Quadratic rational maps: Dynamical limits, disappearing limbs and ideal limit points of $\mathrm{Per}_n(0)$ curves

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Notation and terminology. As usual $\mathbb{N}$ denotes the set of natural numbers, $\mathbb{R}$ the real line and $\mathbb{C}$ the complex plane. We denote by $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ the Riemann sphere and by $d_{\hat{\mathbb{C}}}$ the spherical metric on $\hat{\mathbb{C}}$. We denote $\hat{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ and identify it with the unit circle. A*domain* is an open connected subset U of $\hat{\mathbb{C}}$. By $B_r(z_0)$ we denote the disc of radius r around z_0 in either the Euclidean or spherical metric depending on the context. We also use $\mathbb{D}$ to denote the open unit disc, i.e. $\mathbb{D} = B_1(0)$ and by $\mathbb{D}^*$ the punctured disc $\mathbb{D} - \{0\}$. A *topological disc* is a domain $U \subset \hat{\mathbb{C}}$ that is homeomorphic to $\mathbb{D}$, i.e. U is a simply connected domain with $U \neq \hat{\mathbb{C}}$. A *dynamical system* is a continuous map $f : X \to X$ on some topological space X. The composition of f with itself n times is called the *n-th iterate* and is denoted by f^n. A set $A \subset X$ is *completely invariant* if $f^{-1}(A) \subset A$. Two dynamical systems $f : X \to X$ and $g : Y \to Y$ are *conjugate* if there exists a homeomorphism $h : X \to Y$ such that $h \circ f = g \circ h$. The map h is called a *conjugacy* and depending on its quality it is called a *topological, quasiconformal* or *holomorphic conjugacy.*

List of Symbols

$\hat{\mathbb{C}}$	The Riemann sphere
$d_{\hat{\mathbb{C}}}$	The spherical metric on $\hat{\mathbb{C}}$
$\hat{\mathbb{R}}$	The circle $\mathbb{R} \cup \{\infty\}$
$\mathbb{D}$	The unit disc
$\mathbb{D}^*$	The punctured unit disc $\mathbb{D} - \{0\}$
$\mathrm{Aut}(\hat{\mathbb{C}})$	The group of complex Möbius transformations
Rat_d	The space of degree d rational maps on $\hat{\mathbb{C}}$
$\mathbb{P}^n$	The complex projective n-space
$\mathcal{M}_2$	The moduli space of quadratic rational maps on $\hat{\mathbb{C}}$
$\widehat{\mathcal{M}}_2$	The algebraic compactification of $\mathcal{M}_2$ isomorphic to $\mathbb{P}^2$
$\widetilde{\mathcal{M}}_2$	The dynamical compactification of $\mathcal{M}_2$
$\langle \mu_1, \mu_2, \mu_3 \rangle$	Point $\langle f \rangle$ in $\mathcal{M}_2$ specified by the multipliers at the fixed points of f
$\langle \mu, \mu^{-1}, \infty \rangle$	Ideal point in $\widehat{\mathcal{M}}_2$, $\mu \in \hat{\mathbb{C}}$
$\mathcal{B}_{p/q}$	Sphere of ideal points in $\widetilde{\mathcal{M}}_2$ replacing the ideal point $\langle e^{2\pi i p/q}, e^{-2\pi i p/q}, \infty \rangle$
G_T	The map $G_T(z) = z + \frac{1}{z} + T$
$\langle G_T \rangle_{p/q}$	Ideal point in $\mathcal{B}_{p/q} \subset \widetilde{\mathcal{M}}_2$
$\mathrm{Per}_n(\eta)$	The algebraic curve in $\mathcal{M}_2$ consisting of all $\langle f \rangle \in \mathcal{M}_2$ having a periodic point of period n with multiplier η
$J(f)$	The Julia set of a rational map $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$
$A_f(z_0)$	Basin of an attracting or parabolic fixed point z_0 of f
$K(f, z_0)$	The filled Julia set $K(f, z_0) = \hat{\mathbb{C}} - A_f(z_0)$
P_c	The quadratic polynomial $P_c(z) = z^2 + c$
M	The Mandelbrot set $M = \{c \in \mathbb{C} : \ J(P_c) \text{ is connected}\}$
$M_{p/q}$	The p/q-limb of the Mandelbrot set

$F_{\lambda,T}$	The map $F_{\lambda,T}(z) = \frac{1}{\lambda}(z + \frac{1}{z} + T)$
$\mathcal{F}_\lambda$	The family $\mathcal{F}_\lambda = \{F_{\lambda,T} :\ T \in \mathbb{C}\}$
M^λ	Connectedness locus in $\mathrm{Per}_1(\lambda) \subset \mathcal{M}_2$: $M^\lambda = \{\langle f\rangle \in \mathrm{Per}_1(\lambda) :\ J(f) \text{ is connected}\}$
$M^\lambda_{\mathbb{R}}$	The real connectedness locus in $\mathrm{Per}_1(\lambda)$: $M^\lambda_{\mathbb{R}} = \{\langle f\rangle \in \mathrm{Per}_1(\lambda) :\ J(f) \text{ connected and } \langle f\rangle \text{ has a real representative}\}$ $M^0_{\mathbb{R}}$ is isomorphic to the real Mandelbrot set $[-2, 1/4]$ by $c \mapsto \langle P_c\rangle$ $M^1_{\mathbb{R}}$ is isomorphic to $[-2, 0]$ by $T \mapsto \langle G_T\rangle$
$R_{\lambda,c}$	The map $R_{\lambda,c}(z) = \frac{1}{\lambda}(z + \frac{1}{z} + T_\lambda(c))$, the unique element in the family $\mathcal{F}_\lambda$ that is hybrid equivalent to P_c on its filled Julia set $K(R_{\lambda,c}, \infty)$, and $-1 \in \mathcal{A}_{R_{\lambda,c}}(\infty)$ ($\lambda \in (-1, 1)$, $\lambda \neq 0$ and $c \in M$)
$R_{1,c}$	The map $R_{1,c}(z) = z + \frac{1}{z} + T_{(c)}$, $c \in [-2, 1/4]$, where $T(c)$ is the unique parameter in $[-2, 0]$ such that $R_{1,c} : [0,\infty] \to [0,\infty]$ is real hybrid equivalent to P_c
$\tau_{p/q}$	The tuning map from M into $M_{p/q}$ associated to the unique hyperbolic center of period q in $M_{p/q}$
$k(f)$	Kneading sequence of a unimodal map $f : [a, b] \to [a, b]$

$z \approx 0$	z is a infinitesimal complex number
$z \approx \infty$	z is an unlimited complex number
$z \approx y$	z is infinitely close to y, i.e. $d_{\hat{\mathbb{C}}}(z, y) \approx 0$
${}^\circ z$	The standard part of $z \in \hat{\mathbb{C}}$, i.e. the unique standard point in $y \in \hat{\mathbb{C}}$ with $y \approx z$
$f \approx g$	$f(z) \approx g(z)$ for all $z \in \hat{\mathbb{C}}$
$f \simeq g$	There exists finitely many standard points $z_1, \cdots z_k \in \hat{\mathbb{C}}$ such that $f(z) \approx g(z)$ for all $z \not\approx 0, \infty$
$\mathrm{sh}(f)$	The shadow of $f \in \mathrm{Rat}_d$ — the unique standard rational map, possibly of lower degree, with $\mathrm{sh}(f)(z) \simeq f(z)$
$\langle f\rangle \approx \langle \mu_1, \mu_2, \mu_3\rangle$	$\langle f\rangle \in \mathcal{M}_2$ having fixed points with multipliers infinitely close to μ_1, μ_2 and μ_3

Chapter 1

Introduction

Let $\mathcal{M}_2 = \mathrm{Rat}_2 \,/\, \mathrm{Aut}(\hat{\mathbb{C}}) \cong \mathbb{C}^2$ denote the moduli space of holomorphic conjugacy classes of quadratic rational maps on $\hat{\mathbb{C}}$. Let $\mathrm{Per}_1(\lambda) \cong \mathbb{C}$ denote the slice in $\mathcal{M}_2$ consisting of all conjugacy classes having a fixed point with multiplier λ and let $M^\lambda \subset \mathrm{Per}_1(\lambda)$ denote the set of all $\langle f \rangle \in \mathrm{Per}_1(\lambda)$ whose Julia set $J(f)$ is connected. The slice $\mathrm{Per}_1(0)$ is naturally isomorphic to the quadratic family $P_c(z) = z^2 + c$ with $c \in \mathbb{C}$ and the connectedness locus $M^0 \subset \mathrm{Per}_1(0)$ is naturally isomorphic to the Mandelbrot set $M = \{c \in \mathbb{C} : J(P_c) \text{ is connected}\}$. For all $\lambda \in \mathbb{D}^*$ the connectedness locus M^λ in $\mathrm{Per}_1(\lambda)$ is a homeomorphic copy of M. More precisely, given $\lambda \in \mathbb{D}^*$ and $c \in M$, there exists a unique $\langle R_{\lambda,c} \rangle \in \mathrm{Per}_1(\lambda)$ that has the same dynamical behavior as the quadratic polynomial P_c, but where the superattracting fixed point at ∞ has been replaced by an attracting fixed point with multiplier λ. The map $c \mapsto \langle R_{\lambda,c} \rangle$ maps the Mandelbrot set M homeomorphically onto $M^\lambda \subset \mathrm{Per}_1(\lambda)$. One can think of $R_{\lambda,c}$ as the mating of the quadratic polynomials $P_c(z) = z^2 + c$ and $z \mapsto \lambda z + z^2$.

The Mandelbrot set M consists of the main cardioid together with a collection of limbs $M_{p/q}$, attached to the main cardioid at the point corresponding to the quadratic polynomial with a fixed point of multiplier $e^{2\pi i p/q}$. Petersen proved that for $c \in M_{p/q}$ the map $\langle R_{\lambda,c} \rangle$ tends to infinity in $\mathcal{M}_2$ as λ tends to $e^{-2\pi i p/q}$ radially, i.e. the p/q-limb in M^λ disappears to infinity. Epstein showed that, upon suitable normalization and passing to a subsequence, the q-th iterate $R^q_{\lambda,c}$ converges to a degree two rational map of the form $G_T(z) = z + 1/z + T$ locally uniformly on $\hat{\mathbb{C}} - \{0, \infty\}$.

Our main goal is to determine the limiting map of the q-th iterate $R^q_{\lambda,c}$. How is the limiting map G_T related to c? The answer turns out to depend on whether P^q_c is renormalizable, and leads to a conjectural description of the limbs of the Mandelbrot set that disappear under "illegal mating". We prove this conjecture for the real case, $c \in \mathbb{R}$. We also show that the Julia sets of $R_{\lambda,c}$ converge to the Julia set of the limiting map G_T in the Hausdorff metric on compact subsets of $\hat{\mathbb{C}}$ as λ tends to -1.

As a preliminary consideration, we prove under quite general hypotheses that the quadratic limiting map of the q-th iterate is unique in the sense that it is independent of the choice of normalization of the q-th iterate. This result allows us to give an alternative definition of DeMarco's compactification $\widetilde{\mathcal{M}}_2$ of $\mathcal{M}_2$ in dynamical terms avoiding the geometric invariant theory that appears in the original definition. The compactification

$\widetilde{\mathcal{M}}_2$ records the limiting map G_T of the q-th iterate. The statements formulated above can be reinterpreted as statements about the limiting behavior in the compactification $\widetilde{\mathcal{M}}_2$ of the distorted Mandelbrot sets M^λ as λ tends to $e^{-2\pi ip/q}$. In addition we also determine the ideal limit points of the $\mathrm{Per}_n(0)$ curves in $\widetilde{\mathcal{M}}_2$.

We now give more detailed definitions and state our main conjectures and theorems.

The moduli space $\boldsymbol{\mathcal{M}_2}$. The moduli space $\mathcal{M}_2 = \mathrm{Rat}_2 / \mathrm{Aut}(\hat{\mathbb{C}})$ is isomorphic to $\mathbb{C}^2$. Every conjugacy class in $\mathcal{M}_2$ is determined by the multipliers at its three fixed points. We write the unordered triple $\langle f\rangle = \langle \mu_1, \mu_2, \mu_3\rangle$ for the conjugacy class of the map f having fixed points with multipliers μ_1, μ_2 and μ_3. By the holomorphic fixed point formula the multipliers are subject to the restriction $\mu_1\mu_2\mu_3 - (\mu_1 + \mu_2 + \mu_3) + 2 = 0$. On the other hand, for every triple of complex numbers that satisfies this restriction, there is a quadratic rational map having fixed points with those multipliers and thus every such triple determines an element of $\mathcal{M}_2$. The compactification $\widehat{\mathcal{M}}_2 \cong \mathbb{P}^2$, introduced by Milnor, consists of $\mathcal{M}_2$ together with a sphere of ideal points $\langle \mu, \mu^{-1}, \infty\rangle$, $\mu \in \hat{\mathbb{C}}$, at infinity. A sequence $\langle f_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{M}_2$ converges to an ideal point $\langle \mu, \mu^{-1}, \infty\rangle$ if and only if the multipliers at the three fixed points of f_n tend to μ, μ^{-1} and ∞.

The connectedness locus $\boldsymbol{M^\lambda \subset \mathrm{Per}_1(\lambda) \subset \mathcal{M}_2}$ and the maps $\boldsymbol{R_{\lambda,c}}$. Let $\mathrm{Per}_1(\lambda) \cong \mathbb{C}$ denote the slice in $\mathcal{M}_2$ consisting of all conjugacy classes having a fixed point with multiplier λ and let M^λ denote the connectedness locus in $\mathrm{Per}_1(\lambda)$. For each $|\lambda| < 1$ the connectedness locus $M^\lambda \subset \mathrm{Per}_1(\lambda)$ is a homeomorphic copy of M. The slice $\mathrm{Per}_1(\lambda)$ with $\lambda \neq 0$ is parameterized by the family

$$\mathcal{F}_\lambda = \left\{ F_{\lambda,T}(z) = \frac{1}{\lambda}\left(z + \frac{1}{z} + T\right) \,:\, T \in \mathbb{C} \right\}.$$

The map $F_{\lambda,T}$ has a fixed point at ∞ with multiplier λ and critical points at ± 1. By the theory of polynomial-like maps, for any $c \in M$ and $\lambda \in \mathbb{D}^*$ there exist a unique map

$$R_{\lambda,c}(z) = \frac{1}{\lambda}\left(z + \frac{1}{z} + T_\lambda(c)\right) \in \mathcal{F}_\lambda$$

such that $R_{\lambda,c}$ is hybrid equivalent to P_c on its filled Julia set, the complement of the basin of ∞, and the critical point -1 is in the basin of ∞. The map $c \mapsto \langle R_{\lambda,c}\rangle$ provides a dynamical homeomorphism between M and M^λ.

One can think of $R_{\lambda,c}$ as the mating of the quadratic polynomials P_c and $z \mapsto \lambda z + z^2$ since $R_{\lambda,c}$ is hybrid equivalent to P_c on its filled Julia set and holomorphically conjugate to $z \mapsto \lambda z + z^2$ on the basin of ∞.

It is a well-known conjecture that the connectedness locus $M^1 \subset \mathrm{Per}_1(1)$ is also homeomorphic to the Mandelbrot set M, see e.g. [Mi2, p. 27]:

Conjecture 1 *For any $c \in M$ the limit*

$$R_{1,c} = \lim_{\substack{\lambda\to 1\\ \lambda<1}} R_{\lambda,c}$$

exists in Rat_2 *and the map $M \to M^1 \subset \mathrm{Per}_1(1)$ defined by $c \mapsto \langle R_{1,c}\rangle$ is a homeomorphism.*

The disappearing limbs. In the case that λ tends radially to $e^{-2\pi ip/q}$, $q \geq 2$, Petersen proved that the p/q-limb in M^λ disappears to infinity in $\mathcal{M}_2$, more precisely for c in $M_{p/q}$, the map $R_{\lambda,c}$ tends to the ideal point $\langle e^{2\pi ip/q}, e^{-2\pi ip/q}, \infty\rangle \in \widehat{\mathcal{M}_2}$ [Pe]. This phenomenon is related to the fact that one cannot mate two polynomials corresponding to points in complex conjugate limbs of the Mandelbrot set [Tan].

The dynamical compactification $\widetilde{\mathcal{M}_2}$ of $\mathcal{M}_2$. To get a more detailed picture of the limiting dynamics of $R_{\lambda,c}$ we consider the dynamical compactification $\widetilde{\mathcal{M}_2}$ of $\mathcal{M}_2$, introduced by DeMarco [De2]. In $\widetilde{\mathcal{M}_2}$ the ideal points $\langle e^{2\pi ip/q}, e^{-2\pi ip/q}, \infty\rangle \in \widehat{\mathcal{M}_2}$ with $q \geq 2$ are replaced by spheres of ideal points $\mathcal{B}_{p/q}$ containing as additional information the limiting map of the q-th iterate taking place in the family $G_T(z) = z + \frac{1}{z} + T$ with $T \in \hat{\mathbb{C}}$. A sequence $\langle f_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{M}_2$ converges to an ideal point $\langle G_T\rangle_{p/q} \in \mathcal{B}_{p/q}$ with $T \neq \infty$, if and only if the multipliers at the fixed points of f_n tend to $e^{2\pi ip/q}, e^{-2\pi ip/q}$ and ∞, and f_n can be normalized such that $f_n^q \to G_T$ locally uniformly on $\hat{\mathbb{C}} - \{0, \infty\}$.

The main conjecture. The central goal of this work is to determine the limit of $\langle R_{\lambda,c}\rangle$ as λ tends to a q-th root of unity in the dynamical compactification $\widetilde{\mathcal{M}_2}$. This means we have to determine the limit of the q-th iterate $R^q_{\lambda,c}$. We formulate a precise conjecture and establish it in the real case.

Let $\tau_{p/q} : M \to M_{p/q}$ denote the tuning map corresponding to the unique hyperbolic centerpoint of period q in $M_{p/q}$. The image of tuning $\tau_{p/q}(M)$ is a small copy of M in $M_{p/q}$ attached to the main cardioid. Informally we conjecture that in $\widetilde{\mathcal{M}_2}$ the disappearing p/q-limb converges to the connectedness locus on $\mathcal{B}_{p/q}$, a copy of the connectedness locus $M^1 \subset \mathrm{Per}_1(1)$: The image of tuning $\tau_{p/q}(M)$ maps homeomorphically onto $\{\langle G_T\rangle_{p/q} \in \mathcal{B}_{p/q} :\ J(G_T) \text{ is connected}\} \subset \mathcal{B}_{p/q}$ and the rest of the limb $M_{p/q} - \tau_{p/q}(M)$ is mapped to certain endpoints of the connectedness locus. More precisely:

Conjecture 2 (Main conjecture — Images of disappearing limbs)
For any $c \in M_{p/q}$ the limit

$$L_{p/q}(c) = \lim_{\epsilon\to 0}\langle R_{(1-\epsilon)e^{-2\pi ip/q},c}\rangle$$

exists in the dynamical compactification $\widetilde{\mathcal{M}_2}$ and lies in $\mathcal{B}_{p/q}$. The map

$$L_{p/q} : M_{p/q} \to \mathcal{B}_{p/q} \subset \widetilde{\mathcal{M}_2}$$

has the following properties:

1. *$L_{p/q}$ is continuous.*
2. *$L_{p/q}$ maps $\tau_{p/q}(M)$ homeomorphically onto*

$$\{\langle G_T\rangle_{p/q} \in \mathcal{B}_{p/q} :\ J(G_T) \textit{ is connected}\} \subset \mathcal{B}_{p/q} \subset \widetilde{\mathcal{M}_2}$$

with

$$L_{p/q}(\tau_{p/q}(b)) = \langle R_{1,b}\rangle_{p/q} \in \mathcal{B}_{p/q}.$$

3. *$L_{p/q}$ is locally constant on $M_{p/q} - \tau_{p/q}(M)$.*

Note that Conjecture 2 presumes Conjecture 1, because we assume that the map $R_{1,b}$ exists.

The main theorem.

In this work we prove the above conjectures for the real case. Let $M^1_{\mathbb{R}}$ denote the real connectedness locus in $\mathrm{Per}_1(1)$, i.e. $M^1_{\mathbb{R}} = \{\langle f\rangle \in M^1 : \langle f\rangle \text{ has a real representative}\}$. As a preliminary result we establish a dynamical homeomorphism between the real Mandelbrot set $[-2, 1/4]$ and the real connectedness locus $M^1_{\mathbb{R}}$ according to Conjecture 1:

Theorem 1 *For c in the real Mandelbrot set $[-2, 1/4]$ the limit*

$$R_{1,c} = \lim_{\substack{\lambda\to 1\\ \lambda<1}} R_{\lambda,c}$$

exists in Rat_2. The map $[-2, 1/4] \to M^1_{\mathbb{R}}$ given by $c \mapsto \langle R_{1,c}\rangle$ is a homeomorphism.

Note that $M^1_{\mathbb{R}}$ is naturally isomorphic to $[-2, 0]$, by $T \mapsto \langle G_T\rangle$. The limiting map $R_{1,c}$ is of the form $R_{1,c}(z) = z + \frac{1}{z} + T(c)$ with $T(c) \in [-2, 0]$.

Our main result is a real version of Conjecture 2 :

Theorem 2 (Main theorem — The real case) *For any c in the real $1/2$-limb $[-2, -3/4]$ the limit*

$$L_{1/2}(c) = \lim_{\substack{\lambda\to -1\\ \lambda>-1}} \langle R_{\lambda,c}\rangle$$

exists in the dynamical compactification $\widetilde{\mathcal{M}}_2$ and lies in $\mathcal{B}_{1/2}$. The map

$$L_{1/2} : [-2, -3/4] \to \mathcal{B}_{1/2} \subset \widetilde{\mathcal{M}}_2$$

has the following properties:

1. *$L_{1/2}$ is continuous.*

2. *$L_{1/2}$ maps the image of tuning $\tau_{1/2}([-2, 1/4]) = [-1.54368..., -3/4]$ homeomorphically onto $\{\langle G_T\rangle_{1/2} : T \in [-2, 0]\} \subset \mathcal{B}_{1/2} \subset \widetilde{\mathcal{M}}_2$:*

$$L_{1/2}(\tau_{1/2}(b)) = \langle R_{1,b}\rangle_{1/2} \in \mathcal{B}_{1/2}$$

   for any b in the real Mandelbrot set $[-2, 1/4]$.

3. *For all c not in the image of tuning, $c \in [-2, -1.54368...]$, we have:*

$$L_{1/2}(c) = \langle R_{1,-2}\rangle_{1/2} = \langle G_{-2}\rangle_{1/2} \in \mathcal{B}_{1/2}.$$

The dynamical homeomorphism between the real Mandelbrot set $[-2, 1/4]$ and the real connectedness locus $M^1_{\mathbb{R}}$ is established in Chapter 16 and the main theorem is proved in Chapter 18.

Steps in the proof and complementary theorems. We now summarize some more results that are obtained in this thesis and sketch the proof of the main theorem.

We start with a theorem about dynamical limits of unbounded sequences in $\mathcal{M}_2$ that strengthens a theorem of Epstein [Eps, Proposition 2]. This result is independent from the main conjecture, it does not concern the specific maps $R_{\lambda,c}$.

Theorem 3 (Uniqueness theorem)
Let $\langle \alpha_n, \beta_n, \gamma_n \rangle_{n\in\mathbb{N}}$ be a sequence in $\mathcal{M}_2$ with $\langle \alpha_n, \beta_n, \gamma_n \rangle \to \langle \mu, \mu^{-1}, \infty \rangle$ where μ is a primitive q-th root of unity with $q \geq 2$. Let $f_n \in \langle \alpha_n, \beta_n, \gamma_n \rangle$ with

$$\lim_{n\to\infty} f_n^q = F$$

the convergence taking place locally uniformly outside a finite set. If $\deg(F) = 2$, then $\lim_{n\to\infty} \frac{\alpha_n^q - 1}{\sqrt{1-\alpha_n\beta_n}} = T$ for some choice of square root and some $T \in \mathbb{C}$, and F is holomorphically conjugate to $G_T(z) = z + \frac{1}{z} + T$. Otherwise $\deg(F) \leq 1$.

Epstein proved that, if $\lim_{n\to\infty} \frac{\alpha_n^q - 1}{\sqrt{1-\alpha_n\beta_n}} = T$, then the q-th iterate of one specific normalization converges to G_T.

We prove Theorem 3 in Chapter 10. It turns out that there are essentially q different ways to normalize the sequence of maps in order to get a quadratic limiting map.

The uniqueness theorem allows us to give a new characterization of DeMarco's dynamical compactification $\widetilde{\mathcal{M}}_2$ of $\mathcal{M}_2$ in dynamical terms, where DeMarco uses geometric invariant theory compactifications. We define $\widetilde{\mathcal{M}}_2$ in Chapter 11.

Now we turn to results that are related to the main theorem. An important tool for the proof of Theorem 2 is the theory of S-unimodal maps.

S-unimodal maps. An S-unimodal map is a unimodal maps with negative Schwarzian derivative and a non-degenerate critical point. We also assume that the fixed point on the boundary of the defining interval is not attracting. Two S-unimodal maps are called real hybrid equivalent if they are topologically conjugate and have the same multiplier at their unique attractor, if one exists.

The next result is well-known, see e.g. [LAM, p. 21].

Theorem 4 *Every S-unimodal map is real hybrid equivalent to a unique real quadratic polynomial $P_c(z) = z^2 + c$. Every hybrid class is determined by its kneading sequence and the multiplier at its attractor, in case one exists.*

This theorem is proved in Chapter 14. It follows from well-known but deep results such as the density of hyperbolicity in the real quadratic family.

Real matings. Even though matings of quadratic polynomials have been widely studied, it is problematic to give a definition for the real mating of two quadratic polynomials P_{c_1} and P_{c_2} for arbitrary c_1 and c_2 in the Mandelbrot set. In Chapter 17 we introduce the notion of the mating of two real quadratic polynomials. A conjugacy class $\langle f \rangle \in \mathcal{M}_2$ with a real representative is the mating of two real quadratic polynomials P_{c_1} and P_{c_2} if f can be normalized such that there exist points $a, b \in \hat{\mathbb{R}}$, with $a \neq b$, such that $f : [a,b] \to [a,b]$ is an S-unimodal map that is real hybrid equivalent to P_{c_1}, and $f : [b,a] \to [b,a]$ is an S-unimodal map that is real hybrid equivalent to P_{c_2}.

Examples. Let $P_{c(\lambda)}$ denote the unique quadratic polynomial that has a fixed point with multiplier λ. Let c be in the real Mandelbrot set $[-2, 1/4]$ and $\lambda \in (-1,1)$, $\lambda \neq 0$. Then the map $R_{\lambda,c}(z) = \frac{1}{\lambda}\left(z + \frac{1}{z} + T_\lambda(c)\right)$ is a real rational map that has precisely two S-unimodal restrictions, one real hybrid equivalent to P_c and the other real hybrid equivalent to $P_{c(\lambda)}$. The map $R_{\lambda,c}$ is the real mating of P_c and $P_{c(\lambda)}$.

The map $R_{1,c}(z) = z + \frac{1}{z} + T(c)$ with $c \in [-2, 1/4]$, given by Theorem 1, is the unique map in the family $\{G_T(z) = z + \frac{1}{z} + T : T \in [-2, 0]\}$, such that the S-unimodal restriction $R_{1,c} : [0,\infty] \to [0,\infty]$ is real hybrid equivalent to P_c. Furthermore if $T \in [-2,0]$, then the S-unimodal restrictions $G_T : [\infty, 0] \to [\infty, 0]$ is real hybrid equivalent to $P_{1/4}$. Thus $R_{1,c}$ is the real mating of P_c and $P_{1/4}$.

Theorem 5 (Existence and uniqueness of real matings) *For any c_1 and c_2 in the real Mandelbrot set $[-2, 1/4]$ there exists a real quadratic rational map that is the mating of P_{c_1} and P_{c_2} if and only if c_1 and c_2 are not both in the real 1/2-limb $[-2, -3/4]$. Furthermore this quadratic rational map, if it exists, is unique up to conjugation by real Möbius transformations.*

The maps in $\mathcal{M}_2$ that can be obtained by real mating are therefore:

$$\{\langle f \rangle \in \mathcal{M}_2 : \ \exists\, c \in [-2, 3/4] \text{ such that } f = P_c \text{ or } f = R_{\lambda,c} \text{ for some } \lambda \in (-1,1] - \{0\}\}.$$

Sketch of the proof of Theorem 2. We can now sketch the proof of the main theorem. Let c be in the real 1/2-limb $[-2, -3/4]$ and $\lambda \in (-1, 0)$. Then the map $R_{\lambda,c}(z) = \frac{1}{\lambda}\left(z + \frac{1}{z} + T_\lambda(c)\right)$ has three real fixed points: The attracting fixed point ∞ with multiplier λ and two further fixed points α and β, corresponding to the α- and β-fixed point of P_c. Let $I_\beta = [\beta, 1/\beta]$ and $I_\alpha = [1/\alpha, \alpha]$. We have $I_\alpha \subset I_\beta$. The critical points of $R_{\lambda,c}$ are at ± 1 and the critical point -1 is in the basin of ∞. The map $R_{\lambda,c}$ is the real mating of P_c and $P_{c(\lambda)}$: The S-unimodal restriction $R_{\lambda,c} : I_\beta \to I_\beta$ is real hybrid equivalent to P_c and the S-unimodal restriction to the complementary interval $I_\beta^c = [1/\beta, \beta]$, $R_{\lambda,c} : I_\beta^c \to I_\beta^c$ is real hybrid equivalent to $P_{c(\lambda)}$. Now let $\lambda \approx -1$. Then one can show that $0 \approx \beta \approx 1/\alpha$ and $\alpha \approx 1/\beta \approx \infty$. Furthermore we have that $R_{\lambda,c}^2$ is close to G_T on I_α for some T. If c is in the image of tuning, i.e. P_c^2 is renormalizable hybrid equivalent to P_b for some $b \in [-2, 1/4]$, then the critical point 1 stays in the interval I_α under iteration of $R^2_{\lambda,c}$ and $R^2_{\lambda,c} : I_\alpha \to I_\alpha$ is real hybrid equivalent to P_b. Since $R^2_{\lambda,c}$ is close to G_T on I_α one can show that $G_T : [0,\infty] \to [0,\infty]$ inherits the kneading sequence of $R^2_{\lambda,c}$ and, in case there is one, the multiplier at the attractor of $R^2_{\lambda,c}$, which agree with the kneading sequence and the multiplier at the attractor of P_b. By Theorem 4 this implies that the map $G_T : [0,\infty] \to [0,\infty]$ is real hybrid equivalent to P_b. Thus by Theorem 5 the map $G_T : [0,\infty] \to [0,\infty]$ must be $R_{1,b}$, the real mating of P_b and $P_{1/4}$. If on the other hand c is not in the image of tuning, i.e. P_c^2 is not renormalizable, then the critical point 1 does not stay in the interval I_α under $R^2_{\lambda,c}$, i.e. we have $R^2_{\lambda,c}(1) \in [\beta, 1/\alpha]$. Since $\beta, 1/\alpha \approx 0$ this implies that $G_T(1) \approx R^2_{\lambda,c}(1) \approx 0$ which leads to $T = -2$. q.e.d.

We also prove in Chapter 19 that the Julia sets of $R_{\lambda,c}$ converge to the Julia set of the limiting map G_T as λ tends to -1:

Theorem 6 (Convergence of Julia sets) *Given $c \in [-2, -3/4)$ we have*

$$\lim_{\substack{\lambda\to -1\\ \lambda > -1}} J(R_{\lambda,c}) = J(G_T)$$

in the Hausdorff metric on compact subsets of $\hat{\mathbb{C}}$, where

$$\lim_{\substack{\lambda\to -1\\ \lambda > -1}} \langle R_{\lambda,c}\rangle = \langle G_T\rangle_{1/2} \in \mathcal{B}_{1/2}$$

and $T < 0$.

In Chapter 20 we determine the ideal limit points of the $\mathrm{Per}_n(0)$ curves, consisting of all conjugacy classes having a superattracting period n cycle, in $\widetilde{\mathcal{M}}_2$:

Theorem 7 (Ideal limit points of $\mathrm{Per}_n(0)$ on $\mathcal{B}_{p/q}$) *The ideal point $\langle G_T\rangle_{p/q} \in \mathcal{B}_{p/q}$ is a limit point of $\mathrm{Per}_n(0)$ if and only if $n \geq q \geq 2$ and:*

1. *$G_T^m(1) = 0$ for some $1 \leq m < n/q$ or*
2. *q divides n and G_T has a superattracting periodic cycle of period n/q.*

Note that 1 is a critical point and 0 the preimage of the parabolic fixed point ∞ for G_T.

Nonstandard analysis. We use the language of nonstandard analysis to investigate the dynamical behavior of the considered unbounded sequences in $\mathcal{M}_2$. It seems advantageous instead of considering unbounded sequences of rational maps and analyzing their limiting dynamical behavior, to just consider one map infinitely close to infinity in $\mathcal{M}_2$ and analyze the dynamics of this one specific map - even though it is just a different way of talking about the same thing. Nevertheless we give all our main theorems also in terms of classical mathematics. Any classical theorem that is proved using nonstandard methods can also be proved without them.

Nonstandard analysis is in fact quite useful for the proof of the Uniqueness Theorem. There instead of considering unbounded sequences of rational maps in $\mathcal{M}_2$, conjugating them by unbounded sequences of Möbius transformations, determine the limiting map of higher iterates, we just need to consider one map close to the boundary, conjugate it by one Möbius transformation close to infinity and determine the shadow of higher iterates of this one specific map.

Nonstandard analysis makes rigorous sense of commonly used informal arguments. The classically-minded reader can think roughly speaking as follows: A standard number x, or more generally a standard set x, is just a fixed parameter. Everything that follows is allowed to depend on x, even if the dependence is not explicit. A nonstandard natural number N, or more generally an unlimited number, can be thought of roughly speaking as a sequence of numbers tending to infinity. Similarly one can think of an infinitesimal number as a sequence of numbers tending to 0. More generally a nonstandard number y close to a standard number x can be thought of roughly speaking as a sequence of numbers tending to x.

Remarks and References.

1. The moduli space $\mathcal{M}_2 = \mathrm{Rat}_2 / \mathrm{Aut}(\hat{\mathbb{C}}) \cong \mathbb{C}^2$ and its compactification $\widehat{\mathcal{M}}_2 \cong \mathbb{P}^2$ are discussed in [Mi2].

2. The dynamical compactification $\widetilde{\mathcal{M}}_2$ is introduced in [De2] using geometric invariant theory compactifications.

3. Epstein and Petersen are working independently on questions similar to Conjecture 2. See [EP].

4. Progress on Conjecture 1 by Petersen and Roesch will appear in [PR].

5. The theory of polynomial-like maps is introduced in [DH].

6. The homeomorphism between M and M^λ is also discussed in [Mi2], [GK], [Pe] and [Uh]. The theorem about the disappearing limbs is proved in [Pe].

7. For the theory of renormalization and tuning see [Mc1], [Mi1] and [Ha].

8. Matings of complex quadratic polynomials are discussed in [Tan]. There it is shown that the mating of two critically finite quadratic polynomials P_{c_1} and P_{c_2} exists if and only if c_1 and c_2 do not belong to complex conjugate limbs of the Mandelbrot set M. It is conjectured that the mating of two quadratic polynomials P_{c_1} and P_{c_2} can be well defined as an element of $\mathcal{M}_2$ if and only if c_1 and c_2 do not belong to complex conjugate limbs of M, see [Mi2, Section 7].

9. The limit $L_{1/2}(c)$ in Theorem 2 also exists for $c \in (-3/4, 1/4]$. In that case P_c has a non-repelling fixed point with multiplier $\mu \in (-1, 1]$. And we have

$$L_{1/2}(c) = \lim_{\substack{\lambda \to -1 \\ \lambda > -1}} \langle R_{\lambda,c} \rangle = \langle -1, \mu, \frac{3-\mu}{1+\mu} \rangle$$

which is the mating of $P_{-3/4}$ and $z \mapsto \mu z + z^2$.

10. The only if part of Theorem 7 is due to Epstein. It is a special case of [Eps, Proposition 3]. Our contribution is to show that all these ideal points $\langle G_T \rangle_{p/q} \in \mathcal{B}_{p/q}$ with the above properties actually occur as limit points of $\mathrm{Per}_n(0)$.

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Chapter 2

Nonstandard analysis

This chapter provides a short introduction to nonstandard analysis. We use an axiomatic approach to nonstandard analysis, called *bounded set theory*, see [KR]. It is a modification of Nelson's *internal set theory* [Ne]. Mathematics can be formalized using set theory and bounded set theory is an extension of ordinary set theory that allows us to rigorously introduce infinitesimals. We first give the axioms and discuss some general principles of nonstandard analysis. Then we formulate several topological notions using the language of nonstandard analysis. For more detailed expositions and applications of nonstandard analysis see also [Rob] [Ro] and [Di].

Bounded set theory (BST). BST is an extension of ZFC (Zermelo-Fraenkel set theory with the axiom of choice) obtained by adding to the usual binary predicate $\in$ of set theory the unary predicate *standard*, to distinguish between standard and nonstandard sets, and to the axioms of ZFC four new axioms – Boundedness, Bounded Idealization, Standardization and Transfer – that manage the handling of the new predicate "standard".

A formula is called *internal* if it does not contain the predicate "standard", otherwise it is called *external*. The ZFC axioms only refer to internal formulas. Here are the new axioms:

1. **Axiom of Boundedness.** Every set is an element of a standard set.

$$\forall x\, \exists^{st} X\, x \in X$$

2. **Axiom of Bounded Idealization.** Let Y be a standard set and $\phi(x, y)$ an internal formula with arbitrary parameters. Then, for all standard finite subsets X of Y there is a y such that $\phi(x, y)$ holds for all $x \in X$ if and only if there is a y such that $\phi(x, y)$ holds for all standard $x \in Y$.

$$\forall^{st\, fin} X \subset Y\, \exists y\, \forall x \in X\, \phi(x, y) \;\Leftrightarrow\; \exists y\, \forall^{st} x \in Y\, \phi(x, y)$$

3. **Axiom of Standardization.** Let X be a standard set and $\phi(x)$ be an internal or external formula with arbitrary parameters. Then there exists a standard set Y such that the standard elements of Y are precisely those standard $x \in X$ that satisfy $\phi(x)$.

$$\forall^{st} X\, \exists^{st} Y\, \forall^{st} x\, (x \in Y \;\Leftrightarrow\; x \in X \text{ and } \phi(x))$$

4. **Axiom of Transfer.** Let $\phi(x)$ be an internal formula with only standard parameters. If $\phi(x)$ holds for all standard x, then $\phi(x)$ holds for all x.

$$\forall^{st} x\, \phi(x) \;\Rightarrow\; \forall x\, \phi(x)$$

The axiom of transfer implies that every set defined in terms of traditional mathematics, i.e. defined without using the new predicate "standard", explicitly or implicitly, are standard sets.

By the axiom of idealization a set is standard and finite if and only if all its elements are standard. This implies that every infinite standard set, such as the set of natural numbers $\mathbb{N}$, contains nonstandard elements.

The axioms of ZFC apply to all sets, standard and nonstandard, but they only refer to internal formulas. If ϕ is an internal formula and X a set, then $\{x \in X : \phi(x)\}$ is also a set by the axiom of separation. But if ϕ is an external formula, then $\{x \in X : \phi(x)\}$ is usually not a set, e.g. $\{n \in \mathbb{N} : n \text{ is standard}\}$ is not a set.

However if ϕ is an external formula and X a set we will call $\{x \in X : \phi(x)\}$ an *external set*. An external set can be regarded as a shorthand for the formula itself. If X is a set we denote by ${}^{\sigma}X$ the external set of all its standard elements, i.e. ${}^{\sigma}X = \{x \in X : x \text{ is standard}\}$. By convention all the considered mathematical objects (subsets, sequences, functions, etc.) will always be internal, unless explicitly stated otherwise.

The axiom of standardization gives for any standard set X and any external formula $\phi(x)$ a standard set, that contains precisely those standard elements $x \in X$ with $\phi(x)$. This standard set is by the axiom of transfer unique and we will denote it by ${}^{st}\{x \in X : \phi(x)\}$. Note that the nonstandard elements in ${}^{st}\{x \in X : \phi(x)\}$ do not have to satisfy $\phi(x)$.

All theorems of classical mathematics, i.e. theorems that do not contain the predicate standard, explicitly or implicitly, remain valid in BST. And all internal theorems that are proved using BST can also be proved in terms of traditional mathematics.

When we prove an internal theorem using nonstandard analysis we can, by applying the axiom of transfer, restrict to standard objects.

Ordinary induction does not hold for external formulas, e.g. 0 is standard and if n is standard then $n+1$ is standard, but this does not imply that all natural numbers are standard. But if $P(n)$ is an external property we can by apply ordinary induction to the set ${}^{st}\{n \in \mathbb{N} : P(n)\}$ and get the following external induction principle that also holds for external properties.

Theorem 2.1 (External induction) *Let $P(n)$ be any property, internal or external. If $P(0)$ is true and if $P(n)$ implies $P(n+1)$ for all standard $n \in \mathbb{N}$, then $P(n)$ holds for all standard $n \in \mathbb{N}$.*

Since $\{n \in \mathbb{N} : n \text{ is standard}\}$ is not a set we have the following principle named overspill.

Theorem 2.2 (Overspill) *Let $P(n)$ be an internal property. If $P(n)$ holds for all standard $n \in \mathbb{N}$ then there exists a nonstandard $N \in \mathbb{N}$, such that $P(N)$ holds.*

Infinitesimal and unlimited numbers. A complex number $x \in \mathbb{C}$ is *unlimited* if $|x| > n$ for all standard n, otherwise it is called *limited*. We will use the notation $x \approx \infty$ to indicate that x is unlimited. A number $x \in \mathbb{C}$ is called *infinitesimal*, $x \approx 0$, if $|x| < 1/n$ for all standard $n \in \mathbb{N}$. The existence of such numbers follows from the idealization axiom. Two complex numbers $x, y \in \mathbb{C}$ are *(infinitely) close* to each other if $|x - y| \approx 0$. If x is limited then there exists a unique standard complex number ${}^\circ x$ with ${}^\circ x \approx x$. We refer to ${}^\circ x$ as the *standard part* of x. We also define ${}^\circ x = \infty$ for all unlimited x.

If $\{a_n\}_{n \in \mathbb{N}}$ is a sequence with $a_n \approx 0$ for all standard $n \in \mathbb{N}$ we can apply overspill to the formula $|a_n| < 1/n$ which leads to the following result.

Theorem 2.3 (Robinson's lemma) *Let $\{a_n\}_{n \in \mathbb{N}}$ be a sequence of complex numbers. If $a_n \approx 0$ for all standard $n \in \mathbb{N}$ then there exists a nonstandard $N \in \mathbb{N}$ such that $a_n \approx 0$ for all $n \leq N$.*

Topological notions. Let $(X, \mathcal{O})$ be a standard topological space, i.e. $(X, \mathcal{O})$ is a topological space with the property that the set X and the set $\mathcal{O}$ of all open sets are standard. For all standard $x \in X$ we define the *(topological) halo* of x to be the external set

$$\mathrm{hal}(x) = \bigcap_{\substack{U \in {}^\sigma\mathcal{O} \\ x \in U}} U$$

where ${}^\sigma\mathcal{O}$ denotes the external set of all standard open sets.

We define for $x \in X$ standard and $y \in X$:

$$y \approx x \text{ if and only if } y \in \mathrm{hal}(x).$$

Metric spaces. Let (X, d) be a standard metric space then the topological halo of a standard point $x \in X$ is given by

$$\mathrm{hal}(x) = \{y \in X : \ d(x, y) \approx 0\}.$$

Thus we have $y \approx x$ if and only if $d(x, y) \approx 0$.

Examples:

1. On $\mathbb{C}$ we have
$$\mathrm{hal}(x) = \{y \in \mathbb{C} : \ |x - y| \approx 0\}$$
for every standard $x \in \mathbb{C}$.

2. For $\hat{\mathbb{C}}$ with the topology induced by the spherical metric we have
$$\mathrm{hal}(x) = \{y \in \hat{\mathbb{C}} : \ d_{\hat{\mathbb{C}}}(x, y) \approx 0\}.$$
Note that $\mathrm{hal}(x) = \{y \in \mathbb{C} : \ |x - y| \approx 0\}$ for standard $x \in \mathbb{C}$, and $\mathrm{hal}(\infty) = \{y \in \hat{\mathbb{C}} : y \approx \infty\}$.

Quotient spaces. Let X be a topological space and let $\sim$ be an equivalence relation on X. Let $[x]$ denote the equivalence class of x and $p : X \to X/_\sim$ the projection map from X onto $X/_\sim$ given by $x \mapsto [x]$. Then a set $U \subset X/_\sim$ is open in the quotient topology if $p^{-1}(U) = \bigcup_{[x]\in U}[x]$ is open in X.

Lemma 2.4 *Let X be a standard topological space and let $\sim$ be a standard equivalence relation on X. Let $[x] \in X/_\sim$ be standard. If the projection map $p : X \to X/_\sim$ is open, i.e. the image of an open set is open, then the following properties are equivalent:*

1. $[y] \approx [x]$.
2. *For all standard $x' \in [x]$ there exists a $y' \in [y]$ with $y' \approx x'$.*
3. *There exists a standard $x' \in [x]$ and a $y' \in [y]$ with $y' \approx x'$.*

Proof:

$1 \Rightarrow 2$. Let $x' \in [x]$ be standard. Assume that $y' \not\approx x'$ for all $y' \in [y]$. This means that there is a standard open neighborhoods V of x' with $y' \notin V$. By the axiom of idealization this implies that there exists a standard finite set of open neighborhood of x' such that for all $y' \in [y]$ we have $y' \notin V$ for some neighborhood in that standard finite set. Let $\widetilde{V}$ be the intersection of theses standard finite many neighborhoods. Then we have $y' \notin \widetilde{V}$ for all $y' \in [y]$. Since the projection is open, $p(\widetilde{V})$ is an open, and thus an open set containing $[x]$ but not $[y]$ which implies that $[y] \not\approx [x]$.

$2 \Rightarrow 3$ is trivial.

$3 \Rightarrow 1$. Assume that there exists a standard $x' \in [x]$ and a $y' \in [y]$ with $y' \approx x'$. Let $U \subset X/_\sim$ be a standard open set containing $[x]$. Then $p^{-1}(U)$ is open and standard. Since $[x] \subset p^{-1}(U)$ we have $y' \in \mathrm{hal}(x') \subset p^{-1}(U)$. This implies that $[y] = p(y') \in p(p^{-1}(U)) = U$.

q.e.d.

Example: Let $\mathbb{P}^n$ be the complex projective n-space — the space of all lines through the origin in $\mathbb{C}^{n+1}$. I.e. $\mathbb{P}^n = (\mathbb{C}^{n+1} - \{0\})/_\sim$ with

$$(x_0, \cdots, x_n) \sim (\widetilde{x}_0, \cdots, \widetilde{x}_n) \text{ if and only if } \exists\lambda \in \mathbb{C} - \{0\}\ (x_0, \cdots, x_n) = (\lambda\widetilde{x}_0, \cdots, \lambda\widetilde{x}_n).$$

The projection map $p : \mathbb{C}^{n+1} - \{0\} \to \mathbb{P}^n$ given by $p(x_0, \cdots, x_n) = [x_0, \cdots, x_n]$ is open.

Let $[x_0, \cdots, x_n] \in \mathbb{P}^n$ is standard and $(x_0, \cdots, x_n)$ a standard representative. By Lemma 2.4 we have that

$$[y_0, \cdots, y_n] \approx [x_0, \cdots, x_n] \text{ if and only if } \exists\lambda \in \mathbb{C} - \{0\}\ (\lambda y_0, \cdots, \lambda y_n) \approx (x_0, \cdots, x_n).$$

A standard topological space is determined by the halos of its standard points. If X is a standard set and $\mathcal{O}_1$ and $\mathcal{O}_2$ are two standard topologies on X with $\mathrm{hal}_{\mathcal{O}_1}(x) = \mathrm{hal}_{\mathcal{O}_2}(x)$ for all standard $x \in X$, then $\mathcal{O}_1 = \mathcal{O}_2$.

Now we give external characterizations of some classical topological notions, i.e. the external notion coincides with the classical notion for standard objects.

Open, closed and compact sets. A standard subset U of a standard topological space X is open if and only if $\mathrm{hal}(x) \subset U$ for all standard $x \in U$. A standard set A is closed if and only if $x \in A$ for all standard $x \in X$ with $\mathrm{hal}(x) \cap A \neq \emptyset$ and a standard set $C \subset X$ is compact if and only if it is covered by the by the halos of its standard points.

Limit points. Let X be a standard topological space and M a standard subset of X. Then a standard point $x \in X$ is a limit point of M if and only if there exists a point $y \in M$ with $y \approx x$ and $y \neq x$.

Convergence of sequences. A standard sequence $\{x_n\}_{n \in \mathbb{N}}$ in a standard topological space X converges to $x \in X$ standard if and only if $x_N \approx x$ for all $N \approx \infty$.

S-continuous functions. Let X and Y be standard topological spaces. A map $f : X \to Y$ is called *s-continuous* if

$$f(\mathrm{hal}_X(x)) \subset \mathrm{hal}_Y(f(x))$$

for all standard $x \in X$. I.e. if $y \approx x$ implies $f(y) \approx f(x)$ for all standard $x \in X$.

It is easy to show that this notion of continuity coincides with the classical notion of continuity for standard maps.

Hausdorff space. A standard topological space X is Hausdorff if and only if $\mathrm{hal}(x) \cap \mathrm{hal}(y) = \emptyset$ for all standard $x, y \in X$ with $x \neq y$.

Note that in a standard compact Hausdorff space X, there exist for every $y \in X$ a unique standard $x \in X$ with $y \approx x$. (Let y be a point in X. Because X is compact there is a standard x such that $y \in \mathrm{hal}(x)$ and because it is Hausdorff x is the only standard point in $\mathrm{hal}(x)$.) We will call this unique standard point associated to y the *standard part* of y (as in the case of $\hat{\mathbb{C}}$ above).

Uniform convergence. A standard sequence of functions $f_n : X \to Y$ from a set X to a metric space Y converges uniformly to the standard function $f : X \to Y$ if and only if $d(f_N(x), f(x)) \approx 0$ for all $N \approx \infty$ and all $x \in X$.

Remarks and References.

1. In [KR] the Axiom of "Bounded Idealization" is called "Inner Saturation".
2. BST is a subtheory of the theory *Hrbacek set theory HST*, also introduced in [KR]. One can consider HST as a completion of BST with respect to the separation axiom. In HST the Separation axiom also holds for external formulas.

Chapter 3

Preliminaries from complex analysis

In this chapter we recall some theorems of complex analysis that we will need eventually. For further background we refer to the book by Ahlfors [Ah].

We first recall Rouché's theorem and prove a nonstandard version. We will use this later, see Lemma 4.1, to show that if a holomorphic map g is close to a standard holomorphic map f having a fixed point, then g must also have a fixed point infinitely close to the one of f.

Theorem 3.1 (Rouché's theorem) *Let $f, g : U \to \mathbb{C}$ be holomorphic functions and γ a curve in U homologous to zero, such that*

$$|f(z) - g(z)| < |f(z)| \text{ on } \gamma$$

Then f and g have the same number of zeros, counted with multiplicity, in the bounded domain enclosed by γ.

Corollary 3.2 *Let $U \subset \mathbb{C}$ be a standard domain and $f : U \to \mathbb{C}$ a non-constant standard holomorphic function, and $g : U \to \mathbb{C}$ a holomorphic function with $f(z) \approx g(z)$ for all $z \in U$. If z_0 is a zero for f with multiplicity m, then g has precisely m zeros infinitely close to z_0, counted with multiplicity.*

Proof: Since f is non-constant and standard, there exist a standard $n \in \mathbb{N}$, such that f has no other zeros except z_0 in $B_{1/n}(z_0)$. Since z_0 is the unique zero of f in $B_{1/n}(z_0)$ we have $|f(z)| > 0$ on the boundary of $B_{1/n}(z_0)$ and because this boundary is a standard compact set, we have $|f(z)| > \delta$ for some standard $\delta > 0$. Therefore and because $f(z) \approx g(z)$ we have

$$|f(z) - g(z)| < \delta < |f(z)|$$

for all z on the boundary of $B_{1/n}(z_0)$. And so Rouché's theorem implies that g has the same number of zeros in $B_{1/n}(z_0)$ as f. This is true for every standard $n \in \mathbb{N}$ and so by overspill also for some $N \approx \infty$. Thus g has precisely m zeros in $\mathrm{hal}(z_0)$ counted with multiplicity.

q.e.d.

By the Riemann mapping theorem every simply connected domain $U \subset \hat{\mathbb{C}}$ whose complement contains more than two points can be mapped onto the unit disc $\mathbb{D}$ by a holomorphic homeomorphism.

Theorem 3.3 (Schwarz lemma) *Let $f : \mathbb{D} \to \mathbb{D}$ be a holomorphic map with $f(0) = 0$. Then $|f'(0)| \leq 1$. If $|f'(0)| = 1$, then f is a rotation around the origin, and if $|f'(0)| < 1$ then $|f(z)| < |z|$ for all $z \in \mathbb{D}$.*

Now we cite the Riemann-Hurwitz formula as it can be found in [Ste]. It will be used to calculate the connectivity number of certain domains. A domain $U \subset \hat{\mathbb{C}}$ is called *n-connected* if its complement consists of n connected components. A 1-connected domain is simply connected.

Theorem 3.4 (The Riemann-Hurwitz formula) *Suppose that f is a proper map of degree d of some n-connected domain $U \subset \hat{\mathbb{C}}$ onto some m-connected domain $V \subset \hat{\mathbb{C}}$, f having exactly r critical points in U, counted with multiplicity. Then*

$$n - 2 = d(m - 2) + r.$$

A holomorphic map $f : U \to V$ is a *proper* if the preimage of every compact set in V is compact. In that case f assumes every value in V exactly d times for some $d \geq 1$, counting multiplicity. Note that if $f \in \mathrm{Rat}_d$ and $U \subset \hat{\mathbb{C}}$ a domain, then the map $f : f^{-1}(U) \to U$ is proper.

Corollary 3.5 *Let f be a quadratic rational map and $V \subset \hat{\mathbb{C}}$ a topological disc. If V contains exactly one critical value, then $f^{-1}(V)$ is a topological disc.*

Proof: The map $f : f^{-1}(V) \to V$ is proper, so we can conclude by the Riemann-Hurwitz formula that $f^{-1}(V)$ is n-connected with $n = 2 + 2(1 - 2) + 1 = 1$, thus a topological disc.

q.e.d.

We conclude this section proving a theorem of Robinson, compare [Rob, Chapter 6], about s-continuous holomorphic functions that we will use in Chapter 20.

Let $U \subset \mathbb{C}$ a standard domain and $f : U \to \mathbb{C}$ a holomorphic function with $f(z)$ limited for all standard $z \in U$. Then we define the *shadow* of f, $\mathrm{sh}(f) : U \to \mathbb{C}$ to be the unique standard function with $\mathrm{sh}(f)(z) \approx f(z)$ for all standard $z \in U$.

Theorem 3.6 (Robinson's theorem) *Let $U \subset \mathbb{C}$ standard and $f : U \to \mathbb{C}$ be a s-continuous holomorphic function with $f(z)$ limited for all $z \in U$. Then*

1. *The shadow of f is holomorphic.*

2. *If the shadow of f is not constant, then*

$$f(\mathrm{hal}(z_0)) = \mathrm{hal}(f(z_0))$$

for all standard $z_0 \in U$.

Proof:

1. The shadow of an s-continuous limited function is continuous. This implies in particular that $\mathrm{sh}(f)(z) \approx f(z)$ for all $z \in U$ with ${}^\circ z \in U$. Let γ be a standard closed curve in U. Since γ is compact and thus covered by the halos of its standard points, we have that $\mathrm{sh}(f)(z) \approx f(z)$ for all $z \in \gamma$, which implies that $\int_\gamma \mathrm{sh}(f)(z)dz \approx \int_\gamma f(z)dz = 0$. Because $\int_\gamma \mathrm{sh}(f)(z)dz$ is standard we get $\int_\gamma \mathrm{sh}(f)(z)dz = 0$ which implies that $\mathrm{sh}(f)$ is holomorphic.

2. We only have to show that $\mathrm{hal}(f(z_0)) \subset f(\mathrm{hal}(z_0))$. The other inclusion is just s-continuity.

   Let $z_0 \in U$ be standard.

   We first prove the claim for standard f: Since every non-constant holomorphic map is open, we have that for all standard $n \in \mathbb{N}$ $\mathrm{hal}(f(z_0)) \subset f(B_{1/n}(z_0))$, because $f(B_{1/n}(z_0))$ is a standard open set that contains the standard point $f(z_0)$ and thus also the halo of $f(z_0)$. Thus for all standard n and for all $y \approx f(z_0)$ we have that $y \in f(B_{1/n}(z_0))$. By overspill there is $N \approx \infty$ with $y \in f(B_{1/N}(z_0)) \subset f(\mathrm{hal}(z_0))$. which means that $\mathrm{hal}(f(z_0)) \subset f(\mathrm{hal}(z_0))$.

   Now let f be nonstandard and $y \in \mathrm{hal}(f(z_0)) = \mathrm{hal}(\mathrm{sh}(f)(z_0))$. The function $\mathrm{sh}(f)(z) - {}^\circ y$ is standard and has a zero at z_0. Because $f(z) - y \approx \mathrm{sh}(f)(z) - {}^\circ y$ on a standard neighborhood containing z_0, we can conclude by Corollary 3.2 that there exists a $\widetilde{z}_0 \approx z_0$ with $f(\widetilde{z}_0) - y = 0$. Thus there is a $\widetilde{z}_0 \in \mathrm{hal}(z_0)$ with $f(\widetilde{z}_0) = y$, which implies that $y \in f(\mathrm{hal}(z_0))$.

q.e.d

Chapter 4

Dynamics of rational maps

In this section we introduce the space of rational maps of degree d and recall some facts in rational dynamics. For a detailed exposition see e.g. [Mi3], [Be] and [Ste].

The space $\mathbf{Rat}_d$ of degree d rational maps. Let Rat_d denote the space of rational maps on $\hat{\mathbb{C}}$ with the topology of uniform convergence. A rational map $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ of degree d is by definition the quotient of two polynomials p and q

$$f(z) = \frac{p(z)}{q(z)} = \frac{a_d z^d + \cdots + a_0}{b_d z^d + \cdots + b_0}$$

where $\max\{\deg(p), \deg(q)\} = d$ and p and q having no common zeros. Parameterizing Rat_d by the coefficients of p and q, $(p : q) = (a_d, \cdots, a_0, b_d, \cdots, b_0) \in \mathbb{P}^{2d+1}$, we have

$$\mathrm{Rat}_d \cong \mathbb{P}^{2d+1} \backslash V(\mathrm{Res})$$

where $V(\mathrm{Res})$ is the set of polynomial pairs $(p : q) = (a_d, \cdots, a_0, b_d, \cdots, b_0)$ for which the resultant vanishes.

The topology on $\mathbf{Rat_d}$. Let $f, g \in \mathrm{Rat}_d$, f standard. Then we have

$$g \approx f \ \Leftrightarrow\ g(x) \approx f(x) \text{ for all } x \in \hat{\mathbb{C}}$$

(see Chapter 2 for the external characterization of uniform convergence). This is equivalent to: If $(a_d \cdots a_0, b_d \cdots b_0)$ are standard coefficients representing f, i.e. $f(z) = \frac{a_d z^d + \cdots + a_0}{b_d z^d + \cdots + b_0}$, then there exists $(\widetilde{a}_d \cdots \widetilde{a}_0, \widetilde{b}_d \cdots \widetilde{b}_0) \approx (a_d \cdots a_0, b_d \cdots b_0)$ with $g(z) = \frac{\widetilde{a}_d z^d + \cdots + \widetilde{a}_0}{\widetilde{b}_d z^d + \cdots + \widetilde{b}_0}$. Thus the topology on Rat_d induced by the topology on $\mathbb{P}^{2d+1}$ is the topology of uniform convergence on Rat_d.

Möbius transformations. Note that $\mathrm{Rat}_1 = \mathrm{Aut}(\hat{\mathbb{C}}) \simeq \mathrm{PSL}_2\,\mathbb{C}$ is the group of Möbius transformations.

The moduli space $\boldsymbol{\mathcal{M}}_\mathbf{d}$. Two rational maps $f, g \in \mathrm{Rat}_d$ are *holomorphically conjugate* if there exists a Möbius transformation $h \in \mathrm{Aut}(\hat{\mathbb{C}})$ with $h \circ f = g \circ h$. The quotient space

$\mathcal{M}_d = \mathrm{Rat}_d/\mathrm{Aut}(\hat{\mathbb{C}})$ is called the *moduli space of holomorphic conjugacy classes of degree d rational maps*. We denote by $\langle f \rangle$ the conjugacy class of f. Two maps that belong to the same holomorphic conjugacy class have the same dynamical behavior. All the dynamical notions and properties introduced below are invariant under holomorphic conjugation.

The topology on $\boldsymbol{\mathcal{M}_d}$. Let $\langle f \rangle$ and $\langle g \rangle \in \mathcal{M}_d$ with f standard. Since the projection map is open, the topological halo of f is given by

$$\langle g \rangle \approx \langle f \rangle \;\Leftrightarrow\; \text{there exists a } \widetilde{g} \in \langle g \rangle \text{ such that } \widetilde{g} \approx f.$$

Periodic points. A point z_0 is a *periodic point* for f if $f^n(z_0) = z_0$ for some $n \geq 1$. The least such n is called the *period* of z_0. In the case that the period is 1 z_0 is a *fixed point*. Note that a periodic point of period n for f is a fixed point for the map f^n. The set $\{z_0, f(z_0) \cdots f^{n-1}(z_0)\}$ is called *periodic cycle*. The *multiplier* at a periodic point z_0 of period n is given by $(f^n)'(z_0)$. The point z_0 is

attracting if $|(f^n)'(z_0)| < 1$

repelling if $|(f^n)'(z_0)| > 1$ and

indifferent if $|(f^n)'(z_0)| = 1$.

We call an attracting periodic point z_0 *superattracting* if $(f^n)'(z_0) = 0$ and an indifferent periodic point *parabolic* if $(f^n)'(z_0)$ is a root of unity.

Multiplicity of a fixed point. The multiplicity of a finite fixed point $f(z_0) = z_0$ is defined to be the unique integer $m \geq 1$ for which the Taylor expansion of $f(z) - z$ has the form:

$$f(z) - z = a_m(z - z_0)^m + a_{m+1}(z - z_0)^{m+1} + \cdots$$

with $a_m \neq 0$. If ∞ is a fixed point for f, its multiplicity can be defined by conjugating f by $z \mapsto \frac{1}{z}$, which maps the fixed point ∞ to 0.

Note that z_0 is a multiple fixed point, i.e. $m \geq 2$, if and only if $f'(z_0) = 1$.

The fixed point formula for rational maps. Let f be a rational map of degree d having fixed points $z_0, \cdots, z_d$ with multipliers $\mu_0, \cdots \mu_d \neq 1$. Then we have

$$\sum_{i=0}^{d} \frac{1}{1 - \mu_i} = 1.$$

This formula is proved by applying the residue theorem to $\frac{1}{z - f(z)}$. It plays an essential role in Milnor's description of the moduli space of quadratic rational maps $\mathcal{M}_2$.

Rouché's theorem. The next lemma is an immediate consequence of Rouché's theorem and we will refer to that lemma, which will be used frequently, just as Rouché's theorem. We often have the situation that a holomorphic map g is close to a standard holomorphic map f having a fixed point z_0, and we can then deduce that g must also have a fixed point $\widetilde{z}_0 \approx z_0$ infinitely close to the one of f with multiplier $g'(\widetilde{z}_0) \approx f'(z_0)$.

Lemma 4.1 *Let $f, g : U \to \mathbb{C}$ be holomorphic functions, f standard with $f(z) \approx g(z)$ for all $z \in U$. Let f have a fixed point $f(z_0) = z_0$ of multiplicity m and no other fixed points in U. Then g has precisely m fixed points infinitely close to z_0 counting multiplicity, i.e. there exists $z_1, z_2 \cdots, z_m \approx z_0$ with $g(z_i) = z_i$. Furthermore we have $g'(z_i) \approx f'(z_0)$ for all $1 \leq i \leq m$.*

Note that the multiplier $f'(z_0)$ must be equal to 1 if the multiplicity m is bigger than 1.

Proof: Because $f(z) \approx g(z)$ we have $f(z) - z \approx g(z) - z$, and $f(z) - z$ is non-constant, because z_0 is the only fixed point. So we can conclude by Corollary 3.2 that g has m fixed points close to z_0.

For the multiplier we conclude from the Cauchy integral formula

$$g'(z_i) - f'(z_i) = \frac{1}{2\pi i} \int_\gamma \frac{g(\xi) - f(\xi)}{(\xi - z_i)^2} d\xi \approx 0$$

where γ is a standard curve around z_0. So $g'(z_i) \approx f'(z_i) \approx f'(z_0)$ for all $1 \leq i \leq m$.

q.e.d

On the other hand we have:

Lemma 4.2 *Let $f, g : U \to \mathbb{C}$ be holomorphic functions, f standard with $f(z) \approx g(z)$ for all $z \in U$. If g has a fixed point z_0 with multiplier η then ${}^\circ z_0$ is a fixed point for the nearby standard map f with multiplier ${}^\circ\eta$.*

Apply transfer and use the Cauchy integral formula.

About attracting and fixed points. A repelling fixed point $f(z_0) = z_0$ has the property that there is a neighborhood U of z_0, such that every point in $U - \{z_0\}$ is mapped outside this neighborhood under iteration. An attracting fixed point $f(z_0) = z_0$ has the property that all the nearby points converge to z_0 under iteration. We will need the following more precise statement about attracting points in chapter 19:

Lemma 4.3 *Let $f(z_0) = z_0$ with $|f'(z_0)| < 1$. Then there exists a positive number ϵ and a $0 < c < 1$ such that*

$$|f(z) - z_0| < c|z - z_0|$$

for all $z \in B_\epsilon(z_0)$.

Critical points and critical values. A point ω is a *critical point* of f if $f'(\omega) = 0$. The image of a critical point under f is called a *critical value*.

The Julia set and the Fatou set. A rational map f determines a partition of $\hat{\mathbb{C}}$ into two totally invariant sets, the *Julia set* $J(f)$ and the *Fatou set* $F(f)$. A point $z \in \hat{\mathbb{C}}$ belongs to the *Fatou set* if there is a neighborhood U of z such that the sequence of iterates restricted to that neighborhood $\{f^n|_U\}_{n\in\mathbb{N}}$ is a normal family, i.e. every sequence contains a subsequence which converges locally uniformly. The complement of the Fatou set is called the *Julia set*. If $\deg(f) \geq 2$ then $J(f)$ is a nonempty perfect set.

The Julia set of a rational map f with $\deg(f) \geq 2$ is equal to the closure of the set of all repelling periodic points. The Julia set is the smallest, closed, completely invariant subset of $\hat{\mathbb{C}}$ that contains more than two points.

The Julia set of the n-th iterate f^n is equal to the Julia set of f: $J(f^n) = J(f)$.

Fatou components. The *Fatou components* of f are the connected components of the Fatou set $F(f)$. A Fatou component U is a *periodic component* if $f^n(U) = U$ for some $n \geq 1$.

Fatou components are mapped onto Fatou components. By Sullivan's no wandering domain theorem, every Fatou component is eventually periodic, i.e it maps to a periodic component after finitely many iterations.

A periodic Fatou component is either an attractive or a parabolic periodic component or a Siegel disk or a Hermann ring. We will be concerned with the first two types only.

An *attractive periodic component of period* n contains an attracting fixed point z_0 for f^n, and all the points in the component tend to z_0 under iteration of f^n. A *parabolic periodic component of period* n has a parabolic fixed point z_0 for f^n on its boundary with $(f^n)'(z_0) = 1$ and all the points in the component tend to z_0 under iteration of f^n. A period n component U is called a *Siegel disc* if $f^n|_U$ is holomorphically conjugate to an irrational rotation on the unit disc and a *Hermann ring* if $f^n|_U$ is holomorphically conjugate to an irrational rotation on an annulus.

The Julia set $J(f)$ is connected if and only if each Fatou component is simply connected.

The basin of an attracting periodic cycle. The *basin of an attracting period* n *cycle* is the set of all points that tend to a point in the periodic cycle under iteration of f^n. Note that the basin of an attracting periodic cycle is contained in the Fatou set. The *immediate basin of an attracting periodic cycle* is the union of all the attracting periodic components, belonging to that periodic cycle.

The immediate basin of an attracting periodic cycle of f contains a critical point of f.

The basin of a parabolic periodic cycle. The *basin of a parabolic period* n *cycle* is the set of all points in the Fatou set that tend to a point of the periodic cycle under iteration of f^n (or equivalently all points in $\hat{\mathbb{C}}$ that tend to a point of the periodic cycle, but do not land on the parabolic cycle). Note that the parabolic cycle itself lies in the Julia set and so do all its preimages. The *immediate basin of a parabolic periodic cycle* is the union of all periodic parabolic components, belonging to the parabolic periodic cycle.

In the next theorem we summarize some facts about parabolic basins that we need in Chapter 19. Further details can be found in [Mi3].

Theorem 4.4 (Structure of parabolic basins) *1. If z_0 is a parabolic fixed point with multiplier 1 of multiplicity $n+1$, then the number of connected components of the immediate basin of z_0 (attracting petals) is equal to n. Furthermore each component of the immediate basin contains a critical point of f.*

2. *If z_0 is a parabolic fixed point for f whose multiplier is a primitive q-th root of unity, then the number of connected components of the immediate basin of z_0 (attracting petals) is a multiple of q.*

3. *The number of attracting petals (connected components of the immediate basin) at a parabolic periodic point of period n of a quadratic polynomial $P_c(z) = z^2 + c$ (fixed point of P_c^n) whose multiplier is a primitive q-th root of unity, is exactly q. (The same is true in any other family of maps that have just one free critical point.)*

Proof: Part 1 and 2 can be found in [Mi3, Chapter 10]. Note however that an attracting petal is defined differently here than in the literature, where it is defined to be a certain subset of the components of the immediate basin.

Now consider a quadratic polynomial P_c having a parabolic periodic n cycle, whose multiplier is a primitive q-th root of unity. Then the points in the parabolic cycle are fixed points of P_c^{nq} with multiplier 1. By Part 1 we know that each attracting petal at each fixed point of P_c^{nq} contains a critical point and thus a critical value of P_c^{nq}. The map P_c^{nq} has $2^{nq} - 1$ critical points (in the finite plane), but only nq critical values and thus there can be at most nq attracting petals in the whole period n cycle and thus at most q at each periodic point. By Part 2 we can conclude that the number of attracting petals must be exactly q.

q.e.d

Chapter 5

The shadow of a rational map

In this chapter we introduce the notion of the shadow of a rational map.

A rational map f of degree d corresponds to a point in $[x] \in \mathbb{P}^{2d+1}$. The shadow of f is a rational map that we associate to the unique nearby standard point ${}^\circ[x] \in \mathbb{P}^{2d+1}$, which is of lower degree, if ${}^\circ[x] \in V(\mathrm{Res})$. The shadow is roughly speaking the limiting map: If $\{f_n\}_{n\in\mathbb{N}}$ is a standard sequence of degree d rational maps converging to a rational map f locally uniformly outside a finite set, then we have $\mathrm{sh}(f_N) = f$ for all $N \approx \infty$.

The map $\hat{\Pi}_d : \mathbb{P}^{2d+1} \to \bigcup_{j=0}^{d} \mathrm{Rat}_j$. We now extend the isomorphism between $\mathbb{P}^{2d+1} \backslash V(\mathrm{Res})$ and Rat_d to a map on $\mathbb{P}^{2d+1}$ by associating to the points in $V(\mathrm{Res})$ a rational map of lower degree. Let

$$\Pi_d : \mathbb{P}^{2d+1} \backslash V(\mathrm{Res}) \to \mathrm{Rat}_d$$

denote the isomorphism between Rat_d and $\mathbb{P}^{2d+1} \backslash V(\mathrm{Res})$, i.e.

$$\Pi_d([a_0, \cdots, a_d, b_0, \cdots, b_d]) = \frac{a_d z^d + \cdots + a_0}{b_d z^d + \cdots + b_0}.$$

If $[a_0, \cdots, a_d, b_0, \cdots, b_d] \in V(\mathrm{Res})$ then it does not give coefficients for a rational map of degree d, but still we can associate a rational map, a rational map of lower degree, in the following way: Consider the corresponding polynomials $p(z) = a_d z^d + \cdots + a_0$ and $q(z) = b_d z^d + \cdots + b_0$ and let $z_0, \cdots z_k$ denote the common zeros of p and q if there are any. Then

$$\frac{p(z)}{q(z)} = \frac{\tilde{p}(z)}{\tilde{q}(z)} \prod_{i=0}^{k} \frac{z - z_i}{z - z_i}$$

and we associate to $[a_0, \cdots, a_d, b_0, \cdots, b_d] \in V(\mathrm{Res})$ the rational map

$$f_{[a_0, \cdots, a_d, b_0, \cdots, b_d]}(z) = \frac{\tilde{p}(z)}{\tilde{q}(z)}.$$

Let $\hat{\Pi}_d : \mathbb{P}^{2d+1} \to \bigcup_{j=0}^{d} \mathrm{Rat}_j$ denote the extension of the map Π_d in that sense. Note that $\hat{\Pi}_d$ is not injective.

So we associated to every point $[a_0, \cdots, a_d, b_0, \cdots, b_d]$ in the projective space $\mathbb{P}^{2d+1}$ a rational map $\hat{\Pi}_d([a_0, \cdots, a_d, b_0, \cdots, b_d])$, possibly of lower degree: If $\mathrm{Res}(a_0, \cdots, a_d, b_0, \cdots, b_d) \neq$

0 a rational map of degree d and if $\mathrm{Res}(a_0, \cdots, a_d, b_0, \cdots, b_d) = 0$ a rational map of lower degree by canceling all the common factors from the corresponding polynomials $p(z) = a_d z^d + \cdots + a_0$ and $q(z) = b_d z^d + \cdots + b_0$.

Definition of the shadow of a rational map. Let $d \in \mathbb{N}$ be standard. Then $\mathbb{P}^{2d+1}$ is a standard compact Hausdorff space. Thus every point in $\mathbb{P}^{2d+1}$ is infinitely close to a unique standard point:

$$^\circ[a_0, \cdots, a_d, b_0, \cdots, b_d] = [^\circ a_0, \cdots, ^\circ a_d, ^\circ b_0, \cdots, ^\circ b_d]$$

where $\max\{|a_0|, \cdots, |a_d|, |b_0|, \cdots, |b_d|\} = 1$.

We define the *shadow* of a rational map $f \in \mathrm{Rat}_d$, denoted by $\mathrm{sh}(f)$, to be the rational map given by the standard part of the corresponding point in projective space. This is

$$\mathrm{sh}(f) = \hat{\Pi}_d(^\circ(\Pi_d^{-1}(f))).$$

Examples:

1. Consider

$$f_1(z) = z\frac{z-1+\epsilon}{z-1} = \frac{z^2 + (\epsilon - 1)z}{z-1}$$

which corresponds to the point $[1, \epsilon - 1, 0, 0, 1, -1]$ in $\mathbb{P}^5$.

If $\epsilon \approx 0$ we have $^\circ[1, \epsilon - 1, 0, 0, 1, -1] = [1, -1, 0, 0, 1, -1]$ which leads to

$$\frac{z^2 - z}{z-1} = z\frac{z-1}{z-1}$$

and thus

$$\mathrm{sh}(f_1)(z) = z - 1.$$

We have $\mathrm{sh}(f_1)(z) \approx f_1(z)$ for all $z \not\approx 1$.

2. Now let

$$f_2(z) = \frac{z-1+\epsilon}{z-1}\frac{z-2+\epsilon}{z-2} = \frac{z^2 + (2\epsilon - 3)z + (\epsilon - 1)(\epsilon - 2)}{z^2 - 3z + 2}$$

which corresponds to the point $[1, 2\epsilon - 3, (\epsilon - 1)(\epsilon - 2), 1, -3, 2]$ in $\mathbb{P}^5$.

If $\epsilon \approx 0$ we have $^\circ[1, (2\epsilon - 3), (\epsilon - 1)(\epsilon - 2), 1, -3, 2] = [1, -3, 2, 1, -3, 2]$ which leads to

$$\frac{z^2 - 3z + 2}{z^2 - 3z + 2} = \frac{z-1}{z-1}\frac{z-2}{z-2}$$

and thus

$$\mathrm{sh}(f_2)(z) = 1.$$

We have $\mathrm{sh}(f_2)(z) \approx f_2(z)$ for all $z \not\approx 1, 2$.

3. Now consider
$$f_3(z) = \frac{(z-1)(z-2)}{\epsilon z} = \frac{z^2-3z+2}{\epsilon z}$$
which corresponds to the point $[1,-3,2,0,\epsilon,0]$ in $\mathbb{P}^5$.

   If $\epsilon \approx 0$ we have $^\circ[1,-3,2,0,\epsilon,0] = [1,-3,2,0,0,0]$ which leads to
$$\frac{z^2-3z+2}{0} = \frac{(z-1)(z-2)}{0}$$
and thus
$$\mathrm{sh}(f_3)(z) = \infty$$
and $\mathrm{sh}(f_3)(z) \approx f_3(z)$ for all $z \not\approx 1, 2$.

Definition of $f \simeq g$. Let f and g be rational maps. We define $f \simeq g$ if there exist standard $z_1, \cdots, z_k$ such that $f(z) \approx g(z)$ for all $z \not\approx z_1, \cdots, z_k$.

Lemma 5.1 *Let f be a rational map of standard degree. Then* $\mathrm{sh}(f) \simeq f$ *and* $\mathrm{sh}(f)$ *is the unique standard rational map with that property.*

Proof:

1. Claim: $\mathrm{sh}(f)$ has the property $\mathrm{sh}(f) \simeq f$.

   Let
$$f(z) = \frac{a_d z^d + \cdots + a_0}{b_d z^d + \cdots + b_0}$$
with $\max\{|a_i|, |b_i| : \ 0 \le i \le d\} = 1$ and $p(z) = a_d z^d + \cdots + a_0$, $q(z) = b_d z^d + \cdots + b_0$.

   (a) $\mathrm{sh}(f) \equiv \infty$
   Then we have $b_0 \approx \cdots b_d \approx 0$ and thus $f(z) \approx \infty$ for all limited z that are not close to any of the roots of p. (Note that a polynomial of standard degree with limited coefficients is s-continuous.)

   (b) $\mathrm{sh}(f) \equiv 0$
   Then we have $a_0 \approx \cdots a_d \approx 0$ and thus $f(z) \approx 0$ for all limited z that are not close to any of the roots of q.

   (c) $\mathrm{sh}(f) \not\equiv 0, \infty$
   Let $z_0, \cdots z_k$ denote the common roots of $\mathrm{sh}(p)$ and $\mathrm{sh}(q)$. Then p and q have roots $z_i^{(p)} \approx z_i$ and $z_i^{(q)} \approx z_i$ respectively, $0 \le i \le k$. And we have
$$f(z) = \frac{p(z)}{q(z)} = \frac{\tilde{p}(z)}{\tilde{q}(z)} \prod_{i=0}^{k} \frac{z - z_i^{(p)}}{z - z_i^{(q)}}$$
   with $z_i^{(p)} \approx z_i^{(q)} \approx z_i$. This implies that
$$f(z) \approx \frac{\tilde{p}(z)}{\tilde{q}(z)} \approx \mathrm{sh}(f)(z)$$

for all limited $z \not\approx z_0, \cdots z_k$, because $\prod_{i=0}^{k} \frac{z-z_i^{(p)}}{z-z_i^{(q)}} \approx 1$ and $\tilde{p}$ and $\tilde{q}$ have no further roots close to each other and neither $\tilde{p}$ nor $\tilde{q}$ is close to 0, thus $\tilde{p}$ and $\tilde{q}$ are not both close to 0 for any $z \not\approx z_0, \cdots z_k$.

2. Claim: $\mathrm{sh}(f)$ is the unique standard rational map with this property.

   Assume $\tilde{f}$ is another standard rational map with $f(z) \approx \tilde{f}(z)$ for all z except possibly in the halos of finitely many standard points. This implies that there exists a standard open set U such that for all $z \in U$ we have $\mathrm{sh}(f(z)) \approx \tilde{f}(z)$, this is then especially true for all standard $z \in U$ and thus it follows by transfer that $\mathrm{sh}(f(z)) = \tilde{f}(z)$ holds for all $z \in U$. By the Identity theorem $\mathrm{sh}(f)$ are $\tilde{f}$ are equal. q.e.d.

Lemma 5.2 (Locally uniform convergence outside a finite set) *Let $\{a_1, \cdots, a_k\}$ a standard finite set. A standard sequence of maps $f_n : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ converges locally uniformly to the standard map $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ on $\hat{\mathbb{C}} - \{a_1, \cdots, a_k\}$ if and only if $f_N(z) \approx f(z)$ for all $N \approx \infty$ and all $z \not\approx a_1, \cdots a_k$.*

Proof: The sequence f_n converges locally uniformly on $\hat{\mathbb{C}} - \{a_1, \cdots, a_k\}$ to f if and only if for all $z \neq a_1, \cdots, a_k$ there exists a $\delta > 0$ such that $\lim_{n\to\infty} \sup_{y \in B_\delta(z)} d_{\hat{\mathbb{C}}}(f_n(y), f(y)) = 0$. Applying transfer and the external characterization of uniform convergence this is equivalent to: for all standard $z \neq a_1, \cdots, a_k$ there exists a standard $\delta > 0$ such that $f_N(y) \approx f(y)$ for all $N \approx \infty$ and all $y \in B_\delta(z)$.

First assume that $f_n \to f$ on $\hat{\mathbb{C}} - \{a_1, \cdots, a_k\}$. Let $z \not\approx a_1, \cdots, a_k$. Then ${}^\circ z \neq a_1, \cdots, a_k$. There exists a standard $\delta > 0$ such that $a_i \notin B_\delta({}^\circ z)$ for all $1 \leq i \leq k$. Since $z \in B_\delta({}^\circ z)$ we can conclude by the previous consideration that $f_N(z) \approx f(z)$ for all $N \approx \infty$.

Now assume that $f_N(z) \approx f(z)$ for all $N \approx \infty$ and all $z \not\approx a_1, \cdots a_k$. Let $z \neq a_1, \cdots, a_k$ be standard. Define $\delta = \frac{1}{2} \min_{1 \leq i \leq k} d_{\hat{\mathbb{C}}}(z, a_i)$. Then δ is positive and standard, and $y \not\approx a_i$ for all $y \in B_\delta(z)$ and all $1 \leq i \leq k$. This implies that $f_N(y) \approx f(y)$ for all $N \approx \infty$ and for all $y \in B_\delta(z)$. And thus $f_n \to f$ on $\hat{\mathbb{C}} - \{a_1, \cdots, a_k\}$. q.e.d.

Lemma 5.3 *If a standard sequence $\{f_n\}_{n \in \mathbb{N}}$ of rational maps of degree d converges to a map f locally uniformly outside a finite set, then $\mathrm{sh}(f_N) = f$ for all unlimited $N \in \mathbb{N}$.*

Proof: Since $\{f_n\}_{n \in \mathbb{N}}$ is standard, its limiting map f is standard. Thus there exist standard $a_1, \cdots a_n$, such that $f_n \to f$ locally uniformly on $\hat{\mathbb{C}} - \{a_1, \cdots a_n\}$. By Lemma 5.2 we have $f_N(z) \approx f(z)$ for all $z \not\approx a_1, \cdots a_n$ and all $N \approx \infty$. Thus $f_N \simeq f$ for all $N \approx \infty$. By the Lemma 5.1 we can conclude that $\mathrm{sh}(f_N) = f$ for all $N \approx \infty$. q.e.d.

Remarks and References. We have defined the shadow of a rational map to only keep track of the limiting map and not of the points where the convergence does not take place (the common roots of the polynomials given by the corresponding point in projective space and possibly the point ∞). If we would have done this, there would be a one-to-one correspondence between $\mathbb{P}^{2d+1}$ and the set of pairs of rational maps of degree less or equal than d together with some exceptional points. In [De1] DeMarco associates to every point in $V(\mathrm{Res})$ a rational map of lower degree together with a set of "holes", which are the common roots and possibly the point ∞. Since for our purposes these points are not of dynamical interest, we decided to not keep track of them.

Chapter 6

Quadratic polynomials: Renormalization and tuning

This chapter reviews well-known features of the family of quadratic polynomials. We recall the theory of polynomial-like maps introduced by Douady and Hubbard [DH] and review the idea of renormalization and associated tuning maps. For more detailed expositions see [Mc1] and [Mi1].

Quadratic polynomials. Every quadratic polynomial is holomorphically conjugate to one of the form $P_c(z) = z^2 + c$ for a unique $c \in \mathbb{C}$. A quadratic polynomial, in fact every polynomial, has a superattracting fixed point at ∞. The *Mandelbrot set* is the set of all parameters for which the corresponding Julia set is connected:

$$M = \{c \in \mathbb{C} \ : \ J(P_c) \text{ is connected}\}.$$

Note that 0 is the unique critical point of P_c in $\mathbb{C}$ and that $J(P_c)$ is connected if and only if the critical orbit $\{P_c^n(0)\}_{n\in\mathbb{N}}$ is bounded. The intersection of m with the real axis equals $[-2, 1/4]$.

The limbs of the Mandelbrot set. The *main cardioid* of the Mandelbrot set M is the set of all $c \in \mathbb{C}$ for which P_c has an attracting fixed point. And the *p/q-limb* of M, $M_{p/q}$, is defined to be the set of all $c \in M$ for which P_c has either a fixed point with multiplier $e^{2\pi ip/q}$ or is separated from the main cardioid by the parameter on the boundary of the main cardioid, corresponding to the map having a fixed point with multiplier $e^{2\pi ip/q}$.

The filled Julia set. The filled Julia set of a quadratic polynomial P_c is the complement of the basin of ∞:

$$K(P_c) = \{z \in \mathbb{C} : P_c^n(z) \not\to \infty\}.$$

The α- and β-fixed point. Let $c \in M$. Then P_c has two fixed points in the filled Julia set $K(P_c)$. Let β be the fixed point in $K(P_c)$ that is repelling and does not disconnect the Julia set, i.e. $K(P_c) - \{\beta\}$ is connected. The other fixed point in $K(P_c)$ is called α. If $c \in M_{p/q}$, then α is repelling and $K(P_c) - \{\alpha\}$ consists of q connected components.

Hyperbolic components of the Mandelbrot set. A quadratic polynomial $P_c(z) = z^2 + c$ is *hyperbolic* if the critical point 0 is in the basin of an attracting periodic cycle. This is the case if and only if either 0 is in the basin of ∞ or P_c has an attracting periodic cycle in the finite plane. The hyperbolic parameters $c \in \mathbb{C}$ are an open and conjecturally dense subset of $\mathbb{C}$. The connected components are called *hyperbolic components* of M. All parameters $c \in \mathbb{C}$ for which the critical point 0 is in the basin of ∞ lie in one hyperbolic component, namely the complement of M. In every bounded hyperbolic component the multiplier at the attracting periodic cycle gives a biholomorphic map onto the open unit disk. The parameter for corresponding to the superattracting cycle is called *hyperbolic center*.

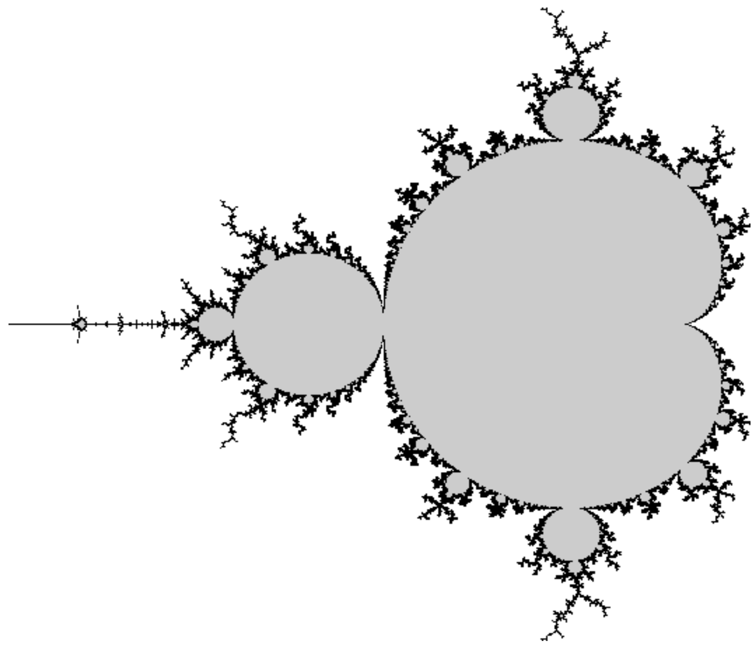


Figure 6.1: The Mandelbrot set, the connectedness locus of the quadratic family $P_c(z) = z^2 + c$.

Polynomial-like maps. A *polynomial-like map* $f : U \to V$ of degree d is a proper map of degree d between topological disks U and V such that $\overline{U}$ is a compact subset of V. Thus every point in V has exactly d preimages in U counting multiplicity. A polynomial-like map of degree 2 is called a *quadratic-like map*.

The *filled Julia set* $K(f)$ of a polynomial-like map $f : U \to V$ consists of all the points that stay in U under iteration:

$$K(f) = \{z \in U : \ f^n(z) \in U \text{ for all } n \in \mathbb{N}\}.$$

The Julia set of the polynomial-like map $f : U \to V$, $J(f)$, is defined to be the boundary of $K(f)$. Note that a quadratic-like map $f : U \to V$ has a unique critical point $\omega \in U$ and that $J(f)$ is connected if and only if $f^n(\omega) \in U$ for all $n \in \mathbb{N}$.

Quasiconformal map. Let $W, V \subset \hat{\mathbb{C}}$. A homeomorphism $\phi : W \to V$ is *quasiconformal* if

$$\limsup_{r\to 0} \frac{\max_{d_{\hat{\mathbb{C}}}(y,z)=r} d_{\hat{\mathbb{C}}}(f(y), f(z))}{\min_{d_{\hat{\mathbb{C}}}(y,z)=r} d_{\hat{\mathbb{C}}}(f(y), f(z))}$$

is uniformly bounded.

Hybrid equivalence. Two polynomial-like maps f and g are *hybrid equivalent* if there is a quasiconformal conjugacy ϕ between f and g, defined on a neighborhood of their filled Julia sets, such that $\frac{\partial\phi}{\partial\bar{z}} = 0$ on $K(f)$. Note that this implies that the conjugacy ϕ is holomorphic in the interior of $K(f)$.

Theorem 6.1 (The straightening theorem) *Every polynomial-like map $f : U \to V$ of degree d is hybrid equivalent to a polynomial of degree d. If the filled Julia set $K(f)$ is connected this polynomial is unique up to affine conjugation.*

See [DH, Theorem 1].

We will also need the following special version of the straightening theorem for real quadratic-like maps.

Theorem 6.2 (The straightening theorem for real quadratic-like maps) *Let $f : U \to V$ be a quadratic-like map with connected Julia set. Suppose that $f(\overline{z}) = \overline{f(z)}$ for all $z \in U$, in particular U has to be symmetric with respect to the real line. Then there is a quasiconformal homeomorphism ϕ defined on a neighborhood of the filled Julia set of $: U \to V$ such that*

$$\phi(\overline{z}) = \overline{\phi(z)}$$

and $\phi^{-1} \circ f \circ \phi$ is a real quadratic polynomial $P_c(z) = z^2 + c$ for some $c \in [-2, 1/4]$.

See [GS, Fact 3.1.2].

Renormalization of quadratic polynomials. The map P_c^n is *renormalizable* if there are open disks U and V such that the critical point $0 \in U$ and $P_c^n : U \to V$ is a quadratic-like map with connected filled Julia set K_n. Note that $K_n \subset K(P_c)$ and that the Julia set is connected if and only if $P_c^{nk}(0) \in U$ for all $k \in \mathbb{N}$. If P_c^n is renormalizable, then any two renormalizations, i.e. choices of suitable topological discs U and V, have the same Julia set.

By the Straightening Theorem 6.1 the renormalization $P_c^n : U \to V$ is hybrid equivalent to a unique quadratic polynomial P_d.

Small Julia sets. Define $K_n(i) = f^i(K_n)$ for $1 \le i \le n$. These sets, who are all subsets of the filled Julia set of P_c are called *small Julia sets*. Note that $K_n(n) = K_n$. We have $K_n(i) \cap K_n(j) = \emptyset$ or $K_n(i) \cap K_n(j) = \{x\}$ where x is a repelling fixed point of P_c^n.

Simply renormalizable. A renormalizable quadratic polynomial is said to be *simply renormalizable* if none of the small Julia sets intersect at the α-fixed point of $P_c^n : K_n \to K_n$.

The tuning map. The limb $M_{p/q}$ contains a unique hyperbolic center c_0 of period q. This is the centerpoint of the hyperbolic component in $M_{p/q}$ attached to the main cardioid. There is a corresponding homeomorphism

$$\tau_{p/q} : M \to \tau_{p/q}(M) \subset M_{p/q}$$

such that $\tau_{p/q}(b) = c$ if and only if P_c^q is simply renormalizable, hybrid equivalent to P_b.

The tuning map $\tau_{p/q}$ has the following properties:

1. $\tau_{p/q}$ maps M onto a small copy of M contained in $M_{p/q}$, in particular $\tau_{p/q}(M) \neq M_{p/q}$.
2. If $H \subset M$ is a hyperbolic component of period n, then $\tau_{p/q}(H)$ is a hyperbolic component of period qn. The multiplier at the attracting periodic cycle is maintained.
3. If $c \in M_{p/q} - \tau_{p/q}(M)$ then there still exist topological discs U and V with $0 \in U \subset V$ such that $P_c^q : U \to V$ is a quadratic-like map, but the Julia set of $P_c^q : U \to V$ is a Cantor set.
4. Let c_0 denote the hyperbolic center of period q in $M_{p/q}$. The filled Julia set $K(P_{\tau_{p/q}(b)})$ is obtained from the filled Julia set $K(P_{c_0})$ by replacing every Fatou component in $K(P_{c_0})$ by a copy of the Julia set $K(P_b)$, in case that P_b is critically finite .

For further details and proofs about tuning see [Mi1] and [Ha].

Chapter 7

The moduli space of quadratic rational maps

In this chapter we summarize the results about the moduli space $\mathcal{M}_2 = \mathrm{Rat}_2 / \mathrm{Aut}(\hat{\mathbb{C}})$ of holomorphic conjugacy classes of quadratic rational maps and its algebraic compactification $\widehat{\mathcal{M}}_2$ following Milnor's exposition [Mi2].

The moduli space $\boldsymbol{\mathcal{M}_2 = \mathrm{Rat}_2 / \mathrm{Aut}(\hat{\mathbb{C}})}$. Every element in $\mathcal{M}_2$ is determined by the multipliers at its three fixed points. The only restriction is

$$\mu_1\mu_2\mu_3 - (\mu_1 + \mu_2 + \mu_3) + 2 = 0.$$

Introducing the elementary symmetric functions of the multipliers:

$$\sigma_1 = \mu_1 + \mu_2 + \mu_3 \quad \sigma_2 = \mu_1\mu_2 + \mu_1\mu_3 + \mu_2\mu_3 \quad \sigma_3 = \mu_1\mu_2\mu_3$$

this is equivalent to

$$\sigma_3 = \sigma_1 - 2.$$

Hence $\mathcal{M}_2$ is isomorphic to $\mathbb{C}^2$ with coordinates σ_1 and σ_2.
We denote by

$$\langle f \rangle = \langle \mu_1, \mu_2, \mu_3 \rangle$$

the conjugacy class of the function f having fixed points with multipliers μ_1, μ_2, μ_3.

Recall that $f(z_0) = z_0$ is a multiple fixed point of f if and only if $f'(z_0) = 1$. If $\mu_1\mu_2 = 1$ then because of $\mu_1\mu_2\mu_3 - (\mu_1 + \mu_2 + \mu_3) + 2 = 0$ we have that $\mu_1 + \mu_2 = 2$ and thus $\mu_1 = \mu_2 = 1$.

The relation $\mu_1\mu_2\mu_3 - (\mu_1 + \mu_2 + \mu_3) + 2 = 0$ follows from the holomorphic fixed point formula

$$\frac{1}{1-\mu_1} + \frac{1}{1-\mu_2} + \frac{1}{1-\mu_3} = 1$$

in case that there are no multiple fixed points. If there is a multiple fixed point, i.e. $\mu_1 = \mu_2 = 1$, then the above relation is satisfied for any $\mu_3 \in \hat{\mathbb{C}}$.

Normal forms. Now we present various forms to normalize a map $\langle\mu_1,\mu_2,\mu_3\rangle \in \mathcal{M}_2$ that will be used throughout this text. For a map in $\mathcal{M}_2$ that is not close to infinity one can always see the same dynamical behavior however the map is normalized. But for a map close to infinity it is essential what normalization we choose in order to make certain dynamical behaviors persist in the limit.

1. **Fixed points at 0 and ∞.** If $\langle f\rangle = \langle\mu_1,\mu_2,\mu_3\rangle \neq \langle 1,1,1\rangle$, i.e f has at least two different fixed points, e.g. $\mu_1 \neq 1$, then we can normalize so that $f(0) = 0$ and $f(\infty) = \infty$:
$$f(z) = z\frac{z+\mu_1}{\mu_2 z+1}.$$
The map f has fixed points at 0 and ∞ with multipliers μ_1 and μ_2 respectively and a third fixed point at $\frac{\mu_1-1}{\mu_2-1}$ with multiplier $\frac{2-\mu_1-\mu_2}{1-\mu_1\mu_2}$. The critical points are at $\frac{-1\pm\sqrt{1-\mu_1\mu_2}}{\mu_2}$.

2. **Fixed points at 0, 1 and ∞.** If there are three different fixed points, i.e. none of the multipliers at the fixed points are 1, we can normalize the map having fixed points at 0, 1, ∞ :
$$f(z) = z\frac{(1-\mu_1)z+\mu_1(1-\mu_2)}{\mu_2(1-\mu_1)z+1-\mu_2} = \frac{z}{\mu_2}(1+\frac{d\epsilon}{z-d})$$
with $\epsilon = 1-\mu_1\mu_2$ and $d = 1-\frac{\epsilon}{\mu_2(1-\mu_1)}$. Here we have $f'(0) = \mu_1$ and $f'(\infty) = \mu_2$. The critical points are at $d(1\pm\sqrt{\epsilon})$.

3. **Fixed points at 0, 1 and -1.** Another way to normalize a map with three different fixed points, is to put the fixed points at 0, 1 and -1. This leads to:
$$f(z) = z\frac{a+bz+1}{a+bz+z^2}.$$
We have $f'(z) = \frac{1}{a+bz+z^2}(a+2bz+1-z(b+2z)\frac{a+bz+1}{a+bz+z^2})$ and thus $f'(0) = 1+\frac{1}{a}$ and $f'(1) = \frac{a+b-1}{a+b+1}$ and $f'(-1) = \frac{a-b-1}{a-b+1}$.

4. **Critical points at ± 1 and fixed point at ∞.** Every class $\langle\mu_1,\mu_2,\mu_3\rangle \in \mathcal{M}_2$ can be represented such that is has a fixed point with multiplier $\mu = \mu_3 \neq 0$ at ∞ and critical points at ± 1:
$$f(z) = \frac{1}{\mu}(z+\frac{1}{z}+T)$$
with $T^2 = \frac{(\mu_1-\mu_2)^2}{1-\mu_1\mu_2}$, if $\mu_1\mu_2 \neq 1$ (i.e. $\mu_1 \neq 1$ or $\mu_2 \neq 1$).

The two finite fixed points are the roots of the equation $(1-\mu)z^2+Tz+1 = 0$, namely $z_{1/2} = \frac{T}{2(\mu-1)} \pm \sqrt{\frac{T^2}{4(\mu-1)^2}+\frac{1}{\mu-1}}$ with $f'(z_{1/2}) = \frac{1}{\mu}(1-\frac{1}{z_{1/2}^2})$. We have $z_1 z_2 = \frac{1}{1-\mu}$ and $z_1+z_2 = \frac{T}{\mu-1}$. This implies that
$$\mu_1+\mu_2 = \frac{1}{\mu}(2-\frac{1}{z_1^2}+\frac{1}{z_2^2}) = \frac{1}{\mu}(2-\frac{(z_1+z_2)^2}{(z_1z_2)^2}+\frac{2}{z_1z_2}) = \frac{1}{\mu}(2-T^2+2(1-\mu))$$

which implies that

$$T^2 = -\mu(\mu_1 + \mu_2) + 4 - 2\mu = \frac{(\mu_1 - \mu_2)^2}{1 - \mu_1\mu_2}$$

because $\mu = \frac{2-\mu_1-\mu_2}{1-\mu_1\mu_2}$.

If $\mu_1 = \mu_2 = 1$ we have $T^2 = 4 - 4\mu$.

The topology on $\boldsymbol{\mathcal{M}_2}$. Let $f \in \mathrm{Rat}_2$ be standard. Since the projection map $p : \mathrm{Rat}_2 \to \mathcal{M}_2$ given by $f \mapsto \langle f \rangle$ is open we have by Lemma 2.4 that $\langle g \rangle \approx \langle f \rangle$ if and only if g can be normalized such that $g \approx f$. This is the case if and only if the multipliers at the fixed points of f and g are close to each other.

The compactification $\boldsymbol{\widehat{\mathcal{M}_2} \cong \mathbb{P}^2}$. Since the complex projective plane $\mathbb{P}^2$ is a natural compactification of $\mathbb{C}^2$ there is a corresponding compactification

$$\widehat{\mathcal{M}_2} \cong \mathbb{P}^2 \cong \mathbb{C}^2 \cup \hat{\mathbb{C}}$$

consisting of $\mathcal{M}_2$ together with a sphere of ideal points at infinity. These ideal points can be identified with unordered triples of the form

$$\langle \mu, \mu^{-1}, \infty \rangle$$

with $\mu \in \hat{\mathbb{C}}$. Because if one of the symmetric functions is close to infinity then at least one of the multipliers is close to infinity.

If only one of the multipliers is close to infinity: Let $\mu_3 \approx \infty$. Then $\mu_1\mu_2 \approx 1$ because $\mu_3 = \frac{2-\mu_1-\mu_2}{1-\mu_1\mu_2}$ and thus $\mu_1 \approx 1/\mu_2$.

If two of the multipliers are close to infinity: Let $\mu_1, \mu_2 \approx \infty$ then $\mu_3 = \frac{2-\mu_1-\mu_2}{1-\mu_1\mu_2} \approx 0$.

We have $\langle f \rangle \approx \langle \mu, \mu^{-1}, \infty \rangle$ if and only if f has fixed points with multipliers close to μ, μ^{-1} and ∞.

Or in classical terms: A sequence $\langle f_n \rangle_{n \in \mathbb{N}}$ converges to an ideal point $\langle \mu, \mu^{-1}, \infty \rangle \in \widehat{\mathcal{M}_2}$ if and only if the multiplier at the fixed points of f_n tend to μ, μ^{-1} and ∞.

We will study quadratic rational maps that are close to these ideal points, i.e. maps that have a fixed point with an unlimited multiplier.

Example: Let $\langle \mu_1, \mu_2, \mu_3 \rangle \approx \langle \mu, \mu^{-1}, \infty \rangle$ with $\mu \neq 0, \infty$ standard. Then this conjugacy class can be represented by

$$f(z) = z\frac{z + \mu_1}{\mu_2 z + 1} = \frac{z}{\mu_2}\frac{z + \mu_1}{z + (1/\mu_2)} \approx \frac{z}{\mu_2} \approx \mu z \quad \text{for all } z \not\approx -\mu.$$

Thus f is close to the degree one standard map $z \mapsto \mu z$ on most of the sphere, i.e. $\mathrm{sh}(f)(z) = \mu z$ and the halo of $-\mu$ is mapped over the whole sphere.

Note that the shadow of a map close to an ideal point in $\widehat{\mathcal{M}_2}$ is of degree at most one, however the map is normalized.

Theorem 7.1 (Totally disconnected Julia sets) *The Julia set of a quadratic rational map is either connected or totally disconnected. It is totally disconnected if and only if either both critical points are attracted to a common attracting fixed point or both critical points are attracted to a common fixed point of multiplicity exactly two and neither of the critical points lands on this point under iteration.*

See [Mi2, Lemma 8.2].

Hyperbolic components in $\boldsymbol{\mathcal{M}_2}$. A rational map f is *hyperbolic* if and only if every critical point tends to some attracting cycle under iteration. The set of all hyperbolic maps is an open and conjecturally dense subset of $\mathcal{M}_2$ and the connected components are called *hyperbolic components*. There are three different types of hyperbolic components in $\mathcal{M}_2$: The *escape locus* where both critical points converge to the same attracting fixed point under iteration. There are infinitely many *capture components* where only one critical point is in the immediate basin of a periodic point, but the other critical point falls into this immediate basin under iteration. We will be especially interested in the hyperbolic components with *disjoint attractors* having two disjoint attracting periodic cycles, each attracting a critical point.

The curves $\mathbf{Per}_{\boldsymbol{n}}\boldsymbol{(\eta)}$. For any integer $n \geq 1$ and any $\eta \in \mathbb{C}$, ($\eta \neq 1$ if $n \geq 2$), we denote by $\mathrm{Per}_n(\eta)$ the set of all conjugacy classes having a periodic cycle of period n with multiplier η:

$$\mathrm{Per}_n(\eta) = \{\langle f \rangle \in M_2 \ : \ f \text{ has a periodic cycle of period } n \text{ with multiplier } \eta\}.$$

Theorem 7.2 $\mathrm{Per}_n(\eta)$ *is an algebraic curve of degree equal to the number of hyperbolic n components in the Mandelbrot set. This number is given by $\vartheta(n)/2$ with $\vartheta(n)$ defined inductively*

$$2^n = \sum_{m|n} \vartheta(m).$$

See [Mi2, Theorem 4.2].

The curve $\mathrm{Per}_1(\eta)$ is a straight line in $\mathcal{M}_2$ with slope $\eta + \frac{1}{\eta}$, if $\eta \neq 0$ and $\mathrm{Per}_1(0)$ is the vertical line $\sigma_1 = 2$. The only ideal limit point of this curve $\mathrm{Per}_n(\eta)$, i.e. the intersection of the closure of $\mathrm{Per}_n(\eta)$ and the sphere (line) at infinity is the ideal point $\langle \eta, \eta^{-1}, \infty \rangle$. The curves $\mathrm{Per}_2(\eta)$ are parallel straight lines in $\mathcal{M}_2$ with slope -2.

Since the degree of a curve is equal to the number of intersections with any line, counted multiplicity, and $\mathrm{Per}_1(\mu)$ is a line in $\mathcal{M}_2$, we have the following corollary:

Corollary 7.3 *Let $n \geq 2$, $\lambda, \eta \in \mathbb{C}$, and $\lambda \neq -1$ if $n = 2$, then the set* $\mathrm{Per}_1(\lambda) \cap \mathrm{Per}_n(\eta)$ *is finite.*

Connectedness locus $\boldsymbol{M^\lambda \subset \mathrm{Per}_1(\lambda)}$. Let M^λ denote the set of all $\langle f \rangle \in \mathrm{Per}_1(\lambda)$ whose Julia set $J(f)$ is connected.

Quadratic polynomials in $\boldsymbol{\mathcal{M}_2}$. The curve $\mathrm{Per}_1(0)$ consists of all elements in $\mathcal{M}_2$ that can be represented as a quadratic polynomial $P_c(z) = z^2 + c$. The connectedness locus $M^0 \subset \mathrm{Per}_1(0)$ is naturally isomorphic to the Mandelbrot set M and the intersection $\mathrm{Per}_n(0) \cap \mathrm{Per}_1(0)$ corresponds to the centerpoints of the hyperbolic n components in M.

Ideal limit points of $\mathrm{Per}_n(\eta)$. We will now prove following Milnor [Mi2] and Epstein [Eps] that the only possible limit points of $\mathrm{Per}_n(\eta)$ are the ideal points $\langle \mu, \mu^{-1}, \infty \rangle$ where μ is a primitive $q-$th root of unity with $1 < q \leq n$. In the following theorem we give an analogous statement using the language of nonstandard analysis. For the external definition of a limit point see Chapter 2. In fact the statement given is slightly stronger, because it is stated for all limited η instead of just for all standard η.

Theorem 7.4 (Possible ideal limit points of $\mathrm{Per}_n(\eta)$ in $\widehat{\mathcal{M}}_2$ – external version) *Let $n \geq 2$ standard and $\eta \in \mathbb{C}$ limited. If $\langle f \rangle \in \mathrm{Per}_n(\eta)$ and $\langle f \rangle \approx \langle \mu, \mu^{-1}, \infty \rangle$ for some $\mu \in \hat{\mathbb{C}}$ standard, then μ is a primitive $q-$th root of unity with $1 < q \leq n$.*

Proof: Let $\langle f \rangle \in \mathrm{Per}_n(\eta)$ and $\langle f \rangle = \langle \mu_1, \mu_2, \mu_3 \rangle \approx \langle \mu, \mu^{-1}, \infty \rangle$ with $\mu \in \hat{\mathbb{C}}$ standard.

1. $\langle \mu, \mu^{-1}, \infty \rangle \neq \langle 0, \infty, \infty \rangle$.

   Let $\mu_1 \approx \mu$ and $\mu_2 \approx \mu^{-1}$. We consider the following normalization:

   $$f(z) = z \frac{z + \mu_1}{\mu_2 z + 1}.$$

   Claim: If $f'(z) \not\approx \infty$ then there exists a $k \approx \mu$ such that $f(z) = k\, z$.

   We have $f(z) = \frac{z}{\mu_2}(1 - \frac{\epsilon}{l(z)})$ with $\epsilon := 1 - \mu_1 \mu_2 \approx 0$ and $l(z) := \mu_2 z + 1$. And $f'(z) = \frac{1}{\mu_2}(1 - \frac{\epsilon}{l(z)^2})$. If $f(z) \neq kz$ for all $k \approx \mu$ then we have $1 - \frac{\epsilon}{l(z)} \not\approx 1$ which implies that $\frac{\epsilon}{l(z)} \not\approx 0$, in particular $\frac{1}{l(z)} \approx \infty$ since $\epsilon \approx 0$. Thus we have $\frac{\epsilon}{l(z)^2} = \frac{\epsilon}{l(z)} \frac{1}{l(z)} \approx \infty$ which implies that $f'(z) \approx \infty$.

   Let $\{z_1, \cdots z_n\}$ denote the periodic cycle. We distinguish several cases:

   (a) If for all $1 \leq i \leq n$ we have $f'(z_i) \not\approx \infty$.

   Then there exist $k_1, \cdots, k_n \approx \mu$ such that

   $$z_1 = f^n(z_1) = k_1 \cdots k_n z_1$$

   which implies, because $z_1 \neq 0$ that

   $$1 = k_1 \cdots k_n \approx \mu^n$$

   and thus by transfer we have that $\mu^n = 1$.

   (b) If there is an $1 \leq i \leq n$ with $f'(z_i) \approx \infty$, but $f'(z_j) \not\approx 0$ for all $1 \leq j \leq n$.

   Then

   $$(f^n)'(z_i) = \underbrace{f'(z_i)}_{\approx\infty} \underbrace{f'(f(z_i)) \cdots f'(f^{n-1}(z_i))}_{\not\approx 0} \approx \infty$$

   which contradicts the fact that $\eta = (f^n)'(z_i)$ is limited.

(c) There are $1 \leq i, j \leq n$ with $f'(z_i) \approx \infty$ and $f'(z_j) \approx 0$. Then z_i and z_j are infinitely close to $-\mu$.

Let q minimal with $f'(f^q(z_j)) \approx \infty$. Then we have $z_j \approx f^q(z_j) \approx \mu^q z_j$, which implies, because $z_j \approx -\mu$ and thus $z_j \not\approx 0$, that $\mu^q \approx 1$ and by transfer that $\mu^q = 1$.

2. $\langle \mu, \mu^{-1}, \infty \rangle = \langle 0, \infty, \infty \rangle$.

We choose a normalization, such that the fixed points are at 1, -1 and ∞:

$$f(z) = z\frac{a + bz + 1}{a + bz + z^2}.$$

Then $f'(0) = 1 + \frac{1}{a}, f'(1) = \frac{a+b-1}{a+b+1}$ and $f'(-1) = \frac{a-b-1}{a-b+1}$. Let $a \approx -1$ and $b \approx 0$, then $f'(0) \approx 0$, in particular 0 is an attracting fixed point for f and $f'(\pm 1) \approx \infty$ and $f(z) \approx 0$ for all $z \not\approx \pm 1$.

Claim: If $f'(z) \not\approx \infty$ then $f(z) \approx 0$.

If $f(z) \not\approx 0$, then $\frac{a+bz+1}{a+bz+z^2} \not\approx 0$ and thus $\frac{1}{a+bz+z^2} \approx \infty$, which implies that

$$f'(z) = \underbrace{\frac{1}{a + bz + z^2}}_{\approx\infty}(\underbrace{a + 2bz + 1}_{\approx 0} - \underbrace{z(b + 2z)\frac{a + bz + 1}{a + bz + z^2}}_{\not\approx 0}) \approx \infty.$$

Furthermore we have that if $|z| < \frac{1}{2}$, then $|f^n(z)| \leq \frac{1}{2^n}|z|$ and thus in particular every point close to 0 is in that basin of the attracting fixed point 0.

So If $f'(z) \not\approx \infty$, then $f(z) \approx 0$, which implies that z is in the basin of 0 and thus in particular not periodic. So we have shown that there are not any maps having a periodic cycle with standard period and limited multiplier close to the ideal point$\langle 0, \infty, \infty \rangle$. Note that in fact all maps close to the point $\langle 0, \infty, \infty \rangle$ have only periodic points with unlimited multiplier except the attracting fixed point.

3. $\langle \mu, \mu^{-1}, \infty \rangle = \langle 1, 1, \infty \rangle$.

Now it remains to show that there are no maps with periodic cycles with standard period and limited multiplier close to the ideal point $\langle 1, 1, \infty \rangle$.

Let $\langle \mu_1, \mu_2, \mu_3 \rangle \approx \langle 1, 1, \infty \rangle$. We conjugate $f(z) = z\frac{z+\mu_1}{\mu_2 z+1}$ with $\mu_1, \mu_2 \approx 1$ in the following way: $0 \mapsto \infty$, $\infty \mapsto \frac{\mu_1}{1-\mu_1} \approx \infty$ and $\frac{1-\mu_1}{1-\mu_2} \mapsto \frac{\mu_1}{\mu_1\mu_3-1} \approx 0$. Thus

$$h(z) = \frac{\frac{\mu_1}{1-\mu_1}z + \frac{\mu_1}{1-\mu_2}(\frac{1-\mu_1}{\mu_1\mu_3-1} - 1)}{z}$$

and

$$g(z) := (h \circ f \circ h^{-1})(z) = \frac{z}{\mu_1} + 1 + \frac{\mu_1 - 1}{\mu_1\mu_3 - 1} - \frac{\mu_1}{(\mu_1\mu_3 - 1)z} \approx z + 1$$

for all $z \not\approx 0$. Furthermore we have

$$g'(z) = \frac{1}{\mu_1} + \frac{\mu_1}{(\mu_1\mu_3 - 1)z^2} \approx 1$$

for all $z \not\approx 0$.

Claim: If $g'(z) \not\approx \infty$ then $g(z) \approx z + 1$.

If $g(z) \not\approx z + 1$ then $\frac{\mu_1}{(\mu_1\mu_3-1)z} \not\approx 0$, which implies, because $z \approx 0$, that $\frac{\mu_1}{(\mu_1\mu_3-1)z^2} \approx \infty$ and thus $g'(z) \approx \infty$.

Now assume that g has a limited periodic cycle with standard period $n \geq 2$. Then this cycle $\{z_1, \cdots, z_n\}$ must contain a point z_j with $g'(z_j) \approx \infty$. Furthermore $g'(z_i) \not\approx 0$ for all $1 \leq i \leq n$, because otherwise $z_i \approx 0$ and $g^n(z_i) \approx n$, and thus not periodic. This implies that

$$(g^n)'(z_j) = \underbrace{g'(z_j)}_{\approx\infty} \underbrace{g'(g(z_j)) \cdots g'(g^{n-1}(z_j))}_{\not\approx 0} \approx \infty.$$

It remains to show that there no periodic cycle with standard period $n \geq 2$ in $\mathrm{hal}(\infty)$. So let $n \geq 2$ be standard. We have $g(z) \approx z + 1$ on $\{z \in \mathbb{C} : |z| > 1\} \cup \{\infty\}$ and $g^n(z) \approx z + n$ on $\{z \in \mathbb{C} : |z| > n + 1\} \cup \{\infty\}$. Since $z \mapsto z + n$ has only a double fixed point at ∞, g^n has by Rouché's theorem, see Lemma 4.1, at most two fixed points on $\{z \in \mathbb{C} : |z| > n + 1\} \cup \{\infty\}$ (note that conjugation by $z \mapsto 1/z$ maps the considered neighborhood of ∞ to a neighborhood of 0, so that we can actually apply Lemma 4.1). Because g has already two fixed points close to infinity, there cannot be any further periodic points.

q.e.d

Corollary 7.5 (Ideal limit points of $\mathrm{Per}_n(\eta)$ in $\widehat{\mathcal{M}_2}$) *The only possible ideal limit points of $\mathrm{Per}_n(\eta)$ for $n \geq 2$ are the points $\langle \mu, \mu^{-1}, \infty \rangle$ where μ is a primitive $q-$th root of unity with $1 < q \leq n$.*

We will see later that all these ideal points actually occur as limit points of $\mathrm{Per}_n(\eta)$.

Chapter 8

The family $G_T(z) = z + \frac{1}{z} + T$

In this chapter we summarize some results about the family $G_T(z) = z + \frac{1}{z} + T$ with $T \in \mathbb{C}$.

The family $G_T(z) = z + \frac{1}{z} + T$ with $T \in \mathbb{C}$ parameterizes the slice $\mathrm{Per}_1(1) \subset \mathcal{M}_2$. Each map G_T has a fixed point with multiplier 1 at ∞. If $T \neq 0$ then G_T has another fixed point $z_0 = -1/T$ with multiplier $1 - T^2$. The critical points are at ± 1. Two maps in the family, G_T and $G_{\widetilde{T}}$, are holomorphically conjugate if and only if $T^2 = \widetilde{T}^2$. G_T is conjugate to G_{-T} by $z \mapsto -z$. Furthermore we have $G_T(1/z) = G_T(z)$. By Theorem 4.4 at least one of the critical points ± 1 is in the basin of the parabolic fixed point ∞.

Examples:

1. The map $G_0(z) = z + \frac{1}{z}$ has a fixed point of multiplicity 3 at ∞. The Julia set of G_0 is the imaginary axis $J(G_0) = \{z \in \mathbb{C} : \mathrm{Re}(z) = 0\} \cup \{\infty\}$.
2. The map $G_{-1}(z) = z + \frac{1}{z} - 1$ has a superattracting fixed point: $G_{-1}(1) = 1$.
3. The map $G_{-3/2}(z) = z + \frac{1}{z} - \frac{3}{2}$ has a superattracting period 2 cycle: $1 \mapsto \frac{1}{2} \mapsto 1$.
4. The map $G_{-2}(z) = z + \frac{1}{z} - 2$ has the property that the critical point 1 is mapped to the parabolic fixed point ∞ after two iterations: $1 \mapsto 0 \mapsto \infty$.

The connectedness locus of the family $G_T(z) = z + \frac{1}{z} + T$. The Julia set of G_T is connected if and only if only one of the critical points is attracted to the parabolic fixed point at ∞ or if both of them lie in the basin of ∞ and one of the critical points lands on ∞ under iteration, or if $T = 0$, in that case ∞ is a fixed point of multiplicity 3. Otherwise the Julia set is totally disconnected. Compare Theorem 7.1.

Lemma 8.1 *Let $g(z) = \frac{1}{\lambda}(z + \frac{1}{z} + iT)$ with $T \in \mathbb{R}$ and $\lambda \in \mathbb{R} - \{0\}$. Then $|g^n(1)| = |g^n(-1)|$ for all $n \in \mathbb{N}$.*

Proof: We have $\mathrm{Re}(g^n(1)) = -\mathrm{Re}(g^n(-1))$ and $\mathrm{Im}(g^n(1)) = \mathrm{Im}(g^n(-1))$. q.e.d.

Corollary 8.2 *If $\mathrm{Re}(T) = 0$ and $T \neq 0$ then $J(G_T)$ is not connected.*

Proof: Since ∞ is a parabolic fixed point for G_T with multiplier 1 at least one of the critical points ± 1 is in its basin, compare Theorem 4.4. The previous lemma implies then that both critical points are in the basin of ∞. By Theorem 7.1 this implies that the Julia set is totally disconnected, unless $T = 0$ (G_0 has a triple fixed point at ∞).

q.e.d.

Note that the imaginary axis $\{z \in \mathbb{C} : \mathrm{Re}(z) = 0\} \cup \{\infty\}$ is a completely invariant set for $g(z) = \frac{1}{\lambda}(z + \frac{1}{z} + iT)$. Thus the Julia set $J(g)$ must be a subset of the imaginary axis.

Note that the connectedness locus $M^1 \subset \mathrm{Per}_1(1) \subset \mathcal{M}_2$ is naturally isomorphic to the connectedness locus of the G_T-family in the T^2 plane:

$$\{T^2 \in \mathbb{C} : J(G_T) \text{ is connected}\}$$

or alternatively to the connectedness locus of the G_T-family in the left half plane $\{T \in \mathbb{C} : \mathrm{Re}(T) \leq 0 \text{ and } J(G_T) \text{ is connected}\}$.

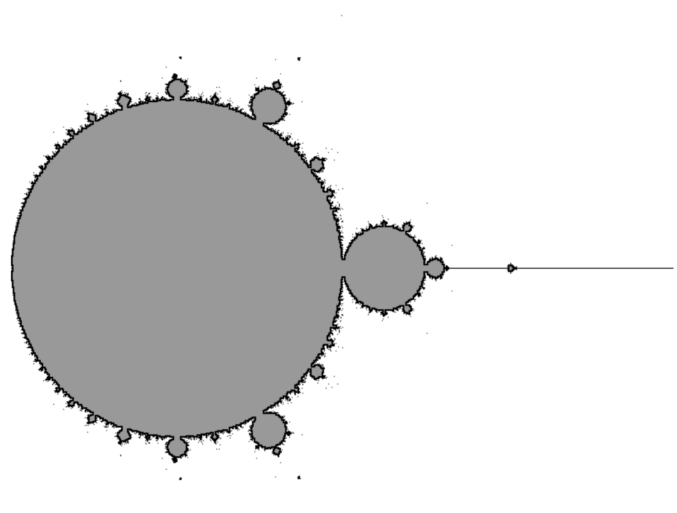


Figure 8.1: The connectedness locus of the family $G_T(z) = z + \frac{1}{z} + T$ in the T^2-plane.

Lemma 8.3 *If $|T| > 3$ then $J(G_T)$ is totally disconnected.*

Proof: Claim: Let $|T| > 3$. The we have $|G_T^n(\pm 1) - nT| \leq n + 1$ for all $n \geq 1$. This implies that $|G_T^n(\pm 1)| \geq n(|T| - 1) - 1 \geq 2n - 1$, in particular $|G_T^n(\pm 1)| \geq 1$.

<u>$n = 1$:</u>

$$|G_T(\pm 1) - T| = 2.$$

<u>$n \to n + 1$:</u>

$$|G_T^{n+1}(\pm 1) - (n + 2)T| = |G_T^n(\pm 1) - \frac{1}{G_T^n(\pm 1)} + T - (n + 1)T|$$

$$\leq \underbrace{|G_T^n(\pm 1) - nT|}_{\leq n+1} + \underbrace{\frac{1}{|G_T^n(\pm 1)|}}_{\leq 1} \leq (n+1) + 1 = n + 2.$$

Thus both critical points are in the basin of the double fixed point ∞.

q.e.d

Lemma 8.4 (The real connectedness locus) *The real connectedness locus* $\{T \leq 0 :$ *$J(G_T)$ is connected*$\}$ *is equal to the interval* $[-2, 0]$.

Proof: We show that for $T \in (-2, 0)$ the critical point 1 is not in the basin of the parabolic fixed point ∞ for G_T. For $T = 2$ it lands on ∞, and for $T = 0$ the point ∞ is a triple fixed point. Thus for $T \in [-2, 0]$ the Julia set $J(G_T)$ is connected. If $T < -2$, both critical points ± 1 are in the basin of ∞ and thus the Julia set is totally disconnected.

For $T \in [-1, 0)$ G_T has an attracting fixed point, which must attract a critical point. Now let $T \in (-2, -1)$. Then $G_T(1) = 2 + T > 0$ is the minimum on $[0, \infty]$ and $G_T([2 + T, 1/(2 + T)]) \subset [2 + T, 1/(2 + T)]$: Let $z \in [2 + T, 1/(2 + T)]$. If $z > 1$, then we have $G_T(z) = z + \frac{1}{z} + T < z$, and if $z \leq 1$, then we have $G_T(z) = z + \frac{1}{z} + T < \frac{1}{z} \leq 1/(2 + T)$. Thus the critical point 1 is not in the basin of ∞.

If $T < 2$, then $G_T(1) = 2 + T < 0$ and every $y < 0$ is in the basin of ∞, because $G_T^n(y) < y + nT$. Thus both critical points are attracted to ∞.

q.e.d

We will see later that the real connectedness locus $[-2, 0]$ is naturally isomorphic to the real connectedness locus $M^1_{\mathbb{R}} = \{\langle f \rangle \in M^1 \subset \mathrm{Per}_1(1) : \ \langle f \rangle$ has a real representative$\}$. See Chapter 15. For this we need Corollary 8.2, because $z \mapsto z + \frac{1}{z} + iT$ is conjugated to the real rational map $z \mapsto z - \frac{1}{z} + T$ by $z \mapsto iz$, but in this case that Julia set is not connected.

It is a well-known conjecture that the connectedness locus $M^1 \subset \mathrm{Per}_1(1)$ is a homeomorphic copy of the Mandelbrot set M [Mi2, p. 27]. In Chapter 16 we establish a dynamical homeomorphism between the real Mandelbrot set $[-2, 1/4]$ and the real connectedness locus $[-2, 0]$ of the G_T-family, which is naturally isomorphic to $M^1_{\mathbb{R}}$.

Chapter 9

Epstein's Theorem

In this chapter we give a variation of a theorem of Epstein. Epstein showed that if $\langle f\rangle = \langle \mu_1, \mu_2, \mu_3\rangle \in \mathcal{M}_2$ is close to the ideal point $\langle e^{2\pi ip/q}, e^{-2\pi ip/q}, \infty\rangle \in \widehat{\mathcal{M}}_2$ for some $q \geq 2$ and $\frac{\mu_1^q - 1}{\sqrt{1-\mu_1\mu_2}} \approx T$ for some $T \in \mathbb{C}$, then the limiting map of the q-th iterate of a certain normalization of f is equal to the quadratic rational map $G_T(z) = z + \frac{1}{z} + T$.

Theorem 9.1 (Epstein's theorem)
Let $\langle \alpha_n, \beta_n, \gamma_n\rangle \in \mathcal{M}_2$ with $\langle \alpha_n, \beta_n, \gamma_n\rangle \to \langle e^{2\pi ip/q}, e^{-2\pi ip/q}, \infty\rangle$ with $q \geq 2$ and $\lim_{n\to\infty} \frac{\alpha_n^q - 1}{\sqrt{1-\alpha_n\beta_n}} = T$ for some $T \in \mathbb{C}$. Then there exists $f_n \in \langle \alpha_n, \beta_n, \gamma_n\rangle$ such that the q-th iterate f_n^q converges to G_T locally uniformly on $\hat{\mathbb{C}} - \{0, \infty\}$.

See [Eps, Proposition 2]. Epstein normalized the maps f_n so that the critical points are at ± 1 and the fixed points with multiplier $\gamma_n \to \infty$ at 0.

In the following theorem, which is just a variation of Epstein's Theorem, we consider a rational map of the form $F_{\lambda,S}(z) = \frac{1}{\lambda}(z + \frac{1}{z} + S)$ with λ close to a q-th root of unity and $S \approx \infty$. Then $\langle F_{\lambda,S}\rangle \approx \langle e^{2\pi ip/q}, e^{-2\pi ip/q}, \infty\rangle$. The map $F_{\lambda,S}$ is suitably normalized to determine the quadratic limiting map of the q-th iterate. We give the parameter for the limiting map of the q-th iterate $F^q_{\lambda,S}$ in terms of S and λ. Let G_∞ denote the constant map ∞.

Theorem 9.2 *Let*

$$F_{\lambda,S}(z) = \frac{1}{\lambda}(z + \frac{1}{z} + S).$$

Then the n-th iterate is given by

$$F^n_{\lambda,S}(z) = \frac{1}{\lambda^n}\left(z + \frac{1}{z} + S\frac{1-\lambda^n}{1-\lambda}\right) + r(z) \text{ with } r(z) = \sum_{i=1}^{n-1} \frac{1}{\lambda^{n-i} F^i_{\lambda,S}(z)}.$$

If $\lambda \approx e^{2\pi ip/q}$ for some $q \geq 2$ standard and $S \approx \infty$. If $1 \leq i < q$ then we have

$$F^i_{\lambda,S}(z) \approx \infty \text{ for all } z \not\approx 0, \infty$$

and

$$F^q_{\lambda,S}(z) \approx G_T(z) \text{ for all } z \not\approx 0, \infty \text{ with } T = {}^\circ(S\frac{1-\lambda^q}{1-\lambda}).$$

Proof: Claim: $F^n_{\lambda,S}(z) = \frac{1}{\lambda^n}\left(z + \frac{1}{z} + S\frac{1-\lambda^n}{1-\lambda}\right) + r(z)$ with $r(z) = \sum_{i=1}^{n-1} \frac{1}{\lambda^{n-i}F^i_{\lambda,S}(z)}$.

For $n = 1$ this is trivial.

$n \to n+1:$

$$F^{n+1}_{\lambda,S}(z) = \frac{1}{\lambda}(F^n_{\lambda,S}(z) + \frac{1}{F^n_{\lambda,S}(z)} + S) =$$

$$\frac{1}{\lambda^{n+1}}\left(z + \frac{1}{z} + S\frac{1-\lambda^n}{1-\lambda} + S\lambda^n\right) + \frac{1}{\lambda}\sum_{i=1}^{n-1}\frac{1}{\lambda^{n-i}F^i_{\lambda,S}(z)} + \frac{1}{\lambda F^n_{\lambda,S}(z)}$$

$$= \frac{1}{\lambda^{n+1}}\left(z + \frac{1}{z} + S\frac{1-\lambda^{n+1}}{1-\lambda}\right) + \sum_{i=1}^{n}\frac{1}{\lambda^{n+1-i}F^i_{\lambda,S}(z)}$$

Now let $\lambda \approx e^{2\pi ip/q}$ for some $q \geq 2$ standard and $S \approx \infty$.

Claim: If $1 \leq i < q$ then $F^i_{\lambda,S}(z) \approx \infty$ for all $z \not\approx 0, \infty$.

This follows by external induction (see Theorem 2.1):
$F_{\lambda,S}(z) \approx \infty$, because $S \approx \infty$. If $i + 1 < q$ then we have

$$F^{i+1}_{\lambda,S}(z) = \frac{1}{\lambda^{i+1}}\left(z + \frac{1}{z} + S\frac{1-\lambda^{i+1}}{1-\lambda}\right) + r_{i+1}(z) \approx \infty$$

because we have $S\frac{1-\lambda^{i+1}}{1-\lambda} \approx \infty$ and by induction hypothesis we have that $r_{i+1}(z) = \sum_{j=1}^{i}\frac{1}{\lambda^{i-j}F^j_{\lambda,S}(z)} \approx 0$ for all $z \not\approx 0, \infty$.

Claim: Let $T = {}^\circ(S\frac{1-\lambda^q}{1-\lambda})$, then $F^q_{\lambda,S}(z) \approx G_T(z)$ for all $z \not\approx 0, \infty$.

$$F^q_{\lambda,S}(z) = \frac{1}{\lambda^q}\left(z + \frac{1}{z} + S\frac{1-\lambda^q}{1-\lambda}\right) + r_q(z) \approx z + \frac{1}{z} + T = G_T(z)$$

because $r_q(z) = \sum_{j=1}^{q-1}\frac{1}{\lambda^{q-j}F^j_{\lambda,S}(z)} \approx 0$ and $\lambda^q \approx 1$. q.e.d.

Let $\langle f\rangle \in \mathcal{M}_2$ having fixed points with multipliers $\mu_1 = \lambda \approx e^{2\pi ip/q}, \mu_2 \approx e^{-2\pi ip/q}$ and $\mu_3 \approx \infty$. Then we have $S^2\frac{(1-\lambda^q)^2}{(1-\lambda)^2} \approx \frac{(1-\mu_1^q)^2}{1-\mu_1\mu_2}$, with $S^2 = \frac{(\mu_2-\mu_3)^2}{1-\mu_2\mu_3}$. Thus the conditions $\frac{\mu_1^q-1}{\sqrt{1-\mu_1\mu_2}} \approx T$ and ${}^\circ(S\frac{1-\lambda^q}{1-\lambda}) = T$ are equivalent.

$$S^2\frac{(1-\lambda^q)^2}{(1-\lambda)^2} = \frac{(\mu_2 - \frac{2-\mu_1-\mu_2}{1-\mu_1\mu_2})^2}{1-\mu_2\frac{2-\mu_1-\mu_2}{1-\mu_1\mu_2}}\frac{(1-\mu_1^q)^2}{(1-\mu_1)^2}$$

$$= \frac{(1-\mu_1^q)^2}{1-\mu_1\mu_2}\underbrace{\frac{(\mu_2(1-\mu_1\mu_2) - 2 + \mu_1 + \mu_2)^2}{1-\mu_1\mu_2 - 2\mu_2 + \mu_1\mu_2 + \mu_2^2}}_{\approx 4}\underbrace{\frac{1}{(1-\mu_1)^2}}_{\approx\frac{1}{4}} \approx \frac{(1-\mu_1^q)^2}{1-\mu_1\mu_2}.$$

Corollary 9.3 (Classical version) *Let*

$$F_m(z) = \frac{1}{\lambda_m}\left(z + \frac{1}{z} + S_m\right)$$

with $\lim_{m\to\infty} \lambda_m = e^{2\pi ip/q}$ *for some* $q \geq 2$ *and* $\lim_{m\to\infty} S_m = \infty$. *Then* F_m^i *converges to the constant function* ∞ *locally uniformly on* $\hat{\mathbb{C}} - \{0, \infty\}$ *for all* $1 \leq i < q$. *If*

$$\lim_{m\to\infty} S_m \frac{1-\lambda_m^q}{1-\lambda_m} = T$$

for some $T \in \hat{\mathbb{C}}$, *then the* q*-th iterate* F_m^q *converges to* G_T *locally uniformly on* $\hat{\mathbb{C}} - \{0, \infty\}$ *as* $m \to \infty$.

Proof: Apply the axiom of transfer and use the external characterization of convergence and locally uniform convergence outside a finite set, see Lemma 5.3.

q.e.d.

Chapter 10

Uniqueness of dynamical limits

In this chapter we prove that the quadratic limiting map in Epstein's theorem (Theorem 9.1) is unique in the sense that it is independent from the choice of representation. To formalize this statement we will use the notion of the standard shadow of a rational map introduced in Chapter 5: If the shadow of the q-th iterate of an arbitrary representation is of degree two then it is conjugate to $G_T(z) = z + \frac{1}{z} + T$. Furthermore we have in the case that the assumptions of Epstein's theorem are not satisfied, that the shadow of all the iterates of every representation of f is of degree at most one.

Theorem 10.1 (Uniqueness theorem)
Let $\langle \mu_1, \mu_2, \mu_3 \rangle \in \mathcal{M}_2$ with $\mu_1 \approx \mu$, $\mu_2 \approx \mu^{-1}$ and $\mu_3 \approx \infty$, where μ is a primitive q-th root of unity, $q \geq 2$. Then we have

1. *There exists a $g \in \langle \mu_1, \mu_2, \mu_3 \rangle$ such that $\deg(\mathrm{sh}(g^q)) = 2$ if and only if $\frac{\mu_1^q - 1}{\sqrt{1-\mu_1\mu_2}}$ is limited.*
2. *If $\frac{\mu_1^q - 1}{\sqrt{1-\mu_1\mu_2}}$ is limited, i.e. $\frac{\mu_1^q - 1}{\sqrt{1-\mu_1\mu_2}} \approx T$ for some standard $T \in \mathbb{C}$, then for all $g \in \langle \mu_1, \mu_2, \mu_3 \rangle$ with $\deg(\mathrm{sh}(g^q)) = 2$ the shadow of g^q is holomorphically conjugate to $G_T(z) = z + \frac{1}{z} + T$.*

We give a classical version of this theorem at the end of this chapter.

The proof of this theorem will show that there are up to conjugation with a s-continuous Möbius transformation q different ways to obtain a limiting quadratic rational map. We will describe those in the following theorem.

Theorem 10.2 (Complementary result)
Let $\langle \mu_1, \mu_2, \mu_3 \rangle \in \mathcal{M}_2$ with $\mu_1 \approx \mu$, $\mu_2 \approx \mu^{-1}$ and $\mu_3 \approx \infty$, where μ is a primitive q-th root of unity, $q \geq 2$, and $\frac{\mu_1^q - 1}{\sqrt{1-\mu_1\mu_2}} \approx T$ for some standard $T \in \mathbb{C}$. Consider the following representative $f \in \langle \mu_1, \mu_2, \mu_3 \rangle$:

$$f(z) = \frac{z}{\mu_2}(1 + \frac{d\epsilon}{z-d})$$

with $\epsilon = 1 - \mu_1\mu_2$ *and* $d = 1 - \frac{\epsilon}{\mu_2(1-\mu_1)}$.
Let $h(z) = \delta z + a$ *with* $\frac{\sqrt{\epsilon}}{\delta} \not\approx 0, \infty$ *and* $\frac{a - d\mu_2^l}{\sqrt{\epsilon}} \not\approx \infty$ *for some* $0 \leq l < q$. *Then the maps* $g \in \langle \mu_1, \mu_2, \mu_3 \rangle$ *whose shadow of the q-th iterate* g^q *is of degree two are precisely those that arise from*

$$h^{-1} \circ f^q \circ h$$

by conjugation with a s-continuous Möbius transformation.

Idea of the proof of the uniqueness theorem. We start with the map $f(z) = \frac{z}{\mu_2}(1 + \frac{d\epsilon}{z-d})$ – this map has the property that the shadow of its q-th iterate is the identity – and conjugate the q-th iterate f^q with Möbius transformations $h(z) = \delta z + a$ with $\delta \approx 0$ that make the dynamics in a small neighborhood of the point a visible and show that either the shadow of the resulting map is of lower degree or conjugate to G_T. We do so by either calculating the q-th iterate of the conjugate map, or in some cases the derivative or nonlinearity, which might include that the shadow must be of lower degree, compare Lemma 10.5. First we give some results about Möbius transformations that justify this procedure – to just conjugate by affine non-s-continuous Möbius transformations.

The normalization with fixed points at 0, ∞, 1. The normalization

$$f(z) = \frac{z}{\mu_2}(1 + \frac{d\epsilon}{z-d})$$

with $\epsilon = 1 - \mu_1\mu_2$ and $d = 1 - \frac{\epsilon}{\mu_2(1-\mu_1)}$ has fixed points at 0, ∞ and 1 with multipliers μ_1, μ_2 and $\frac{2-\mu_1-\mu_2}{1-\mu_1\mu_2}$ respectively. Conjugation with $z \mapsto \frac{1}{z}$ interchanges the fixed points at 0 and ∞. Note that $\epsilon \approx 0$ and $d \approx 1$ if $\langle \mu_1, \mu_2, \mu_3 \rangle \approx \langle \mu, \mu^{-1}, \infty \rangle$.

About Möbius transformations. A Möbius transformation $h \in \mathrm{Aut}(\hat{\mathbb{C}})$ is either s-continuous at every point $z \in \hat{\mathbb{C}}$ or it is not s-continuous at exactly one standard point. If h is a non-s-continuous Möbius transformation a small neighborhood is mapped over almost the whole sphere and everything else is mapped close to one point.

Lemma 10.3 *Let h be a Möbius transformation that is not s-continuous at the standard point* $z_0 \in \mathbb{C}$. *Then h can be written as the composition of a s-continuous Möbius transformation and* $z \mapsto \frac{1}{\delta}(z-a)$ *with* $\delta \approx 0$ *for some* $a \approx z_0$.

Proof: Let $h \in \mathrm{Aut}(\hat{\mathbb{C}})$ not s-continuous at the point $z_0 \in \mathbb{C}$ standard. Assume that h makes the points $z_0 \approx z_1 \approx z_2$ visible, meaning that after applying h there are not close to each other anymore. Define

$$h_1(z) = \frac{z - z_0}{\delta}$$

choosing δ such that $h_1(z_1) = 1$. We have $h_1(z_0) = 0$.
Now it might be that $h_1(z_2) \approx 0 = h_1(z_0)$ (or $h_1(z_2) \approx 1 = h_1(z_1)$). Then define

$$h_2(z) = \frac{z}{\tilde{\delta}}$$

choosing $\tilde{\delta}$ such that $h_2(h_1(z_2)) = 1$. Then we have $h_2(h_1(z_0)) = h_2(0) = 0$ and $h_2(h_1(z_1)) \approx \infty$ — so the three points are not close to each other anymore. And

$$(h_2 \circ h_1)(z) = \frac{1}{\delta\tilde{\delta}}(z - z_0).$$

(or

$$h_2(z) = \frac{z - h_1(z_1)}{\tilde{\delta}}$$

with $\tilde{\delta}$ such that $h_2(h_1(z_2)) = 1$) By applying a s-continuous Möbius transformation one can move the three points to the required places. q.e.d.

Lemma 10.4 *Let $h \in \mathrm{Aut}(\hat{\mathbb{C}})$ be s-continuous and $f \in \mathrm{Rat}_d$. Then we have*

$$\mathrm{sh}(h^{-1} \circ f \circ h) = \mathrm{sh}(h)^{-1} \circ \mathrm{sh}(f) \circ \mathrm{sh}(h).$$

In particular, if $h \in \mathrm{Aut}(\hat{\mathbb{C}})$ is s-continuous, then $\mathrm{sh}(h^{-1} \circ f \circ h)$ is holomorphically conjugate to $\mathrm{sh}(f)$.

Proof: There exist standard $z_1, \cdots z_k$ such that

$$f(z) \approx \mathrm{sh}(f)(z) \quad \text{for all } z \not\approx z_1, \cdots z_k$$

Since $\mathrm{sh}(f)$ is a standard rational map and thus s-continuous we have for all standard $z \neq z_1, \cdots z_n$ that

$$\tilde{z} \approx z \;\Rightarrow\; f(\tilde{z}) \approx f(z).$$

And since h is s-continuous we have

$$h(z) \approx \mathrm{sh}(h(z)) \quad \text{for all } z$$

This implies for all $z \not\approx h^{-1}(z_1), \cdots h^{-1}(z_k)$

$$f(h(z)) \approx f(\mathrm{sh}(h)(z)) \approx \mathrm{sh}(f)(\mathrm{sh}(h)(z)))$$

which implies

$$h^{-1}(f(h(z))) \approx h^{-1}(\mathrm{sh}(f)(\mathrm{sh}(h)(z)))) \approx \mathrm{sh}(h)^{-1}(\mathrm{sh}(f)(\mathrm{sh}(h)(z))).$$

Thus we have $h^{-1} \circ f \circ h \simeq \mathrm{sh}(h)^{-1} \circ \mathrm{sh}(f) \circ \mathrm{sh}(h)$. Lemma 5.1 implies that $\mathrm{sh}(h^{-1} \circ f \circ h) = \mathrm{sh}(h)^{-1} \circ \mathrm{sh}(f) \circ \mathrm{sh}(h)$. q.e.d.

The nonlinearity of a holomorphic function. For a holomorphic map f the nonlinearity of f at a point z is defined as follows:

$$Nf(z) = \frac{f''(z)}{f'(z)}$$

It satisfies the following chain rule:

$$N(f \circ g)(z) = Nf(g(z))g'(z) + Ng(z)$$

Furthermore we have $Nf = 0$ if and only if $f(z) = az + b$.

Lemma 10.5 *Let f be a rational map of standard degree. Then we have:*

1. *If $f' \simeq 0$ and $\mathrm{sh}(f) \not\equiv \infty$ then the shadow of f is constant.*
2. *If $f' \simeq 1$ and $\mathrm{sh}(f) \not\equiv \infty$ then $\mathrm{sh}(f) = z + b$ for some standard $b \in \mathbb{C}$.*
3. *If $f' \simeq \infty$ then the shadow of f is the constant map infinity.*
4. *If $Nf \simeq 0$ and $\mathrm{sh}(f) \not\equiv \infty$ then $\mathrm{sh}(f) = az + b$.*

Proof:

1. There exits a standard domain $U \subset \mathbb{C}$ with $\mathrm{sh}(f)(x) \approx f(x)$ for all $x \in U$. Since $\mathrm{sh}(f) \neq \infty$ this domain can be chosen such that $f(x) \not\approx \infty$ for all $x \in U$. By the Cauchy integral formula for derivatives we get that $\mathrm{sh}(f)'(x) \approx f'(x) \approx 0$ for all standard $x \in U$ and by transfer we can conclude that $\mathrm{sh}(f)'(x) = 0$ for all $x \in U$, which implies that $\mathrm{sh}(f)$ is constant on U and by the Identity theorem it is constant on $\hat{\mathbb{C}}$.
2. Analogously to part 1.
3. Assume that $\mathrm{sh}(f) \not\equiv \infty$. Like before chose a standard neighborhood $U \subset \mathbb{C}$ with $\mathrm{sh}(f)(x) \approx f(x)$ and $f(x) \not\approx \infty$ for all $x \in U$. This implies that $\mathrm{sh}(f)'(x) \not\approx \infty$ for all $x \in U$. By the Cauchy integral formula for derivatives we get that $\mathrm{sh}(f)'(x) \approx f'(x)$ for all standard $x \in U'$ for some standard neighborhood $U' \subset U$. Thus $f'(x) \approx \mathrm{sh}(f)'(x) \not\approx \infty$ for all $x \in U'$ which contradicts $f' \simeq \infty$.
4. Analogously to part 1. Use that $Ng = 0$ implies that $g(z) = az + b$. q.e.d.

Proof of the uniqueness theorem: We will start out with

$$f(z) = \frac{z}{\mu_2}(1 + \frac{d\epsilon}{z - d})$$

and then look at all maps that are holomorphically conjugate.

Note that if $f'(z) \not\approx \infty$ then $f(z) \approx \frac{z}{\mu_2}$.
Because $f'(z) = \frac{1}{\mu_2}(1 - \frac{d^2\epsilon}{(z-d)^2}) \not\approx \infty$ implies that $\frac{\sqrt{\epsilon}}{z-d} \not\approx \infty$ which implies that $\sqrt{\epsilon}\frac{\sqrt{\epsilon}}{z-d} \not\approx \infty = \frac{\epsilon}{z-d} \approx 0$ and thus $f(z) = \frac{z}{\mu_2}(1 + \frac{d\epsilon}{z-d}) \approx \frac{z}{\mu_2}$.

Since $f^q(z) \approx \frac{z}{\mu_2^q} \approx z$ for all $z \not\approx \mu^i, 0 \leq i < q$, the shadow of every conjugate of f^q by an s-continuous transformation is the identity, see lemma 10.4. Thus we just have to consider non-s-continuous transformations. According to lemma it suffices to consider transformations of the following from.

Considered Möbius transformations.

$$h(z) = a + \delta z \quad h^{-1}(z) = \frac{z - a}{\delta} \text{ with } \delta \approx 0 \text{ and } a \in \mathbb{C} \text{ limited.}$$

Note, that we do not have to consider $a \approx \infty$ since conjugation with $z \mapsto 1/z$ interchanges the roles of 0 and∞.

Definition of k_i.

$$k_i(z) = \frac{\sqrt{\epsilon}}{f^i(h(z)) - d} \text{ for } i = 0, \cdots, q-1.$$

The q-th iterate of g.

$$f^q(z) = \frac{z}{\mu_2^q} \prod_{i=0}^{q-1} (1 + \frac{d\epsilon}{f^i(z) - d}).$$

Thus

$$g^q(z) = (h^{-1} \circ f^q \circ h)(z) = \frac{1}{\delta}(f^q(h(z)) - a)$$

$$= \frac{1}{\delta}\left[\left(\frac{\delta z + a}{\mu_2^q} \prod_{i=0}^{q-1} (1 + \frac{d\epsilon}{f^i(h(z)) - d})\right) - a\right] = \frac{1}{\delta}\left[\left(\frac{\delta z + a}{\mu_2^q} \prod_{i=0}^{q-1} (1 + d\sqrt{\epsilon} k_i(z))\right) - a\right]$$

The derivative of g^q.

$$(g^q)'(z) = (h^{-1} \circ f^q \circ h)'(z) = \frac{1}{\delta}(f^q)'(h(z))\delta = (f^q)'(h(z))$$

$$= \prod_{i=0}^{q-1} f'(f^i(h(z)) = \frac{1}{\mu_2^q} \prod_{i=0}^{q-1} (1 - d^2 k_i(z)^2).$$

Since $f'(z) = \frac{1}{\mu_2}(1 - \frac{d^2\epsilon}{(z-d)^2})$ and thus $f'(f^i(h(z))) = \frac{1}{\mu_2}(1 - d^2 k_i(z)^2)$.

The nonlinearity of g^q.

$$Ng^q(z) = N(h^{-1} \circ f^q \circ h)(z)$$

$$= \underbrace{Nh^{-1}((f^q \circ h)(z))(f^q \circ h)'(z)}_{=0} + \sum_{i=0}^{q-1} (Nf((f^i \circ h)(z)))(f^i \circ h)'(z) + \underbrace{Nh(z)}_{=0}$$

Because $Nh^{-1} = 0$ and $Nh = 0$, since the second derivatives of h and h^{-1} are equal to 0, this is equal to

$$\sum_{i=0}^{q-1} \frac{f''(f^i(h(z)))}{f'(f^i(h(z)))}(f^i \circ h)'(z) = \sum_{i=0}^{q-1} \frac{f''(f^i(h(z)))}{f'(f^i(h(z)))}(f^i)'(h(z))h'(z)$$

Since $h'(z) = \delta$ this is equal to

$$\delta \sum_{i=0}^{q-1} \frac{f''(f^i(h(z)))}{f'(f^i(h(z)))}(f^i)'(h(z)).$$

Because $f''(z) = \frac{2d^2\epsilon}{\mu_2(z-d)^3}$ and thus $f''(f^i(h(z))) = \frac{1}{\mu_2}\frac{2d^2\epsilon}{(f^i(h(z))-d)^3} = \frac{1}{\mu_2}\frac{2d^2}{\sqrt{\epsilon}}k_i(z)^3$ we have then

$$Ng^q = \frac{2d^2\delta}{\sqrt{\epsilon}}\sum_{i=0}^{q-1}\left[\frac{k_i(z)^3}{\mu_2^i(1-d^2k_i(z)^2)}\prod_{j=0}^{i-1}(1-d^2k_j(z))\right].$$

Different cases. We will distinguish several cases and show that in each case the standard shadow of g^q is either conjugate to G_T - this can only happen if $\frac{\mu_1^q-1}{\sqrt{1-\mu_1\mu_2}} \approx T$ for some standard $T \in \mathbb{C}$ - or the shadow is of lower degree.

First we note that if $a \not\approx \mu_2^l$ for some $0 \leq l < q$ then $k_i(z) \simeq 0$ for all $0 \leq i < q$ and thus $(g^q)'(z) \simeq 1$ which implies that the shadow is either the constant map infinity or of the form $z \mapsto z + b$ for some standard b, see lemma 10.5. Thus we only have to consider Möbius transformations with $a \approx \mu_2^l$ for some $0 \leq l < q$.

1. $f'(f^i(h(z))) \simeq \infty$ for some $0 \leq i < q$

   (a) $f'(f^j(h(z))) \not\simeq 0$ for all $0 \leq j < q$
   Then we have
   $$(g^q)'(z) = \prod_{i=0}^{q-1} f'(f^i(h(z))) \simeq \infty.$$
   Thus the shadow of g^q is the constant map infinity, see lemma 10.5.

   (b) $f'(f^j(h(z))) \simeq 0$ for some $0 \leq j < q$
   We have $f^i(h(z)) \simeq 1$ and $f^j(h(z)) \simeq 1$, otherwise the derivative at these points would be close to $\frac{1}{\mu_2}$. We also have that $j > i$, because $f(z) \simeq \frac{z}{\mu_2}$ whenever $f'(z) \not\simeq \infty$ and so it requires q iterations of $f^j(h(z)) \simeq 1$ before landing close to 1 again. Thus we have
   $$f^k(\underbrace{f^j(h(z))}_{\simeq 1}) \simeq \frac{1}{\mu_2^k} \quad \text{for all} \quad 0 \leq k < q$$
   because $f(f^j(h(z))) \simeq \frac{1}{\mu_2}$ since $f'(f^i(h(z))) \not\simeq \infty$.
   Recall that $h(z) \simeq \mu_2^l$. We have that $j \neq l$ because otherwise it would take j iterations before landing close to 1, which contradicts that $f^i(h(z)) \simeq 1$ with $i < j$. Using that
   $$f^q(h(z)) = f^{q-i}(f^j(h(z))) \simeq \frac{1}{\mu_2^{q-j}} \approx \mu_2^j$$
   we conclude that
   $$g^q(z) = \frac{f^q(h(z)) - a}{\delta} \simeq \infty$$
   because $a \approx \mu_2^j \not\approx \mu_2^i$. So the standard shadow of g^q is the constant map ∞.

2. $f'(f^i(h(z))) \not\simeq \infty$ for all $0 \le i < q$
   This implies that $f^i(h(z)) \simeq \frac{h(z)}{\mu_2^i}$ for all $0 \le i < q$. And thus we have

$$k_i(z) \simeq 0 \;\text{ for all }\; i \neq l$$

   since $f^i(h(z)) \not\simeq 1$ for all $i \neq l$. This implies

$$\sum_{i=0}^{q-1} k_i(z) \simeq k_l(z) \not\simeq \infty.$$

   Note that $k_l(z) \not\simeq \infty$, because otherwise $f'(f^l(h(z))) = \frac{1}{\mu_2}(1 - d^2 k_l(z)^2) \simeq \infty$.

   (a) $\frac{\sqrt{\epsilon}}{\delta} \not\approx 0, \infty$

      i. $\frac{1-\mu_2^q}{\sqrt{\epsilon}}$ is limited, i.e. $\frac{1-\mu_2^q}{\sqrt{\epsilon}} \approx T$ with $T \in \mathbb{C}$ standard.

$$g^q(z) = \frac{1}{\delta}\left(\frac{\delta z + a}{\mu_2^q}\prod_{i=0}^{q-1}(1 + d\sqrt{\epsilon}k_i(z)) - a\right)$$

$$= \underbrace{\frac{z}{\mu_2^q}}_{\approx z} + \underbrace{\frac{a}{\mu_2^q}}_{\approx \mu_2^l}\frac{\sqrt{\epsilon}}{\delta}\underbrace{\frac{1-\mu_2^q}{\sqrt{\epsilon}}}_{\approx T} + \underbrace{\frac{z}{\mu_2^q} d\sqrt{\epsilon}\sum_{i=0}^{q-1} k_i(z)}_{\simeq 0} + \underbrace{\frac{d\,a}{\mu_2^q}}_{\approx \mu_2^l}\frac{\sqrt{\epsilon}}{\delta}\underbrace{\sum_{i=0}^{q-1} k_i(z)}_{\simeq k_l(z)}$$

$$+\frac{1}{\delta}\frac{\delta z + a}{\mu_2^q}\left[\prod_{i=0}^{q-1}(1 + d\sqrt{\epsilon}k_i(z)) - \left(1 + d\sqrt{\epsilon}\sum_{i=0}^{q-1} k_i(z)\right)\right]$$

$$\simeq z + \mu_2^l\frac{\sqrt{\epsilon}}{\delta}T + \mu_2^l\frac{\sqrt{\epsilon}}{\delta}k_l(z)$$

      Because

$$\prod_{i=0}^{q-1}(1 + d\sqrt{\epsilon}k_i(z)) - \left(1 + d\sqrt{\epsilon}\sum_{i=0}^{q-1} k_i(z)\right) = \sum_{j=2}^{q}(d\sqrt{\epsilon})^j\, K_j(z)$$

      where $K_j(z)$ are products of some $k_i(z)$'s and therefore limited.
      This includes

$$\frac{1}{\delta}\left[\prod_{i=0}^{q-1}(1 + d\sqrt{\epsilon}k_i(z)) - \left(1 + d\sqrt{\epsilon}\sum_{i=0}^{q-1} k_i(z)\right)\right]$$

$$= \sqrt{\epsilon}d\frac{\sqrt{\epsilon}}{\delta}\sum_{j=2}^{q}(d\sqrt{\epsilon})^{j-2}\, K_j(z) \simeq 0$$

      Now we will determine the shadow of $k_l(z)$. We have

$$f^l(h(z)) = \frac{h(z)}{\mu_2^l}\prod_{i=0}^{l-1}(1 + d\sqrt{\epsilon}k_i(z)) = \frac{h(z)}{\mu_2^l} + R(z)$$

with

$$R(z) = \frac{h(z)}{\mu_2^l}\left[\prod_{i=0}^{l-1}(1 + d\sqrt{\epsilon}k_i(z)) - 1\right]$$

and

$$\frac{R(z)}{\sqrt{\epsilon}} = \frac{h(z)}{\mu_2^l}\frac{1}{\sqrt{\epsilon}}\sum_{j=1}^{l}(d\sqrt{\epsilon})^j K_j(z) = \frac{h(z)}{\mu_2^l}\sum_{j=1}^{l}(d\sqrt{\epsilon})^{j-1}K_j(z) \simeq 0$$

because the $K_j(z)$'s are products of $k_i(z)$'s with $i < l$ which are all approximately 0. We conclude

$$k_l(z) = \frac{\sqrt{\epsilon}}{f^l(h(z)) - d} = \frac{\sqrt{\epsilon}}{\frac{a+\delta z}{\mu_2^l} + R(z) - d} = \frac{1}{\frac{\frac{a}{\mu_2^l} - d}{\sqrt{\epsilon}} + \frac{\delta}{\sqrt{\epsilon}}\frac{z}{\mu_2^l} + \frac{R(z)}{\sqrt{\epsilon}}}$$

Let

$$c_1 = {}^\circ\!\left(\frac{\delta}{\sqrt{\epsilon}\mu_2^l}\right) \quad \text{and} \quad c_2 = {}^\circ\!\left(\frac{\frac{a}{\mu_2^l} - d}{\sqrt{\epsilon}}\right)$$

then we have

$$k_l(z) \simeq \frac{1}{c_1\, z + c_2}$$

and

$$g^q(z) \simeq z + \frac{T}{c_1} + \frac{1}{c_1}\frac{1}{c_1\, z + c_2}$$

if $c_2 \neq \infty$.

Note that $z \mapsto z + \frac{T}{c_1} + \frac{1}{c_1}\frac{1}{c_1\, z + c_2}$ is conjugate to G_T by the standard Möbius transformation $z \mapsto c_1 z + c_2$.

In the case that $c_2 = \infty$ (this is the case if and only if $\frac{a - d\mu_2^l}{\sqrt{\epsilon}} \approx \infty$) we get

$$g^q(z) \simeq z + \frac{T}{c_1}.$$

ii. $\frac{1-\mu_2^q}{\sqrt{\epsilon}} \approx \infty$

Then we have

$$g^q \simeq \infty$$

Because

$$g^q(z) \simeq z + \mu_2^l\frac{\sqrt{\epsilon}}{\delta}\frac{1-\mu_2^q}{\sqrt{\epsilon}} + \mu_2^l\frac{\sqrt{\epsilon}}{\delta}k_l(z)$$

and $\mu_2^l\frac{\sqrt{\epsilon}}{\delta}\frac{1-\mu_2^q}{\sqrt{\epsilon}} \approx \infty$, since $\frac{\sqrt{\epsilon}}{\delta} \not\approx 0$ while $\mu_2^l\frac{\sqrt{\epsilon}}{\delta}k_l(z)$ is limited.

(b) $\frac{\sqrt{\epsilon}}{\delta} \approx 0$

Then we have

$$g^q \simeq z + \mu_2^l\frac{1-\mu_2^q}{\delta}$$

and thus:

i. If $\frac{1-\mu_2^q}{\delta} \not\approx \infty$ we have

$$g^q \simeq z + {}^\circ(\mu_2^l \frac{1-\mu_2^q}{\delta}).$$

ii. If $\frac{1-\mu_2^q}{\sqrt{\epsilon}} \approx \infty$ we have

$$g^q \simeq \infty.$$

(c) $\frac{\sqrt{\epsilon}}{\delta} \approx \infty$

i. $f'(f^i(h(z))) \simeq 0$ for some i
This implies

$$(g^q)'(z) = \prod_{i=0}^{q-1} f'(f^i(h(z))) \simeq 0.$$

Thus g^q is close to a constant map.

ii. $f'(f^i(h(z))) \not\simeq 0$ for all $0 \leq i < q$
This implies that $k_i(z)^2 \not\simeq 1$, which includes that

$$Ng^q = \delta \sum_{i=0}^{q-1} \frac{f''(f^i(h(z)))}{f'(f^i(h(z)))} (f^i)'(h(z))$$

$$= \underbrace{\frac{2d^2\delta}{\sqrt{\epsilon}}}_{\approx 0} \underbrace{\sum_{i=0}^{q-1} [\frac{k_i(z)^3}{\mu_2^i(1-d^2k_i(z)^2)} \prod_{j=0}^{i-1}(1-d^2k_j(z))]}_{\not\approx\infty} \simeq 0.$$

q.e.d.

This phenomenon that the limiting map of higher iterates of certain normalizations tend to a quadratic map just happens around ideal points $\langle \mu, \mu^{-1}, \infty \rangle$ where μ is a q-th root of unity with $q \geq 2$.

Theorem 10.6 *Let $\langle \mu_1, \mu_2, \mu_3 \rangle \approx \langle \mu, \mu^{-1}, \infty \rangle$ where μ is not a primitive q-th root of unity with $q \geq 2$. Then for all $g \in \langle \mu_1, \mu_2, \mu_3 \rangle$ and all standard $n \in \mathbb{N}$ we have that* $\deg(\mathrm{sh}(g^n)) \leq 1$.

Proof: Assume that $\deg(\mathrm{sh}(g^n)) \geq 1$. Then there would be a standard rational map F of degree at least two with $f^n \simeq F$, i.e. there exists standard $z_1, \cdots z_k$ such that $f^n(z) \approx g(z)$ for all $z \not\approx z_1, \cdots z_k$. Since F has infinitely many periodic points it has a standard periodic point z_0 that is not close to $z_1, \cdots z_k$. By Rouché's theorem g^n has a periodic point close to z_0 with multiplier close to $F'(z_0)$. Thus f would have a periodic point of standard period with a limited multiplier, i.e. $\langle g \rangle \in \mathrm{Per}_m(\eta)$ for some standard m and some limited η, which would contradict Theorem 7.4, because $\langle g \rangle \approx \langle \mu, \mu^{-1}, \infty \rangle$ with μ not a primitive q-th root of unity where $q \geq 2$.

q.e.d.

Now we give a classical version of the Uniqueness theorem.

Theorem 10.7 (Uniqueness theorem – classical version) *Let $\langle \alpha_n, \beta_n, \gamma_n \rangle_{n \in \mathbb{N}}$ be a sequence in $\mathcal{M}_2$ with $\langle \alpha_n, \beta_n, \gamma_n \rangle \to \langle \mu, \mu^{-1}, \infty \rangle$ where μ is a primitive q-th root of unity with $q \geq 2$. Let $f_n \in \langle \alpha_n, \beta_n, \gamma_n \rangle$ with*

$$\lim_{n \to \infty} f_n^q = F$$

the convergence taking place locally uniformly outside a finite set. If $\deg(F) = 2$, then $\lim_{n \to \infty} \frac{\alpha_n^q - 1}{\sqrt{1 - \alpha_n \beta_n}} = T$ for some choice of square root and for some $T \in \mathbb{C}$, and F is holomorphically conjugate to $G_T(z) = z + \frac{1}{z} + T$. Otherwise $\deg(F) \leq 1$.

Proof: By transfer we can assume that the sequence $\langle \alpha_n, \beta_n, \gamma_n \rangle_{n \in \mathbb{N}}$ as well as the sequence of representatives $\{f_n\}_{n \in \mathbb{N}}$ and its limit of the q-th iterate F are standard. So we have for all $N \approx \infty$

$$\langle \alpha_N, \beta_N, \gamma_N \rangle \approx \langle \mu, \mu^{-1}, \infty \rangle$$

and by Lemma 5.3 we have

$$\mathrm{sh}(f_N^q) = F$$

for all unlimited $N \in \mathbb{N}$.

If the limit does not exists, i.e. there exists $N, N' \approx \infty$ and $T \neq \widetilde{T}$ standard with $\frac{\alpha_N^q - 1}{\sqrt{1 - \alpha_N \beta_N}} \approx T$ and $\frac{\alpha_{N'}^q - 1}{\sqrt{1 - \alpha_{N'} \beta_{N'}}} \approx \widetilde{T}$ and $T^2 \neq \widetilde{T}^2$. Assume that $\deg(F) = 2$. By Theorem 10.1 part 2 this implies that F is conjugate to G_T and $G_{\widetilde{T}}$, which is a contradiction, because $T^2 \neq \widetilde{T}^2$.

If $\frac{(\alpha_N^q - 1)^2}{1 - \alpha_N \beta_N} \approx \infty$ we can conclude by Theorem 10.1 part 1 that $\deg(F) = \deg(\mathrm{sh}(f_N^q)) \leq 1$.

Assume that the limit exists, i.e. there exists a standard $T \in \mathbb{C}$ such that $\frac{\alpha_N^q - 1}{\sqrt{1 - \alpha_N \beta_N}} \approx T$ for all $N \approx \infty$. Let $\deg(F) = 2$. Then we conclude by theorem 10.1 that $F = \mathrm{sh}(f_N^q)$ is holomorphically conjugate to G_T.

q.e.d.

Chapter 11

The dynamical compactification

In this section we define the dynamical compactification $\widetilde{\mathcal{M}}_2$ of $\mathcal{M}_2$.

It is obtained by replacing in the algebraic compactification $\widehat{\mathcal{M}}_2$ the ideal points $\langle e^{2\pi ip/q}, e^{-2\pi ip/q}, \infty\rangle$ with $q \geq 2$, by spheres of ideal points, $\mathcal{B}_{p/q}$, and define a topology on this enlarged set that takes the limiting map of the q-th iterate into account. We define the topology on $\widetilde{\mathcal{M}}_2$ using nonstandard analysis, by defining an external map that associates to every point in the enlarged set $\widetilde{\mathcal{M}}_2$ a unique standard point in $\widetilde{\mathcal{M}}_2$. If this external map satisfies some additional properties, see Lemma 11.1 below, it gives a compactification of $\mathcal{M}_2$. Following the idea that the compactification should take the limiting map of the q-th iterate into account, we define the standard part of a map $\langle f\rangle$ in $\mathcal{M}_2$ to be $\langle G_T\rangle_{p/q} \in \mathcal{B}_{p/q}$ if $\langle f\rangle \approx \langle e^{2\pi ip/q}, e^{-2\pi ip/q}, \infty\rangle$ in $\widehat{\mathcal{M}}_2$ (i.e. the multipliers at the fixed points of f are close to $\langle e^{2\pi ip/q}, e^{-2\pi ip/q}, \infty\rangle$) and f can be normalized such that $\mathrm{sh}(f^q) = G_T$.

This compactification was first introduced by DeMarco in [DeM]. Our approach gives a definition in completely dynamical terms, justified by the uniqueness of the quadratic limiting map, where DeMarco uses geometric invariant theory compactifications.

At the end of this chapter we give a summary of the main features of the dynamical compactification $\widetilde{\mathcal{M}}_2$, that will be used in the following.

External construction of a compactification. We first describe a method to define a compactification, by associating to every point in an enlarged set a unique standard point with certain additional properties. We will use this method to define the topology on $\widetilde{\mathcal{M}}_2$.

Recall that a *compactification* of a topological space X is a compact Hausdorff space Y with $X \subset Y$, such that X is a topological subspace of Y with $\overline{X} = Y$.

Lemma 11.1 *Let $(X, \mathcal{O}_X)$ be a standard topological space and Y a standard set containing X. Let s be an external map that associates to every point $y \in Y$ a unique standard point $s(y) \in Y$ with $s(y) = y$ for all standard $y \in Y$. For $y \in Y$ standard define $u(y)$ to be the following external set:*

$$u(y) = \{x \in Y :\ s(x) = y\}.$$

Then

$$\mathcal{O}_Y = {}^{st}\{U \subset Y : \ \forall^{st} y \in U \ u(y) \subset U\}$$

is a standard topology on Y, that makes Y into a compact space. Furthermore the topological halo

$$\mathrm{hal}_Y(y) = \bigcap_{\substack{U \in^{\sigma} \mathcal{O}_Y \\ y \in U}} U$$

contains $u(y)$ for all standard $y \in Y$.

If in addition:

1. *For all standard $y \in Y$ there exists a point $x \in X$ with $s(x) = y$.*
2. *The topological halo $\mathrm{hal}_Y(y) = u(y)$ for all standard $y \in Y$.*
3. *For all standard $x \in X$, $\mathrm{hal}_Y(x) = \mathrm{hal}_X(x)$.*

Then $(Y, \mathcal{O}_Y)$ is a compactification of $(X, \mathcal{O}_X)$.

Condition 1 guarantees that $\overline{X} = Y$, condition 2 makes Y into a Hausdorff space (note that $u(x) \cap u(y) = \emptyset$ for all standard $x \neq y$) and condition 3 expresses that X is a subspace of Y.

Note that $\mathcal{O}_Y = {}^{st}\{U \subset Y : \ s(x) \in U \Rightarrow x \in U\}$.

Proof:

1. Claim: The set $\mathcal{O}_Y = {}^{st}\{U \subset Y \ : \ \forall^{st} y \in U \ u(y) \subset U\}$ is a standard topology on Y, that makes Y into a compact space.

   The empty set and Y are in $\mathcal{O}_Y$. Let $\{U_i\}_{i \in I}$ a standard family of sets in $\mathcal{O}_Y$ and $y \in \bigcup_{i \in I} U_i$ standard then there exists a standard $i \in I$ such that y is in the standard set I_i, which has the property that $u(y) \subset I_i$ which implies that $u(y) \subset \bigcup_{i \in I} U_i$. So we have $\bigcup_{i \in I} U_i \in \mathcal{O}_Y$, since the union of a standard family is standard. Now let $U, V \in \mathcal{O}_Y$ standard. We have to show that the standard set $U \cap V$ is in $\mathcal{O}_Y$. So let $y \in U \cap V$ be standard. then we have per definition that $u(y) \subset U$ and $u(y) \subset V$, so we also have $u(y) \subset (U \cap V)$, which implies that $U \cap V \in \mathcal{O}_Y$.

   For $y \in Y$ standard we have that $u(y) \subset \mathrm{hal}_Y(y)$, because $u(y)$ is contained in every standard open set that contains y.

   The space Y is compact, because every $x \in Y$ lies in some external set $u(y)$ for some standard y, because we associated to every point x in Y a standard point $s(x)$. Since $u(y) \subset \mathrm{hal}_Y(y)$ (see below), we can conclude that Y is covered by the halos of its standard points, which means that $(Y, \mathcal{O}_Y)$ is compact.

2. Claim: $(X, \mathcal{O}_X)$ is a subspace of $(Y, \mathcal{O}_Y)$, i.e. $\mathcal{O}_X = \{U \cap X \ : \ U \in \mathcal{O}_Y\}$ and the closure of X in $(Y, \mathcal{O}_Y)$ is equal to Y.

   X is a subspace of Y, because for all standard $x \in X$ we have $\mathrm{hal}_Y(x) = \mathrm{hal}_X(x)$. And

   $$\overline{X} = {}^{st}\{y \in Y \ : \ \exists x \in X \ s(x) = y\} = Y$$

   because for all standard $y \in Y$ there exists a $x \in X$ with $s(x) = y$.

3. Claim: $(Y, \mathcal{O}_Y)$ is a Hausdorff space.

   Let $x, y \in Y$ standard with $x \neq y$. Then $u(x) \cap u(y) = \emptyset$ and thus by condition (2) $\mathrm{hal}_Y(x) \cap \mathrm{hal}_Y(y) = \emptyset$, which is the external characterization of a Hausdorff space.

q.e.d.

Definition of the dynamical compactification $\widetilde{\mathcal{M}}_2$.

The enlarged set $\widetilde{\mathcal{M}}_2$. Recall that $\mathrm{Per}_1(1)$ is parameterized by the family $G_T(z) = z + \frac{1}{z} + T$ with $T \in \mathbb{C}$. So we have $\mathrm{Per}_1(1) = \{\langle G_T \rangle : T \in \mathbb{C}\}$. Let G_∞ denote the constant map ∞ and $\widehat{\mathrm{Per}_1(1)} = \{\langle G_T \rangle : T \in \hat{\mathbb{C}}\}$. Let p and q relatively prime with $0 < p/q < 1$. Replace in the algebraic compactification $\widehat{\mathcal{M}}_2$ the ideal points $\langle e^{2\pi i p/q}, e^{-2\pi i p/q}, \infty \rangle$ with $q \geq 2$ by a sphere of ideal points $\mathcal{B}_{p/q}$. Note that $B_{p/q} = B_{1-p/q}$. These spheres are copies of $\widehat{\mathrm{Per}_1(1)}$. We denote its elements by $\langle G_T \rangle_{p/q} \in \mathcal{B}_{p/q}$.

(Note that $\widetilde{\mathcal{M}}_2$ could be defined as a subset of the product $\widehat{\mathcal{M}}_2 \times \widehat{\mathrm{Per}_1(1)}$, by defining that the projection on $\widehat{\mathcal{M}}_2$, $P : \widetilde{\mathcal{M}}_2 \subset \widehat{\mathcal{M}}_2 \times \widehat{\mathrm{Per}_1(1)} \to \widehat{\mathcal{M}}_2$ should have the property that $P^{-1}(\langle \mu_1, \mu_2, \mu_3 \rangle) = (\langle \mu_1, \mu_2, \mu_3 \rangle, \langle G_\infty \rangle)$ if $\langle \mu_1, \mu_2, \mu_3 \rangle \neq \langle e^{2\pi i p/q}, e^{-2\pi i p/q}, \infty \rangle$, and $P^{-1}(\langle \mu_1, \mu_2, \mu_3 \rangle) = \{\langle \mu_1, \mu_2, \mu_3 \rangle\} \times \widehat{\mathrm{Per}_1(1)}$ if $\langle \mu_1, \mu_2, \mu_3 \rangle = \langle e^{2\pi i p/q}, e^{-2\pi i p/q}, \infty \rangle$.)

Definition of the standard part on $\widetilde{\mathcal{M}}_2$. Now we define for every point in this enlarged set its standard part, i.e. we define an external map $s : \widetilde{\mathcal{M}}_2 \to {}^\sigma\widetilde{\mathcal{M}}_2$, where ${}^\sigma\widetilde{\mathcal{M}}_2$ denotes the external set of all standard elements in $\widetilde{\mathcal{M}}_2$. If this external function – associating to every point in $\widetilde{\mathcal{M}}_2$ a unique point in ${}^\sigma\widetilde{\mathcal{M}}_2$ – satisfies some additional requirements, compare Lemma 11.1 – it will give us a standard compactification of $\mathcal{M}_2$.

Let $\hat{s} : \widehat{\mathcal{M}}_2 \to {}^\sigma\widehat{\mathcal{M}}_2$ the external map that associates to every point in the algebraic compactification $\widehat{\mathcal{M}}_2$ its standard part: $\hat{s}(\langle \mu_1, \mu_2, \mu_3 \rangle) = \langle {}^\circ\mu_1, {}^\circ\mu_2, {}^\circ\mu_3 \rangle$.

1. Let $\langle \mu_1, \mu_2, \mu_3 \rangle \in \widehat{\mathcal{M}}_2$ with $\langle \mu_1, \mu_2, \mu_3 \rangle \neq \langle e^{2\pi i p/q}, e^{-2\pi i p/q}, \infty \rangle$, $q \geq 2$.
   (a) We define
   $$s(\langle \mu_1, \mu_2, \mu_3 \rangle) = \hat{s}(\langle \mu_1, \mu_2, \mu_3 \rangle) = \langle {}^\circ\mu_1, {}^\circ\mu_2, {}^\circ\mu_3 \rangle$$
   if $\hat{s}(\langle \mu_1, \mu_2, \mu_3 \rangle) \neq \langle e^{2\pi i p/q}, e^{-2\pi i p/q}, \infty \rangle$.
   (b) If $\hat{s}(\langle \mu_1, \mu_2, \mu_3 \rangle) = \langle e^{2\pi i p/q}, e^{-2\pi i p/q}, \infty \rangle$, then define
      i.
      $$s(\langle \mu_1, \mu_2, \mu_3 \rangle) = \infty \in \mathcal{B}_{p/q}$$
      if either $\langle \mu_1, \mu_2, \mu_3 \rangle = \langle \mu, \mu^{-1}, \infty \rangle$ with $\mu \approx e^{2\pi i \frac{p}{q}}$ or $\langle \mu_1, \mu_2, \mu_3 \rangle \in \mathcal{M}_2$ and $\deg(\mathrm{sh}(f^q)) \leq 1$ for all $f \in \langle \mu_1, \mu_2, \mu_3 \rangle$.
      ii.
      $$s(\langle \mu_1, \mu_2, \mu_3 \rangle) = \langle G_T \rangle_{p/q} \in \mathcal{B}_{p/q}$$
      if $\langle \mu_1, \mu_2, \mu_3 \rangle \in \mathcal{M}_2$ and there exists $f \in \langle \mu_1, \mu_2, \mu_3 \rangle$ with $\mathrm{sh}(f^q) = G_T$.

2. Now let $\langle G_T\rangle_{p/q} \in \mathcal{B}_{p/q}$. Then define

   (a)
   $$s(\langle G_T\rangle_{p/q}) = \langle G_{{}^\circ T}\rangle_{p/q} \in \mathcal{B}_{p/q}$$
   if p/q is standard.

   (b)
   $$s(\langle G_T\rangle_{p/q}) = \infty \in \mathcal{B}_{{}^\circ(p/q)}$$
   if p/q is not standard and ${}^\circ(p/q) \in \mathbb{Q} - \{0,1\}$.

   (c)
   $$s(\langle G_T\rangle_{p/q}) = \langle e^{2\pi i\,{}^\circ(p/q)}, e^{-2\pi i\,{}^\circ(p/q)}, \infty\rangle$$
   if p/q is not standard and ${}^\circ(p/q) \notin \mathbb{Q} - \{0,1\}$.

Verification of the assumptions of Lemma 11.1 for $\widetilde{\mathcal{M}_2}$. Now we will verify the assumptions of Lemma 11.1 for our definition of the standard part on the enlarged set $\widetilde{\mathcal{M}_2}$, which will then guarantee that $\widetilde{\mathcal{M}_2}$ with the from the standard part induced topology

$$\mathcal{O}_{\widetilde{\mathcal{M}_2}} = {}^{st}\{U \subset \widetilde{\mathcal{M}_2} :\ \forall x \in \widetilde{\mathcal{M}_2}\, (s(x) \in U \Rightarrow x \in U)\}$$

is a compactification of $\mathcal{M}_2$.

1. We have to consider an arbitrary ideal standard point in $y \in \widetilde{\mathcal{M}_2}$ and show that there is a point in $\mathcal{M}_2$, whose standard part is y.

   (a) If $y = \langle \mu, \mu^{-1}, \infty\rangle$, then choose
   $$\langle \mu - \epsilon, \mu^{-1}, \frac{2 - (\mu-\epsilon) - \mu^{-1}}{1 - (\mu-\epsilon)\mu^{-1}}\rangle \in \mathcal{M}_2$$
   for some positive infinitesimal ϵ.

   (b) If $y = \infty \in \mathcal{B}_{p/q}$, then choose $\mu \approx e^{2\pi i p/q}$, but $\mu \neq e^{2\pi i p/q}$ and $T \approx \infty$. Let $S = T\frac{1-\mu}{1-\mu^q}$. Then the map
   $$z \mapsto \frac{1}{\mu}(z + \frac{1}{z} + S)$$
   has fixed points with multipliers close to $\langle e^{2\pi i p/q}, e^{-2\pi i p/q}, \infty\rangle$ and the shadow of the q-th iterate is equal to $G_\infty \equiv \infty$, compare Theorem 9.2.

   (c) If $y = \langle G_T\rangle_{p/q} \in \mathcal{B}_{p/q}$ then we define a map in $\mathcal{M}_2$, whose standard part is $\langle G_T\rangle_{p/q} \in \mathcal{B}_{p/q}$ as follows: Let $\mu \approx e^{2\pi i p/q}$, but $\mu \neq e^{2\pi i p/q}$ and $S = T\frac{1-\mu}{1-\mu^q}$. Then the map
   $$z \mapsto \frac{1}{\mu}(z + \frac{1}{z} + S)$$
   has fixed points with multipliers close to $\langle e^{2\pi i p/q}, e^{-2\pi i p/q}, \infty\rangle$ and the shadow of the q-th iterate is equal to G_T, compare Theorem 9.2. So we have $s(\langle z \mapsto \frac{1}{\mu}(z + \frac{1}{z} + S)\rangle) = \langle G_T\rangle_{p/q} \in \mathcal{B}_{p/q}$.

2. We have to show that $\mathrm{hal}_{\widetilde{\mathcal{M}}_2}(y) \subset u(y)$ for all standard $y \in \widetilde{\mathcal{M}}_2$. This is equivalent to: $s(x) \neq y$ implies that there exists a standard open neighborhood U of y that does not contain x. We distinguish several cases:

   (a) First assume that y and $s(x)$ are not points on a common sphere $\mathcal{B}_{p/q}$.

   Let $P : \widetilde{\mathcal{M}}_2 \to \widehat{\mathcal{M}}_2$ denote the *projection* from $\widetilde{\mathcal{M}}_2$ on $\widehat{\mathcal{M}}_2$, i.e. $P(x) = x$, if x is not on any of the to $\widehat{\mathcal{M}}_2$ added ideal spheres $\mathcal{B}_{p/q}$ and $P(\langle G_T\rangle_{p/q}) = \langle e^{2\pi i p/q}, e^{-2\pi i p/q}, \infty\rangle$.

   This projection map P is continuous, because $s(x) = y$ implies that $\hat{s}(P(x)) = P(y)$, which implies that the preimage of closed sets is closed. (Let $A \subset \widehat{\mathcal{M}}_2$ standard, i.e. $y \in A$ implies $\hat{s}(y) \in A$. Now let $x \in P^{-1}(A)$. We have to show that $s(x) \in P^{-1}(A)$. We have $P(x) \in A$ and since A is closed this implies that $\hat{s}(P(x)) \in A$. Thus $s(x) \in P^{-1}(P(s(x))) = P^{-1}(\hat{s}(P(x))) \subset P^{-1}(A)$, because $P(s(x)) = \hat{s}(P(x))$.)

   Since y and $s(x)$ are not on a common sphere we have $P(y) \neq P(s(x))$ and because $\widehat{\mathcal{M}}_2$ is a standard Hausdorff space, we can find disjoint open standard neighborhood U and V in $\widehat{\mathcal{M}}_2$ of $P(y)$ and $P(s(x))$ respectively. Because $P(s(x))$ is in V and since V is a standard open set it also contains $P(x)$. This implies, since P is continuous and standard, that the set $P^{-1}(U)$ is a standard open set containing y, but not x.

   (b) Now assume that y and $s(x)$ are on a common sphere $\mathcal{B}_{p/q}$.

   i. If $y = \infty \in \mathcal{B}_{p/q}$ and $s(x) = \langle G_T\rangle_{p/q} \in \mathcal{B}_{p/q}$, define

   $$A = \{\langle \mu_1, \mu_2, \mu_3\rangle \in \mathcal{M}_2 \ : \ |\mu_3| > 100 \text{ and } |\frac{(\mu_1^q - 1)^2}{1-\mu_1\mu_2} - T^2| < \frac{1}{2}\}$$

   Since A is standard, the closure of A in $\widetilde{\mathcal{M}}_2$, $\overline{A}$ is standard, and thus the complement of $\overline{A}$ is a standard open set containing y. It remains to show that $x \in \overline{A}$.

   A. If $x = \langle \mu_1, \mu_2, \mu_3\rangle \in \mathcal{M}_2$ then $\langle \mu_1, \mu_2, \mu_3\rangle \approx \langle e^{2\pi i p/q}, e^{-2\pi i p/q}, \infty\rangle$ and $\frac{(\mu_1^q-1)^2}{1-\mu_1\mu_2} \approx T^2$, thus $x \in A \subset \overline{A}$.

   B. If $x = \langle G_{\widehat{T}}\rangle_{p/q} \in \mathcal{B}_{p/q}$ with $\widehat{T}^2 \approx T^2$ then $x \in \overline{A}$, because

   $$\{\langle G_S\rangle_{p/q} \in \mathcal{B}_{p/q} \ : \ |S^2 - T^2| < \frac{1}{4}\} \subset \overline{A}.$$

   Per transfer we can assume that S^2 is standard. (Note that a standard z is in $\overline{A}$ if and only if there exists $\tilde{z} \in A$ with $s(\tilde{z}) = z$.) Let $\langle \mu_1, \mu_2, \mu_3\rangle \in \mathcal{M}_2$ with $\langle \mu_1, \mu_2, \mu_3\rangle \approx \langle e^{2\pi i p/q}, e^{-2\pi i p/q}, \infty\rangle$ and $\frac{(\mu_1^q-1)^2}{1-\mu_1\mu_2} \approx S^2$. This implies that $|\mu_3| > 100$ and

   $$|\frac{(\mu_1^q - 1)^2}{1-\mu_1\mu_2} - T^2| \leq \underbrace{|\frac{(\mu_1^q - 1)^2}{1-\mu_1\mu_2} - S^2|}_{\approx 0} + \underbrace{|S^2 - T^2|}_{<1/4} < \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

which implies that $\langle \mu_1, \mu_2, \mu_3 \rangle \in A$ and thus $s(\langle \mu_1, \mu_2, \mu_3 \rangle) = \langle G_S \rangle_{p/q} \in \overline{A}$.

ii. Now let $y = \langle G_{T_1} \rangle_{p/q} \in \mathcal{B}_{p/q}$ and $s(x) = \langle G_{T_2} \rangle_{p/q} \in \mathcal{B}_{p/q}$. Since T_1 and T_2 are standard and $s(x) \neq y$, we have $|T_1^2 - T_2^2| > 0$ standard. Define

$$A = \{\langle \mu_1, \mu_2, \mu_3 \rangle \in \mathcal{M}_2 \ : \ |\mu_3| > 100 \text{ and } |\frac{(\mu_1^q - 1)^2}{1 - \mu_1\mu_2} - T_2^2| < \frac{1}{2}|T_1^2 - T_2^2|\}.$$

Again A is standard and thus is $\overline{A}^c$ is a standard open set. It remains to show that $y \in \overline{A}^c$ and $x \in \overline{A}$. Assume that $y \in \overline{A}$, then there must be a $\langle \mu_1, \mu_2, \mu_3 \rangle \in A$ with $s(\langle \mu_1, \mu_2, \mu_3 \rangle) = y = \langle G_{T_1} \rangle_{p/q}$. I.e. $\mu_3 \approx \infty$ and $\frac{(\mu_1^q - 1)^2}{1-\mu_1\mu_2} \approx T_1^2$ which implies that

$$|\frac{(\mu_1^q - 1)^2}{1 - \mu_1\mu_2} - T_2^2| \geq |T_1^2 - T_2^2| - \underbrace{|\frac{(\mu_1^q - 1)^2}{1 - \mu_1\mu_2} - T_1^2|}_{\approx 0} \geq \frac{1}{2}|T_1^2 - T_2^2|.$$

This is a contradiction to $\langle \mu_1, \mu_2, \mu_3 \rangle \in A$.

A. If $x = \langle \mu_1, \mu_2, \mu_3 \rangle \in \mathcal{M}_2$ then $\langle \mu_1, \mu_2, \mu_3 \rangle \approx \langle e^{2\pi i p/q}, e^{-2\pi i p/q}, \infty \rangle$ and $\frac{(\mu_1^q - 1)^2}{1-\mu_1\mu_2} \approx T_2^2$, thus $x \in A \subset \overline{A}$.

B. If $x = \langle G_T \rangle_{p/q} \in \mathcal{B}_{p/q}$ with $T^2 \approx T_2^2$ then $x \in \overline{A}$, because

$$\{\langle G_S \rangle_{p/q} \in \mathcal{B}_{p/q} \ : \ |S^2 - T^2| < \frac{1}{4}|T_1^2 - T_2^2|\} \subset \overline{A}.$$

The proof of this is analogous to the one above.

3. Let $\langle \mu_1, \mu_2, \mu_3 \rangle \in \mathcal{M}_2$ be standard. The points, whose standard part is $\langle \mu_1, \mu_2, \mu_3 \rangle$ are precisely those maps in $\mathcal{M}_2$ having fixed points with limited multipliers close to μ_1, μ_2 and μ_3 respectively. Thus we have $u(\langle \mu_1, \mu_2, \mu_3 \rangle) = \mathrm{hal}_{\mathcal{M}_2}(\langle \mu_1, \mu_2, \mu_3 \rangle)$.

q.e.d.

Summary of the main features of the dynamical compactification $\widetilde{\mathcal{M}_2}$. In the dynamical compactification $\widetilde{\mathcal{M}}_2$ the ideal points $\langle e^{2\pi i p/q}, e^{-2\pi i p/q}, \infty \rangle$ with $q \geq 2$ in $\widehat{\mathcal{M}}_2$ are replaced by spheres of ideal points $\mathcal{B}_{p/q}$ recording the limiting map of the q-th iterate taking place in the family $G_T(z) = z + \frac{1}{z} + T$ with $T \in \hat{\mathbb{C}}$.

For our purposes it suffices to know when a point in $\mathcal{M}_2$ is close to a standard ideal point $\langle G_T \rangle_{p/q} \in \mathcal{B}_{p/q}$ — or in classical language when a sequence in $\mathcal{M}_2$ converges to an ideal point.

For $\langle G_T \rangle_{p/q} \in \mathcal{B}_{p/q}$ standard (i.e. T and p/q are standard), $T \neq \infty$, we have:

$$\langle f \rangle \approx \langle G_T \rangle_{p/q} \in \mathcal{B}_{p/q} \ \text{ if and only if}$$

$\langle f \rangle \approx \langle e^{2\pi i p/q}, e^{-2\pi i p/q}, \infty \rangle$ in $\widehat{\mathcal{M}}_2$ (i.e. the multipliers at the fixed points of f are close to $e^{2\pi i p/q}$, $e^{-2\pi i p/q}$ and ∞), and f can be normalized such that $\mathrm{sh}(f^q) = G_T$.

Lemma 11.2 *Let f be a rational map having fixed points with multipliers $\mu_1 \approx e^{2\pi ip/q}$, $\mu_2 \approx e^{-2\pi ip/q}$ and $\mu_3 \approx \infty$. Then the following conditions are equivalent:*

1. $\langle f \rangle \approx \langle G_T \rangle_{p/q} \in \mathcal{B}_{p/q}$.
2. $\frac{(\mu_1^q - 1)^2}{1-\mu_1\mu_2} \approx T^2$.
3. *The map* $g(z) = \frac{1}{\mu_1}\left(z + \frac{1}{z} + S\right)$ *with* $S^2 = \frac{(\mu_1-\mu_2)^2}{1-\mu_1\mu_2}$ *has the property*
$$g^q(z) \approx G_T(z)$$
for all $z \not\approx 0, \infty$.

Proof:

(1) $\Rightarrow$ (2): This follows by Theorem 10.1.
(2) $\Rightarrow$ (3): This follows by Theorem 9.2.
(3) $\Rightarrow$ (1): The map g is conjugate to f and $g^q(z) \approx G_T(z)$ for all $z \not\approx 0, \infty$ implies that $\mathrm{sh}(g^q) = G_T$. Thus $\langle f \rangle = \langle g \rangle \approx \langle G_T \rangle_{p/q} \in \mathcal{B}_{p/q}$.

q.e.d

Lemma 11.3 (Convergence in $\widetilde{\mathcal{M}_2}$) *Let $\langle f_n \rangle_{n\in\mathbb{N}}$ be a sequence in $\mathcal{M}_2$ and $T \neq \infty$. We have*
$$\langle f_n \rangle \to \langle G_T \rangle_{p/q} \in \mathcal{B}_{p/q} \quad \textit{if and only if}$$
$\langle f_n \rangle \to \langle e^{2\pi ip/q}, e^{-2\pi ip/q}, \infty \rangle$ in $\widehat{\mathcal{M}_2}$ and the representative maps f_n can be chosen such that f_n^q converges to G_T locally uniformly on $\mathbb{C} - \{0, \infty\}$.

Proof: By the external definition of convergence we have $\langle f_n \rangle \to \langle G_T \rangle_{p/q}$ if and only if $\langle f_N \rangle \approx \langle G_T \rangle_{p/q} \in \mathcal{B}_{p/q}$ for all $N \approx \infty$. This is the case if and only if $\langle f_N \rangle \approx \langle e^{2\pi ip/q}, e^{-2\pi ip/q}, \infty \rangle$ in $\widehat{\mathcal{M}_2}$ and f_N can be normalized such that $\mathrm{sh}(f_N^q) = G_T$. By the previous lemma this is the case if and only if for the normalization $f_N(z) = \frac{1}{\lambda_N}(z + \frac{1}{z} + S_N)$ with $\lambda_N \approx e^{2\pi ip/q}$ we have $f_N(z) \approx G_T(z)$ for all $z \not\approx 0, \infty$. The external characterization of locally uniform convergence outside a finite set implies that this is the case if and only if $\langle f_n \rangle \to \langle e^{2\pi ip/q}, e^{-2\pi ip/q}, \infty \rangle$ in $\widehat{\mathcal{M}_2}$ and the representative maps f_n can be chosen such that f_n^q converges to G_T locally uniformly on $\mathbb{C} - \{0, \infty\}$.

q.e.d.

Remarks and References.

1. By [De2, Theorem 5.4] our definition of $\widetilde{\mathcal{M}_2}$ is equivalent to DeMarco's.
2. Theorem 9.2 allows us to construct a quadratic rational map the is close to a given standard point $\langle G_T \rangle_{p/q} \in \mathcal{B}_{p/q}$. Choose $\lambda \approx e^{2\pi i \frac{p}{q}}$. Define $S = T\frac{1-\lambda}{1-\lambda^q}$. Then we have $\langle F_{\lambda,S} \rangle \approx \langle G_T \rangle_{p/q} \in \mathcal{B}_{p/q}$.

Chapter 12

Homeomorphism between the Mandelbrot set and M^λ

In this chapter we review a dynamical defined homeomorphism between the Mandelbrot set M and the connectedness locus $M^\lambda \subset \mathrm{Per}_1(\lambda) \subset \mathcal{M}_2$ for any $\lambda \in D^*$. We will establish:

For each $c \in M$ and $\lambda \in D^*$ there is a unique $\langle R_{\lambda,c}\rangle \in M^\lambda$ that is hybrid equivalent to P_c on its filled Julia set, the complement of the basin of the attracting fixed point with multiplier λ.

We also discuss a theorem of Petersen, which states that the p/q-limb in M^λ disappears as λ tends to $e^{-2\pi i p/q}$ radially.

The connectedness locus $\mathbf{M}^{\boldsymbol{\lambda}} \subset \mathbf{Per_1}(\boldsymbol{\lambda})$. Recall that $\mathrm{Per}_1(\lambda) \cong \mathbb{C}$ is the slice of all conjugacy classes having a fixed point with multiplier λ. We denote by $M^\lambda \subset \mathrm{Per}_1(\lambda) \subset \mathcal{M}_2$ the connectedness locus in $\mathrm{Per}_1(\lambda)$:

$$M^\lambda = \{\langle f\rangle \in \mathrm{Per}_1(\lambda) : \ J(f) \text{ is connected}\}.$$

Parameterization of $\mathbf{Per_1}(\boldsymbol{\lambda})$. To represent a map in $\mathrm{Per}_1(\lambda)$ with $\lambda \neq 0$ we use the family

$$\mathcal{F}_\lambda = \left\{F_{\lambda,T}(z) = \frac{1}{\lambda}\left(z + \frac{1}{z} + T\right) : T \in \mathbb{C}\right\}.$$

Recall that $F_{\lambda,T}$ is holomorphically conjugate to $F_{\lambda,-T}$ by $z \mapsto -z$ interchanging the roles of the critical points ± 1.

Filled Julia set of a quadratic rational map with a distinguished attracting fixed point. Let f be a rational map of degree two and z_0 an attracting fixed point for f. We define the *filled Julia set* of f relative to z_0 by

$$K(f, z_0) = \{z \in \hat{\mathbb{C}} : \ f^n(z) \text{ does not tend to } z_0 \text{ as } n \to \infty\}.$$

Thus the filled Julia set $K(f, z_0)$ is obtained by filling in all Fatou components except the basin of z_0.

Like in the quadratic family the filled Julia set $K(f, z_0)$ is completely invariant and its complement is connected.

For the maps $F_{\lambda,T}$ with $\lambda \in D^*$ we adopt the convention that the distinguished fixed point is ∞. So in this case we have

$$K(F_{\lambda,T}, \infty) = \{z \in \mathbb{C} : \; F^n_{\lambda,T}(z) \text{ does not tend to } \infty \text{ as } n \to \infty\}.$$

Theorem 12.1 (Quadratic rational maps with an attracting fixed point are quadratic-like) *Let f be a quadratic rational map with connected Julia set and an attracting fixed point z_0. There exist topological discs U and V containing the filled Julia set $K(f, z_0)$, such that $f : U \to V$ is a quadratic-like map.*

By the Straightening Theorem 6.1 this quadratic-like map $f : U \to V$ is hybrid equivalent to a unique polynomial P_c. Note that there is no unique choice of the discs U and V. However for any other choice of discs $\tilde{U}$ and $\tilde{V}$, with $K(f) \subset \tilde{U}$ and $f : \tilde{U} \to \tilde{V}$ a quadratic-like map, $f : \tilde{U} \to \tilde{V}$ is hybrid equivalent to the same quadratic polynomial P_c.

Proof: The basin $\mathcal{A}_f(z_0)$ is a simply connected domain, so by the Riemann mapping theorem there exists a holomorphic homeomorphism ϕ mapping $\mathcal{A}_f(z_0)$ to the unit disc $\mathbb{D}$:

$$\phi : \mathcal{A}_f(z_0) \to \mathbb{D}.$$

Then the map

$$\tilde{f} = \phi \circ f \circ \phi^{-1} : \mathbb{D} \to \mathbb{D}$$

has an attracting fixed point and by conjugating with a Möbius transformation we can assume that this fixed point is at 0. Then we know by the Schwarz Lemma 3.3 that $|\tilde{f}(z)| < |z|$ for all $z \in \mathbb{D}$.

Let $r < 1$ such that the critical value $f(\omega)$, where $\omega \in A_f(z_0)$ is the critical point that is attracted to z_0 under iteration, is in $\phi^{-1}(B_r(0))$.

Define $V = (\overline{\phi^{-1}(B_r(0))})^c$ and $U = f^{-1}(V)$.

Claim: The map $f : U \to V$ is a quadratic-like map.

1. $f : f^{-1}(V) \to V$ is proper of degree 2.
2. $\overline{U} \subset V$, because

$$U = f^{-1}(V) = f^{-1}((\overline{\phi^{-1}(B_r(0))})^c) = (f^{-1}(\overline{\phi^{-1}(B_r(0))})^c \subset (f^{-1}(\phi^{-1}(B_r(0)))^c$$

Note that $(f^{-1}(\phi^{-1}(B_r(0)))^c$ is closed and thus $\overline{U} \subset (f^{-1}(\phi^{-1}(B_r(0)))^c$.

By the Schwarz Lemma we can conclude that $\tilde{f}(\overline{B_r(0)}) \subset B_a(0) \subset B_r(0)$ with $a = \max_{|x|=r} |f(x)|$ and thus $\overline{B_r(0)} \subset \tilde{f}^{-1}(B_r(0))$. So we have

$$\overline{U} \subset (f^{-1}(\phi^{-1}(B_r(0)))^c = (\phi^{-1}(\tilde{f}^{-1}(B_r(0))))^c \subset (\phi^{-1}(\overline{B_r(0)}))^c \subset (\overline{\phi^{-1}(B_r(0))})^c = V.$$

3. V is a topological disc containing exactly one critical value of f and $f : f^{-1}(V) \to V$ a proper degree two map. So we can conclude by the Riemann-Hurwitz formula that U is a topological disc. q.e.d.

By Theorem 12.1 the maps $F_{\lambda,T}$, with $\lambda \in D^*$ and $J(F_{\lambda,T})$ connected, are quadratic-like maps on their filled Julia set, i.e. there exists discs U and V with $K(F_{\lambda,T}) \subset U \subset V$ such that $F_{\lambda,T} : U \to V$ is a quadratic-like map hybrid equivalent to a unique quadratic polynomial $P_c(z) = z^2 + c$.

Theorem 12.2 *Let $\lambda \in \mathbb{D}^*$ and $c \in M$. There exists a unique $\langle F_{\lambda,T} \rangle \in \mathrm{Per}_1(\lambda) \subset \mathcal{M}_2$, such that $F_{\lambda,T}$ is hybrid equivalent to P_c on its filled Julia set $K(F_{\lambda,T}, \infty)$.*

Proof: *Existence:* Let $\lambda \in \mathbb{D}^*$ and $c \in M$. By [DH, Proposition 5] there exists a quadratic-like map f_0 hybrid equivalent to P_c and whose exterior class is $h_\lambda(z) = z\frac{z+\lambda}{\bar{\lambda}z+1}$. Note that $h_\lambda : S^1 \to S^1$ is expanding if $|\lambda| < 1$. The map f_0 can be chosen so it extends to a rational map f, because h_λ extend from the unit circle S^1 to a holomorphic map on the unit disc $\mathbb{D}$. The extended map f can be normalized such that the fixed point of h_λ at $z = 0$ becomes the fixed point of f at $z = \infty$. Then we have $f'(\infty) = \lambda$. Thus $f = F_{\lambda,T}$ for some $T \in \mathbb{C}$. (This is a modification of the proof of [DH, Proposition 4].)

Uniqueness: Let $T_1, T_2 \in \mathbb{C}$ such that F_{λ,T_1} and F_{λ,T_2} are hybrid equivalent to P_c on their filled Julia set. Then F_{λ,T_1} and F_{λ,T_2} restricted to suitable discs containing their filled Julia set are hybrid equivalent and externally equivalent. Thus by [DH, Theorem 6] F_{λ,T_1} and F_{λ,T_2} are holomorphically conjugate on open subset containing their filled Julia set. Thus they have the same multipliers at their fixed points, which implies that they are holomorphically conjugate. q.e.d.

Definition of $R_{\lambda,c}$. Given $\lambda \in \mathbb{D}^*$ and $c \in M$ we define

$$R_{\lambda,c}(z) = \frac{1}{\lambda}\left(z + \frac{1}{z} + T_\lambda(c)\right)$$

where $T_\lambda(c)$ is the unique parameter such that $R_{\lambda,c}$ is real hybrid equivalent to P_c on its filled Julia set $K(R_{\lambda,c}, \infty)$ and the critical point 1 is in $K(R_{\lambda,c}, \infty)$.

This $T_{\lambda(c)}$ exists and is unique by Theorem 12.2. There are two holomorphically conjugate maps in the family $\mathcal{F}_\lambda$ that are hybrid equivalent to P_c on their filled Julia set, but the condition $1 \in K(R_{\lambda,c}, \infty)$ determines that map uniquely.

The fixed point of $R_{\lambda,c}$. The map $R_{\lambda,c}$ has three fixed points. We label them by α, β and γ. By γ we denote the attracting fixed point ∞ with multiplier λ. The other two fixed points are in the filled Julia set $K(R_{\lambda,c}, \infty)$ and correspond to the α- and β-fixed points of P_c. As in the case of quadratic polynomials we define β to be the fixed point in $K(R_{\lambda,c}, \infty)$ that is repelling and does not disconnect the Julia set, i.e. $K(R_{\lambda,c}, \infty) - \{\beta\}$ is connected. The other fixed point that is in $K(R_{\lambda,c}, \infty)$ is called α.

Theorem 12.3 *The map*

$$\mathbb{D}^* \times M \to \mathcal{M}_2 \text{ given by } (\lambda, c) \mapsto \langle R_{\lambda,c} \rangle$$

is continuous, and it maps $\{\lambda\} \times M$ *homeomorphically onto* $M^\lambda \subset \mathrm{Per}_1(\lambda) \subset \mathcal{M}_2$.

This implies in particular that the map $M \to M^\lambda$ given by $c \mapsto \langle R_{\lambda,c} \rangle$ is a homeomorphism.

Proof: Consider the family:

$$\mathcal{P} = \left\{ F_{\lambda,T}(z) = \frac{1}{\lambda}\left(z + \frac{1}{z} + T\right) : \text{ with } (\lambda, T) \in \mathbb{D}^* \times \mathbb{C} \text{ and } 1 \in K(F_{\lambda,T}, \infty) \right\}.$$

Since the critical point 1 is in the filled Julia set $K(F_{\lambda,T}, \infty)$, the Julia set is connected and thus by Theorem 12.1 each map $F_{\lambda,T} \in \mathcal{P}$ is a quadratic-like map that is hybrid equivalent to a unique quadratic polynomial $P_{c(\lambda,T)}(z) = z^2 + c(\lambda, T)$ on its filled Julia set $K(F_{\lambda,T}, \infty)$. Consider the map

$$\Phi : \mathcal{P} \to \mathbb{D}^* \times M \qquad F_{\lambda,T} \mapsto (\lambda, c(\lambda, T)).$$

Claim: The map Φ is continuous.

The first coordinate map $\Phi_1(F_{\lambda,T}) = \lambda$ is obviously continuous. We have to show that $\Phi_2(F_{\lambda,T}) = c(\lambda, T)$ is continuous. Let $F_{\lambda,T} \in \mathcal{P}$. Then there exist discs $K(F_{\lambda,T}) \subset U \subset V$ such that $F_{\lambda,T} : U \to V$ is quadratic-like. There exist $\epsilon > 0$ such that for all $(\tilde{\lambda}, \tilde{T}) \in B_\epsilon(\lambda, T)$ the map $F_{\tilde{\lambda},\tilde{T}} : \tilde{U} \to V$ with $\tilde{U} = F^{-1}_{\tilde{\lambda},\tilde{T}}(V)$ is quadratic-like and the family $\{F_{\tilde{\lambda},\tilde{T}} : \tilde{U} \to V : (\tilde{\lambda}, \tilde{T}) \in B_\epsilon(\lambda, T)\}$ is an analytic family of quadratic-like maps. By [DH, Theorem 2] there exist a continuous straightening map $\{F_{\tilde{\lambda},\tilde{T}} : \tilde{U} \to V : (\tilde{\lambda}, \tilde{a}) \in B_\epsilon(\lambda, T)\} \to \mathbb{C}$, $(\tilde{\lambda}, \tilde{T}) \mapsto c(\tilde{\lambda}, \tilde{T})$. Recall that $c(\tilde{\lambda}, \tilde{T})$ is uniquely determined in case that $J(F_{\tilde{\lambda},\tilde{T}})$ is connected.

Thus $\Phi : \mathcal{P} \to \mathbb{D}^* \times M$ is a continuous and by Theorem 12.2 a bijection. Because $\mathcal{P}$ is locally compact, Φ is a homeomorphism.

Since $\langle \Phi^{-1}(\lambda, c) \rangle = \langle R_{\lambda,c} \rangle$ the map $\mathbb{D}^* \times M \to \mathcal{M}_2$ defined by $(\lambda, c) \mapsto \langle R_{\lambda,c} \rangle$ is continuous and maps $\{\lambda\} \times M$ homeomorphically onto $M^\lambda \subset \mathrm{Per}_1(\lambda) \subset \mathcal{M}_2$.

q.e.d.

The disappearing limbs. The following theorem by Petersen shows that if c is in the p/q-limb $M_{p/q}$ then $\langle R_{\lambda,c} \rangle$ tends to the ideal point $\langle e^{2\pi i p/q}, e^{-2\pi i p/q}, \infty \rangle$ as λ tends to $e^{-2\pi i p/q}$. So in the connectedness locus $M^\lambda \subset \mathrm{Per}_1(\lambda) \subset \mathcal{M}_2$ the whole p/q-limb disappears to infinity as λ tends to $e^{-2\pi i p/q}$.

Theorem 12.4 (Petersen) *Let* $c \in M_{p/q}$ *and* $\lambda = (1 - \epsilon)e^{-2\pi i p/q}$. *Then*

$$\lim_{\epsilon \to 0} \langle R_{\lambda,c} \rangle = \langle e^{2\pi i p/q}, e^{-2\pi i p/q}, \infty \rangle.$$

The multiplier at the β*-fixed point of* $R_{\lambda,c}$ *is tending to infinity as* ϵ *tends to* 0.

See [Pe, Corollary 2]. Note that it suffices to require that λ tends to $e^{-2\pi ip/q}$ non-quadratically.

In the pictures below we see the connectedness locus of the family $F_T(z) = \frac{1}{\lambda}(z+\frac{1}{z}+T)$ in the $1/T^2$ plane for different values of λ tending to -1 and $e^{-2\pi i/3}$ respectively, showing the 1/2-limb and 1/3-limb disappearing.

Remarks and References.

1. Results similar to Theorem 12.3 are also discussed in [Mi2], [GK], [Pe] and [Uh].

2. Given $c \in M_{p/q}$, one can regard $\langle R_{\lambda,c}\rangle$ as the mating of $P_c(z) = z^2 + c$ and $P_{c(\lambda)}(z) = z^2 + c(\lambda)$, the unique quadratic polynomial that has an attracting fixed point with multiplier λ. If $\lambda = (1-\epsilon)e^{-2\pi ip/q}$ the mating $\langle R_{\lambda,c}\rangle$ tends to infinity in $\mathcal{M}_2$ as $\epsilon \to 0$.

   This phenomenon is related to the fact that one cannot mate two polynomials corresponding to points in complex conjugate limbs of the Mandelbrot set. Matings of complex quadratic polynomials are discussed in [Tan]. There it is shown that the mating of two critically finite quadratic polynomials P_{c_1} and P_{c_2} exists if and only if c_1 and c_2 do not belong to complex conjugate limbs of the Mandelbrot set M. It is conjectured that the mating of two quadratic polynomials P_{c_1} and P_{c_2} can be well defined as an element of $\mathcal{M}_2$ if and only if c_1 and c_2 do not belong to complex conjugate limbs of M, see also [Mi2, Section 7].

   The point $c(\lambda)$ corresponds to a point in the main cardioid of the Mandelbrot set tending to the $-p/q$-limb as ϵ tends to 0. If $\epsilon \approx 0$, then $R_{\lambda,c}$ is the result of a nearly illegal mating and so the resulting map is close to infinity in moduli space.

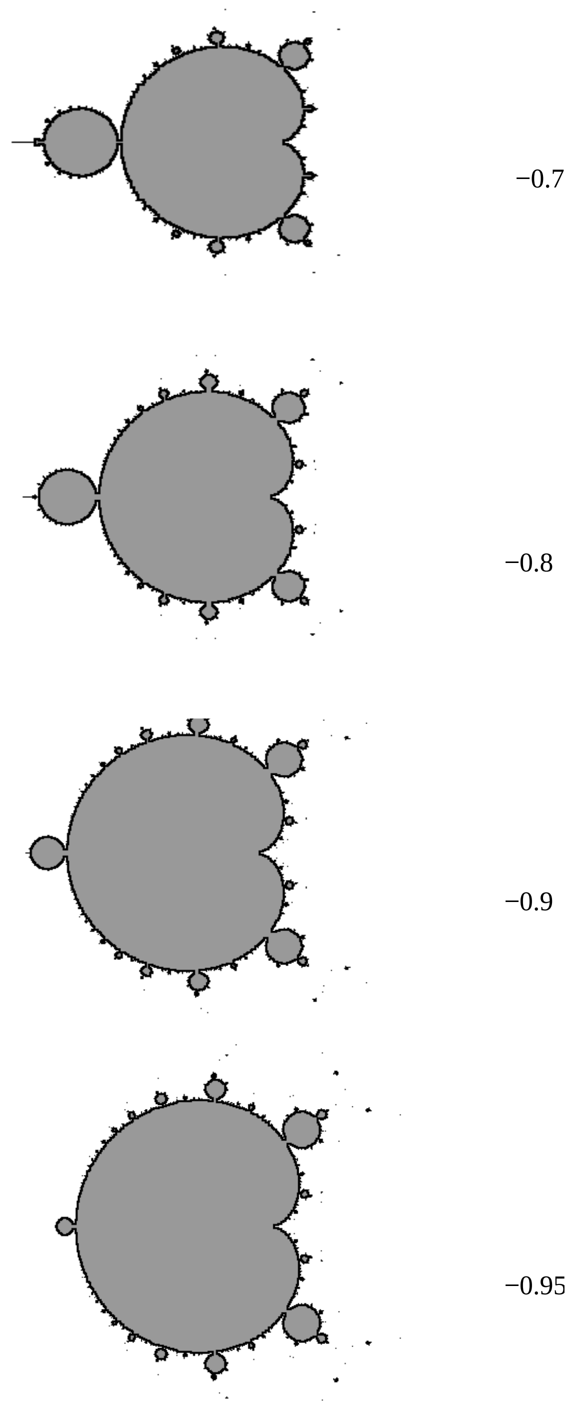


Figure 12.1: The disappearing 1/2-limb.
The connectedness locus for the family $\mathcal{F}_\lambda = \{F_{\lambda,T} = \frac{1}{\lambda}(z + \frac{1}{z} + T) : \ T \in \mathbb{C}\}$ in the $1/T^2$-plane for $\lambda = -0.7, \lambda = -0.8, \lambda = -0.9$ and $\lambda = -0.95$.

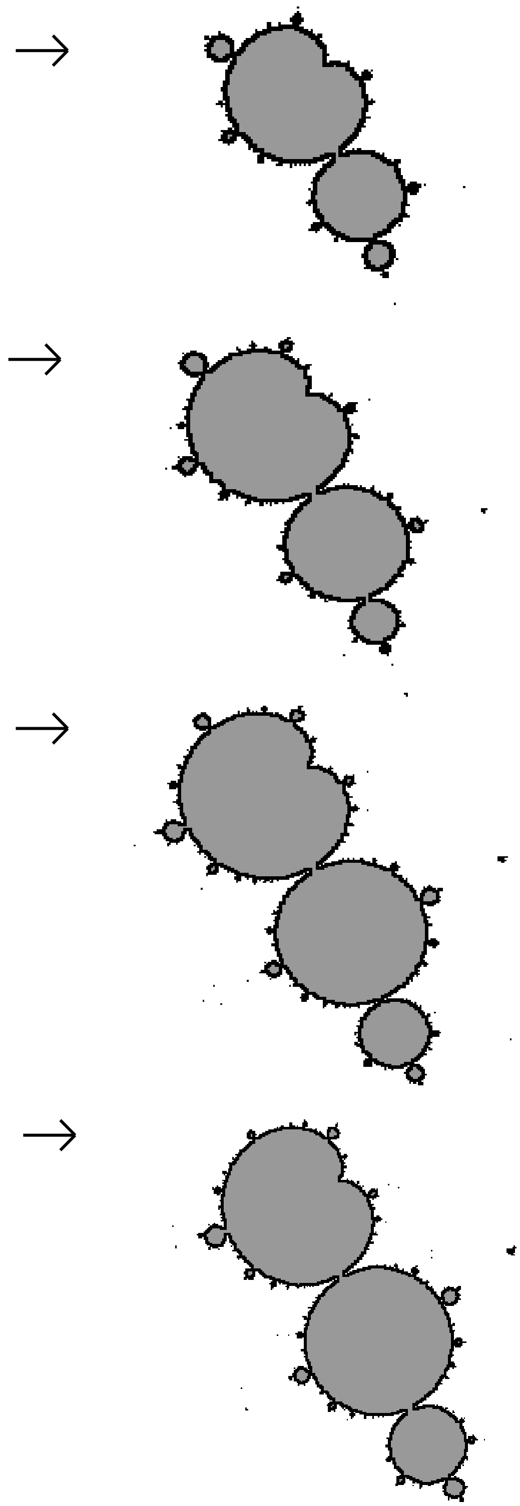


Figure 12.2: The disappearing 1/3-limb.
The connectedness locus for the family $\mathcal{F}_\lambda = \{F_{\lambda,T} = \frac{1}{\lambda}(z + \frac{1}{z} + T) : \ T \in \mathbb{C}\}$ in the $1/T^2$-plane for $\lambda = 0.7\, e^{-2\pi i/3}$, $\lambda = 0.8\, e^{-2\pi i/3}$, $\lambda = 0.9\, e^{-2\pi i/3}$ and $\lambda = 0.95\, e^{-2\pi i/3}$.

Chapter 13

Conjecture about the disappearing limbs

Now we formulate our conjecture about the limit

$$\lim_{\epsilon \to 0} \langle R_{(1-\epsilon)e^{-2\pi i p/q}, c} \rangle$$

in the dynamical compactification $\widetilde{\mathcal{M}}_2$.

If $c \in M_{p/q}$, then Theorem 12.4 implies that $\lim_{\epsilon \to 0} \langle R_{(1-\epsilon)e^{-2\pi i p/q}, c} \rangle = \langle e^{2\pi i p/q}, e^{-2\pi i p/q}, \infty \rangle$ in the algebraic compactification $\widehat{\mathcal{M}}_2$. In the dynamical compactification, $\widetilde{\mathcal{M}}_2$, this ideal point $\langle e^{2\pi i p/q}, e^{-2\pi i p/q}, \infty \rangle$ is replaced by a whole sphere of ideal points $\mathcal{B}_{p/q}$ reflecting the limiting dynamics of the q-th iterate. We conjecture that the limit exists in $\widetilde{\mathcal{M}}_2$ and that the map

$$M_{p/q} \to \mathcal{B}_{p/q} \subset \widetilde{\mathcal{M}}_2 \;\text{ given by }\; c \mapsto \lim_{\epsilon \to 0} \langle R_{(1-\epsilon)e^{-2\pi i p/q}, c} \rangle$$

maps the image of tuning $\tau_{p/q}(M) \subset M_{p/q}$ homeomorphically onto a copy of the connectedness locus M^1 in $\mathcal{B}_{p/q}$ and that the rest of the limb is mapped to certain endpoints of the connectedness locus. This requires a dynamical homeomorphism between the Mandelbrot set and the connectedness locus $M^1 \subset \mathrm{Per}_1(1) \subset \mathcal{M}_2$. It is a well-known conjecture that such a homeomorphism exists, see e.g. [Mi2, p. 27].

Conjecture 13.1 (Homeomorphism between M and M^1) *For any $c \in M$ the limit*

$$R_{1,c} = \lim_{\substack{\lambda \to 1 \\ \lambda < 1}} R_{\lambda, c}$$

exists in Rat_2 and the map $M \to M^1 \subset \mathrm{Per}_1(1)$ defined by $c \mapsto \langle R_{1,c} \rangle$ is a homeomorphism.

Under the assumption of Conjecture 13.1 we formulate our main conjecture.

Conjecture 13.2 (Main conjecture — Images of disappearing limbs) *For any $c \in M_{p/q}$ the limit*

$$L_{p/q}(c) = \lim_{\epsilon \to 0} \langle R_{(1-\epsilon)e^{-2\pi i p/q}, c} \rangle$$

exists in the dynamical compactification $\widetilde{\mathcal{M}}_2$ *and lies in* $\mathcal{B}_{p/q}$. *The map*

$$L_{p/q} : M_{p/q} \to \mathcal{B}_{p/q} \subset \widetilde{\mathcal{M}}_2$$

has the following properties:

1. $L_{p/q}$ *is continuous.*
2. $L_{p/q}$ *maps* $\tau_{p/q}(M)$ *homeomorphically onto*

$$\{\langle G_T \rangle_{p/q} \in \mathcal{B}_{p/q} : \ J(G_T) \text{ is connected}\} \subset \mathcal{B}_{p/q} \subset \widetilde{\mathcal{M}}_2$$

   with

$$L_{p/q}(\tau_{p/q}(b)) = \langle R_{1,b} \rangle_{p/q} \in \mathcal{B}_{p/q}.$$

3. $L_{p/q}$ *is locally constant on* $M_{p/q} - \tau_{p/q}(M)$.

Note that Conjecture 13.2 presumes Conjecture 13.1 , because we assume that the map $R_{1,b}$ exists.

We will prove a real version of Conjecture 13.1 in Chapter 16 and a real version of Conjecture 13.2 in Chapter 18.

Remarks and References.

1. Epstein and Petersen are working independently on questions similar to Conjecture 13.2. See [EP].
2. Progress on Conjecture 13.1 by Petersen and Roesch will appear in [PR].
3. Once $L_{p/q}$ is known to exist it maps $M_{p/q}$ into $\mathcal{B}_{p/q}$ by Theorem 12.4
4. It is easy to show that the limit

$$\lim_{\substack{\lambda \to 1 \\ \lambda < 1}} R_{\lambda,c}$$

   exists in the hyperbolic case. This will be done in the proof of Theorem 16.2.

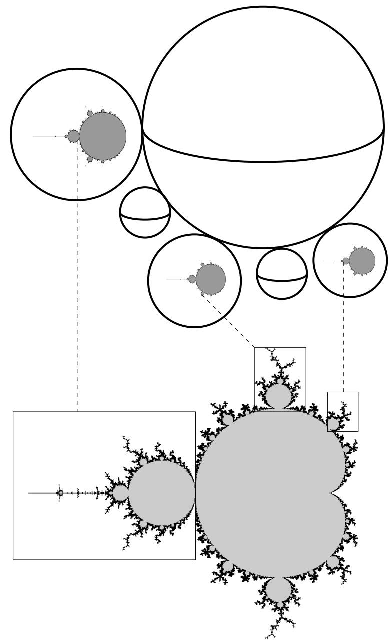


Figure 13.1: The image of the 1/2-limb, 1/3-limb and 1/4-limb on the spheres $\mathcal{B}_{1/2}$, $\mathcal{B}_{1/3}$ and $\mathcal{B}_{1/4}$ in the boundary of the dynamical compactification $\widetilde{\mathcal{M}}_2$ according to the conjecture.

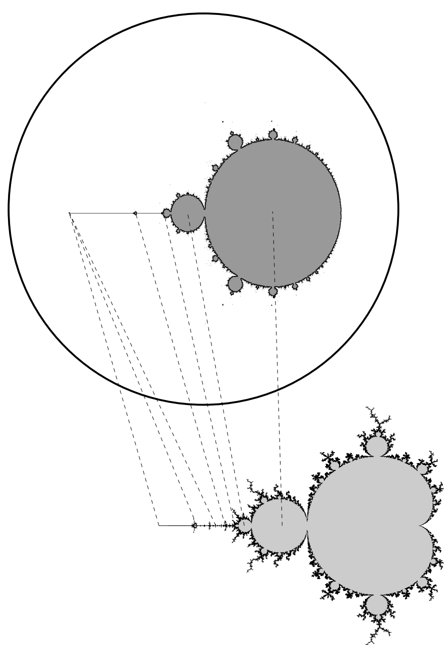


Figure 13.2: The image of the 1/2-limb on the sphere $\mathcal{B}_{1/2}$ in the boundary of the dynamical compactification $\widetilde{\mathcal{M}}_2$.

Chapter 14

Real quadratic polynomials and S-unimodal maps

In this chapter we introduce S-unimodal maps. Our main goal is to establish the following fundamental fact:

> *Every S-unimodal map is real hybrid equivalent to a unique real quadratic polynomial.*

This follows from well-known but deep results such as the density of hyperbolicity in the real quadratic family.

For ease of application we also show that a real quadratic polynomial is determined by its kneading sequence and the multiplier at its non-repelling cycle, if one exists.

Kneading theory of unimodal maps. We now summarize some facts about the kneading theory of unimodal maps. For a detailed exposition we refer to [MT].

Intervals as subsets of $\hat{\mathbb{R}}$. We can identify $\hat{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ with the unit circle S^1 via stereographic projection. Given $a, b \in S^1$ let $[a, b]$ denote the positively oriented subinterval from a to b. When c is in $[a, b]$ we write $a < c < b$. Any pair of points $a, b \in \hat{\mathbb{R}}$ with $a \neq b$ defines two closed interval in $\hat{\mathbb{R}}$: $[a, b]$ and $[b, a]$.

Unimodal maps. Let $[a, b] \subset \hat{\mathbb{R}}$, $a \neq b$, be a closed interval and $f : [a, b] \to [a, b]$ a continuous map. We say that f is *unimodal* if the following conditions are satisfied:

1. There exist a point $\omega \in (a, b)$ such that f is strictly monotone increasing (or decreasing) on $[a, \omega)$ and strictly monotone decreasing (or increasing) on $(\omega, b]$ (ω is called *turning point*).

2. $f([a, b]) \subset [a, b]$ and $f(a) = f(b) \in \{a, b\}$.

Examples:

1. Let $c \in [-2, 1/4]$. Then for the quadratic polynomial $P_c(z) = z^2 + c$ both finite fixed points α and β are real and the map $P_c : [-\beta, \beta] \to [-\beta, \beta]$ is a unimodal map with turning point 0. Also $P_c : [\beta, -\beta] \to [\beta, -\beta]$ is a unimodal map with turning point ∞.

2. Let $-2 \leq T \leq 0$. Then $G_T : [0, \infty] \to [0, \infty]$ given by $G_T(z) = z + 1/z + T$, is a unimodal map with turning point 1. Also $G_T : [\infty, 0] \to [\infty, 0]$ is a unimodal map with turning point -1.

Kneading sequences of unimodal maps. Let $f : [a, b] \to [a, b]$ be some unimodal map with turning point ω monotone increasing on I_1 and monotone decreasing on I_2, such that $[a, b]$ is the disjoint union of I_1, I_2 and $\{\omega\}$. Then the *kneading sequence* $k(f)$ of f is defined by

$$k_n(f) = \widetilde{k}_1(f) \cdots \widetilde{k}_n(f)$$

where

$$\widetilde{k}_i(f) = \begin{cases} 1 & f^i(\omega) \in I_1 \\ 0 & f^i(\omega) = \omega \\ -1 & f^i(\omega) \in I_2 \end{cases}$$

Note that if f is smooth, then $k_n(f)$ is equal to the sign of $(f^n)'(f(\omega))$.

An order relation on the set of all kneading sequences is defined in the following way:

$$k(g) < k(f) \text{ if and only if } \exists n \in \mathbb{N}\ \forall i < n\ k_i(g) = k_i(f) \text{ and } k_n(g) < k_n(f).$$

A kneading sequence is *admissible* if it actually occurs for a unimodal map. The admissible kneading sequences are easily characterized: Let σ denote the shift-operator, i.e. $\sigma(k_1, k_2, k_3 \cdots) = (k_2, k_3 \cdots)$. The sequence $k = (k_1, k_2, k_3 \cdots)$ is admissible if and only if $k \leq \sigma^n(k) \leq \sigma(k)$ for all $n \in \mathbb{N}$.

Two unimodal maps $f : [a, b] \to [a, b]$ and $g : [c, d] \to [c, d]$ are *topologically conjugate* if there exists a homeomorphism $h : [a, b] \to [c, d]$ with $h \circ f = g \circ h$. Topologically conjugate unimodal maps have the same kneading sequence.

Theorem 14.1 (Intermediate value theorem) *Consider a real analytic family of unimodal maps $f_a : I \to I$, $a \in [a_0, a_1]$. Any admissible kneading sequence that lies between the kneading sequence of f_{a_0} and f_{a_1} must actually occur as the kneading sequence of f_a for some $a \in (a_0, a_1)$.*

See [MT, Theorem 12.2.].

The real quadratic family. The intersection of the Mandelbrot set with the real axis, the *real Mandelbrot set*, is the interval $[-2, 1/4]$. Recall that for a quadratic polynomial $P_c(z) = z^2 + c$ with $c \in [-2, 1/4]$, both finite fixed points α_c and β_c are real and $P_c : [-\beta_c, \beta_c] \to [-\beta_c, \beta_c]$ is a unimodal map. By the real quadratic family we mean the family of real quadratic polynomials:

$$P_c : [-\beta_c, \beta_c] \to [-\beta_c, \beta_c] \text{ with } c \in [-2, 1/4].$$

If we just consider one map we write β instead of β_c.

Kneading sequences in the real quadratic family. Because every kneading sequence lies between $(\overline{-1})$ and $(\overline{1})$ and $k(P_{-2}) = (\overline{-1})$ and $k(P_{1/4}) = (\overline{1})$ the Intermediate value Theorem 14.1 implies that all admissible kneading sequences occur in the real quadratic family.

Theorem 14.2 (Monotonicity in the real quadratic family) *For the real quadratic family $P_c(z) = z^2 + c$, $c \in [-2, 1/4]$, the map $c \mapsto k(P_c)$ is monotone increasing.*

See [MT, Theorem 13.1.].

Note that the kneading sequence of a unimodal map f is finite, i.e. it is zero from some point on, if and only if the turning point of f is periodic. The real quadratic polynomials with superattracting cycles are determined by their kneading sequences:

Theorem 14.3 *If P_c and $P_{c'}$ are real quadratic polynomials with superattracting cycles and $k(P_c) = k(P_{c'})$, then $c = c'$.*

See [MT, Lemma 13.2.].

Hyperbolic windows. The set of all real hyperbolic parameters in the Mandelbrot set is an open subset of $[-2, 1/4]$. We call the connected intervals of this open set *hyperbolic windows.* A parameter $c \in [-2, 1/4]$ is called *real hyperbolic center* if P_c has a superattracting periodic cycle. Note that the parabolic parameters are the endpoints of the hyperbolic windows.

Theorem 14.4 (Hyperbolic windows) *Let $H \subset [-2, 1/4]$ be a hyperbolic window. The multiplier at the unique attracting periodic cycle of P_c gives a monotone increasing real analytic diffeomorphism from H onto the interval $(-1, 1)$.*

Proof: If H is a hyperbolic component in the Mandelbrot set for the complex quadratic family, then the multiplier at the attracting periodic cycle gives a biholomorphic map onto the open unit disk. For the proof see [CG, Theorem VIII.2.1.]. Now let H be a hyperbolic component that meets the real axis, then $c \in H$ implies that $\overline{c} \in H$. We have to show that the real multipliers are realized by real parameters. If P_c has an attracting periodic cycle with multiplier $\eta \in \mathbb{R}$ then $P_{\overline{c}}$ has an attracting periodic cycle with multiplier $\overline{\eta} = \eta$. Since their is only one map in H having an attracting periodic cycle with multiplier η, we can conclude that $c = \overline{c}$. The diffeomorphism is monotone increasing, because the kneading sequences are monotone increasing, see Monotonicity Theorem 14.2. q.e.d.

Topologically attracting cycles. Let $f : [a, b] \to [a, b]$ be a unimodal map. A periodic point $f^n(z_0) = z_0$ is called *topologically attracting* if there is an open interval $I \subset [a, b]$ with $z_0 \in I$ such that $(f^n)^m(z) \to z_0$ as $m \to \infty$ for all $z \in I$.

Topologically attracting windows. In the real quadratic family the topologically attracting periodic points that are not attracting are precisely the parabolic periodic points with multiplier -1, see Theorem 4.4. Therefore we add the parameter values c for which P_c has parabolic periodic point with multiplier -1 to the hyperbolic windows and define a *topologically attracting window* to be a hyperbolic window together with the endpoint that corresponds to a quadratic polynomial having a parabolic cycle with multiplier -1. Then every topologically attracting parameter, i.e. every parameter c for which P_c has a topologically attracting periodic cycle, lies in precisely one topologically attracting window.

Next we recall:

Theorem 14.5 (Density of hyperbolicity for the real quadratic family) *In the real quadratic family*

$$P_c(z) = z^2 + c, \quad c \in [-2, 1/4]$$

the set of parameter values c for which P_c has an attracting periodic cycle (and is thus hyperbolic) is open and dense.

See [GS] and [Ly].

Corollary 14.6

1. *If P_c and $P_{c'}$ have no topologically attracting cycles, then P_c and $P_{c'}$ are topologically conjugate if and only if $c = c'$.*

2. *If P_c and $P_{c'}$ have topologically attracting cycles, then P_c and $P_{c'}$ are topologically conjugate if and only if they lie in the same topologically attracting window and the multipliers at their unique topologically attracting cycles have the same sign.*

Proof:

1. Assume P_c and $P_{c'}$ are not topologically attracting and $c \neq c'$. Because hyperbolicity is dense there exists a $c < c_0 < c'$ such that P_{c_0} has a superattracting periodic cycle. Thus $k(P_c) \neq k(P_{c_0})$ since $k(P_{c_0})$ is finite and $k(P_c)$ is infinite. Because of the Monotonicity Theorem 14.2 this implies that $k(P_c) \neq k(P_{c'})$ and thus P_c and $P_{c'}$ cannot be topologically conjugate.

2. If two quadratic polynomials, that have a topologically attracting cycle, belong to different hyperbolic windows, they either have different kneading sequences or multipliers of different sign at their topologically attracting cycle, by Theorem 14.2 and 14.4. Thus two real quadratic polynomials having a topologically attracting cycle that are topologically conjugate must lie in the same topologically attracting window and have a multiplier of the same sign at their attracting cycle.

   On the other hand, maps with the same sign at their topologically attracting cycle in a given topologically attracting window are topologically conjugate. This follows from Guckenheimer's theorem, see [Gu, Theorem 2.10]. q.e.d.

This gives the following description of real topological conjugacy classes of real quadratic polynomials:

Remark 14.7 (Topological cojugacy classes in the real quadratic family)

1. *If two real quadratic polynomials with a topological attracting cycle are topologically conjugate, then they lie in the same topologically attracting window.*

2. *Every topologically attracting window consists of three topological conjugacy classes. The one with positive, zero and negative multiplier at the topologically attracting cycle.*

3. *The topological conjugacy class of a real quadratic polynomial without topologically attracting cycles consists of a single map P_c.*

Example: The set $\{P_c : c \in [-3/4, 0)\}$ is a topological conjugacy class. Note that the map $P_{-3/4}$ belongs to that conjugacy class, even though it has a parabolic fixed point with multiplier -1. The parabolic fixed point is topologically attracting, since it attracts all nearby real points, even though it is not topologically attracting if we consider P_c as a map defined in the complex plane.

Real quadratic polynomials are determined by their kneading sequence and the multiplier at their non-repelling cycle. Now we show that a real quadratic polynomial P_c with $c \in [-2, 1/4]$ is determined by its kneading sequence and, in the case there is a non-repelling cycle, the multiplier at its non-repelling cycle.

Theorem 14.8 1. *If $k(P_c) = k(P_{c'})$ and P_c and $P_{c'}$ both have a non-repelling cycle with multiplier η, then $c = c'$.*

2. *If $k(P_c) = k(P_{c'})$ and P_c has only repelling cycles, then $c = c'$.*

Proof:

1. If P_c and $P_{c'}$ have superattracting cycles and $k(P_c) = k(P_{c'})$, then $c = c'$, by Theorem 14.3. Now let P_c and $P_{c'}$ have an attracting cycle with multiplier $\eta \neq 0$. If c and c' belong to different hyperbolic windows, then there is a hyperbolic center c_0 with $c < c_0 < c'$, because of Theorem 14.4. Since the kneading sequences of the hyperbolic centers are the only ones that have only finitely many nonzero entries it follows by the Monotonicity Theorem 14.2 that $k(P_c) \neq k(P_{c'})$. If c and c' belong to the same hyperbolic window it follows by Theorem 14.4 that $c = c'$. Now assume that P_c and $P_{c'}$ have a parabolic cycle. If $c \neq c'$, then by density of hyperbolicity, Theorem 14.5, there is a hyperbolic center c_0 with $c < c_0 < c'$ and we can conclude by the Monotonicity Theorem 14.2 that $k(P_c) \neq k(P_{c'})$.

2. Let P_c and $P_{c'}$ have only non-repelling cycles with $c \neq c'$. Because hyperbolicity is dense, Theorem 14.5, there is a hyperbolic center c_0 with $c < c_0 < c'$ and we can conclude again by the Monotonicity Theorem that $k(P_c) \neq k(P_{c'})$.

q.e.d

Examples of kneading sequences in the real quadratic family.

1. If $c \in (0, 1/4]$, then $k(P_c) = (\overline{1})$.

2. $k(P_0) = (\overline{0})$.

3. If $c \in (-1, 0)$ then $k(P_c) = (\overline{-1, 1})$.

4. Recall that -1 is the centerpoint of the hyperbolic 2-component. We have $k(P_{-1}) = (-1, \overline{0})$.

5. Let $c_4 = -1.3107...$ denote the centerpoint of the period 4-component on the real axis in the image of tuning $\tau_{1/2}(M)$, $c_4 = \tau_{1/2}(-1)$. Then $k(P_{c_4}) = (-1, -1, 1, \overline{0})$.

6. If $c \in (c_4, -1)$ then $k(P_c) = (\overline{-1, -1, 1, 1})$.

7. Let $c_3 = -1.75488...$ denote the centerpoint of the unique real period 3-component. We have $k(P_{c_3}) = (-1, -1, \overline{0})$.

8. Note that -2 is the tip of the Mandelbrot set on the real axis — in this case the critical point lands on the β-fixed point after two iterations. We have $k(P_{-2}) = (\overline{-1})$.

9. Let $c_0 = -1.54368...$ denote the parameter with $P^2_{c_0}(0) = -\alpha$, i.e. $c_0 = \tau_{1/2}(-2)$, the smallest point on the real axis that is in the image of the tuning map $\tau_{1/2}(M)$. Then the critical point 0 lands on the fixed point α after three iterations, thus $k(P_{c_0}) = (-1, -1, \overline{1, -1})$.

Note that the kneading sequence of a real quadratic map P_c only changes as c passes a hyperbolic centerpoint. In particular, the kneading sequence does not determine the hyperbolic component. E.g. the same kneading sequence occurs in the main cardioid and the period 2 component, but together with the multiplier at the non-repelling cycles it determines the map.

We now review some facts about S-unimodal maps. For a detailed exposition of the theory of S-unimodal maps we refer to [MS].

The Schwarzian derivative. For a real analytic map $f : I \to I$ the Schwarzian derivative is defined as follows:

$$Sf(z) = \frac{f'''(z)}{f'(z)} - \frac{3}{2}\left(\frac{f''(z)}{f'(z)}\right)^2$$

It satisfies the following chain rule:

$$S(f \circ g)(z) = Sf(g(z))(g'(z))^2 + Sg(z).$$

This implies that the composition of maps with negative Schwarzian derivative has negative Schwarzian derivative. In particular iterates of maps with negative Schwarzian derivative have maps with negative Schwarzian derivative. Furthermore we have, $S(f) = 0$ if and only if f is a Möbius transformation.

S-unimodal maps. A real analytic map $f : [a, b] \to [a, b]$ is called *S-unimodal*, if the following conditions hold.

1. The map f is unimodal having a unique critical point ω in (a, b) that is non-degenerate, i.e. $f'(\omega) = 0$ and $f''(\omega) \neq 0$.

2. It has negative Schwarzian derivative , i.e. $Sf(z) < 0$ for all $z \in [a, b]$, $z \neq \omega$.

3. Furthermore we assume that the fixed point on the boundary of $[a, b]$ is non-attracting.

In the case that $\infty \in [a, b]$ these properties can be checked by conjugating with a real Möbius transformation, so that ∞ is not in the interval of definition. All the above properties are invariant under conjugation with a real Möbius transformation.

Example. A real quadratic polynomial P_c has negative Schwarzian derivative: $SP_c(z) = -\frac{3}{2}(\frac{P_c''(z)}{P_c'(z)})^2 < 0$. Furthermore the critical point 0 is non-degenerate since $P_c''(0) = 2 \neq 0$ and the β-fixed point is always repelling. So real quadratic polynomials $P_c : [-\beta, \beta] \to [-\beta, \beta]$ with $c \in [-2, 1/4]$ are S-unimodal maps.

Lemma 14.9 *A real quadratic rational map with real critical points has negative Schwarzian derivative.*

Proof: Every quadratic rational map f can be written as $f(z) = h_2((h_1(z))^2)$ with $h_1, h_2 \in \mathrm{Aut}(\hat{\mathbb{C}})$ — use h_1 to put 0 and ∞ to the critical points of f and h_2 to put the critical values at 0 and ∞, composing $h_2^{-1} \circ f \circ h_1^{-1}$ leads to a map that has fixed critical points at 0 and ∞, thus it must be $P_0(z) = z^2$. If f is a real quadratic rational map with real critical points then h_1 and h_2 are real. And thus we can conclude by the chain rule:

$$Sf(z) = S(h_2((h_1(z))^2))(z) = \underbrace{Sh_2((h_1(z))^2))}_{=0} + \underbrace{SP_0(h_1(z))}_{<0} + \underbrace{Sh_1(z)}_{=0} < 0.$$

q.e.d.

Example. The maps $F_{\lambda,T}(z) = \frac{1}{\lambda}(z + \frac{1}{z} + T)$ with $\lambda, T \in \mathbb{R}$ are real rational maps with critical points at ± 1 and have thus negative Schwarzian derivative.

Attractor. Let $f : [a, b] \to [a, b]$ be a unimodal map. A periodic cycle is called an *attractor* if there is an open interval $J \subset [a, b]$ that is in the basin of the periodic cycle. An attractor is either attracting periodic cycle or parabolic periodic cycle.

Note that a S-unimodal map might have a non-repelling fixed point, that is not an attractor.

Example. The map $G_{-1} : [0, \infty] \to [0, \infty]$ given by $G_T(z) = z + \frac{1}{z} - 1$ has two non-repelling cycles, a superattracting fixed point at 1 and the parabolic fixed point at ∞, but only one attractor. The parabolic fixed point ∞ is not an attractor, since it does not attract an open interval contained in $[0, \infty]$. Note however that it is an attractor for G_{-1} restricted to the interval $[\infty, 0]$.

Theorem 14.10 *Every S-unimodal map has at most one attractor. And in case it has one, its basin contains the unique critical point.*

See [MS, Theorem 6.1 and its Corollary].

Theorem 14.11 *Every S-unimodal map is topologically conjugate to a real quadratic polynomial.*

See [MS, Theorem 6.3 and 6.4 and Remark 1 on p. 164].

Real hybrid equivalence for S-unimodal maps. We say that two S-unimodal maps have *compatible multipliers* if they have the same multiplier at their attractor (in case they both have one). We say that two S-unimodal maps f and g are *real hybrid equivalent* if they are topologically conjugate and have compatible multipliers.

Theorem 14.12 (Real straightening theorem) *Every S-unimodal map is real hybrid equivalent to a unique quadratic polynomial P_c with $c \in [-2, 1/4]$.*

This result is well-known, e.g. it is mentioned in [LAM, p. 21].

Proof: Let f be a S-unimodal map. By Theorem 14.11, f is topologically conjugate to a quadratic polynomial P_c.

1. If f is not topologically attracting then P_c is the unique quadratic polynomial in its topological conjugacy class, see Corollary 14.6. Either P_c and f both have a parabolic attractor with multiplier 1 or they only have repelling cycles. Thus they are real hybrid equivalent.

2. If f has a topologically attracting cycle with multiplier η, then P_c has a topologically attracting cycle with a multiplier of the same sign as η. By Corollary 14.6 and Theorem 14.4 we can conclude that there is a unique quadratic polynomial $P_{c'}$ conjugated to f having a topological attracting cycle with multiplier η. q.e.d.

Combining the Real Straightening Theorem 14.12 with Theorem 14.8, we obtain:

Corollary 14.13 (Alternative characterization of hybrid equivalence) *Two S-unimodal maps are real hybrid equivalent if and only if they have the same kneading sequence and compatible multipliers.*

The following theorem shows that the notion of real hybrid equivalence is compatible with the notion hybrid equivalence of quadratic-like maps.

Theorem 14.14 (Hybrid equivalence and real hybrid equivalence) *Let $f : I \to I$ and $g : J \to J$ be S-unimodal maps that can be extended to quadratic-like maps in $\mathbb{C}$ (i.e. there exists topological discs U, V, U', V' with $I \subset U$ and $J \subset U'$ such that $f : U \to V$ and $g : U' \to V'$ are quadratic-like maps). Then $f : I \to I$ and $g : J \to J$ are real hybrid equivalent if and only if their quadratic-like extensions are hybrid equivalent.*

Proof: Since the critical point stays in I under iteration of f, respectively in J under iteration of g, the Julia set of the polynomial-like maps are connected and thus they are hybrid equivalent by the Straightening Theorem 6.2 for real polynomial-like maps, to real quadratic polynomials P_c and $P_{\tilde{c}}$ respectively by real quasiconformal hoeomorphisms. Thus these conjugacies maintain the kneading sequence and since the conjugacy is holomorphic on the interior of the filled Julia set it also maintains the multiplier at any attracting cycle. q.e.d.

We will also need the following theorem – an extension of the density of hyperbolicity to other families of S-unimodal rational maps.

We say that an S-unimodal map is *hyperbolic* if it has an attracting periodic cycle. Note that the restriction of a rational map to some interval might be hyperbolic as a S-unimodal map, even though it is not hyperbolic as a rational map on the Riemann sphere; e.g. $G_{-1}(z) = z + \frac{1}{z} - 1$ is not a hyperbolic map on $\hat{\mathbb{C}}$, because it has a parabolic fixed point at ∞, but it is a hyperbolic S-unimodal map on $[0, \infty]$.

Theorem 14.15 (Density of hyperbolicity in families of S-unimodal maps) *Let $I \subset \mathbb{R}$ be an interval and $\{f_a\}_{a \in I}$ a real analytic family of S-unimodal maps. If $\{f_a\}_{a \in I}$ is nontrivial in the sense that there exists at least two maps in this family that are not topologically conjugate, then the set of parameter values $a \in I$ for which f_a has an attracting periodic cycle and is thus hyperbolic (as a unimodal map) is open and dense.*

See [Ko, Theorem C].

Example. The family $\{G_T\}_{T \in [-2,0]}$ with $G_T(z) = z + 1/z + T$, $G_T : [0, \infty] \to [0, \infty]$ is a nontrivial analytic family of S-unimodal maps and thus the set of parameter values $T \in [-2, 0]$ for which G_T has an attracting periodic cycle and is thus hyperbolic (as a unimodal map) is open and dense.

Chapter 15

Real rational maps

In this chapter we consider conjugacy classes $\langle f \rangle \in \mathcal{M}_2$ having a representative that is a real rational map. We show that the dynamical homeomorphsim between the Mandelbrot set and the connectedness locus M^λ given by $c \mapsto \langle R_{\lambda,c} \rangle$ (compare Theorem 12.3) maps the real Mandelbrot set $[-2, 1/4]$ homeomorphically onto the real connectedness locus $M^\lambda_{\mathbb{R}}$. If $\lambda \in (-1, 1)$, $\lambda \neq 0$ and $c \in [-2, 1/4]$ then $R_{\lambda,c}$ is a real rational map.

Real rational maps. A *real rational map* is a rational map f with the property $f(\overline{z}) = \overline{f(z)}$ for all $z \in \hat{\mathbb{C}}$. It has the property that $f(\hat{\mathbb{R}}) \subset \hat{\mathbb{R}}$.

Lemma 15.1 *A rational map is real if and only if it has a representative with real coefficients.*

Proof: Choose the representative that has 1 as its leading coefficient on the numerator - this representative is unique. Since $f(z) = \overline{f(\overline{z})}$ it follows that all the coefficients are real. On the other hand a rational map f with real coefficients satisfies $f(\overline{z}) = \overline{f(z)}$. q.e.d.

Conjugacy classes of quadratic rational maps with real representatives. Now we consider conjugacy classes in $\mathcal{M}_2$ that have a real representative.

Lemma 15.2 *An element in $\mathcal{M}_2$ has a real representative if and only if either all three multipliers are real or one is real and the other two are complex conjugate.*

Proof: If f is a real rational map, then $f(z) = \overline{f(\overline{z})}$. Thus if z_0 is a fixed point of f then $\overline{z_0}$ is also a fixed point for f and so either all fixed points are real or one is real and two are complex conjugate. The same is true for their multipliers: If z_0 is a fixed point for f with multiplier μ, then $f'(\overline{z_0}) = \overline{f'(z_0)} = \overline{\mu}$.

Now consider $\langle \mu_1, \mu_2, \mu_3 \rangle$ with $\mu_1 \in \mathbb{R}$ and $\mu_2, \mu_3 \in \mathbb{R}$ or $\mu_2 = \overline{\mu_3}$. If all three multipliers are real then $z \mapsto z\frac{z+\mu_1}{\mu_2 z+1}$ is a real representative. Now let $\mu_2 = \overline{\mu_3}$. If $\mu_1 \neq 0$, then we can normalize $z \mapsto \frac{1}{\mu_1}(z + 1/z + T)$ with $T^2 = \frac{(\mu_2-\mu_3)^2}{1-\mu_2\mu_3} \in \mathbb{R}$. Since either $\mu_2, \mu_3 \in \mathbb{R}$ or $\mu_2 = \mu_3$ we have that either T is real or purely imaginary. So either the above map is real or in case that $T = i\widetilde{T}$ with $\widetilde{T} \in \mathbb{R}$ conjugation with $z \mapsto iz$ leads to a real map $\mapsto \frac{1}{\mu}(z - 1/z - \widetilde{T})$. If $\mu_1 = 0$ then $\mu_3 = 2 - \mu_2$ and thus, because $\mu_3 = \overline{\mu_2}$, $\mu_2 = 1 + ia$ for some $a \in \mathbb{R}$. In this case $P_c(z) = z^2 + c$ with $c = 1/4 + a^2$ is a real representative for this class. q.e.d.

Note that the real representation is unique up to conjugation with a real Möbius transformation except for the conjugacy classes of maps $z \mapsto \frac{1}{\mu}(z+1/z)$ with $\mu \in \mathbb{R}-\{0\}$. These maps $z \mapsto \frac{1}{\mu}(z+1/z)$ are conjugate to $z \mapsto \frac{1}{\mu}(z-1/z)$ by $z \mapsto iz$ [Mi2, p.40].

The complex conjugate map $\overline{f}$. Let f be a rational map. Its conjugate map, $\overline{f}$, is defined by $\overline{f}(z) = \overline{f(\overline{z})}$. If f is holomorphically conjugate to g, then $\overline{f}$ is holomorphically conjugate to $\overline{g}$, because if $f \circ h = h \circ g$, then $\overline{f} \circ \overline{h} = \overline{f \circ h} = \overline{h \circ g} = \overline{h} \circ \overline{g}$. Thus $f \mapsto \overline{f}$ extends to a map on $\mathcal{M}_2$, $\langle f \rangle \mapsto \langle \overline{f} \rangle$, and the fixed points of this map are the conjugacy classes that have a real representative:

Lemma 15.3 *The conjugacy class $\langle f \rangle \in \mathcal{M}_2$ has a real representative if and only if $\langle \overline{f} \rangle = \langle f \rangle$.*

Proof: If $\langle f \rangle = \langle \mu_1, \mu_2, \mu_3 \rangle$ has a real representative then by Lemma 15.2 either all multipliers are real or one is real and two are complex conjugate. Thus $\langle \overline{f} \rangle = \langle \overline{\mu_1}, \overline{\mu_2}, \overline{\mu_3} \rangle = \langle \mu_1, \mu_2, \mu_3 \rangle = \langle f \rangle$.

If on the other hand $\langle \overline{f} \rangle = \langle f \rangle$, then $\langle \mu_1, \mu_2, \mu_3 \rangle = \langle \overline{\mu_1}, \overline{\mu_2}, \overline{\mu_3} \rangle$, because f can be represented in the form $f(z) = z\frac{z+\mu_1}{\mu_2 z+1}$ and then $\overline{f}(z) = z\frac{z+\overline{\mu_1}}{\overline{\mu_2} z+1}$. Thus either all multipliers are real or one is real and two are complex conjugate, which implies that f has a real representative. q.e.d.

The real connectedness locus $M^\lambda_{\mathbb{R}}$. Recall that M^λ is the connectedness locus in the slice $\mathrm{Per}_1(\lambda)$ of all conjugacy classes having a fixed point with multiplier λ. We define

$$M^\lambda_{\mathbb{R}} = \{\langle f \rangle \in M^\lambda : \langle f \rangle \text{ has a real representative}\}.$$

Note that a conjugacy class of a quadratic polynomial $\langle P_c \rangle$ has a real representative if and only if $c \in \mathbb{R}$, because $\langle \overline{P_c} \rangle = \langle P_{\overline{c}} \rangle = \langle P_c \rangle$ if and only if $c = \overline{c}$. So $M^0_{\mathbb{R}}$ is naturally isomorphic to the real Mandelbrot set $M \cap \mathbb{R} = [-2, 1/4]$.

Theorem 15.4 (Dynamical homeomorphism between $[-2, 1/4]$ and $M^\lambda_{\mathbb{R}}$)
Let $\lambda \in (-1,1)$, $\lambda \neq 0$. Then the map

$$[-2, 1/4] \to M^\lambda_{\mathbb{R}} \subset \mathrm{Per}_1(\lambda) \text{ given by } c \mapsto \langle R_{\lambda,c} \rangle$$

is a homeomorphism. In particular the map $R_{\lambda,c}$ is a real rational map.

Proof: The map $\overline{R_{\lambda,c}}(z) = \frac{1}{\overline{\lambda}}(z + \frac{1}{z} + \overline{T_\lambda(c)})$ has a fixed point with multiplier $\overline{\lambda}$ and if ϕ is a quasi-conformal conjugacy between P_c and $R_{\lambda,c}$ on a neighborhood of $K(P_c)$, then $\overline{\phi}$ is a quasi-conformal conjugacy between $P_{\overline{c}}$ and $\overline{R_{\lambda,c}}$ on a neighborhood of $K(P_{\overline{c}})$. Thus we have $\overline{R_{\lambda,c}} = R_{\overline{\lambda},\overline{c}}$.

Now let $\lambda \in (-1,1)$, $\lambda \neq 0$. If $c \in [-2,1/4]$ then $\overline{R_{\lambda,c}} = R_{\overline{\lambda},\overline{c}} = R_{\lambda,c}$ and thus $\overline{T_\lambda(c)} = T_\lambda(c)$ and so $R_{\lambda,c}$ is a real rational map which implies that $\langle R_{\lambda,c} \rangle \in M^\lambda_{\mathbb{R}}$. If on the other hand $c \in M$ and $c \notin [-2,1/4]$ then $c \neq \overline{c}$. Then we have $\langle \overline{R_{\lambda,c}} \rangle = \langle R_{\lambda,\overline{c}} \rangle \neq \langle R_{\lambda,c} \rangle$. Thus $\langle R_{\lambda,c} \rangle \notin M^\lambda_{\mathbb{R}}$ by Lemma 15.3.

Since by Theorem 12.3 the map $M \to M^\lambda$ given by $c \mapsto \langle R_{\lambda,c}\rangle$ is a homeomorphism, it maps $[-2, 1/4]$ homeomorphically onto $M^\lambda_{\mathbb{R}}$.

q.e.d.

The real maps $R_{\lambda,c}$. Recall that for $\lambda \in D^*$ and $c \in M$, $R_{\lambda,c}(z) = \frac{1}{\lambda}\left(z + \frac{1}{z} + T_\lambda(c)\right)$ is the unique rational map in the family $F_{\lambda,T}(z) = \frac{1}{\lambda}\left(z + \frac{1}{z} + T\right)$ such that $1 \in K(R_{\lambda,c}, \infty)$ and $R_{\lambda,c}$ is hybrid equivalent to P_c on $K(R_{\lambda,c}, \infty)$. In the case where λ and c are real the condition $1 \in K(F_{\lambda,T})$ implies that $T_\lambda(c) < 0$. By the Straightening theorem for real quadratic-like maps, Theorem 6.2, the fixed points of $R_{\lambda,c}$ and their immediate preimages are also real. We have $\beta < 1/\alpha < \alpha < 1/\beta$ if $\lambda \in (-1, 0)$ and $1/\beta < \alpha < 1/\alpha < \beta$ if $\lambda \in (0, 1)$.

Note that the real representatives $R_{\lambda,c}$ are unique up to conjugation by a real Möbius transformations, except in the case where c is the value in the main cardioid for which P_c has an attracting fixed point with multiplier λ.

S-unimodal restrictions of quadratic rational maps. Let $[a, b] \subset \hat{\mathbb{R}}$ be an interval. We say that $f : [a, b] \to [a, b]$ is an *S-unimodal restriction* of the real rational map $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$, if $f : [a, b] \to [a, b]$ is an S-unimodal map.

Examples:

1. A quadratic polynomial $P_c(z) = z^2 + c$ with c in the real Mandelbrot set $[-2, 1/4]$ has exactly two S-unimodal restrictions.
   (a) $P_c : [-\beta, \beta] \to [-\beta, \beta]$ is an S-unimodal restriction.
   (b) $P_c : [\beta, -\beta] \to [\beta, -\beta]$ is an S-unimodal restriction which is real hybrid equivalent to $P_0(z) = z^2$ (Because $P_c : [\beta, -\beta] \to [\beta, -\beta]$ and $P_0 : [-1, 1] \to [-1, 1]$ both have a superattracting fixed point and thus the same kneading sequence and the same multiplier at their attractor).
2. If $c \in [-2, 1/4]$ and $\lambda \in (-1, 1)$, $\lambda \neq 0$ then $R_{\lambda,c}(z) = \frac{1}{\lambda}(z + \frac{1}{z} + T_\lambda(c)$ is a real rational map having exactly two S-unimodal restrictions. Let I_β be the finite interval with endpoints β and its preimage $1/\beta$. Then the attracting fixed point ∞ with multiplier λ is in the complementary interval I^c_β (the closure of the complement of I_β).
   (a) $R_{\lambda,c} : I_\beta \to I_\beta$ is an S-unimodal restriction which is real hybrid equivalent to P_c.
   (b) $R_{\lambda,c} : I^c_\beta \to I^c_\beta$ is an S-unimodal restriction which is hybrid equivalent to $z \mapsto \lambda z + z^2$.
3. The map $G_T(z) = z + \frac{1}{z} + T$ with $T \in [-2, 0]$ has exactly two S-unimodal restrictions.
   (a) $G_T : [\infty, 0] \to [\infty, 0]$ is an S-unimodal map which is real hybrid equivalent to $P_{1/4}$.
   (b) Also the restriction $G_T : [0, \infty] \to [0, \infty]$ is an S-unimodal map which is hybrid equivalent to some quadratic polynomial P_c, $c \in [-2, 1/4]$.

Chapter 16

Homeomorphism between the real Mandelbrot set and $M^1_{\mathbb{R}}$

In this chapter we prove the real version of Conjecture 13.1. We show that for c in the real Mandelbrot set $[-2, 1/4]$ the limit

$$R_{1,c} = \lim_{\substack{\lambda \to 1 \\ \lambda < 1}} R_{\lambda,c}$$

exists in Rat_2 and that the map $[-2, 1/4] \to M^1_{\mathbb{R}}$ given by $c \mapsto \langle R_{1,c} \rangle$ is a homeomorphism. Furthermore we show that a suitable S-unimodal restriction of $R_{1,c}$ is real hybrid equivalent to P_c.

Parametrization of $M^1_{\mathbb{R}}$. Recall that $\mathrm{Per}_1(1) \subset \mathcal{M}_2$, the slice of all conjugacy classes having a multiple fixed point with multiplier 1, is parameterized by the family of maps

$$G_T(z) = z + \frac{1}{z} + T$$

with $T \in \mathbb{C}$. The maps G_{T_1} and G_{T_2} are holomorphically conjugate if and only if $T_1 = \pm T_2$. The real connectedness locus $M^1_{\mathbb{R}} \subset \mathrm{Per}_1(1)$ is naturally isomorphic to $[-2, 0]$. Every $\langle f \rangle \in M^1_{\mathbb{R}}$ has a unique representative in the family $\{G_T(z) = z + \frac{1}{z} + T : T \in [-2, 0]\}$.

Lemma 16.1 *The conjugacy class $\langle G_T \rangle$ has a real representative and connected Julia set if and only if $T \in [-2, 2]$.*

Proof: If $\langle G_T \rangle$ has a real representative, then the multiplier μ_3 at the third fixed point has to be real by Lemma 15.2. This implies that $T^2 = 1 - \mu_3 \in \mathbb{R}$ and thus we have that $T \in \mathbb{R}$ or $iT \in \mathbb{R}$. If $iT \in \mathbb{R}$ and $T \neq 0$ or $T \in \mathbb{R} - [-2, 2]$, then $J(G_T)$ is not connected, see Lemma 8.4.

q.e.d.

Recall that for $\lambda \in (-1, 1)$, $\lambda \neq 0$ the map $R_{\lambda,c}(z) = \frac{1}{\lambda}(z + \frac{1}{z} + T_\lambda(c))$ is defined such that $T_\lambda(c) < 0$. Let

$$I^\lambda \subset (\infty, 0) = \{T < 0 : \ J(F_{\lambda,T}) \text{ is connected}\}$$

I^λ is naturally isomorphic to $M_{\mathbb{R}}^\lambda$ and $T_\lambda : [-2,1/4] \to I^\lambda$ defined by $c \mapsto T_\lambda(c)$ is a homeomorphism.

Theorem 16.2 (Homeomorphism between the real Mandelbrot set and $M_{\mathbb{R}}^1$) *For c in the real Mandelbrot set $[-2,1/4]$ the limit*

$$R_{1,c} = \lim_{\substack{\lambda\to 1\\ \lambda<1}} R_{\lambda,c}$$

exists in Rat_2. *The map $[-2,1/4] \to M_{\mathbb{R}}^1$ given by $c \mapsto \langle R_{1,c}\rangle$ is a homeomorphism. Furthermore $R_{1,c} : [0,\infty] \to [0,\infty]$ is real hybrid equivalent to P_c.*

We have the following immediate corollary.

Corollary 16.3 (Characterization of $R_{1,c}$.) *Let c be in the real Mandelbrot set $[-2,1/4]$. Then*

$$R_{1,c}(z) = z + \frac{1}{z} + T(c)$$

where $T(c)$ is the unique parameter in $[-2,0]$, such that $R_{1,c} : [0,\infty] \to [0,\infty]$ is real hybrid equivalent to P_c.

Combining this result with Theorem 14.8 leads to:

Corollary 16.4 *Every map in the family $\{G_T(z) = z + \frac{1}{z} + T : T \in [-2,0]\}$ is determined by the kneading sequence and the multiplier at the attractor, in case one exists, of the S-unimodal restriction $G_T : [0,\infty] \to [0,\infty]$.*

Proof of Theorem 16.2: We first show that the limit exists in the hyperbolic case and this defines homeomorphism between the real hyperbolic parameters in M and the hyperbolic maps in $M_{\mathbb{R}}^1$ (the maps that have an attracting periodic cycle besides the parabolic fixed point). Using the fact that hyperbolicity is dense we can show that the limit exists also in the non-hyperbolic case and that this defines a homeomorphism from $[-2,1/4]$ onto $M_{\mathbb{R}}^1$.

1. Claim: The limit $\lim_{\substack{\lambda\to 1\\ \lambda<1}} \langle R_{\lambda,c}\rangle$ exists for all hyperbolic parameters $c \in M$.

   Let $c \in M$ a standard hyperbolic parameter. Then there exists a standard $n \in \mathbb{N}$ and $\eta \in \mathbb{C}$ such that P_c has an attracting period n cycle with multiplier η, i.e. $\langle P_c\rangle \in \mathrm{Per}_n(\eta)$. Let $\lambda \approx 1$, $\lambda < 1$, then $\langle R_{\lambda,c}\rangle \in \mathrm{Per}_1(\lambda) \cap \mathrm{Per}_n(\eta)$. Since the ideal point $\langle 1,1,\infty\rangle$ is not a limit point of $\mathrm{Per}_n(\eta)$ the map $\langle R_{\lambda,c}\rangle$ must be close to a standard point in $\mathcal{M}_2$, a point in $\mathrm{Per}_1(1) \cap \mathrm{Per}_n(\eta)$, which is a standard finite set, see Theorem 7.2. By Theorem 12.3 the map $(0,1) \to \mathcal{M}_2$, $\lambda \mapsto \langle R_{\lambda,c}\rangle$ is continuous. Let $\epsilon \approx 0$. Since the set $\{\langle R_{\lambda,c}\rangle : \lambda \in (1-\epsilon,1)\}$ is connected, because the image of a connected set under a continuous map is connected, the whole set has to be close to one of the points in $\mathrm{Per}_1(1)\cap\mathrm{Per}_n(\eta)$. This means that there is only one accumulation point and thus the limit exists and is in $\mathrm{Per}_1(1)$.

   Thus the limit $T(c) = \lim_{\substack{\lambda\to 1\\ \lambda<1}} T_\lambda(c)$ exist, $\langle R_{1,c}\rangle = \langle G_{T(c)}\rangle$.

2. Claim: If $c \in [-2, 1/4]$ is a hyperbolic parameter, then $G_{T(c)} : [0, \infty] \to [0, \infty]$ is real hybrid equivalent to P_c.

   Let $c \in [-2, 1/4]$ is a standard hyperbolic parameter. Then P_c has a periodic cycle of period n and multiplier $|\eta| < 1$ with n and $\eta \in \mathbb{C}$ standard. The map $G_{T(c)}$ has two S-unimodal restrictions: $G_{T(c)} : [\infty, 0] \to [\infty, 0]$ is real hybrid equivalent to $P_{1/4}$. Now consider $G_{T(c)} : [0, \infty] \to [0, \infty]$. Let $\lambda \approx 1,\ \lambda < 1$. Since $G_{T(c)}(z) \approx R_{\lambda,c}(z) = \frac{1}{\lambda}(z + \frac{1}{z} + T_\lambda(c))$ for all $z \in \mathbb{C}$ and $R_{\lambda,c}$ has an attracting periodic cycle of period n and multiplier $|\eta| < 1$ in $I_\beta \subset (0, \infty)$, $G_{T(c)}$ has an attracting periodic cycle of period n and multiplier $|\eta| < 1$ in $(0, \infty)$ (note that the period is minimal, because $\eta \neq 1$). Furthermore $G_{T(c)} : [0, \infty] \to [0, \infty]$ has the same kneading sequence as $R_{\lambda,c} : I_\beta \to I_\beta$, because $G^m_{T(c)}(1) \approx R^m_{\lambda,c}(1)$ for all standard m, and $k(R_{\lambda,c}) = k(P_c)$ and thus standard. Theorem 14.13 implies that $G_{T(c)} : [0, \infty] \to [0, \infty]$ is real hybrid equivalent to $P_c : [-\beta, \beta] \to [-\beta, \beta]$.

3. Claim: The map $T(c) = \lim_{\substack{\lambda \to 1 \\ \lambda < 1}} T_\lambda(c)$ maps the hyperbolic parameters in $[-2, 1/4]$ homeomorphically onto the hyperbolic parameters in $[-2, 0]$, i.e. $T : [-2, 1/4]_{hyp} \to [-2, 0]_{hyp}$ is a homeomorphism.

   Monotonicity:

   The map T is monotone, because T_λ is monotone: $c_1 < c_2$ standard hyperbolic parameter in M then $T_\lambda(c_1) < T_\lambda(c_2)$ and thus $T(c_1) \approx T_\lambda(c_1) < T_\lambda(c_2) \approx T(c_2)$. Thus $T(c_1) \leq T(c_2)$.

   Injectivity:

   Since $G_{T(c)}$ is the mating of P_c and $P_{1/4}$, the map is injective. Because if $T(c_1) = T(c_2)$ then $G_{T(c_1)}$ is real hybrid equivalent to P_{c_1} and $G_{T(c_2)}$ is real hybrid equivalent to P_{c_2}, which implies that P_{c_1} is real hybrid equivalent to P_{c_2} and thus by Theorem 14.12 we have that $c_1 = c_2$.

   The hyperbolic components in the family $\{G_T\}_{T \in [-2,0]}$:

   Note that hyperbolicity is an open condition, i.e. if G_T with $T \in \mathbb{R}$ standard has an attracting period n cycle and $\tilde{T} \approx T$ then $G_{\tilde{T}}$ has an attracting periodic cycle by Rouché's theorem. (Note that the attracting periodic cycle has to be real, since the map and its critical points are real.)

   Let η be the map that associates to each hyperbolic S-unimodal map the multiplier at its attracting periodic cycle.

   Let $H = (a, b)$ be a standard real hyperbolic n component in the G_T family. Let $\lambda \approx 1$, $\lambda < 1$. Then for all standard $T \in H$ and all $\tilde{T} \approx T$ the map $F_{\lambda,\tilde{T}} = \frac{1}{\lambda}(z + \frac{1}{z} + \tilde{T})$ has an attracting periodic n cycle with $\eta(F_{\lambda,\tilde{T}}) \approx \eta(G_T)$, so they are all in one and the same hyperbolic component in I^λ. Since G_a and G_b are not hyperbolic the nearby maps have no attracting periodic cycle with multiplier $|\eta| \not\approx 1$. For a hyperbolic component $\tilde{H}$ in I^λ the map $\eta : \tilde{H} \to (-1, 1)$ is a homeomorphism, see Theorem 14.4, thus there must be $\tilde{a} \approx a$ and $\tilde{b} \approx b$ with $\eta(F_{\lambda,\tilde{a}}) \approx -1$ and $\eta(F_{\lambda,\tilde{b}}) \approx 1$.

Furthermore we have that the map $\eta : H \to (-1,1)$ is monotone. Let $T_1 < T_2$ in H, then F_{λ,T_1} and F_{λ,T_2} must be in the same hyperbolic component in M^λ, because otherwise there would be a $T_1 < T < T_2$, such that $F_{\lambda,T}$ has no attracting periodic cycle and thus $G_{\circ T}$ has no attracting periodic cycle, which would contradict the fact that T_1 and T_2 are in the same hyperbolic component. Thus we have $\eta(F_{\lambda,T_1}) < \eta(F_{\lambda,T_2})$ which implies that $\eta(G_{T_1}) \leq \eta(G_{T_2})$.

Surjectivity:

Now we prove surjectivity. Assume that $T : [-2,1/4]_{hyp} \to [-2,0]_{hyp}$ is not surjective. Because all hyperbolic real hybrid classes occur in the image $T([-2,1/4]_{hyp})$ this implies that there exist standard $T_1, T_2 \leq 0$ with $T_1 \neq T_2$ and G_{T_1} is real hybrid equivalent to G_{T_2}.

First assume that T_1 and T_2 are in the same hyperbolic component H. Because $\eta(G_{T_1}) = \eta(G_{T_2})$ and because η is monotone on H this would imply that $\eta(G_T)$ is constant on the interval $[T_1, T_2]$ thus $\langle G_T \rangle \in \mathrm{Per}_1(1) \cap \mathrm{Per}_n(\eta)$ for all $T \in [T_1, T_2]$ which contradicts the fact that $\mathrm{Per}_1(1) \cap \mathrm{Per}_n(\eta)$ is a finite set, see Corollary 7.3.

Now assume T_1 and T_2 are not in the same hyperbolic component. Then F_{λ,T_1} and F_{λ,T_2} are not in the same hyperbolic component in M^λ. Since no two hyperbolic components of the same period have the same kneading sequence we have $k(F_{\lambda,T_1}) \neq k(F_{\lambda,T_2})$. So we can conclude $k(G_{T_1}) = k(F_{\lambda,T_1}) \neq k(F_{\lambda,T_2}) = k(G_{T_2})$. So G_{T_1} and G_{T_2} are not real hybrid equivalent.

Continuity:

Hyperbolic components in $[-2,1/4]_{hyp}$ are mapped bijectively onto hyperbolic components in $[-2,0]_{hyp}$. Since a surjective monotone interval map is continuous the map $T : [-2,1/4]_{hyp} \to [-2,0]_{hyp}$ is continuous and so is its inverse, thus a homeomorphism.

4. Claim: The limit $\lim_{\substack{\lambda\to 1\\ \lambda<1}} \langle R_{\lambda,c} \rangle$ also exists in the non-hyperbolic case.

   Let $c \in (-2,1/4)$ a standard non-hyperbolic parameter.

   There exist standard $c_1 < c < c_2$ such that the limit exists for c_1 and c_2. Since T_λ is monotone this implies that $T_\lambda(c) \neq \infty$.

   Assume that the limit does not exist, i.e. there are $\lambda_1, \lambda_2 \approx 1$ with $T_{\lambda_1}(c) \not\approx T_{\lambda_2}(c)$. Let $T_1 = {}^\circ T_{\lambda_1}(c)$ and $T_2 = {}^\circ T_{\lambda_2}(c)$. Because hyperbolicity is dense in the real G_T family, see 14.15 there exists a $T_1 < T_0 < T_2$ with G_{T_0} having a superattracting periodic cycle. Because the limit exists in the hyperbolic case we have $T_0 \approx T_{\lambda_1}(c_0) \approx T_{\lambda_2}(c_0)$ for some c_0 in $M \cap \mathbb{R}$. If $c < c_0$ this implies because of monotonicity of T_λ that $T_2 \approx T_{\lambda_2}(c) < T_{\lambda_2}(c_0) \approx T_0$ and thus $T_2 < T_0$ which is a contradiction. If $c > c_0$ we would get that $T_0 < T_1$ which would also be a contradiction.

   This implies that the limit exists.

   Furthermore we have $\lim_{\substack{\lambda\to 1\\ \lambda<1}} T_\lambda(-2) = -2$ and $\lim_{\substack{\lambda\to 1\\ \lambda<1}} T_\lambda(1/4) = 0$.

5. Claim: The map $T : [-2, 1/4] \to [-2, 0]$ is a homeomorphism and $G_{T(c)} : [0, \infty] \to [0, \infty]$ is real hybrid equivalent to P_c.

   The map T is monotone, because $T_\lambda : [-2, 1/4] \to I^\lambda$ is monotone.

   We already proved everything for the hyperbolic parameters. Now let $c \in M \cap \mathbb{R}$ be a non-hyperbolic parameter. Let $\lambda \approx 1$ with $\lambda < 1$. Then $R_{\lambda,c}$ has no attracting periodic cycle in the finite plane and thus $G_T(c)$ cannot have an attracting periodic cycle. Furthermore we have that $k(P_c) = k(R_{\lambda,c})$ and $R_{\lambda,c}(z) \approx G_{T(c)}$ for all z. Furthermore we have $R^n_{\lambda,c}(1) \not\approx 1$ for all standard $n \in \mathbb{N}$, because otherwise the limiting map $G_{T(c)}$would have a superattracting periodic cycle and by Rouché's theorem $R_{\lambda,c}$ would have an attracting periodic cycle with multiplier close to 0 — this would contradict the assumption that P_c and thus $R_{\lambda,c}$ is not hyperbolic. This implies that $k(R_{\lambda,c}) = k(G_{T(c)})$ and thus $k(G_{T(c)}) = k(P_c)$, which means that they are real hybrid equivalent. Thus $G_{T(c)}$ is the mating of P_c and $P_{1/4}$. This implies injectivity.

   Now we show surjectivity. Let $T \in M^1 \cap \mathbb{R}$ standard. Since hyperbolicity is dense there exist a hyperbolic parameter $T_1 \approx T$. Let $c_1 \in M \cap \mathbb{R}$ with $T_1 = T(c_1)$. We have that $T(^\circ c_1) = T$, because otherwise there would be a standard hyperbolic parameter T_2 with $T_1 \approx T < T_2 < T_1 = T(c_1)$ and $T_2 = T(c_2)$ for some standard $[-2, 1/4]_{hyp}$. Since $T : [-2, 1/4]_{hyp} \to [-2, 0]_{hyp}$ is a homeomorphism this would imply that $c_1 < c_2 < {^\circ c_1}$ which is impossible, because there is no other standard point close to $^\circ c_1$.

   The map $T : [-2, 1/4] \to [-2, 0]$ is continuous, because it is monotone and surjective and so is its inverse. Thus $T : [-2, 1/4] \to [-2, 0]$ is a homeomorphism.

   This implies that the map $[-2, 1/4] \to M^1_{\mathbb{R}}$, $c \mapsto \langle R_{1,c} \rangle = \langle G_{T(c)} \rangle$ is a homeomorphism.

q.e.d.

Chapter 17

Real matings of real quadratic polynomials

In this chapter we introduce the notion of the real mating of two real quadratic polynomials. We will show:

For $c_1, c_2 \in [-2, 1/4]$ *the mating of* $P_{c_1}(z) = z^2 + c_1$ *and* $P_{c_2}(z) = z^2 + c_2$ *exists and is unique up to conjugation with real Möbius transformations if and only if* c_1 *and* c_2 *do not both belong to the real* $1/2$*-limb* $[-2, -3/4]$*.*

In particular we show that the maps $R_{\lambda,c}$, introduced in Chapter 12, can be interpreted, in case that $\lambda \in (-1, 1)$, $\lambda \neq 0$ and $c \in [-2, 1/4]$, as the real mating of P_c and $P_{c(\lambda)}$, where $c(\lambda)$ is the unique parameter for which $P_{c(\lambda)}$ has an attracting fixed point with multiplier λ.

Recall that a real quadratic polynomial $P_c(z) = z^2 + c$, $c \in [-2, 1/4]$, has two S-unimodal restrictions, to the interval $[-\beta, \beta]$ and to the interval $[\beta, -\beta]$. Convention: If we consider in the following a real quadratic polynomial P_c as an S-unimodal map, we always mean the restriction to the finite interval $[-\beta, \beta]$.

Matings of real quadratic polynomials. A conjugacy class $\langle f \rangle \in \mathcal{M}_2$ with a real representative is the *mating of two real quadratic polynomials* P_{c_1} and P_{c_2} if f can be normalized such that there exist points $a, b \in \hat{\mathbb{R}}$, with $a \neq b$, such that $f : [a, b] \to [a, b]$ is an S-unimodal map that is real hybrid equivalent to P_{c_1} and $f : [b, a] \to [b, a]$ is an S-unimodal map that is real hybrid equivalent to P_{c_2}. In that case we call $\langle f \rangle$ the *real mating* of P_{c_1} and P_{c_2}.

The following theorem shows that the mating of two real quadratic polynomials P_{c_1} and P_{c_2} exists if and only if c_1 and c_2 are not both in the real 1/2-limb. For a general discussion on matings of complex quadratic polynomials see [Tan].

Theorem 17.1 (Existence and uniqueness of real matings)
For any c_1 *and* c_2 *in the real Mandelbrot set* $[-2, 1/4]$ *there exists a real quadratic rational map that is the mating of* P_{c_1} *and* P_{c_2} *if and only if* c_1 *and* c_2 *are not both in the real* $1/2$*-limb* $[-2, -3/4]$*. Furthermore this quadratic rational map, if it exists, is unique up to conjugation by real Möbius transformations.*

The maps that can be obtained by real mating are therefore

$$\{\langle f\rangle \in \mathcal{M}_2 : \ \exists\, c \in [-2, 3/4] \text{ such that } f = P_c \text{ or } f = R_{\lambda,c} \text{ for some } \lambda \in (-1, 1] - \{0\}\}.$$

Proof:

1. Claim: Let $c_1 \in (-3/4, 1/4)$ and $c_2 \in [-2, 1/4]$. Then the mating of P_{c_1} and P_{c_2} exists and is unique.

   The quadratic polynomial P_{c_1} has an attracting fixed point with a real multiplier λ. If $\lambda \neq 0$ then the real quadratic rational map $R_{\lambda,c_2}(z) = \frac{1}{\lambda}(z + \frac{1}{z} + T_\lambda(c_2))$ is the mating of P_{c_1} and P_{c_2}. Let I_β be the finite interval with endpoints β and its preimage $1/\beta$ and I^c_β the complementary interval. Then $R_{\lambda,c} : I_\beta \to I_\beta$ is an S-unimodal restriction which is real hybrid equivalent to P_{c_2} and $R_{\lambda,c} : I^c_\beta \to I^c_\beta$ is an S-unimodal restriction which is real hybrid equivalent to P_{c_1}. R_{λ,c_2} is the only map in $\mathrm{Per}_1(\lambda)$ having that property.

   Note that if f is a real rational maps that is the real mating of two quadratic polynomials, then both critical points and all three fixed points are real. This property is not maintained under conjugation with a non-real Möbius transformation.

2. Claim: Let $c \in [-2, 1/4]$. Then the mating of P_c and $P_{1/4}$ exists and is unique.

   We have shown in Theorem 16.2 that there exists a up to holomorphic conjugation unique real quadratic rational map $G_T(z) = z + \frac{1}{z} + T$ with $T \in [-2, 0]$, whose S-unimodal restriction to $[0, \infty]$ is real hybrid equivalent to P_c. Furthermore $G_T : [\infty, 0] \to [\infty, 0]$ is real hybrid equivalent to $P_{1/4}$. Thus the map G_T is the mating of $P_{1/4}$ and P_c.

3. Claim: Let c_1, c_2 both be in the real 1/2-limb $[-2, -3/4]$. Then there exists no quadratic rational map the is the mating of P_{c_1} and P_{c_2}.

   The α-fixed points of P_{c_1} and P_{c_2} are repelling with a negative multiplier, i.e. $\alpha_1, \alpha_2 < -1$. Assume the mating exists. Since the α-fixed points are in the interior of the S-unimodal restrictions, the mating has two repelling fixed points with multipliers $\tilde\alpha_1, \tilde\alpha_2 < -1$. This implies that the multiplier at the third fixed point $\frac{2-\tilde\alpha_1-\tilde\alpha_2}{1-\tilde\alpha_1\tilde\alpha_2}$ is negative. This contradicts the fact that the map on its S-unimodal restriction is conjugate to P_{c_1}, because for $c_1 < 0$ the multiplier at the β-fixed point is positive.

q.e.d.

Matings of two real quadratic polynomials with non-repelling fixed points. Let $c_1 \in [-3/4, 1/4]$ and $c_2 \in (-3/4, 1/4]$. Then P_{c_1} has a fixed point with multiplier $\lambda_1 \in [-1, 1]$ and P_{c_2} has a fixed point with multiplier $\lambda_2 \in (-1, 1]$. Since real hybrid equivalence maintains the multiplier at any attractor, the mating has two have two fixed points with multipliers λ_1 and λ_2. Thus the mating of P_{c_1} and P_{c_2} equals $\langle \lambda_1, \lambda_2, \frac{2-\lambda_1-\lambda_2}{1-\lambda_1\lambda_2}\rangle$.

Let $P_{c(\lambda)}(z) = z^2 + c(\lambda)$ denote the unique quadratic polynomial that has a fixed point with multiplier λ.

Corollary 17.2 (Characterization of $\langle R_{\lambda,c}\rangle$ in the real case) *Let c be in the real Mandelbrot set $[-2, 1/4]$ and $\lambda \in (-1, 1)$, $\lambda \neq 0$. Then $R_{\lambda,c}$ is the is the mating of P_c and $P_{c(\lambda)}$.*

We conclude this chapter by discussing the notion of renormalization for real quadratic polynomials.

Theorem 17.3 (Renormalization of real quadratic polynomials) *Let $c \in [-2, -3/4]$.*

1. *The map P_c^2 is renormalizable if and only if $P_c^2(0) \in [\alpha, -\alpha]$. This is the case if and only if $c \in [-1.54368..., -3/4]$.*
2. *Let P_c^2 be renormalizable. Then $P_c^2 : [\alpha, -\alpha] \to [\alpha, -\alpha]$ is an S-unimodal map and P_c^2 is hybrid equivalent to P_b if and only if $P_c^2 : [\alpha, -\alpha] \to [\alpha, -\alpha]$ is real hybrid equivalent to P_b.*

Proof:

1. If $P_c^2(0) \notin [\alpha, -\alpha]$ then every topological disc containing $P_c^2(0)$ and the critical point 0 contains points who have more than two preimages under P_c^2. Thus there cannot be a quadratic-like restriction containing $P_c^2(0)$, which means that P_c^2 is not renormalizable.
2. $P_c^2 : [\alpha, -\alpha] \to [\alpha, -\alpha]$ is an S-unimodal map, because iterates of maps with negative Schwarzian derivative have negative Schwarzian derivative and α is non-attracting. By Theorem 14.14 we have that $P_c^2 : [\alpha, -\alpha] \to [\alpha, -\alpha]$ is real hybrid equivalent to P_b if and only P_c^2 is hybrid equivalent to P_b.

q.e.d.

The following theorem is an immediate consequence of the fact that $R_{\lambda,c}$ is the mating of P_c and $P_{c(\lambda)}$.

Theorem 17.4 (Renormalization of $R^2_{\lambda,c}$) *Let $c \in [-2, -3/4]$ and $\lambda \in (-1, 1)$, $\lambda \neq 0$. Let P_c^2 be renormalizable hybrid equivalent to P_b. Then the S-unimodal restriction of $R_{\lambda,c}$ that is real hybrid equivalent to P_c can be further restricted to an interval I_α (the interval whose endpoints are the α fixed point of $R_{\lambda,c}$ and its preimage $1/\alpha$), such that $R^2_{\lambda,c} : I_\alpha \to I_\alpha$ is an S-unimodal map real hybrid equivalent to P_b.*

Chapter 18

Proof of the conjecture in the real case

In this chapter we prove the real version of Conjecture 13.2:

Theorem 18.1 (The real case) *For any c in the real $1/2$-limb $[-2,-3/4]$ the limit*

$$L_{1/2}(c) = \lim_{\substack{\lambda\to -1\\ \lambda>-1}} \langle R_{\lambda,c}\rangle$$

exists in the dynamical compactification $\widetilde{\mathcal{M}}_2$ and lies in $\mathcal{B}_{1/2}$. The map

$$L_{1/2} : [-2,-3/4] \to \mathcal{B}_{1/2} \subset \widetilde{\mathcal{M}}_2$$

has the following properties:

1. *$L_{1/2}$ is continuous.*
2. *$L_{1/2}$ maps the image of tuning $\tau_{1/2}([-2,1/4]) = [-1.54368..., -3/4]$ homeomorphically onto $\{\langle G_T\rangle_{1/2} :\ T \in [-2,0]\} \subset \mathcal{B}_{1/2} \subset \widetilde{\mathcal{M}}_2$:*

$$L_{1/2}(\tau_{1/2}(b)) = \langle R_{1,b}\rangle_{1/2} \in \mathcal{B}_{1/2}$$

   for any b in the real Mandelbrot set $[-2,1/4]$.

3. *For all $c \notin \tau_{1/2}([-2,1/4])$:*

$$L_{1/2}(c) = \langle R_{1,-2}\rangle_{1/2} = \langle G_{-2}\rangle_{1/2} \in \mathcal{B}_{1/2}.$$

Let $\lambda \in (-1,0)$ and $c \in [-2,-3/4]$. Recall that $R_{\lambda,c}$ is of the form

$$R_{\lambda,c}(z) = \frac{1}{\lambda}\left(z + \frac{1}{z} + T_\lambda(c)\right)$$

with $T_\lambda(c) < 0$. The map $R_{\lambda,c}$ has an attracting fixed point γ at infinity with preimage $\widetilde{\gamma} = 0$ and critical points at ± 1. The critical point -1 is always in the basin of ∞. As before we denote by α and β the other two fixed points of $R_{\lambda,c}$, corresponding to the α- and β-fixed point of P_c, and by $\widetilde{\beta} = 1/\beta$ and $\widetilde{\alpha} = 1/\alpha$ their preimages. We have $[\widetilde{\alpha}, \alpha] \subset [\beta, \widetilde{\beta}] \subset [0, \infty]$. Note that α and β depend on c and λ. The map $R_{\lambda,c}$ is the real mating of the quadratic polynomials P_c and $P_{c(\lambda)}$, see Corollary 17.2. In particular, the restriction $R_{\lambda,c} : [\beta, \widetilde{\beta}] \to [\beta, \widetilde{\beta}]$ is an S-unimodal map that is real hybrid equivalent to P_c.

We need the following lemma.

Lemma 18.2 *Let c be in the real 1/2-limb $[-2, -3/4]$ and $\lambda \in (-1, 0)$. If $\lambda \approx -1$, then the map $R_{\lambda,c}(z) = \frac{1}{\lambda}\left(z + \frac{1}{z} + T_\lambda(c)\right)$ has the following properties:*

1. *$T_\lambda(c) \approx \infty$, $0 \approx \beta \approx \widetilde{\alpha}$ and $\alpha \approx \widetilde{\beta} \approx \infty$.*
2. *Let $T = {}^\circ(T_\lambda(c)(1 + \lambda))$. Then we have*

$$R^2_{\lambda,c}(z) \approx z + \frac{1}{z} + T = G_T(z)$$

for all $z \in [\widetilde{\alpha}, \alpha]$.

Proof:

1. By Theorem 12.4 the multipliers at the fixed points of $R_{\lambda,c}$ tend to $-1, -1$ and ∞ as λ tends to -1. Let μ_α denote the multiplier at the α-fixed point and μ_β denote the multiplier at the β-fixed point of $R_{\lambda,c}$. If $\lambda \approx -1$, then we have $\mu_\alpha \approx -1$ and $\mu_\beta \approx \infty$ and thus
$$(T_\lambda(c))^2 = \frac{(\mu_\alpha - \mu_\beta)^2}{1 - \mu_\alpha \mu_\beta} \approx \infty.$$
This implies that $\alpha = \frac{T_\lambda(c)}{2(\lambda-1)} + \sqrt{\frac{T_\lambda(c)^2}{4(\lambda-1)^2} - \frac{1}{1-\lambda}} \approx \infty$ and $\beta = \frac{T_\lambda(c)}{2(\lambda-1)} - \sqrt{\frac{T_\lambda(c)^2}{4(\lambda-1)^2} - \frac{1}{1-\lambda}} = \frac{\beta\alpha}{\alpha} = \frac{1}{(1-\lambda)\alpha} \approx 0$. Thus $\widetilde{\alpha} = 1/\alpha \approx 0$ and $\widetilde{\beta} = 1/\beta \approx \infty$.

2. Since $T_\lambda(c) \approx \infty$, $\lambda^2 \approx 1$ and $T_\lambda(c)(1 + \lambda) \approx T$, we have that
$$R^2_{\lambda,c}(z) = \frac{1}{\lambda^2}\left(z + \frac{1}{z} + T_\lambda(c)(1 + \lambda)\right) + \frac{1}{z + \frac{1}{z} + T_\lambda(c)} \approx z + \frac{1}{z} + T = G_T(z)$$
for all $z \not\approx 0, \infty$.

   Note that since $T_\lambda(c) < 0$ and $(1 + \lambda) > 0$ we have that $T = {}^\circ(T_\lambda(c)(1 + \lambda)) < 0$.

   Now conjugate $R_{\lambda,c}$ by
$$h(z) = \frac{1 - \alpha}{1 - \widetilde{\alpha}} \frac{z - \widetilde{\alpha}}{z - \alpha}.$$
Since h maps 1 to 1, $\alpha \approx \infty$ to ∞, and $\tilde{\alpha} \approx 0$ to 0, the conjugacy h is close to the identity, i.e. $h(z) \approx z$ for all $z \in \hat{\mathbb{C}}$. Let $\widetilde{R}_{\lambda,c} = h \circ R_{\lambda,c} \circ h^{-1}$. Since the resulting map $\widetilde{R}_{\lambda,c}$ has a fixed point with multiplier μ_α at ∞ and critical points at ± 1 it must be of the form
$$\widetilde{R}_{\lambda,c}(z) = \frac{1}{\mu_\alpha}\left(z + \frac{1}{z} + \widetilde{T}_c\right)$$

with $\widetilde{T}_c^2 = \frac{(\lambda-\mu_\beta)^2}{1-\lambda\mu_\beta} \approx \infty$. Then we have

$$\widetilde{R}^2_{\lambda,c}(z) = \frac{1}{\mu_\alpha^2}\left(z + \frac{1}{z} + \widetilde{T}_c(1+\mu_\alpha)\right) + \frac{1}{z+\frac{1}{z}+\widetilde{T}_c}$$

Since the conjugation is close to the identity and $R^2_{\lambda,c}(z) \approx G_T(z)$ for all $z \not\approx 0, \infty$, we also have

$$\widetilde{R}^2_{\lambda,c}(z) \approx G_T(z)$$

for all $z \not\approx 0, \infty$. This implies that $\widetilde{T}_c(1+\mu_\alpha) \approx T$. Since $(1+\mu_\alpha) < 0$, because α is repelling and thus $\mu_\alpha < -1$, we have that $\widetilde{T}_c > 0$. This implies that $\frac{1}{z+\frac{1}{z}+\widetilde{T}_c} \approx 0$ for all $z \geq 0$ and thus

$$\widetilde{R}^2_{\lambda,c}(z) = \underbrace{\frac{1}{\mu_\alpha^2}}_{\approx 1}\left(z + \frac{1}{z} + \underbrace{\widetilde{T}_c(1+\mu_\alpha)}_{\approx T}\right) + \underbrace{\frac{1}{z+\frac{1}{z}+\widetilde{T}_c}}_{\approx 0} \approx z + \frac{1}{z} + T = G_T(z)$$

for all $z \geq 0$. Thus we have

$$\widetilde{R}^2_{\lambda,c}(h(z)) \approx G_T(h(z))$$

for all $z \in h^{-1}([0,\infty])$. Since the conjugation is close to the identity this implies

$$R^2_{\lambda,c}(z) = h^{-1}(\widetilde{R}^2_{\lambda,c}(h(z))) \approx h^{-1}(G_T(h(z))) \approx G_T(z)$$

for all $z \in h^{-1}([0,\infty]) = [\widetilde{\alpha}, \alpha]$. And thus $R^2_{\lambda,c}(z) \approx G_T(z)$ for all $z \in [\widetilde{\alpha}, \alpha]$.

q.e.d.

We now determine the value of the limit $\lim_{\substack{\lambda\to-1 \\ \lambda>-1}} \langle R_{\lambda,c}\rangle$ in the dynamical compactification $\widetilde{\mathcal{M}}_2$ for all c in the real 1/2-limb $[-2, -3/4]$.

Theorem 18.3 (Limit in the dynamical compactification)
Let c be in the real 1/2-limb $[-2, -3/4]$.

1. *If $c \in \tau_{1/2}([-2, 1/4])$, i.e. $c = \tau_{1/2}(b)$ for some $b \in [-2, 1/4]$, then*

$$\lim_{\substack{\lambda\to-1 \\ \lambda>-1}} \langle R_{\lambda,c}\rangle = \langle R_{1,b}\rangle_{1/2} \in \mathcal{B}_{1/2}.$$

2. *If $c \notin \tau_{1/2}([-2, 1/4])$, then*

$$\lim_{\substack{\lambda\to-1 \\ \lambda>-1}} \langle R_{\lambda,c}\rangle = \langle R_{1,-2}\rangle_{1/2} \in \mathcal{B}_{1/2}.$$

Recall that $R_{1,b}$ is the unique map in the family $\{G_T(z) = z + \frac{1}{z} + T : T \in [-2, 0]\}$ with the property that $R_{1,b} : [0, \infty] \to [0, \infty]$ is real hybrid equivalent to P_b. See Corollary 16.3.

Proof: By Theorem 12.4 the multipliers at the fixed points of $R_{\lambda,c}$ tend to $-1, -1$ and ∞. To determine the limit in the dynamical compactification, we also have to determine the limiting map of the second iterate $R^2_{\lambda,c}$ which will give us the limit $\langle G_T \rangle_{1/2} \in \mathcal{B}_{1/2}$ in the dynamical compactification $\widetilde{\mathcal{M}}_2$. Recall that if c is standard, we have $\langle R_{\lambda,c} \rangle \to \langle G_T \rangle_{1/2} \in \mathcal{B}_{1/2}$ as λ tends to -1, if and only if for all $\lambda \approx -1,\ \lambda > -1$, the multipliers at fixed points of $R_{\lambda,c}$ are infinitely close to $-1, -1$ and ∞ and $R^2_{\lambda,c}(z) \approx G_T(z)$ for all $z \not\approx 0, \infty$.

Note that $R_{\lambda,c}$ is already suitably normalized to determine the quadratic limiting map of the second iterate.

Applying the axiom of transfer we can assume that $c \in [-2, -3/4]$ is standard. Then P_c is standard and thus if P_c has an non-repelling periodic cycle its period and multiplier are standard. Recall that $R_{\lambda,c} : [\beta, \widetilde{\beta}] \to [\beta, \widetilde{\beta}]$ is an S-unimodal map that is real hybrid equivalent to P_c. This means that $R_{\lambda,c} : [\beta, \widetilde{\beta}] \to [\beta, \widetilde{\beta}]$ and P_c are topologically conjugate and have the same multiplier (and period) at any attractor, in case one exists.

Let $\lambda \approx -1, \lambda > -1$. Define $T = {}^\circ(T_\lambda(c)(1 + \lambda))$. By Lemma 18.2 the second iterate of the map

$$R_{\lambda,c}(z) = \frac{1}{\lambda}\left(z + \frac{1}{z} + T_\lambda(c)\right)$$

is close to G_T on $[\widetilde{\alpha}, \alpha]$:

$$R^2_{\lambda,c}(z) \approx z + \frac{1}{z} + T = G_T(z) \;\text{ for all } z \in [\widetilde{\alpha}, \alpha].$$

We denote by α_0 and β_0 the α- and β-fixed point of P_c.

We first assume that c not in the image of tuning, $c \notin \tau_{1/2}([-2, 1/4])$. Then the map P^2_c is not renormalizable. By Theorem 17.3 the critical point 0 does not stay in the interval $[\alpha_0, -\alpha_0]$ under iteration of P^2_c. Since $R_{\lambda,c} : [\beta, \widetilde{\beta}] \to [\beta, \widetilde{\beta}]$ is topologically conjugate to P_c the critical point 1 does not stay in the interval $[\widetilde{\alpha}, \alpha]$ under $R^2_{\lambda,c}$. Thus we have $R^2_{\lambda,c}(1) \in [\beta, \widetilde{\alpha}]$. By Lemma 18.2 we have that $\tilde{\alpha} \approx \beta \approx 0$ and $R^2_{\lambda,c}(1) \approx G_T(1)$ which implies that $G_T(1) \approx R^2_{\lambda,c}(1) \approx 0$. Since T is standard we have that $G_T(1) = 0$, which leads to $T = -2$.

Now let c be in the image of tuning, i.e. $c = \tau_{1/2}(b)$ for some $b \in [-2, 1/4]$. Then P^2_c is renormalizable. By Theorem 17.4 $R^2_{\lambda,c} : [\widetilde{\alpha}, \alpha] \to [\widetilde{\alpha}, \alpha]$ is an S-unimodal map that is real hybrid equivalent to P_b. This implies that $R^2_{\lambda,c} : [\widetilde{\alpha}, \alpha] \to [\widetilde{\alpha}, \alpha]$ has the same kneading sequence, $k(R^2_{\lambda,c}|_{[\widetilde{\alpha},\alpha]}) = k(P_b)$, and the same multiplier at its attractor, in case one exists, as the quadratic polynomial P_b.

Since $R^2_{\lambda,c}(z) \approx G_T(z)$ for all $z \in [\widetilde{\alpha}, \alpha]$ with $T = {}^\circ(T_\lambda(c)(1 + \lambda))$, we have $(R^2_{\lambda,c})^m(1) \approx G^m_T(1)$ for all standard $m \in \mathbb{N}$.

We distinguish several cases.

The hyperbolic centers: Let P_c have a superattracting cycle of period n. Note that n is even, because P_c^2 is renormalizable. Since P_c has a superattracting periodic n cycle $R_{\lambda,c}$ has a superattracting periodic n cycle and thus we have $R_{\lambda,c}^n(1) = 1$. We can conclude that

$$1 = R_{\lambda,c}^n(1) = (R_{\lambda,c}^2)^{n/2}(1) \approx G_T^{n/2}(1).$$

By transfer we have that $G_T^{n/2}(1) = 1$. So G_T has a superattracting cycle. And $n/2$ is the minimal period: Assume there exist $m < n/2$ with $G_T^m(1) = 1$. By Rouché's Theorem $R_{\lambda,c}^2$ would have an attracting period m cycle with multiplier close to 0. Then $R_{\lambda,c}$ would have three attracting cycles which is impossible. So $n/2$ must be the minimal period and $R_{\lambda,c}^m(1) \not\approx 1$ for all $1 \leq m < n/2$.

For the kneading sequence we can conclude:

$$k(G_T|_{[0,\infty]}) = k(R_{\lambda,c}^2|_{[\widetilde{\alpha},\alpha]}) = k(P_b).$$

So $G_T : [0,\infty] \to [0,\infty]$ is real hybrid equivalent to P_b, by Theorem 14.13. Thus $G_T = R_{1,b}$.

Now we show that G_T always inherits the kneading sequence of P_b. We already proved that for the hyperbolic centers. Now assume that P_c has no superattracting periodic cycle. Then we have that $(R_{\lambda,c}^2)^m(1) \not\approx 1$ for all standard $m \geq 1$. (Otherwise G_T would have a superattracting periodic cycle and thus by Rouché's theorem $R_{\lambda,c}^2$ would have an attracting periodic cycle with multiplier close to 0 and since P_c and $R_{\lambda,c}$ are real hybrid equivalent and c is standard P_c would have a superattracting periodic cycle.) Because $(R_{\lambda,c}^2)^n(1) \approx G_T^n(1)$ we can conclude that the kneading sequence of G_T and $R_{\lambda,c}^2$ coincide for all standard indices. Since $k(R_{\lambda,c}^2|_{[\widetilde{\alpha},\alpha]}) = k(P_b)$, and G_T and P_b are standard this implies that

$$k(G_T|_{[0,\infty]}) = k(P_b).$$

Only repelling cycles: In case that there are only repelling cycles the hybrid class is determined by its kneading sequence. Since we have $k(G_T|_{[0,\infty]}) = k(P_b)$, we can conclude that $G_T : [0,\infty] \to [0,\infty]$ is real hybrid equivalent to P_b, if P_c and thus P_b have only non-repelling cycles.

The general hyperbolic and parabolic case: Now assume that P_c has an attracting or parabolic period n cycle with multiplier η. Then $R_{\lambda,c}^2$ has an attracting or parabolic periodic $n/2$-cycle $\{z_1, \cdots, z_{n/2}\}$ with multiplier η in the interval $[\widetilde{\alpha},\alpha]$. Since $R_{\lambda,c}^2(z) \approx G_T(z)$ for all $z \in [\widetilde{\alpha},\alpha]$ and $G_T(0) = G_T(\infty) = \infty$ either the whole periodic cycle is infinitely close to ∞ or it stays away from 0 and ∞. Assume that the whole periodic cycle is infinitely to ∞. Because $R_{\lambda,c}^2$ is monotone increasing on $[1,\alpha]$ it would have to be a fixed point for $R_{\lambda,c}^2$. So assume that $z_1 \approx \infty$, $z_1 \in [1,\alpha]$ is a fixed point for $R_{\lambda,c}^2$. Then we have

$$(R_{\lambda,c}^2)'(z_1) = R_{\lambda,c}'(R_{\lambda,c}(z_1))\, R_{\lambda,c}'(z_1) = \left(1 - \frac{1}{R_{\lambda,c}(z_1)}\right)\left(1 - \frac{1}{z_1}\right) \approx 1$$

because $\frac{1}{R_{\lambda,c}(z)} \approx 0$ for all $z \in [1/\alpha, \alpha]$, compare Lemma 18.2. Since the multiplier at the attracting cycle is standard, we have $(R^2_{\lambda,c})'(z_1) = 1$. Since $P_{1/4}$ is the only quadratic polynomial having a fixed point with multiplier 1, we have $b = 1/4$. Thus we have $k(G_T|_{[0,\infty]}) = k(P_{1/4}) = (\overline{1})$. Since T is standard and has thus no attracting fixed points close to ∞, this implies that $T = 0$. And $G_0 = R_{1,1/4}$.

Now assume that $c \neq -3/4$, then the attracting or parabolic periodic cycle for $R^2_{\lambda,c}$ in $[1/\alpha, \alpha]$ has no points infinitely close to 0 or ∞. Then $\{z_1, \cdots, z_{n/2}\}$ is contained in an open standard subset of $[\widetilde{\alpha}, \alpha]$ which implies that $\{{}^\circ z_1, \cdots, {}^\circ z_{n/2}\}$ is a standard subset of $[1/\alpha, \alpha]$. Since $G_T(z) \approx R_{\lambda,c}(z)$, Lemma 4.2 implies that $\{{}^\circ z_1, \cdots, {}^\circ z_{n/2}\}$ is a periodic cycle for G_T with multiplier η. Thus $G_T : [0,\infty] \to [0,\infty]$ is real hybrid equivalent to P_b, which implies that $G_T = R_{1,b}$.

q.e.d.

Proof of Theorem 18.1:
By Theorem 18.3 we know that the limit $L_{1/2}(c) = \lim_{\substack{\lambda\to-1\\ \lambda>-1}} \langle R_{\lambda,c}\rangle$ exists and has the required values. That $L_{1/2}$ maps the image of tuning $\tau_{1/2}([-2, 1/4]) = [-1.54368..., -3/4]$ homeomorphically onto $M^1_{\mathbb{R}} \subset \mathcal{B}_{1/2} \subset \widetilde{\mathcal{M}}_2$ follows from Theorem 16.2. By Theorem 18.3 we know that the value of the limit $L_{1/2}(c)$ equals $\langle R_{1,-2}\rangle_{1/2} \in \mathcal{B}_{1/2}$ for all $c \notin \tau_{1/2}([-2, 1/4])$. Since $L_{1/2}(\tau_{1/2}(-2)) = \langle R_{1,-2}\rangle_{1/2} \in \mathcal{B}_{1/2}$ the map $L_{1/2}$ is continuous.

q.e.d.

Remarks.

If $c \in (-3/4, 1/4]$ then P_c has a non-repelling fixed point with multiplier $\mu \in (-1, 1]$ and the limit

$$\lim_{\substack{\lambda\to-1\\ \lambda>-1}} \langle R_{\lambda,c}\rangle$$

exists in $\mathcal{M}_2$. In that case we have

$$\lim_{\substack{\lambda\to-1\\ \lambda>-1}} \langle R_{\lambda,c}\rangle = \langle -1, \mu, \frac{3-\mu}{1+\mu}\rangle$$

which is the mating of $P_{-3/4}$ and $P_{c(\mu)}$.

Chapter 19

Convergence of Julia sets

In this section we prove a complementary result to Theorem 18.1 concerning the limit of the Julia sets of $R_{\lambda,c}$:

Let c be in the real 1/2-limb $[-2,-3/4)$. Then

$$\lim_{\substack{\lambda\to -1\\ \lambda>-1}} J(R_{\lambda,c}) = J(G_T)$$

in the Hausdorff metric on compact subsets of $\hat{\mathbb{C}}$ where

$$\lim_{\substack{\lambda\to -1\\ \lambda>-1}} \langle R_{\lambda,c}\rangle = \langle G_T\rangle_{1/2} \in \mathcal{B}_{1/2}$$

and $T<0$.

We first discuss convergence in the Hausdorff metric for sequences of compact subsets of $\hat{\mathbb{C}}$.

The Hausdorff metric. Let A and B be compact subsets of $\hat{\mathbb{C}}$. The *distance from a point $x \in \hat{\mathbb{C}}$ to the set A* is defined as follows:

$$d(x,A) = \min_{y\in A} d_{\hat{\mathbb{C}}}(x,y)$$

where $d_{\hat{\mathbb{C}}}(x,y)$ denotes the spherical metric. The *Hausdorff distance* between the compact sets A and B is given by

$$d_H(A,B) = \max\{\max_{x\in A} d(x,B), \max_{x\in B} d(x,A)\}.$$

The halo of a subset of $\hat{\mathbb{C}}$. The *halo of a subset* $A \subset \hat{\mathbb{C}}$ is the external set of points which are infinitely close to A:

$$\mathrm{hal}(A) = \{x \in \hat{\mathbb{C}} :\ d(x,A) \approx 0\}.$$

Lemma 19.1 (External characterization of convergence in the Hausdorff metric) *Let $\{A_n\}_{n\in\mathbb{N}}$ be a standard sequence of compact sets and A a standard compact set, $A_n, A \subset \hat{\mathbb{C}}$. Then*

$$\lim_{n\to\infty} d_H(A_n, A) = 0$$

if and only if

$$\mathrm{hal}(A_N) = \mathrm{hal}(A) \text{ for all } N \approx \infty.$$

Proof: By the external characterization of convergence we have $\lim_{n\to\infty} d_H(A_n, A) = 0$ if and only if $d_H(A_N, A) \approx 0$ for all $N \approx \infty$. And $d_H(A_N, A) \approx 0$ if and only if $d(x, A_N) \approx 0$ for all $x \in A$ and $d(x, A) \approx 0$ for all $x \in A_N$, which means that $A \subset \mathrm{hal}(A_N)$ and $A_N \subset \mathrm{hal}(A)$ and thus $\mathrm{hal}(A_N) = \mathrm{hal}(A)$.

q.e.d.

It is a well-known fact that the Julia set always varies lower semi-continuously in the Hausdorff metric in the space of rational maps, see [Do]. This is also true in the compactified space of rational maps:

Theorem 19.2 (Lower semi-continuity of $J(f)$) *Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of rational maps with $\lim_{n\to\infty} f_n = g$ locally uniformly outside a finite set and $\deg(g) \geq 2$. Then*

$$\lim_{n\to\infty} [\max_{y\in J(g)} d(y, J(f_n))] = 0.$$

Proof: Per Transfer we can assume that $\{f_n\}_{n\in\mathbb{N}}$ and g are standard, and thus $\max_{y\in J(g)} d(y, J(f_n))$ is a standard sequence. Let $N \approx \infty$. We have to show that

$$\max_{y\in J(g)} d(y, J(f_N)) \approx 0$$

(this is equivalent to $J(g) \subset \mathrm{hal}(J(f_N))$).

Since $\lim_{n\to\infty} f_n = g$ locally uniformly outside a finite set, there exists finitely many standard points $a_1, \cdots a_k \in \hat{\mathbb{C}}$ such that $f_N(z) \approx g(z)$ for all $z \not\approx a_1, \cdots a_k$ and all $N \approx \infty$.

Since $J(g)$ is a compact standard set we have that ${}^\circ z \in J(g)$ for all $z \in J(g)$. Thus it suffices to prove that for all standard $z \in J(g)$ there is a $y \in J(f_N)$ such that $d_{\hat{\mathbb{C}}}(z, y) \approx 0$.

So let $z \in J(g)$ standard.

First assume that $z \neq a_1, \cdots a_k$. Since the Julia set is the closure of the repelling periodic points we can find for every standard $n \in \mathbb{N}$ a standard repelling periodic point $p \neq z$ for g with $d_{\hat{\mathbb{C}}}(z, p) < 1/n$. By Rouché's Theorem, see Lemma 4.1, there is a repelling periodic point $\widetilde{p} \approx p$ for f_N, and thus we can find for every standard $n \in \mathbb{N}$ a repelling periodic point $\widetilde{p}$ for f_N with $d_{\hat{\mathbb{C}}}(z, \widetilde{p}) < 1/n$. By overspill this is also true for some $M \approx \infty$ and thus there is repelling periodic point $\widetilde{p} \in J(f_N)$ with $d_{\hat{\mathbb{C}}}(z, \widetilde{p}) \approx 0$.

Now assume that $z = a_i$ for some $1 \leq i \leq k$. Since $J(g)$ is perfect, we can find for every standard $n \in \mathbb{N}$ a point $\widetilde{z} \neq a_1, \cdots a_k$ in $J(g)$ with $d_{\hat{\mathbb{C}}}(z, \widetilde{z}) < 1/n$. By the previous considerations there exists for $\widetilde{z}$ a point $y \in J(f_N)$ with $y \approx \widetilde{z}$ and thus $d_{\hat{\mathbb{C}}}(z, y) < 1/n$. By overspill we get a $y \in J(f_N)$ with $d_{\hat{\mathbb{C}}}(z, y) \approx 0$.

q.e.d.

Note that $\lim_{n\to\infty}[\max_{y\in J(g)} d(y, J(f_n))] = 0$ is equivalent to

$$J(g) \subset \mathrm{hal}(J(f_N)) \text{ for all } N \approx \infty$$

in case $\{f_n\}_{n\in\mathbb{N}}$ and g are standard. Thus this means roughly speaking: "The Julia set of the limit is contained in the limiting Julia sets."

Theorem 19.3 (Convergence of the Julia sets in the real case) *Let c be in the real 1/2-limb $[-2, -3/4)$. Then*

$$\lim_{\substack{\lambda\to -1\\ \lambda>-1}} J(R_{\lambda,c}) = J(G_T)$$

in the Hausdorff metric on compact subsets of $\hat{\mathbb{C}}$ where

$$\lim_{\substack{\lambda\to -1\\ \lambda>-1}} \langle R_{\lambda,c}\rangle = \langle G_T\rangle_{1/2} \in \mathcal{B}_{1/2}$$

and $T < 0$.

Proof: Let $c \in [-2, -3/4)$ with $\lim_{\substack{\lambda\to -1\\ \lambda>-1}} \langle R_{\lambda,c}\rangle = \langle G_T\rangle_{1/2} \in \mathcal{B}_{1/2}$. Applying Transfer we assume that c is standard. This implies that the limiting map G_T is standard. Thus $T < 0$ standard.

Let $\lambda \approx -1$ with $\lambda > -1$.

Then we have $G_T(z) \approx R^2_{\lambda,c}(z)$ for all $z \not\approx 0, \infty$. We will show that

$$\mathrm{hal}(J(R^2_{\lambda,c})) = \mathrm{hal}(J(G_T))$$

Note that $J(R^2_{\lambda,c}) = J(R_{\lambda,c})$. By Lemma 19.1 this is equivalent to the statement of the theorem.

We have to show that $J(R^2_{\lambda,c}) \subset \mathrm{hal}(J(G_T))$. The other inclusion follows from the Theorem 19.2. We show

$$z \notin \mathrm{hal}(J(G_T)) \;\Rightarrow\; z \notin J(R^2_{\lambda,c}).$$

We have that $z \notin \mathrm{hal}(J(G_T))$ if and only if ${}^\circ z \notin J(G_T)$, because $J(G_T)$ is covered by the halos of its standard points. Thus we have to show that if z and ${}^\circ z$ are in the Fatou set for G_T, then z is also in the Fatou set for $R^2_{\lambda,c}$.

The Fatou set of G_T consists of the basin of the parabolic fixed point at ∞ and, if there is one, the basin of an attracting or parabolic periodic cycle in the finite plane. So we have to consider three cases: The basin of the parabolic fixed point at ∞, the basin of an attracting periodic cycle in the finite plane and the basin of a parabolic periodic cycle in the finite plane.

The basin of the parabolic fixed point ∞.

Claim: If z and ${}^\circ z$ are in the basin of the parabolic fixed point ∞ for G_T then z is in the basin of the attracting fixed point ∞ for $R^2_{\lambda,c}$.

Let $z, {}^\circ z \in \mathcal{A}_{G_T}(\infty)$. Then we have

$$\forall N \approx \infty \ \ \mathrm{Re}(G_T^N(z)) \approx -\infty$$

because $G_T(z) \approx z + T$ for all $z \approx \infty$ and $T < 0$. Since $G_T^n({}^\circ z) \neq \infty$ for all $n \in \mathbb{N}$ we have $G_T^n(z) \not\approx 0, \infty$ for all standard $n \in \mathbb{N}$ and thus

$$(R^2_{\lambda,c})^n(z) \approx G_T^n(z) \ \text{ for all standard } n \in \mathbb{N}$$

which by Robinson's Lemma then also holds for some $N \approx \infty$, so that we can conclude

$$\mathrm{Re}((R^2_{\lambda,c})^N(z)) \approx -\infty$$

If $\mathrm{Re}(z) \approx -\infty$ then we have

$$\mathrm{Re}((R^2_{\lambda,c})^n(z)) < \mathrm{Re}(z) + n\frac{T}{2}$$

This implies that $z \in \mathcal{A}_{R^2_{\lambda,c}}(\infty)$.
We prove that by induction over n:

$\underline{n = 1:}$

$$\mathrm{Re}((R^2_{\lambda,c})(z)) = \underbrace{\frac{1}{\lambda^2}}_{>1}(\mathrm{Re}(z) + \underbrace{\mathrm{Re}(\frac{1}{z})}_{\approx 0} + \underbrace{T_\lambda(c)(1+\lambda)}_{\approx T} + \underbrace{\mathrm{Re}(\frac{1}{z + \frac{1}{z} + T_\lambda(c)})}_{\approx 0}) < \mathrm{Re}(z) + \frac{T}{2}$$

$\underline{n \to n+1:}$

$$\mathrm{Re}((R^2_{\lambda,c})^{n+1}(z))$$

$$= \underbrace{\frac{1}{\lambda^2}}_{>1}(\underbrace{\mathrm{Re}((R^2_{\lambda,c})^n(z))}_{<\mathrm{Re}(z)+n\frac{T}{2}} + \underbrace{\mathrm{Re}(\frac{1}{(R^2_{\lambda,c})^n(z)})}_{\approx 0} + \underbrace{(T_\lambda(c)(1+\lambda))}_{\approx T} + \underbrace{\mathrm{Re}(\frac{1}{(R^2_{\lambda,c})^n(z) + \frac{1}{(R^2_{\lambda,c})^n(z)} + T_\lambda(c)})}_{\approx 0})$$

$$< \mathrm{Re}(z) + n\frac{T}{2} + \frac{T}{2} = \mathrm{Re}(z) + (n+1)\frac{T}{2}$$

Attracting periodic cycles. Now we assume that G_T has an attracting periodic cycle. Transfer implies that this cycle is standard, because G_T is standard. Thus it is not close to 0 nor ∞, because these are the parabolic fixed point and its preimage for G_T. By Theorem 18.1 c must be in the image of tuning and thus $R^2_{\lambda,c}$ has an attracting periodic cycle with the same multiplier, since $R^2_{\lambda,c}$ suitably restricted is real hybrid equivalent to $G_T : [\infty, 0] \to [\infty, 0]$. The cycle for $R^2_{\lambda,c}$ is close to the one of G_T.

We now prove a more general statement that applies to our case.

Let f, g be rational maps having attracting fixed points $z_0 \approx \tilde{z_0}$ respectively. Furthermore let g be standard and $f(z) \approx g(z)$ for all $z \in \mathcal{A}_g(z_0)$ with $^\circ z \in \mathcal{A}_g(z_0)$. Then we have: If $z \in \mathcal{A}_g(z_0)$ and $^\circ z \in A_g(z_0)$ then $z \in A_f(\tilde{z_0})$.

Since z_0 is an attracting fixed point for the standard map g there exists according to Lemma 4.3 a standard $\epsilon > 0$ and a standard constant $0 < c < 1$ such that

$$\forall z \ : \ |z - z_0| < \epsilon \ \Rightarrow \ |g(z) - z_0| < c|z - z_0|$$

Because $|g(z) - z_0| \approx |f(z) - \tilde{z_0}|$ and $|z - z_0| \approx |z - \tilde{z_0}|$ we have that if $|z - \tilde{z_0}| < \epsilon/2$ and thus $|f(z) - \tilde{z_0}| \approx |g(z) - z_0| < c|z - z_0| < c\epsilon/2$, and because $c\epsilon/2$ is standard, this implies that $f(B_{\epsilon/2}(\tilde{z_0})) \subset B_{\epsilon/2}(\tilde{z_0})$, which implies that $B_{\epsilon/2}(\tilde{z_0})$ is contained in the Fatou set of f and thus in $A_f(\tilde{z_0})$. Now let $z \in A_g(z_0)$ with $^\circ z \in A_g(z_0)$. Then $g^n(^\circ z) \approx g^n(z) \approx f^n(z)$ for all standard $n \in \mathbb{N}$, since $f(z) \approx g(z)$ for all $z \in \mathcal{A}_g(z_0)$ with $^\circ z \in \mathcal{A}_g(z_0)$. Robinson's Lemma implies that

$$\exists N \approx \infty \ g^N(z) \approx f^N(z)$$

Since $g^N(z) \approx z_0$ we have $f^N(z) \in B_{\frac{\epsilon}{2}}(\tilde{z_0})$ and thus $z \in A_f(\tilde{z_0})$.

Let z_0 be a point in the attracting periodic n cycle for G_T and $\tilde{z_0}$ the nearby point in the attracting periodic n cycle for $R^2_{\lambda,c}$. Since $0, \infty \in J(G_T)$ we have $G^n_T(z) \approx (R^2_{\lambda,c})^n(z)$ for all $z \in \mathcal{A}_{G^n_T}(z_0)$ with $^\circ z \in \mathcal{A}_{G^n_T}(z_0)$, thus $z \in \mathcal{A}_{(R^2_{\lambda,c})^n}(\tilde{z_0})$.

Parabolic periodic cycles in the finite plane. Now we assume that G_T has a parabolic periodic cycle in the finite plane in addition to the parabolic fixed point ∞. Analogously to the previous case we have that this parabolic cycle is not close to 0 nor ∞, and that $R^2_{\lambda,c}$ has a parabolic periodic cycle close to the one of G_T, with the same multiplier. Note that since T is real, the multiplier at the parabolic cycle for G_T must be real, thus it must be 1 or -1. By Theorem 4.4 there are either one or two petals at each periodic point belonging to the parabolic cycle.

We prove again a more general statement that applies to our case.

Claim: Let f, g be rational maps having parabolic fixed points $z_0 \approx \tilde{z_0}$, with multiplier 1 and the same multiplicity (either 2 or 3), so that they have the same number of petals (either 1 or 2), respectively. Compare Theorem 4.4. Furthermore let g be standard and $f(z) \approx g(z)$ for all $z \in \mathcal{A}_g(z_0)$ with $^\circ z \in \mathcal{A}_g(z_0)$ and for all z in a standard neighborhood of z_0. Then we have: If $z \in \mathcal{A}_g(z_0)$ and $^\circ z \in A_g(z_0)$ then $z \in A_f(\tilde{z_0})$.

Conjugate g and f by $h_1 : z \mapsto \frac{1}{z - z_0}$ and $h_2 : z \mapsto \frac{1}{z - \tilde{z_0}}$ respectively, so that the parabolic fixed points are at infinity. Note that the two Möbius transformations h_1 and h_2 are close to each other and so they map nearby points to nearby points.

1. *One petal.* After conjugating such that z_0 and $\tilde{z_0}$ are at ∞ we have

$$g(z) = z + a_0 + \sum_{i=1}^{\infty} \frac{a_i}{z^i}$$

with $a_0 \neq 0$. Conjugate by $z \mapsto a_0 z$ to make $a_0 = 1$. This leads to

$$g(z) = z + 1 + \sum_{i=1}^{\infty} \frac{a_i}{z^i}$$

and

$$f(z) = z + a + \sum_{i=1}^{\infty} \frac{\tilde{a}_i}{z^i}$$

with $a \approx 1$ and $a_i \approx \tilde{a}_i$ in some standard neighborhood of ∞.

Now let z and ${}^\circ z$ in $\mathcal{A}_g(\infty)$ then we have

$$\forall N \approx \infty \ \ \mathrm{Re}(g^N(z)) \approx +\infty$$

because $g(z) \approx z + 1$ for all $z \approx \infty$. Furthermore we have

$$\forall^{st} n \in \mathbb{N} \ \ \mathrm{Re}(g^n({}^\circ z)) \approx \mathrm{Re}(g^n(z)) \approx \mathrm{Re}(f^n(z))$$

By Robinson's lemma there exists a $N \approx \infty$ with

$$\mathrm{Re}(f^N(z)) \approx \mathrm{Re}(g^N(z)) \approx +\infty$$

This implies that $z \in \mathcal{A}_f(\infty)$, because if $\mathrm{Re}(z) \approx +\infty$ then $\mathrm{Re}(f^n(z)) \geq \mathrm{Re}(z) + n/2$ for all $n \in \mathbb{N}$.

2. *Two petals.* After conjugating such that z_0 and $\tilde{z}_0$ are at ∞ we have

$$g(z) = z + \frac{a_1}{z} + \sum_{i=2}^{\infty} \frac{a_i}{z^i}$$

with $a_1 \neq 0$. Conjugate by $z \mapsto a_1 z$ to make $a_1 = 1$. This leads to

$$g(z) = z + \frac{1}{z} + \sum_{i=1}^{\infty} \frac{a_i}{z^i}$$

and

$$f(z) = z + \frac{a}{z} + \sum_{i=1}^{\infty} \frac{\tilde{a}_i}{z^i}$$

with $a \approx 1$ and $a_i \approx \tilde{a}_i$ in a standard neighborhood U of infinity.

Now let z and ${}^\circ z$ in $\mathcal{A}_g(\infty)$ then we have

$$\forall N \approx \infty \ \ \mathrm{Re}((g^N(z))^2) \approx +\infty$$

Furthermore we have

$$\forall^{st} n \in \mathbb{N} \ \ \mathrm{Re}((g^n(z))^2) \approx \mathrm{Re}((f^n(z))^2)$$

By Robinson's lemma there exists a $N \approx \infty$ with

$$\mathrm{Re}((f^N(z))^2) \approx \mathrm{Re}((g^N(z))^2) \approx +\infty$$

Claim: Let $z \approx \infty$ with $\mathrm{Re}(z^2) \approx +\infty$. Then $\mathrm{Re}((f^n(z))^2) \geq \mathrm{Re}(z^2) + n$ for all $n \in \mathbb{N}$.
$\underline{n = 1:}$

$$\mathrm{Re}((f(z))^2) \approx \mathrm{Re}((z + \frac{a}{z})^2) \approx \mathrm{Re}(z^2) + 2a$$

and thus $\mathrm{Re}((f(z))^2) \geq \mathrm{Re}(z^2) + 1$

$\underline{n \to n+1:}$

$$\mathrm{Re}((f^{n+1}(z))^2) \approx \mathrm{Re}((f^n(z) + \frac{a}{f^n(z)})^2) = \mathrm{Re}((f^n(z))^2) + 2a + \mathrm{Re}(\frac{1}{(f^n(z))^2})$$

by the induction hypothesis we have

$$\mathrm{Re}((f^n(z))^2) \geq \mathrm{Re}(z^2) + n$$

and thus

$$\mathrm{Re}((f^{n+1}(z))^2) \geq \mathrm{Re}(z^2) + n + 1.$$

This implies that $z \in \mathcal{A}_f(\infty)$.

Let z_0 be a point of a parabolic periodic n cycle for G_T with multiplier 1 (or with multiplier -1) and $\tilde{z}_0$ the nearby point in the attracting periodic n cycle for $R^2_{\lambda,c}$ with multiplier 1 (or with multiplier -1). By Theorem 4.4 Part 3, G_T and $R^2_{\lambda,c}$ have one petal in case the multiplier is 1 and two petals in case the multiplier is -1. Since $0, \infty \in J(G_T)$ we have $G^n_T(z) \approx (R^2_{\lambda,c})^n(z)$ for all $z \in \mathcal{A}_{G^n_T}(z_0)$ with ${}^\circ z \in \mathcal{A}_{G^n_T}(z_0)$ and for all z in a suitable standard neighborhood of z_0, thus $z \in \mathcal{A}_{(R^2_{\lambda,c})^n}(\tilde{z}_0)$.

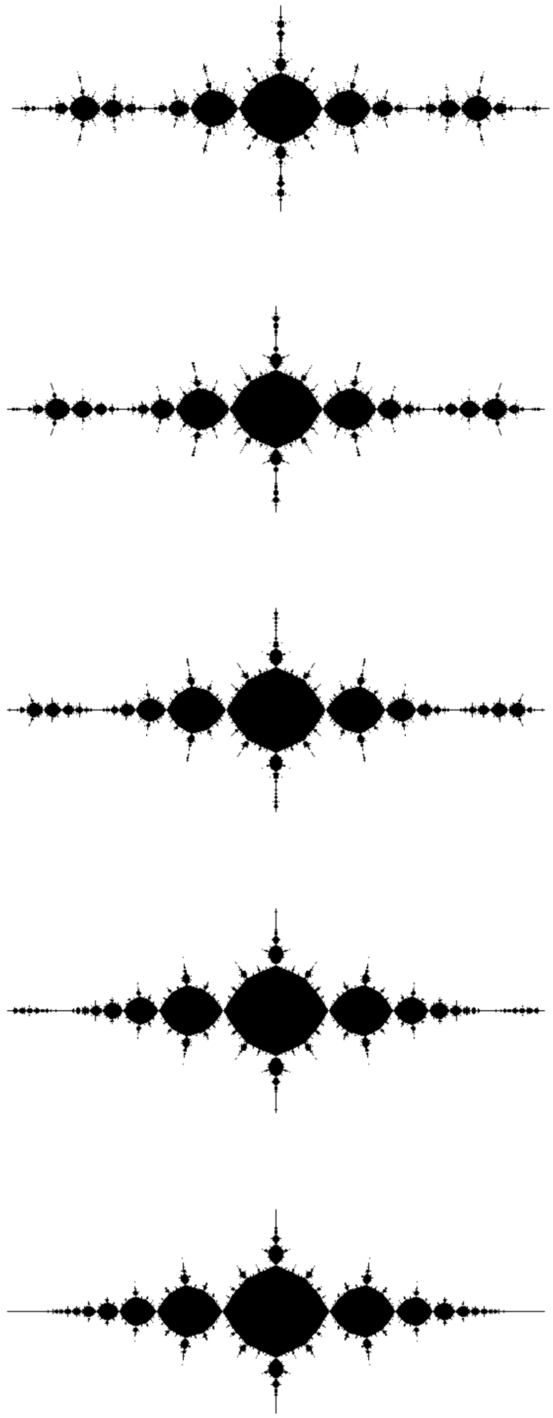
q.e.d.

Remarks and References.

1. We have not proved the convergence of the Julia sets for the special case $c = -3/4$. By Theorem 18.1 we have $\lim_{\substack{\lambda \to -1 \\ \lambda > -1}} \langle R_{\lambda,-3/4} \rangle = \langle G_0 \rangle_{1/2} \in \mathcal{B}_{1/2}$. Our argument requires that $T \neq 0$. Computer picture indicate that the Julia sets of $R_{\lambda,-3/4}$ converge to the limiting Julia set $J(G_0)$. $J(G_0)$ is the imaginary axis and thus all points z with $\mathrm{Re}(z) \neq 0$ are in the basin of the parabolic fixed point ∞ for G_0. Let $\lambda \approx -1$, $\lambda > -1$. It is easy to show that if $\mathrm{Re}(z) < 0$ then z is in the basin of the attracting fixed point ∞ for $R_{\lambda,c}$, if $\mathrm{Re}(z) \not\approx 0$. It remains to show that if $\mathrm{Re}(z) > 0$, $\mathrm{Re}(z) \not\approx 0$, then z is in the basin of the parabolic α-fixed point $\alpha \approx \infty$ for $R_{\lambda,c}$.

2. It is a well-known that $\lim_{n\to\infty}[\max_{y \in J(g)} d(y, J(f_n))] = 0$, if $f_n \to g$, $f_n, g \in \mathrm{Rat}_d$ see [Do]. Theorem 19.2 shows that this is also true if the limiting map g is a rational map of lower degree.

3. Similar results on continuity of Julia sets in a slightly different context occur in [Mc2].



$\lambda = -0.2$

$\lambda = -0.4$

$\lambda = -0.6$

$\lambda = -0.8$

$\lambda = -0.9$

Figure 19.1: The Julia sets of $R_{\lambda,c}$ for $c = -1.3107... = \tau_{1/2}(-1)$, the centerpoint of the period 4-component on the real axis in the image of tuning.

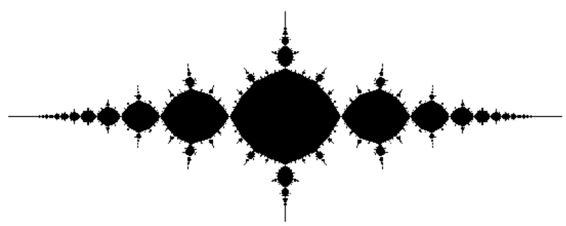
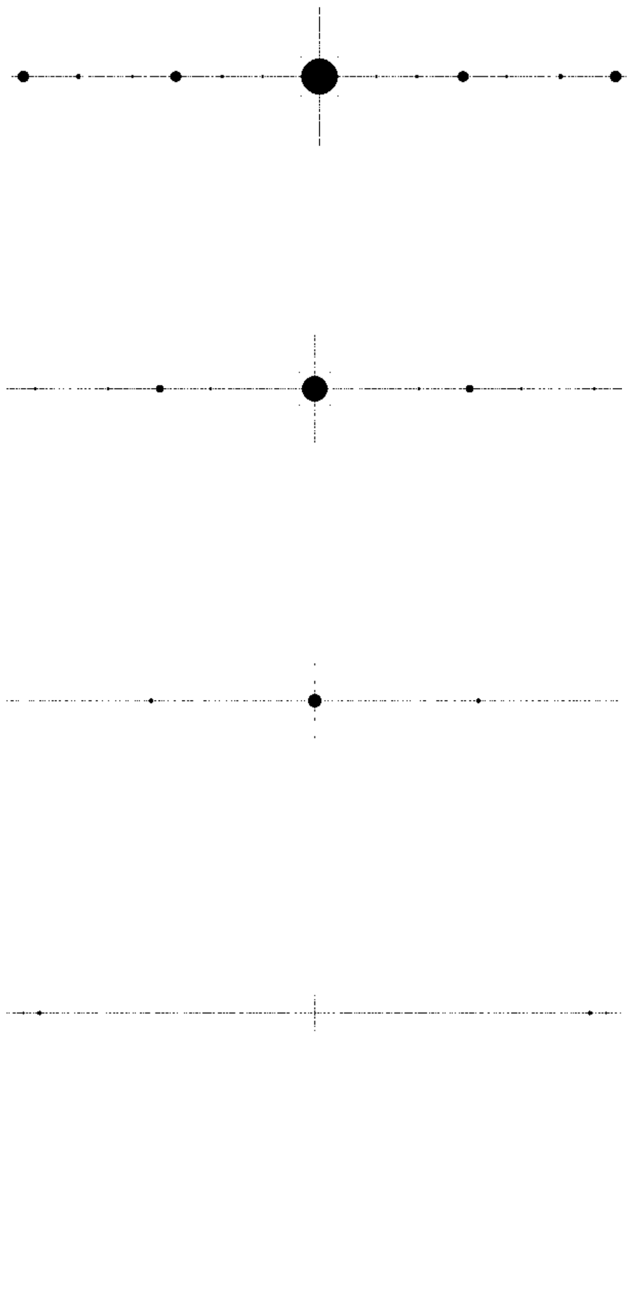


Figure 19.2: The Julia set of the limiting map $G_{-3/2}$.



$\lambda = -0.2$

$\lambda = -0.4$

$\lambda = -0.6$

$\lambda = -0.8$

$J(G_{-2})$

Figure 19.3: The Julia sets of $R_{\lambda,c}$ for $c = -1.75488...$, the centerpoint of the unique real period 3-component (c is not in the image of tuning). And the Julia set of the limiting map G_{-2}.

Chapter 20

Ideal limit points of $\mathrm{Per}_n(0)$ curves

In this section we determine the ideal limit points of the $\mathrm{Per}_n(0)$ curves, consisting of all conjugacy classes having a superattracting periodic n cycle, in the dynamical compactification $\widetilde{\mathcal{M}}_2$.

Theorem 20.1 (Ideal limit points of $\mathbf{Per}_n(0)$ in $\widetilde{\mathcal{M}}_2$)
The ideal point $\langle G_T \rangle_{p/q} \in \mathcal{B}_{p/q}$ is a limit point of $\mathrm{Per}_n(0)$ *if and only if* $n \geq q \geq 2$ *and:*

1. $G_T^m(1) = 0$ *for some* $1 \leq m < n/q$ *or*

2. q *divides* n *and* G_T *has a superattracting periodic cycle of period* n/q*.*

Recall that by Theorem 7.4 all the possible ideal limit points of $\mathrm{Per}_n(0)$ in $\widehat{\mathcal{M}}_2$ are of the form $\langle e^{2\pi ip/q}, e^{-2\pi ip/q}, \infty \rangle$ with $1 < q \leq n$. So all the possible ideal limit points of $\mathrm{Per}_n(0)$ in $\widetilde{\mathcal{M}}_2$ lie on some sphere $\mathcal{B}_{p/q}$. Furthermore if $n < q$ the curve $\mathrm{Per}_n(0)$ has no limit point on $\mathcal{B}_{p/q}$, because $\langle e^{2\pi ip/q}, e^{-2\pi ip/q}, \infty \rangle$ is not a limit point of $\mathrm{Per}_n(0)$ in the algebraic compactification.

The only if part of Theorem 20.1 is due to Epstein. It is a special case of [Eps, Proposition 3].

Our contribution is to show that all these points $\langle G_T \rangle_{p/q} \in \mathcal{B}_{p/q}$ with the above properties actually occur as limit points of $\mathrm{Per}_n(0)$. For a given $\langle G_T \rangle_{p/q} \in \mathcal{B}_{p/q}$ Theorem 9.2 allows us to write down an explicit formula for a map that is infinitely close to this ideal point. We vary this map in order to get one in $\mathrm{Per}_n(0)$.

Proof: Applying transfer we can assume that $q \geq 2$, $n \geq q$ and $\langle G_T \rangle_{p/q} \in \mathcal{B}_{p/q}$ are standard. Then $\langle G_T \rangle_{p/q} \in \mathcal{B}_{p/q}$ is an ideal limit point of $\mathrm{Per}_n(0)$ if and only if there exists a map $\langle f \rangle \in \mathrm{Per}_n(0)$ with $\langle f \rangle \approx \langle G_T \rangle_{p/q} \in \mathcal{B}_{p/q}$ (for the external characterization of a limit point see Chapter 2). Recall that $\langle f \rangle \approx \langle G_T \rangle_{p/q} \in \mathcal{B}_{p/q}$ if and only if the multipliers at the fixed points of f are close to $\langle e^{2\pi ip/q}, e^{-2\pi ip/q}, \infty \rangle$ and f can be normalized such that $\mathrm{sh}(f^q) = G_T$.

We first show that all ideal limit points $\langle G_T \rangle_{p/q} \in \mathcal{B}_{p/q}$ of $\mathrm{Per}_n(0)$ have the property that either $G_T^m(1) = 0$ for some $1 \leq m < n/q$ or G_T has a superattracting n/q cycle.

If $\langle G_T\rangle_{p/q} \in \mathcal{B}_{p/q}$ is a standard ideal limit point of $\mathrm{Per}_n(0)$, then there exists $\langle f\rangle \in \mathcal{M}_2$ with $\langle f\rangle \in \mathrm{Per}_n(0)$ and $\langle f\rangle \approx \langle G_T\rangle_{p/q} \in \mathcal{B}_{p/q}$. Then f has fixed points with multipliers $\mu_1 \approx e^{2\pi ip/q}$, $\mu_2 \approx e^{-2\pi ip/q}$ and $\mu_3 \approx \infty$. Normalize f like in Theorem 9.2:

$$f(z) = \frac{1}{\mu}\left(z + \frac{1}{z} + S\right)$$

with $\mu = \mu_1 \approx e^{2\pi ip/q}$ and $S = \frac{\mu_2-\mu_3}{\sqrt{1-\mu_2\mu_3}} \approx \infty$ such that the periodic critical point is at 1, $f^n(1) = 1$. Then we have $f^q(z) \approx G_T(z)$ with $T = {}^\circ(S\frac{1-\mu^q}{1-\mu})$, and $f^r(z) \approx \infty$ for all $z \not\approx 0, \infty$ and all $r < q$.

First assume that $n/q \notin \mathbb{N}$. Then $n = m\,q + r$ with $1 \le r < q$. Thus we have

$$1 = f^n(1) = f^r((f^q)^m(1)).$$

Since $f^r(z) \approx \infty$ for all $z \not\approx 0, \infty$ it follows that $(f^q)^m(1) \approx 0$ or $(f^q)^m(1) \approx \infty$. This can only be the case, if $G_T^l(1) = 0$ for some $1 \le l \le m$, because $G_T(z) \approx f^q(z)$ for all $z \not\approx 0, \infty$ and $G_T^{-1}(\infty) = \{0, \infty\}$.

Now assume that $n/q \in \mathbb{N}$. Then we have $n = m\,q$ for some $m \ge 1$ and

$$1 = f^n(1) = (f^q)^m(1).$$

We either have that $(f^q)^m(1) \approx G_T^m(1)$ or $G_T^l(1) = 0$ for some $1 \le l < m$. If $1 = (f^q)^m(1) \approx G_T^m(1)$ it follows by transfer that $G_T^m(1) = 1$. Thus G_T has a superattracting cycle and m has to be the minimal period. (Assume that $G_T^l(1) = 1$ for some $1 \le l < m$. Then $(f^q)^k(1) \approx 1$ for some $k < m$ which would imply that $(f^q)^m$ has at least two fixed points infinitely close to 1, so by Corollary 4.1 the point 1 would be a multiple fixed point for G_T, i.e. $(G_T^q)'(1) = 1$, which is a contradiction, since 1 is a critical point for G_T)

It remains to show that all these points actually occur as limit points.

We first show that $\langle G_T\rangle_{p/q} \in \mathcal{B}_{p/q}$ standard with $G_T^m(1) = 1$ (m minimal with this property) is a limit point of $\mathrm{Per}_n(0)$ with $n = mq$.

Let $\mu \approx e^{2\pi ip/q}$ and define

$$f_0(z) = \frac{1}{\mu}\left(z + \frac{1}{z} + T\frac{1-\mu}{1-\mu^q}\right).$$

By Theorem 9.2 we have $f_0^q(z) \approx G_T(z)$ for all $z \not\approx 0, \infty$. Now we vary this map in order to get one with a superattracting periodic cycle. Define

$$f_\delta(z) = \frac{1}{\mu}\left(z + \frac{1}{z} + (T+\delta)\frac{1-\mu}{1-\mu^q}\right)$$

Then we have again by Theorem 9.2 that $f_\delta^q(z) \approx G_{^\circ(T+\delta)}(z)$ for all $z \not\approx 0, \infty$ and in particular $\mathrm{sh}(f_\delta^q) = G_T$ for all $\delta \approx 0$. Let

$$F : B_R(0) \to \mathbb{C}$$

be defined by

$$F(\delta) = f_\delta^n(1) = (f_\delta^q)^m(1)$$

(choose $R > 0$ such that $|G_{T+\delta}^m(1) - 1| < 1$ for all $\delta \in B_R(0)$). F is a holomorphic with $F(\delta) = f_\delta^n(1) = (f_\delta^q)^m(1) \approx G_{\circ(T+\delta)}^m(1)$ and thus s-continuous and the shadow of F is not constant. Furthermore we have $F(0) \approx G_T^m(1) = 1$, so we can conclude by Theorem 3.6 that

$$F(\mathrm{hal}(0)) = \mathrm{hal}(F(0)) = \mathrm{hal}(1)$$

which implies that there is a $\delta \approx 0$ with $F(\delta) = f_\delta^n(1) = 1$, thus $\langle f_\delta \rangle \in \mathrm{Per}_n(0)$. Furthermore we have if $\delta \approx 0$ that $(T+\delta)\frac{1-\mu}{1-\mu^q} \approx \infty$ which implies that $\mu_3 \approx \infty$ and thus $\langle f \rangle \approx \langle e^{2\pi ip/q}, e^{-2\pi ip/q}, \infty \rangle$. Since $\mathrm{sh}(f_\delta^q) = G_T$ this implies that $\langle f_\delta \rangle \approx \langle G_T \rangle_{p/q} \in \mathcal{B}_{p/q}$.

Now we show that $\langle G_T \rangle_{p/q} \in \mathcal{B}_{p/q}$ standard with $G_T^m(1) = 0$ (m minimal with this property) is a limit point of $\mathrm{Per}_n(0)$ with $n = mq + r$ for all standard $r \geq 1$.

Let $\mu \approx e^{2\pi ip/q}$. Define

$$f_\delta(z) = \frac{1}{\mu}\left(z + \frac{1}{z} + (T+\delta)\frac{1-\mu}{1-\mu^q}\right).$$

By Theorem 9.2 we have $f_\delta^q(z) \approx G_{\circ(T+\delta)}(z)$ for all $z \not\approx 0, \infty$ and in particular $f_\delta^q(z) \approx G_T$ for all $z \not\approx 0, \infty$ and all $\delta \approx 0$. We have $f_\delta(z) \approx \infty$ for all $z \not\approx 0, \infty$. Every $z \not\approx \infty$ has two preimages under f_δ one close to 0 and one close to ∞. Both critical values are in $B_{1/4}(\infty)$ and the preimage of $\hat{\mathbb{C}} - B_{1/4}(\infty)$ consists of two connected components, one in hal(0) and one in hal(∞). Let $f_\delta^{-1} : \hat{\mathbb{C}} - B_{1/4}(\infty) \to \hat{\mathbb{C}} - B_{1/4}(\infty)$ be the branch that associates to every $z \in \hat{\mathbb{C}} - B_{1/4}(\infty)$ the preimage under f_δ that is contained in $\hat{\mathbb{C}} - B_{1/4}(\infty)$. Let

$$F : B_R(0) \to \mathbb{C}$$

defined by

$$F(\delta) = f_\delta^{qm}(1) - (f_\delta^{-1})^r(1).$$

(choose $R > 0$ such that $|G_{T+\delta}^m(1)| < 1$ for all $\delta \in B_R(0)$).

F is a holomorphic map with

$$F(\delta) = f_\delta^{qm}(1) - (f_\delta^{-1})^r(1) = \underbrace{(f_\delta^q)^m(1)}_{\approx G_{\circ(T+\delta)}^m(1)} - \underbrace{(f_\delta^{-1})^r(1)}_{\approx 0} \approx G_{\circ(T+\delta)}^m(1)$$

and thus F is an s-continuous holomorphic map whose shadow is not constant. So we can conclude by Theorem 3.6 that

$$F(\mathrm{hal}(0)) = \mathrm{hal}(F(0)) = \mathrm{hal}(0)$$

which implies that there is a $\delta \approx 0$ with $F(\delta) = f_\delta^{qm}(1) - (f_\delta^{-1})^r(1) = 0$. Thus we have $f_\delta^{qm}(1) = (f_\delta^{-1})^r(1) = 0$, which implies that $f_\delta^r(f_\delta^{qm}(1)) = (f_\delta^n)(1) = 1$, thus $\langle f_\delta \rangle \in \mathrm{Per}_n(0)$. Furthermore we have like in the previous case $\langle f_\delta \rangle \approx \langle G_T \rangle_{p/q} \in \mathcal{B}_{p/q}$ if $\delta \approx 0$.

q.e.d.

Remarks and References.

1. Note that the multiplier $\mu \approx e^{2\pi i p/q}$ can be chosen so that the fixed point is attracting. So if $\langle G_T \rangle \in \mathcal{B}_{p/q}$ is an ideal limit point of $\mathrm{Per}_n(0)$, then we can find a map $\langle f_\delta \rangle \in \mathcal{M}_2$ with $\langle f_\delta \rangle \approx \langle G_T \rangle \in \mathcal{B}_{p/q}$ and $\langle f_\delta \rangle = \langle R_{\mu,c} \rangle$ for some hyperbolic center in $c \in M$.

2. That these are the only possible limit points is discussed in a more general setting in [Eps, Proposition 3].

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