MODELING SYSTEMIC RISK CONTRIBUTION USING COPULA

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Abstract

The financial crisis of 2008 highlighted the need for better regulation mechanisms for the stabilization of financial systems. The innovations in financial products and the evolution of financial market technologies and operations observed in the last decades do not only offer new opportunities but also imply **systemic risks**. They contributed to the establishment of interconnected financial systems in which the failure of certain single financial institutions (the so-called systematically important financial institutions: SIFI) can spread through **contagion effects**, thus causing the failures of other financial institutions and threatening the stability of a financial system.

The regulatory authorities reacted to this problem by designing and implementing various new risk management concepts and tasks, such as 1) the estimation of the potential financial loss suffered by the financial system if a given financial institution defaults, 2) the identification of SIFIs 3) the calculation of individual bank's contribution to resolution funds and 4) the elaboration and the performance of bail-in-operations. These tasks necessitate the development of **financial risk measures** that are based not only on individual losses in isolation, as are standard risk measures such as **Value-at-Risk** (VaR), but also consider **loss dependency**. One of the main tools proposed for this purpose is the CoVaR-method of Brunnermeier and Adrian [2011]. The CoVaR-method is based on the statistic CoVaR, which is defined as the VaR of one financial system conditional on the state of a given financial institution.

The main contribution of this thesis is the development of methods for the computation of CoVaR in a wide variety of stochastic settings. We derive, using **copula theory**, a general formula for CoVaR, which takes into account all information on the involved distribution. This allows us to consider not only the normal but also the extreme part of the assumed distributions as well as different types of dependency structure. We make some illustrative applications and related analysis. Also, using the theory of **elliptical distributions** we derive an expression of $CoVaR$ that is more accessible to financial practitioners. Both approaches allow us to consider not only Gaussian- but also **non-Gaussian distribution**. Furthermore, we highlight several inconsistencies in the $CoVaR$-method and suggest **alternative approaches**.

Contents

Chapter 1

Introduction

1.1 Background

The recent financial crisis revealed how financial systems are vulnerable to systemic risk. The structure of contemporary financial systems allows contagion effects across participants of the financial system. As we saw during the last financial crisis, the failure of certain financial institutions, the so called**systemically important financial institutions** (SIFIs), can have an adverse impact on an entire financial system and on the real economy, thus giving rise to a systemic crisis. This forced the regulatory authorities to rethink the ways in which financial systems should be regulated. The redefinition (or the adjustment) of policies aiming to assure the proper functioning and thus the **stability** of **financial systems** is indispensable. We classify these policies into three categories: **risk (or crisis) prevention**, **risk control** and **crisis management**.

1.1.1 Risk and Crisis Prevention

Under risk (or crisis) prevention we can group all policies aiming to prevent of adverse financial impacts **ex ante** and to mitigate risky situations. One example of risk prevention policy is the regulation No. 648/2012 of the European Parliament and of the Council of 4 July 2012 on OTC derivatives, central counterparties and trade repositories[1]. It requires certain classes of **over-the-counter** (OTC) derivatives contracts to be cleared through a **central counterparty** (CCP). This policy aims to reduce counterparty risk in OTC derivative markets. The idea behind this requirement is to mitigate counterparty risk in OTC markets by concentrating (or transferring) the risk

[1] http://eur-lex.europa.eu/legal-content/DE/TXT/?uri=CELEX%3A32012R0648

associated with OTC derivatives to CCPs.

1.1.2 Risk Control

Under risk control we group the policies aiming to **control** the risks taken by financial institutions in their **daily business**. These policies generally impose some restrictions or requirements on the **balance sheet** of financial institutions (balance sheet regulation). The main idea here is to ensure an equilibrium between the risk taken by financial institution and their capital (or risk absorbing capacity).

The balance sheet of a financial institution can be seen as a list of its sources of funds (liabilities) and uses to which these funds are put (assets)[2]. It represents, at a given time, an overview of the assets, liabilities, and **equity** a financial institution posses. Table 1.1 shows the balance sheet of an hypothetical commercial bank. For a detailed description of the items of this balance sheet we refer the reader to Mishkin and Eakins [2012], Chap. 17.

Assets	**Amount**	**Liabilities**	**Amount**
Reserves and cash	20	Deposits	400
		- Short-term	100
Government bonds	100	- Long-term	300
Loans	800	Other liabilities	400
- State and local government	100	- Short-term Financing	100
- Commercial and industrial	360	- Long-term bonds	300
- Real estate	200		
- Interbank	50	Borrowings	100
- Consumer	20		
- Other	70	**Total liabilities**	900
Other assets	80	**Equity**	100
Total	**1000**	**Total**	**1000**

Table 1.1: Bank balance sheet

[2]Financial institutions (financial intermediaries) generate fund by borrowing and by issuing liabilities (e.g. bonds and deposits). The generated funds are then used to finance assets (e.g. loans)

Note that the equity of a bank is defined as the difference between its assets and liabilities:

$$\text{Equity} = \text{Assets} - \text{liabilities}.$$

The primary way to regulate the balance sheet of financial institutions is to impose conditions on its **capital**.

Remark 1. *Note that the capital of a bank consist mainly but not only of its equity. Other liability instruments can also count as bank capital. We have in this context:*

$$\textit{Capital} \geq \textit{Assets} - \textit{Liabilities}.$$

One example of such a requirement imposed on bank capital are **capital adequacy ratios** (CAR):

$$CAR := \frac{C}{A} \geq \alpha$$

where C is the capital, A is the sum of all assets and α is a limit specified by the respective regulatory authority. In this context the balanced sheet of a bank can be reduced to the following representation:

Assets	**Liabilities**
Asset Type 1 (A_1)	
Asset Type 2 (A_2)	Capital (C)
.	
.	
.	Others liabilities
Asset Type n (A_n)	

Table 1.2: Reduced bank balance sheet

The CAR is often defined accordingly to the riskiness of asset types. This is done by assigning to each asset type (A_i) a risk weight (W_i).

$$CAR = \frac{C}{\sum_{i=1}^{n} A_i \times W_i}.$$

This principle is followed by the **Basel Committee on Banking Supervision**(BCBS). The BCBS was created by the Group of Ten (G-10) countries in the wake of the bankruptcy of the Herstatt Bank in 1974. Its mission is to contribute to the stabilization of the international financial system by defining guidelines and recommendations of best practice for financial regulation (the so-called **Basel Accords**). The Basel Accords are not legally binding for single countries, but represent guidelines and recommendations that need to be translated into law.[3] The first Basel Capital Accord is that of 1988 also known as Basel I. It is a **risk-based capital regulation** in the sense that it requires financial institutions to keep a minimum of capital, the so-called **regulatory capital** (RC), depending on the risk they take. This means that, the risks that a financial institution is allowed to take depends on its financial **capital**. Under Basel I the capital instruments are regrouped, depending on their **capacity to absorb losses** (quality) [4] into two categories: (i) Tier 1 capital (Core capital) and (ii) Tier 2 capital (supplementary capital). The total capital is defined as the sum of Tier 1 capital and Tier 2 capital.

- The Tier 1 capital consists of permanent shareholder's equity and disclosed reserves (retained earnings after tax).
- The Tier 2 capital consists of reserves, provisions, hybrid capital, subordinated debt with minimum maturity of 5 years.

Tier 1 capital are high-quality and consist primarily of equity. They are able to absorb losses in a **going-concern** basis, i.e on the assumption that the considered financial institution will still in business for an indefinite period whereas Tier 2 capital is supposed to absorb losses in a **gone-concern**, i.e. basis when the bank becomes insolvent.

Remark 2. *A going-concern financial institution has positive equity capital.*

In the context of the Basel accords the assets of financial institution are splitted into **banking book**, **trading book** and cash. The banking book contains assets that are assume to be held until the maturity (e.g. as loans). The trading book contains assets and instruments that are intentionally held for short-term resale or for **hedging** other instruments of the trading book.

Remark 3. *These trading book assets are typically valuated on a* ***mark-to-market*** *basis using only quoted market prices. while banking books could be valuated at their original cost.*

The Basel I focus **credit risk** arising from both, trading book and banking book. It required the ratio between **capital** of a bank and a weighted sum of

[3] For example, the Basel III is implemented in EU through Capital Requirements Directive IV (CRD IV).

[4] Equity is for the Basel Committee the preferred eligible capital because it is permanent and more reliable.

all its assets (RWA) to be equal or greater than 8% (see Basel Capital Accord [1988]). The corresponding CAR is then:

$$CAR = \frac{\text{Tier 1 capital} + \text{Tier 2 capital}}{RWA = \sum_{i=1}^{n} RWA_i} \geq 0.08$$

$$\text{with } RWA_i = E_i \times RW_i,$$

where n is the number of a bank's assets including **on-balance sheet** and **off-balance sheet** items excluding derivative items, E_i is the financial exposure associated with asset i and RW_i is the risk-weight associated with asset i.

Remark 4. *Off-balance-sheet items doe not appear on the (current) balance sheet of financial institution. However, they could generate a loss in the future, hence affecting the future shape of balance sheet. Example of off-balance-sheet items are options and guarantees.*

The assignment method of RW is defined by the BCBS in such a way to reflect the inherent risk (i.e. the probability of default and the **expected loss** in the case of a default) of the associated asset. In Basel I, they took only 5 values 0%, 10%, 20%, 50% and 100%. Table 1.3 shows a sample of risk weights for certain on-balance-sheet items as specified in Basel I. As we can see, cash and securities issued by governments of OECD[5] countries are considered to be risk free and have then a risk weight of zero. Loans to corporations have a risk weight of 100%. Loans to banks and government agencies in OECD countries have a risk weight of 20%. Residential mortgages have a risk weight of 50%.

Asset Type	**Risk Weight**
Cash, gold, loans to governments in OECD countries	0%
Loans to domestic public-sector entities	0%
Loans to banks in OECD countries	20%
Residential mortgages loans	50%
Loans to corporate-sector, consumer loans, real-estate investments	100%

Table 1.3: Risk weights for certain on-balance-sheet items.

Remark 5. *According to table 1.3, a bank that, has only exposures to OECD sovereign debt does not have to hold any regulatory capital ($RC = 0$), since the RW associated to loans to governments in OECD countries is zero.*

[5]The group of countries that are full members of the Organisation for Economic Cooperation and Development

The RWA of off-balance sheet instruments is generally calculated in two steps

1. the nominal amount E of an off-balance sheet instrument is transformed into an equivalent on-balance sheet loan (credit equivalent amount). This is done by multiplying the principal amount of off-balance sheet instrument by a predefined **credit conversion factor** (CCF).

2. the appropriate risk-weight is assigned to the resulting equivalent credit.

The RWA_i off an off-balance instrument i is thus computed by

$$RWA_i := E_i \times CCF_i \times RW_i.$$

Remark 6. *In practice, the financial exposure E_i associated with an asset i is estimated by the **exposure at default** (EAD). The formula for computing the RWA becomes,*

$$RWA_i = EAD_i \times RW_i.$$

Basel I also requires at least 50% of the required RC to be in Tier 1. This means that the **Tier 1 capital ratio**, which is given by

$$\text{Tier 1 capital ratio} := \frac{\text{Tier 1 capital}}{\text{RWA}},$$

must be at least 4% (Tier 1 capital ratio ≥ 0.04). For a more detailed understanding of the weighting system and the capital definition in Basel I, we refer to Basel Capital Accord [1996]. The Basel accord evolved considerably since 1988. Changes in the Basel accord usually appear in the form of amendments or releases. These changes in the Basel accords aim to **adapt** the regulation framework to changes in financial markets or to **react** to a financial crisis. For example, a partial amendment of the accord of 1988 was made in 1996 (see Basel Capital Accord [1996]). It required banks to additionally allocate capital to cover **market risk**, i.e. risk due to movements in market values, such as interest rate risk, equity position risk, foreign exchange risk and commodities risk. It also defined a new category of eligible capital, that was exclusively eligible to cover market risk only. This capital is called **Tier 3 capital**.[6]

The corresponding CAR was then given by

$$CAR = \frac{\text{Bank's capital}}{\text{Credit risk RWA} + \text{Market risk RWA}} \geq 0.08$$

[6]The Tier 3 capital consists naimly of short-term subordinated debt.

Remark 7. *In practice the risk capital for credit risk (CRC: credit risk charge) and that of market risk (MRC: market risk charge) are computed separately. And the total risk charge (TRC) is computed as the sum of CRC and MRC.*

The release of the Basel accord of 26 June 2004 (see Basel Capital Accord [2004]) also known as Basel II adopts the same philosophy as that of Basel I. It continues to apply a risk weights based CAR. However, it focuses not only on market and credit risk, but also considers **operational risk**, which is defined as the risk resulting from inadequate or failed workflow processes. It also changed the way as risk weights were calculated.

Note that one of the main problems in the Basel I accords is that the assignment method for the risk weights does not take into account the credibility of the borrower. For example, under Basel I, a bank that lent a given amount to a company with a good credit standing were obliged to hold exactly the same amount of regulatory capital as that he would if lent the same amount to a company on the edge of bankruptcy. Basel II has elaborated alternative methods in which the credibility of the borrower is taken into account (or modeled) by its **default probability** (PD).

Basel II provided three different approaches for the determination of risk weights for credit risk: 1) the standardized approach (SA), 2) the foundation internal rating based approach (F-IRB) and 3) the advanced IRB approach. In the standardized approach (SA), the default probability of borrower depend on ratings provided by external specialized financial institutions, the so called **credit rating agencies**. In the IRB approach the default probabilities of borrowers are based on a bank's internal rating system.[7] Figure 1.1 illustrates the link between rating and risk weights for exposures to countries, banks, and corporations under Basel II's standardized approach.

	AAA to AA−	A+ to A−	BBB+ to BBB−	BB+ to BB−	B+ to B−	Below B−	Unrated
Country	0	20	50	100	100	150	100
Banks	20	50	50	100	100	150	50
Corporations	20	50	100	100	150	150	100

Figure 1.1: Rating vs. Risk Weights (in %). Source: Hull [2012]

Another innovation in Basel II is the adoption of a three "Pillar"-structure in the formulation of regulatory policies. Through this structure, the Basel

[7] This need to be approved by relevant regulatory institutions such as the BaFin (Bundesanstalt für Finanzdienstleistungsaufsicht) in Germany.

Committee aims at integrating the three main aspects of financial risk management, namely the quantitative, qualitative, and market discipline (transparency) aspect in the regulation of the financial system.

1. Pillar 1 (quantitative aspect) provides the rule for the calculation of regulatory capital. It consists of similar risk capital ratios as Basel I and additionally considers operational risks.

2. Pillar 2 (qualitative aspect) provides principles for the supervisory review process. Following these principles a bank can estimate, using it own models, the capital that it needs to cover the economic effects of risk-taking activities and to secure the survival of its business on a going concern basis. This capital is called **economic capital**.

3. Pillar 3 (market discipline aspect) aims at promoting discipline and transparency in the financial system by calling on banks to disclose more information about the way they allocate capital and the risks they take.

Remark 8. *Under Basel II, the capital requirements for the banking book was generally higher than that for trading the book. Some financial institutions used this gap and developed practice to reduce their regulatory capitals while holding the same risks (**regulatory arbitrage**). They transfer for this purpose their banking book assets to the trading book. This is done by first transform via secularization techniques the considered asset (e.g. a loan) into a tradable asset (e.g. a bond) then distributes them to other financial institutions.*

As a reaction to the latest financial crisis the BCBS published in December 2010 a revision of the Basel accords, the so- called Basel III. As we can read in Basel Committee on Banking Supervision [2009],

"the objective of Basel III is to improve the banking sector's ability to absorb shocks arising from financial and economic stress, whatever the source, thus reducing the risk of spillover from the financial sector to the real economy".

This statement clearly asserts that the main focus of Basel III is the management of systemic risk. The main aspects of Basel III are:

- The redefinition of the concept of eligible capital[8]. Under Basel III, the Tier 3 capital is eliminated and the Tier 1 capital is splitted into **Common Equity Tier 1**(CET1) and **Additional Tier 1** capital and the relative amount of tier 1 is increasing from 4% to 6%. This increases the

[8]See e.g. CRR article 28

quality of regulatory capital. The augmentation of the regulatory capital is assured by the introduction of three new capital components such as the **Capital Conservation Buffer** , the **Counter-Cyclical Buffer** and the **systemic risk capital buffer for globally systemically important banks** (G-SIBs). The total risk capital is then:

$$\begin{aligned}\text{TRC} = \text{Tier 2} + \text{Tier 1} &+ \text{capital conservation buffer}\\ &+ \text{countercyclical capital buffer}\\ &+ \text{systemic risk capital buffer for G-SIBs}\end{aligned}$$

- The difference in the **treatment of systemically important institutions**. These are for example required to hold extra regulatory capital (systemic risk capital buffer).

- The introduction of liquidity requirements, namely the **liquidity coverage ratio** (LCR) and the **net stable funding ratio** (NSFR). The LCR aims to ensure that banks have enough amount of unencumbered High-Quality Liquid Assets (HQLA)[9] to withstand a period of 30-days of liquidity disruptions while the NSFR aims to ensure that sufficient funding is available in order to cover a period of at least one year.

- The introduction of requirements on leverage. The Basel III **leverage ratio** (LR) is defined by:

$$\text{Leverage Ratio} = \frac{\text{Tier 1 capital}}{\text{Exposure measure}}$$

  The exposure measure can be seen here as the sum of all a bank's exposures (or all assets).

Remark 9. *Contrary to others capital ratios the denominator of the LR is not the RWA.*

1.1.3 Crisis management

With crisis management we refer to procedures aimed at managing financial institutions, being in critical situation, in an orderly way in order to preserve financial stability. Their main objectives are:

- avoid contagion effects,

- protect client funds,

[9]HQLA can be rapidly into cash, with a limited loss in value, in financial markets.

- ensuring that the financial system's banking services remain uninterrupted[10].

The commonly used instruments or tools, for the management of financial institutions being in a critical financial situation are:

- credit,
- asset purchases,
- liquidity facilities,
- guarantees,
- and nationalizations.

The use of these instruments generate costs, which need to be financing. This can be done using taxpayer's money or via special fund called **resolution fund**. One of the prominent examples of crisis management operation are the Troubled Asset Relief Program (TARP) in the US and the **quantitative easing** (QE) program of the European central bank. The TARP was been signed into law in October 2008. The TARP provided to the US treasury a fund of $ 700 billion to **purchase** subprime and other mortgage backed securities from financial institutions in difficulty in order to stabilize the US financial system.[11] The aim of the QE is to increase the money supply in the European financial system by buying securities, such as corporate and government bonds, from financial institutions.

Remark 10. *Note that in the case of TARP and the QE the financial assistance given to financial institutions in difficulty is supported using* ***external funds*** *(provided by the governments and other financial authorities). This procedure corresponds to a* ***Bail-out****.*

The bail-out of private financial institutions by government are very unpopular, because they involve a massive use of taxpayer money to finance the losses caused by financial institutions.[12].

[10] This imposes a continuity of the essential (or critical) financial and economic functions of unsound or failing financial institutions.

[11] Updated information about the recipients, the amount disbursed, the amount returned, the revenues of the TARP program can be seen in the following web-page: https://projects.propublica.org/bailout/list.

[12] Bail-out can be interpreted by the taxpayer as if they have the obligation to participate in the loss (but not the gain) of financial institutions.

However, it is important to note that the cost generated by the collapse of real economies are generally very high[13]. Hence, to avoid the collapse of their financial system and their real economies some governments were forced to bail-out certain financial institutions. In fact, some financial institutions are considered so important (large and highly interconnected with other financial institutions) that their failure could potentially bring down an entire regional (domestic or global) financial system, thereby causing a high social and economic cost for governments and society as a whole. These financial institutions are commonly called '**too big to fail**'.

It is in this context that the German government has set up, in October 2008, a special fund called SoFFin (Sonderfonds Finanzmarktstabilisierung - Special Financial Market Stabilization Fund) for the financing of its bail-out actions. The SoFFin is administrated by the Financial Market Stabilisation Agency (**FMSA**) which was also in charge of the coordination of the bail-out actions taken by the German government.

Table 1.4: Bail-out recipients in Germany: Status: 31.12.2009, in bn EUR

Institution	**Guarantees**	**Financial Aids**
Aareal Bank	2.0	0.5
Bayern LB	5.0	0.0
Commerzbank AG	5.0	18.2
Corealcredit	0.5	0.0
Düsseldorfer Hypothekenbank AG	2.5	0.0
HSH Nordbank AG	17.0	0.0
Hypo Real Estate Holding AG	95.0	6.3
IKB Deutsche Industriebank AG	7.0	0.0
Portigon (WestLB AG)	0.0	2.6
Sicherungseinrichtungsgesellschaft deutscher Banken mbH	5.4	0.0

The table 1.4 shows the actions and the recipients of the German bail-out operation per 31.12.2009.

The bail-out of financial institutions can be seen as an effective way to contain the propagation of financial distress across a financial system and the real economy. However, there are at least three aspects of the governmental bail-out actions that pose a problem:

[13]Example of the effect of crisis in the real economic are: reduction of consumption and investment, rising unemployment and the shortfall in economic growth.

1. They are funded by taxpayer money.

2. The money used by the government for a bail-out action is generally transferred to other financial institutions and not to the real economic. As an illustrative example, the table 1.5 shows some financial charges assumed by the US government in the bail-out of the American International Group (AIG).

Table 1.5: Collateral amounts posted by AIG to its counterparties after it began receiving government assistance. Data source: www.aig.com

Counterparty	**Amount Posted in Billion $**	**Country**
Societe Generale	4.1	France
Deutsche Bank	2.6	Germany
Goldman Sachs	2.5	USA
Merrill Lynch	1.8	USA
Calyon	1.1	USA
Barclays	0.9	UK
UBS	0.8	Swiss
DZ Bank	0.7	Germany
Wachovia	0.7	USA
Rabobank	0.5	Hollande
KFW	0.5	Germany
JPMorgan	0.4	USA
Banco Santander	0.3	Spain
Danske	0.2	Danmark
Reconstruction Finance Corp	0.2	USA
HSBC Bank	0.2	UK
Morgan Stanley	0.2	USA
Bank of America	0.2	USA
Bank of Montreal	0.2	Canada
Royal Bank of Scotland	0.2	UK
Other	4.1	

Remark 11. *Table 1.5 also highlights the internationalization of financial transactions.*

3. The bail-out of financial institutions by their respective governments implicitly implies that the stability of one given financial system (or financial sub-system) depends on the ability of the respective government

to bail-out (or to finance) the respective systemically important financial institutions.[14] This poses a problem when the considered financial system encloses many governments such as the **Eurozone**. Because the stability of such a financial system can only be assured if all its sub-financial system are stable, this requires a harmonized stability mechanism. The lack of harmonized stability mechanisms can lead to a **political crisis**.

Recall that the Eurozone is a currency union with a common monetary policy under the responsibility of the European Central Bank (ECB) and with the Euro as common currency. One main aspect of the Eurozone is the liberalization of all legal financial transactions (capital flows and risk transfers) between financial institutions from different members countries. This promotes the establishment of huge financial system called **euro area**.

The EU recognized these problems and reacted by elaborating harmonized frameworks for the management of failing financial institutions across the euro area. The main frameworks are:

- The **Bank Recovery and Resolution Directive**[15](BRRD).[16].
- The single Resolution Mechanism (**SRM**) and the Single Resolution Fund (**SRF**).

The BRRD provides a minimal[17] harmonized legal framework for the management of failing financial institutions without (or less) contagion effects[18] and without resort to bail-out operations using public funds (resolution). It is builds on the following three main pillars (or tasks)

1. resolution planning: The scope of resolution planning is to develop resolutions strategy and identify the obstacles to resolution operation that need to be addressed in order to facilitate the resolution operation if need be.

2. Mitigation of financial institution's default. This is done using tools such as recovery operation (Sets of measures taken by a financial institutions

[14]From this perspective it is not a surprise why the financial crisis persists in Greece and not in Germany

[15]see http://eur-lex.europa.eu/legal-content/EN/TXT/?uri=celex:32014L0059

[16]The BRRD is implemented in the German law through the Sanierungs- und Abwicklungsgesetz - SAG (see www.gesetze-im-internet.de/bundesrecht/sag/gesamt.pdf)

[17]The BRRD allows state to use their own recovery and resolution strategy, insofar as it is not in conflict with the BRRD

[18]this imposes the maintain of the financial institution's critical functions

in financial dificulty in order to restore its busness operations to a normal condition) and early intervention[19].

3. resolution operation.

It provides to competent authorities legal instruments, which can be used for resolution purpose (**resolution tools**) . Examples of resolution tools are:

- the sale of business unit (Article 38 BRRD).
- the building of bridge institution(Article 40 BRRD). The is to transfer certain asset to a third-party financial institution.[20]
- separation of bank assets (Bad Bank, Article 42 BRRD). The aim of this instrument is to preserve the systematically relevant part of the failing financial institution whilst liquidating the non-systematically relevant part part.
- and bail-in (Article 43 BRRD). In fact, the BRRD stipulates that, should a bank fail, its shareholders, creditors and uninsured depositors should be first in line to assume losses if the bank gets into financial difficulty. This principle is called '**bail-in**'.

In the context of BRRD, the bail-in is used as instrument to ensure the sequential allocation of losses and the write down of the claims of shareholders, subordinated creditors, and senior creditors.

Remark 12. *Depositors below EUR. 100,000 are in general excluded from suffering losses, their claims are protected by national Deposit Guarantee Schemes, such as the Einlagensicherungsgesetz*[21] *in Germany or the Federal Deposit Insurance Corporation (FDIC) in the USA.*

The BRRD is implemented in the euro area through the SRM. Under the SRM, resolution operations are funded by the SRF. The SRF is a resolution fund that is financed through the **ex ante contributions** of banks and certain investment firms established in the countries subject to the SRM. The individual ex ante contribution is calculated, at least annually, by the SRB based on a method[22], which takes into account the **systemic importance** of the focused financial institution and a predefined target level of 55 billion EUR

[19]see Articles 27-30 BRRD
[20] such as the Erste Abwicklungsanstalt (EAA)
[21]URL: http://www.gesetze-im-internet.de/einsig/
[22]URL: http://eur-lex.europa.eu/legal-content/EN/TXT/?uri=CELEX%3A32015R0063

(1% of covered deposits of all credit institutions established in the countries subject to the SRM estimated on 2011 reported data).[23] Following the BRRD requirements, the bail-in instrument has to be used before any public funds or SRF.

Remark 13. *In case that the SRF is insufficient to finance a resolution operation additional* ***ex-post contributions*** *could be collected.*

The individual contributions are collected at a national level by relevant the **relevant national resolution authority** NRA and transferred to the **single resolution Board** (SRB)[24], which is an EU agency entrusted with the task to prepare resolution planning and to ensure the effectiveness of the resolution actions. To perform its tasks, the SRB uses two documents: 1) the **recovery plan** and 2) **the resolution plan**. The recovery plans are elaborated by the financial institutions, its show how they would act during financial turmoil in order restore their financial activities (financial continuity plan) and thus avoid liquidation and default. The resolution plans are elaborated by the resolutions authorities (the SRB and the NRA)[25]. It address how to handle a bank in resolution.[26]

Remark 14. *Recovery plans are concerned by time* ***before the default*** *or resolution.*

Remark 15. *Generally bail-in operations (resolution operations in general) take place when the financial institution is still balance sheet solvent. The starting time and the magnitude of a bail-in operation is decided, based on some preamble analysis, such as the estimation of the potential financial loss suffered by the financial system if the considered financial institution fails and the impact of the planed bail-in operations on the corresponding financial system. For example, the*

Article 44(3) of the BRRD gives resolution authorities the discretion to exclude or partially exclude certain bail-in-liabilities from the bail-in-operation (resolution), in exceptional circumstances, e.g. in the case that the assumed bail-in-operation could lead to a systemic risk.

[23]The data necessary for the calculation are reported to the SRB via the relevant national resolution authority

[24]https://srb.europa.eu/

[25]Following Articles 10-14 of the BRRD resolution plans are elaborated by national resolution authorities under the supervision of the SRB

[26]To facilitate their work, the resolution authorities can require financial institutions to submit information or to suggest ways in which they could be resolved.

The capital generated by a bail-in operation may be insufficient to absorb a loss entirely. The effectivity of a bail-in operation depends thus on the loss absorbing capacity of the involved **Bail-in liabilities** (i.e. the set of liabilities that are eligible to be written down or converted into equity by a bail-in operation). That is the motivation behind the definition of two ratios:

- The **minimum requirement of own funds and eligible liabilities**(MREL) for all financial institutions in the European Union[27]
- and the **total loss-absorbing capacity** (TLAC) for all Global Systematically Important Banks.

The aim of these two measures is to require financial institutions to maintain a minimum amount of liabilities (**bail-inable liabilities**) that should be sufficient to adsorb losses in case of a bail-operation.

There are in the financial market some securities that behave as eligible liabilities, but whose management is not assumed by a resolution authority but by their respective contract terms. The most prominent example of such instruments is the **contingent convertible bond** (CoCo-bond). Traditionally, convertible bonds are usual corporate bonds where the investor has the **right** but **not the obligation** to convert the bond into shares.[28] A CoCo-bond is different in that the conversion of the bond into equity or the write-down of the bond face value is trigger **automatically** when a certain contractual predefined **trigger conditions** is satisfied.[29] The trigger conditions are typically defined based on regulatory capital ratios such as the Common Equity Tier 1 (CET1) ratio. In this context CoCo-bonds can be see as a supplementary capital that adjust (increase) the capital of financial institutions in difficulties[30] in order to allow them to meet the regulatory capital ratios. furthermore, under CRD IV CoCo-bonds are allowed to count as Additional Tier 1 this means that CoCo-bond can account up to 1.5% of a bank RWA.

Remark 16. *Note that, for the CoCo-bonds, the conversion or the haircut takes place when the bank is still a* ***going concern*** *(i.e. has positive equity capital), while a bail-in operation takes place when the bank is almost collapsed.*

[27] Article 45 BRRD

[28] Typically the investor chooses to convert the bond into share when the stock price is high

[29] A CoCo-bond can have more than one trigger condition. In this case the conversion or the write-down is triggered when at least one condition is satisfied

[30] when CET1-ratio falls below a certain level

1.2 Motivation and Contribution

1.2.1 The need for Macro-Prudential Regulatory Policies

One of the main gaps in the pre crisis financial stability policies was that the problem of financial stability had been considered only from a micro perspective (**micro-prudential regulation**). The regulatory authorities tried to ensure the stability of **entire financial system** by reducing the probability of default of **individual financial institutions in isolation**.

This was incorrect. The evolution and internationalization of financial markets and financial services has contributed to the establishment of a globally interconnected and **partially non-regulated**[31] financial system concentrated around a few **big financial institutions**. As an illustrative example, Figure 1.2 shows how the **primary-secondary market design** for government debt is concentrated around a few financial institutions known as **primary dealers**[32], that have the exclusive authorization (privileges) to act as initial buyer for securities issued by the government.

As observed in the last crisis, the links between financial institutions provide a channel through which individual risks or failures can spread across the financial system. This is the macro nature of financial risks (**macro-financial risk**) that was ignored and that the regulatory authorities need to face now in order to stabilize the modern financial system.

In this context, the stability of the financial system can only be assured by regulatory policies that also take into account the potential contagion risk resulting from the interactions of financial institutions within the financial system. Such regulatory policies should aim at meeting the following two objectives:

1. make the failure of individual financial institutions less likely

2. reduce the impact of the failure of a single financial institution on the financial system (reduce the contagion effect).

Direct consequences of the last financial crisis are the measures adopted by governments and regulatory institutions to address the problem of the management of systemic risks. Some important measures are:

[31] Since OTC market was not regulated at this time.

[32] The main activity of a primary dealer is to buy government securities in the primary market and to resell them in a secondary market (typically over-the-counter) to other financial institutions

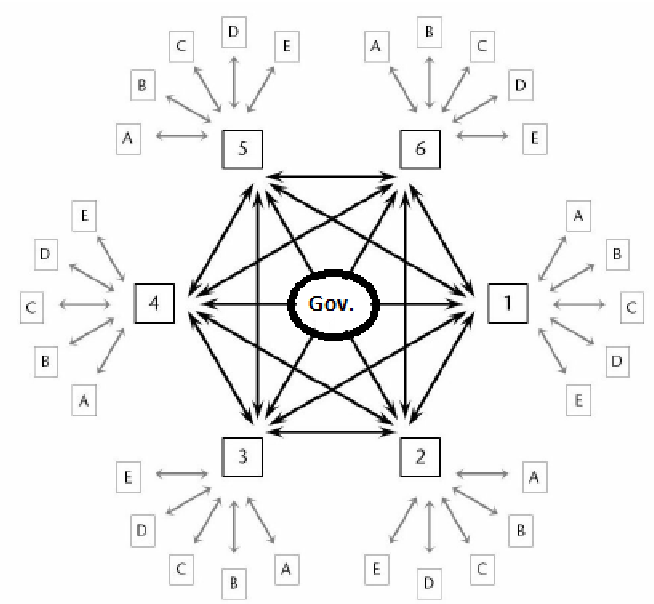


Figure 1.2: Schema of the primary-secondary market system. The government is in the middle and acts as the primary issuer of securities. The dealers are represented by the large boxes containing number. End-users are symbolized by smaller boxes containing letters. The bold arrows represent the primary market and the light arrows represent the secondary market.

1. The treatment of financial institution depending on their systemic importance. three categories of SIFIs are defined for this purpose: 1) global systemically important financial institutions (**G-SIFI**), that is a financial institution whose failure could have a negative impact on the **global** financial system , 2) domestic systemically important financial institutions (**D-SIFIs**) and 3) non systemically important financial institutions (**N-SIFIs**). The systemic importance of a financial institution depends on the effect that its failure could have on a financial system. This can be estimated by analyzing its size, the nature of its activities and the contracts it has entered into with other financial institutions (i.e. its degree of inter-linkage with the rest of the financial system)[33]. Global systemically important banks (G-SIBS) are, due to their system relevance, subject to stricter regulatory rules than other banks. They are for example required to hold a certain amount of bail-in eligible liability (Total Loss Absorbency Capacity: TLAC). This measure aims to ensure an effective bail-in operation in case it is needed.

[33]The Basel Committee uses a scoring methodology to determine which banks are G-SIBs (see Basel Committee on Banking Supervision [2014])

2. The elaboration of resolution plan. Note that, following BRRD the resolution plans of a given financial institution has to elaborated by the respective resolution authority based on an **analysis of the effect of the failure of this institution on the financial system**.

3. The requirement by some governments and financial authorities for financial institution to contribute to the funding of a**Resolution Funds**, for example the **Financial Crisis Responsibility Fee** in the USA and the Single Resolution Fund in the countries subject to the SRM.

4. The requirement standard over-the-counter (OTC) derivatives contracts to be cleared through an eligible central counterparty (**CCP**)[34]. CCPs manage their financial risk by requiring their members to provide adequate amounts of **collateral** (in the form of variation and initial margin) and to make contributions to a so-called **default fund** (or guarantee fund). Default funds are used by CCPs to absorb losses that cannot be covered by the collateral posted by a defaulting CCP member. Doing this, the CCP distributes the losses of defaulting members among the non-defaulting members. For more details on the functions CPPs, we refer the reader to Loader [2002].

   Remark 17. *The use of CCPs contributes to the stabilization of the financial system by managing the financial system losses due to the default of OTC market participants and by simplifying and increasing the transparency of the derivatives market transactions. However, they are, because of their size and function, systematically important.*[35]

Remark 18. *The points 3) and 4) are in some sense methods to mutualize losses from individual defaults. In this context, the risk associated to an individual financial institution should not only depend on its idiosyncratic risk but also on the idiosyncratic of other financial institutions in the same financial system as well as their interdependency (systemic risk contribution with the CCP as financial system).*

An effective estimation of systemic risk contribution in general and in particular an understanding of the way how factors such as the size and interdependency of financial institutions in distress can influence the financial system, is preeminent for solving the problem posed by the implementation of the above measures. The main problem here are:

[34] By the European Market Infrastructure Regulation (EMIR)

[35] Since, the failure of a CCP could leads to a significant systemic disturbance.

- The identification of the systemically important financial institutions (in particular, G-SIBs, D-SIBs and N-SIBs)

- The estimation of individual contribution to a mutual default fund.

This necessitates the definition and computation of new types of measures of financial risk that are able to estimate systemic risk contribution.

This problem poses a new challenge for academics and regulatory institutions, since the common risk measures such as Value-at-Risk (VaR) and Expected Shortfall (ES) only focus on one institution in isolation, thus ignoring the interdependence between financial institution and the related possible contagion effect.

1.2.2 $CoVaR$-Method as a Tool for the estimation of Systemic Risk Contributions

The $CoVaR$-method of Brunnermeier and Adrian [2011] is the most used tool for the analysis of systemic risk. It builds on the term $CoVaR$. This is defined as the Value-at-Risk (VaR) of a financial system conditional on a given state of the considered single financial institution.

Contrary to the traditional financial risk measures, e.g. VaR and ES[36], $CoVaR$ not only involves variables characterizing the univariate behavior of the considered financial institution in isolation (e.g. its loss) but also variables characterizing the univariate behavior of the entire financial system as well as variables characterizing the joint behavior of the considered financial institution and the financial system (e.g. the correlation coefficient between their losses). This allows $CoVaR$ to integrate the interdependency structure (or the link) between the considered financial institutions i and the financial system s, thus enabling it to describe the systemic risk contribution among financial institutions.

However, the calculation methods proposed so far for the estimation of $CoVaR$ are restrictive. Brunnermeier and Adrian [2011] for example adopted a statistical approach (i.e. no closed form or analytical formula) which is based on normal linear quantile regression (cf. Koenker and Bassett [1978]), Jäger-Ambrożewicz [2010] developed a closed formula for the special case where the losses of the financial institution and that of the financial system in focus are modeled by a bivariate normal distribution.

[36]These financial risk measures only consider variables characterizing the financial institution in isolation. For this reason they are called **micro-risk measures**.

The estimation methods cited above have their relative advantages and disadvantages but they share the common restriction that both impose a bivariate normal distribution as a stochastic model. It is well known that the bivariate normal distribution can lead to difficulties relative to the modeling of the single marginal variables of a multi-variate stochastic variables as well as the respective joint stochastic behavior (or **dependence structure**).

In fact, the bivariate normal distribution imposes the univariate normal distribution as a model for univariate margins and the linear correlation coefficient as the unique dependence parameter. Based on the fact that the linear correlation coefficient measures only linear dependence and is controlled by movements **around the mean** of the distribution while movements in the extreme are neglected and considered as abnormal, we think that the linear correlation coefficient is not the appropriate measure of dependence for the analysis of systemic risk contribution.

Note that, in general, institution defaults and systemic crisis can be considered as extreme events. Indeed, the default that produces the contagion effect corresponds generally to a shock (large loss) relative to an expected loss. This can be characterized by an extreme value which appears in the tails of the corresponding loss distributions.

Hence, systemic risk contribution is particularly concerned with the probability of simultaneous large losses and hence with the **tail of the loss distributions**. Therefore, the analysis of systemic risk should be based on models which are able to specify how extreme losses are interdependent.

In this thesis we propose formulas for computing $CoVaR$ in a general stochastic setting. Doing this we provide a flexible framework for an effective analysis of systemic risk contribution. In chapter 4.1, a general formula for the computation of $CoVaR$ in a wide stochastic setting is provided. This formula is derived using the theory of bivariate Copula functions, which is introduced beforehand in chapter 3.

Bivariate copula functions represent a class of bivariate distribution functions defined on the unit square $[0,1]^2$ with uniformly standard distributed margins. The power of copulas is their ability to describe the dependence structure of bivariate random vectors separately of their marginal distributions. In particular the theorem of Sklar [1959] allows to decompose any multidimensional joint distribution function into its univariate marginal distributions and a copula which models the entire joint behavior on a quantile scale. These features give the copula the ability to represent joint distribution with univariate margins of different types and to describe dependence structure precisely in any region of joint distribution. It is for this reason that we use copula in order to model precisely extreme co-movement and hence contagion effects. By connecting the $CoVaR$ concept to copula's theory we develop an analytical formula allowing the analysis and the computation of $CoVaR$ for a more general stochastic setting than only the normal distribution setting. We apply our formula to the Gaussian and the non-Gaussian setting. As non-Gaussian setting we consider the class of elliptical and Archimedean copulas as well as the convex combination of copulas. In chapter 6, we consider the $CoVaR$ under elliptical distribution. The assumption of elliptical distributions, as model for the computation of $CoVaR$, represents a good compromise between the need of pragmatical and practicable risk measure for the definition of regulatory rules on the one hand and the need of flexible and consistent risk measure for the analysis of systemic risk contribution on the other hand.

Both approaches allow to consider not only the normal dependence models, especially those which are appropriate for the modeling of the simultaneous (tail) behavior i.e. model with positive tail dependence coefficient.

One another main contribution of this thesis is the critical analysis of the $CoVaR$-method. By providing an example in which the $CoVaR$-method do not takes into account of the effect tail dependence, we show that the $CoVaR$-method as provided by Brunnermeier and Adrian [2011] is in general not sensible to tail effect. Also, we highlight the fact the $CoVaR$-method is not coherent with the phenomena of contagion effect. We propose alternative models that cover these gaps.

Chapter 2

Modeling Systemic Risk Contribution

An effective modeling of systemic risk contribution requires an understanding of how the modern financial system works, how individual financial institutions and different financial risks are related and how their interaction can lead to systemic crises. For more details about this topic we refer to Claessens and Forbes [2014]. In this chapter, we will give, from a quantitative risk management perspective, a basic understanding of the notion of systemic risk contribution. Then, based on this, we will introduce the $CoVaR$-method as a model for the analysis of systemic risk contribution.

2.1 Financial System and Systemic Risk

The quantitative analysis and modeling of systemic contribution requires an understanding of the notion of a **financial systems**. It is thus important to precise some important notions that are related to financial system and systemic risk.

A financial system can be seen as a network of institutions in which funds and financial services are moving across time and space from one institution to an another institution (e.g. from a bank to a firms). For a detailed analysis of the notions of a financial system, we refer the reader to Schinasi [2005], Neave [2010] and Mishkin and Eakins [2012].

One of the main aspect of the modern financial system is the existence of a huge number of direct-[1] and indirect-[2] contractual linkages between financial

[1]e.g. between buyer and seller of CDS

[2]e.g. when many financial institutions undertake correlated or common risky investments exposing therefor themselves to common risk factors

institutions. This is illustrated in figure 2.1. Hence, financial institutions, that consist a financial system, have exposures to each other. So, in case of the default of one financial institution the other financial institutions could be negatively affected. This provides a channels through which individual failures can spread across the financial system and thus undermine financial stability.

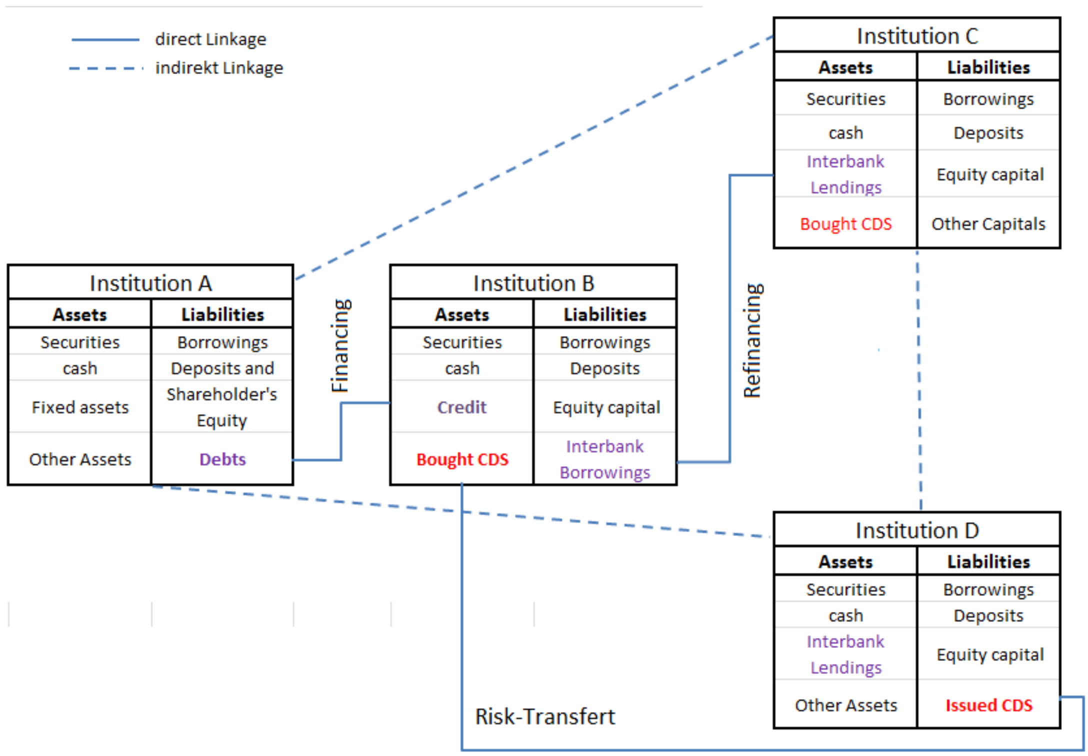


Figure 2.1: Financial linkages

The term systemic risk refers in general to the risk of collapse of an entire complex system as a result of the actions taken by the individual components that comprise this system. Systemic risk in financial systems can be defined as the risk that an initial default by one financial institution threatens the stability of the whole financial system by causing, via different propagation mechanisms, the default of other financial institutions of the system. This description corresponds to the view of systemic risk in a **narrow sense** (cf. De Bandt and Hartmann [2000] and De Bandt, Olivier and Hartmann, Philipp and Peydró, José Luis [2012]). Systemic risk in the **broad sense** is caused by a **common shock** to many financial institutions or an entire financial system. In this thesis i assume the systemic risk in a narrow sense .

The initial default that give rises to systemic risk is called **systemic event** (from a narrow sense perspective). Such a systemic event was the failure of Lehman Brothers on 15th. september 2008, which is assumed to be the systemic event of the recent financial crisis.

Assumption 1. *We consider here the systemic risk in the narrow sense. That is, we assume that systemic risk is caused by an initial default by one financial institution, that then spread in the whole financial system.*

The mechanisms by which the failure of a financial institution spreads in the whole financial system is referred to **contagion effect** (cf. e.g. Allen and Gale [2008] or Allen and Gale [1998]).

Contractual or economical linkages between financial institutions are an important transmission channels of contagion but not the only one. In fact, they are two main classes of contagion channels. The first is the **fundamental channel**. It the part of shock transition that could be completely explained using economic elements such as changes in ECB interest rates or the price of energy. The second is so called **informational channel**. It is the part of shock transition that can not be explained using fundamental economic analysis. It can be seen as the result of actions taken by economic participants or agents (such as fund manager or broker) based on a subjective (emotional) interpretation of financial news. For example, if one systemically important financial institutions defaults, the market will generally expected a negative trend in the whole financial system. This could have the following consequences.

1. A reduction in depositors confidence. As a consequence several severs could close out their accounts causing thus a **liquidity crunch** for the financial institution.

2. The **decreasing in ratings** of financial institutions. As a consequence, the funding cost of financial institutions will increase. This might leads to a situation in which the financial institutions are not able to satisfy their financial obligations and hence become distressed.

Remark 19. *It is important to note that informational contagions leads in general to a liquidity risk and market risk, while fundamental contagions lead in general to a (counterparty) credit risk.*

2.2 Basic Stochastic Model for Systemic Risk

The existing quantitative methods for the analysis of systemic risk can be summarized into two main approaches.

The first approach compares the linkages between financial institutions (or markets) during a relatively **stable period** with the linkages during an **unstable period**. This approach is supported by many empirical studies. For example, Dobric et al. [2007] show that there are differences in the dependence structures of stock returns in **bull and bear markets**. There are also many studies that assert that the dependence between financial returns increases as the market is going down.

The second approach consists to model the direct linkage between financial institutions using network theory. The direct contagion mechanisms are then analyzed via simulations or stress-tests (cf. e.g. Cont et al. [2013] or Reyes and Minoiu [2011]).

The models considered in this thesis follow the first approach. We denote by i the financial institution in focus and by s the corresponding financial system. The loss taken by i and s are modeled following McNeil et al. [2005] (Section 2.1) respectively by the positive random variables L^i and L^s, which are defined on a probability space $(\Omega, \mathcal{F}, Pr)$. The distribution functions of L^i and L^s are denoted by F_i and F_s respectively. The interconnectedness of the financial institution i to the financial system s is modeled by the degree of **dependence** between F_i and F_s. This is done by assuming that the random variables L^i and L^s are stochastically dependent and that their joint behavior is described by a bivariate joint distribution function.

Assumption 2. *We consider only random variables which have strictly positive density function. So, if a bivariate joint distribution function is considered, it is assumed that it has a strictly positive density and that its marginal distributions have strictly positive densities.*

Due to this assumption all distribution functions considered in this thesis are assumed to be absolutely continuous and strictly increasing.

2.3 Systemic Crisis and Financial Extreme Events

Systemic crises are closely related to two kinds of extreme events. First, the default of one systemically relevant financial institution (financial default) and second, the propagation of failures across financial institutions after the failure of one system relevant financial institution (contagion effect).

2.3.1 Financial Default and Extremes Events

From a quantitative risk management view, the failure of one financial institution is the consequence of the realization of a large loss. Intuitively, such an event can be interpreted as an extreme value appearing in the upper tail region of the corresponding loss distribution. Mathematically, the notion of a financial distress can be characterized using special function called **financial risk measure**.

Definition 1. *Let $\mathcal{L}$ be the class of losses defined on a probability space $(\Omega, \mathcal{F}, Pr)$. A mapping $\mathcal{R} : \mathcal{L} \to \mathbb{R}$ is called a* ***monetary measure of risk*** *if it satisfies the following conditions for all $L_1,\ L_2 \in \mathcal{L}$*

1. *Monotonicity: If $L_1 \leq L_2$ a.s., then $\mathcal{R}(L_1) \leq \mathcal{R}(L_2)$*

2. *Cash invariance: If $L \in \mathcal{L}$ and $l \in \mathbb{R}$ then $\mathcal{R}(L + l) = \mathcal{R}(L) + l$*

Then, from the cash invariance property is motivated by the interpretation of $\mathcal{R}(L)$ as regulatory capital. It suggests that the financial risk measure associated to a loss L can be adjusted by an amount l by adding or subtracting a deterministic quantity l to L.

A loss L such that $\mathcal{R}(L) \leq 0$ is called **acceptable** in the sense that a financial institution whit loss L is not required by the regulator to keep any regulatory capital. The set of acceptable losses associated with a risk measure $\mathcal{R}$ is given by

$$\mathcal{A}_{\mathcal{R}} = \{L \in \mathcal{L} \mid \mathcal{R}(L) \leq 0\}.$$

That is, a loss L is **acceptable** with respect to the risk measure $\mathcal{R}$ if $L \in \mathcal{A}_{\mathcal{R}}$.

Let L be a non-acceptable loss i.e. $L \notin \mathcal{A}_{\mathcal{R}}$. By adding to L a positive cash amount of $\mathcal{R}(L)$, we define an adjusted loss (cf. McNeil et al. [2005], Section 6.1)

$$\tilde{L} := L - \mathcal{R}(L).$$

Then, from the cash invariance property of monetary risk measures we have

$$\mathcal{R}(\tilde{L}) = \mathcal{R}(L - \mathcal{R}(L)) = \mathcal{R}(L) - \mathcal{R}(L) = 0,$$

so that $\tilde{L} \in \mathcal{A}_{\mathcal{R}}$. Hence, one can interpret $\mathcal{R}(L)$ as the minimum amount of capital that a financial institution with a loss L should keep as regulatory capital. In this context the monotonicity property implies that financial institutions with higher losses need higher risk capitals.

Recall that, from a purely economic point of view, financial distress may be defined as a situation where a financial institution's operating cash flow are not sufficient to satisfy current obligations (cf. e.g. Ross et al. [1999], A7 3.1). From a quantitative risk management perspective, given a monetary measure of risk RC, we can define distressed financial institutions as follows.

Definition 2 (Distressed financial Institutions)**.** *Let L be the loss of the financial institution B. Let RC be the regulatory capital associated with the loss L. Let l be the realization of L at the time t. We say that the financial institution B is in distress at the time t if l is greater than the associated regulatory capital RC, i.e.*

$$l \ > RC.$$

If we assume that the regulatory capital RC is determined by a risk measure such as the Value-at-Risk (i.e. we assume that RC:=Value-at-Risk), then we say that the financial institution B is in distress at time t if

$$l > \text{Value-at-Risk}. \tag{2.1}$$

2.3.2 Contagion Effect and Extreme dependence

Contagion effect and systemic risk are closely related to the mathematical concept of extreme dependency.

Remark 20. *During the crisis, (many) asset prices, independent of their respective nature, tend to move in the same direction. this increases the probability that single financial institutions fail together with the whole financial system or that a large number of financial institutions fail simultaneously. This phenomenon is well captured by the figure 2.2. It shows three periods of financial crisis (1930-1940, 1980-1994 and 2008-2014). Each period is characterizing by a concentration of massive simultaneous financial institution defaults.*

In fact, as observed by many authors e.g. Chan-Lau, Jorge A. ; Mathieson, Donald J. ; Yao, James Y. [2002], the dependence between asset returns are often higher during the crisis than in normal situation. This can be explained by the fact that in normal situation the dependencies between assets prices are a reflection of their fundamental properties (for example similar assets tend to move in similar ways). But, this situation changes in a crisis or more precisely after a shock. The behavior of the assets prices is in this situation much more affected by the measures taken by the financial market participants as

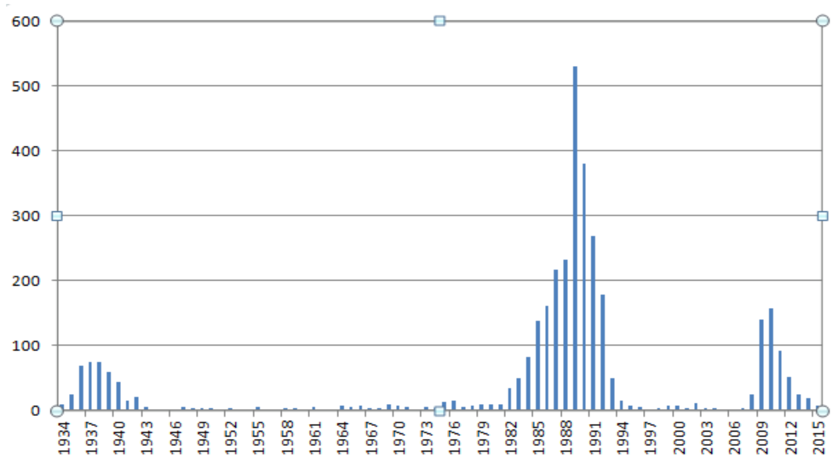


Figure 2.2: Bank Failures in the United States, from 1934 to 2015. (Source: Federal Deposit Insurance Corporation (FDIC)

reaction to the shock. These can be based on fundamental or informational point of view. A typical example, that describes how the actions taken by financial market participants can change the dependence between asset prices, is described in Brunnermeier and Pedersen [2009]. The authors showed how firesale[3] can create new forms of dependence between assets held by similar investors by adversely impacting on the asset prices of other financial institutions (see figure 2.3).

In this context, an increased in the dependency can be see as feature of financial market turmoil and financial crisis. Following this, Forbes and Rigobon [1999] defined contagion as a significant increase in financial market linkages after a shock to one market (or a group of markets). This definition means that, contagion effects are the consequence of a significant increase in the **interdependence** of financial institutions after a systemic event. In the same sense , Brunnermeier and Adrian [2011] argue that "*the main idea of systemic risk measurement is to capture the potential for the spreading of financial distress across institutions by gauging the increase in tail co-movement*".

From a probabilistic point of view, a contagion effect may be seen as a phenomenon in which the failure of one financial institution increases the probability of the failure of other financial institutions. This can be characterized by a raise of the probability of simultaneous large losses. Therefore, in the analysis of contagion effects and hence of systemic risk contributions we are particularly interested in the dependence structure in the upper tails of joint

[3]This can happen because the considered financial institutions suffers from a lack of funding liquidity

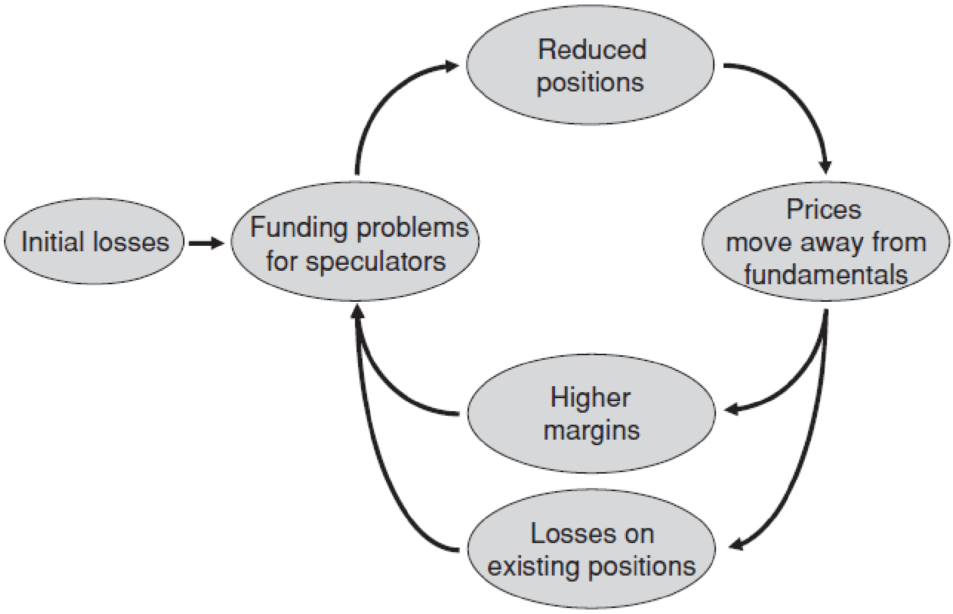


Figure 2.3: Loss spiral: Source Brunnermeier and Pedersen [2009].

loss distributions. Notice that, if losses are independent in the tail of the joint loss distribution, large losses and hence the failures of financial institutions appear to occur independently of each other. Hence, there would be no contagion effect and the systemic risk contribution of the respective element of the system would be equal to zero.

It is thus important to model the extreme (or tail) dependence when analyzing systemic risk contribution. In the next section, we present the usual approaches for the modeling of tail dependence.

2.3.3 Measuring the Dependencies of Extreme Events in Finance

There are two different approaches for modeling extreme dependence. The first approach consist to describe extreme dependence by considering a "conditional" or "local"-version of an existing dependence measure. The second consist to describe the dependence in the tail region of the assumed distribution using conditional probability.

Conditional Correlations Coefficient

The Conditional Correlations Coefficient is the typical example of the first approach. The idea behind conditional correlation is to measure the correlation between financial institutions conditioned on certain extreme events, for example extreme losses (co-exceedance correlation).

Definition 3 (cf. Malevergne and Sornette [2006], Definition 6.2.1)**.** *Let X and Y be two real random variables and C a subset of $\mathbb{R}$ such that $Pr\left(Y \in C\right) > 0$. The conditional correlation coefficient ρ_C of X and Y conditioned on $Y \in C$ by definition, can be expressed as*

$$\rho_C = \frac{Cov\left(X, Y|Y \in C\right)}{\sqrt{Var\left(X|Y \in C\right) \cdot Var\left(Y|Y \in C\right)}}. \tag{2.2}$$

So, by defining $C := [v, +\infty)$, it is possible to examine whether a model is asymptotically dependent by making v tend to ∞. For example, in the case that the variables X and Y have a bivariate normal distribution with an (unconditional) correlation coefficient ρ, we have the following theorem.

Theorem 1 (Boyer et al. [1999], Theorem 1)**.** *Consider a pair of bivariate normal random variables X and Y with variances σ_X^2 and σ_Y^2 respectively, and covariance σ_{XY}. Set $\rho = \frac{\sigma_{XY}}{\sigma_X^2 \sigma_Y^2}$, the unconditional correlation between X and Y. Consider any event $Y \in C$, where $C \in \mathbb{R}$ such that $0 < Pr\left(Y \in C\right) < 1$. The conditional correlation ρ_C between X and Y, conditional on the event $Y \in C$, is equal to*

$$\rho_C := \frac{\rho}{\sqrt{\rho^2 + (1-\rho^2)\frac{Var(X)}{Var(X|Y \in C)}}}. \tag{2.3}$$

Malevergne and Sornette [2006] show that for large v the formula (2.3) may be transformed into the following closed formula (cf. Malevergne and Sornette [2006], Formula (6.3)):

$$\rho_C \sim \lim_{v \to \infty} \frac{\rho}{\sqrt{(1-\rho^2)}} \frac{1}{|v|}. \tag{2.4}$$

This slowly goes to zero as v goes to infinity. That is, the bivariate normal distribution is asymptotically independent and is therefore not a good model for the analysis of systemic risk contributions. A detailed theoretical background about conditional correlation in particular and conditional dependence in general could be found in Malevergne and Sornette [2006] Chapter 6.

Tail dependence coefficient

The main idea of the measurement of extreme dependence via the coefficient of tail dependence[4], is to describe the dependence in the tail region of the distribution through a conditional probability. Let X and Y be random variables

[4]For more details about the measures of dependence between a pair of random variable we refer to Nelsen [2006], Chapter 5 and Balakrishnan and Lai [2009], Chapter 4

with the joint distribution function H and univariate marginal distribution functions F and G, respectively. If $Pr(X > x) > 0$, then the dependence in the upper tail region of the distribution may be expressed by

$$Pr(Y > y|X > x) = \frac{Pr(X > x,\ Y > y)}{Pr(X > x)}. \tag{2.5}$$

By replacing in equation (2.5) x and y by their α-quantiles $F^{-1}(\alpha)$ and $G^{-1}(\alpha)$ respectively, we obtain the tail dependence measure $\chi(\alpha)$ (cf. Coles et al. [1999]).

$$\chi(\alpha) = Pr\left(Y > G^{-1}(\alpha)\,|X > F^{-1}(\alpha)\right) \tag{2.6}$$

$\chi(\alpha)$ measures the probability that Y exceeds $G^{-1}(\alpha)$ given that X exceeds $F^{-1}(\alpha)$. In the context of the analysis of systemic risk contribution the equation (2.6) can be used to express the probability that the financial system s undergoes a large loss given that the single financial institution i also undergoes a large loss. From this perspective the tail dependence measure

$$\chi(\alpha) = Pr\left(L^s > F_s^{-1}(\alpha)\,|L^i > F_i^{-1}(\alpha)\right) \tag{2.7}$$

can be seen as a natural indicator of the potential contagion effect (and hence systemic risk contribution) from financial institution i on the financial system s over a given threshold α.[5]

$\chi(\alpha)$ expresses for $\alpha \to 1$ the probability of extreme co-movements and corresponds to the well-known upper tail dependence coefficient λ_u.

Definition 4 (cf. McNeil et al. [2005] Definition 5.30)**.** *Let (X, Y) be a bivariate random variable with marginal distribution functions F and G, respectively. The upper tail dependence coefficient of X and Y is the limit (if it exists) of the conditional probability that Y is greater than the $100\alpha - th$ percentile of G given that X is greater than the $100\alpha - th$ percentile of F as α approaches 1, i.e.*

$$\lambda_u := \lim_{\alpha \to 1^-} Pr\left(Y > G^{-1}(\alpha)\,|X > F^{-1}(\alpha)\right). \tag{2.8}$$

If $\lambda_u \in (0,\ 1]$ then (X, Y) is said to show upper tail dependence or extremal dependence in the upper tail; if $\lambda_u = 0$, they are asymptotically independent in the upper tail.
Similarly, the lower tail dependence coefficient λ_l is the limit (if it exists) of the conditional probability that Y is less than or equal to the $100\alpha th$ percentile

[5] Note that, the typical value of α in our context are 0.99 or 0.995

of G given that X is less than or equal to the 100α-th percentile of F as α approaches 0, i.e.

$$\lambda_l := \lim_{\alpha \to 0^+} Pr\left(Y \leq G^{-1}(\alpha) \,|X \leq F^{-1}(\alpha)\right). \tag{2.9}$$

λ_u measures the probability that Y exceeds the threshold $G^{-1}(\alpha)$, conditional on that X exceeds the threshold $F^{-1}(\alpha)$. Thus, λ_u measures the tendency for extreme events to occur simultaneously.

Remark 21. *If $F = G$, it follows*

$$\lambda_u = \lim_{z \to \infty} Pr\left(Y > z|X > z\right) = \lim_{z \to \infty} \frac{Pr\left(Y > z, X > z\right)}{1 - Pr\left(X \leq z\right)}, \tag{2.10}$$

$$\lambda_l = \lim_{z \to -\infty} Pr\left(Y \leq z|X \leq z\right) = \lim_{z \to -\infty} \frac{Pr\left(Y \leq z, X \leq z\right)}{Pr\left(X \leq z\right)}. \tag{2.11}$$

Remark 22. *If $(L^i,\ L^s)$ does not exhibit tail dependence the extreme events of L^i and L^s will appear to occur independently in each margin. This would mean that they are no potential systemic risk contribution from i on s.*

Condition 1 (Necessary condition for a systemic risk model)**.** *A suitable model for the quantification and analysis of systemic risk contribution should allow for positive tail dependence coefficients.*

2.4 Measuring Systemic Risk Contribution using CoVaR-Method

The idea behind the $CoVaR$-Method is to measure the systemic risk contribution of a given financial institution by comparing the Value-at-Risk (VaR) of the financial system, under the condition that the considered financial institution realizes a loss corresponding to its expected loss, to the VaR of the financial system when the considered financial institution is supposed to be in distress.

The $CoVaR$-Method builds on the term $CoVaR_\alpha^{s|C\left(L^i\right)}$. This is defined as the Value-at-Risk at the level α of a financial system s conditional on some event $C\left(L^i\right)$ depending on the loss L^i of the financial institution in focus i. $CoVaR_\alpha^{s|C\left(L^i\right)}$ estimates the effect of the loss of the financial institution i to the VaR of the the financial system s. Doing this, it provides a macro-prudential view of the risk of the individuals financial institutions.

$CoVaR$ can thus be used as basic measure for the definition of quantitative rules in a context of a macro-prudential regulation.

As $CoVaR$ is based on Value-at-Risk, it is important to recall the definition of of Value-at-Risk in order to introduce the term $CoVaR_{\alpha}^{s|C\left(L^i\right)}$. We follow for this McNeil et al. [2005] definition 2.10.

Definition 5 (Value at Risk)**.** *Let L be a random variable, defined on a probability space $(\Omega, \mathcal{F}, Pr)$, representing a loss. Given some confidence level $\alpha \in (0,1)$ the VaR of L at the confidence level α is given by the smallest number l such that the probability that the loss L exceeds l is no larger than $(1-\alpha)$. Formally*

$$\begin{aligned} VaR_{\alpha} &:= \inf\left\{l \in \mathbb{R} : Pr\left(L > l\right) \leq 1-\alpha\right\} \\ &= \inf\left\{l \in \mathbb{R} : Pr\left(L \leq l\right) \geq \alpha\right\}. \end{aligned}$$

Recall that, since we assume L to be a positive random variable the number α denotes the **confidence level**[6], it usually assumes the values 0.95 or 0.99.

From a statistical view, the so defined VaR_{α} is the quantile of the loss distribution at the level α (cf. McNeil et al. [2005], Definition 2.12).

Definition 6 (Generalized inverse and quantile function)**.**

a) Given some increasing function $T : \mathbb{R} \to \mathbb{R}$, the generalized inverse of T is defined by $T^{\leftarrow}(y) := \inf\left\{x \in \mathbb{R} : T\left(x\right) \geq y\right\}$.

b) Given some distribution function F, the generalized inverse $F^{\leftarrow}$ is called the quantile function of F. For $\alpha \in (0,1)$ we have

$$q_{\alpha}\left(F\right) = F^{\leftarrow}\left(\alpha\right) := \inf\left\{x \in \mathbb{R} : F\left(x\right) \geq \alpha\right\}.$$

Thus if we assume for the loss L a distribution function F. The Value-at-Risk of L for a given level $\alpha \in (0,1)$ satisfies the relation

$$VaR_{\alpha} = F^{\leftarrow}\left(\alpha\right) = \inf\left\{l \in \mathbb{R} : F\left(l\right) \geq \alpha\right\} \tag{2.12}$$

and its follows

$$F\left(VaR_{\alpha}\right) = \alpha.$$

Note that, if F is continuous and strictly increasing. $F^{\leftarrow}$ is unique and we have

$$F^{\leftarrow}\left(\alpha\right) = F^{-1}\left(\alpha\right),$$

[6] This should not be confused with the significance level $1-\alpha$ which is used when the profit and loss (P&L) instead of the the loss is modeled (cf. McNeil et al. [2005], remark 2.1.).

where F^{-1} is the (ordinary) inverse of F. So, due to Assumption 2 all distribution functions F considered here are continuous and strictly increasing. Hence

$$VaR_\alpha = F^{-1}(\alpha) \text{ and } F(VaR_\alpha) = \int_{-\infty}^{VaR_\alpha} f(x)dx = \alpha,$$

where f is the density function associated to F.

$CoVaR_\alpha^{s|C(L^i)}$ can therefore be implicitly expressed as the α-quantile of the conditional probability of the financial system's loss:

$$Pr\left(L^s \leq CoVaR_\alpha^{s|C(L^i)} | C\left(L^i\right)\right) = \alpha. \tag{2.13}$$

Brunnermeier and Adrian [2011] considered the case in which the condition $C(L^i)$ refers to the loss L^i of the financial institution i being exactly at its Value-at-Risk and at its mean (see Definition 8). This approach can be generalized in order to allow L^i to assumes any value $l \in \mathbb{R}$ (cf. Hakwa, Jäger-Ambrożewicz, and Rüdiger [2015]). Equation (2.13) becomes then

$$Pr\left(L^s \leq CoVaR_\alpha^{s|L^i=l} | L^i = l\right) = \alpha. \tag{2.14}$$

Following Breiman [1992] (Definition 4.7) we can implicitly define, in the context of Assumption 2, a conditional probability of the form $Pr(L^s \leq h | L^i = l)$ for a fixed h as a function of l:

$$Pr\left(L^s \leq h, L^i \leq y\right) = \int_{-\infty}^{y} Pr\left(L^s \leq h | L^i = l\right) f_i(l)\, dl \quad \forall y \in \mathbb{R}. \tag{2.15}$$

Consider, the function

$$R_l(h) := Pr\left(L^s \leq h | L^i = l\right). \tag{2.16}$$

Since $R_l(h)$ is strictly increasing, it follows that it is invertible. Based on this $CoVaR_\alpha^{s|L^i=l}$ can be expresses as follows(cf. Hakwa et al. [2015]).

Definition 7. *Assume that L^i and L^s have densities that satisfy Assumption 2. Then, for a given $\alpha \in (0,1)$ and for a fixed $l \in \mathbb{R}$, $CoVaR_\alpha^{s|L^i=l}$ is defined as:*

$$\begin{aligned} CoVaR_\alpha^{s|L^i=l} &:= \inf\left\{h \in \mathbb{R} : \; Pr\left(L^s > h | L^i = l\right) \leq 1 - \alpha\right\} \\ &= \inf\left\{h \in \mathbb{R} : \; Pr\left(L^s \leq h | L^i = l\right) \geq \alpha\right\} \\ &= R_l^{-1}(\alpha). \end{aligned}$$

For a fixed α we define the function

$$CoVaR_\alpha^{s|i}(l) := CoVaR_\alpha^{s|L^i=l}, \quad \forall\, l \in \mathbb{R}. \tag{2.17}$$

Definition 8. *$\Delta CoVaR_\alpha^{s|i}$ is defined by Brunnermeier and Adrian [2011] as the difference between $CoVaR_\alpha^{s|C\left(L^i\right)}$ conditioned on the loss of the financial institution i being exactly at its Value-at-Risk VaR_α^i (i.e. $C\left(L^i\right) = \{L^i = VaR_\alpha^i\}$) and the $CoVaR_\alpha^{s|C\left(L^i\right)}$ conditioned on the financial institution experiencing the mean loss $\mu^i := E\left[L^i\right]$(i.e. $C\left(L^i\right) = \{L^i = \mu^i\}$):*

$$\Delta CoVaR_\alpha^{s|i} := CoVaR_\alpha^{s|L^i=VaR_\alpha^i} - CoVaR_\alpha^{s|L^i=\mu^i}. \tag{2.18}$$

Definition 9. *For some $l_1, l_2 \in \mathbb{R}$*

$$\Delta CoVaR_\alpha^{s|i}\left(l_1, l_2\right) := CoVaR_\alpha^{s|i}\left(l_1\right) - CoVaR_\alpha^{s|i}\left(l_2\right) \tag{2.19}$$

Especially we have

$$\begin{aligned}\Delta CoVaR_\alpha^{s|i} &= \Delta CoVaR_\alpha^{s|i}\left(VaR_\alpha^i, \mu_i\right)\\ &= CoVaR_\alpha^{s|i}\left(VaR_\alpha^i\right) - CoVaR_\alpha^{s|i}\left(\mu_i\right).\end{aligned}$$

The above definitions show clearly that the main task by the analysis of systemic risk contribution using the $CoVaR$-method (see Definition 8) is the computation of the value

$$CoVaR_\alpha^{s|L^i=l}, \quad \forall\, l \in \mathbb{R}.$$

The computation methods proposed so far present problems with the modeling and the integration of the relevant probabilistic features of the loss distributions. For instance, Brunnermeier and Adrian [2011] proposed an estimation method based on "linear quantile regression"; M. Jäger-Ambrożewicz [2010] developed a closed formula for the special case where the random vector (L^i, L^s) is modeled by a bivariate normal distribution. These two approaches have in common that there are difficulties with the integration of the tail dependence and whit flexibly modeling of the univariate stochastic behaviors of the loss distribution of individual financial institutions. This is due to the fact that these methods assume the bivariate normal distribution for (L^i, L^s).

Our aim is thus to improve the quality of systemic risk analysis by providing a general and flexible framework for the calculation and the theoretical analysis of $CoVaR_\alpha^{s|L^i=l}$ for a large class of stochastic settings. This approach is necessary for a better of the effect of tail dependence as well as the stylized features of marginal losses distribution such as skewness, fat tails.

Chapter 3

Notion of Copula

In this section we introduce the notion of copula and give some basic definitions and fundamental results. Our focus is on properties that will be useful when expressing $CoVaR_{\alpha}^{s|L^i=l}$ in term of Copula. For a detailed analysis of copulas, we refer the reader to Darsow et al. [1992], Joe [1997], McNeil et al. [2005], Nelsen [2006] or Roncalli [2009] and the references therein.

As preliminary we first recall some useful results on the probability integral transform that link copula and and a specific class of joint distribution functions (those with non-standard uniformly distributed margins). Especially probability integral transformation techniques are used to transform univariate continuous distributed random variables into uniformly standard distributed random variables.

Remark 23 (McNeil et al. [2005], Proposition 5.2. and Lemma A.2.). *Assume F is a distribution function such that its inverse function F^{-1} is well defined.*

1. *Consider a random variable X, then*

$$Pr\left(F\left(X\right) \leq F\left(x\right)\right) = Pr\left(X \leq x\right).$$

2. *Let U be a standard uniform distributed random variable (i.e. $U \sim U\left(0,\ 1\right)$), then*

$$Pr\left(F^{-1}\left(U\right) \leq x\right) = F\left(x\right).$$

3. *Let $U \sim U\left(0,\ 1\right)$, then*

$$X := F^{-1}\left(U\right) \sim F. \tag{3.1}$$

4. *Let X be a random variable with distribution function F (i.e. $X \sim F$), then for $0 < u < 1$ it holds*

$$Pr(F(X) \leq u) = Pr\left(X \leq F^{-1}(u)\right) = F\left(F^{-1}(u)\right) = u.$$

5. *Let $X \sim F$, then $F(X)$ has a uniform standard distribution i.e.*

$$F(X) \sim U(0,1).$$

3.1 Definition and Basic Properties

The copula concept was motivated by a (inverse) problem stated by Fréchet [1951]:
"Suppose we have the distribution functions F and G of two random variables X and Y defined on the same probability space $(\Omega, \mathcal{F}, Pr)$ then what can we say about the set of bivariate distribution functions whose marginals are F and G."
Indeed,Fréchet [1951] wanted to know how the margins F and G can be coupled in order to built a bivariate distribution function. This problem led to the definition of the so called **Fréchet class** (denoted by $\mathcal{F}$). $\mathcal{F}$ is defined as the class of multivariate distributions with some given margins (cf. Joe [1997], Chapter 3). That is, a bivariate joint distribution function H is an element of the Fréchet class $\mathcal{F}(F, G)$ if the following two conditions hold:

$$H(x, \infty) = F(x) \qquad \text{and}$$
$$H(\infty, y) = G(y).$$

It is in this context that Sklar [1959] introduced the notion of copula as a partial response to this problem.[1] He defined a bivariate copula as follows:[2]

Definition 10 (Nelsen [2006], Definition 2.2.2).). *A 2-dimensional copula is a (distribution) function $C : [0,1]^2 \to [0,1]$ satisfying:*

- *Boundary conditions:*

 1) For every $u \in [0,1] : C(0,u) = C(u,0) = 0$.

 2) For every $u \in [0,1] : C(1,u) = u$ and $C(u,1) = u$.

[1]"partial" because he assumes the margins to be standard univariate

[2]Original definition from Sklar [1959] (Definition 1, Page 229). Nous appelerons copule à n dimensions toute fonction C continue et non-décroissante au sens employé pour une fonction de répartition à n dimensions définie sur le produit Cartésien de n intervalles fermés [0, 1] et satisfaisant aux conditions $C(0,\ldots,0) = 0$ et $C(1,\ldots,1,\ u,\ 1,\ldots,1)$

- *Monotonicity condition:*

  3) For every $(u_1, u_2), (v_1, v_2) \in [0,1] \times [0,1]$ *with* $u_1 \leq u_2$ *and* $v_1 \leq v_2$:

$$C(u_2, v_2) - C(u_2, v_1) - C(u_1, v_2) + C(u_1, v_1) \geq 0.$$

Conditions (1) and (3) imply that the so defined bivariate copula C is a bivariate joint distribution function (cf. Nelsen [2006] Definition 2.3.2) and condition (2) implies that the copula C has margins that are uniformly standard distributed. That is, a copula is a special multivariate distribution whose margins are standard uniformly distributed.[3] An alternative definition of a bivariate Copula can be formulated using the fact that a bivariate distribution function can be characterized through the notion of 2-increasing and grounded function. Let $S_1, S_2 \in \mathbb{R} \bigcup \{+\infty\} \bigcup \{-\infty\}$. Consider the function $f : S_1 \times S_2 \to \mathbb{R}$.

Definition 11. *If* S_1 *and* S_2 *have a smallest element* s_1 *and* s_2 *respectively, then we say that the function* f *is grounded if and only if*

$$f(x, s_2) = 0, \quad \forall x \in S_1 \quad and$$
$$f(s_1, y) = 0, \quad \forall y \in S_2.$$

Grounded function vanishes on the lower and the upper boundary of its domain (cf.Nelsen [2006], Page 9).

Definition 12. f *is said to be a 2-increasing function if for every* $(x_1, x_2), (y_1, y_2) \in S_1 \times S_2$ *with* $x_1 \leq x_2$ *and* $y_1 \leq y_2$

$$f(x_2, y_2) - f(x_1, y_2) - f(x_2, y_1) + f(x_2, y_2) \geq 0. \tag{3.2}$$

Remark 24. *The 2-increasing condition ensures the non-negativity of the probabilities* $Pr(u_1 \leq U \leq v_1,\ u_2 \leq V \leq v_2)$.

Definition 13 (Embrechts et al. [2003], Definition 2.3)**.** *A bivariate joint distribution function is a function* $H(x, y)$ *with domain* $[-\infty,\ \infty]^2$ *such that*

1. $H(x, y)$ *is 2-increasing,*

2. $H(x, y)$ *is grounded*

3. $H(\infty,\ \infty) = 1.$

[3] Copulas form thus a sub class of the Fréchet class.

The margins F and G of H are given by

$$F(x) = H(x, \infty) \text{ and } G(y) = H(\infty, y).$$

Based on this definition, a bivariate copula can be define as follows:

Definition 14 (Joe [1997], Page 12)**.** *Let $C : [0,1]^2 \to [0,1]$ be a bivariate distribution function on $[0,1]^2$. Then C is called a copula if all its univariate marginals are standard uniformly distributed, i.e. $\forall u, v \in [0,1]$ the following two conditions hold*

- $C(u, v) = Pr(U \leq u, V \leq v)$ *with* $V, U \sim U(0,1)$
- $C(u, 1) = u$ *and* $C(1, v) = v$.

Any bivariate copula has a lower and an upper bound. In fact by setting $u_2 = v_2 = 1$ in the monotonicity condition of copula (see Definition 10) we obtain

$$1 - v_1 - u_1 + C(u_1, v_1) \geq 0.$$

Hence

$$C(u_1,\ v_1) \geq u_1 + v_1 - 1$$

and because of the non-negativity of copula we can write

$$C(u_1,\ v_1) \geq \max(u_1 + v_1 - 1, 0).$$

This means that any bivariate copula $C(u, v)$ is bounded below by $\max(u + v - 1, 0)$.

Consider again the monotonicity condition and set $u_1 = 0,\ v_2 = 1$. We obtain:

$$C(u_2, 1) - C(u_2, v_1) \geq 0 \ \Rightarrow\ C(u_2, v_1) \leq C(u_2, 1) = u_2.$$

Similarly, if we set $u_2 = 1$ and $v_1 = 0$, we obtain

$$C(1, v_2) - C(u_1, v_2) \geq 0. \ \Rightarrow\ C(u_1, v_2) \leq C(1, v_2) = v_2.$$

Hence, for any bivariate copula $C(u, v)$

$$C(u, v) \leq u \text{ and } C(u, v) \leq v.$$

We can write

$$C(u,\ v) \leq \min(u, v).$$

This means that any bivariate copula $C\left(u,v\right)$ is bounded upper by $\min\left(u,v\right)$.

In general, for higher dimensional copulas we have (cf. Nelsen [2006], Theorem 2.10.12):

$$W(u_1,\ldots,u_d) \leq C(u_1,\ldots,u_d) \leq M(u_1,\ldots,u_d),$$

where

$$W(u_1,\ldots,u_d) = \max\left\{1-d+\sum_{i=1}^{d} u_i, 0\right\} \text{ and } M(u_1,\ldots,u_d) = \min\{u_1,\ldots,u_d\}.$$

The functions W and M are called the lower and the upper Fréchet-Hoeffding bound respectively. It is important to note that M is a copula for any dimension but W is a copula in two dimensions only, because it does not satisfy the monotonicity condition in the case of dimension more than two dimensions.
Random vectors that have the upper or the lower Fréchet-Hoeffding bound as copula are called comonotone or countermonotone, respectively. The comonotone copula (countermonote copula) characterizes in some sense the perfect or the deterministic positive (deterministic negative) dependence. If M is the copula associated with the random vector (U,V), then V is an almost surely (a.s.) non-decreasing function of U, i.e.

$$U = f\left(V\right) \text{for some a.s. non-decreasing function } f.$$

Example 1. *The random vector (U,V) with $U=V$ is comonotone*

$$\begin{aligned} C\left(u,v\right) &= Pr\left(U \leq u,\ V \leq v\right) \\ &= Pr\left(U \leq u,\ U \leq v\right) \\ &= Pr\left(U \leq \min\left(u,\ v\right)\right) \\ &= \min\left(u,\ v\right). \end{aligned}$$

Similarly, if W is the copula associated with the random vector (U,V), then V is an almost surely non-increasing function of U, i.e.

$$U = g\left(V\right) \text{ for some a.s. decreasing function } g.$$

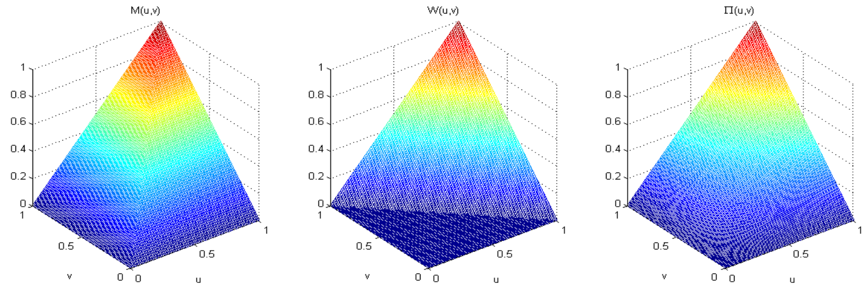
Example 2. *The random vector (U,V) with $U=1-V$ is counter-monotone*

$$\begin{aligned} C\left(u,v\right) &= P\left(U \leq u,\ V \leq v\right) \\ &= P\left(U \leq u,\ 1-U \leq v\right) \\ &= P\left(1-v \leq U \leq u\right) \\ &= \max\left(u-(1-v),\ 0\right) \\ &= \max\left(u+v-1,\ 0\right). \end{aligned}$$

The independence is characterized by the product copula

$$\Pi(u_1, \ldots, u_d) = u_1 \cdot \ldots \cdot u_d.$$

The following figure shows the graph of M, W and Π for $d = 2$ (bivariate copula).



The following lemma is a direct consequence of the monotonicity condition (see Definition 10)

Lemma 2. *For any u_1 and u_2 satisfying $0 \leq u_1 \leq u_2 \leq 1$ the mapping*

$$v \mapsto C(u_2, v) - C(u_1, v) \tag{3.3}$$

is non-decreasing on $[0, 1]$.

Similarly, for any v_1 and v_2 satisfying $0 \leq v_1 \leq v_2 \leq 1$ the mapping

$$u \mapsto C(u, v_2) - C(u, v_1) \tag{3.4}$$

is non-decreasing on $[0, 1]$.

Proof. *The monotonocity condition state that for every $(u_1, u_2), (v_1, v_2) \in [0, 1] \times [0, 1]$ with $u_1 \leq u_2$ and $v_1 \leq v_2$:*

$$C(u_2, v_2) - C(u_2, v_1) - C(u_1, v_2) + C(u_1, v_1) \geq 0 \tag{3.5}$$
$$\Rightarrow C(u_2, v_2) - C(u_1, v_2) \geq C(u_2, v_1) - C(u_1, v_1).$$

This means that the mapping $v \mapsto C(u_2, v) - C(u_1, v)$ is non-decreasing. It also follows from (3.5) that

$$C(u_2, v_2) - C(u_2, v_1) \geq C(u_1, v_2) - C(u_1, v_1).$$

This means that the mapping $u \mapsto C(u, v_2) - C(u, v_1)$ is non-decreasing.

It follows that bivariate copulas are non-decreasing with respect to each of theirs arguments, i.e. for every u and $v \in [0, 1]$ the mappings

$$v \mapsto C(u, v)$$
$$u \mapsto C(u, v)$$

are non-decreasing[4]. The following properties are based on Lemma 2.

[4] To see this set $u_1 = 0$ and $v_1 = 0$ in (3.3) and (3.4) respectively.

Property 1. *For every* $u, v, u_1, u_2, v_1, v_2 \in [0,1]$ *such that if* $u_1 \leq u_2$, $v_1 \leq v_2$, *it holds*

1. $0 \leq C(u_2, v) - C(u_1, v) \leq u_2 - u_1$
2. $0 \leq C(u, v_2) - C(u, v_1) \leq v_2 - v_1$
3. $0 \leq C(u_2, v_2) - C(u_1, v_1) \leq u_2 - u_1 + v_2 - v_1$
4. $|C(u_2, v_2) - C(u_1, v_1)| \leq |u_2 - u_1| + |v_2 - v_1|$

Point *4.* implies that copulas are Lipschitz continuous with Lipschitz constant equal to 1.

Note that the monotonicity condition implies that copulas are almost everywhere differentiable with respect to the Lebesgue measure. Since copulas are increasing (with respect to each parameter) their derivatives are positive where they exist. Because copulas are Lipschitz continuous with Lipschitz constant equal to 1, it follows that their partial derivatives are bounded by 1. All this is summarized in the following theorem.

Theorem 3 (Nelsen [2006], Theorem 2.2.7)**.** *Let* C *be a copula. For any* $u, v \in [0,1]$, *the partial derivative* $\partial C(u,v)/\partial u$ *exists for almost all* u, *and for such* v *and* u

$$0 \leq \frac{\partial C(u,v)}{\partial u} \leq 1.$$

Similarly, the partial derivative $\partial C(u,v)/\partial v$ *exists for almost all* v, *and for such* u *and* v

$$0 \leq \frac{\partial C(u,v)}{\partial v} \leq 1.$$

Furthermore, the functions $u \mapsto \partial C(u,v)/\partial v$ *and* $v \mapsto \partial C(u,v)/\partial u$ *are defined and non-decreasing everywhere on* $[0,1]$.

The following theorem is the main theorem when using Copula for stochastic modeling, because it provides a copula representation of joint distribution function.

Theorem 4 (Sklar's Theorem; cf. Nelsen [2006], Theorem 2.3.3)**.** *Let* H *be a joint distribution function with marginal distribution functions* F *and* G. *Then there exists a copula* C *such that for all* $x, y \in \mathbb{R} \cup \{-\infty\} \cup \{+\infty\}$

$$H(x,y) = C(F(x), G(y)). \tag{3.6}$$

If F *and* G *each have a density, then* C *is unique. Conversely, if* C *is a copula and* F *and* G *are distribution functions, then the function* H *defined by equation* (3.6) *is a joint distribution function with margins* F *and* G.

For any $x, y \in \mathbb{R}$, we have

$$H(x, \infty) = C(F(x), 1) = F(x), \tag{3.7}$$

$$H(\infty, y) = C(1, G(y)) = G(y). \tag{3.8}$$

Corollary 5 (cf. e.g. Nelsen [2006], Corollary 2.3.7)**.** *Let H denotes a bivariate distribution function with margins F and G satisfying assumption 2. Then there exists a unique copula C such that for all $(u, v) \in [0, 1]^2$*

$$C(u, v) = H\left(F^{-1}(u), G^{-1}(v)\right). \tag{3.9}$$

If H satisfies assumption 2, then the transformed random variables $U := F(X)$ and $V := G(Y)$ follow each a standard uniform distribution and $C(u, v)$ is the joint distribution of (U, V). In fact

$$\begin{aligned} C(u, v) &= Pr(U \leq u,\ V \leq v) \\ &= Pr(F(X) \leq u,\ G(Y) \leq v) \\ &= Pr\left(X \leq F^{-1}(u),\ Y \leq G^{-1}(v)\right) \\ &= H\left(F^{-1}(u),\ G^{-1}(v)\right) \\ &= H(x,\ y), \end{aligned}$$

where $x = F^{-1}(u)$ and $y = G^{-1}(v)$.

Sklar's theorem asserts that, joint distributions are formed by coupling together marginal distributions with a copula. Therefore, we can use copulas to extract the dependence structure between the components X and Y of the vector (X, Y) independently from the marginal distributions F and G. This allows us to model the dependence structure and marginals separately, as illustrated in the following figures.

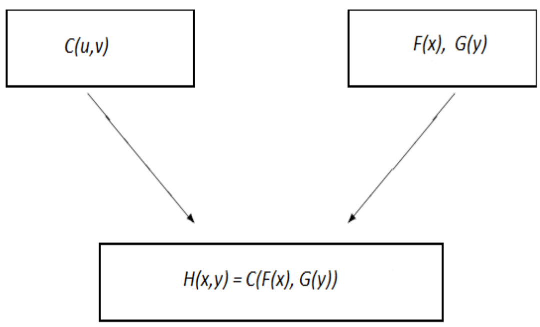


Figure 3.1: Construction of a bivariate distributions function through combination of a given copula with given continuous marginal distributions

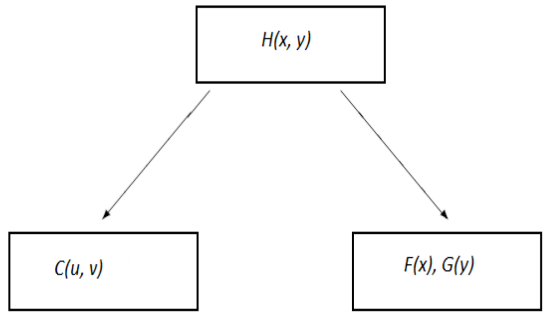


Figure 3.2: Separation of the dependence structure from the margins

Remark 25. *Assume* (X, Y) *is a bivariate random variables with copula* C *and joint distribution* H *satisfying Assumption 2, with corresponding marginals distribution functions* F *and* G. *Then the transformed randoms variables* $U := F(X)$ *and* $V := G(Y)$ *follows each a standard uniform distribution and* $C(U, V)$ *is the joint distribution of* (U, V), *i.e.,*

$$C(u, v) = Pr(U \leq u, V \leq v).$$

As showed by Schweizer and Wolff [1981], the copula of a pair of random variables (X, Y) is invariant under strictly increasing transformations of X and Y and any property of the joint distribution function of X and Y which is invariant under such transformations is solely a function of their copula. The following theorem is a simplified version of the theorem provided by Schweizer and Wolff [1981], Theorem 3.

Theorem 6 (Nelsen [2006], Theorem 2.4.3)**.** *Let* X *and* Y *be two random variables satisfying assumption 2 with copula* C. *Let* C *be the copula of* X *and* Y . *If* f *and* g *are strictly increasing function on the ranges of* X *and* Y, *respectively, then*

$$C(f(X), g(Y)) = C(X, Y),$$

Theorem 6 means that, $C(X, Y)$ is invariant under strictly increasing transformations of X and Y.

Remark 26. *Let* U *and* V *be two standard uniformly distributed random variables. Assume that* $(U,\ V)$ *has the copula* C *as joint distribution function. Let* X *and* Y *be two univariate random variables with distribution functions* F *and* G*satisfying Assumption 2. Then by Theorem 6 we have that:*

$$C(U,\ V) = C\left(F^{-1}(U),\ G^{-1}(V)\right) = C(X,\ Y).$$

This shows that the copula C contain the complete information about the dependence structure of random variables X and Y.

Remark 27. *Due to Assumption 2 any Copula C considered in this thesis is associated with a joint distribution function, that have density. Hence, if we assume a copula C, then there exists a unique function $c : [0,1]^2 \to [0,\infty)$ such that*

$$C(u,v) = \int_0^v \int_0^u c(s,t)\, dsdt.$$

The function c is called copula density of C, with

$$c(u,v) = \frac{\partial^2 C(u,v)}{\partial u \partial v}.$$

Hence, all bivariate copulas considered here are almost everywhere two times differentiable.

Consider equation (3.6) in theorem 4(Sklar's Theorem):

$$H(x,y) = C(F(x), G(y)) \tag{3.10}$$

Given Assumption 2, there exist corresponding joint density function $h(x,y)$ and marginal density functions $f(x)$ and $g(y)$. From basic probability theory, we know that the joint density function $h(x,y)$ can be obtained by differentiating (3.10) with respect to x and y. i.e.

$$\begin{aligned} h(x,y) &= \frac{\partial^2 C(F(x), G(y))}{\partial x \partial y} \\ &= c(F(x), G(y)) \cdot f(x) \cdot g(y). \end{aligned} \tag{3.11}$$

Moreover, for any $u, v \in (0,1)$

$$c(u,v) = \frac{h(F^{-1}(u), G^{-1}(v))}{f(F^{-1}(u))\, g(G^{-1}(v))}. \tag{3.12}$$

3.2 Copula and Tail Dependence Coefficient

As already seen here, copulas are invariant under strictly increasing transformations and they express dependence on a quantile scale. This feature gives copulas the ability to precisely describe the dependencies of extreme events. In particular the tail coefficients can be expressed in terms of copulas.

Recall that the expression of the upper and the lower tail dependence coefficients are definided by equation (2.8) and equation (2.9), respectively

$$\lambda_u := \lim_{\alpha \to 1^-} Pr\left(Y > G^{-1}(\alpha) | X > F^{-1}(\alpha)\right)$$

and

$$\lambda_l := \lim_{\alpha\to 0^+} Pr\left(Y \leq G^{-1}(\alpha)\,|X \leq F^{-1}(\alpha)\right),$$

can be reformulated as follows

$$\lambda_u := \lim_{\alpha\to 1^-} \chi_u(\alpha), \qquad \lambda_l := \lim_{\alpha\to 0^+} \chi_l(\alpha), \tag{3.13}$$

where

$$\chi_u(\alpha) := Pr\left(Y > G^{-1}(\alpha)\,|X > F^{-1}(\alpha)\right)$$

and

$$\chi_l(\alpha) := Pr\left(Y \leq G^{-1}(\alpha)\,|X \leq F^{-1}(\alpha)\right).$$

Assume two random variables X and Y with a joint distribution function H and univariate marginal distribution functions F and G, respectively. Further, assume that C is the copula of X and Y. Then it holds:

$$\begin{aligned}\chi_u(\alpha) &= Pr\left(Y > G^{-1}(u)\,|X > F^{-1}(u)\right)\\ &= Pr\left(V > u|U > u\right)\\ &= \frac{Pr\left(U > u,\ V > v\right)}{Pr\left(U > u\right)}\\ &= \frac{1 - Pr\left(U \leq u\right) - Pr\left(V \leq u\right) + Pr\left(U \leq u,\ V \leq u\right)}{1 - Pr\left(U \leq u\right)}\\ &= \frac{1 - 2u + C\left(u,\ u\right)}{1-u}\end{aligned}$$

Hence, if the limits in equations (3.13) exists, it follows

$$\lambda_u = \lim_{u\to 1^-} \frac{1 - 2u + C(u,u)}{1-u} \tag{3.14}$$

and

$$\lambda_l = \lim_{u\to 0^+} \frac{C(u,u)}{u}. \tag{3.15}$$

Example 3 (Tail Dependence Coefficient of the bivariate W, Π and M)**.**

a) $W(u,v) = \max(u+v-1,0)$*. Hence* $W(u,u) = \max(2u-1,0)$*. Then from equations* (3.14) *and* (3.15) *it follows that*

$$\begin{aligned}\lambda_u &= \lim_{u\to 1^-} \frac{1-2u+W(u,u)}{1-u}\\ &= \lim_{u\to 1^-} \frac{1-2u+\max(2u-1,0)}{1-u}\\ &= \lim_{u\to 1^-} \frac{1-2u+2u-1}{1-u}\\ &= 0\end{aligned}$$

and

$$\begin{aligned}\lambda_l &= \lim_{u\to 0^+} \frac{W(u,u)}{u}\\ &= \lim_{u\to 0^+} \frac{\max(2u-1,0)}{u}\\ &= \lim_{u\to 0^+} \frac{0}{u}\\ &= 0.\end{aligned}$$

b) $\Pi(u,v) = uv$. *Hence* $\Pi(u,u) = u^2$. *from equations* (3.14) *and* (3.15) *it follows that*

$$\begin{aligned}\lambda_u &= \lim_{u\to 1^-} \frac{1-2u+\Pi(u,u)}{1-u}\\ &= \lim_{u\to 1^-} \frac{1-2u+u^2}{1-u}\\ &= \lim_{u\to 1^-} \frac{(1-u)^2}{1-u}\\ &= \lim_{u\to 1^-} 1-u\\ &= 0\end{aligned}$$

and

$$\begin{aligned}\lambda_l &= \lim_{u\to 0^+} \frac{\Pi(u,u)}{u}\\ &= \lim_{u\to 0^+} \frac{u^2}{u}\\ &= \lim_{u\to 0^+} u\\ &= 0.\end{aligned}$$

c) $M(u,v) = \min(u,v)$.*Hence* $\min(u,u) = u$. *From equations* (3.14) *and* (3.15) *it follows that*

$$\begin{aligned}\lambda_u &= \lim_{u\to 1^-} \frac{1-2u+M(u,u)}{1-u}\\ &= \lim_{u\to 1^-} \frac{1-2u+u}{1-u}\\ &= \lim_{u\to 1^-} \frac{1-u}{1-u}\\ &= \lim_{u\to 1^-} 1\\ &= 1\end{aligned}$$

and

$$\begin{aligned}\lambda_l &= \lim_{u\to 0^+} \frac{M(u,u)}{u} \\ &= \lim_{u\to 1^-} \frac{u}{u} \\ &= \lim_{u\to 1^-} 1 \\ &= 1.\end{aligned}$$

Example 4 (Tail Dependence Coefficient of the bivariate Gumbel copula)**.** *The bivariate Gumbel copula function is given by (cf. Nelsen [2006], Chapter 4)*

$$C_\theta^{Gu}(u,v) = \exp\left(-\left[(-\ln(u))^\theta + (-\ln(v))^\theta\right]^{\frac{1}{\theta}}\right), \quad 1 \le \theta < \infty.$$

Hence,

$$C_\theta^{Gu}(u,u) = \exp\left(-\left[(-\ln(u))^\theta + (-\ln(u))^\theta\right]^{\frac{1}{\theta}}\right) = \exp\left(2^{\frac{1}{\theta}}\ln(u)\right) = u^{2^{\frac{1}{\theta}}}.$$

By applying the L'hopital rule to equations (3.14) *and* (3.15) *we obtain*

$$\begin{aligned}\lambda_u &= \lim_{u\to 1^-} \frac{1-2u+C_\theta^{Gu}(u,u)}{1-u} \\ &= \lim_{u\to 1^-} \frac{1-2u+u^{2^{\frac{1}{\theta}}}}{1-u} \\ &\overset{*}{=} \lim_{u\to 1^-} \frac{\left(1-2u+u^{2^{\frac{1}{\theta}}}\right)'}{(1-u)'} \\ &= \lim_{u\to 1^-} \frac{-2+u^{2^{\frac{1}{\theta}}-1}\cdot 2^{\frac{1}{\theta}}}{-1} \\ &= 2-2^{\frac{1}{\theta}}.\end{aligned}$$

and

$$\begin{aligned}\lambda_l &= \lim_{u\to 0^+} \frac{C_\theta^{Gu}(u,u)}{u}. \\ &= \lim_{u\to 0^+} \frac{u^{2^{\frac{1}{\theta}}}}{1-u} \\ &= \lim_{u\to 0^+} \frac{\left(u^{2^{\frac{1}{\theta}}}\right)'}{(1-u)'} \\ &\overset{*}{=} \lim_{u\to 0^+} \frac{-u^{2^{\frac{1}{\theta}}-1}\cdot 2^{\frac{1}{\theta}}}{-1} \\ &= 0.\end{aligned}$$

* *holds because*

$$\begin{aligned}\left(u^{2^{\frac{1}{\theta}}}\right)' =& \frac{\mathrm{d}}{\mathrm{d}u}\left(u^{2^{\frac{1}{\theta}}}\right)\\ =& \left(2^{\frac{1}{\theta}} \cdot u^{2^{\frac{1}{\theta}-1}}\right)\\ =& u^{2^{\frac{1}{\theta}}-1} \cdot 2^{\frac{1}{\theta}}.\end{aligned}$$

Example 5 (Tail Dependence Coefficient of the bivariate Clayton copula)**.** *The bivariate Clayton Copula function is given by (cf. Nelsen [2006], Chapter 4)*

$$C_\theta^C(u,v) = \left(u^{-\theta} + v^{-\theta} - 1\right)^{-\frac{1}{\theta}}. \quad 0 < \theta < \infty. \tag{3.16}$$

Hence,

$$\begin{aligned}C_\theta^C(u,u) &= \left(u^{-\theta} + u^{-\theta} - 1\right)^{-\frac{1}{\theta}}\\ &= \left(2u^{-\theta} - 1\right)^{-\frac{1}{\theta}}.\end{aligned}$$

By applying the L'hopital rule to equations (3.14) *and* (3.15) *we obtain*

$$\begin{aligned}\lambda_u &= \lim_{u\to 1^-} \frac{1 - 2u + C_\theta^C(u,u)}{1-u}\\ &= \lim_{u\to 1^-} \frac{1 - 2u + \left(2u^{-\theta} - 1\right)^{-\frac{1}{\theta}}}{1-u}\\ &= \lim_{u\to 1^-} \frac{\left(1 - 2u + \left(2u^{-\theta} - 1\right)^{-\frac{1}{\theta}}\right)'}{(1-u)'}\\ &\overset{**}{=} \lim_{u\to 1^-} \frac{-2 + 2u^{-\theta-1} \cdot \left(2u^{-\theta} - 1\right)^{\frac{-1}{\theta}-1}}{-1}\\ &= 2 - 2 = 0\end{aligned}$$

and

$$\begin{aligned}\lambda_l &= \lim_{u\to 0^+} \frac{C_\theta^C(u,u)}{u}\\ &= \lim_{u\to 0^+} \frac{\left(2u^{-\theta} - 1\right)^{-\frac{1}{\theta}}}{u}\\ &= \lim_{u\to 0^+} \frac{\left(2u^{-\theta} - 1\right)^{-\frac{1}{\theta}}}{\left(u^{-\theta}\right)^{-\frac{1}{\theta}}}\\ &= \lim_{u\to 0^+} \left(\frac{2u^{-\theta} - 1}{u^{-\theta}}\right)^{-\frac{1}{\theta}}\\ &= \lim_{u\to 0^+} \left(2 - u^{\theta}\right)^{-\frac{1}{\theta}}\\ &= 2^{-\frac{1}{\theta}}.\end{aligned}$$

** holds because

$$
\begin{aligned}
\left(1-\left(2u^{-\theta}-1\right)^{-\frac{1}{\theta}}\right)' &= \frac{\mathrm{d}}{\mathrm{d}u}\left(1-\frac{1}{\left(\dfrac{2}{u^{\theta}}-1\right)^{\frac{1}{\theta}}}\right) \\
&= -\frac{\mathrm{d}}{\mathrm{d}u}\left(\frac{1}{\left(\frac{2}{u^{\theta}}-1\right)^{\frac{1}{\theta}}}\right) \\
&= -\frac{\frac{\mathrm{d}}{\mathrm{d}u}\left(\left(\frac{2}{u^{\theta}}-1\right)^{\frac{1}{\theta}}\right)^{2}}{\left(\left(\frac{2}{u^{\theta}}-1\right)^{\frac{1}{\theta}}\right)} \\
&= \frac{\frac{1}{\theta}\left(\frac{2}{u^{\theta}}-1\right)^{\frac{1}{\theta}-1}\cdot\frac{\mathrm{d}}{\mathrm{d}u}\left(\frac{2}{u^{\theta}}-1\right)}{\left(\frac{2}{u^{\theta}}-1\right)^{\frac{2}{\theta}}} \\
&= \frac{2\cdot\frac{\mathrm{d}}{\mathrm{d}u}\left(\dfrac{1}{u^{\theta}}\right)\cdot\left(\frac{2}{u^{\theta}}-1\right)^{\frac{-1}{\theta}-1}}{\theta} \\
&= \frac{2\cdot\dfrac{-\frac{\mathrm{d}}{\mathrm{d}u}\left(u^{\theta}\right)}{\left(u^{\theta}\right)^{2}}\cdot\left(\dfrac{2}{u^{\theta}}-1\right)^{\frac{-1}{\theta}-1}}{\theta} \\
&= \frac{-2\theta u^{\theta-1}\cdot\left(\frac{2}{u^{\theta}}-1\right)^{\frac{-1}{\theta}-1}}{\theta u^{2\theta}} \\
&= -2u^{-\theta-1}\cdot\left(\frac{2}{u^{\theta}}-1\right)^{\frac{-1}{\theta}-1}.
\end{aligned}
$$

Chapter 4

CoVaR-Method using Copula.

We provide here a flexible and general formula for $CoVaR_\alpha^{s|i}(l)$ using Copula. The essential of the results presented here is published in Hakwa, Jäger-Ambrożewicz, and Rüdiger [2015]. The formula presented here is based on an expression of $CoVaR_\alpha^{s|i}(l)$ in terms of copula. This expression is obtained by using the relation between conditional probability and copula as presented in Darsow et al. [1992]. The so derived formula inherits from copula the ability to represent multivariate distributions as combinations of univariate margins and copula functions. It allows to analyze systemic risk contribution in various stochastic settings, including elliptical models as well as models with complex dependence structures. We did some application, in which we compute and analyze $CoVaR_\alpha^{s|i}(l)$ and $\Delta CoVaR_\alpha^{s|i}$ in different stochastic settings, including Gaussian and non-Gaussian settings. Doing this we highlight several important properties of $CoVaR_\alpha^{s|i}(l)$ and $\Delta CoVaR_\alpha^{s|i}$.

4.1 A General Expression for $CoVaR_\alpha^{s|i}(l)$ using Copula

Let L^i and L^s be two random variables representing the loss of financial institution i and that of the financial system s with univariate distribution functions F_i and F_s, respectively. Assume that the joint distribution H of L^i and L^s satisfies Assumption 2. Let C be the copula associated with H, i.e.

$$H(x,y) = C(F_i(x), F_s(y)).$$

Assumption 2 implies that the copula C has a strictly positive density function c, such that (see Remark 27)

$$C(u,v) = \int_0^v \int_0^u c(s,t)\, ds dt \quad \forall\, u,v \in [0,1]. \tag{4.1}$$

We define the function

$$g(v,u) := \frac{\partial C(u,v)}{\partial u}.$$

Remark 28. *Given Assumption 2, the function $g(v,u)$ is well defined and for each fixed $u \in [0,1]$ invertible with respect to the parameter v.*

In fact, by differentiating (4.1) with respect to u and applying the Fubini's theorem (as in Klenke [2008], Theorem 14.16) we obtain

$$\begin{aligned} g(v,u) =& \frac{\partial C(u,v)}{\partial u} \\ =& \frac{\partial}{\partial u}\int_0^v \int_0^u c(s,t)\, ds dt \\ =& \int_0^v \left(\frac{\partial}{\partial u}\int_0^u c(s,t)\, ds\right) dt \\ =& \int_0^v c(u,t)\, dt. \end{aligned} \tag{4.2}$$

Since the copula density c is strictly positive (due to Assumption 2), it follows that, for a fixed $u \in [0,1]$ the function $g(v,u)$ is strictly increasing and thus invertible with respect to v.

Theorem 7 (Hakwa, Jäger-Ambrożewicz, and Rüdiger [2015])**.** *If we assume Assumption 2. Then for all $l \in \mathbb{R}$ and a given $\alpha \in (0,1)$, $CoVaR_{\alpha}^{s|i}(l)$ is given by*

$$CoVaR_{\alpha}^{s|i}(l) = F_s^{-1}\left(g^{-1}(\alpha, F_i(l))\right). \tag{4.3}$$

Proof. *Recall that the implicit definition of $CoVaR_{\alpha}^{s|L^i=l}$ is given by:*

$$\begin{aligned} &Pr\left(L^s \le CoVaR_{\alpha}^{s|L^i=l} | L^i = l\right) = \alpha \\ \Leftrightarrow &Pr\left(F_s(L^s) \le F_s\left(CoVaR_{\alpha}^{s|L^i=l}\right) | F_i\left(L^i\right) = F_i(l)\right) = \alpha. \end{aligned}$$

We define $V := F_s(L^s)$, $U := F_i(L^i)$, $v := F_s\left(CoVaR_{\alpha}^{s|L^i=l}\right)$ and $u := F_i(l)$. Hence,

$$Pr\left(F_s(L^s) \le F_s\left(CoVaR_{\alpha}^{s|L^i=l}\right) | F_i\left(L^i\right) = F_i(l)\right) = Pr(V \le v | U = u).$$

Given Assumption 2, it follows from Remark 23 that V and U are standard uniform distributed and hence continuous. The conditional probability $Pr(V \le v|U = u)$ can thus be computed in the following way (cf. e.g. (Breiman

[1992], equation. (4.4)) and (Roncalli [2009], Page 263))

$$\begin{aligned}
Pr\left(V \leq v|U = u\right) &= \lim_{\Delta u \to 0^+} \frac{Pr\left(V \leq v, u \leq U \leq u + \Delta u\right)}{Pr\left(u \leq U \leq u + \Delta u\right)} \\
&= \lim_{\Delta u \to 0^+} \frac{Pr\left(V \leq v, U \leq u + \Delta u\right) - Pr\left(V \leq v, U \leq u\right)}{Pr\left(U \leq u + \Delta u\right) - Pr\left(U \leq u\right)} \\
&= \lim_{\Delta u \to 0^+} \frac{C\left(v, u + \Delta u\right) - C\left(v, u\right)}{\Delta u} \\
&= \frac{\partial C\left(v, u\right)}{\partial u} \\
&= g\left(v, u\right).
\end{aligned}$$

consequently,

$$\begin{aligned}
Pr\left(F_s\left(L^s\right) \leq F_s\left(CoVaR_\alpha^{s|L^i=l}\right)|F_i\left(L^i\right) = F_i\left(l\right)\right) &= Pr\left(V \leq v|U = u\right) \\
&= g\left(v, u\right) \\
&= g\left(F_s\left(CoVaR_\alpha^{s|L^i=l}\right), F_i\left(l\right)\right).
\end{aligned}$$

Based on this relation and due to the fact that the function $g\left(v, u\right)$ is invertible with respect to v for any fixed $u \in [0, 1]$ (see Remark 28), we are able to derive explicit expressions for $CoVaR_\alpha^{s|L^i=l}$. We can do this by expressing v as a function of α and u as follow

$$v = g^{-1}\left(\alpha, u\right).$$

By replacing v by $F_s\left(CoVaR_\alpha^{s|L^i=l}\right)$ and u by $F_i\left(l\right)$ we obtain

$$F_s\left(CoVaR_\alpha^{s|L=l}\right) = g^{-1}\left(\alpha, F_i\left(l\right)\right).$$

Hence,

$$CoVaR_\alpha^{s|i}\left(l\right) = F_s\left(CoVaR_\alpha^{s|L=l}\right) = F_s^{-1}\left(g^{-1}\left(\alpha, F_i\left(l\right)\right)\right). \qquad \square$$

Remark 29. *A similar expression of $CoVaR$ was been developed, independently from our work, by Bernard et al. [2012]. One important feature of our formula is that, it allow one to estimate the systemic risk contribution of the financial institution i to the financial system s by modeling the individual losses and the interconnectedness of i to s **separately**. In fact, the expression $CoVaR_\alpha^{s|L^i=l}$, as in Equation (4.3), can be decomppose under **three distinct components**:*

1. *The marginal distributions F_i, which represent the purely univariate features of the single financial institution i.*

2. *The marginal distributions F_s, which represent the purely univariate features of the financial system s, .*

3. *The function g^{-1}, which models the interconnectedness of the single financial institution i to financial system s.*

This feature in the spirit of Sklar's theorem is very important for the analysis of systemic risk contribution, because it allows us to investigate the effects of the marginal distributions F_i and F_s and the assumed copula C (dependence structure) on the systemic risk contribution of the financial institution i.

Remark 30 (Hakwa, Jäger-Ambrożewicz, and Rüdiger [2015]). *We can see from Equation (4.3) that $CoVaR_\alpha^{s|L^i=l}$ is nothing other than a quantile of the loss distribution F_s of the financial system s.*

$$CoVaR_\alpha^{s|L^i=l} = F_s^{-1}(\tilde{\alpha}) \;\; \text{with } \tilde{\alpha} := g^{-1}(\alpha, F_i(l)). \tag{4.4}$$

Hence, $CoVaR_\alpha^{s|i}(l)$ can be seen as the Value-at-Risk of the whole financial system at an adjusted level $\tilde{\alpha}$, with

$$\tilde{\alpha} = g^{-1}(\alpha, F_i(l)). \tag{4.5}$$

This fact motivates the following corollary, which connects the $CoVaR_\alpha^{s|i}(l)$ to a Value-at-Risk of the financial system s.

Corollary 8 (Hakwa, Jäger-Ambrożewicz, and Rüdiger [2015]). *Under Assumption 2, we have that*

$$CoVaR_\alpha^{s|i}(l) = VaR_{\tilde{\alpha}}^s \quad \text{with } \; \tilde{\alpha} = g^{-1}(\alpha, u), \tag{4.6}$$

where $u = F_i(l)$.

It follows that $CoVaR_\alpha^{s|L^i=l}$ as a function of $\tilde{\alpha}$ has the same properties as a comon Value-at-Risk. For example, $CoVaR_\alpha^{s|L^i=l}$ increases when the marginal distribution function F_s exhibits heavy-tail and positive skewness (cf. Alexander [2009], Section IV.2.8.1).

In general, the following corollary of Theorem 7 holds:

Corollary 9 (Hakwa, Jäger-Ambrożewicz, and Rüdiger [2015]). *Under Assumption 2, the risk measure $\Delta CoVaR_\alpha^{s|i}$ is computed using Definition 8 as follows:*

$$\begin{aligned}
\Delta CoVaR_\alpha^{s|i} &= CoVaR_\alpha^{s|L^i=VaR_\alpha^i} - CoVaR_\alpha^{s|L^i=E[L^i]} \\
&= F_s^{-1}\left(g^{-1}\left(\alpha, F_i\left(VaR_\alpha^i\right)\right)\right) - F_s^{-1}\left(g^{-1}\left(\alpha, F_i\left(E\left[L^i\right]\right)\right)\right) \\
&= F_s^{-1}\left(g^{-1}(\alpha, \alpha)\right) - F_s^{-1}\left(g^{-1}(\alpha, F_i(\mu_i))\right),
\end{aligned}$$

where $\mu_i = E[L^i]$.

Remark 31 (Hakwa, Jäger-Ambrożewicz, and Rüdiger [2015]). *If we assume a symmetric distribution for* L^i*, then*

$$\Delta CoVaR_{\alpha}^{s|i} = F_s^{-1}\left(g^{-1}(\alpha,\alpha)\right) - F_s^{-1}\left(g^{-1}(\alpha,0.5)\right). \tag{4.7}$$

Remark 32 (Hakwa, Jäger-Ambrożewicz, and Rüdiger [2015]). *In practice the conditional level* l *for the financial institution* i *is implicitly defined bya given confidence level* $\beta \in (0,1)$ *:*

$$l = F_i^{-1}(\beta). \tag{4.8}$$

The confidence level β *is specified by the respective regulatory institution and represents the probability with which the financial institution* i *remains solvent over a given time horizon.*

Based on this information, we can express $CoVaR_{\alpha}^{s|L^i=l}$ as follows:

$$CoVaR_{\alpha}^{s|L^i=l} = F_s^{-1}\left(g^{-1}(\alpha,\beta)\right). \tag{4.9}$$

We observe that for a given marginal distribution function F_s, $CoVaR_{\alpha}^{s|L^i=l}$ can be expressed as a function of α and β. This motivates the following definition.

Definition 15 (Hakwa, Jäger-Ambrożewicz, and Rüdiger [2015]).

$$\begin{aligned} CoVaR_{\alpha}^{\beta} &:= CoVaR_{\alpha}^{s|L^i=F_i^{-1}(\beta)} \\ \Delta CoVaR_{\alpha}^{\beta} &:= CoVaR_{\alpha}^{s|L^i=F_i^{-1}(\beta)} - CoVaR_{\alpha}^{s|L^i=E\left(L^i\right)} \\ &:= CoVaR_{\alpha}^{\beta} - CoVaR_{\alpha}^{s|L^i=E\left(L^i\right)}. \end{aligned}$$

It follows

$$\Delta CoVaR_{\alpha}^{\beta} = F_s^{-1}\left(g^{-1}(\alpha,\beta)\right) - F_s^{-1}\left(g^{-1}(\alpha,F_i(\mu_i))\right)$$

The subsequent sections are devoted to the application of the formula presented in the Theorem 7.In the course of this applications we investigate the respective qualities of different stochastic models regarding the analysis of systemic risk contribution. We do this by analyzing the abilities, of the considered models, to describe tail dependence.

As we have seen in Section 3.3, the tail dependence coefficients λ_l and λ_u are natural measures for the strength of dependence in the extreme parts of joint distributions. Thus, by considering the tail dependence coefficient of a given stochastic model, we can appreciate its ability to describe extreme comovement and hence systemic risk contribution. In fact, if the joint behavior of L^i and L^s is modeled in such a way that the vector (L^i, L^s) does not show tail dependence (i.e. $\lambda_u = \lambda_u = 0$), then the extreme events (losses) in L^i and L^s would appear to occur independently. In that case there is no possible systemic risk contribution from i to s.

As first application, we consider the Gaussian copula. The Gaussian copula is the most used copula. The use of the Gaussian copula is motivated by its mathematical properties. However, as mentioned by many authors (e.g. Embrechts et al. [1999]), the Gaussian copula may presents some problems in the description of certain specific stochastic behaviors observed in empirical data.

4.2 Application to Gaussian Copula

Assume here that the dependence structure between L^i and L^s is described by a bivariate Gaussian copula.
The bivariate Gaussian copula is defined as follows (cf. Nelsen [2006], Equation 2.3.6):

$$C_\rho(u,v) = \Phi_2\left(\Phi(u)^{-1}, \Phi(v)^{-1}\right),$$

where Φ_2 denotes the bivariate standard normal distribution with linear correlation coefficient ρ, and Φ denotes the univariate standard normal distribution. Hence,

$$C_\rho(u,v) = \int_{-\infty}^{\Phi^{-1}(u)} \int_{-\infty}^{\Phi^{-1}(v)} \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left(\frac{2\rho st - s^2 - t^2}{2(1-\rho^2)}\right) dsdt.$$

Let $X = (U, V)$ be a standard Gaussian random vector with correlation coefficient ρ. Then it follows that

$$\begin{aligned}\Phi_2(u,v) &= Pr(U \le u, V \le v) \\ &= \int_{-\infty}^{u} \int_{-\infty}^{v} \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left(\frac{2\rho su - s^2 - t^2}{2(1-\rho^2)}\right) dsdt\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial \Phi_2(u,v)}{\partial u} &= \int_{-\infty}^{v} \frac{1}{2\pi\sqrt{1-\rho^2}} exp\left(\frac{2\rho ut - u^2 - s^2}{2(1-\rho^2)}\right) ds \\
&= \int_{-\infty}^{v} \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left(\frac{-(s-u\rho)^2 + \rho^2 u^2 - u^2}{2(1-\rho^2)}\right) ds \\
&= \int_{-\infty}^{v} \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left(\frac{-(s-u\rho)^2 - u^2(1-\rho^2)}{2(1-\rho^2)}\right) ds \\
&= \int_{-\infty}^{v} \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left(\frac{-u^2}{2} + \frac{-(s-u\rho)^2}{2(1-\rho^2)}\right) ds \\
&= \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-u^2}{2}\right) \int_{-\infty}^{v} \frac{1}{\sqrt{2\pi}\sqrt{1-\rho^2}} \exp\left(\frac{-(s-u\rho)^2}{2(1-\rho^2)}\right) ds \\
&= \phi(u) \cdot \Phi\left(\frac{v-u\rho}{\sqrt{1-\rho^2}}\right),
\end{aligned}$$

where ϕ denotes the density of the standard univariate normal distribution. Therefore,

$$\Phi_2(u,v) = \int_{-\infty}^{u} \phi(x) \cdot \Phi\left(\frac{v-x\rho}{\sqrt{1-\rho^2}}\right) dx.$$

We have thus

$$\begin{aligned}
C_\rho(u,v) &= \Phi_2\left(\Phi^{-1}(u), \Phi^{-1}(v), \rho\right) \\
&= \int_{-\infty}^{\Phi^{-1}(u)} \phi(x) \cdot \Phi\left(\frac{\Phi^{-1}(v) - x\rho}{\sqrt{1-\rho^2}}\right) dx.
\end{aligned}$$

By substituting s with $\Phi(x)$ $(s = \Phi(x))$, we obtain

$$C_\rho(u,v) = \int_{0}^{u} \Phi\left(\frac{\Phi^{-1}(v) - \rho\Phi^{-1}(s)}{\sqrt{1-\rho^2}}\right) ds$$

and by Theorem 7 we have that

$$g(v,u) = \frac{\partial \Phi_2(u,v)}{\partial u} = \Phi\left(\frac{\Phi^{-1}(v) - \rho\Phi^{-1}(u)}{\sqrt{1-\rho^2}}\right). \tag{4.10}$$

Since the function $g(v,u)$ is invertible with respect to v, we can compute its inverse by setting

$$g(v,u) = \alpha$$

and then solve for v:

$$v = g^{-1}(\alpha,u) = \Phi\left(\rho\Phi^{-1}(u) + \sqrt{1-\rho^2}\Phi^{-1}(\alpha)\right). \tag{4.11}$$

Now, using Theorem 7, we derive the analytical formula for $CoVaR_\alpha^{s|L^i=l}$ for Gaussian copula.

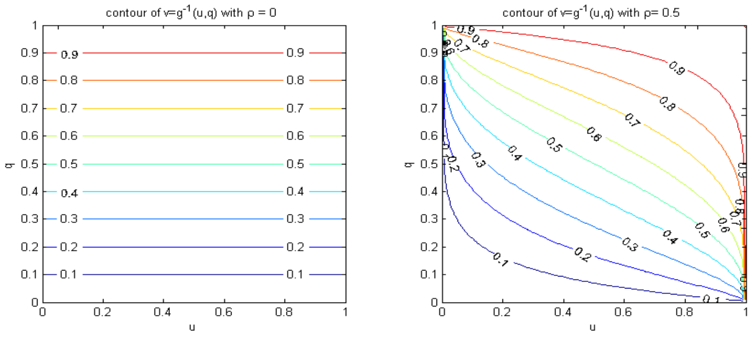


Figure 4.1: g^{-1} of the bivariate Gaussian copula

Corollary 10. *Assume that the copula associated with the joint distribution of L^i and L^s is a Gaussian copula, then*

$$CoVaR_{\alpha}^{s|L^i=l} = F_s^{-1}\left(\Phi\left(\rho\Phi^{-1}\left(F_i\left(l\right)\right) + \sqrt{1-\rho^2}\Phi^{-1}\left(\alpha\right)\right)\right), \tag{4.12}$$

where F^i and F^s represent the univariate distribution functions of L^i and of L^s, respectively.

In the context of Remark 30:

$$CoVaR_{\alpha}^{s|L^i=l} = F_s^{-1}\left(\tilde{\alpha}\right),$$

with $\tilde{\alpha} = \Phi\left(\rho\Phi^{-1}\left(F_i\left(l\right)\right) + \sqrt{1-\rho^2}\Phi^{-1}\left(\alpha\right)\right)$.

Remark 33. *Under the Gaussian copula, the interconnectedness of the financial institution i to the financial system s is modeled only by the correlation coefficient ρ.*

The formula for $\Delta CoVaR_{\alpha}^{s|i}$ according to Definition 8 and using Corollary 10 becomes:

$$\begin{aligned}\Delta CoVaR_{\alpha}^{s|i} =& F_s^{-1}\left(\Phi\left(\rho\Phi^{-1}\left(\alpha\right) + \sqrt{1-\rho^2}\Phi^{-1}\left(\alpha\right)\right)\right) \\ &- F_s^{-1}\left(\Phi\left(\rho\Phi^{-1}\left(F_i\left(\mu_i\right)\right) + \sqrt{1-\rho^2}\Phi^{-1}\left(\alpha\right)\right)\right)\end{aligned}$$

Remark 34. *If $\rho = 0$ then $\tilde{\alpha} = \alpha$, it follows:*

$$\begin{aligned}CoVaR_{\alpha}^{s|L^i=l} &= F_s^{-1}\left(\alpha\right) \\ &= VaR_{\alpha}^{s}.\end{aligned}$$

This is not a surprise because zero correlation means independence under the normal copula setting and consequently

$$\Delta CoVaR_{\alpha}^{s|i} = VaR_{\alpha}^{s} - VaR_{\alpha}^{s} = 0.$$

It is important to note that the distribution functions F_i and F_s can assume any type of univariate distribution function satisfying Assumption 2. In the follows, we consider certain special cases

1. F_i is symmetric, then we have

$$\begin{aligned}\Delta CoVaR_\alpha^{s|i} &= F_s^{-1}\left(\Phi\left(\rho\Phi^{-1}(\alpha)+\sqrt{1-\rho^2}\Phi^{-1}(\alpha)\right)\right)\\ &\quad - F_s^{-1}\left(\Phi\left(\rho\Phi^{-1}(0.5)+\sqrt{1-\rho^2}\Phi^{-1}(\alpha)\right)\right)\\ &= F_s^{-1}\left(\Phi\left(\rho\Phi^{-1}(\alpha)+\sqrt{1-\rho^2}\Phi^{-1}(\alpha)\right)\right) - F_s^{-1}\left(\Phi\left(\sqrt{1-\rho^2}\Phi^{-1}(\alpha)\right)\right).\end{aligned}$$

2. Assume that only L^s (the loss of the financial system s) is normally (univariate) distributed:. Then,

$$\begin{aligned}CoVaR_\alpha^{s|i}(l) &= \sigma_s\Phi^{-1}(\tilde{\alpha})+\mu_s\\ &= VaR_{\tilde{\alpha}}^s,\end{aligned}$$

with $\tilde{\alpha}=\Phi\left(\rho\Phi^{-1}(F_i(l))+\sqrt{1-\rho^2}\Phi^{-1}(\alpha)\right)$. Hence,

$$CoVaR_\alpha^{s|i}(l)=\sigma_s\left(\rho\Phi^{-1}(F_i(l))+\sqrt{1-\rho^2}\Phi^{-1}(\alpha)\right)+\mu_s,$$

and

$$\begin{aligned}\Delta CoVaR_\alpha^{s|i}(l_1,l_2) &= \sigma_s\left(\Phi^{-1}(\tilde{\alpha}_1)-\Phi^{-1}(\tilde{\alpha}_2)\right)\\ &= VaR_{\tilde{\alpha}_1}^s - VaR_{\tilde{\alpha}_2}^s,\end{aligned}$$

where $\tilde{\alpha}_1=g^{-1}(\alpha,F_i(l_1))$ and $\tilde{\alpha}_2=g^{-1}(\alpha,F_i(l_2))$. As special case we have

$$\Delta CoVaR_\alpha^{s|i}=\sigma_s\left(\Phi^{-1}(\tilde{\alpha}_d)-\Phi^{-1}(\tilde{\alpha}_m)\right),$$

with $\tilde{\alpha}_d$ and $\tilde{\alpha}_m$ are the adjusted levels when the financial institution i is under distress and when it has its mean loss respectively. i.e. $\tilde{\alpha}_d = g^{-1}(\alpha,F_i(VaR_\alpha^i))$ and $\tilde{\alpha}_m=g^{-1}(\alpha,F_i(E[L^i]))$.

3. Let us now consider the particular case in which L^i and L^s both assume univariate normal distributions with expected values μ_i, μ_s and standard deviations σ_i, σ_s respectively. W denote by N_i and N_s the distribution function of L^i and L^s, respectively, i.e. $N_i := N(\mu_i,\sigma_i^2)$ and $N_s := N(\mu_s,\sigma_s^2)$.

Corollary 11 (Hakwa, Jäger-Ambrożewicz, and Rüdiger [2015]). *Assume that the copula of (L^i,L^s) is a Gaussian copula and that L^i and L^s both follow univariate normal distribution with expected values μ_i, μ_s and standard deviations σ_i, σ_s, respectively. Then the formula for $CoVaR_\alpha^{s|L^i=l}$ is given by:*

$$CoVaR_\alpha^{s|i}(l)=\rho\frac{\sigma_s}{\sigma_i}(l-\mu_i)+\sqrt{1-\rho^2}\sigma_s\Phi^{-1}(\alpha)+\mu_s. \tag{4.13}$$

Proof. *In fact by Theorem 7 we have:*

$$\begin{aligned} CoVaR_{\alpha}^{s|i}(l) &= N_s^{-1}\left(\Phi\left(\rho\Phi^{-1}(N_i(l)) + \sqrt{1-\rho^2}\Phi^{-1}(\alpha)\right)\right) \\ &= N_s^{-1}\left(N_s\left(\sigma_s\rho\Phi^{-1}(N_i(l)) + \sigma_s\sqrt{1-\rho^2}\Phi^{-1}(\alpha) + \mu_s\right)\right) \\ &= \sigma_s\rho\Phi^{-1}(N_i(l)) + \sigma_s\sqrt{1-\rho^2}\Phi^{-1}(\alpha) + \mu_s \\ &= \sigma_s\rho\Phi^{-1}\left(\Phi\left(\frac{l-\mu_i}{\sigma_i}\right)\right) + \sigma_s\sqrt{1-\rho^2}\Phi^{-1}(\alpha) + \mu_s \\ &= \sigma_s\rho\left(\frac{l-\mu_i}{\sigma_i}\right) + \sigma_s\sqrt{1-\rho^2}\Phi^{-1}(\alpha) + \mu_s \\ &= \rho\frac{\sigma_s}{\sigma_i}(l-\mu_i) + \sqrt{1-\rho^2}\sigma_s\Phi^{-1}(\alpha) + \mu_s. \end{aligned}$$

□

If $\mu_i = \mu_s = 0$, then

$$\begin{aligned} CoVaR_{\alpha}^{s|i}(l) &= \left(\rho\frac{\sigma_s}{\sigma_i}\right)l + \sqrt{1-\rho^2}\sigma_s\Phi^{-1}(\alpha) \\ &= \left(\rho\frac{\sigma_s}{\sigma_i}\right)l + \sqrt{1-\rho^2}VaR_{\alpha}^{s}. \end{aligned}$$

Let l be the Value-at-Risk at the level β of the financial institution i (i.e. $l = VaR_{\beta}^{i} = F_i^{-1}(\beta)$). Then, we have

$$\begin{aligned} CoVaR_{\alpha}^{s|i}\left(VaR_{\beta}^{i}\right) &= \left(\rho\frac{\sigma_s}{\sigma_i}\right)VaR_{\beta}^{i} + \sqrt{1-\rho^2}VaR_{\alpha}^{s} \\ &= \left(\rho\frac{\sigma_s}{\sigma_i}\right)\sigma_i\Phi(\beta) + \sqrt{1-\rho^2}VaR_{\alpha}^{s} \\ &= \rho\sigma_s\Phi(\beta) + \sqrt{1-\rho^2}VaR_{\alpha}^{s} \quad . \end{aligned}$$

Furthermore, if $\beta = \alpha$:

$$\begin{aligned} CoVaR_{\alpha}^{s|i}\left(VaR_{\alpha}^{i}\right) &= \rho\sigma_s\Phi(\alpha) + \sqrt{1-\rho^2}VaR_{\alpha}^{s} \\ &= \rho VaR_{\alpha}^{s} + \sqrt{1-\rho^2}VaR_{\alpha}^{s} \\ &= VaR_{\alpha}^{s}\cdot\left(\rho + \sqrt{1-\rho^2}\right). \end{aligned} \tag{4.14}$$

Remark 35. *Equation* (4.14) *allows us to analyze the effect of the correlation coefficient* ρ *on* $CoVaR_{\alpha}^{s|i}(VaR_{\alpha}^{i})$.

Consider the term $\left(\rho + \sqrt{1-\rho^2}\right)$.

Then

$$\begin{aligned} \frac{\partial\left(\rho + \sqrt{1+\rho^2}\right)}{\partial\rho} &= \frac{\partial\rho}{\partial\rho} + \frac{\partial\left(\rho + \sqrt{1+\rho^2}\right)}{\partial\rho} \\ &= 1 + \frac{1}{2\sqrt{1-\rho^2}}\cdot\frac{\partial(1-\rho^2)}{\partial\rho} \\ &= 1 + \frac{-2\rho}{2\sqrt{1-\rho^2}} \\ &= 1 - \frac{\rho}{\sqrt{1-\rho^2}}. \end{aligned} \tag{4.15}$$

By solving the inequality

$$1 - \frac{\rho}{\sqrt{1-\rho^2}} > 0$$

we obtain as solution $\rho \in \left[-1, \frac{1}{\sqrt{2}}\right]$. This means that the term $\left(\rho + \sqrt{1-\rho^2}\right)$ as function of ρ increases for $\rho \in \left[-1, \frac{1}{\sqrt{2}}\right]$ and decreases for $\rho > \frac{1}{\sqrt{2}}$. This non-monotonic behavior shows that $CoVaR_\alpha^{s|i}\left(VaR_\alpha^i\right)$, under the normal copula setting (i.e. ρ as measure of dependence), is not appropriate for the analyze of systemic risk contribution. .

Remark 36. *The last case considered above, in which a combination of a bivariate Gaussian copula and two univariate Gaussian distributed margins are combined corresponds (bivariate normal distribution setting) was already analyzed in Jäger-Ambrożewicz [2010]. However, differently from the method applied here, Jäger-Ambrożewicz derived a closed formula for* $CoVaR_\alpha^{s|L^i=VaR_\alpha^i}$ *by using the expression of the conditional probability for bivariate normal distribution (cf. e.g. Feller [1968], Equation 2.6).*

Equation (4.13) *coincides with the formula provided by Jäger-Ambrożewicz [2010]. Therefore, the formula proposed by Jäger-Ambrożewicz can be seen as a special case of the formula provided in Theorem 7.*

Corollary 12. *Assume that the copula of* (L^i, L^s) *is a Gaussian copula and that* L^i *and* L^s *both follow univariate normal distributions with expected values* μ_i, μ_s *and standard deviations* σ_i, σ_s*, respectively. Then*

$$a)\ \Delta CoVaR_\alpha^{s|i}\left(l_1, l_2\right) = \rho \frac{\sigma_s}{\sigma_i}\left(l_1 - l_2\right) \tag{4.16}$$

and

$$b)\ \Delta CoVaR_\alpha^{s|i} = \rho\sigma_s\Phi^{-1}\left(\alpha\right) = \rho \cdot \left(VaR_\alpha^s - \mu_s\right). \tag{4.17}$$

Proof. *According to Definition 8 and Definition 11 we have:*

$$a)\ \Delta CoVaR_\alpha^{s|i}\left(l_1, l_2\right) = CoVaR_\alpha^{s|i}\left(l_1\right) - CoVaR_\alpha^{s|i}\left(l_2\right)$$

$$\begin{aligned} \Delta CoVaR_\alpha^{s|i}\left(l_1, l_2\right) &= \rho \frac{\sigma_s}{\sigma_i}\left(l_1 - \mu_i\right) + \sqrt{1-\rho^2}\sigma_s\Phi^{-1}\left(\alpha\right) + \mu_s \\ &\quad - \left[\rho \frac{\sigma_s}{\sigma_i}\left(l_2 - \mu_i\right) + \sqrt{1-\rho^2}\sigma_s\Phi^{-1}\left(\alpha\right) + \mu_s\right] \\ &= \rho \frac{\sigma_s}{\sigma_i}\left(l_1 - l_2\right). \end{aligned}$$

b) $\Delta CoVaR_\alpha^{s|i} = CoVaR_\alpha^{s|L^i=VaR_\alpha^i} - CoVaR_\alpha^{s|L^i=\mu_i}$

$$\begin{aligned}
\Delta CoVaR_\alpha^{s|i} &= CoVaR_\alpha^{s|L^i=VaR_\alpha^i} - CoVaR_\alpha^{s|L^i=\mu_i} \\
&= CoVaR_\alpha^{s|i}\left(VaR_\alpha^i\right) - CoVaR_\alpha^{s|i}\left(\mu_i\right) \\
&= \rho\frac{\sigma_s}{\sigma_i}\left(VaR_\alpha^i - \mu_i\right) \\
&= \rho\frac{\sigma_s}{\sigma_i}\left(\sigma_i\Phi^{-1}\left(\alpha\right) + \mu_i - \mu_i\right) \\
&= \rho\frac{\sigma_s}{\sigma_i}\left(\sigma_i\Phi^{-1}\left(\alpha\right)\right) \\
&= \rho\sigma_s\Phi^{-1}\left(\alpha\right). \qquad \square
\end{aligned}$$

Remark 37. *The ratio* $\rho\frac{\sigma_s}{\sigma_i}$ *in Equation* (4.16) *have not to confound with the beta coefficient* β_{CAPM} *of the capital asset pricing model (CAPM). This is defined as*

$$\beta_{CAPM} = \rho\frac{\sigma_i}{\sigma_s}. \tag{4.18}$$

Recall that under the CAPM framework, the return R_i *of a stock* i *is expressed as a linear combination of a constant* α_i*, a component due to the market (or the system),* R_s*, and a residual* ϵ_i*:*

$$R_i = \alpha_i + \beta_i \cdot R_s + \epsilon_i \tag{4.19}$$

The β_i *coefficient represents in this context the part of the return (or of the risk), that comes from the market (or from the financial system). It is also called* ***systematic risk*** *of the stock* i*.*

4.3 Criticisms on Gaussian Copula as a Model for Systemic Risk Contribution

The main problem with Gaussian copula is its inability to describe tail behaviors. Nevertheless, it still a good model for the center parts of joint distributions[1]. This can be explained by the fact that the linear correlation coefficient is the only dependency parameter in a Gaussian copula model.

Recall hat the linear correlation coefficient ρ_{XY} of two random variables X and Y with means μ_X and μ_Y and standard deviations σ_X and σ_Y can be expressed as:

$$\rho_{X,Y} = E\left[\left(\frac{X-\mu_X}{\sigma_X}\right)\left(\frac{Y-\mu_Y}{\sigma_Y}\right)\right]. \tag{4.20}$$

[1]The most empirical data can be well fitted in their center parts by Gaussian copula

Recall that the covariance Σ_{XY} of the two random variables X and Y, which is defined as

$$\Sigma_{XY} = E\left[\left(X - \mu_X\right)\left(Y - \mu_Y\right)\right], \tag{4.21}$$

describes how X and Y move together around their respective means μ_X and μ_Y. Since the correlation coefficient ρ_{XY} can be seen as a normalization of the covariance Σ_{XY} with the standard deviations σ_X and σ_Y (see equation 4.20). It follows that the linear correlation coefficient measures only the joint dependence of X and Y around their respective means. It is also well know that the linear correlation can only describe the dependence between X and Y when the assume relationship (between X and Y) is linear. This poses a problem when the considered variables are related in a non-linear way.

Let us now consider the tail dependence coefficients of the Gaussian copula. Recall that the Gaussian copula does not have a closed form. therefore, such that the calculation of its tail dependence coefficients using Equation (3.14) and (3.15) pose a problem. Nevertheless, λ_u and λ_l can also be expressed in terms of conditional probabilities. By applying elementary calculus rules (l'Hopital's and differential calculus rule) to equations (3.14) and (3.15), we obtain:

$$\begin{aligned}
\lambda_u &= \lim_{u \to 1^-} \frac{1 - 2u + C\left(u,u\right)}{1-u} \\
&= \lim_{u \to 1^-} \frac{d\left(1 - 2u + C\left(u,u\right)\right)}{d\left(1-u\right)} \\
&= -\lim_{u \to 1^-} -2 + \frac{dC\left(u,u\right)}{du} \\
&= -\lim_{u \to 1^-} \left(-2 + \frac{\partial_1 C\left(u,u\right)}{\partial u} + \frac{\partial_2 C\left(u,u\right)}{\partial u}\right) \\
&= -\lim_{u \to 1^-} \left(-2 + Pr\left(V \leq u | U = u\right) + Pr\left(U \leq u | V = u\right)\right) \\
&= \lim_{u \to 1^-} \left(Pr\left(V > u | U = u\right) + Pr\left(U > u | V = u\right)\right)
\end{aligned} \tag{4.22}$$

and

$$\begin{aligned}
\lambda_l &= \lim_{u \to 0^+} \frac{C\left(u,u\right)}{u} \\
&= \lim_{u \to 0^+} \frac{dC\left(u,u\right)}{du} \\
&= \lim_{u \to 0^+} \left(\frac{\partial_1 C\left(u,u\right)}{\partial u} + \frac{\partial_2 C\left(u,u\right)}{\partial u}\right) \\
&= \lim_{u \to 0^+} \left(Pr\left(V \leq u | U = u\right) + Pr\left(U \leq u | V = u\right)\right).
\end{aligned} \tag{4.23}$$

Remark 38. *If the copula C is symmetric in the sense that* $C(u,\ v) = C(v,\ u)\ \forall u, v$ *then*

$$Pr(V \leq u|U = u) = Pr(U \leq u|V = u) \tag{4.24}$$

Since the Gaussian copula is symmetric, the following expression for the upper tail-dependence coefficients of the Gaussian copula follows:

$$\begin{aligned}\lambda_l = \lambda_u &= 2 \lim_{\alpha \to 1^-} Pr(V > u|U = u)\\ &= 2 \lim_{\alpha \to 1^-} \left[1 - Pr(V \leq u|U = u)\right].\end{aligned}$$

Furthermore, from equation (4.10) it follows that

$$\begin{aligned}\lambda_u &= 2 \lim_{u \to 1^-} \left[1 - \Phi\left(\frac{\Phi^{-1}(u) - \rho\Phi^{-1}(u)}{\sqrt{1-\rho^2}}\right)\right]\\ &= 2 \lim_{u \to 1^-} \left[1 - \Phi\left(\frac{\Phi^{-1}(u)\sqrt{1-\rho}}{\sqrt{1+\rho}}\right)\right].\end{aligned}$$

Hence,

$$\lambda_u = \begin{cases} 0 & \text{if } \rho < 1 \\ 1 & \text{if } \rho = 1. \end{cases}$$

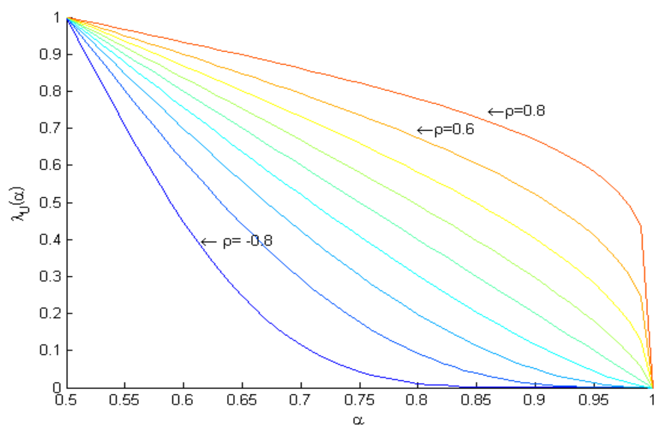


Figure 4.2: Plot of the function $\lambda_u(u) := 1 - \Phi\left(\frac{\Phi^{-1}(u)\sqrt{1-\rho}}{\sqrt{1+\rho}}\right)$.

Therefore if we assume the bivariate Gaussian copula as the dependence model for (L^i, L^s), then as we can see in Figure 4.2 regardless of how high a correlation we choose, if we go far enough into the tail, extreme events appear to occur independently in L^i and L^s. Thus the Gaussian copula is related to the independence in the tail of distributions and hence does not capture tail co-movements. This presents, as already mentioned in this thesis, a significant shortcoming, since tail events and especially tail co-movements, are the main features of systemic financial crises.

Remark 39 (Hakwa, Jäger-Ambrożewicz, and Rüdiger [2015]). *This property of the Gaussian copula is transmitted to the estimation method proposed by Brunnermeier and Adrian [2011] and to the closed formula provided by Jäger-Ambrożewicz [2010]. Both approaches assume that the random vector* $(L^i,\ L^s)$ *follows a bivariate Gaussian distribution, thus implicitly assuming a Gaussian copula as model for the dependence.*

4.4 Application Non-Gaussian Copula

In this section, we use the formula provided in Theorem 7 to analyze and quantify the systemic risk contribution for non Gaussian copulas. Of particular interest are models that allow for tail dependence. We consider the bivariate t-copula as a special case of the class of bivariate elliptical copulas. We furthermore consider the case of Archimedean copula and that of mixtures of copulas.

4.4.1 Application to t-copula

The most used copula besides the Gaussian copula is the t-copulas (Student's copula). Both copulas belong to the class of Elliptical copulas. They are derived from multivariate elliptical distribution functions using Sklar's theorem. IN fact according to Corollary 5, if H is a bivariate distribution function with margins F and G, then the corresponding elliptical copula is given by

$$C(u,v) = H\left(F^{-1}(u),\ G^{-1}(v)\right),\ \forall\ (u,v) \in [0,\ 1]^2. \tag{4.25}$$

In their central parts, the Gaussian copula and the t-copula exhibit the same behavior and properties but show different behaviors in their tail.

The Student copula can be considered as a generalization of the normal copula, that allows the consideration of tail-dependence. It has, in addition to the correlation coefficient ρ, a second dependence parameter, the degree of freedom ν, which controls the heaviness of the tails.

Definition 16. *The distribution function of a bivariate t distributed random variable with correlation coefficient* ρ *degree of freedom* ν *is given by:*

$$t_{\rho,\nu}(u,v) = \int_{-\infty}^{u}\int_{-\infty}^{v} \frac{1}{2\pi\sqrt{1-\rho^2}}\left(1+\frac{s^2+t^2-2\rho^t st}{\nu\left(1-\rho^2\right)}\right)^{-\frac{\nu+2}{2}} dsdt.$$

For $\nu < 3$, the variance does not exist and for $\nu < 5$, the fourth moment does not exist. The t-copula and the Gaussian copula are close to each other in their central parts, and become closer and closer in their tail only when ν increases. Especially both copulas are almost identical when $\nu \to \infty$.

Definition 17. *The bivariate t-copula, $C^t_{\rho,\nu}$, is defined as*

$$\begin{aligned} C^t_{\rho,\nu}(u,v) &= t_{\rho,\nu}\left(t_\nu^{-1}(u), t_\nu^{-1}(v)\right) \\ &= \int_{-\infty}^{t_\nu^{-1}(u)} \int_{-\infty}^{t_\nu^{-1}(v)} \frac{1}{2\pi\sqrt{1-\rho^2}} \left(1 + \frac{s^2+t^2-2\rho^t st}{\nu(1-\rho^2)}\right)^{-\frac{\nu+2}{2}} dsdt, \end{aligned}$$

where t_ν denotes the distribution function of a standard t univariate distributed random variable with ν degrees of freedom.

Proposition 1. *Let (X,Y) be a bivariate standard t-distributed random vector with ν degrees of freedom and linear correlation ρ. Then conditional on $X = x$ we have:*

$$\left(\frac{\nu+1}{\nu+x^2}\right)^{1/2} \frac{Y-\rho x}{\sqrt{1-\rho^2}} \sim t_{\nu+1}. \tag{4.26}$$

The proof of Proposition 1 is done using the transformation rule for density function.

Theorem 13 (Transformation formula (cf. Klenke [2008], Theorem 1.101))**.** *Let μ be a measure on $\mathbb{R}$ that has a continuous (or piecewise continuous) density $f\colon \mathbb{R} \to [0,\infty)$ That is*

$$\mu((-\infty, x]) = \int_{-\infty}^{x} f(t)\,dt, \ \forall\ x \in \mathbb{R}$$

Let $A \subset \mathbb{R}$ be an open or a closed subset of $\mathbb{R}$ with $\mu(\mathbb{R} \setminus A) = 0$. Further, let $B \subset \mathbb{R}$ be open or closed. Finally, assume that $\varphi : A \to B$ is a continuously differentiable bijection with derivative φ' Then the image measure $\mu \circ \varphi^{-1}$ has the density

$$f_\varphi(x) = \begin{cases} \frac{f\left(\varphi^{-1}(x)\right)}{|det(\varphi'(\varphi^{-1}(x)))|}, & if\ x \in B \\ 0, & if\ x \in \mathbb{R} \setminus B. \end{cases}$$

Proof. *Let (X,Y) a random vector following a bivariate t-distribution. Define a new random variable*

$$R := \varphi(Y) = \left(\frac{\nu+1}{(\nu+x^2)(1-\rho^2)}\right)^{1/2} Y - \rho x.$$

Applying Theorem 13, yields:

$$f_{R|x}(r) = \frac{f_{Y|x}\left(r\left(\frac{\nu+1}{(\nu+x^2)(1-\rho^2)}\right)^{-1/2} + \rho x\right)}{\left(\frac{\nu+1}{(\nu+x^2)(1-\rho^2)}\right)^{1/2}}. \tag{4.27}$$

Recall that the conditional density of Y given $X = x$ is given by (cf. Kotz and Nadarajah [2004], Equation (1.15)):

$$f_{Y|X}(y) = \frac{\Gamma\left(\frac{\nu+2}{2}\right)}{\sqrt{\nu\pi(1-\rho^2)}\Gamma\left(\frac{\nu+1}{2}\right)}\left[1 + \frac{(y-\rho x)^2 + x^2(1-\rho^2)}{\nu(1-\rho^2)}\right]^{-\frac{\nu+2}{2}}\left[1+\frac{x^2}{\nu}\right]^{\frac{\nu+1}{2}}. \tag{4.28}$$

By setting $y = \varphi^{-1}(r)$ we obtain

$$\begin{aligned}
f_{Y|X}\left(\varphi^{-1}(r)\right) &= \frac{\Gamma\left(\frac{\nu+2}{2}\right)}{\sqrt{\nu\pi(1-\rho^2)}\Gamma\left(\frac{\nu+1}{2}\right)}\left[1 + \frac{\left(r\left(\frac{(1-\rho^2)(\nu+x^2)}{\nu+1}\right)^{\frac{1}{2}}\right)^2 + x^2(1-\rho^2)}{\nu(1-\rho^2)}\right]^{-\frac{\nu+2}{2}}\left[1+\frac{x^2}{\nu}\right]^{\frac{\nu+1}{2}} \\
&= \frac{\Gamma\left(\frac{\nu+2}{2}\right)}{\sqrt{\nu\pi(1-\rho^2)}\Gamma\left(\frac{\nu+1}{2}\right)}\left[1 + \frac{\frac{r^2(1-\rho^2)(\nu+x^2)}{\nu+1} + x^2(1-\rho^2)}{\nu(1-\rho^2)}\right]^{-\frac{\nu+2}{2}}\left[1+\frac{x^2}{\nu}\right]^{\frac{\nu+1}{2}} \\
&= \frac{\Gamma\left(\frac{\nu+2}{2}\right)}{\sqrt{\nu\pi(1-\rho^2)}\Gamma\left(\frac{\nu+1}{2}\right)}\left[1 + \frac{r^2(\nu+x^2)}{\nu(\nu+1)} + \frac{x^2}{\nu}\right]^{-\frac{\nu+2}{2}}\left[1+\frac{x^2}{\nu}\right]^{\frac{\nu+1}{2}} \\
&= \frac{\Gamma\left(\frac{\nu+2}{2}\right)}{\sqrt{\nu\pi(1-\rho^2)}\Gamma\left(\frac{\nu+1}{2}\right)}\left[\left(1 + \frac{r^2}{\nu+1}\right)\left(1+\frac{x^2}{\nu}\right)\right]^{-\frac{\nu+2}{2}}\left[1+\frac{x^2}{\nu}\right]^{\frac{\nu+1}{2}} \\
&= \frac{\Gamma\left(\frac{\nu+2}{2}\right)}{\sqrt{\nu\pi(1-\rho^2)}\Gamma\left(\frac{\nu+1}{2}\right)}\left(1 + \frac{r^2}{\nu+1}\right)^{-\frac{\nu+2}{2}}\left(1+\frac{x^2}{\nu}\right)^{-\frac{\nu+2}{2}}\left[1+\frac{x^2}{\nu}\right]^{\frac{\nu+1}{2}} \\
&= \frac{\Gamma\left(\frac{\nu+2}{2}\right)}{\sqrt{\nu\pi(1-\rho^2)}\Gamma\left(\frac{\nu+1}{2}\right)}\left(1 + \frac{r^2}{\nu+1}\right)^{-\frac{\nu+2}{2}}\left(1+\frac{x^2}{\nu}\right)^{-\frac{1}{2}} \\
&= \frac{\Gamma\left(\frac{\nu+2}{2}\right)}{\Gamma\left(\frac{\nu+1}{2}\right)}\left(1 + \frac{r^2}{\nu+1}\right)^{-\frac{\nu+2}{2}}\left((\nu\pi(1-\rho^2))\left(1+\frac{x^2}{\nu}\right)\right)^{-\frac{1}{2}} \\
&= \frac{\Gamma\left(\frac{\nu+2}{2}\right)}{\Gamma\left(\frac{\nu+1}{2}\right)}\left(1 + \frac{r^2}{\nu+1}\right)^{-\frac{\nu+2}{2}}\left(\pi(1-\rho^2)(\nu+x^2)\right)^{-\frac{1}{2}} \\
&= \frac{\Gamma\left(\frac{\nu+2}{2}\right)}{\Gamma\left(\frac{\nu+1}{2}\right)}\left(1 + \frac{r^2}{\nu+1}\right)^{-\frac{\nu+2}{2}}\left(\frac{\nu+1}{\pi(\nu+1)(1-\rho^2)(\nu+x^2)}\right)^{\frac{1}{2}} \\
&= \frac{\Gamma\left(\frac{\nu+2}{2}\right)}{\Gamma\left(\frac{\nu+1}{2}\right)}\left(1 + \frac{r^2}{\nu+1}\right)^{-\frac{\nu+2}{2}}\left(\frac{1}{\pi(\nu+1)}\right)^{\frac{1}{2}}\left(\frac{\nu+1}{(1-\rho^2)(\nu+x^2)}\right)^{\frac{1}{2}} \\
&= \frac{\Gamma\left(\frac{\nu+2}{2}\right)}{\sqrt{\pi(\nu+1)}\Gamma\left(\frac{\nu+1}{2}\right)}\left(1 + \frac{r^2}{\nu+1}\right)^{-\frac{\nu+2}{2}}\left(\frac{\nu+1}{(1-\rho^2)(\nu+x^2)}\right)^{\frac{1}{2}}.
\end{aligned}$$

This implies according to equation (4.27) *that*

$$f_{R|X}(y) = \frac{\Gamma\left(\frac{\nu+2}{2}\right)}{\sqrt{\pi(\nu+1)}\Gamma\left(\frac{\nu+1}{2}\right)}\left(1+\frac{r^2}{\nu+1}\right)^{-\frac{\nu+2}{2}}. \tag{4.29}$$

This follows then from the fact that the density function of an univariate t-distribution with degrees of freedom ν is given by:

$$t_\nu(r) = \frac{\Gamma(\frac{\nu+1}{2})}{\sqrt{\nu\pi}\,\Gamma(\frac{\nu}{2})}\left(1+\frac{r^2}{\nu}\right)^{-\frac{\nu+1}{2}}$$

□

Based on Proposition 1, the tail dependence coefficients of the t-copula $C^t_{\rho,\nu}$ can be computed using equation 4.23 (cf. Embrechts et al. [2003])

$$\lambda_l = \lambda_u = 2 - 2t_{\nu+1}\left(\left(\frac{(\nu+1)(1-\rho)}{1+\rho}\right)^{\frac{1}{2}}\right).$$

Hence,

$$\lambda_u \begin{cases} > 0 & \text{if } \rho > -1 \\ = 0 & \text{if } \rho = -1 \end{cases}.$$

So, given $\rho > -1$, the bivariate t-copula captures the dependence of extreme values and is thus appropriate for modeling the analyzing systemic risk contribution.

The t-copula $C^t_{\rho,\nu}(u,v)$ can be expressed as follows (cf. e.g. Roncalli [2009], Page 299):

$$C^t_{\rho,\nu}(u,v) = \int_0^u t_{\nu+1}\left(\left(\frac{\nu+1}{\nu+\left[t_\nu^{-1}(u)\right]^2}\right)^{1/2}\frac{t_\nu^{-1}(v)-\rho t_\nu^{-1}(t)}{\sqrt{1-\rho^2}}\right)dt. \tag{4.30}$$

The expression of $g(v,u)$ is then given by:

$$\begin{aligned} g(v,u) &= \frac{\partial C^t_{\rho,\nu}(u,v)}{\partial u} \\ &= t_{\nu+1}\left(\left(\frac{\nu+1}{\nu+\left[t_\nu^{-1}(u)\right]^2}\right)^{1/2}\frac{t_\nu^{-1}(v)-\rho t_\nu^{-1}(u)}{\sqrt{1-\rho^2}}\right). \end{aligned}$$

The function g is invertible with respect to v and its inverse is obtained by solving the equation

$$g(v,u) = t_{\nu+1}\left(\left(\frac{\nu+1}{\nu+\left[t_\nu^{-1}(u)\right]^2}\right)^{1/2}\frac{t_\nu^{-1}(v)-\rho t_\nu^{-1}(u)}{\sqrt{1-\rho^2}}\right) = \alpha$$

for v:

$$v = g^{-1}(\alpha,u) = t_\nu\left(\rho t_\nu^{-1}(u) + \sqrt{\frac{(1-\rho^2)\left(\nu+\left[t_\nu^{-1}(u)\right]^2\right)}{\nu+1}}t_{\nu+1}^{-1}(\alpha)\right).$$

Using this, we obtain the following formulas for $CoVaR_\alpha^{s|L^i=l}$ and $CoVaR_\alpha^\beta$

Proposition 2 (Hakwa, Jäger-Ambrożewicz, and Rüdiger [2015]). *Let the t-copula be the copula of* (L^i, L^s)*. Then for every* $l \in \mathbb{R}$

$$CoVaR_{\alpha}^{s|L^i=l} = F_s^{-1}\left(t_{\nu}\left(\rho t_{\nu}^{-1}\left(F_i\left(l\right)\right) + \sqrt{\frac{\left(1-\rho^2\right)\left(\nu + \left[t_{\nu}^{-1}\left(F_i\left(l\right)\right)\right]^2\right)}{\nu+1}} t_{\nu+1}^{-1}\left(\alpha\right)\right)\right)$$

and

$$CoVaR_{\alpha}^{\beta} = F_s^{-1}\left(t_{\nu}\left(\rho t_{\nu}^{-1}\left(\beta\right) + \sqrt{\frac{\left(1-\rho^2\right)\left(\nu + \left[t_{\nu}^{-1}\left(\beta\right)\right]^2\right)}{\nu+1}} t_{\nu+1}^{-1}\left(\alpha\right)\right)\right),$$

where F_i *and* F_s *represent the univariate distribution function of* L^i *and* L^s*, respectively, and* β *denotes the regulatory risk level of financial institution* i*.*

Remark 40. *Differently to the Gaussian copula case where the interconnectedness of the financial institution* i *to the financial system* s *is modeled only by the correlation coefficient* ρ *(see Remark 33), the interconnectedness of the financial institution* i *to the financial system* s *is modeled, under the t-copula, by the correlation coefficient* ρ *and the number of degrees of freedom* ν*. system* s*.*

Corollary 14 (Hakwa, Jäger-Ambrożewicz, and Rüdiger [2015]). *Assume that* L^i *and* L^s *both follow univariate standard t distribution with* ν *degrees of freedom. Then for every* $l \in \mathbb{R}$

$$CoVaR_{\alpha}^{s|i}\left(l\right) = \rho l + \sqrt{\frac{\left(1-\rho^2\right)\left(\nu + l^2\right)}{\nu+1}} t_{\nu+1}^{-1}\left(\alpha\right). \tag{4.31}$$

It follows that

$$\begin{aligned}
\Delta CoVaR_{\alpha}^{s|i}\left(l_1, l_2\right) &= CoVaR_{\alpha}^{s|i}\left(l_1\right) - CoVaR_{\alpha}^{s|i}\left(l_2\right) \\
&= \rho l_{+} \sqrt{\frac{\left(1-\rho^2\right)\left(\nu + l_1^2\right)}{\nu+1}} t_{\nu+1}^{-1}\left(\alpha\right) - \rho l_2 - \sqrt{\frac{\left(1-\rho^2\right)\left(\nu + l_2^2\right)}{\nu+1}} t_{\nu+1}^{-1}\left(\alpha\right) \\
&= \rho\left(l_1 - l_2\right) + \sqrt{\frac{1-\rho^2}{\nu+1}} t_{\nu+1}^{-1}\left(\alpha\right)\left[\sqrt{\nu + l_1^2} - \sqrt{\nu + l_2^2}\right]
\end{aligned}$$

Especially, we have that $\Delta CoVaR^{s|i}$ as defined in Brunnermeier and Adrian [2011] (see definition 9), is given by

Corollary 15.

$$\Delta CoVaR^{s|i} = \rho VaR_{\alpha}^{i} + \sqrt{\frac{1-\rho^2}{\nu+1}} t_{\nu+1}^{-1}\left(\alpha\right)\left[\sqrt{\nu + \left(VaR_{\alpha}^{i}\right)^2} - \sqrt{\nu}\right] \tag{4.32}$$

where we use the fact that the standard t-distribution with ν degrees of freedom has a mean equal to zero.

The standard t-distribution can be extended through linear transformation:

$$X := a + bZ, \qquad Z \sim t_\nu.$$

The distribution of X is called generalized t-distribution ($X \sim T\left(a, b^2, \nu\right)$), the mean of X is equal to ($E\left[X\right] = a$) and its variance $V\left[X\right]$ is given by $V\left[X\right] = b^2 V\left[Z\right] = b^2 \frac{\nu}{\nu-2}$. The corresponding density function f_T can be obtained using the transformation formula for density (cf. e.g. Klenke [2008], Theorem. 1.101). Let f_t be the density function of the standard t-distribution, then

$$f_T\left(x\right) = f_t\left(g\left(z\right)\right) = \frac{f_t(g^{-1}(z))}{|g'(g^{-1}(z))|}, \quad \text{with } g\left(z\right) = a + bz.$$

Hence

$$f_T\left(x\right) = f_t\left(\frac{x-a}{b}\right)\left|\frac{1}{b}\right|, \qquad b \neq 0.$$

Note that

$$T\left(x\right) = Pr\left(X \leq x\right) = Pr\left(a + bZ \leq x\right) = Pr\left(Z \leq \frac{x-a}{b}\right) = t_\nu\left(\frac{x-a}{b}\right).$$

Such that

$$\begin{aligned} T\left(x\right) = \alpha \ &\Leftrightarrow \frac{x-a}{b} = t_\nu^{-1}\left(\alpha\right) \\ &\Leftrightarrow x = bt_\nu^{-1}\left(\alpha\right) + a \\ &\Leftrightarrow T^{-1}\left(\alpha\right) = bt_\nu^{-1}\left(\alpha\right) + a. \end{aligned} \tag{4.33}$$

Let us now compute $CoVaR_\alpha^{s|i}\left(l\right)$ and $\Delta CoVaR_\alpha^{s|i}\left(l_1, l_2\right)$ for following cases:

1. Let $\mu_s := E\left(L^s\right)$. If $L^s \sim T\left(\mu_s, \sigma_s^{*2}, \nu\right)$, i.e. $\left(\frac{L^s-\mu_s}{\sigma_s^*}\right)$ follows an univariate standard t-distribution with ν degrees of freedom, then

$$\begin{aligned} CoVaR_\alpha^{s|i}\left(l\right) &= F_s^{-1}\left(t_\nu\left(\rho t_\nu^{-1}\left(F_i\left(l\right)\right) + \sqrt{\frac{\left(1-\rho^2\right)\left(\nu + \left[t_\nu^{-1}\left(F_i\left(l\right)\right)\right]^2\right)}{\nu+1}} t_{\nu+1}^{-1}\left(\alpha\right)\right)\right) \\ &= \sigma_s^* t_\nu^{-1}\left(t_\nu\left(\rho t_\nu^{-1}\left(F_i\left(l\right)\right) + \sqrt{\frac{\left(1-\rho^2\right)\left(\nu + \left[t_\nu^{-1}\left(F_i\left(l\right)\right)\right]^2\right)}{\nu+1}} t_{\nu+1}^{-1}\left(\alpha\right)\right)\right) + \mu_s \\ &= \sigma_s^*\left(\rho t_\nu^{-1}\left(F_i\left(l\right)\right) + \sqrt{\frac{\left(1-\rho^2\right)\left(\nu + \left[t_\nu^{-1}\left(F_i\left(l\right)\right)\right]^2\right)}{\nu+1}} t_{\nu+1}^{-1}\left(\alpha\right)\right) + \mu_s. \end{aligned}$$

2. Let $\mu_i := E(L^i)$ and $\mu_s := E(L^s)$. If $L^i \sim T\left(\mu_i, \sigma_i^{*2}, \nu\right)$ and $L^s \sim T\left(\mu_s, \sigma_s^{*2}, \nu\right)$, i.e. $\left(\frac{L^i-\mu_i}{\sigma_i^*}\right)$ and $\left(\frac{L^s-\mu_s}{\sigma_s^*}\right)$ both follow an univariate standard t distribution with ν degrees of freedom, then

$$
\begin{aligned}
CoVaR_\alpha^{s|i}(l) =& \sigma_s^* \left(\rho t_\nu^{-1}(F_i(l)) + \sqrt{\frac{(1-\rho^2)\left(\nu + \left[t_\nu^{-1}(F_i(l))\right]^2\right)}{\nu+1}} t_{\nu+1}^{-1}(\alpha) \right) + \mu_s \\
=& \sigma_s^* \left(\rho t_\nu^{-1}\left(t_\nu\left(\frac{l-\mu_i}{\sigma_i^*}\right)\right) + \sqrt{\frac{(1-\rho^2)\left(\nu + \left[t_\nu^{-1}\left(t_\nu\left(\frac{l-\mu_i}{\sigma_i^*}\right)\right)\right]^2\right)}{\nu+1}} t_{\nu+1}^{-1}(\alpha) \right) + \mu_s \\
=& \sigma_s^* \left(\rho\left(\frac{l-\mu_i}{\sigma_i^*}\right) + \sqrt{\frac{(1-\rho^2)\left(\nu + \left(\frac{l-\mu_i}{\sigma_i^*}\right)^2\right)}{\nu+1}} t_{\nu+1}^{-1}(\alpha) \right) + \mu_s \\
=& \frac{\sigma_s^* \rho}{\sigma_i^*}(l-\mu_i) + \sigma_s^* t_{\nu+1}^{-1}(\alpha) \sqrt{\frac{(1-\rho^2)\left(\nu + \left(\frac{l-\mu_i}{\sigma_i^*}\right)^2\right)}{\nu+1}} + \mu_s.
\end{aligned}
$$

Recall

$$
\begin{aligned}
\Delta CoVaR_\alpha^{s|i}(l_1, l_2) &= CoVaR_\alpha^{s|i}(l_1) - CoVaR_\alpha^{s|i}(l_2) \\
&= F_s^{-1}\left(t_\nu \left(\rho t_\nu^{-1}(F_i(l_1)) + \sqrt{\frac{(1-\rho^2)\left(\nu + \left[t_\nu^{-1}(F_i(l_1))\right]^2\right)}{\nu+1}} t_{\nu+1}^{-1}(\alpha) \right)\right) \\
&- \left[F_s^{-1}\left(t_\nu \left(\rho t_\nu^{-1}(F_i(l_2)) + \sqrt{\frac{(1-\rho^2)\left(\nu + \left[t_\nu^{-1}(F_i(l_2))\right]^2\right)}{\nu+1}} t_{\nu+1}^{-1}(\alpha) \right)\right)\right].
\end{aligned}
$$

Consider again the previous two cases

1. If $L^s \sim T\left(\mu_s, \sigma_s^{*2}, \nu\right)$, then

$$\begin{aligned}
\Delta CoVaR_\alpha^{s|i}(l_1,l_2) &= \sigma_s^*\left(\rho t_\nu^{-1}(F_i(l_1)) + \sqrt{\frac{(1-\rho^2)\left(\nu+\left[t_\nu^{-1}(F_i(l_1))\right]^2\right)}{\nu+1}}t_{\nu+1}^{-1}(\alpha)\right)+\mu_s \\
&- \left[\sigma_s^*\left(\rho t_\nu^{-1}(F_i(l_2)) + \sqrt{\frac{(1-\rho^2)\left(\nu+\left[t_\nu^{-1}(F_i(l_2))\right]^2\right)}{\nu+1}}t_{\nu+1}^{-1}(\alpha)\right)+\mu_s\right] \\
&= \sigma_s^*\rho\left[t_\nu^{-1}(F_i(l_1)) - t_\nu^{-1}(F_i(l_2))\right] \\
&+ \frac{\sigma_s^* t_{\nu+1}^{-1}(\alpha)\sqrt{1-\rho^2}}{\sqrt{\nu+1}}\left(\left[t_\nu^{-1}(F_i(l_1))\right]^2 - \left[t_\nu^{-1}(F_i(l_2))\right]^2\right) \\
&= \sigma_s^*\rho\left(t_\nu^{-1}(F_i(l_1)) - t_\nu^{-1}(F_i(l_2))\right) \\
&+ \left[\frac{\sigma_s^* t_{\nu+1}^{-1}(\alpha)\sqrt{1-\rho^2}}{\sqrt{\nu+1}}\left(t_\nu^{-1}(F_i(l_1)) - t_\nu^{-1}(F_i(l_2))\right)\left(t_\nu^{-1}(F_i(l_1)) + t_\nu^{-1}(F_i(l_2))\right)\right] \\
&= \sigma_s^*\left[t_\nu^{-1}(F_i(l_1)) - t_\nu^{-1}(F_i(l_2))\right]\left[\rho + \frac{t_{\nu+1}^{-1}(\alpha)\sqrt{1-\rho^2}}{\sqrt{\nu+1}}\left(t_\nu^{-1}(F_i(l_1)) + t_\nu^{-1}(F_i(l_2))\right)\right].
\end{aligned}$$

2. If $L^i \sim T\left(\mu_i, \sigma_i^{*2}, \nu\right)$ and $L^s \sim T\left(\mu_s, \sigma_s^{*2}, \nu\right)$, then

$\Delta CoVaR_\alpha^{s|i}(l_1,l_2)$

$$\begin{aligned}
&= \frac{\sigma_s^*\rho}{\sigma_i^*}(l-\mu_i) + \sigma_s^* t_{\nu+1}^{-1}(\alpha)\sqrt{\frac{(1-\rho^2)\left(\nu+\left(\frac{l-\mu_i}{\sigma_i^*}\right)^2\right)}{\nu+1}}+\mu_s \\
&- \left[\frac{\sigma_s^*\rho}{\sigma_i^*}(l-\mu_i) + \sigma_s^* t_{\nu+1}^{-1}(\alpha)\sqrt{\frac{(1-\rho^2)\left(\nu+\left(\frac{l-\mu_i}{\sigma_i^*}\right)^2\right)}{\nu+1}}+\mu_s\right] \\
&= \frac{\sigma_s^*\rho}{\sigma_i^*}(l_1-l_2) + \frac{\sigma_s^* t_{\nu+1}^{-1}(\alpha)\sqrt{1-\rho^2}}{\sqrt{\nu+1}}\left(\left(\frac{l_1-\mu_i}{\sigma_i^*}\right)^2 - \left(\frac{l_2-\mu_i}{\sigma_i^*}\right)^2\right) \\
&= \frac{\sigma_s^*\rho}{\sigma_i^*}(l_1-l_2) + \frac{\sigma_s^* t_{\nu+1}^{-1}(\alpha)\sqrt{1-\rho^2}}{\sqrt{\nu+1}}\left(\frac{l_1-\mu_i}{\sigma_i^*} - \frac{l_2-\mu_i}{\sigma_i^*}\right)\left(\frac{l_1-\mu_i}{\sigma_i^*} + \frac{l_2-2\mu_i}{\sigma_i^*}\right) \\
&= \frac{\sigma_s^*\rho}{\sigma_i^*}(l_1-l_2) + \frac{\sigma_s^* t_{\nu+1}^{-1}(\alpha)\sqrt{1-\rho^2}}{\sqrt{\nu+1}}\left(\frac{l_1-l_2}{\sigma_i^*}\right)\left(\frac{l_1+l_2-2\mu_i}{\sigma_i^*}\right) \\
&= \frac{\sigma_s^*}{\sigma_i^*}(l_1-l_2)\left[\rho + \frac{t_{\nu+1}^{-1}(\alpha)\sqrt{1-\rho^2}}{\sqrt{\nu+1}}\left(\frac{l_1+l_2-2\mu_i}{\sigma_i^*}\right)\right]. \qquad (4.34)
\end{aligned}$$

Corollary 16 (Hakwa, Jäger-Ambrożewicz, and Rüdiger [2015]). *Let $\mu_i := E(L^i)$ and $\mu_s := E(L^s)$. If $L^i \sim T\left(\mu_i, \sigma_i^{*2}, \nu\right)$ and $L^s \sim T\left(\mu_s, \sigma_s^{*2}, \nu\right)$, i.e. $\left(\frac{L^i-\mu_i}{\sigma_i^*}\right)$ and $\left(\frac{L^s-\mu_s}{\sigma_s^*}\right)$ both follow univariate standard t-distribution with ν degrees of freedom, then*

$$\Delta CoVaR_\alpha^{s|L^i=l} = \sigma_s^* t_\nu^{-1}(\alpha)\left[\rho + t_\nu^{-1}(\alpha)\frac{t_{\nu+1}^{-1}(\alpha)\sqrt{1-\rho^2}}{\sqrt{\nu+1}}\right]. \qquad (4.35)$$

Proof. *From equation* (4.34) *we know that*

$$\begin{aligned}\Delta CoVaR_\alpha^{s|L^i=l} &= \Delta CoVaR_\alpha^{s|i}\left(VaR_\alpha^i, \mu_i\right) \\ &= \frac{\sigma_s^*}{\sigma_i^*}\left(VaR_\alpha^i - \mu_i\right)\left[\rho + \frac{t_{\nu+1}^{-1}(\alpha)\sqrt{1-\rho^2}}{\sqrt{\nu+1}}\left(\frac{VaR_\alpha^i + \mu_i - 2\mu_i}{\sigma_i^*}\right)\right].\end{aligned}$$

Since

$$VaR_\alpha^i = \mu_i + \sigma_i^* t_\nu^{-1}(\alpha),$$

it follows that

$$\begin{aligned}&\Delta CoVaR_\alpha^{s|L^i=l} \\ &= \frac{\sigma_s^*}{\sigma_i^*}\left(\mu_i + \sigma_i^* t\nu^{-1}(\alpha) - \mu_i\right)\left[\rho + \frac{t_{\nu+1}^{-1}(\alpha)\sqrt{1-\rho^2}}{\sqrt{\nu+1}}\left(\frac{\mu_i + \sigma^* t\nu^{-1}(\alpha) + \mu_i - 2\mu_i}{\sigma_i^*}\right)\right] \\ &= \sigma_s^* t_\nu^{-1}(\alpha)\left[\rho + t_\nu^{-1}(\alpha)\frac{t_{\nu+1}^{-1}(\alpha)\sqrt{1-\rho^2}}{\sqrt{\nu+1}}\right].\end{aligned}$$

□

4.4.2 Archimedean copula

Note that the dependence in the Gaussian and t-copulas setting are essentially determined by the correlation coefficient ρ (Because both are elliptical copula).

In contrast to elliptical copulas, the dependence in bivariate Archimedean copula is controlled by a function φ called generator.

Theorem 17. *[Nelsen [2006], Theorem 4.1.4] Let φ be a continuous, strictly decreasing function from $[0,\ 1]$ to $[0,\infty]$ such that $\varphi(1) = 0$ and let $\varphi^{[-1]}(t)$ be the pseudo-inverse of φ defined by*

$$\varphi^{[-1]}(t) = \begin{cases} \varphi^{-1}(t) & \text{if } 0 \le t \le \varphi(0) \\ 0 & \text{if } \varphi(0) < t \le \infty. \end{cases},$$

then the function C from $[0,\ 1]^2$ to $[0,\ 1]$ is given by

$$C(u,v) = \varphi^{[-1]}(\varphi(u) + \varphi(v)) \tag{4.36}$$

is a copula if and only if φ is convex.

Definition 18. *A function φ satisfying the conditions of Theorem 17 is called generator of a copula. A copula constructed through a generator is called Archimedian copula.*

Remark 41.

1. *For any constant $c > 0$, φ and $c \cdot \varphi$ generate the same Archimedean copula. Indeed*

$$C(u,v) = \varphi^{[-1]}(\varphi(u) + \varphi(v)) \\ = \frac{\varphi^{[-1]}}{c}(c \cdot \varphi(u) + c \cdot \varphi(v)).$$

2. $\varphi^{[-1]}(\varphi(t)) = t, \quad \forall\, t \in [0,1].$

If $\varphi(0) = \infty$, the generator is said to be strict and its pseudo-inverse $\varphi^{[-1]}$ coincide with the ordinary functional inverse φ^{-1} (cf. Nelsen [2006] Definition 4.1.1).

The lower and upper tail dependence coefficients of an Archimedean copula can be computed using the following corollary.

Corollary 18 (Nelsen [2006], Corollary. 5.4.3)**.** *Let C be an Archimedean copula with a continuous, strictly decreasing and convex generator φ, then*

$$\lambda_u = 2 - \lim_{x \to 0^+} \frac{1 - \varphi^{-1}(2x)}{1 - \varphi^{-1}(x)} \quad \text{and} \quad \lambda_l = \lim_{x \to \infty} \frac{1 - \varphi^{-1}(2x)}{1 - \varphi^{-1}(x)}$$

In the context of systemic risk analysis, we are particularly interested in Archimedean copulas showing positive tail dependence (e.g. Gumbel and Clayton copula).

Remark 42. *If we assume a copula with positive upper (lower) tail dependence, losses have to be defined as positive (negative) numbers (cf. McNeil et al. [2005]).*

Example 6 (Gumbel Copula)**.** *The generator of the Gumbel copula is defined by*

$$\varphi_\theta(t) = (-\ln(t))^\theta \quad \text{for } \theta \geq 1. \tag{4.37}$$

It holds $\varphi_\theta(0) = \infty$, i.e. φ_θ is strict and its inverse is $\varphi_\theta^{-1}(t) = \exp\left(-t^{\frac{1}{\theta}}\right)$. The Gumbel copula is then according to equation 4.36 given by:

$$C_\theta^{Gu}(u,v) = \exp\left(-\left[(-\ln(u))^\theta + (-\ln(v))^\theta\right]^{\frac{1}{\theta}}\right), \quad 1 \leq \theta < \infty,$$

where θ represents the strength of dependence. Using Corollary 18, we can compute the tail dependence coefficients λ_u and λ_l of the Gumbel copula.

$$\lambda_u = 2 - 2^{\frac{1}{\theta}} \quad \text{and} \quad \lambda_l = 0.$$

The Gumbel copula is thus able to model contagion effects and is therefore a good model for the analysis of systemic risk contribution.

According to Theorem 7, the corresponding function g is given by:

$$\begin{aligned} g_{Gu}(v,u) &:= \frac{\partial C_\theta^{Gu}(u,v)}{\partial u} \\ &= \frac{\partial \exp\left(-\left((-\ln(u))^\theta + (-\ln(v))^\theta\right)^{\frac{1}{\theta}}\right)}{\partial u} \\ &= \exp\left(-\left((-\ln(u))^\theta + (-\ln(v))^\theta\right)^{\frac{1}{\theta}}\right) \cdot \left((-\ln(u))^\theta + (-\ln(v))^\theta\right)^{-\frac{\theta-1}{\theta}} \cdot \frac{(-\ln(u))^{\theta-1}}{u}. \end{aligned}$$

For $u \in (0,1)$ and $\theta > 1$, the function g is strictly increasing with respect to v and therefore invertible. Hence, according to Theorem 7, we can compute $CoVaR_\alpha^{s|L^i=l}$ as

$$CoVaR_\alpha^{s|L^i=l} = F_s^{-1}\left(g_{Gu}^{-1}(\alpha, F_i(l))\right). \tag{4.38}$$

By imposing certain conditions to the generator φ of an Archimedean copula, We derive, using Theorem 7, an explicit expression of $CoVaR_\alpha^{s|L^i=l}$ in terms of φ .

Proposition 3 (Hakwa, Jäger-Ambrożewicz, and Rüdiger [2015]). *Assume that the copula C associated with the joint distribution of $(L^i,\ L^s)$ is a bivariate Archimedean copula with generator φ. If φ is strict and its derivative φ' is invertible, then the explicit formula for $CoVaR_\alpha^{s|L^i=l}$ for a given confidence level $\alpha, \in (0,1)$ is given by*

$$CoVaR_\alpha^{s|L^i=l} = F_s^{-1}\left(\varphi^{-1}\left(\varphi\left(\varphi'^{-1}\left(\frac{\varphi'(F_i(l))}{\alpha}\right)\right) - \varphi(F_i(l))\right)\right). \tag{4.39}$$

Proof. *In fact, let C be an Archimedean copula with a strict generator φ. Then*

$$C(u,v) = \varphi^{-1}(\varphi(u) + \varphi(v))$$

and it holds

$$\varphi(C(u,v)) = \varphi(u) + \varphi(v). \tag{4.40}$$

Hence,

$$\frac{\partial[\varphi(C(u,v))]}{\partial u} = \frac{\partial[\varphi(u) + \varphi(v)]}{\partial u},$$

such that

$$\frac{\partial C(u,v)}{\partial u} \cdot \varphi'(C(u,v)) = \frac{\partial \varphi(u)}{\partial u} = \varphi'(u).$$

Consequently

$$\frac{\partial C(u,v)}{\partial u} = \frac{\varphi'(u)}{\varphi'(C(u,v))} = \frac{\varphi'(u)}{\varphi'\left(\varphi^{-1}\left[\varphi(u)+\varphi(v)\right]\right)}.$$

We have thus

$$g(v,u) = \frac{\partial C(u,v)}{\partial u} = \frac{\varphi'(u)}{\varphi'\left(\varphi^{-1}\left[\varphi(u)+\varphi(v)\right]\right)}.$$

If φ' is invertible, setting $g(v,u)=\alpha$ solving for v yields to:

$$g^{-1}(\alpha,u) = \varphi^{-1}\left(\varphi\left(\varphi'^{-1}\left(\frac{\varphi'(u)}{\alpha}\right)\right) - \varphi(u)\right).$$

It follows then from Theorem 7 that,

$$\begin{aligned} CoVaR_{\alpha}^{s|L^i=l} &= F_s^{-1}\left(g^{-1}(\alpha, F_i(l))\right) \\ &= F_s^{-1}\left(\varphi^{-1}\left(\varphi\left(\varphi'^{-1}\left(\frac{\varphi'(F_i(l))}{\alpha}\right)\right) - \varphi(F_i(l))\right)\right). \end{aligned}$$

□

Corollary 19 (Hakwa, Jäger-Ambrożewicz, and Rüdiger [2015])**.**

$$CoVaR_{\alpha}^{\beta} = F_s^{-1}\left(\varphi^{-1}\left(\varphi\left(\varphi'^{-1}\left(\frac{\varphi'(\beta)}{\alpha}\right)\right) - \varphi(\beta)\right)\right).$$

Remark 43. *Under to the Archimedean copula the interconnectedness of the financial institution i to the financial system s is not modeled by the parameters as in the Gaussian copula (see Remark 33) and the t-copula (see Remark 40) but by a function namely the generator φ.*

Example 7 (Clayton copula)**.** *The generator of the Clayton copula is define as*

$$\varphi(t) = \frac{1}{\theta}\left(t^{-\theta}-1\right), \quad \theta \in [-1,\infty) - \{0\}.$$

For $\theta > 0$, the generator of the Clayton copula is strict and be thus expressed as follows

$$C_{\theta}^{Cl}(u,v) = \left(u^{-\theta}+v^{-\theta}-1\right)^{-\frac{1}{\theta}}, \ \forall\ u,v \in (0,1). \tag{4.41}$$

Furthermore, we have $\varphi_{\theta}^{-1}(s) = (1+\theta s)^{-\frac{1}{\theta}}$, $\varphi_{\theta}'(t) = -t^{-\theta-1}$, $\varphi_{\theta}'^{-1}(z) =$

$-z^{-\frac{1}{\theta+1}}$. *From Proposition 3, follows*

$$
\begin{aligned}
CoVaR_{\alpha}^{s|L^i=l} &= F_s^{-1}\left(\varphi^{-1}\left(\varphi\left(\varphi'^{-1}\left(\frac{\varphi'(F_i(l))}{\alpha}\right)\right)-\varphi(F_i(l))\right)\right)\\
&= F_s^{-1}\left(\left(1+\theta\left(\left(\frac{1}{\theta}\left(-\left(\frac{-F_i(l)^{-(\theta+1)}}{\alpha}\right)^{-\frac{1}{\theta+1}}\right)^{-\theta}\right)-1\right)-\frac{F_i(l)^{-\theta}-1}{\theta}\right)^{-\frac{1}{\theta}}\right)\\
&= F_s^{-1}\left(\varphi_\theta^{-1}\left(\frac{1}{\theta}\left(\left(-\left(\frac{-F_i(l)^{-(\theta+1)}}{\alpha}\right)^{-\frac{1}{\theta+1}}\right)^{-\theta}-1\right)-\frac{F_i(l)^{-\theta}-1}{\theta}\right)\right)\\
&= F_s^{-1}\left(\varphi_\theta^{-1}\left(\frac{1}{\theta}\left(\frac{F_i(l)^{-\theta}}{\alpha^{\frac{\theta}{\theta+1}}}-1\right)-\frac{F_i(l)^{-\theta}-1}{\theta}\right)\right)\\
&= F_s^{-1}\left(\varphi_\theta^{-1}\left(\frac{1}{\theta}\left(\frac{F_i(l)^{-\theta}}{\alpha^{\frac{\theta}{\theta+1}}}-1-F_i(l)^{-\theta}+1\right)\right)\right)\\
&= F_s^{-1}\left(\varphi_\theta^{-1}\left(\frac{F_i(l)^{-\theta}}{\theta}\left(\alpha^{-\frac{\theta}{\theta+1}}-1\right)\right)\right)\\
&= F_s^{-1}\left(\left(1+\theta\left(\frac{F_i(l)^{-\theta}}{\theta}\left(\alpha^{-\frac{\theta}{\theta+1}}-1\right)\right)\right)^{-\frac{1}{\theta}}\right)\\
&= F_s^{-1}\left(\left(1+F_i(l)^{-\theta}\left(\alpha^{-\frac{\theta}{\theta+1}}-1\right)\right)^{-\frac{1}{\theta}}\right).
\end{aligned}
$$

4.4.3 $CoVaR_{\alpha}^{s|L^i=l}$ for Convex Combinations of Copulas

The convex combinations of copulas offer greater flexibility in the description of complex (tail) dependence structures. In particular, in the context of the analysis of systemic risk contribution, it is very important to model different state of dependence in order to be consistent with the real financial market behavior in which the normal and the crisis time are characterized by different state (type) of dependence. Especially as already see in section 2.3.2 the crisis time is characterized by tail dependence.

Since bivariate copulas are bivariate distributions, the convex linear combination of two copulas is again a copula (see e.g. Nelsen [2006], Chapter 2).

Formally, let C_1 and C_2 be two copulas. Then the function C defined by

$$C(u,v) := \alpha C_1(u,v) + (1-\alpha) C_2(u,v), \forall\ u,v,\alpha \in (0,1)$$

is a copula, too. The following remark specifies the effect of tail dependence of the underlying copulas on that of their convex combination.

Remark 44. *Let C be a convex combination of two bivariate copulas C_1 and C_2, i.e.*

$$C(u,v) := \alpha C_1(u,v) + (1-\alpha) C_2(u,v),\ \forall\ \ u,v,\alpha \in (0,1).$$

Denote by λ_u^1 (λ_l^1), λ_u^2 (λ_l^2) and λ_u (λ_l) the upper (lower) tail dependence coefficients of C_1, C_2 and C, respectively. Then

$$\lambda_u = \alpha\lambda_u^1 + (1-\alpha)\,\lambda_u^2 \quad \text{and} \quad \lambda_l = \alpha\lambda_l^1 + (1-\alpha)\,\lambda_l^2. \tag{4.42}$$

Since C is a copula its tail dependence coefficients can be computed using equations (3.14) and (3.15), respectively:

$$\begin{aligned}
\lambda_u &= \lim_{u\to 1^-} \frac{1-2u+C(u,u)}{1-u} \\
&= \lim_{u\to 1^-} \frac{1-2u+\alpha C_1+(1-\alpha)\,C_2}{1-u} \\
&= \alpha \lim_{u\to 1^-} \frac{1-2u+C_1(u,u)}{1-u} + (1-\alpha) \lim_{u\to 1^-} \frac{1-2u+C_2(u,u)}{1-u} \\
&= \alpha\lambda_u^1 + (1-\alpha)\,\lambda_u^2
\end{aligned}$$

and

$$\begin{aligned}
\lambda_l &= \lim_{u\to 0^+} \frac{C(u,u)}{u} \\
&= \lim_{u\to 0^+} \frac{\alpha C_1+(1-\alpha)\,C_2}{u} \\
&= \alpha \lim_{u\to 0^+} \frac{C_1(u,u)}{u} + (1-\alpha) \lim_{u\to 0^+} \frac{C_2(u,u)}{u} \\
&= \alpha\lambda_l^1 + (1-\alpha)\,\lambda_l^2. \qquad \square
\end{aligned}$$

Remark 45. *Let C_1 and C_2 be two copulas satisfying Assumption 2. If we assume that the copula C associated with the joint distribution of $(L^i,\ L^s)$ is a convex combination of C_1 and C_2 such that*

$$C(u,v) := \alpha C_1(u,v) + (1-\alpha)\,C_2(u,v)\,, \qquad u,v,\alpha \in (0,1)\,,$$

then the function $g(v,u) := \alpha g_1(v,u) + (1-\alpha)\,g_2(v,u)$ (where $g_i(v,u) := \frac{\partial C_i(u,v)}{\partial u}$, $i \in \{1,2\}$) is invertible with respect to the parameter v.

From remark 45 we have that

$$g(v,u) = \alpha g_1(v,u) + (1-\alpha)\,g_2(v,u)\,.$$

Given Assumption 2, g_1 and g_2 are strictly increasing with respect to v (see Remark 28). This implies, that $g(v,u)$ is also strictly increasing with respect to v and therefore invertible. Hence by Theorem 7 the formula for $CoVaR_\alpha^{s|L=l}$, under a convex combination of two copulas, is given by:

$$CoVaR_\alpha^{s|L=l} = F_s^{-1}\left(g^{-1}(\alpha, F_i(l))\right), \ \forall l \in \mathbb{R}, \alpha \in (0,1)\,, \tag{4.43}$$

where

$$g(v,u) = \frac{\partial C(u,v)}{\partial u} = \alpha g_1(v,u) + (1-\alpha)\,g_2(v,u)\,.$$

Example 8 (Convex combination of Clayton and Gumbel copula). *Denote with C the convex combination of the Clayton Copula $C^{Cl}_{\theta_1}$ and the Gumbel copula $C^{Gu}_{\theta_2}$*

$$C\left(u,v\right) := \alpha C^{Cl}_{\theta_1}\left(u,v\right) + (1-\alpha) C^{Gu}_{\theta_2}\left(u,v\right), \quad \alpha \in (0,1)$$

For the Clayton copula $C^{Cl}_{\theta_1}\left(u,v\right) = \left(u^{-\theta_1} + v^{-\theta_1} - 1\right)^{-\frac{1}{\theta_1}}$ we have:

$$g_1 = \frac{\partial C^{Cl}_{\theta_1}\left(u,v\right)}{\partial u} = u^{-\theta_1 - 1}\left(u^{-\theta_1} + v^{-\theta_1} - 1\right)^{-\frac{\theta_1+1}{\theta_1}}, \ \lambda_u = 0 \quad and \quad \lambda_l = 2^{-\frac{1}{\theta_1}}.$$

By Lemma 44, the upper tail dependence coefficient of C are given by

$$\begin{aligned}\lambda_u &= \alpha \cdot 0 + (1-\alpha)\left(2 - 2^{\frac{1}{\theta_2}}\right)\\ &= (1-\alpha)\left(2 - 2^{\frac{1}{\theta_2}}\right).\end{aligned}$$

The copula C has thus a positive upper tail dependence coefficient and is hence appropriate for the analysis of systemic risk contribution.

We have

$$\begin{aligned}g\left(v,u\right) &= \frac{\partial C\left(u,v\right)}{\partial u}\\ &= \alpha \frac{\partial C^{Cl}_{\theta_1}\left(u,v\right)}{\partial u} + (1-\alpha)\frac{\partial C^{Gu}_{\theta_2}\left(u,v\right)}{\partial u}\\ &= \alpha\left(u^{-\theta_1-1}\left(u^{-\theta_1} + v^{-\theta_1} - 1\right)^{-\frac{\theta_1+1}{\theta_1}}\right)\\ &\quad + (1-\alpha)\left[\mathrm{e}^{\left(-\left((-\ln(u))^{\theta_2} + (-\ln(v))^{\theta_2}\right)^{\frac{1}{\theta_2}}\right)} \cdot \left((-\ln\left(u\right))^{\theta_2} + (-\ln\left(v\right))^{\theta_2}\right)^{-\frac{\theta_2-1}{\theta_2}} \cdot \frac{(-\ln\left(u\right))^{\theta_2-1}}{u}\right].\end{aligned} \tag{4.44}$$

$g\left(v,u\right)$ is strictly increasing with respect to v and hence invertible. Based on this we derive the following corollary of Theorem 7.

Corollary 20 (Hakwa, Jäger-Ambrożewicz, and Rüdiger [2015]). *If the copula C of $\left(L^i, L^s\right)$ is a convex combination of the Clayton and the Gumbel Copula, namely*

$$C\left(u,v\right) := \alpha C^{Cl}_{\theta_1}\left(u,v\right) + (1-\alpha) C^{Gu}_{\theta_2}\left(u,v\right), \quad \alpha \in (0,1), \quad \theta_1, \theta_2 > 0,$$

then for a given $l \in \mathbb{R}$

$$CoVaR^{s|L=l}_{\alpha} = F^{-1}_{s}\left(\tilde{\alpha}\right), \tag{4.45}$$

where $\tilde{\alpha}$ is the solution of the equation $g\left(\tilde{\alpha}, F^{-1}_{i}\left(l\right)\right) = \alpha$ and g is given by (4.44),

Remark 46 (An Expression of Conditional Quantile in Terms of Copula). *Under assumption 2, Theorem 7 can be rewritten to give an expression of the conditional quantile in terms of copula.*

In fact, let U, V be two standard uniformly distributed random variables. Let C be the copula of (U, V), then for a fixed $u \in (0,1)$ we have for any $\alpha \in (0,1)$

$$Pr\left(V \leq g^{-1}(\alpha, u) \,|U = u\right) = \alpha,$$

with

$$g(v, u) = \frac{\partial C(u, v)}{\partial u}.$$

This means, that $g^{-1}(\alpha, u)$ is the α-(conditional) quantile of the distribution of $V|U = u$.

Now assume two random variables X and Y satisfying Assumption 2 with distribution functions F and G, respectively. Let C be the copula of X and Y. Set $V := G(Y)$ given $U := F(X)$. If q_α^1 is the α-quantile of the distribution of Y given $X = x$, i.e.

$$Pr\left(Y \leq q_\alpha^1 | X = x\right) = \alpha,$$

then from Theorem 7 we have:

$$q_\alpha^1 = G^{-1}\left(g^{-1}(\alpha, u)\right),$$

where $g(v, u) := \frac{\partial C(u,v)}{\partial u}$.

Chapter 5

Alternative Models for the Measurement of Systemic Risk Contribution

5.1 Some Critical Notes on $CoVaR_\alpha^{s|L^i=l}$

In this section, we first highlight certain gaps presented by the $CoVaR$-method. Then, by making some modifications in the initial definition of Brunnermeier and Adrian [2011] in order to cover these gaps, we define alternative risk measures that are more appropriate for the analysis of systemic risk.

5.1.1 $\Delta CoVaR_\alpha^{s|i}$ may not captures Tail Dependence Effects

It is well known that the main idea of the measurement of systemic risk contribution using the $\Delta CoVaR_\alpha^{s|i}$ is to capture the effect of tail dependence (see Brunnermeier and Adrian [2014]). However, the initial definition of $\Delta CoVaR_\alpha^{s|i}$ shows in some cases abnormal responses to the tail dependence. This has been already observed via simulation studies by Jäger-Ambrożewicz [2010]. We demonstrate here that $\Delta CoVaR_\alpha^{s|i}$, as defined by Brunnermeier and Adrian [2011] (see definition refd4) may not be sensitive to tail dependence effects. We do this, by defining a case in which $\Delta CoVaR_\alpha^{s|i}$ does not take into account tail dependence coefficients.

From Proposition 4.4.3 we know that it is possible to construct copula using convex combinations of two or more copulas. Based on this we define the copula C^h as follows:

Definition 19.

$$C^h_{\rho,\theta}(u,v) := (1-\theta)\, C^{Gau}_{\rho}(u,v) + \theta M(u,v). \quad \theta \in (0,1), \tag{5.1}$$

where C^{Gau} and M denote the Gaussian copula and the comonotonicity copula, respectively.

By Equation (4.42) we have that the tail dependence coefficient λ^h_u of C^h is given by:

$$\begin{aligned} \lambda^h_u &= \theta\lambda^{Gau}_u + (1-\theta)\,\lambda^M_u \\ &= (1-\theta)\cdot 0 + \theta \cdot 1 \\ &= \theta. \end{aligned} \tag{5.2}$$

The two parameter ρ and θ of C^h have different functions:

- θ controls the tail dependence while
- ρ controls the linear dependence.

So, using C^h, we can precisely specify the tail dependence as well as the dependence around the means.

From Theorem 7 we know that. for a given copula C, if

$$g(v,u) := \frac{\partial C(u,v)}{\partial u}$$

is invertible with respect to v. Then

$$CoVaR^{s|i}_{\alpha}(l) = F_s^{-1}\left(g^{-1}(\alpha, F_i(l))\right).$$

The function g associated with C^h is given by

$$\begin{aligned} g(v,u) &:= \frac{\partial C^h(u,v)}{\partial u} \\ &= \frac{\partial\left[(1-\theta)\, C^{Gau}_{\rho}(u,v) + \theta M(u,v)\right]}{\partial u} \\ &= \frac{\partial\left[(1-\theta)\, C^{Gau}_{\rho}\right]}{\partial u} + \frac{\partial\left[\theta M(u,v)\right]}{\partial u} \\ &= (1-\theta)\frac{\partial C^{Gau}_{\rho}}{\partial u} + \theta\frac{\partial M(u,v)}{\partial u} \\ &= (1-\theta)\left[\Phi\left(\frac{\Phi^{-1}(v) - \rho\Phi^{-1}(u)}{\sqrt{1-\rho^2}}\right)\right] + \theta\frac{\partial M(u,v)}{\partial u}. \end{aligned}$$

Note that

$$\frac{\partial C^M(u,v)}{\partial u} = \begin{cases} 1 & \text{if } u < v \\ 0 & \text{if } u > v. \end{cases}$$

Assumption 3. *For simplification reason, we only assume the case where* $u < v$.

The function g becomes then:

$$g(v,u) = (1-\theta)\cdot\left[\Phi\left(\frac{\Phi^{-1}(v) - \rho\Phi^{-1}(u)}{\sqrt{1-\rho^2}}\right)\right] + \theta. \tag{5.3}$$

By setting $g(v,u) = \alpha$ and solving for v, we obtain

$$\begin{aligned}
&(1-\theta)\cdot\left[\Phi\left(\frac{\Phi^{-1}(v) - \rho\Phi^{-1}(u)}{\sqrt{1-\rho^2}}\right)\right] + \theta = \alpha \\
&\Leftrightarrow \Phi\left(\frac{\Phi^{-1}(v) - \rho\Phi^{-1}(u)}{\sqrt{1-\rho^2}}\right) = \frac{\alpha-\theta}{1-\theta} \\
&\Leftrightarrow \frac{\Phi^{-1}(v) - \rho\Phi^{-1}(u)}{\sqrt{1-\rho^2}} = \Phi^{-1}\left(\frac{\alpha-\theta}{1-\theta}\right) \\
&\Leftrightarrow \Phi^{-1}(v) - \rho\Phi^{-1}(u) = \Phi^{-1}\left(\frac{\alpha-\theta}{1-\theta}\right)\cdot\sqrt{1-\rho^2} \\
&\Leftrightarrow \Phi^{-1}(v) = \Phi^{-1}\left(\frac{\alpha-\theta}{1-\theta}\right)\cdot\sqrt{1-\rho^2} + \rho\Phi^{-1}(u) \\
&\Leftrightarrow v = \Phi\left[\Phi^{-1}\left(\frac{\alpha-\theta}{1-\theta}\right)\cdot\sqrt{1-\rho^2} + \rho\Phi^{-1}(u)\right].
\end{aligned}$$

This leads to the following corollary of the Theorem 7.

Corollary 21. *Let*

$$C^h_{\rho,\theta}(u,v) := (1-\theta)\,C^{Gau}_{\rho}(u,v) + \theta M(u,v)\,, \quad \theta \in (0,1)$$

be the copula associated with the joint distribution of $(L^i,\ L^s)$. *Then for all* $l \in \mathbb{R}$ *and a given* $\alpha > \theta$, $\alpha \in [0,1]$,

$$CoVaR^{s|i}_{\alpha}(l) = F_s^{-1}\left(\Phi\left[\Phi^{-1}\left(\frac{\alpha-\theta}{1-\theta}\right)\cdot\sqrt{1-\rho^2} + \rho\Phi^{-1}(F_i(l))\right]\right) \tag{5.4}$$

Remark 47. *Equation* (5.4) *shows that* $CoVaR^{s|i}_{\alpha}(l)$ *under* C^h *generalizes the* $CoVaR^{s|i}_{\alpha}(l)$ *under Gaussian copula by accounting for tail dependence through the parameter* θ. *In fact, if* $\theta = 0$ *(i.e. there is no tail dependence) then* $CoVaR^{s|i}_{\alpha}(l)$ *under* C^h *and* $CoVaR^{s|i}_{\alpha}(l)$ *under Gaussian copula coincides (see equation* (4.12)*).*

Assume that the loss L^s of the financial system follows a normal distribution with parameter $\mu_s = E(L^s)$ and $\sigma_s = \sqrt{Var(L^s)}$, i.e. $L^s \sim \mathcal{N}(\mu_s, \sigma_s)$. For that setting

$$
\begin{aligned}
\Delta CoVaR_\alpha^{s|L^i=i} &= CoVaR_\alpha^{s|L^i=VaR_\alpha^i} - CoVaR_\alpha^{s|L^i=\mu_i} \\
&= CoVaR_\alpha^{s|i}\left(VaR_\alpha^i\right) - CoVaR_\alpha^{s|i}\left(\mu_i\right) \\
&= F_s^{-1}\left(\Phi\left[\Phi^{-1}\left(\frac{\alpha-\theta}{1-\theta}\right)\cdot\sqrt{1-\rho^2}+\rho\Phi^{-1}\left(F_i\left(VaR_\alpha^i\right)\right)\right]\right) \\
&\quad - F_s^{-1}\left(\Phi\left[\Phi^{-1}\left(\frac{\alpha-\theta}{1-\theta}\right)\cdot\sqrt{1-\rho^2}+\rho\Phi^{-1}\left(F_i\left(\mu_i\right)\right)\right]\right) \\
&= \Phi^{-1}\left(\frac{\alpha-\theta}{1-\theta}\right)\cdot\sqrt{1-\rho^2}+\rho\Phi^{-1}\left(F_i\left(VaR_\alpha^i\right)\right) - \mu_s \\
&\quad - \left[\Phi^{-1}\left(\frac{\alpha-\theta}{1-\theta}\right)\cdot\sqrt{1-\rho^2}+\rho\Phi^{-1}\left(F_i\left(\mu_i\right)\right)+\mu_s\right] \\
&= \rho\left[\Phi^{-1}\left(F_i\left(VaR_\alpha^i\right)\right)-\Phi^{-1}\left(F_i\left(\mu_i\right)\right)\right]. \qquad (5.5)
\end{aligned}
$$

We see that the parameter θ has no effect on $\Delta CoVaR_\alpha^{s|L^i=i}$. This shows that $\Delta CoVaR_\alpha^{s|L^i=i}$ is, in this setting (copula C^h and normally distributed L^s), insensitive to tail dependence.

Conclusion 1. *From the previous analysis, we conclude that $\Delta CoVaR_\alpha^{s|L^i=l}$ in general does not capture the variations in tail risk.*

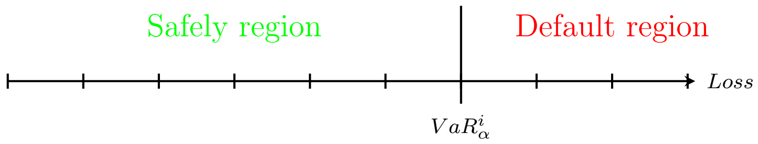
5.1.2 $\Delta CoVaR_\alpha^{s|L^i=l}$ is not Consistent with the Notion Systemic Risk

Recall that the modeling of systemic risk contribution through the $CoVaR$-method builds on the consideration of the notion of systemic risk in a narrow sense. This assume that the initial shock that leads to a systemic risk is the default of one single financial institution (see Assumption 1). That is, in the case we analyze the systemic risk contribution of the financial institution i. The default of the financial institution i is supposed to be the initial shock that leads to systemic risk contribution from the financial institution i to the financial system s.

Furthermore, the $CoVaR$-Method also implicitly assumes the Value-at-Risk as measure for the regulatory capital. Therefore, following Definition 2, the financial institution i is assumed to be in default, if its realized loss l is greater than its Value-at-Risk:

$$l > VaR_\alpha^i. \qquad (5.6)$$

Recall that in Brunnermeier and Adrian [2011] the default of the financial institution is characterized by the situation when $L^i = VaR_\alpha^i$ $(C(L^i) =$



$\{L^i = VaR_\alpha^i\}$ in definition 8). This condition does not satisfy equation 5.6 (default condition). In fact, a loss equal to the Value-at-Risk does not lead to a default. Since, the financial institution i is supposed to hold minimum capital (regulatory capital) equal to its Value-at-Risk (VaR_α^i). Therefore, any losses smaller or equal to VaR_α^i is absorbed. Such losses can not therefore lead to the default of i and hence to a systemic risk contribution from i to s.

Conclusion 2. *The initial definition of $\Delta CoVaR_\alpha^{s|L^i=l}$ (as defined in Brunnermeier and Adrian [2011], here definition 8) can not capture contagion effects. It is thus not an appropriate model for the analysis of systemic risk contribution.*

5.2 Alternative Measures

We present in the following two risk measures that are consistent with the notion of systemic risk contribution. These results are also presented in Hakwa, Jäger-Ambrożewicz, and Rüdiger [2015].

These alternative measures built, like $\Delta CoVaR$, on the term $CoVaR_\alpha^{s|C(L^i)}$, but differ fundamentally in the way the condition $C(L^i)$ is formulated.

As first, we define a measure that is based on the commonly used and well-known risk measure expected shortfall (ES).

Definition 20 (Hakwa, Jäger-Ambrożewicz, and Rüdiger [2015])**.**

$$ECoVaR_\alpha^{s|i} := E\left(CoVaR_\alpha^{s|i}\left(L^i\right)|L^i \geq VaR_\alpha^i\right) \quad (5.7)$$

$$\Delta ECoVaR_\alpha^{s|i} := E\left(CoVaR_\alpha^{s|i}\left(L^i\right)|L^i \geq VaR_\alpha^i\right) - CoVaR_\alpha^{s|L^i=VaR_\alpha^i} \quad (5.8)$$

The conditioning event $L^i \geq VaR_\alpha^i$ ensures the integration of the default condition. The advantage of this model is that it integrates all information about the distribution of losses that can not be hedged with the regulatory capital (loss excess). This allows to estimate the financial system loss induced by the default of the financial institution i.

Remark 48. *The estimated financial system loss induced in case of the default of a given financial institution, can be used as measure for the identification of that financial institution's systemic importance. Furthermore, regulatory* *authorities such as can use the* $ECoVaR_\alpha^{s|i}$ *to make a ranking of financial institutions with respect to their systemic importance.*

Proposition 4 (Hakwa, Jäger-Ambrożewicz, and Rüdiger [2015])**.**

$$ECoVaR_\alpha^{s|i} = \frac{1}{1 - F_i\left(VaR_\alpha^i\right)} \int_{VaR_\alpha^i}^{\infty} CoVaR_\alpha^{s|i}(l)\, f_i(l)\, dl. \tag{5.9}$$

Proof. *From basic probability theories (as in Klenke [2008], Definition 8.9), we know that*

$$\begin{aligned} ECoVaR_\alpha^{s|i} &:= E\left(CoVaR_\alpha^{s|i}\left(L^i\right) | L^i \geq VaR_\alpha^i\right) \\ &= \frac{E\left(CoVaR_\alpha^{s|i}\left(L^i\right) \mathbf{1}_{\{L^i \geq VaR_\alpha^i\}}\right)}{Pr\left(L^i \geq VaR_\alpha^i\right)} \\ &= \frac{1}{1 - F_i\left(VaR_\alpha^i\right)} \int_{VaR_\alpha^i}^{\infty} CoVaR_\alpha^{s|i}(l)\, f_i(l)\, dl. \qquad \square \end{aligned}$$

Remark 49. *We can assume different confidence level for the Value-at-Risk of the financial institution i (e.g.* β*) and for the* $CoVaR$ *(e.g.* α, $\alpha \neq \beta$*). The expression of* $ECoVaR$ *becomes then*

$$ECoVaR_\alpha^{s|i} = \frac{1}{1-\beta} \int_{VaR_\beta^i}^{\infty} CoVaR_\alpha^{s|i}(l)\, f_i(l)\, dl. \tag{5.10}$$

Let us consider the figure 5.1.2 and the initial definition of $\Delta CoVaR_\alpha^{s|i}$, here Definition 8:

$$\Delta CoVaR_\alpha^{s|i} := CoVaR_\alpha^{s|L^i = VaR_\alpha^i} - CoVaR_\alpha^{s|L^i = \mu^i}.$$

If we replace in the first term of "$=$" by a "$>$" we ensures that the considered region is in the distressed region in the Figure 5.6). This means that all the information about the distribution of the losses of the financial institution i , that can not be hedged with the regulatory capital (loss excess) are integrated. This leads to the definition of the second alternative risk measure.

Definition 21 (Hakwa, Jäger-Ambrożewicz, and Rüdiger [2015])**.**

$$\Delta CoVaR_\alpha^{>} = CoVaR_\alpha^{s|L^i > VaR_\alpha^i} - CoVaR_\alpha^{s|L^i = VaR_\alpha^i}. \tag{5.11}$$

The new challenge here is the computing of the value $CoVaR_\alpha^{s|L^i > VaR_\alpha^i}$.

Definition 22 (Hakwa, Jäger-Ambrożewicz, and Rüdiger [2015])**.** *Assume that* L^i *and* L^s *both have densities that satisfy Assumption 2. Then for a given* $\alpha \in (0,1)$ *and for a fixed* l, $CoVaR_\alpha^{s|L^i > l}$ *is defined as:*

$$\begin{aligned} CoVaR_\alpha^{s|L^i > l} &:= \inf\left\{h \in \mathbb{R} : \; Pr\left(L^s > h | L^i > l\right) \leq 1 - \alpha\right\} \\ &= \inf\left\{h \in \mathbb{R} : \; Pr\left(L^s \leq h | L^i > l\right) \geq \alpha\right\} \end{aligned}$$

$CoVaR_\alpha^{s|L^i>VaR_\alpha^i}$ is then implicitly defined by

$$Pr\left(L^s \leq CoVaR_\alpha^{s|L^i>VaR_\alpha^i}|L^i > VaR_\alpha^i\right) = \alpha, \tag{5.12}$$

which is a special case of the generalized $CoVaR_\alpha^{s|C\left(L^i\right)}$ (see equation (2.13)), in which the conditioning event is the default of the financial institution i (i.e. $C\left(L^i\right) = \{L^i > VaR_\alpha^i\}$.

Remark 50 (Hakwa, Jäger-Ambrożewicz, and Rüdiger [2015]). *Given Assumption 2 the function*

$$J\left(u,v\right) := u - C\left(u,v\right) = C\left(u,1\right)$$

is for each fixed $v \in [0,1]$ *strong monotone increasing and hence invertible as a function of* u. *In fact from the boundary condition for 2-dimensional copula (see Definition 10) follows*

$$J\left(u,v\right) = u - C\left(u,v\right) = C\left(u,1\right) - C\left(u,v\right). \tag{5.13}$$

Since we assume Assumption 2, the copula C *has a strictly positive density* c *(see Remark 27). By taking this into account and applying some calculus rules we obtain*

$$\begin{aligned} J\left(u,v\right) &= C(u,1) - C(u,v) \\ &= \int_0^u \int_0^1 c\left(x,y\right) dxdy - \int_0^u \int_0^v c\left(x,y\right) dxdy \\ &= \int_0^u \left(\int_0^1 c\left(x,y\right) dy\right) dx - \int_0^u \left(\int_0^v c\left(x,y\right) dy\right) dx \\ &= \int_0^u \left(\int_0^1 c\left(x,y\right) dy - \int_0^v c\left(x,y\right) dy\right) dx \\ &= \int_0^u \left(\int_v^1 c(x,y)dy\right) dx. \end{aligned} \tag{5.14}$$

So that, for a fixed $v \in [0,1]$, *the function* $J\left(u,v\right)$ *is strictly increasing and thus invertible with respect to* u. *Since because of Assumption 2 the density* c *is strict positive (see Remark 27).*

Define the function

$$j\left(u,v\right) := \frac{J(u,v)}{1-v}. \tag{5.15}$$

$j\left(u,v\right)$ is also for each v fixed invertible as a function of u. Using this result we introduce the following theorem for the calculation of $CoVaR_\alpha^{L^i>l}$.

Theorem 22 (Hakwa, Jäger-Ambrożewicz, and Rüdiger [2015]). *Under Assumption 2,*

$$CoVaR_{\alpha}^{L^i>l} = F_s^{-1}\left(j^{-1}(\alpha, F_i(l))\right). \tag{5.16}$$

Proof. *Recall that for a given $l \in \mathbb{R}$, $CoVaR_{\alpha}^{L^i>l}$ is implicitly defined by*

$$Pr\left(L^s \leq CoVaR_{\alpha}^{L^i>l} | L^i > l\right) = \alpha.$$

By setting $U := F_s(L^s)$, $V := F_i(L^i)$, $u := F_s\left(CoVaR_{\alpha}^{L^i>l}\right)$ and $v := F_i(l)$ we obtain

$$\begin{aligned} Pr\left(L^s \leq CoVaR_{\alpha}^{L^i>l} | L^i > l\right) &= Pr\left(U \leq u | V > v\right) \\ &= \frac{Pr\left(U \leq u, V > v\right)}{Pr\left(V > v\right)} \\ &= \frac{Pr\left(U \leq u\right) - Pr\left(U \leq u, V \leq v\right)}{1 - Pr\left(V \leq v\right)} \\ &= \frac{u - C\left(u, v\right)}{1 - v} \\ &= j\left(u, v\right). \end{aligned}$$

We therefore have $Pr\left(L^s \leq CoVaR_{\alpha}^{L^i>l} | L^i > l\right) = j(u, v) = \alpha$. It follows that

$$u = F_s\left(CoVaR_{\alpha}^{L^i>l}\right) = j^{-1}(\alpha, v).$$

Thus

$$CoVaR_{\alpha}^{L^i>l} = F_s^{-1}\left(j^{-1}(\alpha, F_i(l))\right). \qquad \square \tag{5.17}$$

Remark 51. *Similarly to $CoVaR_{\alpha}^{L^i=l}$, we observe that $CoVaR_{\alpha}^{s|L^i>l}$ is also expressed in form of a quantile of the loss distribution F_s. We have*

$$CoVaR_{\alpha}^{s|L^i>l} = F_s^{-1}(\bar{\alpha}) = VaR_{\bar{\alpha}}^{s}, \tag{5.18}$$

with $\bar{\alpha} := j^{-1}(\alpha, F_i(l))$.

We provide here for some given copula C the function J. For this, we need to express the copula density c of the considered copula C.

Recall that, due to Assumption 2, all copulas considered here have a strictly positive density c (see Remark 27).

$$c(u, v) = \frac{\partial^2 C(u, v)}{\partial u \partial v}. \tag{5.19}$$

Example 9 (Gaussian Copula).

By equation (5.19) the density of the Gaussian copula is given by

$$\begin{aligned} c_\rho^{Gau}(u,v) &= \frac{\partial^2 C_\rho^{Gau}(u,v)}{\partial u \partial v} \\ &= \frac{\partial^2 \left[\int_{-\infty}^{\Phi^{-1}(u)} \int_{-\infty}^{\Phi^{-1}(v)} \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left(\frac{2\rho st - s^2 - t^2}{2(1-\rho^2)}\right) dsdt \right]}{\partial u \partial v} \\ &= \frac{1}{\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2}\left(\frac{x^2 - 2\rho xy + y^2}{1-\rho^2}\right) - x^2 - y^2\right) \end{aligned}$$

with $x = \Phi^{-1}(u)$ and $y = \Phi^{-1}(v)$. We have thus

$$J(u,v) = \int_0^u \int_v^1 \frac{1}{\sqrt{1-\rho^2}} e^{\left(-\frac{1}{2}\left(\frac{\Phi^{-1}(t)^2 - 2\rho\Phi^{-1}(t)\Phi^{-1}(s) + \Phi^{-1}(s)^2}{1-\rho^2}\right) - \Phi^{-1}(t)^2 - \Phi^{-1}(s)^2\right)} dtds.$$

Example 10 (Gumbel Copula)**.** *By equation* (5.19) *the density of the Gumbel copula is given by*

$$\begin{aligned} c_\rho^{Gu}(u,v) &= \frac{\partial^2 C_\rho^{Gu}(u,v)}{\partial u \partial v} \\ &= \frac{\partial^2 \left[\exp\left(-\left[(-ln(u))^\theta + (-ln(v))^\theta\right]^{\frac{1}{\theta}}\right)\right]}{\partial u \partial v} \\ &= \frac{[-\ln(u) - \ln(v)]^{\theta-1} e^{-\left[(-\ln(u))^\theta + (-\ln(v))^\theta\right]^{\frac{1}{\theta}}} \left(\left[(-\ln(u))^\theta + (-\ln(v))^\theta\right]^{\frac{1}{\theta}} + \theta - 1\right)}{uv\left(\left[(-\ln(u))^\theta + (-\ln(v))^\theta\right]^{2-\frac{1}{\theta}}\right)}. \end{aligned}$$

We have thus

$$J(u,v) = \int_0^u \int_v^1 \frac{[-\ln(t) - \ln(s)]^{\theta-1} e^{-\left[(-\ln(t))^\theta + (-\ln(s))^\theta\right]^{\frac{1}{\theta}}} \left(\left[(-\ln(t))^\theta + (-\ln(s))^\theta\right]^{\frac{1}{\theta}} + \theta - 1\right)}{st\left(\left[(-\ln(t))^\theta + (-\ln(s))^\theta\right]^{2-\frac{1}{\theta}}\right)} dtd$$

Example 11 (Clayton Copula)**.** *By equation* (5.19) *the density of the Clayton copula is given by*

$$\begin{aligned} c_\rho^{Cl}(u,v) &= \frac{\partial^2 C_\rho^{Cl}(u,v)}{\partial u \partial v} \\ &= \frac{\partial^2 \left[\left(u^{-\theta} + v^{-\theta} - 1\right)^{-\frac{1}{\theta}}\right]}{\partial u \partial v} \\ &= (\theta+1)(uv)^{-(\theta+1)}\left(u^{-\theta} + v^{-\theta} - 1\right)^{-\frac{2\theta+1}{\theta}}. \end{aligned}$$

We have thus

$$J(u,v) = \int_0^u \int_v^1 (\theta+1)(st)^{-(\theta+1)}\left(s^{-\theta} + t^{-\theta} - 1\right)^{-\frac{2\theta+1}{\theta}} dtds.$$

Chapter 6

Computing $CoVaR_{\alpha}^{s|i}(l)$ under Elliptical Distribution

One of the main challenges for the regulatory authorities in the aftermath of the last financial crisis is to define pragmatical and practicable risk concepts for the control and the regulation of systemic risks. They need for this purpose risk models that on one hand can capture the macro dimension of systemic risk, and on the other hand can be easily implemented and audited. Since regulatory rules or best-practice recommendations require models that can be easily implemented and audited, the consideration of models that are based on $CoVaR_{\alpha}^{(s|i)}(l)$ under elliptical distribution is a reasonable trade-off. Although the class of elliptical distributions can be seen as a sub class of distributions that can be generated using copula (Namely, the class of elliptical copula (see Subsection 4.4.1)), They have the particularity, that they share many of the analytical properties of the multivariate normal distribution and remain flexible enough to model extreme co-movements as those observed during the last financial crisis. This fact is highly important for the consistent use of elliptical distributions as model for the analysis of systemic risk contribution.

As we will see here, the computation of $CoVaR_{\alpha}^{(s|i)}(l)$ under elliptical distributions is based on parameters that are widely used in financial industry. Furthermore, the quantitative risk-management techniques based on elliptical distributions are well studied and in general compatible with the standard concepts used in modern risk management. For example, Embrechts et al. [1999] showed that the Value-at-Risk is coherent for elliptically distributed losses, Kamdem [2004] provided analytic formulas for the computation of the Value-at-Risk and Expected-Shortfall for linear and quadratic portfolios of elliptically distributed risk factors, Chamberlain [1983] showed that portfolio analysis based on the mean-variance analysis of Markowitz works only if the returns assume elliptical distributions.

6.1 Elliptical Distribution: Definition and basis Properties

Definition 23 (cf. Embrechts et al. [2003], Definition 5.1)**.** *Let X be a n-dimensional random vector. For some $\mu \in \mathbb{R}^n$ and some non-negative definite, symmetric matrix $\Sigma \in \mathbb{R}^n$, if the characteristic function $\varphi_{X-\mu}(t)$ of $X-\mu$ is a function of the quadratic form $t'\Sigma t$, i.e.*

$$\varphi_{X-\mu}(t) = \phi(t'\Sigma t), \quad t \in \mathbb{R}^n,$$

then we say that X has an elliptical distribution with location parameter μ, dispersion parameter Σ and characteristic generator ϕ. We denote this by $X \sim E_n(\mu, \Sigma, \phi)$.

Elliptical distributions can also be defined as affine transformations of spherical distributions.

Definition 24 (McNeil et al. [2005], Definition 3.18)**.** *A random vector $Z = (Z_1, \ldots, Z_n)'$ has a spherical distribution, if for every orthogonal matrix $\mathcal{O} \in \mathbb{R}^{n\times n}$ (i.e. $\mathcal{O}'\mathcal{O} = I'$) $\mathcal{O}Z$ and Z have the same distribution. We denote this by*

$$\mathcal{O}Z \stackrel{d}{=} Z. \tag{6.1}$$

Recall that orthogonal matrices can characterize rotations and/or a reflections. Equation (6.1) implies also that spherical distributions are invariant under rotations and reflections. If we assume $\mathcal{O} = -I$, then we have

$$-Z = -IZ \stackrel{d}{=} Z. \tag{6.2}$$

Equation 6.2 shows that Z is symmetric with respect to its origin. This property plays an important role in the characterization of the characteristic function of spherical distributions.

Theorem 23 (Fang et al. [1990], Theorem 2.1)**.** *The random variable Z has a spherical distribution if there exists a functions ϕ of a scalar variable such that its characteristic function $\psi_Z(t)$ satisfies*

$$\psi_Z(t) = \phi(t't), \quad t \in \mathbb{R}^n. \tag{6.3}$$

Note that $t't = t_1^2 + t_2^2 + \cdots + t_n^2 = \|t\|_2^2$, where $\|\|_2$ denotes the Euclidean norm. From this we can rewrite 6.3 as

$$\psi_Z(t) = \phi\left(\|t\|_2^2\right), \quad t \in \mathbb{R}^n.$$

The function ϕ is called characteristic generator of Z. We follow (cf McNeil et al. [2005], Section 3.3) and use the notation

$$Z \sim S_n(\phi),$$

to indicate that Z is a n-variate spherical distributed random variable with characteristic generator ϕ.

A spherical distributions can also be represented as a mixture of standard uniformly distributed random variable. In fact let U be a random variable which is uniformly distributed on the unit sphere surface $S_{L^2}^n := \{x \in \mathbb{R}^n : \|x\|_2 = 1\}$. Then every $Z \sim S_n(\phi)$ can be represented stochastically as follow

$$Z \stackrel{d}{=} RU, \tag{6.4}$$

where $R \geq 0$ is a random variable independent of U (cf. McNeil et al. [2005], Theorem 3.22.).

Definition 25 (Fang et al. [1990], Definition 2.2)**.** *A n-variate random variable X is said to have an elliptical distribution with location parameter $\mu \in \mathbb{R}^n$ and dispersion parameter $\Sigma \in \mathbb{R}^{n\times n}$ if*

$$X \stackrel{d}{=} \mu + A'Z, \quad \text{with } A'A = \Sigma,$$

where $\mu \in \mathbb{R}^n$ and $A \in \mathbb{R}^{k\times n}$ represents a linear shift and a linear transformation of the spherical random variable Z, respectively.

From equation (6.4) we have the following stochastic representation for an elliptical distribution

$$X \stackrel{d}{=} \mu + RA'U \tag{6.5}$$

Furthermore, it holds (cf. Fang et al. [1990], Corollary 2):

$$Q(X) := (X-\mu)'\,\Sigma^{-1}(X-\mu) \stackrel{d}{=} R^2.$$

Note that the existence of the density of a random vector $X \sim E_n(\mu, \Sigma, \phi)$ is not always guaranteed, but if it exists, it must be of the following form(cf. Fang et al. [1990], Equation (2.43))

$$\begin{aligned} f(x) &= |\Sigma|^{-\frac{1}{2}}\, g_n\left((x-\mu)'\,\Sigma^{-1}(x-\mu)\right) \\ &= |\Sigma|^{-\frac{1}{2}}\, g_n(Q(x)), \quad x \in \mathbb{R}^n, \end{aligned} \tag{6.6}$$

The term $|\Sigma|$ denotes here the determinant of Σ. The function $g_n : \mathbb{R}_+ \to \mathbb{R}_+$ is called density generator. Some times the expression $X \sim E_n(\mu, \Sigma, g_n)$ is used instead of $X \sim E_n(\mu, \Sigma, \phi)$ (cf. Fang et al. [1990], Section 2.2.3).

The density generator g_n must satisfy the condition

$$\int_0^\infty x^{n/2-1} g_n(x)\, dx < \infty. \tag{6.7}$$

In fact if R has a density f_R. The relation between g_n and f_R is given by the following theorem.

Theorem 24 (cf. Fang et al. [1990], Theorem 2.9). *If $X \stackrel{d}{=} RU \sim S_n(\phi)$, then X possesses a density generator g_n if and only if R has a density f_R, and the relationship between g_n and f_R is as follows:*

$$f_R(x) = \frac{2\pi^{n/2}}{\Gamma(n/2)} x^{n-1} g_n\left(x^2\right) \quad for \quad x \geq 0. \tag{6.8}$$

From equation (6.8) we have that

$$g_n\left(x^2\right) = x^{1-n} f_R(x) \frac{\Gamma\left(\frac{n}{2}\right)}{2\pi^{n/2}}. \tag{6.9}$$

Let F_R be the distribution function of R, then

$$\begin{aligned} F_R(x) &= \int_0^x f_R(z) dz \\ &= \int_0^x \frac{2\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)} z^{n-1} g_n\left(z^2\right) dz \\ &= \frac{2\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)} \int_0^x z^{n-1} g_n\left(z^2\right) dz \\ &= \frac{\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)} \int_0^x s^{n/2-1} g_n(s)\, ds, \end{aligned}$$

so the function g_n must met the condition (6.7).

Furthermore, it must hold

$$\int_0^\infty \frac{\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)} x^{n/2-1} g_n(x)\, dx = 1.$$

The characteristic generator ϕ and the density generator g_n (and thus the Definition 23 and the Definition 25) can be related by the following equality

$$\int_{\mathbb{R}^n} e^{(itx)} g_n(x'x)\, dx = \phi(t't). \tag{6.10}$$

This is the characteristic function of a n-variate elliptical random vector X with $\mu = 0$, $\Sigma = I_n$ and density generator g_n ($X \sim E_n(0, I_n, g_n)$).

Proposition 5 (cf. McNeil et al. [2005], Proposition 3.28.)**.** *Let $X \sim E_n(\mu, \Sigma, \phi)$ then*

$$Z := \Sigma^{-\frac{1}{2}}(X - \mu) \sim \mathcal{S}_n(\phi)$$

and if the spherical vector Z has a density generator g_n, then $X := \mu + \sqrt{\Sigma} Z$. has a density

$$f(x) = \frac{1}{|\Sigma|^{\frac{1}{2}}} g_n\left((x - \mu)' \Sigma^{-1} (x - \mu)\right).$$

Remark 52. *If $\mu = 0$ and $\Sigma = \boldsymbol{I}_n$ we have*

$$f(x) = g_n(Q(x)) = g_n\left(x^2\right).$$

This corresponds to the expression of the density of a spherical random variable.

Remark 53. *For each fixed constant constant $c > 0$,*

$$\mathcal{E}(c) = \{x \ : Q(x) = c\}$$

is an ellipse centered at μ.

If we assume two constants c_1 and c_2 such that $c_1 > c_2$ then $\mathcal{E}(c_1)$ is inside $\mathcal{E}(c_2)$ because g_n is by its definition decreasing. Also the iso-probability (i.e $\{x : f(x) = c\}$) contours of elliptical distributions are ellipsoids (see. Figure 6.1). It is for this reason that this family of distributions is called "elliptical distribution".

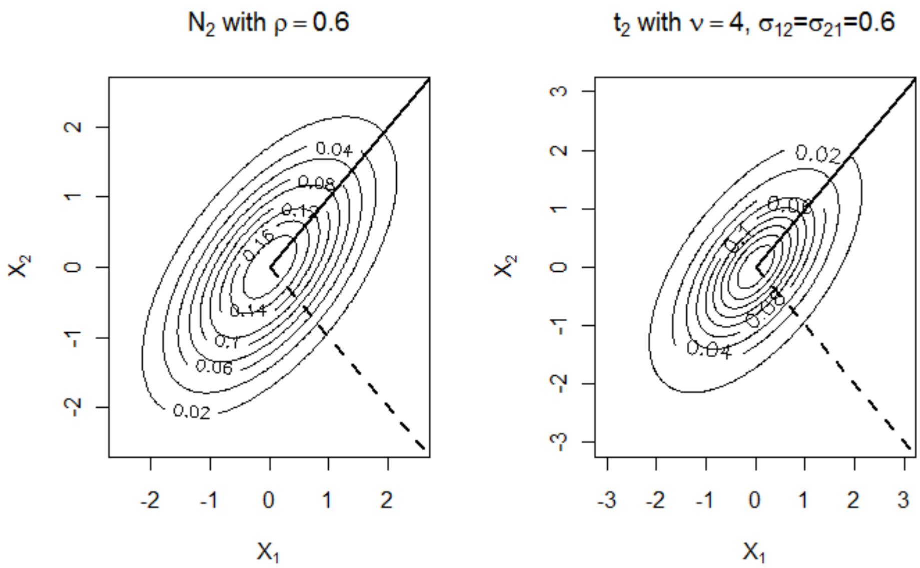


Figure 6.1: Gaussian and t-ellipse

The mean and the covariance vector of an elliptically distributed random variable $X \sim E_n(\mu, \Sigma, \phi)$ is given by (Fang et al. [1990], Theorem 2.17)

$$E[X] = \mu, \quad Cov[X] = \frac{E[R^2]}{n}\Sigma. \tag{6.11}$$

Therefor, a necessary condition for the existence of $Cov[X]$ is that $E[R^2] < \infty$.

Recall that the covariance, $Cov[X_1, X_2]$, of two random variables X_1 and X_2 is given by

$$Cov[X_1, X_2] = E[(X_1 - \mu_1)(X_2 - \mu_2)].$$

Where $\mu_j = E[X_j], \ j \in \{1, 2\}$ denotes the mean of the random variable X_j. The so defined covariance $Cov[X_1, X_2]$ specifies how X_1 and X_2 move together around their respective means μ_1 and μ_2.

By normalizing the covariance $Cov[X_1, X_2]$ with the standard deviations σ_1^2 and σ_2^2, with $\sigma_j^2 = Var[X_j] := E\left[(X_j - \mu_j)^2\right], \ j \in \{1, 2\}$ of X_1 and X_2, we obtain the linear correlation coefficient of X_1 and X_2.

So, let $X = (X_1, X_2) \sim E_2(\mu, \Sigma, g)$ be a bivariate elliptically distributed random vector with $\Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}$. So, if $Cov[X]$ exists and $Var[X_1], Var[X_2] < \infty$, then the linear correlation coefficient ρ of X by can be expressed as follows

$$\rho = \frac{Cov[X]}{\sqrt{Var[X_1]}\sqrt{Var[X_2]}}. \tag{6.12}$$

6.1.1 Examples

Bivariate Normal Distribution

The standard (or spherical) normal distribution has the following stochastic representation (cf. Embrechts et al. [1999], Page 9)

$$X \stackrel{d}{=} RU,$$

where U is a random variable that is uniformly distributed on the unit sphere surface $S_{L^2}^n := \{x \in \mathbb{R}^n : \ \|x\|_2 = 1\}$ and the random variable R satisfies

$$R \stackrel{d}{=} \sqrt{\chi_n^2},$$

where χ_n^2 denotes a chi-squared random variable with degree of freedom n. Recall that the density of χ_n^2 is given by

$$f_{\chi_n^2}(x) = \begin{cases} \dfrac{x^{\frac{n}{2}-1}e^{-\frac{x}{2}}}{2^{\frac{n}{2}}\Gamma(\frac{n}{2})} & , x > 0 \\ 0 & , x \leq 0. \end{cases}$$

We can compute the density f_R of R through the transformation rule for density functions (cf. e.g. Klenke [2008], Theorem 1.101). We obtain

$$f_R(x) = 2x \cdot f_{\chi_n^2}\left(x^2\right).$$

By equation (6.9) we have that

$$g_1\left(x^2\right) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}.$$

This implies that:

$$g_1(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x}{2}}.$$

And, from Remark 52 follows that the density of the univariate spherical distribution associated with the normal distribution is

$$f(x) = g_1\left(x^2\right) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}.$$

Since $E\left[R^2\right] = E\left[\mathcal{X}_n^2\right] = n$, it follows from equation (6.11) that

$$Cov[X] = \Sigma. \tag{6.13}$$

Bivariate t-Distribution

The density generator of the n-multivariate t distribution is given by (cf. McNeil et al. [2005], page 93):

$$g(x) = (\pi\nu)^{-\frac{n}{2}} \frac{\Gamma\left(\frac{\nu+n}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)} \left(1 + \frac{x}{\nu}\right)^{-\frac{\nu+n}{2}}.$$

Hence, the density of a n-multivariate t-distributed random variable X with location parameter $\mu = 0$, dispersion parameter $\Sigma = \mathbf{I}_n$ (spherical or standard t-distribution: $X \sim t(0, \mathbf{I}_n)$) is given by

$$f(x) = (\pi\nu)^{-\frac{n}{2}} \frac{\Gamma\left(\frac{\nu+2}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)} \left(1 + \frac{x^2}{\nu}\right)^{-\frac{\nu+n}{2}}.$$

The density function of a t distributed random variable with location parameter μ and dispersion parameter Σ is

$$f(x) = |\Sigma|^{-\frac{1}{2}} (\pi\nu)^{-\frac{n}{2}} \frac{\Gamma\left(\frac{\nu+n}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)} \left(1 + \frac{(x-\mu)' \Sigma^{-1} (x-\mu)}{\nu}\right)^{-\frac{\nu+n}{2}}.$$

A n-multivariate t distributed random variable Y with location parameter $\mu = 0$ and dispersion parameter $\sigma = \mathbf{I}_n$ and degree of freedom $\nu \in \mathbb{N}$ ($Y \sim$

$t_n(0, \mathbf{I}_n, \nu))$ has the following stochastic representation (cf. Fang et al. [1990], Example 2.5)

$$Y = \frac{\sqrt{\nu}Z}{S} = \frac{\sqrt{\nu}RU}{S} = \hat{R}U, \text{ with } \hat{R} = \frac{\sqrt{\nu}R}{S},$$

where $Z \sim N(0,\ \mathbf{I}_n)$, $R \sim \mathcal{X}_n$ and $S \sim \mathcal{X}_\nu$. And U and s are independent.

Remark 54. *It holds*

$$Y = \frac{\sqrt{\nu}Z}{S} \stackrel{d}{=} \frac{Z}{\sqrt{\frac{S^2}{\nu}}}.$$

Hence, for $\nu \to \infty$ we have $Y \stackrel{d}{=} Z$, i.e. the multivariate t distribution tends to the normal distribution when ν tends to infinity.

Remark 55. *Note that:*

$$\frac{\hat{R}}{\sqrt{n}} = \frac{\sqrt{\frac{R^2}{n}}}{\sqrt{\frac{S^2}{\nu}}} = \sqrt{\frac{\frac{R^2}{n}}{\frac{S^2}{\nu}}}.$$

Since $R^2 \sim \mathcal{X}_n^2$ and $S^2 \sim \mathcal{X}_\nu^2$, we have according to the definition of the F-distribution[1] *that $\frac{\hat{R}^2}{n}$ is F-distributed with parameters n and μ:*

$$\frac{\hat{R}^2}{n} \sim F(n,\ \nu)$$

Hence,

$$E\left(\hat{R}^2\right) = \frac{\nu n}{\nu - 2}, \quad \nu > 2.$$

Furthermore, it follows from equation (6.11) that

$$Cov(Y) = \frac{E\left(\hat{R}^2\right)}{n}\Sigma = \frac{1}{n}\frac{\nu n}{\nu - 2}\Sigma = \frac{\nu}{\nu - 2}\Sigma.$$

Therefore, the covariance of t-distribution is generally not equal to Σ, but it converges to Σ when $\nu \to \infty$.

6.2 Elliptical distribution and Extreme Dependence

Starting from the requirement that a stochastic model has to be able to describe extreme dependency in order to be suitable for the modeling and the

[1] If $X \sim \chi^2_{d_1}$ and $Y \sim \chi^2_{d_2}$ are independent, then $\frac{X/d_1}{Y/d_2} \sim \mathrm{F}(d_1, d_2)$

analysis of systemic risk contribution. It is important to precise when an elliptical distribution is tail-dependent. For this we rely on the work of Schmidt [2002]. In Schmidt [2002], a characterization of tail-dependence for elliptical distribution is provided by using the notion on of regularly varying functions.

Definition 26. *A measurable function $f : (0, \infty) \to (0, \infty)$, is said to be regularly varying with index $\alpha \in \mathbb{R}$ if*

$$\lim_{x \to \infty} \frac{f(tx)}{f(x)} = t^{\alpha} \text{ for any } t > 0.$$

Theorem 25 (Schmidt [2002]). *A bivariate elliptically distributed random vector $(X_1, X_2) \sim E_2(\mu, \Sigma, g)$ with positive definite matrix $\Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}$ and stochastic representation (6.5) is tail-dependent if its density generator g is regularly varying with index $\alpha > 0$ and $\frac{-\alpha}{2} - 1 < 0$. Then the tail dependence coefficient $\lambda = \lambda_u = \lambda_l$(see Definition 4) is given by*

$$\lambda = \frac{\int_0^{r(\rho)} \frac{u^{\alpha}}{\sqrt{1-u^2}} du}{\int_0^1 \frac{u^{\alpha}}{\sqrt{1-u^2}} du},$$

where $\rho = \Sigma_{12}/\sqrt{\Sigma_{11}}\sqrt{\Sigma_{22}}$ and $r(\rho) = \left\{1 + \frac{(1-\rho)^2}{1-\rho^2}\right\}^{-1/2}$.

Here, we have $\lambda = \lambda_u = \lambda_l$, this is a consequence of the fact that elliptical distributions are symmetric.

Furthermore using Theorem 25 Schmidt [2002] showed that the normal distribution does not have positive tail dependence for $\rho < 1$ (Schmidt [2002], Section 6.1). He also shows that all elliptically distributed random variables which have density generator of the form:

$$g(x) = (\pi\nu)^{-\frac{1}{2}n} \frac{\Gamma(N)}{\Gamma\left(N - \frac{n}{2}\right)} \left(1 + \frac{x}{\nu}\right)^{-N}, \quad N > \frac{n}{2}$$

(The so-called Pearson-type VII distribution (see Fang et al. [1990] Section 3.4)), have positive tail dependence (cf. Schmidt [2002], Theorem 6.4).

Since the bivariate t-distribution is a special case of the class of symmetric multivariate Pearson type VII distributions (Namely the case where $N = \frac{\nu+2}{2}$), it follows that the t distribution has positive tail dependence.

From this we can argue that the t distribution is a suitable model for the analysis of systemic risk contribution, while the normal distribution is not.

6.3 Computing $CoVaR_{\alpha}^{s|i}(l)$ when $\left(L^i, L^s\right) \sim E_2(\mu, \Sigma, \phi)$

In this section we present a general formula for the computation a $CoVaR_{\alpha}^{s|i}(l)$ when assumes a bivariate elliptical distribution. We first recall the results of the theory of elliptical distribution that will be use here.

Theorem 26 (cf. Fang et al. [1990]). *Let $X \sim E_n(\mu, \Sigma, \phi)$, such that*

$$X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}, \ \mu = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, \ \Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}, \tag{6.14}$$

where $\mu_1 \in \mathbb{R}^{m \times 1}$, $\Sigma_{11} \in \mathbb{R}^{m \times m}$ and ϕ is the characteristic generator. Then we have

$$X_1 \sim E_m(\mu_1, \Sigma_{11}, \phi) \text{ and } X_2 \sim E_{n-m}(\mu_2, \Sigma_{22}, \phi).$$

This means that the marginal distributions of elliptical distributions are also elliptical with the same characteristic generator.

We introduce in the following the characterization of the conditional distribution of elliptically distributed random vector. We first note that for an elliptical distributed random vector $X = \binom{X_1}{X_2}$, the standard formulation of the distribution function of X_1 given $X_2 = x_2$, $x_2 \in \mathbb{R}$

$$\begin{aligned} F_{X_1|X_2=x_2}(x_1) &= Pr(X_1 \leq x_1 | X_2 = x_2) \\ &= \frac{Pr(X_1 \leq x_1, X_2 = x_2)}{Pr(X_2 = x_2)}, \quad \forall x_1 \in \mathbb{R} \end{aligned} \tag{6.15}$$

cannot be used here, because the condition $\{X_2 = x_2\}$ is a zero-probability event (i.e. $Pr(X_2 = x_2) = 0$). However, we can define, in the context of Assumption 2, $F_{X_1|X_2=x_2}(x_1)$ through a probability density function as follows:

$$F_{X_1|X_2=x_2}(x_1) = \int_{-\infty}^{x_1} f_{X_1|X_2=x_2}(x)\,dx, \tag{6.16}$$

where

$$f_{X_1|X_2=x_2}(x_1) = \frac{f_{X_1,X_2}(x_1, x_2)}{f_{X_2}(x_2)}. \tag{6.17}$$

The denominator $f_{X_2}(x_2) := \int_{-\infty}^{\infty} f_{X_2,X_1}(x_2, v)\,dv$ is the marginal density of X_2, and we must assume it is strictly positive for every x_2.

In fact, under Assumption 2 the conditional distribution function (6.15) can be defined as

$$\begin{aligned} Pr\left(X_1 \le x_1 | X_2 = x_2\right) &= \lim_{\Delta x \to 0} Pr\left(X_1 \le x_1 | x_2 < X_2 \le x_2 + \Delta u\right) \\ &= \lim_{\Delta u \to 0} \frac{Pr\left(X_1 \le x_1, x_2 < X_2 \le x_2 + \Delta u\right)}{Pr\left(x_2 < X_2 \le x_2 + \Delta x\right)} \\ &\approx \lim_{\Delta u \to 0} \frac{\int_{-\infty}^{x_1}\left(\int_{x_2}^{x_2+\Delta u} f_{X_2,X_1}\left(u,v\right) du\right) dv}{f_{X_2}\left(x_2\right)\Delta u} \\ &\approx \lim_{\Delta u \to 0} \frac{\int_{-\infty}^{x_1} f_{X_2,X_1}\left(x_2,v\right)\Delta u dv}{f_{X_2}\left(x_2\right)\Delta u} \\ &= \int_{-\infty}^{x_1} \frac{f_{X_2,X_1}\left(x_2,v\right)}{f_{X_2}\left(x_2\right)} dv. \end{aligned}$$

From now we use the notation $X_1|X_2 = x_2$ to denote a random vector possessing the density $f_{X_1|X_2=x_2}$ (as defined in (6.17)).

The following theorem gives the characterization of the distribution of X_1 given that X_2 assumes a value x (We write $X_1|X_2 = x$) when (X_1, X_2) is an elliptical distributed random vector.

Theorem 27 (cf. Fang et al. [1990] Theorem 2.18). *Let $X \sim E_n(\mu, \Sigma, \phi)$. If we assume the partition* (6.14)*, then we have*

$$(X_1|X_2 = x) \sim E_m\left(\tilde{\mu}, \tilde{\Sigma}, \tilde{\phi}\right),$$

with $\tilde{\mu} = \mu_1 + \Sigma_{12}\Sigma_{22}^{-1}(x - \mu_2)$ and $\tilde{\Sigma} = \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}$.

Notice that the generator $\tilde{\phi}$ is in general not the same as ϕ. Thus from Theorem 27 we have that, if two random variable X_1 and X_2 have a joint elliptical distribution, then the conditional distribution of the random variable X_1 given X_2 take a value x $(X_1|X_2 = x)$[2] is again elliptical, but its type is not necessary the same as that of X.

Remark 56 (Embrechts et al. [2003]). *Notice that for a given elliptically distributed random variable X the representation $E_n(\mu, \Sigma, \phi)$ is not unique. It uniquely determines μ but Σ and ϕ are only determined up to a positive constant. More precisely, if $X \sim E_n(\mu, \Sigma, \phi)$ and $X \sim E_n\left(\tilde{\mu}, \tilde{\Sigma}, \tilde{\phi}\right)$,*

$$\tilde{\mu} = \mu, \quad \tilde{\Sigma} = c\Sigma, \quad \tilde{\phi}(\cdot) = \phi\left(\frac{\cdot}{c}\right),$$

for some constant $c > 0$.

[2] $X_1|X_2 = x$ is a random vector having the distribution function of X_1 under the condition that $X_2 = x$.

Remark 57. *Let ρ be the correlation coefficient of L^i and L^s, then the distribution of $L^s|L^i = l$ is an elliptical distribution with scale parameter μ^* and Σ^* given by*

$$\mu^* = \mu_s + \rho\frac{\sqrt{\Sigma_{ss}}}{\sqrt{\Sigma_{ii}}}(l - \mu_i)$$
$$\Sigma^* = \Sigma_{ss}\left(1 - \rho^2\right)$$

Proof. *In fact from Theorem 27 we have that $L^s|L^i = l$ is univariate elliptically distributed with scale parameter μ^* and Σ^* given by*

$$\mu^* = \mu_s + \Sigma_{si}\Sigma_{ii}^{-1}(x - \mu_s)$$
$$\Sigma^* = \Sigma_{ss} - \Sigma_{si}\Sigma_{ii}^{-1}\Sigma_{is}.$$

Consider the correlation coefficient ρ of L^i and ^{s} (see equation 6.12). It holds

$$\begin{aligned}\rho := & \frac{Cov\left(L^i, L^s\right)}{\sqrt{Var\left[L^i\right]}\sqrt{Var\left[L^s\right]}} \\ = & \frac{h\Sigma_{is}}{\sqrt{h\Sigma_i}\sqrt{h\Sigma_s}} \\ = & \frac{\Sigma_{is}}{\sqrt{\Sigma_{ii}}\sqrt{\Sigma_{ss}}} \\ = & \frac{\Sigma_{si}}{\sqrt{\Sigma_{ii}}\sqrt{\Sigma_{ss}}}.\end{aligned}$$

It follows from this that

$$\Sigma_{is} = \Sigma_{si} = \rho\sqrt{\Sigma_{ii}}\sqrt{\Sigma_{ss}}. \tag{6.18}$$

Furthermore, we have that

$$\begin{aligned}\mu^* &= \mu_s + \Sigma_{si}\Sigma_{ii}^{-1}(x - \mu_s) \\ &= \mu_s + \rho\sqrt{\Sigma_{ii}}\sqrt{\Sigma_{ss}}\Sigma_{ii}^{-1}(x - \mu_s) \\ &= \mu_s + \rho\frac{\sqrt{\Sigma_{ss}}}{\sqrt{\Sigma_{ii}}}(l - \mu_i)\end{aligned}$$

and

$$\begin{aligned}\Sigma^* &= \Sigma_{ss} - \Sigma_{si}\Sigma_{ii}^{-1}\Sigma_{is} \\ &= \Sigma_{ss} - \rho\sqrt{\Sigma_{ii}}\sqrt{\Sigma_{ss}}\Sigma_{ii}^{-1}\rho\sqrt{\Sigma_{ii}}\sqrt{\Sigma_{ss}} \\ &= \Sigma_{ss} - \rho^2\Sigma_{ss} \\ &= \Sigma_{ss}\left(1 - \rho^2\right).\end{aligned}$$

□

Theorem 28. *Assume that the distribution of* $(L^i,\ L^s)$ *is an elliptical distribution with location parameter* $\mu = \binom{\mu_i}{\mu_s}$ *and scale parameter* $\Sigma = \begin{pmatrix} \Sigma_{ii} & \Sigma_{is} \\ \Sigma_{si} & \Sigma_{ss} \end{pmatrix}$. *Let* ρ *be correlation coefficient of* L^i *and* L^s. *If we assume Assumption 2, then*

$$CoVaR_{\alpha}^{s|i}(l) = \mu_s + \rho\frac{\sqrt{\Sigma_{ss}}}{\sqrt{\Sigma_{ii}}}(l-\mu_i) + \sqrt{c\Sigma_{ss}(1-\rho^2)}F_Z^{-1}(\alpha),$$

where F_Z *is the distribution function of a specific spherical random variable* Z *and* $c > 0$ *is a constant.*

Proof. *From Theorem 27 we know that* $L^s|L^i = l$ *is univariate elliptically distributed with scale parameter* μ^* *and* Σ^* *given by:*

$$\mu^* = \mu_s + \Sigma_{si}\Sigma_{ii}^{-1}(x-\mu_s)$$
$$\Sigma^* = \Sigma_{ss} - \Sigma_{si}\Sigma_{ii}^{-1}\Sigma_{is}.$$

That is $L^s|L^i = l \sim E_1\left(\mu^*, \Sigma^*, \tilde{\phi}\right)$.

By Remark 57 we have that

$$L^s|L^i = l \sim E_1\left(\mu_s + \rho\frac{\sqrt{\Sigma_{ss}}}{\sqrt{\Sigma_{ii}}}(l-\mu_i), \Sigma_{ss}\left(1-\rho^2\right), \tilde{\phi}\right). \tag{6.19}$$

Following Remark 56 we can rewrite Equation 6.19 as follows:

$$L^s|L^i = l \sim E_1\left(\mu_s + \rho\frac{\sqrt{\Sigma_{ss}}}{\sqrt{\Sigma_{ii}}}(l-\mu_i), c\Sigma_{ss}\left(1-\rho^2\right), \phi'\right),$$

where $c > 0$ *is a constant and* $\phi' = \tilde{\phi}\left(\frac{\cdot}{c}\right)$.

Furthermore, Proposition 5 allows us to represent stochastically the random variable $L^s|L^i = l$ *as follows:*

$$\left(L^s|L^i = l\right) \overset{d}{=} \mu_s + \rho\frac{\sqrt{\Sigma_{ss}}}{\sqrt{\Sigma_{ii}}}(l-\mu_i) + \sqrt{c\Sigma_{ss}(1-\rho^2)}Z \tag{6.20}$$

where Z *is a spherical random variable.*

Due to Assumption 2 we have that the random variable $(L^s|L^i = l)$ *has a strictly positive density. We denote it by* $f_{s|i}$.

Let g^* *be the generator of* Z, *then by Remark 52 we have that the density function* f_Z *of* Z *is given by*

$$f_Z(x) = g^*\left(x^2\right)$$

and from Proposition 5 we have that

$$\begin{aligned} f_{s|i}(x) &= \frac{1}{\sqrt{c\Sigma_{ss}(1-\rho^2)}}g^*\left(\left[\frac{x-\left[\mu_s+\rho\frac{\sqrt{\Sigma_{ss}}}{\sqrt{\Sigma_{ii}}}(l-\mu_i)\right]}{\sqrt{c\Sigma_{ss}(1-\rho^2)}}\right]^2\right) \\ &= \frac{1}{\sqrt{c\Sigma_{ss}(1-\rho^2)}}f_Z\left(\frac{x-\left[\mu_s+\rho\frac{\sqrt{\Sigma_{ss}}}{\sqrt{\Sigma_{ii}}}(l-\mu_i)\right]}{\sqrt{c\Sigma_{ss}(1-\rho^2)}}\right). \end{aligned}$$

The distribution function $F_{s|i}$ of $(L^s|L^i = l)$ is thus given by:

$$F_{s|i}(x) = \int_{-\infty}^{x} f_{s|i}(u)\, du$$
$$= \int_{-\infty}^{x} \frac{1}{\sqrt{\Sigma^*}} f_Z\left(\frac{u - \left[\mu_s + \rho\frac{\sqrt{\Sigma_{ss}}}{\sqrt{\Sigma_{ii}}}(l - \mu_i)\right]}{\sqrt{c\Sigma_{ss}(1-\rho^2)}}\right) du. \tag{6.21}$$

By setting $t = \frac{x - \left[\mu_s + \rho\frac{\sqrt{\Sigma_{ss}}}{\sqrt{\Sigma_{ii}}}(l-\mu_i)\right]}{\sqrt{c\Sigma_{ss}(1-\rho^2)}}$ *and apply the substitution rule of integration theory to* (6.21) *we obtain*

$$F_{s|i}(x) = \int_{-\infty}^{\frac{x - \left[\mu_s + \rho\frac{\sqrt{\Sigma_{ss}}}{\sqrt{\Sigma_{ii}}}(l-\mu_i)\right]}{\sqrt{c\Sigma_{ss}(1-\rho^2)}}} f_Z(t)\, dt$$
$$= F_Z\left(\frac{x - \left[\mu_s + \rho\frac{\sqrt{\Sigma_{ss}}}{\sqrt{\Sigma_{ii}}}(l - \mu_i)\right]}{\sqrt{c\Sigma_{ss}(1-\rho^2)}}\right). \tag{6.22}$$

F_Z is the distribution function of Z.

Let us consider now Equation 2.14

$$Pr\left(L^s \leq CoVaR_\alpha^{s|L^i=l}|L^i = l\right) = \alpha, \quad l \in \mathbb{R}. \tag{6.23}$$

Given $F_{s|i}$ we can rewrite equation (6.23) *as follows:*

$$F_{s|i}\left(CoVaR_\alpha^{s|L^i=l}\right) = \alpha.$$

From Equation (6.22) *it holds*

$$F_{s|i}\left(CoVaR_\alpha^{s|L^i=l}\right) = F_Z\left(\frac{CoVaR_\alpha^{s|L^i=l} - \left[\mu_s + \rho\frac{\sqrt{\Sigma_{ss}}}{\sqrt{\Sigma_{ii}}}(l - \mu_i)\right]}{\sqrt{c\Sigma_{ss}(1-\rho^2)}}\right) = \alpha$$

Note that because of Assumption 2, we have that F_Z is strictly increasing and thus invertible. Using this, we obtain,

$$F_Z^{-1}(\alpha) = \frac{CoVaR_\alpha^{s|L^i=l} - \left[\mu_s + \rho\frac{\sqrt{\Sigma_{ss}}}{\sqrt{\Sigma_{ii}}}(l - \mu_i)\right]}{\sqrt{c\Sigma_{ss}(1-\rho^2)}}.$$

This implies that

$$CoVaR_\alpha^{s|L^i=l} = \left[\mu_s + \rho\frac{\sqrt{\Sigma_{ss}}}{\sqrt{\Sigma_{ii}}}(l - \mu_i)\right] + \sqrt{c\Sigma_{ss}(1-\rho^2)}F_Z^{-1}(\alpha). \qquad \square$$

Remark 58. *The term $c\Sigma_{ss}(1-\rho^2)$ can be interpreted as a conditional scale of L^s under the condition that $L^i = l$ and may depends on l. In this context, since $\Sigma_{ss}(1-\rho^2)$ is given by the distribution parameters, we see that only the term c can depends on l (and we write c_l).*

Based on this we can rewrite the formula in Theorem 28 as follows:

Corollary 29.

$$CoVaR_{\alpha}^{s|L^i=l} = \mu_s + \rho\frac{\sqrt{\Sigma_{ss}}}{\sqrt{\Sigma_{ii}}}(l-\mu_i) + \sqrt{c_l}\sqrt{\Sigma_{ss}(1-\rho^2)}F_Z^{-1}(\alpha). \tag{6.24}$$

Proposition 6. *Assume that the distribution of* $(L^i,\ L^s)$ *is an elliptical distribution with location parameter* $\mu = \binom{\mu_i}{\mu_s}$, *scale parameter* $\Sigma = \begin{pmatrix}\Sigma_{ii} & \Sigma_{is}\\ \Sigma_{si} & \Sigma_{ss}\end{pmatrix}$ *and correlation coefficient* ρ. *If we assume Assumption 2, then*

$$\Delta CoVaR_{\alpha}^{s|i}(l_1,l_2) = \rho\frac{\sqrt{\Sigma_{ss}}}{\sqrt{\Sigma_{ii}}}(l_1-l_2) + \sqrt{\Sigma_{ss}(1-\rho^2)}F_Z^{-1}(\alpha)\left[\sqrt{c_{VaR_{\alpha}^i}} - \sqrt{c_{\mu_i}}\right].$$

Proof.

$$\begin{aligned}\Delta CoVaR_{\alpha}^{s|i}(l_1,i_2) &= CoVaR_{\alpha}^{s|i}(l_1) - CoVaR_{\alpha}^{s|i}(l_2)\\ &= \left[\mu_s + \rho\frac{\sqrt{\Sigma_{ss}}}{\sqrt{\Sigma_{ii}}}(l_1-\mu_i)\right] + \sqrt{c_{l_1}}\sqrt{\Sigma_{ss}(1-\rho^2)}F_Z^{-1}(\alpha)\\ &\quad - \left[\mu_s + \rho\frac{\sqrt{\Sigma_{ss}}}{\sqrt{\Sigma_{ii}}}(l_2-\mu_i)\right] + \sqrt{c_{l_2}}\sqrt{\Sigma_{ss}(1-\rho^2)}F_Z^{-1}(\alpha)\\ &= \rho\frac{\sqrt{\Sigma_{ss}}}{\sqrt{\Sigma_{ii}}}(l_1-l_2) + \sqrt{\Sigma_{ss}(1-\rho^2)}F_Z^{-1}(\alpha)\left[\sqrt{c_{l_1}} - \sqrt{c_{l_2}}\right] \quad \square\end{aligned}$$

Especially we have that (see definition 9)

$$\begin{aligned}\Delta CoVaR_{\alpha}^{s|i} &:= \Delta CoVaR_{\alpha}^{s|i}\left(VaR_{\alpha}^i, \mu_i\right)\\ &= \rho\frac{\sqrt{\Sigma_{ss}}}{\sqrt{\Sigma_{ii}}}\left(VaR_{\alpha}^i - \mu_i\right) + \sqrt{\Sigma_{ss}(1-\rho^2)}F_Z^{-1}(\alpha)\left[\sqrt{c_{VaR_{\alpha}^i}} - \sqrt{c_{\mu_i}}\right].\end{aligned}$$

6.4 Applications

In this section, we use the Theorem 28 to compute $CoVaR_{\alpha}^{s|i}$ for the normal and the t distribution.

6.4.1 Application to the Bivariate Normal Distribution

Recall that the density generator of the n-variate normal distribution is given by

$$g_n(x) = \frac{1}{(2\pi)^{\frac{n}{2}}}e^{-\frac{x}{2}}.$$

And from 6.6 we have that the density function of the bivariate normal distribution with location parameter $\mu \in \mathbb{R}^2$ and scale parameter $\Sigma \in \mathbb{R}^{2\times 2}$ is given by

$$f(x) = \frac{1}{2\pi|\Sigma|^{\frac{1}{2}}}e^{-\frac{\left((x-\mu)'\Sigma^{-1}(x-\mu)\right)}{2}}, \quad x = \begin{pmatrix}x_1\\ x_2\end{pmatrix} \in \mathbb{R}^2$$

where

$$\mu = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, \ \Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}.$$

It holds

$$\Sigma^{-1} = \frac{1}{\Sigma_{11}\Sigma_{22} - \Sigma_{12}^2} \begin{pmatrix} \Sigma_{22} & -\Sigma_{12} \\ -\Sigma_{21} & \Sigma_{11} \end{pmatrix}.$$

By replacing Σ_{12} and Σ_{21} by $\rho\sqrt{\Sigma_{11}}\sqrt{\Sigma_{22}}$ (see 6.18) we obtain

$$\Sigma^{-1} = \frac{1}{\Sigma_{11}\Sigma_{22} - \Sigma_{12}^2} \begin{pmatrix} \Sigma_{22} & -\Sigma_{12} \\ -\Sigma_{21} & \Sigma_{11} \end{pmatrix} = \frac{1}{\Sigma_{11}\Sigma_{22}\left(1-\rho^2\right)} \begin{pmatrix} \Sigma_{22} & \rho\sqrt{\Sigma_{11}}\sqrt{\Sigma_{22}} \\ -\rho\sqrt{\Sigma_{11}}\sqrt{\Sigma_{22}} & \Sigma_{11} \end{pmatrix}.$$

It follows that

$$\begin{aligned}
&\left(x_1 - \mu_2, x_2 - \mu_2\right) \Sigma^{-1} \begin{pmatrix} x_1 - \mu_1 \\ x_2 - \mu_2 \end{pmatrix} \\
&\qquad = \frac{\Sigma_{22}\left(x_1-\mu_1\right)^2 + \Sigma_{11}\left(x_2-\mu_2\right)^2 - 2\rho\sqrt{\Sigma_{11}}\sqrt{\Sigma_{22}}\left(x_1-\mu_1\right)\left(x_2-\mu_2\right)}{\Sigma_{11}\Sigma_{22}\left(1-\rho^2\right)} \\
&\qquad = \frac{1}{1-\rho^2}\left[\frac{\left(x_1-\mu_1\right)^2}{\Sigma_{11}} + \frac{\left(x_2-\mu_2\right)^2}{\Sigma_{22}} - \frac{2\rho\left(x_1-\mu_1\right)\left(x_2-\mu_2\right)}{\sqrt{\Sigma_{11}}\sqrt{\Sigma_{22}}}\right].
\end{aligned} \tag{6.25}$$

Hence,

$$\begin{aligned}
&f\left(x_1, x_2\right) \\
&= \frac{1}{2\pi\sqrt{\Sigma_{ii}\Sigma_{ss}}\sqrt{1-\rho^2}} e^{\left(-\frac{1}{2(1-\rho^2)}\left[\frac{(x_1-\mu_1)^2}{\Sigma_{11}} + \frac{(x_2-\mu_2)^2}{\Sigma_{22}} - \frac{2\rho(x_1-\mu_1)(x_2-\mu_2)}{\sqrt{\Sigma_{11}}\sqrt{\Sigma_{22}}}\right]\right)}.
\end{aligned}$$

The marginal density of X_2 is then given by

$$\begin{aligned}
f_{X_2}(y) &= \int_{-\infty}^{+\infty} f(x,y)\, dx \\
&= \frac{1}{\sqrt{2\pi\Sigma_{22}}} \exp\left(-\frac{\left(x-\mu_2\right)^2}{2\Sigma_{22}}\right).
\end{aligned}$$

Let $(L^i, \ L^s)$ follows a bivariate normal distribution with location and scale parameter given by:

$$\mu = \begin{pmatrix} \mu_i \\ \mu_s \end{pmatrix}, \text{ and } \Sigma = \begin{pmatrix} \Sigma_{ii} & \Sigma_{is} \\ \Sigma_{si} & \Sigma_{ss} \end{pmatrix}, \quad \mu_i = E\left[L^i\right], \ \mu_s = E\left[L^s\right].$$

The conditional density $f_{s|i}$ of $L^s|L^i = l$ is given by

$$
\begin{aligned}
f_{s|i} &= \frac{f(l_s, l)}{f_{L^i}(l_s)} \\
&= \frac{\frac{1}{2\pi\sqrt{\Sigma_{ss}}\sqrt{\Sigma_{ii}}\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2(1-\rho^2)}\left[\frac{(l_s-\mu_s)^2}{\Sigma_{ss}} + \frac{(l-\mu_i)^2}{\Sigma_{ii}} - \frac{2\rho(l_s-\mu_s)(l-\mu_i)}{\sqrt{\Sigma_{ss}}\sqrt{\Sigma_{ii}}}\right]\right)}{\frac{1}{\sqrt{2\pi\Sigma_{ii}}}\exp\left(-\frac{(l-\mu_i)^2}{2\Sigma_{ii}}\right)} \\
&= \frac{\frac{1}{\sqrt{2\pi}\sqrt{\Sigma_{ss}}\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2(1-\rho^2)}\left[\frac{(l_s-\mu_s)^2}{\Sigma_{ss}} + \frac{(l-\mu_i)^2}{\Sigma_{ii}} - \frac{2\rho(l_s-\mu_s)(l-\mu_i)}{\sqrt{\Sigma_{ss}}\sqrt{\Sigma_{ii}}}\right]\right)}{\exp\left(-\frac{(l-\mu_i)^2}{2\Sigma_{ii}}\right)} \\
&= \frac{\exp\left(-\frac{1}{2(1-\rho^2)}\left[\frac{(l_s-\mu_s)^2}{\Sigma_{ss}} + \frac{(l-\mu_i)^2}{\Sigma_{ii}} - \frac{2\rho(l_s-\mu_s)(l-\mu_i)}{\sqrt{\Sigma_{ss}}\sqrt{\Sigma_{ii}}} - \frac{\left(1-\rho^2\right)(l-\mu_i)^2}{\Sigma_{ii}}\right]\right)}{\sqrt{2\pi}\sqrt{\Sigma_{ss}}\sqrt{1-\rho^2}} \\
&= \frac{\exp\left(-\frac{1}{2(1-\rho^2)}\left[\frac{(l_s-\mu_s)}{\sqrt{\Sigma_{ss}}} - \frac{\rho(l-\mu_i)}{\sqrt{\Sigma_{ii}}}\right]^2\right)}{\sqrt{2\pi}\sqrt{\Sigma_{ss}}\sqrt{1-\rho^2}} \\
&= \frac{\exp\left(-\frac{1}{2(1-\rho^2)}\left[\frac{1}{\sqrt{\Sigma_{ss}}}\left((l_s-\mu_s) - \frac{\rho\sqrt{\Sigma_{ss}}(l-\mu_i)}{\sqrt{\Sigma_{ii}}}\right)\right]^2\right)}{\sqrt{2\pi}\sqrt{\Sigma_{ss}}\sqrt{1-\rho^2}} \\
&= \frac{\exp\left(-\frac{1}{2(1-\rho^2)\Sigma_{ss}}\left[l_s - \left(\mu_s + \frac{\rho\sqrt{\Sigma_{ss}}(l-\mu_i)}{\sqrt{\Sigma_{ii}}}\right)\right]^2\right)}{\sqrt{2\pi}\sqrt{\Sigma_{ss}}\sqrt{1-\rho^2}}. \qquad (6.26)
\end{aligned}
$$

Remark 59. *Equation* (6.26) *corresponds to the expression of the density of an univariate normal distributed random variable with mean equals to* $\mu_s + \frac{\rho\sqrt{\Sigma_{ss}}(l-\mu_i)}{\sqrt{\Sigma_{ii}}}$ *and variance equals to* $(1-\rho^2)\Sigma_{ss}$.

Hence, following equation 6.20 we can represent $(L^s|L^i = l)$ as follows:

$$\left(L^s|L^i = l\right) \stackrel{d}{=} \mu_s + \frac{\rho\sqrt{\Sigma_{ss}}(l-\mu_i)}{\sqrt{\Sigma_{ii}}} + \sqrt{(1-\rho^2)}\sqrt{\Sigma_{ss}}Z_g.$$

Furthermore, we have that, the distribution of Z_g is the univariate spherical distribution associated with the normal distribution (i.e. Z_g is a standard normal distributed random variable) and the constant c is equal to 1.

By using these information and by applying theorem 28 we obtain the following expression for $CoVaR_{\alpha}^{s|L^i=l}$ when (L^i, L^s) assumes a bivariate normal distributions

Corollary 30. *Let* (L^i, L^s) *assumes a bivariate normal distribution with correlation coefficient* ρ, *location parameter* $\mu = \binom{\mu_i}{\mu_s}$ *and scale parameter* $\Sigma = \begin{pmatrix}\Sigma_{ii} & \Sigma_{is} \\ \Sigma_{si} & \Sigma_{ss}\end{pmatrix}$. *Then*

$$CoVaR_{\alpha} = \mu_s + \rho\frac{\sigma_s}{\sigma_i}(l_i - \mu_s) + \sigma_s\sqrt{1-\rho^2}\Phi^{-1}(\alpha) \qquad (6.27)$$

where σ_i and σ_s denote the standard deviation of the financial institution i and s respectively.

Proof. *From theorem 28 we know that (see equation (6.24))*

$$CoVaR_\alpha^{s|L^i=l} = \mu_s + \rho\frac{\sqrt{\Sigma_{ss}}}{\sqrt{\Sigma_{ii}}}(l-\mu_i) + \sqrt{c_l}\sqrt{\Sigma_{ss}(1-\rho^2)}F_Z^{-1}(\alpha).$$

From the previous development we know that Z is standard normal distributed and that $c_l = 1$. This leads to

$$CoVaR_\alpha^{s|L^i=l} = \mu_s + \frac{\rho\sqrt{\Sigma_{ss}}(l-\mu_i)}{\sqrt{\Sigma_{ii}}} + \sqrt{(1-\rho^2)}\sqrt{\Sigma_{ss}}\Phi^{-1}(\alpha), \tag{6.28}$$

where Φ denote the distribution of the standard Gaussian distribution.

Recall that the covariance matrix V of a normally distributed random variable corresponds to its scale matrix Σ (see equation (6.13)). That is, let σ_i and σ_s denote the standard deviation of the financial institution i and s respectively, then

$$\Sigma = \begin{pmatrix}\Sigma_{ii} & \Sigma_{is}\\ \Sigma_{si} & \Sigma_{ss}\end{pmatrix} = V = \begin{pmatrix}V_{ii} & V_{is}\\ V_{si} & V_{ss}\end{pmatrix} = \begin{pmatrix}\sigma_i^2 & \rho\sigma_s^2\sigma_i\\ \rho\sigma_i\sigma_s & \sigma_s^2\end{pmatrix}.$$

Using this, we can rewrite equation (6.28) as follows:

$$CoVaR_\alpha = \mu_s + \rho\frac{\sigma_s}{\sigma_i}(l_i-\mu_i) + \sigma_s\sqrt{1-\rho^2}\Phi^{-1}(\alpha). \qquad \square$$

It follows from proposition 6, that

$$\begin{aligned}\Delta CoVaR_\alpha^{s|i}(l_1,l_2) &= \rho\frac{\sqrt{\Sigma_{ss}}}{\sqrt{\Sigma_{ii}}}(l_1-l_2) + \sqrt{\Sigma_{ss}(1-\rho^2)}\Phi^{-1}(\alpha)[1-1]\\ &= \rho\frac{\sigma_s}{\sigma_i}(l_1-l_2).\end{aligned}$$

Especially, we have that $\Delta CoVaR^{s|i}$ as defined in Brunnermeier and Adrian [2011] (see definition 9), is given by

Corollary 31.

$$\Delta CoVaR^{s|i} = \rho\frac{\sigma_s}{\sigma_i}\left(VaR_\alpha^i - E\left(L^i\right)\right). \tag{6.29}$$

Remark 60. *The formula in corollary 30 (equation (6.27)) coincides with the formula provided by Jäger-Ambrożewicz [2010]. Therefore, the formula proposed by Jäger-Ambrożewicz can be seen as a special case of the formula provided in Theorem 28.*

6.4.2 Application to the Bivariate t-Distribution

Recall that, the density function of a n-multivariate t-distribution random variable X with degree of freedom ν and scale parameter Σ ($X \sim t_d(\nu, \Sigma)$) is given by (McNeil et al. [2005], Example 3.7)

$$f_n(x) = \frac{\Gamma\left(\frac{\nu+n}{2}\right)}{(\nu\pi)^{\frac{n}{2}}\ \Gamma\left(\frac{\nu}{2}\right)|\Sigma|^{\frac{1}{2}}}\left(1 + \frac{x'\Sigma^{-1}x}{\nu}\right)^{-\frac{\nu+n}{2}}.$$

Using (6.25), we can express the density function f_2 of a bivarite t distributed random variable (X_1, X_2) with ν degree of freedom ν and correlation coefficient ρ as follows:

$$f_2(x_1, x_2) = \frac{\Gamma\left(\frac{\nu+2}{2}\right)}{(\nu\pi)\ \Gamma\left(\frac{\nu}{2}\right)\sqrt{(1-\rho^2)}}\left(1 + \frac{\frac{1}{1-\rho^2}\left[\frac{x_1^2}{\Sigma_{11}} + \frac{x_2^2}{\Sigma_{22}} - \frac{2\rho(x_1x_2)}{\sqrt{\Sigma_{11}}\sqrt{\Sigma_{22}}}\right]}{\nu}\right)^{-\frac{\nu+2}{2}}.$$

Without loss of generality, let $(L^i,\ L^s)$ follows a standard bivariate t-distribution (i.e $\Sigma_{ii} = \Sigma ss = 1$) with ν degree of freedom ν. The conditional density $f_{s|i}$ of $L^s|L^i = l$ is given by

$$\begin{aligned}
f_{s|i} &= \frac{\frac{\Gamma\left(\frac{\nu+2}{2}\right)}{(\nu\pi)\Gamma\left(\frac{\nu}{2}\right)\sqrt{(1-\rho^2)}}\left(1 + \frac{\frac{1}{1-\rho^2}\left[l_s^2+l^2-2\rho l_s l\right]}{\nu}\right)^{-\frac{\nu+2}{2}}}{\frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\sqrt{\nu\pi}\,\Gamma\left(\frac{\nu}{2}\right)}\left(1+\frac{l}{\nu}\right)^{-\frac{\nu+1}{2}}} \\
&= \frac{\Gamma\left(\frac{\nu+2}{2}\right)}{\Gamma\left(\frac{\nu+1}{2}\right)\sqrt{\nu\pi}\sqrt{(1-\rho^2)}}\frac{\left(1 + \frac{\frac{1}{1-\rho^2}\left[l_s^2+l^2-2\rho l_s l\right]}{\nu}\right)^{-\frac{\nu+2}{2}}}{\left(1+\frac{l^2}{\nu}\right)^{-\frac{\nu+1}{2}}} \\
&= \frac{\Gamma\left(\frac{\nu+2}{2}\right)}{\Gamma\left(\frac{\nu+1}{2}\right)\sqrt{\nu\pi}\sqrt{(1-\rho^2)}}\frac{\left(\frac{\nu+\frac{1}{1-\rho^2}\left[l_s^2+l^2-2\rho l_s l\right]}{\nu}\right)^{-\frac{\nu+2}{2}}}{\left(\frac{\nu+l^2}{\nu}\right)^{-\frac{\nu+1}{2}}} \\
&= \frac{\Gamma\left(\frac{\nu+2}{2}\right)}{\Gamma\left(\frac{\nu+1}{2}\right)\sqrt{\nu\pi}\sqrt{1-\rho^2}}\left(\frac{\nu+\frac{1}{1-\rho^2}\left[l_s^2+l^2-2\rho l_s l\right]}{\nu}\right)^{-\frac{\nu+2}{2}}\left(\frac{\nu+l^2}{\nu}\right)^{\frac{\nu+1}{2}} \\
&= \frac{\Gamma\left(\frac{\nu+2}{2}\right)}{\Gamma\left(\frac{\nu+1}{2}\right)\sqrt{\nu\pi}\sqrt{(1-\rho^2)}}\left[\left(\frac{1}{\nu}\right)^{-\frac{\nu+2}{2}+\frac{\nu+1}{2}}\left(\nu+\frac{1}{1-\rho^2}\left[l_s^2+l^2-2\rho l_s l\right]\right)^{-\frac{\nu+2}{2}}\left(\nu+l^2\right)^{\frac{\nu+1}{2}}\right] \\
&= \frac{\Gamma\left(\frac{\nu+2}{2}\right)}{\Gamma\left(\frac{\nu+1}{2}\right)\sqrt{\pi}\sqrt{(1-\rho^2)}}\left[\left(\nu+\frac{1}{1-\rho^2}\left[l_s^2+l^2-2\rho l_s l\right]\right)^{-\frac{\nu+2}{2}}\left(\nu+l^2\right)^{\frac{\nu+1}{2}}\right] \\
&= \frac{\Gamma\left(\frac{\nu+2}{2}\right)}{\Gamma\left(\frac{\nu+1}{2}\right)\sqrt{\pi}\sqrt{(1-\rho^2)}}\left[\left(\nu+\frac{1}{1-\rho^2}\left[l^2\left(1-\rho^2\right)+(l_s-\rho l)^2\right]\right)^{-\frac{\nu+2}{2}}\left(\nu+l^2\right)^{\frac{\nu+1}{2}}\right] \\
&= \frac{\Gamma\left(\frac{\nu+2}{2}\right)}{\Gamma\left(\frac{\nu+1}{2}\right)\sqrt{\pi}\sqrt{(1-\rho^2)}}\left(\nu+l^2+\frac{(l_s-\rho l)^2}{(1-\rho^2)}\right)^{-\frac{\nu+2}{2}}\left(\nu+l^2\right)^{\frac{\nu+1}{2}}. \qquad (6.30)
\end{aligned}$$

Remark 61. *Equation* (6.30) *represents the expression of the density function of an univariate t-distributed random variable with* $\nu+1$ *degree of freedom, location parameter equals to* ρl *and scale parameter equals to* $\frac{\sqrt{(1-\rho^2)(\nu+l^2)}}{\sqrt{\nu+1}}$.

Proof. *Recall that density function of the t-distribution with* ν *degree of freedom, location parameter* μ *and scale parameter* σ^2 *can be expressed as:*

$$t_{\nu,\mu,\sigma^2}(x) = \frac{\Gamma(\frac{\nu+1}{2})}{\Gamma(\frac{\nu}{2})\sqrt{\pi\nu\sigma^2}}\left(1+\frac{(x-\mu)^2}{\nu\sigma^2}\right)^{-\frac{\nu+1}{2}}.$$

Hence, the density function of the t-distribution with $\nu+1$ *degree of freedom, location parameter* μ *and scale parameter* σ^2 *is thus given by*

$$t_{\nu+1,\mu,\sigma^2}(x) = \frac{\Gamma(\frac{\nu+2}{2})}{\Gamma(\frac{\nu+1}{2})\sqrt{\pi\,(\nu+1)\,\sigma^2}}\left(1+\frac{(x-\mu)^2}{(\nu+1)\,\sigma^2}\right)^{-\frac{\nu+2}{2}}. \tag{6.31}$$

Now, consider the expression (6.31) *and set*

$$\begin{aligned} x &= l_s, \\ \mu &= \rho l \quad \textit{and} \\ \sigma^2 &= \frac{(1-\rho^2)(\nu+l^2)}{\nu+1}. \end{aligned}$$

We obtain

$$\begin{aligned}
t_{\nu+1,\mu,\sigma^2}(l_s) &= \frac{\Gamma(\frac{\nu+2}{2})}{\Gamma(\frac{\nu+1}{2})\sqrt{\pi(\nu+1)}\sqrt{\frac{(1-\rho^2)(\nu+l^2)}{\nu+1}}}\left(1+\frac{(l_s-\rho l)^2}{(\nu+1)\frac{(1-\rho^2)(\nu+l^2)}{\nu+1}}\right)^{-\frac{\nu+2}{2}} \\
&= \frac{\Gamma(\frac{\nu+2}{2})}{\Gamma(\frac{\nu+1}{2})\sqrt{\pi}}\frac{1}{\sqrt{(\nu+1)}}\frac{1}{\sqrt{\frac{(1-\rho^2)(\nu+l^2)}{\nu+1}}}\left(1+\frac{(l_s-\rho l)^2}{(1-\rho^2)(\nu+l^2)}\right)^{-\frac{\nu+2}{2}} \\
&= \frac{\Gamma(\frac{\nu+2}{2})}{\Gamma(\frac{\nu+1}{2})\sqrt{\pi}}\frac{1}{\sqrt{(1-\rho^2)(\nu+l^2)}}\left(\frac{(1-\rho^2)(\nu+l^2)+(l_s-\rho l)^2}{(1-\rho^2)(\nu+l^2)}\right)^{-\frac{\nu+2}{2}} \\
&= \frac{\Gamma(\frac{\nu+2}{2})}{\Gamma(\frac{\nu+1}{2})\sqrt{\pi}\sqrt{(1-\rho^2)}}\frac{1}{\sqrt{(\nu+l^2)}}\left(\frac{(1-\rho^2)(\nu+l^2)+(l_s-\rho l)^2}{(1-\rho^2)(\nu+l^2)}\right)^{-\frac{\nu+2}{2}} \\
&= \frac{\Gamma(\frac{\nu+2}{2})}{\Gamma(\frac{\nu+1}{2})\sqrt{\pi}\sqrt{(1-\rho^2)}}\frac{1}{\sqrt{(\nu+l^2)}}\left(\frac{(1-\rho^2)\left((\nu+l^2)+\frac{(l_s-\rho l)^2}{(1-\rho^2)}\right)}{(1-\rho^2)(\nu+l^2)}\right)^{-\frac{\nu+2}{2}} \\
&= \frac{\Gamma(\frac{\nu+2}{2})}{\Gamma(\frac{\nu+1}{2})\sqrt{\pi}\sqrt{(1-\rho^2)}}\frac{1}{\sqrt{(\nu+l^2)}}\left(\frac{(\nu+l^2)+\frac{(l_s-\rho l)^2}{(1-\rho^2)}}{(\nu+l^2)}\right)^{-\frac{\nu+2}{2}} \\
&= \frac{\Gamma(\frac{\nu+2}{2})}{\Gamma(\frac{\nu+1}{2})\sqrt{\pi}\sqrt{(1-\rho^2)}}\frac{1}{\sqrt{(\nu+l^2)}}\left((\nu+l^2)+\frac{(l_s-\rho l)^2}{(1-\rho^2)}\right)^{-\frac{\nu+2}{2}}\left(\frac{1}{\nu+l^2}\right)^{-\frac{\nu+2}{2}} \\
&= \frac{\Gamma(\frac{\nu+2}{2})}{\Gamma(\frac{\nu+1}{2})\sqrt{\pi}\sqrt{(1-\rho^2)}}\left((\nu+l^2)+\frac{(l_s-\rho l)^2}{(1-\rho^2)}\right)^{-\frac{\nu+2}{2}}\left(\frac{1}{\nu+l^2}\right)^{-\frac{\nu+2}{2}}\left(\frac{1}{\nu+l^2}\right)^{\frac{1}{2}} \\
&= \frac{\Gamma(\frac{\nu+2}{2})}{\Gamma(\frac{\nu+1}{2})\sqrt{\pi}\sqrt{(1-\rho^2)}}\left((\nu+l^2)+\frac{(l_s-\rho l)^2}{(1-\rho^2)}\right)^{-\frac{\nu+2}{2}}\left(\frac{1}{\nu+l^2}\right)^{-\frac{\nu+1}{2}} \\
&= \frac{\Gamma(\frac{\nu+2}{2})}{\Gamma(\frac{\nu+1}{2})\sqrt{\pi}\sqrt{(1-\rho^2)}}\left(\nu+l^2+\frac{(l_s-\rho l)^2}{(1-\rho^2)}\right)^{-\frac{\nu+2}{2}}\left(\nu+l^2\right)^{\frac{\nu+1}{2}}
\end{aligned}$$

□

Based on remark 61, we can follow equation 6.20 and represent $(L^s|L^i=l)$ as:

$$\left(L^s|L^i=l\right)\overset{d}{=}\rho l+\frac{\sqrt{(1-\rho^2)(\nu+l^2)}}{\sqrt{\nu+1}}Z_t, \tag{6.32}$$

where Z_t follows an univariate t-distribution with $\nu+1$ degree of freedom. Furthermore, we have that the constant c_l is equal to $\frac{\sqrt{\nu+l^2}}{\sqrt{\nu+1}}$. By using these information and by applying theorem 28 we derive the expression of $CoVaR_{\alpha}^{s|L^i=l}$ when (L^i,L^s) assumes a bivariate t-distributions.

Corollary 32. *Assume that (L^i,L^s) follows a standard bivariate t-distribution with ν degree of freedom, and correlation coefficient ρ, then*

$$CoVaR_{\alpha}^{s|L^i=l}=\rho l+\frac{\sqrt{\nu+l^2}}{\sqrt{\nu+1}}\sqrt{1-\rho^2}\;t_{\nu+1}^{-1}(\alpha), \tag{6.33}$$

where $t_{\nu+1}^{-1}$ *denote the distribution function of a standard t distributed radom variable with* $\nu+1$ *degree of freedom.*

Proof. *From theorem 28 we know that (see equation* (6.24)*)*

$$CoVaR_{\alpha}^{s|i}(l) = \mu_s + \rho\frac{\sqrt{\Sigma_{ss}}}{\sqrt{\Sigma_{ii}}}(l-\mu_i) + \sqrt{c_l}\sqrt{\Sigma_{ss}(1-\rho^2)}F_{Z_t}^{-1}(\alpha).$$

From the previous development we know that Z_t *follows a standard t-distribution with* $\nu+1$ *degree of freedom and that* $c_l = \frac{\nu+l^2}{\nu+1}$ *(see equation* (6.32)*). This leads to*

$$CoVaR_{\alpha}^{s|L^i=l} = \rho l + \sqrt{\frac{\nu+l^2}{\nu+1}}\sqrt{1-\rho^2}\; t_{\nu+1}^{-1}(\alpha),$$

where $t_{\nu+1}^{-1}$ *denote the distribution function of a standard t distributed random variable with* $\nu+1$ *degree of freedom.* □

It follows from Proposition 6, that

$$\begin{aligned}\Delta CoVaR_{\alpha}^{s|i}(l_1,l_2) &= \rho(l_1-l_2) + \sqrt{(1-\rho^2)}t_{\nu+1}^{-1}(\alpha)\left[\sqrt{\frac{\nu+l_1^2}{\nu+1}} - \sqrt{\frac{\nu+l_2^2}{\nu+1}}\right] \\ &= \rho(l_1-l_2) + \sqrt{\frac{1-\rho^2}{\nu+1}}t_{\nu+1}^{-1}(\alpha)\left[\sqrt{\nu+l_1^2} - \sqrt{\nu+l_2^2}\right]\end{aligned} \tag{6.34}$$

Especially, we have that $\Delta CoVaR^{s|i}$ as defined in Brunnermeier and Adrian [2011] (see definition 8), is given by

Corollary 33.

$$\Delta CoVaR^{s|i} = \rho VaR_{\alpha}^{i} + \sqrt{\frac{1-\rho^2}{\nu+1}}t_{\nu+1}^{-1}(\alpha)\left[\sqrt{\nu+(VaR_{\alpha}^{i})^2} - \sqrt{\nu}\right] \tag{6.35}$$

Remark 62. *Equation* (6.33) *coincides with the formula in equation* (4.31)*.*

Chapter 7

Conclusion

In this thesis, we deal with the problem of the modeling and estimating of systemic risk contribution. This task is essential for the effective implementation of the new risk management concepts that have been developed in response to the latest financial crisis. These concepts includes:

- The estimation of the potential financial loss suffered by a financial system if a given financial institution fails.
- The determination of the systemic importance of financial institutions.
- the calculation of the individual financial contribution to a **mutual default funds and resolution funds** such as the single bank resolution fund for financial institutions in the countries subject to the SRM and the default fund for CCPs members.
- The preparation of bail-in-operation (cf. Remark 15).

Our starting point was the CoVaR-method proposed by Brunnermeier and Adrian [2011]. The CoVaR-method is one of the most used tools for the analysis of systemic risk. It builds on the statistic $CoVaR_{\alpha}^{s|i}(l)$, which is defined as the Value-at-Risk of a financial system s conditional on the loss of a given financial institution i. Thus, the CoVaR-method is a tools for the investigation of the impact of idiosyncratic losses upon the loss of a financial system.

Brunnermeier and Adrian [2011] estimated the potential financial impact of the failure of a given financial institution i upon a financial system s i.e. the risk contribution of the financial institution i on the system s computed as the difference between the $CoVaR$ conditional on the institution being in distress and the $CoVaR$ conditional on the institution being in a normal situation. They assumed that a financial institution is in distress when its loss is equal to its Value-at-Risk at the level α (VaR_{α}^{i}), and in normal situation when its loss is equal to is equals to its expected loss $E\left[L^{i}\right]$. This difference (between

$CoVaR_\alpha^{s|i}(VaR_\alpha^i)$ and $CoVaR_\alpha^{s|i}(E[L^i]))$ is called $\Delta CoVaR$. Therefor, the main challenge of the estimation of systemic risk contribution using CoVaR-method is the computation of $CoVaR_\alpha^{s|i}(l)$ for any $l \in \mathbb{R}$.

The method for the computation of $CoVaR_\alpha^{s|i}(l)$ proposed sofar (especially those of Brunnermeier and Adrian [2011] and Jäger-Ambrożewicz [2010]) assume the normal distribution. They cannot take into account the stylized behaviors of financial variables during the crisis (such as tail dependence, skewness and fat tails) and are not flexible enough to allows an effective analysis of systemic risk contribution.

We use copula's theory to derive a general formula for $CoVaR_\alpha^{s|i}(l)$ that integrates all information on the assumed distributions. This allows us to consider not only the normal (which describes the non-crisis period) but also the extreme part (which describes the crisis period) of the assumed distribution. Our formula expresses the $CoVaR$ of a given financial institutions as a function of its own loss L^i, its dependence with the financial system, and the loss of the financial system L^s. This makes our formula consistent with the concept of macro-prudential risk measure and allows us to appreciate separately the effect of the interconnectedness of individual financial institution to a financial system (which are modeled by the dependence structure between the considered financial institution and a financial system), of the idiosyncratic losses (which are modeled by the individual loss of the focused financial), and of the systematic risk (which is modeled by the loss of the financial system) on the systemic risk contributions. We use our formula to derive the expressions of $CoVaR_\alpha^{s|i}(l)$, $l \in \mathbb{R}$ for the Gaussian copula and for some Non-Gaussian copula (including the t-copula, Archimedes copula and the convex combinations of copula). We show that the computation method used by Brunnermeier and Adrian [2011] and the formula proposed by Jäger-Ambrożewicz [2010] can be seen as a special case of our formula.

Considering the economic and practical aspects of financial distress, we highlight several gaps in the formulation of the CoVaR-method. We illustrated this with a case in which $\Delta CoVaR_\alpha^{s|i}$ is not sensitive to tail dependence coefficient of the assumed joint distribution functions. We go on and affirm that, in general, $\Delta CoVaR_\alpha^{s|i}$ is not an appropriate measure for extreme comovement. Furthermore, we highlight the fact that $\Delta CoVaR_\alpha^{s|i}$ as defined by Brunnermeier and Adrian [2011] is not consistent with the notions of financial distress and contagion, because it builds on the assumption that a financial institution is in distress when its loss equals its Value-at-Risk. This is not consistent with the economic perception of financial distress.

A significant part of our work consist on the fundamental analysis we are make through this thesis. This was a necessary task as we aimed to produce results that are consistent with the real economic problems that have motivated our research. In particular, in chapter 1, we present from a quantitative risk management perspective, the foundations of the modern financial system in particular those that promote systemic risk. Also, in section 2.1, based on a fundamental analyze of the notion of contagion effect we precise the view of systemic risk that we followed in this thesis. We assume in Assumption 2, that a system risk is due to the propagation of a single financial institution distressed to the other financial institution in the financial system. We also provide in Definition 2 a consistent definition of the notion of financial distress.

Following the fundamental concepts of quantitative risk management we propose alternative methods for a consistent estimation of systemic risk contribution. The alternative models we propose allow us to integrate all information about the assumed loss distributions especially about the part of losses that can not be absorbed by the regulatory capital (**loss excess**) and will be therefor transferred to the financial system in case of the default of the corresponding financial institution. Thus, we can estimate the potential financial system loss induced by the default of a given financial institution.

Finally we consider $CoVaR_{\alpha}^{s|i}(l)$ under elliptical distributions. Our aim here is to develop a computation method, which is accessible easily for financial practitioners and financial regulators. In fact, the financial regulation needs risk concepts that can be easily implemented and audited. Assuming elliptical distributions is a reasonable trade-off between rigorous measurement of systemic risk and effective implementation. Elliptical distributions have the particularity that they share many of the analytical properties of the multivariate normal distribution, but are flexible enough to model extreme co-movements as those observed during financial crises. We present a closed form-formula for the computation of $CoVaR_{\alpha}^{s|i}(l)$ under elliptical distributions, which we use to derive expressions of $CoVaR_{\alpha}^{s|i}(l)$ for Gaussian distribution and t-distribution.

The alternative models we propose allow us to analyze the effect of the main factors of systemic risk (namely, the interconnectedness and the size) on the systemic risk contribution of individual financial institutions in a general stochastic framework that includes the Gaussian and the non-Gaussian world. They provide insights for academic education (cf. Smart [2013]) as well as new perspectives for research and application in the field of the quantitative modeling and analysis of financial risk contribution (cf. for example Fischer et al. [2015], Chen and Khashanah [2015] and Huang et al. [2016]).

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