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Input-to-state stability and stabilizability of infinite-dimensional linear systems

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Introduction

The issues of stability and stabilizability of dynamical systems with external inputs belong to the basic concepts in control theory. In order to illustrate the stability question we take a look at the following time-invariant system of ordinary differential equations given by

$$\dot{x} = f(x, u), \qquad x(0) = x_0, \tag{1}$$

with a function $f\colon \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$. We shall make the assumption that for any initial value $x_0 \in \mathbb{R}^n$ and any essentially bounded function u (called the *input* of the system), this equation has a unique solution x (called the *state* of the system), which is defined on the entire half-axis $[0, \infty)$. Generally there are two kinds of stability behaviours. First we have the *internal stability*, also called *Lyapunov stability*, which is the asymptotic behaviour of the trajectories $t \mapsto x(t)$ for $u = 0$. The second is the *external stability*, which is the stability with respect to the inputs u.
The notion of *input-to-state stability* (ISS) was introduced by E. Sontag in 1989, see [Son89]. For a more recent account of the theory we also refer to compendium [Son08]. It allows a joint description of both internal and external stabilities of a system in a unifying manner. System (1) is called input-to-state stable with respect to L^∞ if for all initial values $x_0 \in \mathbb{R}^n$ and all measurable, essentially bounded functions $u\colon [0, \infty) \to \mathbb{R}^m$ we have

$$\|x(t)\| \leq \beta(\|x_0\|, t) + \gamma(\|u\|_\infty) \tag{2}$$

for all $t \geq 0$. The function $\gamma\colon [0, \infty) \to [0, \infty)$ is continuous, strictly increasing and satisfies $\gamma(0) = 0$. It is called the *gain* and the set of all such functions is denoted by $\mathcal{K}$. The function $\beta\colon [0, \infty) \times [0, \infty) \to [0, \infty)$ is an element of the set $\mathcal{KL}$. This means that we have $\beta(\cdot, t) \in \mathcal{K}$ for all $t \geq 0$ and for every fixed $s > 0$ the function $\beta(s, \cdot)$ is continuous, strictly decreasing and satisfies $\lim_{t\to\infty} \beta(s, t) = 0$.
A further advantage of using ISS lies in its invariance with respect to nonlinear changes of variables. More precisely, assume that we have a homeomorphism $F\colon \mathbb{R}^n \to \mathbb{R}^n$ of the state space with $F(0) = 0$ and a homeomorphism $G\colon \mathbb{R}^m \to \mathbb{R}^m$ of the input space with $G(0) = 0$, both not necessarily linear. Then making the changes of variables $x(t) = F(y(t))$ and $u(t) = G(v(t))$ leads to another system, which is ISS if and only if the original system is ISS, see [Son08]. This invariance does not hold, for example, for exponential stability. This fact makes

the notion of ISS more suitable for scrutinizing questions of stability of nonlinear systems. In fact, the notion of global asymptotic stability, which is loosely speaking the input-to-state stability without the external inputs, and more precisely the estimate (2) with $u = 0$, arises naturally if one starts with an exponentially stable system and performs a nonlinear change of coordinates. Additionally allowing inputs, a nonlinear transformation of the input space leads to the estimate given by (2), see [Son08].

Input-to-state stability has been studied intensively and many important characterisations of ISS have been developed, see e.g. [SW95, SW96]. Later E. Sontag introduced in [Son98] another related notion of stability, the so-called *integral input-to-state stability* (iISS), motivated by the fact that by taking unbounded inputs, the right hand side of the ISS estimate might become infinite and, thus, no relevant information is obtained in this situation. System (1) is called integral input-to-state stable with respect to L^∞ if there exist $\beta \in \mathcal{KL}$, an unbounded function $\theta \in \mathcal{K}$ and $\mu \in \mathcal{K}$ such that

$$\|x(t)\| \leq \beta(\|x_0\|, t) + \theta\left(\int_0^t \mu(\|u(s)\|)\, ds\right) \tag{3}$$

for all $t \geq 0$ and $u \in L^\infty(0, t; \mathbb{R}^m)$. If the system (1) is linear, i.e., we have $f(x, u) = Ax + Bu$ with some linear maps A and B, then it is ISS if and only if it is iISS. In general, for ODE system the notion of iISS is weaker than ISS. For instance, as it follows immediately from the definitions, an iISS system does not necessarily have bounded trajectories if the inputs are bounded. In applications many systems are not ISS but iISS. Therefore, a particular interest in iISS is justified. Both notions can be defined for a more general function space Z, other than L^∞. The definitions have to be adopted accordingly and it is clear that whether or not a certain system is (i)ISS depends on the choice of the topology of Z.

More recently the ISS concept has been adopted for infinite-dimensional systems, see [JLR08, DM13a, KK16, KK17, MP11, MI14, MI16, MW18, MW15, DM13b, Log13, Mir16, MI15]. As we are only standing at the beginning of this development there is no comprehensive ISS theory for infinite-dimensional systems in Banach spaces. As one would expect, many well-known characterisations of ISS and related notions fail to be true in infinite-dimensional settings, see e.g. [MW16] for a series of counterexamples. In [MI16] it is shown that the equivalence between ISS and iISS for linear finite-dimensional systems remains true if we pass on to the following class of linear systems:

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad x(0) = x_0, \tag{4}$$

where A is the generator of a C_0-semigroup $(T(t))_{t\geq 0}$ on a Banach space X, U is another Banach space and $B \in \mathcal{L}(U, X)$. This system is denoted by $\Sigma(A, B)$. In applications, the operator B is typically linear but not necessarily bounded. This class of systems is of particular interest since it includes boundary control problems that are described by evolution partial differential equations. In this situation the notion of *admissibility* plays a crucial role in characterisation of ISS .

The first pivotal question we have here is how ISS and iISS are connected for linear systems with possibly unbounded input operators for $Z = L^p$. We will see that for finite p those notions are equivalent. For $p = \infty$ the problem turns out to be more challenging. One apparent difficulty when dealing with the iISS estimate is that the argument of θ in (3) is in general not a norm of the input function u.

Similar to the ISS situation, the iISS estimate is invariant with respect to nonlinear transformations of the input space and can be obtained by starting with the L^p-norm and then performing such a transformation. This reminds us of the *Orlicz spaces* L_Φ – a generalisation of the usual L^p spaces, where the role in the definition played by the function $t \mapsto t^p$ is replaced by a more general function Φ. So it seems to be natural to try to relate iISS to Orlicz spaces. Indeed, it turns out that iISS with respect to L^∞ is equivalent to ISS with respect to a certain subspace E_Φ of an Orlicz space L_Φ. It is, though, not straightforward to see this connection. The main difficulty in carrying out this relation is that the set of functions Φ allowed in the definition of Orlicz spaces, the so-called Young functions, is not the entire set $\mathcal{K}$. Even if the function μ in the argument of θ in (3) is a Young function, it is still not a norm. Thus it takes some technical preparation. A further result we will obtain is that ISS and iISS are equivalent if both are taken with respect to E_Φ. This generalises the equivalence of both stability notions with respect to L^p spaces with $p < \infty$.

A further advantage of using the Orlicz spaces lies in the fact that, unlike in the case of L^p spaces, their union on a fixed bounded interval is exactly the set L^1. This yields a characterisation of L^1-admissibility.

A particular class of linear ISS systems are the parabolic diagonal systems, which means that the operator A possesses a q-Riesz basis of eigenvectors with eigenvalues lying in a sector in the left half-plane and being uniformly bounded away from the imaginary axis. We will see that for those systems the notions of L^∞-ISS and L^∞-iISS are equivalent if we have scalar inputs, i.e., $U = \mathbb{C}$. Moreover, we will show that every linear operator $B \colon \mathbb{C} \to X_{-1}$ is admissible with respect to L^∞, which adds a further characterisation of ISS for such systems.

In addition, we will study the notion of *strong input-to-state stability* (sISS or strong ISS). It was introduced in [MW18] for bilinear systems. For linear systems it generalises ISS in the sense that the exponential stability of the semigroup is relaxed to the more general strong stability. Our main concern is the connection between sISS and its integral version – *strong integral input-to-state stability* (siISS or strong iISS). We will see that strong iISS with respect to L^∞ is implied by infinite-time admissibility with respect to some Orlicz space E_Φ. But unlike in the ISS situation they are not equivalent. We will construct an example of a system that is L^∞-siISS but not infinite-time admissible with respect to E_Φ for any Young function Φ. Therefore, sISS and siISS cannot be equivalent neither for L^∞ nor E_Φ inputs.

The second issue we will study is the question of *stabilizability* of the linear systems by state feedback. The classical problem of exponential stabilizability is well-documented in the literature, see e.g. [CZ95, Chapter 5] or [JZ12, Chapter 10] and the references therein for more details. Considering a system given by (4) the question is whether there exists a feedback law $u(t) = Fx(t)$ such that the closed-loop system is exponentially stable. For bounded control

operators this means that we can find a bounded feedback operator $F \in \mathcal{L}(X, U)$ such that the system $\Sigma(A+BF, B)$ is input-to-state stable with respect to L^2. Assuming that the input space U is finite-dimensional and the control operator B is bounded, W. Desch and W. Schappacher [DS85], C. A. Jacobson and C. N. Nett [JN88], and S. A. Nefedov and F. A. Sholokhovich [NS86] showed that a system $\Sigma(A, B)$ is exponentially stabilizable if and only if it can be decomposed into two parts: an exponentially stable part and a finite-dimensional controllable part. We will see that similar results hold if we replace the exponential stability by weaker stability concepts, strong or polynomial stability in particular. Our definition of *strong* (or *polynomial*) *stabilizability* of a linear system is motivated by the definition of exponential stabilizability given in [WR00]. In fact, it is a direct generalisation as the exponential stability of the semigroup associated with the system is replaced by a weaker stability notion. We also refer to [OC98] and [CO99], where the notion of strong stability of a linear system is studied. The essential idea is that when weakening the stability requirement of the semigroup associated with the system, one has to tighten the conditions on the entire system by adding input stability, output stability and input-output stability. Roughly speaking, those conditions state that L^2-inputs lead to bounded states, every initial condition leads to outputs, which belong to L^2 and that L^2-inputs lead to L^2-outputs. Those conditions are redundant when the corresponding semigroup is exponentially stable. Thus strong and polynomial stabilizabilities are more general concepts than the exponential stabilizability.
The characterisation of exponential stabilizability was generalised to linear systems with unbounded control operators in [JZ99]. We will see that analogous conditions are sufficient for strong as well as polynomial stabilizability of linear systems with unbounded control. However, it remains an open question if those conditions are also necessary.
This thesis is organised as follows. In Chapter 1 we review some of the definitions and standard facts on Orlicz spaces. Chapter 2 deals with linear systems on Banach spaces. There we set up notation and terminology. As we are interested in ISS and iISS with respect to various function spaces, those notions will be introduced in an abstract way. We axiomatically introduce the class of function spaces we want to work with. Those include, for instance, the Orlicz spaces, the L^p spaces and Sobolev spaces. Most of the results we present in this chapter are well-known in L^p context and their proofs are straight forward generalisations. We still include them for readers' convenience, making the exposition self contained. In Chapter 3 the strong versions of input-to-state stability and integral input-to-state stability are introduced. After breaking down some basic properties of those stability notions we establish the relation between L^∞-siISS and sISS with respect to E_Φ, with some Young function Φ. The question regarding how those notions are related is motivated by the findings of the following chapter, which were established beforehand. But since its concepts and results are more general, we choose to put it first. The main results of this chapter are published in [NS18]. Chapter 4 is devoted to the study of ISS and iISS. We extend the main results from the previous chapter to the situation where the semigroup is exponentially stable. The most important advantage is that in this case, admissibility and infinite-time admissibility become equivalent. This helps us

to derive the characterisation of L^∞-iISS as ISS with respect to an Orlicz space E_Φ. We further study parabolic diagonal systems with scalar inputs. We show that in this situation iISS and ISS are equivalent. At the end of the chapter the main results are applied to an example given by a one-dimensional heat equation with Dirichlet boundary control. The results of Chapter 4 are published in [JNPS18], see also [JNPS16].

In Chapter 5 we shift our focus to stabilizability of infinite-dimensional linear systems. We start by reviewing some of the well-known facts about stabilizability of finite-dimensional systems. We then proceed by summarising without proofs the relevant material on well-posed and regular linear systems. Our main results of this chapter are the sufficient conditions for strong and polynomial stabilizability of linear systems with admissible control operators, see Section 5.5, and equivalent conditions for strong and polynomial stabilizability of linear systems with bounded control operators and finite-dimensional input spaces, see Sections 5.6 and 5.7.

Chapter 1

Orlicz spaces

In this chapter we recall some basic definitions and facts about Orlicz spaces. More details can be found in [KR61], [KJF77, Part II, Chapter 3] and [Ada75, Chapter VIII]. For the generalisation to vector-valued functions see [RR91, VII, Section 7.5].

1.1 Young functions

Let $I \subset \mathbb{R}$ be an interval, U a Banach space and $\Phi\colon [0,\infty) \to [0,\infty)$ a function. We denote by λ the usual Lebesgue measure on $\mathbb{R}$.

Definition 1.1.1. *The* Orlicz class $\tilde{L}_\Phi(I,U)$ *is the set of all equivalence classes (with respect to equality almost everywhere) of Bochner-measurable functions $u\colon I \to U$ such that*

$$\rho_\Phi(u) := \int_I \Phi(\|u(x)\|_U)\,dx < \infty.$$

If $U = \mathbb{K}$ with $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$, then we write $\tilde{L}_\Phi(I) := \tilde{L}_\Phi(I,\mathbb{K})$ for short. In general, $\tilde{L}_\Phi(I,U)$ is not a vector space. Of particular interest are Orlicz classes generated by Young functions.

Definition 1.1.2. *A function $\Phi\colon [0,\infty) \to \mathbb{R}$ is called a* Young function *(or* Young function generated by *φ) if*

$$\Phi(t) = \int_0^t \varphi(s)\,ds$$

for $t \geq 0$, where the function $\varphi\colon [0,\infty) \to \mathbb{R}$ has the following properties: $\varphi(0) = 0$, $\varphi(s) > 0$ for $s > 0$, φ is right continuous at any point $s \geq 0$, φ is nondecreasing on $(0,\infty)$ and $\lim_{s\to\infty} \varphi(s) = \infty$.

We will need the following characterisation of Young functions, see e.g. page 9 in [KR61].

Proposition 1.1.3. *A continuous, increasing, convex function* $\Phi\colon [0,\infty) \to \mathbb{R}$ *with* $\Phi(0) = 0$ *is a Young function if and only if it satisfies* $\lim_{t\searrow 0} \Phi(t)/t = 0$ *and* $\lim_{t\to\infty} \Phi(t)/t = \infty$.

Example 1.1.4. *Using Proposition 1.1.3 it is easy to see that the following holds true:*

(a) For any $p > 1$ *the function* $\Phi(t) = t^p$ *is a Young function.*

(b) The function $\Psi(t) = e^t - t - 1$ *is a Young function.*

For any two Young functions Φ and Ψ their composition $\Phi \circ \Psi$ is again a Young function. More general the following holds:

Lemma 1.1.5. *Let* $\Psi\colon [0,\infty) \to [0,\infty)$ *be a Young function and* $\mu\colon [0,\infty) \to [0,\infty)$ *an unbounded convex function with* $\mu(0) = 0$. *Then the composition* $\Phi := \Psi \circ \mu$ *is a Young function.*

Proof. The function Φ is continuous, increasing and convex since both functions, Ψ and μ have those properties. Using Proposition 1.1.3 we are left to show that Φ satisfies $\lim_{t\searrow 0} \Phi(t)/t = 0$ and $\lim_{t\to\infty} \Phi(t)/t = \infty$. For all $t > 0$ we have

$$0 \le \frac{\Phi(t)}{t} = \frac{\Psi(\mu(t))}{t} = \frac{\Psi(\mu(t))}{\mu(t)} \frac{\mu(t)}{t}. \tag{1.1}$$

Since Ψ is a Young function and μ is continuous with $\mu(0) = 0$, we have by Proposition 1.1.3

$$\lim_{t\to 0} \frac{\Psi(\mu(t))}{\mu(t)} = 0.$$

From the convexity of μ it follows that for each $R > 0$ the map $t \mapsto \mu(t)/t$ is bounded on the interval $(0, R]$. Therefore we have

$$\lim_{t\to 0} \frac{\Psi(\mu(t))}{\mu(t)} \frac{\mu(t)}{t} = 0.$$

Equation (1.1) now yields

$$\lim_{t\to 0} \frac{\Phi(t)}{t} = 0.$$

The convexity of the function μ implies that the map $t \mapsto \mu(t)/t$ is increasing on $(0,\infty)$. Since $\lim_{t\to\infty} \Psi(t)/t = \infty$ and μ is a homeomorphism of $[0,\infty)$, we have

$$\lim_{t\to\infty} \frac{\Psi(\mu(t))}{\mu(t)} = \infty.$$

Therefore we obtain

$$\lim_{t\to\infty} \frac{\Phi(t)}{t} = \lim_{t\to\infty} \frac{\Psi(\mu(t))}{\mu(t)} \frac{\mu(t)}{t} = \infty.$$

Hence, by Proposition 1.1.3, Φ is a Young function. $\square$

In the following lemma we have another construction of Young functions, which will be useful later on.

Lemma 1.1.6. *Let Φ be a Young function. Then there exists some Young function Φ_1 such that $\Phi \leq \Phi_1$ and*

$$\sup_{x>0} \frac{\Phi(cx)}{\Phi_1(x)} < \infty$$

for all $c > 0$.

Proof. We denote by φ the generator of the Young function Φ, i.e., $\Phi(x) = \int_0^x \varphi(t)\,dt$. Let us define two Young functions $\Lambda, \Psi\colon [0,\infty) \to \mathbb{R}$ by

$$\Lambda(x) = \int_0^x \varphi(\sqrt{t})\,dt$$

and $\Psi(x) = \Phi(x^2)$. Then, obviously, $\Phi \leq \Lambda$ holds on the interval $[0,1]$ and $\Phi \leq \Psi$ holds on $[1,\infty)$. Therefore, $\Phi_1\colon [0,\infty) \to \mathbb{R}$,

$$\Phi_1(x) = \begin{cases} \Lambda(x) & \text{for } x < 1, \\ \frac{\Lambda(1)}{\Psi(1)}\Psi(x) & \text{for } x \geq 1, \end{cases}$$

defines a Young function with $\Phi \leq \Phi_1$, since $\Lambda(1) \geq \Psi(1) = \Phi(1)$. We show by checking explicitly the Definition 1.1.2 that Φ_1 is a Young function. Let $\varphi_1\colon [0,\infty) \to \mathbb{R}$ be defined by

$$\varphi_1(t) = \begin{cases} \varphi(\sqrt{t}) & \text{for } t \in [0,1), \\ \frac{2\Lambda(1)}{\Psi(1)} t\varphi(t^2) & \text{for } t \geq 1. \end{cases}$$

Then this function satisfies all the conditions in Definition 1.1.2 and we have $\Phi_1(x) = \int_0^x \varphi_1(t)\,dt$. Indeed, for all $x \in [0,1]$ holds

$$\int_0^x \varphi_1(t)\,dt = \int_0^x \varphi(\sqrt{t})\,dt = \Lambda(x) = \Phi_1(x)$$

and for all $x \in (1,\infty)$ we have

$$\begin{aligned} \int_0^x \varphi_1(t)\,dt &= \int_0^1 \varphi_1(t)\,dt + \int_1^x \varphi_1(t)\,dt \\ &= \Phi_1(1) + \frac{2\Lambda(1)}{\Psi(1)} \int_1^x t\varphi(t^2)\,dt \\ &= \Phi_1(1) + \frac{\Lambda(1)}{\Psi(1)} \int_1^{x^2} \varphi(t)\,dt \\ &= \Phi_1(1) + \frac{\Lambda(1)}{\Psi(1)}(\Psi(x) - \Psi(1)) \\ &= \frac{\Lambda(1)}{\Psi(1)}\Psi(x) \\ &= \Phi_1(x). \end{aligned}$$

We now show that for each $c > 0$ the function $x \mapsto \Phi(cx)/\Phi_1(x)$ is bounded on $(0,\infty)$. For $0 < c \leq 1$ this simply follows from the monotonicity of Φ. Indeed, we have

$$\frac{\Phi(cx)}{\Phi_1(x)} \leq \frac{\Phi(x)}{\Phi_1(x)} \leq 1$$

for all $x > 0$.
Now let $c > 1$. For $x \geq c$ we have

$$\frac{\Phi(cx)}{\Phi_1(x)} = \frac{\Psi(1)\Phi(cx)}{\Lambda(1)\Phi(x^2)} \leq \frac{\Psi(1)}{\Lambda(1)}.$$

For an arbitrary Young function Ω, generated by ω, we have for all $y > 0$

$$\frac{\Omega(y)}{y} = \frac{1}{y}\int_0^y \omega(t)\,dt \geq \frac{1}{y}\int_{y/2}^y \omega(t)\,dt \geq \frac{1}{2}\omega\left(\frac{y}{2}\right)$$

and

$$\frac{\Omega(y)}{y} = \frac{1}{y}\int_0^y \omega(t)\,dt \leq \omega(y).$$

Therefore we have

$$\frac{\Phi(cx)}{\Lambda(x)} = c\frac{\Phi(cx)}{cx}\frac{x}{\Lambda(x)} \leq 2c\frac{\varphi(cx)}{\varphi\left(\sqrt{\frac{x}{2}}\right)} \leq 2c,$$

where the last inequality holds for all $x \in (0, 1/(2c^2)]$. Since the continuous function $x \mapsto \Phi(cx)/\Phi_1(x)$ is bounded on the compact interval $[1/(2c^2), c]$, the claim follows. $\square$

Theorem 1.1.7. *Let Φ be a Young function. Then $\tilde{L}_\Phi(I,U)$ is a convex set. If, additionally, the interval I is bounded, there holds $\tilde{L}_\Phi(I,U) \subset L^1(I,U)$.*

Proof. Let $u, v \in \tilde{L}_\Phi(I,U)$ and $\lambda \in (0,1)$. From the triangle inequality in U and the convexity of Φ we obtain

$$\begin{aligned}\int_I \Phi(\|\lambda u(x) + (1-\lambda)v(x)\|_U)\,dx &\leq \int_I \Phi(\lambda\|u(x)\|_U + (1-\lambda)\|v(x)\|_U)\,dx \\ &\leq \lambda\int_I \Phi(\|u(x)\|_U)\,dx + (1-\lambda)\int_I \Phi(\|v(x)\|_U)\,dx \\ &= \lambda\rho_\Phi(u) + (1-\lambda)\rho_\Phi(v) \\ &< \infty\end{aligned}$$

and hence $\lambda u + (1-\lambda)v \in \tilde{L}_\Phi(I,U)$.
Let I be bounded. Since we have $\lim_{t\to\infty}\Phi(t)/t = \infty$, there is a $t_0 > 0$ such that $\Phi(t)/t > 1$ for all $t > t_0$. Let

$$I_{t_0} = \{x \in I \mid \|u(x)\|_U > t_0\},$$

then for all $x \in I_{t_0}$ we have $\|u(x)\|_U \leq \Phi(\|u(x)\|_U)$ and hence

$$\begin{aligned}\int_I \|u(x)\|_U \,dx &= \int_{I_{t_0}} \|u(x)\|_U \,dx + \int_{I\setminus I_{t_0}} \|u(x)\|_U \,dx \\ &\leq \rho_\Phi(u) + t_0\lambda(I \setminus I_{t_0}) \\ &< \infty.\end{aligned}$$

Thus we have $u \in L^1(I,U)$. □

Theorem 1.1.8. *Assume that the interval I is bounded. Then for every $u \in L^1(I,U)$ there exists a Young function Φ such that $u \in \tilde{L}_\Phi(I,U)$.*

Proof. Let $u \in L^1(I,U)$. For $n \in \mathbb{N}$ let $I_n \subset I$ be the measurable set

$$I_n = \{x \in I \mid \|u(x)\|_U \in [n-1,n)\}.$$

Then $I = \bigcup_{n\in\mathbb{N}} I_n$ holds and the sets I_n, $n \in \mathbb{N}$, are disjoint. Hence we have $\lambda(I) = \sum_{n=1}^\infty \lambda(I_n)$. Now for any $N \in \mathbb{N}$ follows

$$\begin{aligned}\sum_{n=1}^N n\lambda(I_n) &= \sum_{n=1}^N (n-1)\lambda(I_n) + \sum_{n=1}^N \lambda(I_n) \\ &\leq \sum_{n=1}^N \int_{I_n} \|u(x)\|_U \,dx + \sum_{n=1}^N \lambda(I_n) \\ &\leq \int_I \|u(x)\|_U \,dx + \lambda(I) \\ &= \|u\|_{L^1(I,U)} + \lambda(I).\end{aligned}$$

Therefore the series $\sum_{n=1}^\infty n\lambda(I_n)$ converges. By Remark 178 in [Kno28] there is a monotonically increasing unbounded sequence $(\alpha_n) \subset [1,\infty)$ such that the series $\sum_{n=1}^\infty \alpha_n n\lambda(I_n)$ still converges. We define $\varphi\colon [0,\infty) \to \mathbb{R}$ by

$$\varphi(t) = \begin{cases} t & \text{for } t \in [0,1), \\ \alpha_n & \text{for } t \in [n,n+1), n \in \mathbb{N}\setminus\{0\}. \end{cases}$$

Then $\Phi(t) = \int_0^t \varphi(s)\,ds$ is a Young function and for all $n \in \mathbb{N}$ we have $\Phi(n) \leq n\alpha_n$. Hence we have

$$\begin{aligned}\int_I \Phi(\|u(x)\|_U)\,dx &= \sum_{n=1}^\infty \int_{I_n} \Phi(\|u(x)\|_U)\,dx \\ &\leq \sum_{n=1}^\infty \Phi(n)\lambda(I_n) \\ &\leq \sum_{n=1}^\infty \alpha_n n\lambda(I_n) \\ &< \infty,\end{aligned}$$

and therefore $u \in \tilde{L}_\Phi(I, U)$. $\square$

Definition 1.1.9. *Let Φ be the Young function generated by the function φ. We set for $t \geq 0$*

$$\psi(t) = \sup_{\varphi(s) \leq t} s \quad \text{and} \quad \Psi(t) = \int_0^t \psi(s)\, ds.$$

The function Ψ is called the complementary function *to Φ.*

The complementary function of a Young function is again a Young function, i.e., the function ψ has the same properties as the function φ, see Definition 1.1.2. If φ is continuous and strictly increasing in $[0, \infty)$, then ψ is the inverse function φ^{-1} and vice versa. We call Φ and Ψ a *pair of complementary Young functions.*

Lemma 1.1.10. *Assume that the functions $\varphi, \psi \colon [0, \infty) \to \mathbb{R}$ generate two complementary Young functions and $u, v \geq 0$. If $v < \varphi(u)$, then $u > \psi(v)$. If $v > \varphi(u)$, then $u \leq \psi(v)$.*

Proof. If $v < \varphi(u)$, then, by definition of ψ, we have

$$\psi(v) = \sup_{\varphi(s) \leq v} s \leq u,$$

since φ is increasing. The equality $u = \psi(v)$ cannot hold, since it would imply $v \geq \varphi(u)$. If $v > \varphi(u)$, then we have

$$\psi(v) = \sup_{\varphi(s) \leq v} s \geq u$$

by definition of ψ. $\square$

Lemma 1.1.11. *Assume that the functions $\varphi, \psi \colon [0, \infty) \to \mathbb{R}$ generate two complementary Young functions. Then the following are equivalent:*

(i) $\varphi(s) \leq t$ and $\psi(t) \leq s$.

(ii) $\varphi(s) = t$ or $\psi(t) = s$.

Proof. (i) $\Rightarrow$ (ii): Assume that $\varphi(s) < t$ holds. Then, by Lemma 1.1.10, we have $s \leq \psi(t)$. Together with the condition $\psi(t) \leq s$, we obtain $\psi(t) = s$.
Now assume $\psi(t) < s$. We apply Lemma 1.1.10 with exchanged roles for φ and ψ and obtain with the same argument as in the previous case that $\varphi(s) = t$ holds.
(i) $\Rightarrow$ (ii): If $\psi(t) = s$ holds, then we have $\varphi(s) \leq t$, since otherwise $\varphi(s) > t$ holds and, hence, by Lemma 1.1.10, $s > \psi(t)$, which is a contradiction.
If $\varphi(s) = t$ holds, then we have $\psi(t) \leq s$ since otherwise $\psi(t) > s$ holds and, hence, by Lemma 1.1.10, $t > \varphi(s)$. $\square$

Theorem 1.1.12 (Young's inequality)**.** *Let Φ, Ψ be a pair of complementary Young functions and φ, ψ their generating functions. Then for all $u, v \in [0, \infty)$ we have*

$$uv \leq \Phi(u) + \Psi(v).$$

Equality holds if and only if $v = \varphi(u)$ or $u = \psi(v)$.

Remark 1.1.13. *Let Φ, Ψ be a pair of complementary Young functions, $u \in \tilde{L}_\Phi(I)$ and $v \in \tilde{L}_\Psi(I)$. By integrating Young's inequality we obtain*

$$\int_I |u(x)v(x)|\, dx \leq \rho_\Phi(u) + \rho_\Psi(v).$$

Proof of Theorem 1.1.12. For $u, v \geq 0$ denote by R the rectangle $R := [0, u] \times [0, v] \subset \mathbb{R}^2$, by R_1 the part of R below the graph of φ, i.e., $R_1 = R \cap \{(x, y) \in \mathbb{R}^2 \mid x \in [0, u] \text{ and } 0 \leq y \leq \varphi(x)\}$ and by R_2 the part above the graph of φ, i.e., $R_2 = R \cap \{(x, y) \in \mathbb{R}^2 \mid x \in [0, u] \text{ and } \varphi(x) \leq y \leq v\}$. Then $R = R_1 \cup R_2$ and $\Gamma(\varphi) = R_1 \cap R_2$, the graph of φ, is a λ_2-null set. Hence we have

$$uv = \lambda_2(R) = \lambda_2(R_1) + \lambda_2(R_2) = \int_R \chi_{R_1}\, d\lambda_2 + \int_R \chi_{R_2}\, d\lambda_2.$$

By Fubini's theorem we have

$$\int_R \chi_{R_1}\, d\lambda_2 = \int_0^u \int_0^{\min\{\varphi(x), v\}} 1\, dy\, dx \leq \int_0^u \int_0^{\varphi(x)} 1\, dy\, dx = \int_0^u \varphi(x)\, dx = \Phi(u).$$

with equality if and only if $\varphi(x) \leq v$ for almost all $x \in [0, u]$ and, hence, by right continuity and monotonicity of φ, if and only if $\varphi(u) \leq v$. By transformation formula we have

$$\lambda_2(R_2) = \lambda_2(R \cap \{(x, y) \in \mathbb{R}^2 \mid y \in [0, v] \text{ and } 0 \leq y \leq \psi(x)\})$$

and hence, by Fubini's theorem, we have

$$\int_R \chi_{R_2}\, d\lambda_2 = \int_0^v \int_0^{\min\{\psi(x), u\}} 1\, dy\, dx \leq \int_0^v \int_0^{\psi(x)} 1\, dy\, dx = \int_0^v \psi(x)\, dx = \Psi(u)$$

with equality if and only if $\psi(v) \leq u$. Overall we obtain

$$uv \leq \Phi(u) + \Psi(u)$$

and, by Lemma 1.1.11, the equality holds if and only if $\varphi(u) = v$ or $\psi(v) = u$. $\square$

1.2 Orlicz spaces

We are now in the position to define the Orlicz spaces. There are equivalent definitions of Orlicz spaces available. Here we use the so-called *Luxemburg norm.*

Definition 1.2.1. *Let $I \subset \mathbb{R}$ be an interval, U a Banach space and Φ a Young function. The set $L_\Phi(I,U)$ of all equivalence classes (with respect to equality almost everywhere) of Bochner measurable functions $u\colon I \to U$, for which there is a $k > 0$ such that*

$$\int_I \Phi(k^{-1}\|u(x)\|_U)\,dx < \infty,$$

is called the Orlicz space. *The* Luxemburg norm *of $u \in L_\Phi(I,U)$ is defined as*

$$\|u\|_\Phi \coloneqq \|u\|_{L_\Phi(I,U)} \coloneqq \inf\left\{k > 0 \;\middle|\; \int_I \Phi(k^{-1}\|u(x)\|_U)\,dx \le 1\right\}.$$

If $U = \mathbb{K}$ with $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$, then we write $L_\Phi(I) \coloneqq L_\Phi(I,\mathbb{K})$ for short. For the choice $\Phi(t) \coloneqq t^p$, $1 < p < \infty$, the Orlicz space $L_\Phi(I,U)$ is exactly the vector-valued L^p space with the same norm. Next we show that the Orlicz spaces are complete with respect to the Luxemburg norm. The proof we present here mimics the one for the scalar-valued case as it is given in [RR91, pp. 67-68].

Theorem 1.2.2. *The normed space $(L_\Phi(I,U), \|\cdot\|_\Phi)$ is a Banach space.*

Proof. Let $(u_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $L_\Phi(I,U)$. Then, by definition, there are numbers $k_{mn} > 0$ such that

$$\int_I \Phi(k_{mn}\|u_m(x) - u_n(x)\|_U)\,dx \le 1$$

for all $m,n \in \mathbb{N}$. Let $\varepsilon > 0$. From the previous estimate we have that $k_{mn}^{-1} \le \|u_m - u_n\|_{L_\Phi(I,U)}$ and hence for every $R > 0$ there is a natural number m_0 such that $k_{mn} > R$ for all $m,n \ge m_0$. It follows that for all sufficiently large numbers $m,n \in \mathbb{N}$ we have $1/\Phi(k_{mn}\varepsilon) < \varepsilon$. For such $m,n \in \mathbb{N}$ and every measurable set $B \subset I$ with $\lambda(B) < \infty$ we then have

$$\begin{aligned}
\lambda(B \cap \{\|u_m(\cdot) - u_n(\cdot)\|_U \ge \varepsilon\}) &= \lambda(B \cap \{\Phi(k_{mn}\|u_m(\cdot) - u_n(\cdot)\|_U \ge \Phi(k_{mn}\varepsilon)\}) \\
&\le \frac{1}{\Phi(k_{mn}\varepsilon)} \int_B \Phi(k_{mn}\|u_m(x) - u_n(x)\|_U)\,dx \\
&\le \frac{1}{\Phi(k_{mn}\varepsilon)} \\
&< \varepsilon.
\end{aligned}$$

This shows that the sequence $(u_n|_B)_{n\in\mathbb{N}}$ is Cauchy in measure. Since the Lebesgue measure on I is σ-finite, it follows that the sequence $(u_n)_{n\in\mathbb{N}}$ is Cauchy in measure. Therefore it converges in measure. We denote by u its limit. Then there is a subsequence $(u_{n_i})_{i\in\mathbb{N}}$, which converges to u almost everywhere. Since $(u_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $L_\Phi(I,U)$, we obtain, using the reverse triangle inequality, that $(\|u_n\|_{L_\Phi(I,U)})_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{R}$. We denote by ρ its limit. Now by Fatou's lemma we have

$$\int_I \Phi(\rho^{-1}\|u(x)\|_U)\,dx \le \liminf_{i\to\infty}\int_I \Phi\left(\frac{\|u_{n_i}(x)\|_U}{\|u_{n_i}\|_{L_\Phi(I,U)}}\right)dx \le 1.$$

This shows $u \in L_\Phi(I,U)$.
For all fixed $j\in\mathbb{N}$ and $k>0$ the sequence $(\Phi(k\|u_{n_i}(\cdot)-u_{n_j}(\cdot)\|_U))_{i\in\mathbb{N}}$ converges to $\Phi(k\|u(\cdot)-u_{n_j}(\cdot)\|_U)$ as $i\to\infty$ almost everywhere. Let $n_0\in\mathbb{N}$ be such that for all $n_i,n_j\ge n_0$ holds $k_{n_i n_j}\ge k$, then we have

$$\int_I \Phi(k\|u_{n_i}(x)-u_{n_j}(x)\|_U)\,dx \le \int_I \Phi(k_{n_i n_j}\|u_{n_i}(x)-u_{n_j}(x)\|_U)\,dx \le 1.$$

Therefore, Fatou's lemma yields

$$\int_I \Phi(k\|u(x)-u_{n_j}(x)\|_U)\,dx \le \liminf_{i\to\infty}\int_I \Phi(k\|u_{n_i}(x)-u_{n_j}(x)\|_U)\,dx \le 1$$

and hence $\|u-u_{n_j}\|_{L_\Phi(I,U)} \le 1/k$. Since $k>0$ is arbitrary, this means that the sequence $(u_{n_i})_{i\in\mathbb{N}}$ converges to u in $L_\Phi(I,U)$. If $(u_{m_i})_{i\in\mathbb{N}}$ is any other subsequence, which converges to $\tilde{u}$ in $L_\Phi(I,U)$, then we have, since u_n converges to u in measure, $u=\tilde{u}$. Hence we have $u_n\to u$ in $L_\Phi(I,U)$. □

Remark 1.2.3. *Let $U=\mathbb{K}$ and Φ, Ψ a pair of complementary Young functions. We have the following characterisation of Orlicz spaces: A measurable function $u\colon I\to\mathbb{K}$ belongs to $L_\Phi(I)$ if and only if*

$$|||u|||_\Phi := \sup_{\substack{v\in\tilde{L}_\Psi(I)\\ \rho_\Psi(v)\le 1}} \int_I |u(x)v(x)|\,dx < \infty.$$

The function $|||\cdot|||_\Phi$ is called the Orlicz norm *on $L_\Phi(I)$. It defines a norm on $L_\Phi(I)$ with*

$$\|u\|_\Phi \le |||u|||_\Phi \le 2\|u\|_\Phi \tag{1.2}$$

for all $u\in L_\Phi(I)$, i.e., the Luxemburg norm and the Orlicz norm are equivalent (see e.g. Theorem 3.6.4 and Theorem 3.8.5 in [KJF77] [1]).

Remark 1.2.4. *For a measurable $u\colon I\to U$ we have that $u\in L_\Phi(I,U)$ if and only if $f:=\|u(\cdot)\|_U\in L_\Phi(I,\mathbb{R})$. This follows from the fact that*

$$\|u\|_\Phi = \|f\|_\Phi.$$

Thus, a sequence $(u_n)_{n\in\mathbb{N}}\subset L_\Phi(I,U)$ converges to zero if and only if the sequence $(\|u_n(\cdot)\|_U)_{n\in\mathbb{N}}$ converges to zero in $L_\Phi(I,\mathbb{R})$.

[1] Note that in this reference the Luxemburg norm is denoted by $|||\cdot|||_\Phi$ and the Orlicz norm is denoted by $\|\cdot\|_\Phi$.

Remark 1.2.5. *Combining Remarks 1.1.13 and 1.2.4 we obtain for every* $u \in \tilde{L}_\Phi(I,U)$

$$|||\|u(\cdot)\|_U|||_\Phi \le \rho_\Phi(u) + \rho_\Psi(u) \le \rho_\Phi(u) + 1 < \infty.$$

Hence we have

$$\tilde{L}_\Phi(I,U) \subset L_\Phi(I,U).$$

The following Theorem is an extension of Hölder's inequality to Orlicz spaces, see [KJF77, Thm. 3.7.5 and Remark 3.8.6].

Theorem 1.2.6. *Let* Φ, Ψ *be a pair of complementary Young functions. For any* $u \in L_\Phi(I)$ *and* $v \in L_\Psi(I)$ *it holds that* $uv \in L^1(I)$ *and*

$$\int_I |u(s)v(s)|\, ds \le 2\|u\|_{L_\Phi(I)}\|v\|_{L_\Psi(I)}.$$

1.3 The Δ_2-condition

Definition 1.3.1. *A Young function* Φ *satisfies the* Δ_2-condition *if there exist a* $k > 0$ *and* $s_0 \ge 0$ *such that*

$$\Phi(2s) \le k\Phi(s)$$

for all $s \ge s_0$.

Example 1.3.2. *(a) Let* $p > 1$. *The Young function* $\Phi(t) = t^p$ *satisfies the* Δ_2*-condition.*

(b) The Young function $\Psi(t) = e^t - t - 1$ *does not satisfy the* Δ_2*-condition.*

Remark 1.3.3. *We saw in Theorem 1.1.8 that for every bounded interval* I *and every function* $u \in L^1(I)$ *there exists a Young function* Φ *such that* u *belongs to the Orlicz class* $\tilde{L}_\Phi(I)$. *Indeed, this Young function can be chosen such that it satisfies the* Δ_2*-condition. In particular we have then* $u \in E_\Phi(I)$, *see Definition 1.5.2 and Proposition 1.5.4 below. The argument is essentially given on pages 61-62 in [KR61]. There, for any given* $u \in L^1(I)$, *a Young function* Q *is constructed, which satisfies the* Δ'*-condition, that is, there exist* $c, s_0 > 0$ *such that*

$$Q(st) \le cQ(s)Q(t)$$

holds for all $s,t \ge s_0$ *and* $u \in \tilde{L}_\Phi(I)$, *where* $\Phi := Q \circ Q$ *satisfies the* Δ_2*-condition. Indeed let* $k = cQ(cQ(2+s_0)+s_0)$. *Then for all* $s \ge \max\{s_0, Q^{-1}(s_0)\}$ *we have*

$$\begin{aligned}
\Phi(2s) &= Q(Q(2s)) \\
&\le Q(Q((2+s_0)s)) \\
&\le Q(cQ(2+s_0)Q(s)) \\
&\le Q((cQ(2+s_0)+s_0)Q(s)) \\
&\le cQ((cQ(2+s_0)+s_0))Q(Q(s)) \\
&= k\Phi(s),
\end{aligned}$$

where we used twice that the function Q *satisfies the* Δ'*-condition and monotonicity of* Q.

1.4 Convergence in Orlicz spaces

The convergence in Orlicz spaces is understood as convergence with respect to the Luxemburg norm. Besides the norm convergence we will use the following weaker notion.

Definition 1.4.1 (Φ-mean convergence). *A sequence $(u_n)_{n\in\mathbb{N}}$ in $L_\Phi(I)$ is said to converge in Φ-mean to $u \in L_\Phi(I)$ if*

$$\lim_{n\to\infty} \rho_\Phi(u_n - u) = \lim_{n\to\infty} \int_I \Phi(|u_n(x) - u(x)|)\, dx = 0.$$

The convergence in $L_\Phi(I)$ implies mean convergence with the same limit. The converse implication is wrong in general (see p. 75 et seq. in [KR61] for a counterexample). If Φ satisfies the Δ_2-condition, both notions of convergence are equivalent.

Lemma 1.4.2 ([KJF77, Lemma 3.10.4]). *If the Young function Φ satisfies the Δ_2-condition (with $s_0 = 0$ if the interval I is unbounded), then a sequence $(u_n)_{n\in\mathbb{N}} \subset L_\Phi(I)$ converges to u in $L_\Phi(I)$ if and only if it is Φ-mean convergent to u.*

The following Lemma follows from Lemma 3.8.4 in [KJF77] together with Remark 1.2.4.

Lemma 1.4.3. *Let $u \in L_\Phi(I, U)$.*

(a) If $\|u\|_\Phi \leq 1$, then $\rho_\Phi(u) \leq \|u\|_\Phi$.

(b) If $\|u\|_\Phi > 1$, then $\rho_\Phi(u) \geq \|u\|_\Phi$.

Lemma 1.4.4 ([MT50, Lemma 8.1]). *Let $I \subset \mathbb{R}$ be an interval. Let $(u_n)_{n\in N}$ be a sequence in $(u_n)_{n\in N} \subset L_\Phi(I)$ such that for all $r \in \mathbb{N} \setminus \{0\}$ the sequence $(ru_n)_{n\in\mathbb{N}}$ is mean convergent to zero. Then we have $\lim_{n\to\infty} \|u_n\|_\Phi = 0$.*

Proof. Let $r \in \mathbb{N} \setminus \{0\}$. Let Ψ be the complementary Young function to Φ. Then for any $v \in L_\Psi(I)$ with $\|v\|_\Psi \leq 1$ we have, by Lemma 1.4.3, the estimate $\rho_\Psi(v/r) \leq 1/r$. Choose $n_0 \in \mathbb{N}$ such that $\rho_\Phi(ru_n) \leq 1/r$ for all $n \geq n_0$. Then, by Young's inequality,

$$\int_I |u_n(x)v(x)|\, dx = \int_I |ru_n(x)| \left|\frac{v(x)}{r}\right| dx \leq \rho_\Phi(ru_n) + \rho_\Psi\left(\frac{v}{r}\right) \leq \frac{1}{r} + \frac{1}{r} = \frac{2}{r}.$$

Hence we have

$$\|u_n\|_\Phi \leq \sup_{\substack{v\in\tilde{L}_\Psi(I)\\ \rho_\Psi(v)\leq 1}} \int_I |u_n(x)v(x)|\, dx \leq \frac{2}{r}$$

for all $n \geq n_0$. Since r is an arbitrary natural number, the claim follows. $\square$

1.5 The space $E_\Phi(I,U)$

Remark 1.5.1. *Clearly, if I is bounded, then $L^\infty(I,U)$ is a linear subspace of $L_\Phi(I,U)$. From Remark 3.10.7 in [KJF77], together with Remark 1.2.4, we obtain that for any bounded interval $I \subset \mathbb{R}$ the space $L^\infty(I,U)$ is even a dense subspace of $L_\Phi(I,U)$ in the sense of mean convergence.*

Definition 1.5.2. *For bounded intervals I the space $E_\Phi(I,U)$ is defined as*

$$E_\Phi(I,U) = \overline{L^\infty(I,U)}^{\|\cdot\|_{L_\Phi(I,U)}}.$$

The norm $\|\cdot\|_{E_\Phi(I;U)}$ refers to $\|\cdot\|_{L_\Phi(I;U)}$.

Again, we write $E_\Phi(I) \coloneqq E_\Phi(I,\mathbb{K})$ for short if $U = \mathbb{K}$ with $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$.

Lemma 1.5.3. *For every Young function Φ we have the following inclusion*

$$E_\Phi(I,U) \subset \tilde{L}_\Phi(I,U).$$

Proof. Let $u \in E_\Phi(I,U)$. Then there exists a function $u_0 \in L^\infty(I,U)$ with $\|u-u_0\|_{L_\Phi(I,U)} < 1/2$ or, equivalently, $\|2(u-u_0)\|_{L_\Phi(I,U)} < 1$. By Lemma 1.4.3 we have $\rho_\Phi(2(u-u_0)) < 1$, in particular $2(u-u_0) \in \tilde{L}_\Phi(I,U)$. Since $2u_0$ belongs to $L^\infty(I,U)$ and the interval I is bounded, we have $2u_0 \in \tilde{L}_\Phi(I,U)$, and hence, by convexity of $\tilde{L}_\Phi(I,U)$, we obtain

$$u = \frac{2u-2u_0}{2} + \frac{2u_0}{2} \in \tilde{L}_\Phi(I,U). \qquad \square$$

Proposition 1.5.4. *Let $I \subset \mathbb{R}$ be a bounded interval. For every Young function Φ the following inclusions hold:*

$$E_\Phi(I,U) \subset \tilde{L}_\Phi(I,U) \subset L_\Phi(I,U). \tag{1.3}$$

The Young function Φ satisfies the Δ_2-condition if and only if

$$E_\Phi(I,U) = \tilde{L}_\Phi(I,U) = L_\Phi(I,U). \tag{1.4}$$

Proof. The first claim is shown in Remark 1.2.5 and Lemma 1.5.3. Assume that Φ satisfies the Δ_2-condition. Then from Lemma 1.4.2 and Remark 1.5.1 follows that $L^\infty(I,U)$ is a dense subspace of $L_\Phi(I,U)$ with respect to norm convergence. Hence, by definition, follows $E_\Phi(I,U) = L_\Phi(I,U)$ and, therefore, from (1.3) we obtain (1.4). Conversely, if Φ does not satisfy the Δ_2-condition, then, by Theorem 3.5.3 in [KJF77], the set $\tilde{L}_\Phi(I,U)$ is not a linear set. Since $E_\Phi(I,U)$ and $L_\Phi(I,U)$ are always vector spaces, the inclusions in (1.3) have to be strict. $\square$

Remark 1.5.5. *$E_\Phi(I,U)$ is separable, see e.g. [Sch05, Thm. 6.3].*

1.6 Comparison of Young functions and Orlicz spaces

We introduce an ordering for Young functions.

Definition 1.6.1. *Let Φ and Ψ be two Young functions.*

(a) We write $\Phi \prec \Psi$ if there exist two constants $c, t_0 > 0$ such that

$$\Phi(t) \leq \Psi(ct)$$

for all $t \geq t_0$.

(b) We call Φ and Ψ equivalent if $\Phi \prec \Psi$ holds as well as $\Psi \prec \Phi$.

(c) We say that the function Ψ increases essentially more rapidly *than the function Φ if, for arbitrary $s > 0$,*

$$\lim_{t\to\infty} \frac{\Phi(st)}{\Psi(t)} = 0.$$

In this case we write $\Phi \prec\!\prec \Psi$.

Remark 1.6.2. *For $p > 1$ let Ψ_p be the Young function given by $\Psi_p(t) = t^p$. It is shown in [KR61, pp. 24-25] that a Young function Φ satisfies the Δ_2-condition if and only if there exists some $p > 1$ with $\Phi \prec \Psi_p$.*

Theorem 1.6.3 ([KR61, Thm. 13.4])**.** *Let Φ, Φ_1 be Young functions such that Φ_1 increases essentially more rapidly than Φ. If $(u_n)_{n\in\mathbb{N}} \subset \tilde{L}_{\Phi_1}(I)$ converges to zero in Φ_1-mean, then it also converges zero in the norm $\|\cdot\|_\Phi$.*

Remark 1.6.4. *It is well-known that if the interval $I \subset \mathbb{R}$ is bounded and $1 \leq p \leq q \leq \infty$, then the space $L^q(I)$ is contained in $L^q(I)$ and the inclusion is continuous. There is a similar comparison of Orlicz spaces $L_\Phi(I,U)$ and $L_\Psi(I,U)$ using the ordering $\prec$. Let Φ, Ψ be two Young functions. Then we have:*

(a) The inclusion $L_\Phi(I,U) \subset L_\Psi(I,U)$ holds if and only if $\Psi \prec \Phi$. In particular we have $L_\Phi(I,U) = L_\Psi(I,U)$ if and only if Φ and Ψ are equivalent. Moreover, if $L_\Phi(I,U) \subset L_\Psi(I,U)$ holds, then the inclusion $L_\Phi(I,U) \hookrightarrow L_\Psi(I,U)$ is continuous, that is, it is an embedding. More precisely there exists a constant $k > 0$ such that

$$\|u\|_\Psi \leq k\|u\|_\Phi$$

for all $u \in L_\Phi(I,U)$.

(b) Using part (a) and Remark 1.6.2 we obtain that for every Young function the following statements are equivalent:

(i) Φ *satisfies the* Δ_2*-condition.*

(ii) For some $1 < p < \infty$ *holds*

$$L^p(I,U) \hookrightarrow L_\Phi(I,U)$$

and hence

$$L^p(I,U) \hookrightarrow E_\Phi(I,U).$$

(c) If $\Phi \prec\prec \Psi$*, then the space* $L_\Psi(I,U)$ *is continuously contained in* $E_\Phi(I,U)$*.*

We omit the proofs of the statements as they can be found in [KJF77, Section 3.17] for the case $U = \mathbb{K}$*. The proofs given there are easily adopted to the vector-valued case.*

It is well-known that for unbounded intervals $I \subset \mathbb{R}$ there exist bounded functions in $L^p(I)$, $p > 1$, which do not belong to $L^1(I)$. The following Lemma is an Orlicz space version of that fact. For unbounded intervals $I \subset \mathbb{R}$ and any Young function Φ, $L_\Phi(I)$ is not included in $L^1(I)$.

Lemma 1.6.5. *Let* $I \subset \mathbb{R}$ *an unbounded interval. Then for each Young function* Φ *there exists a strictly positive function* $u_0 \in L_\Phi(I) \cap L^\infty(I)$ *with* $u_0 \notin L^1(I)$*.*

Proof. For any Young function Φ holds $\lim_{t\to 0} \Phi(t)/t = 0$. Hence there is a sequence $(t_k)_{k\in\mathbb{N}} \subset (0,1)$ such that for all $k \in \mathbb{N}$ we have

$$\frac{\Phi(t_k)}{t_k} \leq 2^{-k}.$$

Since I is unbounded there is a sequence $(I_k)_{k\in\mathbb{N}}$ of measurable disjoint sets $I_k \subset \mathbb{R}$ with

$$I = \bigcup_{k\in\mathbb{N}} I_k$$

and $\lambda(I_k) = t_k^{-1}$. We define $u_0 \colon I \to \mathbb{R}$ by $u_0 = \sum_{k\in\mathbb{N}} t_k \chi_{I_k}$. Then $u_0 \in L^\infty(I)$. Further we have

$$\int_I |u_0(x)|\, dx = \sum_{k=0}^{\infty} t_k \lambda(I_k) = \sum_{k=0}^{\infty} 1 = \infty$$

and

$$\int_\Omega \Phi(|u_0(x)|)\, dx = \sum_{k=0}^{\infty} \Phi(t_k)\lambda(I_k) \leq \sum_{k=0}^{\infty} 2^{-k} = 2.$$

Hence we have $u_0 \notin L^1(I)$ and $u \in L_\Phi(I)$. By construction holds $0 < u < 1$ on I. $\square$

Chapter 2

Linear Systems

This chapter presents some preliminaries on linear systems and admissibility concepts. We also introduce the notion of comparison functions.

2.1 The general setting

Let X be a Banach space and $A\colon D(A) \supset X \to X$ a closed linear operator, which generates a C_0-semigroup $(T(t))_{t\geq 0}$ on X. We denote by ω_0 the growth bound of $(T(t))_{t\geq 0}$, that is,

$$\omega_0 = \inf_{t>0} \left(\frac{1}{t} \log(\|T(t)\|) \right).$$

Then we have $\omega_0 < \infty$ and for every $\omega > \omega_0$ there exists a constant $M = M_\omega \geq 1$ such that for every $t \geq 0$ we have $\|T(t)\| \leq M e^{\omega t}$. If $s \in \mathbb{C}$ and $\mathrm{Re}(s) > \omega_0$, then $s \in \rho(A)$. In particular, as a generator of a semigroup, the operator A has a nonempty resolvent set $\rho(A)$.

Definition 2.1.1. *Let $\lambda \in \rho(A)$. We denote by X_{-1} the completion of X with respect to the norm*

$$\|x\|_{-1} \coloneqq \|(\lambda I - A)^{-1} x\|$$

for $x \in X$. The space X_1 is defined to be $D(A)$ with the norm

$$\|x\|_1 \coloneqq \|(\lambda I - A)x\|$$

for $x \in X_1$.

The space X_1 is complete and $A \in \mathcal{L}(X_1, X)$ since $\|\cdot\|_1$ is equivalent to the graph norm on $D(A)$. For any $t \geq 0$ the operator $T(t)$ is bounded on X. In fact, it is even bounded with respect to the norm $\|\cdot\|_{-1}$. Indeed, for any $x \in X$ we have

$$\|T(t)x\|_{-1} = \|T(t)(\lambda I - A)^{-1}x\| \leq \|T(t)\| \|(\lambda I - A)^{-1}x\| = \|T(t)\| \|x\|_{-1}.$$

Since X is a dense subspace of X_{-1}, there exists a unique bounded extension of $T(t)$ to X_{-1}. We denote the extended operator by $T_{-1}(t)$.
Let us summarise some basic properties of the extrapolation spaces in the following proposition. This facts are all well-known and can be found, for example, in [EN00, Chapter II] or [TW09, Chapter 2].

Proposition 2.1.2. *With the definitions above we have:*

(a) Different $\lambda \in \rho(A)$ lead to equivalent norms. In particular the spaces X_1 and X_{-1} are well-defined, i.e., the definitions are independent of the choice of a particular $\lambda \in \rho(A)$.

(b) The operators $T_{-1}(t)$ form a C_0-semigroup $(T_{-1}(t))_{t\geq 0}$ on X_{-1}.

(c) The domain of A_{-1} is given by $D(A_{-1}) = X$. The operator A_{-1} is the unique bounded extension of A to an element from $\mathcal{L}(X, X_{-1})$.

Let U be another Banach space and $B \in \mathcal{L}(U, X_{-1})$. We study linear systems $\Sigma(A, B)$ on X given by

$$\dot{x}(t) = Ax(t) + Bu(t), \quad t \geq 0, \quad x(0) = x_0, \tag{2.1}$$

with some fixed $u \in L^1_{\mathrm{loc}}(0,\infty; U)$. Thus we consider special abstract inhomogeneous Cauchy problems. The space X is called the *state space*, U is called the *input space* and B is called the *control operator*. We call B *bounded* if it belongs to $\mathcal{L}(U, X)$ (and *unbounded* otherwise). We call x the *state* and u the *input* of the system. By Proposition 2.1.2, the equation (2.1) may be considered as an abstract inhomogeneous Cauchy problem on the Banach space X_{-1}.

Definition 2.1.3. *For $u \in L^1_{\mathrm{loc}}(0,\infty; U)$ the* (mild) solution *of* (2.1) *is given by the variation of parameters formula*

$$x(t) = T(t)x_0 + \int_0^t T_{-1}(t-s)Bu(s)\,ds \tag{2.2}$$

for $t \geq 0$.

Let $t \geq 0$. For $u \in L^1_{\mathrm{loc}}(0,\infty; U)$ we denote by $P_t u$ the *truncation* of u to $[0, t]$, that is, $P_t u = u_{|[0,t]}$. The *(time-)reflection operator* $R_t \colon L^1_{\mathrm{loc}}(0,\infty; U) \to L^1_{\mathrm{loc}}(0,\infty; U)$ is defined by

$$(R_t u)(s) = \begin{cases} u(t-s) & \text{for } s \in [0, t], \\ 0 & \text{for } s > t. \end{cases}$$

The *left-shift* S^l_t on $L^1_{\mathrm{loc}}(0,\infty; U)$ is defined by $(S^l_t u)(s) = u(s + t)$ for $u \in L^1_{\mathrm{loc}}(0,\infty; U)$. The *right-shift* $S^r_t \colon L^1_{\mathrm{loc}}(0,\infty; U) \to L^1_{\mathrm{loc}}(0,\infty; U)$ is defined by

$$(S^r_t u)(s) = \begin{cases} 0 & \text{for } s \in [0, t), \\ u(s-t) & \text{for } s \geq t. \end{cases}$$

In the subsequent chapters we are going to study various stability concepts for our systems. They can be formulated as continuity properties of the map, which assigns the input u together with the initial value x_0 to the mild solution x of the abstract Cauchy problem. Thus we need some structural conditions for the space of input functions. We consider the following types of function spaces.

Assumption 2.1.4. *For a Banach space U, let $Z \subseteq L^1_{\mathrm{loc}}(0,\infty;U)$ be such that for all $t > 0$ the following conditions hold:*

(a) $Z(0,t;U) := \{f \in Z \mid f|_{[t,\infty)} = 0\}$ becomes a Banach space of functions on the interval $(0,t)$ with values in U (in the sense of equivalence classes w.r.t. equality almost everywhere).

(b) $Z(0,t;U)$ is continuously embedded in $L^1(0,t;U)$, that is, there exists $\kappa(t) > 0$ such that for all $f \in Z(0,t;U)$ it holds that $f \in L^1(0,t;U)$ and

$$\|f\|_{L^1(0,t;U)} \leq \kappa(t)\|f\|_{Z(0,t;U)}.$$

(c) For $u \in Z(0,t;U)$ and $s > t$ we have $\|u\|_{Z(0,t;U)} = \|u\|_{Z(0,s;U)}$.

(d) $Z(0,t;U)$ is invariant under the left-shift and reflection, i.e.,

$$S^l_r Z(0,t;U) \subset Z(0,t;U)$$

for all $r > 0$ and

$$R_t Z(0,t;U) \subset Z(0,t;U).$$

Furthermore, $\|S_r\|_{\mathcal{L}(Z(0,t;U))} \leq 1$ and R_t is an isometry.

(e) For all $u \in Z$ it holds that $P_t u \in Z(0,t;U)$ and

$$\|P_s u\|_{Z(0,s;U)} \leq \|P_t u\|_{Z(0,t;U)}$$

for $s \in (0,t)$.

If, additionally, we have in (b) that

$$\lim_{t\searrow 0} \kappa(t) = 0, \tag{B}$$

then we say that Z satisfies condition (B).

Example 2.1.5. *(a) For fixed $1 \leq p \leq \infty$ and U, $Z = L^p$ refers to the Lebesgue spaces $L^p(0,t;U)$, $t > 0$. If $p > 1$, then L^p satisfies condition (B), thanks to Hölder's inequality. Clearly, L^1 does not satisfy condition (B).*

(b) *For a fixed Young function Φ and a Banach space U, $Z = L_\Phi$ and $Z = E_\Phi$ refer to the Orlicz spaces $L_\Phi(0,t;U)$ and $E_\Phi(0,t;U)$, $t > 0$, respectively. From Hölder's inequality for Orlicz spaces, see Theorem 1.2.6, follows that the condition (B) is satisfied here. Indeed, let $t \geq 0$ and $u \in L_\Phi(0,t;U)$. The Hölder's inequality yields*

$$\|u\|_{L^1(0,t;U)} = \int_0^t \|u(s)\|_U \, ds \leq 2\|\chi_{(0,t)}\|_\Psi \|u\|_\Phi,$$

where Ψ is the complementary Young function for Φ. This shows that $L_\Phi(I,U)$ and $E_\Phi(I,U)$ are continuously embedded in $L^1(I;U)$. We now show that $\kappa(t) := 2\|\chi_{(0,t)}\|_\Psi$ satisfies $\lim_{t\searrow 0} \kappa(t) = 0$. By Example 3.6.9 in [KJF77] we have $\|\chi_{(0,t)}\|_\Psi = t\Phi^{-1}(1/t)$. Using Proposition 1.1.3 we obtain $\lim_{t\searrow 0} t\Phi^{-1}(1/t) = \lim_{t\searrow 0} t/\Phi(t) = 0$. Hence, the claim follows.

Remark 2.1.6. *Further examples of admissible function spaces are Sobolev spaces and Orlicz-Sobolev spaces (see e.g. [Ada75, pp. 246-247]). Our goal is not to include the widest possible range of function spaces. For instance the invariance with respect to the time-reflection excludes some weighted spaces. For our purposes it would be sufficient to consider the cases $Z = L^p$, $1 \leq p \leq \infty$, and $Z = E_\Phi$. But since many properties of linear systems rely on more general conditions rather than specific choices of Z, we choose to formulate them in a more abstract manner.*

2.2 Admissibility

The mild solution is initially defined in X_{-1}. We are interested in those control operators B, for which the mild solution is X-valued.

Definition 2.2.1. *We call the system $\Sigma(A,B)$* (finite-time) admissible with respect to Z *(or* Z-admissible*) if for all $t > 0$ and all $u \in Z(0,t;U)$ it holds that*

$$\int_0^t T_{-1}(s)Bu(s) \, ds \in X. \tag{2.3}$$

An operator $B \in \mathcal{L}(U, X_{-1})$ is called a Z-admissible control operator *for $(T(t))_{t\geq 0}$ if the system $\Sigma(A,B)$ is admissible with respect to Z.*

The following result is well-known for $Z = L^p$, see e.g. Proposition 4.2.2 in [TW09]. The proof presented here for more general spaces of input functions uses basically the same arguments.

Proposition 2.2.2. *If $\Sigma(A,B)$ is admissible with respect to Z, then all mild solutions of* (2.1) *are X-valued and for each $t \geq 0$ there exists a constant $c(t) \geq 0$ such that*

$$\left\| \int_0^t T_{-1}(s)Bu(s) \, ds \right\| \leq c(t)\|u\|_{Z(0,t;U)} \tag{2.4}$$

for all $u \in Z(0,t;U)$. Moreover, $\Sigma(A,B)$ is admissible if (2.3) *holds for some $t_0 > 0$.*

Proof. We choose some $\lambda \in \rho(A)$. Let $\tilde{B} := (\lambda I - A_{-1})^{-1}B$. Since the resolvent $(\lambda I - A_{-1})^{-1}$ belongs to $\mathcal{L}(X_{-1}, X)$ we have $\tilde{B} \in \mathcal{L}(U, X)$. Further we have

$$\int_0^t T_{-1}(s)Bu(s)\,ds = (\lambda I - A_{-1}) \int_0^t T(s)\tilde{B}u(s)\,ds.$$

Hence, for each $t > 0$ the map $Z(0,t;U) \to X$, $u \mapsto \int_0^t T_{-1}(s)Bu(s)\,ds$, being a composition of a bounded and a closed operator, is closed. The closed graph theorem now yields that this map is bounded.
Assume that (2.3) holds for some $t_0 > 0$. Let $t > 0$. We can assume that $t = nt_0$ holds for some $n \in \mathbb{N}$. Otherwise we extend the function $u \in Z(0,t;U)$ to the interval $[0, nt_0]$, where $n = \lceil t/t_0 \rceil$, by zero. Then we have

$$\begin{aligned}
\int_0^t T_{-1}(s)Bu(s)\,ds &= \sum_{k=0}^{n} \int_{kt_0}^{(k+1)t_0} T_{-1}(s)Bu(s)\,ds \\
&= \sum_{k=0}^{n} \int_0^{t_0} T_{-1}(s+k)Bu(s+k)\,ds \\
&= \sum_{k=0}^{n} T_{-1}(k) \int_0^{t_0} T_{-1}(s)Bu_k(s)\,ds,
\end{aligned}$$

where the function $u_k \colon [0, t_0] \to U$ is defined by $u_k(s) = u(s+k)$. By the left-shift invariance of the function space Z we have $u_k \in Z(0, t_0; U)$ for all $k \in \{0, \dots, n\}$ and hence

$$\int_0^{t_0} T_{-1}(s)Bu_k(s)\,ds \in X$$

by assumption. Since the extended semigroup $(T_{-1}(t))_{t \geq 0}$ is invariant with respect to X we obtain

$$T_{-1}(k) \int_0^{t_0} T_{-1}(s)Bu_k(s)\,ds = T(k) \int_0^{t_0} T_{-1}(s)Bu_k(s)\,ds \in X$$

for all $k \in \{0, \dots, n\}$. Hence we have

$$\int_0^t T_{-1}(s)Bu(s)\,ds \in X,$$

i.e., the system $\Sigma(A, B)$ is admissible with respect to Z. □

Definition 2.2.3. *We call the system* $\Sigma(A, B)$ infinite-time admissible with respect to Z *(or* infinite-time Z-admissible*) if the system is Z-admissible and the optimal constants in* (2.4) *satisfy* $c_\infty := \sup_{t>0} c(t) < \infty$.

Remark 2.2.4. *Since the reflection map R_t is an isometry on $Z(0,t;U)$ and*

$$\int_0^t T_{-1}(t-s)Bu(s)\,ds = \int_0^t T_{-1}(s)B(R_t u)(s)\,ds$$

for all $t>0$, the admissibility of the system $\Sigma(A,B)$ with respect to Z means that the mild solution of (2.1), given by (2.2), is X-valued and, for every $t>0$, the so-called input map *$\Phi_t\colon Z(0,t;U)\to X$, defined by*

$$\Phi_t u := \int_0^t T_{-1}(t-s)Bu(s)\,ds,$$

is bounded. The infinite-time admissibility of the system $\Sigma(A,B)$ means that those maps are uniformly bounded, i.e.,

$$\sup_{t>0}\|\Phi_t\|_{\mathcal{L}(Z(0,t;U),X)} < \infty.$$

If $Z=L^p$ with $p\in[1,\infty)$, then this is equivalent to the fact that for each $u\in L^p(0,\infty;U)$ the improper integral

$$\Phi_\infty u := \int_0^\infty T_{-1}(s)Bu(s)\,ds$$

exists in X and defines a bounded linear map $\Phi_\infty\colon L^p(0,\infty;U)\to X$, the so-called extended input map*. Indeed, if $\Phi_\infty\in\mathcal{L}(L^p(0,\infty;U),X)$, then for all $t>0$ we have $\Phi_t u=\Phi_\infty R_t P_t u$. Since the projections $P_t\colon L^p(0,\infty;U)\to L^p(0,t;U)$ satisfy $\|P_t\|\le 1$ for all $t\ge 0$ and the reflections $R_t\colon L^p(0,t;\infty)\to L^p(0,t;\infty)$ are isometries, the claim follows.*
Assume now that the maps Φ_t, $t\ge 0$, are uniformly bounded. Let $u\in L^p(0,\infty;U)$ and $(t_n)_{n\in\mathbb{N}}$ be a sequence of positive numbers such that $\lim_{n\to\infty}t_n=\infty$. Let $\tilde{\Phi}_t=\Phi_\infty P_t$. Then we have

$$\tilde{\Phi}_{t_n}u = \int_0^{t_n} T_{-1}(s)Bu(s)\,ds.$$

Thus for any $m,n\in\mathbb{N}$, assuming without loss of generality $0<t_m<t_n$, we have

$$\|(\tilde{\Phi}_{t_m}-\tilde{\Phi}_{t_n})u\| = \left\|\int_{t_m}^{t_n} T_{-1}(s)Bu(s)\,ds\right\| = \left\|\int_0^{t_n} T_{-1}(s)Bu_m(s)\,ds\right\|,$$

where $u_m := u\chi_{[t_m,\infty)}$. Hence we obtain

$$\|(\tilde{\Phi}_{t_m}-\tilde{\Phi}_{t_n})u\| \le \|\tilde{\Phi}_{t_n}\|\|u_m\|_{L^p(0,t_n;U)} \le c_\infty\|u_m\|_{L^p(0,\infty;U)}.$$

The dominated convergence theorem yields $\lim_{m\to\infty}\|u_m\|_{L^p(0,\infty;U)}=0$. Therefore, $(\tilde{\Phi}_{t_n}u)_{n\in\mathbb{N}}$ is a Cauchy sequence in X and thus converges. The map Φ_∞ is the strong limit of a sequence of continuous maps. Hence, by the uniform boundedness principle, it is bounded.

Remark 2.2.5. *From the definition of admissibility follows immediately that if the inclusion $Z'(0,t;U) \subset Z(0,t;U)$ holds for all $t \geq 0$, then Z-admissibility implies Z'-admissibility. In particular L^p-admissibility implies L^q-admissibility for all $1 \leq p < q \leq \infty$. Furthermore, the inclusions $L^\infty \subset E_\Phi \subset L_\Phi \subset L^1$ yield a corresponding chain of implications of admissibilities. However, the corresponding property for the infinite-time admissibility is not true, as we shall see in Theorem 3.3.5.*

Since the space L^1 is the union of all Orlicz spaces we obtain the following characterisation of L^1-admissibility.

Proposition 2.2.6. *A system $\Sigma(A,B)$ is admissible with respect to L^1 if and only if it is admissible with respect to all Orlicz spaces E_Φ.*

Proof. By Remark 2.2.5 we are left to show that L^1-admissibility is implied by the admissibility with respect to all Orlicz spaces E_Φ. Thus, let $t > 0$ and $u \in L^1(0,t;U)$. By Remark 1.3.3 there exists a Young function Φ satisfying the Δ_2-condition with $\|u(\cdot)\| \in L_\Phi(0,t) = E_\Phi(0,t)$, i.e., $u \in E_\Phi(0,t;U)$. Thus we have $\int_0^t T_{-1}(s)Bu(s)\,ds \in X$ by assumption. This shows that the system $\Sigma(A,B)$ is L^1-admissible. □

Definition 2.2.7. *We call the system $\Sigma(A,B)$* zero-class admissible with respect to Z *(or* Z-zero-class admissible*), if it is admissible with respect to Z and the optimal constants in* (2.4) *satisfy* $\lim_{t\to 0} c(t) = 0$.

Remark 2.2.8. *Clearly, zero-class admissibility and infinite-time admissibility both imply admissibility. Also, if B is a bounded operator from U to X then $\Sigma(A,B)$ is admissible. Admissibility in general does not imply zero-class admissibility as the following simple example illustrates: We take $X = U = \mathbb{C}$, $A = -1$ and $B = 1$. Then the system $\Sigma(A,B)$ is L^1-admissible and $c(t) = 1$ for all $t > 0$. Hence it is not L^1-zero-class admissible.*
We will see in Chapter 4 that admissibility and infinite-time admissibility are equivalent if we additionally assume that A generates an exponentially stable semigroup. In general, Z-admissibility does not imply infinite-time Z-admissibility, not even if B is bounded or if the semigroup is strongly stable, see [DM13a, Ex. 3.1] for an example with $Z = L^\infty$ or [JS07] with $Z = L^2$. In Chapter 3 we will study a counterexample with $Z = E_\Phi$.

2.3 Examples

Example 2.3.1. *Let $X = L^2(0,\infty)$ and $(T(t))_{t\geq 0}$ the right-shift semigroup on X, i.e., $T(t)x = S_t^r x$ for $x \in L^2(0,\infty)$. Its generator is given by*

$$Af = -f'$$

for $f \in D(A) = H_0^1(0,\infty)$. We take $U = \mathbb{C}$ and $B = \delta_0 \in X_{-1}$, where $X_{-1} = H^{-1}(0,\infty)$, the dual of $H^1(0,\infty)$ with respect to the pivot space $L^2(0,\infty)$. We obtain

$$\left(\int_0^t T_{-1}(t-s)Bu(s)\,ds\right)(x) = \begin{cases} u(t-x) & \text{for } x \in [0,t], \\ 0 & \text{for } x \geq t. \end{cases}$$

Thus we have $\int_0^t T_{-1}(t-s)Bu(s)\,ds \in X$ for any $u \in L^2(0,t)$ and, hence, B is admissible with respect to L^2. Further we obtain

$$\left\| \int_0^t T_{-1}(t-s)Bu(s)\,ds \right\| \leq \|u\|_{L^2(0,t)},$$

which shows that $\Sigma(A,B)$ is infinite-time admissible.

Example 2.3.2. *We consider the boundary control system given by the one-dimensional heat equation on the spatial domain $(0,1)$ with Dirichlet boundary control at the boundary point 1:*

$$\begin{aligned} \frac{\partial x}{\partial t}(\xi,t) &= \frac{\partial^2 x}{\partial \xi^2}(\xi,t), \quad \xi \in (0,1),\ t>0, \\ x(0,t) &= 0, \quad x(1,t) = u(t), \quad t > 0, \\ x(\xi,0) &= x_0(\xi). \end{aligned}$$

This system can be written equivalently in the form $\Sigma(A,B)$. The state space here is $X = L^2(0,1)$ and

$$Af = f''$$

for $f \in D(A) = \{f \in H^2(0,1) \mid f(0) = f(1) = 0\}$. The input space is $U = \mathbb{C}$. The extrapolation space is given by $X_{-1} = H^{-2}(0,1)$, c.f. Example 2.10.8 in [TW09]. Thus the state equation can be written as

$$\dot{x}(t) = Ax(t) + \delta_1' u(t).$$

We have $B = \delta_1' \in X_{-1} = \mathcal{L}(\mathbb{C}, X_{-1})$. We will see in Example 4.5.2 that B is an admissible control operator for the heat semigroup.

2.4 Continuity of mild solutions

Since $Z \subseteq L^1_{\mathrm{loc}}(0,\infty;U)$, for any $u \in Z$ and any initial value x_0, the mild solution x of (2.1) is continuous as function from $[0,\infty)$ to X_{-1}. Next we show that zero-class admissibility guarantees that x even lies in $C(0,\infty;X)$.

Proposition 2.4.1. *If $\Sigma(A,B)$ is Z-zero-class admissible, then for every $x_0 \in X$ and every $u \in Z$ the mild solution of (2.1), given by (2.2), satisfies $x \in C(0,\infty;X)$.*

Proof. Since the map $t \mapsto T(t)x_0$ is continuous, it is sufficient to consider the case $x_0 = 0$. Let $u \in Z$. The solution is then given by

$$x(t) = \int_0^t T_{-1}(t-s)Bu(s)\,ds = \int_0^t T_{-1}(s)B(R_t u)(s)\,ds.$$

Hence it is sufficient to show that the map $x\colon [0,\infty) \to X$, given by

$$x(t) = \int_0^t T_{-1}(s)Bu(s)\,ds,$$

is continuous. First we show that this map is right-continuous. Let $t \in [0,\infty)$ and $(t_n)_{n\in\mathbb{N}} \subset [t,\infty)$ a sequence with $\lim_{n\to\infty} t_n = t$. Then we have

$$x(t_n) - x(t) = \int_t^{t_n} T_{-1}(s)Bu(s)\,ds = T(t)\int_0^{t_n - t} T_{-1}(s)Bu(t+s)\,ds$$

and hence, by admissibility,

$$\begin{aligned}\|x(t_n) - x(t)\| &\le c(t_n - t)\|T(t)\|\|S_t u\|_{Z(0,t_n-t;U)}\\ &\le c(t_n - t)\|T(t)\|\|S_t u\|_{Z(0,t;U)}\\ &\le c(t_n - t)\|T(t)\|\|u\|_{Z(0,t;U)}.\end{aligned}$$

Here we used conditions (d) and (e) in Assumption 2.1.4 for the last two steps. From the zero-class admissibility follows $\lim_{n\to\infty} x(t_n) = x(t)$.

Next we show that this map is left continuous on $(0,\infty)$. Let $t \in (0,\infty)$, $(t_n)_{n\in\mathbb{N}} \subset [0,t]$ a sequence with $\lim_{n\to\infty} t_n = t$ and $u \in Z(0,t;U)$. Then we have

$$x(t) - x(t_n) = \int_{t_n}^{t} T_{-1}(s)Bu(s)\,ds = T(t_n)\int_0^{t - t_n} T_{-1}(s)Bu(t_n+s)\,ds$$

and hence, by admissibility, conditions (d) and (e) in Assumption 2.1.4, and the monotonicity of the exponential function we obtain

$$\begin{aligned}\|x(t) - x(t_n)\| &\le c(t - t_n)\|T(t_n)\|\|S_{t_n} u\|_{Z(0,t-t_n;U)}\\ &\le c(t - t_n)Me^{\omega t_n}\|S_{t_n} u\|_{Z(0,t_n;U)}\\ &\le c(t - t_n)Me^{|\omega| t_n}\|u\|_{Z(0,t_n;U)}\\ &\le c(t - t_n)Me^{|\omega| t}\|u\|_{Z(0,t;U)}.\end{aligned}$$

Again, from the zero-class admissibility follows $\lim_{n\to\infty} x(t_n) = x(t)$. □

Remark 2.4.2. *If $\Sigma(A,B)$ is admissible with respect to L^p, $1 \le p < \infty$, then, by Hölder's inequality, $\Sigma(A,B)$ is L^q-zero-class admissible for any $q > p$. Thus, Proposition 2.4.1 implies that the mild solution of* (2.1) *lies in $C(0,\infty;X)$ for all $u \in L^q$. In fact, the mild solution is continuous even for $u \in L^p$ as it is shown in [Wei89a, Prop. 2.3]. It is still unknown whether or not this also holds true for $p = \infty$, c.f. [Wei89a, Problem 2.4]. The Proposition 2.4.1 shows that it is true if we add the zero-class condition.*

2.5 Comparison functions

In this section we introduce the notion of comparison functions. They are very common tools in systems and control theory as they allow for the formulation of various stability properties in a short and elegant way. More information on this topic can be found in the survey [Kel14].

Definition 2.5.1. *We denote by $\mathcal{K}$ the set of all continuous functions $\mu\colon [0,\infty) \to [0,\infty)$, which are strictly increasing and satisfy $\mu(0) = 0$.*

Of particular interest are those functions from $\mathcal{K}$, which are unbounded.

Definition 2.5.2. *We denote by $\mathcal{K}_\infty$ the set of all $\theta \in \mathcal{K}$, which satisfy $\lim_{t\to\infty} \theta(t) = \infty$.*

Evidently, the set $\mathcal{K}_\infty$ consists of all homeomorphisms of $[0,\infty)$ to itself. In particular, $\mathcal{K}_\infty$ is a group with respect to composition as group operation. This means that for any pair of functions $\theta_1, \theta_2 \in \mathcal{K}_\infty$, its composition $\theta_1 \circ \theta_2$ belongs to $\mathcal{K}_\infty$. Further, any function from the class $\mathcal{K}_\infty$ is invertible and its inverse belongs again to $\mathcal{K}_\infty$. If $\theta \in \mathcal{K}$ is bounded, that is, $\theta \in \mathcal{K} \setminus \mathcal{K}_\infty$, then the limit $a := \lim_{t\to\infty} \theta(t)$ exists and θ is a homeomorphism from $[0,\infty)$ onto $[0,a)$.

Definition 2.5.3. *We denote by $\mathcal{L}$ the set of all continuous functions $\gamma\colon [0,\infty) \to [0,\infty)$, which are strictly decreasing and satisfy $\lim_{t\to\infty} \gamma(t) = 0$.*

Similar to the functions from the set $\mathcal{K}$ every function $\gamma \in \mathcal{L}$ is a homeomorphism from $[0,\infty)$ to its range, that is, to $(0,\gamma(0)]$.

Definition 2.5.4. *We denote by $\mathcal{KL}$ the set of all functions $\beta\colon [0,\infty)\times[0,\infty) \to [0,\infty)$ such that $\beta(\cdot,t) \in \mathcal{K}$ for all fixed $t \geq 0$ and $\beta(s,\cdot) \in \mathcal{L}$ for all fixed $s > 0$.*

Next lemma states that every function from $\mathcal{K}_\infty$ is bounded above by another function from $\mathcal{K}_\infty$, which can be written as a composition of a concave and a convex function.

Lemma 2.5.5 ([PW96, Lemma 14])**.** *Let $\mu \in \mathcal{K}_\infty$. Then there exist two continuously differentiable functions $\mu_v, \mu_c \in \mathcal{K}_\infty$ such that μ_v is convex, μ_c is concave and the estimate*

$$\mu(s) \leq \mu_c(\mu_v(s))$$

holds for all $s \in [0,\infty)$.

The following lemma is a special case of Lemma 2.5 in [CLS98].

Lemma 2.5.6. *Let $\theta\colon [0,\infty) \to [0,\infty)$ be a noninreasing function with $\theta(0) = \lim_{t\searrow 0} \theta(t) = 0$. Then there exists a function $\tilde{\theta} \in \mathcal{K}_\infty$ with $\theta \leq \tilde{\theta}$.*

Chapter 3

Strong input-to-state stability

In this chapter we introduce the strong versions of the stability notions we are mainly interested in, that is, the strong input-to-state stability and the strong integral input-to-state stability. Though general spaces of input functions as introduced in Assumption 2.1.4 are allowed, we are mainly interested in inputs from L^∞ and E_Φ for some Young function Φ. Our goal is to understand the connections between those stability notions. The main results of this chapter were published in [NS18].

3.1 Strong input-to-state stability and related notions

Definition 3.1.1. *A C_0-semigroup $(T(t))_{t\geq 0}$ is called* strongly stable *if* $\lim_{t\to\infty} T(t)x = 0$ *holds for all $x \in X$.*

Definition 3.1.2. *The system $\Sigma(A, B)$ is called* strongly input-to-state stable *with respect to Z (or Z-*sISS*) if there exist functions $\mu \in \mathcal{K}$ and $\beta\colon X \times [0,\infty) \to [0,\infty)$ such that*

(a) $\beta(x,\cdot) \in \mathcal{L}$ for all $x \in X$, $x \neq 0$ and

(b) for every $t \geq 0$, $x_0 \in X$ and $u \in Z(0,t;U)$ the state $x(t)$ lies in X and

$$\|x(t)\| \leq \beta(x_0,t) + \mu(\|u\|_{Z(0,t;U)}). \tag{3.1}$$

The system $\Sigma(A, B)$ is called strongly integral input-to-state stable *with respect to Z (or Z-*siISS*) if there exist functions $\theta \in \mathcal{K}_\infty$, $\mu \in \mathcal{K}$ and $\beta\colon X \times [0,\infty) \to [0,\infty)$ such that*

(a) $\beta(x,\cdot) \in \mathcal{L}$ for all $x \in X$, $x \neq 0$ and

(b) for every $t \geq 0$, $x_0 \in X$ and $u \in Z(0,t;U)$ the state $x(t)$ lies in X and

$$\|x(t)\| \leq \beta(x_0,t) + \theta\left(\int_0^t \mu(\|u(s)\|_U)\, ds\right). \tag{3.2}$$

Remark 3.1.3. *(a) The definitions of Z-sISS and Z-siISS given above generalise the standard notions of Z-ISS and Z-iISS, see Definition 4.1.1. We will see in the next chapter that Z-ISS implies Z-sISS and Z-iISS implies Z-siISS.*

(b) The notion of strong input-to-state stability was introduced in [MW18] with the following additional condition: There is a $\sigma \in \mathcal{K}_\infty$ such that

$$\beta(x,t) \leq \sigma(\|x\|)$$

for all $x \in X$ and $t \geq 0$. In our situation of linear systems this condition is redundant. Indeed, Proposition 3.2.1 below shows that strong ISS implies the strong stability of the semigroup $(T(t))_{t\geq 0}$. By the uniform boundedness principle there is some $M > 0$ such that $\|T(t)\|_{\mathcal{L}(X)} \leq M$. Taking $\sigma(s) = Ms$ yields $\sigma \in \mathcal{K}_\infty$ and $\beta(x,t) \leq \sigma(\|x\|)$.

(c) If $Z' \subset Z$ in the sense that $Z'(0,t;U) \subset Z(0,t;U)$ for all $t > 0$, then Z-siISS implies Z'-siISS. The corresponding property for Z-sISS does not hold as we shall see in Theorem 3.3.5.

3.2 Basic properties

Proposition 3.2.1. *(a) The following are equivalent:*

(i) $\Sigma(A,B)$ is Z-sISS.

(ii) $\Sigma(A,B)$ is infinite-time Z-admissible and $(T(t))_{t\geq 0}$ is strongly stable.

(b) If $\Sigma(A,B)$ is Z-siISS, then the system is Z-admissible and $(T(t))_{t\geq 0}$ is strongly stable.

Proof. Clearly, Z-sISS and Z-siISS imply Z-admissibility.
If $\Sigma(A,B)$ is Z-sISS or Z-siISS, then by setting $u = 0$, it follows that for all $x \neq 0$ and $t \geq 0$ we have $\|T(t)x\| \leq \beta(x,t)$ and hence $\lim_{t\to\infty} T(t)x = 0$, which shows that $(T(t))_{t\geq 0}$ is strongly stable. This shows (b). In the case that $\Sigma(A,B)$ is Z-sISS, we get

$$\begin{aligned}\left\|\int_0^t T_{-1}(s)Bu(s)\,ds\right\| &= \left\|\int_0^t T_{-1}(s)B\frac{u(s)}{\|u\|_{Z(0,t;U)}}\,ds\right\| \|u\|_{Z(0,t;U)} \\ &\leq \mu(1)\|u\|_{Z(0,t;U)}\end{aligned}$$

for any $u \in Z(0,t;U)$ with $u \neq 0$. This shows that $\Sigma(A,B)$ is infinite-time Z-admissible and, thus, (i)$\Rightarrow$ (ii) in (a).
Conversely, if the system $\Sigma(A,B)$ is infinite-time Z-admissible and $(T(t))_{t\geq 0}$ is strongly stable we set $\beta(x,t) = \|T(t)x\|$ and $\mu(s) = c_\infty s$. Then μ belongs to $\mathcal{K}$, $\beta(x,\cdot) \in \mathcal{L}$ and $\|x(t)\| \leq \beta(x_0,t) + \mu(\|u\|_{Z(0,t;U)})$ for all $t \geq 0$, $u \in Z(0,t;U)$ and $x_0 \in X$. □

Proposition 3.2.2. *Let $p \in [1,\infty)$. If the system $\Sigma(A,B)$ is L^p-sISS, then it is L^p-siISS.*

Proof. For any $x_0 \in X$ and $u \in L^p(0,t;U)$ we have by the infinite-time L^p-admissibility and strong stability that

$$\begin{aligned} \|x(t)\| &\le \|T(t)x_0\| + c_\infty \|u\|_{L^p(0,t;U)} \\ &= \beta(x_0,t) + c_\infty \left(\int_0^t \|u(s)\|_U^p \, ds \right)^{1/p}, \end{aligned}$$

where $\beta(x_0,t) \coloneqq \|T(t)x_0\|$. This shows the strong integral input-to-state stability with respect to L^p. $\square$

We will see in Theorem 3.3.5 that the converse implication in Proposition 3.2.1 does not hold in general. However, under the additional assumption that the semigroup $(T(t))_{t\ge0}$ is exponentially stable, those conditions are equivalent, see Proposition 4.2.7.

Remark 3.2.3. *Let $1 \le p < \infty$. If the system $\Sigma(A,B)$ is infinite-time L^p-admissible and $(T(t))_{t\ge0}$ is strongly stable, then the system $\Sigma(A,B)$ is L^p-sISS with the following choices for the functions β and μ:*

$$\beta(x,t) \coloneqq \|T(t)x\| \quad \textit{and} \quad \mu(s) \coloneqq c_\infty s,$$

where $c_\infty = \sup_{t\ge0} c(t)$, see Definition 2.2.3. Furthermore, the system is L^p-siISS with the following choices for the functions β, θ and μ:

$$\beta(x,t) \coloneqq \|T(t)x\|, \quad \theta(s) \coloneqq c_\infty s^{1/p} \quad \textit{and} \quad \mu(s) \coloneqq s^p.$$

Proposition 3.2.4. *If $\Sigma(A,B)$ is L^∞-siISS, then $\Sigma(A,B)$ is L^∞-zero-class admissible.*

Proof. If $\Sigma(A,B)$ is L^∞-siISS, then there exist $\theta \in \mathcal{K}_\infty$ and $\mu \in \mathcal{K}$ such that for all $t > 0$, $u \in L^\infty(0,t;U)$, $u \ne 0$,

$$\begin{aligned} \left\| \int_0^t T_{-1}(s)Bu(s)\,ds \right\| &= \left\| \int_0^t T_{-1}(s)Bu(s) \frac{u(s)}{\|u\|_\infty} \, ds \right\| \|u\|_\infty \\ &\le \theta \left(\int_0^t \mu\left(\tfrac{\|u(s)\|_U}{\|u\|_\infty} \right) ds \right) \|u\|_\infty \\ &\le \theta(t\mu(1)) \|u\|_\infty, \end{aligned} \tag{3.3}$$

since the function μ is monotonically increasing and $\|u(s)\|_U \le \|u\|_\infty$ a.e. As $\theta \in \mathcal{K}$, we have $\lim_{t\to0} \theta(t\mu(1)) = 0$. $\square$

3.3 Strong integral input-to-state stability and Orlicz space admissibility

We start by presenting a criterion for a system to be strong iISS with respect to L^∞.

Theorem 3.3.1. *Suppose there is a Young function Φ such that the system $\Sigma(A,B)$ is E_Φ-sISS. Then the system $\Sigma(A,B)$ is L^∞-siISS.*

Proof. Let Φ_1 be a Young function given by Lemma 1.1.6. We define $\theta\colon [0,\infty) \to [0,\infty)$ by $\theta(0) = 0$ and

$$\theta(\alpha) = \sup\left\{ \left\| \int_0^t T_{-1}(s)Bu(s)\,ds \right\| \;\Big|\; u \in L^\infty(0,t;U),\, t \geq 0,\, \int_0^t \Phi_1(\|u(s)\|_U)\,ds \leq \alpha \right\}$$

for $\alpha > 0$. This function is well-defined, since by infinite-time admissibility with respect to E_Φ, Remark 1.2.5 and the inequality $\Phi \leq \Phi_1$ we have

$$\begin{aligned} \left\| \int_0^t T_{-1}(s)Bu(s)\,ds \right\| &\leq c_\infty \|u\|_{E_\Phi(0,t;U)} \\ &\leq c_\infty \left(1 + \int_0^t \Phi(\|u(s)\|_U)\,ds\right) \\ &\leq c_\infty \left(1 + \int_0^t \Phi_1(\|u(s)\|_U)\,ds\right) \end{aligned} \tag{3.4}$$

for all $t \geq 0$ and $u \in L^\infty(0,t;U)$. Clearly, θ is nondecreasing.
We show that θ is continuous at zero. Let $(\alpha_n)_{n\in\mathbb{N}} \subset [0,\infty)$ be a sequence with $\lim_{n\to\infty} \alpha_n = 0$. Then, by construction, for any $n \in \mathbb{N}$ there exists a $u_n \in L^\infty(0,\infty;U)$ with compact essential support such that

$$\int_0^\infty \Phi_1(\|u_n(s)\|_U)\,ds \leq \alpha_n \tag{3.5}$$

and

$$\left| \theta(\alpha_n) - \left\| \int_0^\infty T_{-1}(s)Bu_n(s)\,ds \right\| \right| < \frac{1}{n}. \tag{3.6}$$

From (3.5) follows that the sequence $(\|u_n(\cdot)\|_U)_{n\in\mathbb{N}}$ is Φ_1-mean convergent to zero. According to Lemma 1.1.6 there exists a constant $C > 0$ such that for any $r > 0$ the estimate $\Phi(r\|u(s)\|_U) \leq C\Phi_1(\|u(s)\|_U)$ holds for almost all $s \in [0,t]$. Hence, for all $r > 0$, the sequence $(r\|u_n(\cdot)\|_U)_{n\in\mathbb{N}}$ is Φ-mean convergent to zero. By Lemma 1.4.4 this sequence converges to zero with respect to the norm of the space $L_\Phi(0,\infty)$ and hence $\lim_{n\to\infty} \|u_n\|_{L_\Phi(0,\infty;U)} = 0$. Therefore, by infinite-time admissibility,

$$\left\| \int_0^\infty T_{-1}(s)Bu_n(s)\,ds \right\| \leq c_\infty \|u_n\|_{L_\Phi(0,\infty;U)} \to 0$$

for $n \to \infty$. Hence we obtain

$$\begin{aligned} \theta(\alpha_n) &\leq \left| \theta(\alpha_n) - \left\| \int_0^\infty T_{-1}(s)Bu_n(s)\,ds \right\| \right| + \left\| \int_0^\infty T_{-1}(s)Bu_n(s)\,ds \right\| \\ &\leq \frac{1}{n} + c_\infty \|u_n\|_{L_\Phi(0,\infty;U)} \end{aligned}$$

and thus $\lim_{n\to\infty} \theta(\alpha_n) = 0$.
Applying Lemma 2.5.6 we obtain the existence of a function $\tilde{\theta} \in \mathcal{K}_\infty$ such that $\theta \leq \tilde{\theta}$. The definition of θ yields

$$\left\| \int_0^t T_{-1}(s)Bu(s)\, ds \right\| \leq \theta \left(\int_0^t \Phi(\|u(s)\|_U)\, ds \right) \leq \tilde{\theta} \left(\int_0^t \Phi(\|u(s)\|_U)\, ds \right)$$

for all $t \geq 0$ and $u \in L^\infty(0, t; U)$. This means that the system $\Sigma(A, B)$ is strong iISS with respect to L^∞. □

Lemma 3.3.2. *Let $(T(t))_{t\geq 0}$ be a semigroup and let $\Sigma(A, B)$ be L^∞-siISS. Then there exist $\tilde{\theta}, \Phi \in \mathcal{K}_\infty$ such that Φ is a Young function, which is continuously differentiable on $(0, \infty)$ and*

$$\left\| \int_0^1 T_{-1}(s)Bu(s)\, ds \right\| \leq \tilde{\theta} \left(\int_0^1 \Phi(\|u(s)\|_U)\, ds \right) \tag{3.7}$$

for all $u \in L^\infty(0, 1; U)$.

Proof. By assumption, there exist $\theta \in \mathcal{K}_\infty$ and $\mu \in \mathcal{K}$ such that (3.2) holds for $Z = L^\infty$. Without loss of generality we can assume that μ belongs to $\mathcal{K}_\infty$. By Lemma 2.5.5 there exist a convex function $\mu_v \in \mathcal{K}_\infty$ and a concave function $\mu_c \in \mathcal{K}_\infty$ such that both are continuously differentiable on $(0, \infty)$ and $\mu \leq \mu_c \circ \mu_v$ holds on $[0, \infty)$. Now for any Young function $\Psi\colon [0, \infty) \to [0, \infty)$ we have by Jensen's inequality

$$\begin{aligned}\theta \left(\int_0^1 \mu(\|u(s)\|)\, ds \right) &\leq \theta \left(\int_0^1 \mu_c \circ \mu_v(\|u(s)\|)\, ds \right) \\ &\leq (\theta \circ \mu_c \circ \Psi^{-1}) \left(\int_0^1 (\Psi \circ \mu_v)(\|u(s)\|)\, ds \right).\end{aligned}$$

Taking $\tilde{\theta} := \theta \circ \mu_c \circ \Psi^{-1}$ and $\Phi := \Psi \circ \mu_v$ we obtain the desired estimate. We obviously have $\tilde{\theta}, \Phi \in \mathcal{K}_\infty$ and Φ is a Young function by Lemma 1.1.5. □

The next theorem is a partial converse of Theorem 3.3.1.

Theorem 3.3.3. *Assume that the system $\Sigma(A, B)$ is L^∞-siISS. Then there is a Young function Φ such that the system $\Sigma(A, B)$ is E_Φ-admissible. If, additionally, the function μ in (3.2) can be chosen as a Young function, then $\Sigma(A, B)$ is infinite-time E_μ-admissible and hence E_μ-sISS.*

Proof. Let Φ be a Young function given by Lemma 3.3.2. We show that $\Sigma(A, B)$ is E_Φ-admissible. It is sufficient to show that $\int_0^1 T_{-1}(s)Bu(s)ds \in X$ for all $u \in E_\Phi(0, 1; U)$. By assumption we have that $\int_0^1 T_{-1}(s)Bu(s)\, ds \in X$ if $u \in L^\infty(0, 1; U)$. Let $u \in E_\Phi(0, 1; U)$, then, by definition, there is a sequence $(u_n)_{n\in\mathbb{N}} \subset L^\infty(0, 1; U)$ such that $\lim_{n\to\infty} \|u_n - u\|_{E_\Phi(0,1;U)} = 0$.

Since $(u_n)_{n\in\mathbb{N}}$ is a Cauchy sequence we can assume without loss of generality that $\|u_n - u_m\|_{E_\Phi(0,1;U)} \leq 1$ for all $m, n \in \mathbb{N}$. Lemma 1.4.3 yields

$$\begin{aligned}\left\|\int_0^1 T_{-1}(s)B(u_n(s) - u_m(s))\,ds\right\| &\leq \tilde{\theta}\left(\int_0^1 \Phi(\|u_n(s) - u_m(s)\|)\,ds\right)\\ &\leq \tilde{\theta}\left(\|u_n - u_m\|_{E_\Phi(0,1;U)}\right).\end{aligned}$$

Hence $(\int_0^1 T_{-1}(s)Bu_n(s)\,ds)_{n\in\mathbb{N}}$ is a Cauchy sequence in X and thus converges. Let y denote its limit. Since $E_\Phi(0,1;U)$ is continuously embedded in $L^1(0,1;U)$ it follows that

$$\lim_{n\to\infty}\int_0^1 T_{-1}(s)Bu_n(s)\,ds = \int_0^1 T_{-1}(s)Bu(s)\,ds$$

in X_{-1}. Since X is continuously embedded in X_{-1}, we conclude that

$$y = \int_0^1 T_{-1}(s)Bu(s)\,ds.$$

Thus, we have shown that $\int_0^1 T_{-1}(s)Bu(s)\,ds \in X$ for all $u \in E_\Phi$ and hence $\Sigma(A,B)$ is admissible with respect to E_Φ.

Now assume that the function μ in (3.2) is a Young function. The admissibility with respect to E_μ is now easier to see: For $u \in E_\mu(0,t;U)$ we pick a sequence $(u_n)_{n\in\mathbb{N}} \subset L^\infty(0,t;U)$ such that $\lim_{n\to\infty}\|u_n - u\|_{E_\mu(0,t;U)}$ and $\|u_n - u_m\|_{E_\mu(0,t;U)} \leq 1$ for all $m, n \in \mathbb{N}$. Then the siISS estimate and Lemma 1.4.3 yield

$$\begin{aligned}\left\|\int_0^t T_{-1}(s)B(u_n(s) - u_m(s))\,ds\right\| &\leq \theta\left(\int_0^t \mu(\|u_n(s) - u_m(s)\|_U)\,ds\right)\\ &\leq \theta\left(\|u_n - u_m\|_{E_\mu(0,t;U)}\right).\end{aligned}$$

Hence $(\int_0^t T_{-1}(s)Bu_n(s)\,ds)_{n\in\mathbb{N}}$ is a Cauchy sequence in X and the same argument as above shows that $\int_0^t T_{-1}(s)Bu(s)\,ds \in X$ holds. For all $t \geq 0$, $u \in E_\mu(0,t;U)$, $u \neq 0$, we have by Lemma 1.4.3

$$\begin{aligned}\left\|\int_0^t T_{-1}(s)Bu(s)\,ds\right\| &= \left\|\int_0^t T_{-1}(s)B\frac{u(s)}{\|u\|_{E_\mu(0,t;U)}}\,ds\right\|\|u\|_{E_\mu(0,t;U)}\\ &\leq \theta\left(\int_0^t \mu\left(\frac{\|u(s)\|_U}{\|u\|_{E_\mu(0,t;U)}}\right)ds\right)\|u\|_{E_\mu(0,t;U)}\\ &\leq \theta(1)\|u\|_{E_\mu(0,t;U)}.\end{aligned}$$

Hence the system $\Sigma(A,B)$ is infinite-time E_μ-admissible and thus, by Proposition 3.2.1, E_μ-sISS. □

Given a sequence $(c_n)_{n\in\mathbb{N}} \subset [0,\infty)$ for which the series $\sum_{n\in\mathbb{N}} c_n$ diverges there exists another sequence $(d_n)_{n\in\mathbb{N}} \subset [0,\infty)$ with $\lim_{n\to\infty} d_n = 0$ such that the series $\sum_{n\in\mathbb{N}} c_n d_n$ still diverges. Thus, loosely speaking, there is no real series, which diverges less rapidly than any other. The following Lemma is an integral version of it.

Lemma 3.3.4. *Let $f \in L^\infty(0,\infty) \setminus L^1(0,\infty)$ such that $f > 0$ almost everywhere. Then there exists a bounded, continuously differentiable and decreasing function $h\colon (0,\infty) \to [0,\infty)$ such that $\lim_{t\to\infty} h(t) = 0$ and $\int_0^\infty h(s)f(s)\,ds = \infty$.*

Proof. Let $(c_n)_{n\in\mathbb{N}} \subset \mathbb{R}$ be the sequence defined by

$$c_n = \int_n^{n+1} f(s)\,ds.$$

Then we have $\sum_{n=0}^\infty c_n = \infty$ by assumption. Thus, by a well-known fact, c.f. [Kno28, p. 299], the series $\sum_{n=0}^\infty c_n d_n$ is also divergent, where $d_n := (\sum_{k=0}^n c_k)^{-1}$. Since the function f is positive, the sequence $(d_n)_{n\in\mathbb{N}}$ is strictly decreasing. Therefore there exists a continuously differentiable decreasing function $h\colon (0,\infty) \to [0,\infty)$ such that

$$d_n \le h|_{[n,n+1]} \le d_{n-1} \tag{3.8}$$

for all $n \in \mathbb{N}$, where $d_{-1} := 2/c_0$. From (3.8) follows that $0 \le h(t) \le 2/c_0$ for all $t \in (0,t)$ and $\lim_{t\to\infty} h(t) = 0$. Further we have

$$\int_n^{n+1} h(s)f(s)\,ds \ge d_n \int_n^{n+1} f(s)\,ds = c_n d_n$$

for all $n \in \mathbb{N}$. Thus we obtain

$$\int_0^\infty h(s)f(s)\,ds \ge \sum_{n\in\mathbb{N}} c_n d_n = \infty,$$

which completes the proof. □

The following theorem shows that infinite-time E_Φ-admissibility and strong integral input-to-state stability with respect to L^∞ are not equivalent, i.e., we cannot drop the Young function condition in the second part of Theorem 3.3.3 entirely. Thus, Theorem 3.3.1 and the following result show that sISS with respect to E_Φ is a stronger notion than L^∞-siISS.

Theorem 3.3.5. *There is a system $\Sigma(A,B)$ such that the following holds:*

(a) The semigroup generated by A is strongly stable.

(b) $\Sigma(A,B)$ is infinite-time admissible with respect to L^1.

(c) $\Sigma(A,B)$ is L^1-sISS, L^1-siISS and hence, in particular, L^∞-siISS.

(d) $\Sigma(A,B)$ *is not* E_Φ*-sISS for any Young function* Φ.

(e) $\Sigma(A,B)$ *is not* L^∞*-sISS.*

In particular, Z-siISS does not imply Z-sISS, neither for $Z = E_\Phi$ *nor* $Z = L^\infty$.

Proof. Let $(T(t))_{t\geq 0}$ be the left-translation semigroup on $X = L^1(0,\infty)$, i.e., $(T(t)f)(s) = (S_t^l f)(s) = f(t+s)$, $f \in X$, which is strongly stable. Its generator is given by

$$Af := f'$$

for $f \in D(A)$, where

$$D(A) = \{f \in L^1(0,\infty) \mid f \in AC(0,\infty) \text{ and } f' \in L^1(0,\infty)\},$$

see e.g. p. 51 in [EN00]. We choose $U = X = L^1(0,\infty)$ as input space and $B = I$ as control operator. The system $\Sigma(A,B)$ is infinite-time L^1-admissible since for any $u \in L^1(0,t;X)$ we have

$$\begin{aligned}\left\| \int_0^t T(s)Bu(s)\,ds \right\|_X &\leq \int_0^t \|T(s)Bu(s)\|_X\,ds \\ &\leq \int_0^t \|u(s)\|_X\,ds \\ &= \|u\|_{L^1(0,t;L^1(0,\infty))}.\end{aligned}$$

Hence, by Proposition 3.2.1, $\Sigma(A,B)$ is L^1-sISS. Therefore, Proposition 3.2.2 yields that $\Sigma(A,B)$ is L^1-siISS. By inclusion of the L^p spaces on bounded interval, we get that $\Sigma(A,B)$ is L^∞-siISS, see Remark 3.1.3. Hence we have proved parts (a), (b) and (c) of the theorem.
Now let us fix a Young function Φ. In order to show that $\Sigma(A,B)$ is not infinite-time E_Φ-admissible, we construct a function u in the following way: Let $u_0 \in L_\Phi(0,\infty) \cap L^\infty(0,\infty)$ be a function given by Lemma 1.6.5 with $I = (0,\infty)$ and let h a function given by Lemma 3.3.4 applied to $f := u_0$. Now set $g = -h'$ and define $u\colon (0,\infty) \to L^1(0,\infty)$,

$$[u(s)](r) = g(r)\chi_{[s,\infty)}(r)u_0(s),$$

which is well-defined since for $s \in (0,\infty)$, $\int_s^\infty |g(r)|\,dr = h(s)$ and

$$\|u\|_{L^1(0,t;X)} = \int_0^t u_0(s) \int_s^\infty |g(r)|\,dr\,ds = \int_0^t u_0(s)h(s)\,ds.$$

Hence, the restriction of u to the interval $[0,t]$ belongs to $L^1(0,t;L^1(0,\infty))$ for all $t \geq 0$ but $u \notin L^1(0,\infty;L^1(0,\infty))$. Using that $[u(s)](r) \geq 0$ for all $r,s > 0$ and $[u(s)](r) = 0$ for all $r \in [0,s)$, Fubini's theorem yields

$$\left\| \int_0^t T(s)Bu(s)\,ds \right\|_X = \|u\|_{L^1(0,t;X)}.$$

Since $u_0 \in L^\infty(0,\infty)$ and for all $s > 0$

$$\|u(s)\|_X = \int_0^\infty [u(s)](r)\,dr = u_0(s) \int_s^\infty g(r)\,dr = u_0(s)h(s) \tag{3.9}$$

we have that $u \in L^\infty(0,\infty;X)$ and

$$\|u\|_{L^\infty(0,\infty;X)} \le \|u_0\|_{L^\infty(0,\infty)} \|h\|_{L^\infty(0,\infty)}. \tag{3.10}$$

Therefore $u|_{[0,t]} \in E_\Phi(0,t;X)$ and by (3.9) follows that

$$\|u\|_{E_\Phi(0,t;X)} \le \|h\|_{L^\infty(0,\infty)} \|u_0\|_{E_\Phi(0,t)} \le \|h\|_{L^\infty(0,\infty)} \|u_0\|_{L_\Phi(0,\infty)}. \tag{3.11}$$

If $\Sigma(A,B)$ were infinite-time E_Φ-admissible, (3.11) would lead to

$$\|u\|_{L^1(0,t;X)} = \left\| \int_0^t T(s)Bu(s)\,ds \right\|_X \le c_\infty \|u\|_{E_\Phi(0,t;X)} \le c_\infty \|h\|_{L^\infty(0,\infty)} \|u_0\|_{L_\Phi(0,\infty)}$$

for some $c_\infty > 0$ independent of u and t. Letting $t \to \infty$, this gives a contradiction as $\|u\|_{L^1(0,t;X)}$ tends to ∞.
Using (3.10) instead of (3.11) we obtain, assuming that the system $\Sigma(A,B)$ were infinite-time L^∞-admissible,

$$\begin{aligned}
\|u\|_{L^1(0,t;X)} &= \left\| \int_0^t T(s)Bu(s)\,ds \right\|_X \\
&\le c_\infty \|u\|_{L^\infty(0,t;X)} \\
&\le c_\infty \|u\|_{L^\infty(0,\infty;X)} \\
&\le c_\infty \|h\|_{L^\infty(0,\infty)} \|u_0\|_{L^\infty(0,\infty)}
\end{aligned}$$

for some $c_\infty > 0$ independent of u and t. Letting $t \to \infty$, this gives again the same contradiction as u does not belong to $L^1(0,\infty;X)$. □

The following result generalises Proposition 3.2.2 for $p \in (1,\infty)$.

Theorem 3.3.6. *Let Φ be a Young function, which satisfies the Δ_2-condition with $s_0 = 0$. If the system $\Sigma(A,B)$ is E_Φ-sISS, then it is E_Φ-siISS.*

Proof. Similarly to the proof of Theorem 3.3.1, we consider a nondecreasing function $\theta\colon [0,\infty) \to [0,\infty)$ defined by $\theta(0) = 0$ and

$$\theta(\alpha) = \sup \left\{ \left\| \int_0^t T_{-1}(s)Bu(s)\,ds \right\| \,\Big|\, u \in E_\Phi(0,t;U),\ t \ge 0,\ \int_0^t \Phi(\|u(s)\|_U)\,ds \le \alpha \right\}$$

for $\alpha > 0$. It follows as in the proof of Theorem 3.3.1 that θ is well-defined and nondecreasing. Indeed, since the system $\Sigma(A,B)$ is infinite-time admissible with respect to E_Φ we have

$$\left\| \int_0^t T_{-1}(s)Bu(s)\,ds \right\| \le c_\infty \|u\|_{E_\Phi(0,t;U)} \le c_\infty \left(1 + \int_0^t \Phi(\|u(s)\|_U)\,ds \right)$$

for all $t \geq 0$ and $u \in E_\Phi(0,t;U)$. As in the proof of Theorem 3.3.1, it remains to show that θ is continuous in 0. This follows from the Δ_2-condition. Indeed, let $(\alpha_n)_{n\in\mathbb{N}}$ be a sequence of positive real numbers converging to 0. By the definition of θ, for any $n \in \mathbb{N}$ there exist $t_n \geq 0$ and $u_n \in L_\Phi(0,t_n;U)$ such that

$$\int_0^{t_n} \Phi(\|u_n(s)\|_U)\,ds < \alpha_n$$

and

$$\left|\theta(\alpha_n) - \left\|\int_0^{t_n} T_{-1}(s)Bu_n(s)\,ds\right\|\right| < \frac{1}{n}.$$

By extending the functions u_n to $[0,\infty)$ by 0, we can assume that $(u_n)_{n\in\mathbb{N}} \subset L_\Phi(0,\infty;U)$ and

$$\int_0^{\infty} \Phi(\|u_n(s)\|_U)\,ds < \alpha_n$$

holds, as well as

$$\left|\theta(\alpha_n) - \left\|\int_0^{\infty} T_{-1}(s)Bu_n(s)\,ds\right\|\right| < \frac{1}{n}.$$

It follows that the sequence $(\|u_n(\cdot)\|)_{n\in\mathbb{N}}$ is Φ-mean convergent to zero in $L_\Phi(0,\infty;U)$. Hence, by Lemma 1.4.2, it converges to zero in $L_\Phi(0,\infty;U)$, since Φ satisfies the Δ_2-condition. By infinite-time E_Φ-admissibility we conclude that $\lim_{n\to\infty}\theta(\alpha_n) = 0$. $\square$

3.4 Concluding comments

The fact that the operator A generates a strongly stable semigroup on X is not really significant for the proofs given in this chapter. Of course, looking at the Proposition 3.2.1 it has to be noted that the strong stability is always involved as soon as we assume that the system $\Sigma(A,B)$ has one of stability properties introduced in Definition 3.1.2. A careful scrutiny of the proofs in this chapter shows that $\beta(x,\cdot) \in \mathcal{L}$ is not used in finding and constructing the comparison functions μ and θ, see Definition 3.1.2. The only thing we used in this regard was the boundedness of $\beta(x,\cdot)$ on the interval $[0,\infty)$. So one could replace the first condition in both definitions of sISS and siISS by "$\beta(x,\cdot)$ is bounded on $[0,\infty)$", which is of course weaker than the initial definition. All the results would remain true if "$(T(t))_{t\geq 0}$ is a strongly stable semigroup" was replaced by "$(T(t))_{t\geq 0}$ is a bounded semigroup". This means that for sISS and siISS the intrinsic and the extrinsic stabilities can be studied separately.

The situation changes significantly in the next chapter where we will study the connections between more specific versions of those stability notions, namely input-to-state stability and integral input-to-state stability. These notions involve the exponential stability of the semigroup and in this case the finite-time admissibility is equivalent to the infinite-time admissibility.

Chapter 4

Input-to-state stability

In this chapter we study the concepts of input-to state stability and integral input-to-state stability. As we will see they are special cases of strong input-to state stability and strong integral input-to-state stability, respectively. More precisely they additionally imply the exponential stability of the semigroup associated with the system. This condition implies, as we saw in Chapter 2, that admissibility and infinite-time admissibility are equivalent. Hence we can naturally expect stronger connections between those stability concepts. We will see for instance that admissibility with respect to some Orlicz space is not only sufficient for a system to be integral input-to-state stable, but it is also necessary. The main results of this chapter were published in [JNPS18], see also [JNPS16].

4.1 Stability notions for infinite-dimensional systems

Definition 4.1.1. *The system $\Sigma(A, B)$ is called*

(a) input-to-state stable *with respect to Z (or Z-*ISS*) if there exist functions $\beta \in \mathcal{KL}$ and $\mu \in \mathcal{K}_\infty$ such that for every $t \geq 0$, $x_0 \in X$ and $u \in Z(0, t; U)$ the state $x(t)$ lies in X and*

$$\|x(t)\| \leq \beta(\|x_0\|, t) + \mu(\|u\|_{Z(0,t;U)}), \tag{4.1}$$

(b) integral input-to-state stable *with respect to Z (or Z-*iISS*) if there exist functions $\beta \in \mathcal{KL}$, $\theta \in \mathcal{K}_\infty$ and $\mu \in \mathcal{K}$ such that for every $t \geq 0$, $x_0 \in X$ and $u \in Z(0, t; U)$ the state $x(t)$ lies in X and*

$$\|x(t)\| \leq \beta(\|x_0\|, t) + \theta\left(\int_0^t \mu(\|u(s)\|_U)\, ds\right), \tag{4.2}$$

(c) uniformly bounded energy bounded state *with respect to Z (or Z-*UBEBS*) if there exist functions $\gamma, \theta \in \mathcal{K}_\infty$, $\mu \in \mathcal{K}$ and a constant $c > 0$ such that for every $t \geq 0$, $x_0 \in X$ and*

$u \in Z(0,t;U)$ the state $x(t)$ lies in X and

$$\|x(t)\| \leq \gamma(\|x_0\|) + \theta\left(\int_0^t \mu(\|u(s)\|_U)\, ds\right) + c. \tag{4.3}$$

Remark 4.1.2. *(a) By the inclusion of L^p spaces on bounded intervals we obtain that L^p-ISS (L^p-iISS, L^p-UBEBS) implies L^q-ISS (L^q-iISS, L^q-UBEBS) for all $1 \leq p < q \leq \infty$. Furthermore, the inclusions $L^\infty \subseteq E_\Phi \subseteq L_\Phi \subseteq L^1$ and $Z \subseteq L^1_{\mathrm{loc}}$ yield a corresponding chain of implications of ISS, iISS and UBEBS.*

(b) Note that in general the integral $\int_0^t \mu(\|u(s)\|_U)\, ds$ in the inequalities defining Z-iISS and Z-UBEBS may be infinite. In that case, the inequalities hold trivially. This indicates that the major interest in iISS and UBEBS lies in the case $Z = L^\infty$, in which the integral is always finite.

(c) The difference between (i)ISS and its strong versions is that the function β now belongs to the class $\mathcal{KL}$ and $\beta(\cdot, t)$ only depends on the norm of x_0 which is stronger than the condition that $\beta(x, \cdot) \in \mathcal{L}$ for all $x \in X$, $x \neq 0$. This leads to the uniform convergence to zero of the semigroup $(T(t))_{t\geq 0}$, see Proposition 4.2.2 below.

4.2 Comparison of stability notions

Definition 4.2.1. *The semigroup $(T(t))_{t\geq 0}$ is called* exponentially stable *if there exist constants $M, \omega > 0$ such that*

$$\|T(t)\| \leq Me^{-\omega t} \tag{4.4}$$

for $t \geq 0$.

The following simple characterisations of input-to-state stability and integral input-to-state stability it terms of strong input-to-state stability and strong integral input-to-state stability respectively will be used in the sequel.

Proposition 4.2.2. *(a) The following are equivalent:*

(i) The system $\Sigma(A, B)$ is Z-ISS.

(ii) The system $\Sigma(A, B)$ is Z-sISS and the semigroup $(T(t))_{t\geq 0}$ is exponentially stable.

(b) The following are equivalent:

(i) The system $\Sigma(A, B)$ is Z-iISS.

(ii) The system $\Sigma(A, B)$ is Z-siISS and the semigroup $(T(t))_{t\geq 0}$ is exponentially stable.

Proof. If the system $\Sigma(A, B)$ is Z-ISS or Z-iISS, then, taking $u = 0$ and some $x \in X$, with $x \neq 0$ and $\|x\| \leq 1$ in (4.1) and (4.2), we obtain that $\|T(t)\|_{\mathcal{L}(X)} < 1$ for all sufficiently large $t \geq 0$. This implies the exponential stability of the semigroup.
If the system $\Sigma(A, B)$ is Z-ISS or Z-iISS, then the function $\tilde{\beta}\colon X \times \mathbb{R}_0^+ \to \mathbb{R}_0^+$ defined by $\tilde{\beta}(x, t) = \beta(\|x\|, t)$, where β is the function from the definition of input-to-state stability or integral input-to-state stability respectively, satisfies $\tilde{\beta}(x, \cdot) \in \mathcal{L}$ for all $x \in X$, $x \neq 0$. Now the estimates (3.1) and (3.2) with $\tilde{\beta}$ instead of β follow from (4.1) and (4.2).
If the semigroup $(T(t))_{t\geq 0}$ is exponentially stable, i.e., there exist constants $M, \omega > 0$ such that $\|T(t)\|_{\mathcal{L}(X)} \leq Me^{-\omega t}$ holds for all $t \geq 0$, we define $\tilde{\beta}\colon \mathbb{R}_0^+ \times \mathbb{R}_0^+ \to \mathbb{R}_0^+$ by $\tilde{\beta}(s, t) = Me^{-\omega t}s$. Then we have $\|T(t)x\| \leq \tilde{\beta}(\|x\|, t)$. Now using the sISS estimate we obtain the ISS estimate and from the siSS estimate follows the iISS estimate (with $\tilde{\beta}$ instead of β in both cases). $\square$

Lemma 4.2.3. *Let $(T(t))_{t\geq 0}$ be exponentially stable and $\Sigma(A, B)$ Z-admissible. Then there exists a constant $C = C_A > 0$ such that for any $t > 0$ and $u \in Z(0, t; U)$ there exists a function $\tilde{u} \in Z(0, 1; U)$ with*

$$\left\| \int_0^t T_{-1}(s)Bu(s)\, ds \right\| \leq C_A \left\| \int_0^1 T_{-1}(s)B\tilde{u}(s)\, ds \right\| \tag{4.5}$$

such that

$$\int_0^1 \mu(\|\tilde{u}(s)\|_U)\, ds \leq \int_0^t \mu(\|u(s)\|_U)\, ds$$

for any $\mu \in \mathcal{K}$. Further we have $\|\tilde{u}\|_{Z(0,1;U)} \leq \|u\|_{Z(0,t;U)}$.

Proof. Without loss of generality we can assume that $t \in \mathbb{N} \setminus \{0\}$ (otherwise we extend u to $[0, \lceil t \rceil]$ by zero). Then we have

$$\begin{aligned} \left\| \int_0^t T_{-1}(s)Bu(s)\, ds \right\| &= \left\| \sum_{k=0}^{t-1} \int_k^{k+1} T_{-1}(s)Bu(s)\, ds \right\| \\ &= \left\| \sum_{k=0}^{t-1} T(k) \int_0^1 T_{-1}(s)Bu(s+k)\, ds \right\| \\ &\leq \sum_{k=0}^{t-1} \|T(k)\| \left\| \int_0^1 T_{-1}(s)Bu(s+k_0)\, ds \right\|, \end{aligned} \tag{4.6}$$

where $k_0 \in \{0, \ldots, t-1\}$ is defined by the condition

$$\left\| \int_0^1 T_{-1}(s)Bu(s+k_0)\, ds \right\| = \max_{0\leq k\leq t-1} \left\| \int_0^1 T_{-1}(s)Bu(s+k)\, ds \right\|.$$

Since the semigroup $(T(t))_{t\geq 0}$ is exponentially stable, the series $\sum_{k=0}^{\infty} \|T(k)\|$ converges. Hence its partial sums are bounded, i.e., for all $t \in \mathbb{N} \setminus \{0\}$ holds $\sum_{k=0}^{t-1} \|T(k)\| \leq C_A$ for some $C_A > 0$.

Let $\tilde{u} := u(\cdot + k_0)|_{[0,1]}$. The inequality (4.6) then reads

$$\left\| \int_0^t T_{-1}(s) B u(s)\, ds \right\| \leq C_A \left\| \int_0^1 T_{-1}(s) B \tilde{u}(s)\, ds \right\|.$$

For any $\mu \in \mathcal{K}$ holds

$$\int_0^1 \mu(\|\tilde{u}(s)\|_U)\, ds = \int_{k_0}^{k_0+1} \mu(\|u(s)\|_U)\, ds \leq \int_0^t \mu(\|u(s)\|_U)\, ds.$$

By properties (d) and (e) of the space Z, see Assumption 2.1.4, we have $\tilde{u} \in Z(0,1;U)$ and $\|\tilde{u}\|_{Z(0,1;U)} \leq \|u\|_{Z(0,t;U)}$. □

Lemma 4.2.4. *Assume that the semigroup $(T(t))_{t\geq 0}$ is exponentially stable and the system $\Sigma(A,B)$ is Z-admissible. Then $\Sigma(A,B)$ is Z-iISS if and only if there exist $\theta \in \mathcal{K}_\infty$ and $\mu \in \mathcal{K}$ such that*

$$\left\| \int_0^1 T_{-1}(s) B u(s)\, ds \right\| \leq \theta \left(\int_0^1 \mu(\|u(s)\|_U)\, ds \right)$$

for every $u \in Z(0,1;U)$.

Proof. Using Lemma 4.2.3 and the monotonicity of θ we obtain

$$\begin{aligned} \left\| \int_0^t T_{-1}(s) B u(s)\, ds \right\| &\leq C_A \left\| \int_0^1 T_{-1}(s) B \tilde{u}(s)\, ds \right\| \\ &\leq C_A \theta \left(\int_0^1 \mu(\|\tilde{u}(s)\|_U)\, ds \right) \\ &\leq C_A \theta \left(\int_0^t \mu(\|u(s)\|_U)\, ds \right) \\ &= \tilde{\theta} \left(\int_0^t \mu(\|u(s)\|_U)\, ds \right) \end{aligned}$$

for all $u \in Z(0,t;U)$, where $\tilde{\theta}(s) := C_A \theta(s)$. □

Infinite-time admissibility always implies finite-time admissibility. If the semigroup $(T(t))_{t\geq 0}$ is exponentially stable, those notions are equivalent.

Lemma 4.2.5. *Let $(T(t))_{t\geq 0}$ be exponentially stable. Then the system $\Sigma(A,B)$ is Z-admissible if and only if it is infinite-time Z-admissible.*

Proof. Let $t > 0$ and $u \in Z(0,t;U)$. Then, applying Lemma 4.2.3, the Z-admissibility of $\Sigma(A,B)$ and then Lemma 4.2.3 again, we obtain

$$\left\| \int_0^t T_{-1}(s) B u(s)\, ds \right\| \leq C_A \left\| \int_0^1 T_{-1}(s) B \tilde{u}(s)\, ds \right\| \leq c(1) C_A \|\tilde{u}\|_{Z(0,1;U)} \leq c_\infty \|u\|_{Z(0,t;U)},$$

where $c_\infty := c(1) C_A$ is a positive constant. □

Proposition 4.2.6. *For any function space $Z \subset L^1_{\mathrm{loc}}(0,\infty;U)$ satisfying Assumption 2.1.4 we have:*

(a) The following statements are equivalent:

(i) $\Sigma(A,B)$ is Z-ISS.

(ii) $\Sigma(A,B)$ is Z-admissible and $(T(t))_{t\geq 0}$ is exponentially stable.

(iii) $\Sigma(A,B)$ is infinite-time Z-admissible and $(T(t))_{t\geq 0}$ is exponentially stable.

(b) If $\Sigma(A,B)$ is Z-iISS, then the system is Z-admissible and $(T(t))_{t\geq 0}$ is exponentially stable.

(c) If $\Sigma(A,B)$ is Z-UBEBS, then the system is Z-admissible and $(T(t))_{t\geq 0}$ is bounded.

Proof. From definitions follows directly that Z-ISS, Z-iISS and Z-UBEBS imply Z-admissibility. By Proposition 4.2.2, Z-ISS and Z-iISS each imply the exponential stability of the semigroup $(T(t))_{t\geq 0}$. Taking $u = 0$ in (4.3) we see that Z-UBEBS implies the boundedness of $(T(t))_{t\geq 0}$. By Lemma 4.2.5 the statements (ii) and (iii) in (a) are equivalent. Thus we are left to show that (iii) implies (i). Taking $\beta(s,t) = se^{-\omega t}$ and $\mu(s) = c_\infty s$ we obtain the ISS estimate for the system $\Sigma(A,B)$. □

Proposition 4.2.7. *If $1 \leq p < \infty$, then the following are equivalent:*

(i) $\Sigma(A,B)$ is L^p-ISS.

(ii) $\Sigma(A,B)$ is L^p-iISS.

(iii) $\Sigma(A,B)$ is L^p-UBEBS and $(T(t))_{t\geq 0}$ is exponentially stable.

Proof. Using Proposition 3.2.2 and Proposition 4.2.2 we obtain (i) ⇒ (ii). From Definitions of iISS and UBEBS and Proposition 4.2.6 follows (ii) ⇒ (iii). Also from Proposition 4.2.6 we get (iii) ⇒ (i). □

Remark 4.2.8. *Let $1 \leq p < \infty$. If the system $\Sigma(A,B)$ is L^p-admissible and $(T(t))_{t\geq 0}$ is exponentially stable, then the system $\Sigma(A,B)$ is L^p-ISS with the following choices for the functions β and μ:*

$$\beta(s,t) \coloneqq Me^{-\omega t}s \quad \text{and} \quad \mu(s) \coloneqq c_\infty s,$$

where $c_\infty = \sup_{t\geq 0} c(t)$, cf. Remark 3.2.3. Furthermore, the system is L^p-iISS with the following choices for the functions β, θ and μ:

$$\beta(x,t) \coloneqq Me^{-\omega t}s, \quad \theta(s) \coloneqq c_\infty s^{1/p} \quad \text{and} \quad \mu(s) \coloneqq s^p.$$

Proposition 4.2.9. *If $\Sigma(A,B)$ is L^∞-iISS, then $\Sigma(A,B)$ is L^∞-zero-class admissible.*

Proof. Since L^∞-iISS implies L^∞-siISS the claim follows from Proposition 3.2.4. □

Some of the equivalences in the following proposition were already shown in [MI16] for the case $Z = L^p$.

Proposition 4.2.10. *Let $Z \subset L^1_{\mathrm{loc}}(0,\infty;U)$ be a function space satisfying Assumption 2.1.4 and $B \in \mathcal{L}(U,X)$. Then the following statements are equivalent:*

(i) $(T(t))_{t\geq 0}$ *is exponentially stable.*

(ii) $\Sigma(A,B)$ *is Z-admissible and* $(T(t))_{t\geq 0}$ *is exponentially stable.*

(iii) $\Sigma(A,B)$ *is infinite-time Z-admissible and* $(T(t))_{t\geq 0}$ *is exponentially stable.*

(iv) $\Sigma(A,B)$ *is Z-ISS.*

(v) $\Sigma(A,B)$ *is Z-iISS.*

(vi) $\Sigma(A,B)$ *is Z-UBEBS and* $(T(t))_{t\geq 0}$ *is exponentially stable.*

(vii) $\Sigma(A,B)$ *is L^1_{loc}-admissible and* $(T(t))_{t\geq 0}$ *is exponentially stable.*

If the function space Z additionally satisfies condition (B), then the assertions above are equivalent to:

(viii) $\Sigma(A,B)$ *is Z-zero-class admissible and* $(T(t))_{t\geq 0}$ *is exponentially stable.*

Proof. Since for every bounded control operator B we have $x(t) \in X$ for any $x_0 \in X$ and $u \in L^1(0,t;U)$, the equivalences $(i) \Leftrightarrow (ii) \Leftrightarrow (vii)$ hold. Using Propositions 4.2.6 and 4.2.7 and Remark 4.1.2 we see that $(vii) \Rightarrow (v)$. From Proposition 4.2.6 we obtain $(v) \Rightarrow (iv)$. By Proposition 4.2.6 we have $(iv) \Leftrightarrow (iii)$. The implication $(iii) \Rightarrow (i)$ holds trivially. From definitions of iISS and UBEBS follows directly $(v) \Rightarrow (vi)$. Since trivially $(vi) \Rightarrow (i)$, the proof of the first part is complete.
From definitions we obtain $(viii) \Rightarrow (ii)$. Hence it remains to show that if Condition (B) holds for Z, then $(i) \Rightarrow (viii)$. Since the semigroup $(T(t))_{t\geq 0}$ is exponentially stable there exist constants $M, \omega > 0$ such that (4.4) holds. Let $t \geq 0$ and $u \in Z(0,t;U)$ then

$$\begin{aligned}\left\| \int_0^t T(s)Bu(s)\,ds \right\| &\leq M\|B\| \int_0^t e^{-\omega s}\|u(s)\|_U\,ds \\ &\leq M\|B\| \int_0^t \|u(s)\|_U\,ds \\ &= M\|B\|\|u\|_{L^1(0,t;U)} \\ &\leq M\|B\|\kappa(t)\|u\|_{Z(0,t;U)}.\end{aligned}$$

By Condition (B) we have $\lim_{t\searrow 0}\kappa(t) = 0$ and, hence, the system $\Sigma(A,B)$ is zero-class admissible with respect to Z. □

The following corollary is a simple consequence of Proposition 4.2.10 and Hölder's inequality.

Corollary 4.2.11. *Let $1 \leq p \leq \infty$. If one of the equivalent conditions in Proposition 4.2.10 holds, then the system $\Sigma(A,B)$ is L^p-ISS with the following choices for the functions β and μ:*

$$\beta(s,t) := Me^{-\omega t}s \qquad \text{and} \qquad \mu(s) := \frac{M}{\omega q}\|B\|s,$$

where q is the Hölder conjugate of p. The system $\Sigma(A,B)$ is then also L^p-iISS with the following choices for the functions β, μ and θ:

$$\beta(s,t) := Me^{-\omega t}s, \qquad \mu(s) := s \qquad \text{and} \qquad \theta(s) = M\|B\|s.$$

The constants M and ω are given by (4.4).

Remark 4.2.12. *In Proposition 4.2.10 the assertions are independent of Z as they only rely on exponential stability. In particular, in the situation of Proposition 4.2.10 and Corollary 4.2.11 the system $\Sigma(A,B)$ is L^p-ISS and L^p-iISS for all p if it holds for some p. The choices for the functions μ, however, do depend on p. If the control operator B is unbounded, then the question whether the system $\Sigma(A,B)$ is ISS or iISS with respect to L^p also depends on p.*

4.3 Integral input-to-state stability and Orlicz space admissibility

In this section we show how the integral input-to-state stability can be characterised in terms of admissibility with respect to some Orlicz space E_Φ.

Lemma 4.3.1. *Let $\Sigma(A,B)$ be L^∞-iISS. Then there exist functions $\bar{\theta}, \Phi \in \mathcal{K}_\infty$ such that Φ is a Young function, which is continuously differentiable on $(0,\infty)$ and*

$$\left\| \int_0^t T_{-1}(s)Bu(s),\ ds \right\| \leq \bar{\theta}\left(\int_0^t \Phi(\|u(s)\|_U)\, ds \right) \tag{4.7}$$

for all $t > 0$ and $u \in L^\infty(0,t;U)$.

Proof. It is clear that we only have to consider the case where $t \geq 1$ since for $t \in [0,1)$ the Lemma follows from Lemma 3.3.2. By Lemma 3.3.2 there exist functions $\tilde{\theta}, \Phi \in \mathcal{K}_\infty$ such that (4.7) holds for $t = 1$ with $\tilde{\theta}$ instead of $\bar{\theta}$. Using this and Lemma 4.2.3 we get

$$\begin{aligned}
\left\| \int_0^t T_{-1}(s)Bu(s)\, ds \right\| &\leq C_A \left\| \int_0^1 T_{-1}(s)Bu(s)\, ds \right\| \\
&\leq C_A \tilde{\theta}\left(\int_0^1 \Phi(\|u(s)\|_U)\, ds \right) \\
&\leq C_A \tilde{\theta}\left(\int_0^t \Phi(\|u(s)\|_U)\, ds \right) \\
&= \bar{\theta}\left(\int_0^t \Phi(\|u(s)\|_U)\, ds \right)
\end{aligned}$$

for all $t \geq 1$, where $\bar{\theta} := C_A \tilde{\theta}$. □

Theorem 4.3.2. *The following statements are equivalent:*

(i) There is a Young function Φ such that the system $\Sigma(A,B)$ is E_Φ-ISS.

(ii) $\Sigma(A,B)$ is L^∞-iISS.

(iii) $(T(t))_{t\geq 0}$ is exponentially stable and there is a Young function Φ such that the system $\Sigma(A,B)$ is E_Φ-UBEBS.

Proof. $(i) \Rightarrow (ii)$: By Proposition 4.2.2 the system $\Sigma(A,B)$ is E_Φ-sISS and the semigroup $(T(t))_{t\geq 0}$ is exponentially stable. Theorem 3.3.1 now implies that $\Sigma(A,B)$ strongly iISS with respect to L^∞. Hence, by Proposition 4.2.2, $\Sigma(A,B)$ is L^∞-iISS.
$(ii) \Rightarrow (i)$: The assumption implies that the system $\Sigma(A,B)$ is strongly iISS with respect to L^∞. By Lemma 4.3.1 the function μ in (3.2) can be chosen as a Young function Φ. By Theorem 3.3.3 the system $\Sigma(A,B)$ is E_Φ-sISS. Using Proposition 4.2.2 we obtain that $\Sigma(A,B)$ is E_Φ-ISS.
$(i) \Rightarrow (iii)$: From Proposition 4.2.6 follows that the semigroup $(T(t))_{t\geq 0}$ is exponentially stable. Since, by Lemma 1.5.3, for all $u \in E_\Phi(0,t;U)$ we have $u \in \tilde{L}_\Phi(0,t;U)$ and, by Remark 1.2.5, the following estimate holds

$$\|u\|_{E_\Phi(0,t;U)} \leq 1 + \int_0^t \Phi(\|u(s)\|_U)\, ds,$$

the claim follows.
$(iii) \Rightarrow (i)$: This follows from Proposition 4.2.6. □

Remark 4.3.3. *If the system $\Sigma(A,B)$ is integral ISS with respect to L^∞, then it is not difficult to see that the Young function Φ from statement (i) of Theorem 4.3.2 satisfies the Δ_2-condition if and only if the system $\Sigma(A,B)$ is actually admissible with respect to L^p for some $p < \infty$. Indeed, if the latter is the case, we can choose $\Phi(t) = t^p$ and the claim is shown in Proposition 4.2.7. Conversely, if we assume that our system $\Sigma(A,B)$ is E_Φ-admissible, where Φ is some Young function satisfying the Δ_2-condition, then, by Remark 1.6.4, we have $L^p(I,U) \hookrightarrow E_\Phi(I,U)$ for some $p \in (1,\infty)$ and, hence, $\Sigma(A,B)$ is L^p-admissible, c.f. Remark 4.1.2.*

Next Theorem is a generalisation of Proposition 4.2.7.

Theorem 4.3.4. *If Φ is a Young function that satisfies the Δ_2-condition, then the following are equivalent:*

(i) $\Sigma(A,B)$ is E_Φ-ISS.

(ii) $\Sigma(A,B)$ is E_Φ-iISS.

(iii) $\Sigma(A,B)$ is E_Φ-UBEBS and the semigroup $(T(t))_{t\geq 0}$ is exponentially stable.

Remark 4.3.5. *The proof for* $(i) \Rightarrow (ii)$ *is very similar to the proof of Theorem 3.3.6 but the statement does not follow from this theorem as we additionally assumed there that the Young function* Φ *satisfies the* Δ_2*-condition with* $s_0 = 0$*. This additional assumption is not needed here.*

Proof of Theorem 4.3.4. From definitions of iISS and UBEBS we obtain $(ii) \Rightarrow (iii)$. By Proposition 4.2.6 we have $(iii) \Rightarrow (i)$.
In order to show $(i) \Rightarrow (ii)$ we define $\theta\colon [0,\infty) \to [0,\infty)$ by $\theta(0) = 0$ and

$$\theta(\alpha) = \sup\left\{\left\|\int_0^1 T_{-1}(s)Bu(s)\,ds\right\| \;\middle|\; u \in E_\Phi(0,1;U),\ \int_0^1 \Phi(\|u(s)\|_U)\,ds \leq \alpha\right\}$$

for $\alpha > 0$. The function θ is well-defined since by E_Φ-admissibility and Remark 1.2.5 we have

$$\begin{aligned}\left\|\int_0^1 T_{-1}(s)Bu(s)\,ds\right\| &\leq c(1)\|u\|_{E_\Phi(0,1;U)} \\ &\leq c(1)\left(1 + \int_0^1 \Phi(\|u(s)\|_U)\,ds\right)\end{aligned} \tag{4.8}$$

for all $u \in E_\Phi(0,1;U)$. Clearly, θ is nondecreasing.
We show that θ is continuous at zero. Let $(\alpha_n)_{n\in\mathbb{N}} \subset [0,\infty)$ be a sequence with $\lim_{n\to\infty} \alpha_n = 0$. Then, by construction, there exists a sequence $(u_n)_{n\in\mathbb{N}} \subset E_\Phi(0,1;U)$ with

$$\int_0^1 \Phi(\|u_n(s)\|_U)\,ds \leq \alpha_n \tag{4.9}$$

and

$$\left|\theta(\alpha_n) - \left\|\int_0^1 T_{-1}(s)Bu_n(s)\,ds\right\|\right| < \frac{1}{n}. \tag{4.10}$$

From the estimate (4.9) follows that the sequence $(u_n)_{n\in\mathbb{N}}$ is Φ-mean convergent to zero. Since the Young function Φ satisfies the Δ_2-condition, this sequence converges to zero in $E_\Phi(0,1;U)$[1]. Therefore, the inequality (4.8) applied to u_n yields

$$\left\|\int_0^1 T_{-1}(s)Bu_n(s)\,ds\right\| \leq c(1)\|u_n\|_{E_\Phi(0,1;U)} \to 0,$$

as $n \to \infty$. By (4.10) now follows

$$\begin{aligned}\theta(\alpha_n) &\leq \left|\theta(\alpha_n) - \left\|\int_0^1 T_{-1}(s)Bu_n(s)\,ds\right\|\right| + \left\|\int_0^1 T_{-1}(s)Bu_n(s)\,ds\right\| \\ &\leq \frac{1}{n} + c(1)\|u_n\|_{E_\Phi(0,1;U)}\end{aligned}$$

[1]Here we do not need the Δ_2-condition with $s_0 = 0$ since the interval $(0,1)$ is bounded, c.f. Lemma 1.4.2.

and thus $\lim_{n\to\infty} \theta(\alpha_n) = 0$.
Applying Lemma 2.5.6 we obtain the existence of a function $\tilde{\theta} \in \mathcal{K}_\infty$ such that $\theta \leq \tilde{\theta}$. The definition of θ yields that

$$\left\| \int_0^1 T_{-1}(s)Bu(s)\,ds \right\| \leq \theta\left(\int_0^1 \Phi(\|u(s)\|_U)\,ds\right) \leq \tilde{\theta}\left(\int_0^1 \Phi(\|u(s)\|_U)\,ds\right)$$

for all $u \in E_\Phi(0,1;U)$. By Lemma 4.2.4 the system $\Sigma(A,B)$ is iISS with respect to E_Φ. □

Recall that admissibility and infinite-time admissibility are equivalent if the semigroup $(T(t))_{t\geq}$ is exponentially stable. We are thus led to the following strengthening of Proposition 2.2.6.

Theorem 4.3.6. *The following statements are equivalent:*

(i) $\Sigma(A,B)$ *is* L^1*-ISS.*

(ii) $\Sigma(A,B)$ *is* L^1*-iISS.*

(iii) $\Sigma(A,B)$ *is* E_Φ*-ISS for every Young function* Φ*.*

Proof. By Proposition 4.2.7 we have $(i) \Leftrightarrow (ii)$. As an immediate consequence of Propositions 2.2.6 and 4.2.6 we obtain $(i) \Leftrightarrow (iii)$. □

The following Proposition will be useful for characterising the input-to-state stability of parabolic diagonal systems.

Proposition 4.3.7. *Let* $\Sigma(A,B)$ *be* L^∞*-ISS. Assume that there exist a nonnegative function* $f \in L^1(0,1)$, $\theta \in \mathcal{K}$, *a constant* $c \geq 0$ *and a Young function* μ *such that for every* $u \in L^1(0,1;U)$ *with* $\int_0^1 f(s)\mu(\|u(s)\|_U)\,ds < \infty$ *we have*

$$\left\| \int_0^1 T_{-1}(s)Bu(s)\,ds \right\| \leq c + \theta\left(\int_0^1 f(s)\mu(\|u(s)\|_U)\,ds\right). \tag{4.11}$$

Then $\Sigma(A,B)$ *is* L^∞*-iISS.*

Proof. Since $f \in L^1(0,1)$, Theorem 1.1.8 yields a Young function Ψ such that $f \in \tilde{L}_\Psi(0,1)$. We denote by $\tilde{\Phi}$ its complimentary Young function and define the Young function Φ by $\Phi = \tilde{\Phi} \circ \mu$. Applying the Young's inequality, more precisely Remark 1.1.13, we obtain for any $u \in E_\Phi(0,1;U)$

$$\begin{aligned}
\left\| \int_0^1 T_{-1}(s)Bu(s)\,ds \right\| &\leq c + \theta\left(\int_0^1 f(s)\mu(\|u(s)\|_U)\,ds\right) \\
&\leq c + \theta\left(\int_0^1 \Psi(f(s))\,ds + \int_0^1 \tilde{\Phi}(\mu(\|u(s)\|_U))\,ds\right) \\
&= c + \theta\left(\int_0^1 \Psi(f(s))\,ds + \int_0^1 \Phi(\|u(s)\|_U)\,ds\right)
\end{aligned}$$

and hence $\int_0^1 T_{-1}(s)Bu(s)\,ds \in X$. This shows that the system $\Sigma(A,B)$ is E_Φ-admissible. By Proposition 4.2.6 it is E_Φ-ISS. At last from Theorem 4.3.2 we obtain that $\Sigma(A,B)$ is L^∞-iISS. □

4.4 Stability for parabolic diagonal systems

Definition 4.4.1. *Let X be an infinite-dimensional Banach space. A sequence $(x_n)_{n\in\mathbb{N}} \subset X$ is called a* Schauder basis *for X if for all $x \in X$ there is a unique sequence $(c_n)_{n\in\mathbb{N}} \subset \mathbb{K}$ such that*

$$\lim_{n\to\infty} \left\| x - \sum_{k=0}^{n} c_k x_k \right\|_X = 0.$$

In other words we have $x = \sum_{k=0}^{\infty} c_k x_k$ and the series converges with respect to the norm of X.

Definition 4.4.2. *Let X be an infinite-dimensional Banach space and $1 \le q < \infty$. A sequence $(x_n)_{n\in\mathbb{N}} \subset X$ is called a q-*Riesz basis *for X if it is a Schauder basis for X and there are constants $c_1, c_2 > 0$ such that*

$$c_1 \sum_{k=0}^{\infty} |a_k|^q \le \left\| \sum_{k=0}^{\infty} a_k x_k \right\|_X^q \le c_2 \sum_{k=0}^{\infty} |a_k|^q$$

for all sequences $(a_k)_{k\in\mathbb{N}} \subset \ell^q$.

For this entire section we assume that the input space U is one-dimensional, i.e., $U = \mathbb{C}$, $1 \le q < \infty$ and the state space X possesses a q-Riesz basis of eigenvectors $(e_n)_{n\in\mathbb{N}}$ of A with eigenvalues $(\lambda_n)_{n\in\mathbb{N}}$ such that $\sup\{\operatorname{Re}\lambda_n \mid n \in \mathbb{N}\} < 0$ and there exists a constant $k > 0$ such that $|\operatorname{Im}\lambda_n| \le k|\operatorname{Re}\lambda_n|$ for all $n \in \mathbb{N}$. The latter means that the sequence $(\lambda_n)_{n\in\mathbb{N}}$ lies in a sector in the open left half-plane, i.e., $(-\lambda_n)_{n\in\mathbb{N}} \subset S_\theta$ for some $\theta \in (0, \pi/2)$, where

$$S_\theta = \{z \in \mathbb{C} \setminus \{0\} \mid |\arg z| < \theta\}.$$

As X possesses a q-Riesz basis, we can assume without loss of generality that $X = \ell^q$ and that $(e_n)_{n\in\mathbb{N}}$ is the canonical basis of ℓ^q. Then the linear operator $A\colon D(A) \subset \ell^q \to \ell^q$, given by

$$Ae_n = \lambda_n e_n$$

for $n \in \mathbb{N}$ and $D(A) = \{(x_n)_{n\in\mathbb{N}} \in \ell^q \mid (\lambda_n x_n)_{n\in\mathbb{N}} \in \ell^q\}$, generates an analytic, exponentially stable C_0-semigroup $(T(t))_{t\ge 0}$, which acts on the basis $(e_n)_{n\in\mathbb{N}}$ by $T(t)e_n = e^{t\lambda_n}e_n$. Since 0 belongs to $\rho(A)$ we have

$$\|x\|_{X_{-1}} = \|A^{-1}x\|_{\ell^q}$$

for $x \in \ell^q$. An easy computation shows that the extrapolation space $X_{-1} = (\ell^q)_{-1}$ is given by

$$(\ell^q)_{-1} = \{x = (x_n)_{n\in\mathbb{N}} \subset \mathbb{C} \mid (x_n/\lambda_n)_{n\in\mathbb{N}} \in \ell^q\}.$$

Every linear operator $B\colon \mathbb{C} \to (\ell^q)_{-1}$ is bounded and can be identified with an element from $(\ell^q)_{-1}$, that is, with a sequence $(b_n)_{n\in\mathbb{N}} \subset \mathbb{C}$ for which holds $(b_n/\lambda_n)_{n\in\mathbb{N}} \in \ell^q$. This is equivalent to $(b_n/\operatorname{Re}\lambda_n)_{n\in\mathbb{N}} \in \ell^q$ since the sequence $(\lambda_n)_{n\in\mathbb{N}}$ satisfies the sectoriality condition.
The following theorem shows that in the situation above the system $\Sigma(A,B)$ is integral input-to-state stable with respect to L^∞. This means in particular that under the assumptions above L^∞-iISS is equivalent to L^∞-ISS.

Theorem 4.4.3. *Let $U = \mathbb{C}$ and assume that the state space X possess a q-Riesz basis $(e_n)_{n\in\mathbb{N}}$, which consists of eigenvectors of A with eigenvalues $(\lambda_n)_{n\in\mathbb{N}}$ such that $(-\lambda_n)_{n\in\mathbb{N}} \subset S_\theta$ for some $\theta \in (0,\pi/2)$ and $\sup\{\operatorname{Re}\lambda_n \mid n\in\mathbb{N}\} < 0$. Further let $B \in \mathcal{L}(\mathbb{C}, X_{-1})$. Then the system $\Sigma(A,B)$ is L^∞-iISS.*

Proof. Without loss of generality we may assume that $X = \ell^q$ and $(e_n)_{n\in\mathbb{N}}$ is the standard basis of ℓ^q. Let the function $f\colon (0,1) \to \mathbb{R}$ be defined by

$$f(s) = \sum_{n\in\mathbb{N}} \frac{|b_n|^q}{|\operatorname{Re}\lambda_n|^{q-1}} e^{\operatorname{Re}\lambda_n s}.$$

Then f is nonnegative and belongs to $L^1(0,1)$. Indeed, for $n \in \mathbb{N}$ let $f_n\colon (0,1) \to \mathbb{R}$ be the function given by

$$f_n(s) = \frac{|b_n|^q}{|\operatorname{Re}\lambda_n|^{q-1}} e^{\operatorname{Re}\lambda_n s}.$$

Then each f_n, $n \in \mathbb{N}$, is continuous and, hence, measurable. Further we have

$$\int_0^1 f_n(s)\,ds = \frac{|b_n|^q}{|\operatorname{Re}\lambda_n|^q} e^{\operatorname{Re}\lambda_n} \le \frac{|b_n|^q}{|\operatorname{Re}\lambda_n|^q},$$

as each λ_n has negative real part. Hence the monotone convergence theorem yields

$$\int_0^1 f(s)\,ds \le \sum_{n\in\mathbb{N}} \frac{|b_n|^q}{|\operatorname{Re}\lambda_n|^q} < \infty,$$

since $B \in \mathcal{L}(\mathbb{C}, X_{-1})$ by assumption. Now let $u \in L^1(0,1)$ with $\int_0^1 f(s)|u(s)|^q\,ds < \infty$ and denote by q' the Hölder conjugate of q, i.e., $1/q + 1/q' = 1$. Then we obtain, using Hölder's inequality,

$$\begin{aligned}
\left\|\int_0^1 T_{-1}(s)Bu(s)\,ds\right\|_{\ell^q}^q &= \sum_{n\in\mathbb{N}} |b_n|^q \left|\int_0^1 e^{\lambda_n s}|u(s)|\,ds\right|^q \\
&\le \sum_{n\in\mathbb{N}} |b_n|^q \left(\int_0^1 e^{\operatorname{Re}\lambda_n s}|u(s)|\,ds\right)^q \\
&= \sum_{n\in\mathbb{N}} \frac{|b_n|^q}{|\operatorname{Re}\lambda_n|^q} \left(\int_0^1 |\operatorname{Re}\lambda_n| e^{\operatorname{Re}\lambda_n s}|u(s)|\,ds\right)^q
\end{aligned}$$

$$
\begin{aligned}
&\leq \sum_{n\in\mathbb{N}} \frac{|b_n|^q}{|\operatorname{Re}\lambda_n|^q} \left(\int_0^1 |\operatorname{Re}\lambda_n| e^{\operatorname{Re}\lambda_n s} |u(s)|^q \, ds\right) \left(\int_0^1 |\operatorname{Re}\lambda_n| e^{\operatorname{Re}\lambda_n s} \, ds\right)^{q/q'} \\
&\leq \sum_{n\in\mathbb{N}} \frac{|b_n|^q}{|\operatorname{Re}\lambda_n|^q} \int_0^1 |\operatorname{Re}\lambda_n| e^{\operatorname{Re}\lambda_n s} |u(s)|^q \, ds \\
&= \int_0^1 \sum_{n\in\mathbb{N}} \frac{|b_n|^q}{|\operatorname{Re}\lambda_n|^{q-1}} e^{\operatorname{Re}\lambda_n s} |u(s)|^q \, ds \\
&= \int_0^1 f(s) |u(s)|^q \, ds \\
&< \infty.
\end{aligned}
$$

This shows that the system $\Sigma(A,B)$ is L^∞-ISS and satisfies the estimate (4.11) (with $c = 0$, $\theta(s) = s^{1/q}$ and $\mu(s) = s^q$). Hence Proposition 4.3.7 implies that it is L^∞-iISS. □

Corollary 4.4.4. *In the situation of Theorem 4.4.3 the following are equivalent:*

(i) $\Sigma(A,B)$ *is* L^∞*-ISS.*

(ii) $\Sigma(A,B)$ *is* L^∞*-iISS.*

(iii) $\Sigma(A,B)$ *is* L^∞*-admissible.*

(iv) $\Sigma(A,B)$ *is* L^∞*-zero-class admissible.*

(v) $B \in X_{-1}$.

Remark 4.4.5. *In Theorem 4.4.3 and Corollary 4.4.4 we assumed that the input space U is one-dimensional. The result can actually be generalised to any finite-dimensional Banach space U, see Proposition 4 in [JSZ17].*

Recall that the *support* of a positive Borel measure μ on $\mathbb{R}^n$ is defined as the set

$$\operatorname{supp}(\mu) = \{x \in \mathbb{R}^n \mid \mu(U) > 0 \text{ for each neighbourhood } U \text{ of } x\} \subset \mathbb{R}^n.$$

It is a closed set and its complement $\mathbb{R}^n \setminus \operatorname{supp}(\mu)$ is a μ-null set.

Lemma 4.4.6. *Let μ be a positive regular Borel measure on $\mathbb{C}$ with $\operatorname{supp}(\mu) \subset S_\theta$ for some $\theta \in (0,\pi/2)$ and $1 \leq q < \infty$. Then the Laplace transform $\mathcal{L}\colon L^\infty(0,\infty) \to L^q(\mathbb{C}^+,\mu)$,*

$$\mathcal{L}(f)(s) = \int_0^\infty f(t) e^{-st} \, dt,$$

is bounded if and only if the function $s \mapsto 1/s$ belongs to $L^q(\mathbb{C}^+,\mu)$.

Proof. Assume first that the Laplace transform $\mathcal{L}\colon L^\infty(0,\infty) \to L^q(\mathbb{C}^+, \mu)$ is bounded. Then taking $f(t) = 1$ for $t \geq 0$ yields $(\mathcal{L}f)(s) = 1/s$ and the claim follows.
Conversely, let $f \in L^\infty(0,\infty)$ and $s \in \mathbb{C}^+$. Then we have

$$\left| \int_0^\infty f(t) e^{-st}\, dt \right| \leq \|f\|_\infty \int_0^\infty e^{-\operatorname{Re}(s)t}\, dt = \frac{\|f\|_\infty}{\operatorname{Re}(s)}.$$

Since the measure μ is supported in S_θ for some $\theta \in (0, \pi/2)$ and there exists a constant $M > 0$ such that $|s| \leq M \operatorname{Re}(s)$ for all $s \in S_\theta$, the claim follows. $\square$

Theorem 4.4.7. *Suppose X possesses a q-Riesz basis $(e_n)_{n\in\mathbb{N}}$ consisting of eigenvectors of A with eigenvalues $(\lambda_n)_{n\in\mathbb{N}}$ such that $(-\lambda_n)_{n\in\mathbb{N}} \subset S_\theta$ for some $\theta \in (0, \pi/2)$ and $B = (b_n)_{n\in\mathbb{N}} \in X_{-1}$. Then the following are equivalent:*

(i) $\Sigma(A, B)$ *is infinite-time* L^∞*-admissible.*

(ii) $\sup_{\lambda\in\mathbb{C}^+} \|(\lambda I - A)^{-1} B\| < \infty$.

(iii) The function $s \mapsto 1/s$ *belongs to* $L^q(\mathbb{C}^+, \mu)$*, where* μ *is the measure* $\sum_{k\in\mathbb{N}} |b_k|^q \delta_{-\lambda_k}$*.*

Proof. Theorem 2.1 in [JPP14] applied to $Z = L^\infty(0,\infty)$ yields that the admissibility is equivalent to the boundedness of the Laplace transform $\mathcal{L}\colon L^\infty(0,\infty) \to L^q(\mathbb{C}^+, \mu)$. Therefore (i) and (iii) are equivalent by Lemma 4.4.6.
We have

$$\|(\lambda I - A)^{-1} B\|^q = \sum_{k\in\mathbb{N}} \frac{|b_k|^q}{|\lambda - \lambda_k|^q}.$$

Assume that (ii) holds. Then, letting $\lambda \to 0$, we have $(b_k/\lambda_k)_{k\in\mathbb{N}} \in \ell^q$ and hence (iii) holds. Conversely, if (iii) holds we have $(b_k/\lambda_k)_{k\in\mathbb{N}} \in \ell^q$ and hence, by sectoriality, $(b_k/\operatorname{Re}\lambda_k)_{k\in\mathbb{N}} \in \ell^q$. Since for all $k \in \mathbb{N}$ and $\lambda \in \mathbb{C}^+$ holds $|\operatorname{Re}\lambda_k| \leq |\lambda - \lambda_k|$ we conclude that

$$\sum_{k\in\mathbb{N}} \frac{|b_k|^q}{|\lambda - \lambda_k|^q} \leq \sum_{k\in\mathbb{N}} \frac{|b_k|^q}{|\operatorname{Re}\lambda_k|^q}$$

and therefore $\sup_{\lambda\in\mathbb{C}^+} \|(\lambda I - A)^{-1} B\| < \infty$. $\square$

4.5 Examples

In this section we discuss stability notions on examples of systems, which admit a diagonal representation. For $n \in \mathbb{Z}$ we denote by Q_n the following strip in the complex plane:

$$Q_n = \{ z \in \mathbb{C} \mid 2^{n-1} < \operatorname{Re} z \leq 2^n \}.$$

We need the following characterisation of L^p-admissibility from [JPP14].

Theorem 4.5.1. *Let $1 \leq q < \infty$ and suppose $A\colon D(A) \subset \ell^q \to \ell^q$ is a diagonal operator with eigenvalues $(\lambda_n)_{n\in\mathbb{N}}$ such that $(-\lambda_n)_{n\in\mathbb{N}} \subset S_\theta$ for some $\theta \in (0, \pi/2)$. Let $B \in \mathcal{L}(\mathbb{C}, (\ell^q)_{-1})$ be given by the sequence $(b_n)_{n\in\mathbb{N}}$. Then for any $p \in (q, \infty)$ the following are equivalent:*

(i) $\Sigma(A, B)$ is L^p-admissible.

(ii) The sequence $(2^{-nq/p'}\mu(Q_n))_{n\in\mathbb{Z}}$ belongs to $\ell^{p/(p-q)}(\mathbb{Z})$.

Here p' denotes the Hölder conjugate of p, i.e., $p' = p/(p-1)$ and μ is the measure $\sum_{k\in\mathbb{N}}|b_k|^q\delta_{-\lambda_k}$.

Example 4.5.2. *We consider again the boundary control system, as studied in Example 2.3.2, given by the one-dimensional heat equation on the spatial domain $(0, 1)$ with Dirichlet boundary control at the boundary point 1:*

$$\begin{aligned}
\frac{\partial x}{\partial t}(\xi, t) &= \frac{\partial^2 x}{\partial \xi^2}(\xi, t), \quad \xi \in (0, 1),\ t > 0,\\
x(0, t) &= 0, \quad x(1, t) = u(t), \quad t > 0,\\
x(\xi, 0) &= x_0(\xi).
\end{aligned}$$

We saw in Example 2.3.2 that this system can be written as $\Sigma(A, B)$. The state space here is $X = L^2(0, 1)$ and

$$Af = f''$$

for $f \in D(A)$, where

$$D(A) = \{f \in H^2(0, 1) \mid f(0) = f(1) = 0\}.$$

The input space is $U = \mathbb{C}$. The eigenvalues of A are given by

$$\lambda_n = -\pi^2 n^2$$

for $n \in \mathbb{N} \setminus \{0\}$ and the eigenfunctions $e_n\colon [0, 1] \to \mathbb{C}$ are

$$e_n(t) = \sqrt{2}\sin(n\pi t)$$

for $n \in \mathbb{N} \setminus \{0\}$. The sequence $(e_n)_{n\in\mathbb{N}} \subset L^2(0, 1)$ forms an orthonormal basis of $L^2(0, 1)$. With respect to this basis, the operator $B = \delta_1'$ can be identified with the sequence $(b_n)_{n\in\mathbb{N}} \subset \mathbb{C}$ for

$$b_n = \langle \delta_1', e_n\rangle = -\langle \delta_1, e_n'\rangle = -e_n'(1) = (-1)^n\sqrt{2}n\pi.$$

Hence we have

$$\sum_{n=1}^{\infty} \frac{|b_n|^2}{|\lambda_n|^2} = \frac{2}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{1}{3} < \infty$$

and therefore $B \in (\ell^2)_{-1}$. Hence, Theorem 4.4.3 yields that the system $\Sigma(A, B)$ is L^∞-iISS. Further we have the following L^∞-ISS estimate

$$\|x(t)\|_{L^2(0,1)} \leq e^{-\pi^2 t}\|x_0\|_{L^2(0,1)} + \frac{1}{\sqrt{3}}\|u\|_{L^\infty(0,t)}$$

for all $x_0 \in L^2(0,1)$, $t \geq 0$ and $u \in L^\infty(0,t)$. Using Theorem 4.5.1 we obtain that $\Sigma(A,B)$ is even L^p-admissible for any $p > 2$. Indeed, for any $n \in \mathbb{N}$ we have $\mu(Q_{-n}) = 0$ and $\mu(Q_n) = \mathrm{O}(n^3)$. Hence for any $n \in \mathbb{N}$ and $p > 2$ there holds

$$\left(2^{-2n(p-1)/p}\mu(Q_n)\right)^{p/(p-2)} = 2^{-2n(p-1)/(p-2)}\,\mathrm{O}(n^{3p/(p-2)}),$$

which shows that the sequence $(2^{-2n(p-1)/p}\mu(Q_n))_{n\in\mathbb{Z}}$ belongs to $\ell^{p/(p-2)}(\mathbb{Z})$. Therefore the following L^∞-iISS estimate holds true:

$$\|x(t)\|_{L^2(0,1)} \leq e^{-\pi^2 t}\|x_0\|_{L^2(0,1)} + c\left(\int_0^t |u(s)|^p\,ds\right)^{1/p}$$

for all $x_0 \in L^2(0,1)$, $t \geq 0$ and $u \in L^\infty(0,t)$, where the constant $c = c(p) > 0$ only depends on p.

In the previous example the system $\Sigma(A,B)$ is not only admissible with respect to L^∞ but even L^p-admissible for all $p > 2$. The following example provides a system $\Sigma(A,B)$, which is L^∞-admissible but not L^p-admissible for any $p < \infty$.

Example 4.5.3. *Let $X = \ell^2$. We consider again a parabolic diagonal system $\Sigma(A,B)$ as in Section 4.4. Let us choose $\lambda_n = -2^n$, $n \in \mathbb{N}$, and $b_n = 2^n/n$ for $n \in \mathbb{N}\setminus\{0\}$, $b_0 = 0$. Then we have $b_n/\lambda_n = -1/n$ for all $n \geq 1$ and hence $(b_n/\lambda_n)_{n\in\mathbb{N}} \in \ell^2$. This means that $(b_n)_{n\in\mathbb{N}} \in (\ell^2)_{-1}$, i.e., $B = (b_n)_{n\in\mathbb{N}}$ is an L^∞-admissible control operator. For any $n \in \mathbb{N}\setminus\{0\}$ we have*

$$\mu(Q_n) = \sum_{k\in\mathbb{N}} |b_k|^2 \delta_{-\lambda_k} = \frac{2^{2n}}{n^2}$$

and hence

$$2^{-2n(p-1)/p}\mu(Q_n) = \frac{2^{2n/p}}{n^2}.$$

Thus for any $p > 2$ holds

$$\left(2^{-2n(p-1)/p}\mu(Q_n)\right)^{p/(p-2)} = \frac{2^{2n/(p-2)}}{n^{2p/(p-2)}},$$

which shows

$$\left(2^{-2n(p-1)/p}\mu(Q_n)\right)_{n\in\mathbb{Z}} \notin \ell^{p/(p-2)}.$$

Therefore, by Theorem 4.5.1, the system $\Sigma(A,B)$ is not L^p-admissible for any $p \in (2,\infty)$. Thus, by Remark 4.1.2, it is not L^p-admissible for any $p \in [1,\infty)$. Since $B \in X_{-1}$, Theorem 4.4.3 shows that $\Sigma(A,B)$ is integral ISS with respect to L^∞. Hence, Proposition 4.2.9 shows that it is zero-class admissible with respect to L^∞. By Proposition 2.4.1 the mild solutions are continuous for all $x_0 \in \ell^p$ and $u \in L^\infty(0,\infty)$. By Theorem 4.3.2 there exists a Young function Φ such that $\Sigma(A,B)$ is E_Φ-admissible. By Remark 4.3.3 the Young function Φ cannot satisfy the Δ_2-condition.

4.6 Concluding remarks

In this chapter we studied the notions of input-to-state stability and integral input-to-state stability for infinite-dimensional linear systems as well as the connections between them. We saw that the well-known results concerning the equivalence of ISS and iISS with respect to L^p, with $p < \infty$, admit a generalisation to inputs from any Orlicz space, where the generating Young function satisfies the Δ_2-condition.
Further we have seen that integral input-to-state stability with respect to L^∞ is equivalent to input-to-state stability with respect to some Orlicz space. Since Orlicz spaces on bounded intervals contain L^∞ as a subspace, we conclude that L^∞-iISS is stronger than L^∞-ISS, at least formally. It remains an open question whether or not those conditions are actually equivalent. In the situation of parabolic diagonal systems, those notions are indeed equivalent if the input space is finite-dimensional. More recently B. Jacob, F. Schwenninger and H. Zwart showed, using holomorphic functional calculus, that the equivalence also holds for broader class of linear systems, namely for analytic semigroups on Hilbert spaces, which are equivalent to a contraction semigroup, see [JSZ17].
Other possible questions, which can be addressed in future research, are nonlinear systems, nonanalytic diagonal systems as well as Lyapunov theory for ISS of linear systems.

Chapter 5

Stabilizability of linear systems

In this chapter we continue studying linear systems $\Sigma(A, B)$ given by

$$\dot{x}(t) = Ax(t) + Bu(t), \quad t \geq 0, \quad x(0) = x_0, \tag{5.1}$$

with the restriction that X and U are now Hilbert spaces. The operator A generates a C_0-semigroup $(T(t))_{t\geq 0}$ on X and B is a linear and bounded map from U to the extrapolation space X_{-1}. Recall that for $u \in L^1_{\mathrm{loc}}(0, \infty; U)$ the mild solution of (5.1) is defined by

$$x(t) = T(t)x_0 + \int_0^t T_{-1}(t-s)Bu(s)\, ds \tag{5.2}$$

for $t \geq 0$, where the semigroup $(T_{-1}(t))_{t\geq 0}$ is the extension of $(T(t))_{t\geq 0}$ to X_{-1}, see Section 2.1. For $\lambda \in \mathbb{R}$ let $\mathbb{C}^+_\lambda$ be the open right half-plane

$$\mathbb{C}^+_\lambda = \{z \in \mathbb{C} \mid \operatorname{Re} z > \lambda\}$$

and $\mathbb{C}^-_\lambda$ the open left half-plane

$$\mathbb{C}^-_\lambda = \{z \in \mathbb{C} \mid \operatorname{Re} z < \lambda\}.$$

5.1 Stabilizability of finite-dimensional linear systems

In this section we recall some well-known results concerning controllability and stabilizability of finite-dimensional linear systems. Thus, we consider linear systems

$$\dot{x}(t) = Ax(t) + Bu(t), \quad t \geq 0, \quad x(0) = x_0, \tag{5.3}$$

on a finite-dimensional state space X. The input space U is also assumed to be finite-dimensional. By choosing a basis we can assume without loss of generality that $X = \mathbb{K}^n$ and

$U = \mathbb{K}^m$, both equipped with the usual Euclidean norm. In this situation A and B are matrices, $A \in \mathbb{K}^{n\times n}$, $B \in \mathbb{K}^{n\times m}$, the initial value x_0 is a vector, $x_0 \in \mathbb{K}^n$, and $u \in L^1_{\mathrm{loc}}(0,\infty;\mathbb{K}^m)$. The semigroup generated by A has an explicit representation by its matrix exponential, $(e^{tA})_{t\geq 0}$, where

$$e^{tA} = \exp(tA) = \sum_{k=0}^{\infty} \frac{(tA)^k}{k!}$$

for $t \geq 0$. Thus the mild solution reads in this situation as

$$x(t) = e^{tA}x_0 + \int_0^t e^{(t-s)A}Bu(s)\,ds \tag{5.4}$$

for $t \geq 0$.

Definition 5.1.1. *We call the system* $\Sigma(A,B)$

(a) controllable *if for every* $x_0, x_1 \in \mathbb{K}^n$ *there exists a* $t_1 > 0$ *and a function* $u \in L^1(0,t_1;\mathbb{K}^m)$ *such that the mild solution of* (5.3)*, given by* (5.4)*, satisfies* $x(t_1) = x_1$*,*

(b) controllable in time t_1 *(for a fixed* $t_1 > 0$*) if for every* $x_0, x_1 \in \mathbb{K}^n$ *there exists a function* $u \in L^1(0,t_1;\mathbb{K}^m)$ *such that the mild solution of* (5.3)*, given by* (5.4)*, satisfies* $x(t_1) = x_1$*,*

(c) reachable *if for every* $x_1 \in \mathbb{K}^n$ *there exists a* $t_1 > 0$ *and a function* $u \in L^1(0,t_1;\mathbb{K}^m)$ *such that the mild solution of* (5.3) *with* $x_0 = 0$*, given by* (5.4)*, satisfies* $x(t_1) = x_1$*.*

Definition 5.1.2. *Let* $A \in \mathbb{K}^{n\times n}$ *and* $B \in \mathbb{K}^{n\times m}$*. We define the* controllability matrix $R(A,B)$ *by*

$$R(A,B) = [B, AB, \ldots, A^{n-1}B].$$

It is clear that controllability in time t_1, for some $t_1 > 0$, implies controllability and the latter implies reachability. For finite-dimensional systems we have the following characterisation of controllability.

Theorem 5.1.3. *The following statements are equivalent:*

(i) The system $\Sigma(A,B)$ *is controllable.*

(ii) For every $t_1 > 0$ *the system* $\Sigma(A,B)$ *is controllable in time* t_1*.*

(iii) The system $\Sigma(A,B)$ *is reachable.*

(iv) $\operatorname{rk} R(A,B) = n$*.*

In particular, if the system $\Sigma(A,B)$ is controllable, then it is controllable in arbitrarily small time. The proof can be found in [JZ12], see Theorem 3.1.6 there. This result is no longer true for infinite-dimensional systems.
For a semigroup $(e^{tA})_{t\geq 0}$ on a finite-dimensional space the notions of strong and exponential stability are equivalent. Hence in this situation we call a semigroup simply *stable*.

Definition 5.1.4. *A system* $\Sigma(A,B)$ *is called* stabilizable *if for every* $x_0 \in \mathbb{K}^n$ *there exists a function* $u \in L^1_{\text{loc}}(0,\infty;\mathbb{K}^m)$ *such that* $\lim_{t\to\infty} x(t) = 0$*, where* x *is the unique mild solution of* (5.3)*, given by* (5.4)*.*

It can be shown that if the system $\Sigma(A,B)$ is stabilizable, then the stabilizing control function u can be obtained via a feedback law $u(t) = Fx(t)$ with some $F \in \mathbb{K}^{m\times n}$, see e.g. [JZ12, Sec. 4.3]. The equation (5.3) then becomes

$$\dot{x}(t) = (A+BF)x(t), \quad t \geq 0, \quad x(0) = x_0.$$

Thus the question of stabilizability of a system is related to the so-called *pole placement problem* for $A \in \mathbb{K}^{n\times n}$, $B \in \mathbb{K}^{n\times m}$, which is the following: Given $\lambda_1,\ldots,\lambda_n \in \mathbb{K}$ does there exist a matrix $F \in \mathbb{K}^{m\times n}$ such that $\lambda_1,\ldots,\lambda_n$ are the eigenvalues of the matrix $A+BF$?

Theorem 5.1.5. *Let* $A \in \mathbb{C}^{n\times n}$ *and* $B \in \mathbb{C}^{n\times m}$*. Then the system* $\Sigma(A,B)$ *is controllable if and only if the pole placement problem is solvable.*

The proof can be found in [JZ12], see Corollary 4.2.6 there. This means in particular that if the system $\Sigma(A,B)$ is controllable, then there exists a matrix $F \in \mathbb{K}^{m\times n}$ such that $A+BF$ is a *Hurwitz matrix*, i.e., $\sigma(A+BF) \subset \mathbb{C}_0^-$. The latter condition is obviously weaker than the solvability of the pole placement problem. If we only want to stabilize the system, then this condition is also sufficient.

Theorem 5.1.6. *Let* $A \in \mathbb{C}^{n\times n}$ *and* $B \in \mathbb{C}^{n\times m}$*. Then the system* $\Sigma(A,B)$ *is stabilizable if and only if there exists a matrix* $F \in \mathbb{K}^{m\times n}$ *such that* $A+BF$ *is a Hurwitz matrix.*

This means in particular that every stabilizable system can be stabilized by an input of the form $u(t) = Fx(t)$.

Definition 5.1.7. *Let* $A_1, A_2 \in \mathbb{K}^{n\times n}$ *and* $B_1, B_2 \in \mathbb{K}^{n\times m}$*. The systems* $\Sigma(A_1,B_1)$ *and* $\Sigma(A_2,B_2)$ *are called* similar *if there exists an invertible matrix* $T \in \mathbb{K}^{n\times n}$ *such that* $A_2 = T^{-1}A_1T$ *and* $B_2 = T^{-1}B_1$*.*

We conclude this section on finite-dimensional systems by presenting a condition, which characterises stabilizability. The proof can be found in [JZ12], see Theorem 4.3.3 there.

Theorem 5.1.8. *Let* $A \in \mathbb{C}^{n\times n}$ *and* $B \in \mathbb{C}^{n\times m}$*. Then the following statements are equivalent:*

(i) The system $\Sigma(A,B)$ *is stabilizable.*

(ii) There exist two A*-invariant subspaces* X_s *and* X_u *of* $\mathbb{C}^n$ *such that the following properties hold:*

   (a) $\mathbb{C}^n = X_s \oplus X_u$*.*

(b) The system $\Sigma(A, B)$ *is similar to*

$$\Sigma\left(\begin{pmatrix} A_s & 0 \\ 0 & A_u \end{pmatrix}, \begin{pmatrix} B_s \\ B_u \end{pmatrix}\right).$$

(c) The matrix A_s *is a Hurwitz matrix.*

(d) The system $\Sigma(A_u, B_u)$ *is controllable.*

5.2 Spectral projections

We return to the infinite-dimensional setting. In this section X is a Hilbert space and $(T(t))_{t\geq 0}$ is a C_0-semigroup on X with the generator A.

Definition 5.2.1. *A subspace* V *of* X *is called* $T(t)$-invariant *if* $T(t)V \subset V$ *holds for all* $t \geq 0$.

The $T(t)$-invariance of a subspace $V \subset X$ is equivalent to the fact that the solution of the homogeneous initial value problem $\dot{x}(t) = Ax(t)$, $x(0) = x_0$, stays in V if the initial value x_0 belongs to V.

Definition 5.2.2. *A subspace* V *of* X *is called* A-invariant *if* $A(V \cap D(A)) \subset V$.

It is not difficult to see that the $T(t)$-invariance of a subspace V implies the A-invariance of the same subspace.

Definition 5.2.3. *Let* A *be a closed densely defined operator on* X*. Assume there exists an isolated subset* σ^+ *of* $\sigma(A)$*, the spectrum of* A*. More precisely there exists a rectifiable, closed, simple curve* Γ*, which encloses an open set containing* σ^+ *in its interior and* $\sigma^- := \sigma(A) \setminus \sigma^+$ *in its exterior. The operator* $P_\Gamma \colon X \to X$*, defined by*

$$P_\Gamma x = \int_\Gamma (\lambda I - A)^{-1} x \, d\lambda, \tag{5.5}$$

where Γ *is traversed once in the positive direction, is called the* spectral projection *on* σ^+.

Definition 5.2.4. *Let* $\lambda_0 \in \sigma(A)$ *an isolated eigenvalue. We say that* λ_0 *has* order ν_0 *if* $\lim_{\lambda\to\lambda_0}(\lambda - \lambda_0)^{\nu_0}(\lambda I - A)^{-1}x$ *exists for every* $x \in X$ *and there exists an* $x_0 \in X$ *such that* $\lim_{\lambda\to\lambda_0}(\lambda - \lambda_0)^{\nu_0 - 1}(\lambda I - A)^{-1}x_0$ *does not exist. We say that the order of* λ_0 *is infinity if for all* $\nu \in \mathbb{N}$ *there exists an* $x_\nu \in X$ *such that the limit* $\lim_{\lambda\to\lambda_0}(\lambda - \lambda_0)^{\nu}(\lambda I - A)^{-1}x_\nu$ *does not exist.*

We summarise the main properties of the operator P_Γ in the following theorem. The proofs of the next two theorems can be found in [JZ12, Chapter 8].

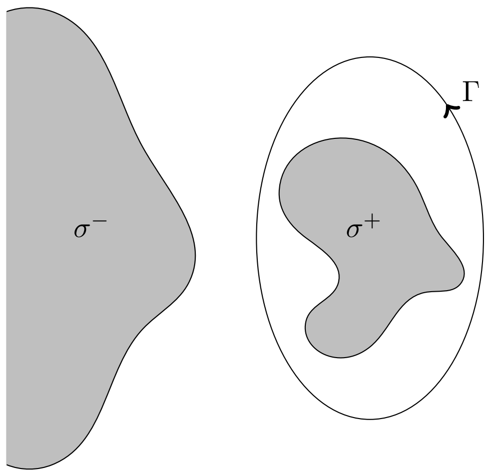

Figure 5.1: Spectral decomposition

Theorem 5.2.5. *The spectral projection P_Γ induces a decomposition of the state space*

$$X = X^+ \oplus X^-, \tag{5.6}$$

where $X^+ = \operatorname{ran} P_\Gamma$ and $X^- = \ker P_\Gamma = \operatorname{ran}(I - P_\Gamma)$. Moreover, the following properties hold:

(a) For all $x \in D(A)$ we have $P_\Gamma A x = A P_\Gamma x$ and for all $\lambda \in \rho(A)$ holds $(\lambda I - A)^{-1} P_\Gamma = P_\Gamma(\lambda I - A)^{-1}$.

(b) The spaces X^+ and X^- are A-invariant and $(\lambda I - A)^{-1}$-invariant for all $\lambda \in \rho(A)$.

(c) $P_\Gamma X \subset D(A)$ and $A^+ := A|_{X^+} \in \mathcal{L}(X^+)$.

(d) $\sigma(A^\pm) = \sigma^\pm$, where $A^- := A|_{X^-}$. Furthermore, for $\lambda \in \rho(A)$ we have that $(\lambda I - A^\pm)^{-1} = (\lambda I - A)^{-1}|_{X^\pm}$.

(e) If σ^+ is finite, $\sigma^+ = \{\lambda_1, \dots, \lambda_n\}$, and each $\lambda_k \in \sigma^+$ has a finite order ν_k, then P_Γ projects onto the space of generalised eigenvectors of the enclosed eigenvalues. Thus we have that

$$\operatorname{ran} P_\Gamma = \sum_{k=1}^{n} \ker(\lambda_k I - A)^{\nu_k} = \sum_{k=1}^{n} \ker(\lambda_k I - A^+)^{\nu_k}.$$

(f) If $\sigma^+ = \{\lambda\}$ and λ is an eigenvalue of multiplicity 1, then

$$P_\Gamma x = \langle x, y \rangle z,$$

where y is the eigenvector of A corresponding to λ and z is an eigenvector of A^* corresponding to $\bar{\lambda}$ with $\langle y, z \rangle = 1$.

Theorem 5.2.6. *Assume that A is the infinitesimal generator of the C_0-semigroup $(T(t))_{t\geq 0}$ and its spectrum is the union of two parts, σ^+ and σ^- as in Theorem 5.2.5. Then X^+ and X^- are $T(t)$-invariant and $(T^+(t))_{t\geq 0}$, $(T^-(t))_{t\geq 0}$, with $T^\pm(t) := T(t)|_{X^\pm}$, define C_0-semigroups on X^+ and X^-, respectively. The infinitesimal generator of $(T^+(t))_{t\geq 0}$ is A^+ and the infinitesimal generator of $(T^-(t))_{t\geq 0}$ is A^-.*

5.3 Exponential stabilizability

In this section we recall some well-known results about exponential stabilizability of infinite-dimensional linear systems. They are a starting point for our study of strong and polynomial stabilizability. We thus consider again linear systems $\Sigma(A, B)$ given by (5.1) on a Hilbert space X. Here A is the generator of a C_0-semigroup $(T(t))_{t\geq 0}$ on X and B is bounded, i.e., $B \in \mathcal{L}(U, X)$.

Definition 5.3.1. *The system $\Sigma(A, B)$ given by* (5.1) *with a bounded control operator B, i.e., $B \in \mathcal{L}(U, X)$, is called* exponentially stabilizable *if there exists an $F \in \mathcal{L}(X, U)$ such that $A + BF$ generates an exponentially stable C_0-semigroup $(T_{BF}(t))_{t\geq 0}$.*

From Theorem 5.1.6 follows that for finite-dimensional systems this definition is equivalent to Definition 5.1.4. If the input space U is finite-dimensional, i.e., $U = \mathbb{C}^m$, then there is a complete characterisation of all systems $\Sigma(A, B)$, which are exponentially stabilizable. We denote by σ^+ the set $\sigma(A) \cap \overline{\mathbb{C}_0^+}$ and by σ^- the set $\sigma(A) \cap \overline{\mathbb{C}_0^-}$.

Definition 5.3.2. *We say that the operator A satisfies the* spectrum decomposition assumption at zero *if there exists a rectifiable, closed, simple curve Γ, which encloses an open set containing σ^+ in its interior and σ^- in its exterior.*

If the spectrum decomposition assumption at zero holds, then, by Theorem 5.2.5, the spectral projection $P_\Gamma\colon X \to X$, given by (5.5), induces a decomposition of the state space X, given by (5.6). We have $B^+ := P_\Gamma B \in \mathcal{L}(U, X^+)$ and $B^- := (I - P_\Gamma)B \in \mathcal{L}(U, X^-)$. Thus, by Theorems 5.2.5 and 5.2.6, we obtain a decomposition of the system $\Sigma(A, B)$ in two subsystems: $\Sigma(A^+, B^+)$ on X^+ and $\Sigma(A^-, B^-)$ on X^-. The following characterisation of stabilizability was obtained by W. Desch and W. Schappacher [DS85], C. A. Jacobson and C. N. Nett [JN88], and S. A. Nefedov and F. A. Sholokhovich [NS86].

Theorem 5.3.3. *For any linear system $\Sigma(A, B)$ given by* (5.1) *with a finite-dimensional input space $U = \mathbb{C}^m$ and a bounded control operator, i.e., $B \in \mathcal{L}(\mathbb{C}^m, X)$, the following assertions are equivalent:*

(i) *$\Sigma(A, B)$ is exponentially stabilizable.*

(ii) *The operator A satisfies the spectrum decomposition assumption at zero, X^+ is finite-dimensional, the semigroup $(T^-(t))_{t\geq 0}$ is exponentially stable and the finite-dimensional system $\Sigma(A^+, B^+)$ is controllable.*

Remark 5.3.4. *Theorem 5.3.3 characterises exponentially stabilizable systems with a finite-dimensional input space and a bounded control operator. Later, a similar characterisation of optimizability of linear systems with admissible control operators was found by B. Jacob and H. Zwart, see [JZ99]. There it is shown that under some additional technical conditions a system is optimizable if and only if it admits a decomposition into two subsystems: an exponentially stable system and an unstable system, which is exactly controllable in finite time.*

5.4 Regular linear systems

In this section we settle the framework for stabilizability questions. We first recall the definitions and some basic properties of Hardy spaces. More information as well as proofs of the statements we mention here can be found in [RR97] and [Dur70]. Then we introduce the so-called regular linear systems – a class of infinite-dimensional linear systems – mainly following the presentation in [Wei94a] and [Wei94b], see also [Wei89b] and [Wei89c].

Definition 5.4.1. *For Banach spaces X, W and $\lambda \in \mathbb{R}$ we define the following* Hardy spaces*:*

$$H^2(X) := \left\{ f \colon \mathbb{C}_0^+ \to X \mid f \text{ is holomorphic and } \sup_{x>0} \int_{\mathbb{R}} \|f(x+iy)\|^2 \, dy < \infty \right\}$$

and

$$H_\lambda^\infty(W) := \left\{ G \colon \mathbb{C}_\lambda^+ \to W \mid G \text{ is holomorphic and } \sup_{s \in \mathbb{C}_\lambda^+} \|G(s)\| < \infty \right\}.$$

The space $H_\lambda^\infty(W)$ is a Banach space with the norm

$$\|G\|_{H_\lambda^\infty} := \|G\|_\infty = \sup_{s \in \mathbb{C}_\lambda^+} \|G(s)\|_W$$

for $G \in H_\lambda^\infty(W)$. The space $H^2(X)$ is a Banach space with the norm

$$\|f\|_{H^2} := \|f\|_2 := \left(\sup_{x>0} \frac{1}{2\pi} \int_{\mathbb{R}} \|f(x+iy)\|^2 \, dy \right)^{1/2}$$

for $f \in H^2(X)$. If X is a separable Hilbert space, then $H^2(X)$ is a Hilbert space with the inner product

$$\langle f, g \rangle_{H^2} := \frac{1}{2\pi} \int_{\mathbb{R}} \langle \tilde{f}(ix), \tilde{g}(ix) \rangle \, dx$$

for $f, g \in H^2(X)$. Here for $f \in H^2(X)$ the function $\tilde{f}$ is the unique element in $L^2(i\mathbb{R}; X)$ such that

$$\lim_{x \searrow 0} f(x+iy) = \tilde{f}(iy)$$

for almost all $y \in \mathbb{R}$ and

$$\lim_{x \searrow 0} \|f(x + \cdot) - \tilde{f}\|_{L^2(i\mathbb{R};X)} = 0,$$

see Theorem 6.5.1 in [Kaw72]. For a function $f \in L^2(0,\infty;X)$ its *Laplace transform* is defined as

$$\mathcal{L}(f)(s) := \hat{f}(s) := \int_0^\infty e^{-st} f(t)\,dt$$

for $s \in \mathbb{C}_0^+$. If X is a Hilbert space, then, by the Paley-Wiener theorem, the Laplace transform is an isometric isomorphism from $L^2(0,\infty;X)$ to $H^2(X)$, see Theorem 1.8.3 in [ABHN11]. Let

$$H_\infty^\infty(W) := \left(\bigcup_{\lambda \in \mathbb{R}} H_\lambda^\infty(W) \right) \Big/ \sim,$$

where the equivalence relation $\sim$ is defined as follows: two functions in $\bigcup_{\lambda\in\mathbb{R}} H_\lambda^\infty(W)$ are equivalent if one of them is a restriction of the other. The set $H_\infty^\infty(W)$ has a natural vector space structure. For any $\lambda \in \mathbb{R}$ we have the embedding $H_\lambda^\infty(W) \hookrightarrow H_\infty^\infty(W)$, with $u \mapsto [u]$, that is, a function u is mapped to its equivalence class in $H_\infty^\infty(W)$. Hence the space $H_\lambda^\infty(W)$, identifying it with its image under the embedding, is a subspace of $H_\infty^\infty(W)$. Furthermore, we have for any $\lambda, \mu \in \mathbb{R}$, with $\lambda \le \mu$, the following inclusions

$$H_\lambda^\infty(W) \subset H_\mu^\infty(W) \subset H_\infty^\infty(W).$$

Definition 5.4.2. *For any $G \in H_\infty^\infty(W)$ its* growth bound*, denoted by $\gamma(G)$, is defined as*

$$\gamma(G) = \inf\{\lambda \in \mathbb{R} \mid G \in H_\lambda^\infty(W)\}.$$

Definition 5.4.3. *Let U, Y be Hilbert spaces. A linear map $\mathcal{F}\colon L^2_{\text{loc}}(0,\infty;U) \to L^2_{\text{loc}}(0,\infty;Y)$ is called* shift-invariant *if it commutes with every right-shift, i.e., $S_t^r \mathcal{F} = \mathcal{F} S_t^r$ for all $t > 0$.*

For $\lambda \in \mathbb{R}$ the space $L^2(0,\infty;W,e^{-2\lambda t}\,dt)$ is denoted by $L^2_\lambda(W)$.

Definition 5.4.4. *Let $\mathcal{F}\colon L^2_{\text{loc}}(0,\infty;U) \to L^2_{\text{loc}}(0,\infty;Y)$ be a shift-invariant linear operator. Then its* growth bound*, denoted by $\gamma(\mathcal{F})$, is defined as*

$$\gamma(\mathcal{F}) = \inf\{\lambda \in \mathbb{R} \mid \mathcal{F} \in \mathcal{L}(L^2_\lambda)\}.$$

The next theorem states that all shift-invariant operators with growth bound $\gamma(\mathcal{F}) < \infty$ have a representation in terms of Laplace transforms.

Theorem 5.4.5 (Thm. 3.1 in [Wei94b])**.** *Let U, Y be Hilbert spaces. Suppose $\mathcal{F}\colon L^2_{\text{loc}}(0,\infty;U) \to L^2_{\text{loc}}(0,\infty;Y)$ is a shift-invariant linear operator with growth bound $\gamma(\mathcal{F}) < \infty$. Then there is a unique $H \in H_\infty^\infty(\mathcal{L}(U,Y))$, which satisfies the following:*

$$\gamma(H) = \gamma(\mathcal{F}) \tag{5.7}$$

and, for any $\lambda > \gamma(\mathcal{F})$ and any $u \in L^2_\lambda(U)$,

$$(\widehat{\mathcal{F}u})(s) = H(s)\hat{u}(s) \tag{5.8}$$

for all $s \in \mathbb{C}^+_\lambda$. Moreover, we have

$$\|H\|_{H^\infty_\lambda} = \|\mathcal{F}\|_{\mathcal{L}(L^2_\lambda)}. \tag{5.9}$$

Conversely, suppose $H \in H^\infty_\infty(\mathcal{L}(U,Y))$. Then there is a unique shift-invariant linear operator $\mathcal{F}\colon L^2_{\mathrm{loc}}(0,\infty;U) \to L^2_{\mathrm{loc}}(0,\infty;Y)$, which satisfies the following: (5.7) *holds and, for any $\lambda > \gamma(H)$ and any $u \in L^2_\lambda(U)$,* (5.8) *and* (5.9) *hold.*

Definition 5.4.6. *Let U, Y be Hilbert spaces. A* well-posed transfer function *from U to Y is an element of $H^\infty_\infty(\mathcal{L}(U,Y))$.*

Definition 5.4.7. *Let U, Y be Hilbert spaces. Suppose H is a well-posed transfer function from U to Y and let $K \in \mathcal{L}(Y,U)$. Then K is an* admissible feedback operator *for H if the equation*

$$H^K - H = HKH^K \tag{5.10}$$

has a unique solution $H^K \in H^\infty_\infty(\mathcal{L}(U,Y))$. H^K is called the closed-loop transfer function *corresponding to H and K.*

We will use the following characterisation of the admissibility of K, see [Wei94a].

Proposition 5.4.8. *Let U, Y and H be as in Definition 5.4.7 and $K \in \mathcal{L}(Y,U)$. Then the following are equivalent:*

(i) $I - KH$ is invertible in $H^\infty_\infty(\mathcal{L}(U))$.

(ii) $I - HK$ is invertible in $H^\infty_\infty(\mathcal{L}(Y))$.

(iii) K is an admissible feedback operator for H.

Definition 5.4.9. *Let U, Y be Hilbert spaces, $v \in U$ and $\mathcal{F}\colon L^2_{\mathrm{loc}}(0,\infty;U) \to L^2_{\mathrm{loc}}(0,\infty;Y)$ a shift-invariant linear operator. The function*

$$y_v = \mathcal{F}(\chi_{[0,\infty)}v)$$

is called the step response *of $\mathcal{F}$ corresponding to v.*

Definition 5.4.10. *Let U, Y be Hilbert spaces and assume $\mathcal{F}\colon L^2_{\mathrm{loc}}(0,\infty;U) \to L^2_{\mathrm{loc}}(0,\infty;Y)$ is a shift-invariant linear operator. Then $\mathcal{F}$ is called* regular *if for any $v \in U$, the corresponding step response y_v has a Lebesgue point at 0, i.e., the following limit*

$$Dv = \lim_{t\to\infty} \frac{1}{t}\int_0^t y_v(s)\,ds \tag{5.11}$$

exists in Y. In that case, the operator $D \in \mathcal{L}(U,Y)$, defined by (5.11)*, is called* feedthrough operator *of $\mathcal{F}$.*

Definition 5.4.11. *Let U, Y be Hilbert spaces and assume H is a well-posed transfer function from U to Y. Then H is called* regular *if the corresponding shift-invariant operator $\mathcal{F}$ is regular. By the* feedthrough operator *of H we mean the feedthrough operator of $\mathcal{F}$.*

Remark 5.4.12. *By Theorem 5.8 in [Wei94b], H is regular if and only if, for any $v \in U$, $H(\lambda)v$ has a limit as $\lambda \to \infty$ with $\lambda \in \mathbb{R}$. In this case we have*

$$\lim_{\substack{\lambda\to\infty \\ \lambda\in\mathbb{R}}} H(\lambda)v = Dv,$$

where D is the feedthrough operator of H.

Recall that a linear map $B \in \mathcal{L}(U, X_{-1})$ is an L^2-admissible control operator for $(T(t))_{t\geq 0}$ if for some (and hence any) $t > 0$ we have $\Phi_\tau \in \mathcal{L}(L^2(0,\infty;U), X)$, where

$$\Phi_t u = \int_0^t T_{-1}(t-s)Bu(s)\,ds$$

for $u \in L^2(0,\infty;U)$. Next we introduce the concept of an admissible observation operator, which is the dual concept of an admissible control operator.

Definition 5.4.13. *An operator $C \in \mathcal{L}(X_1, Y)$ is called an* admissible observation operator *for $(T(t))_{t\geq 0}$ if for some (and hence any) $t > 0$ the operator $\Psi_t \in \mathcal{L}(X_1, L^2(0,\infty;Y))$, defined by*

$$(\Psi_t x)(s) = \begin{cases} CT(s)x & \text{for } s \in [0,t], \\ 0 & \text{for } s > t, \end{cases}$$

has a continuous extension to X.

Remark 5.4.14. *Let X, U be Hilbert spaces. The concepts of an admissible observation operator is dual to the concept of an admissible control operator in the following sense: Let $B \in \mathcal{L}(U, X_{-1})$. Then B is an admissible control operator for $(T(t))_{t\geq 0}$ if and only if B^* is an admissible observation operator for $(T^*(t))_{t\geq 0}$, see Theorem 4.4.3 in [TW09].*

Definition 5.4.15. *Let $u, v \in L^2_{\text{loc}}(0,\infty;U)$ and $t \geq 0$. The* t-concatenation *of u and v is the function $u \diamondsuit_t v \in L^2_{\text{loc}}(0,\infty;U)$ defined by $u \diamondsuit_t v := P_t u + S_t v$, that is,*

$$(u \diamondsuit_t v)(s) = \begin{cases} u(s) & \text{for } s < t, \\ v(s-t) & \text{for } s \geq t. \end{cases}$$

We are now ready to introduce the concept of a well-posed linear system.

Definition 5.4.16. *Let U, X and Y be Hilbert spaces. An* L^2-well-posed linear system Σ *on (Y, X, U) is a quadruple $\Sigma = (T, \Phi, \Psi, \mathcal{F})$ satisfying the following conditions:*

(a) $T = (T(t))_{t\geq 0}$ *is a* C_0*-semigroup of bounded linear operators on* X*.*

(b) $\Phi = (\Phi_t)_{t\geq 0}$ *is a family of bounded linear operators from* $L^2(0,\infty;U)$ *to* X *such that*

$$\Phi_{s+t}(u \diamondsuit_s v) = T(t)\Phi_s u + \Phi_t v$$

for any $u, v \in L^2(0,\infty;U)$ *and any* $s, t \geq 0$*.*

(c) $\Psi = (\Psi_t)_{t\geq 0}$ *is a family of bounded linear operators from* X *to* $L^2(0,\infty;Y)$ *such that* $\Psi_0 = 0$ *and*

$$\Psi_{s+t}x = \Psi_s x \diamondsuit_s \Psi_t T(s)x$$

for any $x \in X$ *and any* $s, t \geq 0$*.*

(d) $\mathcal{F} = (\mathcal{F}_t)_{t\geq 0}$ *is a family of bounded linear operators from* $L^2(0,\infty;U)$ *to* $L^2(0,\infty;Y)$ *such that* $\mathcal{F}_0 = 0$ *and*

$$\mathcal{F}_{s+t}(u \diamondsuit_s v) = \mathcal{F}_s u \diamondsuit_s (\Psi_t \Phi_s u + \mathcal{F}_t v)$$

for any $u, v \in L^2(0,\infty;U)$ *and any* $s, t \geq 0$*.*

The space U *is called the* input space *of* Σ*,* X *is the* state space *of* Σ *and* Y *is the* output space *of* Σ*. The operators* Φ_t*,* $t \geq 0$*, are called* input maps*. The operators* Ψ_t*,* $t \geq 0$*, are called* output maps*. The operators* $\mathcal{F}_t$*,* $t \geq 0$*, are called* input/output maps*.*

Let Σ be a well-posed linear system, then, by Salamon's representation theorem, see Theorem 3.1 in [Sal89], there exists a unique $B \in \mathcal{L}(U, X_{-1})$, called the *control operator* of Σ, such that

$$\Phi_t u = \int_0^t T(t-s)Bu(s)\, ds$$

for any $t \geq 0$. Recall that the system $\Sigma(A, B)$ is infinite-time admissible with respect to L^2 if

$$\sup_{t>0} \|\Phi_t\|_{\mathcal{L}(L^2(0,t;U),X)} < \infty.$$

In this case for each $u \in L^2(0,\infty;U)$ the improper integral

$$\Phi_\infty u := \int_0^\infty T_{-1}(s)Bu(s)\, ds$$

exists in X and defines a bounded linear map $\Phi_\infty\colon L^2(0,\infty;U) \to X$, the so-called extended input map of Σ, see Remark 2.2.4. It can be shown that the families of operators $(\Psi_t)_{t\geq 0}$ and $(\mathcal{F}_t)_{t\geq 0}$ have strong limits as $t \to \infty$ as operators $\Psi_\infty\colon X \to L^2_{\mathrm{loc}}(0,\infty;Y)$ and $\mathcal{F}_\infty\colon L^2_{\mathrm{loc}}(0,\infty;U) \to L^2_{\mathrm{loc}}(0,\infty;Y)$. We have $\Psi_t = P_t\Psi_\infty$ and $\mathcal{F}_t = P_t\mathcal{F}_\infty$ for $t \geq 0$. The operator Ψ_∞ is called the *extended output map* of Σ and $\mathcal{F}_\infty$ is called the *extended input/output* *map* of Σ. By Theorem 3.1 in [Sal89] there is a unique $C \in \mathcal{L}(X_1, Y)$, called the *observation operator* of Σ, such that for any $x \in X_1$,

$$(\Psi_\infty x)(t) = CT(t)x$$

for all $t \geq 0$. We call C *bounded* if it can be extended continuously to X and *unbounded* otherwise.

Definition 5.4.17. *The* Lebesgue extension *of C is defined by*

$$C_L x = \lim_{t\to 0} C \frac{1}{t} \int_0^t T(s)x \, ds, \tag{5.12}$$

with the domain

$$\mathcal{D}(C_L) = \{x \in X \mid \textit{the limit in } (5.12) \textit{ exists}\}.$$

We have the inclusions $X_1 \subset \mathcal{D}(C_L) \subset X$. For any $x \in X$ we have that for almost every $t \geq 0$ holds $T(t)x \in \mathcal{D}(C_L)$ and

$$(\Psi_\infty x)(t) = C_L T(t)x.$$

If we define on $\mathcal{D}(C_L)$ the norm

$$\|x\|_{\mathcal{D}(C_L)} = \|x\|_X + \sup_{t\in(0,1]} \left\| C \frac{1}{t} \int_0^t T(s)x \, ds \right\|_Y,$$

then $\mathcal{D}(C_L)$ becomes a Banach space. Moreover, the inclusions $X_1 \subset \mathcal{D}(C_L) \subset X$ are continuous and $C_L \in \mathcal{L}(\mathcal{D}(C_L), Y)$, see [Wei89b].

Definition 5.4.18. *A well-posed linear system Σ is called* regular *if its extended input/output map $\mathcal{F}_\infty$ is regular.*

It was shown in [Wei89c] that for regular linear system on Hilbert spaces the following representation result holds.

Theorem 5.4.19. *Let $\Sigma = (T, \Phi, \Psi, \mathcal{F})$ be a regular linear system with state space X, input space U and output space Y. Let A be the infinitesimal generator of $T = (T(t))_{t\geq 0}$, B the control operator of Σ, C the observation operator of Σ, C_L the Lebesgue extension of C and D the feedtrough operator of $\mathcal{F}$. Then for any $x_0 \in X$ and $u \in L^2_{\mathrm{loc}}(0, \infty; U)$, the functions $x\colon [0, \infty) \to X$ and $y \in L^2_{\mathrm{loc}}(0, \infty; Y)$, defined by*

$$x(t) = T(t)x_0 + \Phi_t u, \tag{5.13}$$
$$y = \Psi_\infty x_0 + \mathcal{F}_\infty u, \tag{5.14}$$

satisfy

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{5.15}$$

and

$$y(t) = C_L x(t) + Du(t) \tag{5.16}$$

for almost all $t \geq 0$. In particular, $x(t) \in D(C_L)$ for almost all $t \geq 0$. The function x given by (5.13) *is the unique mild solution of* (5.15)*, which satisfies the initial condition $x(0) = x_0$.*

By Proposition 4.1 in [Wei94b] the extended input/output map $\mathcal{F}_\infty$ is shift-invariant and its growth bound $\gamma(\mathcal{F}_\infty)$ satisfies the estimate

$$\gamma(\mathcal{F}_\infty) \leq \omega_0,$$

where ω_0 is the growth bound of the semigroup $(T(t))_{t\geq 0}$. In particular, $\gamma(\mathcal{F}_\infty) < \infty$. Hence, by Theorem 5.4.5, $\mathcal{F}_\infty$ has a well-posed transfer function H, called the *transfer function* of Σ and $\gamma(H) = \gamma(\mathcal{F}_\infty)$. The relationship between H and the operators A, B, C is given by the formula

$$\frac{H(s) - H(t)}{s - t} = -C(sI - A)^{-1}(tI - A)^{-1}B, \tag{5.17}$$

where $s, t \in \mathbb{C}^+_{\omega_0}$ with $s \neq t$. By Theorem 4.7 in [Wei94b] the transfer function H of a regular system Σ is given by

$$H(s) = C_L(sI - A)^{-1}B + D$$

for $s \in \mathbb{C}^+_{\omega_0}$. In particular, $(sI - A)^{-1}Bv \in \mathcal{D}(C_L)$ holds for all $v \in U$ and $s \in \mathbb{C}^+_{\omega_0}$.
The existence and uniqueness of the closed-loop system is shown in [Wei94b].

Theorem 5.4.20. *Let $\Sigma = (T, \Phi, \Psi, \mathcal{F})$ be a well-posed linear system, let H be its transfer function and let K be an admissible feedback operator for H. Then there is a unique well-posed linear system $\Sigma^K = (T^K, \Phi^K, \Psi^K, \mathcal{F}^K)$, called the* closed-loop system*, such that*

$$\begin{aligned} T(t) - T^K(t) &= \Phi_t K \Psi^K_t, \\ \Phi_t - \Phi^K_t &= \Phi_t K \mathcal{F}^K_t, \\ \Psi_t - \Psi^K_t &= \mathcal{F}_t K \Psi^K_t \end{aligned}$$

and

$$\mathcal{F}_t - \mathcal{F}^K_t = \mathcal{F}_t K \mathcal{F}^K_t$$

for all $t \geq 0$. The transfer function of Σ^K is H^K, the closed-loop transfer function corresponding to H and K.

5.5 Strong stabilizability

Definition 5.5.1. *Let $\Sigma = (T, \Phi, \Psi, \mathcal{F})$ be a well-posed linear system, Φ_∞ its extended input map, Ψ_∞ its extended output map and $\mathcal{F}_\infty$ its extended input/output map. The system Σ is called*

(a) input stable *if $\Phi_\infty \in \mathcal{L}(L^2(0,\infty;U), X)$,*

(b) output stable *if $\Psi_\infty \in \mathcal{L}(X, L^2(0,\infty;Y))$,*

(c) input-output stable *if $\mathcal{F}_\infty \in \mathcal{L}(L^2(0,\infty;U), L^2(0,\infty;Y))$.*

The following definition of strongly stable systems is due to R. Curtain and J. C. Oostveen, c.f. [OC98].

Definition 5.5.2. *Let $\Sigma = (T, \Phi, \Psi, \mathcal{F})$ be a well-posed linear system. We call Σ a* strongly stable system *if it is input stable, output stable, input-output stable and the semigroup $T = (T(t))_{t\geq 0}$ is strongly stable.*

Remark 5.5.3. *The input stability of the system Σ is equivalent to B being an infinite-time admissible control operator for $(T(t))_{t\geq 0}$, c.f. Remark 2.2.4. It is also equivalent to the condition $B^*(\cdot I - A^*)^{-1}x \in H^2(U)$ for all $x \in X$. In this case the closed graph theorem implies that $B^*(\cdot I - A^*)^{-1} \in \mathcal{L}(X, H^2(U))$. The output stability is equivalent to C being an infinite-time admissible observation operator for $(T(t))_{t\geq 0}$ and to the condition $C_L(\cdot I - A)^{-1}x \in H^2(Y)$. Again, from the closed graph theorem follows that, in this case, we obtain $C_L(\cdot I - A)^{-1} \in \mathcal{L}(X, H^2(Y))$ for all $x \in X$. The input/output stability of Σ is equivalent to the fact that its transfer function satisfies $H \in H_0^\infty(\mathcal{L}(U,X))$.*

Remark 5.5.4. *If the semigroup $(T(t))_{t\geq 0}$ is exponentially stable and B is an admissible control operator for $(T(t))_{t\geq 0}$, then, by Proposition 4.2.6, the system Σ is input stable. It can be shown in a similar way that in this case it is also output stable and input-output stable if C is an admissible observation operator.*

Remark 5.5.5. *We sometimes write $\Sigma = (A, B, C, D)$ instead of $\Sigma = (T, \Phi, \Psi, \mathcal{F})$, where A, B, C and D are the generating operators, which uniquely determine the system Σ.*
We say that $\Sigma(A, B)$ is a strongly stable system if this holds for the system $\Sigma = (A, B, 0, 0)$, i.e., the operator A generates a strongly stable C_0-semigroup $(T(t))_{t\geq 0}$ and B is an infinite-time admissible control operator for $(T(t))_{t\geq 0}$.

Definition 5.5.6. *Let A be the generator of a C_0-semigroup $(T(t))_{t\geq 0}$ on the Hilbert space X, U another Hilbert space and $B \in \mathcal{L}(U, X_{-1})$. The system $\Sigma(A, B)$ given by (5.1) is called* strongly stabilizable *if there exists an operator $F \in \mathcal{L}(D(A), U)$ such that $\Sigma = (A, B, F, 0)$ is a regular system, I is an admissible feedback operator for Σ and the closed-loop system Σ^I is a strongly stable, regular system.*

Remark 5.5.7. *The Definition 5.5.6 is a natural generalisation of exponential stabilizability as it is defined in [WR00], see Definition 2.1 there.*

Remark 5.5.8. *From Theorem 7.2 in [Wei94a] we obtain that if the system Σ is strongly stabilizable, then A^I, the generator of $(T^I(t))_{t\geq 0}$, is given by*

$$A^I x = (A + BF_L)x$$

for all $x \in D(A^I)$, where

$$D(A^I) = \{x \in D(F_L) \mid F_L x \in U \text{ and } (A + BF_L)x \in X\}.$$

We use the notation A_{BF_L} for A^I and $T_{BF_L}(t)$ for $T^I(t)$.

Let $P \in \mathcal{L}(X)$ be a projection that commutes with the C_0-semigroup $(T(t))_{t\geq 0}$, that is, we have $T(t)P = PT(t)$ for all $t \geq 0$. Then P yields a decomposition of X as $X = \ker P \oplus \operatorname{ran} P$ and both $\ker P$ and $\operatorname{ran} P$ are closed $T(t)$-invariant subspaces of X. By Lemma 4.2 in [JZ99] the restrictions of $(T(t))_{t\geq 0}$ to $\ker P$ and $\operatorname{ran} P$ respectively define C_0-semigroups. We denote by $(T^+(t))_{t\geq 0}$ the restriction of $(T(t))_{t\geq 0}$ to $X^+ := \operatorname{ran} P$ and $(T^-(t))_{t\geq 0}$ the restriction of $(T(t))_{t\geq 0}$ to $X^- := \ker P$. The generators of $(T^-(t))_{t\geq 0}$ and $(T^+(t))_{t\geq 0}$ are denoted by A^- and A^+, respectively.
If B is bounded, then we saw in Section 5.3 that such a projection on X yields a decomposition of the system $\Sigma(A, B)$ in two subsystems $\Sigma(A^+, B^+)$ and $\Sigma(A^-, B^-)$, where $B^+ = PB \in \mathcal{L}(U, X^+)$ and $B^- = (I - P)B \in \mathcal{L}(U, X^-)$. If B is unbounded, then the situation is not as simple since the composition PB is not well-defined. In order to write our system as a decomposition of two subsystems we need an extension of P as an element of $\mathcal{L}(X_{-1})$ that behaves well in a certain sense. The following lemma ensures the existence of such an extension.

Lemma 5.5.9 (Lemma 4.4 in [JZ99])**.** *Let $(T(t))_{t\geq 0}$ be a C_0-semigroup on X and $B \in \mathcal{L}(U, X_{-1})$ an admissible control operator for $(T(t))_{t\geq 0}$. Let $P \in \mathcal{L}(X)$ be a projection, which commutes with $T(t)$ for all $t \geq 0$. Then the following properties hold:*

(a) P has a unique continuous extension $\tilde{P}$ in $\mathcal{L}(X_{-1})$ with $\operatorname{ran} \tilde{P} = (X^+)_{-1} =: X^+_{-1}$ and $\ker \tilde{P} = (X^-)_{-1} =: X^-_{-1}$. The map $\tilde{P}$ is a projection, which commutes with $T_{-1}(t)$, $t \geq 0$, and A_{-1}.

(b) $B^+ := \tilde{P}B \in \mathcal{L}(U, X^+_{-1})$ is an admissible control operator for $(T^+(t))_{t\geq 0}$ on X^+ with the property

$$\int_0^t T^+_{-1}(t-s)B^+ u(s)\, ds = P \int_0^t T_{-1}(t-s)Bu(s)\, ds$$

for $t \geq 0$ and $u \in L^2(0, t; U)$.

(c) $B^- := (I - \tilde{P})B \in \mathcal{L}(U, X_{-1}^-)$ *is an admissible control operator for* $(T^-(t))_{t\geq 0}$ *on* X^- *with the property*

$$\int_0^t T_{-1}^-(t-s)B^-u(s)\,ds = (I-P)\int_0^t T_{-1}(t-s)Bu(s)\,ds$$

for $t \geq 0$ *and* $u \in L^2(0,t;U)$.

Next we present sufficient conditions for strong stabilizability of linear systems with an unbounded control operator.

Proposition 5.5.10. *Consider the system* $\Sigma(A,B)$ *given by* (5.1)*. Assume that there exists a projection* $P \in \mathcal{L}(X)$ *such that the system* $\Sigma(A^-, B^-)$ *on* X^- *is strongly stable and the system* $\Sigma(A^+, B^+)$ *on* X^+ *is strongly stabilizable. Then the system* $\Sigma(A,B)$ *is strongly stabilizable.*

Proof. The system $\Sigma(A^+, B^+)$ is strongly stabilizable and hence, by definition, there exists an operator $F^+ \in \mathcal{L}(D(A^+), U)$ such that $\Sigma_+ := (A^+, B^+, F^+, 0)$ is a regular linear system, I is an admissible feedback operator for Σ_+ and the closed-loop system Σ_+^I is a strongly stable, regular system. Choosing $F := (F^+\, 0)$, we have $F \in \mathcal{L}(D(A), U)$ and it is straight forward to check that $\Sigma = (A, B, F, 0)$ is a well-posed linear system. We denote by H the transfer function of Σ and by H_+ the transfer function of Σ_+. As $I - H_+$ is invertible in $H_\infty^\infty(\mathcal{L}(U))$ by Proposition 5.4.8 and we have $I - H_+ = I - H$ by the choice of F, it follows that $I - H$ is invertible in $H_\infty^\infty(\mathcal{L}(U))$. Therefore, by Proposition 5.4.8, I is an admissible feedback operator for Σ. Hence, by Theorem 5.4.20, there exists a unique well-posed linear system $\Sigma^I = (T^I, \Phi^I, \Psi^I, \mathcal{F}^I)$ such that for all $x \in X$ holds

$$T^I(t)x = (T(t) - \Phi_t\Psi_t^I)x.$$

Using Remark 5.5.8 the last identity can be written equivalently as

$$T^I(t)x = T_{BF_L}(t)x = T(t)x - \int_0^t T_{-1}(t-s)BF_LT_{BF_L}(s)x\,ds. \tag{5.18}$$

Every $x \in X$ has a unique representation as $x = x^- + x^+$, where $x^+ = Px \in X^+$ and $x^- = (I-P)x \in X^-$. Therefore, with respect to the decomposition of the state space given by the projection P, the abstract differential equation $\dot{x}(t) = (A + BF_L)x(t)$, $x(0) = x_0$, is given by

$$\dot{x}(t) = \begin{pmatrix} \dot{x}^+(t) \\ \dot{x}^-(t) \end{pmatrix} = \begin{pmatrix} A^+ + B^+F_L^+ & 0 \\ B^-F_L^+ & A^- \end{pmatrix}\begin{pmatrix} x^+(t) \\ x^-(t) \end{pmatrix}, \quad x(0) = x_0 = \begin{pmatrix} x_0^+ \\ x_0^- \end{pmatrix}.$$

Thus we obtain two differential equations, i.e.,

$$\dot{x}^+(t) = (A^+ + B^+F_L^+)x^+(t), \quad x^+(0) = x_0^+,$$

on X^+ and

$$\dot{x}^-(t) = A^-x^-(t) + B^-F_L^+x^+(t), \quad x^-(0) = x_0^-,$$

on X^-. Integrating both equations we obtain that the semigroup $(T_{BF_L}(t))_{t\geq 0}$ is given by

$$T_{BF_L}(t)x = T^+_{B^+F^+_L}(t)x^+ + \int_0^t T^-_{-1}(t-s)B^-F^+_L T^+_{B^+F^+_L}(s)x^+\,ds + T^-(t)x^- \tag{5.19}$$

for $x \in X$ and $t \geq 0$. Since the function $u\colon [0,\infty) \to U$, given by

$$u(t) = F^+_L T^+_{B^+F^+_L}(s)x^+,$$

belongs to $L^2(0,\infty;U)$, as it is the output function of the well-posed system Σ^I_+, we have by Lemma 12 in [OC98]

$$\lim_{t\to\infty} \int_0^t T^-_{-1}(t-s)B^-F^+_L T^+_{B^+F^+_L}(s)x^+\,ds = 0$$

since, by assumption, the semigroup $(T^-(t))_{t\geq 0}$ is strongly stable. By construction the semigroup $(T^+_{B^+F^+_L}(t))_{t\geq 0}$ is also strongly stable and thus equation (5.19) yields

$$\lim_{t\to\infty} T_{BF_L}(t)x = 0.$$

The system Σ^I is regular since, by construction, its transfer function coincides with the transfer function of the regular system Σ^I_+. Next we show that the system Σ^I is input stable. Applying the Laplace transform to (5.19) we obtain

$$\begin{aligned}(sI - A_{BF_L})^{-1}x =& (sI - A^+_{B^+F^+_L})^{-1}Px + (sI - A^-)^{-1}B^-F^+_L(sI - A^+_{B^+F^+_L})^{-1}Px \\ &+ (sI - A^-)^{-1}(I-P)x\end{aligned}$$

for all $x \in X$ and $s \in \mathbb{C}^+_0$ and hence

$$\begin{aligned}B^*(sI - A^*_{BF_L})^{-1}x =& B^*[(\bar{s}I - A_{BF_L})^{-1}]^*x \\ =& B^*P^*(sI - (A^+_{B^+F^+_L})^*)^{-1}x \\ &+ B^*P^*(sI - (A^+_{B^+F^+_L})^*)^{-1}(F^+_L)^*(B^-)^*(sI - (A^-)^*)^{-1}x \\ &+ B^*(I-P)^*(sI - (A^-)^*)^{-1}x \\ =& (B^+)^*(sI - (A^+_{B^+F^+_L})^*)^{-1}x \\ &+ (B^+)^*(sI - (A^+_{B^+F^+_L})^*)^{-1}(F^+_L)^*(B^-)^*(sI - (A^-)^*)^{-1}x \\ &+ (B^-)^*(sI - (A^-)^*)^{-1}x.\end{aligned}$$

By our assumptions we have $(B^-)^*(\cdot I - (A^-)^*)^{-1}x \in H^2(U)$ and $(B^+)^*(\cdot I - (A^+_{B^+F^+_L})^*)^{-1}x \in H^2(U)$ for all $x \in X$. As the system Σ^I_+ is regular its transfer function satisfies

$$F^+_L(\cdot I - A^+_{B^+F^+_L})^{-1}B^+ \in H^\infty_0(\mathcal{L}(U))$$

and hence

$$(B^+)^*(\cdot I - (A^+_{B^+F_L^+})^*)^{-1}(F_L^+)^* \in H^\infty(\mathcal{L}(U)).$$

Overall we have

$$B_L^*(\cdot I - A_{BF_L}^*)^{-1}x \in H^2(U),$$

which shows that $\Phi_\infty^I \in \mathcal{L}(L^2(0,\infty;U),X)$. The output and input/output stability of Σ^I are not difficult to see as the system Σ_+^I is input and input/output stable and, hence, by the choice of F we have

$$\int_0^\infty \|F_L T_{BF_L}(t)x\|^2\,dt = \int_0^\infty \|F_L^+ T^+_{B^+F_L^+}(t)Px\|^2\,dt \le M\|Px\|^2 \le M\|P\|^2\|x\|^2,$$

which shows $\Psi_\infty^I \in \mathcal{L}(X, L^2(0,\infty;X))$ and

$$\left\| F_L \int_0^\cdot T_{BF_L}(\cdot - s)Bu(s)\,ds \right\|_{L^2} = \left\| F_L^+ \int_0^\cdot T^+_{B^+F_L^+}(\cdot - s)B^+u(s)\,ds \right\|_{L^2} \le M\|u\|_{L^2},$$

which shows $\mathcal{F}_\infty^I \in \mathcal{L}(L^2(0,\infty;U), L^2(0,\infty;X))$. □

Definition 5.5.11. *A system $\Sigma(A,B)$ is called* null controllable in finite time *if for each initial value $x_0 \in X$ there is a time $t_0 > 0$ and an input $u \in L^2(0,t_0,U)$ such that the mild solution of* (5.1)*, given by* (5.2)*, satisfies $x(t_0) = 0$.*

Definition 5.5.12. *We call the system $\Sigma(A,B)$* optimizable *if for every $x_0 \in X$ there exists an input $u \in L^2(0,\infty;U)$ such that the mild solution of* (5.1)*, given by* (5.2)*, satisfies $x \in L^2(0,\infty;X)$.*

The following result generalises Theorem 4.6 in [ABBMS15]. Its proof uses a similar approach.

Proposition 5.5.13. *Consider the system $\Sigma(A,B)$ given by* (5.1)*. Assume that there exists a projection $P \in \mathcal{L}(X)$ such that the system $\Sigma(A^-,B^-)$ on X^- is strongly stable and the system $\Sigma(A^+,B^+)$ on X^+ is null controllable in finite time. Then there exists an operator $F \in \mathcal{L}(D(A),X)$ such that $A + BF_L$ generates a strongly stable semigroup $(T_{BF_L}(t))_{t\ge0}$. Further, the system $\Sigma = (A + BF_L, B, F, 0)$ is input stable, output stable and input-output stable.*

Proof. Since the system $\Sigma(A^+,B^+)$ is null controllable there exists a time $t_0 > 0$ and an input $u \in L^2(0,t_0;U)$ such that the mild solution of (5.1), given by (5.2), satisfies $x(t_0) = 0$. We extend u to $(0,\infty)$ by zero and obtain that the system $\Sigma(A^+,B^+)$ is optimizable. Therefore, by Theorem 2.2 in [FLT88] (see also Propositions 3.2, 3.3 and 3.4 in [WR00]), there exists an exponentially stable semigroup $(T^{opt}(t))_{t\ge0}$ with infinitesimal generator $A^{opt}\colon D(A^{opt}) \to X^+$ and an operator $F^{opt} \in \mathcal{L}(D(A^{opt}),U)$, which is an infinite-time admissible observation operator for $(T^{opt}(t))_{t\ge0}$ such that for every $t > 0$ and $x_0 \in X^+$ there holds

$$T^{opt}(t)x_0 = T^+(t)x_0 + \int_0^t T^+_{-1}(t-s)B^+F^{opt}T^{opt}(s)x_0\,ds.$$

The infinitesimal generator of the semigroup $(T^{opt}(t))_{t\geq 0}$ is given by $A^{opt} = A^+ + B^+F^{opt}$. Let F_L be the Lebesgue extension of the row operator matrix $(F^{opt}\, 0)$. As in the proof of Proposition 5.5.10 we obtain that the semigroup $(T_{BF_L}(t))_{t\geq 0}$ is given by

$$T_{BF_L}(t)x = T^{opt}(t)Px + \int_0^t T^-_{-1}(t-s)B^-F^{opt}T^{opt}(s)Px\,ds + T^-(t)(I-P)x$$

for $x \in X$ and $t \geq 0$. Now all the assertions follow in exactly the same manner as in the proof of Proposition 5.5.10. $\square$

5.6 Polynomial stabilizability

Definition 5.6.1. *Let $\alpha > 0$. The C_0-semigroup $(T(t))_{t\geq 0}$ on the Hilbert space X generated by A is called* polynomially stable with power α *if $(T(t))_{t\geq 0}$ is bounded, $i\mathbb{R} \subset \rho(A)$ and there exists an $M \geq 1$ such that*

$$\|T(t)A^{-1}\| \leq \frac{M}{t^{1/\alpha}} \tag{5.20}$$

for all $t > 0$. We say that the C_0-semigroup $(T(t))_{t\geq 0}$ is polynomially stable *if it is polynomially stable with power α for some $\alpha > 0$.*

Remark 5.6.2. *The estimate (5.20) can be rewritten equivalently as*

$$\|T(t)A^{-1}\| = \mathrm{O}(t^{-1/\alpha})$$

for $t \to \infty$. By Theorem 2.4 in [BT10] a bounded C_0-semigroup $(T(t))_{t\geq 0}$ on the Hilbert space X generated by A with $i\mathbb{R} \subset \rho(A)$ is polynomially stable with power α if and only if

$$\|(isI - A)^{-1}\| = \mathrm{O}(|s|^\alpha) \tag{5.21}$$

for $|s| \to \infty$.

Remark 5.6.3. *From the estimate*

$$\|T(t)A^{-1}\| \leq \|T(t)\|\|A^{-1}\|$$

it is clear that the exponential stability of the semigroup $(T(t))_{t\geq 0}$ on the Hilbert space X implies the polynomial stability of $(T(t))_{t\geq 0}$ for any power $\alpha > 0$. From (5.20) it follows that

$$\lim_{t\to\infty} T(t)x = 0$$

holds for all $x \in \operatorname{ran}(A^{-1}) = D(A)$. Since $D(A)$ is a dense subspace of X and all operators $T(t)$, $t \geq 0$, are uniformly bounded on X we obtain that every polynomially stable semigroup on a Hilbert space X is strongly stable.

Definition 5.6.4. *Let $\Sigma = (T, \Phi, \Psi, \mathcal{F})$ be a well-posed linear system. We call Σ a* polynomially stable system *if it is input stable, output stable, input-output stable and the semigroup $T = (T(t))_{t\geq 0}$ is polynomially stable.*

Definition 5.6.5. *Let A be the generator of a C_0-semigroup $(T(t))_{t\geq 0}$ on the Hilbert space X, U another Hilbert space and $B \in \mathcal{L}(U, X_{-1})$. The system $\Sigma(A, B)$ given by* (5.1) *is called* polynomially stabilizable *if there exists an operator $F \in \mathcal{L}(D(A), U)$ such that $\Sigma = (A, B, F, 0)$ is a regular system, I is an admissible feedback operator for Σ and the closed-loop system Σ^I is a polynomially stable, regular system.*

From Remark 5.6.3 it follows that a system $\Sigma(A, B)$ is strongly stable if it is polynomially stable and it is strongly stabilizable if it is polynomially stabilizable. Next we have a polynomial version of Proposition 5.5.10.

Proposition 5.6.6. *Consider the system $\Sigma(A, B)$ given by* (5.1)*. Assume that there exists a projection $P \in \mathcal{L}(X)$ such that the system $\Sigma(A^-, B^-)$ on X^- is polynomially stable and the system $\Sigma(A^+, B^+)$ on X^+ is polynomially stabilizable. Then the system $\Sigma(A, B)$ is polynomially stabilizable.*

Proof. As the system $\Sigma(A, B)$ is strongly stabilizable we can use the operator $F \in \mathcal{L}(D(A), U)$ constructed in the proof of Proposition 5.5.10 and obtain a strongly stable regular system $\Sigma^I = (T^I, \Phi^I, \Psi^I, \mathcal{F}^I)$. We are left to show that the semigroup $(T^I(t))_{t\geq 0} = (T_{BF}(t))_{t\geq 0}$, given by

$$T_{BF_L}(t)x = T^+_{B^+F^+_L}(t)x^+ + \int_0^t T^-_{-1}(t-s)B^-F^+_L T^+_{B^+F^+_L}(s)x^+\,ds + T^-(t)x^-, \tag{5.22}$$

where $x^+ = Px$ and $x^- = (I - P)x$, is polynomially stable. Taking the Laplace transform on both sides of (5.22) we have

$$\begin{aligned}(sI - A_{BF_L})^{-1}x =& (sI - A^+_{B^+F^+_L})^{-1}Px + (sI - A^-)^{-1}B^-F^+_L(sI - A^+_{B^+F^+_L})^{-1}Px \\ &+ (sI - A^-)^{-1}(I - P)x\end{aligned} \tag{5.23}$$

for all $x \in X$ and $s \in \mathbb{C}^+_0$. Since the semigroups $(T^-(t))_{t\geq 0}$ and $(T^+_{B^+F^+}(t))_{t\geq 0}$ are polynomially stable we have $i\mathbb{R} \subset \rho(A^+_{B^+F^+}) \cap \rho(A^-)$ and hence $i\mathbb{R} \subset \rho(A_{BF})$. Further, there exist positive numbers α, β such that

$$\|(i\omega I - A^-)^{-1}\| = \mathrm{O}(|\omega|^\alpha)$$

for $|\omega| \to \infty$ and

$$\|(i\omega I - A^+_{B^+F^+})^{-1}\| = \mathrm{O}(|\omega|^\beta)$$

for $|\omega| \to \infty$. From (5.23) it follows that for all $\omega \in \mathbb{R}$ and $x \in X$ we have the estimate

$$\begin{aligned}\|(i\omega I - A_{BF})^{-1}x\| \leq& \|(i\omega I - A^+_{B^+F^+})^{-1}Px\| \\ &+ \|(i\omega I - A^-)^{-1}B^-F^+(i\omega I - A^+_{B^+F^+})^{-1}Px\| \\ &+ \|(i\omega I - A^-)^{-1}(I - P)x\|.\end{aligned}$$

For any $\gamma > 0$ we have

$$(i\omega I - A^-)^{-1}B^- = ((i\omega + \gamma)I - A^-)^{-1}B^- + \gamma(i\omega I - A^-)^{-1}((i\omega + \gamma)I - A^-)^{-1}B^-$$

for all $\omega \in \mathbb{R}$. Since B^- is an admissible control operator for the semigroup $(T^-(t))_{t\geq 0}$ we obtain using Proposition 2.3 in [Wei91] that there is a constant $M \geq 0$ such that

$$\|(i\omega I - A^-)^{-1}B^-\| \leq \frac{M}{\sqrt{\gamma}} + \sqrt{\gamma}M\|(i\omega I - A^-)^{-1}\|$$

for all $\omega \in \mathbb{R}$. In a similar way using the admissibility of the observation operator F^+ for the semigroup $(T_{B^+F^+}(t))_{t\geq 0}$ we obtain by the duality between the admissibility concepts that there is a constant $M' \geq 0$ such that

$$\|F^+(i\omega I - A^+_{B^+F^+})^{-1}\| \leq \frac{M'}{\sqrt{\gamma}} + \sqrt{\gamma}M'\|(i\omega I - A^+_{B^+F^+})^{-1}\|.$$

Thus we have $\|(i\omega I - A_{BF})^{-1}\| = \mathrm{O}(|\omega|^{\alpha+\beta})$ for $|\omega| \to \infty$. □

Proposition 5.6.7. *Consider the system $\Sigma(A, B)$ given by (5.1). Assume that there exists a projection $P \in \mathcal{L}(X)$ such that the system $\Sigma(A^-, B^-)$ on X^- is polynomially stable and the system $\Sigma(A^+, B^+)$ on X^+ is null controllable in finite time. Then there exists an operator $F \in \mathcal{L}(D(A), X)$ such that $A + BF_L$ generates a polynomially stable semigroup $(T_{BF}(t))_{t\geq 0}$. Further, the system $\Sigma = (A + BF_L, B, F, 0)$ is input stable, output stable and input-output stable.*

Proof. By Proposition 5.5.13 we are left to show that the semigroup $(T_{BF}(t))_{t\geq 0}$ is polynomially stable. This is done exactly as in the proof of Proposition 5.6.6. □

Remark 5.6.8. *The Proposition 5.6.7 without the part concerning the stability of the system $\Sigma = (A + BF_L, B, F, 0)$ is exactly the Theorem 4.6 in [ABBMS15] and it is generalised by the Proposition 5.5.13. There the authors call a system $\Sigma(A, B)$ stabilizable if there exists a generator A_{BF} of a polynomially stable C_0-semigroup $(T_{BF}(t))_{t\geq 0}$ on X and an admissible observation operator $F \in \mathcal{L}(D(A_{BF}), U)$ for $(T_{BF}(t))_{t\geq 0}$ such that*

$$(\lambda I - A_{BF})^{-1} = (\lambda I - A)^{-1} + (\lambda I - A)^{-1}BF(\lambda I - A_{BF})^{-1}$$

holds for all $\lambda \in \mathbb{C}^+_{\omega_0}$, where $\omega_0 := \max\{\omega_0(A), \omega_0(A_{BF})\}$. In other words, the semigroup $(T_{BF}(t))_{t\geq 0}$ satisfies

$$T_{BF}(t)x = T(t)x - \int_0^t T_{-1}(t-s)BF_LT_{BF}(s)x\, ds$$

for all $x \in X$ and $t \geq 0$. Our definition of polynomial stabilizability is evidently more restrictive, as we additionally pose conditions on the system $\Sigma = (A, B, F, 0)$ such as input stability, output *stability and input-output-stability. On the other hand, as a result we can prove additional properties for the closed-loop system Σ^I, which involve more than the polynomial stability of the semigroup $(T_{BF}(t))_{t\geq 0}$. Thus neither of the both results, Proposition 5.6.7 and Theorem 4.6 in [ABBMS15], can be viewed as a generalisation or a special case of the other one.*

5.7 Stabilizability of systems with bounded control operators

In this section we restrict ourselves to the following systems:

Assumption 5.7.1. *Let $\Sigma(A, B)$ be a linear system given by (5.1) on a Hilbert space X such that:*

(a) The input space U is finite-dimensional, i.e., we have $U = \mathbb{C}^m$ for some $m \in \mathbb{N}$.

(b) The control operator B maps to X, i.e., we have $B \in \mathcal{L}(\mathbb{C}^m, X)$.

(c) There exists an $r > 0$ such that $\{s \in \mathbb{C} \mid 0 \leq \operatorname{Re} s < r\} \subset \rho(A)$.

First we need the following lemma:

Lemma 5.7.2. *Consider the system $\Sigma(A, B)$ with $B \in \mathcal{L}(\mathbb{C}^m, X)$ and let $F \in \mathcal{L}(X, \mathbb{C}^m)$. Then for any $s \in \rho(A + BF)$ we have the following properties:*

(a) The following are equivalent:

(i) $-1 \in \sigma(BF(sI - A - BF)^{-1})$.
(ii) $-1 \in \sigma(F(sI - A - BF)^{-1}B)$.
(iii) $\det(I + F(sI - A - BF)^{-1}B) = 0$.

If $-1 \in \sigma(BF(sI-A-BF)^{-1})$, then it belongs to the point spectrum. The same assertion holds for $F(sI - A - BF)^{-1}B$.

(b) The order and the multiplicity of the eigenvalue -1 of $BF(sI - A - BF)^{-1}$ and $F(sI - A - BF)^{-1}B$ are finite and equal.

(c) We have $s \in \rho(A)$ if and only if $-1 \in \rho(F(sI - A - BF)^{-1}B)$. For any $s \in \rho(A)$ we have

$$\begin{aligned}(sI - A)^{-1} =&(sI - A - BF)^{-1} - (sI - A - BF)^{-1}B \\ &\cdot (I + F(sI - A - BF)^{-1}B)^{-1}F(sI - A - BF)^{-1}.\end{aligned} \tag{5.24}$$

(d) Assume that the holomorphic function on $\rho(A + BF)$, given by

$$s \mapsto \det(I + F(sI - A - BF)^{-1}B),$$

is zero for $s = s_0$, but not identically zero in a neighbourhood of s_0. Then we have $s_0 \in \sigma(A)$ and it is an eigenvalue of A with finite order and finite multiplicity.

The proof can be found in [JZ12], see Lemma 10.4.3 there. Recall that if the operator A satisfies the spectrum decomposition assumption at 0, then, by Theorem 5.2.5, the spectral projection $P_\Gamma \colon X \to X$, given by (5.5), induces a decomposition $X = X^+ \oplus X^-$ of the state space. Furthermore, we have $B^+ := P_\Gamma B \in \mathcal{L}(U, X^+)$ and $B^- := (I - P_\Gamma)B \in \mathcal{L}(U, X^-)$, which leads to a decomposition of the system $\Sigma(A, B)$ in two subsystems: $\Sigma(A^+, B^+)$ on X^+ and $\Sigma(A^-, B^-)$ on X^-.

Lemma 5.7.3. *Assume that the operator A satisfies the spectrum decomposition assumption at zero and the system $\Sigma(A, B)$ is strongly stabilizable. Let $F \in \mathcal{L}(X, \mathbb{C}^m)$ be a stabilizing feedback. Then the system $\Sigma(A^-, B^-, F, 0)$ is output stable.*

Proof. From the spectrum decomposition assumption at zero follows the existence of some $\rho_0 > 0$ such that $\mathbb{C}_0^+ \setminus \mathbb{D}(\rho_0) \subset \rho(A)$. Hence, equation (5.24) yields

$$\begin{aligned} F(sI - A)^{-1} =& F(sI - A - BF)^{-1} - F(sI - A - BF)^{-1}B \\ & \cdot (I + F(sI - A - BF)^{-1}B)^{-1} F(sI - A - BF)^{-1} \end{aligned}$$

for all $s \in \mathbb{C}_0^+ \setminus \mathbb{D}(\rho_0)$. By the input-output stability and regularity of the closed-loop system Σ^I it follows that the maps $(I + F(sI - A - BF)^{-1}B)^{-1}$, $s \in \mathbb{C}_0^+ \setminus \mathbb{D}(\rho_0)$, are uniformly bounded. Thus, there exists a constant $M' > 0$ such that

$$\|F(sI - A)^{-1}x\| \leq M' \|F(sI - A - BF)^{-1}x\| \tag{5.25}$$

for all $x \in X$ and $s \in \mathbb{C}_0^+ \setminus \mathbb{D}(\rho_0)$. Since $\sigma(A^-) \subset \mathbb{C}_0^+$ we have that for every $s \in \overline{\mathbb{C}_0^+}$ the map $F(sI - A^-)^{-1}$ is bounded on X^-. Hence, those maps are uniformly bounded for all s belonging to the compact set $\mathbb{D}(\rho_0) \cap \overline{\mathbb{C}_0^+}$. Therefore we can find a constant $M'' > 0$ such that for all $s \in \mathbb{D}(\rho_0) \cap \overline{\mathbb{C}_0^+}$ and $x \in X^-$ the following estimate holds

$$\|F(sI - A)^{-1}x\| = \|F(sI - A^-)^{-1}x\| \leq M'' \|x\|. \tag{5.26}$$

Now the output stability of the closed-loop system Σ^I together with (5.29) and (5.26) imply the existence of some $M > 0$ such that $\|F(\cdot I - A)^{-1}x\|_{H^2(\mathbb{C}^m)} \leq M\|x\|$ for all $x \in X^-$. □

Theorem 5.7.4. *For any linear system $\Sigma(A, B)$ given by* (5.1) *with a finite-dimensional input space $U = \mathbb{C}^m$ and a bounded control operator, i.e., $B \in \mathcal{L}(\mathbb{C}^m, X)$, the following assertions are equivalent:*

(i) *$\Sigma(A, B)$ is strongly stabilizable.*

(ii) *$\Sigma(A, B)$ satisfies the spectrum decomposition assumption at zero, X^+ is finite-dimensional, $\Sigma(A^-, B^-)$ is a strongly stable system and the finite-dimensional system $\Sigma(A^+, B^+)$ is controllable.*

If $\Sigma(A,B)$ is strongly stabilizable, then a stabilizing feedback operator is given by $F = F^+P_\Gamma$, where F^+ is a stabilizing feedback operator for $\Sigma(A^+,B^+)$.

Proof. By Proposition 5.5.10 we are left to show that the implication (i) $\Rightarrow$ (ii) holds. From Lemma 5.7.2 it follows that for every $s \in \mathbb{C}_0^+$ there holds $s \in \sigma(A)$ if and only if $\det(I + F(sI-A-BF)^{-1}B) = 0$. As the semigroup generated by $A+BF$ is bounded, we have $\mathbb{C}_0^+ \subset \rho(A+BF)$. Thus, the function $\det(I+F(\cdot I-A-BF)^{-1}B)$ is holomorphic on $\mathbb{C}_0^+$. Therefore, by the identity theorem for holomorphic functions, it cannot have an accumulation point of zeros there, unless it is identically zero. As the closed-loop system Σ^I is regular, its transfer function $H^I = F(\cdot I - A - BF)^{-1}B$ satisfies $\lim_{\lambda\to\infty} H^I(\lambda) = 0$. Thus, there exists a positive number ρ_0 such that $I + F(sI - A - BF)^{-1}B$ is invertible for all $s \in \mathbb{C}_0^+ \setminus \mathbb{D}(\rho_0)$. Hence, using part (c) of the Assumption 5.7.1, we have $\overline{\mathbb{C}_0^+} \setminus \mathbb{D}(\rho_0) \subset \rho(A)$. In particular for all $s \in \mathbb{C}_0^+ \setminus \mathbb{D}(\rho_0)$ holds $I+F(sI-A-BF)^{-1}B \neq 0$, which means that this function is not identically zero. Thus it has at most finitely many zeros on the compact set $\mathbb{D}(\rho_0) \cap \overline{\mathbb{C}_0^+}$. Applying Lemma 5.7.2 we see that σ^+ consists of finitely many eigenvalues with finite order and finite multiplicity. Hence the spectrum decomposition assumption at zero holds. Now parts of (d) and (e) of Theorem 5.2.5 imply that $X^+ = \operatorname{ran} P_\Gamma$ is finite-dimensional and $\sigma(A^+) = \sigma^+ \subset \mathbb{C}_0^+$.
Next we show that the semigroup $(T^-(t))_{t\geq 0}$ on X^- is strongly stable. Indeed, for every $x \in X^-$ we have

$$T^-(t)x = T_{BF}(t)x + \int_0^t T_{BF}(t-s)BFT^-(s)x\,ds.$$

Now Lemma 5.7.3 implies that $FT^-(\cdot)x \in L^2(0,\infty,X^-)$. Therefore, Lemma 12 in [OC98] is applicable and we have

$$\lim_{t\to\infty} \int_0^t T_{BF}(t-s)BFT^-(s)x\,ds = 0$$

since the semigroup $(T_{BF}(t))_{t\geq 0}$ is strongly stable. Thus we obtain $T^-(t)x \to 0$ as $t \to \infty$.
We proceed by showing that B^- is an infinite-time admissible control operator for the semigroup $(T^-(t))_{t\geq 0}$, which means that we have to show the existence of a constant $M > 0$ such that the estimate

$$\left\| \int_0^\infty T^-(t)B^-u(t)\,dt \right\|_X \leq M\|u\|_{L^2} \tag{5.27}$$

holds for every $u \in L^2(0,\infty;\mathbb{C}^m)$. Since $B^- = (I-P_\Gamma)B$ and the semigroup $(T(t))_{t\geq 0}$ commutes with the projection P_Γ, (5.27) can be written as

$$\left\| (I-P_\Gamma) \int_0^\infty T(t)Bu(t)\,dt \right\|_X \leq M\|u\|_{L^2}.$$

Using the identity

$$\left\langle x, (I-P_\Gamma) \int_0^\infty T(t)Bu(t)\,dt \right\rangle = \int_0^\infty \langle B^*T^*(t)(I-P_\Gamma)^*x, u(t)\rangle\,dt$$

we obtain that the assertion is equivalent to

$$\int_0^\infty \|B^*T^*(t)(I-P_\Gamma)^*x\| \, dt \leq M\|x\|^2$$

for all $x \in X$. By introducing the space $\tilde{X} := \operatorname{ran}(I-P_\Gamma)^*$ we can rewrite the claim as

$$\int_0^\infty \|B^*T^*(t)x\| \, dt \leq M\|x\|^2 \tag{5.28}$$

for all $x \in \tilde{X}$. From the equation (5.24) follows

$$\begin{aligned} B^*(sI-A^*)^{-1} =& B^*(sI-(A+BF)^*)^{-1} - B^*(sI-(A+BF)^*)^{-1}F^* \\ & \cdot (I+B^*(sI-(A+BF)^*)^{-1}F^*)^{-1}B^*(sI-(A+BF)^*)^{-1} \end{aligned}$$

for all $s \in \overline{\mathbb{C}_0^+} \setminus \mathbb{D}(\rho_0)$. Then, as in the proof of Lemma 5.7.3, by the input-output stability and regularity of the closed-loop system Σ^I the maps $(I+B^*(sI-(A+BF)^*)^{-1}F^*)$, $s \in \mathbb{C}_0^+ \setminus \mathbb{D}(\rho_0)$, are uniformly bounded. Thus, there exists a constant $M' > 0$ such that

$$\|B^*(sI-A^*)^{-1}x\| \leq M'\|B^*(sI-(A+BF)^*)^{-1}x\| \tag{5.29}$$

for all $x \in X$ and $s \in \mathbb{C}_0^+ \setminus \mathbb{D}(\rho_0)$. Lemma 2.15 in [TW14], applied to the decomposition $X = X^+ \oplus X^-$ with the projections $P_1 := I - P_\Gamma$ and $P_2 := P_\Gamma$, yields

$$\operatorname{ran}(I-P_\Gamma)^* = \ker P_\Gamma^* = \operatorname{ran}(P_\Gamma)^\perp = (X^+)^\perp,$$

which means $\tilde{X} = (I-P_\Gamma)^*X = (X^+)^\perp$. Moreover, we get

$$\sigma(A^*|_{(X^+)^\perp}) = \{s \in \mathbb{C} \,|\, \bar{s} \in \sigma(A|_{X^-})\} = \{s \in \mathbb{C} \,|\, \bar{s} \in \sigma(A^-)\}.$$

Since $\sigma(A^*|_{\tilde{X}}) \subset \mathbb{C}_0^-$ we have that for every $s \in \overline{\mathbb{C}_0^+}$ the map $B^*(sI-A^*)^{-1}$ is bounded on $\tilde{X}$ and, hence, they are uniformly bounded for all s belonging to the compact set $\mathbb{D}(\rho) \cap \overline{\mathbb{C}_0^+}$. Thus there exists a constant $M'' > 0$ such that

$$\|B^*(sI-A^*)^{-1}x\| \leq M''\|x\| \tag{5.30}$$

holds for all $x \in \tilde{X}$ and $s \in \mathbb{D}(\rho_0)$. Now the input stability of the closed-loop system Σ^I together with (5.29) and (5.30) imply the existence of some $M > 0$ such that $\|B^*(\cdot I - A^*)^{-1}x\|_{H^2(\mathbb{C}^m)} \leq M\|x\|$ for all $x \in \tilde{X}$. Hence the estimate (5.28) holds, see Remark 5.5.3.
To conclude the proof we are left to show that the finite-dimensional system $\Sigma(A^+, B^+)$ is controllable. For any $x_0 \in X^+$ holds

$$T_{BF}(t)x_0 = T^+(t)x_0 + \int_0^t T^+(t-s)B^+FT_{BF}(s)x_0 \, ds.$$

As the closed-loop system Σ^I is output stable, the function $u\colon [0,\infty) \to \mathbb{C}^m$, given by $u(s) = FT_{BF}(s)x_0$, satisfies $u \in L^2(0,\infty;\mathbb{C}^m)$. The mild solution of the equation

$$\dot{x}(t) = A^+x(t) + B^+u(t), \quad x(0) = x_0, \quad t \geq 0,$$

is given by $x(t) = T_{BF}(t)x_0$ for $t \geq 0$. Thus it satisfies $x(t) \to 0$ as $t \to \infty$. Hence the system $\Sigma(A^+, B^+)$ is stabilizable, c.f. Definition 5.1.4. Since $\sigma(A^+)$ is contained in $\mathbb{C}_0^+$, the system $\Sigma(A^+, B^+)$ is controllable by Theorem 5.1.8. □

Example 5.7.5. *Let $X = \ell^2$ and $U = \mathbb{R}$. Let $\lambda_0 = 1$ and for $n \in \mathbb{N}\setminus\{0\}$ let $\lambda_n = in - 1/n$. We define $A\colon X \supset D(A) \to X$ by $Ae_n = \lambda_n e_n$, where $D(A) = \{x = (x_n)_{n\in\mathbb{N}} \in \ell^2 \mid (\lambda_n x_n)_{n\in\mathbb{N}} \in \ell^2\}$ and $B \in \mathcal{L}(\mathbb{R}, \ell^2)$ by $t \mapsto te_0$. We have $\sigma(A) = \{\lambda_n \mid n \in \mathbb{N}\}$ and hence $\sigma^+ = \{1\}$. Therefore the spectrum decomposition assumption is satisfied and $\dim X^+ = 1$. By the Stability Theorem in [AB88] the operator A^- generates a strongly stable semigroup. Since $B^- = 0$, the system $\Sigma(A^-, B^-)$ is strongly stable. The controllability matrix $R(A^+, B^+) = B^+ = 1$ has the full rank 1 and, thus, the system $\Sigma(A^+, B^+)$ is controllable. Theorem 5.7.4 now implies that the system $\Sigma(A, B)$ is strongly stabilizable.*

For the polynomial stabilizability we have a similar characterisation.

Theorem 5.7.6. *For any linear system $\Sigma(A, B)$ given by (5.1) with a finite-dimensional input space $U = \mathbb{C}^m$ and a bounded control operator, i.e., $B \in \mathcal{L}(\mathbb{C}^m, X)$, the following assertions are equivalent:*

(i) $\Sigma(A, B)$ is polynomially stabilizable.

(ii) $\Sigma(A, B)$ satisfies the spectrum decomposition assumption at zero, X^+ is finite-dimensional, $\Sigma(A^-, B^-)$ is a polynomially stable system and the finite-dimensional system $\Sigma(A^+, B^+)$ is controllable.

If $\Sigma(A, B)$ is polynomially stabilizable, then a stabilizing feedback operator is given by $F = F^+P_\Gamma$, where F^+ is a stabilizing feedback operator for $\Sigma(A^+, B^+)$.

Proof. By Proposition 5.6.6 we are left to show that the implication (i) ⇒ (ii) holds. Since every polynomially stabilizable system $\Sigma(A, B)$ is strongly stabilizable, Theorem 5.7.4 is applicable and it remains to show that the semigroup $(T^-(t))_{t\geq 0}$ generated by the operator A^- is polynomially stable. It is strongly stable and hence bounded. Using the identity (5.24) we obtain $\|(i\omega I - A)^{-1}\| = \mathrm{O}(|\omega|^{3\alpha})$ for $|\omega| \to \infty$, since $\|(i\omega I - A - BF)^{-1}\| = \mathrm{O}(|\omega|^{\alpha})$ for $|\omega| \to \infty$, holds. Thus, Theorem 2.4 in [BT10] is applicable and we obtain the polynomial stability of $(T^-(t))_{t\geq 0}$. □

Example 5.7.7. *Let $X = \ell^2$ and $U = \mathbb{R}$. Let $\lambda_0 = 1$ and for $n \in \mathbb{N}\setminus\{0\}$ let $\lambda_n = in - 1/n$. We define $A\colon X \supset D(A) \to X$ by $Ae_n = \lambda_n e_n$, where $D(A) = \{x = (x_n)_{n\in\mathbb{N}} \in \ell^2 \mid (\lambda_n x_n)_{n\in\mathbb{N}} \in \ell^2\}$ and $B \in \mathcal{L}(\mathbb{R}, \ell^2)$ by $t \mapsto te_0$. By the Stability Theorem in [AB88] the operator A^- generates* *a strongly stable (and hence bounded) semigroup. We have* $i\mathbb{R} \subset \rho(A^-)$*. For any* $\omega \in \mathbb{R}$*,* $x = (x_n)_{n\in\mathbb{N}} \in \ell^2$ *holds*

$$R(i\omega, A^-)x = \sum_{n=1}^{\infty} \frac{x_n}{i\omega - \lambda_n} e_n.$$

Therefore we obtain

$$\|R(i\omega, A^-)\| = \sup_{n\in\mathbb{N}\setminus\{0\}} \frac{1}{|i\omega - \lambda_n|} = \sup_{n\in\mathbb{N}\setminus\{0\}} \frac{1}{\sqrt{(\omega - n)^2 + \frac{1}{n^2}}} \leq |\omega| + 1.$$

In particular we have $\|R(i\omega, A^-)\| = \mathrm{O}(|\omega|)$ *for* $|\omega| \to \infty$*. Thus, Theorem 2.4 in [BT10] is applicable and we obtain the polynomial stability of* $(T^-(t))_{t\geq 0}$*. Since* $B^- = 0$*, the system* $\Sigma(A^-, B^-)$ *is polynomially stable. We have* $\sigma(A) = \{\lambda_n \mid n \in \mathbb{N}\}$ *and hence* $\sigma^+ = \{1\}$*. Therefore the spectrum decomposition assumption is satisfied and* X^+ *is one-dimensional. Since the controllability matrix* $R(A^+, B^+) = B^+ = 1$ *has the full rank 1, the system* $\Sigma(A^+, B^+)$ *is controllable. Theorem 5.7.6 thus implies that the system* $\Sigma(A, B)$ *is polynomially stabilizable.*

5.8 Concluding remarks

In this chapter we studied strong and polynomial stabilizability of linear systems on Hilbert spaces with bounded and unbounded control operators. We found sufficient conditions for both – strong and polynomial – stabilizability of linear systems with unbounded control operators and arbitrary input spaces. For systems with bounded control operators we found a characterisation of all systems with finite-dimensional input spaces, which are strongly or polynomially stabilizable respectively. Those equivalent conditions for stabilizability formally look very similar to those for exponential stabilizability obtained by W. Desch and W. Schappacher [DS85], C. A. Jacobson and C. N. Nett [JN88], and S. A. Nefedov and F. A. Sholokhovich [NS86] independently of each other. It remains an open problem to find a similar characterisation for systems with an unbounded control operator. This is one possible direction for further investigations.

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