A graph based approach to the convergence of some iterative methods for singular M-matrices and Markov chains

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For my parents

Preface

This thesis has been developed between 2002 and 2006 while I was working in the Applied Computer Science Group at the University of Wuppertal. The main results have been worked out in 2005. They cover certain aspects of the convergence of iteration methods for singular M-matrices and Markov Chains and try to find a link between them. The methods to be discussed here are multiplicative and also additive Schwarz methods. Beside this a few results on partially asynchronous iterations have been proven.
Though it was planed to cover only block partially asynchronous iterations, it turned out that the behaviour of partially asynchronous iterations and multiplicative Schwarz iterations is more or less the same. After this observation the focus of this thesis tends more to the Schwarz iterations and, in fact, there are now more results for Schwarz iterations than for the asynchronous ones.
That the number of results for Schwarz iterations outweighs the results for asynchronous iterations is due to the introduced technique. This technique is based on the non-zero patterns of matrices and those patterns are easy to be analysed for Schwarz iterations.
Since all results presented here are more or less based on the same idea, the theory is very homogeneous and that makes it hopefully a useful contribution to the convergence theory of the above mentioned methods.
Anyway, this work is far of being complete and, as usual, just a beginning.

I would like to thank all the members of the Applied Computer Science Group for their colleagueship and especially Prof. Dr. Andreas Frommer who gave me the freedom to act out my curiosity on the topics discussed in this thesis. His comments and suggestions were always a driving force.
Many thanks to Prof. Dr. Daniel B. Szyld who was willing to read this thesis and becoming a member of the exam committee.
Also many thanks to Dipl. Math./Dipl. Ök. Simon Görtz for a lot of fruitful discussions; whatever topic they had.
Finally, many thanks to my partner, Vanessa, who managed to live with me the past four years without becoming a nerve-wreck.

Bochum, May 2006 Stefan Borovac

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Chapter 0

Introduction

This thesis concerns the interaction of two widely known topics in the field of applied mathematics. These are Markov chains (or likewise singular M-matrices) and Schwarz methods.

Markov chains (see, e.g., [9, 20, 62, 65]) as well as singular M-matrices [9, 73] are extensively used, their range of application stretching from stochastic processes over network modeling to the discretisation of partial differential equations.

Schwarz methods are widely used as preconditioners for the numerical solution of partial differential equations and can be classified as domain decomposition methods; see, e.g., [57, 63, 71].

Both topics are brought together for the solution of a consistent square linear system of the form

$$x = Bx + b, \tag{0.1}$$

where a nonnegative matrix B is considered. The matrix B is either column stochastic, i.e. it represents a Markov chain, or $I - B$ is a singular M-matrix; see, e.g., [15, 40, 48]. If

$$(I - B)z = 0, \quad z > 0 \tag{0.2}$$

for some positive vector z, then (0.1) can be regarded as the same problem, whether B is a Markov chain or $I - B$ is a singular M-matrix.

It is a known technique to solve (0.1) using block Jacobi or block Gauss-Seidel iterations; see, e.g., [65]. Schwarz methods can be seen as a generalisation of these techniques. They naturally occur in two setups; see, e.g., [6, 30, 63, 71]. First, the additive Schwarz iterations, which generalise the block Jacobi iteration. Second, the multiplicative Schwarz iterations which extend the block Gauss-Seidel iteration. The generalisation comes mainly from the decomposition which now allows the blocks to have overlap.

Both techniques, the additive and multiplicative Schwarz iteration, have their own advantages. Additive Schwarz methods are easy to parallelise. Multiplicative Schwarz usually converges more rapidly but cannot be parallelised in such an easy way. However, it can be partly parallelised using block asynchronous iterations; see, e.g, [2, 10, 39].

In this work, solutions of (0.1) with the restriction (0.2) using additive and (standard and asynchronous) multiplicative Schwarz iterations are considered.

This is done for several different block updates including the standard (one-level) update (see, e.g., [6, 48]), the two-stage update (see, e.g., [14]), and an update derived from the power iteration. Additionally, relaxed versions of the three block updates are considered. The Schwarz iterations are presented in an algebraic setup which is used in a variety of articles, e.g., [6, 15, 40, 48].

One main goal of this thesis lies in the detailed analysis of the structure of a nonnegative matrix B from (0.1) which satisfies (0.2). It turns out that B possesses some basic pattern which can be exploited to construct convergent Schwarz iterations.

The first investigations are made in the case that $I-B$ has a one dimensional null space. Then B has exactly one strongly connected class and the graph of B^T contains necessarily a spanning tree which is rooted in this strongly connected class. If the null space of B has dimension $r > 1$, then B has exactly r strongly connected classes and the graph of B^T contains r trees, each of which is rooted in another strongly connected class. The union of this trees can be interpreted as a spanning forest which is the natural generalisation to the spanning tree.

Based on these observations, operators for Schwarz iterations can be constructed which have the same null space as $I - B$ (consistency) and are (semi)convergent.

First of all, this is done for non-relaxed block updates. Then it turns out that multiplicative Schwarz iterations do not converge unconditionally; one needs a compatibility condition on how to choose the blocks and the order in which these blocks are updated. The blocks and the order come from the spanning tree within the graph of B^T.

Then it is shown that relaxed block updates eliminate the restrictions of the non-relaxed case and multiplicative Schwarz as well as additive Schwarz can be carried out with nearly no restrictions. Additionally, applications to block asynchronous iterations become possible.

Based on the results for the case that $I-B$ has a one dimensional null space, some extensions are made in the case of higher dimensional null spaces.

Altogether, several new results for additive and multiplicative Schwarz iterations as well as asynchronous iterations are achieved. In contrast to the known theory on this topics (see, e.g., [2, 10, 13, 15, 39, 40, 44, 48, 55, 56]), the theory presented here is completely uniform and based only on the structure which is either a spanning tree or a spanning forest within the graph of B^T.

As a lot of theoretical aspects are covered in here, the thesis is divided into seven chapters.

Chapter 1 introduces some known basics and notation in matrix theory.

Then the two model problems for nonnegative matrices B from (0.1) which satisfy (0.2) are stated (Section 2.1). They are distinguished by the dimension of the null space of $I - B$ which is either one or larger. After this, the necessity and sufficiency of the above mentioned structure is proven for both model problems (Sections 2.2 and 2.3).

The additive and multiplicative Schwarz iterations including all block updates are introduced in Sections 3.2 and 3.3 of Chapter 3. Beside this, the concept of partially asynchronous iterations for linear systems is discussed and their application to the block updates is explained (Section 3.4). After this, the operators which occur within the iterations are analysed in Section 3.5. Based on the analysis, the problems which one must try to overcome for Schwarz methods and partially asynchronous iterations are discussed in Section 3.6.

According to the problems revealed in Chapter 3, a graph based approach is motivated in Section 4.1 of Chapter 4. The approach is restricted to the case that the null space of $I - B$ is one dimensional.

This motivation is followed by a few known theorems which provide a framework for the convergence of inhomogeneous Markov chains (Section 4.2). Then, in Section 4.3, the basic idea is presented, which is based on the structure revealed in Chapter 2. As the first idea is only applicable to single row updates, the generalisation to block updates follows in Section 4.4. With this results, consistent (semi)convergent multiplicative Schwarz operators are constructible. Equipped with the convergence framework and the constructed operators, the first new results for multiplicative Schwarz iterations for the classical one-level update and the two-stage iteration are proved in Section 4.5.

The behaviour of relaxed block updates is examined in Section 4.6. It turns out that relaxation transforms the problem of consistency and (semi)convergence from a multiplicative to an additive one (in a graph theoretical sense). Additionally, some conditions which are needed to guarantee consistency and (semi)convergence are not needed anymore. This leads to some new results for relaxed Schwarz iterations.

Furthermore, the analysis of additive and relaxed multiplicative Schwarz iterations can be carried out in parallel, which immediately delivers new results for additive Schwarz iterations for both, one-level and relaxed two-stage block updates (Section 4.7).

Finally, an application to partially asynchronous iterations becomes possible in the case of relaxed one-level and relaxed two-stage block updates (Section 4.8). The presented results are all new.

As the theory in Chapter 4 concerns the case that the null space of $I - B$ is one dimensional, the more general case is discussed in Chapter 5. There, new generalised results for the one-level update are derived but the convergence for the two-stage update is only achieved for stationary iterations (Sections 5.1). The same holds for relaxed block updates for multiplicative and additive Schwarz iterations (Sections 5.2 and 5.3). Finally, some trivial extensions are discussed in Section 5.4. Additional results for asynchronous iterations are not presented as the convergence theory from Chapter 4 is not strong enough.

After all results are presented, a detailed comparison with known results and other convergence theories is done in Chapter 6. After a survey of some analytical convergence theories in Section 6.1, the results of multiplicative and additive Schwarz are compared with the latest known results (Section 6.2 and 6.3). The comparison of the results for the partially asynchronous iterations are done in a very detailed manner in Section 6.4. The chapter ends with a discussion of other graph based results (Section 6.5).

Chapter 7 summarises the results and, additionally, further questions and open problems are presented.

Chapter 1

Preliminaries

In this chapter all objects of interest which frequently appear in this thesis will be introduced. If not stated otherwise everything being discussed below can be found in [9]. Further background material can be found in, e.g., [43].

1.1 Basic notation and definitions

The identity operator on $\mathbb{R}^n$ is denoted by I_n or simply I, the null operator by 0_n or 0. The vector e is always the vector of all ones, i.e. $(1, 1, \ldots, 1)^T \in \mathbb{R}^n$.

Indexing

For a given $A \in \mathbb{R}^{m\times n}$ the elements of A are denoted by a_{ij} or $a_{i,j}$ for $1 \le i \le m$ and $1 \le j \le n$.

For a given matrix $A \in \mathbb{R}^{n\times n}$ and two subsets $V_1, V_2 \subset \{1, \ldots, n\}$ the matrix $A[V_1, V_2]$ consists of the entries a_{ij} satisfying $i \in V_1$ and $j \in V_2$. If $V_1 = V_2$, then the notation $A[V_1]$ is used, and $A[V_1]$ is also said to be the *principal minor* of A. If the sets V_1 and V_2 can be uniquely identified by their indices, the notation $A[1, 2]$ or A_{12} is used.

If $A \in \mathbb{R}^{n\times m}$ and $B \in \mathbb{R}^{n\times k}$, then $[A \mid B] \in \mathbb{R}^{n\times(m+k)}$ represents the concatenated matrix.

Nonnegativity

A matrix $A \in \mathbb{R}^{m\times n}$ is called *nonnegative* if all $a_{ij} \ge 0$; this is denoted by $A \ge 0$. If all $a_{ij} > 0$, the matrix A is called *positive*. In the latter case one writes $A > 0$. For some compatible B, the relation $A \ge B$ holds if and only if $A - B \ge 0$ and similarly $A > B$ if and only if $A - B > 0$.

The concept carries naturally over to vectors.

Spectra

For $A \in \mathbb{R}^{n\times n}$ the vector spaces $\mathcal{R}(A)$ and $\mathcal{N}(A)$ are the range and the null space, respectively. The set $\sigma(A) \subset \mathbb{C}$ denotes the set of all eigenvalues of A and $\rho(A) \in \mathbb{R}$ the spectral radius, i.e.

$$\rho(A) := \max\{|\lambda| : \lambda \in \sigma(A)\}.$$

The scalar $\gamma(A)$ is defined to be

$$\gamma(A) := \max\{|\lambda| : \lambda \in \sigma(A) \text{ and } \lambda \neq \rho(A)\},$$

and plays an important role if $\rho(A) = 1$. The index of an eigenvalue $\lambda \in \mathbb{C}$ is defined as

$$\operatorname{ind}_\lambda(A) := \min\{k \in \mathbb{N}_0 : \operatorname{rank}((\lambda I - A)^k) = \operatorname{rank}((\lambda I - A)^{k+1})\}.$$

The ranks are equal (cf. [9, 73]) if and only if

$$\mathcal{R}((\lambda I - A)^k) \oplus \mathcal{N}((\lambda I - A)^k) = \mathbb{R}^n. \tag{1.1.1}$$

M-matrices and splittings

Let $A \in \mathbb{R}^{n\times n}$. A is an *M-matrix* if $A = \beta I - B,\ B \geq 0$, and $\rho(B) \leq \beta$. If $\rho(B) = \beta$, the matrix A is singular, otherwise nonsingular; see [9, 73].

The following theorem is well known; see, e.g., [9, 73].

Theorem 1.1 *$A \in \mathbb{R}^{n\times n}$ is a nonsingular M-matrix if and only if $A^{-1} \geq 0$.*

A pair of matrices (M, N) is called a *splitting* of A if $A = M - N$ and M^{-1} exists. A splitting is called

- *weak* (or *nonnegative*) if $M^{-1}N \geq 0$,
- *weak-regular* [53] if $M^{-1}N \geq 0$ and $M^{-1} \geq 0$,
- *regular* [73] if $M^{-1} \geq 0$ and $N \geq 0$, or
- *M-splitting* [61] if M is an M-matrix and $N \geq 0$.

The following theorem can be helpful to create a splitting based on principal minors (cf. [9]).

Theorem 1.2 *Let $A \in \mathbb{R}^{n\times n}$ be a nonsingular M-matrix, then each principal minor is a nonsingular M-matrix.*

Moreover there holds:

Theorem 1.3 *Let $A \in \mathbb{R}^{n\times n}$ be a nonsingular M-matrix, then there exists a (weak) regular splitting $A = M - N$ such that $\rho(M^{-1}N) < 1$. Moreover, each (weak) regular splitting $A = M - N$ satisfies $\rho(M^{-1}N) < 1$.*

Norms and nonexpansivity

The only norm being frequently used here is the *weighted max norm* which is defined as

$$\|x\|_v = \max_{i=1,\ldots,n} \frac{|x_i|}{v_i},$$

for a positive vector $v \in \mathbb{R}^n$; see, e.g., [6]. Therefore, $\|\cdot\|_e = \|\cdot\|_\infty$. The associated matrix norm for given $A \in \mathbb{R}^{n\times n}$ is

$$\|A\|_v := \sup_{\|x\|_v=1} \|Ax\|_v = \max_{i=1,\ldots,n} \frac{(|A|v)_i}{v_i}.$$

A matrix $A \in \mathbb{R}^{n\times n}$ is said to be *nonexpansive* [25] if there exists a norm $\|\cdot\|$ such that

$$\|Ax\| \le \|x\|,$$

for all $x \in \mathbb{R}^n$ or, equivalently, $\|A\| \le 1$. Furthermore, A is called *paracontractive* [25] if there exists a norm such that for all $x \in \mathbb{R}^n$,

$$\|Ax\| < \|x\| \Leftrightarrow Ax \ne x.$$

1.2 Graphs

The basic definitions are taken from [61]. The definitions related to the classification of vertices are due to [9].

Basics

A (directed) *graph* Γ is a pair (V, E) with $E \subseteq V \times V$. Elements of V are called *vertices*, while elements of E are called *edges*.

If the vertex set is $V = \{1, \ldots, n\}$ then sometimes the graph Γ is directly identified with its edge set E.

A *path* from $u \in V$ to $w \in V$ of length k is a sequence $(v_0, \ldots, v_k)$ in V such that $v_0 = u$, $v_k = w$, and $(v_j, v_{j+1}) \in E$ for $j = 0, \ldots, k-1$.

If there exists a nonempty path from u to w one says that u has *access* to w in Γ. This is denoted by $u \to w$. If $u = w$, the path is called *closed.* If all vertices of a closed path are pairwise distinct, the path is called a *circuit.*

A graph is called a *forest* if it does not contain a closed path. A forest is a *tree*, if there is one vertex $v_0 \in V$ which has access to all other vertices. This vertex is called the *root* of the tree.

The union $\Gamma_1 \cup \Gamma_2$ of two graphs Γ_1 and Γ_2 with identical vertex sets V is defined to be the union of the edge sets.

If Γ_1 and Γ_2 are graphs with identical vertex sets V, the product graph $\Gamma_1\Gamma_2 = (V, E)$ is defined by $(u, w) \in E$ if there is a $v \in V$ such that $(u, v) \in E_1$ and $(v, w) \in E_2$. One writes $\Gamma^2 = \Gamma\Gamma$, etc. If V is a given vertex set, the graph $\Delta = (V, E)$ with $E := \{(v, v) : v \in V\}$ denotes the diagonal graph of V.

The *reflexive transitive closure* $\overline{\Gamma}$ of a graph Γ is defined to be

$$\overline{\Gamma} = \Delta \cup \Gamma \cup \Gamma^2 \cup \ldots.$$

Therefore, a $u \in V$ has access to $w \in V$ if and only if (u, w) is an edge of $\overline{\Gamma}$.

The graph of a matrix

For $A \in \mathbb{R}^{n\times n}$, the (directed) graph of A is the pair $\Gamma(A) = (V(A), E(A))$ with $V(A) = \{1, \ldots, n\}$ and $E(A) = \{(i, j) : a_{ij} \neq 0\}$. In this case, the vertices are also called *states* or *indices*.

The first three results in the following lemma are an immediate consequence of the above definitions.

Lemma 1.1 *Let $A, B \in \mathbb{R}^{n\times n}$ be nonnegative, and let $\alpha \in \mathbb{R}$ be positive. Then*

$$\begin{aligned} \Gamma(\alpha A) &= \Gamma(A), \\ \Gamma(A + B) &= \Gamma(A) \cup \Gamma(B), \\ \Gamma(AB) &= \Gamma(A)\Gamma(B). \end{aligned}$$

If A is a nonsingular M-matrix then $\Gamma(A^{-1}) = \overline{\Gamma(A)}$.

Proof: See [61].

Classification of vertices

Now the notion of irreducibility will be introduced [9, 62, 73] and some basic definitions for the classification of indices/vertices are given.

Let a graph Γ be given. If some $u \in V$ has access to $v \in V$ and vice versa, then u *communicates* with v which is denoted by $u \leftrightarrow v$. The relation induced by communication is an equivalence relation. The classes are the strongly connected components (subgraphs) where in each class every vertex communicates with every other vertex. A graph is called *strongly connected* if there is only one equivalence class with respect to the communication relation.

If $A \in \mathbb{R}^{n\times n}$ is some matrix and $\Gamma(A)$ the corresponding graph, then A is called *irreducible* if $\Gamma(A)$ is strongly connected. Otherwise A is called *reducible.*

The concept of irreducibility leads to the following theorem.

Theorem 1.4 *Let $A \in \mathbb{R}^{n\times n}$ be an irreducible M-matrix.*

1) *If A is nonsingular, then $A^{-1} > 0$.*

2) *If A is singular, then each proper principal minor of A is a nonsingular M-matrix.*

Proof: For 1) see [9] or apply Lemma 1.1. For 2) see [9].

For an $A \in \mathbb{R}^{n\times n}$, the *classes* of A are the equivalence classes of $\Gamma(A)$ with respect to the communication relation. A class α has access to a class β if there is an index $i \in \alpha$ and a $j \in \beta$ such that i has access to j. A class is called *final* if it has no access to another class.

If α is a class of A, then $A[\alpha]$ is the submatrix of A based on the indices $j \in \alpha$. With this, a class α is called *basic* if $\rho(A[\alpha]) = \rho(A)$, otherwise α is called *nonbasic.*

1.3 Singularity and convergence

Let $A \in \mathbb{R}^{n\times n}$ be given. It is standard to say that A is convergent if $\lim_{k\longrightarrow\infty} A^k = 0$ and this holds if and only if $\rho(A) < 1$; see [9, 73]. Since the case $\rho(A) = 1$ is to be discussed here, the notion of convergence becomes more general.

Semiconvergence

The matrix A is called *semiconvergent* if $\lim_{k\longrightarrow\infty} A^k = A^*$ exists and additionally $A^* \neq 0$.

The property $A^* \neq 0$ is not necessary but separates convergent from semiconvergent matrices. The following lemma is well known.

Lemma 1.2 *$A \in \mathbb{R}^{n\times n}$ is semiconvergent if and only if A satisfies the following three conditions:*

1) $1 \in \sigma(A)$,

2) $\gamma(A) < 1$, *and*

3) $\mathcal{R}(I - A) \oplus \mathcal{N}(I - A) = \mathrm{I\!R}^n$.

Furthermore, if $A^* = \lim_{k\longrightarrow\infty} A^k$ *then* A^* *is a projection onto* $\mathcal{N}(I - A)$ *along* $\mathcal{R}(I - A)$.

Proof: See [9].

Remark 1.1 *1) Condition 3) of the above lemma is an algebraic formulation, which states that there are no generalised eigenvectors to the eigenvalue 1. This is equivalent to* $\mathrm{ind}_1(A) = 1$ *(cf. (1.1.1)).*

2) By definition, $P \in \mathrm{I\!R}^{n\times n}$ *is a projection if* $P^2 = P$*. For any projection* P *one has* $\mathcal{R}(P) \oplus \mathcal{N}(P) = \mathrm{I\!R}^n$ *and* $I - P$ *is also a projection. The matrix* P *is called a projection onto* $\mathcal{R}(P)$ *along* $\mathcal{N}(P)$*. For more details see [5, 9, 17].*

Spectral decompositions

In contrast to convergent matrices, semiconvergent matrices have an additive decomposition which splits the matrix into a projection and a convergent part. To understand this, the concept of the group generalised inverse is needed.

Let $A \in \mathrm{I\!R}^{n\times n}$ be given. A matrix $X \in \mathrm{I\!R}^{n\times n}$ is the *group inverse* of A (see [5, 9, 17]) if

1) $AXA = A$,

2) $XAX = X$,

3) $AX = XA$.

The group inverse of A is denoted by $A^{\#}$ and is unique if it exists. The existence is equivalent to $\mathcal{R}(A) \oplus \mathcal{N}(A) = \mathrm{I\!R}^n$ or likewise $\mathrm{ind}_1(A) = 1$. Furthermore, the mapping XA is a projection onto $\mathcal{R}(A)$ along $\mathcal{N}(X)$ and AX is a projection onto $\mathcal{R}(X)$ along $\mathcal{N}(A)$. But $AX = XA$, and therefore $\mathcal{R}(A) = \mathcal{R}(X)$.

The conditions for the existence of the group inverse give the final relationship to semiconvergence by the following lemma.

Lemma 1.3 *A matrix* $A \in \mathrm{I\!R}^{n\times n}$ *is semiconvergent if and only if there exists a pair of matrices* (P, Q)*, such that*

1) $A = P + Q$,

2) P is a projection onto $\mathcal{N}(I-A)$,

3) $PQ = QP = 0$, and

4) $\rho(Q) < 1$.

In this case, $\lim_{k\longrightarrow\infty} A^k = P$ with $P = I - (I-A)(I-A)^{\#}$ and $Q = A(I-A)(I-A)^{\#}$. Additionally, $\rho(Q) = \gamma(A)$.

Proof: See [44].

The above lemma allows statements for iterations of the form

$$x^{k+1} = Ax^k + b, \quad k = 0, 1, 2, \ldots,$$

where x^0 is given, $b \in \mathcal{R}(I-A)$, and $A = P + Q$ is a decomposition as above. In this case (see [9])

$$\lim_{k\longrightarrow\infty} x^k = (I-A)^{\#}b + \left(I - (I-A)(I-A)^{\#}\right)x^0 = (I-A)^{\#}b + Px^0.$$

For semiconvergent $A \in \mathbb{R}^{n\times n}$, a pair (P, Q) that satisfies the above conditions is called a *spectral decomposition*. Spectral decompositions have been successfully used by several authors, e.g. [7, 14, 15, 35, 37, 40, 41, 42].

It should be mentioned that for any A with $\mathrm{ind}_1(A) = 1$, there always exists a decomposition

$$A = P + \tilde{Q},$$

where P is a projection onto $\mathcal{N}(I-A)$ and $P\tilde{Q} = \tilde{Q}P = 0$. But in this general case $\rho(\tilde{Q}) < 1$ need not hold since $\gamma(A) < 1$ is not guaranteed.

This section ends with a few results on spectral decomposable matrices.

Lemma 1.4 *Let $B \in \mathbb{R}^{n\times n}$ be given and let (P, Q) be a spectral decomposition of B. Then:*

1) $\mathcal{R}(Q) \subset \mathcal{N}(P)$ and $\mathcal{R}(P) \subset \mathcal{N}(Q)$.

2) $Bv = 0$ for some $0 \neq v \in \mathbb{R}^n$ if and only if $Pv = Qv = 0$.

3) Equality holds in 1) if and only if B is nonsingular.

4) $I - B = (I-P)(I-Q) = (I-Q)(I-P)$.

5) $(I-B)^{\#} = (I-P)(I-Q)^{-1} = (I-Q)^{-1}(I-P) = (I-Q)^{-1} - P$.

Proof: Easy, by direct calculation. □

For the next lemmata, the restriction of an operator $B \in \mathbb{R}^{n\times n}$ to an invariant subspace X is denoted by $B_{|X}$. If $\mathcal{B}$ is a basis of X, then the corresponding coordinate representation is denoted by $(B_{|X})_{\mathcal{B}}$.

Lemma 1.5 *Let $B \in \mathrm{I\!R}^{n\times n}$ be given and let (P,Q) be a spectral decomposition of B. Then:*

1) $P \geq 0$, *if* $B \geq 0$.

2) $\|P\|_z = \|B\|_z = 1$, *if* $Bz = z$ *for a positive* $z \in \mathrm{I\!R}^n$.

3) $\mathcal{R}(P)$ *and* $\mathcal{N}(P)$ *are invariant subspaces of* $I - Q$.

4) $(I-Q)_{|\mathcal{R}(P)} = I_{|\mathcal{R}(P)}$.

Proof: Assertion 1) follows immediately from $0 \leq \lim_{k\longrightarrow\infty} B^k = P$. If $Bz = z$, then $Pz = z$ and $Qz = 0$. Hence, 2) follows. If $x \in \mathcal{N}(P)$, then $(I-Q)x \in \mathcal{N}(P)$ since $PQ = 0$. If $x \in \mathcal{R}(P)$, then $Px = x$ and $Qx = 0$, therefore $(I-Q)x \in \mathcal{R}(P)$. Hence, 3) is proven and 4) is an immediate consequence. □

Lemma 1.6 *Let $B \in \mathrm{I\!R}^{n\times n}$ be nonnegative. Assume there exists a spectral decomposition (P,Q) of B and let the subset $\mathcal{B} = \{e_{l_1},\ldots,e_{l_k}\}$ of the canonical basis of $\mathrm{I\!R}^n$ be a basis of $\mathcal{N}(P)$. Then, with $V = \{l_1,\ldots,l_k\}$:*

1) $(Q_{|\mathcal{N}(P)})_{\mathcal{B}} = Q[V] \geq 0$.

2) $((I-Q)_{|\mathcal{N}(P)})_{\mathcal{B}} = (I-Q)[V]$ *is a nonsingular M-matrix.*

Proof: Assume w.l.o.g. that $V = \{1,\ldots,k\}$. Let Q be partitioned w.r.t. V and its complement, then

$$Q = \begin{pmatrix} Q_{11} & Q_{22} \\ Q_{21} & Q_{22} \end{pmatrix}.$$

Part 3) of Lemma 1.5 implies $Q\mathcal{N}(P) \subset \mathcal{N}(P)$ and therefore $Q_{21} = 0$ and $Q_{11} \geq 0$. Additionally, $\rho(Q_{11}) < 1$ follows from $\rho(Q) < 1$. Since $\mathcal{B} = \{e_1,\ldots,e_k\}$, the restriction of Q to $\mathcal{N}(P)$ w.r.t. $\mathcal{B}$ is necessarily $Q[V] = Q_{11}$ and this proves 1). Part 2) is obvious since $\rho(Q[V]) = \rho(Q_{11}) < 1$ and $I[V] = I_{|\mathcal{N}(P)} = (I_{|\mathcal{N}(P)})_{\mathcal{B}}$. □

Remark 1.2 *The restriction of Lemma 1.6 to a subset of the canonical base has been done for simplicity. There might exist a generalisation of part 2) to arbitrary bases of $\mathcal{N}(P)$ with respect to the invariant cone of P, but cones will not be further considered here; for cones see, e.g., [8, 9, 58].*

1.4 Spectra and eigenvectors of nonnegative matrices

There is a large bibliography on the theory of spectra of nonnegative matrices; see [9, 62, 73] and the references therein. Here some basics are recapitulated which will be helpful in the next chapters.

The first fact presented here deals with the interaction of matrix norms and the index of a maximum modulus eigenvalue.

Theorem 1.5 *Let $B \in \mathbb{C}^{n\times n}$ be given. The following two statements are equivalent:*

1) $\mathrm{ind}_\lambda(B) = 1$ *for all* $\lambda \in \sigma(B)$ *with* $|\lambda| = \rho(B)$.

2) There exists a matrix norm $\|\cdot\|$ *with* $\|B\| = \rho(B)$.

Proof: See [47] or [64].

If $B \in \mathbb{R}^{n\times n}$ is a nonnegative matrix with a positive right eigenvector v with respect to $\rho(B)$, then there holds for the weighted max norm

$$\|B\|_v = \rho(B).$$

For such matrices B, each eigenvalue λ with $|\lambda| = \rho(B)$ has the index one.

Another algebraic ansatz which leads to the same result is based on M-matrices having the so called property **c** (for details see [9, 49, 54]). This property is defined for M-matrices $A \in \mathbb{R}^{n\times n}$ and is equivalent to $\mathrm{ind}_0(A) \le 1$. One consequence is the following lemma which might be a good alternative to Theorem 1.5.

Lemma 1.7 *Let $A \in \mathbb{R}^{n\times n}$ be a singular M-matrix. Suppose there exists a vector $v > 0$ such that $Av \ge 0$, then A has the property* **c** *which implies* $\mathrm{ind}_0(A) = 1$.

Proof: See [49].

The link between Lemma 1.7 and Theorem 1.5 is clear since $A = \rho(B)I - B$ for some nonnegative B and $\mathrm{ind}_0(A) = 1$ if and only if $\mathrm{ind}_{\rho(B)}(B) = 1$.

If $A = I - B$ is an M-matrix with $B \ge 0, \rho(B) = 1$, and there exists some positive $v \in \mathbb{R}^n$ satisfying $Bv = v$, then both, Lemma 1.7 and Theorem 1.5 apply and at least the conditions 1) and 3) of Lemma 1.2 are fulfilled. It should therefore be interesting under which conditions there exists a positive eigenvector corresponding to the spectral radius. One answer for the irreducible case (cf. Section 1.2) is given by the famous Theorem of Perron and Frobenius.

Theorem 1.6 *Let $B \in \mathbb{R}^{n\times n}$ be nonnegative and irreducible. Then:*

1) There exists a $\lambda \in \sigma(B)$ *such that* $\lambda = \rho(B)$.

2) There exists a positive eigenvector z corresponding to $\rho(B)$.

3) $\rho(B)$ *is a simple eigenvalue.*

Proof: See [9, 62, 73].

If the above B is reducible, then a positive vector must exist if the basic classes of B (cf. Section 1.2) behave in a certain manner.

Theorem 1.7 *Let $B \in \mathbb{R}^{n\times n}$ be nonnegative, then there is a positive eigenvector z to the spectral radius if and only if the basic classes of B are exactly its final ones. The latter means that if B has $r > 0$ final classes, then there is a permutation matrix Π such that*

$$\Pi B \Pi^T = \begin{pmatrix} D_1 & 0 & \dots & 0 & 0 \\ 0 & D_2 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & D_r & 0 \\ E_1 & E_2 & \dots & E_r & F \end{pmatrix} \tag{1.4.1}$$

where each D_i is square, irreducible, and its indices belong to a final class. Additionally, $\rho(D_i) = \rho(B)$ and $\rho(F) < \rho(B)$.

Proof: See [9].

If $B \in \mathbb{R}^{n\times n}$, $B \geq 0$ does not meet the assumption of Theorem 1.6 or 1.7, then part 1) of Theorem 1.6 still holds, but the eigenvector v needs not be positive. Anyway, it can be assumed nonnegative as Rothblum has proven.

Theorem 1.8 *Let $B \in \mathbb{R}^{n\times n}$ be nonnegative and let $\mathcal{N}((\rho(B)I - B)^k)$ with $k = \mathrm{ind}_{\rho(B)}(B)$ be the* algebraic eigenspace. *Assume that B has m basic classes $\alpha_1, \dots, \alpha_m$, then the algebraic eigenspace contains nonnegative vectors $v^{(1)}, \dots, v^{(m)}$, such that $v_j^{(i)} > 0$ if and only if the index j has access to α_i in the corresponding graph $\Gamma(B)$. Furthermore, any such collection is a basis of the algebraic eigenspace.*

Proof: See [9, 60].

Finally, consider other eigenvalues whose modulus is equal to $\rho(B)$. The following theorems provide informations about their position in the complex plane and their eigenvectors.

Theorem 1.9 *Let $B \in \mathbb{R}^{n\times n}$, $B \geq 0$. Assume that $\rho(B) = 1$ and there exists a positive eigenvector z satisfying $Bz = z$. Then every other eigenvalue λ with $|\lambda| = 1$ is a root of unity of degree at most n.*

Proof: See [3].

Theorem 1.10 *Let $B \in \mathbb{R}^{n\times n}$ be nonnegative. Suppose there exists an eigenvalue $\lambda \in \sigma(B)$ satisfying $\lambda \in \mathbb{C} \setminus \mathbb{R}_+$ and a corresponding eigenvector $v = y + iz$.*

Then $\operatorname{span}\{y, z\} \cap \mathbb{R}^n_+ = \{0\}$ *and if* $\lambda \notin \mathbb{R}$*, then* y *and* z *are linearly independent.*

Proof: See [3].

For a given nonnegative $B \in \mathbb{R}^{n \times n}$ with $\rho(B) = 1$ there are some simple results which guarantee that there are no complex eigenvalues λ such that $|\lambda| = 1$, i.e. $\gamma(B) < 1$.

Theorem 1.11 *Let* $B \in \mathbb{R}^{n \times n}$ *be nonnegative and suppose* $\rho(B) = 1$*.*

1) *If* B *is irreducible and has at least one positive diagonal entry, then* $\gamma(B) < 1$*.*
2) *If* B *has a positive diagonal, then* $\gamma(B) < 1$*.*
3) *If* $B_\alpha = (1-\alpha)I + \alpha B$ *with* $\alpha \in (0, 1)$*, then* $\gamma(B_\alpha) < 1$ *and* $\mathcal{N}(I - B_\alpha) = \mathcal{N}(I - B)$*.*

Proof: For 1) and 3) see [9, 73]. For 2) see [1].

Remark 1.3 *If a nonnegative* $B \in \mathbb{R}^{n \times n}$ *has a positive eigenvector* z *corresponding to* $\rho(B)$*, then this eigenvector will also be called the* Perron vector *of* B*.*

Chapter 2

The problem classes

This chapter introduces the model problems and the singular linear systems to be dealt with in this thesis. First, there will be two problem classes defined, namely *MP* and *GMP*. The class *GMP* can be regarded as an extension of *MP* because the solution of *GMP* will be partly generated from the solution of *MP* in Chapter 4. Finally, some subclasses of M-matrices and Markov chains will be defined which are derived from the model problems. Those matrices have a structure which will be helpful in Chapters 4 and 5 to solve *MP* and *GMP* with iteration methods provided in Chapter 3.

2.1 The model problems *MP* and *GMP*

Assume that for $A \in \mathbb{R}^{n\times n}$ there holds

$$\begin{aligned} A &= I - B, \\ B &\geq 0, \\ \rho(B) &= 1. \end{aligned}$$

For a given $b \in \mathcal{R}(A)$, the problem to be solved is to find a solution $x^* \in \mathbb{R}^n$ of

$$Ax = b \Leftrightarrow x = Bx + b, \tag{2.1.1}$$

for $x \in \mathbb{R}^n$.

If B fulfils the assumptions of Theorem 1.6, i.e. B is irreducible, there is a positive vector z such that $Az = 0$ and $\dim \mathcal{N}(A) = 1$. The latter implies that z is a basis of $\mathcal{N}(A)$. Since such a situation is of further interest but still too special, assume that there exists a positive vector $z \in \mathbb{R}^n$ such that

$$z \in \mathcal{N}(A). \tag{2.1.2}$$

Condition (2.1.2) is fundamental because it implies some basic structure of the non-zero pattern of B which will be analysed in the following sections. Moreover, this structure exists whether $\dim \mathcal{N}(A) = 1$ (cf. Section 2.2) or not (cf. Section 2.3); but if $\dim \mathcal{N}(A) > 1$ it becomes more general.

The focus of this thesis is to compute a solution of (2.1.1) which satisfies Condition (2.1.2) by iteration.

If $\dim \mathcal{N}(A) = 1$, then the construction of the iteration operators is easier to understand and there will be more convergence results. Thus, the first model problem is given as follows.

Definition 2.1 *Let $A \in \mathbb{R}^{n\times n}$ be given such that $A = I - B$, $B \geq 0$, and $\rho(B) = 1$. Assume $\mathcal{N}(A) = \operatorname{span}\{z\}$, $z > 0$, and let $b \in \mathcal{R}(A)$. Then the model problem* MP *is to find a solution x^* of*

$$Ax = b,\ x \in \mathbb{R}^n, \tag{2.1.3}$$

or equivalently to the (inhomogeneous) fixed point problem

$$x = Bx + b,\ x \in \mathbb{R}^n. \tag{2.1.4}$$

This problem is of course not just a theoretical exercise, but it arises in many applications.

Example 1:

This example is taken from [9] and can also be found in [16].

Let $R := \{x \in \mathbb{R}^2 : a \leq x_1 \leq b, c \leq x_2 \leq d\}$ for finite $a, b, c, d \in \mathbb{R}$. The problem is to find an approximation to the solution of the continuous function $u(x_1, x_2)$ satisfying Poisson's equation

$$\frac{\partial^2 u}{\partial^2 x_1} + \frac{\partial^2 u}{\partial^2 x_2} = -f(x_1, x_2),$$

where f is a given continuous function on R. The periodic boundary conditions are

$$\begin{aligned} u(a, x_2) &= u(b, x_2), \quad c \leq x_2 \leq d, \\ u(x_1, c) &= u(x_1, d), \quad a \leq x_1 \leq b. \end{aligned}$$

A standard discretisation, using a 5-point star and a lexicographical ordering of the unknowns, leads to a system $Au = g$ where g is entirely determined by the right hand side f and the boundary conditions. The matrix A becomes

$$A = \begin{pmatrix} D & -I & 0 & \dots & 0 & -I \\ -I & D & -I & \dots & 0 & 0 \\ 0 & -I & D & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & D & -I \\ -I & 0 & 0 & \dots & -I & D \end{pmatrix},$$

with blocks

$$D = \begin{pmatrix} 4 & -1 & 0 & \dots & 0 & -1 \\ -1 & 4 & -1 & \dots & 0 & 0 \\ 0 & -1 & 4 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 4 & -1 \\ -1 & 0 & 0 & \dots & -1 & 4 \end{pmatrix}.$$

This matrix is symmetric and has rank $n-1$ with $\mathcal{N}(A) = \text{span}\{e\}$. It fits therefore the prerequisites of the model problem *MP*.

Example 2:

For this example suppose $b = 0$ and let B be a finite homogeneous Markov chain, i.e. a nonnegative matrix B with $e^T B = e^T$; see, e.g., [9, 62, 65]. If B is irreducible, then the rank of B is again $n-1$ and there exists a positive solution x^* such that $Bx^* = x^*$, which is also known as the *stationary probability distribution*.

More applications can be found in the literature (see, e.g., [65]), but the above ones are very common and therefore presented here.

Remark 2.1 *Note that nonnegative matrices satisfying* $\mathrm{e}^T B = \mathrm{e}^T$ *are also called* column stochastic, *while nonnegative matrices with* $Be = e$ *are called* row stochastic.

What remains to be defined is the model problem in the case $\dim \mathcal{N}(A) = r \geq 1$.

Definition 2.2 *Let* $A \in \mathbb{R}^{n \times n}$ *be given such that* $A = I - B$, $B \geq 0$, *and* $\rho(B) = 1$. *Assume there exists a vector* $z > 0$ *such that* $z \in \mathcal{N}(A)$ *and* $\dim \mathcal{N}(A) = r \geq 1$. *Let* $b \in \mathcal{R}(A)$. *Then the generalised model problem* GMP *is to find a solution* x^* *of*

$$Ax = b, \; x \in \mathbb{R}^n. \tag{2.1.5}$$

For examples concerning *GMP* see, e.g., [65].

2.2 ST- and STM-matrices

The model problem *MP* has been described for M-matrices A having a null space spanned by a positive vector. This property exhibits some internal structure which will be exploited in Chapter 4 to prove some convergence results based on iterative methods which will be introduced in Chapter 3. The class of matrices to be defined in the sequel is actually the class of matrices that fulfil the requirements of Theorem 1.7 in the case there is only one final and basic class. But first some more notation is needed.

A directed graph $\Gamma = (V, E)$ contains a directed *spanning tree* $\mathcal{T} = (V_\mathcal{T}, E_\mathcal{T})$ (see, e.g., [22] and Section 1.2) if

1) $\mathcal{T}$ is a directed tree,
2) $V = V_\mathcal{T}$, and
3) $E \supset E_\mathcal{T}$.

Since all graphs which will appear (and therefore all spanning trees) are directed, the term "directed" will be omitted in the future.

Definition 2.3 *Assume $B \in \mathbb{R}^{n\times n}$. Let $\Gamma(B) = (V(B), E(B))$ be the corresponding graph and let $\Gamma(B^T)$ be the graph of B^T. Then B is called a* GST-matrix *(GST for "general spanning tree") if*

1) $\Gamma(B^T)$ contains a spanning tree $\mathcal{T}_B$, and

2) if the index $i \in V(B)$ is the root of $\mathcal{T}_B$, then i has access to some $j \in V(B)$ via $(i, j) \in \Gamma(B)$.

The above defined index j will be called guard index.

Remark 2.2 *It might happens that i, i.e. the root, communicates with itself via $(i, i) \in \Gamma(B)$.*

The following examples illustrate the above definition.

$$B_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad B_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

For B_1, every index might act as a root, especially the index 4 with guard 1. The root index of B_2 is obviously the index 4 and the index 1 represents the guard. In B_3, the index 4 also acts as a root and is its own guard entry. The spanning trees for the above examples are illustrated in Figure 2.1.

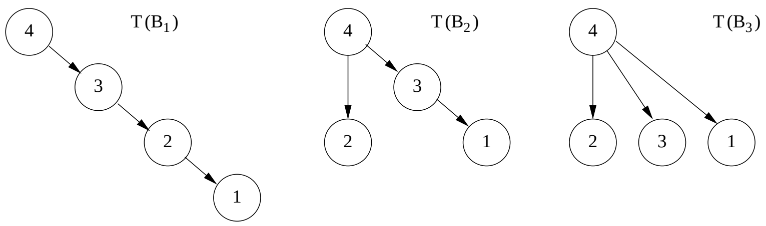

Figure 2.1: Spanning trees of B_1, B_2 and B_3

All examples are somehow minimal because the elimination of any entry destroys the GST property. Clearly, there could exist more than one spanning tree.

The existence of the guard index ensures that there is at least one strongly connected class and the next lemma will show that this class is final (cf. Section 1.2). For the above examples this is illustrated in Figure 2.2.

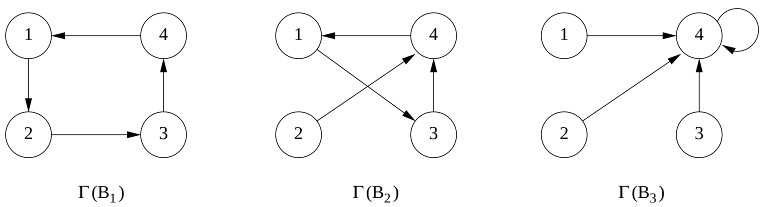

Figure 2.2: Graphs of B_1, B_2 and B_3

Lemma 2.1 *Let B be a GST-matrix, then there is a permutation matrix Π such that*

$$\Pi B \Pi^T = \begin{pmatrix} D & 0 \\ E & F \end{pmatrix}.$$

Furthermore:

1) D is square and irreducible.

2) If B is irreducible, then $\Pi = I$ and $B = D$.

3) If j is the root index of any tree in $\Gamma(B^T)$, then j resides in the index set belonging to D.

Proof: Since there is always at least one final class, $r > 1$ final classes are being assumed. Then there is a permutation matrix Π (see, e.g., [9]) such that

$$\Pi B \Pi^T = \begin{pmatrix} D_1 & 0 & \dots & 0 & 0 \\ 0 & D_2 & \dots & 0 & 0 \\ \vdots & & \ddots & \vdots & \vdots \\ & & & D_r & \\ E_1 & E_2 & \dots & E_r & F \end{pmatrix}.$$

Hence

$$\Pi^T B^T \Pi = \begin{pmatrix} D_1^T & 0 & \dots & 0 & E_1^T \\ 0 & D_2^T & \dots & 0 & E_2^T \\ \vdots & & \ddots & \vdots & \vdots \\ & & & D_r^T & E_r^T \\ 0 & 0 & \dots & 0 & F^T \end{pmatrix},$$

where the D_i are all square, irreducible, and correspond to the final classes. Let j be the root of a spanning tree in $\Gamma(B^T)$. If j is an index belonging to F^T, then j cannot have access to any k belonging to one of the D_i^T in $\Gamma(B^T)$. On the other hand, if j is an index belonging to a D_i^T, then j might have access to elements of F^T, but again F^T has no access to any other D_j^T. In both cases, j cannot be the root of the spanning tree. This is a contradiction, hence $r = 1$ and

$$\Pi B \Pi^T = \begin{pmatrix} D & 0 \\ E & F \end{pmatrix} \tag{2.2.1}$$

or B is irreducible. Furthermore, the root of any tree must reside in the index set belonging to D. □

Now a few corollaries of Lemma 2.1.

Corollary 2.1 *Let $B \in \mathbb{R}^{n \times n}$ be an GST-matrix and let α be the set of indices of the final class of B. Then $\Gamma(B^T)$ contains at least $|\alpha|$ spanning trees with corresponding guard indices, i.e. each index in α might act as a root.*

Proof: Every index $i_0 \in \alpha$ has access to all other indices in $\Gamma(B^T)$. □

The next corollary shows that the set of square irreducible matrices of dimension n is a proper subset of the set of $n \times n$ GST-matrices.

Corollary 2.2 *Let $B \in \mathbb{R}^{n \times n}$ be irreducible. Then $\Gamma(B^T)$ contains at least n spanning trees with corresponding guard indices, i.e. each index can act as a root.*

Proof: Easy, using Corollary 2.1. □

Corollary 2.3 *Let $B \in \mathbb{R}^{n \times n}$ be symmetric, then $\Gamma(B^T)$ contains a spanning tree if and only if B is irreducible.*

Proof: Easy, using again Corollary 2.1 and Corollary 2.2. □

The term symmetric in Corollary 2.3 might be replaced by "symmetric nonzero pattern".

Now consider the following matrix.

$$B = \left(\begin{array}{cccc|cc} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/2 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1/2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right) =: \begin{pmatrix} D & 0 \\ E & F \end{pmatrix}$$

The graph $\Gamma(B^T)$ contains obviously a spanning tree and each index from D can be chosen as a root with corresponding guard index (cf. Corollary 2.1); thus B is a GST-matrix. But B does not have a positive right eigenvector corresponding to $\rho(B) = 1$ since $\rho(D) < 1$. Therefore, Definition 2.3 needs some refinement.

Definition 2.4 *Let $B \in \mathbb{R}^{n \times n}$ be a GST-Matrix such that $B \geq 0$ and $\rho(B) = 1$. Then B is said to be an* ST-matrix *(ST for "spanning tree") if each class of B is final if and only if it is basic.*

Remark 2.3 *If $B \geq 0$ is a GST-matrix whose row sums up to a constant, say $a > 0$, then it is not hard to see that $\frac{1}{a}B$ is an ST-matrix.*

Now the desired existence of a positive right hand fixed point is guaranteed.

Lemma 2.2 *Let $B \in \mathbb{R}^{n \times n}$ be an ST-matrix and let*

$$\Pi B \Pi^T = \begin{pmatrix} D & 0 \\ E & F \end{pmatrix}$$

be the standard form of B from Lemma 2.1, then:

1) $\rho(D) = \rho(B) = 1$.

2) $\rho(F) < 1$.

3) There exists a vector $z > 0$ such that $Bz = z$, $\mathcal{N}(I - B) = \operatorname{span}\{z\}$, and $\operatorname{ind}_1(B) = 1$.

Proof: Assertion 1) is obvious by Definition 2.4. To prove assertion 2) note that there exists a permutation matrix $\tilde{\Pi}$, acting on the indices of F such that

$$\tilde{\Pi}\Pi B \Pi^T \tilde{\Pi}^T = \begin{pmatrix} D & 0 & \dots & 0 \\ E_1 & F_{11} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ E_l & F_{l1} & \dots & F_{ll} \end{pmatrix} =: C,$$

and F_{ii} is either square and irreducible or 1×1 and 0 for all $i = 1, \dots, l$. Now assume $\rho(F) = 1$, then there exists an index i_0 such that $\rho(F_{i_0,i_0}) = 1$. Assume w.l.o.g. that $i_0 = l$, then F_{ll} represent a final class, thus a basic one by Definition 2.4. Consequently, $E_l = 0$ and $F_{lj} = 0$ for $j = 1, \dots, l-1$. But then

$$C = \begin{pmatrix} D & 0 & \dots & 0 \\ E_1 & F_{11} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & F_{ll} \end{pmatrix}.$$

As in the proof of Lemma 2.1, $\Gamma(B^T)$ cannot contain a spanning tree which contradicts the assumptions. Hence, D represents the only final and basic class. To prove the existence of a positive z such that $Bz = z$ let z_0 be any positive fixed point of D. Define inductively

$$z_i := (I - F_{ii})^{-1} \left(E_i z_0 + \sum_{j=1}^{i-1} F_{ij} z_j \right)$$

for $i = 1, \dots, l$. Each z_i is well defined since $\rho(F_{ii}) < 1$. Moreover, $(I - F_{ii})^{-1}$ is a positive matrix (cf. Theorem 1.4) and a simple induction proves each z_i to be positive. Thus, $z^T := (z_0^T, \dots, z_l^T)^T > 0$ and $Bz = z$ follows by a direct calculation. Theorem 1.5 implies $\mathrm{ind}_1(B) = 1$, and thus, the algebraic eigenspace is given by $\mathcal{N}(I - B)$. Since there is only one basic class, the positive vector z is a base for $\mathcal{N}(I - B)$ by Theorem 1.8. □

The property that each class is basic if and only if it is final, follows from Theorem 1.7 because the theorem implies the existence of a positive right hand fixed point. Thus, the following corollary is an easy consequence.

Corollary 2.4 *Consider a nonnegative GST-matrix $B \in \mathbb{R}^{n \times n}$, $\rho(B) = 1$, and a positive vector $z > 0$ such that $Bz = z$, then B is an ST-matrix.*

Proof: As usual let

$$\Pi B \Pi^T = \begin{pmatrix} D & 0 \\ E & F \end{pmatrix},$$

for a proper permutation matrix Π. Then there is a partitioning $(\Pi z)^T = (y_1^T, y_2^T)$ such that $Dy_1 = y_1$, i.e. $\rho(D) = 1$. Thus, D is a final and basic class. The proof of $\rho(F) < 1$ can be done in the same fashion as for Lemma 2.2. □

If every class is final if and only if it is basic, then the existence of a spanning tree is not only sufficient but also necessary.

Theorem 2.1 *Let $B \in \mathbb{R}^{n \times n}$ be nonnegative and $\rho(B) = 1$. Then B is an ST-matrix if and only if the final classes of B are exactly its basic ones and there is only one such class.*

Proof: The sufficient part has been proven by Lemma 2.2. The necessity is shown now.

Again there is a permutation matrix Π such that

$$C := \Pi B \Pi^T = \begin{pmatrix} D & 0 \\ E & F \end{pmatrix} \Rightarrow C^T = \begin{pmatrix} D^T & E^T \\ 0 & F^T \end{pmatrix}. \tag{2.2.2}$$

By Theorem 1.7, D can be assumed irreducible, satisfying $\rho(D) = 1$ and $\rho(F) < 1$. Additionally, there exists a positive vector $z > 0$ such that $Bz = z$. Let the index set $\{1, \ldots, n\}$ be split with respect to (2.2.2) into sets V_1 and V_2 where V_1 corresponds to the indices of D. It remains to show that each index in V_2 is accessible from V_1 in $\Gamma(B^T)$. If this is proven, the existence of a spanning tree is obvious because any index i_0 belonging to D has access to every other index in $\Gamma(B^T)$ (cf. Corollary 2.1).

Assume that there is a nonempty subset W_2 of V_2 containing the indices that are not accessible from V_1. Then there is another permutation matrix $\tilde{\Pi}$ acting on V_2 such that

(2.2.3)

$$\tilde{\Pi} C \tilde{\Pi}^T = \begin{pmatrix} D & 0 & 0 \\ E_1 & F_{11} & F_{12} \\ E_2 & F_{21} & F_{22} \end{pmatrix}, \text{and } \tilde{\Pi}^T C^T \tilde{\Pi} = \begin{pmatrix} D^T & E_1^T & E_2^T \\ 0 & F_{11}^T & F_{21}^T \\ 0 & F_{12}^T & F_{22}^T \end{pmatrix}.$$

Here F_{22} corresponds to the set $W_2 \subset V_2$ of all non-accessible indices, while F_{11} corresponds to $W_1 = V_2 \setminus W_2$.

Since each $j \in W_2$ is not accessible from V_1, one gets $E_2^T = 0$ whereas $E_1^T \neq 0$. But then $F_{21}^T = 0$, since all indices in W_1 are accessible, hence $F_{21}^T \neq 0$ would imply that an index $i \in V_1$ has access to some $j \in W_2$ via a $k \in W_1$. Thus

$$\tilde{\Pi} C \tilde{\Pi}^T = \begin{pmatrix} D & 0 & 0 \\ E_1 & F_{11} & F_{12} \\ 0 & 0 & F_{22} \end{pmatrix}.$$

If z is split into (z_1, z_2, z_3) with respect to V_1, W_1, and W_2, then $F_{22} z_3 = z_3 > 0$. Now, F_{22} is a final and basic class, contradicting the assumptions.

Since there is only one final and basic class, z is a basis for $\mathcal{N}(A)$. $\square$

Sometimes column stochastic matrices, i.e. matrices satisfying $e^T B = e^T$, which have also positive right hand fixed points, are being treated (see, e.g., [40]). The following lemma proves that this class of matrices is not too large.

Lemma 2.3 *Let*

$$ST_n := \{B \in \mathbb{R}^{n \times n} : B \text{ is an ST-matrix}\}$$

and

$$CS_n := \{B \in \mathbb{R}^{n \times n} : B \text{ is column stochastic}\}.$$

If $B \in ST_n \cap CS_n$, then B is irreducible.

Proof: Let $B \in ST_n \cap CS_n$. If B is irreducible, then the lemma is obvious by Theorem 1.6. Thus, assume that B is reducible. Then by Lemma 2.1,

$$\Pi B \Pi^T = \begin{pmatrix} D & 0 \\ E & F \end{pmatrix}$$

for some permutation matrix Π. But then

$$\Pi^T B^T \Pi = \begin{pmatrix} D^T & E^T \\ 0 & F^T \end{pmatrix}$$

and since $B^T e = e$, B^T has a positive fixed point. By Theorem 1.7, $\rho(F) = 1$ and $\rho(D) < 1$ since $E \neq 0$. Thus the indices of D represent a final class which is not basic, i.e. B is not an ST-matrix. □

The above lemma reflects entirely the situation of the example given on page 23. If a column stochastic B contains a spanning tree with a corresponding guard and is reducible, then B cannot have a positive right hand fixed point (this is also obvious by Theorem 1.8).

Remark 2.4 *In the literature dealing with Markov chains, i.e. row stochastic matrices (e.g. [9, 62, 65]), the indices here are called states. In the situation of Lemma 2.2, the states belonging to D are called* ergodic *while those belonging to F are called* transient*. It is known that a transient state must have access to an ergodic one, thus the description of that property using a tree structure seems to be natural. The definition of ST-matrices is also motivated by Theorem 1.8.*

This section will be completed with the introduction of another class of matrices and several properties of its elements. Recall that Δ denotes the diagonal graph of the vertex set $\{1, \ldots, n\}$ (cf. Section 1.2).

Definition 2.5 *A matrix $A \in \mathbb{R}^{n \times n}$ is called an* STM-matrix *(STM for "spanning tree monotone") if $A = I - B$ and B is an ST-matrix.*

Lemma 2.4 *Let $A = I - B$ be an $n \times n$ STM-matrix. If the final class of B contains more than one index, then each a_{ii} is positive. If it contains exactly one index i_0 then the i_0-th row of A is entirely zero and $a_{ii} > 0$ for all $i \neq i_0$.*

Proof: We have $A = I - B$ and $Bz = z$ for a positive vector z. By the existence of a spanning tree in $\Gamma(B^T)$, every non-root row of B has a positive off diagonal element.

If the final class of B contains exactly one element, then the root coincides with the guard index and there is only a positive diagonal element in the corresponding row. Otherwise, the root has access to a guard element which is indicated by a positive off diagonal element. □

Remark 2.5 *If the final class of B contains more than one index, then $\Gamma(A) = \Delta \cup \Gamma(B)$, but otherwise, i.e. the root coincides with the guard index, not. In the latter case, the single edge from the root to itself vanishes and this may cause problems for block iteration methods. However, the whole bunch of results for ST-matrices carries over (in the M-matrix sense) to STM-matrices.*

The next theorem is very important in the context of block iteration methods.

Theorem 2.2 *Let $A = I - B \in \mathbb{R}^{n\times n}$ be an STM-matrix and let α be the subset of indices of the final class of B. If $V \subsetneq \{1, \ldots, n\}$ is given such that $V \cap \alpha \neq \alpha$, then the principal minor $M := A[V]$ is a nonsingular M-matrix. If $B[V]$ is the related principal minor of B, then $\rho(B[V]) < 1$*

Proof: Let $W_1 = V \cap \alpha$ and $W_2 = V \cap (\{1, \ldots, n\} \setminus \alpha)$, then according to (2.2.2), there is a permutation matrix Π such that

$$\Pi M \Pi^T = \begin{pmatrix} M_{11} & 0 \\ M_{21} & M_{22} \end{pmatrix}.$$

Here the indices of W_1 correspond to M_{11} and those of W_2 to M_{22}. If one of these sets is empty, then the corresponding entry vanishes. If $W_1 \neq \emptyset$, M_{11} is a principal minor of an irreducible M-matrix, hence nonsingular by Theorem 1.4. On the other hand, if $W_2 \neq \emptyset$, then M_{22} is the principal minor of a nonsingular M-matrix and also nonsingular by Theorem 1.2. The proposition for $B[V]$ is obvious. □

Remark 2.6 *The assertion of Theorem 2.2 holds also in the case of $r > 1$ final and basic classes (cf. Section 2.3). If α_j are sets of indices corresponding to the basic classes and $V \subset \{1, \ldots, n\}$, then $V \cap \alpha_j \neq \alpha_j$ for all $j = 1, \ldots, r$, leads to the same result.*

Finally, some remarks and a definition.

Remark 2.7 *1) The definition of ST-matrices covers only nonnegative matrices having spectral radius 1. This is for convenience only, since only those matrices are being treated here (cf. Definition 2.1).*

2) The flavor of ST-matrices is comparable to the so called regular matrices *in [31, 62] (cf. Section 4.2). Note that regularity in [31, 62] is different to regularity in the standard sense; see [9, 65, 73].*

An STM-matrix naturally fulfils the requirements of the model problem *MP*. On the other hand, each matrix $A = I - B$ having a positive vector $z > 0$ such that $Az = 0$ and $\dim \mathcal{N}(A) = 1$ is an STM-matrix. Thus, the model problem can be restated.

Definition 2.6 *Let* $A \in \mathbb{R}^{n\times n}$ *be an STM-matrix and* $b \in \mathcal{R}(A)$*. The model problem* MP *is to find a solution* $x^* \in \mathbb{R}^n$ *of*

$$Ax = b,\ x \in \mathbb{R}^n.$$

2.3 SF- and SFM-matrices

The concept of ST-matrices will now be slightly generalised. This generalisation is quite natural in view of Theorems 1.7, 1.8, and 2.1.

Definition 2.7 *Assume* $B \in \mathbb{R}^{n\times n}$ *is nonnegative and* $\rho(B) = 1$*. Let* $\Gamma(B) = (V(B), E(B))$ *be the corresponding graph and let* $\Gamma(B^T)$ *be the graph of* B^T*. Then* B *is called an* SF-matrix of degree r *(SF for "spanning forest") if* $\Gamma(B^T)$ *contains* $r \geq 1$ *trees* $\mathcal{T}_k = (V_k, E_k),\ k = 1, \ldots, r,$ *and if* $i_1, \ldots, i_r$ *are the roots of those trees, then*

1) *any* i_k *has access to at least one* $j \in V(B)$ *in* $\Gamma(B)$ *(possibly itself), and*

2) $i_k \nrightarrow i_l$ *for each* $1 \leq k, l \leq r$ *and* $k \neq l$*, and*

3) *if* $i_k \to j$ *in* $\Gamma(B)$ *for some* $j \in V(B)$*, then also* $j \to i_k$ *for all* $1 \leq k \leq r$*.*

Furthermore

4) $\bigcup_{k=1}^{r} V_k = V(B)$ *and*

5) *each class is final if and only if it is basic.*

Condition 1) guarantees the existence of a guard element for each root and both reside in the same class (i.e. strongly connected component). Point 2) separates these classes and 3) makes them final and irreducible. Condition 4) makes the forest spanning and 5) guarantees the existence of a positive right fixed point.

Remark 2.8 1) *Note that Condition 5) can be replaced by the assumption that the rows of* B *sums up to 1 (cf. Definition 2.4 and Remark 2.3).*

2) *If* $B \in \mathbb{R}^{n\times n}$ *satisfies only Conditions 1) to 4) of Definition 2.7, then* B *is said to be a* GSF-matrix *(cf. Definition 2.3).*

At this point, a result analogous to Lemma 2.1 and Lemma 2.2 can be established. For the sake of completeness we present a proof.

Lemma 2.5 *Let $B \in \mathbb{R}^{n\times n}$ be an SF-matrix of degree r, then there exists a permutation matrix Π such that*

$$\Pi B \Pi^T = \begin{pmatrix} D_1 & 0 & \dots & 0 & 0 \\ 0 & D_2 & \dots & 0 & 0 \\ \vdots & & \ddots & \vdots & \vdots \\ & & & D_r & \\ E_1 & E_2 & \dots & E_r & F \end{pmatrix}.$$

Furthermore:

1) *Each D_k, $k = 1, \dots, r$, is square, irreducible, and $\rho(D_k) = 1$.*

2) *$\rho(F) < 1$.*

3) *If $\mathcal{F} = (\mathcal{T}_1, \dots, \mathcal{T}_r)$ is a spanning forest and $i_1, \dots, i_r$ are the roots of $\mathcal{T}_1, \dots, \mathcal{T}_r$, then each D_k contains exactly one i_l.*

4) *There exists a positive vector z such that $Bz = z$.*

5) *$\dim \mathcal{N}(I - B) = r$.*

Proof: Assume there are $l \in \mathbb{N}$ final classes, then for a proper permutation matrix

$$\Pi^T B^T \Pi = \begin{pmatrix} D_1^T & 0 & \dots & 0 & E_1^T \\ 0 & D_2^T & \dots & 0 & E_2^T \\ \vdots & & \ddots & \vdots & \vdots \\ & & & D_l^T & E_l^T \\ 0 & 0 & \dots & 0 & F^T \end{pmatrix}, \tag{2.3.1}$$

where the D_i are square, irreducible, and final in $\Gamma(B)$. Now let $\alpha_1, \dots, \alpha_l$ be the classes corresponding to $D_1, \dots, D_l$. If $l > r$ then there is a final class, say α_l, which contains no root and is not accessible. Thus the forest is not spanning, which contradicts Condition 4) of Definition 2.7. Hence $l \le r$. If $l < r$, each α_k, $k = 1, \dots, l$, must contain a root (otherwise, with the same argumentation as above, there cannot be a spanning forest). If $i_j \in \alpha_k$, then Condition 2) of Definition 2.7 implies $i_l \notin \alpha_k$ for all $1 \le l \le r$, $l \ne j$. Thus α_k contains exactly one i_j. Hence $m = r - l$ roots reside in the index set given by F. Since the roots are in different classes, one gets with a proper permutation matrix $\tilde{\Pi}$:

$$\tilde{\Pi}^T B^T \tilde{\Pi} = \begin{pmatrix} D_1^T & 0 & \dots & 0 & E_{11}^T & \dots & E_{m1}^T & E_{m+1,1}^T \\ 0 & D_2^T & \dots & 0 & E_{12}^T & \dots & E_{m2}^T & E_{m+1,2}^T \\ \vdots & \vdots & \ddots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & D_l^T & E_{1l}^T & \dots & E_{ml}^T & E_{m+1,l}^T \\ 0 & 0 & \dots & 0 & F_{11}^T & \dots & F_{m1}^T & F_{m+1,1}^T \\ \vdots & \vdots & & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 & & F_{mm}^T & F_{m+1,m}^T \\ 0 & 0 & \dots & 0 & 0 & \dots & 0 & F_{m+1,m+1}^T \end{pmatrix}.$$

The classes the m roots in F belong to are given by the $F_{ii},\ i = 1, \dots, m$. But from Condition 3) of Definition 2.7, the classes containing a root are final, hence $E_{i,j}^T = 0$ for $i = 1, \dots, m$ and $j = 1, \dots, l$, and also $F_{i,j}^T = 0$ for $i = 1, \dots, m-1$ and $i < j < m$. Thus

$$\tilde{\Pi}^T B^T \tilde{\Pi} = \begin{pmatrix} D_1^T & 0 & \dots & 0 & 0 & \dots & 0 & E_{m+1,1}^T \\ 0 & D_2^T & \dots & 0 & 0 & \dots & 0 & E_{m+1,2}^T \\ \vdots & \vdots & \ddots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & D_l^T & 0 & \dots & 0 & E_{m+1,l}^T \\ 0 & 0 & \dots & 0 & F_{11}^T & \dots & 0 & F_{m+1,1}^T \\ \vdots & \vdots & \dots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 & & F_{mm}^T & F_{m+1,m}^T \\ 0 & 0 & \dots & 0 & 0 & \dots & 0 & F_{m+1,m+1}^T \end{pmatrix}.$$

This is exactly the representation (2.3.1) for $l = r$. Moreover, each D_i (i.e. the class of D_i) contains exactly one i_k, which is assertion 3).

Assertion 1) is obvious, using Condition 5) of Definition 2.7. The proof of $\rho(F) < 1$ and the existence of a positive vector z such that $Bz = z$ follows that of Lemma 2.2.

Now assume the representation (2.3.1) and let $z_1, \dots, z_r$ be positive right hand eigenvectors of $D_1, \dots, D_r$ which correspond to the eigenvalue 1. Define

$$z_{r+1} := (I - F)^{-1} \left(\sum_{k=1}^{r} \lambda_k E_k z_k \right), \quad \lambda_1, \dots, \lambda_r \in \mathbb{R},$$

then $y^T := (\lambda_1 z_1^T, \dots, \lambda_r z_r, z_{r+1}^T)^T$ satisfies $By = y$ by direct calculation. Since z_{r+1} is uniquely determined for any linear combination of the $z_1, \dots, z_r$, $\dim \mathcal{N}(I - B) = r$ follows and this is assertion 5). $\square$

Corollary 2.5 *Let $B \in \mathbb{R}^{n \times n}$ be given. Then B is an SF-matrix of degree 1 if and only if B is an ST-matrix.*

The last corollary and the results of Section 2.2 imply that there usually exist a lot of different trees within the graph of a transposed SF-matrix; thus a lot of different forests.

Now an analogous result to Theorem 2.1. Since the proof is almost the same, it will be omitted here.

Theorem 2.3 *Let $B \in \mathbb{R}^{n\times n}$ be nonnegative. Then B is an SF-matrix of degree $r \geq 1$ if and only if the final classes of B are exactly its basic ones and there are exactly r such classes. Furthermore, there exists a positive vector z such that $Bz = z$.*

There holds a proposition which is similar to Lemma 2.3. If $B \in \mathbb{R}^{n\times n}$ is column stochastic and also an SF-matrix, then B has a positive right hand eigenvector if and only if B is completely decomposable into a block diagonal matrix whose blocks are irreducible.

This is a hint how to reduce the number of unknowns of systems of the form $Bx = x$. To understand this let $B \in \mathbb{R}^{n\times n}$ be nonnegative and consider a positive vector y such that $y^T B = y^T$. Assume that for a proper permutation matrix,

$$\Pi B \Pi^T = \begin{pmatrix} D_1 & 0 & \dots & 0 & 0 \\ 0 & D_2 & \dots & 0 & 0 \\ \vdots & & \ddots & \vdots & \vdots \\ & & & D_r & \\ E_1 & E_2 & \dots & E_r & F \end{pmatrix},$$

where as usual each D_i is square and irreducible and $E_i \neq 0$ for all $i = 1, \dots, r$.

The matrix B^T has the positive right eigenvector y and it follows from Theorem 1.7, that each final class of B^T is also basic, i.e. $\rho(F) = 1$. Because all $E_i \neq 0$, the relation $\rho(D_i) < 1$ follows for all $i = 1, \dots, r$. Thus, if $z \in \mathbb{R}^n$ is such that $Bz = z$ and $z = (z_1^T, \dots, z_{r+1}^T)^T$, then $z = (0, \dots, 0, z_{r+1}^T)^T$. Hence, it suffices to find a right hand eigenvector of F instead of B (cf. [9], Chapter 8, Theorem 3.23).

Thus, if complexity permits, it might be a good idea to find the strongly connected components which are final in $\Gamma(B)$ and *not* final in $\Gamma(B^T)$ first. Those classes can be omitted in iterations which compute the Perron vector, i.e. solutions for $Bx = x$.

This section ends with some definitions.

Definition 2.8 *A matrix $A \in \mathbb{R}^{n\times n}$ is called an* SFM-matrix of degree r *(SFM for "spanning forest monotone") if $A = I - B$ and B is an SF-matrix of degree r.*

Definition 2.8 leads to an alternative to Definition 2.2, which is comparable to the situation in Section 2.2. The matrices that fulfil the requirements of the generalised model problem *GMP* are SF-matrices of some degree $r \geq 1$ by Theorem 2.3 and vice versa. For this reason, *GMP* can be restated as follows.

Definition 2.9 *Let $A \in \mathbb{R}^{n \times n}$ be an SFM-matrix of degree $r \geq 1$ and let $b \in \mathcal{R}(A)$. The generalised model problem* GMP *is to find a solution $x^* \in \mathbb{R}^n$ of*

$$Ax = b, \quad x \in \mathbb{R}^n.$$

Chapter 3

Block iterative methods for *MP* and *GMP*

Now some basic iterative methods to solve *MP* (cf. Definition 2.6) and *GMP* (cf. Definition 2.9) will be discussed. At the beginning multiplicative and additive Schwarz methods will be introduced; followed by inexact variants. It follows a discussion of partially asynchronous iterations which are based on the operators being defined for the Schwarz iterations. Further on, the relationships between operators of Schwarz iterations and those of partially asynchronous iterations will be discussed and the main problems to be solved in this thesis are revealed.

Since the above mentioned methods are well known, the discussion will be very short. Anyway, some more notation must be introduced.

3.1 Notation

Let the finite set $S = \{1, \ldots, n\}$ be given. The nonempty sets $S_1, \ldots, S_p$ are called a *partitioning* of S if

$$\bigcup_{i=1}^{p} S_i = S \text{ and } S_i \cap S_j = \emptyset \text{ for all } 1 \leq i, j \leq n,\ i \neq j.$$

The nonempty sets $S_1, \ldots, S_p$ are called a *decomposition* of S if

$$\bigcup_{i=1}^{p} S_i = S.$$

Here, an index $j \in S$ is allowed to appear in more than one set, in which case the so-called overlap occurs. The measure of overlap is the maximum number of sets each index $j \in S$ belongs to. It is given by

$$q = \max_{j=1,\ldots,n} |\{i : j \in S_i\}|, \tag{3.1.1}$$

and $q = 1$ if and only if no overlap occurs; see, e.g., [6].

If $A \in \mathbb{R}^{n\times n}$, then either a partitioning or a decomposition of $\{1, \ldots, n\}$ is called *regular* (w.r.t. A) if $A[S_i]$ is invertible for $i = 1, \ldots, p$ (cf. Theorem 2.2, Remarks 2.5 and 2.6).

3.2 Exact Schwarz methods

In this section an algebraic formulation of exact Schwarz methods will be given. The methods are restricted to SFM-matrices. For more details see [6, 15, 30, 40].

Consider *GMP* (cf. Definition 2.2) and let $S_1, \ldots, S_p$ be a regular decomposition of $\{1, \ldots, n\}$ w.r.t. A.

The idea of multiplicative Schwarz methods is the consecutive application of projections onto the subspaces given by sets $\{1, \ldots, n\} \setminus S_i,\ i = 1, \ldots, p$. For this purpose, restriction operators $R_i \in \mathbb{R}^{|S_i|\times n}$ onto the subspaces are needed. Here, the rows of the R_i are defined to be the rows of the identity corresponding to the indices in S_i. E.g., if $n = 6$ and $S_1 = \{1, 3, 2\}$, then

$$R_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

With the above restriction operators, a projection onto the subspace associated with $\{1, \ldots, n\} \setminus S_i$ is given by

$$H^{(i)} := I - R_i^T (R_i A R_i^T)^{-1} R_i A. \tag{3.2.1}$$

With (3.2.1) and a given $x^0 \in \mathbb{R}^n$ a *(one-level) multiplicative Schwarz method* for *GMP* is defined by

$$x^{k+1} = Tx^k + c, \quad k = 0, 1, 2, \ldots, \tag{3.2.2}$$

where T is given as

$$T := H^{(1)} \cdot H^{(2)} \cdot \ldots \cdot H^{(p)}. \tag{3.2.3}$$

The vector $c \in \mathbb{R}^n$ is given by

$$c = \sum_{i=p}^{1} \left(\prod_{l=1}^{i-1} H^{(l)} \right) b^i, \qquad \text{where } \prod_{l=1}^{0} H^{(l)} =: \mathrm{id}_n \tag{3.2.4}$$

and

$$b^i = R_i^T (R_i A R_i^T)^{-1} R_i b. \tag{3.2.5}$$

The idea of additive Schwarz is to eliminate the error (i.e. the distance to the exact solution) simultaneously on each subspace corresponding to S_i. Since the error of $x \in \mathbb{R}^n$ w.r.t. to S_i is given by $(I - H^{(i)})x$, the *(damped) additive Schwarz method* is defined as

$$x^{k+1} = T_\theta x^k + c_\theta, \quad k = 0, 1, 2, \ldots, \tag{3.2.6}$$

where T_θ is given by

$$T_\theta := I - \theta \sum_{i=1}^{p} (I - H^{(i)}) = I - \theta \sum_{i=1}^{p} R_i^T (R_i A R_i^T)^{-1} R_i A. \tag{3.2.7}$$

Using (3.2.5), the right hand side $c_\theta \in \mathbb{R}^{n \times n}$ is defined as

$$c_\theta = \theta \sum_{i=1}^{p} R_i^T (R_i A R_i^T)^{-1} R_i b = \theta \sum_{i=1}^{p} b^i. \tag{3.2.8}$$

To achieve convergence, $\theta \in (0, 1/q)$ is usually assumed. The number q is the measure of overlap defined in Section 3.1.

First of all it should be noted that the term $R_i A R_i^T = A[S_i]$ in (3.2.1) is a nonsingular M-matrix, since the decomposition is assumed regular. Therefore, the inverse always exists and is nonnegative.

The last observation leads to an alternative representation of (3.2.1). For a given S_i, let $\neg S_i := \{1, \ldots, n\} \setminus S_i$. Furthermore, define $M_i := A[S_i]$ and $N_i := -A[S_i, \neg S_i]$. Then M_i is the principal minor of A corresponding to S_i and the matrix $[M_i | - N_i] \in \mathbb{R}^{|S_i| \times n}$ represents the rows of A corresponding to S_i.

With this, there is a permutation Π_i such that

$$\Pi_i H^{(i)} \Pi_i^T = \begin{pmatrix} 0 & M_i^{-1} N_i \\ 0 & I \end{pmatrix} \in \mathbb{R}^{n \times n}. \tag{3.2.9}$$

The following lemma is easy to prove and summarises the properties of the operators defined above.

Lemma 3.1 *Let $A \in \mathbb{R}^{n \times n}$ be an SFM-matrix and let $S_1, \ldots, S_p$ be a regular decomposition. Let $z \in \mathbb{R}^n$ be the positive vector satisfying $Az = 0$.*

Then, with operators as given in (3.2.1) and (3.2.3):

1) $H^{(i)} \geq 0,\ i = 1, \ldots, p,$	*6)* $T \geq 0,$
2) $H^{(i)} z = z,\ i = 1, \ldots, p,$	*7)* $Tz = z,$
3) $\|H^{(i)}\|_z = 1,\ i = 1, \ldots, p,$	*8)* $\|T\|_z = 1,$
4) $\rho(H^{(i)}) = 1,\ i = 1, \ldots, p,$	*9)* $\rho(T) = 1,$
5) $\mathrm{ind}_1(H^{(i)}) = 1,\ i = 1, \ldots, p,$	*10)* $\mathrm{ind}_1(T) = 1.$

If T_θ is defined by (3.2.7) and $\theta \in (0, 1/q)$, then parts 6) to 10) apply verbatim to T_θ. Moreover, the diagonal of T_θ is positive and therefore, T_θ is semiconvergent.

Proof: To prove 1), note that $N_i \geq 0$ for all $i = 1, \ldots, p$. The regularity of the decomposition implies the existence of M_i^{-1}. It follows by Theorems 1.1 and 2.2 that $M_i^{-1} \geq 0$ for all $i = 1, \ldots, p$. Hence, $H^{(i)} \geq 0,\ i = 1, \ldots, p,$ follows by (3.2.9). Statement 2) is obvious using again (3.2.9). Part 3) follows from 1) and 2), after an application of the weighted max norm (see Section 1.1). Finally, 4) follows directly from 2) and 3); and assertion 5) by applying Theorem 1.5. The above results for T are obvious.

To prove the assertions for T_θ note that

$$\sum_{i=1}^{p} R_i^T R_i \leq qI < \frac{1}{\theta} I.$$

Thus with (3.2.9)

$$\begin{aligned} T_\theta &= I - \theta \sum_{i=1}^{p} (I - H^{(i)}) = I - \theta \sum_{i=1}^{p} \Pi_i^T \begin{pmatrix} I & -M_i^{-1} N_i \\ 0 & 0 \end{pmatrix} \Pi_i \\ &= I - \theta \sum_{i=1}^{p} \Pi_i^T \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \Pi_i + \theta \sum_{i=1}^{p} \Pi_i^T \begin{pmatrix} 0 & M_i^{-1} N_i \\ 0 & 0 \end{pmatrix} \Pi_i \\ &= \underbrace{I - \theta \sum_{i=1}^{p} R_i^T R_i}_{=: I_\theta \geq 0} + \underbrace{\theta \sum_{i=1}^{p} \Pi_i^T \begin{pmatrix} 0 & M_i^{-1} N_i \\ 0 & 0 \end{pmatrix} \Pi_i}_{\geq 0}. \end{aligned}$$

Hence $T_\theta \geq 0$. Since $(I_\theta)_{ii} > 0$ for $i = 1, \ldots, n$, there also holds $(T_\theta)_{ii} > 0$ for $i = 1, \ldots, n$. The rest is obvious and the semiconvergence of T_θ follows from Lemma 1.2 and Theorem 1.11. $\square$

From now on, the operators given by (3.2.1) will be denoted by $H_1^{(i)}$ or simply $P^{(i)}$, the multiplicative Schwarz operator (3.2.3) by T_1, and the additive Schwarz operator (3.2.7) by $T_{\theta,1}$. The $H_1^{(i)}$ are called the *local operators*, while T_1 and $T_{\theta,1}$ are called *global operators*.

3.3 Inexact Schwarz methods

Assume *GMP* with an SFM-matrix A and a regular decomposition $S_1, \ldots, S_p$. The matrices $M_i = A[S_i]$ and $N_i = -A[S_i, \neg S_i]$ are as defined in the foregoing section. For $x \in \mathbb{R}^n$ let $x_i = x[S_i]$ and $x_{\neg i} = x[\neg S_i]$.

An algorithmic description of the multiplicative Schwarz methods leads to the following pseudo code.

Algorithm 3.1 multiplicative Schwarz iteration / Framework

Require: $x^0 \in \mathbb{R}^n$
1: **for** $k = 1, 2, \ldots$ **do**
2: $\quad y \leftarrow x^{k-1}$
3: $\quad$**for** $i = 1, \ldots, p$ **do**
4: $\qquad$update y
5: $\quad$**end for**
6: $\quad x^k \leftarrow y$
7: **end for**

For (one-level) multiplicative Schwarz the update at line 4 is defined as:

Algorithm 3.2 Update 1

$y_i \leftarrow M_i^{-1} N_i y_{\neg i} + M_i^{-1} b_i$

An algorithmic description for additive Schwarz can be given as follows.

Algorithm 3.3 additive Schwarz iteration / Framework

Require: $x^0 \in \mathbb{R}^n,\ \theta \in (0, 1/q)$
1: **for** $k = 1, 2, \ldots$ **do**
2: $\quad$**for** $i = 1, \ldots, p$ **do**
3: $\qquad y^{(i)} \leftarrow x^{k-1}$
4: $\qquad$update $y^{(i)}$
5: $\quad$**end for**
6: $\quad x^k \leftarrow x^{k-1} - \theta \cdot \sum_{i=1}^{p} (x^{k-1} - y^{(i)})$
7: **end for**

The update at line 4 is exactly the same as for exact multiplicative Schwarz, i.e. given by Algorithm 3.2.

Now some alternatives to Algorithm 3.2 will be discussed and they will also apply to both, additive and multiplicative Schwarz iterations. These updates can be regarded as "inexact" because they all behave in a certain manner as will be shown in Section 3.5.

Relaxed (one-level) Schwarz method

The first modification concerns the diagonal. It might be interpreted as an outer relaxation for some $\alpha \in (0,1)$ and will be helpful to ensure convergence later on.

Algorithm 3.4 Update 2

$y_i \leftarrow (1-\alpha)y_i + \alpha \cdot M_i^{-1} N_i y_{\neg i} + \alpha \cdot M_i^{-1} b_i$

The local operators for this scheme are denoted by $H_2^{(i)}$ and have the following shape

$$\Pi_i H_2^{(i)} \Pi_i^T = \begin{pmatrix} (1-\alpha)I & \alpha M_i^{-1} N_i \\ 0 & I \end{pmatrix} \geq 0, \tag{3.3.1}$$

for $1 \leq i \leq p$ and a proper permutation matrix Π_i (cf. 3.2.9). The global operators T_2 and $T_{\theta,2}$ are as defined in (3.2.3) and (3.2.7) (and this will also be the case for the following updates).

Two-stage Schwarz method

The next update is really an inexact one, as the inverse of M_i is approximated. To establish this, a weak regular splitting $M_i = F_i - G_i$ (cf. Section 1.1) is assumed. Such a splitting always exists and is convergent since $\rho(F_i^{-1} G_i) < 1$ (cf. Theorem 1.3).

Algorithm 3.5 Update 3

for $j = 1, \ldots, q(k,i)$ **do**
  $y_i \leftarrow F_i^{-1}\left(G_i y_i + N_i y_{\neg i} + b_i\right)$
end for

Here $1 \leq q(k,i) \in \mathbb{N}$ might depend on the set S_i being updated and on the outer iteration index k. Such two stage iteration schemes have been analysed by several authors in the framework of a multi-splitting iteration (see [51, 52]). For singular systems this has been done in [44, 65].

It should be noted that the whole inner iteration can be described by one assignment which is convenient for theoretical purposes (see [29, 44, 51]). Define $R^{(k,i)} := (F_i^{-1} G_i)^{q(k,i)}$, then the `for`-loop in Algorithm 3.5 can be replaced by

$$y_i \leftarrow R^{(k,i)} y_i + (I - R^{(k,i)}) M_i^{-1} \left(N_i y_{\neg i} + b_i\right).$$

The local operators $H_3^{(i)}$ for this block update are given as

$$\Pi_i H_3^{(i)} \Pi_i^T = \begin{pmatrix} R^{(k,i)} & (I - R^{(k,i)}) M_i^{-1} N_i \\ 0 & I \end{pmatrix} \geq 0. \tag{3.3.2}$$

The nonnegativity of $H_3^{(i)}$ follows from the identity

$$(I - R^{(k,i)})M_i^{-1} = \sum_{j=0}^{q(k,i)-1} (F_i^{-1}G_i)^j F_i^{-1}$$

and the weak regularity of the splitting $M_i = F_i - G_i$.

Relaxed two-stage Schwarz method

The next obvious modification of Algorithm 3.5 is the use of a relaxed version of line 2. For $\omega \in (0,1)$ the update is given by:

Algorithm 3.6 Update 4

1: **for** $j = 1, \ldots, q(k,i)$ **do**
2: $\quad y_i \leftarrow (1-\omega)y_i + \omega \cdot F_i^{-1}(G_i y_i + N_i y_{\neg i} + b_i)$
3: **end for**

Define $R_\omega^{(k,i)} := \big((1-\omega)I + \omega(F_i^{-1}G_i)\big)^{q(k,i)}$, then Algorithm 3.6 reduces to

$$y_i \leftarrow R_\omega^{(k,i)} y_i + (I - R_\omega^{(k,i)})M_i^{-1}(N_i y_{\neg i} + b_i).$$

Since $R_\omega^{(k,i)} \geq 0$ for $\omega \in (0,1)$, there follows

$$\begin{aligned}
0 &\leq \omega \sum_{j=0}^{q(k,i)-1} \big((1-\omega)I + \omega(F_i^{-1}G_i)\big)^j F_i^{-1}N_i \\
&= \omega(I - R_\omega^{(k,i)})\big(I - (1-\omega)I - \omega(F_i^{-1}G_i)\big)^{-1} F_i^{-1}N_i \\
&= \omega(I - R_\omega^{(k,i)})\frac{1}{\omega}\big(I - F_i^{-1}G_i\big)^{-1} F_i^{-1}N_i \\
&= (I - R_\omega^{(k,i)})M_i^{-1}N_i.
\end{aligned}$$

The nonnegative local operators $H_4^{(i)}$ are given by

$$\Pi_i H_4^{(i)} \Pi_i^T = \begin{pmatrix} R_\omega^{(k,i)} & (I - R_\omega^{(k,i)})M_i^{-1}N_i \\ 0 & I \end{pmatrix}. \tag{3.3.3}$$

Furthermore, $\rho(R_\omega^{(k,i)}) < 1$ as is shown in [38].

Remark 3.1 *The notation $H_3^{(i)}$ and $H_4^{(i)}$ (and also $T_3, T_4, T_{\theta,3}$ and $T_{\theta,4}$) is somehow misleading since the number of inner iterations may vary. Hence, if not stated otherwise, the number of inner iterations for each block may be assumed fixed, i.e. they do not depend on k. In the other case the operators are labelled $T_3^{(k)}$ and so on.*

To complete this section, two other updates will be discussed.

Power-like Schwarz method

For Update 1,2,3 and 4, the local operators have been explicitly converted to fixed point operators. Anyway, the fixed point property is still in the original operator A (cf. 2.1.4) and will be exploited now.

Since $A = I - B$, the identities $M_i = A[S_i] = I - B[S_i]$ and $N_i = -A[S_i, \neg S_i] = B[S_i, \neg S_i]$ follow. By Theorem 2.2 and the assumed regularity of the decomposition, there holds $\rho(B[S_i]) < 1$. Furthermore,

$$[I - M_i|N_i] = [\ B[S_i] \mid B[S_i, \neg S_i]\ ] \in \mathbb{R}^{|S_i| \times n}$$

and the following local update seems natural.

Algorithm 3.7 Update 5

$y_i \leftarrow (I - M_i)y_i + N_i y_{\neg i} + b_i$

The local operators for this update are given by

$$\Pi_i H_5^{(i)} \Pi_i^T = \begin{pmatrix} (I - M_i) & N_i \\ 0 & I \end{pmatrix} \geq 0. \tag{3.3.4}$$

Relaxed power-like Schwarz method

Finally, a modification of Algorithm 3.7 is given. It is again an outer relaxation similar to Algorithm 3.4. Let $\alpha \in (0, 1)$ be given and take the following update.

Algorithm 3.8 Update 6

$y_i \leftarrow (1 - \alpha)y_i + \alpha \cdot (I - M_i)y_i + \alpha \cdot N_i y_{\neg i} + \alpha \cdot b_i$

Obviously

$$[(1 - \alpha)I + \alpha \cdot (I - M_i) \mid \alpha \cdot N_i] = [\ (1 - \alpha)I + \alpha B[S_i] \mid \alpha \cdot B[S_i, \neg S_i]\ ]$$

and therefore

$$\Pi_i H_6^{(i)} \Pi_i^T = \begin{pmatrix} I - \alpha M_i & \alpha N_i \\ 0 & I \end{pmatrix} \geq 0. \tag{3.3.5}$$

At this point the relationship between each of the operators $H_l^{(i)}$, $l = 2, \ldots, 6$ and $H_1^{(i)}$ is not clear, since no $H_l^{(i)}$ is a projection. In Section 3.5 a link between these operators will be established with tools provided in Section 1.3. This link will also lead to a "normal form" of the discussed updates.

This section ends with the following lemma whose proof is analogous to that of Lemma 3.1.

Lemma 3.2 *For $l \in \{2, \ldots, 6\}$ let the local operators $H_l^{(i)}$ be given by (3.3.1) – (3.3.5) and let the T_l and $T_{\theta,l}$ be the corresponding global operators, respectively. Then Lemma 3.1 applies verbatim to $H_l^{(i)}$, T_l and $T_{\theta,l}$ for $l = 2, \ldots, 6$ and $i = 1, \ldots, p$. Moreover, T_l is semiconvergent for $l \in \{2, 4, 6\}$ and $T_{\theta,l}$ is semiconvergent for all $l = 2, \ldots, 6$, as soon as $\theta \in (0, 1/q)$.*

Remark 3.2 *The parameters α and ω for Update 2,4 and 6 (Algorithms 3.4, 3.6 and 3.8) can be chosen different on different sets S_i.*

3.4 Asynchronous iterations

In this section the notion of asynchronous iterations will be introduced (see [28] and the extensive bibliography therein, or [10, 27]). The iterations will be defined for linear systems only and the notation is adopted from [4] with some restrictions which are due to [39] (see also [10]).

Basic definitions

Let there be given $A \in \mathbb{R}^{n \times n}$ and a vector $c \in \mathbb{R}^n$. For $k = 0, 1, 2, \ldots$, let $\mathcal{J}_k \subset \{1, \ldots, n\} \neq \emptyset$ and $\mathcal{S}_k = (s_1(k), \ldots, s_n(k)) \in \mathbb{N}_0^n$ such that

$$s_i(k) \leq k, \qquad \text{for } i = 1, \ldots, n,\ k = 0, 1, 2, \ldots, \tag{3.4.1}$$

$$\lim_{k \longrightarrow \infty} s_i(k) = +\infty, \qquad \text{for } i = 1, \ldots, n, \tag{3.4.2}$$

$$|\{k \in \mathbb{N}_0 : i \in \mathcal{J}_k\}| = \infty, \qquad \text{for } i = 1, \ldots, n. \tag{3.4.3}$$

Then, for a given $x^0 \in \mathbb{R}^n$, the assignment

$$x_i^{k+1} := \begin{cases} x_i^k & \text{if } i \notin \mathcal{J}_k, \\ \sum_{j=1}^n a_{ij} \cdot x_j^{s_j(k)} + c_i & \text{if } i \in \mathcal{J}_k, \end{cases} \tag{3.4.4}$$

is said to be an *asynchronous iteration* (for the system $x = Ax + c$).

The set $\mathcal{J}_k$ is called the *set of active components* and $\mathcal{S}_k$ the *strategy*. A sequence $\{(\mathcal{J}_k, \mathcal{S}_k)\}_{k \in \mathbb{N}_0}$ is called a *scenario*.

The term $k - s_i(k)$ might be interpreted as the delay of a component when this component is used in an update. Such delays can be caused by communication delays between processors on parallel computers or load unbalancing. The inequality (3.4.1) states, that only components computed earlier are used. The eventual use of new information is guaranteed by (3.4.2). The identity (3.4.3) guarantees that no component fails to be updated when k tends to infinity.

The index k might be interpreted as a global counter which is incremented after the update of the set of active components. But on parallel computers, this index is most likely unknown to each processor and loses its meaning in the common sense (i.e., as a timestamp or an iteration index).

Partially asynchronous iterations

Since it has been proven that the above iteration might fail to converge for singular systems [10, 18, 27, 67], some restrictions which may yield the convergence have been introduced in [39] (see also [10]). Those restrictions concern the delays, since they are mainly responsible for the convergence. The restrictions are given as follows:

$$(3.4.5) \qquad s_i(k) = k, \qquad \text{if } i \in \mathcal{J}_k,\ k = 0, 1, 2, \dots,$$
$$(3.4.6) \qquad \exists d \in \mathbb{N} : k - d \le s_i(k) \le k, \qquad i = 1, \dots, n,\ k = 0, 1, 2, \dots,$$
$$(3.4.7) \qquad \exists s \in \mathbb{N} : \cup_{l=k}^{k+s} \mathcal{J}_l = \{1, \dots, n\}, \qquad k = 0, 1, 2, \dots.$$

An iteration method (3.4.4) that fulfils (3.4.5) – (3.4.7) is said to be a *partially asynchronous iteration (PAI)* and will be denoted by the triple $(A, x^0, \{\mathcal{J}_k, \mathcal{S}_k\}_{k \in \mathbb{N}_0})$ with the convention $x^{-d} = \dots = x^{-1} = x^0$.

The interpretation of (3.4.5) is, that the values of the components being actually computed are the latest. On parallel computers, this may be achieved in practice by assigning a certain set of variables to one processor only. The relation (3.4.6) bounds the delays while (3.4.7) assures that each component is updated within a fixed interval.

The advantage of PAIs over general asynchronous iterations is the possibility to construct operators which act on the iterates to produce the assignment (3.4.4). This construction goes also back to [39] and will be shown now.

Construction of operators

Assume a matrix $A \in \mathbb{R}^{n \times n}$ and a scenario $\{\mathcal{J}_k, \mathcal{S}_k\}_{k \in \mathbb{N}_0}$ that fulfils (3.4.5)–(3.4.6). Let k be arbitrary but fixed.

Define the diagonal matrix $D^{(k)} \in \mathbb{R}^{n \times n}$ by

$$d_{i,j}^{(k)} := \begin{cases} 1 & \text{if } i \in \mathcal{J}_k \text{ and } i = j, \\ 0 & \text{else,} \end{cases}$$

then $D^{(k)}$ can be viewed as an indicator function of $\mathcal{J}_k$. Another indicator for the delays can be defined as

$$\delta_{\tau, k - s_j(k)} := \begin{cases} 1 & \text{if } \tau = k - s_j(k), \\ 0 & \text{else.} \end{cases}$$

By (3.4.6) there holds $k - d \le s_i(k) \le k$, so the above definition may be restricted to $\tau = 0, \dots, d$. Now define $E^{(k,\tau)} \in \mathbb{R}^{n \times n}$ by

$$e_{i,j}^{(k,\tau)} := d_{i,i}^{(k)} \cdot \delta_{\tau, k - s_j(k)} \cdot a_{i,j},\ \tau = 0, \dots, d,$$

then
$$\sum_{\tau=0}^{d} E^{(k,\tau)} = D^{(k)}A.$$

With this construction, the assignment (3.4.4) can be written as

$$x^{k+1} = (I - D^{(k)})x^k + \sum_{\tau=0}^{d} E^{(k,\tau)}x^{k-\tau} + D^{(k)}c.$$

After at most d steps the iterates x^m with $m < k - d$ have no further influence on x^{k+1}. Hence, the operator $H^{(k)} \in \mathbb{R}^{(d+1)n\times(d+1)n}$ with

$$H_d^{(k)} := \begin{pmatrix} (I - D^{(k)} + E^{(k,0)}) & E^{(k,1)} & \dots & E^{(k,d-1)} & E^{(k,d)} \\ I & 0 & \dots & 0 & 0 \\ 0 & I & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & I & 0 \end{pmatrix} \tag{3.4.8}$$

completely defines the k-th step of a partially asynchronous iteration. Indeed, if one defines $x_d^k = ((x^k)^T, \dots, (x^{k-d})^T)^T \in \mathbb{R}^{(d+1)n}$ and $x^{-1} = \dots = x^{-d} = x^0$, then the PAI is given by

$$x_d^{k+1} = H_d^{(k)} x_d^k + c^k, \quad k = 0, 1, 2, \dots. \tag{3.4.9}$$

Here, $c^k = ((D^{(k)}c)^T, 0^T, \dots, 0^T)^T \in \mathbb{R}^{(d+1)n}$.

Application to *GMP*

Consider *GMP* and a regular decomposition $S_1, \dots, S_p$ w.r.t. A. Since it is natural in real world applications to assign a set S_i to exactly one processor, the set of active components can be regarded as follows.

$$\mathcal{J}_k \in \{S_1, \dots, S_p\}, \; k = 0, 1, 2, \dots. \tag{3.4.10}$$

Now consider a block update as discussed in the last two sections. If at the k-th step of a PAI the set $S_{i(k)}$ is updated, then by (3.4.4) and (3.4.10) the application of a block update leads to

$$I - D^{(k)} + \sum_{\tau=0}^{d} E^{(k,\tau)} = H_l^{(i(k))} \tag{3.4.11}$$

where $H_l^{(i(k))}$ is given by (3.2.9) or (3.3.1)–(3.3.5) depending on $l \in \{1, \dots, 6\}$. Hence there is a relationship between multiplicative Schwarz methods and block PAIs.

Furthermore, by (3.4.5) and a proper permutation matrix $\Pi_{i(k)}$ (cf. (3.2.9))

$$\Pi_{i(k)}\left(I - D^{(k)} + E^{(k,0)}\right)\Pi_{i(k)}^T = \begin{pmatrix} H_l^{(i(k))}[S_{i(k)}] & * \\ 0 & I \end{pmatrix} \tag{3.4.12}$$

where the $*$ might indicate that there are some nonnegative components in the upper right block.

By the regularity of the decomposition and the construction of the local operators (cf. (3.3.1)–(3.3.5))

$$\rho(H_l^{(i(k))}[S_{i(k)}]) < 1. \tag{3.4.13}$$

So even in the case of PAIs, the local updates should make some progress towards a (local) solution.

Remark 3.3 *1) In the case of overlap, (3.4.5) requires a lot of synchronisation, since different processors which are assigned to the same variables must be synchronised.*

2) All PAIs being discussed further on, are to be understood in the sense of (3.4.5)–(3.4.11). They will only be applied to MP with respect to a regular decomposition and the block updates from Sections 3.2 and 3.3 (see Algorithms 3.2– 3.8 and (3.2.9), (3.3.1)–(3.3.5)).

3) Since (3.4.11) holds, the corresponding local operators of a PAI are denoted by $H_{d,l}^{(k)}$, *for* $l \in \{1, \ldots, 6\}$. *Then* $H_{d,l}^{(k)} = H_l^{(i(k))}$ *if* $d = 0$ *and* $H_{d,l}^{(k)}$ *updates the* $i(k)$*-th block.*

4) The identity (3.4.11) reveals another relationship. If $s = n$, $s_i(k) = k$ *for all* $i = 1, \ldots, n$, *and* $\mathcal{J}_k = S_j$ *for* $j = p - (k \mod p)$, *then the iteration (3.4.9) coincides exactly with (3.2.2) for any update given by (3.2.9) and (3.3.1)–(3.3.5). Hence the multiplicative Schwarz iterations are a subset of the partially asynchronous ones for every block update discussed.*

The question that should be discussed now is the following.

Under which conditions converges a PAI applied to MP *for every possible scenario (which include the multiplicative Schwarz methods)?*

Instead of giving an answer to the above question, the problem will be discussed for multiplicative Schwarz methods. This is because results for PAIs can be easily obtained from a generalisation of multiplicative Schwarz methods, as will be shown in Chapter 4.

3.5 Analysis of local operators

As mentioned in the foregoing section some relationships between local multiplicative Schwarz operators and those of PAIs will be presented here. The results of this and the next section will not affect the results to be discussed later, but they should be mentioned because one gets more familiar with the operators and the problems to be solved. Note that some proofs have been moved to the appendix to keep the discussion short.

First of all the links between the different local Schwarz operators will be established.

Throughout this section one may assume *GMP* and a regular decomposition $S_1, \dots, S_p$ w.r.t. A. Furthermore, the operators $H_l^{(i)},\ l = 1, \dots, 6,\ i = 1, \dots, p$ denote again the local Schwarz operators; while the $H_{d,l}^{(k)}$ denote the local operators of a PAI, $(A, x^0, \{\mathcal{J}_k, \mathcal{S}_k\}_{k \in \mathbb{N}_0})$, updating a certain block $i(k) \in \{1, \dots, p\}$ for $k \in \mathbb{N}_0$ and $l = 1, \dots, 6$.

To emphasise the role of the $H_1^{(i)},\ i = 1, \dots, p$, denote them by $P^{(i)},\ i = 1, \dots, p$. Then with (3.2.1) and (3.2.9)

$$P^{(i)} = H_1^{(i)} = I - R_i^T (R_i A R_i^T)^{-1} R_i A = \Pi_i^T \begin{pmatrix} 0 & M_i^{-1} N_i \\ 0 & I \end{pmatrix} \Pi_i,$$

where $M_i = A[S_i] = R_i A R_i^T$, $N_i := -A[S_i, \neg S_i]$, and Π_i is a proper permutation matrix. As usual,

$$R^{(k,i)} := (F_i^{-1} G_i)^{q(k,i)},\ q(k,i) \geq 1 \tag{3.5.1}$$

and

$$R_\omega^{(k,i)} := \left((1-\omega) I + \omega F_i^{-1} G_i\right)^{q(k,i)},\ q(k,i) \geq 1, \omega \in (0,1), \tag{3.5.2}$$

where $M_i = F_i - G_i$ is a weak regular splitting.

Fixed points of local Schwarz operators

Proposition 3.1 *Let $i \in \{1, \dots, p\}$ be arbitrary but fixed. For $l = 2, \dots, 6$ define $Z_l^{(i)} \in \mathbb{R}^{|S_i| \times |S_i|}$ by*

(3.5.3)

$$Z_l^{(i)} := \begin{cases} (1-\alpha) I,\ \alpha \in (0,1) & \text{if } l = 2 \text{ (cf. (3.3.1))}, \\ R^{(k,i)} & \text{if } l = 3 \text{ (cf. (3.3.2) and (3.5.1))}, \\ R_\omega^{(k,i)},\ \omega \in (0,1) & \text{if } l = 4 \text{ (cf. (3.3.3) and (3.5.2))}, \\ I - M_i & \text{if } l = 5 \text{ (cf. (3.3.4))}, \\ I - \alpha M_i,\ \alpha \in (0,1) & \text{if } l = 6 \text{ (cf. (3.3.5))}. \end{cases}$$

Then for a proper permutation matrix Π_i*:*

$$(3.5.4) \qquad (I - H_l^{(i)}) = \Pi_i^T \begin{pmatrix} I - Z_l^{(i)} & 0 \\ 0 & I \end{pmatrix} \Pi_i \cdot (I - P^{(i)}),\ l = 2, \ldots, 6.$$

Moreover

$$(3.5.5) \qquad \mathcal{N}(I - H_l^{(i)}) = \mathcal{N}(I - P^{(i)}) = \mathcal{R}(P^{(i)}),\ l = 2, \ldots, 6.$$

Proof: The representation (3.5.4) follows by direct calculation. Since the decomposition is regular, the relation $\rho(Z_l^{(i)}) < 1$ follows by the construction for $l = 2, \ldots, 6$. Hence $I - Z_l^{(i)}$ is invertible and $z \in \mathbb{R}^n$ is a fixed point of $H_l^{(i)}$ if and only if it is a fixed point of $P^{(i)}$. Since $P^{(i)}$ is a projection, $\mathcal{N}(I - P^{(i)}) = \mathcal{R}(P^{(i)})$, and (3.5.5) follows. □

A consequence of Proposition 3.1 is that the projections of a spectral decomposition of each $H_l^{(i)}$ are the same and equal to $P^{(i)} = H_1^{(i)}$.

Proposition 3.2 *For a fixed* $i \in \{1, \ldots, p\}$*, the spectral decomposition of each* $H_l^{(i)}$*,* $l = 2, \ldots, 6$ *is given by*

$$(3.5.6) \qquad H_l^{(i)} = P^{(i)} + \Pi_i^T \begin{pmatrix} Z_l^{(i)} & 0 \\ 0 & 0 \end{pmatrix} \Pi_i \cdot (I - P^{(i)}) =: P^{(i)} + Q_l^{(i)},$$

wherein $Z_l^{(i)}$ *is given by (3.5.3). Therefore,* $\sigma(H_l^{(i)}) = \{1\} \cup \sigma(Z_l^{(i)})$*, the multiplicity of the eigenvalue 1 is* $n - |S_i|$*, and* $\gamma(H_l^{(i)}) = \rho(Z_l^{(i)}) < 1$*. If* v *is a (possibly generalised) eigenvector to an eigenvalue* $\lambda \neq 1$ *and split as* $v = ((v_i)^T, (v_{\neg i})^T)^T$*, then* $v_{\neg i} = 0$ *and* $(v_i^T, 0)$ *is a (generalised) eigenvector of* $Z_l^{(i)}$*.*

Proof: The decomposition (3.5.6) follows by Proposition 3.1. Since

$$\Pi_i P^{(i)} \Pi_i^T = \begin{pmatrix} 0 & M_i^{-1} N_i \\ 0 & I \end{pmatrix}$$

there holds

$$\begin{pmatrix} 0 & M_i^{-1} N_i \\ 0 & I \end{pmatrix} \begin{pmatrix} Z_l^{(i)} & 0 \\ 0 & 0 \end{pmatrix} = 0$$

and $P^{(i)} Q_l^{(i)} = 0$. The equality $Q_l^{(i)} P^{(i)} = 0$ follows, since $(I - P^{(i)}) P^{(i)} = 0$. With

$$\begin{pmatrix} Z_l^{(i)} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} I & -M_i^{-1} N_i \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} Z_l^{(i)} & -Z_l^{(i)} M_i^{-1} N_i \\ 0 & 0 \end{pmatrix}$$

one gets $\sigma(Q_l^{(i)}) = \sigma(Z_l^{(i)}) \cup \{0\}$. Additionally, $\rho(Z_l^{(i)}) < 1$ follows from Proposition 3.1. This implies that (3.5.6) is a spectral decomposition and $P^{(i)}$ is a projection onto $\mathcal{N}(I-H_l^{(i)})$. The linear independence of the columns of $P^{(i)}$ is obvious, hence the multiplicity of the eigenvalue 1 is $n - |S_i|$.

If v is an eigenvector w.r.t. $\lambda \neq 1,\ \lambda \in \sigma(H_l^{(i)})$, then

$$0 = \Pi_i P^{(i)} \Pi_i^T \Pi_i v = \begin{pmatrix} 0 & M_i^{-1} N_i \\ 0 & I \end{pmatrix} \begin{pmatrix} v_i \\ v_{\neg i} \end{pmatrix} = \lambda \begin{pmatrix} * \\ v_{\neg i} \end{pmatrix}$$

since $P^{(i)}$ is a projection. Thus $v_{\neg i} = 0$ and $Z_l^{(i)} v_i = \lambda v_i$. The assertion for generalised eigenvectors follows by induction on the corresponding Jordan-chain. □

Remark 3.4 *1) Proposition 3.2 implies that*

$$\lim_{k \longrightarrow \infty} \left(H_l^{(i)}\right)^k = P^{(i)}$$

for all $l = 2, \ldots, 6,\ i = 1, \ldots, p$. *Each* $H_l^{(i)}$ *represents an approximation to* $H_1^{(i)} = P^{(i)}$ *(not always a good one). Thus, the operators* $H_l^{(i)},\ l = 2, \ldots, 6$ *can be regarded as* inexact Schwarz operators.

2) Note that each $Z_l^{(i)}$ *given by (3.5.3) coincides with* $(Q_l^{(i)})_{|\mathcal{N}(P^{(i)})}$ *from Lemma 1.6.*

3) With $Z_l^{(i)}$ *given by (3.5.3), define*

$$\Delta_l^{(i)} := \Pi_i^T \begin{pmatrix} Z_l^{(i)} & 0 \\ 0 & 0 \end{pmatrix} \Pi_i.$$

Then, with (3.5.4) and (3.5.6),

$$\begin{aligned} I - H_l^{(i)} &= (I - \Delta_l^{(i)})(I - P^{(i)}), \\ H_l^{(i)} &= P^{(i)} + \Delta_l^{(i)}(I - P^{(i)}), \end{aligned}$$

for all $i = 1, \ldots, p$ *and* $l \in \{2, \ldots, 6\}$. *And this might be interpreted as the "normal forms" of the local Schwarz operators.*

The following result is obvious from equation (3.5.5) of Proposition 3.1.

Proposition 3.3 *There holds*

$$\bigcap_{i=1}^{p} \mathcal{N}(I - H_l^{(i)}) = \mathcal{N}(A)$$

for all $l = 1, \ldots, 6$.

Spectra of local asynchronous operators

Let $k \in \mathbb{N}_0$ be arbitrary but fixed and assume that for some $l \in \{1, \ldots, 6\}$ the operator $H_{d,l}^{(k)} \in \mathbb{R}^{(d+1)n \times (d+1)n}$ updates the $i(k)$-th block and is given as

$$(3.5.7) \qquad H_{d,l}^{(k)} := \begin{pmatrix} \tilde{E}_l^{(k,0)} & E_l^{(k,1)} & \ldots & E_l^{(k,d-1)} & E_l^{(k,d)} \\ I & 0 & \ldots & 0 & 0 \\ 0 & I & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & I & 0 \end{pmatrix},$$

according to (3.4.8) and (3.4.11). Here $\tilde{E}_l^{(k,0)} = I - D^{(k)} + E_l^{(k,0)}$.
First, there is an analogous result to Lemma 3.1 and 3.2.

Proposition 3.4 *Let $z \in \mathbb{R}^n$ be positive such that $Az = 0$. Define $z_d := (z^T, \ldots, z^T)^T \in \mathbb{R}^{(d+1)n}$. Then for each $l = 1, \ldots, 6$ and $k \in \mathbb{N}_0$:*

1) $H_{d,l}^{(k)} \geq 0$, *4)* $\rho(H_{d,l}^{(k)}) = 1$,

2) $H_{d,l}^{(k)} z_d = z_d$, *5)* $\mathrm{ind}_1(H_{d,l}^{(k)}) = 1$,

3) $\|H_{d,l}^{(k)}\|_{z_d} = 1$.

Proof: Condition (3.4.11) together with Lemma 3.1 or 3.2 implies

$$\left(I - D^{(k)} + \sum_{\tau=0}^{d} E_l^{(k,\tau)} \right) z = H_l^{(i(k))} z = z$$

and therefore $H_{d,l}^{(k)} z_d = z_d$, which proves 2). The rest is easy. □

Now a partially analogous result to Proposition 3.2 is presented.

Proposition 3.5 *Let (λ, v) be an eigenpair of $H_{d,l}^{(k)}$, $\lambda \neq 0$. Assume v to be split in $d+1$ parts $v_i \in \mathbb{R}^n$ such that $v = ((v_0)^T, \ldots, (v_d)^T)^T$. Then*

$$(3.5.8) \qquad v_0 = \lambda v_1 = \lambda^2 v_2 = \ldots = \lambda^d v_d.$$

If $\lambda \neq 1$, then v is an eigenvector of $H_{d,l}^{(k)}$ if and only if v_0 is an eigenvector of $H_l^{(i(k))}$ to the eigenvalue λ (and the same holds for generalised eigenvectors). Furthermore, v is an eigenvector to the eigenvalue 1 if and only if

$$(3.5.9) \qquad v_0 = v_1 = \ldots = v_d$$

and v_0 is a fixed point of $H_1^{(i(k))} = P^{(i(k))}$.

Proof: See Appendix A.

The last proposition implies that the non-zero spectrum of $H_{d,l}^{(k)}$ corresponds one to one to the non-zero spectrum of $H_l^{(i(k))}$. Thus the influence of the delays on the spectrum can only concern eigenvalues equal to 0.

Lemma 3.3 *Let $H_{d,l}^{(k)}$ be given, then $\sigma(H_{d,l}^{(k)}) = \sigma(H_l^{(i(k))}) \cup \{0\}$ and the multiplicity of the eigenvalue 0 is at least $d \cdot n$. Furthermore, there holds $\gamma(H_{d,l}^{(k)}) = \gamma(H_l^{(i(k))}) < 1$.*

This one-to-one-correspondence implies that the projection onto $\mathcal{N}(I - H_{d,l}^{(k)})$ must be constant for all $k \in \mathbb{N}_0$ such that $i(k) = i_0 \in \{1, \ldots, p\}$, (as it is for the $H_l^{(i_0)}$). And this is the case, because a simple argumentation using the dual basis of $\mathcal{N}(I - H_{d,l}^{(k)})$ leads to the result that

$$P_d^{(i_0)} := \begin{pmatrix} P^{(i_0)} & 0 & \ldots & 0 \\ \vdots & \vdots & & \vdots \\ P^{(i_0)} & 0 & \ldots & 0 \end{pmatrix} \in \mathbb{R}^{(d+1)n \times (d+1)n} \tag{3.5.10}$$

is the projection onto $\mathcal{N}(I - H_{d,l}^{(k)})$. Here, again, $P^{(i_0)} = H_1^{(i_0)}$.

Lemma 3.4 *For a partially asynchronous iteration let $\{H_{d,l}^{(k)}\}_{i_0}$ be the subset of local operators such that $i(k) = i_0 \in \{1, \ldots, p\}$. Then the spectral decomposition of every $H_{d,l}^{(k)} \in \{H_{d,l}^{(k)}\}_{i_0}$ is $(P_d^{(i_0)}, Q_{d,l}^{(k)})$ where $P_d^{(i_0)}$ is defined by (3.5.10) and $Q_{d,l}^{(k)} = H_{d,l}^{(k)} - P_d^{(i_0)}$. Moreover, the projection $P_d^{(i_0)}$ is independent of any scenario and depends only on d and the block i_0.*

Proof: The relations $P_d^{(i_0)} \cdot Q_{d,l}^{(k)} = Q_{d,l}^{(k)} \cdot P_d^{(i_0)} = 0$ follow by direct computation. The inequality $\rho(Q_{d,l}^{(k)}) < 1$ follows from Proposition 3.5. That $P_d^{(i_0)}$ is a projection onto $\mathcal{N}(I - H_{d,l}^{(k)})$ is obvious by Proposition 3.5. □

Remark 3.5 *The spectral decomposition of an exact local Schwarz operator (cf. (3.2.9)) is given by the operator itself since the convergent part is zero. On the other hand, the convergent part $Q_{d,1}^{(k)}$ of a local operator of an asynchronous operator is nonzero. But the spectra of $Q_{d,1}^{(k)}$ must be entirely zero due to Proposition 3.5. Hence $Q_{d,1}^{(k)}$ is nilpotent.*

To prove an analogous result to Proposition 3.3, assume $\mathcal{N}(A) = \text{span}\{z_1, \ldots, z_m\}$ and define $z_{d,i} := (z_i^T, \ldots, z_i^T)^T \in \mathbb{R}^{(d+1)n}$ for $i = 1, \ldots, m$.

Proposition 3.6 *Let $\{H_{d,l}^{(k)}\}_{k\in\mathbb{N}_0}$ be the sequence of local operators of a PAI. If s is given by (3.4.7), then*

$$\bigcap_{i=k}^{k+s} \mathcal{N}(I - H_{d,l}^{(i)}) = \mathrm{span}\{z_{d,1}, \ldots, z_{d,m}\} \subsetneq \underbrace{\mathcal{N}(A) \times \ldots \times \mathcal{N}(A)}_{(d+1) times}$$

for all $k \in \mathbb{N}_0$ and $l = 1, \ldots, 6$.

Proof: Follows from Proposition 3.3, Proposition 3.5, and (3.4.7). The strict inclusion is due to the shift within each $H_{d,l}^{(k)}$. □

It might happens during a PAI that the same block is updated twice or more times in a row by operators $H_{d,l}^{(k_1)}, \ldots, H_{d,l}^{(k_L)}$, satisfying $i(k_1) = \ldots = i(k_L) = i_0$. If $d = 0$, then

$$\begin{aligned} H_{d,l}^{(k_1)} \cdot \ldots \cdot H_{d,l}^{(k_L)} &= H_l^{(i_0)} \cdot \ldots \cdot H_l^{(i_0)} \\ &= (P^{(i_0)} + Q_l^{(i_0)})^{k_L - k_1 + 1} = P^{(i_0)} + (Q_l^{(i_0)})^{k_L - k_1 + 1}, \end{aligned}$$

since the convergent part, i.e. $Q_l^{(i_0)}$ is constant for the spectral decompositions. This implies that $(P^{(i_0)}, (Q_l^{(i_0)})^{k_L - k_1 + 1})$ is a spectral decomposition for $H_{d,l}^{(k_1)} \cdot \ldots \cdot H_{d,l}^{(k_L)}$. In the case $d \geq 1$ the convergent part will change in general but the result is still the same.

Proposition 3.7 *Let $H_{d,l}^{(k_1)}, \ldots, H_{d,l}^{(k_L)}$ be given such that $i(k_1) = \ldots = i(k_L) = i_0$ and let $(P_d^{(i_0)}, Q_{d,l}^{(k_j)}), j = 1, \ldots, L$ be the spectral decompositions. Then*

$$H_{d,l}^{(k_1)} \cdot \ldots \cdot H_{d,l}^{(k_L)} = P_d^{(i_0)} + \underbrace{Q_{d,l}^{(k_1)} \cdot \ldots \cdot Q_{d,l}^{(k_L)}}_{=:\tilde{Q}}$$

and $(P_d^{(i_0)}, \tilde{Q})$ is a spectral decomposition of $H_{d,l}^{(k_1)} \cdot \ldots \cdot H_{d,l}^{(k_L)}$.

Proof: See Appendix A

3.6 On the way to the main problem

Equipped with the results from the foregoing section, a weak result for Schwarz iterations in the non-overlap case can be proved. This result is not new (since it is restricted to iterations corresponding to exact and inexact Gauss-Seidel), but the technique of the proof will reveal a few problems which will led to the graph based approach in Section 4. Moreover, other relations between Schwarz iterations and partially asynchronous iterations will be presented.

Suppose there are linear operators $H^{(i)} \in \mathbb{R}^{n\times n},\ i = 1,\dots,p$ with spectral decompositions $(P^{(i)}, Q^{(i)})$. Assume further

$$\bigcap_{j=1}^{p} \mathcal{R}(P^{(j)}) \neq \{0\} \quad \text{and} \tag{3.6.1}$$

$$\text{the sum } \mathcal{N}(P^{(1)}) \oplus \dots \oplus \mathcal{N}(P^{(p)}) \subset \mathbb{R}^n \text{ is direct.} \tag{3.6.2}$$

Lemma 3.5 *Let $H^{(1)}, \dots, H^{(p)}$ be given as above, i.e., (3.6.1) and (3.6.2) hold. Then for each $k \in \{1,\dots,p\}$:*

$$\mathcal{N}(I - H^{(1)} \cdot \ldots \cdot H^{(k)}) = \bigcap_{j=1}^{k} \mathcal{N}(I - H^{(j)}) = \bigcap_{j=1}^{k} \mathcal{R}(P^{(j)}). \tag{3.6.3}$$

The following proposition is needed for the proof.

Proposition 3.8 *Let (P, Q) be a spectral decomposition of $H \in \mathbb{R}^{n\times n}$. Then for each $x \in \mathbb{R}^n$ and $y = Hx$ there is a vector $x^\Delta \in \mathcal{N}(P)$ such that $x = y + x^\Delta$.*

Proof: For an arbitrary $x \in \mathbb{R}^n$ there holds

$$x = Hx + (I - H)x = y + x^\Delta.$$

Now (cf. Lemma 1.4)

$$x^\Delta = (I - H)x = (I - Q)(I - P)x = (I - P)(I - Q)x$$

and obviously $x^\Delta \in \mathcal{N}(P)$. □

Proof (of Lemma 3.5): Since $\supset$ is trivial, only $\subset$ will be proven. Furthermore, only the case $k = p$ will be shown since the case $k < p$ follows immediately.

Let $x \in \mathcal{N}(I - H^{(1)} \cdot \ldots \cdot H^{(p)})$, $x \neq 0$, then there are vectors $x_i \in \mathcal{R}(H^{(i)})$ and $x_i^\Delta \in \mathcal{N}(P^{(i)})$ (cf. Proposition 3.8) for $i = 1,\dots,p$, such that

$$\begin{aligned}
H^{(p)}x = x_p &\quad \text{and} \quad x = x_p + x_p^\Delta, \\
H^{(p-1)}x_p = x_{p-1} &\quad \text{and} \quad x_p = x_{p-1} + x_{p-1}^\Delta \Rightarrow x = x_{p-1} + x_{p-1}^\Delta + x_p^\Delta, \\
&\ \ \vdots \\
H^{(2)}x_3 = x_2 &\quad \text{and} \quad x_3 = x_2 + x_2^\Delta \Rightarrow x = x_2 + \sum_{j=2}^{p} x_j^\Delta, \\
H^{(1)}x_2 = x_1 &\quad \text{and} \quad x_2 = x_1 + e_1^\Delta \Rightarrow x = x_1 + \sum_{j=1}^{p} x_j^\Delta.
\end{aligned}$$

But $H^{(1)}x_2 = x_1 = x$, hence

$$\sum_{j=1}^{p} x_j^{\Delta} = 0 \quad \text{with } x_j^{\Delta} \in \mathcal{N}(P^{(j)}),\ j = 1, \ldots, p.$$

The vectors x_j^{Δ} are linearly independent owing to (3.6.2), i.e. $x_j^{\Delta} = 0$ for all $j = 1, \ldots, p$. This implies $x = x_1 = \ldots = x_p$ and $H^{(j)}x = x$ for all $j = 1, \ldots, p$. □

Remark 3.6 *1) It follows from (3.6.2) that the order of the operators does not matter.*

2) The assumption $\rho(Q^{(j)}) < 1$ can be weakened to $1 \notin \sigma(Q^{(j)})$. The lemma is then applicable to more general decompositions (cf. Section 1.3).

Assume now the model problem *GMP* and a regular decomposition $S_1, \ldots, S_p$ without overlap, i.e. a regular partitioning of $\{1, \ldots, n\}$ (cf. Section 3.1).

After an appropriate permutation, A can be assumed to be partitioned by blocks in the following way:

$$A = \begin{pmatrix} A_{11} & A_{12} & \ldots & A_{1p} \\ \vdots & \ddots & & \vdots \\ A_{p1} & A_{p2} & \ldots & A_{pp} \end{pmatrix},$$

with square diagonal blocks A_{ii} of dimension n_i. With the terminology of Section 3.2 and 3.3, the i-th projection of the Schwarz iteration discussed in Sections 3.2 and 3.3 becomes

$$\begin{aligned} H_1^{(i)} &= \begin{pmatrix} I_{n_1} & & & & & & \\ & \ddots & & & & & \\ & & I_{n_{i-1}} & & & & \\ A_{i,i}^{-1}A_{i1} & \ldots & A_{i,i}^{-1}A_{i,i-1} & 0_{n_i} & A_{i,i}^{-1}A_{i,i+1} & \ldots & A_{i,i}^{-1}A_{i,p} \\ & & & & I_{n_{i+1}} & & \\ & & & & & \ddots & \\ & & & & & & I_{n_p} \end{pmatrix} \\ &= \begin{pmatrix} I & 0 & 0 \\ M_i^{-1}N_L & 0 & M_i^{-1}N_R \\ 0 & 0 & I \end{pmatrix} =: P^{(i)} \end{aligned}$$

where N_L and N_R are the "left" and the "right" part of N_i.

These projections fulfil (3.6.1) and (3.6.2), hence the following lemma is an immediate consequence.

Lemma 3.6 *Assume the generalised model problem* GMP *and a regular partitioning* $S_1, \ldots, S_p$ *w.r.t.* A*. For an* $l \in \{1, \ldots, 6\}$ *let* $H_l^{(i)}$ *be the local Schwarz operators for* $i = 1, \ldots, p$*. Then*

$$\begin{aligned} \mathcal{N}(I - H_l^{(1)} \cdot \ldots \cdot H_l^{(p)}) &= \bigcap_{j=1}^{p} \mathcal{N}(I - H_l^{(j)}) \\ &= \bigcap_{j=1}^{p} \mathcal{R}(P^{(j)}) \\ &= \mathcal{N}(A) \end{aligned}$$

for all $l \in \{1, \ldots, 6\}$*.*

Lemma 3.6 states, that if the multiplicative Schwarz iteration

$$x^{k+1} = T_l x^k, \; k = 0, 1, 2, \ldots$$

converges, then the limit is some vector $v \in \mathcal{N}(A)$ for each $l \in \{1, \ldots, 6\}$. Hence the next result follows easily from Lemmas 3.1 and 3.2.

Theorem 3.9 *Assume the generalised model problem* GMP *and a regular partitioning* $S_1, \ldots, S_p$ *w.r.t.* A*. Let* T_l *be the global Schwarz operator for a fixed* $l \in \{2, 4, 6\}$*. Then for any* $x^0 \in \mathbb{R}^n$ *the iteration*

$$x^{k+1} = T_l x^k + c$$

converges to the solution x^* *of* GMP*, i.e.*

$$x^* = Ax^* + b$$

wherein c is given by (3.2.4).

Remark 3.7 *The result of Theorem 3.9 also holds for* T_3 *and* T_5 *(cf. Section 3.3) if the diagonals of each* $F_i^{-1}G_i$ *and* B_i *are positive.*

There is an asynchronous equivalent to Lemma 3.6.

Lemma 3.7 *Assume the generalised model problem* GMP *and a regular partitioning* $S_1, \ldots, S_p$ *w.r.t.* A*. For a fixed* $l \in \{1, \ldots, 6\}$ *let* $\{H_{d,l}^{(k)}\}_{k \in \mathbb{N}_0}$ *be the set of local operators of a partially asynchronous iteration* $(A, x^0, \{\mathcal{J}_k, \mathcal{S}_k\}_{k \in \mathbb{N}_0})$ *and* $i(k)$ *be the index of the corresponding block. Furthermore, let* $\mathcal{N}(A) = \operatorname{span}\{z_1, \ldots, z_m\}$ *and define* $z_{d,i} = (z_i^T, \ldots, z_i^T)^T \in \mathbb{R}^{(d+1)n}$*, for* $i = 1, \ldots, m$*. Then for any set* $\{H_{d,l}^{(k_1)}, \ldots, H_{d,l}^{(k_p)}\}$ *such that*

$$\bigcup_{j=1}^{p} \{i(k_j)\} = \{1, \ldots, p\},$$

there holds

$$\begin{aligned}\mathcal{N}(I - H_{d,l}^{(k_1)} \cdot \ldots \cdot H_{d,l}^{(k_p)}) &= \bigcap_{j=1}^{p} \mathcal{N}(I - H_{d,l}^{(k_j)}) \\ &= \bigcap_{j=1}^{p} \mathcal{R}(P_d^{(i(k_j))}) \\ &= \operatorname{span}\{z_{d,1}, \ldots, z_{d,m}\} \subsetneq \mathcal{N}(A) \times \ldots \times \mathcal{N}(A).\end{aligned}$$

Here, $P_d^{(i(k_j))}$ *is the projection (3.5.10) onto* $\mathcal{N}(I - H_{d,l}^{(k_j)})$ *for* $j = 1, \ldots, p$.

Proof: See Appendix A

There is no equivalent of Theorem 3.9 in the context of PAIs, since the diagonal of any product of local operators $H_{d,l}^{(k)}$ will never be positive in general. Additionally, the order of local operators will be almost chaotic in the sense

$$\bigcup_{j=1}^{s} \{i(k_j)\} = \{1, \ldots, p\}$$

for $s > p$ and

$$\bigcup_{j=1}^{l} \{i(k_j)\} \subsetneq \{1, \ldots, p\}$$

for $l < s$. Therefore, Lemma 3.7 is interesting for theory but not for practice.

In view of Lemma 3.6 and 3.7, there are two questions which arise immediately.

1) What happens if there is overlap?

2) What happens if the order becomes chaotic?

Since overlap causes a lot of synchronisation in PAIs, it will be avoided most likely. Thus, question 1) concerns more the Schwarz type methods. On the other hand, question 2) is to be discussed for PAIs, because the order of the operators is arbitrary in general.

To discuss question 1), consider the situation of Lemma 3.5 and Lemma 3.6. If overlap occurs, there still holds

$$x = T_l x = H_l^{(1)} \cdot \ldots \cdot H_l^{(p)} x \Leftrightarrow \sum_{j=1}^{p} x_j^{\Delta} = 0$$

for each $l = 1, \ldots, 6$. But the vectors x_j^{Δ} need not to be linearly independent anymore. This means that in the situation of Theorem 3.9, the operator T_l is still semiconvergent, but

$$\mathcal{N}(I - T_l) = \mathcal{N}(I - H_l^{(1)} \cdot \ldots \cdot H_l^{(p)}) = \mathcal{N}(A)$$

may not necessarily hold. In this situation, convergence is obtained but convergence to the fixed point z is not guaranteed.

It should be mentioned that the same situation arises in the analysis of additive Schwarz methods. Again, the overlap is to be discussed rather than the order. For $l \in \{1, \ldots, 6\}$ there holds

$$y = T_{\theta,l}x = x - \theta \sum_{j=1}^{p} \underbrace{(I - H^{(j)})x}_{=:y_j^{\Delta}} = x - \theta \sum_{j=1}^{p} y_j^{\Delta}.$$

And again

$$x = T_{\theta,l}x \Leftrightarrow \sum_{j=1}^{p} y_j^{\Delta} = 0.$$

Since $y_j^{\Delta} \in \mathcal{N}(P^{(j)})$ the equation

$$N(I - T_{\theta,l}) = N(I - A)$$

can be guaranteed if no overlap occurs.

For question 2) consider a PAI for $d = 0$ without overlap. Here, the number of local operators to be applied in a sequence such that each block is updated at least once, is at most s (cf. (3.4.7)), and usually $s > p$. Consider a sequence $H_l^{(k_1)}, \ldots, H_l^{(k_m)}$, $s \geq m > p$ such that each block is updated and let

$$x = H_l^{(k_m)} \cdot \ldots \cdot H_l^{(k_1)} x.$$

If the operators which update the same block occur in a consecutive order, i.e.

$$H_l^{(k_m)} \cdot \ldots \cdot H_l^{(k_1)} = \left(H_l^{(\pi(1))}\right)^{t_1} \cdot \ldots \cdot \left(H_l^{(\pi(p))}\right)^{t_p}$$

for a permutation π and numbers $t_j \in \mathbb{N}$, $j = 1, \ldots, p$, then Lemma 3.7 applies by Proposition 3.7.

But if all operators occur in an arbitrary order, then an ansatz according to Lemma 3.5 leads to

$$\sum_{j=1}^{m} x_j^{\Delta} = 0.$$

If, as usually, $P^{(i)} = H_1^{(i)}$ and $I_i := \{j : x_j^{\Delta} \in \mathcal{N}(P^{(i)})\}$, $i = 1, \ldots, p$, then

$$0 = \sum_{j=1}^{m} x_j^{\Delta} = \sum_{i=1}^{p} \sum_{j \in I_i} x_j^{\Delta}$$

where

$$\sum_{j \in I_i} x_j^{\Delta} \in \mathcal{N}(P^{(i)}),\ i = 1, \ldots, p.$$

Since the null spaces $\mathcal{N}(P^{(i)})$ are direct sums there follows

$$0 = \sum_{j \in I_i} x_j^{\Delta}, \; i = 1, \ldots, p.$$

But again, $x_j^{\Delta} = 0$ for $j = 1, \ldots, m$ may not follow and there is no guarantee that

$$\mathcal{N}(I - H_l^{(k_m)} \cdot \ldots \cdot H_l^{(k_1)}) = \mathcal{N}(A)$$

holds.

Hence the algebraic ansatz does not yield general results, but reveals the problems to be solved. In Chapter 4 an approach will be introduced which will eliminate the above problems for Schwarz type methods and PAIs. This approach is based on the combinatorial non-zero structure rather than on invariant subspaces.

Chapter 4

A graph based approach for *MP*

In this chapter a graph based approach will be presented, which is applicable to the model problem *MP* and the iteration schemes discussed in Chapter 3. The approach will eliminate the problems which have been revealed in Section 3.6. The theory given now is somehow in the tradition of [21, 46, 59, 61] (cf. Section 6.5) but not the same. Some parts of it can also be found implicitly in [39, 55] (cf. Section 6.4).

The approach is very basic and uses a convergence theory which has become out of the scientific interest as the impact of analytical methods grew. However, the approach leads to the insight why convergence just happens since it is based on the structure of the iteration operators rather than analytical properties. Moreover, the theory solves the problems in an easy manner and that is what matters.

Note that the comparison with known results takes place in Chapter 6. This has been done to present the theory as a whole.

4.1 Introduction

To start with an easy example, consider the nonnegative matrix

$$B = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

and a multiplicative (one-level) Schwarz iteration for $A = I - B$ with three one-element blocks, one for each component. Then the local operators become

$$H^{(1)} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, H^{(2)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } H^{(3)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Now, it is not hard to see, that each product $H^{(1)}H^{(2)}H^{(3)}$, $H^{(3)}H^{(1)}H^{(2)}$ and $H^{(2)}H^{(3)}H^{(1)}$ has a positive column and represents a convergent operator (actually a rank one projection), e.g.,

$$H^{(3)}H^{(1)}H^{(2)} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

This guarantees the convergence of the multiplicative Schwarz iteration by Lemma 1.2. Each other order leads to a non-convergent operator with the eigenvalues $\{1, -1, 0\}$, e.g.,

$$H^{(1)}H^{(3)}H^{(2)} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

This example can be generalised to arbitrary dimensions.

In view of Theorem 3.9, the application of a relaxed version will solve the problem from the above example; but this is not always a good idea, as the following example will show.

Consider the singular M-matrix

$$A := \begin{pmatrix} 0.7 & -0.6 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.6 & -0.2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.2 & -0.1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.1 & -0.5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.5 & -0.25 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.25 & -0.75 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.75 & -0.5 \\ -0.7 & 0 & 0 & 0 & 0 & 0 & 0 & 0.5 \end{pmatrix}$$

and the regular partitioning $S = \{S_1, S_2, S_3\}$ where

$$S_1 = \{1, 2, 3\}, \quad S_2 = \{4, 5\}, \quad S_3 = \{6, 7, 8\}.$$

The construction of the local Schwarz operators $H_1^{(1)}, H_1^{(2)}$ and $H_1^{(3)}$ leads to the same result as above, i.e. $H_1^{(1)}H_1^{(2)}H_1^{(3)}$ and any cyclic permutation of $\{1, 2, 3\}$ leads to a semiconvergent operator, having a positive column, while each other order leads to a non-convergent operator. To become independent of the order, consider the relaxed Schwarz iteration scheme (Update 2 in Section 3.3) with local operators $H_2^{(1)}, H_2^{(2)}, H_2^{(3)}$. Now, Theorem 3.9

implies the convergence of the relaxed Schwarz iteration, but the number of iterations still depends on the order.

A MATLAB test with the starting vector

$$x^0 = (0.6517 \ -0.4133 \ -0.3918 \ 0.6625 \ 0.3456 \ 0.4923 \ 0.2511 \ -0.464\ )^T$$

and the orders 1–2–3 and 3–2–1 leads to the following number of iterations using different values of the relaxation parameter α and a desired accuracy of 10^{-10}.

	3–2–1	1–2–3
$\alpha = 0.25$	345	24
$\alpha = 0.5$	21735	21
$\alpha = 0.75$	>60000	21
$\alpha = 1$	divergent	21

The result for the order 3–2–1 is to be expected, since the larger α becomes, the more the iteration operator approaches a non-convergent one. The process ends up in a non-convergent operator, whereas the operator with respect to 1–2–3 is always semiconvergent.

This example shows that it is probably very important to think about orderings which yield (fast) convergence rather than enforcing (slow) convergence via relaxation. Therefore, it might be a good idea to think about methods that allow to create a positive column. Here is another good reason:

Theorem 4.1 *Let $B \in \mathbb{R}^{n\times n}$ be nonnegative with $\rho(B) = 1$. Assume that B has a positive column and there exists a $z > 0$ such that $Bz = z$. Then $\gamma(B) < 1$, i.e. B is semiconvergent.*

The above result has been used in [31] and [62] and will be proven in the next section. However, it is easy to prove it in the ST-matrix context.

Proof: Let the j-th column of B be positive. Then B contains a spanning tree with the root index j. Since $Bz = z$ for a positive z, it follows from Corollary 2.4 that B is an ST-matrix. Owing to Lemma 2.2, there is a permutation matrix Π such that

$$\Pi B \Pi^T = \begin{pmatrix} D & 0 \\ E & F \end{pmatrix},$$

where $\rho(F) < 1$, $\rho(D) = 1$ and D is irreducible. But positive columns are invariant under symmetric permutation. Hence D has a positive diagonal element. But then, $\gamma(D) < 1$ from part 1) of Theorem 1.11. Thus $\gamma(B) < 1$ and the semiconvergence follows from Lemma 1.2. □

Another good reason to use operators having a positive column is the following one.

Consider local nonnegative Schwarz operators $H^{(i)},\ i = 1, \ldots, p$ and a positive vector $z \in \mathbb{R}^n$ such that

$$Hz = H^{(1)} \cdot \ldots \cdot H^{(p)} z = z, \ \ \mathcal{N}(I - H) = \operatorname{span}\{z\}.$$

Since $H \geq 0$, there is a left eigenvector $y \geq 0$ corresponding to the eigenvalue 1. If $y^T z = 1$, then

$$z \cdot y^T \in \mathbb{R}^{n \times n}$$

is a nonnegative projection onto span$\{z\}$ and necessarily has a positive column. Thus, if H has a positive column, it is semiconvergent by Theorem 4.1 and each power H^k has also a positive column; hence the limit should have. Moreover,

$$\lim_{k \longrightarrow \infty} H^k = z \cdot y^T,$$

so it seems to be natural to start with an operator having a positive column.

But again, convergence can not be guaranteed if the iteration becomes inhomogeneous, i.e., H is replaced by $H^{(k)}$. Even if each $H^{(k)}$ contains a positive column, an appropriate common norm for each operator is needed to obtain convergence. But in practice norms are not generally available. Hence another tool is needed and will be introduced in the next section.

4.2 A framework for convergence

The convergence theorem presented now is not based on a norm but on algebraic properties of row stochastic matrices and can be easily applied to the model problem *MP*. The theorem is not new and has been successfully used by several authors, e.g., in [39].

Definition 4.1 *Let $B \in \mathbb{R}^{n \times n}$ be row stochastic (cf. Remark 2.1). Then B is said to be a* Markov-matrix *if B has a positive column. B is called* scrambling *if any two rows of B have a positive element in the same position. Finally, B is said to be* ST-regular *if B is an ST-matrix and semiconvergent.*

Remark 4.1 *The above definition is taken from [62]. Note that ST-regular matrices are called regular matrices in [62]. They have been renamed here because today the term regular is used for irreducible semiconvergent nonnegative matrices (see [9, 73]). Anyway, regular matrices are of course ST-regular.*

If $n \in \mathbb{N}$ is fixed, then $\mathcal{M}_n, \mathcal{G}_n^3$ and $\mathcal{G}_n^1$ denote the set of $n \times n$ Markov-matrices, scrambling matrices, and ST-regular matrices respectively.

Theorem 4.2 $\mathcal{M}_n \subsetneq \mathcal{G}_n^3 \subsetneq \mathcal{G}_n^1$. *Moreover $\mathcal{M}_n$ and $\mathcal{G}_n^3$ are closed under multiplication while $\mathcal{G}_n^1$ is not.*

Proof: The proof can be found in [62]. A nice counterexample that $\mathcal{G}_n^1$ is not closed under multiplication is given in [31].

All three sets are connected in the following sense:

Theorem 4.3 *If $A \in \mathcal{G}_n^3$ or $A \in \mathcal{G}_n^1$, then there exists a $k \in \mathbb{N}$ such that $A^k \in \mathcal{M}_n$.*

Proof: See also [62].

The following result is about the spectra of nonnegative matrices with constant row sums.

Theorem 4.4 *([62], Theorem 2.10) Let $B \in \mathbb{R}^{n\times n}$ be a nonnegative matrix with constant row sums a and let $\lambda \neq a$ be an eigenvalue, then*

$$
\begin{aligned}
|\lambda| \leq \tau(B) \quad &:= \quad \frac{1}{2}\max_{1\leq i,j\leq n}\left\{\sum_{k=1}^{n}|b_{ik}-b_{jk}|\right\} \\
&= \quad a - \min_{1\leq i,j\leq n}\left\{\sum_{k=1}^{n}\min\{b_{ik},b_{jk}\}\right\} \\
&\leq \quad a - \max_{1\leq k\leq n}\left\{\min_{1\leq i\leq n}\{b_{ik}\}\right\}.
\end{aligned}
\tag{4.2.1}
$$

Remark 4.2 *Let the assumptions of Theorem 4.4 be fulfilled. Then $\gamma(B) = a$ is possible, if there are two rows $b_{i,*}$ and $b_{j,*}$ such that $\langle b_{i,*}, b_{j,*}\rangle = 0$.*

Corollary 4.1 *Let $B \in \mathbb{R}^{n\times n}$ be row stochastic.*

1) $\tau(B) \leq 1$

2) If either $B \in \mathcal{M}_n$ or $B \in \mathcal{G}_n^3$, then $\tau(B) < 1$.

Markov and scrambling matrices have a very nice contraction property, as will be presented now.

Theorem 4.5 *([62], Theorem 3.1) Assume $B \in \mathbb{R}^{n\times n}$ is row stochastic and let $x \in \mathbb{R}^n$ be arbitrary. Then for $y = Bx$:*

$$\left\{\max_j y_j - \min_j y_j\right\} \leq \tau(B)\left\{\max_j x_j - \min_j x_j\right\}.$$

Moreover, the functional $\tau(\cdot)$ is submultiplicative.

Theorem 4.6 *([62], Theorem 4.3) Let there be given row stochastic matrices $B_1, B_2 \in \mathbb{R}^{n\times n}$, then*

$$\tau(B_1 \cdot B_2) \leq \tau(B_1)\cdot\tau(B_2).$$

Remark 4.3 *Any continuous functional $\tau(\cdot) : \mathbb{R}^{n\times n} \longrightarrow \mathbb{R}$ which satisfies condition 1) of Corollary 4.1 and Theorem 4.6, is called a* coefficient of ergodicity *and there are a few more in the literature (see e.g., [62]). Here, only the functional given by (4.2.1) is used and will play a vital role in the sequel.*

The following theorem will be the key to prove convergence for multiplicative Schwarz iterations and PAIs with block updates defined in Chapter 3. The proof can also be found in [62] in the context of so called "weak ergodicity of backward products". But the part of the theory presented here is much easier and the full range of the results in [62] will not be exploited. The first part of the proof is similar to the proof of Theorem 2 in [39], the second part to Theorem 4.17 in [62]. Hence the proof is given for the sake of completeness.

Theorem 4.7 *Let there be given a sequence $\{H^{(k)}\}_{k\in\mathbb{N}_0}$ of nonnegative matrices. Assume that each $H^{(k)}$ has a positive column and there exists a positive vector z such that $\mathcal{N}(I-H^{(k)}) = \operatorname{span}\{z\}$ for all $k \in \mathbb{N}_0$. If there exists a constant $\kappa > 0$ such that $h_{ij}^{(k)} \geq \kappa$ for each $h_{ij}^{(k)} \neq 0$ and all $k \in \mathbb{N}_0$, then the sequence*

$$x^{k+1} = H^{(k)}x^k, \;\; k = 0, 1, 2, \dots$$

converges to λz for any vector $x^0 \in \mathbb{R}^n$, where $\lambda \in \mathbb{R}$. Moreover, the convergence is obtained at a geometric rate of at least

$$0 < \theta := 1 - \min_{1\leq i,j\leq n} \left\{ \frac{\kappa \cdot z_j}{z_i} \right\} < 1$$

and the limit $H^* := \lim_{k\longrightarrow\infty} H^{(k)} \cdot H^{(k-1)} \cdot \ldots \cdot H^{(0)}$ is a projection onto $\operatorname{span}\{z\}$*.*

Proof: Define $D = \operatorname{diag}(z_1, \dots, z_n)$ and $G^{(k)} := D^{-1}H^{(k)}D$. For an arbitrary $x^0 \in \mathbb{R}^n$ let $y^0 := D^{-1}x^0$ and consider the equivalent iteration

$$y^{k+1} = G^{(k)}y^k, \;\; k = 0, 1, 2, \dots .$$

Then obviously $x^k = Dy^k$ for all $k \in \mathbb{N}_0$. On the other hand, each $G^{(k)}$ is now row stochastic, thus $G^{(k)} \in \mathcal{M}_n$ for all $k \in \mathbb{N}_0$. For an arbitrary $k \in \mathbb{N}_0$ let

$$\eta := \min_{1\leq i,j\leq n} \{g_{ij}^{(k)} \neq 0\} \leq 1,$$

then

$$1 \geq \eta = \min_{1\leq i,j\leq n} \left\{ \frac{h_{ij}^{(k)} \cdot z_j}{z_i}, \; h_{ij}^{(k)} \neq 0 \right\} \geq \min_{1\leq i,j\leq n} \left\{ \frac{\kappa \cdot z_j}{z_i} \right\} > 0,$$

and furthermore

$$\max_{1\le j\le n}\left\{\min_{1\le i\le n}\{g_{ij}^{(k)}\}\right\}\ge\eta. \tag{4.2.2}$$

(Note that the last relation holds especially for matrices in $\mathcal{M}_n$ and is the only point where the assumption that each $H^{(k)}$ has a positive column is needed.) Owing to Theorem 4.4,

$$\begin{aligned}\tau(G^{(k)}) &\le 1-\max_{1\le j\le n}\left\{\min_{1\le i\le n}\{g_{ij}^{(k)}\}\right\}\\ &\le 1-\eta\\ &\le 1-\min_{1\le i,j\le n}\left\{\frac{\kappa\cdot z_i}{z_j}\right\}\\ &= \theta<1,\end{aligned}$$

uniformly for all $k\in\mathbb{N}_0$.

Now apply Theorem 4.5 for a fixed k, then

$$\begin{aligned}\left\{\max_j y_j^k-\min_j y_j^k\right\} &\le \tau(G^{(k-1)})\left\{\max_j y_j^{k-1}-\min_j y_j^{k-1}\right\}\\ &\le \tau(G^{(k-1)})\cdot\ldots\cdot\tau(G^{(0)})\left\{\max_j y_j^0-\min_j y_j^0\right\}\\ &\le \theta^k\left\{\max_j y_j^0-\min_j y_j^0\right\}.\end{aligned}$$

Thus

$$\lim_{k\longrightarrow\infty}\left\{\max_j y_j^k-\min_j y_j^k\right\}=0,$$

hence

$$\lim_{k\longrightarrow\infty} y^k=\lambda\cdot e,\quad \lambda\in\mathbb{R},$$

and finally,

$$\lim_{k\longrightarrow\infty} x^k=\lim_{k\longrightarrow\infty} Dy^k=\lambda\cdot D\cdot e=\lambda\cdot z.$$

To complete the proof let $G^{(0,k)}:=G^{(k)}\cdot\ldots\cdot G^{(0)}$. Then $\tau(G^{(0,k)})\le\theta^k$ and $\lim_{k\longrightarrow\infty}\tau(G^{(0,k)})=0$. Hence, for an $\varepsilon>0$ there is a k_0 such that

$$g_{il}^{(0,k)}-\varepsilon\le g_{jl}^{(0,k)}\le g_{il}^{(0,k)}+\varepsilon$$

for all $i, j, l = 1, \ldots, n$ and $k \geq k_0$ from the definition of $\tau(\cdot)$ (cf. (4.2.1)). But since $G^{(0,k+1)} = G^{(k+1)}G^{(0,k)}$ there holds

$$\begin{aligned} g_{il}^{(0,k)} - \varepsilon &= \sum_{m=1}^{n} g_{jm}^{(k+1)} (g_{il}^{(0,k)} - \varepsilon) \\ &\leq \sum_{m=1}^{n} g_{jm}^{(k+1)} g_{ml}^{(0,k)} \\ &= g_{jl}^{(0,k+1)} \\ &\leq \sum_{m=1}^{n} g_{jm}^{(k+1)} (g_{il}^{(0,k)} + \varepsilon) \\ &= g_{il}^{(0,k)} + \varepsilon. \end{aligned}$$

And by induction

$$g_{il}^{(0,k)} - \varepsilon \leq g_{jl}^{(0,k+r)} \leq g_{il}^{(0,k)} + \varepsilon$$

for all $i, j, l = 1, \ldots, n$ and $r \geq 0$. Thus, $g_{jl}^{(0,k)}$ is a Cauchy sequence in k and

$$\lim_{k \longrightarrow \infty} G^{(0,k)} = G^*$$

exists. Since $\tau(G^*) = 0$ by Theorem 4.4, the limit must be a rank one matrix. Additionally, $G^{(k)}e = e$ for all $k \in \mathbb{N}_0$ implies $G^*e = e$. Thus, G^* is a rank one projection. Now it is obvious that $H^* = DG^*D^{-1}$ and the proof is complete. □

Remark 4.4 *1) If the sequence $\{H^{(k)}\}_{k \in \mathbb{N}_0}$ consists of only finitely many different matrices, then Theorem 4.7 holds without any restriction on the elements, i.e. $h_{ij}^{(k)} \geq \kappa$ holds automatically whenever $h_{ij}^{(k)} \neq 0$.*

2) Theorem 4.7 implies Theorem 4.1 if a constant sequence of operators is considered.

The next corollary is given to emphasise the role of a positive column. Its proof follows easily from (4.2.2) and it plays a major role in the construction of convergent multiplicative Schwarz iterations.

Corollary 4.2 *Let there be given a sequence $\{H^{(k)}\}_{k \in \mathbb{N}_0}$ of nonnegative matrices. Assume that each $H^{(k)}$ has its j_k-th column positive and there exists a positive vector z such that $\mathcal{N}(I - H^{(k)}) = \operatorname{span}\{z\}$ for all $k \in \mathbb{N}_0$. If there exists a constant $\kappa > 0$ such that $h_{ij_k}^{(k)} \geq \kappa$ for each $i = 1, \ldots, n$ and all $k \in \mathbb{N}_0$, then Theorem 4.7 applies to the sequence $\{H^{(k)}\}_{k \in \mathbb{N}_0}$.*

The following result is needed to complete the convergence theory.

Theorem 4.8 *Let $\{H^{(k)}\}_{k\in\mathbb{N}_0}$ be a sequence of nonnegative square matrices such that there exists a vector $z > 0$ satisfying $H^{(k)}z = z$ for all $k \in \mathbb{N}_0$. Suppose $\lim_{k\longrightarrow\infty} H^{(k)} \cdot \ldots \cdot H^{(0)} = P$ and P is a projection onto $\operatorname{span}\{z\}$. If $\{c^k\}_{k\in\mathbb{N}_0}$ is a sequence of vectors such that $c^k \in \mathcal{R}(I-H^{(k)})$ and there exists a $x^* \in \mathbb{R}^n$ such that $(I-H^{(k)})x^* = c^k$ for all $k \in \mathbb{N}_0$, then the sequence*

$$x^{k+1} = H^{(k)}x^k + c^k, \quad k = 0, 1, 2, \ldots$$

converges to $x^* + \lambda z$, $\lambda \in \mathbb{R}$ for every given $x^0 \in \mathbb{R}^n$ and $(I - H^{(k)})(x^* + \lambda z) = c^k$ for all $k \in \mathbb{N}_0$.

Proof: Define $e^k = x^k - x^*$, then $e^{k+1} = H^{(k)}e^k$ and

$$\lim_{k\longrightarrow\infty} e^k = Pe^0 = \lambda z$$

exists for some $\lambda \in \mathbb{R}$. But then

$$\lim_{k\longrightarrow\infty} x^k = x^* + \lambda z$$

and

$$(I - H^{(k)})(x^* + \lambda z) = c^k + (I - H^{(k)})(\lambda z) = c^k$$

for all $k \in \mathbb{N}_0$. □

As a motivation for the sections to follow, consider *MP* and let $T_l = H_l^{(1)} \cdot \ldots \cdot H_l^{(p)}$ be a multiplicative global Schwarz operator for $l \in \{1, \ldots, 6\}$ as defined in Section 3.2 and 3.3. If it is possible to guarantee that T_l has a positive column whose elements are bounded from below, then convergence is obtained from Corollary 4.2. Actually, Theorem 4.1 is sufficient for $l = 1, 2, 5, 6$, but Corollary 4.2 is needed for Update 3 and 4.

For Update 3 and 4 there is a dependence on the iteration index k, since the number of inner iterations may vary, i.e. $T_l = T_l^{(k)}$ for $l = 3, 4$ (and even the right hand side, if any, depends on the iteration index). Thus the iteration becomes

$$x^{k+1} = T_l^{(k)}x^k + c^k$$

for a given $x^0 \in \mathbb{R}^n$ and proper right hand sides c^k.

The problem here is, that the more inner iterations are carried out the smaller some elements of $T_3^{(k)}$ will become (cf. (3.3.2) and (3.3.3)).

Anyway, for Update 3, the identity (3.3.2) implies for each $i \in \{1, \ldots, p\}$

$$\Pi_i H_3^{(k,i)} \Pi_i^T = \begin{pmatrix} R^{(k,i)} & (I - R^{(k,i)})M_i^{-1}N_i \\ 0 & I \end{pmatrix} \geq 0.$$

But

$$(I - R^{(k,i)})M_i^{-1} = \sum_{j=0}^{q(k,i)-1} (F_i^{-1}G_i)^j F_i^{-1} \geq F_i^{-1} \geq 0$$

and therefore

$$\begin{aligned} 0 &\leq \tilde{H}_3^{(i)} := \Pi_i^T \begin{pmatrix} 0 & F_i^{-1}N_i \\ 0 & I \end{pmatrix} \Pi_i \\ &\leq \Pi_i^T \begin{pmatrix} R^{(k,i)} & (I - R^{(k,i)})M_i^{-1}N_i \\ 0 & I \end{pmatrix} \Pi_i = H_3^{(k,i)}, \end{aligned}$$

since the inner splitting is weak regular and $R^{(k,i)} \geq 0$.

If it is possible to find a decomposition such that

$$\tilde{T}_3 := \tilde{H}_3^{(1)} \cdot \ldots \cdot \tilde{H}_3^{(p)}$$

has a positive column, then convergence is obtained from Corollary 4.2. This is because

1) $\tilde{T}_3 \leq T_3^{(k)}$ for all $k \in \mathbb{N}_0$, and

2) each element of $\tilde{T}_3$ is bounded from below by some $\kappa > 0$, since $\tilde{T}_3$ is independent of the number of inner iterations $q(k, i)$, thus independent of k.

An analogous argumentation holds for $T_4^{(k)}$. Hence, if Corollary 4.2 applies, then Theorem 4.8 can be used to obtain convergence.

Now assume a PAI $(A, x^0, \{\mathcal{J}_k, \mathcal{S}_k\}_{k\in\mathbb{N}_0})$ for *MP* with operators $H_{d,l}^{(k)}$ for some scenario and $l \in \{1, \ldots, 6\}$. If it is possible to find a number $r \in \mathbb{N}$ which is independent of $k \in \mathbb{N}_0$ and any scenario, such that

$$T_{d,l}^{(k)} := H_{d,l}^{(k+r)} \cdot \ldots \cdot H_{d,l}^{(k)}$$

has a positive column whose elements are bounded from below, then Corollary 4.2 applies again and convergence for

$$x_d^{k+(m+1)r} = T_{d,l}^{(k+mr)} x_d^{k+mr} + c_d^{k+mr}, \quad m = 0, 1, 2, \ldots,$$

is obtained from Theorem 4.8.

Thus, convergence of multiplicative Schwarz iterations and PAIs can be proven with Corollary 4.2 if it is possible to find conditions that guarantee the existence of a positive column whose elements are bounded from below. This cannot be shown without a consideration of the non-zero pattern of the iteration operators as will be demonstrated in the following sections.

Remark 4.5 *The convergence for multiplicative Schwarz iterations and PAIs is generally obtained on a subsequence* $\{x^{kr}\}_{k=\mathbb{N}_0}$ *for some* $r > 0$*. But since all operators* $H_l^{(k)}$ *and* $H_{d,l}^{(k)}$ *are nonexpansive w.r.t.* $\|\cdot\|_z$ *(cf. Lemma 3.1, Lemma 3.2 and Proposition 3.4), the convergence for the whole sequence is immediately obtained (see, e.g., [13]).*

4.3 The basic idea

In this section the basic idea which will guarantee convergence by the existence of a positive column will be presented. First for operators which update a single row and in the next section for block operators.

Consider the following ST-matrices.

$$B_4 = \begin{pmatrix} & 1 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & 1 & \\ & & & & & 1 \\ 1 & & & & & \end{pmatrix}, B_5 = \begin{pmatrix} & 1 & & & & \\ & & & 1 & & \\ & & & 1 & & \\ & & & & & 1 \\ & & & & & 1 \\ 1 & & & & & \end{pmatrix}, B_6 = \begin{pmatrix} & 1 & & & & \\ & & & & & 1 \\ & & & 1 & & \\ & & & & & 1 \\ & & & & & 1 \\ 1 & & & & & \end{pmatrix}.$$

Assume an inexact multiplicative Schwarz iteration with Update 5 for B_4, B_5, and B_6 such there is only one row updated (this is equivalent to Update 1 for the STM-matrices $A_i = I - B_i, i = 4, 5, 6$ applied to single rows). Then, e.g., the local operator $H^{(3)}$ for B_4 becomes

$$H^{(3)} = \begin{pmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 0 & 1 & & \\ & & & 1 & & \\ & & & & 1 & \\ & & & & & 1 \end{pmatrix}.$$

Now it is easy to see that for each of the above examples the operator

$$T = H^{(1)} \cdot \ldots \cdot H^{(6)}$$

has the first column positive. And this can be explained by a consideration of the spanning trees of the above examples (see Figure 4.1) and an application of an appropriate vector to each B_i.

Consider the vector $x = (1, 0, 0, 0, 0, 0)^T$ and apply it to some B_i. Then the initial information is stored in state 6, the root. Another application of $B_i x$ to B_i reveals that the information is carried to the states which are direct children of the root and so on. Hence the information "flows" through the tree, until it reaches the leaves. The product operator T combines this transport into a single step, i.e. it introduces a shortcut. Moreover, T is

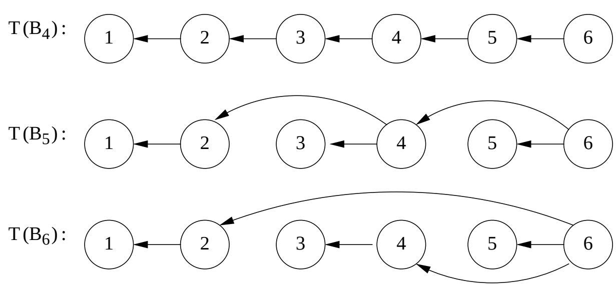

Figure 4.1: Trees of B_4, B_5 and B_6

also an ST-matrix and has exactly one final and basic class. Thus, fixed points are preserved (cf. Section 4.4). It turns out that this simple idea is a key to convergent exact multiplicative Schwarz iterations, since the above mentioned flow will be kept working if the partitions and their order are chosen properly (cf. Section 4.5). This idea will also have a natural extension to inexact multiplicative Schwarz iterations (cf. Section 4.6). Finally, the ansatz will be further improved using relaxation. This improvement will also lead to results on additive Schwarz iterations (cf. Section 4.7) and PAIs (cf. Section 4.8).

Definition 4.2 *Let A be an STM-matrix (or ST-matrix) and let $\mathcal{T}$ be an arbitrary spanning tree in $\Gamma(A^T)$ with some guard index. A* flow compatible numbering (or permutation) *of the vertices of $\mathcal{T}$ is a permutation $\pi : \{1, \ldots, n\} \longrightarrow \{1, \ldots, n\}$ such that if there is a path from $\pi(i)$ to $\pi(j)$ in $\mathcal{T}$, then $i > j$ for each $1 \leq i, j \leq n,\ i \neq j$. Any such permutation is also called flow compatible (w.r.t. $\mathcal{T}$).*

Figure 4.2 shows a flow compatible numbering for an ST-matrix B (guard edges have been left out).

The example defines a numbering $(10, 11, 9, 8, 7, 5, 6, 4, 3, 2, 1)$, but note that $(9, 11, 8, 4, 6, 3, 10, 7, 5, 2, 1)$ is also flow compatible.

Definition 4.2 allows the following theorem.

Theorem 4.9 *Let $B \in \mathbb{R}^{n\times n}$ be an ST-matrix and let π be a flow compatible permutation w.r.t. some tree $\mathcal{T} \subset \Gamma(B^T)$. Consider Update 5, applied to single rows and let $H_5^{(1)}, \ldots, H_5^{(n)}$ be the corresponding operators, then*

$$H_5^{(\pi(1))} \cdot \ldots \cdot H_5^{(\pi(n))}$$

has at least one positive column and therefore it is semiconvergent.

Proof: Since the node $\pi(n)$ is always the root of a tree $\mathcal{T}$ in $\Gamma(B^T)$ (cf. Definition 4.2), there is a guard index j such that $(H_5^{(\pi(n))})_{\pi(n),j} > 0$. It will

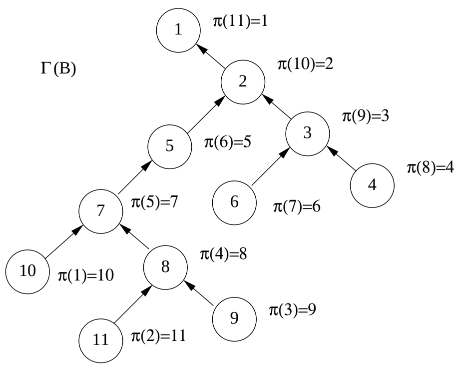

Figure 4.2: A flow compatible numbering

first be shown by induction that

$$(H_5^{(\pi(k))} \cdot \ldots \cdot H_5^{(\pi(n))})_{\pi(k),j} > 0$$

for an arbitrary $k \in \{1, \ldots, n\}$. Since the proposition is obvious for $k = n$ let $1 \leq k < n$ be arbitrary but fixed.

Case 1: $(\pi(k), \pi(k+1)) \in \Gamma(B)$.
In this case, the node $\pi(k+1)$ represents the parent of $\pi(k)$ in $\mathcal{T}$. Thus $(H_5^{(\pi(k))})_{\pi(k),\pi(k+1)} > 0$ and there is an $l \in \mathbb{N},\ k < l \leq n$ such that $(H_5^{(\pi(k+1))})_{\pi(k+1),\pi(l)} > 0$. The latter holds, since there is a path from $\pi(k+1)$ to $\pi(n)$ in $\mathcal{T}$. But then

$$(H_5^{(\pi(k))} \cdot H_5^{(\pi(k+1))})_{\pi(k),\pi(l)} > 0,$$

i.e. the entry $(\pi(k), \pi(k+1))$ has been moved to the $\pi(l)$-th column. Thus $\pi(l)$ is the new parent of $\pi(k)$.

Case 2: $(\pi(k), \pi(k+1)) \notin \Gamma(B)$.
In this case, the index $\pi(k)$ must have access to $\pi(l)$ for some $l \in \mathbb{N},\ k+1 < l \leq n$, since there is also a path from $\pi(k)$ to $\pi(n)$ in $\mathcal{T}$. The construction of the operators implies that $(H_5^{(\pi(k+1))})_{\pi(l),\pi(l)} > 0$; thus

$$(H_5^{(\pi(k))} \cdot H_5^{(\pi(k+1))})_{\pi(k),\pi(l)} > 0.$$

The entry $(\pi(k), \pi(l))$ has been preserved and $\pi(l)$ is still the parent of $\pi(k)$.
A simple induction leads to

$$(H_5^{(\pi(k))} \cdot \ldots \cdot H_5^{(\pi(n-1))})_{\pi(k),\pi(l)} > 0$$

for some $l \in \mathbb{N},\ n-1 < l \leq n$, i.e. $l = n$. But since $(H_5^{(\pi(n))})_{\pi(n),j} > 0$ there holds

$$(H_5^{(\pi(k))} \cdot \ldots \cdot H_5^{(\pi(n))})_{\pi(k),j} > 0.$$

To finish the proof of the theorem observe that $(H_5^{(\pi(j))})_{\pi(k),\pi(k)} > 0$ follows for all $1 \leq j < k$ from the construction of $H_5^{(j)}$. Hence

$$(H_5^{(\pi(1))} \cdot \ldots \cdot H_5^{(\pi(n))})_{\pi(k),j} > 0.$$

This is the theorem since k was arbitrary. $\square$

Remark 4.6 *1) Theorem 4.9 can be interpreted as a mapping from the set of ST-matrices into itself while the fixed points are preserved and the image contains a flat tree (of height 1).*

2) It can be easily shown that each positive entry in the root row (i.e. every guard) creates a positive column.

Now the question is how to find a flow compatible numbering. An easy way is to find the strongly connected component first (see [69] and also [22]) and then apply a depth first search (see [22]) to the transposed graph.

The depth first search algorithm is usually split into two parts. An outer loop (usually called DFS) which runs until each node has been visited and a part that performs the depth first search (DFS_VISIT). For an ST-matrix there is only a need for the second part and the DFS-Algorithm reduces to the pseudo code given in Algorithm 4.1. The algorithm works on a graph $G = (V, E)$ with $V = \{1, \ldots, n\}$, a global array π of length n, and a global counter t which has to be initialised with 0.

Algorithm 4.1 DFS_VISIT

Require: $j \in \{1, \ldots, n\}$

for each $i \in V$ such that $(j, i) \in E$ **do**
    if i is not visited **then**
        mark i as visited
        DFS_VISIT(i)
    end if
end for
$t \leftarrow t + 1$
$\pi(t) \leftarrow j$

Let B be an ST or STM-matrix. If Algorithm 4.1 is applied to $\Gamma(B^T)$, and the initial vertex j is chosen from the final and basic class, then each accessible vertex is marked. But there is at least one tree in $\Gamma(B^T)$, hence each vertex is accessible and there is only the need of one call to DFS_VISIT. Since the numbers t coincide with the finishing times of DFS in [22], the $\pi(t), t = 1, \ldots, n$, represent a flow compatible numbering. That leads to the following lemma.

Lemma 4.1 *Let B be an ST or STM-matrix and let j be any index from the final and basic class. Then DFS_VISIT(j), given by Algorithm 4.1, applied to $\Gamma(B^T)$ delivers a flow compatible permutation π for some tree $\mathcal{T} \subset \Gamma(B^T)$.*

Remark 4.7 *1) None of the algorithms to compute strongly connected components in [22] and [69] delivers a flow compatible numbering, thus the algorithm of [69] should be preferred since it is slightly faster. However, Algorithm 4.1 has to be applied separately.*

2) Let π be a compatible permutation w.r.t. some tree $\mathcal{T} \subset \Gamma(B^T)$, $B \in \mathbb{R}^{n\times n}$. If Π is the permutation matrix corresponding to π, then all edges of $\mathcal{T}$ reside in the upper triangular part of $\Pi^T B \Pi$ by the definition of flow compatibility. A flow compatible numbering might therefore be interpreted as a generalised topological sorting.

3) If an ST-matrix B has a symmetric pattern, then irreducibility can be proven via DFS_VISIT. Moreover, DFS_VISIT can be used to find the strongly connected components.

4.4 Block operators

The idea of the previous section will be extended to show that the "flow" can be preserved for exact multiplicative Schwarz iterations if the blocks are chosen properly.

The following lemma is as fundamental as it is easy, but the proof is a bit stretchy and there is a need for additional notation. To this purpose, let A be an STM-matrix, and let $\mathcal{T}$ be any spanning tree in $\Gamma(A^T)$. Furthermore, let π be a flow compatible permutation corresponding to $\mathcal{T}$. Denote by Π the permutation matrix corresponding to π.

A vertex i has *access to a vertex j along $\mathcal{T}$*, if there is a path $(j = l_1, l_2, \ldots, l_k = i)$ in $\mathcal{T}$. This is denoted by $i \rightarrow_{\mathcal{T}} j$. A vertex i has *direct access to j along $\mathcal{T}$*, if $(j, i) \in \mathcal{T}$. This is denoted by $(i, j)_{\mathcal{T}}$ and the vertex i is said to be *adjacent to j in $\mathcal{T}$*.

Remark 4.8 *1) If $i \rightarrow_{\mathcal{T}} j$ holds, then $(i = l_k, l_{k-1}, \ldots, l_1 = j) \subset \Gamma(A)$.*

2) The term $(i, j)_{\mathcal{T}}$ is to be interpreted as a predicate rather than an edge. However, $(i, j)_{\mathcal{T}}$ implies $(i, j) \in \Gamma(A)$.

3) The access relation along *$\mathcal{T}$ is used to focus on the structure of interest. This structure represents the minimum of positive elements needed to construct convergent multiplicative Schwarz iterations and PAIs. All other positive elements of A can be ignored.*

Lemma 4.2 *Let A be an STM-matrix. Let π be a flow compatible permutation corresponding to a spanning tree $\mathcal{T}$ in $\Gamma(A^T)$. Denote by Π the permutation matrix corresponding to π. Additionally let*

$$V := (\pi(k), \pi(k+1), \ldots, \pi(l)), \;\; 1 \leq k < l \leq n, \; l-k+1 < n$$

such that V satisfies the assumption of Theorem 2.2, i.e. $A[V]^{-1}$ exists. Let the matrix P be defined through

$$\Pi P \Pi^T = \begin{pmatrix} I & 0 & 0 \\ M^{-1}N_L & 0 & M^{-1}N_R \\ 0 & 0 & I \end{pmatrix} \tag{4.4.1}$$

where $M = A[V]$, $N_L = -A[V, (\pi(1), \ldots, \pi(k-1))]$, and $N_R = -A[V, (\pi(l+1), \ldots, \pi(n))]$. Then for each $k \leq j_0 \leq l$, one of the following three conditions hold:

1) If $j_0 = n$, then there exists a $\pi(i_0) \notin V$ such that $\pi(n) \to \pi(i_0)$ in $\Gamma(A)$ and $(\pi(n), \pi(i_0)) \in \Gamma(P)$.

2) If $j_0 \neq n$ and $(\pi(j_0), \pi(i_0))_{\mathcal{T}}$ for a $\pi(i_0) \notin V$, then $(\pi(j_0), \pi(i_0)) \in \Gamma(P)$.

3) If $j_0 \neq n$ and $(\pi(j_0), \pi(o))_{\mathcal{T}}$ for a $\pi(o) \in V$, then there exists $\pi(h) \in V$, $\pi(i_0) \notin V$ such that $\pi(j_0) \to_{\mathcal{T}} \pi(h)$, $(\pi(h), \pi(i_0))_{\mathcal{T}}$, and $(\pi(j_0), \pi(i_0)) \in \Gamma(P)$.

Furthermore the index $\pi(i_0)$ in 1),2) and 3) satisfies:

4) If $l < n$, then $l < i_0$.

5) If $l = n$, then $k > i_0$ and i_0 is given by assertion 1).

In case 4) i_0 is unique (w.r.t. $\mathcal{T}$). In case 5) i_0 depends on the chosen guard.

Remark 4.9 *The lemma has an easy interpretation if one considers (4.4.1). The case $\pi(n) \in V$ plays a special role. Assertion 1) states that there exists a path in $\Gamma(A)$ starting at $\pi(n)$, which leads out of V and this path is replaced by an edge in $\Gamma(P)$ which resides in $M^{-1}N_L$. Assertion 2) says that if $\pi(j_0) \in V$ has direct access to $\pi(i_0) \notin V$ along $\mathcal{T}$ (i.e. $(\pi(j_0), \pi(i_0)) \in \Gamma(A)$), then this edge is still in $\Gamma(P)$, i.e. connections leading out of V are preserved. Assertion 3) says that if $\pi(j_0) \in V$ has direct access to some element in V along $\mathcal{T}$, then there is a path in V along $\mathcal{T}$ leading to some $\pi(i_0) \notin V$. Furthermore, for each such path there will be an edge in $\Gamma(P)$. The latter means that a multiplication with M^{-1} introduces a shortcut in $\mathcal{T}$, but the flow itself is preserved. Assertion 4) states that the edges of interest lie in $M^{-1}N_R$ except, and that is assertion 5), the root is in*

V. Since then each vertex has access to the root, thus access to some vertex outside V due to assertion 1).

Figure 4.3 describes the situation for an STM-matrix $A = I - B$. The figure shows the graph of B and the set $V = \{\pi(4), \pi(5), \pi(6), \pi(7)\} = \{8, 7, 5, 6\}$. $\Gamma(P)$ is the graph of P given by (4.4.1) with respect to V. Note that the dashed edges exist in $\Gamma(A) = \Delta \cup \Gamma(B)$ and have been shortcut in $\Gamma(P)$. The diagonal edges of $\Gamma(P)$ and the guard edges in $\Gamma(A)$ have been left out.

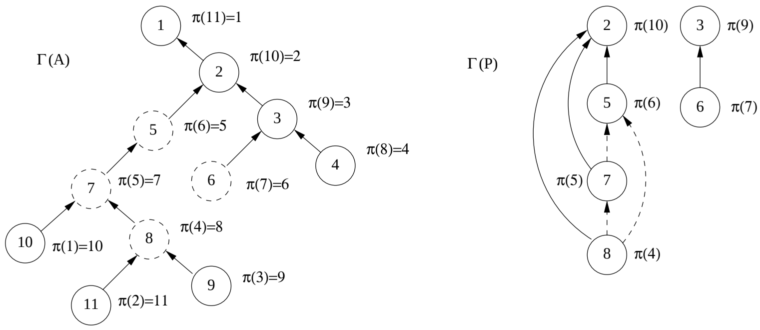

Figure 4.3: Graph of a matrix with corresponding projection

Proof: The proof is a bit technical and has therefore been split into several parts.

Part 1: Classification of vertices in V.

Define *inner* $(I(V))$, *outer* $(O(V))$, and *boundary vertices* $(B(V))$ by

$$\begin{aligned} B(V) &:= \{v \in V : \text{ there exist } w \notin V \text{ such that } (v, w)_{\mathcal{T}}\}, \\ I(V) &:= \{v \in V : \text{ there exist } u \in V \text{ such that } (v, u)_{\mathcal{T}}\}, \\ O(V) &:= V \setminus (B(V) \cup I(V)). \end{aligned}$$

Then there holds:

i) $B(V) \cap I(V) = \emptyset$,

ii) $B(V) \cup O(V) \cup I(V) = V$,

iii) $O(V) \neq \emptyset \Rightarrow O(V) = \{\pi(n)\}$,

iv) $B(V) = \emptyset \Leftrightarrow O(V) \neq \emptyset$.

To prove this let $v \in V$ be arbitrary. Assume there exists a w such that $(w, v) \in \mathcal{T}$, i.e. $(v, w)_{\mathcal{T}}$ holds. Then w is unique since $\mathcal{T}$ is tree. If $w \in V$, then $v \in I(V)$ else $v \in B(V)$. If there exists no w satisfying $(v, w)_{\mathcal{T}}$, then $v = \pi(n)$ and obviously $\pi(n) \in O(V)$. Since the root is unique, this proves i) to iii) and the sets are well defined.

Suppose $B(V) = \emptyset$. Since every vertex, except the root, has a parent but no vertex $v \in V$ has an adjacent vertex $w \notin V$ relative $\mathcal{T}$, $\pi(n) \in V$ holds necessarily. Now assume $O(V) \neq \emptyset$, then $O(V) = \{\pi(n)\}$ follows from iii). If there is a $v \in B(V)$, then there exists a $w \notin V$ satisfying $(v, w)_{\mathcal{T}}$. Additionally, there are numbers $k_1, k_2 \in \{1, \dots, n\}$ such that $v = \pi(k_1)$ and $w = \pi(k_2)$. Since π is flow compatible, $k \leq k_1 < k_2 \leq l = n$. But then $w = \pi(k_2) \in V$, a contradiction and iv) follows.

Part 2: The case $\pi(n) \in V$
Assume $\pi(j_0) = \pi(n) \in V$. Then there is an index $k_0 \in \{1, \dots, n\}$ such that $(A)_{\pi(j_0),\pi(k_0)} > 0$, i.e. k_0 is a guard index. If $\pi(k_0) \notin V$, then we are done ($i_0 := k_0$). Hence assume $\pi(k_0) \in V$ and let α be the final and basic class of A. Denote $\tilde{\alpha} = \pi(\alpha)$, then $\tilde{\alpha} \cap V \neq \tilde{\alpha}$ since $M = A[V]$ is regular (cf. Theorem 2.2) and π is bijective. Owing to the strong connectivity of α, there exists a $\pi(i_0) \in \tilde{\alpha} \setminus V$ and a path $(\pi(n) = \pi(i_k), \dots, \pi(i_1), \pi(i_0))$ from $\pi(n)$ to $\pi(i_0)$ in $\Gamma(A)$ such that $(\pi(i_k), \dots, \pi(i_1)) \subset \Gamma(M)$. Lemma 1.1 implies $\Gamma(M^{-1}) = \overline{\Gamma(M)}$, thus there is an edge $(\pi(n), \pi(i_1)) \in \Gamma(M^{-1})$ and $(\pi(i_1), \pi(i_0)) \in \Gamma(A)$. By the construction, $\pi(i_0) \notin V$ and $i_0 < k < l = n$.

Part 3: Inner vertices
Let $v \in I(V)$, then there exists exactly one vertex $w \in V$ such that $(v, w)_{\mathcal{T}}$. If $w \in I(V)$ then the last argument can be applied inductively until a vertex $u \notin I(V)$ is reached. Now $u \in B(V)$ or from part 1) of assertion iv), $u = \pi(n)$. But in either case there is a unique path $p = (u = \pi(i_k), \pi(i_{k-1}), \dots, \pi(i_1) = v)$ in $\mathcal{T}$, i.e. $v \to_{\mathcal{T}} u$. Furthermore, $l \geq i_k > i_{k-1} > \ldots > i_1 \geq k$ and $p^T = (\pi(i_1), \dots, \pi(i_k))$ is a path in $\Gamma(M)$. As in part 2), there holds $\Gamma(M^{-1}) = \overline{\Gamma(M)}$, i.e. $(v, u) \in \Gamma(M^{-1})$. The latter means, that the path is being shortcut in $\Gamma(M^{-1})$ by $(v, u) = (\pi(i_1), \pi(i_k))$ and $i_k > i_1$.

Part 4: Conclusions
Let $k \leq j_0 \leq l$ be arbitrary but fixed. Write P as follows:

$$\begin{aligned}
\Pi P \Pi^T &= \begin{pmatrix} I & 0 & 0 \\ M^{-1}N_L & 0 & M^{-1}N_R \\ 0 & 0 & I \end{pmatrix} \\
&= \begin{pmatrix} I & 0 & 0 \\ 0 & M^{-1} & 0 \\ 0 & 0 & I \end{pmatrix} \begin{pmatrix} I & 0 & 0 \\ N_L & 0 & N_R \\ 0 & 0 & I \end{pmatrix} =: M_n^{-1} N_n.
\end{aligned}$$

By Lemma 1.1

$$\Gamma(M_n^{-1} N_n) = \Gamma(M_n^{-1}) \Gamma(N_n) = \overline{\Gamma(M_n)} \Gamma(N_n). \tag{4.4.2}$$

Case a: $\pi(j_0) \in B(V)$
There exists exactly one $\pi(i_0)$ satisfying $(\pi(j_0), \pi(i_0))_{\mathcal{T}}$ and $\pi(i_0) \notin V$. But then $(\pi(j_0), \pi(i_0)) \in \Gamma(N_R)$ since $i_0 > j_0$. With $\Delta \subset \overline{\Gamma(M_n)}$ and (4.4.2) the

relation $(\pi(j_0), \pi(i_0)) \in \Gamma(P)$ follows. Hence, assertion 2) combined with 4) is proven.

Case b: $\pi(j_0) \in I(V)$ and $B(V) \neq \emptyset$
In view of part 3) there exists exactly one $\pi(h) \in B(V)$ such that $(\pi(j_0), \pi(h)) \in \overline{\Gamma(M_n)}$ and $h > j_0$. Since $\pi(h)$ satisfies the assumptions of case a) there is exactly one edge $(\pi(h), \pi(i_0)) \in \Gamma(N_R)$. Again, from (4.4.2) there follows $(\pi(j_0), \pi(i_0)) \in \Gamma(P)$ and $i_0 > h > j_0$. This is assertion 3) combined with 4).

Case c: $\pi(j_0) = \pi(n)$
By part 2) there are edges $\pi(i_0) \notin V$ and $\pi(j_1) \in V$ such that $(\pi(j_1), \pi(i_0)) \in \Gamma(A)$ and $(\pi(j_0), \pi(j_1)) \in \Gamma(M^{-1})$. But $i_0 < k < l = n$, thus $(\pi(j_1), \pi(i_0)) \in \Gamma(N_L)$ and again by (4.4.2), $(\pi(j_0), \pi(i_0)) \in \Gamma(P)$, which is assertion 1) combined with 5).

Case d: $\pi(j_0) \in I(V)$ and $\pi(n) \in V$
By part 1) there holds $B(V) = \emptyset$. Part 3) implies that there exists an edge $(\pi(j_0), \pi(n)) \in \overline{\Gamma(M_n)}$ which is unique in $\mathcal{T}$. An application of case c) leads to the existence of edges $(\pi(j_1), \pi(i_0)) \in \Gamma(N_L)$ and $(\pi(j_0), \pi(j_1)) \in \overline{\Gamma(M_n)}$. This implies $(\pi(j_0), \pi(i_0)) \in \Gamma(P)$ and $j_0 \geq k > i_0$. Hence, assertion 2) with 5) is proven.

This finishes the proof of the lemma. □

Suppose $\pi(n) \in V$ and i_0 is the index of assertions 1) and 5). An inspection the proof of Lemma 4.2 shows that each $\pi(k) \in V$ generates a positive entry $(\pi(k), \pi(i_0))$, i.e. $P_{\pi(k),\pi(i_0)} > 0$ for all $\pi(k) \in V$. This is because each vertex has access to the root due to the definition of V. The latter means, that the root always generates a positive column in the block row corresponding to V.

Corollary 4.3 *Let the assumptions of Lemma 4.2 be fulfilled and let $\pi(n) \in V$. Then there exists an i_0 such that $\pi(i_0) \notin V$ and $(\pi(k), \pi(i_0)) \in \Gamma(P)$ for all $\pi(k) \in V$.*

If V is any subset of $\{\pi(1), \ldots, \pi(n)\}$, the assertions 1), 2) and 3) of Lemma 4.2 still hold. But the assertions 4) and 5) can not hold in general since there might be some gaps in the set. If $\pi(n) \notin V$, then assertion 4) still applies but in the other case there holds a somehow weaker condition.

If $\pi(n) \notin V$, the situation can be easily understood by considering different flow compatible numberings of an STM-matrix A. Figure 4.4 shows such a situation (guard edges of A have been left out). Consider the set $V = (\pi(4), \pi(5), \pi(6), \pi(7))$ which satisfies the assumptions of Lemma 4.2 w.r.t. π. In contrast to this, the set $\tilde{V} = (\sigma(3), \sigma(5), \sigma(8), \sigma(9))$ does not fulfil the assumptions of Lemma 4.2 w.r.t. σ. But the access relation in the corresponding projections is of course the same, since the access relation of the reflexive transitive closure of $A[V]^{-1}$ is invariant under permutation.

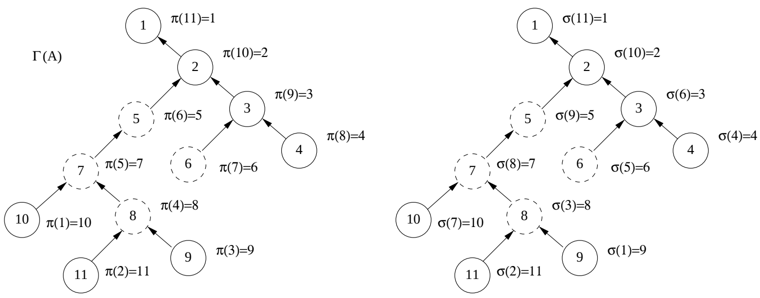

Figure 4.4: Graph of a matrix with different flow compatible numberings

Corollary 4.4 *Let V be any subset of $\{\pi(1), \ldots, \pi(n)\}$ such that $M^{-1} = A[V]^{-1}$ exists (cf. Theorem 2.2) and assume $\pi(n) \notin V$. Then Lemma 4.2 applies with the assertions 2), 3) and 4) to V.*

To understand the situation if $\pi(n) \in V$, consider the STM-matrix A with the graph given by Figure 4.4 and the permutation π. Assume that the guard index of $\pi(11) = 1$ is $\pi(4) = 8$, Π is the permutation matrix corresponding to π, and $V = (\pi(6), \pi(10), \pi(11))$. Then the corresponding projection P_V is given by

$$\Pi P_V \Pi^T = \left(\begin{array}{ccccccccccc} 1 &&&&&&&&&& \\ & 1 &&&&&&&&& \\ && 1 &&&&&&&& \\ &&& 1 &&&&&&& \\ &&&& 1 &&&&&& \\ &&&&& 1 &&&&& \\ &&&&&& 1 &&&& \\ &&& 1 &&&&&&& \\ &&&&&&&& 1 && \\ &&& 1 &&&&&&& \\ &&& 1 &&&&&&& \end{array} \right).$$

Assertion 5) of Lemma 4.2 applies directly with $\pi(i_0) = \pi(4) = 8$ and $i_0 = 4 < 6 = \min\{k : \pi(k) \in V\}$, since each vertex has direct access to the root. But now consider the tuple $U = (\pi(1), \pi(6), \pi(10), \pi(11))$, then

$$\Pi P_U \Pi^T = \left(\begin{array}{ccccccccccc} &&&& 1 &&&&&& \\ & 1 &&&&&&&&& \\ && 1 &&&&&&&& \\ &&& 1 &&&&&&& \\ &&&& 1 &&&&&& \\ &&&&& 1 &&&&& \\ &&&&&& 1 &&&& \\ &&& 1 &&&&&&& \\ &&&&&&&& 1 && \\ &&& 1 &&&&&&& \\ &&& 1 &&&&&&& \end{array} \right),$$

and P_U posses no positive column in the block row corresponding to U. However, Lemma 4.2 and Corollary 4.4 can be applied to parts of U.

Corollary 4.5 *Let V be any subset of $(\pi(1), \ldots, \pi(n))$ such that $M^{-1} = A[V]^{-1}$ exists (cf. Theorem 2.2). Assume $\pi(n) \in V$ and define $\mathcal{T}_V :=$*

$\mathcal{T} \cap \Gamma(M^T)$,

$$V_{\mathcal{T}} := \{\pi(k) \in V : \pi(k) \to_{\mathcal{T}_V} \pi(n)\} \cup \{\pi(n)\},$$

and $V_{\neg\mathcal{T}} := V \setminus V_{\mathcal{T}}$. *Then Corollary 4.4 applies verbatim to* $V_{\neg\mathcal{T}}$. *Moreover, there exists an* $i_0 \in \{1, \ldots, n\}$ *such that* $i_0 < n$ *and* $(\pi(k), \pi(i_0)) \in \Gamma(P)$ *for all* $\pi(k) \in V_{\mathcal{T}}$, *i.e. Lemma 4.2 applies with assertions 1), 3) and 5) to* $V_{\mathcal{T}}$.

Proof: Since $\pi(n) \notin V_{\neg\mathcal{T}}$, Corollary 4.4 applies. Now consider the guard index $\pi(j_0)$ of $\pi(n) \in V_{\mathcal{T}}$. If $\pi(j_0) \notin V$, then Lemma 4.2 applies directly to $V_{\mathcal{T}}$ with $i_0 = j_0$. Thus assume $\pi(j_0) \in V$. Then an argumentation analogous to part 2) of the proof of Lemma 4.2 can be used which gives the proposition for some i_0 satisfying $i_0 < n$ and $\pi(i_0) \notin V$. □

Remark 4.10 *As the proof of Lemma 4.2 and the above discussion have shown, the order within the set* V *does not matter. The reflexive transitive closure is of course not invariant under permutation but, as previously mentioned, the access relation is. Hence,* V *can be interpreted as a set rather than a tuple.*

Definition 4.3 *Let* $A \in \mathbb{R}^{n\times n}$ *be an STM-matrix and let* π *be a flow compatible numbering of a spanning tree* $\mathcal{T} \subset \Gamma(A^T)$. *A regular partitioning* $S_1, \ldots, S_p$ *of* $(\pi(1), \ldots, \pi(n))$ *is termed a* block flow compatible *partitioning if*

$$\max\{j : \pi(j) \in S_k\} < \min\{j : \pi(j) \in S_l\},$$

for all $1 \le k < l \le p$.

Corollary 4.6 *Let* $A \in \mathbb{R}^{n\times n}$ *be an STM-matrix and let* π *be a flow compatible numbering of a spanning tree* $\mathcal{T} \subset \Gamma(A^T)$. *Consider a block flow compatible partitioning* $S_1, \ldots, S_p$ *and define* $M_i := A[S_i]$, $M = \operatorname{diag}(M_1, \ldots, M_p)$, *and* $N = M - A$. *Then* $M^{-1}N$ *is an ST-matrix.*

Proof: Owing to the regularity of the partitioning and Theorem 2.2, each M_i is invertible. By Lemma 4.2, each index $u \in S_i, i < p$ has access to some $v \in S_j, j > i$ in $\Gamma(M^{-1}N)$. Therefore each index $u \in \{1, \ldots, n\} \setminus S_p$ has access to some $v \in S_p$. Furthermore, each index $v \in S_p$ has access to a single index $w \notin S_p$, which is the guard index of $\pi(n) \in S_p$. Hence $\Gamma((M^{-1}N)^T)$ contains a spanning tree. Since $w \notin S_p$, there follows $w \in S_i$ for some $i < p$ and w has access to some index in $S_j, j > i$. Hence there is a path from w to w in $\Gamma(M^{-1}N)$. The vertex w can be considered as the root and there must exist a guard index, since there is a path from w to w. Thus $M^{-1}N$ is a nonnegative GST-matrix. But since A is an STM-matrix, the ST property follows from Corollary 2.4. □

Corollary 4.7 *With the assumptions of Corollary 4.6 there exists a spanning tree* $\tilde{\mathcal{T}}$ *in* $M^{-1}N$ *of height at most* p.

To give an example to Corollary 4.6 and Lemma 4.2, consider the STM-matrix $A = I - B$ where the graph of B is given by Figure 4.3 and the guard index of index 1 is index 3. Then

$$A = \left(\begin{array}{ccccccccccc}
1 & & -1 & & & & & & & & \\
-1 & 1 & & & & & & & & & \\
 & -1 & 1 & & & & & & -1 & & \\
 & & -1 & 1 & & & & & & & \\
 & -1 & & & 1 & & & & & & \\
 & & -1 & & & 1 & & & & & \\
 & & & & -1 & & 1 & & & & \\
 & & & & & & -1 & 1 & & & \\
 & & & & & & & -1 & 1 & & \\
 & & & & & & -1 & & & 1 & \\
 & & & & & & & -1 & & & 1
\end{array}\right).$$

After applying the flow compatible permutation given in Figure 4.3, A becomes

$$\Pi A \Pi^T = \left(\begin{array}{ccc|cccc|cccc}
1 & & & & -1 & & & & & & \\
 & 1 & & -1 & & & & & & & \\
 & & 1 & -1 & & & & & & & \\
\hline
 & & & 1 & -1 & & & & & & \\
 & & & & 1 & -1 & & & & & \\
 & & & & & 1 & & & & -1 & \\
 & & & & & & 1 & & -1 & & \\
\hline
 & & & & & & & 1 & -1 & & \\
 & & & -1 & & & & & 1 & -1 & \\
 & & & & & & & & & 1 & -1 \\
 & & & & & & & & -1 & & 1
\end{array}\right).$$

Now consider the (permuted) partitions $S_1 = \{1, 2, 3\}, S_2 = \{4, 5, 6, 7\}$, $S_3 = \{8, 9, 10, 11\}$, and a splitting according to Corollary 4.6. Then $M^{-1}N$ becomes

$$M^{-1}N = \left(\begin{array}{ccc|cccc|cccc}
 & & & & 1 & & & & & & \\
 & & & 1 & & & & & & & \\
 & & & 1 & & & & & & & \\
\hline
 & & & & & & & & & 1 & \\
 & & & & & & & & & 1 & \\
 & & & & & & & & & 1 & \\
 & & & & & & & & 1 & & \\
\hline
 & & & 1 & & & & & & & \\
 & & & 1 & & & & & & & \\
 & & & 1 & & & & & & & \\
 & & & 1 & & & & & & &
\end{array}\right).$$

The matrix $M^{-1}N$ is obviously an ST-matrix with the root index $\pi(4) = 8$ and the guard index $\pi(10) = 2$. Note that $M^{-1}N$ is not semiconvergent.

Let A and $S_i, i = 1, 2, 3$ be as given in the above example. Let $P^{(1)}, P^{(2)}$ and $P^{(3)}$ be the operators given by (4.4.1) with respect to $S_i, i = 1, 2, 3$. Then the non-trivial block row of $P^{(i)}$ corresponds one to one to the i-th block row of $M^{-1}N$. Additionally, $P^{(1)} \cdot P^{(2)} \cdot P^{(3)}$ contains a positive column and is semiconvergent. This is not surprisingly because now, the block rows behave as single states and the order of the projections, given by the sets S_i, can be interpreted as a flow compatible numbering of a (block) spanning tree of height p (cf. Corollary 4.7).

This observation leads to Theorem 4.10. Since the theorem will be formulated in terms of decompositions, i.e. with overlap, the notion of *block flow compatible decompositions* has to be introduced.

Definition 4.4 *Let $A \in \mathbb{R}^{n\times n}$ be an STM-matrix and let π be a flow compatible numbering of a spanning tree $\mathcal{T} \subset \Gamma(A^T)$. A regular decomposition $S_1,\dots,S_p$ of $(\pi(1),\dots,\pi(n))$ is said to be* block flow compatible *if there exists a block flow compatible partitioning $\tilde{S}_1,\dots,\tilde{S}_p$ such that $\tilde{S}_i \subseteq S_i$ for all $i = 1,\dots,p$.*

Theorem 4.10 *Assume an STM-matrix $A \in \mathbb{R}^{n\times n}$, a flow compatible numbering π of a spanning tree $\mathcal{T} \subset \Gamma(A^T)$, and a block flow compatible decomposition $S_1,\dots,S_p$ such that $\pi(n) \in S_p$ and $\pi(n) \notin S_j,\ j = 1,\dots,p-1$. Let*

$$\Pi^{(i)} P^{(i)} (\Pi^{(i)})^T = \begin{pmatrix} 0 & M_i^{-1}N_i \\ 0 & I \end{pmatrix}$$

where $M_i = A[S_i]$, $N_i = -A[S_i, \neg S_i]$, and $\Pi^{(i)}$ is a suitable permutation matrix. Then the product

$$P := P^{(1)} \cdot \ldots \cdot P^{(p)}$$

has at least one positive column.

Proof: Set

$$\bar{S}_j := \bigcup_{l=j}^{p} \tilde{S}_l,\ j = 1,\dots,p.$$

Furthermore let

$$P^{(j,p)} := P^{(j)} \cdot \ldots \cdot P^{(p)}.$$

The proof will be done by showing for $j = p,\dots,1$:

(*) For all $\pi(k) \in \bar{S}_j$ there holds $(\pi(k),\pi(l_0)) \in \Gamma(P^{(j,p)})$ for a fixed l_0.

To prove this by induction note that $\bar{S}_p = \tilde{S}_p$. With Definition 4.4, $\tilde{S}_p = \{\pi(k_p), \pi(k_p+1),\dots,\pi(n)\}$ and Corollary 4.5 implies that there is an index $l_0 < n$ such that $(\pi(j),\pi(l_0)) \in \Gamma(P^{(p)})$ for all $j = k_p,\dots,n$.

Let $p > j > 1$ and assume (*) for j. The induction step is twofold.

Increasing number of positive elements:
Again from Definition 4.4, $\tilde{S}_{j-1} = \{\pi(k_{j-1}), \pi(k_{j-1}+1),\dots,\pi(k_{j-1}+l)\}$ for some $l \in \mathbb{N}_0$. Due to Lemma 4.2, there exists for each $i = 0,\dots,l$ an $l_i > k_{j-1}+i$ such that $(\pi(k_{j-1}+i),\pi(l_i)) \in \Gamma(P^{(j-1)})$. Since $\pi(n) \notin S_{j-1}$ one gets $l_i > k_{j-1}+l$ for all $i = 0,\dots,l$. But $k_{j-1}+l = k_j - 1$ from Definition 4.4 and thus $n \ge l_i \ge k_j$ for all $i = 0,\dots,l$. By the induction hypothesis there is an edge $(\pi(l_i),\pi(l_0)) \in \Gamma(P^{(j,p)})$ for all $i = 0,\dots,l$. Since $\Gamma(P^{(j-1,p)}) = \Gamma(P^{(j-1)})\Gamma(P^{(j,p)})$, the relation $(\pi(k_{j-1}+i),\pi(l_0)) \in \Gamma(P^{(j-1,p)})$ follows for all $i = 0,\dots,l$.

Positivity preservation:
Let $\pi(k_0) \in \bar{S}_j = \{\pi(k_j),\dots,\pi(n)\}$ be arbitrary but fixed. Then by the induction hypothesis $(\pi(k_0),\pi(l_0)) \in \Gamma(P^{(j,p)})$. There are two possible cases.

Case 1: $\pi(k_0) \notin S_{j-1}$
Considering the construction of $P^{(j-1)}$, there must be an edge $(\pi(k_0), \pi(k_0)) \in \Gamma(P^{(j-1)})$. But then obviously $(\pi(k_0), \pi(l_0)) \in \Gamma(P^{(j-1,p+1)})$.

Case 2: $\pi(k_0) \in S_{j-1}$
Lemma 4.2 implies that there is an edge $(\pi(k_0), \pi(l_1)) \in \Gamma(P^{(j-1)})$, and $l_1 > k_0$. Consequently, $\pi(l_1) \in \bar{S}_j$. By the induction hypothesis there is an edge $(\pi(l_1), \pi(l_0)) \in \Gamma(P^{(j,p)})$ and therefore $(\pi(k_0), \pi(l_0)) \in \Gamma(P^{(j-1,p)})$.

All together this is the induction step.

If $j = 1$, then there follows $\bar{S}_1 = \{\pi(1), \ldots, \pi(n)\}$ and $(\pi(j), \pi(l_0)) \in \Gamma(P^{(1,p)})$ for all $j = 1, \ldots, n$. Hence the theorem is proven. □

Corollary 4.8 *Suppose the assumptions of Theorem 4.10 are fulfilled. Then the product* $P := P^{(1)} \cdot \ldots \cdot P^{(p)}$ *contains at least one spanning tree* $\mathcal{T}$ *of height 1 in* $\Gamma(P^T)$. *If* i_0 *is the root of* $\mathcal{T}$, *then* $i_0 \in \alpha$ *where* α *is the final (and basic) class of* A. *Moreover* $i_0 \leftrightarrow i_0$ *in* $\Gamma(P)$.

Proof: A skillful look on the proofs of Corollary 4.5 and Theorem 4.10 shows that a positive column is either created at the position of a guard index $i_0 \in \alpha$, $i_0 \notin S_p$, or there is a path in $\Gamma(A[\alpha])$ from a guard entry to some entry in $i_1 \in \alpha$, $i_1 \notin S_p$ which is then the index of the positive column. But in either case, i_0 or i_1 are roots of a tree of height one in $\Gamma(P^T)$. □

Corollary 4.9 *With the assumptions of Theorem 4.10, let* α *be the final class of* A *and* β *be the final class of* P, *then* $\beta \subset \alpha$.

The condition $\pi(n) \notin S_j$ for $j \neq p$ might seem somehow unsatisfactory but cannot be omitted in the context of general overlap as the following example shows.

Consider the STM-matrix

$$A = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & -1/2 & 0 & 1 & 0 & -1/2 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 \end{pmatrix}.$$

Assume $S_1 := \{1, 2, 3\}, S_2 := \{4, 5, 7\}$ and $S_3 := \{6, 7\}$. Then

$$P^{(1)} = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, P^{(2)} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix},$$

and

$$P^{(3)} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

Now the product becomes

$$T_1 := P^{(1)}P^{(2)}P^{(3)} = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

This is a rank two semiconvergent matrix. Thus $\mathcal{N}(I - T_1) \neq \mathcal{N}(A)$.
The above problem can only occur if

1) there is more than one set containing the root and those sets have a gap, i.e. there are elements in the set having no access to the root, and

2) the guard index of the sets containing the root changes (in the above example, the guard of S_3 is 5 and that of S_2 is 3).

If one of those points can be avoided, it is not hard to prove that there still is a positive column in the product, but this will be omitted here. Furthermore, the above conditions are probably the only ones to construct a multiplicative Schwarz operator T_2 for Update 2 (cf. (3.3.1) and Algorithm 3.4) w.r.t. some STM-matrix A, such that $\mathcal{N}(I - T_2) \neq \mathcal{N}(A)$. But this has not been proven formally.
This section ends with the following definition.

Definition 4.5 *Let $A \in \mathbb{R}^{n\times n}$ be an STM-matrix and let π be a flow compatible numbering of a spanning tree $\mathcal{T} \subset \Gamma(A^T)$. A block flow compatible decomposition $S_1, \ldots, S_p$ of $(\pi(1), \ldots, \pi(n))$ is said to be* ms-compatible *(ms for "multiplicative Schwarz") if $\pi(n) \notin S_j$, for $j = 1, \ldots, p-1$.*

4.5 Applications to multiplicative Schwarz iterations

The theory developed in Section 4.4 will be applied in order to prove convergence results for exact and inexact multiplicative Schwarz iterations.

For this section, consider *MP* (cf. Definition 2.1 and 2.6) and let π be a flow compatible numbering with respect to a spanning tree $\mathcal{T} \subseteq \Gamma(A^T)$. Furthermore, let $S_1, \ldots, S_p$ be an ms-compatible decomposition.

The following result concerns exact (one-level) Schwarz Iterations, i.e., Algorithm 3.2 in Section 3.3.

Theorem 4.11 *If $H_1^{(1)}, \ldots, H_1^{(p)}$ are the local Schwarz operators given by (3.2.1) w.r.t. $S_1, \ldots, S_p$, $T_1 := H_1^{(1)} \cdot \ldots \cdot H_1^{(p)}$ and c a proper right hand side, then the multiplicative Schwarz iteration*

$$x^{k+1} = T_1 x^k + c, \quad k = 0, 1, 2, \ldots,$$

converges to the solution of $Ax = b$ for every given $x^0 \in \mathbb{R}^n$. The convergence is obtained at a geometric rate.

Proof: The operator T_1 has a positive column which follows from Theorem 4.10. Hence, Corollary 2.4 implies that it is an ST-matrix and the semi-convergence follows from Theorem 4.1. Since $\dim \mathcal{N}(I - T_1) = 1$, one gets $\mathcal{N}(I - T_1) = \mathcal{N}(A)$. $\square$

To apply the results of this chapter to two-stage multiplicative Schwarz iterations, let $M_i = A[S_i]$ and assume a weak regular splitting $M_i = F_i - G_i$. A local operator of a two-stage Schwarz iteration in the k-th step is given as (cf. (3.3.2))

$$\Pi_i H_3^{(k,i)} \Pi_i^T = \begin{pmatrix} R^{(k,i)} & (I - R^{(k,i)}) M_i^{-1} N_i \\ 0 & I \end{pmatrix} \geq 0,$$

where $R^{(k,i)} = (F_i^{-1} G_i)^{q(k,i)}$, $q(k,i) \geq 1$, and Π_i is an appropriate permutation matrix. Following the discussion in Section 4.2 (see page 65),

$$\begin{aligned} (4.5.1) \qquad \Pi_i H_3^{(k,i)} \Pi_i^T &= \begin{pmatrix} R^{(k,i)} & (I - R^{(k,i)}) M_i^{-1} N_i \\ 0 & I \end{pmatrix} \\ &= \begin{pmatrix} R^{(k,i)} & \sum_{j=0}^{q(k,i)-1} (F_i^{-1} G_i)^j F_i^{-1} N_i \\ 0 & I \end{pmatrix} \\ (4.5.2) \qquad &\geq \begin{pmatrix} 0 & F_i^{-1} N_i \\ 0 & I \end{pmatrix} \geq 0. \end{aligned}$$

This observation is essential for the following reason.

Denote by $\mathcal{T}_{|S_i}$ the restriction of the spanning tree $\mathcal{T}$ to S_i. The main property that was exploited in Lemma 4.2 and its corollaries, was that each path $(v_1, \ldots, v_k) \subset \mathcal{T}$ such that $(v_1, \ldots, v_{k-1}) \subset \mathcal{T}_{|S_i} \cap \Gamma(M_i^T)$ and $v_k \notin S_i$, was replaced by an edge (v_1, v_k). But if $\mathcal{T}_{|S_i} \cap \Gamma(F_i^T) = \mathcal{T}_{|S_i} \cap \Gamma(M_i^T)$, then Lemma 4.2 applies verbatim to (4.5.2) and therefore to (4.5.1) since F_i is a nonsingular M-matrix.

Definition 4.6 *Let A be an STM-matrix and let $\mathcal{T}$ be a spanning tree in $\Gamma(A^T)$. For an arbitrary regular decomposition $S_1, \ldots, S_p$ let $A[S_i] = M_i = F_i - G_i$ be a weak regular splitting. Then (F_i, G_i) is called a* flow compatible splitting, *if $\mathcal{T}_{|S_i} \cap \Gamma(F_i^T) = \mathcal{T}_{|S_i} \cap \Gamma(M_i^T)$.*

Theorem 4.12 *Let $H_3^{(k,1)}, \ldots, H_3^{(k,p)}$ be the local Schwarz operators given by (3.3.2). Define $T_3^{(k)} := H_3^{(k,1)} \cdot \ldots \cdot H_3^{(k,p)}$, and let the $c^{(k)}$ be corresponding right hand sides. Additionally, assume that the inner splittings are flow compatible. Then the two-stage multiplicative Schwarz iteration*

$$x^{k+1} = T_3^{(k)} x^k + c^{(k)}, \quad k = 0, 1, 2, \ldots,$$

converges to the solution of $Ax = b$ for every given $x^0 \in \mathbb{R}^n$ and any number of inner iterations. The convergence is obtained at a geometric rate.

Proof: Define (cf. (4.5.2))

$$\tilde{H}_3^{(i)} := \Pi_i^T \begin{pmatrix} 0 & F_i^{-1} N_i \\ 0 & I \end{pmatrix} \Pi_i$$

for $i = 1, \ldots, p$. Then $H_3^{(k,i)} \geq \tilde{H}_3^{(i)}$ for all $i = 1, \ldots, p$ and $k \in \mathbb{N}$. Thus

$$T_3^{(k)} = H_3^{(k,1)} \cdot \ldots \cdot H_3^{(k,p)} \geq \tilde{H}_3^{(1)} \cdot \ldots \cdot \tilde{H}_3^{(p)} =: \tilde{T}_3 \qquad (4.5.3)$$

and Theorem 4.10 implies that $\tilde{T}_3$ has a positive column, say $(\tilde{T}_3)_{*,l}$. Since $\tilde{T}_3$ is independent of k there exists a $\kappa > 0$ such that $(\tilde{T}_3)_{j,l} > \kappa$ for all $j = 1, \ldots, n$. Now, $T_3^{(k)}$ satisfies the assumptions of Corollary 4.2 since (4.5.3) holds and the convergence follows by Theorem 4.8. □

Remark 4.11 *1) Theorem 4.11 and 4.12 apply also to the relaxed versions given by Algorithm 3.4 and 3.6, i.e. Update 2 and 4.*

2) Consider some set S_i according to Theorem 4.12, but now interpret S_i as a tuple $(\pi(k_{i_1}), \pi(k_{i_2}), \ldots, \pi(k_{i_l}))$ such that $k_{i_1} < k_{i_2} < \ldots < k_{i_l}$. Then each edge $v \in \mathcal{T}_{|S_i} \cap \Gamma(M_i^T)$ remains in the upper triangular part of $M_i = A[S_i]$ by the definition of flow compatibility (cf. Remark 4.7). Hence, in this case a simple Gauss-Seidel-Splitting (F_i, G_i), where F_i is the upper triangular part and G_i the lower is flow compatible.

3) Note that it is sufficient to have one inner iteration to guarantee convergence.

4) If the inner splitting is a Jacobi-Splitting, it is not flow compatible. So convergence cannot be deduced from Theorem 4.12.

So far, simple results for multiplicative iteration schemes with Update 1,2,3, or 4 from Section 3.2 and 3.3 have been derived. What remains to be discussed is the Update 5 given by Algorithm 3.7 and (3.3.4).

Theorem 4.9 gives a convergence result in the case that single rows are updated and there seems to be no easy generalisation to the block case. Consider the cyclic matrix

$$B = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Choose the ms-compatible decomposition $S_1 = \{1,2,3\}$ and $S_2 = \{3,4,5\}$, then

$$H_5^{(1)} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, H_5^{(2)} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

and finally

$$H_5^{(1)} \cdot H_5^{(2)} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

which is not semiconvergent. The problem is of course, that all variables belonging to one set are updated simultaneously. If this would be done in a Gauss-Seidel fashion, semiconvergence is achieved by Theorem 4.9.

4.6 The impact of relaxation

In this section the impact of relaxation to multiplicative Schwarz iterations with Update 2,4 and 6 will be discussed. First, the results from Section 4.5 will be slightly generalised but then the concept of relaxation will led to a natural extension of the idea given in Sections 4.3 and 4.4.

With the usual assumptions, consider the local operators of one of Update 2, 4 or 6 given by (3.3.1), (3.3.3), or (3.3.5) and denote them by $H^{(1)}, \ldots, H^{(p)}$. Then each $H^{(i)}$ can be written as $H^{(i)} = D^{(i)} + R^{(i)}$ with $D^{(i)} = \mathrm{diag}(h_{11}^{(i)}, \ldots, h_{nn}^{(i)}) > 0$ and $R^{(i)} = H^{(i)} - D^{(i)} \geq 0$. Obviously, there holds

$$H^{(j)}H^{(i)} = D^{(j)}D^{(i)} + R^{(j)}D^{(i)} + D^{(j)}R^{(i)} + R^{(j)}R^{(i)},$$

and, consequently, from Lemma 1.1

$$\begin{aligned}
\Gamma(H^{(j)}H^{(i)}) &= \Gamma(D^{(j)}D^{(i)} + R^{(j)}D^{(i)} + D^{(j)}R^{(i)} + R^{(j)}R^{(i)}) \\
&= \Delta^2 \cup \Gamma(R^{(j)})\Delta \cup \Delta\Gamma(R^{(i)}) \cup \Gamma(R^{(j)}R^{(i)}) \\
&= \Delta \cup \Gamma(R^{(j)}) \cup \Gamma(R^{(i)}) \cup \Gamma(R^{(j)}R^{(i)}) \\
&= \Gamma(H^{(j)} + H^{(i)}) \cup \Gamma(R^{(j)}R^{(i)}).
\end{aligned}$$

This implies that if $\tilde{p} \geq p$ and $\sigma : \{1, \ldots, \tilde{p}\} \longrightarrow \{1, \ldots, p\}$ is any surjective mapping, then

$$\Gamma(H^{(i)}) \subset \Gamma(H^{(\sigma(1))} \cdot \ldots \cdot H^{(\sigma(\tilde{p}))})$$

for each $i = 1, \ldots, p$, by induction. Thus the graphs of the local operators are preserved. Define

$$\xi := \min_{i=1}^{p}\{h_{jj}^{(i)} : j = 1, \ldots, n\}$$

then

$$H^{(\sigma(1))} \cdot \ldots \cdot H^{(\sigma(\tilde{p}))} \geq \xi^{p-1} H^{(i)} \tag{4.6.1}$$

for each $i = 1, \ldots, p$.

To get back to Schwarz iterations consider *MP* and let $S_1, \ldots, S_p$ be an ms-compatible decomposition with respect to a flow compatible numbering π. Define $H := H^{(\sigma(1))} \cdot \ldots \cdot H^{(\sigma(\tilde{p}))}$. Then (4.6.1) implies,

$$H^k = \left(H^{(\sigma(1))} \cdot \ldots \cdot H^{(\sigma(\tilde{p}))}\right)^k \geq \xi^{k(p-1)} H^{(1)} \cdot \ldots \cdot H^{(k)} \tag{4.6.2}$$

for any $k \leq p$. Hence it follows from (4.6.2) that

1) H^k has a positive column for some $k \leq p$ if Update 2 or 4 is used, and

2) H^k has a positive column for some $k \leq n$ if Update 6 is used.

Point 1) follows from Theorem 4.10 and 2) from Theorem 4.9. This is obvious because the graph of each local operator is preserved by the positive diagonal and an appropriate order of the local operators is forced by (4.6.2). Thus the following results are an immediate consequence.

Theorem 4.13 *Let $H_2^{(1)}, \ldots, H_2^{(p)}$ be the local Schwarz operators given by (3.3.1) and let σ be any permutation on $\{1, \ldots, p\}$. Let $T_2 := H_2^{(\sigma(1))} \cdot \ldots \cdot H_2^{(\sigma(p))}$. Then the relaxed multiplicative Schwarz iteration*

$$x^{k+1} = T_2 x^k + c, \quad k = 0, 1, 2, \ldots,$$

where c is a proper right hand side, converges to the solution of $Ax = b$ for every given $x^0 \in \mathbb{R}^n$ at a geometric rate.

Theorem 4.14 *Let $H_4^{(k,1)}, \dots, H_4^{(k,p)}$ be the local Schwarz operators given by (3.3.3) and let σ be any permutation on $\{1, \dots, p\}$. Define $T_4^{(k)} := H_4^{(k,\sigma(1))} \cdot \ldots \cdot H_4^{(k,\sigma(p))}$ and assume that the inner splittings are flow compatible. Then the two-stage relaxed multiplicative Schwarz iteration*

$$x^{k+1} = T_4^{(k)} x^k + c^{(k)}, \quad k = 0, 1, 2, \dots,$$

where the $c^{(k)}$ are corresponding right hand sides, converges to the solution of $Ax = b$ for every given $x^0 \in \mathbb{R}^n$ and any number of inner iterations. The convergence is obtained at a geometric rate.

Theorem 4.15 *Consider an arbitrary decomposition $S_1, \dots, S_p$. Let $H_6^{(1)}, \dots, H_6^{(p)}$ be the local Schwarz operators given by (3.3.5) and let σ be any permutation on $\{1, \dots, p\}$. Define $T_6 := H_6^{(\sigma(1))} \cdot \ldots \cdot H_6^{(\sigma(p))}$ and let c be a proper right hand side. Then the relaxed power-like multiplicative Schwarz iteration*

$$x^{k+1} = T_6 x^k + c, \quad k = 0, 1, 2, \dots,$$

converges to the solution of $Ax = b$ for every given $x^0 \in \mathbb{R}^n$ at a geometric rate.

Remark 4.12 *1) Note that the iteration of Theorem 4.15 is nothing more than a slightly modified successive under relaxation method.*

2) Theorems 4.13, 4.14 and 4.15 have shown that relaxed iteration schemes are independent of the order of the local updates. But the speed of convergence may not be independent of the order as the example on page 58 has shown.

Theorems 4.13, 4.14 and 4.15 can be further improved. This is due to the fact that since the diagonals of each H are positive, a positive column is no longer needed. If it is possible to prove each H to be an ST-matrix, then H is ST-regular (cf. Definition 4.1) by Theorem 2.1 and Theorem 1.11. The convergence of the multiplicative Schwarz iteration will again be obtained from Theorem 4.7 and Theorem 4.8.

This approach is motivated by Corollary 4.6 and the following lemma.

Lemma 4.3 *Consider nonnegative square matrices $L^{(1)}, \dots, L^{(p)}$ and assume that the diagonals of each $L^{(i)}$ are positive. Then for any permutation $\sigma : \{1, \dots, p\} \longrightarrow \{1, \dots, p\}$,*

$$\Gamma\left(\sum_{j=1}^{p} L^{(\sigma(j))}\right) = \bigcup_{j=1}^{p} \Gamma(L^{(\sigma(j))}) \subset \Gamma(L^{(\sigma(1))} \cdot \ldots \cdot L^{(\sigma(p))}).$$

Proof: Easy, using Lemma 1.1. □

Lemma 4.3 can be interpreted as transforming the multiplicative problem to an additive one.

Additionally, there is no need of ms-compatibility nor of flow compatible inner splittings by the following theorem whose proof is pretty straight forward.

Theorem 4.16 *Suppose $A \in \mathbb{R}^{n\times n}$ is an STM-matrix, α the final and basic class of A, and $j_0 \in \alpha$ is arbitrary. Assume a regular decomposition $S_1, \dots, S_p$ such that*

$$|\{j : j_0 \in S_j\}| = 1. \tag{4.6.3}$$

If $H_2^{(j)}, H_4^{(k,j)}$, and $H_6^{(j)}$ are the corresponding local operators for $j = 1, \dots, p$ and the inner splittings (if any) are M-splittings, then each product

$$\begin{aligned} T_2 &:= H_2^{(\sigma(1))} \cdot \ldots \cdot H_2^{(\sigma(p))}, \\ T_4^{(k)} &:= H_4^{(k,\sigma(1))} \cdot \ldots \cdot H_4^{(k,\sigma(p))}, \\ T_6 &:= H_6^{(\sigma(1))} \cdot \ldots \cdot H_6^{(\sigma(p))} \end{aligned}$$

is ST-regular for any permutation $\sigma : \{1, \dots, p\} \longrightarrow \{1, \dots, p\}$.

Proof: By Lemma 4.3 and the nonnegativity of the operators, it suffices to show that

$$\sum_{j=1}^{p} H_l^{(j)}$$

is an ST-matrix (for Update 4, the k is sometimes omitted).

Part 1: Proof for T_6

For T_6, there is nothing to prove since $A = I - B$ and

$$\Gamma(B) \subset \bigcup_{j=1}^{p} \Gamma(H_6^{(j)}) = \Gamma\left(\sum_{j=1}^{p} H_6^{(j)}\right).$$

Here, one can even skip the condition on j_0.

Thus, T_6 is a GST-matrix and Corollary 2.4 implies that T_6 is an ST-matrix, i.e. $\mathcal{N}(I - T_6) = \mathcal{N}(A)$. Since its diagonal is positive, T_6 is semiconvergent (the same argumentation will also apply to T_2 and T_4).

Part 2: Choose a tree

To prove the result for T_2 and T_4, let $\mathcal{T} \subset \Gamma(A^T)$ be an arbitrary spanning tree with root j_0 (cf. Corollary 2.1). Consider a flow compatible numbering π of $\mathcal{T}$, then $j_0 = \pi(n)$. Assume w.l.o.g. that $\pi(n) \in S_p$ and define

$$\tilde{S}_p := \{\pi(j) \in S_p : \pi(j) \to_{\mathcal{T}} \pi(n)\} \cup \{\pi(n)\}. \tag{4.6.4}$$

The idea is to prove that the "flow" given by $\mathcal{T}$ is preserved for Update 2 and 4, i.e. there exists a spanning tree with a corresponding guard in $\Gamma(T_2^T)$ and $\Gamma(T_4^T)$.

Part 3: Proof for T_2

By Corollary 4.5 there exists an i_0 such that $i_0 < n$ and each $\pi(j) \in \tilde{S}_p$ has access to $\pi(i_0) \notin \tilde{S}_p$ in $\Gamma(H_2^{(p)})$. Thus, it remains to show that each $\pi(l) \in \{1, \ldots, n\} \setminus \tilde{S}_p$ has access to some $\pi(j) \in \tilde{S}_p$ in

$$\Gamma\left(\sum_{j=1}^{p} H_2^{(j)}\right). \tag{4.6.5}$$

Therefore let $\pi(l_0) \in \{1, \ldots, n\} \setminus \tilde{S}_p$ be arbitrary but fixed. Then $\pi(l_0) \in S_{k_0} \setminus \tilde{S}_p$ for some $k_0 \in \{1, \ldots, p\}$. It follows from Lemma 4.2 that $\pi(l_0)$ has access to some $\pi(l_1)$ in $\Gamma(H_2^{(k_0)})$ and $l_1 > l_0$. If $\pi(l_1) \notin \tilde{S}_p$, then $\pi(l_1)$ has access to some $\pi(l_2)$ in $\Gamma(H_2^{(k_1)})$ for some $k_1 \in \{1, \ldots, p\}$ and $l_2 > l_1$. So, inductively, after a maximum of $n-1$ steps, $\pi(l_0)$ has access to some $\pi(j) \in \tilde{S}_p$ in (4.6.5).

Consequently, each $\pi(j) \in \{1, \ldots, n\}$ has access to $\pi(i_0)$ and since $\pi(i_0) \in \{1, \ldots, n\} \setminus \tilde{S}_p$, $\pi(i_0)$ must have access to itself, i.e. T_2 is a GST-matrix. Now the same argumentation as for T_6 completes the proof for T_2.

Part 4: Proof for T_4

For an arbitrary $1 \leq i \leq p$ there holds

$$\Pi_i H_4^{(k,i)} \Pi_i^T = \begin{pmatrix} R_\omega^{(k,i)} & (I - R_\omega^{(k,i)}) M_i^{-1} N_i \\ 0 & I \end{pmatrix}$$

with a proper permutation matrix Π_i. Furthermore

$$\begin{aligned} (I - R_\omega^{(k,i)}) M_i^{-1} N_i &= \omega \sum_{j=0}^{q(k,i)-1} \left((1-\omega)I + \omega(F_i^{-1} G_i)\right)^j F_i^{-1} N_i \\ &\geq \omega F_i^{-1} N_i \geq 0, \end{aligned}$$

for each $q(k,i) \geq 1$. Define

$$R_\omega^{(i)} := (1-\omega)I + \omega(F_i^{-1} G_i).$$

Then $R_\omega^{(i)}$ is nonnegative and has a positive diagonal. Additionally

$$R_\omega^{(k,i)} = (R_\omega^{(i)})^{q(k,i)}$$

and thus

$$\Gamma(R_\omega^{(i)}) \subset \Gamma(R_\omega^{(k,i)}).$$

Let

$$\Pi_i \tilde{H}_4^{(i)} \Pi_i^T = \begin{pmatrix} R_\omega^{(i)} & \omega F_i^{-1} N_i \\ 0 & I \end{pmatrix} \geq 0, \tag{4.6.6}$$

then $\tilde{H}_4^{(i)}$ is the local operator if one inner iteration is carried out and

$$\Gamma(\tilde{H}_4^{(i)}) \subset \Gamma(H_4^{(k,i)}) \text{ for all } q(k,i) \geq 1.$$

Hence it suffices to show that

$$\sum_{j=1}^{p} \tilde{H}_4^{(j)}$$

is a GST-matrix. The prove becomes a bit stretchy, because the resulting spanning tree is not only made up of elements given by $F_i^{-1}N_i$, but also of elements from $R_\omega^{(i)}$ because the inner splittings need not be flow compatible. Consider the M-splitting (cf. Section 1.1) $M_i = F_i - G_i$. Then $\Gamma(M_i) = \Gamma(F_i) \cup \Gamma(G_i)$ (see [61]). In analogy to Lemma 4.2, define inner vertices as

$$I(S_i) := \{\pi(j) \in S_i : \text{ there exists } \pi(l) \in S_i \text{ such that } (\pi(j), \pi(l))_{\mathcal{T}}\}$$

and set $B(S_i) = S_i \setminus I(S_i)$ for all $i = 1, \ldots, p$.

Part 4.1: Preserved flow for S_i, $1 \leq i < p$

Let $1 \leq i < p$ be arbitrary. Then each $\pi(l_0) \in B(S_i)$ has access to some $\pi(l_1) \notin S_i$. But since $\Delta \subset \Gamma(F_i^{-1})$, this relation is also valid in $\tilde{H}_4^{(i)}$ and $l_1 > l_0$. Thus let $\pi(l_0) \in I(S_i)$, then there exists an $\pi(l_{k+1})$ such that $\pi(l_{k+1}) \notin S_i$ and $\pi(l_0) \to_{\mathcal{T}} \pi(l_{k+1})$. If l_{k+1} is chosen as in assertion 3) of Lemma 4.2, then there is a path $p = (\pi(l_0), \pi(l_1), \ldots, \pi(l_k)) \subset \Gamma(M_i)$, $(\pi(l_k), \pi(l_{k+1}))_{\mathcal{T}}$, and $\pi(l_k) \in B(S_i)$. Furthermore, $l_0 < l_1 < \ldots < l_k < l_{k+1}$. To prove that this connection still exists in $\tilde{H}_4^{(i)}$, there are three cases to consider:

Case 1: $p \subset \Gamma(F_i)$

As in the proof of Lemma 4.2, $\pi(l_0)$ has access to $\pi(l_k)$ in $\Gamma(F_i^{-1}) = \overline{\Gamma(F_i)}$. Hence $\pi(l_0)$ has access to $\pi(l_{k+1})$ in $\Gamma(\tilde{H}_4^{(i)})$.

Case 2: $p \subset \Gamma(G_i)$

Here, $p \subset \Gamma(F_i^{-1}G_i)$ follows from $\Delta \subset \Gamma(F_i^{-1})$. Thus $\pi(l_0)$ has access to $\pi(l_{k+1})$ in $\Gamma(\tilde{H}_4^{(i)})$.

Case 3: $p \cap \Gamma(F_i) \neq \emptyset$ and $p \cap \Gamma(G_i) \neq \emptyset$

In this case, p can be split into m subpathes such that $p = (p_1, \ldots, p_m)$.

Since $\pi(l_k) \in p_m$ has access to $\pi(l_{k+1})$ and $\Delta \subset \Gamma(F_i^{-1})$, $p_m \subset \Gamma(F_i)$ can be assumed w.l.o.g. This construction implies that $p_{m-1} \subset \Gamma(G_i)$, $p_{m-2} \subset \Gamma(F_i)$ and so on, and p_ν can be written as $p_\nu = (\pi(l_{\nu_1}), \ldots, \pi(l_{\nu_\mu}))$.

Now consider p_ν for an arbitrary ν, $1 \leq \nu < m$, then $\pi(l_{\nu_\mu})$ has access to $\pi(l_{(\nu+1)_1}) \in p_{\nu+1}$ in $\Gamma(M_i)$.

If $p_\nu \subset \Gamma(F_i)$, then $p_{\nu+1} \subset \Gamma(G_i)$ and each $\pi(l_{\nu_\eta})$, $1 \leq \eta \leq \mu$ has access to $\pi(l_{(\nu+1)_1})$ in $\Gamma(F_i^{-1}G_i) = \overline{\Gamma(F_i)}\Gamma(G_i)$ following the argumentation of case 1).

If $p_\nu \subset \Gamma(G_i)$, then each $\pi(l_{\nu_\eta})$ has access to $\pi(l_{(\nu+1)_1})$ in $\Gamma(F_i^{-1}G_i)$ since $\Delta \subset \Gamma(F_i^{-1})$ as in case 2).

Since $1 \leq \nu \leq m$ was arbitrary, $\pi(l_0)$ has access to $\pi(l_{k+1})$ in $\Gamma(\tilde{H}_4^{(i)})$.

What has been proven so far is that each $\pi(l_0) \in S_i$ has access to some $\pi(l_1) \notin S_i$ and $l_1 > l_0$ in $\Gamma(\tilde{H}_4^{(i)})$, for all $1 \leq i < p$.

Part 4.2: Preserved flow for S_p

Consider S_p. The first thing to note is, that there is nothing further to be discussed for elements $\pi(j) \in S_p \setminus \tilde{S}_p$ (cf. (4.6.4)), because the same argumentation as for the sets $S_1, \ldots, S_{p-1}$ can be applied. Since $\pi(n)$ can be regarded as an element of $\Gamma(F_p)$ let

$$\tilde{S}_p^F := \left\{ \pi(j) \in \tilde{S}_p : \pi(j) \to_{\mathcal{T}} \pi(n) \text{ in } \mathcal{T} \cap \Gamma(F_p^T) \right\} \cup \{\pi(n)\} \tag{4.6.7}$$

and

$$\tilde{S}_p^G = \tilde{S}_p \setminus \tilde{S}_p^F. \tag{4.6.8}$$

Then each element $\pi(j) \in \tilde{S}_p^G$ has access to $\pi(n)$ in $\Gamma(G_p)$ according to the argumentation used for $S_1, \ldots, S_{p-1}$. Additionally, $j < n$.

Part 4.3: Existence of a guard

Let $\pi(i_0)$ be the guard index of $\pi(n)$. If $\pi(i_0) \notin S_p$, then, as usual, each $\pi(j) \in \tilde{S}_p^F$ has access to $\pi(i_0)$ in $\Gamma(\tilde{H}_4^{(p)})$. If $\pi(i_0) \in S_p$, then consider the path $p \subset \Gamma(A)$ as in Lemma 4.2, assertion 1). Then $p = (\pi(n) = \pi(k_\nu), \ldots, \pi(k_1), \pi(k_0))$, $n = k_\nu > k_i$, $i = 0, \ldots, \nu - 1$, and $\pi(k_0) \notin S_p$. If $(\pi(k_\nu), \ldots, \pi(k_1)) \subset \Gamma(F_p)$, then each $\pi(j) \in \tilde{S}_p^F$ has access to $\pi(k_1)$ in $\Gamma(F_p^{-1})$, thus access to $\pi(k_0)$ in $\Gamma(F_p^{-1}N_p)$ and consequently access to $\pi(k_0)$ in $\Gamma(\tilde{H}_4^{(p)})$.

If not, there is a maximum index $\nu > \xi > 1$ such that $(\pi(k_\xi), \pi(k_{\xi-1})) \in \Gamma(G_p)$ and $n = k_\nu > k_\xi$. But then each $\pi(j) \in \tilde{S}_p^F$ has access to $\pi(k_\xi)$ in $\Gamma(F_p^{-1})$ and therefore access to $\pi(k_{\xi-1})$ in $\Gamma(F_p^{-1}G_p)$, i.e. access to $\pi(k_{\xi-1})$ in $\Gamma(\tilde{H}_4^{(p)})$.

Part 4.4: Conclusion

By part 4.1) and 4.2), each $\pi(l_0) \in S_i \setminus \tilde{S}_p^F$ (cf. (4.6.7)) has access to some

$\pi(l_1)$ in $\Gamma(H_4^{(i)})$ for all $i = 1, \ldots, p$ and $l_1 > l_0$. Furthermore, it follows from part 4.3) that each $\pi(l_0) \in \tilde{S}_p^F$ has access to a single index $\pi(l_1)$ in $\Gamma(H_4^{(p)})$ and $l_1 < n$. But now an application of the same argumentation as for T_2 leads to the result that T_4 is a GST-matrix. □

Remark 4.13 *1) The flow compatibility of the inner splitting has been given up for the price of M-splittings. But that should not be a problem in practice.*

2) Note that the inner relaxation for T_4 is mainly responsible to guarantee the ST-property. The same argumentation is not applicable to T_3! Although, there might exist similar results for T_3.

3) If the inner M-splitting is chosen such that the diagonal of each $F_i^{-1}G_i$ is positive, then Theorem 4.16 also holds for T_3.

4) The ST-property is based on elements which might become arbitrarily small as the number of inner iterations grows. Thus the number of inner iterations should be bounded (cf. discussion in Section 4.2).

5) Again, there is the annoying condition (4.6.3) concerning the decomposition. But in the case of an irreducible matrix, this condition reduces to one variable which has no overlap. And that should not be a problem in practice.

Definition 4.7 *Let $A = I - B \in \mathbb{R}^{n\times n}$ be an STM-matrix and denote by α its basic class. A regular decomposition $S_1, \ldots, S_p$ w.r.t. A is said to be* root preserving, *if there exists a $j_0 \in \alpha$ such that $|\{j : j_0 \in S_j\}| = 1$.*

The following theorems are a simple application of Theorem 4.16. Consider *MP* and a root preserving decomposition $S_1, \ldots, S_p$ w.r.t. A.

Theorem 4.17 *Let $H_2^{(1)}, \ldots, H_2^{(p)}$ be the local Schwarz operators given by (3.3.1) and let σ be any permutation on $\{1, \ldots, p\}$. Define $T_2 := H_2^{(\sigma(1))} \cdot \ldots \cdot H_2^{(\sigma(p))}$. Then the relaxed multiplicative Schwarz iteration*

$$x^{k+1} = T_2 x^k + c, \quad k = 0, 1, 2, \ldots$$

where c is a proper right hand side, converges to the solution of $Ax = b$ for every given $x^0 \in \mathbb{R}^n$ at a geometric rate.

Proof: Since T_2 is ST-regular by Theorem 4.16, it is semiconvergent by Theorem 1.11. Furthermore, owing to the ST-regularity, $\mathcal{N}(I - T_2) = \mathcal{N}(A)$.
□

Theorem 4.18 *Let $H_4^{(k,1)}, \ldots, H_4^{(k,p)}$ be the local Schwarz operators given by (3.3.3) and let σ be any permutation on $\{1,\ldots,p\}$. Define $T_4^{(k)} := H_4^{(k,\sigma(1))} \cdot \ldots \cdot H_4^{(k,\sigma(p))}$ and assume that all inner splittings are M-splittings and that the numbers of inner iterations $q(k,i)$ are bounded. Then the relaxed two-stage multiplicative Schwarz iteration*

$$x^{k+1} = T_4^{(k)} x^k + c^{(k)}, \quad k = 0, 1, 2, \ldots$$

where the $c^{(k)}$ are corresponding right hand sides, converges to the solution of $Ax = b$ for every given $x^0 \in \mathbb{R}^n$. The convergence is obtained at a geometric rate.

Proof: By Theorem 4.16, each $T_4^{(k)}$ is ST-regular. But moreover, each $T_4^{(k)}$ contains a pattern of positive elements which is the same for all $k \in \mathbb{N}_0$ and makes $T_4^{(k)}$ ST-regular.
To outline this, note that

$$\Gamma\left(\sum_{j=1}^{p} H_4^{(k,j)}\right) \subset \Gamma(T_4^{(k)})$$

by Lemma 4.3. Consider the operators $\tilde{H}_4^{(j)}$, $j = 1, \ldots, p$ given by (4.6.6), then

$$\Gamma\left(\sum_{j=1}^{p} \tilde{H}_4^{(j)}\right) \subset \Gamma\left(\sum_{j=1}^{p} H_4^{(k,j)}\right) \subset \Gamma(T_4^{(k)}).$$

And the pattern induced by $\tilde{H}_4^{(j)}$, $j = 1, \ldots, p$ is independent of k. Since the number of inner iterations is bounded there exists a $\kappa > 0$ such that

$$(H_4^{(k,j)})_{r,s} > \kappa$$

for all $1 \leq r, s \leq n$ satisfying $(H_4^{(k,j)})_{r,s} > 0$ and all $k \in \mathbb{N}_0$. Theorem 4.3 and the nonnegativity of $T_4^{(k)}$ imply that there exists an $l \in \mathbb{N}$ such that $T_4^{(k+l)} \cdot \ldots \cdot T_4^{(k)}$ has a positive column whose elements are bounded from below by at least κ^l. Thus Corollary 4.2 applies and Theorem 4.8 delivers the desired convergence. □

Remark 4.14 *1) The result for T_6, which can be deduced from Theorem 4.16 is exactly the same as in Theorem 4.15 and has been left out.*

2) Note that the number of inner iterations has to be bounded in Theorem 4.18. This seems unavoidable, because the term $R_\omega^{(k,i)}$ has some influence on the pattern of $T_4^{(k)}$ and becomes arbitrarily small as the number of inner iterations grows. This should not be a problem in practice, *since the smaller $R_\omega^{(k,i)}$ becomes, the more the influence of $M_i^{-1}N_i$ in $(I-R_\omega^{(k,i)})M_i^{-1}N_i$ grows. Clearly, the term $M_i^{-1}N_i$ posses the structure which is necessary for convergence, but it cannot be guaranteed that some elements of $M_i^{-1}N_i$ are not cancelled out in $(I-R_\omega^{(k,i)})M_i^{-1}N_i$ (and this is also the problem for T_3). For an approach to overcome this problem see [2] and also Theorem 6.12 in Section 6.4. There it was assumed, that the number of inner iteration $q(k_j,i)$ tends to infinity on a subsequence k_j.*

4.7 Application to additive Schwarz methods

Based on the results from the previous section, a few propositions for additive Schwarz methods will be proven. They are basically derived from Theorem 4.16.

Consider MP and a root preserving decomposition $S_1,\dots,S_p$ w.r.t. A. Let $T_{\theta,l}$ be the global additive Schwarz operators from Section 3.2 and 3.3. The discussion here will be restricted to Update 1 and 4, i.e. exact and relaxed two-stage inexact additive Schwarz (cf. Algorithm 3.2 and 3.6) for the following reasons.

1) The results for Update 1 will apply verbatim to Update 2.
2) Since the results will be based on Theorem 4.16, they will not apply to Update 3 (cf. Remark 4.13).
3) While Update 5 and 6 could make some theoretical sense in multiplicative Schwarz iterations, they will not make any sense in additive Schwarz.

If $H_l^{(j)}$, $j=1,\dots,p$, $l\in\{1,4\}$ are the local operators given by (3.2.9) and (3.3.3), then by (3.2.7)

$$T_{\theta,l} := I-\theta\sum_{i=1}^{p}(I-H_l^{(i)})$$

where $\theta\in(0,1/q)$ and q is given by (3.1.1). According to the proof of Lemma 3.1, one has

$$\begin{aligned}
T_{\theta,1} &= I-\theta\sum_{i=1}^{p}(I-H_1^{(i)}) = I-\theta\sum_{i=1}^{p}\Pi_i^T\begin{pmatrix} I & -M_i^{-1}N_i\\ 0 & 0\end{pmatrix}\Pi_i\\
&= \underbrace{I-\theta\sum_{i=1}^{p}\Pi_i^T\begin{pmatrix} I_{|S_i|} & 0\\ 0 & 0\end{pmatrix}\Pi_i}_{=:I_\theta\geq 0} +\theta\sum_{i=1}^{p}\underbrace{\Pi^T\begin{pmatrix} 0 & M_i^{-1}N_i\\ 0 & 0\end{pmatrix}\Pi_i}_{=\bar{H}_1^{(i)}\geq 0}
\end{aligned}$$

wherein $\bar{H}_1^{(i)}$ can be written as

$$\begin{aligned}\bar{H}_1^{(i)} &= \Pi_i^T \begin{pmatrix} I_{|S_i|} & 0 \\ 0 & 0 \end{pmatrix} \Pi_i \cdot \Pi^T \begin{pmatrix} 0 & M_i^{-1}N_i \\ 0 & I_{|\neg S_i|} \end{pmatrix} \Pi_i \\ &= \Pi_i^T \begin{pmatrix} I_{|S_i|} & 0 \\ 0 & 0 \end{pmatrix} \Pi_i \cdot H_1^{(i)}\end{aligned}$$

A similar representation is valid for $T_{\theta,4}$. Thus

$$\Gamma(T_{\theta,l}) = \Delta \cup \Gamma\left(\sum_{i=1}^{p} \bar{H}_l^{(i)}\right)$$

for $l \in \{1,4\}$ and obviously

$$\Delta \cup \Gamma\left(\sum_{i=1}^{p} \bar{H}_l^{(i)}\right) = \Delta \cup \Gamma\left(\sum_{i=1}^{p} H_l^{(i)}\right). \tag{4.7.1}$$

The proof of Theorem 4.16 has shown that $\sum_{i=1}^{p} H_l^{(i)}$ is an ST-matrix for $l \in \{2,4\}$. Thus (4.7.1) implies that $T_{\theta,4}$ is an ST-matrix and therefore ST-regular.

The same holds for $T_{\theta,2}$ by Theorem 4.16. But as the argumentation concerns only the off-diagonal pattern, and those patterns are equal for $H_1^{(i)}$ and $H_2^{(i)}$, Theorem 4.16 applies also to $T_{\theta,1}$.

Theorem 4.19 *The (one-level) additive Schwarz iteration*

$$x^{k+1} = T_{\theta,1}x^k + c, \quad k = 0,1,2,\ldots$$

converges to the solution of $Ax = b$ for every given $x^0 \in \mathbb{R}^n$ and $\theta \in (0, 1/q)$ at a geometric rate.

Proof: See Theorem 4.17. □

Theorem 4.20 *Assume for $T_{\theta,4}^{(k)}$ inner M-splittings and the number of inner iterations to be bounded. Then the relaxed two-stage additive Schwarz iteration*

$$x^{k+1} = T_{\theta,4}^{(k)}x^k + c^{(k)}, \quad k = 0,1,2,\ldots$$

converges to the solution of $Ax = b$ for every given $x^0 \in \mathbb{R}^n$ and $\theta \in (0, 1/q)$. Moreover, the convergence is obtained at a geometric rate.

Proof: See Theorem 4.18. □

Remark 4.15 *It should be mentioned that the two-stage-methods in [44] should be analysed in a similar way as above for STM-matrices. The differences between additive Schwarz and multisplittings are small.*

4.8 Application to partially asynchronous iterations

The observations of the last chapters allow some statements for PAIs. The results will be proven in a complete algebraic fashion as in [39], rather than an analytical one as in [2, 55, 56]. The result here is restricted to ms-compatible decompositions. It generalises the results of [2] quite a bit (cf. Section 6.4).

Consider *MP* and let $S_1, \dots, S_p$ be an ms-compatible decomposition with respect to a flow compatible permutation π (see Definition 4.5). Additionally let $\tilde{S}_1, \dots, \tilde{S}_p$ be the block flow compatible core partitioning.

Additionally, consider a PAI $(A, x^0, \{\mathcal{J}_k, \mathcal{S}_k\}_{k\in\mathbb{N}_0})$ with an arbitrary scenario for the constants d and s (cf. (3.4.6) and (3.4.7)), such that $\mathcal{J}_k \in \{S_1, \dots, S_p\}$ (cf. Section 3.4) and the local updates are given by Algorithm 3.4 or Algorithm 3.6. Denote the local operators by $H_{d,2}^{(k)}$ and $H_{d,4}^{(k)}$ respectively (cf. Section 3.4) and let $i(k) \in \{1, \dots, p\}$ be the block which is updated in the k-th step, $k \in \mathbb{N}_0$. Furthermore let the inner splittings for the two-stage method be flow compatible (see Definition 4.6).

With the above assumptions, the behaviour of $H_{d,2}^{(k)}$ and $H_{d,4}^{(k)}$ is almost the same (on the pattern). In order to formulate results for both local updates, the labels $H_d^{(k)}$ and $H^{(i(k))}$ are used.

Theorem 4.21 *Let $k \in \mathbb{N}_0$ be arbitrary and $r = p(d+s)$. Then the product matrix $H_d^{(k,k+r)} := H_d^{(k+r)} \cdot \ldots \cdot H_d^{(k)}$ contains a positive column.*

Remark 4.16 *The proof follows that of Propositions 4 to 6 in [39] and will be illustrated by an example afterwards.*

Proof: The proof is divided into five steps.

Step 1: General properties of $H_d^{(k)}$, $k \in \mathbb{N}_0$
Since the matrix $H_d^{(k)}$ can be constructed from $H^{(i(k))}$, the following relations hold for the graph of $H_d^{(k)}$.

$$(4.8.1) \qquad (j, j) \in \Gamma(H_d^{(k)}), \text{ for } 1 \le j \le n.$$

There exists a unique number $0 \le s(i) \le d$ such that

$$(4.8.2) \qquad (j, i + s(i) \cdot n) \in \Gamma(H_d^{(k)})$$

for all indices $j \in S_{i(k)}$.

$$(4.8.3) \qquad (j, j - n) \in \Gamma(H_d^{(k)}), \text{ for } n < j \le (d+1) \cdot n.$$

Relation (4.8.1) follows from (3.4.5) and the fact that relaxation is used. (4.8.2) is due to (3.4.6) and because complete block columns correspond to the same delay by assumption. The last relation follows directly from the construction of the $H_d^{(k)}$.

Now let $k \in \mathbb{N}_0$ be arbitrary but fixed. Since all local operators are nonnegative, any graph operation is to be interpreted as in Lemma 1.1. Furthermore, for $l \geq k$ let

$$H_d^{(k,l)} := H_d^{(l)} \cdot H_d^{(l-1)} \cdot \ldots \cdot H_d^{(k)}.$$

Step 2: Finding the root column
By (3.4.7) there is an index l_0, $k \leq l_0 \leq k+s$ such that $S_p = S_{i(l_0)} = \mathcal{J}(l_0)$ and $\pi(n) \in \tilde{S}_p \subseteq S_p$. With Corollary 4.5 and (4.8.2) there exists an $i_0 \in \{1, \ldots, n\}$ such that $0 \leq s(\pi(i_0)) \leq d$ and $(\pi(j), \pi(i_0)+s(\pi(i_0))n) \in \Gamma(H_d^{(l_0)})$ for all $\pi(j) \in \tilde{S}_p$. Furthermore $1 \leq \pi(j) \leq n$.

What will be proven now, is the existence of a ξ, $1 \leq \xi \leq (d+1) \cdot n$ such that $(\pi(j), \xi) \in \Gamma(H_d^{(k,l_0)})$ for all $\pi(j) \in \tilde{S}_p$.

If $l_0 = k$, then there is nothing to prove since $H_d^{(k,l_0)} = H_d^{(k)}$, i.e. $\xi = \pi(i_0) + s(\pi(i_0))n$. Therefore let $l_0 > k$. If $s(\pi(i_0)) = 0$, then $(\pi(j), \pi(i_0)) \in \Gamma(H_d^{(k,l_0)})$, i.e. $\xi = \pi(i_0)$ by (4.8.1). So assume $s(\pi(i_0)) > 0$ and note that $n \leq \pi(i_0) + s(\pi(i_0)) \cdot n \leq (d+1) \cdot n$. It follows from (4.8.3) that

$$\begin{aligned} & (\pi(i_0) + s(\pi(i_0)) \cdot n, \pi(i_0) + s(\pi(i_0)) \cdot n - n) \\ = \; & (\pi(i_0) + s(\pi(i_0)) \cdot n, \pi(i_0) + (s(\pi(i_0)) - 1) \cdot n) \in \Gamma(H_d^{(l_0-1)}) \end{aligned}$$

hence with (4.8.2)

$$(\pi(j), \pi(i_0) + (s(\pi(i_0)) - 1) \cdot n) \in \Gamma(H_d^{(l_0-1,l_0)}).$$

If $s(\pi(i_0)) - 1 = 0$, then we are done using (4.8.1). If $s(\pi(i_0)) - 1 > 0$, the above argumentation might be applied inductively. There are two cases to consider. In the first case $s(\pi(i_0)) \leq l_0 - k + 1$. After $s(\pi(i_0))$ multiplications

$$(\pi(j), \pi(i_0)) \in \Gamma(H_d^{(l_0-s(i_0),l_0)})$$

and $\xi = \pi(i_0)$ is proper by (4.8.1).

In the other case $s(\pi(i_0)) > l_0 - k + 1$ and

$$(\pi(j), \pi(i_0) + (s(\pi(i_0)) - l_0 + k - 1) \cdot n) \in \Gamma(H_d^{(k,l_0)}).$$

In this case $\xi = (s(\pi(i_0)) - l_0 + k - 1) \cdot n$ is the choice. Again, taking (4.8.2), the last observation is valid for each $\pi(j) \in \tilde{S}_p$.

Step 3: Positivity preservation and duplication to the past
From step 2), $(\pi(j), \xi) \in \Gamma(H_d^{(k,l_0)})$ and $1 \leq \pi(j) \leq n$ for all $\pi(j) \in S_p$. Hence

$$(\pi(j), \xi) \in \Gamma(H_d^{(k,l_0+t)}),$$

for all $t \in \mathbb{N}$ from (4.8.1). Now $(\pi(j)+n, \pi(j)) \in \Gamma(H_d^{(l_0+1)})$ which follows from (4.8.3) and therefore

$$(\pi(j), \xi) \in \Gamma(H_d^{(k,l_0)}) \Rightarrow (\pi(j)+n, \xi) \in \Gamma(H_d^{(k,l_0+1)}).$$

The same argumentation applied $d-1$ times leads to

$$(\pi(j)+m \cdot d, \xi) \in \Gamma(H_d^{(k,l_0+d)}), \tag{4.8.4}$$

for $m = 0, \ldots, d$ and all $\pi(j) \in \tilde{S}_p$. Moreover, $k < l_0 + d \le k + (s+d)$. Additionally, (4.8.4) holds also for $H_d^{(k,l_0+d+t)}$ for all $t \in \mathbb{N}$.

Step 4: Increasing positivity

According to the definition of PAIs, there exists a $l_0 + d < l_1 \le l_0 + d + s$ such that $S_{p-1} = S_{i(l_1)} = \mathcal{J}(l_1)$. Consider the set $\tilde{S}_{p-1} \subseteq S_{p-1}$. By step 3), (4.8.4) holds for $H_d^{(k,l_1-1)}$ since $l_1 - 1 \ge l_0 + d$. Let h be the ξ-th column of $H_d^{(k,l_1-1)}$ and divide it into $d+1$ blocks $h_i \in \mathbb{R}^n$ such that $h = (h_0^T, \ldots, h_d^T)^T$. Then within each block, $(h_i)_{\pi(j)} > 0$ for all $\pi(j) \in \tilde{S}_p$ and $i = 0, \ldots, d$. Hence the action of h on $H_d^{(k,l_1)}$ is (restricted to the pattern) the same as the action of h_0 on $H^{(p-1)}$, the corresponding local Schwarz type operator. But now the same argumentation as in Theorem 4.10 can be applied, leading to

$$(\pi(j), \xi) \in \Gamma(H_d^{(k,l_1)})$$

for all $\pi(j) \in \tilde{S}_p \cup \tilde{S}_{p-1}$.

Step 5: Conclusion

Step 3 can be applied again, which results in

$$(\pi(j)+m \cdot d, \xi) \in \Gamma(H_d^{(k,l_1+d)})$$

for $m = 0, \ldots, d$ and all $\pi(j) \in \tilde{S}_p \cup \tilde{S}_{p-1}$. Furthermore, $k < l_1 \le k + 2(s+d)$. If steps 3 and 4 are applied $p-2$ times, then the complete ξ-th column of $H_d^{(k,l_{p-1})}$ is positive. Finally, $l_{p-1} \le p(d+s)$. □

What follows is an illustration of the proof:

Consider the STM-matrix

$$A = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ -1 & 0 & 0 & 1 \end{pmatrix},$$

the partitioning $S_1 = \{1, 3\}$ and $S_2 = \{2, 4\}$, a PAI using Update 2, $d = 2$, and the following operators which are assumed fixed

$$H_d^{(1)} = \left(\begin{array}{cccc|cccc|cccc}
\ast & & & & & & & \ast & & & & \\
 & 1 & & & & & & & & & & \\
 & & \ast & & & & & \ast & & & & \\
 & & & 1 & & & & & & & & \\
\hline
1 & & & & & & & & & & & \\
 & 1 & & & & & & & & & & \\
 & & 1 & & & & & & & & & \\
 & & & 1 & & & & & & & & \\
\hline
 & & & & 1 & & & & & & & \\
 & & & & & 1 & & & & & & \\
 & & & & & & 1 & & & & & \\
 & & & & & & & 1 & & & &
\end{array}\right),$$

$$H_d^{(2)} = \left(\begin{array}{cccc|cccc|cccc}
1 & & & & & & & & & & & \\
 & \ast & & & & & & & x & & & \\
 & & 1 & & & & & & & & & \\
 & & & \ast & & & & & x & & & \\
\hline
1 & & & & & & & & & & & \\
 & 1 & & & & & & & & & & \\
 & & 1 & & & & & & & & & \\
 & & & 1 & & & & & & & & \\
\hline
 & & & & 1 & & & & & & & \\
 & & & & & 1 & & & & & & \\
 & & & & & & 1 & & & & & \\
 & & & & & & & 1 & & & &
\end{array}\right).$$

The sequence $(H_d^{(1)})^3(H_d^{(2)})^3H_d^{(1)}$ will be analysed now. Basic entries are marked by an "x".

The root column is obviously in $H_d^{(2)}$; it is column $9 = 1 + s(1)4$ with $s(1) = 2$. Following step two of the proof, the root column must move one block to the left in $H_d^{(2)}H_d^{(1)}$.

$$H_d^{(2)}H_d^{(1)} = \left(\begin{array}{cccc|cccc|cccc}
\ast & & & & & & & \ast & & & & \\
 & \ast & & & x & & & & & & & \\
 & & \ast & & & & & \ast & & & & \\
 & & & \ast & x & & & & & & & \\
\hline
\ast & & & & & & & \ast & & & & \\
 & 1 & & & & & & & & & & \\
 & & \ast & & & & & \ast & & & & \\
 & & & 1 & & & & & & & & \\
\hline
1 & & & & & & & & & & & \\
 & 1 & & & & & & & & & & \\
 & & 1 & & & & & & & & & \\
 & & & 1 & & & & & & & &
\end{array}\right)$$

The root column is now determined and $\xi = 1 + (s(1) - 1)4 = 5$. Here an illustration of step three of the proof; the duplication of entries to the past.

$$(H_d^{(2)})^2H_d^{(1)} = \left(\begin{array}{cccc|cccc|cccc}
\ast & & & & & & & \ast & & & & \\
\ast & \ast & & & x & & & & & & & \\
 & & \ast & & & & & \ast & & & & \\
\ast & & & \ast & x & & & & & & & \\
\hline
\ast & & & & & & & \ast & & & & \\
 & \ast & & & x & & & & & & & \\
 & & \ast & & & & & \ast & & & & \\
 & & & \ast & x & & & & & & & \\
\hline
\ast & & & & & & & \ast & & & & \\
 & 1 & & & & & & & & & & \\
 & & \ast & & & & & \ast & & & & \\
 & & & 1 & & & & & & & &
\end{array}\right),$$

$$(H_d^{(2)})^3H_d^{(1)} = \left(\begin{array}{cccc|cccc|cccc}
\ast & & & & & & & \ast & & & & \\
\ast & \ast & & & x & & & \ast & & & & \\
 & & \ast & & & & & \ast & & & & \\
\ast & & & \ast & x & & & \ast & & & & \\
\hline
\ast & & & & & & & \ast & & & & \\
\ast & \ast & & & x & & & & & & & \\
 & & \ast & & & & & \ast & & & & \\
\ast & & & \ast & x & & & & & & & \\
\hline
\ast & & & & & & & \ast & & & & \\
 & \ast & & & x & & & & & & & \\
 & & \ast & & & & & \ast & & & & \\
 & & & \ast & x & & & & & & &
\end{array}\right)$$

The occurrence of a positive block within column eight is accidental. The basic entries have been moved downward. The following multiplication, which also reflects the flow compatibility, will produce a positive block within the root column. This is step four in the proof.

$$H_d^{(1)}(H_d^{(2)})^3H_d^{(1)} = \left(\begin{array}{cccc|cccc|cccc}
* & & & * & x & & & * & & & & \\
* & * & & & x & & & * & & & & \\
* & & * & * & x & & & * & & & & \\
* & & & * & x & & & * & & & & \\
\hline
* & & & & & & & * & & & & \\
* & * & & & x & & & * & & & & \\
& & * & & & & & * & & & & \\
* & & & * & x & & & * & & & & \\
\hline
* & & & & & & & * & & & & \\
* & * & & & x & & & & & & & \\
& & * & & & & & * & & & & \\
* & & & * & x & & & & & & &
\end{array}\right)$$

The last multiplication is a nice demonstration why Theorem 4.10 is applicable in step four of the proof of Theorem 4.21. Since the necessary information to produce positive elements has been copied, an application of the root column to $H_d^{(1)}$ is the same (on the pattern) as in the case $d = 0$. The inheritance of the positive elements within the root column is now obvious.

To outline the importance of the positive diagonals which are due to Update 2, consider Update 1. Then

$$H_d^{(1)}(H_d^{(2)})^3H_d^{(1)} = \left(\begin{array}{cccc|cccc|cccc}
1 & & & & & & & & & & & \\
& & & & & & & 1 & & & & \\
1 & & & & & & & & & & & \\
& & & & & & & 1 & & & & \\
\hline
& & & & & & & 1 & & & & \\
& & & & & & & 1 & & & & \\
& & & & & & & 1 & & & & \\
& & & & & & & 1 & & & & \\
\hline
& & & & & & & 1 & & & & \\
1 & & & & & & & & & & & \\
& & & & & & & 1 & & & & \\
1 & & & & & & & & & & &
\end{array}\right).$$

This product is a rank-two projection. Thus, it is not guaranteed that the iteration converges to the correct result of $Ax = 0$.

Theorem 4.21 leads to the following statements.

Theorem 4.22 *Consider MP and let $S_1, \ldots, S_p$ be an ms-compatible decomposition w.r.t. a flow compatible permutation π. Then every PAI $(A, x^0, \{\mathcal{J}_k, \mathcal{S}_k\}_{k \in \mathbb{N}_0})$ with Update 2, given by Algorithm 3.4, converges to the solution x^* of*

$$Ax = b, x \in \mathbb{R}^n.$$

Proof: Since the maximum delay is bounded by d there are only finitely many different local operators $H_{d,2}^{(k)}$. Therefore, and since s is bounded, there are only finitely many different operators $H^{(k,k+r)}$ having a positive column for $r \leq p(d + s)$. Hence, Theorem 4.7 is applicable and the convergence follows from Theorem 4.8. $\square$

Theorem 4.23 *Consider MP and let $S_1, \ldots, S_p$ be an ms-compatible decomposition w.r.t. a flow compatible permutation π. Assume Update 4, given by Algorithm 3.6, and let the inner splittings be flow compatible. If the number of inner iterations is bounded, then every PAI $(A, x^0, \{\mathcal{J}_k, \mathcal{S}_k\}_{k\in\mathbb{N}_0})$ converges to the solution x^* of*

$$Ax = b, x \in \mathbb{R}^n.$$

Proof: Since the inner iterations are bounded, there are only finitely many different local operators $H_{d,4}^{(k)}$. Thus, the same argumentation as for Theorem 4.22 applies. □

Remark 4.17 *1) A convergence result for Update 6 has been left out since it is obvious. It can be formulated for general decompositions and holds for STM-matrices. Thus, it generalises the result of [39] and confirms a more general result given in [56].*

2) The number of inner iterations in Theorem 4.23 should be bounded. Otherwise the prove of Theorem 4.21 does not guarantee that the elements of the positive column have some certain size.

Chapter 5

Some extensions for *GMP*

Once the theory has been understood for ST- and STM-matrices, it can be partly extended to SF- and SFM-matrices which will lead to solutions of *GMP*. Though the expansion of the structure is easy (cf. Section 2.3), the extension of the convergence results is not. There will be some results for multiplicative and additive Schwarz, but unfortunately there are no new propositions for PAIs. This is because PAIs imply certain problems which have not yet been solved.

Note that results for Update 5 and 6 (cf. Algorithms 3.7 and 3.8 in Section 3.3) have been left out since they have no relevance in practice.

5.1 Non-relaxed multiplicative Schwarz iterations

The first extension concerns the non-relaxed multiplicative Schwarz iterations, i.e. Update 1 and 3 (cf. Section 3.3, Algorithms 3.2 and 3.5).

According to the theory given in Section 2.3, terms like *block flow compatibility* or *ms-compatibility* can be extended by a localisation.

Consider an SF-matrix $B \in \mathbb{R}^{n\times n}$ of degree r and let $\mathcal{F} = (\mathcal{T}_1, \ldots, \mathcal{T}_r)$ be a spanning forest in $\Gamma(B^T)$. By a simple exchange of certain vertices, the trees $\mathcal{T}_i = (V_i, E_i)$, $i = 1, \ldots, r$ can be chosen such that $E_i \cap E_j = \emptyset$, for all $1 \leq i, j \leq r$, $i \neq j$. This property can always be assumed w.l.o.g., and such a forest $\mathcal{F} = (\mathcal{T}_1, \ldots, \mathcal{T}_r)$ will be called a *disjoint spanning forest*.

Definition 5.1 *A permutation $\pi : \{1, \ldots, n\} \longrightarrow \{1, \ldots, n\}$ is said to be a* locally flow compatible numbering *w.r.t. a disjoint spanning forest $\mathcal{F} = (\mathcal{T}_1, \ldots, \mathcal{T}_r)$, if for each $\mathcal{T}_i$, $i = 1, \ldots, r$, there exist numbers $k_1^i < k_2^i < \ldots < k_{n_i}^i$, $k_j^i \in \{1, \ldots, n\}$, $j = 1, \ldots, n_i$, such that $\pi^{(i)} = (\pi(k_1^i), \ldots, \pi(k_{n_i}^i))$ is a flow compatible numbering of $\mathcal{T}_i$. The tuples $\pi^{(i)} = (\pi(k_1^i), \ldots, \pi(k_{n_i}^i))$, $i = 1, \ldots, r$ are called the* localisations *of π.*

Definition 5.1 is an extension of Definition 4.2 by a localisation of the flow compatibility.

Remark 5.1 *It should be clear that an r-times application of DFS_VISIT (Algorithm 4.1) to an SFM-matrix of degree r, where each run starts within another basic (and final) class, delivers a locally flow compatible numbering of a disjoint forest.*

The following definitions are an immediate consequence of the idea given in Definition 5.1.

Definition 5.2 *Consider a locally flow compatible numbering π with localisations $\pi^{(i)} = (\pi(k_1^i), \ldots, \pi(k_{n_i}^i))$, $i = 1, \ldots, r$.*

1) A regular partitioning $S_1, \ldots, S_p$ of $(\pi(1), \ldots, \pi(n))$ is called locally block flow compatible *if*

$$\max\{j : \pi(k_j^i) \in S_k\} < \min\{j : \pi(k_j^i) \in S_l\}$$

for all $1 \leq k < l \leq p$ and $i = 1, \ldots, r$.

2) A regular decomposition $S_1, \ldots, S_p$ is called locally block flow compatible *if there exists a locally block flow compatible partitioning $(\tilde{S}_1, \ldots, \tilde{S}_p)$, such that $\tilde{S}_i \subseteq S_i$ for all $i = 1, \ldots, p$.*

3) A locally block flow compatible decomposition is called gms-compatible *(gms for "generalised multiplicative Schwarz") if for each $i \in \{1, \ldots, r\}$*

- *there exists exactly one $l_0 \in \{1, \ldots, p\}$ such that $\pi(k_{n_i}^i) \in S_{l_0}$ and*
- *if $\pi(j) \in S_k$ then $k \leq l_0$ for all $j \in \{k_1^i, \ldots, k_{n_i}^i\}$.*

Part 1) of Definition 5.2 extends Definition 4.3 and part 2) Definition 4.4. Finally, part 3) generalises Definition 4.5, since $\pi(k_{n_i}^i)$ is the root of $\mathcal{T}_i$.

Let $A = I - B \in \mathbb{R}^{n \times n}$ be an SFM-matrix of degree r and let $\mathcal{F} = (\mathcal{T}_1, \ldots, \mathcal{T}_r)$ be a disjoint spanning forest. Assume that A has the following block structure

$$A = \begin{pmatrix} D_1 & 0 & \ldots & 0 & 0 \\ 0 & D_2 & \ldots & 0 & 0 \\ \vdots & & \ddots & \vdots & \vdots \\ 0 & 0 & & D_r & 0 \\ E_1 & E_2 & \ldots & E_r & F \end{pmatrix}, \tag{5.1.1}$$

where each D_i contains the root of $\mathcal{T}_i$ (cf. Lemma 2.5). The idea of localisation establishes a new standard representation of A, which is an alternative to (5.1.1).

Lemma 5.1 *Let $D_i, \ldots, D_r$ and F be given as in (5.1.1). There exists a permutation matrix Σ such that*

$$\Sigma A \Sigma^T = \begin{pmatrix} D_1 & 0 & 0 & 0 & \ldots & 0 & 0 \\ E_{11} & F_{11} & E_{12} & F_{12} & \ldots & E_{1r} & F_{1r} \\ 0 & 0 & D_2 & 0 & \ldots & 0 & 0 \\ E_{21} & F_{21} & E_{22} & F_{22} & \ldots & E_{2r} & F_{2r} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & D_r & 0 \\ E_{r1} & F_{r1} & E_{r2} & F_{r2} & \ldots & E_{rr} & F_{rr} \end{pmatrix} \tag{5.1.2}$$

where each submatrix

$$\begin{pmatrix} D_i & 0 \\ E_{ii} & F_{ii} \end{pmatrix}, \quad i = 1, \ldots, r$$

represents an STM-matrix and F_{ii} is a principal minor of F. The minor F_{ii} is of dimension 0×0 if the vertex set V_i of $\mathcal{T}_i$ contains no index from F.

Proof: Since the spanning forest $\mathcal{F} = (\mathcal{T}_1, \ldots, \mathcal{T}_r)$ is assumed disjoint, each index given by F belongs to exactly one tree. Thus, F can be partitioned as follows.

$$F = \begin{pmatrix} F_{\sigma(1),\sigma(1)} & F_{\sigma(1),\sigma(2)} & \cdots & F_{\sigma(1),\sigma(r)} \\ F_{\sigma(2),\sigma(1)} & F_{\sigma(2),\sigma(2)} & \cdots & F_{\sigma(2),\sigma(r)} \\ \vdots & \vdots & \ddots & \vdots \\ F_{\sigma(r),\sigma(1)} & F_{\sigma(r),\sigma(2)} & \cdots & F_{\sigma(r),\sigma(r)} \end{pmatrix}.$$

Here, $\sigma : \{1, \ldots, r\} \longrightarrow \{1, \ldots, r\}$ is a permutation such that the indices of $F_{\sigma(i),\sigma(i)}$ belongs to $\mathcal{T}_i$. If no vertex of $\mathcal{T}_i$ belongs to the indices given by F, $F_{\sigma(i),\sigma(i)}$ might be considered as the 0×0 matrix. The permutation σ induces a partitioning

$$E_i = \begin{pmatrix} E_{\sigma(1),i} \\ E_{\sigma(2),i} \\ \vdots \\ E_{\sigma(r),i} \end{pmatrix}$$

for $i = 1, \ldots, r$. Now, it is easy to see that there exists a permutation matrix Σ, such that $\Sigma A \Sigma^T$ becomes (5.1.2). $\square$

Lemma 5.1 delivers the structure that is needed for the following result.

Lemma 5.2 *Let A be an SFM-matrix of degree r and let $S_1, \ldots, S_p$ be a gms-compatible decomposition w.r.t. to a spanning disjoint forest $\mathcal{F} = (\mathcal{T}_1, \ldots, \mathcal{T}_r)$. If $H_1^{(i)}$, $i = 1, \ldots, p$ are the local Schwarz operators given by (3.2.9) w.r.t. $S_1, \ldots, S_p$, then*

$$T_1 := H_1^{(1)} \cdot \ldots \cdot H_1^{(p)}$$

is a semiconvergent SF-matrix of degree r.

Proof: By Lemma 5.1, assume A to be given in the standard form (5.1.2). Then

$$A \leq \begin{pmatrix} D_1 & 0 & 0 & 0 & \dots & 0 & 0 \\ E_{11} & F_{11} & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & D_2 & 0 & \dots & 0 & 0 \\ 0 & 0 & E_{22} & F_{22} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & D_r & 0 \\ 0 & 0 & 0 & 0 & \dots & E_{rr} & F_{rr} \end{pmatrix} =: \tilde{A}. \tag{5.1.3}$$

Let $M_i = A[S_i]$, then $M_i \leq \tilde{A}[S_i] =: \tilde{M}_i$ and obviously $M_i^{-1} \geq \tilde{M}_i^{-1}$. Let $\tilde{H}_1^{(i)}$ be given by (3.2.9) w.r.t. $\tilde{A}$ and $S_1, \dots, S_p$ for $i = 1, \dots, p$. Then $H_1^{(i)} \geq \tilde{H}_1^{(i)} \geq 0$, $i = 1, \dots, p$, and finally

$$T_1 \geq \tilde{H}_1^{(1)} \cdot \ldots \cdot \tilde{H}_1^{(p)} =: \tilde{T}_1. \tag{5.1.4}$$

The gms-compatibility implies ms-compatibility within each diagonal block of $\tilde{A}$. The representation

$$\tilde{T}_1 = \begin{pmatrix} \tilde{U}_1 & 0 & 0 & 0 & \dots & 0 & 0 \\ \tilde{V}_{11} & \tilde{W}_{11} & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \tilde{U}_2 & 0 & \dots & 0 & 0 \\ 0 & 0 & \tilde{V}_{22} & \tilde{W}_{22} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & \tilde{U}_r & 0 \\ 0 & 0 & 0 & 0 & \dots & \tilde{V}_{rr} & \tilde{W}_{rr} \end{pmatrix}$$

follows from Theorem 4.10 and each diagonal block of $\tilde{T}_1$, i.e.

$$\begin{pmatrix} \tilde{U}_i & 0 \\ \tilde{V}_{ii} & \tilde{W}_{ii} \end{pmatrix}, \quad i = 1, \dots, r,$$

has a positive column. This and (5.1.4) imply that each block

$$T_1^{(i)} := \begin{pmatrix} U_i & 0 \\ V_{ii} & W_{ii} \end{pmatrix}, \quad i = 1, \dots, r,$$

in

$$T_1 = \begin{pmatrix} U_1 & 0 & 0 & 0 & \dots & 0 & 0 \\ V_{11} & W_{11} & V_{12} & W_{12} & \dots & V_{1r} & W_{1r} \\ 0 & 0 & D_2 & 0 & \dots & 0 & 0 \\ V_{21} & W_{21} & V_{22} & W_{22} & \dots & V_{2r} & W_{2r} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & D_r & 0 \\ V_{r1} & W_{r1} & V_{r2} & W_{r2} & \dots & V_{rr} & W_{rr} \end{pmatrix} \geq \tilde{T}_1$$

has a positive column. Thus, $\Gamma(T_1^T)$ contains a disjoint spanning forest which contains r trees of height 1.

Theorem 4.1 implies that each $T_1^{(i)}$ is a semiconvergent ST-matrix for all $i = 1, \ldots, r$, thus $\rho(U_i) = 1$, $\gamma(U_i) < 1$, and also $\rho(W_{ii}) < 1$ since $V_{ii} \neq 0$. Therefore, the indices of the blocks $U_1, \ldots, U_r$ represent the only r final (and also basic) classes. Theorem 2.3 implies that T_1 is an SF-matrix of degree r and the semiconvergence follows since $\gamma(U_i) < 1$ for all $i = 1, \ldots, r$. $\square$

The following theorem is actually a corollary of Lemma 5.2.

Theorem 5.1 *Let A be an SFM-matrix of any degree and let $S_1, \ldots, S_p$ be a gms-compatible decomposition. If $H_1^{(1)}, \ldots, H_1^{(p)}$ are the local Schwarz operators given by (3.2.9) w.r.t. $S_1, \ldots, S_p$, then the multiplicative Schwarz iteration*

$$x^{k+1} = T_1 x^k + c, \quad k = 0, 1, 2, \ldots$$

converges to the solution of $Ax = b$, $b \in \mathcal{R}(A)$ for every given $x^0 \in \mathbb{R}^n$ and a proper right hand side c. The convergence is obtained at a geometric rate.

Proof: Trivial using Lemma 5.2 since T_1 is semiconvergent and $\mathcal{N}(I - T_1) = \mathcal{N}(A)$. $\square$

Theorem 5.1 shows that an extension for exact Schwarz iterations is no problem. Thus, Theorem 5.1 can be regarded as generalisation of Theorem 4.11.

The results for two-stage iteration schemes will be weaker than in the ST-matrix case. To prove them, a generalisation of Definition 4.6 is needed.

Definition 5.3 *Let A be an STM-matrix and $\mathcal{F} = (\mathcal{T}_1, \ldots, \mathcal{T}_r)$ be a spanning disjoint forest in $\Gamma(A^T)$. For an arbitrary regular decomposition $S_1, \ldots, S_p$ let (F_i, G_i) be an M-splitting of $M_i = A[S_i]$. Then (F_i, G_i) is called a* locally flow compatible splitting *if*

$$(\mathcal{T}_j)_{|S_i} \cap \Gamma(F_i^T) = (\mathcal{T}_j)_{|S_i} \cap \Gamma(M_i^T)$$

for all $i = 1, \ldots, p$ and $j = 1, \ldots, r$.

Note that locally flow compatible splittings have been defined for M-splittings instead of weak regular splittings. That should not be a problem in practice and has been done to simplify the proof of the following lemma.

Lemma 5.3 *Let A be an SFM-matrix of degree r and let $S_1, \ldots, S_p$ be a gms-compatible decomposition w.r.t. to a spanning disjoint forest $\mathcal{F} = (\mathcal{T}_1, \ldots, \mathcal{T}_r)$. Let $H_3^{(k,i)}$ be the local Schwarz operators given by (3.3.2) w.r.t. $S_1, \ldots, S_p$ for $i = 1, \ldots, p$ and $k \in \mathbb{N}_0$. Assume that the inner splittings are locally flow compatible. Then*

$$T_3^{(k)} := H_3^{(k,1)} \cdot \ldots \cdot H_3^{(k,p)}$$

is a semiconvergent SF-matrix of degree r.

Proof: Again, define $\tilde{A}$ by (5.1.3) and let $\tilde{A}[S_i] = \tilde{M}_i$. Then $M_i =\leq \tilde{M}_i$. For a locally flow compatible splitting $M_i = F_i - G_i$ define $\tilde{F}_i$ by

$$(\tilde{F}_i)_{k,l} := \begin{cases} (F_i)_{k,l} & \text{if } (\tilde{M}_i)_{k,l} \neq 0, \\ 0 & \text{else,} \end{cases}$$

for all $1 \leq k, l \leq n$ and $i = 1, \ldots, p$. Additionally, let $\tilde{G}_i := \tilde{F}_i - \tilde{M}_i$, then $(\tilde{F}_i, \tilde{G}_i)$ is an M-splitting and $F_i \leq \tilde{F}_i$. This implies $F_i^{-1} \geq \tilde{F}_i^{-1}$. Finally, $\tilde{G}_i \leq G_i$ for all $i = 1, \ldots, p$ since (F_i, G_i) is an M-splitting. Therefore,

$$R^{(k,i)} = (F_i^{-1} G_i)^{q(k,i)} \geq (\tilde{F}_i^{-1} \tilde{G}_i)^{q(k,i)} =: \tilde{R}^{(k,i)}$$

for all $q(k,i) \geq 1$ and

$$\begin{aligned} (I - R^{(k,i)}) M_i^{-1} &= \sum_{j=0}^{q(k,i)-1} (F_i^{-1} G_i)^j F_i^{-1} \\ &\geq \sum_{j=0}^{q(k,i)-1} (\tilde{F}_i^{-1} \tilde{G}_i)^j \tilde{F}_i^{-1} = (I - \tilde{R}^{(k,i)}) \tilde{M}_i^{-1}. \end{aligned}$$

Now define $\tilde{H}_3^{(k,i)}$ by (3.3.2) in terms of $(\tilde{F}_i, \tilde{G}_i)$ and $\tilde{A}$, then

$$0 \leq \tilde{H}_3^{(k,i)} \leq H_3^{(k,i)}$$

for all $i = 1, \ldots, p$ and $k \in \mathbb{N}_0$.

With the above defined matrices $\tilde{H}_3^{(k,i)}$, the same argumentation as used in the proof of Lemma 5.2 can be applied (using Theorem 4.12 instead of Theorem 4.1), leading to the result that $T_3^{(k)}$ is a semiconvergent SF-matrix of degree r. Moreover, the existence of a spanning disjoint forest whose r trees are of height 1 follows. The elements of those trees are bounded from below by some $\kappa > 0$ for any number of inner iterations $q(k,i)$. □

Unfortunately there is no complete generalisation of Theorem 4.12. The problem here is, that there is no proper extension of the coefficient of ergodicity $\tau(\cdot)$ defined in Theorem 4.4 in Section 4.2.

To outline this, consider an SFM-matrix $A \in \mathbb{R}^{n \times n}$ and a two-stage multiplicative Schwarz iteration. Assume w.l.o.g.

$$A = \begin{pmatrix} D_1 & 0 & \ldots & 0 & 0 \\ 0 & D_2 & \ldots & 0 & 0 \\ \vdots & & \ddots & \vdots & \vdots \\ 0 & 0 & & D_r & 0 \\ E_1 & E_2 & \ldots & E_r & F \end{pmatrix}. \tag{5.1.5}$$

Since $Az = 0$ for some positive vector z, assume w.l.o.g. $z = e$. Each global operator $T_3^{(k)}$ is a semiconvergent SF-Matrix by Lemma 5.3. Since $T_3^{(k)}e = e$ for all $k \in \mathbb{N}_0$, each global operator is row stochastic. Moreover

$$T_3^{(k)} = \begin{pmatrix} U_1^{(k)} & 0 & \dots & 0 & 0 \\ 0 & U_2^{(k)} & \dots & 0 & 0 \\ \vdots & & \ddots & \vdots & \vdots \\ 0 & 0 & & U_r^{(k)} & 0 \\ V_1^{(k)} & V_2^{(k)} & \dots & V_r^{(k)} & W^{(k)} \end{pmatrix}. \tag{5.1.6}$$

where each $U_i^{(k)}$ is a row stochastic matrix and the dimension of $U_i^{(k)}$ and D_i in (5.1.5) is the same for all $i = 1, \dots, r$ and $k \in \mathbb{N}_0$.

The elements of the disjoint spanning forest constructed in Lemma 5.3 are bounded from below by some $\kappa > 0$ for all $k \in \mathbb{N}_0$ (cf. Section 4.5). This implies immediately that $\rho(W^{(k)}) \leq \theta < 1$ for all $k \in \mathbb{N}_0$. Moreover, each $U_i^{(k)}$ is row stochastic and has a positive column whose elements are also bounded from below for each $i = 1, \dots, r$ and $k \in \mathbb{N}_0$.

Thus, everything is fine but $\tau(T_3^{(k)}) \equiv 1$ for all $k \in N_0$ if the degree of A exceeds 1. Indeed, the class of row stochastic SF-matrices of degree 1, i.e. row stochastic ST-matrices, is the largest class on which $\tau(\cdot)$ works properly. Hence, a convergence result similar to Theorem 4.7 can not be achieved using $\tau(\cdot)$.

This makes the analysis of the convergence of

$$\lim_{k \longrightarrow \infty} T_3^{(k)} \cdot T_3^{(k-1)} \cdot \ldots \cdot T_3^{(0)}$$

more difficult. According to (5.1.6) let

$$U_j^{(k,0)} := U_j^{(k)} \cdot U_j^{(k-1)} \cdot \ldots \cdot U_j^{(0)},$$

for $j = 1, \dots, r$, and define

$$W^{(k,0)} := W^{(k)} \cdot W^{(k-1)} \cdot \ldots \cdot W^{(0)}.$$

Define for $k \in \mathbb{N}$

$$Y_j^{(k,0)} := V_j^{(k)} U_j^{(k-1,0)} + W^{(k)} Y_j^{(k-1,0)},$$

where

$$Y_j^{(0,0)} := V_j^{(0)}$$

for $j = 1, \ldots, r$. Then

$$(5.1.7) \qquad T_3^{(k)} \cdot \ldots \cdot T_3^{(0)} = \begin{pmatrix} U_1^{(k,0)} & 0 & \ldots & 0 & 0 \\ 0 & U_2^{(k,0)} & \ldots & 0 & 0 \\ \vdots & & \ddots & \vdots & \vdots \\ 0 & 0 & & U_r^{(k,0)} & 0 \\ Y_1^{(k,0)} & Y_2^{(k,0)} & \ldots & Y_r^{(k,0)} & W^{(k,0)} \end{pmatrix}.$$

The convergence of

$$\lim_{k \longrightarrow \infty} U_j^{(k,0)} = U_j^*$$

is obvious by Theorem 4.7 for each $j = 1, \ldots, r$, and the limit is a rank one row stochastic matrix. Additionally

$$\lim_{k \longrightarrow \infty} W^{(k,0)} = 0$$

since $\rho(W^{(k)}) \leq \theta < 1$ for all $k \in \mathbb{N}_0$. But the problem is to prove the convergence of the sequences

$$\lim_{k \longrightarrow \infty} Y_j^{(k,0)} = \lim_{k \longrightarrow \infty} \left(V_j^{(k)} U_j^{(k-1,0)} + W^{(k)} Y_j^{(k-1,0)} \right)$$

whose limits can not be the zero matrix for all $j = 1, \ldots, r$.

The problem comes from the submatrices $V_j^{(k)}$ given in (5.1.6), which reflect the accessibility relation between indices in $U_j^{(k)}$ and $W^{(k)}$. If this relation stays constant, then the convergence can be proven (which will not be done here). But this can not be guaranteed if the number of inner iterations vary. However, if $x^0 \in \mathbb{R}^n$ is an arbitrary vector and partitioned with conformity to (5.1.7) into blocks $(x^0)^T = ((x_1^0)^T, \ldots, (x_{r+1}^0)^T)$, then clearly

$$\lim_{k \longrightarrow \infty} x_j^k = \lim_{k \longrightarrow \infty} U_j^{(k,0)} x_j^0 = \lambda_j e_j$$

for each $j = 1, \ldots, r$. Thus it remains to prove the existence of

$$\lim_{k \longrightarrow \infty} x_{r+1}^k$$

and to discuss the questions, whether its existence is independent of the existence of $\lim_{k \longrightarrow \infty} T_3^{(k)} \cdot \ldots \cdot T_3^{(0)}$ or not. Though it should be feasible, it will be omitted here. Thus the following theorem will be formulated with a constant number of inner iterations, i.e. for a stationary two-stage multiplicative Schwarz iteration.

Theorem 5.2 *Let A be an SFM-matrix of any degree and let $S_1, \ldots, S_p$ be a gms-compatible decomposition. The local Schwarz operators*

$H_3^{(k,1)}, \ldots, H_3^{(k,p)}$ *are given by (3.3.2) w.r.t.* $S_1, \ldots, S_p$. *Assume that the number of inner iterations* $q(k,i) = q(i)$ *is constant for each* $i = 1, \ldots, p$ *and that the inner splittings are locally flow compatible. Then the stationary two-stage multiplicative Schwarz iteration*

$$x^{k+1} = T_3 x^k + c, \quad k = 0, 1, 2, \ldots$$

converges to the solution of $Ax = b$, $b \in \mathcal{R}(A)$ *for every given* $x^0 \in \mathbb{R}^n$ *and a proper right hand side* c. *The convergence is obtained at a geometric rate.*

Proof: Obvious by Lemma 5.3. □

Remark 5.2 *Theorem 5.1 can obviously be applied to Update 2 (relaxed exact Schwarz, cf. (3.3.1)), while Theorem 5.2 can also be applied to Update 4 (relaxed two-stage Schwarz, cf. (3.3.3)).*

5.2 Relaxed multiplicative Schwarz iterations

With the ideas of Section 5.1, results for relaxed multiplicative Schwarz iterations for SFM-matrices can be simply derived. This will led to extensions of Theorems 4.7, 4.17 and 4.18. A revision of Theorems 4.13 and 4.14 will be omitted because it is obvious in view of Theorems 5.1 and 5.2.

At first, a localisation of Definition 4.7.

Definition 5.4 *Let* $A \in \mathbb{R}^{n \times n}$ *be an SFM-matrix of degree* r *and let* $\alpha_1, \ldots, \alpha_r$ *be the basic classes. A regular decomposition* $S_1, \ldots, S_p$ *is called* locally root preserving *if for all* $i = 1, \ldots, r$ *there exists an* $j_i \in \alpha_i$, *such that* $|j : j_i \in S_j| = 1$.

With this definition, Theorem 4.16 can be restated as follows.

Lemma 5.4 *Suppose that* $A \in \mathbb{R}^{n \times n}$ *is an SFM-matrix of degree* r *and assume a locally root preserving decomposition* $S_1, \ldots, S_p$. *If for* $j = 1, \ldots, p$, *the local operators* $H_2^{(j)}$ *and* $H_4^{(k,j)}$ *are given by (3.3.1) and (3.3.3) and the inner splittings (if any) are M-splittings, then each product*

$$\begin{aligned} T_2 &:= H_2^{(\sigma(1))} \cdot \ldots \cdot H_2^{(\sigma(p))}, \\ T_4^{(k)} &:= H_4^{(k,\sigma(1))} \cdot \ldots \cdot H_4^{(k,\sigma(p))}, \end{aligned}$$

is a semiconvergent SF-matrix of degree r *for any permutation* $\sigma : \{1, \ldots, p\} \longrightarrow \{1, \ldots, p\}$.

Proof: Apply the ideas of the proofs of Lemma 5.2 and 5.3 to T_2 and $T_4^{(k)}$, respectively. Then the lemma follows by an easy application of Theorem 4.16. $\square$

The following results are an immediate consequence.

Theorem 5.3 *Let A be an SFM-matrix of any degree. Let $S_1, \ldots, S_p$ be a locally root preserving decomposition and σ be any permutation on $\{1, \ldots, p\}$. For $T_2 := H_2^{(\sigma(1))} \cdot \ldots \cdot H_2^{(\sigma(p))}$ let the local operators $H_2^{(i)}$ be given by (3.3.1). Then the relaxed multiplicative Schwarz iteration*

$$x^{k+1} = T_2 x^k + c, \quad k = 0, 1, 2, \ldots$$

converges to the solution of $Ax = b$, $b \in \mathcal{R}(A)$ for every given $x^0 \in \mathrm{I\!R}^n$ and a proper right hand side c. The convergence is obtained at a geometric rate.

Theorem 5.4 *Let A be an SFM-matrix of any degree. Let $S_1, \ldots, S_p$ be a locally root preserving decomposition and σ be any permutation on $\{1, \ldots, p\}$. For $T_4^{(k)} := H_4^{(k,\sigma(1))} \cdot \ldots \cdot H_4^{(k,\sigma(p))}$ let the local operators $H_4^{(k,i)}$ be given by (3.3.3). Assume that the numbers of inner iterations $q(k,i) = q(i)$ are constant, and the inner splittings are M-splittings for each $i = 1, \ldots, p$. Then the relaxed stationary two-stage multiplicative Schwarz iteration*

$$x^{k+1} = T_4 x^k + c, \quad k = 0, 1, 2, \ldots$$

converges to the solution of $Ax = b$, $b \in \mathcal{R}(A)$ for every given $x^0 \in \mathrm{I\!R}^n$ and a proper right hand side c. The convergence is obtained at a geometric rate.

5.3 Additive Schwarz iterations

Results for additive Schwarz iterations are now easily obtained from the previous section and the discussion in Section 4.7. Actually, there is nothing to prove for the following theorems.

Assume a decomposition $S_1, \ldots, S_p$. Let the local operators $H_l^{(j)}$, $j = 1, \ldots, p$, $l \in \{1, 4\}$ be given by (3.2.9) and (3.3.3), then from (3.2.7),

$$T_{\theta,l} := I - \theta \sum_{i=1}^{p} (I - H_l^{(i)})$$

where $\theta \in (0, 1/q)$ and q is given by (3.1.1).

Theorem 5.5 *Let A be an SFM-matrix of any degree and let $S_1, \ldots, S_p$ be a locally root preserving decomposition. Then the additive Schwarz iteration*

$$x^{k+1} = T_{\theta,1} x^k + c, \quad k = 0, 1, 2, \ldots$$

converges to the solution of $Ax = b$, $b \in \mathcal{R}(A)$ *for every given* $x^0 \in \mathbb{R}^n$, $\theta \in (0, 1/q)$, *and a proper right hand side* c. *The convergence is obtained at a geometric rate.*

Theorem 5.6 *Let* A *be an SFM-matrix of any degree and let* $S_1, \ldots, S_p$ *be a locally root preserving decomposition. If the number of inner iterations* $q(k,i) = q(i)$ *are constant and the inner splittings are M-splittings for each* $i = 1, \ldots, p$, *then the relaxed stationary two-stage additive Schwarz iteration*

$$x^{k+1} = T_{\theta,4} x^k + c, \quad k = 0, 1, 2, \ldots$$

converges to the solution of $Ax = b$, $b \in \mathcal{R}(A)$ *for every given* $x^0 \in \mathbb{R}^n$, $\theta \in (0, 1/q)$, *and a proper right hand side* c. *The convergence is obtained at a geometric rate.*

5.4 Trivial extensions

Finally, it should be mentioned that there are two easy generalisations using the definitions given in Sections 5.1 and 5.2.

Thus, consider an SFM-matrix A given in the normal form (5.1.2) of Lemma 5.1, i.e.

$$(5.4.1) \qquad A = \begin{pmatrix} D_1 & 0 & 0 & 0 & \ldots & 0 & 0 \\ E_{11} & F_{11} & E_{12} & F_{12} & \ldots & E_{1r} & F_{1r} \\ 0 & 0 & D_2 & 0 & \ldots & 0 & 0 \\ E_{21} & F_{21} & E_{22} & F_{22} & \ldots & E_{2r} & F_{2r} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & D_r & 0 \\ E_{r1} & F_{r1} & E_{r2} & F_{r2} & \ldots & E_{rr} & F_{rr} \end{pmatrix}.$$

The first case to consider is the existence of a positive vector y, such that $y^T A = 0$. This case naturally occurs in the analysis of Markov chains and A becomes (cf. discussion in Section 2.3 and also Lemma 2.3)

$$A = \begin{pmatrix} D_1 & 0 & \ldots & 0 \\ 0 & D_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & D_r \end{pmatrix}.$$

For the second case consider every spanning forest within $\Gamma(A^T)$ to be disjoint. Then each index in each F_{ii} from (5.4.1) belongs to exactly one tree. Hence, $E_{i,j} = 0$ for all $1 \leq i, j \leq r$ and $i \neq j$. But also $F_{i,j} = 0$ for all

$1 \leq i, j \leq r$ and $i \neq j$. In this case

$$A = \begin{pmatrix} D_1 & 0 & 0 & 0 & \dots & 0 & 0 \\ E_{11} & F_{11} & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & D_2 & 0 & \dots & 0 & 0 \\ 0 & 0 & E_{22} & F_{22} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & D_r & 0 \\ 0 & 0 & 0 & 0 & \dots & E_{rr} & F_{rr} \end{pmatrix}.$$

Thus, in both cases, A decomposes into a block diagonal matrix, whose blocks are either irreducible or STM-matrices. Hence the extensions are trivial, i.e. with the definitions given in Section 5.1 and 5.2, every theorem from the Sections 4.5, 4.6, 4.7, and 4.8 applies. Since the resulting theorems are obvious they are not restated here.

Chapter 6

Comparison with known results

In this chapter, some known results which are related to the iteration schemes introduced in Chapter 3 will be discussed. This is not only for the sake of completeness but for an assessment of the results which have been presented in Chapters 4 and 5. The differences and similarities between various known results and the approach given here will be examined. Then a closer look at the techniques used elsewhere will also reveal that these techniques are all quite nonuniform, while the ansatz used here is more homogeneous.

6.1 General convergence

There is a lot of theory for iterations

$$x^{k+1} = H^{(k)} x^k, \quad k = 0, 1, 2, \ldots \tag{6.1.1}$$

where $x^0 \in \mathbb{R}^n$ is given and $\{H^{(k)}\}_{k \in \mathbb{N}_0} \subset \mathbb{R}^{n \times n}$ is a sequence of matrices such that $\rho(H^{(k)}) \leq 1$ and $\operatorname{ind}_1(H^{(k)}) = 1$ for all $k \in \mathbb{N}_0$; see, e.g., [11, 13, 23, 24, 31, 32, 33, 44, 50, 62] and for constant sequences, e.g., [9, 21, 46, 61, 59].

For the above iteration it is sufficient to prove the existence of

$$\lim_{k \longrightarrow \infty} H^{(k)} \cdot \ldots \cdot H^{(0)} = P$$

where either $P = 0$ or P is a projection onto some common invariant subspace S.

The theory can be split into two groups. The results of the first group are based on the structure of the $H^{(k)}$ and the operators are usually nonnegative; see [21, 31, 32, 46, 59, 61, 62]. The results of the second group are retrieved from norm conditions; see [11, 13, 23, 24, 33, 44, 50].

The results of [21, 46, 59, 61] will be discussed in detail in Section 6.5. The ansatz in [32] is too special and some results from [31, 62] have been mentioned in Section 4.2. The approach in [50] is comparable to the ideas given in [62]. Thus, what remains to be discussed now are the norm based results from [11, 13, 23, 24, 33, 44].

Since norms are a strong tool, the results based on norms are very general but sometimes hard to verify. Though norms are not used here explicitly, some theory depending on norms should be discussed as the problems are almost the same.

The iteration (6.1.1) is a special case of the so called *LCP set problem* ("LCP" stands for "left convergent products").

Let $\Sigma := \{H^{(i)}\}_{i=0}^{M}$ be a (possibly countable infinite) set of square matrices. Then Σ is called an *LCP* set (or is said to fulfil the the *LCP property*), if for any sequence $\{d_k\}_{k=0}^{\infty}$ of numbers $d_k \in \{1, \ldots, M\}$ the left sided product

$$\lim_{k \longrightarrow \infty} H^{(d_k)} \cdot H^{(d_{k-1})} \cdot \ldots \cdot H^{(d_0)}$$

is (semi)convergent. The *LCP problem* is to decide whether a set has the LCP property or not.

The pioneering paper for the LCP problem is [23]. There it is discussed under which conditions those limits exist. Furthermore, the question whether the limits are continuous (i.e. independent of the sequence $\{d_k\}_{k=0}^{\infty}$) or not is examined; see also [13] or [31]. Note that the limit of (6.1.1) is not continuous in general for *MP* and the iteration methods introduced in Chapter 3.

To discuss one main result from [23], let the subspace $S \subset \mathbb{R}^n$ be a common right-eigenspace to the eigenvalue 1 of each $H^{(k)} \in \Sigma$. A finite product $B := H^{(k)} \cdot \ldots \cdot H^{(1)}$, $H^{(i)} \in \Sigma$ is called a block, if

$$S = \cap_{i=1}^{k} \mathcal{N}(I - H^{(i)}) \text{ and } S \neq \cap_{i=1}^{k-1} \mathcal{N}(I - H^{(i)}). \tag{6.1.2}$$

Denote by Σ_B the set of blocks of Σ. Further define the *joint spectral radius* by

$$\hat{\rho}(\Sigma) := \limsup_{k \longrightarrow \infty} (\hat{\rho}_k(\Sigma, \|\cdot\|))^{1/k}$$

where the norm $\|\cdot\|$ is arbitrary and

$$\hat{\rho}_k(\Sigma, \|\cdot\|) := \sup \left\{ \left\| \prod_{i=k}^{1} H^{(i)} \right\| : H^{(i)} \in \Sigma \right\}.$$

Finally, a set of matrices Σ is said to be product bounded, if there exists a finite number Δ such that

$$\left\| \prod_{i=k}^{1} H^{(i)} \right\| \leq \Delta$$

for all $k < \infty$ and all $H^{(i)} \in \Sigma$.

Theorem 6.1 (Theorem 5.1 in [23]) *A finite set Σ is a product bounded LCP set of $n \times n$ matrices if and only if :*

1) *All strict subsets are product bounded LCP sets.*

2) *All $B \in \Sigma_B$ have $\mathcal{N}(I - B) = S$.*

3) *There is a subspace $V \subset \mathbb{R}^n$ such that $S \oplus V = \mathbb{R}^n$, and the set $P_V \Sigma_B P_V := \{P_V B P_V : B \in \Sigma_B\}$, where P_V is an orthogonal projection onto V, has*

$$\hat{\rho}(P_V \Sigma_B P_V) < 1. \tag{6.1.3}$$

Theorem 6.1 reflects entirely the discussion in Section 3.6. There it has been analysed if blocks have the property (6.1.2). Additionally, (6.1.3) says that each block B satisfies $\gamma(B) < 1$. Thus, each block is semiconvergent. This is the main idea and has been modified by several authors.

In [33] the condition (6.1.3) is modified, i.e. other norm conditions are introduced. In [11, 13] and [24], norm conditions based on paracontractivity (cf. Section 1.1) are used. However, the main idea of [23] is always adapted.

Another condition is given in [44]. There, it is supposed that each $H^{(k)}$ is semiconvergent and the spectral decomposition $(P^{(k)}, Q^{(k)})$ satisfies $\|P^{(k)}\| \le c < \infty$ and $\|Q^{(k)}\| < \theta < 1$ for all $k \in \mathbb{N}_0$ and a certain norm which may not depend on k.

If all projections are equal, then the latter conditions imply paracontractivity for all $H^{(k)}$ w.r.t. the same norm.

Lemma 6.1 *If A is semiconvergent and there exists a norm $\|\cdot\|_1$ such that for the spectral decomposition (P, Q), $\|Q\|_1 < 1$ holds, then A is paracontractive w.r.t. the following norm*

$$\|x\|_2 := \|Px\|_1 + \|(I - P)x\|_1, \quad x \in \mathbb{R}^n.$$

Proof: Following [50], define for $x \in \mathbb{R}^n$

$$\|x\|_2 := \|Px\|_1 + \|(I - P)x\|_1.$$

It is immediately seen that $\|\cdot\|_2$ is a norm since $\|\cdot\|_1$ is one. With the identity $(I - P)Q = Q = Q(I - P)$ one gets

$$\begin{aligned} \|Ax\|_2 &= \|PAx\|_1 + \|(I - P)Ax\|_1 \\ &= \|P(P + Q)x\|_1 + \|(I - P)(P + Q)x\|_1 \\ &= \|Px\|_1 + \|Q(I - P)x\|_1. \end{aligned}$$

If $Px = x$, then $(I - P)x = 0$ and $\|Ax\|_2 = \|x\|_2$. If $Px \neq x$, then $(I - P)x \neq 0$ and

$$\begin{aligned} \|Ax\|_2 &= \|Px\|_1 + \|Q(I-P)x\|_1 \\ &\leq \|Px\|_1 + \|Q\|_1 \|(I-P)x\|_1 \\ &< \|Px\|_1 + \|(I-P)x\|_1 = \|x\|_2 \end{aligned}$$

since $\|Q\|_1 < 1$. □

But the projections in [44] are usually different, i.e. they depend on k, and the lemma can not be applied in such an easy manner. The $H^{(k)}$ in [44] are of course paracontractive, but to different norms. So it is a good (and open) question, whether this is a generalisation of the uniform paracontractivity assumed in [11, 24] or not. Anyway, in Theorem 3.5 of [44], the set of the $H^{(k)}$ was proved to be an LCP set (though the notion of an LCP set was not used). Indeed, Theorem 3.5 of [44] and also Lemma 6.1 build a bridge between LCP set theory and the theory that uses spectral decompositions. Note that there might be a countably infinite number of operators $H^{(k)}$ in [44], rather than finitely many ones as in Theorem 6.1.

All together, the above results are very strong but they also need quite restrictive assumptions based on a norm. Moreover (6.1.2) and (6.1.3), or likewise the conditions of [44], imply several complications for Schwarz iterations or PAIs in practice (cf. Section 3.6) since it is hard to show that finite blocks of iteration operators fulfil some norm conditions.

As the iteration operators discussed in this thesis are nonnegative, it is much easier and somehow more natural to argue on the structure rather than on norms. This has been done in Chapter 4 leading to the convergence of (6.1.1) by Theorem 4.7 which is due to [31] and [62].

6.2 Multiplicative Schwarz methods

The most recent papers for multiplicative Schwarz methods are [40] and [48], whose results will be discussed now.

The main theorem presented in [40] reads as follows.

Theorem 6.2 (Theorem 4.1 in [40]) *Let $A = I - B$, where B is an $n \times n$ column stochastic matrix such that $Bz = z$ with $z > 0$. If $S_1, \dots, S_p$ is a regular decomposition and for each $i = 1, \dots, p$ the splitting*

(6.2.1)
$$\Pi_i \begin{pmatrix} A[S_i] & A[S_i, \neg S_i] \\ 0 & I \end{pmatrix} \Pi_i^T = \begin{pmatrix} \tilde{M}_i & 0 \\ 0 & I \end{pmatrix} - \begin{pmatrix} \tilde{N}_{ii} & \tilde{N}_{i,\neg i} \\ 0 & I \end{pmatrix} =: \bar{M}_i - \bar{N}_i$$

is nonnegative and the diagonals of $H^{(i)} = \bar{M}_i^{-1} \bar{N}_i$ are positive, then the *multiplicative Schwarz iteration*

$$x^{k+1} = Tx^k + c, \quad k = 0, 1, 2, \ldots,$$

where T is given as

$$T := H^{(1)} \cdot H^{(2)} \cdot \ldots \cdot H^{(p)}, \quad H^{(i)} = \bar{M}_i^{-1}\bar{N}_i,$$

converges to a solution of $Ax = b$, $b \in \mathcal{R}(A)$ for every given $x^0 \in \mathbb{R}^n$ and a proper right hand side c.

This result, if correct, is just a bit more general than the results given in Sections 4.5, 4.6, and 5.2 (especially Theorem 5.3). This is because the decomposition might be arbitrary (but of course regular) instead of locally root preserving.

Unfortunately, there is a problem with the proof presented in [40] since it was not shown that $\mathcal{N}(I - T) = \mathcal{N}(A)$ holds (cf. Section 3.6). The semiconvergence itself was shown in [40]; it follows by the theory presented in Chapters 1 and 3.

Note that A is an SFM-matrix and decomposes into a block diagonal matrix, whose diagonal blocks are STM-matrices by the theory given in Sections 2.3 and 5.4.

The splittings (6.2.1) proposed in [40] where

$$\tilde{M}_i := \alpha_i I + A[S_i], \quad \alpha_i > 0. \tag{6.2.2}$$

They have a nice embedding into the normal forms given in Remark 3.5 of Section 3.5. With the notation for M_i, N_i and $P^{(i)}$ given there and

$$\Delta_{ms}^{(i)} := \Pi_i^T \begin{pmatrix} \alpha \cdot \tilde{M}_i^{-1} & 0 \\ 0 & 0 \end{pmatrix} \Pi_i$$

it follows that

$$H^{(i)} = P^{(i)} + \Delta_{ms}^{(i)}(I - P^{(i)})$$

and

$$I - H^{(i)} = (I - \Delta_{ms}^{(i)})(I - P^{(i)}).$$

Note that $I - \Delta_{ms}^{(i)}$ is always nonsingular for every $\alpha_i > 0$ and the iteration scheme can be classified as an inexact Schwarz iteration.

Since

$$\Pi_i H^{(i)} \Pi_i^T = \begin{pmatrix} \alpha_i \tilde{M}_i^{-1} & (I - \alpha_i \tilde{M}_i^{-1}) M_i^{-1} N_i \\ 0 & I \end{pmatrix} = \begin{pmatrix} \alpha_i \tilde{M}_i^{-1} & \tilde{M}_i^{-1} N_i \\ 0 & I \end{pmatrix}$$

and

$$\Gamma(\tilde{M}_i^{-1}) = \Gamma(M_i^{-1}),$$

Theorem 5.3 applies directly for multiplicative Schwarz and Theorem 5.5 for additive Schwarz if a locally root preserving decomposition is used.

Finally, it should be noted, that there are no results for two-stage methods nor exact Schwarz methods in [40] which can be immediately obtained by the theory provided here (cf. Section 5.4).

In [48] symmetric positive semidefinite problems are discussed. The main result for multiplicative Schwarz reads as follows.

Theorem 6.3 (Theorem 4.2 in [48]) *Let A be a symmetric positive semidefinite matrix such that each principal submatrix is positive definite. If $S_1, \ldots, S_p$ is a decomposition, $b \in \mathcal{R}(A)$ and $x^0 \notin \mathcal{N}(A)$ then the (exact) multiplicative Schwarz iteration (3.2.2) converges to a solution x^* of $Ax = b$.*

There is no need of ms-compatibility nor root preserving decompositions in Theorem 6.3. However, the theorem is restricted to symmetric positive semidefinite matrices, as opposed to the results of Sections 4.5 and 5.1. The generality comes from the symmetry, which implies several additional conditions on the error-distribution of Lemma 3.5 in Section 3.6. Note that the proposed positive definiteness of each principal minor is a restriction to A. This holds, e.g., if A is non singular or A is irreducible. Hence, an application of Theorem 6.3 to the problems discussed in this thesis is only possible for the model problem *MP*. In this case A becomes irreducible (cf. Section 2.2).

Additionally, there is also a result for inexact iterations in [48].

Theorem 6.4 (Theorem 5.7 in [48]) *Let A be a symmetric positive semidefinite matrix such that each principal submatrix is positive definite. Let $S_1, \ldots, S_p$ be a decomposition, $b \in \mathcal{R}(A)$ and $x^0 \notin \mathcal{N}(A)$. If $\tilde{M}_i^{-1}$ is an approximation to $M_i^{-1} = A[S_i]^{-1}$ such that $\tilde{M}_i + \tilde{M}_i^T - M_i$ is symmetric positive definite, then the (inexact) multiplicative Schwarz iteration (3.2.2), using $\tilde{M}_i$ instead of M_i, converges to a solution x^* of $Ax = b$.*

This compares to the results from Sections 4.5 and 5.1 in a similar manner as Theorem 6.3. But explicit examples of those inexact methods have not been given. The two-stage methods discussed in [48] were all interpreted as coarse grid corrections, needing an additional step. Hence, they do not coincide with the two-stage methods given in this thesis.

6.3 Additive Schwarz methods

The following results are from [15] and again [48] (cf. Section 6.2).

The main result in [15] can be stated as follows.

Theorem 6.5 (Theorem 3.3 and 4.4 in [15]) *Let $A = I - B$, where B is an $n \times n$ nonnegative matrix such that $Bz = z$ with $z > 0$. If $S_1, \ldots, S_p$ is a regular decomposition and $S_i \cap S_{i+k} = \emptyset$ for $k \geq 2$, i.e. the maximum overlap is $q = 2$, then the (exact) additive Schwarz iteration (3.2.6) converges to a solution x^* of $Ax = b$ if $b \in \mathcal{R}(A)$.*

Again, A is assumed to be an SFM-matrix by Theorem 2.3. Theorem 6.5 gives an alternative to the locally root preserving decompositions used in Theorem 5.5. On the other hand, Theorem 5.5 has no restriction to the amount of overlap as long as the decomposition is locally root preserving. Furthermore, the theory presented here can be applied to inexact methods using Theorem 5.6. Note that the proof in [15] follows the idea given in [34].

As in the previous section, the result for exact additive Schwarz in the symmetric positive semidefinite case does not need further assumptions on the graph of A and the decomposition.

Theorem 6.6 (Theorem 3.3 in [48]) *Let A be a symmetric positive semidefinite matrix such that each principal submatrix is positive definite. If $S_1, \ldots, S_p$ is a decomposition, $b \in \mathcal{R}(A)$ and $x^0 \notin \mathcal{N}(A)$ then the (exact) additive Schwarz iteration (3.2.6) converges to a solution x^* of $Ax = b$ if $0 < \theta < 2/p$.*

In contrast to Theorem 6.5, the maximum overlap is not restricted by $q = 2$ and the condition $A = I - B, Bz = z$ for some positive z can be given up. Compared to Theorem 5.5, there is no need of locally root preservation. The range of the damping factor has been doubled but this is no surprise in the symmetric case (see [30]). This shows that the results in the symmetric positive semidefinite case are quite general. But as mentioned in the previous section, Theorem 6.6 can only be applied to the model problem *MP* in which case A becomes irreducible.

A result for inexact additive Schwarz is also given in [48].

Theorem 6.7 (Theorem 5.1 in [48]) *Let A be a symmetric positive semidefinite matrix such that each principal submatrix is positive definite. Let $S_1, \ldots, S_p$ be a decomposition, $b \in \mathcal{R}(A)$ and $x^0 \notin \mathcal{N}(A)$. If $\tilde{M}_i^{-1}$ is an approximation to M_i^{-1} such that $\tilde{M}_i - M_i$ is symmetric positive semidefinite and $0 < \theta < 2/p$ then the inexact additive Schwarz iteration (3.2.6), using $\tilde{M}_i$ instead of M_i, converges to a solution x^* of $Ax = b$.*

Again, no explicit application for Theorem 6.7 is given in [48]. The proposed two-stage method for additive Schwarz given in [48] is again a coarse grid correction and cannot be compared with the method given here.

6.4 Asynchronous Iterations

The results given for asynchronous iterations will be discussed in the context of [2, 39] and [56].

The theorem of B. Lubachevsky and D. Mitra

To point out the differences between non-block and block iterations the main result from [39] will be discussed first (it can also be found in [10], Section 7.3.2, Proposition 3.2). The main convergence result given in [39] is as follows.

Theorem 6.8 (Theorem 2 in [39]) *Let $B \in \mathbb{R}^{n\times n}$ be irreducible, column stochastic, and assume $b_{i_0,i_0} > 0$ for some $i_0 \in \{1,\ldots,n\}$. Let $0 \le x^0 \in \mathbb{R}^n$ such that $x^0_{i_0} > 0$. Then a PAI $(B, x^0, \{\mathcal{J}_k, \mathcal{S}_k\}_{k\in\mathbb{N}_0})$ such that $\mathcal{J}_k = \{j_k\}$ and $j_k \in \{1,\ldots,n\}$ for all $k \in \mathbb{N}_0$, converges to a positive vector $z > 0$ satisfying $Bz = z$.*

The PAI proposed in [39] is actually Update 5 (cf. (3.3.4)) applied to single rows rather than blocks. The proof of Theorem 6.8 is more or less based on Theorem 4.7 and therefore uses the following proposition.

Proposition 6.9 (Proposition 6 in [39]) *Let $H^{(k)}_{d,5}$ be the operators from Theorem 6.8. Then for any $k \in \mathbb{N}_0$ the product*

$$H^{(k,k+r_{lm})} := H^{(k+r_{lm})}_{d,5} \cdot H^{(k+r_{lm}-1)}_{d,5} \cdot \ldots \cdot H^{(k)}_{d,5}$$

has a positive column for $r_{lm} = 1 + d + (n-1)(d+s)$.

Proposition 6.9 should be analysed in the context of Theorem 4.21.

The first point to be noticed is that Proposition 6.9 does not need the complete diagonal to be positive. This is for the following reasons:

1) The positive diagonal element of B might be interpreted as the root of some spanning tree in B^T (cf. Corollary 2.2) having access to itself. Thus, the information stored in the root remains there during the whole iteration by the construction of the local updates.

   If block iterations are used, i.e. Update 2 or 4, the root will be transported out of the diagonal. Hence an element that stores the information is needed, therefore the relaxation. The technique of the proof of Theorem 4.21 implies that the whole diagonal should be positive, because the root can change (cf. the example in Section 4.4).

2) Then, if a son of the root is updated, it will inherit the information of the root. This information will also be stored during the whole iteration, again by the irreducibility of B and the construction of the local updates $H_{d,5}^{(k)}$ (indeed, a graph based approach was used, cf. Proposition 3 in [39]).

The second main difference between Theorem 4.21 and Proposition 6.9 is the following.

In Theorem 4.21, the upper bound for the positive column to be generated was $r = p(d+s)$. Thus, if the number of blocks p is sufficiently smaller than n, then

$$p(d+s) < 1 + d + (n-1)(d+s) < n(d+s).$$

Hence the speed of convergence should be increased, since semiconvergent operators are obtained faster if block updates and ms-compatible decompositions are used. Note that a better performance cannot be assumed in the case of a block Update 6 according to the discussion in Section 4.6.

The theory of M. Pott

To discuss the results of [55, 56] some basics have to be introduced as the theory is quite different from the classical approaches (cf. [4, 10, 12, 18, 70] and Section 3.4) and generalises some concepts from [25]. In the presentation below, the results from [55] are restricted to linear operators. The discussion is necessary since the approach in [55, 56] is graph based as well.

Let $\mathcal{I} \subset \mathbb{N}, m \in \mathbb{N}, D \subset \mathbb{R}^p$ be closed and let $\mathcal{G} := \{G^i : i \in \mathcal{I}\}$ be a "pool" of operators such that

$$G^i : D^{m_i} \longrightarrow \mathbb{R}^p$$

for $m_i \in \{1, \ldots, m\}$ and all $i \in \mathcal{I}$.

A vector $z \in \mathbb{R}^p$ is said to be a fixed point of $\mathcal{G}$ if

$$G^i(z, \ldots, z) = z$$

for all $i \in \mathcal{I}$. The problem that is tackled in [25, 55, 56], is to find a fixed point for a pool $\mathcal{G}$. In [55] it is analysed how this could be done by using an asynchronous iteration in the following sense.

For $k \in \mathbb{N}, i(k) \in \mathcal{I}$ and $(s_1(k), \ldots, s_{m_{i(k)}}(k)) \in \mathbb{N}_0^{m_{i(k)}}$ the assignment

$$x^{k+1} = G^{i(k)}(x^{s_1(k)}, \ldots, x^{s_{m_{i(k)}}(k)})$$

defines an asynchronous iteration method if

$$\begin{array}{rcl} s_j(k) & \leq & k, \\ \lim\limits_{k \longrightarrow \infty} s_j(k) & = & +\infty, \\ i : \mathbb{N} & \longrightarrow & \mathcal{I}, \end{array}$$

for $j = 1, \ldots, m_{i(k)}$ and $k = 0, 1, 2, \ldots$. Here the notation $(\mathcal{G}, x^0, \{\mathcal{J}_k, \mathcal{S}_k\}_{k \in \mathbb{N}_0})$ is used for a given $x^0 \in \mathbb{R}^p$, $\mathcal{S}_k$ as usual, and $\mathcal{J}_k = \{i(k)\}$.

As it is not possible to prove satisfactory results with such general asynchronous iterations the notion of *confluent* asynchronous iterations (CAI) must be introduced.

The directed graph $G = (V, E)$ of an asynchronous iteration $(\mathcal{G}, x^0, \{\mathcal{J}_k, \mathcal{S}_k\}_{k \in \mathbb{N}_0})$ consists of the vertex set $V = \mathbb{N}$ and the edge set E such that

$$(k, k_0) \in E \Leftrightarrow \text{there exists } 1 \leq l \leq m_{i(k_0-1)} \text{ such that } s_l(k_0 - 1) = k.$$

$(\mathcal{G}, x^0, \{\mathcal{J}_k, \mathcal{S}_k\}_{k \in \mathbb{N}_0})$ is said to be a CAI, if there are numbers $n_0, b, d \in \mathbb{N}$ and a sequence $(b_k)_{k=n_0}^{\infty}$ of natural numbers, such that for all $k \geq n_0$ the following conditions hold:

1) For all $k_0 \geq k$ there exists a directed path from b_k to k_0.
2) $k - b \leq b_k \leq k$.
3) $k - d \leq s_i(k) \leq k, \quad i = 1, \ldots, m_{i(k)}$.
4) For all $i \in \mathcal{I}$ there exists a $c_i \in \mathbb{N}$ such that for each $k \geq n_0$ a vertex $w_k^i \in V$ exists, which is a successor of b_k, a predecessor of b_{k+c_i}, and $i(w_k^i - 1) = i$.

This needs some interpretation.

- The graph of an asynchronous iteration consists of an edge (k, k_0), if the calculation of x^{k_0} needs the approximation of the k-th step as a certain parameter (the l-th).
- Assumption 1) is a connectivity condition for the graph G. The information of x^{b_k} flows into the the calculation of each successor and especially into $x^{b_{k+1}}$.
- Condition 2) is needed to guarantee Assumption 4).
- Point 3) is the same as (3.4.6).
- At first sight, condition 4) seems to be a generalisation of (3.4.7) and seems to be equivalent to (3.4.7) if the pool $\mathcal{G}$ consists of only finitely many operators. But it is actually more restrictive since it connects the iteration and the operators directly. It implies a structure that guarantees all local data to be at least dependent on the other local data generated by the subsequence b_k (and this can be interpreted as a tree structure in certain cases). The exchange of that data is ensured by the sequence $\{b_k\}$ (see also Section 7.3 in [10]).

To give a convergence result for CAIs, a pool $\mathcal{G}$ is said to be *strictly nonexpansive* on D w.r.t. $\|\cdot\|$ if for all $i \in \mathcal{I}$ and $X = (x^1, \dots, x^{m_i}), Y = (y^1, \dots, y^{m_i}) \in D^{m_i}$

$$\begin{aligned} \|G^i(X) - G^i(Y)\| &< \max_j \|x^j - y^j\| \text{ or} \\ G^i(X) - G^i(Y) &= x^j - y^j \text{ for all } j = 1, \dots, m_i. \end{aligned}$$

The following theorem is restricted to strictly nonexpansive pools of operators. The general version is Theorem 4.2 in [56].

Theorem 6.10 *Let $\mathcal{G}$ be a strictly nonexpansive pool on $D \subset \mathbb{R}^p$ and assume that $\mathcal{G}$ has a fixed point $z \in D$, then a CAI $(\mathcal{G}, x^0, \{\mathcal{J}_k, \mathcal{S}_k\}_{k \in \mathbb{N}_0})$ converges to a fixed point of $\mathcal{G}$ in D.*

To embed iterations for linear systems into the theory given above, consider a $B \in \mathbb{R}^{n \times n}$ which is row stochastic, irreducible, and has a positive diagonal element, say b_{i_0,i_0}. Now define for $i = 1, \dots, n$ the compatibility mappings $c_i : \{1, \dots, m_i\} \longrightarrow \{1, \dots, n\}$ such that

$$\begin{aligned} \{c_i(1), \dots, c_i(m_i)\} &= \{j \in \{1, \dots, n\} : b_{ij} > 0\}, \\ c_i(j) < c_i(j_0) &\Leftrightarrow j < j_0. \end{aligned}$$

Then the pool $\mathcal{G} := \{B^i : 1 = 1 \dots, n\}$ with

$$\begin{aligned} B^i &: \mathbb{R}^{m_i} \longrightarrow \mathbb{R} \\ B^i(y^1, \dots, y^{m_i}) &:= \sum_{j=1}^{m_i} b_{i,c_i(j)} y^j, \end{aligned}$$

is strictly nonexpansive w.r.t. $|\cdot|$ on each closed interval on $\mathbb{R}$ (and this is independent of the irreducibility of B). The iteration now becomes

$$y^{k+1} = B^{i(k)}(y^{s_1(k)}, \dots, y^{s_{m_{i(k)}}(k)}).$$

To apply Theorem 6.10, the iteration has to be confluent and this can be obtained (cf. Theorem 5.2 in [56]) by requiring the following conditions

1) There exists a $d \in \mathbb{N}$ such that $k - d \leq s_i(k) \leq k$ for all $k \in \mathbb{N}, i = 1, \dots, m_{i(k)}$ (cf. (3.4.6)).

2) There exists an $r \in \mathbb{N}$ such that $\cup_{l=k}^{k+r} i(l) = \{1, \dots, n\}$ for all $k \in \mathbb{N}$ (cf. (3.4.7)).

3) $s^{c_{i_0}^{-1}(i_0)}(k) = \max\{k_0 < k : i(k_0 - 1) = i_0\}$ for all $k \in \mathbb{N}$ satisfying $i(k) = i_0$ (cf. (3.4.5)).

4) $i(s^l(k)-1) = c_{i(k)}(l)$ for all $k \in \mathbb{N}$ and $l = 1, \dots, m_{i(k)}$ (remapping of the matrix structure).

Here B^{i_0} generates the sequence b_k by condition 3) and, as already mentioned, acts as a root of a spanning tree.

The above setup applies to arbitrary irreducible nonnegative matrices having a positive fixed point, hence Theorem 6.10 is applicable to *MP* in the sense of [39] (cf. Theorem 5.3 in [56]). But there has been shown more in [56]. By a localisation of confluence it was proven that the above setup also applies to general semiconvergent nonnegative matrices rather than irreducible semiconvergent ones (cf. Theorem 5.4 in [56]).

Theorem 6.11 *Let $B \in \mathbb{R}^{n\times n}$ be nonnegative and semiconvergent. Assume B to be in Frobenius normal form, i.e.*

$$B = \begin{pmatrix} B_{11} & 0 & \dots & 0 & 0 \\ B_{21} & B_{22} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ B_{21} & B_{22} & \dots & B_{p,p-1} & B_{pp} \end{pmatrix}.$$

Further, let there be a subset $\mathcal{I}_0 \subset \{1, \dots, n\}$, such that for each $i \in \{1, \dots, p\}$, for which $\rho(B_{ii}) = 1$, there is some index $i_0 \in \mathcal{I}_0$ such that $(B_{ii})_{i_0,i_0} > 0$. In addition assume

1) $k - d \le s_i(k) \le k$ for all $k \in \mathbb{N}, i = 1, \dots, n$, for a $d \in \mathbb{N}$,

2) $\cup_{l=k}^{k+r} i(l) = \{1, \dots, n\}$ for all $k \in \mathbb{N}$, for an $r \in \mathbb{N}$,

3) $s^{i_0}(k) = \max\{k_0 < k : i(k_0-1) = i_0\}$ for all $k \in \mathbb{N}$ satisfying $i(k) = i_0$ and for all $i_0 \in \mathcal{I}_0$.

If $b \in \mathcal{R}(I - B)$, then the iteration

$$x_i^{k+1} := \begin{cases} x_i^k & \text{if } i \neq i(k) \\ \sum_{j=1}^n b_{ij} \cdot x_j^{s^j(k)} + b_i & \text{if } i = i(k) \end{cases}$$

converges to a solution of $(I - B)x = b$.

Theorem 6.11 is in view of Update 5, i.e. in the sense of [39], quite more general than Theorem 6.8, since it does not require irreducibility.

The differences between the approach given in this thesis and in [55] will be discussed now.

First of all, the structure of the operators in [55] is directly embedded into confluent iterations. This is in contrast to the standard models of asynchronous iterations (cf. [4, 10, 12, 18, 39, 70]). The discussions there usually concern the conditions on the operators such that an asynchronous iteration converges for every scenario, i.e. the iteration itself is separated from the operators. Thus, Theorem 6.10 will not apply to any strictly nonexpansive pool of operators, but this is of course not a disadvantage.

In this thesis, the separation of iteration and operators has been kept. Furthermore the question has been answered, how the original flow (which is reflected by condition 4) of confluence) is kept alive under certain transformations, i.e. block iteration methods in various versions. Thus, the motivation here is quite different.

In [55, 56] only Update 5 applied to single rows has been analysed, leading to Theorem 6.11 which looks quite general, as B need not to be an SF-matrix. In this thesis, certain transformations of the original system have been analysed for ST and SF-matrices. Although the class is smaller, the results are more general. Additionally, the techniques used here, give a hint that the speed of convergence might be increased (in the case of ms-compatibility) and that cannot be seen if the iteration operators are interpreted as a pool of linear functionals.

Furthermore, it has been shown that the whole machinery of asynchronous iterations is not needed for additive and multiplicative Schwarz. Though it can be embedded into the classical model and the theory given in [55, 56]. On the contrary, the convergence of asynchronous iterations can be easily derived from the convergence of multiplicative Schwarz methods including some overlap.

The advantage of the theory given in [55, 56] is that it is applicable to nonnegative semiconvergent matrices or reducible singular M-matrices (and even to nonlinear pools of operators which makes this concept to one of the strongest in literature). The graph based approach given here is restricted to ST and STM-matrices.

Finally, it should be mentioned that the approach here is a complete algebraic approach for an algebraic problem and might have further applications. As the nature of the convergence of iterations of nonnegative matrices lies in the flow of information through the graph structure of the iteration operators, a graph based approach is quite natural and cannot be avoided. Therefore, it is used implicitly in [10, 39] and explicitly in [55] and here.

The Theorems of J. Bahi

The last result to be discussed in this section is that of [2]. It was the first paper that has examined block one-level and block two-stage methods for asynchronous iterations.

The main results are based on norms and use some ideas from [12]. The main theorems will be given with all their prerequisites. First two-stage methods will be considered.

As the iteration proposed in [2] is a bit different to the model described in Section 3.4, a few redefinitions must be done.

Let $A \in \mathbb{R}^{n\times n}$ and let $S_1, \dots, S_p$ be a regular partitioning. Additionally, let $M_i = A[S_i]$, $M = \operatorname{diag}(M_1, \dots, M_p)$, $N = M - A$, and consider weak regular splittings $M_i = F_i - G_i$. A vector $x \in \mathbb{R}^n$ is now split as $(x_1^T, \dots, x_p^T)^T$ conforming to the partitioning. With Algorithm 3.5 in mind, let $q(k,i)$ be the number of inner iterations in the k-th step and $R^{(k,i)} = (F_i^{-1}G_i)^{q(k,i)}$.

Theorem 6.12 (Proposition 7 in [2]) *With the assumptions made and $\mathcal{J}_k \subset \{S_1, \dots, S_p\}$, consider the asynchronous two-stage iteration*

(6.4.1)

$$x_i^{k+1} = \begin{cases} x_i^k & \text{if } i \notin \mathcal{J}_k, \\ R^{(k,i)} x_i^{s_i(k)} + & \\ (I - R^{(k,i)}) M_i^{-1} \left(\sum_{j=1, j\neq i}^{p} N[S_i, S_j] x_j^{s_j(k)} + b[S_i] \right) & \text{if } i \in \mathcal{J}_k. \end{cases}$$

Additionally assume:

1) *There exists a $z > 0$ such that $Az = 0$.*

2) *There holds either $Ne > 0$ or $F_i^{-1}G_i e[S_i] > 0$ for each $l = 1, \dots, p$.*

3) *The matrix $M^{-1}N$ is paracontractive w.r.t. $\|\cdot\|_z$.*

4) *There is a subsequence k_j such that $\mathcal{J}_{k_j} = \{S_1, \dots, S_p\}$ and $s_1(k_j) = \dots = s_p(k_j) = k_j$.*

5) *There holds $\lim_{j\longrightarrow\infty} q(k_j, i) = \infty$ for all $i = 1, \dots, p$.*

6) *Condition (3.4.6) holds.*

Then the asynchronous iteration (6.4.1) converges for every starting vector x^0 to the solution x^* of $Ax = b$ for every $b \in \mathcal{R}(A)$.

The differences to the asynchronous iteration given in Section 4.8 are many.

- Theorem 6.12 allows more than one block to be updated simultaneously. But this is only an apparent generalisation.
- Condition 1) is as usual and has been extensively discussed in Chapter 3.
- Condition 2) guarantees that $\|\operatorname{diag}(R^{(k,1)}, \dots, R^{(k,p)})\|_z < 1$ holds (cf. [12]) which causes the iteration operators to converge to $M^{-1}N$ on the subsequence k_j in combination with condition 5). This condition holds automatically for ms-compatible decompositions (cf. Section 4.4 and 4.5).

- Condition 3) is hard to verify in practice.
- Condition 4) replaces (3.4.5) and (3.4.7) and is in fact synchronisation, and thus, very restrictive.
- Condition 5) is somehow unrealistic since one will bound the number of inner iterations usually, instead of letting them grow on a synchronised subsequence.

The matrix A can be chosen more general in Theorem 6.12 as compared to the result of Chapter 4. Condition 2) will usually be fulfilled after some preprocessing only. Thus in the case of A being an STM-matrix, the asynchronous iteration proposed in Section 4.5 should be preferred as neither the paracontractivity of $M^{-1}N$ nor a synchronised subsequence is needed. Therefore the result for STM-matrices provided by Theorem 4.23 is more general than 6.12.

There is also a result for one-level block iterations in [2].

Theorem 6.13 (Proposition 3 in [2]) *With the assumptions of Theorem 6.12 and $\mathcal{J}_k \subset \{S_1, \dots, S_p\}$, consider the block asynchronous iteration*

$$x_i^{k+1} = \begin{cases} x_i^k & \text{if } i \notin \mathcal{J}_k, \\ \sum_{j=1, j\neq i}^{p} M_i^{-1} N[S_i, S_j] x_j^{s_j(k)} + M_i^{-1} b[S_i] & \text{if } i \in \mathcal{J}_k. \end{cases} \tag{6.4.2}$$

Additionally assume:

1) *There exists a $z > 0$ such that $Az \geq 0$.*
2) *The splitting (M, N) is weak regular.*
3) *The matrix $M^{-1}N$ is paracontractive w.r.t. $\|\cdot\|_z$.*
4) *There is a subsequence k_j such that $\mathcal{J}_{k_j} = \{S_1, \dots, S_p\}$ and $s_1(k_j) = \ldots = s_p(k_j) = k_j$.*
5) *The condition (3.4.6) holds.*

Then the asynchronous iteration (6.4.2) converges for every starting vector x^0 to the solution x^* of $Ax = b$ for every $b \in \mathcal{R}(A)$.

There is again the annoying condition that $M^{-1}N$ is paracontractive and also a synchronised subsequence is needed. Anyway, the synchronised subsequence cannot be avoided in the theory given in [2]. Thus, Theorem 6.13 is weaker than Theorem 4.22 since once an ms-compatible decomposition has been found, the iteration can be carried out without any restrictions.

6.5 Other graph based approaches

To end this chapter, some other explicit graph based approaches (rather than those mentioned in the previous section) will be discussed. Those approaches can be found in [21, 46, 59, 61]. The results given in [21, 46, 59, 61] did not include overlap nor inexact iterations. Thus they are not as general as the results given here. But as the ideas are quite similar, they must be discussed.

All the articles mentioned above discuss the question under which conditions a splitting $A = M - N$, where $A \in \mathbb{R}^{n\times n}$ is an M-matrix, is semiconvergent, i.e. $M^{-1}N$ is semiconvergent.

If necessary, A may be considered as partitioned into p blocks, i.e.

$$A = \begin{pmatrix} A_{11} & A_{12} & \dots & A_{1p} \\ A_{21} & A_{22} & \dots & A_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ A_{p1} & A_{p2} & \dots & A_{pp} \end{pmatrix} \tag{6.5.1}$$

where the diagonal blocks A_{ii} are square.

In [59], $A \in \mathbb{R}^{n\times n}$ is assumed to be partitioned with conformity to (6.5.1) and split as $A = D - L - U$, where, as usual, D is the block-diagonal matrix corresponding to the partitioning of A.

Now, define $M = D - L$ and $N \geq 0$ such that

$$A_0 := D - L - U(N) \tag{6.5.2}$$

is irreducible, where

$$N := L(N) + D(N) + U(N) \geq 0 \tag{6.5.3}$$

and $L(N), D(N)$, and $U(N)$ are strictly lower block triangular, block diagonal, and strictly block upper triangular, respectively. Then

$$A = (D - D(N)) - (L + L(N)) - U(N) = M - N.$$

With the above notation, the block splitting $A = M - N$ is said to be an *R-splitting* if

1) $M = D - L$ and N is given by (6.5.3),

2) $D_{ii}^{-1} > 0$ for all $1 \leq i \leq p$, i.e. each D_{ii} is irreducible,

3) $A_0 = D - L - U(N)$ is irreducible,

4) $\Gamma(A_0)$ contains a *monotone cycle* c, i.e. $c = (i_1, \dots, i_l, i_1)$, and $i_j > i_{j+1}$ for $1 \leq j \leq l - 1$.

The above definition implies the irreducibility of A by the irreducibility of A_0. As A_0 is irreducible it contains a cycle (i.e. a circuit) and the monotonicity may be achieved by a reordering of the variables. This is, of course, comparable to the definition of flow compatibility but different, since states not belonging to the circuit can be arranged in any way. The idea is now, to prove that the greatest common divisor of the length of all cycles in $\Gamma(M^{-1}N)$, say $c(M^{-1}N)$, is one, since this imply $\gamma(M^{-1}N) < 1$ (cf. [9] and also Theorem 1.11). That is, each circuit is mapped on a primitive class.

The main results of [59] can be restated as follows.

Proposition 6.14 (Proposition 3 in [59]) *Let A be an irreducible M-matrix and $A = M - N$ be a block splitting such that D is the block diagonal of M. Suppose each D_{ii} is an irreducible regular M-matrix. Then there exists a permutation matrix P such that PAP^T has an R-regular splitting with the same diagonal blocks.*

Theorem 6.15 (Theorem 1 in [59]) *Any R-regular splitting of an irreducible M-matrix A is semiconvergent.*

The techniques used in [59] are somehow comparable to those used here, but the result here is more general, as the irreducibility of the diagonal blocks is not necessary.

Another Theorem that is proven is

Theorem 6.16 (Theorem 2 in [59]) *Any regular splitting of an irreducible M-matrix A with $M^{-1} > 0$ is semiconvergent.*

The proof is based on the fact that $M^{-1}N$ has a positive column. It can be generalised to ST-matrices by the concept of flow-compatibility what has been done in this thesis.

In [61], a result which is similar to Theorem 6.16 is proven together with some more general propositions. E.g.:

Theorem 6.17 *Let $A \in \mathbb{R}^{n\times n}$ be an irreducible M-matrix and $A = M - N$ be an M-splitting. Then in either of the following cases, $\gamma(M^{-1}N) < \rho(M^{-1}N)$, i.e. $c(M^{-1}N) = 1$.*

1) *There exists a circuit α in $\Gamma(M) \cup \Gamma(N)$ which has a single edge in $\Gamma(N) \setminus \Gamma(M)$.*
2) *For some $i, j \in \{1, \ldots, n\}$ there holds $m_{ij} \neq 0$ and $n_{ij} \neq 0$.*
3) *There exist $i, j \in \{1, \ldots, n\}$ such that $m_{ij} \neq 0$ and $n_{ji} \neq 0$.*
4) *There exists an edge $(i, j) \in \Gamma(N)$ such that $\Gamma(M) \cup \{(i, j)\}$ is strongly connected.*

5) M is irreducible.

Those results are pretty strong, but again they did not cover the case of overlap nor inexact splittings. Anyway, in both papers, it is shown that all circuits are mapped on a primitive class.

This has also been done in this thesis by the application of a flow compatible reordering which leads to Theorem 4.10. Additionally, it turns out that such a mapping is not necessary if relaxation is used (cf. Section 4.6). Then it suffices to prove that the structure is preserved, what leads to Theorem 4.16. Clearly, at least Theorem 4.10 is applicable to the above situation as multiplicative Schwarz without overlap leads to a simple Gauss-Seidel-splitting. Thus Theorem 4.10 can be seen as a slight generalisation of the above results.

Another approach has been made in [21] where it has been proven that convergence of the iterates can be achieved even if the iteration matrix is cyclic.

Consider an irreducible M-matrix $A = I - B$ such that $\rho(B) = 1$. Let A be partitioned into p blocks as in (6.5.1). Furthermore, let $A = M - N$ be a block Jacobi or block Gauss-Seidel-splitting.

If $H = M^{-1}N$ is cyclic and irreducible, then there is a permutation matrix (cf. [9]) such that

$$PHP^T = \begin{pmatrix} 0 & H_{12} & 0 & \dots & 0 \\ 0 & 0 & H_{32} & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & H_{(h-1)h} \\ H_{h1} & 0 & 0 & \dots & 0 \end{pmatrix}. \tag{6.5.4}$$

Every right eigenvector $\beta^{(c)}$, $0 \le c \le h-1$, which corresponds to the h unit modulus eigenvalues $1, \lambda, \dots, \lambda^{h-1}$, can be written as (Theorem 1 in [21])

$$\beta^{(c)} = (\lambda^c \xi_1^T, \lambda^{2c}\xi_2^T, \dots, \lambda^{hc}\xi_h^T)^T \tag{6.5.5}$$

where the set of subvectors $\xi_1, \dots, \xi_h$ is unique (up to a multiplicative constant).

Now consider the iteration $x^{k+1} = Hx^k$. Then the argumentation in [21] shows, that the iterates x^k become parallel (in the limit) to some linear combination of the h eigenvectors given by (6.5.5). The iterates are therefore composed of subvectors which are parallel to the ξ_i. The ξ_i can be found if the partitioning of H into its cyclic classes, given by the non-zero blocks in (6.5.4), is known. This is possible if the i-th block of A correspond to the i-th cyclic class, i.e. $h = p$.

Consider the suitably partitioned Perron vector $z^T = (\alpha_1 z_1^T, \ldots, \alpha_p z_p^T)$, where $z_i > 0$ are subvectors and $\alpha_i > 0$ are weights such that $\sum_{j=1}^{p} \alpha_j = 1$. Then

$$\beta^{(0)} = (\xi_1^T, \ldots, \xi_p^T) = (\alpha_1 z_1^T, \ldots, \alpha_p z_p^T) = z.$$

Thus, if the i-th block of the iterate becomes parallel to ξ_i it remains to calculate the weights α_i. This can be done by applying the well known iterative aggregation/disaggregation methods (see [36, 68, 72] and also [65]) to H.

To achieve the one-to-one-correspondence of the blocks of A and the cyclic classes of H, a block j is said to be *connected* if each pair of different indices (v, w) of A_{jj} is connected by an undirected path. That means

$$v \to w \text{ in } \Gamma(A_{jj}) \cup \Gamma(A_{jj}^T).$$

The result now becomes

Theorem 6.18 (Theorem 2 in [21]) *For an irreducible matrix A, when the associated block Jacobi or block Gauss-Seidel iteration matrix H is cyclic and irreducible, the indices of the same connected block of A will belong to the same cyclic class of H.*

This connectivity might obviously be achieved by a reordering of states. The existence of a spanning tree is a harder condition than the above connectivity of the diagonal blocks but the flow compatibility avoids cyclic iteration operators. Thus, there is no need of iterative aggregation/disaggregation. Additionally, iterative aggregation/disaggregation might not be applicable to MP, since the error vector of an iteration with a right hand side need not be nonnegative. Hence it is possible that iterative aggregation/disaggregation cannot be carried out properly.

The approach of [46] (see also [65]) is the one which has the closest connections to the one developed here (unfortunately, this is not clear from the presentation of the results of [46] in [65]). Since the examples and the argumentation given in [46] are based on transition rates and probabilities, the main proposition is a bit hard to find. In short it can be given as follows.

Consider an irreducible M-matrix $A = I - B$ where B is column stochastic and A is given by (6.5.1). As usual, let $A = D - L - U$ be a block Gauss-Seidel decomposition.

The matrix A has *property R* if for each index $i \in \{1, \ldots, n\}$ there is a path to a certain index $i_0 \in \{1, \ldots, n\}$ in $\Gamma(D^{-1}(L + U))$ and $i_0 = n$. The result now is

Theorem 6.19 (Proposition 5.5 in [46]) *If property R holds, then the Gauss-Seidel iteration matrix satisfies*

$$\gamma((D - L)^{-1} U) < 1.$$

This is exactly Theorem 4.10 in the case of no overlap. Moreover, the property R reflects the spanning tree property of the Jacobi-splitting. In the approach used here, this property was proposed for A and it has been shown that it is preserved if a flow compatible ordering is used (cf. Corollary 4.6). Thus the idea is related, but here it has been generalised from "pure" Gauss-Seidel to various iteration schemes and more general classes of matrices.

Chapter 7

Summary

7.1 Results

The main topic of this thesis was the solution of systems of linear equations of the form

$$(7.1.1)\quad (I-B)x=b,\ B\in \mathbb{R}^{n\times n}, B\geq 0, \rho(B)=1, x\in \mathbb{R}^n,\ b\in \mathcal{R}(I-B),$$

using multiplicative and additive Schwarz iterations as well as block partially asynchronous iterations for various block updates.

The system (7.1.1) was not solved in the most general setup, but for the model problems *MP* (cf. Definitions 2.1 and 2.6) and *GMP* (cf. Definitions 2.2 and 2.9). If $z\in \mathbb{R}^n$ denotes a positive vector, then the model problems are defined by the following relations.

$$(7.1.2)\qquad \mathcal{N}(I-B)=\operatorname{span}\{z\},$$

in the case of *MP*, or

$$(7.1.3)\qquad \mathcal{N}(I-B)\ni z$$

in the case of *GMP*.

Both model problems were examined and classified. This led to the concept of ST-matrices in the case of model problem *MP* (Section 2.2). For these matrices there exists a spanning tree in $\Gamma(B^T)$ and $\Gamma(B)$ contains a single basic and final class. It turned out, that this structure is sufficient and necessary if (7.1.2) holds. Additionally, the set of STM-matrices was introduced as $\{A\in \mathbb{R}^{n\times n} : A=I-B,\ B \text{ is an ST-matrix }\}$ (Definition 2.5).

The concept of ST-matrices was generalised using (7.1.3). This led to the class of SF-matrices of a certain degree (Section 2.3). The graph of an SF-matrix of degree r contains r trees, whose union represents a spanning forest.

Additionally, there are r final and basic classes within $\Gamma(B)$. Sufficient and necessary conditions for this structure to exist were also proven. Finally, the set of SFM-matrices was defined analogously to STM-matrices (Definition 2.8).

As (7.1.1) should be solved by iteration methods, exact and inexact multiplicative and additive Schwarz iterations were introduced (Sections 3.2 and 3.3). In addition to the one-level block update (exact update) five different block updates were introduced (inexact updates). These updates are: relaxed one-level update, two-stage update, relaxed two-stage update, power-like update, and relaxed power-like update

Beneath Schwarz iterations, block partially asynchronous iterations have been defined (Section 3.4). By their nature the same block updates as for Schwarz iterations can be used.

All the iteration schemes were analysed within the context of the model problems *MP* and *GMP*. The connections between multiplicative Schwarz iterations and partially asynchronous iterations have been revealed (Section 3.5). Furthermore, the problems to overcome when trying to analyse these iterations were identified (Section 3.6). It turned out that algebraic subspace theory is not an appropriate tool to prove the convergence of the iteration schemes applied to (7.1.1) with the conditions (7.1.2) and (7.1.3).

In Chapter 4, a graph based approach was introduced. This approach deals with the following question leading to the solution of *MP*.

Given an STM-matrix $A = I - B$, is it possible to find a mapping which maps A onto a sequence of ST-matrices $\{T^{(k)}\}_{k\in\mathbb{N}_0}$, that represents a multiplicative Schwarz iteration

$$x^{k+1} = T^{(k)}x^k + c^k,\ c^k \in \mathcal{R}(T^{(k)}), \tag{7.1.4}$$

such that (7.1.4) converges to a solution of (7.1.1) satisfying (7.1.2)?

This means, mapping A onto a sequence $\{T^{(k)}\}_{k\in\mathbb{N}_0}$ which satisfies $\mathcal{N}(A) = \mathcal{N}(I - T^{(k)})$, $T^{(k)} \geq 0$, and $T^{(k)}z = z$, for all $k \in \mathbb{N}_0$, and additionally $\lim_{k\longrightarrow\infty} T^{(k)} \cdot \ldots \cdot T^{(0)} = P$ exists, and P is a projection onto $\mathcal{N}(A)$ (cf. Section 4.2).

This question was successfully answered for several setups, depending on the iteration scheme and the block updates. The idea was always the same, based on a mapping of the basic structure of A into $\{T^{(k)}\}_{k\in\mathbb{N}_0}$. This has been discussed for both, non-block (Section 4.3) and block iterations (Section 4.4). It led to the idea of flow compatibility w.r.t. some spanning tree $\mathcal{T} \subset \Gamma(A^T)$. While flow compatibility was sufficient for single-row updates and non-overlap block updates, the so called ms-compatibility was defined for general multiplicative Schwarz iterations including overlap.

The ms-compatibility implies, that the spanning tree within $\Gamma(A^T)$ can be mapped onto a spanning tree within $\Gamma(T^{(k)})$ for all $k \in \mathbb{N}_0$; i.e. $\mathcal{N}(A) =$

$\mathcal{N}(I - T^{(k)})$ for all $k \in \mathbb{N}_0$. After this, it was proven that the single final and basic class α in $\Gamma(A)$ can be mapped onto a final, basic, and regular class of $T^{(k)}$. And there is again only one such class in $\Gamma(T^{(k)})$, which implies $\gamma(T^{(k)}) < 1$ for all $k \in \mathbb{N}_0$ (Section 4.4). The only two things which must be considered are the order of the block updates and the non-overlap of the root-index of the spanning tree $\mathcal{T}$ of A. To outline the necessity of those conditions a counterexample has been given.

The combination of both, consistency ($\mathcal{N}(A) = \mathcal{N}(I - T^{(k)})$) and convergence ($\gamma(T^{(k)}) < 1$), led to new convergence results for one-level and two-stage multiplicative Schwarz iterations in Section 4.5 via the convergence theorems provided in Section 4.2.

In contrast to the non-relaxed iteration schemes for *MP* from Section 4.5, the analysis of relaxed one-level and relaxed two-stage iteration schemes was much easier (Section 4.6). As relaxed iteration schemes imply $\gamma(T^{(k)}) < 1$, for all $k \in \mathbb{N}_0$ by their nature, only $\mathcal{N}(A) = \mathcal{N}(I - T^{(k)})$ was to verify. This was an easy task, since positive diagonals preserve graphs. Thus, the fixed order that was needed in Section 4.5 could be given up.

A non-consistent multiplicative Schwarz iteration which uses relaxed block updates has not been given. It is still an open question whether it can be constructed.

Those new results on relaxed iteration schemes led directly to new convergence results for additive Schwarz iterations (Section 4.7), again by theorems from Section 4.2. These new results use one-level and relaxed two-stage block updates.

Finally, an application of the developed theory delivered directly some new convergence results for relaxed block partially asynchronous iterations (Section 4.8) applied to *MP*. The block updates were again relaxed one-level and relaxed two-stage block updates.

While there were reliable results for *MP*, their extensions to the model problem *GMP* was only partly successful. As *GMP* is given by (7.1.1) combined with (7.1.3), i.e. given in terms of an SFM-matrix $A = I - B$, some results were carried over to this problem by a localisation. This was, because an SFM-matrix of degree r contains a spanning forest in $\Gamma(A^T)$ containing r trees, whose roots lie in different final and basic classes. Actually, there are r non-overlapping principal minors within A which are STM-matrices. Now, by exploiting the local STM-structure, several extensions were made (Sections 5.1, 5.2, and 5.3).

Indeed, all theorems from Chapter 4, which deliver results for one-level and relaxed one-level multiplicative and additive Schwarz, were generalised. But the convergence theorems on two-stage methods given in Section 4.2 were not applicable anymore. Thus, some restrictions to two-stage iterative methods had to be done, leading to new convergence results, which are partly weaker than those presented in Chapter 4. The restrictions concerned the inner splittings as well as the number of inner iterations for relaxed and non-relaxed multiplicative and also additive Schwarz iterations. Actually, the results were only formulated for stationary two-stage methods but they are new.

Last but not least, two trivial generalisations were analysed (Section 5.4). They were trivial because the systems decompose into a block diagonal form. Thus, all results given in Chapter 4 apply by the localisation given in Section 5.1.

Unfortunately, there are no such generalised results for partially asynchronous iterations so far. An extension for these iterations will be more tricky.

The theory of the solution of the model problems *MP* and *GMP* was accomplished with Chapter 6. There the results given in this thesis were compared to the latest known results in the literature.

The similarities of the algebraic-subspace-ansatz (Section 3.6) and some analytical approaches were examined in Section 6.1. This has been done to delimit the graph based approach from the other ones. Then there followed a brief comparison of the results for multiplicative and additive Schwarz iterations (Sections 6.2 and 6.3).

The results for the partially asynchronous iteration were discussed in a more detailed manner (Section 6.4). This has been done so because the theoretically background is somehow more complex and a comparison of the results is not as easy as for Schwarz iterations.

Finally, other graph based approaches haven been compared with the one presented here. It turned out that there are similarities between the ansatz used here and those of other authors. Actually several authors showed, that strongly connected classes are mapped onto regular ones, resulting in a semiconvergent operator. This is exactly one of the basic conditions which implies semiconvergent Schwarz iterations. But usually, only splittings of irreducible singular M-matrices were examined in other works. Thus, the problem of non-consistency is avoided. Finally, the results given elsewhere did not cover Schwarz iterations nor partially asynchronous iterations.

Altogether, this thesis shows that a lot of different theories can be embedded within a simple theory, which is based on the structure of certain nonnegative matrices. Once this structure is analysed, the conditions needed for semiconvergent iterations are easily obtained. Clearly, there are more general results in the literature, but the homogeneity of the theory presented here and the number of results makes the concept of "flow" a strong tool.

Especially sparse non-symmetric ST- or SF-matrices must be handled carefully as certain examples have shown. If the ST- or SF-matrix is dense or symmetric, convergence will usually happens (cf. Section 7.2), since either a lot of trees exist or the matrix becomes (block) irreducible.

An overview of the achieved results is given in Table 7.1. The results for Update 1,2,3, and 4 are all new. The results for Update 5 and 6, although new, have only a theoretically value.

The results from Section 5.4 have been left out, since they are just simple applications of the theory given in Chapter 4. The results concerning the splitting (6.2.2) in Section 6.2 are also not stated. As previously mentioned, results for multisplitting methods (cf. Section 4.7) from [44, 52] should be obtained in a similar way.

Now a brief description of the contents of Table 7.1.

Column one contains the model problem, which is either *MP* or *GMP*. Column two displays the basic iterative method. This is "MS" for multiplicative Schwarz, "AS" for additive Schwarz (Section 3.2), and "PAI" for partially asynchronous iterations (Section 3.4). The block updates are given in column three by numbers and have been introduced in Section 3.3. The numbers stand for:

Number	**Update**	**Algorithm**	**local update**
1	one-level (exact)	3.2	(3.2.9)
2	relaxed one-level	3.4	(3.3.1)
3	two-stage	3.5	(3.3.2)
4	relaxed two-stage	3.6	(3.3.3)
5	power method	3.7	(3.3.4)
6	relaxed power method	3.8	(3.3.5)

Column four gives an overview of the decompositions to be used. Here, "ms" stands for *ms-compatible* (cf. Definition 4.5), "gms" for *gms-compatible* (cf. Definition 5.2), "rp" for *root preserving* (cf. Definition 4.7), "lrp" for *locally root preserving* (cf. Definition 5.4), and "a" for an arbitrary decomposition. The acronym "gs" stands for Gauss-Seidel and is actually not a decomposition as non-block updates are assumed.

Column five gives some information of the inner splittings if two-stage methods are used. One has "fc" for *flow compatible* inner splittings (cf. Definition 4.6), "lfc" for *locally flow compatible* (cf. Definition 5.3), and "M" for M-splitting (cf. Section 1.1). The number of inner iterations within each block update is given in column six. Here, "a" stands for an arbitrary number, "b" for a bounded number of inner iterations, and "c" for a constant number (i.e. stationary two-stage iterations).

As the order of local updates could be important for non-relaxed multiplicative Schwarz iterations, the dependence on the order is given in column seven. Here, "f" stands for a fixed order, while "a" marks an arbitrary order. If an entry makes no sense in either of the columns five, six, or seven, it has been left out.

Finally, column eight gives the references to the convergence theorems within this thesis.

Pr.	Meth.	Upd.	Decomp.	Spl.	Iter.	Order	Th.
MP	MS	1	ms	–	–	f	4.11
MP	AS	1	rp	–	–	–	4.19
GMP	MS	1	gms	–	–	f	5.1
GMP	AS	1	lrp	–	–	–	5.5
MP	MS	2	ms	–	–	a	4.13
MP	MS	2	rp	–	–	a	4.17
MP	PAI	2	ms	–	–	–	4.22
GMP	MS	2	lrp	–	–	a	5.3
MP	MS	3	ms	fc	a	f	4.12
GMP	MS	3	gms	lfc	c	f	5.2
MP	MS	4	ms	fc	a	a	4.14
MP	MS	4	rp	M	b	a	4.18
MP	AS	4	rp	M	b	–	4.20
MP	PAI	4	ms	fc	b	–	4.23
GMP	MS	4	lrp	M	c	a	5.4
GMP	AS	4	lrp	M	c	–	5.6
MP	MS	5	gs	–	–	f	4.9
MP	MS	6	a	–	–	a	4.15
MP	PAI	6	a	–	–	–	obvious
GMP	MS	6	a	–	–	–	obvious

Table 7.1: Convergence theorems

7.2 Further questions and open problems

Finally, a few open problems and further questions which might be worth to be discussed in the future are presented here.

Reduced flow compatibility

Motivated by the results from Section 6, especially by Theorems 6.15 and 6.19, the following question arises:

Is it sufficient to restrict the flow compatibility to the final and basic class *of an ST-matrix?*

The answer should be yes. Clearly, one loses the positive column in the STM-matrix case, but semiconvergent operators will be obtained anyway.

Reduced relaxation

In [39] and [55], the existence of a positive diagonal element is only proposed for a single element. It has been mentioned in Section 6.4, that the index of this positive element can be interpreted as the root of a spanning tree. As blocks behave similarly to single states (cf. Section 4.4), the following question immediately arises for multiplicative Schwarz methods, based on ms-compatible decompositions $S_1, \ldots, S_p$, i.e. the root lies in S_p.

Is it sufficient to have relaxation only for the p-th block, i.e. the root block?

The answer should be yes, but clearly, this will not have practical impact.

The symmetric case

Consider a symmetric STM-matrix $A \in \mathbb{R}^{n\times n}$.

Does every regular decomposition lead to semiconvergent multiplicative Schwarz iterations with block updates introduced here?

A good hint how this question could be successfully answered is Theorem 6.4, which is Theorem 5.7 in [48]. But in contrast to the methods used there, a proof with the technique given in this thesis would be preferable. This is because the proof will then be based on the pattern. Therefore it would imply results on STM-matrices having a symmetric pattern, rather than being itself symmetric. Probably, those results might also have applications to multi splittings.

Non-consistency

It has been shown in Section 4.4, that the ms-compatibility is sufficient for semiconvergent non-relaxed multiplicative Schwarz iterations for *MP*. Consequently, gms-compatibility is sufficient for *GMP*.

To show the necessity of ms-compatibility, a counterexample has been given for a non-ms-compatible splitting. For an STM-matrix A, there was a flow compatible decomposition such that the global multiplicative Schwarz operator T_1 for Update 1 was semiconvergent but non-consistent, i.e.

$$\mathcal{N}(I - T_1) \neq \mathcal{N}(A).$$

In Section 4.6 the notion of root preservation has been introduced for relaxed iteration schemes as a logical consequence of ms-compatibility. But the necessity of root preservation is an open problem. I.e., is it possible to construct non-consistent non-root-preserving relaxed multiplicative Schwarz-type iteration operators?

Concretely, let A be an STM-matrix. Consider one local update of Update 2, 4, or the update given by (6.2.2). Let $\tilde{T}$ be a corresponding semiconvergent global multiplicative Schwarz operator.

Give an example for an arbitrary regular decomposition $S_1, \ldots, S_p$, such that

$$\mathcal{N}(I - \tilde{T}) \neq \mathcal{N}(A),$$

or prove equality for every arbitrary regular decomposition.

The proof in [40] that every regular decomposition leads to a consistent operator is not correct.

Bounds on the number of inner iterations for *GMP*

Results for additive and multiplicative Schwarz iterations for *GMP* have only been given for a constant number of inner iterations if Update 3 or 4 is used. The problem has been outlined in Section 5.1.

Prove Theorems 5.2 and 5.4 for an arbitrary number of inner iterations.

This should not be a problem if the accessibility relations within each global operator are constant, as mentioned in Section 5.1. But what about a general proof?

Bounds on the inner iterations for PAIs

It should be possible to prove, that the number of inner iterations for PAIs applied to *MP* can be chosen arbitrarily.

Prove Theorem 4.23 for any number of inner iterations.

The only thing to be shown is to find some fixed $\kappa > 0$, which bounds the relevant entries.

PAIs for *GMP*

While some results for multiplicative and additive Schwarz could be extended to *GMP*, the same step was not possible for PAIs. The problem is actually the same as for two-stage multiplicative and additive Schwarz iterations. Results for these iterations have only been proven for a constant number of iterations, i.e. for stationary schemes. This was because the stationary iterations imply a constant sequence of operators. The latter cannot be guaranteed for PAIs, whether the number of inner iterations is constant or not. Indeed, it cannot be guaranteed for every chosen block update.

Anyway, it is not hard to prove a result similar to Theorem 4.21. Let A be an SFM-matrix of degree r and let $\{H_d^{(k)}\}_{k\in\mathbb{N}_0}$ be any sequence for Update 2 or 4, which is based on a gms-compatible splitting. Then, there exists for any $k \in \mathbb{N}_0$ a number $m \in \mathbb{N}$ such that

$$H_d^{(k,k+m)} = H_d^{(k+m)} \cdot \ldots \cdot H_d^{(k)}$$

is an SF-matrix of degree r. The forest contains r trees of height one, i.e. $H_d^{(k,k+m)}$ is semiconvergent. Moreover, any finite product of these semiconvergent matrices is again a semiconvergent SF-matrix of degree r. But the proof of the convergence of the whole sequence seems to be sophisticated without a norm. However, it should be possible to prove it.

Optimal trees / fast convergence

One of the most sophisticated problems in the solution of linear equations using additive and multiplicative Schwarz methods is the domain decomposition. Thus, the question that immediately arises is that of an optimal tree for an ST-matrix (or likewise of an optimal forest for an SF-matrix). This means, a flow compatible numbering of a spanning tree which delivers fast converging iteration operators for a given number of partitions (i.e. small γ-values).

Unfortunately, this work delivers no new answers in which way fast convergent domain decompositions can be achieved.

A few tests have been made, but the results are all unsatisfactory (cf. Appendix B). Indeed, the results achieved with block partitioners (see, e.g., [45, 65]) will usually be better. Looking for a tree that delivers a strong positive column is to crude and the number of iterations depends heavily on the chosen flow compatible numbering.

Note that the authors of [45, 65] do not care about flow compatibility. But

- the iteration operators need not have a positive column to produce a convergent sequence (cf. the above discussion on the reduced flow compatibility),
- although the problems in [45, 65] are almost sparse, it is pretty hard to construct non-consistent iterations (i.e. convergence usually takes place), and
- almost all problems have a symmetric pattern, thus, probably every decomposition is allowed (cf. the above discussion of the symmetric case).

An optimal tree which leads to fast convergence cannot be obtained without considering the whole matrix structure (especially for NCD-matrices) and a good way to obtain one has not been found yet.

Anyway, flow compatibility might be a good tool for sparse matrices which have a non-symmetric pattern.

General problems

Suppose the theory for *MP* has been completely extended to *GMP* and every regular decomposition for relaxed multiplicative and additive iteration schemes has been proven to deliver consistent iteration operators (see above). Then, the following problem can be possibly tackled.
Find a solution of

$$(I-B)x = b,\ x \in \mathbb{R}^n,\ b \in \mathcal{R}(I-B), \tag{7.2.1}$$

where $B \in \mathbb{R}^{n\times n}, B \geq 0$, $\mathcal{R}(I-B) \oplus \mathcal{N}(I-B) = \mathbb{R}^n$, and $\rho(B) = 1$.
This should be possible, because for $A := I - B$ there holds $Av = 0$ for some $v \geq 0$. Thus, A can be permuted into the following form

$$\Pi A \Pi^T = \left(\begin{array}{cccc|ccc} A_{11} & \dots & 0 & 0 & A_{1,r+2} & \dots & A_{1,m} \\ \vdots & \ddots & \vdots & \vdots & \vdots & & \vdots \\ 0 & \dots & A_{r,r} & 0 & A_{r,r+2} & \dots & A_{r,m} \\ A_{r+1,1} & \dots & A_{r+1,r} & A_{r+1,r+1} & A_{r+1,r+2} & \dots & A_{r+1,m} \\ \hline 0 & \dots & 0 & 0 & A_{r+2,r+2} & \dots & A_{r+2,m} \\ \vdots & & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & 0 & 0 & \dots & A_{m,m} \end{array}\right).$$

The upper left part represents an SFM-matrix of degree r and the lower right part a block upper triangular matrix, whose diagonal blocks are irreducible regular M-matrices. Thus, a convergence analysis should be possible.

Epilogue

What remains to be said on the relaxed iteration schemes discussed here . . .

Anything that happens, happens.

Anything that, in happening, causes something else to happen, causes something else to happen.

Anything that, in happening, causes itself to happen again, happens again.

It doesn't necessarily do it in chronological order, though.

Douglas Adams, Mostly Harmless,
The fifth book of the Hitchhiker's trilogy in four parts

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Appendix A

Proofs

Proof of Proposition 3.5 Let (λ, v) be an eigenpair of $H_{d,l}^{(k)}$. Taking (3.5.7) and a proper partitioning $v = (v_0^T, \ldots, v_d^T)^T$,

$$H_{d,l}^{(k)} v = \begin{pmatrix} \tilde{E}_l^{(k,0)} v_0 + \ldots + E_l^{(k,d)} v_d \\ v_0 \\ \vdots \\ v_{d-1} \end{pmatrix} = \lambda \begin{pmatrix} v_0 \\ v_1 \\ \vdots \\ v_d \end{pmatrix}$$

and the representation (3.5.8) follows.

Now let $\lambda \neq 1$, then

$$(I - D^{(k)}) v_0 + \sum_{\tau=0}^{d} E_l^{(k,\tau)} v_\tau = \lambda v_0.$$

Each j-th row of each $E_l^{(k,\tau)}$ is zero if $j \notin \mathcal{J}_k = S_{i(k)}$ since these components are not updated, i.e.

$$\left(\sum_{\tau=0}^{d} E_l^{(k,\tau)} v_\tau \right)_j = 0 \text{ for all } j \notin S_{i(k)}.$$

On the other hand

$$\Big((I - D^{(k)}) v_0 \Big)_j = (v_0)_j = \lambda (v_0)_j \text{ for all } j \notin S_{i(k)}.$$

But then $(v_0)_j = 0$ for all $j \notin S_{i(k)}$ since $0 \neq \lambda \neq 1$ (this is the same result as in Proposition 3.2). With (3.5.8), $(v_\tau)_j = 0$ follows for all $\tau = 0, \ldots, d$, and $j \notin S_{i(k)}$.

It follows from (3.4.5) that the j-th column of $E_l^{(k,\tau)}$ is entirely zero, i.e.

$$E_l^{(k,\tau)} v_\tau = 0$$

for all $j \in S_{i(k)}$ and $\tau = 1, \dots, d$. But then

$$(I - D^{(k)})v_0 + \sum_{\tau=0}^{d} E_l^{(k,\tau)} v_\tau = E_l^{(k,0)} v_0 = \lambda v_0.$$

Assume now v_0 to be split as $v_0 = ((v_0)_{S_i}^T, (v_0)_{\neg S_i}^T)^T = ((v_0)_{S_i}^T, 0)^T$. Then by the construction of the $H_{d,l}^{(k)}$ and a proper permutation matrix $\Pi_{i(k)}$

$$\begin{aligned}
\Pi_{i(k)} \left(E_l^{(k,0)}\right) \Pi_{i(k)}^T \Pi_{i(k)} v_0 &= \begin{pmatrix} H_l^{(i(k))}[S_{i(k)}] & * \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} (v_0)_{S_{i(k)}} \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} Z_l^{(i(k))} & * \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} (v_0)_{S_{i(k)}} \\ 0 \end{pmatrix} \\
&= \lambda \begin{pmatrix} (v_0)_{S_{i(k)}} \\ 0 \end{pmatrix}
\end{aligned}$$

with $Z_l^{(i(k))}$ given by (3.5.3).

Thus, (λ, v_0) is an eigenpair of $H_l^{(i(k))}$ by Proposition 3.2. The assertion for generalised eigenvectors follows by an induction on the corresponding Jordan-chain.

To prove that an eigenvector v_0 of $H_l^{(i(k))}$ is an eigenvector of $H_{d,l}^{(k)}$, construct v as in (3.5.8) and apply the above proof backwards. The claim for eigenvectors to the eigenvalue 1 should be obvious by Proposition 3.4. $\square$

Proof of Proposition 3.7: Since the relations $P_d^{(i_0)} \cdot \tilde{Q} = \tilde{Q} \cdot P_d^{(i_0)} = 0$ are obvious, it remains to prove $\rho(\tilde{Q}) < 1$. Therefore, let $v \in \mathbb{R}^{(d+1)n}$ be an eigenvector of $\tilde{Q}$ to an eigenvalue λ with $1 \neq \lambda \neq 0$. Let $v = (v_0^T, \dots, v_d^T)^T$, then $0 = P_d^{(i_0)} \cdot \tilde{Q} v = \lambda P_d^{(i_0)} v$ and by the structure of $P_d^{(i_0)}$

$$(v_0)_i = 0, \quad \text{for all } i \notin S_{i_0}.$$

This structure is inherited to the other blocks by the implicit shift of the $H_{d,l}^{(k_j)}$ (cf. proof of Proposition 3.5 and take powers of $\tilde{Q}$ if need be). Thus,

$$(v_j)_i = 0, \quad \text{for all } i \notin S_{i_0} \text{ and } j = 0, \dots, d. \tag{A.1}$$

Taking (3.4.5) and (3.5.7), then the 0-th block of the image of $Q_{d,l}^{(k_L)} v$ becomes

$$\begin{aligned}
\left(Q_{d,l}^{(k_L)} v\right)_0 &= \left((H_{d,l}^{(k_L)} - P_d^{(i_0)})v\right)_0 = \left(H_{d,l}^{(k_L)} v\right)_0 \\
&= \tilde{E}_l^{(k_L,0)} v_0 + \sum_{\tau=1}^{d} E_l^{(k_L,\tau)} v_\tau \\
&= \tilde{E}_l^{(k_L,0)} v_0 = \left(I - D^{k_L} + E_l^{(k_L,0)}\right) v_0 \\
&= E_l^{(k_L,0)} v_0.
\end{aligned}$$

And inductively

$$\lambda v_0 = (\tilde{Q}v)_0 = E_l^{(k_1,0)} \cdot \ldots \cdot E_l^{(k_L,0)} v_0.$$

With (3.4.12) and (A.1)

$$\begin{aligned} E^{(k,0)} v_0 &= \Pi_{i_0}^T \begin{pmatrix} H_l^{(k_L,i_0)}[S_{i_0}] & * \\ 0 & 0 \end{pmatrix} \Pi_{i_0} v_0 \\ &= \Pi_{i_0}^T \begin{pmatrix} H_l^{(k_L,i_0)}[S_{i_0}] & 0 \\ 0 & 0 \end{pmatrix} \Pi_{i_0} v_0 \end{aligned}$$

for a proper permutation matrix Π_{i_0}. Again by induction

$$\begin{aligned} \lambda v_0 &= E_l^{(k_1,0)} \cdot \ldots \cdot E_l^{(k_L,0)} v_0 \\ &= \Pi_{i_0}^T \begin{pmatrix} H_l^{(k_1,i_0)}[S_{i_0}] & 0 \\ 0 & 0 \end{pmatrix} \cdot \ldots \cdot \begin{pmatrix} H_l^{(k_L,i_0)}[S_{i_0}] & 0 \\ 0 & 0 \end{pmatrix} \Pi_{i_0} v_0. \end{aligned}$$

But $\rho(H_l^{(k_j,i_0)}[S_{i_0}]) < 1$ for each $j = 1, \ldots, L$ by (3.4.12), hence $\lambda < 1$. $\square$

Proof of Lemma 3.7: Let j with $1 \le j \le p$ be given and consider some $v_j \in \mathcal{N}(P_d^{(i(k_j))})$. From (3.5.10) follows the orthogonal decomposition

$$v_j = \begin{pmatrix} v_0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ v_1 \\ \vdots \\ v_d \end{pmatrix} = \tilde{v}_j + \tilde{v}_j^0,$$

satisfying $\tilde{v}_j \in \mathcal{N}(P^{(i(k_j))})$ and $\tilde{v}_j^0 \in \mathcal{N}(P_d^{(i(k_l))})$ for all $1 \le l \le p$. Thus, the null spaces of all $P_d^{(i(k_l))}$ fulfill

$$\text{(A.2)} \qquad \begin{aligned} \mathcal{N}(P_d^{(i(k_l))}) &= \mathcal{N}^l \oplus \mathcal{N}^0, \quad \text{for all } l = 1, \ldots, p, \\ \mathbb{R}^n &\supseteq \mathcal{N}^0 \oplus \mathcal{N}^1 \oplus \ldots \oplus \mathcal{N}^p, \\ \mathcal{N}^k &\perp \mathcal{N}^j, \quad \text{for all } 0 \le k, j \le p, \quad k \ne j. \end{aligned}$$

Here $\mathcal{N}^j = \mathcal{N}(P^{(i(k_j))})$ and $\mathcal{N}^0 = \cap_{l=1}^p \mathcal{N}(P_d^{(i(k_l))})$.

Since the relation $\supset$ is obvious let $x \in \mathcal{N}(I - H_{d,l}^{(k_1)} \cdot \ldots \cdot H_{d,l}^{(k_p)})$ and assume $x \ne 0$. Denote $x_i = H_{d,l}^{(k_1)} \cdot \ldots \cdot H_{d,l}^{(k_i)} x$.

Now, analogous to the proof of Lemma 3.5, a decomposition

$$x = x_1 + \sum_{j=1}^{p} x_j^{\Delta}$$

such that $x_j^{\Delta} \in \mathcal{N}(P_d^{(i(k_j))})$ follows. With the same argumentation as in Lemma 3.5 one gets

$$0 = \sum_{j=1}^{p} x_j^{\Delta}.$$

It follows from (A.2) that each x_j^{Δ} can be decomposed to $x_j^{\Delta} = \tilde{x}_j + \tilde{x}_j^0$, such that $\tilde{x}_j \in \mathcal{N}^j$ and $\tilde{x}_j^0 \in \mathcal{N}^0$. Thus,

$$0 = \sum_{j=1}^{p} \tilde{x}_j + \sum_{j=1}^{p} \tilde{x}_j^0$$

and again by (A.2)

$$\text{(A.3)} \qquad \tilde{x}_j = 0, \quad \text{for all } j = 1, \dots, p.$$

Let $x^T = (z_0, \dots, z_d)$. Now if each $H_{d,l}^{(k_l)}$ is written as in (3.5.7) then

$$x_p^T = (H_{d,l}^{(k_p)} x)^T = \left(\left(\tilde{E}_l^{(k_p,0)} z_0^T + \sum_{\tau=1}^{d} E_l^{(k_p,\tau)} z_\tau^T \right)^T, z_0, \dots, z_{d-1} \right).$$

The conditions (A.2) and (A.3) imply that $\tilde{E}_l^{(k_p,0)} z_0^T + \sum_{\tau=1}^{d} E_l^{(k_p,\tau)} z_\tau^T = z_0^T$, i.e.

$$x_p^T = (z_0, z_0, z_1, \dots, z_{d-1}).$$

In general

$$x_{p-j}^T = (\underbrace{z_0, \dots, z_0}_{j+1-\text{times}}, z_1, \dots, z_{d-j}).$$

If $d \le p$ then $x_1^T = (z_0, \dots, z_0)$ and therefore $x = (z_0, \dots, z_0)$. Otherwise,

$$x = x_1 = (\underbrace{z_0, \dots, z_0}_{p+1-\text{times}}, z_1, \dots, z_{d-p-1}),$$

and several applications of the same argument lead again to $x^T = (z_0, \dots, z_0)$. But now

$$z_0 = \left(\tilde{E}_l^{(k_j,0)} z_0^T + \sum_{\tau=1}^{d} E_l^{(k_j,\tau)} z_\tau^T \right)^T = \left(H_l^{(i(k_j))} z_0^T \right)^T,$$

for all $j = 1, \dots, p$ and z_0^T is a fixed point of $H_l^{(i(k_1))} \cdot \ldots \cdot H_l^{(i(k_p))}$. Now, from Lemma 3.5, $(I - P^{(i(k_j))}) z_0^T = 0$ for all $j = 1, \dots, p$ which implies $(I - P_d^{(i(k_j))}) x = 0$. The latter means $x_j^{\Delta} = 0$ for all $j = 1, \dots, p$, and the lemma is proved. $\square$

Appendix B

Some simple tests

This section contains a few remarks concerning the calculation of a flow compatible numbering and the dependence of the number of iterations on the chosen root index; thus on the chosen tree.

It has been shown in Section 4.3 that Algorithm 4.1 (see also Lemma 4.1) applied to an ST- or STM-matrix delivers a flow compatible numbering. Now, there will be a few examples and a short discussion on two other approaches.

Following the argumentation of depth first search (DFS), the same result as in Lemma 4.1 can be achieved using a breadth first search (BFS); see, e.g., [22]. It is given by the following algorithm.

The algorithm should be called with some index j_0 lying in the basic class of some ST-matrix B, a field π such that $\pi(1) = j_0$, and an index $t = 1$. As usually, $\Gamma(B^T) = (V, E)$.

Algorithm B.1 BFS_VISIT

Require: $j \in \{1, \ldots, n\}$
for each $i \in V$ such that $(j, i) \in E$ **do**
    if i is not visited **then**
        mark i as visited
        $t \leftarrow t + 1$
        $\pi(t) \leftarrow i$
    end if
end for
for each $i \in V$ such that $(j, i) \in E$ **do**
    BFS_VISIT(i)
end for

The second alternative to be discussed is a modification of Algorithm 4.1. The modification concerns the strategy how to choose the next node. Instead of taking the first possible index, the index that will locally maximize the weight of the resulting spanning tree $\mathcal{T}$ is chosen. The weight of a spanning tree $\mathcal{T}$ for a matrix A is the sum of the modulus of all elements $a_{j,i}$ such that $(j,i) \in E(\mathcal{T})$. I.e., if j_0 is the actual index, then the next index i_0 is chosen such that

$$|a_{i_0,j_0}| = \max_{i=1,\dots,n} |a_{i,j_0}| \tag{B.1}$$

This greedy strategy leads to a method which will be called DFSM. The necessary modification of Algorithm 4.1 should be obvious.

The above mentioned approaches will now be shortly discussed for three examples. The system to be solved is always

$$(I - B)x = 0 \tag{B.2}$$

for some nonnegative ST-matrix $B \geq 0$.

The iteration numbers for the examples have been calculated with a MATLAB implementation of the one-level multiplicative Schwarz iterations, i.e. Algorithm 3.2. The inverse of each principal minor has been calculated with the MATLAB function `inv`. The start vector is always $(1,\dots,1)/n$ and the iterations stopped if $\|x^{k+1} - x^k\|_\infty < 10^{-8}$ or the number of iteration exceeds some specific value. Overlap (if any) has been added to the borders of the partitions. If flow compatible permutations were computed, the systems have always been permuted w.r.t. to this numbering.

Courtois matrix

The first example is the well known 8×8 Courtois matrix (see [20, 65]). It is given by

$$B = \left(\begin{array}{rrr|rr|rrr}
0.85000 & 0.10000 & 0.10000 & 0 & 0.00050 & 0 & 0.00003 & 0 \\
0 & 0.65000 & 0.80000 & 0.00040 & 0 & 0.00005 & 0 & 0.00005 \\
0.14900 & 0.24900 & 0.09960 & 0 & 0.00040 & 0 & 0.00003 & 0 \\
\hline
0.00090 & 0 & 0.00030 & 0.70000 & 0.39900 & 0 & 0.00004 & 0 \\
0 & 0.00090 & 0 & 0.29950 & 0.60000 & 0.00005 & 0 & 0.00005 \\
\hline
0.00005 & 0.00005 & 0 & 0 & 0.00010 & 0.60000 & 0.10000 & 0.19990 \\
0 & 0 & 0.00010 & 0.00010 & 0 & 0.24990 & 0.80000 & 0.25000 \\
0.00005 & 0.00005 & 0 & 0 & 0 & 0.15000 & 0.09990 & 0.55000
\end{array}\right).$$

The Courtois matrix is a nice example of a non-symmetric column stochastic NCD-matrix; for NCD-matrices see [20, 65]. Equation (B.2) with an NCD-matrices B is usually solved by iterative aggregation/disaggregation methods (IAD); see, e.g., [36, 42, 65, 68, 72]. To apply IAD successfully, an optimal partitioning is needed in the sense that the diagonal blocks contain large elements, while the outer diagonal blocks contain only small elements.

Those partitionings can be calculated by tools like MARCA (see [65] and the references therein), TPABLO [19], or XPABLO [26]. However, the optimal partitioning for the Courtois matrix is given by $S_1 = \{1,2,3\}, S_2 = \{4,5\}$, and $S_3 = \{6,7,8\}$, i.e. the diagonal blocks already contain the large elements. An IAD iteration with block Gauss-Seidel smoother needs 3 iterations on the given partitioning (the starting vector is obtained from e by a local normalisation).

Multiplicative Schwarz on the given partitioning (i.e. block Gauss-Seidel) needs 8 iterations. The iteration numbers for the reordered system depend on the chosen root index and are given in the following table.

	1	2	3	4	5	6	7	8
BFS	> 4000	3645	> 4000	> 4000	> 4000	> 4000	> 4000	> 4000
DFS	6	6	1136	> 4000	> 4000	> 4000	> 4000	> 4000
DFSM	6	6	6	7	8	7	7	7

The advantage of DFSM seems obvious, because the larger the elements of the spanning tree become, the larger the elements of a resulting positive column will be (this is not hard to see). Then, the influence of the positive column may outweigh the influence of the rest of the resulting system.

But the truth is that the reordering keep the block structure alive. To see this consider the partitioning $S_1 = \{1,2,3\}, S_2 = \{4,5,6\}$, and $S_3 = \{7,8\}$. As this partitioning does not respect the NCD structure, the number of iterations for IAD is 14 and that of block Gauss-Seidel is 55.

	1	2	3	4	5	6	7	8
BFS	> 4000	> 4000	> 4000	> 4000	> 4000	> 4000	> 4000	> 4000
DFS	7	7	7	> 4000	> 4000	> 4000	> 4000	> 4000
DFSM	7	7	7	7	7	2220	2220	2441

While the number of iterations stay reliable for the first five reordering, the other three become unacceptable. To understand this, consider the DFSM-reordered matrix $B^{(1)}$ with the root index 1.

$$B^{(1)} = \left(\begin{array}{rrr|rrr|rr}
0.55000 & 0.15000 & 0.09990 & 0 & 0 & 0.00005 & 0 & 0.00005 \\
0.19990 & 0.60000 & 0.10000 & 0 & 0.0001 & 0.00005 & 0 & 0.00005 \\
0.25000 & 0.24990 & 0.80000 & 0.0001 & 0 & 0 & 0.0001 & 0 \\
\hline
0 & 0 & 0.00004 & 0.7000 & 0.3990 & 0 & 0.0003 & 0.00090 \\
0.00005 & 0.00005 & 0 & 0.2995 & 0.6000 & 0.00090 & 0 & 0 \\
0.00005 & 0.00005 & 0 & 0.0004 & 0 & 0.65000 & 0.8000 & 0 \\
\hline
0 & 0 & 0.00003 & 0 & 0.0004 & \mathbf{0.24900} & 0.0996 & 0.14900 \\
0 & 0 & 0.00003 & 0 & 0.0005 & 0.10000 & 0.1000 & 0.85000
\end{array}\right)$$

The spanning tree is represented by the upper bidiagonal. Actually, the reordering maps the strong components into the given partitioning.

As the root has access to state/index 7, the boldfaced matrix entry can be regarded as a guard. Indeed, this element causes the iteration operator $T_1^{(1)}$ to have a strong positive column.

$$T_1^{(1)} \approx \begin{pmatrix} 0.0143 & 0.0143 & 0.0116 & 0 & 0.0034 & 1.0242 & 0 & 0 \\ 0.0169 & 0.0169 & 0.0138 & 0 & 0.0040 & 1.2109 & 0 & 0 \\ 0.0391 & 0.0391 & 0.0318 & 0 & 0.0092 & 2.7942 & 0 & 0 \\ 0.0401 & 0.0401 & 0.0324 & 0 & 0.0051 & 1.5097 & 0 & 0 \\ 0.0301 & 0.0301 & 0.0242 & 0 & 0.0038 & 1.1326 & 0 & 0 \\ 0.0002 & 0.0002 & 0.0002 & 0 & 0.0026 & 0.9956 & 0 & 0 \\ 0 & 0 & 0.0001 & 0 & 0.0011 & 0.4348 & 0 & 0 \\ 0 & 0 & 0.0002 & 0 & 0.0041 & 0.9566 & 0 & 0 \end{pmatrix}$$

Additionally, all positive columns within $T_1^{(1)}$ are nearly proportional which seems to be an advantage since the positive column in the limit must be proportional. A MATLAB calculation shows that $\gamma(T_1^{(1)}) \approx 0.06315$ which explains the fast convergence and is maybe caused by the proportionality.

In contrast to the above situation, consider the DFSM-reordering for the root index 8.

$$B^{(8)} = \left(\begin{array}{rrr|rrr|rr} 0.6000 & 0.2995 & 0 & 0 & 0.00090 & 0.00005 & 0 & 0.00005 \\ 0.3990 & 0.7000 & 0.00090 & 0.0003 & 0 & 0 & 0.00004 & 0 \\ 0.0005 & 0 & 0.85000 & 0.1000 & 0.10000 & 0 & 0.00003 & 0 \\ \hline 0.0004 & 0 & \mathbf{0.14900} & 0.0996 & 0.24900 & 0 & 0.00003 & 0 \\ 0 & 0.0004 & 0 & 0.8000 & 0.65000 & 0.00005 & 0 & 0.00005 \\ 0.0001 & 0 & 0.00005 & 0 & 0.00005 & 0.60000 & 0.10000 & 0.19990 \\ \hline 0 & 0.0001 & 0 & 0.0001 & 0 & \mathbf{0.24990} & 0.80000 & 0.25000 \\ 0 & 0 & 0.00005 & 0 & 0.00005 & \mathbf{0.15000} & 0.09990 & 0.55000 \end{array}\right)$$

At first site, everything looks fine. But the iteration operator becomes

$$T_1^{(8)} \approx \begin{pmatrix} 0.0031 & 0.0033 & 1.1705 & 0.0000 & 0.0000 & 0.1124 & 0 & 0 \\ 0.0042 & 0.0044 & 1.5601 & 0.0000 & 0.0000 & 0.1499 & 0 & 0 \\ 0.0027 & 0.0027 & 0.9892 & 0.0000 & 0.0000 & 0.0019 & 0 & 0 \\ 0.0012 & 0.0009 & 0.4498 & 0.0000 & 0.0000 & 0.0004 & 0 & 0 \\ 0.0028 & 0.0031 & 1.0281 & 0.0000 & 0.0000 & 0.0012 & 0 & 0 \\ 0.0003 & 0.0003 & 0.0004 & 0.0002 & 0.0001 & 0.9990 & 0 & 0 \\ 0 & 0.0007 & 0.0002 & 0.0007 & 0.0002 & 2.3061 & 0 & 0 \\ 0 & 0.0002 & 0.0002 & 0.0002 & 0.0002 & 0.8453 & 0 & 0 \end{pmatrix}.$$

Now $\gamma(T_1^{(8)}) \approx 0.99554$, which explains the large number of iterations. The main difference between $T_1^{(8)}$ and $T_1^{(1)}$ is that the proportionality within the positive columns is lost. While the lower right boldfaced entries in $B^{(8)}$ cause column 6 to have some large elements, the marked entry in the upper left part adds large elements to column 3. This may cause the worse γ-value, since such an element does not exist in $B^{(1)}$.

Note that the situation becomes completely worse if the partitioning $S_1 = \{1,2\}, S_2 = \{3,4\}, S_3 = \{5,6\}$, and $S_4 = \{7,8\}$ is used. The number of iterations of IAD is 6225, of block Gauss-Seidel 2055, and for each reordering, the number of iterations exceeds 4000. It is a good and open question if there exists a fast converging reordering within the 40320 possibilities.

Stewart's NCD matrix

The following example is taken from [66]. It is again an NCD-matrix and its dimension is $n = 286$. The number of non-zeros is 1606, thus the matrix is sparse. Additionally, the matrix has a symmetric pattern without being itself symmetric. As the numbers of iterations for BFS and DFS become unacceptable, DFSM is analysed only.

An application of IAD, using a partitioning provided by XPABLO (standard configuration) needs 107 iteration steps to achieve the desired accuracy. A block Gauss-Seidel iteration, based on the same XPABLO partitioning needs 641 iterations without overlap and 494 iterations with an overlap of 3 elements. The number of the partitions XPABLO returned was 8.

The calculations using DFSM have been carried out on 5 partitions. The following plot shows the numbers of iterations needed to achieve the desired accuracy using no overlap. The three dashed lines show the iterations needed by the iteration methods as labeled. This is either block Gauss-Seidel without reordering (GS), Gauss-Seidel using the XPABLO reordering (XPABLO), and iterative aggregation/disaggregation using the same XPABLO ordering (IAD).

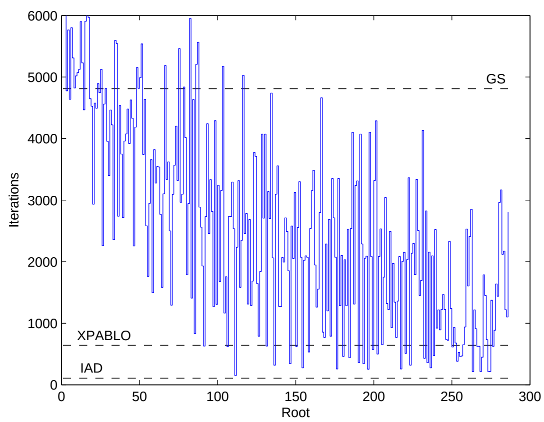

Figure B.1: NCD example: DFSM without overlap

The results for DFSM do not look satisfactory but sometimes the number of iterations are below the XPABLO-iteration.

This behavior seems to be completely accidental because the γ-values of the iteration operators are not as good as the speed of convergence might indicate. E.g., taking index/state 111 as the root, then the number of iterations becomes 149. But the γ-value of the (global) iteration operator T_1 exceeds 0.99, thus is badly large. Additionally, the columns of T_1 are not proportional. The acceptable number of iterations comes from the starting vector which seems to be optimal in this case.

The situation does not become better if an overlap of 3 elements is used.

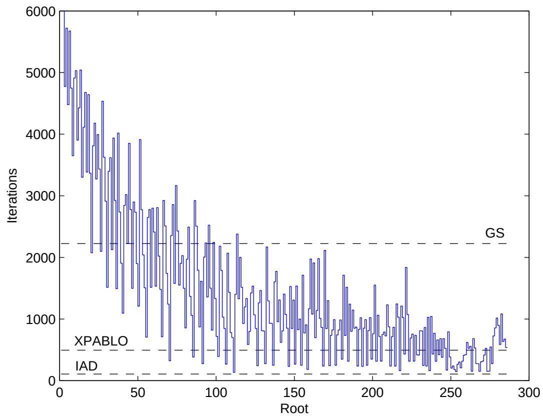

Figure B.2: NCD example: DFSM with overlap

Stewart's TCOMM matrix

The last example is also taken from [66]. The TCOMM matrix is a sparse matrix of dimension 666 having again a symmetric pattern. It can not be classified as an NCD-matrix; thus represents a "normal" problem.

A block Gauss-Seidel iteration using a 14 block standard partitioning takes 125 iterations without overlap and only 28 with an overlap of 6 elements.

An application of XPABLO to the matrix delivers a partitioning with 14 blocks. The block Gauss-Seidel iteration with the given partitioning needs 371 iterations without overlap and 364 with overlap. In this case, a reordering brings nothing. The number of iterations for IAD is also unsatisfactory, it takes 199 iterations. Thus, the TCOMM examples shows that a reordering is not always a good idea and that overlap must not repair a bad reordering.

As in the previous examples, the following figure shows the number of iterations vs. the chosen root in the non-overlap case. The dashed lines show the number of iterations for the alternative methods.

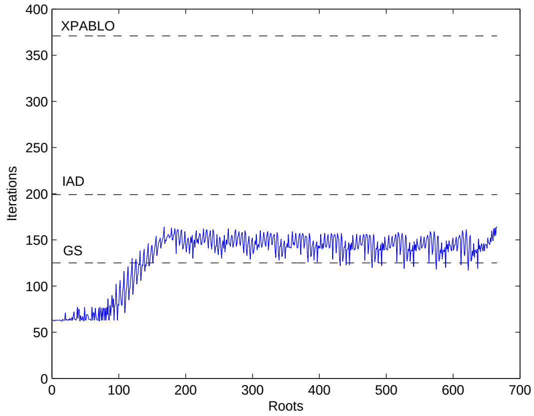

Figure B.3: TCOMM example: DFSM without overlap

Figure B.3 shows that it is possible to lower the number of iterations using DFSM. Although, it is likely to increase it. But the numbers of iterations stay within reliable bounds, which indicates that the problem is really "normal".

The situation changes a bit if overlap is used, as the following sketch shows.

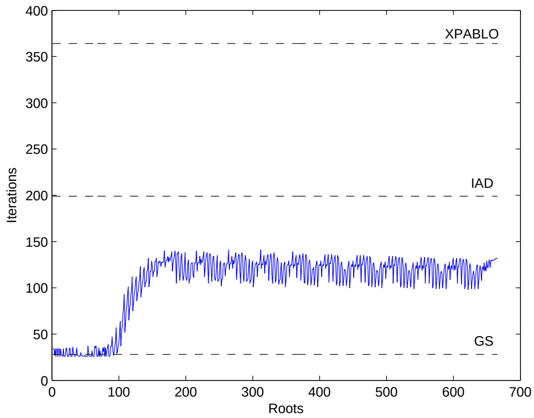

Figure B.4: TCOMM example: DFSM with overlap

Here, the standard partitioning is almost optimal and a bad reordering seems unavoidable. However, a reordering which takes only 26 iterations has been found.

Clearly, the results for IAD and the XPABLO-reordering are unacceptable. Thus the question is, if it is possible to configure XPABLO in such a way that the numbers of iterations can be lowered.

Conclusion

The simple examples show that the DFSM-reordering can deliver fast convergent iteration operators if the problems are not too hard. But it usually fails for the interesting problems which proves it to be unusable. But even XPABLO delivers some unsatisfactory results (within its standard configuration) on an easy problem.

The most remarkable thing is the good column proportionality for the reorderings of the Courtois matrix. Note that XPABLO delivers a reordering such that the iteration operator is nearly a projection. The convergence was obtained after 1 iteration.

So, column proportionality seems to be a key to fast convergence but it is not obtained easily.