On some classes of q-plurisubharmonic functions and q-pseudoconcave sets

Dissertation zur Erlangung des Doktorgrades
Doctor rerum naturalium

im
Fachbereich C - Mathematik und Naturwissenschaften
der
Bergischen Universität Wuppertal

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Juni 2015

Die Dissertation kann wie folgt zitiert werden:

urn:nbn:de:hbz:468-20151210-101726-1
[http://nbn-resolving.de/urn/resolver.pl?urn=urn%3Anbn%3Ade%3Ahbz%3A468-20151210-101726-1]

Dedicated to my family

Gewidmet meiner Familie

Black
then
white are
all I see
in my infancy.
Red and yellow then came to be,
reaching out to me,
lets me see.

(Tool, Lateralus, 2001)

Foreword

The aim of my doctoral thesis is to give new statements on q-plurisubharmonic functions, their subfamilies and the sets generated by them. These include q-pseudoconvex and q-pseudoconcave sets, generalized holomorphically convex hulls and the Shilov boundary for subfamilies of q-plurisubharmonic functions. It was my motivation to generalize the classical results on complex analysis and pluripotential theory such as Lelong's results on the relation of convex and plurisubharmonic functions, Bochner's tube theorem, Hartogs' theorem on separate analyticity, Shcherbina's theorem on foliation of continuous graphs, the Cartan-Thullen theorem and the solution of the Levi problem, the concept of the Bergman-Shilov boundary and Bychkov's geometric characterization of the Shilov boundary. I will summarize the new and most important results in the introduction after giving a short historical overview on q-plurisubharmonicity and q-pseudoconvexity.

Results on q-plurisubharmonic functions and q-pseudoconvex sets can already be found in a vast amount of articles. Therefore, I decided to collect their fundamental properties, extend them by new ones and add them to my thesis. Due to the survey character of some chapters and the concise and elementary proofs of the statements, this thesis can be read by undergraduate students as well.

I shall also mention that some of the achieved results have been already published in form of a survey article [PZ13] by me and E. S. Zeron (from the research institute Cinvestav in Mexico City) in the Journal of Mathematical Analysis and Applications (Vol. 408, Issue I) in December 2013. The second part [PZ15] of that article is also completed and ready for submission to an appropriate journal. Moreover, the contents related to the Shilov boundary for subfamilies of q-plurisubharmonic functions are collected in my article [Paw13] which I placed so far only on arXiv, but which is also ready for submission to a suitable journal. The part about q-pseudoconcave graphs is a joint work of me and my advisor N. V. Shcherbina based on his notes on that topic, but it requires further investigations and is therefore not finished yet. I also plan to extend and release the part about real q-convex functions and find an interesting application for these kinds of functions.

Thomas Patrick Pawlaschyk

(Wuppertal, Juni 2015)

Acknowledgements

Of course, I have to thank Prof. N. V. Shcherbina for his constant support, mathematical discussions and the opportunities to participate in so many interesting conferences and workshops during my years at the University of Wuppertal. Many thanks to all my other colleagues in the complex analysis group in Wuppertal for many discussions on mathematics and especially to my fellow Ph.D. students Tobias Harz and Martin L. Sera for a great and funny time at our university and outside Wuppertal in Albi, Barcelona, Będlewo, Bonn, Cetraro, Grenoble, Lille, Marseille (Luminy), México Ciudad, Oberwohlfach, Poznań, Pisa, Pohang, Reykjavík, Toulouse and Wermelskirchen.

Many thanks to my family, especially to my mother Veronika and my father Christian, who never hesitated in a single moment to support me during my school and study periods and life in general. Being your son makes me very proud every time. [Großen Dank an meine Familie, besonders an meine Mutter Veronika und meinen Vater Christian, die zu keinem einzigen Moment gezögert haben, mich während meiner Schul- und Studienzeit und im Leben allgemein zu unterstützen. Euer Sohn zu sein, erfüllt mich jedesmal mit großem Stolz.] Also thanks to my brothers Damian and Adam and my pals Robert and Martin for so many hilarious moments. I have great respect for your achievements in life. Big thanks to my wonderful and sweet girlfriend Leman who restored me after a horrible period in my life allowing me to continue with my mathematics with a free mind and full concentration. Also many thanks and sportive regards to my non-complex analysis colleague and friend Dr. Lukas Krämer, my colleagues Dr. Jae-Cheon Joo and Dr. Richard Lärkäng, but also to Arne Thielenhaus and Prof. Peter Pflug who helped me with correcting this thesis and/or giving good hints and advices. Finally, many thanks to all the other friends who enjoyed life with me.

Last but not least, I want to mention that I was partially supported by the DFG under the project 'Pluripotential Theory, Hulls and Foliations' (grant SH 456/1-1). The support by the German Academic Exchange Service (DAAD, Deutscher Akademischer Austauschdienst), PPP Proalmex Project No.51240052, permitted me to collaborate with E. S. Zeron in 2011 during a two-week stay in Mexico City. In this context, I would like to thank Prof. Zeron who showed interest in my work which culminated in two articles on q-plurisubharmonicity and q-pseudoconvexity. I also have to thank my colleague and long-year office mate Dr. J. Ruppenthal who suggested this research collaboration and who generously supported it with his DAAD grant mentioned above (and, not to forget, the trip to beautiful Iceland). Since summer 2012, I have been supported by the German Federal Ministry of Education and Research under the project 'Gemeinsames Bund-Länder-Programm für bessere Studienbedingungen und mehr Qualität in der Lehre', grant 01PL12046, which allows me to work as the inofficial assistant of Prof. G. Herbort and to help improve the teaching at the University of Wuppertal. Also big thanks to Prof. K.-T. Kim for his incredible hospitality during my stay at the Postech university (Pohang, Korea) in March 2013, where I was able to develop most of the ideas related to real q-convexity in the inspiring atmosphere of the GAIA research center.

Contents

Part I

Introduction

Historical overview

The classical pluripotential theory is one the most interesting topics in complex analysis. It has its roots in the fundamental works of Kiyoshi Oka (*1901-†1978) and Pierre Lelong (*1912-†2011), who examined independently *domaines pseudoconvexes* and *fonctions plurisousharmonique* in the early 1940s (see, e.g., [Oka42] and [Lel42]). A $\mathcal{C}^2$-smooth plurisubharmonic function f is characterized by demanding that its *Levi matrix* (or *complex Hessian*) $\left(\partial^2 f/\partial z_j \partial \overline{z}_k\right)_{j,k=1}^n$ has no negative eigenvalues. It is then natural to examine functions with the property that the number of negative eigenvalues of their Levi matrix is positive and has a fixed upper bound. It seems that Hans Grauert (*1930-†2011) was one of the first who investigated these type of functions on complex spaces in the 1950s. He called them *q-convex functions*, where q is an integer number greater or equal to one. In his convention, q-convex functions are those $\mathcal{C}^2$-smooth functions whose Levi matrix have at most $q-1$ non-positive eigenvalues. They led to the definition of *q-convex sets* and the Grauert-Andreotti theory, which investigates certain $\overline{\partial}$-cohomology classes of q-convex sets (see [Gra59] or the book [HL88] of G. M. Henkin and J. Leiterer). In the late 1970s, Louis R. Hunt and John J. Murray introduced upper semi-continuous q-plurisubharmonic functions on domains in the complex Euclidean space $\mathbb{C}^n$ in their joint article [HM78]. In the case of $\mathcal{C}^2$-smoothness, they showed that strictly q-plurisubharmonic functions are exactly $(q+1)$-convex functions, so that q-plurisubharmonicity can be viewed as a generalization of q-convexity in the sense of Grauert. Moreover, q-plurisubharmonic functions are strongly linked to *q-holomorphic* functions studied by R. Basener earlier in the late 1970s (see [Bas76]). These functions fulfill the non-linear differential equation $\overline{\partial} f \wedge (\partial\overline{\partial} f)^q = 0$ and obviously generalize the classical holomorphic functions which appear in the case $q = 0$. Similar to the case of holomorphic and plurisubharmonic functions, Hunt and Murray showed that the real part, imaginary part and the logarithm of the absolute value of a q-holomorphic function are indeed q-plurisubharmonic. Further important works, especially related to approximation techniques for q-plurisubharmonic functions, are due to K. Diederich and J. E. Fornæess [DF85], Z. Słodkowski [Sło84] and L. Bungart [Bun90].

The origin of q-pseudoconvexity lies in the works of W. Rothstein [Rot55] and Grauert [Gra59] in the 1950s, both being students of Heinrich Behnke (*1898-†1979). They were motivated by different reasons to introduce q-convex sets. As explained before, Grauert used them to give results on the dimension of certain $\overline{\partial}$-cohomology classes, whereas Rothstein was interested in the extension of sliced analytic sets. Later, further studies of these sets were performed in the 1960s by O. Fujita [Fuj64] and M. Tadokoro [Tad65]. They introduced *pseudoconvex sets of order* $n - q$ and showed that their notion and that of Rothstein and Grauert coincide. In the following decades, the q-convex sets and functions on complex spaces were studied by many different mathematicians like K. Diederich, J. E. Fornæess, T. Peternell, A. Popa-Fischer, M. Colţoiu and V. Vâjâitu, just to name a few. Z. Słodkowski in [Sło86] and O. Fujita [Fuj90] characterized q-pseudoconvexity in terms of boundary distance functions. In [Bas76], R. Basener introduced the *q-holomorphically convex sets* using q-holomorphic functions and compared them with *(strictly) Levi q-pseudoconvex* sets. These are smoothly bounded sets admitting a (strictly) q-plurisubharmonic (or $(q + 1)$-convex) defining function in some neighborhood of its boundary. In [HM78], Hunt and Murray used strictly Levi q-pseudoconvex sets to solve the Dirichlet problem for q-plurisubharmonic functions on Levi q-pseudoconvex sets.

The vast amount of different notions of q-pseudoconvexity and q-plurisubharmonicity with all the confusing changes of the index q motivated us to create one single monograph collecting the knowledge about q-plurisubharmonic and q-pseudoconvex functions and extending it by further properties which were not known so far. The subjects of q-convexity have been already listed and examined in various books (see, e.g., [HL88], [For11] or [Dem12]), surveys (see for example [Col97]) and articles (see [Die06]). Together with E. S. Zeron, we collected properties of q-plurisubharmonic and q-holomorphic functions and sets generated by them, listed most of the different notions of q-pseudoconvexity and gave a proof of their equivalence in a survey article [PZ13]. This article was extended by the second part [PZ15] in which we investigated generalized holomorphically convex hulls and the q-pseudoconvexity of their complements. It is completed and shall be submitted to a journal soon.

Short historical overviews on the Bergman-Shilov boundary and the complex foliation of continuous graphs can be found in the corresponding sections of this thesis.

Chapter overview and new results

This thesis is divided into two parts. In the first one, we introduce upper semi-continuous, real q-convex and q-plurisubharmonic functions and a large amount of their subfamilies. We give their properties and compare them to each other. The main outcome will be approximation techniques for real q-convex, q-plurisubharmonic and q-holomorphic functions. The second part deals with sets generated by q-plurisubharmonic functions like q-pseudoconvex sets, generalized convex hulls of compact sets, complements of q-pseudoconvex sets (i.e., q-pseudoconcave sets) and the Shilov boundary for subfamilies of q-plurisubharmonic functions. In the following, we will describe the content of each chapter and give the most important new results.

In Chapter 1, we recall the definition of upper semi-continuous functions defined on a (compact) Hausdorff spaces with image in $[-\infty, +\infty)$ and present those properties which are relevant to our purposes. More precisely, we are interested in the upper semi-continuous regularization of a locally bounded function, maximal values of upper semi-continuous functions on compact spaces and approximation of upper semi-continuous functions by continuous ones. Monotone sequences of families $\mathcal{A}$ of upper semi-continuous functions lead to the notion of the *monotone closure* $\overline{\mathcal{A}}^{\downarrow}$ of $\mathcal{A}$. It consists of all limits of point-wise decreasing sequences $(f_n)_{n\in\mathbb{N}}$ in $\mathcal{A}$ and will play a useful role in the study of the Shilov boundary in the last two chapters. The reason is that the maximal values $\max_X f_n$ of a decreasing sequence $(f_n)_{n\in\mathbb{N}}$ of upper semi-continuous functions on a compact Hausdorff space X also decreases to the maximal value of its limit function. We include detailed proofs of the statements for the sake of completeness, even though most of them are well-known to experts in this field.

Chapter 2 begins with a very short survey about convex sets and functions. They serve to introduce so-called *real q-convex functions*. These generalize convex functions and share similar properties as their complex relatives, the q-plurisubharmonic functions which we examine in Chapter 3. An upper semi-continuous function on an open set ω in $\mathbb{R}^n$ is called *real q-convex* if for every plane π of dimension $q+1$, every ball B lying relatively compact in $\pi \cap \omega$ and each linear function ℓ satisfying $u \leq \ell$ on the boundary of B, this inequality already holds on the whole of B. It seems that these functions were not studied yet from the viewpoint of complex analysis and pluripotential theory. In fact, all these observations on real q-convex functions come from q-plurisubharmonicity, but it seems that they have never been written down explicitly. For this reason, we give a detailed list of properties of real q-convex functions and develop approximation techniques. For instance, one of the approximation techniques is motivated by Słodkowski's approximation of q-plurisubharmonic functions by continuous ones in [Sło84].

Theorem 2.6.3 *Let $\omega_1 \Subset \omega$ be two open sets in $\mathbb{R}^n$. Then each real q-convex function on ω can be approximated from above by a decreasing sequence of continuous real q-convex functions on ω_1 which are additionally twice differentiable* *almost everywhere on ω_1.*

This enables to show that the sum of a real q-convex and a real r-convex function is real $(q+r)$-convex (see Theorem 2.6.4). The latter fact is a key result in order to show a Bungart-type approximation for real q-convex functions. In fact, Bungart proved a similar result for continuous q-plurisubharmonic functions in [Bun90] solving a certain Dirichlet problem.

Theorem 2.6.8 *Any continuous real q-convex function defined on an open set ω in $\mathbb{R}^n$ can be approximated from above by a decreasing sequence of piecewise smooth real q-convex functions on ω.*

In Chapter 3, we give the definition of *q-plurisubharmonic* functions in the sense of Hunt and Murray [HM78]. An upper semi-continuous function ψ on an open set Ω in $\mathbb{C}^n$ is *q-plurisubharmonic* if for every complex plane π of dimension $q+1$, every ball B lying relatively compact in $\pi \cap \Omega$ and each pluriharmonic function h satisfying $\psi \leq h$ on the boundary of B it also fulfills $\psi \leq h$ on the whole of B. We give their properties, introduce relevant subfamilies of q-plurisubharmonic functions and study their relations to each other. Since they are based on holomorphic, harmonic and plurisubharmonic functions, we devote a very brief section to these classical functions.

We recall Fujita's result [Fuj90] stating that the composition $\psi \circ f$ of a q-plurisubharmonic function ψ on Ω in $\mathbb{C}^n$ and a holomorphic function f defined on an open set G in $\mathbb{C}^k$ with image in Ω remains q-plurisubharmonic on G. We also give a detailed proof which differs from that of Fujita and which is similar to that of N. Dieu in [Die06] (see Theorem 3.7.1). It is also included in our joint article [PZ13]. Dieu's proof is based on Słodkowski's and Bungart's approximation techniques for q-plurisubharmonic functions, whereas that of Fujita uses boundary distance functions and q-pseudoconvex sets. Another application of the approximation techniques are the following statements about q-plurisubharmonic and real q-convex functions.

Theorem 3.6.1 *A real q-convex function u defined on an open set Ω in $\mathbb{C}^n$ is q-plurisubharmonic on Ω.*

A converse result involving real q-convex and rigid q-plurisubharmonic functions generalizes a classical result by P. Lelong [Lel52b].

Theorem 3.6.4 *An upper semi-continuous function u defined on an open set ω in $\mathbb{R}^n$ is real q-convex if and only if the rigid function $\psi(z) := u(\mathrm{Re}(z))$ is q-plurisubharmonic on $\Omega := \omega + i\mathbb{R}^n$.*

Composing q-plurisubharmonic and real r-convex functions yields the subsequent statement.

Theorem 3.5.6 *Let $(\psi_\ell)_{\ell=1,\dots,k}$ be a collection of functions which are q_ℓ-plurisubharmonic on an open set Ω in $\mathbb{C}^n$ such that the mapping $\psi = (\psi_1, \dots, \psi_k)$ on Ω has its image in some open set ω in $\mathbb{R}^k$. If u is real r-convex on ω, then the composition $u \circ \psi$ is $(q+r)$-plurisubharmonic on Ω, where $q = \sum_{\ell=1}^k q_\ell$.*

The q-plurisubharmonic functions can be defined on analytic sets and, thus, on leaves of a certain singular complex foliation by analytic sets.

Theorem 3.9.6 *Let $q \in \{0, \dots, n-1\}$ and $r \geq 0$ be integers and let Ω be an open set in $\mathbb{C}^n$. Given a holomorphic mapping $h : \Omega \to \mathbb{C}^q$, every upper semi-continuous function ψ on Ω, which is r-plurisubharmonic on the fiber $h^{-1}(c)$ for every $c \in h(\Omega)$, is $(q+r)$-plurisubharmonic on Ω.*

Moreover, it is possible to approximate q-plurisubharmonic functions on singular foliations by a decreasing sequence of $\mathcal{C}^\infty$-smooth ones (see Theorem 3.9.11). We also recall the definition of q-holomorphic functions in the sense of R. Basener [Bas76], extend it to functions defined on analytic sets and list their main properties. We obtain a result similar to the previous theorem in terms of holomorphic functions on a singular foliation.

Theorem 3.11.5 *Let Ω be an open set in $\mathbb{C}^n$ and let $q \in \{0, \dots, n-1\}$. Given a holomorphic mapping $h : \Omega \to \mathbb{C}^q$, every $\mathcal{C}^2$-smooth function f on Ω, which is holomorphic on the analytic fiber $h^{-1}(c)$ for every $c \in h(\Omega)$, is q-holomorphic on Ω. Moreover, every continuous function on Ω which is holomorphic on the fiber $h^{-1}(c)$ for every $c \in h(\Omega)$ can be uniformly approximated on compact sets by q-holomorphic functions on Ω.*

The Chapter 3 ends with a Bremermann type approximation for plurisubharmonic functions on a singular foliation (see Theorem 3.11.8). Most of these results will be included in our second joint article [PZ15].

In Chapter 4, we give a list of the different notions of q-pseudoconvexity existing in the literature, extend it by further characterizations and show their equivalence in Theorem 4.3.2. This list is included in the joint article [PZ13]. The study of plurisubharmonic functions on singular foliations enables us to give a new characterization of q-pseudoconvex sets in terms of boundary distance functions. This part can be found in [PZ15].

Theorem 4.3.2 *Let Ω be an open set in $\mathbb{C}^n$ and let $\|\cdot\|$ be any complex norm function on $\mathbb{C}^n$. Define $d_{\|\cdot\|}(z, b\Omega) := \inf\{\|z-w\| : w \in b\Omega\}$. Then Ω is q-pseudoconvex, i.e., it admits a q-plurisubharmonic exhaustion function if and only if $-\log d_{\|\cdot\|}(z, b\Omega)$ is q-plurisubharmonic on Ω. Moreover, the set Ω is q-pseudoconvex if and only if it is Hartogs $(n-q-1)$-pseudoconvex in the sense of Rothstein [Rot55].*

We generalize a classical result by P. Lelong [Lel52b] which goes back to S. Bochner [Boc38] and is nowadays known as *Bochner's tube theorem.*

Theorem 4.4.2 *Given a number* $a > 0$ *and an open set* ω *in* $\mathbb{R}^n$*, the tubular set* $\omega + i(-a, a)^n$ *is* q*-pseudoconvex if and only if its base* ω *is real* q*-convex, i.e., it admits a real* q*-convex exhaustion function.*

We study complements of q-pseudoconvex sets which leads to a duality theorem suggested by N.V. Shcherbina. Here, an open set G is called *strictly Levi* k*-pseudoconvex* inside another open set V in $\mathbb{C}^n$ if G is the sublevel set of a smooth strictly k-plurisubharmonic function ψ on V with non-vanishing gradient on the boundary of G in V.

Theorem 4.6.4 *Let* Ω *be a domain in* $\mathbb{C}^n$ *which is not* q*-pseudoconvex. Then there exist a point* $p \in b\Omega$*, a neighborhood* V *of* p *and a strictly Levi* $(n-q-2)$*-pseudoconvex set* G *in* V *such that* $V \setminus G$ *touches* bG *from the inside of* G *only at* p*.*

This important tool allows to analyze the complex structure of submanifolds and continuous graphs in $\mathbb{C}^n$ whose complement is q-pseudoconvex.

Theorem 4.7.9 *Let* n, k, p *be integers with* $n \geq 1$*,* $p \geq 0$ *and* $k \in \{0, 1\}$ *such that* $N = n+k+p \geq 2$*. Let* G *be a domain in* $\mathbb{C}^n_z \times \mathbb{R}^k_u$ *and let* $f : G \to \mathbb{R}^k_v \times \mathbb{C}^p_\zeta$ *a continuous function such that the complement of the graph* $\Gamma(f)$ *in* $\mathbb{C}^N_{z,u+iv,\zeta}$ *is Hartogs* n*-pseudoconvex in the sense of Rothstein. Then* $\Gamma(f)$ *is locally foliated by* n*-dimensional complex submanifolds.*

This statement generalizes one of the classical Hartogs' theorems (case $n \geq 1$, $k = 0$ and $p = 1$) as well as results by N. V. Shcherbina [Shc93] (case $n = 1$, $k = 1$ and $p = 0$) and E. M. Chirka [Chi01] (case $n \geq 1$, $k = 1$ and $p = 0$). Results on the case of $k \geq 2$ are not known to us yet and needs further investigations.

In Chapter 5, we investigate generalized convex hulls of the form

$$\widehat{K}_{\mathcal{A}} := \{z \in \Omega : \psi(z) \leq \max_K \psi \text{ for every } \psi \in \mathcal{A}\},$$

where $\mathcal{A}$ is some family of upper semi-continuous functions on Ω in $\mathbb{C}^n$ and K is a compact subset of Ω. These hulls allow to characterize q-pseudoconvex domains Ω in $\mathbb{C}^n$ by demanding that for every compact set K in Ω its hull $\widehat{K}_{\mathcal{PSH}_q(\Omega)}$ generated with respect to q-plurisubharmonic functions on Ω has to be compactly contained inside Ω. It is then interesting to compare these hulls to generalized polynomially and rationally convex hulls (studied by R. Basener [Bas78], G. Lupacciolu and E. L. Stout [LS99]) and to the *q-pseudoconvex hull* $\mathcal{H}_q(K)$ of a compact set K in $\mathbb{C}^n$. The last one is defined as the intersection of all q-pseudoconvex sets surrounding K. Similar to the arguments of J. P. Rosay [Ros06] in the case of $q = 0$, we show that the q-plurisubharmonic hull and the q-pseudoconvex hull of K satisfy a generalized version of Rossi's local maximum principle [Ros60] for q-plurisubharmonic functions (see Proposition 5.3.9). Then the techniques developed by Słodkowski in [Sło86] lead to the following statement on the complement of these hulls.

Theorem 5.3.11 *Let Ω be a q-pseudoconvex open set in $\mathbb{C}^n$ and let K be a compact set outside Ω. Then $\Omega \setminus \hat{K}_{\mathcal{PSH}_{n-q-2}(\Omega)}$ and $\Omega \setminus \mathcal{H}_{n-q-2}(K)$ are again q-pseudoconvex.*

The results on the generalized hulls and their complement is collected in the joint article [PZ15].

In Chapter 6, we study the Bergman-Shilov boundary for upper semi-continuous functions defined on a compact Hausdorff space X. It is defined as the smallest closed subset $\check{S}_{\mathcal{A}}$ of X with the property that every function of a given family $\mathcal{A}$ of upper semi-continuous functions attains its maximum on X inside the set $\check{S}_{\mathcal{A}}$. For an additive family of upper semi-continuous functions whose exponentials generate the topology of X, the existence and uniqueness of the Bergman-Shilov boundary is guaranteed by the work of J. Siciak [Sic62]. A much more general result can be found in L. Aĭzenberg's book [Aiz93] in the case of a compact set X in $\mathbb{C}^n$ and a family $\mathcal{A}$ of upper semi-continuous functions which only fulfills the property that for every function f in $\mathcal{A}$ and every complex number c the sum $f + \log|z - c|$ also belongs to $\mathcal{A}$. In this case, no assumption on convexity of the family of upper semi-continuous functions is imposed. Therefore, Aĭzenberg's statement fits perfectly for q-plurisubharmonic functions, since they do not form a convex cone and are stable under sums with plurisubharmonic functions. Anyway, it is still not sufficient in the case of, e.g., smooth q-plurisubharmonic functions. Therefore, we were motivated to establish the following general result on the existence of the Bergman-Shilov boundary.

Theorem 6.2.4 *Let $\mathcal{A}$ be a family of upper semi-continuous functions on a compact Hausdorff space X. If $\mathcal{A}$ contains a subset $\mathcal{A}_0$ which generates the topology of X such that $\mathcal{A} + \mathcal{A}_0 \subset \mathcal{A}$, then the Shilov boundary $\check{S}_{\mathcal{A}}$ exists and is unique.*

In the spirit of Bishop [Bis59] and the case of uniform function algebras, we are also able to obtain generalized peak point theorems for a broader class of continuous and upper semi-continuous functions: one involves unions of uniform subalgebras (see Corollary 6.4.3), and the other one uses families of functions which are stable under small pertubations by continuous ones (see Theorem 6.4.7).

In the final Chapter 7, we apply the results of Chapter 6 to a large amount of families of q-plurisubharmonic and q-holomorphic functions, show the existence of the corresponding (Bergman-) Shilov boundaries and establish peak point properties for these families of functions.

The approximation techniques of Słodkowski and Bungart imply that the Shilov boundary for q-plurisubharmonic functions and that for $\mathcal{C}^2$-smooth q-plurisubharmonic functions coincide (see Proposition 7.1.5). This permits us to characterize the Shilov boundary of a $\mathcal{C}^2$-smoothly bounded relatively compact open set for q-plurisubharmonic functions: it is precisely the closure of all strictly q-pseudoconvex boundary points of that set (see Theorem 7.1.9). We compare the Shilov boundary for subfamilies of r-plurisubharmonic and holomorphic functions on singular foliations to lower dimensional Shilov boundaries.

Proposition 7.3.1 *Let Ω be an open set in $\mathbb{C}^n$ and K be a compact set in Ω. Fix two integers $r \in \mathbb{N}_0$ and $q \in \{1, \ldots, n-1\}$. If $h : \Omega \to \mathbb{C}^q$ is a holomorphic mapping and $A = h^{-1}(c)$ for some $c \in h(\Omega)$, then*

$$\check{S}_{\mathcal{PSH}_r(h,\Omega)}(K) \cap A = \check{S}_{\mathcal{PSH}_r(A)}(K \cap A)$$

and

$$\check{S}_{\mathcal{PSH}_r(h,K)}(K) \cap A = \check{S}_{\mathcal{PSH}_r(K \cap A)}(K \cap A).$$

A similar result is true if we replace r-plurisubharmonic functions on singular foliations ($\mathcal{PSH}_r(h, \Omega)$ and $\mathcal{PSH}_r(h, K)$) by holomorphic functions on singular foliations ($\mathcal{O}(h, \Omega)$ and, respectively, $\mathcal{O}(h, K)$; see Proposition 7.3.2).

In Theorem 7.4.3, we show that the Shilov boundary for some special families of q-holomorphic functions and the Shilov boundary of q-th order investigated by R. Basener in [Bas78] are the same.

Finally, we mention the main goal is of this section, namely the generalization of Bychkov's result [Byč81], which gives a geometric characterization of the Shilov boundary of convex bodies in $\mathbb{C}^n$ for functions holomorphic on the interior of K and continuous up the boundary of K.

Theorem 7.6.7 *Let $q \in \{0, \ldots, n-2\}$ and $D \Subset \mathbb{C}^n$ be a convex open set. Then $bD \setminus \check{S}_{\mathcal{PSH}_q(\overline{D})}$ is exactly the set of all boundary points of D which have a neighborhood U in bD such that U consists only of points each of which is* *contained in some open part of a $(q+1)$-dimensional complex plane lying in bD.*

In fact, one can even replace the q-plurisubharmonic functions by continuous ones which are holomorphic on a regular foliation by complex planes of codimension q. We end the final chapter by giving an estimate on the Hausdorff dimension of the Shilov boundaries of convex bodies for q-plurisubharmonic functions which partially generalizes a result by Bychkov in the two-dimensional case (see Theorem 7.6.13). Most of the results of the Chapters 6 and 7 about the Shilov boundary for subfamilies of upper semi-continuous functions are included in the article [Paw13].

Basic notations

In order to improve the presentation, we fix the following notations and conventions which are valid throughout the whole thesis.

The symbols $\mathbb{N} = \{1, 2, 3, \ldots\}$ and $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$ stand for the set of all positive and, respectively, non-negative integers. Given two sets Y and X, by $Y \Subset X$ we mean that Y *is relatively compact* in X, i.e., the closure $\overline{Y}$ is a compact subset of X. Unless otherwise stated, a *neighborhood* U of a point x in X is always assumed to be an open set U in X containing x. If $f : X \to Y$ is a function defined on a set X with image in another set Y, then for a fixed value $y \in Y$ we sometimes simply write $\{f = y\}$, $\{f > y\}$ or $\{f < y\}$ instead of $\{x \in X : f(x) = y\}$, $\{x \in X : f(x) > y\}$ or, respectively, $\{x \in X : f(x) < y\}$ whenever it is clear from the context how and where these sets are defined. The supremum of a complex valued function f defined on a set X is denoted by $\sup_X |f|$ or $\|f\|_X$. If f is not complex valued but has values in $[-\infty, +\infty) := \mathbb{R} \cup \{-\infty\}$, then we write $\sup_X f$ for the supremum of f on X. If it has values in $(-\infty, +\infty] := \mathbb{R} \cup \{+\infty\}$, we the symbol $\inf_X f$ stands for the infimum of f on X. By convention, the supremum of the constant function $-\infty$ on X equals of course $-\infty$, whereas the infimum of the constant function $+\infty$ is exactly $+\infty$. We also define that $\pm\infty \cdot 0 = 0$. For $k \in \{1, \ldots, \infty\}$, the symbol $\mathcal{C}^k(X)$ stands for the family of complex valued $\mathcal{C}^k$-smooth functions on a normed space X over the complex numbers $\mathbb{C}$. If the image of these functions lies in the real numbers $\mathbb{R}$, we will sometimes write $\mathcal{C}^k(X, \mathbb{R})$. If $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, by *planes in* $\mathbb{K}^n$ we always mean affine linear subspaces which do not necessarily pass through the origin, and *linear functions* always are considered to be affine linear.

More conventions are set throughout this thesis where they will be needed.

Part II

Real q-convexity and q-plurisubharmonicity

Chapter 1

Semi-continuity

Most of the functions which will be defined and studied in the forthcoming chapters have one basic property in common: each of them is upper semi-continuous. It is therefore important to us to recall the notion of upper semi-continuous functions defined on a topological Hausdorff space X and to show some of their properties and examples. We shall only mention those results and examples which we will actually use in later chapters. Even though most of the statements we will present below are already well-known and can be found in the literature (see, e.g., [Bou98], §6.2), we shall present a detailed collection of the statements and give most of their proofs. Especially, we are interested in the upper semi-continuous regularization of an arbitrary function and the approximation of upper semi-continuous functions by a decreasing sequence of continuous ones. This property motivates the notion of the *monotone closure* of a family of upper semi-continuous functions on a Hausdorff space X.

1.1 Upper semi-continuous functions

In the whole first Chapter, the symbol X will always denote a topological Hausdorff space. We give the formal definition of upper semi-continuous functions defined on X.

Definition 1.1.1 Let X be a Hausdorff space.

(1) A function $f : X \to [-\infty, +\infty)$ is called *upper semi-continuous on* X if the sublevel set $\{x \in X : f(x) < c\}$ is open in X for every $c \in \mathbb{R}$. The family of all upper semi-continuous functions on X is denoted by $\mathcal{USC}(X)$.

(2) We say that the function $f : X \to (-\infty, +\infty]$ is *lower semi-continuous on* X if $-f : X \to [-\infty, +\infty)$ is upper semi-continuous on X.

(3) A real valued function f defined on X is *continuous on* X if f is upper semi-continuous and lower semi-continuous on X. The set $\mathcal{C}(X, \mathbb{R}) = \mathcal{C}^0(X, \mathbb{R})$ denotes the family of all continuous functions on X.

(4) A complex valued function f on X is *continuous on* X if its real part and imaginary part are both continuous on X. The family of all complex valued continuous functions on X is given by the set $\mathcal{C}(X, \mathbb{C})$. We also use the notations $\mathcal{C}^0(X, \mathbb{C})$, $\mathcal{C}^0(X)$ or $\mathcal{C}(X)$ for the set $\mathcal{C}(X, \mathbb{C})$.

We can create new upper semi-continuous functions out of subfamilies of upper semi-continuous functions.

Proposition 1.1.2 *Let again X be a Hausdorff space.*

(1) The maximum $f := \max\{f_j : j = 1, \ldots, n\}$ of finitely many upper semi-continuous functions $f_1, \ldots, f_n$ on X is again upper semi-continuous on X.

(2) Let $\{f_j\}_{j \in J}$ be a family of upper semi-continuous functions on X. Then the infimum function $f := \inf\{f_j : j \in J\}$ is also upper semi-continuous on X.

(3) Every decreasing sequence $f_1 \geq f_2 \geq f_3 \geq \ldots$ of upper semi-continuous functions on X converges pointwise to an upper semi-continuous function on X.

(4) The family $\mathcal{USC}(X)$ forms a convex cone, i.e., for every f, g in $\mathcal{USC}(X)$ and two numbers $\lambda, \mu \geq 0$ we have that $\lambda f + \mu g$ lies again in $\mathcal{USC}(X)$. Notice that we use the convention $-\infty \cdot 0 = 0$.

(5) The two families $\mathcal{C}(X, \mathbb{R})$ and $\log |\mathcal{C}(X)| := \{\log |f| : f \in \mathcal{C}(X, \mathbb{C})\}$ lie in $\mathcal{USC}(X)$.

We give our first example of an upper semi-continuous function which will be important later.

Example 1.1.3 Let S be a subset of X. Then the *characteristic functions* χ_S and $\check{\chi}_S$ of S given by

$$\chi_S(x) := \begin{cases} 1, & x \in S \\ 0, & x \in X \setminus S \end{cases} \quad \text{and} \quad \check{\chi}_S(x) := \begin{cases} 0, & x \in S \\ -\infty, & x \in X \setminus S \end{cases}$$

are both upper semi-continuous on X if and only if S is closed.

The following proposition gives two examples of continuous functions created by the supremum.

Proposition 1.1.4 *Let X and Y be two Hausdorff spaces and assume that Y is compact. Let $f_1 \not\equiv -\infty$ be an upper semi-continuous function on X and f_2 be a continuous function on $X \times Y$. Then the function defined by*

$$X \ni x \ \mapsto \ g(x) = \sup\{f_1(y) + f_2(x,y) : y \in Y\}$$

is continuous on X. If, in addition, f_1 and f_2 are non-negative, then

$$X \ni x \ \mapsto \ h(x) = \sup\{f_1(y) f_2(x,y) : y \in Y\}$$

is continuous on X as well.

Proof. Notice first that it will follow from Proposition 1.3.1 that $g(x) < +\infty$ for every x in X, since Y is compact and $y \mapsto f_1(y) + f_2(x,y)$ is upper semi-continuous on Y. We will use this fact already now, since its (simple) proof does not require Proposition 1.1.4.

Let c be a real number such that $X_c := \{x \in X : g(x) < c\}$ is not empty. Let x_0 be a point in X_c. Choose a number $\varepsilon > 0$ so small that it fulfills $g(x_0)+\varepsilon < c$. It follows from the definition of g that

$$f_1(y_0) + f_2(x_0, y_0) \leq g(x_0) \qquad \text{for every } y_0 \in Y.$$

By the upper semi-continuity of f_1 and the continuity of f_2 there are neighborhoods U_{y_0} of x_0 and V_{y_0} of y_0 such that

$$f_1(y) + f_2(x,y) < f_1(y_0) + f_2(x_0, y_0) + \varepsilon \qquad \text{for every } (x,y) \in U_{y_0} \times V_{y_0}.$$

Since y_0 is an arbitrary point in Y and Y is compact, we can find finitely many neighborhoods $V_{y_1}, \ldots, V_{y_n}$ from the covering $\{V_{y_0}\}_{y_0 \in Y}$ such that $\{V_{y_j}\}_{j=1,\ldots,n}$ covers Y. Then the set $U := \bigcap_{j=1}^n U_{y_j}$ is a non-empty neighborhood of x_0. Moreover, if (x,y) is an arbitrary point in $U \times Y$, then $(x,y) \in U_{y_j} \times V_{y_j}$ for some $j \in \{1, \ldots, n\}$ and, therefore,

$$f_1(y) + f_2(x,y) < f_1(y_j) + f_2(x_0, y_j) + \varepsilon \leq g(x_0) + \varepsilon < c.$$

This implies $g(x) < c$ for every x in U, so x_0 is an inner point of X_c. We conclude that X_c is open and, hence, that g is upper semi-continuous on X. On the other hand, the function g is the supremum of a family $\{f_1(y) + f_2(\cdot, y)\}_{y \in Y}$

of lower semi-continuous functions on X, so it is lower semi-continuous on X. Altogether, we have shown that g is continuous on X.

Now assume that f_1 and f_2 are non-negative on X and on $X \times Y$, respectively. Define for a number $\delta > 0$ the function

$$g_\delta(x) := \sup\{\log(f_1(y) + \delta) + \log(f_2(x, y) + \delta) : y \in Y\}.$$

By the previous discussion, g_δ is continuous on X. Therefore,

$$h_\delta := \exp(g_\delta) = \sup\{(f_1(y) + \delta)(f_2(x, y) + \delta) : y \in Y\}$$

is continuous on X. Since the family $\{h_\delta\}_{\delta>0}$ tends uniformly on X to h as δ tends to zero, h is also continuous on X. □

1.2 Upper semi-continuous regularization

We recall the definition of the limes superior in terms of nets in X which serves to characterize upper semi-continuity and produce new upper semi-continuous functions.

Definition & Remark 1.2.1 Let X be a Hausdorff space.

(1) For a fixed point x in X we denote by $\mathcal{U}(x)$ the set of all open neighborhoods of x in X. If f is defined on $X \setminus \{x\}$, we define

$$\limsup_{y \to x} f(x) := \inf\big\{\sup\{f(y) : y \in U \setminus \{x\}\} : U \in \mathcal{U}(x)\big\}.$$

(2) Let Y be a subset of X with non-empty boundary bY containing some point x. For a function f defined on Y we set

$$\limsup_{\substack{y \to x \\ y \in Y}} f(x) := \inf\big\{\sup\{f(y) : y \in U \cap Y\} : U \in \mathcal{U}(x)\big\}.$$

(3) Given a function f defined on X, the *upper semi-continuous regularization f^* of f on X* is given by

$$X \ni x \ \mapsto \ f^*(x) := \limsup_{y \to x} f(x).$$

(4) If f is some locally bounded function on X, the upper semi-continuous regularization is upper semi-continuous on X and can be rewritten as

$$f^* = \inf\{g : g \in \mathcal{USC}(X),\ f \le g\}.$$

This identity follows immediately from the definitions of infimum, supremum and the upper semi-continuity and from Proposition 1.1.2.

(5) Therefore, a locally bounded function f on X is upper semi-continuous if and only if $f = f^*$.

We give an example of an upper semi-continuous regularization.

Example 1.2.2 Let q be a rational number in $(0,1)$. Then there are uniquely determined integers $k, n \in \mathbb{N}$ such that the fraction $q = k/n$ is represented in lowest terms (see, e.g., §405, Theorem V in [Olm62]). In this case the upper semi-continuous regularization f^* of the following function f defined on the interval $[0,1]$ by

$$f(x) := \begin{cases} 1 - 1/n, & x = k/n \in \mathbb{Q} \cap (0,1) \\ 0, & x \in ((0,1) \setminus \mathbb{Q}) \cup \{0,1\} \end{cases}$$

is constantly equal to 1, but $f(x) < 1$ for every x in $[0,1]$. Thus, f fails to be upper semi-continuous on $[0,1]$.

1.3 Maximal values

We continue by giving some properties related to maximal values of upper semi-continuous functions. As before, X stands for a Hausdorff space.

Proposition 1.3.1 *Let X be a compact Hausdorff space and let f be upper semi-continuous on X. Then there exists a point $x_0 \in X$ with* $\sup_X f = f(x_0)$.

Proof. Assume that f does not attain its maximum on X and set $a := \sup\{f(\xi) : \xi \in X\}$. Then for every $x \in X$ we have that $f(x) < a$, so we can find a small positive number $\varepsilon_x > 0$ such that $f(x) + \varepsilon_x < a$. On the other hand, by the upper semi-continuity of f there is a neighborhood U_x of x such that $f(y) < f(x) + \varepsilon_x$ for every $y \in U_x$. Since X is compact, there are finitely many points $x_1, \ldots, x_\ell$ such that $U_{x_1}, \ldots, U_{x_\ell}$ cover X. Now set $b := \max\{f(x_j) + \varepsilon_{x_j} : j = 1, \ldots, \ell\}$. Then we obtain the following contradiction, $f(y) \le b < a$ for every $y \in X$. □

The previous result motivates the following notations.

Definition & Remark 1.3.2 If a function f attains its maximum in X, we write $\max_X f$ rather than $\sup_X f$. Especially, if f is a complex valued function on X, we simply write $\|f\|_X := \sup_X |f|$. Now if X is a compact Hausdorff space, $\|\cdot\|_X$ defines a Banach norm on $\mathcal{C}(X,\mathbb{C})$. Then it is easy to see that if a sequence $(f_n)_{n\in\mathbb{N}}$ of functions $f_n \in \mathcal{C}(X,\mathbb{C})$ converges uniformly on X (i.e., with respect to $\|\cdot\|_X$) to a function f, then f is again in $\mathcal{C}(X,\mathbb{C})$ and $\|f_n\|_X$ converges to $\|f\|_X$ as n tends to $+\infty$.

We are interested in the behavior of the maximal values of decreasing families of upper semi-continuous functions.

Lemma 1.3.3 *Let $\{f_j\}_{j\in J}$ be a family of upper semi-continuous functions on X. Let $f := \inf\{f_j : j \in J\}$ and let g be a lower semi-continuous function on X with $f < g$ on X. Then for every compact set K in X there exists a finite subset I of J such that*

$$f \le \min\{f_i : i \in I\} < g \text{ on } K.$$

Proof. Fix a point $x \in K$. Then there is an index $j(x) \in J$ such that $f(x) \le f_{j(x)}(x) < g(x)$. Since $f_{j(x)} - g$ is upper semi-continuous on X, we can find an open neighborhood U_x of x such that $f_{j(x)}(y) < g(y)$ for every $y \in U_x$. By the compactness of K there are finitely many points $x_1, \dots, x_\ell$ in K such that $U_{x_1}, \dots, U_{x_\ell}$ cover K. We set $I := \{j(x_1), \dots, j(x_\ell)\}$. Then we easily obtain the desired inequalities $f \le \min\{f_i : i \in I\} < g$ on K. □

This observation leads to the following property which is similar to the last one mentioned in Remark 1.3.2.

Proposition 1.3.4 *Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of upper semi-continuous functions on X decreasing to f on a compact set K in X. Then*

$$\lim_{n\to\infty} \max_K f_n = \max_K .$$

Proof. The limit $a := \lim_{n\to\infty} \max_K f_n$ exists because $(\max_K f_n)_{n\in\mathbb{N}}$ is a decreasing sequence which is bounded from below by $\max_K f$. Assume that $a > \max_K f$. By the previous Lemma 1.3.3 we can find a large enough integer

n_0 such that $a > f_{n_0}(y)$ for every $y \in K$. This is a contradiction to the definition of a. $\square$

1.4 Upper semi-continuity in normed spaces

The main purpose of this section is to fix some basic notations on normed spaces and to give a lemma which will turn out to be one the most important tools in the study of the families of functions we will define later.

Definition & Remark 1.4.1 Let $\mathbb{K}$ be one of the fields $\mathbb{R}$ or $\mathbb{C}$.

(1) A normed space $(X, \|\cdot\|)$ over the field $\mathbb{K}$ induces the metric

$$X \times X \ni (x, y) \;\mapsto\; d_{\|\cdot\|}(x, y) := \|x - y\|.$$

(2) Of course, each norm function $x \mapsto \|x\|$ is continuous with respect to the topology induced by $d_{\|\cdot\|}$.

(3) Especially, if $x = (x_1, \ldots, x_n) \in \mathbb{K}^n$, we are interested in the *Euclidean norm* $\|\cdot\|_2$ and the *maximum norm* $\|\cdot\|_\infty$ defined by

$$\|x\|_2 := \sqrt{\sum_{j=1}^{n} |x_j|^2} \quad \text{and} \quad \|x\|_\infty := \max\{|x_j| : j = 1, \ldots, n\}.$$

(4) One can show that the sequence $(\|x_1^k, \ldots, x_n^k\|_2^{1/k})_{k\in\mathbb{N}}$ decreases to to $\|x\|_\infty$ if k tends to $+\infty$.

(5) The *ball* $B_r^n(x_0)$ in $\mathbb{K}^n$ with radius $r > 0$ centered at x_0 is defined by

$$B_r^n(x_0) := \{x \in \mathbb{K}^n : \|x - x_0\|_2 < r\}.$$

We also sometimes write $B_r(x_0)$ instead of $B_r^n(x_0)$.

(6) The *polydisc* $\Delta_r^n(x_0)$ in $\mathbb{C}^n$ with radius $r > 0$ centered at x_0 is given by

$$\Delta_r^n(x_0) := \{x \in \mathbb{C}^n : \|x - x_0\|_\infty < r\}.$$

(7) We already have defined the *uniform norm* $\|f\|_X := \sup_X |f|$ for a complex valued function f on X. If X is compact, the norm $\|\cdot\|_X$ endows $\mathcal{C}(X, \mathbb{C})$ with a Banach space structure.

(8) A vector space X equipped with an inner product $\langle \cdot, \cdot \rangle$ induces a norm $\|\cdot\|$ on X by $\|x\| := \sqrt{\langle x, x \rangle}$.

(9) Clearly, we have the following connection between the inner product and its induced norm,

$$\|x+y\|^2 = \|x\|^2 + 2\mathrm{Re}\langle x, y \rangle + \|y\|^2 \quad \text{for every } x, y \in X.$$

Notice that, if $\mathbb{K} = \mathbb{R}$, then $\mathrm{Re}\langle x, y \rangle = \langle x, y \rangle$ for every $x, y \in X$.

The following lemma is one of the most important tools for our purposes. It is basically Lemma 4.5 in [Sło84], but we have to point out that a similar technique has already been used in the proof of Lemma 2.7 in [HM78].

Lemma 1.4.2 (Striking lemma) *Let X be a vector space over the field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ equipped with the inner product $\langle \cdot, \cdot \rangle$. Let $\|\cdot\|$ denote its induced norm and let u be an upper semi-continuous function on a compact set K in X. Suppose that there is another compact set L in K with $\max_L u < \max_K u$. Then there are a point p in $K \setminus L$, a real number $\varepsilon > 0$ and an $\mathbb{R}$-linear function $\ell : X \to \mathbb{R}$ such that*

$$u(p) + \ell(p) = 0 \quad \text{and} \quad u(x) + \ell(x) < -\varepsilon\|x-p\|^2 \quad \text{for every } x \in K \setminus \{p\}.$$

Proof. For a positive number $\varepsilon > 0$ define $u_\varepsilon(x) := u(x) + \varepsilon\|x\|^2$. Choose $\varepsilon > 0$ so small that $\max_L u_\varepsilon < M := \max_K u_\varepsilon$. Pick a point p in K with $u_\varepsilon(p) = M$. Then p can not lie in L. We define $\ell(x) := \varepsilon\|x\|^2 - \varepsilon\|x-p\|^2 - M$. Since

$$\|x\|^2 - \|x-p\|^2 = -2\mathrm{Re}\langle x, p \rangle - \|p\|^2,$$

the function ℓ is indeed $\mathbb{R}$-linear. Furthermore, we have that

$$u(p) + \ell(p) = u_\varepsilon(p) - M = 0.$$

Since $M := \max_K u_\varepsilon$, we conclude that

$$u(x) + \ell(x) = u_\varepsilon(x) - \varepsilon\|x-p\|^2 - M < -\varepsilon\|x-p\|^2 \quad \text{for every } x \in K \setminus \{p\}.$$

□

1.5 Monotone closures

On a metric space X we have more interesting examples of continuous functions.

Example 1.5.1 (1) Let A be a closed subset of a metric space $X = (X, d)$. Then the *distance function relative to* A defined by

$$x \mapsto d(x, A) := \inf\{d(x, y) : y \in A\}$$

is continuous on X.

(2) If Y is an open subset of $X = (X, d)$ with non-empty boundary, then the *boundary distance function of* Y given by $x \mapsto d(x, bY)$ is continuous on Y.

(3) Given a closed set S in $X = (X, d)$ and a number $\varepsilon > 0$ we set

$$S^{(\varepsilon)} := \{x \in X : d(x, S) < \varepsilon\} \text{ and } \chi_S^\varepsilon(x) := \begin{cases} 1, & x \in S \\ 0, & x \in X \setminus S^{(\varepsilon)} \\ 1 - \frac{1}{\varepsilon} d(x, S), & x \in S^{(\varepsilon)} \setminus S \end{cases} .$$

Then the function χ_S^ε is continuous on X, and the family $\{\chi_S^\varepsilon\}_{\varepsilon > 0}$ decreases to the characteristic function χ_S as ε tends to zero.

(4) For an open set U in $X = (X, d)$ and $\varepsilon > 0$, we define

$$U_{(\varepsilon)} := \{x \in U : d(x, bU) > \varepsilon\} \text{ and } \chi_{U,\varepsilon}(x) := \begin{cases} 1, & x \in U_\varepsilon \\ 0, & x \in X \setminus U \\ \frac{1}{\varepsilon} d(x, bU), & x \in U \setminus U_\varepsilon \end{cases} .$$

The function $\chi_{U,\varepsilon}$ is continuous on X, and the family $\{\chi_{U,\varepsilon}\}_{\varepsilon > 0}$ increases to the characteristic function χ_U of U in X as ε tends to zero.

The last example yields a continuous partition of unity on a metric space which serves to obtain a classical approximation theorem. Recall that a topological space X is called *paracompact* if every open cover of X has an open refinement that is locally finite. Furthermore, X is *locally compact* if every point in X has a compact neighborhood.

Theorem 1.5.2 *Let (X, d) be a metric space which is paracompact and locally compact. Then for every upper semi-continuous function f on X there exists a sequence $(f_n)_{n \in \mathbb{N}}$ of continuous functions f_n on X which decreases to f as n tends to $+\infty$.*

Proof. Since X is paracompact and locally compact, it is easy to see that we can find a locally finite covering $\mathcal{U} = \{U_j\}_{j\in J}$ of X by open relatively compact sets U_j in X. For a fixed index j in J consider the set $U_{j,\varepsilon} := \{x \in U_j : d(x, bU_j) > \varepsilon\}$ and the continuous function $\chi_{U_{j,\varepsilon}}$ from Example 1.5.1 (4). Then for every j in J we can choose $\varepsilon(j) > 0$ so small that the collection $\mathcal{V} := \{V_j := U_{j,\varepsilon(j)}\}_{j\in J}$ is again a locally finite covering of X. It is easy to see that, in a neighborhood of a given point x in X, the sum $S := \sum_{j\in J} \chi_{V_j}$ is positive, finite and, therefore, continuous. Thus, the function $P_j := \frac{1}{S}\chi_{V_j}$ is continuous on X and fulfills $0 \le P_j \le 1$ and $\sum_{j\in J} P_j = 1$, so it is a *partition of unity* on X.

Now fix a number $n \in \mathbb{N}$. Since f is upper semi-continuous and the function $(x, y) \mapsto d(x, y)$ is continuous on $X \times X$ with respect to the product topology induced by the metric on X, the function

$$f_{j,n}(x) := \sup\{f(y) - nd(x, y) : y \in \overline{V_j}\}$$

is bounded from above and continuous on $\overline{V_j}$ according to Propositions 1.3.1 and 1.1.4.

We assert that the sequence $(f_{j,n})_{n\in\mathbb{N}}$ decreases to f on $\overline{V_j}$ if n tends to $+\infty$. Indeed, it is clear that

$$f \le f_{j,n+1} \le f_{j,n} \text{ on } \overline{V_j} \text{ for every } n \in \mathbb{N}.$$

Let x be an arbitrary point in $\overline{V_j}$ and let η be an arbitrary positive number. By the upper semi-continuity of f, there is a number $\delta > 0$ such that $f(y) < f(x)+\eta$ for every y with $d(x, y) < \delta$. Now if $n > M := \sup\{f(y) - f(x) : y \in \overline{V_j}\}/\delta$, for every y in $\overline{V_j}$ it holds that

$$f(y) - nd(x, y) \le \begin{cases} f(x) + \eta, & x \in \overline{V_j},\ d(x, y) < \delta \\ \sup_{\overline{V_j}} f - \delta M = f(x), & x \in \overline{V_j},\ d(x, y) \ge \delta \end{cases}.$$

Therefore, $f_{j,n}(x) < f(x) + \eta$. This proves the assertion.

Now since $\mathcal{V}$ is a locally finite covering of X, the function $f_n := \sum_{j\in J} P_j f_{j,n}$ is continuous on X. Moreover, the sequence $(f_n)_{n\in\mathbb{N}}$ decreases to f on X if n tends to $+\infty$. □

In the last theorem, we were able to give an approximation of an arbitrary upper semi-continuous function by a decreasing family of continuous functions. In this sense, continuous functions are *dense* in the family of upper semi-continuous functions. This motivates to define a notion the *monotone closure* of a subfamily of upper semi-continuous functions. This monotone closure will have a similar meaning to us as the uniform closure in $\mathcal{C}(X)$, which we shall recall at first.

Definition & Remark 1.5.3 Let $\mathcal{B}$ be a subfamily of $\mathcal{C}(X)$ and Y be a subset of X. By $\overline{\mathcal{B}}^Y$ we denote the *uniform closure* of $\mathcal{B}$ in Y, i.e., the set of all complex-valued continuous functions g on Y such that there exists a sequence $(g_n)_{n\in\mathbb{N}}$ of continuous functions g_n in $\mathcal{B}$ which uniformly converges to g on Y.

In order to define the monotone closure, the main idea is to use decreasing sequences of upper semi-continuous functions.

Definition 1.5.4 Let Y be a subset of X and let $\mathcal{A}$ be a family of upper semi-continuous functions on a Hausdorff space X.

(1) The *(monotone) closure* $\overline{\mathcal{A}}^{\downarrow Y}$ of $\mathcal{A}$ in Y is composed by all upper semi-continuous functions f on Y such that there exists a sequence $(f_n)_{n\in\mathbb{N}}$ of functions f_n in $\mathcal{A}$ which decreases pointwise to f on Y, i.e., for every $x \in Y$ it holds that

$$f_n(x) \geq f_{n+1}(x) \text{ for every } n \in \mathbb{N} \quad \text{and} \quad \lim_{n\to\infty} f_n(x) = f(x).$$

(2) If $Y = X$, we also simply write $\overline{\mathcal{A}}^{\downarrow} := \overline{\mathcal{A}}^{\downarrow X}$.

(3) We say that $\mathcal{A}$ is *closed* if $\mathcal{A} = \overline{\mathcal{A}}^{\downarrow}$.

(4) Given an integer $k \in \mathbb{N}$, we define iteratively the *monotone closure of* $\mathcal{A}$ *of order* k by

$$\overline{\mathcal{A}}^{k\downarrow} := \overline{\overline{\mathcal{A}}^{(k-1)\downarrow}}^{\downarrow}.$$

Notice that we use the convention $\overline{\mathcal{A}}^{0\downarrow} := \mathcal{A}$.

(5) The *(monotone) closure of* $\mathcal{A}$ *of infinite order* is the set

$$\overline{\mathcal{A}}^{\infty\downarrow} := \bigcup_{k=0}^{\infty} \overline{\mathcal{A}}^{k\downarrow}.$$

We give a list of simple observations which are easy to verify.

Proposition 1.5.5 *(1) By Proposition 1.1.2 (3), for every $k \in \mathbb{N}_0 \cup \{\infty\}$ the closure $\overline{\mathcal{A}}^{k\downarrow}$ of a family $\mathcal{A}$ of upper semi-continuous functions is in $\mathcal{USC}(X)$.*

(2) *For two given families $\mathcal{A}_1$ and $\mathcal{A}_2$ of upper semi-continuous functions on X with $\mathcal{A}_1 \subset \mathcal{A}_2$, it follows directly from the definition that*

$$\overline{\mathcal{A}_1}^{\downarrow Y} \subset \overline{\mathcal{A}_2}^{\downarrow Y} \text{ for every subset } Y \text{ in } X.$$

(3) *If Y_1 and Y_2 are two subsets of X with $Y_1 \subset Y_2$, then $\overline{\mathcal{A}}^{\downarrow Y_2} \subset \overline{\mathcal{A}}^{\downarrow Y_1}$.*

(4) *It follows from Proposition 1.1.2 (2) that that $\overline{\mathcal{USC}(X)}^{\downarrow} = \mathcal{USC}(X)$.*

(5) *If (X, d) is a paracompact and locally compact metric space, it follows from Theorem 1.5.2 that*

$$\overline{\mathcal{C}(X)}^{\downarrow X} = \mathcal{USC}(X).$$

(6) *Let $\mathcal{B}$ be a subfamily of $\mathcal{C}(X, \mathbb{R})$ and let Y be a compact subspace of X. Then*

$$\mathcal{C}(Y) \cap \overline{\mathcal{B}}^{\downarrow Y} \subset \overline{\mathcal{B}}^{Y}.$$

More precisely, if $(f_n)_{n\in\mathbb{N}}$ is a sequence of continuous functions on Y which decreases to a continuous limit f on X, then this convergence is already uniform on Y.

(7) *It is obvious that $\overline{\mathcal{A}}^{\ell\downarrow}$ lies in $\overline{\mathcal{A}}^{k\downarrow}$ for every $k, \ell \in \mathbb{N}_0$ with $\ell \leq k$.*

(8) *Therefore, $\overline{\mathcal{A}}^{\infty\downarrow} = \bigcup_{k=\ell}^{\infty} \overline{\mathcal{A}}^{k\downarrow}$ for every $\ell \in \mathbb{N}_0$.*

If X is compact and $\mathcal{B}$ is a subfamily of $\mathcal{C}(X)$, then one can show that $\overline{\overline{\mathcal{B}}^{X}}^{X} = \overline{\mathcal{B}}^{X}$. In contrary to this, it is not true, in general, that the closure of order k of $\mathcal{A}$ stays in the closure $\overline{\mathcal{A}}^{\downarrow}$ of $\mathcal{A}$. In other words, the notion of *montone closure* introduced above has not the same meaning as the usual closure in the topological sense. It becomes then an interesting question whether there is a better definition of the closure of $\mathcal{A}$ which yields a closed set in the classical and in our sense at the same time. For instance, a natural candidate would be the intersection of all closed supersets of $\mathcal{A}$, but it is unclear if it is closed itself.

We proceed by giving some examples which underline the difference between the monotone closure and the closure in the topological sense.

Example 1.5.6 (1) Consider the following upper semi-continuous functions on the one-point (or Alexandroff [Ale24]) compactification $K = [0, +\infty]$ of the interval $[0, +\infty)$. For an integer $n \in \mathbb{N}$ we set

$$f_n := \chi_{[1-\frac{1}{n+1}, 1]} \quad \text{and} \quad g_{n,k} := 1/k \cdot \chi_{\{1-\frac{1}{n+1}\}} + f_n.$$

The functions f_n decrease to $f_0 := \chi_{\{1\}}$. Now if $\mathcal{A}$ is the set $\{g_{k,n} : k, n \in \mathbb{N}\}$, then $\overline{\mathcal{A}}^{\downarrow} = \mathcal{A} \cup \{f_n : n \in \mathbb{N}\}$ and $\overline{\mathcal{A}}^{2\downarrow} = \overline{\mathcal{A}}^{\downarrow} \cup \{f_0\}$, but it is easy to see that f_0 can not be the limit of a decreasing sequence of functions from $\mathcal{A}$.

(2) One can think that after closing $\mathcal{A}$ finitely many times we obtain a closed set, but this turns out to be wrong. Given $k \in \mathbb{N}$ and $n_0, \dots, n_k \in \mathbb{N}$ consider the following upper semi-continuous function

$$h_{n_0,\dots,n_k}(x) := \sum_{j=0}^{k-1} g_{n_j,n_{j+1}}(x - j),$$

where $x \in [0, +\infty]$ and $g_{n_j,n_{j+1}}$ are the functions from the previous example. We set $\mathcal{A} := \{h_{n_0,\dots,n_k} : k \in \mathbb{N}, n_0, \dots, n_k \in \mathbb{N}\}$. Then we conclude that $\overline{\mathcal{A}}^{(k+1)\downarrow}$ contains the function $\chi_{\{1,\dots,k\}}$, but not $\chi_{\{1,\dots,k+1\}}$.

(3) Even if we take the closure of infinite order it will not lead to a closed set. Indeed, consider for given integers $k \in \mathbb{N}$ and $n_0, \dots, n_k \in \mathbb{N}$ the functions

$$G_k := \chi_{\{+\infty\}} + \sum_{j=k+1}^{\infty} (1 + 1/j)\chi_{\{j\}} \quad \text{and} \quad H_{n_0,\dots,n_k} := h_{n_0,\dots,n_k} + G_k,$$

where $h_{n_0,\dots,n_k}$ are the functions from the previous example. Now consider the family $\mathcal{A} := \{H_{n_0,\dots,n_k} : k \in \mathbb{N}, n_0, \dots, n_k \in \mathbb{N}\}$. Then by the same argument as before we can derive that $\overline{\mathcal{A}}^{\infty\downarrow}$ contains $\chi_{\{1,\dots,k\}} + g_k$ for every $k \in \mathbb{N}$, but it does not contain $\chi_{\{1,2,\dots,+\infty\}}$. Anyway, the sequence $(\chi_{\{1,\dots,k\}} + g_k)_{k\in\mathbb{N}}$ decreases to $\chi_{\{1,2,\dots,+\infty\}}$ as k tends to $+\infty$. Hence,

$$\chi_{\{1,2,\dots,+\infty\}} \in \overline{\overline{\mathcal{A}}^{\infty\downarrow}}^{\downarrow},$$

but $\chi_{\{1,2,\dots,+\infty\}} \notin \overline{\mathcal{A}}^{\ell\downarrow}$ for every $\ell \in \mathbb{N}$.

Chapter 2

Real q-convexity

Convex sets and functions are simple to define and to visualize. They form the basis for the convexity theory and have strong importance in linear optimization. In the first section, we are mainly interested in upper semi-continuous convex functions, their properties, approximation techniques and their relation to convex sets. In particular, we recall the classical result of Busemann-Feller-Alexandroff (see [BF36] and [Ale39]) which gives regularity properties of convex functions. More precisely, it states that every real-valued convex function is in fact twice differentiable almost everywhere. This fascinating statement serves to develop approximation techniques in the subsequent sections which are based on the supremum convolution (see, e.g, [Sło84]). This sort of convolution is more adequate for the functions we will introduce later rather than the classical integral convolution. Since only a few properties of convex functions and sets are relevant to us, we refer the interested reader to the books of R. T. Rockafellar [Roc70] and H. Busemann [Bus58] for an introduction to convexity theory.

Having the tools of convexity in hand, we introduce the notion of real q-convexity. Here, roughly speaking, q is a non-negative integer which measures in how many directions a real q-convex function fails to be convex. More precisely, if the function is smooth, it is real q-convex if and only if its real Hessian has everywhere at most q negative eigenvalues. Thus, in the case of $q = 0$, the real q-convex functions are exactly the locally convex ones. We shall examine their properties, establish appropriate approximation techniques and compare them later in Chapter 3 to their relatives in the complex setting: the q-plurisubharmonic functions. It seems that the real q-convex functions were not studied yet from the viewpoint of complex analysis. Surprisingly, even though convexity theory is full of various generalizations of convex functions, the real q-convex functions seem not to take a relevant place in its considerations. Nonetheless, studying the signature of the real Hessian of a function is important to Morse theory (to give information about the topology of certain sublevel sets) and to general relativity (using pseudo-Riemannian metrics).

We have to point out that the real q-convex functions resemble in such a strong way the q-plurisubharmonic ones that most of the proofs of their properties can be easily transfered with minor modifications from the existing articles on q-plurisubharmonicity (see [Die06], [HM78], [Sło84] or [Bun90]). Especially, Słodkowski's [Sło84] and Bungart's [Bun90] approximation techniques can be easily adapted to real q-convex functions. For the sake of completeness, we include their proofs in the forthcoming subsections.

2.1 Convex sets and functions

We recall the definition of convex sets in the Euclidean space $\mathbb{R}^n$. We refer to the books of Rockafellar [Roc70] and Busemann [Bus58] for more details on convexity.

Definition & Remark 2.1.1 Let K be a set in $\mathbb{R}^n$.

(1) The set K is called *convex* if for every two points x_1, x_2 contained in K the segment $[x_1, x_2] = \{(1-t)x_1 + tx_2 : 0 \leq t \leq 1\}$ also lies in K.

(2) The dimension of the smallest plane containing K is the *dimension of* K.

(3) If K is a compact convex set with non-empty interior, then it is called a *convex body.*

(4) Let K be a convex body and let p be a boundary point of K. Every hyperplane H in $\mathbb{R}^n$ splits the space $\mathbb{R}^n$ into two open half-spaces H^+ and H^-. The hyperplane H is then said to be *supporting for* K *at* p if H contains p and if K lies in one of the closed half-spaces $H \cup H^+$ or $H \cup H^-$.

We give the definition of convex functions.

Definition 2.1.2 Let K be a convex set and ω be an open set in $\mathbb{R}^n$.

(1) The function $u : K \to [-\infty, +\infty)$ is *convex on* K if for every two points p_1, p_2 in K and every pair of non-negative numbers λ_1, λ_2 with $\lambda_1 + \lambda_2 = 1$, we have the inequality

$$f(\lambda_1 p_1 + \lambda_2 p_2) \leq \lambda_1 f(p_1) + \lambda_2 f(p_2).$$

(2) The function $u : K \to (-\infty, +\infty]$ is called *concave* if $-u$ is convex.

(3) Given a function $u : \omega \to [-\infty, +\infty)$, it said to be *locally convex on* ω if for every point p in ω there is a ball B in ω around p such that u is convex on B.

(4) The function u is called *locally concave on* ω if $-u$ is locally convex on ω.

(5) The set of all locally convex functions on an open set ω in $\mathbb{R}^n$ is denoted by $\mathcal{CVX}(\omega)$.

It is also common to allow that convex functions attain the value $+\infty$. The main reason is that it is then possible to extend convex functions from a domain by the value $+\infty$ to the whole of $\mathbb{R}^n$, which makes it easier to work with those type of convex functions. Nevertheless, since we are mainly interested in upper semi-continuous functions, we consider only those convex functions which omit the value $+\infty$.

We present standard examples of convex functions.

Example 2.1.3 Every norm function on $\mathbb{R}^n$ is convex. Especially, the function $u(x) := \|x\|_2^2$ is convex on $\mathbb{R}^n$. Notice also, that if θ is a $\mathcal{C}^\infty$-smooth function with compact support in $\mathbb{R}^n$, then there is a small positive number $\varepsilon > 0$ such that $u + \varepsilon\theta$ remains convex on $\mathbb{R}^n$. Convex functions with such an property are also called *strongly convex*.

We give another relation between convex functions and convex sets.

Proposition 2.1.4 *Let K be a closed set in $\mathbb{R}^{n+1}_{x,y} = \mathbb{R}^n_x \times \mathbb{R}_y$ with non-empty interior. Denote by π the standard projection of $\mathbb{R}^{n+1}_{x,y}$ to $\mathbb{R}_y$ given by $\pi(x,y) = y$. Then K is convex in $\mathbb{R}^{n+1}$ if and only if $\pi(K)$ is convex and there are convex functions $-f_+$ and f_- on $\pi(K)$ such that*

$$K = \{(x,y) : x \in \pi(K),\ f_-(x) \le y \le f_+(x)\}. \tag{2.1}$$

Proof. Assume first that K is convex. Then for a fixed point x in $\pi(K)$ the slice $S_x = \{x\} \times \mathbb{R}$ is an interval, and we define the functions $f_\pm$ as follows,

$$f_-(x) := \inf S_x \quad \text{and} \quad f_+(x) := \sup S_x.$$

The convexity of K immediately implies that $\pi(K)$ is convex and that the functions f_- and $-f_+$ are convex on $\pi(K)$.

On the other hand, if K is given as in equation (2.1), it follows directly from the definition of convex functions that K is a convex set. □

Simple examples, like $x \mapsto |x|$, show that convex functions fail to be smooth everywhere. Nonetheless, each locally convex function admits a smooth approximation using the integral convolution.

Theorem 2.1.5 *Let ω be an open set in $\mathbb{R}^n$ and fix $k \in \mathbb{N} \cup \{+\infty\}$. Then*

$$\mathcal{CVX}(\omega) \subset \overline{\mathcal{CVX}(\omega_0) \cap \mathcal{C}^k(\omega_0)}^{\downarrow\omega_0} \quad \text{for every} \quad \omega_0 \Subset \omega.$$

Proof. For a positive number $\varepsilon > 0$ we set $\omega_\varepsilon := \{x \in U : d(x, b\omega) > \varepsilon\}$. If $\varepsilon_0 > 0$ is small enough, then ω_0 lies relatively compact in ω_ε for every $\varepsilon \leq \varepsilon_0$. Now let u be a real-valued locally convex function on ω and let θ be a $\mathcal{C}^k$-smooth non-negative function on $\mathbb{R}^n$ with compact support on the unit ball which also fulfills $\theta(x) = \theta(|x|)$ and $\int_{\mathbb{R}^n} \theta(t)dV(t) = 1$. Define $\theta_\varepsilon(x) := \varepsilon^{-n}\theta(x/\varepsilon)$. Then the *integral convolution* of u and θ_ε given by

$$u_\varepsilon(x) := (u * \theta_\varepsilon)(x) := \int\limits_{\mathbb{R}^n} u(x-t)\theta_\varepsilon(t)dV(t)$$

is $\mathcal{C}^k$-smooth, locally convex on ω_ε, and the family $\{u_\varepsilon\}_{\varepsilon \leq \varepsilon_0}$ decreases to u on ω_0 as ε tends to zero. □

2.2 Regularity of convex functions

It is a classical fact that convex functions admit a strong regularity: it is not only possible to show that real-valued convex functions are continuous, but also that they are twice differentiable almost everywhere. Especially, the latter property is relevant to us in the forthcoming sections. It was examined by H. Busemann and W. Feller in [BF36] and then proved by A. D. Alexandroff in [Ale39]. Nowadays, it is known as the Busemann-Feller-Alexandroff theorem. The non-trivial proof of this result will not be repeated here. Instead we refer to the end of chapter 2 in [Bus58] or to the article [BCP96] which gives an alternative proof to that of Alexandroff.

Theorem 2.2.1 (Busemann-Feller-Alexandroff) *Let u be a real-valued locally convex function on an open set ω in $\mathbb{R}^n$. Then, almost everywhere on ω, the function u is twice differentiable and its gradient ∇u is differentiable.*

This important theorem motivates to introduce the following family of functions.

Definition & Remark 2.2.2 Let ω be an open set in $\mathbb{R}^n$ and $L \geq 0$.

(1) The symbol $\mathcal{C}^1_L(\omega)$ is the family of all real valued functions g on ω such that $u(x) := g(x) + \frac{1}{2}L\|x\|_2^2$ is locally convex on ω.

(2) Let g be a function in $\mathcal{C}^1_L(\omega)$. In view of the Busemann-Feller-Alexandroff theorem, the real Hessian $\mathscr{H}_g(x) := \left(\frac{\partial g}{\partial x_i \partial x_j}(x)\right)_{i,j=1}^n$ of g exists at almost every point x in ω. At these points, the smallest eigenvalue is bounded from below by $-L$. It is therefore reasonable to say that functions in $\mathcal{C}^1_L(\omega)$ have a *lower bounded Hessian*.

(3) The collection of all functions on ω with lower bounded Hessian is denoted by $\mathcal{C}^1_\bullet(\omega)$.

(4) It holds that $\mathcal{C}^1_\bullet(\omega) = \bigcup_{L\geq 0} \mathcal{C}^1_L(\omega)$ and $\mathcal{C}^1_\bullet(\omega) \subset \mathcal{C}(\omega)$.

The integral convolution already offers an important method to approximate convex functions, but it will not be appropriate for a certain family of functions we will investigate later. An alternative is given by a convolution method based on taking a supremum rather than an integral. The idea to use this convolution comes directly from section 2 in Słodkowski's article [Sło84], where the reader also finds more properties of functions with lower bounded Hessian.

Definition 2.2.3 Let u, v be two non-negative functions defined on possibly different subsets of $\mathbb{R}^n$. Then for every $x \in \mathbb{R}^n$ the *supremum convolution* of u and v is defined by

$$(u *_s v)(x) := \sup\{\hat{u}(y)\hat{v}(x-y) : y \in \mathbb{R}^n\},$$

where $\hat{u}$ and $\hat{v}$ denote the trivial extensions of u and v by zero into the whole space $\mathbb{R}^n$.

We apply the supremum convolution to functions with lower bounded Hessian (see Proposition 2.6 in [Sło84]).

Proposition 2.2.4 *Let $M > 0$ be a positive number. Let u and g be two non-negative bounded upper semi-continuous functions on $\mathbb{R}^n$. If $g \in \mathcal{C}^1_L(\mathbb{R}^n)$, then $u *_s g$ lies in $\mathcal{C}^1_{ML}(\mathbb{R}^n)$, where $M := \sup_{\mathbb{R}^n} u$. In particular, $u *_s g$ is continuous on $\mathbb{R}^n$ and twice differentiable almost everywhere on $\mathbb{R}^n$.*

Proof. We refer to Proposition 1.1.4 or the original proof of Proposition 2.6 in [Sło84]. □

2.3 Real q-convex functions

We will now generalize the notion of convexity and give the definition of upper semi-continuous real q-convex functions.

Definition 2.3.1 Let ω be an open set in $\mathbb{R}^n$ and let $q \in \{0, \dots, n-1\}$.

(1) We say that an upper semi-continuous function u on ω *fulfills the local maximum property on ω with respect to linear functions on $(q+1)$-dimensional planes* if for every $(q+1)$-dimensional plane π, every ball $B \Subset \omega \cap \pi$ and every linear function ℓ on π with $u \leq \ell$ on bB we already have that $u \leq \ell$ on $\overline{B}$.

(2) An upper semi-continuous function on ω admitting the precedent property is called *real q-convex* on ω.

(3) If $m \geq n$, each upper semi-continuous function is automatically real m-convex by convention.

(4) We denote by $\mathcal{CVX}_q(\omega)$ the set of all real q-convex functions on ω.

The following properties of real q-convex functions follow directly from the definition.

Proposition 2.3.2 *Let all functions mentioned below be defined on an open set ω in $\mathbb{R}^n$ with image in $[-\infty, +\infty)$.*

(1) If u is upper semi-continuous, then it is locally convex if and only if it is real 0-convex.

(2) Every real q-convex function is real $(q+1)$-convex.

(3) If $\lambda \geq 0$, $c \in \mathbb{R}$ and u is real q-convex, then $\lambda u + c$ is also real q-convex.

(4) The limit of a decreasing sequence $(u_k)_{k\in\mathbb{N}}$ of real q-convex functions is again real q-convex.

(5) If $\{u_i : i \in I\}$ is a family of locally bounded real q-convex functions, then the upper semi-continuous regularization u^* of $u := \sup_{i\in I} u_i$ is real q-convex.

In particular, the maximum of finitely many real q-convex functions is again real q-convex.

(6) A real q-convex function remains real q-convex after a linear change of coordinates.

(7) An upper semi-continuous function u is real q-convex if and only if $u + \ell$ is real q-convex for every linear function ℓ on $\mathbb{R}^n$.

In the definition of real q-convex functions we can replace linear by locally concave functions.

Theorem 2.3.3 *Let u be an upper semi-continuous function on an open set ω in $\mathbb{R}^n$. Then u is real q-convex if and only if it fulfills the local maximum property with respect to locally concave functions on $(q+1)$-dimensional planes.*

Proof. If u fulfills the local maximum property on ω with respect to locally concave functions on $(q+1)$-dimensional subspaces, then u is real q-convex on ω, since every linear function is concave.

On the other hand, let u be real q-convex on ω. Fix a $(q+1)$-dimensional linear plane π, a ball $B \Subset \omega \cap \pi$ and a function $f : \omega \to (-\infty, +\infty]$ which is concave in a convex neighborhood of $\overline{B}$ in π such that $u \leq f$ on bB. Let $p_0 = (x_0, f(x_0))$ be a point on the graph $\Gamma(f) = \{(x, f(x)) : x \in B\}$ of f over B with $f(x_0) \neq +\infty$. Notice that if $f(x_0) = +\infty$, then we have automatically $u(x_0) \leq f(x_0)$. Since f is assumed to be concave, the subgraph $\Gamma^-(f) := \{(x, y) \in \overline{B} \times \mathbb{R} : y \leq f(x)\}$ is convex in $\pi \times \mathbb{R}$ in view of Proposition 2.1.4. Let H be a supporting hyperplane of $\Gamma^-(f)$ at p_0 in $\pi \times \mathbb{R}$. Since $f(x_0) \neq +\infty$, x_0 is an interior point of B and f is continuous, the hyperplane H is a graph of a linear function on π, i.e., there is a linear function ℓ on π such that $H = \{(x, \ell(x)) : x \in \pi\}$. The convexity yields $\Gamma^-(f) \subset \Gamma^-(\ell)$, so that $f \leq \ell$ on B. Moreover, we have that $\ell(x_0) = f(x_0)$ and $u \leq f \leq \ell$ on bB. Since u is real q-convex, it follows that $u \leq \ell$ on $\overline{B}$. Thus, $u(x_0) \leq \ell(x_0) = f(x_0)$. Since p_0 was an arbitrary point lying on the graph of f over B, we obtain that $u \leq f$ on B. This shows that u fulfills the local maximum property on ω with respect to concave functions on $(q+1)$-dimensional planes. □

As an immediate consequence, the family of real q-convex functions is stable under summation with locally convex functions.

Corollary 2.3.4 *Let u be upper semi-continuous on an open set ω in $\mathbb{R}^n$. Then u is real q-convex on ω if and only if $u + f$ is real q-convex for every locally convex function f on ω.*

It follows from the striking Lemma 1.4.2 that real q-convexity is a local property.

Theorem 2.3.5 *Let u be upper semi-continuous on an open set ω in $\mathbb{R}^n$. Then u is real q-convex on ω if and only if it is locally real q-convex on ω, i.e., for every point p in ω there is a neighborhood V of p in ω such that u is real q-convex on V.*

Proof. It is obvious that a real q-convex function is locally real q-convex. In order to show the converse, we assume that u is not real q-convex but locally real q-convex on ω. Then there exist a real $(q+1)$-dimensional plane π, a ball B in $\pi \cap \omega$ containing a point p_0 and a linear function ℓ_1 on π such that $u(x) \leq \ell_1(x)$ for every $x \in bB$, but $u(p_0) > \ell_1(p_0)$. Lemma 1.4.2 asserts that there are a point p_1 in B, a positive number $\varepsilon > 0$ and a linear function ℓ_2 on π such that

$$u(p_1) - \ell_1(p_1) - \ell_2(p_1) = 0 \quad \text{and} \quad u(x) - \ell_1(x) - \ell_2(x) < -\varepsilon\|x - p_1\|^2 < 0$$

for every $x \in \overline{B} \setminus \{p_1\}$. Then u cannot be real q-convex in a neighborhood of p_1 according to Proposition 2.3.2 (7), which is a contradiction. □

Lemma 1.4.2 has another important consequence to real q-convex functions.

Theorem 2.3.6 (Local maximum principle) *Let $q \in \{0, \dots, n-1\}$ and let ω be a relatively compact open set in $\mathbb{R}^n$. If u is real q-convex on ω and upper semi-continuous up to the closure of ω, then*

$$\max_{\overline{\omega}} u = \max_{b\omega} u.$$

Proof. Suppose that the statement is false, so that there exists an upper semi-continuous function u on $\overline{\omega}$ which is real q-convex on ω and fulfills $\max_{\overline{\omega}} u >$

$\max_{b\omega} u$. Then it follows from Lemma 1.4.2 that there are a point p in ω, a positive number $\varepsilon > 0$ and a linear function ℓ such that

$$u(p) - \ell(p) = 0 \quad \text{and} \quad u(x) - \ell(x) < -\varepsilon\|x - p\|_2^2 \quad \text{for every } x \in \overline{\omega} \setminus \{p\}.$$

In this case, $u - \ell$ fails to be real q-convex on ω, which contradicts Proposition 2.3.2 (7). □

Two real q-convex functions can be patched together to a new real q-convex function.

Theorem 2.3.7 *Let ω_1 and ω be two open sets in $\mathbb{R}^n$ with $\omega_1 \subset \omega$. Let u be a real q-convex function on ω and u_1 be a real q-convex function on ω_1 such that*

$$\limsup_{\substack{y \to x \\ y \in \omega_1}} u_1(y) \le u(x) \text{ for every } x \in b\omega_1 \cap \omega. \tag{2.2}$$

Then the following function is real q-convex on Ω,

$$\psi(x) := \begin{cases} \max\{u(x), u_1(x)\}, & x \in \omega_1 \\ u(x), & x \in \omega \setminus \omega_1 \end{cases}.$$

Proof. The function ψ is upper semi-continuous on ω due to (2.2). Let π be a real $(q+1)$-dimensional plane in $\mathbb{R}^n$, B be a ball lying relatively compact in $\pi \cap \omega$ and let ℓ be a linear function on π such that $\psi \le \ell$ on bB. Since ψ coincides with u on $\omega \setminus \overline{\omega_1}$ and since it is a maximum of the two real q-convex functions u and u_1 on ω_1, ψ is real q-convex on $\omega \setminus b\omega_1$. Thus, we can assume that $B \cap b\omega_1 \neq \emptyset$. Since u is real q-convex on ω and by the inequalities $u \le \psi \le \ell$ on bB, we obtain that $u \le \ell$ on B. Therefore, we have that $\psi = u \le \ell$ on $B \cap (\omega \setminus \omega_1)$. In particular, we have that $\psi = u \le \ell$ on $B \cap b\omega_1$. This implies that $\psi \le \ell$ on $b(B \cap \omega_1)$. Since ψ is real q-convex on ω_1, the local maximum principle (see Theorem 2.3.6) yields $\psi \le \ell$ on $B \cap \omega_1$. By the previous discussion, we have that $\psi \le \ell$ on B. Finally, we can conclude that ψ is real q-convex on ω. □

2.4 Strictly and smooth real q-convex functions

We are interested in real q-convex functions which are stable under small perturbations by certain smooth functions. We already mentioned relatives of those functions in Example 2.1.3, namely, the strongly convex functions.

Definition & Remark 2.4.1 Let ω be an open set in $\mathbb{R}^n$.

(1) An upper semi-continuous function u on ω is called *strongly real q-convex* on ω if for every $\mathcal{C}^\infty$-smooth function θ with compact support in ω there exists a positive number $\varepsilon_0 > 0$ such that $u + \varepsilon\theta$ is real q-convex for every $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$.

(2) We say that an upper semi-continuous function u on ω is *strictly real q-convex* if for every point p in ω there exist a neighborhood U of p and a positive number $\varepsilon_0 > 0$ such that $x \mapsto u(x) + \varepsilon\|x - p\|_2^2$ is real q-convex on U for every $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$.

(3) In view of the identity in Remark 1.4.1 (9) and Proposition 2.3.2 (7), we can replace $u(x) + \varepsilon\|x - p\|_2^2$ by $u(x) + \varepsilon\|x\|_2^2$ in the previous definition.

The two definitions of strict and strong real q-convexity are equivalent. Therefore, we will always use only the notion *strictly real q-convex.*

Proposition 2.4.2 *An upper semi-continuous function u on an open set ω in $\mathbb{R}^n$ is strongly real q-convex if and only if it is strictly real q-convex.*

Proof. We first show the necessity. Let p be an arbitrary point in ω and let r and R be two positive numbers such that $r < R$ and $B_R(p)$ lies relatively compact in ω. Pick a $\mathcal{C}^\infty$-smooth function θ_0 with compact support in $B_R(p)$ such that $0 \leq \theta \leq 1$ and $\theta_0 \equiv 1$ on $B_r(p)$. Define $\theta := \theta_0\|\cdot\|_2^2$. Then there is a positive number $\varepsilon_0 > 0$ so that $u + \varepsilon\theta$ is real q-convex for every $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$. Since $\theta = \|\cdot\|_2^2$ on $B_r(p)$, we have shown the first part of the statement.

In order to verify the sufficiency, let θ be a $\mathcal{C}^\infty$-smooth function with compact support K in ω. Since K is compact, we may assume that there exist an open neighborhood U of K and a positive number ε_0 such that $u + \varepsilon\|\cdot\|_2^2$ is real q-convex on U for every $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$. Since θ is $\mathcal{C}^\infty$-smooth and has compact support on $\mathbb{R}^n$, we can find a number $\delta_1 > 0$ so large that $\delta_1\|\cdot\|_2^2 + \theta$ and $\delta_1\|\cdot\|_2^2 - \theta$ both remain convex on $\mathbb{R}^n$. Now let $\varepsilon_1 := \varepsilon_0/\delta_1$. Then for every $\eta \in [0, \varepsilon_1)$ we have that $\eta\delta_1$ lies in $[0, \varepsilon_0)$. Thus, the functions

$$u \pm \eta\theta = u - \eta\delta_1\|\cdot\|_2^2 + \eta\big(\delta_1\|\cdot\|_2^2 \pm \theta\big)$$

both are a sum of a real q-convex and a locally convex function on U. According to Corollary 2.3.4, we obtain that $u + \eta\theta$ is real q-convex on U for every $\eta \in (-\varepsilon_1, \varepsilon_1)$. $\square$

In the case of $\mathcal{C}^2$-smooth functions, we have the following characterization of (strict) real q-convexity. Notice that for $v, w \in \mathbb{R}^n$ we use the notation

$$\mathscr{H}_u(p)(v,w) := v^t \mathscr{H}_u(p) w = \sum_{k,\ell=1}^{n} \frac{\partial^2 u}{\partial x_k \partial x_\ell}(p) v_k w_\ell,$$

where v^t is the transpose of the vector v.

Theorem 2.4.3 *Let $q \in \{0, \dots, n-1\}$ and ω be an open set in $\mathbb{R}^n$. A $\mathcal{C}^2$-smooth function u on ω is (strictly) real q-convex if and only if for every point $p \in \omega$ the real Hessian $\mathscr{H}_u(p)$ of u has at most q negative (non-positive) eigenvalues.*

Proof. By Theorem 2.3.5, real q-convexity is a local property, so all considerations can be made in a small neighborhood of some fixed point $p \in \omega$. Due to Proposition 2.3.2 (3) and (6), we can assume without loss of generality that $p = 0$, $u(p) = 0$ and that u has the following Taylor expansion in some neighborhood of the origin,

$$u(x) = A(x) + \frac{1}{2}\mathscr{H}_u(0)(x,x) + o(\|x\|_2^2),$$

where $A(x) = \nabla u(0)x$ is considered as a linear function $\mathbb{R}^n \to \mathbb{R}$. According to Proposition 2.3.2 (7), by replacing u by $u - A$, we can assume without loss of generality that u has the following form near the origin,

$$u(x) = \frac{1}{2}\mathscr{H}_u(0)(x,x) + o(\|x\|_2^2).$$

Now if the real Hessian of u has at least $q+1$ negative eigenvalues at the origin, then we can find a real $(q+1)$-dimensional plane π in $\mathbb{R}^n$ and a ball B inside $\pi \cap \omega$ such that u is strictly negative at every point on the boundary of B but vanishes inside B at the origin. Thus, in view of Theorem 2.3.5, it cannot be real q-convex on ω.

On the other hand, if u is not real q-convex, then there are a point $p_0 \in \omega$, a real $(q+1)$-dimensional plane π, a ball B in $\pi \cap \omega$ containing p_0 and a linear function ℓ_1 on π such that $u(x) \le \ell_1(x)$ for every $x \in bB$, but $u(p_0) > \ell_1(p_0)$. Then by Lemma 1.4.2 there are a point p_1 inside B, a positive number $\varepsilon > 0$ and another linear function ℓ_2 on π such that

$$u(p_1) - \ell_1(p_1) - \ell_2(p_1) = 0 \quad \text{and} \quad u(x) - \ell_1(x) - \ell_2(x) < -\varepsilon\|x - p_1\|_2^2.$$

for every $x \in \overline{B} \setminus \{p_1\}$. Hence, the function $u - \ell_1 - \ell_2$ attains a local maximum at p_1. Therefore, the real Hessian of u at p_1, which corresponds to the real Hessian of $u - \ell_1 - \ell_2$ at p_1, has at least $q+1$ negative eigenvalues. □

The previous condition on real q-convexity in the $\mathcal{C}^2$-smooth case allows us easily to construct examples of real q-convex functions.

Example 2.4.4 Recall the characteristic functions χ_S and $\check{\chi}_S$ of the set $S = \{(x,0) \in \mathbb{R}^2 : x \in \mathbb{R}\}$ on $\mathbb{R}^2$ in Example 1.1.3. They are real 1-convex but obviously not convex. Indeed, they can be approximated from above by the real 1-convex functions $\chi_{n,S}(x,y) := \exp(-nx^2)$ and $\check{\chi}_{n,S}(x,y) := -nx^2$, respectively. This shows that, in general, real q-convex functions are neither continuous nor locally integrable, if $q \geq 1$, whereas every real-valued 0-convex function is continuous (see Theorem 10.1 in [Roc70]).

The previous theorem has the following useful application.

Lemma 2.4.5 *Let ω be an open set in $\mathbb{R}^n$. Assume that u is not real q-convex on ω. Then there is a ball $B \Subset \omega$, a point $x_1 \in B$, a number $\varepsilon > 0$ and a $\mathcal{C}^\infty$-smooth real $(n-q-1)$-convex function v on $\mathbb{R}^n$ such that*

$$(u+v)(x_1) = 0 \quad \text{and} \quad (u+v)(x) < -\varepsilon\|x - x_1\|_2^2 \text{ for every } x \in B \setminus \{x_1\}.$$

Proof. Since u is not real q-convex on ω, there exist a ball $B \Subset \omega$, a point x_0 in B, a $(q+1)$-dimension plane π and a linear function $\ell : \mathbb{R}^n \to \mathbb{R}$ such that $u + \ell < 0$ on $bB \cap \pi$ and $u(x_0) + \ell(x_0) > 0$. Let $h : \mathbb{R}^n \to \mathbb{R}^{n-q-1}$ be a linear mapping such that $\pi = \{h = 0\}$ and fix a number $c > 0$. In view of Theorem 2.4.3, it is easy to verify that the $\mathcal{C}^\infty$-smooth function $v_c(x) := \ell(x) - c\|h(x)\|_2^2$ is real $(n-q-1)$-convex on $\mathbb{R}^n$. Moreover, it equals ℓ on π and tends to $-\infty$ outside π when c goes to $+\infty$. Therefore, if we choose c large enough, then we can arrange that $u + v_c < 0$ on bB and $u(x_0) + v_c(x_0) > 0$. Now it follows from Lemma 1.4.2 that there is another linear function $\ell_1 : \mathbb{R}^n \to \mathbb{R}$, a point $x_1 \in B$ and $\varepsilon > 0$ such that $(u + v_c + \ell_1)(x_1) = 0$, but $(u + v_c + \ell_1)(x) < -\varepsilon\|x - x_1\|_2^2$ for every $x \in B \setminus \{x_1\}$. Finally, $v := v_c + \ell_1$ is the demanded function in view of Proposition 2.3.2 (7). □

2.5 Twice differentiable real q-convex functions

We establish a characterization of twice differentiable real q-convex functions by studying a certain quantity which represents exactly the largest eigenvalue of the real Hessian of a smooth function at a given point. The ideas are directly derived from Chapter 3 in [Sło84], since they perfectly translate to real q-convex functions.

Definition 2.5.1 Let u be a function on an open set ω in $\mathbb{R}^n$ such that its gradient $\nabla u(x)$ exists at some point $x \in \omega$. Then the *maximal eigenvalue of $\mathscr{H}_u$ at x* is defined by

$$\lambda_u(x) := 2 \limsup_{\varepsilon \to 0} (\max\{u(x+\varepsilon h) - u(x) - \varepsilon \nabla u(x) h : h \in \mathbb{R}^n,\ \|h\|_2 = 1\})/\varepsilon^2$$

The following result is an important property of the maximal eigenvalue and has been shown by Słodkowski (see Theorem 3.2 and Corollary 3.5 in [Sło84]).

Theorem 2.5.2 *If u is a locally convex function on an open set ω in $\mathbb{R}^n$ such that $\lambda_u(x) \geq M$ for almost every $x \in \omega$, then $\lambda_u(x) \geq M$ for every $x \in \omega$.*

The precedent theorem allows to generalize Theorem 2.4.3 using twice differentiable real q-convex functions rather than $\mathcal{C}^2$-smooth ones. It is Theorem 4.1 in [Sło84] adapted to real q-convex functions.

Theorem 2.5.3 *Let $q \in \{0, \ldots, n-1\}$ and let u be upper semi-continuous on an open set ω in $\mathbb{R}^n$.*

(1) If u is real q-convex on ω and twice differentiable at a point p in ω, then the real Hessian of u at p has at most q negative eigenvalues.

(2) If $u \in \mathcal{C}^1_L(\omega)$ and the real Hessian at almost every point in ω has at most q negative eigenvalues, then u is real q-convex on ω.

Proof. (1) Pick a point p in ω such that $\mathscr{H}_u(p)$ exists. Let $B_r(p) \Subset \omega$ be a ball centered in p with radius $r > 0$. Then for $t \in (0,1)$ the function u_t given by

$$B_r(0) \ni x \ \mapsto \ u_t(x) := \big(u(p+tx) - u(p) - t\langle \nabla u(p), x\rangle\big)/t^2$$

is real q-convex on $B_r(0)$ due to Proposition 2.3.2, properties (3) and (7). Since u is twice differentiable at p, the family $(u_t)_{t\in(0,1)}$ tends uniformly to $u_0(x) := \mathscr{H}_u(p)(x,x)$ in a small neighborhood of the origin as t tends to zero. Therefore, the function u_0 is real q-convex and $\mathcal{C}^2$-smooth on a neighborhood of the origin. By Theorem 2.4.3 the real Hessian of u_0 at the origin has at most q negative eigenvalues. Since $\mathscr{H}_{u_0}(0) = \mathscr{H}_u(p)$, the proof of the first statement is finished.

(2) If u is not real q-convex on ω, then it follows from Lemma 2.4.5 and Proposition 2.3.2 (7) that, without loss of generality, there exist a ball $B_r(0) \Subset \omega$, a number $\varepsilon > 0$ and a $\mathcal{C}^\infty$-smooth real $(n-q-1)$-convex function v on $\mathbb{R}^n$ which satisfies $(u+v)(0) = 0$ and

$$(u+v)(x) < -\varepsilon\|x\|_2^2 \text{ for every } x \in \overline{B_r(0)} \setminus \{0\}. \tag{2.3}$$

Recall that $u \in \mathcal{C}_L^1(\omega)$ and define

$$f := u + v, \quad M_v := \sup\{\lambda_v(x) : x \in \overline{B_r(0)}\} \quad \text{and} \quad M := L + M_v.$$

Then f is non-positive and belongs to $\mathcal{C}_M^1(\omega)$, so $g(x) := f(x) + \frac{1}{2}M\|x\|_2^2$ is convex on $B_r(0)$. Therefore, for every $x \in \overline{B_r(0)}$ we have that

$$0 = 2g(0) \leq g(x) + g(-x) = f(x) + f(-x) + M\|x\|_2^2 \leq f(x) + M\|x\|_2^2.$$

Thus, $-M\|x\|_2^2 \leq f(x)$. On the other hand, $f(x) \leq -\varepsilon\|x\|_2^2$, so the gradient of f at 0 exists and vanishes there. Of course, the same is also true for the function g. Thus, in view of property (2.3), we may estimate the maximal eigenvalue of g at 0 as follows:

$$\lambda_g(0) = 2\limsup_{\varepsilon\to 0}\big(\max\{g(\varepsilon h) : h \in \mathbb{R}^n,\ \|h\|_2 = 1\}\big)/\varepsilon^2 \leq M - 2\varepsilon. \tag{2.4}$$

By the Busemann-Feller-Alexandroff theorem (see Theorem 2.2.1), the real Hessian of f exists almost everywhere on ω. Moreover, since $\mathscr{H}_u$ has at most q negative and $\mathscr{H}_v$ has at most $n-q-1$ negative eigenvalues, the real Hessian of the sum $f = u + v$ has at least one non-negative eigenvalue almost everywhere on ω. Therefore, since the the largest eigenvalue of the function $\frac{1}{2}M\|\cdot\|_2^2$ is exactly M, we derive the estimate $\lambda_g(x) \geq M$ at almost every point in $B_r(0)$. Then it follows from Theorem 2.5.2 that $\lambda_g \geq M$ everywhere on $B_r(0)$. In particular, $\lambda_g(0) \geq M$, which is a contradiction to (2.4). □

2.6 Approximation of real q-convex functions

We show that any real q-convex function can be approximated from above by a decreasing sequence of real q-convex functions being continuous everywhere and twice differentiable almost everywhere. The idea of this approximation and most of the arguments of the proofs are directly derived from Section 2 in [Sło84].

Theorem 2.6.1 *Let u be a non-negative bounded real q-convex function on an open set ω in $\mathbb{R}^n$. Let $g \in \mathcal{C}^1_L(\mathbb{R}^n)$ be a non-negative function with compact support in some ball $B_r(0)$. Define $\omega_r := \{x \in \omega : \operatorname{dist}(x, b\omega) > r\}$ and $M_r := \sup_{\omega_r} u$. Then $u *_s g$ lies in $\mathcal{C}^1_{LM_r}(\mathbb{R}^n)$ and it is real q-convex on ω_r.*

Proof. Recall that $\hat{u}$ denotes the trivial extension of u by zero to the whole of $\mathbb{R}^n$. The supremum convolution of u and g at $x \in \omega_r$ can be rewritten as follows,

$$\begin{aligned}(u *_s g)(x) &= \sup\{\hat{u}(y)g(x-y) : y \in \mathbb{R}^n\} \\ &= \sup\{\hat{u}(x-t)g(t) : t \in \mathbb{R}^n\} \\ &= \sup\{u(x-t)g(t) : t \in B_r(0)\}.\end{aligned}$$

It follows from Proposition 2.3.2 (3) and (6) that $x \mapsto g(t)u(x-t)$ is real q-convex on ω_r for every $t \in B_r(0)$. Since, in view of Remark 2.2.2 and Proposition 2.2.4, the function $u *_s g$ is continuous, Proposition 2.3.2 (5) implies that $u *_s g$ is real q-convex on ω_r. Finally, it follows directly from Proposition 2.2.4 that $u *_s g$ belongs to $\mathcal{C}^1_{LM_r}(\mathbb{R}^n)$. □

The previous theorem yields the following approximation technique.

Proposition 2.6.2 *Let u be a real q-convex function on an open set ω in $\mathbb{R}^n$ and let D be a relatively compact open set in ω. Assume that f is a continuous function on ω and satisfies $u < f$ on a neighborhood of $\overline{D}$. Then there is a positive number $L > 0$ and a continuous function $\tilde{u} \in \mathcal{C}^1_L(\mathbb{R}^n)$ which is real q-convex in a neighborhood of $\overline{D}$ and which fulfills $u < \tilde{u} < f$ on $\overline{D}$.*

Proof. Let r be a positive number so small that that $\overline{D}$ is contained in $D_r := \omega_r \cap B_{1/r}(0)$, where $\omega_r := \{x \in \omega : \operatorname{dist}(x, b\omega) > r\}$. Given $k \in \mathbb{N}$, we set $v := \max\{u, -k\} + k + 1/k$. Then $u < v - k$ and v is positive. Since the sequence $(v-k)_{k\in\mathbb{N}}$ decreases to u, we can find a large enough integer $k \in \mathbb{N}$ such that $v - k < f$ on $\overline{D}$. By upper semi-continuity of v and compactness of $\overline{D}$, we can choose another radius $r' \in (0, r)$ so small that $D \Subset \omega_{r'}$ and

$$\sup\{v(y) - k : y \in B_{r'}(x)\} < f(x) \quad \text{for every } x \in \overline{D}.$$

Now pick a $\mathcal{C}^\infty$-smooth function g with compact support in the ball $B_{r'}(0)$ such that $0 \le g \le 1$ and $g(0) = 1$. We set $\tilde{u}(x) := (v *_s g)(x) - k$ for $x \in \omega$. Then we obtain for every $x \in \overline{D}$ that

$$\begin{aligned} u(x) &< v(x) - k \\ &= v(x)g(0) - k \\ &\le \sup\{v(y)g(x-y) : y \in B_{r'}(x)\} - k \\ &= (v *_s g)(x) - k \\ &= \tilde{u}(x) \\ &\le \sup\{v(y) : y \in B_{r'}(x)\} - k \\ &= \sup\{v(y) - k : y \in B_{r'}(x)\} < f(x). \end{aligned}$$

The rest of the properties of $\tilde{u}$ follow now from the previous Theorem 2.2.4. □

As an immediate consequence, we get the following approximation for real q-convex functions.

Corollary 2.6.3 *Let ω be an open set in $\mathbb{R}^n$ and let $\varepsilon > 0$. Then for every compact set K in $\omega_\varepsilon = \{z \in \omega : d(z, b\omega) > \varepsilon,\ \|z\|_2 < 1/\varepsilon\}$ it holds that*

$$\mathcal{CVX}_q(\omega) \subset \overline{\mathcal{CVX}_q(\omega_\varepsilon) \cap \mathcal{C}^1_\bullet(\mathbb{R}^n)}^{\downarrow K}.$$

Proof. Let u be a real q-convex function on ω and $k_0 \in \mathbb{N}$ so large that $\omega_\varepsilon \cap B_{1/\varepsilon}(0)$ lies in $\omega_{1/k_0} \cap B_{k_0}(0)$. Then by Proposition 2.6.2 we can inductively construct a sequence $(L_k)_{k \ge k_0}$ of positive numbers L_k and a sequence $(u_k)_{k \ge k_0}$ of functions $u_k \in \mathcal{C}^1_{L_k}(\mathbb{R}^n)$ which are real q-convex on $\omega_{1/k} \cap B_k(0)$ and fulfill $u < u_{k+1} < u_k$ on K for every $k \ge k_0$. Hence, we have shown the desired inclusion of this statement. □

Notice that, in general, the sum of two real q-convex functions will not lead to another real q-convex function. Consider for example the real 1-convex functions $u_1(x, y) = -x^2$ and $u_2(x, y) = -y^2$ in $\mathbb{R}^2$. The real Hessian of their sum $u_1 + u_2$ has two negative eigenvalues at every point in $\mathbb{R}^2$, so it fails to be real 1-convex. Nonetheless, as an application of Theorem 2.5.3 and Corollary 2.6.3 we obtain the following result on sums of real q-convex functions. It has been proved by Słodkowski [Sło84] in the q-plurisubharmonic case.

Theorem 2.6.4 *Given a real q-convex function u_1 and a real r-convex function u_2 on an open set ω in $\mathbb{R}^n$, their sum $u_1 + u_2$ is real $(q + r)$-convex on ω.*

Proof. By the previous Theorem 2.6.3 and since real q-convexity is a local property, we can assume that u_1 and u_2 have lower bounded Hessian and that they are twice differentiable almost everywhere on ω. Then in view of the first statement of Theorem 2.5.3, the real Hessian of u_1 has at most q and the real Hessian of u_2 has at most r negative eigenvalues at almost every point in ω. Now it is easy to verify that the sum of the Hessians of u_1 and u_2 have at most $q + r$ negative eigenvalues almost everywhere. Since the sum $u_1 + u_2$ certainly also has lower bounded Hessian and is twice differentiable almost everywhere on ω, it follows from the second statement in Theorem 2.5.3 that $u_1 + u_2$ is real $(q + r)$-convex on ω. □

In what follows, we present another approximation of real q-convex functions by piecewise smooth real q-convex functions. The methods are derived from Bungart's article [Bun90] and transfered to real q-convex functions.

Definition 2.6.5 Let u be a continuous function on an open set ω in $\mathbb{R}^n$.

(1) The function u is *real q-convex with corners on ω* if for every point p in ω there exists a neighborhood U of p in ω and finitely many $\mathcal{C}^2$-smooth real q-convex functions $u_1, \ldots, u_\ell$ on U such that $u = \max\{u_j : j = 1, \ldots, \ell\}$ on U.

(2) The symbol $\mathcal{CVX}_q^c(\omega)$ stands for the family of all real q-convex functions with corners on ω.

Bungart's approximation technique in [Bun90] is based on the solution of the Dirichlet problem for certain families of continuous functions. We show his result for real q-convex functions and mainly use his arguments.

Theorem 2.6.6 *Let B be a ball in $\mathbb{R}^n$ and let g be a continuous function on $\overline{B}$ which is real q-convex on B. Then the upper-envelope function defined by*

$$E(\overline{B}, g) := \sup\{u : u \in \mathcal{CVX}_q^c(B) \cap \mathcal{C}(\overline{B}), u \leq g\} \tag{2.5}$$

is equal to g on $\overline{B}$. Moreover, for every continuous function f with $g < f$ on $\overline{B}$ there exists a continuous function $\tilde{g}$ on $\overline{B}$ which is real q-convex with corners on B and fulfills $g < \tilde{g} < f$ on $\overline{B}$.

Proof. We verify the identity in (2.5) in five steps.

Step 1. The upper-envelope function $E(\overline{B}, g)$ coincides with g on bB. Indeed, let $\delta > 0$, $k \in \mathbb{N}$ and let p be a boundary point of B. Since B is a ball, it is strictly convex. This implies that there exist a linear function $\ell : \mathbb{R}^n \to \mathbb{R}$ such that $\ell(p) = 0$ and ℓ is negative on B. Then it is obvious that $u_{p,\delta,k}(x) := g(p) - \delta + k\ell(x)$ is a $\mathcal{C}^\infty$-smooth real q-convex function on B which is continuous on $\overline{B}$. Since $u_{p,\delta,k} < g$ on a small neighborhood of p in $\overline{B}$, we can find an integer k so large that $u_{p,\delta,k} < g$ on $\overline{B}$. Hence, by the definition of $E(\overline{B}, g)$, we obtain that

$$u_{p,\delta,k} \leq E(\overline{B}, g) \text{ on } \overline{B} \quad \text{and} \quad u_{p,\delta,k}(p) = g(p) - \delta \leq E(\overline{B}, g)(p) \leq g(p).$$

Since δ is an arbitrary positive number and p is an arbitrary boundary point of B, we conclude that $E(\overline{B}, g) = g$ on bB.

Step 2. The continuity of the upper-envelope $E(\overline{B}, g)$ on $\overline{B}$ is shown exactly by the same arguments as Walsh used in [Wal69]. It implies that $E(\overline{B}, g)$ is real q-convex on B, since it is the supremum of a family of real q-convex functions bounded from above by g.

Step 3. Let $G := \{x \in B : E(\overline{B}, g)(x) < g(x)\}$. Assume in this step that G is not empty. Then we claim that $u := -E(\overline{B}, g)$ is real $(n-q-1)$-convex on G. Indeed, first observe that G is open and lies relatively compact in B, since G does not meet bB in view of Step 1 and since u is continuous on $\overline{B}$ by Step 2.

Assume that u is not real $(n-q-1)$-convex on G. Then by Lemma 2.4.5 there exist a point $x_0 \in G$, a ball $B_r(x_0) \Subset G$, a number $\varepsilon > 0$ and a $\mathcal{C}^\infty$-smooth real q-convex function v on $\mathbb{R}^n$ such that

$$v(x_0) = E(\overline{B}, g)(x_0) \quad \text{and} \quad v(x) + \varepsilon\|x - x_0\|_2^2 < E(\overline{B}, g)(x) \tag{2.6}$$

for every $x \in \overline{B_r(x_0)} \setminus \{x_0\}$. We set $v_0(x) := v(x) + \varepsilon\|x - x_0\|_2^2$. Recall that $E(\overline{B}, g) < g$ on $B_r(x_0) \Subset G$ and choose a positive number δ so small that

$$v_0 + \delta < g \text{ on } \overline{B_r(x_0)} \quad \text{and} \quad v_0 + \delta < E(\overline{B}, g) \text{ on } bB_r(x_0). \tag{2.7}$$

By the definition of $E(\overline{B}, g)$, by the compactness of $\overline{B}$ and according to (2.6) and (2.7), we can find finitely many functions $\varphi_1, \ldots, \varphi_\ell \in \mathcal{CVX}_q^c(B) \cap \mathcal{C}(\overline{B})$ which are dominated by g on $\overline{B}$ and which satisfy for $\varphi := \max\{\varphi_1, \ldots, \varphi_\ell\}$ the inequalities

$$v_0(x_0) + \delta = v(x_0) + \delta > \varphi(x_0) \quad \text{and} \quad v_0 + \delta < \varphi \text{ on } bB_r(x_0). \tag{2.8}$$

Since φ and v both belong to $\mathcal{CVX}_q^c(B) \cap \mathcal{C}(\overline{B})$ and in view of (2.8), Theorem 2.3.7 implies that

$$\psi(x) := \begin{cases} \max\{\varphi(x), v_0(x) + \delta\}, & x \in B_r(x_0) \\ \varphi(x), & x \in \overline{B} \setminus B_r(x_0) \end{cases}$$

is also in $\mathcal{CVX}_q^c(B) \cap \mathcal{C}(\overline{B})$. Furthermore, due to the first inequality in (2.7) and the choice of φ, we have that $\psi \leq g$ on $\overline{B}$. Then it follows from the definition of $E(\overline{B}, g)$ that $\psi \leq E(\overline{B}, g)$ on $\overline{B}$. In view of (2.8), we have that $\varphi(x_0) < v_0(x_0) + \delta$, so $\psi(x_0) = v_0(x_0) + \delta$, but this contradicts to the first identity in (2.6), since then

$$v(x_0) = v_0(x_0) \ < \ v_0(x_0) + \delta = \psi(x_0) \ \leq \ E(\overline{B}, g)(x_0) = v(x_0).$$

This proves the claim we made in the beginning of this step, namely, that $-E(\overline{B}, g)$ is real $(n{-}q{-}1)$-convex on G.

Step 4. Assume that G defined in Step 3 is not empty. Then the function $h := g - E(\overline{B}, g)$ vanishes on the boundary of G. Since h is real $(n{-}1)$-convex by Theorem 2.6.4 and the previous steps, it follows from the local maximum principle (see Theorem 2.3.6) that $h \leq 0$ on G, so $g \leq E(\overline{B}, g)$ on G. This contradicts to the definition of G, so it has to be empty. Therefore, g and $E(\overline{B}, g)$ coincide on $\overline{B}$.

Step 5. It remains to proof the last statement of this theorem. Let f be a continuous function with $g < f$ on $\overline{B}$. Since g is continuous, we can find a positive number $\lambda > 0$ and a point $x_0 \notin \overline{B}$ such that $g < g + \lambda \|x - x_0\|_2^2 < f$. Let p be a point in $\overline{B}$. Then according to the definition of $E(\overline{B}, g)$ and the properties shown above, there are a neighborhood U_p of p in $\overline{B}$ and a real q-convex function g_p with corners on B which is continuous on $\overline{B}$ and satisfies $g - \lambda \|x - x_0\|_2^2 < g_p < g$ on U_p and $g_p < g$ on the whole of $\overline{B}$. Since $\overline{B}$ is compact, there exist finitely many points $p_1, \ldots, p_\ell$ such that the open sets $U_{p_1}, \ldots, U_{p_\ell}$ cover $\overline{B}$. Then the function

$$\tilde{g} := \max\{g_{p_1}, \ldots, g_{p_\ell}\} + \lambda \|x - x_0\|_2^2$$

is continuous on $\overline{B}$, real q-convex with corners on B and admits the desired inequality

$$g < \tilde{g} < g + \lambda \|x - x_0\|_2^2 < f \text{ on } \overline{B}.$$

□

The above solution of the Dirichlet problem leads to the following approximation technique.

Theorem 2.6.7 *Let u be a continuous strictly real q-convex function on an open set ω in $\mathbb{R}^n$. Then for every continuous function f on ω with $u < f$ on ω there exists a real q-convex function $\tilde{u}$ with corners on ω such that $u < \tilde{u} < f$ on ω.*

Proof. Fix collections $(r_i)_{i\in I}$, $(s_i)_{i\in I}$, $(t_i)_{i\in I}$ and $(x_i)_{i\in I}$ of radii $0 < r_i < s_i < t_i$ and points $x_i \in \omega$ such that each collection of balls $\{B_{r_i}(x_i)\}_{i\in I}$, $\{B_{s_i}(x_i)\}_{i\in I}$ and $\{B_{t_i}(x_i)\}_{i\in I}$ forms a locally finite covering of the set ω. Now for each $i \in I$ choose a positive number ε_i such that $u - \varepsilon_i\|x - x_i\|_2^2$ is real q-convex on $\overline{B_{t_i}(x_i)}$ and

$$g_i(x) := u(x) + \varepsilon_i(s_i^2 - \|x - x_i\|_2^2) < f(x) \text{ for every } x \in B_{t_i}(x_i).$$

Then it is obvious that g_i is real q-convex on $B_{t_i}(x_i)$, continuous on $\overline{B_{t_i}(x_i)}$ and fulfills

$$g_i < u \text{ on } bB_{t_i}(x_i) \quad \text{and} \quad f > g_i > u \text{ on } \overline{B_{r_i}(x_i)}.$$

By Theorem 2.6.6, we can find a real q-convex function u_i with corners on $B_{t_i}(x_i)$ which is continuous on $\overline{B_{t_i}(x_i)}$ and satisfies

$$u_i < u \text{ on } bB_{t_i}(x_i) \quad \text{and} \quad f > u_i > u \text{ on } \overline{B_{r_i}(x_i)}.$$

We define $\tilde{u} := \sup\{u_i : i \in I\}$. Since $\{B_{r_i}(x_i)\}_{i\in I}$ is a locally finite covering, the function $\tilde{u}$ is locally a finite maximum of real q-convex functions with corners. Hence, it is itself real q-convex with corners on ω and, by the previous inequalities, it satisfies $u < \tilde{u} < f$ on ω. □

Since every real q-convex function u can be approximated from above by a family $(u_\varepsilon)_{\varepsilon>0}$ of strictly real q-convex functions $u_\varepsilon := u + \varepsilon\|\cdot\|_2^2$, we easily obtain the following approximation for continuous real q-convex functions.

Corollary 2.6.8 *Let ω be an open set in $\mathbb{R}^n$. Then we have that*

$$\mathcal{CVX}_q(\omega) \cap \mathcal{C}(\omega) \;\subset\; \overline{\mathcal{CVX}_q^c(\omega)}^{\downarrow\omega}.$$

Chapter 3

q-Plurisubharmonicity

We now turn to the investigation of q-plurisubharmonic functions defined on open subsets in $\mathbb{C}^n$ and their subfamilies. These functions originate in the q-convex functions in the sense of H. Grauert and generalize the plurisubharmonic functions, which constitute one of the most important functions in complex analysis in several variables. Moreover, they can be regarded as the complex analogue to the real q-convex functions from Chapter 2. We introduce here upper semi-continuous q-plurisubharmonic functions in the sense of H. M. Hunt and J. J. Murray [HM78]. Therefor, we need to repeat the very basics about holomorphic and harmonic functions, which allow to define pluriharmonic and plurisubharmonic functions. In the same way as one usually defines subharmonic functions in terms of a *local maximum property* with respect to harmonic functions, we define subpluriharmonicity by using pluriharmonic functions. In between lie the q-plurisubharmonic functions, where 0-plurisubharmonic functions are the classical plurisubharmonic functions and $(n-1)$-plurisubharmonic functions are exactly the subpluriharmonic ones. We recall the properties of q-plurisubharmonic functions, but omit some of their proofs and refer to the existing literature like [HM78], Słodkowski's articles [Sło84] and [Sło86] and others (see [Fuj90], [Bun90] or [Die06]). Especially, we repeat approximation techniques for plurisubharmonic and q-plurisubharmonic functions developed by H. J. Bremermann [Bre59], R. Richberg [Ric68], Z. Słodkowski [Sło84] and L. Bungart [Bun90]. As an application, we show the relation of real q-convex to (rigid) q-plurisubharmonic functions and recall Fujita's result [Fuj90] on the equivalence of q-plurisubharmonic and weakly q-plurisubharmonic functions. The fact that locally each sufficiently smooth strictly q-plurisubharmonic function induces a foliation by complex submanifolds of codimension q, on which the initial function is plurisubharmonic, motivates to introduce functions which are r-plurisubharmonic on leaves of a given (singular) foliation by analytic subsets of codimension q. Indeed, we show that these functions are $(q+r)$-plurisubharmonic. The last part of this chapter is devoted to q-holomorphic functions in the sense of R. Basener [Bas76]. We derive that $\mathcal{C}^2$-smooth functions which are holomorphic on leaves of a singular foliation by analytic subsets of codimension q are indeed q-holomorphic. Finally, we obtain a Bremermann type approximation technique for functions which are plurisubharmonic on leaves of a given singular foliation by logarithms of finitely many functions being holomorphic on the leaves of this foliation. Most of the parts of this chapter are included in our joint articles [PZ13] and [PZ15].

3.1 Holomorphic and pluriharmonic functions

We recall the definition of holomorphic functions and their properties in the spirit of K. Weierstraß. We refer to the books of S. Krantz [Kra99] and B. V. Shabat [Sha92] for the proofs and more details on holomorphic functions.

Definition 3.1.1 Let Ω be an open set in $\mathbb{C}^n$.

(1) The function $f : \Omega \to \mathbb{C}$ is *holomorphic on* Ω if for each point p in Ω there exists a polydisc $\Delta_r^n(p) = \{z \in \mathbb{C}^n : |z_j - p_j| < r,\ j = 1, \dots, n\}$ in Ω such that
$$f(z) = \sum_{\alpha \in \mathbb{N}_0^n} b_\alpha (z-p)^\alpha$$
is an absolutely convergent power series for every point z in $\Delta_r^n(p)$.

(2) The set of all holomorphic functions on Ω is denoted by $\mathcal{O}(\Omega)$.

(3) Given $q \in \mathbb{N}_0$, a mapping $f : \Omega \to \mathbb{C}^q$ is *holomorphic on* Ω if each component function $f_j : \Omega \to \mathbb{C}$ is holomorphic on Ω. Here, we use the convention $\mathbb{C}^0 = \{0\}$, so that the only holomorphic function $f : \Omega \to \mathbb{C}^0$ is the constant zero function.

(4) The set of all holomorphic mappings on Ω with image in some open set G in $\mathbb{C}^q$ is represented by the symbol $\mathcal{O}(\Omega, G)$.

We have to point out one outstanding result known as *Hartogs' theorem of separate analyticity.* Recall that each holomorphic function f is complex differentiable and has vanishing *Wirtinger derivatives* $\partial f/\partial \overline{z}_j$, $j = 1, \dots, n$. On the contrary, Osgood's lemma states that each continuous holomorphic function with vanishing Wirtinger derivatives is already holomorphic. It was improved to the general case by F. Hartogs [Har06]. For a proof in modern terms we refer to Chapter 2.4 of Krantz's book [Kra99].

Theorem 3.1.2 (Hartogs, 1906) *Let Ω be an open set in $\mathbb{C}^n$ and let f be a complex valued function on Ω with vanishing Wirtinger derivative $\partial f/\partial \overline{z}_j$ for each $j = 1, \dots, n$. Then f is continuous on Ω.*

Real parts of holomorphic functions are related to the following families of functions.

Definition 3.1.3 (1) A $\mathcal{C}^2$-smooth real valued function h defined on an open set V in $\mathbb{C}$ is *harmonic on V* if $\partial^2 h/\partial z \partial \overline{z}$ vanishes identically on V.

(2) A real valued function h defined on an open set Ω in $\mathbb{C}^n$ is *pluriharmonic on Ω* if h is harmonic on $\Omega \cap \mathbb{L}$ for each complex affine line $\mathbb{L}$ intersecting Ω.

(3) We write $\mathcal{PH}(\Omega)$ for the set of all pluriharmonic functions on Ω.

We give the precise relation of harmonic and holomorphic functions. These facts can be found in Chapter 2.2 in [Kra99]

Proposition 3.1.4 *Let Ω be an open set in $\mathbb{C}^n$.*

(1) Let h be a real valued function on Ω. Then h is pluriharmonic on Ω if and only if it is locally the real part of a holomorphic function. In particular, if h is pluriharmonic on a ball B in $\mathbb{C}^n$, then there is a holomorphic function f on B such that $\mathrm{Re}(f) = h$.

(2) Every pluriharmonic function u on Ω is real analytic and, hence, $\mathcal{C}^\infty$-smooth.

(3) If u is $\mathcal{C}^2$-smooth on Ω, then it is pluriharmonic on Ω if and only if its Levi matrix $\mathscr{L}_u(z) := \left(\frac{\partial^2 u}{\partial z_k \partial \overline{z}_\ell}(z)\right)_{k,\ell=1}^n$ vanishes at every point z in Ω.

Using the third property of the previous proposition, we easily obtain the following examples of harmonic and pluriharmonic functions.

Example 3.1.5 (1) The function $f(z) = f(x + iy) = e^z$ is holomorphic on $\mathbb{C}$, so $\mathrm{Re} f(z) = e^x \cos(y)$ and $\mathrm{Im} f(z) = e^x \sin(y)$ are both harmonic on $\mathbb{C}$.

(2) If f is a holomorphic function, then $\pm\log|f|$ are pluriharmonic outside the zero level set of f.

(3) If $\ell : \mathbb{R}^n \to \mathbb{R}^k$ is $\mathbb{R}$-linear, then $h(x+iy) := \|\ell(x)\|_2^2 - \|\ell(y)\|_2^2$ is pluriharmonic on $\mathbb{C}^n_{x+iy} = \mathbb{R}^n_x + i\mathbb{R}^n_y$.

We now present the definitions of one of the most important families of functions in complex analysis and pluripotential theory.

Definition 3.1.6 Let ψ be an upper semi-continuous function on an open set Ω in $\mathbb{C}^n$.

(1) The function ψ is called *plurisubharmonic on* Ω if it fulfills the *local maximum property with respect to harmonic functions on complex lines*, i.e., for every complex line $\mathbb{L}$ in $\mathbb{C}^n$, every relatively compact disc Δ in $\Omega \cap \mathbb{L}$ and every harmonic function h defined on a neighborhood of $\overline{\Delta}$ in $\mathbb{L}$ such that $\psi \le h$ on $b\Delta$ we already have that $\psi \le h$ on $\overline{\Delta}$.

(2) The set of all plurisubharmonic functions on Ω is denoted by $\mathcal{PSH}(\Omega)$.

(3) A lower semi-continuous function $\varphi : \Omega \to (-\infty, +\infty]$ is called *plurisuperharmonic on* Ω if $-\varphi$ is plurisubharmonic on Ω.

Properties of plurisubharmonic functions can be found in most of the books about complex analysis in several variables like [Kra99], [Sha92] or [Hör90]. Since these functions are a special case of functions we will define in the next sections, we do not be repeat their properties here. Nonetheless, we give the most important approximation techniques for plurisubharmonic functions. The first one uses integral convolution similar to the case of convex functions (see Theorem 4.1.4 in [Hör07]).

Theorem 3.1.7 *Let Ω be an open set in $\mathbb{C}^n$. Fix a number $\varepsilon > 0$ and define $\Omega_\varepsilon := \{z \in \Omega : d(z, b\Omega) > \varepsilon\}$. Then for every $k \in \mathbb{N} \cup \{\infty\}$ we have that*

$$\mathcal{PSH}(\Omega) \subset \overline{\mathcal{PSH}(\Omega_\varepsilon) \cap \mathcal{C}^k(\Omega_\varepsilon)}^{\downarrow \Omega_\varepsilon}.$$

Proof. Let ψ be a plurisubharmonic function on Ω. Pick a $\mathcal{C}^k$-smooth non-negative function θ on $\mathbb{C}^n$ with compact support in the unit ball which fulfills $\theta(z) = \theta(|z|)$ for every $z \in \mathbb{C}^n$ and $\int_{\mathbb{C}^n} \theta(\zeta) dV(\zeta) = 1$. Given a number $\delta \in (0, \varepsilon)$ we set $\theta_\delta(z) := \delta^{-2n}\theta(z/\delta)$. Then the convolution of ψ and θ_δ given by

$$\psi_\delta(z) := (\psi * \theta_\delta)(z) := \int_{\mathbb{C}^n} \psi(z-\zeta)\theta_\delta(\zeta) dV(\zeta)$$

is $\mathcal{C}^k$-smooth on $\mathbb{C}^n$ and plurisubharmonic on Ω_δ. Moreover, the family $\{\psi_\delta\}_{\delta>0}$ decreases to ψ on Ω_ε as δ decreases to zero. □

R. Richberg showed in [Ric68] that a continuous plurisubharmonic function on an open set Ω in $\mathbb{C}^n$ admits an approximation from above by smooth ones defined on the whole of Ω. For a simple proof of this result, we refer to §5.E of Demailly's online book [Dem12]. In fact, Richberg's method is more general and works for *strongly* plurisubharmonic functions on *complex spaces*, but we restrict here to the complex Euclidean space.

Theorem 3.1.8 (Richberg, 1968) *Let Ω be an open set in $\mathbb{C}^n$. Then we have that*

$$\mathcal{PSH}(\Omega) \cap \mathcal{C}(\Omega) \subset \overline{\mathcal{PSH}(\Omega) \cap \mathcal{C}^\infty(\Omega)}^{\downarrow\Omega}.$$

On domains of holomorphy, we have another approximation of continuous plurisubharmonic functions by, roughly speaking, logarithms of the absolute value of holomorphic functions. First, we recall some notions.

Definition 3.1.9 (1) Given an open set Ω in $\mathbb{C}^n$, by the symbol $\mathcal{H}(\Omega)$ we denote the family of all *Hartogs functions*. These are upper semi-continuous functions u on Ω which are of the form

$$u = \max\{1/n_j \log|f_j| : j = 1, \ldots, \ell\},$$

where f_j is a holomorphic function on Ω and $n_j > 0$ is a positive integer for every $j = 1, \ldots, \ell$. Notice that the index ℓ depends on u.

(2) A domain Ω in $\mathbb{C}^n$ is called a *domain of holomorphy* if there exists a holomorphic function f on Ω such that for every boundary point $p \in b\Omega$, neighborhood U of p and connected component V of $\Omega \cap U$ the function $f|V$ does not extend holomorphically to U.

The next approximation property is a classical result stated in [Bre58] by H. J. Bremermann. For a concise proof we refer to Theorem 1.3.9 in Stout's book [Sto07].

Theorem 3.1.10 (Bremermann, 1951) *For every compact set K situated in a domain of holomorphy Ω in $\mathbb{C}^n$ we have that*

$$\mathcal{PSH}(\Omega) \cap \mathcal{C}(\Omega) \subset \overline{\mathcal{H}(\Omega)}^{\downarrow K}.$$

3.2 Subpluriharmonic functions

The pluriharmonic functions allow to define the following family of functions.

Definition 3.2.1 An upper semi-continuous function ψ on an open set Ω in $\mathbb{C}^n$ is called *subpluriharmonic on* Ω if it admits the *local maximum property with respect to pluriharmonic functions on balls in* Ω, i.e., if for every ball $B \Subset U$ and every function h which is pluriharmonic on a neighborhood of $\overline{B}$ with $\psi \leq h$ on bB we already have that $\psi \leq h$ on $\overline{B}$.

The subpluriharmonicity is a local property.

Theorem 3.2.2 *Let ψ be an upper semi-continuous function on an open set Ω in $\mathbb{C}^n$. Then ψ is subpluriharmonic on Ω if and only if ψ is locally subpluriharmonic on Ω, i.e., for every point p in Ω there is a neighborhood U of p in Ω such that ψ is subpluriharmonic on U.*

Proof. The proof follows exactly the lines of that of Theorem 2.3.5, but we have to point out that use Lemma 1.4.2 in the complex version. □

The following statement asserts that, in the definition of subpluriharmonic functions, we may replace balls by any collection τ of relatively compact sets in $\mathbb{C}^n$ which forms a basis of the topology of $\mathbb{C}^n$.

Proposition 3.2.3 *An upper semi-continuous function ψ on an open set Ω in $\mathbb{C}^n$ is subpluriharmonic if and only if ψ is τ-subpluriharmonic on Ω, i.e., if it satisfies the local maximum property on Ω with respect to pluriharmonic functions on the sets in τ.*

Proof. It is easy to verify that ψ is τ-subpluriharmonic on Ω if and only if $\psi + h$ is τ-subpluriharmonic on Ω for every pluriharmonic function h on Ω.

Let ψ be subpluriharmonic on Ω. Pick a relatively compact set D in τ and a pluriharmonic function h defined on a neighborhood of $\overline{D}$ such that $\psi \leq h$ on bD. If there exists a point p in D such that $\psi(p) > h(p)$, then according to Lemma 1.4.2 we can find an $\mathbb{R}$-linear function ℓ on $\mathbb{C}^n$ and a point z_0 in D such that $(\psi + \ell)(z_0) = 0$ but $(\psi + \ell)(z) < 0$ for every z in $D \setminus \{z_0\}$. In this case, the function $\psi + \ell$ and cannot be subpluriharmonic in a neighborhood of z_0. Since ℓ is pluriharmonic on $\mathbb{C}^n$, the function ψ is not subpluriharmonic, which is absurd. We conclude that ψ is τ-subpluriharmonic on Ω.

On the other hand, if ψ is τ-subpluriharmonic on Ω, then it is locally subpluriharmonic on Ω. In view of Theorem 3.2.2 it is already subpluriharmonic on Ω. □

In the definition of subpluriharmonicity, we can replace pluriharmonic functions by different subfamilies of plurisubharmonic functions (compare also Lemma 4.4 in [Sło86]).

Proposition 3.2.4 *Let ψ be upper semi-continuous on an open set Ω in $\mathbb{C}^n$. Then the following statements are equivalent:*

(1) ψ admits the local maximum property with respect to plurisuperharmonic functions.

(2) ψ admits the local maximum property with respect to $\mathcal{C}^2$-smooth plurisuperharmonic functions.

(3) ψ is subpluriharmonic on Ω.

(4) ψ admits the local maximum property with respect to real parts of holomorphic functions.

(5) ψ admits the local maximum property with respect to real parts of holomorphic polynomials.

Proof. The implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5) follow from the inclusions

$$\{\mathrm{Re}(g) : g \in \mathbb{C}[z]\} \subset \{\mathrm{Re}(f) : f \in \mathcal{O}(U)\} \subset \mathcal{PH}(U) \dots$$

$$\dots \subset -\mathcal{PSH}(U) \cap \mathcal{C}^2(U) \subset -\mathcal{PSH}(U),$$

where U is some open set in $\mathbb{C}^n$, $\mathbb{C}[z]$ stands for the set of all holomorphic polynomials and $-\mathcal{PSH}(U) := \{-\psi : \psi \in \mathcal{PSH}(U)\}$ is the set of all plurisuperharmonic functions on U.

The implication (5) $\Rightarrow$ (4) follows from the classical Oka-Weil theorem (see, e.g., Theorem 1.5 of Chapter VI in [Ran86]).

Now it remains to show the implication (4) $\Rightarrow$ (1). Indeed, assume that this implication is wrong so that ψ admits the local maximum property with respect to real parts of holomorphic functions, but is not subpluriharmonic on Ω. The latter means that there are a ball B in $\mathbb{C}^n$, a point p in B and a plurisubharmonic function h defined on a neighborhood of $\overline{B}$ with $\psi + h < 0$ on bB but

$(\psi + h)(p) > 0$. In view of Theorem 3.1.7, we can assume that h is $\mathcal{C}^\infty$-smooth on a neighborhood of $\overline{B}$. Then Theorem 3.1.10 yields the existence of a Hartogs function $u = \max\{n_j^{-1} \log |f_j| : j = 1, \dots, \ell\}$ which satisfies $\psi + h < \psi + u < 0$ on bB and $(\psi + u)(p) > 0$. We can find an index $j_0 \in \{1, \dots, \ell\}$ such that $u(p) = n_{j_0}^{-1} \log |f_{j_0}(p)|$. Since f_{j_0} is zero free in a neighborhood of the closure of the ball B, it follows from Proposition 3.1.4 (1) that there exists a holomorphic function g defined on a neighborhood of $\overline{B}$ such that $n_{j_0}^{-1} \log |f_{j_0}| = \mathrm{Re}(g)$. Then we have that $\psi + \mathrm{Re}(g) < 0$ on bB but $(\psi + \mathrm{Re}(g))(p) > 0$. This is a contradiction to the assumption made on ψ. □

3.3 q-Plurisubharmonic functions

In this section, we introduce the most important functions of this thesis: the q-plurisubharmonic functions in the sense of L. R. Hunt and J. J. Murray [HM78].

Definition 3.3.1 Let $q \in \{0, \dots, n-1\}$ and let ψ be an upper semi-continuous function on an open set Ω in $\mathbb{C}^n$.

(1) The function ψ is *q-plurisubharmonic* on Ω if ψ is subpluriharmonic on $\pi \cap \Omega$ for every complex affine plane π of dimension $q+1$.

(2) The set of all q-plurisubharmonic functions on Ω is denoted by $\mathcal{PSH}_q(\Omega)$.

(3) If $m \geq n$, every upper semi-continuous function on Ω is by convention m-plurisubharmonic, i.e., $\mathcal{PSH}_m(\Omega) := \mathcal{USC}(\Omega)$.

We give a list of properties of q-plurisubharmonic functions.

Proposition 3.3.2 *Every below mentioned function is defined on an open set Ω in $\mathbb{C}^n$ unless otherwise stated.*

(1) The 0-plurisubharmonic functions are exactly the plurisubharmonic functions, and the $(n-1)$-plurisubharmonic functions are the subpluriharmonic functions.

(2) Every q-plurisubharmonic function is $(q+1)$-plurisubharmonic.

(3) If $\lambda \geq 0$, $c \in \mathbb{R}$ and ψ is q-plurisubharmonic, then $\lambda\psi + c$ is also q-plurisubharmonic.

(4) *The limit of a decreasing sequence* $(\psi_k)_{k\in\mathbb{N}}$ *of q-plurisubharmonic functions is again q-plurisubharmonic.*

(5) *If* $\{\psi_i : i \in I\}$ *is a family of locally bounded q-plurisubharmonic functions, then the upper semi-continuous regularization* ψ^* *of* $\psi := \sup_{i\in I}\psi_i$ *is q-plurisubharmonic.*

(6) *The minimum* $\min\{\psi_1, \psi_2\}$ *of a q-plurisubharmonic and an r-plurisubharmonic function is* $(q+r+1)$*-plurisubharmonic.*

(7) *An upper semi-continuous function* ψ *is q-plurisubharmonic on* Ω *if and only if it is locally q-plurisubharmonic on* Ω*, i.e., for each point* p *in* Ω *there is a neighborhood* U *of* p *in* Ω *such that* ψ *is q-plurisubharmonic on* U*.*

(8) *Let* ψ *be a q-plurisubharmonic function on* Ω *and* $F : \mathbb{C}^n \to \mathbb{C}^n$ *a* $\mathbb{C}$*-linear isomorphism. Then the composition* $\psi \circ F$ *is q-plurisubharmonic on* $\Omega' := F^{-1}(\Omega)$*. Especially, if* ψ *is subpluriharmonic on* Ω *and* F *is biholomorphic from an open set* G *in* $\mathbb{C}^n$ *onto* Ω*, then* $\psi \circ F$ *is subpluriharmonic on* G*.*

(9) *A function* ψ *is q-plurisubharmonic on* Ω *if and only if for every ball* $B \Subset \Omega$ *and for every plurisubharmonic function* s *on* B *in* Ω *the sum* $\psi + s$ *is q-plurisubharmonic on* B*.*

(10) *Let* Ω_1 *be an open set in* Ω*,* ψ *be a q-plurisubharmonic function on* Ω *and* ψ_1 *be a q-plurisubharmonic function on* Ω_1 *such that*

$$\limsup_{\substack{w\to z\\ w\in\Omega_1}} \psi_1(w) \le \psi(z) \text{ for every } z \in b\Omega_1 \cap \Omega.$$

Then the subsequent function is q-plurisubharmonic on Ω*,*

$$\varphi(z) := \begin{cases} \max\{\psi(z), \psi_1(z)\}, & z \in \Omega_1 \\ \psi(z), & z \in \Omega \setminus \Omega_1 \end{cases}.$$

(11) ***(Local maximum principle)*** *Let* $q \in \{0, \ldots, n-1\}$ *and* Ω *be a relatively compact open set in* $\mathbb{C}^n$*. Then any function* u *which is upper semi-continuous on* $\overline{\Omega}$ *and q-plurisubharmonic on* Ω *fulfills*

$$\max_{\overline{\Omega}} \psi = \max_{b\Omega} \psi.$$

Proof. The properties (1) to (5) can be derived directly from the definition. The property (6) is Lemma 6.2 in [Sło86]. The statement in (7) follows from Theorem 3.2.2 and the definition. The property (8) is an immediate consequence of the definition together with the facts that F is a biholomorphism and that the composition $s \circ F$ is plurisubharmonic whenever s is plurisubharmonic. The property (9) follows from the results in Proposition 3.2.4 and from the fact that the restriction of a plurisubharmonic function to a lower-dimensional complex plane is again plurisubharmonic. The patching method (10) can be found in Proposition 2.2 in [Die06] and can be verified with similar methods as for the proof of Theorem 2.3.7. The local maximum principle (11) is Lemma 2.7 in [HM78]. It can be proved by the same arguments as in Theorem 2.3.6 using Lemma 1.4.2 in the complex setting. □

3.4 Smooth and strictly q-plurisubharmonic functions

In the following, we investigate twice differentiable and $\mathcal{C}^2$-smooth q-plurisubharmonic functions.

Definition 3.4.1 Let Ω be an open set in $\mathbb{C}^n$.

(1) An upper semi-continuous function ψ on Ω is called *strictly q-plurisubharmonic on* Ω if for every point p in Ω there is a neighborhood U of p and a number $\varepsilon > 0$ such that $\psi - \|z-p\|_2^2$ is q-plurisubharmonic on U.

(2) An upper semi-continuous function ψ on Ω is called *strongly q-plurisubharmonic on* Ω if for every $\mathcal{C}^\infty$-smooth non-negative function θ with compact support in Ω there is a positive number ε_0 such that $\psi + \varepsilon\theta$ remains q-plurisubharmonic on Ω for every real number ε with $|\varepsilon| \le \varepsilon_0$.

(3) We denote the family of all strictly q-plurisubharmonic functions on Ω by $\mathcal{SPSH}_q(\Omega)$.

The next remark explains the relation between q-plurisubharmonic, strictly and strongly q-plurisubharmonic functions.

Remark 3.4.2 (1) By the same arguments as the proof of Proposition 2.4.2, we can show that the two notions of strong and strict q-plurisubharmonicity are equivalent.

(2) Every q-plurisubharmonic function ψ on Ω can be approximated from above by the family $(\psi_\varepsilon)_{\varepsilon>0}$ of strictly q-plurisubharmonic functions ψ_ε defined by $\psi_\varepsilon(z) := \psi(z) + \varepsilon\|z\|_2^2$. Thus,

$$\mathcal{PSH}_q(\Omega) \subset \overline{\mathcal{SPSH}_q(\Omega)}^{\downarrow\Omega}.$$

A smooth (strictly) q-plurisubharmonic can be characterized by counting the eigenvalues of its Levi matrix. This matrix is also known in the literature as *complex Hessian*.

Definition 3.4.3 Let ψ be twice differentiable at a point p. For $\zeta, \eta \in \mathbb{C}^n$ we define the *Levi form of ψ at p* by

$$\mathscr{L}_\psi(p)(\zeta,\eta) := \sum_{k,\ell=1}^{n} \frac{\partial^2\psi}{\partial z_k \partial \overline{z}_\ell}(p)\zeta_k\overline{\eta}_\ell.$$

It is an Hermitian form on $\mathbb{C}^n \times \mathbb{C}^n$.

Similarly to Theorem 2.4.3 in the case of real q-convex functions, we have a the following characterization of smooth q-plurisubharmonic functions (see Lemma 2.6 in [HM78]).

Theorem 3.4.4 *Let $q \in \{0,\ldots,n-1\}$ and let ψ be a $\mathcal{C}^2$-smooth function on an open subset Ω in $\mathbb{C}^n$. Then ψ is (strictly) q-plurisubharmonic if and only if the Levi matrix $\mathscr{L}_\psi(p)$ has at most q negative (q non-positive) eigenvalues at every point p in Ω.*

Proof. First, we show that ψ is q-plurisubharmonic on Ω if and only if its Levi matrix has at most q negative eigenvalues at each point in Ω. In order to do so, we make some simplifications. By the definition of q-plurisubharmonicity and since subpluriharmonicity is a local property, it suffices to show that ψ is subpluriharmonic on a neighborhood U of some point p in Ω if and only if its Levi matrix has at least one non-negative eigenvalue at every point of U. It is clear that we can assume $p = 0$ and $\psi(p) = 0$. Moreover, since the property of a function being subpluriharmonic is stable under summation with pluriharmonic functions, by adding a suitable linear function to ψ, we may suppose that the gradient of ψ does not vanish at the origin. Notice also that subpluriharmonic functions remain subpluriharmonic when composed with a biholomorphic mapping (see Proposition 3.3.2 (8)). Following the arguments in the proof of Lemma 3.2.3 in [Kra99], after a local biholomorphic change of coordinates we achieve that the function ψ has the subsequent form on some neighborhood of the point $p = 0$,

$$\psi(z) = \mathrm{Re}(z_n) + \mathscr{L}_\psi(p)(z, z) + o(\|z\|_2^2).$$

Therefore, we can replace ψ by the function $\psi - \mathrm{Re}(z_n)$, since $\mathrm{Re}(z_n)$ is pluriharmonic. Since all these simplifications have neither an effect on the subpluriharmonicity nor on the eigenvalues of the Levi matrix of ψ, the statement reduces to consider the following function defined on some neighborhood of the origin,

$$\psi(z) = \mathscr{L}_\psi(p)(z, z) + o(\|z\|_2^2). \tag{3.1}$$

Then we can follow exactly the arguments in the proof of Theorem 2.4.3. If the Levi matrix of ψ is negative definite at the origin, the equation (3.1) yields the existence of a ball B around the origin such that ψ vanishes at the origin and is negative on the boundary of B. Thus, it cannot be subpluriharmonic according to the definition. On the other hand, if ψ is not subpluriharmonic, then we can use Lemma 1.4.2 in order to show that the Levi matrix of ψ has only negative eigenvalues at some point near the origin.

Finally, it remains to show that a function is strictly q-plurisubharmonic if and only if its Levi matrix has at most q non-positive eigenvalues. But this follows immediately from the previous discussion using the $\mathcal{C}^\infty$-smooth strictly plurisubharmonic function $z \mapsto \|z\|_2^2$ on $\mathbb{C}^n$. □

The previous result was generalized by Z. Słodkowski in [Sło84] to twice differentiable q-plurisubharmonic functions.

Theorem 3.4.5 *Let $q \in \{0, \ldots, n-1\}$ and let ψ be an upper semi-continuous function on an open subset Ω in $\mathbb{C}^n$.*

(1) If ψ is q-plurisubharmonic on Ω and twice differentiable at a point p, then the Levi matrix of u at p has at most q negative eigenvalues.

(2) If ψ has a lower bounded Hessian and the Levi matrix has at most q negative eigenvalues at almost every point in Ω, then ψ is q-plurisubharmonic on Ω.

Proof. We will not include the proof here because we already repeated it with minor adaption for real q-convex functions in Theorem 2.5.3. For the interested reader, we refer to Theorem 4.1 in [Sło84]. □

Theorem 3.4.4 allows us to easily construct examples of q-plurisubharmonic functions.

Example 3.4.6 (1) The function $(z,w) \mapsto |zw|^2$ is strictly 1-plurisubharmonic but fails to be strictly plurisubharmonic on $\mathbb{C}^2$. The eigenvalues of its Levi matrix at a given point (z,w) are zero and $|z|^2 + |w|^2$. Anyway, it is plurisubharmonic.

(2) If $f : \Omega \to \mathbb{C}^m$ is a holomorphic mapping on an open set Ω in $\mathbb{C}^n$, then $-\log \|f\|_2$ and $1/\|f\|_2$ are subpluriharmonic outside $\{f = 0\}$.

(3) For $k \in \mathbb{N}$ consider the 1-plurisubharmonic function $f_k(z,w) := -k|z|^2$ on $\mathbb{C}^2$. Then the sequence $(f_k)_{k\in\mathbb{N}}$ decreases to the characteristic function $\check{\chi}_{\{z=0\}}$ of $\{z = 0\}$ (recall Example 1.1.3). Since the latter function is equal to $-\infty$ almost everywhere on $\mathbb{C}^2$, it fails to be locally integrable, whereas any plurisubharmonic function lies in L^1_{loc}, except the constant function $-\infty$ (see Theorem 4.17 in §4.C.2 of [Dem12]).

(4) Every entire plurisubharmonic function on $\mathbb{C}^n$ admits the Liouville property, i.e., it is constant whenever it is bounded from above on $\mathbb{C}^n$. But the previous example (3) demonstrates that it is no longer true for q-plurisubharmonic functions, if $q \geq 1$. Another example is given by the function $\psi(z,w) = -1/(1 + |z|^2 + |w|^2)$. It has an upper bound and is strictly 1-plurisubharmonic function on $\mathbb{C}^2$, since its Levi matrix $\mathscr{L}_\psi$ at a given point (z,w) has the eigenvalues

$$\frac{1}{(1+|z|^2+|w|^2)^2} \quad \text{and} \quad \frac{1-|z|^2-|w|^2}{(1+|z|^2+|w|^2)^3}.$$

3.5 Approximation of q-plurisubharmonic functions

The idea of the approximation of real q-convex functions involving the supremum convolution was shown in detail in Section 2.6 and was taken from Słodkowski's approximation technique for q-plurisubharmonic functions presented in Section 2 of his paper [Sło84].

Theorem 3.5.1 (Słodkowski, 1984) *Let Ω be an open set in $\mathbb{C}^n$ and let K be a compact set in Ω. Let $\varepsilon > 0$ be a real number so small that the set $\Omega_\varepsilon := \{z \in \Omega : d(z, b\Omega) > \varepsilon,\ \|z\|_2 < 1/\varepsilon\}$ contains K. Then we have that*

$$\mathcal{PSH}_q(\Omega) \subset \overline{\mathcal{PSH}_q(\Omega_\varepsilon) \cap \mathcal{C}^1_\bullet(\mathbb{C}^n)}^{\downarrow K}.$$

Based on this approximation theorem, Słodkowski established the following result on sums of q-plurisubharmonic functions. The proof of Theorem 2.6.4 is based on his arguments verifying Theorem 5.1 in [Sło84].

Theorem 3.5.2 (Słodkowski, 1984) *Let ψ_1 be a q-plurisubharmonic and ψ_2 be an r-plurisubharmonic function. Then their sum $\psi_1 + \psi_2$ is $(q+r)$-plurisubharmonic.*

Due to an example of K. Diederich and J. E. Fornæss (see Theorem 2 in [DF85]), it is in general not possible to approximate q-plurisubharmonic functions from above by a sequence of $\mathcal{C}^2$-smooth ones. Anyway, L. Bungart was able to show in [Bun90] that continuous q-plurisubharmonic functions can be approximated by piecewise smooth (strictly) q-plurisubharmonic functions.

Definition 3.5.3 Let ψ be a continuous function on an open set Ω in $\mathbb{C}^n$.

(1) The function ψ is called *q-plurisubharmonic with corners* on Ω if for every point p in Ω there is an open neighborhood U of p in Ω and finitely many $\mathcal{C}^2$-smooth q-plurisubharmonic functions $\psi_1, \ldots, \psi_\ell$ on U such that $\psi = \max\{\psi_j : j = 1, \ldots, \ell\}$.

(2) The family of all q-plurisubharmonic functions with corners on Ω is denoted by $\mathcal{PSH}^c_q(\Omega)$.

Bungart's approximation method yields the following result (see Corollary 5.4 in [Bun90]).

Theorem 3.5.4 (Bungart, 1990) *Let Ω be an open subset in $\mathbb{C}^n$. Then we have that*

$$\mathcal{PSH}_q(\Omega) \cap \mathcal{C}(\Omega) \ \subset\ \overline{\mathcal{PSH}^c_q(\Omega)}^{\downarrow \Omega}.$$

As a consequence, we obtain another characterization of q-plurisubharmonic functions.

Corollary 3.5.5 *Let Ω be an open subset in $\mathbb{C}^n$ and $q \in \{0, \ldots, n-1\}$. An upper semi-continuous function ψ on Ω is q-plurisubharmonic if and only if for every open set $U \Subset \Omega$ and every $\varphi \in \mathcal{PSH}^c_{n-q-1}(U)$ the sum $\psi + \varphi$ is subpluriharmonic on U.*

The next result allows us to produce q-plurisubharmonic functions by composing them with real r-convex functions (see Proposition 2.11 in [PZ13]).

Theorem 3.5.6 *Let ω be an open set in $\mathbb{R}^\ell$ and let u be a real r-convex function on ω such that u is separately non-decreasing, i.e., $t_j \mapsto u(t_1, \dots, t_j, \dots, t_\ell)$ is non-decreasing with respect to each fixed index $j \in \{1, \dots, \ell\}$. Let $(q_j)_{j=1,\dots,\ell}$ be finitely many integers $q_j \in \{0, \dots, n\}$ and let $\psi := (\psi^j)_{j=1,\dots,\ell}$ be a collection of upper semi-continuous functions ψ^j on an open set Ω in $\mathbb{C}^n$ such that each ψ^j is q_j-plurisubharmonic on Ω. Furthermore, assume that $\psi(\Omega)$ lies in ω. Then the composition $u \circ \psi$ is $(q + r)$-plurisubharmonic on Ω, where $q := \sum_{j=1}^{\ell} q_j$.*

Proof. **Step 1.** Let us first consider the case when u is $\mathcal{C}^2$-smooth on ω and all the components of ψ are twice differentiable at the point p in Ω. Then the entries of the Levi matrix of $u \circ \psi$ at p are as follows,

$$\frac{\partial(u \circ \psi)}{\partial z_j \partial \bar{z}_k}(p) = \sum_{\lambda=1}^{\ell} \frac{\partial u}{\partial t_\lambda}(a) \cdot \frac{\partial^2 \psi^\lambda}{\partial z_j \partial \bar{z}_k}(p) + \sum_{\lambda,\mu=1}^{\ell} \overline{\frac{\partial \psi^\lambda}{\partial z_j}}(p) \cdot \frac{\partial^2 u}{\partial t_\lambda \partial t_\mu}(a) \cdot \frac{\partial \psi^\mu}{\partial z_k}(p),$$

where $j, k = 1, \dots, n$ and $a = \psi(p)$. The Levi matrix of $u \circ \psi$ at p is then given by

$$\mathscr{L}_{u \circ \psi}(p) = \sum_{\lambda=1}^{\ell} \frac{\partial u}{\partial t_\lambda}(a) \cdot \mathscr{L}_{\psi^\lambda}(p) + \mathscr{J}_\psi^h(p) \cdot \mathscr{H}_u(a) \cdot \mathscr{J}_\psi(p), \tag{3.2}$$

where $\mathscr{J}_\psi = \left(\frac{\partial \psi^j}{\partial z_\mu}\right)_{\substack{j=1,\dots,\ell \\ \mu=1,\dots,n}}$ is the complex Jacobian of ψ and $\mathscr{J}_\psi^h = \overline{\mathscr{J}_\psi}^t$ denotes the conjugate transpose of $\mathscr{J}_\psi$. Now observe that, if $\mathscr{H}_u(p)$ has at most r negative eigenvalues, then $\mathscr{J}_\psi^h(p) \cdot \mathscr{H}_u(a) \cdot \mathscr{J}_\psi(p)$ has also at most r negative eigenvalues. Since u is separately non-decreasing, the partial derivatives $\frac{\partial u}{\partial t_j}$ are non-negative on ω. Hence, $\frac{\partial u}{\partial t_j}(a) \cdot \mathscr{L}_{\psi^j}(p)$ has at most q_j negative eigenvalues for each $j = 1, \dots, \ell$. But then the sum of all the matrices in (3.2) has at most $\sum_{j=1}^{\ell} q_j + r = q + r$ negative eigenvalues.

Step 2. In the non-smooth case, we proceed as follows. By Theorem 3.5.1, for each $j = 1, \dots, \ell$ we can approximate the component function ψ^j by a decreasing sequence $(\psi_k^j)_{k\in\mathbb{N}}$ of continuous functions ψ_k^j which are q-plurisubharmonic on $D_{(k)} := \{z \in \Omega : d(z, b\Omega) > 1/k,\ \|z\|_2 < k\}$ and twice differentiable on a dense subset $\tilde{D}_{(k)}$ of $D_{(k)}$. We set $\psi_k := (\psi_k^1, \dots, \psi_k^\ell)$. By the choice of $(D_{(k)})_{k\in\mathbb{N}}$, we can find an integer k_0 so large that $\psi_k(D_{(k)}) \cap \omega$ is not empty for every $k \geq k_0$.

Step 3. We fix an index $k \geq k_0$ and define $W_k := \psi_k^{-1}(\omega) \cap D_{(k)}$. It is an open set since ψ_k is continuous on $D_{(k)}$. In view of the approximation theorems 2.6.3 and 2.6.8, there is a sequence $(u_m)_{m\in\mathbb{N}}$ of real r-convex functions u_m with corners on $G_{(m)} := \{x \in \omega : d(z, b\omega) > 1/m,\ \|x\|_2 < m\}$ which decreases to u on ω. Now if m_0 is large enough, then $W_{k,m} := \psi_k^{-1}(G_{(m)}) \cap D_{(k)}$ is not empty for every $m \geq m_0$. Fix an integer $m \geq m_0$ and pick a point z_0 in $W_{k,m}$. Then $t_0 := \psi_k(z_0)$ lies in $G_{(m)}$, so there are a neighborhood U_0 of t_0 in $G_{(m)}$ and finitely many $\mathcal{C}^2$-smooth real r-convex functions $u_m^1, \ldots, u_m^\mu$ on U_0 such that $u_m = \max\{u_m^1, \ldots, u_m^\mu\}$. It follows from the first step that the real Hessians of the compositions $u_m^1 \circ \psi_k, \ldots, u_m^\mu \circ \psi_k$ have at most $(q+r)$ negative eigenvalues at every point $\psi_k^{-1}(U_0) \cap \tilde{D}_{(k)}$. Therefore, by Theorem 3.4.5 (2), the function $u_m \circ \psi_k$ is $(q+r)$-plurisubharmonic on $\psi_k^{-1}(U_0) \cap D_{(k)}$. Since z_0 was an arbitrary point in $W_{k,m}$, the function $u_m \circ \psi_k$ is $(q+r)$-plurisubharmonic on $W_{k,m}$ for each $m \geq m_0$. Since $W_k = \bigcup_{m\geq m_0} W_{k,m}$ and the sequence $(u_m \circ \psi_k)_{m\geq m_0}$ decreases to $u \circ \psi_k$ on W_k, we conclude that the function $u \circ \psi_k$ is $(q+r)$-plurisubharmonic on W_k. Finally, since $\Omega = \bigcup_{k\geq k_0} W_k$ and u is separately non-decreasing, we deduce that $(u \circ \psi_k)_{k\geq k_0}$ decreases to $u \circ \psi$ on Ω. Hence, $u \circ \psi$ is $(q+r)$-plurisubharmonic on Ω. □

The previous statement is especially important when $\ell = 0$ and the function u is real 0-convex, i.e., locally convex. As an application, we present a useful regularization technique derived from Lemma (5.18) in Chapter 5 in [Dem12].

Definition 3.5.7 Let θ be a non-negative $\mathcal{C}^\infty$-smooth function on $\mathbb{R}$ with compact support in the unit interval $(-1,1)$ such that $\int_{\mathbb{R}} \theta(s)ds = 1$ and $\theta(-t) = \theta(t)$ for every $t \in \mathbb{R}$. Given real numbers $\varepsilon_1 > 0, \ldots, \varepsilon_\ell > 0$ and $t_1, \ldots, t_\ell \in \mathbb{R}$, we define the *regularized maximum* by

$$\widetilde{\max}_{(\varepsilon_1,\ldots,\varepsilon_\ell)}(t_1, \ldots, t_\ell) := \int_{\mathbb{R}^\ell} \max\{t_1 + \varepsilon_1 s_1, \ldots, t_\ell + \varepsilon_\ell s_\ell\}\theta(s)ds.$$

For a single positive number $\varepsilon > 0$ we set $\widetilde{\max}_\varepsilon := \widetilde{\max}_{(\varepsilon,\ldots,\varepsilon)}$.

The regularized maximum has the following properties.

Lemma 3.5.8 *Let $\varepsilon_1, \ldots, \varepsilon_\ell$ be positive real numbers.*

(1) The function $(t_1, \ldots, t_\ell) \mapsto \widetilde{\max}_{(\varepsilon_1,\ldots,\varepsilon_\ell)}(t_1, \ldots, t_\ell)$ is $\mathcal{C}^\infty$-smooth and convex on $\mathbb{R}^\ell$ and separately non-decreasing in each variable $t_1, \ldots, t_\ell$.

(2) Given $t_1, \ldots, t_\ell \in \mathbb{R}$, it holds that

$$\max\{t_1, \ldots, t_\ell\} \leq \widetilde{\max}_{(\varepsilon_1,\ldots,\varepsilon_\ell)}(t_1, \ldots, t_\ell) \leq \max\{t_1 + \varepsilon_1, \ldots, t_\ell + \varepsilon_\ell\}$$

(3) If $t_j + \varepsilon_j < \max_{i\neq j}\{t_i - \varepsilon_i\}$, then we have that

$$\widetilde{\max}_{(\varepsilon_1,\ldots,\varepsilon_\ell)}(t_1, \ldots, t_\ell) = \widetilde{\max}_{(\varepsilon_1,\ldots,\varepsilon_{j-1},\varepsilon_{j+1},\ldots,\varepsilon_\ell)}(t_1, \ldots, t_{j-1}, t_{j+1}, \ldots, t_\ell).$$

We can apply the regularized maximum to q-plurisubharmonic functions.

Lemma 3.5.9 *Let $\psi_1, \ldots, \psi_\ell$ be finitely many $\mathcal{C}^2$-smooth functions on an open set Ω in $\mathbb{C}^n$ such that for each $j \in \{1, \ldots, \ell\}$ the function ψ_j is q_j-plurisubharmonic on Ω. Then for every positive number ε the regularized maximum $\varphi_\varepsilon := \widetilde{\max}_\varepsilon\{\psi_1, \ldots, \psi_\ell\}$ is $\mathcal{C}^\infty$-smooth and $(q_1+\ldots+q_\ell)$-plurisubharmonic on Ω. Moreover, the family $(\varphi_\varepsilon)_{\varepsilon>0}$ decreases to $\max\{\psi_1, \ldots, \psi_\ell\}$ on Ω when $\varepsilon > 0$ goes to zero.*

Proof. This is a consequence of Lemma 3.5.8 and Theorem 3.5.6. □

3.6 Real q-convex and q-plurisubharmonic functions

The real q-convex functions have the same meaning for q-plurisubharmonic ones like the convex functions have for plurisubharmonic ones. The latter relation was already investigated by P. Lelong in [Lel52b] in the case of $q = 0$. We shall give a generalization of his results to the case $q \geq 1$. At first, we show that real q-convex functions are indeed q-plurisubharmonic.

Theorem 3.6.1 *Let Ω be an open subset in $\mathbb{C}^n = \mathbb{R}^{2n}$. Then every real q-convex function u on Ω is q-plurisubharmonic.*

Proof. If $q \geq n$, then the statement is trivial, since every upper semi-continuous function on Ω is q-plurisubharmonic by convention. By Theorem 2.6.3, we can locally approximate u by a sequence of real q-convex functions which are twice differentiable almost everywhere. Thus, since q-plurisubharmonicity is a local property, we can assume without loss of generality that u is twice differentiable almost everywhere on Ω. Since u is q-plurisubharmonic if and only if it is subpluriharmonic on every complex affine plane of dimension $q+1$, and since the restriction of a real q-convex function to an affine plane remains real q-convex, it is enough to prove the statement in the case of $q = n-1$.

Thus, let us assume that $q = n-1$ and that the real Hessian $\mathscr{H}_u(p)$ of u at p exists for some point p in Ω. By Theorem 2.5.3 (1), the real Hessian $\mathscr{H}_u(p)$ of u at p has at least $2n-(n-1) = n+1$ non-negative eigenvalues. This means that there is a real $n+1$ dimensional subspace V of $\mathbb{C}^n = \mathbb{R}^{2n}$ such that $\mathscr{H}_u(p)$ is positive semi-definite on V. Since V is not totally real, there is a vector v in V such that iv also lies in V. Therefore, since $\mathscr{H}_u(p)(v,v)$ and $\mathscr{H}_u(p)(iv,iv)$ are both non-negative by assumption, it follows that the Levi form of u at p is non-negative due to the following identity,

$$\mathscr{L}_u(p)(v,v) = \frac{1}{4}\Big(\mathscr{H}_u(p)(v,v) + \mathscr{H}_u(p)(iv,iv)\Big).$$

Hence, the Levi matrix $\mathscr{L}_u(p)$ of u at p has at least one non-negative eigenvalue. By the choice of p, we deduce that $\mathscr{L}_u$ has at least one non-negative eigenvalue almost everywhere on Ω. Then Theorem 3.4.5 (2) implies that the function u is $(n-1)$-plurisubharmonic on Ω. □

The previous result cannot be improved because of the following examples.

Remark 3.6.2 (1) The converse of the statement in Theorem 3.6.1 is false in general. Consider the function $z \mapsto \mathrm{Re}(z)^2 - \mathrm{Im}(z)^2 = \mathrm{Re}(z^2)$. It is harmonic on $\mathbb{C}$, but not convex.

(2) Since every real q-convex function is real $(q+1)$-convex, we can generalize Theorem 3.6.1 as follows: If $r \leq q$, then every real r-convex function on Ω is q-plurisubharmonic on Ω. But if $r > q$, the statement is false in general due to the following example: the function

$$(z,w) \mapsto \mathrm{Re}(z)^2 - 2\mathrm{Im}(w)^2 + \mathrm{Re}(z)^2 - 2\mathrm{Im}(w)^2$$

is real 2-convex, 2-plurisubharmonic but not 1-plurisubharmonic on $\mathbb{C}^2$.

Anyway, under some additional conditions we obtain a converse statement of Theorem 3.6.1. Therefor, we need functions which are invariant in their imaginary parts.

Definition & Remark 3.6.3 Let ω be an open set in $\mathbb{R}^n$.

(1) A function $\psi = \psi(z)$ on a *tubular set* $\omega + i\mathbb{R}^n$ in $\mathbb{C}^n$ is called *rigid* if $\psi(z) = \psi(\mathrm{Re}(z))$ for every $z \in \omega + i\mathbb{R}^n$.

(2) By the definition, a rigid function ψ on a tubular set $\omega+i\mathbb{R}^n$ can be naturally considered as a function $x \mapsto \psi(x)$ on ω. On the other hand, every function u on ω induces a well defined rigid function on $\omega + i\mathbb{R}^n$ via $z \mapsto u(\mathrm{Re}(z))$ for every $z \in \omega + i\mathbb{R}^n$.

(3) Therefore, it is justified to write $\mathcal{PSH}_q(\omega)$ for the subfamily of upper semicontinuous functions on ω which induce rigid q-plurisubharmonic functions on $\omega + i\mathbb{R}^n$.

The following result is a generalization of Lelong's observation in the case of $q = 0$ stating that every rigid plurisubharmonic function is locally convex.

Theorem 3.6.4 *Let ω be an open set in $\mathbb{R}^n$. Then every rigid function on $\omega + i\mathbb{R}^n$ is q-plurisubharmonic if and only if it is real q-convex on ω, i.e.,*

$$\mathcal{PSH}_q(\omega) = \mathcal{CVX}_q(\omega).$$

Proof. Using the approximation theorems for real q-convex functions, we can easily deduce that, if a function u is real q-convex on ω, then it is also real q-convex on $\omega + i\mathbb{R}^n$. Then the inclusion $\mathcal{CVX}_q(\omega) \subset \mathcal{PSH}_q(\omega)$ follows directly from Theorem 3.6.1.

For the other inclusion, consider a rigid q-plurisubharmonic function ψ on $\Omega := \omega + i\mathbb{R}^n$. Pick a real plane π in $\mathbb{R}^n$ of dimension $q + 1$, a ball $B \Subset \pi \cap \omega$ and a linear function ℓ on π such that $\psi \leq \ell$ on bB. After a linear change of coordinates of the form $z \mapsto \lambda z + p$, where $\lambda \in \mathbb{R}$ and $p \in \mathbb{C}^n$, we may assume that π contains the origin and that $B = B_1^n(0) \cap \pi$. Given a positive number $R > 0$, which will be specified later, and another ball $B_R := B_R^n(0) \cap \pi$ in π, consider the set $D_R := B + iB_R$. Since Ω is tubular, $B \Subset \omega \cap \pi$ and since $0 \in \pi$, the set D_R contains $B + i\{0\}^n$ and lies relatively compact in $\Omega \cap \pi^{\mathbb{C}}$, where $\pi^{\mathbb{C}} := \pi + i\pi$. Moreover, the boundary of D_R in $\pi^{\mathbb{C}}$ splits into two parts,

$$A_1 := bB + i\overline{B_R} \quad \text{and} \quad A_2 := \overline{B} + i(bB_R).$$

Since ℓ is linear, ψ is q-plurisubharmonic on Ω and since $z \mapsto \|x\|_2^2 - \|y\|_2^2 = \sum_{j=1}^n \mathrm{Re}(z_j^2)$ is pluriharmonic on $\mathbb{C}_z^n = \mathbb{R}_x^n + i\mathbb{R}_y^n$, it follows from Proposition 3.3.2 (9) that for every integer $k \in \mathbb{N}$ the function

$$\psi_k(z) := \psi(x) - \ell(x) + \left(\|x\|_2^2 - \|y\|_2^2\right)/k$$

is q-plurisubharmonic on Ω. The assumption $\psi \le \ell$ on bB and the choice of D_R now yield the subsequent estimates for ψ_k on the boundary of D_R,

$$\psi_k \le 1/k \quad \text{on } A_1 \quad \text{and} \quad \psi_k \le \psi - \ell + (1-R^2)/k \quad \text{on } A_2.$$

Thus, if we choose $R > 0$ to be large enough, then ψ_k becomes negative on A_2. Hence, the function ψ_k is bounded by $1/k$ on the boundary of D_R. Since ψ_k is q-plurisubharmonic, the local maximum principle (Proposition 3.3.2 (11)) implies that the function ψ_k is bounded from above by $1/k$ on the closure of D_R in $\pi^{\mathbb{C}}$. In particular, $\psi_k \le 1/k$ on $B + i\{0\}^n$. But the last inequality holds for every integer $k \in \mathbb{N}$. This yields $\psi - \ell \le 0$ on B, and we can conclude that ψ is real q-convex on ω. □

3.7 Weakly q-plurisubharmonic functions

We show that the composition of q-plurisubharmonic with holomorphic functions remains q-plurisubharmonic. This would permit to introduce q-plurisubharmonicity on complex manifolds and spaces, but this topic deserves its own treatise, so we restrict our considerations in this thesis to the complex Euclidean space. A similar proof of the next result can be found in [Die06] (Proposition 2.2) or in [PZ13] (Proposition 2.9).

Theorem 3.7.1 *Let ψ be a q-plurisubharmonic function on an open set Ω in $\mathbb{C}^n$ and let $f : D \to \Omega$ be a holomorphic mapping defined on some other open set D in $\mathbb{C}^\ell$. Then the composition $\psi \circ f$ is q-plurisubharmonic on D.*

Proof. If $q \ge \ell$, then there is nothing to show, so we assume that $q < \ell$.

Suppose first that ψ is $\mathcal{C}^2$-smooth. Then the entries of the Levi matrix of $\psi \circ f$ at a fixed point p in D are given by

$$\frac{\partial^2(\psi \circ f)}{\partial z_j \partial \overline{z}_k}(p) = \sum_{\lambda,\mu=1}^{n} \overline{\frac{\partial f_\lambda}{\partial z_j}}(p) \cdot \frac{\partial^2 \psi}{\partial w_\lambda \overline{w}_\mu}(a) \cdot \frac{\partial f_\mu}{\partial z_k}(p),$$

where $a = f(p)$ and $k, j = 1 \dots, \ell$. Therefore, the Levi matrix of $\psi \circ f$ at p is of the form

$$\mathscr{L}_{\psi \circ f}(p) = \mathscr{J}_f^h(p) \cdot \mathscr{L}_\psi(a) \cdot \mathscr{J}_f(p), \tag{3.3}$$

where $\mathscr{J}_f = (\partial\psi/\partial z_\mu)_{\mu=1}^n$ is the complex gradient of f and $\mathscr{J}_f^h = \overline{\mathscr{J}_f}^t$ denotes its conjugate transpose. Since ψ is q-plurisubharmonic on Ω, its Levi matrix at p has at most q negative eigenvalues. Then Corollary 4.5.11 in [HJ13] implies that the Levi matrix of $\psi \circ f$ also has at most q negative eigenvalues. This means that $\psi \circ f$ is q-plurisubharmonic on D.

If ψ is not necessarily $\mathcal{C}^2$-smooth, by Theorems 3.5.1 and 3.5.4 there exist a collection $(G_m)_{m\in\mathbb{N}}$ of open sets $G_m = \{z \in \Omega : d(z, b\Omega) > 1/m,\ \|z\|_2 < m\}$ in Ω and a family $(\psi_m)_{m\in\mathbb{N}}$ of q-plurisubharmonic functions with corners on G_m decreasing locally to ψ on Ω. Define $D_m := f^{-1}(G_m)$ and let m_0 be so large that $D_m \neq \emptyset$ for every $m \geq m_0$. Fix an integer $m \geq m_0$. Now if z_0 is in D_m, then $w_0 = \psi(z_0)$ lies in G_m and there are a neighborhood U_m of w_0 in G_m and finitely many functions $\psi_m^1, \ldots, \psi_m^{\nu_m}$ which are $\mathcal{C}^2$-smooth and q-plurisubharmonic on U_m and fulfill $\psi_m = \max\{\psi_m^j : j = 1, \ldots, \nu_m\}$ on U_m. In view of the previous discussion, for each $j = 1, \ldots, \nu_m$ the function $\psi_m^j \circ f$ is q-plurisubharmonic on $f^{-1}(U_m)$. Hence, the function $\psi_m \circ f = \max\{\psi_m^j \circ f : j = 1, \ldots, \nu_m\}$ is q-plurisubharmonic on $f^{-1}(U_m)$. Since p was an arbitrary point in D_m, we derive that $\psi_m \circ f$ is q-plurisubharmonic on D_m. Therefore, the sequence $(\psi_m \circ f)_{m\in\mathbb{N}}$ of q-plurisubharmonic functions $\psi_m \circ f$ locally decreases to $\psi \circ f$ on D. We conclude that $\psi \circ f$ is q-plurisubharmonic on D. □

We shall give the following important example.

Example 3.7.2 Let Ω be an open set in $\mathbb{C}^n$ and let $h : \Omega \to \mathbb{C}^q$ be a holomorphic mapping. Define $A := \{h = 0\}$. Then, in view of Theorem 3.7.1, for each $k \in \mathbb{N}$ the function $\check{\chi}_{A,k} := -k\|h\|_2^2$ is q-plurisubharmonic on Ω and decreases to the characteristic function

$$\check{\chi}_A(z) = \begin{cases} 0, & z \in A \\ -\infty, & z \in \Omega \setminus A \end{cases}.$$

Hence, it is q-plurisubharmonic on Ω.

The next definition is based on the following interpretation of q-plurisubharmonicity: an upper semi-continuous function ψ is q-plurisubharmonic on Ω if and only if the composition $\psi \circ L$ is subpluriharmonic for every complex affine linear mapping $L : \mathbb{C}^{q+1} \to \mathbb{C}^n$ restricted to $L^{-1}(\Omega)$. It is then interesting to ask whether one can replace linear mappings by another family of holomorphic mappings with image in Ω.

Definition 3.7.3 An upper semi-continuous function ψ on an open set Ω in $\mathbb{C}^n$ is called *weakly q-plurisubharmonic* if $\psi \circ f$ is subpluriharmonic for every holomorphic mapping $f : D \to \Omega$, where D is a domain in $\mathbb{C}^{q+1}$.

In [Fuj92], O. Fujita showed that q-plurisubharmonicity and weak q-plurisubharmonicity is the same notion using properties of q-pseudoconvex sets, which we will define later in Chapter 2. But in our situation, his result is an immediate consequence of Theorem 3.7.1. Notice that in his notation *pseudoconvex functions of order k* are exactly weakly $(n-k-1)$-plurisubharmonic functions in our sense.

Theorem 3.7.4 (Fujita, 1992) *A function ψ is q-plurisubharmonic on an open set Ω in $\mathbb{C}^n$ if and only if it is weakly q-plurisubharmonic on Ω.*

3.8 q-Plurisubharmonic functions on analytic sets

We recall the definition of analytic subsets in the complex Euclidean space. For more details we refer to E. M. Chirka's book [Chi89] or to the book by H. Grauert and R. Remmert [GR04]. The content of this section will appear in our joint article [PZ15].

Definition & Remark 3.8.1 Let Ω be an open set in $\mathbb{C}^n$.

(1) A subset A of Ω is called *analytic subset* (*of* Ω) if for every point p in Ω there exist an open neighborhood U of p in Ω, an integer $k \geq 1$ and a holomorphic mapping $h : U \to \mathbb{C}^k$ such that $A \cap U = \{h = 0\}$. Notice that by this definition A is a closed set in Ω.

(2) Let A and A' be two analytic subsets of Ω with $A' \subset A$. Then we say that A' is an *analytic subset of* A.

(3) Let A be an analytic subset of Ω and let $p \in A$. If p has an open neighborhood U in Ω such that $A \cap U$ is a complex submanifold, it is called a *regular point of* A. The set of all regular points of A is denoted by A^{reg}. It is a dense subset of A.

(4) For each regular point z of A the *dimension* $\dim_z A$ *of* A *at* z is the complex dimension of the submanifold $A \cap U$.

(5) The set of all *singular points* is defined by $A^{\text{sing}} := A \setminus A^{\text{reg}}$. It is again an analytic subset of A (see Chapter 6, §2.2 of [GR04]).

(6) Given an arbitrary point p in A, the *dimension of A at p* is defined by

$$\dim_p A := \limsup_{\substack{z \to p \\ z \in A^{\text{reg}}}} \dim_z A.$$

(7) We are also interested in the *minimal dimension of A* given by

$$\min\dim A := \inf\{\dim_z A : z \in A\}.$$

(8) The minimal dimension is motivated by the observation that, given an integer $q \in \{0, \dots, n-1\}$ and a holomorphic mapping $h : \Omega \to \mathbb{C}^q$, the rank of the complex Jacobian of h does not exceed q. Therefore, $\{h = 0\}$ is an analytic subset of Ω with minimal dimension at least $n - q$. Such mappings h will play an important role to us in the next Section 3.9.

Our next aim is to define q-plurisubharmonic functions on analytic subsets. Due to Fujita's result there are two possible ways of doing so, but which, a priori, lead to different notions.

Definition 3.8.2 Let A be an analytic subset of an open set Ω in $\mathbb{C}^n$. Let ψ be an upper semi-continuous function on A.

(1) We say that ψ is *(strictly) q-plurisubharmonic on A* if for every point p in A there are an open neighborhood U of p in Ω and a (strictly) q-plurisubharmonic function Ψ on U such that $\psi = \Psi$ on $A \cap U$.

(2) The function ψ is called *(strictly) weakly q-plurisubharmonic on A* if for every holomorphic mapping $f : D \to A$ defined on a domain D in $\mathbb{C}^{q+1}$ the composition $\psi \circ f$ is (strictly) q-plurisubharmonic on D.

(3) The set of all q-plurisubharmonic and weakly q-plurisubharmonic functions on A is denoted by $\mathcal{PSH}_q(A)$ and, respectively, $\mathcal{WPSH}_q(A)$.

J. E. Fornæss and R. Narasimhan showed in [FN80] that these two notions are equivalent on analytic sets A if $q = 0$, i.e.,

$$\mathcal{PSH}_0(A) = \mathcal{WPSH}_0(A).$$

It was generalized by A. Popa-Fischer in [PF02] to the case of continuous functions and $q \geq 1$. It is still an open question whether each upper semi-continuous weakly q-plurisubharmonic function on an analytic set is q-plurisubharmonic. We have to admit that their results are also valid on reduced complex spaces, but we are only interested in analytic subsets in the complex Euclidean space.

Theorem 3.8.3 (Popa-Fischer, 2002) *A continuous function on an analytic subset A is weakly q-plurisubharmonic if and only if it is q-plurisubharmonic, i.e.,*

$$\mathcal{PSH}_q(A) \cap \mathcal{C}(A) = \mathcal{WPSH}_q(A) \cap \mathcal{C}(A).$$

We will show that each r-plurisubharmonic function on an analytic subset extends trivially to the ambient space in the following sense.

Proposition 3.8.4 *Fix numbers $q \in \{0, \ldots, n-1\}$ and $r \in \mathbb{N}_0$. Let Ω be an open set in $\mathbb{C}^n$ and let A be an analytic subset of Ω. We set $q := n - \min\dim A$. If the function ψ is r-plurisubharmonic on A, then ψ extends to a $(q+r)$-plurisubharmonic function Ψ_A on the whole of Ω via the trivial extension*

$$\Psi_A(z) := \begin{cases} \psi(z), & z \in A \\ -\infty, & z \in \Omega \setminus A \end{cases}.$$

Proof. If $q = n$, then by convention ψ is $(n+r)$-plurisubharmonic since it is upper semi-continuous on Ω. Thus, there is nothing to show and we can assume that $q < n$, so that $\min\dim A \geq 1$.

First, we show that Ψ_A is subpluriharmonic on Ω if $\min\dim A = 1$ and $\psi \equiv 0$ on A. In this case, assume that $\check{\chi}_A := \Psi_A$ is not subpluriharmonic. Then it follows from Lemma 1.4.2 and Proposition 3.2.4 that there exist a ball $B \Subset \Omega$ centered at a point p in Ω, a function g holomorphic on some neighborhood of $\overline{B}$ and a number $\varepsilon > 0$ such that $(\check{\chi}_A + \mathrm{Re}(g))(p) = 0$ and

$$(\check{\chi}_A + \mathrm{Re}(g))(z) < -\varepsilon \|z - p\|_2^2 \text{ for every } z \in \overline{B} \setminus \{p\}. \tag{3.4}$$

Notice that, since $\check{\chi}_A$ is identical to $-\infty$ outside of A, the ball B intersects A and p is contained in A. Hence, (3.4) reduces to

$$\mathrm{Re}(g)(p) = 0 \text{ and } \mathrm{Re}(g)(z) < 0 \text{ for every } z \in (A \cap \overline{B}) \setminus \{p\}.$$

But this means that the holomorphic function $\exp(g)$ violates the local maximum modulus principle for holomorphic functions on analytic sets (see Chapter 5, §5.2 in [GR04]). Thus, $\check{\chi}_A$ has to be subpluriharmonic on Ω.

Now consider the more general case of $n - q = \min\dim A \geq 1$, but ψ still vanishing on A. Fix a complex affine plane π of dimension $q+1$ intersecting A. Then for each $z \in A \cap U$ we have that

$$\dim_z A \cap L \geq \dim_z A + \dim_z L - n \geq 1.$$

The last inequality is derived from Proposition 2 in paragraph 3.5, Chapter 1 of [Chi89]. In view of the previous discussion, we obtain that $\check{\chi}_A = \check{\chi}_{A\cap L}$ is subpluriharmonic on $L \cap \Omega$. Since L is an arbitrary complex plane of dimension $q+1$, it follows from definition that $\check{\chi}$ is q-plurisubharmonic on Ω.

We proceed by verifying the general case of ψ being an arbitrary r-plurisubharmonic function on A. By definition, for every point p in A there are an open neighborhood V_p of p in Ω and an r-plurisubharmonic function $\widehat{\psi}_p$ on V_p such that $\widehat{\psi}_p = \psi$ on $A_p := A \cap V_p$. According to Theorem 3.5.2 and the preceding discussion, the sum

$$\Psi_A(z) = (\widehat{\psi}_p + \check{\chi}_A)(z) = \begin{cases} \psi(z), & z \in A_p \\ -\infty, & z \in V_p \setminus A_p \end{cases}$$

is $(q+r)$-plurisubharmonic on V_p. Hence, the function Ψ_A is $(q+r)$-plurisubharmonic on the open neighborhood $V := \bigcup_{p\in A} V_p$ of A in Ω and is identical to $-\infty$ on $V \setminus A$. Therefore, it can be easily extended by $-\infty$ to a $(q+r)$-plurisubharmonic function into the whole of Ω. □

As a consequence of the previous proposition, we obtain a version of the local maximum principle of q-plurisubharmonic functions on analytic set. It was already shown by Słodkowski (see for example Proposition 5.2 and Corollary 5.3 in [Sło86]).

Proposition 3.8.5 (Local maximum principle) *Fix an integer number $q \in \{0, \dots, n-1\}$. Let A be an analytic subset of an open set Ω in $\mathbb{C}^n$ with $\min\dim A = q+1$ and let ψ be a q-plurisubharmonic function on A. Then for every compact set K in A we have that*

$$\max_K \psi = \max_{b_A K} \psi.$$

Here, by $b_A K$ we mean the relative boundary of K in A.

Proof. Let L be a compact set in Ω such that $A\cap L = K$ and $A \cap bL = b_A K$. Since $\min\dim A = q+1$, it follows from Proposition 3.8.4 that its trivial extension Ψ_A from A to Ω by $-\infty$ is $(n-1)$-plurisubharmonic on Ω. Then the local maximum principle for q-plurisubharmonic functions on Ω (see Proposition 3.3.2 (11)) yields the desired identity,

$$\max_K \psi = \max_{A\cap L} \psi = \max_L \Psi_A = \max_{bL} \Psi_A = \max_{A\cap bL} \psi = \max_{b_A K} \psi.$$

□

In the case of $q = 0$ (and even in complex spaces), a stronger extension property is due to M. Colţoiu. We recall some notions related to Colţoiu's result.

Definition & Remark 3.8.6 Let Ω be an open set in $\mathbb{C}^n$.

(1) A compact set $K \subset \Omega$ is called *holomorphically convex in* Ω if it coincides with its *holomorphically convex hull*

$$\widehat{K}^{\Omega}_{\mathcal{O}(\Omega)} = \{z \in \Omega : |f(z)| \leq \|f\|_K \text{ for every } f \in \mathcal{O}(\Omega)\}.$$

(2) The set Ω is a *Stein open set* or *pseudoconvex set* if it admits a continuous plurisubharmonic function φ on Ω such that $\{\varphi < c\} \Subset \Omega$ for every $c \in \mathbb{R}$.

(3) Due to the classical result on the solution of the Levi problem (see, e.g., [Hör90]), we have that a set Ω in $\mathbb{C}^n$ is Stein open if and only if it is a domain of holomorphy (recall Definition 3.1.9 (2)). We will investigate generalized holomorphically convex hulls and Stein open sets later in Part III of this thesis.

We shall repeat Colţoiu's extension theorem (see Proposition 2 in [Col91]). It is still an open question whether his result carries over to the case $q \geq 1$.

Theorem 3.8.7 (Colţoiu, 1990) *Let A be an analytic subset of a Stein open set Ω in $\mathbb{C}^n$. Then every plurisubharmonic function ψ on A extends to a plurisubharmonic function Ψ into the whole of Ω. Moreover, if K is a compact holomorphically convex set in Ω and $\psi < 0$ on $A \cap K$, then $\Psi < 0$ on K.*

Colţoiu's result yields an approximation property of plurisubharmonic functions on analytic subsets.

Proposition 3.8.8 *Let A be an analytic subset of a Stein open set Ω in $\mathbb{C}^n$. Let ψ be a plurisubharmonic function on A. Then for every compact set K in A and every continuous function f on K with $\psi < f$ on K there exists a $\mathcal{C}^\infty$-smooth plurisubharmonic function $\widetilde{\Psi}$ on Ω such that $\psi < \widetilde{\Psi} < f$ on K.*

Proof. Denote by Ψ_0 the plurisubharmonic extension of ψ to the whole of Ω derived from Theorem 3.8.7. Since Ω is Stein open, there is a continuous plurisubharmonic function φ on Ω which fulfills $\{\varphi < c\} \Subset \Omega$ for every $c \in \mathbb{R}$. Without loss of generality we can assume that $\varphi < 0$ on K. In view of Theorem 3.1.7, there exists a $\mathcal{C}^\infty$-smooth plurisubharmonic function Ψ_1 defined on an open neighborhood $U \Subset \Omega$ of $\overline{\{\varphi < 0\}}$ which satisfies $\Psi_0 = \psi < \Psi_1 < f$ on K. Now if we choose a large enough constant $c > 0$, the function

$$\Psi := \begin{cases} \max\{c\varphi, \Psi_1\}, & \text{on } \{\varphi < 0\} \\ c\varphi, & \text{on } \Omega \setminus \{\varphi < 0\} \end{cases}$$

is plurisubharmonic and continuous on the whole of Ω and satisfies the inequalities $\psi < \Psi = \Psi_1 < f$ on K. Then Richberg's theorem 3.1.8 yields the desired $\mathcal{C}^\infty$-smooth plurisubharmonic function $\widetilde{\Psi}$. □

3.9 r-Plurisubharmonic functions on foliations

In this section, we define a special subfamily of q-plurisubharmonic functions. It is motivated by the fact that, locally, every $\mathcal{C}^2$-smooth strictly q-plurisubharmonic function is strictly plurisubharmonic on the leaves of a foliation by complex submanifolds of codimension q. First, we recall the definition of a complex foliation using local coordinate charts (compare the book [CLN85] by C. Camacho and A. Lins Neto and their definition of real foliations). Another good summary on holomorphic (singular) foliations induced by vector fields can be found in the online article [RR11]. All the following considerations in this sections are included in the joint article [PZ15].

Definition & Remark 3.9.1 Let q be an integer in $\{0, \ldots, n-1\}$ and let Ω be an open set in $\mathbb{C}^n$. A *(regular) complex foliation $\mathcal{F}$ on Ω of codimension q (or dimension $n-q$)* is a collection of local charts $(U_j, \varphi_j)_{j \in J}$ such that:

(1) The collection $\{U_j\}_{j \in J}$ is an open covering of Ω.

(2) For every index $j \in J$ the function φ_j is a biholomorphism defined on U_j onto an open set $V_j = V_j' \times V_j'' \subset \mathbb{C}^q \times \mathbb{C}^{n-q}$.

(3) For every pair of indexes j and k in J the composition $\varphi_j \circ \varphi_k^{-1}$ has the form

$$\varphi_j \circ \varphi_k^{-1}(z, w) = \big(g_{jk}(z), h_{jk}(z, w)\big)$$

for $(z,w) \in V_k \cap V_j \subset \mathbb{C}^q \times \mathbb{C}^{n-q}$ and the holomorphic mappings

$$g_{jk} : V_k' \cap V_j' \to \mathbb{C}^q \quad \text{and} \quad h_{jk} : V_k \cap V_j \to \mathbb{C}^{n-q}.$$

We also recall the definition and properties of leaves of a foliation.

Definition & Remark 3.9.2 Let $\mathcal{F} = (U_j, \varphi_j)_{j \in J}$ be a complex foliation on an open set Ω in $\mathbb{C}^n$.

(1) Let $j \in J$ be fixed and let $(z_0, w_0) = \varphi_j(p_0)$ in $\mathbb{C}^q \times \mathbb{C}^{n-q}$ for some point p_0 in U_j. We have that the inverse image $\varphi_j^{-1}(\{z_0\} \times \mathbb{C}^{n-q})$ is a submanifold of U_j of codimension q containing p_0. This submanifold is called a *slice or local leaf of* $\mathcal{F}$ in U_j.

(2) Every chart U_j is a disjoint union of slices of the same codimension q. If $S \subset U_j$ and $T \subset U_k$ are two slices of codimension q, then the condition (3) in the definition above implies that the intersection $A \cap B$ is either empty or a submanifold of $U_j \cap U_k$ of codimension q. Hence, the slices piece together from chart to chart to form maximal connected subsets of Ω, which are called *leaves of* $\mathcal{F}$ in Ω. Every leaf of Ω is the (possible infinite) union of all intersecting slices, and it is naturally endowed with the structure of a complex connected submanifold of Ω of codimension q.

A complex foliation can be defined differently using holomorphic mappings.

Remark 3.9.3 Let Ω be an open set in $\mathbb{C}^n$ and let $\pi_z : \mathbb{C}^q \times \mathbb{C}^{n-q} \to \mathbb{C}^q$ be the trivial projection into the first q entries defined by $\pi_z(z,w) = z$. Given any chart (U_j, φ_j) of a complex foliation $\mathcal{F}$ on Ω, the slices of $\mathcal{F}$ in U_j are the fibres $h_j^{-1}(c)$ of the holomorphic mapping $h_j := \pi_z \circ \varphi_j : U_j \to \mathbb{C}^q$. In particular, the mappings h_j are holomorphic submersions. Thus, every complex foliation $\mathcal{F}$ of codimension q induces a family $H_{\mathcal{F}} := \{h_j\}_{j \in J}$ of holomorphic submersions such that $\{U_j\}_{j \in J}$ covers Ω.

In the literature, one encounters many different ways to define singular (complex) foliations (see e.g. [RR11]). For instance, $\mathcal{F}$ is a *singular foliation* of an open set Ω in $\mathbb{C}^n$ if there exists an analytic subset A of Ω such that A has at least codimension 1 and $\mathcal{F}$ is a (regular) foliation of $\Omega \setminus A$. In this definition, the connection of $\mathcal{F}$ to A is not really clear. In order to obtain more control on the singular part A of the foliation, we will prefer the following definition which is based on the idea to replace $H_{\mathcal{F}}$ in the previous remark by an arbitrary family of holomorphic mappings with image in $\mathbb{C}^q$.

Definition 3.9.4 Let $q \in \{0, \ldots, n-1\}$ and Ω be an open set in $\mathbb{C}^n$. By a *web of singular foliations on* Ω *(of codimension* q*)* we mean a family $H = \{h_j\}_{j \in J}$ of holomorphic mappings $h_j : U_j \to \mathbb{C}^q$, where $\{U_j\}_{j \in J}$ forms an open covering of Ω by open sets U_j in Ω. Each mapping h_j induces a *singular foliation* on Ω_j via its fibers $\{h_j^{-1}(c)\}_{c \in h_j(\Omega_j)}$.

Now we define *r-plurisubharmonic functions on foliations*. We accept by convention that $\mathbb{C}^0 = \{0\}$ and that every upper semi-continuous function is plurisubharmonic on a discrete set. Also recall that, given a holomorphic mapping $h : \Omega \to \mathbb{C}^q$ defined on an open set Ω in $\mathbb{C}^n$, the fiber $h^{-1}(c)$ is an analytic subset of Ω of minimal dimension at least $n - q$ (see Remark 3.8.1 (8)).

Definition 3.9.5 Let $q \in \{0, \ldots, n-1\}$, $r \in \mathbb{N}_0$ and let Ω be an open set in $\mathbb{C}^n$.

(1) Let $H = \{h_j : U_j \to \mathbb{C}^q\}_{j \in J}$ be a web of singular foliations on Ω. An upper semi-continuous function ψ on Ω is *r-plurisubharmonic on the foliations of* H if for every point p in Ω there is an index $j \in J$ such that U_j contains p and for every $c \in h_j(U_j)$ the function ψ is r-plurisubharmonic on the fiber $h_j^{-1}(c)$.

(2) The symbol $\mathcal{PSH}_r(H, \Omega)$ stands for the family of all r-plurisubharmonic functions on the foliations of H.

(3) If $h : \Omega \to \mathbb{C}^q$ is a single holomorphic mapping defined on Ω, then we simply write $\mathcal{PSH}_r(h, \Omega)$ instead of $\mathcal{PSH}_r(\{h\}, \Omega)$ and say that functions in $\mathcal{PSH}_r(h, \Omega)$ are *r-plurisubharmonic on a singular foliation (induced by h)*.

(4) We set $\mathcal{PSH}(H, \Omega) := \mathcal{PSH}_0(H, \Omega)$.

(5) If $q = 0$, then $\mathcal{PSH}_r(H, \Omega) = \mathcal{PSH}_r(\Omega)$.

We assert that each r-plurisubharmonic function on foliations of codimension q is $(q + r)$-plurisubharmonic.

Theorem 3.9.6 *Fix integers* $q \in \{0, \ldots, n-1\}$ *and* $r \in \mathbb{N}_0$. *Let* Ω *be an open set in* $\mathbb{C}^n$ *and let* $H = \{h_j : U_j \to \mathbb{C}^q\}_{j \in J}$ *be a web of singular foliations on* Ω. *Then every function* $\psi \in \mathcal{PSH}_r(H, \Omega)$ *is* $(q + r)$*-plurisubharmonic on* Ω.

Proof. Let p be a point in Ω. By the assumption made on H, there are an open set U in $\{U_j\}_{j \in J}$ containing p and a holomorphic mapping $h : U \to \mathbb{C}^q$ in H such that for every c in $h(U)$ the function ψ is r-plurisubharmonic on the fiber

$A_c := h^{-1}(c)$. Since the rank of h does not exceed q, the fiber A_c is an analytic subset of U of minimal dimension at least $n - q$. In view of Proposition 3.8.4 the function

$$\Psi_{A_c} = \begin{cases} \psi(z), & z \in A_c \\ -\infty, & z \in U \setminus A_c \end{cases} .$$

is $(q + r)$-plurisubharmonic on U. Now consider the function Φ defined by

$$U \ni z \mapsto \Phi(z) := \sup\{\Psi_{A_c}(z) : c \in h(U)\}.$$

Since all fibers in $\{A_c\}_{c\in h(U)}$ are pairwise disjoint and $\Psi_{A_c} = -\infty$ on $U \setminus A_c$, it follows that $\Phi = \psi$ on U. Hence, Φ is upper semi-continuous and, therefore, it coincides with its upper semi-continuous regularization Φ^*. Since it is the regularized supremum of a family $\{\Psi_{A_c}\}_{c\in h(U)}$ of $(q+r)$-plurisubharmonic functions on U, Proposition 3.3.2 (5) implies that the function $\Phi = \psi$ is $(q + r)$-plurisubharmonic on U. Since p is an arbitrary point in Ω and $\{U_j\}_{j\in J}$ forms a covering of Ω, we conclude that the function ψ is $(q + r)$-plurisubharmonic on the whole of Ω. □

As a direct application, we derive the following interesting examples.

Example 3.9.7 (1) Let $n, q \in \mathbb{N}_0$ and let ψ be an upper semi-continuous function defined on an open subset $U := V \times W$ of the product $\mathbb{C}^n \times \mathbb{C}^q$ such that for each fixed entry w in W the function $z \mapsto \psi(z, w)$ is plurisubharmonic on V. Since for every $w \in W$ the function ψ is plurisubharmonic on the q-codimensional fiber $V \times \{w\}$, we derive from Theorem 3.9.6 that ψ is q-plurisubharmonic on U.

(2) Let ψ be a $\mathcal{C}^2$-smooth function defined on an open subset $U = V \times W$ of the product $\mathbb{C}^n \times \mathbb{C}^q$ for some non-negative integers n and q. If for every fixed entry w in W the function $z \mapsto \psi(z, w)$ is strictly plurisubharmonic on U, then the Levi matrix of ψ is positive definite on the subspace $\mathbb{C}^n \times \{0\}$ of $\mathbb{C}^n\times\mathbb{C}^q$ according to Theorem 3.4.4. Therefore, the Levi matrix of ψ has at most q strictly positive eigenvalues at every point (z, w) in U, and so ψ is strictly q-plurisubharmonic on U.

(3) The above result fails to hold if we relax the $\mathcal{C}^2$-smooth condition on ψ. Consider the upper semi-continuous function ϕ defined on $\mathbb{C}^2$ by

$$\phi(z, w) = \begin{cases} |z|^2, & w = 0 \\ |zw|^2, & w \neq 0 \end{cases} .$$

We obviously have that $z \mapsto \phi(z, w)$ is strictly subharmonic on $\mathbb{C}$ for every fixed entry w in $\mathbb{C}$, and so it is 1-plurisubharmonic on the whole space $\mathbb{C}^2$ due to the first above example. Nevertheless, we assert that ϕ cannot be strictly 1-plurisubharmonic near the origin. Indeed, for any fixed constant $\varepsilon > 0$ consider the following upper semi-continuous ε-perturbation of ϕ,

$$\phi_\varepsilon(z, w) := \phi(z, w) - \varepsilon|z|^2 - \varepsilon|w|^2.$$

We have that $\phi_\varepsilon(z, w)$ is equal to $|zw|^2 - \varepsilon|z|^2 - \varepsilon|w|^2$ if $w \neq 0$, and so its Levi matrix has the eigenvalues $-\varepsilon$ and $|z|^2 + |w|^2 - \varepsilon$ at the point (z, w). Hence, if the sum $|z|^2 + |w|^2$ is small enough, then the Levi matrix of ϕ_ε has two negative eigenvalues. Thus, the function ϕ_ε fails to be 1-plurisubharmonic near the origin. This means that ϕ itself is not strictly 1-plurisubharmonic near the origin.

Returning to regular foliations, Theorem 3.9.6 and Remark 3.9.3 yield the following observation.

Corollary 3.9.8 *Let ψ be an upper semi-continuous function defined on an open set Ω in $\mathbb{C}^n$. Assume that Ω admits a regular foliation $\mathcal{F}$ of codimension $q \in \{0, \ldots, n-1\}$. If ψ is r-plurisubharmonic on each leaf of $\mathcal{F}$, then ψ is $(q + r)$-plurisubharmonic on Ω.*

We give an example of a classical foliation of $\mathbb{C}^n \setminus \{0\}$ given by complex lines passing through the origin. It has an important application to complex norms on $\mathbb{C}^n$.

Example 3.9.9 (1) The complex lines in $\Omega := \mathbb{C}^n \setminus \{0\}$ which pass through the origin are the leaves of a complex foliation on Ω of codimension $n - 1$. The local charts $(\varphi_k, U_k)_{k=1,\ldots,n}$ for Ω are defined in the classical way. Fixing $k = 1, \ldots, n$ and $U_k = \{z \in \mathbb{C}^n : z_k \neq 0\}$, the holomorphic mappings φ_k from U_k into $\mathbb{C}^n$ are given by

$$\varphi_k(z) := \Big(\frac{z_1}{z_k}, \ldots, \frac{z_{k-1}}{z_k}, \frac{z_{k+1}}{z_k}, \ldots, \frac{z_n}{z_k}, z_k\Big) \in \mathbb{C}^{n-1} \times \mathbb{C}^*,$$

where $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$. The conditions (1) and (2) in Definition 3.9.1 easily hold. To prove condition (3) we proceed as follows: Assume that $j < k$ and (x, y) lies in $\mathbb{C}^{n-1} \times \mathbb{C}^*$. Then the composition $\varphi_j \circ \varphi_k^{-1}$ has the following form,

$$\varphi_j \circ \varphi_k^{-1}(x, y) = \Big(\frac{x_1}{x_j}, \ldots, \frac{x_{j-1}}{x_j}, \frac{x_{j+1}}{x_j}, \ldots, \frac{x_{k-1}}{x_j}, \frac{1}{x_j}, \frac{x_k}{x_j}, \ldots, \frac{x_{n-1}}{x_j}, yx_j\Big).$$

Finally, it is easy to verify that the leaves are the punctured complex lines that pass through the origin. Indeed, fix an integer $k \in \{1, \ldots, n\}$ and a point $z = (z_1, \ldots, z_n)$ with $z_k \neq 0$. We set $Z_k := (z_1, \ldots, z_{k-1}, 1, z_{k+1}, \ldots, z_n)$ and $Z'_k := (z_1, \ldots, z_{k-1}, z_{k+1}, \ldots, z_n)$. Then for every λ in $\mathbb{C}^*$ we have that $\varphi_k(\lambda \cdot Z_k) = (Z'_k, \lambda)$. This means that the inverse image of $\varphi_k^{-1}(\{Z'_k\} \times \mathbb{C}^*)$ is the punctured complex line in $\mathbb{C}^n$ that passes through the origin and the point Z_k.

(2) Let $\psi : \mathbb{C}^n \setminus \{0\} \to [-\infty, +\infty)$ be an upper semi-continuous function such that the restriction $s \mapsto \psi(sv)$ is subharmonic with respect to the variable s in $\mathbb{C}^*$ and for every fixed vector $v \neq 0$ in $\mathbb{C}^n$. In particular, the function ψ is subharmonic on every complex line of $\mathbb{C}^n$ minus the origin which passes trough the origin. According to Corollary 3.9.8 and the results presented in the previous point (1) above, ψ is subpluriharmonic on $\mathbb{C}^n \setminus \{0\}$.

(3) Let $\beta \in \mathbb{R}$ be fixed and $\psi : \mathbb{C}^n \setminus \{0\} \to [0, +\infty)$ be a continuous function such that

$$\psi(sv) = |s|^\beta \psi(v) \text{ for every } s \in \mathbb{C}^* \text{ and } v \in \mathbb{C}^n \setminus \{0\}.$$

Notice that $s \mapsto \beta \log |s|$ is harmonic on $\mathbb{C}^*$ for every real number β in $\mathbb{R}$. Moreover, Theorem 3.5.6 implies that $s \mapsto |s|^\beta = \exp(\beta \log |s|)$ is subharmonic on $\mathbb{C}^*$. Thus, from the results presented in the point (2) we can deduce that the functions ψ and $\log \psi$ are both subpluriharmonic on $\mathbb{C}^n \setminus \{0\}$ and ψ^{-1} and $-\log \psi$ are subpluriharmonic on $\mathbb{C}^n \setminus \{\psi = 0\}$.

As a consequence of the previous example, we obtain that every complex norm function on $\mathbb{C}^n$ is subpluriharmonic.

Theorem 3.9.10 *Let $\|\cdot\| : \mathbb{C}^n \to [0, +\infty)$ be a complex norm function on $\mathbb{C}^n$. Then the functions $\|\cdot\|$ and $\log \|\cdot\|$ are both plurisubharmonic on $\mathbb{C}^n$. Moreover, $-\log \|\cdot\|$ and $\|\cdot\|^{-1}$ are subpluriharmonic on $\mathbb{C}^n \setminus \{0\}$.*

Proof. Since all complex norms are equivalent on the finite dimensional space $\mathbb{C}^n$, there is a real number $c > 0$ such that $\|z\| \leq c\|z\|_2$ for every vector $z \in \mathbb{C}^n$. This means that $\|\cdot\|$ is continuous on $\mathbb{C}^n$ with respect the topology induced by the Euclidean norm $\|\cdot\|_2$.

We assert that $\|\cdot\|$ and $\log \|\cdot\|$ are both plurisubharmonic on $\mathbb{C}^n$. Denote by $\mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^n, \mathbb{C})$ the family of all $\mathbb{C}$-linear functionals $\Lambda : \mathbb{C}^n \to \mathbb{C}$. By Theorem 4.3 (b) in [Rud91], the following identity holds for each z in $\mathbb{C}^n$,

$$\|z\| = \sup\{|\Lambda(z)| : \Lambda \in \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^n, \mathbb{C}) \text{ with } \|\Lambda\|_* = 1\}, \tag{3.5}$$

where $\|\Lambda\|_* := \sup\{|\Lambda(z)| : z \in \mathbb{C}^n, \ \|z\| = 1\}$ is the dual norm on the space $\mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^n, \mathbb{C})$ induced by $\|\cdot\|$. Observe that each functional Λ in $\mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^n, \mathbb{C})$ is holomorphic on $\mathbb{C}^n$, so that $|\Lambda|$ and $\log|\Lambda|$ are plurisubharmonic on $\mathbb{C}^n$. Therefore, the complex norm $\|\cdot\|$ and its logarithm $\log\|\cdot\|$ each are the supremum of a family of plurisubharmonic functions. Since they both are upper semi-continuous, they are plurisubharmonic on $\mathbb{C}^n$ as well.

Finally, since $\|\lambda z\| = |\lambda| \cdot \|z\|$ for every λ in $\mathbb{C}$ and $z \in \mathbb{C}^n$, the results presented in the Example 3.9.9 imply that the functions $-\log\|\cdot\|$ and $\|\cdot\|^{-1}$ are both subpluriharmonic on $\mathbb{C}^n \setminus \{0\}$. □

We can approximate plurisubharmonic functions on a singular foliation by smooth ones. Eventually, in contrary to the general case $q \geq 1$, it is possible to approximate q-plurisubharmonic functions, whose q-plurisubharmonicity comes from a singular foliation of codimension q, by smooth ones (see [DF85]).

Theorem 3.9.11 *Fix an integer $q \in \{1, \dots, n-1\}$ and let $h : \Omega \to \mathbb{C}^q$ be a holomorphic mapping defined on a Stein open set Ω in $\mathbb{C}^n$. Then for every compact set K in Ω we have that*

$$\mathcal{PSH}(h, \Omega) \ \subset \ \overline{\mathcal{PSH}(h, \Omega) \cap \mathcal{C}^\infty(\Omega)}^{\downarrow K}.$$

Proof. We divide the proof into several steps, since we will use some arguments later for another similar statement.

Step 1. If h is constant, then there is nothing to prove, since in this case $h^{-1}(h(z)) = \Omega$ for every z in Ω. Thus, we can assume that h is not constant. Pick an arbitrary function ψ from the family $\mathcal{PSH}(h, \Omega)$. Let K be a compact set in Ω and f be a continuous function on K such that $\psi < f$ on K. By Proposition 3.8.8, for every $c \in h(K)$ there exists a $\mathcal{C}^\infty$-smooth plurisubharmonic function Ψ_c on Ω satisfying

$$\psi < \Psi_c < f \quad \text{on } K \cap h^{-1}(c). \tag{3.6}$$

Then there exists an open neighborhood U_c of $K \cap h^{-1}(c)$ in Ω such that the inequalities from (3.6) still hold on U_c. Moreover, the collection $\{U_c\}_{c \in h(K)}$ forms an open covering of the compact set K.

Consider the compact set $R_c := K \setminus U_c$ in K. It does not intersect the fiber $h^{-1}(c)$ and, therefore, $h - c$ attains a positive minimum on R_c. More precisely, there exist positive numbers $\varepsilon(c)$ and $\varepsilon'(c)$ such that

$$\min_{R_c} \|h - c\|_2 > \varepsilon(c) > \varepsilon'(c) > 0.$$

This implies that $h^{-1}(B^q_{\varepsilon'(c)}(c)) \cap K$ and $h^{-1}(B^q_{\varepsilon(c)}(c)) \cap K$ both lie in U_c.

Step 2. Since $h(K)$ is compact and the collection of balls $\{B^q_{\varepsilon'(c)}(c)\}_{c \in h(K)}$ forms an open covering of $h(K)$, we can find finitely many points $c_1, \ldots, c_\ell$ in $h(K)$ such that $\{B^q_{\varepsilon'(c_i)}(c_i)\}_{i=1,\ldots,\ell}$ covers K. For each $i = 1, \ldots, \ell$ we define the sets $W'_i := B^q_{\varepsilon'(c_i)}(c_i)$ and $W_i := B^q_{\varepsilon(c_i)}(c_i)$. Then we can pick a small enough neighborhood Ω' of K such that $V'_i := h^{-1}(W'_i) \cap \Omega'$ and $V_i := h^{-1}(W_i) \cap \Omega'$ lie in U_{c_i}. Now let $\{\chi_i\}_{i=1,\ldots,\ell}$ be a family of $\mathcal{C}^\infty$-smooth functions on $\mathbb{C}^q$ with compact support which satisfy

$$0 \leq \chi_i \leq 1 \text{ on } W_i, \quad 0 < \chi_i \leq 1 \text{ on } W'_i, \quad \chi_i = 0 \text{ on } \mathbb{C}^q \setminus \overline{W_i},$$

$$\text{and} \quad \sum_{i=1}^{\ell} \chi_i = 1 \text{ on } L.$$

We define $\varphi_i := \chi_i \circ h$ for $i = 1, \ldots, \ell$. Then $\sum_{i=1}^{\ell} \varphi_i = 1$ on K.

Step 3. We observe that the following function is $\mathcal{C}^\infty$-smooth on Ω,

$$\Psi := \sum_{i=1}^{\ell} \varphi_i \Psi_{c_i},$$

where the functions $\{\Psi_{c_i}\}_{i=1}^{\ell}$ come from Step 1 above. Let us investigate the behavior of Ψ on K.

If z belongs to $K \cap U_{c_i}$ for some $i = 1, \ldots, \ell$, we have that

$$(\varphi_i \psi)(z) \leq (\varphi_i \Psi_{c_i})(z) \leq (\varphi_i f)(z),$$

since $0 \leq \varphi_i(z) \leq 1$ and by the inequalities (3.6). We even have strict inequalities if z is in $K \cap V'_i$. Notice that there is always an index i which guarantees strictness, since $\{V'_i\}_{i=1,\ldots,\ell}$ covers K.

If $z \in K$ does not lie in U_{c_i}, then $h(z)$ is not in W_i. Otherwise, if $h(z) \in W_i$, then we immediately obtain the contradiction

$$z \in h^{-1}(h(z)) \cap K \ \subset \ h^{-1}(W_i) \cap K \ \subset \ U_{c_i}.$$

Hence, $\varphi_i(z) = 0$ in this case. Altogether, we have the following estimates at a point z in K,

$$\psi(z) = \sum_{i=1}^{\ell} (\varphi_i \psi)(z) \ < \ \sum_{i=1}^{\ell} (\varphi_i \Psi_{c_i})(z) = \Psi(z) < \sum_{i=1}^{\ell} (\varphi_i f)(z) = f(z).$$

Step 4. For every c in $h(\Omega)$, the function φ_i is constant on $h^{-1}(c)$, and so the function $\sum_{i=1}^{\ell} \chi_i(c)\Psi_{c_i}$ is a plurisubharmonic extension of $\Psi|h^{-1}(c)$ to the whole of Ω. Hence, we derive that Ψ belongs to the family $\mathcal{PSH}(h,\Omega) \cap \mathcal{C}^\infty(\Omega)$. Moreover, by Step 3 it satisfies $\psi < \Psi < f$ on K. Since f is an arbitrary continuous function with $\psi < f$ on K, we conclude that

$$\mathcal{PSH}(h,\Omega) \subset \overline{\mathcal{PSH}(h,\Omega) \cap \mathcal{C}^\infty(\Omega)}^{\downarrow K}.$$

□

3.10 q-Holomorphic functions

It is natural to ask whether there is a generalization of holomorphic functions which are related to q-plurisubharmonic ones as plurisubharmonic functions are linked to holomorphic ones. An answer was given by L. R. Hunt and J. J. Murray in [HM78] who gave a relation of q-plurisubharmonic and so called *q-holomorphic* functions. Earlier, it was R. Basener who examined these functions in [Bas76] and [Bas78].

This section is part of the joint article [PZ15].

Definition 3.10.1 Let Ω be an open set in $\mathbb{C}^n$, A be an analytic subset of Ω and let $q \in \mathbb{N}_0$.

(1) We say that a complex-valued function f defined is *q-holomorphic on A* if for every point p in A there are a neighborhood U of p in Ω and a $\mathcal{C}^2$-smooth function F on U such that $F = f$ on $A \cap U$ and

$$\overline{\partial} F \wedge (\partial\overline{\partial} F)^q = 0.$$

(2) We denote the family of all q-holomorphic functions on A by $\mathcal{O}_q(A)$.

(3) In the case $q = 0$, we simply write $\mathcal{O}(A)$ instead of $\mathcal{O}_0(A)$.

We give a collection of properties of q-holomorphic functions.

Proposition 3.10.2 *Let Ω be an open set in $\mathbb{C}^n$, A be an analytic subset of Ω and let $q \in \mathbb{N}_0$.*

(1) The family $\mathcal{O}_0(A)$ *is the collection of holomorphic functions on* A*, whereas* $\mathcal{O}_N(A)$ *is simply the family* $\mathcal{C}^2(A)$ *whenever* $N \geq n$*. Here, by* $\mathcal{C}^2(A)$ *we mean the family of all complex valued functions on* A *which locally have a* $\mathcal{C}^2$*-smooth extension into the ambient space.*

(2) We have that $\mathcal{O}_q(A) \subset \mathcal{O}_{q+1}(A)$*.*

(3) Given a constant $\lambda \in \mathbb{C}$ *and a* q*-holomorphic function* f *on* A*, both* λf *and* f^2 *lie in* $\mathcal{O}_q(A)$*.*

(4) Let $r \in \mathbb{N}_0$*. If* f *is* q*-holomorphic on* A *and* g *is* r*-holomorphic on* A*, then the sum* $f+g$ *and the product* fg *are* $(q+r)$*-holomorphic on* A*.*

(5) Let f *be a* q*-holomorphic function on* Ω*. If* $g : D \to \Omega$ *is a holomorphic mapping defined on another open set* D *in* $\mathbb{C}^k$*, then the composition* $f \circ g$ *is* q*-holomorphic on* D*.*

(6) Given a q*-holomorphic function* f *on* A *and a holomorphic function* h *defined on a neighborhood of the image of* f *in* $\mathbb{C}$*, then the composition* $h \circ f$ *is* q*-holomorphic on* A*. In particular, the power* f^m *is again* q*-holomorphic on* A *for every* $m \in \mathbb{N}_0$*.*

(7) The function f *is* q*-holomorphic on* Ω *if and only if the rank of the* extended Levi matrix *of* f *given by*

$$\begin{pmatrix} f_{\bar{z}_1} & \cdots & f_{\bar{z}_n} \\ f_{z_1\bar{z}_1} & \cdots & f_{z_1\bar{z}_n} \\ \vdots & \ddots & \vdots \\ f_{z_n\bar{z}_1} & \cdots & f_{z_n\bar{z}_n} \end{pmatrix}$$

is less or equal to q *at each point in* Ω*.*

(8) If f *is* q*-holomorphic on* A*, then the real part* $\mathrm{Re}(f)$*, imaginary part* $\mathrm{Im}(f)$ *and* $\log|f|$ *are* q*-plurisubharmonic on* A*.*

Proof. The statements (1), (2) and (3) follow directly from the definition. The proofs of (4), (5), (6) and (7) can be found in [Bas76]. The property (8) has been proved in [HM78] (see Theorem 5.3 and Corollary 5.4). We have to point out that the points (4) to (8) have been proved only in the case of $A = \Omega$. But it is easy to verify that these properties carry over directly to q-holomorphic functions on analytic sets using our definition. □

The following examples can be found in [Bas76].

Example 3.10.3 (1) If f is holomorphic, then the complex conjugate $\overline{f}$ and the absolute value $|f|^2$ are both 1-holomorphic.

(2) Let $h : \Omega \to \mathbb{C}^q$ be a holomorphic mapping on an open set Ω in $\mathbb{C}^n$. Let ψ be a complex valued $\mathcal{C}^2$-smooth function defined on a neighborhood of the image of h in $\mathbb{C}^q$. Since ψ is q-holomorphic by the definition an the dimension of $\mathbb{C}^q$, Proposition 3.10.2 (5) implies that the composition $\psi \circ h$ is q-holomorphic on Ω. For instance, the norm $\|h\|_2^2 = \sum_{j=1}^q |h_j|^2$ of h is q-holomorphic on Ω.

(3) If A is an analytic subset of an open set Ω in $\mathbb{C}^n$, then the inclusion mapping $\iota : A \hookrightarrow \Omega$ is holomorphic on Ω. Therefore, every q-holomorphic function f on Ω can be regarded as a q-holomorphic function on A, since $f \circ \iota$ is q-holomorphic on A due to Proposition 3.10.2 (5).

(4) If a $\mathcal{C}^2$-smooth function is locally holomorphic with respect to the first $n-q$ variables $z_1, \dots, z_{n-q}$ of a local chart $z_1, \dots, z_n$, then it is q-holomorphic in view of Proposition 3.10.2 (7).

(5) Fix a positive constant $c > 0$ and let Ω be an open set in $\mathbb{C}^n$. According to the previous examples (2) and (4), for a given holomorphic mapping $h : \Omega \to \mathbb{C}^q$, the function

$$z \mapsto \varphi_c(z) = 1/(1 + c\|h(z)\|_2^2)$$

is q-holomorphic on Ω. Now if c tends to $+\infty$, then $\{\varphi_c\}_{c>0}$ decreases on Ω to the characteristic function χ_A of $A := \{h = 0\}$ given by

$$\chi_A(z) = \begin{cases} 1, & z \in A \\ 0, & z \in \Omega \setminus A \end{cases} . \tag{3.7}$$

The q-holomorphic functions admit the local maximum modulus principle.

Theorem 3.10.4 (Local maximum modulus principle) *Let q be an integer in $\{0, \dots, n-1\}$ and A an analytic subset of an open set Ω in $\mathbb{C}^n$. If f is a q-holomorphic function on A, then for every compact set K in A we have that*

$$\|f\|_K = \|f\|_{b_A K},$$

where $\|f\|_K = \max\{f(z) : z \in K\}$ and $b_A K$ is the boundary of K in A.

Proof. If $A = \Omega$, then this result is due to Theorem 2 in [Bas76]. In the case of an arbitrary analytic set A in Ω, the statement follows from Proposition 3.10.2 (8) and from the local maximum principle for q-plurisubharmonic functions on analytic sets (see Theorem 3.8.5). □

3.11 Holomorphic functions on foliations

R. Basener showed in Theorem 1 in [Bas76] that a 1-holomorphic function defined on some open set in $\mathbb{C}^2$, whose derivative $\overline{\partial} f$ never vanishes, is holomorphic on leaves of a local foliation by holomorphic curves. Later, E. Bedford and M. Kalka generalized this result to q-holomorphic functions and other functions satisfying certain non-linear Cauchy-Riemann equations. We only repeat their result involving q-holomorphic functions (see Theorem 5.3 in [BK77]).

Theorem 3.11.1 *Let $q \in \{1, \dots, n-1\}$. Let f be a $\mathcal{C}^3$-smooth q-holomorphic function on a domain Ω in $\mathbb{C}^n$ which fulfills*

$$\partial \overline{f} \wedge \overline{\partial} f \wedge (\partial\overline{\partial} f)^{q-1} \neq 0 \quad \text{and} \quad (\partial\overline{\partial} f)^q \neq 0.$$

Suppose that $\mathrm{Re}(f)$ *is plurisubharmonic on Ω. Then f is holomorphic on a foliation by complex submanifolds of Ω having codimension q.*

We proceed by explaining what we mean by a *holomorphic function on singular foliations* in order to develop a converse result to that of Kalka and Bedford. Notice that we accept by convention that $\mathbb{C}^0$ is equal to $\{0\}$ and that every continuous function is indeed holomorphic on a discrete set.

Definition 3.11.2 Let r be a non-negative integer and $q \in \{0, \dots, n-1\}$. Let Ω be an open set in $\mathbb{C}^n$.

(1) Let $H = \{h_j : U_j \to \mathbb{C}^q\}_{j \in J}$ be a web of singular foliations on Ω. A continuous function f on Ω is *holomorphic on the (singular) foliations of H* if for every point p in Ω there is an index $j \in J$ such that U_j contains p and for every $c \in h_j(U_j)$, the function f is holomorphic on the fiber $h_j^{-1}(c)$ in the sense of Definition 3.10.1.

(2) The set $\mathcal{O}(H, \Omega)$ stands for the family of all holomorphic functions on the foliations of H.

(3) If $h : \Omega \to \mathbb{C}^q$ is a single holomorphic mapping, then we simply write $\mathcal{O}(h, \Omega)$ rather than $\mathcal{O}(\{h\}, \Omega)$ and say that functions from $\mathcal{O}(h, \Omega)$ are *holomorphic on a (singular) foliation (induced by h)*.

Our next aim is to show that smooth functions which are holomorphic on a singular foliation of codimension q are in fact q-holomorphic.

Theorem 3.11.3 *Fix an integer $q \in \{1, \dots, n-1\}$. Let Ω be an open set in $\mathbb{C}^n$ and let $h : \Omega \to \mathbb{C}^q$ be a holomorphic mapping. Then each $\mathcal{C}^2$-smooth function f in $\mathcal{O}(h, \Omega)$ is q-holomorphic on Ω.*

Proof. The *(complex) rank of h* at a given point z in Ω is the rank of the complex Jacobian $\Big(\partial h_k / \partial z_j\Big)_{\substack{k=1,\dots,q\\ j=1,\dots,n}}$ of h at z. Let $k \leq q$ be the maximal rank of h on Ω and denote by Ω' the set of all points z in Ω such that the rank of h at z equals the maximal rank k. Then $S := \Omega \setminus \Omega'$ is an analytic subset of Ω of dimension strictly less than n. Hence, Ω' is open and relatively dense in Ω. Now it suffices to show that f is q-holomorphic on Ω', since by the $\mathcal{C}^2$-smoothness of f the equation $\overline{\partial} f \wedge (\partial\overline{\partial} f)^q = 0$ on Ω' extends trivially to the whole of Ω.

Let p be a point in Ω'. Without loss of generality we can assume that $p = 0$ and $h(p) = 0$. Then it follows from the complex rank theorem (see Theorem 2 in Appendix A2.2 of [Chi89]) that after a holomorphic change of coordinates near p and $h(p)$, we have that the mapping h is of the form $h(z) = (z_1, \dots, z_k, 0, \dots, 0) \in \mathbb{C}^k \times \mathbb{C}^{q-k}$ in some neighborhood U of the origin. By assumption, for each $c \in h(U)$ the function f is holomorphic on the fiber $h^{-1}(c) = (\{(c_1, \dots, c_k)\} \times \mathbb{C}^{n-k}) \cap U$. According to Example 3.10.3 (4), the function f is k-holomorphic on U. By properties (5) and (6) of Proposition 3.10.2, the q-holomorphicity is invariant under holomorphic changes of coordinates, and of course, it is a local property by definition. Since $k \leq q$, we conclude that f is q-holomorphic on Ω'. Then the arguments in the first part of this proof imply that f is q-holomorphic on the entire set Ω. □

We will need the following extension theorem for holomorphic functions defined on analytic subsets (see Theorem 4 in paragraph 4.2 of chapter V in [GR04]). Recall that, by definition, a Stein open set Ω in $\mathbb{C}^n$ admits a continuous plurisubharmonic function φ on Ω such that $\{\varphi < c\} \Subset \Omega$ for every $c \in \mathbb{R}$.

Theorem 3.11.4 *Let Ω be a Stein open set in $\mathbb{C}^n$ and let A be an analytic subset of Ω. Then any holomorphic function on A extends holomorphically to the whole of Ω.*

Recall that holomorphic functions on foliations are continuous in the first place, whereas q-holomorphic functions are by definition $\mathcal{C}^2$-smooth. It is then interesting to decide whether a continuous function which is holomorphic on the fibers of a singular foliation of codimension q can be approximated by q-holomorphic functions. A positive answer gives the following result (together with the previous Theorem 3.11.3).

Theorem 3.11.5 *Fix an integer $q \in \{1, \ldots, n-1\}$. Let Ω be a Stein open set in $\mathbb{C}^n$ and let $h : \Omega \to \mathbb{C}^q$ be a holomorphic mapping. Then for every compact set K in Ω we have that*

$$\mathcal{O}(h, \Omega) \ \subset \ \overline{\mathcal{O}(h, \Omega) \cap \mathcal{C}^\infty(\Omega)}^K.$$

Proof. The proof is similar to that of Theorem 3.9.11 with some adaption to holomorphic functions.

Step 1. As in Step 1 of the proof of Theorem 3.9.11, we can assume that h is not constant.

Let K be a compact set in Ω, f a function from $\mathcal{O}(h, \Omega)$ and $\varepsilon > 0$ a positive number. Since Ω is a Stein open set, it follows from Theorem 3.11.4 that for every $c \in h(\Omega)$ there exists a holomorphic extension F_c of $f_c := f|h^{-1}(c)$ into the whole of Ω. Then there exists an open neighborhood U_c of $H \cap h^{-1}(c)$ in Ω such that $\|F_c - f\|_{U_c} < \varepsilon$. Obviously, the collection $\{U_c\}_{c \in h(\Omega)}$ forms an open covering of K.

Step 2. For $i \in \{1, \ldots, \ell\}$ let $W_i' \Subset W_i'$, $V_i' \subset V_i \subset U_{c_i}$, χ_i and $\varphi_i = \chi_i \circ h$ be the open sets and, respectively, the cutoff functions from Step 2 of the proof of Theorem 3.9.11. We will use these objects in the following steps. Recall that $\{W_i\}_{i=1,\ldots,\ell}$ and $\{W_i'\}_{i=1,\ldots,\ell}$ cover $h(K)$ and $\{V_i\}_{i=1,\ldots,\ell}$, $\{V_i'\}_{i=1,\ldots,\ell}$ and $\{U_{c_i}\}_{i=1,\ldots,\ell}$ cover K. Also remind that $\sum_{i=1}^{\ell} \varphi_i = 1$ on K, $0 \leq \varphi_i \leq 1$ on Ω and $\varphi_i \equiv 0$ on $h^{-1}(\mathbb{C}^q \setminus \overline{W_i})$ for each $i = 1, \ldots, \ell$.

Step 3. The following function is $\mathcal{C}^\infty$-smooth on Ω,

$$F := \sum_{i=1}^{\ell} F_{c_i} \varphi_i.$$

Then by the same arguments as in Step 3 of the proof of Theorem 3.9.11 we can show the following properties: If $z \in K \cap U_i$ for some $i = 1, \dots, \ell$, then $\varphi_i(z) \cdot |F_c - f|(z) \leq \varphi_i(z) \cdot \varepsilon$. We even have a strict inequality if $z \in K \cap V_i'$. Notice that there exists at least one index i_0 such that $\varphi_{i_0}(z) > 0$, since $\{V_i'\}_{i=1,\dots,\ell}$ covers K. Now if $z \in K \setminus U_i$, then $\varphi_i(z) = 0$. Therefore, since $\varphi_i(z)$ is non-negative and $\sum_{i=1}^{\ell} \varphi_i(z) = 1$ for a point $z \in K$, it holds that

$$|F - f|(z) \ = \ \sum_{i=1}^{\ell} \varphi_i(z) \cdot |F_{c_i} - f|(z) \ < \ \sum_{i=1}^{\ell} \varepsilon \varphi_i(z) = \varepsilon.$$

Step 4. Given $c \in h(\Omega)$, the function φ_i is constant on the fiber $h^{-1}(c)$, and therefore the function $\sum_{i=1}^{\ell} \chi_i(c) F_{c_i}$ is a holomorphic extension of $F|h^{-1}(c)$ to the whole of Ω. By the definition, it means that F belongs to the family $\mathcal{O}(h,\Omega) \cap \mathcal{C}^\infty(\Omega)$. The previous Step 4 yields $\|f - F\|_K < \varepsilon$. In view of Theorem 3.11.3 and since $\varepsilon > 0$ was arbitrarily chosen, we obtain that

$$\mathcal{O}(h,\Omega) \ \subset \ \overline{\mathcal{O}(h,\Omega) \cap \mathcal{C}^\infty(\Omega)}^K .$$

□

The precedent result applies directly to regular foliations. In fact, this observation can be shown in a much easier way using the definition of a complex foliation, the complex version of implicit function theorem and Example 3.10.3 (4).

Corollary 3.11.6 *Let f be a $\mathcal{C}^2$-smooth function defined on an open set Ω in $\mathbb{C}^n$ and let $\mathcal{F}$ be a regular foliation on Ω of codimension q. If f is holomorphic on each leaf of $\mathcal{F}$, then f is q-holomorphic on Ω.*

Another consequence of Theorem 3.11.5 combined with Proposition 3.10.2 (8) is the relation of holomorphic functions on foliations to q-plurisubharmonic functions.

Corollary 3.11.7 *Let $q \in \{1, \dots, n-1\}$, Ω an open set in $\mathbb{C}^n$ and f a holomorphic function on singular foliations of a web $H = \{h_j : \Omega \to \mathbb{C}^q\}_{j \in J}$. Then the real part, imaginary part and the logarithm of the absolute value of f is q-plurisubharmonic on Ω.*

The next statement is a Bremermann type approximation (compare Theorem 3.1.10).

Theorem 3.11.8 *Let $h : \Omega \to \mathbb{C}^q$ be a holomorphic mapping defined on a Stein open set Ω and let ψ be a continuous function in $\mathcal{PSH}(h,\Omega)$. Then for every $\varepsilon > 0$ and every compact set K in Ω there exist positive integers $\alpha_1, \ldots, \alpha_k \in \mathbb{N}$ and functions $f_1, \ldots, f_k$ from $\mathcal{O}(h,\Omega)$ such that*

$$\psi \ < \ \max\{\alpha_j^{-1} \log |f_j| : j = 1, \ldots, k\} \ < \ \psi + \varepsilon \text{ on } K.$$

Proof. Let $c \in h(\Omega)$ and define $\psi_c := \psi|h^{-1}(c)$. By Proposition 3.8.8 there exists a continuous plurisubharmonic function $\widehat{\psi_c}$ on Ω which also satisfies $\psi < \widehat{\psi_c} < \psi + \varepsilon/2$ on $K \cap h^{-1}(c)$. Since every Stein open set is a domain of holomorphy (see later Remark 5.1.4), Bremermann's theorem 3.1.10 implies that there exist positive integers $N_{c,1}, \ldots, N_{c,\mu(c)}$ and holomorphic functions $g_{c,1}, \ldots, g_{c,\mu(c)}$ on Ω which fulfill the subsequent inequalities on K,

$$\widehat{\psi_c} < \Psi_c := \max\{N_{c,\nu}^{-1} \log |g_{c,\nu}| : \nu = 1, \ldots, \mu(c)\} < \widehat{\psi_c} + \varepsilon/2.$$

Therefore, we obtain that $\psi < \Psi_c < \psi + \varepsilon$ on $K \cap h^{-1}(c)$. For an integer $m \in \mathbb{N}$ we set $\chi_{c,m} := 1/(1 + m\|h - c\|_2^2)$. Observe that $\chi_{c,m}$ lies in $\mathcal{O}(h,\Omega)$, since for every $d \in h(\Omega)$ it is constant on the fiber $h^{-1}(d)$. In view of Proposition 3.10.2 (4) and (6), the function $f_{c,\nu,m} := g_{c,\nu}\chi_{c,m}^{N_{c,\nu}}$ belongs to $\mathcal{O}(h,\Omega)$ for every $c \in h(\Omega)$, $\nu \in \{1, \ldots, \mu(c)\}$ and $m \in \mathbb{N}$. Now consider the function $\varrho_{c,m}$ given by

$$\varrho_{c,m} := \Psi_c + \log |\chi_{c,m}| = \max\{N_{c,\nu}^{-1} \log |f_{c,\nu,m}| : \nu = 1, \ldots, \mu(c)\}.$$

Since $(\chi_{c,m})_{m\in\mathbb{N}}$ decreases to the characteristic function of $\{h = c\}$ as m tends to $+\infty$ and since $\psi < \Psi_c < \psi + \varepsilon$ on $K \cap h^{-1}(c)$, we can choose a large enough integer $m(c) \in \mathbb{N}$ such that $\varrho_{c,m(c)} < \psi + \varepsilon$ on K. Since $\varrho_{c,m(c)} = \Psi_c$ on $h^{-1}(c)$, there is an open neighborhood V_c of $K \cap h^{-1}(c)$ with $\psi < \varrho_{c,m(c)}$ on V_c. By the compactness of K, we can pick finitely many sets $V_{c_1}, \ldots, V_{c_\ell}$ which cover K. Finally, we obtain the desired functions and inequalities on K, namely,

$$\begin{aligned} \psi &< \max\{\varrho_{c_j,m(c_j)} : j = 1 \ldots, \ell\} \\ &= \max\{N_{c_j,\nu_j}^{-1} \log |f_{c_j,\nu_j,m(c_j)}| : j = 1, \ldots, \ell, \ \nu_j = 1, \ldots, \mu(c_j)\} \\ &< \psi + \varepsilon. \end{aligned}$$

□

Part III

q-Pseudoconvexity and q-Shilov boundaries

Chapter 4

q-Pseudoconvexity

In the literature, one encounters many different ways to define q-pseudoconvex sets. We collect a large list of equivalent characterizations of this type of sets using q-plurisubharmonic exhaustion functions [Gra59], generalized Hartogs figures ([Rot55] and [Fuj90]), boundary distance functions ([Sło86] and [Fuj90]), a generalized continuity principle (*Kontinuitätssatz*) and generalized convex hulls for q-plurisubharmonic functions (compare [Bas76]). It is needless to say that for $q = 0$ (the pseudoconvex case) these characterizations are classical complex analysis in several variables and can be found in most of the books about this topic (see e.g. [Hör90], [Kra99] or [Sha92]). Interesting examples of q-pseudoconvex sets are given by *real q-convex sets* (i.e., sets admitting a real q-convex exhaustion function) and sublevel sets of q-plurisubharmonic functions. We establish a generalized version of Bochner's tube theorem in the q-pseudoconvex case and study smoothly bounded sets, which leads to the notion of *Levi q-pseudoconvex sets*. They are used to describe a duality principle which, roughly speaking, states that a set is not q-pseudoconvex near a boundary point if and only if some part of its complement near this boundary point touches a strictly Levi $(n - q - 2)$-pseudoconvex set from inside (see also [Bas76]). In the case of $q = 0$, the 0-pseudoconvex sets are exactly the classical pseudoconvex sets. Those are the most important sets for complex analysis of several variables due to the solution of the so-called *Levi problem* (see [Hör90]). More precisely, pseudoconvex sets are domains of holomorphy and vice versa.

Most of the contents of this chapter can be found in the joint articles [PZ13] and [PZ15].

4.1 q-Pseudoconvex sets

The q-pseudoconvex sets are defined similarly as Stein open sets in terms of exhaustion functions (recall Definition 3.8.6 (2)). According to the definition below, Stein open sets are exactly the 0-pseudoconvex sets and are usually known as *pseudoconvex* sets.

Definition 4.1.1 We say that an open set Ω in $\mathbb{C}^n$ is *q-pseudoconvex (in $\mathbb{C}^n$)* if there exists a continuous q-plurisubharmonic *exhaustion function* Φ *for* Ω, i.e., for every $c \in \mathbb{R}$ the set $\{z \in \Omega : \Phi(z) < c\}$ is relatively compact in Ω.

We immediately have the following properties of q-pseudoconvex sets.

Proposition 4.1.2 *(1) Let Ω_1 and Ω_2 be two q-pseudoconvex sets in $\mathbb{C}^n$. Then the intersection $\Omega_1 \cap \Omega_2$ is also q-pseudoconvex.*

(2) If Ω_1 is q-pseudoconvex in $\mathbb{C}^n$ and Ω_2 is q-pseudoconvex in $\mathbb{C}^m$, then $\Omega_1 \times \Omega_2$ is q-pseudoconvex in $\mathbb{C}^{n+m}$.

Proof. For $j = 1, 2$ let Φ_j be a q-plurisubharmonic exhaustion function for Ω_j.
(1) In this case, $\Phi := \max\{\Phi_1, \Phi_2\}$ is a q-plurisubharmonic exhaustion function for $\Omega_1 \cap \Omega_2$.

(2) Since Φ_1 and Φ_2 can be considered as q-plurisubharmonic functions on $\Omega_1 \times \Omega_2$, it is clear that $\Phi(z, w) := \max\{\Phi_1(z), \Phi_2(w)\}$ is a q-plurisubharmonic exhaustion function for the product $\Omega_1 \times \Omega_2$. □

4.2 Boundary distance functions

We recall the definition of boundary distance functions and their relation to each other.

Definition 4.2.1 Let Ω be an open set in $\mathbb{C}^n$ and let $\|\cdot\|$ be some complex norm on $\mathbb{C}^n$.

(1) The *boundary distance (induced by $\|\cdot\|$) between a point z in Ω and its boundary $b\Omega$* is defined by

$$d_{\|\cdot\|}(z, b\Omega) := \inf\big\{\|z - w\| : w \in b\Omega\big\}. \tag{4.1}$$

(2) We obviously define $d_{\|\cdot\|}(z, b\Omega) = +\infty$ whenever $b\Omega$ is the empty set.

(3) As before, we write $d(z, b\Omega)$ instead of $d_{\|\cdot\|}(z, b\Omega)$ whenever $\|\cdot\|$ is the Euclidean norm $\|\cdot\|_2$ and call it the *Euclidean boundary distance*.

(4) Let v be a fixed vector in $\mathbb{C}^n$ with $\|v\|_2 = 1$ and let $z + \mathbb{C}v$ be the complex line in $\mathbb{C}^n$ that passes through z and $z + v$. We define the *(Euclidean) boundary distance in the v-direction from z in Ω to $b\Omega$* by

$$R_{b\Omega,v}(z) := d\big(z, b\Omega \cap (z + \mathbb{C}v)\big). \tag{4.2}$$

The boundary distance functions have the subsequent properties and relations, which are all easy to verify.

Proposition 4.2.2 *Let Ω be an open set in $\mathbb{C}^n$ and z be a point in Ω.*

(1) The boundary distance induced by $\|\cdot\|$ can be described by the boundary distances in all v-directions, namely,

$$d_{\|\cdot\|}(z, b\Omega) = \inf\big\{\|v\| \cdot R_{b\Omega,v}(z) : v \in \mathbb{C}^n,\ \|v\|_2 = 1\big\}. \tag{4.3}$$

(2) Given a vector v in $\mathbb{C}^n$ with $\|v\|_2 = 1$, the boundary distance in v-direction can be rewritten in two ways,

$$\begin{aligned} R_{b\Omega,v}(z) &= \sup\{r > 0 : z + sv \in \Omega \text{ for every } s \in \mathbb{C} \text{ with } |s| < r\} \\ &= d_{\|\cdot\|}(z, b\Omega \cap (z + \mathbb{C}v))/\|v\|. \end{aligned} \tag{4.4}$$

(3) For any complex norm $\|\cdot\|$ on $\mathbb{C}^n$, the boundary distance function $z \mapsto d_{\|\cdot\|}(z, b\Omega)$ is continuous on Ω (with respect to the Euclidean topology), whereas for every vector $v \in \mathbb{C}^n$ with $\|v\|_2 = 1$ the boundary distance function in v-direction $z \mapsto R_{b\Omega,v}(z)$ is only lower semi-continuous on Ω.

We examine the q-plurisubharmonicity of the boundary distance functions.

Proposition 4.2.3 *Let $\|\cdot\|$ be a complex norm on $\mathbb{C}^n$ and Ω be an open set in $\mathbb{C}^n$. Given any vector v in $\mathbb{C}^n$ with $\|v\|_2 = 1$, the following functions both are subpluriharmonic on Ω,*

$$z \mapsto -\log d_{\|\cdot\|}(z, b\Omega) \quad \text{and} \quad z \mapsto -\log R_{b\Omega,v}(z). \tag{4.5}$$

Furthermore, if $z \mapsto -\log R_{b\Omega,v}(z)$ is q-plurisubharmonic on Ω for every vector $v \in \mathbb{C}^n$ with $\|v\|_2 = 1$, then $z \mapsto -\log d_{\|\cdot\|}(z, b\Omega)$ is also q-plurisubharmonic on Ω.

Proof. The results are trivial when $\Omega = \mathbb{C}^n$, because $b\Omega$ is empty and the boundary distance functions are all identically equal to $-\infty$, so we assume in the following that $b\Omega$ is not empty.

In view of Theorem 3.9.10, $-\log\|\cdot\|$ is subpluriharmonic on $\mathbb{C}^n \setminus \{0\}$. Thus, by Proposition 3.3.2 (5), the function

$$z \mapsto -\log d_{\|\cdot\|}(z, b\Omega) = \sup\{-\log\|z-w\| : w \in b\Omega\}$$

is subpluriharmonic on Ω, so this proves the first part of (4.5).

In order to verify the second part of (4.5), suppose that $z \mapsto -\log R_{b\Omega,v}(z)$ is not subpluriharmonic on Ω, i.e., there is a vector v in $\mathbb{C}^n$ with $\|v\|_2 = 1$ such that the function $\psi(z) := -\ln R_{b\Omega,v}(z)$ is not subpluriharmonic in a neighborhood of a fixed point $p_0 \in \Omega$. We can assume without loss of generality that p_0 is the origin. Then according to Proposition 3.2.4 (4) there exist a ball $B = B_r(0)$ compactly contained in Ω and a function g holomorphic on a neighborhood U of $\overline{B}$ such that $\psi < \mathrm{Re}(g)$ on the boundary bB, but $\psi(p_1) > \mathrm{Re}(g)(p_1)$ for some point $p_1 \in B$. Thus,

$$e^{-\mathrm{Re}(g)(p_1)} > R_{b\Omega,v}(p_1) \quad \text{and} \quad e^{-\mathrm{Re}(g)(z)} < R_{b\Omega,v}(z) \text{ for every } z \in bB.$$

Notice that the function $\Phi(z) := z + e^{-g(z)}v$ is holomorphic on U. Since, if $z \in \Omega$, the point $z + sv$ lies in Ω for every complex number s with $|s| \le r$ if and only if $0 < r < R_{b\Omega,v}(z)$, the function Φ satisfies

$$\Phi(p_1) \notin \Omega \quad \text{and} \quad \Phi(bB) \subset \Omega.$$

Now it follows from the compactness of $\Phi(bB)$ that we can find a real number $C > 0$ with $d(\Phi(z), b\Omega) > C$ for every $z \in bB$. Since $\Phi(p_1)$ is not in Ω, there is a point $p_2 \in B \cap \Phi^{-1}(\Omega)$ fulfilling $0 < d(\Phi(p_2), b\Omega) < C$. This gives the inequalities

$$-\log d(\Phi, b\Omega) < -\log C \text{ on } bB \quad \text{and} \quad -\log d(\Phi(p_2), b\Omega) > -\log C. \tag{4.6}$$

Since Φ is holomorphic, we derive from the first part of (4.5) and from Theorem 3.7.1 that the function $z \mapsto -\log d(\Phi(z), b\Omega)$ is subpluriharmonic on U. But then the inequalities in (4.6) contradict to the local maximum property in Proposition 3.3.2 (11). Hence, $z \mapsto -\log R_{b\Omega,v}(z)$ is subpluriharmonic on Ω as proclaimed above.

Finally, assume that $z \mapsto -\log R_{b\Omega,v}(z)$ is q-plurisubharmonic on Ω for every $v \in \mathbb{C}^n$ with $\|v\|_2 = 1$. Then, by the identity in (4.3) and Proposition 3.3.2 (5), we immediately obtain that $z \mapsto -\log d_{\|\cdot\|}(z, b\Omega)$ is q-plurisubharmonic on Ω. $\square$

As a direct consequence, we obtain a characterization of $(n-1)$-pseudoconvex sets. This result can also be found in [Sło86], Proposition 4.6.

Corollary 4.2.4 *Any open set Ω in $\mathbb{C}^n$ is $(n-1)$-pseudoconvex.*

Proof. If $\Omega = \mathbb{C}^n$, simply take $\psi(z) := \|z\|_2^2$ as an exhaustion function for $\mathbb{C}^n$. Otherwise, it follows from Proposition 4.2.3 and Proposition 3.3.2 (9) that $z \mapsto -\log d(z, b\Omega) + \|z\|_2^2$ is subpluriharmonic on Ω. It is easy to see that it is an exhaustion function for Ω. □

4.3 Equivalent notions of q-pseudoconvexity

Several characterizations of q-pseudoconvexity in $\mathbb{C}^n$ can be found in the literature. We may refer, for example, to the works of O. Fujita [Fuj64], Z. Słodkowski [Sło86] and K. Matsumoto [Mat96]. In particular, Matsumoto studied q-pseudoconvexity in Kähler manifolds. First, we present a notion of q-pseudoconvexity which is due W. Rothstein [Rot55].

Definition 4.3.1 (1) We write $\Delta_r^n := \Delta_r^n(0)$ for the polydisc with radius $r > 0$ and $A_{r,R}^n := \Delta_R^n \setminus \overline{\Delta_r^n}$ for the open annulus with radii $r > 0$ and $R > 0$ centered at the origin in $\mathbb{C}^n$.

(2) Let $1 \leq k < n$ be fixed integers, and r and R be real numbers in the interval $(0,1)$. An *Euclidian $(n-k,k)$ Hartogs figure* H_e is the set

$$H_e := \left(\Delta_1^{n-k} \times \Delta_r^k\right) \cup \left(A_{R,1}^{n-k} \times \Delta_1^k\right) \ \subset\ \Delta_1^{n-k} \times \Delta_1^k = \Delta_1^n.$$

(3) A pair (H,P) of domains H and P in $\mathbb{C}^n$ with $H \subset P$ is called a *(general) $(n-k,k)$ Hartogs figure* if there is an Euclidian $(n-k,k)$ Hartogs figure H_e and a biholomorphic mapping F defined from Δ_1^n onto P such that $F(H_e) = H$.

(4) An open set Ω in $\mathbb{C}^n$ is called *Hartogs k-pseudoconvex* if it admits the Kontinuitätssatz with respect to the $(n-k)$-dimensional polydiscs, i.e., given any $(n-k,k)$ Hartogs figure (H,P) such that $H \subset \Omega$, we already have that $P \subset D$.

Now we give a list of equivalent characterizations of q-pseudoconvex sets (see also Proposition 3.10 in [PZ13]).

Theorem 4.3.2 *Let $q \in \{0 \dots, n-2\}$ and Ω be an open set in $\mathbb{C}^n$. Then the following statements are all equivalent.*

(1) The set Ω is Hartogs $(n-q-1)$-pseudoconvex.

(2) For every vector v in $\mathbb{C}^n$ with $\|v\|_2 = 1$ the distance function in v-direction $z \mapsto -\log R_{b\Omega,v}(z)$ is q-plurisubharmonic on Ω.

(3) For every complex norm $\|\cdot\|$ the function $z \mapsto -\log d_{\|\cdot\|}(z, b\Omega)$ is q-plurisubharmonic on Ω.

(4) For some complex norm $\|\cdot\|$ the function $z \mapsto -\log d_{\|\cdot\|}(z, b\Omega)$ is q-plurisubharmonic on Ω.

(5) Ω is q-pseudoconvex.

(6) There exists a (not necessarily continuous) q-plurisubharmonic function ψ on Ω such that $\limsup_{z \to b\Omega} \psi(z) = +\infty$.

(7) For every compact set K in Ω, its q-plurisubharmonic hull

$$\widehat{K}^{\Omega}_{\mathcal{PSH}_q(\Omega)} = \big\{z \in \Omega : \psi(z) \leq \max_K \psi \text{ for every } \psi \in \mathcal{PSH}_q(\Omega)\big\}$$

is compact in Ω, as well.

(8) Let $\{A_t\}_{t\in[0,1]}$ be a family of $(q+1)$-dimensional analytic subsets in some open set U in $\mathbb{C}^n$ that continuously depend on t in the Hausdorff topology. Assume that the closure of $\bigcup_{t\in[0,1]} A_t$ is compact. If Ω contains the boundary bA_1 and the closure $\overline{A_t}$ for each $t \in [0, 1)$, then the closure $\overline{A_1}$ also lies in Ω.

(9) For every point p in $b\Omega$ there is a ball $B = B_r(p)$ centered at p such that $\Omega \cap B$ is q-pseudoconvex.

(10) There exist a neighborhood W of $b\Omega$ and a q-plurisubharmonic function ψ on $W \cap \Omega$ with $\limsup_{z \to b\Omega} \psi(z) = +\infty$.

(11) There is a collection $\{\Omega_j\}_{j\in\mathbb{N}}$ of bounded q-pseudoconvex domains Ω_j in Ω such that $\Omega_j \Subset \Omega_{j+1} \Subset \Omega$ for every $j \in \mathbb{N}$ and $\Omega = \bigcup_{j\in\mathbb{N}} \Omega_j$.

(12) *The intersection $\Omega\cap\pi$ is q-pseudoconvex in π for every $(q+2)$-dimensional complex affine plane π in $\mathbb{C}^n$.*

Proof. We shall follow the classical arguments as in case of pseudoconvex sets and show the following implications,

$$\begin{array}{ccccccccccc} (1) & \Rightarrow & (2) & \Rightarrow & (3) & \Rightarrow & (4) & \Rightarrow & (5) & & (1) \\ & & & & & & \Downarrow & & \Downarrow & & \Uparrow \\ & & & & & & (6) & \Rightarrow & (7) & \Rightarrow & (8) \end{array}$$

$$(5)\Rightarrow(9)\Rightarrow(10)\Rightarrow(11)\Rightarrow(4) \quad \text{and} \quad (12) \iff (4).$$

Notice that if $\Omega=\mathbb{C}^n$, then there is nothing to show. Hence, we assume from now on that $\Omega\subsetneq\mathbb{C}^n$.

(1)⇒(2) Assume that the function $\psi(z):=-\log R_{b\Omega,v}(z)$ is not q-plurisubharmonic on Ω for some fixed vector $v\in\mathbb{C}^n$ with $\|v\|_2=1$. Then there exists a $(q{+}1)$-dimensional complex affine plane π such that ψ is not subpluriharmonic on a neighborhood of a point p in $\pi\cap\Omega$. By Proposition 3.3.2 (8), we can assume without loss of generality that p is the origin and π is equal to $\mathbb{C}^{q+1}\times\{0\}^{n-q-1}$. Let Ω^* be an open subset in π such that $\pi\cap\Omega=\Omega^*\times\{0\}^{n-q-1}$. Consider the function

$$\Omega^*\ni\zeta\mapsto\rho(\zeta):=-\log R_{b\Omega,v}(\zeta,0).$$

By assumption, it is not subpluriharmonic near the origin in Ω^*. According to Proposition 3.2.3 there exist a polydisc $\Delta:=\Delta_r^{q+1}(0)\Subset\Omega^*$ and a holomorphic function f on a neighborhood of $\overline{\Delta}$ such that $\rho<\mathrm{Re}(f)=:h$ on the boundary $b\Delta$, but $\rho(\zeta_0)>h(\zeta_0)$ at some point $\zeta_0\in\Delta$. Then we have that

$$e^{-h(\zeta_0)}>R_{b\Omega,v}(\zeta_0,0) \quad \text{and} \quad e^{-h(\zeta)}<R_{b\Omega,v}(\zeta,0) \text{ for every } \zeta\in b\Delta. \tag{4.7}$$

We claim that $v\notin\pi$. Otherwise, the vector v can be written as $(v',0)$ for some $v'\in\mathbb{C}^{q+1}$, and so the function ρ has the form

$$\rho(\zeta)=-\log R_{b\Omega^*,v'}(\zeta) \text{ for every } \zeta\in\Omega^*\subset\mathbb{C}^{q+1}.$$

But then Proposition 4.2.3 implies that ρ is q-plurisubharmonic on Ω^*, which contradicts to the assumptions made on ψ at the beginning of this step. Hence, $v\notin\pi$, and we can choose linearly independent vectors $w_{q+3},\ldots,w_n$ in $\mathbb{C}^n\backslash\pi$ such that the space $\mathbb{C}^n$ is generated by the vectors $v,w_{q+3},\ldots,w_n$ and by the plane π.

Define for $\zeta \in \mathbb{C}^{q+1}$, $\xi \in \mathbb{C}$, $\eta = (\eta_{q+3}, \dots, \eta_n) \in \mathbb{C}^{n-q-2}$ and $\varepsilon > 0$ the mapping

$$\Phi(\zeta, \xi, \eta) = (\zeta, 0, \dots, 0) \ + \ \xi e^{-f(\zeta)} v \ + \ \varepsilon \sum_{j=q+3}^{n} \eta_j w_j.$$

It is easy to calculate that Φ maps biholomorphically onto its image. Recall that, if $z \in \Omega$, then for every complex number $s \in \Delta_t(0)$ the point $z + sv$ lies in Ω if and only if $0 < t < R_{b\Omega,v}(z)$. In particular, $(\zeta_0, 0) + e^{-f(\zeta_0)} v$ does not belong to Ω. Then, in view of the inequalities in (4.7), we observe that

$$\Phi(\zeta, \xi, 0) \in \Omega \text{ for every } (\zeta, \xi) \in \big(b\Delta_r^{q+1} \times \Delta_1^1\big) \ \cup \ \big(\overline{\Delta_r^{q+1}} \times \{0\}\big),$$

and $(\zeta_0, 1) \in \overline{\Delta_r^{q+1}} \times b\Delta_1^1$. Thus, $\Phi(\zeta_0, 1, 0)$ does not lie in Ω. Now if $\varepsilon > 0$ is small enough, we can arrange that

$$\Phi\Big(\big(b\Delta_r^{q+1} \times \overline{\Delta_1^{n-q-1}}\big) \ \cup \ \big(\overline{\Delta_r^{q+1}} \times \overline{\Delta_\varepsilon^{n-q-1}}\big)\Big) \subset \Omega,$$

but still $\Phi(\zeta_0, 1, 0) \notin \Omega$. Therefore, we can easily construct a $(q{+}1, n{-}q{-}1)$-Hartogs figure (H, P) such that $H \subset \Omega$ but $P \not\subseteq \Omega$. This is a contradiction to the assumption made on Ω. Finally, we have shown the implication (1)⇒(2).

(2)⇒(3) This is a consequence of the last statement in Proposition 4.2.3.

(3)⇒(4) Simply take the Euclidean norm $\|\cdot\| := \|\cdot\|_2$.

(4)⇒(5) The function $\psi(z) = -\log d_{\|\cdot\|}(z, b\Omega) + \|z\|_2^2$ is a continuous q-plurisubharmonic exhaustion function for Ω.

(4)⇒(6) The function $\psi(z) = -\log d_{\|\cdot\|}(z, b\Omega)$ is q-plurisubharmonic on Ω and admits the property that $\psi(z)$ tends to $+\infty$ whenever z tends to $b\Omega$.

(5)⇒(7) Let φ be a q-plurisubharmonic exhaustion function for Ω and K be a compact set in Ω. Then K is contained in $\Omega_c := \{\varphi(z) < c\}$ for some constant $c \in \mathbb{R}$, so the hull $\widehat{K}^{\Omega}_{\mathcal{PSH}_q(\Omega)}$ obviously lies in Ω_c. Since Ω_c is relatively compact in Ω and $\widehat{K}^{\Omega}_{\mathcal{PSH}_q(\Omega)}$ is closed in Ω, the hull is compact in Ω.

(6)⇒(7) Clearly, $\widehat{K}^{\Omega}_{\mathcal{PSH}_q(\Omega)}$ is bounded and closed in Ω. Assume that there is a sequence $(p_n)_{n\in\mathbb{N}}$ in $\widehat{K}^{\Omega}_{\mathcal{PSH}_q(\Omega)}$ which tends to some point $p \in b\Omega$. By the assumption made on ψ in (6) we derive that $\limsup_{n\to+\infty} \psi(p_n) = +\infty$. But on the other hand $\psi(p_n) \leq \max_K \psi$ by the definition of the hull $\widehat{K}^{\Omega}_{\mathcal{PSH}_q(\Omega)}$, which is absurd. Hence, $\widehat{K}^{\Omega}_{\mathcal{PSH}_q(\Omega)}$ is a compact set.

(7)⇒(8) Let $\{A_t\}_{[0,1]}$ be the family as in the assumption of property (7). We obviously have that $b\Omega$ has a positive distance to bA_t for every $t \in [0,1]$. Since the family $\{A_t\}_{t\in[0,1]}$ is continuously parameterized by t and the closure of $\bigcup_{t\in[0,1]} A_t$ is compact, we can choose a compact set $K \Subset \Omega$ that contains bA_t for every $t \in [0,1]$. Then each $\overline{A_t}$ is contained in $\widehat{K}^{\Omega}_{\mathcal{PSH}_q(\Omega)}$ for $t \in [0,1)$ because of the local maximum principle for q-plurisubharmonic functions on analytic subsets (see Proposition 3.8.5). Finally, by the continuity of $t \mapsto A_t$, we deduce that A_1 is also contained in $\widehat{K}^{\Omega}_{\mathcal{PSH}_q(\Omega)}$. By assumption, the q-plurisubharmonic hull $\widehat{K}^{\Omega}_{\mathcal{PSH}_q(\Omega)}$ is entirely contained in Ω, so A_1 also belongs to Ω.

(8)⇒(1) In order to get a contradiction, suppose that Ω is not $(n-q-1)$-Hartogs pseudoconvex, so there is a $(q+1, n-q-1)$-Hartogs figure (H, P) such that $H \subset \Omega$, but $P \not\subseteq \Omega$. Hence, there exist two constants $r, R \in (0,1)$ and a biholomorphism Φ defined from $\Delta^n_1(0)$ onto P such that $\Phi(H_e) = H$, where

$$H_e = \left(\Delta_1^{q+1} \times \Delta_r^{n-q-1}\right) \cup \left(A_{R,1}^{q+1} \times \Delta_1^{n-q-1}\right).$$

By shrinking r and R, if necessary, we can assume that Φ is defined on an open neighborhood of the closure of Δ_1^n and has the same properties as before. Now we can increase the radius $r > 0$ until $\Phi(H_e)$ touches the boundary of Ω for the first time, say at a point $p_0 \in b\Omega$. Then we can find a point $(z_0, w_0) \in \Delta_R^{q+1} \times b\Delta_r^{n-q-1}$ with $p_0 = \Phi(z_0, w_0)$. Now consider the family $\{A_t\}_{t\in[0,1]}$ defined by

$$A_t := \Phi\left(\Delta_1^{q+1} \times \{tw_0\}\right).$$

Since Ψ is biholomorphic from some neighborhood of $\overline{\Delta_1(0)}$ onto its image, the family $\{A_t\}_{t\in[0,1]}$ is continuously parameterized by t in the Hausdorff topology. Moreover, we have that for every $t \in [0,1)$ the compact sets $\overline{A_t}$ and bA_1 belong to Ω, whereas A_1 is not fully contained in Ω. This a contradiction to the assumption made in (8).

(5)⇒(9) This implication follows directly from Proposition 4.1.2 (1).

(9)⇒(10) Let p be a boundary point of Ω and B be a ball centered at p such that $\Omega \cap B$ is q-pseudoconvex. Then for a small enough neighborhood U of p in B we have that $d(z, b\Omega) = d(z, b(\Omega \cap B))$ for every $z \in \Omega \cap U$, and so $-\log d(z, b\Omega)$ is q-plurisubharmonic on $\Omega \cap U$. Since p is an arbitrary boundary point of Ω, there is a neighborhood W of $b\Omega$ such that $-\log d(z, b\Omega)$ is q-plurisubharmonic on W. It is clear that $-\log d(z, b\Omega)$ tends to $+\infty$ when z approaches the boundary of Ω.

(10)⇒(11) For $k \in \mathbb{N}$ let Ω'_k be the intersection of Ω with a ball $B_k(0)$ centered at the origin, so that $\{\Omega'_k\}_{k \in \mathbb{N}}$ is an increasing collection of bounded open sets whose union is Ω. We claim that each Ω'_k is q-pseudoconvex. Indeed, fix an integer $k \in \mathbb{N}$ and define the function $\varphi_k(z) = \log k - \log d(z, bB_k(0))$. By assumption, ψ is q-plurisubharmonic on some neighborhood W of the boundary $b\Omega$ such that $\limsup_{z \to b\Omega} \psi(z) = +\infty$. Notice that ψ is bounded on $\Omega'_k \cap bW$ and that φ_k is non-negative and plurisubharmonic on the ball $B_k(0)$. Then Proposition 3.3.2 (10) implies that for a large enough constant $c_k > 0$ the following function is continuous and q-plurisubharmonic on Ω'_k,

$$\psi_k := \begin{cases} c_k + \varphi_k & \text{on } \Omega'_k \setminus W, \\ \max\{\psi, c_k + \varphi_k\} & \text{on } \Omega'_k \cap W. \end{cases}$$

Observe that $\psi_k(z)$ converges to $+\infty$ when z approaches $b\Omega'_k$. Since we have already shown the implication (6)⇒(4), the set Ω'_k is q-pseudoconvex. Now it is easy to verify that for each $\mu > 0$ the function $u_{k,\mu} := -\log(\mu - \psi_k)$ is q-plurisubharmonic on $\Omega_{k,\mu} := \{z \in \Omega'_k : \psi_k(z) < \mu\}$ such that $u_k(z)$ tends to $+\infty$ whenever z approaches the boundary of $\Omega_{k,\mu}$. Observe that it follows from the property of ψ_k that $\Omega_{k,\mu}$ is a relatively compact q-pseudoconvex subset of Ω'_k. Hence, we can find a sequence $(\mu(k))_{k \in \mathbb{N}}$ such that $\Omega_{k,\mu(k)} \Subset \Omega_{k+1,\mu(k+1)} \Subset \Omega$ and Ω equals $\bigcup_{k \in \mathbb{N}} \Omega_{k,\mu(k)}$.

(11)⇒(4) For every $j \in \mathbb{N}$ the function $f_j(z) = -\log d(z, b\Omega_j)$ is q-plurisubharmonic on Ω_j, since Ω_j is q-pseudoconvex and we have just proved the equivalence (4)⟺(5). The implication follows from the fact that Ω is equal to $\bigcup_{k \in \mathbb{N}} \Omega_k$ and $\Omega_j \Subset \Omega_{j+1} \Subset \Omega$, so that for each fixed $j_0 \in \mathbb{N}$ the sequence $(f_j)_{j \geq j_0}$ decreases to $-\log d(z, b\Omega)$ on Ω_{j_0}. According to Proposition 3.3.2 (4), we conclude that the function $-\log d(z, b\Omega)$ is q-plurisubharmonic on Ω.

(12)⟺(4) The case $q = 0$ is due to P. Lelong in his fundamental work in [Lel52a]. The generalization to $q \geq 1$ has been shown by Z. Słodkowski (see Corollary 4.8 in [Sło86]). We shall not repeat the proofs here, since we will not need this property and included this characterization only for the sake of completeness. □

As a direct application, we are able to extend Proposition 4.1.2.

Proposition 4.3.3

(1) If Ω_1 is q-pseudoconvex and Ω_2 is r-pseudoconvex in $\mathbb{C}^n$, then the union $\Omega_1 \cup \Omega_2$ is $(q + r + 1)$-pseudoconvex.

(2) Let $\{\Omega_j\}_{j\in J}$ be a collection of q-pseudoconvex sets in $\mathbb{C}^n$ such that the set $\Omega := \operatorname{int}\left(\bigcap_{j\in J} \Omega_j\right)$ is not empty. Then Ω is q-pseudoconvex.

(3) If $\{\Omega_n\}_{n\in\mathbb{N}}$ is an increasing collection of q-pseudoconvex sets in $\mathbb{C}^n$ with $\Omega_1 \subset \Omega_2 \subset \ldots$, then the union $\Omega := \bigcup_{n\in\mathbb{N}} \Omega_n$ is q-pseudoconvex.

Proof. (1) Let $\Phi_1(z) := -\log d(z, b\Omega_1)$ and $\Phi_2(z) := -\log d(z, b\Omega_2)$. Then Φ_1 is q-plurisubharmonic on Ω_1 and Φ_2 is r-plurisubharmonic on Ω_2 according to Theorem 4.3.2 (4). In view of Proposition 3.3.2 (6) and (10) the function

$$\Phi := \begin{cases} \Phi_1 & \text{on } \Omega_1 \setminus \Omega_2 \\ \min\{\Phi_1, \Phi_2\} & \text{on } \Omega_1 \cap \Omega_2 \\ \Phi_2 & \text{on } \Omega_2 \setminus \Omega_1 \end{cases}$$

is a continuous $(q+r+1)$-plurisubharmonic function on $\Omega_1 \cup \Omega_2$ such that $\Phi(z)$ tends to $+\infty$ whenever z tends to $b\Omega$. Then Theorem 4.3.2 (6) implies the desired statement.

(2) It is easy to verify that $d(z, b\Omega) = \inf_{j\in J} d(z, b\Omega_j)$ for every $z \in \Omega$. Hence, $z \mapsto -\log d(z, b\Omega)$ is the supremum of a family of q-plurisubharmonic functions on Ω in view of Theorem 4.3.2 (4). Since it is continuous on Ω, it is q-plurisubharmonic there. Hence, Theorem 4.3.2 (4) yields q-pseudoconvexity of Ω.

(3) This follows immediately from Theorem 4.3.2 (11).

□

We can also improve the regularity of the exhaustion functions of a q-pseudoconvex set (see Corollary 6.1. in [Bun90]).

Remark 4.3.4 By Bungart's approximation technique (see Theorem 3.5.4) we can show that an open set Ω in $\mathbb{C}^n$ is q-pseudoconvex if and only if it admits an exhaustion function which is q-plurisubharmonic with corners on Ω. In the case of $q = 0$, due to Richberg's approximation method (see Theorem 3.1.8), we can even obtain that a set Ω is pseudoconvex if and only if it has a $\mathcal{C}^\infty$-smooth plurisubharmonic exhaustion function. If $q \geq 1$, it remains unclear whether a q-pseudoconvex set admits a $\mathcal{C}^2$-smooth q-plurisubharmonic exhaustion function.

4.4 Real q-convex and q-pseudoconvex sets

We examine sets generated by real q-convex functions and give the relation to q-pseudoconvex sets. The main result is a generalized version of P. Lelong's observation in [Lel52b] on pseudoconvexity of sets of the form $\omega + i(-a, a)^n$, where ω is an open set in $\mathbb{R}^n$ and $a > 0$. Earlier results on tubular sets of the form $\omega + i\mathbb{R}^n$ were achieved by S. Bochner [Boc38] and K. Stein [Ste37].

Definition 4.4.1 We say that an open set ω in $\mathbb{R}^n$ is *real q-convex* if the boundary distance function $x \mapsto -\log d(x, b\omega)$ is real q-convex on ω.

Certainly, we could establish a similar collection of different characterizations of real q-convexity like in Proposition 4.3.2 for q-pseudoconvex sets. But this would deserve its own treatise and further research. Anyway, the above definition is sufficient to our purpose to compare the real q-convex sets to q-pseudoconvex sets.

Theorem 4.4.2

(1) Every open real q-convex set Ω in $\mathbb{C}^n = \mathbb{R}^{2n}$ is q-pseudoconvex.

(2) Let ω be an open set in $\mathbb{R}^n$. Then ω is real q-convex if and only if the set $\omega + i(-a, a)^n$ is q-pseudoconvex for some $a \in (0, +\infty]$.

Proof. If $q \geq n - 1$, there is nothing to show, since every open set in $\mathbb{C}^n$ is $(n-1)$-pseudoconvex according to Corollary 4.2.4. Hence, from now on we assume that $q < n - 1$.
(1) By definition, the boundary distance function $z \mapsto -\log d(z, b\Omega)$ is real q-convex on Ω. Thus, due to Theorem 3.6.1, it is q-plurisubharmonic on Ω. In view of Proposition 4.3.2 (4), the set Ω is q-pseudoconvex.

(2) **Case a = +∞.** In this case, we are in the setting of a tubular set of the form $\Omega := \omega + i(-a,a)^n = \omega + i\mathbb{R}^n$. Since $d(z, b\Omega) = d(\mathrm{Re}(z), b\omega)$ for every $z \in \Omega$, the function $z \mapsto d(z, b\Omega)$ is rigid on Ω. Then it follows from Theorem 3.6.4 that the function $x \mapsto -\log d(x, b\omega)$ is real q-convex on ω if and only if $z \mapsto -\log d(z, b\Omega)$ is q-plurisubharmonic on Ω. The result follows now from the Definition 4.4.1 and Proposition 4.3.2 (4).

Case a > 0. Assume that ω is real q-convex. Then, in view of the previous case $a = +\infty$, the set $\omega + i\mathbb{R}^n$ is q-pseudoconvex. Since the set $\mathbb{R}^n + i(-a,a)^n = (\mathbb{R} + i(-a,a))^n$ is pseudoconvex as a product of pseudoconvex sets, it follows from Proposition 4.1.2 (1), that the intersection

$$(\mathbb{R}^n + i(-a,a)^n) \cap (\omega + i\mathbb{R}^n) = \omega + i(-a,a)^n$$

is q-pseudoconvex. This proves the necessity of this statement.

In order to prove the sufficiency, assume first that ω is bounded and that the set $\Omega := \omega + i(-a,a)^n$ is q-pseudoconvex for some $a > 0$. Then there are a small positive number $b < a$ and a neighborhood U of $b\omega$ in $\overline{\omega}$ such that for every $z \in U + i(-b,b)^n$ we have that

$$d(z, b\Omega) = d(z, b\omega + i\mathbb{R}^n) = d(x, b\omega). \tag{4.8}$$

For $\lambda \in \mathbb{R}^n$ consider the mapping $T_\lambda(z) := z - i\lambda$ and the set

$$W_\lambda := \{z \in \mathbb{C}^n : \mathrm{Re}(z) \in \omega \cap U,\ \mathrm{Im}(z_j) \in (-b + \lambda_j, b + \lambda_j),\ j = 1, \ldots, n\}.$$

Then, in view of the equations in (4.8), for $z \in W_\lambda$ we obviously have the following identities,

$$d(z, b\omega + i\mathbb{R}^n) = d(z - i\lambda, b\omega + i\mathbb{R}^n) = d(T_\lambda(z), b\Omega).$$

Since Ω is q-pseudoconvex, the function $z \mapsto -\log d(z, b\Omega)$ is q-plurisubharmonic on Ω. In view of Proposition 3.3.2 (8), the function $-\log d(T_\lambda, b\Omega)$ is q-plurisubharmonic on W_λ. Since $\bigcup_{\lambda \in \mathbb{R}^n} W_\lambda = (\omega \cap U) + i\mathbb{R}^n$ and since q-plurisubharmonicity is a local property, the function $z \mapsto -\log d(z, b\omega + i\mathbb{R}^n)$ is rigid and q-plurisubharmonic on $(\omega \cap U) + i\mathbb{R}^n$. Therefore, according to Theorem 3.6.4, $x \mapsto -\log d(x, b\omega)$ is real q-convex on $\omega \cap U$. Since the last function is continuous and ω is bounded, we can find an appropriate constant $c \in \mathbb{R}$ such that $x \mapsto \max\{-\log d(x, b\omega), c\}$ is real q-convex on ω due to Theorem 2.3.7. Now it follows from Theorem 3.6.4 that the function $z \mapsto \max\{-\log d(z, b\omega + i\mathbb{R}^n), c\}$ is q-plurisubharmonic on $\omega + i\mathbb{R}^n$ and tends to $+\infty$ when z approaches the boundary of $\omega + i\mathbb{R}^n$. Thus, by Proposition 4.3.2 (6), the tubular set $\omega + i\mathbb{R}^n$ is q-pseudoconvex. Then the arguments in the case $a = +\infty$ imply that the set ω is real q-convex.

If ω is not bounded, consider the set $\omega_k := \omega \cap B_k(0)$, where $B_k(0)$ denotes the ball of radius $k \in \mathbb{N}$ around the origin. For a large enough integer $k_0 > 0$, we can assume that ω_k is not empty for every $k \geq k_0$. Since $B_k(0) + i\mathbb{R}^n$ is pseudoconvex, it follows from Proposition 4.1.2 (1) that the intersection

$$(\omega + i(-a,a)^n) \cap (B_k(0) + i\mathbb{R}^n) = \omega_k + i(-a,a)^n =: \Omega_k$$

is q-pseudoconvex. Since ω_k is bounded, we know from the previous discussion that ω_k is real q-convex, i.e., $x \mapsto -\log d(x, \omega_k)$ is real q-convex on ω_k. But for every fixed integer $\ell \geq k_0$ the sequence $(v_k)_{k \geq \ell}$ of real q-convex functions $v_k(x) := -\log d(x, b\omega_k)|(B_\ell(0) \cap \omega)$ is decreasing to $x \mapsto -\log d(x, b\omega)$ on $B_\ell(0) \cap \omega$. Since real q-convexity is a local property, $x \mapsto -\log d(x, b\omega)$ is real q-convex on ω. By the definition, the set ω is real q-convex. □

4.5 Smoothly bounded q-pseudoconvex sets

We recall the definition, the basic properties and some examples of relative q-pseudoconvex sets, which were originally introduced by Z. Słodkowski in chapter 4 of [Sło86]. They will mainly serve to simplify our notations.

Definition 4.5.1 Given two open sets $U \subset V$ in $\mathbb{C}^n$, the set U is said to be *q-pseudoconvex in (or relative to) V* if there is a neighborhood W of $bU \cap V$ in $\mathbb{C}^n$ such that the function $z \mapsto -\log d(z, bU)$ is q-plurisubharmonic on $U \cap W$.

We mention some trivial settings related to relative q-pseudoconvex sets.

Remark 4.5.2 (1) Every open set $U \subset \mathbb{C}^n$ is q-pseudoconvex in itself, but U may not be necessarily q-pseudoconvex in the absolute sense in $\mathbb{C}^n$. For example, $\mathbb{C}^2$ minus any point is pseudoconvex in itself, but not in $\mathbb{C}^2$.

(2) If U is an open subset of an open set V in $\mathbb{C}^n$, then it is always $(n-1)$-pseudoconvex in V according to Proposition 4.2.3.

(3) We proved in Proposition 4.3.2 (9) that the definitions of relative q-pseudoconvex and q-pseudoconvex sets collide in the case of $V = \mathbb{C}^n$, i.e., an open set U is q-pseudoconvex if and only if it is q-pseudoconvex relative to $\mathbb{C}^n$.

The following proposition is a part of Theorem 4.3 and Corollary 4.7 in Słodkowski's article [Sło86]. It gives more characterizations of relative q-pseudoconvex sets also in terms of q-pseudoconvex sets in $\mathbb{C}^n$. More equivalent characterizations of relative q-pseudoconvexity can be found in Theorem 4.3 and Corollary 4.7 of [Sło86].

Proposition 4.5.3 *Let $U \subsetneq V$ be open sets in $\mathbb{C}^n$. Then the following statements are equivalent.*

(1) U is q-pseudoconvex in V.

(2) There exist a neighborhood W of $bU \cap V$ and a q-plurisubharmonic function ψ on W such that $\psi(z)$ tends to $+\infty$ whenever z approaches the relative boundary $bU \cap V$.

(3) For every point p in $V \cap bU$ there exists an open ball $B_r(p)$ centered in p such that the intersection $U \cap B_r(p)$ is q-pseudoconvex in $\mathbb{C}^n$.

We continue by presenting some examples of relative q-pseudoconvex sets.

Example 4.5.4 (1) Let φ be q-plurisubharmonic on an open set V in $\mathbb{C}^n$ and let c be a real number. Then the set $U = \{z \in V : \varphi(z) < c\}$ is q-pseudoconvex in V. If, moreover, the set V is q-pseudoconvex itself, then V is q-pseudoconvex (in $\mathbb{C}^n$).

Indeed, let p be a boundary point in $bU \cap V$ and let $B = B_r(p) \Subset V$ be a ball centered at p and with radius $r > 0$. Then Proposition 3.5.6 implies that $\varphi_0 := -\log(c-\varphi)$ is a q-plurisubharmonic function on U. Furthermore, $\varphi_0(z)$ tends to $+\infty$ as z tends to the boundary of U inside V. Thus, the function $z \mapsto \max\{\varphi_0, -\log(r^2-\|z\|_2^2)\}$ is q-plurisubharmonic on the set $B\cap U$ and fulfills the assumptions of the property (6) in Proposition 4.3.2. This means that $B\cap U$ is q-pseudoconvex. Since p is an arbitrary point in $bU \cap V$, Proposition 4.5.3 implies that U is q-pseudoconvex in V.

Now, if V is additionally q-pseudoconvex, the boundary distance function $\psi(z) := -\log d(z, bV)$ is q-plurisubharmonic on V due to Proposition 4.3.2 (4). Then the function $\max\{\varphi_0, \psi\}$ is q-plurisubharmonic on U and satisfies the property (6) in Proposition 4.3.2. Hence, U is q-pseudoconvex.

(2) Let Ω be an open set in $\mathbb{C}^n$ and let h be a q-holomorphic function on Ω. Let $\Gamma(h) := \{(z, h(z)) \in \mathbb{C}^{n+1} : z \in \Omega\}$ be the graph of f over Ω. By Proposition 3.10.2 (5) and (6), the function $(z, w) \mapsto 1/(h(z) - w)$ is q-holomorphic on $U := (\Omega \times \mathbb{C}) \setminus \Gamma(h)$. Then Proposition 3.10.2 (8) implies that the function

$\psi(z,w) := -\log|h(z)-w|$ is q-plurisubharmonic on U and has the property that $\psi(z,w)$ tends to $+\infty$ whenever (z,w) approaches the graph $\Gamma(h)$. Hence, the open set U is q-pseudoconvex in $V := \Omega \times \mathbb{C}$ by Proposition 4.5.3 (2).

A converse statement is known in the case of $q = 0$ and is one of the classical Hartogs' theorems. We will mention the precise statement of Hartogs and study the holomorphic structure of graphs of functions later in Section 4.7.

The smoothly bounded q-pseudoconvex sets can be characterized in terms of q-plurisubharmonic defining functions.

Definition 4.5.5 Let U be an open set in $\mathbb{C}^n$ and $k \in \mathbb{N} \cup \{\infty\}$.

(1) The set U has a *$\mathcal{C}^k$-smooth boundary at a boundary point* $p \in bU$ if there is a $\mathcal{C}^k$-smooth function ϱ defined on a neighborhood W of p such that $\nabla\varrho(p) \neq 0$ and $U \cap W = \{z \in W : \varrho(z) < 0\}$. In this case, we also say that U is *$\mathcal{C}^k$-smoothly bounded at* p and that ϱ is a *defining function for U at* p.

(2) If U is $\mathcal{C}^k$-smoothly bounded at each of its boundary points, then it is *$\mathcal{C}^k$-smoothly bounded* or it has a *$\mathcal{C}^k$-smooth boundary.*

(3) If there are another open set V in $\mathbb{C}^n$ with $U \subsetneq V$ and a $\mathcal{C}^k$-smooth function ϱ on V satisfying $U = \{z \in V : \varrho(z) < 0\}$ and $\nabla\varrho \neq 0$ on $bU \cap V$, then ϱ is called a *defining function for U (in V).*

(4) Let ϱ be a defining function for U at $p \in bU$. The *holomorphic tangent space $H_p bU$ to bU at* p is then given by

$$H_p bU = \{X \in \mathbb{C}^n : \sum_{j=1}^{n} \frac{\partial\varrho}{\partial z_j}(p) X_j = 0\},$$

and the *Levi form $\widetilde{\mathscr{L}_\varrho}(p)$ of ϱ at* p is defined by

$$H_p bU \ni X, Y \ \mapsto \ \widetilde{\mathscr{L}_\varrho}(p)(X,Y) := \sum_{j,k=1}^{n} \frac{\partial^2\varrho}{\partial z_j \partial \bar{z}_k}(p) X_j \overline{Y}_k.$$

In other words, it is the Hermitian form on $\mathbb{C}^n \times \mathbb{C}^n$ induced by the Levi matrix $\mathscr{L}_\varrho(p)$ of ϱ at p restricted to $H_p bU \times H_p bU$ (recall Definition 3.4.3). For an abbreviation, we also write $\widetilde{\mathscr{L}_\varrho}(p,X) := \widetilde{\mathscr{L}_\varrho}(p)(X,X)$.

(5) The set U is called *Levi q-pseudoconvex* (resp. *strictly Levi q-pseudoconvex*) *at the point* $p \in bU$ if there exists a defining function ϱ for U at p such that its Levi form $\widetilde{\mathscr{L}_\varrho}$ at p has at most q negative (resp. q non-positive) eigenvalues.

(6) Let V be an open neighborhood of U with $U \subsetneq V$. Then U is called *(strictly) Levi q-pseudoconvex in* V if it is (strictly) Levi q-pseudoconvex at every point $p \in bU \cap V$.

(7) A strictly Levi q-pseudoconvex set in $\mathbb{C}^n$ is also simply called *strictly q-pseudoconvex.*

One interesting example is the following version of a Hartogs domain.

Example 4.5.6 (1) Let g be a $\mathcal{C}^2$-smooth (strictly) q-plurisubharmonic function defined on an open set Ω in $\mathbb{C}^n$ and let $V := \Omega \times \mathbb{C}^k$. Then the set

$$U := \left\{(z,w) \in \Omega \times \mathbb{C}^k : \mathrm{e}^{g(z)} < \|w\|_2^2\right\}$$

is (strictly) Levi $(q+k)$-pseudoconvex in V. More precisely, it is (strictly) $(q+k-1)$-pseudoconvex at every point (z,w) in $bU \cap V$ with $\partial g(z) \neq 0$, where ∂g is the complex gradient $(\partial g/\partial z_j)_{j=1}^n$ of g.

To see this, consider the following $\mathcal{C}^2$-smooth function ψ on V defined by,

$$\psi(z,w) := \mathrm{e}^{g(z)} - \|w\|_2^2.$$

Then we have that $U = \{(z,w) \in \Omega \times \mathbb{C}^k : \psi(z,w) < 0\}$. Given a vector $Z = (X,Y) \in \mathbb{C}^n \times \mathbb{C}^k$, the Hermitian form induced by the Levi matrix of ψ at p can be easily calculated,

$$\mathscr{L}_\psi(z,w)(Z,Z) = \mathrm{e}^{g(z)}\mathscr{L}_g(z)(X,X) + \mathrm{e}^{g(z)}\Big|\sum_{j=1}^n \frac{\partial g}{\partial z_j}(z)X_j\Big|^2 - \|Y\|_2^2.$$

Now the vector $(X,Y) \in \mathbb{C}^n \times \mathbb{C}^k$ lies in the holomorphic tangent space to $bU \cap V$ at (z,w) if and only if

$$0 = \sum_{j=1}^n \frac{\partial \psi}{\partial z_j}(z,w)X_j + \sum_{j=1}^k \frac{\partial \psi}{\partial w_j}(z,w)Y_j = e^{g(z)}\sum_{j=1}^n \frac{\partial g}{\partial z_j}(z)X_j - w^h Y,$$

where $w^h = \overline{w}^t$ is the transpose complex conjugate of w. Now if (z,w) is contained in $bU \cap V$, then $\|w\|_2^2 = e^{g(z)}$. According to the previous equations, we obtain that

$$\frac{|w^h Y|^2}{\|w\|_2^2} = \mathrm{e}^{g(z)}\Big|\sum_{j=1}^n \frac{\partial g}{\partial z_j}(z)X_j\Big|^2.$$

Therefore, the Levi form of ψ at (z,w) is as below:

$$\begin{aligned}\widetilde{\mathscr{L}_\psi}\big((z,w),(X,Y)\big) &= \mathrm{e}^{g(z)}\widetilde{\mathscr{L}_g}(z,X) - \|Y\|_2^2 \\ &+ \begin{cases} |w^h Y|^2/\|w\|_2^2, & \text{if } \partial g(z) \neq 0 \\ 0, & \text{if } \partial g(z) = 0 \end{cases}.\end{aligned} \tag{4.9}$$

Since g is (strictly) q-plurisubharmonic on Ω, we have that the Levi matrix of g has at most q negative (resp. non-positive) eigenvalues.

We first consider the case $\partial g(z) = 0$. Since $\mathbb{C}^k \ni Y \mapsto -\|Y\|_2^2$ is always non-positive, the Levi form of ψ at $(z,w) \in bU \cap V$ has at most $q+k$ negative (non-positive) eigenvalues. Hence, U is (strictly) Levi $(q+k)$-pseudoconvex at (z,w).

Now if $\partial g(z) \neq 0$, the Levi form of ψ at (z,w) reduces to

$$\widetilde{\mathscr{L}_\psi}\big((z,w),(X,Y)\big) = \mathrm{e}^{g(z)}\widetilde{\mathscr{L}_g}(z,X) - \left\{ \begin{array}{ll} \|Y\|_2^2, & \text{if } w^h Y = 0 \\ 0, & \text{if } Y = w \end{array} \right\}. \tag{4.10}$$

Since there exist $k-1$ linearly independent vectors Y in $\mathbb{C}^k$ that satisfy $w^h Y = 0$, the Levi form of ψ at $(z,w) \in bU \cap V$ has at most $q+k-1$ negative (resp. non-positive) eigenvalues, and so U is (strictly) Levi $(q+k-1)$-pseudoconvex at (z,w).

(2) With the same function g as in (1), we can achieve similar results for the set

$$U := \{(z,w) \in \Omega \times \mathbb{C}^k : \|w\|_2^2 < e^{-g(z)}\}.$$

We put $\psi(z,w) := \|w\|_2^2 - e^{-g(z)}$ and compute its Levi form at $(z,w) \in bU \cap V$, namely

$$\begin{aligned}\widetilde{\mathscr{L}_\psi}\big((z,w),(X,Y)\big) &= \mathrm{e}^{-g(z)}\widetilde{\mathscr{L}_g}(z,X) \\ &+ \begin{cases} \|Y\|_2^2 - |w^h Y|^2/\|w\|_2^2, & \text{if } \partial g(z) \neq 0 \\ \|Y\|_2^2, & \text{if } \partial g(z) = 0 \end{cases}.\end{aligned}$$

Since g is assumed to be (strictly) q-plurisubharmonic on Ω, by the Cauchy-Schwarz inequality we obtain that the Levi form of ψ at $(z,w) \in bU \cap V$ has at most q negative (non-positive) eigenvalues. Hence, it is (strictly) Levi q-pseudoconvex in V.

We present some facts about Levi q-pseudoconvex sets.

Remark 4.5.7 (1) If an open set $U \subset \mathbb{C}^n$ is (strictly) Levi q-pseudoconvex at a point $p \in bU$, then the Levi form of *every* defining function ϱ for U near p has at most q negative (resp. non-positive) eigenvalues on the holomorphic tangent space at the point p. This means that the definition of Levi q-pseudoconvexity does not depend on the choice of the defining function.

(2) If $U \subset \mathbb{C}^n$ is strictly q-pseudoconvex at a boundary point p and ψ is a defining function for U at p, then for a large enough constant $c > 0$ the function $\exp(c\psi) - 1$ is strictly q-plurisubharmonic on some ball B centered at p and still defines U at p. Therefore, in view of Example 4.5.4 (1), $U \cap B$ is q-pseudoconvex.

(3) Like in the classical case when $q = 0$ (see, e.g., Proposition 3.2.2 in [Kra99]), one can verify that every strictly (Levi) q-pseudoconvex bounded open set U in $\mathbb{C}^n$ admits a *$\mathcal{C}^2$-smooth strictly q-plurisubharmonic global defining function*, i.e., there is a $\mathcal{C}^2$-smooth function ϱ defined on a neighborhood V of $\overline{U}$ such that $U = \{z \in V : \varrho(z) < 0\}$, $\nabla\varrho \neq 0$ on bU and ϱ is strictly q-plurisubharmonic on V. In view of Example 4.5.4 (1), this means that each bounded strictly q-pseudoconvex set is q-pseudoconvex.

(4) The previous result on the global defining function is no longer true in general for unbounded strictly pseudoconvex sets. The reason for this is that they may contain copies of the complex plane on which no upper bounded strictly plurisubharmonic function can exist. This observation can be verified, for example, by considering the set $\Omega = \{(z, w) \in \mathbb{C}^2 : |w| < e^{-|z|^2}\}$.

(5) Nevertheless, it can be proved that every strictly q-pseudoconvex (not necessarily bounded) domain U has a $\mathcal{C}^2$-smooth q-plurisubharmonic global defining function ϱ defined on some neighborhood of $\overline{U}$ which is strictly q-plurisubharmonic on some possibly smaller neighborhood of the boundary of U. This and even more interesting results on unbounded q-pseudoconvex domains can be found in the joint paper [HST14] by T. Harz, N. V. Shcherbina and G. Tomassini.

The next theorem is the main result in G. V. Suria's paper [Sur84].

Theorem 4.5.8 *Every Levi q-pseudoconvex set is q-pseudoconvex. More precisely, it admits a $\mathcal{C}^2$-smooth q-plurisubharmonic exhaustion function.*

4.6 Duality principle of q-pseudoconvex sets

In this section, we are interested in the link between strictly q-pseudoconvex sets and their complements. Their relation leads to two duality theorems. The first one is due to R. Basener (see Proposition 6 in [Bas76]).

Theorem 4.6.1 *If an open set Ω in $\mathbb{C}^n$ is strictly q-pseudoconvex at some point $p \in b\Omega$, then for every neighborhood V of p the set $V \cap (\mathbb{C}^n \setminus \overline{\Omega})$ is not $(n{-}q{-}2)$-pseudoconvex. More precisely, for each w in $b\Omega \cap V$ and every neighborhood $U \Subset V$ of w there is a family $\{A_t\}_{t\in[0,1]}$ of $(n{-}q{-}1)$-dimensional complex submanifolds of U which is continuously parameterized by t and fulfills*

1. $\overline{A}_t \subset (\mathbb{C}^n \setminus \overline{\Omega})$ *for every* $t \in [0, 1)$,
2. $w \in A_1$, *but* $\overline{A}_1 \setminus \{w\} \subset (\mathbb{C}^n \setminus \overline{\Omega})$.

Proof. Fix the point $p \in b\Omega$. By Proposition 6 in [Bas76] there exists a neighborhood V of p in $\mathbb{C}^n$ such that for every point w in $b\Omega \cap V$ the following properties hold after an appropriate holomorphic change coordinates on V,

$$w = 0 \quad \text{and} \quad \mathrm{Re}(z_1) < 0 \;\; \text{for every } z \in V \cap (\overline{\Omega} \setminus \{w\}) \cap (\mathbb{C}^{n-q} \times \{0\}^q). \tag{4.11}$$

Let $U \Subset V$ be any neighborhood of w. Then there are real numbers $\varepsilon > 0$ and $r > 0$ such that for each $t \in [0, 1]$ the submanifold

$$A_t = \{(1-t)\varepsilon\} \times B_r^{n-q-1}(0) \times \{0\}^q$$

is contained in U. Finally, the properties (4.11) imply that the family $\{A_t\}_{t\in[0,1]}$ has the desired properties. □

As an application we can improve Suria's observation (see Theorem 4.5.8) which fully clarifies the relation between Levi q-pseudoconvexity and q-pseudoconvexity.

Corollary 4.6.2 *Let Ω be an open set in $\mathbb{C}^n$ which is $\mathcal{C}^2$-smoothly bounded. Then Ω is Levi q-pseudoconvex if and only if it is q-pseudoconvex.*

Proof. Due to Suria's Theorem 4.5.8, it only remains to prove that, if Ω is q-pseudoconvex, then it is Levi q-pseudoconvex. Suppose that Ω is not Levi q-pseudoconvex at some boundary point p of Ω. Then there are a neighborhood $W \subset U$ of p and a $\mathcal{C}^2$-smooth defining function ϱ for Ω at p defined on W such that its Levi form has at most $n{-}q{-}2$ non-negative eigenvalues on the holomorphic tangent space to $b\Omega$ at p. Hence, $-\varrho$ is a defining function for $D := \mathbb{C}^n \setminus \overline{\Omega}$ at p whose Levi form has at most $n{-}q{-}2$ non-positive eigenvalues on the holomorphic tangent space to $b\Omega$ at p. This means that D is strictly $(n{-}q{-}2)$-pseudoconvex at p. But then Theorem 4.6.1 implies that Ω is not

q-pseudoconvex near p, which is absurd. Therefore, Ω has to be Levi q-pseudoconvex at p. □

In order to establish a converse statement of that in Proposition 4.6.1, we need the following lemma.

Lemma 4.6.3 *Let $q \in \{0, \dots, n-1\}$ and let ψ be a $\mathcal{C}^2$-smooth strictly q-plurisubharmonic function on an open set U in $\mathbb{C}^n$. Assume that U contains two compact sets K and L which fulfill the following properties:*

(1) $K, L \subset \{z \in U : \psi(z) \leq 0\}$

(2) $L \cap \{z \in U : \psi(z) = 0\} = \emptyset$

(3) $K \cap \{z \in U : \psi(z) = 0\} \neq \emptyset$

Under these conditions, there exist a point $z_0 \in bK$, a neighborhood $V \Subset U$ of z_0 and a $\mathcal{C}^2$-smooth strictly q-plurisubharmonic function φ on V satisfying:

(a) $K, L \subset \{z \in V : \varphi(z) \leq 0\}$

(b) $L \cap \{z \in V : \varphi(z) = 0\} = \emptyset$

(c) $K \cap \{z \in V : \varphi(z) = 0\} = \{z_0\}$

(d) $\nabla\varphi \neq 0$ *on* $\{z \in V : \varphi(z) = 0\}$

In other words, the set $G := \{z \in V : \varphi(z) < 0\}$ is strictly q-pseudoconvex in V, contains L, and K touches bG from the inside of G only at the point z_0.

Proof. We proceed similarly as in the proof of Proposition 3.2 in [HST14]. Let $\delta > 0$ and $U_\delta := B_{1/\delta}(0) \cap \{z \in U : d(z, bU) > \delta\}$. We choose $\delta > 0$ so small that the preassumptions (1) to (3) of this lemma still hold if we replace U by $V := U_\delta$.

Let $B := B_\delta(0)$ and consider the function $f : B \to \psi(V)$ defined by $f(w) := \max_{z \in K} \psi(z + w)$. Pick a point $p \in K \cap \{z \in V : \psi(z) = 0\}$. Since ψ is strictly q-plurisubharmonic, it follows from the local maximum principle (see Proposition 3.3.2 (11)) that $\{\psi > 0\} \cap W$ is not empty for any neighborhood W of p. Hence, since p belongs to K and since $f(0) = \psi(p) = 0$, the image $f(B)$ contains a non-empty open interval I around 0. Since $f(B)$ lies in $\psi(V)$, Sard's theorem implies that there exists a regular value $f(w_0)$ inside I which is so close to $\psi(p) = 0$ that the conditions (1) to (3) are still valid for the function $\psi_0(z) := \psi(z + w_0) - f(w_0)$ instead of ψ. Notice that 0 is now a regular value for ψ_0.

Let z_0 be a point in K with $f(w_0) = \psi(z_0 + w_0)$, so that $\psi_0(z_0) = 0$. For $\varepsilon > 0$, we define $\varphi(z) := \psi_0(z) + \varepsilon\|z - z_0\|_2^2$. Then it is easy to see that $K \cap \{z \in V : \varphi(z) = 0\}$ only contains the point z_0, so we obtain property (c). Now if $\varepsilon > 0$ is small enough, then the preassumptions (1) and (2) still hold for the function φ instead of ψ. Hence, the function φ fulfills also the properties (a) and (b). Finally, by the choice of $f(w_0)$, zero is a regular value for φ, so we also gain the property (d). □

We close this section by giving the second duality theorem which is a converse statement to Theorem 4.6.1. It was suggested by N. V. Shcherbina.

Theorem 4.6.4 *Let Ω be a domain in $\mathbb{C}^n$ which is not q-pseudoconvex. Then there exist a point $p \in b\Omega$, a neighborhood V of p and a strictly Levi $(n-q-2)$-pseudoconvex set G in V such that the set $V \setminus \Omega$ is contained in $G \cup \{p\}$ and $\{p\} = bG \cap b\Omega \cap V$, i.e., $V \setminus G$ touches bG from the inside of G only at p.*

Proof. Since Ω is not q-pseudoconvex, there exists a $(q+1, n-q-1)$-Hartogs figure (H, P) and a biholomorphic mapping F on $\Delta := \Delta_1^n(0)$ onto its image such that $H = F(H_e)$ lies in Ω, but $P = F(\Delta_1^n(0))$ is not contained entirely in Ω for the Euclidean Hartogs figure

$$H_e = \left(\Delta_1^{q+1} \times \Delta_r^{n-q-1}\right) \cup \left(A_{R,1}^{q+1} \times \Delta_1^{n-q-1}\right) \ \subset \ \mathbb{C}_z^{q+1} \times \mathbb{C}_w^{n-q-1}.$$

By shrinking Δ if necessary, we can assume that F is defined on a neighborhood of the closure of Δ. We set $M := F^{-1}(\mathbb{C}^n \setminus \Omega) \cap \overline{\Delta}$. Since $\Phi(H_e)$ lies in Ω, we can find appropriate numbers $\alpha, \beta \in (0,1)$ with $\alpha < \beta$ such that

$$K_0 := \left(\overline{\Delta_1^{q+1} \times \Delta_\alpha^{n-q-1}}\right) \cap M \quad \text{and} \quad L_0 := \left(\overline{\Delta_1^{q+1} \times A_{\beta,1}^{n-q-1}}\right) \cap M$$

both are not empty. Recall that $\|w\|_\infty = \max_{j=1,\ldots,n-q-1} |w_j|$ and consider the function $u(w) := -\log\|w\|_\infty$. By the assumptions made on H and P, we can find a large enough number $c \in \mathbb{R}$ such that

$$M \subset \ D_c(u) := \{(z,w) \in \overline{\Delta} : u(w) < c\}. \tag{4.12}$$

Let $k \in \mathbb{N}$ and define the function u_k by

$$u_k(w) := -\frac{1}{k}\log\|(w_1^k, \ldots, w_{n-q-1}^k)\|_2 + \frac{1}{k}\|w\|_2^2.$$

Then by Proposition 4.2.3 the function u_k is $\mathcal{C}^\infty$-smooth and strictly $(n-q-2)$-plurisubharmonic on $\mathbb{C}_w^{n-q-1} \setminus \{0\}$. Moreover, the sequence $(u_k)_{k\in\mathbb{N}}$ converges to u uniformly on compact sets in $\mathbb{C}_w^{n-q-1} \setminus \{0\}$. Therefore, and in view of property (4.12), we can pick an integer k_0 so large that M lies in $D_c(u_k) := \{(z,w) \in \overline{\Delta} : u_k(w) < c\}$ for every $k \geq k_0$. Define

$$c_k := \inf\big\{a \in \mathbb{R} : M \subset D_a(u_k)\big\}.$$

Now we fix an even larger $k \geq k_0$ so that $L_0 \cap D_{c_k}(u_k)$ is empty. Then it is easy to see that K_0 intersects $\{(z,w) \in \overline{\Delta} : u_k(w) = c_k\}$ in a point $\zeta_0 \in \Delta$. Finally, we set $U := F^{-1}(\Delta)$, $K := F^{-1}(K_0)$, $L := F^{-1}(L_0)$ and $\psi := u_k \circ F^{-1}$ and verify that the conditions (1) to (3) in Lemma 4.6.3 all are satisfied. Thus, it follows from this lemma that there are a point p in $b\Omega$, a neighborhood V of p and a strictly $(n-q-2)$-plurisubharmonic function on V such that the set $G := \{z \in V : \varphi(z) < 0\}$ is the desired strictly Levi $(n-q-2)$-pseudoconvex set in V, whose boundary bG shares only a single point with $b\Omega$ in V. □

4.7 q-Pseudoconcave graphs

In this section, we will analyze whether a submanifold or the graph of a continuous function admits a local regular complex foliation under the condition that its complement is q-pseudoconvex. These considerations naturally generalize one of the classical Hartogs' theorems (see, e.g., Chapter III.42, Theorem 2 in [Sha92]).

Theorem 4.7.1 (Hartogs, 1909) *A continuous function $f : G \to \mathbb{C}_\zeta$ is holomorphic on a domain $G \subset \mathbb{C}_z^n$ if and only if the complement of its graph $\Gamma(f) = \{(z, f(z)) : z \in G\}$ is pseudoconvex in $G \times \mathbb{C}_\zeta$.*

In order to simplify our notation, we introduce a generalized version of concavity.

Definition & Remark 4.7.2 Let $q \in \{0, \dots, N\}$ and let S be a closed subset of an open set Ω in $\mathbb{C}^N$.

(1) We say that S is *(Hartogs) q-pseudoconcave* in Ω if $\Omega' := \Omega \setminus S$ is (Hartogs) q-pseudoconvex in Ω, i.e., for every point p in bS there exists a ball B in Ω such that $B \cap \Omega'$ is (Hartogs) q-pseudoconvex.

(2) In view of Theorem 4.3.2 the set S is q-pseudoconcave in Ω if and only if it is Hartogs $(N-q-1)$-pseudoconcave in Ω. For the sake of a better presentation, we shall prefer only in this section the notion of Hartogs q-pseudoconcavity rather than q-pseudoconcavity.

Using the duality theorems of the previous section, we obtain a first relation of foliated sets and q-pseudoconcavity.

Proposition 4.7.3 *Let $q \in \{1, \ldots, N-1\}$ and let S be a closed subset of an open set Ω in $\mathbb{C}^N$. Assume that the boundary $b_\Omega S$ of S in Ω is locally filled by q-dimensional analytic sets, i.e., for every point p in $b_\Omega S$ there is a neighborhood W of p in Ω such that for each point z in $S \cap W$ there exists a q-dimensional analytic subset A_z of W with $z \in A_z \subset S$. Then S is Hartogs q-pseudoconcave in Ω.*

Proof. Assume that the statement is false. Then according to Theorem 4.6.4, there exist a boundary point p of S in Ω, a neighborhood V of p and a strictly $(q-1)$-pseudoconvex set G in V such that $S \cap V$ touches bG from the inside of G exactly in p. In view of Remark 4.5.7 (2), we can construct a strictly $(q-1)$-plurisubharmonic function ψ defined on some neighborhood U of p in $V \cap W$ which is defining for G at p. By the assumption made on $b_\Omega S$, there are a neighborhood W of p and a q-dimensional analytic subset A of W with $p \in A \subset S$. But then $\psi(p) = 0$ and $\psi < 0$ on $A \cap U$, which contradicts the local maximum principle (see Theorem 3.8.5). Therefore, S has to be Hartogs q-pseudoconcave in Ω. □

We present a converse statement on the complex foliation of Hartogs q-pseudoconcave CR-submanifolds and, therefor, need to extend Definition 4.5.5 as follows.

Definition 4.7.4 Let $\Gamma = \{\varphi_1 = \ldots = \varphi_r = 0\}$ be a $\mathcal{C}^2$-smooth submanifold in $\mathbb{C}^N$.

(1) The holomorphic tangent space $H_p\Gamma$ to Γ at some point p in Γ is given by

$$H_p\Gamma := \bigcap_{j=1}^{r} \Big\{ X \in \mathbb{C}^N : (\partial\varphi_j(p), X) = \sum_{\ell=1}^{N} \frac{\partial \varphi_j}{\partial z_\ell}(p) X_\ell = 0 \Big\}.$$

(2) If the complex dimension of $H_p\Gamma$ has the same value d at each point p in Γ, then we say that Γ is a *CR-submanifold* and call d the *CR-dimension* of Γ.

(3) The *Levi null space* of Γ at p is the set

$$\mathcal{N}_p := \bigcap_{j=1}^{r} \Big\{ X \in H_p\Gamma : \mathscr{L}_{\varphi_j}(p)(X,Y) = 0 \text{ for every } Y \in H_p\Gamma \Big\}.$$

Proposition 4.7.5 *Let $\Gamma = \{\varphi_1 = \ldots = \varphi_r = 0\}$ be a real $\mathcal{C}^2$-smooth CR-submanifold of some open set Ω in $\mathbb{C}^N$ of codimension $r \in \{1,\ldots,2N-1\}$ and CR-dimension $q \in \{1,\ldots,n-[n/2]\}$. Assume further that Γ is Hartogs q-pseudoconcave in Ω. Then it is locally foliated by complex q-dimensional submanifolds.*

Proof. Since the Levi null space lies inside the holomorphic tangent space to Γ, it is clear that its complex dimension does not exceed q. We claim that the complex dimension of $\mathcal{N}_p$ is equal to q for each point $p \in \Gamma$, so that $\mathcal{N}_p$ coincides with $H_p\Gamma$.

In order to get a contradiction, suppose that there is a point p in Γ such that $\mathcal{N}_p$ is a proper subspace of $H_p\Gamma$. This implies that there is an index j_0 in $\{1,\ldots,r\}$ and a vector X_0 in $H_p\Gamma$ such that $\widetilde{\mathscr{L}}_{\varphi_{j_0}}(p,X_0) \neq 0$. Indeed, a priori, if $\mathcal{N}_p \subsetneq H_p\Gamma$, there exist two vectors X' and Y' in $H_p\Gamma$ such that

$$\widetilde{\mathscr{L}}_{\varphi_{j_0}}(p)(X',Y') \neq 0.$$

If $\widetilde{\mathscr{L}}_{\varphi_{j_0}}(p,X') \neq 0$ or $\widetilde{\mathscr{L}}_{\varphi_{j_0}}(p,Y') \neq 0$, we are done and proceed by picking $X_0 = X'$ or, respectively, $X_0 = Y'$. Otherwise, if $\widetilde{\mathscr{L}}_{\varphi_{j_0}}(p,X')$ and $\widetilde{\mathscr{L}}_{\varphi_{j_0}}(p,Y')$ both vanish, we can choose an appropriate complex number ν which satisfies

$$\widetilde{\mathscr{L}}_{\varphi_{j_0}}(p,X'+\nu Y') = 2\mathrm{Re}\big(\nu\widetilde{\mathscr{L}}_{\varphi_{j_0}}(p)(X',Y')\big) \neq 0.$$

Then we continue with $X_0 := X'+\nu Y'$. Now without loss of generality we can assume that $j_0 = 1$ and $\widetilde{\mathscr{L}}_{\varphi_1}(p,X_0) > 0$. For a positive constant μ we define another function

$$\varphi := \varphi_1 + \mu\sum_{j=1}^{r}\varphi_j^2.$$

Since we can assume that the gradients $\nabla\varphi_1,\ldots,\nabla\varphi_r$ do not vanish at p, there is a neighborhood U of p such that the set $S := \{z \in U : \varphi(z) = 0\}$ is a real hypersurface containing $\Gamma \cap U$, so that $H_p\Gamma$ becomes a subspace of H_pS.

Moreover, for $X \in H_p\Gamma$ we can easily compute the Levi form of φ at p,

$$\widetilde{\mathscr{L}_\varphi}(p,X) = \widetilde{\mathscr{L}_{\varphi_1}}(p,X) + 2\mu \underbrace{\sum_{j=1}^{r} |(\partial\varphi_j(p),X)|^2}_{=:R(p,X)}. \tag{4.13}$$

We assert that H_pS contains an $(N-q)$-dimensional subspace E on which $\widetilde{\mathscr{L}_\varphi}(p,\cdot)$ is positive. To see this, consider the complex normal space $N_p\Gamma$ to $H_p\Gamma$ in H_pS,

$$N_p\Gamma := \{Y \in H_pS : \sum_{\ell=1}^{N} Y_\ell X_\ell = 0 \ \text{ for every } X \in H_p\Gamma\}.$$

Observe that $N_p\Gamma$ has dimension $d := N-q-1$ and choose a basis $Y_1,\ldots,Y_d$ of $N_p\Gamma$. Let E be the complex span of the vectors X_0 from above and $Y_1,\ldots,Y_d$. Since X_0 belongs to $H_p\Gamma$, but $Y_1,\ldots,Y_d$ do not, the dimension of E equals $N-q$.

We set $E_0 := \{Z \in E : \|Z\|_2 = 1\}$ and $M := \{Z \in E_0 : \widetilde{\mathscr{L}_{\varphi_1}}(p,Z) \leq 0\}$.

If M is empty, then $\widetilde{\mathscr{L}_\varphi}(p,\cdot)$ is positive on E and we can put $\mu = 0$.

If M is not empty, notice first that, if Z lies in M, then $R(p,Z) > 0$ (recall the equation (4.13) for the definition of $R(p,Z)$). Otherwise Z belongs to $H_p\Gamma$ and, therefore, it is a multiple of X_0, i.e., $Z = \lambda X_0$ for some complex number λ. But then $\widetilde{\mathscr{L}_{\varphi_1}}(p,Z) = |\lambda|^2\widetilde{\mathscr{L}_{\varphi_1}}(p,X_0) > 0$ and Z lies in M at the same time, which is absurd. Hence, $R(p,Z) > 0$ for every vector Z in M. Since E_0 is compact and Γ is $\mathcal{C}^2$-smooth, we can find constants $c_0 > 0$ and $c_1 > 0$ such that $R(p,Z) \geq c_0$ for every Z in M and $\widetilde{\mathscr{L}_{\varphi_1}}(p,Z) \geq -c_1$ for every Z in E_0. Now we can choose μ so large that $-c_1 + \mu c_0 > 0$ in order to obtain that $\widetilde{\mathscr{L}_\varphi}(p,Z) > 0$ for each Z in E_0. Since $\widetilde{\mathscr{L}_\varphi}(p,\lambda X) = |\lambda|^2\widetilde{\mathscr{L}_\varphi}(p,X)$ for every X in E and λ in $\mathbb{C}$, we have that $\widetilde{\mathscr{L}_\varphi}(p,\cdot)$ is positive on $E \setminus \{0\}$.

Therefore, in both cases, the Levi form $\widetilde{\mathscr{L}_\varphi}$ at p is positive definite on the $(N-q)$-dimensional space E. Hence, $\{\varphi < 0\}$ is strictly $(q-1)$-pseudoconvex at p. But then, in view of Theorem 4.6.1, the submanifold Γ cannot be Hartogs q-pseudoconcave near p, which is a contradiction. Finally, we can conclude that $\mathcal{N}_p = H_p\Gamma$. By assumption, these two spaces have constant dimension q on Γ, so Freeman's Theorem 1.1 in [Fre74] implies that Γ admits a local foliation by complex q-dimensional submanifolds. □

The next statement is a generalization of Hartogs' theorem to certain graphs of continuous functions. This result is due to N. V. Shcherbina (see [Shc93]) and is based on the work of E. Bedford, B. Gaveau, W. Klingenberg and N. G. Kruzhilin (see [BG83], [BK91] and [Kru91]). Further generalizations are due to E. M. Chirka and Shcherbina (see [CS95] and [Chi01]). Notice that, in contrast to Definition 3.9.1, by a *complex foliation* of a continuous graph we simply mean a disjoint union of complex submanifolds filling locally this graph.

Theorem 4.7.6 (Shcherbina, 1993) *Let G be a domain in $\mathbb{C}_z \times \mathbb{R}_u$. Then the graph of a continuous function $f : G \to \mathbb{R}_v$ on a is locally foliated by complex curves if and only if $\Gamma(f)$ is Hartogs 1-pseudoconcave in $G \times \mathbb{R}_v$ (i.e., its complement is pseudoconvex). More precisely, a local foliation of $\Gamma(f)$ is given by a set of pairwise disjoint complex curves $\{\gamma_\alpha\}_{\alpha\in I}$ which are graphs $\gamma_\alpha = \Gamma(g_\alpha)$ of holomorphic functions $g_\alpha : D_\alpha \to \mathbb{C}_w$, where $\{D_\alpha\}_{\alpha\in I}$ is a collection of simply connected domains D_α lying in the projection of G into $\mathbb{C}_z$. Moreover, there exists a disc Δ in $\mathbb{C}_z$ which is contained in Δ_α for each $\alpha \in I$ and which does not depend on the indexes $\alpha \in I$.*

In what follows, we intend to generalize the theorems of Shcherbina and Hartogs to the subsequent setting.

Setting 4.7.7 Fix integers $n \geq 1$ and $k, p \geq 0$ such that $N = n + k + p \geq 2$. Then $\mathbb{C}^N$ splits into the product

$$\mathbb{C}^N_{z,w,\zeta} = \mathbb{C}^n_z \times \mathbb{C}^k_w \times \mathbb{C}^p_\zeta = \mathbb{C}^n_z \times (\mathbb{R}^k_u + i\mathbb{R}^k_v) \times \mathbb{C}^p_\zeta,$$

where $w = u + iv$. Let G be an open set in $\mathbb{C}^n_z \times \mathbb{R}^k_u$ and let $f = (f_v, f_\zeta)$ be continuous on G with image in $\mathbb{R}^k_v \times \mathbb{C}^p_\zeta$. Then the graph of f is given by

$$\Gamma(f) = \{(z, w, \zeta) \in \mathbb{C}^n_z \times \mathbb{C}^k_w \times \mathbb{C}^p_\zeta : \ (z, u) \in G, \ (v, \zeta) = f(z, u)\}.$$

We are interested in the question whether Γ admits a local foliation by complex submanifolds. In this context, we first have to study the q-pseudoconcavity of the graph of f.

Lemma 4.7.8 *Recall our Setting 4.7.7. Furthermore, pick another integers $m \in \{1, \dots, n\}$ and $r \in \{0, \dots, p\}$ with $k + r \geq 1$. For $\mu_1, \dots, \mu_r \in \{1, \dots, p\}$ and $\mu_1 < \ldots < \mu_r$, we divide the coordinates of ζ into $\zeta' = (\zeta_{\mu_1}, \dots, \zeta_{\mu_r})$ and into the remaining coordinates $\zeta'' = (\zeta_{\mu_j} : j \in \{1, \dots p\} \setminus \{\mu_1, \dots, \mu_r\})$ which we assume to be ordered by their index μ_j, as well. Finally, let π be a complex*

m-dimensional plane in $\mathbb{C}^n_z$. We set $M = m + k + r$ and $\mathbb{C}^M_\bullet := \pi \times \mathbb{C}^k_w \times \mathbb{C}^r_{\zeta'}$, $G_\bullet := G \cap (\pi \times \mathbb{R}^k_u)$ and $f_\bullet := (f_v, f_{\zeta'})|G_\bullet$.

If the graph $\Gamma(f)$ is Hartogs n-pseudoconcave in $G \times \mathbb{R}^k_v \times \mathbb{C}^p_\zeta$, then the graph $\Gamma(f_\bullet)$ is Hartogs m-pseudoconcave in $G_\bullet \times \mathbb{R}^k_v \times \mathbb{C}^r_{\zeta'}$.

Proof. Since the generalized pseudoconcavity is a local property, after shrinking G if necessary and after a biholomorphic change of coordinates we can assume without loss of generality that $\pi = \{0\}^{n-m}_{z'} \times \mathbb{C}^m_{z''} \subset \mathbb{C}^n_z$, where $z' = (z_1, \ldots, z_{n-m})$ and $z'' = (z_{n-m+1}, \ldots, z_n)$, and that the ζ-coordinates are ordered in such a way that $\zeta' = (\zeta_1, \ldots, \zeta_r)$ and $\zeta'' = (\zeta_{r+1}, \ldots, \zeta_p)$.

Assume that $\Gamma(f_\bullet)$ is not Hartogs m-pseudoconvex in $G_\bullet \times \mathbb{R}^k_v \times \mathbb{C}^r_{\zeta'}$ and recall that $M = m + k + r$. Then in view of Theorem 4.5.3 (3) and Theorem 4.3.2, there are a point p in $\Gamma(f_\bullet)$ and a ball $B = B_\varrho(p)$ in $\mathbb{C}^M$ such that the set $(\mathbb{C}^M \setminus \Gamma(f_\bullet)) \cap B$ is not $(M - m - 1) = (k + r - 1)$-pseudoconvex. Since B is pseudoconvex, according to Theorem 4.3.2 (8) there is a family $\{A_t\}_{t\in[0,1]}$ of $(k+r)$-dimensional analytic sets A_t in $\mathbb{C}^M$ which depends continuously on t and which fulfills the following properties:

- The closure of the union $\displaystyle\bigcup_{t\in[0,1]} A_t$ is compact.
- For every $t \in [0, 1)$ the intersection $\overline{A_t} \cap \Gamma(f_\bullet)$ is empty.
- $bA_1 \cap \Gamma(f_\bullet)$ is empty, as well.
- The set A_1 touches $\Gamma(f_\bullet)$ at a point $p_0 = (z_0, w_0, \zeta'_0)$, where $z_0 = (z'_0, z''_0) = (0, z''_0)$ and $w_0 = u_0 + iv_0$.

Given some positive number ρ, consider the $(k + p)$-dimensional analytic sets

$$S_t := \{0\}^{n-m} \times A_t \times \Delta^{p-r}_\rho(f_{\zeta''}(z_0, u_0)) \ \subset\ \mathbb{C}^N.$$

It is easy to verify that the family $\{S_t\}_{t\in[0,1]}$ of $(k+p)$-dimensional analytic sets violates the property (8) of Theorem 4.3.2. According to Theorem 4.3.2 (1), $\Gamma(f)$ cannot be Hartogs n-pseudoconcave, which is a contradiction to the assumption on $\Gamma(f)$. Hence, $\Gamma(f_\bullet)$ has to be Hartogs m-pseudoconcave. □

We are now able to generalize the above presented theorems of Hartogs and Shcherbina to lower dimensional continuous graphs. The idea of the proof of the following theorem is based on notes of Shcherbina and uses results by Hartogs, Shcherbina and Chirka. In remains an open question whether the next statement holds true in the case of $k \geq 3$.

Theorem 4.7.9 *Let n, k, p be integers with $n \geq 1$, $p \geq 0$ and $k \in \{0, 1\}$ such that $N = n + k + p \geq 2$. Let G be a domain in $\mathbb{C}^n_z \times \mathbb{R}^k_u$ and let $f : G \to \mathbb{R}^k_v \times \mathbb{C}^p_\zeta$ a continuous function such that $\Gamma(f)$ is Hartogs n-pseudoconcave. Then $\Gamma(f)$ is locally the disjoint union of n-dimensional complex submanifolds.*

Proof. The statement is of local type, so we can assume that B is an open ball and that $\Gamma(f)$ is bounded. We continue by separating the problem into the subsequent cases.

Case $\mathbf{n \geq 1,\ k = 0,\ p = 1}$. This is the classical Hartogs' theorem which we already stated in Theorem 4.7.1 above. Its proof can be found in [Sha92] (see Chapter III.42, Theorem 2).

Case $\mathbf{n \geq 1,\ k = 0,\ p \geq 1}$. For each $j \in \{1, \ldots, p\}$ the set $\Gamma(f_{\zeta_j})$ is Hartogs n-pseudoconcave by Lemma 4.7.8. Therefore, the function f_j is holomorphic by Hartogs theorem 4.7.1. Therefore, $f = (f_1, \ldots, f_p)$ is holomorphic, and so $\Gamma(f)$ is a complex hypersurface.

Case $\mathbf{n = 1,\ k = 1,\ p = 0}$. The proof of this case is far away from being obvious, so we do not repeat it here. For the interested reader,we refer to Shcherbina's original article [Shc93].

Case $\mathbf{n \geq 1,\ k = 1,\ p = 0}$. This case has been treated by Chirka in [Chi01].

Case $\mathbf{n = 1,\ k = 1,\ p = 1}$. By Lemma 4.7.8 the graph $\Gamma(f_v)$ is Hartogs 1-pseudoconcave. According to Theorem 4.7.6, it is foliated by a family of holomorphic curves $\{\gamma_\alpha\}_{\alpha \in I}$ which are graphs of holomorphic functions g_α which are all defined on a disc D in $\mathbb{C}_z$ which does not depend on the indexes $\alpha \in I$. Denote by π_z the standard projection of points in $\mathbb{C}^2_{z,w}$ into $\mathbb{C}_z$. We define another curves f^α_ζ by the assignment

$$\gamma_\alpha \ni t \ \mapsto \ f^\alpha_\zeta(t) := f_\zeta\big(\pi_z(t), \mathrm{Re}(g_\alpha)(\pi_z(t))\big).$$

Since for each $\alpha \in I$ the curve γ_α is a graph $\Gamma(g_\alpha)$, the function $f^\alpha_\zeta : \gamma_\alpha \to \mathbb{C}_\zeta$ is well-defined, and its graph is given by $\Gamma(f^\alpha_\zeta) = \Gamma(f|\pi_{z,u}(\gamma_\alpha))$. Here, $\pi_{z,u}$

means the standard projection of $\mathbb{C}^2_{z,w}$ to $\mathbb{C}_z \times \mathbb{R}_u$. We claim that the curve f^α_ζ is holomorphic.

Suppose that there is a curve $f^{\alpha_0}_\zeta$ which is not holomorphic in a neighborhood of a point $(z_0, w_0) \in \gamma_{\alpha_0}$. After a local holomorphic change of coordinates, we can assume that $\gamma_{\alpha_0} = \Delta_r(z_0) \times \{w = 0\}$, where $\Delta_r(z_0) \Subset D$ is a disc in $\mathbb{C}_z$ centered in z_0. After a reparametrization we can arrange that $\alpha_0 = 0$ and $(-1, 1) \subset I$. Since the curve γ_0 is of the form $\Delta_r(z_0) \times \{w = 0\}$ near z_0, we can treat f^0_ζ as a function $f^0_\zeta : \Delta_r(z_0) \to \mathbb{C}_\zeta$. Since we assumed that f^0_ζ is not holomorphic, in view of Hartogs' theorem 4.7.1 the set $\Gamma(f^0_\zeta)$ is not Hartogs 1-pseudoconcave in $\mathbb{C}^2_{z,\zeta}$. By Theorem 4.6.4, there exist a point $p_1 = (z_1, \zeta_1) \in \Gamma(f^0_\zeta)$, a small enough open neighborhood V of p_1 in $\mathbb{C}^2_{z,\zeta}$, a $\mathcal{C}^2$-smooth strictly plurisubharmonic function $\varrho_1 = \varrho_1(z, \zeta)$ on V with $\nabla\varrho_1 \neq 0$, and radii $\sigma, r', r'' > 0$ with $r'' < r' < r$ such that

$$\Delta_{r'} \times \Delta_\sigma \Subset V, \qquad \Gamma(f^0_\zeta|\overline{\Delta_{r'}}) \subset \overline{\Delta_{r'} \times \Delta_\sigma},$$

$$\Gamma(f^0_\zeta|\overline{\Delta_{r'}}) \subset \{\varrho_1 \le 0\}, \qquad \Gamma(f^0_\zeta|\overline{\Delta_{r'}}) \cap \{\varrho_1 = 0\} = \{(z_1, \zeta_1)\}, \tag{4.14}$$

$$\text{and} \quad \Gamma(f^0_\zeta|\overline{A_{r',r''}}) \cap \{\varrho_1 = 0\} = \emptyset, \tag{4.15}$$

where each disc Δ_s mentioned above is assumed to be centered in z_0 and $A_{r',r''} := \Delta_{r''} \backslash \overline{\Delta_{r'}}$. For $\alpha \in (-1, 1)$ we set $\gamma'_\alpha := \Gamma(g_\alpha|\overline{\Delta_{r'}})$ and $\Gamma'_\alpha := \Gamma(f^\alpha_\zeta|\gamma'_\alpha)$. Since f is continuous and since the family $\{\gamma_\alpha\}_{\alpha\in I}$ depends continuously on α, it follows from (4.15) that there is a number $\tau \in (0, 1)$ such that

$$K := \bigcup_{\alpha\in[-\tau,\tau]} \Gamma'_\alpha \ \subset \ \Delta_{r'} \times \mathbb{C}_w \times \Delta_\sigma$$

$$\text{and } \varrho_1 < 0 \text{ on } \Gamma'_\alpha \cap \left(\overline{A_{r',r''}} \times \mathbb{C}^2_{w,\zeta}\right) \text{ for every } \alpha \in [-\tau, \tau], \tag{4.16}$$

where ϱ_1 is now considered as a function defined on $\{(z, w, \zeta) \in \mathbb{C}^3 : (z, \zeta) \in V\}$.

Since the curves in $\{\gamma_\alpha\}_{\alpha\in I}$ are holomorphic, the set

$$A := \left(\gamma_{-\tau} \cup \gamma_{\alpha_0} \cup \gamma_\tau\right) \cap (\Delta_r \times \mathbb{C}_w)$$

is a closed analytic subset of the pseudoconvex domain $\Delta_r \times \mathbb{C}_w$. Let h be a holomorphic function on A defined by $h \equiv 0$ on $\gamma_{\pm\tau}$ and $h \equiv 1$ on γ_{α_0}. In view of Theorem 3.11.4, there exists a holomorphic extension $\hat{h}$ of h into the whole of $\Delta_r \times \mathbb{C}_w$. Then the function $\varrho_2(z,w) := \log|\hat{h}(z,w)|$ is plurisubharmonic on $\Delta_r \times \mathbb{C}_w$ and satisfies $\varrho_2 \equiv -\infty$ on $\gamma_{\pm\tau}$. Now for $\varepsilon > 0$ we define

$$\psi_0(z,w,\zeta) := \varrho_1(z,\zeta) + \varepsilon\varrho_2(z,w),$$

where ϱ_1 is the defining function from above. By the inequality (4.16) and the properties of ϱ_2, for a sufficiently small $\varepsilon > 0$ we obtain that

$$\psi_0 < 0 \text{ on } L := \bigcup_{\alpha\in[-\tau,\tau]} \big(\Gamma'_\alpha \cap \big(\overline{A_{r',r''}} \times \mathbb{C}^2_{w,\zeta}\big)\big) \cup \Gamma'_\tau \cup \Gamma'_{-\tau}. \tag{4.17}$$

Recall the point (z_1,ζ_1) from (4.14). By the choice of (z_1,ζ_1) and since $(z_1,0) \in \gamma_{\alpha_0}$, we have that $\varrho_1(z_1,\zeta_1) = 0$, $\varrho_2(z_1,0) = 0$ and, therefore, $\psi_0(z_1,0,\zeta_1) = 0$. Since $(z_1,0,\zeta_1)$ belongs to K, it follows from the inequality (4.17) that ψ_0 attains a non-negative maximal value on K outside L. Since ψ_0 is plurisubharmonic on a neighborhood of K, by using Theorem 3.1.7 and Remark 3.4.2 (3) we can assume without loss of generality that ψ_0 is $\mathcal{C}^\infty$-smooth and strictly plurisubharmonic on a neighborhood of K, satisfies the property (4.17) and still attains its maximum on K outside L.

Now it is easy to verify that K, L and $\psi := \psi_0 - \max_K \psi_0$ fulfill all the conditions (1) to (3) of Lemma 4.6.3. Thus, there exist a point p_2 in $K \setminus L$, a neighborhood U of p_2 containing K and L and a $\mathcal{C}^2$-smooth strictly plurisubharmonic function φ on U so that $G := \{(z,w,\zeta) \in U : \varphi(z,w,\zeta) < 0\}$ is strictly pseudoconvex in U, $\varphi < 0$ on L, $\varphi \le 0$ on K and $\varphi(z,w,\zeta)$ vanishes on K if and only if $(z,w,\zeta) = p_2$. Since G is strictly pseudoconvex at p_2, we derive from Theorem 4.6.1 that the graph $\Gamma(f)$ can not be 1-pseudoconcave, which is a contradiction to the assumption made on $\Gamma(f)$. As a conclusion, the curves in $\{\Gamma(f^\alpha_\zeta)\}_{\alpha\in I}$ have to be holomorphic. This leads to the desired local complex foliation of $\Gamma(f)$.

Case $\mathbf{n \ge 1,\ k = 1,\ p = 1}$. According to Lemma 4.7.8 the graph $\Gamma(f_v)$ is Hartogs n-pseudoconcave in $B \times \mathbb{R}_v$, where B is a ball in $\mathbb{C}^n_z \times \mathbb{R}_u$. Hence by Chirka's result (see the case $n \ge 1$, $k = 1$, $p = 0$), the graph $\Gamma(f_v)$ is foliated by a family $\{A_\alpha\}_{\alpha\in I}$ of holomorphic hypersurfaces A_α. For $\alpha \in I$ define the function

$$f^\alpha_\zeta : A_\alpha \to \mathbb{C}_\zeta \quad \text{by} \quad f^\alpha_\zeta = f_\zeta|\pi_{z,u}(A_\alpha)$$

and identify $\Gamma(f^\alpha_\zeta)$ with $\Gamma(f|\pi_{z,u}(A_\alpha))$. Suppose that some function $f^{\alpha_0}_\zeta$ is not holomorphic. Then by Hartogs' theorem of separate holomorphicity there is a complex one-dimensional curve σ_{α_0} in A_{α_0} on which $f_\zeta^{\alpha_0}$ is not holomorphic near a point $p_0 \in \sigma_{\alpha_0}$. After a change of coordinates we can assume that $p_0 = 0$, $f_\zeta^{\alpha_0}(0) = 0$ and $\sigma_{\alpha_0} = \Delta \times \{z_2 = \ldots = z_n = w = 0\}$ in a neighborhood of 0, where Δ is the unit disc in $\mathbb{C}_{z_1}$. We set $\mathbb{L} := \mathbb{C}_{z_1} \times \{0\}^{n-1}$. By Lemma 4.7.8 the graph $\Gamma(f_\bullet)$ of $f_\bullet := f|(B \cap (\mathbb{L} \times \mathbb{R}_u))$ is Hartogs 1-pseudoconcave in $\mathbb{C}^3_{z_1,w,\zeta}$. Thus, in view of the considered above case $n = k = p = 1$, the graph $\Gamma(f_\bullet)$ is foliated by complex curves of the form

$$(f_\bullet)_\zeta^\beta : \gamma_\beta \to \mathbb{C}_\zeta \quad \text{with} \quad (f_\bullet)_\zeta^\beta = (f_\bullet)_\zeta|\pi_{z,u}(\gamma_\beta),$$

where $\{\gamma_\beta\}_{\beta \in I}$ is a family of holomorphic curves of a foliation of $\pi_{z_1,w}(\Gamma(f_\bullet))$. From the uniqueness of the foliation on $\pi_{z,w}(\Gamma(f_\bullet))$ we deduce that $\pi_{z_1,w}(\sigma_{\alpha_0})$ must coincide at least locally with a curve γ_{β_0} containing 0. Hence, in some neighborhood of 0 we have that $\gamma_{\beta_0} = \Delta \times \{0\}$ and therefore

$$\begin{aligned} f_\zeta^{\alpha_0}|\sigma_0 &= f_\zeta|(\Delta \times \{z_2 = \ldots = z_n = 0\} \times \{u = 0\}) \\ &= (f_\bullet)_\zeta|(\Delta \times \{u = 0\}) = (f_\bullet)_\zeta|\pi_{z,u}(\gamma_{\beta_0}) = (f_\bullet)_\zeta^{\beta_0}. \end{aligned}$$

This means that $f_\zeta^{\alpha_0}$ has to be holomorphic on a neighborhood of 0 in σ_{α_0}, which is a contradiction to the choice of $f_\zeta^{\alpha_0}$ and σ_{α_0}. Hence, $\{\Gamma(f_\zeta^\alpha)\}_{\alpha \in I}$ is the desired foliation of $\Gamma(f)$.

Case $\mathbf{n = 1, k = 1, p \geq 1}$. We derive from Lemma 4.7.8 that the graph $\Gamma(f_v)$ is Hartogs 1-pseudoconcave. Then it follows from Shcherbina's theorem that the graph $\Gamma(f_v)$ is foliated by the family $\{\gamma_\alpha\}_{\alpha \in I}$ of holomorphic curves γ_α. Define similarly to the previous cases for $\alpha \in I$ the mapping

$$f_\zeta^\alpha = (f_{\zeta_1}^\alpha, \ldots, f_{\zeta_p}^\alpha) : \gamma_\alpha \to \mathbb{C}_\zeta^p \quad \text{by} \quad f_\zeta^\alpha := f_\zeta|\pi_{z,u}(\gamma_\alpha). \tag{4.18}$$

Since $\Gamma(f_v, f_j)$ are Hartogs 1-pseudoconcave due to Lemma 4.7.8, it follows by the same arguments as in the case $n = k = p = 1$ that for each $j \in \{1, \ldots, p\}$ the component $f_{\zeta_j}^\alpha : \gamma_\alpha \to \mathbb{C}_{\zeta_j}$ is holomorphic. Hence, the curve f_ζ^α is holomorphic, as well, so that $\Gamma(f)$ is foliated by the family $\{\Gamma(f_\zeta^\alpha)\}_{\alpha \in I}$ of holomorphic curves.

Case $\mathbf{n \geq 1, k = 1, p \geq 1}$. The proof is nearly the same as in the previous case $n = k = 1$, $p \geq 1$. We only need to replace the curves $\{\gamma_\alpha\}_{\alpha \in I}$ in (4.18) by complex hypersurfaces $\{A_\alpha\}_{\alpha \in I}$ obtained from Chirka's result (case $n \geq 1$, $k = 1$, $p = 0$) and to apply the case $n \geq 1$, $k = 1$, $p = 1$ to each $j = 1, \ldots, p$ in order to show that $f_{\zeta_j}^\alpha : A_\alpha \to \mathbb{C}_{\zeta_j}$ is holomorphic on A_α. Then $\{\Gamma(f_\zeta^\alpha)\}_{\alpha \in I}$ is a complex foliation of $\Gamma(f)$.

The proof of the theorem is finally complete. □

We end this section with the following example which shows that it is not always possible to foliate a 1-pseudoconcave real 4-dimensional submanifold in $\mathbb{C}^3$ by complex submanifolds, but at least (in this example) by analytic subsets.

Example 4.7.10 For a fixed integer $k \geq 2$ consider the function

$$f(z_1, z_2) := \begin{cases} \overline{z}_1 {z_2}^{2+k}/\overline{z}_2, & z_2 \neq 0 \\ 0, & z_2 = 0 \end{cases} .$$

It is $\mathcal{C}^k$-smooth on $\mathbb{C}^2$ and holomorphic on complex lines passing through the origin, since $f(\lambda v) = \lambda^{2+k} f(v)$ for every $\lambda \in \mathbb{C}^* := \mathbb{C}\backslash\{0\}$ and each vector $v \in \mathbb{C}$. Therefore, in view of Theorem 3.11.3, the function f is 1-holomorphic on $\mathbb{C}^2$, so $\psi(z_1, z_2, w) := -\log|f(z_1, z_2) - w|$ is 1-plurisubharmonic outside $\{f = w\}$. By Theorem 4.3.2, this means that the graph $\Gamma(f)$ of f is a 1-pseudoconcave real 4-dimensional submanifold of $\mathbb{C}^3$ which does not admit a regular foliation near the origin, but at least a singular one given by the family of holomorphic curves $\{\Gamma(f|\mathbb{C}^* v)\}_{v \in \mathbb{C}^2}$. Of course, the problem arises because the complex Jacobian of f has non-constant rank near the origin.

Chapter 5

Generalized convex hulls

The classical Cartan-Thullen theorem states that a set Ω in $\mathbb{C}^n$ is a domain of holomorphy if and only if it is holomorphically convex, i.e., for each compact set K in Ω its convex hull with respect to holomorphic functions on Ω is still contained in Ω (see Definition 3.8.6 (1)). In this context, we study the link between hulls generated by subfamilies of q-plurisubharmonic (or q-holomorphic) functions and q-pseudoconvex sets (compare Theorem 4.3.2). In the case of q-holomorphic functions, this relation was already analyzed by R. Basener in [Bas78]. In this context, we will introduce several different hulls and study their properties and relation to each other. One of them is the generalized *polynomially convex hull* in the sense of Basener [Bas78], which was also examined by G. Lupacciolu and E. L. Stout in [LS99]. We give a different characterization of this hull in terms of plurisubharmonic functions on a foliation by algebraic analytic sets. We are also interested in a generalized version of Basener's *rationally convex hull* (see [Bas73]). It has been already investigated by G. Corach and F. D. Suárez in [CS89]. We show that this hull has a description in terms of smooth positive forms using the results of J. Duval and N. Sibony in [DS95]. Finally, we present the *q-pseudoconvex hull* of a compact set in $\mathbb{C}^n$, which is defined by the intersection of all q-pseudoconvex neighborhoods of K. It is also sometimes called the *Nebenhülle of* K (see, for example, [DF77]).

At the end of this chapter, we recall the definition and the main properties of *q-maximal* sets originally introduced by Z. Słodkowski in [Sło86]. They can be characterized by the condition that every q-plurisubharmonic function admits the local maximum principle on them. We show that these sets are linked to generalized polynomially convex hulls. Moreover, we deduce that the hull generated by q-plurisubharmonic functions and the q-pseudoconvex hull both are q-maximal. Due to a result by Słodkowski, this implies that these two hulls are $(n{-}q{-}2)$-pseudoconcave.

The whole chapter on generalized convex hulls and q-maximal sets is essentially contained in our article [PZ15]. Especially, the study of these sets was initialized by E. S. Zeron in order to prove Theorem 5.3.11.

5.1 Hulls created by q-plurisubharmonic functions

We have seen in Theorem 4.3.2 (7) that we can characterize q-pseudoconvexity by using hulls created by q-plurisubharmonic functions. A natural question is whether it is possible to generate them by other subfamilies of q-plurisubharmonic functions or to use different generalized convex hulls discussed in the literature.

Definition 5.1.1 Let Ω be an open set in $\mathbb{C}^n$ and let K be a compact set in Ω.

(1) Given a family $\mathcal{A}$ of upper semi-continuous functions on Ω, we define the *generalized convex hull of* K *in* Ω *with respect to* $\mathcal{A}$ or, for short, the $\mathcal{A}$*-hull of* K *in* Ω by

$$\widehat{K}^{\Omega}_{\mathcal{A}} := \big\{z \in \Omega : \psi(z) \le \max_K \psi \text{ for every } \psi \in \mathcal{A}\big\}.$$

(2) If $\Omega = \mathbb{C}^n$, we simply write $\widehat{K}_{\mathcal{A}}$ instead of $\widehat{K}^{\mathbb{C}^n}_{\mathcal{A}}$.

(3) For a subfamily $\mathcal{B}$ of continuous complex valued functions on Ω, the $\mathcal{B}$*-hull of* K *in* Ω is given by

$$\widehat{K}^{\Omega}_{\mathcal{B}} := \big\{z \in \Omega : |f(z)| \le \|f\|_K \text{ for every } f \in \mathcal{B}\big\}.$$

(4) We say that an open set Ω in $\mathbb{C}^n$ is $\mathcal{A}$*-convex* if for every compact set K its $\mathcal{A}$-hull $\widehat{K}^{\Omega}_{\mathcal{A}}$ is compactly contained in Ω.

The generalized convex hulls admit the subsequent properties.

Proposition 5.1.2 *Let* Ω *be an open set in* $\mathbb{C}^n$ *containing a compact set* K.

(1) For a family $\mathcal{A}$ *of upper semi-continuous functions on* Ω, *we have that*

$$\widehat{K}^{\Omega}_{\mathcal{A}^{\downarrow K}} = \widehat{K}^{\Omega}_{\mathcal{A}}.$$

(2) Let $\mathcal{B}$ be a family of complex valued continuous functions on Ω. Then

$$\widehat{K}^{\Omega}_{\overline{\mathcal{B}}^K} = \widehat{K}^{\Omega}_{\mathcal{B}}.$$

(3) If $\mathcal{A}_1 \subset \mathcal{A}_2$ are two subfamilies of upper semi-continuous functions on Ω, then $\widehat{K}^{\Omega}_{\mathcal{A}_2}$ lies in $\widehat{K}^{\Omega}_{\mathcal{A}_1}$. This implies that, if Ω is $\mathcal{A}_1$-convex, then it is also $\mathcal{A}_2$-convex.

(4) Let $\mathcal{A}_1 \subset \mathcal{A}_2$ be two families of upper semi-continuous functions on Ω fulfilling $\overline{\mathcal{A}_2}^{\downarrow K} \subset \overline{\mathcal{A}_1}^{\downarrow K}$. Then $\widehat{K}^{\Omega}_{\mathcal{A}_1} = \widehat{K}^{\Omega}_{\mathcal{A}_2}$.

(5) Given two families $\mathcal{B}_1 \subset \mathcal{B}_2$ of complex valued continuous functions on Ω satisfying $\overline{\mathcal{B}_2}^K \subset \overline{\mathcal{B}_1}^K$, we have that $\widehat{K}^{\Omega}_{\mathcal{B}_1} = \widehat{K}^{\Omega}_{\mathcal{B}_2}$.

Proof. The first statement follows from Proposition 1.3.4. The second one is a direct consequence of the definition of the uniform closure $\overline{\mathcal{B}}^K$. The third one follows from the definition of $\mathcal{A}$-convex sets. The last two properties are immediate consequences of the preceding statements. □

In the next proposition, we examine whether we can characterize q-pseudoconvexity in terms of hulls generated by different subfamilies of q-plurisubharmonic functions.

Proposition 5.1.3 *Let Ω be an open set in $\mathbb{C}^n$. Then we have the following statements.*

(1) Ω is q-pseudoconvex if and only if it is $\mathcal{PSH}^c_q(\Omega)$-convex.

(2) Assume that Ω has a $\mathcal{C}^2$-smooth boundary. Then Ω is q-pseudoconvex if and only if it is $\mathcal{PSH}^2_q(\Omega)$-convex, where $\mathcal{PSH}^2_q(\Omega) := \mathcal{PSH}_q(\Omega) \cap \mathcal{C}^2(\Omega)$.

Proof. (1) If Ω is q-pseudoconvex in $\mathbb{C}^n$, then in view of Theorem 4.3.2 (4) the function $z \mapsto -\log d(z, b\Omega) + \|z\|_2^2$ is a continuous q-plurisubharmonic exhaustion function for Ω. By Bungart's approximation theorem 3.5.4, we can construct a q-plurisubharmonic exhaustion function with corners on Ω. Then we can follow the proof of the implications (4) $\Rightarrow \ldots \Rightarrow$ (7) in Theorem 4.3.2 in order to conclude that Ω is $\mathcal{PSH}^c_q(\Omega)$-convex. On the other hand, since $\mathcal{PSH}^c_q(\Omega)$ is a subfamily of $\mathcal{PSH}_q(\Omega)$, Proposition 5.1.2 (3) and Theorem 4.3.2 (7) imply that a $\mathcal{PSH}^c_q(\Omega)$-convex set is q-pseudoconvex.

(2) According to Corollary 4.6.2 and Theorem 4.3.2 (9), the set Ω is q-pseudoconvex if and only if it is Levi q-pseudoconvex. In view of Vigna Suria's theorem 4.5.8, we derive that Ω is q-pseudoconvex if and only if it admits a $\mathcal{C}^2$-smooth q-plurisubharmonic exhaustion function. The statement follows now from the same arguments as in the previous discussion (1). □

We take a closer look to the case $q = 0$ and recall some classical results in that case.

Remark 5.1.4 (1) Let Ω be a pseudoconvex set in $\mathbb{C}^n$. Then, in view of Bremermann's approximation theorem 3.1.10 and Proposition 5.1.2 (1), the hulls $\widehat{K}^{\Omega}_{\mathcal{O}(\Omega)}$ and $\widehat{K}^{\Omega}_{\mathcal{PSH}(\Omega)\cap\mathcal{C}(\Omega)}$ coincide. Moreover, by Corollary 1.3.10 in [Sto07] the previous hulls are also equal to $\widehat{K}^{\Omega}_{\mathcal{PSH}(\Omega)}$. Then Proposition 5.1.2 (3) gives that an open set Ω is pseudoconvex if and only if it is *holomorphically convex*, i.e., $\mathcal{O}(\Omega)$-convex.

(2) It is obvious that the hull $\widehat{K}_{\mathcal{O}(\mathbb{C}^n)}$ and the *polynomially convex hull* $\widehat{K}_{\mathbb{C}[z]}$ are the same. Here, $\mathbb{C}[z]$ is the set of all holomorphic polynomials. Therefore, it follows from the previous discussion (2) that the hulls $\widehat{K}_{\mathcal{PSH}(\mathbb{C}^n)}$ and $\widehat{K}_{\mathbb{C}[z]}$ coincide (see also Corollary 1.3.11 in [Sto07]).

(3) The classical Cartan-Thullen theorem states that a domain Ω in $\mathbb{C}^n$ is a domain of holomorphy if and only if it is holomorphically convex (see, e.g., Theorem 2.5.5 in [Hör90]). Furthermore, the solution of the Levi problem (see, e.g., [Hör90]) gives that Ω is holomorphically convex if and only if it is pseudoconvex.

The final statement in the previous remark motivates to introduce the following notion.

Definition 5.1.5 An open set Ω in $\mathbb{C}^n$ is called *q-holomorphically convex* if it is $\mathcal{O}_q(\Omega)$-convex.

In general, it is not known so far whether the two notions of q-pseudoconvex and q-holomorphically convex sets are equivalent if $q \geq 1$. In the case of Levi q-pseudoconvex sets, R. Basener derived a partial answer (see Theorem 3 in [Bas76]).

Theorem 5.1.6 *Let Ω be a bounded domain with $\mathcal{C}^2$-smooth boundary.*

1. *If Ω is q-holomorphically convex, then it is Levi q-pseudoconvex.*

2. *If Ω is strictly q-pseudoconvex, then for each $p \in b\Omega$ there is a neighborhood U of p such that $\Omega \cap U$ is q-holomorphically convex.*

There is another relation between q-pseudoconvex sets and the q-plurisubharmonic hulls which will serve as an important tool later. It is mainly Corollary 4.2 in [Die06].

Proposition 5.1.7 *Let Ω be a q-pseudoconvex domain in $\mathbb{C}^n$ and K be a compact subset of Ω. Given any neighborhood V of $\widehat{K}^{\Omega}_{\mathcal{PSH}_q(\Omega)}$ in Ω, there exists a strictly q-plurisubharmonic function ψ with corners on Ω such that $\psi < 0$ on K and $\psi > 0$ on $\Omega \setminus V$. Moreover, ψ is an exhaustion function for Ω.*

In the case of $q = 0$ and $\Omega = \mathbb{C}^n$, the function ψ can be assumed to be $\mathcal{C}^\infty$-smooth. Moreover, it is a classical tool to characterize polynomially convex hulls.

5.2 More generalized convex q-hulls

In [Bas78], R. Basener introduced the following generalized polynomially convex hull.

Definition 5.2.1 Given a compact set K in $\mathbb{C}^n$ and $q \in \{0, \dots, n-1\}$, the *polynomially convex q-hull* is given by

$$h_q(K) = \left\{ z \in \mathbb{C}^n : \begin{array}{c} z \in (K \cap p^{-1}(0))^\wedge \text{ for every polynomial} \\ \text{mapping } p : \mathbb{C}^n \to \mathbb{C}^q \text{ with } p(z) = 0 \end{array} \right\}. \tag{5.1}$$

We obviously fix $\mathbb{C}^0 = \{0\}$. The notation $K^\wedge = \widehat{K} = \widehat{K}_{\mathbb{C}[z]}$ stands for the classical polynomially convex hull of K (see also Remark 5.1.4 (2)).

The polynomially convex q-hull can be expressed in a purely topological way.

Definition 5.2.2 The *topologically convex hull Top-hull*(K) of a compact set $K \subset \mathbb{C}^n$ is the union of K and all the bounded connected components of its complement $\mathbb{C}^n \setminus K$.

Corollary 3.4 in [LS99] yields an alternative definition of the polynomially convex q-hull.

Theorem 5.2.3 *Let K be a compact set in $\mathbb{C}^n$. Then*

$$h_q(K) = \left\{ z \in \mathbb{C}^n : \begin{array}{c} p(z) \in \textit{Top-hull}(p(K)) \text{ for every} \\ \text{polynomial } p : \mathbb{C}^n \to \mathbb{C}^{q+1} \end{array} \right\}. \tag{5.2}$$

We can express the polynomially convex q-hull by the hulls given in Definition 5.1.1, but we have to introduce a new subfamily of functions holomorphic on complex foliations by algebraic subsets. In this context, recall Definition 3.11.2 and Theorem 3.11.3.

Definition 5.2.4 Given $q \in \{0, \ldots, n-1\}$, we denote by $\mathcal{O}_q^{\text{poly}}(\mathbb{C}^n)$ the family $\mathcal{O}(H, \mathbb{C}^n)$, where H is the collection of all holomorphic polynomial mappings $p : \mathbb{C}^n \to \mathbb{C}^q$.

The relation between the different hulls defined above is given by the following result.

Proposition 5.2.5 *For every integer $q \in \{0, \ldots, n-1\}$ and compact set K in $\mathbb{C}^n$ the following inclusions hold,*

$$\widehat{K}_{\mathcal{PSH}_q(\mathbb{C}^n)} \;\subset\; \widehat{K}_{\mathcal{O}_q(\mathbb{C}^n)} \;\subset\; h_q(K) = \widehat{K}_{\mathcal{O}_q^{\text{poly}}(\mathbb{C}^n)}.$$

Proof. We deduce from Remark 5.1.4 (2) and Definition 5.2.1 that $\widehat{K}_{\mathcal{PSH}(\mathbb{C}^n)}$, $\widehat{K}_{\mathcal{O}(\mathbb{C}^n)}$, $\widehat{K}_{\mathcal{O}_0^{\text{poly}}(\mathbb{C}^n)}$ and $h_0(K)$ are all equal to the polynomially convex hull $\widehat{K}$, so we take from now on $q \geq 1$.

We have already seen that $\widehat{K}_{\mathcal{PSH}_q(\mathbb{C}^n)}$ is contained in $\widehat{K}_{\mathcal{O}_q(\mathbb{C}^n)}$, since $\log|f|$ is q-plurisubharmonic on $\mathbb{C}^n$ for every function f in $\mathcal{O}_q(\mathbb{C}^n)$ according to Proposition 3.10.2 (8). Furthermore, Theorem 3.11.5 and Proposition 5.1.2 imply that $\widehat{K}_{\mathcal{O}_q(\mathbb{C}^n)}$ is contained in $\widehat{K}_{\mathcal{O}_q^{\text{poly}}(\mathbb{C}^n)}$. Hence, it remains to show that $\widehat{K}_{\mathcal{O}_q^{\text{poly}}(\mathbb{C}^n)} = h_q(K)$.

Assume that z_0 does not belong to $h_q(K)$. Then there is a polynomial mapping $p_0 : \mathbb{C}^n \to \mathbb{C}^q$ such that $p_0(z_0) = 0$ and z_0 does not lie in the polynomially convex hull of $p_0^{-1}(0) \cap K$. Therefore, there exists another polynomial $p_1 : \mathbb{C}^n \to \mathbb{C}$ such that $p_1(z_0) = 1$ and $|p_1(z)| < 1$ for every point $z \in K \cap p_0^{-1}(0)$. Let $a > 0$ be large enough such that $|p_1(z)| < 1 + a\|p_0(z)\|_2^2$ for every z in K. We define the following function on $\mathbb{C}^n$,

$$f(z) := \frac{p_1(z)}{1 + a\|p_0(z)\|_2^2}.$$

The function f lies in $\mathcal{O}_q^{\mathrm{poly}}(\mathbb{C}^n)$, since p_1 is holomorphic and p_0 is constant on $p_0^{-1}(c)$ for every $c \in p_0(\mathbb{C}^n)$. Moreover, we have that $\|f\|_K < 1$ and $f(z_0) = 1$, so that $z_0 \notin \widehat{K}_{\mathcal{O}_q^{\mathrm{poly}}(\mathbb{C}^n)}$. Hence, the hull $\widehat{K}_{\mathcal{O}_q^{\mathrm{poly}}(\mathbb{C}^n)}$ is contained in $h_q(K)$.

On the other hand, take $z_1 \in h_q(K)$ and $f \in \mathcal{O}_q^{\mathrm{poly}}(\mathbb{C}^n)$. Let $p : \mathbb{C}^n \to \mathbb{C}^q$ be a polynomial mapping such that f is holomorphic on $p^{-1}(c)$ for every $c \in p(\mathbb{C}^n)$. We may assume without loss of generality that $p(z_1) = 0$, so that z_1 lies in the polynomially convex hull of $K \cap p^{-1}(0)$ according to definition (5.1). Let F be an entire holomorphic function on $\mathbb{C}^n$ whose restriction to $p^{-1}(0)$ coincides with f (recall Theorem 3.11.4). Since the generalized convex hulls with respect to polynomials and entire functions coincide (see Remark 5.1.4 (2)) and $z_1 \in (K \cap p^{-1}(0))^\wedge$, the following inequalities hold,

$$|f(z_1)| = |F(z_1)| \leq \|F\|_{K \cap p^{-1}(0)} = \|f\|_{K \cap p^{-1}(0)} \leq \|f\|_K.$$

These inequalities imply that z_1 belongs to the hull $\widehat{K}_{\mathcal{O}_q^{\mathrm{poly}}(\mathbb{C}^n)}$. Finally, we can conclude that $h_q(K)$ is equal to $\widehat{K}_{\mathcal{O}_q^{\mathrm{poly}}(\mathbb{C}^n)}$. □

In the literature, there is also Basener's generalized version of the rationally convex hull (see, e.g., [Bas73] and [CS89]).

Definition 5.2.6 Let K be a compact set in $\mathbb{C}^n$ and let $q \in \{0, \dots, n-1\}$ be an integer.

(1) The *rationally convex q-hull of K* is given by

$$r_q(K) = \left\{ z \in \mathbb{C}^n : \begin{array}{c} p(z) \in p(K) \text{ for every} \\ \text{polynomial } p : \mathbb{C}^n \to \mathbb{C}^{q+1} \end{array} \right\} \tag{5.3}$$

(2) We obviously have have that $r_{n-1}(K) = K$ after taking $p(z) = z$.

J. Duval and N. Sibony have shown that the positive forms are strongly related to the classical *rationally convex hull* $r_0(K)$ (see Theorem 2.1 in [DS95]). We prove below that a similar connection exists between the positive forms and the rationally convex q-hulls. We assume that the reader is familiar with (p, q)-forms, so that we only repeat the notion and the basic properties of positive forms. For the proofs and more properties on differential forms and positive currents we refer to Chapter III.1 in Demailly's online book [Dem12].

Definition & Remark 5.2.7 Let $q \in \{1, \ldots, n\}$ and let Ω be an open set in $\mathbb{C}^n$.

(1) A form u on Ω of bidegree (q, q) is *positive* if for every q-dimensional complex plane π the restriction of u to π is a positive volume form on π.

(2) The *support of u in Ω* is defined by $\operatorname{supp}(u) := \overline{\{z \in \Omega : u_z \neq 0\}}$.

(3) If ψ is a $\mathcal{C}^\infty$-smooth function on Ω, then ψ is plurisubharmonic if and only if the (1,1)-form $i\partial\overline{\partial}\psi$ is positive.

(4) Given finitely many $\mathcal{C}^\infty$-smooth plurisubharmonic functions $\psi_1, \ldots, \psi_q$ on Ω, the following (q, q)-form is positive on Ω,

$$\bigwedge_{j=1}^{q} i\partial\overline{\partial}\psi_j := i\partial\overline{\partial}\psi_1 \wedge \ldots \wedge i\partial\overline{\partial}\psi_q.$$

We obtain the following characterization of rationally convex q-hulls.

Proposition 5.2.8 *Let K be a compact set in $\mathbb{C}^n$. Given an integer q in $\{1, \ldots, n\}$, the point $z_0 \notin r_{q-1}(K)$ if and only if there exists a closed positive smooth form u of bidegree (q, q) such that z_0 lies in* $\operatorname{supp}(u)$*, the set K does not meet* $\operatorname{supp}(u)$ *and $u = \bigwedge_{j=1}^{q} i\partial\overline{\partial}\psi_j$ for finitely many entire $\mathcal{C}^\infty$-smooth plurisubharmonic functions $\psi_1, \ldots, \psi_q$.*

Proof. We follow the arguments of J. Duval and N. Sibony in [DS95] in the case $q = 1$.

We first prove the necessity. Given $z_0 \notin r_{q-1}(K)$, there exists a polynomial mapping $p = (p_1, \ldots, p_q) : \mathbb{C}^n \to \mathbb{C}^q$ such that $p(z_0) = 0$, but $0 \notin p(K)$. We can choose n small perturbations $p_{j,k}$ of every function p_j, $j = 1, \ldots, q$, such that:

- For every $j = 1, \ldots, q$ and $k = 1, \ldots, n$ we have that $p_{j,k}(z_0) = 0$.
- For every $j = 1, \ldots, q$, the functions $p_{j,1}, \ldots, p_{j,n}$ give local coordinates at z_0.
- For every $k = 1, \ldots, n$ it holds that $0 \notin (p_{1,k}, \ldots, p_{q,k})(K)$.

Let $\varrho : \mathbb{C}^n \to \mathbb{R}$ be a non-negative smooth function with compact support in the unit ball $B_1(0)$. Assume that $\varrho(\zeta) = \varrho(|\zeta|)$ on $\mathbb{C}^n$ and that $\int_{\mathbb{C}^n} \varrho(\zeta) dV(\zeta) =$

1. For $\varepsilon > 0$ we additionally define $\varrho_\varepsilon(\zeta) = \varepsilon^{-2n}\varrho(\zeta/\varepsilon)$. Then the form u_ε below has bidegree (q, q) and is closed and positive on $\mathbb{C}^n$,

$$u_\varepsilon := \bigwedge_{j=1}^{q} \left(\sum_{k=1}^{n} i\partial\overline{\partial}\big(\log|p_{j,k}| * \varrho_\varepsilon\big) \right),$$

where $*$ denotes the classical integral convolution. Finally, for $\varepsilon > 0$ small enough we achieve that $(u_\varepsilon)_{z_0} \neq 0$ and the set K does not meet $\operatorname{supp}(u)$. Setting $u := u_\varepsilon$ for this particular $\varepsilon > 0$ finishes the first part of this statement.

On the other hand, in order to prove the sufficiency, we define $u_j := i\partial\overline{\partial}\psi_j$ and $V_{j,\delta} := \{z \in \mathbb{C}^n : d(z, \operatorname{supp}(u_j)) < \delta\}$ for $\delta > 0$ and the index $j = 1, \ldots, q$. Notice that $\operatorname{supp}(u)$ is equal to $\bigcap_{j=1}^{q} \operatorname{supp}(u_j)$. Thus, the given assumptions imply that z_0 lies in each $\operatorname{supp}(u_j)$, $j = 1, \ldots, q$ and the set K does not intersect $\bigcap_{j=1}^{q} V_{j,\delta}$ for $\delta > 0$ small enough. Since the compact set $K \setminus V_{j,\delta}$ does not meet $\operatorname{supp}(u_j)$, we can proceed as in the proof of Theorem 2.1 in [DS95] and deduce the existence of hypersurfaces $H_1, \ldots, H_q$ in $\mathbb{C}^n$ such that for every $j = 1, \ldots, q$ the point z_0 lies in H_j, but H_j does not meet $K \setminus V_{j,\delta}$. After a small perturbations of the hypersurface H_j we can assume that each one of them is algebraic and has the same properties as above. More precisely, for every index $j = 1, \ldots, q$ there is a polynomial mapping $p = (p_1, \ldots, p_q) : \mathbb{C}^n \to \mathbb{C}^q$ such that $H_j = p_j^{-1}(0)$, $p_j(z_0) = 0$ and $0 \notin p_j(K \setminus V_{j,\delta})$. Whence, $p(z_0) = 0$ and $0 \notin p(K \setminus V_{j,\delta})$ for each index $j = 1, \ldots, q$. Finally, since K does not meet $\bigcap_{j=1}^{q} V_{j,\delta}$, we can easily conclude that $0 \notin p(K)$, and so $z_0 \notin r_{q-1}(K)$. □

Based on the results of Duval and Sibony, S. Nemirovski [Nem07] was able to develop the following example related to rationally convex sets.

Example 5.2.9 Let K be the union of finitely many disjoint compact balls in $\mathbb{C}^n$. Then $K = r_0(K)$.

The next hull which we will introduce in this section is related to the *Nebenhülle* of a compact set in the case for $q = 0$ (see, e.g., [DF77]). It is motivated by the fact that a closed set K is convex in $\mathbb{R}^n$ if and only if it is the intersection of all closed convex sets surrounding K.

Definition 5.2.10 Given an integer $q \in \{0, \ldots, n-1\}$ and a compact set K in $\mathbb{C}^n$, the *q-pseudoconvex hull* $\mathcal{H}_q(K)$ denotes the intersection $\bigcap_{j\in J} U_j$ of all q-pseudoconvex neighborhoods U_j of K.

We give some properties of the q-pseudoconvex hull.

Proposition 5.2.11 *Let K be a compact set $\mathbb{C}^n$ and $q \in \{0, \dots, n-1\}$.*

(1) The q-pseudoconvex hull $\mathcal{H}_q(K)$ is compact.

(2) For every neighborhood V of $\mathcal{H}_q(K)$ in $\mathbb{C}^n$ there exists another open neighborhood U of $\mathcal{H}_q(K)$ such that $U \Subset V$ and U is q-pseudoconvex in $\mathbb{C}^n$.

(3) The interior of $\mathcal{H}_q(K)$ is q-pseudoconvex, provided it is not empty.

(4) If K has a neighborhood system of q-pseudoconvex sets, then it coincides with its q-pseudoconvex hull.

(5) We have that $\mathcal{H}_{n-1}(K) = K$ and that $\mathcal{H}_q(\widehat{K}_{\mathcal{A}}) = \widehat{K}_{\mathcal{A}}$ for any subfamily $\mathcal{A}$ of q-plurisubharmonic functions on $\mathbb{C}^n$.

Proof. (1) The hull $\mathcal{H}_q(K) \subset \mathbb{C}^n$ is bounded, since we can always enclose K by a pseudoconvex open ball $B_r(0)$ centered at the origin of $\mathbb{C}^n$ and of radius $r > 0$ large enough. By definition, the hull $\mathcal{H}_q(K)$ is then contained in $B_r(0)$.

We show that the hull $\mathcal{H}_q(K)$ is closed as well. Assume that it is not the case, so that $\overline{\mathcal{H}_q(K)} \setminus \mathcal{H}_q(K)$ contains some point p. By the definition of the hull there exists an open neighborhood W of K which is q-pseudoconvex but which does not contain p. By the definition of q-pseudoconvexity, the set W admits a continuous q-plurisubharmonic exhaustion function ψ. Consider the set

$$D := \{z \in W : \psi(z) < 1 + \max_K \psi\}. \tag{5.4}$$

Since ψ is an exhaustion function for W, we have that $K \subset D \Subset W$. Then Example 4.5.4 (1) yields the q-pseudoconvexity of D in $\mathbb{C}^n$, and so the hull $\mathcal{H}_q(K)$ is contained in D. But this means that $K \subset \overline{\mathcal{H}_q(K)} \subset \overline{D} \subset W$. Since we have assumed that $p \notin W$ but $p \in \overline{\mathcal{H}_q(K)}$, we immediately obtain a contradiction. Thus, the hull $\mathcal{H}_q(K)$ is compact in view of the discussion before.

(2) Pick up an arbitrary neighborhood V of $\mathcal{H}_q(K)$ in $\mathbb{C}^n$. Since the set $\mathcal{H}_q(K)$ is compact, we may assume that V is bounded, so its boundary bV is also compact. Let z be any point in bV. Since z does not lie in $\mathcal{H}_q(K)$, there is an open neighborhood W_z of $\mathcal{H}_q(K)$ which is q-pseudoconvex in $\mathbb{C}^n$, but $z \notin W_z$. We now proceed as in the discussion above by producing a q-plurisubharmonic exhaustion function ψ_z for W_z and an open set D_z as in (5.4), such that D_z is a relatively compact q-pseudoconvex set in $\mathbb{C}^n$, the hull $\mathcal{H}_q(K)$ is contained

in D_z, but D_z does not meet an open ball $B_z := B_r(z)$ centered in $z \in bV$ and of radius $r = r_z > 0$ small enough. Then the collection $\{B_z\}_{z\in bV}$ is a cover for the compact set bV, so we can pick a finite subcover $\{B_{z_k}\}_{k=1}^{\ell}$ of bV. Now Proposition 4.1.2 (1) implies that the finite intersection $U := \bigcap_{k=1}^{\ell} D_{z_k}$ is an open q-pseudoconvex set. Moreover, since each D_{z_k} is an open relatively compact set which contains $\mathcal{H}_q(K)$, the set U is an open relatively compact neighborhood of $\mathcal{H}_q(K)$ lying in V.

(3) This fact follows directly from Proposition 4.3.3 and the definition of the q-pseudoconvex hull.

(4) This observation is a direct consequence of the definition of $\mathcal{H}_q(K)$.

(5) These statements follow from the precedent one using Corollary 4.2.4, Example 4.5.4 (1) and the definition of $\mathcal{A}$-hulls. □

We may now present a more detailed relation between the q-hulls. For more properties of generalized convex hulls, we refer to E. L. Stout's book [Sto07].

Proposition 5.2.12 *Let $q \in \{0, \dots, n-1\}$ and let K be a compact set in $\mathbb{C}^n$. Then we have the following contentions,*

$$\begin{array}{ccccc} \widehat{K}_{\mathcal{PSH}_{q+1}(\mathbb{C}^n)} & \subset & \mathcal{H}_q(K) & \subset & \widehat{K}_{\mathcal{PSH}_q(\mathbb{C}^n)} \\ \cap & & \cap & & \cap \\ h_{q+1}(K) & \subset & r_q(K) & \subset & h_q(K) \end{array} \tag{5.5}$$

Proof. The inclusion $\widehat{K}_{\mathcal{PSH}_q(\mathbb{C}^n)} \subset h_q(K)$ has been already verified in Proposition 5.2.5.

It is easy to see that $r_q(K)$ is contained in $h_q(K)$ according to equations (5.2) and (5.3).

We also have that $h_{q+1}(K)$ is contained in $r_q(K)$. Indeed, given any point $z \notin r_q(K)$, there exists a polynomial mapping $p : \mathbb{C}^n \to \mathbb{C}^{q+1}$ such that $p(z) = 0$ and the intersection of $p^{-1}(0)$ and K is empty, so that z does not lie in $h_{q+1}(K)$ in view of equation (5.1).

We derive from Proposition 5.2.11 (5) that $\mathcal{H}_q(K)$ is contained in the hull $\widehat{K}_{\mathcal{PSH}_q(\mathbb{C}^n)}$.

Now we show that $\mathcal{H}_q(K)$ lies in $r_q(K)$. Indeed, if $z_0 \notin r_q(K)$, there exists a polynomial mapping $p : \mathbb{C}^n \to \mathbb{C}^{q+1}$ such that $p(z_0) = 0$, but $0 \notin p(K)$. We know from Proposition 4.2.3 that $-\log \|\cdot\|_2$ is q-plurisubharmonic on the open set $\mathbb{C}^{q+1} \setminus \{0\}$. Thus, the function $\psi := -\log \|p\|_2$ is q-plurisubharmonic on $\mathbb{C}^n \setminus p^{-1}(0)$ and fulfills $\psi(z_0) = +\infty$. Then the set $U := \{w \in \mathbb{C}^n : \psi(w) < 1 + \max_K \psi\}$ is an open neighborhood of K which does not contain z_0. Since in view of Example 4.5.4 the set U is q-pseudoconvex, the hull $\mathcal{H}_q(K)$ is contained in U, but $z_0 \notin \mathcal{H}_q(K)$.

It remains to verify that $\widehat{K}_{\mathcal{PSH}_{q+1}(\mathbb{C}^n)}$ is a subset of $\mathcal{H}_q(K)$. To see this inclusion, we pick a point $z_1 \notin \mathcal{H}_q(K)$. By Proposition 5.2.11 (2) there is a relatively compact q-pseudoconvex neighborhood W of K in $\mathbb{C}^n$ such that $z_1 \notin W$. By Theorem 4.3.2 (4), the function $\varrho(z) := -\log d(z, bW)$ is q-plurisubharmonic on W, and K lies in $\{\varrho < c\}$ for some $c \in \mathbb{R}$. In view of Proposition 3.3.2 (6), the function $\varphi := \min\{\varrho, c\}$ is $(q+1)$-plurisubharmonic on W. Moreover, since $\varrho(z)$ tends to $+\infty$ if z approaches the boundary of W, the function φ is upper semi-continuous on $\overline{W}$ and satisfies $\varphi(z) = 1$ for every $z \in bW$. Therefore, in view of Proposition 3.3.2 (10), we can extend φ by the constant c to an $(q+1)$-plurisubharmonic function defined on the whole of $\mathbb{C}^n$. We denote this extension again by φ. Since $p \notin W$ and $K \subset \{\varrho < c\}$, we have that $\varphi(p) = c$ but $\varphi = \varrho < c$ on K. Hence, p does not lie in the hull $\widehat{K}_{\mathcal{PSH}_{q+1}(\mathbb{C}^n)}$. □

5.3 q-Maximal sets and q-hulls

We show that the polynomially convex q-hull and the q-pseudoconvex hull satisfy a special version of the local maximal modulus principle for certain holomorphic polynomials. We begin by recalling Słodkowski's definition of 0-maximal sets (see [Sło86]).

Definition 5.3.1 Let X be a non-empty locally closed set in $\mathbb{C}^n$, so that $X = V \cap \overline{X}$ for some open set V in $\mathbb{C}^n$. The set X is said to be 0-*maximal (in V)* if it fulfills the *local maximum modulus principle* on X, i.e., for every compact subset K of X and holomorphic polynomial $p : \mathbb{C}^n \to \mathbb{C}$ it holds that

$$\|p\|_K \leq \|p\|_{b_X K}, \tag{5.6}$$

where $b_X K$ denotes the relative boundary of K in X.

We give an important observation on the compactness of 0-maximal sets.

Remark 5.3.2 An interesting consequence of Definition 5.3.1 is that no compact set X in $\mathbb{C}^n$ can be 0-maximal, since the relative boundary $b_X X$ is empty after taking $K = X$ in equation (5.6). It can be proved in a similar way that no 0-maximal set can have compact connected components.

The previous definition can be reformulated using the closures of a bounded open sets instead of compact sets (compare also Proposition 2.3 in [Sło86]).

Lemma 5.3.3 *Let V be an open set in $\mathbb{C}^n$ and let X be a locally closed set with $V \cap \overline{X}$. The set X is 0-maximal if and only if for each open relatively compact subset W of V and every holomorphic polynomial $p : \mathbb{C}^n \to \mathbb{C}$ we have that*

$$\|p\|_{X\cap\overline{W}} \leq \|p\|_{X\cap bW} \tag{5.7}$$

Proof. The inequality (5.6) immediately implies (5.7) after observing that the following sets are all compact,

$$X \cap \overline{W} = \overline{X} \cap V \cap \overline{W} = \overline{X} \cap \overline{W} \quad \text{and} \quad b_X(X \cap \overline{W}) = X \cap bW.$$

On the other hand, given any compact set $K \subset X$, the interior of K in X is equal to $A \cap X$ for some open set A in $\mathbb{C}^n$ with compact closure $\overline{A} \subset V$. We have that $X \cap \overline{A}$ is contained in K, so that $X \cap bA$ is contained in the relative boundary $b_X K$. Therefore, the inequality (5.7) implies (5.6) after observing that at least one of the following equations holds,

$$\|p\|_K = \|p\|_{b_X K} \quad \text{or} \quad \|p\|_K = \|p\|_{X\cap\overline{A}} \leq \|p\|_{X\cap bA} \leq \|p\|_{b_X K}.$$

The first equation above holds when $|p|$ takes its maximum at the relative boundary $b_X K$ and the second one is fulfilled when the maximum is taken at the relative interior of K in X. □

The polynomially convex hull and the 0-maximal sets are related as follows.

Theorem 5.3.4 *Let $X = V \cap \overline{X}$ be a 0-maximal set in some open set V in $\mathbb{C}^n$. If X is bounded, then $\overline{X} \backslash V$ is compact and non-empty, and X lies in the polynomially convex hull of $\overline{X} \setminus V$.*

Proof. The difference $\overline{X} \backslash V$ is compact, since X is bounded and V is open. We also have that $\overline{X} \backslash V$ is not empty. Otherwise, the set $\overline{X}$ is completely contained inside V. Hence, $X = \overline{X}$ is compact, and so in view of Remark 5.3.2, it cannot be a 0-maximal.

Now take a collection of open bounded subsets $\{W_k\}_{k\in\mathbb{N}}$ in V such that $W_k \Subset W_{k+1}$ and $V = \bigcup_{k=1}^{\infty} W_k$. Let $w \in X$ be a fixed point and $p : \mathbb{C}^n \to \mathbb{C}$ be a holomorphic polynomial. We can suppose without loss of generality that w lies in W_k for every index $k \in \mathbb{N}$. Since for each $k \in \mathbb{N}$ the set $\overline{W}_k$ is compact, Lemma 5.3.3 implies that

$$|p(w)| \leq \|p\|_{X\cap\overline{W}_k} \leq \|p\|_{X\cap bW_k} \leq \|p\|_{\overline{X}\setminus W_k} \text{ for every } k \in \mathbb{N}.$$

Since $V = \bigcup_{k=1}^{\infty} W_k$, we have that $|p(w)| \leq \|p\|_{\overline{X}\setminus V}$. Since w is an arbitrary point in X and $p : \mathbb{C}^n \to \mathbb{C}$ is an arbitrary holomorphic polynomial, we conclude that X lies in the polynomially convex hull of $\overline{X}\setminus V$. □

The precedent result fails if we relax the conditions on X.

Remark 5.3.5 (1) Define $V := \mathbb{C}^n \setminus \overline{B_1(0)}$ and $X := \overline{B_2(0)} \setminus \overline{B_1(0)}$. Then $X = V \cap \overline{X}$, but X fails to be 0-maximal in V. Moreover, the polynomially convex hull of $\overline{X} \setminus V = bB_1(0)$ is the whole of $\overline{B_1(0)}$ by the local maximum modulus principle, so that X cannot lie in the polynomially convex hull of $\overline{X}\setminus V$.

(2) It is easy to see that the real axis $X = \mathbb{R}$ is closed and 0-maximal in the complex plane $\mathbb{C}$, but the difference $\overline{X} \setminus V$ is empty for every open set V that contains X.

Now we recall the definition of *q-maximal sets* originally introduced by Z. Słodkowski in [Sło86].

Definition 5.3.6 Given an integer $q \in \{0, \ldots, n-1\}$, a non-empty locally closed set $X = V\cap\overline{X}$ is said to be *q-maximal* if for every complex q-codimensional affine plane π the intersection $X \cap \pi$ is 0-maximal.

Słodkowski showed that the q-maximal sets can be characterized in various different ways using q-plurisubharmonic functions and q-pseudoconvex sets. For details we refer to Theorem 2.5, Corollary 2.6, Theorem 4.2 and Theorem 5.1 in [Sło86].

Theorem 5.3.7 (Słodkowski, 1986) *Fix an integer $q \in \{0, \ldots, n-1\}$ and let $X = V \cap \overline{X}$ be a locally closed set in $\mathbb{C}^n$. Then X is q-maximal if it admits one of the following properties.*

(1) The intersection $X \cap p^{-1}(0)$ is 0-maximal for every holomorphic polynomial mapping $p : \mathbb{C}^n \to \mathbb{C}^q$.

(2) The following function χ *is* $(n-q-1)$*-plurisubharmonic on* V*,*

$$\chi(z) := \begin{cases} 0, & z \in X \\ -\infty, & z \in V \setminus X \end{cases} .$$

(3) The set X *is* $(n-q-2)$*-pseudoconcave in* V*.*

(4) For every open set U *in* $\mathbb{C}^n$ *with* $U \cap X \neq \emptyset$*, every compact set* K *in* $X \cap U$ *and each* q*-plurisubharmonic function* ψ *on* U *we have that*

$$\max_K \psi = \sup\{\psi(z) : z \in b_{X \cap U} K\},$$

where $b_{X\cap U}K$ *is the relative boundary of* K *in* $X \cap U$*.*

Theorem 5.3.4 can be extended to polynomially convex q-hulls as follows.

Theorem 5.3.8 *Let* $X = V \cap \overline{X}$ *be a bounded* q*-maximal set in an open subset* V *of* $\mathbb{C}^n$*. Then* X *is contained in the polynomially convex* q*-hull* $h_q(\overline{X} \backslash V)$*.*

Proof. The set $\overline{X} \setminus V$ is compact and non-empty according to Theorem 5.3.4. Let z be any fixed point in X and let $p : \mathbb{C}^n \to \mathbb{C}^q$ be a holomorphic polynomial mapping such that $p(z) = 0$. Then we derive from Theorem 5.3.7 (1) that $X \cap p^{-1}(0)$ is 0-maximal. Hence, Theorem 5.3.4 implies that the point z belongs to the polynomially convex hull of the following compact set,

$$\big(\overline{X} \backslash V\big) \cap p^{-1}(0) = \big(\overline{X \cap p^{-1}(0)}\big) \setminus V.$$

Thus, we conclude that X lies inside the hull $h_q(\overline{X} \backslash V)$. □

We show that certain generalized q-hulls are q-maximal.

Proposition 5.3.9 *For any compact set* K *in* $\mathbb{C}^n$*, we have that* $\widehat{K}_{\mathcal{PSH}_q(\mathbb{C}^n)} \setminus K$ *and* $\mathcal{H}_q(K) \setminus K$ *are* q*-maximal.*

Proof. Denote by Π one of the hulls $\widehat{K}_{\mathcal{PSH}_q(\mathbb{C}^n)}$ or $\mathcal{H}_q(K)$. Given a compact set $E \subset \Pi$ and a q-plurisubharmonic function ψ defined on a neighborhood U of E in $\mathbb{C}^n$, we assert that the following equality holds,

$$\sup\{\psi(z) : z \in E\} = \sup\big\{\psi(z) : z \in (E \cap K) \cup b_\Pi E\big\}, \tag{5.8}$$

where $b_\Pi E$ is the boundary of E relative to Π. The identity (5.8) obviously holds when $E \setminus K$ has empty interior with respect to the topology of Π, since in this case $E = b_\Pi E \cup (E \cap K)$. Thus, we assume from now on that $E \setminus K$ has non-empty interior. We follow the proof of Theorem 2.18 in [Sto07], which is originally due to J. P. Rosay for the case of $q = 0$. In order to get a contradiction, suppose that the result is false. Then there exist a constant C_0 and a point p_0 in the interior of $E \setminus K$ relative to Π such that

$$\psi(p_0) > C_0 > \sup\big\{\psi(z) : z \in (E \cap K) \cup b_\Pi E\big\}. \tag{5.9}$$

We can assume without loss of generality that $\psi(p_0) > C_0 > 0$. Choose a small enough open neighborhood U_0 of the compact set E, such that $U_0 \Subset U$ and the inequality $\psi < C_0$ holds on $U_0 \cap K$ and $\Pi \cap bU_0$. Define the function

$$\psi_0 := \begin{cases} C_0, & \text{on } \mathbb{C}^n \setminus U_0 \\ \max\{\psi, C_0\}, & \text{on } U_0 \end{cases} . \tag{5.10}$$

Notice that $\psi_0(p_0) = \psi(p_0) > C_0$, but $\psi_0 \leq C_0$ on K. Since $\psi < C_0$ on the compact set $\Pi \cap bU_0$, we can find a relatively compact open neighborhood V of Π such that $\psi < C_0$ on $V \cap bU_0$ and ψ_0 patches to a q-plurisubharmonic function on V in view of Proposition 3.3.2 (10).

Now if $\Pi = \mathcal{H}_q(K)$, by Proposition 5.2.11 (2) there is a q-pseudoconvex neighborhood W of $\mathcal{H}_q(K)$ with $W \Subset V$. Then Example 4.5.4 implies that $W_0 := \{z \in W : \psi_0(z) < \psi_0(p_0)\}$ is a q-pseudoconvex neighborhood of K which does not contain p_0. Therefore, the hull $\mathcal{H}_q(K)$ lies in W_0 but $p_0 \notin \mathcal{H}_q(K)$, which contradicts the assumption $p_0 \in E \subset \mathcal{H}_q(K)$. Hence, the identity (5.8) holds if Π is the q-pseudoconvex hull of K.

In the case of $\Pi = \widehat{K}_{\mathcal{PSH}_q(\mathbb{C}^n)}$, first recall that V is a neighborhood of Π and $C_0 > 0$. By Proposition 5.1.7, there is a continuous q-plurisubharmonic function φ on $\mathbb{C}^n$ such that $\varphi < 0$ on K and $\varphi > 0$ on $\mathbb{C}^n \setminus V$. We define for a real number $C_2 > 0$ the following function,

$$\psi_1 := \begin{cases} C_2\varphi, & \text{on } \mathbb{C}^n \setminus V \\ \max\{\psi_0, C_2\varphi\}, & \text{on } V \end{cases} .$$

Notice that $\psi_1(p_0) \geq \psi_0(p_0) > C_0$, but $\psi_1 \leq C_0$ on K according to equations (5.9) and (5.10). The property (10) in Proposition 3.3.2 implies that, if C_2 is chosen large enough, then ψ_1 is q-plurisubharmonic on $\mathbb{C}^n$. Therefore, $p_0 \notin \widehat{K}_{\mathcal{PSH}_q(\mathbb{C}^n)}$. This is again a contradiction to the assumption that $p_0 \in E \subset \widehat{K}_{\mathcal{PSH}_q(\mathbb{C}^n)}$. Finally, we can conclude that (5.8) is true also in the case of $\Pi = \widehat{K}_{\mathcal{PSH}_q(\mathbb{C}^n)}$. □

We close this section by mentioning the initial motivation to study complements of generalized convex hulls.

Remark 5.3.10 Let D be a pseudoconvex domain in $\mathbb{C}^2$. It is known that if a compact set K lies in the complement of D, then $D \setminus \widehat{K}_{\mathcal{O}(\mathbb{C}^2)}$ is again pseudoconvex (see e.g., Theorem 5.2.8 in [Sto07]). This result is no longer true for higher dimensions. Indeed, let B be the unit ball $B_1(0)$ in $\mathbb{C}^3$ and consider the compact set $K = \{(z,0,0) \in \mathbb{C}^3 : |z| = 1\}$. Then the hull $\widehat{K}_{\mathcal{O}(\mathbb{C}^3)}$ equals $\Delta := \{(z,0,0) \in \mathbb{C}^3 : |z| \leq 1\}$, but $B \setminus \widehat{K}_{\mathcal{O}(\mathbb{C}^3)}$ is not pseudoconvex, since every holomorphic function on $B \setminus \Delta$ extends holomorphically into the whole of the ball B.

Anyway, in order to get a similar result in the case of higher dimensions we have to use q-pseudoconvex or q-plurisubharmonic hulls.

Theorem 5.3.11 *Fix $q \in \{0, \ldots, n-1\}$. Let K be a compact set in $\mathbb{C}^n$ and let Π be one of the hulls $\mathcal{H}_{n-q-2}(K)$ or $\widehat{K}_{\mathcal{PSH}_{n-q-2}(\mathbb{C}^n)}$. Then $\mathbb{C}^n \setminus \Pi$ is q-pseudoconvex in $\mathbb{C}^n \setminus K$. Moreover, if D is a q-pseudoconvex domain in $\mathbb{C}^n$ such that K lies in the complement of D, then the set $D \setminus \Pi$ is q-pseudoconvex.*

Proof. In view of Proposition 5.3.9, the set $X := \Pi \setminus K$ is $(n-q-2)$-maximal in $V := \mathbb{C}^n \setminus K$. Then Theorem 5.3.7 (3) implies that $V \setminus X = \mathbb{C}^n \setminus \Pi$ is q-pseudoconvex in $V = \mathbb{C}^n \setminus K$. Now if D is a q-pseudoconvex domain in $\mathbb{C}^n$ and $K \subset \mathbb{C}^n \setminus D$, then the intersection $X \cap D = D \setminus \Pi$ is q-pseudoconvex according to Proposition 4.1.2 (1). □

Chapter 6

The Bergman-Shilov boundary

Given a compact Hausdorff space X and a subfamily $\mathcal{A}$ of upper semi-continuous (or continuous complex valued) functions on X, the Shilov boundary for $\mathcal{A}$ is the smallest closed subset of X on which all functions from $\mathcal{A}$ attain their maximum (or, respectively, their absolute maximum). It is interesting to give conditions on X and $\mathcal{A}$ to guarantee the existence and uniqueness of the Shilov boundary. The concept of such a distinguished boundary of certain compact sets K in $\mathbb{C}^2$ for holomorphic functions defined on a neighborhood of K was already introduced by S. Bergman in [Ber31]. Later in the 1940s, it was G. Shilov who studied a similar boundary for Banach algebras of continuous functions on compact sets. This boundary is nowadays known as the *Shilov boundary* and is important for complex analysis as well as for pluripotential theory. By historical reasons, it is legitimated to call it *Bergman-Shilov boundary*, but for the sake of abbreviation we keep using the classical notation.

H. J. Bremermann [Bre59] studied the Shilov boundary based on plurisubharmonic functions on compact sets K in $\mathbb{C}^n$ without showing its existence. This gap was filled by, e.g., J. Siciak in [Sic62]. In particular, he showed that the Shilov boundary exists and is unique if $\mathcal{A}$ has some simple structure, e.g., if $\mathcal{A}$ is additive and if sublevel sets of finitely many exponentials of functions from $\mathcal{A}$ generate the topology of K. Unfortunately, Siciak's result does not apply to q-plurisubharmonic functions, since they are not additive. To overcome this problem, we establish a more general result on the existence of the Shilov boundary under mild conditions on the Hausdorff space X and the family $\mathcal{A}$ (compare also the existence theorems in [Aiz93] for compact sets in $\mathbb{C}^n$).

E. Bishop proved in [Bis59] that, if the compact Hausdorff space X is additionally metrizable, then the closure of the set of peak points and the Shilov boundary for uniform subalgebras of continuous functions coincide (see also [Dal71] and [Hon88]). Using upper semi-continuous functions, similar identities were obtained in [Sic62] and [Wit83b]. We apply these results to unions of uniform algebras and establish additional Bishop-type peak point theorems for special families of upper semi-continuous functions on Hausdorff spaces.

The study of the Shilov boundary for upper semi-continuous functions and the development of generalized Bishop type peak point theorems is the content of the article [Paw13].

6.1 Shilov boundary for upper semi-continuous functions

We will define the Bergman-Shilov or, for short, the Shilov boundary for subfamilies of upper semi-continuous functions and show some of their basic properties.

Definition 6.1.1 Let X be a compact Hausdorff space and $\mathcal{A}$ a family of upper semi-continuous functions on X.

(1) For a given upper semi-continuous function f on X we set

$$S(f) = S_X(f) := \{x \in X : f(x) = \max_X f\}.$$

(2) A subset S of X is called a *boundary for* $\mathcal{A}$ (or $\mathcal{A}$-*boundary*) if the intersection $S \cap S(f)$ is not empty for every $f \in \mathcal{A}$.

(3) We denote by $b_{\mathcal{A}} = b_{\mathcal{A}}(X)$ the set of all closed boundaries for $\mathcal{A}$ (in X).

(4) The set $\check{S}_{\mathcal{A}} = \check{S}_{\mathcal{A}}(X) := \bigcap_{S \in b_{\mathcal{A}}} S$ is called the *Shilov boundary for* $\mathcal{A}$ (in X).

We give some simple examples.

Example 6.1.2 (1) Let $X = [0,2]$ and consider the upper semi-continuous functions $f_1 = \chi_{\{0,1\}}$ and $f_2 = \chi_{\{1,2\}}$. For $\mathcal{A} = \{f_1, f_2\}$ we have that $\{0,2\}, \{1\} \in b_{\mathcal{A}}(X)$, $\check{S}_{\mathcal{A}}(X)$ is empty and that $S_X(f_1) \cap S_X(f_2) = \{1\}$.

(2) Let $X = [0,1]$ and $f_1 = \chi_{\{0\}}$ and $f_2 = \chi_{\{1\}}$. We take $\mathcal{A} = \{f_1, f_2\}$ and observe that $\{0,1\} \in b_{\mathcal{A}}(X)$, $S_X(f_1) \cap S_X(f_2) = \emptyset$ and $\check{S}_{\mathcal{A}}(X) = \{0,1\}$.

(3) Consider the functions $f_1 = \chi_{\{-1,1\}}$ and $f_2 = \chi_{\{0\}}$ defined on $X = [-1,1]$ and set $\mathcal{A} = \{f_1, f_2\}$. Then $\{-1,0\}, \{0,1\} \in b_{\mathcal{A}}(X)$, so $\check{S}_{\mathcal{A}}(X) = \{0\}$, but $\check{S}_{\mathcal{A}}(X)$ cannot be an $\mathcal{A}$-boundary in X since the function f_1 attains its maximum outside of zero.

We have the following properties of Shilov boundaries.

Proposition 6.1.3 *Let X be a compact Hausdorff space and let $\mathcal{A}$ be a family of upper semi-continuous functions on X.*

(1) The set $\check{S}_{\mathcal{A}}(X)$ is closed and possibly empty, whereas $b_{\mathcal{A}}(X)$ is never empty.

(2) $S_X(f)$ is a closed non-empty subset of X.

(3) If the set $T := \bigcap_{f \in \mathcal{A}} S_X(f)$ consists of more than two elements, then $\check{S}_{\mathcal{A}}(X)$ is empty.

(4) If the set T from (3) above consists of one single element $x_0 \in X$ and $\check{S}_{\mathcal{A}}(X) \neq \emptyset$, then $\check{S}_{\mathcal{A}}(X) = \{x_0\}$.

(5) The set $S := \bigcup_{f \in \mathcal{A}} S_X(f)$ is an $\mathcal{A}$-boundary.

(6) If $\mathcal{A}_1 \subset \mathcal{A}_2$ are two families of upper semi-continuous functions on X, then we have the following inclusions,

$$b_{\mathcal{A}_2}(X) \subset b_{\mathcal{A}_1}(X) \quad \text{and} \quad \check{S}_{\mathcal{A}_1}(X) \subset \check{S}_{\mathcal{A}_2}(X).$$

(7) Let $\mathcal{A} = \bigcup_{j \in J} \mathcal{A}_j$, where $\mathcal{A}_j$ are subsets of $\mathcal{USC}(X)$. If for every $j \in J$ the set $\check{S}_{\mathcal{A}_j}(X)$ is an $\mathcal{A}_j$-boundary in X, then $\check{S}_{\mathcal{A}}(X)$ is an $\mathcal{A}$-boundary in X and

$$\check{S}_{\mathcal{A}}(X) = \overline{\bigcup_{j \in J} \check{S}_{\mathcal{A}_j}(X)}.$$

(8) The set of all $\mathcal{A}$-boundaries in X coincides with the set of all $\overline{\mathcal{A}}^{\downarrow}$-boundaries in X, i.e.,

$$b_{\mathcal{A}}(X) = b_{\overline{\mathcal{A}}^{\downarrow}}(X) \quad \text{and} \quad \check{S}_{\mathcal{A}}(X) = \check{S}_{\overline{\mathcal{A}}^{\downarrow}}(X).$$

Proof. (1) The set $\check{S}_{\mathcal{A}}(X)$ is closed, because it is an intersection of closed sets. Example 6.1.2 (1) shows that $\check{S}_{\mathcal{A}}(X)$ might be empty. The set $b_{\mathcal{A}}(X)$ contains at least the whole space X.

(2) Since f is upper semi-continuous on X, the set $\{x \in X : f(x) < \max_X f\}$ is open in X, so the set $S_X(f) = X \setminus \{x \in X : f(x) < \max_X f\}$ is a closed subset of X. It is non-empty because every upper semi-continuous function attains a maximal value on a compact Hausdorff space due to Proposition 1.3.1.

(3) Pick two distinct elements x_0, x_1 from T. By definition $\{x_0\}$ and $\{x_1\}$ are $\mathcal{A}$-boundaries in X and, thus, $\check{S}_{\mathcal{A}}(X) \subset \{x_0\} \cap \{x_1\} = \emptyset$.

(4) Under these assumptions, $\{x_0\}$ is an $\mathcal{A}$-boundary in X. Therefore, $\emptyset \neq \check{S}_{\mathcal{A}}(X) \subset \{x_0\}$ which yields $\check{S}_{\mathcal{A}}(X) = \{x_0\}$.

(5) The set S is an $\mathcal{A}$-boundary in X, since $S \cap S_X(f) = S_X(f) \neq \emptyset$ for every $f \in \mathcal{A}$.

(6) This fact follows directly from the definition.

(7) The previous points (1) and (6) imply that $S := \overline{\bigcup_{j\in J} \check{S}_{\mathcal{A}_j}(X)}$ is in $b_{\mathcal{A}}(X)$. By assumption, the set S and, therefore, the set $\check{S}_{\mathcal{A}}$ are non-empty. Since an arbitrary function $f \in \mathcal{A}$ is contained in $\mathcal{A}_j$ for some $j \in J$ and by the assumption that $\check{S}_{\mathcal{A}_j}(X)$ is an $\mathcal{A}_j$-boundary in X, we obtain that

$$\emptyset \neq S_X(f) \cap \check{S}_{\mathcal{A}_j}(X) \;\subset\; S_X(f) \cap S \;\subset\; S_X(f) \cap \check{S}_{\mathcal{A}}(X).$$

This means that S is an $\mathcal{A}$-boundary in X and so $\check{S}_{\mathcal{A}}(X) \subset S$. Altogether we have that $S = \check{S}_{\mathcal{A}}(X)$ is an $\mathcal{A}$-boundary in X.

(8) This statement is a direct consequence of the corresponding definitions and Proposition 1.3.4.

□

We can easily bring our concept of the Shilov boundary into relation with the classical Shilov boundary for uniform subalgebras of $\mathcal{C}(X)$.

Remark 6.1.4 Let $\mathcal{B}$ be a subset of $\mathcal{C}(X)$. The classical Shilov boundary for $\mathcal{B}$ is the smallest closed subset S of X fulfilling $\max_S |f| = \max_X |f|$ for every $f \in \mathcal{B}$. Clearly, it corresponds to the Shilov boundary for the family $\log|\mathcal{B}| := \{\log|f| : f \in \mathcal{B}\}$. Therefore, it then makes sense to use the notation $b_{\mathcal{B}}(X)$ and $\check{S}_{\mathcal{B}}(X)$ instead of $b_{\log|\mathcal{B}|}(X)$ and $\check{S}_{\log|\mathcal{B}|}(X)$. Moreover, it is obvious that for the uniform closure $\overline{\mathcal{B}}$ of $\mathcal{B}$ in $\mathcal{C}(X)$ we have that $\check{S}_{\mathcal{B}}(X) = \check{S}_{\overline{\mathcal{B}}}(X)$.

6.2 Existence of the Shilov boundary

Now we recall the classical result named after Shilov who worked on the boundaries of Banach algebras in the 1940s.

Theorem 6.2.1 (Shilov, 1940s) *Let X be a compact Hausdorff space and $\mathcal{B}$ a Banach subalgebra of $\mathcal{C}(X)$. Then $\check{S}_{\mathcal{B}}(X)$ is non-empty and, moreover, it is a boundary for the family* $\log|\mathcal{B}|$ *in X.*

In this theorem the Banach algebra structure of $\mathcal{B}$ is heavily involved. We will extract the essential properties from that structure in order to establish similar results for Shilov boundaries for subfamilies of upper semi-continuous functions.

Definition 6.2.2 Let $\mathcal{A}$, $\mathcal{A}_1$ and $\mathcal{A}_2$ be subfamilies of upper semi-continuous functions on a Hausdorff space X.

(1) We set $\mathcal{A}_1 + \mathcal{A}_2 := \{f + g : f \in \mathcal{A}_1, g \in \mathcal{A}_2\}$.

(2) The family $\mathcal{A}$ is a *scalar cone* if $nf + b$ lies in $\mathcal{A}$ for every $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $f \in \mathcal{A}$ and $b \in \mathbb{R}$. Here we use the convention $-\infty \cdot 0 = 0$.

(3) The family $\mathcal{A}$ is a *convex cone* if $af + bg$ is contained in $\mathcal{A}$ for every a, b in $[0, +\infty)$ and f, g in $\mathcal{A}$.

(4) An open set V in X is an $\mathcal{A}$*-polyhedron* if there exist finitely many functions $f_1, \ldots, f_n$ in $\mathcal{A}$ and real numbers $C_1, \ldots, C_n$ such that

$$V = V(f_1, \ldots, f_n) = \{x \in X : f_1(x) < C_1, \ldots, f_n(x) < C_n\}.$$

(5) The family $\mathcal{A}$ *generates the topology of* X if for every point $x \in X$ and every neighborhood U of x in X there is an $\mathcal{A}$-polyhedron V such that $x \in V \subset U$.

Now we are able to show that the Shilov boundary for $\mathcal{A}$ is a non-empty boundary for $\mathcal{A}$ if $\mathcal{A}$ possesses some simple structure. The following two statements are based on standard arguments used in the case of Banach subalgebras of continuous functions (see, e.g., Theorem 9.1 in [AW98]). First, we need the following lemma.

Lemma 6.2.3 *Let X be a compact Hausdorff space and $\mathcal{A} \subset \mathcal{USC}(X)$ be a scalar cone. Assume that there exist an $\mathcal{A}$-boundary $S \in b_{\mathcal{A}}(X)$ and an $\mathcal{A}$-polyhedron $V = V(f_1, \ldots, f_n)$ such that $S \cap V = \emptyset$ and $\mathcal{A} + \{f_j\} \subset \mathcal{A}$ for every $j = 1, \ldots, n$. Given another $\mathcal{A}$-boundary $E \in b_{\mathcal{A}}(X)$, it follows that $E \setminus V \in b_{\mathcal{A}}(X)$.*

Proof. Since $\mathcal{A}$ is a scalar cone and $\mathcal{A} + \{f_j\} \subset \mathcal{A}$ for every $j = 1, \ldots, n$, the constant function 0 and so $f_1, \ldots, f_n$ lie in $\mathcal{A}$. Hence, we can assume that V is of the form $V = \{x \in X : f_1(x) < 0, \ldots, f_n(x) < 0\}$.

Observe first that $E \setminus V$ is non-empty. Otherwise, $E \subset V$, so $\max_E f_j < 0$ for every $j = 1, \ldots, n$. Since S does not meet V, there has to be an index $j_0 \in \{1, \ldots, n\}$ such that $\max_S f_{j_0} \geq 0$. We obtain the contradiction $0 \leq \max_S f_{j_0} = \max_E f_{j_0} < 0$.

Now suppose that the statement of this lemma is false, i.e., there are a point $y \in X$ and a function $f \in \mathcal{A}$ such that $\max_{E \setminus V} f < \max_X f = f(y)$. Since $\mathcal{A}$ is a scalar cone and $S \in b_{\mathcal{A}}(X)$, we can assume that $f(y) = 0$ and $y \in S$. Consider for fixed $m \in \mathbb{N}$ and j=1,...,n the function $g_j := mf + f_j \in \mathcal{A}$. If m is large enough, then $\max_{E \setminus V} g_j < 0$ for each j=1,...,n. Since $\max_X f = 0$ and $f_j < 0$ on V for every j=1,...,n, we have that $g_j(x) < 0$ for every $x \in V$. Hence, $\max_X g_j = \max_E g_j < 0$ for every j=1,...,n. Finally, we assert that $y \in V$. If not, there is an index $j_1 \in \{1, \ldots, n\}$ with $f_{j_1}(y) \geq 0$ and, thus, $g_{j_1}(y) \geq 0$, which is impossible. Thus, $y \in V \cap S = \emptyset$ which is a contradiction. This proves that $E \setminus V$ is an $\mathcal{A}$-boundary. □

We show a version of Shilov's theorem for a quite general family $\mathcal{A}$ of upper semi-continuous functions on X. We have to point out that L. Aĭzenberg already showed this result in the setting where X is a compact subset of $\mathbb{C}^n$ and $\mathcal{A}$ is a family of upper semi-continuous functions on X satisfying $f + \log \|z - c\|_2 \in \mathcal{A}$ for every $f \in \mathcal{A}$ and $c \in \mathbb{C}^n$ (see Chapter III, Theorem 14.1 and Corollary 14.2 in [Aiz93]). Anyway, our version extends L. Aĭzenberg's result to a more general situation in which X is an arbitrary compact Hausdorff space.

Theorem 6.2.4 *Let X be a compact Hausdorff space and $\mathcal{A}$ be a family of upper semi-continuous functions on X. If $\mathcal{A}$ contains a subset $\mathcal{A}_0$ which generates the topology of X such that $\mathcal{A} + \mathcal{A}_0 \subset \mathcal{A}$, then the Shilov $\mathcal{A}$-boundary is an $\mathcal{A}$-boundary in X, i.e.,*

$$\check{S}_{\mathcal{A}}(X) \in b_{\mathcal{A}}(X).$$

Proof. If $\check{S}_{\mathcal{A}}(X) = X$, then there is nothing to show, so we can assume that $\check{S}_{\mathcal{A}} \neq X$. We first treat the case of $\mathcal{A}$ being a scalar cone and $\check{S}_{\mathcal{A}}$ being not empty. In order to get a contradiction, suppose that $\check{S}_{\mathcal{A}}(X)$ is not an $\mathcal{A}$-boundary in X. Then there is a function $f \in \mathcal{A}$ such that $\max_{\check{S}_{\mathcal{A}}(X)} f < \max_X f$. Since f is upper semi-continuous on X, there is an open neighborhood U of $\check{S}_{\mathcal{A}}(X)$ such that $f(x) < \max_X f$ for every $x \in U$. Then, due to the fact that $\mathcal{A}_0$ generates the topology of X, we conclude that for every $y \in L := X \setminus U$ there are an $\mathcal{A}_0$-polyhedron V_y and an $\mathcal{A}$-boundary $S_y \in b_{\mathcal{A}}(X)$ such that $y \in V_y$ and $V_y \cap S_y = \emptyset$. The family $\{V_y\}_{y \in L}$ covers L. Hence, by the compactness of L, there are finitely many points $y_1, \dots, y_\ell \in L$ such that the subfamily $\{V_{y_j}\}_{j=1,\dots,\ell}$ covers L. Since $\mathcal{A} + \mathcal{A}_0 \subset \mathcal{A}$, we can apply iteratively the previous Lemma 6.2.3 in order to obtain that

$$E := (((X \setminus V_{y_1}) \setminus V_{y_2}) \setminus \dots \setminus V_{y_\ell}) = X \setminus \bigcup_{j=1}^{\ell} V_{y_j} \in b_{\mathcal{A}}(X).$$

Notice that, by the construction, the set $\check{S}_{\mathcal{A}}(X)$ lies in E and, therefore, E is non-empty. Moreover, $E \subset U$ and so $\max_E f < \max_X f$, but this contradicts to the fact that $E \in b_{\mathcal{A}}(X)$. Hence, $\check{S}_{\mathcal{A}}(X) \in b_{\mathcal{A}}(X)$.

In the case of $\mathcal{A}$ being a scalar cone and $\check{S}_{\mathcal{A}}(X) = \emptyset$, we pick an arbitrary point $p \in X$ and a neighborhood U of p in X which is an $\mathcal{A}_0$-polyhedron of the form $U = \{x \in X : f_1(x) < 0, \dots, f_k(x) < 0\}$ and satisfies $U \neq X$. Observe that for every $y \in X \setminus U$ there exists an $\mathcal{A}$-boundary S_y in X with $y \notin S_y$, since otherwise $y \in \check{S}_{\mathcal{A}}(X)$, which is absurd. Then we can choose an $\mathcal{A}_0$-polyhedron V_y such that $y \in V_y$, $p \notin V_y$ and $S_y \cap V_y$ is empty. By the same arguments as above we can construct an $\mathcal{A}$-boundary E such that $p \in E \subset U$. Since $U \neq X$, there exists a point $x_0 \in X \setminus U$ and an index $k_0 \in \{1, \dots, k\}$ such that $f_{k_0}(x_0) \geq 0$. This leads to the contradiction

$$0 \leq f_{k_0}(x_0) \leq \max_X f_{k_0} = \max_E f_{k_0} < 0.$$

Thus, $\check{S}_{\mathcal{A}}(X)$ cannot be empty.

If $\mathcal{A}$ is not necessarily a scalar cone, consider the scalar cone generated by $\mathcal{A}$,

$$\mathcal{A}^* := \{nf + c : n \in \mathbb{N}_0,\ f \in \mathcal{A},\ c \in \mathbb{R}\}.$$

Since $\mathcal{A}$ lies in $\mathcal{A}^*$, we have that $b_{\mathcal{A}^*}(X) \subset b_{\mathcal{A}}(X)$ and $\check{S}_{\mathcal{A}}(X) \subset \check{S}_{\mathcal{A}^*}(X)$. Pick an arbitrary $\mathcal{A}$-boundary S in X and a function $nf + c \in \mathcal{A}^*$, where $f \in \mathcal{A}$, $n \in \mathbb{N}$ and $c \in \mathbb{R}$. Since f and $nf + c$ attain their maximum at the same points, we have that

$$S \cap S_X(nf + c) = S \cap S_X(f) \neq \emptyset.$$

This means that S is also an $\mathcal{A}^*$-boundary in X, so $b_{\mathcal{A}}(X) = b_{\mathcal{A}^*}(X)$ and $\check{S}_{\mathcal{A}}(X) = \check{S}_{\mathcal{A}^*}(X)$. Now observe that the family $\mathcal{A}_0^* := \{nf + c : n \in \mathbb{N}_0,\ f \in \mathcal{A}_0,\ c \in \mathbb{R}\}$ generates the topology of X, since it contains $\mathcal{A}_0$. Moreover, we have that $\mathcal{A}^* + \mathcal{A}_0^* \subset \mathcal{A}^*$ and that $\mathcal{A}^*$ is a scalar cone. Thus, by the previous discussions, we conclude that

$$\check{S}_{\mathcal{A}}(X) = \check{S}_{\mathcal{A}^*}(X) \in b_{\mathcal{A}^*}(X) = b_{\mathcal{A}}(X).$$

This finishes the proof. □

6.3 Minimal boundary and peak points

We recall the definition of a *minimal boundary* and *peak points* in the sense of E. Bishop (see [Bis59]), which are closely related to the Shilov boundary.

Definition 6.3.1 Let $\mathcal{A}$ be a subfamily of upper semi-continuous functions on a Hausdorff space X.

(1) We denote by $B_{\mathcal{A}}(X)$ the set of all (possibly non-closed) boundaries for $\mathcal{A}$ in X. By Definition 6.1.1 (1) the set $B_{\mathcal{A}}(X)$ contains $b_{\mathcal{A}}(X)$.

(2) If there exists a subset $m_{\mathcal{A}}(X)$ in $B_{\mathcal{A}}(X)$ such that $m_{\mathcal{A}}(X)$ is contained in every boundary for $\mathcal{A}$, then this set will be called the *minimal boundary for* $\mathcal{A}$ (in X).

(3) A point $x \in X$ is called *peak point for* $\mathcal{A}$ if there is a *(peak) function* $f \in \mathcal{A}$ such that $S(f) = \{x\}$. We say that f *peaks at* x. We denote by $P_{\mathcal{A}}(X)$ the set of all *peak points for* $\mathcal{A}$.

(4) For $\mathcal{B}$ being a subset of $\mathcal{C}(X)$ we use the same simplification of notations as in Remark 6.1.4. Namely, we write $B_{\mathcal{B}}(X)$, $m_{\mathcal{B}}(X)$ and $P_{\mathcal{B}}(X)$ instead of $B_{\log|\mathcal{B}|}(X)$, $m_{\log|\mathcal{B}|}(X)$ and $P_{\log|\mathcal{B}|}(X)$, respectively.

All the sets $m_{\mathcal{A}}(X)$, $P_{\mathcal{A}}(X)$ and $\check{S}_{\mathcal{A}}(X)$ are possibly empty. If $m_{\mathcal{A}}(X)$ is non-empty, it is not necessarily closed, while $\check{S}_{\mathcal{A}}(X)$ is by definition always a closed subset of X. The following examples show that the sets $m_{\mathcal{A}}(X)$, $\check{S}_{\mathcal{A}}(X)$ and $P_{\mathcal{A}}(X)$ may differ or might be empty.

Example 6.3.2 (1) We enumerate the subset $L = [0,1] \cap \mathbb{Q}$ of $X = [0,1]$ by a sequence $(x_n)_{n\in\mathbb{N}}$. For the subfamily $\mathcal{A} = \{\chi_{\{x_n\}} : n \in \mathbb{N}\}$ of upper semi-continuous functions on X, we have that

$$P_{\mathcal{A}}(X) = m_{\mathcal{A}}(X) = L \subsetneq \check{S}_{\mathcal{A}}(X) = [0,1].$$

(2) In contrast to the previous example, by Example 6.1.2 (1) we can see that there is a subfamily $\mathcal{A}$ of $\mathcal{USC}(X)$ such that $\check{S}_{\mathcal{A}}(X)$, $P_{\mathcal{A}}(X)$ and $m_{\mathcal{A}}(X)$ are all empty.

We give some properties of the Shilov boundary, the minimal boundary and the peak points and their mutual relations.

Proposition 6.3.3 *Let $\mathcal{A}$ be a subfamily of upper semi-continuous functions on a compact Hausdorff space X.*

(1) The set $P_{\mathcal{A}}(X)$ lies in every $\mathcal{A}$-boundary S from $B_{\mathcal{A}}(X)$. If $P_{\mathcal{A}}(X)$ is itself an $\mathcal{A}$-boundary, then it is exactly the minimal boundary $m_{\mathcal{A}}(X)$.

(2) Each peak point $x \in X$ for $\mathcal{A}$ belongs to each boundary for $\mathcal{A}$. Moreover, if $m_{\mathcal{A}}(X)$ exists, then

$$P_{\mathcal{A}}(X) \subset m_{\mathcal{A}}(X) \subset \check{S}_{\mathcal{A}}(X).$$

(3) If $m_{\mathcal{A}}(X)$ exists, then $\check{S}_{\mathcal{A}}(X) = \overline{m_{\mathcal{A}}(X)}$.

(4) Let $\mathcal{A}_1 \subset \mathcal{A}_2 \subset \mathcal{USC}(X)$. Then we have the following inclusions,

$$B_{\mathcal{A}_2}(X) \subset B_{\mathcal{A}_1}(X) \quad \text{and} \quad P_{\mathcal{A}_1}(X) \subset P_{\mathcal{A}_2}(X).$$

If $m_{\mathcal{A}_1}(X)$ and $m_{\mathcal{A}_2}(X)$ exist, then $m_{\mathcal{A}_1}(X) \subset m_{\mathcal{A}_2}(X)$.

(5) Let $\mathcal{A} = \bigcup_{j \in J} \mathcal{A}_j$, where $\mathcal{A}_j$ are subsets of $\mathcal{USC}(X)$. Then

$$P_{\mathcal{A}}(X) = \bigcup_{j \in J} P_{\mathcal{A}_j}(X).$$

If $m_{\mathcal{A}_j}(X)$ exists for every $j \in J$, then $m_{\mathcal{A}}(X)$ exists and

$$m_{\mathcal{A}}(X) = \bigcup_{j \in J} m_{\mathcal{A}_j}(X).$$

Proof. (1) Let $x \in P_{\mathcal{A}}(X)$ and $f \in \mathcal{A}$ such that f peaks at x. Given an $\mathcal{A}$-boundary S, it is clear that $S \cap S(f) = \{x\}$. In particular, the point x lies in S. This yields the inclusion $P_{\mathcal{A}}(X) \subset S$. Now if $P_{\mathcal{A}}(X)$ lies in $B_{\mathcal{A}}(X)$, then by the previous discussion and by the definition of the minimal boundary for $\mathcal{A}$, we have that $P_{\mathcal{A}}(X) = m_{\mathcal{A}}(X)$.

(2) Since $m_{\mathcal{A}}(X) \in B_{\mathcal{A}}(X)$, it follows from the property (1) and the definition of $m_{\mathcal{A}}(X)$ that $P_{\mathcal{A}}(X) \subset m_{\mathcal{A}}(X) \subset S$ for every $S \in b_{\mathcal{A}}(X) \subset B_{\mathcal{A}}(X)$. Hence, $m_{\mathcal{A}}(X) \subset \check{S}_{\mathcal{A}}(X)$.

(3) Since $\check{S}_{\mathcal{A}}(X)$ is closed and $m_{\mathcal{A}}(X) \subset \check{S}_{\mathcal{A}}(X)$ by the previous point (2), $\overline{m_{\mathcal{A}}(X)}$ is a subset of $\check{S}_{\mathcal{A}}(X)$. On the other hand, $m_{\mathcal{A}}(X)$ is contained in $B_{\mathcal{A}}(X)$, and therefore $\overline{m_{\mathcal{A}}(X)}$ is a closed $\mathcal{A}$-boundary. By the definition of the Shilov boundary, this means that $\check{S}_{\mathcal{A}}(X) \subset \overline{m_{\mathcal{A}}(X)}$, and so $\overline{m_{\mathcal{A}}(X)} = \check{S}_{\mathcal{A}}(X)$.

(4) These inclusions follow immediately from the definitions of the corresponding sets.

(5) The identity $P_{\mathcal{A}}(X) = \bigcup_{j\in J} P_{\mathcal{A}_j}(X)$ is obvious.

In order to show that $m := \bigcup_{j\in J} m_{\mathcal{A}_j}(X)$ is a minimal $\mathcal{A}$-boundary, pick an arbitrary function $f \in \mathcal{A}$. Then $f \in \mathcal{A}_j$ for some index $j \in J$. By assumption, $m_{\mathcal{A}_j}(X)$ is a minimal boundary for $\mathcal{A}_j$. Thus, we obtain that

$$\emptyset \neq S(f) \cap m_{\mathcal{A}_j}(X) \subset S(f) \cap m,$$

implying that $m \in B_{\mathcal{A}}(X)$. Now let S be an arbitrary $\mathcal{A}$-boundary in X. By point (4), we have that $S \in B_{\mathcal{A}}(X) \subset B_{\mathcal{A}_j}(X)$ for every $j \in J$. Therefore, $m_{\mathcal{A}_j}(X) \subset S$ for every $j \in J$ and, thus, $m \subset S$. This shows the minimality of m, so $m_{\mathcal{A}}(X) = m$. □

6.4 Peak point theorems

We recall some necessary notions which build the setting of Bishop's theorem.

Definition 6.4.1 Let X be a Hausdorff space.

(1) We say that X is *metrizable* if it has a metric which induces the given topology. In this case, its topology admits a countable base.

(2) Let $\mathcal{A}$ be a subset of complex or real valued functions on X. Then we say that $\mathcal{A}$ is *separating* or *separating function of* X if for every $x, y \in X$, $x \neq y$, there exists a function $f \in \mathcal{A}$ such that $f(x) \neq f(y)$.

We recall Bishop's peak point theorem for uniform algebras of continuous functions (see Theorem 1 in [Bis59]). Further generalizations were already obtained by H. G. Dales to Banach function algebras (see [Dal71] and [Hon88]) and by Z. Siciak to certain additive subfamilies of continuous functions (see Theorem 3 in [Sic62]).

Theorem 6.4.2 (Bishop, 1959) *Let X be a compact metrizable Hausdorff space and $\mathcal{B}$ a separating uniform subalgebra of $\mathcal{C}(X)$. Then the minimal boundary of $\mathcal{B}$ in X exists and is exactly the set of all peak points for $\mathcal{B}$ in X, i.e., $m_{\mathcal{B}}(X) = P_{\mathcal{B}}(X)$.*

Bishop's theorem applies to unions of uniform subalgebras.

Corollary 6.4.3 *Suppose $\mathcal{B}$ is a union of separating uniform subalgebras $(\mathcal{B}_j)_{j\in J}$ of $\mathcal{C}(X)$, where X is a metrizable compact Hausdorff space. Then $m_{\mathcal{B}}(X)$ exists and*

$$m_{\mathcal{B}}(X) = P_{\mathcal{B}}(X) \quad \text{and} \quad \check{S}_{\mathcal{B}}(X) = \overline{P_{\mathcal{B}}(X)}.$$

Proof. By Bishop's theorem $m_{\mathcal{B}_j}(X)$ exists and coincides with $P_{\mathcal{A}_j}(X)$ for every $j \in J$. By Proposition 6.3.3 (5), we obtain that $m_{\mathcal{B}}(X)$ is the minimal boundary for $\mathcal{B}$ and

$$P_{\mathcal{B}}(X) = \bigcup_{j\in J} P_{\mathcal{B}_j}(X) = \bigcup_{j\in J} m_{\mathcal{B}_j}(X) = m_{\mathcal{B}}(X).$$

The identity $\check{S}_{\mathcal{B}}(X) = \overline{P_{\mathcal{B}}(X)}$ follows now from Proposition 6.3.3 (3). □

Bishop presented the following examples in [Bis59].

Example 6.4.4 (1) Bishop's theorem fails in general if X is not necessarily metrizable. To see this, consider the compact set

$$X := \big\{x = \{x_\alpha\}_{\alpha\in\mathbb{R}} : x_\alpha \in [0,1] \text{ for every } \alpha \in \mathbb{R}\big\}$$

and the family $\mathcal{A}$ of continuous functions f on X which fulfill the subsequent: if $x, y \in X$ and $x_\alpha = y_\alpha$ for countably many $\alpha \in \mathbb{R}$, then $f(x) = f(y)$. By the Stone-Weierstraß theorem, we have that $\mathcal{A}$ is exactly the set of all continuous functions on X and forms a separating Banach algebra. For $c \in \mathbb{R}$ define the set

$$S_c := \{x \in X : x_\alpha = c \text{ for almost every } \alpha \in \mathbb{R}\}.$$

Then S_c is a boundary for $\mathcal{C}(X)$, but $S_c \cap S_d$ is empty for $c \neq d$. Therefore, $\mathcal{C}(X)$ has no minimal boundary.

(2) Let X be the boundary of the unit disc $\Delta := \Delta_1(0)$ in $\mathbb{C}$. Let $\mathcal{B}$ be the family of all continuous functions f on X which have a holomorphic extension F inside the unit disc Δ and which satisfy $F(0) = f(1)$. Then $\mathcal{B}$ is a separating Banach subalgebra of $\mathcal{C}(\overline{\Delta})$ such that

$$m_{\mathcal{B}}(X) = X \setminus \{1\}, \text{ but } \check{S}_{\mathcal{B}}(X) = X.$$

We will give another peak point theorem. But first, we introduce a useful subfamily of upper semi-continuous functions.

Definition 6.4.5 Let $\mathcal{A}$ be a subfamily of upper semi-continuous functions on a (not necessarily) Hausdorff space X and let Θ be a subset of non-negative continuous functions on X with the following property: for each $x \in X$ and each closed subset S of X with $x \notin S$ there exists a function $\vartheta \in \Theta$ such that $S_X(\vartheta) = \{x\}$ and ϑ vanishes on S. We say that a function $f \in \mathcal{A}$ is a *strictly $\mathcal{A}$-function with respect to* Θ if for every $\vartheta \in \Theta$ there is a number $\varepsilon_0 > 0$ such that $f + \varepsilon\vartheta \in \mathcal{A}$ for every $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$. The subfamily of $\mathcal{A}$ consisting of all strictly $\mathcal{A}$-functions with respect to Θ is denoted by $\mathcal{A}[\Theta]$.

The main motivation for this definition is the following example.

Example 6.4.6 Let Ω be an open set in $\mathbb{C}^n$ and let Θ be the set of all $\mathcal{C}^\infty$-smooth functions on $\mathbb{C}^n$ with compact support in Ω. Then in view of Remark 3.4.2 we easily derive that $\mathcal{PSH}_q(\Omega)[\Theta]$ is exactly the set of all strictly q-plurisubharmonic functions on Ω.

Involving strictly $\mathcal{A}$-functions, we can now present another version of Bishop's theorem which better incorporates the properties of subfamilies of q-plurisubharmonic functions.

Theorem 6.4.7 *Let $\mathcal{A}$ be a subfamily of upper semi-continuous functions on a compact Hausdorff space X. Suppose that there exist a subfamily Θ from Definition 6.4.5 and a positive function $\omega \in \mathcal{A}[\Theta]$ such that $\mathcal{A} + \{\varepsilon\omega\} \in \mathcal{A}[\Theta]$ for every positive number $\varepsilon > 0$. Then*

$$\check{S}_{\mathcal{A}}(X) = \overline{P_{\mathcal{A}}(X)} \in b_{\mathcal{A}}(X).$$

Proof. First, observe that $P_{\mathcal{A}[\Theta]}(X)$ is non-empty. Indeed, the function ω attains its maximum on X, say at a point $x_0 \in X$. Pick a function $\vartheta \in \Theta$ with $S_X(\vartheta) = \{x_0\}$. Then there is a positive number $\delta > 0$ such that $\omega + \delta\vartheta$ belongs to $\mathcal{A}$. Therefore, $2\omega + \delta\vartheta$ is in $\mathcal{A}[\Theta]$ by the assumption made on ω. Moreover, $S_X(2\omega + \delta\vartheta) = \{x_0\}$ and, thus, $x_0 \in P_{\mathcal{A}[\Theta]}(X)$.

The non-empty set $\overline{P_{\mathcal{A}[\Theta]}(X)}$ is obviously a subset of $\check{S}_{\mathcal{A}[\Theta]}(X)$. In order to verify the converse inclusion, we show that $S := \overline{P_{\mathcal{A}[\Theta]}(X)}$ is a boundary for the family $\mathcal{A}[\Theta]$. In order to get a contradiction, suppose that this is not the case. Then there exists a function $f \in \mathcal{A}[\Theta]$ such that $\max_X f > \max_S f$. If $\varepsilon_0 > 0$ is small enough, the function $g := f + \varepsilon_0 \omega$ also fulfills $\max_X g > \max_S g$. Let x_1 be a point in $X \setminus S$ such that $g(x_1) = \max_X g$ and let θ be a function from Θ such that $S_X(\theta) = \{x_1\}$ and θ vanishes on S. In particular, we have that $\theta(x_1) > 0$. Then for a small enough number $\varepsilon_1 > 0$ the function $f + \varepsilon_1 \theta$ is in $\mathcal{A}$. Hence, the function $h := g + \varepsilon_1 \theta = f + \varepsilon_1 \theta + \varepsilon_0 \omega$ lies in $\mathcal{A}[\Theta]$ and fulfills $S_X(h) = \{x_1\}$. Thus, $x_1 \in P_{\mathcal{A}[\Theta]}(X) \subset S$, but this contradicts to

$$\max_S h = \max_S g < \max_X g = g(x_1) < h(x_1) \le \max_S h.$$

Therefore, S has to be a $\mathcal{A}[\Theta]$-boundary, implying that $\check{S}_{\mathcal{A}[\Theta]}(X)$ is contained in S. Hence, Proposition 6.3.3 (1) yields

$$\overline{P_{\mathcal{A}[\Theta]}(X)} = \check{S}_{\mathcal{A}[\Theta]}(X) \in b_{\mathcal{A}[\Theta]}(X).$$

Now let f be an arbitrary function from $\mathcal{A}$. It follows that the sequence $(f_n)_{n\in\mathbb{N}}$ of functions $f_n := f + (1/n)\omega$ in $\mathcal{A}[\Theta]$ decreases to f. This implies that $\mathcal{A}$ lies in the closure of $\mathcal{A}[\Theta]$. Since $\mathcal{A}[\Theta]$ is a subset of $\mathcal{A}$ and, in view of Proposition 6.1.3 (8), we have that $b_{\mathcal{A}}(X) = b_{\mathcal{A}[\Theta]}(X)$ and $\check{S}_{\mathcal{A}[\Theta]}(X) = \check{S}_{\mathcal{A}}(X)$. Finally, the proof is finished due to the following inclusions,

$$\check{S}_{\mathcal{A}}(X) = \check{S}_{\mathcal{A}[\Theta]}(X) = \overline{P_{\mathcal{A}[\Theta]}(X)} \subset \overline{P_{\mathcal{A}}(X)} \subset \check{S}_{\mathcal{A}}(X).$$

□

Chapter 7

The q-Shilov boundaries

The abbreviation *q-Shilov boundary* stands for the Shilov boundary created by subfamilies of q-plurisubharmonic or q-holomorphic functions. We apply the results in the precedent Chapter 6 in order to obtain existence theorems of q-Shilov boundaries and the peak point theorems for a vast number of subfamilies of q-plurisubharmonic and q-holomorphic functions. The monotone closure introduced in Chapter 1 and the approximation techniques achieved in Chapter 3 allow us to restrict the considerations on the q-Shilov boundaries to a smaller number of subfamilies of q-plurisubharmonic. For instance, the Shilov boundary for q-plurisubharmonic functions and that for $\mathcal{C}^2$-smooth ones coincide. In the case of a bounded domain with $\mathcal{C}^2$-smooth boundary, its Shilov boundary for q-plurisubharmonic functions is exactly the closure of all strictly q-pseudoconvex boundary points. The latter statement extends the observations by L. R. Hunt and J. J. Murray [HM78] which are based on results of R. Basener [Bas78] in the case of q-holomorphic functions. We give the relation of q-Shilov boundaries to lower dimensional ones and also compare the q-Shilov boundaries to the q-th order Shilov boundary introduced by R. Basener in [Bas78].

The main result of this chapter is the generalization of S. N. Bychkov's theorem [Byč81] which gives a geometric characterization of the Shilov boundary of a convex body in $\mathbb{C}^n$ in terms of *real* and *complex* boundary points. The latter ones are boundary points of a convex body lying in an open part of a complex plane which is situated inside the boundary of that convex body. If there exists such a plane which has complex dimension at least q, then this boundary point is *q-complex*. Based on Bychkov's result and our observations on lower dimensional Shilov boundaries, we can verify that the complement of the Shilov boundary for q-plurisubharmonic functions is exactly the interior of all $(q+1)$-complex points. We also show that the Shilov boundary of convex bodies for q-plurisubharmonic functions and that for continuous functions which are holomorphic on leaves of a foliation by complex planes of codimension q are identical.

A. G. Vitushkin conjectured that the Hausdorff dimension of the Shilov boundary for holomorphic functions on convex bodies sets in $\mathbb{C}^n$ is greater or equal to n. Bychkov [Byč81] was able to show that this is true for $n = 2$ using the fact that, in the two-dimensional setting, the boundary part of the complement of the Shilov boundary is foliated by parallel complex lines. This is false in general for higher dimensions due to an example by N. Nikolov and P. J. Thomas [NT12]. It seems that Vitushkin's conjecture remains open. Anyway, using Bychkov's result, we give an estimate on the Hausdorff dimension of the q-Shilov boundaries of convex bodies in $\mathbb{C}^n$.

This chapter is mostly contained in the article [Paw13].

7.1 Shilov boundary and q-plurisubharmonicity

We will analyze the relation between the different Shilov boundaries for subfamilies of q-plurisubharmonic.

Definition 7.1.1 Fix $k \in \{*, c, 0, 1, \ldots, \infty\}$ and $q \in \mathbb{N}_0$. Let A be an analytic subset of an open set Ω in $\mathbb{C}^n$.

(1) We set $\mathcal{C}^*(A) := \mathcal{USC}(A)$. Furthermore, for $k \neq c$, the symbol $\mathcal{C}^k(A)$ denotes the family of all real-valued functions f defined on A such that for each point $p \in A$ there exist a neighborhood U of p and a function $F \in \mathcal{C}^k(U)$ with $F = f$ on $A \cap U$.

(2) We define $\mathcal{PSH}_q^k(A) := \mathcal{PSH}_q(A) \cap \mathcal{C}^k(A)$, if $k \neq c$.

(3) For a compact set K in A, the family $\mathcal{PSH}_q^k(K)$ stands for the family of all upper semi-continuous functions f on K such that there are a neighborhood U of K in A and $F \in \mathcal{PSH}_q^k(U)$ with $F = f$ on K.

We also introduce new subfamilies of r-plurisubharmonic functions on singular foliations. In this context, recall Definition 3.9.5.

Definition 7.1.2 Let $k \in \{*, 0, 1, \ldots, \infty\}$, $q \in \{0, \ldots, n-1\}$ and $r \in \mathbb{N}_0$. Assume that $H = \{h_j : U_j \to \mathbb{C}^q\}_{j \in J}$ is a web of singular foliations on an open set Ω in $\mathbb{C}^n$.

(1) We define $\mathcal{PSH}_r^k(H,\Omega) := \mathcal{PSH}_r(H,\Omega) \cap \mathcal{C}^k(\Omega)$.

(2) Let K be a compact set in Ω. An upper semi-continuous function ψ lies in the family $\mathcal{PSH}_r^k(H,K)$ if there exist an open neighborhood V of K and a function Ψ in $\mathcal{PSH}_r^k(H',V\cap\Omega)$, where $H' := \{h_j|V\cap U_j\}_{j\in J}$, such that $\Psi=\psi$ on K.

(3) If $r=0$, we sometimes skip the lower index r and simply write $\mathcal{PSH}^k(H,\Omega)$ and $\mathcal{PSH}^k(H,K)$ instead of $\mathcal{PSH}_0^k(H,\Omega)$ and, respectively, $\mathcal{PSH}_0^k(H,K)$.

The Shilov boundaries for the new families defined above are related as follows.

Remark 7.1.3 With the notations of the preceding two definitions and by Theorem 3.9.6, we have the following scheme of inclusions:

$$\begin{array}{ccccc} \check{S}_{\mathcal{PSH}_r^k(\Omega)}(K) & \subset & \check{S}_{\mathcal{PSH}_r^k(H,\Omega)}(K) & \subset & \check{S}_{\mathcal{PSH}_{q+r}^k(\Omega)}(K) \\ \cup & & \cup & & \cup \\ \check{S}_{\mathcal{PSH}_r^k(K)}(K) & \subset & \check{S}_{\mathcal{PSH}_r^k(H,K)}(K) & \subset & \check{S}_{\mathcal{PSH}_{q+r}^k(K)}(K) \end{array}$$

The same is true for the sets of peak points and the minimal boundaries for the corresponding families.

We show the existence of the Shilov boundary for certain subfamilies of q-plurisubharmonic functions and establish respective peak point properties.

Proposition 7.1.4 *Let A be an analytic subset of an open set Ω in $\mathbb{C}^n$ and let X be a compact subset of A. Given $k\in\{*,c,0,1,\dots,\infty\}$, $q\in\{0,\dots,n-1\}$ and $r\in\mathbb{N}_0$, we have that*

$$\overline{P_{\mathcal{PSH}_q^k(A)}(X)} = \check{S}_{\mathcal{PSH}_q^k(A)}(X) \quad \text{and} \quad \overline{P_{\mathcal{PSH}_q^k(X)}(X)} = \check{S}_{\mathcal{PSH}_q^k(X)}(X).$$

If $k\neq c$ and $H=\{h_j:U_j\to\mathbb{C}^q\}_{j\in J}$ is a web of singular foliations on Ω and K is a compact subset of Ω, then it holds that

$$\overline{P_{\mathcal{PSH}_r^k(H,\Omega)}(K)} = \check{S}_{\mathcal{PSH}_r^k(H,\Omega)}(K) \text{ and } \overline{P_{\mathcal{PSH}_r^k(H,K)}(K)} = \check{S}_{\mathcal{PSH}_r^k(H,K)}(K).$$

Moreover, each of this sets is a boundary for the respective families.

Proof. Let Θ be the set of all $\mathcal{C}^\infty$-smooth functions with compact support on Ω and define $\omega(z) := \|z\|_2^2+1$ for $z\in\mathbb{C}^n$. Denote by $\mathcal{A}$ any of the families mentioned in this proposition. Since $\omega+\varepsilon f$ belongs to $\mathcal{A}$ for every $\varepsilon>0$ and $f\in\mathcal{A}$, we can apply Theorem 6.4.7 to Θ, $\mathcal{A}$ and ω in order to get $\overline{P_{\mathcal{A}}} = \check{S}_{\mathcal{A}} \in b_{\mathcal{A}}$. $\square$

It is natural to ask whether the various q-Shilov boundaries coincide.

Proposition 7.1.5 *Given a compact set K in $\mathbb{C}^n$ we have that*

$$P_{\mathcal{PSH}_q^2(K)}(K) = P_{\mathcal{PSH}_q^c(K)}(K)$$

$$\text{and} \quad \check{S}_{\mathcal{PSH}_q^2(K)}(K) = \check{S}_{\mathcal{PSH}_q^c(K)}(K) = \check{S}_{\mathcal{PSH}_q^0(K)}(K) = \check{S}_{\mathcal{PSH}_q(K)}(K).$$

Proof. Since $\mathcal{PSH}_q^2(K) \subset \mathcal{PSH}_q^c(K) \subset \mathcal{PSH}_q^0(K) \subset \mathcal{PSH}_q(K)$, we derive for the set of peak points of these families that

$$P_{\mathcal{PSH}_q^2(K)}(K) \subset P_{\mathcal{PSH}_q^c(K)}(K) \subset P_{\mathcal{PSH}_q^0(K)}(K) \subset P_{\mathcal{PSH}_q(K)}(K). \qquad (7.1)$$

Then it follows from the first part of Proposition 7.1.4 that

$$\check{S}_{\mathcal{PSH}_q^2(K)}(K) \subset \check{S}_{\mathcal{PSH}_q^c(K)}(K) \subset \check{S}_{\mathcal{PSH}_q^0(K)}(K) \subset \check{S}_{\mathcal{PSH}_q(K)}(K).$$

Assume now that there is a function $\psi \in \mathcal{PSH}_q^c(K)$ which peaks at some point $p \in bK$. Then there exist a bounded open neighborhood U of p and finitely many $\mathcal{C}^2$-smooth q-plurisubharmonic functions $\psi_1, \ldots, \psi_k$ on U such that $\psi = \max_{j=1,\ldots,k} \psi_j$ on U. By picking a slightly smaller neighborhood of p and denoting it again by U, we can arrange that for every $j \in \{1, \ldots, k\}$ the function ψ_j is defined on a neighborhood of $\overline{U}$. Pick an index $j_0 \in \{1, \ldots, k\}$ such that $\psi(p) = \psi_{j_0}(p)$. Since ψ peaks at p, we have that

$$\psi_{j_0}(p) = \psi(p) > \psi(z) \geq \psi_{j_0}(z) \text{ for every } z \in (U \cap K) \setminus \{p\}.$$

Hence, ψ_{j_0} peaks at p in $K \cap U$. Since ψ_{j_0} is continuous on $\overline{U}$, we can choose a suitable constant $c \in \mathbb{R}$ such that

$$\psi_{j_0}(p) > c > \max_{bU \cap K} \psi_{j_0}.$$

By Lemma 3.5.9, for some positive number ε the function $\varphi := \widetilde{\max}_{\varepsilon}\{\psi_{j_0}, c\}$ is $\mathcal{C}^2$-smooth and q-plurisubharmonic on a neighborhood of $\overline{U} \cap K$. In view of Lemma 3.5.8 (3), we can choose $\varepsilon > 0$ so small that the function φ peaks at p in $K \cap \overline{U}$ and fulfills $\varphi \equiv c$ on a neighborhood of $bU \cap K$. Thus, we can extend φ by the constant c into a neighborhood of $K \setminus U$ in order to obtain a function from $\mathcal{PSH}_q^2(K)$ which peaks at p. Since p was an arbitrary peak point for the family $\mathcal{PSH}_q^c(K)$, we obtain the inclusion $P_{\mathcal{PSH}_q^c(K)}(K) \subset P_{\mathcal{PSH}_q^2(K)}(K)$. Since the converse inclusion is obviously true, we conclude that $P_{\mathcal{PSH}_q^c(K)}(K)$ equals $P_{\mathcal{PSH}_q^2(K)}(K)$, so Proposition 7.1.4 yields the identity

$$\check{S}_{\mathcal{PSH}_q^2(K)}(K) = \check{S}_{\mathcal{PSH}_q^c(K)}(K).$$

Furthermore, Słodkowski's approximation theorem 3.5.1 and Bungart's approximation theorem 3.5.4 imply the inclusions

$$\mathcal{PSH}_q(K) \subset \overline{\mathcal{PSH}_q^0(K)}^{\downarrow K} \quad \text{and} \quad \mathcal{PSH}_q^0(K) \subset \overline{\mathcal{PSH}_q^c(K)}^{\downarrow K}.$$

Therefore, Proposition 6.1.3 (8) gives

$$\begin{aligned} \check{S}_{\mathcal{PSH}_q(K)}(K) &\subset \check{S}_{\overline{\mathcal{PSH}_q^0(K)}^{\downarrow K}}(K) = \check{S}_{\mathcal{PSH}_q^0(K)}(K) \\ &\subset \check{S}_{\overline{\mathcal{PSH}_q^c(K)}^{\downarrow K}}(K) = \check{S}_{\mathcal{PSH}_q^c(K)}(K) \\ &\subset \check{S}_{\mathcal{PSH}_q(K)}(K). \end{aligned}$$

Hence, we obtain the remaining identities

$$\check{S}_{\mathcal{PSH}_q(K)}(K) = \check{S}_{\mathcal{PSH}_q^0(K)}(K) = \check{S}_{\mathcal{PSH}_q^c(K)}(K).$$

□

We can derive the following *local-to-global peak point property* from the proof of the previous result.

Proposition 7.1.6 *Let p be a boundary point of a compact set K in $\mathbb{C}^n$. If p is a local peak point for q-plurisubharmonic functions with corners, i.e., there is a neighborhood U of p and a q-plurisubharmonic function ψ with corners on U such that $\psi(p) > \psi(z)$ for every $z \in (U \cap K) \setminus \{p\}$, then p is a (global) peak point for $\mathcal{PSH}_q^2(K)$.*

The following connection between Shilov boundaries for smooth and continuous plurisubharmonic functions on a singular foliation follows directly from Theorem 3.9.11 and Proposition 6.1.3 (8).

Proposition 7.1.7 *Fix an integer* $q \in \{0, \dots, n-1\}$. *Let* $h : \Omega \to \mathbb{C}^q$ *be a holomorphic mapping on a Stein open set* Ω *in* $\mathbb{C}^n$ *and let* K *be a compact subset of* Ω. *Then*

$$\check{S}_{\mathcal{PSH}(h,\Omega)}(K) = \check{S}_{\mathcal{PSH}^\infty(h,\Omega)}(K).$$

If K *has a neighborhood basis of Stein open sets, then*

$$\check{S}_{\mathcal{PSH}(h,K)}(K) = \check{S}_{\mathcal{PSH}^\infty(h,K)}(K).$$

Before we give a precise characterization of the Shilov boundary for q-plurisubharmonic functions in terms of strictly q-pseudoconvex boundary points, we fix the subsequent notation.

Definition 7.1.8 Let D be a bounded domain in $\mathbb{C}^n$ with $\mathcal{C}^2$-smooth boundary. Given an integer $q \in \{0, \dots, n-1\}$, denote by $\mathcal{S}_q(\overline{D})$ the set of all points $p \in bD$ such that D is strictly q-pseudoconvex at p (recall Definition 4.5.5 for the notion of strict q-pseudoconvexity).

We extend the observations by L. R. Hunt and J. J. Murray in [HM78] on the link between the Shilov boundary for q-plurisubharmonic functions and the set of all strict q-pseudoconvex points.

Theorem 7.1.9 *Let* $q \in \{0, \dots, n-1\}$ *and let* D *be a bounded domain in* $\mathbb{C}^n$ *with* $\mathcal{C}^2$*-smooth boundary. Then*

$$\check{S}_{\mathcal{PSH}_q(\overline{D})}(\overline{D}) = \overline{\mathcal{S}_q(\overline{D})}.$$

Proof. It follows from Theorem 5.6 in [HM78] that

$$P_{\mathcal{PSH}^2_q(D)\cap\mathcal{C}^0(\overline{D})}(\overline{D}) \subset \overline{\mathcal{S}_q(\overline{D})} \quad \text{and} \quad \mathcal{S}_q(\overline{D}) \subset P_{\mathcal{PSH}_q(D)\cap\mathcal{C}^0(\overline{D})}(\overline{D}).$$

Since $\mathcal{PSH}^2_q(\overline{D}) \subset \mathcal{PSH}^2_q(D) \cap \mathcal{C}^0(\overline{D})$, we easily obtain the inclusion

$$P_{\mathcal{PSH}^2_q(\overline{D})}(\overline{D}) \subset \overline{\mathcal{S}_q(\overline{D})}. \tag{7.2}$$

Hence, according to Propositions 7.1.5 and 7.1.4, we have that

$$\check{S}_{\mathcal{PSH}^2_q(\overline{D})}(\overline{D}) = \check{S}_{\mathcal{PSH}_q(\overline{D})}(\overline{D}) = \overline{P_{\mathcal{PSH}^2_q(\overline{D})}(\overline{D})} \subset \overline{\mathcal{S}_q(\overline{D})}. \tag{7.3}$$

On the other hand, let $p \in \mathcal{S}_q(\overline{D})$. Then there is a neighborhood U of p and a $\mathcal{C}^2$-smooth strictly q-plurisubharmonic function ρ on U such that ρ vanishes on

$bD \cap U$ and $\rho(z) < 0$ if $z \in U \cap D$. Since ρ is strictly q-plurisubharmonic, there is a positive constant $\varepsilon > 0$ such that $\varphi(z) := \rho(z) - \varepsilon \|z - p\|_2^2$ remains q-plurisubharmonic on U. Moreover, $\varphi(p) = 0$ and $\varphi(z) < 0$ for every $z \in (U \cap \overline{D}) \setminus \{p\}$. In view of Proposition 7.1.6, the point p is also a global peak point for the family $\mathcal{PSH}_q^2(\overline{D})$. Since p is an arbitrary point from $\mathcal{S}_q(\overline{D})$, it follows that $\mathcal{S}_q(\overline{D})$ lies in $P_{\mathcal{PSH}_q^2(\overline{D})}$. Since by Proposition 7.1.4 the set $\check{S}_{\mathcal{PSH}_q^2(\overline{D})}(\overline{D})$ equals the closure of $P_{\mathcal{PSH}_q^2(\overline{D})}(\overline{D})$, the statement follows now from the inclusions (7.3). □

Some parts of the Shilov boundary for q-plurisubharmonic functions on smoothly bounded domains admit a complex foliation. For further results on complex foliations of real submanifolds we refer to [Fre74].

Theorem 7.1.10 *Let $q \in \{1, \ldots, n-1\}$ and let D be a bounded pseudoconvex domain in $\mathbb{C}^n$ with $\mathcal{C}^2$-smooth boundary. Then the following relative open part in bD of the Shilov boundary for $\mathcal{PSH}_q(\overline{D})$,*

$$\mathcal{F}_q(\overline{D}) := \operatorname{int}_{bD}\left(\check{S}_{\mathcal{PSH}_q(\overline{D})}(\overline{D}) \setminus \check{S}_{\mathcal{PSH}_{q-1}(\overline{D})}(\overline{D})\right),$$

locally admits a foliation by complex q-dimensional submanifolds, provided it is not empty.

Proof. Since $\mathcal{S}_q(\overline{D})$ is relatively open in the boundary of D, Theorem 7.1.9 implies that

$$\mathcal{F}_q(\overline{D}) = \operatorname{int}_{bD}\left(\overline{\mathcal{S}_q(\overline{D})} \setminus \overline{\mathcal{S}_{q-1}(\overline{D})}\right) = \mathcal{S}_q(\overline{D}) \setminus \overline{\mathcal{S}_{q-1}(\overline{D})}. \tag{7.4}$$

Given a defining function ϱ of D, we deduce from the definition of the set $\mathcal{S}_q(\overline{D})$, from the pseudoconvexity of D and from the identities in (7.4) that for each point $p \in \mathcal{F}_q(\overline{D})$ the Levi form $\mathscr{L}_\varrho$ of ϱ at p has exactly $n-q-1$ positive and q zero eigenvalues on the holomorphic tangent space $H_p bD$ to bD at p. In particular, for each point p in $\mathcal{F}_q(\overline{D})$ the Levi null space $\mathcal{N}_p$ is q-dimensional (recall Definition 4.7.4). Then it follows from Theorem 1.1 in [Fre74] that the set $\mathcal{F}_q(\overline{D})$ locally admits a foliation by complex q-dimensional submanifolds. □

We close this section by presenting a subfamily of q-plurisubharmonic functions which arises naturally, but which has a trivial Shilov boundary.

Remark 7.1.11 Let K be a convex body in $\mathbb{C}^n$ and let $\mathcal{A}$ be the family of upper semi-continuous functions on K which are q-plurisubharmonic on the interior of K. In view of Theorem 6.2.4, the Shilov boundary $\check{S}_{\mathcal{A}}$ for $\mathcal{A}$ exists. By the local maximum principle in Proposition 3.3.2 (11), the Shilov boundary for $\mathcal{A}$ is contained in the boundary of K. On the other hand, pick a point x in the boundary of K. Then the characteristic function $\chi_{\{x\}}$ of the set $\{x\}$ in K lies in $\mathcal{A}$ and peaks at x. Hence, we have that the whole boundary of K coincides with $P_{\mathcal{A}}(K)$. Since the set of all peak points for $\mathcal{A}$ lies in the Shilov boundary for $\mathcal{A}$, we conclude that $\check{S}_{\mathcal{A}}(K) = bK$.

7.2 Shilov boundary and q-holomorphicity

We introduce new subfamilies of q-holomorphic functions. Most of the families we will define are based on families which already appeared in the literature in the study of the classical Shilov boundary.

Definition 7.2.1 Let $q \in \{0, \ldots, n-1\}$ and K be a compact set in $\mathbb{C}^n$. Denote by $\mathrm{int}(K)$ the interior of K.

(1) We set $A_q(K) := \mathcal{O}_q(\mathrm{int}(K)) \cap \mathcal{C}(K)$. In the case of $q = 0$, we simply write $A(K)$ instead of $A_0(K)$.

(2) A continuous function f on K lies in $\mathcal{O}_q(K)$ if there exist an open neighborhood U of K and a function $F \in \mathcal{O}_q(U)$ with $F = f$ on K. If $q = 0$, it is convenient to us that $\mathcal{O}(K)$ stands for $\mathcal{O}_0(K)$.

(3) By historical reasons [Ber31], the Shilov boundary for the family $\mathcal{O}(K)$ is also called the *Bergman boundary of* K.

We are also interested in the Shilov boundary for the following subfamilies of functions holomorphic on singular foliations.

Definition 7.2.2 Fix $q \in \{0, \ldots, n-1\}$ and let $H := \{h_j : U_j \to \mathbb{C}^q\}_{j \in J}$ be a web of singular foliations of an open set Ω in $\mathbb{C}^n$.

(1) We set $A(H, K) := \mathcal{O}(H, \mathrm{int}(K)) \cap \mathcal{C}(K)$.

(2) A function f belongs to the family $\mathcal{O}(H, K)$ if there are an open neighborhood V of K and a function $F \in \mathcal{O}(H', V \cap \Omega)$, where $H' := \{h_j | V \cap U_j\}_{j \in J}$, such that $F = f$ on K.

We have the following properties of the respective Shilov boundaries for this new subfamilies of q-holomorphic functions.

Proposition 7.2.3 *Let $q \in \{0, \dots, n-1\}$ and let K be a compact set in an open set Ω in $\mathbb{C}^n$. Let $H := \{h_j : U_j \to \mathbb{C}^q\}_{j \in J}$ be a web of singular foliations on Ω. If $\mathcal{B}$ is any of the families from Definitions 7.2.1 and 7.2.2 or even one of the families $\mathcal{O}_q(\Omega)$ or $\mathcal{O}(H,\Omega)$, then the Shilov boundary for $\mathcal{B}$ exists and fulfills $\check{S}_{\mathcal{B}}(K) = \overline{P_{\overline{\mathcal{B}}^K}(K)}$. Moreover, we have the following scheme of inclusions:*

$$\begin{array}{ccccc} \check{S}_{\mathcal{O}_q(\Omega)}(K) & \subset & \check{S}_{\mathcal{O}_q(K)}(K) & \subset & \check{S}_{A_q(K)}(K) \\ \cup & & \cup & & \cup \\ \check{S}_{\mathcal{O}(H,\Omega)}(K) & \subset & \check{S}_{\mathcal{O}(H,K)}(K) & \subset & \check{S}_{A(H,K)}(K) \end{array}$$

Proof. The scheme of inclusions follows directly from the definition of the corresponding families and from Theorem 3.11.5.

Let $\mathcal{B}$ be any of the families $\mathcal{O}_q(\Omega)$, $\mathcal{O}_q(K)$, $A_q(K)$, $\mathcal{O}(H,\Omega)$, $\mathcal{O}(H,K)$ or $A(H,K)$. Pick a function $f \in \mathcal{B}$ and denote by $\mathcal{B}_f$ the uniform closure (in $\mathcal{C}(K)$) of the algebra generated by f and and the family of holomorphic functions $\mathcal{O}(\mathbb{C}^n)$. Now we set $\mathcal{B}_\bullet := \bigcup_{f \in \mathcal{B}} \mathcal{B}_f$. Then it follows from Proposition 3.10.2 (4) that $\mathcal{B}_f \subset \overline{\mathcal{B}}^K$ for every $f \in \mathcal{B}$. Therefore,

$$\mathcal{B} \subset \mathcal{B}_\bullet \subset \overline{\mathcal{B}}^K \quad \text{and} \quad b_{\mathcal{B}}(K) = b_{\overline{\mathcal{B}}^K}(K) \subset b_{\mathcal{B}_\bullet}(K) \subset b_{\mathcal{B}}. \tag{7.5}$$

Therefore, all the Shilov boundaries for $\mathcal{B}$, $\mathcal{B}_\bullet$ and $\overline{\mathcal{B}}^K$ are the same. Since we can apply Bishop's theorem 6.4.2 to each uniform algebra $\mathcal{B}_f$, Proposition 6.4.3 and (7.5) yield

$$\check{S}_{\mathcal{B}}(K) = \check{S}_{\mathcal{B}_\bullet}(K) = \overline{P_{\mathcal{B}_\bullet}(K)} \in b_{\mathcal{B}_\bullet}(K) = b_{\mathcal{B}}(K).$$

Finally, the subsequent chain of contentions gives the remaining peak point property $\check{S}_{\mathcal{B}}(K) = \overline{P_{\overline{\mathcal{B}}^K}(K)}$,

$$\check{S}_{\mathcal{B}}(K) = \check{S}_{\mathcal{B}_\bullet}(K) = \overline{P_{\mathcal{B}_\bullet}(K)} \subset \overline{P_{\overline{\mathcal{B}}^K}(K)} \subset \check{S}_{\overline{\mathcal{B}}^K}(K) = \check{S}_{\mathcal{B}}(K).$$

□

The following remark gives a relation between Shilov boundaries generated by subfamilies of q-holomorphic and q-plurisubharmonic functions.

Remark 7.2.4 (1) Let Ω be an open set in $\mathbb{C}^n$ and fix $k \in \{0, 1, \ldots, \infty\}$ and $q \in \{0, \ldots, n-1\}$. Pick a function $f \in \mathcal{O}_q^k(\Omega) := \mathcal{O}_q(\Omega) \cap \mathcal{C}^k(\Omega)$. Then by Proposition 3.10.2 (8), the function $\log|f|$ is q-plurisubharmonic on Ω and belongs to $\mathcal{C}^k(\Omega \setminus \{f = 0\})$. Now consider the $\psi_m := \widetilde{\max}_{1/m}\{\log|f|, -m\}$ for $m \in \mathbb{N}$. In view of Lemma 3.5.9, it lies in $\mathcal{C}^k(\Omega)$ and is q-plurisubharmonic on the whole of Ω. Moreover, the sequence $\{\psi_m\}_{m\in\mathbb{N}}$ decreases to $\log|f|$ on Ω. Thus, for every compact set K in Ω it holds that

$$\log|\mathcal{O}_q^k(\Omega)| \subset \overline{\mathcal{PSH}_q^k(\Omega)}^{\downarrow K}, \quad \text{and therefore} \quad \check{S}_{\mathcal{O}_q^k(\Omega)}(K) \subset \check{S}_{\mathcal{PSH}_q^k(\Omega)}(K).$$

(2) Let Ω be a Stein open set and K be a compact subset of Ω. Assume that $h : \Omega \to \mathbb{C}^q$ is a holomorphic mapping, where $q \in \{0, \ldots, n-1\}$. Then Theorems 3.11.5 and 3.11.8 yield

$$\check{S}_{\mathcal{PSH}^0(h,\Omega)}(K) = \check{S}_{\mathcal{O}(h,\Omega)}(K) \subset \check{S}_{\mathcal{O}_q(\Omega)}(K)$$

If, in addition, K has a neighborhood basis of Stein open sets, then

$$\check{S}_{\mathcal{PSH}^0(h,K)}(K) = \check{S}_{\mathcal{O}(h,K)}(K) \subset \check{S}_{\mathcal{O}_q(K)}(K).$$

The following example is due to L. Aĭzenberg and can be found in his book [Aiz93] or in Example 1 in §16.3 of Fuks' book [Fuk65]. It shows that the Shilov boundary for $A(K)$ and the Bergman boundary of K may differ in general.

Example 7.2.5 Let $D := \{(z, w) \in \mathbb{C}^2 : 0 < |w| < 1,\ |z| < |w|^{-\log|w|}\}$. Observe that for $m, k \in \mathbb{N}$ the function $f_{m,k}(z, w) := w^{-m} z^k$ belongs to $A(K)$. Then the Shilov boundary for $A(\overline{D})$ is the set $\{|w| \le 1,\ |z| = \varphi(w)\}$, where $\varphi(w) := |w|^{-\log|w|}$ for $0 < |w| < 1$ and $\varphi(0) := 0$. On the other hand, by the local modulus maximum principle for holomorphic functions, we derive that the Bergman boundary of $\overline{D}$ is equal to $\{|z| = 1,\ |w| = 1\}$. Hence, $\check{S}_{\mathcal{O}(\overline{D})} \subsetneq \check{S}_{A(\overline{D})}$. As a consequence, in general, it is impossible to approximate functions from $A(\overline{D})$ uniformly on $\overline{D}$ by functions from $\mathcal{O}(\overline{D})$.

In the case of smoothly bounded domains, it is interesting to describe the Shilov and the Bergman boundaries for certain families of q-holomorphic functions using strictly q-pseudoconvex boundary points of the given domain.

First studies in the case of $q = 0$ were performed by H. J. Bremermann in [Bre59]. It was H. Rossi [Ros61] who proved the subsequent characterization of the Bergman boundary.

Theorem 7.2.6 *Let D be a bounded domain in $\mathbb{C}^n$ which is $\mathcal{C}^2$-smoothly bounded and has a Stein neighborhood basis. Then the Bergman boundary $\check{S}_{\mathcal{O}(\overline{D})}(\overline{D})$ coincides with the closure of the set of all strictly pseudoconvex boundary points of D.*

Remark 7.2.7 The previous theorem becomes false if we drop the assumption on the domain to have a Stein neighborhood basis. A counterexample is given by the so called *worm domain* which was created by K. Diederich and J. E. Fornaess in [DF77]. More precisely, its topological boundary is exactly the closure of the set of all strictly pseudoconvex boundary points, whereas the Bergman boundary is only a proper subset. Initially, the worm domain was created to provide an example of a bounded pseudoconvex domain in $\mathbb{C}^n$ with smooth boundary which does not admit a Stein neighborhood basis. But it turned out that it has more interesting properties. For more details, we refer to the original article [DF77].

Another result for a wider class of holomorphic functions is due to P. Pflug [Pfl79] (see also a later article by R. Basener [Bas77]).

Theorem 7.2.8 *Let D be a domain in $\mathbb{C}^n$ with $\mathcal{C}^\infty$-smooth boundary and let k be a non-negative integer. Then the Shilov boundary of $\overline{D}$ for $\mathcal{O}(D)\cap\mathcal{C}^k(\overline{D})$ coincides with the closure of the set of all strictly pseudoconvex boundary points of D.*

R. Basener examined the relation of the Shilov boundary of a smoothly bounded open set for the family $A_q(\overline{D})$ (see Theorem 5 in [Bas78]).

Theorem 7.2.9 *Let $q \in \{0,\dots,n-1\}$ and let D be a domain in $\mathbb{C}^n$ with $\mathcal{C}^2$-smooth boundary. Then $\check{S}_{A_q(\overline{D})}(\overline{D})$ is contained in the closure of $\mathcal{S}_q(\overline{D})$, i.e., the set of all strictly q-pseudoconvex boundary points of D.*

The $(n-1)$-Shilov boundaries for compact sets are all trivial due to the following remark.

Remark 7.2.10 Consider the following function f derived from Example 5 in [Bas76]. It is $(n-1)$-holomorphic on $\mathbb{C}^n\setminus\{0\}$ and has an isolated non-removable singularity at the origin,

$$f(z)=\frac{\bar{z}_1+\ldots+\bar{z}_n}{|z_1|^2+\ldots+|z_n|^2}.$$

To see this, notice that f is holomorphic in one variable after taking the holomorphic change of coordinates on $\{z\in\mathbb{C}^n : z_j\neq 0\}$ given by $w_j = z_j$ and

$w_i = z_i/z_j$, if $j \neq i$. Now let p be a boundary point of a compact set K in $\mathbb{C}^n$ and let $(p_k)_{k\in\mathbb{N}}$ be a sequence of points $p_k \notin K$ which converges to p outside K. For $k \in \mathbb{N}$ consider the function $f_k(z) := f(z - p_k)$. Of course, it is $(n-1)$-holomorphic on $\mathbb{C}^n \setminus \{p_k\}$. Now if n tends to $+\infty$, the absolute values $|f_k(p)|$ tend to $+\infty$. Hence, for every small enough neighborhood U of p there is an index $k \in \mathbb{N}$ such that U contains p_k and $|f_k|$ attains its maximum on K only inside the set $U \cap K$. By the definition of the Shilov boundary, the set U intersects $\check{S}_{\mathcal{O}_{n-1}(K)}(K)$. Since U is an arbitrary small neighborhood of p, the point p itself belongs to $\check{S}_{\mathcal{O}_{n-1}(K)}(K)$. Therefore, the whole boundary bK of K is contained in $\check{S}_{\mathcal{O}_{n-1}(K)}(K)$. Finally, the local maximum modulus principle 3.10.4 and Remark 7.2.4 (1) imply that

$$\check{S}_{\mathcal{O}_{n-1}(K)}(K) = \check{S}_{\mathcal{PSH}_{n-1}(K)}(K) = bK.$$

7.3 Lower dimensional q-Shilov boundaries

In the next two statements, we compare the Shilov boundary for subfamilies of functions r-plurisubharmonic and holomorphic on singular foliations to lower dimensional q-Shilov boundaries.

Proposition 7.3.1 *Let Ω be an open set in $\mathbb{C}^n$ and K be a compact set in Ω. Fix two integers $r \in \mathbb{N}_0$ and $q \in \{1, \ldots, n-1\}$. If $h : \Omega \to \mathbb{C}^q$ is a holomorphic mapping and $A = h^{-1}(c)$ for some $c \in h(\Omega)$, then*

$$\check{S}_{\mathcal{PSH}_r(h,\Omega)}(K) \cap A = \check{S}_{\mathcal{PSH}_r(A)}(K \cap A) \tag{7.6}$$

and $$\check{S}_{\mathcal{PSH}_r(h,K)}(K) \cap A = \check{S}_{\mathcal{PSH}_r(K\cap A)}(K \cap A). \tag{7.7}$$

Proof. We show the first identity (7.6). Observe that $K \cap A$ is non-empty if and only if $\check{S}_{\mathcal{PSH}_r(h,\Omega)}(K) \cap A$ is non-empty. Indeed, assume that $K \cap A$ is not empty, but A does not intersect $S_0 := \check{S}_{\mathcal{PSH}_r(h,\Omega)}(K)$. Define the function $\check{\chi}_A$ to be identically zero on A and $-\infty$ on $\Omega \setminus A$. It is easy to see that $\check{\chi}_A$ belongs to $\mathcal{PSH}_r(h, \Omega)$. Since S_0 is contained in $\Omega \setminus A$, the function $\check{\chi}_A$ attains its maximum inside $K \cap A$, but not in S_0. This is a contradiction to the definition of the Shilov boundary for the family $\mathcal{PSH}_r(h, \Omega)$ in K. The other direction is obvious, since S_0 is a non-empty subset of K due to Proposition 7.2.3.

We continue by proving the contention

$$\check{S}_{\mathcal{PSH}_r(A)}(K \cap A) \subset \check{S}_{\mathcal{PSH}_r(h,\Omega)}(K) \cap A.$$

To see this, recall that $S_0 = \check{S}_{\mathcal{PSH}_r(h,\Omega)}(K)$. We have to show that

$$\max_{K\cap A}\psi = \max_{S_0\cap A}\psi \quad \text{for every } \psi \in \mathcal{PSH}_r(A).$$

Fix a function $\psi \in \mathcal{PSH}_r(A)$ and extend it by $-\infty$ into the whole of Ω. We denote this extension by Ψ_A. Then Ψ_A belongs obviously to $\mathcal{PSH}_r(h,\Omega)$. Moreover, we have that

$$\max_{K\cap A}\psi = \max_{K}\Psi_A = \max_{S_0}\Psi_A = \max_{S_0\cap A}\psi.$$

By the definition it means that $S_0 \cap A$ is a boundary for the family $\mathcal{PSH}_r(A)$ in $K \cap A$, so

$$\check{S}_{\mathcal{PSH}_r(A)}(K\cap A) \subset \check{S}_{\mathcal{PSH}_r(h,\Omega)}(K)\cap A$$

In order to verify the other inclusion, take a point $p \in P_{\mathcal{PSH}_r(h,\Omega)}(K)\cap A$. By the definition, there is a peak function ψ in $\mathcal{PSH}_r(h,\Omega)$ with $\psi(p) > \psi(z)$ for every $z \in K \setminus \{p\}$. Since the restricted function $\varphi := \psi|A$ lies in $\mathcal{PSH}_r(A)$, we obtain that p is also a peak point for $\mathcal{PSH}_r(A)$. In view of Proposition 7.2.3 we deduce that

$$\begin{aligned}\check{S}_{\mathcal{PSH}_r(h,\Omega)}(K)\cap A &= \overline{P_{\mathcal{PSH}_r(h,\Omega)}(K)}\cap A \\ &\subset \overline{P_{\mathcal{PSH}_r(A)}(K\cap A)} = \check{S}_{\mathcal{PSH}_r(A)}(K\cap A).\end{aligned}$$

By the precedent discussion, we obtain the first identity (7.6).

It remains to show the second identity (7.7). But this is an immediate consequence of the identity (7.6) and Proposition 6.1.3 (7), since $\mathcal{PSH}_r(h,K)$ is the union of all families $\mathcal{PSH}_r(h,V)$, where V varies among all open neighborhoods V of K, and $\mathcal{PSH}_r(K\cap A)$ is the union of all sets $\mathcal{PSH}_r(U)$ with U varying among all open neighborhoods U of $K\cap A$ in A. □

We obtain similar results for certain subfamilies of q-holomorphic functions. Its proof is similar to the previous one.

Proposition 7.3.2 *Fix an integer $q \in \{1,\dots,n-1\}$. Let Ω be a Stein open neighborhood of a compact set K in $\mathbb{C}^n$ and let $h:\Omega\to\mathbb{C}^q$ be a holomorphic mapping. Set $A := h^{-1}(c)$ for some fixed $c \in h(\Omega)$. Then*

$$\check{S}_{\mathcal{O}(h,\Omega)}(K)\cap A = \check{S}_{\mathcal{O}(A)}(K\cap A). \tag{7.8}$$

If, moreover, the set K has a Stein neighborhood basis, then

$$\check{S}_{\mathcal{O}(h,K)}(K) \cap A = \check{S}_{\mathcal{O}(K\cap A)}(K \cap A). \tag{7.9}$$

Proof. We show the identity (7.8). Therefor, we set $S_0 := \check{S}_{\mathcal{O}(h,\Omega)}(K)$. Given $n \in \mathbb{N}$ and $c \in h(\Omega)$ from above, we also define $\chi_{A,n} := 1/(1 + n\|h - c\|_2^2)$. It is obvious that $\chi_{A,n}$ belongs to $\mathcal{O}(h, \Omega)$ and that the sequence $(\chi_{A,n})_{n\in\mathbb{N}}$ decreases to the characteristic function χ_A of A in Ω as n tends to infinity.

Now observe that $K \cap A$ is non-empty if and only if $S_0 \cap A$ is non-empty. Indeed, assume that $K \cap A$ is non-empty, but A does not intersect S_0. Then for large enough integer $n \in \mathbb{N}$, we can arrange that

$$\|\chi_{A,n}\|_K \geq \|\chi_{A,n}\|_{K\cap A} > \|\chi_{A,n}\|_{S_0}.$$

This inequality contradicts the definition of the Shilov boundary for the family $\mathcal{O}(h, \Omega)$ in K. Hence, A meets S_0. The other direction is obvious, because $\check{S}_{\mathcal{O}(h,\Omega)}(K)$ is a non-empty subset of K due to Proposition 7.2.3.

We proceed by showing that $\check{S}_{\mathcal{O}(A)}(K \cap A)$ lies in $\check{S}_{\mathcal{O}(h,\Omega)}(K) \cap A$, so that we need to verify that $\|f\|_{K\cap A} = \|f\|_{S_0\cap A}$ for every function $f \in \mathcal{O}(A)$. Pick an arbitrary holomorphic function f on A and denote by F its holomorphic extension to the whole of Ω (see Theorem 3.11.4). We set $F_n := F \cdot \chi_{A,n}$. Then it is easy to see that F_n belongs to $\mathcal{O}(h, \Omega)$. Since $S_0 = \check{S}_{\mathcal{O}(h,\Omega)}(K)$, we conclude that

$$\|f\|_{K\cap A} = \lim_{n\to\infty} \|F_n\|_K = \lim_{n\to\infty} \|F_n\|_{S_0} = \|f\|_{S_0\cap A}.$$

Now we verify the other inclusion of (7.8). Observe that the uniform closure $\overline{\mathcal{O}(h,\Omega)}^K$ in $\mathcal{C}(K)$ forms a uniform subalgebras of $\mathcal{C}(K)$, so that we obtain by Proposition 7.2.3 that

$$\check{S}_{\mathcal{O}(h,\Omega)}(K) = \check{S}_{\overline{\mathcal{O}(h,\Omega)}^K}(K) = \overline{P_{\overline{\mathcal{O}(h,\Omega)}^K}(K)}.$$

Take a point $p \in P_{\overline{\mathcal{O}(h,\Omega)}^K}(K) \cap A$. Then there is a peak function f in $\overline{\mathcal{O}(h,\Omega)}^K$ such that $S(f) = \{z \in K : \|f\|_K = |f(z)|\} = \{p\}$. Since $p \in A$, it is easy to see that $f|A$ belongs to the family $\overline{\mathcal{O}(A)}^{K\cap A}$ and peaks at p in $K \cap A$. Thus,

$$P_{\overline{\mathcal{O}(h,\Omega)}^K}(K) \cap A \ \subset\ P_{\overline{\mathcal{O}(A)}^{K\cap A}}(K \cap A).$$

Then Proposition 7.2.3 yields

$$\check{S}_{\mathcal{O}(h,\Omega)}(K) \cap A \ \subset\ \check{S}_{\overline{\mathcal{O}(A)}^{K\cap A}}(K \cap A) \ =\ \check{S}_{\mathcal{O}(A)}(K \cap A).$$

Hence, we obtain the identity (7.8).

The final identity (7.9) follows now from Proposition 6.1.3 (7) and the facts that $\mathcal{O}(h,K)$ is the union of all families $\mathcal{O}(h,V)$, where V varies among all Stein open neighborhoods of K in $\mathbb{C}^n$, and that $\mathcal{O}(K\cap A)$ is the collection of all functions from the families $\mathcal{O}(U)$ with U varying among all neighborhoods U of $K\cap A$ in A. □

7.4 Shilov boundary of q-th order

We recall the definition of q-th order Shilov boundary introduced by R. Basener. We use the same terminology as him in [Bas78]. Our aim is to compare this generalized Shilov boundary to the previously defined q-Shilov boundaries.

Definition 7.4.1 Let $\mathcal{A}$ be a uniform algebra of continuous functions on a compact set K in $\mathbb{C}^n$. Fix an integer $q\in\{0,\ldots,n-1\}$.

(1) For a finite subset S of $\mathcal{A}$ denote by $\sharp S$ the number of elements of S and define the set

$$V(S)=V_K(S):=\{z\in K : f(z)=0 \text{ for every } f\in S\}$$

and the family $\mathcal{A}|V(S):=\{f|V(S) : f\in\mathcal{A}\}$.

(2) A closed subset Γ of K is called a *q-th order boundary* for $\mathcal{A}$ if for every finite subset S of $\mathcal{A}$ with $\sharp S\le q$ and $f\in\mathcal{A}$ it holds that

$$\|f\|_{\Gamma\cap V(S)}=\|f\|_{V(S)}.$$

(3) The *q-th order Shilov boundary* or *Shilov boundary of q-th order* for $\mathcal{A}$ is given by

$$\partial_q\mathcal{A}:=\overline{\bigcup_{\substack{S\subset\mathcal{A}\\ \sharp S\le q}}\check{S}_{\mathcal{A}|V(S)}(V(S))}.$$

Of course, $\partial_q\mathcal{A}$ is the smallest q-th order boundary for $\mathcal{A}$.

(4) Let S be a finite subset of $\mathcal{A}$ with $\sharp S=k$. For $I=(i_1,\ldots,i_k)$ and $|I|=i_1+\ldots+i_k$ we also define the set

$$\mathcal{A}(\overline{S}):=\Big\{\sum_{|I|\le\ell} g_I\overline{f}_1^{i_1}\cdots\overline{f}_k^{i_k} : g_I\in\mathcal{A},\ f_1,\ldots,f_k\in S,\ \ell<+\infty\Big\}.$$

The family $\mathcal{A}(\overline{S})$ allows another interpretation of the q-th order boundaries (see Theorem 3 in [Bas78]).

Theorem 7.4.2 *A closed subset Γ of a compact set K in $\mathbb{C}^n$ is a q-th order boundary for $\mathcal{A}$ if and only if for every S in $\mathcal{A}$ with $\sharp S \leq q$ the set Γ is an $\mathcal{A}(\overline{S})$-boundary in K in our sense, i.e., $\Gamma \in b_{\mathcal{A}(\overline{S})}(K)$.*

Basener already studied the relation between the q-th order Shilov boundary and the q-Shilov boundaries. We extend his results by the following list.

Theorem 7.4.3 *Let K be a compact set in $\mathbb{C}^n$ and $q \in \{0, \ldots, n-1\}$.*

(1) We have that $\partial_q A(K) \subset \check{S}_{A_q(K)}(K)$.

(2) It also holds that $\partial_q \mathcal{O}(K) \subset \check{S}_{\mathcal{O}_q(K)}(K)$

(3) Let Ω be a Stein neighborhood of K and set $H := \mathcal{O}(\Omega, \mathbb{C}^q)$. Then

$$\partial_q \mathcal{O}(\Omega)|K = \check{S}_{\mathcal{O}(H,\Omega)}(K),$$

where $\mathcal{O}(\Omega)|K := \{f|K : f \in \mathcal{O}(\Omega)\}$.

(4) Assume that K has a Stein neighborhood basis. Let $H = \{h_j\}_{j \in J}$ be the family of all holomorphic mappings $h_j : \Omega_j \to \mathbb{C}^q$, where Ω_j is some neighborhood of K. Then

$$\partial_q \mathcal{O}(K) = \check{S}_{\mathcal{O}(H,K)}(K).$$

Proof. (1) This inclusion is Theorem 4 in [Bas78]. Anway, we present another proof using our techniques. Let $S = \{f_1, \ldots, f_k\}$ be a subset of $A(K)$ with $\sharp S = k \leq q$ and pick a function $F \in A(K)(\overline{S})$. Then there are an integer $\ell \geq 0$ and a family $g = (g_I)_{|I| \leq \ell}$ of functions $g_I \in A(K)$ such that

$$F = \sum_{|I| \leq \ell} g_I \overline{f}_1^{i_1} \cdots \overline{f}_k^{i_k}.$$

Now consider the function

$$h(\zeta, w_1, \ldots, w_k) := \sum_{|I| \leq \ell} \zeta_I \overline{w}_1^{i_1} \ldots \overline{w}_k^{i_k},$$

where $\zeta = (\zeta_I)_{|I| \leq \ell}$ are complex coordinates in an appropriate $\mathbb{C}^N$. According to Example 3.10.3 (3), this function h is k-holomorphic on $\mathbb{C}^{N+k}$. Therefore, in view of Proposition 3.10.2 (5), the composed function $F = h \circ (g, f_1, \dots, f_k)$ belongs to $A_k(K)$. This shows that $A(K)(\overline{S})$ is a subset of $A_k(K)$. Thus, Theorem 7.4.2 and Proposition 3.10.2 (2) yield

$$\partial_q A(K) = \bigcap_{\substack{S \subset A(K) \\ \sharp S \leq q}} \check{S}_{A(K)(\overline{S})} \subset \check{S}_{A_q(K)}(K)$$

(2) This inclusion can be shown in exactly the same way as the inclusion in (1) by replacing $A_q(K)$ by $\mathcal{O}_q(K)$ and $A(K)$ by $\mathcal{O}(K)$.

(3) Given an integer $k \in \{0, \dots, q\}$, define $H_k := \mathcal{O}(\Omega, \mathbb{C}^k)$. Then we easily verify that

$$\bigcup_{\substack{k \leq q \\ h \in H_k}} \mathcal{O}(h, \Omega) = \mathcal{O}(H, \Omega), \quad \text{hence} \quad \overline{\bigcup_{\substack{k \leq q \\ h \in H_k}} \check{S}_{\mathcal{O}(h,\Omega)}(K)} = \check{S}_{\mathcal{O}(H,\Omega)}(K) \tag{7.10}$$

due to Proposition 6.1.3 (7). For a subset S of $\mathcal{C}(\Omega)$ define also the set

$$V_\Omega(S) := \{z \in \Omega : f(z) = 0 \text{ for every } f \in S\}.$$

and notice the identity

$$V_K(S) = V_\Omega(S) \cap K. \tag{7.11}$$

Now the equality in (3) follows from the subsequent chain of identities:

$$\begin{aligned}
\partial_q \mathcal{O}(\Omega)|K &\overset{\text{Def. 7.4.1}}{=} \overline{\bigcup_{\substack{S \subset \mathcal{O}(\Omega)|K \\ \sharp S \leq q}} \check{S}_{\mathcal{O}(\Omega)|V_K(S)}(V_K(S))} \\
&\overset{(7.11)}{=} \overline{\bigcup_{\substack{S \subset \mathcal{O}(\Omega) \\ \sharp S \leq q}} \check{S}_{\mathcal{O}(\Omega)|V_\Omega(S)}(K \cap V_\Omega(S))} \\
&\overset{\text{Thm. 3.11.4}}{=} \overline{\bigcup_{\substack{S \subset \mathcal{O}(\Omega) \\ \sharp S \leq q}} \check{S}_{\mathcal{O}(V_\Omega(S))}(K \cap V_\Omega(S))} \\
&\overset{\text{Prop. 7.3.2}}{=} \overline{\bigcup_{k \leq q} \bigcup_{h \in H_k} \bigcup_{c \in \mathbb{C}^k} \check{S}_{\mathcal{O}(h,\Omega)}(K) \cap h^{-1}(c)} \\
&= \overline{\bigcup_{k \leq q} \bigcup_{h \in H_k} \check{S}_{\mathcal{O}(h,\Omega)}(K)}
\end{aligned}$$

$$\overset{(7.10)}{=} \quad \check{S}_{\mathcal{O}(H,\Omega)}(K).$$

(4) Notice that if K has a Stein neighborhood basis, we can assume without loss of generality that for every $j \in J$ the holomorphic mapping $h_j \in H$ is defined on a Stein open neighborhood Ω_j of K. Then the identity in (4) follows by the same arguments as in the proof of (3). □

7.5 Real and q-complex points

In [Byč81], S. N. Bychkov gives a geometric characterization of the Shilov boundary of bounded convex domains D in $\mathbb{C}^n$. Our goal in this section is to generalize his result to q-Shilov boundaries (see Theorem 7.6.7). First, we recall some definitions from convexity theory given in Bychkov's article [Byč81] which extend Definition 2.1.1. We also mainly use his notations.

Definition & Remark 7.5.1 Let K be a convex body in $\mathbb{C}^n$, i.e., a compact convex set with non-empty interior.

(1) A subset of the boundary bK which results from an intersection of K with supporting hyperplanes is called a *face of* K. A face is again a lower dimensional convex set. The empty set and K itself are also considered to be faces. A face of a face of K does not need to be a face of K. The intersection of arbitrarily many faces of K is again a face of K.

(2) Given a convex body K, there is a unique minimal face $F_1 := F_{\min}(p, K)$ of $F_0 := K$ in the boundary of K containing the point p. It can be defined as the intersection of K and all supporting hyperplanes for K at p. Now there are two options for p: either it is an inner point of the convex body F_1 or it lies in the boundary of F_1. In the second case, the point p might again lie either in the interior of the minimal face $F_2 := F_{\min}(p, F_1)$ of F_1 or in the boundary of F_2. Inductively, we obtain a finite sequence $(F_j)_{j=0,\ldots,j(p)}$ of convex bodies F_j in K of dimension m_j such that $F_{j+1} := F_{\min}(p, F_j) \supset F_j$ for each $j \in \{0, \ldots, j(p)-1\}$ and such that either, if $m_{j(p)} > 0$, the point p is an interior point of $F_{j(p)}$, or, if $m_{j(p)} = 0$, the minimal face $F_{j(p)}$ consists only of the point p.

(3) The convex body $F_p(K) := F_{j(p)}$ will be called the *face essentially containing p*. It is contained in a plane $E_p(K)$ of dimension $m_{j(p)}$ satisfying $E_{j(p)} \cap K = F_{j(p)}$.

We will give a simple example of a sequence of minimal faces.

Example 7.5.2 Let $\Delta := \Delta_1(0)$ be the unit disc in $\mathbb{R}^2$ and consider the set $K := \overline{\Delta} \cup ([-1,1] \times [-1,0])$. It is a convex body in $\mathbb{R}^2$. The plane $\pi_1 = \{1\} \times \mathbb{R}$ is the only supporting hyperplane of K at $p = (1,0)$ in $\mathbb{R}^2$. Thus, the minimal face of K containing p is the segment $F_1 = \pi_1 \cap K = \{1\} \times [-1,0]$. The point p lies in the boundary of F_1 in $\{1\} \times \mathbb{R}$. Then the set $\pi_2 = \{1\}$ is the only supporting hyperplane of F_1 at p in $\{1\} \times \mathbb{R}$. Hence, the minimal face of F_1 having p inside is the set $F_2 = \pi_2 \cap K = \{p\}$. Therefore, the face essentially containing p is the set $F_2 = \{1\}$.

In the following, let D be always a bounded convex domain in $\mathbb{C}^n$.

Definition 7.5.3 Let p be a boundary point of D.

(1) Denote by $E_p^{\mathbb{C}}(\overline{D})$ the largest complex plane in $E_p(\overline{D})$ containing p.

(2) We define $\nu(p) := \dim_{\mathbb{C}} E_p^{\mathbb{C}}(\overline{D})$.

(3) If $\nu(p) = 0$, then $E_p(\overline{D})$ is totally real and we say that the boundary point p is *real*.

(4) The symbol $\Pi_p(\overline{D})$ denotes the set of all complex planes π in $\mathbb{C}^n$ such that there exists a domain $G \subset \mathbb{C}^n$ with $p \in G \cap \pi \subset bD$.

(5) If $\Pi_p(\overline{D}) \neq \{p\}$, then p is called *complex*.

We restate Lemma 2.5 in [Byč81] and its important corollary.

Lemma 7.5.4 *If $I \subset bD$ is an open segment containing $p \in bD$, then I lies in the face $F_p(\overline{D})$ essentially containing p.*

The previous lemma permits to classify the boundary points of D.

Corollary 7.5.5 *A boundary point $p \in bD$ is either real or complex.*

From this we can derive further consequences.

Corollary 7.5.6 *If $p \in bD$ is complex, then $E_p^{\mathbb{C}}(\overline{D}) \in \Pi_p(\overline{D})$.*

Proof. Since p is complex, it cannot be real due to the previous Corollary 7.5.5. Thus, $E_p^{\mathbb{C}}(\overline{D})$ is not empty and the face essentially containing p cannot be a single point. Then p is an inner point of the convex body $F_p(\overline{D})$ in $E_p(\overline{D})$. Hence, there is an open ball B' with center p inside $F_p(\overline{D})$, and we can find an open ball B in $\mathbb{C}^n$ with center p such that $B \cap E_p(\overline{D}) = B'$. Then we obtain that

$$p \in E_p^{\mathbb{C}}(\overline{D}) \cap B \ \subset \ E_p(\overline{D}) \cap B = B' \ \subset \ F_p(\overline{D}) \ \subset \ bD.$$

Thus, it follows from the definition that $E_p^{\mathbb{C}}(\overline{D})$ lies in $\Pi_p(\overline{D})$. □

Another consequence of Lemma 7.5.4 is that $E_p^{\mathbb{C}}(\overline{D})$ is maximal in $\Pi_p(\overline{D})$.

Corollary 7.5.7 *If $\pi \in \Pi_p(\overline{D})$, then π lies in $E_p^{\mathbb{C}}(\overline{D})$.*

Proof. Let G be an open neighborhood of p in $\mathbb{C}^n$ such that $p \in G \cap \pi \subset bD$. It follows from Lemma 7.5.4 that $G \cap \pi$ lies in $F_p(\overline{D})$. Since $G \cap \pi$ is open in π, we have that π is contained in $E_p(\overline{D})$. Since π is a complex plane containing p and $E_p^{\mathbb{C}}(\overline{D})$ is the largest complex plane inside $E_p(\overline{D})$, we conclude that π lies in $E_p^{\mathbb{C}}(\overline{D})$. □

We specify complex points in the following way.

Definition & Remark 7.5.8 Let p be a complex boundary point of D and $q \in \{1, \ldots, n-1\}$.

(1) The point p is called *q-complex* if $\nu(p) = \dim_{\mathbb{C}} E_p^{\mathbb{C}}(\overline{D}) \geq q$.

(2) The Corollaries 7.5.6 and 7.5.7 yield the following characterization of q-complex points: A boundary point p in bD is q-complex if and only if there is a domain G in $\mathbb{C}^n$ and a complex plane of dimension at least q such that $p \in G \cap \pi \subset bD$.

The next lemma asserts that a complex point p is a lower dimensional real point.

Lemma 7.5.9 *Let p be a complex point in bD. Let π be a complex affine plane of codimension $\nu(p)$ such that $E_p^{\mathbb{C}}(\overline{D}) \cap \pi = \{p\}$. Then p is a real boundary point of $\overline{D} \cap \pi$.*

Proof. If $\nu(p) = n-1$, the statement is obviously true, since every boundary point of $\overline{D} \cap \pi$ is real.

Suppose that $\nu(p) \leq n-2$ and that the statement is false. Then by Corollary 7.5.5 the point p is a complex boundary point of $\overline{D} \cap \pi$. According to the definition, there exist a domain $G \subset \mathbb{C}^n$ and a complex line $\mathbb{L}$ in $\pi \in \Pi_p(\overline{D})$ such that $p \in G \cap \mathbb{L} \subset bD$. By Corollary 7.5.7 the line $\mathbb{L}$ lies in $E_p^{\mathbb{C}}(\overline{D})$. But since $E_p^{\mathbb{C}}(\overline{D}) \cap \pi = \{p\}$ and $\mathbb{L} \subset \pi$, it follows that $\mathbb{L} = \{p\}$, which is absurd. Hence, p has to be a real boundary point of $\overline{D} \cap \pi$. □

7.6 q-Shilov boundaries of convex sets

We recall one of the main results of Bychkov's article [Byč81]. Unless otherwise stated, each Shilov boundary considered in this section is a Shilov boundary of the closure of a convex bounded domain D in $\mathbb{C}^n$. Recall that $A(\overline{D})$ is the family of all holomorphic functions on D which extend to a continuous function on $\overline{D}$.

Theorem 7.6.1 (Bychkov, 1981) *A boundary point $p \in bD$ does not belong to $\check{S}_{A(\overline{D})}$ if and only if there exists a neighborhood U of p in bD such that U consists only of complex points.*

We introduce the subfamily of holomorphic functions on a regular foliation by complex planes.

Definition & Remark 7.6.2 Let $q \in \{0, \ldots, n-1\}$ be an integer and let $H := \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^n, \mathbb{C}^q)$ be the family of all $\mathbb{C}$-linear mappings from $\mathbb{C}^n$ to $\mathbb{C}^q$.

(1) We set $A_q^{\mathrm{hom}}(\overline{D}) := A(H, K)$ and $\mathcal{O}_q^{\mathrm{hom}}(\overline{D}) := \mathcal{O}(H, K)$.

(2) For $q = 0$ we have that $\mathcal{O}_0^{\mathrm{hom}}(\overline{D}) = \mathcal{O}(\overline{D})$ and $A_0^{\mathrm{hom}}(\overline{D}) = A(\overline{D})$.

(3) Since D is bounded and convex, we easily derive that $A_q(\overline{D}) = \mathcal{O}_q(\overline{D})$ and $A_q^{\mathrm{hom}}(\overline{D}) = \mathcal{O}_q^{\mathrm{hom}}(\overline{D})$.

We generalize the important Proposition 2.6 in [Byč81] which states that a real boundary point always lies in the Shilov boundary for the family $A(\overline{D})$.

Proposition 7.6.3 *If $p \in bD$, then $p \in \check{S}_{\mathcal{O}^{\mathrm{hom}}_{\nu(p)}(\overline{D})}$.*

Proof. By Corollary 7.5.5, the point p is either real or complex. If p is real, then by Proposition 2.6 in [Byč81] and Remark 7.6.2 we have that

$$p \in \check{S}_{A(\overline{D})} = \check{S}_{\mathcal{O}(\overline{D})} = \check{S}_{\mathcal{O}_0^{\mathrm{hom}}(\overline{D})}.$$

If p is complex, then there are a domain G and a $\nu(p)$-dimensional complex plane π such that $p \in G \cap \pi \subset bD$. Let L be a complex affine plane of codimension $\nu(p)$ such that $\pi \cap L = \{p\}$ and let $h : \mathbb{C}^n \to \mathbb{C}^{\nu(p)}$ be a $\mathbb{C}$-linear mapping with $L = \{h = 0\}$. In view of Lemma 7.5.9, the point p is a real boundary point of the convex body $\overline{D} \cap L$. Then we conclude by Proposition 2.6 in [Byč81], Remark 7.6.2 and Proposition 7.3.2 that

$$p \in \check{S}_{A(\overline{D}\cap L)}(\overline{D} \cap L) = \check{S}_{\mathcal{O}(\overline{D}\cap L)}(\overline{D} \cap L) \subset \check{S}_{\mathcal{O}(h,\overline{D})} \subset \check{S}_{\mathcal{O}_{\nu(p)}^{\mathrm{hom}}(\overline{D})}.$$

□

As a first consequence, we obtain a characterization of the $(n-1)$-Shilov boundaries.

Corollary 7.6.4 *The Shilov boundaries for the families $\mathcal{O}_{n-1}^{\mathrm{hom}}(\overline{D})$, $\mathcal{O}_{n-1}(\overline{D})$ and $\mathcal{PSH}_{n-1}(\overline{D})$ coincide with the topological boundary of D, i.e.,*

$$\check{S}_{\mathcal{O}_{n-1}^{\mathrm{hom}}(\overline{D})} = \check{S}_{\mathcal{O}_{n-1}(\overline{D})} = \check{S}_{\mathcal{PSH}_{n-1}(\overline{D})} = bD.$$

Proof. If $p \in bD$, then p is real or complex and $0 \leq \nu(p) \leq n-1$. Thus, the Propositions 7.6.3, 7.2.3 and 3.3.2 (10) imply that

$$p \in \check{S}_{\mathcal{O}_{\nu(p)}^{\mathrm{hom}}(\overline{D})} \subset \check{S}_{\mathcal{O}_{n-1}^{\mathrm{hom}}(\overline{D})} \subset \check{S}_{\mathcal{O}_{n-1}(\overline{D})} \subset \check{S}_{\mathcal{PSH}_{n-1}(\overline{D})} \subset bD.$$

□

We will need the following lemma.

Lemma 7.6.5 *Let $p \in bD$ and $q \in \{0, \ldots, n-2\}$. If there exists an analytic subset in bD which contains p and has minimal dimension at least $q+1$, then p is not a peak point for the family $\mathcal{PSH}_q(\overline{D})$. In particular, no $(q+1)$-complex point belongs to $P_{\mathcal{PSH}_q(\overline{D})}$.*

Proof. This follows immediately from the local maximum principle for q-plurisubharmonic functions on analytic sets (see Proposition 3.8.5) and the definition of $(q+1)$-complex points. $\square$

We will need another specification of complex points.

Definition 7.6.6 For $q \in \{1, \ldots, n-1\}$ denote by $\Gamma_q(\overline{D})$ the relative interior of the set of all q-complex boundary points of D in bD.

We are now able to generalize Bychkov's theorem.

Theorem 7.6.7 *Let $q \in \{0, \ldots, n-2\}$. Then*

$$\check{S}_{\mathcal{O}_q^{\hom}(\overline{D})} = \check{S}_{\mathcal{O}_q(\overline{D})} = \check{S}_{\mathcal{PSH}_q(\overline{D})} = bD \setminus \Gamma_{q+1}(\overline{D}).$$

Proof. If $p \in bD \setminus \check{S}_{\mathcal{O}_q^{\hom}(\overline{D})}$, then there is a neighborhood U of p in bD such that $U \cap \check{S}_{\mathcal{O}_q^{\hom}(\overline{D})} = \emptyset$. Thus, if $w \in U$, then $\nu(w) \geq q+1$ due to Proposition 7.6.3. This means that U consists only of $(q+1)$-complex points. Hence, $p \in \Gamma_{q+1}(\overline{D})$, and therefore $bD \setminus \Gamma_{q+1}(\overline{D})$ lies in $\check{S}_{\mathcal{O}_q^{\hom}(\overline{D})}$.

On the other hand, if there is a neighborhood U of p in bD such that U contains only $(q+1)$-complex points, then we know by Lemma 7.6.5 that the intersection $U \cap P_{\mathcal{PSH}_q(\overline{D})}$ is empty. This implies that $p \notin \overline{P}_{\mathcal{PSH}_q(\overline{D})}$. Since, by Proposition 7.1.4, the latter set coincides with $\check{S}_{\mathcal{PSH}_q(\overline{D})}$, we obtain the inclusion

$$\check{S}_{\mathcal{PSH}_q(\overline{D})} \subset bD \setminus \Gamma_{q+1}(\overline{D}).$$

Finally, the statements of Proposition 7.2.3 (2) and Remark 7.2.4 complete the proof. $\square$

Now we give an interesting consequence of the previous theorem related to analytic sets inside the boundary of D.

Remark 7.6.8 Given an integer $q \in \{1, \ldots, n-1\}$, let $\Gamma_q^A(\overline{D})$ be the set of all boundary points p of D such that there exists a neighborhood U of p in bD with the following property: for each point $z \in U$ there is an analytic subset of U which contains z and has minimal dimension at least q. Then

$$\Gamma_q^A(\overline{D}) = \Gamma_q(\overline{D}).$$

Indeed, the inclusion $\Gamma_q(\overline{D}) \subset \Gamma_q^A(\overline{D})$ follows directly from the definition of these two sets and the definition of q-complex points. Now let $p \in \Gamma_q^A(\overline{D})$. Then Lemma 7.6.5 and Proposition 7.1.4 imply that $p \notin \check{S}_{\mathcal{PSH}_{q-1}(\overline{D})}$. Thus, we derive from Theorem 7.6.7 that p is contained in $\Gamma_q(\overline{D})$. This shows the other inclusion.

We study the analytic structure of the q-Shilov boundaries.

Proposition 7.6.9 *Let D be a bounded convex domain in $\mathbb{C}^n$. Fix an integer $q \in \{1, \ldots, n-1\}$ and assume that $\{z \in bD : \nu(z) \geq q+1\}$ is a non-empty open subset of bD. Then the following open part*

$$\mathcal{F}_q(\overline{D}) := \mathrm{int}_{bD} \left(\check{S}_{\mathcal{PSH}_q(\overline{D})} \setminus \check{S}_{\mathcal{PSH}_{q-1}(\overline{D})} \right)$$

of the Shilov boundary for $\mathcal{PSH}_q(\overline{D})$ in bD locally admits a foliation by complex q-dimensional planes in the following sense: for every point p in $\mathcal{F}_q(\overline{D})$ there exists a neighborhood U of p in bD such that for each $z \in U$ there is a domain G_z in $\mathbb{C}^n$ and a unique complex q-dimensional plane π_z with $z \in \pi_z \cap G_z \subset U$. In the special case $q = n-1$, these complex (hyper-)planes are aligned parallelly.

Proof. We set $\Gamma_n := \emptyset$. By Theorem 7.6.7 and by Corollary 7.6.4 we have that $\mathcal{F}_q(\overline{D}) = \Gamma_q(\overline{D}) \setminus \overline{\Gamma_{q+1}(\overline{D})}$. Since the set $\{z \in bD : \nu(z) \geq q+1\}$ is open, it coincides with $\Gamma_{q+1}(\overline{D})$. Thus,

$$\mathcal{F}_q(\overline{D}) = \Gamma_q(\overline{D}) \setminus \overline{\{z \in bD : \nu(z) \geq q+1\}}.$$

Now if p is an arbitrary point from $\mathcal{F}_q(\overline{D})$, then there is a neighborhood W of p in $\mathcal{F}_q(\overline{D})$ such that $\nu(z) = q$ for every point $z \in W$. Hence, the open set $\mathcal{F}_q(\overline{D})$ consists only of *exactly q-complex points*. Then Corollaries 7.5.6 and 7.5.7 imply existence and uniqueness of an open part of a complex q-dimensional plane $\pi_z = E_z^{\mathbb{C}}(\overline{D})$ containing z and lying in U.

In the special case of $q = n-1$, the set $\mathcal{F}_{n-1}(\overline{D})$ is a convex hypersurface foliated by complex hyperplanes. By a result of V. K. Beloshapka and Bychkov in [BB86], they have to be aligned parallelly. □

We give an example of a convex domain D in $\mathbb{C}^3$ such that the part $\mathcal{F}_1(\overline{D})$ does not admit a foliation in the sense of the previous theorem when we drop the assumption on $\{z \in bD : \nu(z) \geq 2\}$ to be open in bD.

Example 7.6.10 Consider the domain G in $\mathbb{C} \times \mathbb{R}$ given by

$$G = \{(x+iy,u) \in \mathbb{C} \times \mathbb{R} : x^2 + (1-y^2)u^2 < (1-y^2),\ |y| < 1\}.$$

It is easy to compute that the function $h(y,u) := \sqrt{(1-y^2)(1-u^2)}$ is concave on $[-1,1]^2$. Since G is the intersection of the sublevel set $\{x < h(y,u)\}$ of the concave function h and the superlevel set $\{x > -h(y,u)\}$ of the convex function $-h$ over $[-1,1]^2$, it is convex in $\mathbb{C} \times \mathbb{R}$ due to Proposition 2.1.4. Moreover, the boundary of G contains the *flat* parts $\{\pm i\} \times (-1,1)$ and $\{0\} \times (-1,1) \times \{\pm 1\}$, whereas the rest of the boundary consists of *strictly convex* points. By putting $D := G \times (-1,1)^3$ we obtain a convex domain D in $\mathbb{C}^3$ such that

$$\{z \in bD : \nu(z) \geq 2\} = \{\pm i\} \times (-1,1)^4.$$

and $\Gamma_1(\overline{D})$ is the whole boundary of D. In particular, $\Gamma_2(\overline{D})$ is empty. Thus, the boundary points z in bD with $\nu(z) \geq 2$ lie in $\Gamma_1(\overline{D})$, but there is no unique foliation by complex one-dimensional planes near these points.

We give some estimates on the Hausdorff dimension of the Shilov boundary for q-plurisubharmonic functions on convex bodies. First, we recall the notion of the Hausdorff dimension.

Definition 7.6.11 Let (X,d) be a metric space.

(1) For a subset U of X denote by $\operatorname{diam}(U)$ the *diameter of* U, i.e.,

$$\operatorname{diam}(U) := \sup\{d(x,y) : x, y \in U\}.$$

(2) Given a subset E of X and positive numbers s and ε we set

$$H^s_\varepsilon(E) := \inf\left\{\sum_{i=1}^{\infty} \operatorname{diam}(U_i)^s : E \subset \bigcup_{i=1}^{\infty} U_i,\ \operatorname{diam}(U_i) < \varepsilon\ \forall\, i \in \mathbb{N}\right\}.$$

The *s-dimensional Hausdorff measure* is then defined by

$$H^s(E) := \lim_{\varepsilon \to 0} H^s_\varepsilon(E).$$

(3) For every subset E of X there is a number $s_0 \in [0,+\infty]$ such that

$$H^s(E) = \infty \text{ for } s \in (0,s_0) \quad \text{and} \quad H^s(E) = 0 \text{ for } s \in (s_0,\infty).$$

The number $\dim_H E := s_0$ is called the *Hausdorff (or metric) dimension of* E.

The next statement can be found in, e.g., [Fal03], Corollary 7.12.

Proposition 7.6.12 *Let I be an m-dimensional cube in $\mathbb{R}^m$, J be an n-dimensional cube in $\mathbb{R}^n$ and F be a subset of $I\times J$. For a given point $x\in I$ consider the slice $F_x := F\cap(\{x\}\times J)$. If $\dim_H F_x \geq \alpha$ for every $x\in I$, then $\dim_H F\geq \alpha+m$.*

Bychkov showed in [Byč81] that the Hausdorff dimension of the Shilov boundary of a convex body in $\mathbb{C}^2$ is not less than 2. We partially generalize this result.

Theorem 7.6.13 *Let D be a convex bounded domain in $\mathbb{C}^n$ and fix an integer $q\in\{0,\ldots,n-2\}$. Suppose that there are a constant $\alpha\geq 0$ and a collection $\{\pi_j\}_{j\in J}$ of disjoint parallel complex affine planes in $\mathbb{C}^n$ satisfying the following properties:*

- *The union $\bigcup\limits_{j\in J}\pi_j$ is open and intersects D.*
- *For every $j\in J$ it holds that $\dim_H \check{S}_{\mathcal{O}(\overline{D}\cap\pi_j)}(\overline{D}\cap\pi_j)\geq\alpha$.*

Then we obtain the estimate

$$\dim_H \check{S}_{\mathcal{O}_q(\overline{D})}(\overline{D})\geq \alpha+2q. \tag{7.12}$$

In particular, $\dim_H \check{S}_{\mathcal{O}_{n-2}(\overline{D})}(\overline{D})\geq 2n-2$.

Proof. By the assumptions made on $\{\pi_j\}_{j\in J}$, we can find a $\mathbb{C}$-linear mapping $h:\mathbb{C}^n\to\mathbb{C}^q$ and a collection $\{c_j\}_{j\in J}$ of points c_j in $\mathbb{C}^q$ with $\pi_j=\{h=c_j\}$ for every $j\in J$. Then by Propositions 7.3.2 and 7.2.3, we have that

$$\bigcup_{j\in J}\check{S}_{\mathcal{O}(\overline{D}\cap\pi_j)}(\overline{D}\cap\pi_j)\ \subset\ \check{S}_{\mathcal{O}(h,\overline{D})}(\overline{D})\ \subset\ \check{S}_{\mathcal{O}_q^{\mathrm{hom}}(\overline{D})}(\overline{D})\ \subset\ \check{S}_{\mathcal{O}_q(\overline{D})}(\overline{D}).$$

Hence, it follows from Proposition 7.6.12 that $\dim_H \check{S}_{\mathcal{O}_q(\overline{D})}(\overline{D})\geq\alpha+2q$.

Now consider the case $q=n-2$. It was shown in Theorem 3.1 in [Byč81] that $\dim_H \check{S}_{\mathcal{O}(\overline{D}\cap\pi)}(\overline{D}\cap\pi)\geq 2$ for every complex two dimensional affine plane π such that $\pi\cap D\neq\emptyset$. Hence, in view of the inequality (7.12) we conclude that

$$\dim_H \check{S}_{\mathcal{O}_{n-2}(\overline{D})}(\overline{D})\geq 2+2(n-2)=2n-2.$$

□

To show that the Hausdorff dimension of $\check{S}_{A(\overline{D})}$ is not less than two if $D \Subset \mathbb{C}^2$ is a convex domain, Bychkov used that $\Gamma_1(\overline{D})$ admits a local foliation by complex lines which are aligned parallelly to each other. More general, if a convex hypersurface (i.e., an open part of the boundary of a convex body) is foliated by complex hyperplanes, then, due to a result of V. K. Beloshapka and Bychkov in [BB86], they are always parallel to each other. Especially, this holds for the open set $\Gamma_{n-1}(\overline{D})$, provided it is not empty (compare Theorem 7.6.9).

The following example in [NT12] shows that, in general, this result fails for lower dimensional complex foliations.

Example 7.6.14 Consider the function $\varrho(z) = (\mathrm{Re} z_2)^2 - (\mathrm{Re} z_1)(\mathrm{Re} z_3)$ for $z \in \mathbb{C}^3$. Then the set $D := \{z \in \mathbb{C}^3 : \mathrm{Re}(z_1) > 0, \varrho(z) < 0\}$ is convex, and an open part of its boundary is foliated by a real 3-dimensional parameter family of open parts of non-parallel complex lines of the form

$$\{(a^2\zeta + ib, a\zeta + ic, \zeta), \zeta \in \mathbb{C}\}, \quad a, b, c \in \mathbb{R}.$$

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Index

"... aaaand it's gone!"