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System Theoretical Properties of linear port-Hamiltonian Systems on Infinite-dimensional Spaces

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Chapter 1

Introduction

The interest in describing dynamics, e.g. the vibration of strings and beams, started in the 17th century by the publication of Newton's *Philosophia Naturalis Principia Mathematica*, [New87]. Since then the questions of how to model a system arises and in [CCD81] the beginning of the vibration theory is described from the first mathematical formulations by Isaac Newton and Leonhard Euler. These developments are summarized by Joseph-Louis de Lagrange in *Mécanique Analytique*, see [Lag11]. The port-Hamiltonian formulation is an extension of the Hamilton formalism, which was introduced by Hamilton. The Hamilton formalism is a further development of the Lagrange formalism. In both formalism the idea is to start from the kinetic and the potential energy to get the partial differential equation model of the system, see [Lan12]. Up to now a model is always just an approximation of the reality and one way, on which this thesis is based, is the port-Hamiltonian way of modelling, see [DMSB09]. Port-based network modeling of complex physical systems leads to port-Hamiltonian systems. Therefore, we introduce modeling in the port-Hamiltonian framework. Here, the idea is to use an energy-based perspective by modeling physical systems. The idea is that a physical system can be viewed as the interconnection of simpler systems, which exchange energy. This structure implies that port-Hamiltonian systems are closed under power conserving interconnections. The important role of the energy is taken into account with the introduction of the *energy norm* and the state space as *energy space*. Therefore, one introduce power conjugated variables, which are connected via the Bond graphs. These are introduced in [Bre82] and lead to the introduction of Dirac structures, see for example [DvdS99]. The power conjugated variables are denoted by flow and effort and their product equals power. In this thesis we restrict ourselves to the introduction of port variables, see the introduction of the *boundary flow* and *boundary effort* in Chapter 3.

The advantages of the port-Hamiltonian approach is on the one hand that the model comes from differential geometry and so it is useful for model reduction and on the other hand it fits for a functional analysis approach and therefore also systems theory.

For finite-dimensional systems there is by now a well-established theory [vdS06, EMvdS07, DMSB09]. The port-Hamiltonian approach has been extended to the

infinite-dimensional situation by a geometric differential approach [vdSM02, MM05, JvdS09, ZLMV10] and also by a functional analytic approach [Vil07, ZLMV10, JZ12, JMZ15, Aug16, JZ18]. In this thesis we take the functional analytic point of view. This approach has been successfully used to derive simple verifiable conditions for well-posedness [LGZM05, Vil07, ZLMV10, JZ12, JMZ15, JK19b], stability [JZ12, AJ14] and stabilization [RZLG17, RLGMZ14, AJ14, SZ18] and robust regulation [HP18].
The port-Hamiltonian systems considered in this thesis can be formulated as a partial differential equation

$$\frac{\partial}{\partial t}x(\zeta,t) = P_1\frac{\partial}{\partial\zeta}(\mathcal{H}x)(\zeta,t) + P_0(\mathcal{H}x)(\zeta,t),\ t \geqslant 0,\ \zeta \in (0,1).$$

Also for the more general class of port-Hamiltonian systems, which we consider in Chapter 6, a similar partial differential equation describes the system.
This class of partial differential equations covers (coupled) wave and beam equations and in particular infinite networks of these equations, that means a network with an infinite number of edges.
A functional analytic approach to the partial differential equation is the formulation as an abstract Cauchy problem.

$$\text{(ACP)}\ \begin{cases} \dot{x}(t) = Ax(t), & t \geqslant 0, \\ x(0) = x_0. \end{cases}$$

There has been an enormous development in the study of the Cauchy problem (ACP) and its well-posedness, see for example [BC16, Eng13, JZ12, LGZM05, vdSM02, Vil07, ZLMV10] and the references therein. These systems are also known as port-Hamiltonian systems, Hamiltonian partial differential equations or systems of linear conservation laws.
For more information we refer to [JZ12, JZ18]. In the following we denote by port-Hamiltonian systems infinite-dimensional linear port-Hamiltonian systems. Thus, having the results of well-posedness, i.e. existence of mild solutions in mind, this thesis answers the further questions:

1. Which port-Hamiltonian systems are exactly controllable?
2. Which port-Hamiltonian operators are discrete Riesz spectral operators?
3. How can well-posedness of infinite-dimensional systems of port-Hamiltonian system characterized? And what is about well-posedness of port-Hamiltonian systems on the semi-axis?

In the following, we give a brief overview on this thesis. To introduce port-Hamiltonian systems and port-Hamiltonian operators in Chapter 3, we recall some basics of functional analysis, strongly continuous semigroups and evolution equations, and systems theory in Chapter 2. This Chapter is mainly based on [Wer00, EN00, TW09, JZ12, RW94], and [Wei94]. In this thesis we focus on port-Hamiltonian systems on a one-dimensional spatial domain.
With this basics in mind, we start in Chapter 3 with the definition of port-Hamiltonian systems and port-Hamiltonian operators following the functional

analytic approach. In Chapter 3, 4 and 5 we focus on port-Hamiltonian systems without internal damping on a finite interval. This is a special class of port-Hamiltonian systems, which however is rich enough to cover in particular the wave equation, the transport equation and the Timoshenko beam equation, and also coupled beam and wave equations each with possibly damping on the boundary. For more information on this class of port-Hamiltonian systems we refer to the monograph [JZ12] and the survey [JZ18].

After introducing the class of port-Hamiltonian systems in 3.1, in 3.2 we formulate generation theorems for this class of systems and in 3.3 we formulate port-Hamiltonian systems as boundary control and observation systems. All these definitions and results are motivated and illustrated by examples as the wave equation. The results listed there can mostly be found in [LGZM05], [Vil07], [ZLMV10],[JZ12], [JMZ15], [Aug16] or [JZ18]. In Section 3.4 we give a new result about the location of the spectrum of a special class of port-Hamiltonian system. In Chapter 4 we consider port-Hamiltonian systems with full boundary control and without internal damping. The main result shows that well-posed port-Hamiltonian systems, with state space $L^2(0,1;\mathbb{C}^d)$ and input space $\mathbb{C}^d$, are exactly controllable and is published in [JK19a].

In Chapter 5 the Riesz basis property of port-Hamiltonian systems is studied. Here, we do not follow the ideas of [Tre00a, Tre00b] as Villegas in [Vil07] but we combine results of systems theory and complex analysis. In Section 5.1 we give a general introduction in bases of Hilbert spaces and in Section 5.2 we give a general characterization of discrete Riesz spectral operators and their properties. In the following Section 5.3 we specify our ideas in the port-Hamiltonian setting and without any technical condition, we give a characterization for the Riesz basis property and show that this is equivalent to the fact that system operator generates a strongly continuous group. Moreover, we get in this situation some more information about the location of the spectrum: Then, the spectrum consists of eigenvalues only, located in a strip parallel to the imaginary axis and they can decomposed into finitely many sets having each a uniform gap. The results of this chapter are published in [JKZ20].

In Chapter 6 we consider generalizations of the port-Hamiltonian systems studied so far. We allow port-Hamiltonian systems with internal damping and consider two kinds of generalizations. In Section 6.1 we consider infinite-dimensional networks of infinite-dimensional port-Hamiltonian systems on a finite interval and in Section 6.2 we consider infinite-dimensional port-Hamiltonian systems on the semi-axis. This class includes in particular infinite networks of transport, wave and beam equations, or even combinations of these. We formulate equivalent conditions for contraction C_0-semigroup generation and these results can be found in [JK19a].

Chapter 2

Preliminaries

In this chapter we introduce some basic notations and ideas of functional analysis, evolution equations and systems theory.
In the following X and Y will always be complex and separable Hilbert spaces. We denote the space of all bounded linear operators from X to Y by $\mathcal{L}(X, Y)$. To shorten notation we write $\mathcal{L}(X) := \mathcal{L}(X, X)$.
We use the notation $s - A := sI - A$, where I denotes the identity operator, and define the resolvent set of a linear operator $A : \mathcal{D}(A) \subset X \to X$ as

$$\rho(A) := \{s \in \mathbb{C} \mid s - A : \mathcal{D}(A) \subset X \to X \text{ is bijective}\}.$$

For each $s \in \rho(A)$ we denote the resolvent operator of A by $(s - A)^{-1} : X \to \mathcal{D}(A)$. The spectrum of A is defined as the set $\sigma(A) := \mathbb{C} \setminus \rho(A)$. The point spectrum $\sigma_p(A)$ is defined by

$$\sigma_p(A) = \{s \in \mathbb{C} \mid \exists x \in \mathcal{D}(A), x \neq 0, Ax = sx\},$$

and consists of eigenvalues of A. We note that in general $\sigma_p(A) \subsetneq \sigma(A)$.

Definition 2.0.1. ([Wer00, Definition V.5.1]) Let $A : \mathcal{D}(A) \subset X \to X$ be a densely defined linear operator. Then the *(Hilbert space) adjoint operator* A^* of A is defined as

$$\begin{aligned} D(A^*) &:= \{y \in X \mid \exists w \in X,\ \forall z \in \mathcal{D}(A) \text{ such that } \langle Az, y\rangle = \langle z, w\rangle\}; \\ A^* y &:= w. \end{aligned}$$

The operator A is called *self-adjoint* if $A^* = A$, and *skew-adjoint* if $A^* = -A$. Note that $A^* = A$ implies $\mathcal{D}(A) = \mathcal{D}(A^*)$ in particular.

A further important property of a linear operator is dissipativity.

Definition 2.0.2. The operator $A : \mathcal{D}(A) \subset X \to X$ is called *dissipative* if $\operatorname{Re}\langle Ax, x\rangle \leqslant 0$ for every $x \in \mathcal{D}(A)$.

Moreover, we introduce the Sobolev spaces $\mathcal{W}^{m,2}(I)$ for $I \subset \mathbb{R}$ open interval and $m \in \mathbb{N}$. For this purpose, we define the weak derivative of a function $f \in L^2(I)$, which is a generalization of the derivative.

Definition 2.0.3. ([Wer00, Definition V.1.11]) Let $I \subset \mathbb{R}$ be an open interval and $m \in \mathbb{N}$. A function $f \in L^2(I)$ is *m-times weakly differentiable* if there exists a function $g \in L^2(I)$, also denoted by $\frac{d^m}{dx^m} f$, such that

$$\langle g, \varphi \rangle = (-1)^m \langle f, \frac{d^m}{dx^m} \varphi \rangle \quad \forall \varphi \in \mathcal{D}(I), \tag{2.1}$$

where

$$\mathcal{D}(I) := \{ \varphi \in C^\infty(I) \mid \operatorname{supp}(\varphi) := \overline{\{x : \varphi(x) \neq 0\}} \subset I \text{ is compact} \} = \mathcal{C}_c^\infty(I)$$

denotes the set of C^∞-functions with compact support. $\mathcal{D}(I)$ is also called the set of test functions.

Now we are in the situation to give the definition of Sobolev spaces.

Definition 2.0.4. Let $I \subset \mathbb{R}$ be an open interval and $m \in \mathbb{N}$. Then the *Sobolev space of m-th order over I* is given by

$$\mathcal{W}^{m,2}(I) := \{ f \in L^2(I) \mid \frac{d^k}{dx^k} f \in L^2(I) \text{ exists for all } k \leqslant m \}$$

with norm

$$\|f\|_{m,2} = \left(\sum_{0 \leqslant k \leqslant m} \left\| \frac{d^k}{dx^k} f \right\|^2_{L^2(I)} \right)^{\frac{1}{2}}.$$

Thus, the Sobolev space of first order on the interval $(0,1)$ with values in $\mathbb{C}^d$ is given by

$$\mathcal{W}^{1,2}(0,1;\mathbb{C}^d) = \{ f \in L^2(0,1;\mathbb{C}^d) \mid \frac{d}{dx} f \in L^2(0,1;\mathbb{C}^d) \text{ exists} \}.$$

A more detailed introduction into Sobolev spaces can be found in [Ada75].

2.1 Preliminaries on evolution equations

Within this section we give a short overview of the theory of evolution equations, i.e., equations which describe the development of a system in time. We consider the following so called abstract Cauchy problem

$$(\mathrm{ACP}) \begin{cases} \dot{x}(t) = Ax(t), & t \geqslant 0, \\ x(0) = x_0, \end{cases} \tag{2.2}$$

where $A : \mathcal{D}(A) \subset X \to X$ denotes a closed and densely defined linear operator. This leads to the question whether (2.2) has a unique solution. This property is also known as the well-posedness of (2.2). For a bounded operator $A \in \mathcal{L}(X)$ or a matrix $A \in \mathbb{C}^{n \times n}$ the solution of (ACP) is given by $x(t) = \mathrm{e}^{tA} x_0$ where

$$\mathrm{e}^{tA} := \sum_{n=1}^{\infty} \frac{(tA)^n}{n!}, \quad t \geqslant 0. \tag{2.3}$$

However, (2.3) does not make sense for general unbounded operators A.
In the following we introduce the concepts for solutions of (ACP), c.f. [EN00, Section II.6].

Definition 2.1.1. Let $x : [0, \infty) \to X$ be a continuous function. Then:

1. The function x is called *classical solution* of (ACP) if x is differentiable, $x(t) \in \mathcal{D}(A)$ for all $t \geqslant 0$ and x satisfies equation (ACP).
2. The function x is called *mild solution* of (ACP) if $\int_0^t x(s)ds \in \mathcal{D}(A)$ and
$$x(t, x_0) := x(t) = x_0 + A \int_0^t x(s)ds, \ t \geqslant 0. \tag{2.4}$$

Using the concepts of solution we can now introduce well-posedness for the abstract Cauchy problem.

Definition 2.1.2. The abstract Cauchy problem (ACP) is *well-posed* if

1. $\mathcal{D}(A)$ is dense in X;
2. For every $x_0 \in \mathcal{D}(A)$ there exists a unique classical solution;
3. For every sequence $(x_n)_{n\in\mathbb{N}} \subset \mathcal{D}(A)$ with $\lim_{n\to\infty} x_n = 0$ it holds $\lim_{n\to\infty} x(t; x_n) = 0$ uniformly on compact intervals $[0; t_0]$.

Well-posedness of (ACP) is closely related to the notion of C_0-semigroups, see also Theorem 2.1.8. They can be seen as a generalization of the exponential function. C_0-semigroups goes back to the work of Hille and Yoshida [Hil48] and [Yos48] and are studied in detail in the monographs by Engel and Nagel [EN00], Pazy [Paz83] and Goldstein [Gol85].

Definition 2.1.3. A family $(T(t))_{t\geqslant 0} \in \mathcal{L}(X)$ of bounded operators is called a *strongly continuous (operator) semigroup*, or C_0-semigroup for short, if it has the following properties:

1. $T(t + s) = T(s)T(t)$ for all $t, s \geqslant 0$,
2. $T(0) = I$,
3. $\lim_{t\searrow 0} \|T(t)x - x\| = 0$ for all $x \in X$.

If property 1. holds for all $t, s \in \mathbb{R}$, the family $(T(t))_{t\geqslant 0} \in \mathcal{L}(X)$ is called a *strongly continuous group*, or C_0-group for short.

The strong continuity implies that $T(\cdot)x \in C(\mathbb{R}_+, X)$ for every $x \in X$.
The following example shows that the notation of C_0-semigroups is also consistent for bounded operators A.

Example 2.1.4. Let $A \in \mathbb{C}^{n\times n}$ or $A \in \mathcal{L}(X)$. Then $T(t) = \mathrm{e}^{tA}, \ t \geqslant 0$, is a C_0-semigroup and even a C_0-group.

For a C_0-semigroup $(T(t))_{t\geqslant 0}$ we define its generator.

Definition 2.1.5. The operator $A : \mathcal{D}(A) \subset X \to X$ with
$$Ax := \lim_{t\searrow 0} \frac{T(t)x - x}{t}$$
$$\mathcal{D}(A) = \{x \in X \mid \lim_{t\searrow 0} \frac{T(t)x - x}{t} \text{ exists in } X\}$$
is called the *(infinitesimal) generator* of the C_0-semigroup $(T(t))_{t\geqslant 0}$.

In the following we mention some of the important properties of generators of C_0-semigroups.

Lemma 2.1.6. ([EN00, Lemma II.1.3]) *Let A be the generator of a C_0-semigroup $(T(t))_{t\geqslant 0}$. Then the following holds:*

1. *$x \in \mathcal{D}(A)$ implies $T(t)x \in \mathcal{D}(A)$ and*
$$\frac{d}{dt}T(t)x = T(t)Ax = AT(t)x \quad \forall t \geqslant 0.$$
2. *For every $x \in X$ and every $t \geqslant 0$ it holds $\int_0^t T(s)x\, ds \in \mathcal{D}(A)$ and*
$$T(t)x - x = A\int_0^t T(s)x\, ds.$$
3. *For every $x \in \mathcal{D}(A)$ and every $t \geqslant 0$ it holds*
$$T(t)x - x = \int_0^t T(s)Ax\, ds.$$

Proposition 2.1.7. ([EN00, Theorem II.1.4]) *Let $(T(t))_{t\geqslant 0}$ be a C_0-semigroup on X with generator A. Then A is linear, closed, densely defined and determines the C_0-semigroup $(T(t))_{t\geqslant 0}$ uniquely.*

The next theorem describes the relationship between the well-posedness of the abstract Cauchy problem (ACP) and the generator of a C_0-semigroup and is a combination of Corollary II.6.9 and Proposition II.6.2 in [EN00].

Theorem 2.1.8. *Let $A : \mathcal{D}(A) \subset X \to X$ be a closed linear operator. Then the following statements are equivalent:*

1. *The abstract Cauchy problem* (2.2) *is well-posed.*
2. *A generates a C_0-semigroup on X.*

In particular, for every $x_0 \in \mathcal{D}(A)$ the unique classical solution of (2.2) *is given by $x(t) := T(t)x_0$.*

In the following, we mention some properties of C_0-semigroups, which can be found in [EN00].

Proposition 2.1.9. ([EN00, Proposition I.5.5]) *For a C_0-semigroup $(T(t))_{t\geqslant 0}$ on X there exist constants $M \geqslant 1$ and $\omega \in \mathbb{R}$ such that*
$$\|T(t)\| \leqslant M\mathrm{e}^{\omega t}, \quad t \geqslant 0. \tag{2.5}$$

Definition 2.1.10. Let $(T(t))_{t\geqslant 0}$ be a C_0-semigroup on X. Then $(T(t))_{t\geqslant 0}$ is

- a *bounded* C_0-semigroup if there exists $M > 0$ such that $\|T(t)\| \leqslant M$,for all $t \geqslant 0$;

- a *contractive* C_0-semigroup, if it is bounded with $M = 1$, i.e., $\|T(t)\| \leqslant 1$.

Moreover, a C_0-group $(T(t))_{t\in\mathbb{R}}$ is called a *unitary group*, if $\|T(t)x\| = \|x\| \ \forall x \in X$ and $t \in \mathbb{R}$.

Definition 2.1.11. Let $(T(t))_{t\geqslant 0}$ be a C_0-semigroup on X with generator A. Then its *growth bound* is defined by

$$\omega_0(A) := \inf\{\omega \in \mathbb{R} \mid \exists M_\omega \geqslant 1 \text{ such that } \|T(t)\| \leqslant M_\omega \mathrm{e}^{\omega t}, t \geqslant 0\}. \tag{2.6}$$

Furthermore,

$$s(A) := \sup\{\operatorname{Re} s \mid s \in \sigma(A)\} \tag{2.7}$$

denotes the *spectral bound* of A.

If the growth bound ω_0 is negative, then the corresponding C_0-semigroup is called *exponentially stable.* We note that for an exponentially stable C_0-semigroup the right half plane $\{s \in \mathbb{C} \mid \operatorname{Re} s > 0\}$ of $\mathbb{C}$ lies in the resolvent set $\rho(A)$ of its generator A, c.f. [EN00, Theorem V.1.11].
Now, we describe some properties of the resolvent operator.

Proposition 2.1.12. ([EN00, Theorem II.1.10]) *Let A be the generator of a C_0-semigroup $(T(t))_{t\geqslant 0}$ with growth bound $\omega_0 \in \mathbb{R}$ and $\omega \in \mathbb{R}$, $M \geqslant 1$ are the constants described in Proposition 2.1.9. Then for all $s \in \mathbb{C}$ with $\operatorname{Re} s > \omega_0$, it holds that s lies in the resolvent set of A, i.e., $s \in \rho(A)$ and the integral*

$$R(s) := \int_0^\infty \mathrm{e}^{-st} T(t)x \, ds \tag{2.8}$$

exists for all $x \in X$ and $R(s) = (s - A)^{-1}$. Moreover, the following estimate for the resolvent holds:

$$\left\|(s - A)^{-1}\right\| \leqslant \frac{M}{\operatorname{Re} s - \omega} \text{ for all } \operatorname{Re} s \geqslant \omega \tag{2.9}$$

Definition 2.1.13. The linear operator $A : \mathcal{D}(A) \subset X \to X$ has *compact resolvent* if there exists an $s \in \rho(A)$ such that the operator $(s - A)^{-1}$ is a compact operator.

Note that by [EN00, Proposition II.4.25] A has compact resolvent if the embedding of $D(A)$ equipped with the graph norm in X is compact.
In Theorem 2.1.8 we have seen the relation between the abstract Cauchy problem (ACP) and the corresponding C_0-semigroup. Thus, the question occurs under which conditions A generates a contraction C_0-semigroup. It is answered by Hille and Yoshida in 1948, c.f. [Hil48] and [Yos48], and reformulated in a more applicable way by Lumer and Phillips in 1961, [LP61], which we state here.

Theorem 2.1.14. ([EN00, Theorem II.3.15]) *Let $A : \mathcal{D}(A) \subset X \to X$ be a linear, densely defined, and closed operator on a Hilbert space X. Then A generates a contraction C_0-semigroup on X if and only if*

1. *A is dissipative and*
2. $s - A : \mathcal{D}(A) \subset X \to X$ *is surjective for one (and then for all)* $s > 0$.

A simpler characterization of generators of contraction C_0-semigroups is given in the following corollary.

Corollary 2.1.15. ([EN00, Corollary II.3.17]) *Let* $A : \mathcal{D}(A) \subset X \to X$ *be a linear, densely defined, and closed operator on a Hilbert space* X. *Then* A *generates a contraction* C_0*-semigroup on* X *if and only if* A *and* A^* *are dissipative.*

We now formulate the Theorem of Stone, which characterizes generators of unitary groups.

Theorem 2.1.16. ([EN00, II.3.24]). *Let* $A : \mathcal{D}(A) \subset X \to X$ *be a linear, densely defined, and closed operator on a Hilbert space* X. *Then the following statements are equivalent:*

1. *A generates a unitary group* $(T(t))_{t\in\mathbb{R}}$ *on* X;
2. *A is skew-adjoint,*
3. A *and* $-A$ *both generate a contraction* C_0*-semigroup.*

We close this section with the result that the generation of C_0-semigroups is inherited on closed subspaces. We recall that a closed subspace $V \subset X$ is called $(T(t))_{t\geqslant 0}$*-invariant* if $T(t)V \subseteq V$ for all $t \geqslant 0$.

Proposition 2.1.17. ([CZ95, Lemma 2.5.3]) *Let* A *generate a* C_0*-semigroup* $(T(t))_{t\geqslant 0}$ *on* X. *In this case the restriction* $(T(t)|_V)_{t\geqslant 0}$ *is again a* C_0*-semigroup with generator* $A|_V$ *on* V, *where* $A|_V = Av$ *for* $v \in \mathcal{D}(A|_V) = \mathcal{D}(A) \cap V$ *and* $(T(t)|_V)_{t\geqslant 0}$ *is generated by the part of* A *in* V.

2.2 Preliminaries on systems theory

The standard formulation of systems in system theory extends the formulation of a partial differential equation as a (homogenous) abstract Cauchy problem taking into account the interaction of the system with its environment. Thus, in addition to the state space X, we need an input space U and an output space Y. In general, X, U, and Y may be Banach spaces, but within this thesis we focus on the Hilbert space setting. Thus, in the following X, U, and Y are supposed to be Hilbert spaces. In this setting the operators in the standard formulation of a system are not necessary bounded in general, but they are bounded in a weaker way. To see this we introduce the extrapolation and the interpolation space.
Therefore, we need to introduce some notation and concepts, which are well-known and can be found in e.g. [EN00, Chapter II] and [TW09, Chapter 2].

Definition 2.2.1. Let $A : \mathcal{D}(A) \subset X \to X$ be the generator of a C_0-semigroup $(T(t))_{t\geqslant 0}$. Hence, $\rho(A) \neq \emptyset$ and let be $s \in \rho(A)$.

1. Then the *extrapolation space* X_{-1} is defined as the completion of X with respect to the norm

$$\|x\|_{-1} = \left\|(s-A)^{-1}x\right\|, \quad x \in X.$$

2. The *interpolation space* X_1 is defined as $\mathcal{D}(A)$ equipped with the norm

$$\|x\|_{1} = \|(s-A)x\|, \quad x \in X.$$

Note that the definitions of X_1 and X_{-1} are independent of the choice of $s \in \rho(A)$, since different $s \in \rho(A)$ lead to equivalent norms. The following inclusions are dense with a continuous embedding:

$$X_1 \subset X \subset X_{-1}.$$

Note that the space X_1 is a Hilbert space and A can be seen as an operator in $\mathcal{L}(X_1, X)$. Then we consider the restriction and the continuous extension of $(T(t))_{t\geqslant 0}$ to the interpolation and the extrapolation space, respectively.

Proposition 2.2.2. *Let A be the generator of a C_0-semigroup on X. Then the following statements hold:*

1. *$(T_1(t))_{t\geqslant 0}$, the restriction of $(T(t))_{t\geqslant 0}$ to X_1, is a C_0-semigroup on X_1, with generator*

$$A_1 x = Ax,\ x \in \mathcal{D}(A_1),\ \mathcal{D}(A_1) = \mathcal{D}(A^2).$$

2. *$(T_{-1}(t))_{t\geqslant 0}$, the continuous extension of $(T(t))_{t\geqslant 0}$ to X_{-1}, is a C_0-semigroup on X_{-1}, whose generator $A_{-1} \in \mathcal{L}(X, X_{-1})$, is the unique bounded extension of A.*

Moreover, we can identify X_{-1} with the dual space of $\mathcal{D}(A^*)$ with respect to the pivot space X, that is $X_{-1} = \mathcal{D}(A^*)'$. Now, let $A_{-1} \in \mathcal{L}(X, X_{-1})$ be the extension of the operator A describing the dynamics of the system, $B \in \mathcal{L}(U, X_{-1})$ denotes the *control operator*, $C \in \mathcal{L}(X_1, Y)$ the *observation operator* and $D \in \mathcal{L}(U, Y)$ the bounded *feedthrough* operator mapping from the input to the output. Then the standard formulation in system theory for a *control system* $\Sigma(A, B, C, D)$ is given by

$$\dot{x}(t) = A_{-1}x(t) + Bu(t), \quad t \geqslant 0, \quad x(0) = x_0, \tag{2.10}$$

$$y(t) = Cx(t) + Du(t), \quad t \geqslant 0. \tag{2.11}$$

Note that we denote by $\Sigma(A, B)$ a control system $\Sigma(A, B, C, D)$ with $C = D = 0$ and by $\Sigma(A, C)$ an observation system $\Sigma(A, B, C, D)$ with $B = D = 0$. We consider the first equation (2.10) as an abstract inhomogeneous Cauchy problem on the extrapolation space X_{-1} and we give its mild solution for $x_0 \in X \subset X_{-1}$.

Definition 2.2.3. For $x_0 \in X$ and $u \in L^2(0,t;U)$ the mild solution of (2.10) is given by the variation of parameters formula

$$x(t) = T(t)x_0 + \int_0^t T_{-1}(t-s)Bu(s)ds, \quad t \geqslant 0. \tag{2.12}$$

We note that even for initial values $x_0 \in X$ the values $x(t)$ of the solution may lie in the extrapolation space X_{-1}: The control operator is a map $B \in \mathcal{L}(U, X_{-1})$, $T_{-1}(t)$ acts on X_{-1}, and thus $\int_0^t T_{-1}(t-s)Bu(s)ds \in X_{-1}$.
To ensure that the solution $x(t)$ lies in X we introduce the idea of admissibility of control operators following Chapters 4, 6, and 11 in [TW09].

Definition 2.2.4. A control operator operator $B \in \mathcal{L}(U, X_{-1})$ is an *admissible control operator for* $(T(t))_{t\geqslant 0}$ if for all $t \geqslant 0$

$$\int_0^t T_{-1}(t-s)Bu(s)\, ds \in X$$

for every $u \in L^2(0,t;U)$.

Admissibility implies that the mild solution x of (2.10) satisfies $x \in C(0,t;X)$ for every initial condition $x_0 \in X$ and every $u \in L^2(0,t;U)$.

Proposition 2.2.5. ([TW14, Proposition 4.4.6]) *Let $B \in \mathcal{L}(U, X_{-1})$ be an admissible control operator for $(T(t))_{t\geqslant 0}$. Then for $\omega > \omega_0(A)$ exists a constant $M_\omega > 0$ such that*

$$\|(s-A_{-1})^{-1}B\|_{\mathcal{L}(U,X)} \leqslant \frac{M_\omega}{\sqrt{\operatorname{Re} s - \omega}} \text{ for } \operatorname{Re} s \geqslant \omega. \tag{2.13}$$

In the same manner as in the motivation of admissibility for control operators, it might happen that for $x \in \mathcal{D}(A)$ the solution x does not lie in the domain of A implying that $Cx(t)$ is not well-defined. To avoid this we introduce admissibility for observation operators.

Definition 2.2.6. An observation operator $C \in \mathcal{L}(X_1, Y)$ is an *admissible observation operator for* $(T(t))_{t\geqslant 0}$ if there exists a positive constant $K > 0$ such that

$$\int_0^\infty \|CT(t)x_0\|^2\, dt \leqslant K\, \|x_0\|^2, \quad x_0 \in \mathcal{D}(A).$$

Definition 2.2.7. A system $\Sigma(A, B)$ with an admissible control operator $B \in \mathcal{L}(U, X_{-1})$ is *exactly controllable*, if there exists a time $\tau > 0$ such that for all $x_1 \in X$ there exists a control function $u \in L^2(0,\tau;U)$ such that the corresponding mild solution satisfies $x(0) = 0$ and $x(\tau) = x_1$.

Note that this definition of exactly controllable is often also denoted as *exactly controllable in finite time*.

Definition 2.2.8. A system $\Sigma(A, C)$ with an admissible observation operator $C \in \mathcal{L}(X_1, Y)$ is *exactly observable*, if there exists a positive constant k such that

$$\int_0^\tau \|CT(t)x_0\|^2\, dt \geqslant k\, \|x_0\|^2, \quad x_0 \in \mathcal{D}(A).$$

This is equivalent to the fact, that every initial state $x_0 \in X$ can be uniquely and continuously reconstructed from the output $y \in L^2(0, \tau; Y)$.
Furthermore, we note that the concepts of controllability and observability are dual in the following sense.

Proposition 2.2.9. ([TW09, Theorem 11.2.1]) *Let $B \in \mathcal{L}(U, X_{-1})$ an admissible control operator for the C_0-semigroup $(T(t))_{\geqslant 0}$ generated by A. Then the system $\Sigma(A, B)$ is exactly controllable if and only if $\Sigma(A^*, B^*)$ is exactly observable.*

In the following we formulate the so-called Hautus test giving a necessary condition for exact observability. In [RW94] the Hautus test is formulated for systems which are exactly observable in infinite time, i.e., there exists a positive constant $\tilde{k}$ such that $\int_0^\infty \|CT(t)x_0\|^2 \ dt \geqslant \tilde{k} \|x_0\|^2, \quad x_0 \in \mathcal{D}(A)$. Of course, this concept of exact observability follows from our notion of exact observability.

Theorem 2.2.10. ([RW94, Theorem 1]) *Let A be the generator of an exponentially stable C_0-semigroup and let $C \in \mathcal{L}(X_1, Y)$ be an admissible observation operator. If the system $\Sigma(A, C)$ is exactly observable, then there exists a positive constant m such that*

$$\frac{1}{|\mathrm{Re}\, s|^2} \|(s - A)x\|^2 + \frac{1}{|\mathrm{Re}\, s|} \|Cx\|^2 \geqslant m \|x\|^2, \quad \mathrm{Re}\, s < 0, \ x \in \mathcal{D}(A). \tag{2.14}$$

Partial differential equations which can be handled via control and observation of the boundary occur frequently in applications. Therefore, we introduce the so-called boundary control and observation systems. In Theorem 2.2.22 we will see that these kind of systems even fit in the standard formulation (2.10)-(2.11). The following is extracted from the Chapters 11 and 13 in [JZ12].

Definition 2.2.11. Let X, U, and Y denote Hilbert spaces and let $\mathfrak{A} : \mathcal{D}(\mathfrak{A}) \subset X \to X$ and $\mathfrak{B} : \mathcal{D}(\mathfrak{A}) \to U$ be linear operators. Then $\mathfrak{S}(\mathfrak{A}, \mathfrak{B})$ is a *boundary control system* if the following hold:

1. The operator $A : \mathcal{D}(A) \subset X \to X$ with $\mathcal{D}(A) = \mathcal{D}(\mathfrak{A}) \cap \ker(\mathfrak{B})$ and $Ax = \mathfrak{A}x$ for $x \in \mathcal{D}(A)$ is the infinitesimal generator of a strongly continuous semigroup on X.

2. There exists a right inverse $\widetilde{B} \in \mathcal{L}(U, X)$ of $\mathfrak{B}$ in the sense that for all $u \in U$ we have $\widetilde{B}u \in \mathcal{D}(\mathfrak{A})$, $\mathfrak{B}\widetilde{B}u = u$ and $\mathfrak{A}\widetilde{B} : U \to X$ is bounded.

If it holds additionally for a linear operator $\mathfrak{C} : \mathcal{D}(\mathfrak{A}) \to Y$ the statement

3. the operator $\mathfrak{C}$ is bounded from $\mathcal{D}(A)$ to Y, where $\mathcal{D}(A)$ is equipped with the graph norm of A,

then the triple $\mathfrak{S}(\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ is a *boundary control and observation system.*

Note that a boundary control system is a boundary control and observation system with $\mathfrak{C} = 0$.

Remark 2.2.12. In the literature, see e.g. Staffans [Sta05] or Tucsnak and Weiss [TW09], there exists a slightly more general formulation of boundary control systems taking into account that $\mathcal{D}(\mathfrak{B}) \neq \mathcal{D}(\mathfrak{A})$ and $\mathcal{D}(\mathfrak{C}) \neq \mathcal{D}(\mathfrak{A})$, but then they have to satisfy $\mathcal{D}(\mathfrak{A}) \subset \mathcal{D}(\mathfrak{A})$ and $\mathcal{D}(\mathfrak{A}) \subset \mathcal{D}(\mathfrak{C})$.

We write operators of Definition 2.2.11 as a system of the following form:

$$\begin{aligned} \dot{x}(t) &= \mathfrak{A}x(t), \quad x(0) = x_0, \\ u(t) &= \mathfrak{B}x(t), \\ y(t) &= \mathfrak{C}x(t), \quad t \geqslant 0. \end{aligned} \tag{2.15}$$

Now we are interested in classical and mild solutions of (2.15).

Definition 2.2.13. Let $\mathfrak{S}(\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ be a boundary control and observation system, with initial value $x_0 \in \mathcal{D}(\mathfrak{A})$ and $u \in C^2(0, \infty; U)$. A function $x : [0, \infty) \to X$ is a *classical solution* of the boundary control and observation system if $x \in C^1(0, \infty; \mathcal{D}(\mathfrak{A}))$ and $x(t)$ satisfies the first two equations of (2.15) for every $t \geqslant 0$.

Lemma 2.2.14. ([JZ12, Lemma 13.1.5]) *Let $\mathfrak{S}(\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ be a boundary control and observation system and $x_0 \in \mathcal{D}(\mathfrak{A})$ and $u \in C^2(0, t; U)$ satisfying $\mathfrak{B}x_0 = u(0)$. Then the unique classical solution on $[0, t]$ of (2.15) is given by*

$$x(t) = T(t)x_0 + \int_0^t T(t-s)\mathfrak{A}\tilde{B}u(s)\, ds - A\int_0^t T(t-s)\tilde{B}u(s)\, ds, \tag{2.16}$$

where $\tilde{B}$ is described in Definition 2.2.11. This implies

$$y(t) = \mathfrak{C}T(t)x_0 + \mathfrak{C}\int_0^t T(t-s)\mathfrak{A}\tilde{B}u(s)\, ds - \mathfrak{C}A\int_0^t T(t-s)\tilde{B}u(s)\, ds.$$

In general, we are also interested in mild solutions, since the initial value x_0 might be an arbitrary element of X, not necessary in the domain of $\mathfrak{A}$, and we also want to allow arbitrary input functions $u \in L^2(0, t; U)$. Nevertheless, the solution x should be a continuous function with values in X and the output y should be a L^2-function. This leads to the definition of well-posedness.

Definition 2.2.15. The boundary control and observation system $\mathfrak{S}(\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ is called *well-posed* if there exist a $t > 0$ and $m_t \geqslant 0$ such that for every initial value $x_0 \in \mathcal{D}(\mathfrak{A})$ and every input function $u \in C^2(0, t; U)$ with $u(0) = \mathfrak{B}x_0$ the classical solution x and y satisfy

$$\|x(t)\|_X^2 + \int_0^t \|y(s)\|^2 ds \leqslant m_t \left(\|x_0\|_X^2 + \int_0^t \|u(s)\|^2 ds \right).$$

There exists a rich literature on well-posed systems, see e.g. Staffans [Sta05] or Tuscnak and Weiss [TW14]. In general, it is not easy to verify well-posedness for a boundary control and observation system. Nevertheless, the following proposition allows us to do so for a special class of systems.

Proposition 2.2.16. ([JZ12, Proposition 13.1.4]) *Let $\mathfrak{S}(\mathfrak{A},\mathfrak{B},\mathfrak{C})$ be a boundary control and observation system* (2.15)*. If every classical solution of* (2.15) *satisfies*

$$\frac{d}{dt}\left\|x(t)\right\|^2 = \left\|u(t)\right\|^2 - \left\|y(t)\right\|^2,$$

then the system is well-posed.

Now we formulate the mild solution of a well-posed boundary control system and observation system.

Definition 2.2.17. Let $\mathfrak{S}(\mathfrak{A},\mathfrak{B},\mathfrak{C})$ be a well-posed boundary control and observation system with initial value $x_0 \in X$ and $u \in L^2(0,t;U)$. Then the function $x(t)$ given in (2.16) is called *mild solution* and $x \in C(0,\infty,X)$.

Lemma 2.2.18. ([JZ12, Lemma 13.1.5]) *Let $\mathfrak{S}(\mathfrak{A},\mathfrak{B},\mathfrak{C})$ be a well-posed boundary control and observation system with initial value $x_0 \in X$ and $u \in L^2(0,t;U)$. Then the unique mild solution x is given by*

$$x(t) = T(t)x_0 + \int_0^t T_{-1}(t-s)(\mathfrak{A}\widetilde{B} - A_{-1}\widetilde{B})u(s)\,ds, \tag{2.17}$$

where $\tilde{B}$ is described in Definition 2.2.11.

For the class of boundary control and observation systems we introduce the concept of transfer functions following Chapter 12 in [JZ12].

Definition 2.2.19. Let $\mathfrak{S}(\mathfrak{A},\mathfrak{B},\mathfrak{C})$ be a boundary control and observation system. For $s \in \rho(A)$ and $u \in U$, $G(s)u$ is the unique solution of

$$\begin{aligned} sx &= \mathfrak{A}x, \ x_0 \in \mathcal{D}(A),\\ u &= \mathfrak{B}x,\\ y &= \mathfrak{C}x. \end{aligned}$$

Then $y = G(s)u$, $G(s) \in \mathcal{L}(U,Y)$, and $G : \rho(A) \to \mathcal{L}(U,Y)$ is called the *transfer function of the system $\mathfrak{S}$.*

For the following proposition we define the weighted L^2-spaces. For any Hilbert space H a function v is an element of the weighted L^2-space $L^2_\mu(0,\infty;H)$ if and only if $\mathrm{e}^{-\mu t}v \in L^2(0,\infty,H)$. $L^2_\mu(0,\infty;H)$ equipped with the norm $\|v\|_{L^2_\mu(0,\infty;H)} := \left\|\mathrm{e}^{-\mu t}v\right\|_{L^2(0,\infty;H)}$ is a Hilbert space.

Proposition 2.2.20. ([JZ12, Theorem 12.1.3] and [Wei94, Proposition 4.1 and Proposition 3.2]) *The transfer function of a well-posed system $\mathfrak{S}(\mathfrak{A},\mathfrak{B},\mathfrak{C})$ is given by*

$$G(s) = \mathfrak{C}(s-A)^{-1}(\mathfrak{A}\widetilde{B} - s\widetilde{B}) + \mathfrak{C}\widetilde{B}, \qquad s \in \rho(A).$$

For a well-posed system there exists a $\mu \geqslant w_0$ such that the transfer function equals the Laplace transform of the linear and bounded mapping

$$\mathbb{L} : L^2_\mu(0,\infty;U) \to L^2_\mu(0,\infty;Y).$$

Furthermore, the transfer function is bounded on the right half plane $\mathbb{C}_\mu := \{s \in \mathbb{C} \mid \operatorname{Re} s > \mu\}$.

However, the boundedness of the transfer function on a right half plane does not imply the convergence along the real axis. But convergence along the real axis implies a suitable representation of the feedthrough operator.

Definition 2.2.21. [JZ12, Definition 13.1.11] A well-posed boundary control and observation system $\mathfrak{S}(\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ with transfer function G is called *regular* if $\lim_{s \in \mathbb{R}, s \to \infty} G(s)$ exists. In this case the *feedthrough operator* D is defined as

$$D := \lim_{s \in \mathbb{R}, s \to \infty} G(s).$$

The next assertion can be found in Chapter 13, Section 1 in [JZ12] and makes the connection between boundary control and observation systems and the standard formulation in system theory.

Theorem 2.2.22. *Every regular well-posed boundary control and observation system* $\mathfrak{S}(\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ (2.15) *can equivalently formulated in the standard formulation* (2.10)-(2.11) *in system theory with a control operator* $B = (\mathfrak{A}\widetilde{B} - A_{-1}\widetilde{B})$.

So far, we have only considered *open-loop systems*, that is, the input $u(t)$ is independent of the output $y(t)$, see Figure 2.1. Systems, where input and output are connected via a feedback law

$$u(t) = Fy(t) + v(t), \tag{2.18}$$

are called *closed-loop systems*, see Figure 2.2. Here F denotes the so called *feedback operator* and $v(t)$ the new input.

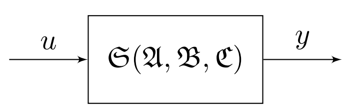

Figure 2.1: open-loop system $\mathfrak{S}(\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$

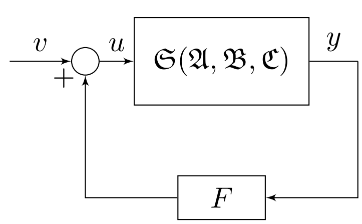

Figure 2.2: closed-loop system $\mathfrak{S}(\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ with feedback F

The proofs of the following results about feedback systems can be found in [Wei94] and [Sta05]. We give a brief overview on closed-loop systems and start with the considerations which operators are admissible as a feedback operator. Then we will see that not only well-posedness is preserved under an admissible feedback operator, but also exact controllability.

Definition 2.2.23. ([Wei94, Proposition 4.9]) Let the system $\mathfrak{S}(\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ be a regular boundary control and observation system (2.15) and we denote by $D \in \mathcal{L}(U, Y)$ the corresponding feedthrough operator. Assume that for the transfer function G holds

$$\lim_{r\to\infty} \sup_{\omega\in\mathbb{R}} \|G(i\omega + r) - D\| = 0. \tag{2.19}$$

Then, an operator $F \in \mathcal{L}(Y, U)$ is called an *admissible feedback operator* for a regular boundary control and observation system (2.15), if $I - DF$ is invertible.

Proposition 2.2.24. ([JZ12, Theorem 13.1.12]) *Let $\mathfrak{S}(\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ be a well-posed boundary control and observation system* (2.15)*. Assume that F is an admissible feedback operator. Then the closed-loop system $\mathfrak{S}(\mathfrak{A}, (\mathfrak{B} - F\mathfrak{C}), \mathfrak{C})$, i.e.,*

$$\begin{aligned}
\dot{x}(t) &= \mathfrak{A}x(t), \quad x(0) = x_0, \\
v(t) &= (\mathfrak{B} - F\mathfrak{C})x(t), \\
y(t) &= \mathfrak{C}x(t), \quad t \geqslant 0.
\end{aligned} \tag{2.20}$$

with input v and output y is a well-posed boundary control and boundary observation system.

Proposition 2.2.25. ([Wei94, cf. Remark 6.5]) *Let $\mathfrak{S}(\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ be a well-posed boundary control and observation system* (2.15)*. Assume that F is an admissible feedback operator. Then the closed-loop system $\mathfrak{S}(\mathfrak{A}, (\mathfrak{B} - F\mathfrak{C}), \mathfrak{C})$ is exactly controllable if and only if the open-loop system $\mathfrak{S}(\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ is exactly controllable.*

Chapter 3

Introduction to port-Hamiltonian systems

Here, we consider a special class of partial differential equation on a one-dimensional space, which has an additional structure motivated by the structure of the port-Hamiltonian systems. Thus, we consider linear port-Hamiltonian systems on infinite-dimensional spaces, which we model as boundary control and observation systems as introduced in Chapter 2. Well-known examples in physics and other applications are the transport equation, the wave equation modelling a vibrating string or a transmission line, and beam equations modelling the Timoshenko beam. This class of systems makes use of the physical structure of the equations. A nice feature of the port-Hamiltonian setting is that it allows us to consider in particular boundary control and observation, which is important in applications, e.g. for the wave equation. Having this at hand we can also consider systems having an input and an output.

3.1 Class of port-Hamiltonian systems

In this first section we introduce port-Hamiltonian systems which have neither input nor output. In the following section we provide the class of port-Hamiltonian systems with an input to control and an output to observe these systems.

Assumption 3.1.1. Let $P_1 \in \mathbb{C}^{d\times d}$ be an invertible Hermitian matrix, $P_0 \in \mathbb{C}^{d\times d}$ a skew-symmetric matrix, $\begin{bmatrix}\widetilde{W}_1 & \widetilde{W}_0\end{bmatrix}$ a full row rank $d \times 2d$ matrix, and $\mathcal{H}(\zeta)$ a positive $d\times d$ Hermitian matrix for a.e. $\zeta \in (0,1)$ satisfying $\mathcal{H}, \mathcal{H}^{-1} \in L^\infty(0,1;\mathbb{C}^{d\times d})$. Since $\mathcal{H}(\zeta)$ is positive definite and P_1 a Hermitian matrix, $P_1\mathcal{H}(\zeta)$ is similar to a Hermitian matrix, and thus, the matrix $P_1\mathcal{H}(\zeta)$ can be diagonalized as $P_1\mathcal{H}(\zeta) = S^{-1}(\zeta)\Delta(\zeta)S(\zeta)$, where $\Delta(\zeta)$ is a diagonal matrix and $S(\zeta)$ is an invertible matrix for almost every $\zeta \in (0,1)$. We suppose the technical assumption that S^{-1}, S, $\Delta : [0,1] \to \mathbb{C}^{d\times d}$ are continuously differentiable.

Note that, for instance, in [Kat95, Chapter II] conditions for $P_1\mathcal{H}$ such that S^{-1}, S, $\Delta : [0,1] \to \mathbb{C}^{d\times d}$ are continuously differentiable, are described.

With this assumption in mind, we introduce that type of partial differential equation which is the main subject of this thesis.

Definition 3.1.2. Let Assumption 3.1.1 be fulfilled. A system which is given by the following partial differential equation

$$\begin{aligned}\frac{\partial x}{\partial t}(\zeta,t) &= \left(P_1\frac{\partial}{\partial\zeta}+P_0\right)(\mathcal{H}(\zeta)x(\zeta,t)),\ \zeta\in(0,1),\ t\geqslant 0\\ x(\zeta,0) &= x_0(\zeta),\\ 0 &= \begin{bmatrix}\widetilde{W}_1 & \widetilde{W}_0\end{bmatrix}\begin{bmatrix}(\mathcal{H}x)(1,t)\\(\mathcal{H}x)(0,t)\end{bmatrix},\ t>0,\end{aligned} \tag{3.1}$$

is called a *port-Hamiltonian system.*

Remark 3.1.3. To shorten the notation, we use the term port-Hamiltonian system instead of linear, first order port-Hamiltonian system.

The energy, also denoted as Hamiltonian, of a port-Hamiltonian system can be described by

$$E(t)=\frac{1}{2}\int_0^1 x(\zeta,t)^*\mathcal{H}(\zeta)x(\zeta,t)d\zeta. \tag{3.2}$$

We choose all states with finite energy as the state space X, i.e., all functions x such that $\frac{1}{2}\int_0^1 x(\zeta,t)^*\mathcal{H}(\zeta)x(\zeta,t)d\zeta$ is finite. Due to the requirements on $\mathcal{H}$ in Assumption 3.1.1 these are all functions, which are square integrable over the unit interval. Thus, we set the state space $X=L^2(0,1;\mathbb{C}^d)$ and we equip it not with the standard L^2-norm, but with the inner product

$$\langle f,g\rangle=\frac{1}{2}\int_0^1 f(\zeta)^*\mathcal{H}(\zeta)g(\zeta)\,d\zeta,\quad f,g\in L^2(0,1;\mathbb{C}^d).$$

Then the squared norm of an element $x\in X$ equals the energy of the state x of the port-Hamiltonian system and therefore, the norm is called energy norm on the energy space X. We point out, that the energy norm and the standard L^2-norm on X are equivalent. For the energy of port-Hamiltonian systems holds the following power balance equation, which can be proved by integration by parts, cf. [JZ12, Lemma 7.1.5].

Proposition 3.1.4. *Let x denote the classical solution of the port-Hamiltonian system* (3.1)*. Then the balance equation*

$$\frac{dE}{dt}(t)=\frac{1}{2}[((\mathcal{H}x)(\zeta,t))^*P_1(\mathcal{H}x)(\zeta,t)]_0^1 \tag{3.3}$$

holds.

The power balance equation (3.3) explains the name of this class of systems by taking into account that the energy can also change via the boundary of the system.

Remark 3.1.5. Without loss of generality it is possible to consider only port-Hamiltonian systems on the unit interval instead of port-Hamiltonian systems on an arbitrary interval $[a, b]$. In fact, there is an isometric isomorphism α between the corresponding state spaces:

$$\begin{aligned} \alpha : L^2(a, b; \mathbb{C}^d) &\to L^2(0, 1; \mathbb{C}^d) \\ x(\cdot) &\mapsto x\left(\frac{\cdot - a}{b - a}\right) \end{aligned}$$

In what follows, we give some examples to illustrate that there are many physical systems which fit in the class of port-Hamiltonian systems. These and further examples can also be found in [Vil07], [JZ12], [JMZ15], [Aug16] or [JZ18], just to mention a few. Examples for a more general class of port-Hamiltonian systems can be found in Chapter 6 of this thesis as well.

Example 3.1.6. The following partial differential equation is called transport equation.

$$\frac{\partial x}{\partial t}(\zeta, t) = \frac{\partial}{\partial \zeta}\left(c(\zeta)x(\zeta, t)\right), \; x(\zeta, 0) = x_0(\zeta), \; \zeta \in (0, 1), \; t > 0, \tag{3.4}$$

$$0 = (cx)(1, t) - \mu(cx)(0, t), \; \mu \in \mathbb{C}, \; t > 0, \tag{3.5}$$

where $c(\zeta) : [0, 1] \to \mathbb{R}$ is a bounded, continuously differentiable function such that $c(\zeta) > 0$ for $\zeta \in [0, 1]$. This is the simplest port-Hamiltonian system with $P_1 = 1$, $P_0 = 0$ and a complex valued function $\mathcal{H}(\zeta) = c(\zeta)$. The boundary conditions 3.5 can reformulated in the port-Hamiltonian setting as

$$\begin{bmatrix} 1 & -\mu \end{bmatrix} \begin{bmatrix} (cx)(1, t) \\ (cx)(0, t) \end{bmatrix} = 0, \; \mu \in \mathbb{C}, \; t > 0.$$

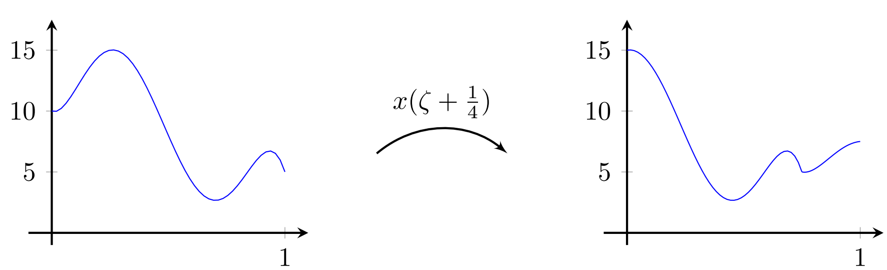

Figure 3.1: Sketch of the translation shift for $c \equiv 1$ and μ=2

The energy of the system is described by

$$E(t) = \frac{1}{2} \int_0^1 c(\zeta) \left| x(\zeta, t) \right|^2 \, d\zeta.$$

Example 3.1.7. The vibrating string can be described by the wave equation. We consider a string which is clamped at the left side and freely vibrating at the right side.

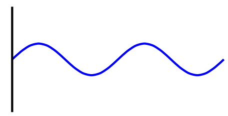

Figure 3.2: vibrating wave clamped at $\zeta = 0$

$$\frac{\partial^2 \omega}{\partial t^2}(\zeta,t) = \frac{1}{\rho(\zeta)}\frac{\partial}{\partial \zeta}\left(T(\zeta)\frac{\partial \omega}{\partial \zeta}(\zeta,t)\right),\ x(\zeta,0)=x_0(\zeta),\ \zeta \in (0,1),\ t>0, \tag{3.6}$$

$$0 = \begin{bmatrix} \frac{\partial \omega}{\partial t}(0,t) \\ T(1)\frac{\partial \omega}{\partial \zeta}(1,t) \end{bmatrix}, \quad t \geqslant 0, \tag{3.7}$$

where $w(\zeta,t)$ is the vertical position of the string at position ζ and time t, $T(\zeta) > 0$ is the Young's modulus of the string, and $\rho(\zeta) > 0$ is the mass density. We introduce as the new state variables

$$\begin{aligned} x_1(\zeta,t) &:= \rho(\zeta)\frac{\partial \omega}{\partial t}(\zeta,t) && \text{the momentum, and} \\ x_2(\zeta,t) &:= \frac{\partial \omega}{\partial \zeta}(\zeta,t) && \text{the strain.} \end{aligned}$$

Hence, we can model the wave equation (3.6) as

$$\frac{\partial}{\partial t}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}(\zeta,t) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\frac{\partial}{\partial \zeta}\left(\begin{bmatrix} \frac{1}{\rho(\zeta)} & 0 \\ 0 & T(\zeta) \end{bmatrix} x(\zeta,t)\right) \tag{3.8}$$

$$0 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} (\mathcal{H}x)(1,t) \\ (\mathcal{H}x)(0,t) \end{bmatrix},$$

where $x = \begin{bmatrix} x_1 & x_2 \end{bmatrix}^\top$.
Moreover, the energy of the system can be written in the chosen state variables as

$$E(t) = \frac{1}{2}\int_0^1 \frac{|x_1(\zeta,t)|^2}{\rho(\zeta)} + T(\zeta)\,|x_2(\zeta,t)|^2\, d\zeta.$$

Example 3.1.8. The Timoshenko beam equations model the effects in a vibrating beam and take into account shear and rotational effects. A beam, which is clamped at both sides, i.e., at $\zeta = 0$ and at $\zeta = 1$, can be modelled by

$$\begin{aligned} \rho(\zeta)\frac{\partial^2 \omega}{\partial t^2}(\zeta,t) &= \frac{\partial}{\partial \zeta}\left(K(\zeta)\left(\frac{\partial \omega}{\partial \zeta}(\zeta,t) - \Phi(\zeta,t)\right)\right),\ \zeta \in (0,1),\ t \geqslant 0, \\ I_{\rho(\zeta)}\frac{\partial^2 \Phi}{\partial t^2} &= \frac{\partial}{\partial \zeta}\left(EI(\zeta)\frac{\partial \Phi}{\partial \zeta}(\zeta,t)\right) + K(\zeta)\left(\frac{\partial \omega}{\partial \zeta}(\zeta,t) - \Phi(\zeta,t)\right), \\ \frac{\partial \omega}{\partial t}(0,t) &= \frac{\partial \Phi}{\partial t}(0,t) = \frac{\partial \omega}{\partial t}(1,t) = \frac{\partial \Phi}{\partial t}(1,t) = 0,\ x(\zeta,0) = x_0(\zeta), \end{aligned}$$

where $\omega(\zeta,t)$ denotes the transverse displacement of the beam and $\Phi(\zeta,t)$ the rotation angle of a filament of the beam. All physical parameters are positive

and continuously differentiable functions of ζ. $K(\zeta)$ denotes the shear modulus, $EI(\zeta)$ is the product of Young's modulus of elasticity and the moment of inertia of a cross section, $\rho(\zeta)$ is the mass per unit length and $I_\rho(\zeta)$ denotes the rotary moment of inertia of a cross section.
In order to model the Timoshenko beam as a port-Hamiltonian system, we introduce new state variables

$$\begin{aligned}
x_1(\zeta,t) &= \frac{\partial \omega}{\partial \zeta}(\zeta,t) - \Phi(\zeta,t) && \text{the shear displacement,}\\
x_2(\zeta,t) &= \rho(\zeta)\frac{\partial \omega}{\partial t}(\zeta,t) && \text{the momentum,}\\
x_3(\zeta,t) &= \frac{\partial \Phi}{\partial \zeta}(\zeta,t) && \text{the angular displacement,}\\
x_4(\zeta,t) &= I_\rho(\zeta)\frac{\partial \Phi}{\partial t}(\zeta,t) && \text{the angular momentum.}
\end{aligned}$$

The Timoshenko beam can be modelled as a port-Hamiltonian system using these new state variables. It can be written in the form of (3.1) with

$$P_1 = \begin{bmatrix} 0 & 1 & 0 & 0\\ 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0 \end{bmatrix}, \ \mathcal{H}(\zeta) = \begin{bmatrix} K(\zeta) & 0 & 0 & 0\\ 0 & \frac{1}{\rho(\zeta)} & 0 & 0\\ 0 & 0 & EI(\zeta) & 1\\ 0 & 0 & 0 & \frac{1}{I_\rho(\zeta)} \end{bmatrix},$$

$$P_0 = \begin{bmatrix} 0 & 0 & 0 & -1\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 1 & 0 & 0 & 0 \end{bmatrix},$$

and

$$\begin{bmatrix} \widetilde{W}_1 & \widetilde{W}_0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

The energy of the system is given by

$$\begin{aligned}
E(t) = \frac{1}{2}\int_0^1 K(\zeta)\left|x_1(\zeta,t)\right|^2 &+ \frac{1}{\rho(\zeta)}\left|x_2(\zeta,t)\right|^2\\
&+ EI(\zeta)\left|x_3(\zeta,t)\right|^2 + \frac{1}{I_\rho(\zeta)}\left|x_4(\zeta,t)\right|^2 \, d\zeta.
\end{aligned}$$

3.2 Generation theorems

To study the whole class of systems instead of considering each example separately, we aim to formulate port-Hamiltonian systems as abstract Cauchy problems (2.2). Hence, we introduce the port-Hamiltonian operator associated to (3.1) and study the question which port-Hamiltonian operators generate (contractive) C_0-semigroups.

Definition 3.2.1. Let $P_0, P_1, \mathcal{H}$ satisfy Assumption 3.1.1 and define $X := L^2(0,1;\mathbb{C}^d)$. Then the operator $A : \mathcal{D}(A) \subset X \to X$ defined by

$$Ax := \left(P_1 \frac{d}{d\zeta} + P_0\right)(\mathcal{H}x), \qquad x \in \mathcal{D}(A), \tag{3.9}$$

$$\mathcal{D}(A) := \left\{ x \in X \mid \mathcal{H}x \in \mathcal{W}^{1,2}(0,1;\mathbb{C}^d) \text{ and } \begin{bmatrix} \widetilde{W}_1 & \widetilde{W}_0 \end{bmatrix} \begin{bmatrix} (\mathcal{H}x)(1) \\ (\mathcal{H}x)(0) \end{bmatrix} = 0 \right\} \tag{3.10}$$

is called *port-Hamiltonian operator.*

For port-Hamiltonian systems the boundary conditions are often equivalently reformulated via the boundary flow and the boundary effort. We introduce them in the following.

Definition 3.2.2. For a port-Hamiltonian system we define the *boundary flow* $f_{\delta,\mathcal{H}x}$ and the *boundary effort* $e_{\delta,\mathcal{H}x}$ as

$$\begin{bmatrix} f_{\delta,\mathcal{H}x} \\ e_{\delta,\mathcal{H}x} \end{bmatrix} := \frac{1}{\sqrt{2}} \begin{bmatrix} P_1 & -P_1 \\ I & I \end{bmatrix} \begin{bmatrix} (\mathcal{H}x)(1) \\ (\mathcal{H}x)(0) \end{bmatrix}. \tag{3.11}$$

Define $\widetilde{W}_B = \begin{bmatrix} \widetilde{W}_1 & \widetilde{W}_0 \end{bmatrix}$. Then

$$\widetilde{W}_B \begin{bmatrix} (\mathcal{H}x)(1,t) \\ (\mathcal{H}x)(0,t) \end{bmatrix} = 0 \Leftrightarrow W_B \begin{bmatrix} f_{\delta,\mathcal{H}x} \\ e_{\delta,\mathcal{H}x} \end{bmatrix} = 0, \tag{3.12}$$

where

$$W_B = \widetilde{W}_B \sqrt{2} \begin{bmatrix} P_1 & -P_1 \\ I & I \end{bmatrix}^{-1} = \widetilde{W}_B \frac{1}{\sqrt{2}} \begin{bmatrix} P_1^{-1} & I \\ -P_1^{-1} & I \end{bmatrix}. \tag{3.13}$$

Thus, we get an equivalent formulation of the boundary conditions of a port-Hamiltonian system, which is useful for the next corollary.
The following assertion can be found in [LGZM05], [JZ12, Thereom 7.2.4], and [Aug16] and characterizes the generation of contraction C_0-semigroups for port-Hamiltonian operators.

Theorem 3.2.3. *Let A be a port-Hamiltonian operator given by* (3.9)-(3.10)*. Let $\widetilde{W}_B$ be a matrix with full row rank. Then the following statements are equivalent.*

1. *A is the generator of a contraction C_0-semigroup on X,*
2. *A is dissipative,*
3. *$W_B \Sigma W_B^* \geqslant 0$, where $\Sigma := \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$.*

If one of the above conditions is fulfilled, then A has a compact resolvent. Furthermore, A generates a unitary C_0-group if and only if $W_B \Sigma W_B^* = 0$ holds true.

Remark 3.2.4. Note that for symmetric matrices $M \in \mathbb{C}^{d\times d}$ we write $M \geqslant mI$ with a real constant m if $\langle v, Mv\rangle \geqslant m\,\|v\|^2$ for all $v \in \mathbb{C}^d$.

The proof of Theorem 3.2.3 can be found in [LGZM05, Theorem 4.1] and [JZ12, Theorem 7.2.4] and is even a Corollary of Theorem 6.1.3 in Chapter 6. Furthermore, the generation of C_0-semigroups for port-Hamiltonian operators is characterized in [JMZ15].

Theorem 3.2.5. *Let A be a port-Hamiltonian operator. Let $Z^+(\zeta)$ be the span of the eigenvectors of $P_1\mathcal{H}(\zeta)$ corresponding to the positive eigenvalues of $P_1\mathcal{H}(\zeta)$ and $Z^-(\zeta)$ be the span of the eigenvectors of $P_1\mathcal{H}(\zeta)$ corresponding to the negative eigenvalues of $P_1\mathcal{H}(\zeta)$. Then the following statements are equivalent:*

1. *A is the generator of a C_0-semigroup on X,*
2. $\widetilde{W}_1\mathcal{H}(1)Z^+(1) \oplus \widetilde{W}_0\mathcal{H}(0)Z^-(0) = \mathbb{C}^d$.

Using Theorem 2.1.16 we formulate the following corollary:

Corollary 3.2.6. *Let A be a port-Hamiltonian operator and let $Z^+(\zeta)$ and $Z^-(\zeta)$ be defined as in Theorem 3.2.5. Then the following statements are equivalent:*

1. *A is the generator of a C_0-group on X,*
2. $\widetilde{W}_1\mathcal{H}(1)Z^+(1) \oplus \widetilde{W}_0\mathcal{H}(0)Z^-(0) = \widetilde{W}_1\mathcal{H}(1)Z^-(1) \oplus \widetilde{W}_0\mathcal{H}(0)Z^+(0) = \mathbb{C}^d$.

With the knowledge of this generation theorems, we consider the Examples 3.1.6-3.1.8 again and study which of these port-Hamiltonian systems are well-posed.

Example 3.2.7. Continuation of Example 3.1.6. For the transport equation we define the associated port-Hamiltonian operator on $X = L^2(0,1;\mathbb{C})$

$$
\begin{aligned}
Ax &= \frac{\partial}{\partial\zeta}(cx), \quad x \in \mathcal{D}(A),\\
\mathcal{D}(A) &= \{x \in X \mid cx \in \mathcal{W}^{1,2}(0,1;\mathbb{C}) \text{ and } \begin{bmatrix}1 & -\mu\end{bmatrix}\begin{bmatrix}(cx)(1)\\(cx)(0)\end{bmatrix} = 0,\ \mu \in \mathbb{R}\}.
\end{aligned}
$$

Since for general port-Hamiltonian systems $\operatorname{Re}\langle Ax, x\rangle$ is not easy to determine to check the dissipativity, Theorem 3.2.3 gives an equivalent easy checkable matrix condition. But for this system we can even determine $\operatorname{Re}\langle Ax, x\rangle$ using integration by parts

$$
\begin{aligned}
\operatorname{Re}\langle Ax, x\rangle = \operatorname{Re}\langle \frac{\partial}{\partial\zeta}(cx), cx\rangle &= \frac{1}{2}\left[\langle cx, \frac{\partial}{\partial\zeta}(cx)\rangle + \langle \frac{\partial}{\partial\zeta}(cx), cx\rangle\right]\\
&= \frac{1}{2}\left[|cx|^2\,|_0^1 - \langle \frac{\partial}{\partial\zeta}(cx), cx\rangle + \langle \frac{\partial}{\partial\zeta}(cx), cx\rangle\right]\\
&= \frac{1}{2}\left[|(cx)(1)|^2 - |(cx)(0)|^2\right].
\end{aligned}
$$

The boundary conditions implies $|(cx)(1)|^2 = \mu^2 |(cx)(0)|^2$ and so

$$\operatorname{Re}\langle Ax, x\rangle = \frac{1}{2}\left[|(cx)(1)|^2 - |(cx)(0)|^2\right] = \frac{1}{2}(\mu^2 - 1)\,|(cx)(0)|^2 .$$

Thus, it holds $\operatorname{Re}\langle Ax, x\rangle \leqslant 0$ if and only if $|\mu| \leqslant 1$ and hence A generates a contraction C_0-semigroup if and only if $|\mu| \leqslant 1$.
Nevertheless, in this example, the matrix condition in Theorem 3.2.3 also answers the question whether A generates a contraction C_0-semigroup much faster: $\widetilde{W}_B = \begin{bmatrix}1 & -\mu\end{bmatrix}$ implies $W_B = \frac{1}{\sqrt{2}}\begin{bmatrix}1+\mu & 1-\mu\end{bmatrix}$ and thus $W_B \Sigma W_B^* = 1 - |\mu| \geqslant 0$ if and only if $|\mu| \leqslant 1$. Furthermore, we can use Theorem 3.2.5 to study whether A generates a C_0-semigroup for $|\mu| > 1$. Since $P_1\mathcal{H}(\zeta) = \mathcal{H}(\zeta) = c(\zeta)$, it holds $Z^+(1) = \mathbb{C}$ and $Z^-(0) = \{0\}$. Thus, A generates a C_0-semigroup even for $|\mu| > 1$.

Example 3.2.8. Continuation of Example 3.1.7. Again, we consider the wave equation which is in Example 3.1.7 written as the port-Hamiltonian system (3.8) and we define the associated port-Hamiltonian operator for $x = \begin{bmatrix}x_1 & x_2\end{bmatrix}^\top$ as

$$\begin{aligned} Ax &= \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\frac{\partial}{\partial\zeta}\left(\begin{bmatrix}\frac{1}{\rho(\zeta)} & 0\\ 0 & T(\zeta)\end{bmatrix}x\right), \qquad x \in \mathcal{D}(A),\\ \mathcal{D}(A) &= \left\{x \in X \mid \mathcal{H}x \in \mathcal{W}^{1,2}(0,1;\mathbb{C}^2) \text{ and } \widetilde{W}_B\begin{bmatrix}(\mathcal{H}x)(1)\\ (\mathcal{H}x)(0)\end{bmatrix} = 0\right\}, \end{aligned} \tag{3.14}$$

where $\mathcal{H} = \begin{bmatrix}\frac{1}{\rho(\zeta)} & 0\\ 0 & T(\zeta)\end{bmatrix}$ and $\widetilde{W}_B = \begin{bmatrix}0 & 0 & 1 & 0\\ 0 & 1 & 0 & 0\end{bmatrix}$ describes the boundary conditions. These boundary conditions of A model the situation where the string is clamped at the left side and free vibrating at the right side. Then it holds due to (3.13)

$$W_B = \frac{1}{2}\begin{bmatrix}0 & -1 & 1 & 0\\ 1 & 0 & 0 & 1\end{bmatrix} \tag{3.15}$$

and thus, $W_B \Sigma W_B^* = 0$. Hence, A generates a contraction C_0-semigroup and moreover a unitary C_0-group. To illustrate that a port-Hamiltonian operator does not always generate a unitary C_0-group, we consider in a slightly modified setting the port-Hamiltonian operator (3.14) with boundary conditions described by

$$\widetilde{W}_B = \begin{bmatrix}0 & 0 & \kappa & 1\\ 0 & 1 & 0 & 0\end{bmatrix}, \quad \kappa > 0,$$

which model a vibrating string with an amplifier at the left end and free at the right end. Then it holds

$$W_B = \frac{1}{\sqrt{2}}\begin{bmatrix}-1 & -\kappa & \kappa & 1\\ 1 & 0 & 0 & 1\end{bmatrix}$$

and $W_B \Sigma W_B^* = \begin{bmatrix}-2\kappa & 0\\ 0 & 0\end{bmatrix}$, which is not a positive semi-definite matrix. Thus, due to Theorem 3.2.3 the wave equation equipped with these boundary conditions does not generate a contraction C_0-semigroup.

In the following, we study the question, whether the operator generates at least a C_0-semigroup. Theorem 3.2.5 gives a helpful characterization for C_0-semigroup generation of port-Hamiltonian operators. This part of the example can also be found in [JMZ15]. Defining $\gamma = \sqrt{T(\zeta)/\rho(\zeta)}$, the matrix function $P_1\mathcal{H}$ can be factorized as

$$P_1\mathcal{H} = \underbrace{\begin{bmatrix} \gamma & -\gamma \\ \rho^{-1} & \rho^{-1} \end{bmatrix}}_{S^{-1}} \underbrace{\begin{bmatrix} \gamma & 0 \\ 0 & -\gamma \end{bmatrix}}_{\Delta} \underbrace{\begin{bmatrix} (2\gamma)^{-1} & \rho/2 \\ (2\gamma)^{-1} & \rho/2 \end{bmatrix}}_{S}. \tag{3.16}$$

Then, $P_1\mathcal{H}$ has eigenvalues γ and $-\gamma$ with corresponding eigenvectors $\left[\begin{smallmatrix} T(\zeta) \\ \gamma(\zeta) \end{smallmatrix}\right]$, and $\left[\begin{smallmatrix} -T(\zeta) \\ \gamma(\zeta) \end{smallmatrix}\right]$, respectively. Since the eigenspaces are one-dimensional $Z^+(\zeta)$ and $Z^-(\zeta)$ are each the span of the corresponding single eigenvector, it equals

$$\begin{aligned} \widetilde{W}_1\mathcal{H}(1)Z^+(1) \oplus \widetilde{W}_0\mathcal{H}(0)Z^-(0) &= \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \gamma(1) \\ T(1) \end{bmatrix} \oplus \begin{bmatrix} \kappa & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} -\gamma(0) \\ T(0) \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ T(1) \end{bmatrix} \oplus \begin{bmatrix} -\kappa\gamma(0) + T(0) \\ 0 \end{bmatrix} = \mathbb{C}^2. \end{aligned}$$

Thus, the port-Hamiltonian system (3.8) is well-posed.

Example 3.2.9. Continuation of example 3.1.8. The port-Hamiltonian operator associated to the port-Hamiltonian system of the Timoshenko beam is given by

$$Ax = \left(\left[\begin{smallmatrix} 0&1&0&0 \\ 1&0&0&0 \\ 0&0&0&1 \\ 0&0&1&0 \end{smallmatrix}\right] \frac{d}{d\zeta} + \left[\begin{smallmatrix} 0&0&0&-1 \\ 0&0&0&0 \\ 0&0&0&0 \\ 1&0&0&0 \end{smallmatrix}\right] \right) \left(\begin{bmatrix} K(\zeta) & 0 & 0 & 0 \\ 0 & \frac{1}{\rho(\zeta)} & 0 & 0 \\ 0 & 0 & EI(\zeta) & 1 \\ 0 & 0 & 0 & \frac{1}{I_\rho(\zeta)} \end{bmatrix} x \right), \qquad x \in \mathcal{D}(A),$$

$$\mathcal{D}(A) = \left\{ x \in X \mid \mathcal{H}x \in \mathcal{W}^{1,2}(0,1;\mathbb{C}^d) \text{ and } \left[\begin{smallmatrix} 0&1&0&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0 \\ 0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&0&1 \end{smallmatrix}\right] \begin{bmatrix} (\mathcal{H}x)(1,t) \\ (\mathcal{H}x)(0,t) \end{bmatrix} = 0 \right\},$$

where $x = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \end{bmatrix}^\top$. Then it holds $W_B = \left[\begin{smallmatrix} 0&1&0&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0 \\ 0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&0&1 \end{smallmatrix}\right]$ and moreover, $W_B \Sigma W_B^* = 0$. Thus, by Theorem 3.2.3 A generates a unitary C_0-group.

3.3 Boundary control and observation port-Hamiltonian systems

Since most of the systems in applications are connected with their environment, we introduce port-Hamiltonian systems with inputs and outputs. In a first step, we add only an input to the port-Hamiltonian system. Thus, we consider infinite-dimensional linear port-Hamiltonian systems on a one-dimensional spatial domain with boundary control of the form

$$\begin{aligned} \frac{\partial x}{\partial t}(\zeta,t) &= \left(P_1 \frac{\partial}{\partial \zeta} + P_0 \right) (\mathcal{H}(\zeta)x(\zeta,t)), \\ x(\zeta,0) &= x_0(\zeta),\ \zeta \in (0,1), \\ u(t) &= \widetilde{W}_B \begin{bmatrix} (\mathcal{H}x)(1,t) \\ (\mathcal{H}x)(0,t) \end{bmatrix},\ t \geqslant 0. \end{aligned} \tag{3.17}$$

Again, Assumption 3.1.1 has to be fulfilled. These systems are called *boundary control port-Hamiltonian systems.* To formulate port-Hamiltonian systems with input as boundary control systems, we introduce a port-Hamiltonian operator without boundary conditions. The following is extracted from Chapters 11 and 13 in [JZ12].

Definition 3.3.1. The operator

$$\mathfrak{A}x := \left(P_1 \frac{d}{d\zeta} + P_0\right)(\mathcal{H}x), \quad x \in \mathcal{D}(\mathfrak{A}), \tag{3.18}$$

on $X := L^2(0,1;\mathbb{C}^d)$ with the domain

$$\mathcal{D}(\mathfrak{A}) := \left\{x \in X \mid \mathcal{H}x \in \mathcal{W}^{1,2}(0,1;\mathbb{C}^d)\right\} \tag{3.19}$$

is called the *(maximal) port-Hamiltonian operator.*

Furthermore, we introduce the input operator $\mathfrak{B} : \mathcal{D}(\mathfrak{A}) \to \mathbb{C}^d$ by

$$\mathfrak{B}x = \widetilde{W}_B \begin{bmatrix} (\mathcal{H}x)(1,t) \\ (\mathcal{H}x)(0,t) \end{bmatrix}. \tag{3.20}$$

Then the partial differential equation (3.17) can be written as a boundary control system

$$\begin{aligned} \dot{x}(t) &= \mathfrak{A}x(t), \quad x(0) = x_0, \\ u(t) &= \mathfrak{B}x(t). \end{aligned}$$

The first important question is whether the port-Hamiltonian system (3.17) is *well-posed* in the sense that for every initial condition $x_0 \in X$ and every $u \in L^2(0,t;\mathbb{C}^d)$ equation (3.17) has a unique mild solution, cf. Definition 2.2.15. Moreover, we see that the port-Hamiltonian operator A associated to the maximal port-Hamiltonian operator with

$$Ax = \mathfrak{A}x, \; x \in \mathcal{D}(A), \tag{3.21}$$

$$\mathcal{D}(A) : = \left\{x \in \mathcal{D}(\mathfrak{A}) \mid \widetilde{W}_B \begin{bmatrix} (\mathcal{H}x)(1) \\ (\mathcal{H}x)(0) \end{bmatrix} = 0\right\} \tag{3.22}$$

equals the port-Hamiltonian operator defined in Definition 3.2.1. The following results gives a characterization for well-posedness of port-Hamiltonian systems.

Theorem 3.3.2. ([Vil07, ZLMV10, JZ12]) *The port-Hamiltonian system* (3.17) *is well-posed if and only if the port-Hamiltonian operator A generates a strongly continuous C_0-semigroup on X.*

We recall, that A generates a contraction C_0-semigroup on X if and only if A is dissipative on X, cf. Theorem 3.2.3. There can be found matrix conditions to guarantee generation of a contraction C_0-semigroup, too. Matrix conditions for the generation of strongly continuous semigroups are given in Theorem 3.2.5. Now we understand well-posedness for boundary control systems and add as next step an output to these systems.

Definition 3.3.3. Systems of the form

$$\begin{aligned}
\frac{\partial x}{\partial t}(\zeta,t) &= \left(P_1\frac{\partial}{\partial\zeta}+P_0\right)(\mathcal{H}(\zeta)x(\zeta,t)),\\
x(\zeta,0) &= x_0(\zeta),\ \zeta\in(0,1),\\
u(t) &= \widetilde{W}_B\begin{bmatrix}(\mathcal{H}x)(1,t)\\(\mathcal{H}x)(0,t)\end{bmatrix},\\
y(t) &= \widetilde{W}_C\begin{bmatrix}(\mathcal{H}x)(1,t)\\(\mathcal{H}x)(0,t)\end{bmatrix},\ t\geqslant 0,
\end{aligned}\tag{3.23}$$

satisfying Assumption 3.1.1 and where $\widetilde{W}_C$ is a full row rank $k\times 2d$-matrix, $k\in\{0,\cdots,d\}$, such that the matrix $\left[\begin{smallmatrix}\widetilde{W}_B\\ \widetilde{W}_C\end{smallmatrix}\right]$ has full row rank are called *boundary control and observation port-Hamiltonian system.*

The case $k=0$ refers to a system without observation, that is, every definition or statement of the port-Hamiltonian system (3.23) also applies to the boundary control port-Hamiltonian system (3.17).
We define $\mathfrak{C}:\mathcal{D}(\mathfrak{A})\to\mathbb{C}^k$ by

$$\mathfrak{C}x=\widetilde{W}_C\begin{bmatrix}(\mathcal{H}x)(1,t)\\(\mathcal{H}x)(0,t)\end{bmatrix}.\tag{3.24}$$

Then we can write the port-Hamiltonian system (3.23) in the following form

$$\begin{aligned}
\dot{x}(t) &= \mathfrak{A}x(t),\quad x(0)=x_0,\\
u(t) &= \mathfrak{B}x(t),\\
y(t) &= \mathfrak{C}x(t).
\end{aligned}\tag{3.25}$$

We recall, that if A, defined by (3.9)-(3.10), generates a strongly continuous semigroup on the state space X, then the port-Hamiltonian system (3.23) is a boundary control and observation system.
We note that for $x_0\in\mathcal{D}(\mathfrak{A})$ and $u\in C^2(0,t;\mathbb{C}^d)$, $t>0$, satisfying $\mathfrak{B}x_0=u(0)$, the boundary control and observation port-Hamiltonian system $\mathfrak{S}(\mathfrak{A},\mathfrak{B},\mathfrak{C})$ possesses a unique classical solution, cf. Lemma 2.2.14.
For technical reasons we formulate the boundary conditions of (3.23) equivalently via the boundary flow and the boundary effort denoted by $f_{\delta,\mathcal{H}x}$ and $e_{\delta,\mathcal{H}x}$. Using Definition 3.2.2 we can write the port-Hamiltonian system (3.23) equivalently as

$$\begin{aligned}
\frac{\partial x}{\partial t}(\zeta,t) &= \left(P_1\frac{\partial}{\partial\zeta}+P_0\right)(\mathcal{H}(\zeta)x(\zeta,t)),\\
x(\zeta,0) &= x_0(\zeta),\ \zeta\in(0,1),\\
u(t) &= W_B\begin{bmatrix}f_{\delta,\mathcal{H}x}\\e_{\delta,\mathcal{H}x}\end{bmatrix},\\
y(t) &= W_C\begin{bmatrix}f_{\delta,\mathcal{H}x}\\e_{\delta,\mathcal{H}x}\end{bmatrix},\ t\geqslant 0,
\end{aligned}\tag{3.26}$$

where analogously to W_B in (3.13) the matrix W_C is defined as

$$W_C = \widetilde{W}_C \frac{1}{\sqrt{2}} \begin{bmatrix} P_1^{-1} & I \\ -P_1^{-1} & I \end{bmatrix}. \tag{3.27}$$

The port-Hamiltonian system (3.23) is uniquely described by $\mathfrak{S}(\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ given by (3.21), (3.22), (3.20) and (3.24). In the following we give an example for a boundary control and observation port-Hamiltonian system.

Example 3.3.4. Continuation of Example 3.1.7 and 3.2.8. We consider the wave equation with an input and an output, namely

$$\begin{aligned} \frac{\partial x}{\partial t}(\zeta, t) &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \frac{\partial}{\partial \zeta} \begin{bmatrix} \frac{1}{\rho(\zeta)} & 0 \\ 0 & T(\zeta) \end{bmatrix} x(\zeta, t),\ x(\zeta, 0) = x_0(\zeta),\ \zeta \in (0,1), \\ u(t) &= \begin{bmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} f_{\delta,\mathcal{H}x} \\ e_{\delta,\mathcal{H}x} \end{bmatrix}, \\ y(t) &= \begin{bmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \end{bmatrix} \begin{bmatrix} f_{\delta,\mathcal{H}x} \\ e_{\delta,\mathcal{H}x} \end{bmatrix},\ t \geqslant 0. \end{aligned} \tag{3.28}$$

Well-posedness is a fundamental property of boundary control and observation systems. In general it is not easy to show that a boundary control and observation system is well-posed, for the port-Hamiltonian system (3.23) well-posedness is already satisfied if A generates a C_0-semigroup, cf. [ZLMV10, Theorem 3.3] and [JZ12, Theorem 13.2.2].

Theorem 3.3.5. *The port-Hamiltonian system* (3.23) *is well-posed if and only if the operator A defined by* (3.9)-(3.10) *generates a strongly continuous semigroup on X.*

There is a special class of port-Hamiltonian systems for which well-posedness follows immediately.

Definition 3.3.6. A port-Hamiltonian systems (3.23) is called *impedance passive* if

$$\operatorname{Re} \langle \mathfrak{A}x, x \rangle \leqslant \operatorname{Re} \langle \mathfrak{B}x, \mathfrak{C}x \rangle \tag{3.29}$$

for every $x \in \mathcal{D}(\mathfrak{A})$. If we have equality in (3.29), then the port-Hamiltonian system is called *impedance energy preserving.*

The fact that a port-Hamiltonian system is impedance energy preserving can be characterized by an easy checkable matrix condition.

Theorem 3.3.7. ([LGZM05, Theorem 4.4]) *The port-Hamiltonian system described in* (3.23) *is impedance energy preserving if and only if it holds*

$$\begin{bmatrix} W_B \Sigma W_B^* & W_B \Sigma W_C^* \\ W_C \Sigma W_B^* & W_C \Sigma W_C^* \end{bmatrix} = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}, \tag{3.30}$$

where $\Sigma = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$.

Remark 3.3.8. Every impedance energy preserving port-Hamiltonian system (3.23) is well-posed; $W_B\Sigma W_B^* = 0$ even implies that A generates a unitary strongly continuous group, cf. [JMZ15, Theorem 1.1].

Example 3.3.9. Continuation of Example 3.1.7, 3.2.8, and 3.3.4. The system (3.28) is impedance energy preserving, since it holds $W_B\Sigma W_B^* = 0$, $W_C\Sigma W_C^* = 0$ and $W_B\Sigma W_C^* = W_C\Sigma W_B^* = I$.

Using the following balance equation we get another property of impedance passive port-Hamiltonian systems.

Lemma 3.3.10. ([JZ12, Theorem 11.3.5]) *Consider the boundary control and observation port-Hamiltonian system* (3.26) *such that the associated port-Hamiltonian operator A generates a C_0-semigroup. If the number of outputs $k = d$, then the following balance equation holds:*

$$\frac{d}{dt}\|x(t)\|^2 = \begin{bmatrix} u^*(t) & y^*(t) \end{bmatrix} \begin{bmatrix} W_B\Sigma W_B^* & W_B\Sigma W_C^* \\ W_C\Sigma W_B^* & W_C\Sigma W_C^* \end{bmatrix}^{-1} \begin{bmatrix} u(t) \\ y(t) \end{bmatrix} \tag{3.31}$$

Remark 3.3.11. For an impedance energy preserving port-Hamiltonian system the balance equation (3.31) becomes

$$\frac{d}{dt}E(t) = \frac{d}{dt}\|x(t)\|^2 = \begin{bmatrix} u(t)^* & y(t)^* \end{bmatrix} \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \begin{bmatrix} u(t) \\ y(t) \end{bmatrix} = 2\mathrm{Re}\,\langle u(t), y(t)\rangle. \tag{3.32}$$

Thus, it is easy to see that for impedance energy preserving systems with input $u(t) = 0$ there is no change of energy.

Well-posedness implies the existence of $\widetilde{B} \in \mathcal{L}(\mathbb{C}^d, X)$ with $\operatorname{ran}\widetilde{B} \subset \mathcal{D}(\mathfrak{A})$ and $\mathfrak{A}\widetilde{B} \in \mathcal{L}(\mathbb{C}^d, X)$. Applying Lemma 2.2.18 we get the mild solution.

Lemma 3.3.12. *The unique mild solution of* (3.23) *with an initial value $x_0 \in L^2(0,1;\mathbb{C}^d)$ and $u \in L^2(0,t;U)$ is given by*

$$x(t) = T(t)x_0 + \int_0^t T_{-1}(t-s)(\mathfrak{A}\widetilde{B} - A_{-1}\widetilde{B})u(s)\,ds.$$

Here the operator $\widetilde{B} : \mathbb{C}^d \to L^2(0,1;\mathbb{C}^d)$ can be defined as

$$(\widetilde{B}u)(\zeta) := (\mathcal{H}(\zeta))^{-1}\,(S_1\zeta + S_2(1-\zeta))\,u,$$

where S_1 and S_2 are $d \times d$-matrices given by

$$\begin{bmatrix} S_1 \\ S_2 \end{bmatrix} := \begin{bmatrix} P_1 & -P_1 \\ I & I \end{bmatrix}^{-1} \widetilde{W}_B^*(\widetilde{W}_B\widetilde{W}_B^*)^{-1}.$$

Then the port-Hamiltonian control system can be written equivalently in the standard control operator formulation (2.10)

$$\dot{x}(t) = A_{-1}x(t) + Bu(t),\ x(0) = x_0,\ t \geqslant 0,$$

where $B \in \mathcal{L}(\mathbb{C}^d, X_{-1})$ is given by

$$B := \mathfrak{A}\widetilde{B} - A_{-1}\widetilde{B}. \tag{3.33}$$

We recall that a boundary control and observation system with transfer function G is regular, if $\lim_{s\in\mathbb{R},s\to\infty} G(s)$ exists. Regularity of this system ensures, among other things, that the feedthrough operator $D \in \mathcal{L}(U,Y)$ exists and can be described via the transfer function, cf. Definition 2.2.21.

Lemma 3.3.13. ([JZ12, Lemma 13.2.2]) *Under the standing assumptions every well-posed port-Hamiltonian system* (3.23) *is regular and it holds*

$$\lim_{\operatorname{Re} s\to\infty} G(s) = \lim_{s\to\infty, s\in\mathbb{R}} G(s). \tag{3.34}$$

Therefore, the conditions in Definition 2.2.23 are fulfilled and we define for regular port-Hamiltonian operators admissibility of feedback operators and recall the properties of the closed-loop system.

Definition 3.3.14. A $d \times d$-matrix F is called an *admissible feedback operator* for a regular port-Hamiltonian system (3.23) with feedthrough operator D, if $I - DF$ is invertible.

Proposition 3.3.15. ([JZ12, Theorem 13.1.12]) *Let* $\mathfrak{S}(\mathfrak{A},\mathfrak{B},\mathfrak{C})$ *be a well-posed port-Hamiltonian system* (3.23)*. Assume that F is an admissible feedback operator. Then the closed-loop system* $\mathfrak{S}(\mathfrak{A},(\mathfrak{B}-F\mathfrak{C}),\mathfrak{C})$*, i.e.,*

$$\begin{aligned}
\frac{\partial x}{\partial t}(\zeta,t) &= \left(P_1\frac{\partial}{\partial\zeta} + P_0\right)(\mathcal{H}(\zeta)x(\zeta,t)),\\
x(\zeta,0) &= x_0(\zeta),\ \zeta\in(0,1),\\
v(t) &= (\mathfrak{B}-F\mathfrak{C})x(t),\\
y(t) &= \mathfrak{C}x(t),\ t\geqslant 0,
\end{aligned} \tag{3.35}$$

with input v and output y is a well-posed port-Hamiltonian system.

We recall that the open-loop system is exactly controllable, if and only if the closed-loop system is exactly controllable. In Chapter 4 we will see that well-posed port-Hamiltonian control systems are always exactly controllable. We close this chapter with a section about the spectrum of port-Hamiltonian systems.

3.4 Spectrum of port-Hamiltonian systems with $P_1\mathcal{H}(\zeta)$ diagonal

Since a well-posed port-Hamiltonian operator A has compact resolvent, the spectrum of A consists of isolated eigenvalues only and every point in the spectrum is an eigenvalue which has finite algebraic as well as finite geometric multiplicity, cf. [GGK90, Theorem XV.2.3].
For arbitrary generators of C_0-semigroups it is well-known that the spectrum lies in a left half-plane, cf. Proposition 2.1.12. Having the generation theorems for C_0-groups in mind, see [EN00, page 79], this implies that the spectrum of operators generating a C_0-group lies in a strip parallel to the imaginary axis.

Next, for port-Hamiltonian operators with $P_1\mathcal{H}(\zeta)$ diagonal and $P_0 = 0$ we prove the same results.
Since the eigenvalues of $P_1\mathcal{H}(\zeta)$ are the same as the eigenvalues of $\mathcal{H}(\zeta)^{\frac{1}{2}}P_1\mathcal{H}(\zeta)^{\frac{1}{2}}$ it follows by Sylvester's law of inertia that the number of positive and negative eigenvalues of $P_1\mathcal{H}(\zeta)$ equal those of P_1. Let d_1 denote the number of positive and $d_2 = d - d_1$ the number of negative eigenvalues of P_1. If $P_1\mathcal{H}(\zeta)$ is diagonal, the matrix $P_1\mathcal{H}(\zeta)$ can be written as $P_1\mathcal{H}(\zeta) = \begin{bmatrix} \Lambda(\zeta) & 0 \\ 0 & \Theta(\zeta) \end{bmatrix}$ without loss of generality, where $\Lambda(\zeta) = \operatorname{diag}(\lambda_i(\zeta)) \in \mathbb{C}^{d_1\times d_1}$ corresponds to the positive eigenvalues and $\Theta(\zeta) = \operatorname{diag}(\theta_i(\zeta)) \in \mathbb{C}^{d_2\times d_2}$ to the negative ones.
We split the variable $x(\zeta) = \begin{bmatrix} x^+(\zeta) \\ x^-(\zeta) \end{bmatrix} \in \mathbb{C}^d$ with $x^+(\zeta) \in \mathbb{C}^{d_1}$ and $x^-(\zeta) \in \mathbb{C}^{d_2}$. Then we formulate the following proposition, which is part of [JZ12, Theorem 13.3.1].

Proposition 3.4.1. *Let A_K be defined as*

$$A_K \begin{bmatrix} x^+(\zeta) \\ x^-(\zeta) \end{bmatrix} = \frac{d}{d\zeta}\left(\begin{bmatrix} \Lambda(\zeta) & 0 \\ 0 & \Theta(\zeta) \end{bmatrix} \begin{bmatrix} x^+(\zeta) \\ x^-(\zeta) \end{bmatrix} \right)$$

$$\mathcal{D}(A_K) = \left\{ \begin{bmatrix} x^+(\zeta) \\ x^-(\zeta) \end{bmatrix} \in \mathcal{W}^{1,2}(0,1;\mathbb{C}^d) \mid K \begin{bmatrix} \Lambda(1)x^+(1) \\ \Theta(0)x^-(0) \end{bmatrix} + Q \begin{bmatrix} \Lambda(0)x^+(0) \\ \Theta(1)x^-(1) \end{bmatrix} \right\},$$

where $K, Q \in \mathbb{C}^{d\times d}$ such that $\begin{bmatrix} K & Q \end{bmatrix}$ has full row rank. Then it holds: A_K generates a C_0-semigroup on $X = L^2(0,1;\mathbb{C}^d)$ if and only if K is invertible.

Proposition 3.4.2. *Let A be a port-Hamiltonian operator* (3.9)-(3.10) *with $P_1\mathcal{H}(\zeta)$ diagonal and $P_0 = 0$. If A generates a C_0-semigroup, then its eigenvalues lie in a strip parallel to the imaginary axis.*

Proof: In the first part of the proof we give a condition under which $s \in \mathbb{C}$ is an eigenvalue. Then, we use this condition to show that there are no eigenvalues in a certain left half-plane. We start with the characterization of the eigenvalues of A.
Let $s \in \mathbb{C}$ be arbitrarily. Then z is a solution of $sz = Az$ if and only if

$$sz(\zeta) = \frac{d}{d\zeta} \begin{bmatrix} \Lambda(\zeta) & 0 \\ 0 & \Theta(\zeta) \end{bmatrix} z(\zeta), \quad \zeta \in [0,1]. \tag{3.36}$$

The solution of (3.36) is given by

$$z_i(\zeta) = \begin{cases} \frac{c_i}{\lambda_i(\zeta)} \mathrm{e}^{s\int_0^\zeta \frac{1}{\lambda_i(y)}dy} & \text{for } 1 \leqslant i \leqslant d_1 \\ \frac{c_i}{\theta_i(\zeta)} \mathrm{e}^{s\int_0^\zeta \frac{1}{\theta_i(y)}dy} & \text{for } d_1 + 1 \leqslant i \leqslant d. \end{cases}$$

The number s is an eigenvalue of A if and only if there exist constants c_i such that $z \in \mathcal{D}(A)$. We split the variable $z(\zeta) = \begin{bmatrix} z^+(\zeta) \\ z^-(\zeta) \end{bmatrix} \in \mathbb{C}^d$ with $z^+(\zeta) \in \mathbb{C}^{d_1}$ and $z^-(\zeta) \in \mathbb{C}^{d_2}$, and we define $\widetilde{W}_1\mathcal{H}(1) =: \begin{bmatrix} V_1 & V_2 \end{bmatrix}$ and $\widetilde{W}_0\mathcal{H}(0) =: \begin{bmatrix} U_1 & U_2 \end{bmatrix}$, where $U_1, V_1 \in \mathbb{C}^{d\times d_1}$ and $U_2, V_2 \in \mathbb{C}^{d\times d_2}$. Then, $z \in \mathcal{D}(A)$ if and only if

$$\begin{aligned}
0 &= \begin{bmatrix} \widetilde{W}_1 & \widetilde{W}_0 \end{bmatrix} \begin{bmatrix} (\mathcal{H}z)(1) \\ (\mathcal{H}z)(0) \end{bmatrix} \\
&= \begin{bmatrix} V_1 & V_2 \end{bmatrix} \begin{bmatrix} z^+(1) \\ z^-(1) \end{bmatrix} + \begin{bmatrix} U_1 & U_2 \end{bmatrix} \begin{bmatrix} z^+(0) \\ z^-(0) \end{bmatrix} \\
&= \begin{bmatrix} V_1 & U_2 \end{bmatrix} \begin{bmatrix} z^+(1) \\ z^-(0) \end{bmatrix} + \begin{bmatrix} U_1 & V_2 \end{bmatrix} \begin{bmatrix} z^+(0) \\ z^-(1) \end{bmatrix} \\
&= K \begin{bmatrix} \Lambda(1) z^+(1) \\ \Theta(0) z^-(0) \end{bmatrix} + Q \begin{bmatrix} \Lambda(0) z^+(0) \\ \Theta(1) z^-(1) \end{bmatrix},
\end{aligned}$$

where $K := \begin{bmatrix} V_1 & U_2 \end{bmatrix} \begin{bmatrix} \Lambda(1)^{-1} & 0 \\ 0 & \Theta(0)^{-1} \end{bmatrix}$ and $Q := \begin{bmatrix} U_1 & V_2 \end{bmatrix} \begin{bmatrix} \Lambda(0)^{-1} & 0 \\ 0 & \Theta(1)^{-1} \end{bmatrix}$. Since A is the generator of a C_0-semigroup, K is invertible, see Proposition 3.4.1. Thus, s is an eigenvalue of A if and only if

$$\begin{aligned}
0 &= \begin{bmatrix} \Lambda(1) z^+(1) \\ \Theta(0) z^-(0) \end{bmatrix} + K^{-1} Q \begin{bmatrix} \Lambda(0) z^+(0) \\ \Theta(1) z^-(1) \end{bmatrix} \\
&= (I + K^{-1} Q G(s)) \begin{bmatrix} \Lambda(1) z^+(1) \\ \Theta(0) z^-(0) \end{bmatrix},
\end{aligned}$$

where it is easy to verify that $G(s) = \operatorname{diag}(g_i(s))$ with

$$g_i(s) = \begin{cases} \mathrm{e}^{-s\int_0^1 \frac{1}{\lambda_i(y)} dy} & \text{for } 1 \leqslant i \leqslant d_1 \\ \mathrm{e}^{s\int_0^1 \frac{1}{\theta_i(y)} dy} & \text{for } d_1 + 1 \leqslant i \leqslant d. \end{cases} \tag{3.37}$$

Summarising, $s \in \mathbb{C}$ is an eigenvalue of A if and only if

$$\det(K G^{-1}(s) + Q) = 0.$$

Thus, in order to prove that all eigenvalues of A lie in a strip, it is sufficient to show that there exists a constant $s_0 \in \mathbb{R}$ such that $\det(KG(s)^{-1} + Q) \neq 0$ for $\operatorname{Re} s \leqslant s_0$. Due to (3.37), we can write $G(s) = \operatorname{diag}(\mathrm{e}^{-h_i s})$ with h_i positive for $i = 1, \ldots, d$. Thus, it yields $G(s)^{-1} = \operatorname{diag}(\mathrm{e}^{h_i s})$ and the determinant of $KG(s)^{-1} + Q$ can be written as a sum of exponentials with $0 \leqslant \tilde{h}_j < \tilde{h}_{j+1}$.

$$\det(KG(s)^{-1} + Q) = \sum_{j=1}^{N} a_j \mathrm{e}^{\tilde{h}_j s},$$

where $N \in \mathbb{N}$ and $a_j \neq 0$. Then

$$\mathrm{e}^{-\tilde{h}_1 s} \det(KG(s)^{-1} + Q) = \sum_{j=1}^{N} a_j \mathrm{e}^{(\tilde{h}_j - \tilde{h}_1)s} \to a_1 \neq 0 \text{ for } \operatorname{Re} s \to -\infty$$

and thus, $\det(KG(s)^{-1} + Q) \neq 0$ for $\operatorname{Re} s < s_0$ and some $s_0 \in \mathbb{R}$. Thus, all eigenvalues of A lie in a strip. □

Chapter 4

Exact controllability of port-Hamiltonian systems

In this chapter, we consider infinite-dimensional linear port-Hamiltonian systems on a one-dimensional spatial domain with boundary control. In Chapter 3, Definition 3.3.3 we have seen that these systems are of the form

$$\begin{aligned} \frac{\partial x}{\partial t}(\zeta, t) &= \left(P_1 \frac{\partial}{\partial \zeta} + P_0\right)(\mathcal{H}(\zeta)x(\zeta, t)), \\ x(\zeta, 0) &= x_0(\zeta),\, \zeta \in (0,1), \\ u(t) &= \widetilde{W}_B \begin{bmatrix} (\mathcal{H}x)(1,t) \\ (\mathcal{H}x)(0,t) \end{bmatrix},\, t \geqslant 0, \end{aligned} \tag{4.1}$$

and Assumption 3.1.1 is fulfilled.
Provided the port-Hamiltonian system (4.1) is well-posed, we aim to characterize exact controllability. Exact controllability is a desirable property of a controlled partial differential equation and has been extensively studied, see for example [Kom94, CZ95, TW09]. Triggiani [Tri91] showed that exact controllability does not hold for many hyperbolic partial differential equations. However, in this chapter we prove, that the port-Hamiltonian system (4.1) is exactly controllable whenever it is well-posed. The main result of this chapter is published in [JK19a].

4.1 Sufficient condition for exact controllability

This section is devoted to the main result of this chapter, that is, we show that every well-posed port-Hamiltonian system (4.1) is exactly controllable. We remind the definition of exact controllability and start this section with a characterization of exact controllability via optimizability. For the definition of exact controllability see Definition 2.2.7. The definition of optimizability and the following statement is extracted from [RW97].

Definition 4.1.1. Let $\mathfrak{S}(\mathfrak{A}, \mathfrak{B})$ denote a boundary control system and let x be its mild solution. Then $\mathfrak{S}(\mathfrak{A}, \mathfrak{B})$ is called *optimizable* if for every initial value

$x_0 \in X$ there exists an input function $u \in L^2(0, \infty; U)$ such that

$$\int_0^\infty \|x(t)\|^2 \, dt < \infty.$$

Note that exact controllability implies optimizability.

Proposition 4.1.2. ([RW97, Corollary 2.2]) *The system $\mathfrak{S}(\mathfrak{A}, \mathfrak{B})$ is exactly controllable if $\mathfrak{S}(\mathfrak{A}, \mathfrak{B})$ is optimizable and $-A$ generates a bounded C_0-semigroup.*

Since impedance energy preserving systems are impedance passive as well, the following statement is useful for the consideration of impedance passive systems.

Proposition 4.1.3. ([Vil07, Theorem 5.1] and [HP18, Lemma 7]) *A boundary control and observation port-Hamiltonian system with boundary conditions described using W_B such that $W_B \Sigma W_B^* > 0$ is exponentially stable. An impedance passive port-Hamiltonian system can be exponentially stabilized via a feedback $u(t) = -ky(t)$, $k > 0$.*

Exact controllability for impedance energy preserving boundary control and observation port-Hamiltonian system has been studied in [JZ18].

Proposition 4.1.4. ([JZ18, Corollary 10.7]) *An impedance energy preserving port-Hamiltonian system* (3.23) *is exactly controllable.*

For completeness we include the proof of Proposition 4.1.4.
Proof: As the port-Hamiltonian system (3.23) is impedance energy preserving the corresponding operator A generates a unitary strongly continuous group, cf. Remark 3.3.8. Thus, $-A$ generates a bounded strongly continuous semigroup and exact controllability is equivalent to optimizability, cf. Proposition 4.1.2. Thus it is sufficient to show that the port-Hamiltonian system (3.23) is optimizable. Let $x_0 \in X$ be arbitrarily. Using Proposition 4.1.3 that for every $k > 0$ the choice $u(t) = -ky(t)$ leads to a mild solution in $L^2(0, \infty; X)$. This shows optimizability of system (3.23) and concludes the proof. □
Now we can formulate the main result of this chapter.

Theorem 4.1.5. *Every well-posed port-Hamiltonian system* (4.1) *is exactly controllable.*

For the proof of this result we need the following lemmas.

Lemma 4.1.6. *Let $\begin{bmatrix} W_1 & W_0 \end{bmatrix} \in \mathbb{C}^{d \times 2d}$ have full row rank with $W_1, W_0 \in \mathbb{C}^{d \times d}$. Then, there exist invertible matrices $\widetilde{R}_1, \widetilde{R}_0 \in \mathbb{C}^{d \times d}$ such that*

$$\begin{bmatrix} W_1 & W_0 \end{bmatrix} \begin{bmatrix} \widetilde{R}_1 \\ \widetilde{R}_0 \end{bmatrix} = I.$$

Proof: Let $\begin{bmatrix} W_1 & W_0 \end{bmatrix}$ have full row rank with $\operatorname{rank} W_1 = d - k$, $k \in \{0, \ldots, d\}$, and $\operatorname{rank} W_0 = d - \ell$ with $\ell \in \{0, \ldots, d\}$. Clearly $d - k + d - \ell \geqslant d$, or equivalently, $k + \ell \leqslant d$.

By W_1^{d-k} we denote the first $d-k$ rows of W_1 and W_1^k denotes the last k rows. Similarly, by $W_0^{d-\ell}$ we denote the last $d-\ell$ rows of W_0 and by W_0^ℓ the first ℓ rows. That is

$$W_1 = \begin{bmatrix} W_1^{d-k} \\ W_1^k \end{bmatrix} \qquad \text{and} \qquad W_0 = \begin{bmatrix} W_0^\ell \\ W_0^{d-\ell} \end{bmatrix}.$$

Without loss of generality, using row reduction and the fact that it yields rank $\begin{bmatrix} W_1 & W_0 \end{bmatrix} = d$, we may assume that $W_1^k = 0$ and that W_1^{d-k} and $W_0^{d-\ell}$ have full row rank.
We choose right inverses $R_1^{d-k} \in \mathbb{C}^{d\times(d-k)}$ for W_1^{d-k} and $R_0^{d-\ell} \in \mathbb{C}^{d\times(d-\ell)}$ for $W_0^{d-\ell}$. Thus,

$$W_1^{d-k} R_1^{d-k} = I \qquad \text{and} \qquad W_0^{d-\ell} R_0^{d-\ell} = I.$$

Clearly, the columns of R_1^{d-k} and $R_0^{d-\ell}$ are linearly independent and are not elements of the kernel of W_1 and W_0, respectively.
Let $R_1^k \in \mathbb{C}^{d\times k}$ consisting of columns spanning the kernel of W_1, and let $R_0^\ell \in \mathbb{C}^{d\times\ell}$ consisting of columns spanning the kernel of W_0. We define $R_1 = \begin{bmatrix} R_1^{d-k} & R_1^k \end{bmatrix} \in \mathbb{C}^{d\times d}$ and $R_0 = \begin{bmatrix} R_0^\ell & R_0^{d-\ell} \end{bmatrix} \in \mathbb{C}^{d\times d}$. Thus, R_1 and R_0 are invertible and it yields

$$\begin{aligned} & W_1 R_1 + W_0 R_0 \\ &= \begin{bmatrix} I_{d-k} & 0_{(d-k)\times k} \\ 0_{k\times(d-k)} & 0_{k\times k} \end{bmatrix} + \begin{bmatrix} 0_{\ell\times\ell} & W_0^l R_0^{d-\ell} \\ 0_{(d-\ell)\times\ell} & I_{d-\ell} \end{bmatrix}. \end{aligned}$$

Thus, $W_1R_1 + W_0R_0 := M$ is invertible as an upper triangular matrix and we define $\widetilde{R}_1 := R_1 M^{-1}$ and $\widetilde{R}_0 := R_0 M^{-1}$ to obtain the assertion of the lemma.□

Lemma 4.1.7. *Let $\alpha \neq 0$ and $\mathfrak{S}(\mathfrak{A}, \mathfrak{B})$ be a well-posed port-Hamiltonian system. Then the port-Hamiltonian system $\mathfrak{S}(\mathfrak{A}, \alpha\mathfrak{B})$ is well-posed as well. Moreover, the system $\mathfrak{S}(\mathfrak{A}, \mathfrak{B})$ is exactly controllable if and only if $\mathfrak{S}(\mathfrak{A}, \alpha\mathfrak{B})$ is exactly controllable.*

Proof: The well-posedness of the scaled system follows immediately. The controllability of the two systems is equivalent, since we can scale the input function u of one system by α or $\frac{1}{\alpha}$ to get an input for the other system without changing the mild solution.
Using the results above, we can now give the proof of the main result of this chapter.

Proof of Theorem 4.1.5: We start with an arbitrary port-Hamiltonian system (4.1) described by the tuple $\mathfrak{S}(\mathfrak{A}, \mathfrak{B})$.
By Lemma 4.1.7, this system is exactly controllable if and only if for some $\alpha > 0$ the system $\mathfrak{S}(\mathfrak{A}, \alpha\mathfrak{B})$ is exactly controllable. We aim to prove that there exists an $\alpha > 0$ such that the system $\mathfrak{S}(\mathfrak{A}, \alpha\mathfrak{B})$ is exactly controllable.
Thus, we aim to write the system $\mathfrak{S}(\mathfrak{A}, \alpha\mathfrak{B})$ as a closed-loop system of an exactly controllable system $\mathfrak{S}(\mathfrak{A}, \mathfrak{B}_o, \mathfrak{C}_o)$. To construct $\mathfrak{S}(\mathfrak{A}, \mathfrak{B}_o, \mathfrak{C}_o)$ we find an impedance energy preserving system $\mathfrak{S}(\mathfrak{A}, \mathfrak{B}_o, \widetilde{\mathfrak{C}})$ which is exactly controllable by Proposition 4.1.4.

By (3.20) and (3.27), the operator $\mathfrak{B}$ is described by a full row rank $d \times 2d$-matrix

$$W_B = \begin{bmatrix} W_1 & W_0 \end{bmatrix}.$$

Using Lemma 4.1.6 there exists a matrix $R = \begin{bmatrix} R_1 \\ R_0 \end{bmatrix} \in \mathbb{C}^{2d \times d}$ such that

$$W_B R = I$$

and $R_1, R_0 \in \mathbb{C}^{d \times d}$ are invertible. If $W_0 = 0$, without loss of generality we may assume that $R_0 = I$ and $R_1 = W_1^{-1}$.
We now consider the port-Hamiltonian system $\mathfrak{S}(\mathfrak{A}, \mathfrak{B}_o, \widetilde{\mathfrak{C}})$, where

$$\mathfrak{B}_o x = \begin{bmatrix} R_1^{-1} & 0 \end{bmatrix} \begin{bmatrix} f_{\delta,\mathcal{H}x} \\ e_{\delta,\mathcal{H}x} \end{bmatrix}$$

and

$$\widetilde{\mathfrak{C}} x = \begin{bmatrix} 0 & R_1^* \end{bmatrix} \begin{bmatrix} f_{\delta,\mathcal{H}x} \\ e_{\delta,\mathcal{H}x} \end{bmatrix}.$$

Obviously, the port-Hamiltonian system $\mathfrak{S}(\mathfrak{A}, \mathfrak{B}_o, \widetilde{\mathfrak{C}})$ is impedance energy preserving. Then it follows from Proposition 4.1.4 that $\mathfrak{S}(\mathfrak{A}, \mathfrak{B}_o, \widetilde{\mathfrak{C}})$ is exactly controllable.
If $W_0 = 0$, then $\mathfrak{S}(\mathfrak{A}, \mathfrak{B}) = \mathfrak{S}(\mathfrak{A}, \mathfrak{B}_o)$ and thus the statement is proved with $\alpha = 1$.
We now assume that $W_0 \neq 0$. In this case we consider the port-Hamiltonian system $\mathfrak{S}(\mathfrak{A}, \mathfrak{B}_o, \mathfrak{C}_o)$, where

$$\mathfrak{C}_o x = \begin{bmatrix} \alpha R_1^{-1} & \alpha R_0^{-1} \end{bmatrix} \begin{bmatrix} f_{\delta,\mathcal{H}x} \\ e_{\delta,\mathcal{H}x} \end{bmatrix}.$$

The constant $\alpha > 0$ will be chosen later. The matrix $\begin{bmatrix} R_1^{-1} & 0 \\ \alpha R_1^{-1} & \alpha R_0^{-1} \end{bmatrix}$ is invertible and the port-Hamiltonian system $\mathfrak{S}(\mathfrak{A}, \mathfrak{B}_o, \mathfrak{C}_o)$ is still exactly controllable, since changing the output does not influence controllability.
The port-Hamiltonian system $\mathfrak{S}(\mathfrak{A}, \mathfrak{B}_o, \mathfrak{C}_o)$ is regular, see Theorem 3.3.5 and Lemma 3.3.13. By D we denote the feedthrough operator of $\mathfrak{S}(\mathfrak{A}, \mathfrak{B}_o, \mathfrak{C}_o)$ and we choose

$$\alpha = \begin{cases} 2\,\|D\|\,\|W_0 R_0\|, & D \neq 0 \\ 1, & D = 0. \end{cases}$$

Then $\alpha > 0$ and the matrix

$$F = \frac{1}{\alpha} W_0 R_0$$

is an admissible feedback operator for $\mathfrak{S}(\mathfrak{A}, \mathfrak{B}_o, \mathfrak{C}_o)$ as $\|DF\| < 1$ (which implies invertibility of $I - DF$).

We now consider the closed-loop system as shown in Figure 4.1 and obtain

$$\begin{aligned}
\dot{x}(t) &= \mathfrak{A}x(t), \quad x(0) = x_0, \\
u_\alpha(t) &= \alpha(u_o(t) - F y_o(t)) \\
&= \alpha(\mathfrak{B}_o - F\mathfrak{C}_o)x(t) \\
&= \left(\alpha \begin{bmatrix} R_1^{-1} & 0 \end{bmatrix} - W_0 R_0 \begin{bmatrix} \alpha R_1^{-1} & \alpha R_0^{-1} \end{bmatrix}\right) \begin{bmatrix} f_{\delta,\mathcal{H}x} \\ e_{\delta,\mathcal{H}x} \end{bmatrix} \\
&= \alpha W_B \begin{bmatrix} f_{\delta,\mathcal{H}x} \\ e_{\delta,\mathcal{H}x} \end{bmatrix}.
\end{aligned}$$

Thus, the closed-loop system equals the port-Hamiltonian system $\mathfrak{S}(\mathfrak{A}, \alpha\mathfrak{B})$. As the open-loop system $\mathfrak{S}(\mathfrak{A}, \mathfrak{B}_o, \mathfrak{C}_o)$ is exactly controllable, by Theorem 2.2.25 the port-Hamiltonian system $\mathfrak{S}(\mathfrak{A}, \alpha\mathfrak{B})$ is exactly controllable.

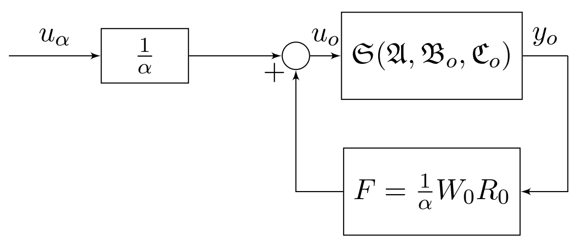

Figure 4.1: $\mathfrak{S}(\mathfrak{A}, \alpha\mathfrak{B})$ as a closed-loop system

Thus, every well-posed port-Hamiltonian system is exactly controllable. □

We close this section with an example, where we apply Theorem 4.1.5.

Example 4.1.8. Continuation of Example 3.1.7 and 3.2.8. In Example 3.1.7 we have seen that an (undamped) vibrating string can be modelled as the port-Hamiltonian system (3.8). Its boundary control is given by

$$\begin{bmatrix} \widetilde{W}_1 & \widetilde{W}_0 \end{bmatrix} \begin{bmatrix} (\mathcal{H}x)(1,t) \\ (\mathcal{H}x)(0,t) \end{bmatrix} = u(t), \tag{4.2}$$

where $\begin{bmatrix} \widetilde{W}_1 & \widetilde{W}_0 \end{bmatrix}$ is a 2×4-matrix with rank 2. Using the diagonalization of $P_1\mathcal{H}$, see equation (3.16) and Theorem 3.2.5, it is easy to see that the port-Hamiltonian system (3.8), (4.2) is well-posed if and only if

$$\widetilde{W}_1 \begin{bmatrix} \gamma(1) \\ T(1) \end{bmatrix} \oplus \widetilde{W}_0 \begin{bmatrix} -\gamma(0) \\ T(0) \end{bmatrix} = \mathbb{C}^2,$$

cf. [JMZ15], or equivalently if the vectors $\widetilde{W}_1 \left[\begin{smallmatrix} \gamma(1) \\ T(1) \end{smallmatrix}\right]$ and $\widetilde{W}_0 \left[\begin{smallmatrix} -\gamma(0) \\ T(0) \end{smallmatrix}\right]$ are linearly independent. By Theorem 4.1.5 the port-Hamiltonian system (3.8), (4.2) is exactly controllable if the vectors $\widetilde{W}_1 \left[\begin{smallmatrix} \gamma(1) \\ T(1) \end{smallmatrix}\right]$ and $\widetilde{W}_0 \left[\begin{smallmatrix} -\gamma(0) \\ T(0) \end{smallmatrix}\right]$ are linearly independent. Here we consider $\widetilde{W}_1 := I$ and $\widetilde{W}_0 := \left[\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$, see also Example 3.2.8. Then the port-Hamiltonian system (3.8), (4.2) is exactly controllable if the vectors $\begin{bmatrix} \gamma(1) \\ T(1) \end{bmatrix}$ and $\begin{bmatrix} \gamma(0) \\ T(0) \end{bmatrix}$ are linearly independent, i.e., it depends not only on the boundary conditions but also on the physical coefficients $T(\zeta)$ and $\rho(\zeta)$ whether the associated port-Hamiltonian operator A generates a C_0-semigroup.

4.2 Closing remarks and open problems

In this chapter we have studied the notion of exact controllability for a class of linear port-Hamiltonian systems on a one dimensional spacial domain with full boundary control and no internal damping. We showed that for this class well-posedness implies exact controllability. Further, we applied the obtained results to the wave equation. By duality a well-posed port-Hamiltonian system $\mathfrak{S}(\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ with state space $L^2(0, \infty; \mathbb{C}^d)$ and output space $\mathbb{C}^d$ is exactly observable. An interesting problem for future research is the characterization of exact controllability for port-Hamiltonian systems with internal damping, i.e., port-Hamiltonian systems where P_0 is not necessarily skew-adjoint. We note, that the condition that $\widetilde{W_B}$ has full rank cannot be neglected, as in general without full boundary control a port-Hamiltonian system is not exact controllable. Further results for the approximate observability of port-Hamiltonian systems with internal damping can be found in [JZ21]. In particular, there is shown port-Hamiltonian systems with internal damping are not exactly controllable in general.
Another open question is the characterization of exact controllability for port-Hamiltonian systems of higher order, see [Vil07]. However, for these systems even the characterization of well-posedness is an open problem.

Chapter 5

Riesz bases of port-Hamiltonian systems

It is well-known that the eigenvectors of a compact self-adjoint operator form an orthonormal basis of the underlying Hilbert space. In the 1960s Dunford and Schwartz [DS71] introduced the more general notion of spectral operators. Further, Curtain [Cur84] analysed discrete spectral operators, i.e., spectral operators with compact resolvent, and the class of Riesz spectral operators was formulated in [CZ95] and extended in [GZ01] to characterize also operators with multiple eigenvalues. For Riesz spectral operators its eigenvectors still form a basis, but this basis is assumed to be Riesz basis. Since a Riesz basis is isomorphic to an orthonormal basis, many of the nice properties of compact self-adjoint operators carry over to Riesz spectral operators. For instance, solutions of the abstract differential equation $\dot{x}(t) = Ax(t) + Bu(t)$, with A a Riesz spectral operator, can still described by an eigenfunction expansion of non-harmonic Fourier series. This enables that many properties of these infinite-dimensional systems such as stability, stabilizability and controllability can be characterized in an elegant manner, see e.g. [CZ95, CZ20].

In this chapter, we investigate the Riesz basis property of a special class of infinite-dimensional systems, namely port-Hamiltonian systems on a one-dimensional spatial domain. Here by the Riesz basis property we mean that the associated system operator is a discrete Riesz spectral operator, see Definition 5.2.2.

First, we start with a short introduction to the concept of bases in a infinite-dimensional vector space and define Riesz bases. Then we give two toy examples in which the Riesz basis consisting of eigenvectors of a port-Hamiltonian operator can be computed exactly and in which we see that these methods are limited to these simplified situations. Finally we give a characterization for discrete spectral operators, where it is not necessary to determine eigenvalues and eigenfunctions. The main result of this chapter is published in [JKZ20] at arXiv and also submitted.

5.1 Preliminaries of bases

Although there is a wide theory of bases in the Banach space setting, we will only consider Hilbert spaces, due to the fact that we study port-Hamiltonian systems on the Hilbert space $X = L^2(0,1,\mathbb{C}^d)$. Nevertheless, we start to recall some notations and definitions, which also hold in the Banach space setting and can for example be found in [GW19, You80, CZ95, AK06].
We recall that a finite sequence of vectors $(x_n)_{n=1}^N$ is linearly independent if the only linear combination for the null is trivial, i.e.,

$$\sum_{n=1}^{N} a_n x_n = 0 \Leftrightarrow a_n = 0 \text{ for } n = 1, \dots, N.$$

A Hamel basis is a sequence of vectors with which each element of the vector space can be represented as a finite linear combination.
In the setting of finite-dimensional vector spaces the term basis usually means a Hamel basis. Due to the axiom of choice also each infinite-dimensional vector space has a Hamel basis, but it consists of more than countable elements and thus, the possibilities of their applications are limited.
In 1927 Julius Schauder introduced an additional type of basis, cf. [Sch27], the so called Schauder basis. This has the advantage that such a basis of an infinite-dimensional separable space is countable.

Definition 5.1.1. Let be X a Banach space. A sequence $(x_n)_{n\in\mathbb{N}}$ in X is a *basis* of X, if every $x \in X$ has a unique representation with a sequence of complex numbers $(a_n)_{n\in\mathbb{N}}$ such that

$$x = \sum_{n=1}^{\infty} a_n x_n. \tag{5.1}$$

Thus, in the following the term basis of infinite-dimensional spaces means a Schauder basis and thus, every $x \in X$ can be uniquely represented as a convergent series. In general, a Schauder basis is not a Hamel basis, since infinite linear combinations are allowed and so the linear span of a Schauder basis must be dense in X, but it may not be the entire space.
In the above definition the convergence in (5.1) holds in the norm topology $\lim_{n\to\infty} \|x - \sum_{i=1}^{n} a_i x_i\| = 0$. In the following definition we introduce the more restrictive concept of unconditional basis.

Definition 5.1.2. An *unconditional bases* is a basis, where (5.1) converges unconditionally, i.e., also all reorderings of the series (5.1) are convergent.

Definition 5.1.3. Let X be a Hilbert space. A sequence $(x_n)_{n\in\mathbb{N}}$ in X is called an *orthonormal basis* of X, if $(x_n)_{n\in\mathbb{N}}$ is a basis and

$$\langle x_n, x_m \rangle = \begin{cases} 1, & \text{if } n = m, \\ 0, & \text{if } n \neq m. \end{cases}$$

Example 5.1.4. It is known that the sequence $(\frac{1}{\sqrt{2\pi}}e^{in\cdot})_{n\in\mathbb{N}}$ is an orthonormal basis of $L^2(0,2\pi;\mathbb{C})$, cf. [Wer00]. Using the variable transformation $x \mapsto \frac{x}{2\pi}$, we see that $(e^{2\pi in\cdot})_{n\in\mathbb{N}}$ is an orthonormal basis of $L^2(0,1;\mathbb{C})$.

We define the term equivalence for bases and introduce Riesz bases afterwards.

Definition 5.1.5. Two bases $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ for a Banach space X are *equivalent* if and only if there exists a boundedly invertible operator $T : X \to X$ such that $Tx_n = y_n$ for all $n \in \mathbb{N}$.

Definition 5.1.6. A *Riesz basis* is a basis which is equivalent to an orthonormal basis.

Remark 5.1.7. Due to Definition 5.1.6 and 5.1.5 a Riesz basis is equivalent to every orthonormal basis.

Definition 5.1.8. A basis $(x_n)_{n\in\mathbb{N}}$ of a Hilbert space X is *bounded* if

$$0 < \inf_n \|x_n\| < \sup_n \|x_n\| < \infty.$$

Remark 5.1.9. A Riesz basis is a bounded basis $(x_n)_{n\in\mathbb{N}}$ since every Riesz basis is obtained from an orthonormal basis $(e_n)_{n\in\mathbb{N}}$ by application of a bounded invertible operator. Therefore, we have

$$\frac{1}{\|T^{-1}\|} \leqslant \|x_n\| \leqslant \|T\| \quad \forall\, n \in \mathbb{N}. \tag{5.2}$$

The following lemma provides an important property of Riesz bases and illustrates the relationship between Riesz bases and unconditional bases.

Lemma 5.1.10. *The following statements are equivalent:*

1. *The sequence $(x_n)_{n\in\mathbb{N}}$ is a Riesz basis of X.*
2. *The sequence $(x_n)_{n\in\mathbb{N}}$ is complete in X and there exist positive constants m_1 and m_2 such that for an arbitrary number $N \in \mathbb{N}$ and arbitrary scalars $a_n \in \mathbb{C}$, $n = 1, \dots, N$, it holds*

$$m_1 \sum_{n=1}^{N} |a_n|^2 \leqslant \left\| \sum_{n=1}^{N} a_n x_n \right\|^2 \leqslant m_2 \sum_{n=1}^{N} |a_n|^2 . \tag{5.3}$$

3. *The sequence $(x_n)_{n\in\mathbb{N}}$ is a bounded unconditional basis.*

Proof: The proof of 1.) ⇔ 2.) can be found for example in [GW19, Theorem 2.2] and the proof of the equivalence 1.) ⇔ 3.) in [Hei11, Theorem 7.13]. □

We close this section with a generalization of the concept of Riesz bases.

Definition 5.1.11. A sequence of closed subspaces $\{X_n\}_{n\in\mathbb{N}}$ in a Hilbert space X is a *Riesz basis of subspaces of* X if $\operatorname{span}\{X_n\}_{n\in\mathbb{N}}$ is dense and there exists an isomorphism $T \in \mathcal{L}(X)$, such that $\{TX_n\}_{n\in\mathbb{N}}$ is a system of pairwise orthogonal subspaces of X.

Remark 5.1.12. If a sequence of vectors $(x_n)_{n\in\mathbb{N}}$ in X is a *Riesz basis of* X, that is, there exists an isomorphism $T \in \mathcal{L}(X)$, such that $(Tx_n)_{n\in N}$ is an orthonormal basis of X, then clearly $\{\operatorname{span} x_n\}_{n\in\mathbb{N}}$ is a Riesz basis of subspaces of X.

The following toy examples show that in special and simplified situations the eigenvalues and eigenfunctions of a port-Hamiltonian operator can be determined exactly. Since this is difficult in general, we give a characterization of the Riesz basis property which is easy to verify in Theorem 5.3.3.

5.1.1 Toy examples

The first example is a port-Hamiltonian equation, i.e., a port-Hamiltonian systems with $d = 1$ and the second one is a wave equation with constant coefficients.

Lemma 5.1.13. *We consider a port-Hamiltonian operator on the interval* $[0, 1]$*, i.e.*

$$Ax = P_1 \frac{\partial}{\partial\zeta}(\mathcal{H}x) + P_0(\mathcal{H}x),$$

on

$$\mathcal{D}(A) = \{x \in L^2(0, 1; \mathbb{C}) | \mathcal{H}x \in \mathcal{W}^{1,2}(0, 1; \mathbb{C}) \ \&\ \widetilde{w}_1(\mathcal{H}x)(1) + \widetilde{w}_0(\mathcal{H}x)(0) = 0\}$$

with $\mathcal{H}, \mathcal{H}^{-1} \in L^\infty(0, 1; \mathbb{C})$ *and* $\mathcal{H}(\zeta) \in (0, \infty)$, $\widetilde{w}_1, \widetilde{w}_0 \in \mathbb{C}$*, such that A generates a C_0-semigroup. Then it holds: The eigenvectors of the operator A form a Riesz basis if and only if A generates a C_0-group.*

Proof: Without loss of generality we may assume $P_1 > 0$ and even $P_1 = 1$. It holds that A generates a C_0-group if and only if $\widetilde{w}_1 \neq 0$ and $\widetilde{w}_0 \neq 0$ cf. Theorem 3.2.5. Since A generates a C_0-semigroup by assumption, we have $\widetilde{w}_1 \neq 0$, cf. Theorem 3.2.5.

The eigenvalue problem $(\mathcal{H}x)'(\zeta) + P_0(\mathcal{H}x)(\zeta) = \mu x(\zeta)$ is equivalent to

$$\frac{d}{d\zeta}(\mathcal{H}x)(\zeta) = (\mu\mathcal{H}^{-1}(\zeta) - P_0)(\mathcal{H}x)(\zeta)$$

and we get the solution

$$(\mathcal{H}x)(\zeta) = ce^{\int_0^\zeta \mu\mathcal{H}^{-1}(s) - P_0 ds} \quad \text{with } c \in \mathbb{R},\ c \neq 0.$$

Since $\mathcal{H}, \mathcal{H}^{-1} \in L^\infty(0, 1; \mathbb{C})$, the integral is well-defined. Thus,

$$x(\zeta) = \mathcal{H}^{-1}(\zeta)(\mathcal{H}x)(\zeta) \in L^2(0, 1; \mathbb{C}). \tag{5.4}$$

Furthermore, (5.4) yields $\mathcal{H}x \in \mathcal{W}^{1,2}(0, 1; \mathbb{C})$. Therefore, to get that $x \in \mathcal{D}(A)$, only the boundary condition $\widetilde{w}_1(\mathcal{H}x)(1) + \widetilde{w}_0(\mathcal{H}x)(0) = 0$ has to be fulfilled. We define $G(\zeta) := \int_0^\zeta \mathcal{H}^{-1}(s)ds$. This yields $G(0) = 0$ and $G(1) > 0$, since $G(\zeta)$ is monotonic increasing. Thus, with $(\mathcal{H}x)(1) = ce^{\mu G(1)}e^{-P_0}$ and $(\mathcal{H}x)(0) = c$ the boundary condition becomes

$$\widetilde{w}_1 ce^{\mu G(1)}e^{-P_0} + \widetilde{w}_0 c = 0.$$

If $\widetilde{w}_0 = 0$, this equation has no solution and thus, we have no eigenvalue and therefore A is not a Riesz operator. We define $k := e^{-P_0}$. For $\widetilde{w}_0 \neq 0$, we obtain eigenvalues $(\mu_n)_{n\in\mathbb{N}}$ with multiplicity one, namely

$$\begin{aligned}
&\widetilde{w}_1 c e^{\mu G(1)} e^{-P_0} + \widetilde{w}_0 c = 0 \\
\Leftrightarrow\ & e^{\mu G(1)} = \frac{-\widetilde{w}_0}{\widetilde{w}_1 k} \\
\Leftrightarrow\ & \ln(e^{\mu G(1)}) = \ln(\frac{-\widetilde{w}_0}{\widetilde{w}_1 k}) \\
\Leftrightarrow\ & \mu_n G(1) = \ln\left(\left|\frac{-\widetilde{w}_0}{\widetilde{w}_1 k}\right|\right) + i\left[\cdot \arg\left(\frac{-\widetilde{w}_0}{\widetilde{w}_1 k}\right) + 2\pi n\right] \\
\Leftrightarrow\ & \mu_n = \frac{\ln\left(\left|\frac{-\widetilde{w}_0}{\widetilde{w}_1 k}\right|\right) + i\left[\cdot \arg\left(\frac{-\widetilde{w}_0}{\widetilde{w}_1 k}\right) + 2\pi n\right]}{G(1)}
\end{aligned}$$

with eigenvectors

$$\begin{aligned}
x_n(\zeta) &= \mathcal{H}^{-1}(\zeta) e^{\mu_n G(\zeta)} e^{-P_0\zeta} && (5.5)\\
&= \mathcal{H}^{-1}(\zeta) e^{(a+ib)G(\zeta)} e^{-P_0\zeta} e^{2\pi i n \frac{G(\zeta)}{G(1)}} && (5.6)
\end{aligned}$$

see (5.4), where $a := \frac{\ln\left(\left|\frac{-\widetilde{w}_0}{\widetilde{w}_1 k}\right|\right)}{G(1)}$ and $b := \frac{\arg \frac{-\widetilde{w}_0}{\widetilde{w}_1 k}}{G(1)}$. Finally, we have to show that these eigenvectors $(x_n)_{n\in\mathbb{N}}$ form a Riesz basis. Since the point spectrum has a uniform gap, we can apply Theorem 1.1 in [Zwa10] and it suffices to prove that the span of eigenvectors $(x_n)_{n\in\mathbb{N}}$ is dense in $L^2(0,1;\mathbb{C})$. Suppose that $x \in \operatorname{span}\{(x_n)_{n\in\mathbb{N}}\}^\perp$, i.e., for every $n \in \mathbb{N}$

$$\begin{aligned}
&\int_0^1 x^*(\zeta) x_n(\zeta) d\zeta = 0 \\
\Leftrightarrow &\int_0^1 x^*(\zeta)\mathcal{H}^{-1}(\zeta) e^{(a+ib)G(\zeta)} e^{-P_0\zeta} e^{2\pi i n\frac{G(\zeta)}{G(1)}} d\zeta = 0 \\
\Leftrightarrow &\int_0^1 x^*(G^{-1}(G(1)z)) e^{(a+ib)G(1)z} e^{-P_0 G^{-1}(G(1)z)} e^{2\pi i n z} G(1) dz = 0 \\
\Leftrightarrow &\int_0^1 \tilde{x}^*(z) e^{2\pi i n z} dz = 0
\end{aligned}$$

with

$$\tilde{x}^*(z) = x(G^{-1}(G(1)z)) e^{(a+ib)G(1)z} e^{-P_0 G^{-1}(G(1)z)} G(1) \text{ for } z \in [0,1].$$

Here we first used the substitution $z = \frac{G(\zeta)}{G(1)}$ and $d\zeta = \frac{G(1)}{\mathcal{H}^{-1}(\zeta)} dz$ since $G'(\zeta) = \mathcal{H}^{-1}(\zeta)$. Since $(e^{2i\pi n z})_{n\in\mathbb{N}}$ is an orthonormal basis of $L^2(0,1;\mathbb{C})$, see Example 5.1.4, $\tilde{x}^*(z) = 0$ for $z \in [0,1]$. Therefore, $x(G^{-1}(G(1)z)) = 0$ for $z \in [0,1]$, which implies $x(\zeta) = 0$ and thus, the eigenvectors of A form a Riesz basis. □

The second example can be found in [DH20].

Example 5.1.14. We consider the wave equation with constant coefficients T and ρ on the one-dimensional spatial domain with viscous damping on the right side and boundary control and boundary observation at the other side.

$$
\begin{aligned}
\frac{\partial^2}{\partial t^2}\omega(\zeta,t) &= \frac{1}{\rho}\frac{\partial}{\partial \zeta}\left(T\omega(\zeta,t)\right) \\
0 &= T\frac{\partial}{\partial \zeta}\omega(1,t) + \frac{\kappa}{\rho}\frac{\partial}{\partial t}\omega(1,t) \\
u(t) &= \frac{\partial}{\partial \zeta}\omega(0,t) \\
y(t) &= \frac{\partial}{\partial t}\omega(0,t),\ \zeta\in(0,1),\ t\geqslant 0,
\end{aligned}
$$

where $\kappa > 0$ describes the damping constant. Then the associated port-Hamiltonian operator is described by

$$
\begin{aligned}
Ax &= \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\frac{\partial}{\partial\zeta}\left(\begin{bmatrix}\frac{1}{\rho} & 0\\ 0 & T\end{bmatrix}x\right), \quad x\in\mathcal{D}(A),\\
\mathcal{D}(A) &= \left\{x\in X \mid \mathcal{H}x\in\mathcal{W}^{1,2}(0,1;\mathbb{C}^2) \text{ and } \begin{bmatrix}-\kappa & 1 & 0 & 0\\ 0 & 0 & 0 & \frac{1}{T}\end{bmatrix}\begin{bmatrix}(\mathcal{H}x)(1)\\ (\mathcal{H}x)(0)\end{bmatrix} = 0\right\}.
\end{aligned}
$$

Solving the equation $A\varphi_n = s_n\varphi_n$ yields the eigenvectors

$$
\varphi_n(\zeta) = \begin{bmatrix}\cosh(\frac{\rho}{T}s_n\zeta)\\ \frac{1}{\rho T}\sinh(\frac{\rho}{T}s_n\zeta)\end{bmatrix}
$$

and the eigenvalues $s_n = s_0 + \sqrt{\frac{T}{\rho}}i\pi n$, where $s_0 = \frac{1}{2}\sqrt{\frac{T}{\rho}}\ln\left(\frac{\sqrt{\rho T}-\kappa}{\sqrt{\rho T}+\kappa}\right)$.
Using the mapping

$$
M := \begin{bmatrix}\cosh(\frac{\rho}{T}s_0\zeta) & -\sqrt{\rho T}\sinh(\frac{\rho}{T}s_0\zeta)\\ i\sinh(\frac{\rho}{T}s_0\zeta) & -i\sqrt{\rho T}\cosh(\frac{\rho}{T}s_0\zeta)\end{bmatrix}
$$

we see that $(\varphi_n)_{n\in\mathbb{N}}$ is a Riesz basis, since

$$
(M\varphi_n)_{n\in\mathbb{N}} = \left(\begin{bmatrix}\cos(n\pi\zeta)\\ \sin(n\pi\zeta)\end{bmatrix}\right)_{n\in\mathbb{N}}
$$

is an orthonormal basis of $X = L^2(0,1;\mathbb{C}^2)$.

5.2 Discrete Riesz spectral operators

The study of the Riesz basis property for infinite-dimensional port-Hamiltonian systems has started with the thesis by Villegas [Vil07, Chapter 4]. Using results on first order eigenvalue problems by Tretter [Tre00a, Tre00b], he obtained a sufficient condition. However, it is not easy to see when this technical sufficient condition is satisfied.
Many systems have a Riesz basis of eigenfunctions, see e.g. [GX04, XG03]. In the monograph [GW19, Section 4.3] Guo and Wang study the Riesz basis property for a closely related class of systems, that is, hyperbolic systems of the form $\frac{\partial x}{\partial t} = K(\zeta)\frac{\partial x}{\partial \zeta} + C(\zeta)x$ with K and C diagonal. Note, that (almost)

every port-Hamiltonian system on a one dimensional spatial domain can be transformed into a hyperbolic system of this form. However, in general not with a diagonal C. Furthermore, the boundary conditions will be more general, see [ZLMV10, JZ12] or the proof of Lemma 5.3.5. Therefore, the main result of this chapter generalizes their theorem [GW19, Theorem 4.11]. We remark, that in this situation the notions of Riesz basis of subspaces and Riesz basis with parentheses are equivalent. Moreover, in [XW11] the Riesz basis property is investigated for operators perturbed by output feedback.
The main result shows that a linear infinite-dimensional port-Hamiltonian system on a one-dimensional spatial domain has the Riesz basis property if and only if the system operator generates a strongly continuous group. Here it is important to note that we do not need constant coefficients, nor extra assumption. Since the group property is equivalent to a simple matrix condition, our results enable us to check very quickly whether the Riesz basis property holds, see also the examples in Section 5.4. Our proof combines methods from complex analysis, differential equations and mathematical systems theory. In particular, we use the fact that every well-posed port-Hamiltonian control system (5.15) is exactly controllable in finite time, cf. Chapter 4.
We start with the definition of discrete Riesz spectral operators.

Definition 5.2.1. For an operator A on X we call $\gamma \subset \sigma(A)$ a *compact spectral set* if γ is a compact subset of $\mathbb{C}$ which is open and closed in $\sigma(A)$. The *spectral projection* on the spectral subset γ is defined as

$$E(\gamma) = \frac{1}{2\pi i} \int_{\Gamma} (s - A)^{-1} ds,$$

where Γ is a closed Jordan curve containing every point of γ and no point of $\sigma(A) \setminus \gamma$.

In this chapter operators with compact resolvent are of particular interest. The spectrum of these operators is a denumerable set of points with no finite accumulation point, cf. [DS71, Lemma XIX.2]. Furthermore, every point in the spectrum is an eigenvalue which has finite algebraic as well as finite geometric multiplicity, cf. [GGK90, Theorem XV.2.3]. If $(s_n)_{n\in\mathbb{N}}$ is the spectrum of an operator with compact resolvent we write $E_n := E((s_n))$, $n \in \mathbb{N}$, for the spectral projection regarding the n-th eigenvalue.

Definition 5.2.2. Let A be an operator with compact resolvent and countable spectrum $\sigma(A) = (s_n)_{n\in\mathbb{N}}$. Then A is a *discrete Riesz spectral operator*, if

1. for every $n \in \mathbb{N}$ there exists $N_n \in \mathcal{L}(X)$ such that $AE_n = (s_n + N_n)E_n$,
2. the sequence of closed subspaces $(E_n(X))_{n\in\mathbb{N}}$ is a *Riesz basis of subspaces of X*.
3. $N := \sum_{n\in\mathbb{N}} N_n$ is bounded and nilpotent.

Remark 5.2.3. If A is a discrete Riesz spectral operator, then clearly E_n commute with A and A is equivalent to the infinite matrix

$$A = \operatorname{diag}(A_1, A_2, \dots, A_n, \dots),$$

where A_n is a square matrix which corresponds to the restriction $A|_{E_nX}$ of A. Then $A = \sum_{n\in\mathbb{N}} i_n A_n E_n$, where i_n is the (natural) inclusion operator and A_n is identified with $(s_n + N_n)|_{E_nX}$.

Remark 5.2.4. Discrete Riesz spectral operators are spectral operators in the sense of Dunford and Schwartz [DS71]. However, we additionally assume that the operator has a compact resolvent, which are discrete operators in the sense of Dunford and Schwartz [DS71, Definition XIX.1]. Every discrete Riesz spectral operator is a Riesz spectral operator in the sense of Guo and Zwart [GZ01] and these are again spectral operator in the sense of Dunford and Schwartz. In Curtain and Zwart [CZ95] a slightly stronger notion is considered, where all eigenvalues have to be simple. However, they do not require that the operator has a compact resolvent.

Furthermore, we emphasize that a port-Hamiltonian operator which generates a C_0-semigroup is closed and that its resolvent is compact, see [Aug16].
Moreover, we introduce a term for a set of complex numbers each of which are not to close together.

Definition 5.2.5. A set $(s_n)_{n\in\mathbb{N}} \subset \mathbb{C}$ has a *uniform gap*, if

$$\inf_{n\neq m} |s_n - s_m| > 0 \text{ for } n, m \in \mathbb{N}.$$

Riesz bases of subspace have the following useful characterizations.

Proposition 5.2.6. ([Zwa10, Definition 1.4]) *Let A be an operator with compact resolvent and $\sigma(A) = (s_n)_{n\in\mathbb{N}}$. Then the sequence of subspaces $(E_n(X))_{n\in\mathbb{N}}$ is a Riesz basis of subspaces of X if and only if there exist positive constants m_1 and m_2 such that it holds*

$$m_1 \|x\|^2 \leqslant \sum_{n\in\mathbb{N}} \|E_n x\|^2 \leqslant m_2 \|x\|^2, \quad x \in X.$$

Lemma 5.2.7, Lemma 5.2.8, Proposition 5.2.9 and Proposition 5.2.10 will be useful for the proof of the main result of this chapter.

Lemma 5.2.7. *Let A be a discrete Riesz spectral operator and $M := \|N\|$, where N is given by Definition 5.2.2. Then there exists a constant $C > 0$ such that for $s \in \rho(A)$ with $d(s, \sigma(A)) > M$ we have*

$$\|(s - A)^{-1}\| \leqslant \frac{C}{d(s, \sigma(A))}, \tag{5.7}$$

where $d(s, \sigma(A))$ denotes the distance from s to the spectrum of A.

Proof: Let $\sigma(A) = (s_n)_{n\in\mathbb{N}}$, E_n, N_n, N as in Definition 5.2.2 and $s \in \rho(A)$ with $d(s, \sigma(A)) > M$ be arbitrary. Since E_n is a spectral projection, it commutes with A and the resolvent of A. By the definition of a discrete spectral operator we have

$$s - A = \sum_{n=1}^{\infty} ((s - s_n) - N_n) E_n$$

and identifying $(s-s_n)-N_n$ with the matrix corresponding to $((s-s_n)-N_n)|_{E_nX}$ we obtain

$$(s-A)^{-1} = \sum_{n=1}^{\infty} ((s-s_n)-N_n)^{-1} E_n.$$

Then it holds for $x \in X$

$$\begin{aligned}
\|(s-A)^{-1}x\|^2 &= \Big\| \sum_{n\in\mathbb{N}} ((s-s_n)-N_n)^{-1} E_n x \Big\|^2 \\
&\leqslant \sum_{n\in\mathbb{N}} \big\| ((s-s_n)-N_n)^{-1} E_n x \big\|^2 \\
&\leqslant \sup_{n\in\mathbb{N}} \big\| ((s-s_n)-N_n)^{-1} \big\|^2 \sum_{n\in\mathbb{N}} \big\| E_n x \big\|^2 \\
&\leqslant m_2 \sup_{n\in\mathbb{N}} \big\| ((s-s_n)-N_n)^{-1} \big\|^2 \|x\|^2 ,
\end{aligned}$$

where m_2 is the positive constant of the Riesz basis of subspaces $(E_n)_{n\in\mathbb{N}}$, cf. Proposition 5.2.6. Using

$$((s-s_n)-N_n) = (s-s_n)\left(I - \frac{1}{(s-s_n)} N_n\right)$$

we get

$$((s-s_n)-N_n)^{-1} = \frac{1}{(s-s_n)} \sum_{j=0}^{k_n} \frac{1}{(s-s_n)^j} N_n^j,$$

where k_n denotes the degree of nilpotency of N_n. Thus, for $s \in \rho(A)$ such that $d(s,\sigma(A)) > M$, it holds

$$\begin{aligned}
\big\| ((s-s_n)-N_n)^{-1} \big\| &\leqslant \sum_{j=1}^{k_n+1} \frac{1}{|(s-s_n)|^j} M^{j-1} \leqslant \sum_{j=1}^{k_n+1} \frac{1}{d(s,\sigma(A))^j} M^{j-1} \\
&\leqslant \frac{1}{d(s,\sigma(A))} \sum_{j=0}^{\infty} \left(\frac{M}{d(s,\sigma(A))} \right)^j
\end{aligned}$$

which concludes the proof. □

Lemma 5.2.8. *Let A be a discrete Riesz spectral operator and generator of a C_0-semigroup, and $P \in \mathcal{L}(X)$. Then there exist constants $K, M > 0$ such that for $s \in \rho(A)$ with $d(s,\sigma(A)) > M$ we have $s \in \rho(A+P)$ and*

$$\big\| (s-(A+P))^{-1} \big\| \leqslant \frac{K}{d(s,\sigma(A))}.$$

Proof: By Lemma 5.2.7 there exists $M_1, C > 0$ such that

$$\|(s-A)^{-1}\| \leqslant \frac{C}{d(s,\sigma(A))},$$

for $s \in \rho(A)$ with $d(s, \sigma(A)) > M_1$. Set $M := \max\{M_1, 2\|P\|C\}$. Let $s \in \rho(A)$ with $d(s, \sigma(A)) > M$. Then $I - P(s-A)^{-1}$ is invertible and we obtain

$$\begin{aligned} \left\|(s-(A+P))^{-1}\right\| &= \|(s-A)^{-1}[I - P(s-A)^{-1}]^{-1}\| \\ &\leqslant \frac{C}{d(s,\sigma(A))} \frac{1}{1 - \|P\| \, \|(s-A)^{-1}\|} \\ &\leqslant \frac{2C}{d(s,\sigma(A))} \end{aligned}$$

which concludes the proof for $K = 2C$. □

Proposition 5.2.9. ([DS71]) *Let A be an operator with $\sigma(A) = (s_n)_{n\in\mathbb{N}}$ such that the family of spectral subspaces is a Riesz basis of subspaces of X. Then A has the representation $A = S + N$, where the scalar part S is defined as*

$$\begin{aligned} Sx &:= \sum_{n\in\mathbb{N}} s_n E_n x, \\ \mathcal{D}(S) &= \{x \in X \mid \sum_{n=1}^{\infty} \left\|s_n E_n x\right\|^2 < \infty\}, \end{aligned}$$

and $N_n := NE_n = (A - s_n)E_n$. Furthermore, N_n is quasi-nilpotent, i.e., $\sigma(N_n) = \{0\}$ for all $n \in \mathbb{N}$.

Proposition 5.2.10. *Let A be a generator of C_0-group on X with compact resolvent. The eigenvalues are counted with algebraic multiplicity. If the following conditions*

I. The span of the (generalized) eigenvectors form a dense set in X,

II. The eigenvalues can be decomposed into finitely many sets each having a uniform gap,

are both fulfilled, then A is a discrete Riesz spectral operator.

Proof: By [Zwa10, Theorem 1.1 and Theorem 1.6] it follows that the family of spectral subspaces is a Riesz basis of subspaces of X.
Thus, A has the representation $A = S + N$, cf. Proposition 5.2.9 where S denotes the scalar part of the spectral operator A and $N := \sum_{n\in\mathbb{N}} N_n$, where $N_n = (A - s_n)E_n$ and N_n is quasi-nilpotent. To prove that A is a discrete Riesz spectral operator it remains to show that N is bounded and nilpotent.
We can identify N_n with a square matrix corresponding to $N_n|_{E_n X}$ and thus N_n is bounded and nilpotent.
Since the eigenvalues of A can be decomposed into finitely many sets each having a uniform gap and their algebraic multiplicity is finite, the degree of the nilpotent matrices N_n is bounded. Thus, N is nilpotent.
Finally, we verify that N is bounded. Without loss of generality we assume that A generates an exponentially stable C_0-group. Then by [LW83], there exists an invertible and positive operator $L \in \mathcal{L}(X)$ such that

$$\langle Ax, Lx\rangle + \langle x, LAx\rangle = -\langle x, x\rangle \quad \forall x \in \mathcal{D}(A). \tag{5.8}$$

We define $A_n := AE_n = (s_n + N_n)E_n$, where E_n denotes the n-th spectral projection, and we identify A_n and N_n with the corresponding matrices on E_nX. From now on we fix n. Then we get for $x \in X$

$$\langle AE_nx, LE_nx\rangle + \langle E_nx, LAE_nx\rangle = -\langle E_nx, E_nx\rangle$$

or equivalently

$$\langle A_nE_nx, L_nE_nx\rangle + \langle E_nx, L_nA_nE_nx\rangle = -\langle E_nx, E_nx\rangle, \tag{5.9}$$

where $L_n := E_n^*LE_n = L_n^*$. As L is self-adjoint, L_n is self-adjoint as well. Again we identify L_n with the corresponding matrix on E_nX and obtain on E_nX

$$(s_n + N_n)^*L_n + L_n(s_n + N_n) = -I.$$

Thus we have

$$N_n^*L_n + L_nN_n = -I + r_nL_n \tag{5.10}$$

with $r_n := -2\mathrm{Re}\, s_n$. Multiplying (5.10) from the right by $N_n^{k_n-j+1}$ and from the left by $(N_n^*)^{k_n-j}$ with $j = 2, 3, \ldots, k_n$ results in

$$\begin{aligned}(N_n^*)^{k_n-j+1}L_nN_n^{k_n-j+1} &+ (N_n^*)^{k_n-j}L_nN_n^{k_n-j+2}\\ &= -(N_n^*)^{k_n-j}N_n^{k_n-j+1} + r_n(N_n^*)^{k_n-j}L_nN_n^{k_n-j+1}\end{aligned}$$

and thus it holds

$$\begin{aligned}\|L_n^{1/2}N_n^{k_n-j+1}\|^2 \leqslant &\|N_n^{k_n-j}\|\|N_n^{k_n-j+1}\| + |r_n|\,\|L_n\|\,\|N_n^{k_n-j}\|\|N_n^{k_n-j+1}\|\\ &+ \|N_n^{k_n-j}\|\|L_n\|\|N_n^{k_n-j+2}\|.\end{aligned} \tag{5.11}$$

Since L_n is boundedly invertible on E_nX, we get

$$m\|N_n^{k_n-j+1}\|^2 \leqslant \|L_n^{1/2}N_n^{k_n-j+1}\|^2 \text{ for some } m \text{ independent of } n. \tag{5.12}$$

For $j = 2$ we use $N_n^{k_n} = 0$ and obtain

$$m\|N_n^{k_n-1}\| \leqslant \|N_n^{k_n-2}\| + |r_n|\,\|L_n\|\,\|N_n^{k_n-2}\|.$$

Since A is the generator of a C_0-group, we have $R := \sup_{n\in\mathbb{N}} |r_n| < \infty$. Moreover, with $M := \sup_{n\in\mathbb{N}} \|L_n\| < \infty$ and $C := 1 + RM$, this implies

$$\|N_n^{k_n-1}\| \leqslant \frac{C}{m}\|N_n^{k_n-2}\|. \tag{5.13}$$

For $j = 3, \ldots, k_n$ we get using (5.12), (5.11), and (5.13) and by induction over j

$$\|N_n^{k_n-j+1}\| \leqslant \|N_n^{k_n-j}\| \sum_{l=1}^{j-1} \frac{C}{m^l}M^{l-1}. \tag{5.14}$$

In particular, for $j = k_n$ and using $k_n \leqslant K$, this implies

$$\|N_n\| \leqslant \sum_{l=1}^{k_n-1} \frac{C}{m^l}M^{l-1} \leqslant \sum_{l=1}^{K-1} \frac{C}{m^l}M^{l-1}.$$

Together with Proposition 5.2.6 this implies that N is bounded. □

5.3 Discrete Riesz spectral port-Hamiltonian operators

We consider first order linear port-Hamiltonian systems on a one-dimensional spatial domain of the form

$$\begin{aligned} \frac{\partial x}{\partial t}(\zeta,t) &= \left(P_1\frac{\partial}{\partial\zeta}+P_0\right)(\mathcal{H}(\zeta)x(\zeta,t)),\\ x(\zeta,0) &= x_0(\zeta), \\ 0 &= \begin{bmatrix}\widetilde{W}_1 & \widetilde{W}_0\end{bmatrix}\begin{bmatrix}(\mathcal{H}x)(1,t)\\(\mathcal{H}x)(0,t)\end{bmatrix}, \end{aligned} \tag{5.15}$$

where $\zeta\in[0,1]$ and $t\geqslant 0$ and the Assumption 3.1.1 is fulfilled. In Chapter 3 and 4 a detailed introduction in these kind of systems is given.
Furthermore, we emphasize that a port-Hamiltonian operator which generates a C_0-semigroup is closed and that its resolvent is compact, see Theorem 3.2.3. Having the Definition 2.2.15 of well-posedness and Theorem 3.3.5 in mind which shows that for a port-Hamiltonian boundary control system (5.15) well-posedness is already satisfied if A generates a C_0-semigroup, we recall the following definition.

Definition 5.3.1. We call the port-Hamiltonian system (5.15) a *well-posed control system* if A generates a C_0-semigroup on X and there exist $\tau>0$ and $m_\tau\geqslant 0$ such that for all $x_0\in\mathcal{D}(\mathfrak{A})$ and $u\in C^2([0,\tau];\mathbb{C}^d)$ with $u(0)=\left[\begin{smallmatrix}(\mathcal{H}x_0)(1,0)\\(\mathcal{H}x_0)(0,0)\end{smallmatrix}\right]$ the classical solution x of (5.15) satisfies

$$\|x(\tau)\|_X^2\leqslant m_\tau\left(\|x_0\|_X^2+\int_0^\tau\|u(t)\|^2dt\right).$$

In the following, we assume that the port-Hamiltonian system (5.15) is a well-posed control system.
Well-posedness implies that for every initial condition $x_0\in X$ and every L^2 control function u the port-Hamiltonian control system has a unique mild solution, cf. Theorem 3.3.12. Due to Theorem 4.1.5 well-posed port-Hamiltonian control systems are always exactly controllable in finite time. As a consequence of exact controllability we obtain that the eigenspaces span the state space.

Proposition 5.3.2. *Consider a well-posed port-Hamiltonian control system* (5.15) *and assume that A generates a C_0-group on X. Let $\sigma(A)=(s_n)_{n\in\mathbb{N}}$. Then*

$$X=\overline{\operatorname{span}_{n\in\mathbb{N}}E((s_n))X},$$

where $E((s_n))$ is introduced in Definition 5.2.1.

Proof: Follows from [JZ99, Lemma 7.3] together with Proposition 4.1.5. □
By $Z^+(\zeta)$, we denote the span of the eigenvectors of $P_1\mathcal{H}(\zeta)$ corresponding to the positive eigenvalues of $P_1\mathcal{H}(\zeta)$ and by $Z^-(\zeta)$ the span of the eigenvectors of $P_1\mathcal{H}(\zeta)$ corresponding to the negative eigenvalues of $P_1\mathcal{H}(\zeta)$.

We are now in the position to formulate our main result.

Theorem 5.3.3. *Let A be a port-Hamiltonian operator and the generator of a C_0-semigroup. Then the following is equivalent:*

1. *A is a discrete Riesz spectral operator.*
2. *A is the generator of a C_0-group.*
3. $\widetilde{W}_1\mathcal{H}(1)Z^+(1) \oplus \widetilde{W}_0\mathcal{H}(0)Z^-(0) = \widetilde{W}_1\mathcal{H}(1)Z^-(1) \oplus \widetilde{W}_0\mathcal{H}(0)Z^+(0) = \mathbb{C}^d$.

If one of the equivalent conditions are satisfied, then $\sigma(A) = \sigma_p(A)$ lie in a strip parallel to the imaginary axis, the eigenvalues (counted according to the algebraic multiplicity) can be decomposed into finitely many sets each having a uniform gap and A satisfies the spectrum determined growth assumption, that is, $\omega_0(A) = s(A)$.

The proof of Theorem 5.3.3 will be given in the next Section 5.3.1.

Remark 5.3.4. It is particularly easy to check if a port-Hamiltonian operator A is the generator of a unitary C_0-group, cf. Theorem 3.2.3. This is actually the case if and only if $W_B \Sigma W_B^* = 0$, where $W_B := \left[\widetilde{W}_1\ \widetilde{W}_0\right]\left[\begin{smallmatrix} P_1 & -P_1 \\ I & I \end{smallmatrix}\right]^{-1}$ and $\Sigma := \left[\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right]$. In this case A is even a skew-adjoint operator by Stone's Theorem, cf. Theorem 2.1.16, which implies that the normalized eigenvectors form an orthonormal basis of X.

5.3.1 Proof of the Main Result

In the following section we give the proof of Theorem 5.3.3. Let us first assume that one and therefore all of the conditions of the theorem are satisfied. As the resolvent of A is compact the spectrum consists of isolated eigenvalues only. That the eigenvalues lie in a strip parallel to the imaginary axis and that they can be decomposed into finitely many sets each having a uniform gap will be shown in the proof of the implication 2) $\Rightarrow$ 1). Finally, $\omega_0(A) = s(A)$ is implied by [GZ01, Theorem 2.12].

5.3.2 Proof of the equivalence 2) $\Leftrightarrow$ 3) of Theorem 5.3.3

The operator A generates a C_0-group, if and only if A and $-A$ generates a C_0-semigroup, [EN00, Section II.3.11]. In Theorem 3.2.5 it is shown that A is the generator of a C_0-semigroup if and only if $\widetilde{W}_1\mathcal{H}(1)Z^+(1) \oplus \widetilde{W}_0\mathcal{H}(0)Z^-(0) = \mathbb{C}^d$. Since $\mathcal{D}(-A) = \mathcal{D}(A)$ and $\sigma_p(P_1\mathcal{H}) = -\sigma_p(-P_1\mathcal{H})$ it follows that $-A$ generates a C_0-semigroup if and only if $\widetilde{W}_1\mathcal{H}(1)Z^-(1) \oplus \widetilde{W}_0\mathcal{H}(0)Z^+(0) = \mathbb{C}^d$.

5.3.3 Proof of the implication 2) $\Rightarrow$ 1) of Theorem 5.3.3

The following lemma will be useful.

Lemma 5.3.5. *Let $s \in \mathbb{C}$ and P_1, P_0 and $\mathcal{H}$ fulfil the conditions for a port-Hamiltonian operator in Assumption 3.1.1. Then the solutions of the system of ordinary differential equations*

$$sx(\zeta) = \left(P_1 \frac{d}{d\zeta} + P_0\right)(\mathcal{H}x)(\zeta), \quad \zeta \in [0,1], \tag{5.16}$$

denoted by $x(\zeta) = \Psi^s(\zeta)x(0)$, satisfy

$$\widetilde{M}e^{-|\mathrm{Re}\, s|\tilde{c}_0\zeta}\,\|v\| \leqslant \|\Psi^s(\zeta)v\| \leqslant Me^{|\mathrm{Re}\, s|c_0\zeta}\,\|v\|, \quad v \in \mathbb{C}^d,\ \zeta \in [0,1],$$

with constants $M, \widetilde{M} > 0$, and $\tilde{c}_0, c_0 \geqslant 0$ independent of s and ζ.

Proof: Writing $\tilde{x} = \mathcal{H}x$, (5.16) can be equivalently formulated as

$$\mathcal{H}(\zeta)P_1\tilde{x}'(\zeta) = s\tilde{x}(\zeta) - \mathcal{H}(\zeta)P_0\tilde{x}(\zeta).$$

We write $s = i\omega + r$ with $\omega, r \in \mathbb{R}$ and diagonalize $P_1\mathcal{H}(\zeta) = S^{-1}(\zeta)\Delta(\zeta)S(\zeta)$, see Assumption 3.1.1. Thus, $\mathcal{H}(\zeta)P_1 = S^*(\zeta)\Delta(\zeta)S^{-*}(\zeta)$ and we get

$$\Delta(\zeta)S^{-*}(\zeta)\tilde{x}'(\zeta) = i\omega S^{-*}(\zeta)\tilde{x}(\zeta) + \left(rI - S^{-*}(\zeta)\mathcal{H}(\zeta)P_0S^*(\zeta)\right)S^{-*}(\zeta)\tilde{x}(\zeta).$$

Using the substitution $z = S^{-*}\tilde{x}$ gives the equivalent differential equation

$$z'(\zeta) = i\omega\Delta^{-1}(\zeta)z(\zeta) + \left(r\Delta^{-1}(\zeta) + Q(\zeta)\right)z(\zeta), \tag{5.17}$$

where

$$Q(\zeta) := -\Delta^{-1}(\zeta)S^{-*}(\zeta)\mathcal{H}(\zeta)P_0S^*(\zeta) - (S^{-*})'(\zeta)S^*(\zeta).$$

Thus, equation (5.16) is equivalent to equation (5.17). Due to the fact that $P_1\mathcal{H}(\zeta)$ has real eigenvalues, $\Delta(\zeta)$ is a diagonal, real matrix and $i\omega\Delta^{-1}(\zeta)$ is a diagonal, purely imaginary matrix. We write $\Delta^{-1}(\zeta) = \mathrm{diag}_{k=1,\dots,n}(\alpha_k(\zeta))$ with $\alpha_k(\zeta) : [0,1] \to \mathbb{R}$ and define $\Phi_\omega(\zeta) = \mathrm{diag}(\exp(-i\omega\int_0^\zeta \alpha_k(\tau)d\tau))$ which satisfies

$$\|\Phi_\omega(\zeta)\|_{\mathcal{L}(\mathbb{C}^d)} = 1, \quad \zeta \in [0,1].$$

Multiplying (5.17) with $\Phi_\omega(\zeta)$, we get

$$\Phi_\omega(\zeta)z'(\zeta) - i\omega\Delta^{-1}(\zeta)\Phi_\omega(\zeta)z(\zeta) = \Phi_\omega(\zeta)\left(r\Delta^{-1}(\zeta) + Q(\zeta)\right)z(\zeta)$$

or equivalently

$$(\Phi_\omega(\zeta)z(\zeta))' = \left(r\Delta^{-1}(\zeta) + \Phi_\omega(\zeta)Q(\zeta)\Phi_\omega^{-1}(\zeta)\right)\Phi_\omega(\zeta)z(\zeta).$$

Using the substitution $y = \Phi_\omega z$, this ordinary differential equation becomes

$$y' = (r\Delta^{-1}(\zeta) + Q_\omega(\zeta))y(\zeta), \tag{5.18}$$

where $Q_\omega(\zeta) := \Phi_\omega(\zeta)Q(\zeta)\Phi_\omega^{-1}(\zeta)$. There exist constants $c_0, c_1 \geqslant 0$, independent of ω, such that

$$2\max_{\zeta\in[0,1]}\|r\Delta^{-1}(\zeta) + Q_\omega(\zeta)\| \leqslant |r|\,c_0 + c_1. \tag{5.19}$$

The solution y of (5.18) satisfies

$$\frac{d}{d\zeta}\|y(\zeta)\|^2 = y(\zeta)^*[(r\Delta^{-1}(\zeta) + Q_\omega(\zeta)) + (r\Delta^{-1}(\zeta) + Q_\omega(\zeta))^*]y(\zeta). \tag{5.20}$$

This together with (5.19) implies

$$-(|r|\, c_0 + c_1)\|y(\zeta)\|^2 \leqslant \frac{d}{d\zeta}\|y(\zeta)\|^2 \leqslant (|r|\, c_0 + c_1)\|y(\zeta)\|^2,$$

and thus

$$e^{-(|r|c_0+c_1)\zeta}\, \|y(0)\|^2 \leqslant \|y(\zeta)\|^2 \leqslant e^{(|r|c_0+c_1)\zeta}\, \|y(0)\|^2 \,.$$

As the mapping $x \mapsto y$ is boundedly invertible on $L^2(0,1;\mathbb{C}^d)$, with norm independent on ω, the statement follows. □

Next we state some results from complex analysis.

Definition 5.3.6. ([AI95, II.1.27]) An entire function f is called an *entire function of exponential type*, if there exist constants C and T such that $|f(s)| \leqslant Ce^{T|s|}$ for all $s \in \mathbb{C}$. Further, an entire function f of exponential type is said to be of *sine type*, if

1. the zeros of f lie in a strip $\{s \in \mathbb{C} \mid |\operatorname{Im} s| \leqslant h\}$ for some $h \geqslant 0$ and
2. there exist $\hat{\omega} \in \mathbb{R}$ and positive constants c and C such that
$$c \leqslant |f(r + i\hat{\omega})| \leqslant C$$
for every $r \in \mathbb{R}$ holds.

Proposition 5.3.7. ([AI95, Proposition II.1.28] (Levin 1961)) *If f is of sine type, then its set of zeros counted with algebraic multiplicity is a finite unification of sets each having a uniform gap.*

Lemma 5.3.8. *A complex number $s \in \mathbb{C}$ is an eigenvalue of a port-Hamiltonian operator A if and only if*

$$\det\left[\widetilde{W}_1\mathcal{H}(1)\Psi^s(1) + \widetilde{W}_0\mathcal{H}(0)\right] = 0,$$

where Ψ^s is described in Lemma 5.3.5.

Proof: For every $x(0) \in \mathbb{C}^d$ there exists a solution of the differential equation $sx(\zeta) = (P_1\frac{d}{d\zeta} + P_0)(\mathcal{H}x)(\zeta)$, $\zeta \in [0,1]$. The complex number s is an eigenvalue of A if and only if $x \in \mathcal{D}(A)$ and $Ax = sx$. Using Lemma 5.3.5 this is equivalent to

$$x(\zeta) = \Psi^s(\zeta)x(0) \text{ and } \begin{bmatrix}\widetilde{W}_1 & \widetilde{W}_0\end{bmatrix}\begin{bmatrix}(\mathcal{H}x)(1)\\(\mathcal{H}x)(0)\end{bmatrix} = 0.$$

Inserting the first equation in the second, we get that s is an eigenvalue of A if and only if $\det\left[\widetilde{W}_1\mathcal{H}(1)\Psi^s(1) + \widetilde{W}_0\mathcal{H}(0)\right] = 0$. □

Now we are in the situation to give the proof of the implication 2) ⇒ 1).

Proof of the implication 2) ⇒ 1) of Theorem 5.3.3: Assertion 2) implies that the eigenvalues lie in a strip parallel to the imaginary axis. Thus, thanks to Proposition 5.3.2 and Proposition 5.2.10 it suffices to show that the eigenvalues $(s_n)_{n\in\mathbb{N}}$ of A (counted according to the algebraic multiplicity) can be decomposed into finitely many sets having each a uniform gap.

Using Proposition 5.3.7, this is implied by the existence of an entire function g of exponential type with

i) g has exactly the zeros $(s_n)_{n\in\mathbb{N}}$ and

ii) there exist $r, c, C > 0$ such that for every $\omega \in \mathbb{R}$: $c \leqslant |g(r+i\omega)| \leqslant C$.

Note that $f : \mathbb{C} \to \mathbb{C}$, defined by $f(s) := g(is)$ is a sine-type function if g is an entire function of exponential type and satisfying the above conditions i) and ii). We define $g : \mathbb{C} \to \mathbb{C}$ by

$$g(s) := \det\left[\widetilde{W}_1\mathcal{H}(1)\Psi^s(1) + \widetilde{W}_0\mathcal{H}(0)\right]. \tag{5.21}$$

By Lemma 5.3.8, a complex number $s \in \mathbb{C}$ is an eigenvalue of the operator A if and only if $g(s) = 0$.
Ψ^s described in Lemma 5.3.5 is an entire function, cf. [Was87, Theorem 24.1] and thus g is an entire function as well. Clearly, g has the zeros $(s_n)_{n\in\mathbb{N}}$. Since the determinant of a matrix equals the product of its eigenvalues and every eigenvalue is smaller or equal the norm of the matrix, it yields

$$|g(s)| \leqslant \left\|\left[\widetilde{W}_1\mathcal{H}(1)\Psi^s(1) + \widetilde{W}_0\mathcal{H}(0)\right]\right\|^n.$$

Using Lemma 5.3.5 it holds $|g(s)| \leqslant c_2 e^{|\operatorname{Re} s| c_3}$ for some constants $c_2, c_3 \geqslant 0$, and thus, g is bounded on lines parallel to the imaginary axis and grows at most exponentially.
Next, we show that g is bounded away from zero on some line parallel to the imaginary axis. Since the control operator B of the port-Hamiltonian system (5.15) is admissible, see Lemma 3.3.12, it yields that for $\omega > \omega_0(A)$ exists a constant $M_\omega > 0$ such that

$$\|(s - A_{-1})^{-1}B\|_{\mathcal{L}(\mathbb{C}^d,X)} \leqslant \frac{M_\omega}{\sqrt{\operatorname{Re} s - \omega}} \quad \text{for } \operatorname{Re} s \geqslant \omega, \tag{5.22}$$

see Proposition 2.2.5. Let $r > \omega_0(A)$ and we assume that g is not bounded away from zero on $r+i\mathbb{R}$, i.e., there exists a sequence $\omega_k \in \mathbb{R}$ such that $g(r+i\omega_k) \to 0$. Since all zeros of g have real part less or equal to the growth bound $\omega_0(T)$ of the C_0-semigroup generated by A, it holds true that $g(r+i\omega_k) \neq 0$. Let u_0 be an arbitrary vector in $\mathbb{C}^d$. By Proposition 2.2.20 the solution $x_{u_0}^{r+i\omega_k}$ of

$$\begin{aligned} (r+i\omega_k)x(\zeta) &= \left(P_1\frac{d}{d\zeta} + P_0\right)(\mathcal{H}x)(\zeta) \\ \begin{bmatrix}\widetilde{W}_1 & \widetilde{W}_0\end{bmatrix}\begin{bmatrix}(\mathcal{H}x)(1)\\(\mathcal{H}x)(0)\end{bmatrix} &= u_0, \end{aligned}$$

is given by

$$\begin{aligned} x_{u_0}^{r+i\omega_k} &= (((r+i\omega_k) - A_{-1})^{-1}(\mathfrak{A}\widetilde{B} - (r+i\omega_k)\widetilde{B}) + \widetilde{B})u_0 \\ &= ((r+i\omega_k) - A_{-1})^{-1}Bu_0 \end{aligned}$$

and $x_{u_0}^{r+i\omega_k}(0)$ fulfils

$$\left[\widetilde{W}_1\mathcal{H}(1)\Psi^{r+i\omega_k}(1) + \widetilde{W}_0\mathcal{H}(0)\right]x_{u_0}^{r+i\omega_k}(0) = u_0.$$

Let $M_k := \left[\widetilde{W}_1\mathcal{H}(1)\Psi^{r+i\omega_k}(1) + \widetilde{W}_0\mathcal{H}(0)\right]$. Since $g(r+i\omega_k) = \det(M_k) \to 0$ and $1 = \det(I) = \det(M_k)\det(M_k^{-1})$, we have $\det(M_k^{-1}) \to \infty$. Hence, M_k^{-1} has an eigenvalue ν_k with $|\nu_k| \to \infty$. Choose $u_{k,0}$ as a normalized eigenvector to ν_k. Then it yields

$$x_{u_{k,0}}^{r+i\omega_k}(0) = M_k^{-1}u_{k,0} = \nu_k u_{k,0},$$

which implies $\|x_{u_{k,0}}^{r+i\omega_k}(0)\|_{\mathbb{C}^d} \to \infty$. We note that the function $x_{u_{k,0}}^{r+i\omega_k}$ is given by

$$x_{u_{k,0}}^{r+i\omega_k}(\zeta) = \Psi^{r+i\omega_k}(\zeta)x_{u_{k,0}}^{r+i\omega_k}(0).$$

Using that the inverse of $\Psi^{r+i\omega_k}$ is a bounded function, see Lemma 5.3.5, we get

$$\|x_{u_{k,0}}^{r+i\omega_k}\|_{L^2((0,1);\mathbb{C}^d)} \to \infty. \tag{5.23}$$

However, since we also have $x_{u_{k,0}}^{r+i\omega_k} = ((r+i\omega_k) - A_{-1})^{-1}Bu_{k,0}$, equation (5.23) is in contradiction with the uniform boundedness of $((r+i\omega_k) - A_{-1})^{-1}Bu_{0,k}$, see equation (5.22). Thus the entire function g is of exponential type and satisfies condition i) and ii). This concludes the proof. $\square$

5.3.4 Proof of the implication 1) $\Rightarrow$ 2) of Theorem 5.3.3

We start with some characterizations of the resolvent and the spectrum of port-Hamiltonian operators.

Lemma 5.3.9. *Let $\Lambda \in C([0,1];\mathbb{C}^d)$ with $\Lambda(\zeta)$ diagonal, invertible and positive definite for every $\zeta \in [0,1]$, $Q \in \mathbb{C}^{d\times d}$ singular and $A : \mathcal{D}(A) \subset L^2(0,1;\mathbb{C}^d) \to L^2(0,1;\mathbb{C}^d)$ defined by $Ax = \Lambda x'$ and $\mathcal{D}(A) = \{x \in \mathcal{W}^{1,2}(0,1;\mathbb{C}^d) \mid x(1) + Qx(0) = 0\}$. Then there exist real constants $\gamma < 0$ and $a, b > 0$ such that $\left\|(s-A)^{-1}\right\| \geqslant ae^{b|s|}$ for real $s \in \rho(A)$ with $s < \gamma$.*

Proof: Let $0 \neq x(0) \in \ker Q$. We define $F_s(\zeta) := s\int_0^\zeta \Lambda^{-1}(\tau)d\tau$ for $s \in \mathbb{R}$ and $\zeta \in [0,1]$. Note that F_s and Λ commute as both are diagonal. There exists $\gamma_0 < 0$ such that $I - e^{2F_s(1)}$ is invertible if $s < \gamma_0$. Further, let $s \in \rho(A)$ with $s < \gamma_0$ and define

$$g(\zeta) = 2se^{F_s(1)-F_s(\zeta)}g_0 = 2se^{s\int_\zeta^1 \Lambda^{-1}(\tau)d\tau}g_0 \tag{5.24}$$

with $g_0 := e^{F_s(1)}(I - e^{2F_s(1)})^{-1}x(0)$. Then the solution of

$$\Lambda x' = sx + g \tag{5.25}$$

is given by

$$\begin{aligned}
x(\zeta) &= e^{F_s(\zeta)}x(0) + \int_0^\zeta e^{F_s(\zeta)-F_s(\tau)}\Lambda^{-1}(\tau)g(\tau)d\tau \\
&= e^{F_s(\zeta)}x(0) - e^{F_s(\zeta)+F_s(1)}\int_0^\zeta (-2s\Lambda^{-1}(\tau))e^{-2F_s(\tau)}d\tau\, g_0 \\
&= e^{F_s(\zeta)}x(0) - e^{F_s(\zeta)+F_s(1)}(e^{-2F_s(\zeta)} - I)g_0 \\
&= e^{F_s(\zeta)}(I - e^{2F_s(1)})e^{-F_s(\zeta)}g_0 - e^{-F_s(\zeta)+F_s(1)}g_0 + e^{-F_s(\zeta)+F_s(1)}g_0 \\
&= e^{F_s(\zeta)-F_s(1)}g_0 - \frac{1}{2s}g(\zeta).
\end{aligned}$$

In particular $x(1) = 0$ and thus $(s-A)^{-1}g = x$. As $\Lambda \in C([0,1];\mathbb{C}^d)$ is a diagonal and invertible matrix-valued function with positive entries on its diagonal, there exists $\lambda_0 > 0$ such that the diagonal elements of Λ are bounded by λ_0. First using (5.24) we can estimate

$$\|g\|_{L^2((0,1);\mathbb{C}^d)} \leqslant c\sqrt{|s|}\,\|g_0\|\,,\ c > 0,$$

and

$$\begin{aligned}
\|x\|_{L^2(0,1;\mathbb{C}^d)} &\geqslant \left\|e^{F_s(\cdot)-F_s(1)}g_0\right\|_{L^2(0,1;\mathbb{C}^d)} - \frac{1}{2|s|}\|g\|_{L^2((0,1);\mathbb{C}^d)} \\
&= \left(\int_0^1 \left\|e^{-s\int_\tau^1 \Lambda^{-1}(\sigma)d\sigma}g_0\right\|^2 d\tau\right)^{\frac{1}{2}} - \frac{1}{2|s|}\|g\|_{L^2((0,1);\mathbb{C}^d)} \\
&\geqslant \left(\int_0^1 e^{2|s|(1-\tau)\lambda_0}\|g_0\|^2 d\tau\right)^{\frac{1}{2}} - \frac{1}{2|s|}\|g\|_{L^2((0,1);\mathbb{C}^d)} \\
&\geqslant \frac{1}{4|s|^2\lambda_0}\left(e^{2|s|\lambda_0}-1\right)\|g\|_{L^2((0,1);\mathbb{C}^d)} - \frac{1}{2|s|}\|g\|_{L^2((0,1);\mathbb{C}^d)}\,.
\end{aligned}$$

This completes the proof of lemma. □

Lemma 5.3.10. *Let A be a port-Hamiltonian operator, which generates a C_0-semigroup. Furthermore, let A be a discrete Riesz spectral operator and let $(s_n)_{n\in\mathbb{N}}$ denote its eigenvalues. Then there exists a constant $K > 0$ such that for every $n \in \mathbb{N}$ within the ball $\{s \in \mathbb{C} \mid |s - s_n| \leqslant K\,|\mathrm{Re}\, s_n|^2\}$ there lie at most d eigenvalues.*

Proof: Without lost of generality we assume that A generates an exponentially stable C_0-semigroup. By Proposition 4.1.5 the corresponding port-Hamiltonian control system (5.15) with control operator B is exactly controllable in finite time. Then the dual system, described by A^* and B^*, is exactly observable and it yields due to the Hautus Test, cf. Theorem 2.2.10, that there exists a positive constant m such that

$$\|(s-A^*)x\|^2 + |\mathrm{Re}\, s|\,\|B^*x\|^2 \geqslant m\,|\mathrm{Re}\, s|^2\,\|x\|^2\,,\quad \mathrm{Re}\, s < 0,\ x \in \mathcal{D}(A^*). \tag{5.26}$$

No matter that there may exist generalized eigenvectors, we consider only eigenvectors corresponding to different eigenvalues of A^*. As $\sigma(A) = \overline{\sigma(A^*)}$, it suffices to prove the statement for A^*. Choose arbitrary $e_1, \ldots e_{d+1}$ normed eigenvectors of the operator A^* to the eigenvalues $\lambda_1, \ldots, \lambda_{d+1}$ with $\lambda_n \neq \lambda_m$ for $n \neq m \in 1, \ldots, d+1$.
Since $B^* \in \mathcal{L}(\mathcal{D}(A^*), \mathbb{C}^d)$ the $d+1$ vectors B^*e_n are linearly dependent in $\mathbb{C}^d$, i.e, there exists scalars $a_1, \ldots, a_{d+1} \in \mathbb{C}$ with $\sum_{n=1}^{d+1}|a_n|^2 = 1$ such that

$$a_1B^*e_1 + \ldots a_{d+1}B^*e_{d+1} = 0. \tag{5.27}$$

Consider $x = \sum_{n=1}^{d+1} a_ne_n$. Then $x \in \mathcal{D}(A^*)$ with $B^*x = 0$, and

$$\|x\|^2 = \left\|\sum_{n=1}^{d+1} a_ne_n\right\| \geqslant m_1 > 0,$$

by Proposition 5.2.6. It follows with the Hautus Test (5.26) at the point $s = \lambda_{d+1}$

$$\begin{aligned} m\, m_1 \left|\operatorname{Re}\lambda_{d+1}\right|^2 &\leqslant \|(\lambda_{d+1} - A^*)x\|^2 = \left\|\sum_{n=1}^{d} a_n(\lambda_{d+1} - \lambda_n)e_n\right\|^2 \\ &\leqslant m_2 \sum_{n=1}^{d} |a_n|^2 \left|\lambda_{d+1} - \lambda_n\right|^2 . \end{aligned}$$

Thus,

$$\frac{d+1}{d+1}\frac{m\, m_1}{m_2} \left|\operatorname{Re}\lambda_{d+1}\right|^2 \leqslant \sum_{n=1}^{d} \left|\lambda_{d+1} - \lambda_n\right|^2 .$$

Since the eigenvalues are arbitrary chosen, this implies that in the ball with radius $\frac{m\, m_1}{(d+1)\, m_2} \left|\operatorname{Re}\lambda_{d+1}\right|^2$ around λ_{d+1} lies at most d eigenvalues. Hence, the statement follows. □

Now we are in the position to prove the implication 1) ⇒ 2).

Proof of the implication 1) ⇒ 2) of Theorem 5.3.3: We assume that A does not generate a C_0-group.

Since A is a port-Hamiltonian operator, we denote by S the matrix-valued function such that $P_1\mathcal{H}(\zeta) = S^{-1}(\zeta)\Delta(\zeta)S(\zeta)$, see Assumption 3.1.1. Since the eigenvalues of $P_1\mathcal{H}(\zeta)$ and $\mathcal{H}(\zeta)^{\frac{1}{2}}P_1\mathcal{H}(\zeta)^{\frac{1}{2}}$ are the same, it follows by Sylvester's law of inertia that the number of positive and negative eigenvalues of $P_1\mathcal{H}(\zeta)$ equal those of P_1. Let d_1 denote the number of positive and $d_2 = d - d_1$ the number of negative eigenvalues of P_1. Thus without loss of generality Δ can be written as $\Delta(\zeta) = \left[\begin{smallmatrix} \Lambda(\zeta) & 0 \\ 0 & \Theta(\zeta)\end{smallmatrix}\right]$, where $\Lambda(\zeta) \in \mathbb{C}^{d_1\times d_1}$ correspond to the positive eigenvalues and $\Theta(\zeta) \in \mathbb{C}^{d_2\times d_2}$ to the negative ones.

Let $\mathcal{S} \in \mathcal{L}(X)$ be the multiplication operator $(\mathcal{S}x)(\zeta) := S(\zeta)x(\zeta)$. By assumption $\mathcal{S}$ is invertible and we obtain

$$\begin{aligned} \mathcal{S}A\mathcal{S}^{-1}z(\zeta) =& \Delta(\zeta)(z(\zeta))' \\ &+ \Delta'(\zeta)z(\zeta) + S(\zeta)(S^{-1})'\Delta(\zeta)z(\zeta) + S(\zeta)P_0\mathcal{H}(\zeta)S^{-1}(\zeta)z(\zeta) \\ \mathcal{D}(\mathcal{S}A\mathcal{S}^{-1}) =& \{z \in \mathcal{W}^{1,2}(0,1;\mathbb{C}^d) \mid \begin{bmatrix}\widetilde{W}_1 & \widetilde{W}_0\end{bmatrix} \left[\begin{smallmatrix} (\mathcal{H}S^{-1}z)(1) \\ (\mathcal{H}S^{-1}z)(0)\end{smallmatrix}\right] = 0\}. \end{aligned}$$

The operator $\mathcal{S}A\mathcal{S}^{-1}$ generates a C_0-semigroup, too. We split the variable $z(\zeta) = \left[\begin{smallmatrix} z^+(\zeta) \\ z^-(\zeta)\end{smallmatrix}\right] \in \mathbb{C}^d$ with $z^+(\zeta) \in \mathbb{C}^{d_1}$ and $z^-(\zeta) \in \mathbb{C}^{d_2}$, and we define $\widetilde{W}_1\mathcal{H}(1)S^{-1}(1) =: \begin{bmatrix} V_1 & V_2\end{bmatrix}$ and $\widetilde{W}_0\mathcal{H}(0)S^{-1}(0) =: \begin{bmatrix} U_1 & U_2\end{bmatrix}$, where $U_1, V_1 \in \mathbb{C}^{d\times d_1}$ and $U_2, V_2 \in \mathbb{C}^{d\times d_2}$. Then, $z \in \mathcal{D}(\mathcal{S}A\mathcal{S}^{-1})$ if and only if $z \in \mathcal{W}^{1,2}(0,1;\mathbb{C}^d)$

and

$$\begin{aligned}
0 &= \begin{bmatrix} \widetilde{W}_1 & \widetilde{W}_0 \end{bmatrix} \begin{bmatrix} (\mathcal{H}S^{-1}z)(1) \\ (\mathcal{H}S^{-1}z)(0) \end{bmatrix} \\
&= \begin{bmatrix} V_1 & V_2 \end{bmatrix} \begin{bmatrix} z^+(1) \\ z^-(1) \end{bmatrix} + \begin{bmatrix} U_1 & U_2 \end{bmatrix} \begin{bmatrix} z^+(0) \\ z^-(0) \end{bmatrix} \\
&= \begin{bmatrix} V_1 & U_2 \end{bmatrix} \begin{bmatrix} z^+(1) \\ z^-(0) \end{bmatrix} + \begin{bmatrix} U_1 & V_2 \end{bmatrix} \begin{bmatrix} z^+(0) \\ z^-(1) \end{bmatrix} \\
&= K \begin{bmatrix} \Lambda(1)z^+(1) \\ \Theta(0)z^-(0) \end{bmatrix} + Q \begin{bmatrix} \Lambda(0)z^+(0) \\ \Theta(1)z^-(1) \end{bmatrix},
\end{aligned}$$

where $K := \begin{bmatrix} V_1 & U_2 \end{bmatrix} \begin{bmatrix} \Lambda(1)^{-1} & 0 \\ 0 & \Theta(0)^{-1} \end{bmatrix}$ and $Q := \begin{bmatrix} U_1 & V_2 \end{bmatrix} \begin{bmatrix} \Lambda(0)^{-1} & 0 \\ 0 & \Theta(1)^{-1} \end{bmatrix}$. Since $\mathcal{S}A\mathcal{S}^{-1}$ is the generator of a C_0-semigroup and this property is invariant under bounded perturbations, K is invertible, see [JZ12, Theorem 13.3.1].
Let $\mathcal{T} \in \mathcal{L}(X)$ be defined by $\mathcal{T}\begin{bmatrix} z^+(\zeta) \\ z^-(\zeta) \end{bmatrix} := \begin{bmatrix} z^+(\zeta) \\ z^-(1-\zeta) \end{bmatrix}$. Clearly, $\mathcal{T}$ is invertible. Then the operator $\mathcal{A} := \mathcal{T}\mathcal{S}A\mathcal{S}^{-1}\mathcal{T}^{-1}$ on X is given by

$$\begin{aligned}
\mathcal{A}z(\zeta) &= \begin{bmatrix} \Lambda(\zeta) & 0 \\ 0 & -\Theta(1-\zeta) \end{bmatrix} z'(\zeta) + R(\zeta)z(\zeta) \\
\mathcal{D}(\mathcal{A}) &= \Big\{ z \in \mathcal{W}^{1,2}(0,1;\mathbb{C}^d) \mid \\
&\qquad K \begin{bmatrix} \Lambda(1) & 0 \\ 0 & \Theta(0) \end{bmatrix} z(1) + Q \begin{bmatrix} \Lambda(1) & 0 \\ 0 & \Theta(0) \end{bmatrix} z(0) = 0 \Big\},
\end{aligned}$$

where $z \mapsto R(\cdot)z(\cdot)$ is a bounded multiplication operator on X. Let $\widetilde{K} := K\begin{bmatrix} \Lambda(1) & 0 \\ 0 & \Theta(0) \end{bmatrix}$ and $\widetilde{Q} := Q\begin{bmatrix} \Lambda(1) & 0 \\ 0 & \Theta(0) \end{bmatrix}$. By assumption, the matrix $\widetilde{K}$ is invertible. As a bounded perturbation of a generator of C_0-group generates again a C_0-group, we obtain that the operator

$$\begin{aligned}
\widetilde{A}z(\zeta) &= \begin{bmatrix} \Lambda(\zeta) & 0 \\ 0 & -\Theta(1-\zeta) \end{bmatrix} z'(\zeta) \\
\mathcal{D}(\widetilde{A}) &= \Big(z \in \mathcal{W}^{1,2}(0,1;\mathbb{C}^d) \mid z(1) = \widetilde{K}^{-1}\widetilde{Q}z(0) \Big\}
\end{aligned}$$

generates a C_0-semigroup, but not a C_0-group. In particular, $Q_1 := \widetilde{K}^{-1}\widetilde{Q}$ is singular.
Since $\mathcal{A}$ is a discrete Riesz spectral operator, due to Lemma 5.3.10, there is a $K > 0$ such that in the ball with radius $K\,|\mathrm{Re}\, s|^2$ around every eigenvalue s of A lie at most d eigenvalues. Thus there exist sequences $(t_n)_{n\in\mathbb{N}} \subset \mathbb{R}$ and $(r_n)_{n\in\mathbb{N}} \subset (0,\infty)$ with $t_n \to -\infty$ and $r_n \to \infty$ such that the ball with center t_n and radius r_n lie in $\rho(A)$. By Lemma 5.2.8 we get

$$\|(t_n - \widetilde{A})^{-1}\| \to 0.$$

However, for the operator $\widetilde{A}$ the Lemma 5.3.9, is applicable which implies that $\|(t_n - \widetilde{A})^{-1}\| \to \infty$. This leads to a contradiction. $\square$

5.4 Examples

5.4.1 Wave equation with boundary feedback

We consider the one-dimensional wave equation with boundary feedback as in [XW11], but we allow for spatial dependent mass density and Young's modulus, given by

$$\begin{aligned} w_{tt}(\zeta,t) &= \frac{1}{\rho(\zeta)}\frac{\partial}{\partial\zeta}(T(\zeta)w_\zeta(\zeta,t)), \quad \zeta\in[0,1],\ t\geqslant 0,\\ w(0,t) &= 0,\\ u(t) &= T(1)w_\zeta(1,t),\\ y(t) &= w_t(1,t),\\ u(t) &= -\kappa y(t), \kappa>0, \end{aligned} \tag{5.28}$$

where $\zeta\in[0,1]$ is the spatial variable, $w(\zeta,t)$ describes the displacement of the point ζ of the string at time t, $T(\zeta)>0$ is the Young's modulus of the string, $\rho(\zeta)>0$ is the mass density, and $\kappa>0$. We model this system as a port-Hamiltonian system. Therefore we introduce the state variable $x=\begin{bmatrix} x_1(\zeta,t)\\ x_2(\zeta,t)\end{bmatrix}$ with $x_1=\rho(\zeta)\frac{\partial w}{\partial t}$ (momentum), $x_2=\frac{\partial w}{\partial\zeta}$ (strain) and the state space $X=L^2(0,1;\mathbb{C}^d)$. Since the meaning of $w(0,t)=0$ is that in the point $\zeta=0$ the string is fixed for all times, we model this boundary condition as $\frac{\partial w}{\partial t}w(0,t)=0$. Then the closed-loop system (5.28) can equivalently be written as

$$\begin{aligned} \frac{\partial}{\partial t}\begin{bmatrix} x_1(\zeta,t)\\ x_2(\zeta,t)\end{bmatrix} &= P_1\frac{\partial}{\partial\zeta}\left(\mathcal{H}(\zeta)\begin{bmatrix} x_1(\zeta,t)\\ x_2(\zeta,t)\end{bmatrix}\right),\\ 0 &= \begin{bmatrix} 0&0&1&0\\ \kappa&1&0&0\end{bmatrix}\begin{bmatrix}\mathcal{H}(1)x(1,t)\\ \mathcal{H}(0)x(0,t)\end{bmatrix}, \end{aligned} \tag{5.29}$$

where $P_1=\begin{bmatrix}0&0\\0&1\end{bmatrix}, \mathcal{H}(\zeta)=\begin{bmatrix}\frac{1}{\rho(\zeta)}&0\\0&T(\zeta)\end{bmatrix}, \widetilde{W}_1=\begin{bmatrix}0&0\\ \kappa&1\end{bmatrix}, \widetilde{W}_0=\begin{bmatrix}1&0\\0&0\end{bmatrix}$ and $\kappa>0$. We define the corresponding port-Hamiltonian operator A

$$\begin{aligned} Ax &:= P_1\frac{\partial}{\partial\zeta}(\mathcal{H}x),\\ \mathcal{D}(A) &= \{x\in X \mid \mathcal{H}x\in\mathcal{W}^{1,2}(0,1;\mathbb{C}^2) \text{ and } \begin{bmatrix}\widetilde{W}_1 & \widetilde{W}_0\end{bmatrix}\begin{bmatrix}(\mathcal{H}x)(1)\\ (\mathcal{H}x)(0)\end{bmatrix}=0\} \end{aligned}$$

and remind that due to Theorem 3.3.5 the system has a unique mild solution if A generates a C_0-semigroup. To show that the port-Hamiltonian operator A is a discrete Riesz spectral operator, it is sufficient due to Theorem 5.3.3 to prove that A generates a C_0-group. So we have only to check that $\widetilde{W}_1\mathcal{H}(1)Z^+(1)\oplus\widetilde{W}_0\mathcal{H}(0)Z^-(0)=\mathbb{C}^d$ and $\widetilde{W}_1\mathcal{H}(1)Z^-(1)\oplus\widetilde{W}_0\mathcal{H}(0)Z^+(0)=\mathbb{C}^d$, where $Z^+(\zeta)$ denotes the span of the eigenvectors of $P_1\mathcal{H}(\zeta)$ corresponding to the positive eigenvalues of $P_1\mathcal{H}(\zeta)$ and $Z^-(\zeta)$ is the span of the eigenvectors of $P_1\mathcal{H}(\zeta)$ corresponding to the negative eigenvalues of $P_1\mathcal{H}(\zeta)$. Defining $\gamma=\sqrt{T(\zeta)/\rho(\zeta)}$, the matrix function $P_1\mathcal{H}$ can be factorized as

$$P_1\mathcal{H} = \underbrace{\begin{bmatrix} \gamma & -\gamma \\ \rho^{-1} & \rho^{-1} \end{bmatrix}}_{S^{-1}} \underbrace{\begin{bmatrix} \gamma & 0 \\ 0 & -\gamma \end{bmatrix}}_{\Delta} \underbrace{\begin{bmatrix} (2\gamma)^{-1} & \rho/2 \\ (2\gamma)^{-1} & \rho/2 \end{bmatrix}}_{S}.$$

It is easy to see that $Z^+(\zeta) = \operatorname{span}\begin{bmatrix} T(\zeta) \\ \gamma(\zeta) \end{bmatrix}$ and $Z^-(\zeta) = \operatorname{span}\begin{bmatrix} -T(\zeta) \\ \gamma(\zeta) \end{bmatrix}$. Then it holds for $\kappa \neq -\frac{T(1)}{\gamma(1)}$ and $\kappa \neq \frac{T(1)}{\gamma(1)}$

$$\widetilde{W}_1\mathcal{H}(1)Z^+(1) \oplus \widetilde{W}_0\mathcal{H}(0)Z^-(0) = \begin{bmatrix} 0 \\ \kappa\gamma(1) + T(1) \end{bmatrix} \oplus \begin{bmatrix} -\gamma(0) \\ 0 \end{bmatrix} = \mathbb{C}^2,$$

$$\widetilde{W}_1\mathcal{H}(1)Z^-(1) \oplus \widetilde{W}_0\mathcal{H}(0)Z^+(0) = \begin{bmatrix} 0 \\ -\kappa\gamma(1) + T(1) \end{bmatrix} \oplus \begin{bmatrix} \gamma(0) \\ 0 \end{bmatrix} = \mathbb{C}^2.$$

Thus, A is a discrete Riesz spectral operator for $\kappa \neq -\frac{T(1)}{\gamma(1)}$ and $\kappa \neq \frac{T(1)}{\gamma(1)}$.
In contrast to [XW11] we do not need determine the eigenvalues exactly, which is only possible if ρ and T are constant. For spacial varying coefficients like $\rho(\zeta) = e^{\zeta}$ and $T(\zeta) = \zeta + 1$ the problem is not analytically solvable, see [JZ12, Exercise 12.1].

5.4.2 Timoshenko beam with boundary damping

In Example 3.1.8 and 3.2.9 it is shown that the Timoshenko beam with boundary damping can be formulated as port-Hamiltonian system: It can be written in the form of (5.15), with

$$P_1 = \begin{bmatrix} 0&1&0&0 \\ 1&0&0&0 \\ 0&0&0&1 \\ 0&0&1&0 \end{bmatrix}, \ \mathcal{H}(\zeta) = \begin{bmatrix} K(\zeta) & 0 & 0 & 0 \\ 0 & \frac{1}{\rho(\zeta)} & 0 & 0 \\ 0 & 0 & EI(\zeta) & 1 \\ 0 & 0 & 0 & \frac{1}{I_\rho(\zeta)} \end{bmatrix}$$

and

$$P_0 = \begin{bmatrix} 0&0&0&-1 \\ 0&0&0&0 \\ 0&0&0&0 \\ 1&0&0&0 \end{bmatrix},$$

where the $K(\zeta)$ denotes the shear modulus, $EI(\zeta)$ is the product of Young's modulus of elasticity and the moment of inertia of a cross section, $\rho(\zeta)$ is the mass per unit length and $I_\rho(\zeta)$ denotes the rotary moment of inertia of a cross section. All these physical parameters are positive and continuously differentiable functions of ζ. To model the fact that the beam is clamped in at $\zeta = 0$ and controlled at $\zeta = 1$ by the force and moment feedback, we add the boundary condition

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \widetilde{W}_1 & \widetilde{W}_0 \end{bmatrix} \begin{bmatrix} (\mathcal{H}x)(1,t) \\ (\mathcal{H}x)(0,t) \end{bmatrix} \text{ with } \begin{bmatrix} \widetilde{W}_1 & \widetilde{W}_0 \end{bmatrix} = \begin{bmatrix} 0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&0&1 \\ 1&\alpha_1&0&0&0&0&0&0 \\ 0&0&1&\alpha_2&0&0&0&0 \end{bmatrix}$$

and α_1, α_2 are given positive gain feedback constants.

For shortness, we define the ζ-depending functions $\gamma_1 = \frac{1}{\sqrt{\rho(\zeta)K(\zeta)}}$ and $\gamma_2 = \frac{1}{\sqrt{I_\rho(\zeta)EI(\zeta)}}$. Then we diagonalize $P_1\mathcal{H}(\zeta)$ and it holds

$$P_1\mathcal{H}(\zeta) = \begin{bmatrix} 0 & \frac{1}{\rho(\zeta)} & 0 & 0 \\ K(\zeta) & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{I_\rho(\zeta)} \\ 0 & 0 & EI(\zeta) & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & \gamma_1 & 0 & -\gamma_1 \\ 0 & 1 & 0 & 1 \\ \gamma_2 & 0 & -\gamma_2 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \sqrt{\frac{EI(\zeta)}{I_\rho(\zeta)}} & 0 & 0 & 0 \\ 0 & \sqrt{\frac{K(\zeta)}{\rho(\zeta)}} & 0 & 0 \\ 0 & 0 & -\sqrt{\frac{EI(\zeta)}{I_\rho(\zeta)}} & 0 \\ 0 & 0 & 0 & -\sqrt{\frac{K(\zeta)}{\rho(\zeta)}} \end{bmatrix} \begin{bmatrix} 0 & 0 & \frac{1}{2\gamma_2} & \frac{1}{2} \\ \frac{1}{2\gamma_1} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & -\frac{1}{2\gamma_2} & \frac{1}{2} \\ -\frac{1}{2\gamma_1} & \frac{1}{2} & 0 & 0 \end{bmatrix}.$$

Thus,

$$Z^+(\zeta) = \operatorname{span}\{\begin{bmatrix} 0 & 0 & \gamma_2(\zeta) & 1 \end{bmatrix}^\top, \begin{bmatrix} \gamma_1(\zeta) & 1 & 0 & 0 \end{bmatrix}^\top\} \text{ and}$$
$$Z^-(\zeta) = \operatorname{span}\{\begin{bmatrix} 0 & 0 & -\gamma_2(\zeta) & 1 \end{bmatrix}^\top, \begin{bmatrix} -\gamma_1(\zeta) & 1 & 0 & 0 \end{bmatrix}^\top\}.$$

Since $\widetilde{W}_1\mathcal{H}(1) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ K(1) & \frac{\alpha_1}{\rho(1)} & 0 & 0 \\ 0 & 0 & EI(1) & \frac{\alpha_2}{I_\rho(1)} \end{bmatrix}$ and $\widetilde{W}_0\mathcal{H}(0) = \begin{bmatrix} 0 & \frac{1}{\rho(0)} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{I_\rho(0)} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, we have

$$\widetilde{W}_1\mathcal{H}(1)Z^+(1) \oplus \widetilde{W}_0\mathcal{H}(0)Z^-(0)$$
$$= \operatorname{span}\{\begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{\gamma_2^{-1}(1)+\alpha_2}{I_\rho(1)} \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ \frac{\gamma_1^{-1}(1)+\alpha_1}{\rho(1)} \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ \frac{1}{I_\rho(0)} \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{1}{\rho(0)} \\ 0 \\ 0 \\ 0 \end{bmatrix}\} = \mathbb{C}^4$$

and

$$\widetilde{W}_1\mathcal{H}(1)Z^-(1) \oplus \widetilde{W}_0\mathcal{H}(0)Z^+(0) =$$
$$\operatorname{span}\{\begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{-\gamma_2^{-1}(1)+\alpha_2}{I_\rho(1)} \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ \frac{-\gamma_1^{-1}(1)+\alpha_1}{\rho(1)} \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ \frac{1}{I_\rho(0)} \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{1}{\rho(0)} \\ 0 \\ 0 \\ 0 \end{bmatrix}\} = \mathbb{C}^4.$$

Recall that $W_B = \begin{bmatrix} \widetilde{W}_1 & \widetilde{W}_0 \end{bmatrix} \begin{bmatrix} P_1 & -P_1 \\ I & I \end{bmatrix}^{-1}$, cf. Equation 3.13. Since $W_B \Sigma W_B \neq 0$, A does not generate a unitary C_0-group, but nevertheless A generates a C_0-group and is by Theorem 5.3.3 a discrete Riesz spectral operator.

Xu and Feng dedicate the paper [XF02] to this example and they proved under the extra assumption that all physical constants are independent of ζ that the eigenvectors and generalized eigenvectors of the operator form a Riesz basis. This example is also revisited in [Vil07] using another approach. Using our main theorem, we can easy verify that the associated system operator is a discrete Riesz spectral operator.

5.5 Closing remarks and open problems

We have shown that a port-Hamiltonian system of the form (5.15) is a Riesz spectral operator if and only if it generates a C_0-group. Many (hyperbolic) systems can be written into this form, with as main exception the Euler-Bernoulli beam equation. Of course the basis property of this equation is well-studied, and many results are known, see e.g. [GW19]. However, we assert that the main result of this chapter does not hold for the Euler-Bernoulli beam equation.
In Theorem 5.3.3 we have shown that if the port-Hamiltonian systems (5.15) is a Riesz spectral operator, then the eigenvalues (counted according to the algebraic multiplicity) can be decomposed into finitely many sets each having a uniform gap. If we count the eigenvalues without multiplicity, then [JZ01a, Theorem 2] shows that they can be decomposed into at most d sets each having a uniform gap. We claim that this results holds true if we count the eigenvalues according to the algebraic multiplicity.
One may ask whether the main theorem of this chapter (Theorem 5.3.3) holds if we drop the assumption that P_0 is skew-symmetric. Our proof uses the fact that every well-posed port-Hamiltonian control system (5.15) is exactly controllable in finite time and this property is only known in the case that P_0 is skew-symmetric. Thus, if P_0 is an arbitrary $d \times d$-matrix, then our proof carries over to this more general case, provided we add the assumption that the corresponding port-Hamiltonian system is exactly controllable in finite-time. However, we assert that even when P_0 is not skew-symmetric, the system (5.15) is exactly controllable in finite time, and thus this extra assumption would not be needed.
Guo and Wang [GW19] studied the Riesz basis property for a closely related class of systems, that is, hyperbolic systems of the form $\frac{\partial x}{\partial t} = K(\zeta)\frac{\partial x}{\partial \zeta} + C(\zeta)x$ with K and C diagonal. They showed that for their class of systems the state space can be split into two parts, one part generated by a C_0-group and the other generated by an operator without spectrum, see [GW19, Theorem 4.10]. In particular, the spectrum of their operators always lies in a strip parallel to the imaginary axis. This result does not generalize to our system class as the following lemma shows.

Lemma 5.5.1. *We consider the port-Hamiltonian system*

$$\frac{\partial}{\partial t}\begin{bmatrix} x_1(\zeta,t) \\ x_2(\zeta,t) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \frac{\partial}{\partial \zeta}\begin{bmatrix} x_1(\zeta,t) \\ x_2(\zeta,t) \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} x_1(\zeta,t) \\ x_2(\zeta,t) \end{bmatrix}, \tag{5.30}$$

$$0 = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} x(1,t) \\ x(0,t) \end{bmatrix}. \tag{5.31}$$

The system operator associated to (5.30)-(5.31) *generates a C_0-semigroup, but there exists a sequence of eigenvalues which real parts converge to $-\infty$.*

Proof: By Theorem 3.2.5 it is easy to see that the operator associated to (5.30)-(5.31) generates a C_0-semigroup. To show that there exists a sequence of eigenvalues which real parts converge to $-\infty$, we characterize the eigenvalues of the port-Hamiltonian operator. If a complex number s is an eigenvalue of the port-Hamiltonian operator associated to the system (5.30)-(5.31), then

$$
\begin{aligned}
sx &= \begin{bmatrix}1 & 0\\ 0 & \frac{1}{2}\end{bmatrix} x' + \begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix} x\\
\Leftrightarrow x' &= s\begin{bmatrix}1 & 0\\ 0 & 2\end{bmatrix} x - \begin{bmatrix}1 & 0\\ 0 & 2\end{bmatrix}\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix} x\\
\Leftrightarrow x' &= s\begin{bmatrix}1 & 0\\ 0 & 2\end{bmatrix} x - \begin{bmatrix}0 & 1\\ -2 & 0\end{bmatrix} x\\
\Leftrightarrow x' &= \left(s\begin{bmatrix}1 & 0\\ 0 & 2\end{bmatrix} + \begin{bmatrix}0 & -1\\ 2 & 0\end{bmatrix}\right) x.
\end{aligned}
$$

Due to Lemma 5.3.8, $s \in \mathbb{C}$ is an eigenvalue if and only if

$$
\det\left(\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\Psi^s(1) + \begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}\right) = 0, \tag{5.32}
$$

where

$$
\Psi^s(1) = \exp\left(s\begin{bmatrix}1 & 0\\ 0 & 2\end{bmatrix} + \begin{bmatrix}0 & -1\\ 2 & 0\end{bmatrix}\right).
$$

To obtain the eigenvalues of the matrix $s\left[\begin{smallmatrix}1 & 0\\ 0 & 2\end{smallmatrix}\right] + \left[\begin{smallmatrix}0 & -1\\ 2 & 0\end{smallmatrix}\right] = \left[\begin{smallmatrix}s & -1\\ 2 & 2s\end{smallmatrix}\right]$ we determine the zeros of

$$
\begin{aligned}
\det\left(\begin{bmatrix}s-\mu & -1\\ 2 & 2s-\mu\end{bmatrix}\right) &= (s-\mu)(2s-\mu) + 2 \qquad (5.33)\\
&= \mu^2 - 3s\mu + 2(s^2+1). \qquad (5.34)
\end{aligned}
$$

Thus the determinant (5.33) has the zeros

$$
\mu_{1,2} = \frac{3s}{2} \pm \sqrt{\left(\frac{-3s}{2}\right)^2 - 2(s^2+1)} = \frac{3s \pm \sqrt{s^2-8}}{2}. \tag{5.35}
$$

These are the eigenvalues of $\left[\begin{smallmatrix}s & -1\\ 2 & 2s\end{smallmatrix}\right]$ to the eigenvectors $v_1 = \left[\begin{smallmatrix}1\\ s-\mu_1\end{smallmatrix}\right]$ and $v_2 = \left[\begin{smallmatrix}1\\ s-\mu_2\end{smallmatrix}\right]$. Note that $\mu_{1,2}$ are s-dependent. Thus, we get

$$
\exp\left(\begin{bmatrix}s & -1\\ 2 & 2s\end{bmatrix}\right) = V(s)\begin{bmatrix}e^{\mu_1} & 0\\ 0 & e^{\mu_2}\end{bmatrix} V(s)^{-1},
$$

where $V(s)$ consists of the eigenvectors v_1 and v_2, i.e., $V(s) = \left[\begin{smallmatrix}1 & 1\\ s-\mu_1 & s-\mu_2\end{smallmatrix}\right]$.

Thus, (5.32) is equivalent to

$$\begin{aligned}
&\det\left(\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} V(s) \begin{bmatrix}\mathrm{e}^{\mu_1} & 0\\ 0 & \mathrm{e}^{\mu_2}\end{bmatrix} V(s)^{-1} + \begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}\right) = 0\\
\Leftrightarrow &\det\left(\left(V(s) \begin{bmatrix}\mathrm{e}^{\mu_1} & 0\\ 0 & \mathrm{e}^{\mu_2}\end{bmatrix} + \begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix} V(s)\right) V(s)^{-1}\right) = 0\\
\Leftrightarrow &\det\left(V(s) \begin{bmatrix}\mathrm{e}^{\mu_1} & 0\\ 0 & \mathrm{e}^{\mu_2}\end{bmatrix} + \begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix} V(s)\right) = 0\\
\Leftrightarrow &\det\left(\begin{bmatrix}1 & 1\\ s-\mu_1 & s-\mu_2\end{bmatrix} \begin{bmatrix}\mathrm{e}^{\mu_1} & 0\\ 0 & \mathrm{e}^{\mu_2}\end{bmatrix} + \begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix} \begin{bmatrix}1 & 1\\ s-\mu_1 & s-\mu_2\end{bmatrix}\right) = 0\\
\Leftrightarrow &\det\left(\begin{bmatrix}\mathrm{e}^{\mu_1} & \mathrm{e}^{\mu_2}\\ (s-\mu_1)\mathrm{e}^{\mu_1} & (s-\mu_2)\mathrm{e}^{\mu_2}\end{bmatrix} + \begin{bmatrix}s-\mu_1 & s-\mu_2\\ 0 & 0\end{bmatrix}\right) = 0\\
\Leftrightarrow &\det\begin{bmatrix}\mathrm{e}^{\mu_1}+s-\mu_1 & \mathrm{e}^{\mu_2}+s-\mu_2\\ (s-\mu_1)\mathrm{e}^{\mu_1} & (s-\mu_2)\mathrm{e}^{\mu_2}\end{bmatrix} = 0\\
\Leftrightarrow &(\mathrm{e}^{\mu_1}+s-\mu_1)(s-\mu_2)\mathrm{e}^{\mu_2} - (\mathrm{e}^{\mu_2}+s-\mu_2)(s-\mu_1)\mathrm{e}^{\mu_1} = 0\\
\Leftrightarrow &(\mu_1-\mu_2)\mathrm{e}^{\mu_2+\mu_1} + (s-\mu_1)(s-\mu_2)\mathrm{e}^{\mu_2} - (s-\mu_1)(s-\mu_2)\mathrm{e}^{\mu_1} = 0. \qquad (5.36)
\end{aligned}$$

By equation (5.33) and (5.35) the determinant of $\begin{bmatrix}s-\mu & -1\\ 2 & 2s-\mu\end{bmatrix}$ is described by a polynomial $p(\mu) = (\mu-\mu_1)(\mu-\mu_2)$. Evaluating $p(\mu)$ at $\mu = s$, we get, using again (5.33),

$$p(s) = (s-\mu_1)(s-\mu_2) = s^2 - 3s^2 + 2s^2 + 2 = 2. \qquad (5.37)$$

Using (5.37), equation (5.36) is equivalent to

$$(\mu_1-\mu_2)\mathrm{e}^{\mu_2+\mu_1} + 2\mathrm{e}^{\mu_2} - 2\mathrm{e}^{\mu_1} = 0. \qquad (5.38)$$

Now, we consider the asymptotic behaviour of the zeros of (5.33). Since it holds

$$\sqrt{s^2-8} - \sqrt{s^2} = \frac{8}{s\left(\sqrt{1-\frac{8}{s^2}}+1\right)},$$

it holds for $s = x + iy$

$$\lim_{x\to-\infty} \frac{8}{(x+iy)\left(\sqrt{1-\frac{8}{(x+iy)^2}}+1\right)} = 0.$$

In the following, the $o(1)$-notation is used for $\operatorname{Re} s \to -\infty$. Thus, $\mu_1 - 2s = o(1)$, i.e., $\lim_{\operatorname{Re} s\to-\infty} \mu_1 - 2s = 0$. Analogously, it holds $\mu_2 - s = o(1)$. This implies

$$\begin{aligned}
\mathrm{e}^{\mu_1} &= \mathrm{e}^{2s+o(1)} = \mathrm{e}^{2s}\mathrm{e}^{o(1)},\\
\mathrm{e}^{\mu_2} &= \mathrm{e}^{s+o(1)} = \mathrm{e}^{s}\mathrm{e}^{o(1)},\\
\mu_1 - s &= o(1) + s,\\
\mu_2 - \mu_1 &= -s + o(1).
\end{aligned}$$

Using equation (5.37) it holds

$$\mu_2 - s = \frac{2}{\mu_1 - s} = \frac{2}{s + o(1)} \text{ and}$$
$$\mu_1 - \mu_2 = (\mu_1 - s) - (\mu_2 - s) = s + o(1) - \frac{2}{s + o(1)}.$$

We aim to apply the Theorem of Rouché, c.f. [BC63, Theorem 12.2].
For a closed contour C in the left halfplane with $\operatorname{Re} s$ very small, it holds

$$\begin{aligned} g_\mu(s) &= \mathrm{e}^{\mu_1}[(\mu_1 - \mu_2)\mathrm{e}^{\mu_2} + 2\mathrm{e}^{\mu_2 - \mu_1} - 2] \\ &= \mathrm{e}^{2s}\mathrm{e}^{o(1)}[(s + o(1) - \frac{2}{s + o(1)})\mathrm{e}^{s}\mathrm{e}^{o(1)} + 2\mathrm{e}^{-s}\mathrm{e}^{o(1)} - 2]. \end{aligned}$$

We have to show that the approximation of $g_\mu(s)$ by

$$g(s) = \mathrm{e}^{2s}[(s - \frac{2}{s})\mathrm{e}^{s} + 2\mathrm{e}^{-s} - 2]$$

is good, i.e.,

$$\frac{|g_\mu(s) - g(s)|}{|g(s)|} < 1.$$

It holds for $\operatorname{Re} s$ very small

$$\begin{aligned} &\frac{|g_\mu(s) - g(s)|}{|g(s)|} \\ &= \frac{\left|\mathrm{e}^{\mu_1}[(\mu_1 - \mu_2)\mathrm{e}^{\mu_2} + 2\mathrm{e}^{\mu_2 - \mu_1} - 2] - \mathrm{e}^{2s}[(s - \frac{2}{s})\mathrm{e}^{s} + 2\mathrm{e}^{-s} - 2]\right|}{\left|\mathrm{e}^{2s}[(s - \frac{2}{s})\mathrm{e}^{s} + 2\mathrm{e}^{-s} - 2]\right|} \\ &= \frac{\left|\mathrm{e}^{2s}\mathrm{e}^{o(1)}[(s + o(1) - \frac{2}{s + o(1)})\mathrm{e}^{s}\mathrm{e}^{o(1)} + 2\mathrm{e}^{-s}\mathrm{e}^{o(1)} - 2] - \mathrm{e}^{2s}[(s - \frac{2}{s})\mathrm{e}^{s} + 2\mathrm{e}^{-s} - 2]\right|}{\left|\mathrm{e}^{2s}[(s - \frac{2}{s})\mathrm{e}^{s} + 2\mathrm{e}^{-s} - 2]\right|} \\ &= \frac{\left|(s + o(1) - \frac{2}{s + o(1)})\mathrm{e}^{o(1)}\mathrm{e}^{o(1)} + 2\mathrm{e}^{-2s}\mathrm{e}^{o(1)} - 2\mathrm{e}^{-s}\mathrm{e}^{o(1)} - \left[(s - \frac{2}{s}) + 2\mathrm{e}^{-2s} - 2\mathrm{e}^{-s}\right]\right|}{\left|(s - \frac{2}{s}) + 2\mathrm{e}^{-2s} - 2\mathrm{e}^{-s}\right|} \\ &= \frac{\left|(s - \frac{2}{s})(1 + o(1)) + 2\mathrm{e}^{-2s}(1 + o(1)) - 2\mathrm{e}^{-s}(1 + o(1)) - (s - \frac{2}{s}) - 2\mathrm{e}^{-2s} + 2\mathrm{e}^{-s}\right|}{\left|(s - \frac{2}{s}) + 2\mathrm{e}^{-2s} - 2\mathrm{e}^{-s}\right|} \\ &= \frac{\left|\left((s - \frac{2}{s}) + 2\mathrm{e}^{-2s} - 2\mathrm{e}^{-s}\right)(1 + o(1)) - (s - \frac{2}{s}) - 2\mathrm{e}^{-2s} + 2\mathrm{e}^{-s}\right|}{\left|(s - \frac{2}{s}) + 2\mathrm{e}^{-2s} - 2\mathrm{e}^{-s}\right|} \\ &= \frac{\left|\left((s - \frac{2}{s}) + 2\mathrm{e}^{-2s} - 2\mathrm{e}^{-s}\right) o(1)\right|}{\left|(s - \frac{2}{s}) + 2\mathrm{e}^{-2s} - 2\mathrm{e}^{-s}\right|} \\ &= o(1). \end{aligned}$$

Thus, it is sufficient to get information about the zeros of

$$\mathrm{e}^{2s}[(s - \frac{2}{s})\mathrm{e}^{s} + 2\mathrm{e}^{-s} - 2].$$

To get some information about the asymptotic behaviour of its zeros we bring the polynomial of exponentials in the standard form introduced in [BC63, Chapter 12]. It holds

$$\begin{aligned} \mathrm{e}^{2s}\left[\left(s-\frac{2}{s}\right)\mathrm{e}^{s}+2\mathrm{e}^{-s}-2\right] &= \frac{1}{s}e^{s}\left[\left(s^2-2\right)\mathrm{e}^{2s}-2s\mathrm{e}^{s}+2s\right] \\ &= \frac{1}{s}e^{s}\sum_{j=0}^{n}p_j(s)\mathrm{e}^{\beta_j s}, \quad 0=\beta_0<\beta_1<\cdots<\beta_n. \end{aligned} \tag{5.39}$$

Let m_j denotes the degree of the polynomial in $p_j(s)$. We aim to draw the distributional diagram. Thus, we need the points with coordinates (β_j, m_j), draw the upper boundary part of the convex hull of (β_j, m_j) and get a polygonal graph. Then it is not possible that points lie above the polygonal graph, but some may lie below it. The points below the polygonal graph does not effect the asymptotics of the zeros, see [BC63, Chapter 12.8]. Thus, due to equation (5.39) we draw the points $(2,2),(0,1),(1,1)$ and get the following distributional diagram with only one line segment with slope $\frac{1}{2}$.

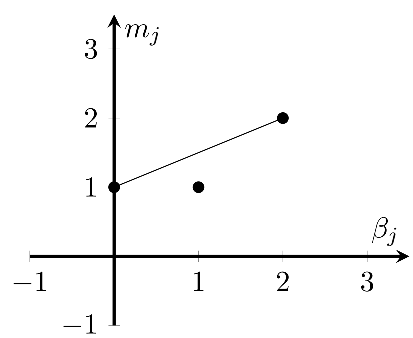

Figure 5.1: Distributional Diagram

Applying Theorem 12.10.d in [BC63], we get that the zeros of (5.39) lie asymptotically along a curve $|s^{\frac{1}{2}}\mathrm{e}^{s}|=c$, where $c\in\mathbb{R}$ denotes a constant and the slope $\frac{1}{2}$ is taken into account. It holds

$$\begin{aligned} |s^{\frac{1}{2}}\mathrm{e}^{s}|=c \Leftrightarrow \left||s|^{\frac{1}{2}}\mathrm{e}^{\frac{1}{2}\cdot i(\arg(s)+2k\pi)}\mathrm{e}^{s}\right| = c \Leftrightarrow |s|^{\frac{1}{2}}\mathrm{e}^{\mathrm{Re}\, s}=c \\ \Leftrightarrow &\mathrm{Re}\, s+\frac{1}{2}\ln(|s|)=\ln(c) \Leftrightarrow \mathrm{Re}\left(s+\frac{1}{2}\ln(|s|)+i\arg(s)\right)=\tilde{c} \\ \Leftrightarrow &\mathrm{Re}\,(s+\frac{1}{2}\ln s)=\tilde{c}, \end{aligned}$$

where $\tilde{c}:=\ln(c)$. We define $s=x+iy$ and by Lemma 12.3 in [BC63] we get that the curve is asymptotic to the curve $x+\frac{1}{2}\ln(|y|)=\tilde{c}$. Then the zeros lies along a curve with $x=\tilde{c}-\frac{1}{2}\ln(|y|)$ and thus, there exists a sequence of eigenvalues which real parts converge to $-\infty$. □

The following example shows that the equivalence 1) $\Leftrightarrow$ 2) in Theorem 5.3.3 does not hold for generators A of C_0-semigroups $(T(t))_{t\geqslant 0}$ on Hilbert spaces even if we additionally assume that there exists a admissible control operator $B \in \mathcal{L}(\mathbb{C}^d, X_{-1})$ for $(T(t))_{t\geqslant 0}$ such that the control system $\dot{x}(t) = Ax(t) + Bu(t)$ is exactly controllable in finite time.

Example 5.5.2. Let $A : \mathcal{D}(A) \subset \ell^2 \to \ell^2$ be defined by $(Ax)_n = (s_n x_n)_n$, $(s_n)_{n\in\mathbb{N}} = (-2^n)_{n\in\mathbb{N}}$, and $\mathcal{D}(A) = \{x \in \ell^2(\mathbb{N}) \mid \sum_{n\in\mathbb{N}} (1 + |s_n|^2) |x_n|^2 < \infty\}$. Clearly, A is a discrete Riesz spectral operator, generates a C_0-semigroup, but not a C_0-group. Here

$$X_{-1} = \ell^2_{-1} = \{(x_n)_{n\in\mathbb{N}} \mid \sum_{n\in\mathbb{N}} \frac{|x_n|^2}{(1 + |s_n|^2)} < \infty\}.$$

Hence, we can identify $B \in \mathcal{L}(\mathbb{C}, \ell^2_{-1})$ with a sequence $(b_n) \in \ell^2_{-1}$. Let $(b_n)_{n\in\mathbb{N}} = (\sqrt{2^n})_{n\in\mathbb{N}}$. Then it holds $(b_n)_{n\in\mathbb{N}} \in \ell^2_{-1}$, since

$$\sum_{n\in\mathbb{N}} \frac{(\sqrt{2^n})^2}{1 + (2^n)^2} = \sum_{n\in\mathbb{N}} \frac{2^n}{1 + 2^{2n}} < \sum_{n\in\mathbb{N}} \frac{2^n}{2^{2n}} < \infty. \tag{5.40}$$

To proof the admissibility of B we use the Carleson measure criterion by Weiss, [Wei88]. Since the eigenvalues $s_n = -2^n$ are real, we only have to check that there exists a constant $M > 0$ independent of h such that

$$\sum_{-s_n \in R(h,0)} |b_n|^2 \leqslant Mh \text{ for any } h > 0, \tag{5.41}$$

where $R(h,0) := \{z \in \mathbb{C} \mid 0 \leqslant \operatorname{Re} z \leqslant h \text{ and } |\operatorname{Im} z| \leqslant h\}$ denotes a rectangle in the right complex half plane at the imaginary axis. It holds

$$\sum_{-s_n \in R(h,0)} \left|\sqrt{2^n}\right|^2 = \sum_{n=1}^{k} 2^n \leqslant 2 \cdot 2^k, \tag{5.42}$$

where $k := \max_i h - 2^i > 0$. To verify that (A, B) is exactly controllable in finite time, we use one of the dual equivalences in [JZ01b, Theorem 2] for one-dimensional output operators and formulate the assertion for diagonal systems with one-dimensional input operators.

Theorem 5.5.3. *The diagonal system (A, B) is exactly controllable in infinite time if and only if the following two conditions hold:*

1. *The eigenvalues s_n are properly spaced, i.e.,*

$$\inf_{n\neq m} \left| \frac{s_n - s_m}{\operatorname{Re} s_n} \right| > 0 \textit{ and} \tag{5.43}$$

2. *there exists a constant $C > 0$ such that $C\,|Res_m| \leqslant |b_m|^2$.*

Here, we use the above theorem for an exponential stable semigroup. Thus, exact controllability in infinite time is equivalent to exact controllability in finite time. It holds

$$\left| \frac{2^n - 2^m}{2^n} \right| \geqslant \left| \frac{2^n - 2^{m-1}}{2^n} \right| = \left| 1 - 2^{-1} \right| > 0. \tag{5.44}$$

Thus, we see that the sequence of eigenvalues is properly spaced and it holds $|\operatorname{Re} s_m| \leqslant |b_m|^2$ as well and therefore, exact controllability in finite time follows.

Chapter 6

Generalization of port-Hamiltonian systems

So far, we have only considered port-Hamiltonian systems of order 1. In the following section we consider port-Hamiltonian systems of order N. Then not only the Timoschenko beam but also the Euler-Bernoulli beam can be modelled as a port-Hamiltonian system, namely a port-Hamiltonian system of order 2. Port-Hamiltonian systems of N-th order on a bounded interval are well-studied, see for example [Vil07],[LGZM05], [AJ14] and [Aug16].

We consider the well-posedness of a class of hyperbolic partial differential equations on a one dimensional spatial domain, i.e., whether the associated operator generates a contraction C_0-semigroup. This class includes coupled wave and beam equations and in particular infinite networks of these equations, that means networks with an infinite number of edges.

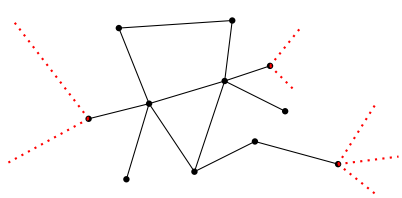

Figure 6.1: Arbitrary infinite-dimensional network

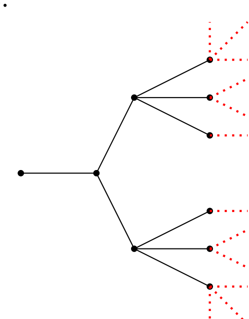

Figure 6.2: Infinite-dimensional tree

In this chapter equivalent conditions for contraction C_0-semigroup generation are derived. We consider these equations on a finite interval as well as on a semi-axis. In particular, contraction C_0-semigroup generation has been studied in Chapter 3 and [LGZM05, JZ12, AJ14, Aug16, JMZ15]. In this chapter we aim to generalize these results to the infinite-dimensional situation and to the semi-axis.

The results of this chapter are published in [JK19b].

6.1 Port-Hamiltonian systems in the infinite-dimensional setting

We consider on the interval $[0,1]$ a system of partial differential equations of the form

$$\frac{\partial x}{\partial t}(\zeta,t) = \left(\sum_{k=0}^{N} P_k \frac{\partial^k}{\partial \zeta^k}\right)(\mathcal{H}(\zeta)x(\zeta,t)), \quad \zeta \in (0,1), t \geqslant 0, \tag{6.1}$$
$$x(\zeta,0) = x_0(\zeta),$$

where P_N is an invertible operator on a Hilbert space H and $P_k \in \mathcal{L}(H)$, $k = 0, \cdots, N$, with $P_k^* = (-1)^{k+1}P_k$, $k = 1, \cdots, N$. Here $\mathcal{L}(H)$ denotes the set of linear bounded operators on H. $\mathcal{H}(\zeta)$ is a positive operator on H for a.e. $\zeta \in (0,1)$ satisfying $\mathcal{H}, \mathcal{H}^{-1} \in L^\infty(0,1;\mathcal{L}(H))$.
To give an example of a port-Hamiltonian system of order 2 we consider the Euler-Bernoulli beam.
The equation of the Euler-Bernoulli beam models the transversal vibration of an elastic beam where the cross section of the beam is also vertical to the neutral axis after the bending. An extension of the Euler-Bernoulli beam is the Timoshenko beam model, which takes shear and rotational inertia effects into account, see Example 3.1.8.

Example 6.1.1. The Euler-Bernoulli beam is described by the partial differential equation

$$\rho(\zeta)\frac{\partial^2 \omega}{\partial t^2}(\zeta,t) + \frac{\partial^2}{\partial \zeta^2}\left(EI(\zeta)\frac{\partial^2 \omega}{\partial \zeta^2}(\zeta,t)\right) = 0, \quad t \geqslant 0, \tag{6.2}$$

where $\omega(\zeta,t)$ describes the transverse displacement of the beam. All physical parameters are positive and continuously differentiable functions of ζ. Here $\rho(\zeta)$ denotes the mass per unit length and $EI(\zeta)$ is the product of Young's modulus of elasticity and the moment of inertia of the cross section. Using the state variables

$$x_1(\zeta,t) := \rho(\zeta)\frac{\partial \omega}{\partial t}(\zeta,t)$$
$$x_2(\zeta,t) := \frac{\partial^2 \omega}{\partial \zeta^2}(\zeta,t)$$

we model the Euler-Bernoulli beam equation (6.2) as a port-Hamiltonian system of order 2.

$$\frac{\partial}{\partial t}\begin{bmatrix} x_1(\zeta,t) \\ x_2(\zeta,t) \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \frac{\partial^2}{\partial \zeta^2}\left(\begin{bmatrix} \frac{1}{\rho(\zeta)} & 0 \\ 0 & EI(\zeta) \end{bmatrix} \begin{bmatrix} x_1(\zeta,t) \\ x_2(\zeta,t) \end{bmatrix} \right), \tag{6.3}$$

There are also examples of port-Hamiltonian systems of higher order. An example for a port-Hamiltonian system of order 3 is the Airy equation, which is described in [MNS18].

Example 6.1.2. The Airy equation is the linear part of the Korteweg-de Vries equation, which describes waves on shallow water. The Airy equation on a one-dimensional spacial domain is given by

$$\frac{\partial \omega}{\partial t}(t,\zeta) + \frac{\partial^3 \omega}{\partial \zeta^3}(\zeta,t) = 0,\ t \geqslant 0,\ \zeta \in (0,1). \tag{6.4}$$

This equation can be written as a port-Hamiltonian system of order 3 with $\mathcal{H} = 1$, $P_0 = P_1 = P_2 = 0$, and $P_3 = -1$.

In order to guarantee unique solutions of equation (6.1), we have to impose boundary conditions, which will be of the form

$$\widetilde{W}_B(\Phi(\mathcal{H}x))(\cdot, t) = 0. \tag{6.5}$$

We assume $\widetilde{W}_B \in \mathcal{L}(H^{2N}, H^N)$ and that the operator Φ is given by

$$\Phi : \mathcal{W}^{N,2}(0,1;H) \to H^{2N}, \qquad \Phi(x) := [\Phi_1(x)\ \Phi_0(x)]^T,$$

where $\Phi_i(x) := \left[x(i)\ \dots\ \frac{d^{N-1}x}{d\zeta^{N-1}}(i)\right]^T$ for $i \in \{0,1\}$ and $\mathcal{W}^{N,2}(0,1;H)$ denotes the Sobolev space of order N, cf. Definition 2.0.4. Clearly, whether or not equation (6.1) possesses unique and non-increasing solutions depend on the boundary conditions, or equivalently on the operator $\widetilde{W}_B$. The partial differential equation (6.1) with the boundary conditions (6.5) can be equivalently written as an abstract Cauchy problem

$$\begin{aligned} \dot{x}(t) &= Ax(t), \\ x(0) &= x_0, \end{aligned}$$

where A is the linear operator on the Hilbert space $X := L^2(0,1;H)$ given by

$$Ax := \sum_{k=0}^{N} P_k \frac{\partial^k}{\partial \zeta^k}(\mathcal{H}x), \quad x \in \mathcal{D}(A), \tag{6.6}$$

$$\mathcal{D}(A) = \left\{ x \in X \mid \mathcal{H}x \in \mathcal{W}^{N,2}(0,1;H) \text{ and } \widetilde{W}_B \Phi(\mathcal{H}x) = 0 \right\}. \tag{6.7}$$

We denote A as *port-Hamiltonian operator of order* N. Again, as in Chapter 3, we equip X not with the standard scalar product of $L^2(0,1;H)$ but with the inner product $\langle f, \mathcal{H}g \rangle$.
We define

$$Q = (Q_{ij})_{\substack{1 \leqslant i \leqslant N \\ 1 \leqslant j \leqslant N}} = \begin{cases} (-1)^{i-1} P_{i+j-1} & \text{if } i+j \leqslant N+1 \\ 0 & \text{else.} \end{cases} \tag{6.8}$$

Clearly, $Q_{ij} \in \mathcal{L}(H)$, i.e. $Q \in \mathcal{L}(H^N)$ and

$$Q = \begin{bmatrix} P_1 & P_2 & P_3 & \cdots & P_{N-1} & P_N \\ -P_2 & -P_3 & -P_4 & \cdots & -P_N & 0 \\ P_3 & P_4 & \iddots & \iddots & 0 & 0 \\ -P_4 & \iddots & \iddots & \iddots & & \vdots \\ \vdots & \iddots & \iddots & & & \vdots \\ (-1)^{N-1} P_N & 0 & \cdots & \cdots & \cdots & 0 \end{bmatrix}.$$

Thus, $Q \in \mathcal{L}(H^N)$ is a selfadjoint block operator matrix and invertible due to the fact that P_N is invertible. Let

$$W_B := \begin{bmatrix} W_1 & W_0 \end{bmatrix} := \widetilde{W}_B \begin{bmatrix} Q & -Q \\ I & I \end{bmatrix}^{-1} \quad \text{and} \quad \Sigma := \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \in \mathcal{L}(H^N \times H^N),$$

where $W_1, W_0 \in \mathcal{L}(H^N)$. Let $P \in \mathcal{L}(H)$. We call P negative semi-definite, in short $P \leqslant 0$, if $\langle x, Px\rangle_H \leqslant 0$ for all $x \in H$. We define $\operatorname{Re} P = \frac{1}{2}(P + P^*)$ and $\operatorname{Im} P = \frac{1}{2i}(P - P^*)$. Thus, $P = \operatorname{Re} P + i\operatorname{Im} P$ and $\operatorname{Re} P \leqslant 0$ if and only if $\langle x, \operatorname{Re} Px\rangle_H = \operatorname{Re}\langle x, Px\rangle_H \leqslant 0$.
The aim of this section is to give equivalent conditions for the fact that A generates a contraction C_0-semigroup on X. Under a weak condition, we show that $A\mathcal{H}$ generates a contraction C_0-semigroup if and only if the operator A is dissipative. Moreover, equivalent conditions in terms of the operator $\widetilde{W}_B$ are presented. We note that the mentioned weak condition is in particular satisfied if the Hilbert space H is finite-dimensional. However, even if H is finite-dimensional, our result contains new equivalent conditions for the contraction C_0-semigroup characterization in [Vil07, LGZM05] and [AJ14].
Thus, we consider the operator A on the Hilbert space $X = L^2(0, 1; H)$, where H is a (possibly infinite-dimensional) Hilbert space.
We start to collect all assertions, before we introduce some technical definitions and lemmas, and give the proofs of the following theorems and corollaries at the end of this section.

Theorem 6.1.3. *Let A be given by* (3.9)-(3.10). *Further, assume*

$$\operatorname{ran}(W_1 - W_0) \subseteq \operatorname{ran}(W_1 + W_0). \tag{6.9}$$

Then the following statements are equivalent:

1. *The operator A generates a contraction C_0-semigroup on X;*
2. *A is dissipative, that is,* $\operatorname{Re}\langle Ax, x\rangle \leqslant 0$ *for every $x \in \mathcal{D}(A)$;*
3. $\operatorname{Re} P_0 \leqslant 0$, *$W_1 + W_0$ is injective and $W_B \Sigma W_B^* \geqslant 0$;*
4. $\operatorname{Re} P_0 \leqslant 0$, *$W_1 + W_0$ is injective and there exists $V \in \mathcal{L}(H)$ with $\|V\| \leqslant 1$ such that $W_B = \frac{1}{2}(W_1 + W_0)\begin{bmatrix} I + V & I - V \end{bmatrix}$;*
5. $\operatorname{Re} P_0 \leqslant 0$ *and $u^*Qu - y^*Qy \leqslant 0$ for every $\left[\begin{smallmatrix} u \\ y \end{smallmatrix}\right] \in \ker \widetilde{W}_B$.*

Remark 6.1.4.
1. Condition (6.9) is in general not satisfied: Let $N = 1$, $H = \ell^2$ and $W_B = \begin{bmatrix} W_1 & W_0 \end{bmatrix} \in \mathcal{L}(\ell^2 \times \ell^2, \ell^2)$ with $W_1 e_i := e_{i+1} + e_i$ and $W_0 e_i := e_{i+1} - e_i$, where $\{e_i\}_{i\in\mathbb{N}}$ is an orthonormal basis of ℓ^2. Then $\operatorname{ran}(W_1 - W_0) = \ell^2$ whereas $e_1 \notin \operatorname{ran}(W_1 + W_0)$.
2. We point out that the implications $1 \Rightarrow 2$, $4 \Rightarrow 3$, and the equivalence $2 \Leftrightarrow 5$ hold even without the additional condition (6.9). Moreover, condition (6.9) is not needed for the fact that 2 implies the injectivity of $W_1 + W_0$.

3. We note that W_B is not uniquely determined, only the kernel of W_B is. However, if W_B does not satisfy condition (6.9), then in general it is not possible to choose another operator instead of W_B with the same kernel such that condition (6.9) holds.

4. If H is finite-dimensional, then A has a compact resolvent, see [AJ14, Theorem 2.3]. However, in general, A does not have a compact resolvent. Take for example $N = 1$, $P_1 = 1$, $P_0 = 0$, $H = \ell^2$, $\widetilde{W}_B = [I\ \ L]$ and $\mathcal{H}(\zeta) = I_{\ell^2}$. Here L denotes the left shift on H, that is, $Le_i = e_{i+1}$. Thus, A generates the left shift semigroup on $X = L^2(0,1;\ell^2)$, which is isometric isomorph to the left shift on $X = L^2(0,\infty;\mathbb{C})$. However, 0 is a spectral point of A, but not in the point spectrum.

As a corollary of Theorem 6.1.3 we obtain the well-known contraction C_0-semigroup characterization for the case of a finite-dimensional Hilbert space H, see [AJ14]. However, we remark that Conditions 3 and 4 are new even in the finite-dimensional situation.

Corollary 6.1.5. *Let A be given by* (3.9)-(3.10) *and assume that H is finite-dimensional. Then, assertions 1 to 5 in Theorem 6.1.3 are equivalent, and, moreover, they are equivalent to*

6. $\operatorname{Re} P_0 \leqslant 0$, *$W_B$ surjective and* $W_B \Sigma W_B^* \geqslant 0$*;*

7. $\operatorname{Re} P_0 \leqslant 0$, *$W_B$ surjective and there exists $V \in \mathcal{L}(H)$ with $\|V\| \leqslant 1$ such that* $W_B = \frac{1}{2}(W_1 + W_0)\begin{bmatrix} I+V & I-V \end{bmatrix}$.

Remark 6.1.6. If H is infinite-dimensional, then in general Conditions 6 and 7 of the previous corollary are not equivalent to the fact that A generates a contraction C_0-semigroup. In the following we give two counterexamples.
Let $H = \ell^2(\mathbb{N})$, $N \in \mathbb{N}$, and P_i and $\mathcal{H}$ are operators satisfying the general assumptions. First, we consider $W_B = \begin{bmatrix} W_1 & W_0 \end{bmatrix}$ with $W_1 := \frac{3}{2}R$ and $W_0 := \frac{1}{2}R$, where R denotes the right shift on $\ell^2(\mathbb{N})$. Then $\operatorname{ran}(W_1 - W_0) = \operatorname{ran}(W_1 + W_0)$, $W_1 + W_0$ is injective and $W_B \Sigma W_B^* \geqslant 0$ but W_B is not surjective. Thus, A generates a contraction C_0-semigroup on X, but Conditions 6 and 7 are not satisfied. Conversely, for the choice $W_B = \begin{bmatrix} I-L & -I-L \end{bmatrix}$, where L denotes the left shift on $\ell^2(\mathbb{N})$, surjectivity of W_B holds, $\operatorname{ran}(W_1 - W_0) \subseteq \operatorname{ran}(W_1 + W_0)$ and $W_B \Sigma W_B^* \geqslant 0$, but $W_1 + W_0$ is not injective. Thus, for these boundary conditions the Conditions 6 and 7 of the previous corollary are satisfied, but A does not generate a contraction C_0-semigroup on X.

Next, we characterize the property of unitary group generation of A.

Theorem 6.1.7. *Let A be given by* (3.9)-(3.10)*. Further assume*

$$\operatorname{ran}(W_1 - W_0) = \operatorname{ran}(W_1 + W_0). \tag{6.10}$$

Then the following statements are equivalent:

1. *A generates a unitary C_0-group on X;*

2. $\operatorname{Re}\langle Ax, x\rangle = 0$ *for every* $x \in \mathcal{D}(A)$*;*

3. $\operatorname{Re} P_0 = 0$, $W_1 + W_0$ *and* $-W_1 + W_0$ *are injective and* $W_B \Sigma W_B^* = 0$;

4. $\operatorname{Re} P_0 = 0$, $W_1 + W_0$ *and* $-W_1 + W_0$ *are injective and there exists* $V \in \mathcal{L}(H)$ *with* $\|V\| = 1$ *such that* $W_B = \frac{1}{2}(W_1 + W_0)\begin{bmatrix} I + V & I - V \end{bmatrix}$;

5. $\operatorname{Re} P_0 = 0$ *and* $u^* Q u - y^* Q y = 0$ *for every* $\left(\begin{smallmatrix} u \\ v \end{smallmatrix}\right) \in \ker \widetilde{W}_B$.

Corollary 6.1.8. *Let A be given by* (3.9)-(3.10) *and assume that H is finite-dimensional. Then, assertions 1 to 5 in Theorem 6.1.7 are equivalent, and, moreover, they are equivalent to*

6. $\operatorname{Re} P_0 = 0$, W_B *surjective and* $W_B \Sigma W_B^* = 0$;

7. $\operatorname{Re} P_0 = 0$, W_B *surjective and there exists* $V \in \mathcal{L}(H)$ *with* $\|V\| = 1$ *such that* $W_B = \frac{1}{2}(W_1 + W_2)\begin{bmatrix} I + V & I - V \end{bmatrix}$.

In order to prove these statements it is convenient to introduce the following linear combinations of the boundary values [LGZM05], which is a generalization of Definition 3.2.2.

Definition 6.1.9. For $x \in \mathcal{H}^{-1}\mathcal{W}^{N,2}(0,1;H)$ we define so called boundary port variables, namely *boundary flow* and *boundary effort*, by

$$\begin{bmatrix} f_{\partial,\mathcal{H}x} \\ e_{\partial,\mathcal{H}x} \end{bmatrix} := \frac{1}{\sqrt{2}} \begin{bmatrix} Q & -Q \\ I & I \end{bmatrix} \Phi(\mathcal{H}x) = R_{ext}\Phi(\mathcal{H}x), \tag{6.11}$$

where Q is defined by (6.8) and $R_{ext} := \frac{1}{\sqrt{2}} \left[\begin{smallmatrix} Q & -Q \\ I & I \end{smallmatrix}\right] \in \mathcal{L}(H^{2N})$.

Remark 6.1.10. Thanks to the invertibility of Q, the operator R_{ext} is invertible. As well as in Chapter 3 we use the boundary port variables to reformulate the domain of the operator A:

$$\begin{aligned} \mathcal{D}(A) &= \left\{ x \in X \mid \mathcal{H}x \in \mathcal{W}^{N,2}(0,1;H) \text{ and } \widetilde{W}_B \Phi(\mathcal{H}x) = 0 \right\} \\ &= \left\{ x \in X \mid \mathcal{H}x \in \mathcal{W}^{N,2}(0,1;H) \text{ and } W_B \begin{bmatrix} f_{\partial,\mathcal{H}x} \\ e_{\partial,\mathcal{H}x} \end{bmatrix} = 0 \right\}, \end{aligned}$$

where $W_B = \widetilde{W}_B R_{ext}^{-1}$.

Next, we determine the adjoint operator of A. We define $\widetilde{Q} = -Q$ and

$$\begin{bmatrix} \tilde{f}_{\partial,\mathcal{H}x} \\ \tilde{e}_{\partial,\mathcal{H}x} \end{bmatrix} = \widetilde{R}_{ext}\Phi(\mathcal{H}x) \text{ with } \widetilde{R}_{ext} = \frac{1}{\sqrt{2}} \begin{bmatrix} \widetilde{Q} & -\widetilde{Q} \\ I & I \end{bmatrix}.$$

Lemma 6.1.11. *The adjoint operator of the operator A defined in (6.6) with domain (6.7) and a boundary operator W_B of the form $W_B = S\begin{bmatrix} I + V & I - V \end{bmatrix}$ where $S, V \in \mathcal{L}(H^N)$ and S is injective, is given by*

$$A^* y = P_0^* y - \sum_{k=1}^{N} P_k \frac{d^k}{d\zeta^k} y, \quad y \in \mathcal{D}(A^*), \tag{6.12}$$

$$\mathcal{D}(A^*) = \left\{ y \in \mathcal{W}^{N,2}(0,1;H) : S\begin{bmatrix} I + V^* & I - V^* \end{bmatrix} \begin{bmatrix} \tilde{f}_{\partial,y} \\ \tilde{e}_{\partial,y} \end{bmatrix} = 0 \right\}. \tag{6.13}$$

Proof: The statement can be proved in a similar manner as Proposition 3.4.3 in [Aug16], where the statement is shown for finite-dimensional Hilbert spaces H. $\square$

Definition 6.1.12. We define the operators $A_0 : \mathcal{D}(A_0) \subseteq X \to X$ and $(A^*)_0 : \mathcal{D}((A^*)_0) \subseteq X \to X$ by

$$\begin{aligned} A_0 x &:= \sum_{k=0}^{N} P_k \frac{\partial^k}{\partial \zeta^k} x, \quad (A^*)_0 y := P_0^* y - \sum_{k=1}^{N} P_k \frac{d^k}{d\zeta^k} y \\ \mathcal{D}(A_0) &= \mathcal{D}(A_0^*) = \mathcal{W}^{N,2}(0,1;H). \end{aligned}$$

Remark, that A_0 and $(A^*)_0$ are extensions of A and A^*, respectively. Integration by parts yields the following lemma.

Lemma 6.1.13. *We have for $x \in \mathcal{W}^{N,2}(0,1;H)$*

$$\begin{aligned} \operatorname{Re}\langle A_0 x, x\rangle &= \operatorname{Re}\langle f_{\partial,x}, e_{\partial,x}\rangle_{H^N} + \operatorname{Re}\langle P_0 x, x\rangle \\ &= \Phi_1(x)^* Q \Phi_1(x) - \Phi_0(x)^* Q \Phi_0(x) + \operatorname{Re}\langle P_0 x, x\rangle, \\ \operatorname{Re}\langle (A^*)_0 x, x\rangle &= \operatorname{Re}\langle \tilde{f}_{\partial,x}, \tilde{e}_{\partial,x}\rangle_{H^N} + \operatorname{Re}\langle P_0 x, x\rangle \\ &= \Phi_1(x)^* \widetilde{Q} \Phi_1(x) - \Phi_0(x)^* \widetilde{Q} \Phi_0(x) + \operatorname{Re}\langle P_0 x, x\rangle. \end{aligned}$$

Furthermore, we need some technical results. First, we give a generalization of the technical Lemma 7.3.2 in [JZ12] for $N \geqslant 1$ and arbitrary Banach spaces Z.

Lemma 6.1.14. *Let Z be a Banach space and $V \in \mathcal{L}(Z)$. Then it holds*

$$\ker \begin{bmatrix} I+V & I-V \end{bmatrix} = \operatorname{ran} \begin{bmatrix} I-V \\ -I-V \end{bmatrix},$$

where $\begin{bmatrix} I+V & I-V \end{bmatrix} \in \mathcal{L}(Z \times Z, Z)$ and $\begin{pmatrix} I-V \\ -I-V \end{pmatrix} \in \mathcal{L}(Z, Z \times Z)$.

Proof: Assume $\begin{pmatrix} x \\ y \end{pmatrix} \in \ker \begin{bmatrix} I+V & I-V \end{bmatrix}$. Thus, it holds

$$x + Vx + y - Vy = 0.$$

For $l := \frac{1}{2}(x-y) \in Z$ we get

$$(I-V)l = \frac{1}{2}(x-y) - \frac{1}{2}V(x-y) = x \text{ and } (-I-V)l = -\frac{1}{2}(x-y) - \frac{1}{2}V(x-y) = y.$$

Thus, it follows $\begin{pmatrix} x \\ y \end{pmatrix} \in \operatorname{ran} \begin{bmatrix} I-V \\ -I-V \end{bmatrix}$. Conversely, assume $\begin{pmatrix} x \\ y \end{pmatrix} \in \operatorname{ran} \begin{bmatrix} I-V \\ -I-V \end{bmatrix}$. Then, we have

$$\begin{bmatrix} I+V & I-V \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{bmatrix} I+V & I-V \end{bmatrix} \begin{bmatrix} I-V \\ -I-V \end{bmatrix} l = 0$$

for some $l \in Z$ and the lemma is proved. $\square$

Lemma 6.1.15. ([KZ15, Lemma 2.4]) Let $W = \begin{bmatrix} W_1 & W_0 \end{bmatrix} \in \mathcal{L}(H^{2N}, H^N)$ such that $W_1 + W_0$ is injective and

$$\operatorname{ran}(W_1 - W_0) \subseteq \operatorname{ran}(W_1 + W_0).$$

Then there exist an unique operator $V \in \mathcal{L}(H^N)$ such that

$$W = \begin{bmatrix} W_1 & W_0 \end{bmatrix} = \frac{1}{2}(W_1 + W_0) \begin{bmatrix} I + V & I - V \end{bmatrix}. \tag{6.14}$$

Moreover,

$$\ker \begin{bmatrix} W_1 & W_0 \end{bmatrix} = \ker \begin{bmatrix} I + V & I - V \end{bmatrix},$$

and

$$\begin{bmatrix} W_1 & W_0 \end{bmatrix} \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \begin{bmatrix} W_1 & W_0 \end{bmatrix}^* \geqslant 0 \Leftrightarrow VV^* \leqslant I.$$

Lemma 6.1.16. *Let A_0 be defined as in Definition 6.1.12. For an arbitrary element $\left(\begin{smallmatrix} u \\ v \end{smallmatrix}\right) \in H^N \times H^N$ there exists a function $x \in \mathcal{D}(A_0)$ such that $\Phi(x) = \left(\begin{smallmatrix} u \\ v \end{smallmatrix}\right)$.*

Proof: We give a constructive proof: Consider $\left(\begin{smallmatrix} u \\ v \end{smallmatrix}\right) \in H^N \times H^N$ where

$$u = \begin{bmatrix} u_1 \\ \vdots \\ u_N \end{bmatrix} \text{ and } v = \begin{bmatrix} v_1 \\ \vdots \\ v_N \end{bmatrix},$$

with entries $u_1, \ldots, u_N, v_1, \ldots, v_N \in H$. To construct a proper function x, we define two polynomials, $P_u(\zeta)$ and $P_v(\zeta)$, by

$$P_u(\zeta) := \sum_{i=0}^{N} \frac{u_{i+1}}{i!}(\zeta - 1)^i \text{ and } P_v(\zeta) := \sum_{i=0}^{N} \frac{v_{i+1}}{i!}\zeta^i.$$

Furthermore, we define the functions $\varphi_0 \in \mathcal{C}^\infty[0,1]$ and $\varphi_1 \in \mathcal{C}^\infty[0,1]$ such that $\varphi_0|_{[0,\varepsilon]} = 0$ and $\varphi_0|_{[1-\varepsilon,1]} = 1$ and analogously $\varphi_1|_{[0,\varepsilon]} = 1$ and $\varphi_1|_{[1-\varepsilon,1]} = 0$ hold. Thus, for

$$x := (\varphi_0 \cdot P_u + \varphi_1 \cdot P_v) I_{H^N} \in \mathcal{C}^\infty([0,1]; H^N) \subseteq \mathcal{D}(A_0)$$

we get $\Phi(x) = \left(\begin{smallmatrix} u \\ v \end{smallmatrix}\right)$. □

Lemma 6.1.17. *Let A be defined by* (3.9)-(3.10)*. Then A is dissipative if and only if $A - P_0$ is dissipative and it holds* $\operatorname{Re} P_0 \leqslant 0$*.*

Proof: "⇒": Let A be dissipative. Hence, the operator $A - P_0$ is dissipative if $\operatorname{Re} P_0 \leqslant 0$ holds. We will prove $\operatorname{Re} \langle P_0 z, z \rangle \leqslant 0$ for all $z \in H$: Let $z \in H$ and $\Psi(\zeta) \in \mathcal{C}_c^\infty(0,1)$ with $\zeta \in [0,1]$ an arbitrary, scalar-valued function with $\Psi \not\equiv 0$. We define

$$x := \Psi(\zeta) z \in \mathcal{C}_c^\infty((0,1); H) \subseteq \mathcal{D}(A)$$

and it yields, since the derivation equals zero at the boundary,

$$\begin{aligned} 0 \geqslant \operatorname{Re}\langle Ax, x\rangle &= \operatorname{Re}\langle P_0 x, x\rangle = \operatorname{Re}\langle P_0 \Psi z, \Psi z\rangle \\ &= \operatorname{Re}\int_0^1 |\Psi(\zeta)|^2 \langle P_0 z, z\rangle_H d\zeta \\ &= \|\Psi\|^2 \operatorname{Re}\langle P_0 z, z\rangle_H. \end{aligned}$$

"$\Leftarrow$": We assume $\operatorname{Re} P_0 \leqslant 0$ and $\operatorname{Re}\langle (A-P_0)x, x\rangle \leqslant 0$ for all $x \in \mathcal{D}(A)$. Thus, we get for $x \in \mathcal{D}(A)$

$$\operatorname{Re}\langle Ax, x\rangle = \operatorname{Re}\langle (A-P_0)x, x\rangle + \operatorname{Re}\langle P_0 x, x\rangle \leqslant 0 .$$

Thus, we get the assertion of the lemma. □

We are now in the position to prove the main results of this section.

Proof of Theorem 6.1.3: The implication $1 \Rightarrow 2$ follows by the Lumer-Phillips Theorem, cf. Theorem 2.1.14, and the equivalence $3 \Leftrightarrow 4$ has been shown in Lemma 6.1.15.

Next, we prove the equivalence $2 \Leftrightarrow 5$: Lemma 6.1.13 implies for $x \in \mathcal{D}(A)$

$$\operatorname{Re}\langle Ax, x\rangle = \Phi_1(x)^* Q \Phi_1(x) - \Phi_0(x)^* Q \Phi_0(x) + \operatorname{Re}\langle P_0 x, x\rangle.$$

Note that $x \in \mathcal{W}^{N,2}(0,1;H)$ satisfies $x \in \mathcal{D}(A)$ if and only if $\left(\begin{smallmatrix}\Phi_1(x)\\ \Phi_0(x)\end{smallmatrix}\right) \in \ker \widetilde{W}_B$. This proves the implication $5 \Rightarrow 2$.

We now assume that 2 holds. Then Lemma 6.1.17 shows that $\operatorname{Re} P_0 \leqslant 0$ and that $A - P_0$ is dissipative, that is,

$$\Phi_1(x)^* Q \Phi_1(x) - \Phi_0(x)^* Q \Phi_0(x) \leqslant 0$$

for every $x \in \mathcal{W}^{N,2}(0,1;H)$ satisfying $\left(\begin{smallmatrix}\Phi_1(x)\\ \Phi_0(x)\end{smallmatrix}\right) \in \ker \widetilde{W}_B$. Further, by Lemma 6.1.16, for an arbitrary element $\left(\begin{smallmatrix}u\\ v\end{smallmatrix}\right) \in \ker \widetilde{W}_B$ there exists a function $x \in \mathcal{D}(A)$ such that $\left(\begin{smallmatrix}\Phi_1(x)\\ \Phi_0(x)\end{smallmatrix}\right) = \left(\begin{smallmatrix}u\\ v\end{smallmatrix}\right)$. This proves 5.

Next, we prove the implication $2 \Rightarrow 4$: Lemma 6.1.17 shows that $\operatorname{Re} P_0 \leqslant 0$ and that $A - P_0$ is dissipative, that is, using Lemma 6.1.13

$$\operatorname{Re}\langle f_{\partial,x}, e_{\partial,x}\rangle_{H^N} \leqslant 0, \quad x \in \mathcal{D}(A). \tag{6.15}$$

For an arbitrary element $\left(\begin{smallmatrix}f\\ e\end{smallmatrix}\right) \in \ker W_B \subseteq H^N \times H^N$ a function $x \in \mathcal{D}(A)$ exists due to Lemma 6.1.16 such that $R_{ext}\Phi(x) = \left(\begin{smallmatrix}f_{\partial,x}\\ e_{\partial,x}\end{smallmatrix}\right) = \left(\begin{smallmatrix}f\\ e\end{smallmatrix}\right)$. By equation (6.15) we get $e^* f + f^* e \leqslant 0$ for all $\left(\begin{smallmatrix}f\\ e\end{smallmatrix}\right) \in \ker W_B$, where $W_B := \begin{bmatrix} W_1 & W_0 \end{bmatrix}$. For $y \in \ker(W_1 + W_0)$ we have $W_B\left(\begin{smallmatrix}y\\ y\end{smallmatrix}\right) = 0$ and thus $y^* y + y y^* \leqslant 0$. Since the norm of an element is non negative, it follows $y = 0$ and therefore $\ker(W_1 + W_0) = \{0\}$, which shows the injectivity of $W_1 + W_0$. Due to this fact, by Lemma 6.1.15 there exists an operator V satisfying (6.14). It remains to show that $\|V\| \leqslant 1$. Let $l \in H^N$ be arbitrarily. By Lemma 6.1.14 we obtain $\left(\begin{smallmatrix}I-V\\ -I-V\end{smallmatrix}\right) l \in \ker W_B$.

From Lemma 6.1.16 we follow that a function $x \in \mathcal{D}(A_0)$ exists, such that $R_{ext}\Phi(x) = \begin{pmatrix} f_{\partial,x} \\ e_{\partial,x} \end{pmatrix} = \begin{pmatrix} I-V \\ -I-V \end{pmatrix} l$. Therefore, $\begin{pmatrix} f_{\partial,x} \\ e_{\partial,x} \end{pmatrix} \in \ker W_B$ and even $x \in \mathcal{D}(A)$. In conclusion, we obtain using (6.15)

$$\begin{aligned} 2\mathrm{Re}\,\langle f_{\partial,x}, e_{\partial,x} \rangle_{H^N} &= \langle f_{\partial,x}, e_{\partial,x} \rangle_{H^N} + \langle e_{\partial,x}, f_{\partial,x} \rangle_{H^N} \\ &= \langle (I-V)l, (-I-V)l \rangle_{H^N} + \langle (-I-V)l, (I-V)l \rangle_{H^N} \\ &= 2\langle l, (-I+V^*V)l \rangle_{H^N} \leqslant 0 \end{aligned} \tag{6.16}$$

and therefore $\|V\| \leqslant 1$.
Finally, we show the implication 4 $\Rightarrow$ 1: A is a closed operator, see [Aug16, Lemma 3.2.2]. To prove that A generates a contraction C_0-semigroup, it is sufficient to verify that A and A^* are dissipative, cf. Theorem 2.1.15. Let $x \in \mathcal{D}(A)$. Then, we have $\begin{pmatrix} f_{\partial,x} \\ e_{\partial,x} \end{pmatrix} \in \ker W_B$ and from Lemma 6.1.14 it follows that there exists an $l \in H^N$ such that $\begin{pmatrix} f_{\partial,x} \\ e_{\partial,x} \end{pmatrix} = \begin{pmatrix} I-V \\ -I-V \end{pmatrix} l$. Using Lemma 6.1.13 and Lemma 6.1.15, we obtain

$$\begin{aligned} 2\mathrm{Re}\,\langle Ax, x \rangle_{L^2} &= 2\mathrm{Re}\,\langle f_{\partial,x}, e_{\partial,x} \rangle_{H^N} + 2\langle P_0 x, x \rangle \\ &\leqslant 2\langle l, (-I+V^*V)l \rangle_{H^N} \leqslant 0. \end{aligned}$$

Now we consider the adjoint operator A^*: Let $y \in \mathcal{D}(A^*)$. By Lemma 6.1.11, we obtain $\begin{pmatrix} \tilde{f}_{\partial,y} \\ \tilde{e}_{\partial,y} \end{pmatrix} \in \ker S \begin{bmatrix} I+V^* & I-V^* \end{bmatrix}$. Applying Lemma 6.1.14 and Lemma 6.1.15 to the operator V^*, there exists $m \in H^N$ such that $\begin{pmatrix} \tilde{f}_{\partial,x} \\ \tilde{e}_{\partial,x} \end{pmatrix} = \begin{pmatrix} I-V^* \\ -I-V^* \end{pmatrix} m$. Using again Lemma 6.1.13 we get

$$2\mathrm{Re}\,\langle A^* y, y \rangle_{L^2} \leqslant 2\langle m, (-I+VV^*)m \rangle_{H^N} \leqslant 0, \tag{6.17}$$

which concludes the proof. $\square$

Proof of Corollary 6.1.5: We want to apply Theorem 6.1.3 for the proof of Corollary 6.1.5. Therefore, we have to check condition (6.9).
If $\dim H < \infty$, then $W_1 + W_0$ injective implies the surjectivity of $W_1 + W_0$ and hence condition (6.9). Due to this and Remark 6.1.4.2 assertions 1, 2, 3, 4 and 5 of Theorem 6.1.5 are equivalent. The implications 3 $\Rightarrow$ 6 and 4 $\Rightarrow$ 7 follows, since we have $W_1 + W_0$ injective, and thus, $W_1 + W_2$ is also surjective. Clearly, it follows that W_B is surjective. A straightforward calculation shows the implication 7 $\Rightarrow$ 6. In order to show 6 $\Rightarrow$ 3 we prove that in the finite-dimensional setting the surjectivity of W_B and $W_B \Sigma W_B^* \geqslant 0$ implies the injectivity of $W_1 + W_0$. From

$$W_B \Sigma W_B^* \geqslant 0 \Leftrightarrow W_0 W_1^* + W_1 W_0^* \geqslant 0,$$

we obtain

$$\begin{aligned} &W_1 W_1^* + W_0 W_1^* + W_0 W_0^* + W_1 W_0^* \\ &= (W_1 + W_0)(W_1 + W_0)^* \geqslant (W_1 - W_0)(W_1 - W_0)^* \geqslant 0. \end{aligned}$$

Let x be in $\ker(W_1 + W_0)^*$. This yields $x \in \ker(W_1 - W_0)(W_1 - W_0)^*$. With

$$\begin{aligned}\|(W_1 - W_0)^* x\|^2 &= \langle (W_1 - W_0)^* x, (W_1 - W_0)^* x\rangle \\ &= \langle x, (W_1 - W_0)(W_1 - W_0)^* x\rangle = \langle x, 0\rangle = 0\end{aligned}$$

we get $x \in \ker(W_1 - W_0)^*$ and thus, $x \in \ker W_1^* \cap W_0^*$. Since W_B is surjective, W_B^* is injective and thus $x = 0$. This implies that $W_1 + W_0$ is injective. □

Proof of Theorem 6.1.7: Without loss of generality again we consider just the case $\mathcal{H} = I$. In the following proof we will often apply Theorem 6.1.3 to the operators A and $-A$. So, first of all, we have to verify, that also the boundary condition operator $\bar{W}_B$ of $-A$ satisfies the condition (6.9).
We define analogously to (6.11) the boundary flow and the boundary effort for $-A$:

$$\begin{bmatrix} \bar{f}_{\partial,x} \\ \bar{e}_{\partial,x} \end{bmatrix} := \frac{1}{\sqrt{2}} \begin{bmatrix} -Q & Q \\ I & I \end{bmatrix} \Phi(\mathcal{H}x). \tag{6.18}$$

Therefore, it yields $\bar{f}_{\partial,x} = -f_{\partial,x}$ and $\bar{e}_{\partial,x} = e_{\partial,x}$. Due to $\mathcal{D}(A) = \mathcal{D}(-A)$, we get

$$\begin{aligned}\mathcal{D}(A) &= \left\{ x \in \mathcal{W}^{N,2}(0,1;H) | W_B \begin{bmatrix} f_{\partial,x} \\ e_{\partial,x} \end{bmatrix} = 0 \right\} \\ &= \left\{ x \in \mathcal{W}^{N,2}(0,1;H) | \bar{W}_B \begin{bmatrix} \bar{f}_{\partial,x} \\ \bar{e}_{\partial,x} \end{bmatrix} = 0 \right\} \\ &= \left\{ x \in \mathcal{W}^{N,2}(0,1;H) | \bar{W}_B \begin{bmatrix} -f_{\partial,x} \\ e_{\partial,x} \end{bmatrix} = 0 \right\}\end{aligned}$$

and thus,

$$\bar{W}_B = \begin{bmatrix} -W_1 & W_0 \end{bmatrix}. \tag{6.19}$$

It is easy to check that under condition (6.10) the operator $\bar{W}_B$ satisfies (6.9). Then the equivalences $1 \Leftrightarrow 2 \Leftrightarrow 5$ follow by Theorem 6.1.3 applied for A and $-A$.
$1 \Rightarrow 4$: Let A be the generator of a unitary group. Then, due to Theorem [JZ12, Theorem 6.2.5] A and $-A$ are generators of contraction C_0-semigroups. It follows $\operatorname{Re} P_0 = 0$, $W_1 + W_0$ and $-W_1 + W_0$ are injective and $\operatorname{Re}\langle Ax, x\rangle = 0$ for all $x \in \mathcal{D}(A)$ by Theorem 6.1.3. Thus, we get with the estimation (6.16)

$$0 = 2\operatorname{Re}\langle Ax, x\rangle = 2\langle l, (-I + V^*V)l)\rangle_{H^N} \text{ for all } l \in H^N \tag{6.20}$$

and therefore $\|V\| = 1$.
$4 \Rightarrow 3$: Let $\operatorname{Re} P_0 = 0$, $\|V\| = 1$, $W_1 + W_0$ and $-W_1 + W_0$ injective. Define $S := \frac{1}{2}(W_1 + W_0)$ and with the technical Lemma 6.1.15 (Lemma 2.4 in [KZ15]) it yields

$$\begin{aligned}W_B \Sigma W_B^* &= S \begin{bmatrix} I + V & I - V \end{bmatrix} \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \left(S \begin{bmatrix} I + V & I - V \end{bmatrix}\right)^* \\ &= S(2I - 2VV^*)S^* = 0.\end{aligned}$$

The implication $3 \Rightarrow 1$ follows analogously to the proof of $3 \Rightarrow 1$ in Theorem 6.1.3 for the operator $-A$. However, instead of the boundary effort and the boundary flow for A we need to consider them for $-A$ and have to determine the boundary condition operator $\bar{W}_B$ for $-A$. □

6.1.1 Examples for port-Hamiltonian systems in the infinite-dimensional setting

In this section we now illustrate our results by a number of examples. Networks of discrete partial differential equations on infinite networks are also considered in [Mug14].
Further examples on the interval $(0,1)$ with a finite-dimensional Hilbert space H can be found in [JZ12] and [Aug16]. In the following we consider examples on the bounded interval $(0,1)$ with an infinite-dimensional Hilbert space H.

Example 6.1.18. Choose $H = \ell^2(\mathbb{N})$ and consider the operator A given by

$$Af = \frac{\partial}{\partial \zeta} f \tag{6.21}$$

on the domain

$$\mathcal{D}(A) = \left\{ f \in \mathcal{W}^{1,2}(0,1;\ell^2(\mathbb{N})) | \begin{bmatrix} I & -L \end{bmatrix} \Phi(f) = 0 \right\}. \tag{6.22}$$

This means that the network is a path graph, see Figure 6.3.

Figure 6.3: Path graph

Clearly, A denotes a port-Hamiltonian operator with $N = 1$, $P_1 = I$, $P_0 = 0$ and $W_B = \left[\begin{smallmatrix} W_1 & W_0 \end{smallmatrix}\right] = \frac{1}{2}\begin{bmatrix} I+L & I-L \end{bmatrix}$. Here L denotes the left shift and $L^* = R$ the right shift, i.e., $L : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ is defined by $L(x_1, x_2, \ldots) \mapsto (x_2, x_3, \ldots)$ and $R : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ is given as $R(x_1, x_2, \ldots) \mapsto (0, x_1, x_2, \ldots)$. Clearly, it yields $W_1 + W_0 = I$, and thus, condition (6.9) is fulfilled. Therefore, we can apply Theorem 6.1.3 and check assertion 3: $W_1 + W_0$ is injective and

$$\begin{aligned} W_B \Sigma W_B^* &= \frac{1}{4}\begin{bmatrix} I+L & I-L \end{bmatrix} \Sigma \begin{bmatrix} I+L & I-L \end{bmatrix}^* \\ &= \frac{1}{4}\begin{bmatrix} I-L & I+L \end{bmatrix} \begin{bmatrix} I+L^* & I-L^* \end{bmatrix} \\ &= \frac{1}{4}((I-L)(I+L^*) + (I+L)(I-L^*)) = \frac{1}{4}(2I - 2LL^*) = 0. \end{aligned}$$

Hence, A generates a contraction C_0-semigroup. In the finite-dimensional setting we would expect that A also generates a unitary C_0-group, since $W_B \Sigma W_B^* = 0$. However, we can apply Theorem 6.1.7, since condition (6.10) is fulfilled: $\operatorname{ran}(L) = \operatorname{ran}(I)$, because the left shift is surjective. Using assertion 3 of Theorem 6.1.7, we can conclude that A does not generate a unitary C_0-group, since $-W_1 + W_0 = -L$ and the left shift is not injective.

Example 6.1.19. We choose again $H = \ell^2(\mathbb{N})$ and consider the operator A given by

$$Af = \frac{\partial}{\partial \zeta} f \tag{6.23}$$

on the domain

$$\mathcal{D}(A) = \left\{ f \in \mathcal{W}^{1,2}(0,1;\ell^2(\mathbb{N})) | \begin{bmatrix} I & T \end{bmatrix} \Phi(f) = 0 \right\}, \tag{6.24}$$

where $T : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ is defined by

$$T(x_1, x_2, \ldots) \mapsto \frac{1}{2}(-x_3 - x_4, -x_5 - x_6, -x_7 - x_8, \ldots).$$

These boundary conditions imply that the network is a binary tree, see Figure 6.4.

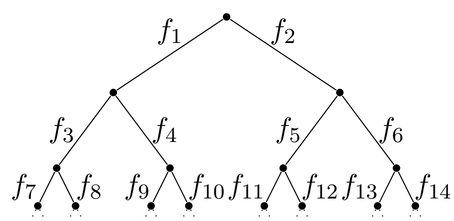

Figure 6.4: Binary tree

We write $f \in \mathcal{W}^{1,2}(0,1;\ell^2(\mathbb{N}))$ as $f = (f_1, f_2, \ldots)^T$, where $f_i \in \mathcal{W}^{1,2}(0,1;\mathbb{C}^d)$ denotes a function on the i-th edge of the binary tree. Clearly, A denotes a port-Hamiltonian operator with $N = 1$, $P_1 = I$, $P_0 = 0$ and $W_B = \frac{1}{2}\begin{bmatrix} I - T & I + T \end{bmatrix}$. It yields $W_1 + W_0 = I$, and thus, condition (6.9) is fulfilled.
$W_1 + W_0$ is injective and $T^* : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ is given by

$$T^*(x_1, x_2, \ldots) \mapsto \frac{1}{2}(0, 0, -x_1, -x_1, -x_2, -x_2, \ldots).$$

We obtain

$$W_B \Sigma W_B^* = \frac{1}{4}(2I - 2TT^*) = \frac{1}{4} I.$$

Hence, A generates a contraction C_0-semigroup.

6.2 Port-Hamiltonian systems on the semi-axis

In this section, we consider port-Hamiltonian systems on the semi-axis, i.e., systems of the form

$$\begin{aligned} \frac{\partial x}{\partial t}(\zeta, t) &= \left(P_1 \frac{\partial}{\partial \zeta} + P_0 \right) (\mathcal{H}(\zeta) x(\zeta, t)), \\ x(\zeta, 0) &= x_0(\zeta),\ \zeta \in (0, \infty),\ t \geqslant 0, && (6.25) \\ 0 &= \widetilde{W}_B(\Phi(\mathcal{H}x))(\cdot, t), && (6.26) \end{aligned}$$

where P_1 is an invertible Hermitian $d \times d$-matrix, $P_0 \in \mathbb{C}^{d\times d}$, $\widetilde{W}_B \in \mathbb{C}^{k\times d}$ with $k \in \{0, 1, \cdots, d\}$ and $\mathcal{H}(\zeta) \in \mathbb{C}^{d\times d}$ is positive definite for a.e. $\zeta \in [0, \infty)$ satisfying $\mathcal{H}, \mathcal{H}^{-1} \in L^\infty(0, \infty; \mathbb{C}^{d\times d})$. Since P_1 is an invertible Hermitian matrix, its eigenvalues are real and nonzero.
Here $\widetilde{W}_B \in \mathcal{L}(H^N, \widetilde{H}^N)$, where $\widetilde{H}$ is a subspace of H, and Φ is given by

$$\Phi : \mathcal{W}^{N,2}(0, \infty; H) \to H^N, \qquad \Phi(x) := \Phi_0(x).$$

The contraction C_0-semigroup property has been shown for some specific examples [EN00, I.4.16], [MNS18], and related results can be found in [BK13], [EKF19], [KPS08], [KS99] and [SSVW15]. In the following we provide a characterization of the contraction C_0-semigroup property of the operator A. Again A generates a contraction C_0-semigroup if and only if the operator A is dissipative. The main difference to the port-Hamiltonian systems on a bounded interval is that the number of boundary conditions depends on P_1.
We consider the port-Hamiltonian operator A, associated to the system (6.25),

$$Ax = P_0\mathcal{H}x + P_1\frac{\partial}{\partial\zeta}(\mathcal{H}x) \text{ with} \tag{6.27}$$

$$\mathcal{D}(A) = \Big\{x \in L^2(0, \infty; \mathbb{C}^d) \mid \mathcal{H}x \in \mathcal{W}^{1,2}(0, \infty; \mathbb{C}^d), \widetilde{W}_B(\mathcal{H}x(0)) = 0\Big\} \tag{6.28}$$

on the space $X = L^2(0, \infty; \mathbb{C}^d)$.
We denote by d_1 the number of positive and by $d_2 = d - d_1$ the number of negative eigenvalues of P_1 and write

$$P_1 = S^{-1}\Delta S = S^{-1}\begin{bmatrix}\Lambda & 0\\ 0 & \Theta\end{bmatrix} S, \tag{6.29}$$

with a unitary matrix $S \in \mathbb{C}^{d\times d}$, a positive definite diagonal matrix $\Lambda \in \mathbb{R}^{d_1\times d_1}$, and a negative definite diagonal matrix $\Theta \in \mathbb{R}^{d_2\times d_2}$. We define $\Delta = \left(\begin{smallmatrix}\Lambda & 0\\ 0 & \Theta\end{smallmatrix}\right)$.
In the following, first we formulate the main results of this section, then we give a technical lemma and finally the proof of the main result.

Theorem 6.2.1. *Assume A is given by (6.27)-(6.28), $\widetilde{W}_B \in \mathbb{C}^{k\times d}$ with $k \leqslant d_2$ has full row rank. Then the following statements are equivalent:*

1. *A generates a contraction C_0-semigroup on X;*

2. *$\operatorname{Re}\langle Ax, x\rangle \leqslant 0$ for every $x \in \mathcal{D}(A)$;*

3. *$\operatorname{Re} P_0 \leqslant 0$ and $y^*P_1y \geqslant 0$ for every $y \in \ker \widetilde{W}_B$;*

4. *$\operatorname{Re} P_0 \leqslant 0$, $k = d_2$ and $\widetilde{W}_B = B\begin{bmatrix}U & I\end{bmatrix} S$, with $B \in \mathbb{C}^{d_2\times d_2}$ invertible, $U \in \mathbb{C}^{d_2\times d_1}$ and $\Lambda + U^*\Theta U \geqslant 0$.*

Further, we are able to characterize the property of unitary group generation for port-Hamiltonian operators on the semi-axis.

Theorem 6.2.2. *Let A be given by (6.27)-(6.28), $\widetilde{W}_B \in \mathbb{C}^{k\times d}$ with $k \leqslant \min\{d_1, d_2\}$ has full row rank. Then the following statements are equivalent:*

1. *A generates a unitary C_0-group on X;*
2. $\operatorname{Re}\langle Ax, x\rangle = 0$ *for every* $x \in \mathcal{D}(A)$*;*
3. $\operatorname{Re} P_0 = 0$ *and* $y^* P_1 y = 0$ *for every* $y \in \ker \widetilde{W}_B$*;*
4. $k = d_1 = d_2$, $\operatorname{Re} P_0 = 0$ *and* $\widetilde{W}_B = \begin{bmatrix} U_1 & U_2 \end{bmatrix} S$*, where* $U_1, U_2 \in \mathbb{C}^{d_1 \times d_1}$ *are invertible with* $\Lambda + U_1^* U_2^{-*} \Theta U_2^{-1} U_1 = 0$.

For the proof of the main statements we need the following technical assertions.

Lemma 6.2.3. 1. *Assume* $\Lambda \in \mathbb{R}^{d_1 \times d_1}$ *is a positive, invertible diagonal matrix and* $y \in L^2(0, \infty; \mathbb{C}^{d_1})$*. Then the function*

$$x(t) := \int_0^\infty e^{-s\Lambda^{-1}} \Lambda^{-1} y(s+t)\, ds, \quad t \geqslant 0, \tag{6.30}$$

satisfies $x \in \mathcal{W}^{1,2}(0, \infty; \mathbb{C}^{d_1})$ *and* $x - \Lambda x' = y$.

2. *Let* $\Theta \in \mathbb{R}^{d_2 \times d_2}$ *be a negative, invertible diagonal matrix,* $y \in L^2(0, \infty; \mathbb{C}^{d_2})$ *and* $x_0 \in \mathbb{C}^{d_2}$*. Then the differential equation*

$$x - \Theta x' = y, \quad x(0) = x_0, \tag{6.31}$$

has a unique solution satisfying $x \in \mathcal{W}^{1,2}(0, \infty; \mathbb{C}^{d_2})$.

Proof Part 1: $\Lambda > 0$ and $y \in L^2(0, \infty; \mathbb{C}^{d_1})$ imply that $x(t)$ is well defined for every $t \geqslant 0$. Minkowski's integral inequality shows $x \in L^2(0, \infty; \mathbb{C}^{d_1})$. Further, the solution of $x - \Lambda x' = y$, or equivalently, of $x' = \Lambda^{-1} x - \Lambda^{-1} y$ is given by

$$x(t) = e^{t\Lambda^{-1}} x(0) - \int_0^t e^{(t-s)\Lambda^{-1}} \Lambda^{-1} y(s)\, ds, \quad t \geqslant 0.$$

The choice of $x(0) = \int_0^\infty e^{-s\Lambda^{-1}} \Lambda^{-1} y(s)\, ds$, implies (6.30). Moreover, $x' = \Lambda^{-1} x - \Lambda^{-1} y$ and hence $x \in \mathcal{W}^{1,2}(0, \infty; \mathbb{C}^{d_1})$.
Part 2: We first note that (6.31) is equivalent to $x' = \Theta^{-1} x - \Theta^{-1} y$. Now the statement of the lemma follows from ODE-Theory for linear stable systems, since $\Theta < 0$ and $y \in L^2(0, \infty; \mathbb{C}^{d_2})$, see [HP05, Proposition 3.3.22]. □

Proof of Theorem 6.2.1: Thanks to the Theorem of Lumer-Phillips, cf. Theorem 2.1.14, it holds 1 implies 2.
Next, we show the implication 2 ⇒ 3. Using integration by parts and $P_1^* = P_1$, it yields $2\operatorname{Re}\langle P_1 \frac{d}{d\zeta} x, x\rangle = -x(0)^* P_1 x(0)$, since $\lim_{\zeta \to \infty} x(\zeta) = 0$ for $x \in \mathcal{W}^{1,2}(0, \infty; \mathbb{C}^d)$. Thus, for $x \in D(A)$ we have

$$2\operatorname{Re}\langle Ax, x\rangle = 2\operatorname{Re}\langle P_1 \frac{d}{d\zeta} x + P_0 x, x\rangle = -x(0)^* P_1 x(0) + 2\operatorname{Re} \int_0^\infty x(\zeta)^* P_0 x(\zeta)\, d\zeta. \tag{6.32}$$

Choosing $x \in \mathcal{W}^{1,2}(0, \infty; \mathbb{C}^d) \setminus \{0\}$ with $x(0) = 0$, we obtain $\operatorname{Re} P_0 \leqslant 0$. For every $y \in \mathbb{C}^d$ and every $\varepsilon > 0$ there exists a function $x \in \mathcal{W}^{1,2}(0, \infty; \mathbb{C}^d)$ such that $x(0) = y$ and the L^2-norm of x is less than ε. Choosing this function in

equation (6.32) and letting ε go to zero implies the second assertion in 3.
In order to prove the implication 3 $\Rightarrow$ 4, for $x \in D(A)$ we define $\left[\begin{smallmatrix} f_1 \\ f_2 \end{smallmatrix}\right] := Sx(0)$.
Using (6.29), the second condition in 3 can be written as

$$\begin{bmatrix} f_1^* & f_2^* \end{bmatrix} \begin{bmatrix} \Lambda & 0 \\ 0 & \Theta \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \geqslant 0, \quad \text{for } \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \in \ker \widetilde{W}_B S^{-1}. \tag{6.33}$$

Since $\widetilde{W}_B S^{-1}$ is a full row rank $k \times d$-matrix with $k \leqslant d_2$, its kernel has dimension $d-k$. By the assumptions on Λ and Θ, we have $d-k \leqslant d_1$, or equivalently, $k \geqslant d_2$. Thus $k = d_2$.
We write $\widetilde{W}_B S^{-1} = \begin{bmatrix} U_1 & U_2 \end{bmatrix}$ with $U_1 \in \mathbb{C}^{d_2 \times d_1}$ and $U_2 \in \mathbb{C}^{d_2 \times d_2}$. Assuming U_2 is not invertible, there exists $u \in \mathbb{C}^{d_2}$ such that $\left[\begin{smallmatrix} 0 \\ u \end{smallmatrix}\right] \in \ker \widetilde{W}_B S^{-1}$ which is in contradiction to (6.33), since $\Theta < 0$. Thus, the matrix $\widetilde{W}_B S^{-1}$ is of the form $B \begin{bmatrix} U & I \end{bmatrix}$, with $U \in \mathbb{C}^{d_2 \times d_1}$ and $B \in \mathbb{C}^{d_2 \times d_2}$ invertible. Hence, (6.33) is equivalent to

$$\begin{bmatrix} f_1^* & f_2^* \end{bmatrix} \begin{bmatrix} \Lambda & 0 \\ 0 & \Theta \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \geqslant 0 \text{ and } Uf_1 + f_2 = 0, \text{ for } \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \in \mathbb{C}^{d_1+d_2} \tag{6.34}$$

which is equivalent to $\Lambda + U^*\Theta U \geqslant 0$. This shows 4.
It remains to show that 4 implies 1. Due to the fact that $\operatorname{Re} P_0 \leqslant 0$, and bounded, dissipative perturbations of generators of contraction C_0-semigroups, again generate a contraction C_0-semigroup, see [EN00, Theorem III.2.7], without loss of generality we may assume $P_0 = 0$.
First, we prove the dissipativity of the operator A. Let $x \in \mathcal{D}(A)$ and define $\left[\begin{smallmatrix} f_1 \\ f_2 \end{smallmatrix}\right] := Sx(0)$, where the unitary matrix S is given by (6.29). This implies $Uf_1 + f_2 = 0$ as $\widetilde{W}_B = B \begin{bmatrix} U & I \end{bmatrix} S$.
Thus, it yields

$$\begin{aligned} \operatorname{Re} \langle Ax, x \rangle &= -\langle x(0), P_1 x(0) \rangle_{\mathbb{C}^d} = -\langle x(0), S^{-1} \begin{bmatrix} \Lambda & 0 \\ 0 & \Theta \end{bmatrix} Sx(0) \rangle_{\mathbb{C}^d} \\ &= -\langle Sx(0), \begin{bmatrix} \Lambda & 0 \\ 0 & \Theta \end{bmatrix} Sx(0) \rangle_{\mathbb{C}^d} = -(f_1^* \Lambda f_1 + f_2^* \Theta f_2) \\ &= -(f_1^* \Lambda f_1 + f_1^* U^* \Theta U f_1) \leqslant 0 \end{aligned}$$

by the last assertion of 4.
Further, thanks to the Theorem of Lumer-Phillips, cf. Theorem 2.1.14, it remains to show that for every $y \in L^2(0,\infty;\mathbb{C}^d)$ there exists $x \in D(A)$ such that $x - Ax = y$. Equivalently, by (6.29) it is sufficient to show that for every $y_1 \in L^2(0,\infty;\mathbb{C}^{d_1})$ and $y_2 \in L^2(0,\infty;\mathbb{C}^{d_2})$ there exist functions $x_1 \in \mathcal{W}^{1,2}(0,\infty;\mathbb{C}^{d_1})$ and $x_2 \in \mathcal{W}^{1,2}(0,\infty;\mathbb{C}^{d_2})$ such that

$$x_1 - \Lambda x_1' = y_1, \;\; x_2 - \Theta x_2' = y_2 \quad \text{and} \quad Ux_1(0) + x_2(0) = 0.$$

Let $y_1 \in L^2(0,\infty;\mathbb{C}^{d_1})$ and $y_2 \in L^2(0,\infty;\mathbb{C}^{d_2})$ be arbitrarily. Lemma 6.2.3.1 implies the existence of $x_1 \in \mathcal{W}^{1,2}(0,\infty;\mathbb{C}^{d_1})$ with $x_1(0) = \int_0^\infty e^{-s\Lambda^{-1}} \Lambda^{-1} y_1(s)\, ds$ and $x_1 - \Lambda x_1' = y_1$. Finally, Lemma 6.2.3.2 shows that there exists a function

$x_2 \in \mathcal{W}^{1,2}(0,\infty;\mathbb{C}^{d_1})$ with $x_2(0) = -Ux_1(0)$ and $x_2 - \Theta x_2' = y_2$. This concludes the proof. $\square$

Proof of Theorem 6.2.2: Since A generates a unitary C_0-group if and only if A and $-A$ generate contraction C_0-semigroups, cf. Theorem 2.1.16, the equivalence of assertions 1, 2, and 3 follows directly from Theorem 6.2.1 for $-A$ and A.
Formulating assertion 4 of Theorem 6.2.1 for $-A$, we get $\operatorname{Re}(-P_0) \leqslant 0$, $k = d_1$,

$$\widetilde{W}_B = \bar{B}\begin{bmatrix} I & \bar{U} \end{bmatrix} S$$

and $\Theta + \bar{U}^*\Lambda\bar{U} \leqslant 0$, where $\bar{B} \in K^{d_1\times d_1}$ is invertible. Thus, assertion 4 of Theorem 6.2.1 for $-A$ and A is equivalent to $\operatorname{Re} P_0 = 0$, $k = d_1 = d_2$ and $\widetilde{W}_B = \bar{B}\begin{bmatrix} I & \bar{U} \end{bmatrix} S = B\begin{bmatrix} U & I \end{bmatrix} S$ with B and $\bar{B}$ invertible. It yields $\bar{B} = BU$ and $B = \bar{B}\bar{U}$ with $B, \bar{B}$ invertible. Therefore, we get $\bar{U}U = I$ and $\bar{U}, U$ invertible. Thus, we have $\Theta + \bar{U}^*\Lambda\bar{U} \leqslant 0 \Leftrightarrow U^*\Theta U + \Lambda \leqslant 0$. Choosing $U_1 = BU$ and $U_2 = B$ we get the assertion. $\square$

6.2.1 Examples for port-Hamiltonian systems on the semi-axis

Example 6.2.4. Let A be given by (6.27)-(6.28) on the semi-axis $(0,\infty)$.

1. Let $P_1 < 0$, that is, $d_2 = d$, and $\operatorname{Re} P_0 \leqslant 0$. In this situation A with domain

   $$\mathcal{D}(A) = \{x \in X \mid \mathcal{H}x \in \mathcal{W}^{1,2}(0,\infty;\mathbb{C}^d) \text{ and } (\mathcal{H}x)(0) = 0\}$$

   generates a contraction C_0-semigroup on X.

2. Let $P_1 > 0$, that is, $d_2 = 0$ and $\operatorname{Re} P_0 \leqslant 0$. Then A with domain

   $$\mathcal{D}(A) = \{x \in X \mid \mathcal{H}x \in \mathcal{W}^{1,2}(0,\infty;\mathbb{C}^d)\}$$

   generates a contraction C_0-semigroup on X.

3. We consider again the wave equation as in Example 3.1.7 but now on the semi-axis. There, an (undamped) vibrating string can be modelled by

   $$\frac{\partial^2 w}{\partial t^2}(\zeta,t) = \frac{1}{\rho(\zeta)}\frac{\partial}{\partial\zeta}\left(T(\zeta)\frac{\partial w}{\partial\zeta}(\zeta,t)\right), \quad t \geqslant 0,\ \zeta \in [0,\infty), \tag{6.35}$$

   where $w(\zeta,t)$ is the vertical position of the string at place ζ and time t, $T(\zeta) > 0$ is the Young's modulus of the string, and $\rho(\zeta) > 0$ is the mass density, which may vary along the string. We assume that T and ρ are positive functions satisfying $\rho, \rho^{-1}, T, T^{-1} \in L^\infty[0,\infty)$. By choosing the state variables $x_1 = \rho\frac{\partial w}{\partial t}$ (momentum) and $x_2 = \frac{\partial w}{\partial\zeta}$ (strain), the partial differential equation (6.35) can equivalently be written as

   $$\begin{aligned}\frac{\partial}{\partial t}\begin{bmatrix} x_1(\zeta,t) \\ x_2(\zeta,t)\end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ 1 & 0\end{bmatrix}\frac{\partial}{\partial\zeta}\left(\begin{bmatrix}\frac{1}{\rho(\zeta)} & 0 \\ 0 & T(\zeta)\end{bmatrix}\begin{bmatrix} x_1(\zeta,t) \\ x_2(\zeta,t)\end{bmatrix}\right) \\ &= P_1\frac{\partial}{\partial\zeta}\left(\mathcal{H}(\zeta)\begin{bmatrix} x_1(\zeta,t) \\ x_2(\zeta,t)\end{bmatrix}\right),\end{aligned} \tag{6.36}$$

where $P_1 = \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$ and $\mathcal{H}(\zeta) = \left[\begin{smallmatrix} \frac{1}{\rho(\zeta)} & 0 \\ 0 & T(\zeta) \end{smallmatrix}\right]$.

The boundary conditions for (6.36) are

$$\widetilde{W}_B(\mathcal{H}x)(0,t) = 0,$$

where $\widetilde{W}_B$ is a $k \times 2$-matrix with rank $k \in \{0,1,2\}$, or equivalently, the partial differential equation (6.35) is equipped with the boundary conditions

$$\widetilde{W}_B \begin{bmatrix} \frac{\partial w}{\partial t}(0,t) \\ T\frac{\partial w}{\partial \zeta}(0,t) \end{bmatrix} = 0.$$

The matrix P_1 can be factorized as

$$P_1 = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{bmatrix},$$

This implies $d_2 = 1$. Thus, by Theorem 6.2.1 the corresponding operator

$$(Ax)(\zeta) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \frac{\partial}{\partial \zeta} \left(\begin{bmatrix} \frac{1}{\rho(\zeta)} & 0 \\ 0 & T(\zeta) \end{bmatrix} x(\zeta) \right),$$
$$D(A) = \left\{ x \in \mathcal{W}^{1,2}(0,1;\mathbb{C}^2) \mid \widetilde{W}_B(\mathcal{H}x)(0,t) = 0 \right\},$$

generates a contraction C_0-semigroup on $L^2(0,1;\mathbb{C}^2)$ if and only if

$$\widetilde{W}_B = \frac{b}{2} \begin{bmatrix} u-1 & u+1 \end{bmatrix}$$

for $b \in \mathbb{C}\backslash\{0\}$ and $u \in \mathbb{C}$. More precisely, the partial differential equation

$$\frac{\partial^2 w}{\partial t^2}(\zeta,t) = \frac{1}{\rho(\zeta)} \frac{\partial}{\partial \zeta} \left(T(\zeta) \frac{\partial w}{\partial \zeta}(\zeta,t) \right), \quad t \geqslant 0,\ \zeta \in [0,\infty),$$
$$(u-1)\frac{\partial w}{\partial t}(0,t) + (u+1)T(0)\frac{\partial w}{\partial \zeta}(0,t) = 0, \quad t \geqslant 0,$$
$$\rho(\zeta)\frac{\partial w}{\partial t}(\zeta,0) = z_0(\zeta), \quad \zeta \geqslant 0,$$
$$\frac{\partial w}{\partial \zeta}(\zeta,0) = z_1(\zeta), \quad \zeta \geqslant 0,$$

where $u \in \mathbb{C}$ and $z_0, z_1 \in L^2[0,\infty)$, possesses a unique solution satisfying

$$\int_0^\infty \rho(\zeta) \left[\frac{\partial w}{\partial t}(\zeta,t) \right]^2 + T(\zeta) \left[\frac{\partial w}{\partial \zeta}(\zeta,t) \right]^2 d\zeta \leqslant \int_0^\infty \frac{z_0^2(\zeta)}{\rho(\zeta)} + T(\zeta) z_1^2(\zeta) d\zeta$$

for $t > 0$, which means that the energy of the system is non-increasing.

6.3 Closing remarks and open problems

The following example illustrates the connection between port-Hamiltonian systems in the infinite-dimensional setting and on the semi-axis.

Example 6.3.1. We consider the port-Hamiltonian operator

$$A_\infty x = \frac{d}{d\zeta}x,$$

$$\mathcal{D}(A_\infty) = \mathcal{W}^{1,2}(0,\infty;\mathbb{R})$$

on the semi-axis, which generates a contraction C_0-semigroup on the space $X = L^2(0,\infty;\mathbb{R})$, namely the translation shift $(T(t))_{t\geqslant 0}$ with $(T(t)f)(\zeta) = f(\zeta+t)$. Since $X \simeq Y := L^2(0,1;\ell^2)$ via the isomorphism J given by

$$J : L^2(0,\infty;\mathbb{R}) \to L^2(0,1;\ell^2)$$
$$f(x) \mapsto g(x) = \begin{pmatrix} g_1(\zeta) \\ \vdots \\ g_n(\zeta) \\ \vdots \end{pmatrix} \text{ with } g_n(\zeta) = f(\zeta + (n-1))$$

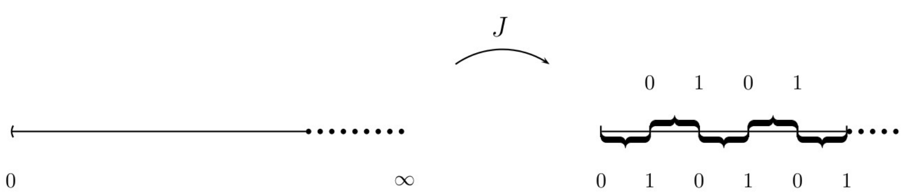

Figure 6.5: Sketch of the isomorphic map J

Therefore, we can conclude that $Ag = JA_\infty J^{-1}g = \frac{d}{d\zeta}g$ with

$$\begin{aligned}\mathcal{D}(A) &= \left\{x \in H^N(0,1;\ell^2(\mathbb{N})) | \widetilde{W}_B\Phi(x) = 0\right\} \\ &= \{g \in H^1(0,1;\ell^2(\mathbb{N})) \text{ with} \\ &\qquad g(\zeta) = \begin{pmatrix} g_1(\zeta) \\ \vdots \\ g_n(\zeta) \\ \vdots \end{pmatrix} | g_n(1) = g_{n+1}(0)\ \forall n \in \mathbb{N}\}\end{aligned}$$

generates also a contraction semigroup on $L^2(0,1;\ell^2)$. Since Ag is a port-Hamiltonian operator with $N=1$, $P_0 = 0$ and $P_1 = I \in \mathcal{L}(\ell^2(\mathbb{N}))$ we may get the same result via Theorem 6.1.3.
On yields $\widetilde{W}_B = \begin{bmatrix} 2I & -2L \end{bmatrix}$ and thus

$$\begin{aligned} W_B &= \begin{bmatrix} 2I & -2L \end{bmatrix} \begin{bmatrix} Q & -Q \\ I & I \end{bmatrix}^{-1} = \begin{bmatrix} 2I & -2L \end{bmatrix} \begin{bmatrix} I & I \\ -I & I \end{bmatrix}^{-1} \\ &= \begin{bmatrix} I & -L \end{bmatrix} \begin{bmatrix} I & I \\ -I & I \end{bmatrix} = \begin{bmatrix} I+L & I-L \end{bmatrix} = \begin{bmatrix} W_1 & W_2 \end{bmatrix}, \end{aligned}$$

where

$L : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$, $L(x_1, x_2, \ldots) \mapsto (x_2, x_3, x_4 \ldots)$ denotes the left shift and $R : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$, $R(x_1, x_2, \ldots) \mapsto (0, x_1, x_2, \ldots)$ denotes the right shift. Thus, we are in the situation of Example 6.1.18 and can verify using Theorem 6.1.3 that A generates a contraction C_0-semigroup.

Remark 6.3.2. Furthermore, Jacob and Wegner give in [JW19] a characterization of the generation of C_0-semigroups on the semi-axis. Results for more general networks of port-Hamiltonian systems are obtained in [WW20]. An open problem is still the characterization of (contraction) C_0-semigroups for port-Hamiltonian systems of higher order.

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