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Ergodicity and parameter estimation for some affine models

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Kurzfassung

Die vorliegende Dissertation thematisiert ergodische Eigenschaften von spezifischen ein- und zweidimensionalen affinen Prozessen. Grob gesagt, besteht die Klasse der affinen Prozesse, eingeführt von Duffie, Filipović, und Schachermayer (2013), aus allen Markov-Prozessen in stetiger Zeit mit Wertebereich $\mathbb{R}^m_{\geqslant 0} \times \mathbb{R}^n$, deren logarithmierte charakteristische Funktion affin vom Anfangszustandsvektor des Prozesses abhängt. Eine wichtige Frage, die im Zusammenhang mit zeithomogenen Markov-Prozessen auftitt, ist deren Langzeitverhalten wie die Ergodizität. Bisher wurde die Ergodizität für affine Prozesse im Allgemeinen noch nicht untersucht. In dieser Dissertation werden zunächst spezifische (nicht-triviale) affine Modelle, wie der Sprung-Diffusions Cox-Ingersoll-Ross Prozess und ein zweidimensionales Modell basierend auf dem α-Wurzel Prozess, bezüglich Ergodizität untersucht. Aufgrund zahlreicher Anwendungen in der Finanzmathematik sind diese Modelle auch für sich genommen von Interesse.

Im ersten Teil dieser Dissertation wird ein affines Zweifaktorenmodell studiert, das auf den α-Wurzel Prozess basiert und von Barczy, Döring, Li und Pap (2014) eingeführt wurde. Eine Komponente dieses zweidimensionalen Modells ist der α-Wurzel Prozess. Es wird die exponentielle Ergodizität für das Zweifaktorenmodell für $\alpha \in (1, 2)$ gezeigt. Die Methodik basiert dabei hauptsächlich auf einer Anwendung des Foster-Lyapunov-Driftkriteriums, das von Meyn und Tweedie (1993) entwickelt wurde. Als ein Hilfsmittel zum Beweisen der Ergodizität und als weiteres Resultat ergibt sich die Existenz von positiven Übergangsdichten des α-Wurzel Prozesses.

Im zweiten Teil dieser Dissertation wird der Sprung-Diffusions Cox-Inger-soll-Ross Prozess vorgestellt, der als eine Erweiterung des klassischen Cox-Ingersoll-Ross Modells verstanden werden kann. Die Sprünge des Sprung-Diffusions Cox-Ingersoll-Ross Prozesses werden durch einen Subordinator beschrieben. Es werden hinreichende Bedingungen an das Lévy-Maß des Subordinators bestimmt, so dass der Sprung-Diffusions Cox-Ingersoll-Ross Prozess ergodisch bzw. exponentiell ergodisch ist. Zudem wird die Existenz der κ-Momente ($\kappa > 0$) des Sprung-Diffusions Cox-Ingersoll-Ross Prozesses charakterisiert durch eine Integrabilitätsbedingung an das Lévy-Maß des Subordinators. Als Konsequenz der Resultate ergibt sich die Konvergenz der Momente für den Sprung-Diffusions Cox-Ingersoll-Ross Prozess. Um eine Anwendung der Ergodizitätsresultate zu veranschaulichen, werden schließlich asymptotische Eigenschaften von bedingten Kleinste-Quadrate-Schätzern der Driftparameter des Sprung-Diffusions Cox-Ingersoll-Ross Prozesses basierend auf zeitdiskreten Beobachtungen untersucht. Im subkritischen Fall wird die Konsistenz und die asymptotische Normalität der Schätzer gezeigt.

Abstract

This thesis is devoted to the study of ergodic properties of some one and two-dimensional affine processes. Roughly speaking, the class of affine processes on the canonical state space, introduced by Duffie, Filipović, and Schachermayer (2013), consists of continuous-time Markov processes taking values in $\mathbb{R}^m_{\geqslant 0} \times \mathbb{R}^n$, whose log-characteristic function depends in an affine way on the initial state vector of the process. A question of interest in the context of time-homogeneous Markov processes is their long-time behavior such as the ergodicity. Unitl now, ergodicity is not very well investigated for general affine processes. This is one reason why we initially started to work on particular (non-trivial) affine models such as a jump-type Cox-Ingersoll-Ross process and a two-factor model based on the α-root process. A further reason is given by the fact that both models discussed in this thesis provide interesting applications in financial mathematics.

In the first part of this thesis we study an affine two-factor model based on the α-root process introduced by Barczy, Döring, Li, and Pap (2014). One component of this two-dimensional model is the so-called α-root process. We manage to prove exponential ergodicity of this two-factor model when $\alpha \in (1, 2)$ mainly by stochastic methods, e.g. a Foster-Lyapunov drift criteria developed by Meyn and Tweedie (1993). As a further result of our considerations, we obtain existence of positive transition densities of the α-root process.

In the second part of the thesis we introduce the jump-diffusion Cox-Ingersoll-Ross process, which is an extension of the Cox-Ingersoll-Ross model and whose jumps are introduced by a subordinator. We provide sufficient conditions on the Lévy measure of the subordinator under which the jump-diffusion Cox-Ingersoll-Ross process is ergodic and exponentially ergodic, respectively. Furthermore, we characterize the existence of the κ-moment ($\kappa > 0$) of the jump-diffusion Cox-Ingersoll-Ross process by an integrability condition on the Lévy measure of the subordinator. As a consequence of our results, we obtain a moment convergence theorem for the jump-diffusion Cox-Ingersoll-Ross process. Eventually, to illustrate the use of our ergodic results, we study asymptotic properties of conditional least squares estimators for the drift parameters of the jump-diffusion Cox-Ingersoll-Ross process based on discrete time observations. In the subcritical case we prove strong consistency and asymptotic normality of our parameter estimators.

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Contents

Introduction

This thesis investigates the ergodic properties of some one and two-dimensional affine processes. The first model $(Y, X) := (Y_t, X_t)_{t\geqslant 0}$, studied in this thesis, is determined by the following stochastic differential equation:

$$\begin{cases} \mathrm{d}Y_t = (a - bY_t)\mathrm{d}t + \sqrt[\alpha]{Y_{t-}}\mathrm{d}L_t, & t \geqslant 0, \quad Y_0 \geqslant 0 \quad \text{a.s.}, \\ \mathrm{d}X_t = (m - \theta X_t)\mathrm{d}t + \sqrt{Y_t}\mathrm{d}B_t, & t \geqslant 0, \end{cases}$$

where $\alpha \in (1, 2)$, $(L_t)_{t\geqslant 0}$ is a spectrally positive α-stable Lévy process with the Lévy measure $(\alpha\Gamma(-\alpha))^{-1} z^{-1-\alpha} \mathbb{1}_{\{z>0\}}\mathrm{d}z$, and $(B_t)_{t\geqslant 0}$ is an independent standard Brownian motion. The process $(Y_t, X_t)_{t\geqslant 0}$ was introduced by Barczy, Döring, Li, and Pap [4]. The second model, this thesis deals with, is the jump-diffusion Cox-Ingersoll-Ross (shorted as JCIR) process. The JCIR process $Z = (Z_t)_{t\geqslant 0}$ is defined as the unique strong solution to the stochastic differential equation

$$\mathrm{d}Z_t = (a - bZ_t)\mathrm{d}t + \sigma\sqrt{Z_t}\mathrm{d}B_t + \mathrm{d}J_t, \quad t \geqslant 0, \quad Z_0 \geqslant 0 \text{ a.s.},$$

where $(B_t)_{t\geqslant 0}$ is a one-dimensional Brownian motion and $(J_t)_{t\geqslant 0}$ is a pure jump Lévy process with its Lévy measure ν concentrating on $(0, \infty)$ and satisfying $\int_0^\infty (z\wedge 1)\nu(\mathrm{d}z) < \infty$. Further assumptions to the parameters of both models are given in the respective chapters hereafter.

Both processes, the two-factor model (Y, X) based on the α-root process and the JCIR process Z, exhibit a log-characteristic function which depends linearly on the initial state vector of the respective process. Roughly speaking, processes arising with this property are called *affine processes*. Strictly speaking, an affine process on $\mathbb{R}^m_{\geqslant 0} \times \mathbb{R}^n$ (for integers $m \geqslant 0$ and $n \geqslant 0$) is a continuous-time and stochastically continuous Markov process taking values in $\mathbb{R}^m_{\geqslant 0} \times \mathbb{R}^n$, whose log-characteristic function depends in an affine way on the initial state vector of the process, i.e. the log-characteristic function is linear with respect to the initial state vector. Affine processes are particularly important in financial mathematics because of their computational tractability. For example, the models of Cox, Ingersoll, and Ross [15], Heston [25] and Vasicek [64] are all based on affine processes. In the case $\alpha = 2$, the two-factor model (Y, X) was used by Chen and Joslin [14] to price defaultable bonds with stochastic recovery rates. As an application of the JCIR process, Barletta and Nicolato [9] recently studied a stochastic volatility model with jumps for the sake of pricing of VIX options, where the volatility (or instantaneous variance process) of the asset price process is modelled via the JCIR process.

The general theory of affine processes on the canonical state space $\mathbb{R}^m_{\geqslant 0} \times \mathbb{R}^n$ was initiated by Duffie, Pan and Singleton [19] and further developed by Duffie, Filipović, and Schachermayer [17]. This type of process unifies the notions of continuous-state branching processes with immigration (shorted as CBI) (see, e.g., [38]) and Ornstein–Uhlenbeck (OU) type processes (e.g.,[59]). Due to Duffie *et al.* [17], the affine processes with state space $\mathbb{R}^m_{\geqslant 0}$ are CBI, and those with state space $\mathbb{R}^n$ are of OU type.

In their seminal article they also established a rigorous mathematical foundation to the theory of affine processes, covering aspects, such as the characterization of an affine process in terms of the *admissible parameters* and properties of the *generalized Riccati equations* that are implied by the process. Apart from these pioneering results, many authors provided further important results regarding affine processes or subclasses of affine processes, respectively. For instance, Keller-Ressel, Schachermayer, and Teichmann [42] proved that the time-differentiability of the characteristic function of the given affine process is implied by its stochastic continuity. Moreover, Keller-Ressel and Mayerhofer [40] investigated the exponential moments of affine processes. For the subclass of CBI processes, an identification as a pathwise unique strong solution of certain stochastic differential equations with jumps has been studied by Barczy, Li, and Pap [5] as well as a moment formula [6]. In addition to these results, the existence of fractional moments of one-dimensional CBI procsess has been investigated by Ji and Li [29]. Another topic of great interest and a rather naturally arising question in the context of Markov processes is the asymptotic behavior of the process. So one may ask under which conditions does the affine process converge with respect to time to a limit distribution. Closely related to this question is the existence of a (unique) invariant measure for the affine process. This question has been investigated by some authors, see, e.g., [24, 39, 43, 59].

However, among all the mentioned properties of affine processes or subclasses of affine processes, the ergodicity property for general affine processes does not seem to have been investigated as yet. For a time-homogeneous Markov process $M = (M_t)_{t\geqslant 0}$ with state space E, let $\mathbf{P}^t(x,\cdot) := \mathbb{P}_x(X_t \in \cdot)$ denotes the distribution of M_t with the initial condition $M_0 = x \in E$. We call M *ergodic* if it admits a unique invariant probability measure π such that

$$\lim_{t\to\infty} \left\| \mathbf{P}^t(x,\cdot) - \pi \right\|_{TV} = 0, \quad \text{for all } x \in E,$$

where $\|\cdot\|_{TV}$ stands for the total variation norm of a signed measure. The Markov process M is called *exponentially ergodic* if it is ergodic and in addition there exists a finite-valued function B on E and a positive constant δ such that

$$\left\| \mathbf{P}^t(x,\cdot) - \pi \right\|_{TV} \leqslant B(x)e^{-\delta t}, \quad \text{for all } x \in E,\ t > 0.$$

One rather general result is due to Masuda [48], who provides sets of conditions under which the OU type process is ergodic and exponentially ergodic as well. Jin, Mandrekar, Rüdiger, and Trabelsi [32], Jin, Rüdiger, and Trabelsi [33, 34] initially started to work on particular CBI processes and managed to prove exponential ergodicity of these models, which arise as extensions of the CIR model. The first affine two-factor model shown to be exponentially ergodic is the affine two-factor model (Y, X) based on the square-root process, i.e., $\alpha = 2$, as recently investigated by Barczy, Döring, Li, and Pap, [4].[1]

Though derivation of the ergodic property is interesting in its own right, this thesis was mainly motivated by statistical analysis for affine processes. Indeed, an important issue for the application of affine processes is the calibration of their parameters. This has been considered for some well known affine models, see e.g. [3, 7, 11, 54, 55]. To study the asymptotic properties of estimators of the parameters, a comprehension of the long-time behavior of the underlying affine processes is very often required. This

[1] That is, the first component Y of the two-factor model is the standard CIR process.

is one of the reasons why the stationary, ergodic and recurrent properties of affine processes have recently attracted many investigations, see e.g. [4, 20, 32–34, 41, 43, 45], and many others.

It was our intention in this work to take up loose ends from both, the article of Barczy *et al.* [4] and Jin *et al.* [33], and to prove (exponential) ergodicity of the model (Y, X) as well as Z in a quiet more general set-up. In part one of this thesis we study the ergodicity problem for the two-factor model (Y, Z) based on the α-root process when $1 < \alpha < 2$. As our main result in Part I of this thesis, we show that $(Y_t, Z_t)_{t\geqslant 0}$ is exponentially ergodic if $\alpha \in (1, 2)$, provided some further assumptions to the parameters, complementing the results of Barczy *et al.* [4], who already proved that the two-factor model has a stationary distribution when $\alpha \in (1, 2)$ and is exponentially ergodic if $\alpha = 2$. We remark that only the case $1 < \alpha < 2$ allows for activity of jumps. In Section 1.1 we briefly introduce the two-factor model as an affine process and derive the Laplace transform of the α-root process Y. Our approach to obtain the exponential ergodicity of the two-factor model (Y, Z) is motivated by that of Jin *et al.* [33]. As a first step, in Section 1.2 we show the existence of positive transition densities of the α-root process $(Y_t)_{t\geqslant 0}$. To achieve this, we calculate explicitly the Laplace transform of it. Through a careful analysis of the decay rate of the Laplace transform of the α-root process at infinity, we manage to show the positivity of the density function of the α-root process using the inverse Fourier transform. The positivity of the density function of Y plays an essential role in the proof of the exponential ergodicity for the two-factor model (Y, X), since it enables us to show that the Lebesgue measure is an irreducibility measure for the skeleton chains of the model. Our method of proving the existence of a positive density function for the α-root process Y is purely analytic. In the second step, we construct a Foster-Lyapunov function for the two-factor model (Y, X), see Section 1.3. Using the general theory of Meyn and Tweedie [50–52] on the ergodicity of Markov processes, we are then able to derive in Section 1.4 the exponential ergodicity of the two-factor model based on the α-root process.

Part II of this thesis is devoted to the study of the JCIR process $(Z_t)_{t\geqslant 0}$, which is an extension of the CIR model and whose jumps are introduced by a subordinator. As mentioned before, to study the fine properties of the estimators, a comprehension of the long-time behavior of the underlying process is required, but, eventually, it turns out that also a knowledge of the moments is necessary for a construction of the different estimators. The purpose of the second part is twofold. Firstly, in Chapter 2 we focus on the ergodicity and moment characterization problem for the JCIR process $(Z_t)_{t\geqslant 0}$ and analyse their subtle dependence on the big jumps of the subordinator. In Section 2.1 we derive the affine property of the JCIR process and, as a first step, using a decomposition of its characteristic function, we show existence of positive transition densities of the JCIR process, which improves a similar result in [33]. Section 2.2 contains our first main result, namely a characterization of the existence of κ-moments ($\kappa > 0$) of the JCIR process Z in terms of the Lévy measure, which is implied by the subordinator. The second aim we pursue in this chapter is to improve the results of Jin *et al.* [33] on the ergodicity of the JCIR process. Sections 2.3 and 2.4 are devoted to the proof of the ergodicity and exponential ergodicity in question of the JCIR process $(Z_t)_{t\geqslant 0}$, respectively. In the second step to achieve this, we construct some Foster-Lyapunov functions for $(Z_t)_{t\geqslant 0}$, which enable us to prove the asserted (exponential) ergodicity by using the results of Meyn and Tweedie [50–52]. For the construction of the Foster-Lyapunov functions we will use some ideas from Masuda [48]. To round out the picture presented by our study of the moments of the JCIR process, in Section 2.5, we present a moment convergence theorem for the JCIR process.

Finally, in Chapter 3 we turn towards the study of asymptotic properties of conditional least squares estimators (CLSEs) for the drift parameters of the JCIR process based on discrete-time observations in order to illustrate an application of our ergodicity result. To achieve this, we start by introducing CLSEs for transformed parameters of the drift of the JCIR process $(Z_t)_{t\geqslant 0}$. As our main results, in Sections 3.1 and 3.2, in the subcritical case (i.e., $b > 0$) we prove that the transformed CLSE is strongly consistent and asymptotic normal. Eventually, we conclude in Section 3.3 with an explicit calculation of a strongly consistent and asymptotic normal CLSE for the original drift of Z based on the analogous properties of the transformed CLSE. Our approach is close to that of Barcy, Pap, and Szabó [8] and Overbeck and Rydén [55], who build some CLSEs based on discrete observations for the original CIR process. The parameter estimation problem for the JCIR process is more complicated, since it has an additional parameter given by the Lévy measure of the driving noise and thus an infinite dimensional object. Nevertheless, based on low frequency observations, Xu [65] proposed some nonparametric estimators for ν, given that ν is absolutely continuous with respect to the Lebesgue measure. Recently, Barczy, Ben-Alaya, Kebaier, and Pap [2] studied also the maximum likelihood estimator for the *growth rate* of the JCIR process based on continuous time observations.

Finally, we provide the reader with a short introduction to two-dimensional affine processes, the basic definitions of properties of Markov chains on uncountable state spaces used in this thesis, as well as a strong law of large numbers and a central limit theorem for discrete time square-integrable martingales in the appendix of the thesis.

Credits. Most of the results in part one of this thesis are established in [31]. This article is a joint work with P. Jin and B. Rüdiger. Apart from the above mentioned we have to refer to other active researchers who are working on this particular model, namely M. Barczy, L. Doering, Z. Li, and G. Pap, [4]. The proofs provided in the first part of this thesis are taken from our joint article [31] with P. Jin and B. Rüdiger and furnished with some details and explanations where we deem it appropriate. In order to present the whole notion of the (two-dimensional) factor model based on the α-root process, we occasionally recall results with proofs or sketches of proofs from Barcy *et al.* [4] and Z. Li and C. Ma [45] (see also the references given in the specific sections).

The stated results about the (exponential) ergodicity of the JCIR process in part two of this thesis are established in a joint work with P. Jin and B. Rüdiger [30].[2] Our considerations about ergodicity of the JCIR process are mainly based on a preparatory work of P. Jin, B. Rüdiger and C. Trabelsi [33]. In our article [30] we managed to improve the results in [33] and additionally to add the characterization of the moments. In this thesis, we add to this chapter some result about the convergence of the moments for the JCIR process which is new, at least to the authors' knowledge.

The parameter estimation of the drift parameters of the JCIR process is motivated by an article of M. Barczy, G. Pap, and T. Szabó [8]. We mimic the proof of [8, Theorem 3.2] with appropriate adjustments where this is necessary in order to transfer into our

[2]Submitted to an international journal.

framework of the JCIR process. The elaboration of this chapter arose out of working with P. Jin and B. Rüdiger. This part of the thesis is also new and not submitted to an international journal yet.

Notation. Throughout this thesis, we use the following notations. Let $\mathbb{N}$, $\mathbb{Z}_{\geqslant 0}$, $\mathbb{R}$, $\mathbb{R}_{\geqslant 0}$, $\mathbb{R}_{>0}$ and $\mathbb{R}_{\leqslant 0}$ denote the sets of positive integers, non-negative integers, real numbers, non-negative real numbers, strictly positive real numbers, and negative real numbers, respectively. Let $\mathbb{C}$ be the set of complex numbers as well as $\mathbb{C}^2$ the set of two-dimensional complex numbers. For $z \in \mathbb{C} \setminus \{0\}$ we denote by $\mathrm{Arg}(z)$ the principal value of its argument and by $\bar{z}$ its conjugate. We define the following subset of $\mathbb{C}^2$:

$$\mathcal{U} := \left\{ u = (u_1, u_2) \in \mathbb{C}^2 \,:\, \mathrm{Re}\, u_1 \leqslant 0 \text{ and } \mathrm{Re}\, u_2 = 0 \right\}.$$

Further, we define the following subsets of $\mathbb{C}$:

$$\mathbb{C}_{\leqslant 0} := \{u \in \mathbb{C} \,:\, \mathrm{Re}\, u \leqslant 0\}, \quad \mathbb{C}_{\geqslant 0} := \{u \in \mathbb{C} \,:\, \mathrm{Re}\, u \geqslant 0\},$$
$$\mathbb{C}_{<0} := \{u \in \mathbb{C} \,:\, \mathrm{Re}\, u < 0\}, \quad \mathbb{C}_{>0} := \{u \in \mathbb{C} \,:\, \mathrm{Re}\, u > 0\},$$

and the set of purely imaginary numbers

$$i\mathbb{R} := \{u \in \mathbb{C} \,:\, \mathrm{Re}\, u = 0\}, \quad \text{together with } \mathcal{O} := \mathbb{C} \setminus \{-x \,:\, x \in \mathbb{R}_{\geqslant 0}\}.$$

With that notation, clearly $\mathcal{U} = \mathbb{C}_{\leqslant 0} \times i\mathbb{R}$. For $z \in \mathbb{C} \setminus \{0\}$ let $\mathrm{Log}(z)$ be the principal value of the complex logarithm of z, i.e., $\mathrm{Log}(z) = \ln(|z|) + i\mathrm{Arg}(z)$. In this thesis, we define $\mathrm{Arg}(x) := \pi$ for $x \in (-\infty, 0)$. For $\beta \in \mathbb{R}$ define the complex power function z^β as

$$z^\beta := \exp(\beta \,\mathrm{Log}\, z), \quad z \in \mathbb{C} \setminus \{0\}. \tag{0.0.1}$$

By $C^2(S, \mathbb{R})$, $C_c^2(S, \mathbb{R})$, $C_b^2(S, \mathbb{R})$, and $C^\infty(S, \mathbb{C})$ we denote the sets of $\mathbb{R}$-valued or $\mathbb{C}$-valued functions on S that are twice continuously differentiable, that are twice continuously differentiable with compact support, that are bounded continuous with bounded continuous first and second order partial derivatives, and that are smooth, respectively, where the space S can be $\mathbb{R}$, $\mathbb{R}_{\geqslant 0}$, $\mathbb{R}_{\geqslant 0} \times \mathbb{R}$ or $\mathbb{R}_{\geqslant 0} \times \mathbb{R}_{\geqslant 0} \times \mathbb{R}$ in this thesis. We denote the Borel σ-algebra on S simply by $\mathcal{B}(S)$. Similarly, we denote by $\mathcal{B}_b(S)$ the set of bounded Borel measurable functions acting on S. We endow the Space S with the inner product $\langle \cdot, \cdot \rangle$. For $a,\ b \in \mathbb{R}$, we denote by $a \wedge b$ and $a \vee b$ the minimum and maximum of a and b, respectively.

Throughout this thesis, we assume that $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geqslant 0}, \mathbb{P})$ is a filtered probability space satisfying the usual conditions, i.e., $(\Omega, \mathcal{F}, \mathbb{P})$ is complete, the filtration $(\mathcal{F}_t)_{t \geqslant 0}$ is right-continuous and $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets in $\mathcal{F}$.

Part I.

Exponential ergodicity of an affine two-factor model based on the alpha-root process

1. The affine two factor model based on the alpha-root process

In this chapter we recall some important properties of the affine process $(Y,X) := (Y_t, X_t)_{t\geqslant 0}$ defined as the (pathwise) unique strong solution of the stochastic differential equation

$$\begin{cases} \mathrm{d}Y_t = (a - bY_t)\mathrm{d}t + \sqrt[\alpha]{Y_{t-}}\mathrm{d}L_t, & t \geqslant 0, \quad Y_0 \geqslant 0 \quad \text{a.s.}, \\ \mathrm{d}X_t = (m - \theta X_t)\mathrm{d}t + \sqrt{Y_t}\mathrm{d}B_t, & t \geqslant 0, \end{cases} \tag{1.0.1}$$

where $a > 0$, $b > 0$, θ, $m \in \mathbb{R}$, $\alpha \in (1,2)$, $(L_t)_{t\geqslant 0}$ is a spectrally positive α-stable Lévy process with the Lévy measure $C_\alpha z^{-1-\alpha}\mathbb{1}_{\{z>0\}}\mathrm{d}z$, with $C_\alpha := (\alpha\Gamma(-\alpha))^{-1}$, and $(B_t)_{t\geqslant 0}$ is an independent standard Brownian motion. The process $(Y_t, X_t)_{t\geqslant 0}$ given by (1.0.1) was introduced by Barczy *et al.* [4]. The strong solution $Y = (Y_t)_{t\geqslant 0}$ of the first stochastic differential equation,

$$\mathrm{d}Y_t = (a - bY_t)\mathrm{d}t + \sqrt[\alpha]{Y_{t-}}\mathrm{d}L_t, \quad t \geqslant 0, \quad Y_0 \geqslant 0 \quad \text{a.s.}, \tag{1.0.2}$$

is sometimes referred as α-root, α-stable or simply stable CIR process. Note, if $\alpha = 2$ in (1.0.2) then $(L_t)_{t\geqslant 0}$ is a standard Brownian motion. In that case, due to the almost surely continuity of the sample paths of a Brownian motion, instead of $\sqrt{Y_{t-}}$ one may write $\sqrt{Y_t}$ in the stochastic differential equation (1.0.2), and Y is nothing but the CIR process.

The following paragraph is intended to give the reader an insight into the notion of Itô type stochastic integrals with respect to α-stable Lévy processes. We shed some light on the notion of a spectrally positve α-stable Lévy process $L := (L_t)_{t\geqslant 0}$ in prior. A non-subordinator[1] is said to be spectrally positive if it has no negative jumps. An $(\mathcal{F}_t)_{t\geqslant 0}$-adapted stochastic process $(L_t)_{t\geqslant 0}$ is said to be an α-stable Lévy process, $\alpha \in (1,2)$, if $L_0 = 0$ almost surely, $L_t - L_s$, $0 \leqslant s < t$, is stable distributed[2] and for any finite time points $0 \leqslant t_0 < t_1 < \cdots < t_n < \infty$, the random variables $L_{t_0}, L_{t_1} - L_{t_0}, \ldots, L_{t_n} - L_{t_{n-1}}$ are independent. Stable Lévy processes form a subclass of Lévy processes which are widely studied in [59, Chapter 3]. In our framework, we assume that $(B_t)_{t\geqslant 0}$ is a standard $(\mathcal{F}_t)_{t\geqslant 0}$-Brownian motion and $(L_t)_{t\geqslant 0}$ is a spectrally positive α-stable $(\mathcal{F}_t)_{t\geqslant 0}$-Lévy process with the Lévy measure $C_\alpha z^{-1-\alpha}\mathbb{1}_{\{z>0\}}\mathrm{d}z$, where $1 < \alpha < 2$.

A consequence of the absence of negative jumps and the stable property of L is that the characteristic function of L_1 reduces to

$$\mathbb{E}\left[e^{iuL_1}\right] = \exp\left\{\int_0^\infty \left(e^{iuz} - 1 - iuz\right) C_\alpha z^{-1-\alpha}\mathrm{d}z\right\}, \quad u \in \mathbb{R}.$$

[1] We call a Lévy process a subordinator if its sample paths are increasing.

[2] The random variable $L_t - L_s$ is said to follow an α-stable distribution with $\alpha \in (1,2)$, if it has characteristic function given by

$$\mathbb{E}\left[e^{iu(L_t - L_s)}\right] = \exp\left\{-(t-s)|u|^\alpha(1 - i\beta \operatorname{sgn}(u)\tan\tfrac{\alpha\pi}{2}\right\}, \quad u \in \mathbb{R},$$

where $\beta \in [-1,1]$. If in addition $\beta = 0$, L_t is called symmetric α-stable.

Let $N(\mathrm{d}s, \mathrm{d}z)$ be a Poisson random measure on $\mathbb{R}^2_{>0}$ with intensity measure given by $C_\alpha z^{-1-\alpha}\mathbb{1}_{\{z>0\}}\mathrm{d}s\mathrm{d}z$ and $\widehat{N}(\mathrm{d}s, \mathrm{d}z)$ be its compensator. Then the Lévy-Itô representation of L takes the form

$$L_t = \gamma t + \int_0^t \int_{\{|z|<1\}} z\widetilde{N}(\mathrm{d}s, \mathrm{d}z) + \int_0^t \int_{\{|z|\geqslant 1\}} zN(\mathrm{d}s, \mathrm{d}z), \quad t \geqslant 0, \tag{1.0.3}$$

where $\gamma := -\mathbb{E}\left[\int_0^1 \int_{\{|z|\geqslant 1\}} zN(\mathrm{d}s, \mathrm{d}z)\right]$ and $\widetilde{N}(\mathrm{d}s, \mathrm{d}z) := N(\mathrm{d}s, \mathrm{d}z) - \widehat{N}(\mathrm{d}s, \mathrm{d}z)$, with $\widehat{N}(\mathrm{d}s, \mathrm{d}z) = C_\alpha z^{-1-\alpha}\mathbb{1}_{\{z>0\}}\mathrm{d}s\mathrm{d}z$, is the compensated Poisson random measure on $\mathbb{R}^2_{>0}$ that corresponds to $N(\mathrm{d}s, \mathrm{d}z)$. We remark that $\gamma t = -\int_0^t \int_{\{|z|\geqslant 1\}} z\widehat{N}(\mathrm{d}s, \mathrm{d}z)$ and

$$\int_0^t \int_{\{|z|\geqslant 1\}} zN(\mathrm{d}s, \mathrm{d}z) + \gamma t, \quad t \geqslant 0,$$

is thus a martingale with respect to the filtration $(\mathcal{F}_t)_{t\geqslant 0}$.

Itô type stochastic integrals with respect to a (symmetric) α-stable Lévy process have some history. It is worth to mention that they are extensively studied in [35], and [36]. By [36, Theorem 3.1], a real-valued $(\mathcal{F}_t)_{t\geqslant 0}$-predictable process X on $\Omega \times \mathbb{R}_{\geqslant 0}$ is integrable with respect to a (symmetric) α-stable Lévy process L_t, that is $\int_0^t X_s \mathrm{d}L_s$ exists for every $t \geqslant 0$, if and only if X satisfies the integrability condition $\int_0^t |X_s|^\alpha \mathrm{d}s < \infty$ almost surely. A construction of stochastic integrals with respect to symmetric α-stable processes is due to Rosinski and Woyczynski [58, Theorem 2.1]. Another way is to consider L as a semimartingale, see e.g. Jacod and Shiryaev [27, Corollary II.4.19], so that Theorems I.4.31 and I.4.40 in Jacod and Shiryaev [27] describe the classes of processes which are integrable with respect to L, see also Remark 1.3 below.

We recall an inequality for moments of stochastic integrals driven by an α-stable Lévy process L_t.

Remark 1.1 (Remark A.8 [45]). *Let $(L_t)_{t\geqslant 0}$ be an α-stable Lévy process with $\alpha \in (0, 2]$ and $(X_t)_{t\geqslant 0}$ a predictable process satisfying almost surely*

$$\int_0^T |X_t|^\alpha \mathrm{d}s < \infty, \quad T \geqslant 0.$$

Let $\beta \in (0, \alpha)$. Then there exists a constant $C = C(\alpha, \beta) \geqslant 0$ such that

$$\mathbb{E}\left[\sup_{t\in[0,T]} \left|\int_0^t X_s \mathrm{d}L_s\right|^\beta\right] \leqslant C\mathbb{E}\left[\left(\int_0^T |X_t|^\alpha \,\mathrm{d}s\right)^{\beta/\alpha}\right]. \tag{1.0.4}$$

The moment inequality (1.0.4) for $\beta = 1$ follows from Rosinksi and Woyczynski [57, Theorem 3.1 and 3.2] in the symmetric case and is extended to the non-symmetric case by Liang [46, Lemma 2.4 and Remark 2.5]. The case $\beta \in (0, \alpha)$ could be considered as a generalization of [46, Remark 2.5], using Hölder's inequality.

As usual for the notion of Markov processes, the notation $\mathbb{E}_{(y,x)}[\cdot]$ means that the process (Y, X) considered under the expectation is with initial condition $(Y_0, X_0) = (y, x)$. The following result is a consequence of Remark 1.1 and yields the existence of moments of Y_t up to a degree $\beta \in (0, \alpha)$. It turns out to play a substantial role in our future considerations.

Proposition 1.2. *Consider the α-CIR process $(Y_t)_{t\geqslant 0}$ with $\alpha \in (1,2)$ defined by* (1.0.2)*. Then for any $\beta \in (0,\alpha)$, there exists a constant $C \geqslant 0$ and a locally bounded function $T \mapsto C(T) \geqslant 0$ such that, for t, $T \geqslant 0$,*

$$\mathbb{E}_y\left[Y_t^{\beta}\right] \leqslant C\left(1 + y^{\beta}e^{-\beta bt/\alpha}\right).$$

and

$$\mathbb{E}_y\left[\sup_{t\in[0,T]} Y_t^{\beta}\right] \leqslant C(T)\left(1 + y^{\beta}\right).$$

For a proof we refer to [45, Proposition 2.8]. A rather direct but important consequence of Proposition 1.2 is the next result.

Remark 1.3. *Following Ikeda and Watanabe* [26, p.61-63]*, we define two classes:*

$$F^1 := \Big\{ f(t,x,\omega) \,:\, f \text{ is } \mathcal{F}_t\text{-predictable and for each } t>0,$$
$$\mathbb{E}\left[\int_0^t \int_{\{|x|\geqslant 1\}} |f(s,x,\cdot)|\widehat{N}(\mathrm{d}s,\mathrm{d}z)\right] < \infty \Big\};$$

$$F^{2,\mathrm{loc}} := \Big\{ f(t,x,\omega) \,:\, f \text{ is } \mathcal{F}_t\text{-predictable and there exists a sequence of}$$
$$\mathcal{F}_t\text{-stopping times } \tau_n \text{ such that } \tau_n \uparrow \infty \text{ almost surely and for each } t>0,$$
$$\mathbb{E}\left[\int_0^{t\wedge\tau_n} \int_{\{|x|<1\}} |f(s,x,\cdot)|^2\widehat{N}(\mathrm{d}s,\mathrm{d}z)\right] < \infty,\, n=1,2,\ldots \Big\}.$$

Let $\tau_n := \inf\{t \in \mathbb{R}_{>0} \,:\, Y_{t-} \geqslant n\}$, $n \in \mathbb{N}$. Noting that $(Y_t)_{t\geqslant 0}$ is predictable as the strong solution of the stochastic differential equation (1.0.2) *(see Theorem 1.4 below), from*

$$\begin{aligned}\mathbb{E}\left[\int_0^{t\wedge\tau_n}\int_{\{|z|<1\}} \left(z\sqrt[\alpha]{Y_{s-}}\right)^2 C_\alpha z^{-1-\alpha}\mathrm{d}s\mathrm{d}z\right] &\leqslant C_\alpha \int_0^1 z^{1-\alpha}\mathrm{d}z \int_0^t \mathbb{E}\left[\mathbb{1}_{\{Y_{s-}\leqslant n\}}Y_{s-}^{2/\alpha}\right]\mathrm{d}s \\ &= \frac{C_\alpha}{2-\alpha} n^{2/\alpha} < \infty, \end{aligned} \tag{1.0.5}$$

it follows that $\mathbb{R}^2_{\geqslant 0}\times\Omega \ni (s,z,\omega) \mapsto \mathbb{1}_{\{|z|<1\}} z\sqrt[\alpha]{Y_{s-}} \in F^{2,\mathrm{loc}}$. Similarly, since for any $0<\varepsilon<\alpha$ we have that $\mathbb{E}_y[Y_t^{\varepsilon}] \leqslant c_1(1+y^{\varepsilon}e^{-\varepsilon bt/\alpha})$ for $t \geqslant 0$ by Proposition 1.2, where $c_1>0$ is some constant, we obtain

$$\begin{aligned}\mathbb{E}\left[\int_0^t\int_{\{|z|\geqslant 1\}} \left|z\sqrt[\alpha]{Y_{s-}}\right| C_\alpha z^{-1-\alpha}\mathrm{d}s\mathrm{d}z\right] &= C_\alpha \int_1^\infty z^{-\alpha}\mathrm{d}z \int_0^t \mathbb{E}\left[\sqrt[\alpha]{Y_{s-}}\right]\mathrm{d}s \\ &\leqslant \frac{C_\alpha}{\alpha-1} c_1 \int_0^t \left(1 + Y_0^{1/\alpha}e^{-bs/\alpha^2}\right)\mathrm{d}s < \infty, \end{aligned} \tag{1.0.6}$$

which verifies that $\mathbb{R}^2_{\geqslant 0}\times\Omega \ni (s,z,\omega) \mapsto \mathbb{1}_{\{|z|\geqslant 1\}} z\sqrt[\alpha]{Y_{s-}} \in F^1$.

We now turn back to the two-dimensional process (Y,X) defined in (1.0.1). The next Theorem indeed ensures the before mentioned (pathwise) uniqueness and existence of a strong solution of the stochastic differential equation (1.0.2). Without any further specification, we always assume that (Y_0, X_0) is a random vector independent of $(L_t)_{t\geqslant 0}$. The proof is very close to that of Barczy *et al.* [4, Theorem 2.2], although we allow the parameter $a = 0$ in (1.0.2), which is different as in [4]. In that case the stochastic differential equation (1.0.2) turns into

$$\mathrm{d}Y_t = -bY_t\mathrm{d}t + \sqrt[\alpha]{Y_{t-}}\mathrm{d}L_t, \quad t \geqslant 0, \quad Y_0 \geqslant 0 \text{ a.s..} \tag{1.0.7}$$

Theorem 1.4. *Let (Y_0, X_0) be a random vector independent of $(L_t, B_t)_{t\geqslant 0}$ satisfying $Y_0 \geqslant 0$ almost surely. Then for all $a \geqslant 0$, $b, m, \theta \in \mathbb{R}$ and $\alpha \in (1,2)$, there is a (pathwise) unique strong solution $(Y_t, X_t)_{t\geqslant 0}$ of the stochastic differential equation* (1.0.1). *If $a \in \mathbb{R}_{>0}$, then $(Y_t)_{t\geqslant 0}$ is almost surely non-negative for all $t \geqslant 0$. Further, we have*

$$Y_t = e^{-bt}Y_0 + \frac{a}{b}\left(1 - e^{-bt}\right) + \int_0^t e^{-b(t-s)}\sqrt[\alpha]{Y_{s-}}\mathrm{d}L_s, \quad t \geqslant 0, \tag{1.0.8}$$

and

$$X_t = e^{-\theta t}X_0 + \frac{m}{\theta}\left(1 - e^{-\theta t}\right) + \int_0^t e^{-\theta(t-s)}\sqrt{Y_s}\mathrm{d}B_s, \quad t \geqslant 0. \tag{1.0.9}$$

Proof. Let $a \in \mathbb{R}_{>0}$. Applying a result of Fu and Li [23, Theorem 6.2 and Corollary 6.3], we get that a (pathwise) unique strong solution $(Y_t)_{t\geqslant 0}$ of the SDE (1.0.2) exists with any initial state value Y_0 satisfying $Y_0 \geqslant 0$ almost surely such that $(Y_t)_{t\geqslant 0}$ stays almost surely non-negative. In case $a = 0$, by [23, Theorem 6.2 and Corollary 6.3], a unique strong solution of (1.0.7) also exists. Furthermore, using Itô's formula to the process $(e^{bt}Y_t)_{t\geqslant 0}$ yields that

$$\begin{aligned}\mathrm{d}\left(e^{bt}Y_t\right) &= be^{bt}Y_t\mathrm{d}t + e^{bt}\mathrm{d}Y_t = be^{bt}Y_t\mathrm{d}t + e^{bt}\left((a - bY_t)\,\mathrm{d}t + \sqrt[\alpha]{Y_{t-}}\mathrm{d}L_t\right)\\ &= ae^{bt}\mathrm{d}t + e^{bt}\sqrt[\alpha]{Y_{t-}}, \quad t \geqslant 0.\end{aligned} \tag{1.0.10}$$

Hence, writing (1.0.10) in integral form, we get

$$e^{bt}Y_t - Y_0 = a\int_0^t e^{bs}\mathrm{d}s + \int_0^t e^{bs}\sqrt[\alpha]{Y_{s-}}\mathrm{d}L_s, \quad t \geqslant 0,$$

yielding (1.0.8). Now, using Itô's formula to $(X_t)_{t\geqslant 0}$ defined in (1.0.9), we obtain

$$\begin{aligned}\mathrm{d}X_t &= -\theta e^{-\theta t}\left(X_0 + m\int_0^t e^{\theta s}\mathrm{d}s + \int_0^t e^{\theta s}\sqrt{Y_s}\mathrm{d}B_s\right)\mathrm{d}t + e^{-\theta t}\left(me^{\theta t}\mathrm{d}t + e^{\theta t}\sqrt{Y_t}\mathrm{d}B_t\right)\\ &= -\theta e^{-\theta t}\left(X_0 + m\int_0^t e^{\theta s}\mathrm{d}s + \int_0^t e^{\theta s}\sqrt{Y_s}\mathrm{d}B_s\right)\mathrm{d}t + m\mathrm{d}t + \sqrt{Y_t}\mathrm{d}B_t, \quad t \geqslant 0.\end{aligned}$$

This implies that $(X_t)_{t\geqslant 0}$ is a strong solution of the second SDE in (1.0.1). As a consequence, with our considerations concerning $(Y_t)_{t\geqslant 0}$, we get the existence of the strong solution of (1.0.1). Finally, assume $(X_t)_{t\geqslant 0}$ is a strong solution of the second SDE in (1.0.1). Then, applying Itô's formula to the process $(e^{\theta t}X_t)_{t\geqslant 0}$, we have

$$\begin{aligned}\mathrm{d}\left(e^{\theta t}X_t\right) &= \theta e^{\theta t}X_t\mathrm{d}t + e^{\theta t}\mathrm{d}X_t = \theta e^{\theta t}X_t\mathrm{d}t + e^{\theta t}\left((m - \theta X_t)\,\mathrm{d}t + \sqrt{X_t}\mathrm{d}B_t\right)\\ &= me^{\theta t}\mathrm{d}t + e^{\theta t}\sqrt{Y_t}\mathrm{d}B_t, \quad t \geqslant 0.\end{aligned} \tag{1.0.11}$$

We rewrite the right-hand side of (1.0.11) into integral form,

$$e^{\theta t}X_t - X_0 = m\int_0^t e^{\theta s}\mathrm{d}s + \int_0^t e^{\theta s}\sqrt{Y_s}\mathrm{d}B_s, \quad t \geqslant 0,$$

yielding (1.0.9), and hence the second SDE in (1.0.1) is pathwise unique. Altogether, it follows that the SDE (1.0.1) has a unique pathwise solution as well. □

1.1. Affine representation of (Y, X)

In this section we derive the Laplace transform of both, the α-stable CIR process $(Y_t)_{t\geqslant 0}$ defined as the strong solution of the stochastic differential equation (1.0.2) and that of the two-dimensional process $(Y_t, X_t)_{t\geqslant 0}$ defined as the strong solution of the stochastic differential equation (1.0.1). However, it turns out that we are able to compute an explicit formula for the Laplace transform of Y but not for the two-dimensional process (Y, X). We leave it as an open problem. Some of the stated results in this section are known, and indeed they go back to the seminal papers of Li and Ma [45] or Barzcy *et al.* [4, Theorems 2.2 and 3.1], respectively. Where it is possible, we outline the proofs so that the reader will not have to hunt for the different references. At least, we will provide also complete proofs, since our approach differs from that of [45] and in comparison with [4] we add explanations where we deem it appropriate, because we use similar arguments to check some further results (see, e.g., Lemma 1.19 below).

The idea to obtain the Laplace transform of $(Y_t, X_t)_{t\geqslant 0}$ is to use its affine representation. For a careful introduction to two-dimensional affine processes with state space $\mathbb{R}_{\geqslant 0} \times \mathbb{R}$ see Appendix A on two-dimensional affine processes below.

We start with a computation of the infinitesimal generator of $(Y_t, X_t)_{t\geqslant 0}$.

Proposition 1.5. *Consider the process $(Y_t, X_t)_{t\geqslant 0}$ with parameter $a \in \mathbb{R}_{\geqslant 0}$, b, m, $\theta \in \mathbb{R}$. Then its infinitesimal generator is given by*

$$(\mathcal{A}f)(y, x) = (a - by)\frac{\partial f}{\partial y}(y, x) + (m - \theta x)\frac{\partial f}{\partial x}(y, x) + \frac{1}{2}y\frac{\partial^2 f}{\partial x^2}(y, x) + y\int_0^\infty \left(f(y+z, x) - f(y, x) - z\frac{\partial f}{\partial y}(y, x)\right) C_\alpha z^{-1-\alpha}\mathrm{d}z, \tag{1.1.1}$$

where $(y, x) \in \mathbb{R}_{\geqslant 0} \times \mathbb{R}$ and $f \in C_c^2(\mathbb{R}_{\geqslant 0} \times \mathbb{R}, \mathbb{R})$.

Proof. The process $(Y_t, X_t)_{t\geqslant 0}$ starts from (y, x), i.e., $(Y_0, X_0) = (y, x) \in \mathbb{R}_{\geqslant 0} \times \mathbb{R}$. Then we can use the Lévy-Itô decomposition of $(L_t)_{t\geqslant 0}$ in (1.0.3) to obtain that for each $t \geqslant 0$,

$$\begin{cases} Y_t = y + \int_0^t \gamma \sqrt[\alpha]{Y_s}\mathrm{d}s + \int_0^t (a - bY_s)\mathrm{d}s \\ \qquad + \int_0^t \int_{\{|z|<1\}} z \sqrt[\alpha]{Y_{s-}}\widetilde{N}(\mathrm{d}s, \mathrm{d}z) + \int_0^t \int_{\{|z|\geqslant 1\}} z \sqrt[\alpha]{Y_{s-}}N(\mathrm{d}s, \mathrm{d}z), \\ X_t = x + \int_0^t (m - \theta X_s)\mathrm{d}s + \int_0^t \sqrt{Y_s}\mathrm{d}B_s, \end{cases}$$

where γ, $N(\mathrm{d}s, \mathrm{d}z)$ and $\widetilde{N}(\mathrm{d}s, \mathrm{d}z)$ are as in (1.0.3). Noting that $\mathbb{R}^2_{\geqslant 0} \times \Omega \ni (t, z, \omega) \mapsto \mathbb{1}_{\{|z|<1\}} z \sqrt[\alpha]{Y_{t-}} \in F^{2,\mathrm{loc}}$ and $\mathbb{R}^2_{\geqslant 0} \times \Omega \ni (t, z, \omega) \mapsto \mathbb{1}_{\{|z|\geqslant 1\}} z \sqrt[\alpha]{Y_{t-}} \in F^1$ (see Remark 1.3) we are able to apply Itô's formula. In particular, by the Lévy-Itô decomposition of $(Y_t, X_t)_{t\geqslant 0}$ and applying Itô's formula for $f \in C_c^2(\mathbb{R}_{\geqslant 0} \times \mathbb{R}, \mathbb{R})$ (see, e.g. Ikeda and Watanabe [26, Theorem 5.1]), we obtain that for each $t \geqslant 0$,

$$\begin{aligned} &f(Y_t, X_t) - f(Y_0, X_0) \\ &\quad = \int_0^t \frac{\partial f}{\partial y}(Y_s, X_s)\gamma \sqrt[\alpha]{Y_s}\mathrm{d}s + \int_0^t \frac{\partial f}{\partial y}(Y_s, X_s)(a - bY_s)\mathrm{d}s \\ &\qquad + \int_0^t \frac{\partial f}{\partial x}(Y_s, X_s)(m - \theta X_s)\mathrm{d}s + \frac{1}{2}\int_0^t \frac{\partial^2 f}{\partial x^2}(Y_s, X_s)Y_s\mathrm{d}s \end{aligned}$$

$$
\begin{aligned}
&+ \int_0^t \frac{\partial f}{\partial x}(Y_s, X_s)\sqrt{Y_s}\mathrm{d}B_s \\
&+ \int_0^t \int_{\{|z|<1\}} \Big(f(Y_{s-} + z\sqrt[\alpha]{Y_{s-}}, X_{s-}) - f(Y_{s-}, X_{s-}) \Big) \widetilde{N}(\mathrm{d}s, \mathrm{d}z) \\
&+ \int_0^t \int_{\{|z|\geqslant 1\}} \Big(f(Y_{s-} + z\sqrt[\alpha]{Y_{s-}}, X_{s-}) - f(Y_{s-}, X_{s-}) \Big) N(\mathrm{d}s, \mathrm{d}z) \\
&+ \int_0^t \int_{\{|z|<1\}} \Big(f(Y_s + z\sqrt[\alpha]{Y_s}, X_s) - f(Y_s, X_s) - z\sqrt[\alpha]{Y_s}\frac{\partial f}{\partial y}(Y_s, X_s) \Big) C_\alpha z^{-1-\alpha}\mathrm{d}s\mathrm{d}z \\
&= \int_0^t (\mathcal{L}f)(Y_s, X_s)\mathrm{d}s + M_t(f),
\end{aligned} \tag{1.1.2}
$$

where

$$
\begin{aligned}
M_t(f) := &\int_0^t \frac{\partial f}{\partial x}(Y_s, X_s)\sqrt{Y_s}\mathrm{d}B_s \\
&+ \int_0^t \int_{\{|z|<1\}} \Big(f(Y_{s-} + z\sqrt[\alpha]{Y_{s-}}, X_{s-}) - f(Y_{s-}, X_{s-}) \Big) \widetilde{N}(\mathrm{d}s, \mathrm{d}z) \\
&+ \int_0^t \int_{\{|z|\geqslant 1\}} \Big(f(Y_{s-} + z\sqrt[\alpha]{Y_{s-}}, X_{s-}) - f(Y_{s-}, X_{s-}) \Big) N(\mathrm{d}s, \mathrm{d}z) \\
&- \int_0^t \int_{\{|z|\geqslant 1\}} \Big(f(Y_s + z\sqrt[\alpha]{Y_s}, X_s) - f(Y_s, X_s) \Big) \widehat{N}(\mathrm{d}s, \mathrm{d}z)
\end{aligned}
$$

and the operator $\mathcal{L}f$ is given by

$$
\begin{aligned}
(\mathcal{L}f)(y,x) := &(a - by)\frac{\partial f}{\partial y}(y,x) + (m - \theta x)\frac{\partial f}{\partial x}(y,x) + \frac{1}{2}y\frac{\partial^2 f}{\partial x^2}(y,x) \\
&+ \int_{\{|z|<1\}} \Big(f(y + z\sqrt[\alpha]{y}, x) - f(y,x) - z\sqrt[\alpha]{y}\frac{\partial f}{\partial y}(y,x) \Big) C_\alpha z^{-1-\alpha}\mathrm{d}z \\
&+ \int_{\{|z|\geqslant 1\}} \left(f(y + z\sqrt[\alpha]{y}, x) - f(y,x) \right) C_\alpha z^{-1-\alpha}\mathrm{d}z \\
&+ \gamma\sqrt[\alpha]{y}\frac{\partial f}{\partial y}(y,x) \\
= &(a - by)\frac{\partial f}{\partial y}(y,x) + (m - \theta x)\frac{\partial f}{\partial x}(y,x) + \frac{1}{2}y\frac{\partial^2 f}{\partial x^2}(y,x) \\
&+ \int_{\{|z|<1\}} \Big(f(y + z\sqrt[\alpha]{y}, x) - f(y,x) - z\sqrt[\alpha]{y}\frac{\partial f}{\partial y}(y,x) \Big) C_\alpha z^{-1-\alpha}\mathrm{d}z \\
&+ \int_{\{|z|\geqslant 1\}} \Big(f(y + z\sqrt[\alpha]{y}, x) - f(y,x) - z\sqrt[\alpha]{y}\frac{\partial f}{\partial y}(y,x) \Big) C_\alpha z^{-1-\alpha}\mathrm{d}z \\
&+ \int_{\{|z|\geqslant 1\}} z\sqrt[\alpha]{y}\frac{\partial f}{\partial y}(y,x) C_\alpha z^{-1-\alpha}\mathrm{d}z + \gamma\sqrt[\alpha]{y}\frac{\partial f}{\partial y}(y,x) \\
= &(a - by)\frac{\partial f}{\partial y}(y,x) + (m - \theta x)\frac{\partial f}{\partial x}(y,x) + \frac{1}{2}y\frac{\partial^2 f}{\partial x^2}(y,x) \\
&+ \int_0^\infty \Big(f(y + z\sqrt[\alpha]{y}, x) - f(y,x) - z\sqrt[\alpha]{y}\frac{\partial f}{\partial y}(y,x) \Big) C_\alpha z^{-1-\alpha}\mathrm{d}z
\end{aligned}
$$

for $(y,x) \in \mathbb{R}_{\geqslant 0} \times \mathbb{R}$. By a change of variable $\widetilde{z} := z\sqrt[\alpha]{y}$, we see that $\mathcal{L}f = \mathcal{A}f$, where $\mathcal{A}$ is given in (1.1.1). As a result, it follows from (1.1.2) that for each $t \geqslant 0$,

$$
f(Y_t, X_t) - f(Y_0, X_0) = \int_0^t (\mathcal{A}f)(Y_s, X_s)\mathrm{d}s + M_t(f). \tag{1.1.3}
$$

We show that $(M_t(f))_{t\geqslant 0}$ is a martingale with respect to the filtration $(\mathcal{F}_t)_{t\geqslant 0}$. To achieve this, we can use the same argument as in [4, Theorem 2.1]. The details are as follows: We define

$$
\begin{aligned}
D_t(f) &:= \int_0^t \frac{\partial f}{\partial x}(Y_s, X_s)\sqrt{Y_s}\mathrm{d}B_s, \quad t \geqslant 0,\\
J_t(f) &:= \int_0^t \int_{\{|z|<1\}} \Big(f(Y_{s-} + z\sqrt[\alpha]{Y_{s-}}, X_{s-}) - f(Y_{s-}, X_{s-})\Big)\, \widetilde{N}(\mathrm{d}s, \mathrm{d}z),\\
&\quad + \int_0^t \int_{\{|z|\geqslant 1\}} \Big(f(Y_{s-} + z\sqrt[\alpha]{Y_{s-}}, X_{s-}) - f(Y_{s-}, X_{s-})\Big)\, N(\mathrm{d}s, \mathrm{d}z)\\
&\quad - \int_0^t \int_{\{|z|\geqslant 1\}} \Big(f(Y_s + z\sqrt[\alpha]{Y_s}, X_s) - f(Y_s, X_s)\Big)\, \widehat{N}(\mathrm{d}s, \mathrm{d}z), \quad t \geqslant 0.
\end{aligned}
$$

We start with the diffusion part $(D_t(f))_{t\geqslant 0}$. Note that the derivative of f with respect to x is bounded as $f \in C_c^2(\mathbb{R}_{\geqslant 0} \times \mathbb{R}, \mathbb{R})$. Hence, there exists a positive constant c_1 such that

$$
\begin{aligned}
\mathbb{E}\left[\left(\int_0^t \frac{\partial}{\partial x} f(Y_s, X_s)\sqrt{Y_s}\mathrm{d}B_s\right)^2\right] &= \int_0^t \mathbb{E}\left[\left(\frac{\partial}{\partial x} f(Y_s, X_s)\right)^2 Y_s\right]\mathrm{d}s\\
&\leqslant c_1 \int_0^t \mathbb{E}[Y_s]\mathrm{d}s < \infty, \qquad t \geqslant 0,
\end{aligned}
$$

where the finiteness of the last integral holds, since there exists some further constant $c_2 > 0$ such that $\mathbb{E}[Y_t] \leqslant c_2(1 + y\exp\{-bt/\alpha\})$ for all $t \geqslant 0$ according to Proposition 1.2. Consequently, we get that $(D_t(f))_{t\geqslant 0}$ is a square integrable martingale with respect to the filtration $(\mathcal{F}_t)_{t\geqslant 0}$.

We continue with the jump part. In order to check that $(J_t(f))_{t\geqslant 0}$ is a martingale with respect to the filtration $(\mathcal{F}_t)_{t\geqslant 0}$, we use a localization approach. Namely, we define

$$
\begin{aligned}
J_{*,t}^n(f) &:= \int_0^t \int_{\{|z|<1\}} \Big(f(Y_{s-} \wedge n + z\sqrt[\alpha]{Y_{s-} \wedge n}, X_{s-}) - f(Y_{s-} \wedge n, X_{s-})\Big)\widetilde{N}(\mathrm{d}s, \mathrm{d}z),\\
J_t^{*,n}(f) &:= \int_0^t \int_{\{|z|\geqslant 1\}} \Big(f(Y_{s-} \wedge n + z\sqrt[\alpha]{Y_{s-} \wedge n}, X_{s-}) - f(Y_{s-} \wedge n, X_{s-})\Big)N(\mathrm{d}s, \mathrm{d}z)\\
&\quad - \int_0^t \int_{\{|z|\geqslant 1\}} \Big((f(Y_s \wedge n + z\sqrt[\alpha]{Y_s \wedge n}, X_s) - f(Y_s \wedge n, X_s)\Big)\widehat{N}(\mathrm{d}s, \mathrm{d}z)
\end{aligned}
$$

for all $t \geqslant 0$ and first prove that $(J_{*,t}^n(f))_{t\geqslant 0}$ and $(J_t^{*,n}(f))_{t\geqslant 0}$ are both martingales with respect to the filtration $(\mathcal{F}_t)_{t\geqslant 0}$, where $n \in \mathbb{N}$ is arbitrary.

We check that $(J_{*,t}^n(f))_{t\geqslant 0}$ is a square integrable martingale with respect to the filtration $(\mathcal{F}_t)_{t\geqslant 0}$. By Taylor's theorem, we get

$$
\begin{aligned}
&\left|f\left((Y_{s-} \wedge n) + z\sqrt[\alpha]{Y_{s-} \wedge n}, X_{s-}\right) - f\left(Y_{s-} \wedge n, X_{s-}\right)\right|\\
&\qquad\qquad \leqslant z\sqrt[\alpha]{Y_{s-} \wedge n} \sup_{(y,x)\in\mathbb{R}_{\geqslant 0}\times\mathbb{R}} \left|\frac{\partial}{\partial y} f(y, x)\right| \qquad (1.1.4)
\end{aligned}
$$

for $z \in \mathbb{R}_{\geqslant 0}$. Thus, since

$$
\mathbb{E}\left[\int_0^t \int_0^1 \Big(f((Y_{s-} \wedge n) + z\sqrt[\alpha]{Y_{s-} \wedge n}, X_{s-}) - f(Y_{s-} \wedge n, X_{s-})\Big)^2 C_\alpha z^{-1-\alpha}\mathrm{d}s\mathrm{d}z\right]
$$

$$
\begin{aligned}
&\leqslant \left(\sup_{(y,x)\in\mathbb{R}_{\geqslant 0}\times\mathbb{R}}\left|\frac{\partial f}{\partial y}(y,x)\right|\right)\int_0^t\int_0^1 \mathbb{E}\left[(Y_{s-}\wedge n)^{2/\alpha}\right] z^{1-\alpha}\mathrm{d}s\mathrm{d}z\\
&\leqslant c_3\int_0^t \mathbb{E}\left[(Y_{s-}\wedge n)^{2/\alpha}\right]\mathrm{d}s\int_0^1 z^{1-\alpha}\mathrm{d}z = c_3 n^{2/\alpha}\frac{t}{2-\alpha} < \infty, \qquad t\geqslant 0,
\end{aligned}
$$

with some constant $c_3 > 0$, by [26, pp.62, 63], we get that $(J^n_{*,t}(f))_{t\geqslant 0}$ is a square integrable martingale with respect to the filtration $(\mathcal{F}_t)_{t\geqslant 0}$.

Next, we prove that $(J^{*,n}_t(f))_{t\geqslant 0}$ is a martingale with respect to the filtration $(\mathcal{F}_t)_{t\geqslant 0}$. We proceed similar as before. Using (1.1.4), we estimate

$$
\begin{aligned}
&\mathbb{E}\left[\int_0^t\int_1^\infty \left|f(Y_{s-}\wedge n) + z\sqrt[\alpha]{Y_{s-}\wedge n}, X_{s-}) - f(Y_{s-}\wedge n, X_{s-})\right| C_\alpha z^{-1-\alpha}\mathrm{d}s\mathrm{d}z\right]\\
&\quad\leqslant \left(\sup_{(y,x)\in\mathbb{R}_{\geqslant 0}\times\mathbb{R}}\left|\frac{\partial f}{\partial y}(y,x)\right|\right)\int_0^t\int_1^\infty \mathbb{E}\left[\sqrt[\alpha]{Y_s\wedge n}\right] C_\alpha z^{-\alpha}\mathrm{d}s\mathrm{d}z\\
&\quad\leqslant c_4\int_0^t \mathbb{E}\left[\sqrt[\alpha]{Y_s\wedge n}\right]\mathrm{d}s\int_1^\infty z^{-\alpha}\mathrm{d}z = C_4 n^{1/\alpha}\frac{t}{\alpha-1} < \infty, \qquad t\geqslant 0,
\end{aligned}
$$

where $c_4 > 0$ is some constant. This implies by [26, Lemma 3.1 in Chapter II and p.62] that $(J^{*,n}_t(f))_{t\geqslant 0}$ is a martingale with respect to the filtration $(\mathcal{F}_t)_{t\geqslant 0}$.

It remains to check that $(J_t(f))_{t\geqslant 0}$ is indeed a martingale with respect to the filtration $(\mathcal{F}_t)_{t\geqslant 0}$. For each $n\in\mathbb{N}$, we define

$$
\eta^n_t(f) := J_t(f) - J^n_{*,t}(f) - J^{*,n}_t(f), \quad t\geqslant 0. \tag{1.1.5}
$$

Similar to (1.1.4), by Taylor's theorem, we have

$$
f(Y_{s-} + z\sqrt[\alpha]{Y_{s-}}, X_{s-}) - f(Y_{s-}, X_{s-}) = \left(\frac{\partial}{\partial y}f\left(Y_{s-} + \zeta z\sqrt[\alpha]{Y_{s-}}, X_{s-}\right)\right) z\sqrt[\alpha]{Y_{s-}},
$$

where $\zeta : \Omega \to \mathbb{R}$ is a function which we will specify later. Furthermore, using the Lévy-Itô decomposition in (1.0.3), for all $t\geqslant 0$, we obtain

$$
\begin{aligned}
\eta^n_t(f) &= \int_0^t\int_{\{|z|<1\}}\left(\frac{\partial}{\partial y}f\left(Y_{s-} + \zeta z\sqrt[\alpha]{Y_{s-}}, X_{s-}\right)\right) z\sqrt[\alpha]{Y_{s-}}\widetilde{N}(\mathrm{d}s,\mathrm{d}z)\\
&\quad + \int_0^t\int_{\{|z|\geqslant 1\}}\left(\frac{\partial}{\partial y}f\left(Y_{s-} + \zeta z\sqrt[\alpha]{Y_{s-}}, X_{s-}\right)\right) z\sqrt[\alpha]{Y_{s-}}N(\mathrm{d}s,\mathrm{d}z)\\
&\quad - \int_0^t\int_{\{|z|\geqslant 1\}}\left(\frac{\partial}{\partial y}f\left(Y_{s-} + \zeta z\sqrt[\alpha]{Y_{s-}}, X_{s-}\right)\right) z\sqrt[\alpha]{Y_{s-}}\widehat{N}(\mathrm{d}s,\mathrm{d}z)\\
&\quad - \int_0^t\int_{\{|z|<1\}}\left(\frac{\partial}{\partial y}f\left(Y_{s-}\wedge n + \zeta z\sqrt[\alpha]{Y_{s-}\wedge n}, X_{s-}\right)\right) z\sqrt[\alpha]{Y_{s-}\wedge n}\widetilde{N}(\mathrm{d}s,\mathrm{d}z)\\
&\quad - \int_0^t\int_{\{|z|\geqslant 1\}}\left(\frac{\partial}{\partial y}f\left(Y_{s-}\wedge n + \zeta z\sqrt[\alpha]{Y_{s-}\wedge n}, X_{s-}\right)\right) z\sqrt[\alpha]{Y_{s-}\wedge n}N(\mathrm{d}s,\mathrm{d}z)\\
&\quad + \int_0^t\int_{\{|z|\geqslant 1\}}\left(\frac{\partial}{\partial y}f\left(Y_{s-}\wedge n + \zeta z\sqrt[\alpha]{Y_{s-}\wedge n}, X_{s-}\right)\right) z\sqrt[\alpha]{Y_{s-}\wedge n}\widehat{N}(\mathrm{d}s,\mathrm{d}z)\\
&= \int_0^t\left(\frac{\partial}{\partial y}f\left(Y_{s-} + \zeta z\sqrt[\alpha]{Y_{s-}}, X_{s-}\right)\right)\sqrt[\alpha]{Y_{s-}}\mathrm{d}L_s\\
&\quad - \int_0^t\left(\frac{\partial}{\partial y}f\left(Y_{s-}\wedge n + \zeta z\sqrt[\alpha]{Y_{s-}\wedge n}, X_{s-}\right)\right)\sqrt[\alpha]{Y_{s-}\wedge n}\mathrm{d}L_s
\end{aligned}
$$

$$= \int_0^t \mathbb{1}_{\{Y_{s-}>n\}} \left(\frac{\partial}{\partial y} f\left(Y_{s-} + \zeta z \sqrt[\alpha]{Y_{s-}}, X_{s-} \right) \right) \sqrt[\alpha]{Y_{s-}} \mathrm{d}L_s$$
$$- \int_0^t \mathbb{1}_{\{Y_{s-}>n\}} \left(\frac{\partial}{\partial y} f\left(Y_{s-} \wedge n + \zeta z \sqrt[\alpha]{Y_{s-} \wedge n}, X_{s-} \right) \right) \sqrt[\alpha]{Y_{s-} \wedge n} \mathrm{d}L_s$$
$$= \int_0^t \mathbb{1}_{\{Y_{s-}>n\}} \left(\frac{\partial}{\partial y} f\left(Y_{s-} + \zeta z \sqrt[\alpha]{Y_{s-}}, X_{s-} \right) \right) \sqrt[\alpha]{Y_{s-}} \mathrm{d}L_s,$$

where we used in the last equality that the integral

$$\int_0^t \mathbb{1}_{\{Y_{s-}>n\}} \left(\frac{\partial}{\partial y} f\left(Y_{s-} \wedge n + z\zeta \sqrt[\alpha]{Y_{s-} \wedge n}, X_{s-} \right) \right) \sqrt[\alpha]{Y_{s-} \wedge n} \mathrm{d}L_s$$

vanishes for n large enough, because f has compact support. Noting that $f(Y_{s-} + z \sqrt[\alpha]{Y_{s-}}, X_{s-}) - f(Y_{s-}, X_{s-})$ is a random variable, we obtain that its derivative $(\partial_y f(Y_{s-} + \zeta z \sqrt[\alpha]{Y_{s-}}, X_{s-}))z \sqrt[\alpha]{Y_{s-}}$ is also a random variable (by equality of both expressions). In the same way we deduce that $(\partial_y f(Y_{s-} + \zeta z \sqrt[\alpha]{Y_{s-}}, X_{s-})_{s\in[0,t]}$ is a predictable process. Thus, we may use Remark 1.1 to deduce that for each $t \geqslant 0$, there exists some constant $c_5 > 0$ such that

$$\mathbb{E}_{(y,x)} \left[\sup_{s\in[0,t]} |\eta_s^n(f)| \right]$$
$$\leqslant c_5 \mathbb{E}_{(y,x)} \left[\left(\int_0^t \mathbb{1}_{\{Y_{s-}>n\}} \left(\frac{\partial}{\partial y} f\left(Y_{s-} + \zeta z \sqrt[\alpha]{Y_{s-}}, X_{s-} \right) \right)^\alpha Y_s \mathrm{d}s \right)^{1/\alpha} \right]$$
$$\leqslant c_6 \left(\int_0^t \mathbb{E}_{(y,x)} \left[\mathbb{1}_{\{Y_{s-}>n\}} Y_s \right] \mathrm{d}s \right)^{1/\alpha},$$

for some further constant $c_6 > 0$, where we used Jensen's inequality together with the fact that $(\partial_y f(Y_{s-} + \zeta z \sqrt[\alpha]{Y_{s-}}, X_{s-}))$ is bounded to get the second inequality. In view of Proposition 1.2 it follows that $\int_0^t \mathbb{E}_{(y,x)}[Y_s]\,\mathrm{d}s < \infty$ and further $\left(\int_0^t \mathbb{E}_{(y,x)}[Y_s]\,\mathrm{d}s\right)^{1/\alpha} < \infty$. Then, by the dominated convergence theorem, we obtain

$$\lim_{n\to\infty} \mathbb{E}_{(y,x)} \left[\sup_{s\in[0,t]} |\eta_s^n(f)| \right] \leqslant c_6 \lim_{n\to\infty} \left(\int_0^t \mathbb{E}_{(y,x)} \left[\mathbb{1}_{\{Y_s>n\}} Y_s \mathrm{d}s \right] \right)^{1/\alpha} = 0. \tag{1.1.6}$$

As shown in the proof of [4, Theorem 2.1], the martingale property of $(J_t(f))_{t\geqslant 0}$ now follows from (1.1.5), (1.1.6)) and the fact that both $(J_{*,t}^n(f))_{t\geqslant 0}$ and $(J_t^{*,n}(f))_{t\geqslant 0}$ are martingales. In particular, for all $0 \leqslant s \leqslant t$ and $A \in \mathcal{F}_s$, by (1.1.6, we have

$$\lim_{n\to\infty} \mathbb{E}\left[|\eta_t^n(f)|\, \mathbb{1}_A\right] \leqslant \lim_{n\to\infty} \mathbb{E}\left[|\eta_t^n(f)|\right] = 0,$$
$$\lim_{n\to\infty} \mathbb{E}\left[|\eta_s^n(f)|\, \mathbb{1}_A\right] \leqslant \lim_{n\to\infty} \mathbb{E}\left[|\eta_s^n(f)|\right] = 0.$$

This yields, for all $n \in \mathbb{N}$, $0 \leqslant s \leqslant t$ and $A \in \mathcal{F}_s$,

$$\begin{aligned} \lim_{n\to\infty} \mathbb{E}\left[J_{*,t}^n(f)\mathbb{1}_A + J_t^{*,n}(f)\mathbb{1}_A \right] &= \mathbb{E}\left[J_t(f)\mathbb{1}_A\right], \\ \lim_{n\to\infty} \mathbb{E}\left[J_{*,s}^n(f)\mathbb{1}_A + J_s^{*,n}(f)\mathbb{1}_A \right] &= \mathbb{E}\left[J_s(f)\mathbb{1}_A\right]. \end{aligned} \tag{1.1.7}$$

Consquently, using that both $(J_{*,t}^n(f))_{t\geqslant}$ and $(J_t^{*,n}(f))_{t\geqslant 0}$ are martingales with respect to the filtration $(\mathcal{F}_t)_{t\geqslant 0}$ together with (1.1.7), for all $n \in \mathbb{N}$, $0 \leqslant s \leqslant t$, and $A \in \mathcal{F}_s$, we get

$$\mathbb{E}\left[J_t(f)\mathbb{1}_A\right] = \lim_{n\to\infty} \mathbb{E}\left[\left(J_{*,t}^n(f) + J_t^{*,n}(f) \right) \mathbb{1}_A \right]$$

$$= \lim_{n\to\infty} \mathbb{E}\left[\left(J^n_{*,s}(f) + J^{*,n}_s(f)\right) \mathbb{1}_A\right] = \mathbb{E}\left[J_s(f)\mathbb{1}_A\right],$$

yielding that $(J_t(f))_{t\geqslant 0}$ is a martingale with respect to the filtration $(\mathcal{F}_t)_{t\geqslant 0}$. It is clear that $(M_t(f))_{t\geqslant 0} = (D_t(f) + J_t(f))_{t\geqslant 0}$ is also a martingale with respect to the filtration $(\mathcal{F}_t)_{t\geqslant 0}$. With this our proof is complete. □

Remark 1.6. *We note that, if one studies the proof of Proposition 1.5, it is easy to see that the Lévy process $(L_t)_{t\geqslant 0}$ is a martingale with respect to the filtration $(\mathcal{F}_t)_{t\geqslant 0}$, using the same strategy. This simple observation leads to the fact that the expectation of the α-root process Y_t is given by*

$$\mathbb{E}_y\left[Y_t\right] = e^{-bt}y + a\int_0^t e^{-bs}\mathrm{d}s = e^{-bt}y + \frac{a}{b}\left(1 - e^{-bt}\right), \quad t \geqslant 0.$$

The following proposition provides the characteristic functions of $(Y_t, X_t)_{t\geqslant 0}$. Namely, we prove that $(Y_t, X_t)_{t\geqslant 0}$ is a (conservative) *regular affine process*[3] with state space $\mathbb{R}_{\geqslant 0} \times \mathbb{R}$. The results about affine processes we are going to apply stem essentially from Duffie *et al.* [17] and are also introduced in the two-dimensional case in the Appendix A on two-dimensional affine processes.

Proposition 1.7. *Let $a \in \mathbb{R}_{\geqslant 0}$, b, m, $\theta \in \mathbb{R}$. Then $(Y_t, X_t)_{t\geqslant 0}$ is a regular affine process with state space $\mathbb{R}_{\geqslant 0} \times \mathbb{R}$.*

Proof. We follow the proof of [4, Theorem 2.1]. Note that we can associate a semigroup $(P_t)_{t\geqslant 0}$ of operators defined in the bounded Borel functions to the time-homogeneous Markov process $(Y_t, X_t)_{t\geqslant 0}$ (see Appendix A for details). In order to obtain that the transition semigroup $(P_t)_{t\geqslant 0}$ with state space $\mathbb{R}_{\geqslant 0} \times \mathbb{R}$ corresponding to $(Y_t, X_t)_{t\geqslant 0}$ is a regular affine semigroup with infinitesimal generator given by (1.1.1) we check that the conditions of Theorem A.1 are satisfied. It only suffices to prove that the parameters of the infinitesimal generator of $(P_t)_{t\geqslant 0}$ are *admissible* in the sense of Definition A.3. We read off the parameters of the infinitesimal generator given by (1.1.1). Here, we see that $(0, \alpha_{ij}, b, \beta_{ij}, 0, \mu)$, $i, j \in \{1, 2\}$, is admissible, since

$$\begin{aligned}
\begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix} &:= \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}; \\
b = (b_1, b_2) &:= (a, m) \in \mathbb{R}_{\geqslant 0} \times \mathbb{R}; \\
\begin{pmatrix} \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \end{pmatrix} &:= \begin{pmatrix} b & 0 \\ 0 & -\theta \end{pmatrix}; \\
\mu(\mathrm{d}y, \mathrm{d}x) &:= C_\alpha y^{-1-\alpha}\mathrm{d}y\delta_0(\mathrm{d}x);
\end{aligned}$$

where δ_0 denotes the Dirac measure concentrated on zero. As shown in the proof of [4, Theorem 2.1] the Lévy measure μ indeed satisfies the admissible condition. For completeness of exposition, we recall the arguments. One simply calculates

$$\begin{aligned}
&\int_{\mathbb{R}_{\geqslant 0}}\int_{\mathbb{R}} \left(|y| \wedge y^2\right)\mu(\mathrm{d}y, \mathrm{d}x) + \int_{\mathbb{R}_{\geqslant 0}}\int_{\mathbb{R}} \left(|x| \wedge x^2\right)\mu(\mathrm{d}y, \mathrm{d}x) \\
&\qquad = C_\alpha \int_0^\infty \left(|y| \wedge y^2\right) y^{-1-\alpha}\mathrm{d}y + C_\alpha \int_{-\infty}^\infty \left(|x| \wedge x^2\right)\delta_0(x)
\end{aligned}$$

[3]see, e.g., Definition A.1

$$
\begin{aligned}
&= C_\alpha \int_0^\infty \left(|y| \wedge y^2\right) y^{-1-\alpha} \mathrm{d}y \\
&= C_\alpha \left(\int_0^1 |y| y^{-1-\alpha} \mathrm{d}y + \int_1^\infty y^2 y^{-1-\alpha} \mathrm{d}y \right) \\
&= C_\alpha \left(\int_0^1 y^{1-\alpha} \mathrm{d}y + \int_1^\infty y^{-\alpha} \mathrm{d}y \right) < \infty.
\end{aligned}
$$

Hence, for this set of admissible parameters one can apply Theorem A.1 to obtain a regular affine semigroup $(Q_t)_{t\geqslant 0}$ with infinitesimal generator given by (1.1.1). It follows also from Theorem A.1 that $C_c^\infty(\mathbb{R}_{\geqslant 0} \times \mathbb{R}, \mathbb{R})$ is a core of the infinitesimal generator corresponding to the affine semigroup $(Q_t)_{t\geqslant 0}$. Since the infinitesimal generators corresponding the semigroups $(P_t)_{t\geqslant 0}$ and $(Q_t)_{t\geqslant 0}$ coincide on $C_c^\infty(\mathbb{R}_{\geqslant 0} \times \mathbb{R}, \mathbb{R})$, by the definition of a core, they actually coincide also on the Banach space of bounded functions on $\mathbb{R}_{\geqslant 0} \times \mathbb{R}$. Consequently, by Theorem A.1, equation (A.0.1) holds, yielding that $(Y_t, X_t)_{t\geqslant 0}$ is a regular affine process with infinitesimal generator given by (1.1.1). □

From Propositions 1.5 and 1.7 we immediately get the following corollary.

Corollary 1.8. *Consider the α-stable CIR process defined by the strong solution of the stochastic differential equation* (1.0.2) *with $a \in \mathbb{R}_{\geqslant 0}$, $b \in \mathbb{R}$. Then $(Y_t)_{t\geqslant 0}$ is a regular affine process with infinitesimal generator*

$$
(\mathcal{A}f)(y) = (a - by)\frac{\partial f}{\partial y}(y) + y \int_0^\infty \left(f(y+z) - f(y) - z\frac{\partial f}{\partial y}(y) \right) C_\alpha z^{-1-\alpha} \mathrm{d}z, \quad (1.1.8)
$$

where $y \in \mathbb{R}_{\geqslant 0}$, and $f \in C_c^2(\mathbb{R}_{\geqslant 0}, \mathbb{R})$.

Up to this point we constantly assumed $\alpha \in (1, 2)$. With the infinitesimal generator $\mathcal{A}$ of Y_t, we would like to clarify why $(Y_t, X_t)_{t\geqslant 0}$ fails to be conservative and therefore, it can not be exponentially ergodic when $\alpha \in (0, 1)$.

Remark 1.9. *If $\alpha \in (0, 1)$, then the process Y_t is no more conservative, namely, it explodes (goes to $+\infty$) in finite time, due to* [21, Theorem 4.11]. *We can understand this phenomenon in the following way: Consider the generator $\mathcal{A}$ of (Y_t) in* (1.1.8) *with $b > 0$. If $\alpha \in (0, 1)$, then the second term (especially the effect coming from the big jumps part) on the right-hand of (1.1.8) dominates the first one (the drift part), and the process $(Y_t)_{t\geqslant 0}$ is thus pushed to $+\infty$ in finite time. Note that the situation reverses if $\alpha \in (1, 2)$, namely, the drift part controls the jump part, and the process is always driven back to a/b.*

We use the affine property to calculate the representation of the joint Laplace transform of $(Y_t, X_t)_{t\geqslant 0}$ as far as possible.

Proposition 1.10. *Let a, $b \in \mathbb{R}_{>0}$, $m \in \mathbb{R}$, $\theta \in \mathbb{R}_{>0}$. Then $v_t(\lambda_1, \lambda_2)$ is the unique non-negative solution of the differential equation*

$$
\begin{cases}
\frac{\partial v_t}{\partial t}(\lambda_1, \lambda_2) = -b v_t(\lambda_1, \lambda_2) - \frac{1}{\alpha}\left(v_t(\lambda_1, \lambda_2)\right)^\alpha + \frac{1}{2} e^{-2\theta t} \lambda_2^2, \quad t \geqslant 0, \\
v_0(\lambda_1, \lambda_2) = \lambda_1,
\end{cases}
\quad (1.1.9)
$$

where $(\lambda_1, \lambda_2) \in \mathbb{R}_{\geqslant 0} \times \mathbb{R}$. Moreover, the Laplace transform of $(Y_t, X_t)_{t\geqslant 0}$ is given by

$$
\mathbb{E}_{(y,x)}\left[e^{-\lambda_1 Y_t + i\lambda_2 X_t} \right]
$$

$$= \exp\left\{-a\int_0^t v_s(\lambda_1,\lambda_2)\mathrm{d}s + im\lambda_2\frac{1-e^{-\theta t}}{\theta} - yv_t(\lambda_1,\lambda_2) + ixe^{-\theta t}\lambda_2\right\}$$

for all $t \in \mathbb{R}_{\geqslant 0}$ and $(\lambda_1, \lambda_2) \in \mathbb{R}_{\geqslant 0} \times \mathbb{R}$.

Proof. Our proof is motivated by the proof of [4, Theorem 3.1]. Since $(Y_t, X_t)_{t\geqslant 0}$ is an affine process, the corresponding characteristic functions of $(Y_t, X_t)_{t\geqslant 0}$ are of affine form, namely, supposing $(Y_t, X_t)_{t\geqslant 0}$ has initial state vector $(Y_0, X_0) = (y, x) \in \mathbb{R}_{\geqslant 0}\times\mathbb{R}$, there exist functions ϕ and ψ on $i\mathbb{R}^2$ such that

$$\int_{\mathbb{R}_{\geqslant 0}\times\mathbb{R}} e^{\langle u,\xi\rangle} P_t\left((y,x),\mathrm{d}\xi\right) = \mathbb{E}_{(y,x)}\left[e^{\langle u,(Y_t,X_t)\rangle}\right] = e^{\phi(t,u)+\langle (y,x),\psi(t,u)\rangle} \tag{1.1.10}$$

for all $u \in \mathcal{U}$, where $(P_t)_{t\geqslant 0}$ denotes the affine semigroup corresponding to $(Y_t, X_t)_{t\geqslant 0}$. The functions ϕ and ψ in turn are given as solutions of the *generalized Riccati equations* (A.0.2), where $F(u)$ and $R(u)$, $u \in \mathcal{U}$, are of Lévy-Khintchine representation (A.0.4) and (A.0.5), respectively. In what follows, we calculate the representation of the complex valued functions F and R first.

The formulas (A.0.4) and (A.0.5) yield $F(u) = au_1 + mu_2$ and

$$R(u) := -bu_1 + \frac{u_2^2}{2} + \int_0^\infty\int_{-\infty}^\infty \left(e^{\langle u,\xi\rangle} - 1 - \langle u,\xi\rangle\right)\mu(\mathrm{d}\xi_1,\mathrm{d}\xi_2), \quad u = (u_1,u_2)\in\mathcal{U},$$

where the Lévy measure μ is given by $\mu(\mathrm{d}\xi_1, \mathrm{d}\xi_2) := C_\alpha \xi_1^{-1-\alpha}\delta_0(\mathrm{d}\xi_2)$, where δ_0 denotes the Dirac measure concentrated on zero. Further, following the method of Sato [59, p.46] (see also Applebaum [1, p.81]), namely applying the trick of writing a repeated integral as a double integral and changing the order of integration, we obtain

$$\begin{aligned}
\int_0^\infty\int_{-\infty}^\infty \left(e^{\langle u,\xi\rangle} - 1 - \langle u,\xi\rangle\right)\mu(\mathrm{d}\xi_1,\mathrm{d}\xi_2) &= C_\alpha\int_0^\infty \left(e^{u_1\xi_1} - 1 - u_1\xi_1\right)\xi_1^{-1-\alpha}\mathrm{d}\xi_1\\
&= C_\alpha\int_0^\infty\left(\int_0^{\xi_1} u\left(e^{u\zeta}-1\right)\mathrm{d}\zeta\right)\xi_1^{-1-\alpha}\mathrm{d}\xi_1\\
&= C_\alpha\int_0^\infty\left(\int_\zeta^\infty \xi_1^{-1-\alpha}\mathrm{d}\xi\right)u_1\left(e^{u_1\zeta}-1\right)\mathrm{d}\zeta\\
&= C_\alpha\frac{-u_1}{\alpha}\int_0^\infty \zeta^{-\alpha}\left(e^{u_1\zeta}-1\right)\mathrm{d}\zeta.
\end{aligned}$$

We employ the same trick once again such that we get

$$\begin{aligned}
C_\alpha\int_0^\infty\left(e^{u_1\xi_1} - 1 - u_1\xi_1\right)\xi_1^{-1-\alpha}\mathrm{d}\xi_1 &= C_\alpha\frac{-u_1}{\alpha}\int_0^\infty\left(-\int_0^\zeta u_1e^{u_1\eta}\mathrm{d}\eta\right)\zeta^{-\alpha}\mathrm{d}\zeta\\
&= C_\alpha\frac{u_1}{\alpha}\int_0^\infty\left(\int_\eta^\infty\zeta^{-\alpha}\mathrm{d}\zeta\right)u_1e^{u_1\eta}\mathrm{d}\eta\\
&= C_\alpha\frac{u_1^2}{\alpha(\alpha-1)}\int_0^\infty\eta^{1-\alpha}e^{u_1\eta}\mathrm{d}\eta\\
&= C_\alpha\frac{u_1^2(-u_1)^{\alpha-2}}{\alpha}\int_0^\infty\eta^{1-\alpha}e^{-\eta}\mathrm{d}\eta\\
&= C_\alpha\frac{(-u_1)^\alpha}{\alpha(\alpha-1)}\Gamma(2-\alpha),
\end{aligned}$$

where we used in the second last equation that $\operatorname{Re} u_1 \leqslant 0$ and $\alpha < 2$. Finally, using that $C_\alpha = (\alpha\Gamma(-\alpha))^{-1} = \alpha(\alpha-1)/(\alpha\Gamma(2-\alpha))$, we obtain the identity

$$C_\alpha \int_0^\infty \left(e^{u_1\xi_1} - 1 - u_1\xi_1\right) \xi_1^{-1-\alpha} \mathrm{d}\xi_1 = \frac{(-u_1)^\alpha}{\alpha}, \quad u_1 \in \mathbb{C}_{\leqslant 0},$$

and summarized it follows that

$$F(u) = au_1 + mu_2 \quad \text{and} \quad R(u) = -bu_1 + \frac{u_2^2}{2} + \frac{(-u_1)^\alpha}{\alpha} \tag{1.1.11}$$

for $u = (u_1, u_2) \in \mathcal{U}$.

As far as possible, we will next solve the generalized Riccati differential equation

$$\begin{cases} \frac{\partial}{\partial t}\psi(t,u) = R\left(\psi(t,u)\right), & \psi(0,u) = u_1 \in \mathbb{C}_{\leqslant 0}, \quad t \geqslant 0, \\ \frac{\partial}{\partial t}\phi(t,u) = F\left(\psi(t,u)\right), & \phi(0,u) = 0, \qquad\qquad\quad t \geqslant 0. \end{cases}$$

Note that $\psi(t,u)$ is vector-valued, i.e., for all $u = (u_1, u_2) \in \mathcal{U}$, we have $\psi(t,u) = (\psi_1(t,u), \psi_2(t,u))$, $t \in \mathbb{R}_{\geqslant 0}$. Since the strong solution $(X_t)_{t\geqslant 0}$ of the second stochastic differential equation in (1.0.1) has state space $\mathbb{R}$, due to Proposition A.2, we have that

$$\psi_2(t,u) = e^{-\theta t}u_2, \quad (t, u_2) \in \mathbb{R}_{\geqslant 0} \times i\mathbb{R},$$

Furthermore, $\psi_1(t,u)$ and $\phi(t,u)$ are now solutions of the generalized Riccati differential equations

$$\begin{cases} \frac{\partial}{\partial t}\psi_1(t,u) = R\left(\psi_1(t,u), e^{-\theta t}u_2\right), & \psi_1(0,u) = u_1 \in \mathbb{C}_{\leqslant 0}, \quad t \geqslant 0, \\ \frac{\partial}{\partial t}\phi(t,u) = F\left(\psi_1(t,u), e^{-\theta t}u_2\right), & \phi(0,u) = 0, \qquad\qquad\quad t \geqslant 0, \end{cases}$$

where the complex-valued functions F and R are given by (1.1.11). Hence, for $u \in \mathcal{U}$, we obtain

$$\begin{cases} \frac{\partial}{\partial t}\psi(t,u) = -b\psi_1(t,u) + \frac{1}{\alpha}\left(\psi^{(1)}(t,u)\right)^\alpha + \frac{1}{2}e^{-2\theta t}u_2, \quad t \geqslant 0, \\ \psi_1(0,u) = u_1 \in \mathbb{C}_{\leqslant 0}, \end{cases}$$

and

$$\begin{aligned} \phi(t,u) = \int_0^t F\left(\psi_1(s,u), e^{-\theta s}u_2\right) \mathrm{d}s &= \int_0^t \left(a\psi_1(s,u) + me^{-\theta s}u_2\right) \mathrm{d}s \\ &= a\int_0^t \psi_1(s,u)\mathrm{d}s + mu_2\frac{1-e^{-\theta t}}{\theta}, \quad t \geqslant 0. \end{aligned}$$

Note, for all $u = (u_1, u_2) \in \mathcal{U}$ and $t \in \mathbb{R}_{\geqslant 0}$, the real part of $\psi_1(t,u)$ is less then or equal to zero by Duffie *et al.* [17, Remark 2.2]. If $u_1 \in \mathbb{R}_{\leqslant 0}$, then $\psi_1(t,u)$ is also less than or equal to zero. Consequently, defining $u_1 := -\lambda_1$, $u_2 := i\lambda_2$, and introducing the notation $v_t(\lambda_1, \lambda_2) := -\psi_1(t, (-\lambda_1, i\lambda_2))$, $(\lambda_1, \lambda_2) \in \mathbb{R}_{\geqslant 0} \times \mathbb{R}$, we conclude with (1.1.9). Furthermore, since $\psi_1(t,u) \leqslant 0$ for $u_1 \in \mathbb{R}_{\leqslant 0}$, we have

$$v_t(\lambda_1, \lambda_2) \geqslant 0 \quad \text{for all } (\lambda_1, \lambda_2) \in \mathbb{R}_{\geqslant 0} \times \mathbb{R}.$$

The uniqueness of the solutions of the differential equation (1.1.9) follows by Duffie *et al.* [17, Proposition 6.1, 6.4, and Lemma 9.2]. Finally, by (1.1.10) and (1.1.9), we get

$$\begin{aligned}
&\mathbb{E}_{(y,x)}\left[e^{-\lambda_1 Y_t + i\lambda_2 X_t}\right] \\
&\qquad = \exp\left\{\phi\left(t,(-\lambda_1, i\lambda_2)\right) + y\psi_1\left(t,(-\lambda_1, i\lambda_2)\right) + x\psi_2\left(t,(-\lambda_1, i\lambda_2)\right)\right\} \\
&\qquad = \exp\left\{-a\int_0^t v_s(\lambda_1,\lambda_2)\mathrm{d}s + im\lambda_2\frac{1-e^{-\theta t}}{\theta} - yv_t(\lambda_1,\lambda_2) + ixe^{-\theta t}\lambda_2\right\},
\end{aligned}$$

for all $t \in \mathbb{R}_{\geqslant 0}$ and $(\lambda_1, \lambda_2) \in \mathbb{R}_{\geqslant 0} \times \mathbb{R}$. □

Since we established explicit representations of $F(u)$ and $R(u)$ in the proof of Proposition 1.10, we now continue to solve the Riccati differential equation in (A.0.2) with respect to the α-stable CIR process Y. In the one-dimensional case we shall obtain an explicit unique solution of the resulting generalized Riccati equations.

However, although our results stated in Proposition 1.10 stems from Barczy *et al.* [4, Theorem 3.1], the explicit form of the solution to the genzeralized Riccati equation (1.1.12) below has not been derived in [4]. In order to study the transition densities of the α-stable CIR process, we will find the explicit form of the solution to (1.1.12) in the following proposition.

Proposition 1.11. *Let $a \in \mathbb{R}_{\geqslant 0}$, $b \in \mathbb{R}_{>0}$. Define $v_t(\lambda_1, 0) := v_t(\lambda) := -\psi(t, -\lambda)$, $\lambda \in \mathbb{R}_{>0}$. Then $v_t(\lambda)$ is the unique non-negative solution of the differential equation*

$$\begin{cases} \frac{\partial}{\partial t} v_t(\lambda) = -bv_t(\lambda) - \frac{1}{\alpha}\left(v_t(\lambda)\right)^\alpha, \quad t \geqslant 0, \\ \quad v_0(\lambda) = \lambda, \end{cases} \tag{1.1.12}$$

where $\lambda \in \mathbb{R}_{>0}$. The unique solution to (1.1.12) *is given by*

$$v_t(\lambda) = \left(\left(\frac{1}{\alpha b} + \lambda^{1-\alpha}\right)e^{b(\alpha-1)t} - \frac{1}{\alpha b}\right)^{1/(1-\alpha)}, \quad t \geqslant 0. \tag{1.1.13}$$

Moreover, supposing $(Y_t)_{t\geqslant 0}$ has initial value $Y_0 = y \in \mathbb{R}_{\geqslant 0}$ almost surely, the Laplace transform of Y_t is given by

$$\begin{aligned}
\mathbb{E}_y\left[e^{-\lambda Y_t}\right] &= \exp\left\{-a\int_0^t v_s(\lambda)\mathrm{d}s - yv_t(\lambda)\right\} \\
&= \exp\left\{-a\int_0^t \left(\left(\frac{1}{\alpha b} + \lambda^{1-\alpha}\right)e^{b(\alpha-1)s} - \frac{1}{\alpha b}\right)^{1/(1-\alpha)} \mathrm{d}s\right. \\
&\qquad\qquad \left. -y\left(\left(\frac{1}{\alpha b} + \lambda^{1-\alpha}\right)e^{b(\alpha-1)t} - \frac{1}{\alpha b}\right)^{1/(1-\alpha)}\right\}
\end{aligned} \tag{1.1.14}$$

for all $t \geqslant 0$ and $\lambda \in \mathbb{R}_{>0}$.

Proof. By Corollary 1.8 the α-CIR process $(Y_t)_{t\geqslant 0}$ is a regual affine process. Therefore, following the proof of Proposition 1.10, it follows that the equation (1.1.12) has a unique non-negative solution. The equation (1.1.12) is a Bernoulli differential equation which can be transformed into a linear differential equation through a change of variables. More precisely, if we write $u_t(\lambda) := \left(v_t(\lambda)\right)^{1-\alpha}$, then

$$\frac{\partial}{\partial t} u_t(\lambda) = (1-\alpha)\left(v_t(\lambda)\right)^{-\alpha}\frac{\partial}{\partial t} v_t(\lambda)$$

$$= (1-\alpha)\,(v_t(\lambda))^{-\alpha}\left(-bv_t(\lambda) - \tfrac{1}{\alpha}\,(v_t(\lambda))^{\alpha}\right)$$
$$= b(\alpha-1)u_t(\lambda) + \left(1-\alpha^{-1}\right) \tag{1.1.15}$$

and $u_0(\lambda) = (v_0(\lambda))^{1-\alpha} = \lambda^{1-\alpha}$. By solving (1.1.15), we obtain

$$u_t(\lambda) = \left(\frac{1}{\alpha b} + \lambda^{1-\alpha}\right) e^{b(\alpha-1)t} - \frac{1}{\alpha b},$$

which leads to

$$v_t(\lambda) = \left(\left(\frac{1}{\alpha b} + \lambda^{1-\alpha}\right) e^{b(\alpha-1)t} - \frac{1}{\alpha b}\right)^{1/(1-\alpha)}$$

for all $t \geqslant 0$ and $\lambda \in \mathbb{R}_{>0}$. By (1.1.10) and (A.0.2) and noting that $v_t(\lambda) = -\psi(t,-\lambda)$, we get

$$\begin{aligned}\mathbb{E}_y\left[e^{-\lambda Y_t}\right] &= \exp\left\{\phi(t,-\lambda) + y\psi(t,-\lambda)\right\}\\ &= \exp\left\{a\int_0^t \psi(s,-\lambda)\mathrm{d}s - yv_t(\lambda)\right\}\\ &= \exp\left\{-a\int_0^t v_s(\lambda)\mathrm{d}s - yv_t(\lambda)\right\}\end{aligned}$$

for all $t \geqslant 0$ and $\lambda \in \mathbb{R}_{>0}$. □

We remark that we have assumed $\lambda \in \mathbb{R}_{>0}$ in Proposition 1.11. However, formula (1.1.14) is true for the trivial case $\lambda = 0$ as well, which can be seen by taking the limit $\lambda \downarrow 0$.

1.2. Transition densities of the alpha-root process Y

In this section we show that the α-root process Y has positive and continuous transition densities. Our approach is essentially based on the inverse Fourier transform. The necessity of that property will become apparent later (see part (b) of the proof of Theorem 1.22).

Recall that the Laplace transform of the α-stable CIR process $(Y_t)_{t\geqslant 0}$ with respect to its initial value $Y_0 = y \in \mathbb{R}_{\geqslant 0}$ is given by

$$\mathbb{E}_y\left[e^{-\lambda Y_t}\right] = \exp\left\{-a\int_0^t v_s(\lambda)\mathrm{d}s - yv_t(\lambda)\right\}, \quad (t,\lambda) \in \mathbb{R}_{\geqslant 0}\times\mathbb{R}_{>0},$$

where the function v_t is given by (1.1.13). In what follows, we give a specification of the Laplace transform of Y_t.

The solution of the stochastic differential equation (1.0.2) as well as the Laplace transform of Y_t depends obviously on its initial value Y_0. From now on, we denote by $(Y_t^y)_{t\geqslant 0}$ the α-stable CIR process starting from a constant initial value $y \in \mathbb{R}_{\geqslant 0}$, i.e., $(Y_t^y)_{t\geqslant 0}$ satisfies

$$\mathrm{d}Y_t^y = (a - bY_t^y)\mathrm{d}t + \sqrt[\alpha]{Y_{t-}^y}\,\mathrm{d}L_t, \quad t \geqslant 0, \quad Y_0^y = y. \tag{1.2.1}$$

and we have

$$\mathbb{E}\left[e^{-\lambda Y_t^y}\right] = \exp\left\{-a\int_0^t v_s(\lambda)\mathrm{d}s - yv_t(\lambda)\right\}, \quad (t,\lambda) \in \mathbb{R}_{\geqslant 0} \times \mathbb{R}_{>0}.$$

Let

$$\varphi_1(t,\lambda,y) := \exp\left\{-y\left(\left(\frac{1}{\alpha b} + \lambda^{1-\alpha}\right)e^{b(\alpha-1)t} - \frac{1}{\alpha b}\right)^{1/(1-\alpha)}\right\},$$
$$\varphi_2(t,\lambda) := \exp\left\{-a\int_0^t \left(\left(\frac{1}{\alpha b} + \lambda^{1-\alpha}\right)e^{b(\alpha-1)s} - \frac{1}{\alpha b}\right)^{1/(1-\alpha)}\mathrm{d}s\right\}.$$

Then

$$\mathbb{E}\left[e^{-\lambda Y_t^y}\right] = \varphi_1(t,\lambda,y)\cdot\varphi_2(t,\lambda). \tag{1.2.2}$$

Keeping this decomposition of the Laplace transform of Y_t^y in mind, we take a closer look at the following two special cases:

Special case i): $a = 0$. To avoid abuse of notations, we use $(Z_t^y)_{t\geqslant 0}$ to denote the strong solution of the stochastic differential equation

$$\mathrm{d}Z_t^y = -bZ_t^y\mathrm{d}t + \sqrt[\alpha]{Z_{t-}^y}\mathrm{d}L_t, \quad t \geqslant 0, \quad Z_0^y = y \geqslant 0. \tag{1.2.3}$$

According to (1.1.14), the corresponding Laplace transform of Z_t^y coincides with $\varphi_1(t,\lambda,y)$. Noting that $b > 0$, we get

$$\lim_{\lambda\to\infty} v_t(\lambda) = \left(\frac{1}{\alpha b}\left(e^{b(\alpha-1)t} - 1\right)\right)^{1/(1-\alpha)} =: d(t) > 0 \tag{1.2.4}$$

where $d(t) \in (0,\infty)$ for all $t > 0$. Furthermore, by dominated convergence theorem, we have

$$\begin{aligned} e^{-yd(t)} &= \lim_{\lambda\to\infty} e^{-yv_t(\lambda)} = \lim_{\lambda\to\infty} \mathbb{E}\left[e^{-\lambda Z_t^y}\right] \\ &= \lim_{\lambda\to\infty}\left(\mathbb{E}\left[e^{-\lambda Z_t^y}\mathbb{1}_{\{Z_t^y=0\}}\right] + \mathbb{E}\left[e^{-\lambda Z_t^y}\mathbb{1}_{\{Z_t^y>0\}}\right]\right) \\ &= \mathbb{P}\left(Z_t^y = 0\right) > 0 \end{aligned} \tag{1.2.5}$$

for all $t > 0$ and $y \geqslant 0$.

Special case ii): $y = 0$. Consider $(Y_t^0)_{t\geqslant 0}$ that satisfies

$$\mathrm{d}Y_t^0 = (a - bY_t^0)\mathrm{d}t + \sqrt[\alpha]{Y_{t-}^0}\mathrm{d}L_t, \quad t \geqslant 0, \quad Y_0^0 = 0. \tag{1.2.6}$$

In view of (1.1.14), we easily see that the Laplace transform of Y_t^0 degenerates to $\varphi_2(t,\lambda)$.

Summarizing the results in case i) and case ii), we have the following proposition.

Proposition 1.12. *Let $a \geqslant 0$ and $b > 0$. Consider the processes $(Y_t^y)_{t\geqslant 0}$ and $(Z_t^y)_{t\geqslant 0}$ defined as the unique strong solutions of the stochastic differential equations* (1.2.1) *and* (1.2.3)*, respectively. Let $\mu_{Y_t^y}$ and $\mu_{Z_t^y}$ be the probability laws of Y_t^y and Z_t^y induced on $(\mathbb{R}_{\geqslant 0}, \mathcal{B}(\mathbb{R}_{\geqslant 0}))$, respectively. Then $\mu_{Y_t^y} = \mu_{Y_t^0} * \mu_{Z_t^y}$, where $*$ denotes the convolution of measures.*

Recall that the function $v_t(\cdot)$ given by (1.1.13) is defined on $\mathbb{R}_{>0}$. By considering the complex power functions, the domain of definition for $v_t(\cdot)$ can be extended to $\mathbb{C}\setminus\{0\}$. Indeed, the function

$$v_t(z)=\left(\left(\frac{1}{\alpha b}+z^{(1-\alpha)}\right)e^{b(\alpha-1)t}-\frac{1}{\alpha b}\right)^{1/(1-\alpha)},\quad z\in\mathbb{C}\setminus\{0\}, \tag{1.2.7}$$

is well-defined, where the complex power function is given by (0.0.1).

We next establish two estimates on $\int_0^t v_s(z)\mathrm{d}s$.

Lemma 1.13. *Let $T>1$. Then there exists a sufficiently small constant $\varepsilon_0>0$ such that*

$$\mathrm{Re}\left(\int_0^t v_s(z)\mathrm{d}s\right)\geqslant -C_1+C_2|z|^{2-\alpha} \tag{1.2.8}$$

when $|\mathrm{Arg}(z)|\in[\pi/2-\varepsilon_0,\pi/2+\varepsilon_0]$ and $T^{-1}\leqslant t\leqslant T$, where C_1, $C_2>0$ are constants depending only on a, b, α, ε_0 and T.

Proof. We will complete the proof in three steps.

"*Step 1*": Consider $\rho\geqslant 2$ and $\vartheta\in[\pi/2-\varepsilon,\pi/2+\varepsilon]$, where $\varepsilon>0$ is a small constant whose exact value will be determined later. We introduce a change of variables

$$z:=\left(\left(\frac{1}{\alpha b}+\left(\rho e^{i\vartheta}\right)^{1-\alpha}\right)e^{b(\alpha-1)s}-\frac{1}{\alpha b}\right)^{1/(1-\alpha)}$$

and define $\Gamma_0:[0,t]\to\mathbb{C}$ by

$$\Gamma_0(s):=\left(\left(\frac{1}{\alpha b}+\left(\rho e^{i\vartheta}\right)^{1-\alpha}\right)e^{b(\alpha-1)s}-\frac{1}{\alpha b}\right)^{1/(1-\alpha)},\quad s\in[0,t].$$

Noting that

$$\begin{aligned}\frac{\partial}{\partial s}\Gamma_0(s)&=-b\left(\frac{1}{\alpha b}+\left(\rho e^{i\vartheta}\right)^{1-\alpha}\right)e^{b(\alpha-1)s}\left(\left(\frac{1}{\alpha b}+\left(\rho e^{i\vartheta}\right)^{1-\alpha}\right)e^{b(\alpha-1)s}-\frac{1}{\alpha b}\right)^{\alpha/(1-\alpha)}\\&=-b\left[\left(\frac{1}{\alpha b}+\left(\rho e^{i\vartheta}\right)^{1-\alpha}\right)e^{b(\alpha-1)s}-\frac{1}{\alpha b}+\frac{1}{\alpha b}\right]z^{\alpha}\\&=-b\left(z^{1-\alpha}+\frac{1}{\alpha b}\right)z^{\alpha}=-b\left(z+\frac{z^{\alpha}}{\alpha b}\right),\end{aligned}$$

we obtain

$$\begin{aligned}\int_0^t v_s\left(\rho e^{i\vartheta}\right)\mathrm{d}s&=\int_0^t\left(\left(\frac{1}{\alpha b}+\left(\rho e^{i\vartheta}\right)^{1-\alpha}\right)e^{b(\alpha-1)s}-\frac{1}{\alpha b}\right)^{1/(1-\alpha)}\mathrm{d}s\\&=-\frac{1}{b}\int_{\Gamma_0}z\left(z+\frac{z^{\alpha}}{\alpha b}\right)^{-1}\mathrm{d}z=-\frac{1}{b}\int_{\Gamma_0}\left(1+\frac{z^{\alpha-1}}{\alpha b}\right)^{-1}\mathrm{d}z.\end{aligned} \tag{1.2.9}$$

Next, we derive a lower bound for $\mathrm{Re}\big(\int_0^t v_s\left(\rho e^{i\vartheta}\right)\mathrm{d}s\big)$.

Let Γ_0^* be the range of Γ_0. Since $\Gamma_0^*\subset\mathcal{O}$ and $z\mapsto\left(1+z^{\alpha-1}/(\alpha b)\right)^{-1}$ is analytic in $\mathcal{O}$, we have

$$\int_{\Gamma_0}\left(1+\frac{z^{\alpha-1}}{\alpha b}\right)^{-1}\mathrm{d}z=\int_{\rho e^{i\vartheta}}^{\left(\left(\frac{1}{\alpha b}+\left(\rho e^{i\vartheta}\right)^{(1-\alpha)}\right)e^{b(\alpha-1)t}-\frac{1}{\alpha b}\right)^{1/(1-\alpha)}}\left(1+\frac{z^{\alpha-1}}{\alpha b}\right)^{-1}\mathrm{d}z. \tag{1.2.10}$$

Here and after, the notation

$$\int_{w_1}^{w_2}\left(1+\frac{z^{\alpha-1}}{\alpha b}\right)^{-1}\mathrm{d}z$$

means the integral $\int_{\Gamma_{[w_1,w_2]}}\left(1+z^{\alpha-1}/(\alpha b)\right)^{-1}\mathrm{d}z$, where $\Gamma_{[w_1,w_2]}$ is the directed segment joining w_1 and w_2 and is defined by

$$\Gamma_{[w_1,w_2]}:[0,1]\to\mathbb{C}\quad\text{with}\quad \Gamma_{[w_1,w_2]}(r):=(1-r)w_1+rw_2,\quad r\in[0,1].$$

By (1.2.9), (1.2.10) and the holomorphicity of $z\mapsto\left(1+z^{\alpha-1}/(\alpha b)\right)^{-1}$ on $\mathcal{O}$, we obtain

$$\begin{aligned}\int_0^t v_s\left(\rho e^{i\vartheta}\right)\mathrm{d}s=&\frac{1}{b}\int_{e^{i\vartheta}}^{\rho e^{i\vartheta}}\left(1+\frac{z^{\alpha-1}}{\alpha b}\right)^{-1}\mathrm{d}z\\&+\frac{1}{b}\int_{\left(\left(\frac{1}{\alpha b}+\left(\rho e^{i\vartheta}\right)^{(1-\alpha)}\right)e^{b(\alpha-1)t}-\frac{1}{\alpha b}\right)^{1/(1-\alpha)}}^{e^{i\vartheta}}\left(1+\frac{z^{\alpha-1}}{\alpha b}\right)^{-1}\mathrm{d}z.\end{aligned}\tag{1.2.11}$$

Since the second term on the right-hand of (1.2.11) is continuous in $(t,\rho,\vartheta)\in[1/T,T]\times[2,\infty)\times[\pi/2-\varepsilon,\pi/2+\varepsilon]$ and converges to

$$\frac{1}{b}\int_{\left(\left(e^{b(\alpha-1)t}-1\right)\frac{1}{\alpha b}\right)^{1/(1-\alpha)}}^{e^{i\vartheta}}\left(1+\frac{z^{\alpha-1}}{\alpha b}\right)^{-1}\mathrm{d}z$$

(uniformly in $(t,\vartheta)\in[1/T,T]\times[\pi/2-\varepsilon,\pi/2+\varepsilon]$) as $\rho\to\infty$, it must be bounded, i.e., we have

$$\left|\frac{1}{b}\int_{\left(\left(e^{b(\alpha-1)t}-1\right)\frac{1}{\alpha b}\right)^{1/(1-\alpha)}}^{e^{i\vartheta}}\left(1+\frac{z^{\alpha-1}}{\alpha b}\right)^{-1}\mathrm{d}z\right|\leqslant c_3\tag{1.2.12}$$

for all $t\in[1/T,T]$, $\vartheta\in[\pi/2-\varepsilon,\pi/2+\varepsilon]$ and $\rho\geqslant 2$, where $c_3=c_3(\varepsilon,T)>0$ is some constant.

Now, define $\Gamma_\vartheta:[0,1]\to\mathbb{C}$ by

$$\Gamma_\vartheta(r):=(1-r)e^{i\vartheta}+r\rho e^{i\vartheta},\quad r\in[0,1],$$

and let Γ_ϑ^* be the range of Γ_ϑ. We can calculate the real part of the first integral appearing on the right-hand side of (1.2.11) by

$$\begin{aligned}&\mathrm{Re}\left(\int_{e^{i\vartheta}}^{\rho e^{i\vartheta}}\left(1+\frac{z^{\alpha-1}}{\alpha b}\right)^{-1}\mathrm{d}z\right)\\&=\mathrm{Re}\left(\int_{\Gamma_\vartheta}\left(1+\frac{z^{\alpha-1}}{\alpha b}\right)^{-1}\mathrm{d}z\right)\\&=\mathrm{Re}\left(\int_0^1\left(1+\frac{(\Gamma_\vartheta(r))^{\alpha-1}}{\alpha b}\right)^{-1}\partial_r\Gamma_\vartheta(r)\mathrm{d}r\right)\\&=\mathrm{Re}\left(\int_0^1\frac{(\rho-1)e^{i\vartheta}}{1+(\Gamma_\vartheta(r))^{\alpha-1}(\alpha b)^{-1}}\mathrm{d}r\right)\end{aligned}$$

$$= \int_0^1 \left| \frac{(\rho-1)e^{i\vartheta}}{1+(\Gamma_\vartheta(r))^{\alpha-1}(\alpha b)^{-1}} \right| \cos\left(\operatorname{Arg}\left(\frac{(\rho-1)e^{i\vartheta}}{1+(\Gamma_\vartheta(r))^{\alpha-1}(\alpha b)^{-1}} \right)\right) \mathrm{d}r. \qquad (1.2.13)$$

For $r \in [0,1]$, we have

$$\begin{aligned} \operatorname{Arg}\left(1+(\Gamma_\vartheta(0))^{\alpha-1}(\alpha b)^{-1}\right) &\leqslant \operatorname{Arg}\left(1+(\Gamma_\vartheta(r))^{\alpha-1}(\alpha b)^{-1}\right) \\ &\leqslant \operatorname{Arg}\left(1+(\Gamma_\vartheta(1))^{\alpha-1}(\alpha b)^{-1}\right). \end{aligned} \qquad (1.2.14)$$

Define δ_ϑ by

$$\begin{aligned} \delta_\vartheta &:= (\alpha-1)\vartheta - \operatorname{Arg}\left(1+(\Gamma_\vartheta(0))^{\alpha-1}(\alpha b)^{-1}\right) \\ &= (\alpha-1)\vartheta - \operatorname{Arg}\left(1+e^{i(\alpha-1)\vartheta}(\alpha b)^{-1}\right) \in (0,(\alpha-1)\vartheta). \end{aligned} \qquad (1.2.15)$$

It is easy to see that

$$\operatorname{Arg}\left(1+(\Gamma_\vartheta(1))^{\alpha-1}(\alpha b)^{-1}\right) < (\alpha-1)\vartheta. \qquad (1.2.16)$$

By (1.2.14), (1.2.15) and (1.2.16), we get

$$\operatorname{Arg}\left(1+(\Gamma_\vartheta(r))^{\alpha-1}(\alpha b)^{-1}\right) \in [(\alpha-1)\vartheta-\delta_\vartheta,(\alpha-1)\vartheta), \quad r\in[0,1].$$

As a result,

$$\operatorname{Arg}\left(\frac{(\rho-1)e^{i\vartheta}}{1+(\Gamma_\vartheta(r))^{\alpha-1}(\alpha b)^{-1}} \right) \in ((2-\alpha)\vartheta,(2-\alpha)\vartheta+\delta_\vartheta], \quad r\in[0,1]. \qquad (1.2.17)$$

Note that $0<\delta_{\pi/2}<(\alpha-1)\pi/2$ by (1.2.15). Since δ_ϑ is continuous in ϑ, we see that

$$0 < \lim_{\vartheta\to\frac{\pi}{2}} \{(2-\alpha)\vartheta+\delta_\vartheta\} = (2-\alpha)\frac{\pi}{2}+\delta_{\frac{\pi}{2}} < \frac{\pi}{2}.$$

Set

$$c_4 := \frac{\pi}{2} - \left((2-\alpha)\frac{\pi}{2}+\delta_{\frac{\pi}{2}}\right) \in \left(0,\frac{\pi}{2}\right).$$

Now, we choose the constant $\varepsilon_0>0$ small enough such that

$$0<(2-\alpha)\vartheta<(2-\alpha)\vartheta+\delta_\vartheta \le \frac{\pi}{2}-\frac{c_4}{2} \qquad (1.2.18)$$

for all $\vartheta\in[\pi/2-\varepsilon_0,\pi/2+\varepsilon_0]$. It follows from (1.2.17) and (1.2.18) that for all $\vartheta\in[\pi/2-\varepsilon_0,\pi/2+\varepsilon_0]$ and $r\in[0,1]$,

$$\cos\left(\operatorname{Arg}\left(\frac{(\rho-1)e^{i\vartheta}}{1+(\Gamma_\vartheta(r))^{\alpha-1}(\alpha b)^{-1}} \right)\right) \geqslant \cos\left(\frac{\pi}{2}-\frac{c_4}{2}\right) =: c_5 > 0. \qquad (1.2.19)$$

In view of (1.2.13) and (1.2.19), we get

$$\begin{aligned} &\operatorname{Re}\left(\int_{e^{i\vartheta}}^{\rho e^{i\vartheta}} \left(1+\frac{z^{\alpha-1}}{\alpha b}\right)^{-1} \mathrm{d}z\right) \\ &\geqslant \cos\left(\frac{\pi}{2}-\frac{c_4}{2}\right)\int_0^1 \left| \frac{(\rho-1)e^{i\vartheta}}{1+(\Gamma_\vartheta(r))^{\alpha-1}(\alpha b)^{-1}} \right| \mathrm{d}r \end{aligned}$$

$$
\begin{aligned}
&= c_5 \int_0^1 \frac{\rho - 1}{\left|1 + (\Gamma_\vartheta(r))^{\alpha-1} (\alpha b)^{-1}\right|} \mathrm{d}r \geqslant c_5 \int_0^1 \frac{\rho - 1}{1 + \left|(\Gamma_\vartheta(r))^{\alpha-1} (\alpha b)^{-1}\right|} \mathrm{d}r \\
&= c_5 \int_0^1 \frac{\rho - 1}{1 + (1 - r + r\rho)^{\alpha-1} (\alpha b)^{-1}} \mathrm{d}r = c_5 \int_0^{\rho-1} \frac{1}{1 + (1 + r)^{\alpha-1} (\alpha b)^{-1}} \mathrm{d}r \\
&\geqslant \frac{c_5}{1 + (\alpha b)^{-1}} \int_0^{\rho-1} \frac{1}{(1+r)^{\alpha-1}} \mathrm{d}r = c_5 \alpha b (1 + \alpha b)^{-1} (2 - \alpha)^{-1} \left(\rho^{2-\alpha} - 1\right). \quad (1.2.20)
\end{aligned}
$$

Combining (1.2.11), (1.2.12) and (1.2.20) yields

$$
\mathrm{Re}\left(\int_0^t v_s\left(\rho e^{i\vartheta}\right) \mathrm{d}s\right) \geqslant c_6 \rho^{2-\alpha} - c_7, \quad \rho \geqslant 2,\ \vartheta \in \left[\frac{\pi}{2} - \varepsilon_0, \frac{\pi}{2} + \varepsilon_0\right],\ t \in [1/T, T], \quad (1.2.21)
$$

where c_6, $c_7 > 0$ are constants that depend only on a, b, α, ε_0 and T.

"*Step 2*": The case with $\rho \geqslant 2$ and $\vartheta \in [-\pi/2 - \varepsilon_0, -\pi/2 + \varepsilon_0]$ can be similarly treated, and we thus get

$$
\mathrm{Re}\left(\int_0^t v_s\left(\rho e^{i\vartheta}\right) \mathrm{d}s\right) \geqslant c_8 \rho^{2-\alpha} - c_9 \quad (1.2.22)
$$

for all $\rho \geqslant 2$, $\vartheta \in [-\pi/2 - \varepsilon_0, -\pi/2 + \varepsilon_0]$ and $t \in [1/T, T]$, where c_8, $c_9 > 0$ are constants depending only on a, b, α, ε_0 and T.

"*Step 3*": Since $\int_0^t v_s\left(\rho e^{i\vartheta}\right) \mathrm{d}s$ is continuous in (t, ρ, ϑ), we can find a constant $c_{10} > 0$ such that

$$
\mathrm{Re}\left(\int_0^t v_s\left(\rho e^{i\vartheta}\right) \mathrm{d}s\right) \geqslant -c_{10} \quad (1.2.23)
$$

for all $0 \leqslant \rho \leqslant 2$, $\vartheta \in [-\pi/2 - \varepsilon_0, -\pi/2 + \varepsilon_0] \cup [\pi/2 - \varepsilon_0, \pi/2 + \varepsilon_0]$ and $t \in [1/T, T]$. The estimate (1.2.8) now follows from (1.2.21), (1.2.22) and (1.2.23). □

Lemma 1.14. *Let ε_0 be as in the previous lemma. Then for each $t \geqslant 0$, we can find constants C_3, $C_4 > 0$, which depend only on a, b, α, ε_0 and t, such that*

$$
\left|\int_0^t v_s(z) \mathrm{d}s\right| \leqslant C_3 + C_4 |z|^{2-\alpha}
$$

when $\mathrm{Arg}(z) \in [\pi/2 + \varepsilon_0, \pi]$ *and* $|z| > 0$.

Proof. Let $\rho \geq 2$ and $\vartheta \in [\pi/2 + \varepsilon_0, \pi]$. Our aim is to show

$$
\left|\int_0^t v_s(\rho e^{i\vartheta}) \mathrm{d}s\right| \leqslant C_3 + C_4 \rho^{2-\alpha} \quad (1.2.24)
$$

for some constants C_3, $C_4 > 0$ that depend only on a, b, α, ε_0 and t. Using the change of variables

$$
z := \left(\frac{1}{\alpha b} + \left(\rho e^{i\vartheta}\right)^{(1-\alpha)}\right) e^{b(\alpha-1)s} - \frac{1}{\alpha b},
$$

we get

$$
\int_0^t v_s(\rho e^{i\vartheta}) \mathrm{d}s = \int_0^t \left(\left(\frac{1}{\alpha b} + \left(\rho e^{i\vartheta}\right)^{1-\alpha}\right) e^{b(\alpha-1)s} - \frac{1}{\alpha b}\right)^{1/(1-\alpha)} \mathrm{d}s \quad (1.2.25)
$$

$$= \frac{1}{b(\alpha-1)} \int_{\left(\rho e^{i\vartheta}\right)^{1-\alpha}}^{\left(\frac{1}{\alpha b}+\left(\rho e^{i\vartheta}\right)^{1-\alpha}\right)e^{b(\alpha-1)t}-\frac{1}{\alpha b}} z^{1/(1-\alpha)} \left(z+\frac{1}{\alpha b}\right)^{-1} \mathrm{d}z. \tag{1.2.26}$$

Since $\vartheta \in [\pi/2+\varepsilon_0, \pi]$, we have $(1-\alpha)\vartheta \in [(1-\alpha)\pi, (1-\alpha)(\pi/2+\varepsilon_0)]$, which implies

$$|\sin((1-\alpha)\vartheta)| \geqslant \min\{\sin((\alpha-1)\pi), \sin((\alpha-1)(\pi/2+\varepsilon_0))\} =: c_1 > 0. \tag{1.2.27}$$

We first consider the case with $0<\rho<2$. Note that for $\rho \in (0,2)$ and $\vartheta \in [\pi/2+\varepsilon_0, \pi]$,

$$\begin{aligned} \left|\left(\frac{1}{\alpha b}+\left(\rho e^{i\vartheta}\right)^{1-\alpha}\right)e^{b(\alpha-1)s}-\frac{1}{\alpha b}\right| &\geqslant \left|\operatorname{Im}\left(\left(\frac{1}{\alpha b}+\left(\rho e^{i\vartheta}\right)^{1-\alpha}\right)e^{b(\alpha-1)s}-\frac{1}{\alpha b}\right)\right| \\ &= \rho^{1-\alpha}e^{b(\alpha-1)s}\sin((\alpha-1)\vartheta) \\ &\geqslant 2^{1-\alpha}e^{b(\alpha-1)s}c_1. \end{aligned} \tag{1.2.28}$$

Then, by (1.2.25) and (1.2.28), we get that for $\rho \in (0,2)$ and $\vartheta \in [\pi/2+\varepsilon_0, \pi]$,

$$\begin{aligned} \left|\int_0^t v_s\left(\rho e^{i\vartheta}\right)\mathrm{d}s\right| &\leqslant \int_0^t \left|\left(\frac{1}{\alpha b}\left(\rho e^{i\vartheta}\right)^{1-\alpha}\right)e^{b(\alpha-1)s}-\frac{1}{\alpha b}\right|^{1/(1-\alpha)}\mathrm{d}s \\ &\leqslant \int_0^t c_1^{1/(1-\alpha)}e^{-bs}\mathrm{d}s = c_1^{1/(1-\alpha)}\frac{1}{b}\left(1-e^{-bt}\right). \end{aligned}$$

We see that the estimate (1.2.24) holds for $0<\rho<2$ and $\vartheta \in [\pi/2+\varepsilon_0, \pi]$.

We now consider $\rho \geqslant 2$. Note that $z \mapsto z^{1/(1-\alpha)}(z+1/(\alpha b))^{-1}$ is holomorphic on $\mathcal{O}$. So we have

$$\begin{aligned} &\int_{(\rho e^{i\vartheta})^{1-\alpha}}^{(\frac{1}{\alpha b}+(\rho e^{i\vartheta})^{1-\alpha})e^{b(\alpha-1)t}-\frac{1}{\alpha b}} z^{1/(1-\alpha)}\left(z+\frac{1}{\alpha b}\right)^{-1}\mathrm{d}z \\ &= \int_{(\rho e^{i\vartheta})^{1-\alpha}}^{(\rho e^{i\vartheta})^{1-\alpha}+2} z^{1/(1-\alpha)}\left(z+\frac{1}{\alpha b}\right)^{-1}\mathrm{d}z \\ &\quad + \int_{(\rho e^{i\vartheta})^{1-\alpha}+2}^{\left(\frac{1}{\alpha b}+(\rho e^{i\vartheta})^{1-\alpha}\right)e^{b(\alpha-1)t}-\frac{1}{\alpha b}} z^{1/(1-\alpha)}\left(z+\frac{1}{\alpha b}\right)^{-1}\mathrm{d}z. \end{aligned} \tag{1.2.29}$$

Since

$$\begin{aligned} \lim_{\rho\to\infty} &\int_{(\rho e^{i\vartheta})^{1-\alpha}+2}^{\left(\frac{1}{\alpha b}+(\rho e^{i\vartheta})^{1-\alpha}\right)e^{b(\alpha-1)t}-\frac{1}{\alpha b}} z^{1/(1-\alpha)}\left(z+\frac{1}{\alpha b}\right)^{-1}\mathrm{d}z \\ &= \int_2^{1/(\alpha b)\left(e^{b(\alpha-1)t}-1\right)} z^{1/(1-\alpha)}\left(z+\frac{1}{\alpha b}\right)^{-1}\mathrm{d}z, \end{aligned}$$

where the convergence is uniform in $\vartheta \in [\pi/2+\varepsilon_0, \pi]$, we can find a constant $c_2>0$ such that

$$\left|\int_{(\rho e^{i\vartheta})^{1-\alpha}+2}^{\left(\frac{1}{\alpha b}+(\rho e^{i\vartheta})^{1-\alpha}\right)e^{b(\alpha-1)t}-\frac{1}{\alpha b}} z^{1/(1-\alpha)}\left(z+\frac{1}{\alpha b}\right)^{-1}\mathrm{d}z\right| \leqslant c_2 \tag{1.2.30}$$

for all $\rho \geqslant 2$ and $\vartheta \in [\pi/2+\varepsilon_0, \pi]$.

We now proceed to estimate the first term on the right-hand side of (1.2.29). Define

$$\Gamma_{\vartheta,\rho}(r) := \left(\rho e^{i\vartheta}\right)^{1-\alpha} + r, \quad r \in [0,2].$$

By (1.2.27), we have

$$|\rho^{1-\alpha}e^{(1-\alpha)i\vartheta}+r| \geqslant \rho^{1-\alpha}|\sin((1-\alpha)\vartheta)| \geqslant c_1\rho^{1-\alpha}, \tag{1.2.31}$$

where $r\in[0,2]$ and $\vartheta\in[\pi/2+\varepsilon_0,\pi]$. If $r\in[2\rho^{1-\alpha},2]$, then

$$|\rho^{1-\alpha}e^{(1-\alpha)i\vartheta}+r| \geqslant r-\rho^{1-\alpha} \geqslant \frac{r}{2}. \tag{1.2.32}$$

It follows from (1.2.31) and (1.2.32) that for $\rho\geqslant 2$ and $\vartheta\in[\pi/2+\varepsilon_0,\pi]$,

$$\begin{aligned}
&\left|\int_{(\rho e^{i\vartheta})^{1-\alpha}}^{(\rho e^{i\vartheta})^{1-\alpha}+2} z^{1/(1-\alpha)}\left(z+\frac{1}{\alpha b}\right)^{-1}\mathrm{d}z\right| \\
&= \left|\int_0^2 (\Gamma_{\vartheta,\rho}(r))^{1/(1-\alpha)}\left(\Gamma_{\vartheta,\rho}(r)+\frac{1}{\alpha b}\right)^{-1}\mathrm{d}r\right| \\
&\leqslant c_3\int_0^2 |\Gamma_{\vartheta,\rho}(r)|^{1/(1-\alpha)}\mathrm{d}r = c_3\int_0^2 \left|\rho^{1-\alpha}e^{(1-\alpha)i\vartheta}+r\right|^{1/(1-\alpha)}\mathrm{d}r \\
&= c_3\int_0^{2\rho^{1-\alpha}} \left|\rho^{1-\alpha}e^{(1-\alpha)i\vartheta}+r\right|^{1/(1-\alpha)}\mathrm{d}r \\
&\quad + c_3\int_{2\rho^{1-\alpha}}^2 \left|\rho^{1-\alpha}e^{(1-\alpha)i\vartheta}+r\right|^{1/(1-\alpha)}\mathrm{d}r \\
&\leqslant c_3\int_0^{2\rho^{1-\alpha}} \left(c_1\rho^{1-\alpha}\right)^{1/(1-\alpha)}\mathrm{d}r + c_3 2^{1/(\alpha-1)}\int_{2\rho^{1-\alpha}}^2 r^{1/(1-\alpha)}\mathrm{d}r \\
&= 2c_3c_1^{1/(1-\alpha)}\rho^{2-\alpha} + c_3 2^{1/(\alpha-1)}\,\tfrac{\alpha-1}{\alpha-2} r^{(2-\alpha)/(1-\alpha)}\Big|_{r=2\rho^{1-\alpha}}^2 \leqslant c_4\rho^{2-\alpha}+c_5,
\end{aligned} \tag{1.2.33}$$

where c_3, c_4, $c_5>0$ are some constants. Combining (1.2.26), (1.2.29), (1.2.30) and (1.2.33) yields (1.2.24). This completes the proof. □

Now, consider the process $(Y_t^0)_{t\geqslant 0}$ given by (1.2.6). The following properties are probably well known in more general framework, but we do not have a reference. The continuity of the function $u\mapsto \mathbb{E}[\exp\{-uY_t^0\}]$ on $\mathbb{C}_{\geqslant 0}$ follows directly from dominated convergence theorem. Let $x_0>0$ be fixed. Consider $u\in\mathbb{C}$ with $\mathrm{Re}(u)>x_0$. Let $(u_n)_{n\in\mathbb{N}}\subset\mathbb{C}$ be a sequence such that u_n converges to u as n tends to infinity and $\mathrm{Re}(u_n)>x_0$. Since $y\mapsto y\exp\{-x_0y\}$ is a bounded function on $\mathbb{R}_{\geqslant 0}$, we obtain $\mathbb{E}[Y_t^0\exp\{-x_0Y_t^0\}]<\infty$. Noting that

$$\left|\frac{e^{-u_nY_t^0}-e^{-uY_t^0}}{u_n-u}\right| \leqslant Y_t^0 e^{-x_0Y_t^0},$$

we can apply dominated convergence theorem to obtain

$$\lim_{n\to\infty}\frac{\mathbb{E}\left[e^{-u_nY_t^0}\right]-\mathbb{E}\left[e^{-uY_t^0}\right]}{u_n-u} = \lim_{n\to\infty}\mathbb{E}\left[\frac{e^{-u_nY_t^0}-e^{-uY_t^0}}{u_n-u}\right] = \mathbb{E}\left[Y_t^0e^{-uY_t^0}\right]. \tag{1.2.34}$$

Hence $u\mapsto\mathbb{E}[\exp\{-uY_t^0\}]$ is holomorphic on $\mathbb{C}_{>0}$.

On the other hand, the function $z\mapsto v_t(z)$ given in (1.2.7) is continuous on $\mathbb{C}_{\geqslant 0}$ and holomorphic on $\mathbb{C}_{>0}$ for each $t\geqslant 0$ as well. Therefore, we have

$$\mathbb{E}\left[e^{-uY_t^0}\right] = \exp\left\{-a\int_0^t v_s(u)\mathrm{d}s\right\}, \quad u\in\mathbb{C}_{\geqslant 0}. \tag{1.2.35}$$

Indeed, the equality (1.2.35) is true at least for $u \in \mathbb{R}_{>0}$ by (1.1.14). This and the identity theorem for holomorphic functions (see e.g. [22, Theorem III.3.2]) imply (1.2.35) for all $u \in \mathbb{C}_{\geqslant 0}$, since both sides of (1.2.35) are functions that are continuous on $\mathbb{C}_{\geqslant 0}$ and holomorphic on $\mathbb{C}_{>0}$. In particular, the characteristic function of Y_t^0 with $t > 0$ is given by

$$\mathbb{E}\left[e^{i\xi Y_t^0}\right] = \exp\left\{-a\int_0^t v_s(-i\xi)\mathrm{d}s\right\}, \quad \xi \in \mathbb{R}.$$

Remark 1.15. *Recall that the process* $(Z_t^y)_{t\geqslant 0}$ *is given in* (1.2.3). *By repeating the same arguments as above for* Z_t^y*, we see that its characteristic function is given by*

$$\mathbb{E}\left[e^{i\xi Z_t^y}\right] = \exp\left\{-yv_t(-i\xi)\right\}, \quad \xi \in \mathbb{R}.$$

In the next lemma we obtain the existence of a density function for Y_t^0 when $t > 0$. Note that by Theorem 1.4, we have $Y_t^0 \geqslant 0$ almost surely for each $t \geqslant 0$.

Lemma 1.16. *Assume* $a > 0$ *and* $b > 0$*. Then for each* $t > 0$*,* Y_t^0 *possesses a density function* $f_{Y_t^0}$ *given by*

$$f_{Y_t^0}(x) := \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ix\xi}\exp\left\{-a\int_0^t v_s(-i\xi)\mathrm{d}s\right\}\mathrm{d}\xi, \quad x \geqslant 0. \tag{1.2.36}$$

Moreover, the function $f_{Y_t^0}(x)$ *is jointly continuous in* $(t, x) \in (0,\infty)\times\mathbb{R}_{\geqslant 0}$*, and* $f_{Y_t^0}(\cdot) \in C^\infty(\mathbb{R}_{\geqslant 0}, \mathbb{C})$ *for each* $t > 0$*.*

Proof. Let $T > 1$ be fixed. By Lemma 1.13, there exist constants c_1, $c_2 > 0$ such that

$$\left|\exp\left\{-a\int_0^t v_s(-i\xi)\mathrm{d}s\right\}\right| = \exp\left\{\mathrm{Re}\left(-a\int_0^t v_s(-i\xi)\mathrm{d}s\right)\right\} \leqslant c_1 e^{-c_2|\xi|^{2-\alpha}} \tag{1.2.37}$$

for all $\xi \in \mathbb{R}$ and $t \in [1/T, T]$, which implies that $\xi \mapsto \exp\{-a\int_0^t v_s(-i\xi)\mathrm{d}s\}$ is integrable on $\mathbb{R}$. Therefore, by the inversion formula of Fourier transform, Y_t^0 has a density $f_{Y_t^0}$ given by (1.2.36). The joint continuity of the density $f_{Y_t^0}(x)$ in (t, x) follows from (1.2.37), (1.2.36) and dominated convergence theorem. The smoothness property of $f_{Y_t^0}(\cdot)$ is a consequence of (1.2.37) and [59, Proposition 28.1]. ☐

We remark that for each $t > 0$, the function $f_{Y_t^0}(x)$ given in (1.2.36) is actually well-defined also for $x < 0$, although $f_{Y_t^0}(x) \equiv 0$ for $x \leqslant 0$, which is due to the fact that $Y_t^0 \geqslant 0$ almost surely. Next, we would like to know if $f_{Y_t^0}(x) > 0$ when $x > 0$. The next lemma partly answers this question.

Lemma 1.17. *Assume* $a > 0$ *and* $b > 0$*. For each* $t > 0$*, the density function* $f_{Y_t^0}(\cdot)$ *of* Y_t^0 *is almost everywhere positive on* $\mathbb{R}_{\geqslant 0}$*.*

Proof. Basically, the idea of the proof is as follows. We will show the following:

Claim. *The function* $x \mapsto f_{Y_t^0}(x)$*,* $x \in \mathbb{R}_{>0}$*, can be extended to a holomorphic function on* $\mathbb{C}_{>0}$*.*

If this claim is true, then the set $A_n := \{x > 1/n : f_{Y_t^0}(x) = 0\}$ with $n \in \mathbb{N}$ must be discrete, that is, for each $x \in A_n$, one can find a neighbourhood of x whose intersection with A_n equals x; otherwise the identity theorem for holomorphic functions (see, e.g. Freitag and Busam [22, Proposition III.3.1]) implies that $f_{Y_t^0}(x) \equiv 0$ for $x > 0$. As a consequence, A_n is countable, which implies that $A := \cup_{n\in\mathbb{N}} A_n$ is also countable and thus has Lebesgue measure 0.

Let $x > 0$ be fixed. We will complete the proof of the above claim in several steps.

"*Step 1*": We derive a simpler representation for $f_{Y_t^0}(x)$. We have

$$\begin{aligned} f_{Y_t^0}(x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ix\xi} \exp\left\{-a\int_0^t v_s(-i\xi)\mathrm{d}s\right\}\mathrm{d}\xi \\ &= \frac{1}{2\pi}\int_{0}^{\infty} e^{-ix\xi} \exp\left\{-a\int_0^t v_s(-i\xi)\mathrm{d}s\right\}\mathrm{d}\xi \\ &\quad + \frac{1}{2\pi}\int_{-\infty}^{0} e^{-ix\xi} \exp\left\{-a\int_0^t v_s(-i\xi)\mathrm{d}s\right\}\mathrm{d}\xi \\ &= \frac{1}{2\pi}\int_{-\infty}^{0} e^{ix\xi} \exp\left\{-a\int_0^t v_s(i\xi)\mathrm{d}s\right\}\mathrm{d}\xi \\ &\quad + \frac{1}{2\pi}\int_{-\infty}^{0} e^{-ix\xi} \exp\left\{-a\int_0^t v_s(-i\xi)\mathrm{d}s\right\}\mathrm{d}\xi. \end{aligned} \tag{1.2.38}$$

For $\xi < 0$, we have

$$\begin{aligned} \overline{v_s(-i\xi)} &= \left(\left(\tfrac{1}{\alpha b} + \overline{(-i\xi)^{1-\alpha}}\right)e^{b(\alpha-1)s} - \tfrac{1}{\alpha b}\right)^{1/(1-\alpha)} \\ &= \left(\left(\tfrac{1}{\alpha b} + (i\xi)^{1-\alpha}\right)e^{b(\alpha-1)s} - \tfrac{1}{\alpha b}\right)^{1/(1-\alpha)} = v_s(i\xi), \end{aligned}$$

which implies

$$\overline{e^{-ix\xi} \exp\left\{-a\int_0^t v_s(-i\xi)\mathrm{d}s\right\}} = e^{ix\xi} \exp\left\{-a\int_0^t v_s(i\xi)\mathrm{d}s\right\}. \tag{1.2.39}$$

By (1.2.38) and (1.2.39), we get

$$f_{Y_t^0}(x) = \mathrm{Re}\left(\frac{1}{\pi}\int_{-\infty}^{0} e^{-ix\xi} \exp\left\{-a\int_0^t v_s(-i\xi)\mathrm{d}s\right\}\mathrm{d}\xi\right). \tag{1.2.40}$$

For simplicity, let

$$I := \frac{1}{\pi}\int_{-\infty}^{0} e^{-ix\xi} \exp\left\{-a\int_0^t v_s(-i\xi)\mathrm{d}s\right\}\mathrm{d}\xi. \tag{1.2.41}$$

"*Step 2*": We calculate I by contour integration. By a change of variables $z := -i\xi$, we get

$$\begin{aligned} I &= \frac{-i}{\pi}\int_{0}^{i\infty} e^{xz} \exp\left\{-a\int_0^t v_s(z)\mathrm{d}s\right\}\mathrm{d}z \\ &= \lim_{K\to\infty}\frac{-i}{\pi}\int_{iK^{-1}}^{iK} e^{xz} \exp\left\{-a\int_0^t v_s(z)\mathrm{d}s\right\}\mathrm{d}z. \end{aligned} \tag{1.2.42}$$

Define two paths $\Gamma_{1,K}$ and $\Gamma_{2,K}$ by

$$\Gamma_{1,K}(\vartheta) := Ke^{i\vartheta}, \quad \vartheta \in \left[\frac{\pi}{2}, \pi\right] \quad \text{and} \quad \Gamma_{2,K}(\vartheta) := K^{-1}e^{i\vartheta}, \quad \vartheta \in \left[\frac{\pi}{2}, \pi\right].$$

According to (1.2.7), we see that the function

$$z \mapsto e^{yz} \exp\left\{-a\int_0^t v_s(z)\mathrm{d}s\right\}, \quad z \in \mathcal{O}_1 := \left\{\rho\exp(i\vartheta) \,:\, \rho > 0, \vartheta \in \left[\frac{\pi}{2}, \pi\right]\right\},$$

can be extended to a holomorphic function on $\mathcal{O}_2 := \{\rho\exp(i\vartheta) \,:\, \rho > 0, \vartheta \in (0, 3\pi/2)\}$. Therefore, we have

$$\begin{aligned}
&\int_{iK^{-1}}^{iK} e^{xz} \exp\left\{-a\int_0^t v_s(z)\mathrm{d}s\right\}\mathrm{d}z \qquad (1.2.43)\\
&\quad = \int_{-K^{-1}}^{-K} e^{xz} \exp\left\{-a\int_0^t v_s(z)\mathrm{d}s\right\}\mathrm{d}z - \int_{\Gamma_{1,K}} e^{xz} \exp\left\{-a\int_0^t v_s(z)\mathrm{d}s\right\}\mathrm{d}z\\
&\qquad + \int_{\Gamma_{2,K}} e^{xz} \exp\left\{-a\int_0^t v_s(z)\mathrm{d}s\right\}\mathrm{d}z.
\end{aligned}$$

Since $\lim_{z\to 0} e^{xz}\exp\left\{-a\int_0^t v_s(z)\mathrm{d}s\right\} = 1$, it follows that

$$\lim_{K\to\infty}\int_{\Gamma_{2,K}} e^{xz} \exp\left\{-a\int_0^t v_s(z)\mathrm{d}s\right\}\mathrm{d}z = 0. \qquad (1.2.44)$$

To estimate the second term on the right-hand side of (1.2.43), we divide the path $\Gamma_{1,K}$ into two parts, namely

$$\Gamma_{11,K}(\vartheta) := Ke^{i\vartheta}, \quad \vartheta \in \left[\frac{\pi}{2}, \frac{\pi}{2} + \varepsilon_0\right] \quad \text{and} \quad \Gamma_{12,K}(\vartheta) =: Ke^{i\vartheta}, \quad \vartheta \in \left[\frac{\pi}{2} + \varepsilon_0, \pi\right],$$

with $\varepsilon_0 > 0$ being the constant appearing in Lemmas 1.13 and 1.14. Then

$$\begin{aligned}
&\int_{\Gamma_{1,K}} e^{xz} \exp\left\{-a\int_0^t v_s(z)\mathrm{d}s\right\}\mathrm{d}z\\
&\quad = \int_{\Gamma_{11,K}} e^{xz} \exp\left\{-a\int_0^t v_s(z)\mathrm{d}s\right\}\mathrm{d}z + \int_{\Gamma_{12,K}} e^{xz} \exp\left\{-a\int_0^t v_s(z)\mathrm{d}s\right\}\mathrm{d}z\\
&\quad := II_1(K) + II_2(K).
\end{aligned}$$

If we can show that $\lim_{K\to\infty} II_1(K) = 0$ and $\lim_{K\to\infty} II_2(K) = 0$, then it follows from (1.2.42), (1.2.43) and (1.2.44) that

$$I = \frac{-i}{\pi}\int_0^{-\infty} e^{xz} \exp\left\{-a\int_0^t v_s(z)\mathrm{d}s\right\}\mathrm{d}z. \qquad (1.2.45)$$

"*Step 3*": We show that $\lim_{K\to\infty} II_1(K) = 0$. If $\vartheta \in [\pi/2, \pi/2 + \varepsilon_0]$, then

$$\left|e^{xKe^{i\vartheta}}\right| = e^{\mathrm{Re}\left(xKe^{i\vartheta}\right)} = e^{xK\cos(\vartheta)} \leqslant 1.$$

By Lemma 1.13, we get

$$\begin{aligned}
|II_1(K)| &= \left|\int_{\frac{\pi}{2}}^{\pi/2+\varepsilon_0} iKe^{i\vartheta}e^{xKe^{i\vartheta}}e^{-a\int_0^t v_s\left(Ke^{i\vartheta}\right)\mathrm{d}s}\mathrm{d}\vartheta\right|\\
&\leqslant K\int_{\frac{\pi}{2}}^{\pi/2+\varepsilon_0}\left|e^{-a\int_0^t v_s\left(Ke^{i\vartheta}\right)\mathrm{d}s}\right|\mathrm{d}\vartheta \leqslant K\varepsilon_0 e^{aC_1 - aC_2K^{2-\alpha}}, \qquad (1.2.46)
\end{aligned}$$

which implies
$$\lim_{K\to\infty} |II_1(K)| \leqslant \lim_{K\to\infty} K\varepsilon_0 e^{aC_1 - aC_2K^{2-\alpha}} = 0.$$

"*Step 4*": We show that $\lim_{K\to\infty} II_2(K) = 0$. In case $\vartheta \in [\pi/2+\varepsilon_0, \pi]$, then
$$\left|e^{xKe^{i\vartheta}}\right| = e^{\mathrm{Re}\left(xKe^{i\vartheta}\right)} = e^{xK\cos(\vartheta)} \leqslant e^{xK\cos\left(\frac{\pi}{2}+\varepsilon_0\right)} = e^{-xK\sin(\varepsilon_0)}. \tag{1.2.47}$$
So
$$\begin{aligned}
|II_2(K)| &= \left|\int_{\frac{\pi}{2}+\varepsilon_0}^{\pi} iKe^{i\vartheta} e^{xKe^{i\vartheta}} \exp\left\{-a\int_0^t v_s\left(Ke^{i\vartheta}\right) \mathrm{d}s\right\} \mathrm{d}\vartheta\right| \\
&\leqslant K\int_{\frac{\pi}{2}+\varepsilon_0}^{\pi} \left|e^{xKe^{i\vartheta}}\right| \left|\exp\left\{-a\int_0^t v_s\left(Ke^{i\vartheta}\right) \mathrm{d}s\right\}\right| \mathrm{d}\vartheta \\
&\leqslant Ke^{-xK\sin(\varepsilon_0)} \int_{\frac{\pi}{2}+\varepsilon_0}^{\pi} \exp\left\{a\left|\int_0^t v_s\left(Ke^{i\vartheta}\right) \mathrm{d}s\right|\right\} \mathrm{d}\vartheta.
\end{aligned}$$
By Lemma 1.14, we get
$$\lim_{K\to\infty} |II_2(K)| \leqslant \lim_{K\to\infty} K\left(\tfrac{\pi}{2} - \varepsilon_0\right) e^{-xK\sin(\varepsilon_0)} e^{aC_3} e^{aC_4K^{2-\alpha}} = 0.$$

"*Step 5*": Finally, we prove that $x \mapsto f_{Y_t^0}(x)$ is holomorphic on $\mathbb{C}_{>0}$. By (1.2.40), (1.2.41) and (1.2.45), we get
$$\begin{aligned}
f_{Y_t^0}(x) &= \mathrm{Re}\left(\frac{-i}{\pi}\int_0^{-\infty} e^{xz} \exp\left\{-a\int_0^t v_s(z)\mathrm{d}s\right\} \mathrm{d}z\right) \\
&= \mathrm{Re}\left(\frac{i}{\pi}\int_0^{\infty} e^{-xz} \exp\left\{-a\int_0^t v_s(-z)\mathrm{d}s\right\} \mathrm{d}z\right) \\
&= -\mathrm{Im}\left(\frac{1}{\pi}\int_0^{\infty} e^{-xz} \exp\left\{-a\int_0^t v_s(-z)\mathrm{d}s\right\} \mathrm{d}z\right) \\
&= \frac{1}{\pi}\int_0^{\infty} e^{-xz} \left\{-\mathrm{Im}\left(\exp\left\{-a\int_0^t v_s(-z)\mathrm{d}s\right\}\right)\right\} \mathrm{d}z.
\end{aligned}$$
Let $x_0 > 0$ be fixed. Consider $x \in \mathbb{C}$ with $\mathrm{Re}(x) > x_0$ and $(x_n) \subset \mathbb{C}$ such that $\mathrm{Re}(x_n) > x_0$ and $x_n \to x$ as $n \to \infty$. Noting that
$$\left|\frac{e^{-x_nz} - e^{-xz}}{x_n - x}\right| \leqslant ze^{-x_0z},$$
we can use Lemma 1.14 to obtain
$$\begin{aligned}
ze^{-x_0z}\left|\mathrm{Im}\left(\exp\left\{-a\int_0^t v_s(-z)\mathrm{d}s\right\}\right)\right| &\leqslant ze^{-x_0z}\left|\exp\left\{-a\int_0^t v_s(-z)\mathrm{d}s\right\}\right| \\
&\leqslant ze^{-x_0z}\exp\left\{aC_3 + aC_4|z|^{2-\alpha}\right\},
\end{aligned} \tag{1.2.48}$$
where the right-hand side of (1.2.48) is an integrable function (with the variable z) on $\mathbb{R}_{\geqslant 0}$. Similarly to (1.2.34), by dominated convergence theorem, we see that the function
$$x \mapsto \frac{1}{\pi}\int_0^{\infty} e^{-xz} \left\{-\mathrm{Im}\left(\exp\left\{-a\int_0^t v_s(-z)\mathrm{d}s\right\}\right)\right\} \mathrm{d}z, \quad x \in \mathbb{C}_{>0},$$

is holomorphic, which means that $x \mapsto f_{Y_t^0}(x)$ has a holomorphic extension on $\mathbb{C}_{>0}$. This completes the proof. □

With the help of the previous lemma, we are now able to prove the main result of this section. Recall that the process $(Y_t^y)_{t\geqslant 0}$ is given by (1.0.2).

Proposition 1.18. *Assume $a > 0$ and $b > 0$. Then for each $y \geqslant 0$ and $t > 0$, Y_t^y possesses a density function $f_{Y_t^y}$ given by*

$$f_{Y_t^y}(x) := \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ix\xi}\exp\left\{-a\int_0^t v_s(-i\xi)\mathrm{d}s - yv_t(-i\xi)\right\}\mathrm{d}\xi, \quad x \geqslant 0, \tag{1.2.49}$$

where $f_{Y_t^y}(\cdot) \in C^\infty(\mathbb{R}_{\geqslant 0}, \mathbb{C})$ and $f_{Y_t^y}(x) > 0$ for all $x > 0$. Moreover, the function $f_{Y_t^y}(x)$ is jointly continuous in $(t, y, x) \in (0,\infty) \times \mathbb{R}_{\geqslant 0} \times \mathbb{R}_{\geqslant 0}$.

Proof. In view of Proposition 1.12, we have

$$\mathbb{E}\left[e^{i\xi Y_t^y}\right] = \mathbb{E}\left[e^{i\xi Y_t^0}\right]\cdot\mathbb{E}\left[e^{i\xi Z_t^y}\right] = \exp\left\{-a\int_0^t v_s(-i\xi)\mathrm{d}s - yv_t(-i\xi)\right\}, \tag{1.2.50}$$

where $\xi \in \mathbb{R}$. It follows from (1.2.37) that

$$\left|\mathbb{E}\left[e^{i\xi Y_t^y}\right]\right| \leqslant \left|\mathbb{E}\left[e^{i\xi Y_t^0}\right]\right| \leq c_1 e^{-c_2|\xi|^{2-\alpha}}$$

for all $\xi \in \mathbb{R}$ and $t \in [1/T, T]$, where $T > 1$ and c_1, $c_2 > 0$ are constants depending on T. It follows that for $t > 0$, Y_t^y has a density $f_{Y_t^y}$ given by (1.2.49). Proceeding in the same way as in Lemma 1.16, we obtain the desired continuity and smoothness properties of $f_{Y_t^y}$.

We next show that if $t > 0$, then $f_{Y_t^y}(x) > 0$ for all $x > 0$. According to (1.2.50), we see that the law of Y_t^y, denoted by $\mu_{Y_t^y}$, is the convolution of the laws of Z_t^y and Y_t^0, which we denote by $\mu_{Z_t^y}$ and $\mu_{Y_t^0}$, respectively. So $\mu_{Y_t^y} = \mu_{Z_t^y} * \mu_{Y_t^0}$. From this we deduce that for all $x > 0$,

$$\begin{aligned} f_{Y_t^y}(x) &= \int_{\mathbb{R}_{\geqslant 0}} f_{Y_t^0}(x - z)\mu_{Z_t^y}(\mathrm{d}z) \\ &= \int_{(0,\infty)} f_{Y_t^0}(x - z)\mu_{Z_t^y}(\mathrm{d}z) + f_{Y_t^0}(x)\mu_{Z_t^y}(\{0\}). \end{aligned} \tag{1.2.51}$$

By Lemma 1.17, the density function $f_{Y_t^0}(x)$ of Y_t^0 is strictly positive for almost all $x > 0$. In the following we consider a fixed $x > 0$ and distinguish between two cases.

"*Case 1*": $f_{Y_t^0}(x) > 0$. It follows from (1.2.51) that

$$f_{Y_t^y}(x) \geqslant f_{Y_t^0}(x)\mu_{Z_t^y}(\{0\}) > 0, \tag{1.2.52}$$

where we used the fact that $\mu_{Z_t^y}(\{0\}) = \mathbb{P}(Z_t^y = 0) > 0$, as shown in (1.2.5).

"*Case 2*": $f_{Y_t^0}(x) = 0$. Then $x \in A_n$ for a large enough n, where the set A_n is the same as in the proof of Lemma 1.17. Since A_n is discrete, we can find a small enough $\delta > 0$ such that

$$f_{Y_t^0}(x - z) > 0, \tag{1.2.53}$$

for all $z \in (0, \delta]$. We next show that $\mu_{Z_t^y}((0, \delta]) > 0$. By (1.2.4), (1.2.5) and L'Hospital's Rule, we get

$$\begin{aligned} \lim_{\lambda\to\infty} &\left(\mathbb{E}\left[e^{-\lambda(Z_t^y-\delta)}\right] - \mathbb{E}\left[e^{-\lambda(Z_t^y-\delta)}\mathbb{1}_{\{Z_t^y=0\}}\right]\right) \\ &= \lim_{\lambda\to\infty} e^{\lambda\delta}\left(\mathbb{E}\left[e^{-\lambda Z_t^y}\right] - \mathbb{P}(Z_t^y = 0)\right) \\ &= \lim_{\lambda\to\infty} e^{\lambda\delta}\left(e^{-yv_t(\lambda)} - e^{-yd}\right) \\ &= \lim_{\lambda\to\infty} \delta^{-1}e^{\lambda\delta}ye^{-yv_t(\lambda)}\left(v_t(\lambda)\right)^{\alpha}e^{b(\alpha-1)t}\lambda^{-\alpha} = \infty. \end{aligned} \tag{1.2.54}$$

Suppose that $\mathbb{P}(Z_t^y \in (0, \delta]) = 0$. Then we can use dominated convergence theorem to get

$$\begin{aligned} &\lim_{\lambda\to\infty}\left(\mathbb{E}\left[e^{-\lambda(Z_t^y-\delta)}\right] - \mathbb{E}\left[e^{-\lambda(Z_t^y-\delta)}\mathbb{1}_{\{Z_t^y=0\}}\right]\right) \\ &\quad = \lim_{\lambda\to\infty}\left(\mathbb{E}\left[e^{-\lambda(Z_t^y-\delta)}\mathbb{1}_{\{0<Z_t^y\leqslant\delta\}}\right] + \mathbb{E}\left[e^{-\lambda(Z_t^y-\delta)}\mathbb{1}_{\{Z_t^y>\delta\}}\right]\right) = 0, \end{aligned}$$

which contradicts (1.2.54). Consequently, the assumption that $\mathbb{P}(Z_t^y \in (0, \delta]) = 0$ is not true and we thus get $\mathbb{P}\left(Z_t^y \in (0, \delta]\right) > 0$. Now, by (1.2.51) and (1.2.53), we get

$$f_{Y_t^y}(x) \geqslant \int_{(0,\delta]} f_{Y_t^0}(x - z)\mu_{Z_t^y}(\mathrm{d}z) > 0. \tag{1.2.55}$$

Summarizing the above two cases, we have $f_{Y_t^y}(x) > 0$ for all $x > 0$. This completes the proof. □

1.3. A Foster-Lyapunov function for (Y, X)

We now turn back to the two-dimensional affine process $(Y, X) = (Y_t, X_t)_{t\geqslant 0}$ defined in (1.0.1). Our aim of this section is to construct a Foster-Lyapunov function for (Y, X).

For a functional $\Phi(Y, X)$ based on the process (Y, X), we use $\mathbb{E}_{(y,x)}[\Phi(Y, X)]$ to indicate that the process (Y, X) considered under the expectation is with the initial condition $(Y_0, X_0) = (y, x)$, where $(y, x) \in \mathbb{R}_{\geqslant 0} \times \mathbb{R}$ is constant. The notation $\mathbb{P}_{(y,x)}(\Phi(Y, X) \in \cdot)$ is similarly defined.

Lemma 1.19. *Let $h \in C^\infty(\mathbb{R}, \mathbb{R})$ be such that $h(x) \geqslant 1$ for all $x \in \mathbb{R}$ and $h(x) = |x|$ whenever $|x| \geqslant 2$. Define*

$$V(y, x) := \beta y + h(x), \quad y \geqslant 0,\ x \in \mathbb{R},$$

where $\beta > 0$ is a constant. If β is sufficiently large, then V is a Foster-Lyapunov function for (Y, X), that is, there exist constants $c, M > 0$ such that

$$\mathbb{E}_{(y,x)}[V(Y_t, X_t)] \leqslant e^{-ct}V(y, x) + \tfrac{M}{c} \tag{1.3.1}$$

for all $(y, x) \in \mathbb{R}_{\geqslant 0} \times \mathbb{R}$ and $t \geqslant 0$.

Remark 1.20. *To see the existence of a function $h \in C^\infty(\mathbb{R},\mathbb{R})$ that fulfills the conditions of Lemma 1.19, we can proceed in the following way: let $\rho \in C^\infty(\mathbb{R},\mathbb{R})$ be such that $\rho(x) = 1$ for $x \geqslant 2$, $\rho(x) = 0$ for $x \leqslant 1$ and $0 \leqslant \rho(x) \leqslant 1$ for $1 \leqslant x \leqslant 2$. Define $F : \mathbb{R} \to \mathbb{R}$ by $F(x) := \int_0^x \rho(r)\mathrm{d}r$, $x \in \mathbb{R}$. Then*

$$F(x) = \begin{cases} 0, & x \leqslant 1, \\ \in [0,1], & 1 < x \leqslant 2, \\ x - 2 + \int_1^2 \rho(r)\mathrm{d}r, & x > 2. \end{cases}$$

We now define $h : \mathbb{R} \to \mathbb{R}$ by $h(x) := F(|x|) + 2 - F(2)$, $x \in \mathbb{R}$. Then h satisfies the conditions required in Lemma 1.19.

Proof of Lemma 1.19. Recall that $\mathbb{R}^2_{\geqslant 0} \times \Omega \ni (s,z,\omega) \mapsto z\mathbb{1}_{\{|z|<1\}} \sqrt[\alpha]{Y_{s-}} \in F^{2,loc}$ and $\mathbb{R}^2_{\geqslant 0} \times \Omega \ni (s,z,\omega) \mapsto z\mathbb{1}_{\{|z|\geqslant 1\}} \sqrt[\alpha]{Y_{s-}} \in F^1$ by Remark 1.3. Define $g(t,y,x) := \exp\{ct\}V(y,x)$, where $c > 0$ is a constant to be determined later. It is easy to see that $g \in C^2(\mathbb{R}_{\geqslant 0} \times \mathbb{R}_{\geqslant 0} \times \mathbb{R}, \mathbb{R})$. We define the functions g_1', g_2', g_3' and $g_{3,3}''$ by

$$g_1'(t,y,x) := \tfrac{\partial}{\partial t} g(t,y,x) = ce^{ct}V(y,x), \quad g_2'(t,y,x) := \tfrac{\partial}{\partial y} g(t,y,x) = \beta e^{ct},$$
$$g_3'(t,y,x) := \tfrac{\partial}{\partial x} g(t,y,x) = e^{ct}\tfrac{\partial}{\partial x}h(x), \quad g_{3,3}''(t,y,x) := \tfrac{\partial^2}{\partial x^2} g(t,y,x) = e^{ct}\tfrac{\partial^2}{\partial x^2}h(x).$$

Proceeding as in the proof of Proposition 1.5, we can apply Itô's formula for g (see [62, Theorem 94]) to obtain that for each $t \geqslant 0$,

$$\begin{aligned}
&g(t,Y_t,X_t) - g(0,Y_0,X_0) \\
&= \int_0^t g_1'(s,Y_s,X_s)\mathrm{d}s + \int_0^t g_2'(s,Y_s,X_s)\gamma\sqrt[\alpha]{Y_s}\mathrm{d}s \\
&\quad + \int_0^t g_2'(s,Y_s,X_s)(a - bY_s)\mathrm{d}s + \int_0^t g_3'(s,Y_s,X_s)(m - \theta X_s)\mathrm{d}s \\
&\quad + \frac{1}{2}\int_0^t g_{3,3}''(s,Y_s,X_s)Y_s\mathrm{d}s + \int_0^t g_3'(s,Y_s,X_s)\sqrt{Y_s}\mathrm{d}B_s \\
&\quad + \int_0^t \int_{\{|z|<1\}} \Big(g(s,Y_{s-} + z\sqrt[\alpha]{Y_{s-}},X_{s-}) - g(s,Y_{s-},X_{s-})\Big)\widetilde{N}(\mathrm{d}s,\mathrm{d}z) \\
&\quad + \int_0^t \int_{\{|z|\geqslant 1\}} \Big(g(s,Y_{s-} + z\sqrt[\alpha]{Y_{s-}},X_{s-}) - g(s,Y_{s-},X_{s-})\Big)N(\mathrm{d}s,\mathrm{d}z) \\
&\quad + \int_0^t \int_{\{|z|<1\}} \Big(g(s,Y_s + z\sqrt[\alpha]{Y_s},X_s) \\
&\qquad\qquad - g(s,Y_s,X_s) - z\sqrt[\alpha]{Y_s}g_2'(s,Y_s,X_s)\Big)C_\alpha z^{-1-\alpha}\mathrm{d}s\mathrm{d}z \\
&= \int_0^t e^{cs}(\mathcal{L}g)(s,Y_s,X_s)\mathrm{d}s + \int_0^t g_1'(s,Y_s,X_s)\mathrm{d}s + M_t(g),
\end{aligned} \tag{1.3.2}$$

where

$$\begin{aligned}
M_t(g) := &\int_0^t g_3'(s,Y_s,X_s)\sqrt{Y_s}\mathrm{d}B_s \\
&+ \int_0^t \int_{\{|z|<1\}} \Big(g(s,Y_{s-} + z\sqrt[\alpha]{Y_{s-}},X_{s-}) - g(s,Y_{s-},X_{s-})\Big)\widetilde{N}(\mathrm{d}s,\mathrm{d}z) \\
&+ \int_0^t \int_{\{|z|\geqslant 1\}} \Big(g(s,Y_{s-} + z\sqrt[\alpha]{Y_{s-}},X_{s-}) - g(s,Y_{s-},X_{s-})\Big)N(\mathrm{d}s,\mathrm{d}z)
\end{aligned}$$

$$-\int_0^t\int_{\{|z|\geqslant 1\}}\Big(g(s,Y_s+z\sqrt[\alpha]{Y_s},X_s)-g(s,Y_s,X_s)\Big)\,\widehat{N}(\mathrm{d}s,\mathrm{d}z)$$

by an easy computation (see the proof of Proposition 1.5), we see that the operator $\mathcal{L}$ corresponds with the infinitesimal generator $\mathcal{A}$ given in (1.1.1). As a result, it follows from (1.3.2) that for each $t\geqslant 0$,

$$\begin{aligned} &g(t,Y_t,X_t)-g(0,Y_0,X_0)\\ &\quad=\int_0^t e^{cs}(\mathcal{A}g)(Y_s,X_s)\mathrm{d}s+\int_0^t g_1'(s,Y_s,X_s)\mathrm{d}s+M_t\,(g)\,. \end{aligned}\tag{1.3.3}$$

The rest of the proof is divided into three steps:

"*Step 1*": We show that $(M_t(g))_{t\geqslant 0}$ is a martingale with respect to the filtration $(\mathcal{F}_t)_{t\geqslant 0}$, where $(\mathcal{F}_t)_{t\geqslant 0}$ is the filtration introduced in Section 1. To achieve this, we can use a similar argument as in Proposition 1.5. Define

$$\begin{aligned} D_t(g)&:=\int_0^t g_3'(s,Y_s,X_s)\sqrt{Y_s}\mathrm{d}B_s,\\ J_t(g)&:=\int_0^t\int_{\{|z|<1\}}\Big(g(s,Y_{s-}+z\sqrt[\alpha]{Y_{s-}},X_{s-})-g(s,Y_{s-},X_{s-})\Big)\,\widetilde{N}(\mathrm{d}s,\mathrm{d}z),\\ &\quad+\int_0^t\int_{\{|z|\geqslant 1\}}\Big(g(s,Y_{s-}+z\sqrt[\alpha]{Y_{s-}},X_{s-})-g(s,Y_{s-},X_{s-})\Big)\,N(\mathrm{d}s,\mathrm{d}z)\\ &\quad-\int_0^t\int_{\{|z|\geqslant 1\}}\Big(g(s,Y_s+z\sqrt[\alpha]{Y_s},X_s)-g(s,Y_s,X_s)\Big)\,\widehat{N}(\mathrm{d}s,\mathrm{d}z), \end{aligned}$$

where $t\geqslant 0$. By noting that $(t,y,x)\mapsto g_2'(t,y,x)$ is bounded for $(t,y,x)\in[0,T]\times\mathbb{R}_{\geqslant 0}\times\mathbb{R}$, where $T>0$ is constant, we can proceed in the same way as in Proposition 1.5 to prove that $(D_t(g))_{t\geqslant 0}$ is a square integrable martingale with respect to the filtration $(\mathcal{F}_t)_{t\geqslant 0}$. Note that $g(s,y+z,x)-g(s,y,x)=\beta\exp\{cs\}z$. Similarly to Remark 1.3 (see equations (1.0.5) and (1.0.6)), we see that

$$\begin{aligned} &\mathbb{1}_{\{|z|<1\}}\Big(g(s,Y_{s-}+z\sqrt[\alpha]{Y_{s-}},X_{s-})-g(s,Y_{s-},X_{s-})\Big)=\beta e^{cs}\mathbb{1}_{\{|z|<1\}}z\sqrt[\alpha]{Y_{s-}}\in F^{2,\mathrm{loc}},\\ &\mathbb{1}_{\{|z|\geqslant 1\}}\Big(g(s,Y_{s-}+z\sqrt[\alpha]{Y_{s-}},X_{s-})-g(s,Y_{s-},X_{s-})\Big)=\beta e^{cs}\mathbb{1}_{\{|z|\geqslant 1\}}z\sqrt[\alpha]{Y_{s-}}\in F^{1}. \end{aligned}$$

Following [26, pp. 62, 63], we obtain that

$$J_{*,t}(g):=\int_0^t\int_{\{|z|<1\}}\Big(g(s,Y_{s-}+z\sqrt[\alpha]{Y_{s-}},X_{s-})-g(s,Y_{s-},X_{s-})\Big)\widetilde{N}(\mathrm{d}s,\mathrm{d}z),\quad t\geqslant 0,$$

is a local square integrable martingale with respect to the filtration $(\mathcal{F}_t)_{t\geqslant 0}$ and

$$\begin{aligned} J_t^*(g)&:=\int_0^t\int_{\{|z|\geqslant 1\}}\Big(g(s,Y_{s-}+z\sqrt[\alpha]{Y_{s-}},X_{s-})-g(s,Y_{s-},X_{s-})\Big)N(\mathrm{d}s,\mathrm{d}z)\\ &\quad-\int_0^t\int_{\{|z|\geqslant 1\}}\Big(g(s,Y_s+z\sqrt[\alpha]{Y_s},X_s)-g(s,Y_s,X_s)\Big)\widehat{N}(\mathrm{d}s,\mathrm{d}z),\quad t\geqslant 0, \end{aligned}$$

is martingale with respect to the filtration $(\mathcal{F}_t)_{t\geqslant 0}$. Therefore, $(J_t(g))_{t\geqslant 0}=(J_{*,t}(g)+J_t^*(g))_{t\geqslant 0}$ is a local martingale with respect to the filtration $(\mathcal{F}_t)_{t\geqslant 0}$. It remains to check that $(J_t(g))_{t\geqslant 0}$ is a martingale with respect to the filtration $(\mathcal{F}_t)_{t\geqslant 0}$. Using the Lévy-Itô decomposition in (1.0.3), we obtain

$$\begin{aligned}J_t(g) &= \int_0^t \int_{\{|z|<1\}} \beta e^{cs} z \sqrt[\alpha]{Y_{s-}} \widetilde{N}(\mathrm{d}s, \mathrm{d}z) \\ &\quad + \int_0^t \int_{\{|z|\geqslant 1\}} \beta e^{cs} z \sqrt[\alpha]{Y_{s-}} N(\mathrm{d}s, \mathrm{d}z) - \int_0^t \int_{\{|z|\geqslant 1\}} \beta e^{cs} z \sqrt[\alpha]{Y_{s-}} \widehat{N}(\mathrm{d}s, \mathrm{d}z) \\ &= \int_0^t \beta e^{cs} \sqrt[\alpha]{Y_{s-}} \mathrm{d}L_s, \qquad t \geqslant 0.\end{aligned}$$

We can use Remark 1.1 and Jensen's inequality to obtain that for each $T > 0$, there exists some constant $c_2 > 0$ such that

$$\mathbb{E}_{(y,x)} \left[\sup_{s\in[0,T]} |J_s(g)| \right] \leqslant c_2 \mathbb{E}_{(y,x)} \left[\left(\int_0^T Y_s \mathrm{d}s \right)^{1/\alpha} \right] \leqslant c_2 \left(\int_0^T \mathbb{E}_{(y,x)} [Y_s] \, \mathrm{d}s \right)^{1/\alpha} < \infty,$$

where finiteness follows from Proposition 1.2. Since $(J_t(g))_{t\geqslant 0}$ is a local martingale with respect to the filtration $(\mathcal{F}_t)_{t\geqslant 0}$, there exists an increasing sequence of stopping times σ_n, $n \in \mathbb{N}$ with $\sigma_n \to \infty$ as n tends to infinity almost surely such that $(J_{t\wedge\sigma_n}(g))_{t\geqslant 0}$ is a martingale with respect to the filtration $(\mathcal{F}_t)_{t\geqslant 0}$. Then, by dominated convergence theorem for conditional expectations, we get that for all $0 \leq s \leq t \leq T$,

$$\begin{aligned}\mathbb{E}\left[J_t(g) \mid \mathcal{F}_s \right] &= \mathbb{E}\left[\lim_{n\to\infty} J_{t\wedge\sigma_n}(g) \,\middle|\, \mathcal{F}_s \right] \\ &= \lim_{n\to\infty} \mathbb{E}\left[J_{t\wedge\sigma_n}(g) \mid \mathcal{F}_s \right] = \lim_{n\to\infty} J_{s\wedge\sigma_n}(g) = J_s(g),\end{aligned}$$

showing that $J_t(g)$ is a martingale with respect to the filtration $(\mathcal{F}_t)_{t\geqslant 0}$. As a result, $(M_t(g))_{t\geqslant 0} = (D_t(g) + J_t(g))_{t\geqslant 0}$ is also a martingale with respect to the filtration $(\mathcal{F}_t)_{t\geqslant 0}$. This completes the proof of step one.

"*Step 2*": We determine the constant $c > 0$ and find another constant $M > 0$ such that

$$(\mathcal{A}V)(y, x) \leqslant -cV(y, x) + M \tag{1.3.4}$$

for all $(y, x) \in \mathbb{R}_{\geqslant 0} \times \mathbb{R}$, where $\mathcal{A}$ is given by (1.1.1). For the function V, we have $V \in C^2(\mathbb{R}_{\geqslant 0} \times \mathbb{R}, \mathbb{R})$,

$$\tfrac{\partial}{\partial y} V(y, x) = \beta, \quad \tfrac{\partial}{\partial x} V(y, x) = \tfrac{\partial}{\partial x} h(x) = \begin{cases} \frac{x}{|x|}, & \text{if } |x| > 2 \\ h'(x), & \text{if } |x| \leqslant 2, \end{cases}$$

and

$$\tfrac{\partial^2}{\partial x^2} V(y, x) = \tfrac{\partial^2}{\partial x^2} h(x) := \begin{cases} 0, & \text{if } |x| > 2, \\ h''(x), & \text{if } |x| \leqslant 2, \end{cases}$$

where h' and h'' denote the first and second order derivatives of the function h, respectively. So

$$\begin{aligned}(\mathcal{A}V)(y, x) &= (a - by)\beta + (m - \theta x) \tfrac{\partial}{\partial x} h(x) + \tfrac{1}{2} y \tfrac{\partial^2}{\partial x^2} h(x) \\ &\quad + y \int_0^\infty \left(\beta(y + z) + h(x) - \beta y - h(x) - z\beta \right) C_\alpha z^{-1-\alpha} \mathrm{d}z\end{aligned}$$

$$= (a - by)\beta + (m - \theta x)\tfrac{\partial}{\partial x}h(x) + \tfrac{1}{2}y\tfrac{\partial^2}{\partial x^2}h(x).$$

By choosing $\beta > 0$ large enough, we obtain that for all $(y, x) \in \mathbb{R}_{\geqslant 0} \times \mathbb{R}$,

$$\begin{aligned}(\mathcal{A}V)(y,x) &= a\beta - \tfrac{by\beta}{2} - \theta x\tfrac{\partial}{\partial x}h(x) + \left(-\tfrac{b\beta}{2} + \tfrac{1}{2}\tfrac{\partial^2}{\partial x^2}h(x)\right)y + m\tfrac{\partial}{\partial x}h(x)\\ &\leqslant a\beta - \tfrac{by\beta}{2} - \theta\left(h(x)\mathbb{1}_{\{x>2\}} + h(x)\mathbb{1}_{\{x<-2\}}\right) + 0 + c_3\\ &\leqslant a\beta - \tfrac{by\beta}{2} - \theta\left(h(x)\mathbb{1}_{\{|x|>2\}} + h(x)\mathbb{1}_{\{|x|\leqslant 2\}}\right) + c_4\\ &= a\beta - \tfrac{by\beta}{2} - \theta h(x) + c_4 = -\tfrac{b\beta}{2}y - \theta h(x) + c_5,\end{aligned} \tag{1.3.5}$$

where we used the boundedness of $|h'|$, $|h''|$ and $|h|\mathbb{1}_{\{|x|\leqslant 2\}}$ to get the first and second inequality. Here c_3, c_4 and c_5 are some positive constants. Now, we see that (1.3.4) holds with $c := \min(b/2, \theta)$ and $M := c_5$.

"*Step 3*": We prove (1.3.1). By (1.3.3), (1.3.4) and the martingale property of $(M_t(g))_{t\geqslant 0}$, we obtain

$$\begin{aligned}&e^{ct}\mathbb{E}_{(y,x)}\left[V(Y_t, X_t)\right] - V(y,x)\\ &\quad= \mathbb{E}_{(y,x)}\left[g(t, Y_t, X_t)\right] - \mathbb{E}_{(y,x)}\left[g(0, Y_0, X_0)\right]\\ &\quad= \mathbb{E}_{(y,x)}\left[\int_0^t \left(e^{cs}(\mathcal{A}V)(Y_s, X_s) + ce^{cs}V(Y_s, X_s)\right)\mathrm{d}s\right]\\ &\quad\leqslant \mathbb{E}_{(y,x)}\left[\int_0^t \left(e^{cs}(-cV(Y_s, X_s) + M) + ce^{cs}V(Y_s, X_s)\right)\mathrm{d}s\right]\\ &\quad= \mathbb{E}_{(y,x)}\left[\int_0^t Me^{cs}\mathrm{d}s\right] \leqslant \frac{M}{c}e^{ct}\end{aligned}$$

for all $(y, x) \in \mathbb{R}_{\geqslant 0} \times \mathbb{R}$ and $t \geqslant 0$, which implies (1.3.1). This completes the proof. □

1.4. Exponential ergodicity of (Y, X)

The aim that we pursue in this section is to check the exponential ergodicity of the affine two factor model $(Y, X) = (Y_t, X_t)_{t\geqslant 0}$. So far, we have derived a lower bound for the transition densities of the α-stable CIR process $(Y_t)_{t\geqslant 0}$ and we have introduced a Foster-Lyapunov function for the two-dimensional process $(Y_t, X_t)_{t\geqslant 0}$ as well.

Certainly, in order to establish the aimed exponential ergodicity for (Y, X), we need the existence of a unique invariant measure for (Y, X) in prior.

Proposition 1.21. *Consider the process $(Y_t, X_t)_{t\geqslant 0}$ with parameters $a > 0$, $b > 0$, $m \in \mathbb{R}$, $\theta > 0$ and $\alpha \in (1, 2)$. Then (Y_t, X_t) converges in law to a unique limit distribution π as $t \to \infty$. Moreover, π is independent of the initial value (Y_0, X_0) and its characteristic function takes the form*

$$\int_0^\infty \int_{-\infty}^\infty e^{\langle(-\lambda_1, i\lambda_2),(y,x)\rangle}\pi(\mathrm{d}y, \mathrm{d}x) = \exp\left\{-a\int_0^\infty v_s(\lambda_1, \lambda_2)\mathrm{d}s + i\frac{m}{\theta}\lambda_2\right\}$$

for all $(\lambda_1, \lambda_2) \in \mathbb{R}_{\geqslant 0} \times \mathbb{R}$.

Proof. The existence of the invariant measure as well as the form of its characteristic function follows by the stationarity, see [4, Theoren 3.1]. According to the discussion in [39, p.80], the limit distribution π is also the unique invariant distribution of $(Y_t, X_t)_{t\geqslant 0}$. □

Let $\|\cdot\|_{TV}$ denote the total variation norm for signed measures on $\mathbb{R}_{\geqslant 0} \times \mathbb{R}$, namely,

$$\|\mu\|_{TV} := \sup\{|\mu(A)|\},$$

where μ is a signed measure on $\mathbb{R}_{\geqslant 0} \times \mathbb{R}$ and the above supremum is running for all Borel sets A in $\mathbb{R}_{\geqslant 0} \times \mathbb{R}$.

Let $\mathbf{P}^t(y, x, \cdot) := \mathbb{P}_{(y,x)}((Y_t, X_t) \in \cdot)$ denote the distribution of $(Y_t, X_t)_{t\geqslant 0}$ with the initial condition $(Y_0, X_0) = (y_0, x_0) \in \mathbb{R}_{\geqslant 0} \times \mathbb{R}$.

Roughly speaking, if for each $(y, x) \in \mathbb{R}_{\geqslant 0} \times \mathbb{R}$, the convergence of the distribution $\mathbf{P}^t(y, x, \cdot)$ to π as $t \to \infty$ is exponentially fast with respect to the total variation norm, then we say that the process $(Y_t, X_t)_{t\geqslant 0}$ is exponentially ergodic.

Theorem 1.22. *Consider the two-dimensional affine process* $(Y, X) = (Y_t, X_t)_{t\geqslant 0}$ *defined by* (1.0.1) *with parameters* $\alpha \in (1, 2)$, $a > 0$, $b > 0$, $m \in \mathbb{R}$ *and* $\theta > 0$. *Then* $(Y_t, X_t)_{t\geqslant 0}$ *is exponentially ergodic, that is, there exist constants* $\delta \in (0, \infty)$ *and* $B \in (0, \infty)$ *such that*

$$\|\mathbf{P}^t(y, x, \cdot) - \pi\|_{TV} \leqslant B\,(V(y, x) + 1)\,e^{-\delta t} \tag{1.4.1}$$

for all $t \geqslant 0$ *and* $(y, x) \in \mathbb{R}_{\geqslant 0} \times \mathbb{R}$.

Proof. We basically follow the proof of [34, Theorem 6.3]. The essential idea is to use the so called Foster-Lyapunov drift criteria developed in [52] for the geometric ergodicity of Markov chains.

We first consider the skeleton chain $(Y_n, X_n)_{n\in\mathbb{Z}_{\geqslant 0}}$, which is a Markov chain on the state space $\mathbb{R}_{\geqslant 0} \times \mathbb{R}$ with transition kernel $\mathbf{P}^n(y, x, \cdot)$. It is easy to see that the measure π is also an invariant probability measure for the chain $(Y_n, X_n)_{n\in\mathbb{Z}_{\geqslant 0}}$.

Let the function V be the same as in Lemma 1.19 and the constant $\beta > 0$ there be sufficiently large. The Markov property together with Lemma 1.19 implies that

$$\begin{aligned}
&\mathbb{E}[V(Y_{n+1}, X_{n+1}) \,|\, (Y_0, X_0), (Y_1, X_1), \ldots, (Y_n, X_n)] \\
&\qquad = \int_{\mathbb{R}_{\geqslant 0}} \int_{\mathbb{R}} V(y, x)\mathbf{P}^1(Y_n, X_n, \mathrm{d}y\mathrm{d}x) \leqslant e^{-c}V(Y_n, X_n) + \frac{M}{c},
\end{aligned}$$

where c and M are the positive constants in Lemma 1.19. If we set $V_0 := V$ and $V_n := V(Y_n, X_n)$, $n \in \mathbb{N}$, then

$$\mathbb{E}[V_1 \,|\, Y_0, X_0] \leqslant e^{-c}V_0(Y_0, X_0) + \frac{M}{c} \tag{1.4.2}$$

and, for all $n \in \mathbb{N}$,

$$\mathbb{E}\left[V_{n+1} \,|\, (Y_0, X_0), (Y_1, X_1), \ldots (Y_n, X_n)\right] \leqslant e^{-c}V_n + \frac{M}{c}. \tag{1.4.3}$$

It follows from (1.4.2) and (1.4.3) that condition (DD4) in [50, p.564] holds. In order to apply [50, Theorem 6.3] for the chain $(Y_n, X_n)_{n\in\mathbb{Z}_{\geqslant 0}}$, it remains to verify the following conditions:

(a) the Lebesgue measure λ on $\mathbb{R}_{\geqslant 0} \times \mathbb{R}$ is an irreducibility measure for the chain $(Y_n, X_n)_{n \in \mathbb{Z}_{\geqslant 0}}$;

(b) the chain $(Y_n, X_n)_{n \in \mathbb{Z}_{\geqslant 0}}$ is aperiodic;

(c) all compact sets of the state space $\mathbb{R}_{\geqslant 0} \times \mathbb{R}$ are petite.

The definitions of '*irreducibility*', '*aperiodicity*', and '*petite sets*' can be found in the Appendix B, see Definitions B.1, B.2, and B.3 therein. We now proceed to prove (a)-(c).

In order to prove (a), we will use the same argument as in [4, Theorem 4.1]. It is enough to check that for each $(y_0, x_0) \in \mathbb{R}_{\geqslant 0} \times \mathbb{R}$, the measure $\mathbf{P}^1(y_0, x_0, \cdot)$ is absolutely continuous with respect to the Lebesgue measure with a density function $p_1(y, x|y_0, x_0)$ that is strictly positive for almost all $(y, x) \in \mathbb{R}_{\geqslant 0} \times \mathbb{R}$. Indeed, let A be a Borel set of $\mathbb{R}_{\geqslant 0} \times \mathbb{R}$ with $\lambda(A) > 0$. Then

$$\mathbb{P}_{(y_0,x_0)}\left(\tau_A < \infty\right) \geqslant \mathbf{P}^1\left(y_0, x_0, A\right) = \iint_A p_0(y, x|y_0, x_0) \mathrm{d}y \mathrm{d}x > 0$$

for all $(y_0, x_0) \in \mathbb{R}_{\geqslant 0} \times \mathbb{R}$, where the stopping time τ_A is defined by $\tau_A := \inf\{n \geqslant 0 : (Y_n, X_n) \in A\}$.

Next, we prove the existence of the density $p_1(y, x|y_0, x_0)$ with the required property. Recall that

$$Y_1 = e^{-b}\left(y_0 + a\int_0^1 e^{bs}\mathrm{d}s + \int_0^1 e^{bs} \sqrt[\alpha]{Y_{s-}}\mathrm{d}L_s\right),$$

and

$$X_1 = e^{-\theta}\left(x_0 + m\int_0^1 e^{\theta s}\mathrm{d}s + \int_0^1 e^{\theta s}\sqrt{Y_s}\mathrm{d}B_s\right),$$

provided that $(Y_0, X_0) = (y_0, x_0) \in \mathbb{R}_{\geqslant 0} \times \mathbb{R}$. For $(\bar{y}, \bar{x}) \in \mathbb{R}_{\geqslant 0} \times \mathbb{R}$, we have

$$\begin{aligned}
\mathbb{P}_{(y_0,x_0)}\left(Y_1 < \bar{y}, X_1 < \bar{x}\right) &= \mathbb{E}_{(y_0,x_0)}\left[\mathbb{P}_{(y_0,x_0)}\left(Y_1 < \bar{y}, X_1 < \bar{x} \mid Y_1\right)\right] \\
&= \mathbb{E}_{(y_0,x_0)}\left[\mathbb{E}_{(y_0,x_0)}\left[\mathbb{1}_{\{Y_1<\bar{y}\}}\mathbb{1}_{\{X_1<\bar{x}\}} \,\middle|\, Y_1\right]\right] \\
&= \mathbb{E}_{(y_0,x_0)}\left[\mathbb{1}_{\{Y_1<\bar{y}\}}\mathbb{E}_{(y_0,x_0)}\left[\mathbb{1}_{\{X_1<\bar{x}\}} \,\middle|\, Y_1\right]\right]. \qquad (1.4.4)
\end{aligned}$$

Note that $(Y_t)_{t\geqslant 0}$ and the Brownian motion $(B_t)_{t\geqslant 0}$ are independent, since $(L_t)_{t\geqslant 0}$ and $(B_t)_{t\geqslant 0}$ are independent and $(Y_t)_{t\geqslant 0}$ is a strong solution. Therefore, the conditional distribution of X_1 given $(Y_t)_{t\in[0,1]}$ is a normal distribution with mean $x_0 \exp\{-\theta\} + m(1 - \exp\{-\theta\})/\theta$ and variance $\exp\{-2\theta\} \int_0^1 Y_s \exp\{2\theta s\}\mathrm{d}s$. Hence, we get that for $\bar{x} \in \mathbb{R}$,

$$\begin{aligned}
&\mathbb{E}_{(y_0,x_0)}\left[\mathbb{1}_{\{X_1<\bar{x}\}} \,\middle|\, Y_1\right] \\
&= \mathbb{E}_{(y_0,x_0)}\left[\mathbb{E}_{(y_0,x_0)}\left[\mathbb{1}_{\{X_1<\bar{x}\}} \,\middle|\, (Y_t)_{0\leqslant t\leqslant 1}\right] \,\middle|\, Y_1\right] \\
&= \mathbb{E}_{(y_0,x_0)}\left[\int_{-\infty}^{\bar{x}} \varrho\left(r - e^{-\theta}x_0 - \tfrac{m}{\theta}\left(1 - e^{-\theta}\right); e^{-2\theta}\int_0^1 e^{2\theta s}Y_s\mathrm{d}s\right)\mathrm{d}r \,\middle|\, Y_1\right], \qquad (1.4.5)
\end{aligned}$$

where $\varrho(r; \sigma^2)$ is the density of the normal distribution with variance $\sigma^2 > 0$, i.e.,

$$\varrho(r; \sigma^2) := \frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{r^2}{2\sigma^2}}, \quad r \in \mathbb{R}.$$

Note that the assumption $a > 0$ ensures that

$$\mathbb{P}_{(y_0,x_0)}\left(\int_0^1 e^{2\theta s} Y_s \mathrm{d}s > 0\right) = 1.$$

By [37, Theorem 6.3] and considering the conditional distribution of $\int_0^1 \exp\{2\theta s\} Y_s \mathrm{d}s$ given Y_1, we can find a probability kernel $K_{(y_0,x_0)}(\cdot,\cdot)$ from $\mathbb{R}_{\geqslant 0}$ to $\mathbb{R}_{\geqslant 0}$ such that

$$\mathbb{P}_{(y_0,x_0)}\left(\int_0^1 e^{2\theta s} Y_s \mathrm{d}s \in \cdot \;\middle|\; Y_1\right) = K_{(y_0,x_0)}(Y_1, \cdot)$$

and

$$K_{(y_0,x_0)}(z, \mathbb{R}_{>0}) = 1, \quad \text{for all } z \geqslant 0. \tag{1.4.6}$$

So

$$\begin{aligned}
&\mathbb{E}_{(y_0,x_0)}\left[\int_{-\infty}^{\bar{x}} \varrho\left(r - e^{-\theta}x_0 - \tfrac{m}{\theta}\left(1 - e^{-\theta}\right); e^{-2\theta}\int_0^1 e^{2\theta s} Y_s \mathrm{d}s\right)\mathrm{d}r \;\middle|\; Y_1\right] \\
&= \int_0^{\infty}\left(\int_{-\infty}^{\bar{x}} \varrho\left(r - e^{-\theta}x_0 - \tfrac{m}{\theta}\left(1 - e^{-\theta}\right); e^{-2\theta}w\right)\mathrm{d}r\right) K_{(y_0,x_0)}(Y_1, \mathrm{d}w) \\
&= \int_{-\infty}^{\bar{x}}\left(\int_0^{\infty} \varrho\left(r - e^{-\theta}x_0 - \tfrac{m}{\theta}\left(1 - e^{-\theta}\right); e^{-2\theta}w\right) K_{(y_0,x_0)}(Y_1, \mathrm{d}w)\right)\mathrm{d}r.
\end{aligned} \tag{1.4.7}$$

It follows from (1.4.4), (1.4.5) and (1.4.7) that for all $(\bar{y}, \bar{x}) \in \mathbb{R}_{\geqslant 0} \times \mathbb{R}$,

$$\begin{aligned}
\mathbb{P}_{(y_0,x_0)}\left(Y_1 < \bar{y}, X_1 < \bar{x}\right) = \int_0^{\bar{y}}\int_{-\infty}^{\bar{x}}\Bigg(\int_0^{\infty} &\varrho\left(r - e^{-\theta}x_0 - \tfrac{m}{\theta}\left(1 - e^{-\theta}\right); e^{-2\theta}w\right) \\
&\cdot K_{(y_0,x_0)}(z, \mathrm{d}w)\Bigg) f_{Y_1^{y_0}}(z)\mathrm{d}r\mathrm{d}z,
\end{aligned} \tag{1.4.8}$$

where $f_{Y_1^{y_0}}$ is given in (1.2.49). Define

$$p_1(y, x|y_0, x_0) := f_{Y_1^{y_0}}(y)\int_0^{\infty} \varrho\left(x - e^{-\theta}x_0 - \tfrac{m}{\theta}\left(1 - e^{-\theta}\right); e^{-2\theta}w\right) K_{(y_0,x_0)}(y, \mathrm{d}w).$$

Since $f_{Y_1^{y_0}}(y) > 0$ for all $y > 0$ and

$$\begin{aligned}
0 &= \mathbb{P}_{(y_0,x_0)}\left(\int_0^1 e^{2\theta s} Y_s \mathrm{d}s = 0\right) \\
&= \int_0^{\infty} K_{(y_0,x_0)}(y, \{0\})\mu_{Y_1^{y_0}}(\mathrm{d}y) = \int_0^{\infty} K_{(y_0,x_0)}(y, \{0\}) f_{Y_1^{y_0}}(y)\mathrm{d}y,
\end{aligned}$$

it follows that $K_{(y_0,x_0)}(y, \{0\}) = 0$ for all $y \in \mathbb{R}_{\geqslant 0} \setminus N$, where N is some null set under the Lebesgue measure. By modifying the definition of the kernel $K_{(y_0,x_0)}(y, \cdot)$ for $y \in N$, we can make sure that $K_{(y_0,x_0)}(y, \{0\}) = 0$ for all $y \in \mathbb{R}_{\geqslant 0}$, or equivalently, $K_{(y_0,x_0)}(y, \mathbb{R}_{>0}) = 1$ for all $y \in \mathbb{R}_{\geqslant 0}$. By (1.4.6) and the fact that $f_{Y_1^{y_0}}(y)$ is strictly positive for all $y > 0$ (see Proposition 1.18), for each $(y_0, x_0) \in \mathbb{R}_{\geqslant 0} \times \mathbb{R}$, the density $p_1(y, x|y_0, x_0)$ is strictly positive for almost all $(y, x) \in \mathbb{R}_{\geqslant 0} \times \mathbb{R}$. Moreover, by (1.4.8), we have

$$\mathbb{P}_{(y_0,x_0)}\left(Y_1 < \bar{y}, X_1 < \bar{x}\right) = \int_0^{\bar{y}}\int_{-\infty}^{\bar{x}} p_1(y, x|y_0, x_0)\mathrm{d}y\mathrm{d}x$$

for all $(\bar{y}, \bar{x}) \in \mathbb{R}_{\geqslant 0} \times \mathbb{R}$. So $p_1(\cdot, \cdot | y_0, x_0)$ is the density function of (Y_t, X_t) given that $(Y_0, X_0) = (y_0, x_0)$.

To prove (b), i.e. the aperiodicity of the skeleton chain $(Y_n, X_n)_{n \in \mathbb{Z}_{\geqslant 0}}$, we use a contradiction argument. Suppose that the period l of the chain $(Y_n, X_n)_{n \in \mathbb{Z}_{\geqslant 0}}$ is greater than 1 (see Definition B.2 for a definition of the period of a Markov chain). Then we can find disjoint Borel sets $A_1, A_2, \cdots, A_l$ such that

$$\lambda(A_i) > 0, \; i = 1, \cdots, l, \quad \cup_{i=1}^{l} A_i = \mathbb{R}_{\geqslant 0} \times \mathbb{R}, \tag{1.4.9}$$

$$\mathbf{P}^1(y_0, x_0, A_{i+1}) = 1 \tag{1.4.10}$$

for all $(y_0, x_0) \in A_i, \; i = 1, \cdots, l-1$, and $\mathbf{P}^1(y_0, x_0, A_1) = 1$ for all $(y_0, x_0) \in A_l$. By (1.4.10), we have

$$\iint_{(A_2)^c} p_1(y, x | y_0, x_0) \mathrm{d}y \mathrm{d}x = 0, \quad (y_0, x_0) \in A_1,$$

and further

$$\iint_{A_1} p_1(y, x | y_0, x_0) \mathrm{d}y \mathrm{d}x = 0, \quad (y_0, x_0) \in A_1.$$

However, since for each $(y_0, x_0) \in \mathbb{R}_{\geqslant 0} \times \mathbb{R}$, the density $p_1(y, x | y_0, x_0)$ is strictly positive for almost all $(y, x) \in \mathbb{R}_{\geqslant 0} \times \mathbb{R}$, we must have $\lambda(A_1) = 0$, which contradicts (1.4.9). Therefore, the assumption that $l \geqslant 2$ is not true. So we have $l = 1$.

In view of [50, Theorem 3.4 (ii)], to prove (c) it is enough to check the Feller property of the skeleton chain $(Y_n, X_n)_{n \in \mathbb{Z}_{\geqslant 0}}$. By Theorem A.1, the two-dimensional process $(Y_t, X_t)_{t \geqslant 0}$, as an affine process, possesses the Feller property. So the skeleton chain $(Y_n, X_n)_{n \in \mathbb{Z}_{\geqslant 0}}$ has also the Feller property.

Since (a), (b), and (c) hold true, we can apply [50, Theorem 6.3] and thus find constants $\delta \in (0, 1)$, $B \in (0, \infty)$ such that

$$\|\mathbf{P}^n(y, x, \cdot) - \pi\|_{TV} \leqslant B \left(V(y, x) + 1\right) e^{-\delta n} \tag{1.4.11}$$

for all $n \in \mathbb{Z}_{\geqslant 0}$, $(y, x) \in \mathbb{R}_{\geqslant 0} \times \mathbb{R}$. For the remainder of the proof, i.e., to extend the inequality (1.4.11) to all $t \geqslant 0$, we can interpolate in the same way as in [52, p.536], and we omit the details. This completes the proof. □

Remark 1.23. *According to the discussion after* [13, Proposition 2.5], *a direct but important consequence of our ergodic result is the following: under the assumptions of Theorem* 1.22, *for all Borel measurable functions* $f : \mathbb{R}_{\geqslant 0} \times \mathbb{R} \to \mathbb{R}$ *satisfying* $\int_{\mathbb{R}_{\geqslant 0}} \int_{\mathbb{R}} |f(y, x)| \pi(\mathrm{d}y, \mathrm{d}x) < \infty$, *it holds*

$$\mathbb{P}\left(\lim_{T \to \infty} \frac{1}{T} \int_0^T f(Y_s, X_s) \mathrm{d}s = \int_0^\infty \int_{-\infty}^\infty f(y, x) \pi(\mathrm{d}y, \mathrm{d}x) \right) = 1. \tag{1.4.12}$$

The convergence (1.4.12) *may be very useful for parameter estimation of* (Y, X).

Part II.

Moments and ergodicity of the jump-diffusion CIR process and parameter estimation for the drift parameters based on discrete time observations

2. The jump-diffusion CIR process

In this chapter, we study the jump-diffusion CIR (shorted as JCIR) process, which is an extension of the Cox-Ingersoll-Ross model introduced in [15]. The JCIR process $X = (X_t)_{t\geqslant 0}$ is defined as the unique strong solution to the stochastic differential equation

$$\mathrm{d}X_t = (a - bX_t)\mathrm{d}t + \sigma\sqrt{X_t}\mathrm{d}B_t + \mathrm{d}J_t, \quad t \geqslant 0, \quad X_0 \geqslant 0 \text{ a.s.}, \tag{2.0.1}$$

where $a \geqslant 0$, $b > 0$, $\sigma > 0$ are constants, $(B_t)_{t\geqslant 0}$ is a one-dimensional Brownian motion and $(J_t)_{t\geqslant 0}$ is a pure jump Lévy process[1] with its Lévy measure ν concentrating on $(0,\infty)$ and satisfying

$$\int_0^\infty (z \wedge 1)\nu(\mathrm{d}z) < \infty. \tag{2.0.2}$$

We assume that X_0, $(B_t)_{t\geqslant 0}$ and $(J_t)_{t\geqslant 0}$ are independent.

The principal aim we pursue in this chapter is to derive the ergodicity and exponential ergodicity of the JCIR process, respectively. Our choice of the approach is the same as for the two-dimensional affine model (Y, X) introduced in Chapter 1. To be precise, we will establish a positive lower bound of the transition densities of $(X_t)_{t\geqslant 0}$, prove existence of the fractional moments under a suitable integrability condition to the Lévy measure ν, and based on this result, we prove the existence of a Foster–Lyapunov function in order to apply the Foster-Lyapunov drift criteria which enables us to show (exponential) ergodicity of X. Before we focus on these, we first recall some elementary properties of the JCIR process in prior.

We let $(B_t)_{t\geqslant 0}$ be a standard $(\mathcal{F}_t)_{t\geqslant 0}$-Brownian motion and $(J_t)_{t\geqslant 0}$ be a one-dimensional $(\mathcal{F}_t)_{t\geqslant 0}$-Lévy process whose characteristic function is given by

$$\mathbb{E}\left[e^{uJ_t}\right] = \exp\left\{t\int_0^\infty \left(e^{uz} - 1\right)\nu(\mathrm{d}z)\right\}, \quad (t,u) \in \mathbb{R}_{\geqslant 0} \times \mathbb{C}_{\leqslant 0},$$

where ν satisfies (2.0.2) (see, e.g., [12, Theorem 1.2] or [56, pp.78-79]). The Lévy-Itô representation of $(J_t)_{t\geqslant 0}$ takes the form

$$J_t = \int_0^t \int_0^\infty zN(\mathrm{d}s, \mathrm{d}z), \quad t \geqslant 0, \tag{2.0.3}$$

where $N(\mathrm{d}t, \mathrm{d}z) = \sum_{s\leqslant t} \delta_{(s,\Delta J_s)}(\mathrm{d}t, \mathrm{d}z)$ is a Poisson random measure on $\mathbb{R}_{\geqslant 0}$, where $\Delta J_s := J_s - J_{s-}$, $s > 0$, $\Delta J_0 := 0$, and $\delta_{(s,x)}$ denotes the Dirac measure concentrated at $(s,x) \in \mathbb{R}^2_{\geqslant 0}$.

Remark 2.1. *Following Filipović* [21] *and Jin et al.* [34]*, the JCIR process defined by* (2.0.1) *includes the so-called basic affine jump-diffusion (BAJD) as a special case, in*

[1] i.e., $(J_t)_{t\geqslant 0}$ is a subordinator.

which the drift takes the form $a(\theta - X_t)$ with parameters $a \in \mathbb{R}_{>0}$ and $\theta \in \mathbb{R}_{\geqslant 0}$, and the Lévy process $(J_t)_{t\geqslant 0}$ is a pure-jump Lévy process with the Lévy measure

$$\nu(\mathrm{d}z) = \begin{cases} cde^{-dz}\mathrm{d}z, & z \geqslant 0, \\ 0, & z < 0, \end{cases} \tag{2.0.4}$$

for some constants $c,\ d \in \mathbb{R}_{>0}$. The measure ν given by (2.0.4) *satisfies the integrability condition* (2.0.2)*, since*

$$\begin{aligned} \int_0^\infty (z \wedge 1)\nu(\mathrm{d}z) &= c\int_0^1 zde^{-dz}\mathrm{d}z + c\int_1^\infty de^{-dz}\mathrm{d}z \\ &= \frac{c\left(1-(d+1)e^{-d}\right)}{d} + ce^{-d} = \frac{c - ce^{-d}}{d} \in \mathbb{R}_{\geqslant 0}. \end{aligned}$$

The BAJD has been introduced by Duffie and Gârleanu [18] *to describe the dynamics of default intensity. It was also used by Filipović* [21] *and Keller-Ressel and Steiner* [43] *as a short-rate model.*

The following proposition ensures the existence and uniqueness of a strong solution of the stochastic differential equation (2.0.1).

Proposition 2.2. *Consider the JCIR process defined by the SDE* (2.0.1) *with parameters $a \in \mathbb{R}_{\geqslant 0}$, $b \in \mathbb{R}$, $\sigma \in \mathbb{R}_{>0}$ and ν satisfying* (2.0.2)*. Then there is a (pathwise) unique strong solution $X = (X_t)_{t\geqslant 0}$ to the SDE* (2.0.1) *such that $(X_t)_{t\geqslant 0}$ is almost surely non-negative for all $t \geqslant 0$. Further, we have*

$$X_t = e^{-bt}\left(X_0 + a\int_0^t e^{bs}\mathrm{d}s + \sigma\int_0^t e^{bs}\sqrt{X_s}\mathrm{d}B_s + \int_0^t e^{bs}\mathrm{d}J_s\right), \quad t \geqslant 0. \tag{2.0.5}$$

Proof. By the Lévy-Itô representation of $(J_t)_{t\geqslant 0}$, we can rewrite the SDE (2.0.1) in

$$X_t = X_0 + \int_0^t (a - bX_s)\mathrm{d}s + \sigma\int_0^t \sqrt{X_s}\mathrm{d}B_s + \int_0^t\int_0^\infty zN(\mathrm{d}s, \mathrm{d}z), \quad t \geqslant 0.$$

It follows from [23, Theorem 5.1] that if X_0 is independent of $(B_t)_{t\geqslant 0}$ and $(J_t)_{t\geqslant 0}$, then there is a unique strong solution $(X_t)_{t\geqslant 0}$ to the SDE (2.0.1). Since the diffusion coefficient in the SDE (2.0.1) is degenerate at zero and only positive jumps are possible, the JCIR process $(X_t)_{t\geqslant 0}$ stays non-negative if $X_0 \geqslant 0$. This fact can be shown rigorously with the help of comparison theorems for SDEs, for more details we refer to [23]. Finally, using Itô's formula to the process $(\exp\{bt\}X_t)_{t\geqslant 0}$, we obtain

$$\begin{aligned} \mathrm{d}\left(e^{bt}X_t\right) &= be^{bt}X_t\mathrm{d}t + e^{bt}\mathrm{d}X_t \\ &= ae^{bt} + e^{bt}\left(\sigma\sqrt{X_t}\mathrm{d}B_t + \mathrm{d}J_t\right), \quad t \geqslant 0, \end{aligned}$$

and hence,

$$e^{bt}X_t - X_0 = a\int_0^t e^{bs}\mathrm{d}s + \sigma\int_0^t e^{bs}\sqrt{X_s}\mathrm{d}B_s + \int_0^t e^{bs}\mathrm{d}J_s, \quad t \geqslant 0,$$

yielding (2.0.5). □

Finally, we introduce some notation. Note that the strong solution $(X_t)_{t\geqslant 0}$ of the stochastic differential equation (2.0.1) obviously depends on its initial value X_0. From now on, we denote by $(X_t^x)_{t\geqslant 0}$ the JCIR process starting from a constant initial value $x \in \mathbb{R}_{\geqslant 0}$, i.e., $(X_t^x)_{t\geqslant 0}$ satisfies

$$\mathrm{d}X_t^x = (a - bX_t^x)\mathrm{d}t + \sigma\sqrt{X_t^x}\mathrm{d}B_t + \mathrm{d}J_t, \quad t \geqslant 0, \quad X_0^x = x \in \mathbb{R}_{\geqslant 0}. \tag{2.0.6}$$

2.1. Affine representation of the JCIR process

In this section we show that the JCIR process belongs to the class of (conservative) regular affine processes with state space $\mathbb{R}_{\geqslant 0}$. We derive the infinitesimal generator as well as the characteristic function of $(X_t^x)_{t\geqslant 0}$.

We first check that the JCIR process is a regular affine process. To do so, we shall calculate the infinitesimal generator of the JCIR process.

Proposition 2.3. *Let $a \in \mathbb{R}_{\geqslant 0}$, $b \in \mathbb{R}$, $\sigma \in \mathbb{R}_{>0}$, and the Lévy measure ν on $\mathbb{R}_{>0}$ satisfying* (2.0.2)*. Then $(X_t^x)_{t\geqslant 0}$ is a regular affine process with infinitesimal generator given by*

$$(\mathcal{A}f)(x) = (a-bx)\frac{\partial f(x)}{\partial x} + \frac{1}{2}\sigma^2 x \frac{\partial^2 f(x)}{\partial x^2} + \int_0^\infty (f(x+z)-f(x))\,\nu(\mathrm{d}z), \qquad (2.1.1)$$

where $f \in C_c^2(\mathbb{R}_{\geqslant 0}, \mathbb{R})$.

Before going to the proof, it is worth noting that the statement of Proposition 2.3 does appear in [33] and [2] but there is no proof stated in [33] and the proof in [2] goes back to some results of [16]. Here is a simple direct proof.

Proof. Let $x \in \mathbb{R}_{\geqslant 0}$ be fixed and assume that $X_0 = x$ almost surely. In view of the Lévy-Itô decomposition of $(J_t)_{t\geqslant 0}$ in (2.0.3), we have

$$X_t = x + \int_0^t (a - bX_s)\mathrm{d}s + \sigma \int_0^t \sqrt{X_s}\mathrm{d}B_s + \int_0^t \int_0^\infty zN(\mathrm{d}s, \mathrm{d}z), \quad t \geqslant 0,$$

where $N(\mathrm{d}s, \mathrm{d}z)$ is defined in (2.0.3). Using Itô's formula to $f \in C_c^2(\mathbb{R}_{\geqslant 0}, \mathbb{R})$, we obtain

$$\begin{aligned} f(X_t^x) - f(X_0) &= \int_0^t (a - bX_s^x)\frac{\partial f}{\partial x}(X_s^x)\,\mathrm{d}s + \sigma \int_0^t \sqrt{X_s^x}\frac{\partial f}{\partial x}(X_s^x)\,\mathrm{d}B_s \\ &\quad + \frac{\sigma^2}{2}\int_0^t X_s^x \frac{\partial^2 f}{\partial x^2}(X_s^x)\,\mathrm{d}s \\ &\quad + \int_0^t \int_0^\infty \left(f(X_{s-}^x + z) - f(X_{s-}^x)\right) N(\mathrm{d}s, \mathrm{d}z) \\ &=: \int_0^t (\mathcal{A}f)(X_s^x)\,\mathrm{d}s + M_t(f), \quad t \in \mathbb{R}_{\geqslant 0}, \end{aligned}$$

where

$$\begin{aligned} M_t(f) := \sigma \int_0^t \sqrt{X_s^x}\frac{\partial f}{\partial x}(X_s^x)\,\mathrm{d}B_s + \int_0^t \int_0^\infty \left(f(X_{s-}^x + z) - f(X_{s-}^x)\right) N(\mathrm{d}s, \mathrm{d}z) \\ - \int_0^t \int_0^\infty \left(f(X_s^x + z) - f(X_s^x)\right)\nu(\mathrm{d}z)\mathrm{d}s, \quad t \geqslant 0, \end{aligned}$$

and

$$(\mathcal{A}f)(x) = (a-bx)\frac{\partial f}{\partial x}(x) + \frac{1}{2}\sigma^2 x \frac{\partial^2 f}{\partial x^2}(x) + \int_0^\infty (f(x+z)-f(x))\,\nu(\mathrm{d}z)$$

for $x \in \mathbb{R}_{\geqslant 0}$ and $f \in C_c^2(\mathbb{R}_{\geqslant 0}, \mathbb{R})$ is precisely corresponding to (2.1.1). Thus, it remains to prove that $(M_t(f))_{t\geqslant 0}$ is a martingale with respect to the filtration $(\mathcal{F}_t)_{t\geqslant 0}$. Defining

$$D_t(f) := \sigma \int_0^t \frac{\partial f(X_s^x)}{\partial x}\sqrt{X_s^x}\mathrm{d}B_s, \quad t \geqslant 0, \quad \text{and}$$

$$J_t(f) := \int_0^t \int_0^\infty \left(f\left(X_{s-}^x + z\right) - f\left(X_{s-}^x\right)\right) N(\mathrm{d}s, \mathrm{d}z) \\ - \int_0^t \int_0^\infty \left(f\left(X_s^x + z\right) - f\left(X_s^x\right)\right) \nu(\mathrm{d}z)\mathrm{d}s, \quad t \geqslant 0,$$

we check that $(M_t(f))_{t\geqslant 0} = (D_t(f) + J_t(f))_{t\geqslant 0}$ is actually a sum of martingales with respect to the filtration $(\mathcal{F}_t)_{t\geqslant 0}$.

We first check that $(D_t(f))_{t\geqslant 0}$ is a square integrable martingale with respect to the filtration $(\mathcal{F}_t)_{t\geqslant 0}$. Indeed, since the derivative of f has compact support, for all $t \in \mathbb{R}_{\geqslant 0}$, we obtain

$$\mathbb{E}\left[\left(\sigma \int_0^t \sqrt{X_s^x} \frac{\partial f}{\partial x}\left(X_s^x\right) \mathrm{d}B_s\right)^2\right] = \sigma^2 \int_0^t \mathbb{E}\left[X_s^x \left(\frac{\partial f}{\partial x}\left(X_s^x\right)\right)^2\right] \mathrm{d}s < \infty, \quad t \geqslant 0,$$

which implies that $(D_t(f))_{t\geqslant 0}$ is a square integrable martingale with respect to the filtration $(\mathcal{F}_{t\geqslant 0})$. Next, we prove that $(J_t(f))_{t\geqslant 0}$ is a martingale with respect to the filtration $(\mathcal{F}_t)_{t\geqslant 0}$. Notice that $f \in C_c^2(\mathbb{R}_{\geqslant 0})$, so $\sup_{x\in\mathbb{R}_{\geqslant 0}} |\partial_x f(x)| < \infty$. By the mean value theorem, we have

$$|f(x+z) - f(x)| \leqslant z \sup_{x\in\mathbb{R}_{\geqslant 0}} \left|\frac{\partial}{\partial x} f(x)\right| < \infty,$$

which in turn yields

$$\begin{aligned} &\mathbb{E}\left[\int_0^t \int_0^\infty \left|f\left(X_s^x + z\right) - f\left(X_s^x\right)\right| \nu(\mathrm{d}z)\mathrm{d}s\right] \\ &\quad = \mathbb{E}\left[\int_0^t \int_{\{z\leqslant 1\}} \left|f\left(X_s^x + z\right) - f\left(X_s^x\right)\right| \nu(\mathrm{d}z)\mathrm{d}s\right] \\ &\quad\quad + \mathbb{E}\left[\int_0^t \int_{\{z>1\}} \left|f\left(X_s^x + z\right) - f\left(X_s^x\right)\right| \nu(\mathrm{d}z)\mathrm{d}s\right] \\ &\quad \leqslant \sup_{x\in\mathbb{R}_{\geqslant 0}} \left|\frac{\partial}{\partial x} f(x)\right| t \int_0^\infty (z \wedge 1)\nu(\mathrm{d}z) < \infty, \quad t \geqslant 0, \end{aligned}$$

where the finiteness of the integral follows by assumption (2.0.2). By [26, Lemma 3.1 in Chapter II and page 62], we get that $(J_t(f))_{t\geqslant 0}$ is a martingale with respect to the filtration $(\mathcal{F}_t)_{t\geqslant 0}$. Consequently, $(M_t)_{t\geqslant 0}$ is a martingale with respect to the filtration $(\mathcal{F}_t)_{t\geqslant 0}$ as asserted.

Noting that $(0, \alpha = 1/2\sigma^2, b = a, \beta = -b, \nu, 0)$ is a set of admissible parameters in the sense of Definition A.3, the rest of the proof goes through as for Proposition 1.7, with hardly any changes. □

The following remark is about the representation of the functions F and R appearing in the generalized Riccati equations (see Appendix A).

Remark 2.4. *Since the JCIR process X is a conservative, regular affine process with state space $\mathbb{R}_{\geqslant 0}$, especially, it is a continuous-time branching process with immigrations (shorted as CBI)[2]. The form of the infinitesimal generator $\mathcal{A}$ of X given in* (2.1.1)

[2]see Appendix A the paragraph after Theorem A.1 for details.

and the formulas (A.0.4) *and* (A.0.5) *yield that X is a CBI process having branching mechanism*

$$R(u) = \frac{\sigma^2 u^2}{2} - bu, \quad u \in \mathbb{C}_{\leqslant 0}, \tag{2.1.2}$$

and immigration mechanism

$$F(u) = au + \int_0^\infty \left(e^{uz} - 1\right) \nu(\mathrm{d}z), \quad u \in \mathbb{C}_{\leqslant 0}. \tag{2.1.3}$$

The next proposition is about the characteristic function of the JCIR process.

Proposition 2.5. *Consider the JCIR process $(X_t^x)_{t\geqslant 0}$ given in* (2.0.6) *with $a \in \mathbb{R}_{\geqslant 0}$, b, $\sigma \in \mathbb{R}_{>0}$, and ν satisfying* (2.0.2)*. Then $\phi(t,u)$ and $\psi(t,u)$ solve the generalized Riccati equations*

$$\begin{cases} \frac{\partial}{\partial t}\phi(t,u) = F\left(\psi(t,u)\right), & \phi(0,u) = 0, \\ \frac{\partial}{\partial t}\psi(t,u) = R\left(\psi(t,u)\right), & \psi(0,u) = u \in \mathbb{C}_{\leqslant 0}, \end{cases}$$

where the functions F and R are given by (2.1.3) *and* (2.1.2)*, respectively. The unique solution of the Riccati equations are given by*

$$\psi(t,u) = \frac{ue^{-bt}}{1 - \frac{\sigma^2 u}{2b}\left(1 - e^{-bt}\right)}, \tag{2.1.4}$$

and

$$\phi(t,u) = -\frac{2a}{\sigma^2}\log\left(1 - \frac{\sigma^2 u}{2b}\left(1 - e^{-bt}\right)\right) + \int_0^t \int_0^\infty \left(e^{z\psi(s,u)} - 1\right)\nu(\mathrm{d}z)\mathrm{d}s,$$

where $(t,u) \in \mathbb{R}_{\geqslant 0} \times \mathbb{C}_{\leqslant 0}$. Moreover, the characteristic function of X_t^x has the form

$$\begin{aligned} \mathbb{E}\left[e^{uX_t^x}\right] = \left(1 - \tfrac{\sigma^2 u}{2b}\left(1 - e^{-bt}\right)\right)^{-\frac{2a}{\sigma^2}} \cdot \exp\left\{\frac{ue^{-bt}x}{1 - \frac{\sigma^2 u}{2b}\left(1 - e^{-bt}\right)}\right\} \\ \cdot \exp\left\{\int_0^t \int_0^\infty \left(e^{z\psi(s,u)} - 1\right)\nu(\mathrm{d}z)\mathrm{d}s\right\} \end{aligned} \tag{2.1.5}$$

for all $t \geqslant 0$ and $u \in \mathbb{C}_{\leqslant 0}$.

Proof. Since X_t^x is affine, its characteristic function is of exponential affine form

$$\mathbb{E}\left[e^{uX_t^x}\right] = \exp\left\{\phi(t,u) + x\psi(t,u)\right\}, \quad (t,u) \in \mathbb{R}_{\geqslant 0} \times \mathbb{C}_{\leqslant 0}. \tag{2.1.6}$$

In view of (2.1.3) and (2.1.2) the functions ϕ and ψ in question solve the differential equations

$$\begin{cases} \partial_t \phi(t,u) = a\psi(s,u) + \int_0^\infty \left(e^{z\psi(t,u)} - 1\right)\nu \mathrm{d}z, & \phi(0,u) = 0, \\ \partial_t \psi(s,u) = \frac{\sigma^2}{2}\psi(t,u)^2 - b\psi(t,u), & \psi(0,u) = u \in \mathbb{C}_{\leqslant 0}. \end{cases}$$

The second equation is a Bernoulli differential equation. It is easy to see that its solution is given explicitly by (2.1.4) and $\phi(t,u)$ is simply obtained by integration once $\psi(t,u)$ is known. After plugging this into (2.1.6), we conclude with (2.1.5). ☐

We recall a decomposition of the characteristic function of X_t^x established in [33]. As mentioned in [33], the product of the first two terms on the right-hand side of (2.1.5) is the characteristic function of the CIR process. More precisely, consider the unique strong solution $(Y_t^x)_{t\geqslant 0}$ of the following stochastic differential equation (2.0.1)

$$\mathrm{d}Y_t^x = (a - bY_t^x)\mathrm{d}t + \sqrt{Y_t^x}\mathrm{d}B_t, \quad t \geqslant 0, \quad Y_0^x = x \in \mathbb{R}_{\geqslant 0} \text{ a.s..} \tag{2.1.7}$$

where $a \in \mathbb{R}_{\geqslant 0}$, and $b, \sigma \in \mathbb{R}_{>0}$. So $(Y_t^x)_{t\geqslant 0}$ is the CIR process starting from x. Note that (2.1.7) is a special case of (2.0.6) with $J_t \equiv 0$ (corresponding to $\nu = 0$). By (2.1.5), we obtain

$$\mathbb{E}\left[e^{uY_t^x}\right] = \left(1 - \tfrac{\sigma^2 u}{2b}\left(1 - e^{-bt}\right)\right)^{-\frac{2a}{\sigma^2}} \exp\left\{\frac{xue^{-bt}}{1-\frac{\sigma^2 u}{2b}\left(1-e^{-bt}\right)}\right\} \tag{2.1.8}$$

for all $t \geqslant 0$ and $u \in \mathbb{C}_{\leqslant 0}$.

We now turn to the third term on the right-hand side of (2.1.5). Let $Z := (Z_t)_{t\geqslant 0}$ be the unique strong solution of the stochastic differential equation

$$\mathrm{d}Z_t = -bZ_t\mathrm{d}t + \sigma\sqrt{Z_t}\mathrm{d}B_t + \mathrm{d}J_t, \quad t \geqslant 0, \quad Z_0 = 0 \text{ a.s.,} \tag{2.1.9}$$

where $\sigma \in \mathbb{R}_{>0}$. It is easy to see that (2.1.9) is also a special case of (2.0.6) with $a = x = 0$. Again by (2.1.5), we have

$$\mathbb{E}\left[e^{uZ_t}\right] = \exp\left\{\int_0^t \int_0^\infty \left(e^{z\psi(s,u)} - 1\right)\nu(\mathrm{d}z)\mathrm{d}s\right\}, \quad (t,u) \in \mathbb{R}_{\geqslant 0} \times \mathbb{C}_{\leqslant 0}. \tag{2.1.10}$$

It follows from (2.1.5), (2.1.8) and (2.1.10) that

$$\mathbb{E}\left[e^{uX_t^x}\right] = \mathbb{E}\left[e^{uY_t^x}\right]\mathbb{E}\left[e^{uZ_t}\right] \tag{2.1.11}$$

for all $t \geqslant 0$ and $u \in \mathbb{C}_{\leqslant 0}$. Let $\mu_{Y_t^x}$ and μ_{Z_t} be the probability laws of Y_t^x and Z_t induced on $(\mathbb{R}_{\geqslant 0}, \mathcal{B}(\mathbb{R}_{\geqslant 0}))$, respectively. Then the probability law $\mu_{X_t^x}$ of X_t^x is given by

$$\mu_{X_t^x} = \mu_{Y_t^y} * \mu_{Z_t}, \tag{2.1.12}$$

where $*$ denotes the convolution of two measures.

2.1.1. Transition densities of the JCIR process

We prove that the JCIR process X has positive transition densities. Our approach is similar to that in Section 1.2 and is based on the representation of the law of X_t^x as the convolution of two probability measures, one of which is the distribution of the normal CIR process.

Proposition 2.6. *Assume $a > 0$. For each $x \in \mathbb{R}_{\geqslant 0}$ and $t \in \mathbb{R}_{>0}$, the random variable X_t^x possesses a density function $f_{X_t^x}(y)$, $y \geqslant 0$ with respect to the Lebesgue measure. Moreover, the density function $f_{X_t^x}(y)$ is strictly positive for all $y \in \mathbb{R}_{>0}$.*

Proof. Recall that $\mu_{X_t^x} = \mu_{Y_t^y} * \mu_{Z_t}$ as shown in (2.1.12). Note that the CIR process $(Y_t^x)_{t\geqslant 0}$ possesses a density function $f_{Y_t^x}(y)$ for $t \in \mathbb{R}_{>0}$, $Y_0 = x \in \mathbb{R}_{>0}$, and $y \in \mathbb{R}_{>0}$. More precisely, in case $x > 0$, we have

$$f_{Y_t^x}(y) = \frac{2be^{bt\frac{2a}{\sigma^2}}}{\sigma^2\left(1 - e^{-bt}\right)}\left(\frac{y}{x}\right)^{\frac{2a}{\sigma^2}-1} \exp\left\{\frac{2b\left(ye^{bt} - x\right)}{\sigma^2\left(e^{bt} - 1\right)}\right\} I_{\frac{2a}{\sigma^2}-1}\left(\frac{4b\sqrt{xye^{bt}}}{\sigma^2\left(e^{bt} - 1\right)}\right), \tag{2.1.13}$$

where $I_{(2a)/(\sigma^2)-1}$ denotes the modified Bessel of the first kind of order $(2a)/\sigma^2-1$, i.e.,

$$I_{\frac{2a}{\sigma^2}-1}(y)=\sum_{m=0}^{\infty}\frac{1}{m!\Gamma(m+2a/\sigma^2)}\left(\frac{y}{2}\right)^{2m+\frac{2a}{\sigma^2}-1},\quad y\in\mathbb{R}_{>0},$$

see, e.g., Cox *et al.* [15, Formula (18)] or Jeanblanc *et al.* [28, Proposition 6.3.2.1]. In case $x=0$, for all $y\in\mathbb{R}$, the density function $f_{Y_t^0}$ is given by

$$f_{Y_t^0}(y)=\frac{1}{\Gamma(2a/\sigma^2)}\left(\frac{2b}{\sigma^2\left(1-e^{-bt}\right)}\right)^{\frac{2a}{\sigma^2}}y^{\frac{2a}{\sigma^2}-1}\exp\left\{\frac{-2by}{\sigma^2\left(1-e^{-bt}\right)}\right\}\mathbb{1}_{(0,\infty)}(y),\quad(2.1.14)$$

due to Ikeda and Watanabe [26], since Y_t^0 has gamma distribution with parameters $2a/\sigma^2$ and $2b/(1-\exp\{-bt\})$. We conclude that $\mu_{X_t^x}$ is also absolutely continuous with respect to the Lebesgue measure and thus possesses a density function denoted by $f_{X_t^x}$ which is given by

$$f_{X_t^x}(y)=\int_{\mathbb{R}_{\geqslant0}}f_{Y_t^x}(y-z)\mu_{Z_t}(\mathrm{d}z),\quad y\geqslant0.$$

We proceed to prove the strict positivity of $f_{X_t^x}(y)$ for all $y\in\mathbb{R}_{>0}$. Let $t>0$ and $y>0$ be fixed. It follows that

$$f_{X_t^x}(y)\geqslant\int_{[0,\delta]}f_{Y_t^x}(y-z)\mu_{Z_t}(\mathrm{d}z),$$

where $\delta>0$ is small enough with $\delta<y$. Noting that $f_{Y_t^x}(y)>0$ for $y>0$ and $f_{Y_t^x}(y)\equiv0$ for $y<0$ (see, formula (2.1.13) in case $x>0$ and formula (2.1.14) in case $x=0$), we have that $f_{Y_t^x}(y-z)>0$ for all $z\in[0,\delta]$. Hence, it is enough to check that $\mu_{Z_t}([0,\delta])>0$. If $\mathbb{P}(Z_t=0)>0$, then we are done. So we now suppose

$$\mathbb{P}(Z_t=0)=0.\quad(2.1.15)$$

Let

$$\Delta_t(u)=\int_0^t\int_0^\infty\left(e^{z\psi(s,u)}-1\right)\nu(\mathrm{d}z)\mathrm{d}s,\quad u\in\mathbb{C}_{\leqslant0},$$

where ψ is given in (2.1.4). By (2.1.15), we conclude

$$\begin{aligned}\mathbb{E}\left[e^{u(Z_t-\delta)}\right]&-\mathbb{E}\left[e^{u(Z_t-\delta)}\mathbb{1}_{\{Z_t=0\}}\right]\\&=e^{-u\delta}\left(\mathbb{E}\left[e^{uZ_t}\right]-\mathbb{E}\left[e^{uZ_t}\mathbb{1}_{\{Z_t=0\}}\right]\right)\\&=e^{-u\delta}\left(e^{\Delta_t(u)}-\mathbb{P}\left(Z_t=0\right)\right)\\&=e^{-u\delta/2}e^{\Delta_t(u)-u\delta/2}.\end{aligned}\quad(2.1.16)$$

For all $u\in(-\infty,-1]$ and $s\in[0,t]$, we have

$$\begin{aligned}\frac{\partial}{\partial u}\left(e^{z\psi(s,u)}-1\right)&=\frac{ze^{-bs}}{\left(1-\frac{\sigma^2u}{2b}\left(1-e^{-bs}\right)\right)^2}\exp\left\{\frac{zue^{-bs}}{1-\frac{\sigma^2u}{2b}\left(1-e^{-bs}\right)}\right\}\\&\leqslant ze^{-bs}\mathbb{1}_{\{z\leqslant1\}}+ze^{-bs}e^{-c_1z}\mathbb{1}_{\{z>1\}}\leqslant c_2e^{-bs}(z\wedge1),\end{aligned}\quad(2.1.17)$$

for some positive constants c_1 and c_2. By the differentiation lemma [10, Lemma 16.2], we see that $\Delta_t(u)$ is differentiable at $u\in(-\infty,-1]$ and

$$\frac{\partial}{\partial u}\left(\Delta_t(u)\right)=\int_0^t\int_0^\infty\frac{\partial}{\partial u}\left(e^{z\psi(s,u)}-1\right)\nu(\mathrm{d}z)\mathrm{d}s,\quad u\in(-\infty,-1].\quad(2.1.18)$$

Note that $\partial/(\partial u)(\exp\{z\psi(s,u)\}-1)>0$ for $z>0$, $u\in(-\infty,-1]$ and $s\in[0,t]$. Therefore, $\Delta_t(u)$ is strictly increasing in u on $(-\infty,-1]$. Moreover, we have

$$\lim_{u\to-\infty}\frac{\partial}{\partial u}\left(e^{z\psi(s,u)}-1\right)=\exp\left\{\frac{-2bz}{\sigma^2\left(e^{bs}-1\right)}\right\}\lim_{u\to-\infty}\frac{ze^{-bs}}{\left(1-\frac{\sigma^2 u}{2b}\left(1-e^{-bs}\right)\right)^2}=0.$$

By (2.1.17), (2.1.18) and the Lebesgue dominated convergence theorem, $\partial/(\partial u)\Delta_t(u)\to 0$ as $u\to-\infty$. So $\partial/(\partial u)(\Delta_t(u)-u\delta/2)\to-\delta/2$ as $u\to-\infty$, which implies that $\Delta_t(u)-u\delta/2$ is monotone in u for sufficiently small u and thus

$$\lim_{u\to-\infty}e^{-u\delta/2}e^{\Delta_t(u)-u\delta/2}=\infty. \tag{2.1.19}$$

It follows from (2.1.16) and (2.1.19) that

$$\lim_{u\to-\infty}\left(\mathbb{E}\left[e^{u(Z_t-\delta)}\right]-\mathbb{E}\left[e^{u(Z_t-\delta)}\mathbb{1}_{\{Z_t=0\}}\right]\right)=\infty.$$

Now, we must have $\mathbb{P}(Z_t\in(0,\delta])>0$, otherwise

$$\begin{aligned}&\lim_{u\to-\infty}\left(\mathbb{E}\left[e^{u(Z_t-\delta)}\right]-\mathbb{E}\left[e^{u(Z_t-\delta)}\mathbb{1}_{\{Z_t=0\}}\right]\right)\\&\qquad=\lim_{u\to-\infty}\left(\mathbb{E}\left[e^{u(Z_t-\delta)}\mathbb{1}_{\{0<Z_t\leqslant\delta\}}\right]+\mathbb{E}\left[e^{u(Z_t-\delta)}\mathbb{1}_{\{Z_t>\delta\}}\right]\right)=0.\end{aligned}$$

This completes the proof. □

2.2. Moments of the JCIR process

In this section we characterize the existence of the κ-moment ($\kappa>0$) of the JCIR process by an integrability condition on the Lévy measure of the subordinator. For these considerations it will be convenient to state beforehand a moment estimation for the Bessel distribution. Moreover, we will calculate the first and second moment of the JCIR process explicitly.

2.2.1. Bessel distribution

Suppose α and β are positive constants. We call a probability measure $m_{\alpha,\beta}$ on $(\mathbb{R}_{\geqslant 0},\mathcal{B}(\mathbb{R}_{\geqslant 0}))$ a Bessel distribution with parameters α and β if

$$m_{\alpha,\beta}(\mathrm{d}x):=e^{-\alpha}\delta_0(\mathrm{d}x)+\beta e^{-\alpha-\beta x}\sqrt{\alpha(\beta x)^{-1}}I_1\left(2\sqrt{\alpha\beta x}\right)\mathrm{d}x,\quad x\in\mathbb{R}_{\geqslant 0}, \tag{2.2.1}$$

where δ_0 denotes the Dirac measure at the origin and I_1 is the modified Bessel function of the first kind, namely,

$$I_1(r)=\frac{r}{2}\sum_{k=0}^{\infty}\frac{\left(\frac{1}{4}r^2\right)^k}{k!(k+1)!},\quad r\in\mathbb{R}. \tag{2.2.2}$$

Let $\widehat{m}_{\alpha,\beta}(u):=\int_{\mathbb{R}_{\geqslant 0}}\exp\{ux\}m_{\alpha,\beta}(\mathrm{d}x)$ for $u\in\mathbb{C}_{\leqslant 0}$ denote the characteristic function of the Bessel distribution $m_{\alpha,\beta}$. It follows from [34, Lemma 3.1] that

$$\widehat{m}_{\alpha,\beta}(u)=\exp\left\{\frac{\alpha u}{\beta-u}\right\},\quad u\in\mathbb{C}_{\leqslant 0}.$$

To study the moments of the JCIR process, the lemma below plays a substantial role.

Lemma 2.7. *Let $\kappa > 0$ and $\delta > 0$ be positive constants. Then*

(i) there exists a positive constant $C_1 = C_1(\kappa)$ such that for all $\alpha > 0$ and $\beta > 0$,

$$\int_{\mathbb{R}_{\geqslant 0}} x^\kappa m_{\alpha,\beta}(\mathrm{d}x) \leqslant C_1 \frac{1+\alpha^\kappa}{\beta^\kappa}.$$

(ii) there exists a positive constant $C_2 = C_2(\kappa, \delta)$ such that for all $\alpha \geqslant \delta$ and $\beta > 0$,

$$\int_{\mathbb{R}_{\geqslant 0}} x^\kappa m_{\alpha,\beta}(\mathrm{d}x) \geqslant C_2 \frac{\alpha^\kappa}{\beta^\kappa}.$$

Proof. (i) If $0 < \kappa \leqslant 1$, then we can use Jensen's inequality to obtain

$$\int_{\mathbb{R}_{\geqslant 0}} x^\kappa m_{\alpha,\beta}(\mathrm{d}x) \leqslant \left(\int_{\mathbb{R}_{\geqslant 0}} x m_{\alpha,\beta}(\mathrm{d}x)\right)^\kappa = \left(\frac{\alpha}{\beta}\right)^\kappa, \tag{2.2.3}$$

where the last identity holds because of

$$\int_{\mathbb{R}_{\geqslant 0}} x m_{\alpha,\beta}(\mathrm{d}x) = \frac{\partial}{\partial u} \widehat{m}_{\alpha,\beta}(u)\bigg|_{u=0} = \frac{\alpha}{\beta}.$$

For $\kappa = n \in \mathbb{N}$ with $n \geqslant 2$, by (2.2.1) and (2.2.2), we have for all $\alpha, \beta > 0$,

$$\begin{aligned}
\int_{\mathbb{R}_{\geqslant 0}} x^n m_{\alpha,\beta}(\mathrm{d}x) &= \int_{\mathbb{R}_{\geqslant 0}} x^n \left(e^{-\alpha}\delta_0(\mathrm{d}x) + \beta e^{-\alpha-\beta x}\sqrt{\alpha(\beta x)^{-1}} I_1\left(2\sqrt{\alpha\beta x}\right)\mathrm{d}x\right) \\
&= e^{-\alpha} \sum_{k=0}^{\infty} \frac{(\alpha\beta)^{k+1}}{k!(k+1)!} \int_0^\infty x^{n+k} e^{-\beta x} \mathrm{d}x \\
&= \frac{e^{-\alpha}}{\beta^n} \sum_{k=0}^{\infty} \frac{\alpha^{k+1}(n+k)!}{k!(k+1)!} \\
&= \frac{e^{-\alpha}}{\beta^n} \sum_{k=0}^{n-2} \frac{\alpha^{k+1}(n+k)!}{k!(k+1)!} \\
&\quad + \frac{e^{-\alpha}\alpha^n}{\beta^n} \sum_{k=n-1}^{\infty} \frac{\alpha^{k+1-n}}{(k+1-n)!} \cdot \frac{(k+1)\cdots(k+n)}{(k+2-n)\cdots(k+1)}.
\end{aligned} \tag{2.2.4}$$

Since

$$\lim_{k\to\infty} \frac{(k+1)\cdots(k+n)}{(k+2-n)\cdots(k+1)} = 1,$$

it follows from (2.2.4) that

$$\begin{aligned}
\int_{\mathbb{R}_{\geqslant 0}} x^n m_{\alpha,\beta}(\mathrm{d}x) &\leqslant c_1 \frac{e^{-\alpha}}{\beta^n} \left(\alpha + \alpha^2 + \cdots + \alpha^{n-1} + \alpha^n \sum_{m=0}^{\infty} \frac{\alpha^m}{m!}\right) \\
&\leqslant c_2 \left(\frac{1}{\beta^n} + \frac{\alpha^n}{\beta^n}\right), \quad \text{for all } \alpha, \beta > 0,
\end{aligned} \tag{2.2.5}$$

where c_1 and c_2 are positive constants depending on n.

For the remaining possible κ, namely, $\kappa > 1$ and $\kappa \notin \mathbb{N}$, we can find $n \in \mathbb{N}$ and $\varepsilon \in (0,1]$ such that $2\kappa = n + \varepsilon$. By (2.2.3), (2.2.5) and Hölder's inequality, we get for all $\alpha, \beta > 0$,

$$\int_{\mathbb{R}_{\geqslant 0}} x^\kappa m_{\alpha,\beta}(\mathrm{d}x) \leqslant \left(\int_{\mathbb{R}_{\geqslant 0}} x^n m_{\alpha,\beta}(\mathrm{d}x)\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}_{\geqslant 0}} x^\varepsilon m_{\alpha,\beta}(\mathrm{d}x)\right)^{\frac{1}{2}}$$

$$\leqslant c_3\left(\frac{1+\alpha^n}{\beta^n}\right)^{\frac{1}{2}}\left(\frac{\alpha}{\beta}\right)^{\frac{\varepsilon}{2}}\leqslant c_4\frac{\alpha^{\varepsilon/2}+\alpha^{(n+\varepsilon)/2}}{\beta^{(n+\varepsilon)/2}}\leqslant c_5\frac{1+\alpha^\kappa}{\beta^\kappa},$$

where c_3, c_4 and c_5 are positive constants depending on κ.

(ii) If $\kappa\geqslant 1$, using again Jensen's inequality, we obtain for all $\alpha,\beta>0$,

$$\int_{\mathbb{R}_{\geqslant 0}} x^\kappa m_{\alpha,\beta}(\mathrm{d}x)\geqslant\left(\int_{\mathbb{R}_{\geqslant 0}} x m_{\alpha,\beta}(\mathrm{d}x)\right)^\kappa=\left(\frac{\alpha}{\beta}\right)^\kappa.$$

Suppose now $0<\kappa<1$ and let $\theta:=1-\kappa\in(0,1)$. Consider a random variable $\eta>0$ such that

$$\eta\sim\left(1-e^{-\alpha}\right)^{-1}\left(m_{\alpha,\beta}(\mathrm{d}x)-e^{-\alpha}\delta_0(\mathrm{d}x)\right). \tag{2.2.6}$$

Then for $u\geqslant 0$, we have

$$\begin{aligned}\mathbb{E}\left[e^{-u\eta}\right]&=\left(1-e^{-\alpha}\right)^{-1}\left(\widehat{m}_{\alpha,\beta}(-u)-e^{-\alpha}\right)\\&=\left(1-e^{-\alpha}\right)^{-1}\left(\exp\left\{\frac{-\alpha u}{\beta+u}\right\}-\exp\left\{-\alpha\right\}\right).\end{aligned}$$

Since, by the Fubini's theorem,

$$\begin{aligned}\int_0^\infty\frac{\partial}{\partial u}\mathbb{E}\left[e^{-u\eta}\right]u^{\theta-1}\mathrm{d}u&=-\int_0^\infty\mathbb{E}\left[Ye^{-u\eta}\right]u^{\theta-1}\mathrm{d}u\\&=-\mathbb{E}\left[\int_0^\infty\eta e^{-u\eta}u^{\theta-1}\mathrm{d}u\right]=-\mathbb{E}\left[\Gamma(\theta)\eta^{1-\theta}\right],\end{aligned}$$

it follows that

$$\begin{aligned}\mathbb{E}\left[\eta^\kappa\right]&=\frac{-1}{\Gamma(\theta)}\int_0^\infty\frac{\partial}{\partial u}\mathbb{E}\left[e^{-u\eta}\right]u^{\theta-1}\mathrm{d}u\\&=\frac{\alpha\beta}{\Gamma(\theta)\left(1-e^{-\alpha}\right)}\int_0^\infty\exp\left\{\frac{-\alpha u}{\beta+u}\right\}\frac{u^{\theta-1}}{(\beta+u)^2}\mathrm{d}u.\end{aligned} \tag{2.2.7}$$

By (2.2.6) and (2.2.7), we see that

$$\int_{\mathbb{R}_{\geqslant 0}} x^\kappa m_{\alpha,\beta}(\mathrm{d}x)=\frac{\alpha\beta}{\Gamma(\theta)}\int_0^\infty\exp\left\{\frac{-\alpha u}{\beta+u}\right\}\frac{u^{\theta-1}}{(\beta+u)^2}\mathrm{d}u,\quad u\in\mathbb{R}_{\geqslant 0}.$$

By a change of variables $w:=\alpha u/\beta$, we get

$$\begin{aligned}\int_{\mathbb{R}_{\geqslant 0}} x^\kappa m_{\alpha,\beta}(\mathrm{d}x)&=\frac{\alpha\beta}{\Gamma(\theta)}\int_0^\infty\exp\left\{-\alpha+\frac{\alpha\beta}{\beta+\frac{\beta w}{\alpha}}\right\}\frac{\left(\frac{\beta w}{\alpha}\right)^{-\kappa}}{\left(\beta+\frac{\beta w}{\alpha}\right)^2}\frac{\beta}{\alpha}\mathrm{d}w\\&=\frac{1}{\Gamma(\theta)}\left(\frac{\alpha}{\beta}\right)^\kappa\int_0^\infty\exp\left\{\frac{-\alpha w}{\alpha+w}\right\}\frac{w^{-\kappa}}{(1+w/\alpha)^2}\mathrm{d}w\\&=:\frac{1}{\Gamma(\theta)}\left(\frac{\alpha}{\beta}\right)^\kappa I(\alpha).\end{aligned} \tag{2.2.8}$$

By Fatou's lemma,

$$\begin{aligned}\liminf_{\alpha\to\infty}I(\alpha)&\geqslant\int_0^\infty\liminf_{\alpha\to\infty}\exp\left\{\frac{-\alpha w}{\alpha+w}\right\}\frac{w^{-\kappa}}{(1+w/\alpha)^2}\mathrm{d}w\\&=\int_0^\infty\exp\left\{-w\right\}w^{-\kappa}\mathrm{d}w=\Gamma(1-\kappa)>0.\end{aligned}$$

On the other hand, the function $(0,\infty)\ni\alpha\mapsto I(\alpha)$ is positive and continuous. So we can find a positive constant c_6 depending on κ and δ such that $I(\alpha)\geq c_6$ for all $\alpha\in[\delta,\infty)$, which, together with (2.2.8), implies the assertion. □

2.2.2. Moment characterization of the JCIR process

Recall that $Z = (Z_t)_{t\geqslant 0}$ is the unique strong solution of the stochastic differential equation (2.1.9) and its characteristic function is given by

$$\mathbb{E}\left[e^{uZ_t}\right] = \exp\left\{\int_0^t \int_0^\infty \left(e^{z\psi(s,u)} - 1\right)\nu(\mathrm{d}z)\mathrm{d}s\right\}, \quad (t,u) \in \mathbb{R}_{\geqslant 0} \times \mathbb{C}_{\leqslant 0},$$

see formula (2.1.10). Note that for all $(t,u) \in \mathbb{R}_{\geqslant 0} \times \mathbb{C}_{\leqslant 0}$,

$$\begin{aligned}\mathbb{E}\left[e^{uZ_t}\right] &= \exp\left\{\int_0^t \int_0^\infty \left(e^{z\psi(s,u)} - 1\right)\nu_1(\mathrm{d}z)\mathrm{d}s\right\}\\ &\quad \cdot \exp\left\{\int_0^t \int_0^\infty \left(e^{z\psi(s,u)} - 1\right)\nu_2(\mathrm{d}z)\mathrm{d}s\right\},\end{aligned} \tag{2.2.9}$$

where $\nu_1(\mathrm{d}z) := \mathbb{1}_{\{z\leqslant 1\}}\nu(\mathrm{d}z)$ and $\nu_2(\mathrm{d}z) := \mathbb{1}_{\{z>1\}}\nu(\mathrm{d}z)$. Similarly to (2.1.9), for $i = 1, 2$, we define $(Z_t^i)_{t\geqslant 0}$ as the unique strong solution of

$$\mathrm{d}Z_t^i = -bZ_t^i\mathrm{d}t + \sigma\sqrt{Z_t^i}\mathrm{d}B_t + \mathrm{d}J_t^i, \quad t \geqslant 0, \quad Z_0^i = 0 \text{ a.s.}, \tag{2.2.10}$$

where $(J_t^i)_{t\geqslant 0}$ is a subordinator of pure jump-type with Lévy measure ν_i. By (2.1.10), we have

$$\mathbb{E}\left[e^{uZ_t^i}\right] = \exp\left\{\int_0^t \int_0^\infty \left(e^{z\psi(s,u)} - 1\right)\nu_i(\mathrm{d}z)\mathrm{d}s\right\}, \quad i = 1, 2, \ (t,u) \in \mathbb{R}_{\geqslant 0} \times \mathbb{C}_{\leqslant 0}. \tag{2.2.11}$$

It follows from (2.2.9) and (2.2.11) that

$$\mu_{Z_t} = \mu_{Z_t^1} * \mu_{Z_t^2}. \tag{2.2.12}$$

Before we turn to check the aimed characterization of the fractional moments for the JCIR process, we preface the proof with a technicality.

Proposition 2.8. *The characteristic function of* $(Z_t^2)_{t\geqslant 0}$ *given by* (2.2.11) *($i = 2$), is the characteristic function of a compound Poisson distribution.*

The proof is a rather lengthy calculation. We just note that $(J_t^2)_{t\geqslant 0}$ has only big jumps. Then we direct the reader to [33, Lemma 2] for details.

Theorem 2.9. *Consider the JCIR process* $X = (X_t)_{t\geqslant 0}$ *defined in* (2.0.1). *Let* $\kappa > 0$ *be a constant. Then the following three conditions are equivalent:*

(i) $\mathbb{E}_x[X_t^\kappa] < \infty$ *for all* $x \in \mathbb{R}_{\geqslant 0}$ *and* $t > 0$,

(ii) $\mathbb{E}_x[X_t^\kappa] < \infty$ *for some* $x \in \mathbb{R}_{\geqslant 0}$ *and* $t > 0$,

(iii) $\int_{\{z>1\}} z^\kappa \nu(\mathrm{d}z) < \infty$.

Proof. "(iii)⇒(i)": Let $\kappa > 0$ be a constant. Suppose that $\int_{\{z>1\}} z^\kappa\nu(\mathrm{d}z) < \infty$. Let $x \in \mathbb{R}_{\geqslant 0}$ and $t > 0$ be arbitrary. We define $f(y) := (|y| \vee 1)^\kappa$, $y \in \mathbb{R}$. Then f is locally bounded and submultiplicative by [59, Proposition 25.4], i.e., there exists a constant $c_1 > 0$ such that $f(y_1 + y_2) \leqslant c_1 f(y_1)f(y_2)$ for all $y_1, y_2 \in \mathbb{R}$. Further, it is easy to see that for any constant $c > 0$, there exists a constant $c_2 > 0$ such that $f(y) \leqslant c_2 \exp\{c|y|\}$, $y \in \mathbb{R}$. By (2.1.12) and (2.2.12), we get

$$\mathbb{E}\left[f\left(X_t^x\right)\right] \leqslant c_1^2\mathbb{E}\left[f\left(Y_t^x\right)\right]\mathbb{E}\left[f\left(Z_t^1\right)\right]\mathbb{E}\left[f\left(Z_t^2\right)\right]$$

$$\leqslant c_1^2 c_2 \mathbb{E}\left[f\left(Y_t^x\right)\right] \mathbb{E}\left[e^{cZ_t^1}\right] \mathbb{E}\left[f\left(Z_t^2\right)\right]. \tag{2.2.13}$$

By [11, Proposition 3], we have $\mathbb{E}[f(Y_t^x)] < \infty$. The finiteness of the exponential moments of Z_t^1, i.e., $\mathbb{E}[\exp\{cZ_t^1\}] < \infty$, follows by [40, Theorem 2.14 (b)], since $(J_t^1)_{t\geqslant 0}$ has only small jumps.

We next show that $\mathbb{E}[f(Z_t^2)] < \infty$. By Proposition 2.8, we know that Z_t^2 is compound Poisson distributed, namely, we can find a probability measure ρ_t on $\mathbb{R}_{\geqslant 0}$ such that

$$\mathbb{E}\left[e^{uZ_t^2}\right] = e^{\lambda_t(\widehat{\rho}_t(u)-1)}, \quad (t,u) \in \mathbb{R}_{>0} \times \mathbb{C}_{\leqslant 0},$$

where $\lambda_t > 0$ and $\widehat{\rho}_t$ denotes the characteristic function of the measure ρ_t. More precisely, according to [33, see p.292], we have

$$\rho_t = \lambda_t^{-1} \int_0^t \int_{\{z>1\}} m_{\alpha(z,s),\beta(z,s)} \nu(\mathrm{d}z)\mathrm{d}s,$$

where $m_{\alpha(z,s),\beta(z,s)}$ is a Bessel distribution with parameters $\alpha(z,s)$ and $\beta(z,s)$ given by

$$\alpha(z,s) := \frac{2bz}{\sigma^2\left(e^{bs}-1\right)} \quad \text{and} \quad \beta(z,s) := \frac{2be^{bs}}{\sigma^2\left(e^{bs}-1\right)},$$

and

$$\lambda_t = \int_0^t \int_{\{z>1\}} \left(1 - e^{-\alpha(z,s)}\right) \nu(\mathrm{d}z)\mathrm{d}s < \infty.$$

By the Fubini's theorem, we obtain

$$\int_{\mathbb{R}_{\geqslant 0}} f(y)\rho_t(\mathrm{d}y) = \lambda_t^{-1} \int_0^t \int_{\{z>1\}} \left(\int_{\mathbb{R}_{\geqslant 0}} f(y) m_{\alpha(z,s),\beta(z,s)}(\mathrm{d}y) \right) \nu(\mathrm{d}z)\mathrm{d}s. \tag{2.2.14}$$

By Lemma 2.7, we have

$$\begin{aligned} \int_{\mathbb{R}_{\geqslant 0}} f(y) m_{\alpha(z,s),\beta(z,s)}(\mathrm{d}y) &\leqslant \int_{\mathbb{R}_{\geqslant 0}} (1+y^\kappa) m_{\alpha(z,s),\beta(z,s)}(\mathrm{d}y) \\ &\leqslant 1 + C_1 \frac{1+\alpha(z,s)^\kappa}{\beta(z,s)^\kappa} \\ &\leqslant 1 + C_1 \sigma^{2\kappa}(2b)^{-\kappa}(1-e^{-bs})^\kappa + C_1 e^{-\kappa bs} z^\kappa. \end{aligned} \tag{2.2.15}$$

It follows from (2.2.14) and (2.2.15) that

$$\int_{\mathbb{R}_{\geqslant 0}} f(y)\rho_t(\mathrm{d}y) < \infty. \tag{2.2.16}$$

Moreover, using (2.2.16) together with the submultiplicativity of f, we get

$$\begin{aligned} \int_{\mathbb{R}_{\geqslant 0}} f(y)\rho_t^{*n}(\mathrm{d}y) &= \int_{\mathbb{R}_{\geqslant 0}} \cdots \int_{\mathbb{R}_{\geqslant 0}} f(y_1+\cdots+y_n)\rho_t(\mathrm{d}y_1)\cdots\rho_t(\mathrm{d}y_n) \\ &\leqslant c_1^n \left(\int_{\mathbb{R}_{\geqslant 0}} f(y)\rho_t(\mathrm{d}y) \right)^n < \infty, \end{aligned} \tag{2.2.17}$$

which implies

$$\mathbb{E}\left[f\left(Z_t^2\right)\right] = \int_{\mathbb{R}_{\geqslant 0}} f(y)\mu_{Z_t^2}(\mathrm{d}y) = e^{-\lambda_t} \sum_{n=0}^\infty \frac{\lambda_t^n}{n!} \int_{\mathbb{R}_{\geqslant 0}} f(y)\rho_t^{*n}(\mathrm{d}y) < \infty. \tag{2.2.18}$$

By (2.2.13) and (2.2.18), we obtain $\mathbb{E}\left[f\left(X_t^x\right)\right] < \infty$. It follows easily that $\mathbb{E}\left[(X_t^x)^\kappa\right] < \infty$.

"(i)⇒(ii)": It is clear.

"(ii)⇒(iii)": Suppose now that $\mathbb{E}\left[(X_t^x)^\kappa\right] < \infty$ for some $x \in \mathbb{R}_{\geqslant 0}$ and $t > 0$. By (2.1.12), we obtain

$$\mathbb{E}\left[(X_t^x)^\kappa\right] = \int_{\mathbb{R}_{\geqslant 0}} \int_{\mathbb{R}_{\geqslant 0}} (y+z)^\kappa \mu_{Y_t^x}(\mathrm{d}y)\mu_{Z_t}(\mathrm{d}z) < \infty.$$

So $\int_{\mathbb{R}_{\geqslant 0}} (y+z)^\kappa \mu_{Z_t}(\mathrm{d}z) < \infty$ for some $y \in \mathbb{R}_{\geqslant 0}$, which implies

$$\mathbb{E}\left[Z_t^\kappa\right] = \int_{\mathbb{R}_{\geqslant 0}} z^\kappa \mu_{Z_t}(\mathrm{d}z) \leqslant \int_{\mathbb{R}_{\geqslant 0}} (y+z)^\kappa \mu_{Z_t}(\mathrm{d}z) < \infty. \tag{2.2.19}$$

Similarly, we can use (2.2.19) and (2.2.12) to conclude that $(Z_t^2)_{t\geqslant 0}$ has finite moment of order κ. Let the function f be as above. Then $\mathbb{E}\left[f\left(Z_t^2\right)\right] \leqslant 1 + \mathbb{E}\left[\left(Z_t^2\right)^\kappa\right] < \infty$. Since now all the summands in the last identity of (2.2.18) are finite, the summand corresponding to $n = 1$ is also finite and thus

$$\int_{\mathbb{R}_{\geqslant 0}} y^\kappa \rho_t(\mathrm{d}y) \leqslant \int_{\mathbb{R}_{\geqslant 0}} f(y)\rho_t(\mathrm{d}y) < \infty.$$

By the Fubini's theorem, we obtain

$$\int_{\mathbb{R}_{\geqslant 0}} y^\kappa \rho_t(\mathrm{d}y) = \lambda_t^{-1} \int_0^t \int_{\{z>1\}} \left(\int_{\mathbb{R}_{\geqslant 0}} y^\kappa m_{\alpha(z,s),\beta(z,s)}(\mathrm{d}y) \right) \nu(\mathrm{d}z)\mathrm{d}s < \infty. \tag{2.2.20}$$

Noting that for all $s \in [0, t]$ and $z > 1$,

$$\alpha(z, s) = \frac{2bz}{\sigma^2\left(e^{bs} - 1\right)} \geqslant \frac{2b}{\sigma^2\left(e^{bt} - 1\right)}.$$

By Lemma 2.7, we can find a constant $c_3 = c_3(t) > 0$ such that

$$\int_{\mathbb{R}_{\geqslant 0}} y^\kappa m_{\alpha(z,s),\beta(z,s)}(\mathrm{d}y) \geqslant c_3 \left(\frac{\alpha(z,s)}{\beta(z,s)}\right)^\kappa = c_3 z^\kappa e^{-\kappa bs}, \quad s \in [0,t],\ z > 1. \tag{2.2.21}$$

It follows from (2.2.20) and (2.2.21) that $\int_{\{z>1\}} z^\kappa \nu(\mathrm{d}z) < \infty$. □

Remark 2.10. *We remark that moments of general 1-dimensional CBI processes were recently studied in* [29]*. If $\kappa > 1$ and $x > 0$, our Theorem 2.9 can be viewed as a special case of* [29, Theorem 2.2]*. However, to the authors' knowledge, the cases $0 < \kappa < 1$ and $\kappa > 0$ with $x = 0$ can not be handled by the approach used in* [29]*.*

Based on the proof of Theorem 2.9 we get the following corollary.

Corollary 2.11. *Let $\kappa > 0$ be a constant. Suppose $\int_{\{z>1\}} z^\kappa \nu(\mathrm{d}z) < \infty$. Then, for all $x \in \mathbb{R}_{\geqslant 0}$ and $T > 0$,*

$$\sup_{t\in[0,T]} \mathbb{E}_x\left[X_t^\kappa\right] < \infty.$$

Proof. Let f, Z_t^1 and Z_t^2 be as in the proof of Theorem 2.9. Note that $|y|^\kappa \leqslant f(y) \leq |y|^\kappa + 1$ for all $y \in \mathbb{R}$. Since $\sup_{t\in\mathbb{R}_{\geqslant 0}} \mathbb{E}[(Y_t^x)^\kappa] < \infty$ due to [11, Proposition 3], by (2.2.13), it suffices to check that

$$\sup_{t\in[0,T]} \mathbb{E}\left[e^{cZ_t^1}\right] < \infty \quad \text{and} \quad \sup_{t\in[0,T]} \mathbb{E}\left[\left(Z_t^2\right)^\kappa\right] < \infty, \quad T > 0,$$

where $c > 0$ is a constant to be chosen. It follows from [40, Theorem 2.14 (b)] that

$$\mathbb{E}\left[e^{cZ_t^1}\right] = \exp\left\{\int_0^t \int_0^1 \left(e^{z\psi(s,c)} - 1\right) \nu_1(\mathrm{d}z)\mathrm{d}s\right\} < \infty, \quad c \in \mathbb{R},$$

where ψ is given in (2.1.4). Now, we choose $c > 0$ sufficiently small such that $\psi(s, c) \geqslant 0$ for all $s \in \mathbb{R}_{\geqslant 0}$. Hence, $\sup_{t\in[0,T]} \mathbb{E}[\exp\{cZ_t^1\}] \leqslant \mathbb{E}[\exp\{cZ_T^1\}] < \infty$. We next show that $\sup_{t\in[0,T]} \mathbb{E}\left[\left(Z_t^2\right)^\kappa\right] < \infty$. By (2.2.14), (2.2.17) and (2.2.18), we have for all $t \in [0, T]$,

$$\begin{aligned}
\mathbb{E}\left[f\left(Z_t^2\right)\right] &\leqslant \exp\left\{-\lambda_t + c_1\lambda_t \int_{\mathbb{R}_{\geqslant 0}} f(y)\rho_t(\mathrm{d}y)\right\} \\
&= \exp\left\{-\lambda_t + c_1 \int_0^t \int_{\{z>1\}} \left(\int_{\mathbb{R}_{\geqslant 0}} f(y) m_{\alpha(z,s),\beta(z,s)}(\mathrm{d}y)\right) \nu(\mathrm{d}z)\mathrm{d}s\right\} \\
&\leqslant \exp\left\{c_1 \int_0^T \int_{\{z>1\}} \left(\int_{\mathbb{R}_{\geqslant 0}} f(y) m_{\alpha(z,s),\beta(z,s)}(\mathrm{d}y)\right) \nu(\mathrm{d}z)\mathrm{d}s\right\} \\
&= \exp\left\{c_1\lambda_T \int_{\mathbb{R}_{\geqslant 0}} f(y)\rho_T(\mathrm{d}y)\right\}. \qquad (2.2.22)
\end{aligned}$$

It follows from (2.2.16) and (2.2.22) that

$$\sup_{t\in[0,T]} \mathbb{E}\left[\left(Z_t^2\right)^\kappa\right] \leqslant \sup_{t\in[0,T]} \mathbb{E}\left[f\left(Z_t^2\right)\right] \leqslant \exp\left\{c_1\lambda_T \int_{\mathbb{R}_{\geqslant 0}} f(y)\rho_T(\mathrm{d}y)\right\} < \infty.$$

This completes the proof. □

In Theorem 2.25 below we will improve the statement of Corollary 2.11 such that $\sup_{t\in\mathbb{R}_{\geqslant 0}} \mathbb{E}_x[X_t^\kappa] < \infty$ using Foster-Lyapunov estimates.

2.2.3. First and second moment of the JCIR process

Finally, we calculate the first and second moment as well as their limits for $t \to \infty$. We need this formulas in Chapter 3 in order to introduce least squares estimators.

Remark 2.12. *It is worth mentioning that the moment formulas* (2.2.23) *and* (2.2.28) *in Propositions 2.13 and 2.15 below are special cases of* [6, Theorem 4.3]*, where an explicit formula of integral moments of general CBI processes has been derived.*

We start with the expectation of the JCIR process.

Proposition 2.13. *Let* $a \in \mathbb{R}_{\geqslant 0}$ *and* $b, \sigma \in \mathbb{R}_{>0}$ *and* ν *satisfying* $\int_{\{z>1\}} z\nu(\mathrm{d}z) < \infty$*. Then, for all* $0 \leqslant s \leqslant t < \infty$*, we have*

$$\mathbb{E}_x[X_t] = e^{-bt}x + \frac{1 - e^{-bt}}{b}\left(a + \int_0^\infty z\nu(\mathrm{d}z)\right), \qquad (2.2.23)$$

and hence

$$\mathbb{E}\left[X_t \mid \mathcal{F}_s\right] = e^{-b(t-s)}X_s + \frac{1 - e^{-b(t-s)}}{b}\left(a + \int_0^\infty z\nu(\mathrm{d}z)\right). \qquad (2.2.24)$$

Proof. Noting that $\mu_{X_t^x} = \mu_{Y_t^x} * \mu_{Z_t}$, we have

$$\mathbb{E}_x\left[X_t\right] = \mathbb{E}_x\left[Y_t\right] + \mathbb{E}\left[Z_t\right].$$

In view of Theorem 2.9, we know that $\mathbb{E}_x[X_t]$ exists and is finite. Therefore, we have

$$\mathbb{E}_x\left[Y_t\right] = \frac{\partial}{\partial u}\mathbb{E}_x\left[e^{uY_t}\right]\Big|_{u=0} = e^{-bt}x + \frac{a}{b}\left(1-e^{-bt}\right), \tag{2.2.25}$$

and we shall proceed to calculate the expectation of Z_t. Recall that for all $(t,u) \in \mathbb{R}_{\geqslant 0} \times \mathbb{R}_{\leqslant 0}$, by (2.1.10),

$$\mathbb{E}\left[Z_t\right] = \frac{\partial}{\partial u}\mathbb{E}\left[e^{uZ_t}\right]\Big|_{u=0} = \frac{\partial}{\partial u}\exp\left\{\int_0^t\int_0^\infty\left(e^{z\psi(s,u)}-1\right)\nu(\mathrm{d}z)\mathrm{d}s\right\}\Big|_{u=0} = \exp\left\{\Delta_t(u)\right\},$$

where $\psi(t,u)$ is given by (2.1.4) and

$$\Delta_t(u) = \int_0^t\int_0^\infty\left(e^{z\psi(s,u)}-1\right)\nu(\mathrm{d}z)\mathrm{d}s, \quad t \geqslant 0.$$

Note that $\psi(t,u) \leqslant 0$ for all $t \in \mathbb{R}_{\geqslant 0}$ and $u \in \mathbb{R}_{\leqslant 0}$, and $\psi(t,0) = 0$. Then, for all $u \in \mathbb{R}_{\leqslant 0}$,

$$\begin{aligned}\frac{\partial}{\partial u}\left(e^{z\psi(s,u)}-1\right) &= ze^{z\psi(s,u)}\frac{\partial}{\partial u}\left(\frac{e^{-bs}u}{1-\frac{\sigma^2 u}{2b}\left(1-e^{-bs}\right)}\right)\\ &= \frac{ze^{-bs}}{\left(1-\frac{\sigma^2 u}{2b}\left(1-e^{-bs}\right)\right)^2} \leqslant ze^{-bs}\end{aligned} \tag{2.2.26}$$

and $\int_0^t\int_0^\infty ze^{-bs}\nu(\mathrm{d}z)\mathrm{d}s < \infty$. We conclude that $\Delta_t(u)$ is differentiable in u and $\partial_u\Delta_t(0) = b^{-1}(1-\exp\{-bt\})\int_0^\infty z\nu(\mathrm{d}z)$ by Lebesgue's differential theorem. We end with

$$\mathbb{E}\left[Z_t\right] = \frac{\partial}{\partial u}\mathbb{E}\left[e^{uZ_t}\right]\Big|_{u=0} = \frac{1-e^{-bt}}{b}\int_0^\infty z\nu(\mathrm{d}z), \tag{2.2.27}$$

and (2.2.23) follows from (2.2.25) together with (2.2.27).

To deduce (2.2.24) from (2.2.23), one simply uses the Markov property of X_t^x. □

We immediately get the following corollary.

Corollary 2.14. *Consider the JCIR process $(X_t)_{t\geqslant 0}$ with parameters $a \in \mathbb{R}_{\geqslant 0}$, b, $\sigma \in \mathbb{R}_{>0}$, and ν satisfying $\int_{\{z>1\}} z\nu(\mathrm{d}z) < \infty$. Then*

$$\lim_{t\to\infty}\mathbb{E}_x\left[X_t\right] = \left(a + \int_0^\infty z\nu(\mathrm{d}z)\right)\frac{1}{b}.$$

Proceeding further in this direction, we compute the second moment.

Proposition 2.15. *Let $a \in \mathbb{R}_{\geqslant 0}$, b, $\sigma \in \mathbb{R}_{>0}$ and ν satisfying $\int_{\{z>1\}} z^2\nu(\mathrm{d}z) < \infty$. Then, for all $t \in \mathbb{R}_{\geqslant 0}$, we have*

$$\begin{aligned}\mathbb{E}_x[X_t^2] = e^{-2bt}x^2 &+ \frac{2a+\sigma^2}{b}e^{-2bt}\left(e^{bt}-1\right)x + \frac{a(2a+\sigma^2)}{2b^2}e^{-2bt}\left(e^{bt}-1\right)^2\\ &+ x\frac{2}{b}e^{-2bt}\left(e^{bt}-1\right)\int_0^\infty z\nu(\mathrm{d}z) + \frac{2a}{b^2}e^{-2bt}\left(e^{bt}-1\right)^2\int_0^\infty z\nu(\mathrm{d}z)\end{aligned}$$

$$+\frac{1-e^{-2bt}}{2b}\int_0^\infty z^2\nu(\mathrm{d}z)+\frac{\sigma^2}{2b^2}e^{-2bt}\left(e^{bt}-1\right)^2\int_0^\infty z\nu(\mathrm{d}z)$$
$$+\frac{(1-e^{-bt})^2}{b^2}\left(\int_0^\infty z\nu(\mathrm{d}z)\right)^2. \tag{2.2.28}$$

Consequently,

$$\lim_{t\to\infty}\mathbb{E}_x\left[X_t^2\right]=\frac{a(2a+\sigma^2)}{2b^2}+\frac{2a}{b^2}\int_0^\infty z\nu(\mathrm{d}z) \tag{2.2.29}$$
$$+\frac{1}{2b}\int_0^\infty z^2\nu(\mathrm{d}z)+\frac{\sigma^2}{2b}\int_0^\infty z\nu(\mathrm{d}z)+\frac{1}{b^2}\left(\int_0^\infty z\nu(\mathrm{d}z)\right)^2.$$

Proof. In view of Theorem (2.9) and (2.1.11) the second moment of the JCIR process X_t^x could be derived by

$$\mathbb{E}_x\left[X_t^2\right]=\left(\frac{\partial^2}{\partial u^2}\mathbb{E}_x\left[e^{uY_t}\right]+2\frac{\partial}{\partial u}\mathbb{E}_x\left[e^{uY_t}\right]\frac{\partial}{\partial u}\mathbb{E}\left[e^{uZ_t}\right]+\frac{\partial^2}{\partial u^2}\mathbb{E}\left[e^{uZ_t}\right]\right)\Bigg|_{u=0},$$

where the first moment of the CIR process Y_t^x is given by (2.2.25), the second moment of Y_t^x is given by

$$\mathbb{E}_x\left[Y_t^2\right]=e^{-2bt}x^2+\frac{2a+\sigma^2}{b}e^{-2bt}\left(e^{bt}-1\right)x+\frac{a(2a+\sigma^2)}{2b^2}e^{-2bt}\left(e^{bt}-1\right)^2, \tag{2.2.30}$$

and the first moment of Z_t is computed in (2.2.27). Hence, by (2.2.25) and (2.2.27), we obtain

$$2\mathbb{E}_x\left[Y_t\right]\mathbb{E}\left[Z_t\right]=2\frac{\partial}{\partial u}\mathbb{E}_x\left[e^{uY_t}\right]\frac{\partial}{\partial u}\mathbb{E}\left[e^{uZ_t}\right]\Bigg|_{u=0}$$
$$=x\frac{2}{b}e^{-2bt}\left(e^{bt}-1\right)\int_0^\infty z\nu(\mathrm{d}z)+\frac{2a}{b^2}e^{-2bt}\left(e^{bt}-1\right)^2\int_0^\infty z\nu(\mathrm{d}z). \tag{2.2.31}$$

Next, we derive the second moment of Z_t. As before, for $t\in\mathbb{R}_{\geqslant 0}$, we define $\Delta_t(u):=\int_0^t\int_0^\infty\left(\exp\{\psi(s,u)\}-1\right)\nu(\mathrm{d}z)\mathrm{d}s$. It follows, for all $u\in\mathbb{R}_{\leqslant 0}$,

$$\frac{\partial^2}{\partial u^2}\left(e^{z\psi(s,u)}-1\right)=ze^{z\psi(s,u)}\left(\frac{\partial^2}{\partial u^2}\psi(s,u)+z\left(\frac{\partial}{\partial u}\psi(s,u)\right)^2\right)$$
$$\leqslant z\left(\frac{\partial^2}{\partial u^2}\psi(s,u)+ze^{-2bs}\right)$$
$$\leqslant z\left(\frac{\sigma^2}{b}e^{-2bs}\left(e^{bs}-1\right)+ze^{-2bs}\right)<\infty, \tag{2.2.32}$$

where we used the estimation (2.2.26) in the first inequality and (2.2.32) indeed holds, since, for all $u\in\mathbb{R}_{\leqslant 0}$,

$$\frac{\partial^2}{\partial u^2}\psi(s,u)=\frac{e^{-bs}\frac{\sigma^2}{b}\left(1-e^{-bs}\right)}{\left(1-\frac{\sigma^2u}{2b}\left(1-e^{-bs}\right)\right)^3}\leqslant\frac{\sigma^2}{b}e^{-2bs}\left(e^{bs}-1\right).$$

Consequently, an application of dominated convergence theorem yields the twice differentiability of $\Delta_t(u)$ with respect to u. Moreover, we obtain

$$\frac{\partial^2}{\partial u^2}\Delta_t(0)=\frac{\partial^2}{\partial u^2}\int_0^t\int_0^\infty\left(e^{z\psi(s,u)}-1\right)\nu(\mathrm{d}z)\mathrm{d}s\Bigg|_{u=0}$$

$$
\begin{aligned}
&= \int_0^t \int_0^\infty \left(\frac{\sigma^2}{b} e^{-2bs} \left(e^{bs} - 1\right) z + e^{-2bs} z^2 \right) \nu(\mathrm{d}z)\mathrm{d}s \\
&= \frac{1 - e^{-2bt}}{2b} \int_0^\infty z^2 \nu(\mathrm{d}z) + \frac{\sigma^2}{2b^2} e^{-2bt} \left(e^{bt} - 1\right)^2 \int_0^\infty z\nu(\mathrm{d}z), \quad t \geqslant 0.
\end{aligned}
$$

Note that

$$
\frac{\partial^2}{\partial u^2} \mathbb{E}\left[e^{uZ_t}\right]\Big|_{u=0} = \exp\left\{\Delta_t(u)\right\}|_{u=0} \left(\frac{\partial^2}{\partial u^2} \Delta_t(u)\Big|_{u=0} + \left(\frac{\partial}{\partial u} \Delta_t(u) \right)^2 \Big|_{u=0} \right).
$$

Using that $\psi(s, 0) = 0$, we conclude $\exp\{\Delta_t(0)\} = 1$. Hence, using (2.2.27) and dominated convergence theorem,

$$
\begin{aligned}
\frac{\partial^2}{\partial u^2} \mathbb{E}\left[e^{uZ_t}\right]\Big|_{u=0} &= \int_0^t \int_0^\infty z^2 e^{-2bs} \nu(\mathrm{d}z)\mathrm{d}s + \int_0^t \int_0^\infty z \frac{\sigma^2}{b} e^{-2bs} \left(e^{bs} - 1\right) \nu(\mathrm{d}z)\mathrm{d}s \\
&\quad + \left(\int_0^t \int_0^\infty z e^{bs} \nu(\mathrm{d}z)\mathrm{d}s \right)^2 \\
&= \frac{1 - e^{-2bt}}{2b} \int_0^\infty z^2 \nu(\mathrm{d}z) + \frac{\sigma^2}{2b^2} e^{-2bt} \left(e^{bt} - 1\right)^2 \int_0^\infty z\nu(\mathrm{d}z) \\
&\quad + \frac{\left(1 - e^{bt}\right)^2}{b^2} \left(\int_0^\infty z\nu(\mathrm{d}z) \right)^2, \quad t \geqslant 0. \qquad (2.2.33)
\end{aligned}
$$

Thus, combining (2.2.30), (2.2.31) and (2.2.33) yield (2.2.28). In view of (2.2.28), one easily checks (2.2.29). □

2.3. Ergodicity of the JCIR process

In this section we prove the ergodicity of the JCIR process X provided that

$$
\int_{\{z>1\}} \log z \nu(\mathrm{d}z) < \infty. \qquad (2.3.1)
$$

Since ergodicity requires existence and uniqueness of an invariant measure for X, we consider this property in prior.

Remark 2.16. *Let a, $b \in \mathbb{R}_{>0}$. If* (2.3.1) *holds, then, by an application of* [43, Theorem 3.16] *(see also* [41, Theorem 2.6]*), the JCIR process X_t converges in law to a limit distribution π which is independent of $X_0 = x$ and whose characteristic function takes the form*

$$
\int_0^\infty e^{ux} \pi(\mathrm{d}x) = \left(1 - \frac{\sigma^2 u}{2b}\right)^{\frac{-2a}{\sigma^2}} \exp\left\{ \int_0^\infty \int_0^\infty \left(e^{z\psi(s,u)} - 1\right) \nu(\mathrm{d}z)\mathrm{d}s \right\}
$$

for all $u \in \mathbb{C}_{\leqslant 0}$. Moreover, by the argument in [39, p.80]*, the limit distribution π is also the unique invariant distribution of the JCIR process.*

Our approach to establish the ergodicity is based on the general theory of Meyn and Tweedie [52] for the ergodicity of Markov processes. The essential step is to find a Foster-Lyapunov function in the sense of [52, condition (CD2)]. Recall that the infinitesimal generator $\mathcal{A}$ of X is given by (2.1.1) as

$$(\mathcal{A}f)(x) = (a - bx)\frac{\partial f(x)}{\partial x} + \frac{1}{2}\sigma^2 x \frac{\partial^2 f(x)}{\partial x^2} + \int_0^\infty (f(x+z) - f(x))\,\nu(\mathrm{d}z),$$

for $f \in C_c^2(\mathbb{R}_{\geqslant 0}, \mathbb{R})$. We introduce a useful decomposition of $\mathcal{A}$. If we write

$$(\mathcal{D}f)(x) = (a - bx)\frac{\partial f(x)}{\partial x} + \frac{1}{2}\sigma^2 x \frac{\partial^2 f(x)}{\partial x^2},$$
$$(\mathcal{J}f)(x) = \int_0^\infty (f(x+z) - f(x))\,\nu(\mathrm{d}z),$$

where $x \in \mathbb{R}_{\geqslant 0}$ and $f \in C_c^2(\mathbb{R}_{\geqslant 0}, \mathbb{R})$, we see that $\mathcal{A}f = \mathcal{D}f + \mathcal{J}f$.

In view of (2.3.1), we choose the Foster-Lyapunov function to be $V(x) = \log(1+x)$, $x \in \mathbb{R}_{\geqslant 0}$. Clearly, V is unbounded. So we first show that this function V is in the domain of the *extended generator* of X which is defined as follows:

Definition 2.1. Let $V : \mathbb{R}_{\geqslant 0} \to \mathbb{R}$ be a measurable function for which there exists a measurable function $U_V : \mathbb{R}_{\geqslant 0} \to \mathbb{R}$ such that for each $x \in \mathbb{R}_{\geqslant 0}$, $t \in \mathbb{R}_{>0}$,

$$\mathbb{E}_x\left[V(X_t)\right] = V(x) + \mathbb{E}_x\left[\int_0^t U_V(X_s)\,\mathrm{d}s\right],$$
$$\mathbb{E}_x\left[\int_0^t |U_V(X_s)|\,\mathrm{d}s\right] < \infty.$$

We adhere to the convention that $\mathcal{A}V := U_V$ and call $\mathcal{A}$ the *extended generator* of the process X_t *associated with* V.

Lemma 2.17. *Suppose* (2.3.1) *is true. Let* $V(x) := \log(1+x)$, $x \in \mathbb{R}_{\geqslant 0}$. *Then for all* $t > 0$ *and* $x \in \mathbb{R}_{\geqslant 0}$, *we have* $\mathbb{E}_x\left[\int_0^t |\mathcal{A}V(X_s)|\,\mathrm{d}s\right] < \infty$ *and*

$$\mathbb{E}_x\left[V(X_t)\right] = V(x) + \mathbb{E}_x\left[\int_0^t \mathcal{A}V(X_s)\,\mathrm{d}s\right], \tag{2.3.2}$$

where $\mathcal{A}$ *is given in* (2.1.1)*. In other words,* V *is in the domain of the extended generator of* X*.*

Proof. It is easy to see that $V \in C^2(\mathbb{R}_{\geqslant 0}, \mathbb{R})$ and

$$V'(x) := \frac{\partial}{\partial x}V(x) = (1+x)^{-1} \quad \text{and} \quad V''(x) := \frac{\partial^2}{\partial x^2}V(x) = -(1+x)^{-2}.$$

Let $x \in \mathbb{R}_{\geqslant 0}$ be fixed and assume that $X_0 = x$ almost surely. In view of the Lévy-Itô decomposition of $(J_t)_{t\geqslant 0}$ in (2.0.3), we have

$$X_t = x + \int_0^t (a - bX_s)\mathrm{d}s + \sigma\int_0^t \sqrt{X_s}\mathrm{d}B_s + \int_0^t\int_0^\infty zN(\mathrm{d}s, \mathrm{d}z), \quad t \geqslant 0,$$

where $N(\mathrm{d}s, \mathrm{d}z)$ is defined in (2.0.3). By Itô's formula, we obtain

$$V(X_t) - V(X_0) = \int_0^t (a - bX_s)\,V'(X_s)\,\mathrm{d}s + \frac{\sigma^2}{2}\int_0^t X_s V''(X_s)\,\mathrm{d}s$$

$$
\begin{aligned}
&+\sigma\int_0^t \sqrt{X_s}V'(X_s)\,\mathrm{d}B_s\\
&+\int_0^t\int_0^\infty (V(X_{s-}+z)-V(X_{s-}))\,N(\mathrm{d}s,\mathrm{d}z)\\
=&\int_0^t (a-bX_s)V'(X_s)\,\mathrm{d}s+\frac{\sigma^2}{2}\int_0^t X_sV''(X_s)\,\mathrm{d}s\\
&+\int_0^t\int_0^\infty (V(X_{s-}+z)-V(X_{s-}))\,\nu(\mathrm{d}z)\mathrm{d}s\\
&+\sigma\int_0^t \sqrt{X_s}V'(X_s)\,\mathrm{d}B_s\\
&+\int_0^t\int_0^\infty (V(X_{s-}+z)-V(X_{s-}))\,\widetilde{N}(\mathrm{d}s,\mathrm{d}z)\\
=&\int_0^t (\mathcal{A}V)(X_s)\,\mathrm{d}s+M_t(V),\quad t\geqslant 0,
\end{aligned}
\tag{2.3.3}
$$

where $\widetilde{N}(\mathrm{d}s,\mathrm{d}z):=N(\mathrm{d}s,\mathrm{d}z)-\nu(\mathrm{d}z)\mathrm{d}s$ and

$$
\begin{aligned}
M_t(V):=&\,\sigma\int_0^t \sqrt{X_s}V'(X_s)\,\mathrm{d}B_s\\
&+\int_0^t\int_{\{z\leqslant 1\}} (V(X_{s-}+z)-V(X_{s-}))\,\widetilde{N}(\mathrm{d}s,\mathrm{d}z)\\
&+\int_0^t\int_{\{z>1\}} (V(X_{s-}+z)-V(X_{s-}))\,\widetilde{N}(\mathrm{d}s,\mathrm{d}z)\\
=&\,D_t+J_{*,t}+J_t^*.
\end{aligned}
$$

Clearly, if $(M_t(V))_{t\geqslant 0}$ is a martingale with respect to the filtration $(\mathcal{F}_t)_{t\geqslant 0}$, by taking the expectation of both sides of (2.3.3), we see that condition (2.3.2) holds.

We start to prove that $(M_t(V))_{t\geqslant 0}$ is a martingale with respect to the filtration $(\mathcal{F}_t)_{t\geqslant 0}$. Since

$$
\mathbb{E}_x\left[(D_t)^2\right]=\sigma^2\int_0^t \mathbb{E}_x\left[X_s(1+X_s)^{-2}\right]\mathrm{d}s\leqslant\sigma^2\int_0^t\mathbb{E}_x\left[(1+X_s)^{-1}\right]\mathrm{d}s\leqslant t\sigma^2<\infty,
$$

it follows that $(D_t)_{t\geqslant 0}$ is a square-integrable martingale. Note that

$$
|V(y+z)-V(y)|\leqslant z\sup_{y\in\mathbb{R}_{\geqslant 0}}|V'(y)|\leqslant z,\quad y,z\in\mathbb{R}_{\geqslant 0}. \tag{2.3.4}
$$

Therefore,

$$
\mathbb{E}_x\left[\int_0^t\int_{\{z\leqslant 1\}}(V(X_{s-}+z)-V(X_{s-}))^2\,\nu(\mathrm{d}z)\mathrm{d}s\right]\leqslant t\int_{\{z\leqslant 1\}} z^2\nu(\mathrm{d}z)<\infty,
$$

which implies that $(J_{*,t})_{t\geqslant 0}$ is also a square-integrable martingale by [26, pp. 62, 63]. If $y\in\mathbb{R}_{\geqslant 0}$ and $z>1$, then

$$
|V(y+z)-V(y)|=\log\left(1+\frac{z}{1+y}\right)\leqslant\log(1+z)\leqslant\log(2)+\log(z). \tag{2.3.5}
$$

So

$$
\mathbb{E}_x\left[\int_0^t\int_{\{z>1\}}|V(X_{s-}+z)-V(X_{s-})|\,\nu(\mathrm{d}z)\mathrm{d}s\right]
$$

$$\leqslant t \int_{\{z>1\}} (\log(2) + \log(z))\, \nu(\mathrm{d}z)$$
$$= t \log(2)\nu(\{z > 1\}) + t \int_{\{z>1\}} \log(z)\nu(\mathrm{d}z) < \infty, \quad t \geqslant 0,$$

and hence, by [26, Lemma 3.1 and p. 62], $(J_t^*)_{t\geqslant 0}$ is a martingale. Consequently, $(M_t(V))_{t\geqslant 0} = (D_t + J_{*,t} + J_t^*)_{t\geqslant 0}$ is a martingale with respect to the filtration $(\mathcal{F}_t)_{t\geqslant 0}$.

Next, we show that $\mathbb{E}_x \left[\int_0^t |\mathcal{A}V(X_s)|\, \mathrm{d}s\right] < \infty$ for all $t \geqslant 0$. By the decomposition of $\mathcal{A}$ into a *diffusion part* $\mathcal{D}$ and a *jump part* $\mathcal{J}$ as introduced in the preamble of this section, we can write $\mathcal{A}V = \mathcal{D}V + \mathcal{J}V$. Concerning the diffusion part $\mathcal{D}V$, it is easy to see that

$$\sup_{y\in\mathbb{R}_{\geqslant 0}} |(\mathcal{D}V)(y)| = \sup_{y\in\mathbb{R}_{\geqslant 0}} \left|(a - by)(1 + y)^{-1} - \frac{\sigma^2}{2} y(1 + y)^{-2}\right| < \infty. \tag{2.3.6}$$

For the jump part $\mathcal{J}V$, we decompose it further as $\mathcal{J}V = \mathcal{J}_*V + \mathcal{J}^*V$, where

$$(\mathcal{J}_*V)(y) = \int_{\{z\leqslant 1\}} (V(y + z) - V(y))\, \nu(\mathrm{d}z), \tag{2.3.7}$$
$$(\mathcal{J}^*V)(y) = \int_{\{z>1\}} (V(y + z) - V(y))\, \nu(\mathrm{d}z). \tag{2.3.8}$$

By (2.3.4), we have

$$|(\mathcal{J}_*V)(y)| \leqslant \int_{\{z\leqslant 1\}} z\nu(\mathrm{d}z) < \infty, \quad y \in \mathbb{R}_{\geqslant 0}. \tag{2.3.9}$$

Concerning $\mathcal{J}^*$, it follows from (2.3.5) that

$$|(\mathcal{J}^*V)(y)| \leqslant \log(2)\nu(\{z > 1\}) + \int_{\{z>1\}} \log z\nu(\mathrm{d}z) < \infty, \quad y \in \mathbb{R}_{\geqslant 0}. \tag{2.3.10}$$

Combining (2.3.6), (2.3.9) and (2.3.10) yields that $|\mathcal{A}V|$ is bounded on $\mathbb{R}_{\geqslant 0}$, which implies $\mathbb{E}_x \left[\int_0^t |\mathcal{A}V(X_s)|\, \mathrm{d}s\right] < \infty$ for all $t \geqslant 0$. □

For the JCIR process X, we let $\mathbf{P}^t(x, \cdot) := \mathbb{P}_x (X_t \in \cdot)$ denote the distribution of X_t with the initial condition $X_0 = x \in \mathbb{R}_{\geqslant 0}$.

We are ready to prove the ergodicity of the JCIR process $(X_t)_{t\geqslant 0}$ under (2.3.1).

Theorem 2.18. *Consider the JCIR process $(X_t)_{t\geqslant 0}$ defined by (2.0.1) with parameters a, b, σ and ν, where ν is the Lévy measure of $(J_t)_{t\geqslant 0}$. Assume $a > 0$. If (2.3.1) is true, then X is ergodic, i.e.,*

$$\lim_{t\to\infty} \left\|\mathbf{P}^t(x, \cdot) - \pi\right\|_{TV} = 0$$

for all $x \in \mathbb{R}_{\geqslant 0}$.

Proof. In view of [52, Theorem 5.1], to prove the ergodicity of the JCIR process $(X_t)_{t\geqslant 0}$, it is enough to check that

(a) $(X_t)_{t\geqslant 0}$ is a Feller process;[3]

[3] Actually, according to [52, Theorem 5.1], it is enough to show that $(X_t)_{t\geqslant 0}$ is a non-explosive (Borel) right process (see, e.g., [60, p.38] or [47, p.67] for a definition of a (Borel) right process). In view of [47, Corollary 4.1.4], the Feller property implies that $(X_t)_{t\geqslant 0}$ is a right process.

(b) all compact sets of the state space $\mathbb{R}_{\geqslant 0}$ are petite for some skeleton chain (see Definition B.3);

(c) there exist positive constants c, M such that

$$(\mathcal{A}V)(x) \leqslant -c + M\mathbb{1}_K(x), \quad x \in \mathbb{R}_{\geqslant 0}, \tag{2.3.11}$$

for some compact subset $K \subset \mathbb{R}_{\geqslant 0}$, where $V(x) = \log(1+x)$, $x \in \mathbb{R}_{\geqslant 0}$.

We proceed to prove (a)-(c).

In view of [17, Theorem 2.7], $(X_t)_{t\geqslant 0}$ possesses the Feller property as an affine process. This proves (a).

To prove (b), according to Proposition 2.6, we can proceed in the very same way as in Jin *et al.* [33, Theorem 1] to see that for each $n \in \mathbb{Z}_{\geqslant 0}$ the δ-skeleton chain $X_{n\delta}$, $\delta > 0$ being a constant, is irreducible with respect to the Lebesgue measure on $\mathbb{R}_{\geqslant 0}$. Indeed, let $A \in \mathcal{B}(\mathbb{R}_{\geqslant 0})$ and $\lambda(A) > 0$. Then it follows from the positivity of the density function of $X_{n\delta}$ that

$$\mathbb{P}_x\left(\tau_A < \infty\right) \geqslant \mathbf{P}^{n\delta}\left(x, A\right) = \int_A f_{X^x_{n\delta}}(y)\mathrm{d}y > 0$$

for all $x \in \mathbb{R}_{\geqslant 0}$ and $y \in \mathbb{R}_{\geqslant 0}$, where the stopping time τ_A is defined by $\tau_A := \inf\{n \geqslant 0 : X_n \in A\}$, since $f_{X^x_{n\delta}}(y) > 0$ for any $x \in \mathbb{R}_{\geqslant 0}$ and $y > 0$ as shown in Proposition 2.6. This implies that the chain $(X_{n\delta})_{n\in\mathbb{Z}_{\geqslant 0}}$ is irreducible with λ being an irreducibility measure. By statement (a), $(X_t)_{t\geqslant 0}$ possesses the Feller property. So the skeleton chain $(X_{n\delta})_{n\in\mathbb{Z}_{\geqslant 0}}$ has also the Feller property. The claim (b) now follows from [49, Proposition 6.2.8].

Finally, we prove (c). As shown in the proof of Lemma 2.17, $|\mathcal{A}V|$ is bounded on $\mathbb{R}_{\geqslant 0}$. Therefore, to get (2.3.11), it suffices to show that $\lim_{x\to\infty} \mathcal{A}V(x)$ exists and is negative. As before, we write $\mathcal{A}V = \mathcal{D}V + \mathcal{J}V$. It is easy to see that

$$\lim_{x\to\infty}(\mathcal{D}V)(x) = \lim_{x\to\infty}\left[(a-bx)(1+x)^{-1} - \frac{\sigma^2}{2}x(1+x)^{-2}\right] = -b.$$

Next, we consider the jump part $\mathcal{J}V$. Note that

$$V(x+z) - V(x) = \log\left(1 + \frac{z}{1+x}\right) \longrightarrow 0 \quad \text{as } x \to \infty.$$

On the other hand, by (2.3.4) and (2.3.5), we have

$$|V(x+z) - V(x)| \leqslant z\mathbb{1}_{\{z\leqslant 1\}} + \left[\log(2) + \log(z)\right]\mathbb{1}_{\{z>1\}},$$

where the function on the right-hand side is integrable with respect to ν. By the dominated convergence theorem, we obtain $\lim_{x\to\infty}(\mathcal{J}V)(x) = 0$. This completes the proof. □

Remark 2.19. *According to the discussion after* [13, Proposition 2.5]*, a direct but important consequence of our ergodic result is the following: under the assumptions of*

Theorem 2.18, for all Borel measurable functions $f : \mathbb{R}_{\geqslant 0} \to \mathbb{R}$ *with* $\int_{\mathbb{R}_{\geqslant 0}} |f(x)|\pi(\mathrm{d}x) < \infty$*, it holds*

$$\mathbb{P}\left(\lim_{T\to\infty} \frac{1}{T}\int_0^T f(X_s)\mathrm{d}s = \int_{\mathbb{R}_{\geqslant 0}} f(x)\pi(\mathrm{d}x)\right) = 1. \tag{2.3.12}$$

The convergence (2.3.12) *may be very useful for parameter estimation of the JCIR process.*

We end this section with a time-discrete version of the statement in Remark 2.19.

Proposition 2.20. *Under the assumptions of Theorem 2.18, for all Borel measurable functions* $f : \mathbb{R}_{\geqslant 0} \to \mathbb{R}$ *with* $\int_{\mathbb{R}_{\geqslant 0}} |f(x)|\pi(\mathrm{d}x) < \infty$*, it holds*

$$\mathbb{P}\left(\lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} f(X_i) = \int_{\mathbb{R}_{\geqslant 0}} f(x)\pi(\mathrm{d}x)\right) = 1. \tag{2.3.13}$$

Proof. We employ the continuous-time ergodicity of the JCIR process established in Theorem 2.18 and [53, Proposition 4.3, pp. 19-20] to get that the tail σ-field $\bigcap_{k\in\mathbb{Z}_{\geqslant 0}} \sigma(X_i : i \geqslant k)$ of the Markov chain $(X_i)_{i\in\mathbb{Z}_{\geqslant 0}}$ is trivial for any initial distribution, i.e., it consists only of events having probability zero or one for any initial distribution on $\mathbb{R}_{\geqslant 0}$. Now, the proof goes along the very same lines as in [8, Theorem 2.4] (see also the discussion after [13, Proposition 2.5]) without any substantial changes. □

2.4. Exponential ergodicity of the JCIR process

Our aim of this section is to show that the JCIR process X is exponentially ergodic if

$$\int_{\{z>1\}} z^\kappa \nu(dz) < \infty \quad \text{for some } \kappa > 0. \tag{2.4.1}$$

As in previous chapter on the two-factor affine model based on the α-root process (see Section 1.3) the following proposition will play an essential role in proving exponential ergodicity of the JCIR process X, provided that (2.4.1) holds.

Proposition 2.21. *Suppose* (2.4.1) *is true. Let* $V \in C^2(\mathbb{R}_{\geqslant 0}, \mathbb{R})$ *be nonnegative and such that* $V(x) = x^{\kappa\wedge 1}$ *for* $x \geqslant 1$*. Then there exist positive constants* c, M *such that*

$$\mathbb{E}_x\left[V(X_t)\right] \leqslant e^{-ct}V(x) + \frac{M}{c} \tag{2.4.2}$$

for all $(t, x) \in \mathbb{R}^2_{\geqslant 0}$.

Proof. If $\kappa \geqslant 1$, then it follows from (2.4.1) that $\int_{\{z>1\}} z\nu(dz) < \infty$, which, together with (2.2.23), implies

$$\mathbb{E}_x\left[X_t\right] \leqslant xe^{-bt} + M_1, \quad t > 0, x \geqslant 0,$$

for some constant $0 < M_1 < \infty$. In this case, we have

$$\begin{aligned}\mathbb{E}_x\left[V(X_t)\right] &= \mathbb{E}_x\left[V(X_t)\mathbb{1}_{\{X_t>1\}}\right] + \mathbb{E}_x\left[V(X_t)\mathbb{1}_{\{X_t\leqslant 1\}}\right]\\ &\leqslant \mathbb{E}_x\left[X_t\right] + \sup_{y\in[0,1]} |V(y)|\end{aligned}$$

$$\begin{aligned}&\leqslant xe^{-bt} + M_1 + \sup_{y\in[0,1]} |V(y)| \\ &\leqslant (V(x)+1)\, e^{-bt} + M_1 + \sup_{y\in[0,1]} |V(y)| \\ &\leqslant V(x)e^{-bt} + M_2,\end{aligned}$$

where $M_2 := 1 + M_1 + \sup_{y\in[0,1]} |V(y)| < \infty$ is a constant. Hence (2.4.2) is true when $\kappa \geqslant 1$. So in the following we assume $0 < \kappa < 1$.

Define $g(t,x) := \exp(ct)V(x)$, where $c \in \mathbb{R}_{>0}$ is a constant to be determined later. Then,

$$\begin{aligned}g'_t(t,x) &:= \frac{\partial}{\partial t} g(t,x) = ce^{ct}V(x), \\ g'_x(t,x) &:= \frac{\partial}{\partial x} g(t,x) = \begin{cases} \kappa e^{ct} x^{\kappa-1}, & x > 1, \\ e^{ct}V'(x), & x \in [0,1], \end{cases} \\ g''_x(t,x) &:= \frac{\partial^2}{\partial x^2} g(t,x) = \begin{cases} \kappa(\kappa-1)e^{ct} x^{\kappa-2}, & x > 1, \\ e^{ct}V''(x), & x \in [0,1]. \end{cases}\end{aligned}$$

Applying Itô's formula for $g(t, X_t)$, we obtain

$$g(t,X_t) - g(0,X_0) = \int_0^t (\mathcal{L}g)(s,X_s)\mathrm{d}s + \int_0^t g'_s(s,X_s)\mathrm{d}s + M_t(g), \quad t \geqslant 0, \tag{2.4.3}$$

where the operator $\mathcal{L}$ is given by $(\mathcal{L}g)(s,X_s) = \exp\{cs\}(\mathcal{A}V)(X_s)$ with $\mathcal{A}$ as in (2.1.1) and

$$\begin{aligned}M_t(g) &:= \sigma \int_0^t \sqrt{X_s} g'_x(s,X_s)\mathrm{d}B_s + \int_0^t \int_0^\infty \left(g(s,X_{s-}+z) - g(s,X_{s-})\right) \widetilde{N}(\mathrm{d}s,\mathrm{d}z) \\ &= D_t(g) + J_t(g), \quad \text{for all } t \geqslant 0.\end{aligned}$$

We will complete the proof in three steps.

"*Step 1*": We check that $(M_t(g))_{t\geqslant 0}$ is a martingale with respect to the filtration $(\mathcal{F}_t)_{t\geqslant 0}$. First, note that

$$D_t(g) := \sigma \int_0^t \frac{\partial}{\partial x} g(s,X_s)\sqrt{X_s}\mathrm{d}B_s, \quad t \geqslant 0,$$

is a square-integrable martingale with respect to the filtration $(\mathcal{F}_t)_{t\geqslant 0}$. Indeed, for each $t \geqslant 0$, we have

$$\begin{aligned}&\mathbb{E}_x\left[\left(\sigma \int_0^t \sqrt{X_s} g'_x(s,X_s)\mathrm{d}B_s\right)^2\right] \\ &\quad = \sigma^2 \int_0^t e^{2cs}\mathbb{E}\left[\mathbb{1}_{\{X_s\leqslant 1\}} X_s V'(X_s)\right]\mathrm{d}s + \sigma^2\kappa^2 \int_0^t e^{2cs}\mathbb{E}\left[\mathbb{1}_{\{X_s>1\}} X_s^{2\kappa-1}\right]\mathrm{d}s.\end{aligned} \tag{2.4.4}$$

Clearly, we have $|\mathbb{1}_{\{X_s\leqslant 1\}} X_s V'(X_s)| \leqslant \sup_{y\in[0,1]} |V'(y)| < \infty$, which implies that the first integral on the right-hand side of (2.4.4) is finite. Since $|\mathbb{1}_{\{X_s>1\}} X_s^{2\kappa-1}| \leqslant |X_s|^\kappa$, by (2.4.1) and Proposition 2.11, we see that the second integral on the right-hand side of (2.4.4) is finite as well. Hence, $(D_t(g))_{t\geqslant 0}$ is a square-integrable martingale with respect to the filtration $(\mathcal{F}_t)_{t\geqslant 0}$.

Next, we prove that $J_t(g)$, $t \geqslant 0$, is a martingale with respect to the filtration $(\mathcal{F}_t)_{t\geqslant 0}$. We define

$$J_{*,t}(V) := \int_0^t \int_{\{z\leqslant 1\}} e^{cs} \left(V(X_{s-}+z) - V(X_{s-})\right) \widetilde{N}(\mathrm{d}s, \mathrm{d}z), \quad t \geqslant 0,$$
$$J_t^*(V) := \int_0^t \int_{\{z>1\}} e^{cs} \left(V(X_{s-}+z) - V(X_{s-})\right) \widetilde{N}(\mathrm{d}s, \mathrm{d}z), \quad t \geqslant 0.$$

So $J_t(g) = J_{*,t}(V) + J_t^*(V)$ for $t \geqslant 0$. In what follows, we establish some elementary inequalities for V. For $y \geqslant 1$, we have

$$\begin{aligned} \mathbb{1}_{\{z\leqslant 1\}}(z)|V(y+z) - V(y)| &= \mathbb{1}_{\{z\leqslant 1\}}(z)\left((y+z)^\kappa - y^\kappa\right) \\ &= \mathbb{1}_{\{z\leqslant 1\}}(z) y^\kappa \left(\left(1+\frac{z}{y}\right)^\kappa - 1\right) \\ &\leqslant \mathbb{1}_{\{z\leqslant 1\}}(z)\kappa y^{\kappa-1} z \leqslant \mathbb{1}_{\{z\leqslant 1\}}(z) z, \end{aligned} \tag{2.4.5}$$

where we used Bernoulli's inequality to obtain the first inequality in (2.4.5). Moreover, it is easy to see that for $y \geqslant 1$,

$$\mathbb{1}_{\{z>1\}}(z)|V(y+z) - V(y)| \leqslant \mathbb{1}_{\{z>1\}}(z)\left(y^\kappa + z^\kappa - y^\kappa\right) \leqslant \mathbb{1}_{\{z>1\}}(z) z^\kappa. \tag{2.4.6}$$

For $y \in [0,1]$, using the mean value theorem, we get

$$\mathbb{1}_{\{z\leqslant 1\}}(z)|V(y+z) - V(y)| \leqslant z \sup_{y\in[0,2]} |V'(y)| \leqslant c_1 z, \tag{2.4.7}$$

for some constant $c_1 > 0$. Finally, for $y \in [0,1]$, again by Bernoulli's inequality, we have

$$\begin{aligned} \mathbb{1}_{\{z>1\}}(z)|V(y+z) - V(y)| &\leqslant \mathbb{1}_{\{z>1\}}(z)\left((y+z)^\kappa + |V(y)|\right) \\ &\leqslant \mathbb{1}_{\{z>1\}}(z)\left(z^\kappa\left(1+\kappa\frac{y}{z}\right) + |V(y)|\right) \\ &\leqslant \mathbb{1}_{\{z>1\}}(z)\left(z^\kappa + 1 + |V(y)|\right) \\ &\leqslant \mathbb{1}_{\{z>1\}}(z)\left(z^\kappa + c_2\right), \end{aligned} \tag{2.4.8}$$

where $c_2 := 1 + \sup_{y\in[0,1]} |V(y)| < \infty$ is a positive constant. Now, from (2.4.5) and (2.4.7), we deduce that

$$\begin{aligned} &\mathbb{E}_x\left[\int_0^t \int_{\{z\leqslant 1\}} e^{cs}|V(X_{s-}+z) - V(X_{s-})|\nu(\mathrm{d}z)\mathrm{d}s\right] \\ &\qquad \leqslant (1+c_1)\int_0^t e^{cs}\mathrm{d}s \int_{\{z\leqslant 1\}} z\nu(\mathrm{d}z) < \infty, \quad t \geqslant 0. \end{aligned}$$

It follows from [26, p.62 and Lemma 3.1] that $(J_{*,t}(V))_{t\geqslant 0}$ is a martingale with respect to the filtration $(\mathcal{F}_t)_{t\geqslant 0}$. Using (2.4.6) and (2.4.8), we obtain

$$\begin{aligned} &\mathbb{E}_x\left[\int_0^t \int_{\{z>1\}} e^{cs}|V(X_{s-}+z) - V(X_{s-})|\nu(\mathrm{d}z)\mathrm{d}s\right] \\ &\quad \leqslant \int_0^t \int_{\{z>1\}} e^{cs}\left(z^\kappa + c_2\right)\nu(\mathrm{d}s)\mathrm{d}s \end{aligned}$$

$$= \int_0^t e^{cs} \mathrm{d}s \left(\int_{\{z>1\}} z^\kappa \nu(\mathrm{d}z) + c_2 \nu(\{z>1\}) \right) < \infty, \quad t \geqslant 0.$$

As a consequence, we see that $(J_t^*(V))_{t\geqslant 0}$ is also a martingale. Clearly, $(M_t(g))_{t\geqslant 0} = (D_t(g) + J_t(g))_{t\geqslant 0}$ is now a martingale with respect to the filtration $(\mathcal{F}_t)_{t\geqslant 0}$.

"*Step 2*": We determine the constant $c \in \mathbb{R}_{>0}$ and find another positive constant $M < \infty$ such that

$$(\mathcal{A}V)(y) = (\mathcal{D}V)(y) + (\mathcal{J}V)(y) \leqslant -cV(y) + M, \quad y \in \mathbb{R}_{\geqslant 0}. \tag{2.4.9}$$

Consider the jump part $\mathcal{J}V = \mathcal{J}_*V + \mathcal{J}^*V$, where $\mathcal{J}_*V$ and $\mathcal{J}^*V$ are defined by (2.3.7) and (2.3.8), respectively. For all $x \in \mathbb{R}_{\geqslant 0}$, using (2.4.5) and (2.4.7), we obtain

$$(\mathcal{J}_*V)(y) = \int_{\{z\leqslant 1\}} |V(y+z) - V(y)| \nu(\mathrm{d}z) \leqslant (1+c_1) \int_{\{z\leqslant 1\}} z \nu(\mathrm{d}z) < \infty.$$

For $\mathcal{J}^*V$, we can use (2.4.6) and (2.4.8) to obtain that for all $y \in \mathbb{R}_{\geqslant 0}$,

$$\begin{aligned}(\mathcal{J}^*V)(y) &= \int_{\{z>1\}} |V(y+z) - V(y)| \nu(\mathrm{d}z) \\ &\leqslant \int_{\{z>1\}} z^\kappa \nu(\mathrm{d}z) + c_2 \nu(\{z>1\}) < \infty.\end{aligned}$$

Next, we estimate $\mathcal{D}V$. Since,

$$V'(x) = \kappa x^{\kappa-1} \quad \text{and} \quad V''(x) = \kappa(\kappa-1)x^{\kappa-2} \quad \text{for } x \geqslant 1,$$

we see that

$$(\mathcal{D}V)(x) = (a - bx)V'(x) + \frac{\sigma^2 x}{2} V''(x) \tag{2.4.10}$$

$$= -b\kappa x^\kappa + \kappa x^{\kappa-1} \left(a + \frac{\sigma^2(\kappa-1)}{2} \right) \leqslant -b\kappa x^\kappa + c_3 \tag{2.4.11}$$

for all $x \geqslant 1$. Here $c_3 < \infty$ is a positive constant. After all we get that for all $x \geqslant 1$,

$$(\mathcal{A}V)(x) \leqslant -b\kappa V(x) + c_4$$

where $c_4 < \infty$ is a positive constant. By noting that $V \in C^2(\mathbb{R}_{\geqslant 0}, \mathbb{R})$, we see that

$$\sup_{y\in[0,1]} |V(y)| < \infty \quad \text{and} \quad \sup_{y\in[0,1]} |(\mathcal{A}V)(y)| < \infty.$$

Consequently, (2.4.9) holds for all $x \geqslant 0$.

"*Step 3*": We prove (2.4.2). Note that $(\mathcal{L}g)(s,x) = \exp\{cs\}(\mathcal{A}V)(x)$. By (2.4.3), (2.4.9) and the martingale property of $(M_t(g))_{t\geqslant 0}$, we obtain that for all $(x,t) \in \mathbb{R}^2_{\geqslant 0}$,

$$\begin{aligned}e^{ct}\mathbb{E}_x[V(X_t)] - V(x) &= \mathbb{E}_x[g(t,X_t) - g(0,X_0)] \\ &= \mathbb{E}_x\left[\int_0^t \left(e^{cs}(\mathcal{A}V)(X_s) + ce^{cs}V(X_s)\right) \mathrm{d}s\right] \\ &\leqslant \mathbb{E}_x\left[\int_0^t \left(e^{cs}\left(-cV(X_s) + M\right) + ce^{cs}V(X_s)\right) \mathrm{d}s\right] \\ &= \mathbb{E}_x\left[\int_0^t e^{cs} M \mathrm{d}s\right] \leqslant \frac{M}{c} e^{ct}.\end{aligned}$$

So (2.4.2) is true. With this our proof is complete. □

Based on Proposition 2.21, we are now ready to prove the exponential ergodicity.

Theorem 2.22. *Consider the JCIR process* $(X_t)_{t\geqslant 0}$ *defined by* (2.0.1) *with parameters* a, b, σ *and* ν*, where* ν *is the Lévy measure of* $(J_t)_{t\geqslant 0}$*. Assume* $a > 0$*. If* (2.4.1) *is true, then* X *is exponentially ergodic, i.e., there exist constants* $\delta \in \mathbb{R}_{>0}$ *and* $B \in \mathbb{R}_{>0}$ *such that*

$$\left\|\mathbf{P}^t(x,\cdot) - \pi\right\|_{TV} \leqslant B\,(V(x)+1)\,e^{-\delta t}$$

for all $t \geqslant 0$ *and* $x \in \mathbb{R}_{\geqslant 0}$*.*

Proof. In view of Proposition 2.6 and Proposition 2.21, to obtain the exponential ergodicity of X, we can follow almost the very same lines as in the proof of [33, Theorem 1]. We remark that the *strong aperiodicity* condition used in the proof of [33, Theorem 1] can be safely replaced by the *aperiodicity* condition, due to [50, Theorem 6.3]. The details are as follows:

We first consider the skeleton chain $(X_n)_{n\in\mathbb{Z}_{\geqslant 0}}$, which is a Markov chain on the state space $\mathbb{R}_{\geqslant 0}$ with transition kernel $\mathbf{P}^n(x,\cdot)$. It is easy to see that the measure π is also an invariant probability measure for the chain $(X_n)_{n\in\mathbb{Z}_{\geqslant 0}}$.

Let the function $V(x)$ be the Foster-Lyapunov function introduced in Proposition 2.21. The Markov property together with Proposition 2.21 implies that

$$\mathbb{E}[V(X_{n+1})\,|\,X_0, X_1, \ldots, X_n] = \int_{\mathbb{R}_{\geqslant 0}} V(x)\mathbf{P}^1(X_n, \mathrm{d}x) \leqslant e^{-c}V(X_n) + \frac{M}{c},$$

where c and M are the positive constants in Proposition 2.21. If we set $V_0 := V$ and $V_n := V(X_n)$, $n \in \mathbb{N}$, then

$$\mathbb{E}[V_1] \leqslant e^{-c}V_0(X_0) + \frac{M}{c}$$

and, for all $n \in \mathbb{N}$,

$$\mathbb{E}\left[V_{n+1}\,|\,X_0, X_1, \ldots, X_n\right] \leqslant e^{-c}V_n + \frac{M}{c}.$$

In order to apply [50, Theorem 6.3] for the chain $(X_n)_{n\in\mathbb{Z}_{\geqslant 0}}$, it remains to verify the following conditions:

(a) the Lebesgue measure λ on $\mathbb{R}_{\geqslant 0}$ is an irreducibility measure for the chain $(X_n)_{n\in\mathbb{Z}_{\geqslant 0}}$;

(b) the chain $(X_n)_{n\in\mathbb{Z}_{\geqslant 0}}$ is aperiodic;

(c) all compact sets of the state space $\mathbb{R}_{\geqslant 0}$ are petite.

By what we have already proved in part (b) of the proof of Theorem 2.18, with $\delta = 1$, we obtain conditions (a) and (c).

To prove (b), i.e., the aperiodicity of the skeleton chain $(X_n)_{n\in\mathbb{Z}_{\geqslant 0}}$, we proceed as in the proof of 1.22 using a contradiction argument. Suppose that the period l of the chain $(X_n)_{n\in\mathbb{Z}_{\geqslant 0}}$ is greater than 1. Then we can find disjoint Borel sets $A_1, A_2, \cdots, A_l$ such that

$$\lambda(A_i) > 0,\ i = 1, \cdots, l, \quad \cup_{i=1}^{l} A_i = \mathbb{R}_{\geqslant 0}, \tag{2.4.12}$$

$$\mathbf{P}^1(x_0, A_{i+1}) = 1 \tag{2.4.13}$$

for all $x_0 \in A_i,\ i = 1, \cdots, l-1$, and

$$\mathbf{P}^1(x_0, A_1) = 1$$

for all $x_0 \in A_l$. By (2.4.13), we have

$$\int_{(A_2)^c} f_{X_1^{x_0}}(x)\mathrm{d}x = 0, \quad x_0 \in A_1,$$

and further

$$\int_{A_1} f_{X_1^{x_0}}(x)\mathrm{d}x = 0, \quad x_0 \in A_1.$$

However, since for each $x_0 \in \mathbb{R}_{\geqslant 0}$, the density $f_{X_1^{x_0}}(x)$ is strictly positive for almost all $x \in \mathbb{R}_{\geqslant 0}$, we must have $\lambda(A_1) = 0$, which contradicts (2.4.12). Therefore, the assumption that $l \geqslant 2$ is not true. So we have $l = 1$.

Now, we can apply [50, Theorem 6.3] and thus find constants $\delta,\ B \in (0, \infty)$ such that

$$\|\mathbf{P}^n(x, \cdot) - \pi\|_{TV} \leqslant B\,(V(x) + 1)\, e^{-\delta n} \tag{2.4.14}$$

for all $n \in \mathbb{Z}_{\geqslant 0}$, $x \in \mathbb{R}_{\geqslant 0}$. For the remainder of the proof, i.e., to extend the inequality (2.4.14) to all $t \geqslant 0$, we can interpolate in the same way as in [52, p.536]. This completes the proof. □

Remark 2.23. *We remark that similar results on the ergodicity of Ornstein-Uhlenbeck type processes were derived by Masuda, see* [48, Theorem 2.6]*. It is also worth mentioning that Jin et al.* [33] *already found a sufficient condition for the exponential ergodicity of the JCIR process, namely, if* $a > 0$, $\int_{\{z\leqslant 1\}} z\log(1/z)\nu(\mathrm{d}z) < \infty$ *and* $\int_{\{z>1\}} z\nu(\mathrm{d}z) < \infty$*. It is seen from our Theorem 2.22 that these conditions can be significantly relaxed.*

2.5. Convergence of moments for the JCIR process

In Corollary 2.14 and Proposition 2.15 we calculated the first and second moments of π explicitly. In this section we study the existence of moments for the unique stationary distribution π of the JCIR process.

Lemma 2.24. *Suppose* $\int_{\{z>1\}} z^{2\kappa-1}\nu(\mathrm{d}z) < \infty$ *for some* $\kappa > 1$*. Then, there exist constants* $c \in \mathbb{R}_{>0}$ *and* $M < \infty$ *such that*

$$\mathbb{E}_x\left[X_t^\kappa\right] \leqslant e^{-ct}x^\kappa + \frac{M}{c} \tag{2.5.1}$$

for all $(t, x) \in \mathbb{R}^2_{\geqslant 0}$.

Proof. We mimic the proof of Proposition 2.21 with appropriate adjustments in the estimates. In particular, let $f(x) = x^\kappa$ and define $g(t, x) := \exp\{ct\}f(x)$, where $c \in \mathbb{R}_{>0}$ is a constant to be determined later. Using Itô's formula for $g(t, X_t)$, we get

$$g(t, X_t) - g(0, X_0) = \int_0^t e^{cs}\left(\mathcal{A}f\right)(X_s)\mathrm{d}s + \int_0^t \frac{\partial}{\partial s} g(s, X_s)\mathrm{d}s + M_t(g), \quad t \geqslant 0,$$

where $(\mathcal{A}f)(X_s)$ is the infinitesimal generator given in (2.1.1), and, for all $t \geqslant 0$,

$$M_t(g) = \sigma \int_0^t g'(s, X_s)\sqrt{X_s}\mathrm{d}B_s + \int_0^t \int_0^\infty \left(g(s, X_{s-} + z) - g(s, X_{s-})\right)\widetilde{N}(\mathrm{d}s, \mathrm{d}z)$$

is a martingale with respect to the filtration $(\mathcal{F}_t)_{t\geqslant 0}$. Indeed, the first stochastic integral on the right-hand side, $\sigma\int_0^t g'(s,X_s))\sqrt{X_s}\mathrm{d}B_s$, $t\geqslant 0$, is a square integrable martingale, since

$$\sigma^2\mathbb{E}\left[\int_0^t X_s g'(s,X_s)^2\mathrm{d}s\right] = \sigma^2\kappa^2\int_0^t e^{2cs}\mathbb{E}\left[X_s^{2\kappa-1}\right]\mathrm{d}s < \infty, \quad t\geqslant 0,$$

where we used that $\sup_{s\in[0,t]}\mathbb{E}_x[X_s^{2\kappa-1}] < \infty$ by Corollary 2.11 to get finiteness. We introduce the following elementary inequality,

$$(x+z)^\kappa \leqslant x^\kappa + \kappa z(x+z)^{\kappa-1} \leqslant x^\kappa + 2^{\kappa-1}\kappa z\left(x^{\kappa-1}+z^{\kappa-1}\right), \tag{2.5.2}$$

which is satisfied for all $x, z\in\mathbb{R}_{\geqslant 0}$ and $\kappa\geqslant 1$. Therefore,

$$\begin{aligned}&\mathbb{E}\left[\int_0^t\int_0^\infty e^{cs}\left|f(X_{s-}+z)-f(X_{s-})\right|\nu(\mathrm{d}z)\mathrm{d}s\right]\\ &\qquad\leqslant 2^{\kappa-1}\kappa\int_0^t\int_0^\infty e^{cs}\left(\mathbb{E}\left[X_s^{\kappa-1}\right]z+z^\kappa\right)\nu(\mathrm{d}z)\mathrm{d}s<\infty, \quad t\geqslant 0,\end{aligned}$$

where the finiteness follows, since $\sup_{s\in[0,t]}\mathbb{E}_x[X_s^{\kappa-1}] < \infty$ by Corollary 2.11 and $\int_0^\infty(z\vee z^\kappa)\nu(\mathrm{d}z)<\infty$ can be obtained by assumption. It follows from [26, Lemma 3.1 and p. 62] that $(M_t(g))_{t\geqslant 0}$ is a martingale with respect to the filtration $(\mathcal{F}_t)_{t\geqslant 0}$.

Further, by (2.1.1) and (2.5.2), we obtain

$$\begin{aligned}(\mathcal{A}f)(x) &= (a-bx)f'(x)+\frac{\sigma^2 x}{2}f''(x)+\int_0^\infty\left(f(x+z)-f(x)\right)\nu(\mathrm{d}z)\\ &\leqslant -b\kappa x^\kappa+\kappa x^{\kappa-1}\left(a+\frac{\sigma^2(\kappa-1)}{2}\right)+2^{\kappa-1}\kappa\int_0^\infty\left(x^{\kappa-1}z+z^\kappa\right)\nu(\mathrm{d}z)\\ &\leqslant -b\kappa x^\kappa + c_1x^{\kappa-1}+2^{\kappa-1}\kappa\left(\int_0^\infty z^\kappa\nu(\mathrm{d}z)+x^{\kappa-1}\int_0^\infty z\nu(\mathrm{d}z)\right)\\ &\leqslant -b\kappa x^\kappa+c_2x^{\kappa-1}+c_3\end{aligned}$$

for all $x\geqslant 1$, with some positive constants c_1, c_2, and c_3 (similar to (2.4.11)). We conclude that for all $x\geqslant 1$,

$$(\mathcal{A}f)(x)\leqslant -b\kappa f(x)+c_4,$$

where $c_4<\infty$ is a positive constant. Notice that $f\in C^\infty(\mathbb{R}_{\geqslant 0},\mathbb{R})$, we get that

$$\sup_{x\in[0,1]}|f(x)|<\infty \quad\text{and}\quad \sup_{x\in[0,1]}\left|\left(\mathcal{A}f\right)(x)\right|<\infty.$$

It follows that there exist constants $c\in\mathbb{R}_{>0}$ and $M<\infty$ such that

$$\left(\mathcal{A}f\right)(x)\leqslant -cf(x)+M$$

holds for all $x\in\mathbb{R}_{\geqslant 0}$.

Finally, the asserted inequality (2.5.1) follows in the very same way as shown in step three of the proof of Proposition 2.21. □

Theorem 2.25. *Consider the JCIR process $(X_t)_{t\geqslant 0}$ with parameters a, b, $\sigma\in\mathbb{R}_{>0}$. Let $\kappa>0$ be a constant. Then the following two statements hold:*

(a) Suppose $\int_1^\infty z^\kappa \nu(\mathrm{d}z) < \infty$ *and* $\mathbb{E}[X_0^\kappa] < \infty$. *Then*

$$\sup_{t \in \mathbb{R}_{\geqslant 0}} \mathbb{E}\left[X_t^\kappa\right] < \infty.$$

Consequently, $\int_0^\infty x^\kappa \pi(\mathrm{d}x) < \infty$.

(b) Suppose $\int_1^\infty z^{\varepsilon+\kappa} \nu(\mathrm{d}z) < \infty$ *and* $\mathbb{E}[X_0^{\varepsilon+\kappa}] < \infty$ *for* $\varepsilon > 0$. *Then*

$$\lim_{t \to \infty} \mathbb{E}\left[X_t^\kappa\right] = \int_0^\infty x^\kappa \pi(\mathrm{d}x).$$

Proof. We start to prove finiteness of $\sup_{t \in \mathbb{R}_{\geqslant 0}} \mathbb{E}[X_t^\kappa]$. If $0 < \kappa \leqslant 1$ there is nothing to do, since then the statement follows immediately by an application of Proposition 2.21 together with the law of total expectation. Indeed, we have that

$$\begin{aligned}
\sup_{t \geqslant 0} \mathbb{E}\left[(X_t)^\kappa\right] &= \sup_{t \geqslant 0} \int_0^\infty \mathbb{E}\left[(X_t)^\kappa \mid X_0 = x\right] \mu_{X_0}(\mathrm{d}x) \\
&\leqslant \sup_{t \geqslant 0} \int_0^\infty \left(e^{-ct} x^\kappa + \frac{M}{c}\right) \mu_{X_0}(\mathrm{d}x) \\
&= \sup_{t \geqslant 0} e^{-ct} \mathbb{E}\left[(X_0)^\kappa\right] + \frac{M}{c} < \infty. \qquad (2.5.3)
\end{aligned}$$

Now, let $\kappa > 1$. The idea to achieve the asserted is very little different from what can already be found in Corollary 2.11. In particular, recall that $(Z_t^i)_{t \geqslant 0}$ is defined as the unique strong solution of the SDE (2.2.10). Similarly, for $i = 1, 2$, we define $(X_t^i)_{t \geqslant 0}$ as the unique strong solution of

$$\mathrm{d}X_t^i = (a - bX_t^i)\mathrm{d}z + \sigma \sqrt{X_t^i} \mathrm{d}B_t + \mathrm{d}J_t^i, \quad t \geqslant 0, \quad X_0 \in \mathbb{R}_{\geqslant 0} \text{ a.s.},$$

where $(J_t^i)_{t \geqslant 0}$ is a subordinator of pure jump-type with Lévy measure ν_i defined by $\nu_1(\mathrm{d}z) := \mathbb{1}_{\{z \leqslant 1\}} \nu(\mathrm{d}z)$ and $\nu_2(\mathrm{d}z) := \mathbb{1}_{\{z > 1\}} \nu(\mathrm{d}z)$, respectively. So, $\mu_{X_t} = \mu_{X_t^1} * \mu_{Z_t^2}$. It follows that

$$\begin{aligned}
\mathbb{E}\left[X_t^\kappa\right] &= \int_0^\infty \int_0^\infty (x+y)^\kappa \mu_{X_t^1}(\mathrm{d}x) \mu_{Z_t^2}(\mathrm{d}y) \\
&\leqslant 2^{\kappa-1} \left(\int_0^\infty x^\kappa \mu_{X_t^1}(\mathrm{d}x) + \int_0^\infty y^\kappa \mu_{Z_t^2}(\mathrm{d}y) \right) \\
&= 2^{\kappa-1} \left(\mathbb{E}\left[\left(X_t^1\right)^\kappa\right] + \mathbb{E}\left[\left(Z_t^2\right)^\kappa\right] \right).
\end{aligned}$$

Hence, it is enough to prove that

$$\sup_{t \in \mathbb{R}_{\geqslant 0}} \mathbb{E}\left[\left(X_t^1\right)^\kappa\right] < \infty \quad \text{and} \quad \sup_{t \in \mathbb{R}_{\geqslant 0}} \mathbb{E}\left[\left(Z_t^2\right)^\kappa\right] < \infty.$$

Finiteness of $\sup_{t \in \mathbb{R}_{\geqslant 0}} \mathbb{E}\left[\left(X_t^1\right)^\kappa\right]$, follows by an application of Proposition 2.24, since $(J_t^1)_{t \geqslant 0}$ has only small jumps, together with the law of total expectation analogous to (2.5.3). We next show that $\sup_{t \in \mathbb{R}_{\geqslant 0}} \mathbb{E}[(Z_t^2)^\kappa] < \infty$. Proceeding as in the proof of Corollary 2.11 (see (2.2.22)), we obtain

$$\sup_{t \geqslant 0} \mathbb{E}\left[\left(Z_t^2\right)^\kappa\right] \leqslant \exp\left\{ c_1 \int_0^\infty \int_{\{z > 1\}} \left(\int_{\mathbb{R}_{\geqslant 0}} y^\kappa m_{\alpha(z,s),\beta(z,s)}(\mathrm{d}y) \right) \nu(\mathrm{d}z)\mathrm{d}s \right\}, \qquad (2.5.4)$$

with some constant $c_1 > 0$. Hence, it suffices to check finiteness of the term on the right-hand side of (2.5.4). Before proceeding, we first note the following simple observation. Namely, the method of proof of Lemma 2.7 part (i) can be easily adapted such that

$$\int_{\mathbb{R}_{\geqslant 0}} x^\kappa m_{\alpha,\beta}(\mathrm{d}x) \leqslant \begin{cases} \frac{\alpha^\kappa}{\beta^\kappa}, & \text{if } 0 < \kappa \leqslant 1, \\ C_1 \frac{\alpha+\alpha^\kappa}{\beta^\kappa}, & \text{if } \kappa > 1, \end{cases} \tag{2.5.5}$$

holds true with some constant $C_1 := C_1(\kappa) > 0$ and for all $\alpha > 0$ and $\beta > 0$. Indeed, if $0 < \kappa \leqslant 1$, the statement follows immediately from (2.2.3). If $\kappa > 1$, in the proof of Lemma 2.7, we can safely replace the estimate (2.2.5) by the following estimate

$$\begin{aligned}\int_{\mathbb{R}_{\geqslant 0}} x^n m_{\alpha,\beta}(\mathrm{d}x) &\leqslant c_1 \frac{e^{-\alpha}}{\beta^n}\left(\alpha + \alpha^2 + \cdots + \alpha^{n-1} + \alpha^n \sum_{m=0}^{\infty} \frac{\alpha^m}{m!}\right) \\ &\leqslant c_2 \left(\frac{\alpha}{\beta^n} + \frac{\alpha^n}{\beta^n}\right), \quad \text{for all } \alpha, \beta > 0.\end{aligned}$$

All the remaining arguments in the proof of Lemma 2.7 work throughout with the appropriate adjustments. Recall that $\alpha(z,s)$ and $\beta(z,s)$ are given by

$$\alpha(z,s) := \frac{2bz}{\sigma^2\left(e^{bs}-1\right)} \quad \text{and} \quad \beta(z,s) := \frac{2be^{bs}}{\sigma^2\left(e^{bs}-1\right)}.$$

Now, for $\kappa > 1$ the preceding observation (2.5.5) leads to

$$\begin{aligned}\int_{\mathbb{R}_{\geqslant 0}} y^\kappa m_{\alpha(z,s),\beta(z,s)}(\mathrm{d}y) &\leqslant C_1 \frac{\alpha(z,s) + \alpha(z,s)^\kappa}{\beta(z,s)^\kappa} \\ &= C_1 z \sigma^{2\kappa-2}(2b)^{1-\kappa} e^{-\kappa bs}(e^{bs}-1)^{\kappa-1} + C_1 e^{-\kappa bs} z^\kappa,\end{aligned}$$

yielding finiteness of the term on the right-hand side of (2.5.4).

Finally, by Remark 2.16 and the continuous mapping theorem, X_t^κ converges in distribution to a random variable X_∞^κ, say, which is distributed according to π, i.e., $\mathbb{E}[X_\infty^\kappa] = \int_0^\infty x^\kappa \pi(\mathrm{d}x)$. Now, statement (a) is a consequence of the moment convergence theorem, e.g., [63, Lemma 2.2.1 formula (2.2.2)], because

$$\int_0^\infty x^\kappa \pi(\mathrm{d}x) \leqslant \liminf_{t\to\infty} \int_0^\infty x^\kappa \mu_{X_t}(\mathrm{d}x) < \infty.$$

To prove (b), note that the assumption $\int_{\{z>1\}} z^{\varepsilon+\kappa}\nu(\mathrm{d}z) < \infty$ for a constant $\kappa > 0$ and $\varepsilon > 0$ together with statement (a) ensures that

$$\sup_{t\in\mathbb{R}_{\geqslant 0}} \int_0^\infty x^{\varepsilon+\kappa}\mu_{X_t}(\mathrm{d}x) < \infty \quad \text{for some } \kappa \in (0,\infty).$$

For this reason and the preceding argument, we are allowed to apply the moment convergence theorem [63, Lemma 2.2.1] from which claim (b) of the theorem follows. □

3. Parameter estimation of the jump-diffusion CIR process

In this chapter we study the asymptotic properties of CLSE for the drift parameters (a, b) of the JCIR process based on discrete time observations $(X_i)_{i\in\mathbb{N}}$ only in the subcritical case, i.e., $b \in (0, \infty)$ is assumed. We will constantly suppose that $\sigma \in (0, \infty)$ and the Lévy measure ν are known.

Remark 3.1. *We remark that we do not estimate the parameter σ, since it could be determined rather than estimated using an arbitrarily short continuous time observation $(X_t)_{t\in[0,T]}$ of X, where $T > 0$, see, e.g.,* [2, Remark 2.6]. *At least, it will turn out that for the calculation of the CLSEs for the drift (a, b), one does not need to know the value of the volatility parameter σ.*

Since the Lévy measure of the driving noise $(J_t)_{t\geqslant 0}$ is an infinite dimensional object, estimation of it can be done using different methods. Nevertheless, based on low frequency observations Xu [65] *proposed some nonparametric estimators for ν, given that ν is absolutely continuous with respect to the Lebesgue measure.*

Since we will deal with moments of higher order, throughout this chapter we assume that

$$\int_{\{z>1\}} z^{4+\delta} \nu(\mathrm{d}z) < \infty \tag{3.0.1}$$

and $\mathbb{E}[X_0^{4+\delta}] < \infty$ for a constant $\delta > 0$ sufficiently small. Recall that condition (3.0.1) together with $\mathbb{E}[X_0^{4+\delta}] < \infty$ yield $\mathbb{E}[X_t^{\kappa}] < \infty$ for any $\kappa \in (0, 4 + \delta)$ by Theorem 2.9 combined with the law of total expectation.

We start with the computation of the CLSEs. Using (2.2.24), for all $i \in \mathbb{N}$,

$$\mathbb{E}\left[X_i \mid \mathcal{F}_{i-1}\right] = e^{-b} X_{i-1} + \frac{1 - e^{-b}}{b} \left(a + \int_0^\infty z\nu(\mathrm{d}z)\right).$$

Further, using that $\sigma(X_1, \ldots, X_{i-1}) \subseteq \mathcal{F}_{i-1}$, $i \in \mathbb{N}$, by the tower rule of conditional expectation, we obtain the first conditional moment with respect to $\sigma(X_1, \ldots, X_{i-1})$ of the JCIR process, namely

$$\mathbb{E}\left[X_i \mid \sigma(X_1, \ldots, X_{i-1})\right] = \mathbb{E}\left[\mathbb{E}\left(X_i \mid \mathcal{F}_{i-1}\right) \mid \sigma(X_1, \ldots, X_{i-1})\right] = \eta_0 X_{i-1} + \eta_1,$$

where

$$\eta_0 := e^{-b} \quad \text{and} \quad \eta_1 := \frac{1 - e^{-b}}{b} \left(a + \int_0^\infty z\nu(\mathrm{d}z)\right),$$

according to Proposition 2.13. Hence, a CLSE of (a, b) based on discrete time observations $(X_i)_{i\in\mathbb{N}}$ could be obtained by solving the extremum problem

$$\underset{(a,b)\in\mathbb{R}^2}{\arg\min} \sum_{i=1}^{n} \left(X_i - \mathbb{E}\left[X_i \mid \mathcal{F}_{i-1}\right]\right)^2 = \underset{(a,b)\in\mathbb{R}^2}{\arg\min} \sum_{i=1}^{n} \left(X_i - \eta_0 X_{i-1} - \eta_1\right)^2. \tag{3.0.2}$$

Moreover, defining

$$f(\eta_0,\eta_1) := \sum_{i=1}^{n} \left(X_i - \eta_0 X_{i-1} - \eta_1\right)^2, \quad (\eta_0,\eta_1) \in \mathbb{R}^2,$$

the first partial derivatives of f with respect to η_0 and η_1 are given by

$$\begin{aligned}
\frac{\partial f(\eta_0,\eta_1)}{\partial \eta_0} &= -2\sum_{i=1}^{n} X_{i-1}\left(X_i - \eta_0 X_{i-1} - \eta_1\right),\\
\frac{\partial f(\eta_0,\eta_1)}{\partial \eta_1} &= -2\sum_{i=1}^{n} \left(X_i - \eta_0 X_{i-1} - \eta_1\right),
\end{aligned}$$

and the second partial derivatives of f with respect to η_0 and η_1 are given by

$$\begin{aligned}
&\frac{\partial^2 f(\eta_0,\eta_1)}{\partial \eta_0^2} = 2\sum_{i=1}^{n} X_{i-1}^2, \qquad \frac{\partial^2 f(\eta_0,\eta_1)}{\partial \eta_1^2} = 2n, \quad \text{and}\\
&\frac{\partial^2 f(\eta_0,\eta_1)}{\partial \eta_0 \partial \eta_1} = \frac{\partial^2 f(\eta_0,\eta_1)}{\partial \eta_1 \partial \eta_0} = 2\sum_{i=1}^{n} X_{i-1}.
\end{aligned}$$

The system of equations of the first order partial derivatives is then equal to zero if and only if

$$\begin{pmatrix} \sum_{i=1}^n X_{i-1}X_i \\ \sum_{i=1}^n X_i \end{pmatrix} = \begin{pmatrix} \sum_{i=1}^n X_{i-1}^2 & \sum_{i=1}^n X_{i-1} \\ \sum_{i=1}^n X_{i-1} & n \end{pmatrix} \begin{pmatrix} \eta_0 \\ \eta_1 \end{pmatrix},$$

yielding that the CLSE $(\widehat{\eta}_{0,n}^{\mathrm{CLSE}}, \widehat{\eta}_{1,n}^{\mathrm{CLSE}})$ of (η_0,η_1) could be obtained by estimating

$$\begin{pmatrix} \widehat{\eta}_{0,n}^{\mathrm{CLSE}} \\ \widehat{\eta}_{1,n}^{\mathrm{CLSE}} \end{pmatrix} = \begin{pmatrix} \sum_{i=1}^n X_{i-1}^2 & \sum_{i=1}^n X_{i-1} \\ \sum_{i=1}^n X_{i-1} & n \end{pmatrix}^{-1} \begin{pmatrix} \sum_{i=1}^n X_{i-1}X_i \\ \sum_{i=1}^n X_i \end{pmatrix} \tag{3.0.3}$$

provided that the Hessian matrix consisting of the second order partial derivatives of f with respect to η_0 and η_1 is positive definite, that is $n\sum_{i=1}^n X_{i-1}^2 > (\sum_{i=1}^n X_{i-1})^2$.

The following Lemma ensures the positive definiteness of the Hessian matrix of f.

Lemma 3.2. *Assume a and $b \in \mathbb{R}_{>0}$. Then, for all $n \geqslant 2$, $n \in \mathbb{N}$, we have*

$$\mathbb{P}\left(\sum_{i=1}^{n} X_{i-1}^2 > 0\right) = 1 \quad \textit{and} \quad \mathbb{P}\left(n\sum_{i=1}^{n} X_{i-1}^2 > \left(\sum_{i=1}^{n} X_{i-1}\right)^2\right) = 1.$$

Proof. We follow the proof of [8, Lemma 3.1]. Note that $\sum_{i=1}^n X_{i-1}^2 \geqslant 0$, and equality holds if and only if $X_0 = X_1 = \ldots = X_{n-1} = 0$. Then, for all $n \geqslant 2$,

$$\mathbb{P}\left(X_0 = X_1 = \ldots = X_{n-1}\right) \leqslant \mathbb{P}\left(X_0 = X_1\right) = \mathbb{P}(X_1 = x) = 0,$$

because the law of X_1 is absolutely continuous as shown in the proof of Proposition 2.6. Consequently, $\sum_{i=1}^n X_{i-1}^2 > 0$ almost surely. Furthermore, an easy calculation yields

$$n\sum_{i=1}^{n} X_{i-1}^2 - \left(\sum_{i=1}^{n} X_{i-1}\right)^2 = n\sum_{i=1}^{n} X_{i-1}^2 - \sum_{i=1}^{n}\sum_{j=1}^{n} X_{i-1}X_{j-1}$$

$$= n \sum_{i=1}^{n} \left(X_{i-1} - \frac{1}{n} \sum_{j=1}^{n} X_{j-1} \right)^2 \geqslant 0,$$

and equality holds if and only if

$$X_{i-1} = \frac{1}{n} \sum_{j=1}^{n} X_{j-1}, \quad i = 1, \ldots, n, \tag{3.0.4}$$

It follows that identity (3.0.4) holds if and only if $X_0 = X_1 = \ldots = X_{n-1}$, $n \geqslant 2$, and for the same reasoning as before, we conclude that

$$\mathbb{P}\left(n \sum_{i=1}^{n} X_{i-1}^2 > \left(\sum_{i=1}^{n} X_{i-1} \right)^2 \right) = 1$$

holds under the given conditions. □

As a consequence of Lemma 3.2, supposing $\eta_0, \eta_1 \in \mathbb{R}$ and $\sigma \in \mathbb{R}_{>0}$, there exists a unique CLSE $(\widehat{\eta}_{0,n}^{\mathrm{CLSE}}, \widehat{\eta}_{1,n}^{\mathrm{CLSE}})$ of (η_0, η_1), with $(\widehat{\eta}_{0,n}^{\mathrm{CLSE}}, \widehat{\eta}_{1,n}^{\mathrm{CLSE}})$ given by (3.0.3).

3.1. Consistency of the LSE

In this section we study the asymptotic behavior of the CLSE $(\widehat{\eta}_{0,n}^{\mathrm{CLSE}}, \widehat{\eta}_{1,n}^{\mathrm{CLSE}})$ of (η_0, η_1).

Recall that X_t converges in distribution to a random variable, say X_∞, which is distributed according to π given in Remark 2.16.

Theorem 3.3. *Consider the JCIR process $(X_t)_{t\geqslant 0}$ defined by (2.0.1) with parameters a, b, σ and ν, where ν is the Lévy measure of $(J_t)_{t\geqslant 0}$. Assume $a > 0$. Then, the CLSE $(\widehat{\eta}_{0,n}^{\mathrm{CLSE}}, \widehat{\eta}_{1,n}^{\mathrm{CLSE}})$ of (η_0, η_1) is strongly consistent, namely,*

$$\mathbb{P}\left(\lim_{n\to\infty} \left(\widehat{\eta}_{0,n}^{\mathrm{CLSE}}, \widehat{\eta}_{1,n}^{\mathrm{CLSE}} \right) = (\eta_0, \eta_1) \right) = 1,$$

where $(\widehat{\eta}_{0,n}^{\mathrm{CLSE}}, \widehat{\eta}_{1,n}^{\mathrm{CLSE}})$ are given by (3.0.3).

Proof. By (3.0.3) the CLSE $(\widehat{\eta}_{0,n}^{\mathrm{CLSE}}, \widehat{\eta}_{1,n}^{\mathrm{CLSE}})$ of (η_0, η_1) is determined by

$$\begin{pmatrix} \widehat{\eta}_{0,n}^{\mathrm{CLSE}} \\ \widehat{\eta}_{1,n}^{\mathrm{CLSE}} \end{pmatrix} = \begin{pmatrix} \sum_{i=1}^{n} X_{i-1}^2 & \sum_{i=1}^{n} X_{i-1} \\ \sum_{i=1}^{n} X_{i-1} & n \end{pmatrix}^{-1} \begin{pmatrix} \sum_{i=1}^{n} X_{i-1} X_i \\ \sum_{i=1}^{n} X_i \end{pmatrix}.$$

By an easy calculation (see also [8, formula (3.5)]),

$$\begin{aligned} \begin{pmatrix} \widehat{\eta}_{0,n}^{\mathrm{CLSE}} \\ \widehat{\eta}_{1,n}^{\mathrm{CLSE}} \end{pmatrix} &= \left(\sum_{i=1}^{n} \begin{pmatrix} X_{i-1} \\ 1 \end{pmatrix} \begin{pmatrix} X_{i-1} \\ 1 \end{pmatrix}^{\top} \right)^{-1} \left(\sum_{i=1}^{n} \begin{pmatrix} X_{i-1} \\ 1 \end{pmatrix} X_i \right) \\ &= \left(\sum_{i=1}^{n} \begin{pmatrix} X_{i-1} \\ 1 \end{pmatrix} \begin{pmatrix} X_{i-1} \\ 1 \end{pmatrix}^{\top} \right)^{-1} \begin{pmatrix} \eta_0 \\ \eta_1 \end{pmatrix} \\ &\quad + \left(\sum_{i=1}^{n} \begin{pmatrix} X_{i-1} \\ 1 \end{pmatrix} \begin{pmatrix} X_{i-1} \\ 1 \end{pmatrix}^{\top} \right)^{-1} \sum_{i=1}^{n} \begin{pmatrix} X_{i-1} \\ 1 \end{pmatrix} (X_i - \eta_0 X_{i-1} - \eta_1) \end{aligned}$$

$$= \begin{pmatrix} \eta_0 \\ \eta_1 \end{pmatrix} + \left(\frac{1}{n} \sum_{i=1}^{n} \begin{pmatrix} X_{i-1} \\ 1 \end{pmatrix} \begin{pmatrix} X_{i-1} \\ 1 \end{pmatrix}^\top \right)^{-1} \frac{1}{n} \sum_{i=1}^{n} \begin{pmatrix} X_{i-1} \\ 1 \end{pmatrix} \varepsilon_i, \tag{3.1.1}$$

where $\varepsilon_i := X_i - \eta_0 X_{i-1} - \eta_1$, $i \in \mathbb{N}$. Our strategy to prove the asserted is now as follows: First we prove that

$$\frac{1}{n} \sum_{i=1}^{n} \begin{pmatrix} X_{i-1} \\ 1 \end{pmatrix} \varepsilon_i \to 0 \quad \text{a.s. as } n \to \infty, \tag{3.1.2}$$

by an application of the strong law of large numbers for discrete time square-integrable martingales (see Theorem C.1). Second, by an application of the discrete time ergodicity, we show that

$$\frac{1}{n} \sum_{i=1}^{n} \begin{pmatrix} X_{i-1} \\ 1 \end{pmatrix} \begin{pmatrix} X_{i-1} \\ 1 \end{pmatrix}^\top$$

converges almost surely to a non-singular constant limit matrix as n tends to infinity. Then it is clear that the product converges almost surely to the zero vector and we obtain the asserted convergence of the CLSE in view of (3.1.1).

We proceed to prove (3.1.2). Since $\mathbb{E}[X_i \,|\, \mathcal{F}_{i-1}] = \eta_0 X_{i-1} + \eta_1$, $i \in \mathbb{N}$, it holds that $\mathbb{E}[\varepsilon_i \,|\, \mathcal{F}_{i-1}] = 0$, which implies that $(\varepsilon_i)_{i\in\mathbb{N}}$ is a sequence of martingale differences with respect to the filtration $(\mathcal{F}_i)_{i\in\mathbb{Z}_{\geqslant 0}}$. From Proposition 2.15 together with the Markov property it follows

$$\mathbb{E}\left[X_i^2 \,|\, \mathcal{F}_{i-1}\right] = \xi_0 X_{i-1}^2 + \xi_1 X_{i-1} + \xi_2,$$

where

$$\begin{aligned}
\xi_0 &= e^{-2b}, \\
\xi_1 &= \frac{e^{-2b}\left(e^b - 1\right)}{b} \left(2a + \sigma^2 + 2\int_0^\infty z\nu(\mathrm{d}z)\right), \quad \text{and} \\
\xi_2 &= \frac{e^{-2b}\left(e^b - 1\right)^2}{2b^2} \left(a(2a + \sigma^2) + (4a + \sigma^2)\int_0^\infty z\nu(\mathrm{d}z)\right) \\
&\quad + \frac{1 - e^{-2b}}{2b} \int_0^\infty z^2 \nu(\mathrm{d}z) + \frac{\left(1 - e^{-b}\right)^2}{b^2} \left(\int_0^\infty z\nu(\mathrm{d}z)\right)^2,
\end{aligned}$$

according to formula (2.2.28). Hence, we derive

$$\begin{aligned}
\mathbb{E}\left[\varepsilon_i^2 \,|\, \mathcal{F}_{i-1}\right] &= \mathbb{E}\left[X_i^2 \,|\, \mathcal{F}_{i-1}\right] - (\eta_0 X_{i-1} + \eta_1)^2 \\
&= \xi_0 X_{i-1}^2 + \xi_1 X_{i-1} + \xi_2 - (\eta_0 X_{i-1} + \eta_1)^2 \\
&= \left(\xi_0 - \eta_0^2\right) X_{i-1}^2 + (\xi_1 - 2\eta_0\eta_1) X_{i-1} + \xi_2 - \eta_1^2 \\
&=: C_1 X_{i-1} + C_2,
\end{aligned}$$

where $C_1 := \xi_1 - 2\eta_0\eta_1$ and $C_2 := \xi_2 - \eta_1^2$. Thus, ε_i is a square integrable martingale with respect to the filtration $(\mathcal{F}_i)_{i\in\mathbb{Z}_{\geqslant 0}}$.

Let $M_n^{(\varepsilon)} := \sum_{i=1}^n \varepsilon_i$, $n \in \mathbb{N}$. By essentially the same argument as before, we see that $(M_n^{(\varepsilon)})_{n\in\mathbb{N}}$ is also a square integrable martingale with respect to the filtration $(\mathcal{F}_i)_{i\in\mathbb{Z}_{\geqslant 0}}$

and $(M_n^{(\varepsilon)})_{n\in\mathbb{N}}$ has quadratic variation process

$$\langle M^{(\varepsilon)}\rangle_n = \sum_{i=1}^n \mathbb{E}\left[\varepsilon_i^2 \,|\, \mathcal{F}_{i-1}\right] = C_1 \sum_{i=1}^n X_{i-1} + nC_2, \quad n \in \mathbb{N},$$

by [61, Chapter VII, Section 1, formula (15)]. Applying the time-discrete ergodicity in Proposition 2.20, where the limit is given due to Corollary 2.14, we obtain

$$\frac{1}{n}\langle M^{(\varepsilon)}\rangle_n = C_1 \frac{1}{n}\sum_{i=1}^n X_{i-1} + C_2 \longrightarrow C_1\mathbb{E}[X_\infty] + C_2 \quad \text{a.s. as } n \to \infty.$$

Note, C_1 and C_2 are strictly positive, since

$$\begin{aligned} C_1 &= \frac{e^{-2b}\left(e^b - 1\right)}{b}\left(2a + \sigma^2 + 2\int_0^\infty z\nu(\mathrm{d}z)\right) - 2e^{-b}\left(a + \int_0^\infty z\nu(\mathrm{d}z)\right)\frac{1-e^{-b}}{b} \\ &= \frac{\sigma^2}{b}e^{-2b}\left(e^b - 1\right) \in \mathbb{R}_{>0}, \end{aligned} \tag{3.1.3}$$

and

$$\begin{aligned} C_2 &= \frac{e^{-2b}\left(e^b-1\right)^2}{2b^2}\left(a(2a+\sigma^2) + (4a+\sigma^2)\int_0^\infty z\nu(\mathrm{d}z)\right) + \frac{1-e^{-2b}}{2b}\int_0^\infty z^2\nu(\mathrm{d}z) \\ &\quad + \frac{\left(1-e^{-b}\right)^2}{b^2}\left(\int_0^\infty z\nu(\mathrm{d}z)\right)^2 - \left(a + \int_0^\infty z\nu(\mathrm{d}z)\right)^2\frac{\left(1-e^{-b}\right)^2}{b^2} \\ &= \frac{e^{-2b}\left(e^b-1\right)^2}{2b^2}\left(a(2a+\sigma^2) + (4a+\sigma^2)\int_0^\infty z\nu(\mathrm{d}z)\right) + \frac{1-e^{-2b}}{2b}\int_0^\infty z^2\nu(\mathrm{d}z) \\ &\quad - \left(a^2 + 2a\int_0^\infty z\nu(\mathrm{d}z)\right)\frac{\left(1-e^{-b}\right)^2}{b^2} \\ &= \frac{e^{-2b}\left(e^b-1\right)^2}{2b^2}\left(a(2a+\sigma^2) + \sigma^2\int_0^\infty z\nu(\mathrm{d}z)\right) + \frac{1-e^{-2b}}{2b}\int_0^\infty z^2\nu(\mathrm{d}z) \\ &\quad - \frac{a^2e^{-2b}\left(e^b-1\right)^2}{b^2} \\ &= \frac{e^{-2b}\left(e^b-1\right)^2}{2b^2}\left(a\sigma^2 + \sigma^2\int_0^\infty z\nu(\mathrm{d}z)\right) + \frac{1-e^{-2b}}{2b}\int_0^\infty z^2\nu(\mathrm{d}z) \in \mathbb{R}_{>0}. \end{aligned} \tag{3.1.4}$$

Therefore, $\langle M^{(\varepsilon)}\rangle_n$ converges almost surely to infinity as n tends to infinity. This allows to apply Theorem C.1, which implies that the following convergence

$$\frac{1}{n}M_n^{(\varepsilon)} = \frac{1}{n}\sum_{i=1}^n \varepsilon_i = \frac{M_n^{(\varepsilon)}}{\langle M^{(\varepsilon)}\rangle_n}\frac{\langle M^{(\varepsilon)}\rangle_n}{n} \longrightarrow 0 \cdot (C_1\mathbb{E}[X_\infty] + C_2) = 0 \tag{3.1.5}$$

holds almost surely as n tends to infinity. Arguing similar, we also obtain

$$\mathbb{E}\left[X_{i-1}^2\varepsilon_i^2 \,|\, \mathcal{F}_{i-1}\right] = X_{i-1}^2\mathbb{E}\left[\varepsilon_i^2 \,|\, \mathcal{F}_{i-1}\right] = C_1X_{i-1}^3 + C_2X_{i-1}^2, \quad i \in \mathbb{N}.$$

Further, by the same reasoning as before,

$$\mathbb{P}\left(\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}X_{i-1}\varepsilon_i=0\right)=1.$$

Finally, applying the time-discrete ergodicity of X_i, $i\in\mathbb{N}$, established in Proposition 2.20, we have

$$\mathbb{P}\left(\lim_{n\to\infty}\left(\frac{1}{n}\sum_{i=1}^{n}\begin{pmatrix}X_{i-1}^2 & X_{i-1}\\ X_{i-1} & 1\end{pmatrix}\right)^{-1}=\begin{pmatrix}\mathbb{E}[X_\infty^2] & \mathbb{E}[X_\infty]\\ \mathbb{E}[X_\infty] & 1\end{pmatrix}^{-1}\right)=1 \tag{3.1.6}$$

provided that the limit matrix is positive definite. Note, the limit matrix is indeed non-singular, since

$$\mathbb{E}\left[X_\infty^2\right]-(\mathbb{E}[X_\infty])^2=\frac{a\sigma^2}{2b^2}+\frac{1}{2b}\int_0^\infty z^2\nu(\mathrm{d}z)+\frac{\sigma^2}{2b}\int_0^\infty z\nu(\mathrm{d}z)\in\mathbb{R}_{>0}. \tag{3.1.7}$$

Thus, by (3.1.1), $(\widehat{\eta}_{0,n}^{\mathrm{CLSE}},\widehat{\eta}_{1,n}^{\mathrm{CLSE}})$ is a strongly consistent CLSE of (η_0,η_1). □

3.2. Asymptotic behavior of least squares estimator

We continue to study the asymptotic behavior of the CLSE $(\widehat{\eta}_{0,n}^{\mathrm{CLSE}},\widehat{\eta}_{1,n}^{\mathrm{CLSE}})$ of (η_0,η_1) and prove asymptotic normality.

Theorem 3.4. *Consider the JCIR process $(X_t)_{t\geqslant 0}$ defined by (2.0.1) with parameters a, b, σ and ν, where ν is the Lévy measure of $(J_t)_{t\geqslant 0}$. Assume $a>0$. Then the CLSE $(\widehat{\eta}_{0,n}^{\mathrm{CLSE}},\widehat{\eta}_{1,n}^{\mathrm{CLSE}})$ of (η_0,η_1) is asymptotically normal, i.e., the convergence*

$$\sqrt{n}\left(\widehat{\eta}_{0,n}^{\mathrm{CLSE}}-\eta_0 \quad \widehat{\eta}_{1,n}^{\mathrm{CLSE}}-\eta_1\right)^\top\longrightarrow\mathcal{N}_2(\mathbf{0},\mathbf{E})$$

holds in distribution as n tends to infinity, where $\mathbf{E}$ is the 2×2 covariance matrix.

Proof. Using (3.1.1), we calculate

$$\sqrt{n}\begin{pmatrix}\widehat{\eta}_{0,n}^{\mathrm{CLSE}}-\eta_0\\ \widehat{\eta}_{1,n}^{\mathrm{CLSE}}-\eta_1\end{pmatrix}=\left(\frac{1}{n}\sum_{i=1}^{n}\begin{pmatrix}X_{i-1}^2 & X_{i-1}\\ X_{i-1} & 1\end{pmatrix}\right)^{-1}\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\begin{pmatrix}X_{i-1}\varepsilon_i\\ \varepsilon_i\end{pmatrix},$$

provided that $\sum_{i=1}^n X_{i-1}^2>1/n(\sum_{i=1}^n X_{i-1})^2$. Recall, provided that $\sum_{i=1}^n X_{i-1}^2>(1/n\sum_{i=1}^n X_{i-1})^2$, by (3.1.6) the first factor on the right hand side converges almost surely, i.e.,

$$\left(\frac{1}{n}\sum_{i=1}^{n}\begin{pmatrix}X_{i-1}^2 & X_{i-1}\\ X_{i-1} & 1\end{pmatrix}\right)^{-1}\longrightarrow\begin{pmatrix}\mathbb{E}[X_\infty^2] & \mathbb{E}[X_\infty]\\ \mathbb{E}[X_\infty] & 1\end{pmatrix}^{-1}=:\mathbf{L}\quad\text{a.s. as } n\to\infty.$$

Basically, the idea of the proof is now as follows. We will show the following:

Claim. *The convergence*

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\begin{pmatrix}X_{i-1}\varepsilon_i\\ \varepsilon_i\end{pmatrix}\longrightarrow\mathcal{N}_2(\mathbf{0},\mathbf{D})$$

holds in distribution, where $\mathbf{D}$ is a 2×2 real-valued, symmetric and positive definite matrix to be determined later.

If this claim is true, then by Slutsky's theorem

$$\sqrt{n}\begin{pmatrix}\widehat{\eta}_{0,n}^{\mathrm{CLSE}}-\eta_0\\ \widehat{\eta}_{1,n}^{\mathrm{CLSE}}-\eta_1\end{pmatrix}\longrightarrow\mathcal{N}_2\left(\mathbf{0},\mathbf{E}\right)$$

holds in distribution as n tends to infinity, where the covariance matrix is given by $\mathbf{E}=\mathbf{L}^{-1}\mathbf{D}\mathbf{L}^{-1}$.

We continue to prove the claim. We apply the martingale central limit theorem, see Theorem C.2 with the following choices: $d=2$, $k_n=n$, $n\in\mathbb{N}$, $\mathcal{F}_{n,k}=\mathcal{F}_k$, $n\in\mathbb{N}$, $k\in\{1,\dots,n\}$ and

$$M_{n,k}=\frac{1}{\sqrt{n}}\sum_{i=1}^{k}\begin{pmatrix}X_{i-1}\varepsilon_i\\ \varepsilon_i\end{pmatrix},\quad n\in\mathbb{N},\ k\in\{1,\dots,n\}.$$

Further, for all $n\in\mathbb{N}$ and $k\in\{1,\dots,n\}$, we have

$$\begin{aligned}
&\mathbb{E}\left[\left(M_{n,k}-M_{n,k-1}\right)\left(M_{n,k}-M_{n,k-1}\right)^{\top}\,\middle|\,\mathcal{F}_{k-1}\right]\\
&=\frac{1}{n}\mathbb{E}\left[\begin{pmatrix}X_{k-1}\varepsilon_k\\ \varepsilon_k\end{pmatrix}\begin{pmatrix}X_{k-1}\varepsilon_k\\ \varepsilon_k\end{pmatrix}^{\top}\,\middle|\,\mathcal{F}_{k-1}\right]\\
&=\frac{1}{n}\mathbb{E}\left[\varepsilon_k^2\,\middle|\,\mathcal{F}_{k-1}\right]\left(\begin{pmatrix}X_{k-1}\\ 1\end{pmatrix}\begin{pmatrix}X_{k-1}\\ 1\end{pmatrix}^{\top}\right)\\
&=\frac{1}{n}\left(C_1X_{k-1}+C_2\right)\left(\begin{pmatrix}X_{k-1}\\ 1\end{pmatrix}\begin{pmatrix}X_{k-1}\\ 1\end{pmatrix}^{\top}\right),
\end{aligned}$$

where C_1 and C_2 are positive constants according to (3.1.3) and (3.1.4). Hence, applying the time-discrete ergodicity (see Proposition (2.20)), the following convergence

$$\begin{aligned}
&\sum_{k=1}^{n}\mathbb{E}\left[\left(M_{n,k}-M_{n,k-1}\right)\left(M_{n,k}-M_{n,k-1}\right)^{\top}\,\middle|\,\mathcal{F}_{k-1}\right]\\
&=\frac{1}{n}\sum_{k=1}^{n}\left(C_1X_{k-1}+C_2\right)\begin{pmatrix}X_{k-1}^2 & X_{k-1}\\ X_{k-1} & 1\end{pmatrix}\\
&=\frac{1}{n}\sum_{k=1}^{n}C_1\begin{pmatrix}X_{k-1}^3 & X_{k-1}^2\\ X_{k-1}^2 & X_{k-1}\end{pmatrix}+\frac{1}{n}\sum_{k=1}^{n}C_2\begin{pmatrix}X_{k-1}^2 & X_{k-1}\\ X_{k-1} & 1\end{pmatrix}\\
&\to C_1\begin{pmatrix}\mathbb{E}\left[X_\infty^3\right] & \mathbb{E}\left[X_\infty^2\right]\\ \mathbb{E}\left[X_\infty^2\right] & \mathbb{E}\left[X_\infty\right]\end{pmatrix}+C_2\begin{pmatrix}\mathbb{E}\left[X_\infty^2\right] & \mathbb{E}\left[X_\infty\right]\\ \mathbb{E}\left[X_\infty\right] & 1\end{pmatrix}=:\mathbf{D}
\end{aligned}$$

holds almost surely as n tends to infinity. Since C_1 and C_2 are strictly positive constants, it holds that $\mathbf{D}\in\mathbb{R}^{2\times2}$ is symmetric and positive definite. Indeed, by (3.1.7), we obtain positive definiteness of the second matrix. For the first matrix, by Hölder's inequality, we have that

$$\left(\mathbb{E}\left[X_\infty\right]\right)^{1/2}\left(\mathbb{E}\left[X_\infty^3\right]\right)^{1/2}\geqslant\mathbb{E}\left[X_\infty^{1/2}X_\infty^{3/2}\right]=\mathbb{E}[X_\infty^2],\tag{3.2.1}$$

yielding the positive semi-definiteness of the first matrix. Consequently, $\mathbf{D}$ is positive definite as desired.

Next, we prove the Lindeberg condition, i.e., we check, for all $\varepsilon > 0$, it holds

$$\lim_{n\to\infty} \mathbb{P}\left(\left|\sum_{k=1}^{k_n} \mathbb{E}\left[\|M_{n,k} - M_{n,k-1}\|^2 \mathbb{1}_{\{\|M_{n,k}-M_{n,k-1}\|\geqslant\varepsilon\}} \,|\, \mathcal{F}_{k-1}\right]\right| > \varepsilon\right) = 0.$$

We fix $\delta > 0$ sufficiently small. Notice that,

$$\|M_{n,k} - M_{n,k-1}\|^2 \, \mathbb{1}_{\left\{\|M_{n,k}-M_{n,k-1}\|\geqslant\varepsilon\right\}} \leqslant \frac{\|M_{n,k} - M_{n,k-1}\|^{2+\delta}}{\varepsilon^\delta},$$

and furthermore, by an elementary inequality,

$$\begin{aligned}\|M_{n,k} - M_{n,k-1}\|^{2+\delta} &= \left\|\frac{1}{\sqrt{n}}\begin{pmatrix} X_{k-1}\varepsilon_k \\ \varepsilon_k \end{pmatrix}\right\|^{2+\delta} = \left(\frac{1}{n}\left(X_{k-1}^2\varepsilon_k^2 + \varepsilon_k^2\right)\right)^{1+\delta/2} \\ &\leqslant \frac{1+\delta/2}{n^{1+\delta/2}}\left(X_{k-1}^{2+\delta}\varepsilon_k^{2+\delta} + \varepsilon_k^{2+\delta}\right), \quad n \in \mathbb{N},\, k \in \{1,\ldots,n\}.\end{aligned}$$

Hence, for $\delta > 0$ sufficiently small, we obtain,

$$\begin{aligned}&\sum_{k=1}^{n} \mathbb{E}\left[\|M_{n,k} - M_{n,k-1}\|^2 \mathbb{1}_{\{\|M_{n,k}-M_{n,k-1}\|\geqslant\varepsilon\}} \,\middle|\, \mathcal{F}_{k-1}\right] \\ &\qquad \leqslant \frac{1+\delta/2}{n^{1+\delta/2}} \sum_{k=1}^{n} \mathbb{E}\left[X_{k-1}^{2+\delta}\varepsilon_k^{2+\delta} + \varepsilon_k^{2+\delta} \,\middle|\, \mathcal{F}_{k-1}\right] \\ &\qquad = \frac{1+\delta/2}{n^{1+\delta/2}} \sum_{k=1}^{n} \left(X_{k-1}^{2+\delta} + 1\right) \mathbb{E}\left[\varepsilon_k^{2+\delta} \,\middle|\, \mathcal{F}_{k-1}\right], \quad n \in \mathbb{N},\, k \in \{1,\ldots,n\}.\end{aligned}$$

Instead of convergence in probability for the Lindeberg condition, we prove L^1 convergence, i.e, it suffices to prove that

$$\sup_{k\in\mathbb{N}} \mathbb{E}\left[\left(X_{k-1}^{2+\delta} + 1\right) \mathbb{E}\left[\varepsilon_k^{2+\delta} \,\middle|\, \mathcal{F}_{k-1}\right]\right] < \infty.$$

By the law of total expectation,

$$\mathbb{E}\left[\left(X_{k-1}^{2+\delta} + 1\right) \mathbb{E}\left[\varepsilon_k^{2+\delta} \,\middle|\, \mathcal{F}_{k-1}\right]\right] = \mathbb{E}\left[\left(X_{k-1}^{2+\delta} + 1\right) \varepsilon_k^{2+\delta}\right],$$

and hence, it is equivalent to check that

$$\sup_{k\in\mathbb{N}} \mathbb{E}\left[\left(X_{k-1}^{2+\delta} + 1\right) \varepsilon_k^{2+\delta}\right] < \infty.$$

Applying the Cauchy Schwarz inequality and the power mean inequality, we get,

$$\begin{aligned}\mathbb{E}\left[\left(X_{k-1}^{2+\delta} + 1\right) \varepsilon_k^{2+\delta}\right] &\leqslant \left(\mathbb{E}\left[\left(X_{k-1}^{2+\delta} + 1\right)^2\right] \mathbb{E}\left[\varepsilon_k^{4+2\delta}\right]\right)^{1/2} \\ &\leqslant \sqrt{2}\left(\mathbb{E}\left[X_{k-1}^{4+2\delta} + 1\right] \mathbb{E}\left[\varepsilon_k^{4+2\delta}\right]\right)^{1/2}, \quad k \in \mathbb{N}.\end{aligned}$$

Using again the power mean inequality, for all $k \in \mathbb{N}$, we estimate

$$\mathbb{E}\left[\varepsilon_k^{4+2\delta}\right] \leqslant \mathbb{E}\left[|X_k - \eta_0 X_{k-1} - \eta_1|^{4+2\delta}\right] \leqslant \mathbb{E}\left[(X_k + \eta_0 X_{k-1} + \eta_1)^{4+2\delta}\right]$$

$$\leqslant 3^{3+2\delta}\mathbb{E}\left[X_k^{4+2\delta} + \eta_0^{4+2\delta}X_{k-1}^{4+2\delta} + \eta_1^{4+2\delta}\right].$$

Consequently, it only suffices to prove that $\sup_{k\in\mathbb{N}}\mathbb{E}\left[X_k^{4+2\delta}\right] < \infty$, and, by our present assumption, this readily follows from Theorem 2.25.

Altogether, by the martingale convergence theorem C.2, we obtain the following convergence

$$M_{n,n} = \frac{1}{\sqrt{n}}\sum_{k=1}^{n}\begin{pmatrix}\varepsilon_k X_{k-1} & \varepsilon_k\end{pmatrix}^\top \longrightarrow \mathcal{N}_2\left(\mathbf{0}, \mathbf{D}\right)$$

in distribution as n tends to infinity. Therefore, our claim is true. □

Remark 3.5. *Note that in Theorem 3.4 it is the assumption* (3.0.1) *which ensures* $\mathbb{E}[X_\infty^3] < \infty$*, by Theorem 2.25.*

Proposition 3.6. *Assume* $a > 0$*. Let* $\mathbf{E} \in \mathbb{R}^{2\times 2}$ *be as in Theorem 3.4. Then* $\mathbf{E}$ *is symmetric and positive definite.*

Proof. By the definition of $\mathbf{L}$ and $\mathbf{D}$, the covariance matrix $\mathbf{E} = \mathbf{L}^{-1}\mathbf{D}\mathbf{L}^{-1}$ takes the form

$$\begin{aligned}
\mathbf{E} &= C_1\begin{pmatrix}\mathbb{E}[X_\infty^2] & \mathbb{E}[X_\infty]\\ \mathbb{E}[X_\infty] & 1\end{pmatrix}^{-1}\begin{pmatrix}\mathbb{E}[X_\infty^3] & \mathbb{E}[X_\infty^2]\\ \mathbb{E}[X_\infty^2] & \mathbb{E}[X_\infty]\end{pmatrix}\begin{pmatrix}\mathbb{E}[X_\infty^2] & \mathbb{E}[X_\infty]\\ \mathbb{E}[X_\infty] & 1\end{pmatrix}^{-1}\\
&\quad + C_2\begin{pmatrix}\mathbb{E}[X_\infty^2] & \mathbb{E}[X_\infty]\\ \mathbb{E}[X_\infty] & 1\end{pmatrix}^{-1}\begin{pmatrix}\mathbb{E}[X_\infty^2] & \mathbb{E}[X_\infty]\\ \mathbb{E}[X_\infty] & 1\end{pmatrix}\begin{pmatrix}\mathbb{E}[X_\infty^2] & \mathbb{E}[X_\infty]\\ \mathbb{E}[X_\infty] & 1\end{pmatrix}^{-1}\\
&= \frac{C_1}{\left(\mathbb{E}[X_\infty^2] - (\mathbb{E}[X_\infty])^2\right)^2}\\
&\begin{pmatrix}(\mathbb{E}[X_\infty])^3 - 2\mathbb{E}[X_\infty^2]\mathbb{E}[X_\infty] + \mathbb{E}[X_\infty^3] & (\mathbb{E}[X_\infty^2])^2 - \mathbb{E}[X_\infty]\mathbb{E}[X_\infty^3]\\ (\mathbb{E}[X_\infty^2])^2 - \mathbb{E}[X_\infty]\mathbb{E}[X_\infty^3] & -\mathbb{E}[X_\infty]\left((\mathbb{E}[X_\infty^2])^2 - \mathbb{E}[X_\infty]\mathbb{E}[X_\infty^3]\right)\end{pmatrix}\\
&\quad + \frac{C_2}{\mathbb{E}[X_\infty^2] - (\mathbb{E}[X_\infty])^2}\begin{pmatrix}1 & -\mathbb{E}[X_\infty]\\ -\mathbb{E}[X_\infty] & \mathbb{E}[X_\infty^2]\end{pmatrix}.
\end{aligned} \tag{3.2.2}$$

To prove that $\mathbf{E}$ is positive definite, it is enough to check that the matrix

$$\begin{pmatrix}(\mathbb{E}[X_\infty])^3 - 2\mathbb{E}[X_\infty^2]\mathbb{E}[X_\infty] + \mathbb{E}[X_\infty^3] & (\mathbb{E}[X_\infty^2])^2 - \mathbb{E}[X_\infty]\mathbb{E}[X_\infty^3]\\ (\mathbb{E}[X_\infty^2])^2 - \mathbb{E}[X_\infty]\mathbb{E}[X_\infty^3] & -\mathbb{E}[X_\infty]\left((\mathbb{E}[X_\infty^2])^2 - \mathbb{E}[X_\infty]\mathbb{E}[X_\infty^3]\right)\end{pmatrix}$$

is positive semi-definite and

$$\begin{pmatrix}1 & -\mathbb{E}[X_\infty]\\ -\mathbb{E}[X_\infty] & \mathbb{E}[X_\infty^2]\end{pmatrix}$$

is positive definite. The positive definiteness of the second matrix readily follows by (3.1.7). Moreover, the determinant of the first matrix is given by

$$-\left((\mathbb{E}[X_\infty])^2 - \mathbb{E}[X_\infty^2]\right)^2\left(\left(\mathbb{E}[X_\infty^2]\right)^2 - \mathbb{E}[X_\infty]\mathbb{E}[X_\infty^3]\right),$$

where the first factor is again positive by (3.1.7) and in (3.2.1) we estimated the negativity of the second factor. Consequently, $\mathbf{E}$ is a symmetric positive definite 2×2-matrix as asserted. □

3.3. Least square estimator of the drift parameters (a, b)

So far in the preamble of this chapter we introduced the CLSE $(\widehat{\eta}_{0,n}^{\mathrm{CLSE}}, \widehat{\eta}_{1,n}^{\mathrm{CLSE}})$ of the transformed parameters (η_0, η_1) as the unique solution of the extremum problem (3.0.2) and proved strong consistency in Theorem 3.3 and asymptotically normality in Theorem 3.4 as well.

Finally, a natural estimator of the drift parameters (a, b) obtained from (3.0.2) and the definition of (η_0, η_1) may be introduced in the same way is in [8, formula (3.17)]. For completeness of exposition, we now recall the steps of [8] and fit them into the framework of the JCIR process. We define the function $g : \mathbb{R}_{>0}^2 \to \mathbb{R}_{>0} \times (0, 1)$ by

$$g(a, b) = \begin{pmatrix} (1 - e^{-b})b^{-1} \left(a + \int_0^\infty z\nu(\mathrm{d}z)\right) \\ e^{-b} \end{pmatrix} = \begin{pmatrix} \eta_1 \\ \eta_0 \end{pmatrix}, \quad (a, b) \in \mathbb{R}_{>0}^2.$$

It is easy to see that g is bijective having inverse

$$g^{-1}(\eta_0, \eta_1) = \begin{pmatrix} (\eta_0 - 1)^{-1}\eta_1 \log(\eta_0) - \int_0^\infty z\nu(\mathrm{d}z) \\ -\log(\eta_0) \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}, \tag{3.3.1}$$

where $(\eta_0, \eta_1) \in (0, 1) \times \mathbb{R}_{>0}$. Thus, by the strong consistency of $(\widehat{\eta}_{0,n}^{\mathrm{CLSE}}, \widehat{\eta}_{1,n}^{\mathrm{CLSE}})$,

$$\mathbb{P}\left(\left(\widehat{\eta}_{0,n}^{\mathrm{CLSE}}, \widehat{\eta}_{1,n}^{\mathrm{CLSE}}\right) \in (0, 1) \times \mathbb{R}_{>0}\right) = 1$$

for n large enough, $n \in \mathbb{N}$. Therefore, a natural estimator of (a, b) based on time-discrete observations $(X_i)_{i\in\{1,\dots,n\}}$ can be obtained by applying g^{-1} to $(\widehat{\eta}_{0,n}^{\mathrm{CLSE}}, \widehat{\eta}_{1,n}^{\mathrm{CLSE}})$, namely

$$\left(\widehat{a}_n, \widehat{b}_n\right) := g^{-1}\left(\widehat{\eta}_{0,n}^{\mathrm{CLSE}}, \widehat{\eta}_{1,n}^{\mathrm{CLSE}}\right)$$

for $n \in \mathbb{N}$ large enough and hence

$$\mathbb{P}\left(\left(\widehat{a}_n, \widehat{b}_n\right) = \underset{(a,b)\in\mathbb{R}_{>0}^2}{\arg\min} \sum_{i=1}^n (X_i - \eta_0 X_{i-1} - \eta_1)^2\right) = 1$$

for sufficiently large $n \in \mathbb{N}$.

The following theorem shows that $(\widehat{a}_n, \widehat{b}_n)$ captures the properties of $(\widehat{\eta}_{0,n}^{\mathrm{CLSE}}, \widehat{\eta}_{1,n}^{\mathrm{CLSE}})$. An analogous theorem has been derived in [8, Theorem 3.4].

Theorem 3.7. *Consider the JCIR process $(X_t)_{t\geqslant 0}$ determined by the SDE* (2.0.1) *with parameters $a, b, \sigma \in \mathbb{R}_{>0}$ and Lévy measure ν satisfying* (3.0.1)*. Then the sequence $(\widehat{a}_n, \widehat{b}_n)$, $n \in \mathbb{N}$, is strongly consistent and asymptotically normal, where the covariance matrix $\mathbf{J}\mathbf{E}\mathbf{J}^\top \in \mathbb{R}^{2\times 2}$, with $\mathbf{E} \in \mathbb{R}^{2\times 2}$ is a symmetric, positive definite matrix given in* (3.2.2) *and*

$$\mathbf{J} := \begin{pmatrix} (\eta_0 - 1)^{-2}\eta_0^{-1}\left(-\eta_1\left(1 - \eta_0 + \eta_0 \log(\eta_0)\right)\right) & (\eta_0 - 1)^{-1}\log(\eta_0) \\ -\eta_0^{-1} & 0 \end{pmatrix}$$

being the Jacobian matrix of g^{-1} with respect to $(\eta_0, \eta_1) \in (0, 1) \times \mathbb{R}_{>0}$.

Proof. We proceed in the same way as in [8, Theorem 3.4]. Note that the inverse function g^{-1} defined in (3.3.1) is continuous on $(0,1) \times \mathbb{R}_{>0}$. Hence, from the strong consistency of the CLSE $(\widehat{\eta}_{0,n}^{\mathrm{CLSE}}, \widehat{\eta}_{1,n}^{\mathrm{CLSE}})$, $n \in \mathbb{N}$, of (η_0, η_1) (see Theorem 3.3), we deduce the strong consistency of $(\widehat{a}_n, \widehat{b}_n)$, $n \in \mathbb{N}$.

With Theorem 3.4 in mind, to prove that $(\widehat{a}_n, \widehat{b}_n)$, $n \in \mathbb{N}$, is asymptotically normal, it is enough to check that the delta method (see, e.g., [44, Theorem 11.2.14]) is applicable. To do so, one simply extends g^{-1} to $\mathbb{R}^2$ by defining

$$g^{-1}(\widehat{\eta}_{0,n}^{\mathrm{CLSE}}, \widehat{\eta}_{1,n}^{\mathrm{CLSE}}) = \mathbb{1}_{\{(0,1)\times\mathbb{R}_{>0}\}}\left(\widehat{\eta}_{0,n}^{\mathrm{CLSE}}, \widehat{\eta}_{1,n}^{\mathrm{CLSE}}\right) \cdot (\widehat{b}_n, \widehat{a}_n), \quad n \in \mathbb{N}.$$

Finally, the representation of the Jacobian matrix $\mathbf{J}$ of g^{-1} with respect to $(\eta_0, \eta_1) \in (0,1) \times \mathbb{R}_{>0}$ could be easily determined. This completes our proof. □

Appendix

A. Two-dimensional affine processes

We recall some important results in the theory of affine processes mainly due to Duffie *et al.* [17]. In their seminal paper affine processes are defined and systematically studied on the $(m+n)$-dimensional state space $\mathbb{R}^m_{\geqslant 0} \times \mathbb{R}^n$. We will simplify notations where this is possible in the two-dimensional case, when the state space is given by $D := \mathbb{R}_{\geqslant 0} \times \mathbb{R}$. In the one-dimensional case, i.e., if D is either $\mathbb{R}_{\geqslant 0}$ or $\mathbb{R}$, all definitions and results stated in this chapter reduces in the obvious way. In the work of Duffie *et al.* the affine processes are allowed to have explosions and killing. Since in this work we only consider *conservative affine processes*, in terms of terminology and notation, we thus follow mainly Keller-Ressel and Mayerhofer [40], where only the conservative case was considered.

We start with a time-homogeneous Markov process with state space D and semigroup (P_t), that is,

$$P_t f(y,x) = \int_D f(\xi) p_t(y,x,d\xi), \quad f \in \mathcal{B}_b(D).$$

Let $((Y,X), (\mathbb{P}_{(y,x)})_{(y,x)\in D})$ be the canonical realization of (P_t) on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geqslant 0})$, where Ω is the set of all càdlàg paths in D and $(Y_t, X_t)(\omega) = \omega(t)$ for $\omega \in \Omega$. Here $(\mathcal{F}_t)_{t\geqslant 0}$ is the filtration generated by (Y,X) and $\mathcal{F} = \bigvee_{t\geqslant 0} \mathcal{F}_t$. The probability measure $\mathbb{P}_x$ on Ω represents the law of the Markov process (Y,X) given $(Y_0, X_0) = (y,x)$.

Definition A.1 (Definition 2.2 [40])**.** The Markov process (Y,X) is called *conservative affine* with state space D, if its transition kernel $p_t(y,x,A) = \mathbb{P}_{(y,x)}((Y_t,X_t)\in A)$ $(t\geqslant 0,\ (y,x)\in D,\ A \in \mathcal{B}(D))$ satisfies the following:

(i) it is stochastically continuous, that is, $\lim_{s\to t} p_s(y,x,\cdot) = p_t(y,x,\cdot)$ weakly for all $t \geqslant 0$, $(y,x) \in D$, and

(ii) (Y,X) is conservative, i.e., $p_t(y,x,D) = 1$ for all $t \geqslant 0$, and

(iii) there exist functions $\phi : \mathbb{R}_{\geqslant 0} \times \mathrm{i}\mathbb{R}^2 \to \mathbb{C}$ and $\psi : \mathbb{R}_{\geqslant 0} \times \mathrm{i}\mathbb{R}^2 \to \mathbb{C}^2$ such that

$$\int_D e^{\langle u,\xi\rangle} p_t(y,x,\mathrm{d}\xi) = \mathbb{E}_x\left[e^{\langle (Y_t,X_t),u\rangle}\right] = \exp\left\{\phi(t,u) + \langle (y,x), \psi(t,u)\rangle\right\} \tag{A.0.1}$$

for all $t \geqslant 0$, $(y,x) \in D$ and $u \in \mathcal{U}$, where $\mathbb{E}_x$ denotes the expectation with respect to $\mathbb{P}_x$.

Definition A.2. An affine process is called *regular* if the right hand derivatives

$$F(u) := \frac{\partial}{\partial t}\phi(t,u)\Big|_{t=0+} \quad \text{and} \quad R(u) := \frac{\partial}{\partial t}\psi(t,u)\Big|_{t=0+}$$

exist for all $u \in \mathcal{U}$, and are continuous at $u = 0$.

Since ϕ is scalar-valued and ψ vector-valued, we have that $F : \mathcal{U} \to \mathbb{C}$ and $R : \mathcal{U} \to \mathbb{C}^2$, respectively. We remark that the stochastic continuity in (i) and the affine property in (iii) together imply the regularity of the functions ϕ and ψ (see [42, Theorem 5.1]) from which it is possible to infer that for $u \in \mathcal{U}$ the *generalized Riccati equations*

$$\begin{cases} \frac{\partial}{\partial t}\phi(t,u) = F(\psi(t,u)), & \phi(0,u) = 0, \\ \frac{\partial}{\partial t}\psi(t,u) = R(\psi(t,u)), & \psi(0,u) = u \in \mathcal{U}, \end{cases} \tag{A.0.2}$$

are satisfied (see [17, Theorem 2.7]).

Duffie *et al.* [17] provide an equivalent characterization of the affine property (iii) in terms of admissible parameters.

Definition A.3. A parameter set $(a, \alpha_{ij}, b, \beta_{ij}, \nu, \mu)$, $i, j \in \{1, 2\}$ is called a *set of admissible parameters* for a conservative affine process with state space D if

- $a \in \mathbb{R}_{\geqslant 0}$ is a constant;
- α_{ij} is a (symmetric) positive semi-definite 2×2-matrix;
- $b = (b_1, b_2) \in D$;
- β_{ij} is a 2×2-matrix with $\beta_{12} = 0$;
- $\nu(\mathrm{d}\xi_1, \mathrm{d}\xi_2)$ is a Lévy measure on D such that

$$\int_{D\setminus\{0\}} \left(\left(\xi_1 + \xi_2^2\right) \wedge 1\right) \nu(\mathrm{d}\xi_1, \mathrm{d}\xi_2) < \infty;$$

- $\mu(\mathrm{d}\xi_1, \mathrm{d}\xi_2)$ is a Lévy measure on D such that

$$\int_{D\setminus\{0\}} \left[\xi_1 + \left(|\xi_2| \wedge \xi_2^2\right)\right] \mu(\mathrm{d}\xi_1, \mathrm{d}\xi_2) < \infty.$$

We remark that our definition of the admissible parameters is a special case of [17, Definition 2.6], since we require here that the set of admissible parameters does not contain parameters corresponding to killing. A sufficient condition for (Y_t, X_t) to be conservative is given by [17, Lemma 9.2] which is included in our definition of the admissible parameters.

The next theorem shows the announced characterization of affine processes through the admissible parameters.

Theorem A.1 (Duffie *et al.* [17], Theorem 2.7)**.** *Suppose $(Y_t, X_t)_{t\geqslant 0}$ is a conservative affine process. Then $(Y_t, X_t)_{t\geqslant 0}$ is a Feller process. Let $\mathcal{A}$ be its infinitesimal generator. Then $C_c^\infty(D)$ is a core of $\mathcal{A}$, $C_c^2(D) \subseteq \mathrm{dom}(\mathcal{A})$, and there exist some admissible parameters $(a, \alpha_{ij}, b, \beta_{ij}, \nu, \mu)$, $i, j \in \{1, 2\}$, such that, for $f \in C_c^2(D)$,*

$$\begin{aligned} (\mathcal{A}f)(y,x) = {} & a\frac{\partial^2 f}{\partial x^2}(y,x) + \alpha_{11} y \frac{\partial^2 f}{\partial y^2}(y,x) + 2\alpha_{12} y \frac{\partial^2 f}{\partial y \partial x}(y,x) + \alpha_{22} y \frac{\partial^2 f}{\partial x^2}(y,x) \\ & + (b_1 + \beta_{11} y)\frac{\partial f}{\partial y}(y,x) + (b_2 + \beta_{21} y + \beta_{22} x)\frac{\partial f}{\partial x}(y,x) \\ & + \int_{D\setminus\{0\}} \left(f(y+\xi_1, x+\xi_2) - f(y,x) - \xi_2 \frac{\partial f}{\partial x}(y,x)\right) \nu(\mathrm{d}\xi) \\ & + y \int_{D\setminus\{0\}} \left(f(y+\xi_1, x+\xi_2) - f(y,x) - \langle \nabla f(y,x), \xi \rangle\right) \mu(\mathrm{d}\xi). \end{aligned} \tag{A.0.3}$$

Moreover, (A.0.1) *holds for all* $(t,u) \in \mathbb{R}_{\geqslant 0} \times \mathcal{U}$ *where* $\phi(t,u)$ *and* $\psi(t,u)$ *solve the generalized Riccati equations* (A.0.2) *with*

$$F(u) = b_1u_1 + b_2u_2 + au_2^2 + \int_{D\setminus\{0\}} \left(e^{u_1\xi_1+u_2\xi_2} - 1 - u_2\xi_2\right) \nu\left(\mathrm{d}\xi_1, \mathrm{d}\xi_2\right), \tag{A.0.4}$$

$$\begin{aligned} R(u) &= \beta_{11}u_1 + \beta_{21}u_2 + \alpha_{11}u_1^2 + 2\alpha_{12}u_1u_2 + \alpha_{22}u_2^2 \\ &\quad + \int_{D\setminus\{0\}} \left(e^{u_1\xi_1+u_2\xi_2} - 1 - u_1\xi_1 - u_2\xi_2\right) \mu\left(\mathrm{d}\xi_1, \mathrm{d}\xi_2\right). \end{aligned} \tag{A.0.5}$$

Conversely, let $(a, \alpha_{ij}, b, \beta_{ij}, \nu, \mu)$*,* $i,j \in \{1,2\}$*, be some admissible parameters. Then there exists a unique, conservative, and regular affine semigroup* $(P_t)_{t\geqslant 0}$ *with infinitesimal generator* (A.0.3)*, and* (A.0.1) *holds for all* $(t,u) \in \mathbb{R}_{\geqslant 0} \times \mathcal{U}$ *where* $\phi(t,u)$ *and* $\psi(t,u)$ *are given by* (A.0.2)*.*

We remark that our preceding formulas are in the spirit of [16]. We can not only characterize the conservative, regular affine processes (Y_t, X_t) with respect to the admissible parameters, but also with respect to the state space (see Duffie *et al.* [17, Corollary 2.10]), roughly speaking. Following Kawazu and Watanabe [38], a conservative affine process with state space $\mathbb{R}_{\geqslant 0}$ is called a *continuous time branching process with immigration* (CBI process). In view of [17, Corollary 2.10], a conservative affine process with state space $\mathbb{R}$ is an OU-type process, that is, X satisfies the Langevin stochastic differential equation

$$\mathrm{d}X_t = \beta_{22}X_t\mathrm{d}t + \mathrm{d}L_t, \quad t \geqslant 0,\ X_0 \in \mathbb{R},$$

where $(L_t)_{t\geqslant 0}$ is a Lévy process.

The latter fact motivates us to decouple the generalized Riccati equation for ψ. On the state space D let us write $\psi = (\psi_1, \psi_2)$ and $R = (R_1, R_2)$, accordingly. We introduce the following useful property.

Proposition A.2. *Let* $(Y_t, X_t)_{t\geqslant 0}$ *be an affine process with state space* D*. Then* $\psi_2(t,u)$ *satisfies*

$$\psi_2(t,u) = e^{\beta_{22}t}u_2$$

for all $t \in \mathbb{R}_{\geqslant 0}$ *and* $u_2 \in i\mathbb{R}$*. Consequently,* $R_2(u) = \beta_{22}u_2$ *for* $u_2 \in i\mathbb{R}$*.*

For a proof we refer to [42] or [17, Theorem 2.7]. As a consequence of Proposition A.2, we see that the two-dimensional affine process $(Y_t, X_t)_{t\geqslant 0}$, for $u = (u_1, u_2) \in \mathcal{U}$, satisfies the following generalized Riccati equations:

$$\begin{aligned} \partial_t\phi(t,u) &= F\left(\psi(t,u)\right), & \phi(0,u) &= 0 \\ \partial_t\psi_1(t,u) &= R_1\left(\psi_1(t,u), e^{\beta_{22}t}u_2\right), & \psi_1(0,u) &= u_1 \\ \psi_2(t,u) &= e^{\beta_{22}t}u_2. \end{aligned}$$

B. Markov chains on uncountable state spaces

We recall definitions of '*irreducibility*', '*aperiodicity*', and '*petite sets*' in the notion of a discrete-time Markov chain on general (uncountable) state spaces mainly due to Meyn and Tweedie [49, 50].

Here, we let $(M_n)_{n\in\mathbb{Z}_{\geqslant 0}}$ be a Markov chain evolving on $(\mathcal{X}, \mathcal{B}(\mathcal{X}))$, where $\mathcal{X}$ is a locally compact separable metric space[1] (in our framework $\mathcal{X} = \mathbb{R}_{\geqslant 0} \times \mathbb{R}$ or $\mathbb{R}_{\geqslant 0}$, respectively) and $\mathcal{B}(\mathcal{X})$ denotes the Borel σ-algebra of $\mathcal{X}$. Let $\mathbf{P}^n(x, \cdot) := \mathbb{P}_n(M_n \in \cdot)$ denote the distribution of M_n with the initial condition $M_0 = x \in \mathcal{X}$.

Intuitively, the classical definition of irreducibility means that the chain has positive probability of eventually reaching any state from any other state. However, since the state space $\mathcal{X}$ may be uncountable, this is impossible. We introduce a weaker definition in the sense of [50].

Definition B.1. The chain $(X_n)_{n\in\mathbb{Z}_{\geqslant 0}}$ is said to be *μ-irreducible*, if for some σ-finite measure μ on $(\mathcal{X}, \mathcal{B}(\mathcal{X}))$

$$\mu(A) > 0 \quad \text{implies} \quad \mathbb{P}_x(\tau_A < \infty) = \mathbb{P}_x(X_n \text{ enters } A) > 0$$

for all initial values $x \in \mathcal{X}$ and $A \in \mathcal{B}(\mathcal{X})$, where the stopping time τ_A is defined for a set $A \in \mathcal{B}(\mathcal{X})$ by $\tau_A = \inf\{n \geqslant 0 : X_n \in A\}$.

If a Markov chain is μ-irreducible, we refer to μ as an *irreducibility measure* for the chain.

Definition B.2. A μ-irreducible Markov chain $(X_n)_{n\in\mathbb{Z}_{\geqslant 0}}$ is said to be *aperiodic* if there do not exist $l \geqslant 2$ and disjoint Borel sets $A_1, A_2, \ldots, A_l \subseteq \mathcal{B}(\mathcal{X})$ with $\mathbf{P}^n(x, A_{i+1}) = 1$ for all $x \in A_i$, $1 \leqslant i \leqslant l-1$, and $\mathbf{P}^n(x, A_1) = 1$ for all $x \in A_l$, such that $\mu(A_1) > 0$ (and hence $\mu(A_i) > 0$ for all i). Otherwise the chain $(X_n)_{n\in\mathbb{Z}_{\geqslant 0}}$ is said to be *periodic* with *period l*.

Recall that a measure μ is the *trivial measure* (or *null measure*) if and only if $\mu(A) = 0$ for all $A \in \mathcal{B}(\mathcal{X})$.

Definition B.3. A nonempty set $C \in \mathcal{B}(\mathcal{X})$ is said to be a *ν_a-petite set* for the chain $(X_n)_{n\in\mathbb{Z}_{\geqslant 0}}$ if there is a probability distribution a on $\mathbb{Z}_{\geqslant 0}$, and a nontrivial measure ν_a such that

$$\sum_{n=0}^{\infty} \mathbf{P}^n(x, A)\, a(n) \geqslant \nu_a(A)$$

for all $x \in C$, $A \in \mathcal{B}(\mathcal{X})$. If the specific measure ν_a is unimportant, we call the set C simply *petite*.

[1] i.e., $\mathcal{X}$ is not necessarily countable

C. Strong law of large numbers and central limit theorem for discrete time square-integrable martingales

Let $(\Omega, \mathcal{F}, (\mathcal{F}_n)_{n\in\mathbb{N}}, \mathbb{P})$ be a filtered probability space.

The following theorem can be considered as a strong law of large numbers for discrete time square-integrable martingales.

Theorem C.1 (Shiryaev [61], Chapter VII, Section 5, Theorem 4; and Barczy et al. [8] Theorem 2.5)**.** *Let $(M_n)_{n\in\mathbb{N}}$ be a square-integrable martingale with respect to the filtration $(\mathcal{F}_n)_{n\in\mathbb{N}}$ such that $M_0 = 0$ almost surely and $\lim_{n\to\infty}\langle M\rangle_n = \infty$ almost surely, where $(\langle M\rangle_n)_{n\in\mathbb{N}}$ denotes the predictable quadratic variation process of M. Then*

$$\frac{M_n}{\langle M\rangle_n} \xrightarrow{a.s} 0 \quad \text{as } n \to \infty.$$

The next theorem is a central limit theorem for discrete time square-integrable martingales.

Theorem C.2 (Jacod and Shiryaev [27], Chapter VII, Theorem 3.33; and Barczy et al. [8] Theorem 2.6)**.** *Let $\{(M_{n,k}, \mathcal{F}_{n,k}) : k = 0, 1, \ldots, k_n\}_{n\in\mathbb{N}}$ be a sequence of d-dimensional square-integrable martingales with $M_{n,0} = \mathbf{0}$ such that there exists some symmetric, positive semi-definite non-random matrix $\mathbf{D} \in \mathbb{R}^{d\times d}$ such that*

$$\sum_{k=1}^{k_n} \mathbb{E}\left[(M_{n,k} - M_{n,k-1})(M_{n,k} - M_{n,k-1})^\top \,\middle|\, \mathcal{F}_{n,k-1}\right] \to \mathbf{D} \quad \text{a.s. as } n \to \infty,$$

and for all $\varepsilon \in \mathbb{R}_{>0}$,

$$\sum_{k=1}^{k_n} \mathbb{E}\left[\|M_{n,k} - M_{n,k-1}\|^2 \mathbb{1}_{\{\|M_{n,k}-M_{n,k-1}\|\geqslant\varepsilon\}} \,\middle|\, \mathcal{F}_{n,k-1}\right] \to 0 \quad \text{a.s. as } n \to \infty.$$

Then $\sum_{k=1}^{k_n}(M_{n,k} - M_{n,k-1}) = M_{n,k_n}$ converges in distribution to a d-dimensional normal distribution with mean vector $\mathbf{0}$ and covariance matrix $\mathbf{D}$.

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