Equivariant Vector Bundles and Rigid Cohomology on Drinfeld's Upper Half Space over a Finite Field

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Introduction

Cohomology theories in algebraic and arithmetic geometry provide an important tool to construct and study representations of a group (for example: an algebraic group, a finite group of Lie type, a p-adic Lie group or a Galois group) which roughly works as follows:

- Find a geometric object (for example an algebraic or analytic variety) on which the group acts.
- Pick a cohomology theory with the desired coefficients in such a way that the group acts on the cohomology of the geometric object.

One very prominent aspect of this approach is the use of ℓ-adic cohomology theories to study representations with ℓ-adic coefficients of finite groups of Lie type (over a finite field k of characteristic $p \neq \ell$, with q elements). According to Lusztig [39, Item 17], this development was started by Tate and Thompson [56] in 1965. They realized a certain irreducible representation of the finite unitary group $U_3(k')$ (of degree 3, over a quadratic extension k' of k) as the first ℓ-adic cohomology of the projective curve over k defined by the equation $X_0^{q+1} + X_1^{q+1} + X_2^{q+1} = 0$.

In 1974 Drinfeld [8] constructed a Langlands correspondence for the general linear group of degree 2 over a function field of positive characteristic. He further observed that all so-called cuspidal representations (with ℓ-adic coefficients) of the finite special linear group of degree 2 over k can be realized in the ℓ-adic cohomology (with compact supports) of the affine curve defined by the equation $XY^q - X^qY = 1$ that now bears his name. This was taken up around the same time by Lusztig [38] and a couple of years later in a joint paper with Deligne [6]: For an arbitrary connected reductive group over k and for each element of the Weyl group of this reductive group, they constructed varieties – today called Deligne-Lusztig varieties – whose ℓ-adic cohomology with compact supports with respect to certain coefficient systems realizes all irreducible representations of the finite group of Lie type associated with the given algebraic group.

Drinfeld's affine curve reappears as a finite principal covering of a certain Deligne-Lusztig variety for the group SL_2 (associated with the Coxeter element of its Weyl group) which is isomorphic to Drinfeld's upper half plane $\mathbb{P}^1_k \setminus \mathbb{P}^1_k(k) \cong \mathbb{A}^1_k \setminus \mathbb{A}^1_k(k)$ over k. This leads to the following generalization: Drinfeld's upper half space over k (of dimension n) is by definition the affine k-variety obtained by removing all k-rational hyperplanes from $\mathbb{P}^n_k$. It was first defined and studied by Drinfeld [9] in 1976. This variety is always isomorphic to the Deligne-Lusztig variety for the group SL_{n+1} (or GL_{n+1}) associated with the standard Coxeter element of its Weyl group. There is an analogous definition – also introduced by Drinfeld – in the case where k is replaced by a finite field extension K of the field $\mathbb{Q}_p$ of p-adic numbers. In this case, the resulting rigid-analytic space is of high interest in the local Langlands program, since the ℓ-adic cohomology of certain coverings of this space realizes the full supercuspidal spectrum of the local Langlands correspondence, as proved by Harris and Taylor in [21] in 2001.

There are of course several other more geometrically motivated reasons why studying any of the spaces mentioned above is of interest. For example, Drinfeld's upper half space over a p-adic field possesses an interpretation as the generic fiber of a formal scheme which parametrizes certain p-divisible groups, cf. [9]. Furthermore, it turns up in p-adic Hodge theory, cf. e.g. the introduction in [5], and its counterpart over a finite field can be viewed as a "toy model" for the p-adic theory, as the geometry involved is considerably easier over a finite field. According to Pink [47], Drinfeld's upper half space over a finite field can be viewed as a moduli space for certain Drinfeld modules which themselves are of particular interest in the Langlands program.

A (partial) list of known results on the cohomologies of Drinfeld's upper half space reads as follows:

- Over a p-adic field, the de Rham cohomology of Drinfeld's upper half space was determined by Schneider and Stuhler [54] in 1991. Different proofs of their result were later given by de Shalit [7], Iovita and Spieß [30], Alon and de Shalit [1], and Orlik [45]. Furthermore, Schneider and Stuhler also computed the ℓ-adic cohomology in this situation.
- The ℓ-adic cohomology of Drinfeld's upper half space over a finite field k was computed by Orlik [42] as a representation of the group of k-rational points of the general linear group over k and as a Galois representation[1]. In particular, Orlik's complex, which will be used extensively in this thesis, appears for the first time in [42]. Partial results were known before due to Kottwitz and Rapoport, cf. [50, 51].
- In the p-adic situation, Orlik considered in [44] equivariant vector bundles for the general linear group over the respective p-adic field on Drinfeld's upper half space. He managed to construct and describe equivariant filtrations on the spaces of global sections of those bundles, generalizing earlier work of Schneider and Teitelbaum [55] and of Pohlkamp [48].
- Große-Klönne studied in [14] the rigid cohomology of Deligne-Lusztig varieties and therefore in particular of Drinfeld's upper half space over a finite field. Among other results, he found that after identifying the respective coefficient fields, the Euler-Poincaré characteristic of this cohomology is the same as the one obtained from ℓ-adic cohomology, seen essentially as a (virtual) module over the finite group of Lie type associated with the respective general linear group over k.

Content of this Thesis

The object of interest in this work is Drinfeld's upper half space[2] $\mathcal{X}^{(n+1)} \subset \mathbb{P}_k^n = \mathbf{Proj}\, k[T_0, \ldots, T_n]$ over a finite field k. As explained above, this space is defined as the complement of the union of all k-rational hyperplanes in $\mathbb{P}_k^n$ and therefore, it carries the structure of an affine k-variety which is Zariski-open in $\mathbb{P}_k^n$. Fix an algebraic action of the algebraic k-group scheme $\mathbf{G}_k = \mathrm{GL}_{n+1,k}$ on $\mathbb{P}_k^n$ by $(g, x) \mapsto g.x = xg^{-1}$ (on closed points). Then this action induces one of the finite group $G = \mathbf{G}_k(k)$ of k-rational points of $\mathbf{G}_k$ on $\mathcal{X}^{(n+1)}$.

Part I: Equivariant Vector Bundles on Drinfeld's Upper Half Space over a Finite Field

In the first part (Chapter 2) of this thesis, the cohomology $\mathrm{H}^*(\mathcal{X}^{(n+1)}, \mathcal{F})$ of $\mathcal{X}^{(n+1)}$ with coefficients in a $\mathbf{G}_k$-equivariant vector bundle $\mathcal{F}$ on $\mathbb{P}_k^n$ is considered. This is the analog over k of the situation studied by Orlik in [44] over a p-adic field. In both situations, $\mathcal{F}$ has no higher cohomology on $\mathcal{X}^{(n+1)}$, i.e.

$$\mathrm{H}^*(\mathcal{X}^{(n+1)}, \mathcal{F}) = \mathrm{H}^0(\mathcal{X}^{(n+1)}, \mathcal{F}),$$

[1]Note that on both the ℓ-adic and the p-adic cohomology there is an induced action of the absolute Galois group of k which comes via functoriality from the Galois action on Drinfeld's upper half space.

[2]See Chapter 1 for a detailed account of the notation used in the sequel.

as $\mathcal{X}^{(n+1)}$ is affine (resp. a Stein space in [44]). As in op. cit., one can construct a complex indexed by certain closed subvarieties of the complement $\mathcal{Y}^{(n+1)}$ of $\mathcal{X}^{(n+1)}$ in $\mathbb{P}^n_k$ with values in the constant sheaf with value $\mathbb{Z}$ over $\mathcal{Y}^{(n+1)}$. This complex – called Orlik's complex from now on – is acyclic and induces a spectral sequence which computes pieces of a G-equivariant filtration on $\mathrm{H}^0(\mathcal{X}^{(n+1)}, \mathcal{F})$. These are then described in terms of the following objects. Let $j \in \{0, \ldots, n-1\}$.

- Denote by $\mathbb{P}^j_k$ the subvariety of $\mathbb{P}^n_k$ characterized by the vanishing of the coordinate functions $T_{j+1}, \ldots, T_n$. This subvariety is stabilized by the standard-parabolic subgroup $\mathbf{P}_{(j+1,n-j),k}$ (with respect to the Borel subgroup $\mathbf{B}_k$ of lower triangular matrices) of $\mathbf{G}_k$ which is associated with the decomposition $n+1 = (j+1) + (n-j)$.
- Let
$$\tilde{\mathrm{H}}^{n-j}_{\mathbb{P}^j_k}(\mathbb{P}^n_k, \mathcal{F}) = \ker\left(\mathrm{H}^{n-j}_{\mathbb{P}^j_k}(\mathbb{P}^n_k, \mathcal{F}) \to \mathrm{H}^{n-j}(\mathbb{P}^n_k, \mathcal{F})\right),$$
where $\mathrm{H}^{n-j}_{\mathbb{P}^j_k}(\mathbb{P}^n_k, \mathcal{F})$ is the local cohomology of $\mathbb{P}^n_k$ with values in $\mathcal{F}$ and support in $\mathbb{P}^j_k$ and the map is the one which appears in the long exact cohomology sequence associated with the closed embedding $\mathbb{P}^j_k \subset \mathbb{P}^n_k$. The module $\tilde{\mathrm{H}}^{n-j}_{\mathbb{P}^j_k}(\mathbb{P}^n_k, \mathcal{F})$ carries naturally the structure of a representation of the group $\mathbf{P}_{(j+1,n-j),k}$.
- Write $\mathrm{St}_{n-j}(k)$ for the Steinberg representation of the factor $\mathrm{GL}_{n-j,k}(k)$ appearing in the group of k-valued points of the Levi subgroup $\mathbf{L}_{(j+1,n-j),k}$ of $\mathbf{P}_{(j+1,n-j),k}$ and denote by $v^G_{\mathbf{P}_{(j+1,1^{n-j}),k}(k)}(k)$ the generalized Steinberg representation of G with respect to the finite standard-parabolic subgroup $\mathbf{P}_{(j+1,1^{n-j}),k}(k)$ (again with respect to $\mathbf{B}_k$) of G associated with the decomposition $n+1 = (j+1) + (n-j) \cdot 1$. Write $v^G_{\mathbf{P}_{(j+1,1^{n-j}),k}(k)}(k)'$ for the k-dual of this module.

The precise result is the following:

Theorem. *(see Theorem 2.1.2.1) On $\mathrm{H}^0(\mathcal{X}^{(n+1)}, \mathcal{F})$ there is a filtration by G-submodules*
$$\mathrm{H}^0(\mathcal{X}^{(n+1)}, \mathcal{F}) = \mathcal{F}(\mathcal{X}^{(n+1)})^0 \supset \mathcal{F}(\mathcal{X}^{(n+1)})^1 \supset \ldots \supset \mathcal{F}(\mathcal{X}^{(n+1)})^n = \mathrm{H}^0(\mathbb{P}^n_k, \mathcal{F})$$
such that the successive quotients $\mathcal{F}(\mathcal{X}^{(n+1)})^j/\mathcal{F}(\mathcal{X}^{(n+1)})^{j+1}$ with $j \in \{0, \ldots, n-1\}$ appear as extensions in short exact sequences of G-modules
$$\begin{aligned}(0) \;&\to\; \mathrm{Ind}^G_{\mathbf{P}_{(j+1,n-j),k}(k)}(\tilde{\mathrm{H}}^{n-j}_{\mathbb{P}^j_k}(\mathbb{P}^n_k, \mathcal{F}) \otimes_k \mathrm{St}_{n-j}(k)) \to \mathcal{F}(\mathcal{X}^{(n+1)})^j/\mathcal{F}(\mathcal{X}^{(n+1)})^{j+1} \\ &\to\; v^G_{\mathbf{P}_{(j+1,1^{n-j}),k}(k)}(k)' \otimes_k \mathrm{H}^{n-j}(\mathbb{P}^n_k, \mathcal{F}) \to (0). \qquad (1)\end{aligned}$$

This theorem is a careful translation of Orlik's result [44, Corollary 2.2.9] and its algebraic content. With the help of this filtration, the problem of describing $\mathrm{H}^0(\mathcal{X}^{(n+1)}, \mathcal{F})$ as a G-module essentially reduces to describing for each $j = 0, \ldots, n-1$ the reduced local cohomology module $\tilde{\mathrm{H}}^{n-j}_{\mathbb{P}^j_k}(\mathbb{P}^n_k, \mathcal{F})$. Under different assumptions, there are given three types of descriptions of these modules, essentially as representations of $\mathbf{L}_{(j+1,n-j),k}$ resp. its subgroup $\mathbf{L}_{(j+1,n-j),k}(k)$ of k-valued points.

The first description is for bundles $\mathcal{F} = \mathcal{F}_\lambda$ which are associated with an integral weight $\lambda = (\lambda_0, \lambda_1, \ldots, \lambda_n)$ of $\mathbf{G}_k$ where $\lambda_0 \in \mathbb{Z}$ and $(\lambda_1, \ldots, \lambda_n)$ is a dominant integral weight of $\mathrm{GL}_{n,k}$ (which is identified as a subgroup of $\mathbf{G}_k$). The result is a direct translation of Orlik's result [44, 1.4.2] and reads as follows.

Proposition. *(see Proposition 2.2.1.1) For $j \in \{0, \dots, n-1\}$ there is an irreducible $\mathbf{L}_{(j+1,n-j),k}$-module $L_{z_{n-j}^{-1}\mu_{n-j,\lambda}}$, depending only on j and λ, such that the $\mathbf{L}_{(j+1,n-j),k}$-module $\tilde{\mathrm{H}}^{n-j}_{\mathbb{P}^j_k}(\mathbb{P}^n_k, \mathcal{F}_\lambda)$ is a quotient of the $\mathbf{L}_{(j+1,n-j),k}$-module*

$$\begin{aligned} &\left(\bigoplus_{l\in\mathbb{N}_0} \mathbf{Sym}^l((k^{j+1})') \boxtimes_k \mathbf{Sym}^l((k^{n-j})')'\right) \otimes_k L_{z_{n-j}^{-1}\mu_{n-j,\lambda}} \\ \cong \quad &\left(\bigoplus_{l\in\mathbb{N}_0} V(l\cdot\epsilon_j)' \boxtimes_k V(l\cdot\epsilon_n)\right) \otimes_k L_{z_{n-j}^{-1}\mu_{n-j,\lambda}}. \end{aligned}$$

Here, the $V(l\cdot\epsilon_j)$ resp. $V(l\cdot\epsilon_n)$ are certain Weyl modules of the factors $\mathrm{GL}_{j+1,k}$ resp. $GL_{n-j,k}$ of $\mathbf{L}_{(j+1,n-j),k}$.

The second description uses the canonical projection $\mathbb{P}^n_k \setminus \mathbb{P}^j_k \to \mathbb{P}^{n-j-1}_k$ and for arbitrary bundles $\mathcal{F}$ it gives the following result:

Proposition. *(see Corollary 2.2.2.4) Let r be the rank of $\mathcal{F}$. Then there exist integers $a_1, \dots, a_r \in \mathbb{Z}$, depending on $\mathcal{F}$, such that the $\mathbf{L}_{(j+1,n-j),k}$-module $\tilde{\mathrm{H}}^{n-j}_{\mathbb{P}^j_k}(\mathbb{P}^n_k, \mathcal{F})$ is a quotient of*

$$\bigoplus_{l=1}^{r} \bigoplus_{\substack{m\in\mathbb{N}_0 \\ m-a_l\geq n-j}} V(m\cdot\epsilon_j)' \boxtimes_k \det{}^{-1} \otimes_k V((m-a_l-n+j)\cdot\epsilon_n).$$

Again, the modules appearing are Weyl modules for the factors of $\mathbf{L}_{(j+1,n-j),k}$. This result (and its proof) yield concrete descriptions for $\tilde{\mathrm{H}}^{n-j}_{\mathbb{P}^j_k}(\mathbb{P}^n_k, \mathcal{F})$ in the cases of twisted structure sheaves $\mathcal{F} = \mathcal{O}_{\mathbb{P}^n_k}(i), i \in \mathbb{Z}$, and sheaves of differential forms $\mathcal{F} = \Omega^i_{\mathbb{P}^n_k/k}, i = \{0, \dots, n\}$.

The third description is for bundles $\mathcal{F}$ such that $\bigoplus_{i\in\mathbb{N}_0} \mathrm{H}^i(\mathbb{P}^n_k, \mathcal{F}(i))$ is a graded $k[T_0, \dots, T_n]$-module generated in degrees ≤ 1 and which are acted upon by a k-algebra $\widehat{U}(\mathfrak{g}_k)_c$ defined as follows: Let K be a finite extension field of $\mathbb{Q}_p$ with valuation ring $\mathcal{V}$ and residue field k. Let $\mathbf{G} := \mathrm{GL}_{n+1,\mathcal{V}}$ (viewed as an algebraic $\mathcal{V}$-group scheme) and similarly write $\mathbf{P}_{(j+1,n-j)}$ resp. $\mathbf{L}_{(j+1,n-j)}$ for the parabolic subgroup scheme associated with the decomposition $n+1 = (j+1)+(n-j)$ resp. its Levi subgroup (with respect to the Borel subgroup of lower triangular matrices). The algebraic k-group schemes used above are then the respective base changes to k of these group schemes. Consider the universal enveloping algebra $U(\mathfrak{g} \otimes_{\mathcal{V}} K)$, where $\mathfrak{g} = \mathrm{Lie}(\mathbf{G})$ is the Lie algebra of $\mathbf{G}$ with standard $\mathcal{V}$-basis $\{L_{(u,v)} \mid 0 \leq u, v \leq n\}$ corresponding to elementary matrices. Define a $\mathcal{V}$-subalgebra $\widehat{U}(\mathfrak{g})$ which is the subalgebra generated by all divided powers

$$\frac{1}{(\sum_{0\leq u\neq v\leq n} m_{u,v})!} \prod_{0\leq u\neq v\leq n} L_{(u,v)}^{m_{u,v}}$$

for $m_{u,v} \in \mathbb{N}_0$, and

$$\frac{L_{(u,u)} \cdot (L_{(u,u)} - 1) \cdot \ldots \cdot (L_{(u,u)} - m + 1)}{m!}$$

for $0 \leq u \leq n$, $m \in \mathbb{N}_0$. This algebra is called the enriched crystalline enveloping algebra associated with $\mathbf{G}$. Such an algebra can be associated with each subgroup scheme of $\mathbf{G}$, in particular with the opposite $\mathbf{U}^+_{(j+1,n-j)}$ of the unipotent radical of $\mathbf{P}_{(j+1,n-j)}$: The enriched crystalline enveloping algebra associated with $\mathbf{U}^+_{(j+1,n-j)}$ is the subalgebra $\widehat{U}(\mathfrak{u}^+_{(j+1,n-j)})$ of $\widehat{U}(\mathfrak{g})$ which is generated by all divided powers

$$\frac{1}{(\sum_{0\leq u\neq v\leq n} m_{u,v})!} \prod_{0\leq u\neq v\leq n} L_{(u,v)}^{m_{u,v}}$$

for $m_{u,v} \in \mathbb{N}_0$ with $m_{u,v} = 0$ if $u \in \{j+1, \dots, n\}$ or $v \in \{0, \dots, j\}$. The algebras $\widehat{U}(\mathfrak{g})$ and $\widehat{U}(\mathfrak{u}^+_{(j+1,n-j)})$ are too big to admit meaningful representations in the context of this work. Therefore, the following subalgebras $\widehat{U}(\mathfrak{g})_c \subset \widehat{U}(\mathfrak{g})$ resp. $\widehat{U}(\mathfrak{u}^+_{(j+1,n-j)})_c \subset \widehat{U}(\mathfrak{u}^+_{(j+1,n-j)})$ are defined:
By definition, $\widehat{U}(\mathfrak{g})_c$ is the subalgebra of $\widehat{U}(\mathfrak{g})$ generated by all divided powers

$$\frac{1}{(\sum_{0 \le u \ne v \le n} m_{u,v})!} \prod_{0 \le u \ne v \le n} L_{(u,v)}^{m_{u,v}}$$

for $m_{u,v} \in \mathbb{N}_0$, such that there exists $w \in \mathbb{N}_0$ with $m_{u,v} = 0$ for all $(u,v) \in \{0, \dots, n\} \times \{0, \dots, n\} \setminus \{w\}$ and by all

$$\frac{L_{(u,u)} \cdot (L_{(u,u)} - 1) \cdot \ldots \cdot (L_{(u,u)} - m + 1)}{m!}$$

for $0 \le u \le n$, $m \in \mathbb{N}_0$. The algebra $\widehat{U}(\mathfrak{u}^+_{(j+1,n-j)})_c$ is then defined as the intersection of $\widehat{U}(\mathfrak{g})_c$ and $\widehat{U}(\mathfrak{u}^+_{(j+1,n-j)})$ in $\widehat{U}(\mathfrak{g})$. Denote by $\widehat{U}(\mathfrak{g}_k)$ the base change to k of $\widehat{U}(\mathfrak{g})$ and similarly for the other algebras just defined. Finally, write ${}^{\mathbf{L}}\widehat{U}(\mathfrak{u}^+_{(j+1,n-j),k})_c$ for the algebra which is generated by all $\mathbf{L}_{(j+1,n-j),k}(k)$-translates of $\widehat{U}(\mathfrak{u}^+_{(j+1,n-j),k})_c$ inside $\widehat{U}(\mathfrak{u}^+_{(j+1,n-j),k})$. The result concerning $\tilde{\mathrm{H}}^{n-j}_{\mathbb{P}^j_k}(\mathbb{P}^n_k, \mathcal{F})$ is then the following.

Proposition. *(see Proposition 2.5.1.3) Under the assumption that $\widehat{U}(\mathfrak{g}_k)_c$ acts on $\mathcal{F}$ (induced by the $\mathbf{G}_k$-action on $\mathcal{F}$) and that the graded $k[T_0, \dots, T_n]$-module $\bigoplus_{i \in \mathbb{N}_0} \mathrm{H}^i(\mathbb{P}^n_k, \mathcal{F}(i))$ is generated in degrees ≤ 1, there is a $\mathbf{P}_{(j+1,n-j),k}$-submodule $N_j \subset \tilde{\mathrm{H}}^{n-j}_{\mathbb{P}^j_k}(\mathbb{P}^n_k, \mathcal{F})$ of finite k-dimension and an epimorphism of $\mathbf{L}_{(j+1,n-j),k}(k)$-modules*

$$\varphi_j : {}^{\mathbf{L}}\widehat{U}(\mathfrak{u}^+_{(j+1,n-j),k})_c \otimes_k N_j \twoheadrightarrow \tilde{\mathrm{H}}^{n-j}_{\mathbb{P}^j_k}(\mathbb{P}^n_k, \mathcal{F}).$$

It should be pointed out here that the direct analogy with Orlik's paper [44] fails: $\tilde{\mathrm{H}}^{n-j}_{\mathbb{P}^j_k}(\mathbb{P}^n_k, \mathcal{F})$ is not finitely generated as a module over either the universal enveloping algebra $U(\mathfrak{g}_k)$ or the distribution algebra $\mathrm{Dist}(\mathbf{G}_k)$. This is shown in Section 2.3.2. Therefore, it seems that the algebra $\widehat{U}(\mathfrak{g}_k)$ and its subalgebras are the proper replacements for $\mathrm{Dist}(\mathbf{G}_k)$ resp. $U(\mathfrak{g}_k)$ in the context of this work. In the first chapter, a notion of a semisimplification $M^{H-ss.}$ of a filtered H-module M for a finite group H is constructed. The above proposition is then used to obtain the following theorem. Write $\mathfrak{d}_j = \ker(\varphi_j)$.

Theorem. *(see Theorem 2.5.1.4) Under the assumptions on $\mathcal{F}$ made above, the $\mathbf{P}_{(j+1,n-j),k}(k)$-semisimplifications $({}^{\mathbf{L}}\widehat{U}(\mathfrak{u}^+_{(j+1,n-j),k})_c \otimes_k N_j/\mathfrak{d}_j)^{\mathbf{P}_{(j+1,n-j),k}(k)-ss.}$ and $(\tilde{\mathrm{H}}^{n-j}_{\mathbb{P}^j_k}(\mathbb{P}^n_k, \mathcal{F}))^{\mathbf{P}_{(j+1,n-j),k}(k)-ss.}$ exist and there is an isomorphism of $\mathbf{P}_{(j+1,n-j),k}(k)$-modules*

$$({}^{\mathbf{L}}\widehat{U}(\mathfrak{u}^+_{(j+1,n-j),k})_c \otimes_k N_j/\mathfrak{d}_j)^{\mathbf{P}_{(j+1,n-j),k}(k)-ss.} \xrightarrow{\sim} (\tilde{\mathrm{H}}^{n-j}_{\mathbb{P}^j_k}(\mathbb{P}^n_k, \mathcal{F}))^{\mathbf{P}_{(j+1,n-j),k}(k)-ss.}.$$

For a graded k-vector space $V = \bigoplus_{i \in \mathbb{Z}} V_i$, denote by $V^\vee$ its graded dual, i.e. $V^\vee = \bigoplus_{i \in \mathbb{Z}} V_i'$. It is shown that the $\mathbf{L}_{(j+1,n-j),k}(k)$-modules $\widehat{U}(\mathfrak{u}^+_{(j+1,n-j),k})$ and $\mathcal{O}(\mathbf{U}^+_{(j+1,n-j),k})$ are in graded duality with each other (Lemma 2.4.2.4). This duality leads to the definition of a functor $\mathrm{Ind}^{G,\mathbf{U}^+-alg}_{\mathbf{P}_{(j+1,n-j),k}(k)}(-)$ from the category of $\mathbf{L}_{(j+1,n-j),k}(k)$-modules (with inflated $\mathbf{P}_{(j+1,n-j),k}(k)$-action) to the category of G-modules, defined by

$$\mathrm{Ind}^{G,\mathbf{U}^+-alg}_{\mathbf{P}_{(j+1,n-j),k}(k)}(-) = \mathrm{Ind}^{G}_{\mathbf{P}_{(j+1,n-j),k}(k)}(\mathcal{O}(\mathbf{U}^+_{(j+1,n-j),k}) \otimes_k -).$$

The above duality for $\widehat{U}(\mathfrak{u}^+_{(j+1,n-j),k})$ is extended to one for the module

$$\tilde{\mathrm{H}}^{n-j}_{\mathbb{P}^j_k}(\mathbb{P}^n_k,\mathcal{F})\otimes_k \mathrm{St}_{n-j}(k)\cong \left({}^{\mathbf{L}}\widehat{U}(\mathfrak{u}^+_{(j+1,n-j),k})_c\otimes_k N_j/\mathfrak{d}_j\right)\otimes_k \mathrm{St}_{n-j}(k)$$

appearing in (1). Under this extended (and then induced) duality, the G-module

$$\mathrm{Ind}^G_{\mathbf{P}_{(j+1,n-j),k}(k)}\left(\left({}^{\mathbf{L}}\widehat{U}(\mathfrak{u}^+_{(j+1,n-j),k})_c\otimes_k N_j\right)/\mathfrak{d}\otimes_k \mathrm{St}_{n-j}(k)\right)$$

corresponds to a subquotient

$$\mathrm{Ind}^{G,\mathbf{U}^+-alg}_{\mathbf{P}_{(j+1,n-j),k}(k)}(N'_j\otimes \mathrm{St}_{n-j}(k)')^{\mathfrak{d}_j\otimes \mathrm{St}_{n-j}(k)}_{\mathrm{red}}$$

of $\mathrm{Ind}^{G,\mathbf{U}^+-alg}_{\mathbf{P}_{(j+1,n-j),k}(k)}(N'_j\otimes \mathrm{St}_{n-j}(k)')$, see Proposition 2.4.2.9. This yields a reinterpretation of the kernels of the short exact sequences for the filtration steps of the above filtration on $\mathrm{H}^0(\mathcal{X}^{(n+1)},\mathcal{F})$.

Theorem. *(see Theorem 2.5.2.4) Let $j\in\{0,\dots,n-1\}$. Under the assumptions on $\mathcal{F}$ made in the last theorem, the G-semisimplifications of the quotients $\mathcal{F}(\mathcal{X}^{(n+1)})^j/\mathcal{F}(\mathcal{X}^{(n+1)})^{j+1}$ exist and appear as extensions in short exact sequences of G-modules*

$$\begin{aligned}(0)\ &\to\ \left(\left(\mathrm{Ind}^{G,\mathbf{U}^+-alg}_{\mathbf{P}_{(j+1,n-j),k}(k)}(N'_j\otimes \mathrm{St}_{n-j}(k)')^{\mathfrak{d}_j\otimes \mathrm{St}_{n-j}(k)}_{\mathrm{red}}\right)^\vee\right)^{G-ss.}\to (\mathcal{F}(\mathcal{X}^{(n+1)})^j/\mathcal{F}(\mathcal{X}^{(n+1)})^{j+1})^{G-ss.}\\ &\to\ (v^G_{\mathbf{P}_{(j+1,1^{n-j}),k}(k)}(k)'\otimes_k \mathrm{H}^{n-j}(\mathbb{P}^n_k,\mathcal{F}))^{G-ss.}\to(0).\end{aligned}$$

It is shown in Lemma 2.6.2.1 that the algebra $\widehat{U}(\mathfrak{g}_k)_c$ acts on the twisted structure sheaves $\mathcal{O}_{\mathbb{P}^n_k}(i), i\in\mathbb{Z}$, and on the sheaves of differential i-forms $\Omega^i_{\mathbb{P}^n_k/k}$, $i=0,\dots,n$, on $\mathbb{P}^n_k$ (induced by the $\mathbf{G}_k$-action on these sheaves). Of these, the sheaves $\mathcal{O}_{\mathbb{P}^n_k}(i), i\geq -1$, satisfy the assumptions made in the theorem.

Part II: Rigid Cohomology of Drinfeld's Upper Half Space over a Finite Field

In the second part (Chapter 3) of this thesis, the rigid cohomology $\mathrm{H}^*_{\mathrm{rig}}(\mathcal{X}^{(n+1)}/K)$ (resp. the rigid cohomology $\mathrm{H}^*_{\mathrm{rig,c}}(\mathcal{X}^{(n+1)}/K)$ "with compact supports") of $\mathcal{X}^{(n+1)}$ with coefficients in a finite field extension $K/\mathbb{Q}_p$ with residue field k is considered as a representation of both G and the absolute Galois group $\mathrm{Gal}(\overline{k}/k)$ of k. The result is the following theorem.

Theorem. *(see Theorem 3.2.4.3 and Theorem 3.3.4.2) The rigid cohomology of $\mathcal{X}^{(n+1)}$ with coefficients in K is*

i) $\mathrm{H}^*_{\mathrm{rig,c}}(\mathcal{X}^{(n+1)}/K)=\bigoplus_{i=0}^n v^G_{\mathbf{P}_{(i+1,1^{n-i}),k}(k)}(K)(-i)[-n-i]$,

ii) $\mathrm{H}^*_{\mathrm{rig}}(\mathcal{X}^{(n+1)}/K)=\bigoplus_{i=0}^n v^G_{\mathbf{P}_{(i+1,1^{n-i}),k}(k)}(K)'(i-2n)[-n+i]$.

The modules appearing in this theorem are again generalized Steinberg representations, this time with coefficients in K. Furthermore, $[-i]$ means that the respective module lives in degree i and (i) determines the Tate twist, i.e. its structure as a $\mathrm{Gal}(\overline{k}/k)$-representation (over K).

As stated above, the Euler-Poincaré characteristic $\sum_{i\in\mathbb{N}_0}(-1)^i\mathrm{H}^i_{\mathrm{rig,c}}(\mathcal{X}^{(n+1)}/K)$ (considered as a virtual G-module) was determined by Große-Klönne [14]. Here, this cohomology is computed once directly as a hypercohomology and once by considering the associated de Rham complex. Both times, modified versions of Orlik's complex from Part I are used. For technical reasons, many computations take place in the category of Huber's adic spaces.

Rigid cohomology computed directly as a hypercohomology

Recall that for a projective k-variety $Y \subset \mathbb{P}_k^n$, its rigid cohomology is simply the de Rham cohomology $\mathrm{H}^*_{\mathrm{dR}}(]Y[_p, K)$ with values in K of its rigid analytic tube $]Y[_P \subset \mathbb{P}_K^{n,\mathrm{rig}}$. Therefore, in this first part, the de Rham cohomology of the rigid analytic tube $]\mathcal{Y}^{(n+1)}[_P$ of $\mathcal{Y}^{(n+1)}$ (the complement of $\mathcal{X}^{(n+1)}$ in $\mathbb{P}_k^n$) is computed and the rigid cohomology of $\mathcal{X}^{(n+1)}$ is then known by applying the long exact cohomology sequence for the pair of inclusions $\mathcal{X}^{(n+1)} \overset{\text{open}}{\subset} \mathbb{P}_k^n \overset{\text{closed}}{\supset} \mathcal{Y}^{(n+1)}$. The key ingredient for the computation of the rigid cohomology $\mathrm{H}^*_{\mathrm{rig}}(\mathcal{Y}^{(n+1)}/K) = \mathrm{H}^*_{\mathrm{dR}}(]\mathcal{Y}^{(n+1)}[_P, K)$ is a modified version of Orlik's complex for $]\mathcal{Y}^{(n+1)}[_P$ with values in the de Rham complex $\Omega^\bullet_{]\mathcal{Y}^{(n+1)}[_P/K}$ on $]\mathcal{Y}^{(n+1)}[_P$. In this situation, Orlik's complex is indexed by the tubes associated with certain projective subvarieties of $\mathcal{Y}^{(n+1)}$ and as in the first part of this thesis, this complex is shown to be acyclic, see Corollary 3.2.1.2. It then induces a spectral sequence converging to $\mathrm{H}^*_{\mathrm{rig}}(\mathcal{Y}^{(n+1)}/K)$ with computable entries on the E_1-page. This spectral sequence degenerates on its E_2-page (Lemma 3.2.3.2) and the evaluation of the associated grading on $\mathrm{H}^*_{\mathrm{rig}}(\mathcal{Y}^{(n+1)}/K)$ then yields this cohomology (Proposition 3.2.4.1).

Rigid cohomology computed from the associated de Rham complex

For a quasi-projective k-variety $X \subset \mathbb{P}_k^n$, its rigid cohomology $\mathrm{H}^*_{\mathrm{rig}}(X/K)$ is defined as the hypercohomology on $\mathbb{P}_K^{n,\mathrm{rig}}$ with values in the direct limit of the de Rham complexes of a system of strict open neighborhoods of $]X[_P$, the rigid-analytic tube of X, in $\mathbb{P}_K^{n,\mathrm{rig}}$. In the first step towards computing $\mathrm{H}^*_{\mathrm{rig}}(\mathcal{X}^{(n+1)}/K)$, a cofinal system $(U^m)_{m\in\mathbb{N}}$ of affinoid strict open neighborhoods of $]\mathcal{X}^{(n+1)}[_P$ in $\mathbb{P}_K^{n,\mathrm{rig}}$ is constructed, see Lemma 3.3.0.4. It is then shown that in order to use the associated de Rham complex for the determination of the above hypercohomology, it is enough to compute the spaces of sections $\mathrm{H}^0(U^m, \Omega^i_{\mathbb{P}_K^{n,\mathrm{rig}}/K})$ of the sheaves of differential i-forms $\Omega^i_{\mathbb{P}_K^{n,\mathrm{rig}}/K}$ on U^m and then take their direct limit (Lemma 3.3.0.5). The spaces U^m are constructed in such a way that one can again make use of an adapted version of Orlik's complex to determine all spaces $\mathrm{H}^0(U^m, \Omega^i_{\mathbb{P}_K^{n,\mathrm{rig}}/K})$. To be precise, Orlik's complex in this situation (which is again acyclic, see Proposition 3.3.1.1) yields a spectral sequence which computes the local cohomology $\mathrm{H}^1_{Y^m}(\mathbb{P}_K^{n,\mathrm{rig}}, \Omega^i_{\mathbb{P}_K^{n,\mathrm{rig}}/K})$ of $\mathbb{P}_K^{n,\mathrm{rig}}$ with supports in the complement $Y^m = \mathbb{P}_K^{n,\mathrm{rig}} \setminus U^m$ of U^m in $\mathbb{P}_K^{n,\mathrm{rig}}$ and values in $\Omega^i_{\mathbb{P}_K^{n,\mathrm{rig}}/K}$. The key to the evaluation of this spectral sequence is again the fact that it has computable entries in its E_1-page, see Lemma 3.3.3.1. The remainder of the evaluation of this spectral sequence then proceeds in analogy with the respective section of Orlik's paper [44]. The result is the following. Recall that $\mathbf{G} = \mathrm{GL}_{n+1,\mathcal{V}}$.

Theorem. *(see Theorem 3.3.3.3) On each* $\mathrm{H}^0(U^m, \Omega^i_{\mathbb{P}_K^{n,\mathrm{rig}}})$ *there exists a filtration*

$$\begin{aligned}\Omega^i_{\mathbb{P}_K^{n,\mathrm{rig}}}(U^m)^\bullet \quad = \quad & \Big(\mathrm{H}^0(U^m, \Omega^i_{\mathbb{P}_K^{n,\mathrm{rig}}}) = \Omega^i_{\mathbb{P}_K^{n,\mathrm{rig}}}(U^m)^0 \supset \Omega^i_{\mathbb{P}_K^{n,\mathrm{rig}}}(U^m)^1 \supset \dots \\ & \dots \supset \Omega^i_{\mathbb{P}_K^{n,\mathrm{rig}}}(U^m)^{n-1} \supset \Omega^i_{\mathbb{P}_K^{n,\mathrm{rig}}}(U^m)^n = \mathrm{H}^0(\mathbb{P}_K^{n,\mathrm{rig}}, \Omega^i_{\mathbb{P}_K^{n,\mathrm{rig}}})\Big)\end{aligned}$$

by $\mathbf{G}(\mathcal{V})$*-submodules such that each filtration step appears in a short exact sequence*

$$\begin{aligned}(0) \quad \to \quad & \mathrm{Ind}^{\mathbf{G}(\mathcal{V})}_{\mathbf{P}_{(j+1,n-j)}(\mathcal{V})(1)}(\tilde{\mathrm{H}}^{n-j}_{]\mathbb{P}_k^j[_{P,\lambda_m}}(\mathbb{P}_K^{n,\mathrm{rig}}, \Omega^i_{\mathbb{P}_K^{n,\mathrm{rig}}}) \otimes \mathrm{St}_{n-j}(K)) \to \mathcal{E}^i(U^m)^j/\mathcal{E}^i(U^m)^{j+1} \\ \to \quad & v^{\mathbf{G}(\mathcal{V})}_{\mathbf{P}_{(j+1,1^{n-j})}(\mathcal{V})(1)}(K)' \otimes_K \mathrm{H}^{n-j}(\mathbb{P}_K^{n,\mathrm{rig}}, \Omega^i_{\mathbb{P}_K^{n,\mathrm{rig}}}) \to (0)\end{aligned}$$

for $j = 0, \dots, n-1$. *For* $j = n$, *there is an identification*

$$\mathcal{E}^i(U^m)^n = \mathrm{H}^0(\mathbb{P}_K^{n,\mathrm{rig}}, \Omega^i_{\mathbb{P}_K^{n,\mathrm{rig}}}).$$

These filtrations are compatible with $\mathbf{G}$*-equivariant morphisms between the involved sheaves.*

Here, $\lambda_m = p^{1/m}$ and $]\mathbb{P}^j_k[_{P,\lambda_m}$ is the tube of radius λ_m of $\mathbb{P}^j_k$ in $\mathbb{P}^{n,\mathrm{rig}}_K$. The groups $\mathbf{P}_{(j+1,n-j)}(\mathcal{V})(1)$ and $\mathbf{P}_{(j+1,1^{n-j})}(\mathcal{V})(1)$ are the respective inverse images under the canonical map $\mathbf{G}(\mathcal{V}) \to G$ of the groups $\mathbf{P}_{(j+1,n-j),k}(k)$ and $\mathbf{P}_{(j+1,1^{n-j}),k}(k)$. Furthermore, the modules $\tilde{\mathrm{H}}^{n-j}_{]\mathbb{P}^j_k[_{P,\lambda_m}}(\mathbb{P}^{n,\mathrm{rig}}_K, \Omega^i_{\mathbb{P}^{n,\mathrm{rig}}_K})$, $\mathrm{St}_{n-j}(K)$ and $v^{\mathbf{G}(\mathcal{V})}_{\mathbf{P}_{(j+1,1^{n-j})}(\mathcal{V})(1)}(K)$ are the analogues of the modules appearing in Part I of this introduction.

The functoriality of the filtrations obtained in the last theorem then imply that the de Rham complex

$$0 \to \mathrm{H}^0(U^m, \mathcal{O}_{\mathbb{P}^{n,\mathrm{rig}}_K}) \to \mathrm{H}^0(U^m, \Omega^1_{\mathbb{P}^{n,\mathrm{rig}}_K}) \to \ldots \to \mathrm{H}^0(U^m, \Omega^n_{\mathbb{P}^{n,\mathrm{rig}}_K}) \to 0$$

is actually a filtered complex. The associated spectral sequence degenerates on its E_1-page with computable entries and taking direct limits one then obtains the desired rigid cohomology. The key result is Lemma 3.3.4.1 which asserts that for each $j = 0, \ldots, n-1$, the complex

$$\begin{aligned}(0) \quad &\to \varinjlim_{m\in\mathbb{N}} \tilde{\mathrm{H}}^{n-j}_{]\mathbb{P}^j_k[_{P,\lambda_m}}(\mathbb{P}^{n,\mathrm{rig}}_K, \Omega^0_{\mathbb{P}^{n,\mathrm{rig}}_K/K}) \to \varinjlim_{m\in\mathbb{N}} \tilde{\mathrm{H}}^{n-j}_{]\mathbb{P}^j_k[_{P,\lambda_m}}(\mathbb{P}^{n,\mathrm{rig}}_K, \Omega^1_{\mathbb{P}^{n,\mathrm{rig}}_K/K}) \to \ldots \\ &\to \varinjlim_{m\in\mathbb{N}} \tilde{\mathrm{H}}^{n-j}_{]\mathbb{P}^j_k[_{P,\lambda_m}}(\mathbb{P}^{n,\mathrm{rig}}_K, \Omega^n_{\mathbb{P}^{n,\mathrm{rig}}_K/K}) \to (0)\end{aligned}$$

consisting of direct limits of reduced local cohomologies with support in $]\mathbb{P}^j_k[_{P,\lambda_m}$ is acyclic.

Structure of this Work

For the convenience of the reader, a short overview of the organization of the content of this work is given in the sequel:

In Chapter 1 the general notation used in the course of this thesis is fixed in Section 1.1. Sections 1.2, 1.3 and 1.4 then contain some reminders on representations of groups, on Drinfeld's upper half space and on sheaves and their cohomology, respectively. In particular, in Section 1.2 a notion of semisimplification for a representation of a group is constructed and some of its properties needed in the sequel are proved.

Chapter 2 deals with the computation of the cohomology of $\mathbf{G}_k$-equivariant vector bundles on $\mathcal{X}^{(n+1)}$. The notion of a $\mathbf{G}_k$-equivariant vector bundle on a $\mathbf{G}_k$-scheme is recalled in an introduction to this chapter. In Section 2.1 the construction of Orlik's complex is reviewed. The description of the spaces $\tilde{\mathrm{H}}^i_{\mathbb{P}^j_k}(\mathbb{P}^n_k, \mathcal{F})$ starts in Section 2.2. In particular, this section contains the first two types of descriptions (Proposition 2.2.1.1 and Corollary 2.2.2.4) as mentioned above. Section 2.3 deals with the actions of the universal enveloping algebra of $\mathrm{Lie}(\mathbf{G}_k)$ and of the distribution algebra $\mathrm{Dist}(\mathbf{G}_k)$ on $\tilde{\mathrm{H}}^i_{\mathbb{P}^j_k}(\mathbb{P}^n_k, \mathcal{F})$. In particular, it is shown in Subsubsections 2.3.2 and 2.3.3 that $\tilde{\mathrm{H}}^{n-j}_{\mathbb{P}^j_k}(\mathbb{P}^n_k, \mathcal{F})$ is not a finitely generated module over either of these two algebras. In Section 2.4 the enriched crystalline enveloping algebra $\widehat{U}(\mathfrak{g})$ and its subalgebras $\widehat{U}(\mathfrak{g})_c$, $\widehat{U}(\mathfrak{g})_r$ are constructed and it is shown under which conditions representations of these algebras can be obtained from a representation of $\mathbf{G}$ resp. of $\mathbf{G}_k$. This is demonstrated in two examples in Subsection 2.4.1. The special case of the algebra $\widehat{U}(\mathfrak{u}^+_{(j+1,n-j),k})$ is considered in Subsection 2.4.2, both as an algebra and as a representation over the group $\mathbf{L}_{(j+1,n-j),k}$. In Section 2.5 the algebra ${}^{\mathbf{L}}\widehat{U}(\mathfrak{u}^+_{(j+1,n-j),k})_c$ is employed to give a description of the module $\tilde{\mathrm{H}}^{n-j}_{\mathbb{P}^j_k}(\mathbb{P}^n_k, \mathcal{F})$. Subsection 2.5.1 contains the result asserting that for bundles $\mathcal{F}$ on which $\widehat{U}(\mathfrak{g}^+_k)_c$ acts and which are of the type described above, there is an epimorphism

$${}^{\mathbf{L}}\widehat{U}(\mathfrak{u}^+_{(j+1,n-j),k})_c \otimes_k N_j \twoheadrightarrow \tilde{\mathrm{H}}^{n-j}_{\mathbb{P}^j_k}(\mathbb{P}^n_k, \mathcal{F})$$

of $\mathbf{L}_{(j+1,n-j),k}(k)$-modules for a $\mathbf{P}_{(j+1,n-j),k}$-module N_j of finite k-dimension (Proposition 2.5.1.3). This gives a description of the $\mathbf{P}_{(j+1,n-j),k}(k)$-semisimplification of $\tilde{\mathrm{H}}^{n-j}_{\mathbb{P}^j_k}(\mathbb{P}^n_k, \mathcal{F})$ as an epimorphic image of the $\mathbf{P}_{(j+1,n-j),k}(k)$-semisimplification of ${}^{\mathbf{L}}\widehat{U}(\mathfrak{u}^+_{(j+1,n-j),k})_c \otimes_k N_j$ in Theorem 2.5.1.4. This description is reinterpreted in terms of $\mathcal{O}(\mathbf{U}^+_{(j+1,n-j),k})$ in Subsection 2.5.2, followed by the definition of the functor $\mathrm{Ind}^{G,\mathbf{U}^+-alg}_{\mathbf{P}_{(j+1,n-j),k}(k)}$ and Theorem 2.5.2.4. Chapter 2 concludes with the study of examples in Section 2.6: First, some classes of G-modules resp. $\mathbf{G}_k$-equivariant vector bundles which carry actions of $\widehat{U}(\mathfrak{g}_k)_c$ are exhibited. Then the cohomologies $\mathrm{H}^0(\mathcal{X}^{(n+1)}, \mathcal{O}(a))$ $(a \in \mathbb{N}_0)$ of positively twisted structure sheaves are studied in light of Theorem 2.5.2.4.

Chapter 3 consists of four sections: In Section 3.1 the construction of rigid cohomology (of a quasi-projective k-variety) is reviewed together with some concepts from p-adic geometry which are needed in the sequel. Sections 3.2 and 3.3, where the computations of the rigid cohomologies in the two ways mentioned are done, then each have the following structure: Orlik's complex is adapted for certain classes of tubes of rigid varieties (Subsections 3.2.1 and 3.3.1), a spectral sequence is set up (Subsections 3.2.2 and 3.3.2), computed (Subsections 3.2.3 and 3.3.3) and the result is used to finally compute the rigid cohomology modules of $\mathcal{X}^{(n+1)}$ (Subsections 3.2.4 and 3.3.4).

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Chapter 1

Preliminaries

In this first brief chapter, the general notation used throughout this work shall be fixed. Furthermore, there are some short reminders concerning the following topics: the representation theories of the groups involved in this thesis, the definition of the main object studied (Drinfeld's upper half space over a finite field), on flasque resolutions arising from the use of Godement sheaves, and on the theory of (local) cohomology of sheaves on a general topological space.

1.1 General Notation

The symbols $\mathbb{Z}, \mathbb{Q}, \mathbb{R}$, have their usual meaning; $\mathbb{N}$ denotes the set of natural numbers (with 0 excluded), $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Furthermore, for a prime p, the field of p-adic numbers is denoted by $\mathbb{Q}_p$. The cardinality of a set X will be denoted by $\#X$.

Let $k = \mathbb{F}_q$, the field with q elements, for $q = p^e$ a prime power and fix an algebraic closure $\overline{k}$ of k. Furthermore, fix a finite extension field $K/\mathbb{Q}_p$ with residue field k and denote by $\mathcal{V}$ its valuation ring, by $\mathfrak{m}$ its unique maximal ideal and let $\pi \in \mathcal{V}$ be a uniformizing element, i.e. $\mathfrak{m} = (\pi)$. Choose a norm $|\ | : K \to \mathbb{R}_{\geq 0}$, normalized such that $|\pi| = q^{-1}$.

For $n \in \mathbb{N}$, let $\mathbf{G}$ be the algebraic group $\mathrm{GL}_{n+1,\mathcal{V}}$ over $\mathcal{V}$ with lower Borel subgroup $\mathbf{B}$ (i.e. for a $\mathcal{V}$-algebra R, the group $\mathbf{B}(R)$ is the subgroup of lower triangular matrices in $\mathbf{G}(R)$) and diagonal torus $\mathbf{T} \subset \mathbf{B}$. Write

$$X(\mathbf{T}) = \mathrm{Hom}(\mathbf{T}, \mathbf{G}_m)$$

for the $\mathbb{Z}$-module of algebraic characters of $\mathbf{T}$. For $i \in \{0, \dots, n\}$, denote by $\epsilon_i \in X(\mathbf{T})$ the character which sends an element $(t_0, \dots, t_n) \in \mathbf{T}(R)$, R a $\mathcal{V}$-algebra, to t_i. For $0 \leq i \neq j \leq n$, write

$$\alpha_{i,j} = \epsilon_i - \epsilon_j.$$

Then

$$\Phi = \{\alpha_{i,j} \mid 0 \leq i \neq j \leq n\}$$

is the set of roots of $\mathbf{G}$ (with respect to $\mathbf{T}$) and

$$\Delta = \{\alpha_0 = \alpha_{1,0}, \dots, \alpha_{n-1} = \alpha_{n,n-1}\}$$

is the set of simple roots (with respect to $\mathbf{B} \supset \mathbf{T}$). Let

$$\Phi = \Phi^+ \cup \Phi^-$$

be the decomposition of Φ into the set of positive roots

$$\Phi^+ = \{\alpha_{i,j} \mid 0 \leq j < i \leq n\}$$

and the set of negative roots

$$\Phi^- = \{\alpha_{i,j} \mid 0 \leq i < j \leq n\}$$

of $\mathbf{G}$. The set of dominant weights of $\mathbf{T}$ (with respect to Φ^+) is

$$X(\mathbf{T})_+ = \{\lambda \in X(\mathbf{T}) \mid \forall \alpha \in \Phi^+ : \langle \lambda, \alpha^\vee \rangle \geq 0\}$$

where $\alpha^\vee$ denotes the coroot associated with $\alpha \in \Phi^+$. With every proper subset $I \subsetneq \Delta$ there is associated a uniquely determined standard-parabolic subgroup $\mathbf{P}_I \supset \mathbf{B}$ with Levi subgroup $\mathbf{L}_I$ and unipotent radical $\mathbf{U}_I$. The respective Lie algebras of the aforementioned groups are denoted by Gothic letters such as

$$\mathfrak{g} = \mathrm{Lie}(\mathbf{G}), \mathfrak{b} = \mathrm{Lie}(\mathbf{B}), \mathfrak{p}_I = \mathrm{Lie}(\mathbf{P}_I), \mathfrak{u}_I = \mathrm{Lie}(\mathbf{U}_I),$$

etc. Whenever it is more convenient, the alternative description of standard-parabolic subgroups in terms of decompositions of $n+1$ will be used: Let $(i_0, \ldots, i_r)$ be a decomposition of $n+1$, i.e. $n+1 = i_0 + \ldots + i_r$ with $i_0, \ldots, i_r \in \mathbb{N}$. Then $(i_0, \ldots, i_r)$ corresponds to a standard-parabolic subgroup $\mathbf{P}_{(i_0,\ldots,i_r)} = \mathbf{P}_I$ with

$$I \quad = \quad \{\alpha_0, \ldots, \alpha_{i_0-2}\} \cup \{\alpha_{i_0}, \ldots, \alpha_{i_0+i_1-2}\} \cup \ldots \cup \{\alpha_{i_0+\ldots+i_{r-1}}, \ldots, \alpha_{i_0+\ldots+i_r-2}\}$$

(where all undefined sets are to be understood as being empty).

For a $\mathcal{V}$-algebra R, write

$$\mathbf{G}_R = \mathbf{G} \times_{\mathbf{Spec}\,\mathcal{V}} \mathbf{Spec}\, R$$

and similarly for the subgroups and their respective Lie algebras mentioned above. In particular, in the case that $R = k$, set

$$G = \mathbf{G}_k(k), B = \mathbf{B}_k(k), P_I = \mathbf{P}_{I,k}(k), L_I = \mathbf{L}_{I,k}(k), U_I = \mathbf{U}_{I,k}(k), T = \mathbf{T}_k(k)$$

for the respective finite groups of k-rational points and similarly in the case where I is replaced by its associated decomposition of $n+1$ in the above sense.

Let

$$S = k[T_0, \ldots, T_n]$$

be the polynomial ring in $n+1$ variables over k with its usual grading $S = \bigoplus_{i \in \mathbb{N}_0} S_i$ (i.e. S_i is the k-vector space of homogeneous polynomials of degree i) and write

$$\mathbb{P}^n_k = \mathbf{Proj}\, S.$$

For $j \in \{0, \ldots, n-1\}$, denote by $\mathbb{P}^j_k$ the closed subvariety $V_+(T_{j+1}, \ldots, T_n) \subset \mathbb{P}^n_k$.

For a field L and an L-vector space M, denote by M' its dual space

$$M' = \mathrm{Hom}_L(M, L).$$

If $M = \bigoplus_{i \in \mathbb{N}_0} M_i$ is a graded L-vector space, then write

$$M^\vee = \bigoplus_{i \in \mathbb{N}_0} M'_i$$

for its graded dual space. Note that $M^\vee$ injects into M' via

$$(f_i)_{i \in \mathbb{N}_0} \mapsto \left(\sum_{i \in \mathbb{N}_0} f_i : (m_i)_{i \in \mathbb{N}_0} \mapsto \sum_{i \in \mathbb{N}_0} f_i(m_i) \right)$$

and that $(M^\vee)^\vee \cong M$ if all M_i are finitely generated L-vector spaces. Suppose that $N = \bigoplus_{i \in \mathbb{N}_0} N_i$ is another graded L-vector space and that $f = (f_i)_{i \in \mathbb{N}_0} : M \to N$ is a graded homomorphism. Dualization of each f_i yields a graded homomorphism $f^\vee : N^\vee \to M^\vee$, called the graded dual of f.

1.2 Representations

For the next few pages, let G be an abstract group. A representation of G over a field L is a left LG-module M, where LG denotes the group algebra of G over L. Usually, in this thesis, the field L will be clear from the context and will be dropped from the terminology, i.e. a representation of G will sometimes be just called a G-module. Furthermore, the effect of a group element $g \in G$ on an element $m \in M$ will be denoted by $g.m$.

The category of LG-modules will be denoted by $\mathrm{rep}_L(G)$. The full subcategory of LG-modules which as L-vector spaces are finite dimensional will be denoted by $\mathrm{rep}_L(G)^f$.

In the particular case that G is an algebraic group, the notion of a representation of G will be used in the sense of Jantzen's book [32].

1.2.1 Induced Representations

Let $H \subset G$ be a subgroup of finite index and let M be an LH-module. The following three descriptions of the induced LG-module $\mathrm{Ind}_H^G(M)$ will be used interchangeably, cf. e.g. [34, § 2.3]:

$$\begin{aligned}\mathrm{Ind}_H^G(M) &= LG \otimes_{LH} M \\ &\cong \bigoplus_{g \in G/H} g * M \\ &\cong \{f : G \to M \mid \forall g \in G, h \in H : f(gh) = h^{-1}.f(g)\}.\end{aligned}$$

Here, $g*M$ is just a formal symbol for a copy of M indexed by the coset g and G acts on $\bigoplus_{g \in G/H} g*M$ by $g.(h*m) = (gh)*m$. Furthermore, G acts on a function $f : G \to M$ as above by left translation, i.e. $(g.f)(x) = f(g^{-1}x)$. In this context, both versions of Frobenius reciprocity hold, i.e. for a G-module N and an H-module M, there are bijections of sets

$$\mathrm{Hom}_{LG}(N, \mathrm{Ind}_H^G(M)) \xrightarrow{\sim} \mathrm{Hom}_{LH}(N, M)$$

resp.

$$\mathrm{Hom}_{LG}(\mathrm{Ind}_H^G(M), N) \xrightarrow{\sim} \mathrm{Hom}_{LH}(M, N),$$

cf. [34, 2.3.8,(2.26)], and Ind_H^G is an exact functor from the category of LH-modules to the category of LG-modules, cf. [53, 10.2]

1.2.2 Composition Series and Semisimplifications

Suppose that a G-module M has a well-ordered ascending composition series of the following type: There is a finite ascending chain

$$(0) = M_0 \subsetneq M_1 \subsetneq M_2 \subsetneq \ldots \subsetneq M_n = M,$$

$n \in \mathbb{N}_0$, of G-submodules such that for $i = 1, \ldots, n$, each submodule M_i possesses itself an ascending chain

$$M_{i-1} = M_{i,0} \subset M_{i,1} \subset M_{i,2} \subset \ldots \subset M_i$$

of G-submodules, possibly of countably infinite length, with

$$M_i = \bigcup_{l \in \mathbb{N}_0} M_{i,l}$$

and such that each subquotient $M_{i,l+1}/M_{i,l}$ is a simple G-module. Then the semisimple G-module

$$M^{G-ss.} := \bigoplus_{i=1}^{n} \bigoplus_{l \in \mathbb{N}_0} M_{i,l+1}/M_{i,l}$$

is called the G-semisimplification of M. Here $(0)^{G-ss.}$ is defined to be (0). By a theorem of Birkhoff (cf. [3, Th. 1], which generalizes the classical theorem of Jordan-Hölder), $M^{G-ss.}$ is well-defined as a G-module up to isomorphism. The following properties hold:

Lemma 1.2.2.1. *Let $H \subset G$ be a subgroup of finite index, let M be a G-module and N an H-module.*

i) If N has a well-ordered ascending composition series of the above type with simple subquotients of finite L-dimension, then so does $\mathrm{Ind}_H^G(N)$. Furthermore, there is an isomorphism of G-modules

$$\mathrm{Ind}_H^G(N^{H-ss.})^{G-ss.} \cong \mathrm{Ind}_H^G(N)^{G-ss.}.$$

ii) Suppose that M' is a G-submodule of M. If M has a well-ordered ascending composition series of the above type, then so do M' and M/M'.

iii) Let

$$(0) \to M' \xrightarrow{\iota} M \xrightarrow{\theta} M'' \to (0)$$

be a short exact sequence of G-modules such that M' and M'' each have a well-ordered ascending composition series with simple subquotients. Then M also has such a series and there is an exact sequence

$$(0) \to M'^{G-ss.} \xrightarrow{\iota} M^{G-ss.} \xrightarrow{\theta} M''^{G-ss.} \to (0)$$

of semisimple G-modules.

Proof. i) Let

$$(0) = N_0 \subsetneq N_1 \subsetneq N_2 \subsetneq \ldots \subsetneq N_n = N$$

with $N_i = \bigcup_{l \in \mathbb{N}_0} N_{i,l}$ for $i = 1, \ldots, n$ be a well-ordered ascending composition series of N in the above sense such that each $N_{i+1,l}/N_{i,l}$ is of finite L-dimension (and H-simple). Application of the induction functor to each $N_{i,l}$ yields G-modules

$$N'_{i,l} = \mathrm{Ind}_H^G(N_{i,l})$$

and a chain

$$(0) = N'_0 \subsetneq N'_1 = \mathrm{Ind}_H^G(N_1) \subsetneq N'_2 = \mathrm{Ind}_H^G(N_2) \subsetneq \ldots \subsetneq N'_n = \mathrm{Ind}_H^G(N_n) = \mathrm{Ind}_H^G(N)$$

of G-submodules of $N' = \mathrm{Ind}_H^G(N)$. By construction, each N'_i has itself an ascending chain of G-submodules

$$N'_{i-1} = N'_{i,0} \subset N'_{i,1} \subset N'_{i,2} \subset \ldots,$$

possibly countably infinite, with

$$N'_i = \bigcup_{l \in \mathbb{N}_0} N'_{i,l}.$$

From exactness of induction it follows that

$$N'_{i,l+1}/N'_{i,l} \cong \mathrm{Ind}_H^G(N_{i,l+1}/N_{i,l})$$

for all i and l. Since the latter module is L-finite dimensional by the assumptions on the $N_{i,l}$ and on the finiteness of the index of H in G, it now follows that each inclusion $N'_{i,l} \subset N'_{i,l+1}$ can be refined into a finite series

$$N'_{i,l} = (N'_{i,l+1})_0 \subsetneq (N'_{i,l+1})_1 \subsetneq \ldots \subsetneq (N'_{i,l+1})_{m_{i,l+1}} = N'_{i,l+1}$$

with simple subquotients $(N'_{i,l+1})_{r+1}/(N'_{i,l+1})_r$. By combining all series so obtained, one now checks that $\mathrm{Ind}_H^G(N)^{G-ss.}$ exists. Furthermore, from the facts that induction and G-semisimplification commute with direct sums, it follows that

$$\begin{aligned}
\left(\mathrm{Ind}_H^G(N^{H-ss.})\right)^{G-ss.} &= \left(\mathrm{Ind}_H^G\left(\bigoplus_{i=1}^n \bigoplus_{l\in\mathbb{N}_0} N_{i,l+1}/N_{i,l}\right)\right)^{G-ss.} \\
&= \left(\bigoplus_{i=1}^n \bigoplus_{l\in\mathbb{N}} \mathrm{Ind}_H^G(N_{i,l+1}/N_{i,l})\right)^{G-ss.} \\
&\cong \bigoplus_{i=1}^n \bigoplus_{l\in\mathbb{N}} \left(N'_{i,l+1}/N'_{i,l}\right)^{G-ss.} \\
&\cong \bigoplus_{i=1}^n \bigoplus_{l\in\mathbb{N}} \bigoplus_{r=0}^{m_{i,l+1}-1} (N'_{i,l+1})_{r+1}/(N'_{i,l+1})_r,
\end{aligned}$$

and the latter module equals $\mathrm{Ind}_H^G(N)^{G-ss.}$ by construction.

ii) Write $M'' = M/M'$. Given chains for M as in the beginning of this subsection, one obtains chains for M' resp. M'' by putting

$$M'_{i,l} = M_{i,l} \cap M$$

resp.

$$M''_{i,l} = (M_{i,l} + M')/M'.$$

The claim follows since $M'_{i,l+1}/M'_{i,l}$ canonically injects into $M_{i,l+1}/M_{i,l}$ on the one hand and $M''_{i,l+1}/M''_{i,l} \cong (M_{i,l+1} + M')/(M_{i,l} + M')$ is canonically isomorphic to an epimorphic image of $M_{i,l+1}/M_{i,l}$ on the other. Therefore, each of these modules is either simple or the zero module and one now obtains $M'^{G-ss.}$ and $M''^{G-ss.}$ in the manner described above.

iii) Given families of submodules $\{M'_{i,l} \mid i = 1, \ldots, n; l \in \mathbb{N}_0\}$ resp. $\{M''_{j,m} \mid j = 1, \ldots, r; m \in \mathbb{N}_0\}$ for M' resp. M'' (and some $n, r \in \mathbb{N}$) which give rise to the respective G-semisimplifications of M' resp. M'', define a family of submodules $\{M_{i,l} \mid i = 1, \ldots, n + r, l \in \mathbb{N}_0\}$ by

$$M_{i,l} = \begin{cases} \iota(M'_{i,l}) & , \text{ for } i = 1, \ldots, n; l \in \mathbb{N}_0 \\ \theta^{-1}(M''_{i-n,l}) & , \text{ for } i = n+1, \ldots, n+r; l \in \mathbb{N}_0 \end{cases}.$$

In this way, one obtains a finite chain

$$\begin{aligned}
(0) = M_0 = M_{1,0} &\subsetneq M_1 = M_{2,0} \subsetneq \ldots \subsetneq M_r = M_{n+1,0} = \iota(M') \\
&\subsetneq \ldots \subsetneq M_{n+r-1} = M_{n+r,0} \subsetneq M_{n+r} := M
\end{aligned}$$

of G-submodules of M such that for each $i = 1, \ldots, n + r$, there is a chain

$$M_{i-1} = M_{i,0} \subset M_{i,1} \subset M_{i,2} \subset \ldots \subset M_i$$

of G-submodules with simple subquotients as in the beginning of this subsection. It follows at once that $M^{G-ss.}$ exists as claimed and that it is by construction isomorphic to $M'^{G-ss.} \oplus M''^{G-ss.}$.

□

1.2.3 (Generalized) Steinberg Representations

Return to the convention that $G = \mathbf{G}(k)$ and suppose that $H = P_I$ is the group of k-points of a standard-parabolic subgroup $\mathbf{P}_I \subset \mathbf{G}$. Equip L with the trivial action of G. For each intermediate group $H \subset H' \subset G$, there are G-equivariant embeddings $\mathrm{Ind}_{H'}^{G}(L) \to \mathrm{Ind}_{H}^{G}(L)$ and the generalized Steinberg representation of G with respect to L and H is defined as

$$v_H^G(L) = \mathrm{Ind}_H^G(L) / \sum_{H \subsetneq H' \subset G} \mathrm{Ind}_{H'}^G(L).$$

If $H = B$, then $v_H^G(L)$ is called Steinberg representation and denoted by $\mathrm{St}_H^G(L)$. This representation is irreducible and self-dual, cf. [28]. The generalized Steinberg representation is irreducible if and only if I is either empty or of the shape $I = \{\alpha_0, \alpha_1, \ldots, \alpha_i\}$, $i \in \{0, \ldots, n-1\}$, cf. [46, Prop. 2.5].

In particular, given a decomposition $n + 1 = r + s$, set

$$\mathrm{St}_s(L) = \mathrm{St}_{\mathrm{GL}_s(k) \cap B}^{\mathrm{GL}_s(k)}(L)$$

resp.

$$\mathrm{St}_r(L) = \mathrm{St}_{\mathrm{GL}_r(k) \cap B}^{\mathrm{GL}_r(k)}(L),$$

where GL_r and GL_s are considered as factors of $L_{(r,s)}$.

1.2.4 The Simple (Algebraic) Representations of $\mathrm{GL}_{n+1,k}$ and $\mathrm{GL}_{n+1}(k)$, and Weyl Modules

Following [32, II.2] (and originally due to Steinberg), one can give a parametrization of the irreducible (algebraic) representations of $\mathbf{G}_k$ over k by $X(\mathbf{T}_k)_+$. For $\lambda \in X(\mathbf{T}_k)$, denote by k_λ the one-dimensional $\mathbf{T}_k$-module over k on which $\mathbf{T}_k$ acts via λ and set

$$H^0(\lambda) = \mathrm{ind}_{\mathbf{B}_k}^{\mathbf{G}_k} k_\lambda.$$

Here, $\mathrm{ind}_{\mathbf{B}_k}^{\mathbf{G}_k}$ is the induction functor from the category of algebraic $\mathbf{B}_k$-modules to the category of algebraic $\mathbf{G}_k$-modules, cf. [32, Ch. 3]. Then $H^0(\lambda) \neq (0)$ if and only if $\lambda \in X(\mathbf{T}_k)_+$ and in this case, $H^0(\lambda)$ has a unique irreducible $\mathbf{G}_k$-submodule $L(\lambda)$ (of highest weight λ and of finite k-dimension). Furthermore, every irreducible $\mathbf{G}_k$-module over k is isomorphic to an $L(\lambda)$ for some $\lambda \in X(\mathbf{T})_+$. From this description, one can derive a parametrization of the irreducible G-modules over k : Write

$$X_e(\mathbf{T}_k) = \{\lambda \in X(\mathbf{T}_k)_+ \mid \forall \alpha \in \Phi^+ : 0 \leq \langle \lambda, \alpha^\vee \rangle < p^e\}.$$

This is the set of p^e-restricted dominant weights of $\mathbf{T}_k$. Let

$$X^0(\mathbf{T}_k) = \{\lambda \in X(\mathbf{T}_k) \mid \forall \alpha \in \Phi : \langle \lambda, \alpha^\vee \rangle = 0\}.$$

Then $X^0(\mathbf{T}_k)$ is a subset of $X_e(\mathbf{T}_k)$ and a system $\mathcal{S}$ of representatives for $X_e(\mathbf{T}_k)/\sim$, where

$$\lambda \sim \mu \Longleftrightarrow \lambda - \mu \in (p^e - 1) X^0(\mathbf{T}_k),$$

gives a system of representatives $\{L(\lambda) \mid \lambda \in \mathcal{S}\}$ for the set of isomorphism classes of irreducible G-modules over k. This last statement (and a proof) can be found for example in [25, 3.10] where it is attributed to Jantzen.

Recall the following identifications of the graded pieces $\mathbf{Sym}^r((k^{n+1})')$ of the symmetric algebra of $(k^{n+1})'$ in terms of Weyl modules. For compatibility with the action of $\mathbf{G}_k$ on $\mathbb{P}_k^n$ specified in the next section, consider the action of $\mathbf{G}_k$ on $\mathbb{A}_k^{n+1}$ given on closed points by

$$(g, x) \mapsto g.x = xg^{-1}.$$

In [32, 2.16] it is shown that for every $r \in \mathbb{N}_0$, there is an isomorphism of $\mathbf{G}_k$-modules

$$\mathbf{Sym}^r((k^{n+1})') \cong H^0(r \cdot \epsilon_0).$$

Denote by w_0 the longest element in the Weyl group of $\mathbf{G}_k$. Then, for any character $\lambda \in X(\mathbf{T}_k)$, its associated Weyl module $V(\lambda)$ is defined as

$$V(\lambda) = H^0(-w_0\lambda)'.$$

In particular, for every dominant $\lambda \in X(\mathbf{T}_k)_+$, the associated simple $\mathbf{G}_k$-module $L(\lambda)$ is the unique simple quotient of $V(\lambda)$. Furthermore, in the language of Weyl modules, the above symmetric powers can be identified as

$$\mathbf{Sym}^r((k^{n+1})') \cong V(-r \cdot \epsilon_n)'.$$

1.3 Drinfeld's Upper Half Space over a Finite Field

The n-dimensional Drinfeld upper half space over k is the affine open subvariety

$$\mathcal{X}^{(n+1)} = \mathbb{P}_k^n \setminus \bigcup_{f \in S_1} V_+(f)$$

of $\mathbb{P}_k^n$ which arises by removing all k-rational hyperplanes from $\mathbb{P}_k^n$. Denote by $\mathcal{Y}^{(n+1)}$ its closed complement in $\mathbb{P}_k^n$, i.e.

$$\mathcal{Y}^{(n+1)} = \bigcup_{f \in S_1} V_+(f),$$

which is thus a projective subvariety of $\mathbb{P}_k^n$.

Consider the action of $\mathbf{G}_k$ on $\mathbb{P}_k^n$ which on closed points is given by

$$(g, [x_0 : \ldots : x_n]) \mapsto [x_0 : \ldots : x_n]g^{-1}.$$

This action restricts to an action of the finite group G on $\mathcal{X}^{(n+1)}$ resp. on $\mathcal{Y}^{(n+1)}$, since G permutes the hyperplanes $V_+(f)$ (with $f \in S_1$).

1.4 Sheaves and Cohomology

The usual notation for sheaves and their (local) cohomology is adopted, cf. e.g. [23, Ch. 3] and [18] resp. [22]. Let X be a topological space, $\mathcal{F}$ a sheaf of abelian groups on X and $U \subset X$ open. Then $\Gamma(U, \mathcal{F}) = \mathcal{F}(U)$ denotes the sections of $\mathcal{F}$ over U and the i-th right derived functor of $\Gamma(X, -)$ is written $\mathrm{H}^i(X, -)$. Furthermore, if $Z = X \setminus U$ is the (closed) complement of U in X, then $\mathrm{H}^i_Z(X, \mathcal{F})$ is the i-th local cohomology module with values in $\mathcal{F}$ and support in Z, i.e. $\mathrm{H}^i_Z(X, -)$ is the i-th right derived functor of the functor $\Gamma_Z(X, -)$ where $\Gamma_Z(X, \mathcal{F})$ are the sections in $\mathcal{F}(X)$ with support in Z. The following facts will be used:

- There is a long exact sequence

$$\ldots \to \mathrm{H}^{i-1}(U,\mathcal{F}) \to \mathrm{H}^i_Z(X,\mathcal{F}) \xrightarrow{\delta_i} \mathrm{H}^i(X,\mathcal{F}) \to \mathrm{H}^i(U,\mathcal{F}) \to \mathrm{H}^{i+1}_Z(X,\mathcal{F}) \to \ldots$$

of cohomology groups. For $i \in \mathbb{Z}$ set

$$\tilde{\mathrm{H}}^i_Z(X,\mathcal{F}) = \ker(\delta^i).$$

- Denote by $\mathbb{Z}$ the constant sheaf on Z with value $\mathbb{Z}$. Then there is an isomorphism

$$\mathrm{H}^*_Z(X,\mathcal{F}) \cong \mathrm{Ext}^*(i_*\mathbb{Z},\mathcal{F}),$$

where $i : Z \hookrightarrow X$ is the inclusion, cf. [18, I.2.3].

Denote by $\mathcal{G}(\mathcal{F})$ the Godement sheaf associated with $\mathcal{F}$, i.e. for $U \subset X$ open, the sections of $\mathcal{G}(\mathcal{F})$ over U are given by

$$\Gamma(U,\mathcal{G}(\mathcal{F})) = \prod_{x\in U} \mathcal{F}_x,$$

where, as usual, $\mathcal{F}_x$ denotes the stalk of $\mathcal{F}$ in x. This is a flasque sheaf and it comes with a natural monomorphism

$$0 \to \mathcal{F} \xrightarrow{d^0} \mathcal{G}(\mathcal{F}).$$

Let

$$\mathcal{G}^0(\mathcal{F}) = \mathcal{G}(\mathcal{F})$$

and inductively define

$$\mathcal{G}^i(\mathcal{F}) = \mathcal{G}(\mathrm{coker}(d^{i-1})),$$

where $d^i : \mathcal{G}^{i-1}(\mathcal{F}) \to \mathcal{G}^i(\mathcal{F})$ is the canonical map. In this way, one obtains a flasque resolution

$$0 \to \mathcal{F} \to \mathcal{G}^\bullet(\mathcal{F})$$

which is functorial in $\mathcal{F}$ (cf. e.g. [52, 6.72-73]). In particular, $\mathrm{H}^i(X,\mathcal{F})$ is isomorphic to the i-th cohomology $h^i(\Gamma(X,\mathcal{G}^\bullet(\mathcal{F})))$ of the complex $\Gamma\left(X,\mathcal{G}^\bullet\left(\mathcal{F}\right)\right) = \left(\Gamma\left(X,\mathcal{G}^i\left(\mathcal{F}\right)\right)\right)_{i\in\mathbb{N}_0}$. Furthermore, given a finite complex

$$\mathcal{F}^\bullet = \mathcal{F}^0 \to \ldots \to \mathcal{F}^r$$

of sheaves of abelian groups on X, its hypercohomology can be computed as

$$\mathbb{H}^*\left(X,\mathcal{F}^\bullet\right) = h^i\left(\Gamma\left(X,\mathcal{G}\left(\mathcal{F}^\bullet\right)\right)\right),$$

cf. [10, Appendix], where $\Gamma\left(X,\mathcal{G}\left(\mathcal{F}^\bullet\right)\right)$ is the complex

$$\Gamma\left(X,\mathcal{G}\left(\mathcal{F}^0\right)\right) \to \Gamma\left(X,\mathcal{G}\left(\mathcal{F}^1\right)\right) \to \ldots \to \Gamma\left(X,\mathcal{G}\left(\mathcal{F}^r\right)\right).$$

Chapter 2

Cohomology of Equivariant Vector Bundles on Drinfeld's Upper Half Space over a Finite Field

Recall the convention that $\mathbf{G}_k$ acts on $\mathbb{P}_k^n$ via

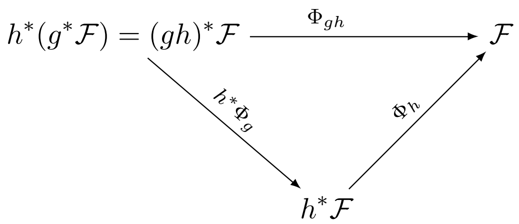
$$\sigma : \begin{array}{ccc} \mathbf{G}_k \times \mathbb{P}_k^n & \to & \mathbb{P}_k^n \\ (g,x) & \mapsto & g.x = xg^{-1} \end{array}$$

for closed points $g \in \mathbf{G}_k$ and $x \in \mathbb{P}_k^n$.

Let $\mathcal{F}$ be a $\mathbf{G}_k$-equivariant algebraic vector bundle on $\mathbb{P}_k^n$. This means that $\mathcal{F}$ is a finite locally free (and hence coherent, cf. [15, 5.4]) $\mathcal{O}_{\mathbb{P}_k^n}$-module of constant rank together with isomorphisms

$$\Phi_g : g^*\mathcal{F} \xrightarrow{\sim} \mathcal{F}$$

for all $g \in \mathbf{G}_k$, considered as morphisms $g : \mathbb{P}_k^n \to \mathbb{P}_k^n$, such that the diagram



commutes for all $g, h \in \mathbf{G}_k$, cf. [40, Ch. 1,§ 3]. Let $U \subset \mathbb{P}_k^n$ be Zariski-open such that U is stabilized by an algebraic subgroup $\mathbf{H}$ of $\mathbf{G}_k$. Restrict σ to $\mathbf{H} \times U$. Then the above condition gives an algebraic action of $\mathbf{H}$ on $\Gamma(U, \mathcal{F})$, as each $h \in H$ induces a morphism

$$\Gamma(U,\mathcal{F}) \to \Gamma(U, h_*h^*\mathcal{F}) = \Gamma(h^{-1}.U, h^*\mathcal{F}) = \Gamma(U, h^*\mathcal{F}) \xrightarrow{\Gamma(U,\Phi_h)} \Gamma(U,\mathcal{F})$$

and the commutativity of the above diagram ensures that one obtains indeed a group representation. Via functoriality, each cohomology module $\mathrm{H}^i(U,\mathcal{F}), i \in \mathbb{Z}$, is then an algebraic $\mathbf{H}$-module.
The structure sheaf $\mathcal{O}_{\mathbb{P}_k^n}$ (and each of its twists $\mathcal{O}_{\mathbb{P}_k^n}(i)$, $i \in \mathbb{Z}$) is always considered as a $\mathbf{G}_k$-equivariant vector bundle with respect to its natural linearization induced by σ.

The goal of this chapter is to describe the cohomology $\mathrm{H}^*(\mathcal{X}^{(n+1)}, \mathcal{F})$ of $\mathcal{F}$ on Drinfeld's upper half space $\mathcal{X}^{(n+1)}$ over k as a representation of the finite group G. As $\mathcal{X}^{(n+1)}$ is affine, this task reduces to describing $\mathrm{H}^0(\mathcal{X}^{(n+1)}, \mathcal{F})$.

2.1 Orlik's Complex for the Cohomology of Equivariant Vector Bundles

In a first step towards determining $\mathrm{H}^0(\mathcal{X}^{(n+1)}, \mathcal{F})$, one can use Orlik's complex and its associated spectral sequence to essentially reduce this problem to determining the local cohomology of $\mathbb{P}_k^n$ with support in certain closed subvarieties. For the sake of completeness, the construction shall be recalled here. The content of this section is an adaption of methods of Orlik, cf. for example [42] or [44].

2.1.1 Construction of the Complex

For a proper subset $I \subsetneq \Delta$ with $i = i(I) = \min\{j \in \mathbb{N}_0 \mid \alpha_j \in \Delta \setminus I\}$ let

$$Y_I = \mathbb{P}_k^{i(I)} \subset \mathbb{P}_k^n.$$

Each inclusion $I \subset J \subsetneq \Delta$ then induces closed embeddings

$$\iota_{I,J} : Y_I \hookrightarrow Y_J$$

and for any two $I, J \subsetneq \Delta$ the identity $Y_{I \cap J} = Y_I \cap Y_J$ holds. The variety Y_I is stabilized by the parabolic subgroup $\mathbf{P}_{I,k}$ under the action of $\mathbf{G}_k$ on $\mathbb{P}_k^n$. By construction, there is then an identification

$$\mathcal{Y}^{(n+1)} = \bigcup_{I \subsetneq \Delta} \bigcup_{g \in G/P_I} g.Y_I.$$

For $I \subsetneq \Delta$ and $g \in G/P_I$ write

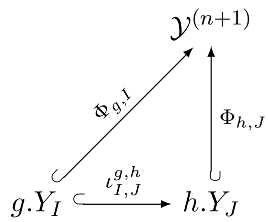
$$\Phi_{g,I} : g.Y_I \hookrightarrow \mathcal{Y}^{(n+1)}$$

for the closed embedding given by inclusion. If furthermore $J \subsetneq \Delta$ and $h \in G/P_J$ with $I \subset J$ and such that gP_I is mapped to hP_J under the canonical map $G/P_I \to G/P_J$, then write

$$\iota_{I,J}^{g,h} : g.Y_I \hookrightarrow h.Y_J$$

for the closed embedding given by inclusion.

Let $\mathbb{Z} = \mathbb{Z}_{\mathcal{Y}^{(n+1)}}$ be the constant sheaf on $\mathcal{Y}^{(n+1)}$ with value $\mathbb{Z}$. The triangle of closed embeddings of algebraic varieties



gives rises to a morphism

$$p_{I,J}^{g,h} : (\Phi_{h,J})_*(\Phi_{h,J})^{-1}\mathbb{Z} \to (\Phi_{g,I})_*(\Phi_{g,I})^{-1}\mathbb{Z}$$

of sheaves on $\mathcal{Y}^{(n+1)}$ via the adjunction property of the involved functors. Define

$$\begin{aligned} \mathbb{Z}_{g,I} &= (\Phi_{g,I})_*(\Phi_{g,I})^{-1}\mathbb{Z}, \\ \mathbb{Z}_I &= \bigoplus_{g \in G/P_I} \mathbb{Z}_{g,I}, \end{aligned}$$

and set

$$\begin{aligned} p_{I,J} &= \bigoplus_{(g,h) \in G/P_I \times G/P_J} p_{I,J}^{g,h}, \\ d_{I,J} &= \begin{cases} (-1)^i p_{I,J} & , \text{ if } J = I \dot{\cup} \{\alpha_i\} \\ 0 & , \text{ else,} \end{cases} \end{aligned}$$

where $p_{I,J}^{g,h} = 0$ if gP_I does not map to hP_J. The maps $d_{I,J}$ now induce differentials so that the following complex is defined:

$$0 \to \mathbb{Z} \to \bigoplus_{\substack{I\subset\Delta \\ \#I=n-1}} \mathbb{Z}_I \to \bigoplus_{\substack{I\subset\Delta \\ \#I=n-2}} \mathbb{Z}_I \to \cdots \to \bigoplus_{\substack{I\subset\Delta \\ \#I=1}} \mathbb{Z}_I \to \mathbb{Z}_\emptyset \to 0. \tag{2.1}$$

This is Orlik's complex and it has the following fundamental property:

Theorem 2.1.1.1. (cf. [42, Satz 5.3], [44, 2.1.1]) *The complex (2.1) is acyclic.*

A variant of the proof will be given in a slightly different setting in the third chapter of this thesis (see Prop. 3.2.1.1) and therefore, in order to keep repetition at a minimum, it is omitted here.

2.1.2 An Equivariant Filtration on the Cohomology of Drinfeld's Upper Half Space

Denote by

$$\iota : \mathcal{Y}^{(n+1)} \hookrightarrow \mathbb{P}_k^n$$

the closed embedding given by inclusion and choose an injective resolution $0 \to \mathcal{F} \to \mathcal{I}^\bullet$ of $\mathcal{F}$. Write the complex (2.1) as

$$0 \to \mathbb{Z} \to \mathcal{Z}^\bullet \to 0$$

with $\mathbb{Z}$ in degree -1. The double complex $\mathrm{Hom}(\iota_*(\mathcal{Z}^\bullet), \mathcal{I}^\bullet)$ then induces a (first quadrant) spectral sequence

$$E_1^{r,s} = \mathrm{Ext}^s(\iota_*(\mathcal{Z}^r), \mathcal{F}) \Longrightarrow \mathrm{Ext}^{r+s}(\iota_*\mathbb{Z}, \mathcal{F}),$$

cf. [20, 4.6]. The relationship between Ext-modules and local cohomology recalled in Section 1.4 as well as the definition of $\mathcal{Z}^\bullet$ imply that this spectral sequence can be rewritten as

$$E_1^{r,s} = \bigoplus_{\substack{I\subset\Delta \\ \#I=n-1-r}} \mathrm{Ind}_{P_I}^G \mathrm{H}_{Y_I}^s(\mathbb{P}_k^n, \mathcal{F}) \Longrightarrow \mathrm{H}_{\mathcal{Y}^{(n+1)}}^{r+s}(\mathbb{P}_k^n, \mathcal{F}).$$

Evaluation of this spectral sequence in the same manner as in [44, 2.2] and avoidance of the use of duals yields the following theorem[1]:

Theorem 2.1.2.1. *(cf. [44, 2.2.9] and [45, Lemma 4])*

i) On $\mathrm{H}^0(\mathcal{X}^{(n+1)}, \mathcal{F})$ there is a filtration by G-submodules

$$\mathrm{H}^0(\mathcal{X}^{(n+1)}, \mathcal{F}) = \mathcal{F}(\mathcal{X}^{(n+1)})^0 \supset \mathcal{F}(\mathcal{X}^{(n+1)})^1 \supset \ldots \supset \mathcal{F}(\mathcal{X}^{(n+1)})^n = \mathrm{H}^0(\mathbb{P}_k^n, \mathcal{F})$$

such that the successive quotients $\mathcal{F}(\mathcal{X}^{(n+1)})^j / \mathcal{F}(\mathcal{X}^{(n+1)})^{j+1}$ with $j \in \{0, \ldots, n-1\}$ appear as extensions in short exact sequences of G-representations

$$\begin{aligned}(0) \quad &\to \quad \mathrm{Ind}_{P_{(j+1,n-j)}}^G (\tilde{\mathrm{H}}_{\mathbb{P}_k^j}^{n-j}(\mathbb{P}_k^n, \mathcal{F}) \otimes_k \mathrm{St}_{n-j}(k)) \to \mathcal{F}(\mathcal{X}^{(n+1)})^j / \mathcal{F}(\mathcal{X}^{(n+1)})^{j+1} \\ &\to \quad v_{P_{(j+1,1^{n-j})}}^G (k)' \otimes_k \mathrm{H}^{n-j}(\mathbb{P}_k^n, \mathcal{F}) \to (0).\end{aligned}$$

ii) The filtration in i) behaves functorially, i.e. any morphism $\mathcal{E} \to \mathcal{F}$ of $\mathbf{G}_k$-equivariant vector bundles induces G-equivariant morphisms

$$\mathcal{E}(\mathcal{X}^{(n+1)})^j \to \mathcal{F}(\mathcal{X}^{(n+1)})^j$$

for $j = 0, \ldots, n$.

[1] Again, in a similar case, some of the arguments will be presented in Section 3.2.1.

(Recall the definition of $\mathrm{St}_{n-j}(k)$ resp. of $\tilde{\mathrm{H}}^{n-j}_{\mathbb{P}^j_k}(\mathbb{P}^n_k, \mathcal{F})$ from Section 1.2 resp. 1.4.)

If one considers the cohomology $\mathrm{H}^*(\mathbb{P}^n_k, \mathcal{F})$ of $\mathcal{F}$ on $\mathbb{P}^n_k$ as known, then the above theorem implies that in order to describe $\mathrm{H}^0(\mathcal{X}^{(n+1)}, \mathcal{F})$ as a G-module, it is sufficient to describe each $\tilde{\mathrm{H}}^{n-j}_{\mathbb{P}^j_k}(\mathbb{P}^n_k, \mathcal{F})$ as a $\mathbf{P}_{(j+1,n-j),k}$-module. This will be the objective of the rest of this chapter.

2.2 Local Cohomology I: First Descriptions

Fix $j \in \{0, \ldots, n-1\}$. First of all,

$$\mathrm{H}^i_{\mathbb{P}^j_k}(\mathbb{P}^n_k, \mathcal{F}) = \begin{cases} (0) & , \text{ if } i < n-j, \\ \mathrm{H}^i(\mathbb{P}^n_k, \mathcal{F}) & , \text{ if } i > n-j, \end{cases} \tag{2.2}$$

see the reasoning in [44, 1.2] which is independent of the ground field and only uses the fact that $\mathbb{P}^j_k$ is smooth in $\mathbb{P}^n_k$. This implies that $\tilde{\mathrm{H}}^i_{\mathbb{P}^j_k}(\mathbb{P}^n_k, \mathcal{F}) = (0)$ for each $i \neq n-j$ and therefore, the remaining case to be studied is $i = n-j$.

A fact that will be used in the following is that the $\mathbf{P}_{(j+1,n-j),k}$-module $\mathrm{H}^{n-j}_{\mathbb{P}^j_k}(\mathbb{P}^n_k, \mathcal{F})$ fits into the $\mathbf{P}_{(j+1,n-j),k}$-equivariant exact sequence

$$(0) \to \mathrm{H}^{n-j-1}(\mathbb{P}^n_k, \mathcal{F}) \to \mathrm{H}^{n-j-1}(\mathbb{P}^n_k \setminus \mathbb{P}^j_k, \mathcal{F}) \to \mathrm{H}^{n-j}_{\mathbb{P}^j_k}(\mathbb{P}^n_k, \mathcal{F}) \to \mathrm{H}^{n-j}(\mathbb{P}^n_k, \mathcal{F}) \to (0).$$

This follows from (2.2) and the fact that $\mathrm{H}^i(\mathbb{P}^n_k \setminus \mathbb{P}^j_k, \mathcal{F}) = (0)$ for all $i \geq n-j$, which is verified by considering the Čech complex with respect to the covering

$$\mathbb{P}^n_k \setminus \mathbb{P}^j_k = \bigcup_{i=j+1}^{n} D_+(T_i)$$

by affine open subvarieties.

In the next two subsections, two attempts to describe the reduced local cohomology $\tilde{\mathrm{H}}^{n-j}_{\mathbb{P}^j_k}(\mathbb{P}^n_k, \mathcal{F})$ appearing in Theorem 2.1.2.1 as quotients of known $\mathbf{L}_{(j+1,n-j),k}$-modules will be presented. The first one is for bundles arising from certain $\mathbf{L}_{(1,n),k}$-representations and is due to Orlik [44, 1.4]. The second attempt is for general bundles and uses the canonical projection $\mathbb{P}^n_k \setminus \mathbb{P}^j_k \to \mathbb{P}^{n-j-1}_k$.

Once and for all, identify the character groups of the standard diagonal tori of the factors $\mathrm{GL}_{j+1,k}$ and $\mathrm{GL}_{n-j,k}$ appearing in $\mathbf{L}_{(j+1,n-j),k}$ as subgroups of $X(\mathbf{T})$ in the obvious way.

2.2.1 Bundles arising from Representations of a Levi Subgroup

In the case where the bundle $\mathcal{F}$ arises from an irreducible representation of the Levi subgroup $\mathbf{L}_{(j+1,n-1),k}$, the reduced cohomology $\tilde{\mathrm{H}}^{n-j}_{\mathbb{P}^j_k}(\mathbb{P}^n_k, \mathcal{F})$ can be written as a quotient of well-known representations in the following way:

Fix a dominant integral weight

$$\lambda' = (\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n) \in \mathbb{Z}^n$$

of $\mathrm{GL}_{n,k}$ and denote by $L(\lambda')$ the irreducible $\mathrm{GL}_{n,k}$-module associated with λ'. Fix $\lambda_0 \in \mathbb{Z}$, set

$$\lambda = (\lambda_0, \lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^{n+1}$$

and then denote by L_λ the $\mathbf{L}_{(1,n),k}$-module $\det^{\lambda_0} \boxtimes L(\lambda')$, i.e. L_λ equals $L(\lambda')$ as a k-vector space, the factor $\mathrm{GL}_{1,k}$ of $\mathbf{L}_{(1,n),k} = \mathrm{GL}_{1,k} \times \mathrm{GL}_{n,k}$ acts by $\det^{\lambda_0}$ and the factor $\mathrm{GL}_{n,k}$ acts as it does on $L(\lambda')$. Via inflation, L_λ is then considered as a $\mathbf{P}_{(1,n),k}$-module. Set $\mathcal{F}_\lambda = \mathcal{F}_{L_\lambda}$. This is the vector bundle on $\mathbb{P}^n_k$ associated with L_λ, cf. [32, 5.8]. For example $\mathcal{F}_{(a,0,\dots,0)} = \mathcal{O}_{\mathbb{P}^n_k}(a)$ for $a \in \mathbb{Z}$. Let W be the Weyl group of $\mathbf{G}$. For $i = 0,\dots,n-1$, denote by s_i the simple reflection in W associated with the simple root $\alpha_i \in \Delta$ and let

$$\begin{aligned} w_{i+1} &= s_i \cdot s_{i-1} \cdot \ldots \cdot s_0, \\ w_0 &= 1. \end{aligned}$$

Let $\rho = \frac{1}{2}\sum_{\alpha \in \Phi^-} \alpha$ and consider the dot action

$$\bullet : \begin{array}{ccc} W \times X(\mathbf{T}) \otimes_{\mathbb{Z}} \mathbb{Q} & \to & X(\mathbf{T}) \otimes_{\mathbb{Z}} \mathbb{Q} \\ (w,\mu) & \mapsto & w \bullet \mu = w(\mu+\rho) - \rho. \end{array}$$

Denote by i_0 either the unique integer $i \in \{0,\dots,n\}$ such that $w_{i_0} \bullet \lambda$ is dominant, if it exists, or else the unique integer in $\{0,\dots,n-1\}$ such that $w_{i_0} \bullet \lambda = w_{i_0+1} \bullet \lambda$ and set

$$\mu_{i,\lambda} = \begin{cases} w_{i-1} \bullet \lambda & , \text{ if } i \le i_0, \\ w_i \bullet \lambda & , \text{ if } i > i_0. \end{cases}$$

This is a dominant weight for the Levi subgroup $\mathbf{L}_{(i,n-i+1),k}$ and the corresponding irreducible module over $\mathbf{L}_{(i,n-i+1),k}$ is denoted by $L_{i,\lambda}$. Consider $L_{i,\lambda}$ as a $\mathbf{P}_{(i,n-i+1),k}$-module via inflation. Let

$$z_i = \begin{pmatrix} 0 & I_i \\ I_{n+1-i} & 0 \end{pmatrix} \in \mathbf{G}_k,$$

where $I_i \in \mathrm{GL}_{i,k}$ denotes the identity element. Using the Grothendieck-Cousin complex associated with the covering of $\mathbb{P}^n_k$ by Schubert cells and translating to Weyl modules, one shows the following result in direct analogy with [44, 1.4.2]. The notable difference is that one has to replace the Lie algebras used in loc. cit. by the respective distribution algebras.

Proposition 2.2.1.1. *For $j \in \{0,\dots,n-1\}$, the $\mathbf{L}_{(j+1,n-j),k}$-module $\tilde{\mathrm{H}}^{n-j}_{\mathbb{P}^j_k}(\mathbb{P}^n_k, \mathcal{F}_\lambda)$ is a quotient of the $\mathbf{L}_{(j+1,n-j),k}$-module*

$$\begin{aligned} &\left(\bigoplus_{l \in \mathbb{N}_0} \mathbf{Sym}^l((k^{j+1})') \boxtimes_k \mathbf{Sym}^l((k^{n-j})')' \right) \otimes_k L_{z_{n-j}^{-1}\mu_{n-j,\lambda}} \\ \cong\ &\left(\bigoplus_{l \in \mathbb{N}_0} V(-l \cdot \epsilon_j)' \boxtimes_k V(-l \cdot \epsilon_n) \right) \otimes_k L_{z_{n-j}^{-1}\mu_{n-j,\lambda}}. \end{aligned}$$

Remarks.

i) In the p-adic setting in [44, 1.4], where k is replaced by a finite extension field $K/\mathbb{Q}_p$, Orlik uses this result to further make explicit the structure of $\tilde{\mathrm{H}}^{n-j}_{\mathbb{P}^j_K}(\mathbb{P}^n_K, \mathcal{F}_\lambda)$. This works by making extensive use of the universal enveloping algebra associated with $\mathfrak{g}_K = \mathrm{Lie}(\mathbf{G}_K)$. In the present case, the action of the universal enveloping algebra (and of the distribution algebra) on $\tilde{\mathrm{H}}^{n-j}_{\mathbb{P}^j_k}(\mathbb{P}^n_k, \mathcal{F}_\lambda)$ is not as well-behaved as in the case of characteristic 0, as will be seen in the next section. Therefore, no further adaption of the results of op. cit. has been made.

ii) Nevertheless, in the particular case of Proposition 2.2.1.1, since the construction is – up to replacing K by k – the same as in [44, 1.4.2], the kernel of the map

$$\left(\bigoplus_{l\in\mathbb{N}_0} V(-l\cdot\epsilon_j)'\boxtimes_k V(-l\cdot\epsilon_n)\right)\otimes_k L_{z_{n-j}^{-1}\mu_{n-j,\lambda}} \twoheadrightarrow \tilde{\mathrm{H}}^{n-j}_{\mathbb{P}^j_k}(\mathbb{P}^n_k,\mathcal{F}_\lambda)$$

obtained from this Proposition has the same description as the one obtained from loc. cit., after replacing the field K with k.

2.2.2 Using the Canonical Projection onto a Projective Subvariety

Denote by $f:\mathbb{P}^n_k\setminus\mathbb{P}^j_k\to V_+(T_0,\ldots,T_j)\cong\mathbb{P}^{n-j-1}_k$ the projection which on closed points is given by

$$[x_0:\ldots:x_n]\mapsto[x_{j+1}:\ldots:x_n].$$

Consider f as a $\mathbf{L}_{(j+1,n-j),k}$-equivariant morphism with respect to the following actions of this group: On $\mathbb{P}^n_k\setminus\mathbb{P}^j_k$, the group $\mathbf{L}_{(j+1,n-j),k}$ acts via restriction of the $\mathbf{G}_k$-action on $\mathbb{P}^n_k$. On $V_+(T_0,\ldots,T_j)$, the factor $\mathrm{GL}_{n-j,k}$ of $\mathbf{L}_{(j+1,n-j),k}$ acts via restriction from the same action and the factor $\mathrm{GL}_{j+1,k}$ acts trivially.

In this subsection, it is shown how this morphism can be used to describe the $\mathbf{L}_{(j+1,n-j),k}$-module $\mathrm{H}^{n-j-1}(\mathbb{P}^n_k\setminus\mathbb{P}^j_k,\mathcal{F})$ as a quotient of direct sums of certain Weyl modules on the one hand and to give explicit descriptions of $\tilde{\mathrm{H}}^{n-j}_{\mathbb{P}^j_k}(\mathbb{P}^n_k,\mathcal{F})$ as an $\mathbf{L}_{(j+1,n-j),k}$-module for $\mathcal{F}=\mathcal{O}_{\mathbb{P}^n_k}(a)$, with $a\in\mathbb{Z}$ and for $\mathcal{F}=\Omega^i_{\mathbb{P}^n_k/k}$, with $i=1,\ldots,n$, on the other. A similar map is considered by Schneider and Stuhler [54] in the analog p-adic situation. For general $\mathcal{F}$, the following holds:

Lemma 2.2.2.1. *For every $i\in\mathbb{Z}$, there is a natural isomorphism of $\mathbf{L}_{(j+1,n-j),k}$-modules*

$$\mathrm{H}^i(f_*):\mathrm{H}^i(\mathbb{P}^n_k\setminus\mathbb{P}^j_k,\mathcal{F})=\mathrm{H}^i(\mathbb{P}^n_k\setminus\mathbb{P}^j_k,\mathcal{F}_{|\mathbb{P}^n_k\setminus\mathbb{P}^j_k})\xrightarrow{\sim}\mathrm{H}^i\left(\mathbb{P}^{n-j-1}_k,f_*\left(\mathcal{F}_{|\mathbb{P}^n_k\setminus\mathbb{P}^j_k}\right)\right),$$

Proof. The existence of the $\mathrm{H}^i(f_*)$ as isomorphisms of k-vector spaces follows from [17, Cor. 1.3.3] since f is an affine morphism (cf. [23, p. 128] for a definition). The sheaf $\mathcal{F}_{|\mathbb{P}^n_k\setminus\mathbb{P}^j_k}$ inherits an $\mathbf{L}_{(j+1,n-j),k}$-linearization via restriction of the $\mathbf{G}_k$-linearization Φ of $\mathcal{F}$, cf. the beginning of this chapter. This linearization then induces a $\mathbf{L}_{(j+1,n-j),k}$-linearization on the sheaf $f_*\left(\mathcal{F}_{|\mathbb{P}^n_k\setminus\mathbb{P}^j_k}\right)$: For $g\in\mathbf{L}_{(j+1,n-j),k}$, define $\Phi'_g:=f_*\Phi_g:f_*g^*\left(\mathcal{F}_{|\mathbb{P}^n_k\setminus\mathbb{P}^j_k}\right)\to f_*\left(\mathcal{F}_{|\mathbb{P}^n_k\setminus\mathbb{P}^j_k}\right)$. From the fact that f_* and g^* commute and from the functoriality of f_*, it follows that this is indeed a linearization. Furthermore, the isomorphisms $\mathrm{H}^i(f_*)$ are then $\mathbf{L}_{(j+1,n-j),k}$-equivariant by functoriality. □

Recall from the introduction of this chapter that $\mathcal{F}$ is in particular a coherent $\mathcal{O}_{\mathbb{P}^n_k}$-module. From the fact that $\mathbb{P}^n_k\setminus\mathbb{P}^j_k$ is noetherian, it follows that the direct image $\mathcal{M}=f_*\left(\mathcal{F}_{|\mathbb{P}^n_k\setminus\mathbb{P}^j_k}\right)$ is a quasi-coherent $\mathcal{O}_{\mathbb{P}^{n-j-1}_k}$-module, cf. [23, II.5.8]. Thus there is now a graded $k[T_{j+1},\ldots,T_n]$-module M such that $\mathcal{M}=M^\sim$ is the $\mathcal{O}_{\mathbb{P}^{n-j-1}_k}$-module associated with M on $\mathbb{P}^{n-j-1}_k$, cf. [23, II.5.13]. In the case where $\mathcal{F}=\mathcal{O}_{\mathbb{P}^n_k}(a)$, $a\in\mathbb{Z}$, is a twist of the structure sheaf, one can explicitly determine the structure of M.

Lemma 2.2.2.2. *Let $a\in\mathbb{Z}$ and let*

$$M_a \;=\; \bigoplus_{m_0,\ldots,m_j\in\mathbb{N}_0} k[T_{j+1},\ldots,T_n]\left(-\sum_{i=0}^{j}m_i+a\right)\cdot T_0^{m_0}\cdot\ldots\cdot T_j^{m_j},$$

considered as a graded $k[T_{j+1},\dots,T_n]$-module for the induced grading of the direct sum and where all $T_i^{m_i}, i=0,\dots,j$, have by definition degree 0. Then the associated $\mathcal{O}_{\mathbb{P}_k^{n-j-1}}$-module is

$$M_a^{\sim} = \bigoplus_{m_0,\dots,m_j\in\mathbb{N}_0} \mathcal{O}_{\mathbb{P}_k^{n-j-1}}\left(-\sum_{i=0}^{j} m_i + a\right)\cdot T_0^{m_0}\cdot\ldots\cdot T_j^{m_j}$$

and there is an isomorphism of $\mathcal{O}_{\mathbb{P}_k^{n-j-1}}$-modules

$$f_*\left(\mathcal{O}_{\mathbb{P}_k^n}(a)_{|\mathbb{P}_k^n\setminus\mathbb{P}_k^j}\right)\cong M_a^{\sim}.$$

Proof. It is enough to consider the case $a=0$ since the general case will then follow by shifting the grading accordingly. Therefore, let

$$M = M_0 = \bigoplus_{m_0,\dots,m_j\in\mathbb{N}_0} k[T_{j+1},\dots,T_n]\left(-\sum_{i=0}^{j} m_i\right)\cdot T_0^{m_0}\cdot\ldots\cdot T_j^{m_j}$$

(as a graded $k[T_{j+1},\dots,T_n]$-module) and consider the standard affine covering

$$\mathbb{P}_k^n\setminus\mathbb{P}_k^j = D_+(T_{j+1})\cup\ldots\cup D_+(T_n).$$

For each $i\in\{j+1,\dots,n\}$, denote by $D_{+,\mathbb{P}_k^{n-j-1}}(T_i)$ the affine subvariety of $\mathbb{P}_k^{n-j-1}$ which is defined by the non-vanishing of the coordinate function T_i. Then the restriction of f to $D_+(T_i)$ induces a morphism

$$f_i : D_+(T_i)\to D_{+,\mathbb{P}_k^{n-j-1}}(T_i)$$

of affine k-varieties. Homogeneous localization of M with respect to T_i yields

$$M_{(T_i)} = \bigoplus_{(m_0,\dots,m_j)\in\mathbb{N}_0^{j+1}} k\left[T_{j+1},\dots,T_n\right]_{(T_i)}\cdot\left(\frac{T_0}{T_i}\right)^{m_0}\cdot\ldots\cdot\left(\frac{T_j}{T_i}\right)^{m_j}$$

which is equal to $S_{(T_i)}$ as a module over $k[T_{j+1},\dots,T_n]_{(T_i)}$. In other words, f_i induces the identity map between the $k[T_{j+1},\dots,T_n]_{(T_i)}$-modules $\Gamma(D_{+,\mathbb{P}_k^{n-j-1}}(T_i), M^{\sim})$ and $\Gamma(D_{+,\mathbb{P}_k^{n-j-1}}(T_i), S_{(T_i)}^{\sim})$ where $S_{(T_i)}$ is again considered as a module over $k[T_{j+1},\dots,T_n]_{(T_i)}$. From [13, 7.24] it follows that the $k[T_{j+1},\dots,T_n]_{(T_i)}^{\sim}$-module $S_{(T_i)}^{\sim}$ is isomorphic to $f_{i*}(S_{(T_i)}^{\sim}) = f_{i*}\left(\left(\mathcal{O}_{\mathbb{P}_k^n\,|\mathbb{P}_k^n\setminus\mathbb{P}_k^j}\right)_{|D_+(T_i)}\right)$, where in the last expression $S_{(T_i)}$ is now considered as a module over itself. Comparing sections, one checks that the identity

$$f_{i*}\left(\left(\mathcal{O}_{\mathbb{P}_k^n\,|\mathbb{P}_k^n\setminus\mathbb{P}_k^j}\right)_{|D_+(T_i)}\right) = \left(f_*\mathcal{O}_{\mathbb{P}_k^n\,|\mathbb{P}_k^n\setminus\mathbb{P}_k^j}\right)_{|D_{+,\mathbb{P}_k^{n-j-1}}(T_i)}$$

holds. Therefore, there is an isomorphism of $(\mathcal{O}_{\mathbb{P}_k^{n-j-1}})_{|D_{+,\mathbb{P}_k^{n-j-1}}(T_i)}$-modules

$$\left(f_*\mathcal{O}_{\mathbb{P}_k^n\,|\mathbb{P}_k^n\setminus\mathbb{P}_k^j}\right)_{|D_{+,\mathbb{P}_k^{n-j-1}}(T_i)} \xrightarrow{\sim} M^{\sim}_{|D_{+,\mathbb{P}_k^{n-j-1}}(T_i)}.$$

The lemma is proved if these isomorphisms are compatible with gluing of the $D_{+,\mathbb{P}_k^{n-j-1}}(T_i)$. To check this, consider $i,l\in\{j+1,\dots,n\}$ with $i\neq l$. Then the gluing isomorphism

$$M^{\sim}_{|D_{+,\mathbb{P}_k^{n-j-1}}(T_i)}\left(D_{+,\mathbb{P}_k^{n-j-1}}(T_i)\cap D_{+,\mathbb{P}_k^{n-j-1}}(T_l)\right)\xrightarrow{\sim} M^{\sim}_{|D_{+,\mathbb{P}_k^{n-j-1}}(T_l)}\left(D_{+,\mathbb{P}_k^{n-j-1}}(T_l)\cap D_{+,\mathbb{P}_k^{n-j-1}}(T_i)\right)$$

is induced by $\frac{T_l}{T_i} \mapsto \frac{T_i}{T_l}$ resp. $\frac{T_m}{T_i} \mapsto \frac{T_m}{T_l} / \frac{T_i}{T_l}$. But the same maps induce the gluing isomorphisms

$$\left(\mathcal{O}_{\mathbb{P}_k^n \setminus \mathbb{P}_k^j}\right)_{|D_+(T_i)} (D_+(T_i) \cap D_+(T_l)) \xrightarrow{\sim} \left(\mathcal{O}_{\mathbb{P}_k^n \setminus \mathbb{P}_k^j}\right)_{|D_+(T_l)} (D_+(T_l) \cap D_+(T_i))$$

and thus the proof of the lemma is finished. □

Endow M_a with the following $\mathbf{L}_{(j+1,n-j),k}$-action: The factors $\mathrm{GL}_{n-j,k}$ resp. $\mathrm{GL}_{j+1,k}$ act on $k[T_{j+1}, \ldots, T_n]$ (and its twists) resp. the variables $T_{j+1}, \ldots, T_n$ in accordance with the convention of the action of $\mathbf{G}_k$ on S. With these preparations, one can now explicitly give the structure of the $\mathbf{L}_{(j+1,n-j),k}$-module $\mathrm{H}^{n-j-1}(\mathbb{P}_k^n \setminus \mathbb{P}_k^j, \mathcal{O}_{\mathbb{P}_k^n}(a))$ with $a \in \mathbb{Z}$. During the proof of the following proposition, identifications of the cohomologies of $\mathcal{O}_{\mathbb{P}_k^n}(a)$ as $\mathbf{G}_k$-modules are needed. They are given by

$$\mathrm{H}^i(\mathbb{P}_k^n, \mathcal{O}_{\mathbb{P}_k^n}(a)) = \begin{cases} \mathbf{Sym}^a((k^{n+1})') & , \text{ if } a \geq 0, i = 0, \\ \det^{-1} \otimes_k \mathbf{Sym}^{-n-1-a}((k^{n+1})')' & , \text{ if } a \leq -n-1, i = n, \\ (0) & , \text{ else,} \end{cases} \tag{2.3}$$

cf. [23, III.5.1]. Recall the action of $\mathbf{G}_k$ on $\mathbb{A}_k^{n+1}$ specified in Subsection 1.2.4.

Proposition 2.2.2.3. *Let $a \in \mathbb{Z}$. Then there is an isomorphism of $\mathbf{L}_{(j+1,n-j),k}$-modules*

$$\mathrm{H}^{n-j-1}(\mathbb{P}_k^n \setminus \mathbb{P}_k^j, \mathcal{O}_{\mathbb{P}_k^n}(a)) \cong \bigoplus_{\substack{m \in \mathbb{N}_0 \\ m-a \geq n-j}} V(-m \cdot \epsilon_j)' \boxtimes_k \det^{-1} \otimes_k V((-m+a+n-j) \cdot \epsilon_n).$$

Proof. With the notation and identifications from above, one computes

$$\begin{aligned} \mathrm{H}^{n-j-1}(\mathbb{P}_k^n \setminus \mathbb{P}_k^j, \mathcal{O}_{\mathbb{P}_k^n}(a)) &= \mathrm{H}^{n-j-1}(\mathbb{P}_k^{n-j-1}, M_a^{\sim}) \\ &= \bigoplus_{m_0,\ldots,m_j \in \mathbb{N}_0} \mathrm{H}^{n-j-1}\left(\mathbb{P}_k^{n-j-1}, \mathcal{O}_{\mathbb{P}_k^{n-j-1}}\left(-\sum_{i=0}^{j} m_i + a\right)\right) \cdot T_0^{m_0} \cdot \ldots \cdot T_j^{m_j} \\ &= \bigoplus_{\substack{m_0,\ldots,m_j \in \mathbb{N}_0 \\ \sum_{i=0}^{j} m_i - a \geq n-j}} \left(\det^{-1} \otimes_k \mathbf{Sym}^{\sum_{i=0}^{j} m_i - a - n + j}((k^{n-j})')'\right) \cdot T_0^{m_0} \cdot \ldots \cdot T_j^{m_j} \\ &= \bigoplus_{m \in \mathbb{N}_0} \mathbf{Sym}^m((k^{j+1})') \boxtimes_k \det^{-1} \otimes_k \mathbf{Sym}^{m-a-n+j}((k^{n-j})')'. \end{aligned}$$

Now use the identification of symmetric powers in terms of Weyl modules, cf. Subsection 1.2.4. □

For general $\mathcal{F}$, one can now derive the following result on the structure of $\mathrm{H}^{n-j-1}(\mathbb{P}_k^n \setminus \mathbb{P}_k^j, \mathcal{F})$ resp. $\tilde{\mathrm{H}}^{n-j}_{\mathbb{P}_k^j}(\mathbb{P}_k^n, \mathcal{F})$.

Corollary 2.2.2.4. *There exist integers $a_1, \ldots, a_r \in \mathbb{Z}$, depending on $\mathcal{F}$, where r is the rank of $\mathcal{F}$, such that the $\mathbf{L}_{(j+1,n-j),k}$-module $\mathrm{H}^{n-j-1}(\mathbb{P}_k^n \setminus \mathbb{P}_k^j, \mathcal{F})$ – and thus $\tilde{\mathrm{H}}^{n-j}_{\mathbb{P}_k^j}(\mathbb{P}_k^n, \mathcal{F})$ – is a quotient of*

$$\bigoplus_{l=1}^{r} \bigoplus_{\substack{m \in \mathbb{N}_0 \\ m-a_l \geq n-j}} V(-m \cdot \epsilon_j)' \boxtimes_k \det^{-1} \otimes_k V((-m+a_l+n-j) \cdot \epsilon_n).$$

Proof. There exist integers $a_l \in \mathbb{Z}$ such that $\mathcal{F}$ can be written as a quotient of $\bigoplus_{l=1}^{r} \mathcal{O}_{\mathbb{P}_k^n}(a_l)$, cf. [23, Cor. II.5.18]. In other words, there exists an $\mathcal{O}_{\mathbb{P}_k^n}$-module $\mathcal{J}$ and an exact sequence of $\mathcal{O}_{\mathbb{P}_k^n}$-modules

$$0 \to \mathcal{J} \to \bigoplus_{l=1}^{r} \mathcal{O}_{\mathbb{P}_k^n}(a_l) \to \mathcal{F} \to 0.$$

Since f is affine (and thus quasi-compact) and the variety $\mathbb{P}^n_k \setminus \mathbb{P}^j_k$ is separated by definition, the sequence

$$0 \to f_*\mathcal{J}_{|\mathbb{P}^n_k\setminus\mathbb{P}^j_k} \to f_* \bigoplus_{l=1}^{r} \mathcal{O}_{\mathbb{P}^n_k}(a_l)_{|\mathbb{P}^n_k\setminus\mathbb{P}^j_k} \to f_*\mathcal{F}_{|\mathbb{P}^n_k\setminus\mathbb{P}^j_k} \to 0$$

is exact as well (cf. [16, Cor. 5.2.2]). This sequence then induces the usual long exact cohomology sequence for the functor $\mathrm{H}^*(\mathbb{P}^{n-j-1}_k, -)$. From the fact that $\mathrm{H}^{n-j}(\mathbb{P}^{n-j-1}_k, f_*\mathcal{J}_{|\mathbb{P}^n_k\setminus\mathbb{P}^j_k}) = 0$ (for dimensional reasons, cf. [23, II.2.7]), it follows then that the module $\mathrm{H}^{n-j-1}(\mathbb{P}^{n-j-1}_k, f_*\mathcal{F}_{|\mathbb{P}^n_k\setminus\mathbb{P}^j_k})$ is a quotient of

$$\mathrm{H}^{n-j-1}(\mathbb{P}^{n-j-1}_k, f_* \bigoplus_{l=1}^{r} \mathcal{O}_{\mathbb{P}^n_k}(a_l)_{|\mathbb{P}^n_k\setminus\mathbb{P}^j_k}) = \bigoplus_{l=1}^{r} \mathrm{H}^{n-j-1}(\mathbb{P}^{n-j-1}_k, f_*\mathcal{O}_{\mathbb{P}^n_k}(a_l)_{|\mathbb{P}^n_k\setminus\mathbb{P}^j_k})$$

which was identified in terms of Weyl modules in the previous proposition. □

As an application, one can now give a complete description of $\tilde{\mathrm{H}}^{n-j}_{\mathbb{P}^j_k}(\mathbb{P}^n_k, \mathcal{F})$ in terms of Weyl modules for Serre twist sheaves $\mathcal{F} = \mathcal{O}_{\mathbb{P}^n_k}(a)$, $a \in \mathbb{Z}$, and for sheaves of differential i-forms $\mathcal{F} = \Omega^i_{\mathbb{P}^n_k/k}$, $i = 1, \ldots, n$. Recall that the cohomologies of the sheaves $\Omega^i_{\mathbb{P}^n_k/k}$ of differential i-forms are given by

$$\mathrm{H}^s(\mathbb{P}^n_k, \Omega^i_{\mathbb{P}^n_k/k}) = \begin{cases} k & , \text{ if } 0 \le s = i \le n, \\ (0) & , \text{ else,} \end{cases} \tag{2.4}$$

with $\mathbf{G}_k$ acting trivially, cf. e.g. [19, 1.1].

Corollary 2.2.2.5.

i) Let $a \in \mathbb{Z}$ and write $\mathcal{F} = \mathcal{O}_{\mathbb{P}^n_k}(a)$. Then, as an $\mathbf{L}_{(j+1,n-j),k}$-module, there is an identification of $\tilde{\mathrm{H}}^{n-j}_{\mathbb{P}^j_k}(\mathbb{P}^n_k, \mathcal{F})$ with either

$$\bigoplus_{\substack{m\in\mathbb{N}_0 \\ m-a-n+j\ge 0}} V(-m\cdot\epsilon_j)' \boxtimes_k \det{}^{-1} \otimes_k V((-m+a+n-j)\cdot\epsilon_n) \Big/ \bigoplus_{l=0}^{a} V(-l\cdot\epsilon_j)' \boxtimes_k V((-a+l)\cdot\epsilon_n)'$$

(if $a \ge 0, j = n-1$) or

$$\bigoplus_{\substack{m\in\mathbb{N}_0 \\ m-a-n+j\ge 0}} V(-m\cdot\epsilon_j)' \boxtimes_k \det{}^{-1} \otimes_k V((-m+a+n-j)\cdot\epsilon_n)$$

(in all other cases).

ii) Let $\mathcal{F} = \Omega^1_{\mathbb{P}^n_k/k}$. There is an identification of $\tilde{\mathrm{H}}^{n-j}_{\mathbb{P}^j_k}(\mathbb{P}^n_k, \mathcal{F})$ with an $\mathbf{L}_{(j+1,n-j),k}$-submodule of

$$\bigoplus_{r=1}^{n+1} \bigoplus_{\substack{m\in\mathbb{N}_0 \\ m+1+j-n\ge 0}} V(-m\cdot\epsilon_j)' \boxtimes_k \det{}^{-1} \otimes_k V((-m-1-j+n)\cdot\epsilon_n)$$

such that the associated quotient is isomorphic to

$$\bigoplus_{\substack{m\in\mathbb{N}_0 \\ m+j-n\ge 0}} V(-m\cdot\epsilon_j)' \boxtimes_k \det{}^{-1} \otimes_k V((-m-j+n)\cdot\epsilon_n).$$

iii) More generally, for each $i = 1, \dots, n$ there is an identification of $\tilde{\mathrm{H}}^{n-j}_{\mathbb{P}^j_k}(\mathbb{P}^n_k, \Omega^i_{\mathbb{P}^n_k/k})$ with an $\mathbf{L}_{(j+1,n-j),k}$-submodule of

$$\bigoplus_{r=1}^{\binom{n+1}{i}} \bigoplus_{\substack{m\in\mathbb{N}_0\\ m+i-n+j\geq 0}} V(-m\cdot\epsilon_j)' \boxtimes_k \det{}^{-1} \otimes_k V((-m-i+n-j)\cdot\epsilon_n)$$

such that the associated quotient is isomorphic to $\mathrm{H}^{n-j-1}(\mathbb{P}^n_k/\mathbb{P}^j_k, \Omega^{i-1}_{\mathbb{P}^n_k/k})$, and the structure of the latter module is computable inductively.

Proof. Recall from the beginning of this section that there is an exact sequence

$$(0) \to \mathrm{H}^{n-j-1}(\mathbb{P}^n_k, \mathcal{F}) \to \mathrm{H}^{n-j-1}(\mathbb{P}^n_k \setminus \mathbb{P}^j_k, \mathcal{F}) \to \mathrm{H}^{n-j}_{\mathbb{P}^j_k}(\mathbb{P}^n_k, \mathcal{F}) \to \mathrm{H}^{n-j}(\mathbb{P}^n_k, \mathcal{F}) \to (0). \tag{2.5}$$

i) Consider first the case that $a \geq 0$. For $j = n-1$ the sequence (2.5) together with (2.3) gives a short exact sequence

$$(0) \to \mathrm{H}^0(\mathbb{P}^n_k, \mathcal{F}) = \mathbf{Sym}^a((k^{n+1})') \to \mathrm{H}^0(\mathbb{P}^n_k \setminus \mathbb{P}^{n-1}_k, \mathcal{F}) \to \mathrm{H}^1_{\mathbb{P}^{n-1}_k}(\mathbb{P}^n_k, \mathcal{F}) \to (0)$$

and for $j < n-1$ it gives an isomorphism

$$\mathrm{H}^{n-j-1}(\mathbb{P}^n_k \setminus \mathbb{P}^j_k, \mathcal{F}) \xrightarrow{\sim} \mathrm{H}^{n-j}_{\mathbb{P}^j_k}(\mathbb{P}^n_k, \mathcal{F}).$$

There are thus isomorphisms

$$\tilde{\mathrm{H}}^1_{\mathbb{P}^{n-1}_k}(\mathbb{P}^n_k, \mathcal{F}) \cong \mathrm{H}^0(\mathbb{P}^n_k \setminus \mathbb{P}^{n-1}_k, \mathcal{F})/\mathbf{Sym}^a((k^{n+1})')$$

and

$$\tilde{\mathrm{H}}^{n-j}_{\mathbb{P}^j_k}(\mathbb{P}^n_k, \mathcal{F}) \cong \mathrm{H}^{n-j-1}(\mathbb{P}^n_k \setminus \mathbb{P}^j_k, \mathcal{F})$$

for $j < n-1$.

Now consider the case that $a < 0$. For $j = 0$ the sequence (2.5) together with (2.3) gives a short exact sequence

$$(0) \to \mathrm{H}^{n-1}(\mathbb{P}^n_k \setminus \mathbb{P}^0_k, \mathcal{F}) \to \mathrm{H}^n_{\mathbb{P}^0_k}(\mathbb{P}^n_k, \mathcal{F}) \to \mathrm{H}^n(\mathbb{P}^n_k, \mathcal{F}) \to (0)$$

and for $j > 0$ it gives an isomorphism

$$\mathrm{H}^{n-j-1}(\mathbb{P}^n_k \setminus \mathbb{P}^j_k, \mathcal{F}) \xrightarrow{\sim} \mathrm{H}^{n-j}_{\mathbb{P}^j_k}(\mathbb{P}^n_k, \mathcal{F}).$$

Both times, there is thus an isomorphism

$$\tilde{\mathrm{H}}^{n-j}_{\mathbb{P}^j_k}(\mathbb{P}^n_k, \mathcal{F}) \cong \mathrm{H}^{n-j-1}(\mathbb{P}^n_k \setminus \mathbb{P}^j_k, \mathcal{F}).$$

Invoking Proposition 2.2.2.3 one then obtains an identification of $\tilde{\mathrm{H}}^{n-j}_{\mathbb{P}^j_k}(\mathbb{P}^n_k, \mathcal{O}_{\mathbb{P}^n_k}(a))$ with either

$$\bigoplus_{\substack{m\in\mathbb{N}_0\\ m-a-n+j\geq 0}} V(-m\cdot\epsilon_j)' \boxtimes_k \det{}^{-1} \otimes_k V((-m+a+n-j)\cdot\epsilon_n) \Big/ \mathbf{Sym}^a((k^{n+1})')$$

(if $a \geq 0, j = n-1$) or

$$\bigoplus_{\substack{m \in \mathbb{N}_0 \\ m-a-n+j \geq 0}} V(-m \cdot \epsilon_j)' \boxtimes_k \det{}^{-1} \otimes_k V((-m+a+n-j) \cdot \epsilon_n)$$

(in all other cases). The claim follows from observing that, as an $\mathbf{L}_{(j+1,n-j),k}$-module and for $a \in \mathbb{N}_0$, the module $\mathbf{Sym}^a((k^{n+1})')$ is isomorphic to

$$\bigoplus_{l=0}^{a} \mathbf{Sym}^l((k^{j+1})') \boxtimes_k \mathbf{Sym}^{a-l}((k^{n-j})') \cong \bigoplus_{l=0}^{a} V(-l \cdot \epsilon_j)' \boxtimes_k V(-(a-l) \cdot \epsilon_n)'.$$

ii) Write $\mathcal{F} = \Omega^1_{\mathbb{P}^n_k/k}$. For $j = n-1$ the sequence (2.5) combined with (2.4) gives a short exact sequence

$$(0) \to \mathrm{H}^0(\mathbb{P}^n_k \setminus \mathbb{P}^{n-1}_k, \mathcal{F}) \to \mathrm{H}^1_{\mathbb{P}^{n-1}_k}(\mathbb{P}^n_k, \mathcal{F}) \to \mathrm{H}^1(\mathbb{P}^n_k, \mathcal{F}) = k \to (0),$$

for $j = n-2$ it gives a short exact sequence

$$(0) \to \mathrm{H}^1(\mathbb{P}^n_k, \mathcal{F}) = k \to \mathrm{H}^1(\mathbb{P}^n_k \setminus \mathbb{P}^{n-2}_k, \mathcal{F}) \to \mathrm{H}^2_{\mathbb{P}^{n-2}_k}(\mathbb{P}^n_k, \mathcal{F}) \to (0),$$

and for $j < n-2$ it gives an isomorphism

$$\mathrm{H}^{n-j-1}(\mathbb{P}^n_k \setminus \mathbb{P}^j_k, \mathcal{F}) \xrightarrow{\sim} \mathrm{H}^{n-j}_{\mathbb{P}^j_k}(\mathbb{P}^n_k, \mathcal{F}).$$

Therefore, if $j = n-1$ or $j < n-2$, there is an isomorphism

$$\tilde{\mathrm{H}}^{n-j}_{\mathbb{P}^j_k}(\mathbb{P}^n_k, \mathcal{F}) \cong \mathrm{H}^{n-j-1}(\mathbb{P}^n_k \setminus \mathbb{P}^j_k, \mathcal{F}), \tag{2.6}$$

and if $j = n-2$, there is an isomorphism

$$\tilde{\mathrm{H}}^{n-j}_{\mathbb{P}^j_k}(\mathbb{P}^n_k, \mathcal{F}) \cong \mathrm{H}^{n-j-1}(\mathbb{P}^n_k \setminus \mathbb{P}^j_k, \mathcal{F})/k. \tag{2.7}$$

The module $\mathcal{F}$ fits into an exact sequence

$$0 \to \mathcal{F} \to \mathcal{O}_{\mathbb{P}^n_k}(-1)^{n+1} \to \mathcal{O}_{\mathbb{P}^n_k} \to 0,$$

cf. [23, II.8.13]. This sequence in turn induces a long exact sequence

$$\begin{aligned} \ldots \;\to\; & \mathrm{H}^{i-1}(\mathbb{P}^n_k \setminus \mathbb{P}^j_k, \mathcal{O}_{\mathbb{P}^n_k}) \to \mathrm{H}^i(\mathbb{P}^n_k \setminus \mathbb{P}^j_k, \mathcal{F}) \to \mathrm{H}^i(\mathbb{P}^n_k \setminus \mathbb{P}^j_k, \mathcal{O}_{\mathbb{P}^n_k}(-1)^{n+1}) \\ \to\; & \mathrm{H}^i(\mathbb{P}^n_k \setminus \mathbb{P}^j_k, \mathcal{O}_{\mathbb{P}^n_k}) \to \ldots, \end{aligned}$$

which abbreviates to

$$\begin{aligned} & (0) = \mathrm{H}^{n-j-2}(\mathbb{P}^n_k, \mathcal{O}_{\mathbb{P}^n_k}) = \mathrm{H}^{n-j-2}(\mathbb{P}^n_k \setminus \mathbb{P}^j_k, \mathcal{O}_{\mathbb{P}^n_k}) \\ \to\; & \mathrm{H}^{n-j-1}(\mathbb{P}^n_k \setminus \mathbb{P}^j_k, \mathcal{F}) \to \mathrm{H}^{n-j-1}(\mathbb{P}^n_k \setminus \mathbb{P}^j_k, \mathcal{O}_{\mathbb{P}^n_k}(-1)^{n+1}) \to \mathrm{H}^{n-j-1}(\mathbb{P}^n_k \setminus \mathbb{P}^j_k, \mathcal{O}_{\mathbb{P}^n_k}) \\ \to\; & \mathrm{H}^{n-j}(\mathbb{P}^n_k \setminus \mathbb{P}^j_k, \mathcal{F}) = (0) \end{aligned}$$

in case $j \neq n-2$ (cf. (2.2) and (2.3)) resp. to

$$\begin{aligned} & (0) = \mathrm{H}^0(\mathbb{P}^n_k, \mathcal{O}_{\mathbb{P}^n_k}(-1)^{n+1}) = \mathrm{H}^0(\mathbb{P}^n_k \setminus \mathbb{P}^{n-2}_k, \mathcal{O}_{\mathbb{P}^n_k}(-1)^{n+1}) \\ \to\; & k = \mathrm{H}^0(\mathbb{P}^n_k, \mathcal{O}_{\mathbb{P}^n_k}) = \mathrm{H}^0(\mathbb{P}^n_k \setminus \mathbb{P}^{n-2}_k, \mathcal{O}_{\mathbb{P}^n_k}) \\ \to\; & \mathrm{H}^1(\mathbb{P}^n_k \setminus \mathbb{P}^{n-2}_k, \mathcal{F}) \to \mathrm{H}^1(\mathbb{P}^n_k \setminus \mathbb{P}^{n-2}_k, \mathcal{O}_{\mathbb{P}^n_k}(-1)^{n+1}) \to \mathrm{H}^1(\mathbb{P}^n_k \setminus \mathbb{P}^{n-2}_k, \mathcal{O}_{\mathbb{P}^n_k}) \\ \to\; & \mathrm{H}^2(\mathbb{P}^n_k \setminus \mathbb{P}^{n-2}_k, \mathcal{F}) = (0) \end{aligned}$$

in case $j = n-2$ (once more using (2.2) and (2.3)). Again reading off from the previous proposition and applying (2.6) resp. (2.7), one can thus realize $\tilde{\mathrm{H}}^{n-j}_{\mathbb{P}^j_k}(\mathbb{P}^n_k, \mathcal{F})$ as a submodule of

$$\bigoplus_{r=0}^{n} \bigoplus_{\substack{m \in \mathbb{N}_0 \\ m+1-n+j \geq 0}} V(-m \cdot \epsilon_j)' \boxtimes_k \det{}^{-1} \otimes_k V(-(m+1-n+j) \cdot \epsilon_n)$$

such that the associated quotient has the asserted structure.

iii) This is just an iteration of ii), using the exact sequence

$$0 \to \Omega^i_{\mathbb{P}^n_k/k} \to \mathcal{O}_{\mathbb{P}^n_k}(-i)^{\binom{n+1}{i}} \to \Omega^{i-1}_{\mathbb{P}^n_k/k} \to 0,$$

which arises by evaluating the filtration on $\bigwedge^i \mathcal{O}_{\mathbb{P}^n_k}(-1)^{n+1} = \mathcal{O}_{\mathbb{P}^n_k}(-i)^{\binom{n+1}{i}}$ which is obtained from [23, Ex. II.5.16]. Alternatively, cf. [19, 1.6(7)].

□

2.3 Local Cohomology II: Failure of "Classical" Lie Algebraic Methods

As was already noted in the last section, in Orlik's paper [44] there is made extensive use of Lie algebraic methods to determine the structure of the algebraic local cohomology modules in the p-adic setting. The objective of this section is to show that those methods are not as powerful in the case of a finite field. First of all, a different description of $\mathrm{H}^{n-j-1}(\mathbb{P}^n_k \setminus \mathbb{P}^j_k, \mathcal{F})$ as a k-space shall be given, namely in terms of generalized fractions.

2.3.1 Generalized Fractions

As $\mathcal{F}$ is locally free of finite rank, it is coherent. Thus there is a graded S-module $F = \bigoplus_{l \in \mathbb{N}_0} F_l$ of finite type, for example

$$F = \bigoplus_{l \in \mathbb{N}_0} \mathrm{H}^0(\mathbb{P}^n_k, \mathcal{F}(l)),$$

such that $F^\sim = \mathcal{F}$. Furthermore, F is an algebraic $\mathbf{G}_k$-module such that $\mathbf{G}_k$ acts homogeneously and compatibly with its action on S, i.e. for any k-algebra R, the formula

$$g.(sf) = (g.s)(g.f)$$

holds for all $g \in \mathbf{G}_k(R), s \in S \otimes_k R, f \in F \otimes_k R$.

Then, for $j \leq n-2$ there is an identification

$$\mathrm{H}^{n-j-1}(\mathbb{P}^n_k \setminus \mathbb{P}^j_k, \mathcal{F}) = \varinjlim_{l \in \mathbb{N}} (F/(T^l_{j+1}, \ldots, T^l_n)F)_{l(n-j)}$$

and for $j = n-1$ there is an exact sequence

$$(0) \to \mathrm{H}^0(\mathbb{P}^n_k, \mathcal{F}) \to \mathrm{H}^0(\mathbb{P}^n_k \setminus \mathbb{P}^j_k, \mathcal{F}) \to \varinjlim_{l \in \mathbb{N}} (F/T^l_n F)_l \to (0),$$

cf. [17, 2.1.5.1-2]. Here, the degree of a coset $f + (T^l_{j+1}, \ldots, T^l_n)F$ is defined as the degree of f and the transition maps are induced by

$$\begin{array}{rcl} \iota_{l,m} : F/(T^l_{j+1}, \ldots, T^l_n)F & \to & F/(T^m_{j+1}, \ldots, T^m_n)F \\ f + (T^l_{j+1}, \ldots, T^l_n)F & \mapsto & T^{m-l}_{j+1} \cdot \ldots \cdot T^{m-l}_n \cdot f + (T^m_{j+1}, \ldots, T^m_n)F \end{array}$$

for $m \geq l$. It is then convenient to use the natural identification of $\varinjlim_{l\in\mathbb{N}}(F/(T_{j+1}^l,\ldots,T_n^l)F)_{l(n-j)}$ as a quotient of $(F_{T_{j+1}\cdot\ldots\cdot T_n})_0$ via

$$\begin{array}{rcl}\rho : (F_{T_{j+1}\cdot\ldots\cdot T_n})_0 & \to & \varinjlim_{l\in\mathbb{N}}(F/(T_{j+1}^l,\ldots,T_n^l)F)_{l(n-j)} \\ \dfrac{f}{T_{j+1}^l\cdot\ldots\cdot T_n^l} & \mapsto & f+(T_{j+1}^l,\ldots,T_n^l)F,\end{array}$$

cf. op. cit., § 2. Under this map, the transition maps $\iota_{l,m}$ are induced by the maps

$$\frac{f}{T_{j+1}^l\cdot\ldots\cdot T_n^l} \mapsto \frac{T_{j+1}^{m-l}\cdot\ldots\cdot T_n^{m-l}\cdot f}{T_{j+1}^m\cdot\ldots\cdot T_n^m}$$

on $(F_{T_{j+1}\cdot\ldots\cdot T_n})_0$. From now on, write

$$(F_{T_{j+1}\cdot\ldots\cdot T_n})_0^\rho = (F_{T_{j+1}\cdot\ldots\cdot T_n})_0/\ker(\rho)$$

and for an element $\frac{f}{T_{j+1}^l\cdot\ldots\cdot T_n^l} \in (F_{T_{j+1}\cdot\ldots\cdot T_n})_0$ denote its image in this quotient under the canonical map by $\left[\frac{f}{T_{j+1}^l\cdot\ldots\cdot T_n^l}\right]$. The above formulas imply that, up to a k-finite-dimensional part, it suffices to describe $(F_{T_{j+1}\cdot\ldots\cdot T_n})_0^\rho$ as a representation of $\mathbf{P}_{(j+1,n-j),k}$ in order to describe $\tilde{\mathrm{H}}^{n-j}_{\mathbb{P}_k^j}(\mathbb{P}_k^n,\mathcal{F})$ as a representation of this group. Therefore, for the rest of this chapter, the focus will lie on $(F_{T_{j+1}\cdot\ldots\cdot T_n})_0^\rho$.

2.3.2 The Action of the Universal Enveloping Algebra

The action of $\mathbf{G}_k$ on $\mathcal{F}$ induces an action of the Lie algebra $\mathfrak{g}_k = \mathrm{Lie}(\mathbf{G}_k) = \mathfrak{g}\otimes_{\mathcal{V}} k$, where $\mathfrak{g} = \mathrm{Lie}(\mathbf{G})$, – and thus also an action of its universal enveloping algebra – on $\mathcal{F}$ which shall now be explained. Denote by $\mathbf{G}_k^{(1)}$ the first infinitesimal neighborhood of the identity element, considered as a group over the algebra $k[\epsilon] = k[T]/(T^2)$ of dual numbers, i.e.

$$\mathbf{G}_k^{(1)} = \mathbf{Spec}\, k[\mathbf{G}_k]/I^2$$

where $I \subset k[\mathbf{G}_k]$ is the ideal defining the identity element $1 \in \mathbf{G}_k$. Then the action of $\mathbf{G}_k$ on $\mathcal{F}$ can be restricted to an action of $\mathbf{G}_k^{(1)}$. This action can be differentiated to an action of $\mathfrak{g}_k$ as follows: For a Zariski-open subset $U \subset \mathbb{P}_k^n$, let $f \in \Gamma(U,\mathcal{F})$ be a section and let $\eta \in \mathfrak{g}_k$. Consider η as an element of $\mathbf{G}_k^{(1)}$ via $\eta \mapsto 1+\epsilon\eta$. Then η acts on f via

$$\eta.f = \frac{d}{d\epsilon}_{\epsilon=0}((1+\epsilon\eta).f),$$

where $\frac{d}{d\epsilon}$ means algebraic differentiation of the morphism

$$\mathbf{G}_k^{(1)} \to \mathrm{GL}(\Gamma(U,\mathcal{F}))$$

whose matrix entries are rational functions. The usual Leibniz rule then applies in this context, i.e. for $s \in \Gamma(U,\mathcal{O}_{\mathbb{P}_k^n})$, $f \in \Gamma(U,\mathcal{F})$ and $\eta \in \mathfrak{g}_k$ the formula

$$\eta.(sf) = s(\eta.f) + (\eta.s)f \tag{2.8}$$

holds. This action extends to an action of the universal enveloping algebra $U(\mathfrak{g}_k)$ of $\mathfrak{g}_k$ by the universal property of this algebra. Furthermore, if U is stabilized by a closed subgroup $\mathbf{H} \subset \mathbf{G}_k$, then both

$\mathbf{H}$ and $\mathfrak{g}_k$ act on $\Gamma(U,\mathcal{F})$ compatibly with respect to the adjoint action of $\mathbf{H}$ on $\mathfrak{g}_k$, i.e. for $f \in \Gamma(U,\mathcal{F}), g \in \mathbf{H}, \eta \in \mathfrak{g}_k$ the formula

$$g.(\eta.f) = (g.\eta).(g.f) \tag{2.9}$$

holds since

$$\begin{aligned} g.(\eta.f) &= g.(\eta.((g^{-1}g).f)) = g.\left(\frac{d}{d\epsilon}_{\epsilon=0}((1+\epsilon\eta).((g^{-1}g).f))\right) = \frac{d}{d\epsilon}_{\epsilon=0}((g(1+\epsilon\eta)g^{-1}).(g.f)) \\ &= \frac{d}{d\epsilon}_{\epsilon=0}((1+\epsilon g\eta g^{-1}).(g.f)) = (g\eta g^{-1}).(g.f) = (g.\eta).(g.f). \end{aligned}$$

In this way, $\Gamma(U,\mathcal{F})$ becomes a representation of the semi-direct product $\mathbf{H} \ltimes \mathfrak{g}_k$ and of the semi-direct product $\mathbf{H} \ltimes U(\mathfrak{g}_k)$ since the adjoint action of $\mathbf{H}$ on $\mathfrak{g}_k$ extends to one on $U(\mathfrak{g}_k)$. All actions mentioned also extend by functoriality to the higher cohomologies $\mathrm{H}^i(U,\mathcal{F})$, $i \in \mathbb{Z}$, and also to the local cohomologies $\mathrm{H}^i_Z(\mathbb{P}^n_k,\mathcal{F})$ for $Z = \mathbb{P}^n_k \setminus U, i \in \mathbb{Z}$. Furthermore, the long exact sequence from local cohomology is equivariant with respect to those actions.

Once and for all fix an ordering of Φ and for each $\alpha_{u,v} \in \Phi$ denote the standard generator of the weight space $\mathfrak{g}_{\alpha_{u,v},k} \subset \mathfrak{g}_k$ by $L_{(u,v)} = L_{\alpha_{u,v}}$. Denote the (ordered) standard basis of $\mathfrak{t}_k$ by $(L_0,\ldots,L_n)$. The k-algebra $U(\mathfrak{g}_k)$ is then generated by

$$\{L_i \mid i = 0,\ldots,n\} \cup \{L_{(u,v)} \mid 0 \le u \ne v \le n\}.$$

In the case of the structure sheaf $\mathcal{F} = \mathcal{O}_{\mathbb{P}^n_k}$ the action of $\mathfrak{g}_k$ and thus of $U(\mathfrak{g}_k)$ on $\Gamma(\mathcal{X}^{(n+1)},\mathcal{F})$ can be written down very explicitly: For

$$\mu = \sum_{i=0}^{n} m_i\epsilon_i \in X^*(\mathrm{T})$$

with $\sum_i m_i = 0$ define $\Xi_\mu \in \Gamma(\mathcal{X}^{(n+1)},\mathcal{O}_{\mathbb{P}^n_k})$ by

$$\Xi_\mu(T_0,\ldots,T_n) = T_0^{m_0}\cdot\ldots\cdot T_n^{m_n}.$$

Then direct calculation gives

$$\begin{aligned} L_{(u,v)}^{l_{u,v}}.\Xi_\mu &= \frac{m_v!}{(m_v - l_{u,v})!}\Xi_{\mu+l_{u,v}\alpha_{u,v}} \\ L_i^l.\Xi_\mu &= m_i^l\Xi_\mu, \end{aligned}$$

for $l_{u,v}, l \in \mathbb{N}_0$ and u,v,i,μ as above.

Failure of using the Universal Enveloping Algebra

In the case of $U = \mathbb{P}^n_k \setminus \mathbb{P}^j_k$ and general $\mathcal{F}$ the action of $\mathfrak{g}_k$ on $\mathrm{H}^{n-j-1}(\mathbb{P}^n_k \setminus \mathbb{P}^j_k,\mathcal{F})$ can be described by using the Čech complex associated with the covering

$$\mathbb{P}^n_k \setminus \mathbb{P}^j_k = \bigcup_{l=j+1}^{n} D_+(T_l)$$

which computes $\mathrm{H}^*(\mathbb{P}^n_k \setminus \mathbb{P}^j_k,\mathcal{F})$. The i-th component of this complex is

$$\bigoplus_{j+1\le l_1<l_2<\ldots<l_{i+1}\le n} \Gamma(D_+(T_{l_1}\cdot\ldots\cdot T_{l_{i+1}}),\mathcal{F}).$$

It is acted upon by $\mathfrak{g}_k$ similarly as the functions Ξ_μ above and the transition maps of this complex are equivariant with respect to this $\mathfrak{g}_k$-action. Let $u \in \{0,\ldots,j\}, v \in \{j+1,\ldots,n\}$ and let $\frac{f}{T_{j+1}^m \cdot \ldots \cdot T_n^m} \in (F_{T_{j+1}\cdot\ldots\cdot T_n})_0$ be a representative of an element of $\mathrm{H}^{n-j-1}(\mathbb{P}_k^n \setminus \mathbb{P}_k^j, \mathcal{F})$. Then

$$L_{(u,v)} \cdot \frac{f}{T_{j+1}^m \cdot \ldots \cdot T_n^m} = \frac{L_{(u,v)} \cdot f}{T_{j+1}^m \cdot \ldots \cdot T_n^m} - m \frac{T_u \cdot T_{j+1} \cdot \ldots \cdot \widehat{T_v} \cdot \ldots \cdot T_n \cdot f}{T_{j+1}^{m+1} \cdot \ldots \cdot T_n^{m+1}}. \tag{2.10}$$

In similar fashion, basis elements $L_{(u,v)}$ with the roles of u and v reversed decrease the degree in the second summand. A toral basis element L_l with $l \in \{0,\ldots,n\}$ acts as

$$L_l \cdot \frac{f}{T_{j+1}^m \cdot \ldots \cdot T_n^m} = \begin{cases} \frac{L_l \cdot f}{T_{j+1}^m \cdot \ldots \cdot T_n^m} & , \text{ if } l \in \{0,\ldots,j\} \\ \frac{L_l \cdot f}{T_{j+1}^m \cdot \ldots \cdot T_n^m} - m \frac{f}{T_{j+1}^m \cdot \ldots \cdot T_n^m} & , \text{ if } l \in \{j+1,\ldots,n\}. \end{cases} \tag{2.11}$$

The algebra $U(\mathfrak{g}_k)$ then acts accordingly by iteration of the above formulas. To simplify later calculations, note that there is no harm done in thinking of the element $\frac{f}{T_{j+1}^m \cdot \ldots \cdot T_n^m}$ as a "completely reduced fraction", i.e.

$$L_{(u,v)} \cdot \frac{T_l^i f}{T_{j+1}^m \cdot \ldots \cdot T_n^m} = L_{(u,v)} \cdot \frac{f}{T_{j+1}^m \cdot \ldots T_{l-1}^m \cdot T_l^{m-i} \cdot T_{l+1}^m \cdot \ldots \cdot T_n^m}$$

for $l \in \{j+1,\ldots,n\}$, $0 \le i < m$, and $f \in F$.

In the analog p-adic situation where k is replaced by a finite extension K of $\mathbb{Q}_p$ and thus the Drinfeld half space over K is considered, Orlik constructs in [44, Lemma 1.2.1] a K-finite dimensional representation of the parabolic group $\mathbf{P}_{(j+1,n-j),K}$ over K which generates the local cohomology module $\tilde{\mathrm{H}}^{n-j}_{\mathbb{P}_K^j}(\mathbb{P}_K^n, \mathcal{F})$ over the universal enveloping algebra $U(\mathfrak{gl}_{n+1,K})$. Trying to copy this verbatim to the situation here does not work, since it follows from the formulas (2.10) and (2.11) that $U(\mathfrak{g}_k).M$ is of finite k-dimension whenever $M \subset \tilde{\mathrm{H}}^n_{\mathbb{P}_k^j}(\mathbb{P}_k^n, \mathcal{F})$ is: Without loss of generality, it can be assumed that M is contained in

$$\left\{ \frac{f}{T_{j+1}^m \cdot \ldots \cdot T_n^m} \mid f \in F_{m(n-j)} \right\}$$

for some $m \in \mathbb{N}$. Suppose that $l \in \mathbb{N}$ is the smallest integer which is larger than m and divisible by p. Then it follows that $U(\mathfrak{g}_k).M \subset \left\{ \frac{f}{T_{j+1}^l \cdot \ldots \cdot T_n^l} \mid f \in F_{l(n-j)} \right\}$.

2.3.3 The Action of the Distribution Algebra

Now consider the the distribution algebra $\mathrm{Dist}(\mathbf{G}_k)$ of $\mathbf{G}_k$ (cf. [32, I.7,II.1]). For $l \in \mathbb{N}$, denote by $\mathbf{G}_k^{(l)}$ the l-th infinitesimal neighborhood of $1 \in \mathbf{G}_k$. By construction, $\mathrm{Dist}(\mathbf{G}_k)$ is equal to the union $\bigcup_{l\in\mathbb{N}} \mathrm{Dist}(\mathbf{G}_k^{(l)})$. For each $l \in \mathbb{N}$, the action of $\mathbf{G}_k^{(l)}$ on $\mathcal{F}$ induces an action of $\mathrm{Dist}(\mathbf{G}_k^{(l)})$ on $\mathcal{F}$. Therefore, $\mathrm{Dist}(\mathbf{G}_k)$ acts on $\mathcal{F}$ as well. Furthermore, compatibility with group actions in the sense of (2.9) and with S-action in the sense of (2.8) hold.

Failure of using the Distribution Algebra

The distribution algebra $\mathrm{Dist}(\mathbf{G}_k)$ of $\mathbf{G}_k$ can be realized as a divided power algebra over k in the following way: Consider $U(\mathfrak{g} \otimes_{\mathcal{V}} K)$, the universal enveloping algebra of $\mathfrak{g} \otimes_{\mathcal{V}} K$ and consider the $\mathcal{V}$-subalgebra $\dot{U}(\mathfrak{g})$ generated by all divided powers

$$L_{(u,v)}^{(l_{u,v})} = \frac{1}{l_{u,v}!} L_{(u,v)}^{l_{u,v}},$$

where $0 \leq u \neq v \leq n,\ l_{u,v} \in \mathbb{N}_0$, and by all expressions of the form

$$\binom{L_i}{l} = \frac{L_i \cdot (L_i - 1) \cdot \ldots \cdot (L_i - l + 1)}{l!},$$

where $0 \leq i \leq n,\ l \in \mathbb{N}_0$. Then there is an isomorphism

$$\mathrm{Dist}(\mathbf{G}_k) \cong \dot{U}(\mathfrak{g}) \otimes_{\mathcal{V}} k,$$

cf. [32, II.1.12]. In this way, one can then read off the action of $\mathrm{Dist}(\mathbf{G}_k)$ on $(F_{T_{j+1}\cdot\ldots\cdot T_n})_0^\rho$ from the previous subsubsection by adjusting for the respective divided powers.

One might suspect that the distribution algebra manages to finitely generate $\tilde{\mathrm{H}}^n_{\mathbb{P}^j_k}(\mathbb{P}^n_k, \mathcal{F})$ in general, but this is wrong. The problem is again the vanishing behavior mod p of the (binomial) coefficients appearing. A counterexample will be given in the following proposition. As this involves calculating binomial coefficients mod p, the reader is reminded of the following result, called Lucas's Theorem, cf. e.g. [11]:

Let $a, b \in \mathbb{N}$ with p-adic expansions $a = \sum_{i=0}^{l} a_i p^i, b = \sum_{i=0}^{l} b_i p^i$, i.e. $0 \leq a_i, b_i < p$ for all $i = 0, \ldots, l$. Then

$$\binom{a}{b} \equiv \prod_{i=0}^{l} \binom{a_i}{b_i} \ (mod\ p).$$

Here and in the following, $\binom{a}{b}$ is defined to be 0 for $a, b \in \mathbb{N}_0$ with $b > a$.

Proposition 2.3.3.1. *Let $p > 2$. The* $\mathrm{Dist}(\mathrm{GL}_{3,k})$*-module* $\tilde{\mathrm{H}}^1_{\mathbb{P}^1_k}(\mathbb{P}^2_k, \mathcal{O}_{\mathbb{P}^2_k})$ *is not generated by a submodule of finite k-dimension.*

Proof. In this special case, $F = S$ and

$$(F_{T_{j+1}\cdot\ldots\cdot T_n})_0^\rho = (S_{T_2})_0^\rho = k\left[\frac{T_0}{T_2}, \frac{T_1}{T_2}\right]/k.$$

Therefore, it is enough to show that the proposition holds true with $k[\frac{T_0}{T_2}, \frac{T_1}{T_2}]$ instead of $\tilde{\mathrm{H}}^1_{\mathbb{P}^1_k}(\mathbb{P}^2_k, \mathcal{O}_{\mathbb{P}^2_k})$. Suppose that there were in fact a k-finite dimensional $\mathrm{Dist}(\mathrm{GL}_{3,k})$-submodule $N \subset k[\frac{T_0}{T_2}, \frac{T_1}{T_2}]$ with $k[\frac{T_0}{T_2}, \frac{T_1}{T_2}] = \mathrm{Dist}(\mathrm{GL}_{3,k}).N$. Without loss of generalization, it can be assumed that $N = k[\frac{T_0}{T_2}, \frac{T_1}{T_2}]_{\leq p^{h+1}}$ for some $h \in \mathbb{N}$. The subalgebra $\mathrm{Dist}(\mathbf{P}_{(2,1),k})$ then stabilizes N and from the PBW decomposition

$$\mathrm{Dist}(\mathrm{GL}_{3,k}) = \mathrm{Dist}(\mathbf{P}_{(2,1),k}) \otimes_k \mathrm{Dist}(\mathbf{U}^+_{(2,1),k})$$

(cf. [32, II.1]) it follows that

$$\mathrm{Dist}(\mathrm{GL}_{3,k}).N = \mathrm{Dist}(\mathbf{U}^+_{(2,1),k}).N,$$

i.e. it is enough to show that $\mathrm{Dist}(\mathbf{U}^+_{(2,1),k}).N \subsetneq k[\frac{T_0}{T_2}, \frac{T_1}{T_2}]$ to contradict the assumption on N. This will be achieved by showing that for $g = p^{h+1}$ and $g' = \frac{g+1}{2}$, the monomial $f = \frac{T_0^{g'} \cdot T_1^{g'}}{T_2^{g+1}}$ is not contained in $\mathrm{Dist}(\mathbf{U}^+_{(2,1),k}).N$. Consider the canonical monomial basis $\left\{\frac{T_0^a T_1^b}{T_2^{a+b}} \mid a, b \in \mathbb{N}_0\right\}$ of $k[\frac{T_0}{T_2}, \frac{T_1}{T_2}]$. Then it suffices to show by decreasing induction on $a + b$ that f does not appear as a summand of any of the elements contained in $\mathrm{Dist}(\mathbf{U}^+_{(2,1),k}).\frac{T_0^a \cdot T_1^b}{T_2^{a+b}}$ for $0 < a+b \leq g$ with respect to this choice of basis. The generators of $\mathrm{Dist}(\mathbf{U}^+_{(2,1),k})$ act on a basis element $\frac{T_0^a \cdot T_1^b}{T_2^{a+b}}$ by

$$\begin{aligned} L^{(l)}_{(0,2)}.\frac{T_0^a T_1^b}{T_2^{a+b}} &= (-1)^l \binom{a+b+l-1}{l} \frac{T_0^{a+l} T_1^b}{T_2^{a+b+l}} \\ L^{(l)}_{(1,2)}.\frac{T_0^a T_1^b}{T_2^{a+b}} &= (-1)^l \binom{a+b+l-1}{l} \frac{T_0^{a} T_1^{b+l}}{T_2^{a+b+l}} \end{aligned}$$

for $l \in \mathbb{N}_0$. Let

$$l = l_0 + l_1 p + l_2 p^2 + \ldots + l_h p^h + \ldots + l_r p^r$$

be the p-adic expansion of l (for some $r \in \mathbb{N}_0$, with coefficients $l_i \in \{0, \ldots, p-1\}$).

i) Suppose that $a+b = g$. Since $g-1$ has p-adic expansion

$$g-1 = (p-1) + (p-1)p + \ldots + (p-1)p^h,$$

the condition $\binom{g+l-1}{l} \not\equiv 0 \pmod p$ requires $l_0 = \ldots = l_h = 0$, according to Lucas's Theorem. This means that either $l = 0$ or $l \geq g$ and $g \mid l$. Therefore,

$$\mathrm{Dist}(\mathbf{U}^+_{(2,1),k}).\frac{T_0^a T_1^b}{T_2^g} \subset k\left[\frac{T_0}{T_2}, \frac{T_1}{T_2}\right]_g \oplus \sum_{\substack{r,s \in \mathbb{N}_0 \\ r+s=2g}} \frac{T_0^r T_1^s}{T_2^{r+s}}\, k\left[\frac{T_0}{T_2}, \frac{T_1}{T_2}\right]$$

and so f is not contained as a summand of any element of this latter set.

ii) Suppose that $1 < a+b < g$. Let

$$a+b-1 = c_0 + c_1 p + c_2 p^2 + \ldots + c_h p^h$$

be the p-adic expansion of $a+b-1$ (with coefficients $c_i \in \{0, \ldots, p-1\}$ and at least one $c_i < p-1$ and at least one $c_j > 0$). For $\binom{a+b+l-1}{l} \bmod p$ to be not equal to 0, the inequality $c_i + l_i \leq p-1$ has to hold for each $i \in \{0, \ldots, h\}$, according to Lucas's Theorem. But this implies that $a+b+l-1$ has p-adic expansion

$$a+b+l-1 = (c_0+l_0) + (c_1+l_1)p + (c_2+l_2)p^2 + \ldots + (c_h+l_h)p^h + l_{h+1}p^{h+1} + \ldots + l_r p^r.$$

In turn, this latter fact means that there are two possibilities:

(1) $a+b+l-1 < g$, if $l_{h+1} = \ldots = l_r = 0$. In this case one applies the induction hypothesis.

(2) $a+b+l-1 > g+1$, if there is at least one $i \in \{h+1, \ldots, r\}$ with $l_i \neq 0$. Recall that at least one c_j is not equal to 0.

This implies that

$$\mathrm{Dist}(\mathbf{U}^+_{(2,1),k}).\frac{T_0^a T_1^b}{T_2^g} \subset k\left[\frac{T_0}{T_2}, \frac{T_1}{T_2}\right]_{\leq g} \oplus \sum_{\substack{r,s \in \mathbb{N}_0 \\ r+s=g+2}} \frac{T_0^r T_1^s}{T_2^{r+s}} k\left[\frac{T_0}{T_2}, \frac{T_1}{T_2}\right]$$

and so f is not contained as a summand of any element of this latter set.

iii) Suppose that $a+b=1$. Without loss of generalization, it can be assumed that $a=1, b=0$, i.e. $\frac{T_0^a \cdot T_1^b}{T_2^{a+b}} = \frac{T_0}{T_2}$. Application of either $L_{(0,2)}^{(l)}$ or $L_{(1,2)}^{(l)}$ to this element with $l \in \mathbb{N}_0$ yields either $\pm\frac{T_0^{l+1}}{T_2^{l+1}}$ or $\pm\frac{T_0 \cdot T_1^l}{T_2^{l+1}}$. According to the induction hypothesis, the element f is not a summand of any element in $(\mathrm{Dist}(\mathbf{U}_{(2,1),k}^+) \cdot L_{(1,2)}^{(l)}).\frac{T_0}{T_2} + (\mathrm{Dist}(\mathbf{U}_{(2,1),k}^+) \cdot L_{(0,2)}^{(l)}).\frac{T_0}{T_2}$ for $l \in \{0,\dots,g-1\}$. On the other hand, since $(l+1,0),(1,l) \neq (g',g')$ for all choices of $l \geq g$, the element f is also not a summand of any element in $(\mathrm{Dist}(\mathbf{U}_{(2,1),k}^+) \cdot L_{(1,2)}^{(l)}).\frac{T_0}{T_2} + (\mathrm{Dist}(\mathbf{U}_{(2,1),k}^+) \cdot L_{(0,2)}^{(l)}).\frac{T_0}{T_2}$ for $l \geq g$.

□

2.4 Enriched Crystalline Enveloping Algebras: Adding more Divided Powers to the Distribution Algebra

Inspired by the purely algebraic construction of $\mathrm{Dist}(\mathbf{G})$ as a divided power algebra over $\mathcal{V}$, one can try to carry this process a bit further in order to solve the problem of finite generation considered in the last section with the help of a "higher divided power algebra" instead of $\mathrm{Dist}(\mathbf{G})$ (and similar for modules over $\mathcal{V}$-algebras R, in particular for $R=k$).

Definition 2.4.0.2. *Define the enriched crystalline enveloping algebra of $\mathfrak{g}$ as the $\mathcal{V}$-algebra $\widehat{U}(\mathfrak{g})$ which is the $\mathcal{V}$-subalgebra of the universal enveloping algebra $U(\mathfrak{g} \otimes_{\mathcal{V}} K) = U(\mathfrak{g}) \otimes_{\mathcal{V}} K$ generated by all expressions of the form*

$$\prod_{\substack{u,v=0\\u\neq v}}^{n} L_{(u,v)}^{[l_{u,v}]} := \frac{1}{\left(\sum_{\substack{0\leq u,v\leq n\\u\neq v}} l_{u,v}\right)!} \prod_{\substack{u,v=0\\u\neq v}}^{n} L_{(u,v)}^{l_{u,v}}$$

with $l_{u,v} \in \mathbb{N}_0$ for all $0 \leq u \neq v \leq n$ and

$$\binom{L_u}{l} := \frac{L_u \cdot (L_u - 1) \cdot \ldots \cdot (L_u - l + 1)}{l!}$$

with $u \in \{0,\dots,n\}$, $l \in \mathbb{N}_0$. For a $\mathcal{V}$-algebra R, define the enriched crystalline enveloping algebra of $\mathfrak{g}_R$ as the R-algebra

$$\widehat{U}(\mathfrak{g}_R) := \widehat{U}(\mathfrak{g}) \otimes_{\mathcal{V}} R.$$

Note that the $\mathcal{V}$-algebra $\widehat{U}(\mathfrak{g})$ is not finitely generated. Furthermore, it is not true in general that a representation of $\mathbf{G}$ induces a representation of $\widehat{U}(\mathfrak{g})$ (compatible with those of $U(\mathfrak{g})$ and $\mathrm{Dist}(\mathbf{G})$), see the examples in Subsection 2.4.1 below. Over k, one can use reduction mod π of $\mathbf{G}$-representations (cf. [32, I.10.1]) to produce $\widehat{U}(\mathfrak{g}_k)$-representations from $\mathbf{G}_k$-representations:

Lemma 2.4.0.3. *Let $V = W \otimes_{\mathcal{V}} k$ be a $\mathbf{G}_k$-representation which is induced by reduction mod π from a $\mathbf{G}$-representation W over $\mathcal{V}$. Suppose that the $\mathbf{G}$-action on W induces a $\widehat{U}(\mathfrak{g})$-action on W. Then the $\mathbf{G}_k$-action on V induces a $\widehat{U}(\mathfrak{g}_k)$-action on V.*

Proof. By assumption, W has the structure of a $\mathrm{Dist}(\mathbf{G})$-module which extends to the structure of a $\widehat{U}(\mathfrak{g})$-module on W. Extension of scalars yields on V the structure of a $\widehat{U}(\mathfrak{g}_k)$-module via

$$V = W \otimes_{\mathcal{V}} k = (W \otimes_{\widehat{U}(\mathfrak{g})} \widehat{U}(\mathfrak{g})) \otimes_{\mathcal{V}} k = W \otimes_{\widehat{U}(\mathfrak{g})} (\widehat{U}(\mathfrak{g}) \otimes_{\mathcal{V}} k) = W \otimes_{\widehat{U}(\mathfrak{g})} \widehat{U}(\mathfrak{g}_k).$$

□

Remark. The adjoint action of $\mathbf{G}$ on $\mathrm{Dist}(\mathbf{G})$ does not extend to an action of $\mathbf{G}$ on $\widehat{U}(\mathfrak{g})$. This can already be verified in the case that $\mathbf{G} = \mathrm{GL}_2$.

2.4.1 First Examples, Row Algebras and Column Algebras

The trivial representation of $\mathbf{G}$ induces the trivial representation of $\widehat{U}(\mathfrak{g})$. In particular, Lemma 2.4.0.3 then applies to the trivial representation of $\mathbf{G}_k$.

Example I

Let $\mathbf{G}$ act on $\mathbb{A}_{\mathcal{V}}^{n+1}$ via $(g,x)\mapsto gx$ for $g\in\mathbf{G}(R), x\in\mathbb{A}_{\mathcal{V}}^{n+1}(R)$, R a $\mathcal{V}$-algebra. The induced action of $\mathbf{G}$ on $\tilde{S}=\mathcal{V}[\mathbb{A}_{\mathcal{V}}^{n+1}]\cong\mathcal{V}[T_0,\dots,T_n]$ is then given by

$$(g,f)\mapsto (g.f : x\mapsto f(g^{-1}x)),$$

hence a non-toral basis element $L_{(u,v)}$ (with $u\neq v$) of $\mathfrak{g}$ acts on a basis element of $\tilde{S}$ by

$$\begin{aligned} L_{(u,v)}.(T_0^{m_0}\cdot\ldots\cdot T_n^{m_n}) &= \frac{d}{d\epsilon}_{\epsilon=0}\left((1+L_{(u,v)}\epsilon).(T_0^{m_0}\cdot\ldots\cdot T_n^{m_n})\right)\\ &= -m_uT_u^{-1}\cdot T_v\cdot T_0^{m_0}\cdot\ldots\cdot T_n^{m_n}\end{aligned}$$

and thus

$$L_{(u,v)}^l.(T_0^{m_0}\cdot\ldots\cdot T_n^{m_n}) = (-1)^l\frac{m_u!}{(m_u-l)!}T_u^{-l}\cdot T_v^l\cdot T_0^{m_0}\cdot\ldots\cdot T_n^{m_n}$$

for $l\in\mathbb{N}_0$. Then, for $w\in\{0,\dots,n\}\setminus\{u,v\}$ and $m\in\mathbb{N}_0$, one gets

$$(L_{(w,v)}^m L_{(u,v)}^l).(T_0^{m_0}\cdot\ldots\cdot T_n^{m_n}) = (-1)^{l+m}\frac{m_u!}{(m_u-l)!}\frac{m_w!}{(m_w-m)!}T_w^{-m}\cdot T_u^{-l}\cdot T_v^{l+m}\cdot T_0^{m_0}\cdot\ldots\cdot T_n^{m_n} \quad (2.12)$$

and

$$\begin{aligned}(L_{(v,w)}^m L_{(v,u)}^l).(T_0^{m_0}\cdot\ldots\cdot T_n^{m_n}) &= (-1)^{l+m}\frac{m_v!}{(m_v-l)!}\frac{(m_v-l)!}{(m_v-l-m)!}T_v^{-l-m}\cdot T_u^l\cdot T_w^m\cdot T_0^{m_0}\cdot\ldots\cdot T_n^{m_n}\\ &= (-1)^{l+m}\frac{m_v!}{(m_v-l-m)!}T_v^{-l-m}\cdot T_u^l\cdot T_w^m\cdot T_0^{m_0}\cdot\ldots\cdot T_n^{m_n}. \quad (2.13)\end{aligned}$$

Therefore, in this case, the full algebra $\widehat{U}(\mathfrak{g})$ does not act on $\tilde{S}$ (compatibly with $\mathrm{Dist}(\mathbf{G})$) since, for example, the element $L_{(w,v)}^{[m]}L_{(u,v)}^{[l]}$ does not: Let $l=m_u, m=m_w$ in (2.12) to see that the expression on the right-hand side is not in general divisible by $(m+l)!$.

Example II

Now consider the action $\mathbf{G}\times\mathbb{A}_{\mathcal{V}}^{n+1}\to\mathbb{A}_{\mathcal{V}}^{n+1}$ given by $(g,x)\mapsto xg^{-1}$ for $g\in\mathbf{G}(R), x\in\mathbb{A}_{\mathcal{V}}^{n+1}$, R a $\mathcal{V}$-algebra. This again induces actions of $\mathbf{G}$ and $\mathfrak{g}$ on $\tilde{S}$. The latter is now given by

$$\begin{aligned} L_{(u,v)}.(T_0^{m_0}\cdot\ldots\cdot T_n^{m_n}) &= \frac{d}{d\epsilon}_{\epsilon=0}\left((1+L_{(u,v)}\epsilon).(T_0^{m_0}\cdot\ldots\cdot T_n^{m_n})\right)\\ &= m_vT_u\cdot T_v^{-1}\cdot T_0^{m_0}\cdot\ldots\cdot T_n^{m_n}\end{aligned}$$

and thus

$$L_{(u,v)}^l.(T_0^{m_0}\cdot\ldots\cdot T_n^{m_n}) = \frac{m_v!}{(m_v-l)!}T_u^l\cdot T_v^{-l}\cdot T_0^{m_0}\cdot\ldots\cdot T_n^{m_n}$$

for $l \in \mathbb{N}_0$. Then, for $w \in \{0,\dots,n\} \setminus \{u,v\}$ and $m \in \mathbb{N}_0$, one gets

$$\begin{aligned}(L_{(w,v)}^{m} L_{(u,v)}^{l}).(T_0^{m_0} \cdot \ldots \cdot T_n^{m_n}) &= \frac{m_v!}{(m_v - l)!}\frac{(m_v - l)!}{(m_v - l - m)!} T_w^m \cdot T_u^l \cdot T_v^{-l-m} \cdot T_0^{m_0} \cdot \ldots \cdot T_n^{m_n} \\ &= \frac{m_v!}{(m_v - l - m)!} T_w^m \cdot T_u^l \cdot T_v^{-l-m} \cdot T_0^{m_0} \cdot \ldots \cdot T_n^{m_n},\end{aligned}$$

and

$$(L_{(v,w)}^{m} L_{(v,u)}^{l}).(T_0^{m_0} \cdot \ldots \cdot T_n^{m_n}) = \frac{m_u!}{(m_u - l)!}\frac{m_w!}{(m_w - m)!} T_v^{m+l} \cdot T_u^{-l} \cdot T_w^{-m} \cdot T_0^{m_0} \cdot \ldots \cdot T_n^{m_n}.$$

By similar reasoning as before, the full algebra $\widehat{U}(\mathfrak{g})$ again does not act on $\tilde{S}$ (compatibly with $\mathrm{Dist}(\mathbf{G})$).

Remark. In particular, Lemma 2.4.0.3 is not applicable to the $\mathbf{G}_k$-representations on $S \cong k[\mathbb{A}_k^n]$ induced by reduction mod π of the above examples.

The study of those examples leads to the following definition.

Definition 2.4.1.1. *Define $\mathcal{V}$-subalgebras $\widehat{U}(\mathfrak{g})_r$ and $\widehat{U}(\mathfrak{g})_c$ of $\widehat{U}(\mathfrak{g})$ as follows:*

i) Let $\widehat{U}(\mathfrak{g})_r$ be the $\mathcal{V}$-subalgebra of $\widehat{U}(\mathfrak{g})$ generated by the union of the sets

$$\left\{ \prod_{\substack{u,v=0\\u\neq v}}^{n} L_{(u,v)}^{[l_{u,v}]} \in \widehat{U}(\mathfrak{g}) \;\middle|\; \forall u,v \in \{0,\dots,n\}, u \neq v : l_{u,v} \in \mathbb{N}_0, \; \exists w \in \{0,\dots,n\} \forall u \in \{0,\dots,n\} \setminus \{w\}, v \in \{0,\dots,n\} : l_{u,v} = 0 \right\}$$

and

$$\left\{ \binom{L_v}{l} \in \widehat{U}(\mathfrak{g}) \;\middle|\; v = 0,\dots,n; l \in \mathbb{N}_0 \right\}.$$

Here, the "r" in the notation $\widehat{U}(\mathfrak{g})_r$ stands for "row". For $w \in \{0,\dots,n\}$, write

$$\prod_{\substack{v=0\\v\neq w}}^{n} L_{(w,v)}^{[l_{w,v}]}$$

for the generator $\prod_{\substack{u,v=0\\u\neq v}}^{n} L_{(u,v)}^{[l_{u,v}]}$ of $\widehat{U}(\mathfrak{g})_r$ for which $l_{u,v} = 0$ for all $u \neq w$.

ii) Let $\widehat{U}(\mathfrak{g})_c$ be the $\mathcal{V}$-subalgebra generated by the union of the sets

$$\left\{ \prod_{\substack{u,v=0\\u\neq v}}^{n} L_{(u,v)}^{[l_{u,v}]} \in \widehat{U}(\mathfrak{g}) \;\middle|\; \forall u,v \in \{0,\dots,n\}, u \neq v : l_{u,v} \in \mathbb{N}_0, \; \exists w \in \{0,\dots,n\} \forall v \in \{0,\dots,n\} \setminus \{w\}, u \in \{0,\dots,n\} : l_{u,v} = 0 \right\}$$

and

$$\left\{ \binom{L_v}{l} \in \widehat{U}(\mathfrak{g}) \;\middle|\; v = 0,\dots,n; l \in \mathbb{N}_0 \right\}.$$

Here, the "c" in the notation $\widehat{U}(\mathfrak{g})_c$ stands for "column". For $w \in \{0,\dots,n\}$, write

$$\prod_{\substack{u=0\\u\neq w}}^{n} L_{(w,v)}^{[l_{w,v}]}$$

for the generator $\prod_{\substack{u,v=0\\u\neq v}}^{n} L_{(u,v)}^{[l_{u,v}]}$ of $\widehat{U}(\mathfrak{g})_c$ for which $l_{u,v}=0$ for all $v\neq w$.

iii) For a $\mathcal{V}$-algebra R, define

$$\widehat{U}(\mathfrak{g}_R)_r := \widehat{U}(\mathfrak{g})_r \otimes_{\mathcal{V}} R$$

and

$$\widehat{U}(\mathfrak{g}_R)_c := \widehat{U}(\mathfrak{g})_c \otimes_{\mathcal{V}} R.$$

Lemma 2.4.1.2.

i) The $\mathcal{V}$-algebra $\widehat{U}(\mathfrak{g})_r$ acts on the $\mathbf{G}$-module $\tilde{S}$ from Example I (compatibly with $\mathrm{Dist}(\mathbf{G})$*).*

ii) The $\mathcal{V}$-algebra $\widehat{U}(\mathfrak{g})_c$ acts on the $\mathbf{G}$-module $\tilde{S}$ from Example II (compatibly with $\mathrm{Dist}(\mathbf{G})$*).*

Proof. It suffices to prove case ii) as i) then follows by symmetry. It has to be checked that the induced action of $U(\mathfrak{g})$ and $\mathrm{Dist}(\mathbf{G})$ on $\tilde{S}$ extends to an action of $\widehat{U}(\mathfrak{g})_c$ on $\tilde{S}$. First of all, this has to be done for generating elements of $\widehat{U}(\mathfrak{g})_c$. For elements of type $\binom{L_v}{l}$ this is clear as those are already contained in $\mathrm{Dist}(\mathbf{G}.)$ Therefore, let $v\in\{0,\dots,n\}$ and consider $\prod_{\substack{u=0\\u\neq v}}^{n} L_{(u,v)}^{l_{u,v}} \in U(\mathfrak{g})$ and $f = T_0^{m_0}\cdot\ldots\cdot T_n^{m_n} \in \tilde{S}$. Then

$$\prod_{\substack{u=0\\u\neq v}}^{n} L_{(u,v)}^{l_{u,v}}.f = \frac{m_v!}{(m_v-\sum_{\substack{u=0\\u\neq v}}^{n} l_{u,v})!} T_v^{m_v-\sum_{\substack{u=0\\u\neq v}}^{n} l_{u,v}} \cdot T_0^{l_{0,v}}\cdot\ldots\cdot \widehat{T_v^{l_{v,v}}}\cdot\ldots\cdot T_n^{l_{n,v}}\cdot f. \tag{2.14}$$

Since the coefficient of this element is divisible by $(\sum_{\substack{u=0\\u\neq v}}^{n} l_{u,v})!$, it follows indeed that the element $\prod_{\substack{u=0\\u\neq v}}^{n} L_{(u,v)}^{[l_{u,v}]} \in U(\mathfrak{g})_c$ acts on f. But this already implies that $\widehat{U}(\mathfrak{g}_k)_c$ acts on $\tilde{S}$ as claimed, since

$$\prod_{\substack{u=0\\u\neq w}}^{n} L_{(u,w)}^{l_{u,w}}.\left(\prod_{\substack{u=0\\u\neq v}}^{n} L_{(u,v)}^{l_{u,v}}.f\right) = \left(\prod_{\substack{u=0\\u\neq w}}^{n} L_{(u,w)}^{l_{u,w}}\cdot\prod_{\substack{u=0\\u\neq v}}^{n} L_{(u,v)}^{l_{u,v}}\right) f$$

and this element then similarly has a coefficient (in $\mathcal{V}$) which is divisible by $(\sum_{\substack{u=0\\u\neq w}}^{n} l_{u,w})!\cdot(\sum_{\substack{u=0\\u\neq v}}^{n} l_{u,v})!$. □

More examples are discussed in Section 2.6.

2.4.2 Some Subalgebras and Duality

For the rest of this chapter, only certain subalgebras of $\widehat{U}(\mathfrak{g})$ (resp. of $\widehat{U}(\mathfrak{g}_k)$) will be of use and therefore of interest. On the one hand, this is due to the fact that the action of $\mathbf{G}_k$ on $\mathbb{P}_k^n$ comes from the action of $\mathbf{G}_k$ on $\mathbb{A}_k^{n+1}$ as described in Example II above. On the other hand, for technical reasons, the Lie algebra $\mathfrak{g}$ (resp. $\mathfrak{g}_k$) has to be replaced by the Lie algebra $\mathfrak{u}_{(j+1,n-j)}^+$ (resp. $\mathfrak{u}_{(j+1,n-j),k}^+$) which is the Lie subalgebra of $\mathfrak{g}$ opposite to the Lie subalgebra $\mathfrak{u}_{(j+1,n-j)}$ (with respect to the decomposition $n+1=(j+1)+(n-j)$).

Definition 2.4.2.1.

i) Let $\widehat{U}(\mathfrak{u}^+_{(j+1,n-j)})$ be the $\mathcal{V}$-subalgebra of $\widehat{U}(\mathfrak{g})$ generated by the set

$$\left\{ \prod_{\substack{0\le u,v\le n\\ u\ne v}} L^{[l_{u,v}]}_{(u,v)} \in \widehat{U}(\mathfrak{g}) \;\middle|\; \forall u,v : l_{u,v}\in\mathbb{N}_0; \forall v\in\{0,\dots,j\}, u\in\{j+1,\dots,n\} : l_{u,v}=0 \right\}.$$

Similarly as above, write $\prod_{\substack{0\le u\le j\\ j+1\le v\le n}} L^{[l_{u,v}]}_{(u,v)}$ for the generator $\prod_{\substack{0\le u,v\le n\\ u\ne v}} L^{[l_{u,v}]}_{(u,v)}$ of $\widehat{U}(\mathfrak{u}^+_{(j+1,n-j)})$ for which $l_{u,v}=0$ for all $u\in\{j+1,\dots,n\}, v\in\{0,\dots,j\}$.

ii) Let

$$\widehat{U}(\mathfrak{u}^+_{(j+1,n-j)})_c = \widehat{U}(\mathfrak{g})_c \cap \widehat{U}(\mathfrak{u}^+_{(j+1,n-j)}),$$

i.e. $\widehat{U}(\mathfrak{u}^+_{(j+1,n-j)})_c$ is the $\mathcal{V}$-subalgebra of $\widehat{U}(\mathfrak{u}^+_{(j+1,n-j)})$ generated by

$$\left\{ \prod_{\substack{0\le u\le j\\ j+1\le v\le n}} L^{[l_{u,v}]}_{(u,v)} \in \widehat{U}(\mathfrak{u}^+_{(j+1,n-j)}) \;\middle|\; \forall u,v : l_{u,v}\in\mathbb{N}_0; \exists v\in\{j+1,\dots,n\}\forall w\in\{j+1,\dots,n\}\setminus\{v\}\ \forall u\in\{j+1,\dots,n\} : l_{u,w}=0 \right\}.$$

Similarly as above, for $v\in\{j+1,\dots,n\}$, write $\prod_{0\le u\le j} L^{[l_{u,v}]}_{(u,v)}$ for the generator of $\widehat{U}(\mathfrak{u}^+_{(j+1,n-j)})_c$ for which $l_{u,w}=0$ for all $u\in\{0,\dots,j\}$ and all $w\in\{j+1,\dots,n\}\setminus\{v\}$.

iii) For a $\mathcal{V}$-algebra R, define the R-algebras

$$\widehat{U}(\mathfrak{u}^+_{(j+1,n-j),R}) = \widehat{U}(\mathfrak{u}^+_{(j+1,n-j)}) \otimes_{\mathcal{V}} R$$

and

$$\widehat{U}(\mathfrak{u}^+_{(j+1,n-j),R})_c = \widehat{U}(\mathfrak{u}^+_{(j+1,n-j)})_c \otimes_{\mathcal{V}} R.$$

Remark. All of the algebras defined above are commutative since $\mathbf{U}^+_{(j+1,n-j)}$ is a commutative group. Furthermore, as a $\mathcal{V}$-module, $\widehat{U}(\mathfrak{u}^+_{(j+1,n-j)})$ is free with basis

$$\left\{ \prod_{\substack{0\le u\le j\\ j+1\le v\le n}} L^{[l_{u,v}]}_{(u,v)} \;\middle|\; l_{u,v}\in\mathbb{N}_0 \right\}.$$

Endow $\widehat{U}(\mathfrak{u}^+_{(j+1,n-j)})$ with the following grading: For $m\in\mathbb{N}_0$, the $\mathcal{V}$-submodule of $\widehat{U}(\mathfrak{u}^+_{(j+1,n-j)})$ of elements of degree m is precisely the $\mathcal{V}$-module generated by the set

$$\left\{ \prod_{\substack{0\le u\le j\\ j+1\le v\le n}} L^{[l_{u,v}]}_{(u,v)} \in \widehat{U}(\mathfrak{u}^+_{(j+1,n-j)}) \;\middle|\; \sum_{u,v} l_{u,v} = m \right\}.$$

The universal enveloping algebra $U(\mathfrak{u}^+_{(j+1,n-j)})$ has a similar grading as a $\mathcal{V}$-module. Consider the adjoint action of $\mathbf{G}$ on $U(\mathfrak{g})$. This action restricts to an action of $\mathbf{L}_{(j+1,n-j)}$ on $U(\mathfrak{g})$. Furthermore, this action of $\mathbf{L}_{(j+1,n-j)}$ on $U(\mathfrak{g})$ stabilizes the universal enveloping algebra $U(\mathfrak{u}^+_{(j+1,n-j)})$ and the resulting action is homogeneous with respect to the natural grading of the $\mathcal{V}$-module $U(\mathfrak{u}^+_{(j+1,n-j)})$ as discussed above.

Lemma 2.4.2.2.

i) The adjoint action of the group $\mathbf{L}_{(j+1,n-j)}$ *on* $U(\mathfrak{u}^+_{(j+1,n-j)})$ *extends to an action of* $\mathbf{L}_{(j+1,n-j)}$ *on* $\widehat{U}(\mathfrak{u}^+_{(j+1,n-j)})$.

ii) The $\mathcal{V}$*-linear map* $U(\mathfrak{u}^+_{(j+1,n-j)}) \to \widehat{U}(\mathfrak{u}^+_{(j+1,n-j)})$ *induced by*

$$\prod_{\substack{0\le u\le j\\ j+1\le v\le n}} L_{(u,v)}^{l_{u,v}} \mapsto \prod_{\substack{0\le u\le j\\ j+1\le v\le n}} L_{(u,v)}^{[l_{u,v}]}$$

is an isomorphism of $\mathbf{L}_{(j+1,n-j)}$*-modules.*

Proof.

i) The follows from the fact that $\mathbf{L}_{(j+1,n-j)}$ acts homogeneously on $U(\mathfrak{u}^+_{(j+1,n-j)})$.

ii) The above map is an isomorphism of $\mathcal{V}$-modules by construction and the fact that the action of $\mathbf{L}_{(j+1,n-j),k}$ is preserved follows at once from the fact that it is homogeneous on both sides, see i).

□

Consider again the $\mathcal{V}$-algebra k.

Lemma 2.4.2.3. *There is an action of* $\mathbf{L}_{(j+1,n-j),k}$ *on* $\widehat{U}(\mathfrak{u}^+_{(j+1,n-j),k})$ *such that, as an* $\mathbf{L}_{(j+1,n-j),k}$*-module,* $\widehat{U}(\mathfrak{u}^+_{(j+1,n-j),k})$ *is isomorphic to* $U(\mathfrak{u}^+_{(j+1,n-j)})$*. Furthermore, there is an isomorphism of* $\mathbf{L}_{(j+1,n-j),k}$*-modules*[2]

$$\widehat{U}(\mathfrak{u}^+_{(j+1,n-j),k}) \cong \bigoplus_{l\in\mathbb{N}_0} \mathbf{Sym}^l((k^{j+1})') \boxtimes_k \mathbf{Sym}^l((k^{n-j})')'. \tag{2.15}$$

Proof. The first two statements of the lemma are obtained by reduction mod π of the corresponding action resp. isomorphism of the last lemma. The isomorphism (2.15) is then obtained from the natural isomorphism of $\mathbf{L}_{(j+1,n-j),k}$-modules

$$U(\mathfrak{u}^+_{(j+1,n-j),k}) \cong \bigoplus_{l\in\mathbb{N}_0} \mathbf{Sym}^l((k^{j+1})') \boxtimes_k \mathbf{Sym}^l((k^{n-j})')'.$$

□

The $L_{(j+1,n-j)}$-module $\widehat{U}(\mathfrak{u}^+_{(j+1,n-j),k})$ can be viewed as the graded k-dual of the $L_{(j+1,n-j)}$-module structure of the k-algebra $\mathcal{O}(\mathbf{U}^+_{(j+1,n-j),k})$ (induced by the conjugation action of $\mathbf{L}_{(j+1,n-j),k}$ on $\mathbf{U}^+_{(j+1,n-j),k}$) of algebraic functions on $\mathbf{U}^+_{(j+1,n-j),k}$ (which is isomorphic to the polynomial k-algebra $k\left[\{T_{u,v} \mid 0\le u\le j; j+1\le v\le n\}\right]$). Consider this module as a graded k-vector space for the grading which corresponds to the degree of polynomials and recall that for a graded k-vector space M, its graded k-dual is denoted by $M^\vee$.

Proposition 2.4.2.4.

i) There exists a non-degenerate $L_{(j+1,n-j)}$*-equivariant pairing*

$$\beta : \widehat{U}(\mathfrak{u}^+_{(j+1,n-j),k}) \times \mathcal{O}(\mathbf{U}^+_{(j+1,n-j),k}) \to k$$

of k*-vector spaces.*

[2] Recall the convention on the action of $\mathbf{G}_k$ on k^{n+1} from the beginning of this chapter.

ii) There is a graded isomorphism of $L_{(j+1,n-j)}$-modules

$$\Gamma : \widehat{U}(\mathfrak{u}^+_{(j+1,n-j),k}) \xrightarrow{\sim} \mathcal{O}(\mathbf{U}^+_{(j+1,n-j),k})^\vee.$$

Proof. i) Define β on basis elements by

$$\beta\Big(\prod_{\substack{0\le u\le j\\ j+1\le v\le n}} L^{[l_{u,v}]}_{(u,v)}, \prod_{\substack{0\le u\le j\\ j+1\le v\le n}} T^{m_{u,v}}_{u,v}\Big) := \begin{cases} 1 & \forall u \in \{0,\dots,j\}, v\in\{j+1,\dots,n\} : l_{u,v} = m_{u,v} \\ 0 & \text{else.}\end{cases}$$

This pairing is well-defined and it is non-degenerate by definition. Furthermore, it is $L_{(j+1,n-j)}$-equivariant since the actions of $L_{(j+1,n-j)}$ on both spaces are dual to each other.

ii) The above pairing induces an $L_{(j+1,n-j)}$-equivariant embedding of $\widehat{U}(\mathfrak{u}^+_{(j+1,n-j),k})$ into the dual space $\mathcal{O}(\mathbf{U}^+_{(j+1,n-j),k})'$ and the image is exactly $\mathcal{O}(\mathbf{U}^+_{(j+1,n-j),k})^\vee$: A basis element $\prod_{\substack{0\le u\le j\\ j+1\le v\le n}} L^{[l_{u,v}]}_{(u,v)}$ of $\widehat{U}(\mathfrak{u}^+_{(j+1,n-j),k})$ is mapped to the dual basis element $\left(\prod_{\substack{0\le u\le j\\ j+1\le v\le n}} T^{m_{u,v}}_{u,v}\right)^* \in \mathcal{O}(\mathbf{U}^+_{(j+1,n-j),k})'$ associated with $\prod_{\substack{0\le u\le j\\ j+1\le v\le n}} T^{m_{u,v}}_{u,v}$.

□

Remarks.

i) The isomorphism Γ is an isomorphism of graded vector spaces with respect to the gradings on both sides as described above.

ii) The natural action of $U(\mathfrak{u}^+_{(j+1,n-j)})$ on $\mathcal{O}(\mathbf{U}^+_{(j+1,n-j)}) = \mathcal{V}[\mathbf{U}^+_{(j+1,n-j)}]$ which is induced by the action of $\mathbf{U}^+_{(j+1,n-j)}$ on itself by left (or right) multiplication does not extend to an action of $\widehat{U}(\mathfrak{u}^+_{(j+1,n-j)})$ on $\mathcal{O}(\mathbf{U}^+_{(j+1,n-j})$.

The subalgebra $^{\mathbf{L}}\widehat{U}(\mathfrak{u}^+_{(j+1,n-j)})_c$ and reduced duality

In general, the $\mathcal{V}$-subalgebra $\widehat{U}(\mathfrak{u}^+_{(j+1,n-j)})_c \subset \widehat{U}(\mathfrak{u}^+_{(j+1,n-j)})$ is not an $\mathbf{L}_{(j+1,n-j)}$-submodule: For example, in the case $n=3, j=1, K=\mathbb{Q}_2, \mathcal{V}=\mathbb{Z}_2$, the element $g = \left(\begin{smallmatrix} 1&0&0&0\\0&1&0&0\\0&0&1&1\\0&0&0&1\end{smallmatrix}\right) \in \mathbf{L}_{(2,2)}(\mathcal{V})$ applied to $L^{[1]}_{(0,2)}L^{[1]}_{(1,2)}$ yields

$$L^{[1]}_{(0,2)}L^{[1]}_{(1,2)} - L^{[1]}_{(0,3)}L^{[1]}_{(1,2)} - L^{[1]}_{(0,2)}L^{[1]}_{(1,3)} + L^{[1]}_{(0,3)}L^{[1]}_{(1,3)} \in \widehat{U}(\mathfrak{u}^+_{(2,2)}),$$

and while $L^{[1]}_{(0,2)}L^{[1]}_{(1,2)}+L^{[1]}_{(0,3)}L^{[1]}_{(1,3)}$ is contained in $\widehat{U}(\mathfrak{u}^+_{(2,2)})_c$, the summand $-\left(L^{[1]}_{(0,3)}L^{[1]}_{(1,2)} + L^{[1]}_{(0,2)}L^{[1]}_{(1,3)}\right)$ is not. This is due to the fact that up to sign, this last expression equals $\frac{1}{2}(L_{(0,3)}L_{(1,2)}+L_{(0,2)}L_{(1,3)})$. By definition, $\widehat{U}(\mathfrak{u}^+_{(2,2)})_c$ as a $\mathcal{V}$-module is generated by the elements $\frac{1}{(a+b)!}\frac{1}{(u+v)!}L^a_{(0,2)}L^b_{(1,2)}L^u_{(0,3)}L^v_{(1,3)}$ with $a,b,u,v\in\mathbb{N}_0$ and since 2 is not invertible in $\mathcal{V}$, the claim follows and thus $g.L^{[1]}_{(0,2)}L^{[1]}_{(1,2)} \notin \widehat{U}(\mathfrak{u}^+_{(2,2)})_c$.

For this reason, the following $\mathbf{L}_{(j+1,n-j)}(\mathcal{V})$-submodule of $\widehat{U}(\mathfrak{u}^+_{(j+1,n-j)})$ will be employed in the next chapter.

Definition 2.4.2.5.

i) Let ${}^{\mathbf{L}}\widehat{U}(\mathfrak{u}^+_{(j+1,n-j)})_c$ *be the* $\mathcal{V}$*-subalgebra of* $\widehat{U}(\mathfrak{u}^+_{(j+1,n-j)})$ *which is generated by all* $\mathbf{L}_{(j+1,n-j)}(\mathcal{V})$*-translates of* $\widehat{U}(\mathfrak{u}^+_{(j+1,n-j)})_c$ *inside* $\widehat{U}(\mathfrak{u}^+_{(j+1,n-j)})$*. In particular, this is an* $\mathbf{L}_{(j+1,n-j)}(\mathcal{V})$*-submodule of* $\widehat{U}(\mathfrak{u}^+_{(j+1,n-j)})$*.*

ii) Set ${}^{\mathbf{L}}\widehat{U}(\mathfrak{u}^+_{(j+1,n-j),k})_c := {}^{\mathbf{L}}\widehat{U}(\mathfrak{u}^+_{(j+1,n-j)})_c \otimes_{\mathcal{V}} k$*. This is a* k*-subalgebra of* $\widehat{U}(\mathfrak{u}^+_{(j+1,n-j),k})$ *which is in particular an* $L_{(j+1,n-j)}$*-submodule.*

Consider all three algebras in the cases $n = 1$ and $n = 2$.

Example 2.4.2.6.

i) Let $n = 1$*. Then, necessarily,* $j = 0$ *and all three algebras* $\widehat{U}(\mathfrak{u}^+_{(1,1)}), \widehat{U}(\mathfrak{u}^+_{(1,1)})_c$ *and* ${}^{\mathbf{L}}\widehat{U}(\mathfrak{u}^+_{(1,1)})_c$ *coincide; they are actually isomorphic to* $\mathrm{Dist}(\mathbf{U}^+_{(1,1)})$*.*

ii) Let $n = 2$*.*

a) Let $j = 1$*. In this case,* $\widehat{U}(\mathfrak{u}^+_{(2,1)})_c = \widehat{U}(\mathfrak{u}^+_{(2,1)})$ *by construction and thus both algebras are equal to* ${}^{\mathbf{L}}\widehat{U}(\mathfrak{u}^+_{(2,1)})_c$*.*

b) Let $j = 0$*. In this case,* $\widehat{U}(\mathfrak{u}^+_{(1,2)})_c$ *is properly contained in* $\widehat{U}(\mathfrak{u}^+_{(1,2)})$ *: If* $p \mid \binom{i+l}{l}$*, the element* $L^{[i]}_{(0,1)}L^{[l]}_{(0,2)} \in \widehat{U}(\mathfrak{u}^+_{(1,2)})$ *is not contained in* $\widehat{U}(\mathfrak{u}^+_{(1,2)})_c$ *since the latter algebra is generated by* $\{L^{[a]}_{(0,1)} \mid a \in \mathbb{N}_0\} \cup \{L^{[b]}_{(0,2)} \mid b \in \mathbb{N}_0\}$ *and the relation (in* $\widehat{U}(\mathfrak{u}^+_{(1,2)})$*)*

$$L^{[a]}_{(0,1)} \cdot L^{[b]}_{(0,2)} = \binom{a+b}{a} L^{[a]}_{(0,1)}L^{[b]}_{(0,2)}$$

holds for all $a, b \in \mathbb{N}_0$*. Still, in this particular case, the module* $\widehat{U}(\mathfrak{u}^+_{(1,2)})_c$ *is stabilized by the* $\mathbf{L}_{(1,2)}$*-action: Let* R *be a* $\mathcal{V}$*-algebra. Application of* $g = \begin{pmatrix} a & 0 & 0 \\ 0 & \alpha & \beta \\ 0 & \gamma & \delta \end{pmatrix} \in \mathbf{L}_{(1,2)}(R)$ *to a product* $L^{[i]}_{(0,1)} \cdot L^{[l]}_{(0,2)} \in \widehat{U}(\mathfrak{u}^+_{(1,2)})_c \otimes_{\mathcal{V}} R$ *yields*

$$\left(\frac{a}{\alpha\delta - \beta\gamma}\right)^{i+l} \cdot \left(\sum_{r=0}^{i} \delta^r(-\beta)^{i-r} L^{[r]}_{0,1} \cdot L^{[i-r]}_{0,2}\right) \cdot \left(\sum_{s=0}^{l} \alpha^{l-s}(-\gamma)^s L^{[s]}_{0,1} \cdot L^{[l-s]}_{0,2}\right)$$

which is contained in $\widehat{U}(\mathfrak{u}^+_{(1,2)})_c \otimes_{\mathcal{V}} R$*.*

Restrict the pairing β from the last proposition to a pairing

$$\beta_c : {}^{\mathbf{L}}\widehat{U}(\mathfrak{u}^+_{(j+1,n-j),k})_c \times \mathcal{O}(\mathbf{U}^+_{(j+1,n-j),k}) \to k$$

and recall from e.g. [36, XIII,§ 5] the notion of non-degeneracy on the left resp. on the right of a pairing.

Lemma 2.4.2.7.

i) The pairing β_c *is* $L_{(j+1,n-j)}$*-equivariant and non-degenerate on the left.*

ii) For an $L_{(j+1,n-j)}$-module N of finite k-dimension, the pairing β_c extends to an $L_{(j+1,n-j)}$-equivariant pairing

$$\begin{array}{rcl} \beta_{c,N}: \ {}^{\mathbf{L}}\widehat{U}(\mathfrak{u}^+_{(j+1,n-j),k})_c \otimes_k N \times \mathcal{O}(\mathbf{U}^+_{(j+1,n-j),k}) \otimes_k N' & \to & k \\ (\eta \otimes x, f \otimes \lambda) & \mapsto & \lambda(x)\beta_c(\eta, f) \end{array},$$

non-degenerate on the left, which induces an $L_{(j+1,n-j)}$-invariant embedding

$${}^{\mathbf{L}}\widehat{U}(\mathfrak{u}^+_{(j+1,n-j),k})_c \otimes_k N \hookrightarrow (\mathcal{O}(\mathbf{U}^+_{(j+1,n-j),k}) \otimes_k N')'.$$

Proof. These statements follow from the previous proposition. □

Let N be an $L_{(j+1,n-j)}$-module of finite k-dimension and let $\mathfrak{d} \subset {}^{\mathbf{L}}\widehat{U}(\mathfrak{u}^+_{(j+1,n-j),k})_c \otimes_k N$ be an $L_{(j+1,n-j)}$-submodule. In the next section, the quotient module ${}^{\mathbf{L}}\widehat{U}(\mathfrak{u}^+_{(j+1,n-j),k})_c \otimes_k N/\mathfrak{d}$ will be used for certain choices of N and $\mathfrak{d}$. In order to extend the duality obtained in Proposition 2.4.2.4 to this module, consider the following $L_{(j+1,n-j)}$-modules:

Definition/Lemma 2.4.2.8. *Let N and $\mathfrak{d}$ as above.*

i) Define

$$\mathcal{O}(\mathbf{U}^+_{(j+1,n-j),k}; N')^{\mathfrak{d}} \ := \ \left\{ X \in \mathcal{O}(\mathbf{U}^+_{(j+1,n-j),k}) \otimes_k N' \ \middle| \ \beta_{c,N}(\mathfrak{d}, X) = 0 \right\}.$$

This is an $L_{(j+1,n-j)}$-submodule of $\mathcal{O}(\mathbf{U}^+_{(j+1,n-j),k}) \otimes_k N'$ and in particular, $\beta_{c,N}$ induces an $L_{(j+1,n-j)}$-equivariant pairing

$$\gamma_{N,\mathfrak{d}} : {}^{\mathbf{L}}\widehat{U}(\mathfrak{u}^+_{(j+1,n-j),k})_c \otimes_k N/\mathfrak{d} \times \mathcal{O}(\mathbf{U}^+_{(j+1,n-j),k}; N')^{\mathfrak{d}} \to k.$$

ii) Denote by $\mathcal{O}(\mathbf{U}^+_{(j+1,n-j),k}; N')^{\mathfrak{d}}_{\mathrm{res}}$ the kernel on the right of $\gamma_{N,\mathfrak{d}}$, i.e.

$$\mathcal{O}(\mathbf{U}^+_{(j+1,n-j),k}; N')^{\mathfrak{d}}_{\mathrm{res}} := \left\{ Y \in \mathcal{O}(\mathbf{U}^+_{(j+1,n-j),k}; N')^{\mathfrak{d}} \ \middle| \ \gamma_{N,\mathfrak{d}}\left({}^{\mathbf{L}}\widehat{U}(\mathfrak{u}^+_{(j+1,n-j),k})_c \otimes_k N/\mathfrak{d}, Y \right) = 0 \right\}$$

and let

$$\mathcal{O}(\mathbf{U}^+_{(j+1,n-j),k}; N')^{\mathfrak{d}}_{\mathrm{red}} := \mathcal{O}(\mathbf{U}^+_{(j+1,n-j),k}; N')^{\mathfrak{d}} / \mathcal{O}(\mathbf{U}^+_{(j+1,n-j),k}; N')^{\mathfrak{d}}_{\mathrm{res}}.$$

This is an $L_{(j+1,n-j)}$-quotient of $\mathcal{O}(\mathbf{U}^+_{(j+1,n-j),k}; N')^{\mathfrak{d}}$ and $\gamma_{N,\mathfrak{d}}$ induces an $L_{(j+1,n-j)}$-equivariant pairing

$$\gamma^{\mathrm{red}}_{N,\mathfrak{d}} : {}^{\mathbf{L}}\widehat{U}(\mathfrak{u}^+_{(j+1,n-j),k})_c \otimes_k N/\mathfrak{d} \times \mathcal{O}(\mathbf{U}^+_{(j+1,n-j),k}; N')^{\mathfrak{d}}_{\mathrm{red}} \to k.$$

By construction, $\gamma^{\mathrm{red}}_{N,\mathfrak{d}}$ is nondegenerate on the right and thus induces an injection of $L_{(j+1,n-j)}$-modules

$$\mathcal{O}(\mathbf{U}^+_{(j+1,n-j),k}; N')^{\mathfrak{d}}_{\mathrm{red}} \hookrightarrow ({}^{\mathbf{L}}\widehat{U}(\mathfrak{u}^+_{(j+1,n-j),k})_c \otimes_k N/\mathfrak{d})'. \tag{2.16}$$

Proof. This follows from general properties of pairings, cf. [36, XIII, § 5]. □

Proposition 2.4.2.9. *If in the above situation N has the additional structure of a graded k-vector space on which the group $L_{(j+1,n-j)}$ acts homogeneously and if $\mathfrak{d}$ is additionally a graded k-subspace of ${}^{\mathbf{L}}\widehat{U}(\mathfrak{u}^+_{(j+1,n-j),k})_c \otimes_k N$, then the embedding obtained in part ii) of the last definition/lemma induces a graded isomorphism of $L_{(j+1,n-j)}$-modules*

$$\mathcal{O}(\mathbf{U}^+_{(j+1,n-j),k}; N')^{\mathfrak{d}}_{\mathrm{red}} \xrightarrow{\sim} ({}^{\mathbf{L}}\widehat{U}(\mathfrak{u}^+_{(j+1,n-j),k})_c \otimes_k N/\mathfrak{d})^{\vee}.$$

(with respect to the induced gradings on the tensor products, submodules and quotients appearing).

Proof. Reconsider the (graded) isomorphism

$$\Gamma : \widehat{U}(\mathfrak{u}^+_{(j+1,n-j),k}) \xrightarrow{\sim} \mathcal{O}(\mathbf{U}^+_{(j+1,n-j),k})^\vee$$

from Proposition 2.4.2.4. Since ${}^{\mathbf{L}}\widehat{U}(\mathfrak{u}^+_{(j+1,n-j),k})_c$ is a graded submodule of $\widehat{U}(\mathfrak{u}^+_{(j+1,n-j),k})$, this isomorphism induces an $L_{(j+1,n-j)}$-equivariant (graded) embedding

$${}^{\mathbf{L}}\widehat{U}(\mathfrak{u}^+_{(j+1,n-j),k})_c \hookrightarrow \left(\mathcal{O}(\mathbf{U}^+_{(j+1,n-j),k})\right)^\vee .$$

Since N is also graded, this embedding extends to an $L_{(j+1,n-j)}$-equivariant embedding

$${}^{\mathbf{L}}\widehat{U}(\mathfrak{u}^+_{(j+1,n-j),k})_c \otimes_k N \hookrightarrow \left(\mathcal{O}(\mathbf{U}^+_{(j+1,n-j),k}) \otimes_k N'\right)^\vee .$$

Graded dualization of this embedding (cf. Section 1.1) yields a surjection

$$\mathcal{O}(\mathbf{U}^+_{(j+1,n-j),k}) \otimes_k N' \twoheadrightarrow \left({}^{\mathbf{L}}\widehat{U}(\mathfrak{u}^+_{(j+1,n-j),k})_c \otimes_k N\right)^\vee .$$

By construction of $\mathcal{O}(\mathbf{U}^+_{(j+1,n-j),k}; N')^{\mathfrak{d}}$ and from the fact that all spaces involved are graded, it follows that this surjection descends to a surjection

$$\mathcal{O}(\mathbf{U}^+_{(j+1,n-j),k}; N')^{\mathfrak{d}} \twoheadrightarrow \left({}^{\mathbf{L}}\widehat{U}(\mathfrak{u}^+_{(j+1,n-j),k})_c \otimes_k N/\mathfrak{d}\right)^\vee .$$

Again by construction, the kernel of this last map is precisely $\mathcal{O}(\mathbf{U}^+_{(j+1,n-j),k}; N')^{\mathfrak{d}}_{\mathrm{res}}$, so that reduction modulo this kernel yields the claimed isomorphism. □

2.5 Local Cohomology III: Description via Enriched Crystalline Enveloping Algebra

The enriched crystalline enveloping algebra can now be used to give descriptions of $(F_{T_{j+1}\cdot\ldots\cdot T_n})^{\rho}_0$ and $\tilde{\mathrm{H}}^{n-j}_{\mathbb{P}^j_k}(\mathbb{P}^n_k, \mathcal{F})$ as representations of the finite group $L_{(j+1,n-j)}$.

From Lemma 2.5.1.3 on through the rest of this chapter, it has to be assumed that, as a graded S-module, F is generated in degrees ≤ 1. This technical restriction is essential and it should be noted that it is not enough to assume that, via the usual procedure of thinning out[3], $\mathcal{F}$ is associated to some module (over some ring) generated in degrees ≤ 1.

2.5.1 Employing the Enriched Crystalline Enveloping Algebra

Write $\tilde{S} = \mathcal{V}[T_0, \ldots, T_n]$ and suppose that there is a finitely generated graded $\tilde{S}$-module $\tilde{F}$ together with an algebraic action of the group $\mathbf{G}$ such that the $\mathbf{G}_k$-module F is the reduction mod π of $\tilde{F}$. Furthermore, assume that the induced action of $\mathrm{Dist}(\mathbf{G})$ on $\tilde{F}$ extends to an action of $\widehat{U}(\mathfrak{g})_c$. In particular, the $\mathbf{G}_k$-action on F then induces an action of $\widehat{U}(\mathfrak{g}_k)_c$ on F in the sense of Lemma 2.4.0.3 (adapted to the algebra $\widehat{U}(\mathfrak{g}_k)_c$). This assumption holds for example in the situation of the following lemma.

[3]Let $f_1, \ldots, f_r \in F$ be homogeneous generators over S, denote by d the least common multiple of the degrees which are nonzero (set $d = 1$ if all generators have degree 0) and let $S^{(d)} = \bigoplus_{i \in \mathbb{N}_0} S_{di}$. Then there is an isomorphism of schemes $\mathbf{Proj}\, S \to \mathbf{Proj}\, S^{(d)}$ induced by the inclusion of $S^{(d)}$ into S. The same procedure applied to F yields a graded $S^{(d)}$-module $F^{(d)} = \bigoplus_{i \in \mathbb{N}_0} F_{di}$ which is now generated in degree ≤ 1 and by construction, $F^\sim \cong (F^{(d)})^\sim$ as modules over $S^\sim \cong (S^{(d)})^\sim$. Finally, $\mathbf{G}_k$ acts homogeneously on S and F and therefore the above isomorphism is even an isomorphism of $\mathbf{G}_k$-equivariant vector bundles.

Lemma 2.5.1.1. *Let V be a $\mathbf{P}_{(1,n)}$-module which is finitely generated as a $\mathcal{V}$-module. Write $V_k = V \otimes_{\mathcal{V}} k$. Suppose that $\mathcal{F} = \mathcal{F}_{V_k}$ is the vector bundle on $\mathbb{P}^n_k$ associated with V_k, cf. [32, I.5.8]. Then the $\mathbf{G}_k$-action on $\mathcal{F}$ induces a $\widehat{U}(\mathfrak{g}_k)_c$-action on $\mathcal{F}$.*

Proof. Since $\mathcal{F}$ is the quasi-coherent $\mathcal{O}_{\mathbb{P}^n_k}$-module associated with the S-module

$$F = \bigoplus_{l \in \mathbb{N}_0} \mathrm{H}^0(\mathbb{P}^n_k, \mathcal{F}(l)),$$

it is enough to show that the $\mathbf{G}_k$-action on each $\mathrm{H}^0(\mathbb{P}^n_k, \mathcal{F}(l))$ induces a $\widehat{U}(\mathfrak{g}_k)_c$-action. The claim will then follow by functoriality.

For each $l \in \mathbb{N}_0$, denote by V_l the 1-dimensional $\mathbf{P}_{(1,n)}$-module over $\mathcal{V}$ on which the factor GL_1 of the Levi subgroup $\mathbf{L}_{(1,n)} \cong \mathrm{GL}_1 \times \mathrm{GL}_n$ acts by $\det^l$ and on which the factor GL_n acts trivially. Write $V_{l,k} = V_l \otimes_{\mathcal{V}} k$. Then $\mathcal{F}_{V_{l,k}} = \mathcal{O}_{\mathbb{P}^n_k}(l)$ and there are thus identifications

$$\begin{aligned}
\mathrm{H}^0(\mathbb{P}^n_k, \mathcal{F}(l)) &= \mathrm{H}^0(\mathbb{P}^n_k, \mathcal{O}_{\mathbb{P}^n_k}(l) \otimes_{\mathcal{O}_{\mathbb{P}^n_k}} \mathcal{F}) \\
&= \mathrm{H}^0(\mathbb{P}^n_k, \mathcal{F}_{V_{l,k}} \otimes_{\mathcal{O}_{\mathbb{P}^n_k}} \mathcal{F}_{V_k}) \\
&= \mathrm{H}^0(\mathbb{P}^n_k, \mathcal{F}_{V_{l,k} \otimes_k V_k}) \\
&= \mathrm{ind}^{\mathbf{G}_k}_{\mathbf{P}_{(1,n),k}}(V_{l,k} \otimes_k V_k),
\end{aligned}$$

cf. [32, I.5.12,II.4.1-3], where the last module is the algebraic induction of the $\mathbf{P}_{(1,n),k}$-module $V_{l,k} \otimes_k V_k$ to $\mathbf{G}_k$, cf. [32, I.3]. According to loc. cit., this last module is isomorphic to the $\mathbf{G}_k$-module of $\mathbf{P}_{(1,n),k}$-invariants $((V_{l,k} \otimes_k V_k) \otimes_k k[\mathbf{G}_k])^{\mathbf{P}_{(1,n),k}}$ on which $\mathbf{G}_k$ acts via its action on $k[\mathbf{G}_k]$. Furthermore, $((V_{l,k} \otimes_k V_k) \otimes_k k[\mathbf{G}_k])^{\mathbf{P}_{(1,n),k}}$ is isomorphic to $((V_l \otimes_{\mathcal{V}} V) \otimes_{\mathcal{V}} \mathcal{V}[\mathbf{G}])^{\mathbf{P}_{(1,n)}} \otimes_{\mathcal{V}} k$, i.e. the $\mathbf{G}_k$-module $\mathrm{ind}^{\mathbf{G}_k}_{\mathbf{P}_{(1,n),k}}(V_{l,k} \otimes_k V_k)$ is isomorphic to the reduction mod π of the $\mathbf{G}$-module $\mathrm{ind}^{\mathbf{G}}_{\mathbf{P}_{(1,n)}}(V_l \otimes_{\mathcal{V}} V)$. In Lemma 2.6.1.1 ii) it is shown that the $U(\mathfrak{g})$-module structure on $\mathcal{V}[\mathbf{G}]$ induced by the natural $\mathbf{G}$-module structure (with respect to the action $\mathbf{G} \times \mathbf{G} \to \mathbf{G}, (g,h) \mapsto hg^{-1}$) extends to a $\widehat{U}(\mathfrak{g})_c$-module structure on $\mathcal{V}[\mathbf{G}]$. Thus the $\mathbf{G}$-module structure on $V_l \otimes_{\mathcal{V}} V \otimes_{\mathcal{V}} \mathcal{V}[\mathbf{G}]$ also induces a $\widehat{U}(\mathfrak{g})_c$-module structure. Since this $\widehat{U}(\mathfrak{g})_c$-action only differs by scalars from the $U(\mathfrak{g})$-action, the submodule of $\mathbf{P}_{(1,n)}$-invariants is stabilized by $\widehat{U}(\mathfrak{g})_c$. This proves the lemma. □

Summarizing, the following assumption is supposed to hold for the rest of this chapter for technical reasons.

Assumption 2.5.1.2. *From now on, it is assumed that $F = \bigoplus_{l \in \mathbb{N}_0} \mathrm{H}^0(\mathbb{P}^n_k, \mathcal{F}_{V_k})$ with V_k as in the previous lemma. As was shown in the proof of this lemma, F is then the reduction mod π of an $\mathbf{G}$-module $\tilde{F}$ as described in the beginning of this subsection. Finally, it is assumed that F is generated in degrees ≤ 1 as an S-module.*

Proposition 2.5.1.3. *There is a $\mathbf{P}_{(j+1,n-j),k}$-submodule $N_j \subset \tilde{\mathrm{H}}^{n-j}_{\mathbb{P}^j_k}(\mathbb{P}^n_k, \mathcal{F})$ of finite k-dimension and an epimorphism of $L_{(j+1,n-j)}$-modules*

$$\varphi_j : {}^{\mathbf{L}}\widehat{U}(\mathfrak{u}^+_{(j+1,n-j),k})_c \otimes_k N_j \twoheadrightarrow \tilde{\mathrm{H}}^{n-j}_{\mathbb{P}^j_k}(\mathbb{P}^n_k, \mathcal{F}).$$

Proof. First of all, it is enough to show that the lemma is true with $(F_{T_{j+1} \cdot \ldots \cdot T_n})^{\rho}_0$ instead of $\tilde{\mathrm{H}}^n_{\mathbb{P}^j_k}(\mathbb{P}^n_k, \mathcal{F})$, see Subsection 2.3.1. Denote by $\tilde{\rho}$ the canonical lift of ρ to $(\tilde{F}_{T_{j+1} \cdot \ldots \cdot T_n})_0$, i.e. let

$$\begin{aligned}
\tilde{\rho} : (\tilde{F}_{T_{j+1} \cdot \ldots \cdot T_n})_0 &\to \varinjlim_{l \in \mathbb{N}} (\tilde{F}/(T^l_{j+1}, \ldots, T^l_n)\tilde{F})_{l(n-j)} \\
\frac{g}{T^l_{j+1} \cdot \ldots \cdot T^l_n} &\mapsto g + (T^l_{j+1}, \ldots, T^l_n)\tilde{F}
\end{aligned}$$

and write

$$(\tilde{F}_{T_{j+1}\cdot\ldots\cdot T_n})_0^{\tilde{\rho}} := (\tilde{F}_{T_{j+1}\cdot\ldots\cdot T_n})_0/\ker(\tilde{\rho}).$$

This is a $\mathbf{P}_{(j+1,n-j)}$-module and the $\widehat{U}(\mathfrak{u}^+_{(j+1,n-j)})_c$-action on $\tilde{F}$ extends to one on this module. Write $\left[\frac{g}{T^l_{j+1}\cdot\ldots\cdot T^l_n}\right]$ for the element of $(\tilde{F}_{T_{j+1}\cdot\ldots\cdot T_n})_0^{\tilde{\rho}}$ induced by the element $\frac{g}{T^l_{j+1}\cdot\ldots\cdot T^l_n} \in (\tilde{F}_{T_{j+1}\cdot\ldots\cdot T_n})_0$ under the residue map. The proposition is proved if it can be shown that the following holds true:

Assertion. *There is a $\mathbf{P}_{(j+1,n-j)}$-submodule $\tilde{N}_j \subset (\tilde{F}_{T_{j+1}\cdot\ldots\cdot T_n})_0^{\tilde{\rho}}$, finitely generated as a $\mathcal{V}$-module, such that as a $\widehat{U}(\mathfrak{u}^+_{(j+1,n-j)})_c$-module, $(\tilde{F}_{T_{j+1}\cdot\ldots\cdot T_n})_0^{\tilde{\rho}}$ is generated by $\tilde{N}_j$.*

If this is the case, then there is an epimorphism of $\mathbf{L}_{(j+1,n-j)}(\mathcal{V})$-modules

$$\tilde{\varphi}_j : ({}^{\mathbf{L}}\widehat{U}(\mathfrak{u}^+_{(j+1,n-j)})_c \otimes_{\mathcal{V}} \tilde{N}_j \twoheadrightarrow (\tilde{F}_{T_{j+1}\cdot\ldots\cdot T_n})_0^{\tilde{\rho}}.$$

But this implies that there is also an epimorphism of $L_{(j+1,n-j)}$-modules

$$\varphi_j = \tilde{\varphi}_j \otimes \mathrm{id}_k : \left({}^{\mathbf{L}}\widehat{U}(\mathfrak{u}^+_{(j+1,n-j)})_c \otimes_{\mathcal{V}} \tilde{N}_j\right) \otimes_{\mathcal{V}} k \twoheadrightarrow (\tilde{F}_{T_{j+1}\cdot\ldots\cdot T_n})_0^{\tilde{\rho}} \otimes_{\mathcal{V}} k \cong (F_{T_{j+1}\cdot\ldots\cdot T_n})_0^{\rho}.$$

Now let $N_j = \tilde{N}_j \otimes_{\mathcal{V}} k$. It follows that

$$\begin{aligned}
{}^{\mathbf{L}}\widehat{U}(\mathfrak{u}^+_{(j+1,n-j),k})_c \otimes_k N_j &\cong \left({}^{\mathbf{L}}\widehat{U}(\mathfrak{u}^+_{(j+1,n-j)})_c \otimes_{\mathcal{V}} k\right) \otimes_k N_j \\
&\cong {}^{\mathbf{L}}\widehat{U}(\mathfrak{u}^+_{(j+1,n-j)})_c \otimes_{\mathcal{V}} (k \otimes_k N_j) \\
&\cong {}^{\mathbf{L}}\widehat{U}(\mathfrak{u}^+_{(j+1,n-j)})_c \otimes_{\mathcal{V}} (N_j \otimes_k k) \\
&\cong \left({}^{\mathbf{L}}\widehat{U}(\mathfrak{u}^+_{(j+1,n-j)})_c \otimes_{\mathcal{V}} N_j\right) \otimes_k k \\
&\cong \left({}^{\mathbf{L}}\widehat{U}(\mathfrak{u}^+_{(j+1,n-j)})_c \otimes_{\mathcal{V}} \left(\tilde{N}_j \otimes_{\mathcal{V}} k\right)\right) \otimes_k k \\
&\cong \left(\left({}^{\mathbf{L}}\widehat{U}(\mathfrak{u}^+_{(j+1,n-j)})_c \otimes_{\mathcal{V}} \tilde{N}_j\right) \otimes_{\mathcal{V}} k\right) \otimes_k k \\
&\cong \left({}^{\mathbf{L}}\widehat{U}(\mathfrak{u}^+_{(j+1,n-j)})_c \otimes_{\mathcal{V}} \tilde{N}_j\right) \otimes_{\mathcal{V}} (k \otimes_k k) \\
&\cong \left({}^{\mathbf{L}}\widehat{U}(\mathfrak{u}^+_{(j+1,n-j)})_c \otimes_{\mathcal{V}} \tilde{N}_j\right) \otimes_{\mathcal{V}} k
\end{aligned}$$

and thus the proposition is proved.

As for the assertion, set

$$\tilde{N}_j = \left\{ \left[\frac{f}{T_{j+1}\cdot\ldots\cdot T_n}\right] \;\middle|\; f \in \tilde{F}_{n-j} \right\}.$$

It is enough to show that all elements of type $\left[\frac{g}{T^m_{j+1}\cdot\ldots\cdot T^m_n}\right] \in (\tilde{F}_{T_{j+1}\cdot\ldots\cdot T_n})_0^{\tilde{\rho}}$ (with $g \in \tilde{F}_{m(n-j)}$, $m \in \mathbb{N}$) are contained in $\widehat{U}(\mathfrak{u}^+_{(j+1,n-j)})_c.\tilde{N}_j$. For this, since $\tilde{F}$ is generated in degrees ≤ 1 by assumption, it can be assumed that either

i) $g = T_0^{i_0} \cdot T_1^{i_1} \cdot \ldots \cdot T_j^{i_j} \cdot T_{j+1}^{i_{j+1}} \cdot \ldots \cdot T_n^{i_n} \cdot f$ with exponents $i_l \in \mathbb{N}_0$ such that $0 \leq i_l < m$ for $l \in \{j+1,\ldots,n\}$ and $\sum_{l=0}^n i_l = m(n-j)-1$ for some $f \in \tilde{F}_1$ or

ii) $g = T_0^{i_0} \cdot T_1^{i_1} \cdot \ldots \cdot T_j^{i_j} \cdot T_{j+1}^{i_{j+1}} \cdot \ldots \cdot T_n^{i_n} \cdot f$ with exponents $i_l \in \mathbb{N}_0$ such that $0 \leq i_l < m$ for $l \in \{j+1,\ldots,n\}$ and $\sum_{l=0}^n i_l = m(n-j)$ for some $f \in \tilde{F}_0$.

Suppose that case $i)$ applies, i.e. $f \in \tilde{F}_1$. Write $\frac{g}{T_{j+1}^m \cdot \ldots \cdot T_n^m}$ in the shape

$$\frac{g}{T_{j+1}^m \cdot \ldots \cdot T_n^m} = \frac{T_0^{i_0} \cdot T_1^{i_1} \cdot \ldots \cdot T_j^{i_j} \cdot f}{T_{j+1}^{m-i_{j+1}} \cdot \ldots \cdot T_n^{m-i_n}}$$

and simultaneously decrease the exponents i_r (for $r = 0, \ldots, j$) and $m - i_s$ (for $s = j+1, \ldots, n$) to obtain an element $\frac{T_0^{a_0} \cdot T_1^{a_1} \cdot \ldots \cdot T_j^{a_j} \cdot f}{T_{j+1} \cdot \ldots \cdot T_n}$ with $\sum_{r=0}^{j} a_r = n - j - 1$. In other words, for $r \in \{0, \ldots, j\}$ and $s \in \{j+1, \ldots, n\}$, there are non-negative numbers $a_r, m_{r,s} \in \mathbb{N}_0$ such that the following conditions are fulfilled:

- $\sum_{l=0}^{j} a_l = n - j - 1$,
- $a_r + \sum_{l=j+1}^{n} m_{r,l} = i_r$,
- $1 + \sum_{d=0}^{j} m_{d,s} = m - i_s$.

In particular, $\left[\frac{T_0^{a_0} \cdot T_1^{a_1} \cdot \ldots \cdot T_j^{a_j} \cdot f}{T_{j+1} \cdot \ldots \cdot T_n}\right]$ is an element of $\tilde{N}_j$. Now compute

$$\begin{aligned} X &= \prod_{l=0}^{j} L_{(l,j+1)}^{[m_{l,j+1}]} \cdot \left(\prod_{l=0}^{j} L_{(l,j+2)}^{[m_{l,j+2}]} \cdot \ldots \cdot \left(\prod_{l=0}^{j} L_{(l,n)}^{[m_{l,n}]} \cdot \left[\frac{T_0^{a_0} \cdot \ldots \cdot T_j^{a_j} \cdot f}{T_{j+1} \cdot \ldots \cdot T_n} \right] \right) \cdots \right) \\ &= \frac{1}{\prod_{r=j+1}^{n} (\sum_{l=0}^{j} m_{l,r})!} \prod_{l=0}^{j} L_{(l,j+1)}^{m_{l,j+1}} \cdot \left(\prod_{l=0}^{j} L_{(l,j+2)}^{m_{l,j+2}} \cdot \ldots \cdot \left(\prod_{l=0}^{j} L_{(l,n)}^{m_{l,n}} \cdot \left[\frac{T_0^{a_0} \cdot \ldots \cdot T_j^{a_j} \cdot f}{T_{j+1} \cdot \ldots \cdot T_n} \right] \right) \cdots \right). \end{aligned}$$

In a first step, using the Leibniz rule, one gets[4]

$$\begin{aligned} &\prod_{l=0}^{j} L_{(l,j+1)}^{m_{l,j+1}} \cdot \left(\cdots \prod_{l=0}^{j} L_{(l,n-1)}^{m_{l,n-1}} \cdot \left(\prod_{l=0}^{j} L_{(l,n)}^{m_{l,n}} \cdot \left[\frac{T_0^{a_0} \cdot \ldots \cdot T_j^{a_j} \cdot f}{T_{j+1} \cdot \ldots \cdot T_n} \right] \right) \cdots \right) \\ = &\prod_{l=0}^{j} L_{(l,j+1)}^{m_{l,j+1}} \cdot \left(\cdots \prod_{l=0}^{j} L_{(l,n-1)}^{m_{l,n-1}} \cdot \left((-1)^{\sum_{l=0}^{j} m_{l,n}} (\sum_{l=0}^{j} m_{l,n})! \left[\frac{T_0^{a_0+m_{0,n}} \cdot \ldots \cdot T_j^{a_j+m_{j,n}} \cdot f}{T_{j+1} \cdot \ldots \cdot T_{n-1} \cdot T_n^{1+\sum_{l=0}^{j} m_{l,n}}} \right] \right. \right. \\ &\left. \left. + \left[\frac{g'}{T_{j+1}^{\sum_{l=0}^{j} m_{l,n}} \cdot \ldots \cdot T_n^{\sum_{l=0}^{j} m_{l,n}}} \right] \right) \cdots \right) \end{aligned}$$

for some $g' \in \tilde{F}_{(n-j)(m-1)}$. Simply applying the operators successively, one obtains in similar fashion

$$\begin{aligned} &\prod_{l=0}^{j} L_{(l,j+1)}^{m_{l,j+1}} \cdot \ldots \cdot \left(\prod_{l=0}^{j} L_{(l,n)}^{m_{l,n}} \cdot \left[\frac{T_0^{a_0} \cdot \ldots \cdot T_j^{a_j} \cdot f}{T_{j+1} \cdot \ldots \cdot T_n} \right] \right) \\ = \; &(-1)^{\sum_{r=j+1}^{n} \sum_{l=0}^{j} m_{l,r}} \prod_{r=j+1}^{n} (\sum_{l=0}^{j} m_{l,r})! \left[\frac{T_0^{i_0} \cdot \ldots \cdot T_j^{i_j} \cdot T_{j+1}^{i_{j+1}} \cdot \ldots \cdot T_n^{i_n} \cdot f}{T_{j+1}^m \cdot \ldots \cdot T_n^m} \right] \\ &+ \left[\frac{g'}{T_{j+1}^{m-1} \cdot \ldots \cdot T_n^{m-1}} \right] \end{aligned}$$

[4]Recall that to simplify the calculations, the generalized fractions can be thought of as completely reduced.

for some $g' \in \tilde{F}_{(n-j)\sum_{l=0}^{j} m_{l,n}}$. Therefore, X equals

$$(-1)^{\sum_{r=j+1}^{n}\sum_{l=0}^{j} m_{l,r}} \left[\frac{T_0^{i_0}\cdot\ldots\cdot T_j^{i_j}\cdot T_{j+1}^{i_{j+1}}\cdot\ldots T_n^{i_n}\cdot f}{T_{j+1}^{m}\cdot\ldots\cdot T_n^{m}}\right] + \left[\frac{g'}{T_{j+1}^{m-1}\cdot\ldots\cdot T_n^{m-1}}\right] \in \widehat{U}(\mathfrak{u}_{(j+1,n-j)})_c.\tilde{N}_j$$

for some $g' \in \tilde{F}_{(n-j)\sum_{l=0}^{j} m_{l,n}}$. This finishes the proof, since the construction for case ii) (i.e. when $f \in \tilde{F}_0$) is completely analogous. □

Denote by $\mathfrak{d}_j$ the kernel of φ_j. This is a $L_{(j+1,n-j)}$-submodule of $\tilde{\mathrm{H}}^{n-j}_{\mathbb{P}^j_k}(\mathbb{P}^n_k, \mathcal{F})$. Let $P_{(j+1,n-j)}$ act on ${}^{\mathbf{L}}\widehat{U}(\mathfrak{u}^+_{(j+1,n-j),k})_c$ by inflation.

Theorem 2.5.1.4. *Under the assumptions made in Assumption 2.5.1.2, the $P_{(j+1,n-j)}$-semisimplifications $({}^{\mathbf{L}}\widehat{U}(\mathfrak{u}^+_{(j+1,n-j),k})_c \otimes_k N_j/\mathfrak{d}_j)^{P_{(j+1,n-j)}-ss.}$ and $(\tilde{\mathrm{H}}^{n-j}_{\mathbb{P}^j_k}(\mathbb{P}^n_k, \mathcal{F}))^{P_{(j+1,n-j)}-ss.}$ exist and there is an isomorphism of $P_{(j+1,n-j)}$-modules*

$$({}^{\mathbf{L}}\widehat{U}(\mathfrak{u}^+_{(j+1,n-j),k})_c \otimes_k N_j/\mathfrak{d}_j)^{P_{(j+1,n-j)}-ss.} \xrightarrow{\sim} (\tilde{\mathrm{H}}^{n-j}_{\mathbb{P}^j_k}(\mathbb{P}^n_k, \mathcal{F}))^{P_{(j+1,n-j)}-ss.}.$$

Proof. The proof proceeds in several steps:

i) The module $M = (F_{T_{j+1}\cdot\ldots\cdot T_n})_0^{\rho}$ has an ascending filtration

$$(0) = M_0 \subsetneq M_1 \subsetneq M_2 \subsetneq \ldots,$$

with

$$M_i = \left\{ \left[\frac{f}{T_{j+1}^{i}\cdot\ldots\cdot T_n^{i}}\right] \;\middle|\; f \in F_{(n-j)i} \right\} \subset (F_{T_{j+1}\cdot\ldots\cdot T_n})_0^{\rho}$$

such that $M = \bigcup_{i\in\mathbb{N}_0} M_i$ and such that each M_i is a finitely generated $\mathbf{L}_{(j+1,n-j),k}$-module. By Lemma 1.2.2.1, there is then an induced filtration

$$(0) = M'_0 \subsetneq M'_1 \subsetneq M'_2 \subsetneq \ldots$$

of the same type on the quotient $M' = \tilde{\mathrm{H}}^{n-j}_{\mathbb{P}^j_k}(\mathbb{P}^n_k, \mathcal{F})$. Therefore $(\tilde{\mathrm{H}}^{n-j}_{\mathbb{P}^j_k}(\mathbb{P}^n_k, \mathcal{F}))^{L_{(j+1,n-j)}-ss.}$ exists.

ii) As was previously mentioned and used in the proof of Proposition 2.4.2.4, the k-vector space $V = \widehat{U}(\mathfrak{u}^+_{(j+1,n-j),k})$ possesses an $\mathbb{N}_0$-grading $V = \bigoplus_{i\in\mathbb{N}_0} V_i$. Since the group $\mathbf{L}_{(j+1,n-j),k}$ acts homogeneously on V, each V_i is stabilized by the $\mathbf{L}_{(j+1,n-j),k}$-action. Finally, each V_i is of finite k-dimension so that one obtains an ascending filtration

$$(0) \subsetneq W_0 = V_0 \subsetneq W_1 = V_0 \oplus V_1 \subsetneq \ldots \subsetneq W_l = \bigoplus_{i=0}^{l} V_i \subsetneq \ldots$$

on $V = \bigcup_{i=0}^{\infty} W_i$ of the type considered in Subsection 1.2.2. For $i \in \mathbb{N}$, set

$$W'_i = (W_{i-1} \cap {}^{\mathbf{L}}\widehat{U}(\mathfrak{u}^+_{(j+1,n-j),k})_c) \otimes_k N_j.$$

Then

$$(0) =: W'_0 \subsetneq W'_1 \subsetneq W'_2 \subsetneq \ldots$$

is an ascending filtration of ${}^{\mathbf{L}}\widehat{U}(\mathfrak{u}^+_{(j+1,n-j),k})_c) \otimes_k N_j = \bigcup_{i\in\mathbb{N}} W'_i$ of the type considered in Subsection 1.2.2. By Lemma 1.2.2.1, the quotient ${}^{\mathbf{L}}\widehat{U}(\mathfrak{u}^+_{(j+1,n-j),k})_c \otimes_k N_j/\mathfrak{d}_j$ then also has such a filtration $W''_\bullet$. Therefore, $({}^{\mathbf{L}}\widehat{U}(\mathfrak{u}^+_{(j+1,n-j),k})_c \otimes_k N_j/\mathfrak{d}_j)^{L_{(j+1,n-j)}-ss.}$ exists.

iii) By construction, φ_j induces an isomorphism $W_i'' \xrightarrow{\sim} M_i'$ (cf. the proof of the last proposition). Therefore, φ_j induces an isomorphism of $L_{(j+1,n-j)}$-modules

$$({}^{\mathbf{L}}\widehat{U}(\mathfrak{u}^+_{(j+1,n-j),k})_c \otimes_k N_j/\mathfrak{d}_j)^{L_{(j+1,n-j)}-ss.} \xrightarrow{\sim} (\tilde{\mathrm{H}}^{n-j}_{\mathbb{P}^j_k}(\mathbb{P}^n_k, \mathcal{F}))^{L_{(j+1,n-j)}-ss.}.$$

iv) The simple $P_{(j+1,n-j)}$-modules over k only depend on their $L_{(j+1,n-j)}$-structure and therefore, the theorem is proved.

□

2.5.2 Functorial Reinterpretation: $\mathbf{U}^+$-Algebraic Induction and Extension of Duality

With the help of the description of $\tilde{\mathrm{H}}^{n-j}_{\mathbb{P}^j_k}(\mathbb{P}^n_k, \mathcal{F})$ found in the last subsection, one can now describe the kernels of the extensions (2.2) appearing in Theorem 2.1.2.1 purely in terms of the representation theory of G and its subgroups by using $\mathbf{U}^+$-algebraic induction and extending the duality developed in Subsection 2.4.2.

First of all, Proposition 2.5.1.3 suggests to consider the following category $\mathcal{C}$. Recall from Section 1.2 the definitions of the categories $\mathrm{rep}_k(H)$ and $\mathrm{rep}_k(H)^f$ for a group H.

Definition 2.5.2.1. *Let $\mathcal{C}$ be the full subcategory of $\mathrm{rep}_k(L_{(j+1,n-j)})$ defined as follows: An $L_{(j+1,n-j)}$-module M over k is an object of $\mathcal{C}$ if there exists an $L_{(j+1,n-j)}$-submodule $W \subset M$ which is an object of $\mathrm{rep}_k(L_{(j+1,n-j)})^f$ and an epimorphism of $L_{(j+1,n-j)}$-modules*

$${}^{\mathbf{L}}\widehat{U}(\mathfrak{u}^+_{(j+1,n-j),k})_c \otimes_k W \twoheadrightarrow M.$$

In the next step, $\mathbf{U}^+$-algebraic induction will be defined.

$\mathbf{U}^+$-Algebraic Induction

Consider $\mathcal{O}(\mathbf{U}^+_{(j+1,n-j),k})$ as a $\mathbf{P}_{(j+1,n-j),k}$-module via inflation with respect to the quotient epimorphism $\mathbf{P}_{(j+1,n-j),k} \twoheadrightarrow \mathbf{L}_{(j+1,n-j),k}$.

Definition 2.5.2.2. *Let W be an object of $\mathrm{rep}_k(L_{(j+1,n-j)})^f$, considered as a $P_{(j+1,n-j)}$-module by inflation.*

i) Set

$$\mathrm{Ind}^{G,\mathbf{U}^+\text{-}alg}_{P_{(j+1,n-j)}}(W) := \mathrm{Ind}^G_{P_{(j+1,n-j)}}(\mathcal{O}(\mathbf{U}^+_{(j+1,n-j),k}) \otimes_k W).$$

The induced functor

$$\mathrm{Ind}^{G,\mathbf{U}^+\text{-}alg}_{P_{(j+1,n-j)}} : \mathrm{rep}_k(L_{(j+1,n-j)})^f \to \mathrm{rep}_k(G)$$

is called $\mathbf{U}^+$-algebraic induction.

ii) Let $\mathfrak{d}$ be an $L_{(j+1,n-j)}$-submodule of ${}^{\mathbf{L}}\widehat{U}(\mathfrak{u}^+_{(j+1,n-j),k})_c \otimes_k W$. Set

$$\mathrm{Ind}^{G,\mathbf{U}^+-alg}_{P_{(j+1,n-j)}}(W)^{\mathfrak{d}}_{\mathrm{red}} := \mathrm{Ind}^G_{P_{(j+1,n-j)}}(\mathcal{O}(\mathbf{U}^+_{(j+1,n-j),k}; W')^{\mathfrak{d}}_{\mathrm{red}}),$$

where $\mathcal{O}(\mathbf{U}^+_{(j+1,n-j),k}; W')^{\mathfrak{d}}_{\mathrm{red}}$ is defined as in Proposition 2.4.2.9.

Extension of Duality

The duality developed in Subsection 2.4.2 can now be extended to the following situation.

Proposition 2.5.2.3. *Let M be an object of $\mathcal{C}$. There exists an epimorphism of $L_{(j+1,n-j)}$-modules*

$$^{\mathbf{L}}\widehat{U}(\mathfrak{u}^+_{(j+1,n-j),k})_c \otimes_k W \twoheadrightarrow M.$$

Let $\mathfrak{d}$ be its kernel. Suppose that M and W each have the additional structure of a graded k-vector space on which $L_{(j+1,n-j)}$ acts homogeneously. If the homomorphism $^{\mathbf{L}}\widehat{U}(\mathfrak{u}_{(j+1,n-j),k})_c \otimes W \twoheadrightarrow M$ is homogeneous, then there is an isomorphism of G-modules

$$\mathrm{Ind}^G_{P_{(j+1,n-j)}}(M) \cong \left(\mathrm{Ind}^{G,\mathbf{U}^+-alg}_{P_{(j+1,n-j)}}(W)^{\mathfrak{d}}_{\mathrm{red}}\right)^\vee.$$

Proof. By assumption, $\mathfrak{d} \subset {}^{\mathbf{L}_{(j+1,n-j),k}}\widehat{U}(\mathfrak{u}^+_{(j+1,n-j),k})\otimes_k W$ is a graded k-subspace. Therefore, it follows from Proposition 2.4.2.9 that there are isomorphisms of $L_{(j+1,n-j)}$-modules

$$M \cong {}^{\mathbf{L}}\widehat{U}(\mathfrak{u}^+_{(j+1,n-j),k}) \otimes_k W/\mathfrak{d} \cong (\mathcal{O}(\mathbf{U}^+_{(j+1,n-j),k}; W')^{\mathfrak{d}}_{\mathrm{red}})^\vee.$$

Application of the induction functor $\mathrm{Ind}^G_{P_{(j+1,n-j)}}$ yields an isomorphism

$$\mathrm{Ind}^G_{P_{(j+1,n-j)}}(M) \cong \mathrm{Ind}^G_{P_{(j+1,n-j)}}((\mathcal{O}(\mathbf{U}^+_{(j+1,n-j),k}; W')^{\mathfrak{d}}_{\mathrm{red}})^\vee).$$

Up to isomorphism, induction in the present setting commutes with duals and in particular with graded duals. Thus the right-hand side of the above isomorphism is isomorphic to the graded dual $(\mathrm{Ind}^G_{P_{(j+1,n-j)}}(\mathcal{O}(\mathbf{U}^+_{(j+1,n-j),k}; W')^{\mathfrak{d}}_{\mathrm{red}}))^\vee$ and the claim follows from the definition of $\mathrm{Ind}^{G,\mathbf{U}^+-alg}_{P_{(j+1,n-j)}}(W)^{\mathfrak{d}}_{\mathrm{red}}$. □

Theorem 2.1.2.1 can now be rewritten. Recall assumption 2.5.1.2.

Theorem 2.5.2.4. *On $\mathrm{H}^0(\mathcal{X}^{(n+1)}, \mathcal{F})$ there is a G-equivariant filtration*

$$\mathrm{H}^0(\mathcal{X}^{(n+1)}, \mathcal{F}) = \mathcal{F}(\mathcal{X}^{(n+1)})^0 \supset \mathcal{F}(\mathcal{X}^{(n+1)})^1 \supset \ldots \supset \mathcal{F}(\mathcal{X}^{(n+1)})^n = \mathrm{H}^0(\mathbb{P}^n_k, \mathcal{F})$$

such that for $j \in \{0, \ldots, n-1\}$, the G-semisimplifications of the quotients $\mathcal{F}(\mathcal{X}^{(n+1)})^j/\mathcal{F}(\mathcal{X}^{(n+1)})^{j+1}$ exist and appear as extensions in short exact sequences of G-modules

$$\begin{aligned}(0) \quad &\to \quad \left(\left(\mathrm{Ind}^{G,\mathbf{U}^+-alg}_{P_{(j+1,n-j)}}(N'_j \otimes \mathrm{St}_{n-j}(k)')^{\mathfrak{d}_j\otimes \mathrm{St}_{n-j}(k)}_{\mathrm{red}}\right)^\vee\right)^{G-ss.} \to (\mathcal{F}(\mathcal{X}^{(n+1)})^j/\mathcal{F}(\mathcal{X}^{(n+1)})^{j+1})^{G-ss.}\\ &\to \quad (v^G_{P_{(j+1,1^{n-j})}}(k)' \otimes_k \mathrm{H}^{n-j}(\mathbb{P}^n_k, \mathcal{F}))^{G-ss.} \to (0).\end{aligned}$$

Proof. Recall from Theorem 2.1.2.1 that for $j = 0, \ldots, n-1$, there are extensions of G-modules

$$\begin{aligned}(0) \quad &\to \quad \mathrm{Ind}^G_{P_{(j+1,n-j)}}(\tilde{\mathrm{H}}^{n-j}_{\mathbb{P}^j_k}(\mathbb{P}^n_k, \mathcal{F}) \otimes_k \mathrm{St}_{n-j}(k)) \to \mathcal{F}(\mathcal{X}^{(n+1)})^j/\mathcal{F}(\mathcal{X}^{(n+1)})^{j+1}\\ &\to \quad v^G_{P_{(j+1,1^{n-j})}}(k)' \otimes_k \mathrm{H}^{n-j}(\mathbb{P}^n_k, \mathcal{F}) \to (0).\end{aligned}$$

Fix $j \in \{0, \ldots, n-1\}$. It was shown in Theorem 2.5.1.4 that $\tilde{\mathrm{H}}^{n-j}_{\mathbb{P}^j_k}(\mathbb{P}^n_k, \mathcal{F})^{P_{(j+1,n-j)}-ss.}$ exists. Since $\mathrm{St}_{n-j}(k)$ is a finite dimensional k-vector space, it follows from the general construction of semisimplifications in Subsection 1.2.2 that $(\tilde{\mathrm{H}}^{n-j}_{\mathbb{P}^j_k}(\mathbb{P}^n_k, \mathcal{F}) \otimes_k \mathrm{St}_{n-j}(k))^{P_{(j+1,n-j)}-ss.}$ exists as well. From Lemma 1.2.2.1 it follows that the G-semisimplification $\mathrm{Ind}^G_{P_{(j+1,n-j)}}(\tilde{\mathrm{H}}^{n-j}_{\mathbb{P}^j_k}(\mathbb{P}^n_k, \mathcal{F}) \otimes_k \mathrm{St}_{n-j}(k))^{G-ss.}$

exists. The G-module $v^G_{P_{(j+1,1^{n-j})}}(k)' \otimes_k \mathrm{H}^{n-j}(\mathbb{P}^n_k, \mathcal{F})$ is of finite k-dimension and therefore has a G-semisimplification $(v^G_{P_{(j+1,1^{n-j})}}(k)' \otimes_k \mathrm{H}^{n-j}(\mathbb{P}^n_k, \mathcal{F}))^{G-ss.}$. Again from Lemma 1.2.2.1 it now follows that $(\mathcal{F}(\mathcal{X}^{(n+1)})^j/\mathcal{F}(\mathcal{X}^{(n+1)})^{j+1})^{G-ss.}$ exists and fits into an exact sequence of G-modules

$$\begin{aligned}(0) \quad &\to \quad (\mathrm{Ind}^G_{P_{(j+1,n-j)}}(\tilde{\mathrm{H}}^{n-j}_{\mathbb{P}^j_k}(\mathbb{P}^n_k, \mathcal{F}) \otimes_k \mathrm{St}_{n-j}(k)))^{G-ss.} \to (\mathcal{F}(\mathcal{X}^{(n+1)})^j/\mathcal{F}(\mathcal{X}^{(n+1)})^{j+1})^{G-ss.} \\ &\to \quad (v^G_{P_{(j+1,1^{n-j})}}(k)' \otimes_k \mathrm{H}^{n-j}(\mathbb{P}^n_k, \mathcal{F}))^{G-ss.} \to (0).\end{aligned}$$

Using Lemma 1.2.2.1 one more time, it follows that $\mathrm{Ind}^G_{P_{(j+1,n-j)}}(\tilde{\mathrm{H}}^{n-j}_{\mathbb{P}^j_k}(\mathbb{P}^n_k, \mathcal{F}) \otimes_k \mathrm{St}_{n-j}(k))^{G-ss.}$ is isomorphic to

$$\left(\mathrm{Ind}^G_{P_{(j+1,n-j)}}\left(\tilde{\mathrm{H}}^{n-j}_{\mathbb{P}^j_k}(\mathbb{P}^n_k, \mathcal{F}) \otimes_k \mathrm{St}_{n-j}(k)\right)^{P_{j+1,n-j}-ss.}\right)^{G-ss.}.$$

Since the simple $P_{j+1,n-j}$-modules only depend on their $L_{j+1,n-j}$-module structure, this last module is isomorphic to

$$\begin{aligned}&\left(\mathrm{Ind}^G_{P_{(j+1,n-j)}}\left(\mathrm{Res}^{P_{(j+1,n-j)}}_{L_{(j+1,n-j)}}\left(\tilde{\mathrm{H}}^{n-j}_{\mathbb{P}^j_k}(\mathbb{P}^n_k, \mathcal{F}) \otimes_k \mathrm{St}_{n-j}(k)\right)\right)^{L_{j+1,n-j}-ss.}\right)^{G-ss.} \\ \cong \quad &\left(\mathrm{Ind}^G_{P_{(j+1,n-j)}}\left(\mathrm{Res}^{P_{(j+1,n-j)}}_{L_{(j+1,n-j)}}\left(\tilde{\mathrm{H}}^{n-j}_{\mathbb{P}^j_k}(\mathbb{P}^n_k, \mathcal{F}) \otimes_k \mathrm{St}_{n-j}(k)\right)\right)\right)^{G-ss.}.\end{aligned}$$

Here the group $P_{(j+1,n-j)}$ now acts by inflation on the argument of the induction. According to Proposition 2.5.1.3, each $L_{(j+1,n-j)}$-module $\tilde{\mathrm{H}}^{n-j}_{\mathbb{P}^j_k}(\mathbb{P}^n_k, \mathcal{F})$ is an object of the category $\mathcal{C}$. It follows that $\tilde{\mathrm{H}}^{n-j}_{\mathbb{P}^j_k}(\mathbb{P}^n_k, \mathcal{F}) \otimes_k \mathrm{St}_{n-j}(k)$ is also an object of $\mathcal{C}$ by virtue of the homomorphism

$$\varphi_j \otimes \mathrm{id} : ({}^{\mathbf{L}}\widehat{U}(\mathfrak{u}_{(j+1,n-j),k})_c \otimes_k N_j) \otimes_k \mathrm{St}_{n-j}(k) \twoheadrightarrow \tilde{\mathrm{H}}^{n-j}_{\mathbb{P}^j_k}(\mathbb{P}^n_k, \mathcal{F}) \otimes_k \mathrm{St}_{n-j}(k).$$

Consider the k-vector space N_j as a graded space which is concentrated in degree 1 and the k-vector space $\mathrm{St}_{n-j}(k)$ as a graded space which is concentrated in degree 0. Give the k-vector space ${}^{\mathbf{L}}\widehat{U}(\mathfrak{u}_{(j+1,n-j),k})_c \otimes_k N_j \otimes_k \mathrm{St}_{n-j}(k)$ the induced grading of the tensor product. It follows from the proof of Proposition 2.5.1.3, that the assumptions made in Proposition 2.5.2.3 apply to the case

$$M = \tilde{\mathrm{H}}^{n-j}_{\mathbb{P}^j_k}(\mathbb{P}^n_k, \mathcal{F}) \otimes_k \mathrm{St}_{n-j}(k), W = N_j \otimes \mathrm{St}_{n-j}(k)$$

i.e. M inherits via $\varphi_j \otimes \mathrm{id}$ a grading and this map then respects this grading by construction. The claim follows from Proposition 2.5.2.3. □

Remark. For a given $P_{(j+1,n-j)}$-module W of finite k-dimension as considered above, it does not seem possible to view $\mathrm{Ind}^{G,\mathbf{U}^+\text{-alg}}_{P_{(j+1,n-j)}}(W)$ as a vector space of functions from $\mathbf{G}_k$ resp. G to $\bar{k} \otimes_k W$ in any reasonable way. Therefore, the analogy with [44] once again fails as in the case of a p-adic ground field K, the resulting induction can be described nicely in terms of locally analytic functions on $\mathbf{G}(K)$, cf. op. cit., Section 2.2. In the present situation, there is only a rather naive interpretation of $\mathrm{Ind}^{G,\mathbf{U}^+\text{-alg}}_{P_{(j+1,n-j)}}(W)$ as functions on a variety or group: Recall from Section 1.2 that there is an isomorphism of G-modules

$$\mathrm{Ind}^{G,\mathbf{U}^+\text{-alg}}_{P_{(j+1,n-j)}}(W) \cong \bigoplus_{g \in G/P_{(j+1,n-j)}} g * (\mathcal{O}(\mathbf{U}^+_{(j+1,n-j),k}) \otimes_k W).$$

Therefore, as a k-vector space, $\mathrm{Ind}^{G,\mathbf{U}^+\text{-alg}}_{P_{(j+1,n-j)}}(W)$ can be identified with

$$\left\{ \coprod_{g\in G/P_{(j+1,n-j)}} \mathbf{U}^+_{(j+1,n-j),k} \to \bar{k}\otimes_k W \;\; k\text{-regular} \right\}$$

with G acting as a permutation group via the canonical epimorphism $G \to G/P_{(j+1,n-j)}$.

2.6 Examples

First of all, it is shown that the natural $U(\mathfrak{g})$-action on the regular representation $\mathcal{V}[\mathbf{G}]$ extends to an action of $\widehat{U}(\mathfrak{g})_c$ resp. $\widehat{U}(\mathfrak{g})_r$ (depending on the action of $\mathbf{G}$ on itself). In particular, the regular representation $k[\mathbf{G}_k]$ is then a module over $\widehat{U}(\mathfrak{g}_k)_c$ resp. $\widehat{U}(\mathfrak{g}_k)_r$. Then it is shown that the same holds true for twisted structure sheaves and sheaves of differential forms (with $\mathbf{G}_k$-linearization). Finally, Theorem 2.5.2.4 is applied for twisted structure sheaves.

2.6.1 The Regular Representation of GL_{n+1}

It was shown in Lemma 2.4.1.2 that the $\mathbf{G}$-module $\tilde{S} \cong \mathcal{V}[\mathbb{A}^{n+1}_{\mathcal{V}}]$ has the structure of either an $\widehat{U}(\mathfrak{g})_r$-module or an $\widehat{U}(\mathfrak{g})_c$-module, depending on the action of $\mathbf{G}$ on $\mathbb{A}^{n+1}_{\mathcal{V}}$. Furthermore, there is the following specific example of the regular representation of $\mathbf{G}$:

Lemma 2.6.1.1.

i) Consider the action of $\mathbf{G}$ *on itself given by*

$$\mathbf{G}\times\mathbf{G} \to \mathbf{G}, (g,h) \mapsto gh.$$

Then the induced $\mathbf{G}$*-module* $M = \mathcal{O}_{\mathbf{G}}(\mathbf{G}) = \mathcal{V}[\mathbf{G}]$ *is a* $\widehat{U}(\mathfrak{g})_r$*-module.*

ii) Consider the action of $\mathbf{G}$ *on itself given by*

$$\mathbf{G}\times\mathbf{G} \to \mathbf{G}, (g,h) \mapsto hg^{-1}.$$

Then the induced $\mathbf{G}$*-module* $M = \mathcal{O}_{\mathbf{G}}(\mathbf{G}) = \mathcal{V}[\mathbf{G}]$ *is a* $\widehat{U}(\mathfrak{g})_c$*-module.*

Proof. It suffices to consider case ii) as i) then follows by symmetry. The induced action of $\mathbf{G}$ on M is in this case given by

$$(g,f) \mapsto g.f$$

where

$$(g.f)(x) = f(xg)$$

for $x \in \mathbf{G}$. The $\mathcal{V}$-algebra $\mathcal{V}[\mathbf{G}]$ is realized as localization $\mathcal{V}[\{T_{u,v} \mid 0 \le u,v \le n\}]_{\det^{-1}}$ of the polynomial algebra $\mathcal{V}[\{T_{u,v} \mid 0 \le u,v \le n\}]$ with respect to the determinant. For $v \in \{0,\dots,n\}$, consider the element $\prod_{\substack{i=0\\ i\neq v}}^n L_{(i,v)}^{l_{i,v}}$ of $U(\mathfrak{g})$. This element acts on an element $T_{u,w}^{m_{u,w}} \in \mathcal{V}[\{T_{u,v} \mid 0 \le u,v \le n\}]$ by

$$\prod_{\substack{i=0\\ i\neq v}}^n L_{(i,v)}^{l_{i,v}}.T_{u,w}^{m_{u,w}} = \begin{cases} \frac{m!}{(m-\sum_{\substack{i=0\\ i\neq v}}^n l_{i,v})!}\left(\prod_{\substack{i=0\\ i\neq v}}^n T_{u,i}^{l_{i,v}}\right) T_{u,w}^{m-\sum_{\substack{i=0\\ i\neq v}}^n l_{u,i}} & \text{if } w = v \\ 0 & \text{if } w \neq v. \end{cases}.$$

As the coefficients appearing are divisible by $(\sum_{\substack{i=0\\ i\neq v}}^n l_{i,v})!$, this action extends to an action of $\widehat{U}(\mathfrak{g})_c$ on $\mathcal{V}[\{T_{u,v} \mid 0 \le u,v \le n\}]$. Using the Leibniz formula for determinants, one now checks that this action extends to one on $\mathcal{V}[\mathbf{G}]$. □

2.6.2 Some Equivariant Vector Bundles on $\mathbb{P}_k^n$ admitting Actions of the Enriched Crystalline Enveloping Algebra

In the last section, it had to be assumed that the algebra $\widehat{U}(\mathfrak{g}_k)_c$ acts on the sheaf $\mathcal{F}$ on $\mathbb{P}_k^n$ under consideration. In the following lemma, it is shown that this holds true for the twisted structure sheaves and for sheaves of differential forms on $\mathbb{P}_k^n$.

Lemma 2.6.2.1.

i) Let $a \in \mathbb{Z}$. Then the algebra $\widehat{U}(\mathfrak{g}_k)_c$ acts on $\mathcal{O}_{\mathbb{P}_k^n}(a)$ (induced by the $\mathbf{G}_k$-action on $\mathcal{O}_{\mathbb{P}_k^n}(a)$).

ii) The algebra $\widehat{U}(\mathfrak{g}_k)_c$ acts on $\Omega^1_{\mathbb{P}_k^n/k}$ (induced by the $\mathbf{G}_k$-action on $\Omega^1_{\mathbb{P}_k^n/k}$).

iii) The algebra $\widehat{U}(\mathfrak{g}_k)_c$ acts on $\Omega^i_{\mathbb{P}_k^n/k}, i = 2, \dots, n$ (induced by the $\mathbf{G}_k$-action on $\Omega^i_{\mathbb{P}_k^n/k}$).

Proof. i) This follows either via functoriality from the fact that $\mathcal{O}_{\mathbb{P}_k^n}(a) = S(a)^\sim$ and from Lemma 2.4.1.2,ii) or from the fact that $\mathcal{O}_{\mathbb{P}_k^n}(a)$ is the $\mathcal{O}_{\mathbb{P}_k^n}$-module associated with the $\mathbf{P}_{(1,n)}$-module $V_{a,k}$ as in (the proof of) Lemma 2.5.1.1.

ii) This follows also from Lemma 2.5.1.1 since $\Omega^1_{\mathbb{P}_k^n/k}$ is the $\mathcal{O}_{\mathbb{P}_k^n}$-module associated with the $\mathbf{P}_{(1,n)}$-module $\mathrm{Lie}(\mathbf{G}_k/\mathbf{P}_{(1,n),k})'$, cf. [32, II.4.2(4)]. Alternatively, one may use the realization of $\Omega^1_{\mathbb{P}_k^n/k}$ as the sheaf associated with the S-module M which appears as the kernel of the S-homomorphism (graded, of degree 0)

$$\lambda : \begin{array}{ccc} E = \bigoplus_{i=0}^n S(-1) & \to & S \\ (f_0, \dots, f_n) & \mapsto & \sum_{i=0}^n T_i f_i, \end{array}$$

cf. [23, II.8.13].

iii) This follows also from Lemma 2.5.1.1 since $\Omega^i_{\mathbb{P}_k^n/k} = \bigwedge^i \Omega^1_{\mathbb{P}_k^n/k}$ is the $\mathcal{O}_{\mathbb{P}_k^n}$-module associated with the $\mathbf{P}_{(1,n)}$-module $\bigwedge^i \left(\mathrm{Lie}(\mathbf{G}_k/\mathbf{P}_{(1,n),k})'\right)$, cf. [32, II.4.1(3)].

□

Remark. The sheaves $\mathcal{F} = \mathcal{O}_{\mathbb{P}_k^n}(a)$ for $a < -1$ and $\mathcal{F} = \Omega^i_{\mathbb{P}_k^n/k}, i = 1, \dots, n$, do not – in general – have the property that they are associated with S-modules generated in degrees ≤ 1. Therefore, Proposition 2.5.1.3 does not apply for them.

2.6.3 The (Twisted) Structure Sheaf

In this subsection, the case of positively twisted structure sheaves is considered once more. Thus, for $a \in \mathbb{N}_0$, let

$$\mathcal{F} = \mathcal{O}_{\mathbb{P}_k^n}(a) = S(a)^\sim,$$

endowed with the natural $\mathbf{G}_k$-linearization corresponding to the action

$$\mathbf{G}_k \times S(a) \to S(a)$$

which is just the shift of the action of $\mathbf{G}_k$ on S considered in Subsection 2.4.1, Example II. In this case,

$$\mathrm{H}^*(\mathbb{P}_k^n, \mathcal{F}) = \mathrm{H}^0(\mathbb{P}_k^n, \mathcal{F}) = S_a = \mathbf{Sym}^a((k^{n+1})')$$

so that

$$\begin{aligned}\tilde{\mathrm{H}}^{n-j}_{\mathbb{P}^j_k}(\mathbb{P}^n_k, \mathcal{F}) &= (S(a)_{T_{j+1}\cdot\ldots\cdot T_n})^{\rho}_0 \\ &\cong \bigoplus_{\substack{l_0,\ldots,l_n \in \mathbb{N}_0 \\ l_{j+1},\ldots,l_n > 0 \\ \sum_{u=0}^{j} l_u - \sum_{v=j+1}^{n} l_v = a}} k \frac{T_0^{l_0}\cdot\ldots\cdot T_j^{l_j}}{T_{j+1}^{l_{j+1}}\cdot\ldots\cdot T_n^{l_n}},\end{aligned}$$

see Subsection 2.3.1. Thus the proof of Proposition 2.5.1.3 gives

$$N_j = \left\{ \left[\frac{f}{T_{j+1}\cdot\ldots\cdot \mathrm{T}_n}\right] \;\middle|\; f \in S(a)_{n-j} = S_{a+n-j} \right\} \subset (S(a)_{T_{j+1}\cdot\ldots\cdot T_n})^{\rho}_0,$$

which, as an $\mathbf{L}_{(j+1,n-j),k}$-module, is isomorphic to $\mathbf{Sym}^{a+n-j}((k^{j+1})') \boxtimes \det^{-1}$, cf. Proposition 2.2.2.3.

Since the higher cohomology groups of $\mathcal{F}$ on $\mathbb{P}^n_k$ vanish, the description of the semisimplifications of the subquotients associated G-filtration $\mathcal{F}(\mathcal{X}^{(n+1)})^{\bullet}$ amounts to giving isomorphisms

$$\begin{aligned} & (\mathcal{F}(\mathcal{X}^{(n+1)})^j / \mathcal{F}(\mathcal{X}^{(n+1)})^{j+1})^{G-ss.} \\ \xrightarrow{\sim} \quad & \mathrm{Ind}^G_{P_{(j+1,n-j)}} \left(\left({}^{\mathbf{L}}\widehat{U}(\mathfrak{u}^+_{(j+1,n-j),k}) \otimes_k N_j/\mathfrak{d}_j \right) \otimes_k \mathrm{St}_{n-j}(k) \right)^{G-ss.}. \end{aligned} \tag{2.17}$$

One can now – at least theoretically – go ahead and first determine the structure of $\mathfrak{d}_j$ as a $L_{(j+1,n-j)}$-module, then determine the $L_{(j+1,n-j)}$-structure of the full argument of the above induction and finally deduce from this the composition factors of $(\mathcal{F}(\mathcal{X}^{(n+1)})^j/\mathcal{F}(\mathcal{X}^{(n+1)})^{j+1})^{G-ss.}$. For example, there is a formula which allows to compute the multiplicity with which the Steinberg representation St_G appears in a given homogeneous component S_l, see [35], but for general simple G-modules, this question is still open (see also [29, 19.5-6] and the comments made there).

A Complete Description in the Case $\mathbf{G} = \mathrm{GL}_2, k = \mathbb{F}_p, \mathcal{F} = \mathcal{O}_{\mathbb{P}^1_k}$

In the case $\mathbf{G} = \mathrm{GL}_2$, $k = \mathbb{F}_p$ one can give a complete description by hand which shall be exemplified for the case $a = 0$.

Assume for simplicity that $p \neq 2$. For

$$(a, b) \in \{0, \ldots, p-2\} \times \{0, \ldots, p-2\}$$

denote by $\chi_{a,b}$ the 1-dimensional T-module over k defined by the character

$$T \to k^{\times}, (t_1, t_2) \mapsto t_1^a t_2^b.$$

For $l \in \mathbb{Z}$, write $\bar{l}$ for the representative in $\{0, \ldots, p-2\}$ of $l + (p-1)\mathbb{Z} \in \mathbb{Z}/(p-1)\mathbb{Z}$. The formula (2.17) collapses to giving an isomorphism

$$\mathrm{H}^0(\mathcal{X}^{(1)}, \mathcal{F})^{G-ss.} \cong \left(\mathrm{Ind}^G_{P_{(1,1)}} \left(\bigoplus_{l \in \mathbb{N}} \chi_{\bar{l}, \overline{-l}} \right) \right)^{G-ss.}:$$

The module $\mathrm{St}_1(k)$ is the trivial representation of $P_{(1,1)}$, the module N_0 equals $\left\{ \left[\frac{\alpha T_0}{T_1}\right] \mid \alpha \in k \right\}$, and the map

$$\varphi_0 : {}^{\mathbf{L}}\widehat{U}(\mathfrak{u}^+_{(1,1),k})_c \otimes_k N_0 \to \tilde{\mathrm{H}}^1_{\mathbb{P}^0_k}(\mathbb{P}^1_k, \mathcal{F})$$

is induced by $L_{(0,1)}^{[l]} \otimes \left[\frac{T_0}{T_1}\right] \mapsto (-1)^l \left[\frac{T_0^{l+1}}{T_1^{l+1}}\right]$ and thus an isomorphism. Note that ${}^{\mathbf{L}}\widehat{U}(\mathfrak{u}_{(1,1),k}^+)_c = \mathrm{Dist}(\mathbf{U}_{(1,1),k}^+)$, cf. Example 2.4.2.6.

The set of isomorphism classes of simple G-modules in this case is parametrized by the set

$$M = \{(u,v) \in \mathbb{N}_0 \times \mathbb{N}_0 \mid 0 \le u \le p-2, 0 \le v-u \le p-1\}.$$

For $(u,v) \in M$ denote the associated simple G-module by $L(u,v)$. It is realized as a submodule of S via

$$L(u,v) = \mathbf{Sym}^{v-u}((k^2)') \otimes \det{}^u.$$

In particular, its dimension is $v-u$. Using Frobenius reciprocity, one can now check that for $a,b \in \{0,\dots,p-2\}, a \le b$, the composition factors of $\mathrm{Ind}_{P_{(1,1)}}^G(\chi_{a,b})$ are $L(a,a) = \det^a$ and $L(p-1+a,a) = \mathrm{St}_G \otimes \det^a$ in case $a=b$ resp. $L(a,b)$ and $L(b,p-1+a)$ in case $a<b$ (see e.g. [24, Lemma 10]). The composition factors of $\mathrm{Ind}_{P_{(1,1)}}^G(\chi_{b,a})$ are then known by duality. This then gives the composition factors of $\mathrm{Ind}_{P_{(1,1)}}^G(\chi_{l,p-1-l})$ for $l = 0,\dots,p-2$ and thus of $\mathrm{H}^0(\mathcal{X}^{(1)},\mathcal{F})^{G-ss.}$:

- If $l=0$, then the composition factors are $L(0,0) = \det^0$ and $L(p-1,0) = \mathrm{St}_G$.
- If $l \in \{1,\dots,\frac{p-1}{2}\}$, then the composition factors are $L(l,p-1-l)$ and $L(p-1-l,p-1+l)$.
- If $l \in \{\frac{p-1}{2}+1,\dots,p-2\}$, then the composition factors are the respective duals of the ones just given.

Each factor appears with infinite multiplicity.

Note that one can also use the identifications $\mathcal{X}^{(1)} = \mathbb{P}_k^1 \setminus \mathbb{P}_k^1(k) \cong \mathbb{A}_k^1 \setminus \mathbb{A}_k^1(k)$ and then rather easily verify the existence of an isomorphism of the above type.

Negatively Twisted Structure Sheaves

There are of course many interesting bundles $\mathcal{F} = F^\sim$ on $\mathbb{P}_k^n$ which do not fulfill the condition that F is generated in degrees ≤ 1. Among them are the negatively twisted structure sheaves $\mathcal{F} = \mathcal{O}(-a) = S(-a)^\sim$ for $a > 1$, as $F = S(-a)$ as an S-module is generated in degree a. The cohomology of $\mathcal{F}$ on $\mathbb{P}_k^n$ is via Serre duality given by

$$\mathrm{H}^*(\mathbb{P}_k^n,\mathcal{F}) = \mathrm{H}^n(\mathbb{P}_k^n,\mathcal{F}) = \det{}^{-1} \otimes \mathbf{Sym}^{a-(n+1)}((k^{n+1})')'.$$

Furthermore, $(F_{T_{j+1}\cdot\ldots\cdot T_n})_0^\rho$ (as a k-vector space) is isomorphic to

$$\bigoplus_{\substack{l\in\mathbb{N} \\ i_0,\dots,i_n\in\mathbb{N}_0 \\ i_{j+1},\dots,i_n<l \\ i_0+\ldots+i_n+a=l(n-j)}} k\frac{T_0^{i_0}\cdot\ldots\cdot T_n^{i_n}}{T_{j+1}^l\cdot\ldots\cdot T_n^l}.$$

For example in the case $(n,j) = (1,0)$, the latter space is isomorphic to $\frac{1}{X_1^a}\cdot k[\frac{X_0}{X_1}]$ and the element $L_{(0,1)} \in \mathfrak{g}$ acts on a basis element via

$$L_{(0,1)}\frac{X_0^i}{X_1^{a+i}} = -(a+i)\frac{X_0^{i+1}}{X_1^{a+i+1}}.$$

Therefore, the enriched enveloping algebra $\widehat{U}(\mathfrak{u}^+_{(1,1),k})_c$ (which is actually equal to $\mathrm{Dist}(\mathbf{U}^+_{(1,1),k})$ in this particular case) would have to be replaced by (the base-change to k of) an algebra[5] which – over $\mathcal{V}$ – is generated by expressions of the form

$$\frac{(a-1)!}{(a+l-1)!}L^l_{(0,1)} \ (l \in \mathbb{N}_0)$$

in order to adapt the proof of Lemma 2.5.1.3. But this algebra does not act on $S(-a)$ (or S, for that matter) compatibly with $\mathbf{G}_k$, $\mathfrak{g}_k$, etc. and therefore, this method does not work.

[5]See the construction of the algebra $\widehat{U}(\mathfrak{g})$ and its subalgebras above for an exact description.

Chapter 3

Rigid Cohomology of Drinfeld's Upper Half Space over a Finite Field

This chapter is devoted to the calculation of the rigid cohomology modules of $\mathcal{X}^{(n+1)}$ with coefficients in K.

First of all, the construction of this cohomology theory shall be recalled briefly. Throughout this chapter, there will be made use of the concepts of rigid-analytic spaces over K (originally due to Tate, cf. e.g. [4, 12]) and of adic spaces over K in the sense of Huber (cf. e.g. [26, 27]). The following notation and facts will be used: For a K-variety X, denote by X^{rig} its associated rigid-analytic K-variety (as in [4, 9.3.4]) and by X^{ad} its associated adic space (as in [26, Par. 4]). If X is a rigid-analytic K-variety, its associated adic space will also be denoted by X^{ad}, cf. loc. cit. To be a little more precise, recall that there are the following functors:

- $(-)^{\text{rig}}$ from the category of K-varieties to the category of rigid analytic K-varieties,
- $(-)^{\text{ad}}$ from the category of rigid analytic K-varieties to the category of adic spaces over K,
- $(-)^{\text{ad}}$ from the category of formal schemes over K to the category of adic spaces over K,

such that for a K-variety X, there is an isomorphism $(X^{\text{rig}})^{\text{ad}} \cong X^{\text{ad}}$ (where, on the right-hand side, X is considered as a formal scheme). Furthermore, each of these functors induces an equivalence of the respective topoi of sheaves, which are, respectively, the topos of sheaves on the Zariski topology of a K-variety, the topos of sheaves on the Grothendieck site associated with a rigid-analytic K-variety, and the topos of sheaves of an adic space over K. So in particular, for a rigid-analytic K-variety X and a sheaf $\mathcal{F}$ on its Grothendieck topology, there is an isomorphism

$$\mathrm{H}^i(X, \mathcal{F}) \cong \mathrm{H}^i(X^{\text{ad}}, \mathcal{F}^{\text{ad}})$$

of cohomology groups, where $\mathcal{F}^{\text{ad}}$ denotes the sheaf on X^{ad} induced by $\mathcal{F}$ via the above equivalence. For all facts mentioned in this paragraph, cf. [26, Par. 4] resp. [27, 1.1.11-12].

Concerning this thesis, the main advantage of working in the category of adic spaces at times instead of the category of rigid analytic spaces is that the former are in fact topological spaces (cf. their definition in, for example, [27, Ch. 1]). In particular, this means that sheaves on adic spaces are well-behaved with respect to localization in points.

Throughout this chapter, fix a completion $\widehat{\overline{K}}$ of an algebraic closure $\overline{K}$ of K and also denote by $|\ | : \widehat{\overline{K}} \to \mathbb{R}_{\geq 0}$ the uniquely determined extension of $|\ |$ to $\widehat{\overline{K}}$.

3.1 Construction of Rigid Cohomology and some Properties

For this section, references are for example [2, 37].

3.1.1 Berthelot's Definition of Rigid Cohomology (with and without Supports)

Let $P = \widehat{\mathbb{P}^n_{\mathcal{V}}}$ be the formal completion of $\mathbb{P}^n_{\mathcal{V}}$ along its special fiber $\mathbb{P}^n_k$. There is then a closed embedding $\mathbb{P}^n_k \hookrightarrow P$, which is a homeomorphism of the underlying topological spaces. Furthermore, there is a map

$$\mathrm{sp}^{\mathrm{rig}} : \mathbb{P}^{n,\mathrm{rig}}_K \to \mathbb{P}^n_k,$$

called specialization map, which on points is given by

$$[x_0 : \ldots : x_n] \mapsto [\bar{x}_0 : \ldots : \bar{x}_n] \in \mathbb{P}^n_k$$

for $[x_0 : \ldots : x_n]$ unimodular, i.e. $|x_i| \leq 1$ for all $i \in \{0, \ldots, n\}$ and $|x_j| = 1$ for at least one $j \in \{0, \ldots, n\}$. Here and in the sequel, $\bar{x}$ denotes the element of $\overline{k}$ defined by an element x in the ring of integers of $\widehat{\overline{K}}$ by reduction modulo its maximal ideal.

Let $X \subset \mathbb{P}^n_k$ be a locally closed (quasi-projective) smooth k-subvariety. By definition, there is a closed subvariety $Y \subset \mathbb{P}^n_k$ together with an open embedding $X \subset Y$. Set

$$]X[_P = (\mathrm{sp}^{\mathrm{rig}})^{-1}(X).$$

This is a rigid-analytic subvariety of $\mathbb{P}^{n,\mathrm{rig}}_K$, called the tube of X (of radius 1). For the purposes of this work, it will suffice to assume that either $Y = \mathbb{P}^n_k$ or $X = Y$, i.e. X is either open or closed in $\mathbb{P}^n_k$.

Suppose first that X is open in $\mathbb{P}^n_k = Y$ and denote by $Z = \mathbb{P}^n_k \setminus X$ its closed complement. A strict open neighborhood of $]X[_P$ in $\mathbb{P}^{n,\mathrm{rig}}_K$ is an admissible open subset $V \subset \mathbb{P}^{n,\mathrm{rig}}_K$ such that $]X[_P \subset V$ and such that $(V,]Z[_P)$ is an admissible covering of $\mathbb{P}^{n,\mathrm{rig}}_K$. The category of coefficients of rigid cohomology is the category of overconvergent F-isocrystals on X/K whose objects can be briefly described as follows: An overconvergent F-isocrystal on X/K is a locally free coherent $\mathcal{O}_V$-module $\mathcal{F}$ on some strict open neighborhood V of $]X[_P$ in $\mathbb{P}^{n,\mathrm{rig}}_K$ together with an integrable connection $\mathcal{F} \to \mathcal{F} \otimes_{\mathcal{O}_V} \Omega^1_V$ such that certain overconvergence conditions are fulfilled and such that locally, there is an isomorphism $F^*\mathcal{F} \xrightarrow{\sim} \mathcal{F}$, where F denotes a local lift of the absolute (q-power) Frobenius[1] on $\mathbb{P}^n_k$ to $\mathbb{P}^{n,\mathrm{rig}}_K$, cf. e.g. [2, § 4] or [37, § 7-8]. Then the rigid cohomology of X with values in $\mathcal{F}$ is defined as the hypercohomology

$$\mathrm{H}^*_{\mathrm{rig}}(X/K, \mathcal{F}) = \mathbb{H}^*\left(\mathbb{P}^{n,\mathrm{rig}}_K, \varinjlim_{V'} (j_{V'*}) \left(j^*_{V',V}\right) (\mathcal{F} \otimes_{\mathcal{O}_V} \Omega^\bullet_V)\right).$$

Here, the limit is taken over all strict open neighborhoods V' of $]X[_P$ in $\mathbb{P}^{n,\mathrm{rig}}_K$ which are contained in V and

$$\begin{aligned} j_{V',V} &: V' \hookrightarrow V, \\ j_{V'} &: V' \hookrightarrow \mathbb{P}^{n,\mathrm{rig}}_K \end{aligned}$$

are the respective embeddings of admissible open subsets given by inclusion. In the case of the trivial isocrystal, i.e. when $\mathcal{F}$ is just the structure sheaf, then write $\mathrm{H}^*_{\mathrm{rig}}(X/K)$ for the resulting rigid cohomology.

If X is closed in $\mathbb{P}^n_k$, then an overconvergent F-isocrystal on X/K is an $\mathcal{O}_{]X[_P}$-module with the above additional properties and the definition of rigid cohomology of X with values in $\mathcal{F}$ simplifies to

$$\mathrm{H}^*_{\mathrm{rig}}(X/K, \mathcal{F}) = \mathbb{H}^*(]X[_P, \mathcal{F} \otimes_{\mathcal{O}_{]X[_P}} \Omega^\bullet_{]X[_P}).$$

In particular, in the case of the trivial isocrystal, its rigid cohomology on X is just the usual de Rham cohomology of $]X[_P$.

[1]Note that in the context of varieties over k, the absolute and relative Frobenius morphisms are identical.

There is a notion of rigid cohomology "with compact supports", written $\mathrm{H}^*_{\mathrm{rig,c}}(X/K,\mathcal{F})$. Here, $\mathcal{F}$ (as above) is replaced by

$$\underline{\Gamma}(\mathcal{F}) = \ker(\mathcal{F} \to \iota_*\iota^*\mathcal{F}),$$

where $\iota : V \cap]Z[_P \hookrightarrow V$ is the inclusion, and then, by definition,

$$\mathrm{H}^*_{\mathrm{rig,c}}(X/K,\mathcal{F}) = \mathbb{H}^*(V, \mathbf{R}\underline{\Gamma}(\mathcal{F} \otimes_{\mathcal{O}_V} \Omega^\bullet_V)).$$

Again, in the case that $\mathcal{F}$ is trivial, the notation reduces to $\mathrm{H}^*_{\mathrm{rig,c}}(X/K)$.

3.1.2 Some Properties of Rigid Cohomology

Here are some facts concerning rigid cohomology[2].

- The above definitions of rigid cohomology and of rigid cohomology with compact supports are essentially (i.e. up to canonical isomorphism) independent of all choices made.
- For each $i \in \mathbb{N}_0$, the K-spaces $\mathrm{H}^i_{\mathrm{rig}}(X/K,\mathcal{F})$ and $\mathrm{H}^i_{\mathrm{rig,c}}(X/K,\mathcal{F})$ are finite-dimensional.
- Via functoriality, the absolute Frobenius morphism of X acts on each $\mathrm{H}^i_{\mathrm{rig}}(X/K,\mathcal{F})$ and each $\mathrm{H}^i_{\mathrm{rig,c}}(X/K,\mathcal{F})$. Therefore, the absolute Galois group $\mathrm{Gal}(\overline{k}/k)$ of k acts on each of the spaces $\mathrm{H}^i_{\mathrm{rig}}(X/K,\mathcal{F})$ and $\mathrm{H}^i_{\mathrm{rig,c}}(X/K,\mathcal{F})$ as well, upon identification of the absolute Frobenius with the arithmetic Frobenius automorphism $\sigma : x \mapsto x^q$ of $\overline{k}$.
- If $X = X_1 \sqcup X_2$ is a disjoint union in the category of k-varieties, then there are canonical isomorphisms

  $$\mathrm{H}^i_{\mathrm{rig}}(X/K,\mathcal{F}) \cong \mathrm{H}^i_{\mathrm{rig}}(X_1/K,\mathcal{F}) \oplus \mathrm{H}^i_{\mathrm{rig}}(X_2/K,\mathcal{F})$$
  $$\mathrm{H}^i_{\mathrm{rig,c}}(X/K,\mathcal{F}) \cong \mathrm{H}^i_{\mathrm{rig,c}}(X_1/K,\mathcal{F}) \oplus \mathrm{H}^i_{\mathrm{rig,c}}(X_2/K,\mathcal{F})$$

  for $i \in \mathbb{Z}$.
- If X is of dimension n, then $\mathrm{H}^i_{\mathrm{rig,c}}(X/K,\mathcal{F}) = 0$ for $i > 2n$.
- For each $i \in \mathbb{Z}$, there is a canonical $\mathrm{Gal}(\overline{k}/k)$-equivariant homomorphism $\mathrm{H}^i_{\mathrm{rig,c}}(X/K,\mathcal{F}) \to \mathrm{H}^i_{\mathrm{rig}}(X/K,\mathcal{F})$ which is an isomorphism if X is proper.
- Let $Z \subset X$ be a closed subvariety and let $U = X \setminus Z$ be its open complement. Then there is a long exact sequence

  $$\ldots \to \mathrm{H}^{i-1}_{\mathrm{rig,c}}(Z/K,\mathcal{F}) \to \mathrm{H}^i_{\mathrm{rig,c}}(U/K,\mathcal{F}) \to \mathrm{H}^i_{\mathrm{rig,c}}(X/K,\mathcal{F}) \to \mathrm{H}^i_{\mathrm{rig,c}}(Z/K,\mathcal{F}) \to \ldots$$

  which is $\mathrm{Gal}(\overline{k}/k)$-equivariant.
- If X is of dimension n, then for each $i \in \{0,\ldots,2n\}$, there is a perfect pairing

  $$\mathrm{H}^i_{\mathrm{rig}}(X/K,\mathcal{F}) \times \mathrm{H}^{2n-i}_{\mathrm{rig,c}}(X/K,\mathcal{F}^\vee) \to K(-n)[-2n]$$

  of K-spaces which is $\mathrm{Gal}(\overline{k}/k)$-equivariant, where $\mathcal{F}^\vee$ is the isocrystal dual to $\mathcal{F}$ (defined in the usual way). Here and in the sequel, $(-)(l)$ means that the Frobenius automorphism σ acts by multiplication with q^{-l} ("l-th Tate twist") and $(-)[j]$ means "shift in degree $-j$". For an arbitrary K-vector space V and an integer $i \in \mathbb{Z}$, set

  $$V(i) = V \otimes_K K(i),$$

  i.e. $\mathrm{Gal}(\overline{k}/k)$ acts on this space through its action on $K(i)$.

[2]For attribution of the respective results, the interested reader should have a look at [37, 9.1], for example.

- If X is affine of dimension n, then $\mathrm{H}^i_{\mathrm{rig}}(X/K, \mathcal{F}) = 0$ for $i \notin \{0, \dots, n\}$.
- Let $n \in \mathbb{N}_0$. Then there are the following identifications:

$$\begin{aligned} \mathrm{H}^*_{\mathrm{rig}}(\mathbb{P}^n_k/K) &= \bigoplus_{i=0}^{n} K(-i)[-2i], \\ \mathrm{H}^*_{\mathrm{rig,c}}(\mathbb{A}^n_k/K) &= K(-n)[-2n], \\ \mathrm{H}^*_{\mathrm{rig}}(\mathbb{A}^n_k/K) &= K(0)[0]. \end{aligned} \tag{3.1}$$

3.2 Rigid Cohomology computed as Hypercohomology

In this section, the rigid cohomology $\mathrm{H}^*_{\mathrm{rig}}(\mathcal{Y}^{(n+1)}/K) = \mathrm{H}^*_{\mathrm{rig,c}}(\mathcal{Y}^{(n+1)}/K)$ of the closed complement $\mathcal{Y}^{(n+1)}$ of $\mathcal{X}^{(n+1)}$ in $\mathbb{P}^n_k$ is computed directly as a hypercohomology. Then, the long exact sequence for rigid cohomology with compact supports for the pair of inclusions

$$\mathcal{X}^{(n+1)} \overset{\text{open}}{\hookrightarrow} \mathbb{P}^n_k \overset{\text{closed}}{\hookleftarrow} \mathcal{Y}^{(n+1)}$$

is used to determine $\mathrm{H}^*_{\mathrm{rig,c}}(\mathcal{X}^{(n+1)}/K)$. The rigid cohomology of $\mathcal{X}^{(n+1)}$ is then known via Poincaré duality.

3.2.1 Adaption of Orlik's Complex

In order to compute $\mathrm{H}^*_{\mathrm{rig}}(\mathcal{Y}^{(n+1)}/K)$, there will be made use of the following modified version of Orlik's complex: Use the same notation for objects Y_I with $I \subsetneq \Delta$ as in Section 2.1.1, associate to each object its tube and then the respective adic space, i.e. there are adic spaces $]\mathcal{Y}^{(n+1)}[^{\mathrm{ad}}_P$ and $]g.Y_I[^{\mathrm{ad}}_P$ for $I \subsetneq \Delta$, $g \in G$, all of which are open in $\mathbb{P}^{n,\mathrm{ad}}_K$. The closed embeddings of k-varieties

$$\begin{aligned} \iota^{g,h}_{I,J} &: g.Y_I \hookrightarrow h.Y_J, \\ \Phi_{g,I} &: g.Y_I \hookrightarrow \mathcal{Y}^{(n+1)} \end{aligned}$$

considered in 2.1.1 for $I \subset J \subsetneq \Delta$ and $g, h \in G$ with $gP_I \mapsto hP_J$ give rise to open embeddings of adic spaces associated with the respective tubes

$$\begin{aligned}]g.Y_I[^{\mathrm{ad}}_P &\hookrightarrow]h.Y_J[^{\mathrm{ad}}_P, \\ ]g.Y_I[^{\mathrm{ad}}_P &\hookrightarrow]\mathcal{Y}^{(n+1)}[^{\mathrm{ad}}_P. \end{aligned}$$

Denote by

$$\mathrm{sp}^{\mathrm{ad}} : \mathbb{P}^{n,\mathrm{ad}}_K \to \mathbb{P}^n_k$$

the adic specialization map. It is continuous, cf. [27, 1.9] resp. [26, Section 4], and for a locally closed subvariety $X \subset \mathbb{P}^n_k$, the adification $]X[^{\mathrm{ad}}_P$ of its rigid-analytic tube identifies with the interior $(\mathrm{sp}^{\mathrm{ad}})^{-1}(X)^o$ of its adic tube, cf. [27, Lemma 5.6.9]. For each $I \subset J$ and $gP_I \mapsto hP_J$ as above, there is then a commutative triangle

$$\begin{array}{ccc} & & (\mathrm{sp}^{\mathrm{ad}})^{-1}(\mathcal{Y}^{(n+1)}) \\ & \overset{\Phi^{\mathrm{ad}}_{g,I}}{\nearrow} & \uparrow \Phi^{\mathrm{ad}}_{h,J} \\ (\mathrm{sp}^{\mathrm{ad}})^{-1}(g.Y_I) & \xrightarrow{\iota^{g,h,\mathrm{ad}}_{I,J}} & (\mathrm{sp}^{\mathrm{ad}})^{-1}(h.Y_J) \end{array} \tag{3.2}$$

of closed embeddings of closed subspaces of $(\mathrm{sp}^{\mathrm{ad}})^{-1}(\mathcal{Y}^{(n+1)})$. Here, the maps appearing in the triangle are again induced by the respective maps of k-varieties above. Let $\mathcal{E}$ be a sheaf of abelian groups on $(\mathrm{sp}^{\mathrm{ad}})^{-1}(\mathcal{Y}^{(n+1)})$. As in the situation of Subsection 2.1.1, one then obtains a complex

$$0 \to \mathcal{E} \to \bigoplus_{\substack{I \subsetneq \Delta \\ \#I=n-1}} \mathcal{E}_I \to \ldots \to \bigoplus_{\substack{I \subsetneq \Delta \\ \#I=1}} \mathcal{E}_I \to \mathcal{E}_\emptyset \to 0 \tag{3.3}$$

of sheaves on $(\mathrm{sp}^{\mathrm{ad}})^{-1}(\mathcal{Y}^{(n+1)})$ where, as in loc. cit.

$$\mathcal{E}_I = \bigoplus_{g \in G/P_I} \mathcal{E}_{g,I}$$

with

$$\mathcal{E}_{g,I} = (\Phi^{\mathrm{ad}}_{g,I})_*(\Phi^{\mathrm{ad}}_{g,I})^{-1}\mathcal{E}.$$

Proposition 3.2.1.1. (cf. [42, Satz 5.3]) *The complex (3.3) is acyclic.*

Proof. For $i \in \{0, \ldots, n-1\}$, set

$$\mathcal{E}_i = \bigoplus_{\substack{I \subsetneq \Delta \\ \#I=n-1-i}} \mathcal{E}_I.$$

As was mentioned in the introduction to this chapter, sheaves on adic spaces are in particular sheaves on topological spaces. Therefore, to prove the proposition, it is sufficient to check that for each point $x \in (\mathrm{sp}^{\mathrm{ad}})^{-1}(\mathcal{Y}^{(n+1)})$, the localized complex

$$(0) \to (\mathcal{E})_x \to (\mathcal{E}_0)_x \to (\mathcal{E}_1)_x \to \ldots \to (\mathcal{E}_{n-1})_x \to (0)$$

is acyclic. By construction, this last complex is equal to

$$(0) \to \mathcal{E}_x \to \bigoplus_{\substack{I \subsetneq \Delta \\ \#I=n-1}} \bigoplus_{\substack{g \in G/P_I \\ x \in (\mathrm{sp}^{\mathrm{ad}})^{-1}(g.Y_I)}} \mathcal{E}_x \to \ldots \to \bigoplus_{\substack{I \subsetneq \Delta \\ \#I=1}} \bigoplus_{\substack{g \in G/P_I \\ x \in (\mathrm{sp}^{\mathrm{ad}})^{-1}(g.Y_I)}} \mathcal{E}_x \to \bigoplus_{\substack{g \in G/\mathbf{P}_\emptyset(k) \\ x \in (\mathrm{sp}^{\mathrm{ad}})^{-1}(g.Y_\emptyset)}} \mathcal{E}_x \to (0). \tag{3.4}$$

Let

$$X = \left\{ gP_I \in G/P_I \;\middle|\; \#I = n-1, x \in (\mathrm{sp}^{\mathrm{ad}})^{-1}(g.Y_I) \right\}$$

and equip X with a partial order structure "$\leq$" by identifying

$$\coprod_{\substack{I \subset \Delta \\ \#I=n-1}} G/P_I$$

with

$$\left\{ U \subset k^{n+1} \ k\text{-subspace} \mid U \neq (0), k^{n+1} \right\}$$

and its natural partial order structure given by inclusion. Then X is identified with the set

$$\left\{ U \subset k^{n+1} \ k\text{-subspace} \;\middle|\; U \neq (0), k^{n+1} \text{ and } x \in (\mathrm{sp}^{\mathrm{ad}})^{-1}(\mathbb{P}(U)) \right\}.$$

Furthermore, X is nonempty since $(\mathrm{sp}^{\mathrm{ad}})^{-1}(\mathcal{Y}^{(n+1)})$ is covered by the union of all $(\mathrm{sp}^{\mathrm{ad}})^{-1}(\mathbb{P}(U))$ with U running through all proper subspaces of k^{n+1}.

For $i \in \{0, \ldots, n-1\}$, let

$$X^i = \{(x_0, \ldots, x_i) \mid \forall j \in \{0, \ldots, i\} : x_j \in X, x_0 < x_1 < \ldots < x_i\}.$$

By construction, $X^\bullet = \bigcup_{i=0}^{n-1} X^i$ has the structure of a simplicial complex with X^i the set of i-simplices and (3.4) is the chain complex with values in $\mathcal{E}_x$ associated with this simplicial complex.

Let $U_0 \in X$ be a subspace of minimal dimension and let $U \in X$ be arbitrary. It follows that

$$x \in (\mathrm{sp}^{\mathrm{ad}})^{-1}(\mathbb{P}(U_0)) \cap (\mathrm{sp}^{\mathrm{ad}})^{-1}(\mathbb{P}(U)) = (\mathrm{sp}^{\mathrm{ad}})^{-1}(\mathbb{P}(U_0) \cap \mathbb{P}(U)) = (\mathrm{sp}^{\mathrm{ad}})^{-1}(\mathbb{P}(U_0 \cap U)).$$

so that $U \cap U_0 \neq (0)$ (as otherwise $\mathbb{P}(U \cap U_0)$ would be empty). By minimality of U_0, this implies that $U_0 \cap U = U_0$. Therefore, the identity map

$$\mathrm{id} : X \to X, U \mapsto U,$$

fulfills

$$U_0, U \leq \mathrm{id}(U)$$

for all $U \in X$ and by Quillen's criterion (cf. [49, 1.5]), the complex (3.4) is then acyclic. This proves the proposition. □

Corollary 3.2.1.2. *Let $\mathcal{E}^\bullet$ be a finite complex of sheaves of abelian groups on $(\mathrm{sp}^{\mathrm{ad}})^{-1}(\mathcal{Y}^{(n+1)})$. Then, the complex*

$$0 \to \mathcal{E}^\bullet \to \bigoplus_{\substack{I \subsetneq \Delta \\ \#I = n-1}} \mathcal{E}_I^\bullet \to \ldots \to \bigoplus_{\substack{I \subsetneq \Delta \\ \#I = 1}} \mathcal{E}_I^\bullet \to \mathcal{E}_\emptyset^\bullet \to 0 \tag{3.5}$$

is acyclic, i.e. (3.5) is an exact sequence in the category of complexes of sheaves of abelian groups on $(\mathrm{sp}^{\mathrm{ad}})^{-1}(\mathcal{Y}^{(n+1)})$.

3.2.2 Construction of a Spectral Sequence

Denote by

$$f : \ ]\mathcal{Y}^{(n+1)}[_P^{\mathrm{ad}} = (\mathrm{sp}^{\mathrm{ad}})^{-1}(\mathcal{Y}^{(n+1)})^o \hookrightarrow (\mathrm{sp}^{\mathrm{ad}})^{-1}(\mathcal{Y}^{(n+1)})$$

the canonical open immersion of the interior of $(\mathrm{sp}^{\mathrm{ad}})^{-1}(\mathcal{Y}^{(n+1)})$. For the rest of this section, let

$$\mathcal{E}^\bullet = \mathcal{O}_{(\mathrm{sp}^{\mathrm{ad}})^{-1}(\mathcal{Y}^{(n+1)})} \to \Omega^1_{(\mathrm{sp}^{\mathrm{ad}})^{-1}(\mathcal{Y}^{(n+1)})} \to \Omega^2_{(\mathrm{sp}^{\mathrm{ad}})^{-1}(\mathcal{Y}^{(n+1)})} \to \ldots \to \Omega^n_{(\mathrm{sp}^{\mathrm{ad}})^{-1}(\mathcal{Y}^{(n+1)})}$$

be the de Rham complex on $(\mathrm{sp}^{\mathrm{ad}})^{-1}(\mathcal{Y}^{(n+1)})$. Plug this complex into (3.5) and set

$$\begin{aligned} \mathcal{E}_{-1}^\bullet &= \mathcal{E}^\bullet, \\ \mathcal{E}_i^\bullet &= \bigoplus_{\substack{I \subsetneq \Delta \\ \#I = n-1-i}} \mathcal{E}_I^\bullet \end{aligned}$$

for $i = 0, \ldots, n-1$. The functor f^* is exact and therefore, the complex

$$0 \to f^*\mathcal{E}_{-1}^\bullet \to f^*\mathcal{E}_0^\bullet \to f^*\mathcal{E}_1^\bullet \to \ldots \to f^*\mathcal{E}_{n-1}^\bullet \to 0$$

(consisting of complexes of sheaves on $]\mathcal{Y}^{(n+1)}[_P^{\mathrm{ad}}$) is still acyclic. Application of the Godement functor $\mathcal{G}$ (cf. Section 1.4) to each sheaf appearing in (3.5) plus application of the global sections functor induces a first quadrant double complex of K-vector spaces

$$
\begin{array}{ccccc}
\Gamma(]\mathcal{Y}^{(n+1)}[_P^{\mathrm{ad}}, \mathcal{G}(f^*\mathcal{E}_0^n)) & \longrightarrow & \Gamma(]\mathcal{Y}^{(n+1)}[_P^{\mathrm{ad}}, \mathcal{G}(f^*\mathcal{E}_1^n)) & \longrightarrow \dots \longrightarrow & \Gamma(]\mathcal{Y}^{(n+1)}[_P^{\mathrm{ad}}, \mathcal{G}(f^*\mathcal{E}_{n-1}^n)) \\
\uparrow & & \uparrow & & \uparrow \\
\vdots & & \vdots & \vdots & \vdots \\
\uparrow & & \uparrow & & \uparrow \\
\Gamma(]\mathcal{Y}^{(n+1)}[_P^{\mathrm{ad}}, \mathcal{G}(f^*\mathcal{E}_0^1)) & \longrightarrow & \Gamma(]\mathcal{Y}^{(n+1)}[_P^{\mathrm{ad}}, \mathcal{G}(f^*\mathcal{E}_1^1)) & \longrightarrow \dots \longrightarrow & \Gamma(]\mathcal{Y}^{(n+1)}[_P^{\mathrm{ad}}, \mathcal{G}(f^*\mathcal{E}_{n-1}^1)) \\
\uparrow & & \uparrow & & \uparrow \\
\Gamma(]\mathcal{Y}^{(n+1)}[_P^{\mathrm{ad}}, \mathcal{G}(f^*\mathcal{E}_0^0)) & \longrightarrow & \Gamma(]\mathcal{Y}^{(n+1)}[_P^{\mathrm{ad}}, \mathcal{G}(f^*\mathcal{E}_1^0)) & \longrightarrow \dots \longrightarrow & \Gamma(]\mathcal{Y}^{(n+1)}[_P^{\mathrm{ad}}, \mathcal{G}(f^*\mathcal{E}_{n-1}^0))
\end{array}
$$

with exact rows.

There is then canonically a quasi-isomorphism of the complex $\Gamma(]\mathcal{Y}^{(n+1)}[_P^{\mathrm{ad}}, \mathcal{G}(f^*\mathcal{E}^\bullet))$ into the associated total complex of the above double complex which implies the existence of a spectral sequence

$$
\begin{array}{rcl}
E_1^{r,s} = h^s(\Gamma(]\mathcal{Y}^{(n+1)}[_P^{\mathrm{ad}}, \mathcal{G}(f^*\mathcal{E}_r^\bullet))) & \Longrightarrow & h^{r+s}(\Gamma(]\mathcal{Y}^{(n+1)}[_P^{\mathrm{ad}}, \mathrm{Tot}(\mathcal{G}(f^*\mathcal{E}_\bullet^\bullet)))) \\
& = & h^{r+s}(\Gamma(]\mathcal{Y}^{(n+1)}[_P^{\mathrm{ad}}, \mathcal{G}(f^*\mathcal{E}^\bullet))) \\
& = & h^{r+s}(\Gamma(]\mathcal{Y}^{(n+1)}[_P^{\mathrm{ad}}, f^*\mathcal{G}(\mathcal{E}^\bullet))) \\
& = & h^{r+s}(\Gamma(]\mathcal{Y}^{(n+1)}[_P^{\mathrm{ad}}, \mathcal{G}(\mathcal{E}^\bullet))).
\end{array}
$$

It follows from the description of hypercohomology in Section 1.4 and from the fact that $\mathcal{Y}^{(n+1)}$ is closed in $\mathbb{P}_k^n$, that

$$h^*(\Gamma(]\mathcal{Y}^{(n+1)}[_P^{\mathrm{ad}}, \mathcal{G}(\mathcal{E}^\bullet))) = \mathrm{H}^*_{\mathrm{rig}}(\mathcal{Y}^{(n+1)}/K),$$

see also Subsection 3.1.1.

3.2.3 Evaluation of the Spectral Sequence

The E_1-Page

Lemma 3.2.3.1. *Let $r, s \in \mathbb{Z}$. Then there is an identification*

$$E_1^{r,s} = \bigoplus_{\substack{I \subsetneq \Delta \\ \#I = n-1-r}} \mathrm{Ind}_{P_I}^G \mathrm{H}^s_{\mathrm{rig}}(Y_I/K).$$

Proof. Using compatibility of Godement resolution with direct and inverse images of sheaves in the present situation, one calculates as follows:

$$
\begin{array}{rcl}
E_1^{r,s} & = & h^s\left(\Gamma\left(]\mathcal{Y}^{(n+1)}[_P^{\mathrm{ad}}, \mathcal{G}\left(f^*\mathcal{E}_r^\bullet\right)\right)\right) \\
& = & h^s\left(\Gamma\left(]\mathcal{Y}^{(n+1)}[_P^{\mathrm{ad}}, \mathcal{G}\left(\mathcal{E}_r^\bullet\right)\right)\right) \\
& = & h^s\left(\Gamma\left(]\mathcal{Y}^{(n+1)}[_P^{\mathrm{ad}}, \mathcal{G}\left(\bigoplus_{\substack{I \subsetneq \Delta, \\ \#I = n-1-r}} \bigoplus_{g \in G/P_I} (\Phi_{g,I}^{\mathrm{ad}})_*(\Phi_{g,I}^{\mathrm{ad}})^*\mathcal{E}^\bullet\right)\right)\right)
\end{array}
$$

$$
\begin{aligned}
&= \bigoplus_{\substack{I\subsetneq\Delta,\\ \#I=n-1-r}} \bigoplus_{g\in G/P_I} h^s\left(\Gamma\left(]\mathcal{Y}^{(n+1)}[^{\mathrm{ad}}_P, \mathcal{G}\left((\Phi^{\mathrm{ad}}_{g,I})_*(\Phi^{\mathrm{ad}}_{g,I})^*\mathcal{E}^\bullet\right)\right)\right)\\
&= \bigoplus_{\substack{I\subsetneq\Delta,\\ \#I=n-1-r}} \bigoplus_{g\in G/P_I} h^s\left(\Gamma\left(]\mathcal{Y}^{(n+1)}[^{\mathrm{ad}}_P, (\Phi^{\mathrm{ad}}_{g,I})_*\mathcal{G}\left((\Phi^{\mathrm{ad}}_{g,I})^*\mathcal{E}^\bullet\right)\right)\right)\\
&= \bigoplus_{\substack{I\subsetneq\Delta,\\ \#I=n-1-r}} \bigoplus_{g\in G/P_I} h^s\left(\Gamma\left(]\mathcal{Y}^{(n+1)}[^{\mathrm{ad}}_P \cap (\mathrm{sp}^{\mathrm{ad}})^{-1}(g.Y_I), \mathcal{G}\left((\Phi^{\mathrm{ad}}_{g,I})^*\mathcal{E}^\bullet\right)\right)\right).
\end{aligned}
$$

By definition, $]\mathcal{Y}^{(n+1)}[^{\mathrm{ad}}_P \cap (\mathrm{sp}^{\mathrm{ad}})^{-1}(g.Y_I)$ is open in $(\mathrm{sp}^{\mathrm{ad}})^{-1}(g.Y_I)$ and therefore, it is contained in $(\mathrm{sp}^{\mathrm{ad}})^{-1}(g.Y_I)^o$. On the other hand, $(\mathrm{sp}^{\mathrm{ad}})^{-1}(g.Y_I)^o =]g.Y_I[^{\mathrm{ad}}_P$ is open in both $]\mathcal{Y}^{(n+1)}[^{\mathrm{ad}}_P$ and $(\mathrm{sp}^{\mathrm{ad}})^{-1}(g.Y_I)$ which implies

$$]\mathcal{Y}^{(n+1)}[^{\mathrm{ad}}_P \cap (\mathrm{sp}^{\mathrm{ad}})^{-1}(g.Y_I) =]g.Y_I[^{\mathrm{ad}}_P$$

for all $I \subsetneq \Delta$ and all $g \in G/P_I$. Again from the description of hypercohomology given in Section 1.4, it then follows that

$$
\begin{aligned}
E_1^{r,s} &= \bigoplus_{\substack{I\subsetneq\Delta,\\ \#I=n-1-r}} \bigoplus_{g\in G/P_I} h^s\left(\Gamma\left(]g.Y_I[^{\mathrm{ad}}_P, \mathcal{G}\left((\Phi^{\mathrm{ad}}_{g,I})^*\mathcal{E}^\bullet\right)\right)\right).\\
&= \bigoplus_{\substack{I\subsetneq\Delta,\\ \#I=n-1-r}} \bigoplus_{g\in G/P_I} \mathbb{H}^s\left(]g.Y_I[^{\mathrm{ad}}_P, \mathcal{E}^\bullet_{|]g.Y_I[^{\mathrm{ad}}_P}\right)\\
&= \bigoplus_{\substack{I\subsetneq\Delta,\\ \#I=n-1-r}} \bigoplus_{g\in G/P_I} \mathbb{H}^s\left(]g.Y_I[_P, \mathcal{E}^\bullet_{|]g.Y_I[_P}\right).
\end{aligned}
$$

For $g \in \mathbf{G}(\mathcal{V})$ also denote by g its induced automorphism on $\mathbb{P}^{n,\mathrm{rig}}_K$ via the action

$$\mathbf{G}(\mathcal{V}) \times \mathbb{P}^{n,\mathrm{rig}}_K \to \mathbb{P}^{n,\mathrm{rig}}_K, (g,x) \mapsto g.x = xg^{-1}.$$

Then the diagram

$$
\begin{array}{ccc}
\mathbb{P}^{n,\mathrm{rig}}_K & \xrightarrow{\;g\;} & \mathbb{P}^{n,\mathrm{rig}}_K \\
\Big\downarrow{\scriptstyle \mathrm{sp}^{\mathrm{rig}}} & & \Big\downarrow{\scriptstyle \mathrm{sp}^{\mathrm{rig}}} \\
\mathbb{P}^n_k & \xrightarrow{\;\overline{g}\;} & \mathbb{P}^n_k
\end{array}
$$

commutes, where $\overline{g}$ is the image of g under the canonical map $\mathbf{G}(\mathcal{V}) \to G$. Therefore,

$$
\begin{aligned}
g.]Y_I[_P &= g.\left\{x \in \mathbb{P}^{n,\mathrm{rig}}_K \mid \mathrm{sp}^{\mathrm{rig}}(x) \in Y_I\right\}\\
&= \left\{g.x \in \mathbb{P}^{n,\mathrm{rig}}_K \mid \mathrm{sp}^{\mathrm{rig}}(x) \in Y_I\right\}\\
&= \left\{x \in \mathbb{P}^{n,\mathrm{rig}}_K \mid \mathrm{sp}^{\mathrm{rig}}(g^{-1}.x) \in Y_I\right\}\\
&= \left\{x \in \mathbb{P}^{n,\mathrm{rig}}_K \mid (\overline{g})^{-1}.\mathrm{sp}^{\mathrm{rig}}(x) \in Y_I\right\}\\
&= \left\{x \in \mathbb{P}^{n,\mathrm{rig}}_K \mid \mathrm{sp}^{\mathrm{rig}}(x) \in \overline{g}.Y_I\right\}\\
&=]\overline{g}.Y_I[_P.
\end{aligned}
$$

Define $\mathbf{P}_I(\mathcal{V})(1)$ to be the preimage of P_I with respect to the above map $\mathbf{G}(\mathcal{V}) \to G$. Then $\mathbf{P}_I(\mathcal{V})(1)$ stabilizes $]Y_I[_P$. For $g \in \mathbf{G}(\mathcal{V})/\mathbf{P}_I(\mathcal{V})(1)$, set

$$g.\mathbb{H}^s(]Y_I[_P, \mathcal{E}^\bullet_{|]Y_I[_P}) = \mathbb{H}^s(g.]Y_I[_P, \mathcal{E}^\bullet_{|g.]Y_I[_P}).$$

This is well-defined and from the identity above, since $\mathbf{G}(\mathcal{V})/\mathbf{P}_I(\mathcal{V})(1) \cong G/P_I$, it then follows that

$$\begin{aligned} E_1^{r,s} &= \bigoplus_{\substack{I \subsetneq \Delta, \\ \#I=n-1-r}} \bigoplus_{g \in \mathbf{G}(\mathcal{V})/\mathbf{P}_I(\mathcal{V})(1)} \mathbb{H}^s\left(g.]Y_I[_P, \mathcal{E}^\bullet_{|g.]Y_I[_P}\right) \\ &= \bigoplus_{\substack{I \subsetneq \Delta, \\ \#I=n-1-r}} \bigoplus_{g \in \mathbf{G}(\mathcal{V})/\mathbf{P}_I(\mathcal{V})(1)} g.\mathbb{H}^s\left(]Y_I[_P, \mathcal{E}^\bullet_{|]Y_I[_P}\right) \\ &= \bigoplus_{\substack{I \subsetneq \Delta, \\ \#I=n-1-r}} \bigoplus_{g \in G/P_I} g.\mathrm{H}^s_{\mathrm{rig}}(Y_I/K) \\ &= \bigoplus_{\substack{I \subsetneq \Delta, \\ \#I=n-1-r}} \mathrm{Ind}^G_{P_I} \mathrm{H}^s_{\mathrm{rig}}(Y_I/K). \end{aligned}$$

The last identity holds since, by functoriality, $\mathrm{H}^s_{\mathrm{rig}}(Y_I/K)$ is a P_I-module and as a K-vector space, $g.\mathrm{H}^s_{\mathrm{rig}}(Y_I/K)$ is isomorphic to $\mathrm{H}^s_{\mathrm{rig}}(Y_I/K)$, cf. the descriptions of induced representations in Subsection 1.2.1. This finishes the proof. □

Directly from the definition, the fact that $Y_I \cong \mathbb{P}^{i_1}_k$ for $i_1 \in \{0, \ldots, n-1\}$ minimal with $\alpha_{i_1} \in \Delta \setminus I$, gives

$$\mathrm{H}^*_{\mathrm{rig}}(Y_I/K) = \bigoplus_{j=0}^{i_1} K(-j)[-2j],$$

see (3.1). For $j \in \{0, \ldots, n-1\}$, set

$$I_j = \begin{cases} \{\alpha_0, \ldots, \alpha_{j-1}\} & \text{if } j \in \{1, \ldots, n-1\} \\ \emptyset & \text{if } j = 0. \end{cases}$$

Then, a row $E_1^{\bullet,s}$ of the first page of the spectral sequence has non-zero entries only for even $s \in \{0, 2, \ldots, 2n-2\}$ and reads

$$\begin{aligned} E_1^{\bullet,s} \quad : \quad E_1^{0,s} = \bigoplus_{\substack{I_{\frac{s}{2}} \subset I \subsetneq \Delta \\ \#I=n-1}} \mathrm{Ind}^G_{P_I} K(-\tfrac{s}{2}) \to \ldots \to \bigoplus_{\substack{I_{\frac{s}{2}} \subset I \subsetneq \Delta \\ \#I=\frac{s}{2}+1}} \mathrm{Ind}^G_{P_I} K(-\tfrac{s}{2}) \\ \to E_1^{n-1-\frac{s}{2},s} = \mathrm{Ind}^G_{P_{I_{\frac{s}{2}}}} K(-\tfrac{s}{2}). \end{aligned}$$

The E_2-Page

For each proper subset $J \subsetneq \Delta$, the sequence

$$(0) \to K \to \bigoplus_{\substack{J \subset I \subsetneq \Delta \\ \#(\Delta \setminus I)=1}} \mathrm{Ind}^G_{P_I} K \to \ldots \to \bigoplus_{\substack{J \subset I \subsetneq \Delta \\ \#(\Delta \setminus I)=n-1-\#J}} \mathrm{Ind}^G_{P_I} K \to v^G_{P_J}(K) \to (0) \qquad (3.6)$$

of G-modules is exact, see e.g. [5, 3.2.5]. Therefore, the E_2-terms of the spectral sequence can be read off:

Lemma 3.2.3.2. *Let $r, s \in \mathbb{Z}$. Then there is an identification*

$$E_2^{r,s} = h^r(E^{\bullet,s}) = \begin{cases} v_{P_{I_{\frac{s}{2}}}}^G(K)(-\frac{s}{2}) & \text{if } s \in \{0, 2, \ldots, 2(n-2)\}, r = n-1-\frac{s}{2} \\ K(-\frac{s}{2}) & \text{if } s \in \{0, 2, \ldots, 2(n-2)\}, r = 0 \\ \mathrm{Ind}_{P_{I_{n-1}}}^G K(-(n-1)) & \text{if } s = 2n-2, r = 0 \\ (0) & \text{else.} \end{cases}$$

Taking into account the fact that there are no nontrivial homomorphisms of Galois modules of different Tate twist, one can now read off from the shape of the E_2-page that all resulting morphisms in E_m, $m \geq 2$, are necessarily trivial and therefore, the spectral sequence degenerates in the E_2-page.

3.2.4 Computation of the Rigid Cohomology Modules

Evaluation of the filtration associated with the spectral sequence now yields the rigid cohomology of $\mathcal{Y}^{(n+1)}$.

Proposition 3.2.4.1. *The rigid cohomology modules of $\mathcal{Y}^{(n+1)}$ have the following shape:*

$$\mathrm{H}_{\mathrm{rig}}^s(\mathcal{Y}^{(n+1)}/K) = \begin{cases} K(-\frac{s}{2}) & \text{if } s \in \{0, \ldots, n-2\} \text{ even} \\ K(-\frac{s}{2}) \oplus v_{P_{I_{n-1-s}}}^G(K)(n-1-s) & \text{if } s \in \{n-1, \ldots, 2n-3\} \text{ even} \\ v_{P_{I_{1+s-n}}}^G(K)(n-1-s) & \text{if } s \in \{n-1, \ldots, 2n-3\} \text{ odd} \\ \mathrm{Ind}_{P_{I_{n-1}}}^G(K)(-(n-1)) & \text{if } s = 2n-2 \\ (0) & \text{else.} \end{cases}$$

During the proof of this proposition, the following fact on extensions of Galois modules is needed. The proof is reproduced from [41].

Lemma 3.2.4.2. *Let $l, m \in \mathbb{Z}$ with $l \neq m$. Then every extension of the* $\mathrm{Gal}(\overline{k}/k)$*-module $K(l)$ by $K(m)$ splits, i.e.*

$$\mathrm{Ext}^1_{\mathrm{Gal}(\overline{k}/k)}(K(m), K(l)) = (0).$$

Proof. By exactness of Tate twist, it is enough to consider the case $l \neq 0$, $m = 0$. By definition of group cohomology as a derived functor cohomology, there is an identification

$$\mathrm{Ext}^1_{\mathrm{Gal}(\overline{k}/k)}(K(0), K(l)) = \mathrm{H}^1(\mathrm{Gal}(\overline{k}/k), K(l)).$$

Recall that σ denotes the standard arithmetic Frobenius automorphism in $\mathrm{Gal}(\overline{k}/k)$ which, in particular, is a topological generator of this group. Then the latter module is isomorphic to the module $K(l)/((\sigma - \mathrm{id})K(l))$, which is trivial for $l \neq 0$. □

Proof of Proposition 3.2.4.1. Recall that

$$h^*(\Gamma(]\mathcal{Y}^{(n+1)}[_P^{\mathrm{ad}}, \mathcal{G}(\mathcal{E}^\bullet))) = \mathrm{H}^*_{\mathrm{rig}}(\mathcal{Y}^{(n+1)}/K).$$

By construction, $E_2 = E_\infty$ now describes steps of descending filtrations of G-modules on each $\mathrm{H}^s_{\mathrm{rig}}(\mathcal{Y}^{(n+1)}/K)$, $s \in \mathbb{N}_0$, via

$$E_2^{r,s} = \mathrm{gr}^r(\mathrm{H}^{r+s}_{\mathrm{rig}}(\mathcal{Y}^{(n+1)}/K)).$$

There are thus descending filtrations on each $\mathrm{H}^s_{\mathrm{rig}}(\mathcal{Y}^{(n+1)}/K)$ with filtration steps

$$\mathrm{gr}^r(\mathrm{H}^s_{\mathrm{rig}}(\mathcal{Y}^{(n+1)}/K)) = E_2^{r,s-r} = \begin{cases} v^G_{P_{I_{\frac{s-r}{2}}}}(K)(-\frac{s-r}{2}) & \text{if } s-r \in \{0,2,\ldots,2(n-2)\}, \\ & \ r = n-1-\frac{s-r}{2} \\ K(-\frac{s-r}{2}) & \text{if } s-r \in \{0,2,\ldots,2(n-2)\}, \\ & \ r=0 \\ \mathrm{Ind}^G_{P_{I_{n-1}}} K(-(n-1)) & \text{if } s-r = 2n-2, r=0 \\ (0) & \text{else.} \end{cases}$$

This shows that for each $s \in \{0, \ldots, 2n-2\}$, the associated graded module of the rigid cohomology module $\mathrm{H}^s_{\mathrm{rig}}(\mathcal{Y}^{(n+1)}/K)$ has the following shape:

$$\mathrm{gr}^\bullet(\mathrm{H}^s_{\mathrm{rig}}(\mathcal{Y}^{(n+1)}/K)) = \begin{cases} K(-\frac{s}{2}) & \text{if } s \in \{0,\ldots,n-2\} \text{ even} \\ K(-\frac{s}{2}) \oplus v^G_{P_{I_{n-1-s}}}(K)(n-1-s) & \text{if } s \in \{n-1,\ldots,2n-3\} \text{ even} \\ v^G_{P_{I_{-n+1+s}}}(K)(n-1-s) & \text{if } s \in \{n-1,\ldots,2n-3\} \text{ odd} \\ \mathrm{Ind}^G_{P_{I_{n-1}}}(-(n-1)) & \text{if } s = 2n-2. \end{cases}$$

Each filtration splits as the above lemma shows that there are no non-split extensions between Galois modules of different Tate twist and therefore,

$$\mathrm{gr}^\bullet(\mathrm{H}^s_{\mathrm{rig}}(\mathcal{Y}^{(n+1)}/K)) \cong \mathrm{H}^s_{\mathrm{rig}}(\mathcal{Y}^{(n+1)}/K) = \mathrm{H}^s_{\mathrm{rig,c}}(\mathcal{Y}^{(n+1)}/K).$$

□

Theorem 3.2.4.3. *The rigid cohomology with compact supports of $\mathcal{X}^{(n+1)}$ is given by*

$$\mathrm{H}^*_{\mathrm{rig,c}}(\mathcal{X}^{(n+1)}/K) = \bigoplus_{i=0}^{n} v^G_{P_{I_i}}(K)(-i)[-n-i].$$

Proof. Using the respective property from Subsection 3.1.2, it follows from the fact that $\mathcal{X}^{(n+1)}$ is a smooth affine variety of dimension n that $\mathrm{H}^i_{\mathrm{rig,c}}(\mathcal{X}^{(n+1)}/K) = 0$ for all $i \notin \{n, \ldots, 2n\}$. Now employ the long exact sequence for rigid cohomology with compact supports for the pair of inclusions

$$\mathcal{X}^{(n+1)} \overset{\text{open}}{\hookrightarrow} \mathbb{P}^n_k \overset{\text{closed}}{\hookleftarrow} \mathcal{Y}^{(n+1)}.$$

For each $i \in \{n, \ldots, 2n\}$, there are thus exact sequences of $G \times \mathrm{Gal}(\overline{k}/k)$-modules

$$\mathrm{H}^{i-1}_{\mathrm{rig}}(\mathbb{P}^n_k/K) \to \mathrm{H}^{i-1}_{\mathrm{rig}}(\mathcal{Y}^{(n+1)}/K) \to \mathrm{H}^i_{\mathrm{rig,c}}(\mathcal{X}^{(n+1)}/K) \to \mathrm{H}^i_{\mathrm{rig}}(\mathbb{P}^n_k/K).$$

Inductive evaluation of those exact sequences (plugging in the result from Proposition 3.2.4.1 and of course the fact that rigid cohomology of $\mathbb{P}^n_k$ is known (see Subsection 3.1.2)) finishes the proof. Here, one has to use the additional fact that, as a Galois module, $\mathrm{H}^i_{\mathrm{rig,c}}(\mathcal{X}^{(n+1)}/K)$ is pure which means that it cannot contain submodules of different Tate twist, cf. e.g. [5, Prop. 3.3.8] (which only uses the fact that – as is rigid cohomology – ℓ-adic cohomology is a Weil cohomology). □

3.3 Rigid Cohomology computed from the Associated De Rham Complex

The goal of this section is to show how – in principle – the rigid cohomology $\mathrm{H}^*_{\mathrm{rig}}(\mathcal{X}^{(n+1)}/K, \mathcal{E})$ of $\mathcal{X}^{(n+1)}$ with values in a $\mathbf{G}$-equivariant overconvergent F-isocrystal $\mathcal{E}$ which is defined on[3] $\mathbb{P}^{n,\mathrm{rig}}_K$ can be computed from its associated de Rham complex. The main tool will again be an adapted version of Orlik's complex.

First of all, a cofinal family of strict open neighborhoods (with respect to the reverse inclusion ordering) of $]\mathcal{X}^{(n+1)}[_P$ in $\mathbb{P}^{n,\mathrm{rig}}_K$ suitable for the purpose of adapting Orlik's complex has to be constructed. Let $X \subset \mathbb{P}^n_k$ be an open subset and write $Z = \mathbb{P}^n_k \setminus X$ for its closed complement. For $\lambda \in (0,1) \cap |\overline{K}^\times|$, let

$$V^\lambda = \mathbb{P}^{n,\mathrm{rig}}_K \backslash]Z[_{P,\lambda}.$$

Here, $]Z[_{P,\lambda}$ is the open tube of Z of radius λ in $\mathbb{P}^{n,\mathrm{rig}}_K$ which can be described as follows, cf. [37, 2.3]: Suppose that the vanishing ideal of Z is generated by the homogeneous polynomials $f_1, \ldots, f_r \in k[T_0, \ldots, T_n]$. For each $l \in \{1, \ldots, r\}$, let $\tilde{f}_l \in \mathcal{V}[T_0, \ldots, T_n]$ be a (homogeneous) lift of f_l. Then

$$]Z[_{P,\lambda} = \left\{ x \in \mathbb{P}^{n,\mathrm{rig}}_K \text{ (unimodular)} \;\middle|\; \forall l \in \{0, \ldots, r\} : |\tilde{f}_l(x)| < \lambda \right\}.$$

According to [37, 3.3.1], V^λ is a strict open neighborhood of $]X[_P$ in $\mathbb{P}^{n,\mathrm{rig}}_K$. Moreover, the system $(V^\lambda)_{\lambda \in (0,1)}$ is even a cofinal system of quasi-compact strict open neighborhoods of $]X[_P$ in $\mathbb{P}^{n,\mathrm{rig}}_K$, cf. [37, 3.3.3]. For $m \in \mathbb{N}$, let

$$\lambda_m \;=\; |\pi|^{1/(m+1)} \in |\overline{K}^\times|.$$

Then the countable system $(V^{\lambda_m})_{m \in \mathbb{N}}$ is cofinal in $(V^\lambda)_{\lambda \in (0,1)}$ and thus it is itself a cofinal system of strict open neighborhoods of $]X[_P$ in $\mathbb{P}^{n,\mathrm{rig}}_K$.

Now specialize to the case $X = \mathcal{X}^{(n+1)}$. A slight technical problem in adapting Orlik's complex is the fact that the "operation" $] - [_{P,\lambda}$ does not commute with taking finite unions. Therefore, there will now be constructed a cofinal system $(U^m)_{m \in \mathbb{N}}$ of strict open neighborhoods of $]\mathcal{X}^{(n+1)}[_P$ in $\mathbb{P}^{n,\mathrm{rig}}_K$ better suited for the task at hand.

For $m \in \mathbb{N}$, set

$$U^m = \mathbb{P}^{n,\mathrm{rig}}_K \setminus \bigcup_{I \subsetneq \Delta} \bigcup_{g \in G/P_I}]g.Y_I[_{P,\lambda_m}.$$

Lemma 3.3.0.4.

i) The set U^m is a strict open neighborhood of $]\mathcal{X}^{(n+1)}[_P$ in $\mathbb{P}^{n,\mathrm{rig}}_K$.

ii) The set U^m is an affinoid subvariety of $\mathbb{P}^{n,\mathrm{rig}}_K$.

iii) The family $\{U^m \mid m \in \mathbb{N}\}$ is cofinal in the family of all strict open neighborhoods of $]\mathcal{X}^{(n+1)}[_P$ in $\mathbb{P}^{n,\mathrm{rig}}_K$.

Proof. i) Since

$$U^m = \bigcap_{I \subsetneq \Delta} \bigcap_{g \in G/P_I} \mathbb{P}^{n,\mathrm{rig}}_K \backslash]g.Y_I[_{P,\lambda_m}$$

[3] By definition, $\mathbb{P}^{n,\mathrm{rig}}_K$ is in particular a strict open neighborhood of $]X[_P$ in $\mathbb{P}^{n,\mathrm{rig}}_K$.

is a finite intersection of admissible open subsets of $\mathbb{P}_K^{n,\mathrm{rig}}$, it is admissible open in $\mathbb{P}_K^{n,\mathrm{rig}}$ itself. Directly from the definition of tubes, it follows that $\bigcup_{I\subsetneq\Delta}\bigcup_{g\in G/P_I}]g.Y_I[_{P,\lambda_m}$ is contained in $]\mathcal{Y}^{(n+1)}[_{P,\lambda_m}$. Therefore, $V^{\lambda_m}=\mathbb{P}_K^{n,\mathrm{rig}}\backslash]\mathcal{Y}^{(n+1)}[_{P,\lambda_m}$ is contained in U^m and the claim follows from [37, 3.1.2].

ii) Denote by $\mathcal{H}$ the set of all n-dimensional k-subspaces of k^{n+1}. Then there is an identification

$$\begin{aligned} U^m &= \mathbb{P}_K^{n,\mathrm{rig}}\setminus\bigcup_{H\in\mathcal{H}}]\mathbb{P}(H)[_{P,\lambda_m} \\ &= \bigcap_{H\in\mathcal{H}}\mathbb{P}_K^{n,\mathrm{rig}}\backslash]\mathbb{P}(H)[_{P,\lambda_m}. \end{aligned}$$

Since a finite intersection of affinoid subvarieties of $\mathbb{P}_K^{n,\mathrm{rig}}$ is again an affinoid subvariety (this is due to the fact that $\mathbb{P}_K^{n,\mathrm{rig}}$ is separated, cf. [12, 4.10.1 and 4.3.4]), it is enough to show that each $\mathbb{P}_K^{n,\mathrm{rig}}\backslash]\mathbb{P}(H)[_{P,\lambda_m}$ is an affinoid subvariety of $\mathbb{P}_K^{n,\mathrm{rig}}$. Possibly after a coordinate transformation, one can reduce to the case that $H=V_+(T_n)$. Then

$$\begin{aligned} \mathbb{P}_K^{n,\mathrm{rig}}\backslash]\mathbb{P}(H)[_{P,\lambda_m} &= \left\{x=[x_0:\ldots:x_n]\in\mathbb{P}_K^{n,\mathrm{rig}}\text{ unimodular}\;\middle|\;|x_n|\geq\lambda_m\right\} \\ &\cong \left\{\left(\frac{x_0}{x_n},\ldots,\frac{x_{n-1}}{x_n}\right)\in(\widehat{\overline{K}})^n\;\middle|\;\forall i=0,\ldots,n-1:|\frac{x_i}{x_n}|\leq\lambda_m^{-1}\right\} \\ &= \left\{(z_0,\ldots,z_{n-1})\in(\widehat{\overline{K}})^n\;\middle|\;\forall i=0,\ldots,n-1:|\pi z_i^{m+1}|\leq 1\right\} \end{aligned}$$

and the last set is an affinoid K-variety.

iii) The family $(V^\lambda)_{\lambda\in(0,1)\cap|\overline{K}^\times|}$ is a cofinal family of strict neighborhoods. Therefore, it is enough to show that for each λ as above, there exists some $m\in\mathbb{N}$ such that $U^m\subset V^\lambda$. By [37, 2.3.6], there is an admissible covering

$$]\mathcal{Y}^{(n+1)}[_{P,\lambda}\subset\bigcup_{I\subsetneq\Delta}\bigcup_{g\in G/P_I}]g.Y_I[_{P,\lambda'}$$

for λ' such that

$$\prod_{I\subsetneq\Delta}\prod_{g\in(\mathbf{G}/\mathbf{P}_I)(k)}\lambda'=(\lambda')^{\left(\sum_{I\subsetneq\Delta}\sum_{g\in G/P_I}1\right)}=\lambda.$$

Choose $m\in\mathbb{N}$ large enough so that $\lambda'\leq\lambda_m$. Then there is an admissible covering

$$]\mathcal{Y}^{(n+1)}[_{P,\lambda}\subset\bigcup_{I\subsetneq\Delta}\bigcup_{g\in G/P_I}]g.Y_I[_{P,\lambda_m},$$

hence

$$V^\lambda=\mathbb{P}_K^{n,\mathrm{rig}}\backslash]\mathcal{Y}^{(n+1)}[_{P,\lambda}\supset\mathbb{P}_K^{n,\mathrm{rig}}\setminus\bigcup_{I\subsetneq\Delta}\bigcup_{g\in G/P_I}]g.Y_I[_{P,\lambda_m}=U^m$$

which finishes the proof of the lemma.

□

Denote by

$$j_m:U^m\hookrightarrow\mathbb{P}_K^{n,\mathrm{rig}}$$

the inclusion. Then the above lemma implies that the rigid cohomology of $\mathcal{X}^{(n+1)}$ with values in $\mathcal{E}$ can be computed as

$$\mathrm{H}^*_{\mathrm{rig}}(\mathcal{X}^{(n+1)}/K,\mathcal{E})=\mathbb{H}^*(\mathbb{P}_K^{n,\mathrm{rig}},\varinjlim_{m\in\mathbb{N}}j_{m*}j_m^*(\mathcal{E}\otimes_{\mathcal{O}_{\mathbb{P}_K^{n,\mathrm{rig}}}}\Omega^\bullet_{\mathbb{P}_K^{n,\mathrm{rig}}})).$$

From now on, for $i = 0, \dots, n$, set

$$\mathcal{E}^i = \mathcal{E} \otimes_{\mathcal{O}_{\mathbb{P}_K^{n,\mathrm{rig}}}} \Omega^i_{\mathbb{P}_K^{n,\mathrm{rig}}}.$$

To make use of the associated de Rham complex for the computation of the above hypercohomology, one has to calculate the cohomology spaces $\mathrm{H}^*(\mathbb{P}_K^{n,\mathrm{rig}}, \varinjlim_{m\in\mathbb{N}} j_{m*}j_m^*\mathcal{E}^i)$ for each $i \in \mathbb{N}_0$.

Lemma 3.3.0.5. *For each $i = 0, \dots, n$, there is an identification*

$$\mathrm{H}^*(\mathbb{P}_K^{n,\mathrm{rig}}, \varinjlim_{m\in\mathbb{N}} j_{m*}j_m^*\mathcal{E}^i) = \varinjlim_{m\in\mathbb{N}} \mathrm{H}^0(U^m, \mathcal{E}^i)[0].$$

Proof. First of all, applying [27, 2.3.13] to the parallel situation of adic spaces yields isomorphisms

$$\mathrm{H}^*(\mathbb{P}_K^{n,\mathrm{rig}}, \varinjlim_{m\in\mathbb{N}} j_{m*}j_m^*\mathcal{E}^i) = \varinjlim_{m\in\mathbb{N}} \mathrm{H}^*(\mathbb{P}_K^{n,\mathrm{rig}}, j_{m*}j_m^*\mathcal{E}^i).$$

The morphism j_m is quasi-Stein in the sense of [37, p. 20] which in particular implies that the higher direct images $\mathbf{R}^l j_{m*}(j_m^*\mathcal{E}^i)$ vanish for $l \geq 1$, cf. loc. cit. From this, it follows that

$$\mathrm{H}^*(\mathbb{P}_K^{n,\mathrm{rig}}, j_{m*}j_m^*\mathcal{E}^i) = \mathrm{H}^*(U^m, j_m^*\mathcal{E}^i) = \mathrm{H}^0(U^m, j_m^*\mathcal{E}^i) = \mathrm{H}^0(U^m, \mathcal{E}^i)$$

since higher coherent cohomology on affinoid spaces vanishes by Kiehl's Theorem B, cf. [33, 2.4]. This finishes the proof. □

So in essence, to compute $\mathrm{H}^*_{\mathrm{rig}}(\mathcal{X}^{(n+1)}/K, \mathcal{E})$, one has to compute $\mathrm{H}^0(U^m, \mathcal{E}^i)$ for all i, apply the direct limit $\varinjlim_{m\in\mathbb{N}}$, plug the resulting spaces into a spectral sequence and describe the associated gradings. This will be done in the next few subsections using the methods of [44, 45] by Orlik. The strategy to determine the spaces $\mathrm{H}^0(U^m, \mathcal{E}^i)$ is the same as in the previous chapter in the case of finite ground fields, namely to use local cohomology, but this time of rigid analytic spaces. For this purpose, this cohomology theory shall be recalled briefly (cf. [57, 1.2-3]):
Let X be a rigid analytic K-space, let $Z \subset X$ be an admissible open subset such that $U = X \setminus Z$ is also admissible open in X. For an abelian sheaf $\mathcal{H}$ on X, set

$$\mathrm{H}^0_Z(X, \mathcal{H}) = \ker(\mathrm{H}^0(X, \mathcal{H}) \to \mathrm{H}^0(U, \mathcal{H})).$$

Then $\mathrm{H}^0_Z(X, -)$ is a left exact functor and therefore it has right derived functors

$$\mathrm{H}^i_Z(X, -) = \mathbf{R}^i\mathrm{H}^0_Z(X, -).$$

The following hold and will be used freely in the sequel:

- There is a long exact sequence

$$\ldots \to \mathrm{H}^i(X, \mathcal{H}) \to \mathrm{H}^i(U, \mathcal{H}) \to \mathrm{H}^{i+1}_Z(X, \mathcal{H}) \to \mathrm{H}^{i+1}(X, \mathcal{H}) \to \ldots,$$

  cf. [57, 1.3], which also holds in the more general situation presented here.

- From the fact that the functor $(-)^{\mathrm{ad}}$ induces an equivalence of topoi combined with the fact that $\mathrm{H}^*_Z(X, \mathcal{H})$ is computed by using an injective resolution of $\mathcal{H}$, it follows that there is an isomorphism

$$\mathrm{H}^*_{X^{\mathrm{ad}} \setminus U^{\mathrm{ad}}}(X^{\mathrm{ad}}, \mathcal{H}^{\mathrm{ad}}) \cong \mathrm{H}^*_Z(X, \mathcal{H}) \tag{3.7}$$

  where the local cohomology for adic spaces is defined as usual for topological spaces.

.

Now set

$$Y^m = \mathbb{P}_K^{n,\mathrm{rig}} \setminus U^m = \bigcup_{I \subsetneq \Delta} \bigcup_{g \in G/P_I}]g.Y_I[_{P,\lambda_m}.$$

Then there are local cohomology groups $\mathrm{H}^*_{Y^m}(\mathbb{P}_K^{n,\mathrm{rig}}, \mathcal{E}^i)$ and an exact sequence

$$(0) \to \mathrm{H}^0(\mathbb{P}_K^{n,\mathrm{rig}}, \mathcal{E}^i) \to \mathrm{H}^0(U^m, \mathcal{E}^i) \to \mathrm{H}^1_{Y^m}(\mathbb{P}_K^{n,\mathrm{rig}}, \mathcal{E}^i).$$

The groups $\mathrm{H}^1_{Y^m}(\mathbb{P}_K^{n,\mathrm{rig}}, \mathcal{E}^i)$ can be computed using an adapted version of Orlik's complex and one can then determine $\mathrm{H}^0(U^m, \mathcal{E}^i)$.

Because of technical reasons concerning the localization of sheaves in points, it is again more convenient to use the framework of adic spaces.

3.3.1 Adaption of Orlik's Complex

For $m \in \mathbb{N}$, set

$$Z^m = \mathbb{P}_K^{n,\mathrm{ad}} \setminus U^{m,\mathrm{ad}}$$

and for $I \subsetneq \Delta$ and $g \in G/P_I$, set

$$Z^m_{g,I} = \mathbb{P}_K^{n,\mathrm{ad}} \setminus \left(\mathbb{P}_K^{n,\mathrm{rig}} \setminus]g.Y_I[_{P,\lambda_m}\right)^{\mathrm{ad}}.$$

Both Z^m and $Z^m_{g,I}$ are closed subspaces in $\mathbb{P}_K^{n,\mathrm{ad}}$. As in Subsection 2.1.1, an inclusion $I \subset J$ of proper subsets of Δ together with a mapping $gP_I \mapsto hP_J$ under the canonical map

$$G/P_I \to G/P_J$$

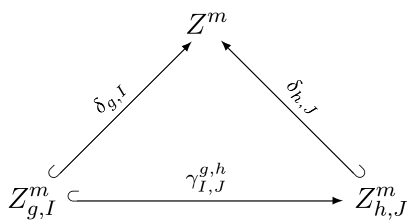
induces a closed embedding of closed subspaces

$$\gamma^{g,h}_{I,J} : Z^m_{g,I} \hookrightarrow Z^m_{h,J}$$

of Z^m. Furthermore, for each I and g as above, there are closed embeddings

$$\delta_{g,I} : Z^m_{g,I} \hookrightarrow Z^m,$$

so that for all g, h and I, J as above, there are commutative triangles



of closed embeddings. Let $\mathcal{F}$ be a sheaf of abelian groups on Z^m. For $I \subsetneq \Delta$ and $g \in G/P_I$, set

$$\begin{aligned} \mathcal{F}_{g,I} &= {\delta_{g,I}}_* \delta_{g,I}^{-1} \mathcal{F}, \\ \mathcal{F}_I &= \bigoplus_{g \in G/P_I} \mathcal{F}_{g,I}. \end{aligned}$$

Proposition 3.3.1.1. *For any sheaf $\mathcal{F}$ of abelian groups on Z^m, there is an acyclic complex*

$$0 \to \mathcal{F} \to \bigoplus_{\substack{I \subsetneq \Delta \\ \#I = n-1}} \mathcal{F}_I \to \bigoplus_{\substack{I \subsetneq \Delta \\ \#I = n-2}} \mathcal{F}_I \to \ldots \to \bigoplus_{\substack{I \subsetneq \Delta \\ \#I = 1}} \mathcal{F}_I \to \mathcal{F}_\emptyset \to 0 \tag{3.8}$$

of sheaves of abelian groups on Z^m.

Proof. (cf. the proof of Proposition 3.2.1.1) The proof is again by localization of the above complex with respect to a point $x \in Z^m$. Consider the set

$$X = \left\{ U \subsetneq k^{n+1} \ k\text{-subspace} \ \middle| \ U \neq (0), x \in \mathbb{P}_K^{n,\mathrm{ad}} \setminus (\mathbb{P}_K^{n,\mathrm{rig}} \backslash]\mathbb{P}(U)[_{P,\lambda_m})^{\mathrm{ad}} \right\}.$$

This set is not empty since, by construction,

$$Z^m = \bigcup_{(0) \subsetneq U \subsetneq k^{n+1}} \mathbb{P}_K^{n,\mathrm{ad}} \setminus (\mathbb{P}_K^{n,\mathrm{rig}} \backslash]\mathbb{P}(U)[_{P,\lambda_m})^{\mathrm{ad}}$$

with U running through all proper non-zero subspaces of k^{n+1}. Choose a minimal subspace $U_0 \in X$ and let $U \in X$ be arbitrary. Directly from the definition of tubes of radius λ (see page 69), one observes that the identity

$$]\mathbb{P}(U_0)[_{P,\lambda_m} \cap]\mathbb{P}(U)[_{P,\lambda_m} =]\mathbb{P}(U_0) \cap \mathbb{P}(U)[_{P,\lambda_m}$$

holds. Both $\mathbb{P}_K^{n,\mathrm{rig}} \backslash]\mathbb{P}(U_0)[_{P,\lambda_m}$ and $\mathbb{P}_K^{n,\mathrm{rig}} \backslash]\mathbb{P}(U)[_{P,\lambda_m}$ are finite unions of affinoid spaces which is seen by using the standard affinoid covering of $\mathbb{P}_K^{n,\mathrm{rig}}$. Therefore, the covering

$$\mathbb{P}_K^{n,\mathrm{rig}} \backslash]\mathbb{P}(U_0) \cap \mathbb{P}(U)[_{P,\lambda_m} = \mathbb{P}_K^{n,\mathrm{rig}} \backslash]\mathbb{P}(U_0)[_{P,\lambda_m} \cup \mathbb{P}_K^{n,\mathrm{rig}} \backslash]\mathbb{P}(U)[_{P,\lambda_m}$$

has a refinement consisting of finitely many affinoid subsets and is thus admissible. By [27, 1.1.11 (c)], this implies that

$$\left(\mathbb{P}_K^{n,\mathrm{rig}} \backslash]\mathbb{P}(U_0) \cap \mathbb{P}(U)[_{P,\lambda_m}\right)^{\mathrm{ad}} = \left(\mathbb{P}_K^{n,\mathrm{rig}} \backslash]\mathbb{P}(U_0)[_{P,\lambda_m}\right)^{\mathrm{ad}} \cup \left(\mathbb{P}_K^{n,\mathrm{rig}} \backslash]\mathbb{P}(U)[_{P,\lambda_m}\right)^{\mathrm{ad}},$$

thus

$$\begin{aligned} x &\in \left(\mathbb{P}_K^{n,\mathrm{ad}} \setminus \left(\mathbb{P}_K^{n,\mathrm{rig}} \backslash]\mathbb{P}(U_0)[_{P,\lambda_m}\right)^{\mathrm{ad}}\right) \cap \left(\mathbb{P}_K^{n,\mathrm{ad}} \setminus \left(\mathbb{P}_K^{n,\mathrm{rig}} \backslash]\mathbb{P}(U)[_{P,\lambda_m}\right)^{\mathrm{ad}}\right) \\ &= \mathbb{P}_K^{n,\mathrm{ad}} \setminus \left(\left(\mathbb{P}_K^{n,\mathrm{rig}} \backslash]\mathbb{P}(U_0)[_{P,\lambda_m}\right)^{\mathrm{ad}} \cup \left(\mathbb{P}_K^{n,\mathrm{rig}} \backslash]\mathbb{P}(U)[_{P,\lambda_m}\right)^{\mathrm{ad}}\right) \\ &= \mathbb{P}_K^{n,\mathrm{ad}} \setminus \left(\mathbb{P}_K^{n,\mathrm{rig}} \backslash]\mathbb{P}(U_0 \cap U)[_{P,\lambda_m}\right)^{\mathrm{ad}}. \end{aligned}$$

It follows that $U_0 \cap U$ cannot be equal to (0) and, by minimality of U_0, this means that $U_0 \cap U = U_0$. Therefore, the identity map $\mathrm{id} : X \to X$ fulfills

$$U_0, U \subset \mathrm{id}(U)$$

for all $U \in X$ and, again by Quillen's criterion, the simplicial complex $X^\bullet$ associated with X is contractible (cf. the construction in the proof of Proposition 3.2.1.1). This implies that the chain complex

$$(0) \to \mathcal{F}_x \to \bigoplus_{\substack{I \subsetneq \Delta \\ \#I = n-1}} \bigoplus_{\substack{g \in G/P_I \\ x \in Z^m_{g,I}}} \mathcal{F}_x \to \ldots \to \bigoplus_{\substack{I \subsetneq \Delta \\ \#I = 1}} \bigoplus_{\substack{g \in G/P_I \\ x \in Z^m_{g,I}}} \mathcal{F}_x \to \bigoplus_{\substack{g \in G/P_\emptyset \\ x \in Z^m_{g,\emptyset}}} \mathcal{F}_x \to (0)$$

associated with $X^\bullet$ with values in X is acyclic. Since this complex is precisely the localization of (3.8) with respect to x, the proposition is proved. $\square$

3.3.2 Construction of a Spectral Sequence

Let $m \in \mathbb{N}$ and consider the case that $\mathcal{F} = \mathbb{Z}_{Z^m}$ is the constant sheaf on Z^m with value $\mathbb{Z}$. Let

$$\iota_m : Z^m \hookrightarrow \mathbb{P}_K^{n,\mathrm{ad}}$$

be the inclusion which is in particular a closed embedding. For $r = 0, \dots, n-1$, set

$$\mathcal{F}_r = \bigoplus_{\substack{I \subsetneq \Delta \\ \# I = n-1-r}} \mathcal{F}_I.$$

As in Subsection 2.1.1, there is then a (second quadrant) spectral sequence

$$E_1^{r,s} = \mathrm{Ext}^s(\iota_{m*}\mathcal{F}_{-r}, \mathcal{E}^{i,\mathrm{ad}}) \Longrightarrow \mathrm{Ext}^{r+s}(\iota_{m*}\mathcal{F}, \mathcal{E}^{i,\mathrm{ad}}) = \mathrm{H}^{r+s}_{Z^m}(\mathbb{P}_K^{n,\mathrm{ad}}, \mathcal{E}^{i,\mathrm{ad}}).$$

This spectral sequence is evaluated in the next subsection. Recall that

$$Y^m = \bigcup_{I \subsetneq \Delta} \bigcup_{g \in G/P_I}]g.Y_I[_{P,\lambda_m}.$$

It follows from (3.7) that

$$\mathrm{H}^{r+s}_{Z^m}(\mathbb{P}_K^{n,\mathrm{ad}}, \mathcal{E}^{i,\mathrm{ad}}) = \mathrm{H}^{r+s}_{Y^m}(\mathbb{P}_K^{n,\mathrm{rig}}, \mathcal{E}^{i}).$$

3.3.3 Evaluation of the Spectral Sequence

The E_1-Page

Lemma 3.3.3.1. *Let $r, s \in \mathbb{Z}$. Then there is an identification*

$$E_1^{r,s} = \bigoplus_{\substack{I \subsetneq \Delta \\ |I| = n-1+r}} \mathrm{Ind}^{\mathbf{G}(\mathcal{V})}_{\mathbf{P}_I(\mathcal{V})(1)} \mathrm{H}^s_{]Y_I[_{P,\lambda_m}}(\mathbb{P}_K^{n,\mathrm{rig}}, \mathcal{E}^i).$$

Proof. For brevity, write $\mathbb{Z} = \mathbb{Z}_{]\mathcal{Y}^{(n+1)}[^{\mathrm{ad}}_{P,\lambda_m}}$. Then

$$\begin{aligned}
E_1^{r,s} &= \mathrm{Ext}^s(\iota_{m*}\mathcal{F}_{-r}, \mathcal{E}^{i,\mathrm{ad}}) \\
&= \bigoplus_{\substack{I \subsetneq \Delta \\ |I| = n-1+r}} \bigoplus_{g \in G/P_I} \mathrm{Ext}^s(\iota_{m*}(\delta_{g,I})_*(\delta_{g,I})^{-1}\mathbb{Z}, \mathcal{E}^{i,\mathrm{ad}}) \\
&= \bigoplus_{\substack{I \subsetneq \Delta \\ |I| = n-1+r}} \bigoplus_{g \in G/P_I} \mathrm{Ext}^s((\iota_m \circ \delta_{g,I})_*\mathbb{Z}_{|Z^m_{g,I}}, \mathcal{E}^{i,\mathrm{ad}}) \\
&= \bigoplus_{\substack{I \subsetneq \Delta \\ |I| = n-1+r}} \bigoplus_{g \in G/P_I} \mathrm{H}^s_{Z^m_{g,I}}(\mathbb{P}_K^{n,\mathrm{ad}}, \mathcal{E}^{i,\mathrm{ad}}) \\
&\cong \bigoplus_{\substack{I \subsetneq \Delta \\ |I| = n-1+r}} \bigoplus_{g \in G/P_I} \mathrm{H}^s_{]g.Y_I[_{P,\lambda_m}}(\mathbb{P}_K^{n,\mathrm{rig}}, \mathcal{E}^i) \qquad (3.9) \\
&= \bigoplus_{\substack{I \subsetneq \Delta \\ |I| = n-1+r}} \bigoplus_{g \in \mathbf{G}(\mathcal{V})/\mathbf{P}_I(\mathcal{V})(1)} \mathrm{H}^s_{g.]Y_I[_{P,\lambda_m}}(\mathbb{P}_K^{n,\mathrm{rig}}, \mathcal{E}^i) \\
&= \bigoplus_{\substack{I \subsetneq \Delta \\ |I| = n-1+r}} \mathrm{Ind}^{\mathbf{G}(\mathcal{V})}_{\mathbf{P}_I(\mathcal{V})(1)} \mathrm{H}^s_{]Y_I[_{P,\lambda_m}}(\mathbb{P}_K^{n,\mathrm{rig}}, \mathcal{E}^i),
\end{aligned}$$

where the isomorphism (3.9) is an application of (3.7). Here, the fact that $g.]Y_I[_{P,\lambda_m}=]\bar{g}.Y_I[_{P,\lambda_m}$ for all $g \in \mathbf{G}(\mathcal{V})$ is used to establish the action of the group $\mathbf{G}(\mathcal{V})$ on $\bigoplus_{g\in G/P_I} \mathrm{H}^s_{]g.Y_I[_{P,\lambda_m}}(\mathbb{P}^{n,\mathrm{rig}}_K, \mathcal{E}^i)$. For this and also for the definition of the subgroup $\mathbf{P}_I(\mathcal{V})(1) \subset \mathbf{G}(\mathcal{V})$, cf. the proof of Lemma 3.2.3.1. Note that the action of $\mathbf{G}(\mathcal{V})$ on $\mathbb{B}^{n+1}(\widehat{\overline{K}}) = \{x = (x_0, \dots, x_n) \mid \forall i = 0, \dots, n : |x_i| \le 1\}$ given by $(g, x) \mapsto xg^{-1}$ preserves the unimodular points of $\mathbb{B}^{n+1}(\widehat{\overline{K}})$. □

By definition, $Y_I = \mathbb{P}^{i_1}_k$ for $\Delta \setminus I = \{\alpha_{i_1}, \dots, \alpha_{i_t}\}$ with $i_1 < \dots < i_t$ (cf. the construction in Subsection 2.1.1) so that

$$\mathrm{H}^s_{]Y_I[_{P,\lambda_m}}(\mathbb{P}^{n,\mathrm{rig}}_K, \mathcal{E}^i) = \mathrm{H}^s_{]\mathbb{P}^{i_1}_k[_{P,\lambda_m}}(\mathbb{P}^{n,\mathrm{rig}}_K, \mathcal{E}^i).$$

In [44, 1.3] it is shown that $\mathrm{H}^s_{]\mathbb{P}^{i_1}_k[_{P,\lambda_m}}(\mathbb{P}^{n,\mathrm{rig}}_K, \mathcal{E}^i)$ carries the structure of a K-Banach space in such a way that the inclusion

$$\mathrm{H}^s_{\mathbb{P}^{i_1,\mathrm{rig}}_K}(\mathbb{P}^{n,\mathrm{rig}}_K, \mathcal{E}^i) \subset \mathrm{H}^s_{]\mathbb{P}^{i_1}_k[_{P,\lambda_m}}(\mathbb{P}^{n,\mathrm{rig}}_K, \mathcal{E}^i)$$

of the algebraic local cohomology space $\mathrm{H}^s_{\mathbb{P}^{i_1,\mathrm{rig}}_K}(\mathbb{P}^{n,\mathrm{rig}}_K, \mathcal{E}^i)$ has dense image. Thus, for $s \neq n - i_1$, the description of the local cohomology modules amounts to

$$\mathrm{H}^s_{]\mathbb{P}^{i_1}_k[_{P,\lambda_m}}(\mathbb{P}^{n,\mathrm{rig}}_K, \mathcal{E}^i) = \begin{cases} (0) & \text{if } s < n - i_1 \\ \mathrm{H}^s(\mathbb{P}^{n,\mathrm{rig}}_K, \mathcal{E}^i) & \text{if } s > n - i_1, \end{cases} \tag{3.10}$$

cf. the remarks at the beginning of Section 2.2. Write

$$M^s_I = \begin{cases} \mathrm{H}^s_{]\mathbb{P}^{n-s}_k[^{\mathrm{rig}}_{P,\lambda_m}}(\mathbb{P}^{n,\mathrm{rig}}_K, \mathcal{E}^i) & \text{if } \alpha_{n-s} \notin I \\ \mathrm{H}^s(\mathbb{P}^{n,\mathrm{rig}}_K, \mathcal{E}^i) & \text{if } \alpha_{n-s} \in I \end{cases}$$

for $I \subsetneq \Delta, s \in \{2, \dots, n\}$. Then, each row $E_1^{\bullet,s}$ with $s \in \{1, \dots, n\}$ of the E_1-page of the spectral sequence has the following shape: For $s \in \{2, \dots, n\}$, one gets

$$\begin{aligned} E_1^{\bullet,s}: \quad & E_1^{-(s-1),s} = \mathrm{Ind}^{\mathbf{G}(\mathcal{V})}_{\mathbf{P}_{\{\alpha_0,\dots,\alpha_{n-1-s}\}}(\mathcal{V})(1)} \mathrm{H}^s_{]\mathbb{P}^{n-s}_k[_{P,\lambda_m}}(\mathbb{P}^{n,\mathrm{rig}}_K, \mathcal{E}^i) \to \\ & E_1^{-(s-2),s} = \bigoplus_{\substack{I \subsetneq \Delta \\ \#I = n-s+1 \\ \alpha_0,\dots,\alpha_{n-1-s} \in I}} \mathrm{Ind}^{\mathbf{G}(\mathcal{V})}_{\mathbf{P}_I(\mathcal{V})(1)} M^s_I \to \dots \to E_1^{0,s} = \bigoplus_{\substack{I \subsetneq \Delta \\ \#I = n-1 \\ \alpha_0,\dots,\alpha_{n-1-s} \in I}} \mathrm{Ind}^{\mathbf{G}(\mathcal{V})(1)}_{\mathbf{P}_I(\mathcal{V})(1)} M^s_I. \end{aligned}$$

For $s = 1$, one gets

$$E_1^{\bullet,1} : E_1^{0,1} = \mathrm{Ind}^{\mathbf{G}(\mathcal{V})}_{\mathbf{P}_{\{\alpha_0,\dots,\alpha_{n-2}\}}(\mathcal{V})(1)} \mathrm{H}^1_{]\mathbb{P}^{n-1}_k[^{\mathrm{rig}}_{P,\lambda_m}}(\mathbb{P}^{n,\mathrm{rig}}_K, \mathcal{E}^i).$$

The E_2-Page

The evaluation of the E_2-page proceeds in complete analogy with [44, 2.2], the notable difference to the present case being the avoidance of duals. The main point – at least in the present case – is that for each $s \in \{2, \dots, n\}$, the complex $E_1^{\bullet,s}$ is acyclic apart from the positions $-(s-1)$ and 0, cf. [44, 2.2.4]. In order to avoid any (more) repetition, the proof of the following proposition is therefore omitted.

Proposition 3.3.3.2. *The E_2-page of the above spectral sequence has the following description:*

i) If $r = 0$ and $s \in \{2, \dots, n\}$, then

$$E_2^{r,s} = \mathrm{H}^s(\mathbb{P}^{n,\mathrm{rig}}_K, \mathcal{E}^i).$$

ii) If $s \in \{2, \ldots, n\}$ and $r = -(s-1)$, then there are short exact sequences of G-modules

$$\begin{aligned} (0) &\to \operatorname{Ind}^{\mathbf{G}(\mathcal{V})}_{\mathbf{P}_{(n+1-s,s)}(\mathcal{V})(1)} \left(\tilde{\mathrm{H}}^s_{]\mathbb{P}^{n-s}_k[_{P,\lambda_m}}(\mathbb{P}^{n,\mathrm{rig}}_K, \mathcal{E}^i) \otimes \mathrm{St}_s(K) \right) \to E_2^{-(s-1),s} \\ &\to v^{\mathbf{G}(\mathcal{V})}_{\mathbf{P}_{(n+1-s,1^s)}(\mathcal{V})(1)}(K)' \otimes_K \mathrm{H}^s(\mathbb{P}^{n,\mathrm{rig}}_K, \mathcal{E}^i) \to (0) \end{aligned}$$

where

- *the $\mathbf{P}_{(s+1-j,s)}(\mathcal{V})(1)$-module $\mathrm{St}_s(K)$ is the inflation to $\mathbf{P}_{(s+1-j,s)}(\mathcal{V})(1)$ of the Steinberg representation (over K) of the factor $\mathrm{GL}_s(\mathcal{V})$ of $\mathbf{L}_{(n+1-s,s)}(\mathcal{V})$, cf. Subsection 1.2.3, and*
- *the $\mathbf{P}_{(s+1-j,s)}(\mathcal{V})(1)$-module $\tilde{\mathrm{H}}^{n-j}_{]\mathbb{P}^j_K[_{P,\lambda_m}}(\mathbb{P}^{n,\mathrm{rig}}_K, \mathcal{E}^i)$ is defined as*

$$\tilde{\mathrm{H}}^{n-j}_{]\mathbb{P}^j_K[_{P,\lambda_m}}(\mathbb{P}^{n,\mathrm{rig}}_K, \mathcal{E}^i) = \ker\left(\mathrm{H}^{n-j}_{]\mathbb{P}^j_K[_{P,\lambda_m}}(\mathbb{P}^{n,\mathrm{rig}}_K, \mathcal{E}^i) \xrightarrow{d} \mathrm{H}^{n-j}(\mathbb{P}^{n,\mathrm{rig}}_K, \mathcal{E}^i) \right),$$

  where d is the ($\mathbf{P}_{(s+1-j,s)}(\mathcal{V})(1)$-equivariant) map appearing in the long exact sequence associated with local cohomology of rigid analytic varieties as defined in the beginning of this subsection.

iii) In all other cases, $E_2^{r,s} = 0$.

Degeneration and the resulting Filtration

With the same arguments as in [44, p. 633], the spectral sequence degenerates on its E_2-page and therefore, $E_2 = E_\infty$ describes filtration steps of a (descending) filtration by $\mathbf{G}(\mathcal{V})$-submodules on $\mathrm{H}^1_{Y^m}(\mathbb{P}^{n,\mathrm{rig}}_K, \mathcal{E}^i)$. This filtration can be pulled back along the $\mathbf{G}(\mathcal{V})$-morphism

$$\mathrm{H}^0(U^m, \mathcal{E}^i) \to \mathrm{H}^1_{Y^m}(\mathbb{P}^{n,\mathrm{rig}}_K, \mathcal{E}^i).$$

Theorem 3.3.3.3. *On each $\mathrm{H}^0(U^m, \mathcal{E}^i)$, there exists a filtration*

$$\mathcal{E}^i(U^m)^\bullet = \left(\mathrm{H}^0(U^m, \mathcal{E}^i) = \mathcal{E}^i(U^m)^0 \supset \mathcal{E}^i(U^m)^1 \supset \ldots \supset \mathcal{E}^i(U^m)^{n-1} \supset \mathcal{E}^i(U^m)^n = \mathrm{H}^0(\mathbb{P}^{n,\mathrm{rig}}_K, \mathcal{E}^i) \right)$$

by $\mathbf{G}(\mathcal{V})$-submodules such that each filtration step appears in a short exact sequence

$$\begin{aligned} (0) &\to \operatorname{Ind}^{\mathbf{G}(\mathcal{V})}_{\mathbf{P}_{(j+1,n-j)}(\mathcal{V})(1)} (\tilde{\mathrm{H}}^{n-j}_{]\mathbb{P}^j_k[_{P,\lambda_m}}(\mathbb{P}^{n,\mathrm{rig}}_K, \mathcal{E}^i) \otimes \mathrm{St}_{n-j}(K)) \to (\mathcal{E}^i(U^m)^j / \mathcal{E}^i(U^m)^{j+1}) \\ &\to v^{\mathbf{G}(\mathcal{V})}_{\mathbf{P}_{(j+1,1^{n-j})}(\mathcal{V})(1)}(K)' \otimes_K \mathrm{H}^{n-j}(\mathbb{P}^{n,\mathrm{rig}}_K, \mathcal{E}^i) \to (0) \end{aligned}$$

for $j = 0, \ldots, n-1$. For $j = n$, there is an identification

$$\mathcal{E}^i(U^m)^n = \mathrm{H}^0(\mathbb{P}^{n,\mathrm{rig}}_K, \mathcal{E}^i).$$

These filtrations are compatible with $\mathbf{G}$-equivariant morphisms between the involved sheaves.

Proof. The proof is in complete analogy with the one of the corresponding result [44, Cor. 2.2.9], the notable exception again being avoidance of the use of duals. The compatibility assertion is proved in [45, Lemma 4]. □

3.3.4 Computation of the Rigid Cohomology Modules

From Theorem 3.3.3.3, one obtains a $\mathbf{G}(\mathcal{V})$-filtration $\varinjlim_{m\in\mathbb{N}} \mathcal{E}^i(U^m)^\bullet$ of each

$$\mathrm{H}^0(\mathbb{P}_K^{n,\mathrm{rig}}, \varinjlim_{m\in\mathbb{N}} j_{m*}j_m^*\mathcal{E}^i) = \varinjlim_{m\in\mathbb{N}} \mathrm{H}^0(\mathbb{P}_K^{n,\mathrm{rig}}, j_{m*}j_m^*\mathcal{E}^i) = \varinjlim_{m\in\mathbb{N}} \mathrm{H}^0(U^m, \mathcal{E}^i),$$

$i \in \{0, \dots, n\}$, (which is also compatible with morphisms between the involved sheaves) such that each filtration step appears in a short exact sequence

$$\begin{aligned}(0) \;&\to\; \varinjlim_{m\in\mathbb{N}} \mathrm{Ind}^{\mathbf{G}(\mathcal{V})}_{\mathbf{P}_{(j+1,n-j)}(\mathcal{V})(1)}(\tilde{\mathrm{H}}^{n-j}_{]\mathbb{P}_k^j[_{P,\lambda_m}}(\mathbb{P}_K^{n,\mathrm{rig}}, \mathcal{E}^i) \otimes \mathrm{St}_{n-j}(K)) \to \varinjlim_{m\in\mathbb{N}} (\mathcal{E}^i(U^m)^j/\mathcal{E}^i(U^m)^{j+1}) \\ &\to\; v^{\mathbf{G}(\mathcal{V})}_{\mathbf{P}_{(j+1,1^{n-j})}(\mathcal{V})(1)}(K)' \otimes_K \mathrm{H}^{n-j}(\mathbb{P}_K^{n,\mathrm{rig}}, \mathcal{E}^i) \to (0)\end{aligned}$$

for $j = 0, \dots, n-1$. Here, one of course uses the fact that taking the direct limit $\varinjlim_{m\in\mathbb{N}}$ in this context preserves exactness and thus is also compatible with taking quotients. For $j = n$, one gets

$$\varinjlim_{m\in\mathbb{N}} \mathcal{E}^i(U^m)^n = \varinjlim_{m\in\mathbb{N}} \mathrm{H}^0(\mathbb{P}_K^{n,\mathrm{rig}}, \mathcal{E}^i) = \mathrm{H}^0(\mathbb{P}_K^{n,\mathrm{rig}}, \mathcal{E}^i).$$

These filtrations can now be used to compute the rigid cohomology modules of $\mathcal{X}^{(n+1)}$ as G-modules. The methods used are those of [45] by Orlik.

First of all, compatibility with G-morphisms (see Theorem 3.3.3.3) gives complexes

$$\begin{aligned}(0) \;&\to\; \varinjlim_{m\in\mathbb{N}} \mathcal{E}^0(U^m)^j/\mathcal{E}^0(U^m)^{j+1} \to \varinjlim_{m\in\mathbb{N}} \mathcal{E}^1(U^m)^j/\mathcal{E}^1(U^m)^{j+1} \to \dots \\ &\to\; \varinjlim_{m\in\mathbb{N}} \mathcal{E}^n(U^m)^j/\mathcal{E}^n(U^m)^{j+1} \to (0)\end{aligned}$$

for $j = 0, \dots, n-1$ (induced by the complex $\mathcal{E}^\bullet$) the totality of which can be considered as the E_0-page of the spectral sequence induced by the filtered complex

$$(0) \to \varinjlim_{m\in\mathbb{N}} \mathcal{E}^0(U^m) \to \varinjlim_{m\in\mathbb{N}} \mathcal{E}^1(U^m) \to \dots \to \varinjlim_{m\in\mathbb{N}} \mathcal{E}^r(U^m) \to (0)$$

computing $\mathrm{H}^*_{\mathrm{rig}}(\mathcal{X}^{(n+1)}/K, \mathcal{E})$, i.e.

$$E_0^{r,s} = \varinjlim_{m\in\mathbb{N}} \mathcal{E}^{r+s}(U^m)^r / \varinjlim_{m\in\mathbb{N}} \mathcal{E}^{r+s}(U^m)^{r+1} \Longrightarrow_r \mathrm{H}^{r+s}_{\mathrm{rig}}(\mathcal{X}^{(n+1)}/K, \mathcal{E}). \tag{3.11}$$

Depending on some cohomological information about $\mathcal{E}$, one might now be able to compute this spectral sequence and thus $\mathrm{H}^*_{\mathrm{rig}}(\mathcal{X}^{(n+1)}/K, \mathcal{E})$ explicitly. As an illustration, consider again

$$\mathcal{E} = \mathcal{O}_{\mathbb{P}_K^{n,\mathrm{rig}}},$$

i.e.

$$\mathcal{E}^i = \Omega^i_{\mathbb{P}_K^{n,\mathrm{rig}}/K},$$

which has the following property:

Lemma 3.3.4.1. *The complex*

$$\begin{aligned}(0) &\to \varinjlim_{m\in\mathbb{N}} \tilde{\mathrm{H}}^{n-j}_{]\mathbb{P}^j_k[_{P,\lambda_m}}(\mathbb{P}^{n,\mathrm{rig}}_K, \Omega^0_{\mathbb{P}^{n,\mathrm{rig}}_K/K}) \to \varinjlim_{m\in\mathbb{N}} \tilde{\mathrm{H}}^{n-j}_{]\mathbb{P}^j_k[_{P,\lambda_m}}(\mathbb{P}^{n,\mathrm{rig}}_K, \Omega^1_{\mathbb{P}^{n,\mathrm{rig}}_K/K}) \to \dots \\ &\to \varinjlim_{m\in\mathbb{N}} \tilde{\mathrm{H}}^{n-j}_{]\mathbb{P}^j_k[_{P,\lambda_m}}(\mathbb{P}^{n,\mathrm{rig}}_K, \Omega^n_{\mathbb{P}^{n,\mathrm{rig}}_K/K}) \to (0)\end{aligned}$$

is acyclic for all $j = 0, \dots, n-1$.

Proof. (cf. the proof of [45, Prop. 5]) Fix $j \in \{0, \dots, n-1\}$. First of all, by construction, there are isomorphisms

$$\tilde{\mathrm{H}}^{n-j}_{]\mathbb{P}^j_k[_{P,\lambda_m}}(\mathbb{P}^{n,\mathrm{rig}}_K, \Omega^i_{\mathbb{P}^{n,\mathrm{rig}}_K/K}) \cong \mathrm{coker}\left(\mathrm{H}^{n-j-1}(\mathbb{P}^{n,\mathrm{rig}}_K, \Omega^i_{\mathbb{P}^{n,\mathrm{rig}}_K/K}) \to \mathrm{H}^{n-j-1}(\mathbb{P}^{n,\mathrm{rig}}_K \backslash]\mathbb{P}^j_k[_{P,\lambda_m}, \Omega^i_{\mathbb{P}^{n,\mathrm{rig}}_K/K})\right)$$

for all $i = 0, \dots, n$. As $n-j-1 < n-j$, it follows from (3.10) that

$$\mathrm{H}^{n-j-1}_{]\mathbb{P}^j_k[_{P,\lambda_m}}(\mathbb{P}^{n,\mathrm{rig}}_K, \Omega^i_{\mathbb{P}^{n,\mathrm{rig}}_K/K}) = (0)$$

for all $i = 0, \dots, n$. Therefore, for each $i \in \{0, \dots, n\}$, using the long exact sequence from local cohomology, one obtains a short exact sequence

$$\begin{aligned}(0) &\to \varinjlim_{m\in\mathbb{N}} \mathrm{H}^{n-j-1}(\mathbb{P}^{n,\mathrm{rig}}_K, \Omega^i_{\mathbb{P}^{n,\mathrm{rig}}_K/K}) = \mathrm{H}^{n-j-1}(\mathbb{P}^{n,\mathrm{rig}}_K, \Omega^i_{\mathbb{P}^{n,\mathrm{rig}}_K/K}) \\ &\to \varinjlim_{m\in\mathbb{N}} \mathrm{H}^{n-j-1}(\mathbb{P}^{n,\mathrm{rig}}_K \backslash]\mathbb{P}^j_k[_{P,\lambda_m}, \Omega^i_{\mathbb{P}^{n,\mathrm{rig}}_K/K}) \to \varinjlim_{m\in\mathbb{N}} \tilde{\mathrm{H}}^{n-j}_{]\mathbb{P}^j_k[_{P,\lambda_m}}(\mathbb{P}^{n,\mathrm{rig}}_K, \Omega^i_{\mathbb{P}^{n,\mathrm{rig}}_K/K}) \to (0)\end{aligned}$$

– again by exactness of $\varinjlim_{m\in\mathbb{N}}$ – which then gives rise to a short exact sequence of complexes

$$\begin{aligned}(0) &\to \left(\mathrm{H}^{n-j-1}(\mathbb{P}^{n,\mathrm{rig}}_K, \Omega^i_{\mathbb{P}^{n,\mathrm{rig}}_K/K})\right)_{i=0,\dots,n} \to \left(\varinjlim_{m\in\mathbb{N}} \mathrm{H}^{n-j-1}(\mathbb{P}^{n,\mathrm{rig}}_K \backslash]\mathbb{P}^j_k[_{P,\lambda_m}, \Omega^i_{\mathbb{P}^{n,\mathrm{rig}}_K/K})\right)_{i=0,\dots,n} \\ &\to \left(\varinjlim_{m\in\mathbb{N}} \tilde{\mathrm{H}}^{n-j}_{]\mathbb{P}^j_k[_{P,\lambda_m}}(\mathbb{P}^{n,\mathrm{rig}}_K, \Omega^i_{\mathbb{P}^{n,\mathrm{rig}}_K/K})\right)_{i=0,\dots,n} \to (0).\end{aligned}$$

Application of the Snake Lemma then yields a long exact sequence of cohomology objects

$$\begin{aligned}\dots &\to h^{l-1}\left(\varinjlim_{m\in\mathbb{N}} \tilde{\mathrm{H}}^{n-j}_{]\mathbb{P}^j_k[_{P,\lambda_m}}(\mathbb{P}^{n,\mathrm{rig}}_K, \Omega^i_{\mathbb{P}^{n,\mathrm{rig}}_K/K})\right)_{i=0,\dots,n} \to h^l\left(\mathrm{H}^{n-j-1}(\mathbb{P}^{n,\mathrm{rig}}_K, \Omega^i_{\mathbb{P}^{n,\mathrm{rig}}_K/K})\right)_{i=0,\dots,n} \\ &\to h^l\left(\varinjlim_{m\in\mathbb{N}} \mathrm{H}^{n-j-1}(\mathbb{P}^{n,\mathrm{rig}}_K \backslash]\mathbb{P}^j_k[_{P,\lambda_m}, \Omega^i_{\mathbb{P}^{n,\mathrm{rig}}_K/K})\right)_{i=0,\dots,n} \\ &\to h^l\left(\varinjlim_{m\in\mathbb{N}} \tilde{\mathrm{H}}^{n-j}_{]\mathbb{P}^j_k[_{P,\lambda_m}}(\mathbb{P}^{n,\mathrm{rig}}_K, \Omega^i_{\mathbb{P}^{n,\mathrm{rig}}_K/K})\right)_{i=0,\dots,n} \to \dots \quad .\end{aligned}$$

Now, because of the fact that

$$h^*\left(\mathrm{H}^{n-j-1}(\mathbb{P}^{n,\mathrm{rig}}_K, \Omega^i_{\mathbb{P}^{n,\mathrm{rig}}_K/K})\right)_{i=0,\dots,n} = K[-(n-j-1)] \tag{3.12}$$

by rigid GAGA, cf. [12, 4.10.5], it is sufficient to show that

$$h^*\left(\varinjlim_{m\in\mathbb{N}} \mathrm{H}^{n-j-1}(\mathbb{P}^{n,\mathrm{rig}}_K \backslash]\mathbb{P}^j_k[_{P,\lambda_m}, \Omega^i_{\mathbb{P}^{n,\mathrm{rig}}_K/K})\right)_{i=0,\dots,n} = K[-(n-j-1)] \tag{3.13}$$

to prove the lemma. This will be done by computing the rigid cohomology $\mathrm{H}^*_{\mathrm{rig}}((\mathbb{P}^n_k \setminus \mathbb{P}^j_k)/K)$ from a system of strict open neighborhoods of $]\mathbb{P}^n_k \setminus \mathbb{P}^j_k[_P$ in $\mathbb{P}^{n,\mathrm{rig}}_K$ and then comparing with the formula which can be obtained from (3.1):

By definition,

$$\mathbb{P}^n_k \setminus \mathbb{P}^j_k = \bigcup_{l=j+1}^{n} D_+(T_l),$$

with closed complement $\mathbb{P}^j_k = \bigcap_{l=j+1}^n V_+(T_l)$ in $\mathbb{P}^n_k$. For each $m \in \mathbb{N}$, set

$$W^m = \mathbb{P}^{n,\mathrm{rig}}_K \backslash]\mathbb{P}^j_k[_{P,\lambda_m}$$

and denote by

$$f_m : W^m \hookrightarrow \mathbb{P}^{n,\mathrm{rig}}_K$$

the inclusion. By construction (see page 69), the system $(W^m)_{m\in\mathbb{N}}$ is a cofinal system of strict open neighborhoods of $]\mathbb{P}^n_k \setminus \mathbb{P}^j_k[_P$ in $\mathbb{P}^{n,\mathrm{rig}}_K$. Thus, by definition,

$$\mathrm{H}^*_{\mathrm{rig}}((\mathbb{P}^n_k \setminus \mathbb{P}^j_k)/K) = \mathbb{H}^*(\mathbb{P}^{n,\mathrm{rig}}_K, \varinjlim_{m\in\mathbb{N}} f_{m*}f_m^*\Omega^\bullet_{\mathbb{P}^{n,\mathrm{rig}}_K/K}).$$

Taking the tube of radius λ commutes with taking finite intersections of closed subspaces (cf. [37, 2.3.5]). Therefore,

$$\begin{aligned} W^m &= \mathbb{P}^{n,\mathrm{rig}}_K \setminus \bigcap_{l=j+1}^{n}]V_+(T_l)[_{P,\lambda_m} \\ &= \bigcup_{l=j+1}^{n} \mathbb{P}^{n,\mathrm{rig}}_K \backslash]V_+(T_l)[_{P,\lambda_m} \end{aligned}$$

and thus each f_m is a quasi-Stein morphism in the sense of [37, p. 20], since W^m is a finite union of affinoid varieties, cf. (the proof of) Lemma 3.3.0.4, ii). This particularly implies that the higher direct image $\mathbf{R}^i f_{m*}$ vanishes for $i > 0$ so that the above hypercohomology can be computed as

$$\begin{aligned} \mathbb{H}^*(\mathbb{P}^{n,\mathrm{rig}}_K, \varinjlim_{m\in\mathbb{N}} f_{m*}{f_m}^*\Omega^\bullet_{\mathbb{P}^{n,\mathrm{rig}}_K/K}) &= \varinjlim_{m\in\mathbb{N}} \mathbb{H}^*(\mathbb{P}^{n,\mathrm{rig}}_K, f_{m*}f_m^*\Omega^\bullet_{\mathbb{P}^{n,\mathrm{rig}}_K/K}) \\ &= \varinjlim_{m\in\mathbb{N}} \mathbb{H}^*(W^m, f_m^*\Omega^\bullet_{\mathbb{P}^{n,\mathrm{rig}}_K/K}) \\ &= \varinjlim_{m\in\mathbb{N}} \mathbb{H}^*(W^m, \Omega^\bullet_{\mathbb{P}^{n,\mathrm{rig}}_K/K}). \end{aligned}$$

Therefore, it can be computed as the cohomology of the total complex associated with the double complex

$$\left(\varinjlim_{m\in\mathbb{N}} \bigoplus_{j+1\le l_0<\ldots<l_r\le n} \Gamma\left(\bigcap_{e=0}^{r} \mathbb{P}^{n,\mathrm{rig}}_K \backslash]V_+(T_{l_e})[_{P,\lambda_m}, \Omega^s_{\mathbb{P}^{n,\mathrm{rig}}_K/K}\right)\right)^{s=0,\ldots,n}_{r=0,\ldots,n-j}$$

which has, say, as r-th row the Čech complex for the sheaf $\Omega^r_{\mathbb{P}^{n,\mathrm{rig}}_K}$, $r = 0, \ldots, n$, associated with the above covering of W^{λ_m}. Taking cohomology along the rows of this double complex yields the E_1-page of a spectral sequence

$$E_1^{r,s} = \varinjlim_{m\in\mathbb{N}} \mathrm{H}^s(\mathbb{P}^{n,\mathrm{rig}}_K \backslash]\mathbb{P}^j_k[_{P,\lambda_m}, \Omega^r_{\mathbb{P}^{n,\mathrm{rig}}_K/K}) \Longrightarrow \mathrm{H}^*_{\mathrm{rig}}((\mathbb{P}^n_k \setminus \mathbb{P}^j_k)/K).$$

Combining (3.10) with the long exact local cohomology sequence and (3.12), one now obtains the desired result (3.13) from computing the E_2-page of this spectral sequence: it can be seen from (3.1), again using the long exact cohomology sequence for rigid cohomology, that

$$\mathrm{H}^*_{\mathrm{rig}}((\mathbb{P}^n_k \setminus \mathbb{P}^j_k)/K) = \bigoplus_{l=0}^{n-j-1} K[-2l].$$

□

It follows that the complex

$$\begin{aligned}(0) \quad \to \quad & \varinjlim_{m\in\mathbb{N}} \mathrm{Ind}^{\mathbf{G}(\mathcal{V})}_{\mathbf{P}_{(j+1,n-j)}(\mathcal{V})(1)} \left(\tilde{\mathrm{H}}^{n-j}_{]\mathbb{P}^j_k[_{P,\lambda_m}}(\mathbb{P}^{n,\mathrm{rig}}_k, \Omega^0_{\mathbb{P}^{n,\mathrm{rig}}_K/K}) \otimes \mathrm{St}_{n-j}(K)\right) \to \dots \\ \to \quad & \varinjlim_{m\in\mathbb{N}} \mathrm{Ind}^{\mathbf{G}(\mathcal{V})}_{\mathbf{P}_{(j+1,n-j)}(\mathcal{V})(1)} \left(\tilde{\mathrm{H}}^{n-j}_{]\mathbb{P}^j_k[_{P,\lambda_m}}(\mathbb{P}^{n,\mathrm{rig}}_k, \Omega^n_{\mathbb{P}^{n,\mathrm{rig}}_K/K}) \otimes \mathrm{St}_{n-j}(K)\right) \to (0)\end{aligned}$$

is acyclic for all $j = 0, \dots, n-1$, hence the E_1-page of the spectral sequence (3.11) computes as

$$\begin{aligned} E_1^{r,s} &= h^s(E_0^{r,\bullet}) \\ &= h^s(\varinjlim_{m\in\mathbb{N}} \mathcal{F}^{r+\bullet}(U^m)^r/\mathcal{F}^{r+\bullet}(U^m)^{r+1}) \\ &= h^s(v^{\mathbf{G}(\mathcal{V})}_{\mathbf{P}_{(r+1,1^{n-r})}(\mathcal{V})(1)}(K)' \otimes_K \mathrm{H}^{n-r}(\mathbb{P}^{n,\mathrm{rig}}_k, \Omega^{r+\bullet}_{\mathbb{P}^{n,\mathrm{rig}}_K/K})) \\ &= \begin{cases} v^{\mathbf{G}(\mathcal{V})}_{\mathbf{P}_{(r+1,1^{n-r})}(\mathcal{V})(1)}(K)' & \text{if } r \in \{0,\dots,n\}, s = n-2r \\ (0) & \text{else.} \end{cases} \end{aligned}$$

From this shape it follows that the spectral sequence degenerates on its E_1-page, i.e.

$$E_1^{r,s} = E_\infty^{r,s} = \mathrm{gr}^r(h^{r+s}(\varinjlim_{m\in\mathbb{N}} \Omega^\bullet_{\mathbb{P}^{n,\mathrm{rig}}_K/K}(U^m))) = \mathrm{gr}^r(\mathrm{H}^{r+s}_{\mathrm{rig}}(\mathcal{X}^{(n+1)}/K)).$$

Furthermore, for each $I \subsetneq \Delta$, the natural identification

$$\mathbf{G}(\mathcal{V})/\mathbf{P}_I(\mathcal{V})(1) \to G/P_I$$

of sets with $\mathbf{G}(\mathcal{V})$-action yields an isomorphism

$$v^{\mathbf{G}(\mathcal{V})}_{\mathbf{P}_I(\mathcal{V})(1)}(K) \xrightarrow{\sim} v^G_{P_I}(K)$$

of $\mathbf{G}(\mathcal{V})$-modules. Therefore, the following theorem is (re)proved:

Theorem 3.3.4.2. *The rigid cohomology of $\mathcal{X}^{(n+1)}$ is given by*

$$\mathrm{H}^*_{\mathrm{rig}}(\mathcal{X}^{(n+1)}/K) = \bigoplus_{i=0}^{n} v^G_{P_{(1+n-i,1^i)}}(K)'[-i].$$

Adding Tate twists then yields the formula that would be obtained from Theorem 3.2.4.3 by using Poincaré duality.

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