On minimal disjoint degenerations with preprojective and preinjective direct summands over tame path algebras

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For my grandma Edith Kirchner,
the best grandmother one can wish for

Contents

Introduction

Let k be an algebraically closed field. Given an algebraic group G and a G-variety X it is an interesting question how to describe the G-orbits in X. Moreover, a standard fact from algebraic geometry guarantees that any G-orbit in X is a locally closed subset of X, whose boundary is a union of G-orbits having strictly smaller dimensions. In this context, if x, y are points in X such that $G.y \subseteq \overline{G.x}$, the G-orbit of y is called a degeneration of the G-orbit of x. This defines a partial order on the set of orbits and poses the problem of a classification, the so called degeneration problem. In general, neither the G-orbits nor their closures can be computed systematically.

One of the most famous solved examples for a degeneration problem is the one of $GL_d(\mathbb{C})$-orbits in $\mathbb{C}^{d\times d}$ where $GL_d(\mathbb{C})$ acts on $\mathbb{C}^{d\times d}$ by conjugation. The orbits are known well in this case. Any orbit has a representative in Jordan normal form J. Therefore, it can be described by means of continuous parameters, the eigenvalues of J, together with some discrete parameters, the partitions that encode the block sizes of J with respect to the different eigenvalues.

To solve the degeneration problem it suffices to understand the degeneration behaviour of the nilpotent orbits. There are two nice criteria, a partition criterion and a rank criterion, to decide whether a nilpotent orbit lies in the closure of another one. Both only use terms of representation theory to reach this decision. If J_p and J_q are nilpotent corresponding to the partitions p, q, then it holds:

$$\begin{aligned} \forall i \in \{1, \ldots, d\} : \sum_{k=1}^{i} p_k \leq \sum_{k=1}^{i} q_k \quad &\Leftrightarrow \quad GL_d(\mathbb{C}).J_p \subseteq \overline{GL_d(\mathbb{C}).J_q} \\ &\Leftrightarrow \quad \forall i \in \{1, \ldots, d\} : Rank(J_p^i) \leq Rank(J_q^i). \end{aligned}$$

From another point of view, $\mathbb{C}^{d\times d}$ can be interpreted as the variety $Mod^d_{\mathbb{C}[X]}(\mathbb{C})$ of d-dimensional modules over $\mathbb{C}[X]$, the path algebra of the extended Dynkin quiver with one point and one loop. The $GL_d(\mathbb{C})$-orbits correspond to the isomorphism classes of $\mathbb{C}[X]$-modules. Thus the just mentioned example is also called the solution of the degeneration problem for isomorphism classes of d-dimensional $\mathbb{C}[X]$-modules.

A natural generalization is to consider a path algebra kQ over any extended Dynkin quiver Q instead of the algebra $\mathbb{C}[X]$. The $GL_d(k)$-orbits in $Mod^d_{kQ}(k)$ were classified independently by Donovan-Freislich and Nazarova (cp. [9], [13]), and the correspondence between the isomorphism classes of d-dimensional kQ-modules and the $GL_d(k)$-orbits in $Mod^d_{kQ}(k)$ allows us again to speak of degenerations of d-dimensional kQ-modules. We write $M \leq_{deg} N$ if the orbit corresponding to N lies in the closure of the orbit corresponding to M.

If kQ is finite dimensional, the partition criterion and the rank criterion fortunately have some kind of generalization, the partial orders $\leq_{ext}$ and $\leq$. Both are defined in terms of representation theory and make the problem accessible for methods of that domain. Furthermore, degenerations are built up of minimal disjoint degenerations. N is called a minimal disjoint degeneration of M, if M and N are adjacent relative to the degeneration order and have no common direct summands. Then N is isomorphic to the direct sum of two indecomposables U and V. Hence, the degeneration problem reduces to the classification of minimal disjoint degenerations, which leads, up to duality and tilting, to the distinction of the following cases:

(a) U projective simple, V preprojective;

(b) U and V regular;

(c) U projective simple, V regular;

(d) U projective simple, V preinjective.

For (a) the combination of a periodicity theorem and computer calculations results in a complete classification (see [6]) while (b) is equivalent to the degeneration problem of nilpotent representations of some oriented cycle and can be found in [11]. The remaining possibilities are investigated in parallel work with Wolters. She treated the case (c) (see [16]).

The present dissertation considers mainly the case (d). On the search of all modules M such that $M <_{deg} U \oplus V$ is minimal there arises one difficulty. Even in small examples so many modules consisting of indecomposables of any connected components of the Auslander-Reiten quiver have to be taken into account that one easily loses track. It is due to the first major result of this work, a reduction theorem, that we get a chance to avoid this complication. Any minimal degeneration $M <_{deg} U \oplus V$ is given by an exact sequence $0 \to U \xrightarrow{\epsilon} M \xrightarrow{\pi} V \to 0$. Choosing a directed decomposition $M = M_1 \oplus M_2$ and denoting the component of ϵ that ends in M_1 by ϵ_1 the reduction theorem says that $Coker(\epsilon_1)$ is again indecomposable and that $M_1 <_{deg} U \oplus Coker(\epsilon_1)$ is also a minimal disjoint degeneration.

A skillful combination of the reduction theorem with tilting theory enables us to deduce an inductive codimension formula, which indicates that the codimension of a general minimal disjoint degeneration $M <_{deg} U \oplus V$ can be written as a finite sum of codimensions of certain minimal degenerations $M' <_{deg} U' \oplus V'$ where M' has no proper directed decomposition. This leads in particular to the consideration of modules M_μ that come from a single regular tube and degenerate into $U \oplus V$. It turns out that this sort of degenerations has bounded codimension.

Exploiting this fact and the codimension formula yields the second main result of this thesis. The codimension of any minimal $M < U \oplus V$ is bounded. Furthermore, the proof points out a method to gain the bound. It suffices to inspect all regular M_μ of "small" dimensions, what can be done with the help of a computer. We achieve an improvement of the second main result: Any minimal degeneration $M <_{deg} U \oplus V$ has codimension one.

This circumstance has two nice consequences. First, from the geometrical point of view a minimal disjoint degeneration is as simple as possible. In particular, results of Bongartz and Zwara insure that the orbit corresponding to M is regular at the point $U \oplus V$ (see [4], [21]).

Second, we obtain as in case (a) a minimality preserving periodicity theorem, which reduces the classification of all minimal disjoint degenerations to a finite problem.

Besides, together with results of Bongartz and Fritzsche in case (a) resp. Wolters for the cases (b) and (c), it follows that the codimension of an arbitrary minimal disjoint degeneration over a tame path algebra is at most 2 (see [6], [16]).

This dissertation consists of 5 chapters and an appendix. The first chapter recalls some basic facts on the representation theory of extended Dynkin quivers that are needed in the sequel. Furthermore, the degeneration problem and results on it for tame path algebras are summarized.

The second chapter adapts some results originating from [6] to our situation. It is shown that in case of a tame path algebra they hold with more generality. The just mentioned reduction theorem is stated and proved. The chapter closes with some remarks on the defect and the position of the preprojective and the preinjective parts of M.

In the third chapter we introduce a general technique to analyse the codimension of minimal disjoint degenerations. The inductive codimension formula is deduced and its consequences for our situation are explained. Then we consider minimal degenerations $M < U \oplus V$ with preprojective M. We show that their codimension is always one.

The fourth part is devoted to the study of the regular part of M. We introduce a test criterion for degenerations and solve the case $-\partial(U) = \partial(V) = 1$ completely by means of theoretical arguments. Subsequently, we concentrate on the case where $-\partial(U) \geq 2$ or $\partial(V) \geq 2$. We derive the periodicity theorem, which preserves codimensions. But notice, its minimality preserving property is not clear at this point. After that, the technically most complicated part of this thesis follows, the consideration of minimal degenerations $M <_{deg} U \oplus V$ where M comes from a single regular tube. If the tube is homogeneous, we describe explicitly the shape of M and show that the codimension is one. For the remaining three non-homogeneous tubes we reduce the problem to the inspection of modules with "small" dimension and prove that the codimension is bounded in this case. The chapter closes with the derivation of the second main result of this dissertation. The codimension of any minimal degeneration $M <_{deg} U \oplus V$ is bounded.

The last chapter deals with the computer program that determines all minimal $M_\mu < U \oplus V$, where M_μ comes from a single regular tube and has "small" dimension. We explain the strategy of the program and summarize the essential results of its calculations. Finally, we reap the benefits of the previous work. Based on the computer results we conclude that the codimension of any minimal degeneration $M <_{deg} U \oplus V$ is one. Moreover, we deduce the above mentioned consequences on the singularities resp. the periodicity theorem and obtain a finite problem.

The appendix contains the lists produced by the computer program of the fifth chapter and some further lists, which are needed in chapter 4.

At this point I wish to thank my advisor Prof. Dr. Klaus Bongartz for suggesting me this interesting topic, for his great support during the research and for many helpful discussions, especially on chapter 3. My special thanks also go to Isabel Wolters for a good cooperation, in particular for providing me parts of her computer program. This saved me much time and efforts. Furthermore, I want to express my gratitude to Prof. Dr. W. Borho for employing me at his chair, to Thomas Konrad for the good collaboration and to all members of the Research Group Algebra/Number Theory for the pleasant working atmosphere.

I am grateful to my parents, my brother and my friends for much moral support and that I always can rely on them. I wish to thank my lovely wife Jenia for checking earlier versions of this dissertation on spelling and grammar and for never giving up to improve my English. Furthermore, I am thankful to her for giving me strength, for her love and her ability to put me into good humor.

Chapter 1

Basic notions and facts

1.1 Extended Dynkin quivers

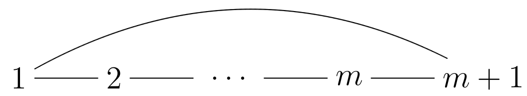
Throughout this thesis let k be an algebraically closed field. We make the following assumptions, that stay valid until the end of this paper:

- The quivers we talk about are always without oriented cycles.
- All considered modules are finite dimensional.

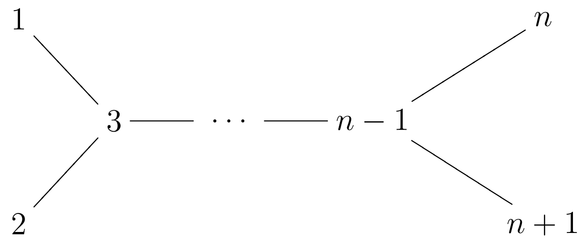
We will investigate degenerations of modules over tame path algebras. To do this we first want to recall some basic facts about the representation theory of extended Dynkin quivers (see [8], [15] or [7]), that will be used in the sequel.

Let Q be an extended Dynkin quiver, i.e. a quiver whose underlying graph $|Q|$ is one of the following:

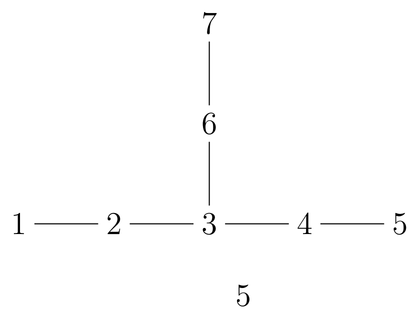
$\tilde{A}_m, m \geq 1$:



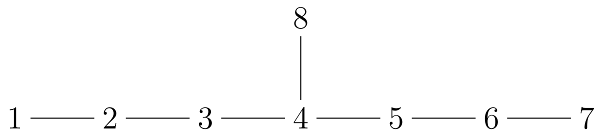
$\tilde{D}_n, n \geq 4$:



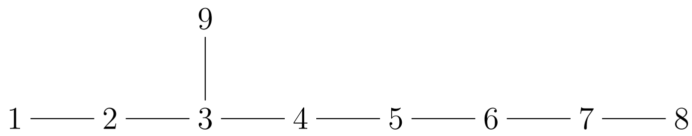
$\tilde{E}_6$:



$\tilde{E}_7$:



$\tilde{E}_8$:



We denote the set of points by Q_0 and the set of arrows by Q_1. The projective resp. injective indecomposable modules corresponding to the points $x \in Q_0$ are denoted by $P(x)$ resp. $I(x)$. With their help, the dimension vector of a kQ-module X can be defined. It is the vector $\underline{\dim}(X) := ([(P(x), X])_{x\in Q_0} = ([X, I(x)])_{x\in Q_0}$. Here and in the following $[X, Y]$ resp. $[X, Y]^1$ are abbreviations of $dim_k Hom_{kQ}(X, Y)$ resp. $dim_k Ext^1_{kQ}(X, Y)$.

Let $\langle \_, \_ \rangle$ be the Euler form of Q and q be the Tits form. q is positive semi-definite and its radical is generated by a uniquely determined vector $\delta \in \mathbb{N}^{|Q_0|}$, with at least one entry equal to one.

Definition 1.1.1 *The defect of a kQ-module X is $\partial(X) := \langle \delta, \underline{\dim}(X) \rangle$.*

Since for modules Y and X there is the relation $\langle \underline{\dim}(Y), \underline{\dim}(X) \rangle = [Y, X] - [Y, X]^1$, we derive $\partial(X) = [E, X] - [E, X]^1$ for all modules E with $\underline{\dim}(E) = \delta$.

The Auslander-Reiten translations DTr resp. TrD are written shortly τ resp. τ^-. We list some of their well known properties. Let X and Y be indecomposable modules.

(a) If X is not projective, then τX is indecomposable, with $\tau^- \tau X \cong X$ and $\partial(\tau X) = \partial(X)$.

(b) If X is not injective, then $\tau^- X$ is indecomposable, with $\tau\tau^- X \cong X$ and $\partial(\tau^- X) = \partial(X)$.

(c) X projective (resp. injective) implies $\tau X = 0$ (resp. $\tau^- X = 0$).

(d) Auslander-Reiten formula: $Hom(X, \tau Y) \cong DExt^1(Y, X) \cong Hom(\tau^- X, Y)$.

The Auslander-Reiten quiver Γ_{kQ} breaks up into a preprojective component $\mathcal{P}$, a $\mathbb{P}^1(k)$-family of regular connected components, the so called regular tubes $\mathcal{T}_\mu$, and a preinjective one $\mathcal{I}$. Therefore a kQ-module X can be decomposed into $X = X_P \oplus \bigoplus_{\mu\in\mathbb{P}^1} X_\mu \oplus X_I$ with $X_P \in add(\mathcal{P})$, $X_\mu \in add(\mathcal{T}_\mu)$ and $X_I \in add(\mathcal{I})$. Recall, if $\mathcal{X}$ is a family of kQ-modules, $add(\mathcal{X})$ denotes the full subcategory of $mod(kQ)$ consisting of all modules that are isomorphic to direct summands of finite direct sums of modules from $\mathcal{X}$.

The preprojective component $\mathcal{P}$ of the Auslander-Reiten quiver is by definition the connected component without oriented cycles whose points are the isomorphism classes of indecomposables X such that $\tau^k X$ is projective for some $k > 0$. But also the defect characterizes the modules of this component. It holds: An indecomposable X is preprojective if and only if $\partial(X) < 0$. Nonzero homomorphisms in $\mathcal{P}$ always go from the left to the right in the following sense: Let X and Y be two preprojective indecomposables. If we have $Hom(X,Y) \neq 0$, then there is a path in the Auslander-Reiten quiver from X to Y.

Dually the preinjective component $\mathcal{I}$ has no oriented cycles and the points are isomorphism classes of indecomposables X such that $\tau^{-k}X$ is injective for a $k > 0$ or equivalently that have positive defect. If X, Y are indecomposable preinjective modules, $Hom(X,Y) \neq 0$ implies that there is a path from X to Y.

An indecomposable module X is called regular if for every $k \in \mathbb{Z}$ the module $\tau^k X$ is neither projective nor injective. Consequently, they can be characterized as those indecomposables whose defects are zero. The full subcategory of regular modules is denoted by $\mathcal{R}$. It is abelian, closed under extensions and the Auslander-Reiten translations are inverse equivalences in this subcategory. Every regular module R can uniquely be written as $R = \oplus_{\mu \in \mathbb{P}^1(k)} R_\mu$, where $R_\mu \in add(\mathcal{T}_\mu)$.

For each μ there exists a $p_\mu \in \mathbb{N}$ such that the full subcategory $add(T_\mu)$ is equivalent to the category $\mathcal{N}(p_\mu)$ of nilpotent representations of the oriented cycle with p_μ points. To be more precise: $add(\mathcal{T}_\mu)$ contains exactly p_μ isomorphism classes of - in this subcategory - simple modules. These modules are called regular simple. They form a τ-orbit and the sum of their dimension vectors is δ.

Every $R \in \mathcal{T}_\mu$ admits a unique composition series in $\mathcal{R}$. There is exactly one regular simple submodule S contained in R. This module is defined to be the regular socle of R, denoted $Soc(R)$. The regular composition factors are then (from the bottom) $S, \tau^- S, \ldots, \tau^{-l} S$, for some $l \in \mathbb{N}$. We call $Top(R) := \tau^l S$ the regular top of R and $l(R) := l + 1$ the regular length of R. In addition, the multiplicity of any $E \in \mathcal{T}_\mu$ in the regular composition series of R is abbreviated by $l_E(R)$. The number p_μ is also called period of the tube $\mathcal{T}_\mu$, since we have $\tau^{p_\mu} R = R$ for all $R \in \mathcal{T}_\mu$ and not only for the simple ones. For every quiver Q there are at most three $\mu \in \mathbb{P}^1(k)$ with $p_\mu \neq 1$. The tubes with $p_\mu = 1$ are called homogeneous.

Between regular indecomposables of different tubes there are no nonzero homomorphisms. To compute the homomorphism space dimensions between regular indecomposables of the same tube there is the following lemma.

Lemma 1.1.2 *Let R_1 and R_2 be regular indecomposable modules belonging to the tube $\mathcal{T}_\mu$ of period p_μ, then*

$$[R_1, R_2] = min(l_{Top(R_1)}(R_2), l_{Soc(R_2)}(R_1)).$$

In particular for homogeneous tubes $[R_1, R_2] = min(l(R_1), l(R_2))$.

Furthermore there are neither nonzero homomorphisms from preinjective to preprojective or regular modules nor from regular to preprojective modules. We can define a transitive relation $\preccurlyeq$ on the set of indecomposable kQ-modules. We denote $X \preccurlyeq Y$ and call X a predecessor of Y if there is a path of non-invertible homomorphisms $X = X_1 \to X_2 \to \ldots \to X_t = Y$. This relation is a partial order on the preprojective and the preinjective connected component of the Auslander-Reiten quiver. Because of the previous we conclude, that $Hom(X, Y) \neq 0$ for indecomposable X, Y implies $X \preccurlyeq Y$.

The additivity of the defect function ∂ on exact sequences gives rise to a property of preprojective (resp. preinjective) modules with defect -1 (resp. 1), which is of convenience.

Lemma 1.1.3 *(a) Suppose $0 \neq \varphi : X \to Y$ is a homomorphism of preprojective modules such that $\partial(X) = -1$, then φ is injective.*

(b) If $\varphi : X \to Y$ is a non-zero homomorphism of preinjective modules such that $\partial(Y) = 1$, then φ is injective.

Finally, let us denote the Coxeter transformation by C. C induces an automorphism of finite order $p(Q)$ on $\mathbb{Z}^{|Q_0|}/\mathbb{Z}\delta$. More details gives the following formula, which holds for every indecomposable module X and which is for a certain natural number $\epsilon(Q)$ of the form

$$C^{p(Q)}(\underline{\dim}(X)) = \underline{\dim}(X) + \epsilon(Q)\partial(X)\delta. \tag{1.1}$$

The positive integer $p(Q)$ is the Coxeter number of Q, which should not be confused with the definition of the Coxeter number in Lie Theory. Since for non-injective indecomposables X there is the equality $\underline{\dim}(\tau^- X) = C^{-1}(\underline{\dim}(X))$, this formula shows that the dimension vectors on the preprojective component have some kind of periodicity. The dual statement is valid for the preinjective component. Besides, we have

Lemma 1.1.4 *Let P be a projective and I be an injective indecomposable. If $k \in \{0, 1, \ldots, p(Q) - 1\}$, then*

$$\underline{\dim}(\tau^{-k}P) \leq \epsilon(Q)(-\partial(P))\delta \quad \text{and} \quad \underline{\dim}(\tau^k I) \leq \epsilon(Q)(-\partial(I))\delta.$$

Proof. To avoid some minus signs the injective case will be shown. Assume $\underline{\dim}(\tau^i I) \not\leq \epsilon(Q)\partial(I)\delta$, then $\eta := \underline{\dim}(\tau^i I) - \epsilon'(Q)\partial(I)\delta \not\leq 0$. On the other hand, η is a root and it were positive for this reason. $\partial(\eta) = \partial(I)$ would imply that η is the dimension vector of an indecomposable preinjective module Z. But then we would have $\underline{\dim}(I) = c^{p(Q)-i}(\underline{\dim}(Z)) = \underline{\dim}(\tau^{p(Q)-i}Z)$, which does not fit with the injectivity of I. □

The following table collects some of the previously mentioned specific dates of the extended Dynkin quivers. In almost all cases the informations are independent from the orientation except of the case, where Q is of type $\tilde{A}_n$. In the case we have to count the numbers p resp. q of arrows that go clockwise resp. anticlockwise. We write $\tilde{A}_n(p,q)$ for this fact.

Q	δ	$\epsilon(Q)$	$p(Q)$	periods of non-hom. tubes
$\tilde{A}_n(p,q)$	$(1,1,\ldots,1,1)$	$\frac{n}{gcd(p,q)}$	$lcm(p,q)$	p,q
$\tilde{D}_{2n+1}$	$(1,1,2,\ldots,2,1,1)$	2	$2(2n-1)$	$2n-1,2,2$
$\tilde{D}_{2n}$	$(1,1,2,\ldots,2,1,1)$	1	$2n-2$	$2n-2,2,2$
$\tilde{E}_6$	$(1,2,3,2,1,2,1)$	1	6	$3,3,2$
$\tilde{E}_7$	$(1,2,3,4,3,2,1,2)$	1	12	$4,3,2$
$\tilde{E}_8$	$(2,4,6,5,4,3,2,1,3)$	1	30	$5,3,2$

1.2 Degenerations of d-dimensional modules

In this section the degeneration problem and some results on it for tame quiver algebras will be explained.

Let A be a finite dimensional k-algebra. We choose a basis $\mathfrak{B} = (a_1 = 1, a_2, \ldots a_l)$ of A with the corresponding structure constants α_{ijk} defined by

$$a_i a_j = \sum_{k=1}^{l} \alpha_{ijk} a_k.$$

To store the structure of a d-dimensional A-module M it suffices to memorize the l-tuple $m = (m_1 = E_d, m_2 \ldots m_l)$ of $d \times d$-matrices whose components m_i are the matrix representations of the left multiplications with a_i after fixing a basis on M.

Definition 1.2.1 *The affine variety of d-dimensional A-modules is the set*

$$Mod_A^d(k) := \left\{ m = (m_1, m_2, \ldots, m_l) \mid m_i m_j = \sum_{k=1}^{l} \alpha_{ijk} m_k \right\} \subseteq (k^{d\times d})^l.$$

The general linear group $GL_d(k)$ acts on $Mod_A^d(k)$ morphically by conjugation. The orbits of this action correspond to the isomorphism classes of d-dimensional A-modules. From algebraic geometry we know that each orbit $GL_d(k).m$ is a locally closed subset of $Mod_A^d(k)$ whose dimension is $d^2 - [M, M]$ and whose boundary is a union of orbits of strictly smaller dimensions. Thus the following defines a partial order on the set of isomorphism classes of d-dimensional modules.

Definition 1.2.2 *Let M and N be A-modules of dimension d.*

(a) We write $M \leq_{deg} N$ and call N a degeneration of M if $\overline{GL_d(k).m} \supseteq GL_d(k).n$. Alternatively we call M a deformation of N.

(b) *A degeneration* $M <_{deg} N$ *is called minimal if there is no chain* $M <_{deg} L <_{deg} N$.

(c) *The codimension of a degeneration* $M <_{deg} N$ *is defined by* $Codim(N, M) := dim(GL_d(k).m) - dim(GL_d(k).n)$ *which is obviously the same as* $[N, N] - [M, M]$.

It would be interesting to understand in which orbit closures the orbit corresponding to a given module N is contained. Zwara gives in [19] a criterion that translates the problem into terms of representation theory.

Theorem 1.2.3 *(Zwara, [19], 1.1, p. 2) Let* M *and* N *be two* A*-modules of dimension* d*. Then* M *degenerates into* N *if and only if there exists some* A*-module* Z *and an exact sequence* $0 \to Z \to Z \oplus M \to N \to 0$.

Nevertheless, for a general algebra A the explicit description of the orbit closures is an unsolved problem since not even the indecomposable modules are classified up to isomorphism. On the contrary, for modules over the path algebra of a Dynkin quiver and for preprojective modules over a tame path algebra all minimal degenerations are known (see [6], [12]). In these cases representation theoretical descriptions of the $\leq_{deg}$-order by the $\leq_{ext}$-order and the $\leq$-order are very helpful.

Definition 1.2.4 *Let* M *und* N *be* A*-modules with the same dimension vector. We define as follows*

(a) $M \leq_{ext} N$ *if there are modules* M_i, U_i, V_i, $1 \leq i \leq t$ *and exact sequences* $0 \to U_i \to M_i \to V_i \to 0$ *such that* $M_1 = M$, $M_{i+1} = U_i \oplus V_i$ *and* $N = U_t \oplus V_t$ *for some* $t \in \mathbb{N}$.

(b) $M \leq N$ *if the inequality* $[M, X] \leq [N, X]$ *holds for all* $X \in A$*-mod.*

$M \leq N$ is equivalent to $[X, M] \leq [X, N]$ for all $X \in A$-mod, since for modules M, N of the same dimension vector and all non-injective modules X there is a formula of Auslander and Reiten (see [1]):

$$[N, X] - [M, X] = [\tau^{-}X, N] - [\tau^{-}X, M]. \tag{1.2}$$

In general we have

$$M \leq_{ext} N \quad \Rightarrow \quad M \leq_{deg} N \quad \Rightarrow \quad M \leq N,$$

but Bongartz showed in [5] that the reverse implications also hold for preprojective modules. For extended Dynkin quivers the situation is even better. There is the following

Theorem 1.2.5 *Let* $A = kQ$ *be a path algebra whose underlying graph is an extended Dynkin diagram. Then:*

(a) *(Bongartz, [3], 5.1, p. 666) The partial orders $\leq_{deg}$ and $\leq$ coincide.*

(b) *(Zwara, [18], p. 72; [19], 1.5, p. 3) The partial orders $\leq_{ext}$ and $\leq_{deg}$ coincide.*

Furthermore Bongartz proved in [4] a theorem that identifies the building blocks of the minimal degenerations provided the three partial orders all agree as in the preprojective and the tame case.

Theorem 1.2.6 *([4], 6.1, p. 593) Let A be an algebra, $\mathcal{C}$ be a full subcategory of A-mod, which is closed under isomorphisms, extensions and direct sums, and M, N be two modules in $\mathcal{C}$. Assume the partial orders $\leq_{ext}$ and $\leq$ coincide on $\mathcal{C}$. Then N is a minimal degeneration of M if and only if there is an exact sequence $0 \to U \to M' \to V \to 0$ with the following properties:*

(a) *U and V are indecomposable such that $M = M' \oplus U^{p-1} \oplus V^{q-1} \oplus X$ and $N = U^p \oplus V^q \oplus X$. Here $U \oplus V$ and $M' \oplus X$ have no common direct summands.*

(b) *$U \oplus V$ is a minimal degeneration of M'.*

(c) *Any common indecomposable T direct summand of M and N that is not isomorphic to V satisfies $[T, N] = [T, M]$.*

(d) *Any common indecomposable T direct summand of M and N that is not isomorphic to U satisfies $[N, T] = [M, T]$.*

The modules U, V, M' and the numbers p, q are uniquely determined by M and N. Moreover, we have $Codim(N, M) = Codim(U \oplus V, M') + h(p+q-2)$, where $h = 1$ for $V \ncong U$ and $h = 2$ for $V \cong U$.

Convention 1.2.7 (a) *Let U and V be indecomposable modules and M be a module such that M and $U \oplus V$ have no common direct summand. If $M <_{deg} U \oplus V$ is minimal, we call $U \oplus V$ a minimal disjoint degeneration of M.*

(b) *If the partial orders $\leq_{ext}$ and $\leq$ are equivalent we simply write $\leq$.*

Dynkin case: In the Dynkin case there are only preprojective modules. The technique of shrinking appropriate arrows (see [12]) reduces the infinite families A_m resp. D_n to the cases $m \leq 3$ resp. $n \leq 6$. The equivalence of $\leq_{ext}$ and $\leq$ makes the problem finite and computable. Markolf determined in [12] all minimal disjoint degenerations with the help of a computer. The codimension of a minimal disjoint degeneration is always 1.

Wild case: In contrast to the Dynkin case, a minimal disjoint degeneration of preprojective modules over a wild path algebra can become any complicated. Olbricht showed in [14] several oddities. One of them is the following: Given any natural number k, there exists a wild quiver with indecomposables U and V such that any $i \in \{2, \ldots, k\}$ occurs as codimension of some minimal deformation of $U \oplus V$. In particular, any natural number occurs as codimension.

Another question that arises in the geometric study of $Mod_A^d(k)$ should only be briefly mentioned. It concerns singularities in the orbit closures. Let $\overline{GL_d(k).m} \subseteq Mod_A^d(k)$ be the orbit closure of some module M. If N is a module with $M \leq_{deg} N$ it is a matter of interest wether the corresponding point $n \in Mod_A^d(k)$ is a singular point of $\overline{GL_d(k).m}$ or not. Moreover, one is interested in identifying the "type" of singularity that occurs in n. For the purpose of such a classification, Hesselink introduced the notion of smoothly equivalence (see [4], [10] or [21]). In case of a representation finite or a tame path algebra there are some results pertaining degenerations of codimension one and two due to Zwara (see [21], [17], [20]). But here we only need the following theorem, which holds in general and is due to Bongartz.

Theorem 1.2.8 *([4], 6.2, p. 597) Let A be an arbitrary finite dimensional k-algebra, M and N be two disjoint modules corresponding to the points $m, n \in Mod_A^d(k)$. If $M \leq_{deg} N$ is of codimension $Codim(N, M) = 1$, then N is either indecomposable or else the direct sum of two indecomposables. In the second case n is a regular point of $\overline{GL_d(k).m}$.*

1.3 Degenerations of modules over tame path algebras

From now on we deal with an extended Dynkin quiver Q unless something else is stated. Let $A = kQ$ be the tame path algebra of Q. In view of the theorems 1.2.5 and 1.2.6 it is convenient to consider minimal disjoint degenerations $M < U \oplus V$ with U, V indecomposable. Exploiting the technique of shrinking and inserting suitable arrows it is moreover sufficient to consider the quivers $\tilde{A}_m$, $\tilde{D}_8$, $\tilde{E}_6$, $\tilde{E}_7$ and $\tilde{E}_8$ where $m \leq 3$.

The structure of the Auslander-Reiten quiver for tame path algebras described in section 1.1 leads, up to duality, to the distinction of the following cases:

(a) U, V preprojective.

(b) U, V regular.

(c) U preprojective, V regular.

(d) U preprojective, V preinjective.

The case (b) is already known for a long time (see [11]). In the cases (a), (c) and (d), i.e. where U is preprojective, a further simplification can be achieved with the aid of tilting theory. Thereto we have to recall some notions and geometric properties of tilting theory, which can originally be found in [4] resp. [15] and are true actually for arbitrary A.

Let T be a tilting module and $B := End_A(T)^{op}$ be the opposite algebra of $End_A(T)$. T induces a torsion theory on the $mod(A)$, the category of finite dimensional A-modules. It is one of the main results of tilting theory that the functors $F := Hom(T_B, \_)$ and $G := T_B \otimes \_$ induce inverse equivalences between the torsion part $\mathcal{T}(T) = Ker(Ext(T, \_))$ of $mod(A)$ and $\mathcal{Y}(T) := Ker(Tor_1^B(T, \_))$ in $mod(B)$. Furthermore the Grothendieck groups of $mod(A)$ and $mod(B)$ are isomorphic under the map $X \mapsto FX - Ext^1(T, X)$. In particular, for all X A-modules with dimension vector $\underline{d}$, the B-module FX has the same dimension vector $\underline{e}$.

Definition 1.3.1 *Using the above notations we define*

$$\mathcal{T}(T, d) := \{M \in \mathcal{T}(M) | \underline{\dim} M = \underline{d}\} \quad and \quad \mathcal{Y}(T, \underline{e}) := \{X \in \mathcal{Y}(T) | \underline{\dim}(X) = \underline{e}\}.$$

Theorem 1.3.2 *([4], 4.2, p. 586) Under the above assumptions there exists a bijection between the $GL_d(k)$-stable subsets of $\mathcal{T}(T, \underline{d})$ and the $GL_e(k)$-stable subsets of $\mathcal{Y}(T, \underline{e})$ that*

- *maps the orbit of $M \in \mathcal{T}(T, \underline{d})$ to the orbit corresponding to FM and*
- *preserves closures, inclusions, codimensions and types of singularities occurring in orbit closures.*

In our special case we choose the slice $\mathcal{S}$ in the preprojective component of Γ_{kQ} that has U as its only sink and define T by $T := \bigoplus_{X \in \mathcal{S}} X$. Part (a) of the following lemma guarantees that any module M that degenerates into $U \oplus V$ belongs to $\mathcal{T}(T, \underline{\dim}(U \oplus V))$.

Lemma 1.3.3 *Let U be preprojective indecomposable, V be indecomposable with $U \preccurlyeq V$ and $M < U \oplus V$ be a degeneration.*

(a) If W is an indecomposable direct summand of M, then $U \prec W \prec V$.

(b) If V is regular, then M has at most one regular direct summand. It is a proper submodule of V.

Proof. To verify (a) it remains to consider the inequalities

$$0 \neq [W, M] \leq [W, U \oplus V] \quad \text{and} \quad 0 \neq [M, W] \leq [U \oplus V, W].$$

Hence, $U \preccurlyeq V$ supplies $U \prec W \prec V$.
(b) Now let V come from the regular tube $\mathcal{T}_\mu$ and $E_1, \ldots, E_{p_\mu}$ be the regular simples of $\mathcal{T}_\mu$. Suppose $Soc(V) = E_1$. Then it holds

$$[E_k, M] \quad \leq \quad [E_k, U \oplus V] = \begin{cases} 1 & \text{if } k = 1 \\ 0 & \text{if } k \geq 2. \end{cases}$$

So M has at most one regular direct summand M_R. M_R has the same regular socle as V. Furthermore it is a submodule of V, because otherwise $Top(M_R)$ would occur with greater multiplicity in the regular composition series of M_R than in the one of V, which is forbidden by

$$l_{Top(M_R)}(M_R) = [M_R, M] \leq [M_R, U \oplus V] \leq l_{Top(M_R)}(V).$$

Finally, $V = M_R$ implies $M_P < U$. This does not fit with theorem 1.2.6. □

Consequently, theorem 1.3.2 furnishes that the problem of finding all modules degenerating to $U \oplus V$ can be transferred via tilting with $F := Hom(T, \_)$ to the equivalent problem of finding all deformations of $FU \oplus FV$. Since the tilting module T was defined with the help of a slice, B is a path algebra whose underlying graph is again $|Q|$. Thus we are allowed to assume that Q has only one source and U is the only projective simple module of kQ.

Preprojective case: Bongartz and Fritzsche showed for (a) in [6] that the periodicity of the preprojective component of the Auslander-Reiten quiver leads to a periodic behaviour of the minimal disjoint degenerations in $\mathcal{P}$. Thus their investigation is a finite problem. They ascertained the minimal disjoint degenerations with a computer program. One further result the computer produced concerns the codimension of a minimal disjoint degeneration: It is at most 2.

The remaining cases: The remaining possibilities are inspected in parallel by Wolters and this thesis. She concentrates on the case (c), see [16].

Here, henceforth mainly the case (d) will be investigated. The determination of all minimal disjoint degenerations of this type is a priori an infinite problem, since

- there is an infinite number of isomorphism classes of indecomposable modules that one has to choose for V.
- there is no upper bound for the number of "candidates" M that are to test on $M < U \oplus V$, that is valid for all V.

At this point, "candidates" are all modules M with the same dimension vector as $U \oplus V$ whose indecomposable direct summands are proper successors of U and proper predecessors of V relative to $\preccurlyeq$. We approach to these problems by deriving suitable necessary conditions on minimal disjoint degenerations of this type.

Chapter 2

Directed decompositions

In the present chapter the notion of a directed decomposition shall be introduced. We generalize resp. adjust some results originated from the preprojective case (see [6]) to our situation and derive some very useful consequences.

2.1 A reduction theorem

First we show a necessary condition on the minimality of a disjoint degenerations that will be the key to get a grip on them. Let U be a projective simple module and V be indecomposable.

Definition 2.1.1 *Suppose M is a kQ-module. A decomposition $M = M_1 \oplus M_2$ is called directed if no indecomposable direct summand of M_2 is a predecessor relative to $\preccurlyeq$ of a direct summand of M_1.*

In particular, the regular indecomposable direct summands of M that come from the same regular tube all either belong to M_1 or to M_2. But notice, the above definition differs slightly from the one in [6].

Theorem 2.1.2 ***(Reduction Theorem)*** *Let U be projective simple, V be indecomposable such that $U \preccurlyeq V$ and M be a module with a directed decomposition $M = M_1 \oplus M_2$ such that $M_1 \neq 0$. Suppose $M < U \oplus V$ is a minimal degeneration provided through the exact sequence*

$$\eta : 0 \longrightarrow U \xrightarrow{\binom{\epsilon_1}{\epsilon_2}} M = M_1 \oplus M_2 \xrightarrow{(\pi_1,\pi_2)} V \longrightarrow 0,$$

then the following holds:

(a) $C := Coker(\epsilon_1)$ is indecomposable.

(b) $M_1 < U \oplus C$ is a minimal degeneration.

(c) *The difference $Codim(U \oplus V, M) - Codim(U \oplus C, M_1)$ of the codimensions of these two degenerations is given by*

$$[V,V] - [C,C] - [M_2, M_2] - [C, M_2] + [C, M_2]^1.$$

The dual statement is also true.

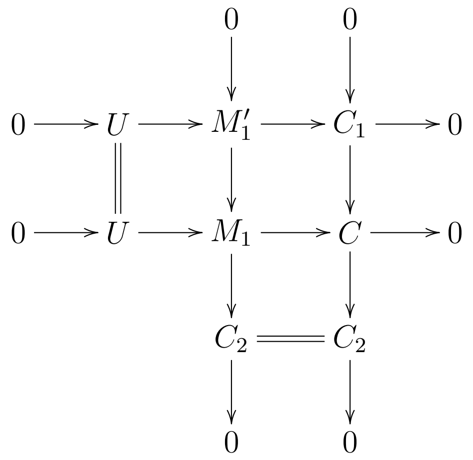
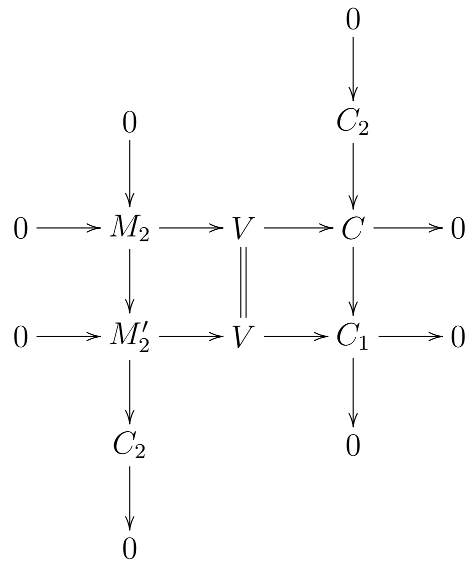
Proof. The proof of (a) is based on the main observation in [6]. If M_2 vanishes, the assertion is clear. From there we may assume $M_2 \neq 0$. Because U is simple, ϵ_1 is zero or injective. If ϵ_1 were zero, then ϵ_2 would be injective, so

$$V \cong M/(\begin{pmatrix}\epsilon_1 \\ \epsilon_2\end{pmatrix}(U)) \cong M_1 \oplus M_2/\epsilon_2(U).$$

But V is indecomposable, so $M_2 \cong \epsilon_2(U) \cong U$ and consequently η would split, which contradicts $M < U \oplus V$. Thus ϵ_1 is injective. The injectivity of ϵ_2 follows analogical.
Regarding η as a pushout and pullback diagram we get that π_1 and π_2 are monomorphisms. For this reason there is the following commutative diagram:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & M_2 & = & M_2 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \rightarrow & U & \rightarrow & M_1 \oplus M_2 & \rightarrow & V & \rightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \rightarrow & U & \rightarrow & M_1 & \rightarrow & C & \rightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array} \qquad (2.1)$$

If we assume that C is not indecomposable, it is possible to write $C = C_1 \oplus C_2$ with C_1 indecomposable and $C_2 \neq 0$. This induces two commutative diagrams, the second one by applying the snake lemma:

In particular there are exact sequences

$$\begin{array}{rl} (i) & 0 \to U \to M_1' \to C_1 \to 0 \\ (ii) & 0 \to M_1' \to M_1 \to C_2 \to 0 \\ (iii) & 0 \to M_2' \to V \to C_1 \to 0 \text{ and} \\ (iv) & 0 \to M_2 \to M_2' \to C_2 \to 0. \end{array}$$

We want to conclude $M_1 \oplus M_2 \le M_1' \oplus M_2' \le U \oplus V$. We use the partial order $\le$ to verify that. Note, that obviously $\underline{\dim}(M) = \underline{\dim}(M_1' \oplus M_2') = \underline{\dim}(U \oplus V)$. Suppose first, T is a non-injective indecomposable predecessor of V. Moreover assume that $T \notin \mathcal{T}_\mu$ in case of $V \in \mathcal{T}_\mu$. Since $[C, T] \le [V, T] = 0$ we can derive the following exact sequences from the above ones

$$\begin{array}{rl} (i') & 0 \to Hom(M_1', T) \to Hom(U, T) \to Ext^1(C_1, T) \to Ext^1(M_1', T) \\ (ii') & 0 \to Hom(M_1, T) \to Hom(M_1', T) \to Ext^1(C_2, T) \to Ext^1(M_1, T) \\ & \qquad \to Ext^1(M_1', T) \to 0 \\ (iii') & 0 \to Hom(V, T) \to Hom(M_2', T) \to Ext^1(C_1, T) \to Ext^1(V, T) \\ (iv') & 0 \to Hom(M_2', T) \to Hom(M_2, T) \to Ext^1(C_2, T) \to Ext^1(M_2', T). \end{array}$$

With the aid of (iv') we have $[M_2', T] \le [M_2, T]$. Hence, if $[M_2, T]$ vanishes, we get

$$[M_1 \oplus M_2, T] = [M_1, T] \overset{(ii')}{\le} [M_1', T] = [M_1' \oplus M_2', T] \overset{(i')}{\le} [U, T] = [U \oplus V, T].$$

Otherwise $[M_2, T] \ne 0$ implies $[M_1, T]^1 = [\tau^- T, M_1]$, since the decomposition is directed. Consequently, using (ii), $[M_1', T] = 0$ and we are able to deduce

$$\begin{array}{rcl} [M_1 \oplus M_2, T] & \overset{(ii')\&(iv)}{\le} & [M_1', T] - [C_2, T]^1 + [M_2', T] + [C_2, T]^1 = [M_1' \oplus M_2', T] \\ & \overset{(i')\&(iii')}{\le} & [U, T] - [C_1, T]^1 + [V, T] + [C_1, T]^1 = [U \oplus V, T]. \end{array}$$

Now let T be a non-injective indecomposable module, which is not a predecessor of V. From the diagrams above we obtain inclusions

$$M_2 \hookrightarrow M_2' \hookrightarrow V \quad \text{and} \quad M_1' \hookrightarrow M_1 \hookrightarrow V,$$

which insure that all indecomposable direct summands of M and M' are predecessors of V. Therefore T is not a predecessor of an indecomposable direct summand of M or M', whence

$$\begin{aligned} &[U \oplus V, T] - [M', T] = [\tau^- T, U \oplus V] - [\tau^- T, M_1' \oplus M_2'] = 0 \quad \text{and} \\ &[M_1' \oplus M_2', T] - [M, T] = [\tau^- T, M_1' \oplus M_2'] - [\tau^- T, M] = 0. \end{aligned}$$

In the remaining case T and V are regular indecomposables of the same tube. Since $U \oplus V$ is a degeneration of M with $M_2 \neq 0$ the regular summand belongs to M_2. Hence M_1 is preprojective and the just stated inclusions yield

$$\begin{aligned} [U \oplus V, T] - [M_1' \oplus M_2', T] &= [\tau^- T, U \oplus V] - [\tau^- T, M_1' \oplus M_2'] \\ &= [\tau^- T, V] - [\tau^- T, M_2'] \\ &\geq [\tau^- T, V] - [\tau^- T, M_2] \\ &= [\tau^- T, U \oplus V] - [\tau^- T, M] \;\geq\; 0 \quad\quad \text{and} \\ [M_1' \oplus M_2', T] - [M, T] &= [\tau^- T, M_1' \oplus M_2'] - [\tau^- T, M] \\ &= [\tau^- T, M_2'] - [\tau^- T, M_2] \;\geq\; 0. \end{aligned}$$

So we have established $M \leq M_1' \oplus M_2' \leq U \oplus V$. The minimality of $M < U \oplus V$ forces $M' \cong M$, since $M' \cong U \oplus V$ would violate the assumption $C_1 \neq 0$. We consider the exact sequence $0 \to M_2 \xrightarrow{i} M_2' \to C_2 \to 0$ with $C_2 \neq 0$. $M \cong M_1' \oplus M_2'$ insures the existence of an indecomposable direct summand X of M_2' which also occurs in M_1. If V is regular, X as a direct summand of M_1 is preprojective. Write $M_2 = X_1 \oplus \ldots \oplus X_t$ with X_j indecomposable and consider for every $j \in \{1, \ldots, t\}$ the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M_2 & \xrightarrow{\;i\;} & M_2' & \longrightarrow & C_2 & \longrightarrow & 0 \\ & & \uparrow{\scriptstyle \epsilon_j} & & \downarrow{\scriptstyle p_X} & & & & \\ & & X_i & \xrightarrow{\;\alpha_j\;} & X & & & & \end{array}$$

By the definition of a directed decomposition we must have $\alpha_j = 0$ for every j. Then $i(M_2) \cap X = 0$ and X is a direct summand of $C_2 \cong M_2'/i(M_2)$. Thus we obtain $X \prec V \preccurlyeq X$, which is absurd. Accordingly, C is indecomposable.

(b) Again we consider the commutative diagram (2.1) with $M_2 \neq 0$. C is indecomposable and $U \oplus C$ is a proper degeneration of M_1. Assume this is not a minimal degeneration. Hence, there exists a module N with $M_1 < N < U \oplus C$ such that $0 \to U \to N \to C \to 0$ is minimal. N can be decomposed as follows:

- N_1 contains all indecomposable direct summands Y of N for which an indecomposable direct summand X of M_1 exists such that $Y \preccurlyeq X$.
- N_2 consists of the remaining direct summands.

$0 < [M, M_1] \leq [N, M_1] = [N_1, M_1]$ guarantees that N_1 is non-zero. According to (a) there is an injection $N_2 \hookrightarrow C$ which induces the following commutative pullback diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M_2 & \longrightarrow & P & \longrightarrow & N_2 & \longrightarrow & 0 \\ & & \| & & \big\downarrow & & \big\downarrow & & \\ 0 & \longrightarrow & M_2 & \longrightarrow & V & \longrightarrow & C & \longrightarrow & 0 \end{array}$$

In particular P degenerates into $M_2 \oplus N_2$.
We claim $M < N_1 \oplus P < U \oplus V$, that is a contradiction. Let T be an indecomposable module. If M_1 contains a direct summand X that is a successor of T we obtain $0 = [M_2, T] = [N_2, T]$. Thus $[P, T]$ vanishes and consequently we get

$$[M, T] = [M_1, T] \leq [N_1 \oplus N_2, T] = [N_1 \oplus P, T] \leq [U \oplus C, T] \leq [U \oplus V, T].$$

If T is not injective and there is no such summand in M_1, then $0 = [\tau^- T, M_1] = [\tau^- T, N_1]$. The injections $M_2 \hookrightarrow P \hookrightarrow V$ imply

$$\begin{array}{rcl} [U \oplus V, T] - [N_1 \oplus P, T] & = & [\tau^- T, U \oplus V] - [\tau^- T, N_1 \oplus P] \\ & \geq & [\tau^- T, V] - [\tau^- T, P] \ \geq \ 0 \quad \text{and} \\ [N_1 \oplus P, T] - [M, T] & = & [\tau^- T, N_1 \oplus P] - [\tau^- T, M] \\ & = & [\tau^- T, P] - [\tau^- T, M_2] \ \geq \ 0. \end{array}$$

Finally, for injective T the equality of the dimension vectors leads to

$$[U \oplus V, T] = [N_1 \oplus P, T] = [M, T].$$

Assume M and $N_1 \oplus P$ were isomorphic. By the definition of the decomposition $N = N_1 \oplus N_2$ the module P would be a direct sum of M_2 and perhaps certain direct summands of M_1. But the injection $M_2 \hookrightarrow P$ would not permit the occurrence of a direct summand of M_1, which would lead to $N_2 = 0$ and $M_1 = N$, a contradiction. On the other hand, under the assumption $N_1 \oplus P \cong U \oplus V$ the following possibilities remain to be considered:
$N_1 \cong U$: Then we would have $N_2 \cong C$, which violates $N < U \oplus V$.
$N_1 \cong V$: This would force $P \cong U$, whence $M_2 \cong U$. This is impossible, since $M < U \oplus V$.
$N_1 \cong U \oplus V$: Then $P = M_2 = 0$. This contradicts the assumption $M_2 \neq 0$.
This proves (b).
(c) From the projectivity of U we obtain $Ext^1(U, V) = Ext^1(U, C) = 0$. Thus

$$[U, V] - [U, C] = \langle \underline{\dim}(U), \underline{\dim}(V) - \underline{\dim}(C) \rangle = \langle \underline{\dim}(U), \underline{\dim}(M_2) \rangle.$$

By definition of a directed decomposition the vector space $Ext^1(M_1, M_2)$ vanishes. So we get $[M_1, M_2] = \langle \underline{\dim}(M_1), \underline{\dim}(M_2)\rangle$. Since the dimension vectors of $U \oplus V$ and M coincide, the difference of these two terms is

$$-\langle \underline{\dim}(C), \underline{\dim}(M_2)\rangle = -[C, M_2] + [C, M_2]^1.$$

The assertion now follows immediately from the codimension formula given in chapter 1.2. □

Supposed for instance that V is preprojective, Bongartz and Fritzsche give a refinement of part (a) of the reduction theorem.

Lemma 2.1.3 *([6], 3.2, p. 2020) Let U be projective simple, V be preprojective with $U \preccurlyeq V$ and $M < U \oplus V$ be a minimal degeneration. If $M = M_1 \oplus M_2$ is a directed decomposition such that $\partial(U) = \partial(M_1)$, it follows that $\underline{\dim}(C) \leq \delta$.*

Let V be preinjective of arbitrary defect and $M < U \oplus V$ a minimal degeneration. Decomposing $M = M_P \oplus M_R \oplus M_I$ into its preprojective, regular and preinjective parts we get

Corollary 2.1.4 *In the situation stated before one has $\partial(M_P) > \partial(U)$ and $\partial(M_I) < \partial(V)$.*

Proof. Choose in theorem 2.1.2 $M_1 = M_P$ and $M_2 = M_R \oplus M_I$. Now consider the sequence

$$0 \longrightarrow U \xrightarrow{\epsilon_1} M_P \longrightarrow C \longrightarrow 0.$$

The module C is a successor of V and thus again preinjective. This shows $\partial(M_P) = \partial(U \oplus C) > \partial(U)$. □

2.2 The root test and its consequences for the preprojective and preinjective parts of minimal deformations

Let U be projective simple and V be indecomposable. The next lemma is also a consequence of part (a) of the reduction theorem. It holds for arbitrary indecomposable module V. But we will apply it only in the case where V is preinjective.

Lemma 2.2.1 ***(Root test)*** *Let U be projective simple, V be indecomposable such that $U \preccurlyeq V$ and M be a module with a directed decomposition $M = M_1 \oplus M_2$. Suppose $\mathcal{S}$ is a slice in the preprojective resp. preinjective component of the Auslander-Reiten quiver. Then:*

(a) The vector $\sigma := ([U, X] - [M_1, X])_{X \in \mathcal{S}}$ is a root of Q. If V is not a predecessor of some $X \in \mathcal{S}$, this root is positive.

(b) *The vector* $\rho := ([U \oplus V, X] - [M, X])_{X \in \mathcal{S}}$ *is a positive root of* Q *or equal to zero.*

Proof. The proof is an adaption of [6], 2.1. As we will see, for tame path algebras it works not only for preprojective V.
(a) Without loss of generality we may assume that each direct summand of M_1 is a predecessor of some module in $\mathcal{S}$ since the remaining summands do not contribute to σ and could consequently also be put to M_2. Denote $T := \bigoplus_{X \in \mathcal{S}} X$. T is a tilting module and $End(T) = k\tilde{Q}$ is a path algebra whose underlying graph is of the same type as Q.
Suppose M_1 vanishes. Thus $\sigma = ([U, X])_{X \in \mathcal{S}}$. Since $\mathcal{S}$ is a slice, there is at least one $X \in \mathcal{S}$ such that $[U, X] \neq 0$. Consequently, there is an injective map $U \hookrightarrow T$. Dualizing the situation provides $DT \twoheadrightarrow DU$, whence $DU \in \mathcal{T}(DT)$. Applying the functor $F := Hom(DT, \_)$ results in

$$\sigma = ([U, X])_{X \in \mathcal{S}} = ([DX, DU])_{X \in \mathcal{S}} = ([FDX, FDU])_{X \in \mathcal{S}} = \underline{\dim}(FDU).$$

The last equality holds, because the indecomposable direct summands of FDX are exactly the indecomposable projective modules of $End(DT)$. This shows that σ is a dimension vector, thus a root of Q.
Suppose now $M_1 \neq 0$. Using Theorem 2.1.2 (a) we obtain an exact sequence

$$0 \to U \to M_1 \to C \to 0$$

where C is indecomposable. All $X \in \mathcal{S}$ come either from the preprojective or from the preinjective component of the Auslander-Reiten quiver. So the assumption that all indecomposable direct summands of M_1 are predecessors of some module in $\mathcal{S}$ provides $Ext^1(M_1, X) = 0$ for all $X \in \mathcal{S}$. Accordingly, for $X \in \mathcal{S}$ the sequence

$$0 \to Hom(C, X) \to Hom(M_1, X) \to Hom(U, X) \to Ext^1(C, X) \to 0$$

is exact. If $[C, X]$ vanishes for all $X \in \mathcal{S}$, then $\tau C \in \mathcal{T}(T)$ and therefore there exists an epimorphism $T^n \twoheadrightarrow \tau C$. We conclude that $[C, T]^1 = [T, \tau C]$ is non-zero and application of the functor $F' := Hom(T, \_)$ delivers

$$\sigma = ([X, \tau C])_{X \in \mathcal{S}} = ([F'X, F'\tau C])_{X \in \mathcal{S}} = \underline{\dim}(F'\tau C).$$

So, σ is a positive root of Q.
If, on the other hand, there is some $X \in \mathcal{S}$ with $[C, X] \neq 0$, then $[C, X']^1 \neq 0$ is impossible for any $X' \in \mathcal{S}$. Assuming the contrary would result in

$$X' \prec \tau^- X' \preccurlyeq C \preccurlyeq X.$$

This does not fit with the definition of a slice. Hence, $DC \in \mathcal{T}(DT)$ and the computation

$$\sigma = (-[C, X])_{X \in \mathcal{S}} = (-[FDX, FDC])_{X \in \mathcal{S}} = -\underline{\dim}(FDC)$$

identifies σ as a negative root.
Suppose V is not a predecessor of a module in $\mathcal{S}$, then $[V, X] = 0$ and consequently $[C, X] = 0$ for all $X \in \mathcal{S}$. Thus σ is positive.
For the proof of (b) decompose $M = M_1 \oplus M_2$ such that all indecomposable direct summands of M that are predecessors of some module in $\mathcal{S}$ belong to M_1. Assume V is not a predecessor of some module in $\mathcal{S}$, then $\rho = [U, X] - [M_1, X]$. We can apply (a) to yield the assertion. Otherwise, obviously we have $M = M_1$, $R = V$ and ρ is zero. □

If V is preinjective, we are able to restrict the number of preprojective resp. preinjective indecomposable modules that can occur as direct summands of a minimal degeneration $M < U \oplus V$. There is an upper bound depending only on the choice of $|Q|$ and not on U and V. With the help of the root test this bound can be sharpened for the direct summands whose defect is not ± 1.

Definition 2.2.2 *(a) The diameter $d(Q)$ of Q is the number of edges in the longest path without cycles in $|Q|$.*

(b) The distance $d(X, Y)$ between two indecomposable modules $X \preccurlyeq Y$ belonging to the same connected component of the Auslander-Reiten quiver is length of a shortest path leading from X to Y. If $X \not\preccurlyeq Y$, we set $d(X, Y) := -\infty$.

Lemma 2.2.3 *Let U be projective simple, V be indecomposable preinjective and M be a minimal deformation of $U \oplus V$. We decompose $M = M_P \oplus M_R \oplus M_I$ into its preprojective, regular and preinjective parts. Then:*

(a) For every direct summand X of M_P there is

$$d(U, X) < \begin{cases} 2(p(Q) + d(Q)), & \partial(X) < -1 \\ 4p(Q) + d(Q), & \partial(X) = -1 \end{cases}.$$

(b) For every direct summand X of M_I we have

$$d(X, V) < \begin{cases} 2(p(Q) + d(Q)), & \partial(X) > 1 \\ 4p(Q) + d(Q), & \partial(X) = 1 \end{cases}.$$

Proof. (a) Note, there is nothing to prove for $\partial(U) = -1$ since there is corollary 2.1.4. Thus we may require that Q is of type $\tilde{D}_8$, $\tilde{E}_6$, $\tilde{E}_7$ or $\tilde{E}_8$. Consequently, we have $\epsilon(Q) = 1$.
Assume there occurs some preprojective direct summand X with $d(U, X) > 4p(Q) + d(Q)$ in M. Hence we can write $M_P = M_P' \oplus X$. Furthermore the existence of the preprojective indecomposable modules $\tau^{p(Q)}X$ and $\tau^{2p(Q)}X$ with the dimension vectors

$$\underline{\dim}(\tau^{p(Q)}X) = \underline{\dim}(X) + \partial(X)\delta \quad \text{and} \quad \underline{\dim}(\tau^{2p(Q)}X) = \underline{\dim}(X) + 2\partial(X)\delta$$

is assured. Since the number of tubes that are involved in M_R is finite, we can find a homogeneous tube $\mathcal{T}_\mu$ such that no module of $\mathcal{T}_\mu$ is a direct summand of M_R. We choose the indecomposable module $R \in \mathcal{T}_\mu$ whose dimension vector is $(-\partial(X))\delta$ and define $\tilde{M} := M'_P \oplus \tau^{p(Q)}X \oplus R \oplus M_R \oplus M_I$. We claim

$$M < \tilde{M} < U \oplus V,$$

which contradicts the minimality of $M < U \oplus V$.
For $M < \tilde{M}$ it is enough to show $X < \tau^{p(Q)}X \oplus R$. To do this it suffices to verify $[T, \tau^{p(Q)}X \oplus R] - [T, X] \geq 0$ for all indecomposable predecessors T of X. Indeed, this is evident since the dimension vectors of $\tau^{p(Q)}X \oplus R$ and X coincide.
Thus it remains to prove $\tilde{M} < U \oplus V$. For this purpose let T be indecomposable. If T is preprojective with $\tau^{p(Q)}X \preccurlyeq T$, we have

$$\begin{aligned}
[U,T] &= \langle \underline{\dim}(U), \underline{\dim}(T) \rangle = \langle \underline{\dim}(U), \underline{\dim}(\tau^{p(Q)}T) - \epsilon(Q)\partial(T)\delta \rangle \\
&= [U, \tau^{p(Q)}T] + \partial(T)\partial(U), \\
[\tau^p X, T] &= \langle \underline{\dim}(\tau^{p(Q)}X), \underline{\dim}(T) \rangle = \langle \underline{\dim}(X), \underline{\dim}(T) \rangle - \epsilon(Q)\partial(T) \\
&= \langle \underline{\dim}(X), \underline{\dim}(\tau^{p(Q)}T) \rangle - 2\partial(X)\partial(T) \\
&\leq [X, \tau^{p(Q)}T] - 2\partial(X)\partial(T), \\
[M'_P, T] &= [M''_p, T] = \langle \underline{\dim}(M''_p, \underline{\dim}(T) \rangle \\
&\leq [M''_P, \tau^{p(Q)}T] + \partial(T)\partial(M''_P).
\end{aligned}$$

Here M''_P consists of those indecomposable direct summands of M'_P which are predecessors of T. Consequently we can compute

$$\begin{aligned}
[U \oplus V, T] - [\tilde{M}, T] &= [U,T] - [M'_P, T] - [\tau^{p(Q)}X, T] \\
&\geq [U, \tau^{p(Q)}T] - [M, \tau^{p(Q)}T] \\
&+ \qquad \underbrace{\partial(T)}_{<0}\underbrace{(\partial(U) - \partial(M''_P) - \partial(X) + 1)}_{\leq 0} \\
&\geq [U, \tau^{p(Q)}T] - [M, \tau^{p(Q)}T] \geq 0.
\end{aligned}$$

Suppose T is not a successor of $\tau^{p(Q)}X$. It follows that $X \not\preccurlyeq T$, whence $[\tau^{p(Q)}X, T] = [X, T] = 0$. This implies

$$[U \oplus V, T] - [\tilde{M}, T] = [U \oplus V, T] - [M, T] \geq 0.$$

For preinjective or regular T that do not come from $\mathcal{T}_\mu$, it is clear that

$$[T, U \oplus V] - [T, \tilde{M}] = [T, U \oplus V] - [T, M] \geq 0.$$

If $T \in \mathcal{T}_\mu$, we obtain

$$\begin{aligned}
[U \oplus V, T] - [\tilde{M}, T] &= \langle \underline{\dim}(U) - \underline{\dim}(M'_P \oplus \tau^{p(Q)}X), \underline{\dim}(T) \rangle - \underbrace{[R, T]}_{=\min(l(T), -\partial(X))} \\
&\geq l(T)\underbrace{(\partial(M'_P) + \partial(X) - \partial(U))}_{\geq 1} - l(T) \geq 0.
\end{aligned}$$

Thus we have excluded preprojective modules X which are too far away from U as direct summands of M.
Now we especially want to examine preprojective indecomposable direct summands of M whose defects are smaller than -1. For them the bound can be improved. But this, unfortunately, has to be checked case by case. We proceed as in the proof of the preprojective case (see [6], 3.3). Nevertheless, corollary 2.1.4 simplifies the situation a bit. We can restrict to the case $\partial(U) \leq -3$, i.e. Q is of type $\tilde{E}$. So assume there is an indecomposable direct summand X of M_P with

$$d(U,X) > 2(p(Q)+d(Q)) \text{ and } \partial(X) \leq -2.$$

Moreover we may assume that X is $\preccurlyeq$-minimal with this property. The idea of the proof is to lookup for trivial or indecomposable summand X' of M_P and an appropriate degeneration Z of $X \oplus X'$ such that $M = M' \oplus X \oplus X'$ and $M < M' \oplus Z < U \oplus V$. For this purpose it suffices to show for all preprojective indecomposable modules T that

$$\beta(T) := [Z,T] - [X \oplus X', T] \leq \gamma(T) := [U \oplus V, T] - [M,T].$$

Recall the root test 2.2.1, which says that β and γ restricted to any slice in the preprojective component of Γ_{kQ} are zero or roots of Q. We define a directed decomposition $M_P = M_P' \oplus M_P''$ as follows:

- M_P'' consists of the direct summands of M_P that are successors of X.
- M_P' is the direct sum of the remaining summands of M_P.

Furthermore we use the following notations ($r \geq s \geq t \geq 1$):

$$\begin{array}{c} c_t \\ | \\ \vdots \\ | \\ c_1 \\ | \\ a_r - \ldots - a_1 - a_0 - b_1 - \ldots - b_s \end{array}$$

In Γ_{kQ} there exists a slice $\mathcal{S}$ that meets X and has only one source, namely in the τ^--orbit the projective indecomposable $P(a_r)$. The module in $\mathcal{S}$ which is a τ^--translated of $P(y)$, $y \in Q_0$, we simply denote by Y. There is a uniquely determined module $Y \prec X$ in $\mathcal{S}$ maximal relative to $\preccurlyeq$ with this property, which we can write as $Y = \tau^{-b}P(y)$, $b \geq p(Q)$. By definition of the decomposition of M_P we obtain

$$\epsilon(Y) = [U,Y] - [M_P', Y].$$

Suppose M_1 vanishes, then (1.1) delivers $\gamma(Y) \geq [U,Y] = [U, \tau^{p(Q)}Y] + \delta_y \geq \delta_y$. On the other hand if $M_1 \neq 0$ we make use of the exact sequence

$$0 \to U \to M_P' \to C \to 0$$

given by theorem 2.1.2 (a). To this sequence we apply $Hom(\_\,, Y)$ and compute

$$[U,Y]-[M'_P,Y]=[C,Y]^1=[\tau^-Y,C]=[P(z),\tau^{a+b+1}I]\geq\delta_z$$

where $C=\tau^a I$ and I is injective. Thus in general we have $\gamma(Y)\geq\delta_z$.
Case (i): $X=A_i$ for some $1\leq i<r$. Suppose first $\gamma(A_j)>0$ for all $j>i$. We choose $X':=0$ and $Z:=A_r\oplus\tau^-A_{i+1}$. Then $\beta(T)=1$ for $T\in\{A_r,\dots A_{i-1}\}$ and zero otherwise, whence $\beta\leq\gamma$. Now we assume the existence of some index $j>i$ with $\gamma(A_j)=0$. The knowledge about $\gamma(Y)\geq\delta_z$, the root test 2.2.1 and a closer look on the roots of Q force the equality $j=r$. Since we have $d(U,X)>2(p(Q)+d(Q))$ the module $\tau^{p(Q)}A_r$ is indecomposable, which guarantees $[U,A_r]\neq 0$ and consequently $M'_P\neq 0$. Moreover using the above exact sequence of theorem 2.1.2 to apply the functor $Hom(\_\,,A_r)$ on it we obtain

$$0=\gamma(A_r)=[U,A_r]-[M'_P,A_r]=[C,A_r]^1-[M'_P,A_r]^1=[\tau^-A_r,C]-[\tau^-A_r,M'_P].$$

From there we have $[\tau^-A_r,M'_P]=[\tau^-A_r,C]=[P(a_r),\tau^cI]\geq 0$ since $c\geq p(Q)$. Accordingly, M'_P has a direct summand Y which is no successor of A_i and satisfies $[\tau^-A_r,Y]\neq 0$. Testing the small number of successors of τ^-A_r which are no successors of A_i on that property delivers $Y=\tau^-A_j$, $j>i+1$. We choose Y such that j is minimal and set $X':=\tau^-A_j$ and $Z:=A_{j-1}\oplus\tau^-A_{i+1}$. Then $\beta(T)=1$ for $T\in\{A_{j-1},\dots A_{i-1}\}$ and zero otherwise, hence $\beta\leq\gamma$.
Case (ii): $|Q|=\tilde{E}_8$, $X=B_1$. In this case we have $Y=A_0$ and $\gamma(Y)\geq 6$. So the root test 2.2.1 and an inspection of the roots of $\tilde{E}_8$ imply $\gamma(\tau B_2)\neq 0$. Hence, it is possible to set $X':=0$ and $Z:=\tau B_2\oplus B_2$.
Case (iii): $|Q|=\tilde{E}_8$, $X=B_2$. Set $\mathcal{S}':=\mathcal{S}\cup\{\tau^{-(k+1)}A_k|0\leq k\leq 5\}$. If $\gamma(T)>0$ for all $T\in\mathcal{S}'$ we can replace X by $Z:=A_5\oplus\tau^{-7}A_5$.
Otherwise, suppose there is some $T\in\mathcal{S}'$ such that $\gamma(T)=0$. This is only possible for $T=A_5$ or $T=\tau^{-6}A_5$ due to $\gamma(B_1)\geq 4$.
It is impossible that γ vanishes in $\tau^{-6}A_5$. Assuming the contrary would imply $[M''_P,\tau^{-6}A_5]\neq 0$, because, if $M'_P\neq 0$ we have

$$[U,\tau^{-6}A_5]-[M'_P,\tau^{-6}A_5]=[C,\tau^{-6}A_5]^1-[M'_P,\tau^{-6}A_5]^1=[\tau^{-7}A_5,C]\neq 0.$$

This contradicts the definition of the directed decomposition $M_P=M'_P\oplus M''_P$.
Suppose $\gamma(A_5)=0$ and $\gamma(\tau^{-6}A_5)\neq 0$. With the same calculation as in the first case we obtain $[\tau^-A_5,M'_P]\neq 0$. Thus M'_P has some τ^-A_j, $j>0$, as direct summand. We take j as small as possible and set $X':=\tau^-A_j$ and $Z:=A_{j-1}\oplus\tau^{-7}A_5$. Thus

$$\beta(T)=\begin{cases}1 & T\in\{A_j,\dots,A_0,B_1,C_1,\tau^-A_0,\dots,\tau^{-6}A_5\}\\ 0 & \text{otherwise}\end{cases}$$

and consequently $\beta\leq\gamma$.
Case (iv): $|Q|=\tilde{E}_7$, $X=C_1$. If $\mathcal{S}$ lies in the support of γ we can choose $X':=0$ and $Z:=A_3\oplus\tau^-B_3$. Otherwise, since $\gamma(A_0)\geq 4$, there remain only two

possibilities, namely either $\gamma(A_3) = 0$ or $\gamma(B_3) = 0$. $\gamma(B_3) = 0$ is forbidden, since otherwise we would obtain $[M''_P, B_3] \neq 0$. If $M'_P = 0$ this is evident and if $M'_P \neq 0$ the computation

$$[U, B_3] - [M'_P, B_3] = [C, B_3]^1 - [M'_P, B_3]^1 = [\tau^- B_3, C] \neq 0$$

verifies it. But this does not fit with the choice of the directed decomposition $M_P = M'_P \oplus M''_P$.
If $\gamma(A_3) = 0$, $\partial(M'_P) > \partial(U) \geq -4$ shows that $\tau^- A_3$ is a direct summand of M'_P. We set $X' := \tau^- A_3$ and $Z := A_3 \oplus \tau^- B_3$.
Case (v): $|Q| = \tilde{E}_8$, $X = C_1$. The proof is similar to the last case.
The proof of (b) is dual. □

It might be possible to improve the bounds of the lemma even more. In the whole proof we were never forced to use our knowledge of corollary 2.1.4 on the defect of the preprojective resp. preinjective part of a minimal deformation of $U \oplus V$ and in considered examples the bounds were always better. A good bound is of practical interest. It reduces the complexity of the determination of all minimal deformations of $U \oplus V$.

Chapter 3

Analysis of the codimension

In this chapter we want to analyse the codimension of minimal disjoint degenerations by means of a more flexible approach. The idea is to permit techniques that do not only vary V but also the orientation of the quiver Q. The method we introduce now works for non-regular V and consists of a skillful combination of the reduction theorem with an application of tilting with slices.

3.1 An inductive codimension formula

Let U be projective simple, V be a non-regular indecomposable and M be a module with directed decomposition $M = M_1 \oplus M_2$ such that $M_1 \neq 0 \neq M_2$. If $M < U \oplus V$ is a minimal degeneration, induced by the exact sequence $0 \to U \to M_1 \oplus M_2 \to V \to 0$, the reduction theorem 2.1.2 provides a commutative diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & M_2 & = & M_2 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \to & U & \to & M_1 \oplus M_2 & \to & V & \to & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \to & U & \to & M_1 & \to & C & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

in which the last row induces another minimal degeneration $M_1 < U \oplus C$ with indecomposable C. Thereby, $\Delta := Codim(U \oplus V, M) - Codim(U \oplus C, M_1)$ equals to

$$[V,V]-[C,C]-[C,M_2]+[C,M_2]^1-[M_2,M_2].$$

Evidently, we have $[V,V]=1$ and $[C,M_2]=[V,M_2]=0$. Supposed V is preinjective, then also C is, which implies $[C,C]=1$. Otherwise, V is preprojective and C can belong to any connected component of Γ_{kQ}. For preprojective or preinjective C we obtain immediately $[C,C]=1$. In the case where C is regular, i.e. M_2 is preprojective of the same defect as V, lemma 2.1.3 insures that $\underline{\dim}(C)\leq\delta$, whence $[C,C]$ is again 1. Thus we have

$$\Delta=[C,M_2]^1-[M_2,M_2]=[V,M_2]^1-[M_2,M_2]^1\geq 0.$$

Apart from that, dualization delivers the minimal degeneration $DM=DM_2\oplus DM_1<DV\oplus DU$ of kQ^{op}-modules. DV is non-regular, DU injective simple and $DM=DM_2\oplus DM_1$ a directed decomposition.

Aiming at a further application of the reduction theorem we choose a slice $\mathcal{S}$ in $\Gamma_{kQ^{op}}$ that has DV as a source. Besides, we additionally assume that DV is the only source of $\mathcal{S}$, unless V is preprojective and no such slice exists. $T:=\bigoplus_{X\in\mathcal{S}}X$ is a tilting module and $B:=End(T)$ is a path algebra with the same underlying graph as Q. We define $F:=Hom(T,\_)$.

Notice, the defect behaves under application of the functor F in the following way. If X is some in kQ-module such that $DX\in\mathcal{T}(T)$, there is

$$\partial(FDX)=\begin{cases}-\partial(X), & \text{if } V\in\mathcal{P}\\ \partial(X), & \text{if } V\in\mathcal{I}.\end{cases}$$

Furthermore, non-homogenenous tubes of period p_μ are mapped into non-homogeneous tubes of period p_μ.

Tilting the above situation via F yields a directed decomposition $FDM=FDM_2\oplus FDM_1$ and a minimal disjoint degeneration $FDM<FDV\oplus FDU$, where FDV is projective simple. In addition, it holds

$$\begin{aligned}\Delta &= [V,M_2]^1-[M_2,M_2]^1\\ &= [DM_2,DV]^1-[DM_2,DM_2]^1\\ &= [FDM_2,FDV]^1-[FDM_2,FDM_2]^1\geq 0.\end{aligned}$$

$FDM<FDV\oplus FDU$ is induced by an exact sequence $0\to FDV\to FDM\to FDU\to 0$, which satisfies the assumptions of the reduction theorem 2.1.2. So, we achieve the following commutative diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & FDM_1 & = & FDM_1 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & FDV & \longrightarrow & FDM_2 \oplus FDM_1 & \longrightarrow & FDU & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & FDV & \longrightarrow & FDM_2 & \longrightarrow & L & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

together with the minimality of $FDM_2 < FDV \oplus L$. The codimension of this degeneration is

$$\begin{aligned}
Codim(FDV \oplus L, FDM_2) &= [FDV \oplus L, FDV \oplus L] - [FDM_2, FDM_2] \\
&= [FDV \oplus L, FDV \oplus L] - [FDM_2, FDV \oplus L] \\
&\qquad + [FDM_2, FDV \oplus L] - [FDM_2, FDM_2] \\
&= 1 + [FDV \oplus L, L] - [FDM_2, L] + [FDM_2, L] - [FDM_2, FDM_2] \\
&= 1 + [FDV \oplus L, L] - [FDM_2, L] + [FDM_2, FDV]^1 - [FDM_2, FDM_2]^1 \\
&= 1 + [L, L]^1 + [FDM_2, FDV]^1 - [FDM_2, FDM_2]^1,
\end{aligned}$$

which implies $\Delta = Codim(FDV \oplus L, FDM_2) - 1 - [L, L]^1$. We have proved the following theorem.

Theorem 3.1.1 (Inductive codimension formula) *Under the above assumptions $Codim(U \oplus V, M)$ is equal to*

$$Codim(U \oplus C, M_1) + Codim(FDV \oplus L, FDM_2) - 1 - [L, L]^1.$$

Definition 3.1.2 *A maximal directed decomposition of a module M is a decomposition $M = \bigoplus_{k=1}^{r} X_k$ such that*

(i) each X_k is non-zero and has no proper directed decomposition and

(ii) $M = (\bigoplus_{k=1}^{i} X_k) \oplus (\bigoplus_{k=i+1}^{r} X_k)$ is a directed decomposition of M for any $1 \leq i < r$.

Remark 3.1.3 *(a) Each module has a maximal directed decomposition, which is uniquely determined up to isomorphism.*

(b) Using the above notation, if $M < U \oplus V$ is minimal, the maximal directed decompositions of M and FDM have the same number of summands.

Preinjective case: Supposed V is preinjective, L is also preinjective, whence $[L, L]^1 = 0$. Thus, if r is the number of summands in the maximal directed decomposition of M, the inductive application of theorem 3.1.1 delivers

$$Codim(U \oplus V, M) = Codim(U_1 \oplus V_1, M_1) + \ldots + Codim(U_r \oplus V_r, M_r) + (r - 1),$$

where $M_i < U_i \oplus V_i$ is a minimal degeneration of kQ_i-modules such that U_i is the only projective simple, V_i is preinjective indecomposable, M_i has no proper directed decomposition and $|Q_i| = |Q|$. There are several possibilities for the M_i, namely

(i) $M_i = X^t$ where X is preprojective indecomposable,
(ii) $M_i = X^t$ where X is preinjective indecomposable or
(iii) $M_i = M_\mu \in add(\mathcal{T}_\mu)$ for some $\mu \in \mathbb{P}^1$.

The codimensions of degenerations of these types cannot be broken up with the aid of the introduced technique. So, we turn our further attention in particular on these degenerations. Notice, the case (ii) is obviously dual to (i).

Preprojective case: If V is preprojective, the modules M_1 and FDM_2 of theorem 3.1.1 are obviously preprojective, while L and C may also be regular or preinjective. Combining arguments of [6], [16], of this and the next section, enables us to show with theoretical arguments what Bongartz and Fritzsche obtained in [6] with the help of computer calculations: If V is preprojective, the codimension of any minimal $M < U \oplus V$ is at most two. The proof will be contained in a forthcoming joint article with Wolters.

3.2 Preprojective deformations of $\mathbf{U \oplus V}$

Let U be projective simple and V be indecomposable. Motivated by the last section, we now want to focus on preprojective modules that degenerate into $U \oplus V$. The first result generalizes a proposition of Bongartz (see [4], 7.1, p. 604).

Proposition 3.2.1 *Let M be a preprojective module such that $M < U \oplus V$ is minimal. Suppose $M = \bigoplus_{k=1}^{s} M_k^{r_k}$ is the decomposition into indecomposables. If $End(V) = k$ and $[M_k, M_j] = 0$ for $k \neq j$, then $Codim(U \oplus V, M) \leq [V, V]^1 + 1$.*

Proof. The proof is essentially the same as in [4]. Only some numbers in the beginning have to be modified. Assume $Codim(U \oplus V, M)$ is not less or equal to $[V, V]^1 + 1$. By minimality of $M < U \oplus V$ there exists an exact sequence $0 \to U \to M \to V \to 0$, which induces

$$0 \to Hom(V, V) \to Hom(M, V) \to Hom(U, V) \to Ext^1(V, V) \to Ext^1(M, V).$$

Since M is preprojective, $Ext^1(M,V)$ vanishes. Furthermore $[U,U]=1$. Accordingly, we have

$$\begin{aligned}
[V,V]^1+2 &\leq [U\oplus V, U\oplus V]-[M,M] \\
&= [U\oplus V, U\oplus V]-[M,U\oplus V]+[M,U\oplus V]-[M,M] \\
&= 1+[U,V]+[V,V]-[M,V]+[M,V]-[M,M] \\
&= 1+[V,V]^1+[M,V]-[M,M],
\end{aligned}$$

which is equivalent to $\sum_{k=1}^{s} r_k[M_k,V]=[M,V]>[M,M]=\sum_{k=1}^{s} r_k[M_k,M]$. Because of $M<U\oplus V$, we already know that $[M_k,V]\geq[M_k,M]$ for any $k\in\{1,\dots,s\}$. Thus there is some index i that satisfies $[M_i,V]>[M_i,M]=r_i$. Without loss of generality $i=1$ may be assumed. This allows us to choose a set of linearly independent homomorphisms

- $f_{1,1},f_{1,2},\dots,f_{1,n_1+1}$ in $Hom(M_1,V)$ resp.
- $f_{k,1},f_{k,2},\dots,f_{k,n_k}$ in $Hom(M_k,V)$, $2\leq k\leq s$.

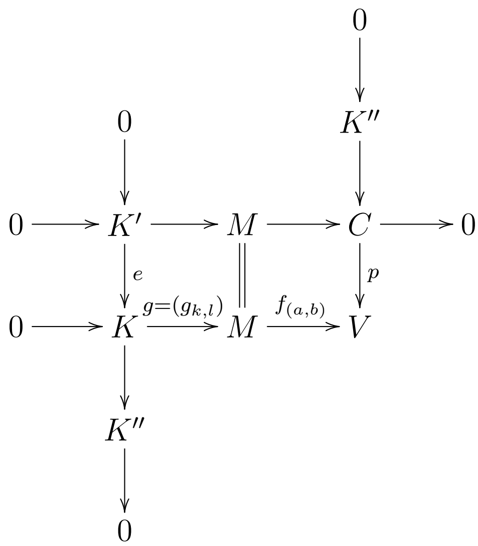
We take two homomorphisms $g_1: M_1^{n_1}\to V,\quad m\mapsto(f_{1,1},\dots,f_{1,n_1})(m)$, $g_2: M_1^{n_1}\to V,\quad m\mapsto(f_{1,2},\dots,f_{1,n_1+1})(m)$. For $(a,b)\in k^2\setminus\{(0,0)\}$ we define $f_{(a,b)}: M=\bigoplus_{k=1}^{s} M_k^{r_k}\to V$ as follows:

$$f_{(a,b)}(m)=\begin{cases}(ag_1+bg_2)(m), & ,m\in M_1^{n_1}\\ (f_{k,1},\dots,f_{k,n_k})(m), & ,m\in M_k^{n_k}, k\geq 2.\end{cases}$$

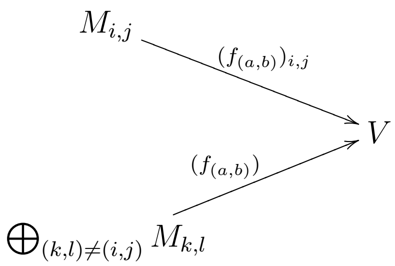
In this context, it is sometimes more convenient to denote the l-th copy of M_k in M by $M_{k,l}$ and components of a map h starting or ending there by $h_{k,l}$. We consider the exact sequence

$$0\to K\xrightarrow{(g_{k,l})} M\xrightarrow{f_{(a,b)}} V$$

where K is the kernel of $f_{(a,b)}$. Let K' be a $\preccurlyeq$-maximal indecomposable direct summand of K and K'' be its complement. We obtain the following commutative diagram with exact rows and columns:



We claim that $M < K' \oplus C \leq U \oplus V$, which implies by minimality $K' \oplus C \leq U \oplus V$ and therefore $K' \cong U$, $C \cong V$. Since $End(V) = k$, it follows that p and consequently also $f_{(a,b)}$ are surjective.
To prove the claim it is important to notice that the choice of the $f_{k,l}$ lets the components of $f_{(a,b)}$ be linearly independent and forbids therefore the existence of homomorphisms making the diagram



commutative. $M \leq K' \oplus C$ is obvious. To see $K' \oplus C \leq U \oplus V$ we take an arbitrary T and apply $Hom(T, \_)$ to the first row resp. the last column of the above diagram. The induced exact sequences are:

$$0 \to Hom(T, K') \to Hom(T, M) \to Hom(T, C) \to Ext^1(T, K'),$$
$$0 \to Hom(T, K'') \to Hom(T, C) \to Hom(T, V).$$

While $[T, K']^1 = 0$ allows us to conclude immediately $[T, M] = [T, K' \oplus C] \leq [T, U \oplus V]$, $[T, K']^1 \neq 0$ leads to $K' \preccurlyeq T$ and therefore to $[T, K'] = 0$ and $[T, K''] = 0$. Thus we also have $[T, K' \oplus C] = [T, C] \leq [T, U \oplus V]$.
The assumption $M \cong K' \oplus C$ would imply that the first row of the diagram splits. In other words, there would exist some homomorphism $h : M \to K'$ such that

$id_{K'} = h \circ g \circ e = \sum_{k,l} h_{k,l} \circ g_{k,l} \circ e$. Accordingly, we would obtain $h_{i,j} \circ g_{i,j} \circ e \neq 0$ for some indices i, j. Since K' and M are preprojective, the invertibility of $g_{i,j} \circ e$ would follow. But then we could express

$$(f_{(a,b)})_{i,j} \circ g_{i,j} \circ e = - \sum_{(k,l) \neq (i,j)} (f_{(a,b)})_{k,l} \circ g_{k,l} \circ e$$

and consequently

$$(f_{(a,b)})_{i,j} = - \sum_{(k,l) \neq (i,j)} (f_{(a,b)})_{k,l} \circ g_{k,l} \circ e \circ (g_{i,j} \circ e)^{-1},$$

which contradicts the above remark. So $K' \oplus C \cong U \oplus V$ and there is an exact sequence

$$0 \to U \to M \xrightarrow{f_{(a,b)}} V \to 0.$$

U is the projective indecomposable $P(x)$ to some point $x \in Q_0$. The simplicity of U insures $\underline{\dim}(U)_x = 1$ and $\underline{\dim}(U)_y = 0$ for all $y \neq x$. Let e_z be the primitive idempotent in kQ belonging to the point $z \in Q_0$. Now regard U, M and V as representations of Q. Thereby, $e_z X$ denotes the vector space of the representation X at the point z and $g(z)$ denotes linear map that is obtained by restricting the homomorphism of representations $g : X \to Y$ to $g(z) : e_z X \to e_z Y$. Since $e_y U = 0$ for $y \neq x$ and therefore $e_y M \cong e_y V$, we can choose suitable bases of $e_y M$ and $e_y V$ to represent the linear maps $(g_1, f')(y) = f_{(1,0)}(y)$ resp. $(g_2, f')(y) = f_{(0,1)}(y)$ by the block matrices

$$\begin{pmatrix} E_p & 0 \\ 0 & E_q \end{pmatrix} \text{ resp. } \begin{pmatrix} A & 0 \\ B & E_q \end{pmatrix}.$$

Thus the matrix representation of a general $f_{(a,b)}(y)$ is

$$\begin{pmatrix} aE_p + bA & 0 \\ bB & (a+b)E_q \end{pmatrix}.$$

But our field k is algebraically closed, so A has at least one eigenvalue. This does not fit with the just shown bijectivity of $f_{(a,-1)}$. Hence, our assumption $Codim(U \oplus V, M) \geq [V, V]^1 + 2$ at the beginning of the proof was wrong. □

Corollary 3.2.2 *Supposed that V has no proper self extensions and M is a preprojective module such that $M = X^r$ with X indecomposable, then $Codim(U \oplus V, M)$ is 1. In particular, if V is preinjective, the codimension is 1.*

As mentioned before, the dual statements are also true. Thus, if V is preinjective and M is a minimal deformation of $U \oplus V$ that has a regular direct summand, we can, roughly speaking, disregard preprojective and preinjective summands for the computation of codimension. Otherwise, if M has no such direct summand, the codimension is 1.

Chapter 4

The regular parts of minimal deformations

From now on, we will concentrate on our main subject and assume that V is preinjective. In the previous chapter we have pointed out that the codimension of a minimal degeneration $M < U \oplus V$ is essentially determined by the regular part of M. Furthermore we have already established some necessary conditions on the preprojective and the preinjective part of M. But until now we know nothing about the regular direct summands. Thus we now want to catch this up and take a closer look on the regular summands.

4.1 A practical degeneration test

First of all, we derive a test criterion for degenerations that will be used in the sequel for several times. Suppose U and V are indecomposable with $\partial(U) < 0$ and $\partial(V) > 0$. Let M be a module with the same dimension vector as $U \oplus V$. M can be written in the form

$$M = M_P \oplus \bigoplus_{\mu \in \mathbb{P}^1} M_\mu \oplus M_I,$$

where $M_P \in add(\mathcal{P})$, $M_\mu \in add(\mathcal{T}_\mu)$ and $M_I \in add(\mathcal{I})$.

Proposition 4.1.1 (Degeneration Test) *Under the above assumptions the module M degenerates into $U \oplus V$ if and only if it satisfies the following conditions:*

(a) $[U,T] - [M_P,T] \geq 0$ *for any indecomposable preprojective T such that there exists some direct summand X of M_P with $d(X,T) \leq 2(p(Q) + d(Q))$.*

(b) $[T,V] - [T,M_I] \geq 0$ *for any indecomposable preinjective T such that there exists some direct summand X of M_I with $d(T,X) \leq 2(p(Q) + d(Q))$.*

(c) *In any tube $\mathcal{T}_\mu$ any regular simple $E \in \mathcal{T}_\mu$ occurs at most $[U, E] - [M_P, E]$-times as regular top of some direct summand of M_μ. In particular, each M_μ has at most $s := \partial(V) - \partial(M_I)$ indecomposable direct summands.*

Proof. We begin with the necessity of these conditions. Since M degenerates into $U \oplus V$, it follows for any preprojective resp. preinjective T that

$$\begin{aligned} [U, T] - [M_P, T] &= [U \oplus V, T] - [M, T] \geq 0 \quad \text{resp.} \\ [T, V] - [T, M_I] &= [T, U \oplus V] - [T, M] = [U \oplus V, \tau T] - [M, \tau T] \geq 0. \end{aligned}$$

Hence, the conditions (a) and (b) hold. For (c), let $M_\mu^1, \ldots, M_\mu^t$ denote the indecomposable direct summands of M_μ. Suppose $E \in \mathcal{T}_\mu$ is regular simple, then lemma 1.1.2 says that

$$[M_\mu^i, E] = \begin{cases} 1 & Top(M_\mu^i) = E \\ 0 & Top(M_\mu^i) \neq E. \end{cases}$$

Accordingly, if E appeared more than $[U, E] - [M_P, E]$-times as regular top of some M_μ^i's, this would lead to

$$[U \oplus V, E] - [M, E] = [U, E] - [M_P, E] - \sum_{k=1}^{t} [M_\mu^k, E] < 0.$$

So we come to the sufficiency of these conditions. We have to verify the inequality $[U \oplus V, T] - [M, T] \geq 0$ for all indecomposable non-injective T. If T is preprojective but no successor of any indecomposable direct summand of M_P, the assertion is clear. Supposed T is preprojective such that there is no indecomposable direct summands X of M_P with $d(X, T) \leq 2(p(Q) + d(Q))$, we choose k minimal with $d(X, T) \leq 2(k + 1)(p(Q) + d(Q))$ for at least one of these X. Then $\tau^{kp(Q)} T$ satisfies $d(X, \tau^{kp(Q)} T) \leq 2(p(Q) + d(Q))$ and by the minimality of k we obtain

$$\begin{aligned} [M, T] = [M_P, T] &= \langle \underline{\dim}(M_P), \underline{\dim}(T) \rangle \\ &= \langle \underline{\dim}(M_P), \underline{\dim}(\tau^{kp(Q)} T) - k\epsilon(Q)\partial(T)\delta \rangle \\ &= [M_P, \tau^{kp(Q)} T] + k\epsilon(Q)\partial(T)\partial(M_P) \\ &\leq [U, \tau^{kp(Q)} T] + k\epsilon(Q)\partial(T)\partial(U) = [U, T] = [U \oplus V, T]. \end{aligned}$$

The dual argument works for preinjective indecomposables. The subsequent application of the Auslander-Reiten formula yields the desired inequality for preinjective but not injective T.
Finally, suppose $T \in \mathcal{T}_\mu$. Let $E_1, \ldots, E_{p_\mu}$ denote the regular simples in $\mathcal{T}_\mu$. Then

$$\begin{aligned} [U \oplus V, T] - [M, T] &= [U, T] - [M_P, T] - [M_\mu, T] \\ &\geq \sum_{k=1}^{p_\mu} l_{E_k}(T)([U, E_k] - [M_P, T]) - \sum_{i=1}^{t} l_{Top(M_\mu^i)}(T). \end{aligned}$$

Because each regular simple E_k occurs at most $[U, E_k] - [M_P, E_k]$-times as the regular top of one of the M_i, we conclude $\sum_{k=1}^{p_\mu} l_{E_k}(T)([U, E_k] - [M_P, E_k]) - \sum_{i=1}^{t} l_{Top(M_\mu^i)}(T) \geq 0$. □

If our module M comes from a single tube $\mathcal{T}_\mu$, it is possible to sharpen up Proposition 4.1.1. Notice, that in this case it holds $-\partial(U) = \partial(V)$.

Corollary 4.1.2 (Degeneration test for regular modules) *Under the requirements stated before, the following conditions are equivalent for a module $M = M_\mu \in add(\mathcal{T}_\mu)$:*

(i) $M < U \oplus V$.

(ii) $\underline{\dim}(M) = \underline{\dim}(U \oplus V)$ *and each E_k occurs at most $[U, E_k]$-times as regular top of some indecomposable direct summand of M.*

4.2 The case $-\partial(\mathbf{U}) = \partial(\mathbf{V}) = \mathbf{1}$

If U is preprojective of defect -1 and V preinjective of defect 1, the deduced degeneration test is sufficient to classify all degenerations into $U \oplus V$. We do not even have to tilt U to a projective simple indecomposable.

Suppose $E_1, \ldots, E_{p_\mu}$ are the regular simples of the regular tube $\mathcal{T}_\mu$. Observe that we have

$$1 = -\partial(U) = \langle \underline{\dim}(U), \delta \rangle = \sum_{k=1}^{p_\mu} [U, E_k], \quad 1 = \partial(V) = \langle \delta, \underline{\dim}(V) \rangle = \sum_{k=1}^{p_\mu} [E_k, V].$$

Therefore, in any $\mathcal{T}_\mu$ there exist uniquely determined regular simple modules S_μ resp. D_μ such that $[S_{\mu_i}, V] = 1 = [U, D_{\mu_i}]$.

Corollary 4.2.1 *Let M be a module with $\underline{\dim}(M) = \underline{\dim}(U \oplus V)$. With the above notation, $U \oplus V$ is a degeneration of M if and only if M has the following properties:*

- $M = M_1 \oplus \ldots \oplus M_r$ *with* $M_i \in \mathcal{T}_{\mu_i}$ *and* $\mathcal{T}_{\mu_i} \neq \mathcal{T}_{\mu_j}$ *for* $i \neq j$.
- $Top(M_i) = D_{\mu_i}$.

The codimension of any degeneration is 1. *In particular, any degeneration is minimal.*

Proof. The first part of the claim follows immediately from corollary 4.1.2. It only remains to compute the codimension. It is

$$\begin{aligned}[U \oplus V, U \oplus V] - [M, M] &= 2 + [U, V] - \sum_{\nu=1}^{r} [M_\nu, M_\nu] = 2 + [U, V] - \sum_{\nu=1}^{r} [U, M_\nu] \\ &= 2 + [U, V] - \langle \underline{\dim}(U), \underline{\dim}(U \oplus V) \rangle = 1. \end{aligned}$$ □

The dimension vectors of the regular simples generate the kernel of the defect function. Thereby only one relation holds: The dimension vectors of the regular simples of any regular tube add up to δ. Consequently, the summands of M belonging to non-homogeneous tubes are uniquely determined modulo δ.

We are able to state the number of direct summands of M which must at least come from non-homogeneous tubes, cp. the following table. It depends on $q(U \oplus V)$ resp. $[U, V] - [V, U]^1$.

$q(\underline{\dim}(U \oplus V))$	$[U, V] - [V, U]^1$	summands in non-hom. tubes
0	-2	≥ 0
1	-1	≥ 1
2	0	≥ 2
3	1	≥ 3

In addition, the corollary also points out some periodicity for degenerations in the above case. If we replace V by V' with $\underline{\dim}(V') + \delta$, then any $M' < U \oplus V'$ can be constructed either by taking a suitable $M < U \oplus V$ and exchanging one of the summands $M_i \in \mathcal{T}_{\mu_i}$ by $M_i' \in \mathcal{T}_{\mu_i}$ with $\underline{\dim}(M_i') = \underline{\dim}(M_i) + \delta$ or by choosing an indecomposable M_{r+1} of a further tube $\mathcal{T}_{\mu_{r+1}}$ such that $\underline{\dim}(M_{r+1}) = \delta$, $[U, Top(M_{r+1})] \neq 0$ and adding it to M. This procedure always delivers a minimal $M' < U \oplus V'$.

4.3 A periodicity theorem

Let U be projective simple and V be preinjective indecomposable. In the previous section we classified all degenerations $M < U \oplus V$ provided that $-\partial(U) = \partial(V) = 1$. Now we concentrate on the case where $-\partial(U) \geq 2$ or $\partial(V) \geq 2$. Hence, we may assume that Q is of type $\tilde{D}_8$, $\tilde{E}_6$, $\tilde{E}_7$ or $\tilde{E}_8$ and consequently $\epsilon(Q)$ is 1. Although this restriction on $\epsilon(Q)$ is irrelevant for the argumentation, it is convenient to simplify the notation and the calculations.

The first theorem establishes a method how new degenerations can be constructed from a given one. This procedure does not work in general, but for a special sort of degenerations. For that purpose, we fix a regular tube $\mathcal{T}_\mu$ with regular simples $E_1, \ldots, E_{p_\mu}$.

Notation 4.3.1 *(a) Let $R \in \mathcal{T}_\mu$ be of regular length $l(R) > p_\mu$. The regular indecomposable with the same regular socle and regular length $l(R) - p_\mu$ is denoted by R/δ. If $l(R) = p_\mu$, we set $R/\delta := 0$.*

(b) Let M_I be a preinjective module. The module consisting of the $\tau^{p(Q)}$-translated indecomposable direct summands of M_I is simply denoted by $\tau^{p(Q)} M_I$.

(c) Suppose $R \in \mathcal{T}_\mu$. The regular indecomposable with the same regular socle and regular length $l(R) + p_\mu$ we denote by $\frac{R}{\delta}$.

Theorem 4.3.2 (Periodicity Theorem) *Let M_P be a preprojective and M_I a preinjective module such that*

$$s = \partial(M_P) - \partial(U) = \partial(V) - \partial(M_I) > 0.$$

We fix indecomposables $R_1, \ldots, R_s \in \mathcal{T}_\mu$ with $l(R_k) \geq p_\mu$ such that each regular simple $E_i \in \mathcal{T}_\mu$ occurs $[U, E_i] - [M_P, E_i]$-times as regular top of some R_k. Suppose M_R is regular with no direct summand belonging to add($\mathcal{T}_\mu$), then the following statements are equivalent.

(i) $M_P \oplus M_R \oplus \bigoplus_{k=1}^{s} R_k \oplus \tau^{p(Q)} M_I < U \oplus \tau^{p(Q)} V$.

(ii) $M_P \oplus M_R \oplus \bigoplus_{k=1}^{s} R_k/\delta \oplus M_I < U \oplus V$.

Thereby, the codimensions coincide.

Proof. First of all, we make the following abbreviations:

$$M := M_P \oplus M_R \oplus \bigoplus_{k=1}^{s} R_k \oplus \tau^{p(Q)} M_I \text{ and } M' := M_P \oplus M_R \oplus \bigoplus_{k=1}^{s} R_k/\delta \oplus M_I.$$

Obviously, $\underline{\dim}(U \oplus V) = \underline{\dim}(M)$ is equivalent to $\underline{\dim}(U \oplus \tau^{p(Q)} V) = \underline{\dim}(M')$. $(i) \Rightarrow (ii)$. We only have to show for all non-injective indecomposable T the relation $[U \oplus V, T] - [M', T] \geq 0$. Observe that, if R_k/δ is non-zero, $T \in \mathcal{T}_\mu$ satisfies

$$[R_k, T] = min(l_{Top(R_k)}(T), l_{Soc(T)}(R_k)) \geq min(l_{Top(R_k)}(T), l_{Soc(T)}(R_k/\delta)) = [R_k/\delta, T].$$

Hence, if T is preprojective or regular, we obtain

$$\begin{aligned} [U \oplus V, T] - [M', T] &= [U, T] - [M_P, T] - [M_R, T] - \sum_{k=1}^{s} [R_k/\delta, T] \\ &\geq [U, T] - [M_P, T] - [M_R, T] - \sum_{k=1}^{s} [R_k, T] \\ &= [U \oplus \tau^{p(Q)} V, T] - [M, T] \geq 0 \end{aligned}$$

Suppose now $T \in \mathcal{I}$, but not injective. Using the formula (1.2) of Auslander and Reiten results in

$$\begin{aligned} [U \oplus V, T] - [M', T] &= [\tau^- T, U \oplus V] - [\tau^- T, M'] = [\tau^- T, V] - [\tau^- T, M_I] \\ &= [\tau^{p(Q)-1} T, \tau^{p(Q)} V] - [\tau^{p(Q)-1} T, \tau^{p(Q)} M_I] \\ &= [\tau^{p(Q)-1} T, U \oplus \tau^{p(Q)} V] - [\tau^{p(Q)-1} T, M] \geq 0. \end{aligned}$$

$(ii) \Rightarrow (i)$. To prove the reverse direction we make use of the degeneration test 4.1.1. If T is preprojective indecomposable, we simply have $[U \oplus \tau^{p(Q)} V, T] - [M, T] =$

$[U \oplus V, T] - [M', T] \geq 0$.
For preinjective T with $T \preccurlyeq \tau^{p(Q)}V$ we find

$$\begin{aligned}[T, U \oplus \tau^{p(Q)}V] - [M, T] &= [T, \tau^{p(Q)}V] - [\tau^{-}T, \tau^{p(Q)}M_I] \\ &= [\tau^{-p(Q)}T, V] - [\tau^{-p(Q)}T, M_I] \geq 0.\end{aligned}$$

It remains to confirm that any regular simple E occurs at most $[U, E] - [M_P, E]$-times as regular top of some indecomposable summand of M. Supposed that E does not belong to $\mathcal{T}_\mu$, this is guaranteed by 4.1.1 applied to $M' < U \oplus V$. Otherwise, E is isomorphic to one of the $E_i \in \mathcal{T}_\mu$. For them we have assumed that property.
Now we come to the codimensions. We have to show

$$[U \oplus \tau^{p(Q)}V, U \oplus \tau^{p(Q)}V] - [M, M] = [U \oplus V, U \oplus V] - [M', M'].$$

Of course, we can leave out terms that appear on both sides. The obvious ones are $[U, U]$ and $[M_P \oplus M_R, M_P \oplus M_R]$. But observe that

$$\begin{aligned}[U, \tau^{p(Q)}V] &= \langle \underline{\dim}(U), \underline{\dim}(\tau^{p(Q)}V)\rangle = \langle \underline{\dim}(U), \underline{\dim}(V) + \partial(V)\delta\rangle \\ &= [U, V] - \partial(U)\partial(V).\end{aligned}$$

In addition we have

$$\begin{aligned}[M_P, \bigoplus_{k=1}^{s} R_k \oplus \tau^{p(Q)}M_I] &= \langle \underline{\dim}(M_P), \sum_{k=1}^{s} \underline{\dim}(R_k) + \underline{\dim}(\tau^{p(Q)}M_I)\rangle \\ &= \langle \underline{\dim}(M_P), \sum_{k=1}^{s} \underline{\dim}(R_k/\delta) + s\delta)\rangle \\ &+ \langle \underline{\dim}(M_P), \underline{\dim}(\tau^{p(Q)}M_I) + \partial(M_I)\delta\rangle \\ &= [M_P, \bigoplus_{k=1}^{s} R_k/\delta \oplus M_I] - \partial(M_P)(s + \partial(M_I)), \\ [M_R, \tau^{p(Q)}M_I] &= [\tau^{-p(Q)}M_R, M_I] = [M_R, M_I]\end{aligned}$$

and

$$\begin{aligned}[\bigoplus_{j=1}^{s} R_j, \bigoplus_{k=1}^{s} R_k \oplus \tau^{p(Q)}M_I] &= [\bigoplus_{j=1}^{s} R_j, \bigoplus_{k=1}^{s} R_k] + \langle \sum_{k=1}^{s} \underline{\dim}(R_k), \underline{\dim}(\tau^{p(Q)}M_I)\rangle \\ &= \sum_{j,k=1}^{s} ([R_j/\delta, R_k/\delta]+) \\ &+ \langle \sum_{k=1}^{s} \underline{\dim}(R_k)/\delta, \underline{\dim}(M_I) + s\partial(M_I)\rangle \\ &= [\bigoplus_{j=1}^{s} R_j/\delta, \bigoplus_{k=1}^{s} R_k/\delta \oplus M_I] + s(s + \partial(M_I)).\end{aligned}$$

Therefore, the equation that remains to be verified is simply

$$-\partial(U)\partial(V) + \partial(M_P)(s + \partial(M_I)) - s(s + \partial(M_I)) = 0.$$

Indeed, this follows from $s = \partial(M_P) - \partial(U) = \partial(V) - \partial(M_I)$. □

The requirements on the multiplicities of the regular tops of the R_k's are only needed to come from (ii) to (i) provided that some of the R_k/δ vanish. Otherwise, this condition is already included in the statements (i) and (ii).

Notice, if we want to apply the periodicity theorem on a minimal degeneration $M < U \oplus V$, we are in trouble for two reasons. It is not clear how many minimal M actually satisfy the assumptions of the theorem. Furthermore, the periodicity theorem says nothing about the preservation of minimality. We aim to solve these problems. A first result is the following lemma.

Lemma 4.3.3 *Suppose $s := \partial(V) = -\partial(U)$. Let $R_1, \ldots, R_s \in \mathcal{T}_\mu$ be indecomposable of regular length $l(R_i) \geq p_\mu$. If $\bigoplus_{k=1}^s R_k < U \oplus \tau^{p(Q)}V$ is a minimal degeneration, then $\bigoplus_{k=1}^s R_k/\delta < U \oplus V$ is also minimal.*

Proof. If $R' := \bigoplus_{k=1}^s R_k/\delta < U \oplus V$ is not minimal, then there exists a minimal degeneration N' of $\bigoplus_{k=1}^s R_k/\delta$ which has $U \oplus V$ in its orbit closure. According to theorem 1.2.6 we know that there exist indecomposable modules U' and V' such that N' is, up to a renumbering of the R_k/δ, of the form

$$N' = U' \oplus V' \oplus \bigoplus_{k=t+1}^{s} R_k/\delta, \quad (t \geq 1).$$

Moreover, the theorem says that $\bigoplus_{k=1}^t R_k/\delta < U' \oplus V'$ is a minimal disjoint degeneration.
Suppose U' is preprojective. This is the case if and only if V' is preinjective and $\partial(V') = -\partial(U')$. In view of proposition 4.1.1 $N' < U \oplus V$ implies $(s-t) + \partial(V') \leq \partial(V)$ and therefore $\partial(V') \leq t$. On the other hand, since $\bigoplus_{k=1}^t R_k/\delta$ degenerates into $U' \oplus V'$ we must have $t \leq \partial(V')$. Thus we have $t = \partial(V')$. Applying the periodicity theorem 4.3.2 now yields the two degenerations

$$\bigoplus_{k=1}^{t} R_k < U' \oplus \tau^{p(Q)}V' \quad \text{and} \quad U' \oplus \bigoplus_{k=t+1}^{s} R_k \oplus \tau^{p(Q)}V' < U \oplus \tau^{p(Q)}V,$$

whence $\bigoplus_{k=1}^s R_k < U' \oplus \bigoplus_{k=t+1}^s R_k \oplus \tau^{p(Q)}V' < U \oplus \tau^{p(Q)}V$. This violates the assumption.
Otherwise, if U' and V' are regular, we find by means of a similar calculation $t = 2$. But from $R_1/\delta \oplus R_2/\delta < U' \oplus V'$ it is easy to conclude $R_1 \oplus R_2 < \frac{U'}{\delta} \oplus \frac{V'}{\delta}$. Apart from that, the periodicity theorem insures $\frac{U'}{\delta} \oplus \frac{V'}{\delta} \oplus \bigoplus_{k=3}^s R_k < U \oplus \tau^{p(Q)}V$. Hence, like in the first case, we get a contradiction to minimality, namely $\bigoplus_{k=1}^s R_k < \frac{U'}{\delta} \oplus \frac{V'}{\delta} \oplus \bigoplus_{k=3}^s R_k < U \oplus \tau^{p(Q)}V$. □

4.4 Regular deformations of $\mathbf{U} \oplus \mathbf{V}$

Suppose Q is a quiver of type $\tilde{D}_8$, $\tilde{E}_6$, $\tilde{E}_7$ or $\tilde{E}_8$. Let U be the only projective simple module of kQ and V be preinjective indecomposable such that $-\partial(U) = \partial(V) \geq 2$. This section is devoted to the study of regular minimal deformations of $U \oplus V$. As mentioned before, the study of this type of degenerations is also of interest for the codimension of the general case.

Besides, by reason of substantially the same arguments as for the codimension in chapter 3, the consideration of regular minimal deformations gives helpful informations on the regular parts of minimal degenerations in general. We want to explain this phenomenon.

Let $M < U \oplus V$ be a minimal degeneration. In particular, there is an exact sequence

$$0 \to U \xrightarrow{\epsilon} M \to V \to 0.$$

M can be decomposed into $M = M_P \oplus \bigoplus_{\mu \in \mathbb{P}^1} M_\mu \oplus M_I$, its preprojective, regular and preinjective parts. Assume there is some ν with $M_\nu \neq 0$. Defining

$$M_1 := M_P \oplus M_\nu \quad \text{resp.} \quad M_2 := \bigoplus_{\mu \neq \nu} M_\mu \oplus M_I$$

we obtain a directed decomposition $M = M_1 \oplus M_2$. Hence, we can write $\epsilon = \binom{\epsilon_1}{\epsilon_2}$, where $\epsilon_i : U \to M_i$. An application of the reduction theorem 2.1.2 delivers a new minimal degeneration $M_P \oplus M_\nu < U \oplus C$, where $C := Coker(\epsilon_1)$.

Supposed M_P is nonzero, our next aim is to discard the preprojective part with the help of the dual version of the reduction theorem. To achieve this, we take the slice $\mathcal{S}$ in the preinjective component of Γ_{kQ} whose only sink is C. Let T be the tilting module defined by $T := \bigoplus_{X \in \mathcal{S}} X$ and $F := Hom(T, \_)$. $B := End(T)$ is again a path algebra with the same underlying graph as Q. $FM_P \oplus FM_\nu < FU \oplus FC$ is a minimal degeneration of B-modules given by an exact sequence

$$0 \to FU \to FM_P \oplus FM_\nu \xrightarrow{(\pi_1, \pi_2)} FC \to 0.$$

Thereby, FU and FM_P are preprojective, FM_ν is regular and FC is injective simple. This allows us to apply the dual version of theorem 2.1.2, which delivers a minimal degeneration $FM_\nu < Ker(\pi_1) \oplus FC$. Tilting once again, by means of an appropriate tilting module in the preprojective component of Γ_B, we may assume that $Ker(\pi_1)$ is projective simple. During the whole procedure only the orientation of the arrows in Q changed, but not the underlying graph. In this way, the study of regular deformations yields inferences on the regular parts of degenerations in general.

We fix a regular tube $\mathcal{T}_\mu$ of period p_μ with regular simples $E_1, \ldots, E_{p_\mu}$. Let $M = M_\mu$ be a module coming from $add(\mathcal{T}_\mu)$.

Notation 4.4.1 *Let R_1, R_2 be two regular indecomposable modules of the tube $\mathcal{T}_\mu$ such that the regular top of R_1 is the τ-translated of the regular socle of R_2. Then $\substack{R_2\\R_1}$ denotes the regular indecomposable module R of regular length $l(R_1)+l(R_2)$ and submodule R_1.*

Lemma 4.4.2 *Suppose A, B and C are regular indecomposable modules belonging to the tube $\mathcal{T}_\mu$ and $D \in add(\mathcal{T}_\mu)$ such that $\substack{C\\B\\A} \oplus B \oplus D < U \oplus V$ is a degeneration, then*

$$\substack{C\\B\\A} \oplus B \oplus D < \substack{B\\A} \oplus \substack{C\\B} \oplus D < U \oplus V.$$

Proof. $\substack{C\\B\\A} \oplus B \oplus D < \substack{B\\A} \oplus \substack{C\\B} \oplus D$ follows from the tubular structure of $\mathcal{T}_\mu$ since there is an exact sequence of the form $0 \longrightarrow \substack{B\\A} \longrightarrow \substack{C\\B\\A} \oplus B \longrightarrow \substack{C\\B} \longrightarrow 0$. $\substack{B\\A} \oplus \substack{C\\B} \oplus D < U \oplus V$ is due to corollary 4.1.2, the degeneration test for regular modules. □

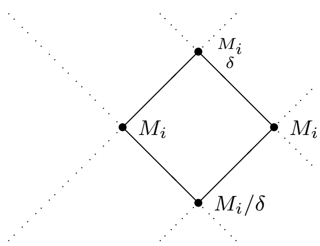
Assume $M = M_\mu \in add(\mathcal{T}_\mu)$ such that $M < U \oplus V$ is minimal. We know already that $M = M_1 \oplus \ldots \oplus M_t$ with M_k indecomposable $t \leq \partial(V)$. Moreover, the minimality insures that the regular lengths of the M_k differ only a little.

Notation 4.4.3 *Let X be regular indecomposable of the tube $\mathcal{T}_\mu$. As in [18] we denote by $\varphi^- X$ the quotient $X/Soc(X)$ and by ψX the regular indecomposable with regular radical X.*

Lemma 4.4.4 *If $M = \bigoplus_{k=1}^t M_k < U \oplus V$, $M_k \in \mathcal{T}_\mu$, is minimal, then for each pair (M_i, M_j) of direct summands of M it holds $M_i = \varphi^{-\alpha}\psi^{\beta} M_j$ with $0 \leq \alpha, \beta \leq p_\mu$.*

Proof. Suppose for instance, there would occur two indecomposable direct summands M_1, M_2 of M such that $M_1 = \varphi^{-\alpha}\psi^{p_\mu + t} M_2$, $t > 0$, $0 \leq \alpha \leq p_\mu$. Then we could use lemma 4.4.2 and replace M_1 by $M_1' := \varphi^{-\alpha}\psi^{p_\mu} M_2$ resp. M_2 by $M_2' := \psi^t M_2$ to obtain a degeneration between M and $U \oplus V$. For the other possibilities we proceed similarly. □

The statement of the above lemma can be visualized as follows. For each M_i the remaining summands M_j do not lie outside the square of the picture down below (resp. the part of the square that actually exists in case of $l(M_i) < p_\mu$).



Corollary 4.4.5 *Let* $M = \bigoplus_{k=1}^{\partial(V)} M_k < U \oplus V$*,* $M_k \in \mathcal{T}_\mu$*, be minimal such that* $l(M_i) < p_\mu$ *for at least one* $1 \leq i \leq \partial(V)$*, then* $l(M_k) < 2p_\mu$ *for all* k*. In particular* $\underline{\dim}(U \oplus V) < 2\partial(V)\delta$*.*

Proof. The assertion follows immediately from lemma 4.4.4. □

It cannot be expected that arguments, as above, within the tube $\mathcal{T}_\mu$ are sufficient to disprove the minimality of a degeneration $M = M_\mu < U \oplus V$. But there is another effective method, which uses only modules of defect ± 1 and is relatively easy to handle.

Lemma 4.4.6 *Let* P *be a preprojective module of defect* -1 *and* $i : U \hookrightarrow P$ *be an injection. Then* $Coker(i)$ *has no preprojective direct summand.*

Proof. Consider the exact sequence $0 \to U \xrightarrow{i} P \xrightarrow{\pi} Coker(i) \to 0$ and assume $Coker(i)$ had a preprojective direct summand X and $p : Coker(i) \to X$ were the projection on X. Then the composition $p\pi$ would be surjective. Since $\partial(P) = -1$, lemma 1.1.3 says that $p\pi$ were injective and therefore $P \cong X$. But U is non-zero, so the assumption is absurd. □

Lemma 4.4.7 *Let* P *be preprojective of defect* -1 *and* $R \in \mathcal{T}_\mu$ *such that* $\underline{\dim}(R) \geq \underline{\dim}(P)$*. If* $[P, Top(R)]$ *is nonzero, then there is an exact sequence* $0 \to P \to R \to I \to 0$*, where* I *is preinjective of defect* 1*.*

Proof. We denote the regular radical of R by R'. Since $[P, Top(R)] \neq 0$, there exists a homomorphism $\psi : P \to R$ such that $Im(\psi) \not\subseteq R'$. ψ cannot be surjective due to $\underline{\dim}(R) \geq \underline{\dim}(P)$. For this reason $Im(\psi)$ has a preprojective direct summand. Because $\partial(P) = -1$, P must be isomorphic to it. Thus $Ker(\psi) = 0$, i.e. ψ is injective. We consider the exact sequence

$$0 \longrightarrow P \xrightarrow{\psi} R \longrightarrow I := Coker(\psi) \longrightarrow 0$$

and assume I has a regular direct summand $X \in \mathcal{T}_\mu$. Since X is a quotient of R, it is of the form $X = R/\tilde{R}$, where $\tilde{R}$ is a proper regular submodule of R. Hence, we get the following diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & P & \overset{\psi}{\hookrightarrow} & R & \overset{\binom{\pi_1}{\pi_2}}{\twoheadrightarrow} & I = X \oplus W & \longrightarrow & 0 \\ & & & & & & \downarrow{\scriptstyle pr} & & \\ & & & & & & X & & \end{array}$$

The exactness of the above sequence implies $\pi_1 \circ \psi = 0$. Thus $(pr \circ \pi_1) \circ \psi = 0$, i.e. $\psi(P) \subseteq \tilde{R} \subseteq R'$, which does not fit with the choice of ψ. So I is preinjective of defect -1. □

Lemma 4.4.8 *Let $M = M_1 \oplus \ldots \oplus M_t < U \oplus V$, $M_k \in \mathcal{T}_\mu$ be a degeneration. Suppose P is a preprojective module of defect -1 with $[U, P] \neq 0$. If*

(i) there is an exact sequence $0 \to P \to M_1 \to I \to 0$ such that I is preinjective with $[I, V] \neq 0$ and

(ii) P satisfies $[P, E] \leq [U, E]$ for any regular simple module E of an arbitrary tube,

then $M < P \oplus M_2 \oplus \ldots \oplus M_t \oplus I < U \oplus V$.

Proof. Due to the exact sequence (i) $M < P \oplus M_2 \oplus \ldots \oplus M_t$ is obvious. So, we only have to show $P \oplus M_2 \oplus \ldots \oplus M_t \oplus I < U \oplus V$. For that purpose, we use the degeneration test 4.1.1.
Since $[U, P] \neq 0$, the simplicity of U guarantees the existence of injection $i : U \hookrightarrow P$ whose cokernel has no preprojective direct summand, whence we conclude for any preprojective T: $[P, T] \leq [U, T]$.
In the case where T is preinjective and $T \preccurlyeq V$, we obtain

$$[T, U \oplus V] - [T, P \oplus M_2 \oplus \ldots \oplus M_t \oplus I] \quad = \quad [T, V] - [T, I].$$

To compute this difference we tilt V and I with the help of the slice in the preinjective component that has V as its only sink. Therefore we may assume that V is injective simple. Thus, $[I, V] \neq 0$ implies that there is a surjection whose kernel has by the dual version of lemma 4.4.6 no preinjective direct summand. This allows us to conclude $[T, V] \geq [T, I]$. If T is not a predecessor of V, there is nothing to prove.
Finally, since $M < U \oplus V$, we know that each regular simple $E_i \in \mathcal{T}_\mu$ is the regular top of at most $[U, E_i]$ indecomposable summands of M. Furthermore $[P, E_i] \neq 0$ if and only if $E_i = Top(M_1)$. Hence, the substitution of M_1 by $P \oplus I$ insures that each E_i appears at most $[U, E_i] - [P, E_i]$-times as regular top of some M_j with $j \geq 2$. □

4.4.1 Minimal deformations in homogeneous tubes

We write $U = P(x)$ where $x \in Q_0$. We aim to classify all minimal degenerations $M < U \oplus V$ with $M = M_\mu \in add(\mathcal{T}_\mu)$ for homogeneous tubes $\mathcal{T}_\mu$.

To do this, we make an interesting observation on the dimension vectors of the regular simples of the non-homogeneous tubes whose correctness can easily be checked by a glance at these dimension vectors, cp. A.1.

Observation 4.4.9 *Let $\mathcal{T}_\nu$ be a non-homogeneous tube and $E \in \mathcal{T}_\nu$ be regular simple. The dimension vector of E has the following property. If $y \in Q_0$ is an extension point of Q, i.e. a point with $\delta_y = 1$, then $\underline{\dim}(E)_y \leq \underline{\dim}(E)_x$.*

Proposition 4.4.10 *Let $\mathcal{T}_\mu$ be a homogeneous tube and $\underline{\dim}(U \oplus V) = k\delta$. Then $M = M_\mu < U \oplus V$, $M_\mu \in add(\mathcal{T}_\mu)$, is a minimal degeneration if and only if $\partial(V)|k$ and $M_\mu = \bigoplus_{k=1}^{\partial(V)} R$ where $R \in \mathcal{T}_\mu$ is the uniquely determined indecomposable with $\underline{\dim}(R) = \frac{k}{\partial(V)}\delta$. In this case $Codim(U \oplus V, M_\mu)$ is one.*

Proof. Suppose $M = \bigoplus_{k=1}^{\partial(V)} R$. Referring to lemma 1.1.2 it follows at once that we have for any $T \in \mathcal{T}_\mu$

$$[U,T] - [M,T] \geq l(T)\partial(V) - l(T)\partial(V) = 0,$$

whence $M < U \oplus V$. We compute the codimension. It is

$$\begin{aligned} [U \oplus V, U \oplus V] - [M,M] &= 1 + [U, U \oplus V] - \sum_{j,k=1}^{\partial(V)} [R,R] \\ &= 1 + k\partial(V) - \partial(V)^2 \frac{k}{\partial(V)} = 1. \end{aligned}$$

Hence, the degeneration is minimal.
Conversely, let $M = M_\mu < U \oplus V$ be minimal. Assume $M \neq \bigoplus_{k=1}^{\partial(V)} R$. Because $\mathcal{T}_\mu$ is homogeneous, there is a direct summand R' of M of maximal length such that $\partial(V)\underline{\dim}(R')_k > \underline{\dim}(V)_k$ holds for any k. We choose an arbitrary projective module $P = P(y)$ of defect -1. We observe $\underline{\dim}(P) \leq \underline{\dim}(R')$ and $[P, Top(R')] = -\partial(P) = 1$. Therefore, lemma 4.4.7 implies the existence of an exact sequence $0 \to P \to R' \to I \to 0$. Due to this we obtain with $\underline{\dim}(V) = k\delta - \underline{\dim}(U)$:

$$\begin{aligned} [I,V] &\geq [I,V] - [I,V]^1 = [R',V] - [P,V] = l(R')\partial(V) - k \\ &= \partial(V)\underline{\dim}(R')_y - \underline{\dim}(V)_y > 0. \end{aligned}$$

We want to apply lemma 4.4.8 for $M_1 := R'$. Of course, $[U,P]$ is non-zero. Besides, we can write $U = P(x)$, $x \in Q_0$, and get by means of observation 4.4.9

$$[P,E] = \underline{\dim}(E)_y \leq \underline{\dim}(E)_x = [U,E]$$

for all regular simples E belonging to a non-homogeneous tube. □

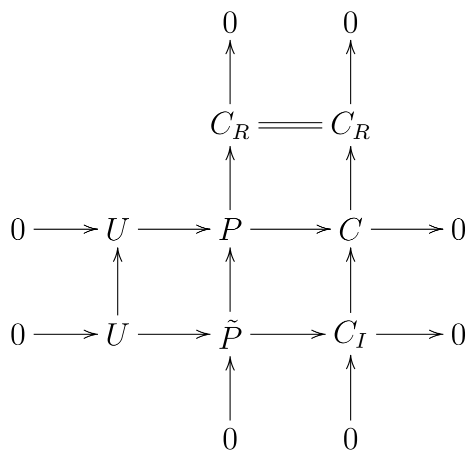
4.4.2 Minimal deformations in non-homogeneous tubes

For non-homogeneous tubes the combinatorics is much more involved. Let $\mathcal{T}_\mu$ be such a tube of period p_μ with regular simples $E_1, \dots, E_{p_\mu}$.

Lemma 4.4.11 *(a) For any regular simple module $E_i \in \mathcal{T}_\mu$ with $[U, E_i] \neq 0$ there exists a preprojective module P of defect -1 such that $\underline{\dim}(P) \leq \delta$, $[P, E_i] \neq 0$ and $[U,P] \neq 0$.*

(b) *If P is of minimal dimension with the properties in (a), then there is an exact sequence $0 \to U \to P \to C \to 0$ where C is preinjective. In particular, $[P, E] \leq [U, E]$ for any regular simple modules E of any tube.*

Proof. (a) Let Q be a projective module of defect -1. Then there exists exactly one index k such that $[Q, E_k] = 1$. We can write $E_k = \tau^l E_i$ with $0 \leq l < p_\mu$. Therefore, the modules $\tau^{-l}Q, \tau^{-l-p_\mu}Q, \ldots, \tau^{-l-(p(Q)-p_\mu)}Q$ are those τ^--translates of Q with $[\tau^{-k}Q, E_i] = 1$ and $\underline{\dim}(\tau^{-k}Q) \leq \delta$. Among these modules one can find per case by case inspection some P that already satisfies $[U, P] \neq 0$, e.g. using the starting functions of [2], 1.3, p. 122 or by computing the dimension vectors of $\tau^{-l}Q, \tau^{-l-p_\mu}Q, \ldots, \tau^{-l-(p(Q)-p_\mu)}Q$. Anyway, in each case there is some P of minimal dimension with the desired properties given in the appendix A.2 because we will use these modules in the sequel.
(b) If P is of minimal dimension satisfying the requirements of (a), we obtain an exact sequence $0 \to U \to P \to C \to 0$. C has no preprojective direct summand, so we can decompose $C = C_R \oplus C_I$ and consider the following commutative diagram:



We claim that $C_R = 0$. Otherwise, we would have $\underline{\dim}(\tilde{P}) < \underline{\dim}(P)$ and apply the functor $Hom(\_\,, E)$ to the diagram. If $[C, E] = 0$, then the middle column induces $0 = Hom(C_R, E) \to Hom(P, E) \to Hom(\tilde{P}, E)$. Because $Hom(P, E)$ is non-zero, this also holds for $Hom(\tilde{P}, E)$, which violates the minimality of $\underline{\dim}(P)$.
In the case $[C, E]^1 = 0$ there are surjections $Hom(P, E) \to Hom(U, E) \to 0$ and $Hom(P, E) \to Hom(\tilde{P}, E) \to 0$. Thanks to $Hom(U, E) \neq 0$ we obtain $Hom(P, E) \neq 0$ and $Hom(\tilde{P}, E) \neq 0$. This is impossible.
Finally, suppose $[C, E] \neq 0 \neq [C, E]^1$. We decompose $C_R = C' \oplus C''$ into C', the part belonging to $add(\mathcal{T}_\mu)$, and C'', the remaining part, and deduce

$$[C', E_k] = [C, E_k] \leq [P, E_k] = \begin{cases} 1 & k = i \\ 0 & k \neq i. \end{cases}$$

For this reason C' is indecomposable with $Top(C') = E_i$. On the other hand, $0 \neq [C', E] = [\tau^- E, C']$ means that $Soc(C') = \tau^- E_i$. Hence $\underline{\dim}(C') \geq \delta$, which contradicts $\underline{\dim}(P) \leq \delta$. □

Since the proof of lemma 4.4.11 (a) requires anyway a closer look on the first $p(Q)-1$ τ^{-}-translated of the projective indecomposables having defect -1, it does not make much more effort to determine the set of all P that are of minimal dimension satisfying (a). We denote this set by $\mathcal{H}(E_i)$. For the convenience, all regular simples E_i with $[U, E_i] \neq 0$ are collected in the appendix. There is also given at least one preprojective module P of $\mathcal{H}(E_i)$. Notice, since $\underline{\dim}(P) \leq \delta$, it follows that $[U, \tau P] \leq \partial(V)$. But during the examination of the table in the appendix we make an interesting

Observation 4.4.12 *Let $\mathcal{T}_\mu$ be a non-homogeneous tube.*

(a) If $-\partial(U) = \partial(V) = 2$, then for any regular simple $E_i \in \mathcal{T}_\mu$ we can take a projective $P \in \mathcal{H}(E_i)$ with $[U, \tau P] = 0$.

(b) In the remaining cases, we can always find a preprojective $P \in \mathcal{H}(E_i)$ that satisfies moreover $[U, \tau P] \leq 1$.

(c) Except for four cases of type $\tilde{E}_8$ there exists some $P \in \mathcal{H}(E_i)$ that already satisfies $\underline{\dim}(P) \leq \underline{\dim}(E_i)$. In the exceptional cases we find some $\underline{\dim}(P) \leq \underline{\dim}\left(\begin{smallmatrix} E_i \\ E_{i-1} \end{smallmatrix}\right)$.

In view of the requirements of the periodicity theorem we achieve the following worth knowing result.

Proposition 4.4.13 *Let $\mathcal{T}_\mu$ be a non-homogeneous tube and $M = \bigoplus_{k=1}^{t} M_k < U \oplus V$, $M_k \in \mathcal{T}_\mu$, be minimal. Suppose $t < \partial(V)$, then for all $k \in \{1, \dots, t\}$*

$$[M_k, V] < \begin{cases} \partial(V) & \text{if } \partial(V) - t \geq 2 \text{ or } -\partial(U) = \partial(V) = 2, \\ 2\partial(V) & \text{if } \partial(V) - t = 1 \text{ and } -\partial(U) = \partial(V) > 2. \end{cases}$$

In particular, we have $l(M_k) < p_\mu$ and $\underline{\dim}(U \oplus V) < \partial(V)\delta$ provided $-\partial(U) = \partial(V) = 2$ resp. $l(M_k) < 2p_\mu$ and $\underline{\dim}(U \oplus V) < 2\partial(V)\delta$ otherwise.

Proof. Without loss of generality we may assume that M_1 is a direct summand of M such that $[M_1, V]$ is maximal. Suppose furthermore $l(M_1) \geq 2$ in the exceptional cases of 4.4.12 (c). Let $P \in \mathcal{H}(Top(M_1))$ be as in the observation. Since $\underline{\dim}(P) \leq \underline{\dim}(M_1)$, we can consider the exact sequence $0 \to P \to M_1 \to I \to 0$ given by Lemma 4.4.7. $[I, V] \neq 0$ would not allow $M < U \oplus V$ to be minimal. We show now that $[I, V] = 0$ forces M to have the claimed shape.
There exists an exact sequence $0 \to U \to M \to V \to 0$, which induces by application of $Hom(P, \_)$ the sequence

$$0 \to Hom(P, M) \to Hom(P, V) \to Ext^1(P, U) \to 0.$$

We conclude $[P,V] = \sum_{k=1}^{t}[P,M_i] + [U,\tau P]$. By reason of the exactness of $0 \to P \to M_1 \to I \to 0$ it follows that

$$0 = [I,V] \geq [I,V] - [I,V]^1 = [M_1,V] - [P,V].$$

Write $[M_1,V] = l_1\partial(V)+r_1$ with $l_1 \in \mathbb{N}_0$ and $0 \leq r_1 < \partial(V)$. Notice, $[P,M_k] = l_1 + 1$ means that the regular top of M_1 occurs l_1+1-times as composition factor in the regular composition series. But then $l_{Soc(M_k)}(M_k) = l_1+1$, whence $l_{Soc(M_k)}(M_1) = l_1+1$. According to proposition 4.1.1 there are at most r_1 direct summands of M with this property. We deduce

$$\begin{aligned} 0 \geq [M_1,V] - [P,V] &= [M_1,V] - \sum_{k=1}^{t}[P,M_i] - [U,\tau P] \\ &\geq l_1\partial(V) + r_1 - \sum_{k=1}^{t} l_1 - r_1 - [U,\tau P] \\ &= l_1(\partial(V) - t) - [U,\tau P], \end{aligned}$$

whence $[U,\tau P] \geq l_1(\partial(V)-t)$. Supposed that $\partial(V)-t \geq 2$ this inequality implies $l_1 \leq 1$, while $l_1 \leq 2$ has to be provided in the case $\partial(V)-t = 1$ and $[U,\tau P] = 1$. □

Unfortunately, the combinatorics of the non-homogeneous tubes is too complicated to derive more detailed informations on the direct summands of a minimal deformation $M = M_\mu \in add(\mathcal{T}_\mu)$ of $U \oplus V$ using general arguments.

Though, with the current state of knowledge we are prepared to give an upper bound for the codimension. Recall that the choice of the only projective simple module U determines uniquely the orientation of Q.

Proposition 4.4.14 *There exists some smallest natural number $c \in \mathbb{N}$ such that for any choice of U, any preinjective indecomposable V, any non-homogeneous tube $\mathcal{T}_\mu$ in Γ_{kQ} and any minimal degeneration $M < U \oplus V$ with $M = M_\mu \in add(\mathcal{T}_\mu)$ it holds: $Codim(U \oplus V, M) \leq c$.*

Proof. Let U, V, $\mathcal{T}_\mu$ and $M = M_\mu < U \oplus V$ be given. Suppose there is some indecomposable direct summand X of M with $l(X) \geq 2p_\mu$. Then proposition 4.4.13 and corollary 4.4.5 guarantee that M can be written in the form $M = \bigoplus_{k=1}^{\partial(V)} \frac{R_k}{\delta}$ for some indecomposable or zero R_k. We can apply lemma 4.3.3 and obtain a new minimal degeneration $\bigoplus_{k=1}^{\partial(V)} R_k < U \oplus \tau^{p(Q)}V$ of the same codimensions. Iteration of this procedure reduces the problem to the consideration of the codimensions of minimal degeneration $M' < U \oplus V'$ such that the regular lengths of all summands of M' are smaller than $2p_\mu$, i.e. $\underline{\dim}(V') \leq 2\partial(V)\delta$. But there is only a finite number of such degenerations. Furthermore the numbers of choices for U and $\mathcal{T}_\mu$ are finite. Thus we can take c as the maximal codimension of all these degenerations. □

4.5 Boundedness of the codimension in the general case

Now we come back to the analysis of the codimension in the general case. Let Q be a quiver of type $\tilde{D}_8$, $\tilde{E}_6$, $\tilde{E}_7$ or $\tilde{E}_8$, U be projective simple, V be preinjective indecomposable with $-\partial(U) \geq 2$ or $\partial(V) \geq 2$. Suppose $M < U \oplus V$ is a minimal degeneration and $M = \bigoplus_{k=1}^{r} X_k$ is the maximal directed decomposition of M.

As explained in chapter 3, the codimension of $U \oplus V$ in M can be written by means of theorem 3.1.1 in the form

$$Codim(U \oplus V, M) = Codim(U_1 \oplus V_1, M_1) + \ldots + Codim(U_r \oplus V_r, M_r) + (r-1),$$

where $M_i < U_i \oplus V_i$ is a minimal degeneration of kQ_i-modules with $|Q_i| = |Q|$. Each M_i arises from X_i, just as U_i from V and V_i from U, by dualization and tilting with the slice in the preprojective component of $\Gamma_{kQ^{op}}$ that has DV as its only source. This procedure preserves indecomposability, directed decompositions, the membership to homogeneous or non-homogeneous tubes of all involved modules, but changes the sign of their defects.

Accordingly, U_i is the only projective simple, V_i is preinjective indecomposable and M_i is of one of the following types:

(i) $M_i = X^t$ where X is preprojective indecomposable,
(ii) $M_i = X^t$ where X is preinjective indecomposable or
(iii) $M_i = M_\mu \in add(\mathcal{T}_\mu)$ for some $\mu \in \mathbb{P}^1$.

In the first two cases, corollary 3.2.2 says that $Codim(U_i \oplus V_i, M_i) = 1$. If $\mathcal{T}_\mu$ is homogeneous and $M_i = M_\mu \in add(\mathcal{T}_\mu)$ or if $-\partial(U_i) = \partial(V_i) = 1$, we also have $Codim(U_i \oplus V_i, M_i) = 1$ thanks to corollary 4.4.10 and corollary 4.2.1. Since Q has exactly three non-homogeneous tubes $\mathcal{T}_1$, $\mathcal{T}_2$ and $\mathcal{T}_3$, proposition 4.4.14 furnishes

Theorem 4.5.1 (Bounded codimension) *The codimension of a minimal degeneration $M < U \oplus V$ is bounded. To be more precise: If c is the natural number of proposition 4.4.14, then $Codim(U \oplus V, M) \leq 3c - 2$.*

Since we now know that the codimension of a minimal $M < U \oplus V$ is bounded by $3c-2$, we are certainly interested in the value of c. The proof of proposition 4.4.14 points out a way to gain c. For any choice of U and any preinjective indecomposable V such that $-\partial(U) = \partial(V)$ and $\underline{\dim}(U \oplus V) \leq 2\partial(V)\delta$ it suffices to determine all minimal deformations in the three non-homogeneous tubes and to compute their codimensions. The computer program presented in the next chapter serves that purpose.

Chapter 5

Computer calculations and their consequences

5.1 Strategy of the computer program

First, we give a short description of the strategy that was pursued in the computer program. Let Q be a quiver of type $\tilde{D}_8$, $\tilde{E}_6$, $\tilde{E}_7$ or $\tilde{E}_8$ with only one sink. Let U be the only projective simple kQ-module. and $\mathcal{T}_\mu$ be an arbitrary non-homogeneous tube of period p_μ and regular simples $E_1, \ldots, E_{p_\mu}$.

The aim of the program is to compute for any indecomposable preinjective V with $-\partial(U) = \partial(V)$ and $\underline{\dim}(V) \leq 2\partial(V)\delta$ all $M = M_\mu \in add(\mathcal{T}_\mu)$ such that $M < U \oplus V$ is minimal. Notice first, lemma 4.4.4 insures that all indecomposable direct summands of such a minimal M have regular length less or equal to $3p_\mu$. To understand the program we first need two easy lemmas that are used in it.

Lemma 5.1.1 *Let R and S be regular modules of $add(\mathcal{T}_\mu)$ such that the length of all indecomposable direct summands is at most $3p_\mu$. Then the following statements are equivalent.*

(i) $R \leq S$.

(ii) $\underline{\dim}(R) = \underline{\dim}(S)$ *and* $[R, T] \leq [S, T]$ *for all* $T \in \mathcal{T}_\mu$ *with* $l(T) \leq 4\epsilon(Q)p_\mu$.

Proof. We only have to show that (ii) implies (i). Let $T \in \mathcal{T}_\mu$ be given such that $4 < \epsilon(Q)p_\mu l(T) \leq 5\epsilon(Q)p_\mu$. Our requirement on the lengths of R and S guarantees

$$[R, T] = l_{Soc(T)}(R) = [R, T/\delta] \leq [S, T/\delta] = l_{Soc(T)}(S) = [S, T].$$

Iteration of this procedure yields the claim. □

Lemma 5.1.2 *Let $R_1, \ldots, R_t \in \mathcal{T}_\mu$ such that $R_1 \oplus \ldots \oplus R_t < U \oplus V$ is a degeneration. Suppose there exist some preprojective indecomposable P and some preinjective indecomposable I with the properties*

- $s := -\partial(P) = \partial(I) \leq min(t, \partial(V) - 1)$,
- $d(U, P) \leq 2p(Q)$ *resp.* $d(I, V) \leq 2p(Q)$ *and*
- $R_1 \oplus \ldots \oplus R_s < P \oplus I$.

Then $P \oplus R_{s+1} \oplus \ldots \oplus R_t \oplus I < U \oplus V$ *holds if and only if the following conditions are satisfied:*

(a) $[P, T] \leq [U, T]$ *for all indecomposable* T *with* $d(U, T) \leq 2(p(Q) + d(Q))$.

(b) $[T, I] \leq [T, I]$ *for all indecomposable* T *with* $d(T, V) \leq 2(p(Q) + d(Q))$.

Proof. The necessity of these conditions is trivial. It remains to show that they are sufficient. For this purpose we use the degeneration test 4.1.1. We set $M' := P \oplus R_{s+1} \oplus \ldots \oplus R_t \oplus I$. The dimension vectors of M' and $U \oplus V$ coincide by the assumption. Furthermore the criteria (a) and (b) of the degeneration are fulfilled because of $d(U, P) \leq 2p(Q)$ resp. $d(I, V) \leq 2p(Q)$ together with the conditions (a) and (b) of the requirement.
Thus it remains to show that each $E_i \in \mathcal{T}_\mu$ occurs at most $[U, E_i] - [P, E_i]$-times as regular top of $R_{s+1}, \ldots, R_t$. Thanks to $R_1 \oplus \ldots \oplus R_t < U \oplus V$, it follows that each E_i is at most $[U, E_i]$-times the regular top of $R_1, \ldots, R_t$. On the other hand, we have also presupposed that $R_1 \oplus \ldots \oplus R_s < P \oplus I$. But $s = -\partial(P)$, whence the occurrence of E_i as regular top of $R_1 \oplus \ldots \oplus R_s$ must be equal to $[P, E_i]$. So the assertion is clear. □

Notation 5.1.3 *For the regular indecomposable module with regular socle* E *and regular length* l *we simply write* $E[l]$ *below.*

Now we come to the computer program. We fix V with the above properties and describe how the program proceeds to compute all minimal deformations of $U \oplus V$ that belong to $add(\mathcal{T}_\mu)$.

Step 1: Initialization. Two lists are initialized. The first one is called `Codim1` and will be filled with those $M_\mu < U \oplus V$ that are of codimension one and therefore obviously minimal. The second list is called `Potmin`. This list contains in each step those deformations $M_\mu < U \oplus V$ whose minimality was not yet disproven and whose codimension is not one. We call the entries of the first list "codimension one deformations" and those of the second one shortly "potentially minimal deformations".

Step 2: Filling of the lists. To fill the lists the program tests each module consisting of up to $\partial(V)$ indecomposables with maximal length $3p_\mu$ belonging $\mathcal{T}_\mu$ whether it is a deformation of $U \oplus V$ or not. The test is done with the aid of of lemma 4.1.2. More formal: For each $n \in \{1, \ldots, \partial(V)\}$, each $i_1, \ldots, i_n \in \{1, \ldots, p_\mu\}$ with $i_1 \leq \ldots \leq i_n$ and each $j_1, \ldots, j_n \in \{1, \ldots, 3\epsilon(Q)p_\mu\}$ such that $j_k \leq j_{k+1}$ whenever

$i_k = i_{k+1}$ the module $E_{i_1}[j_1] \oplus \ldots \oplus E_{i_n}[j_n]$ is tested on the criteria of lemma 4.1.2. If it is a deformation of $U \oplus V$, then the program computes the codimension and adds the module to the appropriate list.

Step 3: Comparison of the list entries among each other. In this step the program compares any potentially minimal M_μ in `Potmin` with all other members of `Potmin` and `Codim1`. It uses lemma 5.1.1 for this comparison. The procedure stops when the program has found some M'_μ with $M_\mu < M'_\mu$, removes M_μ from `Potmin` and goes on with the next entry of `Potmin`.

If $-\partial(U) = \partial(V) = 2$, the list `Potmin` is already empty after these three steps. In the other cases there remains at most one potentially minimal deformation for fixed V and $\mathcal{T}_\mu$. It has the very special form

$$M_\mu = E_{i_1}[j] \oplus \ldots \oplus E_{i_s}[j] \oplus E_{i_{s+1}}[j'] \oplus \ldots \oplus E_{i_{\partial(V)}}[j'],$$

where $j > j' \geq 0$, and its codimension is at most $\partial(V) - 1$. The last step of the program is devoted to this deformation.

Step 4: Replacement of regular summands by non-regular ones. For each M_μ in `Potmin`, which has the shape as above, the program looks for indecomposables P and I of defect $\pm s$ that satisfy the requirements of lemma 5.1.2. To decide if $E_{i_1}[j] \oplus \ldots \oplus E_{i_s}[j] < P \oplus I$ it uses as in step 2 lemma 4.1.2, while $P \oplus E_{i_{s+1}}[j'] \oplus \ldots \oplus E_{i_{\partial(V)}}[j'] \oplus I < U \oplus V$ is tested by means of lemma 5.1.2.

In all cases, there are no potentially minimal deformations left in `Potmin` after these four steps. The defect of possible I in step 4 reaches independently from the codimension of the corresponding M_μ also up to $\partial(V) - 1$. The computed minimal degenerations are given in the appendix A.3.

5.2 Results and consequences

Now we come to the result of the computer calculations. Not to interrupt the reading flow, the list with the computed minimal degenerations is shifted to the appendix, cp. A.3. At this point only the most interesting results will be summarized.

Observation 5.2.1 *Let U be projective simple, V be preinjective indecomposable with $\underline{\dim}(U \oplus V) \leq 2\epsilon(Q)\partial(V)\delta$ and $\mathcal{T}_\mu$ be a non-homogeneous tube. Then:*

(a) There is at most one minimal degeneration $M < U \oplus V$ such that $M = M_\mu \in add(\mathcal{T}_\mu)$.

(b) *If there exists some minimal degeneration as in (a), then* M *can be written in the form* $M = \bigoplus_{k=1}^{\partial(V)} R_k/\delta$ *with* $R_1, \ldots, R_{\partial(V)}$ *as in the periodicity theorem.*

(c) *If there exists some minimal degeneration as in (a), then its codimension is one.*

Needless to say that especially observation 5.2.1 (c) has far reaching consequences. Using the observation in the proof of proposition 4.4.14 delivers the codimension of any minimal deformation in $\mathcal{T}_\mu$ and consequently the codimension of any minimal degeneration in general.

Proposition 5.2.2 *Suppose* U *is projective simple and* V *is preinjective indecomposable of arbitrary dimension. Then the codimension of any minimal degeneration* $M = M_\mu < U \oplus V$ *with* $M \in add(\mathcal{T}_\mu)$ *is one.*

Theorem 5.2.3 (Codimension 1) *Let* U *be projective simple,* V *be preinjective indecomposable and* $M < U \oplus V$*. Then* $M < U \oplus V$ *is minimal if and only if* $Codim(U \oplus V, M) = 1$.

Hence, from geometrical point of view the minimal disjoint degenerations are as simple as possible. This has consequences for their singularities. In regard to theorem 1.2.8 we obtain

Corollary 5.2.4 *Let* U *be projective simple,* V *be preinjective indecomposable and* M *be a module such that* $M < U \oplus V$ *is minimal. Then the orbit closure of* M *is regular at* $U \oplus V$*.*

Recall, concerning the utility of the periodicity theorem 4.3.2 for minimal degenerations we had two problems. It was not clear whether the theorem preserves minimality or not and how many minimal M actually comply with the requirement of the periodicity theorem. At this point, we are able to answer these questions.

Corollary 5.2.5 *The periodicity theorem* 4.3.2 *preserves minimality.*

Corollary 5.2.6 *Let* M *be an arbitrary module and* $M < U \oplus V$ *be a minimal degeneration. If* $\mathcal{T}_\mu$ *is a non-homogeneous tube such that* M_μ *is non-zero or if* $\mathcal{T}_\mu$ *is homogeneous, then* M *satisfies the requirements (ii) of the periodicity theorem for* $\mathcal{T}_\mu$*.*

Proof. Referring to the arguments at the beginning of chapter 4.4, the claim follows for homogeneous $\mathcal{T}_\mu$ from proposition 4.4.10, whilst proposition 4.4.13 and observation 5.2.1 (b) imply it in the case where $\mathcal{T}_\mu$ is non-homogeneous. □

Using this minimality preserving property of the periodicity theorem and the assertion of corollary 2.2.3 on the distances of the preprojective resp. preinjective indecomposable direct summands of M to U resp. V we draw our final conclusion.

Corollary 5.2.7 *The classification of all minimal disjoint degenerations $M < U \oplus V$ over a tame path algebra kQ with U preprojective indecomposable and V preinjective indecomposable is a finite problem.*

Proof. Due to tilting theory we may assume that U is the only projective simple kQ-module. Let $\mathcal{T}_1$, $\mathcal{T}_2$, and $\mathcal{T}_3$ be the three non-homogeneous tubes of Γ_{kQ}. It suffices to compute all minimal degeneration $M < U \oplus V$ such that

$$M = M_P \oplus M_1 \oplus M_2 \oplus M_3 \oplus M_I$$

where $M_P \in add(\mathcal{P})$, $M_i \in add(\mathcal{T}_i)$ and $M_I \in add(\mathcal{I})$. The remaining minimal degenerations can be obtained by application of the periodicity theorem, which preserves minimality. For the same reason we may assume that any indecomposable direct summand X of M_i satisfies $l(X) \leq 2p_i$. Since M_i has at most $\partial(V)$ direct summands, we conclude $\underline{\dim}(M_1 \oplus M_2 \oplus M_3) \leq 6\partial(V)\delta \leq 36\delta$.
Thanks to lemma 2.2.3 any summand X of M_P satisfies $d(U, X) < 4p(Q) + d(Q)$, whence we generously estimate $\underline{\dim}(M_P) \leq -4\partial(M_P)\delta \leq 20\delta$.
Provided that $t\partial(V)\delta \leq \underline{\dim}(V) < (t+1)\partial(V)\delta$ for some $t \in \mathbb{N}$, the same lemma implies $\underline{\dim}(M_I) \leq (t+4)\partial(M_I)\delta \leq (t+4)(\partial(V)-1)\delta$. If we choose t such that $t > 78$ there is no $M < U \oplus V$ of the presumed type. So, the numbers of V and M that are to test on $M < U \oplus V$ are finite. □

Appendix

Now we have to append some tables that allow to comprehend the observations 4.4.9, 4.4.12 and 5.2.1 concerning the extended Dynkin quivers of type $\tilde{D}_8$, $\tilde{E}_6$, $\tilde{E}_7$ or $\tilde{E}_8$ and their non-homogeneous tubes. Throughout the whole appendix let Q be a quiver of this type. Recall, we may reduce to the case that Q has only one sink i, i.e. there exists exactly one projective simple $U := P(i)$. Thus Q is uniquely determined by the type of underlying graph $|Q|$ together with the only projective simple U. In the following lists Q is always given by these informations. For the numbering of the points we refer to chapter 1.1

A.1 Lists of regular simples

Γ_{kQ} has three non-homogeneous tubes $\mathcal{T}_1$, $\mathcal{T}_2$ and $\mathcal{T}_3$. The first table collects basic dates of each $\mathcal{T}_k$, i.e. the period and the dimension vectors of the regular simple modules of $\mathcal{T}_k$. Thereby, the regular simples E whose dimension vector are printed in bold are those with $[U, E] \neq 0$.

$\lvert Q\rvert$	U	k	p_k	Regular simples		
$\tilde{D}_8$	$P(3)$	1	6	$S_1=\left(\begin{smallmatrix}0\\0\end{smallmatrix}00010\begin{smallmatrix}0\\0\end{smallmatrix}\right)$	$S_2=\left(\begin{smallmatrix}0\\0\end{smallmatrix}00001\begin{smallmatrix}0\\0\end{smallmatrix}\right)$	$\boldsymbol{S_3=\left(\begin{smallmatrix}0\\0\end{smallmatrix}11111\begin{smallmatrix}1\\1\end{smallmatrix}\right)}$
				$\boldsymbol{S_4=\left(\begin{smallmatrix}1\\1\end{smallmatrix}10000\begin{smallmatrix}0\\0\end{smallmatrix}\right)}$	$S_5=\left(\begin{smallmatrix}0\\0\end{smallmatrix}01000\begin{smallmatrix}0\\0\end{smallmatrix}\right)$	$S_6=\left(\begin{smallmatrix}0\\0\end{smallmatrix}00100\begin{smallmatrix}0\\0\end{smallmatrix}\right)$
		2	2	$\boldsymbol{S'_1=\left(\begin{smallmatrix}1\\0\end{smallmatrix}11111\begin{smallmatrix}1\\0\end{smallmatrix}\right)}$	$\boldsymbol{S'_2=\left(\begin{smallmatrix}0\\1\end{smallmatrix}11111\begin{smallmatrix}0\\1\end{smallmatrix}\right)}$	
		3	2	$\boldsymbol{S''_1=\left(\begin{smallmatrix}1\\0\end{smallmatrix}11111\begin{smallmatrix}0\\1\end{smallmatrix}\right)}$	$\boldsymbol{S''_2=\left(\begin{smallmatrix}0\\1\end{smallmatrix}11111\begin{smallmatrix}1\\0\end{smallmatrix}\right)}$	
	$P(4)$	1	6	$S_1=\left(\begin{smallmatrix}0\\0\end{smallmatrix}00010\begin{smallmatrix}0\\0\end{smallmatrix}\right)$	$S_2=\left(\begin{smallmatrix}0\\0\end{smallmatrix}00001\begin{smallmatrix}0\\0\end{smallmatrix}\right)$	$\boldsymbol{S_3=\left(\begin{smallmatrix}0\\0\end{smallmatrix}01111\begin{smallmatrix}1\\1\end{smallmatrix}\right)}$
				$S_4=\left(\begin{smallmatrix}0\\0\end{smallmatrix}10000\begin{smallmatrix}0\\0\end{smallmatrix}\right)$	$\boldsymbol{S_5=\left(\begin{smallmatrix}1\\1\end{smallmatrix}11000\begin{smallmatrix}0\\0\end{smallmatrix}\right)}$	$S_6=\left(\begin{smallmatrix}0\\0\end{smallmatrix}00100\begin{smallmatrix}0\\0\end{smallmatrix}\right)$
		2	2	$\boldsymbol{S'_1=\left(\begin{smallmatrix}1\\0\end{smallmatrix}11111\begin{smallmatrix}1\\0\end{smallmatrix}\right)}$	$\boldsymbol{S'_2=\left(\begin{smallmatrix}0\\1\end{smallmatrix}11111\begin{smallmatrix}0\\1\end{smallmatrix}\right)}$	
		3	2	$\boldsymbol{S''_1=\left(\begin{smallmatrix}1\\0\end{smallmatrix}11111\begin{smallmatrix}1\\0\end{smallmatrix}\right)}$	$\boldsymbol{S''_2=\left(\begin{smallmatrix}0\\1\end{smallmatrix}11111\begin{smallmatrix}0\\1\end{smallmatrix}\right)}$	
	$P(5)$	1	6	$S_1=\left(\begin{smallmatrix}0\\0\end{smallmatrix}00010\begin{smallmatrix}0\\0\end{smallmatrix}\right)$	$S_2=\left(\begin{smallmatrix}0\\0\end{smallmatrix}00001\begin{smallmatrix}0\\0\end{smallmatrix}\right)$	$\boldsymbol{S_3=\left(\begin{smallmatrix}0\\0\end{smallmatrix}00111\begin{smallmatrix}1\\1\end{smallmatrix}\right)}$
				$S_4=\left(\begin{smallmatrix}0\\0\end{smallmatrix}01000\begin{smallmatrix}0\\0\end{smallmatrix}\right)$	$S_5=\left(\begin{smallmatrix}0\\0\end{smallmatrix}10000\begin{smallmatrix}0\\0\end{smallmatrix}\right)$	$\boldsymbol{S_6=\left(\begin{smallmatrix}1\\1\end{smallmatrix}11100\begin{smallmatrix}0\\0\end{smallmatrix}\right)}$
		2	2	$\boldsymbol{S'_1=\left(\begin{smallmatrix}1\\0\end{smallmatrix}11111\begin{smallmatrix}1\\0\end{smallmatrix}\right)}$	$\boldsymbol{S'_2=\left(\begin{smallmatrix}0\\1\end{smallmatrix}11111\begin{smallmatrix}0\\1\end{smallmatrix}\right)}$	

$\lvert Q\rvert$	U	k	p_k	Regular simples		
		3	2	$\boldsymbol{S''_1=\left(\begin{smallmatrix}1\\0\end{smallmatrix}\,1\,1\,1\,1\,1\,\begin{smallmatrix}0\\1\end{smallmatrix}\right)}$	$\boldsymbol{S''_2=\left(\begin{smallmatrix}0\\1\end{smallmatrix}\,1\,1\,1\,1\,1\,\begin{smallmatrix}1\\0\end{smallmatrix}\right)}$	
$\tilde{E}_6$	$P(2)$	1	3	$\boldsymbol{S_1=\left(\begin{smallmatrix}&&0&&\\&&0&&\\1&1&1&1&1\end{smallmatrix}\right)}$	$S_2=\left(\begin{smallmatrix}&&0&&\\&&1&&\\0&0&1&0&0\end{smallmatrix}\right)$	$\boldsymbol{S_3=\left(\begin{smallmatrix}&&1&&\\&&1&&\\0&1&1&1&0\end{smallmatrix}\right)}$
		2	3	$\boldsymbol{S'_1=\left(\begin{smallmatrix}&&1&&\\&&1&&\\1&1&1&0&0\end{smallmatrix}\right)}$	$S'_2=\left(\begin{smallmatrix}&&0&&\\&&0&&\\0&0&1&1&0\end{smallmatrix}\right)$	$\boldsymbol{S'_3=\left(\begin{smallmatrix}&&0&&\\&&1&&\\0&1&1&1&1\end{smallmatrix}\right)}$
		3	2	$\boldsymbol{S''_1=\left(\begin{smallmatrix}&&0&&\\&&1&&\\1&1&1&1&0\end{smallmatrix}\right)}$	$\boldsymbol{S''_2=\left(\begin{smallmatrix}&&1&&\\&&1&&\\0&1&2&1&1\end{smallmatrix}\right)}$	
	$P(3)$	1	3	$\boldsymbol{S_1=\left(\begin{smallmatrix}&&0&&\\&&0&&\\1&1&1&1&0\end{smallmatrix}\right)}$	$\boldsymbol{S_2=\left(\begin{smallmatrix}&&0&&\\&&1&&\\0&0&1&1&1\end{smallmatrix}\right)}$	$\boldsymbol{S_3=\left(\begin{smallmatrix}&&1&&\\&&1&&\\0&1&1&0&0\end{smallmatrix}\right)}$
		2	3	$\boldsymbol{S'_1=\left(\begin{smallmatrix}&&0&&\\&&1&&\\1&1&1&0&0\end{smallmatrix}\right)}$	$\boldsymbol{S'_2=\left(\begin{smallmatrix}&&1&&\\&&1&&\\0&0&1&1&0\end{smallmatrix}\right)}$	$\boldsymbol{S'_3=\left(\begin{smallmatrix}&&0&&\\&&0&&\\0&1&1&1&1\end{smallmatrix}\right)}$
		3	2	$\boldsymbol{S''_1=\left(\begin{smallmatrix}&&0&&\\&&1&&\\0&1&1&1&0\end{smallmatrix}\right)}$	$\boldsymbol{S''_2=\left(\begin{smallmatrix}&&1&&\\&&1&&\\1&1&2&1&1\end{smallmatrix}\right)}$	
$\tilde{E}_7$	$P(2)$	1	4	$S_1=\left(\begin{smallmatrix}&&&0&&&\\0&0&0&1&1&0&0\end{smallmatrix}\right)$	$\boldsymbol{S_2=\left(\begin{smallmatrix}&&&1&&&\\0&1&1&1&1&1&0\end{smallmatrix}\right)}$	$\boldsymbol{S_3=\left(\begin{smallmatrix}&&&0&&&\\1&1&1&1&1&1&1\end{smallmatrix}\right)}$
						$S_4=\left(\begin{smallmatrix}&&&1&&&\\0&0&1&1&0&0&0\end{smallmatrix}\right)$
		2	3	$S'_1=\left(\begin{smallmatrix}&&&0&&&\\0&0&1&1&1&1&0\end{smallmatrix}\right)$	$\boldsymbol{S'_2=\left(\begin{smallmatrix}&&&1&&&\\0&1&1&2&1&1&1\end{smallmatrix}\right)}$	$\boldsymbol{S'_3=\left(\begin{smallmatrix}&&&1&&&\\1&1&1&1&1&0&0\end{smallmatrix}\right)}$
		3	2	$\boldsymbol{S''_1=\left(\begin{smallmatrix}&&&1&&&\\0&1&2&2&2&1&1\end{smallmatrix}\right)}$	$\boldsymbol{S''_2=\left(\begin{smallmatrix}&&&1&&&\\1&1&1&2&1&1&0\end{smallmatrix}\right)}$	
	$P(3)$	1	4	$S_1=\left(\begin{smallmatrix}&&&0&&&\\0&0&0&1&1&0&0\end{smallmatrix}\right)$	$\boldsymbol{S_2=\left(\begin{smallmatrix}&&&1&&&\\0&0&1&1&1&1&0\end{smallmatrix}\right)}$	$\boldsymbol{S_3=\left(\begin{smallmatrix}&&&0&&&\\0&1&1&1&1&1&1\end{smallmatrix}\right)}$
						$\boldsymbol{S_4=\left(\begin{smallmatrix}&&&1&&&\\1&1&1&1&0&0&0\end{smallmatrix}\right)}$
		2	3	$\boldsymbol{S'_1=\left(\begin{smallmatrix}&&&0&&&\\1&1&1&1&1&1&0\end{smallmatrix}\right)}$	$\boldsymbol{S'_2=\left(\begin{smallmatrix}&&&1&&&\\0&0&1&2&1&1&1\end{smallmatrix}\right)}$	$\boldsymbol{S'_3=\left(\begin{smallmatrix}&&&1&&&\\0&1&1&1&1&0&0\end{smallmatrix}\right)}$
		3	2	$\boldsymbol{S''_1=\left(\begin{smallmatrix}&&&1&&&\\0&1&1&2&1&1&0\end{smallmatrix}\right)}$	$\boldsymbol{S''_2=\left(\begin{smallmatrix}&&&1&&&\\1&1&2&2&2&1&1\end{smallmatrix}\right)}$	
	$P(4)$	1	4	$\boldsymbol{S_1=\left(\begin{smallmatrix}&&&1&&&\\0&0&0&1&1&1&0\end{smallmatrix}\right)}$	$\boldsymbol{S_2=\left(\begin{smallmatrix}&&&0&&&\\0&0&1&1&1&1&1\end{smallmatrix}\right)}$	$\boldsymbol{S_3=\left(\begin{smallmatrix}&&&1&&&\\0&1&1&1&0&0&0\end{smallmatrix}\right)}$
						$\boldsymbol{S_4=\left(\begin{smallmatrix}&&&0&&&\\1&1&1&1&1&0&0\end{smallmatrix}\right)}$
		2	3	$\boldsymbol{S'_1=\left(\begin{smallmatrix}&&&1&&&\\0&0&1&1&1&0&0\end{smallmatrix}\right)}$	$\boldsymbol{S'_2=\left(\begin{smallmatrix}&&&0&&&\\0&1&1&1&1&1&0\end{smallmatrix}\right)}$	$\boldsymbol{S'_3=\left(\begin{smallmatrix}&&&1&&&\\1&1&1&2&1&1&1\end{smallmatrix}\right)}$
		3	2	$\boldsymbol{S''_1=\left(\begin{smallmatrix}&&&1&&&\\0&1&1&2&2&1&1\end{smallmatrix}\right)}$	$\boldsymbol{S''_2=\left(\begin{smallmatrix}&&&1&&&\\1&1&2&2&1&1&0\end{smallmatrix}\right)}$	
	$P(8)$	1	4	$S_1=\left(\begin{smallmatrix}&&&0&&&\\0&0&0&1&1&1&0\end{smallmatrix}\right)$	$\boldsymbol{S_2=\left(\begin{smallmatrix}&&&1&&&\\0&0&1&1&1&1&1\end{smallmatrix}\right)}$	$S_3=\left(\begin{smallmatrix}&&&0&&&\\0&1&1&1&0&0&0\end{smallmatrix}\right)$
						$\boldsymbol{S_4=\left(\begin{smallmatrix}&&&1&&&\\1&1&1&1&1&0&0\end{smallmatrix}\right)}$
		2	3	$S'_1=\left(\begin{smallmatrix}&&&0&&&\\0&0&1&1&1&0&0\end{smallmatrix}\right)$	$\boldsymbol{S'_2=\left(\begin{smallmatrix}&&&1&&&\\0&1&1&1&1&1&0\end{smallmatrix}\right)}$	$\boldsymbol{S'_3=\left(\begin{smallmatrix}&&&1&&&\\1&1&1&2&1&1&1\end{smallmatrix}\right)}$
		3	2	$\boldsymbol{S''_1=\left(\begin{smallmatrix}&&&1&&&\\0&1&1&2&2&1&1\end{smallmatrix}\right)}$	$\boldsymbol{S''_2=\left(\begin{smallmatrix}&&&1&&&\\1&1&2&2&1&1&0\end{smallmatrix}\right)}$	
$\tilde{E}_8$	$P(1)$	1	5	$S_1=\left(\begin{smallmatrix}&&0&&&&&\\0&0&1&1&1&0&0&0\end{smallmatrix}\right)$	$\boldsymbol{S_2=\left(\begin{smallmatrix}&&1&&&&&\\1&1&1&1&1&1&0&0\end{smallmatrix}\right)}$	$S_3=\left(\begin{smallmatrix}&&0&&&&&\\0&1&1&1&1&1&1&0\end{smallmatrix}\right)$
					$\boldsymbol{S_4=\left(\begin{smallmatrix}&&1&&&&&\\1&1&2&1&1&1&1&1\end{smallmatrix}\right)}$	$S_5=\left(\begin{smallmatrix}&&1&&&&&\\0&1&1&1&0&0&0&0\end{smallmatrix}\right)$
		2	3	$S'_1=\left(\begin{smallmatrix}&&1&&&&&\\0&1&2&1&1&1&0&0\end{smallmatrix}\right)$	$\boldsymbol{S'_2=\left(\begin{smallmatrix}&&1&&&&&\\1&1&2&2&1&1&1&0\end{smallmatrix}\right)}$	$\boldsymbol{S'_3=\left(\begin{smallmatrix}&&1&&&&&\\1&2&2&2&2&1&1&1\end{smallmatrix}\right)}$
		3	2	$\boldsymbol{S''_1=\left(\begin{smallmatrix}&&1&&&&&\\1&2&3&3&2&2&1&1\end{smallmatrix}\right)}$	$\boldsymbol{S''_2=\left(\begin{smallmatrix}&&2&&&&&\\1&2&3&2&2&1&1&0\end{smallmatrix}\right)}$	

$\lvert Q\rvert$	U	k	p_k	Regular simples		
	$P(2)$	1	5	$S_1=\left(\begin{smallmatrix}&&0\\0&0&1&1&1&0&0&0\end{smallmatrix}\right)$	$\boldsymbol{S_2=\left(\begin{smallmatrix}&&1\\0&1&1&1&1&1&0&0\end{smallmatrix}\right)}$	$\boldsymbol{S_3=\left(\begin{smallmatrix}&&0\\1&1&1&1&1&1&1&0\end{smallmatrix}\right)}$
					$\boldsymbol{S_4=\left(\begin{smallmatrix}&&1\\0&1&2&1&1&1&1&1\end{smallmatrix}\right)}$	$\boldsymbol{S_5=\left(\begin{smallmatrix}&&1\\1&1&1&1&0&0&0&0\end{smallmatrix}\right)}$
		2	3	$\boldsymbol{S'_1=\left(\begin{smallmatrix}&&1\\1&2&2&2&2&1&1&1\end{smallmatrix}\right)}$	$\boldsymbol{S'_2=\left(\begin{smallmatrix}&&1\\1&1&2&1&1&1&0&0\end{smallmatrix}\right)}$	$\boldsymbol{S'_3=\left(\begin{smallmatrix}&&1\\0&1&2&2&1&1&1&0\end{smallmatrix}\right)}$
		3	2	$\boldsymbol{S''_1=\left(\begin{smallmatrix}&&1\\1&2&3&3&2&2&1&1\end{smallmatrix}\right)}$	$\boldsymbol{S''_2=\left(\begin{smallmatrix}&&2\\1&2&3&2&2&1&1&0\end{smallmatrix}\right)}$	
	$P(3)$	1	5	$\boldsymbol{S_1=\left(\begin{smallmatrix}&&1\\0&0&1&1&1&1&0&0\end{smallmatrix}\right)}$	$\boldsymbol{S_2=\left(\begin{smallmatrix}&&0\\0&1&1&1&1&1&1&0\end{smallmatrix}\right)}$	$\boldsymbol{S_3=\left(\begin{smallmatrix}&&1\\1&1&2&1&1&1&1&1\end{smallmatrix}\right)}$
					$\boldsymbol{S_4=\left(\begin{smallmatrix}&&1\\0&1&1&1&0&0&0&0\end{smallmatrix}\right)}$	$\boldsymbol{S_5=\left(\begin{smallmatrix}&&0\\1&1&1&1&1&0&0&0\end{smallmatrix}\right)}$
		2	3	$\boldsymbol{S'_1=\left(\begin{smallmatrix}&&1\\1&2&2&1&1&1&0&0\end{smallmatrix}\right)}$	$\boldsymbol{S'_2=\left(\begin{smallmatrix}&&1\\1&1&2&2&1&1&1&0\end{smallmatrix}\right)}$	$\boldsymbol{S'_3=\left(\begin{smallmatrix}&&1\\0&1&2&2&2&1&1&1\end{smallmatrix}\right)}$
		3	2	$\boldsymbol{S''_1=\left(\begin{smallmatrix}&&1\\1&2&3&3&2&2&1&1\end{smallmatrix}\right)}$	$\boldsymbol{S''_2=\left(\begin{smallmatrix}&&2\\1&2&3&2&2&1&1&0\end{smallmatrix}\right)}$	
	$P(4)$	1	5	$\boldsymbol{S_1=\left(\begin{smallmatrix}&&1\\0&0&1&1&1&1&1&0\end{smallmatrix}\right)}$	$\boldsymbol{S_2=\left(\begin{smallmatrix}&&0\\0&1&1&1&1&1&1&1\end{smallmatrix}\right)}$	$\boldsymbol{S_3=\left(\begin{smallmatrix}&&1\\1&1&2&1&0&0&0&0\end{smallmatrix}\right)}$
					$\boldsymbol{S_4=\left(\begin{smallmatrix}&&1\\0&1&1&1&1&0&0&0\end{smallmatrix}\right)}$	$\boldsymbol{S_5=\left(\begin{smallmatrix}&&0\\1&1&1&1&1&1&0&0\end{smallmatrix}\right)}$
		2	3	$\boldsymbol{S'_1=\left(\begin{smallmatrix}&&1\\1&1&2&2&2&1&1&1\end{smallmatrix}\right)}$	$\boldsymbol{S'_2=\left(\begin{smallmatrix}&&1\\0&1&2&1&1&1&0&0\end{smallmatrix}\right)}$	$\boldsymbol{S'_3=\left(\begin{smallmatrix}&&1\\1&2&2&2&1&1&1&0\end{smallmatrix}\right)}$
		3	2	$\boldsymbol{S''_1=\left(\begin{smallmatrix}&&2\\1&2&3&3&2&2&1&1\end{smallmatrix}\right)}$	$\boldsymbol{S''_2=\left(\begin{smallmatrix}&&1\\1&2&3&2&2&1&1&0\end{smallmatrix}\right)}$	
	$P(5)$	1	5	$\boldsymbol{S_1=\left(\begin{smallmatrix}&&1\\0&0&1&1&1&1&1&1\end{smallmatrix}\right)}$	$S_2=\left(\begin{smallmatrix}&&0\\0&1&1&1&0&0&0&0\end{smallmatrix}\right)$	$\boldsymbol{S_3=\left(\begin{smallmatrix}&&1\\1&1&2&1&1&0&0&0\end{smallmatrix}\right)}$
					$\boldsymbol{S_4=\left(\begin{smallmatrix}&&1\\0&1&1&1&1&1&0&0\end{smallmatrix}\right)}$	$\boldsymbol{S_5=\left(\begin{smallmatrix}&&0\\1&1&1&1&1&1&1&0\end{smallmatrix}\right)}$
		2	3	$\boldsymbol{S'_1=\left(\begin{smallmatrix}&&1\\1&1&2&2&1&1&0&0\end{smallmatrix}\right)}$	$\boldsymbol{S'_2=\left(\begin{smallmatrix}&&1\\0&1&2&1&1&1&1&0\end{smallmatrix}\right)}$	$\boldsymbol{S'_3=\left(\begin{smallmatrix}&&1\\1&2&2&2&2&1&1&1\end{smallmatrix}\right)}$
		3	2	$\boldsymbol{S''_1=\left(\begin{smallmatrix}&&2\\1&2&3&3&2&1&1&0\end{smallmatrix}\right)}$	$\boldsymbol{S''_2=\left(\begin{smallmatrix}&&1\\1&2&3&2&2&2&1&1\end{smallmatrix}\right)}$	
	$P(6)$	1	5	$S_1=\left(\begin{smallmatrix}&&0\\0&1&1&1&0&0&0&0\end{smallmatrix}\right)$	$\boldsymbol{S_2=\left(\begin{smallmatrix}&&1\\1&1&2&1&1&1&0&0\end{smallmatrix}\right)}$	$\boldsymbol{S_3=\left(\begin{smallmatrix}&&1\\0&1&1&1&1&1&1&0\end{smallmatrix}\right)}$
					$\boldsymbol{S_4=\left(\begin{smallmatrix}&&0\\1&1&1&1&1&1&1&1\end{smallmatrix}\right)}$	$S_5=\left(\begin{smallmatrix}&&1\\0&0&1&1&1&0&0&0\end{smallmatrix}\right)$
		2	3	$\boldsymbol{S'_1=\left(\begin{smallmatrix}&&1\\1&1&2&2&1&1&1&0\end{smallmatrix}\right)}$	$\boldsymbol{S'_2=\left(\begin{smallmatrix}&&1\\0&1&2&1&1&1&1&1\end{smallmatrix}\right)}$	$\boldsymbol{S'_3=\left(\begin{smallmatrix}&&1\\1&2&2&2&2&1&0&0\end{smallmatrix}\right)}$
		3	2	$\boldsymbol{S''_1=\left(\begin{smallmatrix}&&2\\1&2&3&3&2&2&1&1\end{smallmatrix}\right)}$	$\boldsymbol{S''_2=\left(\begin{smallmatrix}&&1\\1&2&3&2&2&1&1&0\end{smallmatrix}\right)}$	
	$P(7)$	1	5	$S_1=\left(\begin{smallmatrix}&&0\\0&1&1&1&0&0&0&0\end{smallmatrix}\right)$	$\boldsymbol{S_2=\left(\begin{smallmatrix}&&1\\1&1&2&1&1&1&1&0\end{smallmatrix}\right)}$	$\boldsymbol{S_3=\left(\begin{smallmatrix}&&1\\0&1&1&1&1&1&1&1\end{smallmatrix}\right)}$
					$S_4=\left(\begin{smallmatrix}&&0\\1&1&1&1&1&1&0&0\end{smallmatrix}\right)$	$S_5=\left(\begin{smallmatrix}&&1\\0&0&1&1&1&0&0&0\end{smallmatrix}\right)$
		2	3	$\boldsymbol{S'_1=\left(\begin{smallmatrix}&&1\\1&2&2&2&2&1&1&0\end{smallmatrix}\right)}$	$\boldsymbol{S'_2=\left(\begin{smallmatrix}&&1\\1&1&2&2&1&1&1&1\end{smallmatrix}\right)}$	$S'_3=\left(\begin{smallmatrix}&&1\\0&1&2&1&1&1&0&0\end{smallmatrix}\right)$
		3	2	$\boldsymbol{S''_1=\left(\begin{smallmatrix}&&2\\1&2&3&3&2&2&1&0\end{smallmatrix}\right)}$	$\boldsymbol{S''_2=\left(\begin{smallmatrix}&&1\\1&2&3&2&2&1&1&1\end{smallmatrix}\right)}$	
	$P(9)$	1	5	$S_1=\left(\begin{smallmatrix}&&0\\0&1&1&1&0&0&0&0\end{smallmatrix}\right)$	$\boldsymbol{S_2=\left(\begin{smallmatrix}&&1\\1&1&1&1&1&0&0&0\end{smallmatrix}\right)}$	$S_3=\left(\begin{smallmatrix}&&0\\0&0&1&1&1&1&0&0\end{smallmatrix}\right)$
					$\boldsymbol{S_4=\left(\begin{smallmatrix}&&1\\0&1&1&1&1&1&1&0\end{smallmatrix}\right)}$	$\boldsymbol{S_5=\left(\begin{smallmatrix}&&1\\1&1&2&1&1&1&1&1\end{smallmatrix}\right)}$
		2	3	$\boldsymbol{S'_1=\left(\begin{smallmatrix}&&1\\1&1&2&2&1&1&1&0\end{smallmatrix}\right)}$	$\boldsymbol{S'_2=\left(\begin{smallmatrix}&&1\\0&1&2&2&2&1&1&1\end{smallmatrix}\right)}$	$\boldsymbol{S'_3=\left(\begin{smallmatrix}&&1\\1&2&2&1&1&1&0&0\end{smallmatrix}\right)}$
		3	2	$\boldsymbol{S''_1=\left(\begin{smallmatrix}&&2\\1&2&3&3&2&2&1&1\end{smallmatrix}\right)}$	$\boldsymbol{S''_2=\left(\begin{smallmatrix}&&1\\1&2&3&2&2&1&1&0\end{smallmatrix}\right)}$	

The following lists all refer to the choices made in the above table.

A.2 Lists of some preprojective modules of defect -1

Now, for each regular simple $E \in \mathcal{T}_k$ with $[U, E] \neq 0$, i.e. the bold printed regular simples of the last table, we need some preprojective module $P \in \mathcal{H}(E)$. If Q is of type D one easily finds a projective $P \in \mathcal{H}(E)$.

For Q of type E, the next list contains not all $P \in \mathcal{H}(E)$, but at least one of them. The first two columns determine the quiver, the third resp. fourth column give the preprojective module P of defect -1 together resp. the dimension of $Hom(U, \tau P)$ and the last column contains all bold printed E with $P \in \mathcal{H}(E)$.

$\lvert Q\rvert$	U	P	$[U,\tau P]$	Reg. simples S with $P \in \mathcal{H}(S)$, $[U,S] \neq 0$	
$\tilde{E}_6$	$P(2)$	$P(1)=\begin{pmatrix} &&0&& \\ &&0&& \\ 1&1&0&0&0 \end{pmatrix}$	0	$S_1=\begin{pmatrix} &&0&& \\ &&0&& \\ 1&1&1&1&1 \end{pmatrix}$	$S_1'=\begin{pmatrix} &&1&& \\ &&1&& \\ 1&1&1&0&0 \end{pmatrix}$
					$S_1''=\begin{pmatrix} &&0&& \\ &&1&& \\ 1&1&1&1&0 \end{pmatrix}$
		$P(5)=\begin{pmatrix} &&0&& \\ &&0&& \\ 0&1&1&1&1 \end{pmatrix}$	0	$S_3'=\begin{pmatrix} &&0&& \\ &&1&& \\ 0&1&1&1&1 \end{pmatrix}$	$S_2''=\begin{pmatrix} &&1&& \\ &&1&& \\ 0&1&2&1&1 \end{pmatrix}$
		$P(7)=\begin{pmatrix} &&1&& \\ &&1&& \\ 0&1&1&0&0 \end{pmatrix}$	0	$S_3=\begin{pmatrix} &&1&& \\ &&1&& \\ 0&1&1&1&0 \end{pmatrix}$	$S_2''=\begin{pmatrix} &&1&& \\ &&1&& \\ 0&1&2&1&1 \end{pmatrix}$
	$P(3)$	$P(1)=\begin{pmatrix} &&0&& \\ &&0&& \\ 1&1&1&0&0 \end{pmatrix}$	0	$S_1=\begin{pmatrix} &&0&& \\ &&0&& \\ 1&1&1&1&0 \end{pmatrix}$	$S_1'=\begin{pmatrix} &&0&& \\ &&1&& \\ 1&1&1&0&0 \end{pmatrix}$
					$S_2''=\begin{pmatrix} &&1&& \\ &&1&& \\ 1&1&2&1&1 \end{pmatrix}$
		$P(5)=\begin{pmatrix} &&0&& \\ &&0&& \\ 0&0&1&1&1 \end{pmatrix}$	0	$S_2=\begin{pmatrix} &&0&& \\ &&1&& \\ 0&0&1&1&1 \end{pmatrix}$	$S_3'=\begin{pmatrix} &&0&& \\ &&0&& \\ 0&1&1&1&1 \end{pmatrix}$
					$S_2''=\begin{pmatrix} &&1&& \\ &&1&& \\ 1&1&2&1&1 \end{pmatrix}$
		$P(7)=\begin{pmatrix} &&1&& \\ &&1&& \\ 0&0&1&0&0 \end{pmatrix}$	0	$S_3=\begin{pmatrix} &&1&& \\ &&1&& \\ 0&1&1&0&0 \end{pmatrix}$	$S_2'=\begin{pmatrix} &&1&& \\ &&1&& \\ 0&0&1&1&0 \end{pmatrix}$
					$S_2''=\begin{pmatrix} &&1&& \\ &&1&& \\ 1&1&2&1&1 \end{pmatrix}$
		$\tau^- P(1)=\begin{pmatrix} &&0&& \\ &&1&& \\ 0&0&1&1&0 \end{pmatrix}$	1	$S_2=\begin{pmatrix} &&0&& \\ &&1&& \\ 0&0&1&1&1 \end{pmatrix}$	$S_2'=\begin{pmatrix} &&1&& \\ &&1&& \\ 0&0&1&1&0 \end{pmatrix}$
					$S_1''=\begin{pmatrix} &&0&& \\ &&1&& \\ 0&1&1&1&0 \end{pmatrix}$
$\tilde{E}_7$	$P(2)$	$P(1)=\begin{pmatrix} &&&0&&& \\ 1&1&0&0&0&0&0 \end{pmatrix}$	0	$S_3=\begin{pmatrix} &&&0&&& \\ 1&1&1&1&1&1&1 \end{pmatrix}$	$S_3'=\begin{pmatrix} &&&1&&& \\ 1&1&1&1&1&0&0 \end{pmatrix}$
					$S_2''=\begin{pmatrix} &&&1&&& \\ 1&1&1&2&1&1&0 \end{pmatrix}$
		$P(7)=\begin{pmatrix} &&&0&&& \\ 0&1&1&1&1&1&1 \end{pmatrix}$	0	$S_2'=\begin{pmatrix} &&&1&&& \\ 0&1&1&2&1&1&1 \end{pmatrix}$	
		$\tau^{-3}P(1)=\begin{pmatrix} &&&1&&& \\ 0&1&1&1&1&0&0 \end{pmatrix}$	0	$S_2=\begin{pmatrix} &&&1&&& \\ 0&1&1&1&1&1&0 \end{pmatrix}$	$S_1''=\begin{pmatrix} &&&1&&& \\ 0&1&2&2&2&1&1 \end{pmatrix}$
	$P(3)$	$P(1)=\begin{pmatrix} &&&0&&& \\ 1&1&1&0&0&0&0 \end{pmatrix}$	0	$S_4=\begin{pmatrix} &&&1&&& \\ 1&1&1&1&0&0&0 \end{pmatrix}$	$S_1'=\begin{pmatrix} &&&0&&& \\ 1&1&1&1&1&1&0 \end{pmatrix}$
					$S_2''=\begin{pmatrix} &&&1&&& \\ 1&1&2&2&2&1&1 \end{pmatrix}$

$\lvert Q\rvert$	U	P	$[U,\tau P]$	Reg. simples S with $P\in\mathcal{H}(S)$, $[U,S]\neq 0$	
		$P(7)=\left(\begin{smallmatrix}&&&0&&&\\0&0&1&1&1&1&1\end{smallmatrix}\right)$	0	$S_3=\left(\begin{smallmatrix}&&&0&&&\\0&1&1&1&1&1&1\end{smallmatrix}\right)$	$S_2'=\left(\begin{smallmatrix}&&&1&&&\\0&0&1&2&1&1&1\end{smallmatrix}\right)$
		$\tau^-P(7)=\left(\begin{smallmatrix}&&&1&&&\\0&1&1&1&0&0&0\end{smallmatrix}\right)$	1	$S_3'=\left(\begin{smallmatrix}&&&1&&&\\0&1&1&1&1&0&0\end{smallmatrix}\right)$	$S_1''=\left(\begin{smallmatrix}&&&1&&&\\0&1&1&2&1&1&0\end{smallmatrix}\right)$
		$\tau^{-2}P(1)=\left(\begin{smallmatrix}&&&1&&&\\0&0&1&1&1&0&0\end{smallmatrix}\right)$	0	$S_2=\left(\begin{smallmatrix}&&&1&&&\\0&0&1&1&1&1&0\end{smallmatrix}\right)$	$S_3'=\left(\begin{smallmatrix}&&&1&&&\\0&1&1&1&1&0&0\end{smallmatrix}\right)$
	$P(4)$	$P(1)=\left(\begin{smallmatrix}&&&0&&&\\1&1&1&1&0&0&0\end{smallmatrix}\right)$	0	$S_4=\left(\begin{smallmatrix}&&&0&&&\\1&1&1&1&1&0&0\end{smallmatrix}\right)$	$S_3'=\left(\begin{smallmatrix}&&&1&&&\\1&1&1&2&1&1&1\end{smallmatrix}\right)$
		$P(7)=\left(\begin{smallmatrix}&&&0&&&\\0&0&0&1&1&1&1\end{smallmatrix}\right)$	0	$S_2=\left(\begin{smallmatrix}&&&0&&&\\0&0&1&1&1&1&1\end{smallmatrix}\right)$	$S_3'=\left(\begin{smallmatrix}&&&1&&&\\1&1&1&2&1&1&1\end{smallmatrix}\right)$
		$\tau^-P(1)=\left(\begin{smallmatrix}&&&1&&&\\0&0&0&1&1&0&0\end{smallmatrix}\right)$	1	$S_1=\left(\begin{smallmatrix}&&&1&&&\\0&0&0&1&1&1&0\end{smallmatrix}\right)$	$S_1'=\left(\begin{smallmatrix}&&&1&&&\\0&0&1&1&1&0&0\end{smallmatrix}\right)$ $S_1''=\left(\begin{smallmatrix}&&&1&&&\\0&1&1&2&2&1&1\end{smallmatrix}\right)$
		$\tau^-P(7)=\left(\begin{smallmatrix}&&&1&&&\\0&0&1&1&0&0&0\end{smallmatrix}\right)$	1	$S_3=\left(\begin{smallmatrix}&&&1&&&\\0&1&1&1&0&0&0\end{smallmatrix}\right)$	$S_1'=\left(\begin{smallmatrix}&&&1&&&\\0&0&1&1&1&0&0\end{smallmatrix}\right)$ $S_2''=\left(\begin{smallmatrix}&&&1&&&\\1&1&2&2&1&1&0\end{smallmatrix}\right)$
		$\tau^{-2}P(1)=\left(\begin{smallmatrix}&&&0&&&\\0&0&1&1&1&1&0\end{smallmatrix}\right)$	1	$S_2=\left(\begin{smallmatrix}&&&0&&&\\0&0&1&1&1&1&1\end{smallmatrix}\right)$	$S_2'=\left(\begin{smallmatrix}&&&0&&&\\0&1&1&1&1&1&0\end{smallmatrix}\right)$
	$P(8)$	$P(1)=\left(\begin{smallmatrix}&&&1&&&\\1&1&1&1&0&0&0\end{smallmatrix}\right)$	0	$S_4=\left(\begin{smallmatrix}&&&1&&&\\1&1&1&1&1&0&0\end{smallmatrix}\right)$	$S_3'=\left(\begin{smallmatrix}&&&1&&&\\1&1&1&2&1&1&1\end{smallmatrix}\right)$ $S_2''=\left(\begin{smallmatrix}&&&1&&&\\1&1&2&2&1&1&0\end{smallmatrix}\right)$
		$P(7)=\left(\begin{smallmatrix}&&&1&&&\\0&0&0&1&1&1&1\end{smallmatrix}\right)$	0	$S_2=\left(\begin{smallmatrix}&&&1&&&\\0&0&1&1&1&1&1\end{smallmatrix}\right)$	$S_3'=\left(\begin{smallmatrix}&&&1&&&\\1&1&1&2&1&1&1\end{smallmatrix}\right)$ $S_1''=\left(\begin{smallmatrix}&&&1&&&\\0&1&1&2&2&1&1\end{smallmatrix}\right)$
		$\tau^{-2}P(1)=\left(\begin{smallmatrix}&&&1&&&\\0&0&1&1&1&1&0\end{smallmatrix}\right)$	0	$S_2=\left(\begin{smallmatrix}&&&1&&&\\0&0&1&1&1&1&1\end{smallmatrix}\right)$	$S_2'=\left(\begin{smallmatrix}&&&1&&&\\0&1&1&1&1&1&0\end{smallmatrix}\right)$ $S_2''=\left(\begin{smallmatrix}&&&1&&&\\1&1&2&2&1&1&0\end{smallmatrix}\right)$
$\tilde{E}_8$	$P(1)$	$P(8)=\left(\begin{smallmatrix}&&0&&&&&\\1&1&1&1&1&1&1&1\end{smallmatrix}\right)$	0	$S_4=\left(\begin{smallmatrix}&&1&&&&&\\1&1&2&1&1&1&1&1\end{smallmatrix}\right)$	$S_1''=\left(\begin{smallmatrix}&&1&&&&&\\1&2&3&3&2&2&1&1\end{smallmatrix}\right)$
		$\tau^{-3}P(8)=\left(\begin{smallmatrix}&&1&&&&&\\1&1&1&1&1&0&0&0\end{smallmatrix}\right)$	0	$S_2=\left(\begin{smallmatrix}&&1&&&&&\\1&1&1&1&1&1&0&0\end{smallmatrix}\right)$	$S_3'=\left(\begin{smallmatrix}&&1&&&&&\\1&2&2&2&2&1&1&1\end{smallmatrix}\right)$ $S_2''=\left(\begin{smallmatrix}&&2&&&&&\\1&2&3&2&2&1&1&0\end{smallmatrix}\right)$
		$\tau^{-5}P(8)=\left(\begin{smallmatrix}&&1&&&&&\\1&1&2&1&1&1&1&0\end{smallmatrix}\right)$	0	$S_2'=\left(\begin{smallmatrix}&&1&&&&&\\1&1&2&2&1&1&1&0\end{smallmatrix}\right)$	
	$P(2)$	$P(8)=\left(\begin{smallmatrix}&&0&&&&&\\0&1&1&1&1&1&1&1\end{smallmatrix}\right)$	0	$S_4=\left(\begin{smallmatrix}&&1&&&&&\\0&1&2&1&1&1&1&1\end{smallmatrix}\right)$	
		$\tau^-P(8)=\left(\begin{smallmatrix}&&1&&&&&\\1&1&1&0&0&0&0&0\end{smallmatrix}\right)$	1	$S_5=\left(\begin{smallmatrix}&&1&&&&&\\1&1&1&1&0&0&0&0\end{smallmatrix}\right)$	$S_2'=\left(\begin{smallmatrix}&&1&&&&&\\1&1&2&1&1&1&0&0\end{smallmatrix}\right)$ $S_2''=\left(\begin{smallmatrix}&&2&&&&&\\1&2&3&2&2&1&1&0\end{smallmatrix}\right)$
		$\tau^{-3}P(8)=\left(\begin{smallmatrix}&&1&&&&&\\0&1&1&1&1&0&0&0\end{smallmatrix}\right)$	0	$S_2=\left(\begin{smallmatrix}&&1&&&&&\\0&1&1&1&1&1&0&0\end{smallmatrix}\right)$	$S_1'=\left(\begin{smallmatrix}&&1&&&&&\\1&2&2&2&2&1&1&1\end{smallmatrix}\right)$
		$\tau^{-4}P(8)=\left(\begin{smallmatrix}&&0&&&&&\\1&1&1&1&1&1&0&0\end{smallmatrix}\right)$	1	$S_3=\left(\begin{smallmatrix}&&0&&&&&\\1&1&1&1&1&1&1&0\end{smallmatrix}\right)$	$S_1''=\left(\begin{smallmatrix}&&1&&&&&\\1&2&3&3&2&2&1&1\end{smallmatrix}\right)$
		$\tau^{-5}P(8)=\left(\begin{smallmatrix}&&1&&&&&\\0&1&2&1&1&1&1&0\end{smallmatrix}\right)$	1	$S_3'=\left(\begin{smallmatrix}&&1&&&&&\\0&1&2&2&1&1&1&0\end{smallmatrix}\right)$	
	$P(3)$	$P(8)=\left(\begin{smallmatrix}&&0&&&&&\\0&0&1&1&1&1&1&1\end{smallmatrix}\right)$	0	$S_3=\left(\begin{smallmatrix}&&1&&&&&\\1&1&2&1&1&1&1&1\end{smallmatrix}\right)$	
		$\tau^-P(8)=\left(\begin{smallmatrix}&&1&&&&&\\0&1&1&0&0&0&0&0\end{smallmatrix}\right)$	1	$S_4=\left(\begin{smallmatrix}&&1&&&&&\\0&1&1&1&0&0&0&0\end{smallmatrix}\right)$	$S_1'=\left(\begin{smallmatrix}&&1&&&&&\\1&2&2&1&1&1&0&0\end{smallmatrix}\right)$ $S_2''=\left(\begin{smallmatrix}&&2&&&&&\\1&2&3&2&2&1&1&0\end{smallmatrix}\right)$

$\lvert Q\rvert$	U	P	$[U,\tau P]$	Reg. simples S with $P\in\mathcal{H}(S)$, $[U,S]\neq 0$	
		$\tau^{-2}P(8)=\left(\begin{smallmatrix}&&0\\1&1&1&1&0&0&0&0\end{smallmatrix}\right)$	1	$S_5=\left(\begin{smallmatrix}&&0\\1&1&1&1&1&0&0&0\end{smallmatrix}\right)$	$S_2'=\left(\begin{smallmatrix}&&1\\1&1&2&2&1&1&1&0\end{smallmatrix}\right)$
					$S_1''=\left(\begin{smallmatrix}&&1\\1&2&3&3&2&2&1&1\end{smallmatrix}\right)$
		$\tau^{-3}P(8)=\left(\begin{smallmatrix}&&1\\0&0&1&1&1&0&0&0\end{smallmatrix}\right)$	1	$S_1=\left(\begin{smallmatrix}&&1\\0&0&1&1&1&1&0&0\end{smallmatrix}\right)$	$S_3'=\left(\begin{smallmatrix}&&1\\0&1&2&2&2&1&1&1\end{smallmatrix}\right)$
		$\tau^{-4}P(8)=\left(\begin{smallmatrix}&&0\\0&1&1&1&1&1&0&0\end{smallmatrix}\right)$	1	$S_2=\left(\begin{smallmatrix}&&0\\0&1&1&1&1&1&1&0\end{smallmatrix}\right)$	
	$P(4)$	$P(8)=\left(\begin{smallmatrix}&&0\\0&0&0&1&1&1&1&1\end{smallmatrix}\right)$	0	$S_2=\left(\begin{smallmatrix}&&0\\0&1&1&1&1&1&1&1\end{smallmatrix}\right)$	$S_1'=\left(\begin{smallmatrix}&&1\\1&1&2&2&2&1&1&1\end{smallmatrix}\right)$
		$\tau^{-2}P(8)=\left(\begin{smallmatrix}&&1\\0&1&1&1&0&0&0&0\end{smallmatrix}\right)$	0	$S_4=\left(\begin{smallmatrix}&&1\\0&1&1&1&1&0&0&0\end{smallmatrix}\right)$	$S_3'=\left(\begin{smallmatrix}&&1\\1&2&2&2&1&1&1&0\end{smallmatrix}\right)$
					$S_1''=\left(\begin{smallmatrix}&&2\\1&2&3&3&2&2&1&1\end{smallmatrix}\right)$
		$\tau^{-3}P(8)=\left(\begin{smallmatrix}&&0\\1&1&1&1&1&0&0&0\end{smallmatrix}\right)$	1	$S_5=\left(\begin{smallmatrix}&&0\\1&1&1&1&1&1&0&0\end{smallmatrix}\right)$	$S_1'=\left(\begin{smallmatrix}&&1\\1&1&2&2&2&1&1&1\end{smallmatrix}\right)$
					$S_2''=\left(\begin{smallmatrix}&&1\\1&2&3&2&2&1&1&0\end{smallmatrix}\right)$
		$\tau^{-4}P(8)=\left(\begin{smallmatrix}&&1\\0&0&1&1&1&1&0&0\end{smallmatrix}\right)$	1	$S_1=\left(\begin{smallmatrix}&&1\\0&0&1&1&1&1&1&0\end{smallmatrix}\right)$	$S_2'=\left(\begin{smallmatrix}&&1\\0&1&2&1&1&1&0&0\end{smallmatrix}\right)$
		$\tau^{-6}P(8)=\left(\begin{smallmatrix}&&1\\1&1&2&2&1&1&1&1\end{smallmatrix}\right)$	1	$S_3=\left(\begin{smallmatrix}&&1\\1&1&2&1&0&0&0&0\end{smallmatrix}\right)$	
	$P(5)$	$P(8)=\left(\begin{smallmatrix}&&0\\0&0&0&0&1&1&1&1\end{smallmatrix}\right)$	0	$S_1=\left(\begin{smallmatrix}&&1\\0&0&1&1&1&1&1&1\end{smallmatrix}\right)$	$S_3'=\left(\begin{smallmatrix}&&1\\1&2&2&2&2&1&1&1\end{smallmatrix}\right)$
					$S_2''=\left(\begin{smallmatrix}&&1\\1&2&3&2&2&2&1&1\end{smallmatrix}\right)$
		$\tau^{-3}P(8)=\left(\begin{smallmatrix}&&1\\0&1&1&1&1&0&0&0\end{smallmatrix}\right)$	0	$S_4=\left(\begin{smallmatrix}&&1\\0&1&1&1&1&1&0&0\end{smallmatrix}\right)$	$S_1''=\left(\begin{smallmatrix}&&2\\1&2&3&3&2&1&1&0\end{smallmatrix}\right)$
		$\tau^{-4}P(8)=\left(\begin{smallmatrix}&&0\\1&1&1&1&1&1&0&0\end{smallmatrix}\right)$	1	$S_5=\left(\begin{smallmatrix}&&0\\1&1&1&1&1&1&1&0\end{smallmatrix}\right)$	$S_1'=\left(\begin{smallmatrix}&&1\\1&1&2&2&1&1&0&0\end{smallmatrix}\right)$
		$\tau^{-5}P(8)=\left(\begin{smallmatrix}&&1\\0&0&1&1&1&1&1&0\end{smallmatrix}\right)$	1	$S_2'=\left(\begin{smallmatrix}&&1\\0&1&2&1&1&1&1&0\end{smallmatrix}\right)$	
		$\tau^{-7}P(8)=\left(\begin{smallmatrix}&&1\\1&1&2&2&1&0&0&0\end{smallmatrix}\right)$	1	$S_3=\left(\begin{smallmatrix}&&1\\1&1&2&1&1&0&0&0\end{smallmatrix}\right)$	
	$P(6)$	$P(8)=\left(\begin{smallmatrix}&&0\\0&0&0&0&0&1&1&1\end{smallmatrix}\right)$	0	$S_4=\left(\begin{smallmatrix}&&0\\1&1&1&1&1&1&1&1\end{smallmatrix}\right)$	$S_2'=\left(\begin{smallmatrix}&&1\\0&1&2&1&1&1&1&1\end{smallmatrix}\right)$
					$S_1''=\left(\begin{smallmatrix}&&2\\1&2&3&3&2&2&1&1\end{smallmatrix}\right)$
		$\tau^{-4}P(8)=\left(\begin{smallmatrix}&&1\\0&1&1&1&1&1&0&0\end{smallmatrix}\right)$	0	$S_3=\left(\begin{smallmatrix}&&1\\0&1&1&1&1&1&1&0\end{smallmatrix}\right)$	$S_3'=\left(\begin{smallmatrix}&&1\\1&2&2&2&2&1&0&0\end{smallmatrix}\right)$
		$\tau^{-5}P(8)=\left(\begin{smallmatrix}&&0\\1&1&1&1&1&1&1&0\end{smallmatrix}\right)$	1	$S_1'=\left(\begin{smallmatrix}&&1\\1&1&2&2&1&1&1&0\end{smallmatrix}\right)$	$S_2''=\left(\begin{smallmatrix}&&1\\1&2&3&2&2&1&1&0\end{smallmatrix}\right)$
		$\tau^{-8}P(8)=\left(\begin{smallmatrix}&&1\\1&1&2&2&1&1&0&0\end{smallmatrix}\right)$	0	$S_2=\left(\begin{smallmatrix}&&1\\1&1&2&1&1&1&0&0\end{smallmatrix}\right)$	
	$P(7)$	$P(8)=\left(\begin{smallmatrix}&&0\\0&0&0&0&0&0&1&1\end{smallmatrix}\right)$	0	$S_3=\left(\begin{smallmatrix}&&1\\0&1&1&1&1&1&1&1\end{smallmatrix}\right)$	$S_2'=\left(\begin{smallmatrix}&&1\\1&1&2&2&1&1&1&1\end{smallmatrix}\right)$
					$S_2''=\left(\begin{smallmatrix}&&1\\1&2&3&2&2&1&1&1\end{smallmatrix}\right)$
		$\tau^{-5}P(8)=\left(\begin{smallmatrix}&&1\\0&1&1&1&1&1&1&0\end{smallmatrix}\right)$	0	$S_1'=\left(\begin{smallmatrix}&&1\\1&2&2&2&2&1&1&0\end{smallmatrix}\right)$	$S_1''=\left(\begin{smallmatrix}&&2\\1&2&3&3&2&2&1&0\end{smallmatrix}\right)$
		$\tau^{-9}P(8)=\left(\begin{smallmatrix}&&1\\1&1&2&2&1&1&1&0\end{smallmatrix}\right)$	0	$S_2=\left(\begin{smallmatrix}&&1\\1&1&2&1&1&1&1&0\end{smallmatrix}\right)$	
	$P(9)$	$P(8)=\left(\begin{smallmatrix}&&1\\0&0&1&1&1&1&1&1\end{smallmatrix}\right)$	0	$S_5=\left(\begin{smallmatrix}&&1\\1&1&2&1&1&1&1&1\end{smallmatrix}\right)$	$S_2'=\left(\begin{smallmatrix}&&1\\0&1&2&2&2&1&1&1\end{smallmatrix}\right)$
		$\tau^{-2}P(8)=\left(\begin{smallmatrix}&&1\\1&1&1&1&0&0&0&0\end{smallmatrix}\right)$	0	$S_2=\left(\begin{smallmatrix}&&1\\1&1&1&1&1&0&0&0\end{smallmatrix}\right)$	$S_1'=\left(\begin{smallmatrix}&&1\\1&1&2&2&1&1&1&0\end{smallmatrix}\right)$
					$S_1''=\left(\begin{smallmatrix}&&2\\1&2&3&3&2&2&1&1\end{smallmatrix}\right)$
		$\tau^{-4}P(8)=\left(\begin{smallmatrix}&&1\\0&1&1&1&1&1&0&0\end{smallmatrix}\right)$	0	$S_4=\left(\begin{smallmatrix}&&1\\0&1&1&1&1&1&1&0\end{smallmatrix}\right)$	$S_3'=\left(\begin{smallmatrix}&&1\\1&2&2&1&1&1&0&0\end{smallmatrix}\right)$

$\|Q\|$	U	P	$[U,\tau P]$	Reg. simples S with $P \in \mathcal{H}(S)$, $[U,S] \neq 0$	
		$\tau^{-7}P(8) = \begin{pmatrix} & & 1 & & & & & \\ 1 & 2 & 2 & 1 & 1 & 0 & 0 & 0 \end{pmatrix}$	1	$S_2'' = \begin{pmatrix} & & 1 & & & & & \\ 1 & 2 & 3 & 2 & 2 & 1 & 1 & 0 \end{pmatrix}$	

Inspection of this list yields observation 4.4.12. As stated in the observation, only in four cases, in which Q is of type $\tilde{E}_8$, the given $P \in \mathcal{H}(E)$ does not satisfy $\underline{\dim}(P) \leq \underline{\dim}(E)$. These exceptions are each to find in the last row of the cases $U = P(4)$, $U = P(5)$, $U = P(6)$ and $U = P(7)$. But then we have $\underline{\dim}(P) \leq \underline{\dim}(\begin{smallmatrix} E \\ \tau E \end{smallmatrix})$.

A.3 Lists of the computed minimal degenerations

The following lists contain the results of the computer program described in chapter 5.1. A closer look on them provides observation 5.2.1. We explain how the lists have to be read. The lists include column by column the following informations:

1. the only projective simple U, which also determines the orientation of Q;
2. the preinjective indecomposables V and $\tau^{p(Q)}V$, separated by a "/";
3. the number t of the non-homogeneous tube $\mathcal{T}_k$ (for its period see table A.1);
4. the list of all modules M_k resp. M_k' of $add(\mathcal{T}_k)$ that degenerate into $U \oplus V$ resp. $U \oplus \tau^{p(Q)}V$, separated by a "/";
5. the codimensions of the degenerations of the fourth column, separated by a "/".

V and $\tau^{p(Q)}V$ are given in the same row to point out the effect of the periodicity theorem. But notice, V is only listed in the table if some $M_k < U \oplus V$ exists.

Type $\tilde{D}_8$				
U	V	k	M_k	c
$P(3)$	$\tau^0 I(3)/\tau^6 I(3)$	1	$S_3[1] \oplus S_4[1]/S_3[7] \oplus S_4[7]$	1/1
	$\tau^1 I(3)/\tau^7 I(3)$	1	$S_2[2] \oplus S_3[2]/S_2[8] \oplus S_3[8]$	1/1
	$\tau^2 I(3)/\tau^8 I(3)$	1	$S_1[3] \oplus S_2[3]/S_1[9] \oplus S_2[9]$	1/1
	$\tau^3 I(3)/\tau^9 I(3)$	1	$S_1[4] \oplus S_6[4]/S_1[10] \oplus S_6[10]$	1/1
	$\tau^4 I(3)/\tau^{10} I(3)$	1	$S_5[5] \oplus S_6[5]/S_5[11] \oplus S_6[11]$	1/1
	$\tau^5 I(3)/\tau^{11} I(3)$	1	$S_4[6] \oplus S_5[6]/S_4[12] \oplus S_5[12]$	1/1
	$\tau^5 I(3)/\tau^{11} I(3)$	2	$S_1'[2] \oplus S_2'[2]/S_1'[4] \oplus S_2'[4]$	1/1
	$\tau^5 I(3)/\tau^{11} I(3)$	3	$S_1''[2] \oplus S_2''[2]/S_1''[4] \oplus S_2''[4]$	1/1
	$\tau^0 I(4)/\tau^6 I(4)$	1	$S_3[1]/S_3[7] \oplus S_5[6]$	1/1

Type $\tilde{D}_8$				
U	V	k	M_k	c
	$\tau^1 I(4)/\tau^7 I(4)$	1	$S_2[2] \oplus S_4[1]/S_2[8] \oplus S_4[7]$	1/1
	$\tau^2 I(4)/\tau^8 I(4)$	1	$S_1[3] \oplus S_3[2]/S_1[9] \oplus S_3[8]$	1/1
	$\tau^3 I(4)/\tau^9 I(4)$	1	$S_2[3] \oplus S_6[4]/S_2[9] \oplus S_6[10]$	1/1
	$\tau^4 I(4)/\tau^{10} I(4)$	1	$S_1[4] \oplus S_5[5]/S_1[10] \oplus S_5[11]$	1/1
	$\tau^5 I(4)/\tau^{11} I(4)$	1	$S_4[6] \oplus S_6[5]/S_4[12] \oplus S_6[11]$	1/1
	$\tau^6 I(5)$	1	$S_3[7] \oplus S_6[5]$	1
	$\tau^1 I(5)/\tau^7 I(5)$	1	$S_2[2]/S_2[8] \oplus S_5[6]$	1/1
	$\tau^2 I(5)/\tau^8 I(5)$	1	$S_1[3] \oplus S_4[1]/S_1[9] \oplus S_4[7]$	1/1
	$\tau^3 I(5)/\tau^9 I(5)$	1	$S_3[2] \oplus S_6[4]/S_3[8] \oplus S_6[10]$	1/1
	$\tau^4 I(5)/\tau^{10} I(5)$	1	$S_2[3] \oplus S_5[5]/S_2[9] \oplus S_5[11]$	1/1
	$\tau^5 I(5)/\tau^{11} I(5)$	1	$S_1[4] \oplus S_4[6]/S_1[10] \oplus S_4[12]$	1/1
	$\tau^6 I(6)$	1	$S_1[4] \oplus S_3[7]$	1
	$\tau^7 I(6)$	1	$S_2[8] \oplus S_6[5]$	1
	$\tau^2 I(6)/\tau^8 I(6)$	1	$S_1[3]/S_1[9] \oplus S_5[6]$	1/1
	$\tau^3 I(6)/\tau^9 I(6)$	1	$S_4[1] \oplus S_6[4]/S_4[7] \oplus S_6[10]$	1/1
	$\tau^4 I(6)/\tau^{10} I(6)$	1	$S_3[2] \oplus S_5[5]/S_3[8] \oplus S_5[11]$	1/1
	$\tau^5 I(6)/\tau^{11} I(6)$	1	$S_2[3] \oplus S_4[6]/S_2[9] \oplus S_4[12]$	1/1
	$\tau^6 I(7)$	1	$S_2[3] \oplus S_3[7]$	1
	$\tau^7 I(7)$	1	$S_1[4] \oplus S_2[8]$	1
	$\tau^8 I(7)$	1	$S_1[9] \oplus S_6[5]$	1
	$\tau^3 I(7)/\tau^9 I(7)$	1	$S_6[4]/S_5[6] \oplus S_6[10]$	1/1
	$\tau^4 I(7)/\tau^{10} I(7)$	1	$S_4[1] \oplus S_5[5]/S_4[7] \oplus S_5[11]$	1/1
	$\tau^4 I(7)/\tau^{10} I(7)$	2	$S'_1[1] \oplus S'_2[1]/S'_1[3] \oplus S'_2[3]$	1/1
	$\tau^4 I(7)/\tau^{10} I(7)$	3	$S''_1[1] \oplus S''_2[1]/S''_1[3] \oplus S''_2[3]$	1/1
	$\tau^5 I(7)/\tau^{11} I(7)$	1	$S_3[2] \oplus S_4[6]/S_3[8] \oplus S_4[12]$	1/1
$P(4)$	$\tau^0 I(4)/\tau^6 I(4)$	1	$S_3[1] \oplus S_5[1]/S_3[7] \oplus S_5[7]$	1/1
	$\tau^1 I(4)/\tau^7 I(4)$	1	$S_2[2] \oplus S_4[2]/S_2[8] \oplus S_4[8]$	1/1
	$\tau^2 I(4)/\tau^8 I(4)$	1	$S_1[3] \oplus S_3[3]/S_1[9] \oplus S_3[9]$	1/1
	$\tau^3 I(4)/\tau^9 I(4)$	1	$S_2[4] \oplus S_6[4]/S_2[10] \oplus S_6[10]$	1/1
	$\tau^4 I(4)/\tau^{10} I(4)$	1	$S_1[5] \oplus S_5[5]/S_1[11] \oplus S_5[11]$	1/1

Type $\tilde{D}_8$				
U	V	k	M_k	c
	$\tau^5 I(4)/\tau^{11} I(4)$	1	$S_4[6] \oplus S_6[6]/S_4[12] \oplus S_6[12]$	1/1
	$\tau^5 I(4)/\tau^{11} I(4)$	2	$S'_1[2] \oplus S'_2[2]/S'_1[4] \oplus S'_2[4]$	1/1
	$\tau^5 I(4)/\tau^{11} I(4)$	3	$S''_1[2] \oplus S''_2[2]/S''_1[4] \oplus S''_2[4]$	1/1
	$\tau^0 I(3)/\tau^6 I(3)$	1	$S_5[1]/S_4[6] \oplus S_5[7]$	1/1
	$\tau^1 I(3)/\tau^7 I(3)$	1	$S_3[1] \oplus S_4[2]/S_3[7] \oplus S_4[8]$	1/1
	$\tau^2 I(3)/\tau^8 I(3)$	1	$S_2[2] \oplus S_3[3]/S_2[8] \oplus S_3[9]$	1/1
	$\tau^3 I(3)/\tau^9 I(3)$	1	$S_1[3] \oplus S_2[4]/S_1[9] \oplus S_2[10]$	1/1
	$\tau^4 I(3)/\tau^{10} I(3)$	1	$S_1[5] \oplus S_6[4]/S_1[11] \oplus S_6[10]$	1/1
	$\tau^5 I(3)/\tau^{11} I(3)$	1	$S_5[5] \oplus S_6[6]/S_5[11] \oplus S_6[12]$	1/1
	$\tau^0 I(5)/\tau^6 I(5)$	1	$S_3[1]/S_3[7] \oplus S_6[6]$	1/1
	$\tau^1 I(5)/\tau^7 I(5)$	1	$S_2[2] \oplus S_5[1]/S_2[8] \oplus S_5[7]$	1/1
	$\tau^2 I(5)/\tau^8 I(5)$	1	$S_1[3] \oplus S_4[2]/S_1[9] \oplus S_4[8]$	1/1
	$\tau^3 I(5)/\tau^9 I(5)$	1	$S_3[3] \oplus S_6[4]/S_3[9] \oplus S_6[10]$	1/1
	$\tau^4 I(5)/\tau^{10} I(5)$	1	$S_2[4] \oplus S_5[5]/S_2[10] \oplus S_5[11]$	1/1
	$\tau^5 I(5)/\tau^{11} I(5)$	1	$S_1[5] \oplus S_4[6]/S_1[11] \oplus S_4[12]$	1/1
	$\tau^6 I(6)$	1	$S_1[5] \oplus S_3[7]$	1
	$\tau^1 I(6)/\tau^7 I(6)$	1	$S_2[2]/S_2[8] \oplus S_6[6]$	1/1
	$\tau^2 I(6)/\tau^8 I(6)$	1	$S_1[3] \oplus S_5[1]/S_1[9] \oplus S_5[7]$	1/1
	$\tau^3 I(6)/\tau^9 I(6)$	1	$S_4[2] \oplus S_6[4]/S_4[8] \oplus S_6[10]$	1/1
	$\tau^3 I(6)/\tau^9 I(6)$	2	$S'_1[1] \oplus S'_2[1]/S'_1[3] \oplus S'_2[3]$	1/1
	$\tau^3 I(6)/\tau^9 I(6)$	3	$S''_1[1] \oplus S''_2[1]/S''_1[3] \oplus S''_2[3]$	1/1
	$\tau^4 I(6)/\tau^{10} I(6)$	1	$S_3[3] \oplus S_5[5]/S_3[9] \oplus S_5[11]$	1/1
	$\tau^5 I(6)/\tau^{11} I(6)$	1	$S_2[4] \oplus S_4[6]/S_2[10] \oplus S_4[12]$	1/1
	$\tau^6 I(7)$	1	$S_2[4] \oplus S_3[7]$	1
	$\tau^7 I(7)$	1	$S_1[5] \oplus S_2[8]$	1
	$\tau^2 I(7)/\tau^8 I(7)$	1	$S_1[3]/S_1[9] \oplus S_6[6]$	1/1
	$\tau^3 I(7)/\tau^9 I(7)$	1	$S_5[1] \oplus S_6[4]/S_5[7] \oplus S_6[10]$	1/1
	$\tau^4 I(7)/\tau^{10} I(7)$	1	$S_4[2] \oplus S_5[5]/S_4[8] \oplus S_5[11]$	1/1
	$\tau^5 I(7)/\tau^{11} I(7)$	1	$S_3[3] \oplus S_4[6]/S_3[9] \oplus S_4[12]$	1/1
$P(5)$	$\tau^0 I(5)/\tau^6 I(5)$	1	$S_3[1] \oplus S_6[1]/S_3[7] \oplus S_6[7]$	1/1

Type $\tilde{D}_8$				
U	V	k	M_k	c
	$\tau^1 I(5)/\tau^7 I(5)$	1	$S_2[2] \oplus S_5[2]/S_2[8] \oplus S_5[8]$	1/1
	$\tau^2 I(5)/\tau^8 I(5)$	1	$S_1[3] \oplus S_4[3]/S_1[9] \oplus S_4[9]$	1/1
	$\tau^2 I(5)/\tau^8 I(5)$	2	$S_1'[1] \oplus S_2'[1]/S_1'[3] \oplus S_2'[3]$	1/1
	$\tau^2 I(5)/\tau^8 I(5)$	3	$S_1''[1] \oplus S_2''[1]/S_1''[3] \oplus S_2''[3]$	1/1
	$\tau^3 I(5)/\tau^9 I(5)$	1	$S_3[4] \oplus S_6[4]/S_3[10] \oplus S_6[10]$	1/1
	$\tau^4 I(5)/\tau^{10} I(5)$	1	$S_2[5] \oplus S_5[5]/S_2[11] \oplus S_5[11]$	1/1
	$\tau^5 I(5)/\tau^{11} I(5)$	1	$S_1[6] \oplus S_4[6]/S_1[12] \oplus S_4[12]$	1/1
	$\tau^5 I(5)/\tau^{11} I(5)$	2	$S_1'[2] \oplus S_2'[2]/S_1'[4] \oplus S_2'[4]$	1/1
	$\tau^5 I(5)/\tau^{11} I(5)$	3	$S_1''[2] \oplus S_2''[2]/S_1''[4] \oplus S_2''[4]$	1/1
	$\tau^6 I(3)$	1	$S_5[5] \oplus S_6[7]$	1
	$\tau^1 I(3)/\tau^7 I(3)$	1	$S_5[2]/S_4[6] \oplus S_5[8]$	1/1
	$\tau^2 I(3)/\tau^8 I(3)$	1	$S_3[1] \oplus S_4[3]/S_3[7] \oplus S_4[9]$	1/1
	$\tau^3 I(3)/\tau^9 I(3)$	1	$S_2[2] \oplus S_3[4]/S_2[8] \oplus S_3[10]$	1/1
	$\tau^4 I(3)/\tau^{10} I(3)$	1	$S_1[3] \oplus S_2[5]/S_1[9] \oplus S_2[11]$	1/1
	$\tau^5 I(3)/\tau^{11} I(3)$	1	$S_1[6] \oplus S_6[4]/S_1[12] \oplus S_6[10]$	1/1
	$\tau^0 I(4)/\tau^6 I(4)$	1	$S_6[1]/S_4[6] \oplus S_6[7]$	1/1
	$\tau^1 I(4)/\tau^7 I(4)$	1	$S_3[1] \oplus S_5[2]/S_3[7] \oplus S_5[8]$	1/1
	$\tau^2 I(4)/\tau^8 I(4)$	1	$S_2[2] \oplus S_4[3]/S_2[8] \oplus S_4[9]$	1/1
	$\tau^3 I(4)/\tau^9 I(4)$	1	$S_1[3] \oplus S_3[4]/S_1[9] \oplus S_3[10]$	1/1
	$\tau^4 I(4)/\tau^{10} I(4)$	1	$S_2[5] \oplus S_6[4]/S_2[11] \oplus S_6[10]$	1/1
	$\tau^5 I(4)/\tau^{11} I(4)$	1	$S_1[6] \oplus S_5[5]/S_1[12] \oplus S_5[11]$	1/1
	$\tau^0 I(6)/\tau^6 I(6)$	1	$S_3[1]/S_1[6] \oplus S_3[7]$	1/1
	$\tau^1 I(6)/\tau^7 I(6)$	1	$S_2[2] \oplus S_6[1]/S_2[8] \oplus S_6[7]$	1/1
	$\tau^2 I(6)/\tau^8 I(6)$	1	$S_1[3] \oplus S_5[2]/S_1[9] \oplus S_5[8]$	1/1
	$\tau^3 I(6)/\tau^9 I(6)$	1	$S_4[3] \oplus S_6[4]/S_4[9] \oplus S_6[10]$	1/1
	$\tau^4 I(6)/\tau^{10} I(6)$	1	$S_3[4] \oplus S_5[5]/S_3[10] \oplus S_5[11]$	1/1
	$\tau^5 I(6)/\tau^{11} I(6)$	1	$S_2[5] \oplus S_4[6]/S_2[11] \oplus S_4[12]$	1/1
	$\tau^6 I(7)$	1	$S_2[5] \oplus S_3[7]$	1
	$\tau^1 I(7)/\tau^7 I(7)$	1	$S_2[2]/S_1[6] \oplus S_2[8]$	1/1
	$\tau^2 I(7)/\tau^8 I(7)$	1	$S_1[3] \oplus S_6[1]/S_1[9] \oplus S_6[7]$	1/1

Type $\tilde{D}_8$				
U	V	k	M_k	c
	$\tau^3 I(7)/\tau^9 I(7)$	1	$S_5[2] \oplus S_6[4]/S_5[8] \oplus S_6[10]$	1/1
	$\tau^4 I(7)/\tau^{10} I(7)$	1	$S_4[3] \oplus S_5[5]/S_4[9] \oplus S_5[11]$	1/1
	$\tau^5 I(7)/\tau^{11} I(7)$	1	$S_3[4] \oplus S_4[6]/S_3[10] \oplus S_4[12]$	1/1

$\tilde{E}_6$				
U	V	k	M_k	c
$P(2)$	$\tau^2 I(2)/\tau^8 I(2)$	1	$S_1[1] \oplus S_2[2]/S_1[4] \oplus S_2[5]$	1/1
	$\tau^2 I(2)/\tau^8 I(2)$	2	$S'_1[1] \oplus S'_2[2]/S'_1[4] \oplus S'_2[5]$	1/1
	$\tau^2 I(2)/\tau^8 I(2)$	3	$S''_1[1] \oplus S''_2[1]/S''_1[3] \oplus S''_2[3]$	1/1
	$\tau^5 I(2)/\tau^{11} I(2)$	1	$S_1[3] \oplus S_2[3]/S_1[6] \oplus S_2[6]$	1/1
	$\tau^5 I(2)/\tau^{11} I(2)$	2	$S'_1[3] \oplus S'_2[3]/S'_1[6] \oplus S'_2[6]$	1/1
	$\tau^5 I(2)/\tau^{11} I(2)$	3	$S''_1[2] \oplus S''_2[2]/S''_1[4] \oplus S''_2[4]$	1/1
	$\tau^1 I(4)/\tau^7 I(4)$	1	$S_3[1]/S_2[3] \oplus S_3[4]$	1/1
	$\tau^2 I(4)/\tau^8 I(4)$	2	$S'_1[1] \oplus S'_3[1]/S'_1[4] \oplus S'_3[4]$	1/1
	$\tau^4 I(4)/\tau^{10} I(4)$	1	$S_2[2] \oplus S_3[2]/S_2[5] \oplus S_3[5]$	1/1
	$\tau^5 I(4)/\tau^{11} I(4)$	2	$S'_1[3] \oplus S'_3[2]/S'_1[6] \oplus S'_3[5]$	1/1
	$\tau^1 I(6)/\tau^7 I(6)$	2	$S'_3[1]/S'_2[3] \oplus S'_3[4]$	1/1
	$\tau^2 I(6)/\tau^8 I(6)$	1	$S_1[1] \oplus S_3[1]/S_1[4] \oplus S_3[4]$	1/1
	$\tau^4 I(6)/\tau^{10} I(6)$	2	$S'_2[2] \oplus S'_3[2]/S'_2[5] \oplus S'_3[5]$	1/1
	$\tau^5 I(6)/\tau^{11} I(6)$	1	$S_1[3] \oplus S_3[2]/S_1[6] \oplus S_3[5]$	1/1
$P(3)$	$\tau^1 I(3)/\tau^7 I(3)$	1	$S_1[1] \oplus S_2[1] \oplus S_3[1]/S_1[4] \oplus S_2[4] \oplus S_3[4]$	1/1
	$\tau^1 I(3)/\tau^7 I(3)$	2	$S'_1[1] \oplus S'_2[1] \oplus S'_3[1]/S'_1[4] \oplus S'_2[4] \oplus S'_3[4]$	1/1
	$\tau^2 I(3)/\tau^8 I(3)$	3	$S''_1[1] \oplus S''_2[1] \oplus S''_2[1]/S''_1[3] \oplus S''_2[3] \oplus S''_2[3]$	1/1
	$\tau^3 I(3)/\tau^9 I(3)$	1	$S_1[2] \oplus S_2[2] \oplus S_3[2]/S_1[5] \oplus S_2[5] \oplus S_3[5]$	1/1
	$\tau^3 I(3)/\tau^9 I(3)$	2	$S'_1[2] \oplus S'_2[2] \oplus S'_3[2]/S'_1[5] \oplus S'_2[5] \oplus S'_3[5]$	1/1
	$\tau^5 I(3)/\tau^{11} I(3)$	1	$S_1[3] \oplus S_2[3] \oplus S_3[3]/S_1[6] \oplus S_2[6] \oplus S_3[6]$	1/1
	$\tau^5 I(3)/\tau^{11} I(3)$	2	$S'_1[3] \oplus S'_2[3] \oplus S'_3[3]/S'_1[6] \oplus S'_2[6] \oplus S'_3[6]$	1/1
	$\tau^5 I(3)/\tau^{11} I(3)$	3	$S''_1[2] \oplus S''_1[2] \oplus S''_2[2]/S''_1[4] \oplus S''_1[4] \oplus S''_2[4]$	1/1

$\tilde{E}_7$				
U	V	k	M_k	c
$P(2)$	$\tau^2 I(2)/\tau^{14} I(2)$	1	$S_1[2]/S_1[6] \oplus S_4[4]$	1/1
	$\tau^3 I(2)/\tau^{15} I(2)$	2	$S_2'[1] \oplus S_3'[1]/S_2'[4] \oplus S_3'[4]$	1/1
	$\tau^5 I(2)/\tau^{17} I(2)$	1	$S_1[2] \oplus S_2[2]/S_1[6] \oplus S_2[6]$	1/1
	$\tau^7 I(2)/\tau^{19} I(2)$	2	$S_1'[2] \oplus S_2'[2]/S_1'[5] \oplus S_2'[5]$	1/1
	$\tau^8 I(2)/\tau^{20} I(2)$	1	$S_2[2] \oplus S_3[4]/S_2[6] \oplus S_3[8]$	1/1
	$\tau^{11} I(2)/\tau^{23} I(2)$	1	$S_3[4] \oplus S_4[4]/S_3[8] \oplus S_4[8]$	1/1
	$\tau^{11} I(2)/\tau^{23} I(2)$	2	$S_1'[3] \oplus S_3'[3]/S_1'[6] \oplus S_3'[6]$	1/1
	$\tau^{11} I(2)/\tau^{23} I(2)$	3	$S_1''[2] \oplus S_2''[2]/S_1''[4] \oplus S_2''[4]$	1/1
	$\tau^{13} I(6)$	1	$S_1[3] \oplus S_2[5]$	1
	$\tau^3 I(6)/\tau^{15} I(6)$	2	$S_2'[1]/S_1'[3] \oplus S_2'[4]$	1/1
	$\tau^4 I(6)/\tau^{16} I(6)$	1	$S_2[1] \oplus S_3[1]/S_2[5] \oplus S_3[5]$	1/1
	$\tau^7 I(6)/\tau^{19} I(6)$	1	$S_3[1] \oplus S_4[3]/S_3[5] \oplus S_4[7]$	1/1
	$\tau^7 I(6)/\tau^{19} I(6)$	2	$S_1'[2] \oplus S_3'[1]/S_1'[5] \oplus S_3'[4]$	1/1
	$\tau^7 I(6)/\tau^{19} I(6)$	3	$S_1''[1] \oplus S_2''[1]/S_1''[3] \oplus S_2''[3]$	1/1
	$\tau^{10} I(6)/\tau^{22} I(6)$	1	$S_1[3] \oplus S_4[3]/S_1[7] \oplus S_4[7]$	1/1
	$\tau^{11} I(6)/\tau^{23} I(6)$	2	$S_2'[2] \oplus S_3'[3]/S_2'[5] \oplus S_3'[6]$	1/1
	$\tau^2 I(8)/\tau^{14} I(8)$	1	$S_2[1]/S_2[5] \oplus S_4[4]$	1/1
	$\tau^5 I(8)/\tau^{17} I(8)$	1	$S_1[2] \oplus S_3[1]/S_1[6] \oplus S_3[5]$	1/1
	$\tau^8 I(8)/\tau^{20} I(8)$	1	$S_2[2] \oplus S_4[3]/S_2[6] \oplus S_4[7]$	1/1
	$\tau^{11} I(8)/\tau^{23} I(8)$	1	$S_1[3] \oplus S_3[4]/S_1[7] \oplus S_3[8]$	1/1
$P(3)$	$\tau^1 I(3)/\tau^{13} I(3)$	1	$S_2[1] \oplus S_3[1]/S_1[4] \oplus S_2[5] \oplus S_3[5]$	1/1
	$\tau^3 I(3)/\tau^{15} I(3)$	1	$S_1[2] \oplus S_3[1] \oplus S_4[1]/S_1[6] \oplus S_3[5] \oplus S_4[5]$	1/1
	$\tau^3 I(3)/\tau^{15} I(3)$	2	$S_1'[1] \oplus S_2'[1] \oplus S_3'[1]/S_1'[4] \oplus S_2'[4] \oplus S_3'[4]$	1/1
	$\tau^5 I(3)/\tau^{17} I(3)$	1	$S_1[2] \oplus S_2[2] \oplus S_3[2]/S_1[6] \oplus S_2[6] \oplus S_3[6]$	1/1
	$\tau^7 I(3)/\tau^{19} I(3)$	1	$S_1[3] \oplus S_3[2] \oplus S_4[3]/S_1[7] \oplus S_3[6] \oplus S_4[7]$	1/1
	$\tau^7 I(3)/\tau^{19} I(3)$	2	$S_1'[2] \oplus S_2'[2] \oplus S_3'[2]/S_1'[5] \oplus S_2'[5] \oplus S_3'[5]$	1/1
	$\tau^9 I(3)/\tau^{21} I(3)$	1	$S_1[3] \oplus S_2[3] \oplus S_3[4]/S_1[7] \oplus S_2[7] \oplus S_3[8]$	1/1
	$\tau^{11} I(3)/\tau^{23} I(3)$	1	$S_1[4] \oplus S_3[4] \oplus S_4[4]/S_1[8] \oplus S_3[8] \oplus S_4[8]$	1/1
	$\tau^{11} I(3)/\tau^{23} I(3)$	2	$S_1'[3] \oplus S_2'[3] \oplus S_3'[3]/S_1'[6] \oplus S_2'[6] \oplus S_3'[6]$	1/1

$\tilde{E}_7$				
U	V	k	M_k	c
	$\tau^{11}I(3)/\tau^{23}I(3)$	3	$S_1''[2]\oplus S_1''[2]\oplus S_2''[2]/S_1''[4]\oplus S_1''[4]\oplus S_2''[4]$	1/1
	$\tau^{1}I(5)/\tau^{13}I(5)$	1	$S_2[1]/S_1[4]\oplus S_2[5]\oplus S_4[4]$	1/1
	$\tau^{3}I(5)/\tau^{15}I(5)$	1	$S_2[1]\oplus S_3[1]\oplus S_4[1]/S_2[5]\oplus S_3[5]\oplus S_4[5]$	1/1
	$\tau^{5}I(5)/\tau^{17}I(5)$	1	$S_1[2]\oplus S_2[2]\oplus S_4[1]/S_1[6]\oplus S_2[6]\oplus S_4[5]$	1/1
	$\tau^{6}I(5)/\tau^{18}I(5)$	3	$S_1''[1]\oplus S_2''[1]\oplus S_2''[1]/S_1''[3]\oplus S_2''[3]\oplus S_2''[3]$	1/1
	$\tau^{7}I(5)/\tau^{19}I(5)$	1	$S_2[2]\oplus S_3[2]\oplus S_4[3]/S_2[6]\oplus S_3[6]\oplus S_4[7]$	1/1
	$\tau^{9}I(5)/\tau^{21}I(5)$	1	$S_1[3]\oplus S_2[3]\oplus S_4[3]/S_1[7]\oplus S_2[7]\oplus S_4[7]$	1/1
	$\tau^{11}I(5)/\tau^{23}I(5)$	1	$S_2[3]\oplus S_3[4]\oplus S_4[4]/S_2[7]\oplus S_3[8]\oplus S_4[8]$	1/1
$P(4)$	$\tau^{2}I(4)/\tau^{14}I(4)$	1	$S_1[1]\oplus S_2[1]\oplus S_3[1]\oplus S_4[1]/S_1[5]\oplus S_2[5]\oplus S_3[5]\oplus S_4[5]$	1/1
	$\tau^{3}I(4)/\tau^{15}I(4)$	2	$S_1'[1]\oplus S_2'[1]\oplus S_3'[1]\oplus S_3'[1]/S_1'[4]\oplus S_2'[4]\oplus S_3'[4]\oplus S_3'[4]$	1/1
	$\tau^{5}I(4)/\tau^{17}I(4)$	1	$S_1[2]\oplus S_2[2]\oplus S_3[2]\oplus S_4[2]/S_1[6]\oplus S_2[6]\oplus S_3[6]\oplus S_4[6]$	1/1
	$\tau^{5}I(4)/\tau^{17}I(4)$	3	$S_1''[1]\oplus S_1''[1]\oplus S_2''[1]\oplus S_2''[1]/S_1''[3]\oplus S_1''[3]\oplus S_2''[3]\oplus S_2''[3]$	1/1
	$\tau^{7}I(4)/\tau^{19}I(4)$	2	$S_1'[2]\oplus S_2'[2]\oplus S_2'[2]\oplus S_3'[2]/S_1'[5]\oplus S_2'[5]\oplus S_2'[5]\oplus S_3'[5]$	1/1
	$\tau^{8}I(4)/\tau^{20}I(4)$	1	$S_1[3]\oplus S_2[3]\oplus S_3[3]\oplus S_4[3]/S_1[7]\oplus S_2[7]\oplus S_3[7]\oplus S_4[7]$	1/1
	$\tau^{11}I(4)/\tau^{23}I(4)$	1	$S_1[4]\oplus S_2[4]\oplus S_3[4]\oplus S_4[4]/S_1[8]\oplus S_2[8]\oplus S_3[8]\oplus S_4[8]$	1/1
	$\tau^{11}I(4)/\tau^{23}I(4)$	2	$S_1'[3]\oplus S_1'[3]\oplus S_2'[3]\oplus S_3'[3]/S_1'[6]\oplus S_1'[6]\oplus S_2'[6]\oplus S_3'[6]$	1/1
	$\tau^{11}I(4)/\tau^{23}I(4)$	3	$S_1''[2]\oplus S_1''[2]\oplus S_2''[2]\oplus S_2''[2]/S_1''[4]\oplus S_1''[4]\oplus S_2''[4]\oplus S_2''[4]$	1/1
$P(8)$	$\tau^{1}I(8)/\tau^{13}I(8)$	2	$S_2'[1]/S_1'[3]\oplus S_2'[4]$	1/1
	$\tau^{2}I(8)/\tau^{14}I(8)$	1	$S_2[1]\oplus S_4[1]/S_2[5]\oplus S_4[5]$	1/1
	$\tau^{3}I(8)/\tau^{15}I(8)$	2	$S_2'[1]\oplus S_3'[1]/S_2'[4]\oplus S_3'[4]$	1/1
	$\tau^{5}I(8)/\tau^{17}I(8)$	1	$S_1[2]\oplus S_3[2]/S_1[6]\oplus S_3[6]$	1/1
	$\tau^{5}I(8)/\tau^{17}I(8)$	2	$S_1'[2]\oplus S_3'[1]/S_1'[5]\oplus S_3'[4]$	1/1
	$\tau^{5}I(8)/\tau^{17}I(8)$	3	$S_1''[1]\oplus S_2''[1]/S_1''[3]\oplus S_2''[3]$	1/1
	$\tau^{7}I(8)/\tau^{19}I(8)$	2	$S_1'[2]\oplus S_2'[2]/S_1'[5]\oplus S_2'[5]$	1/1
	$\tau^{8}I(8)/\tau^{20}I(8)$	1	$S_2[3]\oplus S_4[3]/S_2[7]\oplus S_4[7]$	1/1
	$\tau^{9}I(8)/\tau^{21}I(8)$	2	$S_2'[2]\oplus S_3'[3]/S_2'[5]\oplus S_3'[6]$	1/1
	$\tau^{11}I(8)/\tau^{23}I(8)$	1	$S_1[4]\oplus S_3[4]/S_1[8]\oplus S_3[8]$	1/1
	$\tau^{11}I(8)/\tau^{23}I(8)$	2	$S_1'[3]\oplus S_3'[3]/S_1'[6]\oplus S_3'[6]$	1/1
	$\tau^{11}I(8)/\tau^{23}I(8)$	3	$S_1''[2]\oplus S_2''[2]/S_1''[4]\oplus S_2''[4]$	1/1
	$\tau^{2}I(2)/\tau^{14}I(2)$	1	$S_2[1]/S_1[4]\oplus S_2[5]$	1/1

$\tilde{E}_7$				
U	V	k	M_k	c
	$\tau^5 I(2)/\tau^{17} I(2)$	1	$S_2[1] \oplus S_3[2]/S_2[5] \oplus S_3[6]$	1/1
	$\tau^8 I(2)/\tau^{20} I(2)$	1	$S_3[2] \oplus S_4[3]/S_3[6] \oplus S_4[7]$	1/1
	$\tau^{11} I(2)/\tau^{23} I(2)$	1	$S_1[4] \oplus S_4[3]/S_1[8] \oplus S_4[7]$	1/1
	$\tau^2 I(6)/\tau^{14} I(6)$	1	$S_4[1]/S_3[4] \oplus S_4[5]$	1/1
	$\tau^5 I(6)/\tau^{17} I(6)$	1	$S_1[2] \oplus S_4[1]/S_1[6] \oplus S_4[5]$	1/1
	$\tau^8 I(6)/\tau^{20} I(6)$	1	$S_1[2] \oplus S_2[3]/S_1[6] \oplus S_2[7]$	1/1
	$\tau^{11} I(6)/\tau^{23} I(6)$	1	$S_2[3] \oplus S_3[4]/S_2[7] \oplus S_3[8]$	1/1

$\tilde{E}_8$				
U	V	k	M_k	c
$P(1)$	$\tau^2 I(1)/\tau^{32} I(1)$	1	$S_2[1]/S_2[6] \oplus S_5[5]$	1/1
	$\tau^4 I(1)/\tau^{34} I(1)$	2	$S'_2[1]/S'_1[3] \oplus S'_2[4]$	1/1
	$\tau^5 I(1)/\tau^{35} I(1)$	1	$S_2[1] \oplus S_4[1]/S_2[6] \oplus S_4[6]$	1/1
	$\tau^8 I(1)/\tau^{38} I(1)$	1	$S_1[2] \oplus S_4[1]/S_1[7] \oplus S_4[6]$	1/1
	$\tau^9 I(1)/\tau^{39} I(1)$	2	$S'_2[1] \oplus S'_3[1]/S'_2[4] \oplus S'_3[4]$	1/1
	$\tau^{11} I(1)/\tau^{41} I(1)$	1	$S_1[2] \oplus S_3[2]/S_1[7] \oplus S_3[7]$	1/1
	$\tau^{14} I(1)/\tau^{44} I(1)$	1	$S_3[2] \oplus S_5[3]/S_3[7] \oplus S_5[8]$	1/1
	$\tau^{14} I(1)/\tau^{44} I(1)$	2	$S'_1[2] \oplus S'_3[1]/S'_1[5] \oplus S'_3[4]$	1/1
	$\tau^{14} I(1)/\tau^{44} I(1)$	3	$S''_1[1] \oplus S''_2[1]/S''_1[3] \oplus S''_2[3]$	1/1
	$\tau^{17} I(1)/\tau^{47} I(1)$	1	$S_2[3] \oplus S_5[3]/S_2[8] \oplus S_5[8]$	1/1
	$\tau^{19} I(1)/\tau^{49} I(1)$	2	$S'_1[2] \oplus S'_2[2]/S'_1[5] \oplus S'_2[5]$	1/1
	$\tau^{20} I(1)/\tau^{50} I(1)$	1	$S_2[3] \oplus S_4[4]/S_2[8] \oplus S_4[9]$	1/1
	$\tau^{23} I(1)/\tau^{53} I(1)$	1	$S_1[4] \oplus S_4[4]/S_1[9] \oplus S_4[9]$	1/1
	$\tau^{24} I(1)/\tau^{54} I(1)$	2	$S'_2[2] \oplus S'_3[3]/S'_2[5] \oplus S'_3[6]$	1/1
	$\tau^{26} I(1)/\tau^{56} I(1)$	1	$S_1[4] \oplus S_3[5]/S_1[9] \oplus S_3[10]$	1/1
	$\tau^{29} I(1)/\tau^{59} I(1)$	1	$S_3[5] \oplus S_5[5]/S_3[10] \oplus S_5[10]$	1/1
	$\tau^{29} I(1)/\tau^{59} I(1)$	2	$S'_1[3] \oplus S'_3[3]/S'_1[6] \oplus S'_3[6]$	1/1
	$\tau^{29} I(1)/\tau^{59} I(1)$	3	$S''_1[2] \oplus S''_2[2]/S''_1[4] \oplus S''_2[4]$	1/1
	$\tau^{32} I(7)$	1	$S_1[4] \oplus S_2[6]$	1

$\tilde{E}_8$				
U	V	k	M_k	c
	$\tau^5 I(7)/\tau^{35} I(7)$	1	$S_4[1]/S_3[5] \oplus S_4[6]$	1/1
	$\tau^8 I(7)/\tau^{38} I(7)$	1	$S_1[2]/S_1[7] \oplus S_5[5]$	1/1
	$\tau^{11} I(7)/\tau^{41} I(7)$	1	$S_2[1] \oplus S_3[2]/S_2[6] \oplus S_3[7]$	1/1
	$\tau^{14} I(7)/\tau^{44} I(7)$	1	$S_4[1] \oplus S_5[3]/S_4[6] \oplus S_5[8]$	1/1
	$\tau^{17} I(7)/\tau^{47} I(7)$	1	$S_1[2] \oplus S_2[3]/S_1[7] \oplus S_2[8]$	1/1
	$\tau^{20} I(7)/\tau^{50} I(7)$	1	$S_3[2] \oplus S_4[4]/S_3[7] \oplus S_4[9]$	1/1
	$\tau^{23} I(7)/\tau^{53} I(7)$	1	$S_1[4] \oplus S_5[3]/S_1[9] \oplus S_5[8]$	1/1
	$\tau^{26} I(7)/\tau^{56} I(7)$	1	$S_2[3] \oplus S_3[5]/S_2[8] \oplus S_3[10]$	1/1
	$\tau^{29} I(7)/\tau^{59} I(7)$	1	$S_4[4] \oplus S_5[5]/S_4[9] \oplus S_5[10]$	1/1
$P(2)$	$\tau^2 I(2)/\tau^{32} I(2)$	1	$S_2[1] \oplus S_3[1]/S_1[5] \oplus S_2[6] \oplus S_3[6] \oplus S_5[5]$	1/1
	$\tau^5 I(2)/\tau^{35} I(2)$	1	$S_2[1] \oplus S_3[1] \oplus S_4[1] \oplus S_5[1]/S_2[6] \oplus S_3[6] \oplus S_4[6] \oplus S_5[6]$	1/1
	$\tau^8 I(2)/\tau^{38} I(2)$	1	$S_1[2] \oplus S_2[2] \oplus S_4[1] \oplus S_5[1]/S_1[7] \oplus S_2[7] \oplus S_4[6] \oplus S_5[6]$	1/1
	$\tau^9 I(2)/\tau^{39} I(2)$	2	$S'_1[1] \oplus S'_1[1] \oplus S'_2[1] \oplus S'_3[1]/S'_1[4] \oplus S'_1[4] \oplus S'_2[4] \oplus S'_3[4]$	1/1
	$\tau^{11} I(2)/\tau^{41} I(2)$	1	$S_1[2] \oplus S_2[2] \oplus S_3[2] \oplus S_4[2]/S_1[7] \oplus S_2[7] \oplus S_3[7] \oplus S_4[7]$	1/1
	$\tau^{14} I(2)/\tau^{44} I(2)$	1	$S_1[3] \oplus S_3[2] \oplus S_4[2] \oplus S_5[3]/S_1[8] \oplus S_3[7] \oplus S_4[7] \oplus S_5[8]$	1/1
	$\tau^{14} I(2)/\tau^{44} I(2)$	3	$S''_1[1] \oplus S''_1[1] \oplus S''_2[1] \oplus S''_2[1]/S''_1[3] \oplus S''_1[3] \oplus S''_2[3] \oplus S''_2[3]$	1/1
	$\tau^{17} I(2)/\tau^{47} I(2)$	1	$S_1[3] \oplus S_2[3] \oplus S_3[3] \oplus S_5[3]/S_1[8] \oplus S_2[8] \oplus S_3[8] \oplus S_5[8]$	1/1
	$\tau^{19} I(2)/\tau^{49} I(2)$	2	$S'_1[2] \oplus S'_2[2] \oplus S'_3[2] \oplus S'_3[2]/S'_1[5] \oplus S'_2[5] \oplus S'_3[5] \oplus S'_3[5]$	1/1
	$\tau^{20} I(2)/\tau^{50} I(2)$	1	$S_2[3] \oplus S_3[3] \oplus S_4[4] \oplus S_5[4]/S_2[8] \oplus S_3[8] \oplus S_4[9] \oplus S_5[9]$	1/1
	$\tau^{23} I(2)/\tau^{53} I(2)$	1	$S_1[4] \oplus S_2[4] \oplus S_4[4] \oplus S_5[4]/S_1[9] \oplus S_2[9] \oplus S_4[9] \oplus S_5[9]$	1/1
	$\tau^{26} I(2)/\tau^{56} I(2)$	1	$S_1[4] \oplus S_2[4] \oplus S_3[5] \oplus S_4[5]/S_1[9] \oplus S_2[9] \oplus S_3[10] \oplus S_4[10]$	1/1
	$\tau^{29} I(2)/\tau^{59} I(2)$	1	$S_1[5] \oplus S_3[5] \oplus S_4[5] \oplus S_5[5]/S_1[10] \oplus S_3[10] \oplus S_4[10] \oplus S_5[10]$	1/1
	$\tau^{29} I(2)/\tau^{59} I(2)$	2	$S'_1[3] \oplus S'_2[3] \oplus S'_2[3] \oplus S'_3[3]/S'_1[6] \oplus S'_2[6] \oplus S'_2[6] \oplus S'_3[6]$	1/1
	$\tau^{29} I(2)/\tau^{59} I(2)$	3	$S''_1[2] \oplus S''_1[2] \oplus S''_2[2] \oplus S''_2[2]/S''_1[4] \oplus S''_1[4] \oplus S''_2[4] \oplus S''_2[4]$	1/1
	$\tau^2 I(5)/\tau^{32} I(5)$	1	$S_2[1]/S_1[5] \oplus S_2[6] \oplus S_4[5] \oplus S_5[5]$	1/1
	$\tau^5 I(5)/\tau^{35} I(5)$	1	$S_2[1] \oplus S_3[1] \oplus S_4[1]/S_1[5] \oplus S_2[6] \oplus S_3[6] \oplus S_4[6]$	1/1
	$\tau^8 I(5)/\tau^{38} I(5)$	1	$S_1[2] \oplus S_3[1] \oplus S_4[1] \oplus S_5[1]/S_1[7] \oplus S_3[6] \oplus S_4[6] \oplus S_5[6]$	1/1
	$\tau^{11} I(5)/\tau^{41} I(5)$	1	$S_1[2] \oplus S_2[2] \oplus S_3[2] \oplus S_5[1]/S_1[7] \oplus S_2[7] \oplus S_3[7] \oplus S_5[6]$	1/1
	$\tau^{14} I(5)/\tau^{44} I(5)$	1	$S_2[2] \oplus S_3[2] \oplus S_4[2] \oplus S_5[3]/S_2[7] \oplus S_3[7] \oplus S_4[7] \oplus S_5[8]$	1/1
	$\tau^{17} I(5)/\tau^{47} I(5)$	1	$S_1[3] \oplus S_2[3] \oplus S_4[2] \oplus S_5[3]/S_1[8] \oplus S_2[8] \oplus S_4[7] \oplus S_5[8]$	1/1

$\tilde{E}_8$				
U	V	k	M_k	c
	$\tau^{20}I(5)/\tau^{50}I(5)$	1	$S_1[3]\oplus S_2[3]\oplus S_3[3]\oplus S_4[4]/S_1[8]\oplus S_2[8]\oplus S_3[8]\oplus S_4[9]$	1/1
	$\tau^{23}I(5)/\tau^{53}I(5)$	1	$S_1[4]\oplus S_3[3]\oplus S_4[4]\oplus S_5[4]/S_1[9]\oplus S_3[8]\oplus S_4[9]\oplus S_5[9]$	1/1
	$\tau^{26}I(5)/\tau^{56}I(5)$	1	$S_1[4]\oplus S_2[4]\oplus S_3[5]\oplus S_5[4]/S_1[9]\oplus S_2[9]\oplus S_3[10]\oplus S_5[9]$	1/1
	$\tau^{29}I(5)/\tau^{59}I(5)$	1	$S_2[4]\oplus S_3[5]\oplus S_4[5]\oplus S_5[5]/S_2[9]\oplus S_3[10]\oplus S_4[10]\oplus S_5[10]$	1/1
$P(3)$	$\tau^{5}I(3)/$	1	$S_1[1]\oplus S_2[1]\oplus S_3[1]\oplus S_3[1]\oplus S_4[1]\oplus S_5[1]/$	1/
	$\tau^{35}I(3)$		$S_1[6]\oplus S_2[6]\oplus S_3[6]\oplus S_3[6]\oplus S_4[6]\oplus S_5[6]$	1
	$\tau^{9}I(3)/$	2	$S'_1[1]\oplus S'_1[1]\oplus S'_2[1]\oplus S'_2[1]\oplus S'_3[1]\oplus S'_3[1]/$	1/
	$\tau^{39}I(3)$		$S'_1[4]\oplus S'_1[4]\oplus S'_2[4]\oplus S'_2[4]\oplus S'_3[4]\oplus S'_3[4]$	1
	$\tau^{11}I(3)/$	1	$S_1[2]\oplus S_2[2]\oplus S_2[2]\oplus S_3[2]\oplus S_4[2]\oplus S_5[2]/$	1/
	$\tau^{41}I(3)$		$S_1[7]\oplus S_2[7]\oplus S_2[7]\oplus S_3[7]\oplus S_4[7]\oplus S_5[7]$	1
	$\tau^{14}I(3)/$	3	$S''_1[1]\oplus S''_1[1]\oplus S''_1[1]\oplus S''_2[1]\oplus S''_2[1]\oplus S''_2[1]/$	1/
	$\tau^{44}I(3)$		$S''_1[3]\oplus S''_1[3]\oplus S''_1[3]\oplus S''_2[3]\oplus S''_2[3]\oplus S''_2[3]$	1
	$\tau^{17}I(3)/$	1	$S_1[3]\oplus S_1[3]\oplus S_2[3]\oplus S_3[3]\oplus S_4[3]\oplus S_5[3]/$	1/
	$\tau^{47}I(3)$		$S_1[8]\oplus S_1[8]\oplus S_2[8]\oplus S_3[8]\oplus S_4[8]\oplus S_5[8]$	1
	$\tau^{19}I(3)/$	2	$S'_1[2]\oplus S'_1[2]\oplus S'_2[2]\oplus S'_2[2]\oplus S'_3[2]\oplus S'_3[2]/$	1/
	$\tau^{49}I(3)$		$S'_1[5]\oplus S'_1[5]\oplus S'_2[5]\oplus S'_2[5]\oplus S'_3[5]\oplus S'_3[5]$	1
	$\tau^{23}I(3)/$	1	$S_1[4]\oplus S_2[4]\oplus S_3[4]\oplus S_4[4]\oplus S_5[4]\oplus S_5[4]/$	1/
	$\tau^{53}I(3)$		$S_1[9]\oplus S_2[9]\oplus S_3[9]\oplus S_4[9]\oplus S_5[9]\oplus S_5[9]$	1
	$\tau^{29}I(3)/$	1	$S_1[5]\oplus S_2[5]\oplus S_3[5]\oplus S_4[5]\oplus S_4[5]\oplus S_5[5]/$	1/
	$\tau^{59}I(3)$		$S_1[10]\oplus S_2[10]\oplus S_3[10]\oplus S_4[10]\oplus S_4[10]\oplus S_5[10]$	1
	$\tau^{29}I(3)/$	2	$S'_1[3]\oplus S'_1[3]\oplus S'_2[3]\oplus S'_2[3]\oplus S'_3[3]\oplus S'_3[3]/$	1/
	$\tau^{59}I(3)$		$S'_1[6]\oplus S'_1[6]\oplus S'_2[6]\oplus S'_2[6]\oplus S'_3[6]\oplus S'_3[6]$	1
	$\tau^{29}I(3)/$	3	$S''_1[2]\oplus S''_1[2]\oplus S''_1[2]\oplus S''_2[2]\oplus S''_2[2]\oplus S''_2[2]/$	1/
	$\tau^{59}I(3)$		$S''_1[4]\oplus S''_1[4]\oplus S''_1[4]\oplus S''_2[4]\oplus S''_2[4]\oplus S''_2[4]$	1
$P(4)$	$\tau^{5}I(4)/$	1	$S_1[1]\oplus S_2[1]\oplus S_3[1]\oplus S_4[1]\oplus S_5[1]/$	1/
	$\tau^{35}I(4)$		$S_1[6]\oplus S_2[6]\oplus S_3[6]\oplus S_4[6]\oplus S_5[6]$	1
	$\tau^{9}I(4)/$	2	$S'_1[1]\oplus S'_1[1]\oplus S'_2[1]\oplus S'_3[1]\oplus S'_3[1]/$	1/
	$\tau^{39}I(4)$		$S'_1[4]\oplus S'_1[4]\oplus S'_2[4]\oplus S'_3[4]\oplus S'_3[4]$	1
	$\tau^{11}I(4)/$	1	$S_1[2]\oplus S_2[2]\oplus S_3[2]\oplus S_4[2]\oplus S_5[2]/$	1/
	$\tau^{41}I(4)$		$S_1[7]\oplus S_2[7]\oplus S_3[7]\oplus S_4[7]\oplus S_5[7]$	1

$\tilde{E}_8$				
U	V	k	M_k	c
	$\tau^{14}I(4)/$	3	$S''_1[1]\oplus S''_1[1]\oplus S''_1[1]\oplus S''_2[1]\oplus S''_2[1]/$	1/
	$\tau^{44}I(4)$		$S''_1[3]\oplus S''_1[3]\oplus S''_1[3]\oplus S''_2[3]\oplus S''_2[3]$	1
	$\tau^{17}I(4)/$	1	$S_1[3]\oplus S_2[3]\oplus S_3[3]\oplus S_4[3]\oplus S_5[3]/$	1/
	$\tau^{47}I(4)$		$S_1[8]\oplus S_2[8]\oplus S_3[8]\oplus S_4[8]\oplus S_5[8]$	1
	$\tau^{19}I(4)/$	2	$S'_1[2]\oplus S'_2[2]\oplus S'_2[2]\oplus S'_3[2]\oplus S'_3[2]/$	1/
	$\tau^{49}I(4)$		$S'_1[5]\oplus S'_2[5]\oplus S'_2[5]\oplus S'_3[5]\oplus S'_3[5]$	1
	$\tau^{23}I(4)/$	1	$S_1[4]\oplus S_2[4]\oplus S_3[4]\oplus S_4[4]\oplus S_5[4]/$	1/
	$\tau^{53}I(4)$		$S_1[9]\oplus S_2[9]\oplus S_3[9]\oplus S_4[9]\oplus S_5[9]$	1
	$\tau^{29}I(4)/$	1	$S_1[5]\oplus S_2[5]\oplus S_3[5]\oplus S_4[5]\oplus S_5[5]/$	1/
	$\tau^{59}I(4)$		$S_1[10]\oplus S_2[10]\oplus S_3[10]\oplus S_4[10]\oplus S_5[10]$	1
	$\tau^{29}I(4)/$	2	$S'_1[3]\oplus S'_1[3]\oplus S'_2[3]\oplus S'_2[3]\oplus S'_3[3]/$	1/
	$\tau^{59}I(4)$		$S'_1[6]\oplus S'_1[6]\oplus S'_2[6]\oplus S'_2[6]\oplus S'_3[6]$	1
	$\tau^{29}I(4)/$	3	$S''_1[2]\oplus S''_1[2]\oplus S''_2[2]\oplus S''_2[2]\oplus S''_2[2]/$	1/
	$\tau^{59}I(4)$		$S''_1[4]\oplus S''_1[4]\oplus S''_2[4]\oplus S''_2[4]\oplus S''_2[4]$	1
$P(5)$	$\tau^{2}I(5)/\tau^{32}I(5)$	1	$S_3[1]\oplus S_4[1]/S_1[5]\oplus S_2[5]\oplus S_3[6]\oplus S_4[6]$	1/1
	$\tau^{5}I(5)/\tau^{35}I(5)$	1	$S_1[1]\oplus S_3[1]\oplus S_4[1]\oplus S_5[1]/S_1[6]\oplus S_3[6]\oplus S_4[6]\oplus S_5[6]$	1/1
	$\tau^{8}I(5)/\tau^{38}I(5)$	1	$S_1[1]\oplus S_2[2]\oplus S_3[2]\oplus S_5[1]/S_1[6]\oplus S_2[7]\oplus S_3[7]\oplus S_5[6]$	1/1
	$\tau^{9}I(5)/\tau^{39}I(5)$	2	$S'_1[1]\oplus S'_2[1]\oplus S'_3[1]\oplus S'_3[1]/S'_1[4]\oplus S'_2[4]\oplus S'_3[4]\oplus S'_3[4]$	1/1
	$\tau^{11}I(5)/\tau^{41}I(5)$	1	$S_2[2]\oplus S_3[2]\oplus S_4[2]\oplus S_5[2]/S_2[7]\oplus S_3[7]\oplus S_4[7]\oplus S_5[7]$	1/1
	$\tau^{14}I(5)/\tau^{44}I(5)$	1	$S_1[3]\oplus S_2[3]\oplus S_4[2]\oplus S_5[2]/S_1[8]\oplus S_2[8]\oplus S_4[7]\oplus S_5[7]$	1/1
	$\tau^{14}I(5)/\tau^{44}I(5)$	3	$S''_1[1]\oplus S''_1[1]\oplus S''_2[1]\oplus S''_2[1]/S''_1[3]\oplus S''_1[3]\oplus S''_2[3]\oplus S''_2[3]$	1/1
	$\tau^{17}I(5)/\tau^{47}I(5)$	1	$S_1[3]\oplus S_2[3]\oplus S_3[3]\oplus S_4[3]/S_1[8]\oplus S_2[8]\oplus S_3[8]\oplus S_4[8]$	1/1
	$\tau^{19}I(5)/\tau^{49}I(5)$	2	$S'_1[2]\oplus S'_2[2]\oplus S'_2[2]\oplus S'_3[2]/S'_1[5]\oplus S'_2[5]\oplus S'_2[5]\oplus S'_3[5]$	1/1
	$\tau^{20}I(5)/\tau^{50}I(5)$	1	$S_1[4]\oplus S_3[3]\oplus S_4[3]\oplus S_5[4]/S_1[9]\oplus S_3[8]\oplus S_4[8]\oplus S_5[9]$	1/1
	$\tau^{23}I(5)/\tau^{53}I(5)$	1	$S_1[4]\oplus S_2[4]\oplus S_3[4]\oplus S_5[4]/S_1[9]\oplus S_2[9]\oplus S_3[9]\oplus S_5[9]$	1/1
	$\tau^{26}I(5)/\tau^{56}I(5)$	1	$S_2[4]\oplus S_3[4]\oplus S_4[5]\oplus S_5[5]/S_2[9]\oplus S_3[9]\oplus S_4[10]\oplus S_5[10]$	1/1
	$\tau^{29}I(5)/\tau^{59}I(5)$	1	$S_1[5]\oplus S_2[5]\oplus S_4[5]\oplus S_5[5]/S_1[10]\oplus S_2[10]\oplus S_4[10]\oplus S_5[10]$	1/1
	$\tau^{29}I(5)/\tau^{59}I(5)$	2	$S'_1[3]\oplus S'_1[3]\oplus S'_2[3]\oplus S'_3[3]/S'_1[6]\oplus S'_1[6]\oplus S'_2[6]\oplus S'_3[6]$	1/1
	$\tau^{29}I(5)/\tau^{59}I(5)$	3	$S''_1[2]\oplus S''_1[2]\oplus S''_2[2]\oplus S''_2[2]/S''_1[4]\oplus S''_1[4]\oplus S''_2[4]\oplus S''_2[4]$	1/1
	$\tau^{2}I(2)/\tau^{32}I(2)$	1	$S_3[1]/S_1[5]\oplus S_2[5]\oplus S_3[6]\oplus S_5[5]$	1/1

$\tilde{E}_8$				
U	V	k	M_k	c
	$\tau^5 I(2)/\tau^{35} I(2)$	1	$S_3[1] \oplus S_4[1] \oplus S_5[1]/S_2[5] \oplus S_3[6] \oplus S_4[6] \oplus S_5[6]$	1/1
	$\tau^8 I(2)/\tau^{38} I(2)$	1	$S_1[1] \oplus S_2[2] \oplus S_4[1] \oplus S_5[1]/S_1[6] \oplus S_2[7] \oplus S_4[6] \oplus S_5[6]$	1/1
	$\tau^{11} I(2)/\tau^{41} I(2)$	1	$S_1[1] \oplus S_2[2] \oplus S_3[2] \oplus S_4[2]/S_1[6] \oplus S_2[7] \oplus S_3[7] \oplus S_4[7]$	1/1
	$\tau^{14} I(2)/\tau^{44} I(2)$	1	$S_1[3] \oplus S_3[2] \oplus S_4[2] \oplus S_5[2]/S_1[8] \oplus S_3[7] \oplus S_4[7] \oplus S_5[7]$	1/1
	$\tau^{17} I(2)/\tau^{47} I(2)$	1	$S_1[3] \oplus S_2[3] \oplus S_3[3] \oplus S_5[2]/S_1[8] \oplus S_2[8] \oplus S_3[8] \oplus S_5[7]$	1/1
	$\tau^{20} I(2)/\tau^{50} I(2)$	1	$S_2[3] \oplus S_3[3] \oplus S_4[3] \oplus S_5[4]/S_2[8] \oplus S_3[8] \oplus S_4[8] \oplus S_5[9]$	1/1
	$\tau^{23} I(2)/\tau^{53} I(2)$	1	$S_1[4] \oplus S_2[4] \oplus S_4[3] \oplus S_5[4]/S_1[9] \oplus S_2[9] \oplus S_4[8] \oplus S_5[9]$	1/1
	$\tau^{26} I(2)/\tau^{56} I(2)$	1	$S_1[4] \oplus S_2[4] \oplus S_3[4] \oplus S_4[5]/S_1[9] \oplus S_2[9] \oplus S_3[9] \oplus S_4[10]$	1/1
	$\tau^{29} I(2)/\tau^{59} I(2)$	1	$S_1[5] \oplus S_3[4] \oplus S_4[5] \oplus S_5[5]/S_1[10] \oplus S_3[9] \oplus S_4[10] \oplus S_5[10]$	1/1
$P(6)$	$\tau^{31} I(6)$	1	$S_1[4] \oplus S_2[6] \oplus S_3[6]$	1
	$\tau^3 I(6)/\tau^{33} I(6)$	1	$S_1[2]/S_1[7] \oplus S_4[5] \oplus S_5[5]$	1/1
	$\tau^5 I(6)/\tau^{35} I(6)$	1	$S_2[1] \oplus S_3[1] \oplus S_4[1]/S_2[6] \oplus S_3[6] \oplus S_4[6]$	1/1
	$\tau^7 I(6)/\tau^{37} I(6)$	1	$S_1[2] \oplus S_2[2]/S_1[7] \oplus S_2[7] \oplus S_5[5]$	1/1
	$\tau^9 I(6)/\tau^{39} I(6)$	1	$S_3[1] \oplus S_4[1] \oplus S_5[3]/S_3[6] \oplus S_4[6] \oplus S_5[8]$	1/1
	$\tau^9 I(6)/\tau^{39} I(6)$	2	$S'_1[1] \oplus S'_2[1] \oplus S'_3[1]/S'_1[4] \oplus S'_2[4] \oplus S'_3[4]$	1/1
	$\tau^{11} I(6)/\tau^{41} I(6)$	1	$S_1[2] \oplus S_2[2] \oplus S_3[2]/S_1[7] \oplus S_2[7] \oplus S_3[7]$	1/1
	$\tau^{13} I(6)/\tau^{43} I(6)$	1	$S_1[3] \oplus S_4[1] \oplus S_5[3]/S_1[8] \oplus S_4[6] \oplus S_5[8]$	1/1
	$\tau^{14} I(6)/\tau^{44} I(6)$	3	$S''_1[1] \oplus S''_1[1] \oplus S''_2[1]/S''_1[3] \oplus S''_1[3] \oplus S''_2[3]$	1/1
	$\tau^{15} I(6)/\tau^{45} I(6)$	1	$S_2[2] \oplus S_3[2] \oplus S_4[4]/S_2[7] \oplus S_3[7] \oplus S_4[9]$	1/1
	$\tau^{17} I(6)/\tau^{47} I(6)$	1	$S_1[3] \oplus S_2[3] \oplus S_5[3]/S_1[8] \oplus S_2[8] \oplus S_5[8]$	1/1
	$\tau^{19} I(6)/\tau^{49} I(6)$	1	$S_3[2] \oplus S_4[4] \oplus S_5[4]/S_3[7] \oplus S_4[9] \oplus S_5[9]$	1/1
	$\tau^{19} I(6)/\tau^{49} I(6)$	2	$S'_1[2] \oplus S'_2[2] \oplus S'_3[2]/S'_1[5] \oplus S'_2[5] \oplus S'_3[5]$	1/1
	$\tau^{21} I(6)/\tau^{51} I(6)$	1	$S_1[3] \oplus S_2[3] \oplus S_3[5]/S_1[8] \oplus S_2[8] \oplus S_3[10]$	1/1
	$\tau^{23} I(6)/\tau^{53} I(6)$	1	$S_1[4] \oplus S_4[4] \oplus S_5[4]/S_1[9] \oplus S_4[9] \oplus S_5[9]$	1/1
	$\tau^{25} I(6)/\tau^{55} I(6)$	1	$S_2[3] \oplus S_3[5] \oplus S_4[5]/S_2[8] \oplus S_3[10] \oplus S_4[10]$	1/1
	$\tau^{27} I(6)/\tau^{57} I(6)$	1	$S_1[4] \oplus S_2[6] \oplus S_5[4]/S_1[9] \oplus S_2[11] \oplus S_5[9]$	1/1
	$\tau^{29} I(6)/\tau^{59} I(6)$	1	$S_3[5] \oplus S_4[5] \oplus S_5[5]/S_3[10] \oplus S_4[10] \oplus S_5[10]$	1/1
	$\tau^{29} I(6)/\tau^{59} I(6)$	2	$S'_1[3] \oplus S'_2[3] \oplus S'_3[3]/S'_1[6] \oplus S'_2[6] \oplus S'_3[6]$	1/1
	$\tau^{29} I(6)/\tau^{59} I(6)$	3	$S''_1[2] \oplus S''_2[2] \oplus S''_2[2]/S''_1[4] \oplus S''_2[4] \oplus S''_2[4]$	1/1
	$\tau^{31} I(9)$	1	$S_1[4] \oplus S_2[6] \oplus S_4[5]$	1

$\tilde{E}_8$				
U	V	k	M_k	c
	$\tau^3 I(9)/\tau^{33} I(9)$	1	$S_2[1]/S_2[6] \oplus S_4[5] \oplus S_5[5]$	1/1
	$\tau^5 I(9)/\tau^{35} I(9)$	1	$S_2[1] \oplus S_3[1]/S_2[6] \oplus S_3[6] \oplus S_5[5]$	1/1
	$\tau^7 I(9)/\tau^{37} I(9)$	1	$S_1[2] \oplus S_3[1]/S_1[7] \oplus S_3[6] \oplus S_5[5]$	1/1
	$\tau^9 I(9)/\tau^{39} I(9)$	1	$S_1[2] \oplus S_3[1] \oplus S_4[1]/S_1[7] \oplus S_3[6] \oplus S_4[6]$	1/1
	$\tau^{11} I(9)/\tau^{41} I(9)$	1	$S_1[2] \oplus S_2[2] \oplus S_4[1]/S_1[7] \oplus S_2[7] \oplus S_4[6]$	1/1
	$\tau^{13} I(9)/\tau^{43} I(9)$	1	$S_2[2] \oplus S_4[1] \oplus S_5[3]/S_2[7] \oplus S_4[6] \oplus S_5[8]$	1/1
	$\tau^{15} I(9)/\tau^{45} I(9)$	1	$S_2[2] \oplus S_3[2] \oplus S_5[3]/S_2[7] \oplus S_3[7] \oplus S_5[8]$	1/1
	$\tau^{17} I(9)/\tau^{47} I(9)$	1	$S_1[3] \oplus S_3[2] \oplus S_5[3]/S_1[8] \oplus S_3[7] \oplus S_5[8]$	1/1
	$\tau^{19} I(9)/\tau^{49} I(9)$	1	$S_1[3] \oplus S_3[2] \oplus S_4[4]/S_1[8] \oplus S_3[7] \oplus S_4[9]$	1/1
	$\tau^{21} I(9)/\tau^{51} I(9)$	1	$S_1[3] \oplus S_2[3] \oplus S_4[4]/S_1[8] \oplus S_2[8] \oplus S_4[9]$	1/1
	$\tau^{23} I(9)/\tau^{53} I(9)$	1	$S_2[3] \oplus S_4[4] \oplus S_5[4]/S_2[8] \oplus S_4[9] \oplus S_5[9]$	1/1
	$\tau^{25} I(9)/\tau^{55} I(9)$	1	$S_2[3] \oplus S_3[5] \oplus S_5[4]/S_2[8] \oplus S_3[10] \oplus S_5[9]$	1/1
	$\tau^{27} I(9)/\tau^{57} I(9)$	1	$S_1[4] \oplus S_3[5] \oplus S_5[4]/S_1[9] \oplus S_3[10] \oplus S_5[9]$	1/1
	$\tau^{29} I(9)/\tau^{59} I(9)$	1	$S_1[4] \oplus S_3[5] \oplus S_4[5]/S_1[9] \oplus S_3[10] \oplus S_4[10]$	1/1
$P(7)$	$\tau^{32} I(7)$	1	$S_1[7] \oplus S_5[4]$	1
	$\tau^4 I(7)/\tau^{34} I(7)$	2	$S'_1[1]/S'_1[4] \oplus S'_3[3]$	1/1
	$\tau^5 I(7)/\tau^{35} I(7)$	1	$S_2[1] \oplus S_3[1]/S_2[6] \oplus S_3[6]$	1/1
	$\tau^8 I(7)/\tau^{38} I(7)$	1	$S_5[3]/S_4[5] \oplus S_5[8]$	1/1
	$\tau^9 I(7)/\tau^{39} I(7)$	2	$S'_1[1] \oplus S'_2[1]/S'_1[4] \oplus S'_2[4]$	1/1
	$\tau^{11} I(7)/\tau^{41} I(7)$	1	$S_1[2] \oplus S_2[2]/S_1[7] \oplus S_2[7]$	1/1
	$\tau^{14} I(7)/\tau^{44} I(7)$	1	$S_3[1] \oplus S_4[4]/S_3[6] \oplus S_4[9]$	1/1
	$\tau^{14} I(7)/\tau^{44} I(7)$	2	$S'_2[1] \oplus S'_3[2]/S'_2[4] \oplus S'_3[5]$	1/1
	$\tau^{14} I(7)/\tau^{44} I(7)$	3	$S''_1[1] \oplus S''_2[1]/S''_1[3] \oplus S''_2[3]$	1/1
	$\tau^{17} I(7)/\tau^{47} I(7)$	1	$S_1[3] \oplus S_5[3]/S_1[8] \oplus S_5[8]$	1/1
	$\tau^{19} I(7)/\tau^{49} I(7)$	2	$S'_1[2] \oplus S'_3[2]/S'_1[5] \oplus S'_3[5]$	1/1
	$\tau^{20} I(7)/\tau^{50} I(7)$	1	$S_2[2] \oplus S_3[5]/S_2[7] \oplus S_3[10]$	1/1
	$\tau^{23} I(7)/\tau^{53} I(7)$	1	$S_4[4] \oplus S_5[4]/S_4[9] \oplus S_5[9]$	1/1
	$\tau^{24} I(7)/\tau^{54} I(7)$	2	$S'_1[2] \oplus S'_2[3]/S'_1[5] \oplus S'_2[6]$	1/1
	$\tau^{26} I(7)/\tau^{56} I(7)$	1	$S_1[3] \oplus S_2[6]/S_1[8] \oplus S_2[11]$	1/1
	$\tau^{29} I(7)/\tau^{59} I(7)$	1	$S_3[5] \oplus S_4[5]/S_3[10] \oplus S_4[10]$	1/1

$\tilde{E}_8$				
U	V	k	M_k	c
	$\tau^{29}I(7)/\tau^{59}I(7)$	2	$S'_2[3] \oplus S'_3[3]/S'_2[6] \oplus S'_3[6]$	1/1
	$\tau^{29}I(7)/\tau^{59}I(7)$	3	$S''_1[2] \oplus S''_2[2]/S''_1[4] \oplus S''_2[4]$	1/1
	$\tau^{32}I(1)$	1	$S_2[6] \oplus S_5[4]$	1
	$\tau^{5}I(1)/\tau^{35}I(1)$	1	$S_2[1]/S_2[6] \oplus S_4[5]$	1/1
	$\tau^{8}I(1)/\tau^{38}I(1)$	1	$S_1[2]/S_1[7] \oplus S_4[5]$	1/1
	$\tau^{11}I(1)/\tau^{41}I(1)$	1	$S_1[2] \oplus S_3[1]/S_1[7] \oplus S_3[6]$	1/1
	$\tau^{14}I(1)/\tau^{44}I(1)$	1	$S_3[1] \oplus S_5[3]/S_3[6] \oplus S_5[8]$	1/1
	$\tau^{17}I(1)/\tau^{47}I(1)$	1	$S_2[2] \oplus S_5[3]/S_2[7] \oplus S_5[8]$	1/1
	$\tau^{20}I(1)/\tau^{50}I(1)$	1	$S_2[2] \oplus S_4[4]/S_2[7] \oplus S_4[9]$	1/1
	$\tau^{23}I(1)/\tau^{53}I(1)$	1	$S_1[3] \oplus S_4[4]/S_1[8] \oplus S_4[9]$	1/1
	$\tau^{26}I(1)/\tau^{56}I(1)$	1	$S_1[3] \oplus S_3[5]/S_1[8] \oplus S_3[10]$	1/1
	$\tau^{29}I(1)/\tau^{59}I(1)$	1	$S_3[5] \oplus S_5[4]/S_3[10] \oplus S_5[9]$	1/1
$P(9)$	$\tau^{1}I(9)/\tau^{31}I(9)$	1	$S_4[1]/S_1[5] \oplus S_3[5] \oplus S_4[6]$	1/1
	$\tau^{3}I(9)/\tau^{33}I(9)$	1	$S_2[1] \oplus S_4[1]/S_1[5] \oplus S_2[6] \oplus S_4[6]$	1/1
	$\tau^{5}I(9)/\tau^{35}I(9)$	1	$S_2[1] \oplus S_4[1] \oplus S_5[1]/S_2[6] \oplus S_4[6] \oplus S_5[6]$	1/1
	$\tau^{7}I(9)/\tau^{37}I(9)$	1	$S_2[1] \oplus S_3[2] \oplus S_5[1]/S_2[6] \oplus S_3[7] \oplus S_5[6]$	1/1
	$\tau^{9}I(9)/\tau^{39}I(9)$	1	$S_1[2] \oplus S_3[2] \oplus S_5[1]/S_1[7] \oplus S_3[7] \oplus S_5[6]$	1/1
	$\tau^{9}I(9)/\tau^{39}I(9)$	2	$S'_1[1] \oplus S'_2[1] \oplus S'_3[1]/S'_1[4] \oplus S'_2[4] \oplus S'_3[4]$	1/1
	$\tau^{11}I(9)/\tau^{41}I(9)$	1	$S_1[2] \oplus S_3[2] \oplus S_4[2]/S_1[7] \oplus S_3[7] \oplus S_4[7]$	1/1
	$\tau^{13}I(9)/\tau^{43}I(9)$	1	$S_1[2] \oplus S_2[3] \oplus S_4[2]/S_1[7] \oplus S_2[8] \oplus S_4[7]$	1/1
	$\tau^{14}I(9)/\tau^{44}I(9)$	3	$S''_1[1] \oplus S''_1[1] \oplus S''_2[1]/S''_1[3] \oplus S''_1[3] \oplus S''_2[3]$	1/1
	$\tau^{15}I(9)/\tau^{45}I(9)$	1	$S_2[3] \oplus S_4[2] \oplus S_5[3]/S_2[8] \oplus S_4[7] \oplus S_5[8]$	1/1
	$\tau^{17}I(9)/\tau^{47}I(9)$	1	$S_2[3] \oplus S_3[3] \oplus S_5[3]/S_2[8] \oplus S_3[8] \oplus S_5[8]$	1/1
	$\tau^{19}I(9)/\tau^{49}I(9)$	1	$S_1[4] \oplus S_3[3] \oplus S_5[3]/S_1[9] \oplus S_3[8] \oplus S_5[8]$	1/1
	$\tau^{19}I(9)/\tau^{49}I(9)$	2	$S'_1[2] \oplus S'_2[2] \oplus S'_3[2]/S'_1[5] \oplus S'_2[5] \oplus S'_3[5]$	1/1
	$\tau^{21}I(9)/\tau^{51}I(9)$	1	$S_1[4] \oplus S_3[3] \oplus S_4[4]/S_1[9] \oplus S_3[8] \oplus S_4[9]$	1/1
	$\tau^{23}I(9)/\tau^{53}I(9)$	1	$S_1[4] \oplus S_2[4] \oplus S_4[4]/S_1[9] \oplus S_2[9] \oplus S_4[9]$	1/1
	$\tau^{25}I(9)/\tau^{55}I(9)$	1	$S_2[4] \oplus S_4[4] \oplus S_5[5]/S_2[9] \oplus S_4[9] \oplus S_5[10]$	1/1
	$\tau^{27}I(9)/\tau^{57}I(9)$	1	$S_2[4] \oplus S_3[5] \oplus S_5[5]/S_2[9] \oplus S_3[10] \oplus S_5[10]$	1/1
	$\tau^{29}I(9)/\tau^{59}I(9)$	1	$S_1[5] \oplus S_3[5] \oplus S_5[5]/S_1[10] \oplus S_3[10] \oplus S_5[10]$	1/1

$\tilde{E}_8$				
U	V	k	M_k	c
	$\tau^{29}I(9)/\tau^{59}I(9)$	2	$S_1'[3] \oplus S_2'[3] \oplus S_3'[3]/S_1'[6] \oplus S_2'[6] \oplus S_3'[6]$	1/1
	$\tau^{29}I(9)/\tau^{59}I(9)$	3	$S_1''[2] \oplus S_2''[2] \oplus S_2''[2]/S_1''[4] \oplus S_2''[4] \oplus S_2''[4]$	1/1
	$\tau^{31}I(6)$	1	$S_2[4] \oplus S_3[5] \oplus S_4[6]$	1
	$\tau^{3}I(6)/\tau^{33}I(6)$	1	$S_2[1]/S_1[5] \oplus S_2[6] \oplus S_5[5]$	1/1
	$\tau^{5}I(6)/\tau^{35}I(6)$	1	$S_4[1] \oplus S_5[1]/S_3[5] \oplus S_4[6] \oplus S_5[6]$	1/1
	$\tau^{7}I(6)/\tau^{37}I(6)$	1	$S_2[1] \oplus S_3[2]/S_1[5] \oplus S_2[6] \oplus S_3[7]$	1/1
	$\tau^{9}I(6)/\tau^{39}I(6)$	1	$S_1[2] \oplus S_4[1] \oplus S_5[1]/S_1[7] \oplus S_4[6] \oplus S_5[6]$	1/1
	$\tau^{11}I(6)/\tau^{41}I(6)$	1	$S_2[1] \oplus S_3[2] \oplus S_4[2]/S_2[6] \oplus S_3[7] \oplus S_4[7]$	1/1
	$\tau^{13}I(6)/\tau^{43}I(6)$	1	$S_1[2] \oplus S_2[3] \oplus S_5[1]/S_1[7] \oplus S_2[8] \oplus S_5[6]$	1/1
	$\tau^{15}I(6)/\tau^{45}I(6)$	1	$S_3[2] \oplus S_4[2] \oplus S_5[3]/S_3[7] \oplus S_4[7] \oplus S_5[8]$	1/1
	$\tau^{17}I(6)/\tau^{47}I(6)$	1	$S_1[2] \oplus S_2[3] \oplus S_3[3]/S_1[7] \oplus S_2[8] \oplus S_3[8]$	1/1
	$\tau^{19}I(6)/\tau^{49}I(6)$	1	$S_1[4] \oplus S_4[2] \oplus S_5[3]/S_1[9] \oplus S_4[7] \oplus S_5[8]$	1/1
	$\tau^{21}I(6)/\tau^{51}I(6)$	1	$S_2[3] \oplus S_3[3] \oplus S_4[4]/S_2[8] \oplus S_3[8] \oplus S_4[9]$	1/1
	$\tau^{23}I(6)/\tau^{53}I(6)$	1	$S_1[4] \oplus S_2[4] \oplus S_5[3]/S_1[9] \oplus S_2[9] \oplus S_5[8]$	1/1
	$\tau^{25}I(6)/\tau^{55}I(6)$	1	$S_3[3] \oplus S_4[4] \oplus S_5[5]/S_3[8] \oplus S_4[9] \oplus S_5[10]$	1/1
	$\tau^{27}I(6)/\tau^{57}I(6)$	1	$S_1[4] \oplus S_2[4] \oplus S_3[5]/S_1[9] \oplus S_2[9] \oplus S_3[10]$	1/1
	$\tau^{29}I(6)/\tau^{59}I(6)$	1	$S_1[5] \oplus S_4[4] \oplus S_5[5]/S_1[10] \oplus S_4[9] \oplus S_5[10]$	1/1

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