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Positivity in perturbation theory and infinite-dimensional systems

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Chapter 1

Introduction

In natural sciences one often has a great interest to describe motions in time mathematically. This interest can be traced back to Galileo Galilei (1564-1642), and way beyond him, who said: "the book of nature is written in mathematical language". A big breakthrough to describe motions in time was done by the *Philosophia Naturalis Principia Mathematica* a publication from Sir Isaac Newton (1643-1727). His work is the foundation of classical mechanics and up to now it is used in macroscopic physics and mechanical engineering. However, the early work of Newton was stated in a somehow raw mathematical description. The more analytical formalism of Newton's laws as well as analytical methods and mathematical developments by Alexis Clairaut, Jean le Rond d'Alembert, Leonhard Euler, and Johann and Jacob Bernoulli where put together by Joseph-Louis de Lagrange in his work *Mécanique Analytique.* Since this publication it was possible to state a great amount of mechanical questions in one single algebraic equation.
An umbrella term for equations that describe motions in time analytically are the so-called evolution equations. These equations can be understood as the differential law of the evolution in time of some system with an initial condition. A typical class of evolution equations is given by one-dimensional ordinary differential equations such as

$$\dot{u}(t) = f(t, u) \qquad u(0) = u_0,$$

where f may be a non-linear function. Other types of evolution equations are partial differential equations (**PDE**) such as the linear transport equation

$$\begin{gathered} \frac{\partial}{\partial t} u(t, x) = \frac{\partial}{\partial x} u(t, x), \\ u(0, x) = u_0(x), \end{gathered}$$

or the heat equation, which is given by the equations

$$\begin{gathered} \frac{\partial}{\partial t} u(t, x) = \left(\frac{\partial}{\partial x}\right)^2 u(t, x), \\ u(0, x) = u_0(x). \end{gathered}$$

The above PDEs are linear and the spacial domain is one-dimensional. Throughout this dissertation we will restrict ourselves to linear evolution equations that can be transformed and expressed by the so-called abstract Cauchy problem (**ACP**), that is

$$\begin{cases} \dot{x}(t) = Ax(t), & t \geqslant 0 \\ x(0) = x_0. \end{cases} \tag{1.1}$$

In the case where A is defined as the first derivative one can express the transport equation via the ACP and if A is defined as the second derivative we get the heat equation. At first glance it seems that one simplified the PDE to an ordinary differential equation, but now the operator A is acting on a function space, which is almost always an infinite-dimensional space. However, the advantage of this setting is that we can adopt a huge background on linear functional analysis, because A is a linear operator and by assumption closed and densely defined. It turns out that this assumption is satisfied in many applications, e.g. the above PDEs still fit in this setting. The ACP and the existence of its solutions is intimately connected to strongly continuous one-parameter semigroups (in the following we say strongly continuous semigroups), because if the operator A is a so-called generator of a strongly continuous semigroup, then the ACP is solvable and moreover this is even a necessary condition to solvability of the ACP. We will give more explanation to semigroups in Chapter II and refer the reader to the monographs by Goldstein [27], Engel and Nagel [24], Pazy [59], Butzer and Berens [13] for more information on semigroup theory and its connection to the ACP.

Since in the early twentieth century S. Banach in cooperation with H. Hahn and E. Helly introduced the Banach space, it evolved a need to give Banach spaces more structure, because in functional analysis most of the applications include function spaces such as the continuous functions, L^p spaces or sequence spaces. Especially if these spaces are real valued it is obvious to add more structure via a pointwise ordering. E.g. take two real-valued continuous functions f and g on some compact set Ω, then the ordering is defined via

$$f \leqslant g \quad \Longleftrightarrow \quad f(t) \leqslant g(t) \qquad \text{for all } t \in \Omega.$$

It was F. Riesz who invented vector lattices, which are ordered vector spaces such that the supremum with respect to the ordering over any two functions exists in this space. These spaces are also called Riesz spaces and it was just the beginning of a great research concerning vector lattices, positive operators, ideals, bands and much more. We refer the reader to the monographs by Schäfer [66], Aliprantis and Burkinshaw [3], Aliprantis and Border [2] for more information on vector lattices and positive operators.
Of course it was only a matter of time that positivity was combined with semigroup theory. We refer to Nagel [56], Banasiak and Arlotti [7] and Bátkai, Fijavž and Rhandi [8] for the theory of positive semigroups.

The first main topic in this thesis is perturbation theory. The idea behind this theory is very simple. We have some equation or problem, that is too difficult

to solve, but one part of it is simple or one already knows the solution. Then, if the other part of the problem, called the perturbation, is "small" or does not highly affect the first part one may get solvability to the whole problem.
Here, in the field of semigroup and operator theory, we interpret perturbation theory in the following way. Given an abstract Cauchy problem (1.1), where we already know that A is a generator of a strongly continuous semigroup, we add a perturbing operator G, i.e. we consider the equations

$$\begin{cases} \dot{x}(t) &= Ax(t) + Gx(t), \qquad \text{for } t \geqslant 0, \\ x(0) &= x_0. \end{cases} \tag{1.2}$$

Now we want to answer the question: does $A+G$ generate a strongly continuous semigroup again? In the simple case were G is a bounded perturbation operator the research goes back to Phillips [60] and was continued by Pazy [58]. But many of the problems in physics cannot be treated with bounded perturbations. For example if we take the point evaluation in some L^2-space as state space. Here in this thesis we consider two kinds of unbounded perturbing operators.

The first one is a so called Miyadera-Voigt perturbation. That is, the domain of A lies in the domain of the perturbing operator G, the operator G is a bounded mapping from the domain of A equipped with the graph norm to the state space and there exists a $\tau > 0$ such that

$$\int_0^\tau \|GT(t)x\| \, \mathrm{d}t \leqslant M\|x\|$$

holds for all x in the domain of A and some constant $0 < M < 1$, where $(T(t))_{t\geqslant 0}$ is the semigroup generated by A. That $A + G$ generates a strongly continuous semigroup was first proved by Miyadera [53] with a slightly different integral condition for the operator G as above. The result from Miyadera was developed further by Voigt [77] including the above integral estimation for G. He later gave applications to this in [78] and [79]. The Miyadera-Voigt perturbation result for locally convex state spaces and equicontinuous semigroups was treated by Dembart [18].
The other kind of unbounded perturbation we will look at is the perturbation of Desch and Schappacher [21]. Here we look at perturbations that have range "larger" than the state space or more precisely we extend the state space such that the perturbing operator is bounded from the state space to the extended space. Further assumptions on these perturbations are made on the following convolution

$$\int_0^\tau T_{-1}(\tau - s)Gf(s) \, \mathrm{d}x,$$

where f is a function with values in a Banach space and $(T_{-1}(t))_{t\geqslant 0}$ the continuously extended semigroup from $(T(t))_{t\geqslant 0}$ to the extended state space. As in the Miyadera-Voigt case there exists a result for Desch-Schappacher perturbations with equicontinuous semigroups on locally convex spaces from Jacob, Wegner and Wintermayr [39]. Also we mention that Nagel and Engel used the so-called

Volterra operator and proved herewith both perturbation results in a more abstract way (see [24], Chapter III, Section 3a and 3c). In Chapter III we consider both of the above discussed perturbation cases with positive perturbations.

Another huge field for strongly continuous semigroups can be found in system theory and control theory. Here we extend the abstract Cauchy equation (1.1) to the following two systems

$$\Sigma(A,B) \quad \begin{cases} \dot{x}(t) = Ax(t) + Bu(t), & t \geqslant 0, \\ x(0) = x_0, \end{cases} \tag{1.3}$$

and

$$\Sigma(A,C) \quad \begin{cases} \dot{x}(t) = Ax(t), & t \geqslant 0, \\ y(t) = Cx(t), & t \geqslant 0, \\ x(0) = x_0. \end{cases} \tag{1.4}$$

The operator B is called the control operator and $u(\cdot)$ the input function, whereas C is the observation operator and $y(\cdot)$ the output function. In $\Sigma(A,C)$ we observe the behaviour of the system via the observation operator C. Here, we will not consider the simple case were C is a bounded operator. Instead of this we look at unbounded observation operators, which have a great similarity to Miyadera-Voigt perturbations, if we check admissibility for the observation operator, that is, the output function $y(t) = CT(t)$ should be in L^p. A lot of research on unbounded observation operators was done by Lasiecka and Triggiani [46], Pritchard and Salamon [61], Pritchard and Wirth [62], Salamon [64] [65], Seidman [68], Yamamoto [87], Weiss [83] and many more.

On the other side if we look at the system $\Sigma(A,B)$ we can control the behaviour of this system via the control operator. As above we drop the simple case for bounded control operators and consider unbounded ones. In contrast to the observation operators, unbounded control operators with values in an extrapolation space are highly related to Desch-Schappacher perturbations, if we investigate such control operators on admissibility, i.e. the state remains in the state space for each time. In the literature there is a huge amount of publications that investigate systems with unbounded control operators, e.g. Curtain and Pritchard [14], Curtain and Salomon [15], Desch, Lasiecka and Schappacher [20], Ho and Russel [32], Lasiecka [44], Lasiecka and Triggiani [45], [47], [48], Pritchard and Wirth [62], Salomon [64], [65] Weiss [82] and many others.

In this thesis we will look at positive unbounded observation and especially positive unbounded control operators and we will simplify conditions on admissibility that already exist for these operators. Moreover, we have a closer look at zero-class admissibility. This topic is a more recent study and can be found in the work of Jacob, Partington and Pott [36], Jacob, Schwenninger and Zwart [37], Nabiullin and Schwenninger [55] and Jacob, Nabiullin, Partington and Schwenninger [33], where the last two deal with Orlicz spaces.

Clearly, there is a big interest to combine the two systems $\Sigma(A,B)$ and $\Sigma(A,C)$, such that one can observe and control the behaviour of this new linear system. If B and C are bounded operators one can consider

$$\Sigma(A,B,C,D) \quad \begin{cases} \dot{x}(t) = Ax(t) + Bu(t), & t \geqslant 0 \\ y(t) = Cx(t) + Du(t), & t \geqslant 0, \\ x(0) = x_0, \end{cases} \tag{1.5}$$

where we added the so-called feedthrough operator $D \in \mathcal{L}(U,Y)$. However, if B and C are unbounded operators we need the concept of well-posedness for such systems. The research on well-posedness for linear systems with unbounded control and observation operators goes back to Salomon [65] and since then was developed further by Arov and Nudelman [6], Staffans [70], [71], Weiss [81], [85], Curtain and Weiss [16], Staffans and Weiss [73] and many others.
Also we mention that well-posedness of the above system $\Sigma(A,B,C,D)$ can be characterized via a semigroup approach (see K.-J. Engel and M. Bombieri [11]). However, we will follow the book of Staffans [72] and adopt his notation and results for regular and well-posed systems. Then in the last part of this dissertation we will define positivity of such systems and give necessary and sufficient conditions to this positivity.
For more general information about the theory for linear systems we refer to the monographs by Curtain and Pritchard [14], Curtain and Zwart [17], Zabczyk [88], Fuhrmann [26], Staffans [72] and Bensoussan, Da Prato, Delfour and Mitter [10].

In the following we give a short description of how this thesis is structured. After the introduction we begin with the collection of a few facts and results concerning semigroup theory. This collection, which can be almost completely found in [24], will contain the fundamentals on semigroups, generators of strongly continuous semigroups and the important link to the resolvent of such generator. Moreover, we will have a short look at inter- and extrapolation spaces for semigroups, sectorial operator, bounded analytic semigroups, the dual and the sun dual semigroup and finally the zero-one law for semigroups. Subsequently, we give an explanation to the so called well-posed abstract Cauchy system, which leads directly to generators of strongly continuous semigroups. After that, we introduce the concept of an ordering on vector spaces and see that most function spaces are vector lattices, also known as Riesz spaces. Banach and Fréchet lattices can be defined if the ordering is compatible with the norm, respectively the topology, on the vector space. If one has an ordering on a vector space it is self-evident to define positive operators and also positive semigroups. The statements for vector lattices and positive semigroups are mainly collected from [66] and [8]. We close the preliminaries on positivity with the development of an ordering on the extrapolation space for a positive semigroup. This is the result of a recent study on positive semigroups and is contained in [9]. At the end of this chapter we give a short overview to Orlicz spaces.
In Chapter III we begin with indicating some known results for perturbations of Miyadera-Voigt and Desch-Schappacher type. Next, we introduce resolvent positive operators and their properties, where we only look at positive perturbation

operators of the form $C \in \mathcal{L}(E_1, E)$ and $B \in \mathcal{L}(E, E_{-1})$. Here E_1 is the interpolation space and E_{-1} the extrapolation space for the semigroup. The first one, operator C, will be treated for the perturbation type of Miyadera-Voigt and the second one, operator B, for the perturbation type of Desch-Schappacher. In the Miyadera-Voigt case one has to assume that the positive semigroup and the positive perturbation are acting on an AL-space. This result is not new and is due to Desch [19] and developed by Voigt [80] in the late 80's. The other perturbation type requires AM-spaces and is a recent work from Batkai, Jacob, Voigt and Wintermayr [9]. We end this chapter with examples. The first one treats a resolvent positive operator mapping $D(A)$ to an AM-space. Then we give an application to our main perturbation result for AM-spaces. The last of these examples shows that a Desch-Schappacher perturbation is not necessarily positive, if the composition of the resolvent (of the generator of a positive semigroup) and this perturbation is positive.
In the last chapter we focus on system theory with linear operators (cf. (1.3) and (1.4)). We will always start with a well-posed abstract Cauchy system, that is, we assume that the operator A from the equations (1.1) is a generator of a strongly continuous semigroup. Then we add a control or an observation operator to this setting and describe what we mean by admissibility. Our research in this chapter starts with range conditions for control operators. Then we adapt the results from perturbation theory to this setting. Here the type of Miyadera-Voigt perturbation will cover the case for positive observation operator with an AL-space as observation space and the other type, the Desch-Schappacher perturbation, will cover the case for positive control operators with an AM-space as control space. Next, we prove that every generator that is also L^∞-admissible is bounded if we assume that the state space is the sequence space c_0. After this, we state equivalent conditions for admissibility on Hilbert spaces and show that the constant for the admissibility condition for control operators is bounded by the constant for the Weiss conjecture. These are the dual parts to already known results concerning observation operators in Hilbert spaces (see [36]). Further, we specify admissibility to zero-class admissibility and show that on finite-dimensional control spaces, positive control operators are also zero-class admissible for positive semigroups. Moreover, we will see that regulated admissibility does not imply regulated zero-class admissibility. This will be followed by an example to zero-class admissibility for a class of Orlicz functions. Then we will develop again the dual part to the paper [36] to conditions for zero-class admissible control operators. Next, we give an intermezzo to interpolation theory and define the degree of unboundedness for control and observation spaces. Finally, we treat linear systems that can be expressed by (1.5). First we have to introduce a lot of known results regarding well-posed linear systems and linear systems that are regular. The facts are all collected by the monograph from Staffans [72]. At the very end we define positive systems in this setting.

Chapter 2

Preliminaries

In this chapter we will give a short description over all mathematical knowledge which will be needed for this thesis. First we look at strongly continuous semigroups and their connection to the abstract Cauchy problem.

2.1 Strongly continuous semigroups

We begin this section with the following functional equations

$$T(t)T(s) = T(t+s) \tag{2.1}$$
$$T(0) = I \tag{2.2}$$

for all $t, s > 0$ and a family of linear operators $(T(t))_{t\geqslant 0}$. The functional equations are just algebraic properties and clearly not enough to handle evolution equations and especially partial differential equations. The exponential function is probably the most common function in analysis that satisfies equations (2.1) and (2.2). Moreover, we know that it is also differentiable, even if we consider

$$e^{At} := \sum_{k=1}^{\infty} \frac{(At)^k}{k!}$$

for some bounded operator $A \in \mathcal{L}(X)$ on a Banach space X (cf. [24, Proposition 3.5, Chapter I]). However, in order to add differentiability to the functional equations (2.1) and (2.2) we will make a more general assumption, namely the strong continuity in zero for the family of operators $(T(t))_{t\geqslant 0}$:

$$\lim_{t\searrow 0} \|T(t)x - x\| = 0 \text{ for all } x \in X. \tag{2.3}$$

This weaker property allows us to consider more applications that fit in the setting of such families, called strongly continuous semigroups.

Definition 2.1.1 *Let $(T(t))_{t\geqslant 0}$ be a family of bounded operators on a Banach space X, we call $(T(t))_{t\geqslant 0}$ a* strongly continuous semigroup *if it satisfies the functional equations* (2.1) *and* (2.2) *and the family is strongly continuous in zero for each element of X, i.e.* (2.3) *holds.*

If $(T(t))_{t\in\mathbb{R}}$ is a family of bounded operators on a Banach space X. We call $(T(t))_{t\in\mathbb{R}}$ a strongly continuous group *if it satisfies the functional equations* (2.1) *and* (2.2) *for all $t, s \in \mathbb{R}$ and the family is strongly continuous in zero (for the left and right limit) for each element of X, i.e.*

$$\lim_{t\to 0} \|T(t)x - x\| = 0 \text{ for all } x \in X \tag{2.4}$$

holds.

For any linear operator G on some Banach space X, we denote by $D(G)$ the *domain* of G.

Definition 2.1.2 *Let $(T(t))_{t\geqslant 0}$ be a strongly continuous semigroup on a Banach space X. Then we define the operator*

$$Ax := \lim_{t\searrow 0} \frac{1}{h}\left(T(h)x - x\right),$$
$$D(A) = \left\{x \in X \,:\, \lim_{t\searrow 0} \frac{1}{h}\left(T(h)x - x\right) \text{ exists in } X\right\},$$

where the limit is taken in the norm topology of X. The operator $(A, D(A))$ is called the generator *of the semigroup $(T(t))_{t\geqslant 0}$.*

Remark 2.1.3 *It is also quite common that $(A, D(A))$ is called the* infinitesimal generator *of a strongly continuous semigroup. However, we always say generator in the following.*

There are a lot of properties concerning strongly continuous semigroups, which one can collect only using the functional equations, the strong continuity and the definition of the generator. Some of these important properties are listed next.

Proposition 2.1.4 *Let $(T(t))_{t\geqslant 0}$ be a strongly continuous semigroup on a Banach space X with generator $(A, D(A))$. Then the following statements hold:*

i) There exist constants $M \geqslant 1$and $\omega \in \mathbb{R}$ such that for all $t \geqslant 0$

$$\|T(t)\| \leqslant Me^{\omega t}. \tag{2.5}$$

ii) The operator $(A, D(A))$ is linear, closed, densely defined and determines its semigroup uniquely.

iii) For every $t \geqslant 0$ and $x \in X$ we have

$$\int_0^t T(s)x \,\mathrm{d}x \in D(A) \quad \text{and} \quad T(t)x - x = A\int_0^t T(s)x\,\mathrm{d}x.$$

iv) If $x \in D(A)$, then $T(t)x \in D(A)$ and for all $t \geqslant 0$ we have

$$\frac{\mathrm{d}}{\mathrm{d}t}T(t)x = AT(t)x = T(t)Ax \quad \text{and} \quad T(t)x - x = \int_0^t T(s)Ax\,\mathrm{d}s.$$

Proof: See [24, Proposition 5.5, Chapter I] and [24, Lemma 1.3 and Theorem 1.4, Chapter II]. ■

Remark 2.1.5 *The integrals in the above proposition are defined as* Bochner integrals. *We refer to [2, Chapter 11] or [22] on this topic. Throughout this dissertation we will interpret every vector valued integral as a Bochner integral unless we state it otherwise.*

As commonly used we denote by $\rho(A)$ the *resolvent set* of a linear operator A and by $\sigma(A)$ its *spectrum*. For each $\lambda \in \rho(A)$ we denote the *resolvent* of A by $R(\lambda, A) := (\lambda - A)^{-1}$.

Definition 2.1.6 *We define*

$$\omega_0 := \inf\{\omega \in \mathbb{R} : \text{ such that Inequality (2.5) holds for some } M \geqslant 1\}$$

and call ω_0 the growth bound *for the semigroup $(T(t))_{t\geqslant 0}$. If $\omega_0 < 0$ holds, then the corresponding semigroup is called* exponentially stable*. Further, we denote by*

$$s(A) := \sup\{\operatorname{Re}\lambda : \lambda \in \sigma(A)\}$$

the spectral bound *for any linear operator A.*

The resolvent of a generator can be explicitly calculated with the semigroup using the Laplace transform.

Proposition 2.1.7 *Let $(T(t))_{t\geqslant 0}$ be a strongly continuous semigroup with generator $(A, D(A))$ and growth bound ω_0. Then for all $\operatorname{Re}\lambda > \omega_0$ we have $\lambda \in \rho(A)$ and*

$$R(\lambda, A)x = \int_0^\infty e^{-\lambda s} T(s)x \, \mathrm{d}s \tag{2.6}$$

exists for all $x \in X$. Moreover, we have

$$\|R(\lambda, A)\| \leqslant \frac{M}{\operatorname{Re}\lambda - \omega},$$

where M and $\omega < \operatorname{Re}\lambda$ are the constants from Inequality (2.5).

Proof: See [24, Theorem 1.10, Chapter II]. ■

It is also possible to construct the semigroup via the resolvent of the generator.

Proposition 2.1.8 *Let $(T(t))_{t\geqslant 0}$ be a strongly continuous semigroup with generator $(A, D(A))$. Then, we have for every $x \in X$*

$$T(t)x = \lim_{n\to\infty} \left(\frac{n}{t} R\left(\frac{n}{t}, A\right)\right)^n x = \lim_{n\to\infty} \left(I - \frac{t}{n} A\right)^{-n} x, \tag{2.7}$$

uniformly for t in compact intervals.

Proof: See [24, Corollary 5.5, Chapter III]. ∎

In semigroup theory it often arises the question: given a densely defined closed linear operator $(A, D(A))$, does this operator generate a semigroup? One of the first famous result answering this question was proven by Hille and Yosida for contraction semigroups (cf. [24, Theorem 3.5, Chapter II]). Here, we will not go into detail for contraction semigroups and present the more general case next. The following version is due to Feller, Miyadera and Phillips (see [25], [52] and [60]).

Theorem 2.1.9 *Let $(A, D(A))$ be a linear operator on a Banach space X and let $\omega \in \mathbb{R}$, $M \geqslant 1$ be constants. Then the following properties are equivalent.*

i) $(A, D(A))$ generates a strongly continuous semigroup satisfying

$$\|T(t)\| \leqslant Me^{\omega t} \qquad \text{for } t \geqslant 0.$$

ii) $(A, D(A))$ is closed, densely defined, and for every $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > \omega$ one has $\lambda \in \rho(A)$ and

$$\|R(\lambda, A)^n\| \leqslant \frac{M}{(\operatorname{Re}\lambda - \omega)^n}$$

holds for all $n \in \mathbb{N}$.

Proof: See [24, Theorem 3.8, Chapter II]. ∎

Lemma 2.1.10 *Let $(A, D(A))$ be a closed, densely defined operator. Assume that there exists $M \geqslant 1$ such that for all $\lambda > 0$, we have $\lambda \in \rho(A)$ and $\|\lambda R(\lambda, A)\| \leqslant M$. Then*

(i) for all $x \in X$ we have $\lambda R(\lambda, A)x \to x$ for $\lambda \to \infty$,

(ii) for all $x \in D(A)$ we have $\lambda AR(\lambda, A)x \to Ax$ for $\lambda \to \infty$,

where the limits are taken in the norm topology.

Proof: See [24, Lemma 3.4, Chapter II]. ∎

Definition 2.1.11 *Let $(T(t))_{t\geqslant 0}$ be a strongly continuous semigroup on a Banach space X with generator $(A, D(A))$. We define the* interpolation space X_1 *for this semigroup via*

$$X_1 := (D(A), \|\cdot\|_1),$$

where

$$\|x\|_1 := \|x\| + \|Ax\|,$$

for all $x \in D(A)$ and the operators $T(t)$ restricted to the space X_1 by

$$T_1(t) := T(t)_{|X_1} \qquad \text{for all } t \geqslant 0.$$

On the other hand we define the extrapolation space X_{-1} *for the semigroup* $(T(t))_{t\geqslant 0}$ *as the completion*

$$X_{-1} := (X, \|\cdot\|_{-1})^{\sim},$$

where the norm is given by

$$\|x\|_{-1} := \|R(\lambda, A)x\|,$$

for every $x \in X_{-1}$ *and some* $\lambda \in \rho(A)$ *and we denote the continuous extension of the operators* $T(t)$ *to the space* X_{-1} *by* $T_{-1}(t)$ *for all* $t \geqslant 0$.

Proposition 2.1.12 *Let* $(T(t))_{t\geqslant 0}$ *be a strongly continuous semigroup on a Banach space* X *with generator* $(A, D(A))$ *and let the inter- and extrapolation spaces* X_1 *and* X_{-1} *be given as above. Then, we obtain the following statements.*

i) X_1 *and* X_{-1} *are Banach spaces.*

ii) The families of operators $(T_1(t))_{t\geqslant 0}$ *and* $(T_{-1}(t))_{t\geqslant 0}$ *form strongly continuous semigroups on the spaces* X_1, X_{-1} *respectively.*

iii) The generator A_1 *of* $(T_1(t))_{t\geqslant 0}$ *is given by the part of* A *in* X_1, *i.e.*

$$\begin{aligned} A_1 x &= Ax \qquad \text{for } x \in D(A_1) \text{ with} \\ D(A_1) &= \{x \in X_1 \,:\, Ax \in X_1\}. \end{aligned}$$

iv) The generator A_{-1} *of* $(T_{-1}(t))_{t\geqslant 0}$ *has domain* $D(A_{-1}) = X$ *and is the unique continuous extension of* $A : D(A) \to X$ *to an isometry from* X *onto* X_{-1}.

Proof: See [24, Proposition 5.2 and Theorem 5.5, Chapter II]. ■

Remark 2.1.13 *The inter- and extrapolation spaces can be inductively extended to spaces* X_n *for all* $n \in \mathbb{Z}$. *This construction is called* Sobolev towers *and we refer the reader to [24, Section 5, Chapter II] for more information on this topic.*
If $\omega_0 < 0$, *then we can define* $\|x\|_1 = \|Ax\|$ *and* $\|x\|_{-1} := \|A_{-1}^{-1}x\|$ *for* $x \in D(A)$, $x \in X_{-1}$ *respectively. This follows from the fact that for every* $\mu \in \rho(A)$ *the norms given in the above definition are equivalent to* $\|\cdot\|_{1,\mu} := \|(\mu - A)\cdot\|$, $\|\cdot\|_{-1,\mu} := \|R(\mu, A_{-1})\cdot\|$ *respectively (cf. [24, Exercise 5.9, Chapter II]).*

The next statement gives the opportunity to rescale a semigroup. Especially, one can shift the spectrum of a generator and therefore the growth bound of its semigroup with this rescaling procedure.

Proposition 2.1.14 *If* $(A, D(A))$ *is the generator of a strongly continuous semigroup* $(T(t))_{t\geqslant 0}$, *then for all* $\lambda \in \mathbb{R}$ *we have that* $(A - \lambda, D(A))$ *is the generator of the strongly continuous semigroup* $(e^{-\lambda t}T(t))_{t\geqslant 0}$. *In particular, we have* $\sigma(A - \lambda) = \sigma(A) - \lambda$.

Definition 2.1.15 *Let $(T(t))_{t\geqslant 0}$ be a strongly continuous semigroup on a Banach space X. The family $(T(t)')_{t\geqslant 0}$ consisting of all adjoint operators $T(t)'$ for all $t \geqslant 0$ on the dual space X' is called the* adjoint semigroup*. Because the adjoint semigroup is not always strongly continuous, we define the* sun dual semigroup *via $T(t)^{\odot} := T(t)'_{|X^{\odot}}$ for all $t \geqslant 0$, where*

$$X^{\odot} := \left\{ x' \in X' \,:\, \lim_{t\searrow 0} \left\| T(t)'x' - x' \right\| = 0 \right\}.$$

Proposition 2.1.16 *If $(T(t))_{t\geqslant 0}$ is a strongly continuous semigroup on a reflexive Banach space X. Then the adjoint semigroup is again strongly continuous, i.e. $X' = X^{\odot}$.*

Proof: See proof of [24, Corollary 5.21, Chapter II]. ■

Definition 2.1.17 *A closed linear densely defined operator $(G, D(G))$ on a Banach space X is called* sectorial *(of angle δ) if there exists $0 < \delta \leqslant \frac{\pi}{2}$ such that the sector*

$$\Sigma_{\frac{\pi}{2}+\delta} := \left\{ \lambda \in \mathbb{C} \,:\, |\arg \lambda| < \frac{\pi}{2} + \delta \right\} \setminus \{0\}$$

is contained in the resolvent set $\rho(G)$, and if for each $\epsilon \in (0, \delta)$ there exists $M_\epsilon \geqslant 1$ such that

$$\|R(\lambda, G)\| \leqslant \frac{M_\epsilon}{|\lambda|}$$

for all $0 \neq \lambda \in \overline{\Sigma}_{\frac{\pi}{2}+\delta-\epsilon}$.

Remark 2.1.18 *We mention that in the literature, especially in interpolation theory, sectorial operators were defined such that their spectrum lies in the right half plane of $\mathbb{C}$ (see e.g. [31] page 19 or [59] page 69). In the above definition it is obvious that the spectrum of a sectorial operator lies in the left half plane of $\mathbb{C}$, this fits to the setting of generators. If one choose the other definition for a sectorial operator, then one gets that $-A$ is a generator, respectively if A generates a bounded analytic semigroup then $-A$ is sectorial.*

Definition 2.1.19 *Let $(A, D(A))$ be a sectorial operator with $0 \in \rho(A)$. Then for every $\alpha > 0$ we define for $0 < \lambda$ the* fractional power

$$(-A)^{-\alpha} := \frac{1}{2\pi i} \int_\gamma \lambda^{-\alpha} R(\lambda, A) \,\mathrm{d}\lambda, \tag{2.8}$$

where the path γ runs in the resolvent set of $(A, D(A))$ from $\lim_{r\to\infty} re^{-i\psi}$ to $\lim_{r\to\infty} re^{i\psi}$, for $\omega < \psi < \pi$, avoiding the negative real axis and the origin. Moreover, we define

$$(-A)^{\alpha} := \left((-A)^{-\alpha}\right)^{-1}.$$

Remark 2.1.20 *The bounded operator in equation* (2.8) *can be defined for a more general class of closed operators, namely operators* $(A, D(A))$ *with* $(0, \infty) \subset \rho(A)$ *and*

$$\|R(\lambda, A)\| \leqslant \frac{M}{1+\lambda}$$

for $\lambda \in (0, \infty)$ *and some constant* $M > 0$. *See e.g. [24, page 137].*

Theorem 2.1.21 *Let* $(A, D(A))$ *be a sectorial operator, then the following holds.*

i) $(-A)^\alpha$ *is a closed operator with domain* $D((-A)^\alpha) = \mathrm{Ran}((-A)^{-\alpha})$ *for every* $\alpha > 0$,

ii) $\alpha \geqslant \beta > 0$ *implies* $D((-A)^\alpha) \subset D((-A)^\beta)$,

iii) $\overline{D(-A)^\alpha)} = X$ *for every* $\alpha \geqslant 0$.

Proof: See [59, Theorem 6.8, Chapter 2] and consider $-A$ instead of A (cf. Remark 2.1.18). ■

Definition 2.1.22 *A family of operators* $(T(z))_{z \in \Sigma_\delta \cup \{0\}} \subset \mathcal{L}(X)$ *is called an* analytic semigroup *(of angle* $\delta \in (0, \frac{\pi}{2}]$*) if the following statements are true.*

i) $T(0) = I$ *and* $T(z_1 + z_2) = T(z_1)T(z_2)$ *for all* $z_1, z_2 \in \Sigma_\delta$.

ii) The map $z \mapsto T(z)$ *is analytic in* Σ_δ.

iii) $\lim\limits_{\substack{z \in \Sigma_{\delta'} \\ z \to 0}} T(z)x = x$ *for all* $x \in X$ *and* $0 < \delta' < \delta$.

We call $(T(z))_{z \in \Sigma_\delta \cup \{0\}}$ *a* bounded analytic semigroup, *if in addition* $\|T(z)\|$ *is bounded for all* $z \in \Sigma_{\delta'}$ *and every* $0 < \delta' < \delta$.

A direct connection between bounded analytic semigroups and sectorial operators is given by the theorem below.

Theorem 2.1.23 *Let* $(A, D(A))$ *be a linear operator on a Banach space* X. *Then* A *is sectorial if and only if* A *generates a bounded analytic semigroup* $(T(z))_{z \in \Sigma_\delta \cup \{0\}}$ *on* X.

Proof: See [24, Theorem 4.6, Chapter II]. ■

Theorem 2.1.24 *Let* $(A, D(A)$ *be the generator of an analytic semigroup* $(T(t))_{t \geqslant 0}$. *If* $0 \in \rho(A)$ *and* $0 < \alpha \leqslant 1$ *we have*

i) $\mathrm{Ran}(T(t)) \subset D((-A)^\alpha)$ *for all* $t > 0$.

ii) For every $x \in D((-A)^\alpha)$ *we have* $T(t)(-A)^\alpha x = (-A)^\alpha T(t)x$.

iii) For every $t > 0$ *the operator* $(-A)^\alpha T(t)$ *is bounded and there exists a constant* $\delta > 0$, *such that*

$$\|(-A)^\alpha T(t)\| \leqslant M_\alpha t^{-\alpha} e^{-\delta t}.$$

Proof: See [59, Theorem 6.13, Chapter 2] and take $-A$ instead of A (cf. Remark 2.1.18). ∎

At last, we state the so called zero-one-law for semigroups.

Proposition 2.1.25 *Let $(A, D(A))$ be the generator of a strongly continuous semigroup $(T(t))_{t\geqslant 0}$ on a Banach space X. Define $L := \limsup_{t\to 0} \|T(t) - I\|$. If $L < 1$, then $L = 0$.*

Proof: The proof can be found in [56, Chapter A-II, Lemma 3.1]. Because it is a very nice one we state it here, too. We calculate

$$(T(t) - I)^2 = T(2t) - 2T(t) + I \quad \Longleftrightarrow \quad 2T(t) = T(2t) + I - (T(t) - I)^2.$$

Keeping this in mind we get

$$2(T(t) - I) = 2T(t) - 2I = T(2t) - I - (T(t) - I)^2.$$

It follows with the triangular inequality

$$2\|T(t) - I\| \leqslant \|T(2t) - I\| + \|T(t) - I\|^2.$$

Then with the definition of L we get

$$2L \leqslant L + L^2.$$

Finally, this shows that either $L \geqslant 1$ or $L = 0$. ∎

Remark 2.1.26 *If $L := \limsup_{t\to 0} \|T(t) - I\| = 0$, then the semigroup is uniformly (norm) continuous for all $t \geqslant 0$, because of*

$$\|T(s) - T(t)\| \leqslant \|T(t)\| \cdot \|T(s-t) - I\| \text{ for } s > t > 0. \tag{2.9}$$

Therefore its generator is bounded (see e.g. [24, Theorem 3.7]).

2.1.1 Abstract Cauchy problem

Recall the equations from the introduction for the abstract Cauchy problem.

$$(ACP) \quad \begin{cases} \dot{u}(t) & = Au(t), \qquad \text{for } t \geqslant 0, \\ u(0) & = u_0. \end{cases}$$

It is evident to require existence and uniqueness of the solutions for the above equations. Here, we add that the solutions should depend continuously on the initial value. This leads us to the definition of well-posedness.

Definition 2.1.27 *The abstract Cauchy problem (ACP) is called* well-posed *if the associated closed operator $(A, D(A))$ fulfills the following conditions*

i) For every $x \in D(A)$, there exists a unique solution $u(\cdot, x)$ of (ACP).

ii) $D(A)$ is dense in X.

iii) For every sequence $(x_n)_{n\in\mathbb{N}} \subset D(A)$ *converging to zero, i.e.* $\lim_{n\to\infty} x_n = 0$, *we have* $\lim_{n\to\infty} u(t, x_n) = 0$ *uniformly in compact intervals* $[0, t_0]$.

The next statement connects every well-posed abstract Cauchy system to a generator of a strongly continuous semigroup.

Theorem 2.1.28 *Let* $(A, D(A))$ *be a closed operator. The associated abstract Cauchy problem (ACP) is well-posed if and only if* $(A, D(A))$ *is the generator of a strongly continuous semigroup on* X.

Proof: See [24, Corollary 6.9, Chapter II]. ∎

2.2 Positivity

This section is dedicated to the theory of vector lattices and positive operators. We start with basic facts from an ordered set and end with Banach and Fréchet lattices. After that we will focus on positive semigroups and positive resolvents. We end this section with introducing an ordering on the extrapolation space for a positive strongly continuous semigroup.

2.2.1 Vector lattices

Let M be a set and "$\leqslant$" be a binary relation for which we assume that it is

transitive:	$x \leqslant y$ and $y \leqslant z \Rightarrow x \leqslant z$ for all $x, y, z \in M$,
reflexive:	$x \leqslant x$ for all $x \in M$,
anti-symmetric:	$x \leqslant y$ and $y \leqslant x \Rightarrow x = y$ for all $x, y \in M$.

We call $(M, \leqslant)$ an *ordered set.* If we write $x \geqslant y$, we mean $y \leqslant x$ and with $x < y$ we express that $x \leqslant y$ and $x \neq y$ holds. Similarly for $x > y$ we mean $y < x$.

Definition 2.2.1 *We call* E *an* ordered vector space, *if* E *is a vector space over* $\mathbb{R}$ *endowed with an order relation "*$\leqslant$*" which is compatible with the structure of the vector space, i.e. the following axioms are satisfied:*

a) $x \leqslant y \Rightarrow x + z \leqslant y + z$ *for all* $x, y, z \in E$,

b) $x \leqslant y \Rightarrow \lambda x \leqslant \lambda y$ *for all* $x, y \in E$ *and* $\lambda \in \mathbb{R}_+$.

If additionally $x \vee y := \sup\{x, y\}$ *and* $x \wedge y := \inf\{x, y\}$ *exist in* E *for all* $x, y \in E$, *then* E *is called a* vector lattice. *In a vector lattice, we can define*

$$x_+ := \sup\{x, 0\}, \quad x_- := \sup\{-x, 0\} \quad \textit{and} \quad |x| := \sup\{x, -x\}.$$

We denote by $E_+ := \{x \geqslant 0 : x \in E\}$ *the* positive cone *of* E *and we call elements* $x \in E_+$ positive.

Proposition 2.2.2 *Let E be a vector lattice. Then we have*

$$|x| = x_+ + x_- \quad \textit{and} \quad x = x_+ - x_-$$
$$|x| = 0 \Leftrightarrow x = 0; \quad |\lambda x| = |\lambda| \cdot |x|; \quad |x+y| \leqslant |x| + |y|$$
$$x + y = \sup\{x, y\} + \inf\{x, y\}$$
$$|x - y| = \sup\{x, y\} - \inf\{x, y\}$$

for all $x, y \in E$ and $\lambda \in \mathbb{R}$.

Proof: See [66, Proposition 1.4, Chapter II]. ■

Remark 2.2.3 *The positive cone is a convex cone (recall that a set C of a vector space is a convex cone if $C + C \subseteq C$ and $\alpha C \subseteq C$ for every $\alpha \geqslant 0$ hold). It is easy to show that any convex cone C defines an ordering via*

$$x \leqslant y \Longleftrightarrow y - x \in C. \tag{2.10}$$

The positive cone E_+ is generating if $E_+ - E_+ = E$. In a vector lattice the positive cone is generating. This is a consequence of the above proposition. On the other hand, if for a convex cone C in a vector space we have additionally $C \cap (-C) = \{0\}$, then the order induced by the convex cone defines a vector lattice.

We mention that vector lattices are also known as Riesz spaces. Moreover, it is also possible to define an order on complex-valued vector space (cf. [56, page 243] or [66, Paragraph 11, Chapter II]), but then we do not have that for every element $x \in X$ in a complex Banach lattice there exists a unique decomposition $x = x_+ - x_-$, where x_+ and x_- are positive. Since the main proofs in this dissertation use this decomposition, we consider throughout this thesis real-valued vector lattices.

Definition 2.2.4 *Let E be a vector lattice and A a subset of E. We call A* solid*, if $|x| \leqslant |y|$ for $y \in A$, $x \in E$ implies $x \in A$. A solid vector subspace I is called an* ideal *(or* lattice ideal*) of E.*

Clearly, any ideal is intersection invariant and we denote by $I(B)$ for some subset B of E the smallest ideal of E that contains B, called the *ideal generated by B*.

Definition 2.2.5 *The ideal generated by one single element $u \in E$ is called* principal ideal *and is denoted by E_u. If $E_u = E$ holds for some $u \in E_+$ we call u an* order unit*.*

2.2.2 Banach and Fréchet lattices

Definition 2.2.6 *Let E be a Banach space with an ordering, such that $(E, \leqslant)$ is a vector lattice. If the norm is compatible with the ordering "$\leqslant$", i.e. for all elements $x, y \in X$ we have*

$$|x| \leqslant |y| \Rightarrow \|x\| \leqslant \|y\|, \tag{2.11}$$

then X is called a Banach lattice*. Every norm in E for which* (2.11) *holds for every $x, y \in E$ is called a* lattice norm*.*

Proposition 2.2.7 *If E is a Banach lattice, then the dual space E' is again a Banach lattice with the order*

$$f' \leqslant g' \Longleftrightarrow f'(x) \leqslant g'(x)$$

for all $x \in E_+$ and all $f', g' \in E'$.

Proof: See [2, Theorem 8.48 and Theorem 9.11]. ■

Next we define special classes of Banach lattices which will play an important role for future chapters.

Definition 2.2.8 *If the norm on a Banach lattice E satisfies*

$$\| \sup\{f, g\}\| = \sup\{\|f\|, \|g\|\} \tag{2.12}$$

for all $f, g \in E_+$, then the Banach lattice E is called an abstract M-space *or an* AM-space*. If the norm on a Banach lattice E satisfies*

$$\|f + g\| = \|f\| + \|g\| \tag{2.13}$$

for all $f, g \in E_+$, then the Banach lattice E is called an abstract L-space *or an* AL-space*.*

Example 2.2.9 *Let K be a compact set and (Ω, Σ, μ) a measure space. Then the function spaces $C(K)$ and $L^\infty(\Omega)$ are AM-spaces with order unit. Let Ω be a locally compact Hausdorff space and define*

$$C_0(\Omega) := \left\{ f \in C(\Omega) \; : \; \begin{array}{l} \text{for all } \epsilon > 0 \text{ exists a compact set } K_\epsilon \subset \Omega \\ \text{such that } |f(s)| < \epsilon \text{ holds for all } s \in \Omega \setminus K_\epsilon \end{array} \right\}.$$

Then $C_0(\Omega)$ is an AM-space without order unit. Also c_0, the sequence space of all null-sequences, is an AM-space without order unit, whereas l^∞, the space of all bounded sequences, is an AM-space with order unit.
$\mathrm{L}^1(\Omega)$ *and l^1 are AL-spaces, where (Ω, Σ, μ) is a measure space with σ-finite measure.*

Theorem 2.2.10 *The dual of an AM-space is an AL-space and the dual of an AL-space is an AM-space with order unit.*

Proof: See [66, Proposition 9.1, Chapter II]. ■

Remark 2.2.11 *From basic functional analysis it is well-known that a* Fréchet space *is a completely metrizable locally convex Hausdorff space. Recall, that a* locally convex space *is a topological vector space in which every neighborhood of zero includes a convex neighborhood of zero. A* topological vector space *is a vector space with a topology τ such that addition and scalar multiplication are τ-continuous. In this case τ is called a* linear topology*. It can be shown* *that in a locally convex space every linear topology τ is generated by a family of seminorms. Furthermore, one can show that for every Fréchet space there exists a sequence of seminorms that generate its topology. We refer the reader to [50, Chapter III], [42, Chapter II] or [2, Chapter V] for more details on topological vector spaces and especially Fréchet spaces.*

Definition 2.2.12 *Let E be a vector lattice and τ be a linear topology on E. If τ has a base at zero consisting of solid neighborhoods we say that τ is* locally solid. *In this case we call (E, τ) a* locally solid vector lattice.
A Fréchet lattice *is a completely metrizable locally solid vector lattice.*

Theorem 2.2.13 (Fischer-Riesz) *Let (Ω, Σ, μ) be a measure space with σ-finite measure μ. Then $L^p(\Omega)$ is a Banach lattice for every $p \in [1, \infty]$.*

Proof: See [2, Theorem 13.5]. ■

Theorem 2.2.14 *Let (Ω, Σ, μ) be a measure space with finite measure and E be a Banach lattice. Then $L^p(\Omega, E)$ is a Banach lattice for every $p \in [1, \infty]$.*

Proof: This is a special case for [28, Theorem 3.2, Chapter 3]. Here $L_\rho(E)$ is a Banach lattice if and only if L_ρ is a Banach lattice, where E is a Banach lattice and ρ a generalized function norm on all Bochner measurable functions mapping Ω to E. Obviously, the L^p-norm is such a generalized function norm. ■

Definition 2.2.15 *Let (Ω, Σ, μ) be a measure space and X be a Banach space. We denote by $L^p_{loc}(\Omega; X)$ all functions defined on Ω with range in X and such that they are locally of type $L^p(\Omega; X)$. We denote by $Reg(\mathbb{R}; X)$ all regulated functions with range in X. Recall that regulated functions are bounded right continuous functions which have a left hand limit at each finite point. Moreover, $Reg_{loc}(\mathbb{R}; X)$ include all functions defined on $\mathbb{R}$ with range in X and such that they are locally of type $Reg(\mathbb{R}; X)$, i.e. for every bounded subset $J \subset \mathbb{R}$ they belong to $Reg(J; X)$.*

Theorem 2.2.16 *Let X be a Banach lattice and (Ω, Σ, μ) be a measure space such that there are countable compact sets $(K_j)_{j\in\mathbb{N}}$ with $K_j \subsetneq K_{j+1}$ for each $j \in \mathbb{N}$ and $\Omega = \cup_{j=1}^\infty K_j$. Then $L^p_{loc}(\Omega; X)$ is a Fréchet lattice.*

Proof: Clearly, $L^p_{loc}(\Omega; X)$ is a Fréchet space and we only have to show that it is a solid Riesz space. Let now $p < \infty$ and $f, g \in L^p_{loc}(\Omega; X)$, then $f(t) \vee g(t)$ exists in X for almost every $t \in \Omega$. Moreover, for each compact set $K_j \subset \mathbb{R}$, we have from Theorem 2.2.14 that $L^p(K_j, X)$ is a Banach lattice for all $j \in \mathbb{N}$. This gives for every compact set $K_j \subset \mathbb{R}$

$$\int_{K_j} \|(f \vee g)(s)\|^p \,\mathrm{d}s < \infty. \tag{2.14}$$

At last we have to show that $L^p_{loc}(\Omega; X)$ is solid. Let $f \in L^p_{loc}(\Omega; X)$ and $g : \Omega \to X$ be a function, such that $|g| \leqslant |f|$, which is per definition

$|g(t)| \leqslant |f(t)|$ for a.e. $t \in \Omega$. Because X is a Banach lattice we get $\|g(t)\| \leqslant \|f(t)\|$ for a.e. $t \in \Omega$ and immediately $\|g(t)\|^p \leqslant \|f(t)\|^p$ for a.e. $t \in \mathbb{R}$ and for all $p \in [1, \infty)$. Therefore, we have for every compact set $K_j \subset \Omega$

$$\int_{K_j} \|g(s)\|^p \,\mathrm{d}s \leqslant \int_{K_j} \|f(s)\|^p \,\mathrm{d}s < \infty. \tag{2.15}$$

This implies $g \in L^p_{loc}(\Omega; X)$. The case for $p = \infty$ follows in a similar way. ■

Remark 2.2.17 *If we have $\Omega = \mathbb{R}$ with the Lebesgue measure, then one can define such compact sets via $K_j := [-j, j] \subset \mathbb{R}$. Note that this construction gives an inductive system for the Fréchet space $L^p_{loc}(\Omega; X)$ (cf. [50, Lemma 24.6]). In the same way one shows that the vector space $Reg_{loc}(\Omega, X)$ is a Fréchet lattice, too.*

Theorem 2.2.18 *The positive cone of a locally solid Hausdorff Riesz space is τ-closed.*

Proof: See [2, Theorem 8.43]. ■

Remark 2.2.19 *Here τ-closed represent closed with respect to the locally solid topology τ (cf. Remark 2.2.11 and Definition 2.2.12).*

An immediate consequence of this theorem is the following statement.

Corollary 2.2.20 *The positive cone of a Banach lattice or a Fréchet lattice is closed (norm closed or τ-closed).*

2.2.3 Positive operators

Definition 2.2.21 *Let E, F be Banach lattices. A linear operator $T : E \to F$ is called* positive *if T maps positive elements to positive elements, i.e. $T(E_+) \subset F_+$.*

A very strong consequence for positive operators is given by the following theorem.

Theorem 2.2.22 *Let E and F be Banach lattices. Every positive linear operator $T : E \to F$ is continuous.*

Proof: See [8, Theorem 10.20]. ■

Proposition 2.2.23 *Let E, F be Banach lattices and $T : E \to F$ be a positive operator. Then the dual operator $T' : F' \to E'$ is positive, too.*

Proof: First, note that E' and F' are again Banach lattices (see Proposition 2.2.7). Let $0 \leqslant f' \in F'$, then for all $0 \leqslant x \in F$, we get $(T' \circ f')(x) = f'(Tx) \geqslant 0$. ■

For any linear bounded operator G we denote by $r(G)$ the *spectral radius* for G.

Lemma 2.2.24 *Let T be a positive linear operator on E. Then*

$$r(T) < 1 \quad \Longleftrightarrow \quad 1 \in \rho(T) \text{ and } R(1,T) \geqslant 0. \tag{2.16}$$

Proof: See [8, Lemma 10.25]. ■

2.2.4 Positive semigroups

From Definition 2.2.21 it follows that a strongly continuous semigroup $(T(t))_{t\geqslant 0}$ on E is positive if and only if $T(t)E_+ \subset E_+$ for all $t \geqslant 0$. We show that the positivity of the semigroup is equivalent to the generator having a positive resolvent.

Proposition 2.2.25 *Let $(A, D(A))$ be the generator of a strongly continuous semigroup $(T(t))_{t\geqslant 0}$ with growth bound ω_0. Then $(T(t))_{t\geqslant 0}$ is positive if and only if the resolvent $R(\lambda, A)$ is positive for some $\lambda > \omega_0$.*

Proof: Let $(T(t))_{t\geqslant 0}$ be positive. By the Laplace transform (2.6)

$$R(\lambda, A)x = \int_0^\infty e^{-\lambda s} T(s)x \, \mathrm{d}s \tag{2.17}$$

and the fact that the positive cone is closed we see that $R(\lambda, A)$ is positive for every λ for which the Laplace transform exists.
The other direction is proven by the Post-Widder Inversion Formula (2.7):

$$T(t) = \lim_{n\to\infty} \left(I - \frac{t}{n}A\right)^{-n} = \left(\frac{n}{t}R\left(\frac{n}{t}, A\right)\right)^n \qquad \text{for every } t \geqslant 0. \tag{2.18}$$

Again the positive cone is closed and we have that $\left(\frac{n}{t}R\left(\frac{n}{t}, A\right)\right)^n$ is positive for sufficiently large $n \in \mathbb{N}$. This completes the proof. ■

Lemma 2.2.26 *Let $(A, D(A))$ generate a positive strongly continuous semigroup on E and let $\lambda \in \rho(A)$. Then $R(\lambda, A)$ is positive if and only if $\lambda > s(A)$.*

Proof: See [8, Corollary 12.10]. ■

Proposition 2.2.27 *If $(A, D(A))$ is the generator of a positive strongly continuous semigroup $(T(t))_{t\geqslant 0}$, then for all $\lambda \in \mathbb{R}$ we have that $(A - \lambda, D(A))$ is the generator of the positive strongly continuous semigroup $(e^{-\lambda t}T(t))_{t\geqslant 0}$.*

Proof: This follows from Proposition 2.1.14 and the positivity of $(e^{-\lambda t}T(t))_{t\geqslant 0}$ is obvious. ■

Proposition 2.2.28 *If $(A, D(A))$ is the generator of a positive strongly continuous semigroup $(T(t))_{t\geqslant 0}$, then the adjoint semigroup from Definition 2.1.15 is again positive.*

Proof: Let $0 \leqslant x' \in X'$ be arbitrary, then we have for all $x \in X_+$ and all $t \geqslant 0$,

$$(T'(t) \circ x')(x) = x'(T(t)x) \geqslant 0.$$

■

Remark 2.2.29 *From the proof above we see immediately that the sun dual semigroup is positive, if the semigroup $(T(t))_{t \geqslant 0}$ is positive.*

2.2.5 Positivity on extrapolation spaces

Extrapolation spaces are an important tool in semigroup theory. For example if one considers a Desch-Schappacher perturbation. However, in general it is hard to determine extrapolation spaces (from Definition 2.1.11), because they are defined as a completion. So in most applications we only want to apply the theory for such extrapolation spaces, but not calculate these spaces. Now the question arises, how to define positivity on spaces which we don't know in general. To answer this question consider the left shift group

$$T(t)f(\xi) = f(t + \xi)$$

for all $t, \xi \in \mathbb{R}$ on the Banach space $L^2(\mathbb{R})$. The extrapolation space is given by the Sobolev space $H^{-1}(\mathbb{R})$ (see [24, Example 5.8, Chapter II]). From [1, page 51 and Corollary 3.19, Chapter III] we know that $H^{-1}(\mathbb{R})$ is the dual space of $H^1(\mathbb{R})$ and further that this space is embedded in the bounded continuous functions on $\mathbb{R}$ (see page 97, [1]). Summarizing this, we get that the point evaluation, also called the Dirac delta distribution, is an element of the extrapolation space $H^{-1}(\mathbb{R})$ and it is well-known that this distribution can be approximated (in the space $H^{-1}(\mathbb{R})$) by the functions

$$\delta_\epsilon(x) = \frac{1}{\sqrt{2\pi\epsilon}} \exp\left(-\frac{x^2}{2\epsilon}\right). \tag{2.19}$$

However, these functions are elements of $L^2(\mathbb{R})$ for every $\epsilon > 0$ and it is easy to see that they are positive in the usual ordering.
This motivates the idea to define positivity on extrapolation spaces with positive sequences from the original space with respect to the extrapolation space, i.e. $x \in X_{-1}$ is positive if there exists a positive sequence $(f_n)_{n \in \mathbb{N}} \subset X$ with $f_n \to x$ for $n \to \infty$ in X_{-1}. Since X is dense in X_{-1} by definition (see Definition 2.1.11), this is equivalent to the following.

Definition 2.2.30 *Let E be a Banach lattice and E_{-1} the extrapolation space for the positive strongly continuous semigroup $(T(t))_{t \geqslant 0}$. We say that $f \in E_{-1}$ is* positive, *if f belongs to the closure of E_+ in E_{-1}. We denote by $E_{-1,+}$ the set of all positive elements in E_{-1}.*

From the definition the set of positive elements satisfies $E_+ \subseteq E_{-1,+}$.

Remark 2.2.31 *In the context of Definition 2.2.30, for* $\lambda > \mathrm{s}(A_{-1}) = \mathrm{s}(A)$ *it is easy to see that* $R(\lambda, A_{-1}) \in \mathcal{L}(E_{-1})$ *is a positive operator (i.e., maps positive elements to positive elements). In fact, it will follow from Proposition 2.2.32 that* $R(\lambda, A_{-1})$ *is also positive as an operator in* $\mathcal{L}(E_{-1}, E)$*, see Remark 2.2.33 below.*
Let $B \in \mathcal{L}(E, E_{-1})$*. If* B *is positive, i.e.* $Bf \geqslant 0$ *for all* $f \in E_+$*, then* $R(\lambda, A_{-1})B$ *is positive as an operator in* $\mathcal{L}(E)$*, for all* $\lambda > \mathrm{s}(A)$*. Conversely, if there exists a sequence* $(\lambda_n)_{n\in\mathbb{N}}$ *in* $(\mathrm{s}(A), \infty)$ *tending to* ∞*, and such that* $R(\lambda_n, A_{-1})B \geqslant 0$ *for all* $n \in \mathbb{N}$*, then* $B \geqslant 0$*. Indeed, if* $f \in E_+$*, then* $R(\lambda_n, A_{-1})Bf \in E_+$ *for all* $n \in \mathbb{N}$*, and the convergence* $\lambda_n R(\lambda_n, A_{-1})Bf \to Bf$ *in* E_{-1} $(n \to \infty)$ *implies* $Bf \in E_{-1,+}$*.*
In Example 3.2.8 we will show that positivity of $R(\lambda, A_{-1})B$ *for only a single* $\lambda > \mathrm{s}(A_{-1})$ *does not imply the positivity of* B*.*

Next we establish some basic properties of the ordering on E_{-1}.

Proposition 2.2.32 *Let* E *be a real Banach lattice and* $(T(t))_{t\geqslant 0}$ *a positive strongly continuous semigroup on* E*. The set* $E_{-1,+}$ *is a closed convex cone in* E_{-1}*, satisfying*
$$E_+ = E_{-1,+} \cap E.$$

Proof: Taking closures in the inclusions $E_+ + E_+ \subseteq E_+$ and $\alpha E_+ \subseteq E_+$ for $\alpha \geqslant 0$, one obtains the corresponding inclusions for $E_{-1,+}$, i.e. $E_{-1,+}$ is a convex cone. Also, $E_{-1,+}$ is closed as the closure of E_+. To show $E_{-1,+} \cap (-E_{-1,+}) = \{0\}$, let $f \in E_{-1,+}$ and assume also that $-f \in E_{-1,+}$. Then there exist sequences $(f_n)_{n\in\mathbb{N}}, (g_n)_{n\in\mathbb{N}}$ in E_+ such that $f_n \to f$ and $g_n \to -f$ in E_{-1}, and thereby $f_n + g_n \to 0$ in E_{-1}, as $n \to \infty$. Choose $\lambda > \mathrm{s}(A)$ and let the norm $\|\cdot\|_{-1}$ be defined in terms of this λ. Note that $0 \leqslant f_n \leqslant f_n + g_n$, and hence $0 \leqslant R(\lambda, A)f_n \leqslant R(\lambda, A)(f_n + g_n)$, by the positivity of the semigroup. Therefore
$$\|f_n\|_{-1} = \|R(\lambda, A)f_n\| \leqslant \|R(\lambda, A)(f_n + g_n)\| = \|f_n + g_n\|_{-1} \to 0$$
as $n \to \infty$. This shows that $f = 0$. Finally, to show that the definition of positivity in the extrapolation space is compatible with the original ordering, we note that $E_+ \subseteq E_{-1,+} \cap E$ is immediate from the definition.
To prove the reverse inclusion let $f \in E_{-1,+} \cap E$. Then there exists a sequence $(f_n)_{n\in\mathbb{N}}$ in E_+ such that $\|f - f_n\|_{-1} \to 0$ $(n \to \infty)$. Recalling that the norm $\|\cdot\|_{-1}$ can be defined using any $\lambda \in \rho(A)$ we obtain
$$\|R(\lambda, A)f - R(\lambda, A)f_n\| \to 0 \text{ as } n \to \infty,$$
for all $\lambda > \mathrm{s}(A)$. Because of $R(\lambda, A)f_n \in E_+$ for all $n \in \mathbb{N}$ this implies that $R(\lambda, A)f \in E_+$ for all $\lambda > \mathrm{s}(A)$. From $\lambda R(\lambda, A)f \to f$ (in E) as $\lambda \to \infty$ we therefore obtain $f \in E_+$. ∎

Remark 2.2.33 *It is important to keep in mind the following simple consequence of the properties shown in Proposition 2.2.32. In the context of this proposition, let* $C\colon E_{-1} \to E$ *be an operator. Then* C *is positive if and only if* C *is positive as an operator from* E_{-1} *to* E_{-1}*.*

Lemma 2.2.34 *Let E be a real Banach lattice, E_{-1} the extrapolation space for a positive strongly continuous semigroup $(T(t))_{t\geqslant 0}$ on E, let $B \in \mathcal{L}(E, E_{-1})$ be a positive operator, and let $\tau > 0$. Then we have:*

(i) $(T_{-1}(t))_{t\geqslant 0}$ *is positive.*

(ii) For each step function $u \in \mathrm{L}^\infty([0,\tau]; E)$ we have

$$\int_0^\tau T_{-1}(s)Bu(s)\,\mathrm{d}s \in E.$$

(iii) For all $f \in E_+$ we have $\int_0^\tau T_{-1}(s)Bf\,\mathrm{d}s \in E_+$.

(iv) If in addition $(T(t))_{t\geqslant 0}$ is exponentially stable, then we have

$$\int_0^\tau T_{-1}(s)Bf\,\mathrm{d}s \leqslant \int_0^\infty T_{-1}(s)Bf\,\mathrm{d}s$$

in E, for all $f \in E_+$.

Proof: (i) This follows because $T_{-1}(t)$ is the continuous extension of $T(t)$, for all $t \geqslant 0$.
(ii) Let $u \in \mathrm{L}^\infty([0,\tau]; E)$ be a step function, i.e., $u(t) = \sum_{n=1}^N u_n\chi_{I_n}(t)$ where $u_1, \ldots, u_N \in E$, $I_1, \ldots, I_N \subseteq [0,\tau]$ are pairwise disjoint intervals with $\bigcup_{n=1}^N I_n = [0,\tau]$, and where χ_{I_n} denotes the indicator function of I_n. It suffices to show that

$$\int_{I_n} T_{-1}(s)Bu_n\,\mathrm{d}s = \int_{t_{n-1}}^{t_n} T_{-1}(s)Bu_n\,\mathrm{d}s \in E,$$

where $(t_{n-1}, t_n) \subseteq I_n \subseteq [t_{n-1}, t_n]$. With the substitution $s' = s - t_{n-1}$ we get

$$\begin{aligned}\int_{t_{n-1}}^{t_n} T_{-1}(s)Bu_n\,\mathrm{d}s &= \int_0^{t_n - t_{n-1}} T_{-1}(s + t_{n-1})Bu_n\,\mathrm{d}s \\ &= T_{-1}(t_{n-1})\int_0^{t_n - t_{n-1}} T_{-1}(s)Bu_n\,\mathrm{d}s.\end{aligned}$$

Because $(T_{-1}(t))_{t\geqslant 0}$ is a strongly continuous semigroup on E_{-1} with generator A_{-1}, we have from Proposition 2.1.4 that $\int_0^{t_n - t_{n-1}} T_{-1}(s)Bu_n\,\mathrm{d}s$ belongs to $D(A_{-1}) = E$, and the assertion follows. Statements (iii) and (iv) follow directly from Proposition 2.2.32. ∎

2.3 Orlicz spaces

Here we collect some basic facts for Orlicz spaces, which we will need in Example 4.2.13. The notation and results are taken from the monograph [43].

Definition 2.3.1 *Let $p(t)$ be a real-valued function which is right continuous for $t \geqslant 0$, positive for $t > 0$, non-decreasing and such that the following holds*

$$p(0) = 0 \qquad \lim_{t\to\infty} p(t) = \infty.$$

We call $M\colon \mathbb{R} \to \mathbb{R}$ *an* N-function *if this function admits the representation*

$$M(u) = \int_0^{|u|} p(t)\,\mathrm{d}t. \tag{2.20}$$

Further define $q(s)$ *for* $s \geqslant 0$ *by the equality*

$$q(s) = \sup_{p(t) \leqslant s} t. \tag{2.21}$$

We say that $M(u)$ *from Equation* (2.20) *and*

$$N(v) = \int_0^{|v|} q(s)\,\mathrm{d}s \tag{2.22}$$

are mutually complementary *N-functions.*
A function $\Phi : [0, \infty) \to \mathbb{R}$ *is called a* Young function *if*

$$\Phi(t) = \int_0^t p(s)\,\mathrm{d}s$$

for $t \geqslant 0$*, where* $p(\cdot)$ *is defined as above.*

Proposition 2.3.2 *Every N-function $M(u)$ is even, continuous, convex, $M(0) = 0$ holds and $M(u)$ increases for positive values of the argument.*

Proof: See [43, page 7] ∎

An N-function is sometimes defined via the properties from the lemma below.

Lemma 2.3.3 *$M\colon \mathbb{R} \to \mathbb{R}$ is an N-function if and only if $M(u)$ is a continuous, even and convex function such that*

$$\lim_{u \to 0} \frac{M(u)}{u} = 0 \quad \text{and} \quad \lim_{u \to \infty} \frac{M(u)}{u} = \infty.$$

Proof: See [43, page 7 and 9] ∎

Remark 2.3.4 *It is easy to see that the above statement holds for Young functions, too, if one restricts the domain to $[0, \infty)$ and drop the assumption of an even function.*

Definition 2.3.5 *Let $M(u)$ be a N-function. We say that $M(u)$ satisfies the Δ_2-condition if there exist constants $k > 0$, $u_0 \geqslant 0$ such that for all $u \geqslant u_0$*

$$M(2u) \leqslant kM(u)$$

holds.

Definition 2.3.6 *Let G be a bounded closed set in a finite-dimensional Euclidean space, on which we consider the usual Lebesgue measure. Define*

$$\rho(u; M) := \int_G M[u(x)]\,\mathrm{d}x \tag{2.23}$$

and denote by $L_M(G)$ the classes of all real-valued functions, defined on G, for those $\rho(u; M) < \infty$ holds. We call the classes $L_M(G)$ Orlicz classes. *Further, we define the* Luxemburg norm *of $u \in L_M(G)$ by*

$$\|u\|_M := \inf\left\{k > 0 \,:\, \rho\left(\frac{u}{k}; M\right) = \int_G M\left[\frac{u(x)}{k}\right]\mathrm{d}x \leqslant 1\right\} \tag{2.24}$$

and define for mutually complementary N-functions M and N,

$$L_{M^*}(G) := \left\{u : G \to \mathbb{R} \,:\, \int_G u(x)v(x)\,\mathrm{d}x < \infty \text{ for all } v \in L_N(G)\right\}. \tag{2.25}$$

Further, we define the Orlicz space

$$E_M(G) = \overline{L^\infty(G, \mathbb{R})}^{\|\cdot\|_M}. \tag{2.26}$$

Proposition 2.3.7 *For every N-function we have*

$$E_M(G) \subset L_M(G) \subset L_{M^*}(G) \tag{2.27}$$

Proof: See [43, page 67 and 81]. ■

Next we state an expansion of the famous Hölder inequality.

Theorem 2.3.8 *For any pair of functions $u \in L_{M^*}(G)$ and $v \in L_{N^*}(G)$ the following estimation is true*

$$\left|\int_G u(x)v(x)\,\mathrm{d}x\right| \leqslant \|u\|_M \cdot \|v\|_N$$

Proof: See [43, Theorem 9.3, Chapter II] ■

Proposition 2.3.9 *Let $\Phi\colon [0,\infty) \to [0,\infty)$ be a Young function and $f : [0,\infty) \to [0,\infty)$ an unbounded convex function with $f(0) = 0$. Then the composition $\Psi := \Phi \circ f$ is a Young function.*

Proof: See [54, Lemma 1.1.5] ■

Chapter 3

Perturbation theory

In the field of semigroup theory one often deals with the following kind of perturbations. Given the abstract Cauchy problem (ACP) on a Banach space X (see Subsection 2.1.1), one adds a perturbing operator to the setting, where it is assumed that $(A, D(A))$ is already a generator of a strongly continuous semigroup. We denote this perturbing operator by B with domain $D(B)$ and describe this setting with the extended abstract Cauchy problem via the equations

$$\begin{cases} \dot{x}(t) &= Ax(t) + Bx(t), \quad \text{for } t \geqslant 0, \\ x(0) &= x_0. \end{cases} \tag{3.1}$$

If B is a bounded operator, i.e. $B \in \mathcal{L}(X)$, then it is not hard to check that the solution is given by the strongly continuous semigroup generated by the operator $C := A + B$ with domain $D(A)$ (see e.g. [24, Theorem 1.3, Chapter III]). In the following we will restrict ourselves to the case where B is a continuous operator from X_1 to X or continuous from X to X_{-1}. Recall that X_1 and X_{-1} are the inter- and extrapolation spaces of X for the semigroup with generator $(A, D(A))$ introduced in Definition 2.1.11. Another typical class of perturbation operators are the so called A-bounded operators.

Definition 3.0.1 *Let A be a linear operator on a Banach space X with domain $D(A)$. An operator $(B, D(B))$ is said to be* A-bounded, *if $D(A) \subseteq D(B)$ and if there exist constants $a, b \in \mathbb{R}_+$ such that*

$$\|Bx\| \leqslant a\|Ax\| + b\|x\| \tag{3.2}$$

for all $x \in D(A)$.

Remark 3.0.2 *In the literature it is also common that operators described by the above definition are called* relatively A-bounded.

Such A-bounded operator can be characterised by the composition with the resolvent of $(A, D(A))$.

Lemma 3.0.3 *Let X be a Banach space and $(A, D(A))$ be a closed linear operator on X. If $\rho(A) \neq \varnothing$, then B is A-bounded if and only if $BR(\lambda, A) \in \mathcal{L}(X)$ for some $\lambda \in \rho(A)$.*

Proof: See [8, Lemma 13.2]. ■

We now look at perturbations that are continuous from X_1 to X. An important class of such operators are the so called Miyadera-Voigt perturbations.

Definition 3.0.4 *Let $(A, D(A))$ be the generator of a strongly continuous semigroup $(T(t))_{t\geqslant 0}$ on a Banach space X and $C \in \mathcal{L}(X_1, X)$. If there exist $\tau > 0$ and $\gamma \in [0, 1[$ such that for all $f \in D(A)$ the following estimation holds*

$$\int_0^\tau \|CT_1(t)f\| \,\mathrm{d}t \leqslant \gamma\|f\|, \tag{3.3}$$

we call the operator C a Miyadera-Voigt perturbation *of A.*

Remark 3.0.5 *We mention that in the literature the Miyadera-Voigt perturbation is also defined via an abstract Volterra operator (cf. [24], page 195 and 196).*

For Miyadera-Voigt perturbations it is true that Equation (3.1) has a solution and this solution is explicitly given by the Dyson-Phillips series.

Theorem 3.0.6 *Let $(A, D(A))$ be the generator of a strongly continuous semigroup $(T(t))_{t\geqslant 0}$ on a Banach space X and C a Miyadera-Voigt perturbation. Then $A + C$ with domain $D(A + C) = D(A)$ generates a strongly continuous semigroup $(S(t))_{t\geqslant 0}$ on X. Moreover, this semigroup is given by the Dyson-Phillips series*

$$S(t) = \sum_{k=0}^{\infty} S_k(t), \tag{3.4}$$

where the operators $S_k(t) \in \mathcal{L}(X)$ satisfy

$$S_0(t) := T(t) \qquad \text{and} \qquad S_{k+1}(t)x := \int_0^t S_k(t-s)CT(s)x \,\mathrm{d}s \tag{3.5}$$

for $t \geqslant 0$, $x \in D(A)$ and for all $k \in \mathbb{N}$.

Proof: See [8, Theorem 13.6]. ■

Next we consider perturbation operators which have X as their domain, but they are only bounded if we consider them as operators mapping to X_{-1}. For operators of the form $B \in \mathcal{L}(X, X_{-1})$ there exists again a special class of operators solving equation (3.1), called Desch-Schappacher perturbations.

Definition 3.0.7 *Let $(A, D(A))$ be the generator of a strongly continuous semigroup $(T(t))_{t\geqslant 0}$ on a Banach space X and let $B \in \mathcal{L}(X, X_{-1})$. Moreover, assume that there exist $\tau > 0$ and $K \in [0, 1)$ such that*

(i) $\displaystyle\int_0^\tau T_{-1}(\tau - s)Bu(s) \,\mathrm{d}s \in X,$

(ii) $\displaystyle\left\| \int_0^\tau T_{-1}(\tau - s)Bu(s) \,\mathrm{d}s \right\| \leqslant K\|u\|_\infty,$

hold for all continuous functions $u \in \mathrm{C}([0,\tau], X)$. *Then we call the operator* B *a* Desch-Schappacher perturbation *of* A.

Remark 3.0.8 *Again (cf. Remark 3.0.5), there is another way to define such Desch-Schappacher perturbation, namely via an abstract Volterra operator (cf. [24], page 182 and 183).*

As we have seen that for every Miyadera-Voigt perturbation equation (3.1) has a solution, we are also able to do this for Desch-Schappacher perturbations.

Theorem 3.0.9 *Let* $(A, D(A))$ *be the generator of a* C_0*-semigroup* $(T(t))_{t\geqslant 0}$ *on a Banach space* X *and let* B *be a Desch-Schappacher perturbation of* A. *Then the operator* $A_{-1} + B$ *with* $D(A_{-1} + B) = \{f \in X : A_{-1}f + Bf \in X\}$ *generates a* C_0*-semigroup* $(S(t))_{t\geqslant 0}$ *on* X. *Furthermore, this semigroup is given by the Dyson–Phillips series*

$$S(t) = \sum_{n=0}^{\infty} S_n(t), \text{ for all } t \geqslant 0, \tag{3.6}$$

where $S_0(t) := T(t)$ *and*

$$S_n(t)f := \int_0^t T_{-1}(t-s)BS_{n-1}(s)f \,\mathrm{d}s \text{ for all } f \in X. \tag{3.7}$$

Proof: See [24, Chapter III, Corollaries 3.2 and 3.3]. ■

There is another nice way to show that an operator $B \in \mathcal{L}(X, X_{-1})$ is a Desch-Schappacher perturbation.

Corollary 3.0.10 *Let* $(A, D(A))$ *be the generator of a strongly continuous semigroup* $(T(t))_{t\geqslant 0}$ *on a Banach space* X *and let* $B \in \mathcal{L}(X, X_{-1})$. *Moreover, assume that there exist* $\tau > 0$ *and* $p \in [1, \infty)$ *such that for every* $f \in L^p([0,\tau]; X)$ *we have*

$$\int_0^\tau T_{-1}(\tau - s)Bf(s) \,\mathrm{d}s \in X. \tag{3.8}$$

Then B *is a Desch-Schappacher perturbation and therefore* $A_{-1}+B$ *with domain* $D(A_{-1} + B) = \{x \in X_{-1} : (A_{-1} + B)x \in X\}$ *generates a strongly continuous semigroup on* X.

Proof: See [24, Chapter III, Corollory 3.4]. ■

Remark 3.0.11 *The above corollary is the pendant to admissibility, if* f *is replaced by the so called control function* u *which maps* $[0,\tau]$ *into a Banach space* U, *called the control space (cf. Section 4.1). Therefore, this corollary shows (if we assume* $X = U$*) that* L^p*-admissibility for a control operator* $B \in \mathcal{L}(U, X_{-1})$ *for* $p \in [1, \infty)$ *is stronger than the conditions i) and ii) from Definition 3.0.7 for the type of a Desch-Schappacher perturbation. We mention that it is important that the case* $p = \infty$ *is excluded, here. However, zero-class* L^∞*-admissibility for the control operator* B *would imply (if* $X = U$*) that it is a Desch-Schappacher perturbation (cf. Section 4.2).*

3.1 Resolvent positive operators

Now we start to work with positive operators and especially operators whose resolvents are positive.

Definition 3.1.1 *An operator A on a Banach lattice E is called* resolvent positive *if there exists $\omega \in \mathbb{R}$ such that $(\omega, \infty) \subset \rho(A)$ and $R(\lambda, A) \geqslant 0$ for all $\lambda > \omega$.*

For the generator $(A, D(A))$ of a strongly continuous semigroup, it is a weaker assumption to assume that $(A, D(A))$ is resolvent positive instead of $(A, D(A))$ generates a positive semigroup, because in the latter case the resolvent is always positive for all $\lambda > s(A)$ (cf. Lemma 2.2.26).
Next we establish two theorems, which we will need in the ensuing section. An important tool to prove these statements is the famous Neumann series. The following theorem was developed by Voigt [80] and one direction of this theorem was stated earlier by Arendt [4, Theorem 3.1]. We start with an A-bounded perturbation operator.

Theorem 3.1.2 *Let $(A, D(A))$ be a resolvent positive operator in E, and $\lambda > s(A)$. Let $C : D(A) \to E$ be a positive and linear operator. Then the following conditions are equivalent:*

(i) $r(CR(\lambda, A)) < 1$,

(ii) $\lambda \in \rho(A + C)$ *and* $R(\lambda, A + C) \geqslant 0$.

If one of these condition is satisfied, then $A + C$ is resolvent positive, $\lambda > s(A + C)$, and

$$R(\lambda, A + C) = R(\lambda, A) \sum_{n=0}^{\infty} (CR(\lambda, A))^n \geqslant R(\lambda, A). \tag{3.9}$$

Proof: First, observe that $CR(\lambda, A)$ is positive, because C and $R(\lambda, A)$ are positive. Therefore, Theorem 2.2.22 implies that $CR(\lambda, A) \in \mathcal{L}(E)$.

Now assume that $r(CR(\lambda, A)) < 1$ holds for a fixed $\lambda > s(A)$ and let $f \in D(A)$. Then, we calculate

$$(\lambda - (A + C))f = (\lambda - A)f - Cf = (\lambda - A)f - CR(\lambda, A)(\lambda - A)f.$$

This leads to the identity

$$(\lambda - (A + C))f = (I - CR(\lambda, A))(\lambda - A)f. \tag{3.10}$$

Since for the spectral radius $r(CR(\lambda, A)) < 1$ holds, we can apply the Neumann series:

$$S_\lambda := (I - CR(\lambda, A))^{-1} = \sum_{n=0}^{\infty} (CR(\lambda, A))^n. \tag{3.11}$$

Moreover, $CR(\lambda, A)$ is positive and this implies $S_\lambda \geqslant 0$. Now, we multiply $R(\lambda, A)S_\lambda$ to the left side to Equality (3.10). This gives

$$R(\lambda, A)S_\lambda(\lambda - (A + C))f = f. \tag{3.12}$$

Secondary, $(\lambda - A)$ is an isomorphism from $D(A)$ to E. This implies that for each $g \in E$ we find an element $f \in D(A)$ with $f = R(\lambda, A)g$. Applying this to the identity (3.10), we get

$$(\lambda - (A + C))R(\lambda, A)g = (I - CR(\lambda, A))g. \tag{3.13}$$

Finally we multiply S_λ to the right side of the above equation and we get

$$(\lambda - (A + C))R(\lambda, A)S_\lambda g = g. \tag{3.14}$$

Equations (3.12) and (3.14) show that $(\lambda - (A + C))$ is invertible with inverse $R(\lambda, A + C) = R(\lambda, A)S_\lambda$. Clearly, this implies $\lambda \in \rho(A + C)$ and since S_λ and $R(\lambda, A)$ are positive we have $R(\lambda, A + C) \geqslant 0$.
The Estimation (3.9) follows, because of

$$\begin{aligned} R(\lambda, A + C) = R(\lambda, A)S_\lambda &= R(\lambda, A)\sum_{n=0}^{\infty}(CR(\lambda, A))^n \\ &= R(\lambda, A) + \sum_{n=1}^{\infty} R(\lambda, A)(CR(\lambda, A))^n \geqslant R(\lambda, A) \geqslant 0. \end{aligned}$$

For the converse direction let $g \in E$ be arbitrary. We calculate for a fixed $\lambda \in \rho(A + C)$

$$\begin{aligned} (I - CR(\lambda, A))(\lambda - A)R(\lambda, A + C)g &= ((\lambda - A) - C)R(\lambda, A + C)g \\ &= (\lambda - (A + C))R(\lambda, A + C)g = g. \end{aligned}$$

On the other side we have

$$\begin{aligned} &(\lambda - A)R(\lambda, A + C)(I - CR(\lambda, A))g \\ &= (\lambda - A)R(\lambda, A + C)((\lambda - A)R(\lambda, A) - CR(\lambda, A))g \\ &= (\lambda - A)R(\lambda, A + C)(\lambda - A - C)R(\lambda, A)g \\ &= (\lambda - A)R(\lambda, A)g = g. \end{aligned}$$

Thus, $1 \in \rho(CR(\lambda, A))$ and

$$(I - CR(\lambda, A))^{-1} = (\lambda - A)R(\lambda, A + C).$$

Additionally we get

$$\begin{aligned} (\lambda - A)R(\lambda, A + C) &= (\lambda - (A + C) + C)R(\lambda, A + C) \\ &= I + CR(\lambda, A + C) \geqslant 0. \end{aligned}$$

This shows that $R(1, CR(\lambda, A))$ is positive and therefore the last part of the proof follows by Lemma 2.2.24. ∎

We will use the above result for Miyadera-Voigt perturbations later on. Our next theorem is in some sense the dual part of the above statement and will be needed for Desch-Schappacher perturbations. For the sake of completeness we will give the proof, too. However, it is very similar to the proof of Theorem 3.1.2. This result can be found in [9, Theorem 3.2].

Theorem 3.1.3 *Let $(A, D(A))$ be a resolvent positive operator in E, and $\lambda > s(A)$. Let $B\colon E \to E_{-1}$ be a positive and linear operator. Then the following conditions are equivalent:*

(i) $r(R(\lambda, A_{-1})B) < 1$,

(ii) $\lambda \in \rho(A_{-1} + B)$ *and* $R(\lambda, A_{-1} + B) \geqslant 0$.

(In condition (ii) we consider A_{-1} and B as operators in E_{-1}, with domain E.)
If one of these properties are satisfied, then $A_{-1} + B$ is resolvent positive, $\lambda > s(A_{-1} + B)$ and

$$R(\lambda, A_{-1} + B) = \left(\sum_{n=0}^{\infty} (R(\lambda, A_{-1})B)^n \right) R(\lambda, A_{-1}) \geqslant R(\lambda, A_{-1}). \tag{3.15}$$

Proof: Assume that $r(R(\lambda, A_{-1})B) < 1$ holds. Since $R(\lambda, A_{-1})\colon E_{-1} \to E$, we have $R(\lambda, A_{-1})B \in \mathcal{L}(E)$. Because of $r(R(\lambda, A_{-1})B) < 1$ we can apply the Neumann series:

$$T_\lambda := (I - R(\lambda, A_{-1})B)^{-1} = \sum_{n=0}^{\infty} (R(\lambda, A_{-1})B)^n. \tag{3.16}$$

Then, we calculate for an arbitrary $g \in E$

$$\begin{aligned} & T_\lambda R(\lambda, A_{-1})(\lambda - (A_{-1} + B))g \\ &= (I - R(\lambda, A_{-1})B)^{-1} R(\lambda, A_{-1})(\lambda - A_{-1} + (\lambda - A_{-1})R(\lambda, A_{-1})B)g \\ &= (I - R(\lambda, A_{-1})B)^{-1}(I - R(\lambda, A_{-1})B)g = g. \end{aligned}$$

Secondary, we have

$$\begin{aligned} & (\lambda - (A_{-1} + B))T_\lambda R(\lambda, A_{-1})g \\ &= (\lambda - A_{-1} - (\lambda - A_{-1})R(\lambda, A_{-1})B)(I - R(\lambda, A_{-1})B)^{-1} R(\lambda, A_{-1})g \\ &= (\lambda - A_{-1})(I - R(\lambda, A_{-1})B)(I - R(\lambda, A_{-1})B)^{-1} R(\lambda, A_{-1})g = g. \end{aligned}$$

This shows $\lambda \in \rho(A_{-1} + B)$ and that $R(\lambda, A_{-1} + B) = T_\lambda R(\lambda, A_{-1})$ is positive, because $R(\lambda, A_{-1})$ and B are positive. Finally, the last Estimation (3.15) follows from

$$\begin{aligned} R(\lambda, A_{-1} + B) = T_\lambda R(\lambda, A_{-1}) &= \left(\sum_{n=0}^{\infty} (R(\lambda, A_{-1})B)^n \right) R(\lambda, A_{-1}) \\ &= \left(I + \sum_{n=1}^{\infty} (R(\lambda, A_{-1})B)^n \right) R(\lambda, A_{-1}) \geqslant R(\lambda, A_{-1}). \end{aligned}$$

For the other direction let $g \in E$ be arbitrary. We calculate

$$\begin{aligned} &(I - R(\lambda, A_{-1})B)R(\lambda, A_{-1} + B)(\lambda - A_{-1})g \\ &= (R(\lambda, A_{-1})(\lambda - A_{-1}) - R(\lambda, A_{-1})B)R(\lambda, A_{-1} + B)(\lambda - A_{-1})g \\ &= R(\lambda, A_{-1})(\lambda - A_{-1} - B)R(\lambda, A_{-1} + B)(\lambda - A_{-1})g = g. \end{aligned}$$

On the otherside we have for all $g \in E$

$$\begin{aligned} &R(\lambda, A_{-1} + B)(\lambda - A_{-1})(I - R(\lambda, A_{-1})B)g \\ &= R(\lambda, A_{-1} + B)(\lambda - A_{-1})(R(\lambda, A_{-1})(\lambda, A_{-1} - R(\lambda, A_{-1})B)g \\ &= R(\lambda, A_{-1} + B)(\lambda - A_{-1})R(\lambda, A_{-1})(\lambda - A_{-1} - B)g = g. \end{aligned}$$

Thus $1 \in \rho(R(\lambda, A_{-1})B)$ and the inverse of $I - R(\lambda, A_{-1})B$ is given by

$$(I - R(\lambda, A_{-1})B)^{-1} = R(\lambda, A_{-1} + B)(\lambda - A_{-1}).$$

Further we see with the calculation

$$\begin{aligned} R(\lambda, A_{-1} + B)(\lambda - A_{-1}) &= R(\lambda, A_{-1} + B)(\lambda - A_{-1} - B + B) \\ &= I + R(\lambda, A_{-1} + B)B \geqslant 0 \end{aligned}$$

that $R(1, R(\lambda, A_{-1})B)$ is positive and therefore Lemma 2.2.24 completes the proof. ∎

3.2 Perturbation theory with positive operators

In this section we consider Miyadera-Voigt and Desch-Schappacher perturbations on the special Banach lattices AL- and AM-spaces, which we introduced in the first chapter.

3.2.1 Miyadera-Voigt perturbation on AL-spaces

The next proposition is originally from Desch [19] and was refined by Voigt [80, Lemma 2.1].

Proposition 3.2.1 *Let $(A, D(A))$ be the generator of a positive strongly continuous semigroup $(T(t))_{t\geqslant 0}$ on a real AL-space E. Moreover, let $C\colon D(A) \to E$ be a positive and linear operator, and $G\colon D(A) \to E$ be a linear operator satisfying $|Gf| \leqslant Cf$ for all $0 \leqslant f \in D(A)$. If there exists $\lambda > s(A)$ such that $\|CR(\lambda, A)\| < 1$, then we have $\|GR(\lambda, A)\| < 1$ and the operator $A + G$ with domain $D(A)$ generates a strongly continuous semigroup on E. Additionally, $A + C$ with domain $D(A)$ generates a positive strongly continuous semigroup on E.*

Proof: Note that $|Gf| \leqslant Cf$ for all $f \in D(A)_+$ implies for $\lambda > s(A)$

$$|GR(\lambda, A)g| \leqslant CR(\lambda, A)g$$

for all $g \in E_+$, since $R(\lambda, A)$ is positive and it maps E to $D(A)$ bijectively. Moreover, we have for every $0 \leqslant f \in D(A)$

$$\|Gf\| \leqslant \|Cf\|,$$

because E is a Banach lattice and $|Gf| \leqslant Cf = |Cf|$ holds. Now let $f \in E$ be arbitrary, then there exist $f^+, f^- \in E_+$ such that $f = f^+ - f^-$ and we calculate

$$\begin{aligned} |GR(\lambda, A)f| &= |GR(\lambda, A)f^+ - GR(\lambda, A)f^-| \\ &\leqslant |GR(\lambda, A)f^+| + |GR(\lambda, A)f^-| \\ &\leqslant CR(\lambda, A)(f^+ + f^-) = CR(\lambda, A)|f|. \end{aligned}$$

It follows that

$$\|GR(\lambda, A)\| \leqslant \|CR(\lambda, A)\| < 1. \tag{3.17}$$

Then Lemma 3.0.3 shows that both operators G and C are A-bounded. Next, we have for all $f \in D(A)_+$ and every $\tau > 0$

$$\begin{aligned} \int_0^\tau \|e^{-\lambda t} GT(t)f\| \, \mathrm{d}t &\leqslant \int_0^\tau \|e^{-\lambda t} CT(t)f\| \, \mathrm{d}t \\ &= \left\| C \int_0^\tau e^{-\lambda t} T(t) f \, \mathrm{d}t \right\| \\ &\leqslant \left\| C \int_0^\infty e^{-\lambda t} T(t) f \, \mathrm{d}t \right\| = \|CR(\lambda, A)f\| \\ &\leqslant \|CR(\lambda, A)\| \|f\| =: \gamma \|f\|. \end{aligned} \tag{3.18}$$

For arbitrary $f \in D(A)$ we have $f = f^+ - f^-$, but it is not clear if $f^-, f^+ \in D(A)$ holds. Therefore we define the sequences $f_n^+ := nR(n, A)f^+ \in D(A)$ and $f_n^- := nR(n, A)f^- \in D(A)$ for all $n > \lambda$. With the help of Lemma 2.1.10, we see that

$$\lim_{n\to\infty} \|A(f_n^+ - f_n^-) - Af\| = 0$$

holds. Using the same lemma, we also get

$$\lim_{n\to\infty} \|f_n^+ - f^+\| = 0 \quad \text{and} \quad \lim_{n\to\infty} \|f_n^- - f^-\| = 0.$$

The next calculation follows by Estimation (3.18) and the triangular inequality.

$$\begin{aligned} &\int_0^\tau \|e^{-\lambda t} GT(t)(f_n^+ - f_n^-)\| \, \mathrm{d}t \\ &\leqslant \int_0^\tau \|e^{-\lambda t} GT(t) f_n^+\| \, \mathrm{d}t + \int_0^\tau \|e^{-\lambda t} GT(t) f_n^-\| \, \mathrm{d}t \\ &\leqslant \gamma(\|f_n^+\| + \|f_n^-\|). \end{aligned}$$

We define $g := e^{-\lambda t} T(t) f$, $g_n^+ := e^{-\lambda t} T(t) f_n^+$ and $g_n^- := e^{-\lambda t} T(t) f_n^-$. Since G is A-bounded there exists $a, b > 0$ such that for all $x \in D(A)$

$$\|Gx\| \leqslant a\|Ax\| + b\|x\|$$

holds. Keeping this in mind we calculate

$$\begin{aligned}|\|G(g_n^+ - g_n^-)\| - \|Gg\|| &\leqslant \|Gg_n^+ - Gg_n^- - Gg\| \\ &\leqslant a\|Ag_n^+ - Ag_n^- - Ag\| + b\|g_n^+ - g_n^- - g\| \\ &\leqslant \tilde{a}\|A(f_n^+ - f_n^-) - Af\| + \tilde{b}\|(f_n^+ - f_n^-) - f\|,\end{aligned}$$

where we have used in the last line that $e^{-\lambda t}T(t)$ is bounded on compact intervals for t. Therefore, we see that $Ge^{-\lambda t}T(t)(f_n^+ - f_n^-)$ converges uniformly to $Ge^{-\lambda t}T(t)f$ on compact intervals for $n \to \infty$ and this implies with the additivity of the norm for every $f \in D(A)$

$$\int_0^\tau \|e^{-\lambda t}GT(t)f\| \,\mathrm{d}t \leqslant \gamma(\|f^+\| + \|f^-\|) = \gamma\|f\|. \tag{3.19}$$

Using the same arguments as above, we also get

$$\int_0^\tau \|e^{-\lambda t}CT(t)f\| \,\mathrm{d}t \leqslant \gamma\|f\| \tag{3.20}$$

for all $f \in D(A)$. Because $\gamma < 1$, the operator $(G, D(G))$ is a Miyadera-Voigt perturbation of $A - \lambda$. Therefore, Theorem 3.0.6 implies that $A + G - \lambda$ is the generator of a strongly continuous semigroup and hence by Proposition 2.1.14 $A+G$ is a generator of a strongly continuous semigroup, too. Moreover, Estimation (3.20) and Proposition 2.1.14 show that $A + C$ is a generator of a strongly continuous semigroup $(S(t))_{t\geqslant 0}$, which is also a positive semigroup. This follows by the Dyson-Phillips series (3.4) and (3.5), because C and $(T(t))_{t\geqslant 0}$ are positive. ■

Remark 3.2.2 *The perturbing operator G with $|Gf| \leqslant Cf$ was not stated in the original perturbation result form Voigt, Desch respectively. This additional perturbation result can be found in [8, Proposition 13.7].*

Using Theorem 3.1.2, we can develop the above proposition to this next theorem.

Theorem 3.2.3 *Let $(A, D(A))$ be the generator of a positive strongly continuous semigroup on a real AL-space E, $C : D(A) \to E$ be a positive linear operator and $G : D(A) \to E$ be a linear operator with $|Gf| \leqslant Cf$ for any $f \in D(A)_+$. If for some $\lambda > s(A)$ we have $\lambda \in \rho(A + C)$ and the resolvent $R(\lambda, A + C)$ is positive, then $A + G$ with domain $D(A)$ generates a strongly continuous semigroup on E and $A + C$ with domain $D(A)$ generates a positive strongly continuous semigroup on E.*

Proof: Theorem 3.1.2 implies $r(CR(\lambda, A)) < 1$ and in the proof of this theorem we have seen that

$$R(\lambda, A + C) = R(\lambda, A)\sum_{k=0}^{\infty}(CR(\lambda, A))^k \tag{3.21}$$

holds. Moreover, the operator $CR(\lambda, A+C)\colon E \to E$ is positive and Theorem 2.2.22 states that $CR(\lambda, A+C)$ is bounded. Hence there exists a constant $M \in \mathbb{N}$ such that

$$\|CR(\lambda, A+C)\| < M. \tag{3.22}$$

In the proof of Theorem 3.1.2 we have already seen that

$$0 \leqslant R(\lambda, A) \leqslant R(\lambda, A+C) \tag{3.23}$$

holds for some $\lambda > s(A)$. Further, we have for any $j = 0, \dots, M$

$$R(\lambda, A) \sum_{k=0}^{\infty} \left(\frac{j}{M}\right)^k (CR(\lambda, A))^k \leqslant R(\lambda, A) \sum_{k=0}^{\infty} (CR(\lambda, A))^k. \tag{3.24}$$

This and identity (3.21) give

$$0 \leqslant R(\lambda, A) \leqslant R\left(\lambda, A + \frac{j}{M} C\right) \leqslant R(\lambda, A+C), \tag{3.25}$$

which implies

$$\left\| \frac{1}{M} CR\left(\lambda, A + \frac{j}{M} C\right) \right\| < 1, \tag{3.26}$$

for all $j \in \{0, \dots, M-1\}$. Now Proposition 3.2.1 shows that $A + \frac{1}{M} C$ generates a positive strongly continuous semigroup. Repeating this step with $A + \frac{1}{M} C$ and C we obtain that $A + \frac{1}{M} C + \frac{1}{M} C = A + \frac{2}{M} C$ generates a positive strongly continuous semigroup. Iterating this process M-times yields that $A + \frac{M-1}{M} C + \frac{1}{M} C = A + C$ generates a positive strongly continuous semigroup on E. By the Hille-Yosida Theorem 2.1.9 we know that there has to be a constant $\omega \in \mathbb{R}$ such that

$$\sup_{\lambda > \omega, k \in \mathbb{N}} \| \left[(\lambda - \omega)^k R(\lambda, A+C) \right]^k \| < \infty \tag{3.27}$$

holds.

For the last part of the proof take $\lambda > \max\{s(A), s(A+C)\}$. We have seen in the proof of Proposition 3.2.1 that for arbitrary $f \in E$

$$|GR(\lambda, A) f| \leqslant CR(\lambda, A)|f|, \tag{3.28}$$

holds. Iterating the above inequality we obtain for $f \in E$ and all $k \in \mathbb{N}$

$$|(GR(\lambda, A))^k f| \leqslant (CR(\lambda, A))^k |f|. \tag{3.29}$$

Applying Lemma 2.2.26 to the generator $A + C$ gives $R(\lambda, A+C) \geqslant 0$ for all $\lambda > s(A+C)$. Then Proposition 3.2.1 implies $r(CR(\lambda, A)) < 1$. Thus, estimation (3.29) shows $r(GR(\lambda, A)) < 1$ and we see that the identity (3.21) is also true for $R(\lambda, A+G)$. We calculate for $f \in E$

$$\begin{aligned} |R(\lambda, A+G) f| &= \left| R(\lambda, A) \sum_{k=0}^{\infty} (GR(\lambda, A))^k f \right| \\ &\leqslant R(\lambda, A) \sum_{k=0}^{\infty} (CR(\lambda, A))^k |f| \\ &= R(\lambda, A+C)|f|. \end{aligned}$$

Repeating the above estimation we get for all $f \in E$ and all $k \in \mathbb{N}$

$$|R(\lambda, A+G)^k f| \leqslant R(\lambda, A+C)^k |f|. \tag{3.30}$$

Finally, Inequalities (3.30) and (3.27) imply

$$\sup_{\lambda>\omega, k\in\mathbb{N}} \|(\lambda-\omega)^k R(\lambda, A+G)^k\| < \infty. \tag{3.31}$$

Using again the Hille-Yosida Theorem 2.1.9, we have that $A+G$ generates a strongly continuous semigroup on E. ■

3.2.2 Desch-Schappacher perturbation on AM-spaces

Analogously to Proposition 3.2.1 we state the result for positive Desch-Schappacher perturbations on AM-spaces. This result is due to Batkai, Jacob, Voigt and Wintermayr and can be found in [9, Proposition 4.2]. Recall that the dual space of an AL-space is an AM-space and vice versa. This fact was significant for the development of the next statement, because we tried to design the proof with the help of the AM-property $\|\sup\{x,y\}\| = \sup\{\|x\|, \|y\|\}$ for positive elements x, y instead of using the additivity of the norm, which holds for positive elements on AL-spaces.

Proposition 3.2.4 *Let E be a real AM-Space, $(T(t))_{t\geqslant 0}$ a positive strongly continuous semigroup on E with generator $(A, D(A))$. Let $B \in \mathcal{L}(E, E_{-1})$ be a positive operator and suppose further that there exists $\lambda > \mathrm{s}(A)$ such that $K := \|R(\lambda, A_{-1})B\| < 1$. Then*

$$A_{-1} + B \quad \textit{with} \quad D(A_{-1}+B) := \{f \in E : (A_{-1}+B)f \in E\}$$

is the generator of a positive strongly continuous semigroup $(S(t))_{t\geqslant 0}$, and the extrapolation space E_{-1} for this semigroup is the same as for $(T(t))_{t\geqslant 0}$.

Proof: In the following we write $(A_{-1}+B)_{|E}$ instead of $A_{-1}+B$ with domain $D(A_{-1}+B) = \{f \in E : (A_{-1}+B)f \in E\}$. For the first part of the proof we will assume that the given semigroup is exponentially stable, and that $\lambda = 0$. Let $\tau > 0$ be arbitrary and let us denote by $\mathrm{T}([0,\tau]; E)$ the vector space of E-valued step functions. In fact, $\mathrm{T}([0,\tau]; E)$ is a normed vector lattice, a sublattice of $\mathrm{L}^\infty([0,\tau]; E)$. We define a linear operator $R\colon \mathrm{T}([0,\tau]; E) \to E$ by

$$Ru := \int_0^\tau T_{-1}(\tau - s) B u(s)\, \mathrm{d}s.$$

Note that Lemma 2.2.34 implies that R is a positive operator. We show that

$$\|Ru\|_E \leqslant K \|u\|_\infty, \tag{3.32}$$

for all $u \in \mathrm{T}([0,\tau]; E)$. First, let u be a positive step function, $u = \sum_{n=1}^N u_n \chi_{I_n}$ as above, with $u_1, u_2, \ldots, u_N \geqslant 0$. Then $0 \leqslant u \leqslant z\chi_{[0,\tau]}$, where $z := \sup_n u_n$

exists in E. We conclude with the help of Proposition 2.2.32 and Lemma 2.2.34 that

$$\begin{aligned}\|Ru\| &\leqslant \left\| \int_0^\tau T_{-1}(\tau-s)Bz \,\mathrm{d}s \right\| \\ &\leqslant \left\| \int_0^\infty T_{-1}(\tau)Bz \,\mathrm{d}s \right\| \leqslant \|A_{-1}^{-1}B\| \|z\| \\ &= K\|z\| = K\|\sup_n u_n\| = K \sup_n \|u_n\| = K\|u\|_\infty,\end{aligned}$$

where we have used the AM-property of E in the last line. If u is an arbitrary E-valued step function, then $u = u^+ - u^-$, $|Ru| = |Ru^+ - Ru^-| \leqslant Ru^+ + Ru^- = R|u|$, hence $\|Ru\| \leqslant \|R|u|\| \leqslant K\||u|\|_\infty = K\|u\|_\infty$. The Estimate (3.32) implies that R possesses a unique linear continuous extension – still denoted by R – to the closure of $\mathrm{T}([0,\tau];E)$ in $\mathrm{L}^\infty([0,\tau];E)$. This closure contains $\mathrm{C}([0,\tau];E)$, and the Estimate (3.32) carries over to all u in the closure. If $u \in \mathrm{C}([0,\tau];E)$, and $(u_n)_{n\in\mathbb{N}}$ is a sequence in $\mathrm{T}([0,\tau];E)$ converging to u uniformly on $[0,\tau]$, then $Ru_n \to Ru$ in E. But also

$$Ru_n = \int_0^\tau T_{-1}(\tau-s)Bu_n(s)\,\mathrm{d}s \to \int_0^\tau T_{-1}(\tau-s)Bu(s)\,\mathrm{d}s$$

in E_{-1}, because $B\colon E \to E_{-1}$ is continuous and $(T_{-1}(t))_{t\geqslant 0}$ is bounded on $[0,\tau]$. This implies that $\int_0^\tau T_{-1}(\tau-s)Bu(s)\,\mathrm{d}s = Ru \in E$, and that

$$\left\| \int_0^\tau T_{-1}(\tau-s)Bu(s)\,\mathrm{d}s \right\| \leqslant K\|u\|_\infty.$$

Therefore both conditions in Definition 3.0.7 are satisfied; hence, by Theorem 3.0.9, $(A_{-1}+B)_{|E}$ generates a strongly continuous semigroup $(S(t))_{t\geqslant 0}$ which is given by the Dyson–Phillips series (3.6) and (3.7). Using Lemma 2.2.34 and Remark 2.2.33 we conclude that the iterates $S_n(t)$ as well as the semigroup operators $S(t)$ are positive. This shows all the statements for the present case, except for the assertion concerning the extrapolation spaces. For the general case we note that from Proposition 2.2.27 we know that A is the generator of the positive strongly continuous semigroup $(T(t))_{t\geqslant 0}$ if and only if $A-\lambda$ is the generator of the positive strongly continuous semigroup $(\mathrm{e}^{-\lambda t}T(t))_{t\geqslant 0}$. The function $(\mathrm{s}(A),\infty) \ni \lambda \mapsto \|R(\lambda, A_{-1})B\|$ is decreasing, because $R(\lambda, A_{-1})$ is decreasing and B is positive. Now choose $\lambda > \mathrm{s}(A)$ such that $A-\lambda$ generates a positive exponentially stable strongly continuous semigroup and such that $\|R(\lambda, A_{-1})B\| < 1$. Then the case treated so far implies that $(A_{-1}-\lambda+B)_{|E}$ generates a positive strongly continuous semigroup. Now we show the equality of the extrapolation spaces. We choose

$$\lambda > \max\bigl\{\mathrm{s}(A), \mathrm{s}\bigl((A_{-1}+B)_{|E}\bigr)\bigr\}$$

and such that $\|R(\lambda, A_{-1}))B\| < 1$. From the identity

$$(\lambda - A_{-1} - B) = (\lambda - A_{-1})\bigl(I - (\lambda - A_{-1})^{-1}B\bigr)$$

we obtain

$$(\lambda - A_{-1} - B)^{-1} = \big(I - (\lambda - A_{-1})^{-1}B\big)^{-1}(\lambda - A_{-1})^{-1}.$$

Restricting this equality to E we conclude that

$$(\lambda - (A_{-1} - B)_{|E})^{-1} = \big(I - (\lambda - A_{-1})^{-1}B\big)^{-1}(\lambda - A)^{-1}.$$

In view of the continuous invertibility of the first operator on the right hand side, this equality shows that the $\|\cdot\|_{-1}$-norms corresponding to $(A_{-1} - B)_{|E}$ and A are equivalent on E, and therefore the completions are the same. ∎

Next we give the main result for positive Desch-Schappacher perturbations, which is in some sense the dual part to Theorem 3.2.3. This is given in the paper of Batkai, Jacob, Voigt, Wintermayr [9, Theorem 1.2].

Theorem 3.2.5 *Let $(A, D(A))$ be the generator of a positive strongly continuous semigroup $(T(t))_{t\geqslant 0}$ on a real AM-space E. Further, $B\colon E \to E_{-1}$ a linear positive operator and $F \in \mathcal{L}(E, E_{-1})$ such that $|R(\lambda, A_{-1})Fx| \leqslant R(\lambda, A_{-1})Bx$ for all $x \in E_+$. If for some $\lambda > s(A)$ we have $\lambda \in \rho(A_{-1}+B)$ and the resolvent $R(\lambda, A_{-1}+B)$ is positive, then $A_{-1}+B$ with domain $D(A_{-1}+B) := \{f \in E : (A_{-1}+B)f \in E\}$ generates a positive strongly continuous semigroup on E and $A_{-1}+F$ with domain $D(A_{-1}+F) := \{f \in E : (A_{-1}+F)f \in E\}$ generates a strongly continuous semigroup on E.*

Proof: The following is an adaptation of the proof given in Voigt [80, Proof of Theorem 0.1]. Let $\lambda > s(A)$ such that $\lambda \in \rho(A_{-1}+B)$, $R(\lambda, A_{-1}+B) \geqslant 0$. By Theorem 3.1.3 this is equivalent to $r\big(R(\lambda - A_{-1})^{-1}B\big) < 1$. From the proof of Theorem 3.1.3 we have the identity

$$R(\lambda, A_{-1}+B) = \sum_{n=0}^{\infty}(R(\lambda, A_{-1})B)^n R(\lambda, A_{-1}). \tag{3.33}$$

This gives for any $s \in (0,1)$

$$\begin{aligned} R(\lambda, A_{-1}) &\leqslant R(\lambda, A_{-1}) + \sum_{n=1}^{\infty} s^n (R(\lambda, A_{-1})B)^n R(\lambda, A_{-1}) \\ &= \sum_{n=0}^{\infty} s^n (R(\lambda, A_{-1})B)^n R(\lambda, A_{-1}) \\ &\leqslant \sum_{n=0}^{\infty}(R(\lambda, A_{-1})B)^n R(\lambda, A_{-1}) \end{aligned}$$

and therefore

$$R(\lambda, A_{-1}) \leqslant R(\lambda, A_{-1}+sB) \leqslant R(\lambda, A_{-1}+B)$$

for all $s \in (0,1)$. Moreover, Theorem 2.2.22 and the positivity of B and $R(\lambda, A_{-1}+B)$ implies the boundedness of $R(\lambda, A_{-1}+B)B$. Hence, there exists a constant $M > 0$ such that

$$\|R(\lambda, A_{-1}+B)B\| < M. \tag{3.34}$$

This implies

$$\left\| R\left(\lambda, A_{-1} + \left(\frac{j}{M}\right) B\right) \left(\frac{1}{M}\right) B \right\| < 1$$

for all $j = 0, \ldots, M-1$. Applying Proposition 3.2.4 successively to the operators

$$A, \left(A_{-1} + \left(\frac{1}{M}\right) B\right)_{|E}, \ldots, \left(A_{-1} + \left(\frac{(M-1)}{M}\right) B\right)_{|E},$$

with the perturbation $\frac{1}{M}B$, the desired result is obtained (cf. proof of Theorem 3.2.3). Recall from the Proof of Proposition 3.2.4 that we write $(A_{-1}+B)_{|E}$ instead of $A_{-1}+B$ with domain $D(A_{-1}+B) = \{f \in E : (A_{-1}+B)f \in E\}$. An important point in this sequence of steps is that the extrapolation space E_{-1} does not change; this issue is taken care of by the last statement of Proposition 3.2.4.

The last part of the proof follows by iterating the assumption

$$|R(\lambda, A_{-1})Fx| \leqslant R(\lambda, A_{-1})B|x|.$$

This leads to

$$|(R(\lambda, A_{-1})F)^k x| \leqslant (R(\lambda, A_{-1})B)^k |x| \tag{3.35}$$

for any $k \in \mathbb{N}$ and for all $x \in E_+$. Since

$$\|R(\lambda, A_{-1})F\| \leqslant \|R(\lambda, A_{-1})B\|, \tag{3.36}$$

this gives for $R(\lambda, A_{-1})F$ the same Identity (3.33) (with F instead of B). It follows with Inequality (3.35)

$$\begin{aligned}|R(\lambda, A_{-1} + F)x| &= \left|\sum_{n=0}^{\infty}(R(\lambda, A_{-1})F)^k R(\lambda, A_{-1})x\right| \\ &\leqslant \sum_{n=0}^{\infty}(R(\lambda, A_{-1})B)^k R(\lambda, A_{-1})|x| = R(\lambda, A_{-1} + B)|x|.\end{aligned}$$

for all $x \in E_+$. By iterating the above result we get

$$\|R(\lambda, A_{-1} + F)^k\| \leqslant \|R(\lambda, A_{-1} + B)^k\| \tag{3.37}$$

for all $k \in \mathbb{N}$. Further, $(A_{-1} + B)_{|E}$ generates a strongly continuous semigroup and the Hille-Yosida Theorem implies that there exists a constant $\omega \in \mathbb{R}$ such that

$$\sup_{\lambda > \omega, k \in \mathbb{N}} \|(\lambda - \omega)^k R(\lambda, A_{-1} + B)^k\| < \infty. \tag{3.38}$$

holds. Combining this with estimation (3.37) we get

$$\sup_{\lambda > \omega, k \in \mathbb{N}} \|(\lambda - \omega)^k R(\lambda, A_{-1} + F)^k\| < \infty. \tag{3.39}$$

Finally, the operator $(A_{-1} + F)_{|E}$ generates a strongly continuous semigroup by applying the Hille-Yosida Theorem again. ■

3.2.3 Examples

First we state a counterexample on an AM-space with a positive perturbing A-bounded operator. This case is in some sense between our statements of Theorem 3.2.3 and Theorem 3.2.5. The example is given in the book of Bátkai, Kramer and Rhandi [8, Example 13.17.].

Example 3.2.6 *Let* $X := \{f \in C([0,1]) : f(0) = 0\}$ *and*

$$Lf = -f' \qquad \text{for} \qquad f \in D(L) := \{f \in C^1([0,1]) : f'(0) = f(0) = 0\}.$$

We know from [24, Chapter II, Corollary 3.18 and Example 3.19] that $(L, D(L))$ *generates a positive strongly continuous contraction semigroup* $(T(t))_{t \geqslant 0}$ *on* X*. This semigroup can be identified as the nilpotent right translation semigroup, which is defined via*

$$T(t)f(s) = \begin{cases} f(s-t) & \text{for } t \leqslant s, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, $(T(t))_{t \geqslant 0}$ *is a positive semigroup on* X*. Now, we define the perturbing operator*

$$Cf(s) = \begin{cases} \frac{1}{s} f(s) & \text{if } s \in]0,1], \\ 0 & \text{otherwise.} \end{cases}$$

on the domain $D(C) = D(L) = \{f \in C^1([0,1]) : f'(0) = f(0) = 0\}$*. It is easy to see that* $(C, D(C))$ *is a positive operator, too. Let now* $E := X \times X$*, then we define the operators*

$$A := \begin{pmatrix} L & 0 \\ 0 & L \end{pmatrix} \qquad \text{and} \qquad B := \begin{pmatrix} 0 & 0 \\ C & 0 \end{pmatrix}$$

on E*. Next, let*

$$G_\lambda := \begin{pmatrix} R(\lambda, L) & 0 \\ R(\lambda, L)CR(\lambda, L) & R(\lambda, L) \end{pmatrix}.$$

An easy calculation shows that $G_\lambda(\lambda - A - B) = I$ *and also* $(\lambda - A - B)G_\lambda = I$*. Therefore, the operator* G_λ *is the resolvent for* $\lambda - (A + B)$*, i.e.*
$G_\lambda = R(\lambda, A + B)$*. Moreover, with the Laplace transform we get for some* $\lambda > s(L)$

$$\begin{aligned} (R(\lambda, L)f)(s) &= \left(\int_0^\infty e^{-\lambda r} T(r) f(\cdot) \, \mathrm{d}r \right)(s) = \int_0^s e^{-\lambda r} f(s-r) \, \mathrm{d}r \\ &= \int_0^s e^{-\lambda(s-r)} f(r) \, \mathrm{d}r = e^{-\lambda s} \int_0^s e^{\lambda r} f(r) \, \mathrm{d}r. \end{aligned}$$

This shows that L *is resolvent positive. Since* C *is a positive operator, we have* *that $A+B$ is resolvent positive. Now we calculate*

$$\begin{aligned} R(\lambda,L)CR(\lambda,L)f(s) &= e^{-\lambda s}\int_0^s e^{\lambda r}CR(\lambda,L)f(r)\,\mathrm{d}r \\ &= e^{-\lambda s}\int_0^s e^{\lambda r}\frac{1}{r}e^{-\lambda r}\int_0^r e^{\lambda t}f(t)\,\mathrm{d}t\,\mathrm{d}r \\ &= e^{-\lambda s}\int_0^s e^{\lambda t}f(t)\int_t^s \frac{1}{r}\,\mathrm{d}r\,\mathrm{d}t \\ &= \int_0^s e^{-\lambda(s-t)}f(t)\log\left(\frac{s}{t}\right)\mathrm{d}t \\ &= \int_0^s e^{-\lambda u}W(u)f(s)\,\mathrm{d}u, \end{aligned}$$

where we have used Fubini's theorem in the third line, substituted with $u = s-t$ in the last line and where

$$W(u)f(s) = \begin{cases} \log\left(\frac{s}{s-u}\right)f(s-u) & \text{if } s > u, \\ 0 & \text{otherwise.} \end{cases}$$

The uniqueness of the Laplace transform (see [5, Theorem 1.7.3]) guarantees that the semigroup generated by $A+B$ is given by

$$S(t)(f,g) = \begin{pmatrix} T(t)f & 0 \\ W(t)f & T(t)g \end{pmatrix}, \qquad (f,g)\in E.$$

But the operators $W(t)$ for each $t>0$ are not bounded and this implies that B is not a Miyadera-Voigt perturbation.

The following examples are given in Bátkai, Jacob, Voigt and Wintermayr [9, Example 3.3.2-3.3.5]. First we give an application of Theorem 3.2.5.

Example 3.2.7 *Let $h \in \mathrm{L}^1(0,1)_+$. Consider the partial differential equation*

$$\begin{aligned} \frac{\partial}{\partial t}u(t,x) &= \frac{\partial}{\partial x}u(t,x) + \int_0^1 u(t,y)\,\mathrm{d}y\cdot h(x), \qquad x\in[0,1], t\geqslant 0, \\ u(0,x) &= u_0(x), \quad u(t,1) = 0, \qquad x\in[0,1], t\geqslant 0. \end{aligned}$$

We interpret this equation as an abstract Cauchy problem on the AM-space $E := \{f \in \mathrm{C}([0,1]) : f(1) = 0\}$ with norm $\|f\|_\infty := \sup_{x\in[0,1]} |f(x)|$, this leads to the equations

$$\begin{aligned} \dot{u}(t) &= Au(t) + Bu(t) \\ u(0) &= u_0, \end{aligned}$$

where the operator $(A, D(A))$ is defined by

$$Af = f', \qquad D(A) = \{f \in \mathrm{C}^1[0,1] : f(1) = f'(1) = 0\}. \tag{3.40}$$

In Engel and Nagel [24, Chapter II, Example 3.19(i)], it is shown that $(A, D(A))$ *is the generator of the nilpotent positive left-shift semigroup* $(T(t))_{t\geqslant 0}$ *with* $\mathrm{s}(A) = -\infty$*, given by*

$$(T(t)f)(x) = \begin{cases} f(s+t) & \text{if } x+t \leqslant 1, \\ 0 & \text{otherwise.} \end{cases} \tag{3.41}$$

Obviously, there exists $h \in L^1(0,1)_+$ *with* $h \notin E$ *and so we cannot realise* B *as a linear operator in* E*. However, our aim is to show* $B \in \mathcal{L}(E, E_{-1})$ *and we claim that the extrapolation space of* E *for the generator* $(A, D(A))$ *is given by*

$$E_{-1} = \{g \in \mathscr{D}(0,1)' : \, g = \partial f \text{ for some } f \in E\}, \tag{3.42}$$

where $\mathscr{D}(0,1) = C_{\mathrm{c}}^{\infty}(0,1)$ *denotes the usual space of 'test functions', with the inductive limit topology. Further we denote by* $\mathscr{D}(0,1)'$ *its dual space, and* ∂ *denotes the differentiation on distributions. The 'standard embedding'* $j \colon E \hookrightarrow \mathscr{D}(0,1)'$ *can be extended to a mapping*

$$j_{-1} \colon E_{-1} \to \mathscr{D}(0,1)',$$

defined by

$$\langle j_{-1}(g), \varphi \rangle := \big\langle A_{-1}^{-1} g, -\varphi' \big\rangle = -\int_0^1 \big(A_{-1}^{-1} g\big)(x) \varphi'(x) \, \mathrm{d}x.$$

Indeed, if $g \in E$*, then*

$$\begin{aligned} \langle j_{-1}(g), \varphi \rangle &= -\int_0^1 (A^{-1} g)(x) \varphi'(x) \, \mathrm{d}x \\ &= -\big[(A^{-1}g)(x)\varphi(x)\big]_0^1 + \int_0^1 (A^{-1}g)'(x)\varphi(x) \, \mathrm{d}x \\ &= 0 + \int_0^1 g(x)\varphi(x) \, \mathrm{d}x, \end{aligned}$$

which shows that j_{-1} *is an extension of* j*. Next, we prove that* j_{-1} *is injective, which is necessary for an embedding. Let* $g \in E_{-1}$ *such that* $j_{-1}(g) = 0$*, then* $\int_0^1 \big(A_{-1}^{-1} g\big)(x)\varphi'(x) \, \mathrm{d}x = 0$ *for all* $\varphi \in \mathscr{D}(0,1)$*, which implies by the fundamental Lemma of the variation calculation (see e.g. [41, Lemma 1.2.1.]) that the continuous function* $A_{-1}^{-1} g$ *is constant and this constant has to be zero because* $\big(A_{-1}^{-1} g\big)(1) = 0$*. Then the injectivity of* A_{-1}^{-1} *implies* $g = 0$*. In fact, we see that the following diagram commutes*

$$\begin{array}{ccc} E & \underset{A_{-1}^{-1}}{\overset{A_{-1}}{\rightleftarrows}} & E_{-1} \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle j_{-1}} \\ \mathscr{D}' & \xrightarrow[\partial]{} & \mathscr{D}' \end{array}$$

and we have

$$j_{-1} = \partial \circ j \circ A_{-1}^{-1}. \tag{3.43}$$

This formula shows that j_{-1} maps E_{-1} continuously to $\mathscr{D}(0,1)'$, because j_{-1} is the composition of the continuous operators $\partial : \mathscr{D}' \to \mathscr{D}'$, $j : E \to \mathscr{D}'$ and $A_{-1}^{-1} : E_{-1} \to E$. So far, we have shown

$$E_{-1} \subseteq \{g \in \mathscr{D}(0,1)' : g = \partial f \text{ for some } f \in E\}.$$

For the other direction let $g \in \mathscr{D}(0,1)'$ such that there exists $f \in E$ with $g = \partial f$. Since A_{-1} is bijective there exists exactly one $h \in E_{-1}$ such that $h = A_{-1}f$ and also $A_{-1}^{-1}h = f$. Then Formula (3.43) gives

$$j_{-1}(h) = \partial(j(A_{-1}^{-1}h)) = \partial(j(f)),$$

which implies $h = \partial f = g$ (in the sense of distribution) and therefore

$$E_{-1} \supseteq \{g \in \mathscr{D}(0,1)' : g = \partial f \text{ for some } f \in E\}.$$

Moreover, we see that in the image of E in $\mathscr{D}(0,1)'$ the operator A_{-1} acts as differentiation ∂.
Next we are going to show that, in the image of E_{-1} in $\mathscr{D}(0,1)'$, one has

$$E_{-1,+} = \{g \in \mathscr{D}(0,1)' : g = \partial f \text{ for some increasing function } f \in E\}. \quad (3.44)$$

So, let $g \in E_{-1,+}$. Then there exists per definition a sequence $(g_n)_{n\in\mathbb{N}}$ in E_+ such that $g_n \to g$ in E_{-1}, and thus in $\mathscr{D}(0,1)'$. Hence

$$\int_0^1 g_n(x)\varphi(x)\,\mathrm{d}x \to \langle g, \varphi\rangle \qquad (n \to \infty),$$

for all $\varphi \in \mathscr{D}(0,1)$, and therefore $\langle g, \varphi\rangle \geqslant 0$ for all $0 \leqslant \varphi \in \mathscr{D}(0,1)$, i.e., g is a 'positive distribution'. It is known that this implies that g is a positive Borel measure; see Schwartz [67, Chap. I, Théorème V]. As g is also the distributional derivative of a function $f \in E$, it follows that f is increasing. For the reverse inclusion, let $f \in E$ be an increasing function. Then one shows by standard methods of Analysis that f can be approximated in E by a sequence $(f_n)_{n\in\mathbb{N}}$ in $E \cap \mathrm{C}^1[0,1]$, all f_n increasing and vanishing in a neighbourhood of 1. This implies that $f_n' \in E_+$ for all $n \in \mathbb{N}$, and that $f_n' \to A_{-1}f = \partial f$ $(n \to \infty)$ in E_{-1}; hence $g := \partial f \in E_{-1,+}$.

Now we come back to treating the initial value problem stated at the beginning. Since $0 \leqslant h \in L^1(0,1)$, we can define $f(x) = -\int_x^1 h(s)\,\mathrm{d}s$. Then $f \in E$ is an increasing function such that $h = \partial f$ (in the sense of distribution), and therefore $h \in E_{-1,+}$. Hence the operator $B \in \mathcal{L}(E, E_{-1})$, defined by $g \mapsto \int_0^1 g(x)\,\mathrm{d}x \cdot h$, is a positive operator. We calculate

$$\begin{aligned} \|R(\lambda, A_{-1})B\| &= \sup_{\|g\|=1} \|R(\lambda, A_{-1})Bg\| \\ &= \sup_{\|g\|=1} \left\| \left(\int_0^1 g(x)\,\mathrm{d}x\right) \cdot R(\lambda, A_{-1})h \right\| \\ &= \|R(\lambda, A_{-1})h\|_E. \end{aligned}$$

From Theorem 2.1.9 it follows that $\|R(\lambda, A_{-1})h\|_E \to 0$ *as* $\lambda \to \infty$, *for all* $h \in E_{-1}$, *because* A_{-1} *is again a generator. Hence* $\|R(\lambda, A_{-1})B\| < 1$ *for large* λ. *Therefore, Proposition 3.2.5 implies that* $(A_{-1} + B)_{|E}$ *is the generator of a positive semigroup.*

Our final example shows that, for an operator $B \in \mathcal{L}(E, E_{-1})$ to be positive it is not sufficient that $R(\lambda, A_{-1})B$ is positive in $\mathcal{L}(E)$ for some $\lambda > \mathrm{s}(A)$.

Example 3.2.8 *Let* E *and* $(A, D(A))$ *be as in Example 3.2.7. Define*

$$h := -\chi_{[0,1/2)} + \chi_{[1/2,1]}.$$

Then the description of $E_{-1,+}$ *in Example 3.2.7 shows that* h *is not positive in* E_{-1}, *because* $h = \partial g$, *for the function* $g \in E$ *given by*

$$g(x) = \begin{cases} -x & \text{if } x \in \left[0, \frac{1}{2}\right[, \\ x - 1 & \text{if } x \in \left[\frac{1}{2}, 1\right], \end{cases}$$

which is not increasing. However, $R(0, A_{-1})h = (-A_{-1})^{-1}h = -g$ *belongs to* E_+. *Defining the operator* $B \in \mathcal{L}(E, E_{-1})$ *by*

$$Bf := \int_0^1 f(x)\,\mathrm{d}x \cdot h$$

we see that $R(0, A_{-1})B \in \mathcal{L}(E)$ *is positive, but* B *is not positive.*

Chapter 4

Linear systems theory

A huge amount of infinite-dimensional systems can be described on Banach spaces via the following equations (cf. introduction of this thesis)

$$\Sigma(A,B,C,D) \quad \begin{cases} \dot{x}(t) = Ax(t) + Bu(t), & t \geqslant s, \\ y(t) = Cx(t) + Du(t), & t \geqslant s, \\ x(s) = x_s. \end{cases} \tag{4.1}$$

Here $x(t) \in X$ denotes the state of this system which lies in a Banach space X, called the *state space*. Further we have the input $u(t) \in U$ and the output $y(t) \in Y$, where U is called the *control space* and Y the *observation space*, both are assumed to be Banach spaces.

In Section 4.4 we will see that the equations (4.1) can be described by an abstract linear system (under additional assumptions) and we give a definition for such a system to be well-posed and to be positive. But first we restrict $\Sigma(A,B,C,D)$ to the case of positive time, i.e. $t \geqslant 0$ with initial time zero and initial value $x(0) = x_0$. Further we set $D = 0$ and additionally $B = 0$ or $C = 0$. Summarising this, we consider in the next section the system

$$\Sigma(A,B) \quad \begin{cases} \dot{x}(t) = Ax(t) + Bu(t), & t \geqslant 0 \\ x(0) = x_0, \end{cases} \tag{4.2}$$

or

$$\Sigma(A,C) \quad \begin{cases} \dot{x}(t) = Ax(t), & t \geqslant 0, \\ y(t) = Cx(t), & t \geqslant 0, \\ x(0) = x_0. \end{cases} \tag{4.3}$$

4.1 Admissibility

We begin this section with the definition of admissibility for the operators B and C. Here we assume $C \in \mathcal{L}(X_1, Y)$ and $B \in \mathcal{L}(U, X_{-1})$.

The vector valued function space $Z(I,U)$ will always refer to either $\mathrm{L}^p(I,U)$ for $p \in [1,\infty]$, $Reg(I,U)$ (the vector space of all regulated functions), $C(I,U)$ or an U-valued Orlicz space $E_\Phi(I,U)$ for some interval $I \subseteq [0,\infty)$ and a Young function Φ that satisfies the Δ_2 condition.

Definition 4.1.1 *Let U and X be Banach spaces, let $(T(t))_{t\geqslant 0}$ be a semigroup on X, let $B \in \mathcal{L}(U, X_{-1})$. We call B a* (finite-time) Z-admissible control operator *for $U, X, (T(t))_{t\geqslant 0}$, if for some $\tau > 0$ the operator*

$$\Phi_\tau \colon Z([0,\tau],U) \to X_{-1}, \quad u \mapsto \int_0^\tau T_{-1}(\tau - t)Bu(t)\,\mathrm{d}t \tag{4.4}$$

has range in X, i.e. $\mathrm{Ran}(\Phi_\tau) \subset X$.
If additionally $\sup_{\tau>0} \|\Phi_\tau\| < \infty$ *holds, then B is called* infinite-time Z-admissible. *In the case $Z([0,\tau],U) = \mathrm{L}^p(0,\tau),U)$ for $p \in [1,\infty]$ we call B* L^p-admissible *or* infinite-time L^p-admissible, *if it is Z-admissible or infinite-time Z-admissible and we denote by*

$$\mathbb{B}_p(U, X, (T(t))_{t\geqslant 0})$$

the vector space of all L^p-admissible control operators B for $U, X, (T(t))_{t\geqslant 0}$ and p. In the case $Z([0,\tau],U) = Reg([0,\tau],U)$ we call B regulated admissible *or* infinite-time regulated admissible, *if it is Z-admissible or infinite-time Z-admissible.*
At last if $Z([0,\tau],U) = C([0,\tau],U)$ we call B continuous admissible *or* infinite-time continuous admissible, *if it is Z-admissible or infinite-time Z-admissible.*

Definition 4.1.2 *Let Y and X be Banach spaces, let $(T(t))_{t\geqslant 0}$ be a semigroup on X and let $C \in \mathcal{L}(X_1, Y)$. We call C a* (finite-time) Z-admissible observation operator *for $Y, X, (T(t))_{t\geqslant 0}$, if for some $\tau > 0$ the operator*

$$\Psi_\tau \colon X_1 \to Z([0,\tau],Y), \quad x \mapsto CT_1(\cdot)x \tag{4.5}$$

has a bounded extension to X, which we denote again by Ψ_τ.
If additionally $\sup_{\tau>0} \|\Psi_\tau\| < \infty$ *holds, then C is called* infinite-time Z-admissible. *In the case $Z([0,\tau],U) = \mathrm{L}^p(0,\tau),U)$ for $p \in [1,\infty]$ we call C* L^p-admissible *or* infinite-time L^p-admissible, *if it is Z-admissible or infinite-time Z-admissible and we denote by*

$$\mathbb{C}_p(X, Y, (T(t))_{t\geqslant 0})$$

the vector space of all L^p-admissible observation operator for $X, Y, (T(t))_{t\geqslant 0}$ and p. In the case $Z([0,\tau],U) = Reg([0,\tau],U)$ we call C regulated admissible *or* infinite-time regulated admissible, *if it is Z-admissible or infinite-time Z-admissible.*
At last if $Z([0,\tau],U) = C([0,\tau],U)$ we call C continuous admissible *or* infinite-time continuous admissible, *if it is Z-admissible or infinite-time Z-admissible.*

Remark 4.1.3 *By the Closed-Graph Theorem we have that B is Z-admissible if and only if there exists some constant $K_\tau > 0$ such that for all $u \in Z([0,\tau],U)$ we have*

$$\left\| \int_0^\tau T_{-1}(\tau - s)Bu(s)\,\mathrm{d}s \right\| \leqslant K_\tau \|u\|_{Z([0,\tau],U)}. \tag{4.6}$$

We define the best possible constant of all choices of K_τ via

$$K_{\tau,B} := \sup_{\|u\|_Z \leqslant 1} \left\| \int_0^\tau T_{-1}(\tau - s)Bu(s)\,\mathrm{d}s \right\|. \tag{4.7}$$

In the same manner we have that an observation operator C is Z-admissible if and only if there exists some constant $K_\tau > 0$ such that for all $x \in D(A)$ we have

$$\|CT(t)x\|_{Z([0,\tau],Y)} \leqslant K_\tau \|x\|.$$

The best constant of the above inequality is given by

$$K_{\tau,C} := \sup_{\|x\|_{X_1} \leqslant 1} \|CT(t)x\|_{Z([0,\tau],Y)}. \tag{4.8}$$

Remark 4.1.4 *Clearly, because $C(I,U) \subset Reg(I,U) \subset \mathrm{L}^\infty(I,U)$ we have that every L^∞-admissible control operator is regulated admissible and every regulated admissible control operator is continuous admissible, too.*

In the next proposition we exclude the U-valued Orlicz space for the function space $Z([0,\tau],U)$, i.e. we specify by Z-admissible the cases L^p-admissible (for $p \in [1,\infty]$), continuous admissible or regulated admissible.

Proposition 4.1.5 *Let X, Y, and U be Banach spaces, $(T(t))_{t\geqslant 0}$ be a strongly continuous semigroup, $B \in \mathcal{L}(U, X_{-1})$, $C \in \mathcal{L}(X_1, Y)$ and $\alpha \in \mathbb{R}$ be arbitrary. Then B is Z-admissible for $(T(t))_{t\geqslant 0}$ if and only if B is Z-admissible for the rescaled semigroup $(\mathrm{e}^{\alpha t}T(t))_{t\geqslant 0}$.*
Moreover, we have that C is Z-admissible for $(T(t))_{t\geqslant 0}$ if and only if C is Z-admissible for the rescaled semigroup $(\mathrm{e}^{\alpha t}T(t))_{t\geqslant 0}$.

Proof: First assume that B is a Z-admissible control operator for $(T(t))_{t\geqslant 0}$. Let $u \in Z([0,\tau],U)$ be arbitrary, then we have $e^{\alpha\cdot} \otimes u \in Z([0,\tau],U)$ for all $\alpha \in \mathbb{R}$. Define $M := |e^{\alpha t}|$ and $\tilde{u}(s) := e^{-\alpha s}u(s)$. We get

$$\|\tilde{u}\|_{Z([0,\tau],U)} \leqslant e^{|\alpha|\tau}\|u\|_{Z([0,\tau],U)} \tag{4.9}$$

and we calculate

$$\begin{aligned}
\left\| \int_0^\tau e^{\alpha(\tau-s)} T_{-1}(\tau - s)Bu(s)\,\mathrm{d}s \right\| &= |e^{\alpha\tau}| \left\| \int_0^\tau T_{-1}(\tau - s)Be^{-\alpha s}u(s)\,\mathrm{d}s \right\| \\
&= M \left\| \int_0^\tau T_{-1}(\tau - s)B\tilde{u}(s)\,\mathrm{d}s \right\| \\
&\leqslant MK_\tau \|\tilde{u}\|_{Z([0,\tau],U)} \leqslant Me^{|\alpha|\tau}K_\tau\|u\|_{Z([0,\tau],U)}.
\end{aligned}$$

For the other direction assume that B is Z-admissible for the rescaled semigroup $(e^{\alpha t}T(t))_{t\geqslant 0}$. Define $M := |e^{-\alpha t}|$ and $\tilde{u}(s) := e^{\alpha s}u(s)$. Then we calculate

$$\begin{aligned}
\left\| \int_0^\tau T_{-1}(\tau - s)Bu(s)\,\mathrm{d}s \right\| &= \left\| e^{-\alpha\tau} \int_0^\tau e^{\alpha(\tau-s)} T_{-1}(\tau - s)Be^{\alpha s}u(s)\,\mathrm{d}s \right\| \\
&= M \left\| \int_0^\tau e^{\alpha(\tau-s)} T_{-1}(\tau - s)B\tilde{u}(s)\,\mathrm{d}s \right\| \\
&\leqslant MK_\tau\|\tilde{u}\|_{Z([0,\tau],U)} \leqslant Me^{|\alpha|\tau}K_\tau\|u\|_{Z([0,\tau],U)}.
\end{aligned}$$

The proof of the statement for the observation C follows in a very similar way.
■

Lemma 4.1.6 *Let U and X be Banach spaces, let $(T(t))_{t\geqslant 0}$ be a semigroup on X, let $B \in \mathcal{L}(U, X_{-1})$ and $p \in [1,\infty)$. Then B is a L^p-admissible control operator for $U, X, (T(t))_{t\geqslant 0}$, if for some $\tau > 0$, $K > 0$ and for any step function $u \in T([0,\tau], U)$*

$$\left\| \int_0^\tau T_{-1}(\tau - t) B u(t) \, \mathrm{d}t \right\| \leqslant K \|u\|_{\mathrm{L}^p([0,\tau],U)} \tag{4.10}$$

holds.

Proof: It is well-know that the vector space of step functions is dense in $\mathrm{L}^p([0,\tau], U)$ for every $p \in [1, \infty)$. Hence, for every function $v \in \mathrm{L}^p([0,\tau], U)$ there exists a sequence of step functions $(u_n)_{n\in\mathbb{N}} \subset T([0,\tau], U)$ converging to v in $\mathrm{L}^p([0,\tau], U)$, i.e.

$$\|v - u_n\|_{\mathrm{L}^p([0,\tau],U)} \longrightarrow 0 \quad \text{for } n \to \infty.$$

We define

$$\mathcal{B}u := \int_0^\tau T_{-1}(\tau - t) B u(t) \, \mathrm{d}t.$$

Because $B \colon X \to X_{-1}$ is continuous and $(T_{-1}(t))_{t\geqslant 0}$ is bounded on $[0,\tau]$ in X_{-1}, we have that $\mathcal{B}u_n \to \mathcal{B}v$ in X_{-1} for $n \to \infty$. Further, we know from Proposition 2.2.34 ii) (for this item the positivity of the semigroup is not needed) that for each step function $u_n \in T([0,\tau], U)$ we have

$$\int_0^\tau T_{-1}(\tau - t) B u_n(t) \, \mathrm{d}t \in X. \tag{4.11}$$

With estimation (4.10) and the linearity of the integral and the operators we get

$$\left\| \int_0^\tau T_{-1}(\tau - t) B u_n(t) \, \mathrm{d}t - \int_0^\tau T_{-1}(\tau - t) B u_m(t) \, \mathrm{d}t \right\| \leqslant K \|u_n - u_m\|_{\mathrm{L}^p([0,\tau],U)}.$$

This shows that $(\mathcal{B}u_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in X, since $(u_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathrm{L}^p([0,\tau], U)$. Because X is a Banach space and the limit is unique we get

$$\mathcal{B}v = \int_0^\tau T_{-1}(\tau - t) B v(t) \, \mathrm{d}t \in X.$$

■

Remark 4.1.7 *Because the vector space of all step functions $\mathrm{T}([0,\tau], U)$ is dense in the vector space of all regulated functions $Reg([0,\tau], U)$, the above result is true for regulated admissible control operators, too.*

For $Z = \mathrm{L}^p([0,\tau], U)$ and $Z = E_\Phi([0,\tau], U)$, the following is a result from Jacob, Nabiullin, Partington and Schwenninger [33, Lemma 2.8].

Lemma 4.1.8 *If the semigroup is exponentially stable and the control operator B is Z-admissible, then B is infinite-time Z-admissible.*

G. Weiss published a duality result between control and observation operators, which we state below.

Theorem 4.1.9 ([83], Theorem 6.9) *Let U, X, Y be Banach spaces, let $(T(t))_{t\geqslant 0}$ be a strongly continuous semigroup on X such that $(T'(t))_{t\geqslant 0}$ is also a strongly continuous semigroup and let $p, q \in [1, \infty]$ with $\frac{1}{p} + \frac{1}{q} = 1$. Then we have the following duality relations:*

(i) For any $C \in \mathcal{L}(X_1, Y)$,

$$C \in \mathbb{C}_q \left(X, Y, (T(t))_{t\geqslant 0}\right) \Longleftrightarrow C' \in \mathbb{B}_p \left(Y', X', (T'(t))_{t\geqslant 0}\right),$$

(ii) For any $B \in \mathcal{L}(U, X_{-1})$, if $p < \infty$ or if X is reflexive,

$$B \in \mathbb{B}_p \left(U, X, (T(t))_{t\geqslant 0}\right) \Longleftrightarrow B' \in \mathbb{C}_q \left(X', U', (T'(t))_{t\geqslant 0}\right).$$

4.1.1 Range condition for admissible control operators

Our aim in this paragraph is to show that a control operator is admissible if its range lies in some range of an admissible control operator. To do this we use quotient spaces, the properties of the quotient mapping and we restrict the control space to Hilbert spaces. We start with a few preparations for the main result, Theorem 4.1.13, in this paragraph.

Proposition 4.1.10 *Let X be a Banach space and U, V be Hilbert spaces, $(T(t))_{t\geqslant 0}$ be a strongly continuous semigroup on X with generator $(A, D(A))$, $B \in \mathcal{L}(U, X_{-1})$ be L^p-admissible for some $p \in [1, \infty]$ and $\tilde{B} \in \mathcal{L}(V, X_{-1})$. Further there exists a linear, continuous and surjective mapping $q : U \to V$ such that $\tilde{B} \circ q = B$ holds. Then we have that $\tilde{B}$ is also L^p-admissible (with the same p).*

Proof: There exists a linear continuous and injective mapping $\tilde{q} : V \to U$ such that $q\tilde{q} = Id$, i.e. the mapping q has a right inverse (see e.g. [74], page 94). Now let $v \in \mathrm{L}^p(0, t; V)$, we have to show that $\int_0^t T_{-1}(t-s)\tilde{B}v(s)\,\mathrm{d}s \in X$. Define $u(s) := \tilde{q}(v(s))$, since $\tilde{q}$ is continuous we get that the function u is measurable and if $p < \infty$ we have

$$\int_0^t \|u(s)\|^p \,\mathrm{d}s = \int_0^t \|\tilde{q}(v(s))\|^p \,\mathrm{d}s \leqslant M^p \int_0^t \|v\|^p \,\mathrm{d}s < \infty$$

and in the case where $p = \infty$ we have

$$\operatorname*{ess\,sup}_{s\in[0,t]} \|u(s)\| = \operatorname*{ess\,sup}_{s\in[0,t]} \|\tilde{q}(v(s))\| \leqslant M \operatorname*{ess\,sup}_{s\in[0,t]} \|(v(s))\| < \infty, \tag{4.12}$$

where $M > 0$ is a constant with $\|\tilde{q}\| \leqslant M$. This shows $u \in \mathrm{L}^p(0,t;U)$. Further we have $qu(s) = q\tilde{q}v(s) = v(s)$ and we can calculate

$$\begin{aligned}\int_0^t T_{-1}(t-s)\tilde{B}v(s)\,\mathrm{d}s &= \int_0^t T_{-1}(t-s)\tilde{B}qu(s)\,\mathrm{d}s \\ &= \int_0^t T_{-1}(t-s)Bu(s)\,\mathrm{d}s \in X.\end{aligned}$$

This shows that $\tilde{B}$ is L^p-admissible, if B is L^p-admissible. ∎

From basic functional analysis we know that the quotient mapping is a linear, continuous and surjective mapping. This leads to the next corollary.

Corollary 4.1.11 *Let X be a Banach space and U be Hilbert spaces, $(T(t))_{t\geqslant 0}$ be a strongly continuous semigroup on X with generator $(A, D(A))$, $B \in \mathcal{L}(U, X_{-1})$ be L^p-admissible for some $p \in [1,\infty]$. Then there exists an operator $\hat{B} : \hat{U} \to X_{-1}$ which is L^p-admissible, where*

$$\hat{U} := {}^{U}\!/_{N(B)}$$

is the quotient space for $N(B) := \{y \in U \,:\, By = 0\}$.

Proof: The quotient mapping $q : U \to \hat{U}$ is surjective, linear and continuous. Further there exists an operator $\hat{B} \in L\left(\hat{U}, X_{-1}\right)$ with $\mathrm{Ran}(\hat{B}) = \mathrm{Ran}(B)$ and $\hat{B}q = B$. The conclusion follows from the proposition above. ∎

Before we state our main result of this paragraph, we need one further preparation.

Lemma 4.1.12 *Let X be a Banach space, U, V be Hilbert spaces, $(T(t))_{t\geqslant 0}$ be a strongly continuous semigroup on X with generator $(A, D(A))$, $B \in \mathcal{L}(U, X_{-1})$ be L^p-admissible for some $p \in [1,\infty]$ and $G \in \mathcal{L}(V, X_{-1})$. Further there exists a continuous and linear mapping $S : V \to U$ with $G = BS$. Then G is a L^p-admissible operator (with the same p).*

Proof: Let $v \in \mathrm{L}^p(0,t;V$ be arbitrary and define $u(s) := S(v(s)) \in \mathrm{L}^p(0,t;U)$. Then we get

$$\begin{aligned}\int_0^t T_{-1}(t-s)Gv(s)\,\mathrm{d}s &= \int_0^t T_{-1}(t-s)BSv(s)\,\mathrm{d}s \\ &= \int_0^t T_{-1}(t-s)Bu(s)\,\mathrm{d}s \in X\end{aligned}$$

∎

Combining the above statements, we can state the main theorem of this paragraph.

Theorem 4.1.13 *Let X be a Banach space, U, V be Hilbert spaces, $(T(t))_{t\geqslant 0}$ be a strongly continuous semigroup on X with generator $(A, D(A))$, $B \in \mathcal{L}(U, X_{-1})$ be L^p-admissible for some $p \in [1,\infty]$ and $F \in \mathcal{L}(V, X_{-1})$ such that* $\mathrm{Ran}(F) \subseteq \mathrm{Ran}(B)$. *Then F is L^p-admissible (with the same p).*

Proof: From the corollary above we know that $\hat{B} : \hat{U} \to X_{-1}$ is L^p-admissible and also that $\mathrm{Ran}(F) \subseteq \mathrm{Ran}(\hat{B})$ holds (recall that $\hat{U}$ is the quotient space from Corollary 4.1.11). Further $\hat{B}$ is injective and has an inverse on $\mathrm{Ran}(\hat{B})$. This implicates that $\hat{B}^{-1}F$ is well-defined on V. Since $\hat{B}^{-1}F$ is a closed operator it follows from the closed graph theorem that $\hat{B}^{-1}F \in \mathcal{L}(V, \hat{U})$. Now define $S := \hat{B}^{-1}F$. Then we get $F = \hat{B}\hat{B}^{-1}F = \hat{B}S$ and using the above lemma the statement follows. ■

A closer look to the proofs of all statements above lead to the awareness that the given results should not only hold for L^p-admissible control operators. For our further studies we want to consider regulated admissible control operators, too.

Proposition 4.1.14 *Let X be a Banach space, U, V be Hilbert spaces, $(T(t))_{t\geqslant 0}$ be a strongly continuous semigroup on X with generator $(A, D(A))$, $B \in \mathcal{L}(U, X_{-1})$ be regulated admissible and $\tilde{B} \in \mathcal{L}(V, X_{-1})$. Further assume that there exists a linear, continuous and surjective mapping $q : U \to V$ such that $\tilde{B}q = B$ holds. Then we have that $\tilde{B}$ is also regulated admissible.*

Proof: As in the proof of Proposition 4.1.10 there exists a linear continuous and injective mapping $\tilde{q} : V \to U$ such that $q\tilde{q} = Id$. Let $v \in Reg([0, t]; U)$ and define $u := \tilde{q}v$. First we show that $u \in Reg([0, t]; U)$ holds. Since v is a regulated function, there exists a sequence of step functions $(f_n)_{n\in\mathbb{N}} \subset T([0, t]; V)$ such that

$$\sup_{s\in[0,t]} \|v(s) - f_n(s)\| \longrightarrow 0 \text{ for } n \to \infty$$

holds. Now define the sequence $(g_n)_{n\in\mathbb{N}}$ with $g_n := \tilde{q}f_n$ for each $n \in \mathbb{N}$. Since $\tilde{q}$ is continuous, it is obvious that $(g_n)_{n\in\mathbb{N}} \subset T([0, t]; U)$. Next we calculate

$$\begin{aligned}\sup_{s\in[0,t]} \|u(s) - g_n(s)\| &= \sup_{s\in[0,t]} \|\tilde{q}v(s) - \tilde{q}f_n(s)\| \\ &\leqslant M \sup_{s\in[0,t]} \|v(s) - f_n(s)\| \longrightarrow 0 \text{ for } n \to \infty.\end{aligned}$$

This shows $u \in Reg([0, t]; U)$ and the rest of the proof follows equivalently as in the end of the proof given in Proposition 4.1.10. ■

With the above proposition we can reformulate Theorem 4.1.13 for regulated admissible control operators.

Corollary 4.1.15 *Let X be a Banach space, U, V be Hilbert spaces, $(T(t))_{t\geqslant 0}$ be a strongly continuous semigroup on X with generator $(A, D(A))$, $B \in \mathcal{L}(U, X_{-1})$ be regulated admissible and $F \in \mathcal{L}(V, X_{-1})$ such that* $\mathrm{Ran}(F) \subseteq \mathrm{Ran}(B)$. *Then F is regulated admissible.*

Proof: It is easy to see that one can use the proofs given in Corollary 4.1.11, Lemma 4.1.12 and Theorem 4.1.13 identically. ■

Remark 4.1.16 *It is obvious that Corollary 4.1.15 holds for continuous admissible control operators, too (in the sense that if B is continuous admissible,*

then F is continuous admissible).
Moreover, we mention that the content of this paragraph is independent of the theory of positive operators and it may already exist in the literature, since the proofs are basic and with less requirement. However, we did not find such results in any literature we looked at. But we mention that there are indeed results concerning a range condition. For instance have a closer look at [24, Corollary 3.6, Chapter III], where it is assumed that $\operatorname{Ran}(B)$ *lies in the Farvard space*

$$F_0 := \left\{ x \in X_{-1} : \sup_{t>0} \left\| \frac{1}{t} \left(T_{-1}(t)x - x \right) \right\|_{X_{-1}} \right\}.$$

In the proof the authors showed, that if

$$\operatorname{Ran}(B) \subset F_0 \tag{4.13}$$

holds, then the assumption (for some $\tau > 0$)

$$\int_0^\tau T_{-1}(\tau - s)Bu(s)\,\mathrm{d}s \in X \quad \text{for all} \quad u \in L^1([0,\tau],U),$$

of a previous corollary, [24, Corollary 3.4, Chapter III], is fulfilled. But this is nothing else then L^1*-admissibility. Summarising these facts, we have that a control operator is* L^1*-admissible if condition* (4.13) *holds.*

4.1.2 Admissible observation operator on AL-spaces

Now we give a statement for observation operators in AL-spaces. This result is closely related to the perturbation result, which we stated in the previous chapter (cf. Proposition 3.2.1 and Theorem 3.2.3). The result is originally due to Desch [19] and developed further by Voigt [80].

Theorem 4.1.17 *Let X be a Banach lattice and $(A, D(A))$ a generator of a positive strongly continuous semigroup $(T(t))_{t\geqslant 0}$ in X. Furthermore let Y be a real AL-space, $G \in \mathcal{L}(X_1, Y)$ be a positive operator and $C \in \mathcal{L}(X_1, Y)$ an operator satisfying $|Cf| \leqslant Gf$ for all $0 \leqslant f \in X_1$. Then G and C are L^1-admissible observation operators for Y, X and $(T(t))_{t\geqslant 0}$.*

The proof for the above theorem is similar to the proof of Theorem 3.2.1. But for the sake of completeness we state it anyway.

Proof: First we can assume that $(T(t))_{t\geqslant 0}$ is an exponentially stable semigroup, because otherwise we can consider the rescaled semigroup $(e^{-\lambda t}T(t))_{t\geqslant 0}$ for $\lambda > \omega_0$ (recall that ω_0 is defined as the growth bound of the semigroup) and use Proposition 4.1.5. It is easy to see that the rescaled semigroup $(e^{-\lambda t}T(t))_{t\geqslant 0}$ is positive, too.
Now let $x \in D(A)$ with $x \geqslant 0$. Since Y is an AL-space and $|Cx| \leqslant Gx = |Gx|$ implies $\|Cx\| \leqslant \|Gx\|$, we get

$$\int_0^\infty \|CT(t)x\|\,\mathrm{d}t \leqslant \int_0^\infty \|GT(t)x\|\,\mathrm{d}t = \left\| \int_0^\infty GT(t)x\,\mathrm{d}t \right\| = \left\| GA^{-1}x \right\| \leqslant M\|x\|.$$

It follows,

$$\int_0^\tau \|CT(t)x\| \,\mathrm{d}t \leqslant M\|x\| \text{ for all } \tau \geqslant 0.$$

Now let $x \in D(A)$ arbitrary and $x = x_+ - x_-$. We define

$$x_{n,\pm} := n \int_0^{\frac{1}{n}} T(t)x_\pm \,\mathrm{d}t,$$

because it is possible that x_+ and x_- lie in $X \setminus D(A)$. We have $x_{n,\pm} \in D(A)_+$ and $x_{n,\pm} \to x_\pm$ in X for $n \to \infty$. We calculate for all $n \in \mathbb{N}$

$$\begin{aligned}
&\int_0^\tau \|CT(t)(x_{n,+} - x_{n,-})\| \,\mathrm{d}t \\
&\leqslant \int_0^\tau \|CT(t)x_{n,+}\| \,\mathrm{d}t + \int_0^\tau \|CT(t)x_{n,-}\| \,\mathrm{d}t \\
&\leqslant M\left(\|x_{n,+}\| + \|x_{n,-}\|\right).
\end{aligned}$$

Define $y_n := A(x_{n,+} - x_{n,-})$. Then we have

$$y_n = An\int_0^{\frac{1}{n}} T(t)x_+ \,\mathrm{d}t - An\int_0^{\frac{1}{n}} T(t)x_- \,\mathrm{d}t = n\int_0^{\frac{1}{n}} T(t)Ax \,\mathrm{d}t$$

and therefore $y_n \to Ax =: y$ in X. Since $0 \in \rho(A)$ we can calculate

$$\begin{aligned}
&\big|\|CT(t)(x_{n,+} - x_{n,-})\| - \|CT(t)x\|\big| \\
&= \big|\|CT(t)A^{-1}y_n\| - \|CT(t)A^{-1}y\|\big| \\
&\leqslant \|CA^{-1}T(t)y_n - CA^{-1}T(t)y\| \\
&\leqslant \|CA^{-1}\| \cdot \|T(t)(y_n - y)\| \\
&\leqslant \|CA^{-1}\| \cdot \|T(t)(y_n - y)\| \leqslant K\|y_n - y\|,
\end{aligned}$$

for some $K > 0$ and where the last expression converge to 0 for $n \to \infty$. This shows uniform convergence for the integrand and thus

$$\int_0^\tau \|CT(t)x\| \,\mathrm{d}t \leqslant M\left(\|x_+\| + \|x_-\|\right) = M\||x|\| = M\|x\| \text{ for all } x \in D(A).$$

With the same calculation above one can show

$$\int_0^\tau \|GT(t)x\| \,\mathrm{d}t \leqslant M\|x\| \text{ for all } x \in D(A)$$

and the conclusion follows. ■

4.1.3 Admissible control operator on AM-spaces

Via duality the result in the previous paragraph, Theorem 4.1.17, carries over to control operators on a reflexive state space and an AM-space as control space.

Theorem 4.1.18 *Let X be a reflexive Banach lattice, $(A, D(A))$ be the generator of a positive strongly continuous semigroup $(T(t))_{t\geqslant 0}$, U a real AM-space and $B \in \mathcal{L}(U, X_{-1})$ a positive operator. Then B is L^∞-admissible.*

Proof: Since X is a reflexive space, we have $X' = X^\odot$ and $(X_{-1})' = (X')_1$ (see e.g. [24, Lemma 5.18, Chapter II]) and also that $B' \in \mathcal{L}((X')_1, U')$ is a positive operator (see Proposition 2.2.23). Moreover, $(T'(t))_{t\geqslant 0}$ is a positive strongly continuous semigroup semigroup with generator $(A', D(A'))$ and U' is a real AL-Space (see Proposition 2.2.28, Remark 2.2.29 and Theorem 2.2.10). Using Theorem 4.1.9 and Theorem 4.1.17 the conclusion follows. ■

If the state space is not reflexive we give a direct proof, which is similar to the proof of the perturbation result for AM-spaces given in Proposition 3.2.4. Here we only get regulated admissibility instead of L^∞-admissibility on reflexive state spaces, because step functions are not dense in $\mathrm{L}^\infty(I, U)$.

Theorem 4.1.19 *Let X be Banach lattice and U a real AM space. Let $(A, D(A))$ be the generator of a positive strongly continuous semigroup $(T(t))_{t\geqslant 0}$ and $B \in \mathcal{L}(U, X_{-1})$ be a positive operator. Then B is a regulated admissible control operator.*

The proof is similar to the proof of the proposition for the Desch-Schappacher perturbation (cf. Proposition 3.2.4).

Proof: Without loss of generality we assume that the semigroup $(T(t))_{t\geqslant 0}$ is exponentially stable (cf. proof of Theorem 4.1.17).

Now let $\tau > 0$. Recall that $\mathrm{T}([0, \tau]; U)$ denotes the vector space of U-valued step functions and $Reg([0, \tau]; U)$ the vector space of all regulated functions (cf. beginning of Section 4.1). We define the linear operator $R\colon \mathrm{T}([0, \tau]; U) \to X$ by

$$Ru := \int_0^\tau T_{-1}(\tau - s)Bu(s)\,\mathrm{d}s.$$

We indicate that R maps into X, because of Lemma 2.2.34 ii). Further we note that the operator R is positive. Now for the operator R, we show that

$$\|Ru\|_X \leqslant K\|u\|_\infty \tag{4.14}$$

holds for all $u \in \mathrm{T}([0, \tau]; U)$.

First, let u be a positive step function, $u = \sum_{n=1}^N u_n \chi_{I_n}$ as above, with $u_1, u_2, \ldots, u_N \geqslant 0$. Then $0 \leqslant u \leqslant z\chi_{[0,\tau]}$, where $z := \sup_n u_n$ exists in U. We conclude that

$$\begin{aligned}\|Ru\| &\leqslant \left\|\int_0^\tau T_{-1}(\tau - s)Bz\,\mathrm{d}s\right\| \\ &\leqslant \left\|\int_0^\infty T_{-1}(\tau)Bz\,\mathrm{d}s\right\| \leqslant \|A_{-1}^{-1}B\|\|z\| \\ &= K\|z\| = K\|\sup_n u_n\| = K\sup_n \|u_n\| = K\|u\|_\infty,\end{aligned}$$

where we have used the AM-property of U in the last line. If u is an arbitrary U-valued step function, then $u = u^+ - u^-$, $|Ru| = |Ru^+ - Ru^-| \leqslant Ru^+ + Ru^- = R|u|$, hence $\|Ru\| \leqslant \|R|u|\| \leqslant K\||u|\|_\infty = K\|u\|_\infty$.

The Estimate (4.14) implies that R possesses a (unique linear) continuous extension – still denoted by R – to $Reg([0,\tau];U)$ and the Estimate (4.14) carries over to this closure.
If $u \in Reg([0,\tau];U)$, and $(u_n)_{n\in\mathbb{N}}$ is a sequence in $\mathrm{T}([0,\tau];U)$ converging to u uniformly on $[0,\tau]$, then $Ru_n \to Ru$ in X. But also

$$Ru_n = \int_0^\tau T_{-1}(\tau - s)Bu_n(s)\,\mathrm{d}s \to \int_0^\tau T_{-1}(\tau - s)Bu(s)\,\mathrm{d}s$$

in X_{-1}, because $B\colon U \to X_{-1}$ is continuous and $(T_{-1}(t))_{t\geqslant 0}$ is bounded on $[0,\tau]$. This implies that $\int_0^\tau T_{-1}(\tau - s)Bu(s)\,\mathrm{d}s = Ru \in X$, and that

$$\left\|\int_0^\tau T_{-1}(\tau - s)Bu(s)\,\mathrm{d}s\right\| \leqslant K\|u\|_\infty$$

holds for all $u \in Reg([0,\tau];U)$. ■

4.1.4 Admissibility on the sequence space c_0

In this paragraph we consider the operator A_{-1} with domain $D(A_{-1}) = X$ as control operator, where the control space is chosen as X. It is known that the generator $(A, D(A))$ of a strongly continuous semigroup is bounded, if X contains no subspace which is isomorphic to c_0 and A_{-1} is L^∞-admissible (this is a consequence of Baillon's Theorem, see [23]). The most common Banach spaces in this situation (which contains no subspace isomorphic to c_0) are any reflexive Banach space, like L^p, ℓ^p for $p \in (1,\infty)$, and L^1, ℓ^1 (see remark after Lemma 3.4 in [75]). In this paragraph our main statement shows that A is still bounded, if A_{-1} is L^∞-admissible and $X = c_0$.

Proposition 4.1.20 *Let $(T(t))_{t\geqslant 0}$ be a strongly continuous semigroup with generator $(A, D(A))$ on a Banach space X. Moreover, we assume that A_{-1} is L^∞-admissible. Then for every $r > 0$ and every continuous function $f : [0,r] \to X$ the convolution*

$$(\mathcal{T} * f)(t) := \int_0^t T(t-s)f(s)\,\mathrm{d}s \tag{4.15}$$

is a continuous function from $[0,r]$ to X_1, i.e. $\mathcal{T} * f \in C([0,r];X_1)$.

Proof: It is easy to see that $L(f) := (\mathcal{T} * f)$ is a bounded operator from $C([0,r];X)$ to X. Moreover, L^∞-admissibility implies admissibility for continuous functions and therefore we have

$$A_{-1}\int_0^t T_{-1}(t-s)f(s)\,\mathrm{d}s = \int_0^t T_{-1}(t-s)A_{-1}f(s)\,\mathrm{d}s \in X$$

for all $f \in C([0,r],X)$ and $0 < t \leqslant r$. This is equivalent to

$$\int_0^t T_{-1}(t-s)f(s)\,\mathrm{d}s \in D(A),$$

because $(A, D(A))$ is bijective from X_1 to X. We see with the closedness of $(A, D(A))$ and the argument from the beginning that $(A \circ L)$ is a bounded operator from $C([0,r];X)$ to X (cf. [75], proof of Proposition 3.1). Keeping this in mind we define the functions

$$u(s) = \begin{cases} x & \text{if } s \in [0,\tau], \\ y & \text{if } s \in (\tau, r], \end{cases} \quad \text{and} \quad \tilde{u}(s) = \begin{cases} x & \text{if } s \in [0,\tau), \\ y & \text{if } s \in [\tau, r], \end{cases}$$

for some arbitrary $\tau \in (0,r)$ and $x, y \in X$. Now the integral from the convolution exists in X as a Bochner integral and therefore we have

$$\int_0^r T_{-1}(r-s)u(s)\,\mathrm{d}s = \int_0^r T_{-1}(r-s)\tilde{u}(s)\,\mathrm{d}s$$

in X. Because the values of both integrals lie in X_1, we have equality in X_1, too. We now show the continuity of $\mathcal{T} * f$ in τ as function mapping to X_1. We calculate for $\epsilon > 0$

$$\begin{aligned}
&\int_0^\tau T(\tau-s)u(s)\,\mathrm{d}s - \int_0^{\tau-\epsilon} T(\tau-\epsilon-s)u(s)\,\mathrm{d}s \\
&= \int_0^{\tau-\epsilon} T(\tau-\epsilon-s)T(\epsilon)x\,\mathrm{d}s - \int_0^{\tau-\epsilon} T(\tau-\epsilon-s)x\,\mathrm{d}s + \int_{\tau-\epsilon}^{\tau} T(\tau-s)x\,\mathrm{d}s \\
&= (T(\epsilon)-I)\int_0^{\tau-\epsilon} T(s)x\,\mathrm{d}s + \int_0^\epsilon T(s)x\,\mathrm{d}s \\
&= (T(\epsilon)-I)\int_0^{\tau} T(s)x\,\mathrm{d}s - (T(\epsilon)-I)\int_{\tau-\epsilon}^{\tau} T(s)x\,\mathrm{d}s + \int_0^\epsilon T(s)x\,\mathrm{d}s \\
&= (T(\epsilon)-I)\int_0^{\tau} T(s)x\,\mathrm{d}s - (T(\tau)-T(\tau-\epsilon))\int_0^{\epsilon} T(s)x\,\mathrm{d}s + \int_0^\epsilon T(s)x\,\mathrm{d}s \\
&= (T(\epsilon)-I)\int_0^{\tau} T(s)x\,\mathrm{d}s - (T(\tau)-T(\tau-\epsilon)-I)\int_0^{\epsilon} T(s)x\,\mathrm{d}s.
\end{aligned}$$

It follows

$$\begin{aligned}
&\left\|(T(\epsilon)-I)\int_0^{\tau} T(s)x\,\mathrm{d}s - (T(\tau)-T(\tau-\epsilon)-I)\int_0^{\epsilon} T(s)x\,\mathrm{d}s\right\|_{X_1} \\
&\leqslant \left\|A(T(\epsilon)-I)\int_0^{\tau} T(s)x\,\mathrm{d}s\right\| + \left\|A(T(\tau)-T(\tau-\epsilon)-I)\int_0^{\epsilon} T(s)x\,\mathrm{d}s\right\| \\
&= \left\|(T(\epsilon)-I)\int_0^{\tau} T_{-1}(s)A_{-1}x\,\mathrm{d}s\right\| + \|(T(\tau)-T(\tau-\epsilon)-I)\| \cdot \|T(\epsilon)x - x\|,
\end{aligned}$$

which tend to zero if $\epsilon \to 0$. This shows the left-side continuity in τ, for the right-side continuity we calculate

$$\begin{aligned}
&\int_0^{\tau+\epsilon} T_{-1}(\tau+\epsilon-s)u(s)\,\mathrm{d}s - \int_0^{\tau} T_{-1}(\tau-s)u(s)\,\mathrm{d}s \\
&= \int_0^{\tau+\epsilon} T_{-1}(\tau+\epsilon-s)\tilde{u}(s)\,\mathrm{d}s - \int_0^{\tau} T_{-1}(\tau-s)\tilde{u}(s)\,\mathrm{d}s \\
&= (T(\epsilon)-I)\int_0^{\tau} T(s)y\,\mathrm{d}s + \int_0^{\epsilon} T(s)y\,\mathrm{d}s.
\end{aligned}$$

and

$$\begin{aligned}&\left\|\int_0^{\tau+\epsilon} T_{-1}(\tau+\epsilon-s)u(s)\,\mathrm{d}s-\int_0^{\tau} T_{-1}(\tau-s)u(s)\,\mathrm{d}s\right\|_{X_1}\\ &=\left\|(T(\epsilon)-I)\int_0^{\tau} T_{-1}(s)A_{-1}y\,\mathrm{d}s\right\|+\|T(\epsilon)y-y\|\,.\end{aligned}$$

Obviously, in every other point from the interval $[0, r]$ the convolution $\mathcal{T} * u : [0, r] \to X_1$ is continuous. Using linearity and the above argumentation we see that for each step function g (with finite steps) we have $\mathcal{T} * u \in C([0, r]; X_1)$. Now let $f \in C([0, r]; X)$ be arbitrary and choose a sequence of step functions $(g_n)_{n\in\mathbb{N}}$ such that g_n converges uniformly in $[0, r]$ to f for $n \to \infty$. Then

$$\begin{aligned}&\left\|\int_0^{t} T(t-s)f(s)\,\mathrm{d}s-\int_0^{t} T(t-s)g_n(s)\,\mathrm{d}s\right\|_{X_1}\\ &=\|(A\circ L)\,(f-g_n)\|_X\leqslant M\|f-g_n\|_\infty\to 0,\end{aligned}$$

for $n \to \infty$. Finally, this gives for every $\tau \in [0, r)$

$$\begin{aligned}&\left\|\int_0^{\tau+\epsilon} T_{-1}(\tau+\epsilon-s)f(s)\,\mathrm{d}s-\int_0^{\tau} T_{-1}(\tau-s)f(s)\,\mathrm{d}s\right\|_{X_1}\\ &=\left\|\int_0^{\tau+\epsilon} T_{-1}(\tau+\epsilon-s)g_n(s)\,\mathrm{d}s-\int_0^{\tau} T_{-1}(\tau-s)g_n(s)\,\mathrm{d}s\right\|_{X_1}\\ &+\left\|\int_0^{\tau+\epsilon} T_{-1}(\tau+\epsilon-s)(f(s)-g_n(s))\,\mathrm{d}s\right\|_{X_1}\\ &+\left\|\int_0^{\tau} T_{-1}(\tau-s)(f(s)-g_n(s))\,\mathrm{d}s\right\|_{X_1}.\end{aligned}$$

The previous argumentations show that all terms tend to zero if $\epsilon \to 0$ and $n \to \infty$. Clearly, the left-side continuity in every $\tau \in (0, r]$ follow by similar calculations. Thus $\mathcal{T} * f \in C([0, r]; X_1)$ for every $r > 0$ and every $f \in C([0, r]; X)$. ■

Theorem 4.1.21 *Let $(T(t))_{t\geqslant 0}$ be a strongly continuous semigroup with generator $(A, D(A))$ on $X = c_0$, such that the evaluation map is continuous on the extrapolation space X_{-1}. Moreover, we assume that A_{-1} is L^∞-admissible. Then the generator is a bounded operator in $\mathcal{L}(X)$.*

Proof: The proof follows partially the proof of Baillon's Theorem in [23]. We split the proof in three steps.

Step 1:
In the first step we show that there exists a divergent series in X, which is unconditionally bounded, that is, there exists a constant $M > 0$ such that

$$\left\|\sum_{j=0}^{m}\delta_j x_j\right\|\leqslant M$$

whenever $m \in \mathbb{N}_0$ and $\delta_j \in \{0,1\}$, $j \in \{0, \cdots, m\}$.

Now from Proposition 4.1.20 we have $\mathcal{T} * f \in C([0,1], X_1)$. It follows from [75, Proposition 3.1] and [75, Lemma 3.3] that the semigroup $(T(t))_{t\geqslant 0}$ is analytic. Then, we have with [24, Theorem 4.6 c), Chapter II] that

$$t\|AT(t)\| \leqslant C \tag{4.16}$$

holds for all $t > 0$ and some constant $C > 0$. Now assume that $A \notin \mathcal{L}(X)$, then [23, Theorem 0.2] implies

$$\limsup_{t\to 0} t\|AT(t)\| \geqslant \frac{1}{e}.$$

Thus, there exist a sequence $(t_i)_{i\in\mathbb{N}_0} \subset [0,1]$ such that

$$t_0 = 1, \quad t_{i+1} < \frac{1}{2^{i+1}} t_i, \qquad i \in \mathbb{N}_0$$

and

$$t_i\|AT(t_i)\| > \frac{1}{2e}, \qquad i \in \mathbb{N}_0.$$

Further, there exists a sequence $(y_i)_{i\in\mathbb{N}_0}$ in X such that $\|y_i\| = 1$ and

$$t_i\|AT(t_i)y_i\| > \frac{1}{2e}, \qquad i \in \mathbb{N}_0. \tag{4.17}$$

Define $x_i := t_i AT(t_i)y_i$, $i \in \mathbb{N}_0$. We get with (4.16) and (4.17) the estimation

$$C \geqslant \|x_i\| \geqslant \frac{1}{2e}, \qquad i \in \mathbb{N}_0.$$

Therefore, the series $\sum_{n=0}^{\infty} x_n$ is divergent in X. Next we show that the series $\sum_{n=0}^{\infty} x_n$ is unconditionally bounded. In order to show this we define for $m \in \mathbb{N}$ and $\delta_j \in \{0,1\}$, $j \in \{0, \cdots, m\}$ the function

$$f(s) := \begin{cases} \delta_i T(s + t_i - 1)y_i & 1 - t_i \leqslant s < 1 - t_{i+1}, \quad i \in \{0, \cdots, m\} \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, $f \in \mathrm{L}^\infty([0,1]; X)$ and since A_{-1} is L^∞-admissible we have

$$\left\| \int_0^1 A_{-1}T(1-s)f(s)\,\mathrm{d}s \right\| \leqslant M\|f\|_{L^\infty} \leqslant \tilde{M}, \tag{4.18}$$

where $M, \tilde{M} > 0$ are independent of m and δ_i, $i \in \mathbb{N}_0$. We define

$$g(s) := A_{-1}T(1-s)f(s) = \delta_i AT(t_i)y_i = \frac{\delta_i}{t_i} x_i$$

for $s \in [1 - t_i, 1 - t_{i+1})$ and $i \in \{0, \cdots, m\}$ and $g(s) = 0$ otherwise. Next we estimate

$$\left\| \sum_{i=0}^{m} \delta_i \frac{t_{i+1}}{t_i} x_i \right\| \leqslant \sum_{i=0}^{m} \frac{t_{i+1}}{t_i} \|x_i\| \leqslant C \sum_{i=0}^{\infty} \frac{1}{2^{i+1}} = C.$$

This implies

$$\begin{aligned}\left\|\sum_{i=0}^{m}\delta_i x_i\right\| &= \left\|\sum_{i=0}^{m}\frac{t_i-t_{i+1}}{t_i}\delta_i x_i+\sum_{i=0}^{m}\frac{t_{i+1}}{t_i}\delta_i x_i\right\|\\ &\leqslant \left\|\int_0^1 A_{-1}T(1-s)f(s)\,\mathrm{d}s\right\|+C\\ &\leqslant \tilde{M}+C.\end{aligned}$$

Thus, we have shown that the series $\sum_{n=0}^{\infty} x_n$ is unconditionally bounded and divergent in X.

Step 2:
Next we show that there exists a sequence $(z_j)_{j\in\mathbb{N}}$ in X, such that for all $m\in\mathbb{N}$ and all $\beta_0,\dots,\beta_m\in\mathbb{C}$

$$\frac{1}{2}\max_{0\leqslant j\leqslant m}|\beta_j| \leqslant \left\|\sum_{j=0}^{m}\beta_j z_j\right\| \leqslant \frac{3}{2}\max_{0\leqslant j\leqslant m}|\beta_j| \tag{4.19}$$

holds and that $\sum_{j=0}^{\infty} z_j$ is an element of X. Following the proof of Lemma D.2 in [5] we define

$$\gamma_k := \sup\left\{\left\|\sum_{n=k+1}^{m}\alpha_n x_n\right\| \;:\; m>k, \alpha_n\in\mathbb{C}, |\alpha_n|\leqslant 1\right\}.$$

Obviously, γ_k is decreasing and finite by [5, Lemma D.1]. We have $\gamma := \lim_{k\to\infty}\gamma_k > 0$, because $\sum_{n=0}^{\infty} x_n$ is divergent. Replacing x_n by $\frac{5}{4}\frac{1}{\gamma}x_n$ and setting $\tilde{x}_n := \frac{5}{4}\frac{1}{\gamma}x_n$, it is easy to see that $\sum_{n=0}^{\infty}\tilde{x}_n$ is still unconditionally bounded and divergent in X. Now we can assume $\gamma=\frac{5}{4}$ (for the series $\sum_{n=0}^{\infty}\tilde{x}_n$) and we can choose $1\leqslant k_0\in\mathbb{N}$ such that $\gamma_{k_0}<\frac{3}{2}$. Because $\gamma_{k_0}>1$, there exist $k_2>k_1$ and $\alpha_n\in\mathbb{C}$ for $k_0<n\leqslant k_1$ with $|\alpha_n|\leqslant 1$ and

$$\nu_1 := \left\|\sum_{n=k_0+1}^{k_2}\alpha_n\tilde{x}_n\right\| > 1.$$

By iteration, there exists a strictly increasing sequence $(k_j)_{j\in\mathbb{N}_0}$ in $\mathbb{N}_0$ and a sequence $(\alpha_n)_{n\in\mathbb{N}}$ in $\mathbb{C}$ with $|\alpha_n|\leqslant 1$ for $n>k_0$, such that

$$\nu_j := \left\|\sum_{n=k_j+1}^{k_{(j+1)}}\alpha_n\tilde{x}_n\right\| > 1$$

for all $j\in\mathbb{N}$. Choose $\alpha_0=\cdots=\alpha_{k_0}=0$ and define $(z_j)_{j\in\mathbb{N}}$ via

$$z_j := \nu_j^{-1}\sum_{n=k_j+1}^{k_{(j+1)}}\alpha_n\tilde{x}_n.$$

Then, $\|z_j\| = 1$ for all $j \in \mathbb{N}_0$. Next we prove that (4.19) holds. As it is shown in the proof of Lemma D.2 and Theorem D.3 in [5], choose j_n with $k_{j_n} < n \leqslant k_{j_n+1}$. Then for all $m \in \mathbb{N}_0$ and $\beta_0, \cdots, \beta_m \in \mathbb{C}$ we have

$$\left\|\sum_{j=0}^{m} \beta_j z_j\right\| = \left\|\sum_{j=0}^{m} \beta_j \nu_j^{-1} \sum_{n=k_j+1}^{k_{(j+1)}} \alpha_n \tilde{x}_n\right\| = \left\|\sum_{n=k_0+1}^{k_{(m+1)}} \beta_{j_n} \nu_{j_n}^{-1} \alpha_n \tilde{x}_n\right\|$$
$$= \left\|\sum_{n=k_0+1}^{k_{(m+1)}} |\beta_{j_n} \nu_{j_n}^{-1} \alpha_n| \frac{\beta_{j_n} \nu_{j_n}^{-1} \alpha_n}{|\beta_{j_n} \nu_{j_n}^{-1} \alpha_n|} \tilde{x}_n\right\| \leqslant \frac{3}{2} \max_{0 \leqslant j \leqslant m} |\beta_j|.$$

For the first estimation of (4.19) choose $k \in \mathbb{N}_0$ such that $|\beta_k| = \max_{0 \leqslant j \leqslant m} |\beta_j|$, $x' \in X'$ with $\|x'\| = 1$ and $\beta_k \langle z_k, x' \rangle = |\beta_k|$. Define

$$\tilde{\beta}_j := \begin{cases} \beta_j & j \neq k, \\ -\beta_k & j = k. \end{cases} \tag{4.20}$$

Then

$$\left\|\sum_{j=0}^{m} \beta_j y_j\right\| \geqslant \operatorname{Re}\left\langle \sum_{j=0}^{m} \beta_j z_j, x' \right\rangle = 2|\beta_k| + \operatorname{Re}\left\langle \sum_{j=0}^{m} \tilde{\beta}_j z_j, x' \right\rangle \tag{4.21}$$

$$\geqslant 2|\beta_k| - \left\|\sum_{j=0}^{m} \beta_j z_j\right\| \geqslant 2|\beta_k| - \frac{3}{2} \max_{0 \leqslant j \leqslant m} |\beta_j| = \frac{1}{2} \max_{0 \leqslant j \leqslant m} |\beta_j|. \tag{4.22}$$

We prove next that $\sum_{n=0}^{\infty} z_n$ is an element of X. In order to show this we define the function $h : [0,1] \to X$ by

$$h(1) = 0, \qquad h(s) := \frac{\alpha_i}{\nu_{j_i}} T(s + t_i - 1)\tilde{y}_i, \quad 1 - t_i \leqslant s < 1 - t_{i+1},\ i \in \mathbb{N}_0.$$

where $\tilde{y}_i$ is such that $\tilde{x}_i = t_i A T(t_i) \tilde{y}_i$ for all $i \in \mathbb{N}$ and where $(j_i)_{j_i \in \mathbb{N}}$ is chosen as the subsequence $(j_n)_{j_n \in \mathbb{N}}$ from above. Clearly, $h \in \mathrm{L}^\infty([0,1]; X)$ and since A_{-1} is L^∞-admissible we have

$$\int_0^1 A_{-1} T(1-s) h(s) \, \mathrm{d}s \in X. \tag{4.23}$$

Note, that the integral is a Bochner integral of a X_{-1}-valued function. For $s \in [1 - t_i, 1 - t_{i+1})$ we have

$$AT(1-s)h(s) = \frac{\alpha_i}{\nu_{j_i}} AT(t_i)\tilde{y}_i = \frac{\alpha_i}{\nu_{j_i} t_i} \tilde{x}_i.$$

Thus, the integrand is constant on each interval $[1 - t_i, 1 - t_{i+1})$. Next we estimate

$$\left\|\sum_{i=0}^{\infty} \frac{\alpha_i}{\nu_{j_i}} \frac{t_{i+1}}{t_i} \tilde{x}_i\right\| \leqslant \sum_{i=0}^{\infty} \frac{t_{i+1}}{t_i} \|\tilde{x}_i\| \leqslant C \sum_{i=0}^{\infty} \frac{1}{2^{i+1}} = C.$$

This shows that $\sum_{i=0}^{\infty} \frac{\alpha_i}{\nu_{j_i}} \frac{t_{i+1}}{t_i} \tilde{x}_i$ converges absolute in X and combining this with (4.23) we get

$$\int_0^1 A_{-1} T(1-s) h(s) \, \mathrm{d}s + \sum_{i=0}^{\infty} \frac{\alpha_i}{\nu_{j_i}} \frac{t_{i+1}}{t_i} \tilde{x}_i \in X. \tag{4.24}$$

We calculate

$$\begin{aligned}
&\int_0^1 A_{-1} T(1-s) h(s) \, \mathrm{d}s + \sum_{i=0}^{\infty} \frac{\alpha_i}{\nu_{j_i}} \frac{t_{i+1}}{t_i} \tilde{x}_i \\
&= \sum_{i=0}^{\infty} \int_{1-t_i}^{1-t_{i+1}} A_{-1} T(1-s) h(s) \, \mathrm{d}s + \sum_{i=0}^{\infty} \frac{\alpha_i}{\nu_{j_i}} \frac{t_{i+1}}{t_i} \tilde{x}_i \\
&= \sum_{i=0}^{\infty} \left[(t_i - t_{i+1}) \frac{\alpha_i}{\nu_{j_i}} A T(t_i) \tilde{y}_i + \frac{\alpha_i}{\nu_{j_i}} \frac{t_{i+1}}{t_i} \tilde{x}_i \right] \\
&= \sum_{i=0}^{\infty} \frac{\alpha_i}{\nu_{j_i}} \tilde{x}_i = \sum_{j=0}^{\infty} \nu_j^{-1} \sum_{i=k_j+1}^{k_{(j+1)}} \alpha_i \tilde{x}_i \\
&= \sum_{j=0}^{\infty} z_j.
\end{aligned}$$

Thus $\sum_{j=0}^{\infty} z_j \in X$.

Step 3:
We finish the proof via the contradiction $\sum_{j=0}^{\infty} z_j \notin c_0$, because we have shown $\sum_{j=0}^{\infty} z_j \in X$ and assumed that $X = c_0$. Let $e_n \in c_0$ be the sequence $(\delta_{jn})_{n \in \mathbb{N}_0}$. Then we can write

$$z_j = \sum_{n=0}^{\infty} z_{jn} e_n.$$

As $\|z_j\| = 1$, it yields

$$\forall j \in \mathbb{N}_0 \, \exists n_j \in \mathbb{N}_0 : \qquad |z_{j n_j}| = 1. \tag{4.25}$$

We have already shown that for every $m \in \mathbb{N}_0$ and $\beta_0, \cdots, \beta_m \in \mathbb{C}$

$$\frac{1}{2} \max_{0 \leqslant j \leqslant m} |\beta_j| \leqslant \left\| \sum_{j=0}^{m} \beta_j z_j \right\| \leqslant \frac{3}{2} \max_{0 \leqslant j \leqslant m} |\beta_j| \tag{4.26}$$

holds. Moreover

$$\begin{aligned}
\left\| \sum_{j=0}^{m} \beta_j z_j \right\| &= \left\| \sum_{j=0}^{m} \beta_j \sum_{n=0}^{\infty} z_{jn} e_n \right\| \\
= \left\| \sum_{n=0}^{\infty} \sum_{j=0}^{m} \beta_j z_{jn} e_n \right\| &= \sup_{n \in \mathbb{N}_0} \left| \sum_{j=0}^{m} \beta_j z_{jn} \right|.
\end{aligned}$$

Now let $k \in \mathbb{N}$ be arbitrary, then we choose for each $m \in \mathbb{N}$, $\beta_j = \frac{\overline{z_{jk}}}{|z_{jk}|}$ if $z_{jk} \neq 0$ and $\beta_j = 0$ otherwise for all $j \in \{0, \dots, m\}$. It follows with the upper bound from (4.26)

$$\sum_{j=0}^{m} |z_{jk}| = \left| \sum_{j=0}^{m} \frac{\overline{z_{jk}}}{|z_{jk}|} z_{jk} \right| \leqslant \sup_{n \in \mathbb{N}_0} \left| \sum_{j=0}^{m} \frac{\overline{z_{jk}}}{|z_{jk}|} z_{jn} \right| \leqslant \frac{3}{2}.$$

Since this holds for all $m \in \mathbb{N}$ we get for all $k \in \mathbb{N}$

$$\sum_{j=0}^{\infty} |z_{jk}| \leqslant \frac{3}{2}. \tag{4.27}$$

Because of (4.25) there exists for each $l \in \mathbb{N}$, $n_l \in \mathbb{N}$ with $|z_{l,n_l}| = 1$. Applying this to (4.27) leads to

$$\sum_{j=0}^{\infty} |z_{jn_l}| = \sum_{\substack{j=0 \\ j \neq l}}^{\infty} |z_{jn_l}| + |z_{ln_l}| \leqslant \frac{3}{2}$$

and we get

$$\sum_{\substack{j=0 \\ j \neq l}}^{\infty} |z_{jn_l}| \leqslant \frac{1}{2}.$$

Combining this, for each $l \in \mathbb{N}$ there exists $n_l \in \mathbb{N}$ such that

$$\begin{aligned} \left| \sum_{j=0}^{\infty} z_{jn_l} \right| &= \left| \sum_{\substack{j=0 \\ j \neq l}}^{\infty} z_{jn_l} + z_{ln_l} \right| \\ &\geqslant |z_{ln_l}| - \sum_{\substack{j=0 \\ j \neq l}}^{\infty} |z_{jn_l}| \geqslant \frac{1}{2}. \end{aligned}$$

Now assume that there exist $k, l \in \mathbb{N}$ with $k \neq l$ such that $n_k = n_l$ and $|z_{kn_k}| = |z_{ln_l}| = |z_{ln_k}| = 1$ hold. This leads with (4.27) to the contradiction

$$2 = |z_{ln_k}| + |z_{kn_k}| \leqslant \sum_{j=0}^{\infty} |z_{jn_k}| \leqslant \frac{3}{2}.$$

The above argumentations imply that there has to be a strictly monotone increasing sequence $(n_l)_{l \in \mathbb{N}_0}$ such that

$$\left| \sum_{j=0}^{\infty} z_{jn_l} \right| \geqslant \frac{1}{2}, \qquad \text{for all } l \in \mathbb{N}_0.$$

Now the series

$$z := \sum_{j=0}^{\infty} z_j = \lim_{J \to \infty} \sum_{j=0}^{J} z_j \in c_0$$

converges in X_{-1}. Under the assumption that the evaluation map

$$f_k \colon X \to \mathbb{C}, \qquad x \mapsto x_k$$

is continuous on the space X_{-1}, we can calculate

$$\begin{aligned} |f_{n_l}(z)| &= \left| f_{n_l}\left(\lim_{J\to\infty} \sum_{j=0}^{J} z_j \right) \right| \\ &= \left| \lim_{J\to\infty} \sum_{j=0}^{J} f_{n_l}(z_j) \right| = \left| \sum_{j=0}^{\infty} z_{jn_l} \right| \geqslant \frac{1}{2} \end{aligned}$$

for all $l \in \mathbb{N}$. This implies

$$\lim_{l\to\infty} f_{n_l}(z) \neq 0$$

which is a contradiction to $z = \sum_{j=0}^{\infty} z_j \in c_0$.

■

Remark 4.1.22 *If the generator is a diagonal operator on c_0 (and therefore the semigroup, too) the evaluation map is continuous on X_{-1}.*
Theorem 4.1.21 cannot be restricted to regulate admissibility, because the constructed function $h(s)$ in the above proof is not regulated (this follows from the construction with infinite many intervals $[1-t_i, 1-t_{i+1})$ for $i \in \mathbb{N}$).
Clearly, there is still an open question: if $X \neq c_0$, but X contain a subspace isomorphic to c_0 and A_{-1} is L^∞-admissible, does this imply $A \in \mathcal{L}(X)$?

4.1.5 Equivalent conditions for admissibility in Hilbert spaces

The following is the dual part to observation operators on Hilbert spaces, which is already stated in [36]. Here we give three equivalent conditions for control operators.

Theorem 4.1.23 *Let H and U be Hilbert spaces, $(A, D(A))$ the generator of a strongly continuous bounded semigroup $(T(t))_{t\geqslant 0}$ and $B \in \mathcal{L}(U, X_{-1})$ a control operator. Then, the following conditions are equivalent.*

(A1) There exists a constant $m > 0$ such that

$$\left\| (sI - A_{-1})^{-1} Bu \right\| \leqslant \frac{m}{\sqrt{\operatorname{Re} s}} \|u\| \qquad u \in U, s \in \mathbb{C}_+.$$

(A2a) There exists a constant $K > 0$ such that

$$\left\| \frac{1}{\sqrt{\tau}} \int_0^{\tau} e^{i\omega t} T_{-1}(t) Bu \,\mathrm{d}t \right\| \leqslant K\|u\| \qquad u \in U, \tau > 0, \omega \in \mathbb{R}.$$

(A2b) There exists a constant $K > 0$ such that

$$\left\| \frac{1}{\sqrt{\tau}} \int_{\tau}^{2\tau} e^{i\omega t} T_{-1}(t) Bu \,\mathrm{d}t \right\| \leqslant K\|u\| \qquad u \in U, \tau > 0, \omega \in \mathbb{R}.$$

In Subsection 4.2.3 we will modify the above conditions, such that we can use these for zero-class admissible control operators and we will prove that these new conditions (see Definition 4.2.15) are equivalent, too. Therefore, we will drop the proof of Theorem 4.1.23, since it is a simplification to the proofs given in Proposition 4.2.16 and Theorem 4.2.17. Also, the dual part to the above Theorem, equivalent conditions for observation operators, is proven in [35].
Next we state the following proposition.

Proposition 4.1.24 *Let $N \in \mathbb{N}$, $M \geqslant 1$, $\alpha > 0$ and $\omega \in (0, \frac{\pi}{2})$, let $(T(t))_{t\geqslant 0}$ be a bounded strongly continuous semigroup with infinitesimal generator $(A, D(A))$ and $B \in \mathcal{L}(U, H_{-1})$. We assume further that one of the following conditions hold:*

1. *$(T(t))_{t\geqslant 0}$ is a contraction semigroup and $\dim U \leqslant N$.*
2. *$-A$ is an ω-sectorial operator and $(-A)^{\frac{1}{2}}$ is infinite-time admissible.*
3. *$(T(t))_{t\geqslant 0}$ is an exponentially stable left-invertible semigroup with $\|T(t)\| \leqslant Me^{-\alpha t}$.*

Then B is an infinite-time admissible control operator if and only if Property (A1) holds. Further define $K := \sup_{\nu>0} K_{\nu,B}$, where $K_{\nu,B}$ is the best constant given in (4.7) and define by m the best constant in Property (A1). Then, there exists a constant $c > 0$, only dependent on N, M, α and ω, such that $K \leqslant cm$.

Proof: It is well-known that B is an infinite-time admissible control operator if and only if Property (A1) holds under the assumption of this proposition (see [84], [34] and [51]).

Now assume that K is not bounded by an absolute multiple of m. Then there exists for each $n \in \mathbb{N}$, a semigroup $(T_n(t))_{t\geqslant 0}$ with infinitesimal generator $(A_n, D(A_n))$ on a Hilbert space H_n and a control operator $B_n \in \mathcal{L}(U, (H_n)_{-1})$ such that B_n satisfy the assumption of the proposition with $m_n = 1$ (otherwise take $B_n' := \frac{1}{m_n} B_n$) and an unbounded sequence $(K_n)_{n\in\mathbb{N}}$, where $K_n := \sup_{\nu>0} (K_n)_{\nu,B}$ and $(K_n)_{\nu,B}$ is the constant defined in (4.7) for the semigroup $(T_n(t))_{t\geqslant 0}$. Without loss of generality we assume that each semigroup $(T_n(t))_{t\geqslant 0}$ satisfies the same Condition (1), (2) or (3) and this implies that
$\hat{\omega}_0 := \sup_{n\in\mathbb{N}} \omega_0^n < \infty$, where ω_0^n is the growth bound for the semigroup $(T_n(t))_{t\geqslant 0}$.
Now we construct the product semigroup $(\mathcal{T}(t))_{t\geqslant 0}$ on the l^2 direct sum H of the spaces H_n, i.e.

$$\mathcal{T}(t) := \operatorname{diag}(T_1(t), T_2(t), \dots, T_k(t), \dots)$$

for every $t \geqslant 0$ defined on the Hilbert space

$$\mathcal{H} := \left\{ (x_n)_{n\in\mathbb{N}} \subset \prod_{n=1}^{\infty} H_n : \sum_{n=1}^{\infty} \|x_n\|^2 < \infty \right\}$$

with norm

$$\|x\|_{\mathcal{H}} = \sqrt{\sum_{n=1}^{\infty} \|x_n\|^2}.$$

It is easy to see, that the generator of this semigroup $(\mathcal{T}(t))_{t\geqslant 0}$ is given by

$$\mathcal{A} = \operatorname{diag}(A_1, A_2, \dots, A_k, \dots)$$
$$D(\mathcal{A}) = \{(x_n)_{n\in\mathbb{N}} \in \mathcal{H} :\ x_n \in D(A_n) \text{ for all } n \in \mathbb{N}\}$$

and that for the resolvent we have for each $s > \hat{\omega}_0$

$$(sI - \mathcal{A})^{-1} = \operatorname{diag}\left((sI - A_1)^{-1}, (sI - A_2)^{-1}, \dots, (sI - A_n)^{-1}, \dots\right).$$

Next, for every l^2 sequence $(\alpha_n)_{n\in\mathbb{N}}$ we define the control operator via

$$\mathcal{B}\colon U \longrightarrow \mathcal{H}_{-1} \qquad \mathcal{B}u := (\alpha_1 B_1 u, \alpha_2 B_2 u, \dots, \alpha_n B_n u, \dots),$$

where

$$\mathcal{H}_{-1} = \left\{ (x_n)_{n\in\mathbb{N}} \in \prod_{n=1}^{\infty} H_{-1,n} :\ \sum_{n=1}^{\infty} \|x_n\|^2 < \infty \right\}$$

and $H_{-1,n}$ is the extrapolation space for H_n and the semigroup $(T_n(t))_{n\in\mathbb{N}}$. We can choose $(\alpha_n)_{n\in\mathbb{N}}$ such that $\sum_{n=1}^{\infty} \alpha_n^2 = 1$ and $(K_n\alpha_n)_{n\in\mathbb{N}}$ is unbounded. Further, we have that $(\mathcal{T}(t))_{t\geqslant 0}$ fulfills the same condition – (1), (2) or (3) – that holds for all semigroups $(T_n(t))_{t\geqslant 0}$. Therefore, $\mathcal{B}$ is an infinite-time admissible control operator if and only if condition (A1) holds.
Now we calculate

$$
\begin{aligned}
&(sI - \mathcal{A}_{-1})^{-1}\mathcal{B}u \\
&= \operatorname{diag}\left((sI - A_{-1,1})^{-1}, (sI - A_{-1,2})^{-1}, \dots, (sI - A_{-1,n})^{-1}, \dots\right) \cdot \begin{pmatrix} \alpha_1 B_1 u \\ \alpha_2 B_2 u \\ \dots \\ \alpha_n B_n u \\ \dots \end{pmatrix} \\
&= \begin{pmatrix} \alpha_1 (sI - A_{-1,1})^{-1} B_1 u \\ \alpha_2 (sI - A_{-1,2})^{-1} B_2 u \\ \dots \\ \alpha_n (sI - A_{-1,n})^{-1} B_n u \\ \dots \end{pmatrix},
\end{aligned}
$$

where $A_{-1,n}$ is the extension of the generator A_n to $H_{-1,n}$. This leads to

$$
\begin{aligned}
\|(sI - \mathcal{A}_{-1})\mathcal{B}u\|_{\mathcal{H}}^2 &= \sum_{n=1}^{\infty} \|\alpha_n (sI - A_{-1,n})^{-1} B_n u\|^2 \\
&\leqslant \sum_{n=1}^{\infty} \alpha_n^2 \frac{\|u\|^2}{\operatorname{Re} s} = \frac{\|u\|^2}{\operatorname{Re} s},
\end{aligned}
$$

for every $s > \hat{\omega}_0$ and we obtain

$$\|(sI - \mathcal{A}_{-1})\mathcal{B}u\|_{\mathcal{H}} \leqslant \frac{\|u\|}{\sqrt{\operatorname{Re} s}}.$$

On the other hand we have

$$\begin{aligned}
&\mathcal{T}_{-1}(\tau-s)\mathcal{B}u(s)\\
&= \operatorname{diag}(T_{-1,1}(\tau-s), T_{-1,2}(\tau-s), \dots, T_{-1,n}(\tau-s), \dots) \cdot \begin{pmatrix} \alpha_1 B_1 u(s) \\ \alpha_2 B_2 u(s) \\ \dots \\ \alpha_n B_n u(s) \\ \dots \end{pmatrix}\\
&= \begin{pmatrix} \alpha_1 T_{-1,1}(\tau-s) B_1 u(s) \\ \alpha_2 T_{-1,2}(\tau-s) B_2 u(s) \\ \dots \\ \alpha_n T_{-1,n}(\tau-s) B_n u(s) \\ \dots \end{pmatrix}.
\end{aligned}$$

This gives

$$\begin{aligned}
\left\| \int_0^\tau \mathcal{T}_{-1}(\tau-s)\mathcal{B}u(s)\,\mathrm{d}s \right\|_{\mathcal{H}}^2 &= \sum_{n=1}^{\infty} \left\| \int_0^\tau \alpha_n T_{-1,n}(\tau-s) B_n u(s)\,\mathrm{d}s \right\|^2\\
&= \sum_{n=1}^{\infty} |\alpha_n|^2 \left\| \int_0^\tau T_{-1,n}(\tau-s) B_n u(s)\,\mathrm{d}s \right\|^2\\
&\geqslant |\alpha_n|^2 \left\| \int_0^\tau T_{-1,n}(\tau-s) B_n u(s)\,\mathrm{d}s \right\|^2
\end{aligned}$$

and finally we get

$$\sup_{\|u\|\leqslant 1} \left\| \int_0^\tau \mathcal{T}_{-1}(\tau-s)\mathcal{B}u(s)\,\mathrm{d}s \right\|_{\mathcal{H}} \geqslant |\alpha_n| K_n.$$

Since $(\alpha_n K_n)$ is an unbounded sequence the operator $\mathcal{B}$ is not admissible, but satisfies the resolvent condition. This is a contradiction to our assumption, that K is not bounded by an absolute multiple of m. ∎

4.2 Zero-class admissibility

In this section we look at a specific definition regarding admissibility for control and observation operators, which is important for our further research. As in Section 4.1 the vector valued function space $Z(I,U)$ will always refer to either $\mathrm{L}^p(I,U)$ for $p \in [1,\infty]$, $Reg(I,U)$ (the vector space of all regulated functions), $C(I,U)$ or an U-valued Orlicz space $E_\Phi(I,U)$ for some interval $I \subset [0,\infty)$ and some Young function Φ.

Definition 4.2.1 *Let X and U be Banach spaces, $(T(t))_{t\geqslant 0}$ be a strongly continuous semigroup with generator $(A, D(A))$ and $B \in \mathcal{L}(U, X_{-1})$ be a control operator. We call B a* zero-class Z-admissible control operator *for X, U and $(T(t))_{t\geqslant 0}$, if there exists constants $K_B(\tau) \geqslant 0$, such that for all $u \in Z([0,\tau],U)$*

$$\left\| \int_0^\tau T_{-1}(s) B u(s)\,\mathrm{d}s \right\| \leqslant K_B(\tau)\, \|u\|_{Z([0,\tau];U)} \tag{4.28}$$

and $\lim_{\tau \to 0} K_B(\tau) = 0$ holds.

Remark 4.2.2 *Recall Remark 4.1.3 where we had the constant K_τ. This constant was not supposed to tend to zero if τ did.*
Obviously, if the control operator $B \in \mathcal{L}(U, X_{-1})$ is zero-class Z-admissible, then the operator B is Z-admissible.

In this section we only consider control operators. However, we state the definition for zero-class admissible observation operator anyway.

Definition 4.2.3 *Let Y and X be Banach spaces, let $(T(t))_{t\geqslant 0}$ be a semigroup on X and let $C \in \mathcal{L}(X_1, Y)$. We call C a* zero-class Z-admissible observation operator *for Y, X and $(T(t))_{t\geqslant 0}$, if there exists constants $K_C(\tau) \geqslant 0$, such that for all $x \in D(A)$*

$$\|CT(t)x\|_{Z([0,\tau],Y)} \leqslant K_C(\tau)\|x\| \tag{4.29}$$

and $\lim_{\tau \to 0} K_C(\tau) = 0$ holds.

In the next proposition we exclude the U-valued Orlicz space for the function space $Z([0,\tau], U)$, i.e. we specify by Z-admissible the cases L^p-admissible (for $p \in [1, \infty]$), continuous admissible or regulated admissible.

Proposition 4.2.4 *Let X, Y, and U be Banach spaces, $(T(t))_{t\geqslant 0}$ be a strongly continuous semigroup, $B \in \mathcal{L}(U, X_{-1})$, $C \in \mathcal{L}(X_1, Y)$ and $\alpha \in \mathbb{R}$ be arbitrary.*
Then B is zero-class Z-admissible for $(T(t))_{t\geqslant 0}$ if and only if B is zero-class Z-admissible for the rescaled semigroup $(e^{\alpha t}T(t))_{t\geqslant 0}$.
Moreover, we have that C is zero-class Z-admissible for $(T(t))_{t\geqslant 0}$ if and only if C is zero-class Z-admissible for the rescaled semigroup $(e^{\alpha t}T(t))_{t\geqslant 0}$.

Proof: The proof follows in the same manner as in the proof of Proposition 4.1.5. ■

Next, we state an interesting result from Jacob, Schwenninger and Zwart. Here A_{-1} is considered as control operator on the control space $U = X$.

Proposition 4.2.5 *([37, Proposition 16]) Let $(A, D(A))$ be the generator of a strongly continuous semigroup $(T(t))_{t\geqslant 0}$. If A_{-1} is a zero-class L^∞-admissible operator, then A is bounded.*

This result remains true if we restrict the assumption for admissibility to regulated functions.

Proposition 4.2.6 *Let $(A, D(A))$ be the generator of a strongly continuous semigroup $(T(t))_{t\geqslant 0}$. If A_{-1} is a zero-class regulated admissible operator, then A is bounded.*

Proof: This proof is almost identical to the one given in [37, Proposition 16]. Let $x \in X$ with $\|x\| = 1$ we define $u(s) = x$ as the identity, which is a regulated function. We get

$$\|T(\tau)x - x\| = \left\| \int_0^\tau A_{-1}T_{-1}(s)u(s)\,\mathrm{d}s \right\|_X \leqslant k(\tau)\|u\|_\infty$$

where $k(\tau) \to 0$ for $\tau \to 0$, since A_{-1} is regulated zero-class admissible. It follows

$$\limsup_{t\to 0} \|T(t) - I\| = \limsup_{\substack{t\to 0\\ \|x\|=1}} \|T(t)x - x\| = 0. \tag{4.30}$$

Therefore, A is bounded by the zero-one law for semigroups (see Proposition 2.1.25 and Remark 2.1.26). ■

4.2.1 Zero-class admissibility for finite-dimensional control spaces

If U is a finite-dimensional vector space, we are able to show that L^∞-admissibility implies zero-class L^∞-admissibility for positive control operators.

Theorem 4.2.7 *Let X be a Banach lattice and $U = \mathbb{R}^n$. Let $(A, D(A))$ be the generator of a positive strongly continuous semigroup $(T(t))_{t\geqslant 0}$, $B \in \mathcal{L}(U, X_{-1})$ a positive operator and let B be* L^∞*-admissible (resp. regulated admissible, resp. continuous admissible). Then B is zero-class* L^∞*-admissible (resp. regulated admissible, resp. continuous admissible).*

Proof: For each $z \in U$ with $z = \sum_{i=1}^n z_i e_i$ we have

$$\begin{aligned}
\left\| \int_0^t T_{-1}(s)Bz \,\mathrm{d}s \right\|_X &= \left\| A_{-1} \int_0^t T_{-1}(s)Bz \,\mathrm{d}s \right\|_{X_{-1}} \\
&= \|T_{-1}(t)Bz - Bz\|_{X_{-1}} \\
&= \left\| T_{-1}(t)B \sum_{i=1}^n z_i \cdot e_i - B \sum_{i=1}^n z_i \cdot e_i \right\|_{X_{-1}} \\
&\leqslant \sum_{i=1}^n |z_i| \, \|T_{-1}(t)Be_i - Be_i\|_{X_{-1}} \\
&\leqslant \max |z_i| \sum_{i=1}^n \|T_{-1}(t)Be_i - Be_i\|_{X_{-1}} = k(t) \max |z_i|
\end{aligned}$$

where $k(t) := \sum_{i=1}^n \|T_{-1}(t)Be_i - Be_i\|_{X_{-1}}$. Let now $u \in L_+^\infty([0,t];U)$ and set $z_i := \operatorname{ess\,sup}_s u_i(s)$, where the supremum is taken coordinate wise in the usual ordering of $\mathbb{R}$. Then,

$$\begin{aligned}
\left\| \int_0^t T_{-1}(t-s)Bu(s) \,\mathrm{d}s \right\|_X &\leqslant \left\| \int_0^t T_{-1}(s)Bz \,\mathrm{d}s \right\|_X \\
&\leqslant k(t) \max_i |\operatorname*{ess\,sup}_s u_i(s)| \leqslant k(t) \max_i \operatorname*{ess\,sup}_s |u_i(s)| \\
&= k(t) \operatorname*{ess\,sup}_s \max_i |u_i(s)| \leqslant k(t) M \|u\|_{\mathrm{L}^\infty([0,t];U)} \\
&= \tilde{k}(t) \|u\|_{\mathrm{L}^\infty([0,t];U)},
\end{aligned}$$

where $\tilde{k}(t) = k(t)M$ and the constant $M > 0$ depend on the norm chosen for U (recall that in any finite-dimensional space all norms are equivalent). For arbitrary $u \in \mathrm{L}^\infty(0,t;U)$ set $u = u_+ - u_-$ and we get

$$\begin{aligned}\left\|\int_0^t T_{-1}(t-s)Bu(s)\,\mathrm{d}s\right\|_X &\leqslant \left\|\int_0^t T_{-1}(t-s)B|u(s)|\,\mathrm{d}s\right\|_X \\ &\leqslant \left\|\int_0^t T_{-1}(s)B|z|\,\mathrm{d}s\right\|_X \leqslant \tilde{k}(t)\||u|\|_{\mathrm{L}^\infty([0,t];U)} \\ &= \tilde{k}(t)\|u\|_{\mathrm{L}^\infty([0,t];U)}.\end{aligned}$$

Since $\lim_{t\to 0}\|T_{-1}(t)Be_i - Be_i\| = 0$ for all $i \in \{1,\dots,n\}$, we have that B is zero-class L^∞-admissible (using the same calculations: resp. regulated admissible, resp. continuous admissible). ∎

Remark 4.2.8 *Recall that $\mathrm{L}^\infty([0,t];U)$ is a Banach lattice (cf. Theorem 2.2.13), because U is a Banach lattice under its canonical ordering. Therefore, the decomposition $u = u_+ - u_-$ exists in $\mathrm{L}^\infty([0,t];U)$.*

Putting the above results and the ones from Subsection 4.1.3 together, we can state the following corollaries.

Corollary 4.2.9 *Let X be a reflexive Banach lattice and U a finite-dimensional real vector space. Let $(A, D(A))$ be a generator of a positive strongly continuous semigroup $(T(t))_{t\geqslant 0}$ and $B \in \mathcal{L}(U, X_{-1})$ a positive operator. Then B is zero-class L^∞-admissible.*

Proof: Every finite-dimensional vector space is an AM-space. Therefore, the conclusion follows with Theorem 4.1.18 and Theorem 4.2.7. ∎

Corollary 4.2.10 *Let X be a Banach lattice and U a finite-dimensional real vector space. Let $(A, D(A))$ be a generator of a positive strongly continuous semigroup $(T(t))_{t\geqslant 0}$ and $B \in \mathcal{L}(U, X_{-1})$ a positive operator. Then B is zero-class regulated admissible.*

Proof: This is a consequence of Theorem 4.1.19 and Theorem 4.2.7. ∎

4.2.2 Zero-class admissibility on the sequence space c_0

Next, we restrict us to multiplication operators that are generator of strongly continuous semigroups on the sequence space c_0. The big advantage of this setting is, that we can easily construct examples and in particular counterexamples. The following one shows, that there exist regulated admissible control operators, which are not zero-class regulated admissible. Therein, we use the results for positive control operators on AM-spaces.
To deal with positive operators we restrict ourselves exclusively to the case where each element in c_0 is real-valued.

Example 4.2.11 *Let* $X = U = c_0(\mathbb{N})$ *and* $Ax = \sum_{n=1}^{\infty} -nx_n e_n$ *with* $D(A) = \{x \in X : \sum_{n=1}^{\infty} -nx_n e_n \in c_0(\mathbb{N})\}$*. It is well known that* $(A, D(A))$ *generates a exponentially stable strongly continuous semigroup* $(T(t))_{t\geqslant 0}$ *given by*

$$T(t)x = \sum_{n=1}^{\infty} e^{-nt} x_n e_n$$

(see e.g. [24, Example 4.7 iii), Chapter I]). Obviously, this semigroup is positive. Let now $B = -A_{-1} \in \mathcal{L}(U, X_{-1})$*. Again it is easy to see that the operator* B *is positive. Define* $u \in \mathrm{L}^\infty([0,\tau], c_0(\mathbb{N}))$*,* $u(s) = \sum_{n=1}^{\infty} (u(s))_n e_n$ *via*

$$(u(s))_n := \begin{cases} 1 & \text{if } s \in [\tau - \frac{1}{n}, \tau - \frac{1}{2n}] \text{ and } \frac{1}{n} < \tau, \\ 0 & \text{otherwise.} \end{cases}$$

The element $f = \int_0^\tau T_{-1}(\tau - s)Bu(s)\,\mathrm{d}s = \int_0^\tau T_{-1}(s)Bu(\tau - s)\,\mathrm{d}s$ *is a sequence and we can calculate for all* $n \in \mathbb{N}$ *with* $\frac{1}{n} < \tau$*:*

$$f_n e_n = \int_0^\tau e^{-ns} n(u(\tau - s))_n \,\mathrm{d}s = \int_{\frac{1}{2n}}^{\frac{1}{n}} ne^{-ns}\,\mathrm{d}s = -(e^{-1} - e^{-\frac{1}{2}}) > \frac{1}{5}.$$

This shows $f \notin c_0(\mathbb{N})$ *and therefore* B *is not* L^∞*-admissible, but regulated admissible by Theorem 4.1.19. Furthermore, Proposition 4.2.6 shows that* B *is not zero-class regulated admissible.*

Example 4.2.12 *Let* $X = U = c_0(\mathbb{N})$ *and* $Ax = \sum_{n=1}^{\infty} -2^n x_n e_n$ *with* $D(A) = \{x \in X : (2^n x_n)_{n\in\mathbb{N}} \in c_0(\mathbb{N})\}$*. As in the previous example, we get that* $(A, D(A))$ *generates a positive exponentially stable strongly continuous semigroup* $(T(t))_{t\geqslant 0}$ *given by*

$$T(t)x = \sum_{n=1}^{\infty} e^{-2^n t} x_n e_n.$$

Define $B \in \mathcal{L}(U, X_{-1})$ *via*

$$Bu = \sum_{n=1}^{\infty} \frac{2^n}{n} u_n e_n.$$

Obviously, the operator B *is positive. Let* $u \in \mathrm{L}^\infty(0, \tau; c_0(\mathbb{N}))$ *be arbitrary, then we obtain*

$$\begin{aligned} \left| \left(\int_0^\tau T(t)Bu(t)\,\mathrm{d}t \right)(n) \right| &\leqslant \int_0^\tau \left| e^{-2^n t} \frac{2^n}{n} \right| \mathrm{d}t \sup_{t\in[0,\tau]} |u_n(t)| \\ &\leqslant \frac{1}{n} \left(1 - e^{-2^n \tau}\right) \cdot \|u\|_{\mathrm{L}^\infty(0,\tau;c_0(\mathbb{N}))}. \end{aligned}$$

This shows that B *is* L^∞*-admissible.*
Taking suprema over the equations above we get

$$\left\| \int_0^\tau T(t)Bu(t)\,\mathrm{d}t \right\|_X \leqslant \sup_{n\in\mathbb{N}} \left\{ \int_0^\tau \left| e^{-2^n t} \frac{2^n}{n} \right| \mathrm{d}t \sup_{t\in[0,1]} |u_n(t)| \right\}$$

$$\leqslant \sup_{n\in\mathbb{N}} \left\{ \frac{1}{n}\left(1-e^{-2^n\tau}\right) \right\} \|u\|_{\mathrm{L}^\infty(0,\tau;c_0(\mathbb{N}))}.$$

Let $\epsilon > 0$ be arbitrary, take $N \in \mathbb{N}$ such that $\frac{1}{n} < \epsilon$ for all $n \geqslant N$, then for all $\tau > 0$ we have

$$\frac{1}{n}\left(1-e^{-2^n\tau}\right) < \epsilon.$$

For $n \in \{1,\dots,N\}$, we find $\tau_n > 0$ such that for all $\tau \in [0,\tau_n]$ we have

$$\frac{1}{n}\left(1-e^{-2^n\tau}\right) < \epsilon$$

Define $\tau_0 := \min\{\tau_1,\dots,\tau_N\}$. Then we get for all $\tau \in [0,\tau_0]$ and for all $n \in \mathbb{N}$:

$$\frac{1}{n}\left(1-e^{-2^n\tau}\right) < \epsilon.$$

This shows

$$\lim_{\tau\to 0} \sup_{n\in\mathbb{N}} \frac{1}{n}\left(1-e^{-2^n\tau}\right) = 0$$

and we see that B is zero-class L^∞-admissible.
Now define

$$u(s) = \begin{cases} n & \text{if } s \in [\frac{1}{2^{n+1}}, \frac{1}{2^n}], \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, $u \notin \mathrm{L}^\infty(0,1;c_0(\mathbb{N}))$, but we have for $p \in [1,\infty)$

$$\|u\|_p^p = \int_0^1 (\sup_n |u_n|)^p \,\mathrm{d}t = \sum_{n=1}^\infty \frac{n^p}{2^n} < \infty$$

and hence $u \in \mathrm{L}^p(0,1;c_0(\mathbb{N}))$ for all $p \in [1,\infty)$. Furthermore, we have for all $n \in \mathbb{N}$

$$\left|\left(\int_0^1 T(t)Bu(t)\,\mathrm{d}t\right)(n)\right| = \int_{\frac{1}{2^{n+1}}}^{\frac{1}{2^n}} e^{-2^n t}\frac{2^n}{n}\cdot n\,\mathrm{d}t$$
$$= \left[-e^{-2^n t}\right]_{\frac{1}{2^{n+1}}}^{\frac{1}{2^n}} = -e^{-1} + e^{-\frac{1}{2}} > 0.$$

This shows that B is not L^p-admissible for all $p \in [1,\infty)$.

The above example shows that one can find a control operator which is L^∞-admissible, but not L^p-admissible for all $1 \leqslant p \lneqq \infty$. To find control operator which are between these two cases (in some sense) we use the class of Orlicz spaces. These spaces were introduced in Section 2.3. Continuing the above example, we see that there are control operator, which are E_Φ-admissible for some Young function Φ, but not L^p-admissible. The following example is evolved by a joint work with F. Schwenninger.

Example 4.2.13 *Let $X = U = c_0(\mathbb{N})$ and $(A, D(A))$ the generator from Example 4.2.12. Next, we define the function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ via*

$$\varphi(x) := x\log(\log(x+e))$$

and calculate for all $n \in \mathbb{N} \setminus \{0\}$

$$\begin{aligned}
\varphi\left(\frac{2^n}{n}e^{-2^n s}\right) &= \frac{2^n}{n}e^{-2^n s} \cdot \log\left(\log\left(\frac{2^n}{n}e^{-2^n s} + e\right)\right) \\
&\leqslant \frac{2^n}{n}e^{-2^n s} \cdot \log\left(\log\left(\frac{2^n}{n}\left(e^{-2^n s} + e\right)\right)\right) \\
&\leqslant \frac{2^n}{n}e^{-2^n s} \cdot \log\left(n\log(2) - \log(n) + \log(2e)\right) \\
&\leqslant \frac{2^n}{n}e^{-2^n s} \cdot \log\left(n\log(2) + \log(2e)\right) \\
&\leqslant \frac{2^n}{n}e^{-2^n s} \cdot \log\left(n \cdot (\log(2) + \log(2e))\right) \\
&= \frac{2^n \cdot \log(n \cdot C)}{n} \cdot e^{-2^n s}
\end{aligned}$$

where $C = \log(2) + \log(2e) > 1$. *Moreover, the function* φ *is convex, continuous and*

$$\lim_{x\to 0}\frac{\varphi(x)}{x} = 0 \qquad \lim_{x\to\infty}\frac{\varphi(x)}{x} = \infty$$

holds. This shows that φ *is a Young function (see Lemma 2.3.3 and Remark 2.3.4). Let now* ψ *the complementary Young function for* φ*, then we get with the extended Hölder Inequality, Theorem 2.3.8,*

$$\begin{aligned}
&\left|\left(\int_0^\tau T_{-1}(s)Bu(s)\,\mathrm{d}s\right)(n)\right| \\
&= \left|\int_0^\tau e^{-2^n s}\frac{2^n}{n}(u(s))_n\,\mathrm{d}s\right| \leqslant \left\|\frac{2^n}{n}e^{-2^n s}\right\|_\varphi \cdot \|u_n\|_\psi \\
&= \int_0^\tau \varphi\left(e^{-2^n s}\frac{2^n}{n}\right)\mathrm{d}s \cdot \int_0^\tau \psi((u(s))_n)\,\mathrm{d}s \\
&\leqslant \int_0^\tau \frac{2^n \cdot \log(n \cdot C)}{n}e^{-2^n s}\,\mathrm{d}s \cdot \int_0^\tau \psi\left(\sup_{n\in\mathbb{N}}|(u(s))_n|\right)\mathrm{d}s \\
&\leqslant \frac{\log(n \cdot C)}{n}\left(1 - e^{-2^n \tau}\right)\|u\|_\psi.
\end{aligned}$$

It follows with the same calculations from Example 4.2.12 that

$$\lim_{\tau\to 0}\sup_{n\in\mathbb{N}}\frac{\log(n \cdot C)}{n}\left(1 - e^{-2^n \tau}\right) = 0$$

holds. This implies L_ψ *zero-class admissibility for* B*. Therefore,* B *is an* E_ψ *zero-class admissible control operator, because we know from Proposition 2.3.7 that* $E_\psi \subset L_\psi$ *holds.*

Next, we generalize Example 4.2.12 to those multiplication operator, that are generator of exponentially strongly continuous semigroups on c_0.

Proposition 4.2.14 *Let* $X = U = c_0(\mathbb{N})$ *and* $(M_q, D(M_q))$ *be a multiplication operator defined by a sequence* $q = (q_n)_{n\in\mathbb{N}}$ *with* $\mathrm{Re}(q_n) < -\epsilon$ *for all* $n \in \mathbb{N}$ *and* *some fixed $\epsilon > 0$. Then, M_q generates an exponentially stable positive strongly continuous semigroup $(T(t))_{t\geqslant 0}$ given by*

$$T(t)x = (e^{q_n t}x_n)_{n\in\mathbb{N}}$$

Now, take an element $\boldsymbol{b} = (b_n)_{n\in\mathbb{N}} \in X = c_0(\mathbb{N})$ and define $B \in \mathcal{L}(U, X_{-1})$ as

$$By := \boldsymbol{b}M_q y = (b_n \cdot q_n \cdot y_n)_{n\in\mathbb{N}}$$

for each $y \in U$. Next, let $u \in \mathrm{L}^\infty(0, \tau; c_0(\mathbb{N}))$ be arbitrary. We have

$$x_n := \sup_{t\in[0,\tau]} |u_n(t)| \leqslant \sup_{t\in[0,\tau]} \sup_{n\in\mathbb{N}} |u_n(t)| = \|u\|_{\mathrm{L}^\infty}$$

and therefore $(x_n)_{n\in\mathbb{N}} \in l^\infty(\mathbb{N})$. However, we have $x = (x_n)_{n\in\mathbb{N}} \notin c_0(\mathbb{N})$ for arbitrary $u \in \mathrm{L}^\infty(0, \tau, c_0(\mathbb{N}))$. For example take $v \in \mathrm{L}^\infty(0, \tau; c_0(\mathbb{N}))$, defined by

$$v(t) = \begin{cases} 1 & \text{if } t \in [\frac{1}{2^{n+1}}, \frac{1}{2^n}], \\ 0 & \text{otherwise}, \end{cases}$$

then we get for all $n \in \mathbb{N}$, $\sup_{t\in[0,\tau]} |v_n(t)| = 1$. Nevertheless, we continue with the calculation

$$\left|\int_0^\tau T_{-1}(t)Bu(t)\,\mathrm{d}t\right| = \left|\int_0^\tau \left(e^{q_n t} b_n q_n u_n(t)\right)_{n\in\mathbb{N}} \mathrm{d}t\right|$$
$$= \left(\left|b_n \int_0^\tau e^{q_n t} q_n u_n(t)\,\mathrm{d}t\right|\right)_{n\in\mathbb{N}} \leqslant \left(|b_n| \left|\int_0^\tau e^{q_n t} q_n\,\mathrm{d}t\right| \sup_{t\in[0,\tau]} |u_n(t)|\right)_{n\in\mathbb{N}}$$
$$= (|b_n| \cdot |x_n|\, |e^{q_n\tau} - 1|)_{n\in\mathbb{N}} \leqslant (|b_n| \cdot |e^{q_n\tau} - 1|\, \|u\|_{\mathrm{L}^\infty})_{n\in\mathbb{N}} \in c_0(\mathbb{N}) = X.$$

This shows that B is a L^∞-admissible operator. Additionally, we have that B is a zero-class L^∞-admissible operator. This follows in the same way as in Example 4.2.12, because we have

$$\lim_{\tau\to 0} \left(\sup_{n\in\mathbb{N}} \{b_n \cdot |e^{q_n\tau} - 1|\}\right) = 0.$$

4.2.3 Equivalent conditions for zero-class admissibility

In this subsection we develop the theory from Subsection 4.1.5. First we introduce conditions for control operators on Hilbert spaces. These conditions are the dual part to those conditions given in [36] (also denoted with B1, B2a and B2b) for observation operators on Hilbert spaces. Moreover, these are modifications to the conditions (A1), (A2a) and (A2b) stated in Theorem 4.1.23.
We mention that all statements in this subsection are the dual part for observation operators on Hilbert spaces (see [36]).

Definition 4.2.15 *Let $(T(t))_{t\geqslant 0}$ be a bounded strongly continuous semigroup with infinitesimal generator $(A, D(A))$ on a Hilbert space H and $B \in \mathcal{L}(U, H_{-1})$ be infinite-time admissible, where U is a Hilbert space. Then we define the following conditions:*

(B1) For each $r > 0$ there exists a constant m_r such that the constants $(m_r)_{r>0}$ are uniformly bounded and $m_r \to 0$ as $r \to \infty$, and

$$\left\|(sI - A_{-1})^{-1}Bu\right\| \leqslant \frac{m_{\operatorname{Re} s}}{\sqrt{\operatorname{Re} s}}\|u\| \qquad u \in U, s \in \mathbb{C}_+.$$

(B2a) For each $\tau > 0$ there exists a constant K_τ such that the constants $(K_\tau)_{\tau>0}$ are uniformly bounded and $K_\tau \to 0$ as $\tau \to 0$, and

$$\left\|\frac{1}{\sqrt{\tau}}\int_0^\tau e^{i\omega t}T_{-1}(t)Bu\,\mathrm{d}t\right\| \leqslant K_\tau\|u\| \qquad u \in U, \tau > 0, \omega \in \mathbb{R}.$$

(B2b) For each $\tau > 0$ there exists a constant K_τ such that the constants $(K_\tau)_{\tau>0}$ are uniformly bounded and $K_\tau \to 0$ as $\tau \to 0$, and

$$\left\|\frac{1}{\sqrt{\tau}}\int_\tau^{2\tau} e^{i\omega t}T_{-1}(t)Bu\,\mathrm{d}t\right\| \leqslant K_\tau\|u\| \qquad u \in U, \tau > 0, \omega \in \mathbb{R}.$$

We will always assume that m_r is a decreasing function of r and that K_τ is an increasing function of τ. This can be achieved via

$$m_r := \inf_{r' \in (0,r]}\{m_{r'}\} \text{ and } K_\tau := \sup_{\tau' \in (0,\tau]}\{K_{\tau'}\}.$$

In the next two statements we will show that conditions (B1), (B2a) and (B2b) are equivalent.

Proposition 4.2.16 *Under the assumption of Definition 4.2.15 conditions (B2a) and (B2b) are equivalent.*

Proof: First assume that (B2a) holds. Then we calculate

$$\begin{aligned}
&\left\|\frac{1}{\sqrt{\tau}}\int_\tau^{2\tau} e^{i\omega t}T_{-1}(t)Bu\,\mathrm{d}t\right\| \\
&= \left\|\frac{1}{\sqrt{\tau}}\int_0^{2\tau} e^{i\omega t}T_{-1}(t)Bu\,\mathrm{d}t - \frac{1}{\sqrt{\tau}}\int_0^{\tau} e^{i\omega t}T_{-1}(t)Bu\,\mathrm{d}t\right\| \\
&\leqslant \left\|\frac{\sqrt{2}}{\sqrt{\tau'}}\int_0^{\tau'} e^{i\omega t}T_{-1}(t)Bu\,\mathrm{d}t\right\| + \left\|\frac{1}{\sqrt{\tau}}\int_0^{\tau} e^{i\omega t}T_{-1}(t)Bu\,\mathrm{d}t\right\| \\
&\leqslant 2\max\left\{\sqrt{2}K_{\tau'}; K_\tau\right\}\|u\|,
\end{aligned}$$

where we have substituted in the first integrand with $\tau = \frac{\tau'}{2}$. With easy calculations we can show the other direction.

$$\begin{aligned}
\left\|\frac{1}{\sqrt{\tau}}\int_0^\tau e^{i\omega t}T_{-1}(t)Bu\,\mathrm{d}t\right\| &= \left\|\frac{1}{\sqrt{\tau}}\sum_{n=0}^\infty\int_{2^{-(n+1)}\tau}^{2^{-n}\tau} e^{i\omega t}T_{-1}(t)Bu\,\mathrm{d}t\right\| \\
&\leqslant \sum_{n=0}^\infty\left\|\frac{1}{\sqrt{\tau}}\int_{2^{-(n+1)}\tau}^{2^{-n}\tau} e^{i\omega t}T_{-1}(t)Bu\,\mathrm{d}t\right\| \\
&= \sum_{n=0}^\infty\left\|2^{\frac{-(n+1)}{2}}\frac{1}{\sqrt{\tau'}}\int_{\tau'}^{2\tau'} e^{i\omega t}T_{-1}(t)Bu\,\mathrm{d}t\right\| \\
&\leqslant \sum_{n=0}^\infty 2^{\frac{-(n+1)}{2}}K_{\frac{\tau}{2^{n+1}}}\|u\| \leqslant C\cdot K_\tau\|u\|
\end{aligned}$$

where $C > 0$ is some constant and where we have substituted with $\tau = \tau' \cdot 2^{n+1}$ and used that K_τ is an increasing function of τ. ■

Theorem 4.2.17 *Let H and U be Hilbert spaces, let $(T(t))_{t\geqslant 0}$ be a bounded strongly continuous semigroup with infinitesimal generator $(A, D(A))$ on H and let $B \in \mathcal{L}(U, H_{-1})$ be infinite-time admissible. Then conditions (B1) and (B2a) are equivalent.*

Proof: First assume that condition (B2a) holds and let $s = \sigma + i\omega \in \mathbb{C}_+$. Then

$$\begin{aligned}(sI - A_{-1})^{-1}Bu &= \int_0^\infty e^{-st}T_{-1}(t)Bu\,\mathrm{d}t = \int_0^\infty e^{-\sigma t}e^{-i\omega t}T_{-1}(t)Bu\,\mathrm{d}t \\ &= \int_0^\infty T_{-1}(t)Bue^{-i\omega t}\int_t^\infty \sigma e^{-\sigma y}\,\mathrm{d}y\,\mathrm{d}t \\ &= \int_0^\infty \int_0^y T_{-1}(t)Bue^{-i\omega t}\sigma e^{-\sigma y}\,\mathrm{d}t\,\mathrm{d}y,\end{aligned}$$

where we used Fubini in the last equality. Let $\tilde{\omega} = -\omega$, then with the above calculations and with condition (B2a) we obtain

$$\begin{aligned}\|(sI - A_{-1})^{-1}Bu\| &\leqslant \int_0^\infty \left\| \int_0^y e^{i\tilde{\omega}t}T_{-1}(t)Bu\,\mathrm{d}t \right\| \sigma e^{-\sigma y}\,\mathrm{d}y \\ &\leqslant \int_0^\infty K_y\sqrt{y}\sigma e^{-\sigma y}\,\mathrm{d}y\|u\|.\end{aligned}$$

Recall that $\sigma > 0$ holds. Then we substitute $y = \frac{v}{\sigma}$ and get

$$\int_0^\infty K_y\sqrt{y}\sigma e^{-\sigma y}\,\mathrm{d}y = \frac{1}{\sqrt{\sigma}}\int_0^\infty K_{\frac{v}{\sigma}}\sqrt{v}e^{-v}\,\mathrm{d}v.$$

Since B is infinite-time admissible we have that $K_{\frac{v}{\sigma}}$ is uniformly bounded for v and σ and further the integral $\int_0^\infty \sqrt{v}e^{-v}\,\mathrm{d}v$ is finite. Therefore, we find for every $\epsilon > 0$ a constant $N > 0$, such that

$$\begin{aligned}\frac{1}{\sqrt{\sigma}}\int_0^\infty K_{\frac{v}{\sigma}}\sqrt{v}e^{-v}\,\mathrm{d}v &\leqslant \frac{1}{\sqrt{\sigma}}\left(\int_0^N K_{\frac{v}{\sigma}}\sqrt{v}e^{-v}\,\mathrm{d}v + \epsilon\right) \\ &\leqslant \frac{1}{\sqrt{\sigma}}\left(\sup_{v\in[0,N]}\{K_{\frac{v}{\sigma}}\}\cdot\int_0^N \sqrt{v}e^{-v}\,\mathrm{d}v + \epsilon\right) \\ &\leqslant \frac{1}{\sqrt{\sigma}}\left(C\cdot\sup_{v\in[0,N]}\{K_{\frac{v}{\sigma}}\} + \epsilon\right),\end{aligned}$$

for some constant $C > 0$. We know by (B2a) that $K_{\frac{v}{\sigma}} \to 0$ for $\sigma \to \infty$ holds for bounded v. So we can find for every $\epsilon > 0$ a $\sigma > 0$ big enough that

$$\left(C\cdot\sup_{v\in[0,N]}\{K_{\frac{v}{\sigma}}\} + \epsilon\right) \leqslant 2\epsilon$$

This shows that there exists a constant M_σ with $M_\sigma \to 0$ as $\sigma \to \infty$ such that

$$\frac{1}{\sqrt{\sigma}} \int_0^\infty K_{\frac{v}{\sigma}} \sqrt{v} e^{-v} \, \mathrm{d}v \leqslant \frac{M_\sigma}{\sqrt{\sigma}}$$

holds. Summarising the above results we have

$$\|(sI - A_{-1})^{-1} B u\| \leqslant \frac{M_\sigma}{\sqrt{\sigma}} \|u\|.$$

For the other direction of the proof let $\tau > 0$, $0 < \rho < \frac{1}{2}$, and set $s = \frac{\rho}{\tau} + i\omega$. We calculate

$$\begin{aligned}
&\int_0^\tau T_{-1}(t) B u e^{-(\frac{\rho}{\tau} + i\omega)t} \, \mathrm{d}t \\
&= \int_0^\infty T_{-1}(t) B u e^{-(\frac{\rho}{\tau} + i\omega)t} \, \mathrm{d}t - \int_\tau^\infty T_{-1}(t) B u e^{-(\frac{\rho}{\tau} + i\omega)t} \, \mathrm{d}t \\
&= (sI - A_{-1})^{-1} B u - \int_0^\infty T_{-1}(t+\tau) B u e^{-(\frac{\rho}{\tau} + i\omega)(t+\tau)} \, \mathrm{d}t \\
&= (sI - A_{-1})^{-1} B u - e^{-\rho - i\omega\tau} T(\tau) (sI - A_{-1})^{-1} B u.
\end{aligned}$$

Now we can estimate

$$\begin{aligned}
&\left\| \int_0^\tau T_{-1}(t) B u e^{-(\frac{\rho}{\tau} + i\omega)t} \, \mathrm{d}t \right\| \\
&\leqslant \left\| (sI - A_{-1})^{-1} B u \right\| + |e^{-\rho - i\omega\tau}| \cdot \|T(\tau)\| \cdot \left\| (sI - A_{-1})^{-1} B u \right\| \qquad (4.31) \\
&\leqslant m_{\frac{\rho}{\tau}} (1 + e^{-\rho} M) \frac{\sqrt{\tau}}{\sqrt{\rho}} \|u\|,
\end{aligned}$$

where M is the constant for the norm estimation of the semigroup in $\mathcal{L}(H)$ (see Equation (2.5)).
We know from [35, Lemma 2.4] that there exists a decomposition of the characteristic function $\chi_{[0,\tau)}$ via

$$\chi_{[0,\tau)} = \sum_{I \in \mathfrak{D}_\tau} \alpha_I \psi_I,$$

where $\mathfrak{D}_\tau$ is the collection of dyadic subintervals $I \subseteq [0, \tau)$, with left endpoint $l(I)$ and $\alpha_I \geqslant 0$ for all I. Moreover we have

$$\psi_I = \frac{1}{\sqrt{|I|}} \chi_I e^{-\frac{\rho}{|I|}(t - l(I))}$$

and the sum $\sum_{I \in \mathfrak{D}_\tau} \alpha_I \psi_I$ is converging uniformly to $\chi_{[0,\tau)}$ on the interval $[0, \tau)$. Here χ_I denotes the indicator function for the interval I. Further we have $\sum_{I \in \mathfrak{D}_\tau} \alpha_I \leqslant K\sqrt{\tau}$, where the constant K can be taken independent of τ. With this preparation we can calculate

$$
\begin{aligned}
&\left\|\int_0^\tau T_{-1}(t)Bue^{i\omega t}\,\mathrm{d}t\right\| \\
&\leqslant \sum_{I\in\mathfrak{D}_\tau}\alpha_I\left\|\int_I \psi_I(t)e^{i\omega t}T_{-1}(t)Bu\,\mathrm{d}t\right\| \\
&\leqslant \sum_{I\in\mathfrak{D}_\tau}\alpha_I\frac{1}{\sqrt{|I|}}\left\|\int_I e^{-\frac{\rho}{|I|}(t-l(I))}e^{i\omega t}T_{-1}(t)Bu\,\mathrm{d}t\right\| \\
&= \sum_{I\in\mathfrak{D}_\tau}\alpha_I\frac{1}{\sqrt{|I|}}\left\|\int_0^{|I|} e^{-\frac{\rho}{|I|}s}e^{i\omega(s+l(I))}T_{-1}(s)T_{-1}(l(I))Bu\,\mathrm{d}s\right\| \\
&\leqslant \sum_{I\in\mathfrak{D}_\tau}\alpha_I\frac{1}{\sqrt{|I|}}M\left\|\int_0^{|I|} e^{-\frac{\rho}{|I|}s}e^{i\omega(s+l(I))}T_{-1}(s)Bu\,\mathrm{d}s\right\| \\
&\leqslant \sum_{I\in\mathfrak{D}_\tau}\alpha_I\frac{1}{\sqrt{|I|}}Mm_{\frac{\rho}{|I|}}(1+e^{-\rho}M)\frac{\sqrt{|I|}}{\sqrt{\rho}}\|u\| \\
&\leqslant KMm_{\frac{\rho}{\tau}}(1+e^{-\rho}M)\frac{\sqrt{\tau}}{\sqrt{\rho}}\|u\|,
\end{aligned}
$$

where we substituted with $s = t - l(I)$ in the third line and used estimation (4.31) in the second to last line. ■

In the next statement we show that each of the three condition (B1), (B2a) or (B2b) is equivalent to infinite-time zero-class admissible for the control operator $B \in \mathcal{L}(U, H_{-1})$ under some restrictions to the semigroup, the generator and/or the control space.

Theorem 4.2.18 *Let H and U be Hilbert spaces and $(T(t))_{t\geqslant 0}$ be a bounded strongly continuous semigroup with infinitesimal generator $(A, D(A))$ on H and let $B \in \mathcal{L}(U, H_{-1})$. Further, assume that one of the following condition hold:*

(1) $(T(t))_{t\geqslant 0}$ is a contraction semigroup and U is finite-dimensional.

(2) $(T(t))_{t\geqslant 0}$ is an exponentially stable right-invertible semigroup.

(3) $(T(t))_{t\geqslant 0}$ is an analytic semigroup and $(-A)^{\frac{1}{2}}$ is infinite-time admissible.

Then B is an infinite-time zero-class admissible control operator if and only if condition (B1) holds.

Proof: Assume that condition (B1) holds and let $\nu > 0$ and $\lambda = \frac{1}{\nu}$. We can estimate for $s \in \mathbb{C}_+$ and $u \in U$,

$$\|((s+\lambda)I - A_{-1})^{-1}Bu\| \leqslant \frac{m_{\operatorname{Re} s+\lambda}}{\sqrt{\operatorname{Re} s+\lambda}}\|u\| \leqslant \frac{m_\lambda}{\sqrt{\operatorname{Re} s}}\|u\|.$$

This shows that for the operator B condition (A1) holds and therefore Proposition 4.1.24 implies that B is infinite-time admissible for the semigroup

$(e^{-\lambda t}T(t))_{t\geqslant 0}$ generated by $A-\lambda I$. Additionally, because of condition $(B1)$, we have $m_\lambda \longrightarrow 0$ as $\lambda \to \infty$. We now calculate

$$\begin{aligned}
&\left\|\int_0^\nu T_{-1}(t)Bu(t)\,\mathrm{d}t\right\| \\
&\leqslant e\left\|\int_0^\nu e^{-\lambda t}T_{-1}(t)Bu(t)\,\mathrm{d}t\right\| \\
&= e\left\|\int_0^\infty e^{-\lambda t}T_{-1}(t)Bu(t)\,\mathrm{d}t - \int_\nu^\infty e^{-\lambda t}T_{-1}(t)Bu(t)\,\mathrm{d}t\right\| \\
&\leqslant e\left(K_{\infty,B,\lambda}\|u\|_{L^2} + \|T(\nu)\|\cdot\left\|\int_0^\infty e^{-\lambda(s+\nu)}T_{-1}(s)Bu(s+\nu)\,\mathrm{d}s\right\|\right) \\
&\leqslant (e+M)K_{\infty,B,\lambda}\|u\|_{L^2},
\end{aligned}$$

where M is the constant from Equation (2.5) and $K_{\infty,B,\lambda}$ is the best constant for $\tau = \infty$ from Equation (4.7) for the semigroup $(e^{-\lambda t}T(t))_{t\geqslant 0}$. Then by Proposition 4.1.24 $K_{\infty,B,\lambda}$ is bounded by an absolute multiple m_λ. Recalling that $\nu = \frac{1}{\lambda}$ holds, we have $K_{\infty,B,\lambda} \to 0$ as $\lambda \to \infty$.
For the converse direction we show that condition (B2a) holds and then by applying Theorem 4.2.17 we know that (B1) holds, too. Now assume that B is an infinite-time zero-class admissible control operator. An easy calculation shows that for any fixed $u \in U$

$$\begin{aligned}
\left\|\frac{1}{\sqrt{\tau}}\int_0^\tau e^{i\omega t}T_{-1}(t)Bu\,\mathrm{d}t\right\| &= \left\|\int_0^\tau T_{-1}(t)B\left(\frac{e^{i\omega t}}{\sqrt{\tau}}u\right)\mathrm{d}t\right\| \\
&\leqslant K_\tau\left(\int_0^\tau \left\|\frac{e^{i\omega t}}{\sqrt{\tau}}u\right\|^2 \mathrm{d}t\right)^{\frac{1}{2}} = K_\tau\|u\|
\end{aligned}$$

holds. Since B is zero-class admissible, we have $K_\tau \to 0$ as $\tau \to 0$. ■

Theorem 4.2.19 *Let* $(T(t))_{t\geqslant 0}$ *be an exponentially stable strongly continuous semigroup with infinitesimal generator* $(A, D(A))$ *and let* $B := (-A)^\alpha S$ *where* $S \in \mathcal{L}(U,X)$ *and* $\alpha \in]0,\frac{1}{2}[$. *Then* B *is a zero-class admissible observation operator.*

Proof: For analytic semigroups we have $\mathrm{Ran}(T(t)) \subset D(A)$ for $t > 0$ (see Theorem 2.1.24 and recall that for exponentially stable semigroups we have $0 \in \rho(A)$) and $D(A) = D(-A) \subset D((-A)^\alpha)$ (see Theorem 2.1.21). Thus we have

$$\begin{aligned}
\left\|\int_0^\tau T_{-1}(\tau-s)Bu(s)\,\mathrm{d}s\right\| &= \left\|\int_0^\tau (-A)^\alpha T(s)Su(\tau-s)\,\mathrm{d}s\right\| \\
&\leqslant \int_0^\tau \|(-A)^\alpha T(s)\|\cdot\|Su(\tau-s)\|\,\mathrm{d}s.
\end{aligned}$$

Using the Cauchy-Schwarz inequality and the fact that
$\|(-A)^{\alpha}T(s)\| \leqslant Mt^{-\alpha}$ holds (see Theorem 2.1.24) we get with $M' = \|S\|$

$$\left\| \int_0^{\tau} T_{-1}(\tau - s)Bu(s)\,\mathrm{d}s \right\| \leqslant \left(\int_0^{\tau} M^2 t^{-2\alpha}\,\mathrm{d}t \right)^{\frac{1}{2}} \left(\int_0^{\tau} M'^2 \|u(\tau - s)\|^2\,\mathrm{d}s \right)^{\frac{1}{2}}$$
$$= \tilde{M} K_{\tau} \|u\|_{L^2}$$

for some constant $\tilde{M} > 0$ and where $K_{\tau} \longrightarrow 0$ for $\tau \to 0$. ∎

In the following we will show that there exists control operators $B \in \mathcal{L}(U, H_{-1})$ which fulfill condition (B1), but these control operators are not admissible for any $\tau > 0$, even if $(A, D(A))$ generates a bounded analytic semigroup. Before we state this result we introduce some knowledge on Riesz sequences.

Definition 4.2.20 *Let H be a Hilbert space and $(\varphi_n)_{n\in\mathbb{N}}$ a conditional basis in H.*

i) $(\varphi_n)_{n\in\mathbb{N}}$ is called Besselian *if there exists a constant $C > 0$ such that*

$$\sum_{k=1}^{N} |\alpha_k|^2 \leqslant C \left\| \sum_{k=1}^{N} \alpha_k \varphi_k \right\|^2$$

for every finite sequence of scalars $\alpha_1, \ldots, \alpha_N$.

ii) $(\varphi_n)_{n\in\mathbb{N}}$ is called Hilbertian *if there exists a constant $C > 0$ such that*

$$\left\| \sum_{k=1}^{N} \alpha_k \varphi_k \right\|^2 \leqslant C \sum_{k=1}^{N} |\alpha_k|^2$$

for every finite sequence of scalars $\alpha_1, \ldots, \alpha_N$.

Remark 4.2.21 *Recall that a basis is called conditional if the sum $\sum_{n=0}^{\infty} x_n \varphi_n$ converges conditional. A Riesz sequence is a sequence in H such that condition i) and ii) hold (with probably different constants $C > 0$). Moreover, a Riesz sequence is a Riesz basis if we have additional that the linear span of this sequence is dense in H.*

Lemma 4.2.22 *Let H be a Hilbert space. Then for every Riesz sequence its biorthogonal sequence is again a Riesz sequence.*

Proof: Let $(g_n)_{n\in\mathbb{N}}$ be a Riesz sequence in a Hilbert space H. Then $(g_n)_{n\in\mathbb{N}}$ is a Riesz basis on its norm closed linear span. Moreover, from [69, Theorem 11.1, §11, Chapter II] it follows that the biorthogonal sequence of a Besselian basis is Hilbertian and the biorthogonal sequence of a Hilbertian basis is Besselian. ∎

Theorem 4.2.23 *There exists an analytic, exponentially stable semigroup $(T(t))_{t\geqslant 0}$ with infinitesimal generator $(A, D(A))$ on a separable Hilbert space H and a control operator $B \in \mathcal{L}(\mathbb{C}, H_{-1})$, such that*

1. *Condition (B1) holds;*
2. *there exists a sequence of positive numbers* $(b_K)_{K\in\mathbb{N}}$ *and a control function* $u_K \in L^2(0,\tau)$ *with* $\|u_K\|_{L^2} = 1$ *such that* $(b_K)_{K\in\mathbb{N}}$ *is unbounded and*
$$\left\| \int_0^1 T(t)Bu_K(t)\,\mathrm{d}t \right\| \geqslant b_K \|u_K\|_{L^2} \text{ for all } K \in \mathbb{N}.$$

Before we give a proof to the above theorem, we state the following lemma.

Lemma 4.2.24 *Let H be an infinite-dimensional separable Hilbert space, $(A, D(A))$ a generator of an analytic, exponentially stable strongly continuous semigroup. Then there exists a control operator $B \in \mathcal{L}(\mathbb{C}, H_{-1})$, a sequence of positive numbers $(C_N)_{N\in\mathbb{N}}$ with $C_N \to \infty$ for $N \to \infty$, a sequence of control functions $(u_N)_{N\in\mathbb{N}} \subset \mathrm{L}^2(0,1)$ with $\|u_N\| = 1$, and sequences $(m_{N,r})_{N\in\mathbb{N}}$ with $m_{N,r} \to 0$ as $r \to 0$ for each $N \in \mathbb{N}$ and $m_{N,r}$ uniformly bounded in N and r, such that*
$$\|(sI - A_{-1})^{-1}B_N\| \leqslant m_{N.\operatorname{Re} s}\frac{1}{\sqrt{\operatorname{Re} s}}$$
and
$$\left\| \int_0^1 T(t)B_N u_N(t)\,\mathrm{d}t \right\| \geqslant C_N \|u_N\|_{\mathrm{L}^2}.$$

Proof: Let $(\varphi_n)_{n\in\mathbb{N}}$ be a non-Hilbertian conditional basis of H with $\sup_{n\in\mathbb{N}} \|\varphi_n\| < \infty$ and $\mu_n := -4^n$ for each $n \in \mathbb{N}$. Further, we define the operator $(A, D(A))$ via $A\varphi_n = \mu_n\varphi_n$ for $n \in \mathbb{N}$. This gives that A is a generator of an analytic, exponentially stable strongly continuous semigroup $(T(t))_{t\geqslant 0}$ given by $T(t)\varphi_n = e^{\mu_n t}\varphi_n$ for all $t \geqslant 0$ (see e.g. [38, Example 2.3]). Further, we can identify an operator $B \in \mathcal{L}(\mathbb{C}, H_{-1})$ as an element of H_{-1}. Therefore, define
$$B_N := \sum_{n=1}^{N} \sqrt{-\mu_n}\varphi_n.$$
With this definition we can calculate
$$\begin{aligned}
\left\|(sI - A_{-1})^{-1}B_N\right\| &= \left\| \sum_{n=1}^{N} \frac{2^n}{\operatorname{Re} s + 4^n}\varphi_n \right\| \\
&\leqslant \sup_n \|\varphi_n\| \sum_{n=1}^{N} \frac{2^n}{\operatorname{Re} s + 4^n} \\
&\leqslant 2K \sum_{n=1}^{2^N} \frac{1}{\operatorname{Re} s + n^2} = 2K\sqrt{\operatorname{Re} s} \sum_{n=1}^{2^N} \frac{1}{\operatorname{Re} s + n^2} \cdot \frac{1}{\sqrt{\operatorname{Re} s}},
\end{aligned}$$
where $K = \sup_n \|\varphi_n\|$ and where we used the calculations in [40, Proposition 3.2]. Define $m_{N,r} := 2K\sqrt{r}\sum_{n=1}^{2^N} \frac{1}{r+n^2}$. Then we have for fixed $N \in \mathbb{N}$
$$m_{N,r} = 2K\frac{1}{\sqrt{r}} \sum_{n=1}^{2^N} \frac{1}{1 + \frac{n^2}{r}} \longrightarrow 0 \text{ for } r \to \infty.$$

Further, we can estimate with the Cauchy's integral convergence test

$$\begin{aligned} m_{N,r} &= 2K\sqrt{r}\sum_{n=1}^{2^N}\frac{1}{r+n^2} \leqslant 2K\sqrt{r}\sum_{n=1}^{\infty}\frac{1}{r+n^2} \\ &\leqslant 2K\sqrt{r}\int_0^{\infty}\frac{1}{r+s^2}\,\mathrm{d}s = 2K\sqrt{r}\left(\left[\frac{1}{\sqrt{r}}\arctan\frac{s}{\sqrt{r}}\right]_0^{\infty}\right) \\ &= 2K\sqrt{r}\frac{\pi}{\sqrt{r}2} = K\pi. \end{aligned}$$

This gives

$$\left\|(sI - A_{-1})^{-1}B_N\right\| \leqslant \frac{m_{N,r}}{\sqrt{\operatorname{Re} s}}$$

and B_N satisfies (B1) for the semigroup $(T(t))_{t\geqslant 0}$ and for each $N \in \mathbb{N}$, where the constants $m_{N,r}$ are uniformly bounded in N.
From [57, Corollary 4.5.2] we have that $\left(\sqrt{-\mu_n}e^{\mu_n t}\chi_{(0,1)}\right)_{n\in\mathbb{N}}$ is a Riesz sequence in $L^2(0,1)$. Let $(f_n(t))_{n\in\mathbb{N}}$ be the biorthogonal Riesz sequence to $\left(\sqrt{-\mu_n}e^{\mu_n t}\chi_{(0,1)}\right)_{n\in\mathbb{N}}$ (see Lemma 4.2.22). Then we define $u_N(t) := \sum_{k=1}^{N}\alpha_{N,k}\cdot f_k(t)$ were we choose the sequence $(\alpha_{N,k})_{k\in\{1,\dots,N\}}$ such that $\|u_N\|_{\mathrm{L}^2} = 1$. This leads to

$$\begin{aligned} &\left\|\int_0^1 T_{-1}(t)B_N u_N(t)\,\mathrm{d}t\right\|^2 = \left\|\int_0^1\sum_{n=1}^{N}\sqrt{-\mu_n}e^{\mu_n t}\varphi_n\sum_{k=1}^{N}\alpha_{N,k}\cdot f_k(t)\,\mathrm{d}t\right\|^2 \\ &= \left\|\sum_{n=1}^{N}\varphi_n\sum_{k=1}^{N}\alpha_{N,k}\int_0^1\sqrt{-\mu_n}e^{\mu_n t}\cdot f_k(t)\,\mathrm{d}t\right\|^2 = \left\|\sum_{n=1}^{N}\alpha_{N,n}\varphi_n\right\|^2 \\ &\geqslant \tilde{C}_N\sum_{n=1}^{N}|\alpha_{N,n}|^2 \geqslant K\cdot\tilde{C}_N\left\|\sum_{n=1}^{N}\alpha_{N,n}\cdot f_n(\cdot)\right\|^2_{L^2(0,1)} = C_N^2\|u_N\|^2_{L^2}, \end{aligned}$$

where $C_N^2 := K\cdot\tilde{C}_N$ and where we have used in the last line that $(\varphi_n)_{n\in\mathbb{N}}$ is a non-Hilbertian basis and that $(f_n)_{n\in\mathbb{N}}$ is a Riesz sequence. ■

Finally, we are able to prove the theorem.
Proof of Theorem 4.2.23.
Take the generator A with strongly continuous semigroup $(T(t))_{t\geqslant 0}$ on the Hilbert space H from the previous lemma and define the space with the l^2-direct sum of countable Hilbert spaces $H_N := H$ via $\mathcal{H} := \prod_{N=1}^{\infty}H_N$. We denote by $\mathcal{A} = \operatorname{diag}(A, A, \dots, A, \dots)$ the generator with its semigroup $\mathcal{T}(t) := \operatorname{diag}(T(t), T(t), \dots, T(t), \dots)$ as it is defined in Proposition 4.1.24. Obviously, the semigroup $(\mathcal{T}(t))_{t\geqslant 0}$ is again analytic and exponentially stable. Then, take $\gamma \in]0, \frac{1}{2}[$ and we choose a sequence of nonnegative numbers $(\alpha_N)_{N\in\mathbb{N}}$ such that $\sum_{N=1}^{\infty}\alpha_N^2 = 1$ and $(\alpha_N^2 C_N^{1-2\gamma}) =: b_N$ is unbounded. Here, $(C_N)_{N\in\mathbb{N}}$ is the sequence from Lemma 4.2.24. Now define

$$\mathcal{B}\colon \mathbb{C} \longrightarrow \mathcal{H}_{-1} \qquad \mathcal{B} := \left(\frac{\alpha_1}{C_1^{1-\gamma}}B_1, \frac{\alpha_2}{C_2^{1-\gamma}}B_2, \dots, \frac{\alpha_N}{C_N^{1-\gamma}}B_N, \dots\right),$$

For $s \in \mathbb{C}_+$ we calculate

$$\begin{aligned}
\|(sI-\mathcal{A}_{-1})^{-1}\mathcal{B}\|_{\mathcal{H}}^2 &= \sum_{N=1}^{\infty} \left\| \frac{\alpha_N}{C_N^{\gamma}} (sI-A_{-1})^{-1} B_N \right\|_{H_N}^2 \\
&= \sum_{N=1}^{\infty} \frac{\alpha_N^2}{C_N^{2\gamma}} \left\| (sI-A_{-1})^{-1} B_N \right\|_{H_N}^2 \\
&\leqslant \sup_{N\in\mathbb{N}} \frac{1}{C_N^{2\gamma}} \|(sI-A_{-1})^{-1}B_N\|_{H_N}^2 \leqslant \sup_{N\in\mathbb{N}} \frac{1}{C_N^{2\gamma}} \left(\frac{m_{N,\operatorname{Re} s}}{\sqrt{\operatorname{Re} s}} \right)^2,
\end{aligned}$$

where $m_{N,\operatorname{Re} s}$ is the constant from Lemma 4.2.24. Obviously, $M_r := m_{N,r} \cdot C_N^{-\gamma}$ tends to zero as $r \to \infty$ for each $N \in \mathbb{N}$. Therefore, $\mathcal{B}$ satisfies condition (B1) for the semigroup $(\mathcal{T}(t))_{t\geqslant 0}$. To prove that $\mathcal{B}$ is not infinite-time admissible, take $u_K(\cdot) := \sum_{j=1}^{K} \alpha_{K,j} \cdot f_j(\cdot) \in L^2(0,1)$ where the sequence $(\alpha_{K,j})_{j\in\{1,\dots,K\}}$ is chosen such that $\|u_K\|_{\mathrm{L}^2} = 1$ (cf. Lemma 4.2.24). Then, using the computation from the above lemma, we get

$$\begin{aligned}
\left\| \int_0^1 \mathcal{T}_{-1}(t)\mathcal{B}u_K(t)\,\mathrm{d}t \right\|_{\mathcal{H}}^2 &= \sum_{N=1}^{\infty} \left\| \int_0^1 T_{-1}(t) \left(\frac{\alpha_N}{C_N^{\gamma}} \right) B_N u_K(t)\,\mathrm{d}t \right\|_{H_N}^2 \\
&\geqslant \alpha_K^2 C_K^{-2\gamma} \left\| \int_0^1 T_{-1}(t) B_K u_K(t)\,\mathrm{d}t \right\|_{H_K}^2 \\
&\geqslant \alpha_K^2 C_K^{1-2\gamma} \|u_K\|_{\mathrm{L}^2} = b_K \|u_K\|_{\mathrm{L}^2}
\end{aligned}$$

which proves the assumption from the theorem. ■

4.3 Interpolation theory and admissibility

In this section we study the connection between the degree of unboundedness and the admissibility for observation and control operators. To this end we need some results from interpolation theory. Instead of introducing basic notions of this theory, e.g. the definition of a real interpolation space, we only state the results we use for our purpose, because otherwise it would go beyond the scope of this dissertation. For more details on this topic we refer to Triebel [76] and Lunardi [49]. Interpolation theory combined with semigroups and especially analytic semigroups can be found in the book of Haase [31].
Let $\mathrm{L}_*^p(0,\infty)$ denote the space of all p-integrable functions on the interval $(0,\infty)$ with measure $\mathrm{d}t/t$.
Here we state for observation operators our

Definition 4.3.1 *Let X_1, Y be given as above and $C \in \mathcal{L}(X_1, Y)$. The number $\alpha(C)$ defined by*

$$\alpha(C) := \inf \left\{ \alpha \geqslant 0 \mid \sup_{\lambda \geqslant 0} \left\| \lambda^{1-\alpha} C R(\lambda, A) \right\| < \infty \right\} \tag{4.32}$$

is called the degree of unboundedness of C with respect to A.

This definition is equal to those given in [86]. Similarly, we give the definition for control operators.

Definition 4.3.2 *Let X_{-1}, U, be given as above and $B \in \mathcal{L}(U, X_{-1})$. The number $\beta(B)$ defined by*

$$\beta(B) := \sup \left\{ \beta \geqslant 0 \,|\, \sup_{\lambda \geqslant 0} \left\| \lambda^\beta R(\lambda, A_{-1}) B \right\| < \infty \right\} \tag{4.33}$$

is called the degree of unboundedness of B with respect to A.

In contrast to the above definition, this one differs from those given in [63]. Here the authors defined the degree of unboundedness via the infimum over all numbers $\beta \geqslant 0$ such that $B \in \mathcal{L}(U, X_{-\alpha})$ holds, where $X_{-\alpha}$ is the completion of X with respect to the norm $\|\cdot\|_{-\alpha} := \|(-A)^{-\alpha}\|$. This is equal to our definition in the case were X is a Hilbert space and A has a sequence of eigenvectors, which form a Riesz basis for X. These definitions do not coincide in the general case. The results in this chapter concerning the degree of unboundedness will be achieved by using interpolation theory. For our purpose we use the findings in the articles from Haak and Kunstmann [30] and [29] were the last one is a joint work with Haase.

Theorem 4.3.3 *Let $(A, D(A))$ be the generator of an exponentially stable strongly continuous semigroup. Further, let $B \in \mathcal{L}(U, X_{-1})$ be L^p-admissible and $C \in \mathcal{L}(X_1, U)$ be $\mathrm{L}^{p'}$-admissible for some $p, p' \in (1, \infty)$. Then the degree of unboundedness for B is at least $\frac{1}{p}$, i.e. $\beta(B) \geqslant \frac{1}{p}$ and for C it is at most $\frac{1}{p'}$, i.e. $\alpha(C) \leqslant \frac{1}{p'}$.*

Proof: Let $x \in D(A)$ and $\lambda > 0$, then we get for a $\mathrm{L}^{p'}$-admissible observation operator with the help of the Laplace-transform

$$\|C(\lambda - A)^{-1} x\| \leqslant \int_0^\infty |e^{-\lambda t}| \cdot \|CT(t)x\| \,\mathrm{d}t$$

$$\leqslant \|CT(t)x\|_{L^{p'}(\mathbb{R}_+, U)} \cdot \left(\int_0^\infty e^{-\lambda q t} \,\mathrm{d}t \right)^{\frac{1}{q}} \leqslant M \left(\frac{1}{\lambda q} \right)^{\frac{1}{q}} \|x\|,$$

where we have used Hölder inequality with $\frac{1}{q} + \frac{1}{p'} = 1$ and where $M > 0$ is the constant for $\mathrm{L}^{p'}$-admissibility. Since $D(A)$ is dense in X we get

$$\|\lambda^{1 - \frac{1}{p'}} C(\lambda - A)^{-1} x\| \leqslant \tilde{M} \|x\|$$

for all $x \in X$. Now for a L^p-admissible control operator we have

$$\|\lambda^{\frac{1}{p}} (\lambda - A_{-1})^{-1} B u\| = \left\| \int_0^\infty \lambda^{\frac{1}{p}} e^{-\lambda t} T_{-1}(t) B u \,\mathrm{d}t \right\|$$

$$= \left\| \int_0^\infty T_{-1}(t) B \left[\lambda^{\frac{1}{p}} e^{-\lambda t} u \right] \mathrm{d}t \right\| \leqslant K \|\lambda^{\frac{1}{p}} e^{-\lambda t} u\|_{\mathrm{L}^p(\mathbb{R}_+, U)}$$

$$\leqslant \tilde{K}\lambda^{\frac{1}{p}-\frac{1}{p}}\|u\| = \tilde{K}\|u\|$$

Summarising these results, we have for p and p'

$$\sup_{0<\lambda<\infty} \|\lambda^{1-\frac{1}{p'}}C(\lambda - A)^{-1}\| < \infty, \tag{4.34}$$

$$\sup_{0<\lambda<\infty} \|\lambda^{\frac{1}{p}}(\lambda - A_{-1})^{-1}B\| < \infty. \tag{4.35}$$

■

Theorem 4.3.4 *Let $(A, D(A))$ be the generator of an exponentially stable strongly continuous semigroup. If $B \in \mathcal{L}(U, X_{-1})$ and $C \in \mathcal{L}(X_1, U)$ are* L^p*-admissible for all $p \in (1, \infty)$. Then there exists a continuous extension $\tilde{C}$ of C with $D(C) \subset D(\tilde{C})$ such that $\tilde{C}A_{-1}^{-1}B \in \mathcal{L}(U)$.*

Proof: Since $(A, D(A))$ generates an exponentially stable semigroup, this semigroup is bounded and $(-A, D(-A) = D(A))$ is a sectorial operator of angle $\frac{\pi}{2}$ (see e.g. [31], page 24). We do not give the definition of a real interpolation space, but here we state a characterisation of such spaces (see e.g. [31], Chapter 6, page 141)

$$(X, X_1)_{\theta,p} = \{x \in X \mid t^{\theta}A(t - A)^{-1}x \in \mathrm{L}^p_*(0, \infty)\}$$
$$(X_{-1}, X)_{\theta,p} = \{x \in X_{-1} \mid t^{\theta}A_{-1}(t - A_{-1})^{-1}x \in \mathrm{L}^p_*(0, \infty)\},$$

where we have used the properties of the inter- and extrapolation spaces for semigroups (see Proposition 2.1.12) in the last characterisation. P. Kunstmann and B. Haak showed in [30], Theorem 1.9 (or [29], Corollary 4.8) that the results from Theorem 4.3.3, namely the expressions (4.34) and (4.35), are equivalent to $B \in \mathcal{L}(U, X_{-1})$ having $\mathrm{Ran}(B) \subset (X_{-1}, X)_{\frac{1}{p},\infty}$ and $C \in \mathcal{L}(X_1, U)$ has a continuous extension $\tilde{C}$ that is bounded from $(X, X_1)_{\frac{1}{p},1}$ to U.
Using these facts, let $x \in (X_{-1}, X)_{\frac{1}{p},\infty}$, then we have

$$\sup_{0<t<\infty} \|t^{\frac{1}{p}}A_{-1}(t - A_{-1})^{-1}x\|_{-1} < \infty,$$

using the definition of the norm $\|\cdot\|_{-1} = \|A_{-1}^{-1}\cdot\|$ the above statement is equivalent to

$$\sup_{0<t<\infty} \|t^{\frac{1}{p}}(t - A_{-1})^{-1}x\| < \infty. \tag{4.36}$$

To finish the proof we have to show that $y := A_{-1}^{-1}x \in (X, X_1)_{\frac{1}{p'},1}$, where $\frac{1}{p'}$ can be chosen differently from $\frac{1}{p}$. Calculating

$$\int_0^\infty \|t^{\frac{1}{p'}}A(t - A)^{-1}y\|\frac{\mathrm{d}t}{t} = \int_0^\infty \|t^{\frac{1}{p'}-1}(t - A_{-1})^{-1}x\|\,\mathrm{d}t$$

$$= \int_0^1 t^{\frac{1}{p'}-1}\|(t - A_{-1})^{-1}x\|\,\mathrm{d}t + \int_1^\infty t^{\frac{1}{p'}-1-\frac{1}{p}}\|t^{\frac{1}{p}}(t - A_{-1})^{-1}x\|\,\mathrm{d}t$$

we have for the first integral $\int_0^1 t^{\frac{1}{p'}-1}\,\mathrm{d}t < \infty$ and that $\sup_{0<t<1} \|(t-A)^{-1}x\|$ is bounded since $x \in X_{-1}$, $(t-A_{-1})^{-1}x \in X$ and the mapping $t \mapsto \|(t-A_{-1})^{-1}x\|$ is continuous on the compact interval $[0,1]$. The second integral is bounded whenever $\frac{1}{p} > \frac{1}{p'}$ or equivalently $p < p'$, because of equation (4.36) and the integrable function

$$t \mapsto t^{\frac{1}{p'}-1-\frac{1}{p}}$$

on the interval $[1,\infty]$. ■

The proof shows even the following result.

Corollary 4.3.5 *Let $(A, D(A))$ be the generator of an exponentially stable strongly continuous semigroup. If $B \in \mathcal{L}(U, X_{-1})$ is L^p-admissible for some $p \in (1,\infty)$ and $C \in \mathcal{L}(X_1, U)$ is p'-admissible for $p' \in (1,\infty)$, such that $p < p'$. Then there exists a continuous extension $\tilde{C}$ of C with $D(C) \subset D(\tilde{C})$ such that $\tilde{C}A_{-1}^{-1}B \in \mathcal{L}(U)$ and there exist a Banach space Z such that $Z \subseteq D(\tilde{C})$, $\tilde{C} \in \mathcal{L}(Z,U)$ and $X_1 \overset{c}{\hookrightarrow} Z \overset{c}{\hookrightarrow} X$, i.e. with continuous embeddings and we have*

$$\operatorname{Ran}(A_{-1}^{-1}B) \subset Z.$$

Proof: Define $Z := (X, X_1)_{\frac{1}{p'},1}$ and use the calculations in the above proof. ■

Moreover, we can formulate the following result by the degree of unboundedness.

Corollary 4.3.6 *Let $(A, D(A))$ be the generator of an exponentially stable strongly continuous semigroup, $B \in \mathcal{L}(U, X_{-1})$ and $C \in \mathcal{L}(X_1, U)$. If $\beta(B) > \alpha(C)$, then all conclusions from Corollary 4.3.5 are valid.*

Remark 4.3.7 *We mention that the results in this section may also be accomplished with the help of Farvard spaces. Since for exponentially stable semigroups the Farvard space for some $\alpha \in (0,1]$ is given by*

$$F^\alpha(A) = \left\{ x \in X \;:\; \sup_{\lambda>0} \|\lambda^\alpha AR(\lambda, A)x\| < \infty \right\}$$

(see e.g. [24], Definition 5.10, Chapter II). Moreover, H. Bounit and A. Fadili showed in [12] that (infinite-time) L^p-admissible control operators have range in $F^{\frac{1}{p}}(A_{-1})$ (see in particular Theorem 5.3 in [12]).

4.4 Linear systems

Recall the system of equations $\Sigma(A,B,C,D)$ from (4.1). Next, we introduce the so-called "integral representation" of such systems, which is given by the equations

$$\begin{cases} x(t) = T(t-s)x_s + \mathcal{B}_s^t u, & t \geqslant s \\ \quad y = \mathcal{C}_s x_s + \mathcal{D}_s u. \end{cases} \tag{4.37}$$

Throughout this section we denote by $\mathcal{T} := (T(t))_{t\geqslant 0}$ the strongly continuous semigroup generated by $(A, D(A))$ with growth bound ω_0. In the following we will give a meaning to these equations, operators respectively.

Remark 4.4.1 *We mention that the above linear system is time invariant and therefore one can set $s = 0$ (cf. [72, Theorem 2.2.11]). However, this section is highly orientated to the book of Staffans ([72]) and therefore we adopt the notation with arbitrary $s \in \mathbb{R}$.*

4.4.1 Well-posed linear systems

First we denote $\mathbb{R}^+ := (0, \infty)$, $\mathbb{R}^- := (-\infty, 0)$ and define the operators

$$(\pi_J u)(s) := \begin{cases} u(s), & s \in J, \\ 0, & s \notin J, \end{cases} \qquad \text{for all } J \subset \mathbb{R}, \tag{4.38}$$

$$\pi_+ u := \pi_{[0,\infty)}, \qquad \pi_- u := \pi_{(-\infty,0)}, \tag{4.39}$$

$$(\tau^t u)(s) := u(t+s), \quad -\infty < t, s < \infty. \tag{4.40}$$

Let X be a Banach space. The space $\mathrm{L}^p_{c,loc}(\mathbb{R}; X)$ consists of all the functions that are locally in $\mathrm{L}^p(\mathbb{R}; X)$ and whose support is bounded to the left (recall that the function space $\mathrm{L}^p(\mathbb{R}; X)$ is defined via the Bochner integral). Let $\mathrm{L}^p_{loc}(\mathbb{R}^+; X)$ be the subspace of $\mathrm{L}^p_{c,loc}(\mathbb{R}; X)$ for which its elements vanish on $\mathbb{R}^-$ and let $\mathrm{L}^p_{loc}(\mathbb{R}^-; X)$ be the subspace of $\mathrm{L}^p_{c,loc}(\mathbb{R}; X)$ for which its elements vanish on $\mathbb{R}^+$.
The space $Reg(\mathbb{R}; X)$ is the vector space of all bounded regulated functions acting on $\mathbb{R}$ and mapping into the Banach space X (cf. Definition 2.2.15). We define by $Reg_{c,loc}(\mathbb{R}; X)$ the space of all functions that are locally in $Reg(\mathbb{R}; X)$ and whose support is bounded to the left.
We define by $\mathrm{L}^p_\omega(\Omega; U)$ the function space of all functions u that satisfy $(t \mapsto e^{-\omega t} u(t)) \in \mathrm{L}^p(\Omega; U)$ and with $\mathrm{L}^p_{\omega,loc}(\mathbb{R}; U)$ all functions $u \in \mathrm{L}^p_{loc}(\mathbb{R}; U)$ which satisfy $\pi_- u \in \mathrm{L}^p_\omega(\mathbb{R}^-; U)$.
Moreover, we define by $Reg_0(\mathbb{R}; U)$ all regulated functions which tend to zero at $-\infty$ and $+\infty$, by $Reg_\omega(\mathbb{R}; U)$ the set of functions u for which $(t \mapsto e^{-\omega t} u(t)) \in Reg(\mathbb{R}; U)$ holds, by $Reg_{0,\omega}(\mathbb{R}; U)$ the set of functions u for which $(t \mapsto e^{-\omega t} u(t)) \in Reg_0(\mathbb{R}; U)$ holds and by $Reg_{0,\omega,loc}(\mathbb{R}; U)$ all functions $u \in Reg_{loc}(\mathbb{R}; U)$ which satisfy $\pi_- u \in Reg_{0,\omega}(\mathbb{R}^-; U)$.
Now we are in the position to declare what we mean by a well-posed linear system. This next definition is a combination of [72, Definition 2.2.1, 2.2.3, 2.2.6 and 2.2.7].

Definition 4.4.2 *Let X, Y and U be Banach space and let $p \in [1, \infty]$. An* L^p-well-posed linear system *on (X, Y, U) is a quadruple $\Sigma(\mathcal{T}, \mathcal{B}, \mathcal{C}, \mathcal{D})$ of continuous linear operators such that the following conditions hold:*

i) $\mathcal{T} := (T(t))_{t \geqslant 0}$ *is a strongly continuous semigroup on X with generator $(A, D(A))$;*

ii) $\mathcal{B} : \mathrm{L}^p_{c,loc}(\mathbb{R}; U) \to X$ *satisfies* $T(t)\mathcal{B}\pi_- u = \mathcal{B}\tau^t \pi_- u$, *for all* $u \in \mathrm{L}^p_{c,loc}(\mathbb{R}; U)$ *and all* $t \geqslant 0$*;*

iii) $\mathcal{C} : X \to \mathrm{L}^p_{c,loc}(\mathbb{R}; Y)$ *satisfies* $\pi_+ \mathcal{C} T(t) x = \pi_+ \tau^t \mathcal{C} x$, *for all* $x \in X$ *and all* $t \geqslant 0$*;*

vi) $\mathcal{D} : \mathrm{L}^p_{c,loc}(\mathbb{R};U) \to \mathrm{L}^p_{c,loc}(\mathbb{R};Y)$ *satisfies* $\tau^t\mathcal{D}u = \mathcal{D}\tau^t u$, $\pi_-\mathcal{D}\pi_+u = 0$ *and* $\pi_+\mathcal{D}\pi_-u = \pi_+\mathcal{C}\mathcal{B}\pi_-u$, *for all* $u \in \mathrm{L}^p_{c,loc}(\mathbb{R};U)$ *and all* $t \in \mathbb{R}$.

We call $\mathcal{B}$ *the controllability map,* $\mathcal{C}$ *the observability map and* $\mathcal{D}$ *the input/output map.*
The quadruple $\Sigma(\mathcal{T},\mathcal{B},\mathcal{C},\mathcal{D})$ *of continuous linear operators is* Reg-well-posed *if the conditions i),ii),iii),iv) are fulfilled for the space* $Reg_{c,loc}(\mathbb{R};U)$ *and* $Reg_{c,loc}(\mathbb{R};Y)$ *instead of* $\mathrm{L}^p_{c,loc}(\mathbb{R};U)$ *and* $\mathrm{L}^p_{c,loc}(\mathbb{R};Y)$*, where the operators* $\mathcal{B},\mathcal{C},\mathcal{D}$ *defined for the function spaces* $Reg_{c,loc}(\mathbb{R};U)$ *and* $Reg_{c,loc}(\mathbb{R};Y)$ *are given the same names.*
Next, we give the definition of the operators which we used in the equations (4.37)*:*

$$\mathcal{B}^t_s := \mathcal{B}\tau^t\pi_{[s,t)}, \qquad \mathcal{C}_s := \tau^{-s}\mathcal{C}, \qquad \mathcal{D}_s u := \mathcal{D}\pi_{[s,\infty)}. \tag{4.41}$$

Further, for each $s \in \mathbb{R}$, $x_s \in X$, $t \geqslant s$, *and* $u \in \mathrm{L}^p_{c,loc}([s,\infty);U)$ *respectively* $u \in Reg_{c,loc}([s,\infty);U)$ *we denote by* $x(t)$ *the* state trajectory at time t *with* initial time s, initial state x_s *and we denote by* y *the* output function *and by* u *the* input function. *Finally, we define the linear system*

$$\Sigma(\mathcal{T},\mathcal{B},\mathcal{C},\mathcal{D}) \quad \begin{cases} x(t) = T(t-s)x_s + \mathcal{B}^t_s u, & t \geqslant s \\ \quad y = \mathcal{C}_s x_s + \mathcal{D}_s u. \end{cases} \tag{4.42}$$

Remark 4.4.3 *In [72, Definition 2.2.1] the mapping* $\mathcal{B}$ *is defined on the space* $\mathrm{L}^p_c(\mathbb{R}_-;U)$ *and* $\mathcal{C}$ *is defined as an operator mapping to* $\mathrm{L}^p_{loc}(\mathbb{R}_+;Y)$*. Here we use in both cases* $\mathrm{L}^p_{c,loc}(\mathbb{R};U)$ *resp.* $\mathrm{L}^p_{c,loc}(\mathbb{R};Y)$*, which is more convenient for the following statements. However, for the algebraic equations in the above definitions, we have to add the projections* π_+, π_- *respectively (cf. [72, Definition 2.2.6 i) and ii)]).*

Lemma 4.4.4 *Let* $\Sigma(\mathcal{T},\mathcal{B},\mathcal{C},\mathcal{D})$ *be an* L^p*-well-posed linear or Reg-well-posed system. For all* $x \in X$ *and all* $u \in \mathrm{L}^p_{c,loc}(\mathbb{R};U)$ *or all* $Reg_{c,loc}(\mathbb{R};U)$ *we have*

$$\mathcal{B}\pi_-u = \lim_{s\to-\infty} \mathcal{B}^0_s u, \qquad \mathcal{C}x = \mathcal{C}_0 x, \qquad \mathcal{D}u = \lim_{s\to-\infty} \mathcal{D}_s u. \tag{4.43}$$

Proof: See [72, Lemma 2.2.10]. ∎

Theorem 4.4.5 *If* $\Sigma(\mathcal{T},\mathcal{B},\mathcal{C},\mathcal{D})$ *is an* L^∞*-well-posed linear system on* (X,Y,U)*, and if we restrict the domains of* $\mathcal{B}$ *and* $\mathcal{D}$ *to* $Reg_{c,loc}(\mathbb{R};U)$*, then the resulting system is Reg-well-posed.*

Proof: See [72, Theorem 4.5.4] ∎

Denote $\mathbb{C}^+_{\omega_0} := \{z \in \mathbb{C} : \ \mathrm{Re}\, z > \omega_0\}$.

Definition 4.4.6 *We define the function*

$$\widehat{\mathcal{D}} : \mathbb{C}^+_{\omega_0} \to \mathcal{L}(U,Y) \qquad z \mapsto (u \mapsto (\mathcal{D}(e^{z\cdot}u))(0)) \tag{4.44}$$

and call $\widehat{\mathcal{D}}$ *the* transfer function.

Lemma 4.4.7 *The transfer function is an analytic $\mathcal{L}(U,Y)$-valued function on $\mathbb{C}^+_{\omega_0}$.*

Proof: See [72, Lemma 4.6.2] ∎

The following statement is the union of [72, Theorem 4.2.1, 4.2.4, 4.4.2 and Corollary 4.6.6].

Theorem 4.4.8 *Let $1 \leqslant p < \infty$ and $\omega > \omega_0$. An L^p-well-posed or Reg-well-posed linear system is determined uniquely by its semigroup generator A, its control operator B, its observation operator C, and its transfer function $\widehat{\mathcal{D}}$, evaluated at one point $\alpha \in \mathbb{C}^+_{\omega_0}$ (where ω_0 is the growth bound of the semigroup). The representation for the observation operator is given by*

$$(\mathcal{C}x)(t) = CT(t)x \text{ for all } x \in X_1 \text{ and all } t \geqslant 0. \tag{4.45}$$

If the system is L^p-well-posed ($p < \infty$), then the representation for the control operator is given by

$$\mathcal{B}\pi_- u = \int_{-\infty}^{0} T_{-1}(-v)Bu(v)\,\mathrm{d}v, \tag{4.46}$$

for all $u \in \mathrm{L}^p_{\omega,loc}(\mathbb{R};U)$ and $\omega > \omega_0$. If the system is Reg-well-posed, then (4.46) holds for all $u \in Reg_{0,\omega,loc}(\mathbb{R};U)$ ($\omega > \omega_0$).

Corollary 4.4.9 *Let $\Sigma(\mathcal{T},\mathcal{B},\mathcal{C},\mathcal{D})$ be an L^p-well-posed system for $1 \leqslant p < \infty$ and let $t, s \in \mathbb{R}$ with $t > s$. The operator $\mathcal{B}^t_s$ from Definition 4.4.2 has the representation*

$$\mathcal{B}^t_s u = \int_s^t T_{-1}(t-v)Bu(v)\,\mathrm{d}v \text{ for all } u \in \mathrm{L}^p([s,t);U) \tag{4.47}$$

If the system $\Sigma(\mathcal{T},\mathcal{B},\mathcal{C},\mathcal{D})$ is L^∞-well-posed or Reg-well-posed, then (4.47) holds for all $u \in Reg([s,t);U)$.

Proof: This is shown in [72, Corollary 4.2.3]. ∎

Theorem 4.4.10 *C is the observation operator of an L^p-well-posed system $\Sigma(\mathcal{T},\mathcal{B},\mathcal{C},\mathcal{D})$ for $1 \leqslant p < \infty$ if and only if $C \in \mathcal{L}(X_1,Y)$ and the map $\mathcal{C} : X_1 \to C(\overline{\mathbb{R}}_+,Y)$ defined by*

$$(\mathcal{C}x)(t) = CT(t)x, \qquad t \geqslant 0,$$

can be extended to a continuous map $X \to \mathrm{L}^p_{loc}(\mathbb{R}_+,Y)$.

Proof: See [72, Theorem 4.4.7 i)]. ∎

The next result is a combination of [82, Theorem 4.8] and [83, Proposition 6.5].

Proposition 4.4.11 *If the linear system $\Sigma(\mathcal{T},\mathcal{B},\mathcal{C},\mathcal{D})$ is L^∞-well-posed or Reg-well-posed, then the observation operator C has a bounded continuous extension $C_{|X} \in \mathcal{L}(X,Y)$. If $\Sigma(\mathcal{T},\mathcal{B},\mathcal{C},\mathcal{D})$ is L^1-well-posed and X is reflexive, then the control operator B is bounded, i.e. $B \in \mathcal{L}(U,X)$.*

Remark 4.4.12 *In the literature for well-posed-linear systems there are different notations to define a linear system, such that it is the "integral representation" of* (4.1)*. For example compare the notations of Weiss [82], [83], [85] and Curtain [16]. Many other workers in this field use the Weiss notation, too. The operator* Φ_τ *and* Ψ_τ *from Definition 4.1.1 and 4.1.2 can be described by* $\mathcal{B}_0^\tau = \Phi_\tau$ *and* $\mathcal{C}_0 = \Psi_\tau$.

4.4.2 Regular linear systems

Definition 4.4.13 *The system* $\Sigma(\mathcal{T}, \mathcal{B}, \mathcal{C}, \mathcal{D})$ *is called* regular*, if the limit* $\lim_{\alpha\to+\infty} \widehat{\mathcal{D}}(\alpha)u$ *exists in* Y *for all* $u \in U$ *(in the norm topology). In this case we have*

$$Du = \lim_{\alpha\to+\infty} \widehat{\mathcal{D}}(\alpha)u$$

and D *is called the* feedthrough operaotor *of* $\mathcal{D}$.

Remark 4.4.14 *We mention that in the literature there are three different kind of regular systems: weakly, strong and uniformly regularity. Here we restrict ourselves to strong regular systems and use only the terminology regular instead. (see e.g. [72, Definition 5.6.1])*

The controllability map of a linear well-posed system has its range "at least" in X, but this is not enough to apply the observation operator to it, which is needed if we want to give a formulae for the output function in terms of C and D. Because of this we introduce extensions for the observation operator.

Definition 4.4.15 *The* Lebesgue extension *is defined by*

$$C_L x := \lim_{t\to 0} \frac{1}{t} C \int_0^t T(s)x \,\mathrm{d}s \tag{4.48}$$

with domain

$$D(C_L) = \{x \in X : \text{ the limit in (4.48) exists}\}, \tag{4.49}$$

where the limit is taken in the norm topology. We define a norm on $D(C_L)$ *via*

$$\|x\|_{D(C_L)} := \|x\| + \sup_{t\in(0,1]} \left\| \frac{1}{t} C \int_0^t T(s)x \,\mathrm{d}s \right\|.$$

Another extension for the observation operator is the so called Yosida extension.

Definition 4.4.16 *The* Yosida-extension *is defined by*

$$\tilde{C}x = \lim_{\alpha\to+\infty} \alpha C(\alpha - A)^{-1}x \tag{4.50}$$

with domain

$$D(\tilde{C}) = \{x \in X : \text{ the limit in (4.50) exists}\}, \tag{4.51}$$

where the limit is taken in the norm topology. We define a norm on $D(\tilde{C})$ *via*

$$\|x\|_{D(\tilde{C})} := \|x\| + \sup_{\alpha > 1+\omega_0} \left\| \alpha C(\alpha - A)^{-1}x \right\|.$$

Proposition 4.4.17 *The spaces $(D(C_L), \|\cdot\|_{D(C_L)})$ and $(D(\tilde{C}), \|\cdot\|_{D(\tilde{C})})$ are Banach spaces and we have*

$$X_1 \subset D(C_L) \subset D(\tilde{C}) \subset X, \tag{4.52}$$

all with continuous embeddings. Moreover, $C_L : D(C_L) \to X$ and $\tilde{C} : D(\tilde{C}) \to X$ are bounded linear operators.

Proof: See [83, Proposition 4.3] and [72, Theorem 5.4.3 ii)]. ∎

Remark 4.4.18 *In the context of system theory both extensions are due to G. Weiss (see [83, Section 4] and [85, Definition 5.6]), where the Yosida extension is denoted by C_Λ. In the book of O. Staffans the Lebesgue extension is a special case for the Cesáro extension (see [72, Definition 5.3.4]). Moreover, in this book the extensions for the observation operator are separated to the cases where the limits are taken in the strong and weak sense. Here we restrict ourselves to the strong limits.*

Before we state the next theorem, we define for all $u \in U$ the function

$$(1_+u)(t) := \begin{cases} u & \text{for } t \geqslant 0, \\ 0 & \text{otherwise.} \end{cases} \tag{4.53}$$

Theorem 4.4.19 *Let $\Sigma(\mathcal{T}, \mathcal{B}, \mathcal{C}, \mathcal{D})$ be a regular and L^p-well-posed linear system for $p < \infty$. Then we have*

i) the limit $Du = \lim_{h\to 0} \frac{1}{h} \int_0^h (\mathcal{D}1_+u)(s)\,\mathrm{d}s$ exists in Y (for the norm topology) for all $u \in U$;

ii) $\operatorname{Ran}\big((\alpha - A_{-1})^{-1}B\big) \subset D(C_L)$ *and* $\widehat{\mathcal{D}}(\alpha) = C_L(\alpha - A_{-1})^{-1}B + D$ *for all $\alpha \in \rho(A)$;*

iii) $\operatorname{Ran}\big((\alpha - A_{-1})^{-1}B\big) \subset D(\tilde{C})$ *and* $\widehat{\mathcal{D}}(\alpha) = \tilde{C}(\alpha - A_{-1})^{-1}B + D$ *for all $\alpha \in \rho(A)$.*

iv) For all $s \in \mathbb{R}$, $x_s \in X$ and all $u \in \mathrm{L}^p_{loc}(\mathbb{R}^+; U)$ we have that for almost all $t \in [s, \infty)$ the state trajectory x of $\Sigma(\mathcal{T}, \mathcal{B}, \mathcal{C}, \mathcal{D})$ with initial time s, initial state x_s and input function u satisfies $x(t) \in D(\tilde{C})$ and the corresponding output function y is given by $y(t) = \tilde{C}x(t) + Du(t)$.

Proof: See [72, Theorem 5.6.5 iii), v), v') and vi')]. ∎

Using the fact that for L^∞- or Reg-well-posed systems the observation operator has a continuous bounded extension we get the following nice result.

Theorem 4.4.20 *Let $\Sigma(\mathcal{T}, \mathcal{B}, \mathcal{C}, \mathcal{D})$ be a L^∞- or Reg-well-posed linear system. Then $\Sigma(\mathcal{T}, \mathcal{B}, \mathcal{C}, \mathcal{D})$ is regular, the observation operator C has a bounded extension $C_{|X} \in \mathcal{L}(X, Y)$ and the input/output map is given by*

$$(\mathcal{D}u)(t) = C_{|X} \int_0^t T_{-1}(t-s)Bu(s)\,\mathrm{d}s + Du(t) \tag{4.54}$$

for all $u \in Reg_{0,\omega,loc}(\mathbb{R};U)$. Moreover, in this situation the feedthrough operator is uniquely determined by the input/output map $\mathcal{D}$.

Proof: Use Theorem 4.4.5, Proposition 4.4.11, [72, Theorem 4.5.2] and [72, Lemma 5.7.1 i)]. ■

Now we are ready to adapt positivity to the theory of well-posed linear systems.

4.4.3 Positive linear systems

To use positive operators in the setting of linear systems we now assume that the state space X, the control space U and the observation space Y are (real-valued) Banach lattices throughout this whole paragraph.

Next we give our definition for well-posed linear systems to be positive. This differs from the definition given in [8, Definition 8.4] where the system $\Sigma(A,B,C)$ is finite-dimensional, i.e. A, B and C are matrices, $D = 0$ and X, U and Y are finite-dimensional vector spaces. Therefore B and C are bounded and this implies that the system is well-posed and especially regular. Moreover, in this case a linear system is positive if and only if A generates a positive semigroup and the matrices B and C are positive (see [8, Proposition 8.5]). In our case we use the integral representation, since B and C may be unbounded.

Definition 4.4.21 *Let $1 \leqslant p \leqslant \infty$. An L^p-well-posed or Reg-well-posed linear system $\Sigma(\mathcal{T},\mathcal{B},\mathcal{C},\mathcal{D})$ is said to be* positive*, if there exists an initial time s such that the initial value $x_s \in X$ is positive and for every positive input function $0 \leqslant u \in \mathrm{L}^p_{loc}([s,\infty);U)$, $0 \leqslant u \in Reg_{loc}([s,\infty);U)$ respectively, the state and the output defined in (4.42) are positive for every $t \geqslant s$, i.e. for every situation $u \geqslant 0$ and $x_s \geqslant 0$, we have $x(t) \geqslant 0$ for all $t \geqslant s$ and $y(t) \geqslant 0$ for almost all $t \geqslant s$.*

Remark 4.4.22 *Note that by Theorem 2.2.16 (and Remark 2.2.17), we have that $\mathrm{L}^p_{loc}([s,\infty);U)$ and $Reg_{loc}([s,\infty);U)$ are Fréchet lattices. Also it is obvious that $\mathrm{L}^p_{c,loc}(\mathbb{R});U)$ is a Fréchet lattice.*

Proposition 4.4.23 *Let $\Sigma(\mathcal{T},\mathcal{B},\mathcal{C},\mathcal{D})$ be an L^p-well-posed or Reg-well-posed system for $1 \leqslant p < \infty$ with initial time $s \in \mathbb{R}$, the observation operator $C \in \mathcal{L}(X_1,X)$ and the control operator $B \in \mathcal{L}(U,X_{-1})$. If the semigroup $\mathcal{T}$ is positive, the following statements are true:*

i) B is positive if and only if $\mathcal{B}$ is positive if and only if $\mathcal{B}^t_s$ is positive for all $t \in \mathbb{R}$ with $t \geqslant s$ if and only if $\mathcal{B}^t_{s'}$ is positive for all $t, s' \in \mathbb{R}$ with $t \geqslant s'$.

ii) C is positive if and only if $\mathcal{C}$ is positive if and only if $\mathcal{C}_s$ is positive if and only if $\mathcal{C}_{s'}$ is positive for every $s' \in \mathbb{R}$.

iii) $\mathcal{D}$ is positive if and only if $\mathcal{D}_s$ is positive if and only if $\mathcal{D}_{s'}$ is positive for every $s' \in \mathbb{R}$.

Proof: Let $B \in \mathcal{L}(U, X_{-1})$ be a positive operator. Then for all $0 \leqslant u \in \mathrm{L}^p_{c,loc}(\mathbb{R}, U)$ we have

$$\mathcal{B}\pi_- u = \int_{-\infty}^{0} T_{-1}(-v)Bu(v)\,\mathrm{d}v \geqslant 0,$$

and further for all $t, s' \in \mathbb{R}$ with $t > s'$ and $0 \leqslant u \in \mathrm{L}^p([s', t); U)$ we have

$$\mathcal{B}^t_{s'} u = \int_{s'}^{t} T_{-1}(t-v)Bu(v)\,\mathrm{d}v \geqslant 0,$$

because the extended semigroup $(T_{-1}(t))_{t\geqslant 0}$ is positive. Note that this positivity is primarily seen in the ordering of X_{-1} (see Definition 2.2.30). Then Proposition 2.2.32 and the fact that $\mathcal{B}$ and $\mathcal{B}^t_{s'}$ have its range in X guarantee that the above positivity holds in X, too.
If $\mathcal{B}^t_{s'}$ is positive (for every $s' \in \mathbb{R}$), then it is obvious that $\mathcal{B}^t_s$ (for fixed initial time $s \in \mathbb{R}$) is positive. Now let $\mathcal{B}^t_s$ be positive for some fixed initial time $s \in \mathbb{R}$, take $0 \leqslant u \in U$ arbitrary and consider

$$\frac{1}{t-s}\mathcal{B}^t_s u = \frac{1}{t-s}\int_s^t T_{-1}(t-v)Bu\,\mathrm{d}v = \frac{1}{t-s}\int_0^{t-s} T_{-1}(v)Bu\,\mathrm{d}v \geqslant 0$$

for each $t > s$. The integral is evaluated in X_{-1}, but its value lies in X. With the positivity of the extended semigroup $(T_{-1}(t))_{t\geqslant 0}$ and Proposition 2.2.32 (use $X_+ = X \cap (X_{-1})_+$) we get that $\frac{1}{t-s}\mathcal{B}^t_s u$ is a positive element in X_+ for each $u \in U_+$ and $t > 0$. Next, take the following limit in X_{-1}

$$0 \leqslant \lim_{t\to s}\frac{1}{t-s}\mathcal{B}^t_s u = \lim_{t\to s}\frac{1}{t-s}\int_0^{t-s} T_{-1}(v)Bu\,\mathrm{d}v = Bu.$$

Thus, Bu is a positive element in X_{-1} per definition. Because this holds for all $u \in U_+$, we have that $B \in \mathcal{L}(U, X_{-1})$ is a positive operator. If $\mathcal{B}$ is positive, then for all $u \geqslant 0$ we have $\mathcal{B}^t_s u = \mathcal{B}\tau^t \pi_{[s,t)} u \geqslant 0$, because τ^t and $\pi_{[s,t)}$ are positive operators and therefore $\mathcal{B}^t_s$ is positive for every $s \in \mathbb{R}$.

Assume that $\mathcal{C}_{s'}$ is a positive operator for any $s' \in \mathbb{R}$. Then $\mathcal{C}_0 = \mathcal{C}$ is positive, too. Moreover, we have with Theorem 4.4.10 that $\pi_+\mathcal{C} : X_1 \to C(\overline{\mathbb{R}}_+, Y)$ and therefore we can evaluate $(\mathcal{C}x)(\cdot)$ in each point (in $\overline{\mathbb{R}}_+$). This gives for every $0 \leqslant x \in D(A)$

$$0 \leqslant (\mathcal{C}x)(0) = CT(0)x = Cx.$$

For the other direction, let C be a positive operator. Then for every $0 \leqslant x \in X_1$ we have for all $t, s' \in \mathbb{R}$ with $t \geqslant s$

$$\mathcal{C}_{s'} x = CT(t-s')x \geqslant 0.$$

Further, for every $x \in X_+$ there exists a positive sequence $(x_n)_{n\in\mathbb{N}}$ with $0 \leqslant x_n \in X_1$ such that $x_n \to x$ in X for $n \to \infty$ (e.g. take the sequence $x_n := n\int_0^{1/n} T(s)x\,\mathrm{d}s$). From Proposition 2.2.16 and Corollary 2.2.20 (and Corollary 4.4.22) we know that the space $\mathrm{L}^p_{c,loc}(\mathbb{R}; Y)$ is a Fréchet lattice with closed positive cone. Because the mapping $\mathcal{C}_{s'}$ is continuous from X to $\mathrm{L}^p_{c,loc}(\mathbb{R}, Y)$, we have $\mathcal{C}_{s'}x_n \to \mathcal{C}_{s'}x$ and $0 \leqslant \mathcal{C}_{s'}x \in \mathrm{L}^p_{c,loc}(\mathbb{R}; Y)$. Clearly, if $\mathcal{C}_{s'}$ is positive for every $s' \in \mathbb{R}$, then $\mathcal{C}_s$ is positive for the initial time $s \in \mathbb{R}$.

Let $\mathcal{D}$ be a positive operator. Then $\mathcal{D}_{s'} := \mathcal{D}\pi_{[s',\infty)}$ is a positive operator for all $s' \in \mathbb{R}$, because the projection $\pi_{[s',\infty)}$ is positive. Clearly, this implies the positivity of $\mathcal{D}_s$ (for some fixed $s \in \mathbb{R}$).
Now assume that $\mathcal{D}_s$ is a positive operator (for some fixed $s \in \mathbb{R}$). Then, from [72, Lemma 2.2.10 iii)] we have that for all $s, h \in \mathbb{R}$ the equality

$$\mathcal{D}_{s+h} = \tau^{-h}\mathcal{D}_s\tau^h$$

holds. Since τ^h is a positive operator for all $h \in \mathbb{R}$ it follows with the limit

$$\mathcal{D}u = \lim_{s\to-\infty} \mathcal{D}_s u \geqslant 0$$

for all $0 \leqslant u \in \mathrm{L}^p_{c,loc}(\mathbb{R}; U)$ from Lemma 4.4.4 that $\mathcal{D}$ is a positive operator, too. ■

Remark 4.4.24 *Note that the operators B and C are uniquely defined by $\mathcal{B}$ and $\mathcal{C}$ (see Theorem 4.4.8), because we have $p = \infty$ excluded. If we have an L^∞-well-posed system we can restrict the input functions $\mathrm{L}^\infty_{c,loc}(\mathbb{R}; U)$ to the space $Reg_{c,loc}(\mathbb{R}; U)$ (see Theorem 4.4.5). Then B is uniquely defined by $\mathcal{B}$ (for all $u \in Reg_{c,loc}(\mathbb{R}; U)$) and the conclusions of the above proposition hold.*

Remark 4.4.25 *It is not trivial that the limit $\lim_{s\to-\infty} \mathcal{D}_s u$ is still positive, because the limit is taken in the function space $\mathrm{L}^p_{c,loc}(\mathbb{R}; U)$. However, from Remark 4.4.22 we know that this function space is a Fréchet lattice and this implies the closedness of the positive cone (see Corollary 2.2.20).*

Theorem 4.4.26 *Let $1 \leqslant p \leqslant \infty$. An L^p-well-posed or Reg-well-posed linear system $\Sigma(\mathcal{T}, \mathcal{B}, \mathcal{C}, \mathcal{D})$ is positive if and only if $\mathcal{T}$ is a positive strongly continuous semigroup and the operators $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$ are positive.*

Proof: First note that the operators (4.38) and (4.40) defined at the beginning of this section are positive operators.
Now assume that $\Sigma(\mathcal{T}, \mathcal{B}, \mathcal{C}, \mathcal{D})$ is a positive (for an initial time $s \in \mathbb{R}$) and a L^p-well-posed system for some $p \in [1, \infty]$. Then take $u \equiv 0$ and $0 \leqslant x_s \in X$ which gives $T(t-s)x_s = x(t) \geqslant 0$ for almost every $t \geqslant s$ and every $x_s \in X_+$. This implies, with the strong continuity, the positivity of the semigroup $\mathcal{T} = (T(t))_{t\geqslant 0}$. Moreover, in this situation we have $\mathcal{C}_s x_s = y \geqslant 0$ for all $x_s \in X_+$, which shows that $\mathcal{C}_s$ is positive and therefore $\mathcal{C}$ is a positive mapping, because of Proposition 4.4.23.
Now let $t \in \mathbb{R}$ be arbitrary, set $x_s = 0 \in X$ and take $u \geqslant 0$. Then $\mathcal{B}^t_s u = x(t) \geqslant 0$ for each $t \geqslant s$ (and fixed $s \in \mathbb{R}$) and therefore $\mathcal{B}$ is positive, because of Proposition 4.4.23. Further, we have $\mathcal{D}_s u = y \geqslant 0$ for each $u \geqslant 0$, because $x_s = 0$. Then Proposition 4.4.23 implies the positivity of $\mathcal{D}$.

To prove the other direction, we assume that the operators $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$ are positive and that the semigroup $\mathcal{T}$ is positive, too. Let $s \in \mathbb{R}$ be arbitrary, $x_s \in X_+$ and $0 \leqslant u \in \mathrm{L}^p_{c,loc}(\mathbb{R}; U)$. Then, with the positivity of the operators τ^t and $\pi_{[s,t)}$ (for each $t \in \mathbb{R}$ with $t \geqslant s$) we have $\mathcal{B}^t_s u = \mathcal{B}\tau^t \pi_{[s,t)} u \geqslant 0$. This gives

$$x(t) = T(t-s)x_s + \mathcal{B}^t_s u \geqslant 0.$$

Moreover, we have $\mathcal{C}_s x_s = \tau^{-s}\mathcal{C}x_s \geqslant 0$ and $\mathcal{D}_s u = \mathcal{D}\pi_{[s,\infty)} u \geqslant 0$ and therefore

$$y = \mathcal{C}_s x_s + \mathcal{D}_s u \geqslant 0.$$

In the case were $\Sigma(\mathcal{T}, \mathcal{B}, \mathcal{C}, \mathcal{D})$ is *Reg*-well-posed, take $u \in Reg_{c,loc}(\mathbb{R}; U)$ instead of $\mathrm{L}^p_{c,loc}(\mathbb{R}; U)$ and the proof follows in the same way as above. ∎

Remark 4.4.27 *The above theorem and Proposition 4.4.23 implies that if an* L^p*-well-posed or Reg-well-posed system* $\Sigma(\mathcal{T}, \mathcal{B}, \mathcal{C}, \mathcal{D})$ *is positive for some initial time* $s \in \mathbb{R}$*, then this system is positive for every initial time.*

Lemma 4.4.28 *Let* $\Sigma(\mathcal{T}, \mathcal{B}, \mathcal{C}, \mathcal{D})$ *be an* L^p*-well-posed and regular sytem for* $1 \leqslant p < \infty$*. Further, let* $\mathcal{T}$ *be a positive semigroup. Then the following statements hold:*

i) C *is positive* $\Longleftrightarrow$ $\tilde{C}$ *is positive,*

ii) C *is positive* $\Longleftrightarrow$ C_L *is positive,*

iii) $\mathcal{D}$ *is positive* $\Longrightarrow$ D *is positive,*

Proof: Let $(C, D(C))$ be a positive operator and $0 \leqslant x \in D(C)$ (recall that we have $D(A) \subset D(C)$). Then for each $t \geqslant 0$

$$\frac{1}{t} C \int_0^t T(s)x \, \mathrm{d}s \geqslant 0$$

and for each $\alpha \geqslant \omega_0$

$$\alpha C(\alpha - A)^{-1}x \geqslant 0.$$

Because both terms have their values in the Banach lattice Y, the limit is still positive if it exists, i.e. $\tilde{C}$ and C_L are positive operators. For the converse direction we have for all $0 \leqslant x \in D(C)$ that $\tilde{C}x = C_L x = Cx$ and this implies the positivity of the operator $(C, D(C))$. Thus we have proven i) and ii).
If $\mathcal{D}$ is a positive operator we have for each $h > 0$ and every $0 \leqslant u \in U$

$$\frac{1}{h} \int_0^h (\mathcal{D}1_+ u)(s) \, \mathrm{d}s \geqslant 0.$$

Again, the above term has its values in the Banach lattice Y and therefore its limit for $h \to 0$ is still positive. This implies the positivity of $D \in \mathcal{L}(U, Y)$ with the formulae from Theorem 4.4.19 i). Note that 1_+ from (4.53) is a positive operator. ∎

Theorem 4.4.29 *Let $\Sigma(\mathcal{T},\mathcal{B},\mathcal{C},\mathcal{D})$ be a L^p-well-posed and regular system for $1 \leqslant p < \infty$. Then $\Sigma(\mathcal{T},\mathcal{B},\mathcal{C},\mathcal{D})$ is positive if and only if $\mathcal{T}$ is a positive semigroup and the operators $C \in \mathcal{L}(X_1, X)$, $B \in \mathcal{L}(U, X_{-1})$, $D \in \mathcal{L}(U, Y)$ are positive.*

Proof: First assume that $\Sigma(\mathcal{T},\mathcal{B},\mathcal{C},\mathcal{D})$ is positive (for every initial time $s \in \mathbb{R}$, cf. Remark 4.4.27). Then the semigroup $\mathcal{T}$ is positive, this implication is already shown in Theorem 4.4.26. From Proposition 4.4.23, Theorem 4.4.26 and Lemma 4.4.28 it follows that C, B and D are positive operators.
For the other implication assume that the semigroup $\mathcal{T}$ is positive and the operators B, C and D, too. From Proposition 4.4.23 and Lemma 4.4.28 we have that the mappings $\tilde{C}$, $\mathcal{B}$ and $\mathcal{B}_s^t$ are positive for every $t, s \in \mathbb{R}$ with $t \geqslant s$. Moreover, Theorem 4.4.19 iv) allows that $\tilde{C}$ can be applied to $x(t)$, i.e. $x(t) \in D(\tilde{C})$ for all $t \geqslant s$. This gives for every $0 \leqslant u \in \mathrm{L}^p_{c,loc}(\mathbb{R}; U)$, $s \in \mathbb{R}$ and $x_s \in X_+$

$$\begin{aligned} x(t) &= T(t-s)x_s + \mathcal{B}_s^t u \geqslant 0 \\ y(t) &= \tilde{C}x(t) + Du(t) = \tilde{C}\left(T(t-s)x_s + \mathcal{B}_s^t u\right) + Du(t) \geqslant 0 \end{aligned}$$

for almost all $t \in \mathbb{R}$ with $t \geqslant s$, which implies that the system is positive. ■

Corollary 4.4.30 *Let $\Sigma(\mathcal{T},\mathcal{B},\mathcal{C},\mathcal{D})$ be a Reg-well-posed sytem. Then $\Sigma(\mathcal{T},\mathcal{B},\mathcal{C},\mathcal{D})$ is positive if and only if $\mathcal{T}$ is a positive semigroup and the uniquely defined operators $C \in \mathcal{L}(X_1, X)$, $B \in \mathcal{L}(U, X_{-1})$, $D \in \mathcal{L}(U, Y)$ are positive.*

Proof: From Theorem 4.4.20 it follows that $\Sigma(\mathcal{T},\mathcal{B},\mathcal{C},\mathcal{D})$ is regular. Then we can either adapt the proof of the above theorem or one uses that in this case the input/output map is given by

$$(\mathcal{D}u)(t) = C_{|X} \int_0^t T_{-1}(t-s)Bu(s)\,\mathrm{d}s + Du(t), \tag{4.55}$$

where D is uniquely determined by $\mathcal{D}$. Note that $C_{|X}$ is positive if and only if C is positive, because Y_+ is closed since Y is a Banach lattice (cf. proof of Lemma 4.4.28 i) and ii)). ■

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