}_{k,\alpha}, d_H)$ of admissible shapes and let $J_1, \ldots, J_l$ be local cost functionals on the Graph $\mathcal{G} := \{(\Omega, u_\Omega, \phi_\Omega) : \Omega \in \mathcal{O}_{k,\alpha},\ u_\Omega \text{ solves (4.5) on } \Omega_B,\ \phi_\Omega \text{ solves (2.13) on } D\backslash\Omega\}$. The multi-physics shape optimization problem is defined by:

$$\begin{cases} \text{Find } \Omega^* \in \mathcal{O}_{k,\alpha} \text{ such that} \\ (\Omega^*, u_{\Omega^*}, \phi_{\Omega^*}) \text{ is Pareto optimal with respect to } J. \end{cases} \tag{4.14}$$

4.4 Existence of Pareto Optimal Shapes

A well-posed shape optimization problems must characterize the underlying engineering task as well as possible while, at the same time, being solvable at all. Showing that the multi-physics shape optimization problem (4.14) possesses at least one Pareto optimal solution includes, besides proving the lower semicontinuity of the local cost functionals (see Definition (4.3.2)), proving the compactness of the graph $\mathcal{G}$. As $\mathcal{G}$ depends on the solution to the multi-physics boundary value problem, uniform bounds for the corresponding solution spaces are needed. In the following subsection, we derive uniform bounds for the unique solutions u_Ω of (4.5) and ϕ_Ω of (2.13), which holds for every $\Omega \in \mathcal{O}_{k,\alpha}$.

4.4.1 Uniform Bounds for Solution Spaces

For both, the elasticity equation and the potential flow equation, the approach to derive a uniform bound is the same. We further investigate estimate (4.6) and (4.7) in order to obtain a uniform bound in the shape space $\mathcal{O}_{k,\alpha}$.

Lemma 4.4.1. *Let ϕ_Ω be the unique solution to the potential flow problem* (2.13) *on $\Omega \in \mathcal{O}_{k,\alpha}$, with $k \geqslant 2$ and $\alpha \in]0,1]$. Then, there exists a constant $M_\phi > 0$ such that for every $\Omega \in \mathcal{O}_{k,\alpha}$ we have*

$$\|\phi_\Omega\|_{C^{2,\alpha}(D\backslash\Omega,\mathbb{R})} \leqslant M_\phi.$$

Proof. We consider Schauder estimate (4.7):

$$\|\phi_\Omega\|_{C^{2,\alpha}(D\backslash\Omega,\mathbb{R})} \leqslant C\left(\|g\|_{C^{1,\alpha}(\partial D\backslash\partial(D\cap\Omega),\mathbb{R})} + \|\phi_\Omega\|_{C^{0,\alpha}(D\backslash\Omega,\mathbb{R})}\right).$$

In general, the constant C potentially depends on the domain Ω; see, e.g., [55, Theorem 6.30]. However, the dependency of C on Ω is through the ellipticity of the differential operator, which depends on the hemisphere transforms that are used to straighten $\partial\Omega$ in order to prove (4.7) for neighbourhoods near the boundary. The constant C reflects the upper bound of the hemisphere transforms and since the shape space $\mathcal{O}_{k,\alpha}$ possesses a uniform hemisphere condition (see Lemma 4.3.1), this upper bound is independent with respect to $\mathcal{O}_{k,\alpha}$.

Next, $\|g\|_{C^{1,\alpha}(\partial D\backslash\partial(D\cap\Omega),\mathbb{R})}$ is obviously bounded by $\|g\|_{C^{1,\alpha}(\partial D,\mathbb{R})}$. Moreover, since $\mathcal{O}_{k,\alpha}$ satisfies a uniform cone condition (see Lemma 4.3.1), Lemma 2.2.2 implies that for every $\varepsilon > 0$ there is a constant $C_\varepsilon > 0$ such that

$$\|\phi_\Omega\|_{C^{0,\alpha}(D\backslash\Omega,\mathbb{R})} \leqslant \varepsilon\|\phi_\Omega\|_{C^{2,\alpha}(D\backslash\Omega,\mathbb{R})} + C_\varepsilon\|\phi_\Omega\|_{H^1(D\backslash\Omega,\mathbb{R})}.$$

Combining all of the above considerations and choosing $\varepsilon < \frac{1}{C}$ yields

$$\begin{aligned}\|\phi\|_{C^{2,\alpha}(D\backslash\Omega,\mathbb{R})} &\leqslant C\left(\|g\|_{C^{1,\alpha}(\partial D\backslash\partial(D\cap\Omega),\mathbb{R})} + \|\phi_\Omega\|_{C^{0,\alpha}(D\backslash\Omega,\mathbb{R})}\right)\\ &\leqslant \frac{1}{1-\varepsilon C}\left(C\|g\|_{C^{1,\alpha}(\partial D,\mathbb{R})} + C_\varepsilon\|\phi_\Omega\|_{H^1(D\backslash\Omega,\mathbb{R})}\right).\end{aligned}$$

Lastly, with the same approach as in the proof to Theorem 2.4.17, one can show that the a-priori estimate

$$\|\phi_\Omega\|_{H^1(D\backslash\Omega,\mathbb{R})} \leqslant C_P|D|\|g\|_{C^1(\partial D,\mathbb{R})}$$

holds for all shapes $\Omega \in \mathcal{O}_{k,\alpha}$. Hence, there exists a constant $M > 0$ such that

$$\|\phi_\Omega\|_{C^{2,\alpha}(D\backslash\Omega,\mathbb{R})} \leqslant M_\phi,$$

for all $\Omega \in \mathcal{O}_{k,\alpha}$. □

Lemma 4.4.2. *Let u_Ω be the unique solution to (4.5) on $\Omega \in \mathcal{O}_{k,\alpha}$, with $k \geqslant 2$ and $\alpha \in]0,1]$. Then, there exists a constant $M_u > 0$ such that for every $\Omega \in \mathcal{O}_{k,\alpha}$ we have*

$$\|u_\Omega\|_{C^{2,\alpha'}(\Omega_B,\mathbb{R}^3)} \leqslant M_u,$$

Proof. Following the same steps as in the proof to Lemma 4.4.1, we first state Schauder estimate (4.6):

$$\|u_\Omega\|_{C^{2,\alpha}(\Omega_B,\mathbb{R}^3)} \leqslant C \left(\|f\|_{C^{0,\alpha}(\Omega_B,\mathbb{R}^3)} + \|g_{\mathrm{s}}\|_{C^{1,\alpha}(\partial\Omega,\mathbb{R}^3)} + \|u_\Omega\|_{C^0(\Omega_B,\mathbb{R}^3)}\right),$$

where, as explained above, the constant C can be choosen independently of the shape $\Omega \in \mathcal{O}_{k,\alpha}$. The norm $\|f\|_{C^{0,\alpha}(\Omega,\mathbb{R}^3)}$ of volume force f is bounded by $\|f\|_{C^{0,\alpha}(\Omega^{\mathrm{ext}},\mathbb{R}^3)}$ and the norm $\|g_{\mathrm{s}}\|_{C^{1,\alpha}(\partial\Omega,\mathbb{R}^3)}$ can be further estimated as follows. Recalling that the surface force is given by $g_{\mathrm{s}} = \left(\frac{1}{2}\rho|\nabla\phi|^2 - p_{\mathrm{st}}\right) n$, we estimate

$$\begin{aligned} \|g_{\mathrm{s}}\|_{C^{1,\alpha}(\partial\Omega,\mathbb{R}^3)} &= \left\| \left(\frac{1}{2}\rho|\nabla\phi_\Omega|^2 - p_{\mathrm{st}}\right) n \right\|_{C^{1,\alpha}(\partial\Omega,\mathbb{R}^3)} \\ &\leqslant \frac{1}{2}\rho\||\nabla\phi_\Omega|^2 n\|_{C^{1,\alpha}(\partial\Omega,\mathbb{R}^3)} + \|p_{\mathrm{st}} n\|_{C^{1,\alpha}(\partial\Omega,\mathbb{R}^3)}. \end{aligned} \tag{4.15}$$

In the proof of Lemma 4.3.1 we constructed, for $\Omega \in \mathcal{O}_{k,\alpha}$, $C^{k,\alpha}$-continuous hemisphere transforms which are uniformly bounded in their respective norm. As we can use these diffeomorphisms as chart mappings to describe the two-dimensional submanifold $\partial\Omega$, we can conclude that the unitary normal vector n of $\partial\Omega$ is uniformly bounded in $\mathcal{O}_{k,\alpha}$ with respect to $\|\cdot\|_{C^{1,\alpha}(\partial\Omega,\mathbb{R}^3)}$. Since p_{st} is constant and by Lemma 4.4.1 $\nabla\phi_\Omega$ is uniformly upper bounded in $\Omega \in \mathcal{O}_{k,\alpha}$ with respect to $\|\cdot\|_{C^{1,\alpha}(\partial\Omega,\mathbb{R}^3)}$, it follows that (4.15) is uniformly upper bounded in $\mathcal{O}_{k,\alpha}$ by some constant $L > 0$.

Next, by Lemma 4.3.1 the shape space $\mathcal{O}_{k,\alpha}$ possesses a uniform cone condition, and, hence, we can apply Lemma 2.2.2 which implies that for any $\varepsilon > 0$ there is a constant $C_\varepsilon > 0$ such that

$$\|u_\Omega\|_{C^{0,\alpha}(\Omega_B,\mathbb{R}^3)} \leqslant \varepsilon\|u_\Omega\|_{C^{2,\alpha}(\Omega_B,\mathbb{R}^3)} + C_\varepsilon\|u_\Omega\|_{H^1(\Omega_B,\mathbb{R}^3)}.$$

Combining the above and choosing $\varepsilon < \frac{1}{C}$ gives

$$\|u_\Omega\|_{C^{2,\alpha}(\Omega_B,\mathbb{R}^3)} \leqslant \frac{1+C}{1-\varepsilon C}\left(\|f\|_{C^{0,\alpha}(\Omega^{\mathrm{ext}},\mathbb{R}^3)} + L + C_\varepsilon\|u_\Omega\|_{H^1(\Omega_B,\mathbb{R})}\right)$$

Lastly, it remains to show that $\|u_\Omega\|_{H^1(\Omega_B,\mathbb{R}^3)}$ is uniformly bounded. For this purpose, we consider the weak formulation of (4.5)

$$\int_\Omega \boldsymbol{\varepsilon}(u) : \boldsymbol{\sigma}(v)\,\mathrm{d}x = \int_\Omega \langle f, v\rangle\,\mathrm{d}x + \int_{\Gamma_N} \langle g_{\mathrm{s}}, v\rangle\,\mathrm{dA}$$

on $H^1_D(\Omega_B,\mathbb{R}^3) = \{u \in H^1(\Omega_B,\mathbb{R}^3) : u = 0 \text{ along } \partial B\}$. Now, for every $v \in H^1_D(\Omega_B,\mathbb{R}^3)$ we can find a constant $\tilde{C} > 0$ that is uniform with respect to $\mathcal{O}_{k,\alpha}$ such that

$$\tilde{C}^{-1}\|\varepsilon(v)\|^2_{L^2(\Omega_B,\mathbb{R}^{3\times 3})} \leqslant \int_{\Omega_B} \boldsymbol{\sigma}(v) : \boldsymbol{\varepsilon}(v)\,\mathrm{d}x,$$

and, as f and g_s are uniformly bounded, also

$$\left|\int_\Omega \langle f, v\rangle\,\mathrm{d}x + \int_{\Gamma_N} \langle g_\mathrm{s}, v\rangle\,\mathrm{d}A\right| \leqslant \tilde{C}\|v\|_{H^1(\Omega_B,\mathbb{R}^3)}.$$

Applying Korn's second inequality with some suitable constant $C_K > 0$ then yields

$$\tilde{C}^{-1}\|\boldsymbol{\varepsilon}(u_\Omega)\|^2_{L^2(\Omega_B,\mathbb{R}^{3\times 3})} \leqslant \tilde{C}\|u_\Omega\|_{H^1(\Omega_B,\mathbb{R}^3)} \leqslant C_K\|\boldsymbol{\varepsilon}(u_\Omega)\|_{L^2(\Omega_B,\mathbb{R}^{3\times 3})},$$

which gives that $\|u_\Omega\|_{H^1(\Omega_B,\mathbb{R}^3)}$ is bounded by $\tilde{C}C_K$, where the constant C_K may depends on the shape Ω. Examining the proof to Korn's second inequality (see, e.g., [88]), one can see that the dependence of the constant C_K on Ω is through the cone of the uniform cone condition satisfied by Ω. As the shape space $\mathcal{O}_{k,\alpha}$ satisfies itself a uniform cone conditon, we have that C_K is uniform with respect to $\mathcal{O}_{k,\alpha}$. □

4.4.2 Existence Theorem for Pareto Optimal Shapes

Given that $k \geqslant 2$, we can derive from Lemma 4.4.1 and 4.4.2 that for any $0 < \alpha' < \alpha \leqslant 1$ the solution spaces $\mathcal{P} := \{\phi_\Omega : \Omega \in \mathcal{O}_{k,\alpha}\}$ and $\mathcal{E} := \{u_\Omega : \Omega \in \mathcal{O}_{k,\alpha}\}$, formed by solving potential flow equation (2.13) and elasticity equation (4.5) on $\mathcal{O}_{k,\alpha}$, are compact metric spaces with respect to the norm $\|\cdot\|_{C^{k,\alpha'}}$ as well as the shape space $\mathcal{O}_{k,\alpha}$ with respect to the Hausdorff distance d_H. This subsections first provides a proof to each of these assertions. Afterwards, we show that the local cost functionals defined in Definition 4.3.2 are continuous functionals on the graph $\mathcal{G} := \{(\Omega, \phi_\Omega, u_\Omega) : \Omega \in \mathcal{O}_{k,\alpha}\}$. Then, from Theorem 4.2.3 we can conclude that the multi-physics shape optimization problem (4.14) possesses at least one Pareto optimal solution and that the corresponding Pareto front is closed.

Lemma 4.4.3. *Let $k \geqslant 1$ and $\alpha \in]0,1]$, then the space of admissible variables $U^{ad}_{k,\alpha}(\Omega^{ext})$ defines a compact space with respect to $\|\cdot\|_{C^{k,\alpha'}(\Omega^{ext},\mathbb{R}^3)}$ for all $0 < \alpha' < \alpha \leqslant 1$.*

Proof. Due to its definition, $U^{\mathrm{ad}}_{k,\alpha}$ is bounded with respect to $\|\cdot\|_{C^{k,\alpha}(\Omega^{\mathrm{ext}},\mathbb{R}^3)}$ and therefore, by Lemma 2.2.8, a precompact subset of $C^{k,\alpha'}(\Omega^{\text{ext}}, \mathbb{R}^3)$ for any $0 < \alpha' < \alpha \leqslant 1$. Since $C^{k,\alpha'}(\Omega^{\text{ext}}, \mathbb{R}^3)$ is a Banach space, for any sequence $(\psi_n)_{n\in\mathbb{N}} \subset U^{\text{ad}}_{k,\alpha}$ there is a subsequence $(\psi_{n_l})_{l\in\mathbb{N}}$ that converges against some function $\psi \in C^{k,\alpha'}(\Omega^{\text{ext}}, \mathbb{R}^3)$ with respect to $\|\cdot\|_{C^{k,\alpha'}(\Omega^{\text{ext}},\mathbb{R}^3)}$. Using that $\|\psi_{n_l}\|_{C^{k,\alpha}(\Omega^{\text{ext}},\mathbb{R}^3)} \leqslant K$ for all $l \in \mathbb{N}$, we have for all $\beta \in \mathbb{N}$ with $|\beta| = k$ that

$$\begin{aligned}
[D^\beta\psi]_{C^{0,\alpha}(\Omega^{\text{ext}},\mathbb{R}^3)} &\leqslant 2\|D^\beta\psi - D^\beta\psi_{n_l}\|_{C^0(\Omega^{\text{ext}},\mathbb{R}^3)} + [D^\beta\psi_{n_l}]_{C^{0,\alpha}(\Omega^{\text{ext}},\mathbb{R}^3)} \\
&\leqslant 2\|D^\beta\psi - D^\beta\psi_{n_l}\|_{C^0(\Omega^{\text{ext}},\mathbb{R}^3)} + K - \|D^\beta\psi_{n_l}\|_{C^0(\Omega^{\text{ext}},\mathbb{R}^3)} \\
&\xrightarrow{n\to\infty} K - \|D^\beta\psi\|_{C^0(\Omega^{\text{ext}},\mathbb{R}^3)}.
\end{aligned}$$

Therefore, ψ is in $C^{k,\alpha}(\Omega^{\text{ext}}, \mathbb{R}^3)$ with $\|\tilde{\psi}\|_{C^{k,\alpha}(\Omega^{\text{ext}},\mathbb{R}^3)} \leqslant K$.

In addition, by the same arguments, it follows that the sequence of inverse $(\psi^{-1}_{n_l})_{l\in\mathbb{N}}$ converges to some function $\tilde{\psi} \in C^{k,\alpha}(\Omega^{\text{ext}}, \mathbb{R}^3)$ with $\|\psi\|_{C^{k,\alpha}(\Omega^{\text{ext}},\mathbb{R}^3)} \leqslant K$ with respect to $\|\cdot\|_{C^{k,\alpha'}(\Omega^{\text{ext}},\mathbb{R}^3)}$. It is straightforward to show that any bounded set of Hölder continuous functions is equicontinuous, which gives that $\tilde{\psi} = \psi^{-1}$ and hence $\psi \in U^{\text{ad}}_{k,\alpha}$. □

Corollary 4.4.4. *Let $k \geqslant 1$ and $\alpha \in]0, 1]$, then the space of admissible shapes $\mathcal{O}_{k,\alpha}$ defines a compact space with respect to the Hausdorff distance d_{H}.*

Proof. Let $(\Omega_n)_{n\in\mathbb{N}}$ be a sequence of shapes in $\mathcal{O}_{k,\alpha}$ with associated sequence of admissible variables $(\psi_n)_{n\in\mathbb{N}} \subset U^{\text{ad}}_{k,\alpha}$ such that $\psi_n(\Omega_0) = \Omega_n$. Since $U^{\text{ad}}_{k,\alpha}$ is compact (see Lemma 4.4.3) we can find a subsequence $(\psi_{n_l})_{l\in\mathbb{N}}$ that converges against some variable $\psi \in U^{\text{ad}}_{k,\alpha}$ with respect to $\|\cdot\|_{C^{k,\alpha'}(\Omega^{\text{ext}},\mathbb{R}^3)}$ for any $0 < \alpha' < \alpha \leqslant 1$. Let $(\Omega_{n_l})_{l\in\mathbb{N}}$ be the corresponding subsequence of shapes, then

$$\begin{aligned}
d_H(\Omega_{n_l}, \Omega) &= \max\{\sup_{x\in\Omega_{n_l}} \inf_{y\in\Omega} |x - y|, \sup_{y\in\Omega} \inf_{x\in\Omega_{n_l}} |x - y|\} \\
&= \max\{\sup_{x\in\Omega_0} \inf_{y\in\Omega_0} |\psi_{n_l}(x) - \psi(y)|, \sup_{y\in\Omega_0} \inf_{x\in\Omega_0} |\psi_{n_l}(x) - \psi(y)|\} \\
&\leqslant \sup_{x\in\Omega_0} |\psi_{n_l}(x) - \psi(x)| \\
&\xrightarrow{n\to\infty} 0.
\end{aligned}$$

Hence, each sequence in $\mathcal{O}_{k,\alpha}$ has a subsequences that converges to a limit shape in $\mathcal{O}_{k,\alpha}$ which implies the compactness of $\mathcal{O}_{k,\alpha}$. □

While investigating the compactness of $\mathcal{P}$ and $\mathcal{E}$, the question arises of how to compare two functions defined on different domains. In order to solve this issue we use the extension operator introduced in Lemma 2.2.9 to extend the Hölder continuous solutions of the multi-physics shape optimization problem to a common domain. The topology of the solution spaces is then defined through the topology induced by the resulting extensions with respect to the Hölder norm on the extended domain.

Definition 4.4.5. *Let either $V = \mathcal{P}$ or $V = \mathcal{E}$, and let p be the extension operator from Lemma 2.2.9. With $v_\Omega^{ext} := p(v_\Omega)$ we denote the extension of a solution $v_\Omega \in V$ from Ω to Ω^{ext} (or from $D\backslash\Omega$ to D) and with V^{ext} the space consisting of all such extensions v_Ω^{ext} for all $v_\Omega \in V$. For a sequence of solutions $(v_{\Omega_n})_{n\in\mathbb{N}} \subset V$, the expression $v_{\Omega_n} \rightsquigarrow v_\Omega$ as $n \to \infty$ is defined by $v_{\Omega_n}^{ext} \to v_\Omega^{ext}$ in V^{ext} with respect to $\|\cdot\|_{C^{k,\alpha}(\Omega^{ext},\mathbb{R}^3)}$ (or $\|\cdot\|_{C^{k,\alpha}(D,\mathbb{R})}$).*

Lemma 4.4.6. *Let $k \geqslant 2$, then the extension space $\mathcal{P}^{ext}$ to the solutions of potential flow equation (2.13) is a compact subspace of $C^{k,\alpha'}(D,\mathbb{R})$ for any $0 < \alpha < \alpha' \leqslant 1$.*

Proof. Using Lemma 2.2.9 and 4.4.1 gives

$$\|\phi_\Omega^{\text{ext}}\|_{C^{2,\alpha}(D,\mathbb{R})} \leqslant C\|\phi_\Omega\|_{C^{2,\alpha}(D\backslash\Omega,\mathbb{R})} \leqslant CK,$$

where the constants C and K are independent of $\Omega \in \mathcal{O}_{k,\alpha}$ as $\mathcal{O}_{k,\alpha}$ satisfies a uniform cone property. Hence, $\mathcal{P}^{\text{ext}}$ is bounded in $C^{k,\alpha}(D,\mathbb{R})$ and therefore, by Lemma 2.2.8, a precompact subset of $C^{k,\alpha'}(D,\mathbb{R})$ for any $0 < \alpha' < \alpha \leqslant 1$. As $C^{k,\alpha'}(D,\mathbb{R})$ is a Banach space, for any sequence $(\phi_{\Omega_n}^{\text{ext}})_{n\in\mathbb{N}} \subset \mathcal{P}^{\text{ext}}$ we can find a subsequence $(\phi_{\Omega_{n_l}}^{\text{ext}})_{l\in\mathbb{N}}$ that converges against some function $\phi \in C^{k,\alpha'}(D,\mathbb{R})$. In addition, the corresponding subsequence of shapes $(\Omega_{n_l})_{l\in\mathbb{N}} \subset \mathcal{O}_{k,\alpha}$ converges, as $\mathcal{O}_{k,\alpha}$ is compact (see Corollary 4.4.4), to some shape $\Omega \in \mathcal{O}_{k,\alpha}$. In the proof to Lemma 4.4.3, we have seen that $\phi \in C^{k,\alpha}(D,\mathbb{R})$ and since the convergence in $\|\cdot\|_{C^{k,\alpha'}(D,\mathbb{R})}$ implies pointwise convergence, the function ϕ is the extension to the unique solution $\phi_\Omega \in \mathcal{P}$ and belongs therefore to $\mathcal{P}^{\text{ext}}$. Hence, $\mathcal{P}^{\text{ext}}$ is a closed subspace of $C^{k,\alpha}(D,\mathbb{R})$ with respect to $\|\cdot\|_{C^{k,\alpha'}(D,\mathbb{R})}$ and thereby compact. □

Lemma 4.4.7. *Let $k \geqslant 2$, then the extension space $\mathcal{E}^{ext}$ to the solutions of elasticity equation (4.5) is a compact subspace of $C^{k,\alpha'}(\Omega^{ext},\mathbb{R}^3)$ for any $0 < \alpha < \alpha' \leqslant 1$.*

Proof. The proof is the same as that of Lemma 4.4.6 and is therefore omitted. □

Lemma 4.4.8. *We consider the multi-physics shape optimization problem (4.14) with boundary regularity of class $C^{k,\alpha}$, with $k \geqslant 2$. Then, the graph $\mathcal{G}$ is compact with respect to the corresponding product metric.*

Proof. By Corollary 4.4.4, Lemma 4.4.6 and Lemma 4.4.7 the product space $\mathcal{O}_{k,\alpha} \times \mathcal{P} \times \mathcal{E}$ is compact. Let $((\Omega_n, \phi_{\Omega_n}, u_{\Omega_n}))_{n\in\mathbb{N}}$ be a sequence in $\mathcal{G}$ that converges to some $(\Omega, \phi, u) \in \mathcal{O}_{k,\alpha} \times \mathcal{P} \times \mathcal{E}$. Since the convergence of ϕ_{Ω_n} in $\|\cdot\|_{C^{k,\alpha'}(D,\mathbb{R})}$ and of u_{Ω_n} in $\|\cdot\|_{C^{k,\alpha'}(\Omega^{\text{ext}},\mathbb{R}^3)}$, with $0 < \alpha' < \alpha \leqslant 1$, implies pointwise convergence, the limit functions $\phi = \phi_\Omega$ and $u = u_\Omega$ solve the potential flow equation and the elasticity equation on $D\backslash\Omega$ and Ω_B, respectively. Hence, $(\Omega, \phi, u) \in \mathcal{G}$ and thereby $\mathcal{G}$ is compact. □

Lemma 4.4.9 (Continuity of Local Cost Functionals [60, Lemma 6.3])**.** *Let $\mathcal{F}_{vol}$, $\mathcal{F}_{sur} \in C^0(\mathbb{R}^r, \mathbb{R})$ (with r as in Definition 4.3.2), and for $\Omega \in \mathcal{O}_{k,\alpha}$ and $v_i \in C^k(\Omega, \mathbb{R}^{d_i})$ for all $i = 1, \ldots, m$, consider the volume integral $J_{vol}(\Omega, v)$ and the surface integral $J_{sur}(\Omega, v)$. Let $\Omega_n \subset \mathcal{O}_{k,\alpha}$ with $\Omega_n \xrightarrow{\tilde{O}} \Omega$ as $n \to \infty$ and let $(v_n)_{n\in\mathbb{N}} \in C^k(\Omega_n, \mathbb{R}^d)$, with $d = \sum_{i=1}^m d_i$, such that $v_n \rightsquigarrow v$ as $n \to \infty$ for some $v \in C^k(\Omega, \mathbb{R}^d)$. Then,*

(i) $J_{vol}(\Omega_n, v_n) \longrightarrow J_{vol}(\Omega, v)$ as $n \to \infty$, and

(ii) if $k \geqslant 1$, we also have $J_{sur}(\Omega_n, v_n) \longrightarrow J_{sur}(\Omega, v)$ as $n \to \infty$.

Proof. For assertion (i), we first consider the volume integral on a shape Ω_n

$$J_{\text{vol}}(\Omega_n, v_n) = \int_{\Omega^{\text{ext}}} \mathbb{1}_{\Omega_n} \cdot \mathcal{F}_{\text{vol}}(x, v_n^{\text{ext}}, \nabla v_n^{\text{ext}}, \ldots, \nabla^k v_n^{\text{ext}})\, \mathrm{d}x.$$

Because of $\mathcal{F}_{\text{vol}} \in C^0(\mathbb{R}^r, \mathbb{R})$ and $v_n \rightsquigarrow v$ as $n \to \infty$, there exists a constant $C > 0$ such that $|\mathbb{1}_{\Omega_n} \cdot \mathcal{F}_{\text{vol}}(x, v_n^{\text{ext}}, \nabla v_n^{\text{ext}}, \ldots, \nabla^k v_n^{\text{ext}})| \leqslant C$ is valid for all $n \in \mathbb{N}$ almost everywhere in Ω_{ext}. Moreover, $\Omega_n \xrightarrow{\tilde{O}} \Omega$ and $v_n^{\text{ext}} \to v^{\text{ext}}$ in $C_c^k(\Omega^{\text{ext}}, \mathbb{R}^d)$ ensures the existence of

$$\lim_{n\to\infty} \mathbb{1}_{\Omega_n} \cdot \mathcal{F}_{\text{vol}}(x, v_n^{\text{ext}}, \nabla v_n^{\text{ext}}, \ldots, \nabla^k v_n^{\text{ext}}) = \mathbb{1}_{\Omega} \cdot \mathcal{F}_{\text{vol}}(x, v^{\text{ext}}, \nabla v^{\text{ext}}, \ldots, \nabla^k v^{\text{ext}}),$$

for all $x \in \Omega^{\text{ext}}$. Therefore, using Lebesgue‘s dominated convergence theorem yields

$$\begin{aligned}
\lim_{n\to\infty} J_{\text{vol}}(\Omega_n, v_n) &= \lim_{n\to\infty} \int_{\Omega^{\text{ext}}} \mathbb{1}_{\Omega_n} \cdot \mathcal{F}_{\text{vol}}(x, v_n^{\text{ext}}, \nabla v_n^{\text{ext}}, \ldots, \nabla^k v_n^{\text{ext}})\, \mathrm{d}x \\
&= \int_{\Omega^{\text{ext}}} \lim_{n\to\infty} \mathbb{1}_{\Omega_n} \cdot \mathcal{F}_{\text{vol}}(x, v_n^{\text{ext}}, \nabla v_n^{\text{ext}}, \ldots, \nabla^k v_n^{\text{ext}})\, \mathrm{d}x \\
&= \int_{\Omega^{\text{ext}}} \mathbb{1}_{\Omega} \cdot \mathcal{F}_{\text{vol}}(x, v^{\text{ext}}, \nabla v^{\text{ext}}, \ldots, \nabla^k v^{\text{ext}})\, \mathrm{d}x \\
&= J_{\text{vol}}(\Omega, v).
\end{aligned}$$

For assertion (ii), we first recall that for each shape $\Omega \in \mathcal{O}_{k,\alpha}$ can be considered, by its definition, as a differentiable submanifold which is locally embeddable into $\mathbb{R}^2$. Let $A_n^i \subset \partial\Omega_n$ for $1 \leqslant i \leqslant l$, with $\cup_{i=1}^l A_i = \partial\Omega_n$ and $A_i \cap A_j = \varnothing$ for $i \neq j$, be a disjoint decomposition of the boundary $\partial\Omega_n$. We can use the hemisphere transforms we constructed proving Lemma 4.3.1 as, in i and n uniformly bounded, chart mappings $\mathbb{T}_n^i : A_n^i \to \tilde{A}_i$ with $\tilde{A}_i \subset \mathbb{R}^2$ in order to straighten the boundary of Ω_n. This gives

$$\begin{aligned} J_{\text{sur}}(\Omega_n, v_n) &= \int_{\partial\Omega_n} \mathcal{F}_{\text{sur}}\,(x, v_n, \nabla v_n, \dots, \nabla^k v_n)\,\mathrm{d}A \\ &= \sum_{i=1}^{l} \int_{A_n^i} \mathcal{F}_{\text{sur}}\,(x, v_n, \nabla v_n, \dots, \nabla^k v_n)\,\mathrm{d}A \\ &= \sum_{i=1}^{l} \int_{\tilde{A}_i} \mathcal{F}_{\text{sur}}\,\left(\mathbb{T}_n^i(s), v_n(\mathbb{T}_n^i(s)), \nabla v_n(\mathbb{T}_n^i(s)), \dots, \nabla^k v_n(\mathbb{T}_n^i(s))\right) \sqrt{g^{\mathbb{T}_n^i}(s)}\,\mathrm{d}s, \end{aligned}$$

which is a volume integral with corresponding Gram determinants $g^{\mathbb{T}_n^i}$. Due to the fact that the chart mappings $\mathbb{T}_n^i$ are uniformly bounded and since $\tilde{A}_i$ is independent of n, one can see that, similarly to (i), we can apply Lebesgue's Theorem which proves the assertion. □

Theorem 4.4.10. *Given that $k \geqslant 2$, the multi-physics shape optimization problem (4.14) possesses at least one Pareto optimal solution $(\Omega^*, \phi_{\Omega^*}, u_{\Omega^*}) \in \mathcal{G}$ and the associated Pareto front covers all non-dominated points in $\mathcal{Y}$, i.e., $\mathcal{Y}_N = \mathrm{cl}(\mathcal{Y})_N$.*

Proof. Lemma 4.4.8 provides the compactness of the graph $\mathcal{G}$ and Lemma 4.4.9 the continuity of the local cost functionals. Then, Theorem 4.2.3 gives the existence of an optimal shape and the closeness of the set of optimal shapes. □

4.5 Scalarization and Multi-Physics Optimization

Scalarizing is the traditional approach to solving a multi-criteria optimization problem. This includes formulating a single objective optimization problem that is related to the original Pareto optimality problem by means of a real-valued scalarizing function typically being a function of the objective function, auxiliary scalar or vector variables, and/or scalar or vector parameters. Additionally, scalarization techniques sometimes further restrict the feasible set of the problem with new variables or/and restriction functions. In this section, we investigate the stability of the parameter-dependent optimal shapes to different types of scalarization techniques with underlying design problem (4.14).

First, let us define the scalarization methods we consider. This involves a certain class of real-valued functions $S_\theta : \mathbb{R}^l \to \mathbb{R}$, referred to as scalarization function that possibly depends on a parameter θ which lies in a parameter space Θ. The scalarization problem is given by

$$\begin{aligned} \min S_\theta\left(J(\Omega, u_\Omega, \phi_\Omega)\right) \\ \text{subject to } (\Omega, u_\Omega, \phi_\Omega) \in \mathcal{G}_\theta, \end{aligned} \tag{4.16}$$

where $\mathcal{G}_\theta \subseteq \mathcal{G}$. For the sake of notational convenience, we sometimes identify an element $(\Omega, u_\Omega, \phi_\Omega) \in \mathcal{G}_\theta$ only by its distinct shape Ω. If we assume that $\mathcal{G}_\theta$ is closed and the scalarization $S_\theta(J)$ is lower semicontinuous on $\mathcal{G}_\theta \times \{\theta\}$, then, by the results of Section 4.4.2, (4.16) obviously has an optimal solution for $\theta \in \Theta$. For a fixed $\theta \in \Theta$ we shall denote the set of all optimal shapes to an achievement function problem by $\zeta_\theta = \arg\min_{\Omega \in \mathcal{G}_\theta} S_\theta\left(J(\Omega, u_\Omega, \phi_\Omega)\right)$. We assume that $\Theta \subset \mathbb{R}^l$ is closed and equip the space $\mathcal{Z} := \{\zeta_\theta \,:\, \theta \in \Theta\}$ with the Hausdorff distance, which in this setting defines, due to the closeness of the optimal shapes sets, a metric (see Lemma 4.5.2 and Corollary 4.5.3.

In the following, we gather some definitions and results from Chapter 4 of [16]. We define the optimal set mapping $\chi : \Theta \rightrightarrows \mathcal{Z}$, the optimal value mapping $\tau : \Theta \longrightarrow \mathbb{R}$, and the graph mapping $G : \Theta \rightrightarrows 2^{\mathcal{G}}$ which maps a parameter $\theta \in \Theta$ to the corresponding set of optimal shapes ζ_θ, the corresponding optimal value $\min_{\Omega \in \mathcal{G}_\theta} S_\theta(J)$, and the corresponding graph $\mathcal{G}_\theta$, respectively.

Definition 4.5.1 (Closed point-to-set mappings)**.** *Let (Θ, d_Θ) and (X, d_x) be metric spaces and Γ a point-to-set mapping of Θ into 2^X. We say that Γ is closed at a point $\theta^* \in \Theta$ if for each pair of sequences $(\theta_n)_{n\in\mathbb{N}} \subseteq \Theta$ and $(x_n)_{n\in\mathbb{N}} \subseteq X$ with*

(i) $\theta_n \to \theta^*$, *as* $n \to \infty$,

(ii) $x_n \in \Gamma(\theta_n)$, *for all* $n \in \mathbb{N}$,

(iii) $x_n \to x^*$, *as* $n \to \infty$,

it follows that $x^* \in \Gamma(\theta^*)$.

With these definitions at hand, we can describe the stability of the optimal shapes for a wide range of scalarization methods. First, we state a lemma that shows that $(\mathcal{Z}, d_{\mathrm{H}})$ is indeed a metric space.

Lemma 4.5.2. *[16, Theorem 4.2.1 (3)] If the graph mapping G is closed at some parameter $\theta^* \in \Theta$, the optimal value mapping τ is upper semicontinuous at θ^* and the scalarization $S_{\theta^*}(J)$ is lower semicontinuous on $\mathcal{G} \times \{\theta^*\}$, then the optimal set mapping χ is closed at θ^**

Corollary 4.5.3. *If G is closed, and the scalarization function S_θ is lower semicontinuous on $\mathbb{R}^l \times \{\theta\}$, for every $\theta \in \Theta$, and uniformly continuous on $\{x\} \times \Theta$, for each $x \in \mathbb{R}^l$, then the Hausdorff distance d_H defines a metric on $\mathcal{Z}$.*

Proof. Due to the continuity of J (see Lemma 4.4.9) and the uniform continuity of S_θ on $\{x\} \times \Theta$, the optimal value maping τ is upper semicontinuous for every $x \in \mathbb{R}^l$, and therefore, by Lemma 4.5.2, the optimal set mapping χ is closed. Since d_H defines a metric on $F(\mathcal{G})$ (the set of all closed subsets of $\mathcal{G}$), $(\mathcal{Z}, d_H)$ defines a metric space. □

Since the sclarization solution is not necessarly unique, we need some sort of continuity property of point-to-set mappings in order to discuss the stability of sets of optimal shapes. The literature describes serveral definitions which vary in the statement. We investigate the stability according to Hausdorff and Berge (for Berge see [16, Section 2.2]) which, in this setting, are equivalent.

Definition 4.5.4 (Upper semicontiniuity according to Hausdorff)**.** *Let (Θ, d_Θ) and (X, d_x) be metric spaces and Γ a point-to-set mapping of Θ into X. Γ is called upper semicontinuous in θ^* if for each sequence $(\theta_n)_{n\in\mathbb{N}} \subseteq \Theta$ with $\theta_n \longrightarrow \theta^*$, for $n \to \infty$, we have*

$$\sup_{x\in\Gamma(\theta_n)} \inf_{x'\in\Gamma(\theta^*)} d_X(x, x') \longrightarrow 0. \tag{4.17}$$

Γ is called upper semicontinuous if Γ is upper semicontinuous in each $\theta \in \Theta$. For this type of continuity we simply write u.s.c.-H.

The last theorem of this chapter states stability conditions for scalarization function problems.

Theorem 4.5.5 ([16, Theorem 4.2.2 (3)])**.** *Assume that G is u.s.c.-H at θ^* and $G(\theta^*)$ is compact. Further, let τ be upper semicontinuous at θ^* and S_{θ^*} lower semicontinuous on $G(\theta^*) \times \{\theta^*\}$. Then, the optimal set mapping χ is u.s.c.-H at θ^*.*

The following two corollaries demonstrate continuity properties of shapes under changes of preferences for two commonly used scalarization techniques. In particular, the results apply to the multi-physics shape optimization problem (4.14).

Corollary 4.5.6 (Weighted Sum Method)**.** *Consider cost functionals* $J = (J_1, \dots, J_l)$ *and let* $\Theta \subset \mathbb{R}^l$ *be a closed subset. Then, the weighted sum scalarization method (with* $\theta \in \Theta$*), which is given by*

$$\min \textstyle\sum_{i=1}^{l} \theta_i J_i(\Omega, u_\Omega, \phi_\Omega) \\ \text{subject to } (\Omega, u_\Omega, \phi_\Omega) \in \mathcal{G},$$

fulfills all conditions of Theorem 4.5.5 due to the compactness of $\mathcal{G}$ *(see Lemma 4.4.8) and the continuity of* J *(see Lemma 4.4.9).*

Corollary 4.5.7 (ε-Constraint Method)**.** *Let* $J = (J_1, \dots, J_l)$ *be cost functionals. We optimize cost functional* J_j *and constrain the other functionals by* $J_i \leqslant \varepsilon_i \in \mathbb{R}$*, for* $1 \leqslant i \leqslant n$ *and* $i \neq j$*. If each* ε_i *converges monotonically decreasing to some* ε_i^**, then the* ε*-Constraint Method*

$$\min J_j(\Omega, u_\Omega, \phi_\Omega) \\ \text{subject to } J_i \leqslant \varepsilon_i,$$

fulfills all conditions of Theorem 4.5.5.

Proof. Let $\varepsilon = (\varepsilon_1, \dots, \varepsilon_l)$ and $\mathcal{G}_\varepsilon = \{\Omega \in \mathcal{G} \ : \ J_i(\Omega) \leqslant \varepsilon_i, \ i \neq j\}$. The u.s.c.-H of G is given due to the continuity of J. The continuity of J_j, the u.s.c.-H of G, and the fact that $\mathcal{G}_\varepsilon \subseteq \mathcal{G}_{\varepsilon'}$ for all $\varepsilon^* \leqslant \varepsilon' \leqslant \varepsilon$ gives that $\tau(\varepsilon)$ converge to $\tau(\varepsilon^*)$ for $\varepsilon \searrow \varepsilon^*$. Hence, the optimal sets $\chi(\varepsilon)$ converge against $\chi(\varepsilon^*)$ for $\varepsilon \searrow \varepsilon^*$ in the sense of u.s.c.-H. □

Remark 4.5.8. *Whenever the scalarized problem (4.16) possesses a unique solution* $\zeta_\theta = \{\Omega_\theta\}$ *for all* θ *in some neighborhood of* $\theta^* \in \Theta$*, then* $\Omega_{\theta_n} \longrightarrow \Omega_{\theta^*}$ *in the Hausdorff distance (for subsets in* $\mathbb{R}^d$*) if* $\theta_n \to \theta^*$*.*

Chapter 5

Integrability and Approximability of Solutions to the Stationary Diffusion Equation with Lévy Coefficient

The second part of this thesis considers diffusion equations with random diffusion coefficient given by Lévy fields [42]. Modeling physical phenomena occuring in real world application with differential equation using deterministic coefficient functions is often an inadequate approach as it is not always possible to determine the precise values of these coefficients. Random differential equations treat this problem by describing the uncertain data with random variables or stochastic processes. As illustration, we introduce in this chapter randomness in the diffusion equation (2.15) which is often used as model problem for a variety of numerical approximation methods.

In many application, e.g., when we model the groundwater flow in a porous medium governed by Darcy's law, the precise value of the conductivity a in (2.15) is typically uncertain as it is often derived from sparse information based on limited observations. In order to model this randomness, we introduce a probability distribution on the set of admissible coefficient functions a which gives a random partial differential equation. Typically, the stochastic conductivity model $a = a(x, \omega)$ is chosen to be a lognormal random field, with the mean and covariance structure of the underlying Gaussian random field $\log a$ estimated using geostatistical methods.

For the resulting elliptic linear differential equation, early results on the existence and uniqueness of solution in a finite-dimensional setting focusing on numerical approximation methods are presented in e.g., [11, 12, 13, 37, 47, 82]. An extension of these results to random fields characterized by infinite-dimensional parameters can be found in [1, 14, 15, 30, 33, 34, 43, 68, 89]. Most of these works employ random model based on transformed Gaussian random field as they provide a strong instrument which describes a wide range of effects. However, there are also effects which cannot be captured by Gaussian fields, e.g. discontinuities and heavy-tail behaviour as they occur in applications such as flow in fractured media, anomalous diffusion and the modeling of heterogeneous material; see, e.g., [29, 109]. In order to model these blind spots of the Gaussian model, an extension is needed.

The work Sarkis et al. [50, 51, 52] considers the diffusion equation (2.15) in the absence of uniform ellipticity and boundness, allowing for a diffusion coefficient which is a smooth transformation of a Gaussian white noise. Our work uses the approach of chapter 3 where we extensively analyzed the notion of noise as generalized random field in the sense of Minlos [53, 85]. We provide a more general stochastic approach and extend naturally the Gaussian coefficient to one which follows a Lévy distribution [9, 62, 76].

For possible numerical treatments, this chapter gives a proof of integrability of the corresponding random solution to (2.15). This proof is based on a priori estimate of elliptic partial differential equations which crucially depends on the minimal value of the coefficient "$a(x)$" and leads to an extremal value problem for Lévy fields. These kind of problems have been studied extensively in the field of empirical processes by exploiting metric entropy estimates and concentration phenomena [56, 108, 111]. In order to investigate the tail of the Lévy field we decompose it in its Gaussian part, which we control with a metric entropy estimate provided by Talagrand [107], and its Poisson part, whose tail is described by a Chernoff-like bound under the assumption that the Lévy measure defining the Poisson contribution has a Laplace transform.

Furthermore, we present here an adaptation of the Karhunen-Loève (KL) expansion for smoothed Lévy noise fields. Unlike Gaussian fields, Lévy fields are not determined by their covariance function. Therefore, we choose to expand the smoothing kernel $k(x-y)$ instead of the covariance function. Due to translation invariance, the kernel always has a continuous spectrum as an integral operator on $L^2(\mathbb{R}^d, \mathrm{d}x)$. Hence, we have to cut off the noise field $Z(y)$ and restrict it to some sufficient large domain Λ. The solution u_N arising out of this finite-dimensional approximation of the random diffusion coefficient then converges, under sufficient conditions, to the real solution u in $L^n((\Omega, \mathfrak{A}, \mathbf{P}), H^1(D, \mathbb{R}))$. In addition, we provide convergence rates for the combined decay of the cut-off to Λ and the truncation of series given by the Mercer expansion of the smoothing kernel.

We begin with modeling a as a transformed smoothed random field, hence $a(x) = T(Z_k(x))$, where T is a suitable Borel-measurable real-valued function and Z_k is a noise field smoothed by a window function $k \in L^1(\mathbb{R}^d, \mathbb{R}) \cap L^2(\mathbb{R}^d, \mathbb{R})$. We intend to estimate quantities of interest connected to the solution u of the resulting random boundary value problem. Further, we are interested in the probability of certain events and the expected or maximum flow through a subdomain or boundary. Assuming for each $\omega \in \Omega$ that

$$0 < \operatorname*{ess\,inf}_{x \in D} a(x, \omega) \leqslant \operatorname*{ess\,sup}_{x \in D} a(x, \omega) < \infty \tag{5.1}$$

ensures the strict ellipticity of the differential operator of boundary value problem (2.15) with conductivity function a given by a realization $a(\cdot, \omega)$ of the random field $a = T \circ Z_k$. Therefore, there exists a unique $u = u(\cdot, \omega) \in H^1(D, \mathbb{R})$ which solves (2.15) with $a = a(\cdot, \omega)$ and which satisfies consequently, by Theorem 2.4.17, the a-priori estimate

$$\|u\|_{H^1(D,\mathbb{R})} \leqslant C \, \frac{1 + \|a\|_{C^0(D,\mathbb{R})}}{\operatorname{ess\,inf}_{x \in D} a(x)} \left(\|f\|_{L^2(D,\mathbb{R})} + \|g_D\|_{H^{\frac{1}{2}}(\Gamma_D,\mathbb{R})} + \|g_N\|_{H^{-\frac{1}{2}}(\Gamma_N,\mathbb{R})} \right). \tag{5.2}$$

Lemma 5.0.1. *(i) Let Z be a $|||\cdot|||$-continuous random field and $k \in L^1(\mathbb{R}^d, \mathbb{R}) \cap L^2(\mathbb{R}^d, \mathbb{R})$ be a window function such that the random field $(Z_k(x))_{x \in \mathbb{R}^d}$ has almost surely continuous paths. Then, for a strictly positive and locally Lipschitz continuous function T on $\mathbb{R}$, we have for the random conductivity $a := T \circ Z_k \in L^\infty(D, \mathbb{R})$ as well as $\operatorname{ess\,inf} a > 0$ almost surely. Denoting with $u(\cdot, \omega)$ the solution of (2.15) with conductivity function $a(\cdot, \omega)$, the mapping $\omega \mapsto u(\cdot, \omega)$ is an $H^1(D, \mathbb{R})$-valued, Borel-measurable random variable.*

(ii) For a $|||\cdot|||$-continuous generalized random field and a Matérn kernel $k_{\alpha,m}$, with $\alpha > d + \max\{0, \frac{3d-12}{8}\}$, the random field $(Z_{k_{\alpha,m}}(x))_{x \in \mathbb{R}^d}$ has almost surely continuous paths. If Z is a Gaussian field, the same holds already for $\alpha > d/2$.

Proof. To prove (i), we first show that the mapping $(\Omega, \mathfrak{A}, \mathbf{P}) \to C^0(\mathrm{cl}(D), \mathbb{R})$, $\omega \to Z_k(\cdot, \omega)$ is measurable with respect to the Borel σ-algebra generated by the $\|\cdot\|_{C^0(\mathrm{cl}(D),\mathbb{R})}$-norm. As $Z_k(x) \in L^0(\Omega, \mathfrak{A}, \mathbf{P})$, $x \in \mathbb{R}^d$, we have for any $q \in C^0(\mathrm{cl}(D), \mathbb{R})$ and $r > 0$

that the set

$$Z_k^{-1}\left(\mathrm{cl}(B_r(q))\right) = \{\|Z_k - q\|_{C^0(\mathrm{cl}(D),\mathbb{R})} \leqslant r\} = \bigcap_{x\in\bar{D}\cap\mathbb{Q}^d} \{|Z_k(x) - q(x)| \leqslant r\}$$

is almost measurable. Since $(C^0(\mathrm{cl}(D),\mathbb{R}), \|\cdot\|_{C^0(\mathrm{cl}(D),\mathbb{R})})$ is separable, every open set $U \subset C^0(\mathrm{cl}(D),\mathbb{R})$ is a countable union of open balls $B_r(q)$, which gives that the above implies $\{Z_k \in U\}$ is measurable for any open $U \subset C(\mathrm{cl}(D),\mathbb{R})$. Moreover, due to the local Lipschitz continuity of T, the mapping $q \mapsto T \circ q$ is continuous on $C^0(\mathrm{cl}(D),\mathbb{R})$ with respect to $\|\cdot\|_{C^0(\mathrm{cl}(D),\mathbb{R})}$, and thus $\|\cdot\|_{C^0(\mathrm{cl}(D),\mathbb{R})}$-Borel measurable. To see that for fixed $f \in L^2(D,\mathbb{R})$, $g_D \in H^{\frac{1}{2}}(\Gamma_D,\mathbb{R})$ and $g_N \in H^{-\frac{1}{2}}(\Gamma_N,\mathbb{R})$ the solution map

$$C_{>0}(\mathrm{cl}(D),\mathbb{R}) := \{a \in C^0(\mathrm{cl}(D),\mathbb{R}) \,:\, \inf a > 0\} \to H^1(D,\mathbb{R}), \quad a \mapsto u_a$$

is continuous, where u_a denotes the unique solution to (2.15), we refer to [64] or to the methods used in Section 5.2. Thus, $u \in L^0(\Omega, H^1(D,\mathbb{R}))$ and therefore (i) holds. Lastly, (ii) follows directly from Theorem 3.2.10 and Remark 3.5.1. □

5.1 Integrability of Solutions

In this section, we study the integrability of solutions to diffusion equation (2.15) with transformed and smoothed Lévy diffusion coefficient $a = T \circ Z_k$. More precisely, we perform extreme value estimation on the random diffusion coefficient in order to show the existence of moments of the Sobolev norm of solutions. The first result serves to connect the existence of moments with extremal value theory for random fields.

Lemma 5.1.1. *Let Z be a $|||\cdot|||$-continuous generalized random field and let $k \in L^1(\mathbb{R}^d,\mathbb{R}) \cap L^2(\mathbb{R}^d,\mathbb{R})$ be a window function such that $(Z_k(x))_{x\in\mathbb{R}^d}$ has almost surely continuous paths. In addition, let T be a function on $\mathbb{R}$ that is locally Lipschitz continuous such that for some $h \geqslant 0$ and $B, \rho > 0$ it holds that $B^{-1}\mathrm{e}^{-\rho|z|^h} \leqslant T(z) \leqslant B\mathrm{e}^{\rho|z|^h}$ for all $z \in \mathbb{R}$.*

Then, for the random conductivity $a = T \circ Z_k$ there exists a constant $C \geqslant 1$ so that for all $f \in L^2(D,\mathbb{R})$, $g_D \in H^{\frac{1}{2}}(\Gamma_D,\mathbb{R})$, and $g_N \in H^{-\frac{1}{2}}(\Gamma_N,\mathbb{R})$ the solution u to the boundary value problem (2.15) satisfies

$$\mathbf{E}\left[\|u\|^n_{H^1(D,\mathbb{R})}\right] \leqslant \tilde{C}^n 2^{n-1}(B^n + B^{2n}) \sum_{j=0}^{\infty} \mathrm{e}^{2n\rho(j+1)^h}\, \mathbf{P}(\sup_{x\in D}|Z_k(x)| \geqslant j), \quad \textit{for all } n \in \mathbb{N}$$

with $\tilde{C} = C(\|f\|_{L^2(D,\mathbb{R})} + \|g_D\|_{H^{\frac{1}{2}}(\Gamma_D,\mathbb{R})} + \|g_N\|_{H^{-\frac{1}{2}}(\Gamma_N,\mathbb{R})})$.

Proof. By assumption, the smoothed random field Z_k has continuous paths on the complement N^c of some $\mathbf{P}$-null set $N \in \mathfrak{A}$. By setting the functions

$$\|a\|_{C^0(D,\mathbb{R})} = \sup_{x \in D} T(Z_k(x)) = \sup_{x \in D \cap \mathbb{Q}^d} T(Z_k(x))$$

and

$$\operatorname{ess\,inf} a = \inf_{x \in D} T(Z_k(x)) = \inf_{x \in D \cap \mathbb{Q}^d} T(Z_k(x))$$

to zero on N, they are both measurable. Applying (5.2) and the law of total probability yields

$$\begin{aligned}
&\tilde{C}^{-n}\mathbf{E}\left[\|u\|^n_{H^1(D,\mathbb{R})}\right] \leqslant \mathbf{E}\left[\left(\frac{1+\sup_{x\in D} T(Z_k(x))}{\inf_{x\in D} T(Z_k(x))}\right)^n\right] \\
&\leqslant \mathbf{E}\left[\frac{(1+B\sup_{x\in D} \mathrm{e}^{\rho|Z_k(x)|^h})^n}{(B\inf_{x\in D}\mathrm{e}^{\rho|Z_k(x)|^h})^{-n}}\right] \\
&\leqslant \mathbf{E}\left[\frac{2^{n-1}+2^{n-1}B^n\mathrm{e}^{n\rho\sup_{x\in D}|Z_k(x)|^h}}{B^{-n}\mathrm{e}^{-n\rho\sup_{x\in D}|Z_k(x)|^h}}\right] \leqslant 2^{n-1}(B^n+B^{2n})\mathbf{E}\left[\mathrm{e}^{2n\rho\sup_{x\in D}|Z_k(x)|^h}\right] \\
&\leqslant 2^{n-1}(B^n+B^{2n})\sum_{j=0}^{\infty}\mathbf{E}\left[\mathrm{e}^{2n\rho(j+1)^h}|j \leqslant \sup_{x\in D}|Z_k(x)| < j+1\right]\mathbf{P}(\sup_{x\in D}|Z_k(x)| \geqslant j) \\
&\leqslant 2^{n-1}(B^n+B^{2n})\sum_{j=0}^{\infty}\mathrm{e}^{2n\rho(j+1)^h}\mathbf{P}(\sup_{x\in D}|Z_k(x)| \geqslant j).
\end{aligned}$$

□

The above Lemma shows that we need exponential bounds for the extreme values of the smoothed and transformed Lévy field. We obtain the bounds by decomposing Z_k into its Gaussian part G_k and its Poisson part P_k and then seperately estimating the extreme values of each field. For this purpose, the next lemma from Talagrand gives an exponential bound for Gaussian random fields.

Lemma 5.1.2. *[107, Theorem 2.4] Let* $(G(x))_{x\in D}$ *be a centered Gaussian field with almost surely continuous paths and let* $\bar{\sigma}^2 = \sup_{x\in D}\mathbf{E}\left[G(x)^2\right]$. *Consider the canonical distance* $d_c(x,y) := \mathbf{E}\left[(G(x)-G(y))^2\right]^{\frac{1}{2}}$ *on* D *and let* $N(D,d_c,\varepsilon)$ *be the smallest number of* d_c*-open balls with* d_c*-radius* ε *needed to cover* D. *Assume that for some constant* $A > \bar{\sigma}$, *some* $v > 0$ *and* $0 \leqslant \varepsilon_0 \leqslant \bar{\sigma}$, *the number* $N(D,d_c,\varepsilon)$ *is bounded above by* $(\frac{A}{\varepsilon})^v$ *whenever* $\varepsilon \in]0,\varepsilon_0[$.

Then, there is a uniform constant $K > 0$ such that for $g \geqslant \bar{\sigma}^2(1+\sqrt{v})/\varepsilon_0$ we have

$$\mathbf{P}(\sup_{x\in D}|G(x)| \geqslant g) \leqslant 2\left(\frac{KAg}{\sqrt{v}\bar{\sigma}^2}\right)^v \Phi\left(-\frac{g}{\bar{\sigma}}\right) \leqslant \left(\frac{KAg}{\sqrt{v}\bar{\sigma}^2}\right)^v \mathrm{e}^{-\frac{g^2}{2\bar{\sigma}^2}}, \tag{5.3}$$

where Φ denotes the cumulative distribution function of the standard normal distribution. If $\varepsilon_0 = \bar{\sigma}$, the condition on g is $g \geqslant \bar{\sigma}(1+\sqrt{v})$.

We continue with deriving properties of Matérn kernel functions which are needed below.

Lemma 5.1.3. *Let $D \subset \mathbb{R}^d$ be open and bounded, $\alpha > \frac{d}{2}$, and $m > 0$. Then the following holds:*

(i) For $0 < \eta < 2\alpha - d$ there exists $C = C(m,\eta,\alpha) > 0$ such that for all $x, y \in \mathbb{R}^d$

$$|k_{\alpha,m}(x) - k_{\alpha,m}(y)| \leqslant C(m,\eta,\alpha)\,|x-y|^\eta.$$

(ii) The absolute value $|k_{\alpha,m}|$ is bounded, decreases like $\mathrm{e}^{-m|x|}$, and the mapping $y \mapsto \sup_{x\in D} |\tau_y (k_{\alpha,m}(x))|$ lies in $L^1(\mathbb{R}^d,\mathbb{R}) \cap L^\infty(\mathbb{R}^d,\mathbb{R})$.

Proof. For (i), we first note that for a fixed $\eta \in (0,1)$ and for all $w, z \in \mathbb{C}$ with $|w-z| \leqslant 2$ we have $|w-z| \leqslant 2^{1-\eta}|w-z|^\eta$. Further, we have $|\mathrm{e}^{-i\xi x} - \mathrm{e}^{-i\xi y}| \leqslant 2$ and $|\mathrm{e}^{-i\xi x} - \mathrm{e}^{-i\xi y}| \leqslant |\xi(x-y)|$ for all $x, y, \xi \in \mathbb{R}^d$. Therefore,

$$\begin{aligned}|k_{\alpha,m}(x) - k_{\alpha,m}(y)| &= \frac{1}{(2\pi)^d}\left|\int_{\mathbb{R}^d} \frac{\mathrm{e}^{-i\xi x} - \mathrm{e}^{-i\xi y}}{(|\xi|^2+m^2)^\alpha}\,\mathrm{d}\xi\right| \\ &\leqslant \frac{2^{1-\eta}}{(2\pi)^d}|x-y|^\eta \int_{\mathbb{R}^d} \frac{|\xi|^\eta}{(|\xi|^2+m^2)^\alpha}\,\mathrm{d}\xi,\end{aligned}$$

where the last integral converges if $0 < \eta < 2\alpha - d$.

Proving (ii), one can apply the Hankel transform to see that

$$k_{\alpha,m}(x) = \mathscr{F}^{-1}(\hat{k}_{\alpha,m})(x) = \frac{(|x|/m)^{\alpha-d/2}K_{\alpha-d/2}(|x|m)}{2^{\alpha-1}\Gamma(\alpha)(2\pi)^{d/2}},$$

where K_v is the modified Bessel function of second kind; see, e.g., [2]. For a fixed $v > 0$, $K_v(|x|) \sim \frac{1}{2}\Gamma(v)(\frac{1}{2}|x|)^{-v}$ for $|x| \to 0$ and $K_v(|x|) \sim \sqrt{\pi/(2|x|)}\mathrm{e}^{-|x|}$ for $|x| \to \infty$. This implies that $|k_{\alpha,m}|$ is bounded and decreases as $\mathrm{e}^{-m|x|}$. Therefore, since D is relatively compact, $\sup_{x\in D} |\tau_y k_{\alpha,m}(x)|$ is bounded and exponentially decreasing as well, which implies the assertion. □

Next, we further specify the Lévy measure ν of the Lévy characteristic (3.2). This assumption will be used repeatedly below. Recall that for a Borel measure ν on $\mathbb{R}\backslash\{0\}$ we denote by ν_+ its image measure on $\mathbb{R}_{\geqslant 0}$ under $|\cdot|$.

Assumptions 5.1.4. *Let Z be a $|||\cdot|||$-continuous Lévy field with characteristic triplet (b, σ^2, ν), such that ν is a Lévy measure satisfies $\int_{\mathbb{R}\backslash\{0\}} |s|\, \nu(\mathrm{d}s) < \infty$ and $\int_{\mathbb{R}} (\mathrm{e}^{\beta s} - 1)\, \nu_+(\mathrm{d}s) < \infty$ for some $\beta > 0$.*

The Lemma below provides an exponential upper bound of Chernov-type for the extreme values of the Poisson part of a Lévy random field.

Lemma 5.1.5. *Let P be a compound Poisson field, i.e., a Lévy field with characteristic triplet $(\int_{\{0<|s|\leqslant 1\}} s\, \nu(\mathrm{d}s), 0, \nu)$ with a finite measure ν, which satisfies Assumption 5.1.4. Moreover, let $D \subset \mathbb{R}^d$ be open, and bounded and let $k_\iota : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}, \iota \in I$, be a family of smoothing functions such that with $\tilde{k}_\iota(y) := \sup_{x\in D} |k_\iota(x, y)|, y \in \mathbb{R}^d, \iota \in I$, the following conditions hold:*

i) for all $\iota \in I : \tilde{k}_\iota \in L^1(\mathbb{R}^d, \mathbb{R}) \cap L^\infty(\mathbb{R}^d, \mathbb{R})$.

ii) $\kappa := \sup_{\iota\in I} \|\tilde{k}_\iota\|_{L^\infty(\mathbb{R}^d,\mathbb{R})} < \infty$ as well as $\kappa_1 := \sup_{\iota\in I} \|\tilde{k}_\iota\|_{L^1(\mathbb{R}^d,\mathbb{R})} < \infty$.

Then, for all $\tau \in]0, 1[$, $\iota \in I$ and $p > 0$ we have

$$\mathbf{P}(\sup_{x\in D} |P_{k_\iota}(x)| \geqslant p) \leqslant \mathrm{e}^{f_\tau - p\frac{\tau\beta}{\kappa}},$$

where f_τ is given by

$$f_\tau := \frac{\tau\beta\kappa_1}{\kappa}\left(\mathrm{e}^{\tau\beta} \int\limits_{\{0<s\leqslant 1\}} |s|\nu_+(\mathrm{d}s) + \int\limits_{\{s>1\}} \frac{1}{(1 - \tau\kappa_\iota/\kappa)\beta\mathrm{e}} \int_{\{s>1\}} \mathrm{e}^{\beta s}\nu_+(\mathrm{d}s)\right).$$

Proof. For $\iota \in I$ we define $\kappa_\iota := \|\tilde{k}_\iota\|_{L^\infty(\mathbb{R}^d,\mathbb{R})}$ as well as

$$f_\iota :]0, \infty[\to [0, \infty], \quad f_\iota(\vartheta) := \int_{\mathbb{R}^d} \int_{\mathbb{R}_{\geqslant 0}} (\mathrm{e}^{\vartheta s \tilde{k}_\iota(y)} - 1)\nu_+(\mathrm{d}s)\mathrm{d}y$$

and

$$\theta_\iota :]0, \infty[\to \mathbb{R} \cup \{\infty\}, \quad \theta_\iota(p) := \sup_{\vartheta>0} \vartheta p - f_\iota(\vartheta).$$

Then, f_ι is a convex, increasing function and θ_ι is its Legendre transform (Fenchel transform, conjugate function). Using the notation from Remark 3.5.2 we can derive, for $\vartheta > 0$ and abbreviating $P_\iota(x) := P_{k_\iota}(x), \iota \in I, x \in D$, analogously to (3.7)

$$\begin{aligned}\mathbf{E}[\mathrm{e}^{\vartheta \sup_{x\in D} |P_\iota(x)|}] &\leqslant \mathbf{E}[\mathrm{e}^{\vartheta \sup_{x\in D} |P|_{|k_\iota|}(x)}] \leqslant \mathbf{E}[\mathrm{e}^{\vartheta \sum_j \sum_{l=1}^{N_{\Lambda_j}} |S_l^{(j)}| \tilde{k}_\iota(X_l^{(j)})}] \\ &= \mathrm{e}^{\int_{\mathbb{R}^d}\int_{\mathbb{R}_+} (\mathrm{e}^{\vartheta s \tilde{k}_\iota(y)}-1)\,\nu_+(\mathrm{d}s)\,\mathrm{d}y}.\end{aligned}$$

Applying Markov's inequality, this yields for $p > 0$

$$\begin{aligned}\mathbf{P}(\sup_{x\in D}|P_\iota(x)| \geqslant p) &= \inf_{\vartheta>0} \mathbf{P}\left(\mathrm{e}^{\vartheta \sup_{x\in D}|P_\iota(x)|} \geqslant \mathrm{e}^{\vartheta p}\right) \leqslant \inf_{\vartheta>0} \frac{\mathbf{E}[\mathrm{e}^{\vartheta \sup_{x\in D} |P_\iota(x)|}]}{\mathrm{e}^{\vartheta p}} \\ &\leqslant \inf_{\vartheta>0} \mathrm{e}^{\int_{\mathbb{R}^d}\int_{\mathbb{R}_+} (\mathrm{e}^{\vartheta s \tilde{k}_\iota(y)}-1)\,\nu_+(\mathrm{d}s)\,\mathrm{d}y - \vartheta p} \\ &\leqslant \mathrm{e}^{-\sup\limits_{\vartheta>0}\{\vartheta p - \int_{\mathbb{R}^d}\int_{\mathbb{R}_+} (\mathrm{e}^{\vartheta s \tilde{k}_\iota(y)}-1)\,\nu_+(\mathrm{d}s)\,\mathrm{d}y\}} = \mathrm{e}^{-\theta_\iota(p)}.\end{aligned} \tag{5.4}$$

Using Assumption 5.1.4, for $0 \leqslant \vartheta < \frac{\beta}{\kappa_\iota}$ we get

$$\begin{aligned}f_\iota(\vartheta) &= \int_{\mathbb{R}^d}\int_{\mathbb{R}_+} \left(\mathrm{e}^{\vartheta s \tilde{k}_\iota(y)} - 1\right)\nu_+(\mathrm{d}s)\,\mathrm{d}y \\ &\leqslant \int_{\mathbb{R}^d}\left(\int_{\{0<s\leqslant 1\}} + \int_{\{s>1\}}\right) \mathrm{e}^{\vartheta s \tilde{k}_\iota(y)}\vartheta s \tilde{k}_\iota(y)\,\nu_+(\mathrm{d}s)\,\mathrm{d}y \\ &\leqslant \vartheta \|\tilde{k}_\iota\|_{L^1(\mathbb{R}^d,\mathbb{R})}\left(\mathrm{e}^{\vartheta\kappa_\iota}\int_{\{0<s\leqslant 1\}} |s|\,\nu_+(\mathrm{d}s) + \int_{\{s>1\}} \mathrm{e}^{\theta s \kappa_\iota} s\,\nu_+(\mathrm{d}s)\right) \\ &\leqslant \vartheta\kappa_1\left(\mathrm{e}^{\vartheta\kappa}\int_{\{0<s\leqslant 1\}} |s|\,\nu_+(\mathrm{d}s) + \int_{\{s>1\}} \mathrm{e}^{\beta s}\mathrm{e}^{(\theta\kappa_\iota-\beta)s} s\,\nu_+(\mathrm{d}s)\right) \\ &\leqslant \vartheta\kappa_1\left(\mathrm{e}^{\vartheta\kappa}\int_{\{0<s\leqslant 1\}} |s|\,\nu_+(\mathrm{d}s) + \frac{1}{(\beta-\theta\kappa_\iota)\mathrm{e}}\int_{\{s>1\}} \mathrm{e}^{\beta s}\,\nu_+(\mathrm{d}s)\right),\end{aligned} \tag{5.5}$$

where we have used that for any $\alpha > 0$ the function

$$f_\alpha : (0,\infty) \to \mathbb{R}, \quad s \mapsto \mathrm{e}^{-s\alpha}s$$

attains its maximum at $s = \frac{1}{\alpha}$. Therefore, $f_\iota\restriction_{[0,\beta/\kappa_\iota[}$ is finite. For any arbitrary $\tau \in]0,1[$ and $\vartheta = \tau\frac{\beta}{\kappa}$, it follows from the definition of θ_ι and the fact that $\kappa \geqslant \kappa_\iota$

$$\theta_\iota(p) \geqslant \tau\frac{\beta}{\kappa}p - f_\tau(\tau)$$

for every $p > 0$. Thus, from (5.4) and the previous inequality, the claim follows. □

Now, by using both exponential bounds, we can prove the main result of this subsection.

Theorem 5.1.6. *Let Z be a Lévy field such that Assumption 5.1.4 is satisfied and let $k = k_{\alpha,m}$ be a Matérn kernel with $2\alpha > d$. Moreover, we assume that T is locally Lipschitz such that for $h \in [0,1]$, $B, \rho > 0$ we have $B^{-1}\mathrm{e}^{-\rho|z|^h} \leqslant T(z) \leqslant B\mathrm{e}^{\rho|z|^h}$ for all $z \in \mathbb{R}$.*

Then, for the solution u to boundary value problem (2.15) with random conductivity function $a = T \circ Z_k$ we have $u \in L^n(\Omega, H^1(D,\mathbb{R}))$, for any $n \in \mathbb{N}$ if $h < 1$ and for $n < \beta/2\kappa\rho$ if $h = 1$, where $\kappa := \sup_{x\in D, y\in\mathbb{R}^d} |k_{\alpha,m}(x-y)|$.

In particular, all moments of u exist if $h \leqslant 1$ and $\int_{\mathbb{R}_+} (\mathrm{e}^{\beta s} - 1)\, \nu_+(\mathrm{d}s) < \infty$ for all $\beta > 0$.

Proof. First, we show that without loss of generality, we may assume that Z has the characteristic triplet (b', σ^2, ν) with $b' := \int_{\{0<|s|\leqslant 1\}} s\, \nu(\mathrm{d}s)$. By Proposition 3.3.3, the Lévy noise field $\tilde{Z}$ associated with characteristic triplet (b', σ^2, ν) is $|||\cdot|||$-continuous. In addition, for any $\alpha \in \mathbb{R}$, $T_\alpha(z) := T(z + \alpha)$ is locally Lipschitz, and with $\tilde{\rho} := \max\{1, 2^{h-1}\}\rho$, $\tilde{B} := B\mathrm{e}^{\tilde{\rho}|\alpha|^h}$ we have

$$\tilde{B}^{-1}\mathrm{e}^{-\tilde{\rho}|z|^h} \leqslant T_\alpha(z) \leqslant \tilde{B}\mathrm{e}^{\tilde{\rho}|z|^h}.$$

In the case of $\alpha_k := (b - b') \int_{\mathbb{R}^d} k(y)\, \mathrm{d}y$ we obtain $a = T \circ Z_k = T_{\alpha_k} \circ \tilde{Z}_k$. Thus, by replacing T with T_{α_k} and Z with $\tilde{Z}$, we may indeed assume that Z has the characteristic triplet (b', σ^2, ν). Therefore, we have $Z = G + P$, where G is the $|||\cdot|||$-continuous generalized centered Gaussian field with characteristic triplet $(0, \sigma^2, 0)$ and P is the $|||\cdot|||$-continuous Lévy field with characteristic triplet $(b', 0, \nu)$.

Let d_c denote the canonical distance of the centered Gaussian field $(G_k(x))_{x\in D}$ which, by Theorem 3.2.10, has almost surely continuous paths. We fix $\eta \in]0, 2\alpha - d[$ and $\delta > \operatorname{diam}(D)$ and set $\bar{\sigma}^2 := \sup_{x\in D} \mathbf{E}\left[G_k(x)^2\right] = \sigma^2 \|k\|^2_{L^2(\mathbb{R}^d,\mathbb{R})}$.

Using Lemma 5.1.3 (i), with some suitable constant $C_1 = C_1(m, \eta, 2\alpha) > 0$, we have for arbitrary $x, y \in D$

$$\begin{aligned} \mathrm{d}_c(x,y)^2 &= \mathbf{Var}(G_k(x) - G_k(y)) \\ &= \mathbf{Var}(G_k(x)) - \mathbf{Var}(G_k(y)) - 2\,\mathbf{Cov}(G_k(x), G_k(y)) \\ &= 2\sigma^2(k_{2\alpha,m}(0) - k_{2\alpha,m}(x-y)) \leqslant 2\sigma^2 C_1 |x-y|^\eta. \end{aligned} \tag{5.6}$$

Then, with $C'^2 := 2\sigma^2 C_1$, we have for all $\varepsilon > 0$ and $x \in \mathbb{R}^d$

$$\begin{aligned} B_{|\cdot|,(\frac{\varepsilon^2}{C'^2})^{\frac{1}{\eta}}}(x) :=& \left\{ y \in \mathbb{R}^d \ : \ |x-y| < \left(\frac{\varepsilon^2}{C'^2}\right)^{\frac{1}{\eta}} \right\} \\ \subseteq& \ \{ y \in \mathbb{R}^d \ : \ d_c(x,y) < \varepsilon \} =: B_{\mathrm{d}_c,\varepsilon}(x). \end{aligned} \tag{5.7}$$

Since D is bounded, we can cover D with a finite number N of open balls $B_{|\cdot|,(\frac{\varepsilon^2}{C'^2})^{\frac{1}{\eta}}}(x)$. By the choice of δ, the minimal number N be bounded by $(C'^{\frac{2}{\eta}}\delta/\varepsilon^{\frac{2}{\eta}})^d = (C'\delta^{\eta/2}/\varepsilon)^{2d/\eta}$. By (5.7) we thus obtain for all $\varepsilon > 0$

$$N(D, d_c, \varepsilon) \leqslant (C'\delta^{\eta/2}/\varepsilon)^{2d/\eta},$$

so that the canonical distance d_c satisfies the covering property of Talagrand's Lemma 5.1.2 with $v := 2d/\eta$ and $A := \max\{C'\delta^{\eta/2}, \bar{\sigma}+1\}$ for every $\varepsilon > 0$. Therefore, by Talagrand's Lemma 5.1.2, there is a uniform constant $K > 0$ such that for every $g \geqslant \bar{\sigma}(1+\sqrt{v})$ we can estimate

$$\mathbf{P}\left(\sup_{x \in D} |G_k(x)| \geqslant g\right) \leqslant \left(\frac{KAg}{\sqrt{v}\bar{\sigma}^2}\right)^v \mathrm{e}^{-\frac{g^2}{\bar{\sigma}^2}}. \tag{5.8}$$

Next, Lemma 5.1.3 (ii), gives

$$\tilde{k}(y) := \sup_{x \in D} |k(x,y)| = \sup_{x \in D} |k_{\alpha,m}(x-y)| \in L^1(\mathbb{R}^d, \mathbb{R}) \cap L^\infty(\mathbb{R}^d, \mathbb{R}),$$

so that by Lemma 5.1.5 applied to a family of smoothing functions consisting only of Matérn functions $k_{\alpha,m}$, for arbitrary $\tau \in]0,1[$ there is a constant C_τ depending only on τ, $\|\tilde{k}\|_{L^1(\mathbb{R}^d,\mathbb{R})}$, $\|\tilde{k}\|_{L^\infty(\mathbb{R}^d,\mathbb{R})}$, β, and ν such that for every $p > 0$

$$\mathbf{P}(\sup_{x \in D} |P_k(x)| \geqslant p) \leqslant C_\tau \mathrm{e}^{-\frac{\beta}{\kappa}(1-\tau)p}. \tag{5.9}$$

Considering that $Z = G + P$, it follows from Lemma 5.1.1 together with (5.8), (5.9) that for every $\tau \in]0,1[$ we have with

$$D_\tau := \max\left\{ \left(\frac{KA}{\sqrt{v}}\right)^v, C_\tau, 1 \right\}$$

that

$$
\begin{aligned}
\mathbf{E}\left[\|u\|^n_{H^1(D,\mathbb{R})}\right] &\leqslant \tilde{C}^n 2^{n-1}(B^n + B^{2n}) \sum_{z=0}^{\infty} \mathrm{e}^{2n\rho(z+1)^h}\, \mathbf{P}(\sup_{x\in D}|Z_k(x)| \geqslant z) \\
&\leqslant \tilde{C}^n 2^{n-1}(B^n + B^{2n}) D_\tau \Bigg\{ \sum_{z=0}^{\lfloor\bar{\sigma}(1+\sqrt{v})/\tau\rfloor} \mathrm{e}^{2n\rho(z+1)^h} \qquad (5.10)\\
&\quad + \sum_{z=\lfloor\bar{\sigma}(1+\sqrt{v})/\tau\rfloor+1}^{\infty} \mathrm{e}^{2n\rho(z+1)^h}\left(z^v \mathrm{e}^{-\frac{\tau^2 z^2}{2\bar{\sigma}^2}} + \mathrm{e}^{-\frac{\beta}{\kappa}(1-\tau)z}\right)\Bigg\}.
\end{aligned}
$$

Thus, in the case of $h < 1$ the above series converges. Choosing $h = 1$, the above series converges if $n < (1-\tau)\beta/2\kappa\rho$. Hence, by choosing τ sufficiently close to zero, the case $h = 1$ converges for all $n < \beta/2\kappa\rho$. □

Remark 5.1.7. *(i) By Theorem 5.1.6, in the case of $h = 1$ we get all moments up to an order that depends on β. The larger β is, the more moments u has with respect to the $\|\cdot\|_{H^1(D,\mathbb{R})}$ norm.*

(ii) If we assume the existence of the Laplace transform for ν, i.e., $\int_{\mathbb{R}_{\geqslant 0}} \mathrm{e}^{\beta s}\,\nu_+(\mathrm{d}s) < \infty$ for some $\beta > 0$, we exclude noises with infinite activity like Gamma noise. That is the reason we employ the more general condition $\int_{\mathbb{R}_{\geqslant 0}} (\mathrm{e}^{\beta s} - 1)\,\nu_+(\mathrm{d}s) < \infty$.

(iii) In the special case, where the smoothed Lévy noise field Z_k is a Gaussian field without a compound Poisson noise component, we have $\theta(p) = \infty$ for all $p > 0$ so that (5.10) gives us the existence of all moments if $h < 2$. Moreover, in case of $h = 2$, we then obtain the existence of moments up to order $n < 1/(4\rho\sigma^2\|k\|^2_{L^2(\mathbb{R}^d,\mathbb{R})})$. This improves [26], where this result was shown for $h = 1$.

5.2 Approximability of Solutions of the Random Diffusion Equation

In this section, we approximate the random diffusion coefficient a by a finite dimensional modal expansion and thus reducing it from an infinite-dimensional Lévy field to a finite-dimensional Lévy vector. Under similar assumptions as for integrability, we show that the resulting solution to (2.15) with approximated diffusion coefficient converges to that of the original equation in the Bochner space $L^n((\Omega, \mathfrak{A}, \mathbf{P}), H^1(D,\mathbb{R}))$.

5.2.1 Approximation Scheme for the Random Solution

In order to investigate the approximability of the solution u to the random diffusion equation, we need to control the change in the solution that stems from a change in the coefficients. The results we discuss in this section can be applied in various context of interest as, e.g., to control the error in statistical estimation of the law and smoothing function of the random field. In addition, one can easily generalize these results to arbitrary continuous random fields and differentiable transformations $T(z)$ which are exponentially bounded from below and above.

Let Z_k be a smoothed Lévy random field with a.s. continuous paths and smoothing function $k : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$. Let $k = k_N + r_N$ be any decomposition of k such that $\lim_{N\to\infty} k_N(x, \cdot) = k(x, \cdot)$ with respect to $|||\cdot|||$ for every $x \in \mathbb{R}^d$. We define the corresponding approximation of Z_k with $Z_N(x) := Z_{k_N}(x)$, $N \in \mathbb{N}$ and the corresponding remainder with $R_N(x) := Z_{r_N}(x)$ such that $Z_k(x) = Z_N(x) + R_N(x)$. Assuming that Z_N and R_N have continuous paths on $\mathrm{cl}(D)$, this yields an approximating diffusion coefficient $T(Z_N(x))$ in equation (2.15) with associated random solution u_N to the corresponding weak problem.

We prove the convergence of the approximating solution u_N to the solution u in $L^n((\Omega, \mathfrak{A}, \mathbf{P}), H^1(D, \mathbb{R}))$, for $n \in \mathbb{N}$, as $N \to \infty$. For this purpose, we first derive an estimate based on an interpolated diffusion equation with diffusion coefficient $T(Z_{N,t}(x))$ where $Z_{N,t}(x) := Z_{k_N + tr_N} = Z_N + tR_N$ with $t \in [0, 1]$. The resulting weak form of (2.15) with approximating diffusion coefficient and homogenized Dirichlet boundary condition with weak solution $u_{0_{N,t}} \in H^1_D(D, \mathbb{R})$ is characterized by

$$b_{N,t}(u_{0_{N,t}}, v) = \ell_{N,t}(v) \quad \forall v \in H^1_D(D, \mathbb{R}),$$

with

$$b_{N,t}(u, v) := \int_D T(Z_{N,t}(x))\nabla u(x) \cdot \nabla v(x)\, \mathrm{d}x, \quad u, v \in H^1_D(D, \mathbb{R}),$$

and

$$\ell_{N,t}(v) := \int_D [f(x)v(x) - T(Z_{N,t}(x))\nabla Eg_D(x) \cdot \nabla v(x)]\, \mathrm{d}x + \int_{\Gamma_N} g_N(x)v(x)\, \mathrm{d}\sigma,$$

where $Eg_D \in H^1(D, \mathbb{R})$ is an extension of g_D. The weak solution to (2.15) with inhomogeneous Dirichlet boundary condition then is given by $u_{N,t} = u_{0_{N,t}} + Eg_D$. For the transformation T we additionally aussume that it is continuously differentiable.

Then, it can be shown that the mapping $t \mapsto u_{0_{N,t}}$ (and thus $t \mapsto u_{N,t} = u_{0_{N,t}} + Eg_D$) is differentiable with respect to the weak topology [18, 75]. We denote the derivative by $\dot{u}_{0_{N,t}}$ and $\dot{u}_{N,t}$, respectively. Moreover, setting

$$\begin{aligned} \dot{b}_{N,t}(u,v) &:= \int_D T'(Z_{N,t}(x))R_N(x)\nabla u(x)\nabla v(x)\mathrm{d}x \\ \dot{\ell}_{N,t}(v) &:= -\int_D T'(Z_{N,t}(x))R_N(x)\nabla Eg_D(x)\cdot\nabla v(x)\,\mathrm{d}x \end{aligned}$$

for $u, v \in H_D^1(D,\mathbb{R})$, one can show that

$$b_{N,t}(\dot{u}_{0_{N,t}}, v) = \dot{\ell}_{N,t}(v) - \dot{b}_{N,t}(u_{0_{N,t}}, v) \qquad \forall v \in H_D^1(D,\mathbb{R}). \tag{5.11}$$

Using (5.11), we can prove that $t \mapsto \dot{u}_{N,t}$ is continuous with respect to the strong H^1-topology, so we conclude that $t \mapsto u_{N,t}$ is also differentiable with respect to the strong topology with derivative $\dot{u}_{N,t}$; see [18].

Lemma 5.2.1. *Let Z be a $|||\cdot|||$-continuous generalized random field and $k : \mathbb{R}^d \to \mathbb{R}$ a smoothing function such that $(Z_k(x))_{x\in D}$ has a.s. continuous paths. Further, let $Z_{N,t}$ and T be given as described above. Then, for the solution $u_{N,t}$ to equation (2.15) with random conductivity $a := T \circ Z_{N,t}$ we have*

$$\begin{aligned} \|\dot{u}_{N,t}\|_{H^1(D,\mathbb{R})} \leqslant C \sup_{x\in D}|T'(Z_{N,t}(x))| \sup_{x\in D}|R_N(x)| \bigg(&\frac{1+\sup_{x\in D}|T(Z_{N,t}(x))|}{(\inf_{x\in D}|T(Z_{N,t}(x))|)^2} \\ &+ \frac{1}{\inf_{x\in D}|T(Z_{N,t}(x))|}\bigg) \big(\|f\|_{L^2(D,\mathbb{R})} + \|g_D\|_{H^{\frac{1}{2}}(\Gamma_D,\mathbb{R})} + \|g_N\|_{H^{-\frac{1}{2}}(\Gamma_N,\mathbb{R})}\big), \end{aligned}$$

where $C = (1 + C_P^2)^2 \max\{1, 2\|E\|, \|\mathrm{tr}\|\}$ with $E : H^{\frac{1}{2}}(\Gamma_D,\mathbb{R}) \to H^1(D,\mathbb{R})$ denoting an extension operator, $\mathrm{tr} : H^1(D,\mathbb{R}) \to H^{\frac{1}{2}}(\Gamma_D,\mathbb{R})$ *the trace operator (see Theorem 2.3.6), and where $C_P > 0$ only depends on D and Γ_D.*

Proof. Since we can write the weak solution to (2.15) as $u_{N,t} = u_{0_{N,t}} + Eg_D$, we have $\|\dot{u}_{N,t}\|_{H^1(D,\mathbb{R})} = \|\dot{u}_{0_{N,t}}\|_{H^1(D,\mathbb{R})}$. Therefore, by setting $\tilde{C} := (1+C_P^2)\max\{1, 2\|E\|, \|\mathrm{tr}\|\}$ by (a generalization of) Poincaré's inequality (cf. (2.17)), (5.11), the definition of $\dot{b}_{N,t}$ and $\dot{\ell}_{N,t}$, as well as inequality (2.16) we have

$$\begin{aligned} &\frac{\inf_{x\in D}|T(Z_{N,t}(x))|}{(1+C_P^2)}\|\dot{u}_{0_{N,t}}\|^2_{H^1(D,\mathbb{R})} \leqslant b_{N,t}(\dot{u}_{0_{N,t}}, \dot{u}_{0_{N,t}}) = |\dot{\ell}_{N,t}(\dot{u}_{0_{N,t}}) - \dot{b}_{N,t}(u_{0_{N,t}}, \dot{u}_{0_{N,t}})| \\ &\leqslant \int_D |T'(Z_{N,t}(x))R_N(x)\nabla(Eg_D + u_{0_{N,t}})(x)\cdot\nabla\dot{u}_{0_{N,t}}(x)|\,\mathrm{d}x \end{aligned}$$

$$
\begin{aligned}
&\leqslant \sup_{x\in D}|T'(Z_{N,t}(x))| \sup_{x\in D}|R_N(x)|\, \|\dot u_{0_{N,t}}\|_{H^1(D,\mathbb{R})}\|u_{0_{N,t}} + Eg_D\|_{H^1(D,\mathbb{R})} \\
&\leqslant \sup_{x\in D}|T'(Z_{N,t}(x))| \sup_{x\in D}|R_N(x)|\, \|\dot u_{0_{N,t}}\|_{H^1(D,\mathbb{R})}\Big(\|E\|\, \|g_D\|_{H^{\frac{1}{2}}(\Gamma_D,\mathbb{R})} \\
&\quad + \tilde C\,\frac{1+\sup_{x\in D}|T(Z_{N,t}(x))|}{\inf_{x\in D}|T(Z_{N,t}(x))|}\big(\|f\|_{L_2(D,\mathbb{R})} + \|g_D\|_{H^{\frac{1}{2}}(\Gamma_D,\mathbb{R})} + \|g_N\|_{H^{-\frac{1}{2}}(\Gamma_N,\mathbb{R})}\big)\Big).
\end{aligned}
$$

The assertion now follows on dividing by $\frac{\inf_{x\in D}|T(Z_{N,t}(x))|}{(1+C_P^2)}\|\dot u_{0,t}\|_{H^1(D,\mathbb{R})}$. □

The next lemma provides an estimate which we use to describe the effects of the perturbation R_N by a term that is exponentially growing in the extreme values of $Z_{N,t}(x)$ and a moment in the perturbation. To derive this estimate, we need an additional assumption on the derivative of the function T.

Assumptions 5.2.2. *For the continuously differentiable function $T:\mathbb{R}\to\mathbb{R}_{\geqslant 0}$ there exist $\rho, B>0, h\in\,]0,1]$ such that for all $z\in\mathbb{R}$ it holds*

$$
B^{-1}\mathrm{e}^{-\rho|z|^h} \leqslant T(z) \leqslant B\mathrm{e}^{\rho|z|^h} \quad \textit{and} \quad |T'(z)| \leqslant B\mathrm{e}^{\rho|z|^h}. \tag{5.12}
$$

Lemma 5.2.3. *Assuming that, in addition to the assumptions of Lemma 5.2.1, assumption 5.2.2 is satisfied for any $\varrho>1$, $\frac{1}{\varrho}+\frac{1}{\varrho'}=1$, and $n\geqslant 1$ we have*

$$
\mathbf{E}\left[\|u-u_N\|^n_{H^1(D,\mathbb{R})}\right] \leqslant \bar C\,\mathbf{E}\left[\sup_{x\in D}|R_N(x)|^{n\varrho'}\right]^{\frac{1}{\varrho'}} \sup_{t\in[0,1]}\mathbf{E}\left[\mathrm{e}^{4\varrho\rho n\sup_{x\in D}|Z_{N,t}(x)|}\right]^{\frac{1}{\varrho}}, \tag{5.13}
$$

where $\bar C = C^n(B^2+B^3)^n(\|f\|_{L^2(D,\mathbb{R})} + \|g_D\|_{H^{\frac{1}{2}}(\Gamma_D,\mathbb{R})} + \|g_N\|_{H^{-\frac{1}{2}}(\Gamma_N,\mathbb{R})})^n$ with C from Lemma 5.2.1.

Proof. Considering the properties of the Bochner integral for Banach space-valued functions and Jensen's inequality for the ordinary integral over $[0,1]$, we get

$$
\begin{aligned}
\mathbf{E}\left[\|u-u_N\|^n_{H^1(D,\mathbb{R})}\right] &= \mathbf{E}\left[\left\|\int_0^1 \dot u_{N,t}\,\mathrm{d}t\right\|^n_{H^1(D,\mathbb{R})}\right] \leqslant \mathbf{E}\left[\left(\int_0^1 \|\dot u_{N,t}\|_{H^1(D,\mathbb{R})}\,\mathrm{d}t\right)^n\right] \\
&\leqslant \mathbf{E}\left[\int_0^1 \|\dot u_{N,t}\|^n_{H^1(D,\mathbb{R})}\,\mathrm{d}t\right] = \int_0^1 \mathbf{E}\left[\|\dot u_{N,t}\|^n_{H^1(D,\mathbb{R})}\right]\mathrm{d}t,
\end{aligned}
$$

where we used that we can interchange the order of integration for non-negative integrands. Applying now Lemma 5.2.1 and Hölder's inequality, we easily obtain (5.13). □

5.2.2 Convergence of the Solution Moments

Lemma 5.2.3 implies that proving the convergence $u_N \to u$ in $L^n((\Omega, \mathfrak{A}, \mathbf{P}), H^1(D, \mathbb{R}))$ can be achieved in two steps:

(i) First, we establish a bound, based on the Laplace transform $\mathbf{E}\left[e^{4\varrho\rho n \sup_{x\in D}|Z_{N,t}(x)|}\right]$, of the extreme values of $Z_{N,t}$, which is uniform in N and t.

(ii) Then, we prove that $\mathbf{E}\left[\sup_{x\in D}|R_N(x)|^{n\varrho'}\right] \to 0$ as $N \to \infty$.

In this subsection, we first identify suitable conditions on k and k_N which imply (i) and (ii). Afterwards, we apply these conditions to the natural generalization of the Karhunen-Loève expansion to smoothed Lévy fields. The first lemma of this section is a uniform version of Talagrand's Lemma 5.1.2.

Lemma 5.2.4. *Let G be a generalized centered Gaussian field, i.e., a $|||\cdot|||$-continuous Lévy field with characteristic triplet $(0, \sigma^2, 0)$, with $\sigma > 0$. Further, let D be an open and bounded subset of $\mathbb{R}^d$ and $k_\iota : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$, $\iota \in I$ be a family of smoothing functions such that for another smoothing function $k : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ the following hold:*

(i) For all $\iota \in I : \sup_{x\in D}\|k_\iota(x,\cdot)\|_{L^2(\mathbb{R}^d,\mathbb{R})} \leqslant \sup_{x\in D}\|k(x,\cdot)\|_{L^2(\mathbb{R}^d,\mathbb{R})}$.

(ii) The canonical distances $d_\iota, \iota \in I$ and d_c of the centered Gaussian random fields $(G_{k_\iota}(x))_{x\in D}, \iota \in I$, and $(G_k(x))_{x\in D}$, respectively, satisfy $d_\iota \leqslant d_c, \iota \in I$, and d_c satisfies the covering property in Talagrand's Lemma 5.1.2.

(iii) The centered Gaussian fields $(G_{k_\iota}(x))_{x\in D}, \iota \in I$, and $(G_k(x))_{x\in D}$ all have almost surely continuous paths.

Then, for $\bar{\sigma}_\iota^2 := \sigma^2 \sup_{x\in D}\|k_\iota(x,\cdot)\|_{L^2(\mathbb{R}^d,\mathbb{R})}$ and $\bar{\sigma}^2 := \sigma^2 \sup_{x\in D}\|k(x,\cdot)\|_{L^2(\mathbb{R}^d,\mathbb{R})}$, there are constants $A > \bar{\sigma}^2$, $K, v, \gamma > 0$, and $\varepsilon_0 \in]0, \bar{\sigma}[$ such that

$$\forall\, \iota \in I, g > \bar{\sigma}_\iota(1+\sqrt{v})/\varepsilon_0 : \ \mathbf{P}(\sup_{x\in D}|G_{k_\iota}(x)| \geqslant g) \leqslant \left(\frac{KAg}{\sqrt{v}\bar{\sigma}_\iota^2}\right)^v \exp\left(-\frac{g^2}{2\bar{\sigma}_\iota^2}\right) \quad (5.14)$$
$$\leqslant \gamma\left(\frac{KAg}{\sqrt{v}\bar{\sigma}^2}\right)^v \exp\left(-\frac{g^2}{2\bar{\sigma}^2}\right).$$

Proof. As k is a smoothing function, it follows that $\bar{\sigma}^2 > 0$ and by assumption (i), with abbreviating $G_\iota(x) := G_{k_\iota}(x), \iota \in I, x \in D$, we have for all $\iota \in I$

$$\begin{aligned}\bar{\sigma}_\iota^2 &:= \sup_{x\in D} \mathbf{E}\left[G_\iota(x)^2\right] = \sup_{x\in D} \sigma^2 \|k_\iota(x,\cdot)\|_{L^2(\mathbb{R}^d,\mathbb{R})} \leqslant \sup_{x\in D} \sigma^2 \|k(x,\cdot)\|_{L^2(\mathbb{R}^d,\mathbb{R})} \\ &= \sup_{x\in D} \mathbf{E}\left[G_k(x)^2\right] = \bar{\sigma}^2.\end{aligned}$$

By assumption (ii), the d_c-ball centered at $x \in \mathbb{R}^d$ with d_c-radius $\varepsilon > 0$ is contained in the d_ι-ball centered at x with d_ι-radius ε. Therefore, using the notation in Talagrand's Lemma 5.1.2, we have $N(D, d_\iota, \varepsilon) \leqslant N(D, d_c, \varepsilon)$. Again using assumption (ii) there are thus $A > \bar{\sigma}^2$, $v > 0$, $\varepsilon_0 \in]0, \bar{\sigma}[$ such that

$$\forall\, \iota \in I, \varepsilon \in]0, \varepsilon_0[:\; N(D, d_\iota, \varepsilon) \leqslant \left(\frac{A}{\varepsilon}\right)^v.$$

Since $f : [1, \infty[\to \mathbb{R},\ f(x) := x^v \exp(-\bar{\sigma}^2 \frac{(1+\sqrt{v})^2}{2\varepsilon_0}(x-1))$ is bounded from above, the assertion follows by setting $\gamma := \sup_{x\geqslant 1} f(x)$. □

The first step of the two-steps apprach outlined above involves a uniform estimate of the Laplace transform of the extreme values of $Z_{N,t}$ under suitable assumptions on the smoothing kernel k. For the sake of convenience we introduce the following abbrevation for the bivariate kernel function $k = k(x, y)$

$$\tilde{k}(y) := \sup_{x\in D} |k(x,y)|, \quad y \in \mathbb{R}^d,$$

and define the following.

Definition 5.2.5. *A smoothing function $k : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ has an orthogonal approximation sequence $k = k_N + r_N$, $N \in \mathbb{N}$, if k_N and r_N are smoothing functions with*

(i) $\int_{\mathbb{R}^d} k_N(x_1, y) r_N(x_2, y)\, \mathrm{d}y = 0$ *for all* $x_1, x_2 \in \mathbb{R}^d$*;*

(ii) $\max\{\|\tilde{r}_N\|_{L^1(\mathbb{R}^d,\mathbb{R})}, \kappa_{r,N}\} \to 0$ *as* $N \to \infty$*, where* $\kappa_{r,N} := \sup_{x\in D, y\in\mathbb{R}^d} |r_N(x,y)|$ *and* $\tilde{r}_N$ *is defined as* $\tilde{k}$ *is above.*

Lemma 5.2.6. *Consider a Lévy field Z that satisfies Assumption 5.1.4 and let $k : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ be a smoothing function such that $\tilde{k} \in L^1(\mathbb{R}^d, \mathbb{R}) \cap L^\infty(\mathbb{R}^d, \mathbb{R})$ and such that the canonical distance d_c of $(G_k(x))_{x\in D}$ satisfies the covering property in*

Talagrand's Lemma 5.1.2, where G is the centered Gaussian part of Z, i.e., the $|||\cdot|||$-continuous Lévy field with characteristic triplet $(0,\sigma^2,0)$. Furthermore, let $k = k_N + r_N$, $N \in \mathbb{N}$ be an orthogonal approximation sequence for which the centered Gaussian fields $(G_{k_N})_{x\in D}$ and $(G_{r_N}(x))_{x\in D}$, $N \in \mathbb{N}$, all have a.s. continuous paths and for which $\tilde{k}_N \in L^1(\mathbb{R}^d,\mathbb{R}) \cap L^\infty(\mathbb{R}^d,\mathbb{R})$, with $N \in \mathbb{N}$. Additionally, let $\varrho > 1$, $\rho > 0$, and $n \in]0, \frac{\beta}{4\varrho\kappa\rho}[$, where $\kappa := \|\tilde{k}\|_{L^\infty(\mathbb{R}^d,\mathbb{R})}$. Then, there is $M \in \mathbb{N}$ such that

$$\sup_{N \geqslant M, t\in[0,1]} \mathbf{E}\left[\mathrm{e}^{4\varrho\rho n \sup_{x\in D}|Z_{N,t}(x)|}\right] < \infty.$$

If $k_N(x,\cdot)$ and $r_N(x,\cdot)$, $x \in \mathbb{R}^d$, have disjoint supports for every $N \in \mathbb{N}$, one can choose $M = 1$.

Proof. We define $b' := \int_{\{0<s\leqslant 1\}} s\nu(\mathrm{d}s)$ and denote the Lévy characteristic associated with the characteristic triplet $(b',0,\nu)$ with P. Then, P is $|||\cdot|||$-continuous and for an arbitrary smoothing function $l : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ the smooth field Z_l satisfies

$$Z_l(x) = (b-b')\int_{\mathbb{R}^d} l(x,y)\mathrm{d}y + G_l(x) + P_l(x), \quad x \in \mathbb{R}^d.$$

With $G_{N,t}(x) := G_{k_N+tr_N}(x)$, $P_{N,t}(x) := P_{k_N+tr_N}(x), N \in \mathbb{N}, t \in [0,1]$ it follows for arbitrary $B > 2|b-b'|\,\|\tilde{k}\|_{L^1(\mathbb{R}^d,\mathbb{R})}$ and each $N \in \mathbb{N}, t \in [0,1]$ and every $\lambda \in]0,1[$:

$$\begin{aligned}
\mathbf{E}\left[\mathrm{e}^{4\varrho\rho n \sup_{x\in D}|Z_{N,t}(x)|}\right] &\leqslant \sum_{j=0}^{\infty} \mathrm{e}^{4\varrho\rho n(j+1)B}\mathbf{P}(\sup_{x\in D}|Z_{N,t}(x)| \geqslant jB) \\
&\leqslant \mathrm{e}^{4\varrho\rho nB} + \sum_{j=1}^{\infty} \mathrm{e}^{4\varrho\rho n(j+1)B}\mathbf{P}\Big(\sup_{x\in D}|G_{N,t}(x)| + \sup_{x\in D}|P_{N,t}(x)| \geqslant (j-\frac{1}{2})B\Big) \\
&\leqslant \mathrm{e}^{4\varrho\rho nB}\left(1 + \sum_{j=1}^{\infty} \mathrm{e}^{4\varrho\rho njB}\left[\mathbf{P}(\sup_{x\in D}|G_{N,t}(x)| \geqslant (j-\frac{1}{2})\lambda B)\right.\right. \\
&\left.\left.+\mathbf{P}(\sup_{x\in D}|P_{N,t}(x)| \geqslant (j-\frac{1}{2})(1-\lambda)B)\right]\right).
\end{aligned} \tag{5.15}$$

In order to apply Lemma 5.2.4, we next verify its assumptions for the family of smoothing functions $k_N + tr_N$, $N \in \mathbb{N}$, $t \in [0,1]$ and the smoothing function k. By using property (i) of an orthogonal approximation sequence, we set

$$\bar{\sigma}^2_{N,t} := \sup_{x\in D}\mathbf{E}\left[G_{N,t}(x)^2\right] = \sup_{x\in D}\sigma^2\left(\int_{\mathbb{R}^d}|k_N(x,y)|^2\,\mathrm{d}y + t^2\int_{\mathbb{R}^d}|r_N(x,y)|^2\,\mathrm{d}y\right)$$

$$\leqslant \sup_{x\in D} \sigma^2 \left(\int_{\mathbb{R}^d} |k_N(x,y)|^2 \,\mathrm{d}y + \int_{\mathbb{R}^d} |r_N(x,y)|^2 \,\mathrm{d}y\right) = \sup_{x\in D} \mathbf{E}\left[G_k(x)^2\right] =: \bar{\sigma}^2,$$

which implies assumption (i) of Lemma 5.2.4. For the centered Gaussian random fields $(G_k(x))_{x\in D}$ and $(G_{N,t}(x))_{x\in D}$ we denote with d_c and $d_{N,t}$ the canonical distances, respectively. For arbitrary $N \in \mathbb{N}$, $t \in [0,1]$ and each $x_1, x_2 \in \mathbb{R}^d$, we have by using property (i) of an orthogonal approximation sequence

$$\begin{aligned}
d_{N,t}(x_1,x_2) &= \left(\mathbf{E}\left[(G_{N,t}(x_1) - G_{N,t}(x_2))^2\right]\right)^{\frac{1}{2}} \\
&= \sigma \left(\int_{\mathbb{R}^d} (k_{N,t}(x_1,y) - k_{N,t}(x_2,y))^2 \,\mathrm{d}y\right)^{\frac{1}{2}} \\
&= \sigma \left(\int_{\mathbb{R}^d} (k_N(x_1,y) - k_N(x_2,y))^2 \,\mathrm{d}y + t^2 \int_{\mathbb{R}^d} (r_N(x_1,y) - r_N(x_2,y))^2 \,\mathrm{d}y\right)^{\frac{1}{2}} \\
&\leqslant \sigma \left(\int_{\mathbb{R}^d} (k_N(x_1,y) - k_N(x_2,y))^2 \,\mathrm{d}y + \int_{\mathbb{R}^d} (r_N(x_1,y) - r_N(x_2,y))^2 \,\mathrm{d}y\right)^{\frac{1}{2}} \\
&= \sigma \left(\int_{\mathbb{R}^d} (k(x_1,y) - k(x_2,y))^2 \,\mathrm{d}y\right)^{\frac{1}{2}} = d_c(x_1,x_2),
\end{aligned}$$

which implies assumption (ii) of Lemma 5.2.4. Assumption (iii) of Lemma 5.2.4 is given by the assumptions on k_N and r_N. Therefore, it follows from Lemma 5.2.4 that there exist constants $A > \bar{\sigma}^2$, $K, v, \gamma > 0$ and $\varepsilon_0 \in]0, \bar{\sigma}[$ such that for all $N \in \mathbb{N}$, $t \in [0,1]$ and $\lambda \in]0,1[$

$$\mathbf{P}\Big(\sup_{x\in D} |G_{k_\iota}(x)| \geqslant (j - \frac{1}{2})\lambda B\Big) \leqslant \gamma \left(\frac{KA(j-\frac{1}{2})\lambda B}{\sqrt{v}\bar{\sigma}^2}\right)^v \exp\left(-\frac{((j-\frac{1}{2})\lambda B)^2}{2\bar{\sigma}^2}\right), \tag{5.16}$$

whenever $j > \frac{1}{2} + \frac{\bar{\sigma}(1+\sqrt{v})}{\lambda B \varepsilon_0}$. Next, due to

$$\forall\, N \in \mathbb{N}, t \in [0,1] : \|\tilde{k}_{N,t}\|_{L^1(\mathbb{R}^d,\mathbb{R})} \leqslant \|\tilde{k}\|_{L^1(\mathbb{R}^d,\mathbb{R})} + \|\tilde{r}_N\|_{L^1(\mathbb{R}^d,\mathbb{R})}$$

and property (ii) of an orthogonal approximation sequence, we have

$$\kappa_1 := \sup_{N\in\mathbb{N}, t\in[0,1]} \|\tilde{k}_{N,t}\|_{L^1(\mathbb{R}^d,\mathbb{R})} < \infty.$$

Further, property (ii) of an orthogonal approximation gives in addition, since for all

$N \in \mathbb{N}$, $t \in [0,1]$

$$\kappa_{N,t} = \sup_{x\in D} \|k_N(x,\cdot) + tr_N(x,\cdot)\|_{L^\infty(\mathbb{R}^d,\mathbb{R})} \leqslant \kappa + (1-t)\kappa_{r,N} \leqslant \kappa + \kappa_{r,N}, \tag{5.17}$$

that for all $\varepsilon > 0$ there exists M_ε such that

$$\forall\, N \in \mathbb{N}, N \geqslant M_\varepsilon, t \in [0,1] :\ \kappa_{N,t} \leqslant \kappa + \varepsilon.$$

Moreover, if $k_N(x,\cdot)$ and $r_N(x,\cdot)$ have disjoint supports for every $N \in \mathbb{N}, x \in \mathbb{R}^d$, it follows that in (5.17) we even have $\kappa_{N,t} \leqslant \kappa$. Now, using Lemma 5.1.5 for some fixed $\varepsilon > 0$ to the family of smoothing functions $(k_N + tr_N)_{N\geqslant M_\varepsilon, t\in[0,1]}$ (respectively to $(k_N + tr_N)_{N\in\mathbb{N}, t\in[0,1]}$) gives that for every $\tau \in]0,1[$ there is a constant C_τ such that

$$\mathbf{P}\Big(\sup_{x\in D} |P_{N,t}(x)| \geqslant (j - \frac{1}{2})(1-\lambda)B\Big) \leqslant C_\tau\, \mathrm{e}^{-\frac{\beta}{\kappa+\varepsilon}(j-\frac{1}{2})(1-\lambda)\tau B} \tag{5.18}$$

for all $j \in \mathbb{N}, \lambda \in]0,1[$, whenever $N \geqslant M_\varepsilon$ (respectively $N \in \mathbb{N}$), $t \in [0,1]$ and where C_τ is given by

$$C_\tau := \frac{\beta\kappa_1}{\kappa}\Bigg(\mathrm{e}^\beta \int_{\{0<s\leqslant 1\}} |s|\nu_+(\mathrm{d}s) + \frac{1}{(1-\tau)\beta\mathrm{e}} \int_{\{s>1\}} \mathrm{e}^{\beta s}\nu_+(\mathrm{d}s)\Bigg).$$

Since $n \in (0, \frac{\beta}{4\varrho\kappa\rho})$ there are $\lambda_0 \in (0,1)$ and $\varepsilon > 0$ such that $n < \beta(1-\lambda_0)/(\kappa+\varepsilon)4\varrho\rho$. Then, with $B > 2|b - b'|\|\tilde{k}\|_{L^1(\mathbb{R}^d,\mathbb{R})}$ large enough so that $2\bar{\sigma}(1+\sqrt{v})/\varepsilon_0 B\lambda_0 < \frac{1}{2}$ it follows from (5.15), (5.16), and (5.18) that for every $\tau \in]0,1[$ and for all $N \geqslant M_\varepsilon$ (respectively $N \in \mathbb{N}$), $t \in [0,1]$

$$\mathbf{E}\left[\mathrm{e}^{4\varrho\rho n \sup_{x\in D} |Z_{N,t}(x)|}\right] \leqslant \mathrm{e}^{4\varrho\rho n B}\left(1 + \sum_{j=1}^\infty \mathrm{e}^{4\varrho\rho n j B}\left[C_\tau \mathrm{e}^{-\frac{\beta}{\kappa+\varepsilon}(j-\frac{1}{2})(1-\lambda_0)\tau B}\right.\right.$$
$$\left.\left. + \gamma\left(\frac{KA(j-\frac{1}{2})\lambda_0 B}{\sqrt{v}\bar{\sigma}^2}\right)^v \exp\left(-\frac{((j-\frac{1}{2})\lambda_0 B)^2}{2\bar{\sigma}^2}\right)\right]\right).$$

With the same arguments we employed in the proof of Theorem 5.1.6, the series converges as $4\varrho\rho n < \beta(1-\lambda_0)/(\kappa+\varepsilon)$. Thus, the assertion follows. □

Lemma 5.2.7. *Consider Z, k, k_N, and $r_N, N \in \mathbb{N}$ as given in Lemma 5.2.6 and let further $\varrho' > 1$ and $n \geqslant 1/\varrho'$. Then, for every $\delta \in]0,1[$ there exists a constant $C > 0$*

depending only on δ, Z, k, and $n\varrho'$ such that

$$\forall\, N \in \mathbb{N}:\ \|\sup_{x\in D}|R_N(x)|^{\varrho'}\|_{L^n((\Omega,\mathfrak{A},\mathbf{P}),\mathbb{R})} \leqslant C\left(\max\{\alpha_N, \alpha_N^{1-\delta}\}\right)^{\varrho'}$$

where $\alpha_N := \max\{\|\tilde{r}_N\|_{L^1(\mathbb{R}^d,\mathbb{R})}, \kappa_{r,N}\}$. *In particular,* $\lim_{N\to\infty} \mathbf{E}\left[\sup_{x\in D}|R_N(x)|^{n\rho'}\right] = 0$.

Proof. Using the notation of the proof to Lemma 5.2.6 we have

$$R_N(x) = (b-b')\int_{\mathbb{R}^d} r_N(x,y)\,\mathrm{d}y + G_{r_N}(x) + P_{r_N}(x),$$

and with Jensen's inequality it follows

$$\begin{aligned}\mathbf{E}\left[\sup_{x\in D}|R_N(x)|^{n\varrho'}\right] \leqslant 3^{n\varrho'-1}\Bigg(&|b-b'|^{n\varrho'}\|\tilde{r}_N\|^{n\varrho'}_{L^1(\mathbb{R}^d,\mathbb{R})} + \mathbf{E}\left[\sup_{x\in D}|G_{r_N}(x)|^{n\varrho'}\right] \\ &+ \mathbf{E}\left[\sup_{x\in D}|P_{r_N}(x)|^{n\varrho'}\right]\Bigg).\end{aligned} \tag{5.19}$$

The Gaussian part can be estimated as above. From property (i) of an orthogonal approximation sequence we get

$$\forall\, N\in\mathbb{N}, x\in D:\ \|r_N(x,\cdot)\|_{L^2(\mathbb{R}^d,\mathbb{R})} \leqslant \|k(x,\cdot)\|_{L^2(\mathbb{R}^d,\mathbb{R})} \quad \text{and} \quad d_N \leqslant d_c,$$

where d_N and d_c denote the canonical distances associated with $(G_{r_N}(x))_{x\in D}$ and $(G_k(x))_{x\in D}$, respectively. Using the first inequality of (5.14), there are constants

$$A > \bar{\sigma}^2 := \sigma^2\sup_{x\in D}\|k(x,\cdot)\|^2_{L^2(\mathbb{R}^d,\mathbb{R})} \geqslant \sigma^2\sup_{x\in D}\|r_N(x,\cdot)\|^2_{L^2(\mathbb{R}^d,\mathbb{R})} =: \bar{\sigma}^2_{r,N},$$

K, $v>0$ and $\varepsilon_0\in\,]0,\bar{\sigma}[$ such that for every $N\in\mathbb{N}$, $g\geqslant\bar{\sigma}_{r,N}(1+\sqrt{v})/\varepsilon_0$ we have

$$\mathbf{P}(\sup_{x\in D}|G_{r_N}(x)|\geqslant g) \leqslant \left(\frac{KAg}{\sqrt{v}\bar{\sigma}^2_{r,N}}\right)^v \exp\left(-\frac{g^2}{2\bar{\sigma}^2_{r,N}}\right).$$

Since for $j\in\mathbb{N}$ with $j>\bar{\sigma}^\delta(1+\sqrt{v})/\varepsilon_0$ it holds $j\bar{\sigma}^{1-\delta}_{r,N} > \sigma_{r,N}(1+\sqrt{v})/\varepsilon_0$, we have with $M := \max\{\lceil\bar{\sigma}^\delta(1+\sqrt{v})/\varepsilon_0\rceil, \lceil\bar{\sigma}^\delta\sqrt{v\frac{1+\delta}{\delta}}\rceil\}$ that

$$\mathbf{E}\left[\sup_{x\in D}|G_{r_N}(x)|^{n\varrho'}\right] \leqslant \sum_{j=0}^{\infty}\left((j+1)\bar{\sigma}^{1-\delta}_{r,N}\right)^{n\varrho'}\mathbf{P}(\sup_{x\in D}|G_{r_N}(x)|\geqslant j\bar{\sigma}^{1-\delta}_{r,N})$$

$$
\begin{aligned}
&\leqslant \sum_{j=0}^{M} \left((j+1)\bar{\sigma}_{r,N}^{1-\delta}\right)^{n\varrho'} \\
&\quad + \sum_{j=M+1}^{\infty} \left((j+1)\bar{\sigma}_{r,N}^{1-\delta}\right)^{n\varrho'} \left(\frac{KAj}{\sqrt{v}\bar{\sigma}_{r,N}^{1+\delta}}\right)^{v} \exp(-\frac{j^2}{2\bar{\sigma}_{r,N}^{2\delta}}) \\
&\leqslant \bar{\sigma}_{r,N}^{(1-\delta)n\varrho'} \left(\sum_{j=0}^{M}(j+1)^{n\varrho'}\right. \\
&\quad \left. +2^{n\varrho'}\left(\frac{KA}{\sqrt{v}}\right)^{v} \sum_{j=M+1}^{\infty} j^{n\varrho'} \left(\frac{j}{\bar{\sigma}_{r,N}^{1+\delta}}\right)^{v} \exp\left(-\frac{j^2}{2\bar{\sigma}_{r,N}}^{2\delta}\right)\right) \\
&\leqslant \bar{\sigma}_{r,N}^{(1-\delta)n\varrho'} \left(\sum_{j=0}^{M}(j+1)^{n\varrho'}\right. \\
&\quad \left. +2^{n\varrho'}\left(\frac{KA}{\sqrt{v}}\right)^{v} \sum_{j=M+1}^{\infty} j^{n\varrho'} \left(\frac{j}{\bar{\sigma}^{1+\delta}}\right)^{v} \exp\left(-\frac{j^2}{2\bar{\sigma}^{2\delta}}\right)\right),
\end{aligned}
$$

where in the last step we have used that for every $j > \lceil \bar{\sigma}^{\delta}\sqrt{v\frac{1+\delta}{\delta}}\rceil$ the functions $f_j : [0,\infty[\to \mathbb{R}$, $f_j(x) := (jx^{-(1+\delta)})^{v}\exp(-j^2/2x^{2\delta})$ are strictly increasing on $[0,\bar{\sigma}]$ and that $\bar{\sigma}_{r,N} \in [0,\bar{\sigma}]$ for all $N \in \mathbb{N}$. As the above series converges, denoting with C_1 the expression in brackets on the right-hand side of the above inequality and taking into account that

$$
\forall\, N \in \mathbb{N} :\ \bar{\sigma}_{r,N}^2 = \sigma^2 \sup_{x\in D}\int_{\mathbb{R}^d} |r_N(x,y)|^2\,\mathrm{d}y \leqslant \sigma^2\kappa_{r,N}\|\tilde{r}_N\|_{L^1(\mathbb{R}^d,\mathbb{R})} \leqslant \sigma^2\alpha_N^2,
$$

we derive

$$
\forall\, N \in \mathbb{N} :\ \mathbf{E}\left[\sup_{x\in D}|G_{r_N}(x)|^{n\varrho'}\right] \leqslant \sigma^{1-\delta}C_1\alpha_N^{(1-\delta)n\varrho'}. \tag{5.20}
$$

Now, it remains to estimate the Poisson part in (5.19). By Hölder's inequality, $|P|_{|r_N|}(x) \leqslant |P|(\tilde{r}_N)$ (cf. (3.8)), Lemma 3.3.7, and $\|\tilde{r}_N\|_{L^1(\mathbb{R}^d,\mathbb{R})}^k\kappa_{r,N}^{l-k} \leqslant \alpha_N^l$ for all $0 \leqslant k \leqslant l$, we have for every $N \in \mathbb{N}$

$$
\begin{aligned}
\mathbf{E}\left[\sup_{x\in D}|P_{r_N}(x)|^{n\varrho'}\right] &\leqslant \left(\mathbf{E}\left[\sup_{x\in D}|P_{r_N}(x)|^{\lceil n\varrho'\rceil}\right]\right)^{\frac{n\varrho'}{\lceil n\varrho'\rceil}} \qquad (5.21)\\
\leqslant \left(\mathbf{E}\left[|P|(\tilde{r}_N(x)|^{\lceil n\varrho'\rceil}\right]\right)^{\frac{n\varrho'}{\lceil n\varrho'\rceil}} &= \left(\sum_{\substack{I\in\mathscr{P}(\lceil n\varrho'\rceil)\\ I=\{I_1,\dots,I_k\}}}\prod_{\ell=1}^{k} c_{|I_\ell|}^{+}\int_{\mathbb{R}^d}\tilde{r}_N^{|I_\ell|}\,\mathrm{d}x\right)^{\frac{n\varrho'}{\lceil n\varrho'\rceil}}
\end{aligned}
$$

$$\leqslant \Bigg(\sum_{\substack{I\in\mathscr{P}^{(\lceil n\varrho'\rceil)}\\ I=\{I_1,\ldots,I_k\}}} \prod_{\ell=1}^{k} c^+_{|I_\ell|} \|\tilde{r}_N\|_{L^1(\mathbb{R}^d,\mathbb{R})} \kappa_{r,N}^{|I_l|-1} \Bigg)^{\frac{n\varrho'}{\lceil n\varrho'\rceil}}$$

$$= \Bigg(\sum_{\substack{I\in\mathscr{P}^{(\lceil n\varrho'\rceil)}\\ I=\{I_1,\ldots,I_k\}}} \|\tilde{r}_N\|^k_{L^1(\mathbb{R}^d,\mathbb{R})} \kappa_{r,N}^{\lceil n\varrho'\rceil - k} \prod_{\ell=1}^{k} c^+_{|I_\ell|} \Bigg)^{\frac{n\varrho'}{\lceil n\varrho'\rceil}} \leqslant \Bigg(\sum_{\substack{I\in\mathscr{P}^{(\lceil n\varrho'\rceil)}\\ I=\{I_1,\ldots,I_k\}}} \prod_{\ell=1}^{k} c^+_{|I_\ell|} \Bigg)^{\frac{n\varrho'}{\lceil n\varrho'\rceil}} \alpha_N^{n\varrho'},$$

where $\mathscr{P}^{(\lceil n\varrho'\rceil)}$ denotes the collection of all partitions on $\{1,\ldots,\lceil n\varrho'\rceil\}$ into non-intersecting, non-empty sets $I_1,\ldots,I_k, 1 \leqslant k \leqslant \lceil n\varrho'\rceil$, and $c^+_{|I_l|}$ are suitable non-negative numbers. Further, note that the constants $c^+_{|I_\ell|}$ are given with respect to the modified Lévy measure ν_+ associated with $|P|$ instead of ν associated with P. We set the constant C_2 to be equal with the factor in front of $\alpha_N^{n\varrho'}$ from the previous inequality. Then, the previous inequality gives

$$\forall\, N \in \mathbb{N} : \mathbf{E}\left[\sup_{x\in D} |P_{r_N}(x)|^{n\varrho'}\right] \leqslant C_2 \alpha_N^{n\varrho'}. \tag{5.22}$$

At last, by combining (5.19), (5.20) and (5.22) we obtain

$$\forall\, N \in \mathbb{N} : \mathbf{E}\left[\sup_{x\in D} |R_N(x)|^{n\varrho'}\right] \leqslant 3^{n\varrho'-1}(|b-b'| + \sigma^\delta C_1 + C_2)\left(\max\{\alpha_N, \alpha_N^{1-\delta}\}\right)^{n\varrho'}$$

wich proves the assertion. □

Now, combining Lemma 5.2.3, 5.2.6 and 5.2.7 yields the following convergence result.

Theorem 5.2.8. *Consider a Lévy field Z that satisfies Assumption 5.1.4 and a smoothing function $k : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ such that $\tilde{k} \in L^1(\mathbb{R}^d,\mathbb{R}) \cap L^\infty(\mathbb{R}^d,\mathbb{R})$ and such that the canonical distance d_c of $(G_k(x))_{x\in D}$ satisfies the covering property in Talagrand's Lemma 5.1.2, where G is the centered part of Z. Furthermore, let $k = k_N + r_N$, $N \in \mathbb{N}$, be an orthogonal approximation sequence for which the centered Gaussian fields $(G_{k_N}(x))_{x\in D}$ and $(G_{r_N}(x))_{x\in D}$, all have a.s. continuous paths and for which $\tilde{k}_N \in L^1(\mathbb{R}^d,\mathbb{R}) \cap L^\infty(\mathbb{R}^d,\mathbb{R})$ for all $N \in \mathbb{N}$.*

Let u and u_N, with $N \in \mathbb{N}$, be the solution to (2.15) with random conductivity $T \circ Z_k$ and $T \circ Z_{k_N}$, respectively, where T satisfies Assumption 5.2.2. Assuming that with $\kappa := \|\tilde{k}\|_{L^\infty(\mathbb{R}^d,\mathbb{R})}$ we have $\beta > 4\kappa\rho$, for all $n \in [1, \frac{\beta}{4\kappa\rho}[$, $\varrho \in]1, \frac{\beta}{4\kappa\rho n}[$, and $\delta \in]0,1[$

there exist constants $C' > 0$ and $M \in \mathbb{N}$ such that for all $N \geqslant M$ we have

$$\|u - u_N\|_{L^n((\Omega,\mathfrak{A},\mathbf{P}),H^1(D,\mathbb{R}))} \leqslant C' \max\{\alpha_N, \alpha_N^{1-\delta}\},$$

where $\alpha_N = \max\{\|\tilde{r}_N\|_{L^1(\mathbb{R}^d,\mathbb{R})}, \kappa_{r,N}\}$. If $k_N(x,\cdot)$ and $r_N(x,\cdot)$ have disjoint support for all $x \in \mathbb{R}^d$ and all $N \in \mathbb{N}$, one can choose $M = 1$.

In particular, $(u_N)_{N\in\mathbb{N}}$ converges to u in $L^n((\Omega,\mathfrak{A},\mathbf{P}),H^1(D,\mathbb{R}))$. The constant C' depends only on B, Z, k, $\frac{n\varrho}{\varrho-1}$, $\|f\|_{L^2(D,\mathbb{R})}$, $\|g_D\|_{H^{\frac{1}{2}}(\Gamma_D,\mathbb{R})}$, $\|g_N\|_{H^{\frac{1}{2}}(D,\mathbb{R})}$, and C, the constant from Lemma 5.2.1.

5.2.3 Series Expansion of Lévy Coefficients

We approximate the smoothed Lévy coefficient and therefore the solution to (2.15) in two steps. With our approximation scheme we have to consider that only x is restricted to the domain D, whereas y transport the effect of noise source terms from locations $y \notin D$ into D. Let $(\Lambda_N)_{N\in\mathbb{N}}$ denote a compact exhaustion of $\mathbb{R}^d$, i.e., $(\Lambda_N)_{N\in\mathbb{N}}$ is a sequence of compact subsets of $\mathbb{R}^d$ with $\Lambda_N \subseteq \operatorname{int}(\Lambda_{N+1})$, $N \in \mathbb{N}$, and $\cup_N \Lambda_N = \mathbb{R}^d$. In the first step, we restrict the second argument of the kernel $k(x,y)$ to a sufficiently large domain Λ_N from this sequence. Since now both x and y are restricted to sufficiently large compact domains, we can apply Mercer's expansion onto the restricted version of the smoothing function k in the second step of our two-steps approximation approach. This yields a finite-dimensional approximation of Z_k and is a natural generalization of the Karhunen-Loève expansion for Lévy fields.

The first step, induces for a $|||\cdot|||$-continuous Lévy field Z directly a $|||\cdot|||$-continuous generalized random field $Z^N(f) := Z(\mathbb{1}_{\Lambda_N} f)$, $N \in \mathbb{N}$, which, however, is no longer stationary. From Theorem 3.2.10 it follows for Matérn kernel, with $k_{\alpha,m}$, $m > 0$, and $\alpha > d + \max\{0, \frac{3d-12}{8}\}$, that the smoothed fields $(Z_k^N(x))_{x\in D}$, $N \in \mathbb{N}$, with smoothing function $k(x,y) = k_{\alpha,m}(x-y)$, have a.s. continuous paths. Furthermore, in the proof of Theorem 5.1.6 we have seen that the canonical distance d_c associated with the Gaussian field $(G_k(x))_{x\in D}$ (where G denotes the centered Gaussian part of Z) fulfills the covering property in Talagrand's Lemma 5.1.2 and, by Lemma 5.1.3 (ii), we have $\tilde{k} \in L^1(\mathbb{R}^d,\mathbb{R}) \cap L^\infty(\mathbb{R}^d,\mathbb{R})$ with $\lim_{|y|\to\infty} \tilde{k}(y) = 0$. Therefore, the assumptions of Theorem 5.2.8 are directly satisfied for smoothing functions given by Matérn kernels.

Corollary 5.2.9. *Let Z, T and u be given as in Theorem 5.2.8 and let $k_{\alpha,m}$ be a Matérn kernel with $\alpha > d$. For a compact exhaustion $(\Lambda_N)_{N\in\mathbb{N}}$ of $\mathbb{R}^d$ with $D \subseteq \Lambda_1$, we set $k_N(x,y) := k_{\alpha,m}(x,y)\mathbb{1}_{\Lambda_N}(y)$ and denote by u_N the solution to diffusion equation (2.15) with random conductivity $T \circ Z_{k_N}$.*

Then, for all $n \in [1, \frac{\beta}{4\kappa\rho}[$, $\varrho \in]1, \frac{\beta}{4\kappa\rho n}[$, and $0 < m' < m$, there is a constant $C > 0$ such that for all $N \in \mathbb{N}$ we have

$$\|u - u_N\|_{L^n((\Omega,\mathfrak{A},\mathbf{P}),H^1(D,\mathbb{R}))} \leqslant C\mathrm{e}^{-m'\mathrm{d}_e(D,\Lambda_N^c)}, \tag{5.23}$$

where d_e denotes the Euclidean distance between D and Λ_N^c.

Proof. We first note that $k = k_N + r_N$ is an orthogonal approximation sequence in the sense of Definition 5.2.5, since, first, as the y-domain of k_N and r_N are disjoint by definition, it follows that condition (i) is satisfied, and, second, the decay rate $k_{\alpha,m}(x,y) \leqslant C\mathrm{e}^{-m|x-y|}$ for $|x-y| \to \infty$ from Lemma 5.1.3 (ii) implies for $0 < m' < m$

$$\|\tilde{r}_N\|_{L^1(\mathbb{R}^d,\mathbb{R})} \leqslant C \int_{\Lambda_N^c} \sup_{x\in D} \mathrm{e}^{-m|x-y|}\,\mathrm{d}y = C\mathrm{e}^{-m\mathrm{d}_e(D,\Lambda_N^c)} \int_{\Lambda_N^c} \sup_{x\in D} \mathrm{e}^{-(m-m')|x-y|}\,\mathrm{d}y < \infty,$$

where the last integral can be estimated by a constant as the integration area gets smaller for $N \to \infty$. Since we have

$$\kappa_{r,N} = \|\tilde{r}_N\|_{L^\infty(\mathbb{R}^d,\mathbb{R})} = \sup_{x\in D, y\in\Lambda_N^c} C\mathrm{e}^{-m|x-y|} = C\mathrm{e}^{-md_e(D,\Lambda_N^c)}$$

it follows

$$\alpha_N = \max\{\|\tilde{r}_N\|_{L^1(\mathbb{R}^d,\mathbb{R})}, \kappa_{r,N}\} \leqslant Ce^{-m'd_e(D,\Lambda_N^c)}.$$

and therefore the second condition of Definition 5.2.5 is satisfied as well. We now can use Theorem 5.2.8 and obtain

$$\|u - u_N\|_{L^n((\Omega,\mathfrak{A},\mathbf{P}),H^1(D,\mathbb{R}))} \leqslant C' \max\{\alpha_N, \alpha_N^{1-\delta}\} \leqslant C'C\mathrm{e}^{-m'd_e(D,\Lambda_N^c)},$$

where we merged m' and δ as for $m' \in]0,m[$ and $\delta \in]0,1[$ we again have $(1-\delta)m' \in]0,m[$. Redefining the constant C gives (5.23). ☐

Remark 5.2.10. *(i) The discontinuous cut-off $\mathbb{1}_{\Lambda_N}(y)$ perhaps seems to be a contradiction to the assumptions needed for the continuity of the paths of $Z_{k_N}(x)$, which is part of the prerequisites of Theorem 5.2.8. However, as it is equiva-*

lent to apply the noise Z to $k_N(x,\cdot)$ or to apply the noise $\mathbb{1}_{\Lambda_N} Z$ to $k(x,\cdot)$ we may still obtain continuous realizations of $Z_{k_N}(x)$ from Theorem 3.2.10. As $\varphi_N(f) = \mathrm{e}^{\int_{\Lambda_N} \psi(f)\,\mathrm{d}x}$ (and equally for $R_N(y)$ with Λ_N replaced by Λ_N^c), we see that this functional still is $|||\cdot|||$-continuous and therefore the results of Theorem 3.2.10 are compatible with the cut-off Λ_N.

(ii) In a similar way, the Hölder continuity of the covariance function $k_{2\alpha,m}$ of the Gaussian part (see (5.6)) also holds for the truncated fields $G_{k_N}(x)$, $Z_{k_{N,t}}(x)$ and $R_N(x)$, as by Definition 5.2.5 (i) the canonical distances of all these fields are dominated by that of $G_k(x)$.

After we restricted both arguments of the smoothing kernel k to a sufficient large compact domain Λ at the cost of a small controllable error, we are now able to apply Mercer's expansion to k on the domain Λ.

Theorem 5.2.11 (Mercer's Theorem, [57, Theorem 1.80])**.** *Let Λ be a compact subset of $\mathbb{R}^d$ and $k : \Lambda \times \Lambda \to \mathbb{R}$ be a continuous and positive definite kernel. Consider the compact linear operator $K : L^2(\Lambda, \mathbb{R}) \to L^2(\Lambda, \mathbb{R})$,*

$$[K\phi](x) = \int_\Lambda k(x,y)\phi(y)\,\mathrm{d}y$$

associated with k. Then, there exists an orthonormal basis $\{e_j\}_{j\in\mathbb{N}}$ of $L^2(\Lambda,\mathbb{R})$ consisting of eigenfunctions of K such that the associated sequence of eigenvalues $\{\lambda_j\}_{j\in\mathbb{N}}$ is non-negative with zero as its only possible point of accumulation. The eigenfunctions corresponding to positive eigenvalues are continuous on Λ and k has the representation

$$k(x,y) = \sum_{j=1}^{\infty} \lambda_j e_j(x) e_j(y), \quad x, y \in \Lambda,$$

where the convergence is absolute and uniform.

Remark 5.2.12. *(i) In the following, instead of applying Mercer's expansion onto the covariance function $k^\vee * k$ induced by the smoothing kernel k as in a Karhunen-Loève (KL) expansion, we obtain a finite-dimensional approximation of the smoothed random field by expanding the smoothing kernel k itself. Whereas the standard KL-expansion expands the covariance function $\int_{\mathbb{R}^d} k(x-z)k(y-z)\,\mathrm{d}z$, our approach expands the covariance of the truncated noise $\int_\Lambda k(x-z)k(y-z)\,\mathrm{d}z$,* *where $x, y \in \Lambda$. We could easily show that the eigenvalues obtained for the expansion of the second covariance function are given by λ_i^2 where $e_i(x)$ are the eigenfunctions of the integral operator defined by $k(x-y)$ in $L^2((\Lambda, \mathbb{R}), \mathrm{d}y)$.*

(ii) *In principle, one could also expand the paths of $Z_k(x)$ in eigenfunctions of the first covariance operator, by expanding the smoothing kernel $k(x-y)$ in $x \in \Lambda$ (or D). In addition, a proof for the uniformity and decay properties of this expansion in $y \in \mathbb{R}^d$ are needed. This approach seems more involved than the cut-off method used here, as the spectral properties of the integral operator induced by $k(x-y)$ can not be used. In addition, the cut-off method seems not to loose any efficiency for Matérn kernels as it does not lead to a worsening of rates of convergence as shown in Theorem 5.2.16 below.*

(iii) *The assumptions of Theorem 5.2.11 is clearly satisfied for $k(x,y) = k_{\alpha,m}(x-y)$ for $2\alpha > d$ when restricted to Λ in both arguments x, y; see, Lemma 5.1.3. Note that the positive definiteness of the kernel $k_{\alpha,m}(x-y)$ is given as the Fourier transform of $k_{\alpha,m}(x)$ is positive (see Definition 3.2.8).*

(iv) *As the eigenfunctions e_i and the eigenvalues λ_i depend on Λ, in what follows we use for $\Lambda = \Lambda_N$ the notation $\lambda_{N,i}$ and $e_{N,i}$.*

Corollary 5.2.13. *We assume in addition to the assumptions of Theorem 5.2.8 that k is a positive definite kernel. Then, for fixed $N \in \mathbb{N}$ the decomposition*

$$k_N(x,y) = k(x,y)\mathbb{1}_{\Lambda_N}(y) = k_{N,N'}(x,y) + r_{N,N'}(x,y), \quad x, y \in \Lambda_N,$$

where the truncated Mercer expansion $k_{N,N'}(x,y) = \sum_{j=1}^{N'} \lambda_{N,j} e_{N,j}(x) e_{N,j}(y)$ from Theorem 5.2.11 with remainder $r_{N,N'}$ represents an orthogonal approximation sequence in the sense of Definition 5.2.5 with respect to the approximation parameter $N' \in \mathbb{N}$. For the solution u_N of (2.15) with smoothing kernel truncated in the y variable (see Corollary 5.2.9) and the solution $u_{N,N'}$ associated with $Z_{k_{N,N'}}$ we have

$$\|u_N - u_{N,N'}\|_{L^n((\Omega,\mathfrak{A},\mathbf{P}),H^1(D,\mathbb{R}))} \leqslant \hat{C}|\Lambda_N|\kappa_{r,N,N'} \to 0, \qquad \text{as } N' \to \infty,$$

where $\kappa_{r,N,N'} = \sup_{x \in D, y \in \Lambda_N} |\sum_{j=N'+1}^{\infty} \lambda_{N,j} e_{N,j}(x) e_{N,j}(y)|$ and $\Lambda_N > 1$.

Proof. Mercer's Theorem 5.2.11 provides

$$0 \leqslant \kappa_{r,N,N'} \leqslant \sup_{x\in\Lambda_N, y\in\Lambda_N} \left| \sum_{j=N'+1}^{\infty} \lambda_{N,j}\, e_{N,j}(x)\, e_{N,j}(y) \right| \to 0 \quad \text{as } N' \to \infty.$$

Additionally, we have $\|\tilde{r}_{N,N'}\|_{L^1(\Lambda_N,\mathbb{R})} \leqslant |\Lambda_N|\kappa_{r,N,N'}$ as Λ_N is bounded. Now, the assertion follows by Theorem 5.2.8 □

In order to establish a convergence rate for Matérn kernels, we first have to prove the following auxiliary lemma.

Lemma 5.2.14. *Consider the compact operator $K : L^2(\Lambda,\mathbb{R}) \to L^2(\Lambda,\mathbb{R})$ on a compact set $\Lambda \subset \mathbb{R}^d$ such that $\mathrm{cl}(\mathrm{int}(\Lambda)) = \Lambda$, where*

$$[Kf](x) = \int_\Lambda k_{\alpha,m}(x-y) f(y)\,\mathrm{d}y, \quad f \in L^2(\Lambda,\mathbb{R}),\, x \in \mathbb{R}^d,$$

and where $k_{\alpha,m}$ is the Matérn kernel with parameters $\alpha > \frac{d}{2}$, $m > 0$. Let $(\lambda_{\Lambda,j})_{j\in\mathbb{N}}$ be the eigenvalues of K with normalized eigenfunctions $(e_{\Lambda,j})_{j\in\mathbb{N}}$, and let $\varepsilon \in]0, \frac{\alpha}{d} - \frac{1}{2}[$, $\delta > 0$ and $\chi > \max\{1, \frac{1}{\delta}\}$. For $\mathrm{diam}(\Lambda) \leqslant \delta$ and every $\gamma > \chi^\delta$ we have

$$\sqrt{\lambda_{\Lambda,j}} \|e_{\Lambda,j}\|_{L^\infty(\Lambda,\mathbb{R})} \leqslant C\gamma^{2(\alpha-\frac{d}{2}-\varepsilon)} j^{-\frac{\alpha}{d}+\frac{1}{2}+\varepsilon}, \quad \textit{for all } j \in \mathbb{N}, \tag{5.24}$$

where the constant C only depends on α, m, ε, and χ.

Proof. This proof is very much inspired by [78] where the author prove a similar bound which, however, depends on the domain Λ.

As $k_{\alpha,m}$ is a real-valued function with positive Fourier transform, K is a positive, self-adjoint operator. The eigenfunctions $(e_{\Lambda,j})_{j\in\mathbb{N}} =: (e_j)_{j\in\mathbb{N}}$ define an orthonormal basis of $L^2(\Lambda,\mathbb{R})$ and since K is positive, $\lambda_j \geqslant 0$, $j \in \mathbb{N}$. As usual we assume that the eigenvalue sequence is decreasing.

By extending every $f \in L^2(\Lambda,\mathbb{R})$ by zero to $\mathbb{R}^d$ and denoting this extension of f to $\mathbb{R}^d$ again by f, we interpret $L^2(\Lambda,\mathbb{R})$ as a closed subspace of $L^2(\mathbb{R}^d,\mathbb{R})$. Since Λ is compact, we also have $L^2(\Lambda,\mathbb{R}) \subset L^1(\mathbb{R}^d,\mathbb{R})$, therefore

$$k_{\alpha,m} * f \in L^1(\mathbb{R}^d,\mathbb{R}) \quad \text{for all } f \in L^2(\Lambda,\mathbb{R})$$

and $Kf = (k_{\alpha,m} * f)|_\Lambda$. Clearly, K is the compression to $L^2(\Lambda,\mathbb{R})$ of the convolution operator on $L^2(\mathbb{R}^d,\mathbb{R})$ with convolution kernel $k_{\alpha,m}$. Therefore, $k_{\alpha,m} * f \in H^\alpha(\mathbb{R}^d,\mathbb{R})$, for $f \in L^2(\mathbb{R}^d, \mathbb{R})$, and since $\alpha > \frac{d}{2}$ we have $\widehat{k_{\alpha,m} * f} \in L^1(\mathbb{R}^d, \mathbb{R})$, for $f \in L^2(\mathbb{R}^d, \mathbb{R})$. Because

$$\lambda_j e_j = (k_{\alpha,m} * e_j)\mathbb{1}_\Lambda, \quad \text{for all } j \in \mathbb{N}$$

it follows $\lambda_j \neq 0$ for $j \in \mathbb{N}$ as well as $e_j \in H^\alpha(\text{int}(\Lambda), \mathbb{R}) \cap C^0(\Lambda, \mathbb{R})$.

Let $\frac{d}{2} < s < \alpha$. By the fact that due to $s > \frac{d}{2}$ the Fourier transform of every $H^s(\mathbb{R}^d, \mathbb{R})$ function belongs to $L^1(\mathbb{R}^d, \mathbb{R})$ (see, e.g. [74, Corollary 7.9.4]), the Fourier inversion formula gives for every $U \in H^s(\mathbb{R}^d, \mathbb{R})$ with $U\restriction_{\text{int}(\Lambda)} = e_j$ that

$$\|e_j\|_{L^\infty(\Lambda,\mathbb{R})} \leqslant (2\pi)^{-d}\|\hat{U}\|_{L^1(\mathbb{R}^d,\mathbb{R})} \leqslant \|(1+|\xi|^2)^{-s}\|_{L^1(\mathbb{R}^d,\mathbb{R})}\|U\|_{H^s(\mathbb{R}^d,\mathbb{R})}$$

so that with $c_s := \|(1+|\xi|^2)^{-s}\|_{L^1(\mathbb{R}^d,\mathbb{R})}$ we have

$$\|e_j\|_{L^\infty(\Lambda,\mathbb{R})} \leqslant c_s\|e_j\|_{H^s(\text{int}(\Lambda),\mathbb{R})}, \quad \text{for all } j \in \mathbb{N}.$$

By applying an interpolation inequality (see, e.g. [83, Theorem B.8 and Lemma B.1]) we have for all $j \in \mathbb{N}$

$$\begin{aligned} \|e_j\|_{L^\infty(\Lambda,\mathbb{R})} &\leqslant c_s\alpha\left(\frac{\sin(\frac{s\pi}{\alpha})}{\pi s(\alpha-s)}\right)^{\frac{1}{2}} \|e_j\|^{1-\frac{s}{\alpha}}_{L^2(\Lambda,\mathbb{R})}\|e_j\|^{\frac{s}{\alpha}}_{H^\alpha(\text{int}(\Lambda),\mathbb{R})} \\ &= c_s\alpha\left(\frac{\sin(\frac{s\pi}{\alpha})}{\pi s(\alpha-s)}\right)^{\frac{1}{2}} \|e_j\|^{\frac{s}{\alpha}}_{H^\alpha(\text{int}(\Lambda),\mathbb{R})}. \end{aligned} \tag{5.25}$$

From (2.3) and Plancherel's Theorem [83, Theorem 3.12] we conclude

$$\begin{aligned} \lambda_j^2\|e_j\|^2_{H^\alpha(\text{int}(\Lambda),\mathbb{R})} \leqslant \|k_{\alpha,m} * e_j\|^2_{H^\alpha(\mathbb{R}^d,\mathbb{R})} &= (2\pi)^{-2d}\int_{\mathbb{R}^d}(1+|\xi|^2)^\alpha(m^2+|\xi|^2)^{-2\alpha}|\hat{e_j}|^2\,\mathrm{d}\xi \\ &\leqslant \max\{1, m^{-2\alpha}\}(2\pi)^{-2d}\int_{\mathbb{R}^d}\widehat{k_{\alpha,m} * e_j}\,\overline{\hat{e_j}}\,\mathrm{d}x \\ &= \max\{1, m^{-2\alpha}\}(2\pi)^{-2d}(Ke_j, e_j)_{L^2(\Lambda)} = \max\{1, m^{-2\alpha}\}(2\pi)^{-2d}\lambda_j, \end{aligned}$$

where $\overline{\hat{e_j}}$ denotes the complex conjugate of $\hat{e_j}$ and where $(\cdot,\cdot)_{L^2(\Lambda)}$ is the L^2 inner product on Λ. Combining the previous inequality with (5.25) we obtain for every $s \in (\frac{d}{2}, \alpha)$ and for all $j \in \mathbb{N}$:

$$\sqrt{\lambda_j}\|e_j\|_{L^\infty(\Lambda,\mathbb{R})} \leqslant c_s\alpha\left(\frac{\sin(\frac{s\pi}{\alpha})}{\pi s(\alpha-s)}\right)^{\frac{1}{2}} \max\{1, m^{-s}\}(2\pi)^{-\frac{ds}{\alpha}}\lambda_j^{\frac{1}{2}-\frac{s}{2\alpha}}. \tag{5.26}$$

Next we derive estimates for the eigenvalue sequence $(\lambda_j)_{j\in\mathbb{N}}$. Let $\delta \geqslant \operatorname{diam}(\Lambda)$. Without loss of generality we assume that $\Lambda \subseteq [-\frac{\delta}{2}, \frac{\delta}{2}]^d$. Then, the integral operator K is determined by $k_{\alpha,m}\restriction_{[-\delta,\delta]^d}$. For arbitrary $\gamma > \delta$ we can consider $L^2(\Lambda,\mathbb{R})$ as a subspace of $L^2([-\gamma,\gamma]^d,\mathbb{R})$, by extending the functions in $L^2(\Lambda,\mathbb{R})$ by zero to $[-\gamma,\gamma]^d$. Again, we do not distinguish notationally between functions from $L^2(\Lambda,\mathbb{R})$ and their extensions.

If k is continuous, real-valued, and an even extension of $k_{\alpha,m}\restriction_{[-\delta,\delta]^d}$ to $[-\gamma,\gamma]^d$ – note that $k_{\alpha,m}$ is a radially symmetric function, so in particular even – it follows that

$$\tilde{K} : L^2([-\gamma,\gamma]^d,\mathbb{R}) \to L^2([-\gamma,\gamma]^d,\mathbb{R}), \quad [\tilde{K}f](x) := \int_{[-\gamma,\gamma]^d} k(x-y)f(y)\,\mathrm{d}y$$

is a self-adjoint, compact operator which satisfies $Kf = \tilde{K}f\restriction_\Lambda$, for $f \in L^2(\Lambda,\mathbb{R})$. As $\tilde{K}$ is self-adjoint and compact, there exists an orthonormal basis $(f_j)_{j\in\mathbb{N}}$ of $L^2([-\gamma,\gamma]^d,\mathbb{R})$ consisting of eigenfunctions of $\tilde{K}$ and a real sequence of corresponding eigenvalues $(\tilde{\lambda}_j)_{j\in\mathbb{N}}$ which, without loss of generality, have decreasing moduli.

Clearly, for every $j \in \mathbb{N}$ the operators

$$B_j : L^2(\Lambda,\mathbb{R}) \to L^2(\Lambda,\mathbb{R}), \quad f \mapsto \sum_{l=1}^{j} \lambda_l (f, e_l)_{L^2(\Lambda)} e_l,$$
$$C_j : L^2(\Lambda,\mathbb{R}) \to L^2(\Lambda,\mathbb{R}), \quad f \mapsto \sum_{l=1}^{j} \tilde{\lambda}_l (f, f_l\restriction_\Lambda)_{L^2(\Lambda)} f_l\restriction_\Lambda$$

and

$$\tilde{C}_j : L^2([-\gamma,\gamma]^d,\mathbb{R}) \to L^2([-\gamma,\gamma]^d,\mathbb{R}), \quad f \mapsto \sum_{l=1}^{j} \tilde{\lambda}_l (f, f_l)_{L^2([-\gamma,\gamma]^d)} f_l$$

are continuous and linear operators with at most j-dimensional range.

Then, $\tilde{C}_j f\restriction_\Lambda = C_j f$ for $f \in L^2(\Lambda,\mathbb{R})$ and $j \in \mathbb{N}_0$. By denoting the associated operator norms with $\|\cdot\|_\Lambda$ and $\|\cdot\|_\gamma$, we have

$$\|Kf - C_j f\|_\Lambda = \|(\tilde{K}f - \tilde{C}_j f)\restriction_\Lambda\|_\Lambda \leqslant \|\tilde{K}f - \tilde{C}_j f\|_\gamma, \quad \text{for all } f \in L^2(\Lambda,\mathbb{R})$$

so that

$$\|K - C_j\|_\Lambda \leqslant \|\tilde{K} - \tilde{C}_j\|_\gamma, \quad \text{for all } j \in \mathbb{N}. \tag{5.27}$$

Since K is positive and both K as well as $\tilde{K}$ are self-adjoint and compact operators, by a well-known result (see, e.g., [84, Lemma 16.5 and its proof]), using (5.27) gives

$$\lambda_j = \|K - B_{j-1}\|_\Lambda \leqslant \|K - C_{j-1}\|_\Lambda \leqslant \|\tilde{K} - \tilde{C}_{j-1}\|_\gamma = |\tilde{\lambda}_j|, \quad \text{for all } j \in \mathbb{N}. \tag{5.28}$$

Up until this point we did not specify the extension k of $k_{\alpha,m}\restriction_{[-\delta,\delta]^d}$. For this purpose we fix $\chi > \max\{1, \frac{1}{\delta}\}$ and a real-valued, even $\phi_{1,\chi} \in \mathscr{D} = C_c^\infty(\mathbb{R}^d, \mathbb{R})$ with $\phi_{1,\chi}\restriction_{[-\frac{1}{\chi},\frac{1}{\chi}]^d} = 1$ and $\operatorname{supp}\phi_{1,\chi} \subseteq [-1,1]^d$ (the support of $\phi_{1,\chi}$). For $\gamma \geqslant \chi\delta$ we define $\phi_{\gamma,\chi}(x) := \phi_{1,\chi}(\frac{1}{\gamma}x)$ so that $\phi_{\gamma,\chi}\restriction_{[-\delta,\delta]^d} = 1$ and $\operatorname{supp}\phi_{\gamma,\chi} \subseteq [-\gamma,\gamma]^d$. To simplify the notation we write ϕ_1 and ϕ_γ instead of $\phi_{1,\chi}$ and $\phi_{\gamma,\chi}$, respectively.

Then, $k_\gamma := k_{\alpha,m}\phi_\gamma$ is an even extension of $k_{\alpha,m}\restriction_{[-\delta,\delta]^d}$ whose support lies in $[-\gamma,\gamma]^d$, with $\gamma \geqslant \chi\delta$. We define the γ-periodic extension k_p of k_γ by

$$k_p(x) := \sum_{n\in\mathbb{Z}^d} k_\gamma(x + 2\gamma n), \quad \text{for all } x \in \mathbb{R}^d.$$

Then, for the integral operator $\tilde{K}$ corresponding to k_γ, we have for all $x \in \mathbb{R}^d$ and $n \in \mathbb{Z}^d$ that

$$\begin{aligned}[\tilde{K}\mathrm{e}^{-i\frac{\pi}{\gamma}n\cdot y}](x) &= \int_{[-\gamma,\gamma]^d} k_\gamma(x-y)\mathrm{e}^{-i\frac{\pi}{\gamma}n\cdot y}\,\mathrm{d}y = \int_{[-\gamma,\gamma]^d} k_p(x-y)\mathrm{e}^{-i\frac{\pi}{\gamma}n\cdot y}\,\mathrm{d}y \\ &= \int_{[-\gamma,\gamma]^d} k_p(z)\mathrm{e}^{-i\frac{\pi}{\gamma}n\cdot z}\,\mathrm{d}z\, e^{-i\frac{\pi}{\gamma}n\cdot x}.\end{aligned}$$

Since $\{e^{-i\frac{\pi}{\gamma}n\cdot x}, n \in \mathbb{Z}^d\}$ is an orthogonal basis of $L^2([-\gamma,\gamma]^d, \mathbb{R})$ it follows that the Fourier coefficients $c_n(k_p)$ of k_p

$$c_n(k_p) := \int_{[-\gamma,\gamma]^d} k_p(z)\mathrm{e}^{-i\frac{\pi}{\gamma}n\cdot z}\,\mathrm{d}z = \int_{[-\gamma,\gamma]^d} k_\gamma(z)\mathrm{e}^{-i\frac{\pi}{\gamma}n\cdot z}\,\mathrm{d}z, \quad \text{for all } n \in \mathbb{Z}^d$$

(which are real since k_γ is even) are the eigenvalues of $\tilde{K}$, i.e., a suitable enumeration of $(c_n(k_p))_{n\in\mathbb{Z}^d}$ yields the eigenvalue sequence $(\tilde{\lambda}_j)_{j\in\mathbb{N}}$. Since $\operatorname{supp} k_\gamma \subseteq [-\gamma,\gamma]^d$ we have for all $n \in \mathbb{Z}^d$

$$\begin{aligned}|c_n(k_p)| &= \left|\int_{[-\gamma,\gamma]^d} k_\gamma(z)\mathrm{e}^{-i\frac{\pi}{\gamma}n\cdot z}\,\mathrm{d}z\right| = \left|\int_{\mathbb{R}^d} k_\gamma(z)\mathrm{e}^{-i\frac{\pi}{\gamma}n\cdot z}\,\mathrm{d}z\right| \\ &= \left|\widehat{k_{\alpha,m}\cdot\phi_\gamma}\left(-\frac{\pi}{\gamma}n\right)\right| = \left|\hat{k}_{\alpha,m} * \hat{\phi}_\gamma\left(-\frac{\pi}{\gamma}n\right)\right|.\end{aligned} \tag{5.29}$$

Furthermore, for $\xi \in \mathbb{R}^d$ it holds

$$\begin{aligned}\left|\hat{k}_{\alpha,m} * \hat{\phi}_\gamma(\xi)\right| &\leqslant \left|\int_{|\eta|\leqslant\frac{|\xi|}{2}} \hat{k}_{\alpha,m}(\eta)\hat{\phi}_\gamma(\xi-\eta)\,\mathrm{d}\eta\right| + \left|\int_{|\eta|\geqslant\frac{|\xi|}{2}} \hat{k}_{\alpha,m}(\eta)\hat{\phi}_\gamma(\xi-\eta)\,\mathrm{d}\eta\right| \\ &\leqslant \max_{|\zeta|\geqslant\frac{|\xi|}{2}} \left|\hat{\phi}_\gamma(\zeta)\right| \|\hat{k}_{\alpha,m}\|_{L^1(\mathbb{R}^d,\mathbb{R})} + \max_{|\zeta|\geqslant\frac{|\xi|}{2}} \left|\hat{k}_{\alpha,m}(\zeta)\right| \|\hat{\phi}_\gamma\|_{L^1(\mathbb{R}^d,\mathbb{R})}.\end{aligned} \tag{5.30}$$

Since $\hat{\phi}_\gamma(\xi) = \gamma^d \hat{\phi}_1(\gamma\xi)$, it follows

$$\|\hat{\phi}_\gamma\|_{L^1(\mathbb{R}^d,\mathbb{R})} = \|\hat{\phi}_1\|_{L^1(\mathbb{R}^d,\mathbb{R})}. \tag{5.31}$$

Moreover, due to $\gamma \geqslant \chi\delta \geqslant 1$ and $\alpha > \frac{d}{2}$ we have

$$\begin{aligned}|\hat{\phi}_\gamma(\xi)| &= \gamma^d \left(1+|\gamma\xi|^2\right)^{-\lceil\alpha\rceil} \left(1+|\gamma\xi|^2\right)^{\lceil\alpha\rceil} \left|\hat{\phi}_1(\gamma\xi)\right| \\ &= \gamma^d \left(1+|\gamma\xi|^2\right)^{-\lceil\alpha\rceil} \left|\widehat{(1-\Delta)^{\lceil\alpha\rceil}\phi_1}(\gamma\xi)\right| \\ &\leqslant \gamma^{2\alpha} \left(1+|\xi|^2\right)^{-\alpha} \left\|(1-\Delta)^{\lceil\alpha\rceil}\phi_1\right\|_{L^1(\mathbb{R}^d,\mathbb{R})}.\end{aligned}$$

Applying this and (5.31) on (5.30) gives for $\xi \in \mathbb{R}^d$

$$\begin{aligned}|\hat{k}_{\alpha,m} * \hat{\phi}_\gamma(\xi)| &\leqslant \gamma^{2\alpha}\left(1+\frac{|\xi|^2}{4}\right)^{-\alpha} \left\|(1-\Delta)^{\lceil\alpha\rceil}\phi_1\right\|_{L^1(\mathbb{R}^d,\mathbb{R})} \|\hat{k}_{\alpha,m}\|_{L^1(\mathbb{R}^d,\mathbb{R})} \\ &\quad + \left(m^2+\frac{|\xi|^2}{4}\right)^{-\alpha} \|\hat{\phi}_1\|_{L^1(\mathbb{R}^d,\mathbb{R})} \\ &\leqslant \gamma^{2\alpha}\max\{1,m^{-2\alpha}\}\left(1+\frac{|\xi|^2}{4}\right)^{-\alpha} \Big[\left\|(1-\Delta)^{\lceil\alpha\rceil}\phi_1\right\|_{L^1(\mathbb{R}^d,\mathbb{R})} \\ &\quad \cdot \|\hat{k}_{\alpha,m}\|_{L^1(\mathbb{R}^d,\mathbb{R})} + \|\hat{\phi}_1\|_{L^1(\mathbb{R}^d,\mathbb{R})}\Big].\end{aligned}$$

Thus, by using (5.29) we get for every $n \in \mathbb{Z}^d$

$$\begin{aligned}|c_n(k_p)| &\leqslant \max\{1,m^{-2\alpha}\}\left[\|(1-\Delta)^{\lceil\alpha\rceil}\phi_1\|_{L^1(\mathbb{R}^d,\mathbb{R})}\|\hat{k}_{\alpha,m}\|_{L^1(\mathbb{R}^d,\mathbb{R})}+\|\hat{\phi}_1\|_{L^1(\mathbb{R}^d,\mathbb{R})}\right] \\ &\quad \cdot \gamma^{2\alpha}\max\left\{1,\frac{2\gamma}{\pi}\right\}^{2\alpha}\left(1+|n|^2\right)^{-\alpha}.\end{aligned}$$

For $\gamma \geqslant \chi\delta$ we follow that for every $n \in \mathbb{Z}^d$ and $0 < \eta \leqslant |c_n(k_p)|$ we have

$$\begin{aligned}|n| &< \max\{1,m^{-1}\}\left[\|(1-\Delta)^{\lceil\alpha\rceil}\phi_1\|_{L^1(\mathbb{R}^d,\mathbb{R})}\|\hat{k}_{\alpha,m}\|_{L^1(\mathbb{R}^d,\mathbb{R})}+\|\hat{\phi}_1\|_{L^1(\mathbb{R}^d,\mathbb{R})}\right]^{\frac{1}{2\alpha}} \\ &\quad \cdot \max\left\{1,\frac{2\gamma}{\pi}\right\}\gamma\eta^{-\frac{1}{2\alpha}}.\end{aligned}$$

Therefore, for $\gamma \geqslant \chi\delta$ and $\eta > 0$ we obtain

$$\#\{n \in \mathbb{Z}^d : |c_n(k_p)| \geqslant \eta\} \leqslant 2^d \max\{1, m^{-d}\}$$
$$\cdot \left[\|(1-\Delta)^{\lceil\alpha\rceil}\phi_1\|_{L^1(\mathbb{R}^d,\mathbb{R})}\|\hat{k}_{\alpha,m}\|_{L^1(\mathbb{R}^d,\mathbb{R})} + \|\hat{\phi}_1\|_{L^1(\mathbb{R}^d,\mathbb{R})}\right]^{\frac{d}{2\alpha}} \left(\max\left\{1, \frac{2\gamma}{\pi}\right\}\right)^d \gamma^d \eta^{-\frac{d}{2\alpha}}.$$

For the eigenvalue sequence $(\tilde{\lambda}_j)_{j\in\mathbb{N}}$ of the operator $\tilde{K}$ associated with k_γ, $\gamma \geqslant \chi\delta$, it thus follows for all $j \in \mathbb{N}$ that

$$|\tilde{\lambda}_j| \leqslant 4^\alpha \max\{1, m^{-2\alpha}\} \left[\|(1-\Delta)^{\lceil\alpha\rceil}\phi_1\|_{L^1(\mathbb{R}^d,\mathbb{R})}\|\hat{k}_{\alpha,m}\|_{L^1(\mathbb{R}^d,\mathbb{R})} + \|\hat{\phi}_1\|_{L^1(\mathbb{R}^d,\mathbb{R})}\right]$$
$$\cdot \left(\max\left\{1, \frac{2\gamma}{\pi}\right\}\right)^{2\alpha} \gamma^{2\alpha} j^{-\frac{2\alpha}{d}}.$$

By taking (5.28) and (5.26) into account, we finally obtain that for every $s \in (\frac{d}{2}, \alpha)$, $\gamma \geqslant \chi\delta$ and each $j \in \mathbb{N}$

$$\sqrt{\lambda_j}\|e_j\|_{L^\infty(\Lambda,\mathbb{R})} \leqslant c_s\alpha \left(\frac{\sin(\frac{s\pi}{\alpha})}{\pi s(\alpha - s)}\right)^{\frac{1}{2}} \max\{1, m^{-s}\}(2\pi)^{-\frac{ds}{\alpha}} 2^{\alpha-s} \max\{1, m^{-(\alpha-s)}\}$$
$$\cdot \left[\|(1-\Delta)^{\lceil\alpha\rceil}\phi_1\|_{L^1(\mathbb{R}^d,\mathbb{R})}\|\hat{k}_{\alpha,m}\|_{L^1(\mathbb{R}^d,\mathbb{R})} + \|\hat{\phi}_1\|_{L^1(\mathbb{R}^d,\mathbb{R})}\right]^{\frac{\alpha-s}{2\alpha}} \gamma^{\alpha-s} j^{-\frac{\alpha}{d}+\frac{s}{d}}.$$

It follows that for every $\delta > 0$ and each $\varepsilon \in (0, \frac{\alpha}{d} - \frac{1}{2})$ (with $s = \frac{d}{2} + \varepsilon$ in the previous inequality) for every $\chi > \max\left\{1, \frac{1}{\delta}\right\}$ there is a constant $C > 0$, depending only on α, m, ε, and χ, such that for every compact subset $\Lambda \subseteq \mathbb{R}^d$ with $\mathrm{cl}(\mathrm{int}(\Lambda)) = \Lambda$ and $\mathrm{diam}(\Lambda) \leqslant \delta$ and every $\gamma \geqslant \chi\delta$, there holds

$$\sqrt{\lambda_{\Lambda,j}}\|e_{\Lambda,j}\|_{L^\infty(\Lambda,\mathbb{R})} \leqslant C\gamma^{\alpha-\frac{d}{2}-\varepsilon} j^{-\frac{\alpha}{d}+\frac{1}{2}+\varepsilon}, \quad \text{for all } j \in \mathbb{N}.$$

□

Corollary 5.2.15. *Let a kernel function $k = k_{\alpha,m}$ be given by a Matérn function with $\alpha > d$ and let $(\Lambda_N)_{N\in\mathbb{N}}$ be a compact exhaustion of $\mathbb{R}^d$ with $D \subseteq \Lambda_1$ and $\mathrm{diam}(\Lambda_1) \geqslant 1$. Then, for every $\varepsilon \in]0, \frac{\alpha}{d} - \frac{1}{2}[$ there exists a constant $C > 0$ such that for each $N \in \mathbb{N}$ the uniform bound $\kappa_{r,N,N'} = \sup_{x\in D, y\in\Lambda_N} |\sum_{j=N'+1}^{\infty} \lambda_{N,j} e_{N,j}(x) e_{N,j}(y)|$ on the remainder of the Mercer series of the restriction k_N of k to $\Lambda_N \times \Lambda_N$ satisfies*

$$\kappa_{r,N,N'} \leqslant C\,\mathrm{diam}(\Lambda_N)^{2(\alpha-\frac{d}{2}-\varepsilon)} N'^{-2(\frac{\alpha}{d}-1-\varepsilon)} \quad \forall\, N' \in \mathbb{N}.$$

In particular, if $\varepsilon < \frac{\alpha}{d} - 1$, the right-hand side converges as $-\frac{\alpha}{d} + 1 + \varepsilon < 0$.

Proof. Consider the compact operator $K : L^2(\Lambda, \mathbb{R}) \to L^2(\Lambda, \mathbb{R})$ on a compact set $\Lambda \subset \mathbb{R}^d$, where

$$[Kf](x) = \int_\Lambda k_{\alpha,m}(x-y)f(y)\,\mathrm{d}y, \quad f \in L^2(\Lambda,\mathbb{R}),\, x \in \mathbb{R}^d,$$

and where $k_{\alpha,m}$ is the Matérn kernel with parameters $\alpha > \frac{d}{2}$, $m > 0$. Let $\varepsilon \in \left]0, \frac{\alpha}{d} - \frac{1}{2}\right[$, $\delta > 1$ and $\chi > \max\{1, \frac{1}{\delta}\}$. Lemma 5.2.14 provides a bound on the eigenvalues $\{\lambda_{\Lambda,j}\}_{j\in\mathbb{N}}$ and eigenfunctions $\{e_{\Lambda,j}\}_{j\in\mathbb{N}}$ of K which holds for $\operatorname{diam}(\Lambda) \leqslant \delta$ and every $\gamma > \chi^\delta$ and is given by

$$\sqrt{\lambda_{\Lambda,j}}\|e_{\Lambda,j}\|_{L^\infty(\Lambda,\mathbb{R})} \leqslant C\gamma^{\alpha-\frac{d}{2}-\varepsilon} j^{-\frac{\alpha}{d}+\frac{1}{2}+\varepsilon}, \quad j \in \mathbb{N}, \tag{5.32}$$

where $\varepsilon \in \left]0, \frac{\alpha}{d} - 1\right[$, and where the constant C only depends on α, m, ε, and χ.

Using (5.32) gives

$$\begin{aligned}
\kappa_{r,N,N'} &\leqslant \sum_{j=N'+1}^\infty \sup_{x\in D, y\in\Lambda_N} |\lambda_{N,i}e_{N,i}(x)e_{N,i}(y)| \leqslant \sum_{j=N'+1}^\infty \|\sqrt{\lambda_{N,j}}e_{N,j}\|^2_{L^\infty(\Lambda_N,\mathbb{R})} \\
&\leqslant C^2 \operatorname{diam}(\Lambda_N)^{2(\alpha-\frac{d}{2}-\varepsilon)} \sum_{j=N'+1}^\infty j^{-\frac{2\alpha}{d}+1+2\varepsilon} \\
&\leqslant C^2 \operatorname{diam}(\Lambda_N)^{2(\alpha-\frac{d}{2}-\varepsilon)} \int_{N'}^\infty x^{-\frac{2\alpha}{d}+1+2\varepsilon}\,\mathrm{d}x \\
&\leqslant C^2(2\alpha/d - 2 - 2\varepsilon)^{-1} \operatorname{diam}(\Lambda_N)^{2(\alpha-\frac{d}{2}-\varepsilon)} N'^{-2(\frac{\alpha}{d}-1-\varepsilon)}.
\end{aligned}$$

Note that the series converges, since $-2\frac{\alpha}{d} + 1 + 2\varepsilon < -1$. Redefining C yields the assertion. □

Now, combining Corollaries 5.2.9 and 5.2.13 provides the second main result of this chapter.

Theorem 5.2.16. *Let the assumption of Theorem 5.2.8 hold and let the smoothing function $k = k_{\alpha,m}$ be given by a Matérn function with $\alpha > d$. Let $\delta :=$ diam(D) and fix $x_0 \in D$ with $D \subseteq x_0 + \left[-\frac{\delta}{2}, \frac{\delta}{2}\right]^d$. For fixed $0 < \tilde{m} < m$ let*

$$\delta_N := \frac{\delta+1}{2} + \frac{2}{\tilde{m}}\left(\frac{\alpha}{d} - 1\right)\log N \text{ and } \Lambda_N := x_0 + [-\delta_N, \delta_N]^d, N \in \mathbb{N}.$$

Further, we denote with u the solution to (2.15) with random conductivity $T \circ Z_k$ and with $u_{N,N'}$ the solution associated with random conductivity $T \circ Z_{k_{N,N'}}$, where $k_{N,N'}$ is given by the truncated Mercer expansion of k_N on $\Lambda_N \times \Lambda_N$.

Then, for every $\upsilon \in (0, 2\alpha/d - 2)$ there is a constant $C > 0$ such that

$$\forall\, N \in \mathbb{N} : \; \|u - u_{N,N}\|_{L^n((\Omega,\mathfrak{A},\mathbf{P}),H^1(D,\mathbb{R}))} \leqslant C N^{-\upsilon}.$$

Proof. Let $\upsilon \in]0, 2\frac{\alpha}{d} - 2[$ and $\varepsilon \in]0, \frac{\alpha}{d} - 1[$ so that $\upsilon < \frac{2\alpha}{d} - 2 - 2\varepsilon$. Under the assumption that $\tilde{m} \in]0, m[$ we define

$$m' := \frac{\tilde{m}\left(\frac{\alpha}{d} - 1 - \varepsilon\right)}{\left(\frac{\alpha}{d} - 1\right)} \in]0, m[.$$

It follows that

$$\frac{2}{m'}\left(\frac{\alpha}{d} - 1 - \varepsilon\right) = \frac{2}{\tilde{m}}\left(\frac{\alpha}{d} - 1\right) \quad \Rightarrow \quad \delta_N = \frac{\delta + 1}{2} + \frac{2}{m'}\left(\frac{\alpha}{d} - 1 - \varepsilon\right)\log N,$$

as well as

$$d_e(D, \Lambda_N^c) > \frac{2\left(\frac{\alpha}{d} - 1 - \varepsilon\right)}{m'}\log N, \quad \text{for all } N \in \mathbb{N},$$

where d_e denotes the Euclidean distance. Furthermore, we have $|\Lambda_N| = 2^d \delta_N^d$, $\mathrm{diam}(\Lambda_1) > 1$, and $\mathrm{diam}(\Lambda_N) = \sqrt{2d\delta_N}$. Combining Corollaries 5.2.9, 5.2.13 and Corollary 5.2.15, and abbreviating $L^1((\Omega, \mathfrak{A}, \mathbf{P}), H^1(D, \mathbb{R}))$ by $L^1(\Omega, H^1(D))$ we thus obtain

$$\begin{aligned}
\|u - u_{N,N}\|_{L^1(\Omega,H^1(D))} &\leqslant \|u - u_N\|_{L^1(\Omega,H^1(D))} + \|u_N - u_{N,N}\|_{L^1(\Omega,H^1(D))} \\
&\leqslant C' \mathrm{e}^{-m' d_e(D,\Lambda_N^c)} + \hat{C}|\Lambda_N| C \left(\mathrm{diam}(\Lambda_N)\right)^{2\left(\alpha - \frac{d}{2} - \varepsilon\right)} N^{-2\left(\frac{\alpha}{d} - 1 - \varepsilon\right)} \\
&\leqslant C' N^{-2\left(\frac{\alpha}{d} - 1 - \varepsilon\right)} + \hat{C} C\, 2^d \sqrt{2d}\, \delta_N^{\alpha + \frac{d}{2} - \varepsilon} N^{-2\left(\frac{\alpha}{d} - 1 - \varepsilon\right)}
\end{aligned}$$

Next, we make use of the fact that for an arbitrary $\varepsilon' > 0$ there exists a constant $C'' > 0$, depending on $\delta, \alpha, m, m', \varepsilon$ and ε', such that $\delta_N^{\alpha + \frac{d}{2} - \varepsilon} \leqslant C'' N^{\varepsilon'}$, for all $N \in \mathbb{N}$. Choosing ε' such that $\upsilon + \varepsilon' < 2(\frac{\alpha}{d} - 1 - \varepsilon)$ we therefore get

$$\begin{aligned}
\|u - u_{N,N}\|_{L^1((\Omega,\mathfrak{A},\mathbf{P}),H^1(D,\mathbb{R}))} &\leqslant C' N^{-2\left(\frac{\alpha}{d} - 1 - \varepsilon\right)} + \hat{C} C\, 2^d \sqrt{2d}\, \delta_N^{\alpha + \frac{d}{2} - \varepsilon} N^{-2\left(\frac{\alpha}{d} - 1 - \varepsilon\right)} \\
&\leqslant C' N^{-2\left(\frac{\alpha}{d} - 1 - \varepsilon\right) + \varepsilon'} + \hat{C} C C'' 2^d \sqrt{2d} N^{-2\left(\frac{\alpha}{d} - 1 - \varepsilon\right) + \varepsilon'} \\
&\leqslant \left(C' + \hat{C} C C'' 2^d \sqrt{2d}\right) N^{-\upsilon}
\end{aligned}$$

which proves the assertion. $\square$

Remark 5.2.17. *(i) By combining Theorem 5.2.8 and Corollary 5.2.9 we obtain the convergence of the approximated solution for any positive definite kernel function satisfying the assumptions of Theorem 5.2.8. However, in order to derive a convergence rate, additional informations on the the remainder* r_N *are necessary.*

(ii) As $Z_{k_{N,N}}(x) = \sum_{j=1}^{N} \lambda_{N,j} e_{N,j}(x) Z(e_{N,j})$ *only depends on the finite-dimensional Lévy distribution of* $(Z(e_{N,1}), \ldots, Z(e_{N,N}))$*, we can follow out of Theorem 5.2.16 an approximation scheme for* u *provided by the coefficients from an infinite-dimensional distribution by* $u_{N,N}$ *obtained from a finite-dimensional distributions.*

Chapter 6

Conclusion

In this thesis, we presented two applications in different fields of mathematics which includes randomness in the physical modeling as common denominator. The first work extended the well-known shape optimization framework [27, 65] into a multi-criteria setting. We presented conditions for the existence of Pareto optimal solutions and the completeness of the corresponding Pareto front. Further, we demonstrated this framework with a simple multi-physics toy model, where we coupled fluid-dynamical and mechanical systems, motivated by the optimization process of a vane in the context of gas-turbines. Our choice of cost functionals represented the friction effects on the surface of the vane exerted by a fluid flowing through the shroud in which the vane lies, and the integrity of the mechanical component described by the probability of failure under low cycle fatigue. In multi-criteria optimization, the Pareto front contains points which are optimal with respect to different preferences of a decision maker. We investigated if a small variation in the preference parameter leads to a small variation in the design. As, in fact, the Pareto optimal shape need not be unique, we studied the convergence behaviour of the sets of optimal shapes of the presented analytical model in terms of the Hausdorff metric. Considering the weighted sum method or an ε-constraint scalarization, we derived certain continuity properties in the preference parameter.

At this point it seems natural to consider multi-criteria shape optimization also from an algorithmic point of view, using the theory of shape derivatives and gradient based optimization; For a first step towards that direction, see, e.g., [22, 39]. For the numerical approach of shape optimization, it would be worth to discuss if (a) the optima of the discretized problem are close to the optima of the continuous problem and, if (b) the same holds for shape gradients for non-optimal solutions, as, e.g., used in multi-criteria descent algorithms. In particular, this should be true for the objective values of discretized and continuous solutions, respectively. Using iso-geometric finite elements [35, 48, 113] could potentially be an approach to avoid spoiling the $C^{k,\alpha}$ domain regularity that is built into our framework by the need of $C^{k,\alpha}$-classical solutions necessary for the evaluation of the objectives in multi-criteria shape optimization problems like the one presented here.

The second application considered the random diffusion equation and extended the model of Gaussian diffusion coefficients into its natural generalization of Lévy type. We comprehensively described the theory of generalized random fields in the terms of multi-Hilbertian spaces and Minlos' theorem. We provided continuity conditions for the smoothed random fields, which are sufficient enough to apply them as diffusion coefficients in equation (2.15). For prospective numerical treatments, we established integrability results of the Sobolev norm of the random solution to (2.15) with smoothed Lévy noise field diffusion coefficient, by investigating the decreasing rate of the probability of the extreme values of Lévy fields. Furthermore, we provided a two-steps approximation scheme for the random solutions, where the first step is a cut-off of the random diffusion coefficient to a sufficient large domain and the second step is a truncated Mercer expansion of the smoothing kernel of the field. By applying the same methods of extreme value estimation of Lévy fields, we proved the convergence of the resulting finite dimensional approximation to the real solution and provided the corresponding convergence rate. In order to further specify the relevance of Lévy models, statistical investigations of the actual distribution of, e.g., hydraulic conductivity in groundwater problems, are necessary.

For further research, it would be of interest to proceed with numerical treatments of the random diffusion equation using the proposed stochastic approximation scheme. However, since the uncorrelated Lévy random variables $(Z(e_1), \ldots, Z(e_n))$ of the finite-dimensional approximation are not necessarily independent, we cannot apply standard (sparse) tensor quadrature formulae for numerically computing the expected value of quantities of interest without any modification.

Moreover, in this thesis we only have considered Lévy fields with Poisson part corresponding to finite activity, i.e. $\int_{\{|s|<1\}} |s|\nu(\mathrm{d}s) < \infty$. This allowed us to shift the compensator term $its\mathbb{1}_{\{|s|\leqslant 1\}}(s)$ for small jumps in the Lévy characteristic to the constant b (cf. 3.5.2). For Lévy fields associated to Lévy measures which only provides that $\int_{\{|s|<1\}} |s|^2\nu(\mathrm{d}s) < \infty$, the presented tail estimates are not applicable. In addition, it would be of interest to weaken the integrability conditions for the Lévy measure and allow thicker tails for $\nu(\mathrm{d}s)$ in order to include, e.g., α-stable Lévy fields with extremely fat tails. Furthermore, random fields with positive paths as, e.g., Gamma noises with positive kernel functions $k(z)$, would make the transformation $T(z)$ unnecessary and turn the maximum value problem for $Z_k(x)$ into a minumum value problem, which requires a new set of estimates.

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