Itô formula and Differentials for mild solutions of Stochastic Partial Differential Equations with Gaussian and compensated Poisson Lévy noise

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Zusammenfassung in deutscher Sprache

In dieser Arbeit behandeln wir die Itô-Formel für milde Lösungen von stochastischen partiellen Differentialgleichungen (SPDG) bzgl. das Gaußschen sowie des Nicht-Gaußschen Rauschens (kompensiertes Poisson-Zufallsmaß). Hierbei betrachten wir die Funktionen $\Psi \in C^{1,2}([0,T] \times H)$, wobei $\Psi : [0,T] \times H \to \mathbb{R}$ und H ein reeller separabler Hilbertraum ist. Wir schreiben zuerst die Itô-Formel für die starken Lösungen der SPDG auf und leiten daraus die Itô-Formel für milde Lösungen mit Hilfe der Yosida-Approximationen ab. Dann zeigen wir, wie man die Techniken für die Itô-Formel für milde Lösungen benutzen kann, um Resultate über exponentielle Stabilität und Exponentially Ultimate Boundedness im quadratischen Mittel für milde Lösungen zu beweisen. Auch bringen wir diese Itô-Formel in Zusammenhang mit einer Itô-Formel für milde Lösungen, die von Ichikawa für das Gaußsche Rauschen eingeführt wurde. Ferner verallgemeinern wir die Itô-Formel von Ichikawa für milde Lösungen der SPDG mit Lévy-Rauschen. Wir zeigen auch, dass die milde Itô-Formel von Da Prato, Jentzen und Röckner, die bisher nur für das Gaußsche Rauschen gezeigt worden ist, auch für Nicht-Gaußsches Rauschen gilt. Dann geben wir einige Beispiele, auf die wir unsere Theorie anwenden. Außerdem untersuchen wir die Stetigkeit und Differenzierbarkeit der milden Lösungen bezüglich des Anfangswertes für SPDG, sowohl mit Gaußschem Rauschen als auch mit Nicht-Gaußschem Rauschen.

Abstract

In this thesis we study the Itô formula for mild solutions of SPDEs with respect to the Gaussian and non-Gaussian noise (compensated Poisson random measure). We consider the functions $\Psi \in C^{1,2}([0,T] \times H)$, where $\Psi : [0,T] \times H \to \mathbb{R}$ and H is a real separable Hilbert space. First we write the Itô formula for the strong solutions of SPDEs, then by Yosida approximation we obtain our Itô formula for mild solutions. Then we show how we can apply the arguments of our Itô formula for mild solutions for proving the results of exponential stability and exponentially ultimate boundedness in the mean square sense for the mild solutions. We also relate this Itô formula to an Itô formula for mild solutions provided by Ichikawa for the Gaussian noise. We generalize Ichikawa's Itô formula for mild solutions to SPDEs with Lévy noise. Then we extend Da Prato, Jentzen and Röckner's mild Itô formula for the Gaussian case to the non-Gaussian case. Then we present a set of examples where we apply our theory. We also study the continuity and differentiability results of the mild solutions with respect to the initial condition for SPDEs which contain both Gaussian and non-Gaussian noise.

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Contents

Chapter 1

Introduction

The Itô formula for the strong solutions of SPDEs can be derived similarly as for the SDEs, see e.g. [16], [28] for the Gaussian case and [23], [22] for the non-Gaussian case. In [8]; Da Prato, Jentzen and Röckner showed that under suitable conditions one can obtain a mild Itô formula for the mild solutions of SPDEs driven by Brownian Motion. They transform the mild solution to a standard Itô process by using the techniques of [14], through the works of Nagy [33], [34], [35]. However this relation between the two processes does not allow us to study the asymptotics of the solution as done in [24] for the non-Gaussian case and in [16] for the Gaussian case. Here, we studied the asymptotic properties of the mild solutions for both the Gaussian and non-Gaussian noise as applications of the Itô formula for mild solutions. In [1], our purpose is two fold. In the first place we obtain through Yosida approximation an Itô formula for mild solutions to SPDEs driven by Wiener process and general Lévy processes, which is different from the mild Itô formula of [8]. We also relate this idea to the original Itô formula for mild solutions provided by Ichikawa in [19], for the Gaussian case and show its applications in this work.

In Section 2.1 we present required definitions, inequalities, existence and uniqueness of mild solutions and discuss under what assumptions the mild solution is also a strong solution. In Section 2.2 we present our results of [1], where we show how we can approximate the mild solution by a sequence of strong solutions by using Yosida approximation technique and obtain our Itô formula for mild solutions, for those functions $\Psi \in C^{1,2}([0,T] \times H)$. This is proved in Theorem 6, ([1]). Then in the next Section 2.3, we present some applications of the Itô formula for mild solutions of Theorem 6 to prove the results of exponential stability and exponentially ultimate boundedness in the mean square sense of the mild solutions. In [19], Ichikawa obtained an Itô formula for mild solutions and w.r.t. the Gaussian noise and discussed also how to relate the generator of a semigroup with the generator of a Markov process. This will be recalled in Section 2.4 w.r.t. both Gaussian and non-Gaussian noise. Moreover in [1], we also present an Itô formula for mild solutions following Ichikawa [19] w.r.t. both Gaussian and non-Gaussian noise, for those functions $\Psi \in C^{1,2}([0,T] \times H)$ for which the function $\mathcal{L}\Psi$ can be extended to a continuous function, where $\mathcal{L}$ denotes the infinitesimal generator of the homogeneous Markov process $\{X^x(t), t \geq 0\}$. In [1], we also relate the semigroup with the operator $\mathcal{L}$, this is presented in Section 2.4. As a consequence we get, in [1], the Kolmogorov's backward equation for mild solutions w.r.t. Gaussian and non-Gaussian noise, this is in Corollary 2 (equation (2.79)). However, in Ichikawa's Itô formula for mild soltions, the assumption that- the function $\mathcal{L}\Psi$ can be extended to a continuous function, is rather restrictive to study the stability theory of Lyapunov functions. So, following Ichikawa's result for the Gaussian case, in [1], we introduce Corollary 4, which is useful in applications since continuous extension of the function $\mathcal{L}\Psi$ is not required. In Corollary 4, we assume that the function $\mathcal{L}\Psi(x) \leq \mathcal{U}(x)$, where $\mathcal{U}(x)$ is a continuous function. As a result we obtain an inequality instead of an Itô formula but we can also apply the Itô formula of Theorem 6 for those functions, where $\mathcal{L}\Psi(x)$ is controlled by a continuous function $\mathcal{U}(x)$. This allows us to study the exponential stability in the mean square sense of the solution (this is explained through the last example). In [1], we are able to obtain Da Prato, Jentzen and Röckner's mild Itô formula, but w.r.t. the non-Gaussian noise, this is presented in Section 2.5. Here we used the transformation of [14]. In [1], we also present a set of examples where we apply our theory. These examples are presented here in Section 2.6. Through some of these examples, we also relate our Itô formula for mild solutions with that of Ichikawa's.

In Chapter 3, we study the continuity and differentiability results of the mild solutions with respect to the initial condition for SPDEs which contain both Gaussian and non-Gaussian noise. These results are also shown in [22], [2] for the non-Gaussian case and in [16], [9] for the Gaussian case seperately. We unify these results.

In Chapter 4 (Appendix), we write the explanation of- how we can write the Itô formula for strong solutions w.r.t. both Gaussian and non-Gaussian noise.

Chapter 2

The Itô formula for mild solutions

2.1 Preliminaries

The contents of this Chapter is mainly from our paper, [1]. First we introduce some basic definations and ideas, which are related to our thesis. We will start our discussions with Semigroup theory and Abstract Cauchy problem, and how they are inter-related (for details, see [16], [12]).

Definition 1. *A family $S(t) \in \mathcal{L}(X)$, $t \geq 0$, of bounded linear operators on a Banach space X is called a strongly continuous semigroup (or a C_0-semigroup) if*
(S1) $S(0) = I$,
(S2) (Semigroup property) $S(t+s) = S(t)S(s)$ for every $t, s \geq 0$,
(S3) (Strong continuity property) $\lim_{t\to 0^+} S(t)x = x$ for every $x \in X$.

Let $S(t)$ be a C_0-semigroup on a Banach space X. Then there exist constants $\alpha \geq 0$ and $M \geq 1$ such that $\|S(t)\|_{\mathcal{L}(X)} \leq Me^{\alpha t}$, $t \geq 0$.

If $M = 1$, then $S(t)$ is called a *pseudo-contraction semigroup*. If $\alpha = 0$, then $S(t)$ is called *uniformly bounded*, and if $\alpha = 0$ and $M = 1$ (i.e. $\|S(t)\|_{\mathcal{L}(X)} \leq 1$), then $S(t)$ is called a *semigroup of contractions*.

For any C_0-semigroup $S(t)$ and arbitrary $x \in X$, the mapping $t \to S(t)x \in X$, $t \in \mathbb{R}_+$, is continuous.

Definition 2. *Let $S(t)$ be a C_0-semigroup on a Banach space X. The linear operator A with domain*

$$\mathcal{D}(A) := \{x \in X : \lim_{t\to 0^+} \frac{S(t)x - x}{t} \text{ exists}\}$$

defined by

$$Ax := \lim_{t\to 0^+} \frac{S(t)x - x}{t}$$

is called the infinitesimal generator of the semigroup $S(t)$.

A semigroup $S(t)$ is called *uniformly continuous* if $\lim_{t\to 0^+}\|S(t)-I\|_{\mathcal{L}(X)} = 0$. A linear operator A is the infinitesimal generator of a uniformly continuous semigroup $S(t)$ on a Banach space X iff $A \in \mathcal{L}(X)$. We have $S(t) = e^{tA} = \sum_{n=0}^{\infty} \frac{(tA)^n}{n!}$, the series convergeing in norm for every $t \geq 0$.

We will however be mostly interested in the case where $A \notin \mathcal{L}(X)$.

Let A be an infinitesimal generator of a C_0-semigroup $S(t)$ on a Banach space X. Then the following properties hold-
i) For $x \in X$, $\lim_{h\to 0} \frac{1}{h}\int_t^{t+h} S(t)xds = S(t)x$.
ii) For $x \in \mathcal{D}(A)$, $S(t)x \in \mathcal{D}(A)$ and $\frac{d}{dt}S(t)x = AS(t)x = S(t)Ax$.
iii) For $x \in X$, $\int_0^t S(s)xds \in \mathcal{D}(A)$, and $A\int_0^t S(s)xds = S(t)x - x$. If $x \in \mathcal{D}(A)$ then $\int_0^t S(s)Axds = S(t)x - x$.
iv) For $x \in \mathcal{D}(A)$, $S(t)x - S(s)x = \int_s^t S(u)Axdu = \int_s^t AS(u)xdu$.
v) $\mathcal{D}(A)$ is dense in X, and A is a closed linear operator.

Definition 3. *The resolvent set $\rho(A)$ of a closed linear operator A on a Banach space X is the set of all complex numbers λ for which $\lambda I - A$ has a bounded inverse, i.e., the operator $(\lambda I - A)^{-1} \in \mathcal{L}(X)$. The family of bounded linear operators*

$$R(\lambda, A) = (\lambda I - A)^{-1}, \quad \lambda \in \rho(A),$$

is called the resolvent of A.

We note that $R(\lambda, A)$ is a one-to-one transformation of X onto $\mathcal{D}(A)$, i.e.,

$$\begin{aligned} (\lambda I - A)R(\lambda, A)x &= x, \qquad x \in X, \\ R(\lambda, A)(\lambda I - A)x &= x, \qquad x \in \mathcal{D}(A). \end{aligned}$$

In particular, if $x \in \mathcal{D}(A)$ then $AR(\lambda, A)x = R(\lambda, A)Ax$.

Let $S(t)$ be a C_0-semigroup with infinitesimal generator A on a Banach space X. If $\alpha_0 = \lim_{t\to\infty} t^{-1}\ln\|S(t)\|_{\mathcal{L}(X)}$, then any real number $\lambda > \alpha_0$ belongs to the resolvent set $\rho(A)$, and $R(\lambda, A)x = \int_0^\infty e^{-\lambda t}S(t)xdt$, $x \in X$. Furthermore, for each $x \in X$, $\lim_{\lambda\to\infty}\|\lambda R(\lambda, A)x - x\|_X = 0$.

Theorem 1. *(Hille-Yosida) Let $A : \mathcal{D}(A) \subset X \to X$ be a linear operator on a Banach space X. Necessary and sufficient conditions for A to generate a C_0-semigroup $S(t)$ are*
(1) A is closed and $\overline{\mathcal{D}(A)} = X$.
(2) There exist real numbers M and α such that for every $\lambda > \alpha$, $\lambda \in \rho(A)$ (the resolvent set) and $\|(R(\lambda, A))^r\|_{\mathcal{L}(X)} \leq M(\lambda - \alpha)^{-r}$, for $r = 1, 2, \cdots$. In this case, $\|S(t)\|_{\mathcal{L}(X)} \leq Me^{\alpha t}$, $t \geq 0$.

We will now introduce Yosida approximation of an operator A and of the C_0-semigroup it generates. For $\lambda \in \rho(A)$, consider the family of operators

$$R_\lambda = \lambda R(\lambda, A).$$

Since the range $\mathcal{R}(R(\lambda, A)) \subset \mathcal{D}(A)$, we can define the *Yosida approximation* of A by

$$\begin{aligned} A_\lambda x &= AR_\lambda x, \quad x \in X; \\ A_\lambda x &= R_\lambda Ax, \quad x \in \mathcal{D}(A). \end{aligned}$$

Let $S_\lambda(t)$ denote the (uniformly continuous) semigroup generated by A_λ,

$$S_\lambda(t)x = e^{tA_\lambda}x, \qquad x \in X.$$

Theorem 2. *(Yosida Approximation) Let A be an infinitesimal generator of a C_0-semigroup $S(t)$ on a Banach space X. Then*

$$\begin{aligned} &\lim_{\lambda\to\infty} R_\lambda x = x, \quad x \in X; \\ &\lim_{\lambda\to\infty} A_\lambda x = Ax, \quad x \in \mathcal{D}(A); \\ &\lim_{\lambda\to\infty} S_\lambda(t)x = S(t)x, \quad x \in X. \end{aligned}$$

The convergence in the last eq. i.e. $\lim_{\lambda\to\infty} S_\lambda(t)x = S(t)x$, for $x \in X$, is uniform on compact subsets of $\mathbb{R}_+$. The following estimate holds:

$$\|S_\lambda(t)\|_{\mathcal{L}(X)} \leq M \exp\{t\lambda\alpha/(\lambda - \alpha)\}$$

with the constants M, α determined by the Hille-Yosida theorem.

Abstract Cauchy Problem

Basically we can see the semigroups as a solution to PDEs. Let A be a linear operator on a real separable Hilbert space H. Let us consider the abstract Cauchy problem given by

$$\begin{cases} \frac{du(t)}{dt} = Au(t), & 0 < t < T \\ u(0) = x, & x \in H. \end{cases} \tag{2.1}$$

Definition 4. *A function $u : [0, T[:\to H$ is a (classical) solution of the problem (2.1) on $[0, T[$ if u is continuous on $[0, T[$, continuously differentiable and $u(t) \in \mathcal{D}(A)$ for $t \in [0, T[$, and (2.1) is satisfied on $[0, T[$.*

If A is an infintesimal generator of a C_0 semigroup $\{S(t), t \geq 0\}$, then for any $x \in \mathcal{D}(A)$, the function $u^x(t) = S(t)x$, $t \geq 0$, is a solution of (2.1).

If $x \notin \mathcal{D}(A)$, then $u^x(t) = S(t)x$ is not a solution in the usual sense, but it can be viewed as a "generalized solution", which will be called a "mild solution". Infact, the concept of mild solutions can be introduced to study the following nonhomogeneous initial-value problem:

$$\begin{cases} \frac{du(t)}{dt} = Au(t) + f(t), & 0 < t < T \\ u(0) = x, & x \in H, \end{cases} \tag{2.2}$$

where $f : [0, T[\to H$.

Definition 5. *Let A be an infintesimal generator of a C_0 semigroup $S(t)$ on H, $x \in H$ and $f \in L^1([0,T],H)$ be the space of Bochner-integrable functions on $[0,T]$ with values in H. The function $u \in C([0,T],H)$ given by*

$$u^x(t) = S(t)x + \int_0^t S(t-s)f(s)ds, \qquad 0 \leq t \leq T,$$

is the mild solution of the initial-value problem (2.2) *on* $[0,T]$.

Note that for $x \in H$ and $f \equiv 0$, the mild solution is $S(t)x$, which is not in general a classical solution.

Now we will discuss about Hilbert space valued Wiener process (for details, see [16], [9]).

We assume that a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$, satisfying the "*usual hypothesis*", is given by:
(i) $\mathcal{F}_t$ contains all null sets of $\mathcal{F}$, for all t s.t. $0 \leq t < \infty$.
(ii) $\mathcal{F}_t = \mathcal{F}_t^+$, where $\mathcal{F}_t^+ = \cap_{u>t}\mathcal{F}_u$, for all t s.t. $0 \leq t < \infty$, i.e. the filtrartion is right continuous.
(iii) the filtration $\mathcal{F}_0$ is independent of $(\mathcal{F}_t)_{t>0}$.

First we define K-valued Gaussian random variable, where K is a real separable Hilbert space. Let $\mathcal{L}_1(K)$ be the space of trace-class operators on K,

$$\mathcal{L}_1(K) = \{L \in \mathcal{L}(K) : \tau(L) := tr((LL^*)^{1/2}) < \infty\},$$

where the trace of the operator $[L] = (LL^*)^{1/2}$ is defined by

$$tr([L]) = \sum_{j=1}^{\infty} \langle [L]f_j, f_j\rangle_K$$

for an ONB $\{f_j\}_{j=1}^{\infty} \subset K$. $tr([L])$ is independent of the choice of ONB and $\mathcal{L}_1(K)$ is equipped with the trace norm τ is a Banach space. Let $Q : K \to K$ be a symmetric nonnegative definite trace-class operator.

Assume that $X : K \to L^2(\Omega, \mathcal{F}, P)$ satisfies the following conditions:
(1) The mapping X is linear.
(2) For an arbitrary $k \in K$, $X(k)$ is a Gaussian random variable with mean zero.
(3) For arbitrary $k, k' \in K$, $E(X(k)X(k')) = \langle Qk, k'\rangle_K$.

Let $\{f_j\}_{j=1}^{\infty}$ be an ONB in K diagonalizing Q, and let the eigenvalues corresponding to the eigenvectors f_j be denoted by λ_j, so that $Qf_j = \lambda_j f_j$. We define

$$X(\omega) = \sum_{j=1}^{\infty} X(f_j)(\omega) f_j. \tag{2.3}$$

Since $\sum_{j=1}^{\infty} \lambda_j < \infty$, the series converges in $L^2(\Omega, \mathcal{F}, P)$ and hence P-a.s.. In this case, P-a.s.

$$\langle X(\omega), k\rangle_K = X(k)(\omega),$$

so that $X : \Omega \to K$ is $\mathcal{F}/\mathcal{B}(K)$-measurable, where $\mathcal{B}(K)$ denotes the Borel σ-field on K. We can show that, (2.3), converges in $L^2(\Omega, \mathcal{F}, P)$ in the following way

$$\begin{aligned} E[\|X(\omega)\|_K^2] &= E[\|\sum_{j=1}^{\infty} X(f_j)(\omega) f_j\|_K^2] \\ &= \sum_{j=1}^{\infty} E(X(f_j)(\omega))^2 \quad \text{[by Parseval's identity]} \\ &= \sum_{j=1}^{\infty} \langle Qf_j, f_j\rangle_K \quad \text{[since, } E(X(k)X(k')) = \langle Qk, k'\rangle_K] \\ &= \sum_{j=1}^{\infty} \lambda_j < \infty. \text{ [since, } Q \text{ is trace-class operator]} \end{aligned}$$

Definition 6. *We call $X : \Omega \to K$ defined above a K-valued Gaussian random variable with covariance Q.*

Definition 7. *Let Q be a nonnegative definite symmetric trace-class operator on a separable Hilbert space K, $\{f_j\}_{j=1}^{\infty}$ be an ONB in K diagonalizing Q, and let the corresponding eigenvalues be $\{\lambda_j\}_{j=1}^{\infty}$. Let $\{w_j(t)\}_{t\geq 0}$, $j = 1, 2, \cdots$, be a sequence of independent Brownian motions defined on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_t, P)$. The process*

$$W_t = \sum_{j=1}^{\infty} \lambda_j^{1/2} w_j(t) f_j \tag{2.4}$$

is called a Q-Wiener process in K.

We can show that, how (2.4) is connected to (2.3), by an identification of the coefficients via the covariance operator Q. Let

$$W_t = \sum_{j=1}^{\infty} c_j w_j(t) f_j.$$

Now for the covariance matrix Q, when $i = j$

$$\begin{aligned} E[(c_j w_j(t))^2] - (E[c_j w_j(t)])^2 &= c_j^2 E[(w_j(t))^2] - c_j^2 (E[w_j(t)])^2 \\ &= c_j^2 t - 0 = c_j^2 t. \end{aligned}$$

Since $w_j(t)$ is a Brownian motion $E[w_j(t)] = 0$ and $E[w_j^2(t)] = t$.

When $i \neq j$, then

$$\begin{aligned} E[c_i w_i(t) c_j w_j(t)] - E[c_i w_i(t)] E[c_j w_j(t)] &= c_i c_j E[w_i(t)] E[w_j(t)] - c_i c_j E[w_i(t)] E[w_j(t)] \\ &= 0. \end{aligned}$$

Since $w_i(t)$ and $w_j(t)$ are independent Brownian motion.

Since Q is the covariance matrix, symmetric nonnegative definite, diagonalizable and λ_j are the eigenvalues, therefore

$$c_j^2 t = \lambda_j t \Longrightarrow c_j = \lambda_j^{1/2}.$$

Therefore,

$$W_t = \sum_{j=1}^{\infty} \lambda_j^{1/2} w_j(t) f_j.$$

Now we will give a brief introduction about Itô integral w.r.t. Jump process (for details, see [22], [23]). Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ be a filtered probability space satisfying the usual hypothesis. Let $(F, \|.\|)$ denote a separable Banach space. Suppose $(E, \mathcal{E})$ be a measurable space which we assume to be a Blackwell space, for example every Polish space with its Borel σ-field is a Blackwell space. Let N be a time-homogeneous Poisson random measure on $\mathbb{R}_+ \times E$. Then its compensator is $\nu(dt, dx) = dt\beta(dx)$, where β is a σ-finite measure on $(E, \mathcal{E})$. $q(dt, dx) := N(dt, dx) - \nu(dt, dx)$ is the associated compensated Poisson random measure (cPrm). We fix an arbitrary $T \in \mathbb{R}_+$. Let us consider the set of progressively measurable functions on the time interval $[0, T]$, i.e.

$$\begin{aligned} M^T(E/F) :=& \{f : \Omega \times [0,T] \times E \to F : f \text{ is } \mathcal{B}[0,T] \otimes \mathcal{E} \otimes \mathcal{F}_T - \text{measurable and} \\ & f(t,x) \text{ is } \mathcal{F}_t - \text{measurable for all } t \in [0,T] \text{ and } x \in E\}. \end{aligned}$$

We define

$$M_\nu^{T,2}(E/F) := \{f \in M^T(E/F) : \int_0^T \int_E E[\|f(t,x)\|^2]\nu(dt,dx) < \infty\}.$$

Definition 8. *A function $f \in M^T(E/F)$ belongs to the set $\sum_T(E/F)$ of simple functions, if there exist $n, m \in \mathbb{N}$ such that*

$$f(t,x) = \sum_{k=1}^{n-1} \sum_{l=1}^{m} 1\!\!1_{A_{k,l}}(x) 1\!\!1_{F_{k,l}} 1\!\!1_{(t_k, t_{k+1}]}(t) a_{k,l},$$

where $\beta(A_{k,l}) < \infty$, $t_k \in (0,T]$, $t_k < t_{k+1}$, $F_{k,l} \in \mathcal{F}_{t_k}$, $a_{k,l} \in F$, and for all $k \in 1, \cdots, n-1$ we have $A_{k,l_1} \times F_{k,l_1} \cap A_{k,l_2} \times F_{k,l_2} = \emptyset$ if $l_1 \neq l_2$.

The set $\sum_T(E/F)$ of simple functions is dense in the Banach space $M_\nu^{T,2}(E/F)$ with norm

$$\|f\|_2 := \sqrt{\int_0^T \int_E \mathbb{E}[\|f(t,u)\|^2]\nu(dt,dx)}.$$

The Itô integral of simple functions is defined as usual pathwise in a very natural way, for $f \in \sum_T(E/F)$

$$\int_0^T \int_A f(t,x) q(dt,dx) = \sum_{k=1}^{n-1} \sum_{l=1}^{m} a_{k,l} 1\!\!1_{F_{k,l}} q((t_k, t_{k+1}] \cap (0,T] \times A_{k,l} \cap A).$$

Let $\mathcal{M}_T^2(F)$ be the linear space of all F-valued square integrable martingales $M = (M_t)_{t\in[0,T]}$ with norm

$$\|M\|_{\mathcal{M}_T^2} = (\mathbb{E}[\|M_T\|^2])^{1/2}.$$

The Itô integral for functions $f \in M_\nu^{T,2}(E/F)$ is well defined, if the linear operator

$$\sum_T(E/F) \to \mathcal{M}_T^2(F), \quad f \mapsto \left(\int_0^t \int_E f(s,x)q(ds,dx)\right)_{t\in[0,T]} \tag{2.5}$$

can be uniquely extended to a continuous linear operator

$$M_\nu^{T,2}(E/F) \to \mathcal{M}_T^2(F), \quad f \mapsto \left(\int_0^t \int_E f(s,x)q(ds,dx)\right)_{t\in[0,T]}. \tag{2.6}$$

In particular, for all $f \in M_\nu^{T,2}(E/F)$ there is sequence $(f_n)_{n\in\mathbb{N}} \subset \sum_T(E/F)$ s.t. $\lim_{n\to\infty}\|f - f_n\|_2 = 0$ and

$$\lim_{n\to\infty} \mathbb{E}\left[\|\int_0^T \int_E (f(s,x) - f_n(s,x))q(ds,dx)\|^2\right] = 0.$$

Let, $\mathcal{K}_{T,\beta}^2(E/F)$ denotes the linear space of all progressively measurable $f \in M^T(E/F)$ such that

$$P\left(\int_0^T \int_E \|f(s,x)\|^2 ds\beta(dx) < \infty\right) = 1.$$

If (2.6) is well defined, then the definition of the Itô integral can be extended to all $f \in \mathcal{K}_{T,\beta}^2(E/F)$. For all $f \in \mathcal{K}_{T,\beta}^2(E/F)$ we define the sequence of stopping times

$$\tau_n := \inf\left\{t \in [0,T] : \int_0^t \int_E \|f(s,x)\|^2 ds\beta(dx) \geq n\right\}, \quad n \in \mathbb{N}.$$

Note that $f\mathbb{1}_{[0,\tau_n]} \in M_\nu^{T,2}(E/F)$ for all $n \in \mathbb{N}$. Hence, we can define the Itô integral

$$\int_0^t \int_E f(s,x)q(ds,dx) := \lim_{n\to\infty} \int_0^t \int_E f(s,x)\mathbb{1}_{[0,\tau_n]}q(ds,dx), \quad t \in [0,T]$$

which is a local martingale.

Now, we gradually proceed towards our main results of this Chapter. Let K and H be real separable Hilbert spaces. Let $(H\setminus\{0\}, \mathcal{B}(H\setminus\{0\}), \beta)$ be a σ-finite measurable space, with $\mathcal{B}(H\setminus\{0\})$ denoting the Borel sets of $H\setminus\{0\}$ and β a positive measure on $\mathcal{B}(H\setminus\{0\})$ with

$$\int_{H\setminus\{0\}} (\|u\|_H^2 \wedge 1)\beta(du) < \infty.$$

We refer to this as a Lévy measure on $H\setminus\{0\}$.

We shall denote the compensated Poisson random measure (cPrm) by $q(ds,du) := N(ds,du)(\omega) - ds\beta(du)$ on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\leq T}, P)$ satisfying the usual hypothesis. ds denotes the Lebesgue measure on $\mathcal{B}(\mathbb{R}_+)$ and $N(ds,du)(\omega)$ is a Poisson distributed σ-finite measure on the σ-algebra $\mathcal{B}(\mathbb{R}_+, H\setminus\{0\})$, generated by the product semiring $\mathcal{B}(\mathbb{R}_+)\times\mathcal{B}(H\setminus\{0\})$ of the Borel σ-algebra $\mathcal{B}(\mathbb{R}_+)$ and the trace borel σ-algebra $\mathcal{B}(H\setminus\{0\})$. Then $E(q(A\times B))^2 = \beta(A)\lambda(B)$, for any $A\in\mathcal{B}(H\setminus\{0\})$, $B\in\mathcal{B}(\mathbb{R}_+)$, $0\notin\bar{A}$, $\lambda(B)$ is the Lebesgue measure of B. (For more details we refer section 1 of [2]).

The definition of stochastic integral with respect to compensated Poisson random measure and their properties are given in, e.g. [3], [22], [2], [4], [29], [30], [20], [32].

Consider the following stochastic partial differential equation with values in H,

$$dX(t) = (AX(t) + F(X(t)))dt + B(X(t))dW_t + \int_{H\setminus\{0\}} f(v, X(t))q(dv,dt);$$

$$X(0) = \xi. \tag{2.7}$$

Where ξ is an $\mathcal{F}_0$-measurable random variable. We assume that, the terms in (2.7) satisfy the following conditions:

(A1) A is the infinitesimal generator of a pseudo-contraction semigroup $\{S(t), t\geq 0\}$ on H. This means in particular that there exists a constant $\alpha\in\mathbb{R}_+$ s.t. $\|S(t)\|\leq e^{\alpha t}$.

(A2) $(W_t)_{t\geq 0}$ is a K-valued $\mathcal{F}_t$-Wiener process with covariance Q on a complete filtered probability space $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\leq T}, P\right)$ satisfying the usual hypothesis, where Q is a nonnegative definite symmetric trace-class operator on the real separable Hilbert space K. $q(ds,du) := N(ds,du)(\omega) - ds\beta(du)$ is a compensated Poisson random measure (cPrm) on a complete filtered probability space $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\leq T}, P\right)$ satisfying the usual hypothesis. $(W_t)_{t\geq 0}$ is assumed to be independent of cPrm $q(dv,dt)$.

(A3) $F: H\to H$, $B: H\to\mathcal{L}(K,H)$, $f: H\setminus\{0\}\times H\to H$ are continuous, and Bochner measurable functions satisfying:

$$\|F(x)\|_H^2 + tr(B(x)QB^*(x)) + \int_{H\setminus\{0\}} \|f(v,x)\|_H^2\,\beta(dv) \leq l(1+\|x\|_H^2);$$

and

$$\left\|F(x)-F(y)\right\|_H^2 + tr((B(x)-B(y))Q(B(x)-B(y))^*)$$

$$+\int_{H\setminus\{0\}}\|f(v,x)-f(v,y)\|_H^2\beta(dv)\le\mathcal{K}\|x-y\|_H^2;$$

for all $x, y \in H$. Where l, $\mathcal{K}$ are positive constants.

Let $\mathcal{L}(K,H)$ be the space of all linear bounded operators from K to H. Let $\{f_j\}_{j=1}^{\infty}$ be an ONB in K diagonailizing Q and let the corresponding eigenvalues be $\{\lambda_j\}_{j=1}^{\infty}$. (For more details we refer to Section 2.1.2 of Chapter 2 of [16]). The space $K_Q = Q^{1/2}K$ equipped with the scalar product

$$\langle u,v\rangle_{K_Q}=\sum_{j=1}^{\infty}\frac{1}{\lambda_j}\langle u,f_j\rangle_K\langle v,f_j\rangle_K$$

is a separable Hilbert space with an ONB $\left\{\lambda_j^{1/2}f_j\right\}_{j=1}^{\infty}$.

$\mathcal{L}_2(K_Q,H)$ is the space of Hilbert-Schmidt operators from K_Q to H. If $\{e_j\}_{j=1}^{\infty}$ is an ONB in H, then the Hilbert -Schimdt norm of an operator $L \in \mathcal{L}_2(K_Q,H)$ is given by,

$$\begin{aligned}\|L\|^2_{\mathcal{L}_2(K_Q,H)}&=\sum_{j,i=1}^{\infty}\left\langle L(\lambda_j^{1/2}f_j),e_i\right\rangle_H^2\\&=\sum_{j,i=1}^{\infty}\left\langle LQ^{1/2}f_j,e_i\right\rangle_H^2\\&=\left\|LQ^{1/2}\right\|^2_{\mathcal{L}_2(K,H)}\\&=tr((LQ^{1/2})(LQ^{1/2})^*).\end{aligned}$$

The scalar product between two operators $L, M \in \mathcal{L}_2(K_Q,H)$ is defined by,

$$\langle L,M\rangle_{\mathcal{L}_2(K_Q,H)}=tr((LQ^{1/2})(MQ^{1/2})^*).$$

Since the Hilbert spaces K_Q and H are separable, then the space $\mathcal{L}_2(K_Q,H)$ is also separable.

Let $L \in \mathcal{L}(K,H)$. If $k \in K_Q$, then

$$k=\sum_{j=1}^{\infty}\left\langle k,\lambda_j^{1/2}f_j\right\rangle_{K_Q}\lambda_j^{1/2}f_j,$$

and L, considered as an operator norm from K_Q to H, defined as

$$Lk=\sum_{j=1}^{\infty}\left\langle k,\lambda_j^{1/2}f_j\right\rangle_{K_Q}\lambda_j^{1/2}Lf_j,$$

has a finite Hilbert-Schmidt norm. (For more details we refer section 2.2 of chapter 2 of [16], Chapter 4 of [9], or [27]).

Let $\Lambda_2(K_Q, H)$ be the class of $\mathcal{L}_2(K_Q, H)$ valued measurable processes as mapping from $([0,T]\times\Omega, \mathcal{B}([0,T])\otimes\mathcal{F})$ to $(\mathcal{L}_2(K_Q, H), \mathcal{B}(\mathcal{L}_2(K_Q, H)))$, adapted to the filtration $\{\mathcal{F}_t\}_{\{t\leq T\}}$, and satisfying the condition

$$E\Big[\int_0^T \|\phi(t)\|^2_{\mathcal{L}_2(K_Q,H)}\, dt\Big] < \infty.$$

$\Lambda_2(K_Q, H)$ when equipped with the norm

$$\|\phi\|_{\Lambda_2(K_Q,H)} = \left(E\Big[\int_0^T \|\phi(t)\|^2_{\mathcal{L}_2(K_Q,H)}\, dt\Big]\right)^{1/2},$$

is a Hilbert space.

Definition 9. *A stochastic process $\{X(t), t \geq 0\}$ is called a mild solution of (2.7) in $[0,T]$, if for all $t \leq T$*
(i) $X(t)$ is $\mathcal{F}_t$-adapted on the filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\leq T}, P)$,
(ii) $\{X(t), t \geq 0\}$ is jointly measurable and $\int_0^T E[\|X(t)\|^2_H]dt < \infty$,
(iii)

$$X(t) = S(t)\xi + \int_0^t S(t-s)F(X(s))ds + \int_0^t S(t-s)B(X(s))dW_s + \int_0^t\int_{H\setminus\{0\}} S(t-s)f(v, X(s))q(dv, ds)$$

holds in $[0,T]$ a.s..

Definition 10. *A stochastic process $\{X(t), t \geq 0\}$ is called a strong solution of (2.7) in $[0,T]$, if for all $t \leq T$*
(i) $X(t)$ is $\mathcal{F}_t$-adapted on the filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\leq T}, P)$,
(ii) $X(t)$ is càdlàg with probability one,
(iii) $X(t) \in \mathcal{D}(A)$, $dt \otimes dP$ a.e., $\int_0^T \|AX(t)\|_H\, dt < \infty$ P a.s.,
(iv)

$$X(t) = \xi + \int_0^t (AX(s) + F(X(s)))ds + \int_0^t B(X(s))dW_s + \int_0^t\int_{H\setminus\{0\}} f(v, X(s))q(dv, ds)$$

holds in $[0,T]$ a.s..

Lemma 1. *Let $L^2_{T,\beta}(H)$ be the space s.t. $L^2_{T,\beta}(H) := \{\varphi : (H\backslash\{0\}) \times [0,T] \times \Omega \to H$, such that φ is jointly measurable and $\mathcal{F}_t$-adapted for all $v \in H\backslash\{0\}$, $t \in [0,T]$ with $E[\int_0^T \int_{H\backslash\{0\}} \|\varphi(v,t)\|_H^2 \beta(dv)dt] < \infty\}$.*
a) Let $B(s) \in \Lambda_2(K_Q, H)$ with $E[\int_0^T \|B(s)\|^2_{\mathcal{L}_2(K_Q,H)} ds] < \infty$. Then for any stopping time τ, there exists a constant C_1, depending on α, T s.t.

$$E\left[\sup_{0\le t\le T\wedge\tau} \|\int_0^t S(t-s)B(s)dW_s\|_H^2\right] \le C_1 E\left[\int_0^{T\wedge\tau} \|B(s)\|^2_{\mathcal{L}_2(K_Q,H)} ds\right].$$

b) Let $\varphi \in L^2_{T,\beta}(H)$ and τ be a stopping time. Then, there exists a constant C_2, depending on α, T s.t.

$$\begin{aligned} E&\left[\sup_{0\le t\le T\wedge\tau} \|\int_0^t \int_{H\backslash\{0\}} S(t-s)\varphi(v,s)q(dv,ds)\|_H^2\right] \\ &\le C_2 E\left[\int_0^{T\wedge\tau}\int_{H\backslash\{0\}} \|\varphi(v,s)\|_H^2 \beta(dv)ds\right]. \end{aligned}$$

Proof. For the proof of the first inequality we refer Lemma 3.3(b) of [16]. And for the second inequality we refer Lemma 5.1.9(1) of [22], [5].

□

Let

$$\begin{aligned} I(t,\xi(t)) = \int_0^t S(t-s)F(\xi(s))ds + \int_0^t S(t-s)B(\xi(s))dW_s \\ + \int_0^t \int_{H\backslash\{0\}} S(t-s)f(v,\xi(s))q(dv,ds). \end{aligned} \tag{2.8}$$

Lemma 2. *Let $\{S(t), t \ge 0\}$ be a pseudo-contraction semigroup. Assume that $E[\sup_{0\le s\le T} \|\xi(s)\|_H^2] < \infty$ and the coefficients F, B, f satisfy (A1), (A2), (A3). Then for any stopping time τ*

$$E\left[\sup_{0\le s\le t\wedge\tau} \|I(s,\xi(s))\|_H^2\right] \le C_3\left(t + \int_0^t E[\sup_{0\le u\le s\wedge\tau} \|\xi(u)\|_H^2]ds\right),$$

where C_3 is a constant depending on α, T and l. $I(s,\xi(s))$ is defined in (2.8).

Proof. Here we followed the proof of, Lemma 3.4 of [16] and Theorem 5.2.1 of [22].

$$\sup_{0\le s\le t\wedge\tau}\|I(s,\xi(s))\|_H^2 \le 2\sup_{0\le s\le t\wedge\tau}\left\|\int_0^s S(s-u)F(\xi(u))ds\right\|_H^2 + 2\sup_{0\le s\le t\wedge\tau}\left\|\int_0^s S(s-u)B(\xi(u))dW_u\right\|_H^2 + 2\sup_{0\le s\le t\wedge\tau}\left\|\int_0^s\int_{H\setminus\{0\}} S(s-u)f(v,\xi(u))q(dv,du)\right\|_H^2.$$

Now from (A3) we can write

$$E\left[\sup_{0\le s\le t\wedge\tau}\left\|\int_0^s S(s-u)F(\xi(u))ds\right\|_H^2\right] \le E\sup_{0\le s\le t\wedge\tau} lC_{\alpha,t}\int_0^s(1+\sup_{0\le r\le u}\|\xi(r)\|^2)du \le C_{\alpha,T,l}\left(t+\int_0^t E\sup_{0\le u\le s\wedge\tau}\|\xi(u)\|^2\,ds\right).$$

From Lemma 1 and (A3) we can write

$$E\left[\sup_{0\le s\le t\wedge\tau}\left\|\int_0^s S(s-u)B(\xi(u))dW_u\right\|_H^2\right] \le C_{\alpha,t}E\int_0^{t\wedge\tau}\|B(\xi(s))\|^2\,ds \le E\sup_{0\le s\le t\wedge\tau} lC_{\alpha,t}\int_0^s(1+\sup_{0\le r\le u}\|\xi(r)\|^2)du \le C_{\alpha,T,l}\left(t+\int_0^t E\sup_{0\le u\le s\wedge\tau}\|\xi(u)\|^2\,ds\right).$$

Again from Lemma 1 and (A3) we write

$$E\left[\sup_{0\le s\le t\wedge\tau}\left\|\int_0^s\int_{H\setminus\{0\}} S(s-u)f(v,\xi(u))q(dv,du)\right\|_H^2\right] \le C_{\alpha,t}E\int_0^{t\wedge\tau}\int_{H\setminus\{0\}}\|f(v,\xi(s))\|^2\beta(dv)ds \le E\sup_{0\le s\le t\wedge\tau} lC_{\alpha,t}\int_0^s(1+\sup_{0\le r\le u}\|\xi(r)\|^2)du \le C_{\alpha,T,l}\left(t+\int_0^t E\sup_{0\le u\le s\wedge\tau}\|\xi(u)\|^2\,ds\right).$$

Now combining the last 3 inequalities we get our desired inequality. □

Lemma 3. *Let $\{S(t),t\ge 0\}$ be a pseudo-contraction semigroup. Assume that $E[\sup_{0\le s\le T}\|\xi(s)\|_H^2]<\infty$ and the coefficients F, B, f satisfy (A1), (A2), (A3). Then*

$$E\left[\sup_{0\le s\le t}\|I(s,\xi_1(s))-I(s,\xi_2(s))\|_H^2\right] \le C_4\int_0^t E[\sup_{0\le u\le s}\|\xi_1(u)-\xi_2(u)\|_H^2]ds,$$

where C_4 is a constant depending on α, T and $\mathcal{K}$. $I(s,\xi(s))$ is defined in (2.8).

Proof. Here we followed the proof of, Lemma 3.5 of [16] and Lemma 5.2.2 of [22].

$$\begin{aligned}
&E\left(\sup_{0\le s\le t}\|I(s,\xi_1(s))-I(s,\xi_2(s))\|_H^2\right)\\
&\le 2E\sup_{0\le s\le t}\left\|\int_0^s S(s-u)(F(u,\xi_1(u))-F(u,\xi_2(u)))du\right\|^2\\
&+2E\sup_{0\le s\le t}\left\|\int_0^s S(s-u)(B(u,\xi_1(u))-B(u,\xi_2(u)))dW_u\right\|^2\\
&+2E\sup_{0\le s\le t}\left\|\int_0^s\int_{H\setminus\{0\}} S(s-u)(f(v,\xi_1(u))-f(v,\xi_2(u)))q(dv,du)\right\|^2.
\end{aligned}$$

Now by (A3)

$$\begin{aligned}
&E\sup_{0\le s\le t}\left\|\int_0^s S(s-u)(F(u,\xi_1(u))-F(u,\xi_2(u)))du\right\|^2\\
&\le \mathcal{K}C_{\alpha,T}E\sup_{0\le s\le t}\int_0^s\sup_{0\le r\le u}\|\xi_1(r)-\xi_2(r)\|^2\,du\\
&=C_{\alpha,T,\mathcal{K}}\int_0^t E\sup_{0\le u\le s}\|\xi_1(u)-\xi_2(u)\|^2\,ds.
\end{aligned}$$

From Lemma 1 and (A3)

$$\begin{aligned}
&E\sup_{0\le s\le t}\left\|\int_0^s S(s-u)(B(u,\xi_1(u))-B(u,\xi_2(u)))dW_u\right\|^2\\
&\le C_{\alpha,T}E\int_0^t\|B(u,\xi_1(u))-B(u,\xi_2(u))\|^2\,du\\
&\le C_{\alpha,T,\mathcal{K}}\int_0^t E\sup_{0\le u\le s}\|\xi_1(u)-\xi_2(u)\|^2\,ds.
\end{aligned}$$

Again from Lemma 1 and (A3)

$$\begin{aligned}
&E\sup_{0\le s\le t}\left\|\int_0^s\int_{H\setminus\{0\}} S(s-u)(f(v,\xi_1(u))-f(v,\xi_2(u)))q(dv,du)\right\|^2\\
&\le C_{\alpha,T}E\int_0^t\int_{H\setminus\{0\}}\|f(v,\xi_1(u))-f(v,\xi_2(u))\|^2\,\beta(dv)du\\
&\le C_{\alpha,T,\mathcal{K}}\int_0^t E\sup_{0\le u\le s}\|\xi_1(u)-\xi_2(u)\|^2\,ds.
\end{aligned}$$

Now combining the last 3 inequalities we get our desired inequality. □

2.1.1 Existence and uniqueness of the mild solutions

Let $(D[0,T],H)$ be the space of càdlàg functions defined on $[0,T]$ and with values in H, with the sup norm $\|.\|_\infty := \sup_{t\in[0,T]}\|.\|_H$. Let $\mathcal{H}_2^T$ denote the space of $(D[0,T],H)$-valued random processes $\xi(t)$, which are jointly measurable, adapted to the filtration $\{\mathcal{F}_t\}_{t\in[0,T]}$, with $E[\sup_{0\le s\le T}\|\xi(s)\|_H^2]<\infty$. The space $\mathcal{H}_2^T$, equipped with the norm $\|X\|_{\mathcal{H}_2^T} := \left(E[\sup_{0\le s\le T}\|X(s)\|_H^2]\right)^{1/2}$ is a Banach space (see Section 4.1 of [22]).

Theorem 3. *Let the coefficients F, B, f satisfy (A1), (A2), (A3); let $\{S(t), t\ge 0\}$ be a pseudo-contraction semigroup generated by A and assume that $E[\|X(0)\|_H^2]<\infty$. Then equation (2.7) has a unique mild solution $X(t)\in(D[0,T],H)$ satisfying $E[\sup_{0\le s\le T}\|X(s)\|_H^2]<\infty$, i.e. the mild solution is in $\mathcal{H}_2^T$.*

Proof. Here we followed the proof of, Theorem 3.3 of [16] and Theorem 3.3 of [2].

We have $E\|X(0)\|_H^2<\infty$. Let, $I(t,X)$ be defined similarly as (2.8), and consider $I(X)(t)=I(t,X)$. Then by Lemma 2, $I:\mathcal{H}_2^T\to\mathcal{H}_2^T$. The solution can be approximated by the following sequence:

$$\begin{aligned}X_0(t) &= S(t)X(0),\\ X_{n+1}(t) &= S(t)X(0)+I(t,X_n), \qquad n=0,1,\cdots\end{aligned}$$

Indeed, let $v_n(t)=E\sup_{0\le s\le t}\|X_{n+1}(s)-X_n(s)\|_H^2$. Then from previous Lemma 2, we have a constant $V_{\alpha,l,T}$ s.t.

$$v_0(t)=E\sup_{0\le s\le t}\|X_1(s)-X_0(s)\|_H^2\le V_{\alpha,l,T}.$$

Similarly by Lemma 3, there exists a constant $C_{\alpha,\mathcal{K},T}$ s.t.

$$\begin{aligned}v_1(t)=E\sup_{0\le s\le t}\|X_2(s)-X_1(s)\|_H^2 &= E\sup_{0\le s\le t}\|I(s,X_1)-I(s,X_0)\|_H^2\\ &\le C_{\alpha,\mathcal{K},T}\int_0^t E\sup_{0\le u\le s}\|X_1(u)-X_0(u)\|_H^2\,ds\\ &\le C_{\alpha,\mathcal{K},T}V_{\alpha,l,T}t.\end{aligned}$$

And in general,

$$v_n(t)\le C_{\alpha,\mathcal{K},T}\int_0^t v_{n-1}(s)ds\le\frac{V_{\alpha,l,T}(C_{\alpha,\mathcal{K},T}t)^n}{n!}.$$

Let

$$\epsilon_n=\left(\frac{V_{\alpha,l,T}(C_{\alpha,\mathcal{K},T}T)^n}{n!}\right)^{1/3}.$$

Then by applying Chebychev's inequality we get

$$P(\sup_{0\le t\le T}\|X_{n+1}(t)-X_n(t)\|_H>\epsilon_n)\le\frac{\frac{V_{\alpha,l,T}(C_{\alpha,\mathcal{K},T}T)^n}{n!}}{\left(\frac{V_{\alpha,l,T}(C_{\alpha,\mathcal{K},T}T)^n}{n!}\right)^{2/3}}=\epsilon_n.$$

Because $\sum_{n=1}^{\infty}\epsilon_n<\infty$, by the Borel-Cantelli lemma, $\sup_{0\le t\le T}\|X_{n+1}(t)-X_n(t)\|_H<\epsilon_n$ P-a.s.. Thus, the series

$$\sum_{n=1}^{\infty}\sup_{0\le t\le T}\|X_{n+1}(t)-X_n(t)\|_H$$

converges P-a.s., showing that X_n converges to some X a.s. in $(D[0,T],H)$.

Moreover

$$\begin{aligned}E\sup_{0\le t\le T}\|X(t)-X_n(t)\|_H^2&=E\lim_{m\to\infty}\sup_{0\le t\le T}\|X_{n+m}(t)-X_n(t)\|_H^2\\&=E\lim_{m\to\infty}\sup_{0\le t\le T}\left\|\sum_{k=n}^{n+m-1}(X_{k+1}(t)-X_k(t))\right\|_H^2\\&\le E\lim_{m\to\infty}\left(\sum_{k=n}^{n+m-1}\sup_{0\le t\le T}\|X_{k+1}(t)-X_k(t)\|_H\right)^2\\&=\lim_{m\to\infty}E\left(\sum_{k=n}^{n+m-1}\sup_{0\le t\le T}\|X_{k+1}(t)-X_k(t)\|_H k\frac{1}{k}\right)^2\\&\le\sum_{k=n}^{\infty}E\sup_{0\le t\le T}\|X_{k+1}(t)-X_k(t)\|_H^2 k^2\left(\sum_{k=n}^{\infty}\frac{1}{k^2}\right).\end{aligned}$$

This converges to 0 as $n\to\infty$, because the second series converges and the first series is bounded by: $\sum_{k=n}^{\infty}v_k(t)k^2\le\sum_{k=n}^{\infty}\frac{V_{\alpha,l,T}(C_{\alpha,\mathcal{K},T}t)^k}{k!}k^2\to 0$ as $n\to\infty$. Therefore $X_n(t)$ converges to $X(t)$ in $\mathcal{H}_2^T$, as $n\to\infty$ and $Esup_{0\le s\le T}\|X(s)\|_H^2<\infty$.

Now, we will prove the uniqueness. Suppose $X(t)$ and $Y(t)$ are two solutions. Let

$$\vartheta_t=E\sup_{0\le s\le t}\|X(s)-Y(s)\|_H^2.$$

Then by the similar calculation as above, we get

$$\vartheta_t\le C_{\alpha,\mathcal{K},T}\int_0^t\vartheta_s ds$$

and by induction

$$\vartheta_t \leq \frac{(C_{\alpha,\mathcal{K},T}t)^n}{n!} E \sup_{0\leq s\leq T} \|X(s) - Y(s)\|_H^2 \to 0$$

when $n \to \infty$, i.e. $\vartheta_t = 0$; for all $t \in [0, T]$.

□

2.1.2 When a mild solution is a strong solution

Theorem 4. *Suppose,*
(a) $S(t)$ is a pseudo-contraction semigroup, $\xi \in \mathcal{D}(A)$, $S(t-r)F(y) \in \mathcal{D}(A)$, $S(t-r)B(y) \in \mathcal{D}(A)$, $S(t-r)f(v,y) \in \mathcal{D}(A)$;
$X(t) \in \mathcal{D}(A)$ $dt \otimes dP$ a.e., for $r < t$, $y \in H$ and $v \in H\setminus\{0\}$.

(b)

$$E\int_0^T \|B(X(t))\|^2_{\mathcal{L}_2(K_Q,H)}\, dt < \infty,$$

and

$$\int_0^T \int_0^T \int_{H\setminus\{0\}} E\|f(v, X(s))\|_H^2 \beta(dv) ds dt < \infty,$$

(c)

$$\|AS(t-r)F(y)\|_H \leq g_1(t-r)(1 + \|y\|_H);$$

with $g_1 \in L_1(0,T)$ and

$$\|AS(t-r)B(y)\|_H \leq g_2(t-r)(1 + \|y\|_H);$$

with $g_2 \in L_2(0,T)$,

(d)

$$\int_{H\setminus\{0\}} \|AS(t-r)f(v,y)\|^2 \beta(dv) \leq g_3(t-r)(1 + \|y\|_H^2);$$

with $g_3 \in L_1(0,T)$.
Then any mild solution of (2.7) (if it exists) is a strong solution.

Proof. Here we follow the methods provided in [18] and [24]. In [18] it is done for the Gaussian case and in [24] it is done for non-Gaussian case.

From the definition of mild solution we have,

$$X(s) = S(s)\xi + \int_0^s S(s-r)F(X(r))dr + \int_0^s S(s-r)B(X(r))dW_r$$

$$+ \int_0^s \int_{H\setminus\{0\}} S(s-r)f(v, X(r))q(dv, dr).$$

From the assumptions we can write,

$$\int_0^t AX(s)ds = \int_0^t AS(s)\xi ds + \int_0^t \int_0^s AS(s-r)F(X(r))drds + \int_0^t \int_0^s AS(s-r)B(X(r))dW_r ds$$

$$+ \int_0^t \int_0^s \int_{H\setminus\{0\}} AS(s-r)f(v,X(r))q(dv,dr)ds.$$

Since, we know

$$\begin{aligned}\int_0^t \int_0^s f(s-r)g(r)drds &= \int_0^t \int_0^t f(s-r)g(r)\chi_{[0,s]}(r)drds \\ &= \int_0^t \int_0^t f(s-r)g(r)\chi_{[0,s]}(r)dsdr \\ &= \int_0^t \int_0^t f(s-r)g(r)\chi_{[r,t]}(s)dsdr \\ &= \int_0^t \int_r^t f(s-r)g(r)dsdr.\end{aligned}$$

And, by the given conditions we have

$$\int_0^T \int_0^t \|AS(t-r)F(X(r))\| \, drdt < \infty$$

with probability one,

$$\int_0^T \int_0^t \|AS(t-r)B(X(r))\|^2 \, drdt < \infty$$

with probability one and

$$\int_0^T \int_0^t \int_{H\setminus\{0\}} E\,\|AS(t-r)f(v,X(r))\|^2 \, \beta(dv)drdt < \infty$$

with probability one. Hence, by applying Fubini theorem we get,

$$\int_0^t AX(s)ds = \int_0^t AS(s)\xi ds + \int_0^t \int_r^t AS(s-r)F(X(r))dsdr + \int_0^t \int_r^t AS(s-r)B(X(r))dsdW_r$$

$$+ \int_0^t \int_{H\setminus\{0\}} \int_r^t AS(s-r)f(v,X(r))dsq(dv,dr).$$

(For stochastic Fubini theorem we refer theorem (2.8) of [16] and theorem (3.1) of [24]). Now we apply the formula,

$$\int_0^t AS(s)\xi ds = S(t)\xi - \xi;$$

when $\xi \in \mathcal{D}(A)$. Hence $AX(t)$ is integrable with probability one and

$$\int_0^t AX(s)ds = S(t)\xi - \xi + \int_0^t S(t-r)F(X(r))dr - \int_0^t F(X(r))dr$$

$$+ \int_0^t S(t-r)B(X(r))dW_r - \int_0^t B(X(r))dW_r$$

$$+ \int_0^t \int_{H\setminus\{0\}} S(t-r)f(v, X(r))q(dv, dr) - \int_0^t \int_{H\setminus\{0\}} f(v, X(r))q(dv, dr).$$

Hence

$$\int_0^t AX(r)dr = X(t) - \xi - \int_0^t F(X(r))dr - \int_0^t B(X(r))dW_r$$

$$- \int_0^t \int_{H\setminus\{0\}} f(v, X(r))q(dv, dr).$$

Therefore

$$X(t) = \xi + \int_0^t AX(r)dr + \int_0^t F(X(r))dr + \int_0^t B(X(r))dW_r$$

$$+ \int_0^t \int_{H\setminus\{0\}} f(v, X(r))q(dv, dr).$$

By Definition 10, $\{X(t), t \geq 0\}$ is a strong solution of equation (2.7). □

2.2 Main Theorems

Now we consider the approximating system of equation (2.7),

$$dX(t) = (AX(t) + R_nF(X(t)))dt + R_nB(X(t))dW_t + \int_{H\setminus\{0\}} R_nf(v, X(t))q(dv, dt);$$

$$X(0) = \xi \in \mathcal{D}(A). \tag{2.9}$$

Here A generates a pseudo-contraction semigroup. Let $R(n, A) = (nI - A)^{-1}$ denote the resolvent of A evaluted at n where $R_n = nR(n, A)$, with $n \in \rho(A)$ the resolvent set of A. We have $R_n : H \to \mathcal{D}(A)$ and $A_n = AR_n$ are the Yosida approximations of A (see Chapter 1 of [16]). We assume that F, B, f satisfy conditions (A1), (A2), (A3).

By applying Theorem 3, we can conclude that equation (2.9) has a unique mild solution, denoted by $X_n^\xi(t)$. Then

$$X_n^\xi(t) = S(t)\xi + \int_0^t S(t-s)R_nF(X_n^\xi(s))ds + \int_0^t S(t-s)R_nB(X_n^\xi(s))dW_s$$

$$+\int_0^t \int_{H\setminus\{0\}} S(t-s)R_n f(v, X_n^\xi(s))q(dv, ds). \tag{2.10}$$

Since the range $\mathcal{R}(R(n,A)) \subset \mathcal{D}(A)$ (see Chapter 1 of [16]) and the conditions of Theorem 4 are satisfied, therefore we can conclude that $X_n^\xi(t) \in \mathcal{D}(A)$ is also a strong solution.

Now we are in a position to approximate the mild solution of equation (2.7) with respect to the strong solutions of equation (2.9). The mild solution of equation (2.7), say $X^\xi(t)$, satisfies

$$X^\xi(t) = S(t)\xi + \int_0^t S(t-s)F(X^\xi(s))ds + \int_0^t S(t-s)B(X^\xi(s))dW_s$$

$$+\int_0^t \int_{H\setminus\{0\}} S(t-s)f(v, X^\xi(s))q(dv, ds). \tag{2.11}$$

2.2.1 Approximating a mild solution by the strong solutions

Theorem 5. *Let $S(t)$ be the pseudo-contraction semigroup and the coefficients F, B, f satisfy (A1), (A2), (A3). The stochastic partial differential equation (2.9) has a unique strong solution $\left\{X_n^\xi(t), t \geq 0\right\}$ in $D([0,T], L_2((\Omega, \mathcal{F}, P), H)$ for T finite and*

$$\lim_{n\to\infty} E\left[\sup_{0\leq t\leq T} \left\|X_n^\xi(t) - X^\xi(t)\right\|_H^2\right] = 0, \tag{2.12}$$

where $\left\{X^\xi(t), t \geq 0\right\}$ is the mild solution of equation (2.7).

Proof. We proved this result in [1].

In Theorem 3 we have already proved that there exists a unique solution of (2.9) in $D([0,T], L_2((\Omega, \mathcal{F}, P), H)$ and by Theorem 4 this is also a strong solution. Now we will prove (2.12).

$$E\sup_{0\leq t\leq T} \left\|X_n^\xi(t) - X^\xi(t)\right\|_H^2$$

$$= E\sup_{0\leq t\leq T} \Big\| \int_0^t S(t-s)(R_n F(X_n^\xi(s)) - F(X^\xi(s)))ds$$

$$+\int_0^t S(t-s)(R_n B(X_n^\xi(s)) - B(X^\xi(s)))dW_s$$

$$+\int_0^t \int_{H\setminus\{0\}} S(t-s)(R_n f(v, X_n^\xi(s)) - f(v, X^\xi(s)))q(dv, ds)\Big\|_H^2$$

$$= E \sup_{0\le t\le T} \Big\| \int_0^t S(t-s)R_n(F(X_n^\xi(s)) - F(X^\xi(s)))ds + \int_0^t S(t-s)(R_n - I)F(X^\xi(s))ds$$

$$+ \int_0^t S(t-s)R_n(B(X_n^\xi(s)) - B(X^\xi(s)))dW_s + \int_0^t S(t-s)(R_n - I)B(X^\xi(s))dW_s$$

$$+ \int_0^t \int_{H\setminus\{0\}} S(t-s)R_n(f(v, X_n^\xi(s)) - f(v, X^\xi(s)))q(dv, ds)$$

$$+ \int_0^t \int_{H\setminus\{0\}} S(t-s)(R_n - I)f(v, X^\xi(s))q(dv, ds)\Big\|_H^2$$

$$\le C\big\{E \sup_{0\le s\le t} \Big\| \int_0^s S(s-r)R_n(F(X_n^\xi(r)) - F(X^\xi(r)))dr\Big\|_H^2$$

$$+E \sup_{0\le s\le t} \Big\| \int_0^s S(s-r)(R_n - I)F(X^\xi(r))dr\Big\|_H^2$$

$$+E \sup_{0\le s\le t} \Big\| \int_0^s S(s-r)R_n(B(X_n^\xi(r)) - B(X^\xi(r)))dW_r\Big\|_H^2$$

$$+E \sup_{0\le s\le t} \Big\| \int_0^s S(s-r)(R_n - I)B(X^\xi(r))dW_r\Big\|_H^2$$

$$+E \sup_{0\le s\le t} \Big\| \int_0^s \int_{H\setminus\{0\}} S(s-r)R_n(f(v, X_n^\xi(r)) - f(v, X^\xi(r)))q(dv, dr)\Big\|_H^2$$

$$+E \sup_{0\le s\le t} \Big\| \int_0^s \int_{H\setminus\{0\}} S(s-r)(R_n - I)f(v, X^\xi(r))q(dv, dr)\Big\|_H^2\big\}$$

$$\le C\big\{E \sup_{0\le s\le t} \Big\| \int_0^s S(s-r)R_n(F(X_n^\xi(r)) - F(X^\xi(r)))dr\Big\|_H^2$$

$$+E \sup_{0\le s\le t} \Big\| \int_0^s S(s-r)(R_n - I)F(X^\xi(r))dr\Big\|_H^2$$

$$+E \sup_{0\le s\le t} \Big\| \int_0^s S(s-r)R_n(B(X_n^\xi(r)) - B(X^\xi(r)))dW_r\Big\|_H^2$$

$$+C_1 E \int_0^t \big\|(R_n - I)B(X^\xi(r))\big\|_{\mathcal{L}_2(K_Q,H)}^2 dr$$

$$+E\sup_{0\le s\le t}\Big\|\int_0^s\int_{H\setminus\{0\}}S(s-r)R_n(f(v,X_n^\xi(r))-f(v,X^\xi(r)))q(dv,dr)\Big\|_H^2$$

$$+C_2E\int_0^t\int_{H\setminus\{0\}}\big\|(R_n-I)f(v,X^\xi(r))\big\|_H^2\beta(dv)dr\Big\},$$

where C_1, C_2 are constants depending on α and T. By Lemma 3, the first, third and fifth summands are bounded by $\mathcal{G}_1\mathcal{K}\int_0^t E\sup_{0\le r\le s}\left\|X_n^\xi(r)-X^\xi(r)\right\|_H^2 dr$ for $n>n_0$ (n_0 sufficiently large), where $\mathcal{G}_1$ is a constant which depends on $\sup_{0\le t\le T}\|S(t)\|_{\mathcal{L}(H)}$ and $\sup_{n>n_0}\|R_n\|_{\mathcal{L}(H)}$ and $\mathcal{K}$ is the Lipschitz constant.

By the properties of R_n, the integrands in the second, fourth and sixth summands converge to zero. The integrands are bounded by $\mathcal{G}_2 l(1+\left\|X^\xi(r)\right\|_H^2)$ (by condition (A3)) for some constant $\mathcal{G}_2$ depending on $\|S(t)\|_{\mathcal{L}(H)}$ and $\|R_n\|_{\mathcal{L}(H)}$, and the constant l is the linear growth condition. So by Lebesgue DCT the integrals converge to zero as $n\to\infty$. Therefore there exists $\epsilon>0$ s.t. for sufficiently large n each of the three summands are less or equal ϵ. So for sufficiently large n,

$$E\sup_{0\le t\le T}\left\|X_n^\xi(t)-X^\xi(t)\right\|_H^2\le 3\mathcal{G}_1\mathcal{K}\int_0^t E\sup_{0\le r\le s}\left\|X_n^\xi(r)-X^\xi(r)\right\|_H^2 dr+3\epsilon.$$

By Gronwall's lemma (for sufficiently large n),

$$E\sup_{0\le t\le T}\left\|X_n^\xi(t)-X^\xi(t)\right\|_H^2\le 3\epsilon e^{3\mathcal{G}_1\mathcal{K}t}.$$

Hence we can conclude that

$$\lim_{n\to\infty}E\sup_{0\le t\le T}\left\|X_n^\xi(t)-X^\xi(t)\right\|_H^2=0.$$

□

$\left\{X_n^\xi(t)\right\}$ in the above theorem are the Yosida approximation of the mild solution of (2.7).

Definition 11. *We call a continuous, non-decreasing function $h:\mathbb{R}_+\to\mathbb{R}_+$ quasi-sublinear if there is a constant $C>0$ such that*

$$h(x+y)\le C(h(x)+h(y)),\qquad x,y\in\mathbb{R}_+,$$

$$h(xy)\le Ch(x)h(y),\qquad x,y\in\mathbb{R}_+.$$

Let $C^{1,2}([0,T]\times H)$ denote the class of real valued continuous functions Ψ on $[0,T]\times H$ with continuous Fréchet derivatives $\partial_s\Psi(s,x)$, $\partial_x\Psi(s,x)$, $\partial_s\partial_x\Psi(s,x)$, $\partial_x\partial_s\Psi(s,x)$ and $\partial_x\partial_x\Psi(s,x)$. From (2.9), $\xi\in\mathcal{D}(A)$ and $X_n^\xi(t)\in\mathcal{D}(A)$, where $X_n^\xi(t)$ denotes the strong solution of equation (2.10). Let $\Psi\in C^{1,2}([0,T]\times H)$

and $\Psi : [0,T] \times H \to \mathbb{R}$. Moreover assume that the following conditions (a) and (b) hold:

(a)

$$\|\Psi_x(s,x)\|_H \leq h_1(\|x\|_H)$$

and

$$\|\Psi_{xx}(s,x)\|_{\mathcal{L}(H)} \leq h_2(\|x\|_H).$$

(b)

$$\int_0^T \|F(s)\|_H ds < \infty \quad P\text{-a.s.}, \quad P\Big\{\int_0^T \|B(s)\|^2_{\mathcal{L}_2(K_Q,H)} ds < \infty\Big\} = 1$$

and let $h_1, h_2 : \mathbb{R}_+ \to \mathbb{R}_+$ be quasi-sublinear functions such that,

$$\int_{H\setminus\{0\}} \|f(v,s)\|^2 \beta(dv) + \int_{H\setminus\{0\}} h_1(\|f(v,s)\|)^2 \|f(v,s)\|^2 \beta(dv) + \int_{H\setminus\{0\}} h_2(\|f(v,s)\|) \|f(v,s)\|^2 \beta(dv) < \infty$$

P-a.s. for all $s \in [0,T]$. Then due to the results of [23], [16], the Itô formula for strong solutions is well defined:

$$\Psi(t, X_n^\xi(t)) - \Psi(0,\xi) = \int_0^t (\Psi_s(s, X_n^\xi(s)) + \mathcal{L}_n\Psi(s, X_n^\xi(s)))ds \tag{2.13}$$

$$+ \int_0^t \left\langle \Psi_x(s, X_n^\xi(s)), R_n B(X_n^\xi(s)) dW_s \right\rangle_H$$

$$+ \int_0^t \int_{H\setminus\{0\}} \left[\Psi(s, X_n^\xi(s) + R_n f(v, X_n^\xi(s))) - \Psi(s, X_n^\xi(s))\right] q(dv, ds),$$

where

$$\mathcal{L}_n\Psi(s, X_n^\xi(s)) = \left\langle \Psi_x(s, X_n^\xi(s)), A X_n^\xi(s) + R_n F(X_n^\xi(s)) \right\rangle_H \tag{2.14}$$

$$+\frac{1}{2} tr(\Psi_{xx}(s, X_n^\xi(s))(R_n B(X_n^\xi(s))) Q (R_n B(X_n^\xi(s)))^*)$$

$$+ \int_{H\setminus\{0\}} \left[\Psi(s, X_n^\xi(s) + R_n f(v, X_n^\xi(s))) - \Psi(s, X_n^\xi(s)) - \left\langle \Psi_x(s, X_n^\xi(s)), R_n f(v, X_n^\xi(s)) \right\rangle_H \right] \beta(dv).$$

P-a.s. for all $s \in [0,T]$. See Appendix, 4.

2.2.2 The Itô formula for mild solutions

Here we will prove our main theorem.

Theorem 6. *Assume that $S(t)$ is a pseudo-contraction semigroup and $\Psi \in C^{1,2}([0,T]\times H)$. Let the coefficients F, B, f satisfy (A1), (A2), (A3). Moreover assume that*
(a)

$$\|\Psi_x(s,x)\|_H \leq h_1(\|x\|_H)$$

and

$$\|\Psi_{xx}(s,x)\|_{\mathcal{L}(H)} \leq h_2(\|x\|_H),$$

(b)

$$\int_0^T \|F(s)\|_H ds < \infty \quad P\text{-a.s.}, \quad P\Big\{\int_0^T \|B(s)\|^2_{\mathcal{L}_2(K_Q,H)} ds < \infty\Big\} = 1$$

and let $h_1, h_2 : \mathbb{R}_+ \to \mathbb{R}_+$ be quasi-sublinear functions such that,

$$\int_{H\setminus\{0\}} \|f(v,s)\|^2 \beta(dv) + \int_{H\setminus\{0\}} h_1(\|f(v,s)\|)^2 \|f(v,s)\|^2 \beta(dv) \\ + \int_{H\setminus\{0\}} h_2(\|f(v,s)\|) \|f(v,s)\|^2 \beta(dv) < \infty. \tag{2.15}$$

P-a.s. for all $s \in [0,T]$. Then the following Itô Formula for mild solutions hold P-a.s. for all $t \in [0,T]$

$$\lim_{n\to\infty} \int_0^t \left\langle \Psi_x(s, X_n^\xi(s)), AX_n^\xi(s)\right\rangle_H ds \tag{2.16}$$

$$= \Psi(t, X^\xi(t)) - \Psi(0,\xi) - \int_0^t (\Psi_s(s, X^\xi(s)))ds$$

$$- \int_0^t \left\langle \Psi_x(s, X^\xi(s)), F(X^\xi(s))\right\rangle_H ds$$

$$- \int_0^t \frac{1}{2} tr(\Psi_{xx}(s, X^\xi(s))(B(X^\xi(s)))Q(B(X^\xi(s)))^*)ds$$

$$- \int_0^t \int_{H\setminus\{0\}} \left[\Psi(s, X^\xi(s) + f(v, X^\xi(s))) - \Psi(s, X^\xi(s)) - \left\langle \Psi_x(s, X^\xi(s)), f(v, X^\xi(s))\right\rangle_H\right] \beta(dv)ds$$

$$- \int_0^t \left\langle \Psi_x(s, X^\xi(s)), B(X^\xi(s))dW_s\right\rangle_H$$

$$- \int_0^t \int_{H\setminus\{0\}} \left[\Psi(s, X^\xi(s) + f(v, X^\xi(s))) - \Psi(s, X^\xi(s))\right] q(dv, ds).$$

Proof. We proved this result in [1].

First, we rewrite eq. (2.13) as follows

$$\Psi(t, X_n^\xi(t)) - \Psi(0, \xi)$$

$$= \int_0^t (\Psi_s(s, X_n^\xi(s)) + \mathcal{L}_n \Psi(s, X_n^\xi(s)))ds + \int_0^t \left\langle \Psi_x(s, X_n^\xi(s)), R_n B(X_n^\xi(s)) dW_s \right\rangle_H$$

$$+ \int_0^t \int_{H\setminus\{0\}} \left[\Psi(s, X_n^\xi(s) + R_n f(x, X_n^\xi(s))) - \Psi(s, X_n^\xi(s))\right] q(dx, ds)$$

$$= \int_0^t (\Psi_s(s, X_n^\xi(s)))ds + \int_0^t \mathcal{L}_n(\Psi(s, X_n^\xi(s)))ds + \int_0^t \left\langle \Psi_x(s, X_n^\xi(s)), R_n B(X_n^\xi(s)) dW_s \right\rangle_H$$

$$+ \int_0^t \int_{H\setminus\{0\}} \left[\Psi(s, X_n^\xi(s) + R_n f(v, X_n^\xi(s))) - \Psi(s, X_n^\xi(s))\right] q(dv, ds)$$

Now we substite $\mathcal{L}_n(\Psi(s, X_n^\xi(s)))$,

$$= \int_0^t (\Psi_s(s, X_n^\xi(s)))ds + \int_0^t \left\langle \Psi_x(s, X_n^\xi(s)), A X_n^\xi(s) + R_n F(X_n^\xi(s)) \right\rangle_H ds$$

$$+ \int_0^t \frac{1}{2} tr(\Psi_{xx}(s, X_n^\xi(s))(R_n B(X_n^\xi(s))) Q (R_n B(X_n^\xi(s)))^*) ds$$

$$+ \int_0^t \int_{H\setminus\{0\}} \left[\Psi(s, X_n^\xi(s) + R_n f(v, X_n^\xi(s))) - \Psi(s, X_n^\xi(s)) - \left\langle \Psi_x(s, X_n^\xi(s)), R_n f(v, X_n^\xi(s)) \right\rangle_H\right] \beta(dv) ds$$

$$+ \int_0^t \left\langle \Psi_x(s, X_n^\xi(s)), R_n B(X_n^\xi(s)) dW_s \right\rangle_H$$

$$+ \int_0^t \int_{H\setminus\{0\}} \left[\Psi(s, X_n^\xi(s) + R_n f(v, X_n^\xi(s))) - \Psi(s, X_n^\xi(s))\right] q(dv, ds). \quad (2.17)$$

Now our task is to show that the above equation converges P-a.s.(term by term) and also to find the limit.

The convergence in Theorem 5 (equation (2.12)) allows us to choose a subsequence $X_{n_k}^\xi$ such that,

$$X_{n_k}^\xi(t) \to X^\xi(t),\ 0 \le t \le T,\ P\text{-a.s.}$$

We will denote such a subsequence again by X_n^ξ.

In fact, we can say that

$$\sup_{0\le t\le T} \|X_n(t) - X(t)\|_H \to 0, \tag{2.18}$$

P a.s.. This implies that the set

$$S = \{X_n(t),\ X(t) :\ n = 1, 2..., 0 \le t \le T\} \tag{2.19}$$

is bounded in H, hence all the values of Ψ and its derivatives evaluated on S are bounded by some constant. Now we are ready to show the term by term convergence of equation (2.17).

First consider the first term of the L.H.S. of eq. (2.17). Since Ψ is continuous, from (2.18) we can conclude that

$$\Psi(t, X_n^\xi(t)) \to \Psi(t, X^\xi(t)),$$

P-a.s.

Now consider the first term of the R.H.S. of eq. (2.17). Ψ_s is continuous, $\Psi_s(s, X_n^\xi(s)) < C$ by equation (2.18). So by applying Lebesgue DCT we can conclude that

$$\int_0^t (\Psi_s(s, X_n^\xi(s)))ds \to \int_0^t (\Psi_s(s, X^\xi(s)))ds,$$

P-a.s.

Now consider the second term of the R.H.S. of eq. (2.17),

$$\int_0^t \left\langle \Psi_x(s, X_n^\xi(s)), AX_n^\xi(s) + R_nF(X_n^\xi(s))\right\rangle_H ds$$

$$= \int_0^t \left\langle \Psi_x(s, X_n^\xi(s)), AX_n^\xi(s)\right\rangle_H ds + \int_0^t \left\langle \Psi_x(s, X_n^\xi(s)), R_nF(X_n^\xi(s))\right\rangle_H ds.$$

Since Ψ_x is continuous, by (2.18) we get $\Psi_x(s, X_n^\xi(s)) \to \Psi_x(s, X^\xi(s))$. Since F is continuous and $R_n(F(X_n^\xi(s))$ is a double sequence, therefore we have

$$\begin{aligned} \|R_n\left(F\left(X_n(s)\right)\right) - F(X(s))\|_H &\le \|R_n\left(F\left(X_n(s)\right) - F(X(s))\right)\|_H + \|R_n\left(F\left(X(s)\right)\right) - F(X(s))\|_H \\ &\le \|R_n\|_H \left\|F\left(X_n(s)\right) - F(X(s))\right\|_H + \|\left(R_n - I\right)F(X(s))\|_H \end{aligned} \tag{2.20}$$

Therefore $R_n\left(F\left(X_n(s)\right)\right) \to F(X(s))$ because of the uniform boundedness of $\|R_n\|_{\mathcal{L}(H)}$, and the convergence of $(R_n - I)x \to 0$. So, by (2.18) and Lebesgue DCT,

$$\int_0^t \left\langle \Psi_x(s, X_n^\xi(s)), R_nF(X_n^\xi(s))\right\rangle_H ds \to \int_0^t \left\langle \Psi_x(s, X^\xi(s)), F(X^\xi(s))\right\rangle_H ds$$

P-a.s..

We will discuss the convergence of the term,

$$\int_0^t \left\langle \Psi_x(s, X_n^\xi(s)), AX_n^\xi(s) \right\rangle_H ds$$

at the end.

Now consider the third term of the R.H.S. of eq. (2.17),

$$\int_0^t \frac{1}{2} tr(\Psi_{xx}(s, X_n^\xi(s))(R_n B(X_n^\xi(s)))Q(R_n B(X_n^\xi(s)))^*)ds.$$

We have

$$tr(\Psi_{xx}(s, X_n^\xi(s))(R_n B(X_n^\xi(s)))Q(R_n B(X_n^\xi(s)))^*)$$

$$= tr((R_n B(X_n^\xi(s)))^* \Psi_{xx}(s, X_n^\xi(s))(R_n B(X_n^\xi(s)))Q)$$

$$= \sum_{j=1}^{\infty} \lambda_j \left\langle \Psi_{xx}(s, X_n^\xi(s))(R_n B(X_n^\xi(s)))f_j, (R_n B(X_n^\xi(s)))f_j \right\rangle_H .$$

Here we used the property that, for a symmetric operator $T \in \mathcal{L}(H)$ and $\phi \in \mathcal{L}(K, H)$,

$$tr(T\phi Q\phi^*) = tr(\phi^* T\phi Q).$$

Ψ_{xx} being continuous, B is continuous, $\|R_n\|_{\mathcal{L}(H)}$ is uniformly bounded and having the convergence of $(R_n - I)x \to 0$, by a similar calculation as in (2.20) we can deduce that

$$\left\langle \Psi_{xx}(s, X_n^\xi(s))(R_n B(X_n^\xi(s)))f_j, (R_n B(X_n^\xi(s)))f_j \right\rangle_H$$

$$\to \left\langle \Psi_{xx}(s, X^\xi(s))(B(X^\xi(s)))f_j, (B(X^\xi(s)))f_j \right\rangle_H .$$

Hence,

$$tr(\Psi_{xx}(s, X_n^\xi(s))(R_n B(X_n^\xi(s)))Q(R_n B(X_n^\xi(s)))^*)$$

$$\to tr(\Psi_{xx}(s, X^\xi(s))(B(X^\xi(s)))Q(B(X^\xi(s)))^*).$$

Also we have,

$$\begin{aligned} tr(\Psi_{xx}(s, X_n^\xi(s))(R_n B(X_n^\xi(s)))Q(R_n B(X_n^\xi(s)))^*) &\le \|\Psi_{xx}(s, X_n^\xi(s))\| \|R_n B(X_n^\xi(s))\|^2 \\ \text{by (A3)} \quad &\le \|\Psi_{xx}(s, X_n^\xi(s))\| \|R_n\|^2 l(1 + \|X_n^\xi(s)\|^2). \end{aligned}$$

So by (2.18) and Lebesgue DCT we can conclude that,

$$\int_0^t tr(\Psi_{xx}(s, X_n^\xi(s))(R_n B(X_n^\xi(s)))Q(R_n B(X_n^\xi(s)))^*)ds$$

$$\to \int_0^t tr(\Psi_{xx}(s, X^\xi(s))(B(X^\xi(s)))Q(B(X^\xi(s)))^*)ds,$$

P-a.s..

Now consider the fourth term of the R.H.S. of eq. (2.17),

$$\int_{H\setminus\{0\}} \left[\Psi(s, X_n^\xi(s) + R_n f(v, X_n^\xi(s))) - \Psi(s, X_n^\xi(s)) - \left\langle \Psi_x(s, X_n^\xi(s)), R_n f(v, X_n^\xi(s)) \right\rangle_H \right] \beta(dv).$$

Using Theorem 5, (2.18), the continuity of Ψ, Ψ_x, f and $(R_n - I)x \to 0$, we can conclude

$$\left[\Psi(s, X_n^\xi(s) + R_n f(v, X_n^\xi(s))) - \Psi(s, X_n^\xi(s)) - \left\langle \Psi_x(s, X_n^\xi(s)), R_n f(v, X_n^\xi(s)) \right\rangle_H \right]$$

converges to

$$\left[\Psi(s, X^\xi(s) + f(v, X^\xi(s))) - \Psi(s, X^\xi(s)) - \left\langle \Psi_x(s, X^\xi(s)), f(v, X^\xi(s)) \right\rangle_H \right]$$

P-a.s.. Again by Taylor's theorem, the Cauchy Schwarz inequality and assumption (a) of the theorem we get

$$\int_{H\setminus\{0\}} \left\| \Psi(s, X_n^\xi(s) + R_n f(v, X_n^\xi(s))) - \Psi(s, X_n^\xi(s)) - \left\langle \Psi_x(s, X_n^\xi(s)), R_n f(v, X_n^\xi(s)) \right\rangle_H \right\| \beta(dv)$$

$$= \int_{H\setminus\{0\}} \left\| \int_0^1 \Psi_{xx}(s, X_n^\xi(s) + \theta R_n f(v, X_n^\xi(s))) \left\langle R_n f(v, X_n^\xi(s)), R_n f(v, X_n^\xi(s)) \right\rangle d\theta \right\| \beta(dv)$$

$$\leq \int_{H\setminus\{0\}} \int_0^1 \left\| \Psi_{xx}(s, X_n^\xi(s) + \theta R_n f(v, X_n^\xi(s))) \right\| \left\| R_n f(v, X_n^\xi(s)) \right\|^2 d\theta \beta(dv)$$

$$\leq \int_{H\setminus\{0\}} \int_0^1 h_2\left(\left\| X_n^\xi(s) + \theta R_n f(v, X_n^\xi(s)) \right\| \right) \left\| R_n f(v, X_n^\xi(s)) \right\|^2 d\theta \beta(dv)$$

$$\leq C \int_{H\setminus\{0\}} \int_0^1 \left(h_2(\left\| X_n^\xi(s) \right\|) + C h_2(\theta) h_2(\left\| R_n f(v, X_n^\xi(s)) \right\|) \right) \left\| R_n f(v, X_n^\xi(s)) \right\|^2 d\theta \beta(dv)$$

$$\leq C \int_{H\setminus\{0\}} h_2(\left\| X_n^\xi(s) \right\|) \left\| R_n f(v, X_n^\xi(s)) \right\|^2 \beta(dv)$$
$$+ C^2 h_2(1) \int_{H\setminus\{0\}} h_2(\left\| R_n f(v, X_n^\xi(s)) \right\|) \left\| R_n f(v, X_n^\xi(s)) \right\|^2 \beta(dv) < \infty,$$

P-a.s. by the condition (2.15). Since $\|R_n\|_{\mathcal{L}(H)}$ is uniformly bounded, therefore by Lebesgue DCT

$$\int_0^t \int_{H\setminus\{0\}} \left[\Psi(s, X_n^\xi(s) + R_n f(v, X_n^\xi(s))) - \Psi(s, X_n^\xi(s)) - \left\langle \Psi_x(s, X_n^\xi(s)), R_n f(v, X_n^\xi(s)) \right\rangle_H\right] \beta(dv)ds$$

converges to

$$\int_0^t \int_{H\setminus\{0\}} \left[\Psi(s, X^\xi(s) + f(v, X^\xi(s))) - \Psi(s, X^\xi(s)) - \left\langle \Psi_x(s, X^\xi(s)), f(v, X^\xi(s)) \right\rangle_H\right] \beta(dv)ds$$

P-a.s..

Now consider the fifth term of the R.H.S. of eq. (2.17),

$$\int_0^t \left\langle \Psi_x(s, X_n^\xi(s)), R_n B(X_n^\xi(s))dW_s \right\rangle_H .$$

Now,

$$E\Big\| \int_0^t \left\langle \Psi_x(s, X_n^\xi(s)), R_n B(X_n^\xi(s))dW_s \right\rangle_H - \int_0^t \left\langle \Psi_x(s, X^\xi(s)), B(X^\xi(s))dW_s \right\rangle_H \Big\|^2$$

$$\leq C \int_0^t E\big\|(B(X^\xi(s)))^*(\Psi_x(s, X_n^\xi(s) - \Psi_x(s, X^\xi(s)))\big\|^2_{\mathcal{L}_2(K_Q,H)} ds$$

$$+C \int_0^t E\big\|((B(X^\xi(s)))^* - (R_n B(X_n^\xi(s)))^*)\Psi_x(s, X_n^\xi(s))\big\|^2_{\mathcal{L}_2(K_Q,H)} ds$$

$$\leq C \int_0^t E(\big\|(B(X^\xi(s))\big\|^2_{\mathcal{L}_2(K_Q,H)} \big\|\Psi_x(s, X_n^\xi(s)) - \Psi_x(s, X^\xi(s))\big\|^2_H) ds$$

$$+C \int_0^t E(\big\|(B(X^\xi(s)))^* - (R_n B(X_n^\xi(s)))^*\big\|^2_{\mathcal{L}_2(K_Q,H)} \big\|\Psi_x(s, X_n^\xi(s))\big\|^2_H) ds.$$

Here, the first integral converges to zero, since the first factor is an integrable process, and the second factor converges to zero almost surely, so we can apply Lebesgue DCT. The second integral is bounded by $M\big\|(B(X^\xi(s)))^* - (R_n B(X_n^\xi(s)))^*\big\|^2_{\Lambda_2(K_Q,H)}$ for some constant M (from (2.18), Ψ_x is bounded by some constant), since $R_n B(X_n^\xi(s)) \to B(X^\xi(s))$ in the space $\Lambda_2(K_Q, H)$, so the second integral also converges to zero by Lebesgue DCT. Hence we can conclude that,

$$\int_0^t \left\langle \Psi_x(s, X_n^\xi(s)), R_n B(X_n^\xi(s))dW_s \right\rangle_H \to \int_0^t \left\langle \Psi_x(s, X^\xi(s)), B(X^\xi(s))dW_s \right\rangle_H$$

in mean square, therefore in probability.

Now consider the sixth term of the R.H.S. of eq. (2.17),

$$\int_0^t \int_{H\setminus\{0\}} \left[\Psi(s, X_n^\xi(s) + R_n f(v, X_n^\xi(s))) - \Psi(s, X_n^\xi(s))\right] q(dv, ds).$$

Now,

$$\left\|\left\{\Psi(s, X_n^\xi(s) + R_n f(v, X_n^\xi(s))) - \Psi(s, X_n^\xi(s))\right\} - \left\{\Psi(s, X^\xi(s) + f(v, X^\xi(s))) - \Psi(s, X^\xi(s))\right\}\right\|^2$$

$$= \left\|\left[\Psi(s, X_n^\xi(s) + R_n f(v, X_n^\xi(s))) - \Psi(s, X^\xi(s) + f(v, X^\xi(s)))\right] + \left[\Psi(s, X^\xi(s)) - \Psi(s, X_n^\xi(s))\right]\right\|^2$$

$$\leq 2\left\|\Psi(s, X_n^\xi(s) + R_n f(v, X_n^\xi(s))) - \Psi(s, X^\xi(s) + f(v, X^\xi(s)))\right\|^2 + 2\left\|\Psi(s, X^\xi(s)) - \Psi(s, X_n^\xi(s))\right\|^2$$

$$\leq 2\left\|X_n^\xi(s) + R_n f(v, X_n^\xi(s)) - \left\{X^\xi(s) + f(v, X^\xi(s))\right\}\right\|^2 \sup_{0<\theta\leq 1} \|\Psi_x(s, \eta_1(\theta))\|^2 + 2\left\|X^\xi(s) - X_n^\xi(s)\right\|^2 \sup_{0<\theta\leq 1} \|\Psi_x(s, \eta_2(\theta))\|^2$$

to obtain the above inequality, we used the following inequality

$$\|\Psi(x) - \Psi(y)\| \leq \|x - y\| \sup_{0<\theta\leq 1} \|\Psi_x(y + \theta(x - y))\|.$$

Where

$$\eta_1(\theta) = X^\xi(s) + f(v, X^\xi(s)) + \theta\left(X_n^\xi(s) - X^\xi(s) + R_n f(v, X_n^\xi(s)) - f(v, X^\xi(s))\right)$$

and

$$\eta_2(\theta) = X_n^\xi(s) + \theta\left(X^\xi(s) - X_n^\xi(s)\right).$$

Therefore, by using condition (a) of the theorem, we can write the above inequality is

$$\leq 4\left\{\left\|X_n^\xi(s) - X^\xi(s)\right\|^2 + \left\|R_n f(v, X_n^\xi(s)) - f(v, X^\xi(s))\right\|^2\right\} \sup_{0<\theta\leq 1} \{h_1(\|\eta_1(\theta)\|)\}^2 + 2\left\|X^\xi(s) - X_n^\xi(s)\right\|^2 \sup_{0<\theta\leq 1} \{h_1(\|\eta_2(\theta)\|)\}^2.$$

Now as $n \to \infty$, the R.H.S. of the above inequality converges to 0 P-a.s.. Therefore

$$\lim_{n\to\infty} \int_0^t \int_{H\setminus\{0\}} \left\| \left\{\Psi(s, X_n^\xi(s) + R_n f(v, X_n^\xi(s))) - \Psi(s, X_n^\xi(s))\right\} - \left\{\Psi(s, X^\xi(s) + f(v, X^\xi(s))) - \Psi(s, X^\xi(s))\right\} \right\|^2 \beta(dv)ds = 0 \tag{2.21}$$

P-a.s.. Again by Taylor's theorem, the Cauchy Schwarz inequality and assumption (a) of the theorem we get

$$\int_0^t \int_{H\setminus\{0\}} \left\|\Psi(s, X_n^\xi(s) + R_n f(v, X_n^\xi(s))) - \Psi(s, X_n^\xi(s))\right\|^2 \beta(dv)ds$$

$$= \int_0^t \int_{H\setminus\{0\}} \left\| \int_0^1 \Psi_x(s, X_n^\xi(s) + \theta R_n f(v, X_n^\xi(s))) R_n f(v, X_n^\xi(s)) d\theta \right\|^2 \beta(dv)ds$$

$$\leq \int_0^t \int_{H\setminus\{0\}} \int_0^1 \left\|\Psi_x(s, X_n^\xi(s) + \theta R_n f(v, X_n^\xi(s)))\right\|^2 \left\|R_n f(v, X_n^\xi(s))\right\|^2 d\theta \beta(dv)ds$$

$$\leq \int_0^t \int_{H\setminus\{0\}} \int_0^1 h_1(\left\|X_n^\xi(s) + \theta R_n f(v, X_n^\xi(s))\right\|)^2 \left\|R_n f(v, X_n^\xi(s))\right\|^2 d\theta \beta(dv)ds$$

$$\leq C^2 \int_0^t \int_{H\setminus\{0\}} \int_0^1 \left\{h_1(\left\|X_n^\xi(s)\right\|) + C h_1(\theta) h_1(\left\|R_n f(v, X_n^\xi(s))\right\|)\right\}^2 \left\|R_n f(v, X_n^\xi(s))\right\|^2 d\theta \beta(dv)ds$$

$$\leq 2C^2 \int_0^t \int_{H\setminus\{0\}} h_1(\left\|X_n^\xi(s)\right\|)^2 \left\|R_n f(v, X_n^\xi(s))\right\|^2 \beta(dv)ds \qquad (2.22)$$
$$+ 2C^4 h_1(1) \int_0^t \int_{H\setminus\{0\}} h_1(\left\|R_n f(v, X_n^\xi(s))\right\|)^2 \left\|R_n f(v, X_n^\xi(s))\right\|^2 \beta(dv)ds < \infty,$$

P-a.s. by the condition (2.15). Therefore from (2.21) and (2.22) we can conclude that

$$\lim_{n\to\infty} \int_0^t \int_{H\setminus\{0\}} \left[\Psi(s, X_n^\xi(s) + R_n f(v, X_n^\xi(s))) - \Psi(s, X_n^\xi(s))\right] q(dv, ds)$$

$$= \int_0^t \int_{H\setminus\{0\}} \left[\Psi(s, X^\xi(s) + f(v, X^\xi(s))) - \Psi(s, X^\xi(s))\right] q(dv, ds)$$

in probability.

Thus we have showed the term by term convergence of left- and right- hand sides of eq. (2.17) except for the term $\int_0^t \left\langle \Psi_x(s, X_n^\xi(s)), AX_n^\xi(s)\right\rangle_H ds$. Now since all the terms of the eq. (2.17) converge, so the term $\int_0^t \left\langle \Psi_x(s, X_n^\xi(s)), AX_n^\xi(s)\right\rangle_H ds$ has to converge. Where the nonstochastic integrals converge in P-a.s. sense and stochastic integrals converge in probability. In conclusion, possibly for a subsequence of left- and right- hand sides of eq. (2.17) converges in P-a.s. sense for all $t \in [0, T]$. Hence we can conclude that eq. (2.13) converges P-a.s. and we can write it as,

$$\lim_{n\to\infty} \int_0^t \left\langle \Psi_x(s, X_n^\xi(s)), AX_n^\xi(s)\right\rangle_H ds$$

$$= \Psi(t, X^{\xi}(t)) - \Psi(0, \xi) - \int_0^t (\Psi_s(s, X^{\xi}(s)))ds$$

$$- \int_0^t \left\langle \Psi_x(s, X^{\xi}(s)), F(X^{\xi}(s)) \right\rangle_H ds$$

$$- \int_0^t \frac{1}{2} tr(\Psi_{xx}(s, X^{\xi}(s))(B(X^{\xi}(s)))Q(B(X^{\xi}(s)))^*)ds$$

$$- \int_0^t \int_{H \setminus \{0\}} \left[\Psi(s, X^{\xi}(s) + f(v, X^{\xi}(s))) - \Psi(s, X^{\xi}(s)) - \left\langle \Psi_x(s, X^{\xi}(s)), f(v, X^{\xi}(s)) \right\rangle_H \right] \beta(dv)ds$$

$$- \int_0^t \left\langle \Psi_x(s, X^{\xi}(s)), B(X^{\xi}(s))dW_s \right\rangle_H$$

$$- \int_0^t \int_{H \setminus \{0\}} \left[\Psi(s, X^{\xi}(s) + f(v, X^{\xi}(s))) - \Psi(s, X^{\xi}(s)) \right] q(dv, ds).$$

This completes the proof.

□

Remark 1: When $X^{\xi}(t) \in \mathcal{D}(A)$, then $AX^{\xi}(t)$ is well defined. Hence,we get back the same Itô formula for strong solutions as in eq. (2.13), as by Theorem 4 when $X^{\xi}(t) \in \mathcal{D}(A)$, then it is also a strong solution.

Definition 12. *Let $X_n^{\xi}(s)$ be the strong solution of* (2.9) *defined in* (2.10) *and $X^{\xi}(s)$ be the mild solution of* (2.7) *defined in* (2.11). *Let us define the processes $\mathcal{L}_n \Psi(s, X_n^{\xi}(s))$ and $\mathcal{L}\Psi(s, X^{\xi}(s))$ respectively as follows-*

$$\mathcal{L}_n \Psi(s, X_n^{\xi}(s)) := \left\langle \Psi_x(s, X_n^{\xi}(s)), AX_n^{\xi}(s) + R_n F(X_n^{\xi}(s)) \right\rangle_H \tag{2.23}$$

$$+ \frac{1}{2} tr(\Psi_{xx}(s, X_n^{\xi}(s))(R_n B(X_n^{\xi}(s)))Q(R_n B(X_n^{\xi}(s)))^*)$$

$$+ \int_{H \setminus \{0\}} \left[\Psi(s, X_n^{\xi}(s) + R_n f(v, X_n^{\xi}(s))) - \Psi(s, X_n^{\xi}(s)) - \left\langle \Psi_x(s, X_n^{\xi}(s)), R_n f(v, X_n^{\xi}(s)) \right\rangle_H \right] \beta(dv)$$

and

$$\mathcal{L}\Psi(s, X^{\xi}(s)) := \lim_{n \to \infty} \left\langle \Psi_x(s, X_n^{\xi}(s)), AX_n^{\xi}(s) \right\rangle + \left\langle \Psi_x(s, X^{\xi}(s)), F(X^{\xi}(s)) \right\rangle_H \tag{2.24}$$

$$+ \frac{1}{2} tr(\Psi_{xx}(s, X^{\xi}(s))(B(X^{\xi}(s)))Q(B(X^{\xi}(s)))^*)$$

$$+ \int_{H \setminus \{0\}} \left[\Psi(s, X^{\xi}(s) + f(v, X^{\xi}(s))) - \Psi(s, X^{\xi}(s)) - \left\langle \Psi_x(s, X^{\xi}(s)), f(v, X^{\xi}(s)) \right\rangle_H \right] \beta(dv),$$

where $X_n^{\xi}(s) \in \mathcal{D}(A)$.

Corollary 1. *Assume that F, B, f satisfy (A1), (A2), (A3) and conditions (a), (b) of Theorem 6 hold. Let $X_n^\xi(s) \in \mathcal{D}(A)$ be the strong solution of* (2.9) *defined in* (2.10) *and $X^\xi(s)$ be the mild solution of* (2.7) *defined in* (2.11). *Let $\mathcal{L}_n\Psi(s, X_n^\xi(s))$ and $\mathcal{L}\Psi(s, X^\xi(s))$ be defined as in Definition 12. Then $\mathcal{L}\Psi(s, X^\xi(s)) - \mathcal{L}_n\Psi(s, X_n^\xi(s)) \to 0$, P-a.s. as $n \to \infty$.*

Proof. We proved this result in [1].

$$\mathcal{L}\Psi(s, X^\xi(s)) - \mathcal{L}_n\Psi(s, X_n^\xi(s)) = \lim_{n\to\infty} \left\langle \Psi_x(s, X_n^\xi(s)), AX_n^\xi(s) \right\rangle$$

$$+ \left\langle \Psi_x(s, X^\xi(s)), F(X^\xi(s)) \right\rangle_H + \frac{1}{2} tr(\Psi_{xx}(s, X^\xi(s))(B(X^\xi(s)))Q(B(X^\xi(s)))^*)$$

$$+ \int_{H\setminus\{0\}} \left[\Psi(s, X^\xi(s) + f(v, X^\xi(s))) - \Psi(s, X^\xi(s)) - \left\langle \Psi_x(s, X^\xi(s)), f(v, X^\xi(s)) \right\rangle_H \right] \beta(dv)$$

$$- \left\langle \Psi_x(s, X_n^\xi(s)), AX_n^\xi(s) \right\rangle - \left\langle \Psi_x(s, X_n^\xi(s)), R_n F(X_n^\xi(s)) \right\rangle_H$$

$$-\frac{1}{2} tr(\Psi_{xx}(s, X_n^\xi(s))(R_n B(X_n^\xi(s)))Q(R_n B(X_n^\xi(s)))^*)$$

$$- \int_{H\setminus\{0\}} \left[\Psi(s, X_n^\xi(s) + R_n f(v, X_n^\xi(s))) - \Psi(s, X_n^\xi(s)) - \left\langle \Psi_x(s, X_n^\xi(s)), R_n f(v, X_n^\xi(s)) \right\rangle_H \right] \beta(dv).$$

Now as $n \to \infty$,

$$\mathcal{L}\Psi(s, X^\xi(s)) - \mathcal{L}_n\Psi(s, X_n^\xi(s))$$

$$= \lim_{n\to\infty} \left\langle \Psi_x(s, X_n^\xi(s)), AX_n^\xi(s) \right\rangle - \lim_{n\to\infty} \left\langle \Psi_x(s, X_n^\xi(s)), AX_n^\xi(s) \right\rangle$$
$$= 0$$

P-a.s.. As, all other terms converge to the respective terms P-a.s. from Theorem 6.

□

2.3 Applications of the Itô formula for mild solutions

In this section we will present some applications of the Itô formula for mild solutions of Theorem 6 to prove the results of exponential stability and exponentially ultimate boundedness in the mean square sense (m.s.s.).

First, we will show how we use the limiting argument of Theorem 6 and Corollary 1 to prove the results of exponential stability in the mean square sense of the mild solutions. These results are proved, for the Gaussian case in chapter 6 (Theorem 6.4 and Theorem 6.5) of [16] and for the non-Gaussian case in [24] (Theorem 4.2 and Theorem 4.3). But here we prove the same results in much shorter way.

Consider the following stochastic partial differential equation with values in H,

$$dX(t) = (AX(t) + F(X(t)))dt + B(X(t))dW_t + \int_{H\setminus\{0\}} f(v, X(t))q(dv, dt);$$

$$X(0) = x \in H. \tag{2.25}$$

Which satisfy conditions (A1), (A2) and (A3).

Definition 13. *Let $\{X^x(t), t \geq 0\}$ be a mild solution of (2.25). We say that $X^x(t)$ is exponentially stable in the mean square sense (m.s.s.) if for all $t \geq 0$ and $x \in H$,*

$$E\|X^x(t)\|_H^2 \leq ce^{-\beta t}\|x\|_H^2; \quad c, \beta > 0. \tag{2.26}$$

Let $C^2(H)$ be the space of continuous functions on $\Lambda : H \to \mathbb{R}$, with continuous partial Fréchet derivatives $\Lambda'(x)$ and $\Lambda''(x)$ exists for $x \in H$. Let $C^2_{2p}(H)$, with $p \geq 1$, denote the subspace of $C^2(H)$ consisting of functions $\Lambda : H \to \mathbb{R}$ whose first two derivatives satisfy the following growth conditions:

$$\|\Lambda'(x)\|_H \leq C\|x\|_H^{2p} \quad \text{and} \quad \|\Lambda''(x)\|_{\mathcal{L}(H)} \leq C\|x\|_H^{2p}$$

for some constant $C \geq 0$.

Theorem 7. *Let us assume that F, B, f satisfy (A1), (A2), (A3) and the conditions (a), (b) of Theorem 6 hold. The mild solution of (2.25) is exponentially stable in the m.s.s. if there exists a function $\Lambda : H \to \mathbb{R}$ satisfying the following conditions:*
(I) $\Lambda \in C^2_{2p}(H)$.
(II) There exist finite constants c_1, $c_2 > 0$ such that; for all $x \in H$

$$c_1\|x\|_H^2 \leq \Lambda(x) \leq c_2\|x\|_H^2$$

(III) There exists a constants $c_3 > 0$ such that

$$\mathcal{L}\Lambda(x) \leq -c_3\Lambda(x) \quad \textit{for all} \quad x \in \mathcal{D}(A)$$

with $\mathcal{L}\Lambda(x)$ defined in (2.14).

Proof. The detailed proof is in Theorem 6.4 of [16], Theorem 4.2 of [24]. Here we show, how we use the the limiting argument of Theorem 6 and Corollary 1 in the proof.

First we will apply the Itô formula for strong solution to the function $e^{c_3 t}\Lambda(X_n^x(t))$. Where $X_n^x(t) \in \mathcal{D}(A)$ are the sequence of strong solutions which approximate the mild solution $X^x(t)$. Now after applying the Itô formula to the function $e^{c_3 t}\Lambda(X_n^x(t))$, we take expectations on both sides and obtain

$$e^{c_3 t} E\Lambda(X_n^x(t)) - \Lambda(X_n^x(0)) = E\int_0^t e^{c_3 s}\left(c_3\Lambda(X_n^x(s)) + \mathcal{L}_n\Lambda(X_n^x(s))\right) ds. \tag{2.27}$$

Now from condition (III),

$$c_3\Lambda(X_n^x(s)) + \mathcal{L}_n\Lambda(X_n^x(s)) \leq -\mathcal{L}\Lambda(X_n^x(s)) + \mathcal{L}_n\Lambda(X_n^x(s))$$

$$\Rightarrow e^{c_3 t} E\Lambda(X_n^x(t)) - \Lambda(X_n^x(0)) \leq E\int_0^t e^{c_3 s}\left(-\mathcal{L}\Lambda(X_n^x(s)) + \mathcal{L}_n\Lambda(X_n^x(s))\right) ds. \tag{2.28}$$

Now from (2.23) and (2.24)

$$-\mathcal{L}\Lambda(X_n^x(s)) + \mathcal{L}_n\Lambda(X_n^x(s)) \tag{2.29}$$

$$= \langle \Lambda'(X_n^x(s)), AX_n^x(s)\rangle - \lim_{n\to\infty}\langle \Lambda'(X_n^x(s)), AX_n^x(s)\rangle$$

$$+ \langle \Lambda'(X_n^x(s)), (R_n - I)F(X_n^x(s))\rangle_H$$

$$+\frac{1}{2} tr\left\{(\Lambda''(X_n^x(s))[(R_n B(X_n^x(s)))Q(R_n B(X_n^x(s)))^*) - (B(X_n^x(s)))Q(B(X_n^x(s)))^*)]\right\}$$

$$+ \int_{H\setminus\{0\}} [\Lambda(X_n^x(s) + R_n f(v, X_n^x(s))) - \Lambda(X_n^x(s) + f(v, X_n^x(s)))]\beta(dv)$$

$$+ \int_{H\setminus\{0\}} \langle \Lambda'(X_n^x(s)), (R_n - I)f(v, X_n^x(s))\rangle_H \,\beta(dv).$$

Now by Cauchy-Schwarz inequality and condition (II) we get,

$$\begin{aligned}|\langle \Lambda'(X_n^x(s)), (R_n - I)F(X_n^x(s))\rangle_H| &\leq \|\Lambda'(X_n^x(s))\| \cdot \|(R_n - I)F(X_n^x(s))\| \\ &\leq c_4 \|X_n^x(s)\| \cdot \|(R_n - I)F(X_n^x(s))\|.\end{aligned}$$

By (2.18), $\|R_n\|$ is uniformly bounded, $(R_n - I)x \to 0$ and Λ', F are continuous, hence we get by Theorem 5 and Lebesgue DCT

$$E\int_0^t e^{c_3 s}\langle \Lambda'(X_n^x(s)), (R_n - I)F(X_n^x(s))\rangle_H \, ds \to 0. \tag{2.30}$$

With the similar argument, we can conclude

$$E\int_0^t \int_{H\setminus\{0\}} e^{c_3 s}\left\langle \Lambda'(X_n^x(s)), (R_n - I)f(v, X_n^x(s))\right\rangle_H \beta(dv)ds \to 0. \tag{2.31}$$

Now

$$\begin{aligned} &tr\left\{(\Lambda''(X_n^x(s))(R_nB(X_n^x(s)))Q(R_nB(X_n^x(s)))^*)\right\} \\ &\quad \to tr\left\{(\Lambda''(X^x(s))(B(X^x(s)))Q(B(X^x(s)))^*)\right\} \end{aligned}$$

and

$$\begin{aligned} &tr\left\{(\Lambda''(X_n^x(s))(B(X_n^x(s)))Q(B(X_n^x(s)))^*)\right\} \\ &\to tr\left\{(\Lambda''(X^x(s))(B(X^x(s)))Q(B(X^x(s)))^*)\right\}. \end{aligned}$$

Again,

$$\begin{aligned} tr\left\{(\Lambda''(X_n^x(s))(R_nB(X_n^x(s)))Q(R_nB(X_n^x(s)))^*)\right\} &\le \|\Lambda''(X_n^x(s))\|\,\|R_nB(X_n^x(s))\|^2 \\ \text{[by (2.18), for some constant } c_5>0] \quad &\le c_5\|R_nB(X_n^x(s))\|^2 \\ \text{[by assumption on } B \text{, in (A3)]} \quad &\le c_5\|R_n\|^2\, l(1+\|X_n^x(s)\|^2). \end{aligned}$$

Similarly

$$tr\left\{(\Lambda''(X_n^x(s))(B(X_n^x(s)))Q(B(X_n^x(s)))^*)\right\} \le c_5 l(1+\|X_n^x(s)\|^2).$$

By (2.18) and $\|R_n\|^2$ is uniformly bounded, Λ'', B are continuous, hence we get by Theorem 5 and Lebesgue DCT

$$\begin{aligned} E\int_0^t e^{c_3 s}\frac{1}{2}tr\{(\Lambda''(X_n^x(s))[(R_nB(X_n^x(s)))Q(R_nB(X_n^x(s)))^*) \\ -(B(X_n^x(s)))Q(B(X_n^x(s)))^*)]\}ds \to 0. \end{aligned} \tag{2.32}$$

Now consider the term

$$\int_{H\setminus\{0\}} [\Lambda(X_n^x(s)+R_nf(v,X_n^x(s))) - \Lambda(X_n^x(s)+f(v,X_n^x(s)))]\beta(dv).$$

Now

$$\Lambda(X_n^x(s)+R_nf(v,X_n^x(s))) \to \Lambda(X^x(s)+f(v,X^x(s)))$$

and

$$\Lambda(X_n^x(s)+f(v,X_n^x(s))) \to \Lambda(X^x(s)+f(v,X^x(s))).$$

Again

$$\|\Lambda(X_n^x(s) + R_n f(v, X_n^x(s))) - \Lambda(X_n^x(s) + f(v, X_n^x(s)))\|$$

$$\leq \|(R_n - I)f(v, X_n^x(s))\| \sup_{0<\theta\leq 1} \|\Lambda'\{X_n^x(s) + \theta(R_n - I)f(v, X_n^x(s))\}\| .$$

By (2.18), $\|R_n\|$ is uniformly bounded, $(R_n - I)x \to 0$ and Λ, f are continuous, hence by Theorem 5 and Lebesgue DCT

$$E\int_0^t \int_{H\backslash\{0\}} e^{c_3 s}[\Lambda(X_n^x(s) + R_n f(v, X_n^x(s))) - \Lambda(X_n^x(s) + f(v, X_n^x(s)))]\beta(dv)ds \to 0. \tag{2.33}$$

And also

$$\lim_{n\to\infty} \langle \Lambda'(X_n^x(s)), AX_n^x(s)\rangle - \lim_{n\to\infty} \langle \Lambda'(X_n^x(s)), AX_n^x(s)\rangle \tag{2.34}$$
$$= 0.$$

Therefore from (2.30), (2.31), (2.32), (2.33) and (2.34) we can conclude as $n \to \infty$, the R.H.S. of (2.28) converges to 0. Hence by Lebesgue DCT and using the continuity of Λ, from (2.27) we obtain

$$e^{c_3 t} E\Lambda(X^x(t)) \leq \Lambda(x) \tag{2.35}$$

$$\Rightarrow c_1 E\|X^x(t)\|_H^2 \leq E\Lambda(X^x(t)) \leq e^{-c_3 t}\Lambda(x) \leq c_2 e^{-c_3 t}\|x\|_H^2 \quad \text{[from condition (ii)]}$$

$$\Rightarrow E\|X^x(t)\|_H^2 \leq \frac{c_2}{c_1} e^{-c_3 t}\|x\|_H^2 . \tag{2.36}$$

Since the mild solution $X^x(t)$ depends continuously on the initial condition x, therefore (2.36) holds for all $x \in H$. Now choosing $c = \frac{c_2}{c_1}$ and $\beta = c_3$, we can conclude that, the mild solution $X^x(t)$ is exponentially stable in the mean square sense (m.s.s.). □

Definition 14. *The function Λ satisfying conditions (I)-(III) of Theorem 7, is called a Lyapunov function.*

Now we consider the linear case of eq. (2.25) with $F(x) = 0$, $B(x) = B_0 x$ and $f(v, x) = f_0(v)x$. We consider the solution of the equation,

$$dX(t) = AX(t)dt + B_0 X(t)dW_t + \int_{H\backslash\{0\}} f_0(v)X(t)q(dv, dt); \tag{2.37}$$

$$X(0) = x \in H$$

Where $B_0 \in \mathcal{L}(H, \mathcal{L}(K, H))$, $f_0(v) : H\backslash\{0\} \to \mathbb{R}$, $\|B_0 x\| \le d_1 \|x\|_H$ and $\int_{H\backslash\{0\}} \|f_0(v)x\|_H^2 \beta(dv) \le d_2 \|x\|_H^2$ for $x \in H$.

Mild solutions are solutions of the corresponding integral equation

$$X(t) = S(t)x + \int_0^t S(t-s)B_0 X(s)dW_s + \int_0^t \int_{H\backslash\{0\}} S(t-s)f_0(v)X(s)q(dv, ds). \tag{2.38}$$

We can show that the existence of a Lyapunov function is a necessary condition for exponential stability in the m.s.s. of the mild solutions of (2.37). The following notation will be used:

$$\mathcal{L}_0\Psi(x) = \langle \Psi'(x), Ax\rangle_H + \frac{1}{2} tr(\Psi''(x)(B_0x)Q(B_0x)^*) \tag{2.39}$$

$$+ \int_{H\backslash\{0\}} [\Psi(x + f_0(v)x) - \Psi(x) - \langle \Psi'(x), f_0(v)x\rangle_H]\, \beta(dv)$$

for $x \in \mathcal{D}(A)$.

Theorem 8. *Let us assume that F, B, f satisfy (A1), (A2), (A3) and the conditions (a), (b) of Theorem 6 hold. Assume that A generates a pseudo-contraction semigroup of operators $\{S(t), t \ge 0\}$ on H and that the mild solution of (2.37) is exponentially stable in the m.s.s. Then there exists a function $\Lambda_0(x)$ satisfying conditions (I) and (II) of Theorem 7 and the condition and $\mathcal{L}_0\Lambda_0(x) \le -c_3\Lambda_0(x)$, $x \in \mathcal{D}(A)$, for some $c_3 > 0$.*

Proof. The detailed proof is in Theorem 6.5 of [16], Theorem 4.3 of [24]. Here we show, how we use the the limiting argument of Theorem 6 and Corollary 1 in the proof.

Let

$$\Lambda_0(x) = \int_0^\infty E\|X^x(t)\|_H^2 dt + \alpha\|x\|_H^2, \tag{2.40}$$

where the value of the constant $\alpha > 0$ will be determined later. Note that $X^x(t)$ depends on x linearly. The exponential stability in the m.s.s. implies that

$$\int_0^\infty E\|X^x(t)\|_H^2 dt < \infty.$$

Hence, by the Schwarz inequality,

$$T(x, y) = \int_0^\infty E\, \langle X^x(t), X^y(t)\rangle_H\, dt$$

defines a continuous bilinear form on $H \times H$, and there exists a symmetric bounded linear operator $\tilde{T} : H \to H$ such that

$$\left\langle \tilde{T}x, x \right\rangle_H = \int_0^\infty E\|X^x(t)\|_H^2 dt.$$

Let

$$\Psi(x) = \left\langle \tilde{T}x, x \right\rangle_H .$$

Using the same arguments, we define bounded linear operators on H by

$$\left\langle \tilde{T}(t)x, x \right\rangle_H = \int_0^t E\|X^x(s)\|_H^2 ds.$$

Consider solutions $\{X_n^x(t), t \geq 0\}$ to the following equation:

$$dX(t) = A_n X(t)dt + B_0 X(t)dW_t + \int_{H\setminus\{0\}} f_0(v)X(t)q(dv, dt),$$

$$X(0) = x \in H,$$

obtained by using the Yosida approximations of A. Just as above, we have continuous bilinear forms T_n, symmetric linear operators $\tilde{T}_n(t)$, and real-valued continuous functions $\Psi_n(t)$, defined for X_n,

$$T_n(t)(x, y) = \int_0^t E\left\langle X_n^x(u), X_n^y(u)\right\rangle_H du,$$

$$\left\langle \tilde{T}_n(t)x, x \right\rangle_H = \int_0^t E\|X_n^x(u)\|_H^2 du,$$

$$\Psi_n(t)(x) = \left\langle \tilde{T}_n(t)x, x \right\rangle_H = \int_0^t E\|X_n^x(u)\|_H^2 du. \tag{2.41}$$

Let $\{P_t\}_{t\geq 0}$ be the Markov semigroup associated with the stochastic process $X^x(t)$. Let $\varphi : H \to \mathbb{R}$; $\varphi(h) = \|h\|_H^2$ and $(P_t\varphi)(x) = E\varphi(X^x(t))$, $x \in H$. Using the Markov property we have,

$$E\Psi_n(t)(X_n^x(s)) = \Psi_n(t + s)(x) - \Psi_n(s)(x). \tag{2.42}$$

With t and n fixed, we use the Ito formula for the function $\Psi_n(t)(x)$, then take the expectation of both sides to arrive at

$$E(\Psi_n(t)(X_n^x(s))) = \Psi_n(t)(x) + \int_0^s E(\mathcal{L}_n\Psi_n(t)(X_n^x(u)))du. \tag{2.43}$$

Where

$$\mathcal{L}_n\Psi_n(t)(x) = 2\left\langle \tilde{T}_n(t)x, A_n x\right\rangle_H + tr(\tilde{T}_n(t)(B_0x)Q(B_0x)^*) \tag{2.44}$$

$$+\int_{H\setminus\{0\}}[\left\langle \tilde{T}_n(t)(x+f_0(v)x),(x+f_0(v)x)\right\rangle_H - \left\langle \tilde{T}_n(t)x,x\right\rangle_H - 2\left\langle \tilde{T}_n(t)x,f_0(v)x\right\rangle_H]\beta(dv).$$

From (2.42) and (2.43) we get,

$$\Psi_n(t+s)(x) - \Psi_n(s)(x) = \Psi_n(t)(x) + \int_0^s E(\mathcal{L}_n\Psi_n(t)(X_n^x(u)))du. \tag{2.45}$$

From (2.41), $\lim_{s\to 0}\frac{\Psi_n(s)(x)}{s} = \lim_{s\to 0}\frac{1}{s}\int_0^s E\|X_n^x(u)\|_H^2 du = \|x\|_H^2$. Hence from (2.45) we get,

$$\frac{d}{dt}\Psi_n(t)(x) = \mathcal{L}_n\Psi_n(t)(x) + \|x\|_H^2. \tag{2.46}$$

Now $x \in \mathcal{D}(A)$,

$$\mathcal{L}_n\Psi_n(t)(x) = 2\left\langle \tilde{T}_n(t)x, A_n x\right\rangle_H + tr(\tilde{T}_n(t)(B_0x)Q(B_0x)^*)$$

$$+\int_{H\setminus\{0\}}[\left\langle \tilde{T}_n(t)(x+f_0(v)x),(x+f_0(v)x)\right\rangle_H - \left\langle \tilde{T}_n(t)x,x\right\rangle_H - 2\left\langle \tilde{T}_n(t)x,f_0(v)x\right\rangle_H]\beta(dv).$$

This converges to

$$\mathcal{L}_0\Psi(t)(x) = 2\left\langle \tilde{T}(t)x, Ax\right\rangle_H + tr(\tilde{T}(t)(B_0x)Q(B_0x)^*)$$

$$+\int_{H\setminus\{0\}}[\left\langle \tilde{T}(t)(x+f_0(v)x),(x+f_0(v)x)\right\rangle_H - \left\langle \tilde{T}(t)x,x\right\rangle_H - 2\left\langle \tilde{T}(t)x,f_0(v)x\right\rangle_H]\beta(dv).$$

P-a.s. for $x \in \mathcal{D}(A)$ by the similar limiting argument as in Theorem 6.

Again from (2.41) we get,

$$\frac{d}{dt}\Psi_n(t)(x) = E\|X_n^x(t)\|_H^2 \to E\|X^x(t)\|_H^2 = \frac{d}{dt}\Psi(t)(x). \tag{2.47}$$

Therefore we can write when $x \in \mathcal{D}(A)$

$$\frac{d}{dt}\Psi(t)(x) = \mathcal{L}_0\Psi(t)(x) + \|x\|_H^2 . \tag{2.48}$$

$$\Rightarrow \frac{d}{dt}\left\langle \tilde{T}(t)x, x\right\rangle_H = \mathcal{L}_0\left\langle \tilde{T}(t)x, x\right\rangle_H + \|x\|_H^2 . \tag{2.49}$$

Now, when $t \to \infty$, then from the exponential stability condition we get

$$\frac{d}{dt}\Psi(t)(x) = E\|X^x(t)\|_H^2 \to 0 \tag{2.50}$$

and, since $\left\langle \tilde{T}(t)x, x\right\rangle_H \to \left\langle \tilde{T}x, x\right\rangle_H$ so by the weak convergence of $\tilde{T}(t)x$ to $\tilde{T}x$ we conclude

$$\mathcal{L}_0\left\langle \tilde{T}(t)x, x\right\rangle_H \to \mathcal{L}_0\left\langle \tilde{T}x, x\right\rangle_H = \mathcal{L}_0\Psi(x). \tag{2.51}$$

Hence from (2.49)

$$\mathcal{L}_0\Psi(x) = -\|x\|_H^2 ; \quad x \in \mathcal{D}(A). \tag{2.52}$$

Therefore by construction of Λ_0,

$$\begin{aligned}\Lambda_0(x) &= \Psi(x) + \alpha\|x\|^2 \\ &= \left\langle \tilde{T}x, x\right\rangle + \alpha\|x\|^2 \\ &\le \left\|\tilde{T}x\right\| \cdot \|x\| + \alpha\|x\|^2 \\ &\le \rho_1\|x\|^2 + \alpha\|x\|^2 . \qquad \text{[since, } \tilde{T} \text{ is bounded linear operator]}\end{aligned} \tag{2.53}$$

for some constatnt ρ_1. Therefore, we can conclude that $\Lambda_0 \in C_{2p}^2(H)$ satisfy conditions (I) and (II) of Lyapunov function.

Now $x \in \mathcal{D}(A)$

$$\begin{aligned}\mathcal{L}_0\|x\|_H^2 &= 2\langle x, Ax\rangle + tr((B_0x)Q(B_0x)^*) + \int_{H\setminus\{0\}} \|f_0(v)x\|^2\beta(dv) \\ &\le (2\lambda + d_1^2 tr(Q) + d_2)\|x\|_H^2 .\end{aligned} \tag{2.54}$$

Hence, $x \in \mathcal{D}(A)$

$$\mathcal{L}_0\Lambda_0(x) \le -\|x\|_H^2 + \alpha(2\lambda + d_1^2 tr(Q) + d_2)\|x\|_H^2 \le -c_3\Lambda_0(x)$$

$c_3 > 0$, by choosing α small enough.

Therefore Λ_0 satisfies all the properties of a Lyapunov function. □

Then we will show how we use the limiting argument of Theorem 6 and Corollary 1 to prove the results of exponentially ultimate boundedness in the m.s.s. of the mild solutions. These results are proved, for the Gaussian case in chapter 7 (Theorem 7.1 and Theorem 7.2) of [16] and for the non-Gaussian case in [24] (Theorem 5.2 and Theorem 5.5). But here we prove the same results in much shorter way.

Definition 15. *We say that the mild solution of (2.25) is exponentially ultimately bounded in the mean square sense (m.s.s.) if there exist positive constants* c, β, M *such that*

$$E\|X^x(t)\|_H^2 \le ce^{-\beta t}\|x\|_H^2 + M; \quad \text{for all } x \in H. \tag{2.55}$$

Theorem 9. *Let us assume that* F, B, f *satisfy (A1), (A2), (A3) and the conditions (a), (b) of Theorem 6 hold. The mild solution* $\{X^x(t), t \ge 0\}$ *of (2.25) is exponentially ultimately bounded in the m.s.s. if there exists a function* $\Lambda \in C_{2p}^2(H)$ *satisfying the following conditions:*

(i) $c_1\|x\|_H^2 - k_1 \le \Lambda(x) \le c_2\|x\|_H^2 - k_2$; *for all* $x \in H$

(ii) $\mathcal{L}\Lambda(x) \le -c_3\Lambda(x) + k_3$; *for* $x \in \mathcal{D}(A)$,

Where c_1, c_2, c_3, k_1, k_2 *and* k_3 *are finite, positive constants.*

Proof. The detailed proof is in Theorem 7.1 of [16], Theorem 5.2 of [24]. Here we show, how we use the the limiting argument of Theorem 6 and Corollary 1 in the proof.

Consider the function $e^{c_3t}\Lambda(X_n^x(t))$, then we write the Itô formula for strong solution for this function and taking expectation, we get

$$e^{c_3t}E\Lambda(X_n^x(t)) - \Lambda(X_n^x(0)) = E\int_0^t e^{c_3s}\left(c_3\Lambda(X_n^x(s)) + \mathcal{L}_n\Lambda(X_n^x(s))\right)ds. \tag{2.56}$$

Now from condition (ii) we have, when $X_n^x(s) \in \mathcal{D}(A)$

$$c_3\Lambda(X_n^x(s)) + \mathcal{L}_n\Lambda(X_n^x(s)) \le -\mathcal{L}\Lambda(X_n^x(s)) + k_3 + \mathcal{L}_n\Lambda(X_n^x(s)).$$

$$\Rightarrow e^{c_3t}E\Lambda(X_n^x(t)) - \Lambda(X_n^x(0)) \le E\int_0^t e^{c_3s}\left(-\mathcal{L}\Lambda(X_n^x(s)) + k_3 + \mathcal{L}_n\Lambda(X_n^x(s))\right)ds \tag{2.57}$$

$$= E\int_0^t e^{c_3s}\left(-\mathcal{L}\Lambda(X_n^x(s)) + \mathcal{L}_n\Lambda(X_n^x(s))\right)ds + \int_0^t e^{c_3s}k_3 ds.$$

Now by similar calculation as in Example 2 we can prove

$$E\int_0^t e^{c_3 s}\left(-\mathcal{L}\Lambda(X_n^x(s))+\mathcal{L}_n\Lambda(X_n^x(s))\right)ds\to 0.$$

Therefore as $n\to\infty$, from (2.57), using the continuity of Λ and Lebesgue DCT, we get

$$\begin{aligned} e^{c_3 t}E\Lambda(X^x(t)) &\le \Lambda(x)+\int_0^t e^{c_3 s}k_3 ds \qquad (2.58)\\ &= \Lambda(x)+\frac{k_3}{c_3}(e^{c_3 t}-1). \end{aligned}$$

$$\Rightarrow E\Lambda(X^x(t))\le e^{-c_3 t}\Lambda(x)+\frac{k_3}{c_3}(1-e^{-c_3 t}). \qquad (2.59)$$

Now from condition (i) and (2.59), for all $x\in H$

$$\begin{aligned} c_1 E\left\|X^x(t)\right\|_H^2-k_1 \le E\Lambda(X^x(t)) &\le e^{-c_3 t}\left(c_2\left\|x\right\|_H^2-k_2\right)+\frac{k_3}{c_3}(1-e^{-c_3 t}) \qquad (2.60)\\ &\le c_2 e^{-c_3 t}\left\|x\right\|_H^2+\frac{k_3}{c_3}(1-e^{-c_3 t}). \end{aligned}$$

$$\Rightarrow c_1 E\left\|X^x(t)\right\|_H^2\le c_2 e^{-c_3 t}\left\|x\right\|_H^2+\frac{k_3}{c_3}(1-e^{-c_3 t})+k_1,$$

$$\Rightarrow E\left\|X^x(t)\right\|_H^2\le \frac{c_2}{c_1}e^{-c_3 t}\left\|x\right\|_H^2+\frac{1}{c_1}\left(k_1+\frac{k_3}{c_3}\right).$$

Now choosing $c=\frac{c_2}{c_1}$, $\beta=c_3$ and $M=\frac{1}{c_1}\left(k_1+\frac{k_3}{c_3}\right)$ we can conclude that the mild solution $X^x(t)$ is exponentially ultimately bounded in the m.s.s.. Since the mild solution $X^x(t)$ depends continuously on the initial condition x, therefore (2.60) holds for all $x\in H$. □

Theorem 10. *Let us assume that F, B, f satisfy (A1), (A2), (A3) and the conditions (a), (b) of Theorem 6 hold. Assume that A generates a pseudo-contraction semigroup of operators $\{S(t), t\ge 0\}$ on H. If the solution of the linear equation (2.37) is exponentially ultimately bounded in the m.s.s., then there exists a function $\Lambda_0\in C^2_{2p}(H)$ which satisfies condition (i) of Theorem 9 and $\mathcal{L}_0\Lambda_0(x)\le -c_3\Lambda_0(x)+k_3$; for $x\in\mathcal{D}(A)$.*

Proof. The detailed proof is in Theorem 7.2 of [16], Theorem 5.5 of [24]. Here we show, how we use the the limiting argument of Theorem 6 and Corollary 1 in the proof.

Let

$$\Lambda_0(x) = \int_0^T E\|X^x(t)\|_H^2 dt + \alpha\|x\|_H^2, \tag{2.61}$$

where T and α are positive constants, to be determined later. Let

$$\Psi_0(x) = \int_0^T E\|X^x(t)\|_H^2 dt, \tag{2.62}$$

which is finite for $T < \infty$ by the exponential ultimate boundedness in m.s.s.. Infact we can show

$$\begin{aligned}\Psi_0(x) &\leq \int_0^T (ce^{-\beta t}\|x\|_H^2 + M)dt \\ &= \frac{c}{\beta}(1 - e^{-\beta T})\|x\|_H^2 + MT \\ &\leq \frac{c}{\beta}\|x\|_H^2 + MT.\end{aligned} \tag{2.63}$$

If $\|x\|_H^2 = 1$, then $\Psi_0(x) \leq \frac{c}{\beta} + MT$.

Since $X^x(t)$ is linear in x (i.e. $X^{kx}(t) = kX^x(t)$, for any positive constant k), therefore

$$\Psi_0(kx) = \int_0^T E\|X^{kx}(t)\|_H^2 dt = k^2\int_0^T E\|X^x(t)\|_H^2 dt = k^2\Psi_0(x). \tag{2.64}$$

Let $c' = \frac{c}{\beta} + MT$, then for all $x \in H$,

$$\Psi_0(x) = \|x\|_H^2\,\Psi_0\left(\frac{x}{\|x\|_H}\right) \leq \left(\frac{c}{\beta} + MT\right)\|x\|_H^2 = c'\|x\|_H^2. \tag{2.65}$$

Then there exists a bounded linear form on $H \times H$. Hence there exists a symmetric operator C, $\|C\|_{\mathcal{L}(H)} \leq c'$ such that, for all x, $y \in H$

$$\langle Cx, y\rangle = \int_0^T E\langle X^x(t), X^y(t)\rangle_H\, dt.$$

Therefore $\Psi_0(x) = \langle Cx, x\rangle$. Hence $\Psi_0'(x) = 2Cx$ and $\Psi_0''(x) = 2C$. So, by construction

$$\Lambda_0(x) = \Psi_0(x) + \alpha\|x\|_H^2 = \langle Cx, x\rangle + \alpha\|x\|_H^2 \leq c'\|x\|_H^2 + \alpha\|x\|_H^2, \tag{2.66}$$

So, we can conclude that $\Lambda_0 \in C_{2p}^2(H)$ satisfies condition (i). To prove the second part, we can use our limiting argument of Theorem 6, to show that $\lim_{n\to\infty}\mathcal{L}_{0,n}\Psi_0^n(x) = \mathcal{L}_0\Psi_0(x)$ for $x \in \mathcal{D}(A)$. Where $\Psi_0^n(x) = \int_0^T E\|X_n^x(t)\|_H^2 dt$.

From the Markov property,

$$E\Psi_0^n(X_n^x(r)) = \int_r^{T+r} E\|X_n^x(u)\|_H^2 du. \tag{2.67}$$

And we have

$$\begin{aligned}
\mathcal{L}_{0,n}\Psi_0^n(x) &= \frac{d}{dr}\left(E\Psi_0^n(X_n^x(r))\right)\Big|_{r=0} \qquad (2.68)\\
&= \lim_{r\to 0}\frac{E\Psi_0^n(X_n^x(r)) - E\Psi_0^n(x)}{r}\\
&= \lim_{r\to 0}\left\{\frac{1}{r}\int_r^{T+r} E\|X_n^x(u)\|_H^2 du - \frac{1}{r}\int_0^T E\|X_n^x(u)\|_H^2 du\right\} \quad [\text{by } (2.67)]\\
&= \lim_{r\to 0}\left\{-\frac{1}{r}\int_0^r E\|X_n^x(u)\|_H^2 du + \frac{1}{r}\int_0^{T+r} E\|X_n^x(u)\|_H^2 du - \frac{1}{r}\int_0^T E\|X_n^x(u)\|_H^2 du\right\}\\
&= \lim_{r\to 0}\left\{-\frac{1}{r}\int_0^r E\|X_n^x(u)\|_H^2 du + \frac{1}{r}\int_T^{T+r} E\|X_n^x(u)\|_H^2 du\right\}\\
&= -\|x\|_H^2 + E\|X_n^x(T)\|_H^2.
\end{aligned}$$

Therefore

$$\begin{aligned}
\mathcal{L}_0\Psi_0(x) &= -\|x\|_H^2 + E\|X^x(T)\|_H^2 \qquad (2.69)\\
&\le -\|x\|_H^2 + ce^{-\beta T}\|x\|_H^2 + M \qquad [\text{by } (2.55)]\\
&= (-1 + ce^{-\beta T})\|x\|_H^2 + M.
\end{aligned}$$

Again from (2.54), we have $\mathcal{L}_0\|x\|_H^2 \le (2\lambda + d_1^2 tr(Q) + d_2)\|x\|_H^2$. Hence for $x \in \mathcal{D}(A)$

$$\begin{aligned}
\mathcal{L}_0\Lambda_0(x) &= \mathcal{L}_0\Psi_0(x) + \alpha\mathcal{L}_0\|x\|_H^2 \qquad (2.70)\\
&\le (-1 + ce^{-\beta T})\|x\|_H^2 + \alpha(2\lambda + d_1^2 tr(Q) + d_2)\|x\|_H^2 + M.
\end{aligned}$$

Now taking $T > \frac{\ln c}{\beta}$ and α small enough, we get the desired result. □

2.4 Ichikawa's Itô formula for the mild solutions

In this section we prove that Ichikawa's Itô formula for the mild solutions obtained by Ichikawa for SPDE driven by Gaussian noise in [19] can also be generalized to the case of SPDE driven by non-Gaussian noise. Let A be the generator of a pseudo-contraction semigroup, then for $x \in \mathcal{D}(A)$, let us define

$$\begin{aligned}
\mathcal{L}\Psi(s,x) &:= \langle\Psi_x(s,x), Ax + F(x)\rangle_H + \frac{1}{2}tr(\Psi_{xx}(s,x)(B(x))Q(B(x))^*)\\
&\quad + \int_{H\setminus\{0\}} \left[\Psi(s,x+f(v,x)) - \Psi(s,x) - \langle\Psi_x(s,x), f(v,x)\rangle_H\right]\beta(dv).
\end{aligned}$$

Let $\Psi \in C^{1,2}([0,T] \times H)$, $\Psi : [0,T] \times H \to \mathbb{R}$ and conditions (a), (b) of Theorem 6 hold. Then for $X^{\xi}(t) \in \mathcal{D}(A)$, the Itô formula is well defined:

$$\Psi(t, X^{\xi}(t)) - \Psi(0, \xi) = \int_0^t (\Psi_s(s, X^{\xi}(s)) + \mathcal{L}\Psi(s, X^{\xi}(s)))ds$$

$$+ \int_0^t \left\langle \Psi_x(s, X^{\xi}(s)), B(X^{\xi}(s))dW_s \right\rangle_H$$

$$+ \int_0^t \int_{H\setminus\{0\}} \left[\Psi(s, X^{\xi}(s) + f(v, X^{\xi}(s))) - \Psi(s, X^{\xi}(s))\right] q(dv, ds),$$

where,

$$\mathcal{L}\Psi(s, X^{\xi}(s)) = \left\langle \Psi_x(s, X^{\xi}(s)), AX^{\xi}(s) + F(X^{\xi}(s)) \right\rangle_H \tag{2.71}$$

$$+\frac{1}{2} tr(\Psi_{xx}(s, X^{\xi}(s))(B(X^{\xi}(s)))Q(B(X^{\xi}(s)))^*)$$

$$+ \int_{H\setminus\{0\}} \left[\Psi(s, X^{\xi}(s) + f(v, X^{\xi}(s))) - \Psi(s, X^{\xi}(s)) - \left\langle \Psi_x(s, X^{\xi}(s)), f(v, X^{\xi}(s)) \right\rangle_H\right] \beta(dv).$$

Let $\tilde{C}^{1,2}([0,T] \times H)$ be the class of functions $\Psi \in C^{1,2}([0,T] \times H)$ with the properties:

(I1) The function $\mathcal{L}\Psi(s,x)$ can be extended to a continuous function $\overline{\mathcal{L}\Psi}(s,x)$ on $[0,T] \times H$ for $x \in H$.

(I2) $\|\Psi(s,x)\| + \|\Psi_x(s,x)\| + \|\Psi_{xx}(s,x)\| + \|\overline{\mathcal{L}\Psi}(s,x)\| \leq k(1 + \|x\|^2)$, for $x \in H$, $s \in [0,T]$ and for some $k > 0$.

Since the function $\mathcal{L}\Psi(s,x)$ can be extended to a continuous function $\overline{\mathcal{L}\Psi}(s,x)$ in $\tilde{C}^{1,2}([0,T] \times H)$, therefore it follows $\overline{\mathcal{L}\Psi}(s, X^{\xi}(s)) = \lim_{n\to\infty} \mathcal{L}_n\Psi(s, X_n^{\xi}(s))$, where $\mathcal{L}_n\Psi(s, X_n^{\xi}(s))$ is defined in (2.14).

2.4.1 Ichikawa's Itô formula

Theorem 11. *Assume that $S(t)$ is a pseudo-contraction semigroup and $\Psi \in \tilde{C}^{1,2}([0,T] \times H)$. Moreover assume that the following conditions are satisfied:*
(a)

$$\|\Psi_x(s,x)\|_H \leq h_1(\|x\|_H)$$

and

$$\|\Psi_{xx}(s,x)\|_{\mathcal{L}(H)} \leq h_2(\|x\|_H).$$

(b)

$$\int_0^T \|F(s)\|_H ds < \infty \quad P\text{-a.s.}, \quad P\Big\{\int_0^T \|B(s)\|^2_{\mathcal{L}_2(K_Q,H)} ds < \infty\Big\} = 1$$

and let $h_1, h_2 : \mathbb{R}_+ \to \mathbb{R}_+$ be quasi-sublinear functions such that,

$$\int_{H\setminus\{0\}} \|f(v,s)\|^2 \beta(dv) + \int_{H\setminus\{0\}} h_1(\|f(v,s)\|)^2 \|f(v,s)\|^2 \beta(dv) + \int_{H\setminus\{0\}} h_2(\|f(v,s)\|) \|f(v,s)\|^2 \beta(dv) < \infty.$$

P-a.s. for all $s \in [0,T]$. Let the coefficients F, B, f satisfy (A1), (A2), (A3). Then the following Itô Formula for mild solutions hold P-a.s. for all $t \in [0,T]$

$$\Psi(t, X^\xi(t)) - \Psi(0,\xi) = \int_0^t (\Psi_s(s, X^\xi(s)) + \overline{\mathcal{L}\Psi}(s, X^\xi(s)))ds \tag{2.72}$$

$$+ \int_0^t \big\langle \Psi_x(s, X^\xi(s)), B(X^\xi(s))dW_s \big\rangle_H$$

$$+ \int_0^t \int_{H\setminus\{0\}} \big[\Psi(s, X^\xi(s) + f(v, X^\xi(s))) - \Psi(s, X^\xi(s))\big] q(dv, ds).$$

Proof. We proved this result in [1].

Here we assumed that $\Psi \in \tilde{C}^{1,2}([0,T] \times H)$. So, it satisfied the conditions (I1) and (I2). From (I1) the continuous extension of $\mathcal{L}\Psi(s,x)$ exists in $[0,T]\times H$, which is $\overline{\mathcal{L}\Psi}(s,x)$. Therefore we can write,

$$\begin{aligned}\overline{\mathcal{L}\Psi}(s, X^\xi(s)) &= \lim_{n\to\infty} \mathcal{L}_n\Psi(s, X_n^\xi(s)) \\ &= \big\langle \Psi_x(s, X^\xi(s)), F(X^\xi(s))\big\rangle_H + \lim_{n\to\infty} \big\langle \Psi_x(s, X_n^\xi(s)), AX_n^\xi(s)\big\rangle_H \\ &+ \frac{1}{2} tr(\Psi_{xx}(s, X^\xi(s))(B(X^\xi(s)))Q(B(X^\xi(s)))^*)\end{aligned}$$

$$+ \int_{H\setminus\{0\}} \big[\Psi(s, X^\xi(s) + f(v, X^\xi(s))) - \Psi(s, X^\xi(s)) - \big\langle \Psi_x(s, X^\xi(s)), f(v, X^\xi(s))\big\rangle_H\big] \beta(dv)$$

P-a.s., where $X_n^\xi(s) \in \mathcal{D}(A)$ and $X^\xi(s) \in H$. $\mathcal{L}_n\Psi(s, X_n^\xi(s))$ is defined in (2.14). By (I2) $\overline{\mathcal{L}\Psi}(s, X^\xi(s))$ is bounded by integrable function, so by applying Lebesgue DCT, we can conclude that, $\int_0^t \mathcal{L}_n\Psi(s, X_n^\xi(s))ds \to \int_0^t \overline{\mathcal{L}\Psi}(s, X^\xi(s))ds$. We can show the term by term convergence of all terms similarly as in Theorem 6. Hence we can conclude the Itô Formula of (2.72). □

Remark 2: From Theorem 11 we can remark that whenever $\Psi \in \tilde{C}^{1,2}([0,T]\times H)$ i.e. $\overline{\mathcal{L}\Psi}(s,x)$ exists and conditions of Theorem 11 are satisfied, then we can interchange the limit with the integral in the Itô formula for mild solutions of Theorem 6. Then the Itô formula for mild solutions of Theorem 6 (eq. 2.16) can be rewritten as follows-

$$\int_0^t \lim_{n\to\infty} \left\langle \Psi_x(s, X_n^\xi(s)), AX_n^\xi(s) \right\rangle_H ds \tag{2.73}$$
$$= \Psi(t, X^\xi(t)) - \Psi(0,\xi) - \int_0^t (\Psi_s(s, X^\xi(s)))ds - \int_0^t \left\langle \Psi_x(s, X^\xi(s)), F(X^\xi(s)) \right\rangle_H ds$$
$$- \int_0^t \frac{1}{2} tr(\Psi_{xx}(s, X^\xi(s))(B(X^\xi(s)))Q(B(X^\xi(s)))^*)ds$$
$$- \int_0^t \int_{H\setminus\{0\}} \left[\Psi(s, X^\xi(s) + f(v, X^\xi(s))) - \Psi(s, X^\xi(s)) - \left\langle \Psi_x(s, X^\xi(s)), f(v, X^\xi(s)) \right\rangle_H\right] \beta(dv)ds$$
$$- \int_0^t \left\langle \Psi_x(s, X^\xi(s)), B(X^\xi(s))dW_s \right\rangle_H$$
$$- \int_0^t \int_{H\setminus\{0\}} \left[\Psi(s, X^\xi(s) + f(v, X^\xi(s))) - \Psi(s, X^\xi(s))\right] q(dv, ds).$$

2.4.2 Relating semigroup with the generator of the solution process

Now we will relate the semigroup associated with the stochastic process with the operator $\mathcal{L}$. Let A be the generator of a pseudo-contraction semigroup, then for $x \in \mathcal{D}(A)$, let us define

$$\mathcal{L}\Psi(x) := \langle \Psi_x(x), Ax + F(x) \rangle_H + \frac{1}{2} tr(\Psi_{xx}(x)(B(x))Q(B(x))^*)$$
$$+ \int_{H\setminus\{0\}} [\Psi(x + f(v,x)) - \Psi(x) - \langle \Psi_x(x), f(v,x) \rangle_H] \beta(dv).$$

Here we put an additional restriction that the coefficients F, B and f are independent of time t, depend only on $x \in H$. The mild solution $X^x(t) \in H$ of (2.11) satisfies the Markov property, where $X^x(t) = X(t, 0; x)$ with deterministic initial condition $x \in H$ (see, section 3.4 of [16] and section 6 of [2]). Let us denote the semigroup associated with the process $X^x(t)$ by P_t, where P_t is the bounded linear operator on H. We will relate P_t with $\mathcal{L}$. We also assume that $\Psi \in C_b^2(H)$, the space of bounded continuous functions on H, with continuous bounded partial Fréchet derivatives of $\Psi_x(x)$ and $\Psi_{xx}(x)$ exists for $x \in H$. Since $\Psi \in C_b^2(H)$ hence $P_t(\Psi) \in C_b^2(H)$. We define for a bounded measurable function Ψ on H,

$$[P_t\Psi](x) = E[\Psi(X^x(t))]$$

for $x \in H$ (for more detailed discussion we refer to section 3.4 of [16] and section 6 of [2]).

Theorem 12. *Assume that the solution of SPDE* (2.7)*,* $X^x(t) \in \mathcal{D}(A)$ *and* P_t *be the semigroup associated to* $X^x(t)$*. Let* F*,* B *and* f *be independent of time* t *and satisfy (A1), (A2) and (A3). Assume that* $\Psi \in C_b^2(H)$ *and the condition (b) of Theorem 11 holds. Then for* $x \in \mathcal{D}(A)$

$$[P_t\Psi](x) - \Psi(x) = \int_0^t [P_s\mathcal{L}\Psi](x)ds \tag{2.74}$$

and

$$\lim_{t\downarrow 0} \frac{[P_t\Psi](x) - \Psi(x)}{t} = [\mathcal{L}\Psi](x). \tag{2.75}$$

Proof. We refer sections 4.1, 4.2 of [13] or sections 3.2, 3.3 of [4] for related theory.

First we rewrite (2.13) when it is independent of t and $X^x(t) \in \mathcal{D}(A)$,

$$\Psi(X^x(t)) - \Psi(x) = \int_0^t \mathcal{L}\Psi(X^x(s))ds \tag{2.76}$$

$$+ \int_0^t \langle \Psi_x(X^x(s)), B(X^x(s))dW_s\rangle_H$$

$$+ \int_0^t \int_{H\setminus\{0\}} \left[\Psi(X^x(s) + f(v, X^x(s))) - \Psi(X^x(s))\right] q(dv, ds).$$

Now take the expectation on both sides of (2.76). Second and third term of R.H.S. of (2.76) will be zero, because of martingale and we get,

$$E[\Psi(X^x(t))] - E[\Psi(x)] = \int_0^t E[\mathcal{L}\Psi(X^x(s))]ds.$$

Then we substitute $[P_t\Psi](x) = E[\Psi(X^x(t))]$, and we get

$$[P_t\Psi](x) - \Psi(x) = \int_0^t [P_s\mathcal{L}\Psi](x)ds.$$

Again to prove (2.75), we can rewrite (2.76) as,

$$\begin{aligned} d\Psi(X^x(t)) = &\left\langle \frac{d\Psi(X^x(t))}{dx}, AX^x(t) + F(X^x(t)) \right\rangle_H dt \\ &+ \frac{1}{2} tr\left(\frac{d^2\Psi(X^x(t))}{dx^2} (B(X^x(t))Q^{1/2})(B(X^x(t))Q^{1/2})^* \right) dt \end{aligned}$$

$$+ \int_{H\setminus\{0\}} \left[\Psi(X^x(t) + f(v, X^x(t))) - \Psi(X^x(t)) - \left\langle \frac{d\Psi(X^x(t))}{dx}, f(v, X^x(t)) \right\rangle_H \right] \beta(dv)dt$$

$$+ \left\langle \frac{d\Psi(X^x(t))}{dx}, B(X^x(t))dW_t \right\rangle_H$$

$$+\int_{H\setminus\{0\}} [\Psi(X^x(t) + f(v, X^x(t))) - \Psi(X^x(t))]\, q(dv, dt).$$

Now, since $[P_t\Psi](x) = E[\Psi(X^x(t))]$, so by Lebesgue DCT

$$\lim_{t\downarrow 0} \frac{[P_t\Psi](x) - \Psi(x)}{t}$$

$$= E \lim_{t\downarrow 0} \frac{1}{t} \int_0^t \left\langle \frac{d\Psi(X^x(s))}{dx}, AX^x(s) + F(X^x(s)) \right\rangle_H ds$$

$$+\frac{1}{2} E \lim_{t\downarrow 0} \frac{1}{t} \int_0^t tr\left(\frac{d^2\Psi(X^x(s))}{dx^2} (B(X^x(s))Q^{1/2})(B(X^x(s))Q^{1/2})^* \right) ds$$

$$+E \lim_{t\downarrow 0} \frac{1}{t} \int_0^t \int_{H\setminus\{0\}} \left[\Psi(X^x(s) + f(v, X^x(s))) - \Psi(X^x(s)) - \left\langle \frac{d\Psi(X^x(s))}{dx}, f(v, X^x(s)) \right\rangle_H \right] \beta(dv)ds$$

$$= \left\langle \frac{d\Psi(x)}{dx}, Ax + F(x) \right\rangle_H + \frac{1}{2} tr\left(\frac{d^2\Psi(x)}{dx^2} (B(x)Q^{1/2})(B(x)Q^{1/2})^* \right)$$

$$+\int_{H\setminus\{0\}} \left[\Psi(x + f(v, x)) - \Psi(x) - \left\langle \frac{d\Psi(x)}{dx}, f(v, x) \right\rangle_H \right] \beta(dv)$$

$$= [\mathcal{L}\Psi](x).$$

Hence the proof. □

Now we consider also the case when $X^x(t) \notin \mathcal{D}(A)$. Ψ is independent of time t and $\Psi \in \tilde{C}^2(H) \cap C_b^2(H)$. Where $C^2(H)$ denote the space of real valued continuous functions on H, with continuous partial Fréchet derivatives $\Psi_x(x)$ and $\Psi_{xx}(x)$ are well defined for $x \in H$. Let $\tilde{C}^2(H)$ be the class of functions $\Psi \in C^2(H)$ with the properties:

($\tilde{1}$) The function $\mathcal{L}\Psi(x)$ can be extended to a continuous function $\overline{\mathcal{L}\Psi}(x)$ on H for $x \in H$.

($\tilde{2}$) $\|\Psi(x)\| + \|\Psi_x(x)\| + \|\Psi_{xx}(x)\| + \|\overline{\mathcal{L}\Psi}(x)\| \leq k(1 + \|x\|^2)$, for $x \in H$ and for some $k > 0$.

Corollary 2. *Let P_t be the semigroup associated to $X^x(t)$. Let F, B and f be independent of time t and satisfy (A1), (A2) and (A3). Assume that $\Psi \in \tilde{C}^2(H) \cap C_b^2(H)$ is independent of t and the condition (b) of Theorem 11 holds. Moreover assume that $\overline{\mathcal{L}\Psi}$ satisfies the condition*

$$\|\overline{\mathcal{L}\Psi}(y) - \overline{\mathcal{L}\Psi}(z)\| \leq k\|y - z\|^2(\|y\|^2 + \|z\|^2) \tag{2.77}$$

for some $k > 0$. Then for $x \in H$ we can write

$$[P_t\Psi](x) - \Psi(x) = \int_0^t [P_s\overline{\mathcal{L}\Psi}](x)ds \tag{2.78}$$

and

$$\lim_{t\downarrow 0} \frac{[P_t\Psi](x) - \Psi(x)}{t} = [\overline{\mathcal{L}\Psi}](x). \tag{2.79}$$

Proof. We proved this result in [1].

First we rewrite (2.72), when it is independent of t, as

$$\Psi(X^x(t)) - \Psi(x) = \int_0^t \overline{\mathcal{L}\Psi}(X^x(s))ds \tag{2.80}$$

$$+ \int_0^t \langle \Psi_x(X^x(s)), B(X^x(s))dW_s\rangle_H$$

$$+ \int_0^t \int_{H\setminus\{0\}} [\Psi(X^x(s) + f(v, X^x(s))) - \Psi(X^x(s))]\, q(dv, ds).$$

Then we take the expectation on both sides of (2.80) and obtain (2.78) similarly as in the proof of Theorem 12.

The condition (2.77) assures the continuity of $P_s\overline{\mathcal{L}\Psi}$ in s. Hence we can write

$$\lim_{t\downarrow 0} \frac{[P_t\Psi](x) - \Psi(x)}{t} = E \lim_{t\downarrow 0} \frac{1}{t} \int_0^t \overline{\mathcal{L}\Psi}(X^x(s))ds = [\overline{\mathcal{L}\Psi}](x).$$

Hence the proof is completed. □

Now $x \in H$, $\Psi \in \tilde{C}^2(H) \cap C_b^2(H)$ satisfying the conditions of Corollary 2, we define

$$[\mathcal{A}\Psi](x) := [\overline{\mathcal{L}\Psi}](x). \tag{2.81}$$

Where $\mathcal{A}$ is defined to be the weak generator of the Markov process $X^x(t)$ (to show that the mild solution $X^x(t)$ is a Markov process we refer to Section 3.4 of [16] and Section 6 of [2]). The existence of $\overline{\mathcal{L}\Psi}$ is rather restrictive. So, we introduce a class of functions which is larger than $\tilde{C}^2(H)$, where continuous extension of the function $\mathcal{L}\Psi$ is not required. We say that $\Psi \in V$ if $\Psi \in \tilde{C}^2(H)$ and satisfies:

(i) The function $\mathcal{L}\Psi(x) \leq \mathcal{U}(x)$, for $x \in \mathcal{D}(A)$, where $\mathcal{U}(x)$ is a continuous function on H,

(ii) $\|\mathcal{U}(x)\| + \|\Psi(x)\| + \|\Psi_x(x)\| + \|\Psi_{xx}(x)\| \leq k(1 + \|x\|^2)$ for some $k > 0$.

Corollary 3. *Assume that F, B, f satisfy (A1), (A2), (A3) and conditions (a), (b) of Theorem 11 hold. Let $\mathcal{L}_n\Psi(x) \leq \mathcal{U}(x)$. Then $\mathcal{L}\Psi(x) \leq \mathcal{U}(x)$ for $x \in \mathcal{D}(A)$. Where $\mathcal{L}_n$ and $\mathcal{L}$ are defined as in Definition 12.*

Proof. We proved this result in [1].

For $x \in \mathcal{D}(A)$,

$$\begin{aligned}\mathcal{L}_n\Psi(x) =& \langle \Psi_x(x), Ax + R_nF(x)\rangle + \frac{1}{2}tr(\Psi_{xx}(x)(R_nB(x))Q(R_nB(x))^*)\\ &+ \int_{H\setminus\{0\}} [\Psi(x + R_nf(v,x)) - \Psi(x) - \langle \Psi_x(x), R_nf(v,x)\rangle]\beta(dv)\\ &\leq \mathcal{U}(x).\end{aligned}$$

Now taking limit $n \to \infty$ of the both sides of the inequality, we get

$$\begin{aligned}\mathcal{L}\Psi(x) =& \langle \Psi_x(x), Ax + F(x)\rangle + \frac{1}{2}tr(\Psi_{xx}(x)(B(x))Q(B(x))^*)\\ &+ \int_{H\setminus\{0\}} [\Psi(x + f(v,x)) - \Psi(x) - \langle \Psi_x(x), f(v,x)\rangle]\beta(dv)\\ &\leq \mathcal{U}(x). \qquad [\text{ by Yosida approximation } (R_n - I)x \to 0]\end{aligned}$$

Hence the result. □

Corollary 4. *Let $\Psi(x) \in V$ with properties:*
(i) The function $\mathcal{L}\Psi(x) \leq \mathcal{U}(x)$, for $x \in \mathcal{D}(A)$, where $\mathcal{U}(x)$ is a continuous function on H;
(ii) $\|\mathcal{U}(x)\| + \|\Psi(x)\| + \|\Psi_x(x)\| + \|\Psi_{xx}(x)\| \leq k(1 + \|x\|^2)$ for some $k > 0$.
Moreover assume that F, B, f satisfy (A1), (A2), (A3) and conditions (a), (b) of Theorem 11 hold. Then

$$\Psi(X^x(t)) - \Psi(x) \leq \int_0^t \mathcal{U}(X^x(s))ds \tag{2.82}$$

$$+ \int_0^t \langle \Psi_x(X^x(s)), B(X^x(s))dW_s\rangle_H$$

$$+ \int_0^t \int_{H\setminus\{0\}} [\Psi(X^x(s) + f(v, X^x(s))) - \Psi(X^x(s))]\, q(dv, ds).$$

If, in particular, $\mathcal{U}(x) = 0$, then $\Psi(X^x(t))$ is a supermartingale.

Proof. We proved this result in [1].

From (i) we have $\mathcal{L}_n\Psi(X_n^x(s)) \leq \mathcal{U}(X_n^x(s))$, when $X_n^x(s) \in \mathcal{D}(A)$. By Theorem 6, we have $\lim_{n\to\infty} \mathcal{L}_n\Psi(X_n^x(s))$ exists P-a.s. and $\mathcal{U}(X_n^x(s)) \to \mathcal{U}(X^x(s))$ P-a.s.. Therefore $\lim_{n\to\infty} \mathcal{L}_n\Psi(X_n^x(s)) \leq \mathcal{U}(X^x(s))$. Now we have already proved in Theorem 5

$$\lim_{n\to\infty} E \sup_{0\le t\le T} \|X_n^x(t) - X^x(t)\|_H^2 = 0.$$

Therefore,

$$\sup_{0\le t\le T} \|X_n(t) - X(t)\|_H \to 0,$$

P a.s.. This implies that the set

$$S = \{X_n(t),\ X(t):\ n = 1, 2..., 0 \le t \le T\}$$

is bounded in H. Therefore continuous function $\mathcal{U}$ evaluated on S is bounded by some constant. Hence $\int_0^t \mathcal{U}(X^x(s))ds$ exists. Hence

$$\int_0^t \lim_{n\to\infty} \mathcal{L}_n \Psi(X_n^x(s))ds \le \int_0^t \mathcal{U}(X^x(s))ds.$$

Therefore we can conclude (2.82).

For the second part, we take the conditional expectation in both sides of (2.82), then we get

$$E[\Psi(X^x(t))|\mathcal{F}_0^X] - E[\Psi(x)|\mathcal{F}_0^X] \le 0,$$

since $\mathcal{U}(x) = 0$ and terms containing the Gaussian and non-Gaussian noise in the R.H.S. of (2.82) are martingales.

$$\Rightarrow E[\Psi(X^x(t))|\mathcal{F}_0^X] \le \Psi(x).$$

Therefore a supermartingale. □

2.5 [Da Prato, Jentzen, Röckner]'s; mild Itô formula w.r.t. cPrm

Here we obtain the mild Itô formula of [8] for Lévy noise. In [8], they did it for the Gaussian case. They transformed the mild Itô process to a standard Itô process and then they apply standard Itô formula over this transformed standard Itô process. At the end by relating this transformed standard Itô process with the original mild Itô process with a suitable relation, they obtained their mild Itô formula.

We consider the SPDE with values in H as

$$dX(t) = (AX(t) + F(X(t)))dt + \int_{H\setminus\{0\}} f(v, X(t))q(dv, dt);$$

with initial condition $X(0) \in H$. Where H is a real separable Hilbert space and conditions (A1), (A2), (A3) are satisfied (but without Gaussian term). Then from definition 9, the milld solutions are defined as

$$X(t) = S(t)X(0) + \int_0^t S(t-s)F(X(s))ds + \int_0^t \int_{H\setminus\{0\}} S(t-s)f(v,X(s))q(dv,ds)$$

with probability one for $t \in [0,T]$. Then we have the following mild Itô formula.

Theorem 13. *Assume that $\varphi \in C^{1,2}([0,T]\times H)$, $\varphi : [0,T]\times H \to \mathbb{R}$. $S(t)$ is the pseudo-contraction semigroup. Also assume that the conditions (a) and (b) of Theorem 6 hold. Where H is a real separable Hilbert space. Then the following mild Itô formula holds*

$$\varphi(t,X(t)) = \varphi(S(t)X(0)) + \int_0^t (\partial_1\varphi)(s, S(t-s)X(s))ds \qquad (2.83)$$
$$+ \int_0^t (\partial_2\varphi)(s,S(t-s)X(s))S(t-s)F(X(s))ds$$

$$+ \int_0^t \int_{H\setminus\{0\}} [\varphi(s, S(t-s)X(s) + S(t-s)f(v,X(s))) - \varphi(s,S(t-s)X(s))$$
$$- \langle(\partial_2\varphi)(s,S(t-s)X(s)), S(t-s)f(v,X(s))\rangle]\beta(dv)ds$$

$$+ \int_0^t \int_{H\setminus\{0\}} [\varphi(s, S(t-s)X(s) + S(t-s)f(v,X(s))) - \varphi(s,S(t-s)X(s))]\, q(dv,ds)$$

P-a.s. for all $t \in [0,T]$. Here $(\partial_1\varphi)(t,x) = (\frac{\partial\varphi}{\partial t})(t,x)$ and $(\partial_2\varphi)(t,x) = (\frac{\partial\varphi}{\partial x})(t,x)$. $(\partial_1\varphi) \in C([0,T]\times H, \mathbb{R})$ and $(\partial_2\varphi) \in C([0,T]\times H, \mathcal{L}(H,\mathbb{R}))$.

Proof. We proved this result in [1].

Here we use the transformation technique, given in [14]. For existence and uniqueness of the mild solutions w.r.t. cPrm we refer [2].

Let $U_t \in \mathcal{L}(H)$, $t \in [0,\infty)$, is a strongly continuous pseudo-contractive semigroup on H and $S(t-s) = U_{(t-s)} \in \mathcal{L}(H)$ for all $0 \le s \le t \le T$.

Then there exists a separable $\mathbb{R}$-Hilbert space $(\mathcal{H}, \langle .,.\rangle_{\mathcal{H}}, \|.\|_{\mathcal{H}})$ with $H \subset \mathcal{H}$ and $\|v\|_H = \|v\|_{\mathcal{H}}$ for all $v \in H$ and a strongly continuous group $\mathcal{U}_t \in \mathcal{L}(\mathcal{H})$, $t \in \mathbb{R}$,(Here we use the fact that, the strongly continuous pseudo-contracitve semigroup can be dilated to strongly continuous group); such that

$$U_t(v) = P(\mathcal{U}_t(v)) \qquad (2.84)$$

for all $v \in H \subset \mathcal{H}$ and all $t \in [0,\infty)$ where $P : \mathcal{H} \to H$ is the orthogonal projection from $\mathcal{H}$ to H.

Now we transform our mild Itô process $X : [0,T]\times\Omega \to H$ to a standard Itô process $\bar{X}$, by following the technique of [14], roughly speaking by multiplying the mild Itô process with $\mathcal{U}_{-t}$ for $t \in [0,T]$. Let, $\bar{X} : [0,T]\times\Omega \to \mathcal{H}$ be the unique adapted, càdlàg stochastic process such that,

$$\bar{X}_t = X(0) + \int_0^t \mathcal{U}_{-s}F(X(s))ds + \int_0^t \int_{\mathcal{H}\setminus\{0\}} \mathcal{U}_{-s}f(v,X(s))q(dv,ds). \quad (2.85)$$

P-a.s. for all $t \in [0,T]$. Here we use the following transformation,

$$P(\mathcal{U}_t(\bar{X}_s)) = P(\mathcal{U}_t(X(0))) + \int_0^s P\mathcal{U}_{(t-u)}F(X(u))du + \int_0^s \int_{H\setminus\{0\}} P\mathcal{U}_{(t-u)}f(v,X(u))q(dv,du) \quad (2.86)$$

$$= S(t)X(0) + \int_0^s S(t-u)F(X(u))du + \int_0^s \int_{H\setminus\{0\}} S(t-u)f(v,X(u))q(dv,du)$$

$$= S(t-s)\big(S(s)X(0) + \int_0^s S(s-u)F(X(u))du + \int_0^s \int_{H\setminus\{0\}} S(s-u)f(v,X(u))q(dv,du)\big)$$

$$= S(t-s)X(s).$$

$$\Rightarrow P(\mathcal{U}_t(\bar{X}_t)) = X(t) \quad \text{and} \quad P(\mathcal{U}_t(\bar{X}_0)) = S(t)X(0).$$

P-a.s. for all $s,t \in [0,T]$ with $s \le t$.

Now we will apply the Itô formula for strong solution of [23] to the test function $\varphi(s, P(\mathcal{U}_t(v)))$ for $s \in [0,t]$, $v \in \mathcal{H}$

$$\varphi(t,X(t)) = \varphi(t,P(\mathcal{U}_t(\bar{X}_t))) = \varphi(P(\mathcal{U}_t(\bar{X}_0))) + \int_0^t (\partial_1\varphi)(s,P(\mathcal{U}_t(\bar{X}_s)))ds \quad (2.87)$$

$$+ \int_0^t (\partial_2\varphi)(s,P(\mathcal{U}_t(\bar{X}_s)))P\mathcal{U}_{(t-s)}F(X(s))ds$$

$$+ \int_0^t \int_{H\setminus\{0\}} [\varphi(s,P(\mathcal{U}_t(\bar{X}_s)) + P\mathcal{U}_{(t-s)}f(v,X(s))) - \varphi(s,P(\mathcal{U}_t(\bar{X}_s))) - \big\langle(\partial_2\varphi)(s,P(\mathcal{U}_t(\bar{X}_s))), P\mathcal{U}_{(t-s)}f(v,X(s))\big\rangle]\beta(dv)ds$$

$$+ \int_0^t \int_{H\setminus\{0\}} \big[\varphi(s,P(\mathcal{U}_t(\bar{X}_s)) + P\mathcal{U}_{(t-s)}f(v,X(s))) - \varphi(s,P(\mathcal{U}_t(\bar{X}_s))\big]\, q(dv,ds)$$

P-a.s. for all $\varphi \in C^{1,2}([0,T]\times H)$. Now substituting (2.84) and (2.86) in (2.87), we get our Mild Itô formula of (2.83).

□

2.6 Examples

These examples are done in [1].

Example 1:-

Let $\Lambda : H \to \mathbb{R}$. Assume that the conditions of Theorem 6 hold. Now if we apply the Itô formula for mild solutions, of Theorem 6, for the function $e^{ct}\Lambda(x)$ where $c > 0$, $t \geq 0$ then it will be

$$\lim_{n\to\infty}\int_0^t e^{cs}\left\langle \Lambda'(X_n^\xi(s)), AX_n^\xi(s)\right\rangle_H ds$$

$$= e^{ct}\Lambda(X^\xi(t)) - \Lambda(\xi) - \int_0^t ce^{cs}(\Lambda(X^\xi(s)))ds$$

$$-\int_0^t e^{cs}\left\langle \Lambda'(X^\xi(s)), F(X^\xi(s))\right\rangle_H ds$$

$$-\int_0^t e^{cs}\frac{1}{2}tr(\Lambda''(X^\xi(s))(B(X^\xi(s)))Q(B(X^\xi(s)))^*)ds$$

$$-\int_0^t\int_{H\setminus\{0\}} e^{cs}\left[\Lambda(X^\xi(s)+f(v,X^\xi(s))) - \Lambda(X^\xi(s)) - \left\langle \Lambda'(X^\xi(s)), f(v,X^\xi(s))\right\rangle_H\right]\beta(dv)ds$$

$$-\int_0^t e^{cs}\left\langle \Lambda'(X^\xi(s)), B(X^\xi(s))dW_s\right\rangle_H$$

$$-\int_0^t\int_{H\setminus\{0\}} e^{cs}\left[\Lambda(X^\xi(s)+f(v,X^\xi(s))) - \Lambda(X^\xi(s))\right]q(dv,ds)$$

P-a.s..

We can also wrtie the Itô formula for mild solutions, of Theorem 6, for the functions $\Psi(x) = \|x\|^2$, $\Psi(x) = e^{ct}\|x\|^2$; for $x \in H$ and *Lyapunov function* in stability theory.

Example 2:-

Let A be a symmetric linear operator and the conditions of Theorem 6 hold. Assume that, for a fixed $l > 0$, $l \in (0,\infty)$, $\Psi(s,x) = e^{lA}\mathit{\Lambda}(s,x)$ and $\mathit{\Lambda}_x(s,x) \in \mathcal{D}(A)$. Where, $0 \leq s \leq t \leq T$, $x \in H$. Also assume that Ψ satisfies the following property,

(i) There exists constants $c_1, c_2, c_3 > 0$ s.t.

$$\|\Psi(s,x)\| \leq c_1 \|x\|_H^2 \,; \quad \|\Psi_x(s,x)\| \leq c_2 \|x\|_H \,; \quad \|\Psi_{xx}(s,x)\| \leq c_3 \|x\|_H$$

for all $x \in H$.

Hence, from the given conditions, there exists a constant $\mathcal{C} > 0$ s.t.

$$\left\langle \Psi_x(s, X_n^\xi(s)), AX_n^\xi(s) \right\rangle_H \leq \mathcal{C} \left\| X_n^\xi(s) \right\|_H^2 . \tag{2.88}$$

Here,

$$\left\langle \Psi_x(s, X_n^\xi(s)), AX_n^\xi(s) \right\rangle_H$$

$$= \left\langle e^{lA} \Lambda_x(s, X_n^\xi(s)), AX_n^\xi(s) \right\rangle_H$$

$$= \left\langle A^* e^{lA} \Lambda_x(s, X_n^\xi(s)), X_n^\xi(s) \right\rangle_H$$

$$\leq \mathcal{C} \left\| X_n^\xi(s) \right\|_H^2 .$$

The last inequality we get from (2.88). We can write from second to third equality, due to the fact that, $e^{lA}\Lambda_x(s, X_n^\xi(s)) \in \mathcal{D}(A) \subset \mathcal{D}(A^*)$ as we assumed $\Lambda_x(s,x) \in \mathcal{D}(A)$ and A to be a symmetric linear operator. By Theorem 5 we can choose a subsequence s.t. $X_n^\xi(s) \to X^\xi(s)$ P a.s.. So by (2.18) and Lebesgue DCT $\int_0^t \left\| X_n^\xi(s) \right\|_H^2 ds \to \int_0^t \left\| X^\xi(s) \right\|_H^2 ds$. Therefore by applying Generalized Lebesgue DCT (Theorem 3.4 of [16]) we can conclude that,

$$\int_0^t \left\langle e^{lA} \Lambda_x(s, X_n^\xi(s)), AX_n^\xi(s) \right\rangle_H ds \to \int_0^t \left\langle A^* e^{lA} \Lambda_x(s, X^\xi(s)), X^\xi(s) \right\rangle_H ds$$

P-a.s.. Therefore we can write the Itô formula for mild solutions of Theorem 6 for the function $\Psi(s,x) = e^{lA}\Lambda(s,x)$, for $x \in H$, as follows

$$e^{lA}\Lambda(t, X^\xi(t)) - e^{lA}\Lambda(0,\xi) = \int_0^t e^{lA}\Lambda_s(s, X^\xi(s)) ds \tag{2.89}$$

$$+ \int_0^t \left\langle e^{lA}\Lambda_x(s, X^\xi(s)), F(X^\xi(s)) \right\rangle_H ds + \int_0^t \left\langle A^* e^{lA}\Lambda_x(s, X^\xi(s)), X^\xi(s) \right\rangle_H ds$$

$$+ \int_0^t \frac{1}{2} tr(e^{lA}\Lambda_{xx}(s, X^\xi(s))(B(X^\xi(s)))Q(B(X^\xi(s)))^*) ds$$

$$+ \int_0^t \int_{H\setminus\{0\}} e^{lA} \left[\Lambda(s, X^\xi(s) + f(v, X^\xi(s))) - \Lambda(s, X^\xi(s)) - \left\langle \Lambda_x(s, X^\xi(s)), f(v, X^\xi(s)) \right\rangle_H \right] \beta(dv) ds$$

$$+\int_0^t \left\langle e^{lA}\Lambda_x(s,X^\xi(s)), B(X^\xi(s))dW_s\right\rangle_H$$

$$+\int_0^t\int_{H\setminus\{0\}} e^{lA}\left[\Lambda(s,X^\xi(s)+f(v,X^\xi(s)))-\Lambda(s,X^\xi(s))\right]q(dv,ds)$$

P-a.s. and by Theorem 11, we obtain

$$\overline{\mathcal{L}\Psi}(s,X^\xi(s))=\left\langle e^{lA}\Lambda_x(s,X^\xi(s)),F(X^\xi(s))\right\rangle_H+\left\langle A^*e^{lA}\Lambda_x(s,X^\xi(s)),X^\xi(s)\right\rangle_H$$

$$+\frac{1}{2}tr(e^{lA}\Lambda_{xx}(s,X^\xi(s))(B(X^\xi(s)))Q(B(X^\xi(s)))^*)$$

$$+\int_{H\setminus\{0\}} e^{lA}\left[\Lambda(s,X^\xi(s)+f(v,X^\xi(s)))-\Lambda(s,X^\xi(s))-\left\langle \Lambda_x(s,X^\xi(s)),f(v,X^\xi(s))\right\rangle_H\right]\beta(dv).$$

Example 3:-

In Example 2, if we consider $l=0$, then we have, $\Psi(s,x)=\Lambda(s,x)$ for $x\in H$. And assume that $\Lambda_x(s,X^\xi(s))\in\mathcal{D}(A^*)$ (Here we don't need the assumption that A is a symmetric linear operator). Then we can also write the Itô formula for mild solutions by Theorem 6 as,

$$\Lambda(t,X^\xi(t))-\Lambda(0,\xi)=\int_0^t\Lambda_s(s,X^\xi(s))ds \tag{2.90}$$

$$+\int_0^t\left\langle \Lambda_x(s,X^\xi(s)),F(X^\xi(s))\right\rangle_H ds+\int_0^t\left\langle A^*\Lambda_x(s,X^\xi(s)),X^\xi(s)\right\rangle_H ds$$

$$+\int_0^t\frac{1}{2}tr(\Lambda_{xx}(s,X^\xi(s))(B(X^\xi(s)))Q(B(X^\xi(s)))^*)ds$$

$$+\int_0^t\int_{H\setminus\{0\}}\left[\Lambda(s,X^\xi(s)+f(v,X^\xi(s)))-\Lambda(s,X^\xi(s))-\left\langle \Lambda_x(s,X^\xi(s)),f(v,X^\xi(s))\right\rangle_H\right]\beta(dv)ds$$

$$+\int_0^t\left\langle \Lambda_x(s,X^\xi(s)),B(X^\xi(s))dW_s\right\rangle_H$$

$$+\int_0^t\int_{H\setminus\{0\}}\left[\Lambda(s,X^\xi(s)+f(v,X^\xi(s)))-\Lambda(s,X^\xi(s))\right]q(dv,ds)$$

P-a.s. and by Theorem 11, we obtain

$$\overline{\mathcal{L}\Psi}(s,X^{\xi}(s)) = \left\langle \Lambda_x(s,X^{\xi}(s)), F(X^{\xi}(s))\right\rangle_H + \left\langle A^*\Lambda_x(s,X^{\xi}(s)), X^{\xi}(s)\right\rangle_H$$

$$+\frac{1}{2}tr(\Lambda_{xx}(s,X^{\xi}(s))(B(X^{\xi}(s)))Q(B(X^{\xi}(s)))^*)ds$$

$$+\int_{H\setminus\{0\}}\left[\Lambda(s,X^{\xi}(s)+f(v,X^{\xi}(s))) - \Lambda(s,X^{\xi}(s)) - \left\langle \Lambda_x(s,X^{\xi}(s)), f(v,X^{\xi}(s))\right\rangle_H\right]\beta(dv).$$

Now we discuss two specific examples of this case. If we consider $\Psi(x) = \Lambda(x) = \langle x,h\rangle^2$; $x,h\in H$. Then $\Lambda_x(x) = 2\langle x,h\rangle h$. And we can say,

$$\int_0^t \left\langle \Lambda_x(X_n^{\xi}(s)), AX_n^{\xi}(s)\right\rangle_H ds \to \int_0^t \left\langle 2\langle X^{\xi}(s),h\rangle A^*h, X^{\xi}(s)\right\rangle_H ds$$

P-a.s.. Hence we can write the Itô formula for mild solutions following (2.90). In [7], we have this sort of Itô formula for mild solutions.

But, If we consider, $\Psi(x) = \Lambda(x) = \|x\|^2$; $x\in H$. Then we can not write the Itô formula for mild solutions by following (2.90). Because in this case $\Lambda_x(X^{\xi}(s)) = 2X^{\xi}(s) \notin \mathcal{D}(A^*)$. Where as by applying Theorem 6, we can write the Itô formula for mild solutions as follows-

$$\lim_{n\to\infty}\int_0^t \langle 2X_n^{\xi}(s), AX_n^{\xi}(s)\rangle ds \tag{2.91}$$

$$= \|X^{\xi}(t)\|^2 - \|\xi\|^2 - \int_0^t \langle 2X^{\xi}(s), F(X^{\xi}(s))\rangle ds - \int_0^t tr((B(X^{\xi}(s)))Q(B(X^{\xi}(s)))^*)ds$$

$$-\int_0^t\int_{H\setminus\{0\}}[\|X^{\xi}(s)+f(v,X^{\xi}(s))\|^2 - \|X^{\xi}(s)\|^2 - \langle 2X^{\xi}(s), f(v,X^{\xi}(s))\rangle]\beta(dv)ds$$

$$-\int_0^t \langle 2X^{\xi}(s), B(X^{\xi}(s))dW_s\rangle - \int_0^t\int_{H\setminus\{0\}}[\|X^{\xi}(s)+f(v,X^{\xi}(s)\|^2 - \|X^{\xi}(s)\|^2]q(dv,ds)$$

P-a.s..

Example 4:-

Let A be a symmetric linear operator and the conditions of Theorem 6 hold. Now, we consider $\Psi(s,x) = e^{(t-s)A}\Gamma(x)$ for $x\in H$. Assume that Ψ satisfies the condition (i) of Example 2. Also assume that $\Gamma,\Gamma_x\in\mathcal{D}(A)$. Now,

$$\Psi_s(s,X_n^{\xi}(s)) = \left(-Ae^{(t-s)A}\Gamma(X_n^{\xi}(s))\right) \to \left(-Ae^{(t-s)A}\Gamma(X^{\xi}(s))\right)$$

P-a.s.. The convergence of all other terms are similar as in Example 2. So, we can write the Itô formula of Theorem 6 as,

$$\Gamma(X^\xi(t)) - e^{tA}\Gamma(\xi) = \int_0^t \left(-Ae^{(t-s)A}\Gamma(X^\xi(s))\right) ds \tag{2.92}$$

$$+\int_0^t \left\langle e^{(t-s)A}\Gamma_x(X^\xi(s)), F(X^\xi(s))\right\rangle_H ds + \int_0^t \left\langle A^* e^{(t-s)A}\Gamma_x(X^\xi(s)), X^\xi(s)\right\rangle_H ds$$

$$+\int_0^t \frac{1}{2} tr(e^{(t-s)A}\Gamma_{xx}(X^\xi(s))(B(X^\xi(s)))Q(B(X^\xi(s)))^*)ds$$

$$+\int_0^t \int_{H\setminus\{0\}} [e^{(t-s)A}\Gamma(X^\xi(s) + f(v, X^\xi(s))) - e^{(t-s)A}\Gamma(X^\xi(s))$$
$$- \left\langle e^{(t-s)A}\Gamma_x(X^\xi(s)), f(v, X^\xi(s))\right\rangle_H]\beta(dv)ds$$

$$+\int_0^t \left\langle e^{(t-s)A}\Gamma_x(X^\xi(s)), B(X^\xi(s))dW_s\right\rangle_H$$

$$+\int_0^t \int_{H\setminus\{0\}} \left[e^{(t-s)A}\Gamma(X^\xi(s) + f(v, X^\xi(s))) - e^{(t-s)A}\Gamma(X^\xi(s))\right] q(dv, ds)$$

P-a.s. and by Theorem 11, we obtain

$$\overline{\mathcal{L}\Psi}(s, X^\xi(s)) = \left\langle e^{(t-s)A}\Gamma_x(X^\xi(s)), F(X^\xi(s))\right\rangle_H + \left\langle A^* e^{(t-s)A}\Gamma_x(X^\xi(s)), X^\xi(s)\right\rangle_H$$

$$+\frac{1}{2} tr(e^{(t-s)A}\Gamma_{xx}(X^\xi(s))(B(X^\xi(s)))Q(B(X^\xi(s)))^*)$$

$$+\int_{H\setminus\{0\}} [e^{(t-s)A}\Gamma(X^\xi(s) + f(v, X^\xi(s))) - e^{(t-s)A}\Gamma(X^\xi(s))$$
$$- \left\langle e^{(t-s)A}\Gamma_x(X^\xi(s)), f(v, X^\xi(s))\right\rangle_H]\beta(dv).$$

Example 5:-

Let A be a symmetric linear operator and the conditions of Theorem 6 hold. Let us consider the function $\Psi(s, x) = \langle S(t-s)y, x\rangle$, where $y \in \mathcal{D}(A)$.

Therefore, for $y \in \mathcal{D}(A)$,

$$\Psi_s(s, X_n^\xi(s)) = -\left\langle AS(t-s)y, X_n^\xi(s)\right\rangle, \tag{2.93}$$

$$\Psi_x(s, X_n^\xi(s)) = S(t-s)y$$

and

$$\Psi_{xx}(s, X_n^\xi(s)) = 0.$$

Hence

$$\left\langle \Psi_x(s, X_n^\xi(s)), AX_n^\xi(s) \right\rangle = \left\langle S(t-s)y, AX_n^\xi(s) \right\rangle.$$

Since, A is a symmetric operator

$$\left\langle \Psi_x(s, X_n^\xi(s)), AX_n^\xi(s) \right\rangle = \left\langle AS(t-s)y, X_n^\xi(s) \right\rangle. \tag{2.94}$$

So from (2.93) and (2.94)

$$\Psi_s(s, X_n^\xi(s)) + \left\langle \Psi_x(s, X_n^\xi(s)), AX_n^\xi(s) \right\rangle = 0.$$

Therefore we can write the Itô formula for mild solutions of Theorem 6 for the above function as follows,

$$\left\langle y, X^\xi(t) \right\rangle - \langle S(t)y, \xi \rangle = \int_0^t \left\langle S(t-s)y, F(X^\xi(s)) \right\rangle_H ds$$

$$+ \int_0^t \int_{H\setminus\{0\}} [\left\langle S(t-s)y, X^\xi(s) + f(v, X^\xi(s)) \right\rangle - \left\langle S(t-s)y, X^\xi(s) \right\rangle$$
$$- \langle S(t-s)y, f(v, X^\xi(s)) \rangle_H] \beta(dv) ds$$

$$+ \int_0^t \left\langle S(t-s)y, B(X^\xi(s)) dW_s \right\rangle_H$$

$$+ \int_0^t \int_{H\setminus\{0\}} \left[\left\langle S(t-s)y, X^\xi(s) + f(v, X^\xi(s)) \right\rangle - \left\langle S(t-s)y, X^\xi(s) \right\rangle \right] q(dv, ds)$$

P-a.s. and by Theorem 11, we obtain

$$\begin{aligned}\overline{\mathcal{L}\Psi}(s, X^\xi(s)) = &\left\langle S(t-s)y, F(X^\xi(s)) \right\rangle_H \\ &+ \int_{H\setminus\{0\}} [\left\langle S(t-s)y, X^\xi(s) + f(v, X^\xi(s)) \right\rangle - \left\langle S(t-s)y, X^\xi(s) \right\rangle \\ &- \langle S(t-s)y, f(v, X^\xi(s)) \rangle_H] \beta(dv).\end{aligned}$$

Example 6:-

Consider the stochastic heat equation

$$dX(x,t) = \frac{\partial^2}{\partial x^2} X(x,t)dt + B(X(x,t))dW_t + \int_{H\setminus\{0\}} f(v)X(x,t)q(dv,dt), \tag{2.95}$$

with

$$0 < x < 1.$$

$$X(0,t) = X(1,t) = 0; \quad X(x,0) = X_0(x); \quad X_0, B, f \in L_2(0,1).$$

Here we take $H = L_2(0,1)$, $A = d^2/dx^2$ and

$$\mathcal{D}(A) = \{f \in H | f', f'' \in H; f(0) = f(1) = 0\}.$$

Since A has eigenvectors $\{\sqrt{2}\sin n\pi x\}$ and eigenvalues $\{-n^2\pi^2\}$ for $n \in \mathbb{N}$. Then $X \in \mathcal{D}(A)$, $\langle AX, X\rangle \leq -\pi^2 |X|^2$ (see example 6.1 of [19]).

Now consider the function $\Psi(x) = \|x\|^2$. Therefore $\Psi_x(x) = 2x$ and $\Psi_{xx} = 2$. Hence, for $x \in \mathcal{D}(A)$

$$\langle \Psi_x(x), Ax\rangle \leq -2\pi^2 \|x\|^2,$$

$$\frac{1}{2} tr(\Psi_{xx}(x)(B(x))Q(B(x))^*) = tr((B(x))Q(B(x))^*) \leq l(1 + \|x\|^2),$$

by (A3) and

$$\begin{aligned}
&\int_{H\setminus\{0\}} [\Psi(x + f(v)x) - \Psi(x) - \langle \Psi_x(x), f(v)x\rangle]\beta(dv) \\
&\leq \int_{H\setminus\{0\}} \|f(v)x\|^2 \times \sup_{0\leq\theta\leq 1} |\Psi_{xx}(x + \theta f(v)y)|\, \beta(dv) \\
&\leq 2l(1 + \|x\|^2) \qquad\qquad [\text{since, } |\Psi_{xx}| = 2.]
\end{aligned}$$

by (A3). To get the above inequality we followed the argument used in the proof of Theorem 6. Therefore for $x \in \mathcal{D}(A)$,

$$\begin{aligned}
\mathcal{L}\Psi(x) = \mathcal{L}\|x\|^2 &\leq -2\pi^2 \|x\|^2 + (l + 2l)(1 + \|x\|^2) \\
&= (-2\pi^2 + 3l)\|x\|^2 + 3l
\end{aligned}$$

$$\Rightarrow \Psi(x) \in V.$$

The description of V is given in Corollary 4 and the inequality of Corollary 4 will be-

$$\begin{aligned}\|X(t)\|^2 - \|X(0)\|^2 \leq & \int_0^t \{(-2\pi^2 + 3l)\|X(s)\|^2 + 3l\}ds \\ & + \int_0^t \langle 2X(s), B(X(s))dW_s\rangle \\ & + \int_0^t \int_{H\setminus\{0\}} [\|X(s) + f(v)X(s)\|^2 - \|X(s)\|^2]q(dv, ds).\end{aligned} \tag{2.96}$$

So, from Corollary 4, we obtain the inequality of (2.96) for the function $\Psi(x) = \|x\|^2$ but from (2.91) we are able to write the Itô formula for mild solutions for the function $\Psi(x) = \|x\|^2$.

Now whenever $\pi^2 > \frac{3}{2}l$, then for some constant $k > 0$

$$\begin{aligned}\|X(t)\|^2 - \|X(0)\|^2 \leq & \int_0^t \{-k\|X(s)\|^2 + 3l\}ds + \int_0^t \langle 2X(s), B(X(s))dW_s\rangle \\ & + \int_0^t \int_{H\setminus\{0\}} [\|X(s) + f(v)X(s)\|^2 - \|X(s)\|^2]q(dv, ds).\end{aligned} \tag{2.97}$$

Now applying expectation on both sides of (2.97)

$$\begin{aligned}E[\|X(t)\|^2] & \leq -k \int_0^t E[\|X(s)\|^2]ds + 3lt + \|X(0)\|^2 \\ & \leq -k \int_0^t E[\|X(s)\|^2]ds + \lambda\|X(0)\|^2 \quad \text{[for sufficiently large } \lambda > 0].\end{aligned} \tag{2.98}$$

Therefore, by Gronwall's lemma

$$E[\|X(t)\|^2] \leq \lambda e^{-kt}\|X(0)\|^2,$$

hence by definition 13, we can conclude that the solution $X(t)$ is exponentially stable in the mean square sense.

Chapter 3

Differentials of SPDEs

In this Chapter, we study the continuity and differentiability results of the mild solutions with respect to the initial condition for SPDEs which contain both Gaussian and non-Gaussian noise.

Consider the following stochastic differential equation with values in H (where H is a real separable Hilbert space),

$$dX(t) = (AX(t) + F(t, X(t)))dt + B(t, X(t))dW_t + \int_{H\setminus\{0\}} f(t, v, X(t))q(dv, dt);$$

$$X(0) = \xi_0 \in H. \tag{3.1}$$

The initial condition ξ_0 is an $\mathcal{F}_0$-measurable H-valued random variable. Where $X(t) \in D([0,T], H)$ such that F, B nad f do not depend on ω. Where $(D[0,T], H)$ is the space of càdlàg functions defined on $[0,T]$ and with values in H, with the sup norm $\|.\|_\infty := \sup_{t\in[0,T]} \|.\|_H$. The terms in (3.1) satisfying the following conditions:

(K1) A is the infinitesimal generator of a pseudo-contraction semigroup $\{S(t), t \geq 0\}$ on H. This means in particular that there exists a constant $\alpha \in \mathbb{R}_+$ s.t. $\|S(t)\| \leq e^{\alpha t}$.

(K2) W_t is a K-valued $\mathcal{F}_t$-Wiener process with covariance Q on a complete filtered probability space $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\leq T}, P\right)$ satisfying the usual hypothesis, where Q is a nonnegative definite symmetric trace-class operator on the real separable Hilbert space K. $q(ds, du) := N(ds, du)(\omega) - ds\beta(du)$ is a compensated Poisson random measure (cPrm) on a complete filtered probability space $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\leq T}, P\right)$ satisfying the usual hypothesis. $(W_t)_{t\geq 0}$ is assumed to be independent of cPrm $q(dv, dt)$.

(K3) $F : [0,T] \times H \to H$, $B : [0,T] \times H \to \mathcal{L}_2(K_Q, H)$, $f : [0,T] \times H\setminus\{0\} \times H \to H$ are continuous, and jointly measurable functions satisfying:

$$\|F(t,x)\|_H^2 + tr(B(t,x)QB^*(t,x)) + \int_{H\setminus\{0\}} \|f(t,v,x)\|_H^2 \,\beta(dv) \le l(1+\|x\|_H^2);$$

and,

$$\big\|F(t,x) - F(t,y)\big\|_H^2 + tr((B(t,x) - B(t,y))Q(B(t,x) - B(t,y))^*)$$

$$+ \int_{H\setminus\{0\}} \big\|f(t,v,x) - f(t,v,y)\big\|_H^2 \beta(dv) \le \mathcal{K}\big\|x-y\big\|_H^2;$$

for all $x, y \in H$, $t \in [0,T]$ and where l, $\mathcal{K}$ are positive constants.

Let $\tilde{\mathcal{H}}_2$ denote the class of H-valued stochastic processes X that are measurable as mappings from $([0,T]\times\Omega, \mathcal{B}([0,T])\otimes\mathcal{F})$ to $(H,\mathcal{B}(H))$ adapted to the filtration $\{\mathcal{F}_t\}_{t\le T}$, and satisfying $E[\sup_{0\le s\le T}\|X(s)\|_H^2] < \infty$. Then $\tilde{\mathcal{H}}_2$ is a Banach space (see Section 4.1 of [22]) with the norm

$$\|X\|_{\tilde{\mathcal{H}}_2} = \big(E[\sup_{0\le t\le T}\|X(t)\|_H^2]\big)^{\frac{1}{2}}.$$

In Theorem 3 of earlier chapter, we proved that- if $\{S(t), t\ge 0\}$ is a pseudo-contraction semigroup and F, B, f satisfy (K1), (K2), (K3) with $E[\|X(0)\|_H^2] < \infty$. Then equation (3.1) has a unique mild solution $X(t) \in (D[0,T],H)$ satisfying $E[sup_{0\le s\le T}\|X(s)\|_H^2] < \infty$.

Here we assume that the coefficients F, B and f are independent of ω.

Lemma 4. *Let $\{S(t), t\ge 0\}$ be a pseudo-contraction semigroup, $\xi \in L^2(\Omega, H)$ and $X \in D([0,T],H)$ with $E[\sup_{0\le s\le T}\|X(s)\|^2] < \infty$. Let*

$$\tilde{I}(\xi,X)(t) = S(t)\xi + \int_0^t S(t-s)F(s,X(s))ds + \int_0^t S(t-s)B(s,X(s))dW_s$$

$$+ \int_0^t \int_{H\setminus\{0\}} S(t-s)f(s,v,X(s))q(dv,ds) \tag{3.2}$$

and F, B, f satisfy conditions (K1), (K2), (K3). Then, for $0\le t\le T$,
(I)

$$E\left[\sup_{0\le t\le T}\left\|\tilde{I}(\xi,X)(t) - \tilde{I}(\eta,X)(t)\right\|_H^2\right] \le C_{1,T}E\left[\sup_{0\le t\le T}\|\xi-\eta\|_H^2\right],$$

(II)

$$E\left[\sup_{0\le t\le T}\left\|\tilde{I}(\xi,X)(t) - \tilde{I}(\xi,Y)(t)\right\|_H^2\right] \le C_{2,T}\int_0^T E\left[\sup_{0\le s\le T}\|X(s)-Y(s)\|_H^2\right] ds,$$

for some positive constants $C_{1,T}$, $C_{2,T}$.

Proof. Here we followed the proof of Lemma 3.7 of [16] as well as Lemma 5.5.6 of [22].

(I) Let $X^{\xi}(t)$ and $X^{\eta}(t)$ be mild solutions of (3.1) with initial conditions ξ and η, respectively. Then, by applying Lemma 1

$$
\begin{aligned}
E\sup_{0\le t\le T}\left\|X^{\xi}(t)-X^{\eta}(t)\right\|^{2} \le 4C_{\alpha,T}\big(E\left\|\xi-\eta\right\|^{2} & \\
&+E\Big\{\int_0^T \left\|F(s,X^{\xi}(s))-F(s,X^{\eta}(s))\right\|^{2}ds \\
&+\int_0^T \left\|B(s,X^{\xi}(s))-B(s,X^{\eta}(s))\right\|^{2}ds \\
&+\int_0^T\int_{H\setminus\{0\}} \left\|f(s,v,X^{\xi}(s))-f(s,v,X^{\eta}(s))\right\|^{2}\beta(dv)ds\Big\}\big)
\end{aligned}
$$

by (K3),

$$
\le 4C_{\alpha,T}\left(E\left\|\xi-\eta\right\|^{2}+\mathcal{K}\int_0^T E\sup_{0\le s\le T}\left\|X^{\xi}(s)-X^{\eta}(s)\right\|^{2}ds\right).
$$

Now, by using the Gronwall's lemma, we obtain our result.

(II)

$$
\begin{aligned}
&E\sup_{0\le t\le T}\left\|\tilde{I}(\xi,X)(t)-\tilde{I}(\xi,Y)(t)\right\|_{H}^{2} \\
&\le C\Big\{E\sup_{0\le t\le T}\left\|\int_0^t S(t-s)(F(s,X(s))-F(s,Y(s)))ds\right\|^{2} \\
&+E\sup_{0\le t\le T}\left\|\int_0^t S(t-s)(B(s,X(s))-B(s,Y(s)))dW_s\right\|^{2} \\
&+E\sup_{0\le t\le T}\left\|\int_0^t\int_{H\setminus\{0\}} S(t-s)(f(s,v,X(s))-f(s,v,Y(s)))q(dv,ds)\right\|^{2}\Big\}
\end{aligned}
$$

by applying Lemma 1,

$$
\begin{aligned}
\le C_{\alpha,T}\Big\{&E\int_0^T \left\|F(s,X(s))-F(s,Y(s))\right\|^{2}ds \\
&+E\int_0^T \left\|B(s,X(s))-B(s,Y(s))\right\|^{2}ds \\
&+E\int_0^T\int_{H\setminus\{0\}} \left\|f(s,v,X(s))-f(s,v,Y(s))\right\|^{2}\beta(dv)ds\Big\}
\end{aligned}
$$

by (K3),

$$
\le C_{\alpha,T}\mathcal{K}\int_0^T E\sup_{0\le s\le T}\left\|X(s)-Y(s)\right\|^{2}ds.
$$

Hence the proof.

□

Lemma 5. *Let $\tilde{I} : H \times \tilde{\mathcal{H}}_2 \to \tilde{\mathcal{H}}_2$ be such that its projection at time $t \in [0, T]$ is given by $\tilde{I}(\xi, X)(t)$. Assume that F, B, f satisfy conditions (K1), (K2), (K3). Then there exists a constant α_T, depending on T, s.t. $\alpha_T \in (0, 1)$ and*

$$\|\tilde{I}(\xi, X) - \tilde{I}(\xi, Y)\|_{\tilde{\mathcal{H}}_2} \leq \alpha_T \|X - Y\|_{\tilde{\mathcal{H}}_2}. \tag{3.3}$$

Proof. Here we follow the proof of Lemma 5.5.7 of [22].

Let $\mathbf{S}X := \tilde{I}(\xi, X)(t)$. We will prove that, $\mathbf{S}^n$ is a contraction operator on $\tilde{\mathcal{H}}_2$, for sufficiently large $n \in \mathbb{N}$. From Lemma 4(II), it follows by induction

$$\begin{aligned} E \sup_{0 \leq t \leq T} \|\mathbf{S}^n X(t) - \mathbf{S}^n Y(t)\|_H^2 &\leq C_{2,T}^n \int_0^T ds_1 \int_0^T ds_2 \cdots \int_0^T E \sup_{0 \leq s \leq T} \|X(s_n) - Y(s_n)\|_H^2 ds_n \\ &\leq C_{2,T}^n \frac{T^{n-1}}{(n-1)!} \int_0^T E \sup_{0 \leq s \leq T} \|X(s) - Y(s)\|_H^2 ds. \end{aligned}$$

Therefore, for sufficiently large $n \in \mathbb{N}$, $\mathbf{S}^n$ is a contraction operator on $\tilde{\mathcal{H}}_2$. Hence, has a unique fixed point. Suppose that $\mathbf{S}^{n_0}$ is a contraction operator on $\tilde{\mathcal{H}}_2$. Therefore

$$\begin{aligned} E \sup_{0 \leq t \leq T} \|\mathbf{S}X(t) - \mathbf{S}Y(t)\|_H^2 &= E \sup_{0 \leq t \leq T} \|\mathbf{S}^{kn_0+1} X(t) - \mathbf{S}^{kn_0+1} Y(t)\|_H^2 \\ &\leq C_{2,T}^{kn_0+1} \frac{T^{kn_0}}{(kn_0)!} \int_0^T E \sup_{0 \leq s \leq T} \|X(s) - Y(s)\|_H^2 ds \\ &\to 0, \quad \text{when} \quad k \to \infty. \end{aligned}$$

This completes the proof. □

We now prove the continuity of the mild solutions w.r.t. the initial value.

Theorem 14. *Let X_n be the mild solutions to the sequence of equations (3.1) with coefficients F_n, B_n, f_n and initial conditions ξ_n, so that the following equations hold P-a.s. for $t \in [0, T]$:*

$$\begin{aligned} X_n(t) = S(t)\xi_n &+ \int_0^t S(t-s) F_n(s, X_n(s)) ds + \int_0^t S(t-s) B_n(s, X_n(s)) dW_s \\ &+ \int_0^t \int_{H \setminus \{0\}} S(t-s) f_n(s, v, X_n(s)) q(dv, ds). \end{aligned}$$

Assume that F_n, B_n, f_n satisfy conditions (K1), (K2), (K3). And in addition, let the following conditions hold:
I)

$$\sup_{n \in \mathbb{N}_0} E[\|\xi_n\|^2] < \infty,$$

II) with $n \to \infty$

$$\|F_n(t,x) - F_0(t,x)\|_H^2 + \|B_n(t,x) - B_0(t,x)\|_{\mathcal{L}_2(K_Q,H)}$$

$$+ \int_{H\setminus\{0\}} \big\|f_n(t,v,x) - f_0(t,v,x)\big\|_H^2 \beta(dv) \to 0,$$

III)

$$E[\|\xi_n - \xi_0\|_H^2] \to 0.$$

Then

$$\lim_{n\to\infty} E \sup_{0\le t\le T} [\|X_n(t) - X_0(t)\|_H^2] = 0.$$

Proof. Here we followed the proof of Theorem 3.7 of [16] and Theorem 8.1 of [2].

For any $t \le T$,

$$\begin{aligned}
&E \sup_{0\le t\le T} \|X_n(t) - X_0(t)\|^2 \\
&\le 4\big\{E \sup_{0\le t\le T} \|S(t)(\xi_n - \xi_0)\|^2 \\
&+ 2E \sup_{0\le t\le T} \left\| \int_0^t S(t-s)(F_n(s,X_n(s)) - F_n(s,X_0(s)))ds \right\|^2 \\
&+ 2E \sup_{0\le t\le T} \left\| \int_0^t S(t-s)(F_n(s,X_0(s)) - F_0(s,X_0(s)))ds \right\|^2 \\
&+ 2E \sup_{0\le t\le T} \left\| \int_0^t S(t-s)(B_n(s,X_n(s)) - B_n(s,X_0(s)))dW_s \right\|^2 \\
&+ 2E \sup_{0\le t\le T} \left\| \int_0^t S(t-s)(B_n(s,X_0(s)) - B_0(s,X_0(s)))dW_s \right\|^2 \\
&+ 2E \sup_{0\le t\le T} \left\| \int_0^t \int_{H\setminus\{0\}} S(t-s)(f_n(s,v,X_n(s)) - f_n(s,v,X_0(s)))q(dv,ds) \right\|^2 \\
&+ 2E \sup_{0\le t\le T} \left\| \int_0^t \int_{H\setminus\{0\}} S(t-s)(f_n(s,v,X_0(s)) - f_0(s,v,X_0(s)))q(dv,ds) \right\|^2 \big\}
\end{aligned}$$

by (K3) and Lemma 1,

$$\begin{aligned}
&\le 4e^{2\alpha T}\big\{E\|\xi_n - \xi_0\|^2 + \mathcal{K}\int_0^T E \sup_{0\le s\le T} \|X_n(s) - X_0(s)\|^2 ds \\
&+ 2E\int_0^T \|F_n(s,X_0(s)) - F_0(s,X_0(s))\|^2 ds \\
&+ 2E\int_0^T \|B_n(s,X_0(s)) - B_0(s,X_0(s))\|^2 ds \\
&+ 2E\int_0^T \int_{H\setminus\{0\}} \|f_n(s,v,X_0(s)) - f_0(s,v,X_0(s))\|^2 \beta(dv)ds\big\}.
\end{aligned} \tag{3.4}$$

Now from (III), we have $E\|\xi_n - \xi_0\|_H^2 \to 0$ and from (II)

$$\|F_n(t,x) - F_0(t,x)\|_H^2 + \|B_n(t,x) - B_0(t,x)\|_{\mathcal{L}_2(K_Q,H)} + \int_{H\setminus\{0\}} \left\|f_n(t,v,x) - f_0(t,v,x)\right\|_H^2 \beta(dv) \to 0.$$

Also from (K3)

$$\begin{aligned}\|F_n(t,X_0(s))\|_H^2 &+ \|B_n(t,X_0(s))\|_{\mathcal{L}_2(K_Q,H)}^2 \\ &+ \int_{H\setminus\{0\}} \|f_n(t,v,X_0(s))\|_H^2 \beta(dv) \\ &\leq l(1 + \|X_0(s)\|_H^2) \\ &= l(1 + \|\xi_0\|_H^2).\end{aligned}$$

Therefore as $n \to \infty$,

$$E\int_0^T \|F_n(s,X_0(s)) - F_0(s,X_0(s))\|^2 ds + E\int_0^T \|B_n(s,X_0(s)) - B_0(s,X_0(s))\|^2 ds + E\int_0^T \int_{H\setminus\{0\}} \|f_n(s,v,X_0(s)) - f_0(s,v,X_0(s))\|^2 \beta(dv)ds \to 0.$$

Therefore by applying Gronwall's lemma to (3.4), we can conclude that

$$\lim_{n\to\infty} E \sup_{0\leq t\leq T} \|X_n(t) - X_0(t)\|_H^2 = 0.$$

Hence the proof.

□

Theorem 15. *Assume that $F : [0,T] \times H \to H$, $B : [0,T] \times H \to \mathcal{L}_2(K_Q,H)$, $f : [0,T] \times H\setminus\{0\} \times H \to H$ and satisfy conditions (K1), (K2) and (K3).*
(a) If the Fréchet derivatives $DF(t,.)$, $DB(t,.)$ and $Df(t,v,.)$ are continuous in H and bounded,

$$||DF(t,x)y||_H^2 + ||DB(t,x)y||_{\mathcal{L}_2(K_Q,H)}^2 + \int_{H\setminus\{0\}} ||Df(t,v,x)y||_H^2 \beta(dv) \leq M_1\|y\|_H^2 \tag{3.5}$$

uniformly for x, $y \in H$, $t \in [0,T]$ and $M_1 \geq 0$ is a constant. Then the first-order Fréchet partial derivatives of $\tilde{I} : H \times \tilde{\mathcal{H}}_2 \to \tilde{\mathcal{H}}_2$ are given by

$$\left(\frac{\partial \tilde{I}(x,\xi)}{\partial x} y\right)(t) = S(t)y,$$

$$\left(\frac{\partial \tilde{I}(x,\xi)}{\partial \xi}\eta\right)(t) = \int_0^t S(t-s)DF(s,\xi(s))\eta(s)ds + \int_0^t S(t-s)DB(s,\xi(s))\eta(s)dW_s$$

$$+\int_0^t \int_{H\backslash\{0\}} S(t-s)Df(s,v,\xi(s))\eta(s)q(dv,ds) \tag{3.6}$$

P-a.s.; with ξ, $\eta \in \tilde{\mathcal{H}}_2$; x, $y \in H$; $0 \le t \le T$.

(b) If in addition, the second-order Fréchet derivatives $D^2F(t,.)$, $D^2B(t,.)$ and $D^2f(t,v,.)$ are continuous in H and bounded,

$$||D^2F(t,x)(y,z)||_H^2 + ||D^2B(t,x)(y,z)||^2_{\mathcal{L}_2(K_Q,H)} + \int_{H\backslash\{0\}} ||D^2f(t,v,x)(y,z)||_H^2\beta(dv) \le M_2\|y\|_H^2\|z\|_H^2 \tag{3.7}$$

uniformly for x, y, $z \in H$, $t \in [0,T]$ and $M_2 \ge 0$ is a constant. Then the second-order Fréchet partial derivative of $\tilde{I} : H \times \tilde{\mathcal{H}}_2 \to \tilde{\mathcal{H}}_2$ is given by

$$\left(\frac{\partial^2 \tilde{I}(x,\xi)}{\partial \xi^2}(x,\xi)(\eta,\zeta)\right)(t) = \int_0^t S(t-s)D^2F(s,\xi(s))(\eta(s),\zeta(s))ds$$
$$+\int_0^t S(t-s)D^2B(s,\xi(s))(\eta(s),\zeta(s))dW_s$$
$$+\int_0^t \int_{H\backslash\{0\}} S(t-s)D^2f(s,v,\xi(s))(\eta(s),\zeta(s))q(dv,ds) \tag{3.8}$$

P-a.s.; with ξ, η, $\zeta \in \tilde{\mathcal{H}}_2$; $x \in H$; $0 \le t \le T$.

Proof. Here we followed the proof of Theorem 3.8 of [16].

Consider

$$\frac{\tilde{I}(x+h,\xi)(t) - \tilde{I}(x,\xi)(t) - S(t)h}{\|h\|_H} = \frac{S(t)(x+h) - S(t)x - S(t)h}{\|h\|_H} = 0,$$

proving the first equality of (a). To prove the second equality, let

$$r_F(t,x,h) = F(t,x+h) - F(t,x) - DF(t,x)h,$$

$$r_B(t,x,h) = B(t,x+h) - B(t,x) - DB(t,x)h,$$

$$r_f(t,v,x,h) = f(t,v,x+h) - f(t,v,x) - Df(t,v,x)h.$$

Here, we will use the following result- Let H_1, H_2 be two Hilbert spaces. For a Fréchet differentiable function $L : H_1 \to H_2$, define $r_L(x,h) = L(x+h) - L(x) - DL(x)h$ and $r_{DL}(x,h_1) = DL(x+h_1)h - DL(x)h - D^2L(x)(h,h_1)$. Then

$$\|r_L(x,h)\|_{H_2} \leq 2 \sup_{x\in H_1} \|DL(x)\|_{\mathcal{L}(H_1,H_2)} \|h\|_{H_1} \tag{3.9}$$

and

$$\|r_{DL}(x,h_1)\|_{H_2} \leq 2 \sup_{x\in H_1} \left\|D^2L(x)\right\|_{\mathcal{L}(H_1\times H_1,H_2)} \|h\|_{H_1} \|h_1\|_{H_1} \tag{3.10}$$

Now with $\frac{\partial \tilde{I}(x,\xi)}{\partial \xi}$ as given by the r.h.s. of (3.6), we have

$$\begin{aligned}
r_{\tilde{I}}(x,\xi,\eta)(t) &= \tilde{I}(x,\xi+\eta)(t) - \tilde{I}(x,\xi)(t) - \left(\frac{\partial \tilde{I}(x,\xi)}{\partial \xi}\eta\right)(t) \\
&= \int_0^t S(t-s) r_F(s,\xi(s),\eta(s))ds + \int_0^t S(t-s) r_B(s,\xi(s),\eta(s))dW_s \\
&+ \int_0^t \int_{H\setminus\{0\}} S(t-s) r_f(s,v,\xi(s),\eta(s)) q(dv,ds) \\
&= I_1 + I_2 + I_3.
\end{aligned}$$

We need to show that, as $\|\eta\|_{\tilde{\mathcal{H}}_2} \to 0$,

$$\frac{\|r_{\tilde{I}}(x,\xi,\eta)\|_{\tilde{\mathcal{H}}_2}}{\|\eta\|_{\tilde{\mathcal{H}}_2}} \to 0.$$

Consider

$$\frac{\left(E\sup_{0\leq t\leq T}\left\|\int_0^t S(t-s) r_F(s,\xi(s),\eta(s))ds\right\|_H^2\right)^{1/2}}{\|\eta\|_{\tilde{\mathcal{H}}_2}}$$

$$\leq C\left(E\int_0^T \frac{\|r_F(s,\xi(s),\eta(s))\|_H^2}{\|\eta(s)\|_H^2}\frac{\|\eta(s)\|_H^2}{\|\eta\|_{\tilde{\mathcal{H}}_2}^2} 1_{\{\|\eta(s)\|_H\neq 0\}} ds\right)^{1/2}.$$

Since F is Fréchet differentiable, therefore as $\|\eta(s)\|_H^2 \to 0$

$$\frac{\|r_F(s,\xi(s),\eta(s))\|_H^2}{\|\eta(s)\|_H^2} \to 0,$$

by (3.9) and (3.5), this is bounded by some constant. In addition, the factor

$$\frac{\|\eta(s)\|_H^2}{\|\eta\|_{\tilde{\mathcal{H}}_2}^2} 1_{\{\|\eta(s)\|_H\neq 0\}} \leq 1.$$

Consequently, by the Lebesgue DCT, $\frac{\|I_1\|_{\tilde{\mathcal{H}}_2}}{\|\eta\|_{\tilde{\mathcal{H}}_2}} \to 0$ as $\|\eta\|_{\tilde{\mathcal{H}}_2} \to 0$.

Again, by Lemma 1

$$\left(E \sup_{0\le t\le T} \left\| \int_0^t S(t-s) r_B(s,\xi(s),\eta(s)) dW_s \right\|^2 \right)^{1/2}$$
$$\le C \left(E \int_0^T \|r_B(s,\xi(s),\eta(s))\|^2 ds \right)^{1/2},$$

and by Lemma 5.1.9(1) of [22]

$$\left(E \sup_{0\le t\le T} \left\| \int_0^t \int_{H\setminus\{0\}} S(t-s) r_f(s,v,\xi(s),\eta(s)) q(dv,ds) \right\|^2 \right)^{1/2}$$
$$\le C \left(E \int_0^T \int_{H\setminus\{0\}} \|r_f(s,v,\xi(s),\eta(s))\|^2 \beta(dv) ds \right)^{1/2}.$$

Therefore, by doing similar calculation as I_1, we obtain $\frac{\|I_2\|_{\tilde{\mathcal{H}}_2}}{\|\eta\|_{\tilde{\mathcal{H}}_2}} \to 0$ and $\frac{\|I_3\|_{\tilde{\mathcal{H}}_2}}{\|\eta\|_{\tilde{\mathcal{H}}_2}} \to 0$, as $\|\eta\|_{\tilde{\mathcal{H}}_2} \to 0$.

This concludes the proof of (a). Proof of (b) can be carried out by using similar arguments. Let

$$r_{DF}(t,x,h_1) = DF(t,x+h_1)h - DF(t,x)h - D^2F(t,x)(h,h_1),$$

$$r_{DB}(t,x,h_1) = DB(t,x+h_1)h - DB(t,x)h - D^2B(t,x)(h,h_1),$$

$$r_{Df}(t,v,x,h_1) = Df(t,v,x+h_1)h - Df(t,v,x)h - D^2f(t,v,x)(h,h_1).$$

Now with $\frac{\partial^2 \tilde{I}(x,\xi)}{\partial \xi^2}$ as given by the r.h.s. of (3.8), we have

$$\begin{aligned} r_{\frac{\partial \tilde{I}}{\partial \xi}}(x,\xi,\eta)(t) &= \left(\frac{\partial \tilde{I}(x,\xi+\eta)}{\partial \xi} \zeta \right)(t) - \left(\frac{\partial \tilde{I}(x,\xi)}{\partial \xi} \zeta \right)(t) - \left(\frac{\partial^2 \tilde{I}(x,\xi)}{\partial \xi^2}(x,\xi)(\eta,\zeta) \right)(t) \\ &= \int_0^t S(t-s) r_{DF}(s,\xi(s),\eta(s)) ds + \int_0^t S(t-s) r_{DB}(s,\xi(s),\eta(s)) dW_s \\ &\quad + \int_0^t \int_{H\setminus\{0\}} S(t-s) r_{Df}(s,v,\xi(s),\eta(s)) q(dv,ds) \\ &= J_1 + J_2 + J_3. \end{aligned}$$

We need to show that, as $\|\eta\|_{\tilde{\mathcal{H}}_2} \to 0$,

$$\frac{\left\| r_{\frac{\partial \tilde{I}}{\partial \xi}}(x,\xi,\eta) \right\|_{\tilde{\mathcal{H}}_2}}{\|\eta\|_{\tilde{\mathcal{H}}_2}} \to 0.$$

Consider

$$\frac{\left(E\sup_{0\le t\le T}\left\|\int_0^t S(t-s)r_{DF}(s,\xi(s),\eta(s))ds\right\|_H^2\right)^{1/2}}{\|\eta\|_{\tilde{\mathcal{H}}_2}}$$

$$\le C\left(E\int_0^T \frac{\|r_{DF}(s,\xi(s),\eta(s))\|_H^2}{\|\eta(s)\|_H^2}\frac{\|\eta(s)\|_H^2}{\|\eta\|_{\tilde{\mathcal{H}}_2}^2}1_{\{\|\eta(s)\|_H\neq 0\}}ds\right)^{1/2}.$$

Since DF is Fréchet differentiable, therefore as $\|\eta(s)\|_H^2 \to 0$

$$\frac{\|r_{DF}(s,\xi(s),\eta(s))\|_H^2}{\|\eta(s)\|_H^2} \to 0,$$

by (3.10) and (3.7), this is bounded by some constant. In addition, the factor

$$\frac{\|\eta(s)\|_H^2}{\|\eta\|_{\tilde{\mathcal{H}}_2}^2}1_{\{\|\eta(s)\|_H\neq 0\}} \le 1.$$

Consequently, by the Lebesgue DCT, $\frac{\|J_1\|_{\tilde{\mathcal{H}}_2}}{\|\eta\|_{\tilde{\mathcal{H}}_2}} \to 0$ as $\|\eta\|_{\tilde{\mathcal{H}}_2} \to 0$.

Again, by Lemma 1

$$\left(E\sup_{0\le t\le T}\left\|\int_0^t S(t-s)r_{DB}(s,\xi(s),\eta(s))dW_s\right\|^2\right)^{1/2}$$
$$\le C\left(E\int_0^T \|r_{DB}(s,\xi(s),\eta(s))\|^2 ds\right)^{1/2},$$

and by Lemma 5.1.9(1) of [22]

$$\left(E\sup_{0\le t\le T}\left\|\int_0^t\int_{H\setminus\{0\}} S(t-s)r_{Df}(s,v,\xi(s),\eta(s))q(dv,ds)\right\|^2\right)^{1/2}$$
$$\le C\left(E\int_0^T\int_{H\setminus\{0\}} \|r_{Df}(s,v,\xi(s),\eta(s))\|^2\beta(dv)ds\right)^{1/2}.$$

Therefore, by doing similar calculation as J_1, we obtain $\frac{\|J_2\|_{\tilde{\mathcal{H}}_2}}{\|\eta\|_{\tilde{\mathcal{H}}_2}} \to 0$ and $\frac{\|J_3\|_{\tilde{\mathcal{H}}_2}}{\|\eta\|_{\tilde{\mathcal{H}}_2}} \to 0$, as $\|\eta\|_{\tilde{\mathcal{H}}_2} \to 0$. This completes the proof.

□

Now we will use the following lemma on contractions depending on a parameter, for proving the next theorem i.e. Theorem 16.

Lemma 6. *Let X, U be Banach spaces, and $f : X \times U \to U$ be a contraction with respect to the second variable, i.e., for some $0 \le \alpha < 1$,*

$$\|f(x,u) - f(x,v)\|_U \le \alpha \|u - v\|_U, \quad x \in X, \quad u, v \in V \tag{3.11}$$

and let for every $x \in X$, $\varphi(x)$ denote the unique fixed point of the contraction $f(x,.) : U \to U$. Then the unique transformation $\varphi : X \to U$ defined by

$$f(x, \varphi(x)) = \varphi(x) \quad \textit{for every} \quad x \in X \tag{3.12}$$

is of class $C^k(X)$ whenever $f \in C^k(X \times U)$, $k = 0, 1, \cdots$. The derivatives of φ can be calculated using the chain rule; in particular,

$$D\varphi(x)y = \left[I - \frac{\partial f(x,\varphi(x))}{\partial u}\right]^{-1} \left(\frac{\partial f(x,\varphi(x))}{\partial x} y\right), \tag{3.13}$$

$$\begin{aligned} D^2\varphi(x)(y,z) = \left[I - \frac{\partial f(x,\varphi(x))}{\partial u}\right]^{-1} &\Big(\frac{\partial^2 f(x,\varphi(x))}{\partial x^2}(y,z) \\ &+ \frac{\partial^2 f(x,\varphi(x))}{\partial x \partial u}(D\varphi(x)y, z) + \frac{\partial^2 f(x,\varphi(x))}{\partial u \partial x}(y, D\varphi(x)z) \\ &+ \frac{\partial^2 f(x,\varphi(x))}{\partial u^2}(D\varphi(x)y, D\varphi(x)z)\Big). \end{aligned} \tag{3.14}$$

Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of mappings in $C^l(X \times U)$ satisfying condition (3.11)*, denote by $\varphi_n : X \to U$ the unique transformations satisfying condition* (3.12)*, and assume that for all $x, x_1, \cdots, x_k \in X$, $u, u_1 \cdots, u_j \in U$, $0 \le k + j \le l$,*

$$\lim_{n\to\infty} \frac{\partial^{k+j} f_n(x,u)}{\partial x^k \partial u^j}(u_1, \cdots, u_j, x_1, \cdots, x_k) = \frac{\partial^{k+j} f(x,u)}{\partial x^k \partial u^j}(u_1, \cdots, u_j, x_1, \cdots, x_k). \tag{3.15}$$

Then

$$\lim_{n\to\infty} D^l \varphi_n(x)(x_1, \cdots, x_l) = D^l \varphi(x)(x_1, \cdots, x_l). \tag{3.16}$$

Proof. For the proof of the lemma, we refer to Appendix C (Proposition C.0.3 and Proposition C.0.5) of [6]. Also see Lemma 3.8 of [16]. □

Now we consider the approximating system of equation (3.1),

$$dX(t) = (AX(t) + R_nF(X(t)))dt + R_nB(X(t))dW_t + \int_{H\setminus\{0\}} R_n f(v, X(t)) q(dv, dt);$$

$$X(0) = x \in \mathcal{D}(A). \tag{3.17}$$

Here A generates a pseudo-contraction semigroup. Let $R(n, A) = (nI - A)^{-1}$ denote the resolvent of A evaluted at n where $R_n = nR(n, A)$, with $n \in \rho(A)$ the resolvent set of A. We have $R_n : H \to \mathcal{D}(A)$ and $A_n = AR_n$ are the Yosida approximations of A. We assume that F, B, f satisfy conditions (K1), (K2), (K3).

By applying Theorem 3, we can conclude that equation (3.17) has a unique mild solution, denoted by $X_n^x(t)$. Then

$$X_n^x(t) = S(t)x + \int_0^t S(t-s)R_nF(X_n^x(s))ds + \int_0^t S(t-s)R_nB(X_n^x(s))dW_s$$

$$+ \int_0^t \int_{H\backslash\{0\}} S(t-s)R_nf(v, X_n^x(s))q(dv, ds). \qquad (3.18)$$

Since the range $\mathcal{R}(R(n,A)) \subset \mathcal{D}(A)$ and the conditions of Theorem 4 are satisfied, therefore we can conclude that $X_n^x(t) \in \mathcal{D}(A)$ is also a strong solution.

Now we are in a position to approximate the mild solution of equation (3.1) with respect to the strong solutions of equation (3.17). The mild solution of equation (3.1), say $X^x(t)$ (with initial condition $x \in H$), satisfies

$$X^x(t) = S(t)x + \int_0^t S(t-s)F(X^x(s))ds + \int_0^t S(t-s)B(X^x(s))dW_s$$

$$+ \int_0^t \int_{H\backslash\{0\}} S(t-s)f(v, X^x(s))q(dv, ds) \qquad (3.19)$$

and $\lim_{n\to\infty} E\left[\sup_{0\le t\le T} \|X_n^x(t) - X^x(t)\|_H^2\right] = 0$. This follows from Theorem 5.

We are now ready to prove a result on differentiability of the solution w.r.t. the initial condition.

Theorem 16. *Assume that $F : [0,T] \times H \to H$, $B : [0,T] \times H \to \mathcal{L}_2(K_Q, H)$, $f : [0,T] \times H\backslash\{0\} \times H \to H$ satisfy conditions (K1), (K2) and (K3). Let the Fréchet derivatives $DF(t,.)$, $DB(t,.)$, $Df(t,v,.)$, $D^2F(t,.)$, $D^2B(t,.)$ and $D^2f(t,v,.)$ be continuous in H and satisfy conditions (3.5) and (3.7). Then the solution X^x of (3.1) with initial condition $x \in H$, viewed as a mapping $X : H \to \tilde{\mathcal{H}}_2$, is twice continuously Fréchet differentiable in x and for any y, $z \in H$, the first and second derivative process $DX^x(.)y$ and $D^2X^x(.)(y,z)$ are mild solutions of the equations*

$$dZ(t) = (AZ(t) + DF(t, X^x(t))Z(t))dt + DB(t, X^x(t))Z(t)dW_t$$

$$+ \int_{H\backslash\{0\}} Df(t, v, X^x(t))Z(t)q(dv, dt), \qquad (3.20)$$

$$Z(0) = y,$$

and

$$dZ(t) = (AZ(t) + DF(t, X^x(t))Z(t) + D^2F(t, X^x(t))(DX^x(t)y, DX^x(t)z))dt$$

$$+(DB(t,X^x(t))Z(t)+D^2B(t,X^x(t))(DX^x(t)y,DX^x(t)z))dW_t \tag{3.21}$$

$$+\int_{H\backslash\{0\}}(Df(t,v,X^x(t))Z(t)+D^2f(t,v,X^x(t))(DX^x(t)y,DX^x(t)z))q(dv,dt),$$

$$Z(0)=0.$$

If X_n is the solution to the approximating system of (3.1) with deterministic initial condition $x\in H$, then for y, $z\in H$, we have the following approximations for the first and second derivative processes:

$$\lim_{n\to\infty}\|(DX_n^x(.)-DX^x(.))\,y\|_{\tilde{\mathcal{H}}_2}=0, \tag{3.22}$$

$$\lim_{n\to\infty}\left\|\left(D^2X_n^x(.)-D^2X^x(.)\right)(y,z)\right\|_{\tilde{\mathcal{H}}_2}=0. \tag{3.23}$$

Proof. Here we will follow the proof of Theorem 3.9 of [16].

$\tilde{\mathcal{H}}_2$ is a Banach space. By Lemma 5, $\tilde{I}$ is a contraction. Therefore, from the unique fixed point theorem, we get $X^x=\tilde{I}(X^x)$. Now we will apply (3.13) and (3.6) respectively. Therefore

$$DX^x(.)y=\left[I-\frac{\partial\tilde{I}(x,X^x(.))}{\partial X^x(.)}\right]^{-1}\left(\frac{\partial\tilde{I}(x,X^x(.))}{\partial x}y\right)$$

$$\begin{aligned}
\Rightarrow DX^x(.)y&=\frac{\partial\tilde{I}(x,X^x(.))}{\partial x}y+\frac{\partial\tilde{I}(x,X^x(.))}{\partial X^x(.)}DX^x(.)y\\
&=S(t)y+\int_0^t S(t-s)DF(s,X^x(s))DX^x(.)yds\\
&\quad+\int_0^t S(t-s)DB(s,X^x(s))DX^x(.)ydW_s\\
&\quad+\int_0^t\int_{H\backslash\{0\}}S(t-s)Df(s,v,X^x(s))DX^x(.)yq(dvds).
\end{aligned}$$

Therefore we can conclude that $DX^x(.)y$ is a mild solution of the equation (3.20). Similarly, we can prove that, $D^2X^x(.)(y,z)$ is a mild solution of the equation (3.21) in the following way. From (3.14)

$$\begin{aligned}
D^2X^x(.)(y,z)&=\left[I-\frac{\partial\tilde{I}(x,X^x(.))}{\partial X^x(.)}\right]^{-1}\Big(\frac{\partial^2\tilde{I}(x,X^x(.))}{\partial x^2}(y,z)\\
&\quad+\frac{\partial^2\tilde{I}(x,X^x(.))}{\partial x\partial X^x(.)}(DX^x(.)y,z)+\frac{\partial^2\tilde{I}(x,X^x(.))}{\partial X^x(.)\partial x}(y,DX^x(.)z)\\
&\quad+\frac{\partial^2\tilde{I}(x,X^x(.))}{\partial X^x(.)^2}(DX^x(.)y,DX^x(.)z)\Big).
\end{aligned}$$

$$\Rightarrow D^2X^x(.)(y,z) = \frac{\partial^2 \tilde{I}(x,X^x(.))}{\partial x^2}(y,z) + \frac{\partial^2 \tilde{I}(x,X^x(.))}{\partial x \partial X^x(.)}(DX^x(.)y,z) + \frac{\partial^2 \tilde{I}(x,X^x(.))}{\partial X^x(.) \partial x}(y,DX^x(.)z) + \frac{\partial^2 \tilde{I}(x,X^x(.))}{\partial X^x(.)^2}(DX^x(.)y,DX^x(.)z) + \frac{\partial \tilde{I}(x,X^x(.))}{\partial X^x(.)}D^2X^x(.)(y,z). \tag{3.24}$$

Now,

$$\frac{\partial^2 \tilde{I}(x,X^x(.))}{\partial x^2}(y,z) = \frac{\partial^2 \tilde{I}(x,X^x(.))}{\partial x \partial X^x(.)}(DX^x(.)y,z) = \frac{\partial^2 \tilde{I}(x,X^x(.))}{\partial X^x(.) \partial x}(y,DX^x(.)z) = 0. \tag{3.25}$$

Because, from Theorem 15,

$$\left(\frac{\partial \tilde{I}(x,\xi)}{\partial x}y\right)(t) = S(t)y,$$

$$\left(\frac{\partial \tilde{I}(x,\xi)}{\partial \xi}\eta\right)(t) = \int_0^t S(t-s)DF(s,\xi(s))\eta(s)ds + \int_0^t S(t-s)DB(s,\xi(s))\eta(s)dW_s + \int_0^t \int_{H\setminus\{0\}} S(t-s)Df(s,v,\xi(s))\eta(s)q(dv,ds).$$

Therefore

$$\begin{aligned} &\lim_{h\to 0} \frac{\|\frac{\partial \tilde{I}(x+h,\xi)}{\partial x}y - \frac{\partial \tilde{I}(x,\xi)}{\partial x}y - 0.h\|}{\|h\|_H} \\ &= \lim_{h\to 0} \frac{\|S(t)y - S(t)y - 0.h\|}{\|h\|_H} \\ &= 0 \\ &\Rightarrow \frac{\partial^2 \tilde{I}(x,\xi)}{\partial x^2} = 0, \end{aligned}$$

and

$$\begin{aligned} &\lim_{h\to 0} \frac{\|\frac{\partial \tilde{I}(x+h,\xi)}{\partial \xi}\eta - \frac{\partial \tilde{I}(x,\xi)}{\partial \xi}\eta - 0.h\|}{\|h\|_H} \\ &= 0 \\ &\Rightarrow \frac{\partial^2 \tilde{I}(x,\xi)}{\partial x \partial \xi} = 0, \end{aligned}$$

and

$$\lim_{\|\eta\|_{\tilde{\mathcal{H}}_2}\to 0} \frac{\|\frac{\partial \tilde{I}(x,\xi+\eta)}{\partial x}y - \frac{\partial \tilde{I}(x,\xi)}{\partial x}y - 0.\eta\|}{\|\eta\|_{\tilde{\mathcal{H}}_2}}$$
$$= \lim_{\|\eta\|_{\tilde{\mathcal{H}}_2}\to 0} \frac{\|S(t)y - S(t)y - 0.\eta\|}{\|\eta\|_{\tilde{\mathcal{H}}_2}}$$
$$= 0$$
$$\Rightarrow \frac{\partial^2 \tilde{I}(x,\xi)}{\partial \xi \partial x} = 0.$$

Hence, we can conclude (3.25). Again, from Theorem 15,

$$\frac{\partial^2 \tilde{I}(x, X^x(.))}{\partial X^x(.)^2}(DX^x(.)y, DX^x(.)z) \tag{3.26}$$
$$= \int_0^t S(t-s)D^2F(s, X^x(s))(DX^x(.)y, DX^x(.)z)ds$$
$$+ \int_0^t S(t-s)D^2B(s, X^x(s))(DX^x(.)y, DX^x(.)z)dW_s$$
$$+ \int_0^t \int_{H\setminus\{0\}} S(t-s)D^2f(s, v, X^x(s))(DX^x(.)y, DX^x(.)z)q(dv, ds)$$

and

$$\frac{\partial \tilde{I}(x, X^x(.))}{\partial X^x(.)} D^2X^x(.)(y,z) \tag{3.27}$$
$$= \int_0^t S(t-s)DF(s, X^x(s))D^2X^x(.)(y,z)ds$$
$$+ \int_0^t S(t-s)DB(s, X^x(s))D^2X^x(.)(y,z)dW_s$$
$$+ \int_0^t \int_{H\setminus\{0\}} S(t-s)Df(s, v, X^x(s))D^2X^x(.)(y,z)q(dvds).$$

Now substituting (3.25), (3.26) and (3.27) into (3.24) we get

$$D^2X^x(.)(y,z)$$
$$= \int_0^t S(t-s)(DF(s, X^x(s))D^2X^x(.)(y,z) + D^2F(s, X^x(s))(DX^x(.)y, DX^x(.)z))ds$$
$$+ \int_0^t S(t-s)(DB(s, X^x(s))D^2X^x(.)(y,z) + D^2B(s, X^x(s))(DX^x(.)y, DX^x(.)z))dW_s$$
$$+ \int_0^t \int_{H\setminus\{0\}} S(t-s)(Df(s, v, X^x(s))D^2X^x(.)(y,z) + D^2f(s, v, X^x(s))(DX^x(.)y, DX^x(.)z))q(dvds).$$

Therefore we can conclude that $D^2X^x(.)(y,z)$ is a mild solution of the equation (3.21).

Now we will prove, $\lim_{n\to\infty} E\big[\sup_{0\le t\le T}\|(DX_n^x(.) - DX^x(.))y\|_H^2\big] = 0$. Consider the Yosida Approximation of (3.20),

$$\begin{aligned} dZ(t) =&(AZ(t) + R_nDF(t, X^x(t))Z(t))dt + R_nDB(t, X^x(t))Z(t)dW_t \\ &+ \int_{H\setminus\{0\}} R_nDf(t, v, X^x(t))Z(t)q(dv, dt), \end{aligned}$$

$$Z(0) = y,$$

that is,

$$\begin{aligned} DX_n^x(.)y = S(t)y &+ \int_0^t S(t-s)R_nDF(s, X_n^x(s))DX^x(.)yds \\ &+ \int_0^t S(t-s)R_nDB(s, X_n^x(s))DX^x(.)ydW_s \\ &+ \int_0^t\int_{H\setminus\{0\}} S(t-s)R_nDf(s, v, X_n^x(s))DX^x(.)yq(dvds). \end{aligned}$$

Now,

$$E\big[\sup_{0\le t\le T}\|(DX_n^x(.) - DX^x(.))y\|_H^2\big]$$

$$\begin{aligned} =&E[\sup_{0\le t\le T}\Big\|\int_0^t S(t-s)(R_nDF(s, X_n^x(s)) - DF(s, X^x(s)))DX^x(.)yds \\ &+ \int_0^t S(t-s)(R_nDB(s, X_n^x(s)) - DB(s, X^x(s)))DX^x(.)ydW_s \\ &+ \int_0^t\int_{H\setminus\{0\}} S(t-s)(R_nDf(s, v, X_n^x(s)) - Df(s, v, X^x(s)))DX^x(.)yq(dv, ds)\Big\|_H^2] \end{aligned}$$

$$\begin{aligned} =&E[\sup_{0\le t\le T}\Big\|\int_0^t S(t-s)R_n(DF(s, X_n^x(s)) - DF(s, X^x(s)))DX^x(.)yds \\ &+ \int_0^t S(t-s)(R_n - I)DF(s, X^x(s))DX^x(.)yds \\ &+ \int_0^t S(t-s)R_n(DB(s, X_n^x(s)) - DB(s, X^x(s)))DX^x(.)ydW_s \\ &+ \int_0^t S(t-s)(R_n - I)DB(s, X^x(s))DX^x(.)ydW_s \\ &+ \int_0^t\int_{H\setminus\{0\}} S(t-s)R_n(Df(s, v, X_n^x(s)) - Df(s, v, X^x(s)))DX^x(.)yq(dv, ds) \\ &+ \int_0^t\int_{H\setminus\{0\}} S(t-s)(R_n - I)Df(s, v, X^x(s))DX^x(.)yq(dv, ds)\Big\|_H^2] \end{aligned}$$

by Lemma 1,

$$
\begin{aligned}
\leq C\{ & E[\int_0^T \left\|S(t-s)R_n(DF(s,X_n^x(s))-DF(s,X^x(s)))DX^x(.)y\right\|_H^2 ds] \\
&+ E[\int_0^T \left\|S(t-s)(R_n-I)DF(s,X^x(s))DX^x(.)y\right\|_H^2 ds] \\
&+ E[\int_0^T \left\|S(t-s)R_n(DB(s,X_n^x(s))-DB(s,X^x(s)))DX^x(.)y\right\|_{\mathcal{L}_2(K_Q,H)}^2 ds] \\
&+ E[\int_0^T \left\|(R_n-I)DB(s,X^x(s))DX^x(.)y\right\|_{\mathcal{L}_2(K_Q,H)}^2 ds] \\
&+ E[\int_0^T \int_{H\setminus\{0\}} \left\|S(t-s)R_n(Df(s,v,X_n^x(s))-Df(s,v,X^x(s)))DX^x(.)y\right\|_H^2 \beta(dv)ds] \\
&+ E[\int_0^T \int_{H\setminus\{0\}} \left\|(R_n-I)Df(s,v,X^x(s))DX^x(.)y\right\|_H^2 \beta(dv)ds]\}.
\end{aligned}
$$

Now from Theorem 5 of Chapter 2, we have $\lim_{n\to\infty} E\left[\sup_{0\leq t\leq T}\|X_n^x(t)-X^x(t)\|_H^2\right] = 0$. This allows us to choose a subsequence $X_{n_k}^x$ such that,

$$X_{n_k}^x(t) \to X^x(t),\ 0\leq t\leq T,\ P\text{-a.s..}$$

We will denote such a subsequence again by X_n^x. In fact, we can say that

$$\sup_{0\leq t\leq T}\|X_n(t)-X(t)\|_H \to 0, \tag{3.28}$$

P a.s.. This implies that the set

$$S = \{X_n(t),\ X(t):\ n=1,2...,\ 0\leq t\leq T\}$$

is bounded in H, hence any continuous function evaluated on S are bounded by some constant. Since $DF(t,.)$, $DB(t,.)$, $Df(t,v,.)$ are continuous, therefore by (3.28) $DF(s,X_n^x(s)) \to DF(s,X^x(s))$, $DB(s,X_n^x(s)) \to DB(s,X^x(s))$ and $Df(s,v,X_n^x(s)) \to Df(s,v,X^x(s))$ P-a.s.. $\sup_{0\leq t\leq T}\|S(t)\|_{\mathcal{L}(H)}$ and $\sup_{n>n_0}\|R_n\|_{\mathcal{L}(H)}$ (for n_0 sufficiently large) are uniformly bounded. Therefore, by (3.28) and Lebesgue DCT, we can conclude that 1st, 3rd and 5th integral of the R.H.S. of above equation converge to zero. Again $(R_n-I)x \to 0$, Therefore by the Lebesgue DCT we can conclude that 2nd, 4th and 6th integral of the R.H.S. of above equation also converge to zero. Therefore $E\left[\sup_{0\leq t\leq T}\|(DX_n^x(.)-DX^x(.))y\|_H^2\right] \to 0$, which means $\lim_{n\to\infty}\|(DX_n^x(.)-DX^x(.))\,y\|_{\tilde{\mathcal{H}}_2} = 0$. Similarly, we can prove $\lim_{n\to\infty}\left\|\left(D^2X_n^x(.)-D^2X^x(.)\right)(y,z)\right\|_{\tilde{\mathcal{H}}_2} = 0$.

□

Chapter 4

Appendix

Discussion about the Itô formula for strong solutions w.r.t. Gaussian and non-Gaussian noise.

Itô formula for strong solutions w.r.t. Gaussian noise as well as non-Gaussian noise is well known (see [16], [9], [22], [23]). Here we give an short description of, how we can write the Itô formula for strong solutions, when the stochastic process contains both Gaussian and non-Gaussian noise.

•Let H be a real separable Hilbert space.
•Let $F \in (C^{1,2}[0,T] \times H)$; $F : [0,T] \times H \to \mathbb{R}$ such that

$$\|\partial_y F(s,y)\| \le h_1(\|y\|), \qquad (s,y) \in \mathbb{R}_+ \times F \tag{4.1}$$

$$\|\partial_{yy} F(s,y)\| \le h_2(\|y\|), \qquad (s,y) \in \mathbb{R}_+ \times F \tag{4.2}$$

for quasi-sublinear functions $h_1, h_2 : \mathbb{R}_+ \to \mathbb{R}_+$.
•Let $\psi : \Omega \times \mathbb{R}_+ \to H$ be a $\mathcal{F}_s$-measurable P-a.s. Bochner-integrable process on $[0,T]$, s.t. $\int_0^T \|\psi(s)\| ds < \infty$ P-a.s..
•Let $f : \Omega \times \mathbb{R}_+ \times H\backslash\{0\} \to H$ be a progressively measurable process such that for all $t \in \mathbb{R}_+$ we have P-a.s.

$$\begin{aligned}\int_0^t \int_{H\backslash\{0\}} \|f(v,s)\|^2 \beta(dv) ds + \int_0^t \int_{H\backslash\{0\}} h_1(\|f(v,s)\|)^2 \|f(v,s)\|^2 \beta(dv) ds \\ + \int_0^t \int_{H\backslash\{0\}} h_2(\|f(v,s)\|) \|f(v,s)\|^2 \beta(dv) ds < \infty\end{aligned} \tag{4.3}$$

•So whenever we have the H valued Itô process

$$Y_t = Y_0 + \int_0^t \psi(s) ds + \int_0^t \int_{H\backslash\{0\}} f(v,s) q(dv,ds).$$

Then we can write the Itô formula for strong solution, according to [23] as,

$$\begin{aligned}F(t,Y_t) =&F(0,Y_0) + \int_0^t \partial_s F(s,Y_s)ds + \int_0^t \langle \partial_y F(s,Y_s), \psi(s)\rangle ds \\ &+ \int_0^t \int_{H\setminus\{0\}} \big(F(s,Y_s+f(v,s)) - F(s,Y_s) - \langle \partial_y F(s,Y_s), f(v,s)\rangle\big)\beta(dv)ds \\ &+ \int_0^t \int_{H\setminus\{0\}} \big(F(s,Y_{s-}+f(v,s)) - F(s,Y_{s-})\big)q(dv,ds)\end{aligned}$$

P-a.s., for $t \geq 0$.

Again, let K, H are real separable Hilbert spaces. W_t is K-valued Q-Wiener process on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0\leq t\leq T}, P)$. Consider $\mathcal{L}_2(K_Q, H)$, the space of Hilbert-Schemidt operators from K_Q to H. Let $\mathcal{P}(K_Q, H)$ denote the class of $\mathcal{L}_2(K_Q, H)$-valued stochastic processes adapted to the filtration $\{\mathcal{F}_t\}_{t\leq T}$, measurable as mappings from $([0,T]\times\Omega, \mathcal{B}([0,T])\otimes\mathcal{F}_T)$ to $(\mathcal{L}_2(K_Q,H), \mathcal{B}(\mathcal{L}_2(K_Q,H)))$, and satisfying the condition $P\{\int_0^T \|B(s)\|^2_{\mathcal{L}_2(K_Q,H)} ds < \infty\} = 1$. If, we have the H valued Itô process

$$X_t = X_0 + \int_0^t \psi(s)ds + \int_0^t B(s)dW_s.$$

Where $\psi(s)$ is a $\mathcal{F}_s$-measurable P-a.s. Bochner-integrable process on $[0,T]$ s.t. $\int_0^T \|\psi(s)\|ds < \infty$ P-a.s. and $B(s) \in \mathcal{P}(K_Q, H)$. Then for $F \in (C^{1,2}[0,T]\times H)$; $F : [0,T]\times H \to \mathbb{R}$, we can write the Itô's formula, as Theorem 2.9 of [16], as following

$$\begin{aligned}F(t,X_t) =&F(0,X_0) + \int_0^t \partial_s F(s,X_s)ds + \int_0^t \langle \partial_x F(s,X_s), \psi(s)\rangle ds \\ &+ \int_0^t \frac{1}{2} tr[\partial_{xx} F(s,X_s)(B(s)Q^{1/2})(B(s)Q^{1/2})^*]ds \\ &+ \int_0^t \langle \partial_x F(s,X_s), B(s)dW_s\rangle\end{aligned}$$

P-a.s., for $t \geq 0$.

So, whenever we have a H valued Itô process as

$$Z_t = Z_0 + \int_0^t \psi(s)ds + \int_0^t B(s)dW_s + \int_0^t \int_{H\setminus\{0\}} f(v,s)q(dv,ds),$$

ψ, B and f satisfying all the above conditions. Then we can write the Itô formula for strong solution as

$$\begin{aligned}F(t,Z_t) =&F(0,Z_0)+\int_0^t \partial_s F(s,Z_s)ds+\int_0^t \langle \partial_z F(s,Z_s),\psi(s)\rangle ds\\&+\int_0^t \frac{1}{2}tr[\partial_{zz}F(s,Z_s)(B(s)Q^{1/2})(B(s)Q^{1/2})^*]ds\\&+\int_0^t\int_{H\setminus\{0\}}\big(F(s,Z_s+f(v,s))-F(s,Z_s)-\langle\partial_z F(s,Z_s),f(v,s)\rangle\big)\beta(dv)ds\\&+\int_0^t\langle\partial_z F(s,Z_s),B(s)dW_s\rangle\\&+\int_0^t\int_{H\setminus\{0\}}\big(F(s,Z_{s-}+f(v,s))-F(s,Z_{s-})\big)q(dv,ds)\end{aligned}$$

P-a.s., for $t\geq 0$. Where $F\in(C^{1,2}[0,T]\times H)$; $F:[0,T]\times H\to\mathbb{R}$.

Because, we can show that the cross variation of two stochastic processes X_t and Y_t is zero i.e. $[X,Y](t)=0$. Here we only prove that, the cross variation of Wiener process and cPrm is zero. Let us define,

$$M(t):=\int_0^t B(s)dW_s$$

and

$$N(t):=\int_0^t\int_{H\setminus\{0\}} f(v,s)q(dv,ds).$$

Now we will evaluate the cross variation of the process $M(t)$ and $N(t)$ for $t\in[0,T]$. We denote $\Pi=\{0=t_0<t_1\cdots<t_n=T\}$ be the set of times on the time interval $[0,T]$. Let us define,

$$C_\Pi(M,N):=\sum_{j=0}^{n-1}(M(t_{j+1})-M(t_j))(N(t_{j+1})-N(t_j))$$

Where $\|\Pi\|:=\max_j(t_{j+1}-t_j)$. Now the cross variation of M and N on $[0,T]$ is defined to be,

$$[M,N](T):=\lim_{\|\Pi\|\to 0}C_\Pi(M,N)$$

Now,

$$M(t_{j+1})-M(t_j)$$

$$=\int_0^{t_{j+1}}B(s)dW_s-\int_0^{t_j}B(s)dW_s$$

$$= \int_{t_j}^{t_{j+1}} B(s)dW_s.$$

Similarly,

$$N(t_{j+1}) - N(t_j)$$

$$= \int_{t_j}^{t_{j+1}} \int_{H\setminus\{0\}} f(v,s)q(dv,ds).$$

So,

$$[M,N](T) = \lim_{\|\Pi\|\to 0} C_\Pi(M,N)$$

$$= \lim_{\|\Pi\|\to 0} \sum_{j=0}^{n-1} (M(t_{j+1}) - M(t_j))(N(t_{j+1}) - N(t_j))$$

$$= \lim_{\|\Pi\|\to 0} \sum_{j=0}^{n-1} \left(\int_{t_j}^{t_{j+1}} B(s)dW_s \right) \left(\int_{t_j}^{t_{j+1}} \int_{H\setminus\{0\}} f(v,s)q(dv,ds) \right)$$

$$\leq \lim_{\|\Pi\|\to 0} \max_{0\leq j\leq n-1} \left\| \int_{t_j}^{t_{j+1}} B(s)dW_s \right\| \left\| \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} \int_{H\setminus\{0\}} f(v,s)q(dv,ds) \right\|$$

$$= \lim_{\|\Pi\|\to 0} \max_{0\leq j\leq n-1} \left\| \int_{t_j}^{t_{j+1}} B(s)dW_s \right\| \left\| \int_0^T \int_{H\setminus\{0\}} f(v,s)q(dv,ds) \right\|$$

We will show that $[M,N](T) = 0$. As $\|\Pi\| \to 0$; $\max_{0\leq j\leq n-1} \left\| \int_{t_j}^{t_{j+1}} B(s)dW_s \right\| \to 0$. So, we have to show that $\left\| \int_0^T \int_{H\setminus\{0\}} f(v,s)q(dv,ds) \right\| < \infty$.

Now, for $t \in [0,T]$

$$E[\| \int_0^t \int_{H\setminus\{0\}} f(v,s)q(dv,ds)\|^2] = \int_0^t \int_{H\setminus\{0\}} E[\|f(v,s)\|^2]\beta(dv)ds < \infty. \tag{4.4}$$

Here the first equality holds due to Itô Isometry and second inequality holds, becuase Itô integral w.r.t. cPrm to be well defined (for this we refer to the introductory sections of [2] and [23]).

Now, by Jensen's inequality

$$\left(E[\| \int_0^t \int_{H\setminus\{0\}} f(v,s)q(dv,ds)\|]\right)^2 \leq E[\| \int_0^t \int_{H\setminus\{0\}} f(v,s)q(dv,ds)\|^2] \quad (4.5)$$

So, combining (4.4) and (4.5) we get

$$\left(E[\| \int_0^t \int_{H\setminus\{0\}} f(v,s)q(dv,ds)\|]\right)^2 < \infty$$

$$\Rightarrow E[\| \int_0^t \int_{H\setminus\{0\}} f(s,x)q(dv,ds)\|] < \infty.$$

$\Rightarrow$ For $t \in [0,T]$; for almost every ω, $\| \int_0^t \int_{H\setminus\{0\}} f(v,s)q(dv,ds)\| < \infty$.

Hence, we can conclude that, $[M,N](T) = 0$.

Bibliography

[1] Albeverio, S., Gawarecki, L., Mandrekar, V., Rüdiger, B. and Sarkar, B. :*Itô formula for mild solutions w.r.t. Gaussian and non-Gaussian Noise*, preprint(2016).

[2] Albeverio, S., Mandrekar, V. and Rüdiger, B.:*Existence of mild solutions for SDE's and semilinear equations with non-Gaussian noise*, Stochastic proc. and Appl. 119(2009)835-863.

[3] Albeverio, S., Rüdiger, B. :*Stochastic integrals and the Lévy-Ito decomposition theorem on separable Banach spaces*, Stoch. Anal. Appl. 23 (2005), no. 2, 217-253.

[4] Applebaum, D. :*Levy Process and Stochastic Calculus*, Cambridge University Press, (2004).

[5] Brzeźniak, Z., Hausenblas, E., Zhu, J. :*Maximal inequalities for Stochastic convolutions driven by Compensated Poisson Random Measures in Banach spaces*, arXiv, (2010).

[6] Cerrai, S. :*Second Order PDE's in Finite and Infinite Dimension; A Probabilistic Approach*, Springer-Verlag Berlin Heidelberg, (2001).

[7] Curtain, R.F. :*Markov process generated by linear stochastic evolution equations*, Stochastics, 5(1982), 135-165.

[8] Da Prato, G., Jentzen, A., Röckner, M. :*A mild Ito formula for SPDEs*, arXiv, (September 6,2011).

[9] Da Prato, G., Zabczyk, J. :*Stochastic Equations in Infinite Dimensions*, Cambridge University Press, (1992).

[10] Da Prato, G., Zabczyk, J. :*Second Order Partial Differential Equations in Hilbert Spaces*, Cambridge University Press, (2004).

[11] Eberlein, E., Glau, K. :*PIDEs for pricing European options in Lévy models- a fourier approach*, preprint(2011).

[12] Engel, K.J. and Nagel, R.:*One-Parameter Semigroups for Linear Evoulution Equations*,Springer-Verlag New York, Inc.(2000).

[13] Ethier, S.N. and Kurtz, T.G. :*Markov Processes, Characterization and Convergence*, John Wiley and Sons, Inc.(1986).

[14] Fillipović, D., Tappe, S., Teichmann, J.:*Term Structure Models Driven by Wiener Process and Poisson Measures: Existence and Positivity*, SIAM J. Finan. Math. 1(1), 523-554, (2010).

[15] Fillipović, D., Tappe, S., Teichmann, J. :*Jump-diffusions in Hilbert spaces: existence, stability and numerics*, Stochastics 82, 5(2010), 475-520.

[16] Gawarecki,L. and Mandrekar,V.:*Stochastic Differential Equation in Infinite Dimension*,Springer Heidelberg Dordrecht London New York,(2011).

[17] Glau, K. :*Sobolev index: A classification of Lévy processes via their symbols*, preprint(2012).

[18] Ichikawa, A. :*Stability of semilinear stochastic evolution equations*, J. Math. Anal. Appl. 90, 12-44(1982).

[19] Ichikawa, A. :*Semilinear stochastic evolution equations: boundedness, stability and invariant measures*, Stochastics 12, 1-39(1984).

[20] Ikeda, N., Watanabe, S. :*Stochastic Differential Equations and Diffusion Processes*, North-Holland Mathematical Library (1989).

[21] Kallenberg, O. :*Foundations of Modern Probability(Second Edition)*, by the Applied Probability Trust, 2002.

[22] Mandrekar, V. and Rüdiger, B.:*Stochastic Integration in Banach Spaces*, Springer Cham Heidelberg New York Dordrecht London,(2015).

[23] Mandrekar, V., Rüdiger, B., Tappe, S. :*Ito's Formula for Banach Space valued Jump Processes driven by Poisson Random Measures*, Seminar on Stochastic Analysis, Random Fields and Applications, Centro Stefano Franscini, Ascona (2011), Birkhäuser, May 2013.

[24] Mandrekar, V. and Wang, L.:*Asymptotic Properties of SPDE in Hilbert Spaces driven by Non-Gaussian noise*,Communications on Stochastic Analysis,Vol. 5,No. 2(2011) 309-331.

[25] Øksendal, B.:*Stochastic Differential Equations, An Introduction with Applications*, Sixth Edition, Universitext, Springer-Verlag Berlin Heidelberg ,(2007).

[26] Pascucci, A.:*PDEs and Martingale Methods in Option Pricing*,Springer Milan Dordrecht Heidelberg London New York,(2011).

[27] Peszat, S., Zabczyk, J. :*Stochastic Partial Differential Equations with Lévy Noise*, Cambridge University Press (2007).

[28] Revuz, D., Yor, M. :*Continuous Martingales and Brownian Motion*, Springer-Verlag Berlin Heidelberg 1991, 1994, 1999.

[29] Rüdiger, B.:*Stochastic integration with respect to Poisson random measure on seperable Banach space*,Stoch. stoch. Rep. 76(2004)213-242.

[30] Rüdiger, B., Ziglo, G. :*Ito Formula for Stochastic Integrals w.r.t. Compensated Poisson Random Measures on Seperable Banach Spaces*, No.187, (October 2004).

[31] Shreve, Steven E.:*Stochastic Calculus for Finance II*,Springer-Verlag New York,LLC(2004).

[32] Skorokhod, A.V. :*Studies in the Theory of Random Processes*, Addison-Wesley, Reading, Massachusetts (1965).

[33] Sz.-Nagy, B. :*Sur les contractions de l'espace de Hilbert*, Acta Sci. Math. Szeged 15 (1953), 87-92.

[34] Sz.-Nagy, B. :*Transfotmations de l'espace de Hilbert, functions de type positif sur un groupe*, Acta Sci. Math. Szeged 15 (1954), 104-114.

[35] Sz.-Nagy, B., Foias, C. :*Harmonic analysis of operators on Hilbert space*, Translated from the French and revised. North-Holland Publishing Co., Amsterdam, 1970.