Notions of L^2-Dolbeault Cohomology with Values in Vector Bundles on Singular Complex Spaces

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Contents

Chapter 1
Introduction

There is a close connection between properties of differential operators and the geometry of manifolds. On complex manifolds, this relation between analysis and geometry is represented best by the Dolbeault operator $\overline{\partial}$, which can be seen as a generalization of the Wirtinger derivatives. $\overline{\partial}$ represents the Cauchy-Riemann equations and is called as well Cauchy-Riemann operator.

Particularly, the L^2-theory for the Dolbeault operator is a crucial part of complex analysis and has become indispensable for the subject area after the fundamental work of L. Hörmander on L^2-estimates and existence theorems for the $\overline{\partial}$-operator (see [Hör65] and [Hör66]) and the related work of A. Andreotti and E. Vesentini (see [AV63]). One should also mention J. Kohn's solution of the $\overline{\partial}$-Neumann problem (see [Koh63, Koh64] and also [KN65], joint with L. Nirenberg), which also implies existence and regularity results of the $\overline{\partial}$-complex (see Chap. III.1 in [FK72], joint with G. Folland). Some important applications of the L^2-methods are for instance the Ohsawa-Takegoshi extension theorem (see [OT87]), analyticity of level sets of Lelong numbers and invariance of plurigenera proven by Y.-T. Siu (see [Siu74] or [Siu98], respectively).

Whereas the theory is very well-developed on complex manifolds, there are many open questions and problems in singular settings. Hence, we are especially interested in studying the Dolbeault operator on singular complex spaces by transferring the mentioned L^2-methods.

The first problem here is that there is no canonical way to define differential forms on complex spaces. Let us recall three possibilities (among others):

1. intrinsic: differential forms defined on the regular part of the complex space.
2. extrinsic: differential forms which are (locally) given by the embedding of the complex space in the complex number space, and
3. as differential forms on a resolution.

Holomorphic differential forms of top-degree in the second and third category are used to define canonical sheaves of complex spaces (see Section 2.4).

For an appropriate definition of the Dolbeault operator, we choose the intrinsic setup. Since we are interested in L^2-methods, we need to measure the modulus of differential forms. For this, we choose a Hermitian metric on the regular part of the variety which extends smoothly to the singular points. Here, the next problem occurs: Since the Hermitian metric restricted to the regular part is not complete, densely defined differential operators – especially, the Dolbeault operator – between L^2-spaces have various closed extensions. Actually, this will be the essential difference between Dolbeault theory on manifolds and complex spaces and the reason why the theory gets significantly more complicated.

Let us briefly introduce the definitions and notations in the singular setting. For more details, please see Chapter 5. Let X be a (reduced) complex space with a Hermitian structure on the regular part $X^*{:=}X_{\text{reg}}$ which extends smoothly to the singular points. Let $\overline{\partial}_{\text{cpt}}\colon \mathscr{D}^{p,q}(X^*) \to \mathscr{D}^{p,q+1}(X^*)$ denote the Dolbeault operator with compact support on the manifold X^*. On the Hilbert space $L^2_{p,q}(X^*)$ of square-integrable differential forms on X^*, $\overline{\partial}_{\text{cpt}}$ is a densely defined differential operator, which is not closed. Let $\overline{\partial}_s$ denote the strong / minimal extension which is defined by the closure of the graph of $\overline{\partial}_{\text{cpt}}$, and let $\overline{\partial}_w$ be the weak / maximal extension, i. e. $\overline{\partial}$ in the sense of distributions. Like $\overline{\partial}_{\text{cpt}}$, the two operators $\overline{\partial}_s$ and $\overline{\partial}_w$ induce chain complexes, i. e. $\overline{\partial}_e \circ \overline{\partial}_e = 0$ for $e \in \{\text{cpt}, w, s\}$. The associated so-called Dolbeault cohomologies

$$H_s^{p,q}(X) := \ker\left(\overline{\partial}_s\colon L^2_{p,q}(X^*) \to L^2_{p,q+1}(X^*)\right) \big/ \, \overline{\partial}_s L^2_{p,q-1}(X^*) \quad \text{and}$$
$$H_w^{p,q}(X) := \ker\left(\overline{\partial}_w\colon L^2_{p,q}(X^*) \to L^2_{p,q+1}(X^*)\right) \big/ \, \overline{\partial}_w L^2_{p,q-1}(X^*)$$
are crucial indicators to study the geometry of the complex space X.

In [CGM82], J. Cheeger, M. Goresky and R. MacPherson were hoping that the Hodge decomposition of the de Rham cohomology generalizes to the singular case considering L^2-cohomologies. This motivated W. Pardon and M. Stern to study the arithmetic genus of projective varieties with respect to the above mentioned L^2-Dolbeault cohomologies. In [Par89], W. Pardon proved relations between the L^2-arithmetic genus of an algebraic surface X, i. e. the alternating sum of the dimensions of L^2-Dolbeault cohomology groups, and the (classical) arithmetic Todd genus of a resolution of X. In [PS91], W. Pardon and M. Stern proved that $\overline{\partial}_s$-Dolbeault cohomology groups of low-degree (i. e. $p = 0$) of a projective variety are isomorphic to the sheaf cohomology groups of the structure sheaf on a resolution of the variety, and that $\overline{\partial}_w$-Dolbeault cohomology groups of low-degree of a projective surface with isolated singularities are isomorphic to cohomology groups of the sheaf of holomorphic functions on a resolution with values in a certain divisor (cf. (1.2) below). For projective varieties with isolated singularities of arbitrary dimension, they proved the conjecture about the Hodge decomposition mentioned above in [PS01]. In [BS02], B. Berndtsson and N. Sibony studied the solvability of the strong and weak extensions of the Dolbeault operator on currents.

Just recently, considerable progress has been made in understanding the $\overline{\partial}_w$-Dolbeault cohomology: Let X be a compact complex space and let $\pi\colon M \to X$ be a resolution of singularities such that the unreduced exceptional divisor $Z := \pi^{-1}(X_{\mathrm{sing}})$ is supported on the union of smooth hypersurfaces with normal crossings. Then, it has been shown by N. Øvrelid and S. Vassiliadou in [ØV13] and J. Ruppenthal in [Rup11, Rup14a] by different approaches that there is a natural isomorphism

$$H_w^{n,q}(X) \cong H^q(M, \Omega^n_M) \cong H^{n,q}(M), \tag{1.1}$$

and, if X has only isolated singularities, there exists an effective divisor $D \geq Z - |Z|$ on M such that

$$H_w^{0,q}(X) \cong H^q(M, \mathcal{O}(D)) \,/\, H^q_{|Z|}(M, \mathcal{O}(D)) \tag{1.2}$$

for all $0 \leq q \leq n$. Here, Ω^n_M denotes the sheaf of holomorphic n-forms on M and $H^q_{|Z|}$ denotes the cohomology with support on $|Z|$. If $\dim X \leq 2$, then (1.2) holds with the divisor $D = Z - |Z|$ and $H^q_{|Z|}\big(M, \mathcal{O}(Z - |Z|)\big) = 0$, so that (1.1) and (1.2) give a smooth representation of the L^2-cohomology groups $H_w^{0,q}(X)$. J. Ruppenthal conjectured that (1.2) holds with $D = Z - |Z|$ for arbitrary dimension of X (see [Rup11]).

However, the L^2-theory for the $\overline{\partial}$-operator developed in [ØV13] and [Rup11, Rup14a] applies only to $\dim X \geq 2$ (for $\dim X = 1$, (1.1) and (1.2) have been known before, see [Par89, PS91]). Hence, we present a complete L^2-theory for the $\overline{\partial}$-operator on a singular complex curve in Chapter 6. There, we will comprehend the appearance of the divisor $Z - |Z|$ for $\dim X = 1$ in very detail. This is done by use of the Puiseux normalization for complex curves. In particular, we present an L^2-Dolbeault version of the Riemann-Roch theorem, which connects the topological invariant genus to the dimension of the Dolbeault cohomology groups with values in line bundles (cf. L^2-arithmetic genus mentioned above). Furthermore, we will prove that for a complex curve X,

$$H^{0,q}_{s,\mathrm{loc}}(X) \cong H^q(X, \widehat{\mathcal{O}}_X), \quad q = 0, 1,$$

where $\widehat{\mathcal{O}}_X$ denotes the sheaf of weakly holomorphic functions on X and $H^{p,q}_{s,\mathrm{loc}}$ denotes the Dolbeault cohomology with respect to a localized version of $\overline{\partial}_s$ (see Theorem 6.3 and Section 6.4).

A further aim of this thesis is the study of Dolbeault cohomologies with values in vector bundles. We obtain the following generalization of (1.1) (for a line-bundle-version, see [Rup14a, Thm. 1.5]).

Theorem 1.3. *Let X be a Hermitian complex space of pure dimension n, let $\mathscr{S}$ be a coherent analytic sheaf on X, and let $\pi\colon M \to X$ be a resolution of singularities such that the torsion-free preimage $\pi^T\mathscr{S}$ is locally free. We set $X' := X_{\mathrm{reg}} \setminus \operatorname{Sing}\mathscr{S}$ and denote the vector bundle associated to $\mathscr{S}_{X'}$ as $F \to X'$. Then, for all $q \geq 0$,*

$$H^{n,q}_{w,\mathrm{loc}}(X, F) \cong H^q(M, \Omega^n_M \otimes \pi^T\mathscr{S}).$$

If either X is compact or $X \Subset Y$, where Y is a Hermitian complex space such that ∂X is smooth, strictly pseudoconvex and contained in X_{reg}, then

$$H^{n,q}_w(X, F) \cong H^q(M, \Omega^n_M \otimes \pi^T\mathscr{S}).$$

Here, $H^{p,q}_{w,\mathrm{loc}}(X, F)$ denotes the Dolbeault cohomology with respect to a localized version of $\overline{\partial}_w$ and values in F, defined in Section 5.1. Note that the index 'loc' means local on X, i. e. the cohomology group with respect to differential forms with on X locally square-integrable coefficients, and not on X'. In general, we have $H^{p,q}_{w,\mathrm{loc}}(X) \neq H^{p,q}_{w,\mathrm{loc}}(X_{\mathrm{reg}})$, where the latter cohomology group is defined by $\overline{\partial}_{w,\mathrm{loc}}\colon L^{2,\mathrm{loc}}_{p,q}(X_{\mathrm{reg}}) \to L^{2,\mathrm{loc}}_{p,q}(X_{\mathrm{reg}})$ in the usual way for the manifold X_{reg}.

Additionally, we get the following slight variation:

Theorem 1.3'. *Let X be a Hermitian complex space of pure dimension n, let $\pi\colon M \to X$ be a resolution of singularities, let $M' \subset M$ be the complement of the exceptional set of π, and let $E \to M$ be a Hermitian vector bundle on M which is Nakano semi-positive on $\pi^{-1}(U)$ for all $U \subset X$ small enough. Let $F := \left(\pi|_{M'}^{-1}\right)^* E$ denote the pullback bundle of E on $\pi(M')$. Then, for all $q \geq 0$,*

$$H^{n,q}_{w,\mathrm{loc}}(X, F) \cong H^q(M, \Omega^n_M(E)).$$

For locally free sheaves, we get the following corollary:

Corollary 1.4. *Let X be a Hermitian complex space of pure dimension n, and let E be a Hermitian vector bundle on X. Then, for all $q \geq 0$,*

$$H^{n,q}_{w,\mathrm{loc}}(X, E) \cong H^q(X, \mathscr{K}_X(E)).$$

Here, $\mathscr{K}_X$ denotes the Grauert-Riemenschneider canonical sheaf (see Section 2.4 or [GR70]) and $\mathscr{K}_X(E) := \mathscr{K}_X \otimes \mathcal{O}(E)$.

A crucial ingredient of the proof of the theorems above is the relative vanishing theorem of K. Takegoshi in [Tak85]. More precisely, we will prove and use its generalization for Nakano semi-positive vector bundles on manifolds in Chapter 7. In Chapter 8, we present further generalizations.

Let us picture the historical background of the relative vanishing theorem. K. Kodaira proved in [Kod53], that for a compact Kähler manifold M of dimension n and a positive (holomorphic) line bundle L on M, all higher sheaf cohomology groups of the canonical bundle Ω^n_M with values in L vanish, i. e.

$$H^q(X, \Omega^n_M(L)) = 0 \quad \text{for } q \geq 1.$$

There exist many generalizations of this theorem – for instance, the so-called Nakano vanishing theorem for Nakano positive vector bundles (cf. [Nak55]). H. Grauert and O. Riemenschneider studied the generalization of Kodaira's vanishing theorem to complex spaces. In [GR70], they proved that, if X is a Moishezon space, i. e. an irreducible compact complex space of dimension n with n independent (globally defined) meromorphic functions, and if $\mathscr{E}$ is a quasi-positive locally free sheaf on X, then

$$H^q(X, \mathscr{E} \otimes \mathscr{K}_X) = 0 \quad \text{for } q \geq 1.$$

K. Takegoshi gave a relative version of the Grauert-Riemenschneider vanishing theorem in [Tak85]. He proved that for a proper surjective holomorphic map $f\colon M \to Z$ where M is a manifold of dimension n bimeromorphic to a Kähler manifold, the higher direct images $f_{(q)}(\Omega^n_M)$ vanish for $q > n - \dim Z$. The key ingredient is an L^2-vanishing theorem for weakly 1-complete Kähler manifolds. These results have many applications, in particular in the study of singular complex spaces (see e. g. [CS95, CR09]). Therefore, we are interested in generalizations of these so-called relative vanishing theorems.

Applying the Grauert-Riemenschneider vanishing theorem (Satz 2.1 in [GR70]) and K. Takegoshi's L^2-vanishing theorem (see Theorem 7.9), we get the following two corollaries of Theorem 1.3.

Corollary 1.5. *Let X be a compact Moishezon space, and let $\mathscr{S}$ be a torsion-free quasi-positive sheaf. We set $X' := X_{\text{reg}} \setminus \operatorname{Sing} \mathscr{S}$ and denote the vector bundle associated to $\mathscr{S}_{X'}$ as F. Then, for each $q > 0$,*

$$H^{n,q}_{w}(X, F) = 0.$$

Corollary 1.6. *Let X be a holomorphically convex irreducible Kähler space of dimension n, let Φ be a smooth plurisubharmonic exhaustion function of X whose complex Hessian has in at least one point r strict positive eigenvalues, and let $\mathscr{S}$ be a Nakano semi-positive torsion-free sheaf on X. We set $X' := X_{\text{reg}} \setminus \operatorname{Sing} \mathscr{S}$ and denote the vector bundle associated to $\mathscr{S}_{X'}$ as F. Then, for each $q > n - r$,*

$$H^{n,q}_{w,\text{loc}}(X, F) = 0.$$

Let Y be a complex space, and $X \Subset Y$ have a smooth, strictly pseudoconvex boundary contained in Y_{reg}. Then, for all $q > 0$,

$$H^{n,q}_{w}(X, F) = 0.$$

A key ingredient of the computation of the Dolbeault cohomology groups of top degree, i. e. for $p = n$, is the following L^2-variant of a canonical sheaf with values in a coherent analytic sheaf $\mathscr{S}$:

$$\mathcal{K}_X(\mathscr{S}) := \mathscr{K}\!\mathit{er}\left(\overline{\partial}_{w,\text{loc}} \colon L^{2,\text{loc}}_{n,0}(X, F) \to L^{2,\text{loc}}_{n,1}(X, F)\right),$$

where F is the vector bundle on $X' := X \setminus \operatorname{Sing} \mathscr{S}$ associated to $\mathscr{S}_{X'}$ (for more details see Section 5.1). $\mathcal{K}_X(\mathscr{S})$ can be interpreted as the Grauert-Riemenschneider canonical sheaf $\mathscr{K}_X$ with values in $\mathscr{S}$. In Chapter 10, we will prove

$$\mathcal{K}_X(\mathscr{S}) \cong \mathscr{S} \cdot \mathscr{K}_X$$

using the notation of H. Grauert and O. Riemenschneider in [GR70]. For locally free sheaves $\mathscr{E}$ on X, the projection formula (see Theorem 4.10) implies

$$\mathcal{K}_X(\mathscr{E}) \cong \mathscr{E} \otimes \mathscr{K}_X.$$

For arbitrary coherent analytic sheaves $\mathscr{S}$, $\mathcal{K}_X(\mathscr{S})$ and $\mathscr{S}\otimes\mathscr{K}_X$ do not coincide. Since $\mathscr{S}\otimes\mathscr{K}_X$ is a natural way to define n-forms with values in $\mathscr{S}$, we are interested in understanding $\mathscr{S}\otimes\mathscr{K}_X$ and in the question under which circumstances the projection formula does generalize for non-locally-free sheaves.

To study these questions, we investigate proper modifications of coherent analytic sheaves in Chapter 4. More precisely, we study the direct image of the preimage sheaf and vice versa. Among other results, we obtain the following statement (see Theorem 4.2). Let $\mathscr{S}$ be a torsion-free coherent analytic sheaf on a locally irreducible complex space X. If the linear space associated to $\mathscr{S}$ is normal, then

$$\mathscr{S} \cong \pi_*\pi^T\mathscr{S} \tag{1.7}$$

for all proper modifications $\pi\colon Y \to X$ of X. Using this result and Theorem 4.8, we get the following answer to the question from above: If additionally $\mathscr{K}_X$ and $\mathscr{K}_Y$ are locally free, then there exists an effective Cartier divisor D on Y with support on the exceptional divisor of π such that

$$\mathscr{S} \otimes \mathscr{K}_X \cong \pi_*(\pi^T\mathscr{S} \otimes \mathscr{K}_Y(D)).$$

A further motivation for the study of modifications of coherent analytic sheaves is as follows. H. Rossi proved that for all coherent analytic sheaves $\mathscr{S}$, there exists a proper modification π such that the torsion-free preimage $\pi^T\mathscr{S}$ is locally free. Using this, we will prove a generalization of Takegoshi's relative vanishing theorem (see Theorem 8.2 and Theorem 8.4) and of the vanishing theorem of H. Grauert and O. Riemenschneider itself (see Corollary 8.3). Furthermore, we will show that the normality assumption is necessary to obtain (1.7) for coherent analytic sheaves of rank one whose associated linear spaces are irreducible and Cohen-Macaulay (see Remark 4.20).

A linear space is a fibre space which has vector spaces as fibres. For a proper definition and crucial properties, see Section 3.2. Whereas all fibres of a vector bundle have the same dimension, the dimension of the fibres of a linear space depends on the base point. The category of linear spaces is dual to the category of coherent analytic sheaves. Motivated by the results on modifications of coherent analytic sheaves mentioned above, we will study linear spaces in Chapter 3 comprehensively. In particular, we present criteria for coherent analytic sheaves whose associated linear spaces are normal (see Theorem 3.1 and Corollary 3.20).

Let us summarize the structure of this thesis and clarify the connection to the author's articles. We start by recalling basic definitions and preliminaries in Chapter 2. Then, we discuss linear spaces in Chapter 3. Most of its results are already presented in Sect. 3 of [RS13]. In Chapter 4, we study direct image and preimage of coherent analytic sheaves under proper modifications, covering results of [RS13] and Sect. 4 in [Ser15]. Chapter 5 contains the definitions of the studied Dolbeault operators and basic properties of them, for instance an L^2-extension theorem (see Theorem 5.6). This is followed by the study of singular complex curves in Chapter 6 as in [RS15]. The generalization of Takegoshi's relative vanishing theorem is done in Chapter 7 and Chapter 8. Here, we follow mainly the lines of [Ser15]. Chapter 9 contains applications to ideal sheaves on holomorphically convex manifolds as studied in Sect. 7 of [RS13] and in Sect. 5 of [Ser15]. Finally, we conclude this thesis with the proofs of Theorem 1.3 and Theorem 1.3' in Chapter 10.

Chapter 2
Preliminaries

In this chapter, we recall basic definitions and propositions about modifications, coherent analytic sheaves and plurisubharmonic functions, which will be used in the other sections of this thesis.

2.1 Proper modifications

In complex analysis, proper modifications belong to the most important tools, especially in the study of bimeromorphic geometry on complex spaces. The best example is the σ-process, which rises to be useful for resolutions of singularities of complex spaces. We recall the definition:

Def. 2.1. A proper surjective holomorphic map $\varphi\colon X \to Y$ of complex spaces X and Y is called a *(proper) modification of* X if there are closed analytic sets $A \subset X$ and $B \subset Y$ such that
(1) $B = \varphi(A)$,
(2) $\varphi|_{X\setminus A}\colon X \setminus A \to Y \setminus B$ is biholomorphic, and
(3) A and B are analytically rare.
If A and B are minimal with the properties $(1-3)$, then A is called the *exceptional set of* φ and B the *centre of the modification.*

Let us also recall the definition of rare and thin:

Def. 2.2. Let X be a complex space and A a subspace of X. If the restriction $\mathcal{O}_X(U) \to \mathcal{O}_X(U\setminus A)$ is injective for all open $U \subset X$, then we call A *rare in* X. If $\overline{X\setminus A} = X$, then A is called *thin.*

If X is reduced, then rare and thin are equivalent. In general, rare does not imply thin and vice versa.

Monoidal transformations. One of the most popular modifications is the monoidal transformation along a submanifold in a manifold, also called σ-process or blow-up: Let M be a complex manifold and Σ be a (complex) submanifold of M of codimension $m+1$. Then, there exists a proper modification $\sigma\colon \widetilde{M} \to M$ with Σ as centre such that the exceptional set $\sigma^{-1}(\Sigma)$ is a smooth hypersurface in $\widetilde{M}$, locally (with respect to M) $\widetilde{M}$ is embedded in $M\times\mathbb{CP}^m$, σ is the restriction of the projection $\mathrm{pr}|_{\widetilde{M}}\colon \widetilde{M} \to M$ and $\sigma^{-1}(\Sigma)=\Sigma\times\mathbb{CP}^m$. The monoidal transformation is quite useful. Among other, one application is the generalization of results about hypersurfaces to submanifolds of arbitrary codimension, e. g. the theorem of M. Schneider (see [Sch73]), which answered a conjecture of Hartshorne: If M is compact and Σ has a 'metrisch q-konkaves' normal bundle (i. e. outside of the zero section, the hermitian form is fibrewise a q-convex function), then $M\backslash\Sigma$ is $(m+q)$-convex in the sense of Andreotti-Grauert.

If $\pi\colon M \to X$ is a proper modification of X with smooth M, then π is called a *resolution of singularities of* X. Note that there exists no minimal resolution of singularities, i. e. in general, it can not satisfy a universal property (except for $\dim X = 1$, see next section). Nevertheless, H. Hironaka proved that a resolution of singularities always exists by using the blow-up procedure in an inductive procedure (see [Hir64] or [Hir77, Thm. 7.1]):

Theorem 2.3 (Resolution of singularities)**.** *Let X be a reduced complex space. Then, there exist a smooth manifold M and a proper modification $\pi\colon M \to X$ such that the centre of π is the singular set X_{sing} of X and the exceptional set $\pi^{-1}(X_{\mathrm{sing}})$ is the union of smooth hypersurfaces with only normal crossings (snc), i. e. the intersections of the hypersurfaces are transversal.*

2.2 Resolution of complex curves

Def. 2.4. We call a reduced complex space of pure-dimension one a *(singular) complex curve.*

Let X be a compact complex curve. Then, a resolution of the singularities of X is given just by the normalization of the curve, and it is unique up to biholomorphism: Let $\pi_1 \colon M_1 \to X$ and $\pi_2 \colon M_2 \to X$ be two resolutions of X. Then, $\psi := \pi_2^{-1} \circ \pi_1 \colon M_1 \backslash \pi_1^{-1}(X_{\text{sing}}) \to M_2 \backslash \pi_2^{-1}(X_{\text{sing}})$ is biholomorphic and bounded in the singular locus. Yet, $\pi_i^{-1}(X_{\text{sing}})$ consist of isolated points. Therefore, ψ has a (bi-) holomorphic extension.

Let $\pi \colon M \to X$ be a resolution of a compact complex curve X. We define the *genus* of X by the genus of the resolution

$$g(X) := h^1(M) = \dim H^1(M, \mathcal{O}).$$

If X has more than one irreducible component, then M is not connected and $h^1(M)$ is the sum of the genera of the connected components. Since the resolution is unique, it is well-defined.

Throughout Chapter 6 (except of Section 6.5.1), we will work with divisors on compact Riemann surfaces only. Therefore, there are no differences between Cartier and Weil divisors, and we can associate to each line bundle a divisor.

Let $L \to X$ be a holomorphic line bundle. Then, the pullback $\pi^* L \to M$ is well-defined by the pullback of the transition functions of the line bundle. There is a divisor D on M associated to $\pi^* L$ such that $\mathcal{O}(\pi^* L) \cong \mathcal{O}(D)$, where $\mathcal{O}(D)$ denotes the sheaf of germs of holomorphic functions f such that $\operatorname{div}(f) + D \geq 0$, and $\deg \pi^* L = \deg D$. The uniqueness of the resolution (up to biholomorphism) implies the independence of $\deg \pi^* L$ from π, so that

$$\deg L := \deg \pi^* L$$

is also well-defined.

For any divisor D on M, there exists a holomorphic line bundle $L_D \to M$ associated to D such that $\mathcal{O}(L_D) \cong \mathcal{O}(D)$. Since the construction of L_D allows different possible choices, we will define square-integrable sections via the identification with meromorphic functions on M:

Since M is a Riemann surface, we can assume $D = \sum_{i\in I} a_i p_i$, where $\{p_i\}_{i\in I}$ is a locally finite subset of M and p_i denotes the reduced divisor in the point $p_i \in M$. Choose small enough neighbourhoods $V_i \Subset U_i \Subset M$ of p_i with charts $\psi_i \colon U_i \to \mathbb{C}, \psi_i(p_i) = 0$ such that the U_i are pairwise disjoint, and set $U_0 := M \setminus \bigcup V_i$. For each holomorphic section $s\colon M \to L_D$, there exists a meromorphic function $\Psi(s)\colon M \to \mathbb{C}$, with $\psi_i^{a_i}\Psi(s) \in \mathcal{O}(U_i)$. By definition, $\Psi(s)$ can be defined for any section $s\colon M \to L_D$ even for non-holomorphic sections. Hence, we define

$$L^{2,\mathrm{loc}}(M,D) := \left\{h\colon M \to \mathbb{C} : h \in L^{2,\mathrm{loc}}(U_0), \psi_i^{a_i} \cdot h \in L^2(U_i)\right\}$$
$$L^2(M,D) := \left\{h\colon M \to \mathbb{C} : h \in L^2(U_0), \textstyle\sum_{i\in I} \int_{U_i} |\psi_i^{a_i} \cdot h|^2 < \infty\right\}.$$

Obviously, the definition does not depend on the choice of U_i, V_i and ψ_i. A section $s\colon M \to L_D$ is called square-integrable if $\Psi(s) \in L^2(M,D)$. If M is compact, then this coincides with the usual notation of square-integrable sections with respect to an arbitrary smooth Hermitian metric on the line bundle (all such metrics are equivalent). The set of (p,q)-forms with values in L_D and square-integrable coefficients will be denoted by $L^{2,\mathrm{loc}}_{p,q}(M, L_D)$. For an effective divisor Y (i. e. $Y \geq 0$), we get $L^2(M) = L^2(M,0) \subset L^2(M,Y)$. For arbitrary Hermitian metrics on M and on the line bundles, this induces the embedding / inclusion $L^{2,\mathrm{loc}}_{p,q}(M, M\times\mathbb{C}) \subset L^{2,\mathrm{loc}}_{p,q}(M, L_Y)$ and

$$L^{2,\mathrm{loc}}_{p,q}(M, L_D) \subset L^{2,\mathrm{loc}}_{p,q}(M, L_{D+Y}), \text{ via } s \mapsto [\Psi^{-1}](\Psi(s)+Y), \quad (2.5)$$

and, if M is compact, then

$$L^2_{p,q}(M, L_D) \subset L^2_{p,q}(M, L_{D+Y}). \quad (2.6)$$

Let $Z := \pi^{-1}(X_{\mathrm{sing}})$ be the unreduced exceptional divisor of the resolution $\pi\colon M \to X$ and $|Z|$ the underlying reduced divisor. Then,

$\deg(Z - |Z|)$ is independent of the resolution as well. We will discuss some alternative ways to compute $\deg Z$.

Locally, the resolution is given by the Puiseux parametrization: Let A be an analytic set of dimension one in $\Omega \Subset \mathbb{C}^n$ with $A_{\text{sing}} = \{0\}$ which is irreducible at 0. Shrinking Ω, there are coordinates $z, w_1, ..., w_{n-1}$ around 0 such that A is contained in the cone $\|w\| \leq C|z|$, $w = (w_1, .., w_{n-1})$. The projection $\mathrm{pr}_z \colon A \to \mathbb{C}_z$ on the z-coordinate is a finite ramified covering. Let s be the number of the sheets of pr_z. Generic choices of the coordinates give the same number of sheets s, called the multiplicity $\mathrm{mult}_0 A$ of A in $\{0\}$. There exists a parametrization $\pi \colon \Delta \to A, t \mapsto (t^s, w_1(t), ..., w_{n-1}(t))$, where $\Delta := \{t \in \mathbb{C} : |t| < 1\}$; cf. e. g. [Chi89, Sect. 6.1]. π is called the *Puiseux parametrization*. The unreduced exceptional divisor is just $Z = (\pi^{-1}(z)) = (t^s)$, and so $\deg Z = s$.

The number of sheets of the covering pr_z is also equal to the Lelong number $\nu([A],0)$ of the positive current $[A]$ given by integration over A (see [Chi89, Prop. 2 in § 3.15], [Dem12, Thm. 7.7] or [GH78, § 3.2]).

The tangent cone gives another way to compute $\mathrm{mult}_0 A$. For a holomorphic function f on Ω, let $f = \sum_{k=k_0}^{\infty} f_k$ be the decomposition in homogeneous polynomials f_k of degree k with $f_{k_0} \neq 0$ (choosing a smaller Ω) and $f^* := f_{k_0} \neq 0$ be the *initial homogeneous polynomial of* f. If A is given by the ideal sheaf $\mathscr{J}_A$, then

$$C_0(A) = \{\alpha \in \mathbb{C}^n : f^*(\alpha) = 0 \; \forall f \in \mathscr{J}_{A,0}\} \subset T_0\mathbb{C}^n$$

is called the *tangent cone* of A in 0 (cf. [Chi89, Sect. 8.4]). The natural projection $\mathbb{C}^n \backslash 0 \to \mathbb{CP}^{n-1}$ maps $C_0(A)$ on a projective variety $\widetilde{C_0}(A)$. The *degree* $\deg Y$ of a projective variety Y in $\mathbb{CP}^{n-1}$ of dimension p is defined as the class of Y in $H_{2p}(\mathbb{CP}^{n-1}, \mathbb{Z}) \cong \mathbb{Z}$, and $\mathrm{mult}_0 A = \deg \widetilde{C_0}(A)$ (see e. g. Sect. 2 of [GH78, § 1.3]). In the case of an irreducible complex curve A, note that $\widetilde{C_0}(A)$ is just a point of multiplicity $\mathrm{mult}_0 A$.

All in all, we have

$$\deg Z = \mathrm{mult}_0 A = \nu([A], 0) = \deg \widetilde{C_0}(A).$$

2.3 Coherent analytic sheaves

In this section, we recall some basic properties about coherent analytic sheaves.

Let X be a complex space. Throughout this thesis, we will always denote the associated structure sheaf with $\mathcal{O}_X$. An $\mathcal{O}_X$-module sheaf is called analytic. If M is an open subset or an analytic subspace, then $\mathscr{S}_M$ shall denote the restriction of a sheaf $\mathscr{S}$ to M.

Since $\mathcal{O}_X$ is $\mathcal{O}_X$-coherent, we obtain that an analytic sheaf is coherent (with respect to $\mathcal{O}_X$) if and only if, for every $p \in X$, there exists a neighbourhood $U \subset X$ and numbers s, t such that

$$\mathcal{O}_U^s \to \mathcal{O}_U^t \to \mathscr{S}_U \to 0$$

is exact. Actually, the definition of coherence is represented in this sequence: If there exists a number t such that $\mathcal{O}_p^t \to \mathscr{S}_p$ is surjective, then this means $\mathscr{S}_p$ is generated by less than t elements as $\mathcal{O}_p$-module. In particular, it is finitely generated. The number s gives an estimate of the relations which are needed to define $\mathscr{S}_p$ as a quotient of $\mathcal{O}_U^t$. This shows that the kernel of $\mathcal{O}_p^t \to \mathscr{S}_p$ is finitely generated and from that one can conclude that $\mathscr{S}$ is relation finite.

If $\mathfrak{m}_p$ denotes the maximal ideal of $\mathcal{O}_p$, then $\mathscr{S}_p/\mathfrak{m}_p\mathscr{S}_p$ is a finite dimensional vector space. We define the *rank of* $\mathscr{S}$ *in* p by

$$\operatorname{rk}_p \mathscr{S} := \dim \mathscr{S}_p/\mathfrak{m}_p\mathscr{S}_p$$

and obtain that there exist a neighbourhood $U \subset X$ of p and a surjection $\mathcal{O}_U^{\operatorname{rk}_p \mathscr{S}} \twoheadrightarrow \mathscr{S}_U$. In particular, $p \mapsto \operatorname{rk}_p \mathscr{S}$ is an upper semi-continuous function. The *rank of* $\mathscr{S}$ is defined as $\operatorname{rk} \mathscr{S} := \min_{p \in X} \operatorname{rk}_p \mathscr{S}$. In points p, where $\operatorname{rk}_p \mathscr{S} = \operatorname{rk} \mathscr{S}$, $\mathscr{S}$ is locally free, i. e. there exists a neighbourhood U such that $\mathscr{S}_U \cong \mathcal{O}_U^{\operatorname{rk} \mathscr{S}}$. The *singular locus*

$$\operatorname{Sing} \mathscr{S} := \{p \in X : \operatorname{rk}_p \mathscr{S} \geq \operatorname{rk} \mathscr{S}\}$$

of $\mathscr{S}$ is analytic in X. If X is irreducible, then $\operatorname{Sing} \mathscr{S}$ is thin in X (see e. g. Prop. 3.1 in [Ros68]).

Torsion of sheaves. Let $s_p \in \mathscr{S}_p$ be a germ of the coherent analytic sheaf $\mathscr{S}$ on X, $p \in X$. If there exits an $r_p \in \mathcal{O}_{X,p}$, $r_p \neq 0$, such that $r_p \cdot s_p = 0$, then s_p is called torsion element of $\mathscr{S}_p$. The subsheaf

$$\mathscr{T}(\mathscr{S}) := \dot{\bigcup}_{p \in X} \{s_p \in \mathscr{S}_p : s_p \text{ is torsion element}\}$$

of $\mathscr{S}$ is called *torsion sheaf*. Actually, $\mathscr{T}(\mathscr{S})$ is the kernel of the canonical map $\mathscr{S} \to (\mathscr{S}^*)^*$. In particular, $\mathscr{T}(\mathscr{S})$ is coherent. If X is irreducible, then the support of $\mathscr{T}(\mathscr{S})$ is thin. Furthermore, we get $\mathscr{S} = \mathscr{T}(\mathscr{S})$ if and only if $\operatorname{rk} \mathscr{S} = 0$.

Def. 2.7. $\mathscr{S}$ is called *torsion-free* if $\mathscr{T}(\mathscr{S}) = 0$.

Remark 2.8. Let $\mathscr{S}$ be a torsion-free coherent analytic sheaf on a complex space X which is normal or (more weakly) satisfies $\operatorname{codim} X_{\text{sing}} \geq 2$. The torsion-freeness of $\mathscr{S}$ implies that for all small enough open sets $U \subset X_{\text{reg}}$, there is an inclusion

$$\mathscr{S}_U \hookrightarrow \mathcal{O}_U^r,$$

where r denotes the rank of $\mathscr{S}$, i. e. $\mathscr{S}_{X_{\text{reg}}}$ is a first syzygy sheaf. Then, the syzygy theorem implies $\operatorname{codim} \operatorname{Sing} \mathscr{S}_{X_{\text{reg}}} \geq 2$ (see e. g. Lem. 1.1.8 and its corollary in [OSS11, Chap. 2]). Since $\operatorname{codim} X_{\text{sing}} \geq 2$ and $\operatorname{Sing} \mathscr{S} \subset X_{\text{sing}} \cup \operatorname{Sing} \mathscr{S}_{X_{\text{reg}}}$, we get

$$\operatorname{codim} \operatorname{Sing} \mathscr{S} \geq 2.$$

Remark 2.9. The tensor product of two torsion-free sheaves do not need to be torsion-free. E. g., let $\mathscr{I}$ be the ideal sheaf generated by (z^2, zw) on $\mathbb{C}^2_{z,w}$ and $\mathscr{J}$ be the ideal sheaf generated by (w^2, zw). Then, $z^2 \otimes w^2 - zw \otimes zw \in \mathscr{I} \otimes \mathscr{J}$ is not zero. Yet, $z \cdot (z^2 \otimes w^2 - zw \otimes zw) = z^3 \otimes w^2 - z^2 \otimes zw^2 = 0$.

Let $\pi \colon Y \to X$ be a proper modification of a complex space X. Then, it is easy to see that the direct image $\pi_* \mathscr{F}$ of a torsion-free coherent analytic sheaf $\mathscr{F}$ remains torsion-free. But, the analytic inverse image sheaf $\pi^* \mathscr{S}$ of a torsion-free coherent analytic sheaf $\mathscr{S}$ is not torsion-free in general. For a counterexample, see the example in [GR70, Sect. 1],

i. e. the pullback of the maximal ideal sheaf of the origin in $\mathbb{C}^2$ under blow-up of the origin is not torsion-free. One can say more or less that $\pi^*\mathscr{S}$ is torsion-free in a point $y \in Y$ if and only if $\mathscr{S}$ is locally free in $\pi(y)$ (see [Rab79] or Remark 3.15 below). This motivates the following definition:

Def. 2.10. Let $f\colon Y \to X$ be a holomorphic map between complex spaces such that Y is locally irreducible. Let $\mathscr{S}$ be a coherent analytic sheaf on X. Then,

$$f^T\mathscr{S} := f^*\mathscr{S}/\mathscr{T}(f^*\mathscr{S})$$

is called the *torsion-free preimage sheaf* of $\mathscr{S}$ under f.

Torsion-free preimages under proper modifications have been first studied by H. Rossi in [Ros68], H. Grauert and O. Riemenschneider in [GR70] and [Rie71] (they denoted it as $\mathscr{S} \circ f$).

Grauert's direct image theorem. Let X and Y be complex spaces, $f\colon X \to Y$ a proper holomorphic map and $\mathscr{S}$ a coherent analytic sheaf on X. Then, $U \to H^q(f^{-1}(U), \mathscr{S})$, $U \overset{\text{open}}{\subset} X$, is a presheaf. We call the associated sheaf the *higher direct image sheaf* $f_{(q)}\mathscr{S}$. $f_{(0)}\mathscr{S}$ coincides with the classical direct image $f_*\mathscr{S}$. In [Gra60, Thm. I], H. Grauert has proven for all $q \geq 0$:

If f is proper, then $f_{(q)}\mathscr{S}$ is coherent.

With Grauert's direct image theorem, the proofs of the following famous theorems of R. Remmert get much more simplified.

Theorem 2.11 (Remmert reduction, [Rem56])**.** *Let X be a holomorphically convex complex space. Then, there is a Stein space Y and a proper, holomorphic and surjective $\pi\colon X \to Y$ such that the sheaf homomorphism $\mathcal{O}_Y \to \pi_*(\mathcal{O}_X)$ is an isomorphism.*

Theorem 2.12 (Remmert's mapping theorem, [Rem57])**.** *Let $f\colon X \to Y$ be a proper holomorphic map of complex spaces, and let $A \subset X$ be analytic. Then, the image $f(A)$ is analytic in Y.*

Normalization sheaf. If X is a pure dimensional complex space, let $\widehat{\mathcal{O}} = \widehat{\mathcal{O}}_X$ denote the normalization sheaf of $\mathcal{O}_X$ which is defined stalkwise by the integral closure of $\mathcal{O}_{X,x}$ in the sheaf $\mathcal{M}_{X,x}$ of meromorphic functions for all $x \in X$ (cf. e. g. [GR84, § VI.4]). A function in $\widehat{\mathcal{O}}(U)$, $U \subset X$ open, is called *weakly holomorphic*. Weakly holomorphic functions are holomorphic in regular points of X and bounded in singular points. If X is locally irreducible, then weakly holomorphic functions are continuous in X_{sing}.

The classical Riemann extension theorem generalizes to the following result (see e. g. [GR84, Sect. VII.4.1]):

Theorem 2.13 (Riemann extension)**.** *Let X be a pure dimensional complex space. Every holomorphic function on X_{reg} which is bounded in points of X_{sing} is weakly holomorphic on X.*

2.4 Canonical sheaves on singular spaces

Def. 2.14. Let X be a complex space of pure dimension n and $\pi\colon M \to X$ be a resolution of singularities. Let Ω^n_M denote the canonical sheaf of holomorphic n-forms on M. Then,

$$\mathscr{K}_X := \pi_* \Omega^n_M$$

is the so-called *Grauert-Riemenschneider canonical sheaf.*

H. Grauert and O. Riemenschneider introduced this canonical sheaf and proved that the definition is independent of the resolution in [GR70, § 2.1]. In § 2.2 of [GR70], it is shown that for normal spaces X, $\mathscr{K}_X$ is given by the presheaf

$$\mathscr{K}_X(U) = \{\alpha \in \Omega^n_{X_{\text{reg}}}(U_{\text{reg}}) : \int_{V_{\text{reg}}} \alpha \wedge \overline{\alpha} < \infty \ \forall\, V \Subset U\}, \tag{2.15}$$

where the definition is obviously independent of the chosen Hermitian metric on U_{reg}. We will see that (2.15) holds even for non-normal X (see Theorem 10.1).

In the following, we recall the definition and basic results of the tangent and the cotangent sheaf on complex spaces. For more details, see e. g. [PR94, § 1].

Let X be a complex subspace of an open set $D \subset \mathbb{C}^N$ of pure-dimension n, given by the ideal sheaf $\mathscr{J}$. The map $\mathscr{J} \to \Omega^1_D, f \mapsto df$ induces a morphism $\alpha\colon \mathscr{J}/\mathscr{J}^2 \to \Omega^1_D/\mathscr{J}\Omega^1_D$. We call $\Omega^1_X := \mathscr{C}oker\, \alpha$ the *cotangent sheaf of* X. The *tangent sheaf* $\mathscr{T}_X$ *of* X is the dual of the cotangent sheaf, i. e. $\mathscr{T}_X := \mathscr{H}om(\Omega^1_X, \mathcal{O}_X)$. For all points $x \in X$, let $\mathfrak{m}_x$ denote the maximal ideal of $\mathcal{O}_{X,x}$. Then, there exists a bijection

$$\Omega^1_{X,x}/\mathfrak{m}_{X,x}\Omega^1_{X,x} \xrightarrow{\sim} \mathfrak{m}_{X,x}/\mathfrak{m}^2_{X,x}, \tag{2.16}$$

which allows to compute $\Omega^1_{X,x}$ and $\mathscr{T}_{X,x}$.

The *dualizing canonical sheaf* in sense of A. Grothendieck is defined as

$$\omega_X := \mathscr{E}xt^{N-n}_{\mathcal{O}_D}(\mathcal{O}_X, \Omega^N_D).$$

In general, the n^{th} exterior power $\Omega^n_X := \Lambda^n \Omega^1_X$ and the dualizing sheaf ω_X do not coincide. The first one is in some sense too singular. If X is a locally complete intersection, the adjunction formula gives the canonical isomorphism

$$\omega_X \cong \Omega^N_D|_X \otimes \det \mathscr{N}_{X/D},$$

where $\mathscr{N}_{X/D} := \mathscr{H}om_{\mathcal{O}_D}(\mathscr{J}/\mathscr{J}^2, \mathcal{O}_X)$ denotes the *normal sheaf of* X *in* D. For regular X, $\mathscr{N}_{X/D}$ is the sheaf of sections of the normal bundle to X.

For arbitrary complex spaces X, the local definitions can be glued together. Hence, Ω^1_X and ω_X are well-defined. If X is a manifold, the notations coincide with the usual one and $\omega_X = \Omega^n_X = \mathscr{K}_X$.

For normal spaces X, we obtain that ω_X is a subsheaf of $\mathscr{K}_X$ (see § 3.1 in [GR70]), i. e.

$$\omega_X \subset \mathscr{K}_X.$$

If ω_X is locally free and X a Cohen-Macaulay space, then X is called *Gorenstein*. For these complex spaces, we get the following criterion for $\mathscr{K}_X$ being locally free (see Thm. 5.3 in [Rup14b]).

Theorem 2.17. *Let X be a Gorenstein space. If and only if X has (only) canonical singularities, the dualizing sheaf coincides with the Grauert-Riemenschneider canonical sheaf, i. e.*

$$\omega_X = \mathscr{K}_X$$

and $\mathscr{K}_X$ is locally free.

2.5 Monoidal transformations w. r. t. sheaves

As mentioned above, monoidal transformation are an especially useful tool in bimeromorphic complex analysis. In this section, we will introduce the monoidal transformations of a complex spaces with respect to a coherent analytic sheaf.

H. Rossi showed in [Ros68, Sect. 3] that coherent analytic sheaves can be made locally free by the use of modifications. This process has been treated more systematically by O. Riemenschneider in [Rie71]. Following [Rie71, § 2], we define:

Def. 2.18. Let X be a complex space and $\mathscr{S}$ a coherent analytic sheaf on X. Then, a pair $(X_{\mathscr{S}}, \varphi_{\mathscr{S}})$ of a complex space $X_{\mathscr{S}}$ and a proper modification $\varphi_{\mathscr{S}} \colon X_{\mathscr{S}} \to X$ is called the *monoidal transformation of X with respect to $\mathscr{S}$* if the following two conditions are fulfilled:

(1) the torsion-free preimage $\varphi_{\mathscr{S}}^T \mathscr{S} = \varphi_{\mathscr{S}}^* \mathscr{S} / \mathscr{T}(\varphi_{\mathscr{S}}^* \mathscr{S})$ is locally free on $X_{\mathscr{S}}$,

(2) if $\pi \colon Y \to X$ is any proper modification with (1), then there is a unique holomorphic mapping $\psi \colon Y \to X_{\mathscr{S}}$ such that $\pi = \varphi_{\mathscr{S}} \circ \psi$.

Thus, if $X_{\mathscr{S}}$ exists, it is uniquely determined up to biholomorphism by (2). But its existence was first proven by H. Rossi (see Thm. 3.5 in [Ros68]) and then studied further by O. Riemenschneider (see Thm. 2 in [Rie71]):

Theorem 2.19. *Let X be an (irreducible) complex space, $\mathscr{S}$ a coherent analytic sheaf on X and $A = \operatorname{Sing}\mathscr{S}$ the singular locus of $\mathscr{S}$. Then, there exists the monoidal transformation $(X_{\mathscr{S}}, \varphi_{\mathscr{S}})$ of X with respect to $\mathscr{S}$. $X_{\mathscr{S}}$ is a reduced (irreducible) complex space and $\varphi_{\mathscr{S}}$ is a projective proper modification such that*

$$\varphi_{\mathscr{S}} \colon X_{\mathscr{S}} \setminus \varphi_{\mathscr{S}}^{-1}(A) \to X \setminus A$$

is biholomorphic. If $U \subset X$ is an open subset, then $(\varphi_{\mathscr{S}}^{-1}(U), \varphi_{\mathscr{S}})$ is the monoidal transformation of U with respect to $\mathscr{S}_U$. $\varphi_{\mathscr{S}}$ is a projective morphism.

We also recall the following simple observation (see the Korollar in § 1.3 of [GR70]).

Lemma 2.20. *Let $\rho\colon Z \to Y$ and $\pi\colon Y \to X$ be two holomorphic mappings where Z and Y have the same dimension n such that the preimage of analytic sets of dimension $< n$ in Y under ρ have dimension $< n$ in X. Then, $\rho^T \pi^T \mathscr{F} = (\pi \circ \rho)^T \mathscr{F}$ for any coherent analytic sheaf $\mathscr{F}$ on X.*

2.6 Plurisubharmonic functions on complex spaces

Def. 2.21. Let X be a complex space. An upper semi-continuous function $\phi\colon X \to [-\infty, \infty)$ is called *plurisubharmonic* if there exists a holomorphic embedding $\iota\colon U \hookrightarrow D \subset \mathbb{C}^N$ for each small enough open set $U \subset X$ such that ϕ admits a plurisubharmonic extension $\widehat{\phi}$ on D, i. e. $\widehat{\phi} \circ \iota = \phi$. If there exits a smooth function $\widetilde{\phi}\colon D \to \mathbb{R}$ with $\widetilde{\phi} \circ \iota = \phi$, then ϕ is called *smooth*. Note that for a smooth plurisubharmonic function ϕ on X, the plurisubharmonic extension $\widehat{\phi}$ does not need to coincide with the smooth extension $\widetilde{\phi}$, i. e. in general, there does not need to exist a smooth plurisubharmonic extension of ϕ.

There exists also an intrinsic definition of plurisubharmonicity (so-called weak plurisubharmonicity), which coincides with the one from above (see Thm. 5.3.1 in [FN80]):

Theorem 2.22. *Let X be a complex space and ϕ an upper semi-continuous function on X. If $\phi \circ f$ is subharmonic for all analytic discs $f\colon \Delta \to X$, then ϕ is plurisubharmonic.*

Corollary 2.23. *An upper semi-continuous function ϕ on a complex space X is plurisubharmonic if it is plurisubharmonic on each irreducible component of X.*

Since a plurisubharmonic function can be extended plurisubharmonic over pluripolar sets where it is locally bounded (see e. g. Prop. 6 and Cor. 2 in [Vâj99]), we get:

Theorem 2.24. *Let X be a complex space and $\phi \in \mathscr{C}^0(X,\mathbb{R})$ be plurisubharmonic on X_{reg}. Then, ϕ is plurisubharmonic on the whole of X.*

Furthermore, we get for all continuous functions $\phi\colon X \to \mathbb{R}$ and for all proper modifications $\pi\colon M \to X$: ϕ is plurisubharmonic if and only if $\phi \circ \pi$ is plurisubharmonic.

Weakly 1-complete spaces. A complex space with a smooth plurisubharmonic exhaustion function is called *weakly 1-complete*. This property is stable under proper holomorphic maps. If the exhaustion is strictly plurisubharmonic, then X is Stein, i. e. X is holomorphically convex and globally defined holomorphic functions separate points (this is obviously not stable under modifications, not to mention under proper holomorphic maps).

Every holomorphically convex space X is weakly 1-complete: Using the Remmert reduction (see Theorem 2.11), we get a Stein space Y and a proper holomorphic map $\pi\colon X \to Y$ (with further properties). Then, Y admits a strictly plurisubharmonic exhaustion function Φ

(see [Nar62, Thm. II]). Hence, $\Phi \circ \pi$ is a plurisubharmonic exhaustion function of X.

The converse is not true. In [Gra63], H. Grauert constructed an example of a weakly 1-complete complex manifold / space which is not holomorphically convex.

For a ($\mathscr{C}^2$-) smooth function ϕ on a complex space X, let $H(\phi)_x$ denote the complex Hessian of ϕ at a point $x \in X_{\mathrm{reg}}$. We set

$$\sigma(\phi) := \max_{x \in X_{\mathrm{reg}}} (\operatorname{rk} H(\phi)_x). \tag{2.25}$$

$\sigma(\phi)$ is equal to the maximal number of positive eigenvalues of $H(\phi)$. For a smooth plurisubharmonic exhaustion function Φ, we obtain $\sigma(\Phi) > 0$ since an exhaustion function can not be pluriharmonic (contradiction to the Maximum Principle).

Chapter 3
Linear Spaces

Coherent analytic sheaves are an essential ingredient of complex analysis. Especially, to study singular complex spaces, it is crucial to understand coherent analytic sheaves, though sections of sheaves are quite abstract objects to study. A section of a coherent analytic sheaf can be written locally as a tuple of holomorphic functions (finiteness of the sheaf). However, it is difficult to figure out the relations between these tuples, which are essential in order to obtain the complete picture since coherent analytic sheaves are in general not locally free. From this point of view, the duality between coherent analytic sheaves and linear spaces is very interesting. As a fibre space, the linear space associated to a coherent analytic sheaf can sometimes be handled more easily: The duality allows to identify the sections of the sheaf with holomorphic mappings on a complex space.

Therefore, this chapter treats the study of linear spaces. After recalling the basic facts about fibre spaces in Section 3.1, we introduce the notion of linear spaces in the sense of G. Fischer (see [Fis66, Fis67]) in Section 3.2. Furthermore, we will define the primary component of a linear space and prove some basic facts in Section 3.3. Assumptions on the corank of a linear space / sheaf (minimal number of generators minus rank) will be helpful to prove deeper relations between coherent analytic sheaves and linear spaces (see Section 3.4 and Section 3.5). Section 3.3 and Section 3.5 are mainly based on Sect. 3 in [RS13].

Let us emphasize the following theorem, which can be seen as one of the main results of this chapter (cf. Thm. 1.3 in [RS13], proven here in Section 3.5).

Theorem 3.1. *Let X be a connected factorial Cohen-Macaulay space and $\mathscr{S}$ a coherent analytic sheaf on X, generated by $\operatorname{rk}\mathscr{S}+m$ sections, $m\leq 2$, such that the singular locus of $\mathscr{S}$ is at least of codimension $m+1$ in X. Then, the following is equivalent:*

(1) $\mathscr{S}$ is torsion-free.

(2) The linear space $L(\mathscr{S})$ associated to $\mathscr{S}$ is (globally) irreducible (i. e. it consists only of its primary component).

(3) $L(\mathscr{S})$ is locally irreducible.

(4) For all $p \in X$, there is a neighbourhood $U \subset X$ such that

$$0 \to \mathcal{O}_U^m \to \mathcal{O}_U^{\operatorname{rk}\mathscr{S}+m} \to \mathscr{S}_U \to 0$$

is exact, i. e. the homological dimension of $\mathscr{S}$ is at most one.

If (1 – 4) is fulfilled and

(5) if $\operatorname{codim}\operatorname{Sing}\mathscr{S} \geq m+2$, then $L(\mathscr{S})$ is normal.

For coherent analytic sheaves with homological dimension less or equal 1, (5) holds as well for $m > 2$ (see Corollary 3.20).

3.1 Fibre spaces

Let X be a complex space. A pair (Y, π) of a complex space Y and a holomorphic map $\pi\colon Y \to X$ is called a *complex space (or fibre space) over X*. A holomorphic map $\varphi\colon Y_1 \to Y_2$ between complex spaces $\pi_1\colon Y_1 \to X, \pi_2\colon Y_2 \to X$ over X is called a *holomorphic map over X* if $\pi_1 = \pi_2 \circ \varphi$, i. e. φ preserves fibres. The complex spaces over X with holomorphic maps over X as morphisms form a category, which we denote as C_X. The product in C_X is called the fibre product over X:

Def. 3.2. For two complex spaces $\pi_1\colon Y_1 \to X$ and $\pi_2\colon Y_2 \to X$ in C_X, we define the *fibre product of Y_1 and Y_2 over X* as the complex space $Y_1 \times_X Y_2 \in C_X$ together with holomorphic maps $\operatorname{pr}_i\colon Y_1 \times_X Y_2 \to Y_i$ over X, $i = 1, 2$, which satisfy the following universal property: for any complex space Z over X with holomorphic maps $\psi_i\colon Z \to Y_i$ over X, $i = 1, 2$, (in particular $\pi_1 \circ \psi_1 = \pi_2 \circ \psi_2$), there exists a holomorphic map $\psi\colon Z \to Y_1{\times_X}Y_2$ over X such that the following diagram commutes:

$$\begin{tikzcd} Z \arrow[dr, dashed, "\psi"] \arrow[drr, bend left, "\psi_1"] \arrow[ddr, bend right, "\psi_2"'] & & \\ & Y_1{\times_X}Y_2 \arrow[r, "\mathrm{pr}_1"'] \arrow[d, "\mathrm{pr}_2"] & Y_1 \arrow[d, "\pi_1"] \\ & Y_2 \arrow[r, "\pi_2"] & X\,, \end{tikzcd}$$

i. e. ψ is a holomorphic map over Y_1 and over Y_2.

It is well known that the fibre product exists (see e. g. Cor. 0.32 in [Fis76]). Actually, it is the closed subspace of the Cartesian product $Y_1{\times}Y_2$, given by the pullback under (π_1, π_2) of the ideal sheaf which defines the diagonal in $X{\times}X$.

The fibre product of two reduced complex spaces over X need not be reduced. Yet, under the assumption that the universal property is only satisfied for reduced spaces Z, one gets a reduced version, as well.

Let $f\colon Y \to X$ be a holomorphic map (i. e. Y is a complex space over X via f), and let Z be a complex space over X. Then, we define the *pullback of Z under f* by $f^*(Z) := Y{\times_X}Z$. With $\mathrm{pr}_Y\colon f^*(Z) \to Y$, we get that $f^*(Z)$ is a complex space over Y.

Complex subspaces (whether closed or open) of X are complex spaces over X via the embedding. If A is a closed subspace of X and $f\colon Y \to X$ a holomorphic map, then the (unreduced) *preimage of A* can be defined as $f^{-1}(A) := f^*(A) = A{\times_X}Y$. It is easy to see that $f^{-1}(A)$ is a closed subspace of Y. The (unreduced) *intersection* of two closed complex subspaces $A, B \subset X$ can be defined as $A{\cap}B := A{\times_X}B$, which is as well a closed subspace of X. Per definition, this coincides with the preimage of B under the embedding of A (and vice versa). If A and B are reduced, i. e. analytic sets, the intersection is reduced and coincides with the set-intersection.

3.2 Definition of linear spaces and preliminaries

Let X be a complex space. The category of complex vector bundles on X is equivalent to the category of locally free sheaves on X. Unfortunately, vector bundles do not behave well under direct images and, correspondingly, the direct image of a locally free sheaf is not locally free any more. Yet, H. Grauert's direct image theorem says that it is still a coherent analytic sheaf. The equivalence between vector bundles and locally free sheaves can be generalized in the following sense: The category of coherent analytic sheaves is dual to the category of linear spaces in the sense of A. Grothendieck and G. Fischer, which is a subcategory of C_X. In this section, we will define linear spaces, sketch the construction of the mentioned duality and recall some basic properties. For the proofs and more crucial properties, we recommend [Gro61, Fis66, Fis67, Fis76, PR94].

Def. 3.3. Let X be a complex space. A complex space $L \in C_X$ with a holomorphic map $\lambda\colon L \to X$ is called *linear space over X* if there exist holomorphic maps

$$+\colon L \times_X L \to L \text{ (addition)},$$
$$\boldsymbol{\cdot}\colon \mathbb{C} \times L \to L \text{ (scalar multiplication) and}$$
$$0\colon X \to L \text{ (zero section)}$$

with the following properties:

(i) $+$ and $\boldsymbol{\cdot}$ commute with λ (e. g. $\lambda \circ + = + \circ (\lambda, \lambda)$) and $\lambda \circ 0 = \mathrm{id}$. Particularly, $+$, $\boldsymbol{\cdot}$ and 0 can be understood as holomorphic maps over X.

(ii) They satisfy the usual vector space / module axioms given by commutative diagrams, e. g.

$$\boldsymbol{\cdot} \circ (\mathrm{id}_{\mathbb{C}}, +) = + \circ (\boldsymbol{\cdot}, \boldsymbol{\cdot}) \circ (\mathrm{pr}_{\mathbb{C}}, \mathrm{pr}_1; \mathrm{pr}_{\mathbb{C}}, \mathrm{pr}_2)$$

(distributive property) where $\mathrm{pr}_{\mathbb{C}}\colon \mathbb{C} \times L \times_X L \to \mathbb{C}$ denotes the projection on $\mathbb{C}$ and pr_i the projections on the factors of $L \times_X L$.

λ is called the *projection of L to X*.

This means there is a linear structure on L: Each fibre $L_p := \lambda^{-1}(p)$ is a $\mathbb{C}$-vector space of a dimension which depends on $p \in X$. We call $\operatorname{rk}_p L := \dim L_p$ the *rank of L in p* and $\operatorname{rk} L := \min_{p\in X} \operatorname{rk}_p L$ the *rank of L*. The rank $\operatorname{rk}_p L$ is an upper semi-continuous function on X and the set $\{p \in X : \operatorname{rk}_p L \geq r\}$ is analytic in X for all $r \in \mathbb{N}_0$. We call

$$\operatorname{Sing} L := \{p \in X : \operatorname{rk}_p L \geq \operatorname{rk} L\}$$

the *singular locus of L*. Please distinguish it from the singular set (denoted as L_{sing}) of L as complex space. If X is reduced and $\operatorname{rk}_p L$ is constant in p, then L is a vector bundle over X (see Satz 3 in [Fis66]). In particular, L is a vector bundle over $X \setminus \operatorname{Sing} L$.

For an analytic or open subset M of X the *restriction of L to M* is defined as $L_M := \lambda^{-1}(M)$. If $\iota\colon M \to X$ is a holomorphic (closed or open, respectively) embedding of M in X, we get that $L_M = \iota^* L = M \times_X L$.

A *(homo-) morphism ξ between linear spaces* $L_1 \to L_2$ is a holomorphic map $\xi\colon L_1 \to L_2$ which commutes with the addition $+$ and the scalar multiplication $\cdot$. We denote the set of homomorphism $\operatorname{Hom}(L_1, L_2)$. In particular, the restriction $\xi_p\colon L_{1,p} \to L_{2,p}$ of ξ is a homomorphism of vector spaces for all $p \in X$.

For a vector space V of dimension r, $X \times V$ with the projection on X is the trivial linear space of rank r. Locally all linear spaces can be embedded in $U \times \mathbb{C}^N$ for N big and open $U \subset X$ small enough. For all Stein U, N can be chosen less or equal $\dim U + \operatorname{rk} L_U - 1$ if $\operatorname{rk}_p L < \infty$ for all $p \in X$ (see Satz 4 in [Fis67, Sect. 3]). Moreover, we get (see Lem. 1 and 2 in [Fis67]):

Theorem 3.4. *Let X be a complex space, and let L be a linear space over X. For all $p_0 \in X$, there exists a neighbourhood $U \subset X$ such that L_U can be embedded in $U \times \mathbb{C}^N_z$ with $N = \operatorname{rk}_{p_0} L$ and the embedding is the analytic subspace given by holomorphic functions $h_1, \ldots, h_m \in$*

$\mathcal{O}(U\times\mathbb{C}^N)$ over U which are fibrewise linear, i. e. $(p,z)\mapsto(p,h_i(p,z)) \in$ $\mathrm{Hom}(U\times\mathbb{C}^N, U\times\mathbb{C})$.

There is a duality between the category of linear spaces and coherent analytic sheaves on X (see Satz 2 in [Fis67]). Let us sketch the construction of the functors:

For a linear space $L \to X$, the presheaf $\mathscr{L} = \mathscr{L}(L)$ given by $\mathscr{L}(U) :=$ $\mathrm{Hom}(L_U,\ U\times\mathbb{C})$ is canonical, i. e. it defines a sheaf. For an open set $U \subset X$, a holomorphic $r \in \mathcal{O}_X(U)$ and a section $s \in \mathscr{L}(U) =$ $\mathrm{Hom}(L_U, U\times\mathbb{C})$, the scalar multiplication $\cdot$ induces a section $r \cdot s \in$ $\mathrm{Hom}(L_U, U\times\mathbb{C})$. This means $\mathscr{L}(L)$ is an analytic sheaf.
A homomorphism $\xi\colon L_1 \to L_2$ between linear spaces induces a homomorphism $\xi^* := \mathscr{L}(\xi)\colon \mathscr{L}(L_2) \to \mathscr{L}(L_1)$ between the associated coherent analytic sheaves defined by

$$\xi_U^*(s) := s \circ \xi|_U \in \mathrm{Hom}(L_{1,U}, U\times\mathbb{C}) = \mathscr{L}(L_1)(U)$$

for all open $U \subset X$ and $s \in \mathscr{L}(L_2)(U) = \mathrm{Hom}(L_{2,U}, U\times\mathbb{C})$ (using $\xi|_U \in \mathrm{Hom}(L_{1,U}, L_{2,U})$).
Theorem 3.4 implies the coherence of $\mathscr{L}$: For small enough $U \subset X$, L_U can be embedded in $U\times\mathbb{C}^N$ such that $L_U = \{h_1{=}...{=}h_m{=}0\}$ (in the unreduced sense) for holomorphic functions h_i in $\mathcal{O}(U\times\mathbb{C}^N)$ which are fibrewise linear. This means $L_U = \ker\alpha$ where

$$\alpha(p,z) := (p; h_1(p,z), .., h_m(p,z)) \in \mathrm{Hom}(U\times\mathbb{C}^N, U\times\mathbb{C}^m),$$

where p denotes the coordinates of U and z of $\mathbb{C}^N$, i. e.

$$0 \to L_U \hookrightarrow U\times\mathbb{C}^N \xrightarrow{\alpha} U\times\mathbb{C}^m$$

is an exact sequence. We get the exact sequence

$$\mathcal{O}_U^m \overset{\alpha^*}{\to} \mathcal{O}_U^N \to \mathscr{L}(L)_U \to 0.$$

Vice versa, the linear space associated to a coherent analytic sheaf is constructed as follows:

Let $\mathscr{S}$ be a coherent analytic sheaf on X. For a small enough open set $U \subset X$, the coherence of $\mathscr{S}$ gives us an exact sequence $\mathcal{O}_U^s \overset{\alpha}{\to}$

$\mathcal{O}_U^t \to \mathscr{S}_U \to 0$. The holomorphic morphism α can be interpreted as matrix with holomorphic entries. Hence, the transposed of α induces a homomorphism $\alpha^*\colon U\times\mathbb{C}^t \to U\times\mathbb{C}^s \in \mathrm{Hom}(U\times\mathbb{C}^t, U\times\mathbb{C}^s)$. Let S_U be the kernel of α^*, i. e.

$$S_U := \{(p,z) \in U\times\mathbb{C}^t \colon \alpha^*(p,z) = (p,0)\}.$$

The linear space $S := L(\mathscr{S})$ associated to $\mathscr{S}$ can be obtained by patching the constructed S_U from above. For the construction, the coherence of $\mathscr{S}$ is obviously necessary.

For a sheaf morphism $\xi\colon \mathscr{S}_1 \to \mathscr{S}_2$, $L(\xi)$ can be constructed as follows: Let $\mathcal{O}_U^{s_i} \overset{\alpha_i}{\to} \mathcal{O}_U^{t_i} \overset{\beta_i}{\to} \mathscr{S}_{i,U} \to 0$ be exact sequences for a small enough $U \subset X$. Per construction of the linear space, $L(\mathscr{S}_i) = \ker \alpha_i^*$. There exist an extension $\hat{\xi}\colon \mathcal{O}_U^{t_1} \to \mathcal{O}_U^{t_2}$ of ξ, i. e. $\xi \circ \beta_1 = \beta_2 \circ \hat{\xi}$, and a morphism $\tilde{\xi}\colon \mathcal{O}_U^{s_1} \to \mathcal{O}_U^{s_2}$ such that $\hat{\xi} \circ \alpha_1 = \alpha_2 \circ \tilde{\xi}$ (see e. g. Lem. 20.3 in [Eis95]), i. e. the following exact diagram commutes:

$$\begin{tikzcd}
\mathcal{O}_U^{s_1} \arrow[r, "\alpha_1"] \arrow[d, "\tilde{\xi}"] & \mathcal{O}_U^{t_1} \arrow[r, "\beta_1"] \arrow[d, "\hat{\xi}"] & \mathscr{S}_{1,U} \arrow[r] \arrow[d, "\xi"] & 0 \\
\mathcal{O}_U^{s_2} \arrow[r, "\alpha_2"] & \mathcal{O}_U^{t_2} \arrow[r, "\beta_2"] & \mathscr{S}_{2,U} \arrow[r] & 0 .
\end{tikzcd}$$

As homomorphisms, $\hat{\xi}$ and $\tilde{\xi}$ can be interpreted as matrices with holomorphic entries (like α_i, see above), i. e. the transposition gives homomorphisms

$$(\hat{\xi})^* \in \mathrm{Hom}(U\times\mathbb{C}^{t_2}, U\times\mathbb{C}^{t_1}) \quad \text{and} \quad (\tilde{\xi})^* \in \mathrm{Hom}(U\times\mathbb{C}^{s_2}, U\times\mathbb{C}^{s_1}).$$

Let $L(\xi)$ be the restriction of $(\hat{\xi})^*$ to $L(\mathscr{S}_2) \subset U\times\mathbb{C}^{t_2}$. Since $L(\mathscr{S}_2) = \ker \alpha_2^*$, we get

$$\alpha_1^* \circ (\hat{\xi})^* \, (L(\mathscr{S}_2)) = (\tilde{\xi})^* \circ \alpha_2^*(L(\mathscr{S}_2)) = 0.$$

Hence, $\mathrm{Im}\,(\hat{\xi})^*|_{L(\mathscr{S}_2)} \subset \ker \alpha_1^* = L(\mathscr{S}_1)$ (in the unreduced sense) and $L(\xi) = (\hat{\xi})^* \in \mathrm{Hom}(L(\mathscr{S}_2), L(\mathscr{S}_1))$ is well-defined.

Let $\xi\colon \mathscr{S}_1 \to \mathscr{S}_2$ be a morphism between sheaves. Since the functor L gives a duality between categories, $L(\xi)$ is an epimorphism with respect to the category of linear spaces if and only if ξ is a monomorphism. Here, one should be careful because an epimorphism of linear spaces must not be surjective as holomorphic map: For a counterexample, we consider the injective sheaf morphism $\iota\colon \mathcal{O}_{\mathbb{C}} \hookrightarrow \mathcal{O}_{\mathbb{C}}$ defined by $\iota(r) = p{\cdot}r$ on the $\mathbb{C}_p$. Then, $\iota^*{=}L(\iota)\colon \mathbb{C}^2 \to \mathbb{C}^2$ is given by $\iota^*(p,z) = (p, p\cdot z)$ and an epimorphism with respect to the category of linear spaces. Yet, ι^* is not surjective since $\iota^*(0,z) = (0,0)$.

On the other hand, for a surjective $\xi\colon \mathscr{S}_1 \to \mathscr{S}_2$ (i. e. an epimorphism), one can show that $L(\xi)\colon L(\mathscr{S}_2) \to L(\mathscr{S}_1)$ is a closed immersion, i. e. $L(\mathscr{S}_2)$ is (can be seen as) a closed complex subspace of $L(\mathscr{S}_1)$. Actually, this follows from the construction of $L(\xi)$ from above.

If $\mathscr{E}$ is a locally free sheaf, then $L(\mathscr{E})$ is a vector bundle. Please do not mix $L(\mathscr{E})$ up with the vector bundle $R(\mathscr{E})$ which has $\mathscr{E}$ as sheaf of sections, i. e. $\mathcal{O}(R(\mathscr{E})) \cong \mathscr{E}$. Actually, $R(\mathscr{E})$ is dual to $L(\mathscr{E})$. For an arbitrary coherent analytic sheaf $\mathscr{S}$ on X, $\mathscr{S}$ is locally free on $X' := X \setminus \operatorname{Sing}\mathscr{S}$. Let us define $R(\mathscr{S})$ as the vector bundle over X' which has $\mathscr{S}_{X'}$ as sheaf of sections, i. e. $R(\mathscr{S}) := R(\mathscr{S}_{X'}) = L(\mathscr{S}_{X'})^*$.

Let $f\colon Y \to X$ be a holomorphic map between complex spaces, and let $\lambda\colon L \to X$ be a linear space over X. Then, the *pullback f^*L of L under f* is defined as the fibre product of Y and L over X:

$$f^*L := Y \times_X L.$$

We obtain $f^*(L(\mathscr{S})) = L(f^*\mathscr{S})$. The pullback (as functor on linear spaces) is covariant and left-exact. If $L \subset U_p{\times}\mathbb{C}^N_z$ is given by holomorphic fibrewise linear $h_1, ..., h_m$, then f^*L is given by $f^*h_1, ..., f^*h_m$ with $f^*h_i(p,z) := h_i(f(p),z)$. We obtain:

Lemma 3.5. *Let $L \subset U{\times}\mathbb{C}^N$ be a linear space over a complex space U, and let $f\colon V \to U$ be a holomorphic map. Then, the pullback $f^*(L) = V{\times}_U L$ can be embedded in $V{\times}\mathbb{C}^N$.*

Linear spaces in sense of Grauert. Let X be a reduced complex space. There exists the (sub-) category C_X^{red} of reduced complex spaces over X. Let $\oplus_X$ denote the (fibre) product in C_X^{red}, then it coincides with the set-theoretic fibre product. In [Gra62, § 3.6], H. Grauert defined linear spaces with respect to C_X^{red}: In contrast to G. Fischer's notion of a linear space, where it is required that $+\colon L\times_X L\to L$ is a holomorphic map, H. Grauert requires that the addition $+\colon L\oplus_X L\to L$ is holomorphic between reduced complex spaces. That gives a different category of linear spaces (which is no longer dually equivalent to the category of coherent analytic sheaves).

3.3 Primary component of a linear space

In the following, we will always assume that X is a locally irreducible complex space. Thus, X decomposes into disjoint connected components, which can be considered separately. So, we can assume that X is connected, thus also globally irreducible. For a coherent analytic sheaf $\mathscr{S}$, let $A := \operatorname{Sing}\mathscr{S} = \operatorname{Sing} S$ be the singular locus of $\mathscr{S}$ and $S = L(\mathscr{S})$, respectively, which is a thin analytic set. As X is irreducible, $X' := X\setminus(A\cup X_{\mathrm{sing}})$ and $A^c := X\setminus A$ are connected. $S_U \cong U\times\mathbb{C}^r$, for small open sets $U\subset A^c$, implies that S_{A^c} is also connected. The set S_A is an analytic subset of S. Let E be the irreducible component of $\operatorname{red}(S)$ which contains S_{A^c}. $\operatorname{PC}(S) := E$ will be called the ***primary component*** *of* S (following the notation of J. Rabinowitz in [Rab78]). We get the decomposition $S = E\cup S_A$.

Remark 3.6. Let S be the linear space associated to a coherent analytic sheaf $\mathscr{S}$. Let $s\in\operatorname{Hom}(S_U, U\times\mathbb{C})\cong\mathscr{S}(U)$ be a section. Then, the primary component $E=\operatorname{PC}(S)$ of S determines s up to torsion:

$$\text{If } s|_E = 0, \text{ then } s\in\mathscr{T}(\mathscr{S}).$$

This is clear as $s|_E = 0$ implies that s is supported only on an analytically thin set.

Lemma 3.7. *Let X be a locally irreducible complex space and $\mathscr{S}$ a coherent analytic sheaf on X. Let $S = L(\mathscr{S})$ be the linear space associated to $\mathscr{S}$ and E its primary component. If $\mathscr{S}$ has a torsion element, then $E\neq S$. In particular, S is reducible.*

Proof: Assume that $\mathscr{S}$ has a torsion element, i. e. there are an open set $U \subset X$, an $s \in \mathscr{S}(U) \cong \mathrm{Hom}(S_U, U \times \mathbb{C})$ (see Section 3.2) and an $r \in \mathcal{O}_X(U)$ such that $s, r \neq 0$ but $r \bullet s = r \cdot s = 0$ on S_U. As X is locally irreducible, we can assume that U is irreducible. So, there is a dense open set $V \subset U$ such that $r \in \mathcal{O}_X^*(V)$. Thus $s|_{S_V} = 0$. But $V \cap A^c$ is also open and dense in U. So, $s|_E = 0$ by the identity theorem as E is irreducible. Since $s \neq 0$, S_U has to contain (parts of) other irreducible components than E. □

Remark 3.8. The converse of Lemma 3.7 is not true: Let $\mathscr{J}$ be the ideal sheaf generated by x^2, xy^2, y^4 on $\mathbb{C}^2_{x,y}$ and $S := L(\mathscr{J})$ the linear space associated to $\mathscr{J}$. Since $\mathscr{J}$ can not be generated by 2 elements, we get $\mathrm{rk}\, S_0 = \mathrm{rk}\, \mathscr{J}_0 = 3$. Hence, S_0 is a 3-dimensional analytic subset of S. On the other hand, the primary component has dimension $2 + \mathrm{rk}\, S = 3$. Hence, S is not irreducible. Furthermore, one can compute that S is even not reduced: S is given in $\mathbb{C}^2_{x,y} \times \mathbb{C}^3_z$ by the ideal sheaf $\mathscr{J}_S$ generated by $h_1(x, y; z) := y^2 z_1 - x z_2$ and $h_2(x, y; z) := y^2 z_2 - x z_3$ where $z = (z_1, z_2, z_3)$. Since $y^2(z_2^2 - z_1 z_3) = z_2 h_2 - z_3 h_1$, we get $(y(z_2^2 - z_1 z_3))^2 \in \mathscr{J}_S$. Yet, we have $y(z_2^2 - z_1 z_3) \notin \mathscr{J}_S$.

The primary component $\mathrm{PC}(S)$ is defined by the functions h_1, h_2 and $z_2^2 - z_1 z_3$. Then, the fibre over the origin is just $\{z_2^2 = z_1 z_3\}$ and not a vector space. So, the fibres of the primary component do not need to be linear and the primary component is in general not a linear space (in the sense of Fischer). In [Rab78, p. 238], J. Rabinowitz claims that the primary component of a linear space is a linear space in the sense of Grauert, but not in the sense of Fischer. Our example shows that even this is not the case.

At least, we obtain that the smallest linear space containing the primary component induces the torsion-free part of a sheaf. More precisely:

Lemma 3.9. *For a coherent analytic sheaf $\mathscr{S}$ on a complex space X, the linear space associated to the torsion-free sheaf $\mathscr{S}/\mathscr{T}(\mathscr{S})$ can be* *realized locally as the unreduced intersection over all linear spaces L containing the primary component of $L(\mathscr{S})$ as analytic set, i. e.*

$$L(\mathscr{S}/\mathscr{T}(\mathscr{S})) = \bigcap_{L \supset \mathrm{PC}(L(\mathscr{S}))} L\,.$$

Proof: Let X' be the complement of $\operatorname{Sing}\mathscr{S}$ in X, let S be the linear space associated to $\mathscr{S}$. We define $\mathscr{T} := \mathscr{T}(\mathscr{S})$ and $F := L(\mathscr{S}/\mathscr{T})$. We consider a small open set so that S_U is embedded in $U\times\mathbb{C}^N$. In $U\times\mathbb{C}^N$, let $L_{\min}$ be the unreduced intersection $\bigcap_{L\supset \mathrm{PC}(S_U)} L$, where the intersection is over all linear spaces L over U embedded in $U\times\mathbb{C}^N$ that contain $\mathrm{PC}(S_U)$. Theorem 3.4 implies that $L_{\min}$ is a linear space over U. Since $F_{X'} = S_{X'}$, we get $\mathrm{PC}(S) = \mathrm{PC}(F) \subset F$, i. e. $L_{\min} \subset F_U$ in the unreduced sense. In order to produce a contradiction, assume that $L \neq F$. Since the ideal sheaf associated to $L_{\min}$ is generated by fibrewise linear holomorphic function on $U\times\mathbb{C}^N$ with $L_{\min} \subset F_U \subset U\times\mathbb{C}^N$ (see Theorem 3.4, shrinking U), there exists a holomorphic map $s \in \mathrm{Hom}(U\times\mathbb{C}^N, U\times\mathbb{C})$ vanishing on $L_{\min}$ but not on F_U (in the unreduced sense). Since s vanishes on $\mathrm{PC}(S_U) = \mathrm{PC}(F_U)$, we get that s is a torsion element of $\mathscr{L}(F) = \mathscr{S}/\mathscr{T}$. This is a contradiction. □

Using Rossi's monoidal transformation, we can make the following observation about the primary component:

Theorem 3.10. *Let X be a locally irreducible complex space and $\mathscr{S}$ a coherent analytic sheaf on X. Then, the primary component E of the linear space S associated to $\mathscr{S}$ is locally irreducible.*

Proof: As above, we can assume that X is connected, i. e. irreducible. Let

$$\varphi := \varphi_{\mathscr{S}} \colon X_{\mathscr{S}} \to X$$

be the monoidal transformation of X with respect to $\mathscr{S}$. This implies that φ is biholomorphic on $X_{\mathscr{S}} \setminus \varphi^{-1}(A)$ with $A := \operatorname{Sing}\mathscr{S}$. Then,

$$\varphi^* S = X_{\mathscr{S}} \times_X S$$

is the linear space associated to $\varphi^*\mathscr{S}$, and there is a proper holomorphic projection

$$\mathrm{pr}\colon \varphi^* S \to S.$$

Now consider the natural surjective homomorphism

$$\varphi^*\mathscr{S} \longrightarrow \varphi^T\mathscr{S} = \varphi^*\mathscr{S}/\mathscr{T}(\varphi^*\mathscr{S}),$$

which induces a closed embedding of the linear space $V := L(\varphi^T\mathscr{S})$ into $\varphi^* S$. Note that V coincides with $\varphi^* S$ on $X_{\mathscr{S}} \setminus \varphi^{-1}(A)$. Thus, the vector bundle V is just the primary component of $\varphi^* S$, and V is locally and globally irreducible since the base space $X_{\mathscr{S}}$ is connected and locally irreducible.

As pr is a proper holomorphic mapping, we have that $\mathrm{pr}(V)$ is an irreducible analytic subset of S by Remmert's proper mapping theorem (see Theorem 2.12) and the fact that holomorphic images of irreducible sets are again irreducible (see § 1.3 in [GR84, Chap. 9]). But $\mathrm{pr}(V)$ coincides with the primary component E of S over $X \setminus A$. Thus:

$$\mathrm{pr}(V) = E, \tag{3.11}$$

and so $\mathrm{pr}|_V\colon V \to E$ is a proper modification. Using this and the fact that V is clearly locally irreducible, it is easy to see that E is also locally irreducible: For an open connected set $W \subset E$, $\mathrm{pr}|_V^{-1}(W) \subset V$ is again open and connected, thus irreducible since φ is a proper modification of the irreducible X. But then, $W = \mathrm{pr}|_V\big(\mathrm{pr}|_V^{-1}(W)\big)$ is also irreducible by the same argument as above (holomorphic images of irreducible sets are irreducible). □

Using Theorem 3.10, we can now show:

Lemma 3.12. *Let X be a locally irreducible complex space and $\mathscr{S}$ a torsion-free coherent analytic sheaf on X. Then, $S = L(\mathscr{S})$ is locally irreducible if and only if the primary component of S is a linear space.*

Proof: Let $E \subset S$ denote the primary component of S. Theorem 3.10 implies that E is locally irreducible.

Assume first that E is a linear space. For all points in X, there is a neighbourhood $U \subset X$ such that E_U and S_U are linear spaces in $U \times \mathbb{C}^N$. Lem. 1 in [Fis67] implies that E is defined by holomorphic functions $h_1, ..., h_m \in \mathcal{O}(U \times \mathbb{C}^N)$ that are fibrewise linear. The restriction of h_j to S gives a section in $\mathrm{Hom}(S_U, U \times \mathbb{C}) \cong \mathscr{S}(U)$. Since h_j vanishes on the primary component E of S, we get $h_j \in \mathscr{T}(\mathscr{S}) = 0$ (see Remark 3.6), i. e. $S \subset E$. This shows that actually $E = S$. The converse of the statement is trivial. $\square$

As we have seen in the counterexample Remark 3.8, the primary component does not need to be a linear space. Yet, it has a homogeneous structure, i. e. it is a cone (fibre) space in the sense of G. Fischer [Fis76, Sect. 1.2]:

Lemma 3.13. *Let X be a (locally irreducible) complex space and $\mathscr{S}$ a coherent analytic sheaf on X. Then, the primary component E of the linear space $S = L(\mathscr{S})$ associated to $\mathscr{S}$ is fibrewise homogeneous and E is locally defined as analytic set in $U \times \mathbb{C}^N$ by holomorphic fibrewise homogeneous functions for $U \subset X$ small enough.*

Proof: As above, let $A \subset X$ be the singular locus of $\mathscr{S}$. So, E and S coincide over $X \setminus A$, and we only have to show that the fibres of E are homogeneous over points of A.

The question is local, so consider a point $p \in A$ and a Stein neighbourhood U of p in X such that S_U can be realized as a closed linear subspace of $U \times \mathbb{C}^N$. Now $E_U \subset U \times \mathbb{C}^N$ is a closed component of S_U, which is linear in the second component over $U \setminus A$. Let $f_1, ..., f_k$ be a set of defining functions for E_U in $U \times \mathbb{C}^N$ (U is chosen to be Stein). For f_j, $j = 1, ..., k$, we define

$$f_j^{\bullet}(\lambda, x, z) := f_j(x, \lambda \cdot z)$$

on $\mathbb{C} \times U \times \mathbb{C}^N$. Since $E_{U \setminus A}$ is linear, the $f_j^{\bullet}$ vanish on $(\mathbb{C} \times E)_{U \setminus A}$ and on its closure. By definition, the closure is the irreducible set $\mathbb{C} \times E_U$. Hence, the fibres of E_U are homogeneous. Therefore, $\bullet \colon \mathbb{C} \times E_U \to E_U$ given by the restriction of $\bullet \colon \mathbb{C} \times U \times \mathbb{C}^N \to U \times \mathbb{C}^N$ is a holomorphic map. Using this, we obtain that the ideal sheaf defining E_U (as analytic set) is generated by fibrewise homogeneous functions (see the first step in the proof of Lem. 1 in [Fis67]). $\square$

If $E \times_X E$ is irreducible, one can prove in the same way that the primary component E is a linear space. Yet, for an irreducible fibre space $E \to X$, the fibre product of $E \times_X E$ does not need to be reduced (not to mention irreducible; for a counterexample, see Sect. 4 in [Fis66]). Therefore, the restriction of the addition does not need to be holomorphic.

More preliminaries on torsion.

We will use the following observation without referring to it explicitly, again. Let $\psi\colon \mathscr{F} \to \mathscr{G}$ be a morphism of analytic sheaves on a (locally irreducible) complex space $(X, \mathcal{O}_X)$. Then, ψ induces a canonical map

$$\widehat{\psi}\colon \mathscr{F}/\mathscr{T}(\mathscr{F}) \to \mathscr{G}/\mathscr{T}(\mathscr{G})$$

because the torsion sheaf $\mathscr{T}(\mathscr{F})$ of $\mathscr{F}$ is mapped by ψ into the torsion sheaf $\mathscr{T}(\mathscr{G})$ of $\mathscr{G}$: $r_x\psi(s_x) = \psi(r_x s_x) = 0$ for germs $r_x \in \mathcal{O}_{X,x}$, $s_x \in \mathscr{T}_x(\mathscr{F})$ with $r_x s_x = 0$. Note that particularly $\mathscr{T}(\mathscr{F}) \subset \ker\psi$ if $\mathscr{G}$ is torsion-free. Additionally, if ψ is an epimorphism, then $\widehat{\psi}$ is as well.

Lemma 3.14. *Let X be a locally irreducible complex space and let $\mathscr{F}$ and $\mathscr{G}$ be coherent analytic sheaves on X such that there exists a morphism $\psi\colon \mathscr{F} \to \mathscr{G}$ which is a monomorphism on an open dense subset of X. If $\mathscr{F}$ is torsion-free, then ψ is a monomorphism. If not, ψ induces a monomorphism $\widehat{\psi}\colon \mathscr{F}/\mathscr{T}(\mathscr{F}) \hookrightarrow \mathscr{G}/\mathscr{T}(\mathscr{G})$.*

Proof: The second statement follows from the considerations above and the torsion-free case. Hence, we can assume that $\mathscr{F}$ is torsion-free. Let F and G denote the linear spaces associated to $\mathscr{F}$ and $\mathscr{G}$, respectively. Theorem 3.10 implies that $\mathrm{PC}(F)$ and $\mathrm{PC}(G)$ are locally irreducible. Let ψ be a monomorphism on the open dense subset W of X with $W \subset X \backslash (\operatorname{Sing}\mathscr{F} \cap \operatorname{Sing}\mathscr{G})$. Thus, ψ induces a holomorphic fibrewise linear map $\psi^*\colon G \to F$ such that $\psi^*_W\colon G_W \to F_W$ is a surjective map of vector bundles. Let s be a section in $\mathscr{K}\!\mathit{er}\,\psi$, i. e. $\psi^* \circ s$ vanishes on G. We get that s vanishes on F_W and, hence, on $\mathrm{PC}(F)$. Since $\mathscr{F}$ is torsion free, we obtain $s = 0$ (cf. Remark 3.6). □

Alternatively, one can prove this lemma by using only sheaf-theoretical terminology and arguments (cf. the proof of Lemma 4.4).

Remark 3.15. Let $\mathscr{S}$ be a coherent analytic sheaf over a locally irreducible complex space X. Let $\varphi = \varphi_{\mathscr{S}}\colon X_{\mathscr{S}} \to X$ be the monoidal transformation of X with respect to $\mathscr{S}$, i. e. $\mathscr{E} := \varphi^T\mathscr{S}$ is locally free. Note that $X_{\mathscr{S}}$ is again locally irreducible. Then, $\varphi^*\mathscr{S}$ has torsion in a point q if and only if $\mathscr{S}$ is not locally free in $\varphi(q)$ (see [Rab79]). We will give a short, alternative proof with the statements from above. Let S, S^* and E denote the linear complex spaces associated to $\mathscr{S}$, $\varphi^*\mathscr{S}$ and $\mathscr{E}$, respectively. If $\mathscr{S}$ is not locally free in $\varphi(q)$, then $\dim E_q < \dim S_{\varphi(q)}$ (as $\dim E_q = \mathrm{rk}\, E = \dim S_{\widetilde{q}}$ in all points $\widetilde{q}$ where $\mathscr{S}$ is locally free, see § 1.1 in [GR70]). Lemma 3.5 implies $\dim S^*_q = \dim S_{\varphi(q)} > \dim E_q$. Since $\mathrm{PC}(S^*) = E$ and E is a vector bundle, we obtain that S^* is reducible in $(q, 0)$, i. e. $\varphi^*\mathscr{S}$ has torsion in q by Lemma 3.12. The other implication of the claim is trivial.

3.4 Corank and Cohen-Macaulay linear spaces

In this section, we introduce the notion of corank of linear spaces and coherent analytic sheaves. It will be helpful to characterize linear spaces and to give a criterion whether the linear space associated to a coherent analytic sheaf is normal.

Def. 3.16. Let X be a complex space and S a linear space over X. For a point $p \in X$, we define the *corank of S in p* as the number

$$\mathrm{cork}_p\, S := \mathrm{rk}_p\, S - \mathrm{rk}\, S$$

and $\operatorname{cork} S := \sup_{p\in X} \operatorname{cork}_p S$. For a coherent analytic sheaf $\mathscr{S}$ on X, we define the *corank of* $\mathscr{S}$ *in a point* $p \in X$, $\operatorname{cork}_p \mathscr{S}$, as the difference of the minimal number of generators of $\mathscr{S}_p$ and the rank of $\mathscr{S}$, and the global corank as $\operatorname{cork} \mathscr{S} := \sup_{p\in X} \operatorname{cork}_p \mathscr{S}$. The corank of a linear space coincides with the corank of the associated coherent analytic sheaf (using $S \overset{\text{loc}}{\subset} X \times \mathbb{C}^{\operatorname{rk} S}$, see Theorem 3.4) and is upper semi-continuous.

Lemma 3.17. *Let X be a complex space and $\mathscr{S}$ a coherent analytic sheaf on X. Then, for all $p \in X$, the following is equivalent:*

(1) There exists a neighbourhood U of p such that the following sequence is exact:

$$\mathcal{O}_U^{\operatorname{cork}_p \mathscr{S}} \xrightarrow{\alpha} \mathcal{O}_U^{\operatorname{rk}\mathscr{S} + \operatorname{cork}_p \mathscr{S}} \to \mathscr{S}_U \to 0.$$

(2) The homological dimension of $\mathscr{S}$ in p is less or equal 1, i. e. (per definition) there exists a neighbourhood U of p such that

$$0 \to \mathcal{O}_U^m \overset{\alpha}{\to} \mathcal{O}_U^N \to \mathscr{S}_U \to 0$$

is exact for suitable m and N.

Proof: For the implication (1) $\Rightarrow$ (2), we just need to show that α is injective: In points where $\mathscr{S}$ is locally free, α is injective (due to the rank / dimension). Hence, $\mathscr{K\!e\!r}\,\alpha$ has support on a proper analytic set in U, i. e. is a torsion sheaf or the zero sheaf. Since $\mathcal{O}_U$ does not contain any torsion sheaf, α is a monomorphism. (Alternatively, one can apply Lemma 3.14.)

(2) $\Rightarrow$ (1): By the uniqueness of the minimal resolution (see e. g. Thm. 20.2 in [Eis95]), we can assume that N is equal to the minimal number of generators of $\mathscr{S}$ in p, i. e.

$$N = \operatorname{rk}_p \mathscr{S} \overset{\text{def}}{=} \operatorname{cork}_p \mathscr{S} + \operatorname{rk} \mathscr{S}.$$

The injectivity of α implies $N - m = \operatorname{rk}(\mathcal{O}^N/\alpha(\mathcal{O}^m)) = \operatorname{rk}\mathscr{S}$, i. e.

$$m = N - \operatorname{rk}\mathscr{S} = \operatorname{cork}_p \mathscr{S}.$$

□

Lemma 3.17 can be interpreted as follows: If and only if the homological dimension of a coherent analytic sheaf is less or equal 1, then the minimal number of holomorphic fibrewise linear functions defining the associated linear space is equal to the codimension of the linear space (locally) embedded in the trivial vector bundle, i. e. the linear space is locally some kind of complete intersection. Actually, estimates on the corank of a linear space will be quite useful to show that a linear space is a Cohen-Macaulay space (which is close to complete intersection). Let us recall the definition and crucial properties of Cohen-Macaulay.

Remark 3.18. Let X be a complex space. Then, X is called *Cohen-Macaulay* (or *perfect*) if $\operatorname{codh}\mathcal{O}_{X,x} = \dim_x X$ for all $x \in X$, where $\operatorname{codh} M$ denotes the homological codimension of a module M (for more details, see e. g. § 11 in [Rem94]). We will use the following facts about Cohen-Macaulay spaces (see e. g. § 5 in [PR94]):

(i) Every Cohen-Macaulay space is locally pure dimensional.

(ii) X is Cohen-Macaulay in $p \in X$ if and only if for any (or at least one) non-zero-divisor f in the maximal ideal sheaf $\mathfrak{m}_p$, $\{f = 0\}$ is Cohen-Macaulay in p.

(iii) If X is Cohen-Macaulay and A is an analytic subset of X with $\operatorname{codim} A \geq 2$, then $\mathcal{O}(X) \to \mathcal{O}(X \setminus A)$ is bijective.

(iv) A Cohen-Macaulay space is normal if and only if its singular set is at least 2-codimensional.

The following theorem will be the first example to show how useful the corank estimate is to characterize linear spaces:

Theorem 3.19. *Let $S \subset U \times \mathbb{C}^N$ be a linear space over an irreducible Cohen-Macaulay space U of rank r defined by m holomorphic fibrewise linear functions $h_1, .., h_m \in \mathcal{O}(U \times \mathbb{C}^N)$ (i. e. $(p, h_i(p, z)) \in \operatorname{Hom}(U \times \mathbb{C}^N, U \times \mathbb{C})$) such that $N = r + m$, i. e. $m \geq \operatorname{cork} S$.*

(1) If $\operatorname{codim}\operatorname{Sing} S \geq m$, then S is Cohen-Macaulay. If (additionally) U is a complete intersection, then S is a complete intersection, as well.

(2) If $\operatorname{codim}\operatorname{Sing} S \geq m+1$, then S is (locally) irreducible.

(3) If U is normal and $\operatorname{codim}\operatorname{Sing} S \geq m+2$, then S is normal.

Proof: Since $S=\{h_1=...=h_m=0\}$, we get $\mathrm{codim}_{(p,z)}S \le m$ for all $(p,z)\in S\subset U_p{\times}\mathbb{C}^N_z$.

Let $A\subset U$ denote the singular locus of S, i. e. the set where S is not locally free, and let $E := \mathrm{PC}(S)$ denote the primary component of S. Then, $\dim E=\dim U+r$, i. e. $\mathrm{codim}E=m$. We set $T:=(A{\times}\mathbb{C}^N)\cap S$.

(1) By the assumption and by $T\subset A{\times}\mathbb{C}^N$,

$$\mathrm{codim}T \ge \mathrm{codim}(A{\times}\mathbb{C}^N)\ge m.$$

Hence, $\mathrm{codim}_{(p,z)}S{=}m$ for all $(p,z){\in}S$. Since $\mathcal{O}_S{=}\mathcal{O}_{U\times\mathbb{C}^N}/(h_1,..,h_m)$, we get that S is Cohen-Macaulay (see Remark 3.18 (ii)). Additionally, S is a locally complete intersection if U is a locally complete intersection.

(2) Let us assume that S is not irreducible, i. e. $T\backslash E\neq\emptyset$. For all $(p,z)\in T\backslash E$, there is a neighbourhood V of (p,z) such that $T\cap V=\{h_1=...=h_m=0\}$, i. e. $\mathrm{codim}_{(p,z)}T\le m$. We get $\mathrm{codim}_U A\le \mathrm{codim}_{U\times\mathbb{C}^N}T\le m$. This proves the second claim.

(3) By the assumption, we obtain $\mathrm{codim}T\ge\mathrm{codim}(A{\times}\mathbb{C}^N)\ge m{+}2$. Since U is normal, i. e. $\mathrm{codim}_U U_{\mathrm{sing}}\ge 2$, we get $\mathrm{codim}_{S_{U\backslash A}}E_{U_{\mathrm{sing}}\backslash A}\ge 2$. Since the singular set S_{sing} of S is contained in $T\cup E_{U_{\mathrm{sing}}}$ and $E_A\subset T$, we get that

$$\mathrm{codim}_S S_{\mathrm{sing}}\ge\min\{\mathrm{codim}T,\mathrm{codim}_S E_{U_{\mathrm{sing}}}\}\ge 2.$$

Remark 3.18 (iv) implies that S is normal (by use of (1)). □

Using (2) ⇒ (1) of Lemma 3.17 and Lemma 3.7, we conclude:

Corollary 3.20. *Let $\mathscr{S}$ be a coherent analytic sheaf of homological dimension less or equal than 1 on an irreducible Cohen-Macaulay space X.*

(1) If $\mathrm{cork}\,\mathscr{S}\le\mathrm{codim}\,\mathrm{Sing}\,\mathscr{S}$, then the linear space $L(\mathscr{S})$ associated to $\mathscr{S}$ is Cohen-Macaulay.

(2) If $\mathrm{cork}\,\mathscr{S}+1\le\mathrm{codim}\,\mathrm{Sing}\,\mathscr{S}$, then $L(\mathscr{S})$ is irreducible and $\mathscr{S}$ torsion-free.

(3) If $\mathrm{cork}\,\mathscr{S}+2\le\mathrm{codim}\,\mathrm{Sing}\,\mathscr{S}$ and if X is normal, then $L(\mathscr{S})$ is normal.

3.5 Linear spaces of small corank

For a linear space to be (locally) irreducible, it is necessary that the associated coherent analytic sheaf is torsion-free (see Lemma 3.7). In the following, we will prove that this is a sufficient criterion under certain additional assumptions, including the proof of Theorem 3.1 at the end of the section.

Lemma 3.21. *Let X be a normal or Cohen-Macaulay space and $S \subset X{\times}\mathbb{C}^N$ be a linear space over X with at least 2-codimensional singular locus in X and defined by one fibrewise linear function $h \in \mathcal{O}(X{\times}\mathbb{C}^N)$. Then, S is locally irreducible. In particular, the coherent analytic sheaf associated to S is torsion-free.*

Proof: Let $A \subset X$ denote the singular locus of S (as linear space) and E denote the primary component of S. Lemma 3.13 implies that E is given by the ideal sheaf $(h, g_1, ..., g_m)$ with g_i holomorphic on $X{\times}\mathbb{C}^N$ and fibrewise homogeneous (shrink X if necessary). On the regular part $X' := X \backslash A$ of S, we get $S_{X'} = E_{X'}$, i. e. $g_{i,(p,z)} \in (h)_{(p,z)}$ $\forall (p,z) \in X'{\times}\mathbb{C}^N$. Therefore, $f_i := g_i/h$ is a holomorphic function on $X'{\times}\mathbb{C}^N$. Since we assumed X to be normal or Cohen-Macaulay and A is of codimension 2 in X, f_i can be extended to a holomorphic function on $X{\times}\mathbb{C}^N$. We obtain $g_i \in (h)$ and $E = S$. Now, Lemma 3.7 implies the second statement. □

Note that for the proof of Lemma 3.21, we hardly used the fact that S is given by a principal ideal sheaf. If S is defined by more than two functions while the corank of S is 1, it can happen that $\mathscr{S} {=} \mathscr{L}(S)$ has torsion elements with support on a 2-codimensional set.

Since the singular locus of a torsion-free coherent analytic sheaf on a normal complex space is at least 2-codimensional (see Remark 2.8), we get the following corollary:

Corollary 3.22. *Let $\mathscr{S}$ be a torsion-free coherent analytic sheaf on a normal complex space X such that $\mathcal{O}_X \to \mathcal{O}_X^N \to \mathscr{S} \to 0$ is exact (i. e. the homological dimension of $\mathscr{S}$ is at most 1, see Lemma 3.17). Then, the linear space associated to $\mathscr{S}$ is locally irreducible.*

We call a normal complex space *factorial* if its structure sheaf is factorial (also called unique factorization domain). In this case, hypersurfaces are (locally) given as the zero set of one holomorphic function. The most simple examples for factorial spaces are manifolds.

Theorem 3.23. *Let S be a linear space over a factorial complex space X which is locally defined by one holomorphic fibrewise linear function in $X \times \mathbb{C}^{\operatorname{rk} S+1}$ (i. e. the associated ideal sheaf is a principal ideal sheaf). Then, the primary component of S is a linear space.*

Proof: Let $S \subset X \times \mathbb{C}^N$ be given by the fibrewise linear $h \in \mathcal{O}(X \times \mathbb{C}^N)$. The primary component E of S is an irreducible hypersurface. Since X (and, hence, $X \times \mathbb{C}^N$) is factorial, the ideal sheaf $\mathscr{I}_E$ is generated by one element. By Lemma 3.13, we get that g is fibrewise homogeneous. Moreover, g divides h. Hence, it has to be fibrewise linear. This implies that E is a linear space. □

Lemma 3.24. *Let X be a factorial complex space and $S \subset X \times \mathbb{C}^N$ a linear space associated to a torsion-free coherent analytic sheaf on X. Then, S can be defined by locally irreducible holomorphic fibrewise linear functions.*

Proof: Let the linear space S be defined by fibrewise linear $h_1, ..., h_m \in \mathcal{O}(X \times \mathbb{C}^N)$. Let $S_i := \mathrm{PC}(\{h_i = 0\})$ be defined by the fibrewise linear $g_i \in \mathcal{O}(X \times \mathbb{C}^N)$ (using Theorem 3.23). We will prove $S = \bigcap S_i$, i. e. $(h_i)_{i=1}^m = (g_j)_{j=1}^m$:

Since $g_j | h_j$, we get $(h_i)_{i=1}^m \subset (g_j)_{j=1}^m$. On the other hand, g_j vanishes on S_j. Hence, it vanishes on $\mathrm{PC}(S)$, as well. Since the coherent analytic sheaf $\mathscr{L}(S)$ associated to S is torsion-free, we get $g_j = 0$ on S (in the non-reduced sense; see Remark 3.6), i. e. $g_j \in (h_i)_{i=1}^m$. □

Theorem 3.25. *Let $\mathscr{S}$ be a torsion-free coherent analytic sheaf of corank 1 on a factorial complex space X. Then, the linear space associated to $\mathscr{S}$ is locally irreducible and, for small enough open $U \subset X$, there exists an exact sequence*

$$0 \to \mathcal{O}_U \to \mathcal{O}_U^{\operatorname{rk}\mathscr{S}+1} \to \mathscr{S}_U \to 0,$$

i. e. the homological dimension of $\mathscr{S}$ is at most 1.

Proof: For small enough open $U \subset X$, Lemma 3.24 implies that the linear space S associated to $\mathscr{S}_U$ can be defined by irreducible fibrewise linear $h_1, ..., h_m \in \mathcal{O}(U \times \mathbb{C}^N)$ with $N = \operatorname{rk}\mathscr{S}+1$. Yet, the primary component E of S is already an irreducible hypersurface in $U \times \mathbb{C}^N$. Hence, E coincides with $S_i = \{h_i = 0\}$ and is a linear space. Lemma 3.12 implies $S = E = S_i$.

We obtain the exact sequence $\mathcal{O}_U \xrightarrow{h_i^*} \mathcal{O}_U^N \to \mathscr{S}_U \to 0$. Lemma 3.17, (1) $\Rightarrow$ (2) or Lemma 3.14 give the injectivity of h_i^*. □

Let us generalize this for sheaves of corank 2:

Theorem 3.26. *Let X be a factorial Cohen-Macaulay space and let S be a linear space of corank 2 on X such that* $\operatorname{Sing} S$ *has at least codimension 3 in X and the coherent analytic sheaf $\mathscr{L}(S)$ associated to S is torsion-free. Then, S is locally irreducible.*

Proof: The proof is similar to the proof of Lemma 3.21. Let $S \subset U \times \mathbb{C}^N$ be defined by the fibrewise linear $h_1, ..., h_m \in \mathcal{O}(U \times \mathbb{C}^N)$ for an open subset $U \subset X$ with $N = 2 + \operatorname{rk}\mathscr{S}$. Because of Lemma 3.24, we can assume that $S_i := \{h_i = 0\}$ is locally irreducible. In particular, S_i is Cohen-Macaulay. Let us assume $h_1, h_2 \neq 0$ and $h_2 \notin (h_1)$. Since h_1 is irreducible, $S_{12} := S_1 \cap S_2$ is a linear space with the same rank as S. We will prove that $S_1 \cap S_2$ coincides with $E := \operatorname{PC}(S)$:

Through Lemma 3.13, E is defined by fibrewise homogeneous holomorphic functions $g_1, ..., g_k$ (in particular, $h_i \in (g_1, ..., g_k)$). Let A denote the singular locus of S and $U' = U \backslash A$. Since $E \subset S_{12}$ and $\dim E_{U'} = \dim S_{12,U'}$, we get $E = S_{12}$ over U'. Hence, we obtain

$g_{j,(p,z)} \in (h_1, h_2)_{(p,z)}$ for all $(p, z) \in S_{1,U'}$. This means $\frac{g_j}{h_2}$ is a holomorphic function on $S_{1,U'}$. By assumption,

$$\operatorname{codim}_{S_1}((A\times\mathbb{C}^N) \cap S_1) \geq \operatorname{codim}_{U\times\mathbb{C}^N}(A\times\mathbb{C}^N) - 1 \geq 2.$$

Taking into account that S_1 is Cohen-Macaulay, we conclude that $f_j := g_j/h_2$ is holomorphic on S_1. Since $g_j - f_j \cdot h_2 \in \mathcal{O}(U\times\mathbb{C}^N)$ vanishes on S_1, we get $g_j \in (h_1, h_2)$, i. e. $E = S_{12}$. Lemma 3.12 implies the claim. □

Corollary 3.27. *Let X be a factorial Cohen-Macaulay space and $\mathscr{S}$ be a torsion-free coherent analytic sheaf $\mathscr{S}$ of corank 2 on X with at least 3-codimensional singular locus. Then, the linear space associated to $\mathscr{S}$ is locally irreducible and, for small enough open $U \subset X$, there exists an exact sequence*

$$0 \to \mathcal{O}_U^2 \to \mathcal{O}_U^{\operatorname{rk}\mathscr{S}+2} \to \mathscr{S}_U \to 0,$$

i. e. the homological dimension of $\mathscr{S}$ is at most 1.

Proof: In the proof of Theorem 3.26, we have seen that the linear space associated to $\mathscr{S}$ is given by two holomorphic fibrewise linear functions on $U\times\mathbb{C}^{\operatorname{rk}\mathscr{S}+2}$. We get the exact sequence $\mathcal{O}^2 \to \mathcal{O}^{\operatorname{rk}\mathscr{S}+2} \to \mathscr{S}$. Lemma 3.17 implies the claim. □

The counterexample Remark 3.8 shows that the assumption on the codimension is necessary in Corollary 3.27.

We can now put together the proof of the main result of this section:

Proof of Theorem 3.1: Lemma 3.7 yields the implication (2) ⇒ (1) and Theorem 3.25 and Corollary 3.27 yield the implication (1) ⇒ (3, 4). Using Corollary 3.20 (2, 3), we get the implication (4) ⇒ (3, 5). It only remains to show (3) ⇒ (2):

Assume that (3) is satisfied, i. e. that $L(\mathscr{S})$ is locally irreducible. But $L(\mathscr{S})$ is connected. So, there can be just one irreducible component, i. e. (2) holds, too. □

Chapter 4
Proper Modifications of Coherent Analytic Sheaves

In bimeromorphic geometry, the use of locally free coherent analytic sheaves is limited: The direct image of a locally free sheaf under a proper modification is not locally free any more. Instead, it is reasonable to consider the wider category of torsion-free coherent analytic sheaves. The restriction to torsion-free sheaves makes sense for bimeromorphic considerations as the torsion of a coherent analytic sheaf is supported on analytically thin subsets. To exemplify the use of torsion-free sheaves, just recall that an irreducible reduced compact space X is a Moishezon space if and only if it carries a positive torsion-free coherent analytic sheaf $\mathscr{S}$ with $\operatorname{supp}(\mathscr{S}) = X$ (see e. g. Thm. 6.14 in [Pet94a]).

The main motivation is as follows: Let $\mathscr{S}$ be a torsion-free coherent analytic sheaf on an irreducible complex space X. Then, H. Rossi showed that there exists a proper modification $\varphi = \varphi_{\mathscr{S}} \colon Y \to X$ such that the torsion-free preimage $\varphi^T \mathscr{S}$ is locally free (see Section 2.5 for details). Combining this with a resolution of singularities $\sigma \colon M \to Y$, which exists due to H. Hironaka, we obtain a resolution of singularities $\pi{=}\varphi \circ \sigma \colon M \to X$ such that $\pi^T \mathscr{S}$ is locally free. Thus, it is possible to study coherent analytic sheaves modulo torsion by reducing the problem in order to study vector bundles on manifolds.

In view of this idea, it seems very interesting to study the connection between $\mathscr{S}$ and its torsion-free preimage $\pi^T \mathscr{S}$ closer, and we found the following theorem, which to our knowledge has not been covered in the literature, yet:

Theorem 4.1. *Let $\pi : Y \to X$ be a proper modification of a complex space X, and let $\mathscr{F}$ and $\mathscr{G}$ be torsion-free coherent analytic sheaves on X and Y respectively.*

(i) If $\mathscr{F} = \pi_*\mathscr{G}$, then

$$\mathscr{F} \cong \pi_*\pi^T\mathscr{F}.$$

(ii) If $\mathscr{G} = \pi^T\mathscr{F}$, then

$$\pi^T\pi_*\mathscr{G} \cong \mathscr{G}.$$

To prove this, we will show that the natural maps

$$\begin{aligned} \mathscr{F} &\to \pi_*\pi^T\mathscr{F}, \\ \pi^T\pi_*\mathscr{G} &\to \mathscr{G} \end{aligned}$$

both are injective (Theorem 4.12 (i) and Lemma 4.4; the proof of Lemma 4.4 presented here is due to M. Toma). We give also counter-examples to show that these injections are not bijective in general (Remark 4.6 and Remark 4.14).

The assumption in Theorem 4.1 (i) can be replaced by a normality assumption on the linear space associated to the sheaf:

Theorem 4.2. *Let X be a locally irreducible complex space, $\mathscr{S}$ a torsion-free coherent analytic sheaf on X such that the linear space associated to $\mathscr{S}$ is normal, and $\pi\colon Y \to X$ a proper modification of X. Then, the canonical homomorphism $\mathscr{S} \to \pi_*(\pi^T\mathscr{S})$ is bijective, i. e.*

$$\mathscr{S} \cong \pi_*(\pi^T\mathscr{S}).$$

In particular, $\mathscr{S}$ can actually be represented as the direct image of a locally free sheaf. In Chapter 3, we presented some criteria for normality of the linear space associated to $\mathscr{S}$ (see Theorem 3.1 and Corollary 3.20). Additionally, we will see that the assumptions in Theorem 4.2 are necessary to obtain that $\mathscr{S}$ is the direct image of a locally free sheaf on a normal space (see Remark 4.20). For this, we show that a locally free sheaf on a non-normal complex space X can not be the direct image of a sheaf on a normal modification of X (cf. Theorem 4.21). In particular, the Grauert-Riemenschneider canonical sheaf $\mathscr{K}_X$ can not be locally free on a non-normal space X.

Furthermore, we obtain the following two results for specific proper modifications: First, we show that the torsion-free preimage of the direct image of an arbitrary torsion-free coherent analytic sheaf $\mathscr{F}$ with respect to a 1:1 modification is canonically isomorphic to $\mathscr{F}$ (see Theorem 4.9). Second, we prove that the direct image of the torsion-free preimage of a suitable sheaf $\mathscr{S}$ of rank 1 under the monoidal transformation with respect to $\mathscr{S}$ is canonically isomorphic to $\mathscr{S}$ (see Theorem 4.17).

Let us consider two applications of Theorem 4.1. First, let X be a locally irreducible complex space of pure dimension n, and $\mathscr{K}_X$ the Grauert-Riemenschneider canonical sheaf on X (as introduced in Section 2.4). Then, there exists a resolution of singularities $\pi\colon M \to X$ (with only normal crossings) such that $\pi^T\mathscr{K}_X$ is locally free, and so there is an effective divisor D with support on the exceptional set of the modification such that

$$\mathscr{K}_X \cong \pi_*\pi^T\mathscr{K}_X = \pi_*\Omega^n_M(-D) = \pi_*\big(\Omega^n_M \otimes \mathcal{O}(-D)\big) \tag{4.3}$$

(see Theorem 4.24). Let us explain briefly the meaning of (4.3). By definition of the Grauert-Riemenschneider canonical sheaf, we know already that $\mathscr{K}_X \cong \pi_*\Omega^n_M$. (4.3) tells us that we can as well consider the push-forward of holomorphic n-forms which vanish to the order of D on the exceptional set. This is an important detail, particularly if π is explicitly given so that D can be calculated explicitly. An example: If X is already a manifold (i. e. $\mathscr{K}_X = \Omega^n_X$) and $\pi\colon M \to X$ is the blow-up along a submanifold of codimension s in X with exceptional set E, then (see e. g. Prop. VII.12.7 in [Dem12]):

$$\pi^T\mathscr{K}_X = \pi^*\mathscr{K}_X = \Omega^n_M\big(-(s-1)E\big),$$

and so

$$\Omega^n_X = \mathscr{K}_X \cong \pi_*\Omega^n_M\big(-(s-1)E\big).$$

Considerations of this kind are particularly important in the study of canonical sheaves on singular complex spaces (see [Rup14a]). We will also set up the relation (4.3) for holomorphic n-forms with values in locally free coherent analytic sheaves (see Theorem 4.28).

Second, we will study reduced ideal sheaves in Section 9.1. Let $\pi\colon Y \to X$ be a proper modification of a locally irreducible complex space X, $A \subset X$ an analytic subset with (reduced) ideal sheaf $\mathscr{I}_A$, and $B := \pi^{-1}(A)$ the reduced analytic preimage with (reduced) ideal sheaf $\mathscr{I}_B$. Then, $\pi^T \mathscr{I}_A = \mathscr{I}_B$ (cf. Lemma 9.2), and Theorem 4.1 yields that $\mathscr{I}_B \cong \pi^T \pi_* \mathscr{I}_B$.

If we assume moreover that X is normal and that A is either a locally complete intersection or a normal analytic set and that $\sigma\colon Y \to X$ is the monoidal transformation with respect to $\mathscr{I}_A$, then $\mathscr{I}_B = \sigma^T \mathscr{I}_A$ (is locally free) and we have (see Lemma 9.5):

$$\mathscr{I}_A \cong \sigma_* \mathscr{I}_B \cong \sigma_* \sigma^T \mathscr{I}_A.$$

Using Theorem 4.2, we are able to generalize Takegoshi's relative version [Tak85] of the Grauert-Riemenschneider vanishing theorem in several directions. This is elaborated in Chapter 8 (cf. [Ser15]).

This chapter is organized as follows. We study torsion-free analytic preimages of direct image sheaves in Section 4.1 and direct images of (torsion-free) analytic preimage sheaves (including the proof of Theorem 4.2) in Section 4.2. In Section 4.3, we show that the analytic inverse image functor preserves monomorphisms and epimorphisms and use this fact in combination with the previous considerations to prove Theorem 4.1. Section 4.4 contains the first application described above. We complement the chapter by analogous considerations on the non-analytic inverse image functor in Section 4.5. The presented results of this chapter and their proofs can be found in [RS13] and [Ser15, Sect. 4], as well.

4.1 Torsion-free preimages of direct image sheaves

In this section, we study the torsion-free preimage of direct image sheaves under proper modifications.

Lemma 4.4. *Let $\pi : Y \to X$ be a proper modification between complex spaces Y, X, and $\mathscr{E}$ a torsion-free coherent analytic sheaf on Y. Then, the canonical homomorphism $\pi^*\pi_*\mathscr{E} \to \mathscr{E}$ induces a canonical injection*

$$\pi^T\pi_*\mathscr{E} \hookrightarrow \mathscr{E}, \tag{4.5}$$

where $\pi^T\pi_*\mathscr{E} = \pi^*\pi_*\mathscr{E}/\mathscr{T}(\pi^*\pi_*\mathscr{E})$ is the torsion-free preimage of $\pi_*\mathscr{E}$.

The following proof was communicated to us by Matei Toma. Alternatively, Lemma 4.4 follows also from Lemma 3.14.

Proof: Let $\mathscr{T}$ denote the torsion sheaf of $\pi^*\pi_*\mathscr{E}$, and let $\psi\colon \pi^*\pi_*\mathscr{E} \to \mathscr{E}$ denote the natural map. Since $\mathscr{E}$ is torsion-free, $\psi(\mathscr{T}) = 0$ and, hence, ψ factors through $\widehat{\psi}\colon \pi^T\pi_*\mathscr{E} \to \mathscr{E}$. Since ψ is an isomorphism outside of a thin analytic set $A \subset X$, an element in the kernel of ψ has support in A. Therefore, the kernel is a subset of $\mathscr{T}$, i. e. $\widehat{\psi}$ is injective. $\square$

Remark 4.6. Let us give a counterexample showing that (4.5) is in general not an isomorphism. Consider a modification $\pi : M \to \mathbb{C}^n$ where M is a complex manifold with canonical sheaves Ω^n_M and $\Omega^n_{\mathbb{C}^n} \cong \mathcal{O}_{\mathbb{C}^n}$. Then, $\pi_*\Omega^n_M = \Omega^n_{\mathbb{C}^n} \cong \mathcal{O}_{\mathbb{C}^n}$ so that $\pi^T\pi_*\Omega^n_M \cong \pi^T\mathcal{O}_{\mathbb{C}^n} \cong \mathcal{O}_M$. But, $\mathcal{O}_M \neq \Omega^n_M$ in general.

However, we can be a bit more precise in Lemma 4.4 by use of the following observation if $\mathscr{E}$ is locally free of rank 1:

Lemma 4.7. *Let X be a complex space and $i : \mathscr{F} \hookrightarrow \mathscr{G}$ an injective morphism between two coherent locally free sheaves of rank 1 over X. Then, there exists an effective Cartier divisor, $D \geq 0$, such that*

$$i(\mathscr{F}) = \mathscr{G} \otimes \mathcal{O}_X(-D).$$

In particular, i is an isomorphism precisely on $X \setminus |D|$.

Proof: Let $\{X_\alpha\}_\alpha$ be a locally finite open cover of X such that both, $\mathscr{F}$ and $\mathscr{G}$, are free over each X_α. So, there are trivializations

$$\begin{aligned}\phi_\alpha\colon \mathscr{F}_{X_\alpha} &\xrightarrow{\sim} \mathcal{O}_{X_\alpha},\\ \psi_\alpha\colon \mathscr{G}_{X_\alpha} &\xrightarrow{\sim} \mathcal{O}_{X_\alpha},\end{aligned}$$

and, for $X_{\alpha\beta} := X_\alpha \cap X_\beta \neq \emptyset$, we have transition functions $F_{\beta\alpha} := \phi_\beta \circ \phi_\alpha^{-1} \in \mathcal{O}^*(X_{\alpha\beta})$ and $G_{\beta\alpha} := \psi_\beta \circ \psi_\alpha^{-1} \in \mathcal{O}^*(X_{\alpha\beta})$ satisfying the cocycle conditions. In trivializations,

$$\psi_\alpha \circ i|_{X_\alpha} \circ \phi_\alpha^{-1}\colon \mathcal{O}_{X_\alpha} \to \mathcal{O}_{X_\alpha}$$

is given by a holomorphic function $i_\alpha \in \mathcal{O}(X_\alpha)$, vanishing nowhere identically with (unreduced) divisor (i_α). On $X_{\alpha\beta}$, we get

$$G_{\beta\alpha} \cdot i_\alpha = \psi_\beta \circ \psi_\alpha^{-1} \circ \psi_\alpha \circ i|_{\alpha\beta} \circ \phi_\alpha^{-1} = \psi_\beta \circ i|_{\alpha\beta} \circ \phi_\beta^{-1} \circ \phi_\beta \circ \phi_\alpha^{-1} = i_\beta \cdot F_{\beta\alpha}$$

so that $i_\alpha / i_\beta = F_{\beta\alpha} / G_{\beta\alpha} \in \mathcal{O}^*(X_{\alpha\beta})$. Thus, $D := \{(X_\alpha, i_\alpha)\}_\alpha$ defines in fact an effective Cartier divisor with support $|D|$.

To see that $i(\mathscr{F}) = \mathscr{G} \otimes \mathcal{O}_X(-D)$, note that $\mathscr{G} \otimes \mathcal{O}_X(-D)$ is a coherent subsheaf of $\mathscr{G}$ because $\mathcal{O}_X(-D)$ is a sheaf of ideals in $\mathcal{O}_X$, and that

$$\psi_\alpha \otimes 1\colon\ \mathscr{G} \otimes \mathcal{O}_X(-D)|_{X_\alpha} \xrightarrow{\sim} \mathcal{O}_{X_\alpha} \otimes \mathcal{O}_{X_\alpha}(-(i_\alpha)).$$ □

So, we can deduce the following direct consequence of Lemma 4.4:

Theorem 4.8. *Let $\pi\colon Y \to X$ be a proper modification of X, $\mathscr{E}$ a locally free analytic sheaf of rank 1 on Y and assume that $\pi^T \pi_* \mathscr{E}$ is also locally free. Then, there exists an effective Cartier divisor D on Y such that the following holds: The canonical homomorphism $\pi^* \pi_* \mathscr{E} \to \mathscr{E}$ induces a canonical injection*

$$i\colon \pi^T \pi_* \mathscr{E} \hookrightarrow \mathscr{E}$$

and

$$i(\pi^T \pi_* \mathscr{E}) = \mathscr{E} \otimes \mathcal{O}_Y(-D).$$

In particular, i is an isomorphism precisely on $Y - |D|$, and $|D|$ is contained in the exceptional set of π.

For the torsion-free inverse image of the direct image sheaf under a 1:1 modification, we obtain:

Theorem 4.9. *Let X be a locally irreducible complex space, let $\psi\colon \widehat{X} \to X$ be the normalization of X, and let $\mathscr{F}$ be a torsion-free coherent analytic sheaf on $\widehat{X}$. Then, the canonical morphism $\psi^*\psi_*\mathscr{F} \to \mathscr{F}$ induces an isomorphism*

$$\psi^T\psi_*\mathscr{F} \cong \mathscr{F}.$$

Proof: Since the 1-sheeted covering ψ is a homeomorphism (X is locally irreducible), we get for all $q \in \widehat{X}$:

$$\mathscr{F}_q = (\psi_*\mathscr{F})_{\psi(q)} \stackrel{\text{def}}{=} (\psi^{-1}\psi_*\mathscr{F})_q.$$

By the definition of the (analytic) inverse image sheaf, we obtain

$$\psi^*\psi_*\mathscr{F} \stackrel{\text{def}}{=} \psi^{-1}\psi_*\mathscr{F} \otimes_{\psi^{-1}\mathcal{O}_X} \mathcal{O}_{\widehat{X}} = \mathscr{F} \otimes_{\psi^{-1}\mathcal{O}_X} \mathcal{O}_{\widehat{X}}.$$

Yet, the injective map $\psi^{-1}\mathcal{O}_X \hookrightarrow \mathcal{O}_{\widehat{X}}$ gives us the surjectivity of the canonical morphism:

$$\psi^*\psi_*\mathscr{F} = \mathscr{F} \otimes_{\psi^{-1}\mathcal{O}_X} \mathcal{O}_{\widehat{X}} \twoheadrightarrow \mathscr{F} \otimes_{\mathcal{O}_{\widehat{X}}} \mathcal{O}_{\widehat{X}} = \mathscr{F}, s \otimes r \mapsto r \cdot s.$$

With Lemma 4.4, we also have

$$\psi^T\psi_*\mathscr{F} \hookrightarrow \mathscr{F}, s \otimes r + \mathscr{T}(\psi^*\psi_*\mathscr{F}) \mapsto r \cdot s.$$

□

4.2 Direct image sheaves of torsion-free preimage sheaves

In this section, we study the direct image of the torsion-free preimage sheaf under a proper modification. In particular, we will prove Theorem 4.2 (see Theorem 4.12 (iii)). Furthermore, we obtain more results for sheaves of rank one. Let us first recall the elementary projection formula (cf. e. g. Ex. II.5.1 in [Har77]).

Theorem 4.10 (Projection Formula). *Let $f\colon Y \to X$ be a holomorphic map between complex spaces, let $\mathscr{E}$ be a locally free sheaf on X, and let $\mathscr{F}$ be a coherent analytic sheaf on Y. Then,*

$$\mathscr{E} \otimes f_*\mathscr{F} \cong f_*\left(f^*\mathscr{E} \otimes \mathscr{F}\right). \tag{4.11}$$

Proof: Let r denote the rank of $\mathscr{E}$. Since the direct product is invariant under the direct image, we conclude

$$\begin{aligned}\mathcal{O}_U^{\oplus r}\otimes f_*\mathscr{F} &\cong (f_*\mathscr{F})^{\oplus r} \cong f_*(\mathscr{F}^{\oplus r}) \cong f_*(\mathcal{O}_{f^{-1}(U)}^{\oplus r}\otimes\mathscr{F})\\ &\cong f_*(f^*(\mathcal{O}_U^{\oplus r})\otimes\mathscr{F})\end{aligned}$$

for all $U \subset X$. Let α_U denote the map from left to right. Since all used isomorphisms are natural, α itself is natural.

Choose an open covering $\{X_\alpha\}_\alpha$ of X such that there exist trivializations $\phi_\alpha\colon \mathscr{E}_{X_\alpha} \xrightarrow{\sim} \mathcal{O}_{X_\alpha}^r$ of $\mathscr{E}$ over X_α. Let $F_{\beta\alpha} := \phi_\beta \circ \phi_\alpha^{-1}$ denote the transition function on $X_{\alpha\beta} := X_\alpha \cap X_\beta$. The linear space L_1 associated to $\mathscr{E}\otimes f_*\mathscr{F}$ is given by $L(\mathcal{O}_{X_\alpha}^{\oplus r}\otimes f_*\mathscr{F})$ glued together via $L(F_{\beta\alpha}\otimes\mathrm{id}_{f_*\mathscr{F}})$. Also, $L_2 := L(f_*(f^*(\mathscr{E})\otimes\mathscr{F}))$ is given by $L(f_*(f^*(\mathcal{O}_{X_\alpha}^{\oplus r})\otimes\mathscr{F}))$ glued together via $L(f_*(f^*(F_{\beta\alpha})\otimes\mathrm{id}_{\mathscr{F}}))$. Since $\alpha_{X_{\beta\alpha}}$ is natural, we get $\alpha_{X_{\beta\alpha}} \circ (F_{\beta\alpha}\otimes\mathrm{id}_{f_*\mathscr{F}}) = (f_*(f^*(F_{\beta\alpha})\otimes\mathrm{id}_{\mathscr{F}})) \circ \alpha_{X_{\beta\alpha}}$. Hence, L_1 and L_2 are obtained by isomorphic gluing data, i. e. $L_1 \cong L_2$. □

Theorem 4.12. *Let X be a locally irreducible complex space, $\mathscr{S}$ a torsion-free coherent analytic sheaf on X and $\pi\colon Y \to X$ a proper modification of X.*

(i) Then, the canonical homomorphisms $\mathscr{S} \to \pi_*(\pi^*\mathscr{S})$ and $\mathscr{S} \to \pi_*(\pi^T\mathscr{S})$ both are injective.

(ii) If the linear space $L(\mathscr{S})$ associated to $\mathscr{S}$ is irreducible and $L(\pi^*\mathscr{S})$ is reduced, then $\pi_*(\pi^*\mathscr{S}) \to \pi_*(\pi^T\mathscr{S})$ is injective.

(iii) If the linear space $L(\mathscr{S})$ is normal, then

$$\mathscr{S} \cong \pi_*(\pi^T\mathscr{S}).$$

Proof: We can assume that X is connected. Let S denote the linear space associated to $\mathscr{S}$, $A \subset Y$ the set where π is not biholomorphic and A^c the complement. $S^* = Y\times_X S$ is the linear space associated to $\mathscr{S}^* := \pi^*\mathscr{S}$. Let $\mathrm{pr}\colon S^* \to S$ denote the projection, let E be the linear space associated to $\mathscr{E} := \pi^T\mathscr{S}$, let U be an open Stein set in X, and let $V := \pi^{-1}(U)$. The construction of the linear spaces implies

$$\begin{aligned}
&\operatorname{Hom}(S_U, U\times\mathbb{C}) \cong \mathscr{S}(U),\\
&\operatorname{Hom}(S_V^*, V\times\mathbb{C}) \cong \pi^*\mathscr{S}(V) = \big(\pi_*(\pi^*\mathscr{S})\big)(U) \text{ and}\\
&\operatorname{Hom}(E_V, V\times\mathbb{C}) \cong \mathscr{E}(V) = (\pi_*\mathscr{E})(U).
\end{aligned}$$

Let N be an integer big enough so that S_U can be realized as a subset of $U\times\mathbb{C}^N$. We obtain closed embeddings $E_V \subset S_V^* \subset V\times\mathbb{C}^N$, and (q,z) is in S_V^* if and only if $(\pi(q), z) \in S$. Since obviously $\operatorname{pr}(E_{A^c}) = S_{\pi(A^c)}$ and proper holomorphic images of irreducible sets are irreducible (using Theorem 2.12), we obtain (cf. (3.11))

$$\operatorname{pr}(\operatorname{PC}(E_V)) = \operatorname{PC}(S_U), \tag{4.13}$$

where PC denotes the primary component of a linear space (see Section 3.3).

(i) $\mathscr{S} \hookrightarrow \pi_*(\pi^*\mathscr{S})$ and $\mathscr{S} \hookrightarrow \pi_*(\pi^T\mathscr{S})$ follows from Lemma 3.14.

(ii) Assume S is irreducible and S^* is reduced. To prove that the natural map given by the restriction $\operatorname{Hom}(S_V^*, V\times\mathbb{C}) \to \operatorname{Hom}(E_V, V\times\mathbb{C})$ is injective, we use $\operatorname{pr}(\operatorname{PC}(E_V)) = S_U$ (using (4.13) and that S is irreducible):

Let $s \in \operatorname{Hom}(S_V^*, V\times\mathbb{C})$ with $s|_E = 0$, i.e. $s|_{\operatorname{PC}(E)} = 0$. Since S is irreducible, it is reduced, as well. Let $s(q,z) = (q, f(q,z)) \in \operatorname{Hom}(S_V^*, V\times\mathbb{C})$ be not the zero section, i.e. there is a point $(q', z_0) \in S^*$ with $f(q', z_0) \neq 0$ (S^* is reduced). There is a $q'' \in \pi^{-1}(\pi(q'))$ such that $(q'', z_0) \in E$. Since $\pi^{-1}(\pi(q'))\times\{z_0\}$ is a compact analytic set in S_V^*, we get $f(q', z_0) = f(q'', z_0)$, i.e. $f|_E \neq 0$ and $s|_E \neq 0$.

(iii) Assume that S is normal. In particular, S is irreducible. Fix a section $s(q,z) = (q, f(q,z)) \in \operatorname{Hom}(E_V, V\times\mathbb{C})$. Since $\operatorname{pr}\colon \operatorname{PC}(E_V) \to \operatorname{PC}(S_U) = S_U$ is a proper modification (surjectivity is (4.13) and S_U is irreducible), the map $\widetilde{f} := f\circ\operatorname{pr}^{-1}\colon S_U \to \mathbb{C}$ is a bounded meromorphic function, i.e. it is weakly holomorphic. Since S_U is normal, $\widetilde{f}$ is holomorphic. Obviously, it is linear in the second argument. Hence, pr^{-1} gives a map $(\operatorname{pr}^{-1})^*\colon \operatorname{Hom}(E_V, V\times\mathbb{C}) \to \operatorname{Hom}(S_U, U\times\mathbb{C}), s \mapsto \widetilde{s}$ with $\widetilde{s}(p,z) = (p, \widetilde{f}(p,z))$. Since $f\circ\operatorname{pr}^{-1} = \widetilde{f} = 0$ implies $f = 0$, this map is injective. It is now easy to see that $(\mathrm{pr}^{-1})^*\colon \pi_*\pi^T\mathscr{S} \hookrightarrow \mathscr{S}$ is the inverse to the natural mapping $\mathscr{S} \hookrightarrow \pi_*\pi^T\mathscr{S}$. □

Remark 4.14. Without the additional assumption about normality, the natural map $\mathscr{S} \hookrightarrow \pi_*\pi^T\mathscr{S}$ is not necessarily bijective. The following counterexample is derived from one due to Mircea Mustaţă.

Let $\mathscr{S} = (x^3, y^3)$ be the ideal sheaf on $\mathbb{C}^2_{x,y}$, generated by the functions x^3 and y^3, and let $\pi\colon M \to \mathbb{C}^2$ be the blow up of the origin, i. e.

$$M = \{(x, y; [t_1 : t_2]) \in \mathbb{C}^2 \times \mathbb{CP}^1 \colon xt_2 = yt_1\}.$$

$\mathscr{S}$ has the following sequence as resolution:

$$\begin{array}{ccccccc} \mathcal{O} & \longrightarrow & \mathcal{O}^2 & \longrightarrow & \mathscr{S} & \longrightarrow & 0\,. \\ & & (f_1, f_2) & \longmapsto & x^3 f_1 + y^3 f_2 & & \\ g & \longmapsto & (y^3 g, -x^3 g) & & & & \end{array}$$

Hence,

$$S = L(\mathscr{S}) = \{(x, y; z_1, z_2) \in \mathbb{C}^2 \times \mathbb{C}^2 \colon z_2 x^3 = z_1 y^3\},$$

and sections in $\mathscr{S}$ correspond to sections in $\mathrm{Hom}(S, \mathbb{C}^2 \times \mathbb{C})$ via the assignment $x^3 \mapsto [(x, y; z_1, z_2) \mapsto z_1]$, $y^3 \mapsto [(x, y; z_1, z_2) \mapsto z_2]$. Now,

$$S^* = L(\pi^*\mathscr{S}) = \{(x, y; [t_1 : t_2]; z_1, z_2) \in M \times \mathbb{C}^2 \colon z_2 x^3 = z_1 y^3\},$$

$$E = L(\pi^T\mathscr{S}) = \{(x, y; [t_1 : t_2]; z_1, z_2) \in M \times \mathbb{C}^2 \colon z_2 t_1^3 = z_1 t_2^3\}.$$

Thus, $S^* = E \cup T$ with

$$T = \{(x, y; [t_2 : t_2]; z_1, z_2) \in M \times \mathbb{C}^2 \colon x = y = 0\} = 0 \times \mathbb{CP}^1 \times \mathbb{C}^2.$$

In $\mathrm{Hom}(E, M \times \mathbb{C})$, we have now also the section

$$\big\{(\tfrac{t_2}{t_1} z_1 \colon t_1 \neq 0); (\tfrac{t_1^2}{t_2^2} z_2 \colon t_2 \neq 0)\big\},$$

corresponding to $x^2 y$ in $\pi^T\mathscr{S}$, and the section

$$\big\{(\tfrac{t_2^2}{t_1^2} z_1 : t_1 \neq 0); (\tfrac{t_1}{t_2} z_2 \colon t_2 \neq 0)\big\},$$

corresponding to xy^2, but these two do not extend to $S^* = E \cup T$ because there is no relation between z_1 and z_2 on T. In $\mathrm{Hom}(S^*, M \times \mathbb{C})$, however, we have the section

$$\big\{(y \tfrac{t_2}{t_1} z_1 \colon t_1 \neq 0); (y \tfrac{t_1^2}{t_2^2} z_2 \colon t_2 \neq 0)\big\},$$

corresponding to x^2y^2 in $\pi^*\mathscr{S}$. Moreover, it is easy to verify that x, x^2, y, y^2 and xy are neither contained in $\pi_*\pi^*\mathscr{S}$ nor in $\pi_*\pi^T\mathscr{S}$. Hence, we have:

$$\mathscr{S} \subsetneq (x^3, x^2y^2, y^3) = \pi_*\pi^*\mathscr{S} \subsetneq (x^3, x^2y, xy^2, y^3) = \pi_*\pi^T\mathscr{S}.$$

Remark 4.15. Let us present a counterexample for (ii) in Theorem 4.12 if S is not irreducible:

Let $X = \{x^3 = y^2\} \subset \mathbb{C}^2$ be the cusp, $\pi\colon \mathbb{C} \to X$, $\pi(t) := (t^2, t^3)$ the normalization and let $\widehat{\mathcal{O}}_X$ denote the sheaf of weakly holomorphic functions on X. Then, one can compute that $\pi_*\pi^*\widehat{\mathcal{O}}_X$ has torsion elements with support in 0. Yet, $\pi_*\pi^T\widehat{\mathcal{O}}_X = \pi_*\mathcal{O}_{\mathbb{C}} = \widehat{\mathcal{O}}_X$ is torsion-free. Hence, there can not exist an injective morphism $\pi_*\pi^*\widehat{\mathcal{O}}_X \to \pi_*\pi^T\widehat{\mathcal{O}}_X$.

Remark 4.16. In order to get the isomorphism $\mathscr{S} = \pi_*\pi^T\mathscr{S}$ for all proper modifications, the normality of the linear space of $\mathscr{S}$ is a natural assumption. For example, if $\mathscr{E}$ is locally free, we obtain $\pi_*\pi^*\mathscr{E} \cong \mathscr{E}$ if and only if $\pi_*\mathcal{O}_Y \cong \mathcal{O}_X$. Yet, this is only satisfied for all proper modifications if it is satisfied for the normalization, i. e. this is equivalent to X being normal.

Additionally, the normality assumption is required for the following purpose (see Remark 4.20): If $\varphi = \varphi_{\mathscr{S}}\colon Y \to X$ is the monoidal transformation with respect to $\mathscr{S}$, then Y does not have to be normal (under some additional assumptions this is equivalent to $L(\mathscr{S})$ being normal, see Remark 4.19). The question is now, does there exist a (locally free) sheaf on the normalization of Y such that its direct image is $\varphi^T(\mathscr{S})$. Under the same assumptions as in Remark 4.19, we will see that this is equivalent to Y being normal. This means that the normality assumption on $L(\mathscr{S})$ is necessary to obtain $\mathscr{S} \cong \pi_*\pi^T\mathscr{S}$ for normal Y and locally free $\pi^T\mathscr{S}$.

If we choose the monoidal transformation for the proper modification (see Section 2.5), we get an alternative result to Theorem 4.12 (iii), where the normality assumption can be replaced to show that the composition of the direct image and preimage functor is an isomorphism (cf. Sect. 4 in [Ser15]).

Theorem 4.17. *Let X be an irreducible Cohen-Macaulay space, and let $\mathscr{S}$ be a coherent analytic sheaf of rank 1 on X such that, for each $p \in X$, there exist a neighbourhood U and a free resolution $\mathcal{O}_U^m \to \mathcal{O}_U^{m+1} \to \mathscr{S}_U \to 0$ (i. e. the homological dimension of $\mathscr{S}$ is at most 1, see Lemma 3.17) and the singular locus of $\mathscr{S}$ is at least $(m{+}1)$-codimensional, or such that (more weakly) the linear space $L(\mathscr{S})$ associated to $\mathscr{S}$ is Cohen-Macaulay and irreducible. Let $\varphi{=}\varphi_{\mathscr{S}}\colon Y \to X$ denote the monoidal transformation of X with respect to $\mathscr{S}$. Then, the canonical morphism $\mathscr{S} \to \varphi_*\varphi^*\mathscr{S}$ induces an isomorphism*

$$\mathscr{S} \xrightarrow{\sim} \varphi_*\varphi^T\mathscr{S}.$$

For $m \leq 2$, the assumption on the free resolution of $\mathscr{S}_U$ in Theorem 4.17 is always satisfied if X is factorial and Cohen-Macaulay and if $\mathscr{S}$ is torsion-free with $\operatorname{cork}\mathscr{S} \leq m$ (see Theorem 3.1).

Proof: Let S denote the linear space associated to $\mathscr{S}$. With Theorem 3.19 or by the assumption, we get that S is Cohen-Macaulay and irreducible (in particular, $\mathscr{S}$ is torsion-free, see Lemma 3.7). Let $E := L(\varphi^T\mathscr{S}) \subset \varphi^*S = L(\varphi^*\mathscr{S})$ denote the linear space associated to the torsion-free preimage of $\mathscr{S}$ and $\mathrm{pr}\colon E \to S$ be the restriction of the projection of the fibre product $Y{\times_X}S = \varphi^*S$ to S. Then, E is irreducible and the proper mapping theorem implies $\mathrm{pr}(E) = S$, i. e. pr is a proper modification of S.

The biholomorphism between $\mathbb{C}^{m+1}\backslash 0$ and the universal line bundle without zero section $\mathcal{O}_{\mathbb{CP}^m}(1)\backslash(\mathbb{CP}^m{\times}0)$ defined by $z \mapsto ([z], z)$ induces a biholomorphic map from $S_p\backslash 0 \xrightarrow{\sim} E_{\varphi^{-1}(p)}\backslash(\varphi^{-1}(p){\times}0)$ for each $p \in X$, which is the inverse map of

$$\mathrm{pr}\colon E\backslash(Y{\times}0) \to S\backslash(X{\times}0)$$

(in this respect, recall the construction of $\varphi{=}\varphi_{\mathscr{S}}$ in [Rie71, § 2]).

Let A denote the singular locus of $\mathscr{S}$, i. e. the set where $\mathscr{S}$ is not locally free (and S is not a line bundle), and set $B := \varphi^{-1}(A)$. Since

φ is biholomorphic outside of B, $\mathrm{pr} = (\varphi, \mathrm{id}_{\mathbb{C}^{m+1}})|_E$ is already a biholomorphic map on the complement of B and A:

$$\mathrm{pr}\colon E \backslash (B \times 0) \xrightarrow{\sim} S \backslash (A \times 0). \tag{4.18}$$

Since $A \times 0$ is at least of codimension 2 in S and S is Cohen-Macaulay, every holomorphic function on $S \backslash (A \times 0)$ extends to S (see Cor. 5.9 in [PR94]). Hence, for all open sets $U \subset X$, we get

$$\begin{aligned}(\varphi_* \varphi^T \mathscr{S})(U) &\overset{\text{def}}{=} (\varphi^T \mathscr{S})(\varphi^{-1}(U)) \cong \mathrm{Hom}(E_{\varphi^{-1}(U)}, \varphi^{-1}(U) \times \mathbb{C}) \\ &\cong \mathrm{Hom}(S_U, U \times \mathbb{C}) \cong \mathscr{S}(U),\end{aligned}$$

where the second and the last isomorphism are given by the construction of the linear spaces associated to $\varphi^T \mathscr{S}$ and $\mathscr{S}$, resp. □

The following two remarks show that the assumption on normality is just implicit:

Remark 4.19. For the irreducible Cohen-Macaulay space S of rank 1, we get that S is normal if and only if $S \backslash (A \times 0)$ is normal (recall that a Cohen-Macaulay space is normal if and only if its singular locus is at least of codimension 2). Since $E \backslash (B \times 0) \xrightarrow{\sim} S \backslash (A \times 0)$ (4.18), and since E is a vector bundle, this is furthermore equivalent to Y being normal.

Remark 4.20. Let $\mathscr{S}$ be a coherent analytic sheaf of rank 1 on a complex manifold M such that the linear space $S = L(\mathscr{S})$ is Cohen-Macaulay and irreducible, but *not normal*, and let $\varphi : Y \to M$ be the monoidal transformation of M with respect to $\mathscr{S}$. We obtain with Theorem 4.17:

$$\mathscr{S} \otimes \mathscr{K}_M \cong \varphi_*(\varphi^T \mathscr{S} \otimes \varphi^* \mathscr{K}_M).$$

The sheaf $\mathscr{E} := \varphi^T \mathscr{S} \otimes \varphi^* \mathscr{K}_M$ is locally free. For our purpose of generalizing Takegoshi's vanishing theorem (in Chapter 8), we need a proper modification $\pi\colon Z \to Y$ and a locally free sheaf $\widetilde{\mathscr{E}}$ with $\mathscr{E} \cong \pi_*(\widetilde{\mathscr{E}} \otimes \mathscr{K}_Z)$. With the projection formula for locally free sheaves and a normalization, we can assume that Z is normal. Hence, we can apply the following theorem, which maintains a contradiction to the assumption that S and, hence, Y are not normal (see Remark 4.19). This means that the assumption on normality of $L(\mathscr{S})$ is necessary for a generalization of Takegoshi's vanishing theorem by use of a monoidal transformation.

Theorem 4.21. *If $\mathscr{E}$ is a locally free sheaf with positive rank on a locally irreducible complex space Y such that there exist a proper modification $\pi\colon Z \to Y$ with normal Z and a coherent analytic sheaf $\mathscr{F}$ with $\pi_*\mathscr{F} \cong \mathscr{E}$, then Y is normal.*

Proof: Let $\psi\colon \widehat{Y} \to Y$ be a normalization of Y. Since Z is normal, π factorizes over the normalization, i. e. $\exists\, \widehat{\pi}\colon Z \to \widehat{Y}$ with $\pi = \psi \circ \widehat{\pi}$ (see e. g. Sect. 8.4.3 in [GR84]). Therefore, $\mathscr{E} \cong \psi_*\widehat{\pi}_*\mathscr{F}$. For $\widehat{\mathscr{F}} := \widehat{\pi}_*\mathscr{F}$, Theorem 4.9 implies

$$\begin{aligned} \mathscr{E} &\cong \psi_*\widehat{\mathscr{F}} \cong \psi_*\psi^T\psi_*\widehat{\mathscr{F}} \cong \psi_*\psi^T\mathscr{E} \\ &= \psi_*(\psi^*\mathscr{E}\otimes\mathcal{O}_{\widehat{Y}}) \overset{(4.11)}{\cong} \mathscr{E}\otimes\psi_*\mathcal{O}_{\widehat{Y}} \cong \mathscr{E}\otimes\widehat{\mathcal{O}}_Y. \end{aligned}$$

Since $\mathscr{E}$ is locally free of positive rank, we obtain $\widehat{\mathcal{O}}_Y \cong \mathcal{O}_Y$, i. e. Y is normal. $\square$

We deduce immediately:

Corollary 4.22. *The Grauert-Riemenschneider canonical sheaf on a non-normal locally irreducible complex space is not locally free.*

4.3 Proof of Theorem 4.1

In preparation, we make the following observation:

Lemma 4.23. *Let $\pi\colon Y \to X$ be a proper modification of a complex space X such that Y is locally irreducible. Let*

$$\psi\colon \mathscr{F} \to \mathscr{G}$$

be a morphism of coherent analytic sheaves. If ψ is an epimorphism, then the induced mapping

$$\pi^T\psi\colon \pi^T\mathscr{F} \to \pi^T\mathscr{G}$$

is also an epimorphism. If ψ is a monomorphism, then π^T is also a monomorphism.

Note that π^T is not exact. A simple counterexample is the following. Let $\mathfrak{m}_0$ be the maximal ideal sheaf of the origin in $\mathbb{C}^2$. Then,

$$0 \to \mathfrak{m}_0 \hookrightarrow \mathcal{O}_{\mathbb{C}^2} \to \mathcal{O}_{\mathbb{C}^2}/\mathfrak{m}_0 \to 0$$

is exact. Let π be just the identity on $\mathbb{C}^2$. So, we have $\pi^T\mathfrak{m}_0 = \mathfrak{m}_0$, $\pi^T\mathcal{O}_{\mathbb{C}^2} = \mathcal{O}_{\mathbb{C}^2}$ and $\pi^T\big(\mathcal{O}_{\mathbb{C}^2}/\mathfrak{m}_0\big) = 0$. The resulting sequence $0 \to \mathfrak{m}_0 \hookrightarrow \mathcal{O}_{\mathbb{C}^2} \to 0$ is clearly not exact.

Proof: Let ψ be an epimorphism, i. e. surjective. Recall that π^* is right-exact. So, $\pi^*\psi\colon \pi^*\mathscr{F} \to \pi^*\mathscr{G}$ is still surjective. But then, it is easy to see that the induced mapping $\pi^T\psi\colon \pi^T\mathscr{F} \to \pi^T\mathscr{G}$ is also surjective.

To prove the second statement in the lemma, let ψ be injective. Let $f_x \in (\pi^T\mathscr{F})_x$ such that $\pi^T\psi(f_x) = 0$. This means that there is an open set $U \subset Y$ and a representative $f \in \pi^T\mathscr{F}(U)$ such that $\pi^T\psi(f) = 0$. But, $\pi^T\psi$ is injective on a dense open subset $W \subset X$. Thus, $f = 0$ on $U \cap W$, i. e. f has support on a thin set. Since $\pi^T\mathscr{F}$ is torsion-free, we get $f_x = 0$ and $f = 0$. □

Proof of Theorem 4.1: Let $\pi : Y \to X$ be a proper modification between locally irreducible complex spaces. Let $\mathscr{F}$ and $\mathscr{G}$ be coherent analytic sheaves on X and Y, respectively.

(i) The case $\mathscr{F} = \pi_*\mathscr{G}$: By Theorem 4.12 (i), the natural map $\mathscr{F} \to \pi_*\pi^T\mathscr{F}$ is injective. Moreover, Lemma 4.4 yields injectivity of the natural map

$$\pi^T\mathscr{F} = \pi^T\pi_*\mathscr{G} \to \mathscr{G}.$$

Since π_* is left-exact, we obtain the second natural injection

$$\pi_*\pi^T\mathscr{F} \hookrightarrow \pi_*\mathscr{G} = \mathscr{F}.$$

(ii) The case $\mathscr{G} = \pi^T\mathscr{F}$: As above, Lemma 4.4 gives $\pi^T\pi_*\mathscr{G} \hookrightarrow \mathscr{G}$. By Theorem 4.12 (i) we have also the natural injection

$$\mathscr{F} \hookrightarrow \pi_*\pi^T\mathscr{F} = \pi_*\mathscr{G}.$$

But, π^T preserves injectivity (Lemma 4.23) so that we obtain the injection

$$\mathscr{G} = \pi^T\mathscr{F} \hookrightarrow \pi^T\pi_*\mathscr{G}.$$

□

4.4 Holomorphic n-forms on singular spaces

As a consequence of Theorem 4.1 and Theorem 4.8, we have the following application to holomorphic n-forms:

Theorem 4.24. *Let X be a complex space of pure dimension n and $\mathscr{K}_X$ the Grauert-Riemenschneider canonical sheaf on X. Then, there exist a resolution of singularities $\pi : M \to X$ and an effective divisor, $D \geq 0$, with support on the exceptional set of the resolution such that*

$$\pi^T\mathscr{K}_X \cong \Omega^n_M(-D) = \Omega^n_M \otimes \mathcal{O}_M(-D), \tag{4.25}$$

where Ω^n_M is the canonical sheaf of holomorphic n-forms on M, and (4.25) is induced by the natural mapping $\pi^*\mathscr{K}_X = \pi^*\pi_*\Omega^n_M \to \Omega^n_M$. Moreover, we get

$$\pi_*\Omega^n_M = \mathscr{K}_X \cong \pi_*\Omega^n_M(-D).$$

Proof: Let $\pi : M \to X$ be a resolution of singularities such that $\pi^T \mathscr{K}_X$ is locally free. Such a resolution exists due to H. Rossi and H. Hironaka (apply first Rossi's Theorem 2.19 and, then, Hironaka's resolution of singularities). Recall that $\mathscr{K}_X = \pi_* \Omega^n_M$ by definition of the Grauert-Riemenschneider canonical sheaf. So, the assertion follows directly from Theorem 4.8 and Theorem 4.1. □

The following observation is also useful:

Lemma 4.26. *Let $\mathscr{F}$, $\mathscr{G}$ be torsion-free coherent analytic sheaves on a locally irreducible complex space X, and let $\pi : Y \to X$ be a proper modification of X such that $\pi^T \mathscr{G}$ is locally free. Then,*

$$\pi^T \big(\mathscr{F} \otimes \mathscr{G} \big) = \pi^T \mathscr{F} \otimes \pi^T \mathscr{G}$$

and there is a natural injection

$$\mathscr{F} \otimes \mathscr{G} \hookrightarrow \pi_* \big(\pi^T \mathscr{F} \otimes \pi^T \mathscr{G} \big).$$

Proof: Note that Y is also locally irreducible. Consider the two natural surjections $\pi^* \mathscr{F} \to \pi^T \mathscr{F}$ and $\pi^* \mathscr{G} \to \pi^T \mathscr{G}$. These yield a natural surjection

$$\pi^* (\mathscr{F} \otimes \mathscr{G}) = \pi^* \mathscr{F} \otimes \pi^* \mathscr{G} \longrightarrow \pi^T \mathscr{F} \otimes \pi^T \mathscr{G},$$

which is an isomorphism on an open dense subset of Y. Since the tensor product of a torsion-free and a locally free sheaf is torsion-free (this is not the case if both are not locally free, see Remark 2.9), we obtain by use of Lemma 3.14 a natural isomorphism

$$\pi^T \big(\mathscr{F} \otimes \mathscr{G} \big) = \frac{\pi^* (\mathscr{F} \otimes \mathscr{G})}{\mathscr{T} \big(\pi^* (\mathscr{F} \otimes \mathscr{G}) \big)} \xrightarrow{\sim} \pi^T \mathscr{F} \otimes \pi^T \mathscr{G}. \tag{4.27}$$

The second statement follows by taking the direct image of (4.27) under π and Theorem 4.12 (i). □

Using this lemma and the projection formula (see Theorem 4.10), we obtain with an analogue proof the following variant of Theorem 4.24.

Theorem 4.28. *Let X be a complex space of pure dimension n, $\mathscr{F}$ be a torsion-free coherent analytic sheaf on X and $\pi : M \to X$ be a* *resolution of singularities such that $\pi^T(\mathscr{K}_X)$ is locally free. Then, there exists an effective divisor, $D \geq 0$, with support on the exceptional set of the resolution such that*

$$\pi^T(\mathscr{F} \otimes \mathscr{K}_X) \cong \pi^T \mathscr{F} \otimes \Omega^n_M(-D).$$

If $\mathscr{F}$ is locally free, then

$$\pi_*(\pi^* \mathscr{F} \otimes \Omega^n_M) \cong \mathscr{F} \otimes \mathscr{K}_X \cong \pi_*\left(\pi^* \mathscr{F} \otimes \Omega^n_M(-D)\right).$$

4.5 Non-analytic preimages and direct images

In this section, we will finally study non-analytic preimages of direct image sheaves and vice versa. For our purpose, the following definition is useful:

Def. 4.29. Let $\mathscr{F}$ be a sheaf on a complex space X. We say that $\mathscr{F}$ satisfies the property (id) if the following holds: For any irreducible open set $W \subset X$ and sections $s, t \in \mathscr{F}(W)$, the equality $s = t$ on a non-empty open subset of W implies that $s = t$ on W.

Property (id) means that the identity theorem generalizes to sections of $\mathscr{F}$. Actually, the identity theorem for irreducible complex spaces (cf. e. g. § 1.3 in [GR84, Chap. 9]) implies that the structure sheaf $\mathcal{O}_X$ of a complex space satisfies (id). Moreover, we have:

Lemma 4.30. *Let X be a locally irreducible complex space. Then, a coherent analytic sheaf $\mathscr{F}$ on X satisfies the property (id) if and only if it is torsion-free.*

Proof: Let $\mathscr{F}$ be a torsion-free coherent sheaf on X and $F := L(\mathscr{F})$ the associated linear space. Then, by Remark 3.6, a section of $\mathscr{F}$ is uniquely defined by it values on the locally irreducible primary component $E := \mathrm{PC}(F)$ of F. I. e. for $W \subset X$ and $s, t \in \mathscr{F}(W) = \mathrm{Hom}(F_W, W \times \mathbb{C})$, $s|_E = t|_E$ is equivalent to $s = t$. So, the desired property follows by the identity theorem applied to E.

Conversely, it is clear that sheaves with torsion on a locally irreducible space do not satisfy (id). □

For non-coherent sheaves, the equivalence of Lemma 4.30 does not hold in general: The sheaf $\mathscr{C}$ of continuous functions on an irreducible complex space X is torsion-free as $\mathcal{O}_X$-module sheaf, but it does not satisfy (id):

Torsion-freeness: Choose $r_x \in \mathcal{O}_{X,x}$ and $s_x \in \mathscr{C}_{X,x}$ with $r_x \neq 0$ and $r_x \cdot s_x = 0$. We obtain $rs = 0$ on small enough neighbourhoods U of x and $r(y) \neq 0$ outside of a thin analytic set $A \subset U$. This implies $s = 0$ on $U \backslash A$. Since s is continuous, we get $s_x = 0$.

On the other hand, it is obvious that $\mathscr{C}$ does not satisfy (id).

The property (id) is useful in the context of non-analytic preimages:

Lemma 4.31. *Let $\pi\colon Y \to X$ be a proper modification of a locally irreducible complex space X, and $\mathscr{F}$ a sheaf on X satisfying (id). Then, for $U \subset Y$ open:*

$$\pi^{-1}\mathscr{F}(U) = \varinjlim_{V \supset \pi(U)} \mathscr{F}(V), \tag{4.32}$$

where the limit runs over the open neighbourhoods of $\pi(U)$.

Proof: As X is locally irreducible, we can assume that X and Y are connected. Recall that $\pi^{-1}\mathscr{F}$ is the sheaf associated to the presheaf

$$U \mapsto F(U) := \varinjlim_{V \supset \pi(U)} \mathscr{F}(V)$$

where $U \subset Y$ is open and the limit runs over the open neighbourhoods of $\pi(U)$. We have to show that the presheaf F is canonical (i. e. it is already a sheaf).

i) Existence/Gluing-axiom: Let $U \subset Y$ be covered by open sets U_i, $i \in I$, and let $s_i \in F(U_i)$ satisfy $s_i = s_j$ on $U_{ij} := U_i \cap U_j$. By definition of the inductive limit, $s_i \in F(U_i)$ means there are an open set $V_i \supset \pi(U_i)$ and a section $f_i \in \mathscr{F}(V_i)$ with $s_i = [f_i]$ (s_i is represented by f_i). A priori, we just get $f_i = f_j$ on $\pi(U_{ij}) \subset V_{ij}$, where

$V_{ij} = V_i \cap V_j$. Without loss of generality, we can assume that each connected component of V_{ij} contains an open subset of $\pi(U_{ij})$ (π is a modification). So, (id) for $\mathscr{F}$ implies that $f_i = f_j$ on V_{ij}. As $\mathscr{F}$ is a sheaf, there is a section $f \in \mathscr{F}(V)$ with $f|_{V_i} = f_i$, where $V := \bigcup_{i\in I} V_i \supset \pi(U)$. f represents an $s \in F(U)$ with $s|_{U_i} = s_i$.

ii) Uniqueness-axiom: Let $U \subset Y$ be a connected open set, covered by open sets U_i, $i \in I$, and let $s, t \in F(U)$ satisfy $s = t$ on U_i for all $i \in I$. By definition of the inductive limit, there are a connected open set $V \supset \pi(U)$ and sections $f, g \in \mathscr{F}(V)$ with $s = [f]$ and $t = [g]$. We get $f = g$ on $\pi(U_i)$. Since $\pi(U_i)$ contains an open subset of V, (id) implies that $f = g$ on V. (We have not directly used the uniqueness-axiom for $\mathscr{F}$ because it is contained in (id)). □

As a special case, we have:

Lemma 4.33. *Let $\pi : Y \to X$ be a proper modification of a locally irreducible complex space X and $\mathscr{G}$ a sheaf on Y satisfying (id). Then, for $U \subset Y$ open:*

$$\pi^{-1}\pi_*\mathscr{G}(U) = \varinjlim_{V\supset\pi(U)} \mathscr{G}(\pi^{-1}(V)), \tag{4.34}$$

where the limit runs over the open neighbourhoods of $\pi(U)$. Furthermore, the canonical homomorphism $\pi^{-1}\pi_*\mathscr{G} \to \mathscr{G}$ is injective so that $\pi^{-1}\pi_*\mathscr{G}$ is a subsheaf of $\mathscr{G}$.

Proof: As X is locally irreducible, we can assume that X and Y are connected. Here, $\pi^{-1}\pi_*\mathscr{G}$ is the sheaf associated to the presheaf

$$U \mapsto F(U) := \varinjlim_{V\supset\pi(U)} \mathscr{G}(\pi^{-1}(V))$$

where $U \subset Y$ is open and the limit runs over the open neighbourhoods of $\pi(U)$. The canonical homomorphism $\pi^{-1}\pi_*\mathscr{G} \to \mathscr{G}$ is, then, induced by the restrictions $\cdot|_U : \mathscr{G}(\pi^{-1}(V)) \to \mathscr{G}(U)$.

By Lemma 4.31, F is canonical, i.e. (4.34) holds. It is now easy to see that the canonical homomorphism $\psi\colon \pi^{-1}\pi_*\mathscr{G} \to \mathscr{G}$ is injective: Let $s_x \in \left(\pi^{-1}\pi_*\mathscr{G}\right)_x$. Then, (4.34) implies that s_x is represented by a section $s \in \mathscr{G}(U)$, where U is an open neighbourhood of $K_x := \pi^{-1}(\pi(x))$. But, our assumptions yield that K_x is connected and so we can assume that U is a connected neighbourhood of K_x. Assume that $\psi(s_x) = 0$. This means that s is vanishing on a neighbourhood of the point x. But then, $s = 0$ as U is connected (and $\mathscr{G}$ satisfies (id)). $\square$

Lemma 4.33 allows for the following interpretation of $\pi^{-1}\pi_*\mathscr{G}$: The sections of $\pi^{-1}\pi_*\mathscr{G}$ are the sections of $\mathscr{G}$ which extend along fibres of the modification π. This is of particular interest for the choices $\mathscr{G} = \mathcal{O}_M$ or $\mathscr{G} = \Omega_M^n$ when $\pi\colon M \to X$ is a resolution of singularities, giving the useful injections $\pi^{-1}\pi_*\mathcal{O}_M \hookrightarrow \mathcal{O}_M$ and $\pi^{-1}\pi_*\Omega_M^n \hookrightarrow \Omega_M^n$, respectively.

Let $\mathscr{F}$ be a sheaf on an irreducible complex space X satisfying (id), let π be a proper modification of X, and let U be an open subset of X. For the direct image of a non-analytic inverse image sheaf, Lemma 4.31 implies

$$(\pi_*\pi^{-1}\mathscr{F})(U) = (\pi^{-1}\mathscr{F})(\pi^{-1}(U)) = \varinjlim_{V \supset U} \mathscr{F}(V) = \mathscr{F}(U),$$

i. e. we obtain:

Corollary 4.35. *Let $\pi\colon Y \to X$ be a proper modification of a locally irreducible complex space X, and $\mathscr{F}$ a sheaf on X satisfying (id). Then,*

$$\pi_*\pi^{-1}\mathscr{F} = \mathscr{F}.$$

Chapter 5
The Dolbeault Operator on Complex Spaces

In this chapter, we introduce the Dolbeault operator on complex spaces and different types of extensions for its L^2-version (see e. g. Section 5.2). Additionally, we define the Dolbeault operator as sheaf homomorphism between $L^{2,\mathrm{loc}}$-spaces and the sheaf $\mathcal{K}_X(F)$ of holomorphic square-integrable n-forms with values in a vector bundle F, which will be a key ingredient to understand L^2-Dolbeault cohomologies on complex spaces (see Chapter 10). In Section 5.3, we recall an extension theorem for the Dolbeault operator on manifolds.

Def. 5.1. A *Hermitian complex space* (X, γ) is a reduced complex space X with a Hermitian metric γ on the regular part X_{reg} such that the following holds: If $x \in X_{\mathrm{sing}}$ is an arbitrary point, there exist a neighbourhood $U = U(x) \subset X$, a holomorphic embedding of U into a domain D in $\mathbb{C}^N$ and an ordinary smooth Hermitian metric in D whose restriction to U is $\gamma|_U$.

Note that two Hermitian metrics on X are equivalent on all compact subsets of X.

Let (X, γ) be a Hermitian complex space. Let $A \subset X$ be an analytic set which contains the singular set of X, i. e. $X_{\mathrm{sing}} \subset A$. Let $F \to X'$ be a Hermitian vector bundle on $X' := X \backslash A \subset X_{\mathrm{reg}}$. Then, the set of smooth forms (with compact support) with values in F is well-defined:

$$\mathscr{E}^{p,q}(X', F) := \mathscr{C}^\infty(X', \Lambda^{p,q} T^* X' \otimes F),$$
$$\mathscr{D}^{p,q}(X', F) := \mathscr{C}^\infty_{\mathrm{cpt}}(X', \Lambda^{p,q} T^* X' \otimes F).$$

Further, the space of square-integrable (p,q)-forms with values in F on X is denoted $L^2_{p,q}(X,F) := L^2_{p,q}(X',F)$. This definition is independent of A since square-integrable forms extend over analytic sets.

The Dolbeault operator $\overline{\partial}$ is well-defined for differential forms with values in a holomorphic vector bundle since the transition functions commute with $\overline{\partial}$. Let $\overline{\partial}_{\text{cpt}}\colon \mathscr{D}^{p,q}(X',F) \to \mathscr{D}^{p,q+1}(X',F)$ denote the Dolbeault operator on differential forms with values in F and compact support. Then, $\overline{\partial}_{\text{cpt}}$ is a densely defined operator on $L^2_{p,q}(X,F)$, which is (in general) not closed. Let $\overline{\partial}_s\colon L^2_{p,q}(X,F) \to L^2_{p,q+1}(X,F)$ denote the minimal closed extension of $\overline{\partial}_{\text{cpt}}$ which is given by the closure of the graph

$$\Gamma_{\overline{\partial}_{\text{cpt}}} := \{(u, \overline{\partial}_{\text{cpt}} u)\colon u \in \mathscr{D}^{p,q}(X',F)\} \subset L^2_{p,q}(X,F) \times L^2_{p,q+1}(X,F).$$

Let $\overline{\partial}_w\colon L^2_{p,q}(X,F) \to L^2_{p,q+1}(X,F)$ denote the maximal extension of $\overline{\partial}_{\text{cpt}}$, i.e. $\overline{\partial}_w$ is defined in the sense of distribution:

$$(u \in \operatorname{dom} \overline{\partial}_w \text{ and } \overline{\partial}_w u = v) :\Longleftrightarrow$$
$$\int_{X'} u \wedge \overline{\partial}\alpha = (-1)^{p+q+1} \int_{X'} v \wedge \alpha \quad \forall\, \alpha \in \mathscr{D}^{n-p,n-q-1}(X', F^*).$$

In Remark 5.10, we will see that the definitions of $\overline{\partial}_w$ and $\overline{\partial}_s$ are independent of the choice of A (as long as $X_{\text{sing}} \subset A$ and F is defined on $X \setminus A$).

The extensions $\overline{\partial}_s$ and $\overline{\partial}_w$ correspond to some kind of boundary conditions on A and at the boundary of X or at infinity of X, respectively. If there is no boundary condition made, then we obtain $\overline{\partial}_w$. It is also called the weak extension and it is the closed (L^2-) extension with the largest domain of definition. If all boundary conditions are made, then we get $\overline{\partial}_s$, which is also called the strong extension. More precisely, $u \in \operatorname{dom} \overline{\partial}_s$ if and only if there exits a sequence $\{u_j\} \subset \mathscr{D}^{p,q}(X',F)$ (in particular, $\operatorname{supp} u_j \cap \partial X' = \emptyset$) such that

$$u_j \to u \text{ and } \overline{\partial}_w u_j \to \overline{\partial}_w u \text{ in } L^2.$$

$\overline{\partial}_s$ is the closed extension with the smallest domain of definition. In Section 5.2 and Section 6.1, we also study other extensions which are needed in Section 6.4.

We use the following notations of Dolbeault cohomology:
$H^{p,q}_s(X,F) := \ker\big(\overline{\partial}_s\colon L^2_{p,q}(X,F)\to L^2_{p,q+1}(X,F)\big)/\overline{\partial}_s(L^2_{p,q-1}(X,F))$,
$H^{p,q}_w(X,F) := \ker\big(\overline{\partial}_w\colon L^2_{p,q}(X,F)\to L^2_{p,q+1}(X,F)\big)/\overline{\partial}_w(L^2_{p,q-1}(X,F))$.
Let $h^{p,q}_e(X',F)$ denote the dimension of $H^{p,q}_e(X',F)$ for $e=w$ or $e=s$. Since $\overline{\partial}_w$ and $\overline{\partial}_s$ are independent of A, we get that the cohomologies are independent, as well. A priori, they depend on the metric γ and the Hermitian metric of F.

Let $\vartheta_{\text{cpt}}\colon \mathscr{D}^{p,q+1}(X',F)\to\mathscr{D}^{p,q}(X',F)$ be the formal adjoint of $\overline{\partial}_{\text{cpt}}$ and $\vartheta_s := \overline{\partial}_w^*$ and $\vartheta_w := \overline{\partial}_s^*$ the Hilbert-space adjoints of $\overline{\partial}_w$ and $\overline{\partial}_s$, respectively. This notation makes sense as $\vartheta_{w/s}$ is in fact the maximal (weak) or minimal (strong), respectively, L^2-extension of ϑ_{cpt}. Let $\overline{*}\colon L^2_{p,q}(X,F)\to L^2_{n-p,n-q}(X,F^*)$ be the conjugated Hodge-$*$-operator with respect to the metric γ of X and the Hermitian metric of F defined by

$$\langle u,v\rangle_{\gamma,F}\,dV = u\wedge\overline{*}\,v \quad\text{for } u,v\in L^2_{p,q}(X,F),$$

where $dV=dV_\gamma$ denotes the volume form with respect to γ. Then, we have

$$\vartheta_e = -\,\overline{*}\,\overline{\partial}_e\,\overline{*} \tag{5.2}$$

for each $e\in\{\text{cpt},w,s\}$.

5.1 Local version of the weak extension, definition of $\mathcal{K}_X(F)$

In this section, we introduce a local version of the Dolbeault operator and obtain a sheaf homomorphism on the sheaves of locally (square-) integrable forms. Actually, there exists a localized version of the $\overline{\partial}_s$-operator, as well (cf. Section 6.4).

Let X be a Hermitian complex space, $A\supset X_{\text{sing}}$ an analytic set in X and F a Hermitian vector bundle on $X' := X\setminus A$. We define locally square-integrable (p,q)-forms with values in F on an open set $U \subset X$ by

$$L^{2,\mathrm{loc}}_{p,q}(U,F) := \left\{ u \in L^{2,\mathrm{loc}}_{p,q}(U\setminus A, F)\colon u \in L^2_{p,q}(V\setminus A, F)\ \forall\, V \Subset U \right\}.$$

The presheaf $\mathcal{L}^{p,q}_F$ defined by $\mathcal{L}^{p,q}_F(U) := L^{2,\mathrm{loc}}_{p,q}(U,F)$ is canonical, i. e. $\mathcal{L}^{p,q}_F$ is already a sheaf.

Let $\overline{\partial}_{w,\mathrm{loc}}\colon L^{2,\mathrm{loc}}_{p,q}(U,F) \to L^{2,\mathrm{loc}}_{p,q+1}(U,F)$ denote the Dolbeault operator in the sense of distributions on locally square-integrable differential forms with values in F. Then, the extension $\overline{\partial}_{w,\mathrm{loc}}$ is locally defined, i. e. for open $V \subset U$, $\overline{\partial}_{w,\mathrm{loc}}\colon L^{2,\mathrm{loc}}_{p,q}(V,F) \to L^{2,\mathrm{loc}}_{p,q+1}(V,F)$ is given by the restriction of $\overline{\partial}_{w,\mathrm{loc}}\colon L^{2,\mathrm{loc}}_{p,q}(U,F) \to L^{2,\mathrm{loc}}_{p,q+1}(U,F)$. In particular, the presheaf

$$\mathcal{C}^{p,q}_F(U) = \mathcal{C}^{p,q}(U,F) := \mathrm{dom}\left(\overline{\partial}_{w,\mathrm{loc}}\colon L^{2,\mathrm{loc}}_{p,q}(U,F) \to L^{2,\mathrm{loc}}_{p,q+1}(U,F)\right)$$

defines a sheaf and $\overline{\partial}_{w,\mathrm{loc}}$ defines a sheaf homomorphism $\mathcal{C}^{p,q}_F \to \mathcal{C}^{p,q+1}_F$. Since $\mathcal{C}^{p,q}_F$ admits a partition of the unity, it is fine. This implies that it is acyclic, i. e. $H^r(U,\mathcal{C}^{p,q}_F) = 0$ for all $r \geq 1$.

We define

$$\mathcal{K}_X(F) := \mathscr{K}\!er(\overline{\partial}_{w,\mathrm{loc}}\colon \mathcal{C}^{n,0}_F \to \mathcal{C}^{n,1}_F)$$

and obtain the following complex of sheaves

$$0 \to \mathcal{K}_X(F) \to \mathcal{C}^{n,0}_F \to \mathcal{C}^{n,1}_F \to \mathcal{C}^{n,2}_F \to \dots \tag{5.3}$$

Under some additional assumptions, (5.3) is a fine resolution of $\mathcal{K}_F$ (see Theorem 10.5). Hence, we can use it to compute the Dolbeault cohomology of locally square-integrable differential forms with values in F which is defined as follows:

$$H^{p,q}_{w,\mathrm{loc}}(X,F) := \frac{\ker(\overline{\partial}_{w,\mathrm{loc}}\colon L^{2,\mathrm{loc}}_{p,q}(X,F) \to L^2_{p,q+1}(X,F))}{\overline{\partial}_{w,\mathrm{loc}}(L^{2,\mathrm{loc}}_{p,q-1}(X,F))}.$$

This does not depend on A and the Hermitian metric γ of X. If the Hermitian metric of F does not extend over A, then $H^{p,q}_{w,\mathrm{loc}}(X,F)$ does not coincide with $H^{p,q}_{w,\mathrm{loc}}(X',F)$ which is usually defined as

$$\ker\left(\overline{\partial}_{w,\mathrm{loc}}\colon L^{2,\mathrm{loc}}_{p,q}(X',F) \to L^2_{p,q+1}(X',F)\right)/\overline{\partial}_{w,\mathrm{loc}}(L^{2,\mathrm{loc}}_{p,q-1}(X',F)).$$

Furthermore, $H^{p,q}_{w,\mathrm{loc}}(X,F)$ does depend on the Hermitian metric of F unless the metric extends over A.

This motivates the following definitions.

Def. 5.4. Let $\mathscr{S}$ be a coherent analytic sheaf on a complex space X, let $S := L(\mathscr{S})$ denote the associated linear space. For each $x \in X$, there exist a neighbourhood $U = U(x) \subset X$ and an embedding of S_U in $U \times \mathbb{C}^{N(x)}$. We call $h = \{h_x\}_{x \in X}$ a *(smooth) Hermitian form on* S if all h_x are Hermitian forms on S_x and, for an open covering $\{U_j\}$ of X with embeddings $S_{U_j} \subset U_j \times \mathbb{C}^{N_j}$, there exist smooth Hermitian forms h_j on $U_j \times \mathbb{C}^{N_j}$ with $h_j|_{S_{U_j}} = h$. We call $\mathscr{S}$ *Hermitian* if there exits a smooth Hermitian metric h on S. The dual g of h induces a Hermitian metric on the vector bundle $R(\mathscr{S})$. We say *g is induced by the Hermitian structure of $\mathscr{S}$.*

Recall that $R(\mathscr{S})$ denotes the vector bundle on $X' := X \setminus \operatorname{Sing} \mathscr{S}$ associated to the locally free sheaf $\mathscr{S}_{X'}$, i. e. $R(\mathscr{S}) = L(\mathscr{S}^*_{X'})$ (see Section 3.2).

The construction of a Hermitian metric on an arbitrary vector bundle generalizes to linear spaces, i. e. all coherent analytic sheaves are Hermitian. Yet, not every Hermitian metric on $R(\mathscr{S})$ is induced by a Hermitian structure of $\mathscr{S}$. For us, this special kind of metrics are of interest because of the following fact: If g_1, g_2 are two Hermitian metrics induced by two Hermitian structures of $\mathscr{S}$, then g_1 and g_2 are equivalent on $K \setminus \operatorname{Sing} \mathscr{S}$ for all compact subsets K of X. In particular, $L^{2,\mathrm{loc}}_{p,q}(U, R(\mathscr{S}))$ and $\overline{\partial}_{w,\mathrm{loc}}$ are independent of it.

For a Hermitian sheaf $\mathscr{S}$, we set

$$\mathcal{K}_X(\mathscr{S}) := \mathcal{K}_X(R(\mathscr{S})).$$

We will see that (5.3) is a fine resolution of $\mathcal{K}_X(\mathscr{S})$ (see Theorem 10.5 with independence of the choice of the Hermitian metric on $\mathscr{S}$). Therefore, we can use it to compute the Dolbeault cohomology of locally square-integrable differential forms with values in $\mathscr{S}$

$$H^{p,q}_{w,\mathrm{loc}}(X, \mathscr{S}) := H^{p,q}_{w,\mathrm{loc}}(X, R(\mathscr{S}))$$

which is independent of A, γ and the Hermitian structure of $\mathscr{S}$.

5.2 The extensions $\overline{\partial}_{w,s}$ and $\overline{\partial}_{s,w}$

In Section 6.1, we will study the following closed extensions of $\overline{\partial}_{\mathrm{cpt}}$, which are different from the minimal $\overline{\partial}_s$ and the maximal $\overline{\partial}_w$. Let $X \Subset Y$ be a relative-compact domain in a Hermitian complex space Y. We can interpret $\overline{\partial}_s$ as $\overline{\partial}_w$ with certain boundary conditions. The boundary of $X^* := X \backslash Y_{\mathrm{sing}}$ consists of two parts, the singular set X_{sing} and the boundary at ∂X, i. e. $\partial X^* = \partial X \cup X_{\mathrm{sing}}$. Therefore, we have to deal with two kinds of boundary conditions. Let $\overline{\partial}_{s,w}$ denote the closed L^2-extension which satisfies the boundary condition at X_{sing}, i. e.

$$\operatorname{dom} \overline{\partial}_{s,w} := \{u \in \operatorname{dom} \overline{\partial}_w \colon \exists \{u_j\} \subset \operatorname{dom} \overline{\partial}_w \text{ with } X_{\mathrm{sing}} \cap \operatorname{supp} u_j = \emptyset, \\ u_j \to u \;\&\; \overline{\partial}_w u_j \to \overline{\partial}_w u \text{ in } L^2\}.$$

$\overline{\partial}_{w,s}$ denotes the extension which satisfies the boundary condition at ∂X, i. e.

$$\operatorname{dom} \overline{\partial}_{s,w} := \{u \in \operatorname{dom} \overline{\partial}_w \colon \exists \{u_j\} \subset \operatorname{dom} \overline{\partial}_w \text{ with } \partial X \cap \operatorname{supp} u_j = \emptyset, \\ u_j \to u \;\&\; \overline{\partial}_w u_j \to \overline{\partial}_w u \text{ in } L^2\}.$$

Here, we refer to the support with respect to the ambient space Y. The first index represents the inner boundary X_{sing}, the second the outer boundary ∂X. $H^{*,*}_{s,w}$ and $H^{*,*}_{w,s}$ denote the cohomologies with respect to $\overline{\partial}_{s,w}$ and $\overline{\partial}_{w,s}$ respectively.

We define the formal adjoint operators (cf. (5.2))

$$\vartheta_{s,w} := -\overline{*}\,\overline{\partial}_{s,w}\,\overline{*} \quad \text{and} \quad \vartheta_{w,s} := -\overline{*}\,\overline{\partial}_{w,s}\,\overline{*},$$

which we can realize as Hilbert-space adjoint operators in special cases (cf. [Rup14a, Lem. 5.1]):

Lemma 5.5. *If ∂X does not intersect Y_{sing}, the Hilbert-space adjoints $\overline{\partial}^*_{s,w}$ and $\overline{\partial}^*_{w,s}$ satisfy the representations $\overline{\partial}^*_{s,w} = \vartheta_{w,s} = -\overline{*}\,\overline{\partial}_{w,s}\,\overline{*}$ and $\overline{\partial}^*_{w,s} = \vartheta_{s,w} = -\overline{*}\,\overline{\partial}_{s,w}\,\overline{*}$, respectively.*

Proof: We will show that $\operatorname{dom} \overline{\partial}^*_{s,w} = \operatorname{dom} \vartheta_{w,s}$. The other identity $\operatorname{dom} \overline{\partial}^*_{w,s} = \operatorname{dom} \vartheta_{s,w}$ can be shown in the same way.

(i) $\operatorname{dom} \vartheta_{w,s} \subset \operatorname{dom} \overline{\partial}^*_{s,w}$: Choose a $u \in \operatorname{dom} \vartheta_{w,s}$, i. e. there is a sequence $u_j \in \operatorname{dom}\vartheta_w$ with $u_j \to u$, $\vartheta u_j \to \vartheta u$, and $\operatorname{supp} u_j \cap \partial X = \emptyset$. We have to prove

$$(u, \overline{\partial} v) \lesssim \|v\| \ \forall v \in \operatorname{dom}\overline{\partial}_{s,w}.$$

Let v be in $\operatorname{dom}\overline{\partial}_{s,w}$ and v_j a sequence associated to v in $\operatorname{dom}\overline{\partial}_w$, i. e. $v_j \to v$, $\overline{\partial} v_j \to \overline{\partial} v$, and $X_{\text{sing}} \cap \operatorname{supp} v_j = \emptyset$. Since a product of u_j and v_j has compact support in X^* (using $\partial X \cap Y_{\text{sing}} = \emptyset$), integration by parts implies

$$(u_j, \overline{\partial} v_j) = \int u_j \wedge \overline{*}\, \overline{\partial} v_j = -\int u_j \wedge \vartheta\, \overline{*}\, v_j = \int \vartheta u_j \wedge \overline{*}\, v_j = (\vartheta u_j, v_j).$$

Passing to the limit, we get

$$|(u, \overline{\partial} v)| = \lim_{j\to\infty} |(v_j, \vartheta u_j)| \le \lim_{j\to\infty} \|v_j\| \cdot \|\vartheta u_j\| = \|v\| \|\vartheta u\|.$$

(ii) $\operatorname{dom}\overline{\partial}^*_{s,w} \subset \operatorname{dom}\vartheta_{w,s}$: Let u be in $\operatorname{dom}\overline{\partial}^*_{s,w}$. This implies that $u \in \operatorname{dom}\vartheta_w$, and we have to show that $u \in \operatorname{dom}\vartheta_{w,s}$. For this, we can choose a cut-off function $\chi \in \mathscr{C}^\infty\big(\overline{X}; [0,1]\big)$ with $\chi = 0$ on a neighbourhood $U(\overline{X}_{\text{sing}})$ of the singular set and $\chi = 1$ on a neighbourhood $U(\partial X)$ of the boundary since ∂X does not intersect $\overline{X}_{\text{sing}} \subset Y_{\text{sing}}$. We will show that both, $\chi u \in \operatorname{dom}\vartheta_{w,s}$ and $(1-\chi)u \in \operatorname{dom}\vartheta_{w,s}$, separately.

First, it is clear that $\chi u, (1-\chi)u \in \operatorname{dom}\vartheta_w$. But $(1-\chi)u$ has support away from the boundary ∂X, and so $(1-\chi)u \in \operatorname{dom}\vartheta_{w,s}$.

Let us now consider χu. For each $v \in \operatorname{dom}\overline{\partial}_w$, we have

$$(\chi u, \overline{\partial} v) = (u, \chi \overline{\partial} v) = \big(u, \overline{\partial}(\chi v) - \overline{\partial}\chi \wedge v\big) = (\overline{\partial}^* u, \chi v) - (u, \overline{\partial}\chi \wedge v),$$

where we infer the last equation from the assumption $u \in \operatorname{dom}\overline{\partial}^*_{s,w}$ and $\chi v \in \operatorname{dom}\overline{\partial}_{s,w}$. We conclude

$$|(\chi u, \overline{\partial} v)| \le \|\overline{\partial}^* u\| \|\chi v\| + \|u\| \|\overline{\partial}\chi\| \|v\| \lesssim \|v\|$$

and, hence, χu is in $\operatorname{dom}\overline{\partial}^*_w = \operatorname{dom}\vartheta_s$. Therefore, $\chi u \in \operatorname{dom}\vartheta_s \subset \operatorname{dom}\vartheta_{w,s}$. □

Note that we say that $a(x) \lesssim b(x)$ for functions $a, b : S \to \mathbb{R}$ on a set S if there is a constant $C > 0$ with $a(x) \le C \cdot b(x)$ for all $x \in S$. $a(x) \sim b(x)$ means that $a(x) \lesssim b(x)$ and $b(x) \lesssim a(x)$, i. e. there exists a constant $C > 0$ such that $\frac{1}{C} b(x) \le a(x) \le C \cdot b(x)$ for all $x \in S$.

5.3 L^2-extension theorem

In this section, we recall the following L^2-version of Riemann's extension theorem:

Theorem 5.6. *Let M be a Hermitian manifold, let $A \subset M$ be a thin analytic set, let F be a Hermitian vector bundle on M, and let $P\colon \mathscr{D}^{p,q}(M,F) \to \mathscr{D}^{p',q'}(M,F)$ be a partial differential operator of first order with continuous coefficients. If $u \in L^{2,\mathrm{loc}}_{p,q}(M,F)$ and $v \in L^{1,\mathrm{loc}}_{p',q'}(M,F)$ satisfy $Pu = v$ on $M\backslash A$ in the sense of distributions, then $Pu = v$ in the sense of distribution on M.*

This theorem is a simple generalization of Thm. 3.2 in [Rup09]. We will present the same proof as in the author's diploma thesis [Ser10].

Proof: We want to simplify the problem. We keep in mind that it is a local statement.

The proof is conducted recursively by dividing A in the regular part A_{reg} and singular part A_{sing}. If we assume that the proposition holds for A_{reg} in place of A and $M\backslash A_{\mathrm{sing}}$ in place of M, then it remains to show the proposition for A_{sing} in M. Since A_{sing} is an analytic set in M with smaller dimension than A, we can repeat the argument by separating A_{sing}. Hence, the problem can be reduced to the case that A is regular.

With a partition of unity and a biholomorphic transformation, we can consider $M \Subset \mathbb{C}^n$, F trivial and $A = M \cap (\mathbb{C}^m \times 0)$ since A is regular. In addition, we can see P as a matrix if we interpret the (p,q)-forms and (p',q')-forms as vector spaces. One entry of P looks like $P_{ij} = \sum_{k=1}^{2n} \alpha_k \frac{\partial}{\partial x_k} + \beta$ if we set $z_j = x_j + \mathrm{i}x_{n+j}$.

It remains to show that, if $\alpha \in \mathscr{C}^0(M)$, $f \in L^{2,\mathrm{loc}}(M)$ and $g \in L^{1,\mathrm{loc}}(M)$ satisfy

$$\int_{M\backslash A} f\alpha \tfrac{\partial\varphi}{\partial x_k} dV = -\int_{M\backslash A} g\varphi dV \quad \forall \varphi \in \mathscr{D}(M\backslash A), \tag{5.7}$$

then this equation (5.7) holds for all $\varphi \in \mathscr{D}(M)$.

Let φ be in $\mathscr{D}(M)$, and let d be the diameter of M. We choose $\chi \in \mathscr{D}(\mathbb{R}_0^+, [0,1])$ such that $\chi(x) = 1$ for $x \leq \frac{1}{4}$, $\chi(x) = 0$ for $x \geq \frac{3}{4}$ and $\chi' \leq C$. We set $\chi_\delta \colon M \to \mathbb{R}, \chi_\delta(z', z'') := \chi\big(\frac{\|z''\|}{\delta}\big)$ with $z' = (z_1, ..., z_m)$ and $z'' = (z_{m+1}, ..., z_n)$. Then, we get $\varphi_\delta := \varphi \cdot (1 - \chi_\delta) \in \mathscr{D}(M \backslash A)$ and $\operatorname{supp} \chi_\delta \varphi \subset U_\delta := U_\delta(A) = \{(z', z'') \in M \colon \|z''\| < \delta\}$. We infer from (5.7) that

$$\begin{aligned} \left| \int_M f\alpha \tfrac{\partial \varphi}{\partial x_k} + g\varphi \; dV \right| &\leq \left| \int_M f\alpha \tfrac{\partial \varphi_\delta}{\partial x_k} + g\varphi_\delta \, dV \right| + \int_M \left| f\alpha \tfrac{\partial(\chi_\delta \varphi)}{\partial x_k} + g\chi_\delta \varphi \right| dV \\ &= \int_M \left| f\alpha \tfrac{\partial(\chi_\delta \varphi)}{\partial x_k} + g\chi_\delta \varphi \right| dV \\ &\leq \int_M \left| f\alpha\varphi \tfrac{\partial \chi_\delta}{\partial x_k} \right| + \chi_\delta \left| f\alpha \tfrac{\partial \varphi}{\partial x_k} \right| + \chi_\delta \, |g\varphi| \;\; dV. \end{aligned}$$

With $\chi_\delta \, |g\varphi| \leq |g\varphi|$, we conclude $\chi_\delta \, |g\varphi| \in L^{1,\mathrm{loc}}(M)$ and

$$\int_M \chi_\delta \, |g\varphi| \, dV \leq \int_{U_\delta} |g\varphi| \, dV \to 0 \;\; \text{if } \delta \to 0$$

(cf. Lem. A1.17 in [Alt06]). Analogously, we get

$$\int_M \chi_\delta \left| f\alpha \tfrac{\partial \varphi}{\partial x_k} \right| dV \to 0 \;\; \text{if } \delta \to 0.$$

Hölder's inequality implies

$$\begin{aligned} \int_M \left| f\alpha\varphi \tfrac{\partial \chi_\delta}{\partial x_k} \right| dV &= \int_{U_\delta} |f\alpha\varphi| \left| \tfrac{\partial \chi_\delta}{\partial x_k} \right| dV \\ &\leq \left(\int_{U_\delta} |f\alpha\varphi|^2 \, dV \cdot \int_{U_\delta} \left| \tfrac{\partial \chi_\delta}{\partial x_k} \right|^2 dV \right)^{\frac{1}{2}}. \end{aligned}$$

With $\left| \frac{\partial \chi_\delta}{\partial x_k} \right| \leq \frac{C}{\delta}$, we can estimate

$$\int_{U_\delta} \left| \tfrac{\partial \chi_\delta}{\partial x_k} \right|^2 dV \leq \int_{U_\delta} \tfrac{C^2}{\delta^2} dV = \operatorname{vol}(U_\delta) \cdot \tfrac{C^2}{\delta^2} \leq d^{2m} (2\delta)^{2(n-m)} \cdot \tfrac{C^2}{\delta^2}.$$

Since $m < n$, we deduce that $\int_{U_\delta} \left| \frac{\partial \chi_\delta}{\partial x_k} \right|^2 dV$ is bounded and

$$\int_M \left| f\alpha\varphi \tfrac{\partial \chi_\delta}{\partial x_k} \right| dV \lesssim \left(\int_{U_\delta} |f\alpha\varphi|^2 \, dV \right)^{\frac{1}{2}} \to 0 \;\; \text{if } \delta \to 0.$$

□

Corollary 5.8. *Let M be a Hermitian manifold, and let F be a Hermitian vector bundle on M. Then,*

$$\overline{\partial}_s^{M \backslash A} := \overline{\partial}_s \colon L^2_{p,q}(M \backslash A, F_{M \backslash A}) \to L^2_{p,q+1}(M \backslash A, F_{M \backslash A})$$

coincides with

$$\overline{\partial}_s^M := \overline{\partial}_s \colon L^2_{p,q}(M,F) \to L^2_{p,q+1}(M,F).$$

More precisely, if $u \in \operatorname{dom} \overline{\partial}_s^M$*, then there exits a sequence* $\{u_j\}$ *of smooth forms with values in* F *and compact support in* $M \backslash A$ *such that* $u_j \to u$ *and* $\overline{\partial}_w u \to \overline{\partial}_w u$ *in* L^2*.*

Proof: We use the notation with the superscript for the adjoint operator ϑ of $\overline{\partial}$, as well, and set $M' := M \backslash A$. Since the Hilbert-space adjoint of $\overline{\partial}_s^{M'}$ is $\vartheta_w^{M'}$, we get

$$\left(\vartheta_w^{M'} v, \varphi\right) = \left(v, \overline{\partial}\varphi\right) \tag{5.9}$$

for all $\varphi \in \mathscr{D}^{p,q}(M', F)$. With Theorem 5.6, we get the equality (5.9) also for all $\varphi \in \mathscr{D}^{p,q}(M,F)$. This means

$$\operatorname{dom} \vartheta_w^{M'} = \operatorname{dom} \vartheta_w^{M}.$$

In particular the Hilbert-space adjoints coincide:

$$\operatorname{dom} \overline{\partial}_s^{M'} = \operatorname{dom} \overline{\partial}_s^{M}.$$

□

Remark 5.10. Let X be a Hermitian complex space, let A, B be analytic sets in X with $X_{\text{sing}} \subset A \subset B$, let F' be a Hermitian vector bundle on $X' := X \backslash A$, and let F'' be the restriction of F' to $X'' := X \backslash B$. Since $\overline{\partial}_w$ is defined in the sense of distributions, Theorem 5.6 implies that $\overline{\partial}_w \colon L^2_{p,q}(X,F') \to L^2_{p,q+1}(X,F')$ and $\overline{\partial}_w \colon L^2_{p,q}(X,F'') \to L^2_{p,q+1}(X,F'')$ have essentially the same domain of definition, i. e. they coincide. The same holds for $\overline{\partial}_{w,\text{loc}}$, i. e. $\mathcal{C}_{F'} = \mathcal{C}_{F''}$. In particular,

$$H^{p,q}_w(X,F') = H^{p,q}_w(X,F'') \quad \text{and} \quad H^{p,q}_{w,\text{loc}}(X,F') = H^{p,q}_{w,\text{loc}}(X,F'').$$

As elaborated in Corollary 5.8, we obtain that the domains of definition of $\overline{\partial}_s$ in $L^2(X,F')$ and $L^2(X,F'')$ coincide, as well. This implies

$$H^{p,q}_s(X,F') = H^{p,q}_s(X,F'').$$

Chapter 6
L^2-Riemann-Roch for Singular Complex Curves

The purpose of this chapter is to give a comprehensive L^2-theory for the $\overline{\partial}$-operator on a singular complex curve, including L^2-versions of the Riemann-Roch theorem as it is presented in [RS15].

Let us explain some of our results in detail. Let X be a compact singular Hermitian complex space of dimension 1, i. e. a Hermitian complex curve, and $L \to X$ a Hermitian holomorphic line bundle. Note that the genus $g = g(X)$ of X and the degree $\deg(L)$ of L are well-defined, even in the presence of singularities (see Section 2.2). For a singular point $x \in X_{\text{sing}}$, we define its modified multiplicity $\operatorname{mult}'_x X$ as follows: Let X_j, $j = 1, ..., m$, be the irreducible components of X in the singular point x. Then,

$$\operatorname{mult}'_x X := \sum\nolimits_{j=1}^{m} \big(\operatorname{mult}_x X_j - 1 \big).$$

Note that regular irreducible components of X do not contribute to $\operatorname{mult}'_x X$. In Section 2.2, we recalled the definition of the multiplicity $\operatorname{mult}_x X_j$ and presented different ways to compute it. We obtain the following L^2-version of the Riemann-Roch theorem:

Theorem 6.1 ($\overline{\partial}_w$-Riemann-Roch)**.** *Let X be a compact Hermitian complex curve with m irreducible components and $L \to X$ a holomorphic line bundle. Then,*

$$h_w^{0,0}(X, L) - h_w^{0,1}(X, L) = m - g + \deg(L) + \sum_{x \in X_{\text{sing}}} \operatorname{mult}'_x X, \quad (6.2)$$

and

$$h_w^{1,1}(X, L) - h_w^{1,0}(X, L) = m - g - \deg(L).$$

Theorem 6.1 is a corollary of Theorem 6.15, which we prove in Section 6.2. There, we also consider an L^2-dual version, i. e. an L^2-Riemann-Roch theorem for the minimal closed L^2-extension of the $\overline{\partial}$-operator $\overline{\partial}_s$.

On singular complex curves, the $\overline{\partial}_s$-operator is of particular importance because of its relation to weakly holomorphic functions.

Theorem 6.3. *Let X be a Hermitian complex curve. Then,*

$$\begin{aligned} H^0(X, \widehat{\mathcal{O}}_X) &= H^{0,0}_{s,\mathrm{loc}}(X), \\ H^1(X, \widehat{\mathcal{O}}_X) &\cong H^{0,1}_{s,\mathrm{loc}}(X). \end{aligned}$$

If X is irreducible and compact, then $\dim H^0(X, \widehat{\mathcal{O}}_X) = 1$, $g(X) = \dim H^1(X, \widehat{\mathcal{O}}_X)$. We prove Theorem 6.3 in Section 6.4.

To exemplify the use of L^2-theory for the $\overline{\partial}$-operator on a singular complex space, in particular the L^2-Riemann-Roch theorem, we give two applications in Section 6.5. There, we use our L^2-theory to give alternative proofs of two well-known facts. First, we show that each compact complex curve can be realized as a ramified covering of $\mathbb{CP}^1$. Second, we show that a positive holomorphic line bundle over a compact complex curve is ample, yielding that any compact complex curve is projective.

Let us clarify the relation to the previous work of others. One can deduce parts of Theorem 6.15 and the second statement of Corollary 6.19 from W. Pardon's work [Par89] by some additional arguments on the regularity of the $\overline{\partial}$-operator. The first part of Corollary 6.19 was discovered by P. Haskell [Has89], from which one can deduce the second statement of Theorem 6.1 by the use of L^2-Serre duality. Moreover, Theorem 6.1 was proven in essence by J. Brüning, N. Peyerimhoff and H. Schröder in [BPS90] and [Sch89] by computing the indices of the $\overline{\partial}_w$-and the $\overline{\partial}_s$-operator.

The new point in the present chapter is that we can put all the partial results mentioned above in the general framework of a comprehensive

L^2-theory. From that, we draw also a new understanding of weakly holomorphic functions (Theorem 6.3) and of the divisor $Z - |Z|$, which was mentioned in the introduction (see (1.2)). Moreover, all the previous work has been done only for forms with values in the trivial bundle (except of [Sch89]), whereas we incorporate line bundles. This is essential for the application as we will illustrate by the examples mentioned above.

6.1 Local L^2-theory of complex curves

In this section, we study the local L^2-theory of (locally) irreducible analytic curves in $\mathbb{C}^n$. By the remarks on the local structure of singular complex curves in Section 2.2 and Section 6.2, it follows that the studied situation is general enough. We will compute the L^2-Dolbeault cohomology by use of the Puiseux parametrization and will see why the term $\sum_{x\in X_{\text{sing}}} \text{mult}'_x X$ occurs in (6.2).

Let A be an irreducible analytic curve in $\Delta^n \subset \mathbb{C}^n_{zw_1\ldots w_{n-1}}$, given by the Puiseux parametrization

$$\pi\colon \Delta \to \mathbb{C}^n \ , \ \pi(t) := (t^s, w(t)),$$

where $w = (w_1, ..., w_{n-1})\colon \Delta \to \Delta^{n-1}$ is a holomorphic map such that each component w_i vanishes at least of the order $s+1$ in the origin. Here, Δ is the unit disk $\{t \in \mathbb{C}\colon |t| < 1\}$. We can assume that π is bijective, in particular, a resolution / normalization of A such that $\text{mult}_0 A = s$. Further, we can assume that 0 is the only singular point of A.

For a regular point $(z_0, w_0) \in A^* := A_{\text{reg}}$, let $t_0 \in \Delta^* := \Delta\setminus\{0\}$ be the preimage under π. Since π is biholomorphic on Δ^*, $d\pi_{t_0}(\frac{\partial}{\partial t}) = st_0^{s-1}\frac{\partial}{\partial z} + \sum_{k=1}^{n-1} w'_k(t_0)\frac{\partial}{\partial w_k}$ is a non-vanishing tangent vector of A^* in (z_0, w_0), i. e.

$$(1 + \|\tfrac{1}{s}t_0^{1-s}w'(t_0)\|^2)^{-1/2}\left(\frac{\partial}{\partial z} + \frac{1}{s}t_0^{1-s}\sum_{k=1}^{n-1} w'_k(t_0)\frac{\partial}{\partial w_k}\right)$$

is a normalized generator of $T_{(z_0,w_0)}A^*$ and $(1+\|\frac{1}{s}t_0^{1-s}w'(t_0)\|^2)^{1/2}dz$ is a normalized generator of $T^*_{(z_0,w_0)}A^*$. Since w'_k vanishes at least of order s in the origin, we obtain $1+\|\frac{1}{s}t^{1-s}w'(t)\|^2 \sim 1$ on Δ and $dV_{A^*} \sim i dz\wedge d\overline{z}$, where dV_{A^*} denotes the volume form on A^* induced by the standard Euclidean metric of $\mathbb{C}^n$. Using $\pi^*dz = d(\pi^*z) = dt^s = st^{s-1}dt$ and $\pi^*(dz\wedge d\overline{z}) = s^2|t|^{2(s-1)}dt\wedge d\overline{t}$, we get

$$\pi^* dV_{A^*} \sim |t|^{2(s-1)}dV_\Delta.$$

Let $\iota\colon A^* \to \Delta^*$ be the inverse of π. Then, $\iota(z,w)$ is the root $t = \sqrt[s]{z}$ with $w = w(t)$. We get $\iota^*(dt) = \frac{1}{s}z^{1/s-1}dz$ and $\iota^*(dt\wedge d\overline{t}) = \frac{1}{s^2}|z|^{2(1/s-1)}dz\wedge d\overline{z}$, i. e.

$$\iota^* dV_{\Delta^*} \sim |z|^{2(1/s-1)}dV_{A^*}.$$

If g is a measurable function on A^*, we obtain

$$\int_{A^*}|g|^2 dV_{A^*} = \int_\Delta |\pi^*g|^2\pi^*dV_{A^*} \sim \int_\Delta |\pi^*g|^2\cdot|t|^{2(s-1)}dV_\Delta.$$

Hence,

$$g\in L^2_{0,0}(A^*) \Leftrightarrow t^{s-1}\pi^*g \in L^2_{0,0}(\Delta).$$

For $(0,1)$-forms and $(1,1)$-forms, we have

$$\pi^*(gd\overline{z}) = \pi^*g\cdot\pi^*(d\overline{z}) = \overline{t}^{s-1}\pi^*(g)d\overline{t},$$
$$\pi^*(gdz\wedge d\overline{z}) = |t|^{2(s-1)}\pi^*(g)dt\wedge d\overline{t}.$$

Thus,

$$\begin{aligned} f\in L^2_{0,0}(A^*) &\Leftrightarrow t^{s-1}\cdot\pi^*f \in L^2_{0,0}(\Delta),\\ f\in L^2_{1,0}(A^*) &\Leftrightarrow \pi^*f\in L^2_{1,0}(\Delta),\\ f\in L^2_{0,1}(A^*) &\Leftrightarrow \pi^*f\in L^2_{0,1}(\Delta), \quad \text{and}\\ f\in L^2_{1,1}(A^*) &\Leftrightarrow t^{1-s}\cdot\pi^*f\in L^2_{1,1}(\Delta). \end{aligned} \tag{6.4}$$

On the other hand, if $v\in L^2_{0,0}(\Delta)$, we get

$$\infty > \int_\Delta |v|^2 dV_\Delta = \int_{A^*}|\iota^*v|^2\iota^*dV_\Delta \sim \int_{A^*}|\iota^*v|^2\cdot|z|^{2(1/s-1)}dV_{A^*}.$$

Thus, $|z|^{1/s-1}\iota^*v$ is square-integrable on A^*. For each $(0,1)$-form $vd\overline{t}\in L^2_{0,1}(\Delta)$, we get $s\iota^*(vd\overline{t}) = \overline{z}^{1/s-1}\iota^*(v)d\overline{z}\in L^2_{0,1}(A^*)$, and for each $(1,1)$-form $vdt\wedge d\overline{t}\in L^2_{1,1}(\Delta)$, we get $|z|^{1-\frac{1}{s}}\iota^*(vdt\wedge d\overline{t})\in L^2_{1,1}(A^*)$.

So, if $f\in L^2_{0,1}(A^*)$, then $u := \pi^*f$ is in L^2, too. Since $\dim\Delta = 1$, there exists $v\in L^2_{0,0}(\Delta)$ with $\overline{\partial}_w v = u$. We set $g := \iota^*v$. Since $|z|^{1/s-1}g$

is in L^2 and $|z|^{2(1-1/s)}$ is bounded,

$$\begin{aligned}\|g\|_{L^2}^2 &= \int_{A^*} |z^{1/s-1}g|^2 \cdot |z|^{2(1-1/s)} dV_{A^*} \\ &\leq \|z^{1/s-1}g\|_{L^2} \cdot \|z^{2(1-1/s)}\|_{L^\infty} < \infty.\end{aligned}$$

Hence, we get an L^2-solution for $\overline{\partial}_w g = f$ and so

$$H_w^{0,1}(A) = L_{0,1}^2(A^*)/\mathcal{R}(\overline{\partial}_w) = 0.$$

In the same way, it is easy to compute

$$H_w^{1,1}(A) = 0.$$

We will now determine $H_w^{p,0}(A) = \ker(\overline{\partial}_w : L_{p,0}^2 \to L_{p,1}^2)$ by use of the L^2-extension theorem (Theorem 5.6). For this, let $\mathcal{O}_{L^2}(\Delta)$ be the square-integrable holomorphic functions on Δ, and let $\Omega_{L^2}^1(\Delta)$ be the holomorphic 1-forms with square-integrable coefficient. If $g \in L_{0,0}^2(A^*)$ and $\overline{\partial}_w g = 0$, then $v := \pi^* g \in |t|^{1-s} L_{0,0}^2(\Delta)$ and $\overline{\partial}_w v = 0$ on Δ^*. Therefore, $\overline{\partial}(t^{s-1}v) = 0$ on Δ^* and $t^{s-1}v \in L_{0,0}^2(\Delta)$. The L^2-extension theorem implies $\overline{\partial}(t^{s-1}v) = 0$ on Δ, i. e. v is a meromorphic function with a pole of order $s-1$ or less at the origin. We say $v \in t^{1-s}\mathcal{O}_{L^2}(\Delta)$. Since, on the other hand, $\iota^*(t^{1-s}\mathcal{O}_{L^2}(\Delta)) \subset \ker \overline{\partial}_w$, we conclude

$$H_w^{0,0}(A) \cong t^{1-s}\mathcal{O}_{L^2}(\Delta). \tag{6.5}$$

If $f \in L_{1,0}^2(A^*)$ and $\overline{\partial}_w f = 0$, then $u := \pi^* f \in L_{1,0}^2(\Delta)$ and $\overline{\partial}_w u = 0$ on Δ (using the L^2-extension theorem again). Hence, u is holomorphic on Δ and

$$H_w^{1,0}(A) \cong \Omega_{L^2}^1(\Delta).$$

To compute the cohomology groups $H_s^{*,*}(A)$, we use L^2-duality:

Lemma 6.6. *Let $\overline{\partial}_e$ denote either the weak or the strong closed extension of $\overline{\partial}$, and $\overline{\partial}_{e^c}$ the other one. For $p \in \{0,1\}$, let the range $\mathcal{R}(\overline{\partial}_e)$ of $\overline{\partial}_e \colon L_{p,0}^2 \to L_{p,1}^2$ be closed. Then,*

$$H_e^{p,1}(A) \cong H_{e^c}^{1-p,0}(A).$$

For the proof see e. g. [Rup14a, Thm. 2.3].

Lemma 6.7. *For $p \in \{0,1\}$,*

$$H_s^{p,0}(A) \cong H_w^{1-p,1}(A) = 0 \quad \textit{and}$$
$$H_s^{p,1}(A) \cong H_w^{1-p,0}(A).$$

Proof: Recall that $H_w^{1-p,1}(A) = 0$. This implies $L^2_{1-p,1}(A^*) = \mathcal{R}(\overline{\partial}_w)$ and, particularly, that the range of $\overline{\partial}_w \colon L^2_{1-p,0} \to L^2_{1-p,1}$ is closed. As $\vartheta_w = -\overline{*}\,\overline{\partial}_w\,\overline{*}$ and $\overline{*}$ is an isometric isomorphism, we conclude that the range of $\vartheta_w \colon L^2_{p,1} \to L^2_{p,0}$ is closed as well. This is equivalent to the range of $\overline{\partial}_s = \vartheta_w^* \colon L^2_{p,0} \to L^2_{p,1}$ being closed (standard functional analysis). Lemma 6.6 implies both isomorphisms. □

To get the complete picture, we also need to understand the Dolbeault cohomology groups of the closed extensions $\overline{\partial}_{s,w}$ and $\overline{\partial}_{w,s}$, which was defined in Section 5.2.

Lemma 6.8. *For $p \in \{0,1\}$,*

$$H_{w,s}^{p,0}(A) = 0.$$

Proof: Let $f \in \ker \overline{\partial}_{w,s} = H_{w,s}^{p,0}(A)$. We have shown that $\omega \cdot u := \omega \cdot \pi^* f \in L^2_{p,0}(\Delta)$ with $\omega(t) = t^{s-1}$ if $p = 0$ and $\omega(t) \equiv 1$ if $p = 1$. By the L^2-extension theorem, we conclude $\overline{\partial}_s(\omega \cdot u) = 0$ on Δ. The generalized Cauchy condition implies that the trivial extension of ωu to the complex plane is a holomorphic p-form with compact support (cf. [LM02, § V.3]). We deduce that $\omega u = 0$ and, hence, $f = 0$. □

Lemma 6.9.

$$H_{s,w}^{0,0}(A) \cong \mathcal{O}_{L^2}(\Delta) \quad \textit{and}$$
$$H_{s,w}^{1,0}(A) \cong t^{s-1}\Omega^1_{L^2}(\Delta).$$

As $\mathcal{O}_{L^2}(\Delta) \cong \widehat{\mathcal{O}}_{L^2}(A)$, the first isomorphism implies that the $\overline{\partial}_{s,w}$-holomorphic functions on a singular complex curve are precisely the square-integrable weakly holomorphic functions.

Proof: First, we prove that $\mathcal{O}_{L^2}(\Delta) = \pi^* \ker\big(\overline{\partial}_{s,w} \colon L^2_{0,0}(A^*) \to L^2_{0,1}(A^*)\big)$.

(i) For $v \in \mathcal{O}_{L^2}(\Delta)$, we claim that $g := \iota^* v \in \ker \overline{\partial}_{s,w}$. To show this, choose smooth functions $\widetilde{\chi}_k \colon \mathbb{R} \to [0,1]$ with $\widetilde{\chi}_k|_{(-\infty,k]} = 0$,

$\widetilde{\chi}_k|_{[k+1,\infty)} = 1$ and $|\widetilde{\chi}_k'| \leq 2$. We get $(\widetilde{\chi}_k \circ \log \circ |\log|)'(\rho) = \frac{\widetilde{\chi}_k'(\log|\log\rho|)}{\rho\log\rho}$. We define $\chi_k\colon A^* \to [0,1], (z,w) \mapsto \widetilde{\chi}_k(\log|\log|z||)$ (which is inspired by [PS91, p. 617]) and get $\operatorname{supp}\overline{\partial}\chi_k \subset A^* \cap \Delta^n_{\varepsilon_k}$, where $\varepsilon_k := \exp(-\exp(k)) \to 0$ if $k \to \infty$. As $v \in L^2_{0,0}(\Delta)$, we have

$$g \in z^{1-\frac{1}{s}} L^2_{0,0}(A^*) \subset L^2_{0,0}(A^*).$$

Then, $g \cdot \chi_k \to g$ in L^2. As a holomorphic function, v is bounded in a neighbourhood of 0. Therefore,

$$\begin{aligned}
\|g\overline{\partial}\chi_k\|^2_{A^*} &= \left\| g \cdot \frac{\widetilde{\chi}_k'(\log|\log|z||)}{|z|\log|z|}\overline{\partial}|z| \right\|^2_{A^*\cap\Delta^n_{\varepsilon_k}} \\
&\lesssim \left\| g \cdot \frac{1}{|z|\log|z|} \right\|^2_{A^*\cap\Delta^n_{\varepsilon_k}} \sim \left\| v \cdot \frac{|t|^{s-1}}{|t|^s\log|t|^s} \right\|^2_{\Delta_{\varepsilon_k}} \\
&\lesssim \left\| \frac{1}{|t|\log|t|} \right\|^2_{\Delta_{\varepsilon_k}} = \int_{\Delta_{\varepsilon_k}} \frac{1}{|t|^2\log^2|t|} dV \\
&= 2\pi \int_0^{\varepsilon_k} \frac{\rho}{\rho^2\log^2\rho} d\rho \sim \left[-\frac{1}{\log\rho} \right]_0^{\varepsilon_k} \to 0 \quad \text{if } k \to \infty.
\end{aligned}$$

Hence, $\overline{\partial}(g\chi_k) = g\overline{\partial}\chi_k \to 0 = \overline{\partial}_w g$ in L^2. So, $g \in \operatorname{dom}\overline{\partial}_{s,w}$.

(ii) $\pi^*(\ker\overline{\partial}_{s,w}) \subset \mathcal{O}_{L^2}(\Delta)$ (cf. the proof of Lem. 6.2 in [Rup14a]): Let g be in $\ker\overline{\partial}_{s,w}$, i. e. there are g_j in $L^2(A^*)$ with $g_j \to g$, $\overline{\partial}g_j \to 0$ in $L^2(A^*)$ and $0 \notin \operatorname{supp} g_j$. Let $\chi \in \mathscr{C}^\infty_{\mathrm{cpt}}(\Delta,[0,1])$ be identically 1 on $\Delta_{1/2}$. We define $u := \chi\pi^*g$ and $u_j := \chi\pi^*g_j$. It follows that $t^{s-1}u_j \to t^{s-1}u$ and $\overline{\partial}u_j \to \overline{\partial}u$ in $L^2(\Delta)$. Let $P\colon L^2(\Delta) \to L^2(\Delta)$ be the Cauchy operator on the punctured disc, i. e.

$$[P(h)](t) := \frac{1}{2\pi \mathrm{i}} \int_{\Delta^*} \frac{h(\zeta)}{\zeta - t} d\zeta \wedge d\overline{\zeta}.$$

Since the support of u_j is away from 0 and $\partial\Delta$, we get $u_j = P\big(\frac{\partial u_j}{\partial\overline{\zeta}}\big)$. The L^2-continuity of P and $\overline{\partial}u_j \to \overline{\partial}u$ in L^2 imply that

$$u_j = P\left(\frac{\partial u_j}{\partial\overline{\zeta}}\right) \to P\left(\frac{\partial u}{\partial\overline{\zeta}}\right)$$

in L^2. Since t^{s-1} is bounded, we obtain $t^{s-1}u_j \to t^{s-1}P\big(\frac{\partial u}{\partial\overline{\zeta}}\big)$ and, hence, $u = P\big(\frac{\partial u}{\partial\overline{\zeta}}\big)$ in L^2. That yields $\pi^*g \in L^2(\Delta)$. With $\pi^*g \in t^{1-s}\mathcal{O}_{L^2}(\Delta)$ and the L^2-extension theorem, we conclude $\pi^*g \in \mathcal{O}_{L^2}(\Delta)$.

Second, we claim that $\ker\left(\overline{\partial}_{s,w}\colon L^2_{1,0}(A^*) \to L^2_{1,1}(A^*)\right) \cong t^{s-1}\Omega^1_{L^2}(\Delta)$: $f = gdz$ is in $\ker\overline{\partial}_{s,w}$ if and only if $g \in \ker\overline{\partial}_{s,w}$. This is equivalent to $\pi^* g \in \mathcal{O}_{L^2}(\Delta)$. Since $\pi^*(dz) = t^{s-1}dt$, we infer that

$$\pi^*\colon H^{1,0}_{s,w}(A) \to t^{s-1}\Omega^1_{L^2}(\Delta), \pi^* f = t^{s-1}\pi^* g dt$$

is an isomorphism.

The Riemann extension theorem (see Theorem 2.13) implies $\mathcal{O}_{L^2}(\Delta) \cong \widehat{\mathcal{O}}_{L^2}(A)$ and the last statement. □

Lemma 6.10. *For $p \in \{0,1\}$, $\mathcal{R}(\overline{\partial}_{w,s}\colon L^2_{p,0}(A^*) \to L^2_{p,1}(A^*))$ and $\mathcal{R}(\overline{\partial}_{s,w}\colon L^2_{p,0}(A^*) \to L^2_{p,1}(A^*))$ both are closed, and*

$$H^{p,1-q}_{w,s}(A) \cong H^{1-p,q}_{s,w}(A) \quad \textit{for } q \in \{0,1\}.$$

Proof: Since $\overline{\partial}^*_{w,s} = \vartheta_{s,w}$, $\overline{\partial}^*_{s,w} = \vartheta_{w,s}$, $\vartheta_{s,w} = -\overline{*}\,\overline{\partial}_{s,w}\overline{*}$ and $\vartheta_{w,s} = -\overline{*}\,\overline{\partial}_{w,s}\overline{*}$ (see Lemma 5.5), it is easy to see that Lemma 6.6 holds for $\overline{\partial}_{w,s}$ and $\overline{\partial}_{s,w}$. We remark (cf. the proof of Lemma 6.7) that

$$\begin{aligned}\mathcal{R}(\overline{\partial}_{s,w}) \text{ is closed } &\Leftrightarrow \mathcal{R}(\vartheta_{w,s}) \text{ is closed}\\ &\Leftrightarrow \mathcal{R}(\overline{\partial}_{w,s}) \text{ is closed.}\end{aligned}$$

Therefore, it is enough to show that $\mathcal{R}(\overline{\partial}_{w,s})$ is closed.

Let $\varphi\colon \Delta^* \to \mathbb{R}$ be the smooth function defined by $\varphi(t) := (1-s)\log|t|^2$. Then, we get

$$t^{1-s}L^2_{p,q}(\Delta) = L^2_{p,q}(\Delta,\varphi) := \left\{u \in L^{2,\mathrm{loc}}_{p,q}(\Delta^*)\colon \int |u|^2 e^{-\varphi} < \infty\right\}$$

for the L^2-space with weight $e^{-\varphi}$.

We set $T_1 := \pi^*\overline{\partial}_{w,s}\iota^*\colon L^2_{0,0}(\Delta,\varphi) \to L^2_{0,1}(\Delta)$. The L^2-extension theorem implies that T_1 is the (strong) closure of $\overline{\partial}_{\mathrm{cpt}}\colon L^2_{0,0}(\Delta,\varphi) \to L^2_{0,1}(\Delta)$. Therefore, T_1^* is the weak closed extension of $\vartheta^{\varphi}_{\mathrm{cpt}}\colon L^2_{0,1}(\Delta) \to L^2_{0,0}(\Delta,\varphi)$ which is defined by

$$(\overline{\partial}_{\mathrm{cpt}}\alpha,\beta) = (\alpha,\vartheta^{\varphi}_{\mathrm{cpt}}\beta)_{\varphi} := \int \langle\alpha,\vartheta^{\varphi}_{\mathrm{cpt}}\beta\rangle e^{-\varphi} dV.$$

We set $\overline{*}_\varphi := e^{-\varphi}\,\overline{*}$. Then, $T_2 := -\overline{*}_\varphi\, T_1^*\,\overline{*}$ is the weak closed extension of $\overline{\partial}_{\mathrm{cpt}}\colon L^2_{1,0}(\Delta) \to L^2_{1,1}(\Delta, -\varphi)$ because integration by parts implies $\vartheta^\varphi_{\mathrm{cpt}} = -\overline{*}_{-\varphi}\,\overline{\partial}_{\mathrm{cpt}}\,\overline{*}$:

$$(\alpha, \overline{*}_{-\varphi}\,\overline{\partial}_{\mathrm{cpt}}\,\overline{*}\,\beta)_\varphi = \int \alpha \wedge \overline{*}_\varphi\,\overline{*}_{-\varphi}\,\overline{\partial}_{\mathrm{cpt}}\,\overline{*}\,\beta = \int \alpha \wedge \overline{\partial}_{\mathrm{cpt}}\,\overline{*}\,\beta$$
$$= -\int \overline{\partial}_{\mathrm{cpt}}\alpha \wedge \overline{*}\,\beta = -(\overline{\partial}_{\mathrm{cpt}}\alpha, \beta) = -(\alpha, \vartheta^\varphi_{\mathrm{cpt}}\beta)_\varphi.$$

Hence, T_2 is $\overline{\partial}_w\colon L^2_{1,0}(\Delta) \to L^2_{1,1}(\Delta, -\varphi)$ in the sense of distributions. Since, for each $u \in L^2_{1,1}(\Delta, -\varphi) = t^{s-1}L^2_{1,1}(\Delta) \subset L^2_{1,1}(\Delta)$, there is a $v \in L^2_{1,0}(\Delta)$ with $T_2 v = \overline{\partial}_w v = u$, the range of T_2 is closed. Thus, the range of T_1^* and the range of $\overline{\partial}_{w,s} = \iota^* T_1 \pi^*\colon L^2_{0,0}(A^*) \to L^2_{0,1}(A^*)$ are closed, as well.

Analogously, we set $S_1 := \pi^*\overline{\partial}_{w,s}\iota^*\colon L^2_{1,0}(\Delta) \to L^2_{1,1}(\Delta, -\varphi)$ and $S_2 := -\overline{*}\,S_1^*\,\overline{*}_\varphi$. Then, S_2 is the weak closure of $\overline{\partial}_{\mathrm{cpt}}\colon L^2_{0,0}(\Delta, \varphi) \to L^2_{0,1}(\Delta)$.

$$\mathcal{R}(S_2) = \{u \in L^2_{0,1}(\Delta)\colon \exists v \in L^2_{0,0}(\Delta, \varphi) = t^{1-s}L^2_{0,0}(\Delta) \text{ with } S_2 v = u\}$$
$$\supset \{u \in L^2_{0,1}(\Delta)\colon \exists v \in L^2_{0,0}(\Delta) \text{ with } \overline{\partial}_w v = u\} = L^2_{0,1}(\Delta).$$

Therefore, $\mathcal{R}(S_2) = L^2_{0,1}(\Delta)$ is closed. This implies the claim. □

Summarizing, we computed (with $s = \mathrm{mult}_0 A$):

$$\begin{aligned}
H^{0,0}_w(A) &\cong H^{1,1}_s(A) &&\cong t^{1-s}\mathcal{O}_{L^2}(\Delta),\\
H^{1,0}_w(A) &\cong H^{0,1}_s(A) &&\cong \Omega^1_{L^2}(\Delta),\\
H^{p,1}_w(A) &= H^{1-p,0}_s(A) &&= 0,\\
H^{0,0}_{s,w}(A) &\cong H^{1,1}_{w,s}(A) &&\cong \mathcal{O}_{L^2}(\Delta),\\
H^{1,0}_{s,w}(A) &\cong H^{0,1}_{w,s}(A) &&\cong t^{s-1}\Omega^1_{L^2}(\Delta), \quad \text{and}\\
H^{p,1}_{s,w}(A) &= H^{1-p,0}_{w,s}(A) &&= 0.
\end{aligned} \tag{6.11}$$

6.2 L^2-cohomology of compact complex curves

We will prove Theorem 6.1 in this section. As a preparation, we consider the following local situation: Let A be a locally irreducible analytic set of dimension one in a domain $\Omega \Subset \mathbb{C}^n_{zw_1\ldots w_{n-1}}$ with $A_{\mathrm{sing}} = \{0\}$, let dV denote the volume form on $A^* := A_{\mathrm{reg}}$ which is induced by the Euclidean metric, and let $z\colon A \to \mathbb{C}_z$ be the projection on the first coordinate. Let us mention (cf. e. g. Prop. in [Chi89, Sect. 8.1]):

Theorem 6.12. *The set of all tangent vectors at a point of a one-dimensional irreducible analytic set in $\mathbb{C}^n$ is a complex line.*

Thus, we can assume that $C_0(A) = \mathbb{C}_z \times \{0\} \subset \mathbb{C}_z \times \mathbb{C}^{n-1}_{w_1\ldots w_{n-1}}$, and, therefore, $dV \sim dz \wedge d\overline{z}$ (by shrinking Ω if necessary).

Let $\pi\colon M \to A$ be a resolution of A, $x_0 := \pi^{-1}(0)$. Then, $Z = (\pi^*(z))$ is the unreduced exceptional divisor of the resolution. After shrinking A and M again, we can assume that M is covered by a single chart $\psi\colon M \to \mathbb{C}$ with $x_0 \in M$ and $\psi(x_0) = 0$. We set $\zeta := \pi^*(z)$ and get $Z = (\zeta)$. $|Z| = (\psi)$ implies $Z - |Z| = \left(\frac{\zeta}{\psi}\right)$. We obtain

$$\pi^*(dz) = d(\pi^* z) = \frac{\partial \zeta}{\partial \psi} d\psi \sim \frac{\zeta}{\psi} d\psi.$$

Therefore, $\pi^*(dV) \sim \left|\frac{\zeta}{\psi}\right|^2 d\psi \wedge d\overline{\psi}$, and we conclude (recall the definition of line bundles L_D from Section 2.2):

$$\begin{aligned} f \in L^2_{p,q}(A^*) &\Leftrightarrow \left|\frac{\zeta}{\psi}\right|^{1-p-q} \cdot \pi^* f \in L^2_{p,q}(M) \\ &\Leftrightarrow \pi^* f \in L^2_{p,q}(M, L_{(1-p-q)(Z-|Z|)}). \end{aligned} \tag{6.13}$$

M. Nagase stated this equivalence already in Lem. 5.1 of [Nag90]. By use of the L^2-extension Theorem 5.6, we get:

$$\begin{aligned} & f \in \mathrm{dom}\left(\overline{\partial}_w\colon L^2_{p,0}(A^*) \to L^2_{p,1}(A^*)\right) \\ \Leftrightarrow\ & \pi^* f \in \mathrm{dom}\left(\overline{\partial}_w\colon L^2_{p,0}(M, L_{(1-p)(Z-|Z|)}) \to L^2_{p,1}(M, L_{p(|Z|-Z)})\right). \end{aligned} \tag{6.14}$$

The essential observation for the proof of Theorem 6.1 is the following:

Theorem 6.15. *Let X be a compact complex curve and $L \to X$ a holomorphic line bundle. Let $\pi\colon M \to X$ be a resolution of X with exceptional divisor Z, and D a divisor on M such that $\pi^*L \cong L_D$, i. e. $\mathcal{O}(\pi^*L) \cong \mathcal{O}(D)$. Then,*

$$\begin{aligned}
&H^{0,0}_w(X,L) \cong H^0(M,\mathcal{O}(Z-|Z|+D)),\\
&H^{0,1}_w(X,L) \cong H^1(M,\mathcal{O}(Z-|Z|+D)),\\
&H^{1,0}_w(X,L) \cong H^0(M,\Omega^1(D)) \cong H^1(M,\mathcal{O}(-D)), \quad \textit{and}\\
&H^{1,1}_w(X,L) \cong H^1(M,\Omega^1(D)) \cong H^0(M,\mathcal{O}(-D)).
\end{aligned}$$

In [Par89, § 5], Pardon proved that $H^{0,q}_{(2),\mathrm{sm}}(X^*) \cong H^q(M,\mathcal{O}(Z-|Z|)$, where $H^{p,q}_{(2),\mathrm{sm}}(X^*)$ denotes the $\overline{\partial}$-cohomology groups with respect to smooth L^2-forms. We will use similar arguments here.

Proof: Let x_0 be in X_{sing}, and let A be an open neighbourhood of $x_0 = 0$ in X embedded locally in $\mathbb{C}^n$. We assume that $A = A_1 \cup ... \cup A_m$ with at x_0 irreducible analytic sets A_i. For $M_i := \pi^{-1}(A_i)$, we obtain the resolutions $\pi_i := \pi|_{M_i}\colon M_i \to A_i$ of A_i. The sets M_i are pairwise disjoint in M and so are the supports of the exceptional divisors $Z_i = Z|_{M_i}$ of the resolutions π_i. We get $Z|_{\pi^{-1}(A)} = \sum_{i=1}^m Z_i$ and $|Z||_{\pi^{-1}(A)} = \sum_{i=1}^m |Z_i|$. Therefore, the consideration in the local case (see (6.14)) implies that $\overline{\partial}_w\colon L^2_{p,0}(X^*,L) \to L^2_{p,1}(X^*,L)$ can be identified with

$$\overline{\partial}_w\colon L^2_{p,0}(M,L_{(1-p)(Z-|Z|)+D}) \to L^2_{p,1}(M,L_{p(|Z|-Z)+D}).$$

Hence,

$$\begin{aligned}
H^{0,0}_w(X,L) &\cong \ker(\overline{\partial}_w\colon L^2_{0,0}(M,L_{Z-|Z|+D}) \to L^2_{0,1}(M,L_D))\\
&\cong H^0(M,\mathcal{O}(Z-|Z|+D))
\end{aligned}$$

and

$$\begin{aligned}
H^{1,0}_w(X,L) &\cong \ker(\overline{\partial}_w\colon L^2_{1,0}(M,L_D) \to L^2_{1,1}(M,L_{|Z|-Z+D}))\\
&\cong H^0(M,\Omega^1(D)).
\end{aligned}$$

Serre duality (see Thm. 2 in [Ser55, § 3.10]) implies

$$H^{1,0}_w(X,L) \cong H^0(M,\Omega^1(D)) \cong H^1(M,\mathcal{O}(-D)).$$

To prove the other two isomorphisms, consider the following general situation: Let E be a divisor on M, L_E the associated bundle, and let $\mathcal{L}^{p,q}_E$ denote the sheaf on M which is defined by $\mathcal{L}^{p,q}_E(U) := L^{2,\text{loc}}_{p,q}(U, L_E)$ for each open set $U \subset M$. Let $E' \leq E$ be another divisor. Consider the $\overline{\partial}$-operator in the sense of distributions $\overline{\partial}_{w,\text{loc}} \colon \mathcal{L}^{p,0}_E \to \mathcal{L}^{p,1}_{E'}$. Let $\mathcal{C}^{p,0}_{E,E'}$ denote the sheaf defined by

$$\mathcal{C}^{p,0}_{E,E'}(U) := \operatorname{dom}\left(\overline{\partial}_{w,\text{loc}} \colon L^{2,\text{loc}}_{p,0}(U, L_E) \to L^{2,\text{loc}}_{p,1}(U, L_{E'})\right).$$

Then, $\mathcal{C}^{p,0}_{E,E'}$ is fine and, in particular, $H^1(M, \mathcal{C}^{p,0}_{E,E'}) = 0$. We get the sequence

$$0 \to \Omega^p(E) \to \mathcal{C}^{p,0}_{E,E'} \xrightarrow{\overline{\partial}_{w,\text{loc}}} \mathcal{L}^{p,1}_{E'} \to 0, \tag{6.16}$$

which is exact by the usual Grothendieck-Dolbeault lemma because there is an embedding $\mathcal{L}^{p,q}_{E'} \subset \mathcal{L}^{p,q}_E$ (induced by the natural inclusion $\mathcal{O}(E') \subset \mathcal{O}(E)$, see (2.5) in Section 2.2).

This induces the long exact sequence of cohomology groups

$$\begin{aligned} 0 \longrightarrow \Omega^p(M, E) \longrightarrow \mathcal{C}^{p,0}_{E,E'}(M) \xrightarrow{\overline{\partial}_w} \mathcal{L}^{p,1}_{E'}(M) \to \\ \to H^1(M, \Omega^p(E)) \to H^1(M, \mathcal{C}^{p,0}_{E,E'}) = 0\,. \end{aligned}$$

Hence, $\mathcal{L}^{p,1}_{E'}(M)/\overline{\partial}_w \mathcal{C}^{p,0}_{E,E'}(M) \cong H^1(M, \Omega^p(E))$. We conclude

$$\begin{aligned} H^{0,1}_w(X, L) &\cong \mathcal{L}^{0,1}_D(M)/\overline{\partial}_w \mathcal{C}^{0,0}_{Z-|Z|+D,D}(M) \\ &\cong H^1(M, \mathcal{O}(Z - |Z| + D)) \end{aligned}$$

and, using Serre duality again,

$$\begin{aligned} H^{1,1}_w(X, L) &\cong \mathcal{L}^{1,1}_{|Z|-Z+D}(M)/\overline{\partial}_w \mathcal{C}^{1,0}_{D,|Z|-Z+D}(M) \\ &\cong H^1(M, \Omega^1(D)) \cong H^0(M, \mathcal{O}(-D)). \end{aligned}$$

□

Theorem 6.1 follows now as a simple corollary by the use of the classical Riemann-Roch theorem for each connected component of the Riemann surface M, keeping in mind that by definition $g(M) = g(X)$, $\deg L = \deg \pi^* L = \deg D$ and $\operatorname{mult}'_x X = \sum_{p \in \pi^{-1}(x)} \deg_p(Z - |Z|)$ (see Section 2.2).

To deduce also a Riemann-Roch theorem for the $\overline{\partial}_s$-cohomology, we can use the following L^2-version of Serre duality:

Theorem 6.17. *For each $p \in \{0,1\}$, the range of $\overline{\partial}_w \colon L^2_{p,0}(X^*, L) \to L^2_{p,1}(X^*, L)$ is closed. In particular, we get*

$$H^{p,q}_w(X, L) \cong H^{1-p,1-q}_s(X, L^{-1}).$$

Proof: Recall the following well-known fact. If $P \colon H_1 \to H_2$ is a densely defined closed operator between Hilbert spaces with range $\mathcal{R}(P)$ of finite codimension, then the range $\mathcal{R}(P)$ is closed in H_2 (see e. g. [HL84], App. 2.4).

As M is compact, Theorem 6.15 implies particularly that the range of $\overline{\partial}_w$ is finite codimensional and, therefore, closed. Since $\overline{\partial}_s$ is the adjoint of $-\overline{*}\,\overline{\partial}_w\,\overline{*}$, the range of $\overline{\partial}_s \colon L^2_{1-p,0}(X^*, L^{-1}) \to L^2_{1-p,1}(X^*, L^{-1})$ is closed, as well. That both ranges are closed implies the L^2-duality (cf. Lemma 6.6)

$$\begin{aligned} H^{p,q}_w(X, L) &\cong \mathscr{H}^{p,q}_w(X, L) \cong \mathscr{H}^{1-p,1-q}_s(X, L^{-1}) \\ &\cong H^{1-p,1-q}_s(X, L^{-1}), \end{aligned}$$

where $\mathscr{H}^{p,q}_e(X, L) := \ker \overline{\partial}_e \cap \ker \overline{\partial}^*_e$ denotes the space of $\overline{\partial}_e$-harmonic forms with values in L for $e \in \{w, s\}$. $\square$

Therefore, Theorem 6.15 yields:

$$\begin{aligned} H^{0,0}_s(X, L) &\cong H^{1,1}_w(X, L^{-1}) \cong H^0(M, \mathcal{O}(D)), \\ H^{0,1}_s(X, L) &\cong H^{1,0}_w(X, L^{-1}) \cong H^1(M, \mathcal{O}(D)), \\ H^{1,0}_s(X, L) &\cong H^{0,1}_w(X, L^{-1}) \cong H^1(M, \mathcal{O}(Z - |Z| - D)), \quad \text{and} \\ H^{1,1}_s(X, L) &\cong H^{0,0}_w(X, L^{-1}) \cong H^0(M, \mathcal{O}(Z - |Z| - D)). \end{aligned} \tag{6.18}$$

Haskell computed $H^{0,q}_{\text{cpt}}(X^*) \cong H^q(M, \mathcal{O}_M)$, where $H^{p,q}_{\text{cpt}}(X^*)$ denotes the $\overline{\partial}$-cohomology groups with respect to smooth forms with compact support (see Thm. 3.1 in [Has89]). From (6.18), we obtain the dual version of Theorem 6.1, i. e. the Riemann-Roch theorem for the $\overline{\partial}_s$-cohomology:

Corollary 6.19 ($\overline{\partial}_s$-Riemann-Roch)**.** *Let X be a compact complex curve with m irreducible components, $L \to X$ be a holomorphic line bundle and $\pi\colon M \to X$ be a resolution of X. Then,*

$$h_s^{0,0}(X, L) - h_s^{0,1}(X, L) = m - g + \deg L, \quad \textit{and}$$
$$h_s^{1,1}(X, L) - h_s^{1,0}(X, L) = m - g + \deg(Z - |Z|) - \deg L,$$

where Z is the exceptional divisor of the resolution.

In [BPS90], J. Brüning, N. Peyerimhoff, and H. Schröder proved that $h_s^{0,0}(X) - h_s^{0,1}(X) = m - g$ and $h_w^{0,0}(X) - h_w^{0,1}(X) = m - g + \deg Z - |Z|$ by computing the indices of the differential operators $\overline{\partial}_s$ and $\overline{\partial}_w$. H. Schröder generalized this result to vector bundles in [Sch89].

6.3 Singular metrics

In this section, we will present the L^2-Riemann-Roch theorem generalized to the case that the regular part of the complex curve carries a singular metric.

Let X be a compact complex curve and $\pi\colon M \to X$ a resolution of X. We set $X^* := X_{\mathrm{reg}}$ and $M^* := \pi^{-1}(X^*)$. Let σ be a Hermitian metric on M^*, σ_0 a standard metric on M (all metrics are equivalent) and E a divisor on M with $|E| \subset M\backslash M^*$. We say that σ behaves like E if

$$dV_\sigma \sim |t|^{2k} dV_{\sigma_0}$$

close to each point $p_0 \in M\backslash M^*$, where $t\colon U(p_0) \to \mathbb{C}$ is a chart, $k = E(p_0)$ and dV_σ and dV_{σ_0} denote the volume forms which are given by σ and σ_0, respectively. If $E \geq 0$, then σ is a pseudo-metric on M, i. e. σ is positive semi-definite on M. Consider the metric $\gamma_E := (\pi^{-1})^*\sigma$ on X^* and the Dolbeault cohomology groups $H^{p,q}_{w,\gamma_E}(X, L)$ given by the square-integrable forms on X^* with respect to the metric γ_E. We already computed that the standard metric γ_{std} on X^* coincides with $\gamma_{Z-|Z|}$, where Z is the exceptional divisor of the resolution π.

Theorem 6.20. *Let X be a compact complex curve and $L \to X$ a holomorphic line bundle. Let $\pi\colon M \to X$ be a resolution of X, γ_E a Hermitian metric on X^*, where $\pi^*\gamma_E$ behaves like the divisor $E \geq 0$ on M whose support is contained in $\pi^{-1}(X_{\mathrm{sing}})$, and D is the divisor associated to π^*L. Then,*

$$\begin{aligned} H^{0,0}_{w,\gamma_E}(X,L) &\cong H^0(M,\mathcal{O}(E+D)),\\ H^{0,1}_{w,\gamma_E}(X,L) &\cong H^1(M,\mathcal{O}(E+D)),\\ H^{1,0}_{w,\gamma_E}(X,L) &\cong H^1(M,\mathcal{O}(-D)), \quad \textit{and}\\ H^{1,1}_{w,\gamma_E}(X,L) &\cong H^0(M,\mathcal{O}(-D)). \end{aligned}$$

In particular,

$$h^{0,0}_{w,\gamma_E}(X,L) - h^{0,1}_{w,\gamma_E}(X,L) = 1 - g + \deg E + \deg L.$$

The proof of this is the same as the proof of Theorem 6.15. The assumption $E \geq 0$ is necessary to see that (6.16) is exact.

6.4 Weakly holomorphic functions

In this section, we will prove Theorem 6.3 by studying weakly holomorphic functions and a localized version of the $\overline{\partial}_s$-operator.

Recalling the arguments at the beginning of Section 6.2, it is easy to see that the results of Section 6.1 generalize to arbitrary complex curves. In particular, the $\overline{\partial}_{s,w}$-holomorphic functions on a singular complex curve are precisely the square-integrable weakly holomorphic functions (cf. Lemma 6.9), and the $\overline{\partial}_{s,w}$-equation is locally solvable in the L^2-sense (combine Lemma 6.8 and Lemma 6.10).

Let X be a singular complex curve, $\mathcal{L}^{p,q}_X$ the sheaf of locally square-integrable forms, and let $\overline{\partial}_w\colon \mathcal{L}^{p,0}_X \to \mathcal{L}^{p,1}_X$ be the $\overline{\partial}$-operator in the sense of distributions. For each open set $U \subset X$, we define $\overline{\partial}_{s,\mathrm{loc}}$ on $L^{2,\mathrm{loc}}_{p,0}(U)$ as the restriction of $\overline{\partial}_w$ to

$$\begin{aligned} \operatorname{dom}\overline{\partial}_{s,\mathrm{loc}} := \{ & f \in \operatorname{dom}\overline{\partial}_w : \\ & f \in \operatorname{dom}\big(\overline{\partial}_{s,w} : L^2_{p,0}(V) \to L^2_{p,1}(V)\big) \ \forall V \Subset U\} \end{aligned}$$

(for more details, see Section 5.2 and [Rup14a, Sect. 6]). Let $\mathcal{F}^{p,0}_X$ be the sheaf of germs defined by

$$\mathcal{F}^{p,0}_X(U) := \operatorname{dom}\left(\overline{\partial}_{s,\mathrm{loc}} : L^{2,\mathrm{loc}}_{p,0}(U) \to L^{2,\mathrm{loc}}_{p,1}(U)\right)$$

Then, Lemma 6.9 implies that the sheaf of germs of weakly holomorphic functions $\widehat{\mathcal{O}}_X$ coincides with $\mathscr{K}er(\overline{\partial}_{s,\mathrm{loc}} : \mathcal{F}^{0,0}_X \to \mathcal{L}^{0,1}_X)$. Lemma 6.8 and Lemma 6.10 yield an exact sequence

$$0 \to \widehat{\mathcal{O}}_X = \mathscr{K}er\,\overline{\partial}_{s,\mathrm{loc}} \hookrightarrow \mathcal{F}^{0,0}_X \xrightarrow{\overline{\partial}_{s,\mathrm{loc}}} \mathcal{L}^{0,1}_X \to 0. \tag{6.21}$$

The sheaves $\mathcal{F}^{0,0}_X$ and $\mathcal{L}^{0,1}_X$ are fine and so (6.21) is a fine resolution of $\widehat{\mathcal{O}}_X$. Let $H^{p,q}_{s,\mathrm{loc}}(X)$ denote the $L^{2,\mathrm{loc}}$-Dolbeault cohomology on X^* with respect to the $\overline{\partial}_{s,\mathrm{loc}}$-operator. Using $\widehat{\mathcal{O}}_X = \pi_*\mathcal{O}_M$ and the first direct image $\pi_{(1)}\mathcal{O}_M = 0$, we deduce from (6.21) (by use of the Leray spectral sequence, cf. Theorem 8.5):

$$\begin{aligned} H^0(M,\mathcal{O}_M) &\cong H^0(X,\widehat{\mathcal{O}}_X) = H^{0,0}_{s,\mathrm{loc}}(X), \\ H^1(M,\mathcal{O}_M) &\cong H^1(X,\widehat{\mathcal{O}}_X) \cong H^{0,1}_{s,\mathrm{loc}}(X), \end{aligned}$$

where $\pi : M \to X$ is a resolution of X. This proves Theorem 6.3.

6.5 Applications

There are many applications of the classical Riemann-Roch theorem. We will transfer two of them to our situation to exemplify how the L^2-Riemann-Roch theorem can substitute the classical one on singular spaces.

6.5.1 Compact complex curves as covering spaces of $\mathbb{CP}^1$

Let X be a compact irreducible complex curve with $X_{\text{sing}} = \{x_1, .., x_k\}$, let $(h_i)_{x_i} \in \mathcal{O}_{x_i}$ be chosen such that $(h_i)_{x_i} \widehat{\mathcal{O}}_{x_i} \subset \mathcal{O}_{x_i}$ and let $U_i \subset X$ be a (Stein) neighbourhood of x_i with $h_i \cdot \widehat{\mathcal{O}}(U_i) \subset \mathcal{O}(U_i)$ (for the existence of the h_i, see e. g. Thm. 6 and its Cor. in [Nar66, § III.2]). Choose an $x_0 \in X^* := X_{\text{reg}}$ and a (Stein) neighbourhood U_0 of x_0. We can assume that U_0, ..., U_k are pairwise disjoint.

We define a line bundle $L \to X$ as follows. Let $U_{k+1} = X^* \setminus \{x_0\}$ and choose $f_0 \in \mathcal{O}(U_0)$ such that f_0 is vanishing to the order $r := \text{ord}_{x_0} f_0 \geq 1$ in x_0, which we will determine later, but has no other zeros. We also set $f_i := 1/h_i$ for $i = 1, ..., k$ and $f_{k+1} = 1$, and consider the Cartier divisor $\{(U_i, f_i)\}_{i=0,...,k+1}$ on X. Let $L \to X$ be the line bundle associated to this divisor. As the f_i have no zeros for $i > 0$, there exists a non-negative integer δ such that $\deg L = r - \delta$. Now choose $r := g(X) + \delta + 1$. It follows that $\deg L = g(X) + 1$. Give L an arbitrary smooth Hermitian metric.

There is a canonical way to identify holomorphic sections of L with meromorphic functions on X. A holomorphic section $s \in \mathcal{O}(L)$ is represented by a tuple $\{s_i\}_i$ where $s_j/f_j = s_l/f_l$ on $U_j \cap U_l$. This gives a meromorphic function $\Psi(s)$ by setting $\Psi(s) := s_j/f_j$ on U_j. Note that $\Psi(s)$ has zeros in the singular points $x_1, ..., x_k$ and may have a pole of order r at $x_0 \notin X_{\text{sing}}$.

We can now apply our L^2-Riemann-Roch theorem. The $\overline{\partial}_s$-Riemann-Roch theorem, Corollary 6.19, implies $\dim H^{0,0}_s(X,L) \geq 1 - g(X) + \deg L = 2$. Therefore, there is a section $\tau \in L^2(X^*, L)$ with $\overline{\partial}_s \tau = 0$ and τ_{k+1} is non-constant, where $\tau = \{\tau_i\}_{i=0,..,k+1}$ is written in the trivialization as above. This means that $\tau_i \in L^2(X^* \cap U_i)$, $\overline{\partial}_s \tau_i = 0$, and $\tau_j / f_j = \tau_{k+1}$ is non-constant on $U_j \cap U_{k+1}$. Theorem 6.3 implies that $\tau_i \in \widehat{\mathcal{O}}(U_i)$, $i = 1, ..., k+1$. Now consider $\Psi(\tau)$ as defined above, i. e. $\Psi(\tau) = \tau_i / f_i$ on U_i. We conclude that $\Psi(\tau)/h_i \in \widehat{\mathcal{O}}(U_i)$, thus $\Psi(\tau) \in \mathcal{O}(U_i)$ for $i = 1, ..., k$. Moreover, $\Psi(\tau)$ is non-constant, so it cannot be holomorphic on the whole compact space X, thus, must have a pole of some order $\leq r$ in x_0. Hence, we get

$$\Psi(\tau)\colon X \setminus \{x_0\} \to \mathbb{C}, \quad \text{and}$$
$$\widetilde{\Psi}(\tau)\colon X \to \mathbb{CP}^1, x \mapsto \begin{cases} [\Psi(\tau)(x) : 1], & x \neq x_0 \\ [1 : \frac{1}{\Psi(\tau)(x)}], & x \in U_0 \end{cases}$$

are finite, open and, therefore, analytic ramified coverings (see e. g. Covering Lemma in [GR84, Sect. VII.2.2]). In particular, $X \setminus \{x_0\}$ is Stein (use e. g. Thm. 1 in [GR79, § V.1]).

6.5.2 Projectivity of compact complex curves

A line bundle $L \to X$ on a compact complex space is called very ample if its global holomorphic sections induce a holomorphic embedding into the projective space $\mathbb{CP}^N$, i. e. if $s_0, ..., s_N$ is a basis of the space of holomorphic sections $\mathcal{O}(X, L)$, then the map

$$\Phi\colon X \to \mathbb{CP}^N \ , \ x \mapsto [s_0(x) : ... : s_N(x)], \tag{6.22}$$

given in local trivializations of the s_i, defines a holomorphic embedding of X in $\mathbb{CP}^N$. If some positive power of the line bundle has this property, then we say that it is ample. A compact complex space is called projective if there is an ample (and, hence, a very ample) line bundle on it.

A classical application of the Riemann-Roch theorem is that any compact Riemann surface is projective, and a line bundle on a Riemann surface is ample if its degree is positive (cf. e. g. [Nar92, Sect. 10]). This generalizes to singular complex curves:

Theorem 6.23. *Let X be a compact locally irreducible complex curve. If $L \to X$ is a holomorphic line bundle with $\deg L \gg 0$, then L is very ample. In particular, X is projective and each holomorphic line bundle on X is ample if its degree is positive.*

Clearly, this result is well-known and follows from more general sheaf-theoretical methods (vanishing theorems) once one knows that L is positive if and only if $\deg L > 0$ (cf. e. g. Thm. 4.4 in [Pet94b] or Satz 2 in [Gra62, § 3]). Nevertheless, it seems interesting to us to present another proof of Theorem 6.23 which is based on the L^2-Riemann-Roch of singular complex curves. The assumption that X must be locally irreducible in Theorem 6.23 is not necessary. One can prove the result without this assumption easily by the same technique. Yet, to keep the notation simple, we present here only the locally irreducible case.

Let us make some preparations for the proof of Theorem 6.23. Let X be a connected complex curve and $\pi\colon M \to X$ a resolution of X. We choose a point $x_0 \in X_{\text{sing}}$ and a small neighbourhood $U \subset X$ of x_0 with $U^* := U \backslash \{x_0\} \subset X_{\text{reg}}$. Assume X is irreducible at x_0. We define $p_0 := \pi^{-1}(x_0)$, $V := \pi^{-1}(U)$, and $V^* := V\backslash\{p_0\}$. We can assume that there is a chart $t\colon V \to \mathbb{C}$ such that the image of t is bounded.

The Riemann extension theorem implies that $\pi^{-1}\colon U \to V$ is weakly holomorphic or, briefly, $\tau := t \circ \pi^{-1} \in \widehat{\mathcal{O}}(U)$ (see Theorem 2.13). We show that τ generates the weakly holomorphic functions at x_0 in the following sense: Let f be in $\widehat{\mathcal{O}}(U)$. Then, $f \circ \pi$ is holomorphic on V^* and bounded in p_0. This implies that $f \circ \pi$ is holomorphic on V, $f \circ \pi(t) = \sum_{\iota=0}^{\infty} a_\iota t^\iota$, and

$$f(x) = \sum_{\iota=0}^{\infty} a_\iota \tau(x)^\iota$$

(by shrinking U and V if necessary). This allows to define the order $\operatorname{ord}_{x_0} f$ of vanishing of f in x_0 by $r \in \mathbb{N}_0$ if $a_r \neq 0$ and $a_\iota = 0$ for $\iota < r$. In particular,

$$\operatorname{ord}_{x_0} f = \operatorname{ord}_{p_0}(f \circ \pi).$$

Note that this definition does not depend on the resolution as different resolutions are biholomorphically equivalent (see Section 2.2).

The L^2-extension theorem (see Theorem 5.6) and (6.14) imply

$$\begin{aligned} f \in H_w^{0,0}(U) &\Leftrightarrow t^{r_0} \cdot \pi^* f \in \mathcal{O}_{L^2}(V) \\ &\Leftrightarrow \big(\tau^{r_0} \cdot f \in \widehat{\mathcal{O}}(U) \text{ and } f \in L^2(U)\big), \end{aligned} \tag{6.24}$$

where Z denotes the exceptional divisor of the resolution and $r_0 := \deg_{p_0}(Z - |Z|)$. In particular, we get the representation $f(x) = \sum_{\iota \geq -r_0} a_\iota \tau(x)^\iota$ and $\operatorname{ord}_{x_0} f := \operatorname{ord}_{p_0} \pi^* f \geq -r_0$ is again well-defined. f is weakly holomorphic if and only if $\operatorname{ord}_{x_0} f \geq 0$.

We denote by L_{x_0} the holomorphic line bundle on X which is trivial on $X \setminus \{x_0\}$ and is given by τ on U, i. e. the line bundle on X given by the open covering $U_1 := X \setminus \{x_0\}, U_0 := U$ and the transition function $g_{01} := \tau \colon U_0 \cap U_1 \to \mathbb{C}$. Then, $\pi^* L_{x_0} \cong L_{p_0}$, where L_{p_0} is the holomorphic line bundle $L_{p_0} \to M$ associated to the divisor $\{p_0\}$.

Let $L \to X$ be any holomorphic line bundle, $L' := L \otimes L_{x_0}^{-1}$, and let s' be a section in $H_w^{0,0}(X, L')$. We can assume that L and L' are given by divisors $\{(U_j, f_j)\}$ and $\{(U_j, f'_j)\}$, respectively, where $\{U_j\}$ is an open covering of X with $U_0 = U$ and $x_0 \notin U_j$ for $j \neq 0$ and where $f_j, f'_j \in \mathcal{M}(U_j)$ with $g_{jk} := f_j/f_k$ and $g'_{jk} := f'_j/f'_k$ in $\mathcal{O}(U_j \cap U_k)$ ($g_{j,k}$ and g'_{jk} are the transition functions of L and L', respectively).

We get $f_0 = f'_0 \cdot \tau$ and $f_j = f'_j$ for $j \neq 0$. There is a meromorphic function $\widetilde{s} := \Psi(s') \in \mathcal{M}(X)$ representing s'. This meromorphic function is defined by $\widetilde{s} = s'_j/f'_j$ on U_j, where s'_j is the trivialization of s' on U_j. We can define a section $s = \{s_j\} \in H_w^{0,0}(X, L)$ by $s_j = \widetilde{s} \cdot f_j$. Thus, $s_0 = s'_0 \cdot \tau$ and $s_j = s'_j$ for $j \neq 0$. Hence, $\operatorname{ord}_{x_0} s_0 = \operatorname{ord}_{x_0} s'_0 + 1$. Summarizing, we get an injective linear map

$$T \colon H_w^{0,0}(X, L \otimes L_{x_0}^{-1}) \to H_w^{0,0}(X, L),\ s' \mapsto s,$$

which we call the *natural inclusion*. It follows from the construction above and by use of (6.24) that each section $s \in H^{0,0}_w(X,L)$ with $\operatorname{ord}_{x_0} s_0 > -r_0$ is in the image of T.

As $H^1(M,\mathcal{O}(D')) = 0$ for a divisor D' with $\deg D' > 2g-2$ by the classical Riemann-Roch theorem (cf. e. g. [Nar92, Sect. 10]), Theorem 6.15 (more precisely, $H^{0,1}_w(X,L) \cong H^1(M,\mathcal{O}(Z-|Z|+D))$) implies the following vanishing theorem.

Theorem 6.25. *If $L \to X$ is a holomorphic line bundle on an irreducible compact complex curve X with $\deg L > 2g-2-\sum_{x\in X_{\text{sing}}} \operatorname{mult}'_x X$, then $H^{0,1}_w(X,L) = 0$.*

As a preparation for the proof of Theorem 6.23, we get our main ingredient:

Lemma 6.26. *Let $L \to X$ be a holomorphic line bundle on a connected compact locally irreducible complex curve X with $\deg L > 2g-1-\sum_{x\in X_{\text{sing}}} \operatorname{mult}'_x X$. Then, the natural inclusion*

$$T\colon H^{0,0}_w(X, L\otimes L^{-1}_{x_0}) \to H^{0,0}_w(X,L)$$

is not surjective. If $\deg L > 2g+r_0-1-\sum_{x\in X_{\text{sing}}} \operatorname{mult}'_x X$, then there is a section $s \in H^{0,0}_w(X,L)$ which is weakly holomorphic on $U(x_0)$ and does not vanish in x_0.

Recall that $r_0 = \operatorname{mult}_{x_0} X - 1 = \deg_{p_0}(Z-|Z|)$.

Proof: (i) As $\pi^*(L\otimes L^{-1}_{x_0}) \cong \pi^* L \otimes L^{-1}_{p_0}$, we get $\deg L\otimes L^{-1}_{x_0} = \deg L - 1 > 2g-2-\deg(Z-|Z|)$. The $\overline{\partial}_w$-Riemann-Roch theorem and $h^{0,1}_w(X,L) = 0 = h^{0,1}_w(X,L\otimes L^{-1}_{x_0})$ (using Theorem 6.25) yield

$$\begin{aligned} h^{0,0}_w(X,L\otimes L^{-1}_{x_0}) &= 1-g+\deg(Z-|Z|)+\deg L\otimes L^{-1}_{x_0} \\ &< 1-g+\deg(Z-|Z|)+\deg L = h^{0,0}_w(X,L). \end{aligned}$$

Therefore, the natural inclusion T cannot be surjective.

(ii) The image of $T^{r_0}\colon H^{0,0}_w(X,L\otimes L^{-r_0}_{x_0}) \to H^{0,0}_w(X,L)$ are the sections s with $\operatorname{ord}_{x_0} s_0 \geq 0$, i. e. s_0 is weakly holomorphic on $U(x_0)$, where s_0 denotes the trivialization of s over $U(x_0)$. As $H^{0,0}_w(X,L\otimes L^{-r_0-1}_{x_0}) \to H^{0,0}_w(X,L\otimes L^{-r_0}_{x_0})$ is not surjective (use $\deg L\otimes L^{-r_0}_{x_0} =$

$\deg L - r_0 > 2g - 1 - \deg(Z - |Z|)$ and part (i)), there is a section $s' \in H_w^{0,0}(X, L \otimes L_{x_0}^{-r_0})$ with $\operatorname{ord}_{x_0} s'_0 = -r_0$ and $\operatorname{ord}_{x_0}(T^{r_0}(s'))_0 = 0$. So, $s := T^{r_0}(s')$ is the section of $H_w^{0,0}(X, L)$ that we were looking for. $\square$

Proof of Theorem 6.23: Let X be a connected compact locally irreducible complex curve with $X_{\text{sing}} = \{x_1, ..., x_k\}$ and $L \to X$ a line bundle with $\deg L \gg 0$. Following the classical arguments in order to show that the map Φ in (6.22) is a well-defined holomorphic embedding (see e. g. [Pet94b, Thm. 4.4]), we have to prove:

(i) Φ is well-defined: For $x \in X$, there exists an $s \in \mathcal{O}(X, L)$ such that $s(x) \neq 0$.
(ii) Φ is injective: For $x, y \in X$, $x \neq y$, there exists an $s \in \mathcal{O}(X, L)$ such that $s(x) \neq 0$ and $s(y) = 0$.
(iii) Φ is an immersion: For $x \in X$, the differential $T_x\Phi$ is injective.

Since (obviously) Φ is closed, (ii) and (iii) imply that Φ is an embedding (see e. g. Sect. 1.2.7 in [GR84]).

We will prove the statements (i) and (iii) for singular points $x \in X_{\text{sing}}$. The case of regular points is simpler and follows easily with the natural inclusion and Lemma 6.26. The statement (ii) can be seen as (i) by imposing the additional condition that $s(y) = 0$.

Let $\pi\colon M \to X$ be a resolution of singularities. Set $X^* = X_{\text{reg}}$, $M^* = \pi^{-1}(X^*)$, $p_j := \pi^{-1}(x_j)$, and $r_j := \deg_{p_j}(Z - |Z|)$, where Z is the unreduced exceptional divisor of the resolution.

Fix a $\mu \in \{1, ..., k\}$, choose a neighbourhood U_μ of x_μ such that there exists a chart $t\colon V_\mu \to U_\mu$ with $V_\mu \subset \mathbb{C}$ and $t_\mu(0) = x_\mu$, and set $\tau := t^{-1}$.

For each singularity x_j, we can choose a function $h_j \in \mathcal{O}(U_j)$ such that $h_j \cdot \widehat{\mathcal{O}}(U_j) \subset \mathcal{O}(U_j)$ for a neighbourhood U_j of x_j small enough (see [Nar66, § III.2]). The number

$$\eta_j := \operatorname{ord}_{x_j} h_j$$

is important for our considerations because of the following fact: If f is a function on U_j with $\operatorname{ord}_{x_j} f \geq \eta_j$, then f/h_j is bounded at x_j ($\operatorname{ord}_{x_j} f/h_j \geq 0$). This implies $f/h_j \in \widehat{\mathcal{O}}(U_j)$ and, hence, $f \in \mathcal{O}(U_j)$. For the maximal ideal in $\mathcal{O}_{X,x_j}$, we get $\mathfrak{m}_{x_j} = \{f \in \mathcal{O}_{X,x_j} \colon \operatorname{ord}_{x_j} f > 0\}$ and $\{f \in \mathcal{O}_{X,x_j} \colon \operatorname{ord}_{x_j} f \geq 2\eta_j\} \subset \mathfrak{m}_{x_j}^2$.

We can choose a weakly holomorphic section $\sigma \in H_w^{0,0}(X, L)$ such that σ does not vanish in x_μ and $\operatorname{ord}_{x_j} \sigma \geq \eta_j$ for $j \neq \mu$. This section σ exists as we have the natural inclusion (see the construction above)

$$H_w^{0,0}\Big(X, L \otimes L_{x_\mu}^{-r_\mu} \otimes \bigotimes_{j\neq\mu} L_{x_j}^{-\eta_j - r_j}\Big) \to H_w^{0,0}(X, L)$$

and $\deg L \gg 0$ implies by Lemma 6.26 that the natural inclusion

$$H_w^{0,0}\Big(X, L \otimes L_{x_\mu}^{-r_\mu - 1} \otimes \bigotimes_{j\neq\mu} L_{x_j}^{-\eta_j - r_j}\Big) \to H_w^{0,0}\Big(X, L \otimes L_{x_\mu}^{-r_\mu} \otimes \bigotimes_{j\neq\mu} L_{x_j}^{-\eta_j - r_j}\Big)$$

is not surjective.

Note that σ is holomorphic on $X - \{x_\mu\}$ but just weakly holomorphic in x_μ. We will now modify σ so that it becomes holomorphic and non-vanishing in x_μ. Shrink U_μ such that $\sigma = \sum_{\iota \geq 0} a_\iota \tau^\iota$ on U_μ with $a_0 \neq 0$. Let $\sigma' := \sigma/a_0$ so that $\operatorname{ord}_{x_\mu}(\sigma' - 1) \geq 1$, i. e. $\sigma' - 1 = \sum_{\iota\geq 1} a'_\iota \tau^\iota$ on U_μ. Choose as above a $\widetilde{\sigma} \in H_w^{0,0}(X, L)$ with $\operatorname{ord}_{x_\mu} \widetilde{\sigma} = 1$ and $\operatorname{ord}_{x_j} \widetilde{\sigma} \geq \eta_j$ for $j \neq \mu$. Let $\widetilde{\sigma} = \sum_{\iota\geq 1} \widetilde{a}_\iota \tau^\iota$ close to x_μ with $\widetilde{a}_1 \neq 0$. We define $\sigma'' := \sigma' - \frac{a'_1}{\widetilde{a}_1}\widetilde{\sigma}$. Then, $\operatorname{ord}_{x_\mu}(\sigma'' - 1) \geq 2$ and $\operatorname{ord}_{x_j} \sigma'' \geq \eta_j$ for $j \neq \mu$. We repeat this procedure recursively to get a section $\xi = \{\xi_j\} \in H_w^{0,0}(X, L)$ with $\operatorname{ord}_{x_\mu}(\xi_\mu - 1) \geq \eta_\mu$ and $\operatorname{ord}_{x_j} \xi_j \geq \eta_j$ for $j \neq \mu$. Thus, ξ is a holomorphic section on X, non-vanishing in x_μ. That shows (i) for $x = x_\mu$.

We will prove (iii) for x_μ. Let $v \in T_{x_\mu}X \overset{(2.16)}{=} (\mathfrak{m}_{x_\mu}/\mathfrak{m}_{x_\mu}^2)^*$ satisfy $v \neq 0$, i. e. there exists an $f \in \mathfrak{m}_{x_\mu}$ with $v(f + \mathfrak{m}_{x_\mu}^2) \neq 0$. We claim there exists a $g \in \mathfrak{m}_{\Phi(x_\mu)}$ with $g \circ \Phi - f \in \mathfrak{m}_{x_\mu}^2$. Then, $v(g\circ\Phi + \mathfrak{m}_{x_\mu}^2) = v(f + \mathfrak{m}_{x_\mu}^2) \neq 0$, i. e. $T_x\Phi(v) \neq 0$.

Proof of the claim: Replacing 1 with $f = \sum_{\iota\geq 1} f_\iota \tau^\iota$, we can repeat the procedure in (i) to construct a section $\xi = \{\xi_j\} \in H_w^{0,0}(X, L)$

with $\operatorname{ord}_{x_\mu}(\xi_\mu - f) \geq 2\eta_\mu$ and $\operatorname{ord}_{x_j} \xi_j \geq \eta_j$ for $j \neq \mu$. We get ξ is holomorphic, $\xi_\mu \in \mathfrak{m}_{x_\mu}$ and

$$\xi_\mu - f \in \mathfrak{m}_{x_\mu}^2.$$

Let Φ be defined by $\Phi(x) = [s_0(x){:}...{:}s_N(x)]$ with holomorphic sections $s_i = \{s_{ij}\}$ (see (6.22)). Hence, we can choose a vector $(g_0, ..., g_N) \in \mathbb{C}^{N+1}$ such that $\xi = \sum_i g_i s_i$. Because of (i), there exits an i_0 such that $c := s_{i_0\mu}(x_\mu) \neq 0$ – we can assume $i_0 = 0$. We set $U := \{x \in U_\mu \colon s_{0\mu}(x) \neq 0\}$ and identify $\{[t_0{:}...{:}t_N]\colon t_0 = 1\} \subset \mathbb{CP}^N$ with $\mathbb{C}^N$ such that $\Phi|_U \colon U \to \mathbb{C}^N$ is defined by $\Phi(x) = \left(\frac{s_{1\mu}(x)}{s_{0\mu}(x)}, ..., \frac{s_{N\mu}(x)}{s_{0\mu}(x)}\right)$. Let $g \colon \mathbb{C}^N \to \mathbb{C}$ denote the holomorphic function $g(t_1, ..., t_N) := c \cdot (g_0 + \sum_{i=1}^N g_i t_i)$, i. e.

$$s_{0\mu} \cdot (g \circ \Phi|_U) = c \sum\nolimits_{i=0}^N g_i s_{i\mu} = c \cdot \xi_\mu$$

on U. Since $c = s_{0\mu}(x_\mu) \neq 0$ and since f and $\frac{c}{s_{0\mu}} - 1$ are in $\mathfrak{m}_{x_\mu}$, we get $g \in \mathfrak{m}_{\Phi(x_\mu)}$ and

$$g \circ \Phi - f = \frac{c}{s_{0\mu}}(\xi_\mu - f) + f \cdot \left(\frac{c}{s_{0\mu}} - 1\right) \in \mathfrak{m}_{x_\mu}^2. \qquad \square$$

For this proof, L has to satisfy

$$\begin{aligned} \deg L &> 2g + \max\{\eta_j\} + \sum\nolimits_{j=1}^k (\eta_j + r_j) - \deg(Z - |Z|) \\ &= 2g + \max\{\eta_j\} + k + \sum\nolimits_{j=1}^k \eta_j. \end{aligned}$$

Chapter 7
Nakano Semi-positive Vector Bundles on Complex Manifolds

In this chapter, we first generalize K. Takegoshi's vanishing theorem [Tak85, Thm. 2.1] to Nakano semi-positive vector bundles, using mainly the argumentation as in Takegoshi's original proof (also presented in [Ser15, Sect. 2]). In Section 7.1, we recall the definition of positivity for vector bundles. The key lemmata (positivity statements and an a-priori-estimate) are achieved by L^2-methods elaborated by J.-P. Demailly in [Dem02] (see Section 7.2). In Section 7.3, we state L^2-vanishing theorems for Nakano semi-positive vector bundles on complex manifolds including generalizations of Takegoshi and Donnelly-Fefferman-Ohsawa.

7.1 Positivity for vector bundles

Def. 7.1. Let M be a Hermitian manifold of dimension n, and let $E \to M$ be a holomorphic vector bundle of rank r with Hermitian metric $\langle \cdot, \cdot \rangle_E$. We call a tensor $u \in T_xM \otimes E_x$ of rank m if $m \geq 0$ is the smallest integer such that u is the sum $\sum_{j=1}^{m} \xi_j \otimes s_j$ of m pure / simple tensors $\xi_j \otimes s_j$ with $\xi_j \in T_xM$ and $s_j \in E_x$. A Hermitian form h on $T_xM \otimes E_x$ is called *m-(semi-)positive* if $h(u,u) > 0$ (or ≥ 0 resp.) for every tensor $u \in T_xM \otimes E_x \backslash \{0\}$ of rank $\leq m$. We say E is m-(semi-)positive if the Hermitian form associated to the Chern curvature $\mathrm{i}\Theta(E)$ is m-(semi-)positive in each point $x \in M$ and write $E >_m 0$ (or $E \geq_m 0$ resp.). We call E *Griffiths (semi-) positive* if E is

1-(semi-)positive, and *Nakano (semi-) positive* if E is $\min\{n,r\}$-(semi-) positive, i. e. $\mathrm{i}\Theta(E)$ is (semi-) positive in the classical sense.

If ω denotes the Hermitian form on M, then Λ denotes the formal adjoint of the operator $\omega\wedge\cdot$, and $\langle\cdot,\cdot\rangle_{\omega,E}$ is the metric on $\Lambda^{s,t}T^*M\otimes E$ given by ω and the Hermitian metric $\langle\cdot,\cdot\rangle_E$.

As a more or less immediate consequence of Def. 7.1, we get (see e. g. Lem. VII.7.2 in [Dem12]):

Lemma 7.2. *Let E be an m-semi-positive holomorphic vector bundle on a Kähler manifold M of rank r. If $t\geq 1$ and $m\geq\min\{n-t+1,r\}$, then the Hermitian operator $\mathrm{i}\Theta(E)\wedge\Lambda$ is semi-positive definite on $\Lambda^{n,t}T^*M\otimes E$ and, particularly,*

$$\langle \mathrm{i}\Theta(E)\wedge\Lambda w, w\rangle_{\omega,E}\geq 0 \quad \forall\ w\in\mathscr{D}^{n,t}(M,E).$$

Sketch of the proof: The proof is more or less straight forward but very technical. Hence, we will just present a sketch of it to emphasize where the assumptions are used.

Let ω denote the Kähler form of X and fix a point $x\in M$. We can choose coordinates $(z_1,..,z_n)$ of M around x with orthonormal $(\frac{\partial}{\partial z_1},..,\frac{\partial}{\partial z_n})$ and such that $\omega=\omega(x)=\mathrm{i}\sum_{j=1}^n dz_j\wedge d\overline{z}_j$. Let $(e_1,..,e_r)$ be an orthonormal frame of E. Then, there exist coefficients $c_{jk\lambda\mu}\in\mathbb{C}$ with

$$\mathrm{i}\Theta(E)=\mathrm{i}\Theta(E)(x)=\mathrm{i}\sum_{j,k,\lambda,\mu} c_{jk\lambda\mu}\, dz_j\wedge d\overline{z}_k\otimes e_\lambda^*\otimes e_\mu$$

and $c_{kj\mu\lambda}=\overline{c_{jk\lambda\mu}}$. In particular,

$$\theta_E\Big(\sum_{j,\lambda}u_{j\lambda}\tfrac{\partial}{\partial z_j}\otimes e_\lambda,\sum_{k,\mu}v_{k\mu}\tfrac{\partial}{\partial z_k}\otimes e_\mu\Big):=\sum_{j,k,\lambda,\mu}c_{jk\lambda\mu}\cdot u_{j\lambda}\overline{v}_{k\mu}$$

is a Hermitian form on $T_xM\otimes E_x\cong\mathbb{C}^{nr}$. The m-semi-positivity of E can be characterized by: $\theta_E(u,u)$ is semi-positive for all tensors $u\in T_xM\otimes E_x$ of rank $\leq m$.

For all (n,t)-forms $w=\sum_{K,\lambda}w_{K,\lambda}\,dz_{\{1,..,n\}}\wedge d\overline{z}_K\otimes e_\lambda$, we get

$$\langle\mathrm{i}\Theta(E)\wedge\Lambda w,w\rangle=\sum_{|S|=t-1}\sum_{j,k,\lambda,\mu}c_{jk\lambda\mu}w_{jS,\lambda}\overline{w}_{kS,\mu}.$$

Actually, a similar formula can be computed for forms of arbitrary degree. Yet, there occur summands whose signs will not be manageable. Fortunately, these summands vanish for (n,t)-forms.

Since $w_{jS} = 0$ for $j \in S$, we obtain

$$\langle \mathrm{i}\Theta(E) \wedge \Lambda w, w \rangle = \sum_{|S|=t-1} \theta_E(\widetilde{w}_S, \widetilde{w}_S),$$

where $\widetilde{w}_S := \sum_{j \notin S, \lambda \in \{1,..,r\}} w_{jS,\lambda}$. Yet, $\widetilde{w}_S$ is a tensor of rank less or equal to both, r and $n-|S| = n-t+1$. Since E is r- and $(n-t+1)$-semi-positive, we get $\theta_E(\widetilde{w}_S, \widetilde{w}_S) \geq 0$. □

7.2 A-priori-estimates for the $\overline{\partial}$-operator

Let M be a weakly 1-complete Kähler manifold of dimension n, ω the Kähler form on M, and let Φ be a plurisubharmonic smooth exhaustion function of M. Let $\lambda\colon \mathbb{R} \to \mathbb{R}$ be an increasing convex smooth function with $\lambda(t) = 0$ for $t \leq 0$ and $\int \sqrt{\lambda''(t)}dt = +\infty$. By replacing, firstly, Φ by $\lambda \circ \exp \circ \Phi$ and, secondly, ω by $\omega + \mathrm{i}\partial\overline{\partial}\Phi$, we can assume that $\Phi > 0$ and that the Kähler metric associated to ω is complete (see e. g. Prop. 12.10 in [Dem02]).

Let dV be the volume form on M induced by ω. We define the weighted product of u and v in $\mathscr{D}^{s,t}(M,E)$ by

$$(u,v)_\Phi := \int_M \langle u, v \rangle_{\omega,E} \cdot e^{-\Phi} dV$$

and set $\|u\|_\Phi := \sqrt{(u,u)_\Phi}$. Let $\vartheta_\Phi\colon \mathscr{E}^{s,t+1}(M,E) \to \mathscr{E}^{s,t}(M,E)$ denote the formal adjoint of $\overline{\partial}\colon \mathscr{E}^{s,t}(M,E) \to \mathscr{E}^{s,t+1}(M,E)$ with respect to $(\,\cdot\,,\,\cdot\,)_\Phi$.

We obtain the following a-priori-estimate for the $\overline{\partial}$-operator:

Lemma 7.3. *For all $t \geq 1$ and $w \in \mathscr{D}^{n,t}(M,E)$:*

$$(\mathrm{i}(\Theta(E) + \partial\overline{\partial}\Phi) \wedge \Lambda w, w)_\Phi \leq \|\overline{\partial} w\|_\Phi^2 + \|\vartheta_\Phi w\|_\Phi^2.$$

Proof: Let L denote the trivial line bundle $M\times\mathbb{C}$ with the Hermitian metric $\langle\,\cdot\,,\cdot\,\rangle_L := \langle\,\cdot\,,\cdot\,\rangle_{\mathbb{C}} e^{-\Phi}$. Then, $\mathrm{i}\Theta(L) = \mathrm{i}\partial\overline{\partial}\Phi$. Let $F := E\otimes L$ denote the vector bundle with the metric $\langle\,\cdot\,,\cdot\,\rangle_F$ induced by $\langle\,\cdot\,,\cdot\,\rangle_E$ and $\langle\,\cdot\,,\cdot\,\rangle_L$. We can assume $\mathscr{D}^{s,t}(M,E) = \mathscr{D}^{s,t}(M,F)$ by identifying $u\otimes g = (g\wedge u)\otimes 1$ with $g\wedge u$ for $u\in\mathscr{D}^{s,t}(M,E), g\in\mathcal{O}(L)$. Thus, we get $\langle\,\cdot\,,\cdot\,\rangle_F = \langle\,\cdot\,,\cdot\,\rangle_E\cdot e^{-\Phi}$. With

$$\Theta(F) = \Theta(E)\otimes\mathrm{id}_L + \mathrm{id}_E\otimes\Theta(L)$$

(cf. [Dem12, § V.4]), we conclude for $u\in\mathscr{D}^{s,t}(M,E)$:

$$\begin{aligned}\Theta(F)\wedge u = \Theta(F)\wedge(u\otimes 1) &= (\Theta(E)\wedge u)\otimes 1 + u\otimes\Theta(L)\\ &= (\Theta(E)+\Theta(L))\wedge u\otimes 1 = (\Theta(E)+\partial\overline{\partial}\Phi)\wedge u.\end{aligned}$$

Let $\Delta_F^{(i)} := D_F^{(i)}D_F^{(i)*} + D_F^{(i)*}D_F^{(i)}$ denote the Laplace-Beltrami operators, where $D_F = D'_F + D''_F$ is the Chern connection with respect to the metric $\langle\,\cdot\,,\cdot\,\rangle_E e^{-\Phi}$. Note that $D''_F = \overline{\partial}$ and $(D''_F)^* = \vartheta_\Phi$. Recall the Bochner-Kodaira-Nakano identity (see e. g. [Dem02, Sect. 13.2]):

$$\Delta''_F = \Delta'_F + [\mathrm{i}\Theta(F),\Lambda],$$

where $[\,\cdot\,,\cdot\,]$ denotes the graded Lie bracket, i. e. $[P_1,P_2] = P_1P_2 + (-1)^{p_1\cdot p_2}P_2P_1$ for differential operators P_i with degree p_i, $i=1,2$.

Integration by parts yields

$$(\Delta_F^{(i)}u,u)_\Phi = \|D_F^{(i)}u\|_\Phi^2 + \|(D_F^{(i)})^*u\|_\Phi^2 \quad \forall\, u\in\mathscr{D}^{s,t}(M,E).$$

Altogether, we conclude

$$\begin{aligned}\|\overline{\partial}w\|_\Phi^2 + \|\vartheta_\Phi w\|_\Phi^2 &= (\Delta''_F w,w)_\Phi\\ &= \|D'_F w\|_\Phi^2 + \|(D'_F)^*w\|_\Phi^2 + ([\mathrm{i}\Theta(F),\Lambda]w,w)_\Phi\\ &\geq \int_M \langle\mathrm{i}\Theta(F)\wedge\Lambda w,w\rangle_{\omega,E}e^{-\Phi}dV\\ &= (\mathrm{i}(\Theta(E)+\partial\overline{\partial}\Phi)\wedge\Lambda w,w)_\Phi\end{aligned}$$

for all $w\in\mathscr{D}^{n,t}(M,E)$. □

We will combine the a-priori-estimate with the following positivity statement:

Lemma 7.4. *Choose some integers q, t with $t \geq q \geq 1$. Then, there is a non-negative (bounded) continuous function δ on M such that $\delta(x) > 0$ for all points $x \in M$ satisfying $\operatorname{rk} H(\Phi)_x > n - q$, and*

$$\delta \cdot \langle w, w \rangle_{\omega,E} \leq \langle \mathrm{i}\partial\overline{\partial}\Phi \wedge \Lambda w, w \rangle_{\omega,E} \quad \forall\ w \in \mathscr{D}^{n,t}(M,E).$$

Proof: Fix a point x in M. There is a base $\left\{\frac{\partial}{\partial z_j}\right\}_{j=1}^{n}$ of $T_x^{\mathbb{C}}M$ such that

$$\omega(x) = \mathrm{i}\sum_{j=1}^{n} dz_j \wedge d\overline{z}_j \quad \text{and} \quad \mathrm{i}\partial\overline{\partial}\Phi(x) = \mathrm{i}\sum_{j=1}^{n} \delta_j dz_j \wedge d\overline{z}_j$$

with $\delta_1, ..., \delta_n \in \mathbb{R}$. For $u = \sum u_{JK} dz_J \wedge d\overline{z}_K \in \mathscr{D}^{s,t}(M)$, we get (see e. g. Prop. 6.8 in [Dem02, Sect. 6.B])

$$\left[\mathrm{i}\partial\overline{\partial}\Phi, \Lambda\right] u(x) = \sum_{J,K} \left(\sum_{j\in J} \delta_j + \sum_{j \in K} \delta_j - \sum_{j=1}^{n} \delta_j \right) u_{J,K}(x) dz_J \wedge d\overline{z}_K. \tag{7.5}$$

Let us define $\delta_{J,K} := \sum_{j\in J}\delta_j + \sum_{j\in K}\delta_j - \sum_{j=1}^{n}\delta_j$. Choose a frame $\{e_1, ..., e_r\}$ of E on a small open neighbourhood of x. Let $(h_{\lambda\mu})$ denote the Hermitian matrix associated to the Hermitian metric $\langle\,\cdot\,,\cdot\,\rangle_E$ on E such that $\langle u, v\rangle_{\omega,E} = \sum_{\lambda,\mu=1}^{r} h_{\lambda\mu}\langle u_\lambda, v_\mu\rangle_\omega$ for $u = \sum u_\lambda \otimes e_\lambda$, $v = \sum v_\lambda \otimes e_\lambda \in \mathscr{D}^{s,t}(M,E)$. For $u = \sum u_{J,K,\lambda} dz_J \wedge d\overline{z}_K \otimes e_\lambda \in \mathscr{D}^{s,t}(M,E)$, we obtain (in the point x):

$$\begin{aligned}
\left\langle \left[\mathrm{i}\partial\overline{\partial}\Phi, \Lambda\right] u, u \right\rangle_{\omega,E} &= \sum_{\lambda,\mu} h_{\lambda\mu} \left\langle \left[\mathrm{i}\partial\overline{\partial}\Phi, \Lambda\right] u_\lambda, u_\mu \right\rangle_\omega \\
&\overset{(7.5)}{=} \sum_{\lambda,\mu} h_{\lambda\mu} \left\langle \sum_{J,K} \delta_{J,K} u_{J,K,\lambda} dz_J \wedge d\overline{z}_K\,,\, u_\mu \right\rangle_\omega \\
&= \sum_{J,K,\lambda,\mu} \delta_{J,K} h_{\lambda\mu} \Big\langle u_{J,K,\lambda} dz_J \wedge d\overline{z}_K\,,\, u_{J,K,\mu} dz_J \wedge d\overline{z}_K \Big\rangle_\omega \\
&= \sum_{J,K} \delta_{J,K} \left\langle u_{J,K}, u_{J,K} \right\rangle_E = \sum_{J,K} \delta_{J,K} |u_{J,K}|_E^2,
\end{aligned}$$

where $u_\lambda := \sum_{J,K} u_{J,K,\lambda} dz_J \wedge d\overline{z}_K$ and $u_{J,K} := \sum_\lambda u_{J,K,\lambda} \otimes e_\lambda$.

Now, we define $\delta(x)$ as the maximum of the q smallest δ_j (but not greater than 1). Moreover, we get $\delta_{\{1,..,n\},K} = \sum_{j\in K}\delta_j \geq \delta(x)$ if

$|K| \geq q$. Hence,

$$\left\langle \left[\mathrm{i}\partial\overline{\partial}\Phi, \Lambda \right] w, w \right\rangle_{\omega,E}(x) \geq \delta(x) \sum_K |w_K|^2_E(x) = \delta(x)\langle w, w\rangle_{\omega,E}(x)$$

for all $w = \sum w_K dz_{\{1,..,n\}} \wedge d\overline{z}_K \in \mathscr{D}^{n,t}(M,E)$ and $t \geq q$. Obviously, δ is continuous in x and positive where $\operatorname{rk} H(\Phi)_x > n - q$. □

The three lemmata 7.2, 7.3 and 7.4 together imply, assuming that E is an m-semi-positive holomorphic vector bundle of rank r on M, that

$$(\delta \cdot w, w)_\Phi \leq \|\overline{\partial} w\|^2_\Phi + \|\vartheta_\Phi w\|^2_\Phi \tag{7.6}$$

for all $w \in \mathscr{D}^{n,t}(M,E)$ and $t \geq 1$ if $m \geq \min\{n, r\}$, i. e. E is Nakano semi-positive, or $t \geq \min\{n - m + 1, r\}$ else.

7.3 L^2-vanishing theorems on complete manifolds

We keep the setting and notations of the former section. Further, let $L^2_{s,t}(M,E;\Phi)$ denote the square-integrable forms u with respect to the norm $\|\cdot\|_\Phi$, and let $\overline{\partial} = \overline{\partial}_w$ be the weak extension of $\overline{\partial}_{\mathrm{cpt}}$ (cf. Chapter 5). Since the metric on M is complete, we obtain that the weak extension and the strong extension of the $\overline{\partial}_{\mathrm{cpt}}$-operator coincide. Therefore, $\vartheta_\Phi = \vartheta_{\Phi,w}$ in the sense of distributions coincides with the Hilbert-space adjoint of $\overline{\partial}$. Let $\overline{*}_\Phi \colon L^2_{s,t}(M,E;\Phi) \to L^2_{n-s,n-t}(M,E^*;-\Phi)$ be the weighted conjugated Hodge-$*$-operator defined by

$$\langle u, v\rangle_{\omega,E} \cdot e^{-\Phi} = u \wedge \overline{*}_\Phi v \quad \text{for } u, v \in L^2_{s,t}(M,E;\Phi).$$

Recall $\sigma(\Phi) := \max_{x\in M}(\operatorname{rk} H(\Phi)_x)$, where $H(\Phi)_x$ denotes the complex Hessian of Φ at x.

Theorem 7.7. *Let M be a complete Kähler manifold of dimension n, let Φ be a smooth plurisubharmonic exhaustion function of M, and let*

$E \to M$ be a Nakano semi-positive holomorphic vector bundle. Then, the following groups of harmonic forms are zero for all $q > n - \sigma(\Phi)$:

$$\mathscr{H}^{n,q}_{L^2(\Phi)}(M,E) := \{u \in L^2_{n,q}(M,E;\Phi) \colon \overline{\partial} u = 0, \vartheta_\Phi u = 0\} = 0 \quad \textit{and}$$
$$\mathscr{H}^{0,n-q}_{L^2(-\Phi)}(M,E^*) = 0.$$

Proof: Using that the given metric is complete, it is well-known that $\overline{*}_\Phi$ induces the L^2-duality $\mathscr{H}^{n,q}_{L^2(\Phi)}(M,E) \cong \mathscr{H}^{0,n-q}_{L^2(-\Phi)}(M,E^*)$. Let h be a harmonic form in $\mathscr{H}^{n,q}_{L^2(\Phi)}(M,E)$. Then, h is in the kernel of the elliptic weighted Laplace operator $\Box_\Phi := \Delta''_\Phi$ so that h is smooth, i. e. $h \in L^2_{n,q}(M,E;\Phi) \cap \mathscr{E}^{n,q}(M,E)$. As the metric is complete, $\mathscr{D}^{n,q}(M,E)$ is dense in $\operatorname{dom}\overline{\partial} \cap \operatorname{dom}\vartheta_\Phi$ with respect to the graph norm $u \mapsto \|u\|_\Phi + \|\overline{\partial} u\|_\Phi + \|\vartheta_\Phi u\|_\Phi$. Hence, there is a sequence $\{h_k\}$ in $\mathscr{D}^{n,q}(M,E)$ with $h_k \to h$, $\overline{\partial} h_k \to \overline{\partial} h = 0$ and $\vartheta_\Phi h_k \to \vartheta_\Phi h = 0$ in $L^2_{n,\cdot}(M,E;\Phi)$. With (7.6), we obtain

$$\begin{aligned}(\delta\cdot h, h)_\Phi &= (\delta\cdot(h-h_k), h)_\Phi + (\delta\cdot h_k, h-h_k)_\Phi + (\delta\cdot h_k, h_k)_\Phi \\ &\lesssim \|h-h_k\|_\Phi\cdot\|h\|_\Phi + \|h_k\|_\Phi\cdot\|h-h_k\|_\Phi + \|\overline{\partial} h_k\|^2_\Phi + \|\vartheta_\Phi h_k\|^2_\Phi \\ &\to 0 \text{ if } k \to \infty.\end{aligned}$$

Therefore, the harmonic form h vanishes on the open set

$$\{x \in M \colon \delta(x) > 0\} \supset \{x \in M \colon \operatorname{rk} H(\Phi)_x > n-q\}.$$

But, $\{x \in M \colon \operatorname{rk} H(\Phi)_x > n-q\}$ is not empty for $q > n - \sigma(\Phi)$. So, the unique continuation theorem (see [Aro57]) implies that h vanishes on M. $\square$

Before proving Takegoshi's vanishing theorem for vector bundles on complex manifolds, let us recall the following criterion of A. Andreotti and E. Vesentini (see [AV63, Prop. 41 and Lem. 12]). For a differential form $v \in \mathscr{D}^{n-s,n-t}(M,E^*)$, we define the distribution

$$T_v \colon \mathscr{E}^{s,t}(M,E) \to \mathbb{C} \text{ by } T_v u := \int_M v \wedge u.$$

Theorem 7.8. *Let M be a complex manifold of dimension n, let $E \to M$ be a holomorphic vector bundle on M, let the Dolbeault* *operator* $\overline{\partial}\colon \mathscr{E}^{s,t}(M,E) \to \mathscr{E}^{s,t+1}(M,E)$ *be a topological homomorphism, and let* $v \in \mathscr{D}^{n-s,n-t}(M,E^*)$ *be a* $\overline{\partial}$*-closed form with values in the dual vector bundle* E^**. Then, the equation* $\overline{\partial}w = v$ *has a solution* $w \in \mathscr{D}^{n-s,n-t-1}(M,E^*)$ *if and only if* $T_v u = 0$ *for all* $\overline{\partial}$*-closed* $u \in \mathscr{E}^{s,t}(M,E)$*.*

Theorem 7.9. *Let* M *be a weakly 1-complete complex manifold of dimension* n*, and let* $E \to M$ *be a Nakano semi-positive holomorphic vector bundle on* M*. Assume that* M *admits a smooth plurisubharmonic exhaustion function* Φ *such that the sublevel sets* $M_l := \{x \in M\colon \Phi(x) < l\} \Subset M$ *are Kähler,* $l \in \mathbb{N}$ *(i. e.* M *is Kähler on relative compact sets). Then, for all* $q > n - \sigma(\Phi)$*:*

$$H^{n-q}_{\mathrm{cpt}}(M, \mathcal{O}_M(E^*)) \cong H^{0,n-q}_{\mathrm{cpt}}(M,E^*) = 0$$

if and only if $H^{q+1}(M, \Omega^n_M(E))$ *is Hausdorff. In this case, the following is equivalent:*

(1) $H^q(M, \Omega^n_M(E))$ *is Hausdorff.*

(2) $H^{n-q+1}_{\mathrm{cpt}}(M, \mathcal{O}_M(E^*))$ *is Hausdorff.*

(3) $H^{n,q}(M,E) \cong H^q(M, \Omega^n_M(E)) = 0$*.*

If M *is holomorphically convex, then all mentioned cohomology spaces vanish.*

Proof: The Dolbeault isomorphism theorem (see e. g. § 1.3 in [GR79, Chap. B]) yields

$$\begin{aligned} H^t(M, \Omega^s_M(E)) &\cong H^{s,t}(M,E) \\ &:= \{u \in \mathscr{E}^{s,t}(M,E)\colon \overline{\partial}u = 0\} \big/ \overline{\partial}\mathscr{E}^{s,t-1}(M,E) \end{aligned}$$

and

$$\begin{aligned} H^t_{\mathrm{cpt}}(M, \Omega^s_M(E^*)) &\cong H^{s,t}_{\mathrm{cpt}}(M,E^*) \\ &:= \{u \in \mathscr{D}^{s,t}(M,E^*)\colon \overline{\partial}u = 0\} \big/ \overline{\partial}\mathscr{D}^{s,t-1}(M,E^*). \end{aligned}$$

Let us first prove the implication $H^{q+1}(M, \Omega^n_M(E))$ Hausdorff $\Rightarrow$ $H^{0,n-q}_{\mathrm{cpt}}(M,E^*) = 0$:

Since $H^{q+1}(M, \Omega^n_M(E))$ is Hausdorff, $\overline{\partial}\colon \mathscr{E}^{n,q}(M,E) \to \mathscr{E}^{n,q+1}(M,E)$ is a topological homomorphism on Fréchet spaces (see e. g. Prop. 6

of [Ser55]). Hence, the assumptions of Theorem 7.8 are satisfied for $(s,t) = (n,q)$ and we can use it to show that $H^{0,n-q}_{\text{cpt}}(M, E^*) = 0$:

So, let $v \in \mathscr{D}^{0,n-q}(M, E^*)$ be $\overline{\partial}$-closed. We have to show that v is $\overline{\partial}$-exact. Choose an $l \in \mathbb{N}$ such that $\operatorname{supp} v \subset M_l$ and $\sigma(\Phi|_{M_l}) = \sigma(\Phi)$, i. e. M_l contains a point x where $\operatorname{rk} H(\Phi)_x$ is maximal. By Theorem 7.8, it suffices to show that $T_v u = 0$ for all $u \in \mathscr{E}^{n,q}(M_l, E)$ with $\overline{\partial} u = 0$. Fix such a u.

Choosing an appropriate smooth increasing convex positive function $\lambda\colon (-\infty, l) \to \mathbb{R}^+$ with $\lim_{t\to l} \lambda(t) = \infty$, we get

(i) a smooth plurisubharmonic exhaustion function $\Psi{:=}\lambda\circ\Phi$ of M_l,

(ii) the Kähler metric given by ω is complete on M_l (replace ω by $\omega + \mathrm{i}\partial\overline{\partial}\Psi$), and

(iii) $u \in L^2_{n,q}(M_l, E; \Psi) \cap \mathscr{E}^{n,q}(M_l, E)$.

Thus, $g := (-1)^{n+q}\,\overline{*}_\Psi\, u \in L^2_{0,n-q}(M_l, E^*; -\Psi) \cap \mathscr{E}^{0,n-q}(M_l, E^*)$ and $\vartheta_{-\Psi} g = 0$. Since $\ker \vartheta_{-\psi} \cap \ker \overline{\partial} = \mathscr{H}^{0,n-q}_{L^2(-\Psi)}(M_l, E^*) = 0$ (see Theorem 7.7), we get $g \in \ker \vartheta_{-\psi} = (\ker \overline{\partial})^\perp = \overline{\operatorname{Im}(\vartheta_{-\Psi})}$. Hence, there is a sequence $\{f_k\}$ in $\mathscr{D}^{0,n-q+1}(M_l, E^*)$ with

$$\|g - \vartheta_{-\Psi} f_k\|_{-\Psi} \to 0 \text{ if } k \to \infty.$$

Finally, we infer

$$\begin{aligned} T_v u = \int_{M_l} v \wedge u &= \int_{M_l} v \wedge \overline{*}_{-\Psi}\, g = (v, g)_{-\Psi} \\ &= \lim_{k\to\infty} (v, \vartheta_{-\Psi} f_k)_{-\Psi} = \lim_{k\to\infty} (\overline{\partial} v, f_k)_{-\Psi} \overset{\overline{\partial} v = 0}{=} 0. \end{aligned}$$

This shows that, indeed, $H^{n-q}_{\text{cpt}}(M, \mathcal{O}_M(E^*)) \cong H^{0,n-q}_{\text{cpt}}(M, E^*) = 0$.

To prove the other implications, we use the following result of H. Laufer (see Thm. 3.2 in [Lau67]): There exist linear topological spaces $R = R^{q+1}(M, \Omega^n_M(E))$ and $R_{\text{cpt}} = R^{n-q+1}_{\text{cpt}}(M, \mathcal{O}_M(E^*))$ such that

$$H^q(M, \Omega^n_M(E)) \cong H^{n-q}_{\text{cpt}}(M, \mathcal{O}_M(E^*))^* \oplus R_{\text{cpt}}, \tag{7.10}$$

$$H^{n-q}_{\text{cpt}}(M, \mathcal{O}_M(E^*)) \cong H^q(M, \Omega^n_M(E))^* \oplus R, \tag{7.11}$$

$$R = 0 \quad \Leftrightarrow H^{q+1}(M, \Omega^n_M(E)) \text{ is Hausdorff} \tag{7.12}$$

$$R_{\text{cpt}} = 0 \quad \Leftrightarrow H^{n-q+1}_{\text{cpt}}(M, \mathcal{O}_M(E^*)) \text{ is Hausdorff.} \tag{7.13}$$

Using (7.11), $H^{n-q}_{\mathrm{cpt}}(M, \mathcal{O}_M(E^*)) = 0$ implies that $R = 0$ and, hence, $H^{q+1}(M, \Omega^n_M(E))$ is Hausdorff.

If $H^{q+1}(M, \Omega^n_M(E))$ is Hausdorff, then (7.11) implies

$$H^q(M, \Omega^n_M(E))^* \cong H^{n-q}_{\mathrm{cpt}}(M, \mathcal{O}_M(E^*)) = 0.$$

If $H^q(M, \Omega^n_M(E))$ is Hausdorff, then $H^q(M, \Omega^n_M(E))$ has to vanish, i. e. (1) $\Rightarrow$ (3). The converse (3) $\Rightarrow$ (1) is trivial. Finally, (7.10) and (7.13) give us (2) $\Leftrightarrow$ (3), immediately. Actually, the equivalence (1) $\Leftrightarrow$ (2) can directly be proven with functional analysis tools.

It is well-known that the sheaf cohomology groups for coherent analytic sheaves on holomorphically convex manifolds are Hausdorff (see Lem. II.1 in [Pri71]). □

The result [Lau67, Thm. 3.2] of H. Laufer is a generalization of the Serre duality (see [Ser55, Thm. 2]). He treated the case where $\overline{\partial}$ is not necessarily a topological homomorphism.

Donnelly-Fefferman-Ohsawa vanishing theorem. For the proof of Theorem 1.3 in Chapter 10, we will need the following simple generalization of a theorem of H. Donnelly and C. Fefferman (see Thm. 1.1 and Prop. 2.1 in [DF83]) which was simplified by T. Ohsawa in [Ohs87, Thm. 1.1].

Theorem 7.14. *Let M be a complete Kähler manifold of dimension n, and let $E \to M$ be a Nakano semi-positive holomorphic vector bundle on M. Assume that the Kähler form is given by $\omega = \mathrm{i}\partial\overline{\partial}G$ for a smooth $G\colon M \to \mathbb{R}$ with bounded $|\partial G|_\omega \leq C$ (self-bounded gradient). If u is a $\overline{\partial}_w$-closed square-integrable (n,q)-form on M with values in E, then there exists a $v \in L^2_{n,q-1}(M,E)$ with $\overline{\partial}_w v = u$ and $\|v\|_\omega \leq 4C\|u\|_\omega$, i. e. $H^{n,q}_w(M,E) = 0$ for all $q > 0$.*

J. Ruppenthal proved this theorem for semi-positive line bundles (see Thm. 3.2 in [Rup14a]). Using Lemma 7.2, the proof generalizes easily to Nakano semi-positive vector bundles.

Chapter 8
The Relative Vanishing Theorem

In this section, we will present further generalizations of K. Takegoshi's vanishing theorem besides Theorem 7.9 above. Especially, we are interested in a generalization for torsion-free coherent analytic sheaves.

Let us first describe our setting and explain the notation. Let X be a weakly 1-complete complex space with Φ as plurisubharmonic exhaustion function. Recall that $\sigma(\Phi) := \max_{x\in X_{\text{reg}}}(\operatorname{rk} H(\Phi)_x)$ is the maximal number of positive eigenvalues of Φ's complex Hessian. We will assume that the Grauert-Riemenschneider canonical sheaf $\mathscr{K}_X$ is locally free. This is for example the case if X is Gorenstein and has canonical singularities (see Thm. 5.3 in [Rup14b] or Theorem 2.17). Recall that local freeness of $\mathscr{K}_X$ implies that X is normal (see Theorem 4.21).

We say a complex space X is Kähler if there exists a Kähler form on the regular part X_{reg} such that each singular point $x \in X_{\text{sing}}$ has an open neighbourhood $U = U(x) \subset X$ which can be embedded in $V \subset \mathbb{C}^{d(x)}$ and a Kähler form η on V with $\eta|_{U_{\text{reg}}} = \omega$.

Let $\mathscr{S}$ be a coherent analytic sheaf on X. Recall that there exists a duality $L(\,\cdot\,)$ between the coherent analytic sheaves and the linear (fibre) spaces over X (see Section 3.2). We call $\mathscr{S}$ Nakano semi-positive if there exists a smooth Hermitian form on $L(\mathscr{S})$ which induces that $R(\mathscr{S}) = L(\mathscr{S}^*_{X\setminus \operatorname{Sing}\mathscr{S}})$ is Nakano semi-positive (for more details, see Def. 8.11 or [GR70, § 1.2]).

Def. 8.1. We say that a coherent analytic sheaf $\mathscr{S}$ on X fulfils condition (+) if there exist a projective morphism $\pi\colon \widetilde{X} \to X$ with locally free $\pi^T\mathscr{S}$ and a semi-positive invertible coherent analytic sheaf $\mathscr{L}$ on $\widetilde{X}$ such that $\pi^T\mathscr{K}_X \cong \mathscr{L} \otimes \mathscr{K}_{\widetilde{X}}$. If these exist on all relatively compact holomorphically convex open subsets of X, we say that $\mathscr{S}$ satisfies $(+)_{\mathrm{loc}}$.

We can now state our first main result of this chapter:

Theorem 8.2. *Let X be a weakly 1-complete connected (normal) Kähler space of dimension n with locally free canonical sheaf, let Φ be a smooth plurisubharmonic exhaustion function of X, and let $\mathscr{S}$ be a Nakano semi-positive torsion-free sheaf on X with $(+)$ such that $L(\mathscr{S})$ is normal. Then, for each $q > n - \sigma(\Phi)$,*

$$H^q(X, \mathscr{S} \otimes \mathscr{K}_X) = 0$$

if $H^q(X, \mathscr{S} \otimes \mathscr{K}_X)$ and $H^{q+1}(X, \mathscr{S} \otimes \mathscr{K}_X)$ are Hausdorff.

If X is holomorphically convex, then the Hausdorff assumption is always satisfied, [Pri71, Lem. II.1]. We remark also that the Kähler structure of X and the Nakano semi-positivity of $\mathscr{S}$ are needed only on relatively compact holomorphically convex subsets of X (cf. Theorem 7.9).

The proof of Theorem 8.2 uses that the isomorphism induced by the Leray spectral sequence is already topological (see Theorem 8.5). Actually, this is easy to prove, although, to the knowledge of the author, it has not been observed in the literature (yet, [Pri71, Lem. II.1] and its proof is interesting in this context).

Obviously, locally free sheaves satisfy $(+)$ and their associated linear spaces are normal. In this case, we get the vanishing theorem for arbitrary irreducible complex spaces (see Theorem 8.9). In Section 7.3, we already proved the result in the regular case (for vector bundles on manifolds). By use of the projection formula (see Theorem 4.10), we obtain Theorem 7.9 for singular complex spaces (see Section 8.1).

Since there is no projection formula for non-locally-free sheaves, the situation is much more complicated for such sheaves. Then, we need to assume that $L(\mathscr{S})$ is normal, and the additional semi-positivity property (+). If the linear space associated to $\mathscr{S} \otimes \mathscr{K}_X$ is normal, there is a canonical isomorphism

$$\mathscr{S} \otimes \mathscr{K}_X \cong \pi_*(\pi^T \mathscr{S} \otimes \pi^T \mathscr{K}_X)$$

for all modifications π of X (see Theorem 4.2). In Remark 4.20, we have seen that the normality assumption on $L(\mathscr{S})$ is necessary for a generalization of Takegoshi's vanishing theorem that makes use of the monoidal transformation. Furthermore, one needs a suitable connection between $\pi^T \mathscr{K}_X$ and $\mathscr{K}_{\widetilde{X}}$. If (+) holds, then we obtain $\mathscr{S} \otimes \mathscr{K}_X$ as the direct image of a Nakano semi-positive locally free sheaf tensored with the canonical sheaf. Using the Leray spectral sequence, we can, then, prove Theorem 8.2 (see Section 8.2.1). The last step was inspired by [GR70] of H. Grauert and O. Riemenschneider. Analogously, one can prove the following corollary of Satz 2.1 in [GR70]:

Corollary 8.3. *Let X be a compact normal Moishezon space with locally free canonical sheaf, and let $\mathscr{S}$ be a torsion-free quasi-positive sheaf with (+) such that $L(\mathscr{S})$ is normal. Then, for each $q > 0$,*

$$H^q(X, \mathscr{S} \otimes \mathscr{K}_X) = 0.$$

Let us add a few words on how to verify that $\mathscr{S}$ satisfies (+). H. Rossi proved that there exists a projective morphism $\varphi = \varphi_{\mathscr{S}} \colon X_{\mathscr{S}} \to X$ such that $\varphi^T \mathscr{S}$ is locally free (see Thm. 3.5 in [Ros68] or Theorem 2.19). We called φ monoidal transformation with respect to $\mathscr{S}$. Moreover, we have the following useful fact (see Theorem 4.24): For any resolution $\pi \colon M \to X$ of singularities (such that $\pi^T \mathscr{K}_X$ is locally free), there exists an effective Cartier divisor $D \geq 0$ (with support on the exceptional set of the resolution) such that

$$\pi^T \mathscr{K}_X \cong \mathscr{K}_M \otimes \mathcal{O}_M(-D).$$

Hence, for the property (+) to hold, it is just needed that $\mathcal{O}(-D)$ is semi-positive.

In Section 9.2, we give an example where the assumption (+) holds for a non-locally-free sheaf. More precisely, we consider a semi-positive (reduced) ideal sheaf $\mathscr{J}$ given by a submanifold on a holomorphically convex manifold and prove that $\mathscr{J}$ satisfies $(+)$. This is obtained by the semi-positivity of $\mathscr{J}$ itself, which is an indication for a link between (Nakano) semi-positivity of a sheaf and $(+)$. Using Theorem 8.2, we obtain a vanishing theorem for globally defined submanifolds (see Corollary 9.12).

The following result (a generalization of Thm. I in [Tak85]) is a conclusion of Theorem 8.2 proven in Section 8.2.2; the presented proof is derived from Takegoshi's.

Theorem 8.4. *Let X be a normal complex space with a locally free canonical sheaf which is bimeromorphic to a Kähler space, let $f\colon X \to Z$ be a proper surjective holomorphic map onto a complex space Z, and let $\mathscr{S}$ be a semi-globally (i. e. on relatively compact holomorphically convex sets) Nakano semi-positive torsion-free sheaf on X satisfying $(+)_{\mathrm{loc}}$ such that $L(\mathscr{S})$ is normal. Then, the higher direct images of $\mathscr{S} \otimes \mathscr{K}_X$ under f vanish for all $q > \dim X - \dim Z$:*

$$f_{(q)}(\mathscr{S} \otimes \mathscr{K}_X) = 0.$$

In Section 8.3, we finally study coherent analytic sheaves with torsion. We prove a generalization of Theorem 8.2 for q strictly greater than the dimension of the support of the associated torsion sheaf. For smaller q, we give a counterexample.

8.1 Irreducible complex spaces

In this section, we prove Takegoshi's vanishing theorem for locally free sheaves on irreducible complex spaces. For this, we indicate how a vanishing theorem as Theorem 7.9 yields vanishing of some higher direct image sheaves (relative vanishing theorem). We will need this observation later in the proof of Theorem 8.2 and Theorem 8.4, as well.

Theorem 8.5. *Let X be a complex space of pure dimension n, and let $\mathscr{F}$ be a coherent analytic sheaf on X such that the following property is satisfied: For every relatively compact holomorphically convex open $U \subset X$ with a smooth plurisubharmonic exhaustion function Φ, we have $H^r(U, \mathscr{F}) = 0$ for all $r > n - \sigma(\Phi)$. Further, we assume there is a proper surjective holomorphic map $f\colon X \to Z$ to a complex space Z. For each $r > n - \dim Z$, we get*

$$f_{(r)}(\mathscr{F}) = 0.$$

If $\dim Z = n$, the isomorphism

$$H^q(X, \mathscr{F}) \cong H^q(Z, f_*\mathscr{F})$$

induced by the Leray spectral sequence is topological for all q.

Proof: Let r be a number greater than $n - \dim Z$, let z be in Z, and let $V \subset Z$ be a relatively compact Stein neighbourhood of z, i. e. there is a smooth strictly plurisubharmonic exhaustion function Φ of V. Then, $\Phi \circ f$ is a smooth plurisubharmonic exhaustion function of the relatively compact set $U := f^{-1}(V)$, which is holomorphically convex (using that f is proper). Since f is surjective, we obtain $\sigma(\Phi \circ f) = \sigma(\Phi) = \dim Z$. So, the assumption says $H^r(U, \mathscr{F}) = 0$ since $r > n - \dim Z$. Yet, the direct image sheaf $f_{(r)}(\mathscr{F})$ is the sheaf associated to the presheaf defined by $V \mapsto H^r(f^{-1}(V), \mathscr{F}) = 0$. That proves

$$f_{(r)}(\mathscr{F}) = 0.$$

If $\dim Z = n$, the Leray spectral sequence (see [Ler50, Chap. II]) implies

$$H^q(X, \mathscr{F}) \cong H^q(Z, f_*\mathscr{F}).$$

Let $\mathfrak{V} = \{V_i\}_{i \in I}$ be a Leray Covering of Z, i. e.

$$H^q(Z, f_*\mathscr{F}) \cong \check{H}^q(\mathfrak{V}, f_*\mathscr{F}). \tag{8.6}$$

Actually, the latter space defines the topology on the first one. If it is Hausdorff, it is independent of $\mathfrak{V}$ (see Lem. 4.2 in [Kau67]). For $\mathfrak{U} := \{f^{-1}(V_i)\}_{i \in I}$, the definition of Čech cohomology implies $\check{H}^q(\mathfrak{V}, f_*\mathscr{F}) = \check{H}^q(\mathfrak{U}, \mathscr{F})$ with the same topology. Yet, we know that

$\mathfrak{U}$ is already a Leray covering of X, i. e.

$$H^q(X, \mathscr{F}) \cong \check{H}^q(\mathfrak{U}, \mathscr{F}) = \check{H}^q(\mathfrak{V}, f_*\mathscr{F}) \cong H^q(Z, f_*\mathscr{F})$$

is topological as well. □

Remark 8.7. If X is regular and $\mathscr{F}$ locally free, Thm. 2.1 in [Lau67] says that the different topologies of a cohomology group given by Leray coverings, differential forms or currents coincide.

Let us recall the projection formula (see Theorem 4.10). If $f\colon Y \to X$ is a holomorphic map between complex spaces, if $\mathscr{E}$ is a locally free sheaf on X, and if $\mathscr{F}$ is a coherent analytic sheaf on Y, then

$$f_*\mathscr{F} \otimes \mathscr{E} \cong f_*\left(\mathscr{F} \otimes f^*\mathscr{E}\right). \tag{8.8}$$

Using this fact, we obtain the following generalization of Takegoshi's vanishing theorem.

Theorem 8.9. *Let X be a weakly 1-complete irreducible complex space of dimension n which is Kähler on relatively compact sets, let Φ be a smooth plurisubharmonic exhaustion function of X, and let $\mathscr{E}$ be a Nakano semi-positive locally free sheaf on X. Then, for each $q > n - \sigma(\Phi)$:*

$$H^q(X, \mathscr{E} \otimes \mathscr{K}_X) = 0$$

if $H^q(X, \mathscr{E} \otimes \mathscr{K}_X)$ and $H^{q+1}(X, \mathscr{E} \otimes \mathscr{K}_X)$ are Hausdorff.

Proof: Let $\pi\colon M \to X$ be a resolution of the singularities of X (cf. [Hir64, Hir77]). Since X is irreducible, M is connected. We can assume that π is projective. This implies that M is Kähler on relatively compact open sets (cf. e. g. Lem. 4.4 in [Fuj78]). Since $\Phi \circ \pi$ is a smooth plurisubharmonic exhaustion function of M with $\sigma(\Phi \circ \pi) = \sigma(\Phi)$, Theorem 7.9 implies: For each $q > n - \sigma(\Phi)$,

$$H^q(M, \pi^*\mathscr{E} \otimes \Omega^n_M) = 0$$

if $H^{q/q+1}(M, \pi^*\mathscr{E} \otimes \Omega^n_M)$ are Hausdorff. Theorem 7.9 also implies that the assumptions of Theorem 8.5 are satisfied for $\pi^*\mathscr{E} \otimes \Omega^n_M$ over M and π, i. e. for each $q > n - \sigma(\Phi)$,

$$H^q(X, \pi_*(\pi^*\mathscr{E} \otimes \Omega^n_M)) \cong H^q(M, \pi^*\mathscr{E} \otimes \Omega^n_M) = 0$$

if $H^{q/q+1}(X, \pi_*(\pi^*\mathscr{E} \otimes \Omega^n_M))$ are Hausdorff. With the projection formula (8.8) / Theorem 4.10, we obtain the claimed. □

Theorem 8.5 and Theorem 8.9 immediately imply the following variation of Theorem 8.4:

Corollary 8.10. *Let X be a Kähler space, let $f\colon X \to Z$ be a proper surjective holomorphic map onto a complex space Z, and let $\mathscr{E}$ be a locally free Nakano semi-positive torsion-free sheaf on X. Then, for all $q > \dim X - \dim Z$,*

$$f_{(q)}(\mathscr{E} \otimes \mathscr{K}_X) = 0.$$

8.2 Vanishing theorems for torsion-free sheaves

In this section, we will prove the main theorems. Let us first recall the definition of Nakano semi-positive coherent analytic sheaves in the sense of H. Grauert and O. Riemenschneider (see [GR70, § 1.2]).

Def. 8.11. Let $\mathscr{S}$ be a coherent analytic sheaf on a complex space X, let $S := L(\mathscr{S})$ denote the associated linear space, let h be a Hermitian form on S (recall the definition of Hermitian form on a linear space in Def. 5.4). On the manifold $X' := X_{\mathrm{reg}} \setminus \operatorname{Sing} \mathscr{S}$, S is a vector bundle. We call $\mathscr{S}$ *Nakano semi-positive* if there is a smooth Hermitian form on S which induces a Nakano semi-positive hermitian metric on the vector bundle $R(\mathscr{S}) = S^*_{X'}$ (cf. Def. 7.1).

If, furthermore, there exists a Zariski open set $X'' \subset X'$ (the complement of a thin analytic set) such that $R(\mathscr{S}_{X''})$ is Nakano positive, then $\mathscr{S}$ is called *quasi-positive*.

If $\mathscr{S}$ is locally free, then the linear space $L(\mathscr{S})$ is dual to the vector bundle associated to $\mathscr{S}$. Hence, in this case, the notions of Nakano semi-positivity coincide. We will need only the following fact: Let $\mathscr{S}$ be a Nakano semi-positive sheaf on a complex space, and let $\pi\colon Y \to X$ be a proper modification. Then, $L(\pi^*\mathscr{S}) = \pi^* L(\mathscr{S})$ and $L(\pi^T\mathscr{S})$ is embedded in $L(\pi^*\mathscr{S})$ because of $\pi^*\mathscr{S} \twoheadrightarrow \pi^T\mathscr{S}$. With the pullback on $L(\pi^*\mathscr{S})$ of the Hermitian metric on $L(\mathscr{S})$ and the restriction to $L(\pi^T\mathscr{S})$, we get that both, $\pi^*\mathscr{S}$ and $\pi^T\mathscr{S}$, are Nakano semi-positive sheaves on Y. Actually, this property is equivalent to the Nakano semi-positivity of $\mathscr{S}$ because of the following fact (see [GR70, § 1.2]):

Lemma 8.12. *Let M be a complex manifold and $(E, h) \to M$ a Hermitian vector bundle on M. If (E, h) is Nakano-semi positive on a Zariski open set M', then (E, h) is Nakano semi-positive on the whole of M.*

8.2.1 Proof of Theorem 8.2

Let X be a weakly 1-complete normal connected complex Kähler space with a smooth plurisubharmonic exhaustion function Φ and locally free $\mathscr{K}_X$. Let $\mathscr{S}$ be a Nakano semi-positive torsion-free coherent analytic sheaf on X with normal $L(\mathscr{S})$ and (+), i. e. there is a projective $\pi\colon \widetilde{X} \to X$ and a semi-positive locally free analytic sheaf $\mathscr{L}$ of rank 1 such that $\pi^T\mathscr{S}$ is locally free and $\pi^*\mathscr{K}_X = \pi^T\mathscr{K}_X \cong \mathscr{L} \otimes \mathscr{K}_{\widetilde{X}}$.

The locally free sheaf $\mathscr{E} := \pi^T\mathscr{S} \otimes \mathscr{L}$ is Nakano semi-positive.

The composition $\Phi \circ \pi$ is a plurisubharmonic exhaustion function of $\widetilde{X}$ because π is proper and holomorphic, and $\sigma(\Phi \circ \pi) = \sigma(\Phi)$ since π is biholomorphic on a dense open set. Recall that

$$\sigma(\Phi) = \max_{x \in X_{\mathrm{reg}}} (\operatorname{rk} H(\Phi)_x),$$

where $H(\Phi)_x$ denotes the complex Hessian of Φ at x. As X is Kähler and π is projective, the irreducible complex space $\widetilde{X}$ is Kähler on relatively compact open sets (cf. e. g. Lem. 4.4 in [Fuj78]). Theorem 8.9 yields: For each $q > n - \sigma(\Phi)$,

$$H^q(\widetilde{X}, \mathscr{E} \otimes \mathscr{K}_{\widetilde{X}}) = 0$$

if $H^q(\widetilde{X}, \mathscr{E} \otimes \mathscr{K}_{\widetilde{X}})$ and $H^{q+1}(\widetilde{X}, \mathscr{E} \otimes \mathscr{K}_{\widetilde{X}})$ are Hausdorff.

Theorem 8.9 also implies that the assumptions of Theorem 8.5 are satisfied for $\mathscr{E} \otimes \mathscr{K}_{\widetilde{X}}$ and π. Therefore, the suitable Hausdorff assumption implies

$$H^q(X, \pi_*(\mathscr{E} \otimes \mathscr{K}_{\widetilde{X}})) \cong H^q(\widetilde{X}, \mathscr{E} \otimes \mathscr{K}_{\widetilde{X}}) = 0 \quad \forall\, q > n - \sigma(\Phi).$$

Since $\mathscr{K}_X$ is locally free and $L(\mathscr{S})$ is normal, we get $L(\mathscr{S} \otimes \mathscr{K}_X)$ is normal. Therefore, Theorem 4.2 implies

$$\mathscr{S} \otimes \mathscr{K}_X \cong \pi_*(\pi^T \mathscr{S} \otimes \pi^* \mathscr{K}_X) \cong \pi_*(\mathscr{E} \otimes \mathscr{K}_{\widetilde{X}}). \tag{8.13}$$

□

8.2.2 Proof of Theorem 8.4

We will now use Theorem 8.2 to prove Theorem 8.4 with the help of Theorem 8.5. We also need the following fact.

Lemma 8.14. *Let X be a reduced complex space bimeromorphic to a Kähler space and U a relatively compact open set in X. Then, there is a Kähler manifold M and a proper modification $g\colon M \to U$ of U such that $g_{(q)}(\mathscr{F} \otimes \Omega^n_M) = 0$ for all $q > 0$ and all Nakano semi-positive torsion-free sheaves $\mathscr{F}$ on M with $(+)$ and normal $L(\mathscr{F})$.*

Proof: By assumption, there exists a Kähler space Y and a bimeromorphic map $\alpha\colon Y \dashrightarrow X$ given by its graph $\Gamma_\alpha \subset Y \times X$ as analytic set. Let $\mathrm{pr}_Y : \Gamma_\alpha \to Y$ and $\mathrm{pr}_X : \Gamma_\alpha \to X$ be the holomorphic projections such that $\alpha = \mathrm{pr}_X \circ \mathrm{pr}_Y^{-1}$. H. Hironaka's version of the Chow lemma (see Cor. 2 of the Flattening Theorem in [Hir75]) gives a projective, particularly, proper bimeromorphic morphism $\beta\colon \widetilde{M} \to Y$ which dominates pr_Y, i. e. there is a holomorphic $h\colon \widetilde{M} \to \Gamma_\alpha$ with $\mathrm{pr}_Y \circ h = \beta$. We can assume that $\widetilde{M}$ is smooth by using a resolution of singularities. We obtain the following commutative diagram:

$$\begin{tikzcd} \widetilde{M} \arrow[r, "h"] \arrow[dr, dashed, "\beta"'] & \Gamma_\alpha \arrow[r, "\mathrm{pr}_X"] \arrow[d, "\mathrm{pr}_Y"] & X \\ & Y \arrow[ur, dashed, "\alpha"'] & \end{tikzcd} .$$

Then, $\widetilde{g} := \alpha \circ \beta = \mathrm{pr}_X \circ h$ is a proper modification of X. Moreover, $M := \widetilde{g}^{-1}(U)$ is a Kähler manifold – using [Fuj78, Lem. 4.4] for the projective $\beta\colon \widetilde{M} \to Y$ and the relatively compact set $\alpha^{-1}(U)$ in the Kähler space Y – and $g := \widetilde{g}|_M\colon M \to U$ is a proper modification of U.

To prove $g_{(q)}(\mathscr{F} \otimes \Omega_M^n) = 0$, we will use Theorem 8.2: Let x be a point in U and W an open Stein neighbourhood of x in U, i. e. there is a smooth strictly plurisubharmonic exhaustion function Ψ of W. Since g is a proper modification, we get a plurisubharmonic exhaustion function $\Psi \circ g$ of $g^{-1}(W)$ with $\sigma(\Psi \circ g) = \sigma(\Psi) = \dim M$. Hence, the assumptions of Theorem 8.2 are satisfied for the holomorphically convex Kähler manifold $g^{-1}(W)$ and any Nakano semi-positive torsion-free sheaf $\mathscr{F}$ with $(+)$ and normal $L(\mathscr{F})$. So, we obtain $H^q(g^{-1}(W), \mathscr{F} \otimes \Omega_M^n) = 0$ for all $q > \dim X - \dim M = 0$ and, finally, $g_{(q)}(\mathscr{F} \otimes \Omega_M^n) = 0$. $\square$

Proof of Theorem 8.4: Let X be a normal complex space of pure dimension n with locally free $\mathscr{K}_X$ which is bimeromorphic to a Kähler space, let $\mathscr{S}$ be a (semi-globally) Nakano semi-positive torsion-free coherent analytic sheaf on X with $(+)_{\mathrm{loc}}$ and normal $L(\mathscr{S})$, and let $f\colon X \to Z$ be a proper surjective holomorphic map to a complex space Z. To prove the vanishing of the higher direct images of $\mathscr{S} \otimes \mathscr{K}_X$, we have to check that the assumptions of Theorem 8.5 are satisfied, i. e. if a relatively compact open set $U \subset X$ possesses a smooth plurisubharmonic exhaustion function Φ, then $H^r(U, \mathscr{S} \otimes \mathscr{K}_X) = 0$ for $r > n - \sigma(\Phi)$.

Let $U \subset X$ be a relatively compact holomorphically convex open set with smooth plurisubharmonic exhaustion function Φ. By assumption, $\mathscr{S}_U$ satisfies $(+)$, i. e. there is a proper modification $\pi\colon \widetilde{U} \to U$ and a semi-positive locally free sheaf $\mathscr{L}$ on $\widetilde{U}$ of rank one such that $\pi^T \mathscr{S}_U$ is locally free and $\pi^* \mathscr{K}_U \cong \mathscr{L} \otimes \mathscr{K}_{\widetilde{U}}$. In particular, the sheaf $\mathscr{E} := \pi^T \mathscr{S}_U \otimes \mathscr{L}$ is locally free and Nakano semi-positive. Since $L(\mathscr{S}_U \otimes \mathscr{K}_U)$ is normal, Theorem 4.2 implies

$$\mathscr{S}_U \otimes \mathscr{K}_U \cong \pi_*(\mathscr{E} \otimes \mathscr{K}_{\widetilde{U}}). \tag{8.15}$$

Since $\widetilde{U}$ is bimeromorphic to U, it is bimeromorphic to a Kähler space, as well. Therefore, Lemma 8.14 gives a Kähler manifold M and a proper modification $g\colon M \to \widetilde{U}$ with $g_{(q)}(\mathscr{F} \otimes \Omega^n_M) = 0$ for $q \geq 1$ and $\mathscr{F} := g^*\mathscr{E}$.

For all holomorphically convex open $V \subset \widetilde{U}$ with smooth plurisubharmonic exhaustion function Ψ and $W := g^{-1}(V)$, we get

$$\begin{aligned} 0 &\overset{\text{Thm. 8.2}}{=} H^r(W, \mathscr{F} \otimes \Omega^n_W) \overset{\text{Leray}}{\cong} H^r(V, g_*(g^*\mathscr{E} \otimes \Omega^n_W)) \\ &\overset{(8.8)}{\cong} H^r(V, \mathscr{E} \otimes \mathscr{K}_V) \;\; \forall r > n - \sigma(\Psi). \end{aligned} \tag{8.16}$$

i. e. the assumptions of Theorem 8.5 hold for $\mathscr{E} \otimes \mathscr{K}_{\widetilde{U}}$ and π. Therefore, Theorem 8.5 and (8.15) imply

$$H^r(\widetilde{U}, \mathscr{E} \otimes \mathscr{K}_{\widetilde{U}}) \cong H^r(U, \pi_*(\mathscr{E} \otimes \mathscr{K}_{\widetilde{U}})) \cong H^r(U, \mathscr{S}_U \otimes \mathscr{K}_U). \tag{8.17}$$

Using (8.16) for $V = \widetilde{U}$ and $\Psi = \Phi$, we obtain

$$H^r(U, \mathscr{S}_U \otimes \mathscr{K}_U) = 0 \;\; \forall r > n - \sigma(\Phi).$$

□

In the proof, the Nakano semi-positivity of $\mathscr{S}$ is just needed on preimages of small Stein sets in Z under f / on relatively compact weakly 1-complete subsets of X (cf. Def. 8.1 of $(+)_{\text{loc}}$).

8.3 Sheaves with torsion

Let X be a holomorphically convex normal Kähler space of dimension n, Φ a smooth plurisubharmonic exhaustion function of X, and let $\mathscr{S}$ be a Nakano semi-positive coherent analytic sheaf on X satisfying $(+)$. We define $\mathscr{T} := \mathscr{T}(\mathscr{S})$ as the torsion sheaf of $\mathscr{S}$ and obtain the exact sequence

$$0 \to \mathscr{T} \to \mathscr{S} \to \mathscr{S}/\mathscr{T} \to 0.$$

Assuming that the Grauert-Riemenschneider canonical sheaf $\mathscr{K}_X$ is locally free, we get the exact sequence

$$0 \to \mathscr{T} \otimes \mathscr{K}_X \to \mathscr{S} \otimes \mathscr{K}_X \to (\mathscr{S}/\mathscr{T}) \otimes \mathscr{K}_X \to 0.$$

This yields the long exact sequence of cohomology:

$$0 \longrightarrow (\mathscr{T} \otimes \mathscr{K}_X)(X) \longrightarrow (\mathscr{S} \otimes \mathscr{K}_X)(X) \longrightarrow (\mathscr{S}/\mathscr{T} \otimes \mathscr{K}_X)(X) \longrightarrow \ldots$$

$$\ldots \longrightarrow H^q(X, \mathscr{T} \otimes \mathscr{K}_X) \longrightarrow H^q(X, \mathscr{S} \otimes \mathscr{K}_X) \longrightarrow H^q(X, \mathscr{S}/\mathscr{T} \otimes \mathscr{K}_X) \longrightarrow \ldots \tag{8.18}$$

Since the restriction of the Hermitian metric on $L(\mathscr{S})$ induces a Hermitian metric on the embedded space $L(\mathscr{S}/\mathscr{T})$, the torsion-free coherent analytic sheaf $\mathscr{S}/\mathscr{T}$ is Nakano semi-positive. $\mathscr{S}/\mathscr{T}$ inherits the property (+) from $\mathscr{S}$ because $\pi^T(\mathscr{S}) = \pi^T(\mathscr{S}/\mathscr{T})$. Assuming that $L(\mathscr{S}/\mathscr{T})$ is normal, we obtain $H^q(X, (\mathscr{S}/\mathscr{T}) \otimes \mathscr{K}_X) = 0$ for all $q > n - \sigma(\Phi)$ by Theorem 8.2. Thus, the long exact sequence (8.18) gives isomorphisms

$$H^q(X, \mathscr{T} \otimes \mathscr{K}_X) \cong H^q(X, \mathscr{S} \otimes \mathscr{K}_X) \quad \forall q > n - \sigma(\Phi) + 1$$

and the surjective homomorphism

$$H^{n-\sigma(\Phi)+1}(X, \mathscr{T} \otimes \mathscr{K}_X) \twoheadrightarrow H^{n-\sigma(\Phi)+1}(X, \mathscr{S} \otimes \mathscr{K}_X).$$

On the other hand, $\mathscr{T}$ and, hence, $\mathscr{T} \otimes \mathscr{K}_X$ have support on an analytic set $A \subset X$ with $r := \dim A = \sup_{x \in A} \dim_x A < n$. Let $\iota \colon A \hookrightarrow X$ denote the embedding. As $\mathscr{T} \otimes \mathscr{K}_X$ is a coherent analytic sheaf with support in A, we have

$$\mathscr{T} \otimes \mathscr{K}_X = \iota_* \iota^* \big(\mathscr{T} \otimes \mathscr{K}_X\big). \tag{8.19}$$

This is easy to see by working in the category of linear (fibre) spaces associated to coherent analytic sheaves: Let $L := L(\mathscr{T} \otimes \mathscr{K}_X)$ denote the associated linear space. For linear spaces, ι^* means nothing else but restriction of the linear space to the subvariety A, i. e. $\iota^* L = L_A$. On the other hand, ι_* means just a trivial extension of the space over A to X. Since L is trivial outside of A, we get $\iota_*(L_A) = L$. Note that (8.19) is not true for sheaves which are not coherent.

We get (cf. e. g. Prop. 5.2 in [Ive84, Chap. II])

$$H^q(A, \iota^*(\mathscr{T} \otimes \mathscr{K}_X)) \cong H^q(X, \iota_* \iota^*(\mathscr{T} \otimes \mathscr{K}_X)) \overset{(8.19)}{=} H^q(X, \mathscr{T} \otimes \mathscr{K}_X).$$

Since $H^q(A, \iota^*(\mathscr{T} \otimes \mathscr{K}_X)) = 0$ for $q > r$ (see e. g. Thm. 10.2 in [Ive84, Chap. II]), we conclude:

Theorem 8.20. *Let X be a holomorphically convex normal connected Kähler space of dimension n such that $\mathscr{K}_X$ is locally free, let Φ denote a smooth plurisubharmonic exhaustion function of X, and let $\mathscr{S}$ be a Nakano semi-positive sheaf on X with $(+)$ and normal $L(\mathscr{S}/\mathscr{T}(\mathscr{S}))$. Then, we get, for $q > \max\{n - \sigma(\Phi), \dim \operatorname{supp} \mathscr{T}(\mathscr{S})\}$,*

$$H^q(X, \mathscr{S} \otimes \mathscr{K}_X) = 0.$$

We give a counterexample to show that this result is sharp (with respect to the dimension). Let M be a holomorphically convex Kähler manifold of dimension n which is not Stein and admits a smooth plurisubharmonic exhaustion function which is in (at least) one point strictly plurisubharmonic (consider e. g. the blow up of $\mathbb{C}^n$ in a point). Let A be a compact analytic subset of M and $\iota\colon A \hookrightarrow M$ the embedding of A. For any $0 < q \leq \dim A$, one can find such spaces M and A admitting a coherent analytic sheaf $\mathscr{F}$ on A such that $H^q(A, \mathscr{F}) \neq 0$. We set

$$\mathscr{S} := \iota_* \mathscr{F} \otimes (\Omega^n_M)^*.$$

$\mathscr{S}$ is Nakano semi-positive as it vanishes outside a thin set. Since $\mathrm{id}^T_M \mathscr{S} = \mathscr{S}/\mathscr{T}(\mathscr{S}) = 0$, $\mathscr{S}$ satisfies $(+)$, and $L(\mathscr{S}/\mathscr{T}(\mathscr{S})) = M \times 0$ is normal. Yet, we have

$$H^q(M, \mathscr{S} \otimes \Omega^n_M) = H^q(M, \iota_* \mathscr{F}) \cong H^q(A, \mathscr{F}) \neq 0.$$

Chapter 9
Ideal Sheaves

In this chapter, we apply the generalized Takegoshi theorem to reduced ideal sheaves on manifolds. First, we study proper modifications of such sheaves by applying results from Chapter 4 in Section 9.1 (cf. Sect. 7 in [RS13]). In Section 9.2, we will use Theorem 8.2 to derive vanishing results for submanifolds of holomorphically convex manifolds (cf. Sect. 5 in [Ser15]).

9.1 Proper modifications of reduced ideal sheaves

In this section, we discuss the application of Theorem 4.1 to reduced ideal sheaves. As a preparation, note the following:

Lemma 9.1. *Let $\pi\colon Y \to X$ be a surjective holomorphic mapping between complex spaces Y, X. Then, $\pi^*\mathcal{O}_X = \mathcal{O}_Y$.*

Proof: As $\pi^{-1}\mathcal{O}_X \subset \mathcal{O}_Y$, we have that $\pi^*\mathcal{O}_X = \pi^{-1}\mathcal{O}_X \otimes_{\pi^{-1}\mathcal{O}_X} \mathcal{O}_Y = \mathcal{O}_Y$ because $\pi^{-1}\mathcal{O}_X$ contains (the germ of) the function 1 at any point of Y. □

Coming to reduced ideal sheaves, let us start with the following observation:

Lemma 9.2. *Let X be a locally irreducible complex space and $A \subset X$ an (unreduced) analytic subspace with ideal sheaf $\mathscr{J}_A$. Let $\pi : Y \to X$*

be a proper modification and $B := \pi^{-1}(A)$ the unreduced analytic preimage with ideal sheaf $\mathscr{J}_B$. Then:

$$\pi^T \mathscr{J}_A = \mathscr{J}_B.$$

Proof: Consider the short exact sequence of sheaves over X:

$$0 \to \mathscr{J}_A \xrightarrow{\alpha} \mathcal{O}_X \longrightarrow \mathcal{O}_X/\mathscr{J}_A \to 0.$$

By right-exactness of π^*, we deduce the exact sequence

$$\pi^* \mathscr{J}_A \xrightarrow{\pi^*\alpha} \pi^*\mathcal{O}_X \longrightarrow \pi^*\big(\mathcal{O}_X/\mathscr{J}_A\big) \to 0.$$

Now, we use Lemma 9.1 twice: $\pi^*\mathcal{O}_X = \mathcal{O}_Y$ and $(\pi|_B)^*\mathcal{O}_A = \mathcal{O}_B$ (which implies that $\pi^*(\mathcal{O}_X/\mathscr{J}_A) = \mathcal{O}_Y/\mathscr{J}_B$ using the definition of the analytic preimage, see e. g. Prop. 0.27 in [Fis76]). As $\mathcal{O}_Y$ is torsion-free, it is clear that

$$\mathscr{T}(\pi^* \mathscr{J}_A) \subset \mathscr{K}\!er\, \pi^*\alpha. \tag{9.3}$$

Consider $\pi^T\alpha\colon \pi^T \mathscr{J}_A \to \pi^T\mathcal{O}_X = \pi^*\mathcal{O}_X = \mathcal{O}_Y$. By (9.3), it follows that $\mathscr{I}\!m(\pi^*\alpha) = \mathscr{I}\!m(\pi^T\alpha)$, and Lemma 4.23 tells us that π^T preserves injectivity, i. e. $\pi^T\alpha$ is injective. So, we obtain the short exact sequence

$$0 \to \pi^T \mathscr{J}_A \xrightarrow{\pi^T\alpha} \mathcal{O}_Y \longrightarrow \mathcal{O}_Y/\mathscr{J}_B \to 0,$$

telling us that in fact $\pi^T \mathscr{J}_A = \mathscr{J}_B$. □

It is clear that $\mathscr{J}_A$ and $\pi^T \mathscr{J}_A = \mathscr{J}_B$ are torsion-free, and so we obtain from Theorem 4.1 that:

$$\mathscr{J}_B \cong \pi^T\pi_* \mathscr{J}_B. \tag{9.4}$$

Under some additional assumptions, we have also:

Lemma 9.5. *Let X be a normal complex space, and let A be a locally complete intersection or a normal analytic set in X with (reduced) ideal sheaf $\mathscr{J}_A$. Let $\sigma\colon \widetilde{X} \to X$ denote the blow up of X with centre A, i. e. the monoidal transformation with respect to $\mathscr{J}_A$, and let $\mathscr{J}_B$ be the (reduced) ideal sheaf associated to the reduced exceptional set $B := \sigma^{-1}(A)$. Then:*

$$\mathscr{J}_A \cong \sigma_* \mathscr{J}_B. \tag{9.6}$$

Proof: The statement is local with respect to X, so we can assume that A is the zero-set of reduced holomorphic functions $f_0, ..., f_m$. $(\mathscr{J}_A)_p$ is generated by the germs $f_{0,p}, .., f_{m,p}$, and $\widetilde{X} \subset X \times \mathbb{CP}^m$ (see e. g. § 2.5 in [Rie71] for the monoidal transformation of ideal sheaves). We show that

$$\mathcal{O}_A \cong \sigma_* \mathcal{O}_B.$$

I) A is a complete intersection, i. e. $m + 1 = \operatorname{codim} A$: We claim that $B = A \times \mathbb{CP}^m$. This implies, for all open subsets $U \subset A$,

$$\mathcal{O}_A(U) \cong \mathcal{O}_{A \times \mathbb{CP}^m}(U \times \mathbb{CP}^m) = \mathcal{O}_B(\sigma^{-1}(U)).$$

Let us now prove $B = A \times \mathbb{CP}^m$. Since A is a complete intersection, the Koszul complex implies that the monoidal transformation $\sigma \colon \widetilde{X} \to X$ is given by

$$\widetilde{X} = \{(p; [t_0 : ... : t_m]) \in X \times \mathbb{CP}^m \colon t_i f_j(p) = t_j f_i(p), i, j = 0, .., m\}$$

and $\sigma = \operatorname{pr}_1|_{\widetilde{X}} \colon \widetilde{X} \to X$ (see e. g. [GR71, Sect. III.2.7]). Since $t_i f_j(p) = t_j f_i(p)$ for all i, j if and only if all $f_i(p)$ vanish in p, we get the claimed.

II) A is normal: In this case, B is an analytic subset of $A \times \mathbb{CP}^m$. By the surjectivity $\sigma(B) = A$, we get the injection $\mathcal{O}_A \cong \sigma_* \mathcal{O}_{A \times \mathbb{CP}^m} \hookrightarrow \sigma_* \mathcal{O}_B$. On the other hand, a section in $\mathcal{O}_B(\sigma^{-1}(U))$ gives a weakly holomorphic function on A: With part (I) applied on the regular part A_{reg} of A, we get a holomorphic function on A_{reg} which is bounded in points of A_{sing}. Since we assumed A to be normal, we get $\sigma_* \mathcal{O}_B \cong \widehat{\mathcal{O}}_A \cong \mathcal{O}_A$.

Thus, $\mathcal{O}_A \cong \sigma_* \mathcal{O}_B$, as desired. In other words: $\mathcal{O}_X / \mathscr{J}_A \cong \sigma_*(\mathcal{O}_{\widetilde{X}} / \mathscr{J}_B)$. We obtain the following exact and commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & \sigma_* \mathscr{J}_B \arrow[r] & \sigma_* \mathcal{O}_{\widetilde{X}} \arrow[r] & \sigma_*(\mathcal{O}_{\widetilde{X}} / \mathscr{J}_B) & \\
0 \arrow[r] & \mathscr{J}_A \arrow[r] & \mathcal{O}_X \arrow[r] \arrow[u, equal, "\wr"'] & \mathcal{O}_X / \mathscr{J}_A \arrow[r] \arrow[u, equal, "\wr"'] & 0.
\end{tikzcd}$$

It follows that in fact $\sigma_* \mathscr{J}_B \cong \mathscr{J}_A$. □

In the situation of Lemma 9.5, we can now apply Theorem 4.1 to $\mathscr{J}_A \cong \sigma_* \mathscr{J}_B$ and obtain:

$$\mathscr{J}_A \cong \sigma_* \sigma^T \mathscr{J}_A. \tag{9.7}$$

9.2 Submanifolds of holomorphically convex manifolds

In this section, we give an example of a torsion-free coherent analytic non-locally-free sheaf which satisfies (+) (see Def. 8.1). This yields vanishing results for reduced ideal sheaves by Theorem 8.2.

Let M be a complex manifold of dimension n, let Σ be a (connected) submanifold of M of codimension m, and let $\mathscr{J}=\mathscr{J}_\Sigma$ be the (reduced) ideal sheaf of Σ. If $m>1$, then $\mathscr{J}$ is not locally free. The monoidal transformation with respect to $\mathscr{J}$ of M is given by the blow up $\varphi\colon \widetilde{M}\to M$ of M with centre in Σ such that $\varphi^T\mathscr{J}$ is locally free. Let $Z=\varphi^{-1}(\Sigma)$ denote the exceptional divisor / set of φ, and let $\mathcal{O}(-Z)$ denote the ideal sheaf on $\widetilde{M}$ of holomorphic functions vanishing on Z. (9.4) and (9.6) imply

$$\varphi^T\mathscr{J}=\mathcal{O}(-Z) \text{ and } \mathscr{J}\cong\varphi_*\mathcal{O}(-Z). \tag{9.8}$$

Hence, $\mathscr{J}\cong\varphi_*\varphi^T\mathscr{J}$, which is already the statement of Theorem 4.2. Therefore, one need not verify the normality of $L(\mathscr{J})$ to prove Theorem 8.2 for $\mathscr{J}$: one can use the second isomorphism of (9.8) combined with the projection formula, Theorem 4.10, to get (8.13).

On the other hand, for $m=2$, $L(\mathscr{J})$ is a hypersurface and, hence, a Cohen-Macaulay space. Remark 4.20 and (9.8) imply that $L(\mathscr{J})$ is normal, which can be observed by computing the codimension of the singular set without using (9.8), as well.

For the canonical sheaf on $\widetilde{M}$, we have (see e. g. Prop. VII.12.7 in [Dem12])

$$\Omega^n_{\widetilde{M}}=\varphi^*\Omega^n_M\otimes\mathcal{O}((m-1)Z).$$

Combining this with (9.8), we get

$$\varphi^T(\mathscr{J}\otimes\Omega^n_M)=\Omega^n_{\widetilde{M}}\otimes\mathcal{O}(-mZ).$$

Under the assumption that $\mathscr{J}$ is semi-positive (e. g. Σ is the zero set of finitely many globally defined holomorphic functions, see Lemma 9.13

below), we get that $\varphi^T\mathscr{J} \cong \mathcal{O}(-Z)$ is semi-positive, too. Let L denote the line bundle on $\widetilde{M}$ associated to $\mathcal{O}(-Z)$ such that $L^{\otimes k}$ is the line bundle associated to $\mathcal{O}(-kZ)$. Since $\Theta(L^{\otimes k}) = k\Theta(L)$, semi-positivity of $\varphi^T\mathscr{J} = \mathcal{O}(-Z)$ gives us the semi-positivity of $\mathcal{O}(-(m-1)Z)$. Hence, $\mathscr{J}$ satisfies (+) (with $\mathscr{L} = \mathcal{O}(-(m-1)Z)$). In particular, this is an example where the property (+) is derived from the semi-positivity of $\mathscr{J}$. Applying Theorem 8.2, we get:

Corollary 9.9. *Let M be a holomorphically convex Kähler manifold of dimension n, let Φ be a smooth plurisubharmonic exhaustion function of M, let $\mathscr{E}$ be a Nakano semi-positive locally free analytic sheaf on M, and let $\mathscr{J}$ be a semi-positive ideal sheaf (e. g. generated by finitely many globally defined holomorphic functions) given by a submanifold of M. Then, for each $q > n - \sigma(\Phi)$:*

$$H^q(X, \mathscr{J} \otimes \mathscr{E} \otimes \Omega^n_M) = 0.$$

Further, we obtain a vanishing result for submanifolds of holomorphically convex manifolds: Keeping the notation from above, the short exact sequence

$$0 \to \mathscr{J} \to \mathcal{O}_M \to \mathcal{O}_M/\mathscr{J} \to 0$$

gives the short exact sequence

$$0 \to \mathscr{J} \otimes \Omega^n_M \to \Omega^n_M \to \Omega^n_M \otimes \mathcal{O}_M/\mathscr{J} \to 0. \tag{9.10}$$

For all $q > n-\sigma(\Phi)$, the long exact sequence of cohomology associated to (9.10) tensored with a Nakano semi-positive locally free $\mathscr{E}$

$$\ldots \to \underset{\substack{\| \text{Theorem 7.9} \\ 0}}{H^q(M,\mathscr{E}\otimes\Omega^n_M)} \to H^q\left(M,\mathscr{E}\otimes\Omega^n_M\otimes\frac{\mathcal{O}_M}{\mathscr{J}}\right) \to \underset{\substack{\| \text{Corollary 9.9} \\ 0}}{H^{q+1}(M,\mathscr{J}\otimes\mathscr{E}\otimes\Omega^n_M)} \to \ldots$$

implies that

$$H^q(X, \mathscr{E} \otimes \Omega^n_M \otimes \mathcal{O}_M/\mathscr{J}) = 0. \tag{9.11}$$

Let $\mathscr{N}_{\Sigma/M}$ denote the sheaf of sections of the normal bundle of Σ. Since

$$(\Omega^n_M \otimes \mathcal{O}_M/\mathscr{J})|_\Sigma = \Omega^n_M|_\Sigma \otimes \mathcal{O}_\Sigma$$

and

$$\Omega_\Sigma^{n-m} = \Omega_M^n|_\Sigma \otimes \det \mathscr{N}_{\Sigma/M}$$

(adjunction formula, see e.g. (5.26a) in [PR94]), we obtain

$$(\Omega_M^n \otimes \mathcal{O}_M/\mathscr{J})|_\Sigma \cong \Omega_\Sigma^{n-m} \otimes \det \mathscr{N}^*_{\Sigma/M}.$$

Since $\Omega_M^n \otimes \mathcal{O}_M/\mathscr{J}$ is only supported on Σ, this means $\iota_*(\Omega_\Sigma^{n-m} \otimes \det \mathscr{N}^*_{\Sigma/M}) = \Omega_M^n \otimes \mathcal{O}_M/\mathscr{J}$, where ι denotes the embedding of Σ in M. In particular,

$$H^q(M, \Omega_M^n \otimes \mathcal{O}_M/\mathscr{J}) \cong H^q(\Sigma, \Omega_\Sigma^{n-m} \otimes \det \mathscr{N}^*_{\Sigma/M})$$

(cf. e.g. Prop. 5.2 in [Ive84, Chap. II]). Applying Corollary 9.9 on this and on (9.11), we get:

Corollary 9.12. *Let M be a holomorphically convex Kähler manifold of dimension n, let Φ be a smooth plurisubharmonic exhaustion function of M, and let Σ be a submanifold of M with a semi-positive ideal sheaf and of dimension r. Then, for each $q > n - \sigma(\Phi)$:*

$$H^q(\Sigma, \Omega_\Sigma^r \otimes \det \mathscr{N}^*_{\Sigma/M}) = 0.$$

In the case that the normal bundle of Σ (or the determinant of it) is the restriction of a Nakano semi-positive vector bundle, we get for each $q > n - \sigma(\Phi)$:

$$H^q(\Sigma, \Omega_\Sigma^r) = 0.$$

The following observation shows that the presented corollaries can be applied for globally defined submanifolds.

Lemma 9.13. *Let M be a complex manifold and let $f_0, ..., f_m$ be holomorphic functions on M. Then, the ideal sheaf $\mathscr{J}$ generated by $f_0, ..., f_m$ is semi-positive.*

Proof: Let S denote the linear space associated to $\mathscr{J}$. The surjection $\mathcal{O}^{m+1} \twoheadrightarrow \mathscr{J}, (r_i)_{i=0}^m \mapsto \sum r_i f_i$ induces an embedding $S \hookrightarrow M \times \mathbb{C}^{m+1}$. The restriction of the flat hermitian metric of $M \times \mathbb{C}^{m+1}$ defines a (Nakano) semi-negative metric on S, i.e. $\mathscr{J}$ is semi-positive. $\square$

Chapter 10
Fine Resolutions of $\mathcal{K}_X(\mathscr{S})$

In this chapter, we prove the theorems presented in the introduction. Since the proofs are close to the line bundle case as presented by J. Ruppenthal in [Rup14a], we will skip some details.

Let X be a Hermitian complex space of pure dimension n, let $\mathscr{S}$ be a (Hermitian) coherent analytic sheaf on X, and let $S := L(\mathscr{S})$ be the associated linear space. We set $A := \operatorname{Sing}\mathscr{S}$, $X' := X_{\mathrm{reg}} \setminus A$, and $F := R(\mathscr{S}) = S^*_{X'}$. In Section 5.1, we defined

$$\mathcal{K}_X(\mathscr{S}) = \mathcal{K}_X(F) := \mathscr{K}\!\mathit{er}(\overline{\partial}_{w,\mathrm{loc}} \colon \mathcal{C}_F^{n,0} \to \mathcal{C}_F^{n,1}).$$

Let $\pi\colon M \to X$ be a resolution of singularities such that $\mathscr{E} := \pi^T(\mathscr{S})$ is locally free (using Hironaka and Rossi, Theorem 2.19). We denote by E the vector bundle on M such that $\mathscr{E} = \mathcal{O}(E)$. For small enough open Stein subsets U of X, Theorem 7.9 (Takegoshi's vanishing theorem) implies that the $\overline{\partial}_{w,\mathrm{loc}}$-equation is solvable on $V := \pi^{-1}(U)$. For this, choose a Hermitian metric on U which is Kähler, a Hermitian metric on S such that $\mathscr{S}_U$ is Nakano semi-positive, and recall that $\overline{\partial}_{w,\mathrm{loc}}$ and $\mathcal{K}_X(\mathscr{S})$ are independent of the choice of the Hermitian metrics. We conclude

$$\pi_*(\mathcal{C}_E^{n,0}) \xrightarrow{\pi_*\overline{\partial}_{w,\mathrm{loc}}} \pi_*(\mathcal{C}_E^{n,1}) \xrightarrow{\pi_*\overline{\partial}_{w,\mathrm{loc}}} \pi_*(\mathcal{C}_E^{n,2}) \xrightarrow{\pi_*\overline{\partial}_{w,\mathrm{loc}}} \dots$$

is exact. Fix (again) a small enough open Stein set $U \subset X$ and set $V := \pi^{-1}(U)$. If $u \in \mathcal{C}_F^{n,0}(U)$ with $\overline{\partial}_{w,\mathrm{loc}} u = 0$, then the pullback $v := \pi^* u \in L^{2,\mathrm{loc}}_{n,0}(V, E)$ is obviously holomorphic on $\pi^{-1}(U \setminus A)$. Using the L^2-extension theorem (see Theorem 5.6), we get $\overline{\partial}_{w,\mathrm{loc}} v = 0$ on the whole of V, i. e. $v \in \Omega_M^n(V, E)$ and $u \in \mathscr{K}\!\mathit{er}\, \pi_*\overline{\partial}_{w,\mathrm{loc}}$. This implies that

$$\mathcal{K}_X(\mathscr{S}) = \mathscr{K}er\left(\pi_*\overline{\partial}_{w,\mathrm{loc}}\colon \pi_*(\mathcal{C}_E^{n,0}) \to \pi_*(\mathcal{C}_E^{n,1})\right).$$

Altogether, we obtain:

Theorem 10.1. *Let X be a Hermitian complex space of pure dimension n, $\mathscr{S}$ a coherent analytic sheaf on X and $\pi\colon M \to X$ a resolution of singularities of X such that $\pi^T\mathscr{S}$ is locally free. Then, the complex $(\pi_*\mathcal{C}^{n,\cdot}_{\pi^T\mathscr{S}}, \pi_*\overline{\partial}_{w,\mathrm{loc}})$ is a free resolution of $\mathcal{K}_X(\mathscr{S})$. In particular,*

$$\mathcal{K}_X(\mathscr{S}) \cong \pi_*(\Omega_M^n \otimes \pi^T\mathscr{S}). \tag{10.2}$$

In the notation of H. Grauert and O. Riemenschneider in [GR70], (10.2) means

$$\mathcal{K}_X(\mathscr{S}) \cong \mathscr{S} \cdot \mathscr{K}_X,$$

where $\mathscr{K}_X$ is the Grauert-Riemenschneider canonical sheaf. In general, $\mathscr{S} \cdot \mathscr{K}_X := \pi_*(\Omega_M^n \otimes \pi^T\mathscr{S})$ is not isomorphic to $\mathscr{S} \otimes \mathscr{K}_X$. Only for locally free sheaves $\mathscr{E}$ on X, the projection formula (8.8) implies

$$\mathcal{K}_X(\mathscr{E}) \cong \mathscr{K}_X \otimes \mathscr{E}. \tag{10.3}$$

In particular, $\mathcal{K}_X := \mathcal{K}_X(\mathbb{C}) \cong \mathscr{K}_X$ even for non-normal complex spaces.

Let us get back to the setting of Theorem 10.1: Since $\mathscr{S}$ is locally Nakano semi-positive, we get $\pi_{(q)}(\Omega_M^n \otimes \pi^T\mathscr{S}) = 0$ for all $q > 0$ (see Corollary 8.10). The Leray spectral implies: For all $U \subset X$ and $q > 0$,

$$H^q(U, \mathcal{K}_X(\mathscr{S})) \cong H^q(\pi^{-1}(U), \Omega_M^n \otimes \pi^T\mathscr{S}). \tag{10.4}$$

For the proof of Theorem 10.1 – more precisely, for the $\mathcal{K}_X(\mathscr{S})$ independence of the Hermitian structure of $\mathscr{S}$ – we used that the Hermitian metric has a smooth extension to singular points of $\mathscr{S}$. Is this not the case, we need to assume π-relative Nakano semi-positivity (cf. Thm. 2.1 in [Rup14a]):

Theorem 10.1'. *Let X be a Hermitian complex space of pure dimension n, let $\pi\colon M \to X$ be a resolution of singularities of X, let $X' \subset X_{\mathrm{reg}}$ be the complement of the centre of π, and let $E \to M$ be a Hermitian vector bundle on M. Let $F := \pi^{-1}|_{X'}^* E$ be Nakano* *semi-positive on $U' := U \cap X'$ for all small enough $U \subset X$, i. e. E is Nakano semi-positive on $\pi^{-1}(U)$. Then,*

$$\mathcal{K}_X(F) \cong \pi_*(\Omega^n_M(E))$$

and, for all $U \subset X$ and $q > 0$,

$$H^q(U, \mathcal{K}_X(F)) \cong H^q(\pi^{-1}(U), \Omega^n_M(E)).$$

The following theorem connects $\mathcal{K}_X(F)$ with the $L^{2,\mathrm{loc}}$-Dolbeault cohomologies on X' (cf. Thm. 3.1 in [Rup14a]).

Theorem 10.5. *Let X be a Hermitian complex space of pure dimension n, let $F \to X'$ be a Hermitian vector bundle on a Zariski open set $X' \subset X_{\mathrm{reg}}$ which is Nakano semi-positive on $U' := U \cap X'$ for small enough $U \subset X$. Then, the complex*

$$0 \to \mathcal{K}_X(F) \to \mathcal{C}_F^{n,0} \longrightarrow \mathcal{C}_F^{n,1} \longrightarrow \mathcal{C}_F^{n,2} \longrightarrow \dots$$

given by $\overline{\partial}_{w,\mathrm{loc}}$ is exact, i. e. a fine resolution of $\mathcal{K}_X(F)$.

The formal de Rham lemma (see e. g. Sect. B.1.3 in [GR79]) implies

$$H^q(X, \mathcal{K}_X(F)) \cong H^q(\mathcal{C}_F^{n,\cdot}(X)) \overset{\mathrm{def}}{=} H^{n,q}_{w,\mathrm{loc}}(X, F) \quad \forall\, q \geq 0. \tag{10.6}$$

Before we prove Theorem 10.5, let us show the first theorem of the introduction:

Theorem 1.3. *Let X be a Hermitian complex space of pure dimension n, let $\mathscr{S}$ be a coherent analytic sheaf on X, and let $\pi\colon M \to X$ be a resolution of singularities such that the torsion-free preimage $\pi^T\mathscr{S}$ is locally free. We set $X' := X_{\mathrm{reg}} \setminus \operatorname{Sing} \mathscr{S}$ and denote the vector bundle associated to $\mathscr{S}_{X'}$ as $F \to X'$. Then, for all $q \geq 0$,*

$$H^{n,q}_{w,\mathrm{loc}}(X, F) \cong H^q(M, \Omega^n_M \otimes \pi^T\mathscr{S}).$$

If either X is compact or $X \Subset Y$, where Y is a Hermitian complex space such that ∂X is smooth, strictly pseudoconvex and contained in X_{reg}, then

$$H^{n,q}_w(X, F) \cong H^q(M, \Omega^n_M \otimes \pi^T\mathscr{S}).$$

Proof: Since Theorem 10.1 implies $\mathcal{K}_X(\mathscr{S}) \cong \pi_*(\Omega^n_M \otimes \pi^T \mathscr{S})$ and the equation (10.4), we get with (10.6) that

$$\begin{aligned} H^{n,q}_{w,\mathrm{loc}}(X,F) &\cong H^q(X, \mathcal{K}_X(F)) \cong H^q(X, \mathcal{K}_X(\mathscr{S})) \\ &\cong H^q(M, \Omega^n_M \otimes \pi^T \mathscr{S}). \end{aligned}$$

The second isomorphism for the compact case is trivial. In the other case, it follows from

$$H^{p,q}_w(\Omega, F) \cong H^{p,q}_{w,\mathrm{loc}}(\Omega, F). \tag{10.7}$$

For a proof, see e. g. Thm. 4.1 in [LM02, Chap. VIII] or Thm. 2.7 in [Ser10]. The proof involves Grauert's bumping method developed in [Gra58]. □

Using Theorem 10.1' with (10.6), we obtain Theorem 1.3', as well. Furthermore, (10.3) now implies Corollary 1.4.

If an incomplete Hermitian metric can be approximated in an appropriate way by complete metrics, then the vanishing of (n, q)-Dolbeault cohomology groups can also be obtained with respect to the incomplete metric (see e. g. Lem. 2.3 in [PS91]):

Theorem 10.8. *Let M be a complex manifold, $E \to M$ a Hermitian vector bundle, and let $\{\gamma_k\}$ be a pointwise decreasing sequence of Hermitian metrics on M which converges pointwise to a Hermitian metric γ_0 on M. If the $\overline{\partial}_w$-equation is solvable for $\overline{\partial}_w$-closed (n, q)-forms in $L^2(M, E; \gamma_k)$ with an estimate independent of k, then the $\overline{\partial}_w$-equation is solvable for $\overline{\partial}_w$-closed (n, q)-forms in $L^2(M, E; \gamma_0)$ with the same estimate.*

The proof in [PS91] straight forwardly generalizes to forms with values in vector bundles without any crucial changes.

Sketch of the proof of Theorem 10.5: Since the proof works more or less the same as the proof presented by J. Ruppenthal in [Rup14a, Sect. 3.1], we will sketch it:

Let p be a point in X and let $U \subset X$ be a Stein neighbourhood of p such that U can be embedded in $\mathbb{C}^N$ for an $N \geq n$, and such that F is Nakano semi-positive on $U' := U \cap X'$. We can assume $p = 0$ and $U = B_c \cap X$ for a small $r > 0$ such that

$$A := (X \backslash X') \cap B_c = \{f_1(z) = ... = f_m(z) = 0\},$$

where B_c denotes the ball in $\mathbb{C}^N$ with radius c around 0 and f_i are holomorphic functions on B_c. We define

$$\varphi_0(z) := -\log(c^2 - |z|^2) \quad \text{and} \quad \varphi_k := \varphi_0 - \tfrac{1}{k} \log\left(-\log \textstyle\sum |f_j|^2\right).$$

Then, the Kähler forms $\omega_k := \mathrm{i}\partial\overline{\partial}\varphi_k$, $k \geq 1$, give complete metrics γ_k on $U \backslash A$, which decrease pointwise monotonically to the metric γ_0 given by $\omega_0 := \mathrm{i}\partial\overline{\partial}\varphi_0$, and $|\partial\varphi_k|_{\omega_k}$ is bounded by a constant C independently of k (see Lem. 2.4 in [PS91]). Obviously, γ_0 is not complete on U' – yet, γ_k is complete for $k \geq 1$. Hence, we can apply the Donnelly-Fefferman-Ohsawa vanishing theorem (see Theorem 7.14) for γ_k. More precisely, if $u \in L^2_{n,q}(U', F; \gamma_k)$ is $\overline{\partial}_w$-closed, then there exists a $v \in L^2_{n,q-1}(U', F; \gamma_k)$ with $\overline{\partial}_w v = u$ and $\|v\|_{\omega_k} \leq 4C\|u\|_{\omega_k}$. Since C is independent of k, we can use Theorem 10.8 and obtain that the $\overline{\partial}_w$-equation is solvable in the L^2-sense with respect to γ_0, as well.

If γ denotes the metric given on X, we get $\gamma \sim \gamma_0$ on $X \cap B_r$ for all $r < c$. Therefore, we conclude that $\overline{\partial}_{w,\mathrm{loc}}$ is locally solvable, i. e. $(\mathcal{C}^{n,\cdot}_F, \overline{\partial}_{w,\mathrm{loc}})$ is a fine resolution of $\mathcal{K}_X(F)$. $\square$

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"... aaaand it's gone!"