# Defining Functions and Cores of Unbounded Domains

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Tobias Harz

betreut von Prof. Dr. Nikolay Shcherbina

Bergische Universität Wuppertal
Fachbereich C
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Solange der Virtuose Anschläge fasset, Ideen sammlet, wählet, ordnet, in Plane verteilet: so lange genießt er die sich selbst belohnenden Wollüste der Empfängnis. Aber sobald er einen Schritt weiter gehet und Hand anleget, seine Schöpfung auch außer sich darzustellen: sogleich fangen die Schmerzen der Geburt an, welchen er sich selten ohne alle Aufmunterung unterziehet.

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*(G.E. Lessing)*

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# Introduction

## 1. Background and motivation

Let $\Omega \subset \mathbb{R}^n$ be a domain. It is well known from classical results in convex analysis that the following conditions are equivalent characterizations for convexity of the set $\Omega$ (see, for example, [Kr15]):

(1) There exists a convex function $\varphi \colon \Omega \to \mathbb{R}$ such that for every $c \in \mathbb{R}$ the set $\varphi^{-1}((-\infty, c])$ is relatively compact in $\Omega$.

(2) For every compact set $K \subset \Omega$, the linearly convex hull

$$\hat{K}^{\mathcal{L}(\Omega)} := \Big\{x \in \Omega : |L(x)| \leq \max_K |L| \text{ for every } \\ \text{affine linear function } L \colon \Omega \to \mathbb{R}\Big\}$$

is relatively compact in $\Omega$.

For domains $\Omega \subset \mathbb{C}^n$, the solution of the so-called Levi problem (see [Ok53], [Br54] and [No54]) shows that, in analogy to the case of convex domains, the following assertions on $\Omega$ are equivalent:

(1′) There exists a plurisubharmonic function $\varphi \colon \Omega \to \mathbb{R}$ such that for every $c \in \mathbb{R}$ the set $\varphi^{-1}((-\infty, c])$ is relatively compact in $\Omega$ (i.e., $\Omega$ is *pseudoconvex*).

(2′) For every compact set $K \subset \Omega$, the holomorphically convex hull

$$\hat{K}^{\mathcal{O}(\Omega)} := \Big\{z \in \Omega : |f(z)| \leq \max_K |f| \text{ for every } \\ \text{holomorphic function } f \colon \Omega \to \mathbb{C}\Big\}$$

is relatively compact in $\Omega$ (i.e., $\Omega$ is *holomorphically convex*).

It is a fundamental observation in the theory of several complex variables that the existence of holomorphic objects (functions, differential forms, sections in vector bundles) on $\Omega \subset \mathbb{C}^n$ is closely related to the convexity properties of $\Omega$ which are described in (1′) and (2′). Indeed, a classical result due to Cartan and Thullen shows that $\Omega$ is holomorphically convex if and only if it is the domain of existence of a holomorphic function (see, for example, [Hö90]). Moreover, it follows from works of Morrey, Kohn and Hörmander (see [M59], [Ko63] and [Hö65]) that pseudoconvexity of $\Omega$ is equivalent to the vanishing of all Dolbeault cohomology groups $H^{p,q}(\Omega)$, $q \geq 1$ (i.e., solvability of the equation $\bar{\partial}u = f$ for every $\bar{\partial}$-closed $f \in \Lambda^{p,q}(\Omega)$, $q \geq 1$); solving inhomogeneous $\bar{\partial}$-equations is one of the strongest known techniques for constructing holomorphic objects.

For smoothly bounded domains, convexity and pseudoconvexity can also be characterized by means of local defining functions: a domain $\Omega \subset \mathbb{R}^n$ (resp. $\Omega \subset \mathbb{C}^n$) with $\mathcal{C}^2$-smooth boundary is convex (resp. pseudoconvex) if and only if for every $p \in b\Omega$ there exists a $\mathcal{C}^2$-smooth function $\varphi\colon U \to \mathbb{R}$ on an open neighbourhood $U$ of $p$ such that $\Omega \cap U = \{\varphi < 0\}$, $d\varphi \neq 0$ on $b\Omega \cap U$ and for every $q \in b\Omega \cap U$ one has

$$\sum_{j,k=1}^{n} \frac{\partial^2 \varphi}{\partial x_j \partial x_k}(q)\xi_j \xi_k \geq 0 \text{ for every } \xi \in \mathbb{R}^n \text{ such that } \sum_{j=1}^{n} \frac{\partial \varphi}{\partial x_j}(q)\xi_j = 0$$

$$\Bigg(\text{resp. } \sum_{j,k=1}^{n} \frac{\partial^2 \varphi}{\partial z_j \partial \bar{z}_k}(q)\xi_j \bar{\xi}_k \geq 0 \text{ for every } \xi \in \mathbb{C}^n \text{ such that } \sum_{j=1}^{n} \frac{\partial \varphi}{\partial z_j}(q)\xi_j = 0\Bigg).$$

The above results are of considerable practical use, since they provide local characterizations of (pseudo-)convexity in terms of differential conditions on functions. Moreover, they also make it possible to define in a natural way the notions of strict (pseudo-)convexity, by requiring the involved inequalities to be strict: a domain $\Omega \subset \mathbb{R}^n$ (resp. $\Omega \subset \mathbb{C}^n$) with $\mathcal{C}^2$-smooth boundary is called *strictly convex* (resp. *strictly pseudoconvex*) if for every $p \in b\Omega$ there exists a $\mathcal{C}^2$-smooth function $\varphi\colon U \to \mathbb{R}$ on an open neighbourhood $U$ of $p$ such that $\Omega \cap U = \{\varphi < 0\}$, $d\varphi \neq 0$ on $b\Omega \cap U$ and for every $q \in b\Omega \cap U$ one has

$$\sum_{j,k=1}^{n} \frac{\partial^2 \varphi}{\partial x_j \partial x_k}(q)\xi_j \xi_k > 0 \quad \text{for every } \xi \in \mathbb{R}^n \setminus \{0\} \text{ such that } \sum_{j=1}^{n} \frac{\partial \varphi}{\partial x_j}(q)\xi_j = 0$$

$$\Bigg(\text{resp. } \sum_{j,k=1}^{n} \frac{\partial^2 \varphi}{\partial z_j \partial \bar{z}_k}(q)\xi_j \bar{\xi}_k > 0 \quad \text{for every } \xi \in \mathbb{C}^n \setminus \{0\} \text{ such that } \sum_{j=1}^{n} \frac{\partial \varphi}{\partial z_j}(q)\xi_j = 0\Bigg).$$

Strict pseudoconvexity turns out to be an important and extremely useful concept. In fact, while general pseudoconvexity provides the most natural setting for developing function theory in higher dimensions, many important results can only be established on strictly pseudoconvex domains. In some cases this may be merely due to the fact that sufficiently strong techniques for extending results to the general setting are missing, and maybe stronger statements can still be obtained in the future (for example, it is still an open question whether every biholomorphic map $\Phi\colon \Omega_1 \to \Omega_2$ between smoothly bounded pseudoconvex domains $\Omega_1, \Omega_2 \subset\subset \mathbb{C}^n$ extends to a diffeomorphism $\Phi\colon \bar{\Omega}_1 \to \bar{\Omega}_2$; this is always the case if the domains $\Omega_j$, $j = 1, 2$, are assumed to be strictly pseudoconvex, see [Fe74]). However, many results which can be proved in the strictly pseudoconvex case are known to fail for general pseudoconvex domains (for example, if $\Omega \subset\subset \mathbb{C}^n$ is strictly pseudoconvex with smooth boundary, then $\Omega$ admits a Stein neighbourhood basis (see the proof of Proposition 3 in [Gr58]; see also [To83]), each $f \in \mathcal{O}(\Omega) \cap \mathcal{C}(\bar{\Omega})$ can be approximated uniformly on $\bar{\Omega}$ by functions $\{f_j\} \subset \mathcal{O}(\bar{\Omega})$ (see [He69], [Ke71] and [Li69]), and it is possible to prove subelliptic estimates for the $\bar{\partial}$-Neumann problem (see [Ca87]); none of these assertions hold true for arbitrary pseudoconvex domains $\Omega \subset\subset \mathbb{C}^n$ with smooth boundary, see [DF77b], [Ca83] and [Ca87]).

As it is pointed out in [DF77a] (see also [Li07]), the main reasons why one observes significant differences between the weakly and strictly pseudoconvex cases are the following three elementary facts:

(i) Strict pseudoconvexity is stable under small $\mathcal{C}^2$-perturbations.

(ii) Strictly pseudoconvex domains are locally biholomorphically equivalent to strictly convex domains.

(iii) If $\Omega \subset\subset \mathbb{C}^n$ is strictly pseudoconvex with $\mathcal{C}^2$-smooth boundary, then there exists an open neighbourhood $U \subset \mathbb{C}^n$ of $\bar{\Omega}$ and a $\mathcal{C}^2$-smooth strictly plurisubharmonic function $\varphi\colon U \to \mathbb{R}$ such that $\Omega = \{\varphi < 0\}$ and $d\varphi \neq 0$ on $b\Omega$.

The first two properties are of purely local nature. Thus they hold true for arbitrary strictly pseudoconvex domains. However, the third statement is formulated only for $\Omega \subset\subset \mathbb{C}^n$.

*The main goal of this thesis is to investigate how the above result (iii) on existence of global strictly plurisubharmonic defining functions can be generalized to the case of unbounded strictly pseudoconvex domains.*

It should be noted that complex analysis in several variables is studied so far more extensively, and is generally better understood, on bounded domains than on unbounded ones. This is to a large extent due to the fact that some techniques which are typical for higher-dimensional complex analysis use boundedness of the domains on which they are applied in various ways. For example, solution operators for $\bar{\partial}$ with $L^p$-estimates, $1 \leq p \leq \infty$, are constructed by using integral formulas (see [He69], [Ke71], [Li69], [Øv71]; the case $p = 2$ does not depend on integral formulas, see [Ko63]); if $\bar{\Omega}$ is not compact, the possibilities for applying integral formulas are essentially limited to local problems. Also, boundary regularity properties of biholomorphic mappings between bounded strictly pseudoconvex domains are proved by means of the Bergman kernel and Bergman metric (see [Fe74], [BL80] and [Be81]); on unbounded domains the Bergman space of $L^2$-holomorphic functions may be trivial. Moreover, Kobayashi hyperbolicity of bounded pseudoconvex domains $\Omega \subset \mathbb{C}^n$ and the closed range property of the $\bar{\partial}$-operator on $\Omega$ can be seen to ultimately depend on the existence of bounded (uniformly) strictly plurisubharmonic functions on $\Omega$ (see [Si81] and [Hö65]; for more details on the second assertion see also [HM14]); such functions do not exist in general on unbounded domains.

Another reason for the distinguished role of bounded domains is the fact that in higher-dimensional complex analysis there exist phenomena that can be observed only on unbounded domains, but which do not occur in the bounded case (note that, while every smoothly bounded domain $\Omega \subsetneq \mathbb{C}$ is biholomorphically equivalent to a bounded subset of $\mathbb{C}$, the analoguous statement is not longer true for domains $\Omega \subsetneq \mathbb{C}^n$, $n \geq 2$). Examples with respect to bounded plurisubharmonic functions and holomorphic extension of $CR$ functions will be given below.

*The second goal of this thesis is to explore some of these phenomena, especially those which are related to the existence of global defining functions. In particular, we will introduce and investigate core sets of unbounded domains.*

# 2. Main Results

## A. Global plurisubharmonic defining functions and the core

The main topic of this thesis is the problem of existence of defining functions for strictly pseudoconvex domains $\Omega$ with smooth boundary $b\Omega$ in a complex manifold $\mathcal{M}$. More precisely, we are interested in the existence of global defining functions, namely, defining functions that are defined in a neighbourhood of the closure $\bar{\Omega}$ (we will also be concerned with the more general situations of strictly $q$-pseudoconvex domains in complex manifolds and strictly hyper-$q$-pseudoconvex domains in complex spaces). In what follows, a real-valued function $\varphi$ will be called a *defining function* for $\Omega$ if it has the following properties:

(I) $\varphi$ is a smooth function on an open neighbourhood $U \subset \mathcal{M}$ of $b\Omega$.

(II) $\varphi$ is strictly plurisubharmonic in U.

(III) $\Omega \cap U = \{\varphi < 0\}$ and $d\varphi \neq 0$ on $b\Omega$.

It is well known that defining functions always exist whenever $\Omega \subset \mathcal{M}$ is relatively compact, and there are different proofs available for this fact, see, for example, [FG02], [FSt87], [Gr62], [MR75]. In fact, a careful investigation of the corresponding proof shows that the method presented in [Gr62] still works, with only minor changes, even without assuming relative compactness of $\Omega$. In particular, every strictly pseudoconvex domain with smooth boundary in a complex manifold admits a defining function.

If $\Omega$ is a relatively compact domain in a Stein manifold $\mathcal{M}$, then in fact more is known. In this case one can choose $\varphi$ to be defined not only near $b\Omega$ but on a neighbourhood of the whole of $\bar{\Omega}$ (see, for example, Lemma 1.3 in [MR75]). For arbitrary domains and manifolds this is not longer true in general, as it is illustrated by the following examples.

**Example I.** Let $\mathcal{M}$ be the blow-up of $\mathbb{C}^{n+1}$ at the origin, i.e., $\mathcal{M} := \{(z, x) \in \mathbb{C}^{n+1} \times \mathbb{CP}^n : z \in l(x)\}$, where $l(x) \subset \mathbb{C}^{n+1}$ denotes the complex line determined by $x \in \mathbb{CP}^n$. Then

$$\Omega := \big\{(z, x) \in \mathcal{M} : \|z\| < 1\big\} \subset\subset \mathcal{M}$$

is a strictly pseudoconvex domain with smooth boundary in $\mathcal{M}$ containing the compact analytic set $E := \{0\} \times \mathbb{CP}^n$. Let $\varphi$ be a plurisubharmonic function defined on a neighbourhood of $\bar{\Omega}$ such that $\Omega = \{\varphi < 0\}$. Then $\varphi$ is bounded from above on $E$, hence it is a constant by the maximum principle. In particular, $\varphi$ is not strictly plurisubharmonic at the points of $E$.

**Example II.** Let $f \colon \mathbb{C} \to \mathbb{C}$ be an entire function and

$$\Omega := \big\{(z, w) \in \mathbb{C}^2 : \log|w - f(z)| + C_1\big(|z|^2 + |w|^2\big) < C_2\big\} \subset \mathbb{C}^2,$$

where $C_1$ and $C_2$ are constants and $C_1 > 0$. For almost all constants $C_2$, $\Omega$ is an unbounded strictly pseudoconvex domain with smooth boundary in $\mathbb{C}^2$ containing the complex line $L := \{(z, f(z)) \in \mathbb{C}^2 : z \in \mathbb{C}\}$. Let $\varphi$ be a plurisubharmonic function defined on a neighbourhood of $\bar{\Omega}$ such that $\Omega = \{\varphi < 0\}$. Then $\varphi$ is subharmonic and bounded from above on $L$, hence it is a constant by Liouville's theorem. In particular, $\varphi$ is not strictly plurisubharmonic at the points of $L$.

As the above examples show, we cannot longer expect $\varphi$ to be strictly plurisubharmonic on a neighbourhood of the whole of $\bar{\Omega}$ as soon as $\mathcal{M}$ fails to be Stein, or

$\Omega$ fails to be relatively compact in $\mathcal{M}$. Hence we will call a real-valued function $\varphi$ a *global defining function* for $\Omega$ if it has the following properties:

(I) $\varphi$ is a smooth function on an open neighbourhood $U \subset \mathcal{M}$ of $\bar{\Omega}$.

(II) $\varphi$ is plurisubharmonic in $U$ and strictly plurisubharmonic near $b\Omega$.

(III) $\Omega = \{\varphi < 0\}$ and $d\varphi \neq 0$ on $b\Omega$.

Observe that instead of imposing (I) and (II), it is equivalent to claim that $\varphi$ is a smooth plurisubharmonic function on $\bar{\Omega}$ such that $\varphi$ is strictly plurisubharmonic near $b\Omega$. Moreover, after possibly shrinking $U$ and composing $\varphi$ with a suitable convex function, we can always assume that $\varphi$ is bounded.

As the main result of this thesis we will prove that every strictly pseudoconvex domain $\Omega$ with smooth boundary in a complex manifold $\mathcal{M}$ admits a global defining function. In view of this result and the examples above, it is then meaningful to consider the set of all points in $\Omega$ where every global defining function for $\Omega$ fails to be strictly plurisubharmonic. We will show that this set coincides with the core of $\Omega$, which we introduce in the following definition.

**Definition.** Let $\mathcal{M}$ be a complex manifold and let $\Omega \subset \mathcal{M}$ be a domain. Then the set

$$\begin{aligned}\mathfrak{c}(\Omega) := \big\{ z \in \Omega : &\text{ every smooth plurisubharmonic function on } \Omega \text{ that is} \\ &\text{bounded from above fails to be strictly plurisubharmonic in } z \big\}\end{aligned}$$

will be called the *core* of $\Omega$.

**Remark.** Similar definitions in different settings have also been introduced in [HaL12], [HaL13] and [SlT04].

It is easy to see that every domain $\Omega \subset \mathcal{M}$ admits a smooth and bounded plurisubharmonic function that is strictly plurisubharmonic precisely in $\Omega \setminus \mathfrak{c}(\Omega)$. In the special case of global defining functions we get the following version of our main theorem:

**Main Theorem.** *Every strictly pseudoconvex domain $\Omega$ with smooth boundary in a complex manifold $\mathcal{M}$ admits a bounded global defining function that is strictly plurisubharmonic outside $\mathfrak{c}(\Omega)$.*

The problem of existence of global defining functions is thus reduced to the study of the core. A major part of the present thesis is devoted to this topic.

The two questions, which will be investigated in most detail, are motivated by the following observation: In all elementary examples of domains $\Omega$ with nonempty core which we are going to construct (see also Example I and Example II above), every connected component $Z$ of $\mathfrak{c}(\Omega)$ has the following two properties:

- $Z$ satisfies a Liouville type theorem, i.e., every smooth and bounded from above plurisubharmonic function on $\Omega$ is constant on $Z$.
- $Z$ possesses an analytic structure, i.e., there exists a dense subset of $Z$ that is the union of nonconstant holomorphic discs contained in $Z$ (in fact, in the above examples Z is always a complex manifold).

Moreover, every set $Z \subset \Omega$ with the above two properties has to be contained in $\mathfrak{c}(\Omega)$. We thus want to know whether it is true in general that, first, every connected component of $\mathfrak{c}(\Omega)$ has a Liouville type property, and, second, $\mathfrak{c}(\Omega)$ possesses an analytic structure. With respect to analytic structure of the core, we obtain the following two results, which give a rather complete answer to the second question (see Theorem 3.3.1 and Theorem 3.3.2):

**Theorem I.** *For every $n \geq 2$, there exists an unbounded strictly pseudoconvex domain $\Omega \subset \mathbb{C}^n$ with smooth boundary such that $\mathfrak{c}(\Omega)$ is nonempty and contains no analytic variety of positive dimension.*

**Theorem II.** *Let $\mathcal{M}$ be a complex manifold and let $\Omega \subset \mathcal{M}$ be a domain. Then $\mathfrak{c}(\Omega)$ is 1-pseudoconcave in $\Omega$. In particular, $\mathfrak{c}(\Omega)$ is pseudoconcave in $\Omega$ if $\dim_{\mathbb{C}} \mathcal{M} = 2$.*

Regarding Liouville type properties of $\mathfrak{c}(\Omega)$, we will prove the following results (for more details see Example 9, Example 10 and Theorem 3.4.2):

**Theorem III.** *For every $n \geq 3$, there exist an unbounded strictly pseudoconvex domain $\Omega \subset \mathbb{C}^n$ with smooth boundary and a smooth plurisubharmonic function $\varphi\colon \Omega \to \mathbb{R}$ which is bounded from above such that $\mathfrak{c}(\Omega)$ is nonempty and connected but $\varphi$ is not constant on $\mathfrak{c}(\Omega)$.*

**Theorem IV.** *Let $\mathcal{M}$ be a 2-dimensional complex manifold and let $\Omega \subset \mathcal{M}$ be a domain. Assume that there exist a smooth plurisubharmonic function $\varphi\colon \Omega \to \mathbb{R}$ which is bounded from above and a connected component $Z$ of $\mathfrak{c}(\Omega)$ such that $\varphi$ is not constant on $Z$. Then there exist uncountably many pairwise disjoint connected immersed complex curves $\gamma_\alpha \subset Z$, $\alpha \in \mathcal{A}$, such that for every $\alpha \in \mathcal{A}$ and every smooth plurisubharmonic function $\psi\colon \Omega \to \mathbb{R}$ which is bounded from above it follows that $\psi$ is constant on $\gamma_\alpha$.*

It should be noted that the failure of Liouville type properties for connected components of $\mathfrak{c}(\Omega)$ can already be observed in very simple cases, if we do not require $\Omega$ to be strictly pseudoconvex (for example, if $\Omega := \Omega' \times \mathbb{C}_w \subset \mathbb{C}^2$ for some domain $\Omega' \subset\subset \mathbb{C}_z$, then $\mathfrak{c}(\Omega) = \Omega$ is connected, but $\varphi(z, w) := |z|^2$ is a smooth and bounded from above plurisubharmonic function on $\Omega$ which is not constant on $\mathfrak{c}(\Omega)$). However, to construct a strictly pseudoconvex domain as in Theorem III is much harder, and, in fact, we do not know if there exists such a domain in $\mathbb{C}^2$.

In turns out to be an interesting problem in general to understand how properties of $\Omega$ are related to properties of $\mathfrak{c}(\Omega)$. A very precise result on this question in the special situation where the core is assumed to have a certain product structure is obtained in the following theorem (see Theorem 3.2.1):

**Theorem V.** *The following assertions hold true for domains $\Omega \subset \mathbb{C}^n$, $n \geq 2$:*

*(1) There exists a domain $\Omega \subset \mathbb{C}^n$ such that $\mathfrak{c}(\Omega) = E \times \mathbb{C}^{n-1}$, where $E \subset \mathbb{C}$ is the set $E = [0, 1] \times \mathbb{R}$.*

*(2) Let $\Omega \subset \mathbb{C}^n$ be a pseudoconvex domain such that $\mathfrak{c}(\Omega) = E \times \mathbb{C}^k$ for some $k \in \{1, 2, \dots, n-1\}$ and some set $E \subset \mathbb{C}^{n-k}$. Then either $E$ is locally complete pluripolar or $E$ is open. In the latter case $\Omega = E \times \mathbb{C}^k$.*

*(3) Let $k \in \{1, 2, \dots, n-1\}$ be arbitrary but fixed. Then there exists a strictly pseudoconvex domain $\Omega \subset \mathbb{C}^n$ such that $\mathfrak{c}(\Omega) = E \times \mathbb{C}^k$ for a set $E \subset \mathbb{C}^{n-k}$ if and only if $E$ is closed and complete pluripolar.*

It is natural to also introduce a series of stronger notions of the core of a domain $\Omega \subset \mathcal{M}$, by requiring not only failure of strict plurisubharmonicity of smooth and bounded from above plurisubharmonic functions $\varphi$ on $\Omega$, but instead by prescribing an upper bound $k$ for the rank of the Levi form of $\varphi$, with $k$ possibly different from $\dim_{\mathbb{C}} \mathcal{M} - 1$.

**Definition.** Let $\mathcal{M}$ be a complex manifold of complex dimension $n$ and let $\Omega \subset \mathcal{M}$ be a domain. Then for every $q = 1, 2, \dots, n$ the set

$$\mathfrak{c}_q(\Omega) := \big\{ z \in \Omega : \operatorname{rank} \operatorname{Lev}(\varphi)(z, \cdot\,) \leq n - q \text{ for every smooth plurisub-} \\ \text{harmonic function } \varphi\colon \Omega \to \mathbb{R} \text{ that is bounded from above} \big\}$$

is called the *core of order* $q$ of $\Omega$.

The introduction of these higher order cores leads to a further slight improvement of the Main Theorem (see Theorem 3.5.2). Moreover, in view of 1-pseudoconcavity of the core $\mathfrak{c}_1(\Omega) = \mathfrak{c}(\Omega)$, pseudoconcavity properties of higher order cores are a natural object of study. It will be shown, however, that in general the higher order cores do not possess any such properties at all (see Theorem 3.5.1):

**Theorem VI.** *For every $n \geq 2$ and every $q = 1, 2, \ldots, n$, $q' = 0, 1, \ldots, n-1$ such that $(q, q') \neq (1, 0)$, there exists a strictly pseudoconvex domain $\Omega \subset \mathbb{C}^n$ with smooth boundary such that $\mathfrak{c}_q(\Omega)$ is $q'$-pseudoconcave but not $(q'+1)$-pseudoconcave.*

## B. Holomorphic extension of CR functions and the CR-core

One more situation, where new phenomena can be observed after generalizing the setting from the case of bounded sets to unbounded ones, is related to the extension problem of $CR$ functions defined on the boundary $b\Omega$ of a domain $\Omega$ in $\mathbb{C}^n$, $n \geq 2$. When $\Omega$ is bounded with a connected smooth boundary (no hypothesis of pseudoconvexity) holomorphic extension of $CR$ functions to the whole of $\bar{\Omega}$ is granted by the classical result of Bochner (see, for example, Theorem 2.3.2$'$ in [Hö90]). In particular, if $\Omega$ is a domain of holomorphy, the envelope of holomorphy $E(b\Omega)$ of $b\Omega$ (i.e. the envelope of $b\Omega$ with respect to the algebra of continuous $CR$ functions on $b\Omega$ (for details see, for example, [J95], [MP06] and [St07])) coincides with $\bar{\Omega}$. For unbounded domains such an extension result is not longer true in general, even for strictly pseudoconvex domains, as it is shown by the following example.

**Example III.** Let $f$ be an entire function in $\mathbb{C}^2$ and

$$\Omega := \left\{ z \in \mathbb{C}^2 : \log|f(z)| + C_1\|z\|^2 < C_2 \right\}$$

where $C_1$ and $C_2$ are constants and $C_1 > 0$. For almost all constants $C_2$, $\Omega$ is an unbounded strictly pseudoconvex open set with smooth boundary in $\mathbb{C}^2$ containing the divisor $\{f = 0\}$. We are going to show that $E(b\Omega)$ is one-sheeted, contained in $\Omega$ and

$$\bar{\Omega} \setminus E(b\Omega) = \left\{ z \in \mathbb{C}^2 : f(z) = 0 \right\}.$$

Fix an exhaustion $V_1 \subset\subset V_2 \subset\subset \cdots \subset\subset b\Omega$ of $b\Omega$ by relatively compact subsets. Intersecting $\Omega$ by balls $B^2(0, R_k) \subset \mathbb{C}^2$ centered at the origin of radius $R_k$ in such a way that $V_k \subset\subset b\Omega \cap B^2(0, R_k)$ and then smoothing the edges as in [To], we can find strictly pseudoconvex bounded open sets $\Omega_k$ in $\mathbb{C}^2$ such that $V_k \subset b\Omega_k \cap b\Omega$ for every $k \in \mathbb{N}$. Let $\Gamma_k := b\Omega_k \setminus V_k$. Then, in view of Theorem A from [J95], one has

$$E(V_k) = E(b\Omega_k \setminus \Gamma_k) = \bar{\Omega}_k \setminus \widehat{\Gamma_k}^{\mathcal{A}(\Omega_k)} \subset \mathbb{C}^2,$$

where $\widehat{\Gamma_k}^{\mathcal{A}(\Omega_k)}$ is the $\mathcal{A}(\Omega_k)$-hull of $\Gamma_k$, i.e., the hull of $\Gamma_k$ with respect to the algebra of holomorphic functions on $\Omega_k$ which are continuous up to the boundary. In particular, every continuous $CR$ function on $b\Omega$ has a single-valued holomorphic extension to $\bigcup_{k=1}^{\infty} E(V_k) = \bar{\Omega} \setminus \bigcap_{k=1}^{\infty} \widehat{\Gamma_k}^{\mathcal{A}(\Omega_k)}$. By construction, there exists a sequence $\{c_k\}$ of positive constants such that $\widehat{\Gamma_k}^{\mathcal{A}(\Omega_k)} \subset \{|f| < c_k\}$ for every $k \in \mathbb{N}$ and $\lim_{k\to\infty} c_k = 0$. It follows that $\bigcup_{k=1}^{\infty} E(V_k) = \bar{\Omega} \setminus \{f = 0\}$, and hence that $\bar{\Omega} \setminus \{f = 0\} \subset E(b\Omega)$. Since the $CR$ function $1/f$ on $b\Omega$ does not extend to $\{f = 0\}$, it follows that $E(b\Omega) = \bar{\Omega} \setminus \{f = 0\}$.

If $\Omega \subset \mathbb{C}^n$ is strictly pseudoconvex, then each continuous $CR$ function on $b\Omega$ extends holomorphically to a one-sided neighbourhood $U \subset \bar{\Omega}$ of $b\Omega$ (see, for example, [Bo91]). In view of the previous example, it is thus meaningful to introduce the following definition.

**Definition.** Let $\Omega \subset \mathbb{C}^n$ be a strictly pseudoconvex domain such that $E(b\Omega)$ is one-sheeted. Then

$$\mathfrak{c}_{CR}(\Omega) := \bar{\Omega} \setminus E(b\Omega)$$

is called the *CR-core* of $\Omega$.

If $\Omega$ is a domain as in Example III, then the above arguments show that $\mathfrak{c}_{CR}(\Omega)$ coincides with the divisor $\{f = 0\}$. In this context, we have to mention Trépreau's theorem [Tr86] stating that, given a point $z$ in a smooth hypersurface $M \subset \mathbb{C}^n$, the homomorphism

$$\mathcal{O}_z \to \varinjlim_{U \ni z} \mathcal{O}(U \setminus M)$$

is onto if and only if no germ of a complex hypersurface passing through $z$ is contained in $M$. Recall also Chirka's generalization [Ch01] of Trépreau's result (in the case $n = 1$ this generalization can also be obtained from the earlier work [Sh93]): Let $\Gamma \subset \mathbb{C}^{n+1}$ be a continuous graph over a convex domain $D \subset \mathbb{C}^n \times \mathbb{R}$ and $z \in \Gamma$ be a point such that none of the connected components of $(D \times \mathbb{R}) \setminus \Gamma$ is extendable holomorphically to $z$. Then, $z$ is contained in an $n$-dimensional holomorphic graph lying on and closed in $\Gamma$.

A natural question arises: Let $\Omega$ be an unbounded strictly pseudoconvex domain in $\mathbb{C}^n$, $n \geq 2$, such that $E(b\Omega)$ is one-sheeted and $\bar{\Omega} \setminus E(b\Omega) \neq \varnothing$; does $\bar{\Omega} \setminus E(b\Omega)$ possess an analytic structure? We will show that this is not always the case, by proving the following result (see Theorem 4.1.1):

**Theorem VII.** *For each $n \in \mathbb{N}$, $n \geq 2$, there exist an unbounded strictly pseudoconvex domain $\Omega \subset \mathbb{C}^n$ and a smooth $CR$ function $f$ on $b\Omega$ such that:*

*(1) The envelope of holomorphy $E(b\Omega)$ of the set $b\Omega$ is one-sheeted.*

*(2) $\mathfrak{c}_{CR}(\Omega)$ is nonempty and contains no analytic variety of positive dimension.*

*(3) $f$ has a single-valued holomorphic extension exactly to $\Omega \setminus \mathfrak{c}_{CR}(\Omega)$.*

However, observe that it follows immediately from the Kontinuitätssatz and the definition of the $CR$-core, that $\mathfrak{c}_{CR}(\Omega)$ is always pseudoconcave in $\Omega$.

## C. Unbounded Wermer type sets

In Theorem I and Theorem VII above we want to construct strictly pseudoconvex domains $\Omega \subset \mathbb{C}^n$, $n \geq 2$, with smooth boundary such that the core $\mathfrak{c}(\Omega)$ and the $CR$-core $\mathfrak{c}_{CR}(\Omega)$ do not possess any analytic structure, respectively. The main step in the proof of these results is the following theorem on existence of unbounded Wermer type sets in $\mathbb{C}^n$ (see Lemma 1.1.2, Lemma 1.2.2, Lemma 1.2.3, Lemma 1.3.6, Lemma 1.4.3 and Corollary 1.4.2):

**Theorem VIII.** *For each $n \in \mathbb{N}$, $n \geq 2$, there exist a nonempty connected closed set $\mathcal{E} \subset \mathbb{C}^n$ and a plurisubharmonic function $\varphi \colon \mathbb{C}^n \to [-\infty, +\infty)$ such that*

*(1) The set $\mathcal{E}$ contains no analytic variety of positive dimension;*

*(2) $\mathcal{E} = \{z \in \mathbb{C}^n : \varphi(z) = -\infty\}$;*

*(3) The function $\varphi$ is pluriharmonic on $\mathbb{C}^n \setminus \mathcal{E}$;*

*(4) The domain $\mathbb{C}^n \setminus \mathcal{E}$ is pseudoconvex;*

*(5) For every constant $R > 0$, one has that $\widehat{bB^n(0,R) \cap \mathcal{E}} = \bar{B}^n(0,R) \cap \mathcal{E}$, where $\widehat{bB^n(0,R) \cap \mathcal{E}}$ denotes the polynomial hull of the set $bB^n(0,R) \cap \mathcal{E}$.*

The set $\mathcal{E}$ is obtained as a limit in the Hausdorff metric of a sequence $\{E_\nu\}$ of algebraic hypersurfaces in $\mathbb{C}^n = \mathbb{C}^{n-1}_z \times \mathbb{C}_w$ such that the union of the corresponding sets of ramification points with respect to the projection $\mathbb{C}^n \to \mathbb{C}^{n-1}_z$ is an everywhere dense subset of $\mathbb{C}^{n-1}_z$. For $n = 2$ this idea goes back to Wermer in [We82], where an example of a compact set $K$ in $\mathbb{C}^2$ with nontrivial polynomial hull $\hat{K}$ such that $\hat{K} \setminus K$ has no analytic structure is given. Wermer's construction was then further exploited and developed in a series of articles [Al96], [Du10], [DS95], [EM08], [Le88], [Sl99]. Note also that, first, our construction of $\mathcal{E}$ is slightly different from Wermer's one (the main idea being the same) and, second, that, in the general case $n > 2$, the situation is substantially more difficult from the technical point of view than that considered by Wermer.

One more property of the Wermer type set $\mathcal{E}$, which will be needed in the construction of cores with no analytic structure, and which is also of independent interest, is described in the following Liouville type result (see Theorem 1.5.1):

**Theorem IX.** *Let $\varphi$ be a plurisubharmonic function defined on an open neighbourhood $U \subset \mathbb{C}^n$ of $\mathcal{E}$. If $\varphi$ is bounded from above, then $\varphi \equiv C$ on $\mathcal{E}$ for some $C \in \mathbb{R}$.*

The existence of unbounded Wermer type sets will also play an important role in the proof of Theorem VI on higher order cores with arbitrary pseudoconcavity properties. However, for this purpose, a further generalization of our construction of Wermer type sets is necessary. The corresponding results are summarized in the following theorem (see the results in Chapter 2):

**Theorem X.** *For each $n \in \mathbb{N}$, $n \geq 2$, and for every $q = 1, 2, \ldots, n-1$, there exist a nonempty connected closed set $\mathcal{E} \subset \mathbb{C}^n$ and a plurisubharmonic function $\varphi\colon \mathbb{C}^n \to [-\infty, +\infty)$ such that*

*(1) The set $\mathcal{E}$ contains no analytic variety of positive dimension;*

*(2) $\mathcal{E} = \{z \in \mathbb{C}^n : \varphi(z) = -\infty\}$;*

*(3) The set $\mathcal{E}$ is $q$-pseudoconcave but not $(q+1)$-pseudoconcave;*

*(4) If $\psi\colon U \to \mathbb{R}$ is a smooth plurisubharmonic function defined on an open neighbourhood of $\mathcal{E}$ such that $\psi$ is bounded from above on $\mathcal{E}$, then the Levi form of $\psi$ vanishes identically at every point of $\mathcal{E}$.*

*(5) For every constant $R > 0$, one has that $\widehat{bB^n(0,R) \cap \mathcal{E}} = \bar{B}^n(0,R) \cap \mathcal{E}$, where $\widehat{bB^n(0,R) \cap \mathcal{E}}$ denotes the polynomial hull of the set $bB^n(0,R) \cap \mathcal{E}$.*

The set $\mathcal{E}$ is obtained as a limit in the Hausdorff metric of a sequence $\{E_\nu\}$ of $q$-dimensional algebraic varieties in $\mathbb{C}^n = \mathbb{C}^q_z \times \mathbb{C}^{n-q}_w$ such that for every $k = q+1, q+2, \ldots, n$ the union of the corresponding sets of ramification points with respect to the projection $\mathbb{C}^q_z \times \mathbb{C}_{w_k} \to \mathbb{C}^q_z$ is an everywhere dense subset of $\mathbb{C}^q_z$.

## 3. Organization of the content

The present thesis consists of two parts. In the first part, we describe the construction of unbounded Wermer type sets $\mathcal{E} \subset \mathbb{C}^n$. Chapter 1 contains the results on Wermer type sets which are limits of algebraic hypersurfaces in $\mathbb{C}^{n-1}_z \times \mathbb{C}_w$. Theorem VIII on the existence and general properties of $\mathcal{E}$ is proven in Sections 1.1-1.4, the Theorem IX on Liouville type properties is contained in Section 1.5. Chapter 2 is devoted to the construction of generalized Wermer type sets which are limits of $q$-dimensional analytic varieties in $\mathbb{C}^q_z \times \mathbb{C}^{n-q}_w$. The additional properties (3) and (4) of Theorem X on the order of pseudoconcavity and degeneration of plurisubharmonic functions are contained in Section 2.3.

The second part of this thesis deals with core sets of unbounded domains and the phenomena which are related to it. The focus lies on the construction of global plurisubharmonic defining functions and the investigation of properties of the core $\mathfrak{c}(\Omega)$. The corresponding results are contained in Chapter 3, which can also be regarded as the main part of this thesis. Chapter 4 is devoted to the study of holomorphic extension of $CR$ functions and the $CR$-core $\mathfrak{c}_{CR}(\Omega)$.

In Section 3.1 we prove the existence of global defining functions in a number of different settings. We first consider in Section 3.1.1 the case of strictly $q$-pseudoconvex domains in complex manifolds; in the special case $q = 0$ we obtain the described above Main Theorem. Later on we deal in Section 3.1.2 with the situation of strictly hyper-$q$-pseudoconvex domains in complex spaces. Section 3.2 contains several examples of unbounded domains $\Omega \subset \mathbb{C}^n$ such that $\mathfrak{c}(\Omega) \neq \varnothing$. We also prove here Theorem V. The question of existence of analytic structure in $\mathfrak{c}(\Omega)$ and the related to it Theorem I and Theorem II, are considered in Section 3.3. Section 3.4 deals with Liouville type properties of the core and the corresponding Theorems III and IV. The Theorem VI on pseudoconcavity properties of higher order cores is proven in Section 3.5.

Section 4.1 contains an example of a $CR$-core with no analytic structure, as it is described in Theorem VII. Finally, we show in Section 4.2 that in general $\mathfrak{c}(\Omega) \neq \mathfrak{c}_{CR}(\Omega)$, even for strictly pseudoconvex domains $\Omega \subset \mathbb{C}^2$.

**Remark.** Some of the results of the present dissertation were already obtained in the author's diploma thesis, and have been published earlier in [HST12]. In particular, Theorem VII and Theorem VIII are already contained, with slightly different formulations, in the above mentioned work. More precisely, the following parts of the present thesis were taken from [HST12]: Section 1.1, Section 1.3, Lemma 1.4.1, Lemma 1.4.2, Lemma 1.4.3, Corollary 1.4.2 and Theorem 4.1.1. Furthermore, the largest part of Section 2.B as well as some parts of Section 2.C from this Introduction also appear in [HST12].

# Part I

# Unbounded Wermer type sets

# 1 Construction of Wermer type sets in codimension 1

We construct a class of unbounded Wermer type sets in $\mathbb{C}^n$, which are limits in the Hausdorff metric of sequences of algebraic varities of codimension 1.

In Section 1.1 we describe the general method of construction. We explicitly choose a sequence $\{P_\nu\}$ of holomorphic polynomials, such that each $P_\nu$ does only depend on finitely many positive constants $\varepsilon_1, \ldots, \varepsilon_\nu$. We then show that for $\{\varepsilon_l\}$ decreasing to zero fast enough, the algebraic varieties $E_\nu = \{P_\nu = 0\}$ converge in the Hausdorff metric to a closed unbounded set $\mathcal{E} \subset \mathbb{C}^n$. Section 1.2 contains some elementary geometric properties of the set $\mathcal{E}$, which will be occasionally needed later on. In Section 1.3 we show that for $\{\varepsilon_l\}$ decreasing to zero fast enough, the set $\mathcal{E}$ contains no analytic variety of positive dimension. Moreover, in Section 1.4 we show that for $\{\varepsilon_l\}$ decreasing to zero fast enough, the set $\mathcal{E}$ is complete pluripolar. The last Section 1.5 contains a Liouville theorem for plurisubharmonic functions on Wermer type sets.

## 1.1 The general construction

Let $(z, w) = (z_1, \ldots, z_{n-1}, w)$ denote the coordinates in $\mathbb{C}^n$ and for each $\nu \in \mathbb{N}$ let $\mathbb{N}_\nu := \{1, 2, \ldots, \nu\}$. For each $p \in \mathbb{N}_{n-1}$ fix an everywhere dense subset $\{a_l^p\}_{l=1}^\infty$ of $\mathbb{C}$ such that $a_l^p \neq a_{l'}^p$ if $l \neq l'$. Further, fix a bijection $\Phi := ([\,\cdot\,], \phi)\colon \mathbb{N} \to \mathbb{N}_{n-1} \times \mathbb{N}$ and define a sequence $\{a_l\}_{l=1}^\infty$ in $\mathbb{C}$ by letting $a_l := a_{\phi(l)}^{[l]}$. Moreover, let $\{\varepsilon_l\}_{l=1}^\infty$ be a decreasing sequence of positive numbers converging to zero that we consider to be fixed, but that will be further specified later on. Then for every $\nu \in \mathbb{N}$ we define $g_\nu$ to be the algebraic function

$$g_\nu(z) := \sum_{l=1}^{\nu} \varepsilon_l \sqrt{z_{[l]} - a_l}$$

and let

$$E_\nu := \big\{(z, w) \in \mathbb{C}^n : w = g_\nu(z)\big\}.$$

By definition, $g_\nu$ is a multi-valued function that takes $2^\nu$ values at each point $z \in \mathbb{C}^{n-1}$ (counted with multiplicities). Therefore we can always choose single-valued functions $w_1^{(\nu)}, \dots, w_{2^\nu}^{(\nu)}$ on $\mathbb{C}^{n-1}$ such that

$$g_\nu(z) = \{w_j^{(\nu)}(z) : j = 1, \dots, 2^\nu\}$$

for all $z \in \mathbb{C}^{n-1}$. Note that these functions are not continuous and that they are not uniquely determined, even though the set $g_\nu(z)$ is well-defined for each $z \in \mathbb{C}^{n-1}$. Indeed we may freely change the numeration of the values $w_1^{(\nu)}(z), \dots, w_{2^\nu}^{(\nu)}(z)$ for each $z \in \mathbb{C}^{n-1}$.

Define for each $\nu \in \mathbb{N}$ a function $P_\nu \colon \mathbb{C}^n \to \mathbb{C}$ as

$$P_\nu(z, w) := \big(w - w_1^{(\nu)}(z)\big) \cdots \big(w - w_{2^\nu}^{(\nu)}(z)\big).$$

**Lemma 1.1.1.** *The sequence $\{P_\nu\}_{\nu=1}^\infty$ consists of holomorphic polynomials on $\mathbb{C}^n$ and has the following properties:*

*(1) $E_\nu = \{(z, w) \in \mathbb{C}^n : P_\nu(z, w) = 0\}$.*

*(2) $P_{\nu+1} \to P_\nu^2$ uniformly on compact subsets of $\mathbb{C}^n$ as $\varepsilon_{\nu+1} \to 0$.*

**Proof.** First note that if for each $p \in \mathbb{N}_{n-1}$ we let $U_p$ be an open convex subset of $\mathbb{C}$ not meeting $A_\nu^p := \{a_l : l \in \mathbb{N}_\nu, [l] = p\}$, then after possibly renumbering the values $w_j^{(\nu)}(z)$ for $z \in U := U_1 \times \dots \times U_{n-1}$ we can always assume the functions $w_1^{(\nu)}, \dots, w_{2^\nu}^{(\nu)}$ to be holomorphic on $U$. Since the value $P_\nu(z, w)$ is independent of the numeration of the $w_j^{(\nu)}(z)$, this shows that $P_\nu$ is a holomorphic function outside the set $\mathcal{A}_\nu := \{(z, w) \in \mathbb{C}^n : z_p \in A_\nu^p \text{ for some } p \in \mathbb{N}_{n-1}\}$. Observing that $P_\nu$ is locally bounded near each point of $\mathcal{A}_\nu$ and applying Riemann's removable singularities theorem we conclude that $P_\nu$ is actually holomorphic in the whole of $\mathbb{C}^n$. Then estimating $|P_\nu|$ outside some ball $B^n(0, R) \subset \mathbb{C}^n$ from above by a suitable scalar multiple of $|w^{2^\nu}| + \sum_{p=1}^{n-1} |z_p^{2^{\nu-1}}|$ one can easily see that $P_\nu$ is in fact a holomorphic polynomial. To prove the second part of the lemma we observe that $P_{\nu+1}(z, w)$ is in fact the product of the $2^\nu$ factors $\big((w - w_j^{(\nu)}(z))^2 - \varepsilon_{\nu+1}^2(z_{[\nu+1]} - a_{\nu+1})\big)$, $j \in \mathbb{N}_{2^\nu}$, and hence equals

$$\sum_{p=0}^{2^\nu} (-1)^p \Big[\big(\varepsilon_{\nu+1}^2(z_{[\nu+1]} - a_{\nu+1})\big)^{2^\nu - p} \cdot \sum_{1 \le j_1 < \dots < j_p \le 2^\nu} \big(w - w_{j_1}^{(\nu)}(z)\big)^2 \cdots \big(w - w_{j_p}^{(\nu)}(z)\big)^2\Big].$$

Note that for $p = 2^\nu$ the inner sum equals $P_\nu^2(z, w)$. Since $w_1^{(\nu)}, \dots, w_{2^\nu}^{(\nu)}$ are independent of $\varepsilon_{\nu+1}$ and bounded on compact subsets of $\mathbb{C}^{n-1}$, we conclude that $P_{\nu+1} \to P_\nu^2$ uniformly on compact subsets as $\varepsilon_{\nu+1} \to 0$. □

**Remark.** A more careful consideration shows that one has the following explicit formula for $P_\nu$,

$$P_\nu(z,w) = \sum_{d=0}^{2^{\nu-1}} (-1)^d \Big( \sum_{l=1}^{\nu} \varepsilon_l^2 (z_{[l]} - a_l) \Big)^d w^{2^\nu - 2d}.$$

**Lemma 1.1.2.** *Let $\{\varepsilon_l\}$ be chosen in such a way that $\varepsilon_l \sqrt{|z_{[l]} - a_l|} < 1/2^l$ on $B^{n-1}(0,l) \subset \mathbb{C}_z^{n-1}$ for every $l \in \mathbb{N}$. Then the following assertions hold true:*

*(1) For every $R > 0$ and $\nu, \mu \in \mathbb{N}$, $\nu \geq R$, the Hausdorff distance between $E_\nu \cap \bar{B}^n(0,R)$ and $E_{\nu+\mu} \cap \bar{B}^n(0,R)$ is less than $1/2^\nu$. In particular the sequence $\{E_\nu \cap \bar{B}^n(0,R)\}_{\nu=1}^\infty$ converges in the Hausdorff metric to a closed set $\mathcal{E}_{(R)} \subset \bar{B}^n(0,R)$.*

*(2) The union $\mathcal{E} := \bigcup_{R>0} \mathcal{E}_{(R)}$ of all $\mathcal{E}_{(R)}$ is a nonempty closed unbounded subset of $\mathbb{C}^n$ and a point $(z,w) \in \mathbb{C}^n$ lies in $\mathcal{E}$ if and only if there exists a sequence of complex numbers $w_\nu$ converging to $w$ such that $(z, w_\nu) \in E_\nu$ for every $\nu \in \mathbb{N}$.*

*(3) For each $z \in \mathbb{C}^{n-1}$, the set $\mathcal{E}_z := \{w \in \mathbb{C} : (z,w) \in \mathcal{E}\}$ has zero 2-dimensional Lebesgue measure.*

**Proof.** Let $\Delta_R := \bar{B}^{n-1}(0,R) \times \mathbb{C}$. For every $\big(z, w_j^{(\nu+\mu)}(z)\big) \in E_{\nu+\mu} \cap \bar{\Delta}_R$ there exists $\big(z, w_k^{(\nu)}\big) \in E_\nu \cap \bar{\Delta}_R$ such that for suitably chosen signs one has

$$w_j^{(\nu+\mu)}(z) = w_k^{(\nu)}(z) + \sum_{l=\nu+1}^{\nu+\mu} \pm \varepsilon_l \sqrt{z_{[l]} - a_l}$$

(here, by some abuse of notation, $\sqrt{\cdot}$ denotes a single-valued branch of the multi-valued function $\sqrt{\cdot}$). By assumption, we have $\varepsilon_l \big|\sqrt{z_{[l]} - a_l}\,\big| = \varepsilon_l \sqrt{|z_{[l]} - a_l|} < 1/2^l$ on $\bar{B}^{n-1}(0,R)$ for each $l > \nu$. Hence $|w_j^{(\nu+\mu)}(z) - w_k^{(\nu)}(z)| < 1/2^\nu$ and it follows that the Hausdorff distance between $E_{\nu+\mu} \cap \bar{\Delta}_R$ and $E_\nu \cap \bar{\Delta}_R$ is less than $1/2^\nu$. In particular, $\{E_\nu \cap \bar{B}^n(0,R)\}_{\nu=1}^\infty$ is a Cauchy sequence in the Hausdorff metric and thus converges to a nonempty closed subset $\mathcal{E}_{(R)} \subset \mathbb{C}^n$. Since $\mathcal{E} \cap \bar{B}^n(0,R) = \mathcal{E}_{(R)}$ for all $R > 0$, we conclude that $\mathcal{E}$ is closed. Obviously, it is also unbounded and nonempty. The characterization of $(z,w) \in \mathcal{E}$ as a limit of points $(z, w_\nu) \in E_\nu$ follows immediately from the facts that in each bounded neighbourhood of $(z,w)$ the set $\mathcal{E}$ is the limit of $\{E_\nu\}$ in the Hausdorff metric and that $E_\nu \cap \big(\{z\} \times \mathbb{C}\big) \neq \varnothing$ for all $z \in \mathbb{C}^{n-1}$. Finally, by what we have already proven, we know that the Hausdorff distance between $E_\nu \cap \bar{\Delta}_R$ and $\mathcal{E}_{(R)}$ is not greater than $1/2^\nu$. Hence if $z \in \mathbb{C}^{n-1}$ is fixed, the set $\mathcal{E}_z$ is contained in $\bigcup_{j=1}^{2^\nu} \bar{\Delta}^1(w_j^{(\nu)}(z), 1/2^\nu)$ for every

$\nu \in \mathbb{N}$ big enough (here $\bar{\Delta}^1(a,r) \subset \mathbb{C}$ denotes the closed disc centered at the point $a$ of radius $r$). But the volume of the later set is not greater than $\pi/2^\nu$, thus $\mathcal{E}_z$ has zero 2-dimensional Lebesgue measure. $\square$

From now on we always assume that $\{\varepsilon_l\}$ is decreasing to zero so fast that

$$\varepsilon_l \sqrt{|z_{[l]} - a_l|} < 1/2^l \text{ on } B^{n-1}(0,l) \text{ for every } l \in \mathbb{N}.$$

By the previous lemma, the analytic sets $E_\nu$ then determine a limit set $\mathcal{E}$. We want to use this set in the constructions of Part II below. In order to do so, we need to have three specific properties of this set. Namely, we want to ensure that $\mathcal{E}$ has no analytic structure, we need to guarantee that $\mathcal{E}$ is complete pluripolar, and we have to prove a Liouville theorem for plurisubharmonic functions on $\mathcal{E}$. In the remaining parts of this chapter we will show that we indeed can assure $\mathcal{E}$ to have these properties, provided that $\{\varepsilon_l\}$ is converging to zero fast enough.

## 1.2 Some geometric properties

Recall that a map $f : (X_1, d_1) \to (X_2, d_2)$ between metric spaces is called $(M,\alpha)$-Hölder continuous if $d_2(f(x), f(y)) \le M d_1(x,y)^\alpha$ for every $x, y \in X_1$. Here $M, \alpha > 0$ are positive constants. Moreover, observe that the Wermer type set $\mathcal{E}$ defines a map $\underline{\mathcal{E}}$ from the metric space $\mathbb{C}^{n-1}$ of all $(n-1)$-tupels of complex numbers with the standard euclidean metric $d_{\|\cdot\|}$ to the metric space $\mathcal{F}(\mathbb{C})$ of all nonempty compact subsets of $\mathbb{C}$ with the Hausdorff metric $d_H$, namely $\underline{\mathcal{E}}\colon (\mathbb{C}^{n-1}, d_{\|\cdot\|}) \to (\mathcal{F}(\mathbb{C}), d_H)$, $\underline{\mathcal{E}}(z) := \mathcal{E}_z := \{w \in \mathbb{C} : (z,w) \in \mathcal{E}\}$.

**Lemma 1.2.1.** *There exists a constant $M > 0$ such that the map $\underline{\mathcal{E}}\colon \mathbb{C}^{n-1} \to \mathcal{F}(\mathbb{C})$ is $(M, 1/2)$-Hölder continuous.*

**Proof.** We have to show that there exists $M > 0$ such that

$$d_H\big(\mathcal{E}_{z_1}, \mathcal{E}_{z_2}\big) \le M \sqrt{\|z_1 - z_2\|} \quad \text{for all } z_1, z_2 \in \mathbb{C}^{n-1}. \tag{1.1}$$

To prove (1.1), consider the set-valued functions $e_l(z) := \varepsilon_l \sqrt{z_{[l]} - a_l}$, $l \in \mathbb{N}$, $e_l\colon \mathbb{C}^{n-1} \to \mathcal{F}(\mathbb{C})$. Observe that $\underline{\mathcal{E}} = \sum_{l=1}^\infty e_l$, by definition of $\mathcal{E}$, where the sum of the functions $e_l$ is taken pointwise and the sum of two elements $K_1, K_2 \in \mathcal{F}(\mathbb{C})$ is defined as $K_1 + K_2 := \{w \in \mathbb{C} : w = k_1 + k_2 \text{ for some } k_1 \in K_1, k_2 \in K_2\}$. For each $l \in \mathbb{N}$, choose $e_l^*\colon \mathbb{C}^{n-1} \to \mathbb{C}$ such that $e_l(z) = \{e_l^*(z), -e_l^*(z)\}$ for every

$z \in \mathbb{C}^{n-1}$. Then for every $z_1, z_2 \in \mathbb{C}^{n-1}$ we have

$$\begin{aligned}\varepsilon_l\sqrt{|z_1 - z_2|} \geq \varepsilon_l\sqrt{|z_{1,[l]} - z_{2,[l]}|} &= \sqrt{\left|\varepsilon_l^2(z_{1,[l]} - a_l) - \varepsilon_l^2(z_{2,[l]} - a_l)\right|} \\ &= \sqrt{\left|\big(\pm e_l^*(z_1) - e_l^*(z_2)\big)\big(\pm e_l^*(z_1) + e_l^*(z_2)\big)\right|} \\ &\geq \sup_{\zeta_1 \in e_l(z_1)} \inf_{\zeta_2 \in e_l(z_2)} |\zeta_1 - \zeta_2|\end{aligned}$$

and similarly

$$\begin{aligned}\varepsilon_l\sqrt{|z_1 - z_2|} \geq \varepsilon_l\sqrt{|z_{1,[l]} - z_{2,[l]}|} &= \sqrt{\left|\varepsilon_l^2(z_{1,[l]} - a_l) - \varepsilon_l^2(z_{2,[l]} - a_l)\right|} \\ &= \sqrt{\left|\big(e_l^*(z_1) \pm e_l^*(z_2)\big)\big(- e_l^*(z_1) \pm e_l^*(z_2)\big)\right|} \\ &\geq \sup_{\zeta_2 \in e_l(z_2)} \inf_{\zeta_1 \in e_l(z_1)} |\zeta_1 - \zeta_2|.\end{aligned}$$

This shows that $d_H\big(e_l(z_1), e_l(z_2)\big) \leq \varepsilon_l\sqrt{|z_1 - z_2|}$, i.e., $e_l$ is $(\varepsilon_l, 1/2)$-Hölder continuous. Observe now that for any two functions $f, g\colon \mathbb{C}^{n-1} \to \mathcal{F}(\mathbb{C})$ we have $d_H(f(z_1) + g(z_1), f(z_2) + g(z_2)) \leq d_H(f(z_1), f(z_2)) + d_H(g(z_1), g(z_2))$, hence if $f$ is $(M_1, 1/2)$-Hölder continuous and $g$ is $(M_2, 1/2)$-Hölder continuous, then $f + g$ is $(M_1 + M_2, 1/2)$-Hölder continuous. Applying this to the sequence $\{e_l\}$, we conclude that

$$d_H\Big(\sum_{l=1}^{\nu} e_l(z_1), \sum_{l=1}^{\nu} e_l(z_2)\Big) \leq \sum_{l=1}^{\nu} \varepsilon_l\sqrt{|z_1 - z_2|}$$

for every $\nu \in \mathbb{N}$, and for $\nu \to \infty$ this yields (1.1) with $M := \sum_{l=1}^{\infty} \varepsilon_l$. □

**Lemma 1.2.2.** *The map $\underline{\mathcal{E}}\colon \mathbb{C}^{n-1} \to \mathcal{F}(\mathbb{C})$ is an analytic multifunction.*

**Proof.** Assume, to get a contradiction, that $\mathcal{E}$ is not pseudoconcave. Then there exists a Hartogs figure $H = \{(\zeta, \eta) \in \Delta^1 \times \Delta^{n-1} : |\zeta|_\infty > r_1 \text{ or } \|\eta\|_\infty < r_2\}$ and an injective holomorphic mapping $\Phi\colon \hat{H} \to \mathbb{C}^n$ such that $\Phi(H) \subset \mathbb{C}^n \setminus \mathcal{E}$ but $\Phi(\hat{H}) \cap \mathcal{E} \neq \varnothing$; here $\|z\|_\infty = \max_{1 \leq j \leq n}|z_j|$ and $\hat{H} := \Delta^n := \{z \in \mathbb{C}^n : \|z\|_\infty < 1\}$. After possibly shrinking $H$, one can easily see that for $\nu \in \mathbb{N}$ large enough the pure $(n-1)$-dimensional varieties $E_\nu$ will also satisfy the conditions $\Phi(\bar{H}) \subset \mathbb{C}^n \setminus E_\nu$ and $\Phi(\hat{H}) \cap E_\nu \neq \varnothing$. Then $V := \Phi(\hat{H})$ is a relatively compact subset of $\mathbb{C}^n$ such that the $(n-2)$-plurisubharmonic function $\varphi := -\log\|\eta\| \circ \Phi^{-1}$ satisfies $\max_{E_\nu \cap V} \varphi > \max_{E_\nu \cap bV} \varphi$. This contradicts the local maximum property of $(n-2)$-plurisubharmonic functions on $(n-1)$-dimensional analytic varieties, see Corollary 5.3 in [Sl86]. □

**Remark.** The statement of Lemma 1.2.2 will also follow from formula (1.11) of Lemma 1.4.2 below.

**Lemma 1.2.3.** *The set $\mathcal{E}$ is connected.*

**Proof.** Recall that $\{\varepsilon_l\}$ is chosen in such a way that $\varepsilon_l\sqrt{|z_{[l]}-a_l|} < 1/2^l$ on $B^{n-1}(0,l)$ for every $l \in \mathbb{N}$. Assume, to get a contradiction, that there exist two open sets $U_1, U_2 \subset \mathbb{C}^n$ such that $\mathcal{E} \cap U_1 \neq \varnothing$, $\mathcal{E} \cap U_2 \neq \varnothing$, $\mathcal{E} \subset U_1 \cup U_2$ and $U_1 \cap U_2 = \varnothing$. Then we conclude from continuity of $\underline{\mathcal{E}}$, see Lemma 1.2.1 above, that $\pi_z(\mathcal{E} \cap U_1)$ and $\pi_z(\mathcal{E} \cap U_2)$ are open in $\mathbb{C}^{n-1}$, where $\pi_z\colon \mathbb{C}^n \to \mathbb{C}^{n-1}_z$ denotes the canonical projection. Since $\pi_z(\mathcal{E}\cap U_1) \cup \pi_z(\mathcal{E}\cap U_2) = \pi_z(\mathcal{E}) = \mathbb{C}^{n-1}$, it follows that $D := \pi_z(\mathcal{E} \cap U_1) \cap \pi_z(\mathcal{E} \cap U_2)$ is open and nonempty. Thus we can choose $z_0 \in D$ such that $z_{0,p} \notin \{a_l\}_{l=1}^{\infty}$ for every $p \in \mathbb{N}_{n-1}$ and $\arg(a_l - z_{0,[l]}) \neq \arg(a_{l'} - z_{0,[l']})$ for every $l, l' \in \mathbb{N}$, $[l] = [l']$, $l \neq l'$. Set $U_j(z_0) := \{w \in \mathbb{C} : (z_0, w) \in U_j\}$, $j = 1, 2$, and choose $\delta > 0$ so small that $\operatorname{dist}(\mathcal{E}_{z_0}, b(U_1(z_0) \cup U_2(z_0))) > \delta$. Fix $\nu_0 \in \mathbb{N}$ such that $\sum_{l=\nu_0+1}^{\infty} \varepsilon_l \sqrt{|z_{0,[l]} - a_l|} < \delta/2$. Then $E_{\nu,z_0} \cap U_1(z_0) \neq \varnothing$, $E_{\nu,z_0} \cap U_2(z_0) \neq \varnothing$ and $E_{\nu,z_0} \subset U_1(z_0) \cup U_2(z_0)$, where $E_{\nu,z_0} := \{w \in \mathbb{C} : (z_0, w) \in E_\nu\}$. For every $l \in \mathbb{N}$, let $\sigma_l := \{z \in \mathbb{C}^{n-1} : \arg(z_{[l]} - z_{0,[l]}) = \arg(a_l - z_{0,[l]}), |z_{[l]} - z_{0,[l]}| > |a_l - z_{0,[l]}|\}$, and let $h_l\colon \mathbb{C}^{n-1} \setminus \sigma_l \to \mathbb{C}$ be a continuous branch of $\varepsilon_l\sqrt{z_{[l]} - a_l}$. Fix $p_1 = (z_0, w_1) \in E_{\nu_0} \cap U_1$ and $p_2 = (z_0, w_2) \in E_{\nu_0} \cap U_2$. Then there exist functions $\tau_1, \tau_2\colon \mathbb{N}_{\nu_0} \to \{0, 1\}$ such that $w_j = \sum_{l=1}^{\nu_0} (-1)^{\tau_j(l)} h_l(z_0)$, $j = 1, 2$. Set $\hat{p}_j := (z_0, w_j + \sum_{l=\nu_0+1}^{\infty} h_l(z_0))$ and observe that, by the choice of $\delta$ and $\nu_0$, one has $\hat{p}_j \in \mathcal{E} \cap U_j$, $j = 1, 2$. Now define a continuous curve $\gamma_z\colon [0, \nu_0] \to \mathbb{C}^{n-1}_z \setminus \bigcup_{l=1}^{\infty} \sigma_l$ as

$$\gamma_z(t) := \begin{cases} \big(z_{0,1}, \ldots, z_{0,[\nu]-1}, z_{0,[\nu]} + 2(t-\nu+1)(a_\nu - z_{0,[\nu]}), \\ \qquad\qquad z_{0,[\nu]+1}, \ldots, z_{n-1}\big), \quad t \in [\nu-1, \nu-1/2] \\ \big(z_{0,1}, \ldots, z_{0,[\nu]-1}, a_\nu + 2(t-\nu+1/2)(z_{0,[\nu]} - a_\nu), \\ \qquad\qquad z_{0,[\nu]+1}, \ldots, z_{n-1}\big), \quad t \in [\nu-1/2, \nu] \end{cases} \quad (\nu \in \mathbb{N}_{\nu_0}).$$

and let $\gamma\colon [0, \nu_0] \to \mathcal{E}$ be given as

$$\gamma(t) := \begin{cases} \big(\gamma_z(t), \sum_{l=1}^{\nu-1} (-1)^{\tau_2(l)} h_l(\gamma_z(t)) + \sum_{l=\nu}^{\nu_0} (-1)^{\tau_1(l)} h_l(\gamma_z(t)) \\ \qquad\qquad + \sum_{l=\nu_0+1}^{\infty} h_l(\gamma_z(t))\big), \quad t \in [\nu-1, \nu-1/2] \\ \big(\gamma_z(t), \sum_{l=1}^{\nu} (-1)^{\tau_2(l)} h_l(\gamma_z(t)) + \sum_{l=\nu+1}^{\nu_0} (-1)^{\tau_1(l)} h_l(\gamma_z(t)) \\ \qquad\qquad + \sum_{l=\nu_0+1}^{\infty} h_l(\gamma_z(t))\big), \quad t \in [\nu-1/2, \nu] \end{cases} \quad (\nu \in \mathbb{N}_{\nu_0}).$$

Then it is easy to see that $\gamma$ is a continuous curve in $\mathcal{E}$ such that $\gamma(0) = \hat{p}_1 \in U_1$ and $\gamma(1) = \hat{p}_2 \in U_2$. This is a contradiction. □

**Remark.** The connectedness of the set $\mathcal{E}$ will also follow from Theorem 1.5.1 below.

## 1.3 Choice of the sequence $\{\varepsilon_l\}$ - Part I. Absence of analytic structure

In this section we want to show that, for $\{\varepsilon_l\}$ decreasing fast enough, the set $\mathcal{E}$ contains no analytic varieties of positive dimension. In order to do so, it obviously suffices to show that $\mathcal{E}$ contains no analytic disc, i.e., there exists no (nonconstant) holomorphic mapping $f\colon \mathbb{D} \to \mathbb{C}^n$ from the unit disc $\mathbb{D} \subset \mathbb{C}$ to $\mathbb{C}^n$ with image completely contained in $\mathcal{E}$. For analytic discs with constant $z$-coordinates this is immediately clear, since we know that $\mathcal{E}_z$ has zero two-dimensional Lebesgue measure for every $z \in \mathbb{C}^{n-1}$. The hard part is to show that there exists no analytic disc $f(\mathbb{D}) \subset \mathcal{E}$ such that the projection $f_z := \pi_z \circ f$ onto $\mathbb{C}^{n-1}_z$ is not constant. The general idea is the following: Let $f\colon \mathbb{D} \to \mathbb{C}^n$ be an analytic disc lying in the analytic hypersurface $w = \sqrt{z_p - a}$, $a \in \mathbb{C}$, and such that $f_z\colon \mathbb{D} \to \mathbb{C}^{n-1}_z$ is a biholomorphic embedding of $\mathbb{D}$ into $\mathbb{C}^{n-1}_z$. Then $f_z(\mathbb{D})$ is either completely contained in the slice $S^p_a := \{z \in \mathbb{C}^{n-1} : z_p = a\}$ or does not intersect $S^p_a$ at all. This is due to the fact that if $S^p_a \cap f_z(U) = \{z_0\}$, $U \subset \mathbb{D}$ open and small enough, then for the canonical parametrization $g\colon f_z(U) \to \mathbb{C}_w$ of $f(U)$ and for $\zeta^+, \zeta^- \in \mathbb{C}^{n-1}$ such that $z_0 + \zeta^+, z_0 - \zeta^- \in f_z(U)$, the slope $|g(z_0 + \zeta^+) - g(z_0 - \zeta^-)|/\|\zeta^+ + \zeta^-\|$ becomes unbounded as $\zeta^+, \zeta^- \to 0$, which contradicts the holomorphicity of $g$. Since each set $E_\nu$ is defined by a sum of terms of the form $\sqrt{z_{[l]} - a_l}$, and since, moreover, the subsequence $\{a^p_l\}_{l=1}^\infty$ of $\{a_l\}$ is dense in $\mathbb{C}$, this will enable us to show that for $\{\varepsilon_l\}$ decreasing fast enough, every analytic disc $f(\mathbb{D}) \subset \mathcal{E}$ must have constant $z_p$-coordinate. Due to the fact that $p \in \mathbb{N}_{n-1}$ here is arbitrary, our assertion will be proved.

There arise some technical difficulties, the most important of which is the following: while for every above-described analytic disc in the analytic hypersurface $w = \sqrt{z_p - a}$ the projection $f_z(\mathbb{D})$ cannot intersect $S^p_a$ (at least if its $z_p$-coordinate is not already constant), this property might get spoiled when adding further terms $\sqrt{z_{[l]} - a_l}$, $l \in \mathbb{N}$, and thus does not carry over necessarily to the limit set $\mathcal{E}$. In general this problem can be easily handled, except, however, at points $z_0 \in S^p_a$ that are contained in $S^{[l]}_{a_l}$ for more than one $l \in \mathbb{N}$. In this situation there are root branches originating from $z_0$ in different directions $p_1, \ldots, p_T \in \mathbb{N}_{n-1}$, and in general their slopes near the point $z_0$ may cancel out each other. To deal with this problem, we will show that we can at least guarantee the following: for every $z_0 \in S^{[l]}_{a_l} \cap B^{n-1}(0, l)$, $l \in \mathbb{N}$, there does not exist any analytic disc $f(\mathbb{D}) \subset \mathcal{E}$ such that $f_z(\mathbb{D}) \cap S^{[l]}_{a_l} = \{z_0\}$ and such that $f_z(\mathbb{D})$ is contained in the cone $z_0 + \bigcap_{t=1}^T \Gamma^{p_t}(\alpha)$; here

$$\Gamma^p(\alpha) := \{\zeta \in \mathbb{C}^{n-1} : \zeta_p \neq 0 \text{ and } \frac{|\zeta_q|}{|\zeta_p|} < \alpha, \text{ for all } q \in \mathbb{N}_{n-1},\, q \neq p\},$$

where $\alpha$ is a positive number that will depend on the choice of $\{\varepsilon_l\}$ (note that if $\zeta \in \Gamma^p(\alpha)$, then also $\lambda\zeta \in \Gamma^p(\alpha)$ for every $\lambda \in \mathbb{C}^*$). In fact, the faster $\{\varepsilon_l\}$ decreases, the larger we will be able to choose $\alpha$. It turns out that this weaker assertion is sufficient for our purpose, since locally for every analytic disc $f(\mathbb{D}) \subset \mathcal{E}$ the projection $f_z(\mathbb{D})$ lies in $\bigcap_{t=1}^T \Gamma^p(\alpha)$ for suitable $p_1, \ldots, p_T \in \mathbb{N}_{n-1}$ and $\alpha > 0$ large enough.

The above complications, as well as most of the other technical difficulties for choosing the sequence $\{\varepsilon_l\}$, do not occur in the case $n = 2$. In fact, in this case the proof becomes relatively simple, and most of the work of this section is not needed. Hence in what follows we will often implicitly assume that $n \geq 3$, though this will not have any influence on the course and correctness of our arguments (for example, the set $\Gamma^p(\alpha) = \mathbb{C}^*$ is still well-defined for $n = 2$, though it is obviously not needed in this case).

**Remark.** Many of the statements in this section involve the function $\sqrt{\cdot}\colon \mathbb{C} \to \mathbb{C}$, which is multivalued. In general, whenever such a statement is made, we will implicitly mean it to hold true for every choice of a single-valued branch $(\sqrt{\cdot}\,)_b\colon \mathbb{C} \to \mathbb{C}$ of $\sqrt{\cdot}$ (no assumptions on continuity). However, there will be cases when we will have to deal with particular single-valued branches of $\sqrt{\cdot}$. By some abuse of notation, they will be denoted by the same symbol $\sqrt{\cdot}$. We will always point out when $\sqrt{\cdot}$ denotes a particular single-valued branch whenever such a situation first occurs.

**Lemma 1.3.1.** *There exists a constant $0 < C < 1$ such that for all $z, z', \zeta \in \mathbb{C}$,*

$$\sqrt{|\zeta|} \leq \big|\sqrt{z+\zeta} - \sqrt{z'-\zeta}\,\big| \leq 2\sqrt{|\zeta|} \quad \textit{if} \quad |z|, |z'| \leq C|\zeta|.$$

**Proof.** This is immediately clear, since

$$\frac{\big|\sqrt{z+\zeta} - \sqrt{z'-\zeta}\,\big|}{\sqrt{|\zeta|}} = \Big|\sqrt{(z/\zeta)+1} - \sqrt{(z'/\zeta)-1}\,\Big| \xrightarrow{z/\zeta,\, z'/\zeta \to 0} \sqrt{2}\,. \qquad \square$$

**Lemma 1.3.2.** *For every $p \in \mathbb{N}_{n-1}$ and $\alpha > 0$, one has*

$$\lim_{\zeta \to 0} \frac{\big|\sqrt{\zeta_p} - \sqrt{-\zeta_p}\big|}{2\|\zeta\|} = +\infty \quad \textit{on} \quad \Gamma^p(\alpha).$$

**Proof.** Indeed, with $c_\alpha := \max\{1, \alpha\}$ we have

$$\frac{|\sqrt{\zeta_p} - \sqrt{-\zeta_p}\,|}{2\|\zeta\|} = \frac{1}{\sqrt{2}}\frac{\sqrt{|\zeta_p|}}{\|\zeta\|} = \frac{1}{\sqrt{2}}\Big(\sum_{q=1}^{n-1}\frac{|\zeta_q|^2}{|\zeta_p|}\Big)^{-1/2} \geq \frac{1}{\sqrt{2}}\Big(\sum_{q=1}^{n-1} c_\alpha|\zeta_q|\Big)^{-1/2}$$

on $\Gamma^p(\alpha)$, and the last term tends to $+\infty$ as $\zeta \to 0$. $\square$

**Lemma 1.3.3.** *Let $P := \{p_t\}_{t=1}^T \subset \{1,\dots,n-1\}$, $p_t \neq p_{t'}$ if $t \neq t'$, and $\{e_t\}_{t=1}^T \subset (0,\infty)$, $T \geq 2$. Define a constant $\alpha > 0$ by $\alpha := \min\big\{\frac{1}{9}(e_m/e_{m+1})^2 : 1 \leq m \leq T-1\big\}$. Then for every $\nu > 0$, there exists a positive number $\delta > 0$ such that*

$$\frac{\big|\sum_{m=1}^T e_m\big(\sqrt{z_{p_m} + (\zeta_{p_m} + \zeta'_{p_m})} - \sqrt{z_{p_m} - (\zeta_{p_m} + \zeta''_{p_m})}\,\big)\big|}{2\|\zeta\|} > \nu$$

*for every $\zeta \in \big(\bigcap_{m=1}^T \Gamma^{p_m}(\alpha)\big) \cap B^{n-1}(0,\delta)$ and $\zeta', \zeta'', z \in \Delta^{n-1}\big(0,(C/2)|\zeta|_P\big)$. Here $C$ is the constant from Lemma 1.3.1, $|\zeta|_P \in [0,\infty]^{n-1}$ is defined by $(|\zeta|_P)_p = |\zeta_p|$ if $p \in P$, $(|\zeta|_P)_p = \infty$ if $p \in \complement P := \mathbb{N}_{n-1} \setminus P$, and $\Delta^{n-1}\big(0,(r_1,\dots,r_{n-1})\big) := \{z \in \mathbb{C}^{n-1} : |z_p| < r_p,$ if $r_p > 0$, or $z_p = 0$, if $r_p = 0$, $p \in \mathbb{N}_{n-1}\}$ for $r \in [0,\infty]^{n-1}$.*

**Remark.** The statement of this lemma is interesting and will be used only in the case when $\alpha > 1$ (otherwise the intersection $\bigcap_m \Gamma^{p_m}(\alpha)$ is empty).

**Proof.** For every $m \in \mathbb{N}_{T-1}$ we define $\alpha_m := \frac{1}{9}(e_m/e_{m+1})^2$, and for every $m \in \mathbb{N}_T$ we let $D_m(\zeta) := \{z \in \mathbb{C}^{n-1} : |z_{p_m}| \leq C|\zeta_{p_m}|\}$. We will show by induction that for every $t = 1,\dots,T$, the inequality

$$\Big|\sum_{m=1}^t e_m\big(\sqrt{z'_{p_m} + \zeta_{p_m}} - \sqrt{z''_{p_m} - \zeta_{p_m}}\,\big)\Big| \geq e_t\sqrt{|\zeta_{p_t}|} \tag{1.2}$$

holds true for $\zeta \in \bigcap_{m=1}^{t-1} \Gamma^{p_m}(\alpha_m)$ and $z', z'' \in \bigcap_{m=1}^t D_m(\zeta)$. Indeed, the case $t = 1$ is already proven by Lemma 1.3.1. For the step $t \to t+1$, let $H_{t+1}$ denote the left term in (1.2) where the sum is taken up to $t+1$. Using the induction hypothesis and applying Lemma 1.3.1, we see that

$$\begin{aligned} H_{t+1} &\geq \Big|\sum_{m=1}^t e_m\big(\sqrt{z'_{p_m} + \zeta_{p_m}} - \sqrt{z''_{p_m} - \zeta_{p_m}}\,\big)\Big| \\ &\qquad - e_{t+1}\Big|\sqrt{z'_{p_{t+1}} + \zeta_{p_{t+1}}} - \sqrt{z''_{p_{t+1}} - \zeta_{p_{t+1}}}\,\Big| \\ &\geq e_t\sqrt{|\zeta_{p_t}|} - 2e_{t+1}\sqrt{|\zeta_{p_{t+1}}|} \end{aligned}$$

for $\zeta \in \bigcap_{m=1}^{t-1} \Gamma^{p_m}(\alpha_m)$ and $z', z'' \in \bigcap_{m=1}^{t+1} D_m(\zeta)$. Observe that there is nothing to show in the case $\zeta_{p_{t+1}} = 0$. Hence we can assume $\zeta_{p_{t+1}} \neq 0$ and write

$$e_t\sqrt{|\zeta_{p_t}|} - 2e_{t+1}\sqrt{|\zeta_{p_{t+1}}|} = 2e_{t+1}\sqrt{|\zeta_{p_{t+1}}|}\Big(\frac{e_t}{2e_{t+1}}\frac{\sqrt{|\zeta_{p_t}|}}{\sqrt{|\zeta_{p_{t+1}}|}} - 1\Big).$$

One immediately checks that the term between the brackets is not less than 1/2 precisely if $|\zeta_{p_{t+1}}|/|\zeta_{p_t}| \le \alpha_t$; hence

$$e_t\sqrt{|\zeta_{p_t}|} - 2e_{t+1}\sqrt{|\zeta_{p_{t+1}}|} \ge e_{t+1}\sqrt{|\zeta_{p_{t+1}}|} \qquad \text{for } \zeta \in \Gamma^{p_t}(\alpha_t).$$

This completes the induction and proves (1.2). But from Lemma 1.3.2 we know that

$$\lim_{\zeta\to 0} \frac{|\sqrt{\zeta_{p_T}} - \sqrt{-\zeta_{p_T}}|}{2\|\zeta\|} = +\infty \qquad \text{on } \Gamma^{p_T}(\alpha_T),$$

where $\alpha_T := \alpha$. Combining this with the estimate (1.2) in the case $t = T$, we conclude that for every $\nu > 0$ there exists $\delta > 0$ such that

$$\frac{\big|\sum_{m=1}^T e_m\big(\sqrt{z'_{p_m} + \zeta_{p_m}} - \sqrt{z''_{p_m} - \zeta_{p_m}}\big)\big|}{2\|\zeta\|} > \nu$$

for $\zeta \in \bigcap_{m=1}^T \Gamma^{p_m}(\alpha_m) \cap B^{n-1}(0,\delta)$ and $z', z'' \in \bigcap_{m=1}^T D_m(\zeta) = \Delta^{n-1}(0, C|\zeta|_P)$. Since $\alpha \le \alpha_m$ for all $m \in \mathbb{N}_T$ and $\Gamma^p(\alpha) \subset \Gamma^p(\alpha')$ for $\alpha \le \alpha'$, this concludes the proof. Indeed, for $\zeta', \zeta'', z \in \Delta^{n-1}\big(0, (C/2)|\zeta|_P\big)$, the points $z' := z + \zeta'$ and $z'' := z - \zeta''$ always satisfy $z', z'' \in \Delta^{n-1}(0, C|\zeta|_P)$. $\square$

We want to estimate the slope between two points of the set $E_\nu$ when their projection to $\mathbb{C}_z^{n-1}$ lies near the zero set of one of the functions $\sqrt{z_{[l]} - a_l}$, $l = 1, \dots, \nu$. For this we need some notations: For every $\nu \in \mathbb{N}$ and $p \in \mathbb{N}_{n-1}$ we define

$$S_\nu := \{\zeta \in \mathbb{C}^{n-1} : \zeta_{[\nu]} = a_\nu\}, \qquad S^p := \{\zeta \in \mathbb{C}^{n-1} : \zeta_p = 0\},$$

and

$$L_\nu^p := \{l \in \mathbb{N} : 1 \le l \le \nu,\ [l] = p\}, \qquad A_\nu^p := \{a_l \in \mathbb{C} : l \in L_\nu^p\}.$$

Obviously, $\bigcup_{p=1}^{n-1} L_\nu^p = \mathbb{N}_\nu$. Moreover, if $z \in \mathbb{C}^{n-1}$, we define

$$L_\nu^p(z) := \{l \in L_\nu^p : z_p = a_l\}.$$

Note that $L_\nu^p(z)$ consists of at most one element. Further, for $P \subset \mathbb{N}_{n-1}$ such that $[\nu] \in P$ and $z \in S_\nu$ we let

$$\mathscr{L}_\nu^P(z) := \bigcup_{p\in P} L_\nu^p(z).$$

Observe that under the assumptions on $P$ and $z$, we always have $\nu \in \mathscr{L}_\nu^P(z)$. As mentioned before, the case $|\mathscr{L}_\nu^P(z)| > 1$ is of special interest and leads us to consider the sets $\bigcap_p \Gamma^p(\alpha)$ for $\alpha > 1$. Here $\alpha$ was claimed to depend on $\{\varepsilon_l\}$, and we now clarify this dependence by the following definition: for every $\nu \in \mathbb{N}$,

$P \subset \mathbb{N}_{n-1}$ such that $[\nu] \in P$ and every $z \in S_\nu$, let $\alpha_\nu^P(z)$ be the positive number

$$\alpha_\nu^P(z) := \begin{cases} \nu+1 & \text{if } \mathscr{L}_\nu^P(z) = \{\nu\} \\ \min\left\{\frac{1}{9}(\varepsilon_l/\varepsilon_{l'})^2 : l, l' \in \mathscr{L}_\nu^P(z), l' > l\right\} & \text{if } \mathscr{L}_\nu^P(z) \supsetneq \{\nu\}. \end{cases}$$

Observe that, since the sequence $\{\varepsilon_l\}$ is still in our hands, we can always assume that $\alpha_\nu^P(z) > 1$. Finally, for each $P \subset \mathbb{N}_{n-1}$ and $\alpha > 0$ we let

$$\gamma(P,\alpha) := \Big(\bigcap_{p\in P} \Gamma^p(\alpha)\Big) \cap \Big(\bigcap_{p\in \complement P} S^p\Big).$$

**Lemma 1.3.4.** *Suppose $\varepsilon_1, \dots, \varepsilon_\nu$ have already been chosen. Let $\delta > 0$. Then for every $z_0 \in S_\nu$ and $P \subset \mathbb{N}_{n-1}$ such that $[\nu] \in P$, there exist $r^P(z_0) > 0$ and $\delta^P(z_0) \in (0,\delta)$ such that for every $j, k \in \mathbb{N}_{2^\nu}$ the inequality*

$$\frac{\big|w_j^{(\nu)}\big(z + (\zeta+\zeta')\big) - w_k^{(\nu)}\big(z - (\zeta+\zeta'')\big)\big|}{2\|\zeta\|} > \nu \tag{1.3}$$

*holds for every $z \in B^{n-1}\big(z_0, r^P(z_0)\big)$, $\zeta \in \gamma\big(P, \alpha_\nu^P(z_0)\big) \cap bB^{n-1}\big(0, \delta^P(z_0)\big)$ and $\zeta', \zeta'' \in \Delta^{n-1}\big(0, (C/2)|\zeta|\big)$; here $|\zeta| = (|\zeta_1|, \dots, |\zeta_{n-1}|)$.*

**Proof.** Fix $z_0 \in S_\nu$ and $P \subset \mathbb{N}_{n-1}$ such that $[\nu] \in P$. For each $p \in \mathbb{N}_{n-1}$, let $U_p \subset \mathbb{C}$ be an open convex neighbourhood of $z_{0,p}$ such that

$$U_p \cap A_\nu^p = \begin{cases} \varnothing & \text{if } L_\nu^p(z_0) = \varnothing \\ \{z_{0,p}\} & \text{if } L_\nu^p(z_0) \neq \varnothing \end{cases}$$

and let $U := U_1 \times \cdots \times U_{n-1}$. Choose $r > 0$ so small that $B^{n-1}(z_0, 2r) \subset U$. For each $l \in \mathbb{N}_\nu$, consider a single-valued branch of the multi-valued function $\sqrt{z_{[l]} - a_l}$ which will also be denoted here by $\sqrt{z_{[l]} - a_l}$. Since for every $l \in \mathbb{N}_\nu \setminus \bigcup_{p=1}^{n-1} L_\nu^p(z_0)$ we have $a_l \notin U_{[l]}$, we can assume that $\sqrt{z_{[l]} - a_l}$ is holomorphic on $U$ for these $l$. After possibly changing the numeration of the roots of $P_\nu(z, \cdot)$ for $z \in U$, we may further assume for every $h \in \mathbb{N}_{2^\nu}$ that $w_h^{(\nu)}(z) = \sum_{l=1}^{\nu} \pm\varepsilon_l \sqrt{z_{[l]} - a_l}$ on $B^{n-1}(z_0, 2r)$ for suitably chosen signs depending only on $l$ and $h$. Now define $\tilde{w}_h \colon B^{n-1}(z_0, 2r) \to \mathbb{C}$ as

$$\tilde{w}_h(z) := \sum_{p\in P} \sum_{l \in L_\nu^p \setminus L_\nu^p(z_0)} \pm\varepsilon_l \sqrt{z_{[l]} - a_l} + \sum_{p \in \complement P} \sum_{l \in L_\nu^p} \pm\varepsilon_l \sqrt{z_{[l]} - a_l}\,. \tag{1.4}$$

Since $\mathbb{N}_\nu = \bigcup_{p=1}^{n-1} L_\nu^p$, we have $w_h^{(\nu)}(z) = \tilde{w}_h(z) + \sum_{l \in \mathscr{L}_\nu^P(z_0)} \pm\varepsilon_l \sqrt{z_{[l]} - a_l}$ on $B^{n-1}(z_0, 2r)$. Let $\mathbb{N}_{2^\nu}^2 := \mathbb{N}_{2^\nu} \times \mathbb{N}_{2^\nu}$ and $\mathbb{N}_{2^\nu}^2(z_0) := \{(j,k) \in \mathbb{N}_{2^\nu}^2 : \tilde{w}_j(z_0) = \tilde{w}_k(z_0)\}$.

Step 1: *We show that there exist* $r' > 0$ *and* $\delta' \in (0,\delta)$ *such that* (1.3) *holds for every* $\zeta \in B^{n-1}(0,\delta')$, $\zeta',\zeta'' \in \Delta^{n-1}\big(0,(C/2)|\zeta|\big)$, $z \in B^{n-1}(z_0,r')$ *and* $(j,k) \in \mathbb{N}^2_{2^\nu} \setminus \mathbb{N}^2_{2^\nu}(z_0)$.

Proof. For $l \in L^p_\nu(z_0)$, we have $z_{0,[l]} = a_l$ and $\sqrt{\cdot}$ is continuous at the origin; hence we conclude from (1.4) and the holomorphicity of $\sqrt{z_{[l]} - a_l}$ for $l \in L^p_\nu \setminus L^p_\nu(z_0)$ that $\tilde{w}_h$ is continuous at $z_0$ for every $h \in \mathbb{N}_{2^\nu}$. Thus there exist $M > 0$ and $r_1 > 0$ such that $|\tilde{w}_j(z + (\zeta + \zeta')) - \tilde{w}_k(z - (\zeta + \zeta''))| > M$ for every $\zeta \in B^{n-1}(0,r_1)$, $\zeta',\zeta'' \in \Delta^{n-1}\big(0,(C/2)|\zeta|\big)$, $z \in B^{n-1}(z_0,r_1)$ and $(j,k) \in \mathbb{N}^2_{2^\nu} \setminus \mathbb{N}^2_{2^\nu}(z_0)$. Moreover, since again $z_{0,[l]} = a_l$ for $l \in \mathscr{L}^P_\nu(z_0)$ and $\sqrt{\cdot}$ is continuous at the origin, there exists $r_2 > 0$ such that $\sqrt{|(z_{[l]} \pm (\zeta_{[l]} + \tilde{\zeta}_{[l]})) - a_l|} < M/\big(4(n-1)\varepsilon_l\big)$, where $\tilde{\zeta} \in \{\zeta',\zeta''\}$, for every $\zeta \in B^{n-1}(0,r_2)$, $\zeta',\zeta'' \in \Delta^{n-1}\big(0,(C/2)|\zeta|\big)$, $z \in B^{n-1}(z_0,r_2)$ and $l \in \mathscr{L}^P_\nu(z_0)$. Let $r' := \min\{r,r_1,r_2\}$ and $\delta' := \min\{\delta,r,r_1,r_2,M/4\nu\}$. Then the following estimate holds true for every $\zeta \in B^{n-1}(0,\delta')$, $\zeta',\zeta'' \in \Delta^{n-1}\big(0,(C/2)|\zeta|\big)$, $z \in B^{n-1}(z_0,r')$ and $(j,k) \in \mathbb{N}^2_{2^\nu} \setminus \mathbb{N}^2_{2^\nu}(z_0)$:

$$
\begin{aligned}
&\frac{\big|w^{(\nu)}_j\big(z + (\zeta + \zeta')\big) - w^{(\nu)}_k\big(z - (\zeta + \zeta'')\big)\big|}{2\|\zeta\|}\\
&\geq \frac{\big|\tilde{w}_j\big(z + (\zeta + \zeta')\big) - \tilde{w}_k\big(z - (\zeta + \zeta'')\big)\big|}{2\|\zeta\|}\\
&\quad - \frac{\sum_{l \in \mathscr{L}^P_\nu(z_0)} \varepsilon_l\Big(\sqrt{\big|\big(z_{[l]} + (\zeta_{[l]} + \zeta'_{[l]})\big) - a_l\big|} + \sqrt{\big|\big(z_{[l]} - (\zeta_{[l]} + \zeta''_{[l]})\big) - a_l\big|}\Big)}{2\|\zeta\|}\\
&> \frac{M - \sum_{l \in \mathscr{L}^P_\nu(z_0)} 2\varepsilon_l M/\big(4(n-1)\varepsilon_l\big)}{2\|\zeta\|} \geq \frac{M - M/2}{2\|\zeta\|} > \nu.
\end{aligned}
$$

Step 2: *We show that there exist* $r'' \in (0,r')$ *and* $\delta'' \in (0,\delta')$ *such that* (1.3) *holds for every* $\zeta \in \gamma\big(P,\alpha^P_\nu(z_0)\big) \cap K^{n-1}(\delta''/2,\delta'')$, $\zeta',\zeta'' \in \Delta^{n-1}\big(0,(C/2)|\zeta|\big)$, $z \in \Delta^{n-1}\big(z_0,(C/2)|\zeta|_P\big) \cap B^{n-1}(z_0,r'')$ *and* $(j,k) \in \mathbb{N}^2_{2^\nu}(z_0)$, *where for* $R_1,R_2 \geq 0$ *we put* $K^{n-1}(R_1,R_2) := \{z \in \mathbb{C}^{n-1} : R_1 < \|z\| < R_2\}$.

Proof. Observe that the first term in (1.4) is holomorphic in $B^{n-1}(z_0,2r)$ and the second term is constant on the set $z_0 + \bigcap_{p \in \complement P} S^p$. Therefore we can find $M > 0$ and $\tilde{r} > 0$ such that

$$
\frac{\big|\tilde{w}_j\big(z_0 + (\zeta + \zeta')\big) - \tilde{w}_k\big(z_0 - (\zeta + \zeta'')\big)\big|}{2\|\zeta\|} < M
$$

for all $\zeta \in \big(\bigcap_{p \in \complement P} S^p\big) \cap B^{n-1}(0,\tilde{r})$, $\zeta',\zeta'' \in \Delta^{n-1}\big(0,(C/2)|\zeta|\big)$ and $(j,k) \in \mathbb{N}^2_{2^\nu}(z_0)$. Moreover, since, by definition, $z_{0,[l]} = a_l$ for every $l \in \mathscr{L}^P_\nu(z_0)$, we have $\sqrt{\big(z_{[l]} \pm (\zeta_{[l]} + \tilde{\zeta}_{[l]})\big) - a_l} = \sqrt{(z_{[l]} - z_{0,[l]}) \pm (\zeta_{[l]} + \tilde{\zeta}_{[l]})}$, where $\tilde{\zeta} \in \{\zeta',\zeta''\}$.

Hence, using Lemma 1.3.1 and 1.3.2 if $\mathscr{L}_\nu^P(z_0) = \{\nu\}$ and Lemma 1.3.3 if $\mathscr{L}_\nu^P(z_0) \supsetneq \{\nu\}$, there exists $\tilde{\delta} > 0$ such that

$$\frac{\left| \sum_{l\in\mathscr{L}_\nu^P(z_0)} \varepsilon_l \Big( \sqrt{(z_{[l]} + (\zeta_{[l]} + \zeta'_{[l]})) - a_l} - \sqrt{(z_{[l]} - (\zeta_{[l]} + \zeta''_{[l]})) - a_l} \Big) \right|}{2\|\zeta\|} > \nu + M$$

for all $\zeta \in \big[\bigcap_{p\in P} \Gamma^p\big(\alpha_\nu^P(z_0)\big)\big] \cap B^{n-1}(0,\tilde{\delta})$, $\zeta',\zeta'' \in \Delta^{n-1}\big(0,(C/2)|\zeta|_P\big)$ and $z \in \Delta^{n-1}\big(z_0,(C/2)|\zeta|_P\big)$ (recall the definition of $\alpha_\nu^P(z_0)$). Now choose $\delta''$ such that $0 < \delta'' < \min\{\tilde{r},\tilde{\delta},\delta'\}$. Observe that $\tilde{w}_h$ is continuous in $z_0 + \big[\big(\bigcap_{p\in\complement P} S^p\big) \cap B^{n-1}(0,2r)\big]$ for every $h \in \mathbb{N}_{2^\nu}$. Hence there exists some $r'' \in (0,r')$ such that the following estimate holds true for every $\zeta \in \big(\bigcap_{p\in\complement P} S^p\big) \cap K^{n-1}(\delta''/2,\delta'')$, $\zeta',\zeta'' \in \Delta^{n-1}\big(0,(C/2)|\zeta|\big)$, $z \in B^{n-1}(z_0,r'')$ and $(j,k) \in \mathbb{N}^2_{2^\nu}(z_0)$:

$$\frac{\big|\tilde{w}_j\big(z+(\zeta+\zeta')\big) - \tilde{w}_k\big(z-(\zeta+\zeta'')\big)\big|}{2\|\zeta\|} < M.$$

Thus for every $\zeta \in \gamma\big(P,\alpha_\nu^P(z_0)\big) \cap K^{n-1}(\delta''/2,\delta'')$, $\zeta',\zeta'' \in \Delta^{n-1}\big(0,(C/2)|\zeta|\big)$, $z \in \Delta^{n-1}\big(z_0,(C/2)|\zeta|_P\big) \cap B^{n-1}(z_0,r'')$ and $(j,k) \in \mathbb{N}^2_{2^\nu}(z_0)$ we get

$$\begin{aligned}
&\frac{|w_j^{(\nu)}\big(z{+}(\zeta+\zeta')\big) - w_k^{(\nu)}\big(z{-}(\zeta+\zeta'')\big)|}{2\|\zeta\|} \\
&\geq \frac{\Big| \sum_{l\in\mathscr{L}_\nu^P(z_0)} \varepsilon_l \Big( \sqrt{(z_{[l]} + (\zeta_{[l]} + \zeta'_{[l]})) - a_l} - \sqrt{(z_{[l]} - (\zeta_{[l]} + \zeta''_{[l]})) - a_l} \Big) \Big|}{2\|\zeta\|} - \\
&- \frac{\big|\tilde{w}_j\big(z+(\zeta+\zeta')\big) - \tilde{w}_k\big(z-(\zeta+\zeta'')\big)\big|}{2\|\zeta\|} > \nu.
\end{aligned}$$

STEP 3: *We show that there exist $r^P(z_0) > 0$ and $\delta^P(z_0) \in (0,\delta)$ such that* (1.3) *holds for every $\zeta \in \gamma\big(P,\alpha_\nu^P(z_0)\big) \cap bB^{n-1}\big(0,\delta^P(z_0)\big)$, $\zeta',\zeta'' \in \Delta^{n-1}\big(0,(C/2)|\zeta|\big)$, $z \in B^{n-1}\big(z_0,r^P(z_0)\big)$ and $j,k \in \mathbb{N}_{2^\nu}$.*

PROOF. We know that (1.3) holds for every $\zeta \in \gamma\big(P,\alpha_\nu^P(z_0)\big) \cap K^{n-1}(\delta''/2,\delta'')$, $\zeta',\zeta'' \in \Delta^{n-1}\big(0,(C/2)|\zeta|\big)$, $z \in \Delta^{n-1}\big(z_0,(C/2)|\zeta|_P\big) \cap B^{n-1}(z_0,r'')$ and $j,k \in \mathbb{N}_{2^\nu}$. It only remains to make proper choices for the constants $r^P(z_0)$ and $\delta^P(z_0)$. First, choose any $\delta^P(z_0)$ such that $\delta'' > \delta^P(z_0) > \delta''/2$. Then there exists $K > 0$ such that

$$|\zeta_p| > K \quad \text{for all} \quad \zeta \in \gamma\big(P,\alpha_\nu^P(z_0)\big) \cap bB^{n-1}\big(0,\delta^P(z_0)\big),\ p \in P.$$

Indeed, let $p \in P$. Then for $\zeta \in \gamma\big(P,\alpha_\nu^P(z_0)\big)$ we have in particular $\zeta \in \Gamma^p\big(\alpha_\nu^P(z_0)\big)$

and hence $|\zeta_q|/|\zeta_p| < \alpha_\nu^P(z_0)$ for every $q \in \mathbb{N}_{n-1}$ (assuming that $\alpha_\nu^P(z_0) > 1$, which is the only interesting case). Thus $\|\zeta\| < \alpha_\nu^P(z_0)\sqrt{n-1}\,|\zeta_p|$. Since also $\zeta \in bB^{n-1}\big(0, \delta^P(z_0)\big)$, we conclude that $|\zeta_p| > \delta^P(z_0)/\big(\alpha_\nu^P(z_0)\sqrt{n-1}\big) =: K$. Now choose $\rho > 0$ such that $|z_p - z_{0,p}| < (CK)/2$ for all $z \in B^{n-1}(z_0, \rho)$ and $p \in P$, i.e., $B^{n-1}(z_0, \rho) \subset \Delta^{n-1}\big(z_0, (C/2)|\zeta|_P\big)$ for all $\zeta \in \gamma\big(P, \alpha_\nu^P(z_0)\big) \cap bB^{n-1}\big(0, \delta^P(z_0)\big)$. Then $r^P(z_0) := \min\{r'', \rho\}$ is a desired constant. □

Fix $\nu \in \mathbb{N}$. By the previous lemma, we have assigned positive numbers $r^P(z_0)$, $\delta^P(z_0)$ to every $z_0 \in S_\nu$. As we shall see in the proof of Lemma 1.3.5, the choice of $\varepsilon_{\nu+1}$ will depend on the numbers $\delta^P(z_0)$, $z_0 \in S_\nu$; in fact, we will need a positive lower bound for the set $\{\delta^P(z_0) : z_0 \in S_\nu\}$. However, such a bound does not always exist. Hence from now on we restrict our attention to the compact subset $S_\nu \cap \bar{B}^{n-1}(0, \nu)$ of $S_\nu$. This set can be covered by finitely many balls $B^{n-1}\big(z_0, r^P(z_0)\big)$, $z_0 \in S_\nu$, and thus leads to a finite set $\{\delta^P(z_1), \dots, \delta^P(z_m)\} \subset (0, \infty)$ (which of course has a positive minimum). On the way, we have to choose the numbers $r^P(z_0)$ in the covering $\big\{B^{n-1}\big(z_0, r^P(z_0)\big)\big\}_{z_0 \in S_\nu}$ small enough in order to limit the influence of points $z_0 \in S_\nu$ with small value $\alpha_\nu^P(z_0)$. For this purpose, we need some further notations: Fix a decreasing sequence $\{\rho_\nu\}$ of positive numbers converging to zero, such that

$$\max_{1 \le p \le n-1} \operatorname{vol}\Big(\bigcup_{l \in L_\nu^p} \Delta^1(a_l, \rho_\nu)\Big) \to 0 \quad \text{for} \quad \nu \to \infty.$$

Then for every $\nu \in \mathbb{N}$, $p \in \mathbb{N}_{n-1}$ and $z \in \mathbb{C}^{n-1}$ we let

$$\tilde{L}_\nu^p(z) := \{l \in L_\nu^p : |z_p - a_l| \le \rho_\nu\}.$$

Moreover, if $z \in S_\nu$ and $P \subset \mathbb{N}_{n-1}$ such that $[\nu] \in P$ we let

$$\tilde{\mathscr{L}}_\nu^P(z) := \bigcup_{p \in P} \tilde{L}_\nu^p(z).$$

Note that under the assumptions on $P$ and $z$ we always have $\nu \in \tilde{\mathscr{L}}_\nu^P(z)$. Hence

$$\tilde{\alpha}_\nu^P(z) := \begin{cases} \nu + 1 & \text{if } \tilde{\mathscr{L}}_\nu^P(z) = \{\nu\} \\ \min\big\{\nu + 1, \min\{\frac{1}{9}(\varepsilon_l/\varepsilon_{l'})^2 : l, l' \in \tilde{\mathscr{L}}_\nu^P(z), l' > l\}\big\} & \text{if } \tilde{\mathscr{L}}_\nu^P(z) \supsetneq \{\nu\} \end{cases}$$

is a well-defined positive number.

**Corollary 1.3.1.** *Suppose $\varepsilon_1, \dots, \varepsilon_\nu$ have already been chosen. Let $\delta > 0$. Then there exists a finite subset $D_\nu := \{\delta_\nu^1, \dots, \delta_\nu^{d_\nu}\} \subset (0, \delta)$ such that for every $z \in S_\nu \cap B^{n-1}(0, \nu)$ and $P \subset \mathbb{N}_{n-1}$ such that $[\nu] \in P$, there exists some $\sigma \in \{1, \dots, d_\nu\}$*

*such that for every* $j, k \in \mathbb{N}_{2^\nu}$ *the inequality*

$$\frac{\big|w_j^{(\nu)}\big(z+(\zeta+\zeta')\big)-w_k^{(\nu)}\big(z-(\zeta+\zeta'')\big)\big|}{2\|\zeta\|}>\nu \tag{1.5}$$

*holds true for all* $\zeta \in \gamma\big(P, \tilde{\alpha}_\nu^P(z)\big) \cap bB^{n-1}(0, \delta_\nu^\sigma)$ *and* $\zeta', \zeta'' \in \Delta^{n-1}\big(0, (C/2)|\zeta|\big)$.

**Proof.** By the previous lemma, for every $z_0 \in S_\nu$ and $P \subset \mathbb{N}_{n-1}$, $[\nu] \in P$, there exist positive numbers $r^P(z_0) \in (0, \rho_\nu)$ and $\delta^P(z_0) \in (0, \delta)$ such that (1.5) holds for every $j, k \in \mathbb{N}_{2^\nu}$, $z \in B^{n-1}\big(z_0, r^P(z_0)\big)$, $\zeta \in \gamma\big(P, \alpha_\nu^P(z_0)\big) \cap bB^{n-1}\big(0, \delta^P(z_0)\big)$ and $\zeta', \zeta'' \in \Delta^{n-1}\big(0, (C/2)|\zeta|\big)$. Let

$$r(z_0) := \min\big\{r^P(z_0) : P \subset \mathbb{N}_{n-1} \text{ such that } [\nu] \in P\big\}.$$

By compactness of $S_\nu \cap \bar{B}^{n-1}(0, \nu)$, there exist finitely many points $z_1, \ldots, z_M \in S_\nu$ such that $S_\nu \cap \bar{B}^{n-1}(0, \nu) \subset \bigcup_{m=1}^M B^{n-1}\big(z_m, r(z_m)\big)$. Let

$$D_\nu := \big\{\delta^P(z_m) : P \subset \mathbb{N}_{n-1} \text{ such that } [\nu] \in P,\, m = 1, \ldots, M\big\}.$$

Then for every $z \in S_\nu \cap B^{n-1}(0, \nu)$ and $P \subset \mathbb{N}_{n-1}$, $[\nu] \in P$, there exist $\sigma \in \{1, \ldots, d_\nu\}$ and $m \in \mathbb{N}_M$ such that $|z - z_m| \le \rho_\nu$ and such that (1.5) holds for every $j, k \in \mathbb{N}_{2^\nu}$, $\zeta \in \gamma\big(P, \alpha_\nu^P(z_m)\big) \cap bB^{n-1}(0, \delta_\nu^\sigma)$ and $\zeta', \zeta'' \in \Delta^{n-1}\big(0, (C/2)|\zeta|\big)$. It remains to observe that we herein can replace $\alpha_\nu^P(z_m)$ by $\tilde{\alpha}_\nu^P(z)$. Indeed, since $|z - z_m| \le \rho_\nu$, we have $L_\nu^p(z_m) \subset \tilde{L}_\nu^p(z)$ for all $p \in \mathbb{N}_{n-1}$ and thus $\mathscr{L}_\nu^P(z_m) \subset \tilde{\mathscr{L}}_\nu^P(z)$. Recalling the definitions of $\alpha_\nu^P(z_m)$ and $\tilde{\alpha}_\nu^P(z)$, we conclude that $\tilde{\alpha}_\nu^P(z) \le \alpha_\nu^P(z_m)$. In particular, we get $\gamma\big(P, \tilde{\alpha}_\nu^P(z)\big) \subset \gamma\big(P, \alpha_\nu^P(z_m)\big)$. □

We are now able to specify the choice of the sequence $\{\varepsilon_l\}$:

**Lemma 1.3.5.** *If* $\{\varepsilon_l\}$ *is decreasing fast enough, then for every fixed* $\nu \in \mathbb{N}$ *and for every* $z \in S_\nu \cap B^{n-1}(0, \nu)$ *and* $P \subset \mathbb{N}_{n-1}$ *such that* $[\nu] \in P$*, there exists* $\delta \in (0, 1/\nu)$ *such that*

$$\frac{w' - w''}{\|\zeta' + 2\zeta + \zeta''\|} \ge \frac{\nu - 1}{1 + (C/2)} \quad \textit{for all} \quad w' \in \mathcal{E}_{z+(\zeta+\zeta')},\, w'' \in \mathcal{E}_{z-(\zeta+\zeta'')} \tag{1.6}$$

*and all choices of* $\zeta \in \gamma\big(P, \tilde{\alpha}_\nu^P(z)\big) \cap bB^{n-1}(0, \delta)$ *and* $\zeta', \zeta'' \in \Delta^{n-1}\big(0, (C/2)|\zeta|\big)$. *Moreover,* $\frac{1}{9}(\varepsilon_l/\varepsilon_{l+1})^2 > l$ *and* $\varepsilon_l\sqrt{|z_{[l]} - a_l|} < \frac{1}{2^l}$ *on* $B^{n-1}(0, l)$.

**Proof.** We proceed by induction on $l$ and simultaneously choose a sequence $(D_l)$ of finite subsets $D_l = \{\delta_l^1, \ldots, \delta_l^{d_l}\} \subset (0, 1/l)$ such that $\varepsilon_l\sqrt{|z_{[l]} - a_l|} < \frac{1}{2^l}\min\{\delta' \in D_\nu : 1 \le \nu \le l - 1\}$ for every $z \in B^{n-1}(0, l + 1)$. First let $\varepsilon_1 := 1$

and let $D_1 \subset (0,1)$ be the set provided by Corollary 1.3.1 in the case $\nu, \delta = 1$. If $\varepsilon_1, \dots, \varepsilon_l$ and $D_1, \dots, D_l$ have already been chosen, we choose $\varepsilon_{l+1} > 0$ so small that $\frac{1}{9}(\varepsilon_l/\varepsilon_{l+1})^2 > l$ and $\varepsilon_{l+1}\sqrt{|z_{[l+1]} - a_{l+1}|} < \frac{1}{2^{l+1}} \min\{\delta' \in D_\nu : 1 \leq \nu \leq l\}$ for $z \in B^{n-1}(0, l+2)$. Observe that every $\varepsilon'_{l+1} \in (0, \varepsilon_{l+1})$ would also be a proper choice for $\varepsilon_{l+1}$. We then take for $D_{l+1}$ the set provided by Corollary 1.3.1 in the case $\nu = l+1$ and $\delta = 1/(l+1)$.

Fix $\nu \in \mathbb{N}$, $z \in S_\nu \cap B^{n-1}(0,\nu)$ and $P \subset \mathbb{N}_{n-1}$ such that $[\nu] \in P$. Then by choice of $D_\nu$, there exists $\delta \in D_\nu$ such that estimate (1.5) holds true for all $j, k \in \mathbb{N}_{2^\nu}$ and all considered $\zeta, \zeta', \zeta''$. By choice of the sequence $(\varepsilon_l)$, if for abbrevation we write $z^+ := z + (\zeta + \zeta')$ and $z^- := z - (\zeta + \zeta'')$, we thus get the following estimate for all $\mu > \nu$ and $j', k' \in \mathbb{N}_{2^\mu}$ (for suitable $j, k \in \mathbb{N}_{2^\nu}$ depending on $j', k'$):

$$\frac{\left|w^{(\mu)}_{j'}\big(z + (\zeta + \zeta')\big) - w^{(\mu)}_{k'}\big(z - (\zeta + \zeta'')\big)\right|}{\|\zeta' + 2\zeta + \zeta''\|} \geq \frac{\left|w^{(\mu)}_{j'}(z^+) - w^{(\mu)}_{k'}(z^-)\right|}{(2+C)\|\zeta\|}$$

$$\geq \frac{\left|w^{(\nu)}_{j}(z^+) - w^{(\nu)}_{k}(z^-)\right|}{(2+C)\|\zeta\|} - \frac{1}{(2+C)\|\zeta\|} \sum_{l=\nu+1}^{\mu} \varepsilon_l \Big(\sqrt{|z^+_{[l]} - a_l|} + \sqrt{|z^-_{[l]} - a_l|}\Big)$$

$$\geq \frac{\nu}{1 + (C/2)} - \frac{1}{(2+C)\,\delta} \sum_{l=\nu+1}^{\mu} \frac{\delta}{2^{l-1}} \geq \frac{\nu - 1}{1 + (C/2)}.$$

Since by Lemma 1.1.2 each $(z,w) \in \mathcal{E}$ is a limit of points $\big(z, w^{(\mu)}_{j_\mu}\big)$, this proves (1.6). □

**Lemma 1.3.6.** *If $\{\varepsilon_l\}$ is decreasing fast enough, then $\mathcal{E}$ contains no analytic variety of positive dimension.*

**Proof.** Let $\{\varepsilon_l\}$ be decreasing so fast that the assertions of Lemma 1.3.5 hold true. To get a contradiction, assume that $\mathcal{E}$ contains an analytic variety of positive dimension. Then in particular $\mathcal{E}$ contains a nonconstant analytic disc, i.e., there exists a nonconstant holomorphic mapping $f = (f_1, f_2, \dots, f_n)\colon \mathbb{D}_r(0) \to \mathbb{C}^n$ such that $f\big(\mathbb{D}_r(0)\big) \subset \mathcal{E}$, where $\mathbb{D}_r(\xi_0) = \{\xi \in \mathbb{C} : |\xi - \xi_0| < r\}$. Let $P \subset \mathbb{N}_{n-1}$ be the set of all coordinate directions in $\mathbb{C}^{n-1}$ such that $f_p$ is not constant. Since by the choice of $\{\varepsilon_l\}$ and Lemma 1.1.2 the set $\mathcal{E}_z$ has zero 2-dimensional Lebesgue measure for every $z \in \mathbb{C}^{n-1}$, we see that $P \neq \varnothing$. Without loss of generality, we can assume that $P = \{1, \dots, T\}$ for some $T \leq n-1$. After possibly passing to a subset $\mathbb{D}_{r'}(\xi_0) \subset \mathbb{D}_r(0)$, we can assume by the implicit function theorem that there exist an open subset $U \subset \mathbb{C}$ and some

$$\phi\colon U \to \mathbb{C}^n \text{ holomorphic}, \;\; \phi(U) = f\big(\mathbb{D}_{r'}(\xi_0)\big)$$

such that $\phi(\xi) = \big(\xi, \phi_2(\xi), \ldots, \phi_T(\xi), q_{T+1}, \ldots, q_{n-1}, \phi_n(\xi)\big) =: \big(\phi_*(\xi), \phi_n(\xi)\big)$ with suitable constants $q_{T+1}, \ldots, q_{n-1} \in \mathbb{C}$. After a possible shrinking of $U$, we can assume that there exist positive numbers $\sigma, \theta > 0$ such that on $U$

$$\theta < |\phi_p'| \quad \text{for all } p \in P, \qquad |\phi_p'| < \sigma \quad \text{for all } p \in \mathbb{N}_n. \tag{1.7}$$

Indeed, $\theta$ exists since the zero set of each $|\phi_p'|$ is discret, and we use Cauchy's estimates to find $\sigma$. Thus, after possibly shrinking $U$ again, we can assume that for $z, z' \in \phi_*(U)$ and $1 \leq s, t \leq T$ we have $\theta < |z_t' - z_t|/|z_1' - z_1|$ and $|z_s' - z_s|/|z_1' - z_1| < \sigma$, i.e., $|z_s' - z_s|/|z_t' - z_t| < \sigma/\theta$. In particular, we see that there exists $\alpha := \sigma/\theta > 1$ such that

$$D\phi_*(z_1)(\mathbb{C}) \subset \gamma(P, \alpha) \quad \text{for all } z_1 \in U.$$

Moreover (after possibly further shrinking $U$), we can assume that for every $p \in \mathbb{N}_n$

$$\frac{\big|\phi_p(a+\xi) - \phi_p(a) - \phi_p'(a)\xi\big|}{|\xi|} < (C/2)\,\theta \qquad \begin{array}{l}\text{for all } a \in U,\ \xi \in \mathbb{C}\\ \text{such that } a + \xi \in U.\end{array} \tag{1.8}$$

Since we can assume $f\big(\mathbb{D}_r(0)\big)$ to be bounded, and since $\frac{1}{9}(\varepsilon_l/\varepsilon_{l+1})^2 > l$, we can choose $\nu_0 \in \mathbb{N}$ so large that $\phi_*(U) \subset B^{n-1}(0, \nu_0)$,

$$\nu_0 + 1 > \alpha \quad \text{and} \quad \tfrac{1}{9}(\varepsilon_l/\varepsilon_{l+1})^2 > \alpha \ \text{ for all } \ l \geq \nu_0. \tag{1.9}$$

Further, since $\max_{1 \leq p \leq n-1} \operatorname{vol}\big(\bigcup_{l \in L_\nu^p} \Delta^1(a_l, \rho_\nu)\big) \to 0$ for $\nu \to \infty$ and $\{\rho_\nu\}$ is decreasing, we can assume (after possibly enlarging $\nu_0$ and then shrinking of $U$) that $\phi_p(U) \cap \bigcup_{l \in L_{\nu_0}^p} \Delta^1(a_l, \rho_\nu) = \varnothing$ for all $p \in P$ and $\nu \geq \nu_0$. But then $\tilde{\mathscr{L}}_\nu^P(z) \cap \mathbb{N}_{\nu_0} = \varnothing$ for all $\nu \geq \nu_0$, $z \in S_\nu \cap \phi_*(U)$, $[\nu] \in P$. By definition of $\tilde{\alpha}_\nu^P(z)$ and from (1.9), we therefore get

$$\tilde{\alpha}_\nu^P(z) > \alpha \quad \text{for all } \ \nu \geq \nu_0,\ z \in S_\nu \cap \phi_*(U),\ [\nu] \in P.$$

After these preparations, we now choose a strictly increasing sequence $\{\nu_k\}$ of natural numbers such that for each $\nu$ from this sequence we have

$$\nu \geq \nu_0, \quad [\nu] = 1, \quad B^{n-1}\big(\phi_*(a_\nu), 1/\nu\big) \subset U \times \mathbb{C}^{n-2}.$$

Let $\nu$ be an arbitrary fixed member of this sequence. Since $\phi_*(U) \subset B^{n-1}(0, \nu_0)$, we see that $z := \phi_*(a_\nu) \in S_\nu \cap B^{n-1}(0, \nu)$. Hence we can use Lemma 1.3.5 to find a $\delta \in (0, 1/\nu)$ such that

$$\frac{w' - w''}{\|\zeta' + 2\zeta + \zeta''\|} \geq \frac{\nu - 1}{1 + (C/2)} \quad \text{for all} \quad w' \in \mathcal{E}_{z+(\zeta+\zeta')},\ w'' \in \mathcal{E}_{z-(\zeta+\zeta'')}$$

and all choices of

$$\zeta \in \gamma\big(P, \tilde{\alpha}_\nu^P(z)\big) \cap bB^{n-1}(0,\delta) \quad \text{and} \quad \zeta', \zeta'' \in \Delta^{n-1}\big(0, (C/2)|\zeta|\big). \tag{1.10}$$

By the choice of $U$ and $\nu_0$, we have $D\phi_*(a_\nu)(\xi - a_\nu) \in \gamma(P,\alpha)$ for all $\xi \in U \setminus \{a_\nu\}$ and $\tilde{\alpha}_\nu^P(z) > \alpha$; hence $D\phi_*(a_\nu)(\xi - a_\nu) \in \gamma\big(P, \tilde{\alpha}_\nu^P(z)\big)$. Moreover, $B^{n-1}(z,\delta) \subset U \times \mathbb{C}^{n-2}$. Thus

$$\begin{aligned}\Sigma :=& \big[z + \gamma\big(P, \tilde{\alpha}_\nu^P(z)\big)\big] \\ &\cap \big[z + \{D\phi_*(a_\nu)(\xi - a_\nu) : \xi \in \mathbb{C} \setminus \{0\} \text{ such that } a_\nu + \xi \in U\}\big] \cap bB^{n-1}(z,\delta)\end{aligned}$$

is nonempty. Therefore we can choose $\zeta \in \gamma\big(P, \tilde{\alpha}_\nu^P(z)\big)$ such that $z \pm \zeta \in \Sigma$, and $\xi \in \mathbb{C}$ such that $a_\nu \pm \xi \in U$ and $D\phi_*(a_\nu)(\pm\xi) = \pm\zeta$. Now applying (1.8) in the case $a = a_\nu$ and using (1.7) yields

$$\big|\phi_p(a_\nu + \xi) - z_p - \zeta_p\big| < (C/2)\,\theta|\xi| < (C/2)|\phi_p'(a_\nu)\xi| = (C/2)|\zeta_p|$$

for every $p \in P$. Since also $\phi_p(a_\nu + \xi) = z_p + \zeta_p$ for $p \in \mathbb{N}_{n-1} \setminus P$ and $\phi_1(z_1) = z_1$, this shows that there exist uniquely determined $\zeta', \zeta'' \in \Delta^{n-1}\big(0, (C/2)|\zeta|\big)$ such that $z + (\zeta + \zeta') = \phi_*(a_\nu + \xi)$, $z - (\zeta + \zeta'') = \phi_*(a_\nu - \xi)$ and $\zeta_1', \zeta_1'' = 0$. In particular, we see from $\phi(U) \subset f\big(\mathbb{D}_{r'}(\xi_0)\big) \subset f\big(\mathbb{D}_r(0)\big) \subset \mathcal{E}$ that

$$w := \phi_n(a_\nu + \xi) \in \mathcal{E}_{z+(\zeta+\zeta')}, \qquad w' := \phi_n(a_\nu - \xi) \in \mathcal{E}_{z-(\zeta+\zeta'')}.$$

Observe that $\zeta, \zeta'\zeta''$ satisfy the conditions in (1.10). Since $a_\nu \in U$ and $\phi_1' \equiv 1$ on $U$, we get $\|\zeta' + 2\zeta + \zeta''\| \geq (2 - C)\|\zeta\| = (2 - C)\|D\phi_*(a_\nu)(\xi)\| \geq (2 - C)|\xi|$ and thus, in view of Lemma 1.3.5, can finally make the following estimate:

$$\frac{\big|\phi_n(a_\nu + \xi) - \phi_n(a_\nu - \xi)\big|}{2|\xi|} \geq (1 - C/2) \cdot \frac{|w - w'|}{\|\zeta' + 2\zeta + \zeta''\|} \geq \frac{1 - C/2}{1 + C/2} \cdot (\nu - 1)\,.$$

This holds true for every member $\nu$ of the strictly increasing sequence $(\nu_k)$, and the right term becomes unbounded as $\nu \to +\infty$. Since for each fixed $\nu$ the number $\xi$ was chosen such that $a_\nu \pm \xi \in U$, this contradicts the fact that $\phi_n$ has a bounded derivate on $U$. □

## 1.4 Choice of the sequence $\{\varepsilon_l\}$ - Part II. Complete pluripolarity

Recall that $E_\nu = \{P_\nu = 0\}$, $\nu \in \mathbb{N}$. We show that for $\{\varepsilon_l\}$ decreasing fast enough we can guarantee nice convergence properties of the sequence $\{P_\nu\}$ as well as certain relations between the limit set $\mathcal{E}$ of $\{E_\nu\}$ and the sublevel sets of the defining polynomials $P_\nu$. These results will then be seen to imply complete pluripolarity of $\mathcal{E}$.

**Lemma 1.4.1.** *The sequence* $\{|P_\nu|^{1/2^\nu}\}$ *converges uniformly on compact subsets of* $\mathbb{C}^n \setminus \mathcal{E}$*, and* $\lim_{\nu\to\infty}|P_\nu|^{1/2^\nu} > 0$ *on* $\mathbb{C}^n \setminus \mathcal{E}$*.*

**Proof.** Recall that $\{\varepsilon_l\}$ is chosen in such a way that $\varepsilon_l\sqrt{|z_{[l]} - a_l|} < 1/2^l$ on $B^{n-1}(0,l)$ for every $l \in \mathbb{N}$. Fix $(z_0, w_0) \in \mathbb{C}^n \setminus \mathcal{E}$ and choose $R > 0$ such that $(z_0, w_0) \in \Delta_R := B^{n-1}(0,R) \times \mathbb{C}$. Since $\mathcal{E}$ is closed and $E_\nu \cap \bar{\Delta}_R \to \mathcal{E} \cap \bar{\Delta}_R$ in the Hausdorff metric, there exist a ball $B := B^n\big((z_0,w_0),\delta\big) \subset \Delta_R$ and positive numbers $r > 0$, $N_r > 0$ such that $\operatorname{dist}(B, E_\nu) > r$ for all $\nu \geq N_r$. Now for every $\nu, \mu \in \mathbb{N}$, $j \in \mathbb{N}_{2^\nu}$ and $z \in \mathbb{C}^{n-1}$ we denote the $2^\mu$ values of $w_j^{(\nu)}(z) + \sum_{l=\nu+1}^{\nu+\mu} \varepsilon_l\sqrt{z_{[l]} - a_l}$ by $w_1^{(\mu)}(\nu, j; z), \ldots, w_{2^\mu}^{(\mu)}(\nu, j; z)$. Observe that with this notation we have

$$|P_{\nu+\mu}(z,w)|^{1/2^{\nu+\mu}} = \prod_{l=1}^{2^{\nu+\mu}} |w - w_l^{(\nu+\mu)}(z)|^{1/2^{\nu+\mu}} = \prod_{j=1}^{2^\nu}\prod_{k=1}^{2^\mu} |w - w_k^{(\mu)}(\nu,j;z)|^{1/2^{\nu+\mu}};$$

thus passing from $|P_\nu(z,w)|^{1/2^\nu}$ to $|P_{\nu+\mu}(z,w)|^{1/2^{\nu+\mu}}$ amounts to replace each term $|w - w_j^{(\nu)}(z)|$ occuring in the product expansion of $|P_\nu(z,w)|^{1/2^\nu}$ by the mean value $\prod_{k=1}^{2^\mu}|w - w_k^{(\mu)}(\nu,j;z)|^{1/2^\mu}$. Since for $\nu \geq R$ one has $|w_j^{(\nu)}(z) - w_k^{(\mu)}(\nu,j;z)| \leq \sum_{l=\nu+1}^{\nu+\mu} \varepsilon_l\sqrt{|z_{[l]} - a_l|} < 1/2^\nu$ for all $z \in B^{n-1}(0,R)$, we can estimate the resulting error, by means of

$$\prod_{k=1}^{2^\mu}|w - w_k^{(\mu)}(\nu,j;z)|^{1/2^\mu} > \prod_{k=1}^{2^\mu}\Big(|w - w_j^{(\nu)}(z)| - 1/2^\nu\Big)^{1/2^\mu} = |w - w_j^{(\nu)}(z)| - 1/2^\nu,$$

$$\prod_{k=1}^{2^\mu}|w - w_k^{(\mu)}(\nu,j;z)|^{1/2^\mu} < \prod_{k=1}^{2^\mu}\Big(|w - w_j^{(\nu)}(z)| + 1/2^\nu\Big)^{1/2^\mu} = |w - w_j^{(\nu)}(z)| + 1/2^\nu,$$

to be less than $1/2^\nu$ for all $(z,w) \in B \subset B^{n-1}(0,R) \times \mathbb{C}$ (obviously the first inequality is trivial if $|w - w_j^{(\nu)}(z)| < 1/2^\nu$). In particular, whenever $|w - w_j^{(\nu)}(z)| \geq 1/2^\nu$ on $B$ and $\nu \geq R$, we get

$$\prod_{j=1}^{2^\nu}\Big(|w - w_j^{(\nu)}(z)| - 1/2^\nu\Big)^{1/2^\nu} \leq |P_{\nu+\mu}(z,w)|^{1/2^{\nu+\mu}} \leq \prod_{j=1}^{2^\nu}\Big(|w - w_j^{(\nu)}(z)| + 1/2^\nu\Big)^{1/2^\nu}$$

on $B$. But $|w - w_j^{(\nu)}(z)| > r$ on $B$ for all $\nu \geq N_r$, where $r$ does not depend on $\nu$. Since $|P_\nu(z,w)| = \prod_{j=1}^{2^\nu}|w - w_j^{(\nu)}(z)|$, this shows that $\big\{|P_\nu(z,w)|^{1/2^\nu}\big\}_{\nu\geq 1}$ is a

Cauchy sequence for every $(z, w) \in B$ and in fact that $\left\{|P_\nu|^{1/2^\nu}\right\}_{\nu \geq 1}$ converges uniformly on $B$. Moreover, $\lim_{\nu\to\infty}|P_\nu|^{1/2^\nu} > 0$ on $B$, since the above estimates hold true for all $\mu \in \mathbb{N}$. $\square$

**Lemma 1.4.2.** *If $\{\varepsilon_l\}$ is decreasing fast enough, then*

$$\mathcal{E} = \bigcap_{\nu\in\mathbb{N}} \bigcup_{\mu\geq\nu} \{|P_\mu| < (\tfrac{1}{\mu})^{2^\mu}\}. \tag{1.11}$$

*Moreover, the following relations hold true for every $\mu \geq \nu \geq R$:*

*(1)* $\{|P_\mu| < (\frac{1}{\nu+1})^{2^\mu}\} \cap \bar{B}^n(0, R) \subset\subset \{|P_\nu| < (\frac{1}{\nu})^{2^\nu}\}$.

*(2)* $\{|P_\nu| < (\frac{1}{\nu})^{2^\nu}\} \cap \bar{B}^n(0, R) \subset\subset \{|P_\mu| < (\frac{1}{\nu-1})^{2^\mu}\}$.

**Proof.** For $M \subset \mathbb{C}^n$ and $R, \delta > 0$ we let $M_{(R)} := M \cap \bar{B}^n(0, R)$ and

$$M^{\langle\delta\rangle} := M \cup \bigcup_{x\in bM} B^n(x, \delta) \quad \text{and} \quad M^{\langle-\delta\rangle} := M \setminus \bigcup_{x\in bM} B^n(x, \delta).$$

One easily verifies the following relations for all $M, N \subset \mathbb{C}^n$ and $R, \delta, \delta_1, \delta_2 > 0$:

(A) $M \subset N \Rightarrow M^{\langle\delta\rangle} \subset N^{\langle\delta\rangle}$ and $M \subset N \Rightarrow M^{\langle-\delta\rangle} \subset N^{\langle-\delta\rangle}$.

(B) $M_{(R)} \subset N \Rightarrow M_{(R)} \subset\subset N^{\langle\delta\rangle}$ and $M_{(R)} \subset N \Rightarrow [M^{\langle-\delta\rangle}]_{(R)} \subset\subset N$.

(C) $[M^{\langle\delta_1\rangle}]^{\langle\delta_2\rangle} = M^{\langle\delta_1+\delta_2\rangle}$ and $[M^{\langle-\delta_1\rangle}]^{\langle-\delta_2\rangle} = M^{\langle-(\delta_1+\delta_2)\rangle}$.

(D) $[M^{\langle\delta\rangle}]_{(R-\delta)} \subset [M_{(R)}]^{\langle\delta\rangle}$ and $[M^{\langle-\delta\rangle}]_{(R-\delta)} \subset [M_{(R)}]^{\langle-\delta\rangle}$.

Moreover, $M_{(R)}^{\langle\pm\delta\rangle}$ will denote the set $M^{\langle\pm\delta\rangle} \cap \bar{B}^n(0, R)$. We can choose sequences $\{\varepsilon_l\}, \{\delta_l\}$ of positive numbers converging to zero such that for all $\nu \in \mathbb{N}$ the following relations hold true:

$(1_\nu)$ $\quad \varepsilon_\nu\sqrt{|z_{[\nu]} - a_\nu|} < \frac{1}{2^\nu}$ on $B^{n-1}(0, \nu)$.

$(2_\nu)$ $\quad \left[\left\{|P_\nu| < (\frac{1}{\nu+1})^{2^\nu}\right\}_{(\nu+1)} \cup \left\{|P_\nu| > (\frac{1}{\nu-1})^{2^\nu}\right\}_{(\nu+1)}\right] \cap \left\{|P_\nu| = (\frac{1}{\nu})^{2^\nu}\right\}^{\langle\delta_\nu\rangle} = \varnothing.$

$(3_{\nu+1})$ $\quad \left\{|P_{\nu+1}| < (\frac{1}{\lambda})^{2^{\nu+1}}\right\}_{(\nu+1)} \subset \left\{|P_\nu| < (\frac{1}{\lambda})^{2^\nu}\right\}^{\langle\delta_\nu/2^\nu\rangle}$ for $\lambda = 1, \ldots, \nu + 1$.

$(3'_{\nu+1})$ $\quad \left\{|P_\nu| < (\frac{1}{\lambda})^{2^\nu}\right\}^{\langle-\delta_\nu/2^\nu\rangle}_{(\nu+1)} \subset \left\{|P_{\nu+1}| < (\frac{1}{\lambda})^{2^{\nu+1}}\right\}$ for $\lambda = 1, \ldots, \nu - 1$.

Indeed, we can choose $\varepsilon_1$ to satisfy $(1_1)$. After fixing such $\varepsilon_1$, the polynomial $P_1$ is fixed, and we can choose $\delta_1 < 1/2$ to satisfy $(2_1)$. Suppose now that $\varepsilon_l, \delta_l$

are already chosen for $l = 1, 2, \ldots, \nu$ such that $(1_\nu)$-$(3'_\nu)$ hold true. By Lemma 1.1.1, we know that $P_{\nu+1} \to P_\nu^2$ uniformly on compact subsets as $\varepsilon_{\nu+1} \to 0$; hence we can find $\varepsilon > 0$ such that for $\varepsilon_{\nu+1} < \varepsilon$ the polynomial $P_{\nu+1}$ satisfies $(3_{\nu+1})$ and $(3'_{\nu+1})$. Moreover, we can find $\varepsilon' > 0$ such that for $\varepsilon_{\nu+1} < \varepsilon'$ the inequality $(1_{\nu+1})$ holds true. We choose $\varepsilon_{\nu+1} < \min\{\varepsilon, \varepsilon'\}$, and we point out that every $\varepsilon'_{\nu+1} \in (0, \varepsilon_{\nu+1})$ would also be a proper choice for $\varepsilon_{\nu+1}$. For $P_{\nu+1}$ now being fixed, we can find $\delta_{\nu+1} < \delta_\nu/2$ satisfying $(2_{\nu+1})$.

(i) *We now prove statement (1) of the lemma. In order to do this, we need the following*

Claim 1. For $\mu > \nu \geq R$, one has

$$\left\{|P_\mu| < (\tfrac{1}{\nu+1})^{2^\mu}\right\}_{(R)} \subset \left\{|P_\nu| < (\tfrac{1}{\nu+1})^{2^\nu}\right\}^{\langle \sum_{l=\nu}^{\mu-1} \delta_l/2^l \rangle}.$$

Proof. Let $\mu > \nu \geq R$ be fixed. For proving the statement of the claim, we use reverse induction on $\rho$ to show that

$$\left\{|P_\mu| < (\tfrac{1}{\nu+1})^{2^\mu}\right\}_{(R)} \subset \left\{|P_\rho| < (\tfrac{1}{\nu+1})^{2^\rho}\right\}^{\langle \sum_{l=\rho}^{\mu-1} \delta_l/2^l \rangle} \quad \text{for } \rho = \mu-1, \ldots, \nu. \tag{1.12}$$

The case $\rho = \mu - 1$ follows immediately from $(3_\mu)$ with $\lambda = \nu + 1$. Suppose that property (1.12) holds for some $\rho \in \mathbb{N}$ such that $\mu > \rho > \nu \geq R$. Then one also has

$$\left\{|P_\mu| < (\tfrac{1}{\nu+1})^{2^\mu}\right\}_{(R)} \subset \left\{|P_\rho| < (\tfrac{1}{\nu+1})^{2^\rho}\right\}_{(R)}^{\langle \sum_{l=\rho}^{\mu-1} \delta_l/2^l \rangle}. \tag{1.13}$$

Hence applying $(3_\rho)$ with $\lambda = \nu + 1$, we can conclude that

$$\begin{aligned}
&\left\{|P_\rho| < (\tfrac{1}{\nu+1})^{2^\rho}\right\}_{(\rho)} \subset \left\{|P_{\rho-1}| < (\tfrac{1}{\nu+1})^{2^{\rho-1}}\right\}^{\langle \delta_{\rho-1}/2^{\rho-1} \rangle} \\
&\quad \Rightarrow \left[\left\{|P_\rho| < (\tfrac{1}{\nu+1})^{2^\rho}\right\}_{(\rho)}\right]^{\langle \sum_{l=\rho}^{\mu-1} \delta_l/2^l \rangle} \\
&\qquad\qquad \subset \left[\left\{|P_{\rho-1}| < (\tfrac{1}{\nu+1})^{2^{\rho-1}}\right\}^{\langle \delta_{\rho-1}/2^{\rho-1} \rangle}\right]^{\langle \sum_{l=\rho}^{\mu-1} \delta_l/2^l \rangle} \\
&\quad \Rightarrow \left[\left\{|P_\rho| < (\tfrac{1}{\nu+1})^{2^\rho}\right\}^{\langle \sum_{l=\rho}^{\mu-1} \delta_l/2^l \rangle}\right]_{(\rho - \sum_{l=\rho}^{\mu-1} \delta_l/2^l)} \\
&\qquad\qquad \subset \left\{|P_{\rho-1}| < (\tfrac{1}{\nu+1})^{2^{\rho-1}}\right\}^{\langle \sum_{l=\rho-1}^{\mu-1} \delta_l/2^l \rangle} \\
&\quad \Rightarrow \left[\left\{|P_\rho| < (\tfrac{1}{\nu+1})^{2^\rho}\right\}^{\langle \sum_{l=\rho}^{\mu-1} \delta_l/2^l \rangle}\right]_{(R)} \\
&\qquad\qquad \subset \left\{|P_{\rho-1}| < (\tfrac{1}{\nu+1})^{2^{\rho-1}}\right\}^{\langle \sum_{l=\rho-1}^{\mu-1} \delta_l/2^l \rangle}.
\end{aligned}$$

This and (1.13) completes our argument by induction and proves Claim 1.

Observe that, since $\{\delta_l\}$ is monotonically decreasing, we get from Claim 1 and $(B)$ the following property:

$$\big\{|P_\mu| < (\tfrac{1}{\nu+1})^{2^\mu}\big\}_{(R)} \subset\subset \big\{|P_\nu| < (\tfrac{1}{\nu+1})^{2^\nu}\big\}^{\langle\delta_\nu\rangle}. \tag{1.14}$$

Fix now some $\nu \geq R$. We are going to show that

$$\big\{|P_\nu| < (\tfrac{1}{\nu+1})^{2^\nu}\big\}^{\langle\delta_\nu\rangle}_{(R)} \subset \big\{|P_\nu| < (\tfrac{1}{\nu})^{2^\nu}\big\}. \tag{1.15}$$

Note that (1.14) and (1.15) together prove (1). By definition, we have

$$\big\{|P_\nu| < (\tfrac{1}{\nu+1})^{2^\nu}\big\}^{\langle\delta_\nu\rangle}_{(R)} = \big\{|P_\nu| < (\tfrac{1}{\nu+1})^{2^\nu}\big\}_{(R)} \cup \bigcup_{x\in b\{|P_\nu|<(\frac{1}{\nu+1})^{2^\nu}\}} B^n(x,\delta_\nu)_{(R)}$$

Obviously,

$$\big\{|P_\nu| < (\tfrac{1}{\nu+1})^{2^\nu}\big\}_{(R)} \subset \big\{|P_\nu| < (\tfrac{1}{\nu})^{2^\nu}\big\}.$$

Let $\zeta \in B^n(x,\delta_\nu)_{(R)}$ for some $x \in b\{|P_\nu| < (\frac{1}{\nu+1})^{2^\nu}\}$. Then in particular $x \in \{|P_\nu| = (\frac{1}{\nu+1})^{2^\nu}\}_{(\nu+1)}$. Assume, to get a contradiction, that $\zeta \in \{|P_\nu| \geq (\frac{1}{\nu})^{2^\nu}\}$. Since $x \in \{|P_\nu| < (\frac{1}{\nu})^{2^\nu}\}_{(\nu+1)}$, we then can find $t \in (0,1]$ such that $\tilde{x} := (1-t)x + t\zeta \in \{|P_\nu| = (\frac{1}{\nu})^{2^\nu}\}$. Now obviously $\|\tilde{x} - x\| < \delta_\nu$, which shows that $x \in \{|P_\nu| = (\frac{1}{\nu+1})^{2^\nu}\}_{(\nu+1)} \cap \{|P_\nu| = (\frac{1}{\nu})^{2^\nu}\}^{\langle\delta_\nu\rangle}$. In particular, we conclude that $\{|P_\nu| < (\frac{1}{\nu+1})^{2^\nu}\}_{(\nu+1)} \cap \{|P_\nu| = (\frac{1}{\nu})^{2^\nu}\}^{\langle\delta_\nu\rangle} \neq \varnothing$, which contradicts $(2_\nu)$. This proves that

$$\bigcup_{x\in b\{|P_\nu|<(\frac{1}{\nu+1})^{2^\nu}\}} B^n(x,\delta_\nu)_{(R)} \subset \big\{|P_\nu| < (\tfrac{1}{\nu})^{2^\nu}\big\},$$

and hence (1.15). The proof of statement (1) of the lemma is now complete.

(ii) *We now prove statement (2) of the lemma. For being able to do this, we need the following*

Claim 2. For $\mu > \nu \geq R$, one has

$$\big\{|P_\nu| < (\tfrac{1}{\nu-1})^{2^\nu}\big\}^{\langle -\sum_{l=\nu}^{\mu-1}\delta_l/2^l\rangle}_{(R)} \subset \big\{|P_\mu| < (\tfrac{1}{\nu-1})^{2^\mu}\big\}.$$

Proof. Let $\mu > \nu \geq R$ be fixed. For proving the statement of the claim, we use induction on $\rho$ to show that

$$\big\{|P_\nu| < (\tfrac{1}{\nu-1})^{2^\nu}\big\}^{\langle -\sum_{l=\nu}^{\rho-1}\delta_l/2^l\rangle}_{(\nu+1-\sum_{l=\nu}^{\rho-1}\delta_l/2^l)} \subset \big\{|P_\rho| < (\tfrac{1}{\nu-1})^{2^\rho}\big\}, \quad \text{for } \rho = \nu+1,\dots,\mu. \tag{1.16}$$

The case $\rho = \nu + 1$ follows immediately from $(3'_{\nu+1})$ with $\lambda = \nu - 1$. Suppose that property (1.16) holds for some $\rho \in \mathbb{N}$ such that $\mu > \rho > \nu \geq R$. Then we also have

$$\begin{aligned}
&\Big[\big\{|P_\nu| < (\tfrac{1}{\nu-1})^{2^\nu}\big\}^{\langle -\sum_{l=\nu}^{\rho-1} \delta_l/2^l\rangle}\Big]_{(\nu+1-\sum_{l=\nu}^{\rho-1} \delta_l/2^l)} \subset \big\{|P_\rho| < (\tfrac{1}{\nu-1})^{2^\rho}\big\} \\
&\quad\Rightarrow \bigg[\Big[\big\{|P_\nu| < (\tfrac{1}{\nu-1})^{2^\nu}\big\}^{\langle -\sum_{l=\nu}^{\rho-1} \delta_l/2^l\rangle}\Big]_{(\nu+1-\sum_{l=\nu}^{\rho-1} \delta_l/2^l)}\bigg]^{\langle -\delta_\rho/2^\rho\rangle} \\
&\qquad\qquad \subset \big\{|P_\rho| < (\tfrac{1}{\nu-1})^{2^\rho}\big\}^{\langle -\delta_\rho/2^\rho\rangle} \\
&\quad\Rightarrow \Big[\big\{|P_\nu| < (\tfrac{1}{\nu-1})^{2^\nu}\big\}^{\langle -\sum_{l=\nu}^{\rho} \delta_l/2^l\rangle}\Big]_{(\nu+1-\sum_{l=\nu}^{\rho} \delta_l/2^l)} \\
&\qquad\qquad \subset \big\{|P_\rho| < (\tfrac{1}{\nu-1})^{2^\rho}\big\}^{\langle -\delta_\rho/2^\rho\rangle} \\
&\quad\Rightarrow \Big[\big\{|P_\nu| < (\tfrac{1}{\nu-1})^{2^\nu}\big\}^{\langle -\sum_{l=\nu}^{\rho} \delta_l/2^l\rangle}\Big]_{(\nu+1-\sum_{l=\nu}^{\rho} \delta_l/2^l)} \\
&\qquad\qquad \subset \big\{|P_\rho| < (\tfrac{1}{\nu-1})^{2^\rho}\big\}^{\langle -\delta_\rho/2^\rho\rangle}_{(\nu+1)}
\end{aligned}$$

while from $(3'_{\rho+1})$ with $\lambda = \nu - 1$, we get

$$\big\{|P_\rho| < (\tfrac{1}{\nu-1})^{2^\rho}\big\}^{\langle -\delta_\rho/2^\rho\rangle}_{(\nu+1)} \subset \big\{|P_{\rho+1}| < (\tfrac{1}{\nu-1})^{2^{\rho+1}}\big\}.$$

This completes our argument by induction and, since $\nu + 1 - \sum_{l=\nu}^{\mu-1} \delta_l/2^l > R$, proves Claim 2.

Observe that, since $\{\delta_l\}$ is monotonically decreasing, we get from Claim 2 and $(B)$ the following property:

$$\big\{|P_\nu| < (\tfrac{1}{\nu-1})^{2^\nu}\big\}^{\langle -\delta_\nu\rangle}_{(R)} \subset\subset \big\{|P_\mu| < (\tfrac{1}{\nu-1})^{2^\mu}\big\}. \tag{1.17}$$

Fix now some $\nu \geq R$. We are going to show that

$$\big\{|P_\nu| < (\tfrac{1}{\nu})^{2^\nu}\big\}_{(R)} \subset \big\{|P_\nu| < (\tfrac{1}{\nu-1})^{2^\nu}\big\}^{\langle -\delta_\nu\rangle}_{(R)}. \tag{1.18}$$

Note that (1.17) and (1.18) together prove (2). By definition, we have

$$\big\{|P_\nu| < (\tfrac{1}{\nu-1})^{2^\nu}\big\}^{\langle -\delta_\nu\rangle}_{(R)} = \big\{|P_\nu| < (\tfrac{1}{\nu-1})^{2^\nu}\big\}_{(R)} \setminus \bigcup_{x \in b\{|P_\nu| < (\frac{1}{\nu-1})^{2^\nu}\}} B^n(x, \delta_\nu)_{(R)}.$$

Obviously,

$$\big\{|P_\nu| < (\tfrac{1}{\nu})^{2^\nu}\big\}_{(R)} \subset \big\{|P_\nu| < (\tfrac{1}{\nu-1})^{2^\nu}\big\}_{(R)}.$$

Let $\zeta \in B^n(x, \delta_\nu)_{(R)}$ for some $x \in b\{|P_\nu| < (\frac{1}{\nu-1})^{2^\nu}\}$. Then, in particular,

$x \in \{|P_\nu| = (\frac{1}{\nu-1})^{2^\nu}\}_{(\nu+1)}$. In order to get a contradiction, assume that $\zeta \in \{|P_\nu| < (\frac{1}{\nu})^{2^\nu}\}_{(R)}$. Since $x \in \{|P_\nu| > (\frac{1}{\nu})^{2^\nu}\}$, we then can find $t \in (0,1)$ such that $\tilde{x} := (1-t)x + t\zeta \in \{|P_\nu| = (\frac{1}{\nu})^{2^\nu}\}$. Now obviously $\|\tilde{x} - x\| < \delta_\nu$, which shows that $x \in \{|P_\nu| = (\frac{1}{\nu-1})^{2^\nu}\}_{(\nu+1)} \cap \{|P_\nu| = (\frac{1}{\nu})^{2^\nu}\}^{\langle \delta_\nu \rangle}$. In particular, we conclude that $\{|P_\nu| > (\frac{1}{\nu-1})^{2^\nu}\}_{(\nu+1)} \cap \{|P_\nu| = (\frac{1}{\nu})^{2^\nu}\}^{\langle \delta_\nu \rangle} \neq \varnothing$, which contradicts $(2_\nu)$. This proves that

$$\left\{|P_\nu| < (\tfrac{1}{\nu})^{2^\nu}\right\}_{(R)} \cap \bigcup_{x \in b\{|P_\nu| < (\frac{1}{\nu-1})^{2^\nu}\}} B^n(x, \delta_\nu)_{(R)} = \varnothing,$$

and hence (1.18). The proof of statement (2) of the lemma is now complete.

(iii) *Finally, we show that the representation* (1.11) *holds true.*

Let $(z,w) \in \mathbb{C}^n$ and choose $R > 0$ such that $(z,w) \in B^n(0,R)$. Assume that $(z,w) \in \mathcal{E}$. Let $\mu \geq R$. Applying (1), we get

$$(E_{\mu+l})_{(R)} \subset \left\{|P_{\mu+l}| < (\tfrac{1}{\mu+l})^{2^{\mu+l}}\right\}_{(R)} \subset\subset \left\{|P_\mu| < (\tfrac{1}{\mu})^{2^\mu}\right\}$$

for all $l \in \mathbb{N}$. But since $(1_\rho)$ holds true for all $\rho \in \mathbb{N}$, we can apply Lemma 1.1.2 to see that $\mathcal{E}_{(R)} = \lim_{l\to\infty}(E_{\mu+l})_{(R)}$ in the Hausdorff metric. Hence $\mathcal{E}_{(R)} \subset \{|P_{\mu+1}| \leq (\frac{1}{\mu+1})^{2^{\mu+1}}\}_{(R)} \subset \{|P_\mu| < (\frac{1}{\mu})^{2^\mu}\}$. Since this holds true for all $\mu \geq R$, it follows $(z,w) \in \bigcap_{\nu\in\mathbb{N}} \bigcup_{\mu\geq\nu} \{|P_\mu| < (\frac{1}{\mu})^{2^\mu}\}$. Conversely, assume that $(z,w) \notin \mathcal{E}$. Then by Lemma 1.4.1, the sequence $\{|P_\nu(z,w)|^{1/2^\nu}\}$ is converging to a positive real number; hence there exist $\delta > 0$ and $\mu_0 \in \mathbb{N}$ such that $|P_\mu(z,w)|^{1/2^\mu} > \delta$ for all $\mu \geq \mu_0$. In particular, $(z,w) \notin \{|P_\mu| < (\frac{1}{\mu})^{2^\mu}\}$ for $\mu \geq \max\{\mu_0, 1/\delta\}$, which shows that $(z,w) \notin \bigcap_{\nu\in\mathbb{N}} \bigcup_{\mu\geq\nu} \{|P_\mu| < (\frac{1}{\mu})^{2^\mu}\}$. □

For each $\nu \in \mathbb{N}$, define a function $\varphi_\nu \colon \mathbb{C}^n \to [-\infty, +\infty)$ as

$$\varphi_\nu(z,w) := \frac{1}{2^\nu} \log|P_\nu(z,w)|.$$

Then $\varphi_\nu$ is a plurisubharmonic function on $\mathbb{C}^n$, pluriharmonic on $\mathbb{C}^n \setminus E_\nu$, and $\varphi_\nu(z,w) = -\infty$ if and only if $(z,w) \in E_\nu$.

**Lemma 1.4.3.** *If $\{\varepsilon_l\}$ is decreasing fast enough, then the sequence $\{\varphi_\nu\}$ converges uniformly on compact subsets of $\mathbb{C}^n \setminus \mathcal{E}$ to a pluriharmonic function $\varphi \colon \mathbb{C}^n \setminus \mathcal{E} \to \mathbb{R}$, and $\lim_{(z,w)\to(z_0,w_0)} \varphi(z,w) = -\infty$ for every $(z_0,w_0) \in \mathcal{E}$. In particular, $\varphi$ has a unique extension to a plurisubharmonic function on $\mathbb{C}^n$ and the set $\mathcal{E}$ is complete pluripolar.*

**Proof.** Let $\{\varepsilon_l\}$ to be converging to zero so fast that the conclusions of Lemma 1.4.2 hold true. Then, applying Lemma 1.4.1, we immediately see that $\{\varphi_\nu\}$ converges uniformly on compact subsets of $\mathbb{C}^n \setminus \mathcal{E}$. In particular, $\varphi$ is pluriharmonic in $\mathbb{C}^n \setminus \mathcal{E}$. Let $(z_0, w_0) \in \mathcal{E}$ and let $\{(z_j, w_j)\}_{j\geq 1}$ be an arbitrary sequence of points converging to $(z_0, w_0)$. Let $R \in \mathbb{N}$ be such that $(z_0, w_0) \in B^n(0, R)$. From part (1) of Lemma 1.4.2 we know that

$$\{|P_{\mu+1}| < (\tfrac{1}{\mu+1})^{2^{\mu+1}}\} \cap \bar{B}^n(0, R) \subset \{|P_\mu| < (\tfrac{1}{\mu})^{2^\mu}\}$$

for every $\mu \geq R$; thus it follows from

$$\mathcal{E} = \bigcap_{\nu\in\mathbb{N}} \bigcup_{\mu\geq\nu} \{|P_\mu| < (\tfrac{1}{\mu})^{2^\mu}\}$$

that $\mathcal{E} \cap \bar{B}^n(0, R) \subset \{|P_\nu| < (\frac{1}{\nu})^{2^\nu}\}$ for all $\nu \geq R$. Hence for every $\nu \geq R$ there exists $j(\nu) \in \mathbb{N}$ such that $(z_j, w_j) \in \{|P_\nu| < (\frac{1}{\nu})^{2^\nu}\} \cap B^n(0, R)$ for all $j \geq j(\nu)$. But whenever $(z_j, w_j) \in \{|P_\nu| < (\frac{1}{\nu})^{2^\nu}\} \cap \bar{B}^n(0, R)$ we know from part (2) of Lemma 1.4.2 that also $(z_j, w_j) \in \{|P_\mu| < (\frac{1}{\nu-1})^{2^\mu}\}$ for each $\mu \geq \nu$. This means that $\varphi_\mu(z_j, w_j) < -\log(\nu - 1)$ for each $\mu \geq \nu$. Hence $\varphi(z_j, w_j) \leq -\log(\nu - 1)$ for each $j \geq j(\nu)$. This shows that $\lim_{j\to\infty} \varphi(z_j, w_j) = -\infty$. □

Complete pluripolarity of the set $\mathcal{E}$ will play an important role when we are going to apply the existence of Wermer type sets to the study of global plurisubharmonic defining functions in Part II below. In fact, the precise property of $\mathcal{E}$ that will be needed is the existence of a smooth plurisubharmonic function $\Phi\colon \mathbb{C}^n \to [0, \infty)$ such that $\mathcal{E} = \{\Phi = 0\}$ and such that $\Phi$ is strictly plurisubharmonic outside $\mathcal{E}$. The next corollary shows that we can easily construct a function $\Phi$ as desired, by smoothing up $\varphi + \|\cdot\|^2$ along the set $\mathcal{E} = \{\varphi = -\infty\}$, where $\varphi$ is the function from the previous lemma.

**Corollary 1.4.1.** *If $\{\varepsilon_l\}$ is decreasing fast enough, then there exists a smooth plurisubharmonic function $\Phi\colon \mathbb{C}^n \to [0, \infty)$ such that $\mathcal{E} = \{\Phi = 0\}$ and such that $\Phi$ is strictly plurisubharmonic outside $\mathcal{E}$.*

**Proof.** For every $j \in \mathbb{N}$, let $\chi_j : \mathbb{R} \to \mathbb{R}$ be a smooth convex increasing function such that $\chi|_{(-\infty,-j)} \equiv -j$. Let $\{\eta_j\}_{j=1}^\infty$ be a sequence of positive numbers that converges to zero fast enough. Then $\Phi := \sum_{j=1}^\infty \eta_j \chi_j(\varphi + \|\cdot\|^2)$ is a function as desired. □

Recall that a compact set $K \subset \mathbb{C}^n$ is polynomially convex if and only if there exists a smooth plurisubharmonic function $\Phi\colon \mathbb{C}^n \to [0, \infty)$ such that $K = \{\Phi = 0\}$ and such that $\Phi$ is strictly plurisubharmonic outside $K$ (see, for example, Theorem 1.3.8 in [St07]). In this context, observe that we also have the following additional property of the Wermer type set $\mathcal{E}$.

**Corollary 1.4.2.** *If $\{\varepsilon_l\}$ is decreasing fast enough, then for every $R > 0$ one has $\widehat{bB^n(0,R) \cap \mathcal{E}} = \bar{B}^n(0,R) \cap \mathcal{E}$, where $\widehat{bB^n(0,R) \cap \mathcal{E}}$ denotes the polynomial hull of the set $bB^n(0,R) \cap \mathcal{E}$.*

**Proof.** Using (1.11) and part (1) of Lemma 1.4.2, we see that, for every $(z,w) \in \mathbb{C}^n \setminus \mathcal{E}$, there exists $\nu \in \mathbb{N}$ such that $\bar{B}^n(0,R) \cap \mathcal{E} \subset \{|P_\nu| < (\frac{1}{\nu})^{2^\nu}\}$ but $|P_\nu(z,w)| \geq (\frac{1}{\nu})^{2^\nu}$, i.e., $(z,w) \notin \widehat{bB^n(0,R) \cap \mathcal{E}}$. In particular, since clearly $\widehat{bB^n(0,R) \cap \mathcal{E}} \subset \bar{B}^n(0,R)$, this shows that $\widehat{bB^n(0,R) \cap \mathcal{E}} \subset \bar{B}^n(0,R) \cap \mathcal{E}$. Concerning the other direction, note that $\widehat{bB^n(0,R) \cap E_\nu} = \bar{B}^n(0,R) \cap E_\nu$ for every $\nu \in \mathbb{N}$ by the maximum modulus principle and the fact that $E_\nu$ is the zero set of the polynomial $P_\nu$. Since on bounded subsets of $\mathbb{C}^n$ the sequence $\{E_\nu\}$ converges to $\mathcal{E}$ in the Hausdorff metric, we thus conclude that $\bar{B}^n(0,R) \cap \mathcal{E} = \lim_{\nu\to\infty} \bar{B}^n(0,R) \cap E_\nu = \lim_{\nu\to\infty} \widehat{bB^n(0,R) \cap E_\nu} \subset \widehat{bB^n(0,R) \cap \mathcal{E}}$. □

At the end of this section, we want to use Hölder continuity of $\underline{\mathcal{E}}$ to give an explicit form for the function $\Phi$ from Corollary 1.4.1: Let $\varphi$ be the plurisubharmonic function defined in Lemma 1.4.3 such that $\mathcal{E} = \{\varphi = -\infty\}$, and let $\phi := \varphi + \|\cdot\|^2$. We then want to make an explicit choice of a function $\Lambda\colon [-\infty,\infty) \to [0,\infty)$ such that $\Phi := \Lambda \circ \phi$ is smooth and plurisubharmonic on $\Omega$ and strictly plurisubharmonic outside $\mathcal{E}$. In order to do so, consider first the function $e^\phi\colon \mathbb{C}^n \to [0,\infty)$. Observe that $e^\phi$ is a continuous plurisubharmonic function on $\mathbb{C}^n$ that is smooth and strictly plurisubharmonic in the complement of $\mathcal{E}$. Thus this function has all the properties we seek except, possibly, for smoothness in points of $\mathcal{E}$. Now the general idea to obtain $\Lambda$ and $\Phi$ as desired is to compose $e^\phi$ with a smooth, strictly increasing and strictly convex function that vanishes at 0 of infinite order. In fact, we will take $\Lambda\colon [-\infty,\infty) \to [0,\infty)$ such that $\Lambda(x) = e^{-1/e^x}$ for small values of $x$. To actually prove smoothness of $\Phi$, we proceed as follows: We show that for each point $(z,w) \in \mathbb{C}^n \setminus \mathcal{E}$ there exists a polycylinder around $(z,w)$ that does not intersect $\mathcal{E}$, the size of which depends uniformly on the vertical distance $d(w,\mathcal{E}_z) := \inf_{w' \in \mathcal{E}_z} |w - w'|$ of $(z,w)$ to $\mathcal{E}$. Moreover, we estimate the value of $e^\varphi$ by means of the vertical distance $d(w,\mathcal{E}_z)$ to the set $\mathcal{E}$. We then use the Poisson integral formula and pluriharmonicity of $\varphi$ outside $\mathcal{E}$ to derive Cauchy type estimates for the derivatives of $\phi$, and apply the above results to conclude that each $D^\alpha\Phi(z,w)$ tends to zero when $(z,w)$ approaches $\mathcal{E}$.

We first prove the existence of uniformly large polycylinders in the complement of $\mathcal{E}$, which follows easily from Hölder continuity of the map $\underline{\mathcal{E}}$.

**Lemma 1.4.4.** *There exists a constant $C > 0$ such that*

$$\bar{\Delta}^n_{\mathcal{E}}(z,w) := \bar{\Delta}^{n-1}\Big(z, C\Big(\frac{d(w,\mathcal{E}_z)}{2}\Big)^2\Big) \times \bar{\Delta}^1\Big(w, \frac{d(w,\mathcal{E}_z)}{2}\Big) \subset \mathbb{C}^n \setminus \mathcal{E} \tag{1.19}$$

*for every $(z,w) \in \mathbb{C}^n \setminus \mathcal{E}$.*

**Proof.** Fix $(z,w) \in \mathbb{C}^n \setminus \mathcal{E}$. Then, in view of Lemma 1.2.1, for any given $(z',w') \in \mathcal{E} \cap \big[\bar{\Delta}^{n-1}\big(z, (1/(\sqrt{n}M^2))(d(w,\mathcal{E}_z)/2)^2\big) \times \mathbb{C}_w\big]$ we can find $(z,\tilde{w}) \in \mathcal{E}_z$ such that $\|(z,\tilde{w}) - (z',w')\| \leq M\sqrt{\|z-z'\|} < d(w,\mathcal{E}_z)/2$. Hence $|w-w'| \geq |w-\tilde{w}| - |\tilde{w}-w'| > d(w,\mathcal{E}_z) - d(w,\mathcal{E}_z)/2 > d(w,\mathcal{E}_z)/2$, which proves (1.19) for $C := 1/(\sqrt{n}M^2)$. □

We now want to estimate the growth of $e^{\varphi(z,w)}$ in terms of the vertical distance $d(w,\mathcal{E}_z)$ to $\mathcal{E}$. For every $\nu,\mu \in \mathbb{N}$, $j \in \mathbb{N}_{2^\nu}$ and $z \in \mathbb{C}^{n-1}$, we denote the $2^\mu$ values of $w_j^{(\nu)}(z) + \sum_{l=\nu+1}^{\nu+\mu} \varepsilon_l \sqrt{z_{[l]} - a_l}$ by $w_1^{(\mu)}(\nu,j;z), \ldots, w_{2^\mu}^{(\mu)}(\nu,j;z)$. Moreover, for every set $K \subset \mathbb{C}^n$ and every positive number $\delta > 0$ we let $K^{(\delta)} := \bigcup_{\zeta\in K} B^n(\zeta,\delta)$.

**Lemma 1.4.5.** *If $\{\varepsilon_l\}$ is decreasing fast enough, then there exists an increasing sequence $\{L_N\}_{N=1}^\infty$ of positive numbers such that for every $N \in \mathbb{N}$, $\nu \geq N$ and $j \in \mathbb{N}_{2^\nu}$*

$$|P_{\nu+\mu}(z,w)|^{1/2^{\nu+\mu}} \leq L_N \cdot \prod_{k=1}^{2^\mu} |w - w_k^{(\mu)}(j,\nu;z)|^{1/2^{\nu+\mu}} \text{ on } \Delta_N \cap \mathcal{E}^{(1)} \tag{1.20}$$

*for every $\mu \in \mathbb{N}$, where $\Delta_N := B^{n-1}(0,N) \times \mathbb{C}$.*

**Proof.** Fix $N \in \mathbb{N}$, $\nu \geq N$ and $j \in \mathbb{N}_{2^\nu}$. Recall that the sequence $\{\varepsilon_l\}$ is chosen in such a way that $\varepsilon_l\sqrt{|z_{[l]} - a_l|} < 1/2^l$ on $B^{n-1}(0,l)$ for every $l \in \mathbb{N}$. Thus $|w_{j'}^{(\nu)}(z) - w_k^{(\mu)}(j',\nu;z)| \leq \sum_{l=\nu+1}^{\nu+\mu} \varepsilon_l \sqrt{|z_{[l]} - a_l|} < 1/2^\nu$ on $B^{n-1}(0,N)$ for every $j' \in \mathbb{N}_{2^\nu}$ and $\mu \in \mathbb{N}$, $k \in \mathbb{N}_{2^\mu}$. Hence $d_H(E_{\nu,z},\mathcal{E}_z) \leq 1/2^\nu$ for every $z \in B^{n-1}(0,N)$, where $E_{\nu,z} := \{w \in \mathbb{C} : (z,w) \in E_\nu\}$, and

$$\prod_{k=1}^{2^\mu} |w - w_k^{(\mu)}(j',\nu;z)|^{1/2^\mu} < |w - w_{j'}^{(\nu)}(z)| + 1/2^\nu \tag{1.21}$$

for every $(z,w) \in \Delta_N$, $j' \in \mathbb{N}_{2^\nu}$ and $\mu \in \mathbb{N}$. Let $\{\varepsilon_l\}$ be decreasing so fast that $\sum_{l=1}^\infty \varepsilon_l \sqrt{|a_l|} < 1/2$ and $\varepsilon_l < 1/2^{l+1}$ for every $l \in \mathbb{N}$. Since $\big|\sqrt{|z_{[l]} - a_l|} - \sqrt{|z_{[l]}|}\big| \leq \sqrt{\big||z_{[l]} - a_l| - |z_{[l]}|\big|} \leq \sqrt{|a_l|}$, we have $\sqrt{|z_{[l]} - a_l|} \leq \sqrt{|z_{[l]}|} + \sqrt{|a_l|} \leq \sqrt{\|z\|} +$

$\sqrt{|a_l|}$. Thus, by definition of the sets $E_\nu$, it follows that $|w| \leq \sum_{l=1}^{\nu} \varepsilon_l \sqrt{|z_{[l]} - a_l|} \leq \sum_{l=1}^{\nu} \varepsilon_l (\sqrt{\|z\|} + \sqrt{|a_l|}) < (1/2)(\sqrt{\|z\|} + 1)$ for every $(z,w) \in E_\nu$. Taking the limit $\nu \to \infty$ we get that $\mathcal{E} \subset \{(z,w) \in \mathbb{C}^n : |w| \leq (1/2)(\sqrt{\|z\|} + 1)\}$. For fixed $(z,w) \in \Delta_N \cap \mathcal{E}^{(1)}$ and $j' \in \mathbb{N}_{2^\nu}$ choose $\tilde{w}_1, \tilde{w}_2 \in \mathcal{E}_z$ such that $|w - \tilde{w}_1| = d(w, \mathcal{E}_z)$ and $|\tilde{w}_2 - w_{j'}^{(\nu)}(z)| = d_H(E_{\nu,z}, \mathcal{E}_z)$. Then $|w - w_{j'}^{(\nu)}(z)| \leq |w - \tilde{w}_1| + |\tilde{w}_1 - \tilde{w}_2| + |\tilde{w}_2 - w_{j'}^{(\nu)}(z)| \leq 1 + (1 + \sqrt{N}) + 1/2^N$, hence there exists a constant $L_N > 1$ such that

$$|w - w_{j'}^{(\nu)}(z)| + 1/2^\nu \leq L_N \quad \text{on} \quad \Delta_N \cap \mathcal{E}^{(1)}. \tag{1.22}$$

We conclude that for every $(z,w) \in \Delta_N \cap \mathcal{E}^{(1)}$ and $\mu \in \mathbb{N}$ we have

$$\begin{aligned}
|P_{\nu+\mu}(z,w)|^{1/2^{\nu+\mu}} &= \prod_{l=1}^{2^{\nu+\mu}} |w - w_l^{(\nu+\mu)}(z)|^{1/2^{\nu+\mu}} = \prod_{j'=1}^{2^\nu} \prod_{k=1}^{2^\mu} |w - w_k^{(\mu)}(j', \nu; z)|^{1/2^{\nu+\mu}} \\
&= \prod_{\substack{1 \leq j' \leq 2^\nu \\ j' \neq j}} \Big( \prod_{k=1}^{2^\mu} |w - w_k^{(\mu)}(j', \nu; z)|^{1/2^\mu} \Big)^{1/2^\nu} \cdot \prod_{k=1}^{2^\mu} |w - w_k^{(\mu)}(j, \nu; z)|^{1/2^{\nu+\mu}} \\
&\leq L_N \cdot \prod_{k=1}^{2^\mu} |w - w_k^{(\mu)}(j, \nu; z)|^{1/2^{\nu+\mu}},
\end{aligned}$$

where the last inequality follows from (1.21) and (1.22). □

**Lemma 1.4.6.** *The following assertions hold true:*

(1) *If $\{\varepsilon_l\}$ is decreasing fast enough, then*

$$d(w, \mathcal{E}_z) \leq e^{\varphi(z,w)} \leq d(w, \mathcal{E}_z) + (1 + \sqrt{\|z\|}) \text{ on } \mathbb{C}^n.$$

(2) *Let $\{\delta_l\}_{l=1}^\infty$ be a decreasing sequence of positive numbers converging to 0. If $\{\varepsilon_l\}$ is decreasing fast enough, then for every $N \in \mathbb{N}$ and $\nu \geq N$ we have*

$$d(w, \mathcal{E}_z) \leq e^{\varphi}(z,w) \leq (L_N + 1) d(w, \mathcal{E}_z)^{1/2^\nu} \text{ on } B^n(0,N) \cap \big(\mathcal{E}^{(1)} \setminus \mathcal{E}^{(\delta_\nu)}\big),$$

*where $L_N$ are the constants from Lemma 1.4.5.*

**Proof.** 1) Observe that $e^\varphi = \lim_{\nu \to \infty} e^{\varphi_\nu} = \lim_{\nu \to \infty} |P_\nu|^{1/2^\nu}$. Hence the first inequaltiy follows from $|P_\nu(z,w)|^{1/2^\nu} = \prod_{j=1}^{2^\nu} |w - w_j^{(\nu)}(z)|^{1/2^\nu} \geq d(w, E_{\nu,z})$ and the fact that $E_{\nu,z} \to \mathcal{E}_z$ in the Hausdorff metric for $\nu \to \infty$.

Let $\{\varepsilon_l\}$ be decreasing so fast that $\sum_{l=1}^\infty \varepsilon_l \sqrt{|a_l|} < 1/2$ and $\varepsilon_l < 1/2^{l+1}$ for every $l \in \mathbb{N}$. As in the proof of Lemma 1.4.5 we conclude that $E_\nu \subset \{(z,w) \in \mathbb{C}^n :$

$|w| < (1/2)(\sqrt{\|z\|} + 1)\}$ for every $\nu \in \mathbb{N}$. Now for arbitrary fixed $\nu \in \mathbb{N}$ and $(z, w) \in \mathbb{C}^n$ choose $\tilde{w} \in E_{\nu,z}$ such that $|w - \tilde{w}| = d(w, E_{\nu,z})$. Then $|w - w_j^{(\nu)}(z)| \leq |w - \tilde{w}| + |\tilde{w} - w_j^{(\nu)}(z)| < d(w, E_{\nu,z}) + (\sqrt{\|z\|} + 1)$ for every $j \in \mathbb{N}_{2^\nu}$, hence $|P_\nu(z, w)|^{1/2^\nu} = \prod_{j=1}^{2^\nu} |w - w_j^{(\nu)}(z)|^{1/2^\nu} < d(w, E_{\nu,z}) + (\sqrt{\|z\|} + 1)$ and the second inequality follows for $\nu \to \infty$.

2) We only need to show the second inequality. Let $\{\varepsilon_l\}$ be decreasing so fast that the assertion of Lemma 1.4.5 holds, and observe that (1.20) remains true with the same constants $\{L_N\}$ if later on we choose $\{\varepsilon_l\}$ to be converging to zero even faster. For every $N \in \mathbb{N}$, let $\gamma_N$ be a positive constant such that

$$\gamma_N L_N \leq d(w, \mathcal{E}_z)^{1/2^N} \text{ on } B^n(0, N) \setminus \mathcal{E}^{(\delta_N)}. \tag{1.23}$$

Let $\{\varepsilon_l\}$ be decreasing so fast that $\sum_{l=\nu+1}^{\infty} \varepsilon_l \sqrt{|z_{[l]} - a_l|} \leq r_\nu := 2^{\nu-1} \gamma_\nu \delta_\nu^{(2^\nu - 1)/2^\nu}$ on $B^{n-1}(0, \nu)$ for every $\nu \in \mathbb{N}$. Fix $N \in \mathbb{N}$ and let $\nu \geq N$ be arbitrary. Let $(z, w) \in B^n(0, N) \cap (\mathcal{E}^{(1)} \setminus \mathcal{E}^{(\delta_\nu)})$. Choose $\tilde{w} \in \mathcal{E}_z$ such that $|w - \tilde{w}| = d(w, \mathcal{E}_z)$ and choose $j \in \mathbb{N}_{2^\nu}$ such that $|\tilde{w} - w_j^{(\nu)}(z)| \leq d_H(\mathcal{E}_z, E_{\nu,z})$. Then for every $\mu \in \mathbb{N}$ and $k \in \mathbb{N}_{2^\mu}$ we get $|w - w_k^{(\mu)}(j, \nu; z)| \leq |w - \tilde{w}| + |\tilde{w} - w_j^{(\nu)}(z)| + |w_j^{(\nu)} - w_k^{(\mu)}(j, \nu; z)| \leq d(w, \mathcal{E}_z) + r_\nu + r_\nu$. Hence, by the choice of $r_\nu$, and since $\delta_\nu \leq d(w, \mathcal{E}_z)$, it follows that $\prod_{k=1}^{2^\mu} |w - w_k^{(\mu)}(j, \nu; z)|^{1/2^{\nu+\mu}} \leq (d(w, \mathcal{E}_z) + 2r_\nu)^{1/2^\nu} \leq (d(w, \mathcal{E}_z)^{2^\nu/2^\nu} + 2^\nu d(w, \mathcal{E}_z)^{(2^\nu-1)/2^\nu} \gamma_\nu)^{1/2^\nu} \leq d(w, \mathcal{E}_z)^{1/2^\nu} + \gamma_\nu$. Applying Lemma 1.4.5, monotonicity of $\{L_N\}$ and (1.23), we conclude that

$$|P_{\nu+\mu}(z, w)|^{1/2^{\nu+\mu}} \leq L_N \big( d(w, \mathcal{E}_z)^{1/2^\nu} + \gamma_\nu \big) \leq (L_N + 1) d(w, \mathcal{E}_z)^{1/2^\nu}.$$

Since here $\mu \in \mathbb{N}$ and $(z, w) \in B^n(0, N) \cap (\mathcal{E}^{(1)} \setminus \mathcal{E}^{(\delta_\nu)})$ are arbitrary, the claim follows from the fact that $e^{\varphi(z,w)} = \lim_{\mu \to \infty} |P_{\nu+\mu}(z, w)|^{1/2^{\nu+\mu}}$. □

Finally, we prove Cauchy type estimates for the derivatives of $e^\varphi$. Let $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. For multiindices $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_{2n}) \in \mathbb{N}_0^{2n}$ and $\beta = (\beta_1, \beta_2, \ldots, \beta_{2n}) \in \mathbb{N}_0^{2n}$ we write $\alpha \leq \beta$ if and only if $\alpha_\nu \leq \beta_\nu$ for every $\nu = 1, 2, \ldots, 2n$. Moreover, for every $\alpha \in \mathbb{N}_0^{2n}$ and $r = (r_1, r_2, \ldots, r_{2n}) \in [0, \infty)^{2n}$ we let $r^\alpha := r_1^{\alpha_1} r_2^{\alpha_2} \cdots r_{2n}^{\alpha_{2n}}$.

**Lemma 1.4.7.** *Let $\Delta^n(a, r) \subset\subset \mathbb{C}^n$ be a polycylinder with polyradius $r \in [0, \infty)^n$, let $u\colon \bar{\Delta}^n(a, r) \to \mathbb{R}$ be a continuous function such that $u$ is pluriharmonic on $\Delta^n(a, r)$ and let $\alpha \in \mathbb{N}_0^{2n}$. Then*

$$\big| D^\alpha u(\zeta) \big| \leq C_{|\alpha|} \cdot \frac{\sup_{\xi \in b\Delta^n(a,r)} |u(\xi)|}{\hat{r}^\alpha} \tag{1.24}$$

*for $\zeta \in \Delta^n(a, r/2)$, where $\hat{r} := (r_1, r_1, r_2, r_2, \dots, r_n, r_n)$ and $C_{|\alpha|} > 0$ is a constant that depends only on $|\alpha|$.*

**Proof.** The proof of Lemma 1.4.7 will be divided in two steps.

STEP 1. *Let $v\colon \bar{B}^2(a, \rho) \to \mathbb{R}$ be a continuous function such that $v$ is harmonic on $B^2(a, \rho)$. Then for arbitrary fixed $\theta \in (0, 1)$ the inequality*

$$\Big|\frac{\partial v}{\partial x_j}(x)\Big| \leq C\,\frac{\sup_{y \in bB^2(a,\rho)}|v(y)|}{\rho} \tag{1.25}$$

*holds true for every $x \in \bar{B}^2(a, \theta\rho)$ and $j = 1, 2$. Here $C = C_\theta$ is a positive constant.*

PROOF. Without loss of generality we can assume that $a = 0$. Applying the Poisson integral formula to the function $v$, we see that

$$v(x) = \frac{1}{2\pi\rho}\int_{y \in bB^2(0,\rho)} \frac{\rho^2 - \|x\|^2}{\|y - x\|^2} v(y) d\sigma(y).$$

Since for $x \in \bar{B}^2(0, \theta\rho)$ and $y \in bB^2(0, \rho)$ we have

$$\begin{aligned}\Big|\frac{\partial}{\partial x_j}\Big(\frac{\rho^2 - \|x\|^2}{\|y - x\|^2}\Big)\Big| &= \Big|\frac{-2x_j\|y - x\|^2 + 2(\rho^2 - \|x\|^2)(y_j - x_j)}{\|y - x\|^4}\Big| \\ &\leq \frac{8\rho^3 + 4\rho^3}{((1-\theta)\rho)^4} =: \frac{C_\theta}{\rho},\end{aligned}$$

it follows that

$$\Big|\frac{\partial v}{\partial x_j}(x)\Big| \leq \frac{C_\theta}{\rho}\cdot\frac{1}{2\pi\rho}\int_{y \in bB^2(0,\rho)}|v(y)| d\sigma(y) \leq C_\theta\,\frac{\sup_{y \in bB^2(0,\rho)}|v(y)|}{\rho}.$$

STEP 2. *Let $\Delta_1 \supset\supset \Delta_2 \supset\supset \cdots$ be defined as $\Delta_m := \Delta^n(a, (1 - \sum_{j=1}^m 1/2^{j+1})r)$, $m \in \mathbb{N}$. We show that (1.24) holds true for every $\zeta \in \bar{\Delta}_{|\alpha|}$. Since $\Delta^n(a, r/2) \subset \Delta_m$ for every $m \in \mathbb{N}$, this proves the claim of the lemma.*

PROOF. We proceed by induction on $k := |\alpha|$. Since $u$ is pluriharmonic, the case $k = 1$ is an immediate consequence of (1.25) with $\theta = 3/4$. For the step $k \to k+1$, write $\alpha = \tilde{\alpha} + e_\nu$ for some $\nu \in \mathbb{N}$, where $e_\nu = (0, \dots, 1, \dots, 0)$ is the $\nu$-th canonical unit vector and $\tilde{\alpha} \in \mathbb{N}_0^{2n}$ satisfies $|\tilde{\alpha}| = k$. Without loss of generality we can assume that $\nu = 1$. Applying (1.25) to $v := D^{\tilde{\alpha}}u$, $\rho := \big(1 - \sum_{j=1}^k 1/2^{j+1}\big)r_1 > r_1/2$ and

$\theta = \big(1 - \sum_{j=1}^{k+1} 1/2^{j+1}\big)/\big(1 - \sum_{j=1}^{k} 1/2^{j+1}\big)$ yields

$$\begin{aligned} \big|D^\alpha u(\zeta)\big| = \Big|\frac{\partial v}{\partial \zeta_1}(\zeta)\Big| &\le C\,\frac{\sup_{y\in bB^2((a_1,a_2),\rho)\times\{\zeta_3,\dots,\zeta_{2n}\}}|v(y)|}{\rho} \\ &< 2C\,\frac{\sup_{y\in b\Delta_k}|v(y)|}{r_1} \end{aligned}$$

for $\zeta \in \Delta_{k+1}$. But, by induction hypothesis, $|v(\zeta)| \le C_k \sup_{\xi\in b\Delta^n(a,r)}|u(\xi)|/\hat{r}^{\tilde{\alpha}}$ for $\zeta \in \bar{\Delta}_k$. Thus we get

$$\big|D^\alpha u(\zeta)\big| \le 2CC_k \frac{\sup_{\xi\in b\Delta^n(a,r)}|u(\xi)|}{\hat{r}^{\tilde{\alpha}} r_1} = C_{|\alpha|}\frac{\sup_{\xi\in b\Delta^n(a,r)}|u(\xi)|}{\hat{r}^{\alpha}}$$

with $C_{|\alpha|} = C_{k+1} := 2CC_k$ as desired. This proves Lemma 1.4.7. □

Now fix a smooth strictly increasing and strictly convex function $\Lambda\colon [-\infty,\infty) \to [0,\infty)$ such that $\Lambda(x) = e^{-1/e^x}$ for small values of $x$. Observe that the function $\phi = \varphi + \|\cdot\|^2\colon \mathbb{C}^n \to [-\infty,\infty)$ is plurisubharmonic on $\mathbb{C}^n$, smooth and strictly plurisubharmonic on $\mathbb{C}^n \setminus \{\varphi = -\infty\}$. Further, the function $\Lambda$ is smooth strictly increasing and strictly convex. Hence the function $\Phi := \Lambda \circ \phi$ is plurisubharmonic on $\mathbb{C}^n$, smooth and strictly plurisubharmonic on $\mathbb{C}^n \setminus \{\varphi = -\infty\}$. In the remaining part of this section we will show that $\Phi$ is also smooth at the points of $\mathcal{E} = \{\varphi = -\infty\}$.

Step 1. *For every $\alpha \in \mathbb{N}_0^{2n}$ and $N \in \mathbb{N}$, there exist constants $\rho_N, C_{\alpha,N} > 0$ and $m_\alpha \in \mathbb{N}$ such that*

$$\big|D^\alpha \phi(z,w)\big| \le C_{\alpha,N}\frac{\log\big(1/d(w,\mathcal{E}_z)\big)}{d(w,\mathcal{E}_z)^{m_\alpha}} \quad \textit{for} \quad (z,w) \in B^n(0,N)\cap\big(\mathcal{E}^{(\rho_N)}\setminus\mathcal{E}\big).$$

Proof. We know from Lemma 1.4.6 that $d(w,\mathcal{E}_z) \le e^\varphi \le d(w,\mathcal{E}_z) + (1+\sqrt{\|z\|})$, hence $|\varphi(z,w)| \le \max\{|\log d(w,\mathcal{E}_z)|, |\log(d(w,\mathcal{E}_z)+(1+\sqrt{\|z\|}))|\}$. Choose $\rho_N > 0$ so small that $-\log d(w,\mathcal{E}_z) > 1 + |\log(d(w,\mathcal{E}_z) + (1+\sqrt{\|z\|}))|$ on $\bar{\Delta}^n_{\mathcal{E}}(z,w)$ for every $(z,w) \in B^n(0,N)\cap(\mathcal{E}^{(\rho_N)}\setminus\mathcal{E})$, see Lemma 1.4.4. Then

$$|\varphi(z,w)| \le \log\big(1/d(w,\mathcal{E}_z)\big) \quad \text{on} \quad \bar{\Delta}^n_{\mathcal{E}}(z,w) \subset \mathbb{C}^n\setminus\mathcal{E} \tag{1.26}$$

for $(z,w) \in B^n(0,N)\cap(\mathcal{E}^{(\rho_N)}\setminus\mathcal{E})$. Since $\varphi$ is pluriharmonic in $\mathbb{C}^n\setminus\mathcal{E}$, and in view of Lemma 1.4.4, we conclude from Lemma 1.4.7 that

$$\big|D^\alpha\varphi(z,w)\big| \le C_{|\alpha|}\frac{\sup_{\xi\in b\Delta^n_{\mathcal{E}}(z,w)}|\varphi(\xi)|}{\hat{r}_{\mathcal{E}}(z,w)^\alpha} \le C'_{|\alpha|}\frac{\sup_{\xi\in b\Delta^n_{\mathcal{E}}(z,w)}|\varphi(\xi)|}{d(w,\mathcal{E}_z)^{m_\alpha}}$$

on $\mathbb{C}^n \setminus \mathcal{E}$ for suitable constants $C'_{|\alpha|} > 0$ and $m_\alpha \in \mathbb{N}$, where $r_{\mathcal{E}}(z,w) \in (0,\infty)^n$ is defined as $r_{\mathcal{E}}(z,w) := \big(C(d(w,\mathcal{E}_z)/2)^2, \dots, C(d(w,\mathcal{E}_z)/2)^2, d(w,\mathcal{E}_z)/2\big)$. Using (1.26), we get

$$\big|D^\alpha \varphi(z,w)\big| \le C'_{|\alpha|} \frac{\log\big(1/d(w,\mathcal{E}_z)\big)}{d(w,\mathcal{E}_z)^{m_\alpha}} \tag{1.27}$$

for $(z,w) \in B^n(0,N) \cap (\mathcal{E}^{(\rho_N)} \setminus \mathcal{E})$. Moreover, for every $N \in \mathbb{N}$, there exists a constant $C''_N > 0$ such that $|D^\alpha \|\cdot\|^2| \le C''_N$ on $B^n(0,N)$ for every $\alpha \in \mathbb{N}_0^{2n}$. Since, by the choice of $\rho_N$, we have $\log(1/d(w,\mathcal{E}_z)) > 1$ on $B^n(0,N) \cap (\mathcal{E}^{(\rho_N)} \setminus \mathcal{E})$, it follows together with (1.27) that

$$\big|D^\alpha \phi(z,w)\big| \le \big|D^\alpha \varphi(z,w)\big| + \big|D^\alpha \|\cdot\|^2(z,w)\big| \le C_{\alpha,N} \frac{\log\big(1/d(w,\mathcal{E}_z)\big)}{d(w,\mathcal{E}_z)^{m_\alpha}}$$

for $(z,w) \in B^n(0,N) \cap (\mathcal{E}^{(\rho_N)} \setminus \mathcal{E})$ and $C_{\alpha,N} := C'_{|\alpha|} + C''_N$.

Step 2. *For every $\alpha \in \mathbb{N}_0^{2n}$ and $N \in \mathbb{N}$, there exists a polynomial $P_{\alpha,N} \in \mathbb{R}[x]$ with nonnegative coefficients such that*

$$\big|D^\alpha \Phi(z,w)\big| \le P_{\alpha,N}\big(1/d(w,\mathcal{E}_z)\big)e^{-1/e^{\phi(z,w)}} \quad \textit{for} \quad (z,w) \in B^n(0,N) \cap \big(\mathcal{E}^{(\rho_N)} \setminus \mathcal{E}\big).$$

Proof. Recall that $\Phi = e^{-1/e^\phi}$ (since the smoothness of $\Phi$ in $\mathcal{E}$ depends only on the values $\Lambda(x)$ for $0 < x << 1$, we can assume here for simplicity that $\Lambda(x) \equiv e^{-1/e^x}$). Let $\beta_2, \beta_3, \dots, \beta_{\langle\alpha\rangle} \in \mathbb{N}_0^{2n}$ be pairwise distinct multiindices such that $\{\beta \in \mathbb{N}_0^{2n} : 0 \le \beta \le \alpha, \beta \ne 0\} = \{\beta_2, \beta_3, \dots, \beta_{\langle\alpha\rangle}\}$. Then an easy induction on $|\alpha|$ shows that there exists a polynomial $Q_\alpha \in \mathbb{R}[x_1, \dots, x_{\langle\alpha\rangle}]$, $Q_\alpha(x) = \sum_{\gamma \in \mathbb{N}_0^{\langle\alpha\rangle}} a_\gamma x^\gamma$, such that $D^\alpha \Phi = Q_\alpha(1/e^\phi, D^{\beta_2}\phi, \dots, D^{\beta_{\langle\alpha\rangle}}\phi)e^{-1/e^\phi}$. Define $\tilde{Q}_\alpha \in \mathbb{R}[x_1, \dots, x_{\langle\alpha\rangle}]$ as $\tilde{Q}_\alpha(x) = \sum_{\gamma \in \mathbb{N}_0^{\langle\alpha\rangle}} |a_\gamma| x^\gamma$. Then

$$\big|D^\alpha \Phi(z,w)\big| \le \tilde{Q}_\alpha\big(1/e^\phi, \big|D^{\beta_2}\phi\big|, \dots, \big|D^{\beta_{\langle\alpha\rangle}}\phi\big|\big)(z,w) \cdot e^{-1/e^{\phi(z,w)}}.$$

From Lemma 1.4.6 we know that $1/e^{\phi(z,w)} = 1/(e^{\|(z,w)\|^2} e^{\varphi(z,w)}) \le 1/d(w,\mathcal{E}_z)$ for every $(z,w) \in \mathbb{C}^n$. Applying this and Step 1 to the above formula, we get

$$\begin{aligned}
&\big|D^\alpha \varphi^*(z,w)\big| \\
&\le \tilde{Q}_\alpha\bigg(\frac{1}{d(w,\mathcal{E}_z)}, C_{\beta_2,N}\frac{\log\big(1/d(w,\mathcal{E}_z)\big)}{d(w,\mathcal{E}_z)^{m_{\beta_2}}}, \dots, C_{\beta_{\langle\alpha\rangle},N}\frac{\log\big(1/d(w,\mathcal{E}_z)\big)}{d(w,\mathcal{E}_z)^{m_{\beta_{\langle\alpha\rangle}}}}\bigg)e^{-1/e^{\phi(z,w)}} \\
&= \tilde{P}_{\alpha,N}\bigg(\frac{1}{d(w,\mathcal{E}_z)}, \log\Big(\frac{1}{d(w,\mathcal{E}_z)}\Big)\bigg)e^{-1/e^{\phi(z,w)}} \le P_{\alpha,N}\big(1/d(w,\mathcal{E}_z)\big)e^{-1/e^{\phi(z,w)}}
\end{aligned}$$

on $B^n(0,N)\cap\big(\mathcal{E}^{(\rho_N)}\setminus\mathcal{E}\big)$ for suitable polynomials $\tilde{P}_{\alpha,N}\in\mathbb{R}[x_1,x_2]$ and $P_{\alpha,N}\in\mathbb{R}[x]$ with nonnegative coefficients.

Step 3. *For every $(z_0,w_0)\in\mathcal{E}$ and $\alpha\in\mathbb{N}_0^{2n}$, one has $\lim_{(z,w)\to(z_0,w_0)} D^\alpha\Phi(z,w)=0$.*

Proof. By a standard application of l'Hospital's rule, $\lim_{x\to\infty}P(x)e^{-cx^{1/m}}=0$ for every polynomial $P\in\mathbb{R}[x]$, $c>0$ and $m\in\mathbb{N}$. Hence for every $\nu\in\mathbb{N}$ there exists a constant $\delta_\nu>0$ such that

$$P_{\alpha,N}\big(1/d(w,\mathcal{E}_z)\big)e^{-1/[e^{N^2}(L_N+1)d(w,\mathcal{E}_z)^{1/2^\nu}]}<1/\nu \quad\text{for}\quad (z,w)\in\overline{\mathcal{E}^{(\delta_\nu)}} \tag{1.28}$$

for every $N\in\mathbb{N}$ and $\alpha\in\mathbb{N}_0^{2n}$ such that $N,|\alpha|\le\nu$, where $\{L_N\}$ are the constants from Lemma 1.4.5. Clearly, we can assume that $\delta_\nu<\min\{\rho_\nu,1\}$ and $\delta_{\nu+1}<\delta_\nu$ for every $\nu\in\mathbb{N}$ and that $\lim_{\nu\to\infty}\delta_\nu=0$. Let $\{\varepsilon_l\}$ be decreasing so fast that

$$e^{\varphi(z,w)}\le(L_N+1)d(w,\mathcal{E}_z)^{1/2^\nu}\quad\text{for}\quad (z,w)\in B^n(0,N)\cap\big(\mathcal{E}^{(1)}\setminus\mathcal{E}^{(\delta_{\nu+1})}\big) \tag{1.29}$$

for every $N\in\mathbb{N}$ and $\nu\ge\mathbb{N}$. This is always possible as is shown in the second part of Lemma 1.4.6. Now fix $N\in\mathbb{N}$ and $\alpha\in\mathbb{N}_0^{2n}$. Then, since $\phi=\varphi+\|\cdot\|^2$, we conclude from (1.28), (1.29) and Step 2 that

$$\begin{aligned}\big|D^\alpha\Phi(z,w)\big| &\le P_{\alpha,N}\big(1/d(w,\mathcal{E}_z)\big)e^{-1/e^{\phi(z,w)}}\\ &\le P_{\alpha,N}\big(1/d(w,\mathcal{E}_z)\big)e^{-1/[e^{N^2}(L_N+1)d(w,\mathcal{E}_z)^{1/2^\nu}]}<1/\nu\end{aligned}$$

for every $(z,w)\in B^n(0,N)\cap(\overline{\mathcal{E}^{(\delta_\nu)}}\setminus\mathcal{E}^{(\delta_{\nu+1})})$. Thus it follows from $\lim_{\nu\to\infty}\delta_\nu=0$ that $\lim_{(z,w)\to(z_0,w_0)}D^\alpha\Phi(z,w)=0$ for every $(z_0,w_0)\in B^n(0,N)\cap\mathcal{E}$. Since this holds true for every $N\in\mathbb{N}$ and $\alpha\in\mathbb{N}_0^{2n}$, the proof is complete.

## 1.5 A Liouville theorem for Wermer type sets

In this section we prove a Liouville type theorem for plurisubharmonic functions on $\mathcal{E}$. We start by recalling some facts from potential theory, which will be needed in the course of the proof, mainly to fix our notation: Let $M$ be a Riemann surface and let $D\subset\subset M$, $D\neq M$, be a relatively compact open subset. For every $f\colon bD\to\mathbb{R}$, the associated Perron function

$$\begin{aligned}H_Df:=\sup\{&u\colon D\to[-\infty,\infty): u \text{ subharmonic}\\ &\text{and } \limsup_{z\to\zeta}u(z)\le f(\zeta)\text{ for every }\zeta\in bD\}\end{aligned}$$

is harmonic in $D$. If $z \in D$ is fixed, then the assignment $z \mapsto (H_D f)(z)$ is a positive linear functional on $\mathcal{C}^0(bD)$. Hence there exists a unique Radon measure $\omega_D(z, \cdot)$ on the Borel $\sigma$-algebra of $bD$, called the harmonic measure with respect to $D$ and $z$, such that

$$(H_D f)(z) = \int_{bD} f(\zeta)\, d\omega_D(z, \zeta) \tag{1.30}$$

for every $f \in \mathcal{C}^0(bD)$. It turns out that (1.30) remains true for arbitrary bounded Borel measurable functions on $bD$. In particular, it holds for characteristic functions $\chi_E \colon bD \to \{0, 1\}$ of Borel sets $E \subset bD$. Thus

$$\omega_D(z, E) = (H_D \chi_E)(z)$$

for every Borel set $E \subset bD$ and $z \in D$, and $\omega_D(\,\cdot\,, E) \colon D \to \mathbb{R}$ is harmonic. If $D$ is regular with respect to the Dirichlet problem (this is always satisfied if $bD$ is smooth), and if $f$ is continuous at $\zeta \in bD$, then

$$\lim_{z \to \zeta} (H_D f)(z) = f(\zeta).$$

**Theorem 1.5.1.** *Let $\varphi$ be a plurisubharmonic function defined on an open neighbourhood $U \subset \mathbb{C}^n$ of $\mathcal{E}$. If $\varphi$ is bounded from above, then $\varphi \equiv C$ on $\mathcal{E}$ for some $C \in \mathbb{R}$.*

**Remark.** Observe that the above theorem implies, in particular, that the set $\mathcal{E}$ is connected. A more geometric proof of this fact was already given in Lemma 1.2.3.

**Proof.** We proceed in two steps.

Step 1. *The theorem holds true in the case $n = 2$.*

Proof. Since $\mathbb{C}^2 \setminus \mathcal{E}$ is pseudoconvex, the function $\phi(z) := \sup_{w \in \mathcal{E}_z} \varphi(z, w)$, where for every $z \in \mathbb{C}$ the set $\mathcal{E}_z := \{w \in \mathbb{C} : (z, w) \in \mathcal{E}\}$ denotes the fiber of $\mathcal{E}$ over $z$, is subharmonic on $\mathbb{C}$ (see [Sl81], Theorem II). Moreover, by assumption on $\varphi$, this function is bounded from above. Hence, by the classical Liouville theorem, it is constant, i.e., there exists $C \in [-\infty, \infty)$ such that $\phi \equiv C$ on $\mathbb{C}$. Observe that for $C = -\infty$ this already proves our claim, hence without loss of generality we can assume that $C \in \mathbb{R}$. We want to show that $\varphi \equiv C$ on $\mathcal{E}$, so, in order to get a contradiction, assume that $\varphi \not\equiv C$ on $\mathcal{E}$. Clearly, in this case there exists a point $\tilde{p} = (\tilde{z}, \tilde{w}) \in \mathcal{E}$ such that $\varphi(\tilde{p}) < C$. By continuity of $\underline{\mathcal{E}}$, and by upper semicontinuity of $\varphi$, we can assume that $\tilde{z} \notin \{a_l\}_{l=1}^{\infty}$ and that for some positive numbers $\delta, \rho > 0$ we have $\varphi < C - \delta$ on the ball $B^2(\tilde{p}, \rho)$. Fix $\nu_0 \in \mathbb{N}$ such that

$\sum_{l=\nu_0+1}^{\infty} \varepsilon_l \sqrt{|z-a_l|} < \rho/3$ for $z \in \Delta(\tilde{z}, \rho)$. Further, choose $p_0 = (z_0, w_0) \in \mathcal{E}$ such that $\varphi(p_0) = C$, and a bounded smoothly bounded domain $U \subset \mathbb{C}_z$ such that $a_1, a_2, \ldots, a_{\nu_0}, z_0 \in U$, $\tilde{z} \in bU$ and $bU \subset \mathbb{C} \setminus \{a_l\}_{l=1}^{\infty}$. Now the general idea of the proof is to show that the harmonic measure of $\mathcal{E} \cap b(U \times \mathbb{C}_w) \cap B^2(\tilde{p}, \rho)$ with respect to the set $\mathcal{E} \cap (U \times \mathbb{C}_w)$ and the point $p_0 \in \mathcal{E} \cap (U \times \mathbb{C}_w)$ is positive, and hence, since $\varphi \leq C$ on $\mathcal{E}$ and $\varphi < C$ on $B^2(\tilde{p}, \rho)$, that $\varphi(p_0) < C$. However, in order to have a decent notion of harmonic measure available, we need to approximate $\mathcal{E}$ by the analytic varieties $E_\nu$ and, by performing desingularizations $\pi_\nu \colon F_\nu \to E_\nu$, translate the situation into a problem on Riemann surfaces $F_\nu$. The setup is as follows:

For every $\nu \in \mathbb{N}$, let $f_\nu \colon \mathbb{C}^{\nu+1} \to \mathbb{C}^\nu$ be the holomorphic mapping

$$f_\nu(z, w'_1, \ldots, w'_\nu) := \big(w'^2_1 - \varepsilon_1^2(z-a_1), \ldots, w'^2_\nu - \varepsilon_\nu^2(z-a_\nu)\big).$$

Then, using the fact that $a_l \neq a_{l'}$ for $l \neq l'$, one immediately sees that $F_\nu := \{f_\nu = 0\}$ is a one-dimensional complex submanifold of $\mathbb{C}^{\nu+1}$. For every $\nu \geq \nu_0$, define holomorphic projections

$$\pi_\nu \colon F_\nu \to E_\nu, \quad \pi_\nu(z, w'_1, \ldots, w'_\nu) := \big(z, \sum\nolimits_{l=1}^{\nu} w'_l\big)$$
$$P_\nu \colon F_\nu \to F_{\nu_0}, \quad P_\nu(z, w'_1, \ldots, w'_\nu) := (z, w'_1, \ldots, w'_{\nu_0}).$$

Since $(z,w) \in E_\nu$ if and only if $w = \sum_{l=1}^{\nu} \varepsilon_l \sqrt{z-a_l}$, and since $(z, w'_1, \ldots, w'_\nu) \in F_\nu$ if and only if $w'_l = \varepsilon_l \sqrt{z-a_l}$ for every $l \in \mathbb{N}_\nu$ (with the obvious abuse of notation), these maps are indeed well defined. Let $V_\nu := E_\nu \cap (U \times \mathbb{C}_w)$ and $W_\nu := F_\nu \cap (U \times \mathbb{C}^\nu_{w'})$, $\nu \in \mathbb{N}$. Then $V_\nu$ is an analytic subvariety of $E_\nu$ with boundary $bV_\nu = E_\nu \cap b(U \times \mathbb{C}_w)$, and $W_\nu$ is a relatively compact open subset of $F_\nu$ with boundary $bW_\nu = F_\nu \cap b(U \times \mathbb{C}^\nu_{w'})$. Since $bU \subset \mathbb{C} \setminus \{a_l\}_{l=1}^{\infty}$, it is easy to see that for every $\nu \in \mathbb{N}$ the Riemann surface $F_\nu$ intersects the corresponding set $b(U \times \mathbb{C}^\nu_{w'})$ transversally. In particular, the boundary of $W_\nu$ is smooth and hence $W_\nu$ is regular with respect to the Dirichlet problem. Finally, for $\nu \geq \nu_0$ choose points $q_\nu = (z_0, w'(\nu)) \in W_\nu$ such that $P_\nu(q_\nu) = q_{\nu_0}$ for every $\nu \geq \nu_0$ and $\lim_{\nu\to\infty} \pi_\nu(q_\nu) = p_0$.

We consider now the sets

$$X := \pi_{\nu_0}^{-1}\big(bV_{\nu_0} \cap B^2(\tilde{p}, \rho/3)\big) \quad \text{and} \quad X_\nu := \pi_\nu^{-1}\big(bV_\nu \cap B^2(\tilde{p}, 2\rho/3)\big),\ \nu > \nu_0.$$

Since the complex curve $F_\nu$ intersects $b(U \times \mathbb{C}^\nu_{w'})$ transversally, and since $W_\nu$ is relatively compact in $F_\nu$, it follows that $bW_\nu$ is a compact smooth manifold of real dimension 1 for every $\nu \geq \nu_0$. The mappings $\pi_\nu \colon F_\nu \to E_\nu$ are continuous and satisfy $\pi_\nu(bW_\nu) = bV_\nu$, thus $X$ is open in $bW_{\nu_0}$ and $X_\nu$ is open in $bW_\nu$ for every $\nu > \nu_0$. Moreover, applying Sard's theorem simultaneously to the smooth functions $r_\nu\colon bW_\nu \to \mathbb{R}$, $r_\nu := \|\pi_\nu(\,\cdot\,) - \tilde{p}\|^2$, $\nu \geq \nu_0$, we see that, after a slight perturbation of $\rho$, we can assume that the relative boundary $b_{bW_{\nu_0}} X$ of $X$ in $bW_{\nu_0}$, and the relative boundaries $b_{bW_\nu} X_\nu$ of $X_\nu$ in $bW_\nu$, $\nu > \nu_0$, consist of at most finitely many points. Since one sees easily that the map $P_\nu$ is finite, it then follows that $P_\nu^{-1}(b_{bW_{\nu_0}} X) \cup b_{bW_\nu} X_\nu \subset bW_\nu$ is a finite set for every $\nu > \nu_0$.

We come to the main point of the proof. We claim that $P_\nu^{-1}(X) \subset X_\nu$ for every $\nu > \nu_0$. Indeed, let $q = (z, w_1', \ldots, w_\nu') \in P_\nu^{-1}(X)$. Then $(\pi_{\nu_0} \circ P_\nu)(q) \in bV_{\nu_0} \subset b(U \times \mathbb{C}_w)$. In particular, $z \in bU$, i.e., $\pi_\nu(q) \in E_\nu \cap b(U \times \mathbb{C}_w) = bV_\nu$. Moreover, since $\big(z, \sum_{l=1}^{\nu_0} w_l'\big) = (\pi_{\nu_0} \circ P_\nu)(q) \in B^2(\tilde{p}, \rho/3)$, and since $|w_l'| = \varepsilon_l \sqrt{|z - a_l|}$ for every $l \in \mathbb{N}$ and $\sum_{l=\nu_0+1}^{\infty} \varepsilon_l \sqrt{|z - a_l|} < \rho/3$, we also have $\pi_\nu(q) = \big(z, \sum_{l=1}^{\nu} w_l'\big) \in B^2(\tilde{p}, 2\rho/3)$. This shows that $P_\nu^{-1}(X) \subset X_\nu$ and, therefore, also that $\chi_{X_\nu} \geq \chi_X \circ P_\nu$. It follows now from regularity of $W_\nu$ and $W_{\nu_0}$ with respect to the Dirichlet problem, and from continuity of the functions $\chi_{X_\nu}\colon bW_\nu \to \{0,1\}$ and $\chi_X\colon bW_{\nu_0} \to \{0,1\}$ outside the finite sets $b_{bW_\nu} X_\nu$ and $b_{bW_{\nu_0}} X$, respectively, that $H_{W_\nu} \chi_{X_\nu} \geq (H_{W_{\nu_0}} \chi_X) \circ P_\nu$. Evaluating this inequality at the point $q_\nu \in W_\nu$ and using $P_\nu(q_\nu) = q_{\nu_0}$, we see that

$$\omega_{W_\nu}(q_\nu, X_\nu) \geq \omega_{W_{\nu_0}}(q_{\nu_0}, X) \quad \text{for} \quad \nu > \nu_0. \tag{1.31}$$

Since the condition $a_1, a_2, \ldots, a_{\nu_0} \in U$ implies that $W_{\nu_0}$ is connected, we obviously have that

$$\omega_{W_{\nu_0}}(q_{\nu_0}, X) > 0.$$

Moreover, since for $\nu \in \mathbb{N}$ large enough the function $\varphi \circ \pi_\nu$ is well defined and subharmonic near $W_\nu \subset F_\nu$, we also have that

$$\begin{aligned}\varphi(\pi_\nu(q_\nu)) &\leq H_{W_\nu}(\varphi \circ \pi_\nu|_{bW_\nu})(q_\nu) = \int_{bW_\nu} (\varphi \circ \pi_\nu)(\zeta)\, d\omega_{W_\nu}(q_\nu, \zeta) \\ &= \int_{X_\nu} (\varphi \circ \pi_\nu)(\zeta)\, d\omega_{W_\nu}(q_\nu, \zeta) + \int_{bW_\nu \setminus X_\nu} (\varphi \circ \pi_\nu)(\zeta)\, d\omega_{W_\nu}(q_\nu, \zeta).\end{aligned}$$

Observe now that, since $\varphi$ is upper semicontinuous, $\varphi \leq C$ on $\mathcal{E}$ and $\lim_{\nu\to\infty} V_\nu = \mathcal{E} \cap (U \times \mathbb{C}_w)$ in the Hausdorff metric, there exists a sequence $\{\delta_\nu\}$ of positive numbers such that $\varphi < C + \delta_\nu$ on $V_\nu$ and $\lim_{\nu\to\infty} \delta_\nu = 0$. Moreover, $\varphi < C - \delta$ on $B^2(\tilde{p}, \rho)$. Hence we get from the above estimate that

$$\varphi(\pi_\nu(q_\nu)) \leq (C - \delta)\omega_{W_\nu}(q_\nu, X_\nu) + (C + \delta_\nu)\big(1 - \omega_{W_\nu}(q_\nu, X_\nu)\big),$$

and then, in view of (1.31), we conclude that for $\nu \geq \nu_0$

$$\varphi(\pi_\nu(q_\nu)) \leq (C - \delta)\omega_{W_{\nu_0}}(q_{\nu_0}, X) + (C + \delta_\nu)\big(1 - \omega_{W_{\nu_0}}(q_{\nu_0}, X_{\nu_0})\big).$$

Since $\omega_{W_{\nu_0}}(q_{\nu_0}, X) > 0$, and since $\lim_{\nu\to\infty} \delta_\nu = 0$, this implies that $\varphi(\pi_\nu(q_\nu)) < C$

for every $\nu \geq \nu_0'$ if $\nu_0' \geq \nu_0$ is large enough. Finally, there exists $\mu \geq \nu_0'$ such that $\|p_0 - \pi_\mu(q_\mu)\| < \rho/3$, hence applying the same reasoning as above to the translated variety $V_\mu' := V_\mu + (p_0 - \pi_\mu(q_\mu))$ for large enough $\mu$ we see that also $\varphi(p_0) < C$ (here we use that $X_\mu + (p_0 - \pi_\mu(q_\mu)) \in B^2(\tilde{p}, \rho)$). This contradicts the fact that $\varphi(p_0) = C$ and completes the proof of Step 1.

Step 2. *The theorem holds true in the case $n > 2$.*

Proof. We first show that $\varphi(z_0, w) = \varphi(z_0, w')$ for every $z_0 \in \mathbb{C}^{n-1}$ and every $w, w' \in \mathcal{E}_{z_0}$. To do so, fix $z_0 \in \mathbb{C}^{n-1}$ and consider for every $p \in \mathbb{N}_{n-1}$ the set

$$\tilde{\mathcal{E}}_p := \big\{(z, w) \in \mathbb{C}^{n-1} \times \mathbb{C} : z_j = z_{0,j} \text{ for } j \in \mathbb{N}_{n-1} \setminus \{p\}, w = \sum_{l=1,[l]=p}^{\infty} \varepsilon_l \sqrt{z_p - a_l}\,\big\}.$$

Observe that for $\tilde{\mathcal{E}}_p(z_0) := \{w \in \mathbb{C} : (z_0, w) \in \tilde{\mathcal{E}}_p\}$ we have

$$\mathcal{E}_{z_0} = \tilde{\mathcal{E}}_1(z_0) + \cdots + \tilde{\mathcal{E}}_{n-1}(z_0).$$

Thus for arbitrary fixed $w, w' \in \mathcal{E}_{z_0}$ we can write

$$\begin{matrix} w = w[1] + \cdots + w[n-1] \\ w' = w'[1] + \cdots + w'[n-1] \end{matrix}, \text{ where } w[p], w'[p] \in \tilde{\mathcal{E}}_p(z_0).$$

Define

$$\begin{aligned} w_p &:= w'[1] + \cdots + w'[p-1] + w[p] + \cdots + w[n-1],\ p \in \mathbb{N}_n, \\ \tilde{w}_p &:= w'[1] + \cdots + w'[p-1] + w[p+1] + \cdots + w[n-1],\ p \in \mathbb{N}_{n-1}, \end{aligned}$$

and observe that

$$(z_0, w_p), (z_0, w_{p+1}) \in \tilde{\mathcal{E}}_p + (0, \tilde{w}_p) \subset \mathcal{E} \text{ for every } p \in \mathbb{N}_{n-1}.$$

Since, up to a suitable embedding $i_{z_0, \tilde{w}_p} \colon \mathbb{C}^2_{z_p, w} \hookrightarrow \mathbb{C}^n$, the set $\tilde{\mathcal{E}}_p + (0, \tilde{w}_p)$ is a Wermer type set in $\mathbb{C}^2$, it follows from Step 1 that $\varphi$ is constant on $\tilde{\mathcal{E}}_p + (0, \tilde{w}_p)$, and hence $\varphi(z_0, w_p) = \varphi(z_0, w_{p+1})$ for each $p \in \mathbb{N}_{n-1}$. But $w_1 = w$ and $w_n = w'$. Thus $\varphi(z_0, w) = \varphi(z_0, w')$, as claimed.

Now for every choice of $p \in \mathbb{N}_{n-1}$ and $\xi = (\xi', \xi'') \in \mathbb{C}^{p-1} \times \mathbb{C}^{n-p-1}$ consider the set

$$\mathcal{E}_{p,\xi} := \mathcal{E} \cap \big[(\{\xi'\} \times \mathbb{C}_{z_p} \times \{\xi''\}) \times \mathbb{C}_w\big].$$

Observe that, up to embedding of $\mathbb{C}^2_{z_p, w}$ into $\mathbb{C}^n$, each set $\mathcal{E}_{p,\xi}$ is of the form $\mathcal{E}_{p,\xi} = \bigcup_{w \in W} [\mathcal{E}_p + (0, w)]$ for a Wermer type set $\mathcal{E}_p \subset \mathbb{C}^2_{z_p, w}$ and a suitable set $W = W(p, \xi) \subset \mathbb{C}_w$. Since, by Step 1, the function $\varphi$ is constant on $\mathcal{E}_p + (0, w)$

for every $w \in W$, and since we have already shown that the values of $\varphi$ on $\mathcal{E}$ depend only on the $z$-coordinate, it follows that $\varphi \equiv C_{p,\xi}$ on $\mathcal{E}_{p,\xi}$ for some constant $C_{p,\xi} \in [-\infty, \infty)$. Now if $(z, w)$ and $(z', w')$ are arbitrary points of $\mathcal{E}$, let $\xi_p := (z'_1, \ldots, z'_{p-1}, z_{p+1}, \ldots, z_{n-1})$ for every $p \in \mathbb{N}_{n-1}$ and observe that in the sequence

$$(z, w) \in \mathcal{E}_{1,\xi_1}, \mathcal{E}_{2,\xi_2}, \ldots, \mathcal{E}_{n-2,\xi_{n-2}}, \mathcal{E}_{n-1,\xi_{n-1}} \ni (z', w')$$

we have $\mathcal{E}_{j,\xi_j} \cap \mathcal{E}_{j+1,\xi_{j+1}} \neq \varnothing$ for every $j \in \mathbb{N}_{n-2}$. It follows that $C_{p,\xi_p} = C_{p',\xi_{p'}}$ for every $p, p' \in \mathbb{N}_{n-1}$, and thus $\varphi(z, w) = \varphi(z', w')$. Since $(z, w), (z', w') \in \mathcal{E}$ were arbitrary, the proof is complete. □

# 2 Construction of Wermer type sets in codimension k

We construct a class of unbounded Wermer type sets in $\mathbb{C}^n$, which are limits in the Hausdorff metric of sequences of algebraic varities of codimension $k$. This generalizes the constructions from Chapter 1.

Section 2.1 describes the general method of construction, and in Section 2.2 we extend the results from Chapter 1 to the new setting of codimension $k$. Section 2.3 contains the main results of this chapter. Namely, we investigate pseudoconcavity properties of the generalized Wermer type sets $\mathcal{E} \subset \mathbb{C}^n$ and we prove a lemma on degeneration of plurisubharmonic functions along $\mathcal{E}$.

## 2.1 The general construction

Let $(z, w) = (z_1, \dots, z_{n-k}, w_1, \dots, w_k)$ denote the coordinates in $\mathbb{C}^n$ and for each $\nu \in \mathbb{N}$ let $\mathbb{N}_\nu := \{1, 2, \dots, \nu\}$. For each $(p, q) \in \mathbb{N}_{n-k} \times \mathbb{N}_k$, fix an everywhere dense subset $\{a_l^{p,q}\}_{l=1}^\infty$ of $\mathbb{C}$ such that $a_l^{p,q} \neq a_{l'}^{p,q'}$ if $(q, l) \neq (q', l')$. Further, fix a bijection $\Phi := ([\,\cdot\,], \langle\,\cdot\,\rangle, \phi)\colon \mathbb{N} \to \mathbb{N}_{n-k} \times \mathbb{N}_k \times \mathbb{N}$ and define a sequence $\{a_l\}_{l=1}^\infty$ in $\mathbb{C}$ by letting $a_l := a_{\phi(l)}^{[l],\langle l\rangle}$. Moreover, let $\{\varepsilon_l\}_{l=1}^\infty$ be a decreasing sequence of positive numbers converging to zero that we consider to be fixed, but that will be further specified later on. For every $\nu \in \mathbb{N}$ and $q \in \mathbb{N}_k$, we define sets $E_{\nu,q}, E_\nu \subset \mathbb{C}^n$ as

$$E_{\nu,q} := \Big\{(z, w) \in \mathbb{C}^n : w_q = \sum_{l \in L_\nu^{*,q}} \varepsilon_l \sqrt{z_{[l]} - a_l}\Big\},$$

$$E_\nu := \Big\{(z, w) \in \mathbb{C}^n : w = \Big(\sum_{l \in L_\nu^{*,1}} \varepsilon_l \sqrt{z_{[l]} - a_l}, \dots, \sum_{l \in L_\nu^{*,k}} \varepsilon_l \sqrt{z_{[l]} - a_l}\Big)\Big\},$$

where $L_\nu^{*,q} := \{l \in \mathbb{N}_\nu : \langle l\rangle = q\}$. Observe that $E_\nu = \bigcap_{q=1}^k E_{\nu,q} = \{(z, w) \in \mathbb{C}^n : w = \sum_{l=1}^\nu \varepsilon_l \mathbf{e}_{\langle l\rangle} \sqrt{z_{[l]} - a_l}\}$, where for every $q \in \mathbb{N}_k$ we denote by $\mathbf{e}_q := (0, \dots, 1, \dots, 0)$ the $q$-th unit vector in $\mathbb{C}^k$. Note further that $\sum_{l=1}^\nu \varepsilon_l \mathbf{e}_{\langle l\rangle} \sqrt{z_{[l]} - a_l}$ takes $2^\nu$ values at each $z \in \mathbb{C}^{n-k}$ (counted with multiplicities). Thus there exist single-valued maps $w_1^{(\nu)}, \dots, w_{2^\nu}^{(\nu)} : \mathbb{C}_z^{n-k} \to \mathbb{C}_w^k$ such that $\sum_{l=1}^\nu \varepsilon_l \mathbf{e}_{\langle l\rangle} \sqrt{z_{[l]} - a_l} =$

$\{w_j^{(\nu)}(z) : j = 1, \ldots, 2^\nu\}$ for every $z \in \mathbb{C}^{n-k}$. For every $\nu \in \mathbb{N}$ and $q \in \mathbb{N}_k$, define maps $P_{\nu,q} \colon \mathbb{C}^n \to \mathbb{C}$, $P_\nu \colon \mathbb{C}^n \to \mathbb{C}^k$ as

$$P_{\nu,q}(z,w) := \big(w_q - w_1^{(\nu)}(z)_q\big) \cdots \big(w_q - w_{2^\nu}^{(\nu)}(z)_q\big),$$
$$P_\nu(z,w) := \big(P_{\nu,1}(z,w), \ldots, P_{\nu,k}(z,w)\big),$$

where for every $j \in \mathbb{N}_{2^\nu}$ we denote by $w_j^{(\nu)}(z)_q$ the $q$-th coordinate of $w_j^{(\nu)}(z) \in \mathbb{C}^k$. Observe that $E_{\nu,q} = \{P_{\nu,q} = 0\}$ and $E_\nu = \{P_\nu = 0\}$. As in Lemma 1.1.1, we see that each $P_{\nu,q}$ is a holomorphic polynomial. Moreover, one easily proves the following lemma.

**Lemma 2.1.1.** *If $\{\varepsilon_l\}$ is decreasing fast enough, then for every $R > 0$ the sequences $\{E_{\nu,q} \cap \bar{B}^n(0,R)\}_{\nu=1}^\infty$ and $\{E_\nu \cap \bar{B}^n(0,R)\}_{\nu=1}^\infty$ converge in the Hausdorff metric to closed sets $\mathcal{E}_{(R),q}$ and $\mathcal{E}_{(R)}$, $q \in \mathbb{N}_k$, respectively. The sets $\mathcal{E}_q := \bigcup_{R>0} \mathcal{E}_{(R),q}$ and $\mathcal{E} := \bigcup_{R>0} \mathcal{E}_{(R)}$ are unbounded closed subsets of $\mathbb{C}^n$ and $\mathcal{E} = \bigcap_{q=1}^k \mathcal{E}_q$. Moreover, $\mathcal{E}_z := \{w \in \mathbb{C}^k : (z,w) \in \mathcal{E}\}$ is compact for every $z \in \mathbb{C}^{n-k}$.*

**Proof.** This follows immediately from Lemma 1.1.2 and the equality $E_\nu = \bigcap_{q=1}^k E_{\nu,q}$. □

## 2.2 Generalization of results to the case of codimension k

**Lemma 2.2.1.** *If $\{\varepsilon_l\}$ is decreasing fast enough, then $\mathcal{E}$ contains no analytic variety of positive dimension.*

**Proof.** By Lemma 1.3.6, we can choose $\{\varepsilon_l\}$ such that $\pi_q(\mathcal{E}) = \mathcal{E}_q \cap (\mathbb{C}_z^{n-k} \times \mathbb{C}_{w_q})$ contains no analytic variety of positive dimension for every $q \in \mathbb{N}_k$, where $\pi_q \colon \mathbb{C}^n \to \mathbb{C}_z^{n-k} \times \mathbb{C}_{w_q}$ is the canonical projection. Thus for every analytic set $A \subset \mathcal{E}$ all projections $\pi_q(A)$, $q = 1, \ldots, k$, consist of only one point, i.e., $A = \{P\}$ for some $P \in \mathcal{E}$. □

**Lemma 2.2.2.** *If $\{\varepsilon_l\}$ is decreasing fast enough, then for every $q \in \mathbb{N}_k$ the sequence $\{\frac{1}{2^\nu} \log|P_{\nu,q}|\}_{\nu=1}^\infty$ converges uniformly on compact subsets of $\mathbb{C}^n \setminus \mathcal{E}_q$ to a pluriharmonic function $\varphi_q \colon \mathbb{C}^n \setminus \mathcal{E}_q \to \mathbb{R}$ and $\lim_{(z,w)\to(z_0,w_0)} \varphi_q(z,w) = -\infty$ for every $(z_0,w_0) \in \mathcal{E}_q$. In particular, $\varphi_q$ has a unique extension to a plurisubharmonic function on $\mathbb{C}^n$ and the set $\mathcal{E}_q$ is complete pluripolar.*

**Proof.** See Lemma 1.4.3. □

Since $\mathcal{E} = \bigcap_{q=1}^{k} \mathcal{E}_q$, we conclude from Lemma 2.2.2 that the set $\mathcal{E}$ is complete pluripolar. Moreover, it follows from the results in Section 1.4 that

$$\Phi(z,w) := \Lambda(\varphi_1(z,w) + \|z\|^2 + |w_1|^2) + \cdots + \Lambda(\varphi_k(z,w) + \|z\|^2 + |w_k|^2)$$

is a smooth plurisubharmonic function $\Phi\colon \mathbb{C}^n \to [0,\infty)$ such that $\Phi$ is strictly plurisubharmonic outside $\mathcal{E} = \{\Phi = 0\}$; here $\Lambda\colon [-\infty,\infty) \to [0,\infty)$ is any smooth, strictly increasing and strictly convex function such that $\Lambda(x) = e^{-1/e^x}$ for small values of $x$. Also, since it is easy to see that Lemma 1.4.2 remains true in the more general setting where $k \geq 1$, the same arguments as in the proof of Corollary 1.4.2 show that $\widehat{bB^n(0,R) \cap \mathcal{E}} = \bar{B}^n(0,R) \cap \mathcal{E}$ for every $R > 0$, where $\widehat{bB^n(0,R) \cap \mathcal{E}}$ denotes the polynomial hull of $bB^n(0,R) \cap \mathcal{E}$. (The last assertion can also be seen to follow from complete pluripolarity and 1-pseudoconcavity of the set $\mathcal{E}$ (for 1-pseudoconcavity of $\mathcal{E}$, see Lemma 2.3.1 below). We do not give the details here.)

As before, the set $\mathcal{E}$ defines a map $\underline{\mathcal{E}}$ from the metric space $\mathbb{C}^{n-k}$ of all $(n-k)$-tupels of complex numbers with the standard euclidean metric $d_{\|\cdot\|}$ to the metric space $\mathcal{F}(\mathbb{C}^k)$ of all nonempty compact subsets of $\mathbb{C}^k$ with the Hausdorff metric $d_H$, namely $\underline{\mathcal{E}}\colon (\mathbb{C}^{n-1}, d_{\|\cdot\|}) \to (\mathcal{F}(\mathbb{C}), d_H)$, $\underline{\mathcal{E}}(z) := \mathcal{E}_z := \{w \in \mathbb{C}^k : (z,w) \in \mathcal{E}\}$.

**Lemma 2.2.3.** *There exists a constant $M > 0$ such that the map $\underline{\mathcal{E}}$ is $(M, 1/2)$-Hölder continuous.*

**Proof.** The proof is essentially the same as in Lemma 1.2.1. □

**Lemma 2.2.4.** *The set $\mathcal{E}$ is connected.*

**Proof.** The proof is essentially the same as in Lemma 1.2.3. □

**Theorem 2.2.1.** *Let $\varphi$ be a continuous plurisubharmonic function defined on an open neighbourhood $U \subset \mathbb{C}^n$ of $\mathcal{E}$. If $\varphi$ is bounded from above, then $\varphi \equiv C$ on $\mathcal{E}$ for some $C \in \mathbb{R}$.*

**Proof.** Using the same argument as in Step 2 of the proof of Theorem 1.5.1, we can restrict ourselves to the case $k = n-1$. Choose an increasing sequence $\{B_\nu\}_{\nu=\nu_0}^{\infty}$ of open sets $B_\nu \subset \mathbb{C}$ such that $\bigcup_{\nu=\nu_0}^{\infty} B_\nu = \mathbb{C}$ and such that $E_\nu \cap (B_\nu \times \mathbb{C}^{n-1}) \subset U$ for every $\nu \geq \nu_0$. Moreover, define functions $\phi_\nu\colon B_\nu \to \mathbb{R}$, $\nu \geq \nu_0$, as $\phi_\nu(z) := \max_{1 \leq j \leq 2^\nu} \varphi(z, w_j^{(\nu)}(z))$ and let $\phi(z) := \sup_{w \in \mathcal{E}(z)} \varphi(z,w)$. Since on compact subsets of $\mathbb{C}^n$ the sequence $\{E_\nu\}$ converges in the Hausdorff metric to $\mathcal{E}$, and since $\varphi$ is continuous, one easily sees that $\lim_{\nu\to\infty} \phi_\nu = \phi$ uniformly on compact subsets of $\mathbb{C}$. Moreover, every function $\varphi_\nu$ is subharmonic, since on each convex set in the complement of the polar set $\{a_1, \ldots, a_\nu\}$ the functions $w_1^{(\nu)}, \ldots, w_{2^\nu}^{(\nu)}$ can be chosen to be holomorphic. In particular, $\phi$ is a subharmonic function on $\mathbb{C}$ that is bounded from above, hence, in view of Liouville's theorem, $\phi \equiv C$ for some $C \in \mathbb{R}$. The proof can now be completed in the same way as in Step 1 of Theorem 1.5.1.□

**Remark.** In the two-dimensional situation, which is considered in Step 1 of the proof of Theorem 1.5.1, the subharmonicity of the function $\phi$ was obtained by using Theorem II from [Sl81]. A more general version of this result, which also works for the case $n > 2$, was claimed in Theorem 2.3 of [Sl83], but since it does not have a proof, and since we were not able to find a reference with the proof, we have included the above argument. Observe that if we replace our argument by the result from [Sl83], then we can drop the assumption on continuity of the function $\varphi$.

## 2.3 Pseudoconcavity and degeneration of plurisubharmonic functions

We briefly recall the notion of $q$-pseudoconcavity: Let $\Delta^n := \{z \in \mathbb{C}^n : \|z\|_\infty < 1\}$, where $\|z\|_\infty = \max_{1\le j\le n}|z_j|$. An $(n-q,q)$ Hartogs figure $H$ is a set of the form

$$H = \big\{(\zeta,\eta) \in \Delta^{n-q} \times \Delta^q : \|\zeta\|_\infty > r_1 \text{ or } \|\eta\|_\infty < r_2\big\}$$

where $0 < r_1, r_2 < 1$, and we write $\hat{H} := \Delta^n$. A domain $\Omega$ in a complex manifold $\mathcal{M}$, $\dim_{\mathbb{C}} \mathcal{M} =: n$, is called $q$-pseudoconvex in $\mathcal{M}$, $q = 1, \dots, n-1$, if it satisfies the Kontinuitätssatz with respect to $(n-q)$ polydiscs in $\mathcal{M}$, i.e., if for every $(n-q,q)$ Hartogs figure and every injective holomorphic mapping $\Phi\colon \hat{H} \to \mathcal{M}$ such that $\Phi(H) \subset \Omega$ we have $\Phi(\hat{H}) \subset \Omega$ (for details see [Rot55]; a good presentation of this topic can also be found in [Rie67]). In particular, $(n-1)$-pseudoconvexity is usual pseudoconvexity, and every $q$-pseudoconvex domain is $q'$-pseudoconvex for every $q' < q$. A closed set $A \subset \mathcal{M}$ is called $q$-pseudoconcave in $\mathcal{M}$ if $\mathcal{M} \setminus A$ is $q$-pseudoconvex in $\mathcal{M}$. For technical reasons, the last definition is extended to the cases $q \in \{0, n\}$, by insisting that every closed set $A$ is 0-pseudoconcave in $\mathcal{M}$ and that $A$ is $n$-pseudoconcave in $\mathcal{M}$ if and only if $A$ is a union of connected components of $\mathcal{M}$. (The above definition of $q$-pseudoconvexity is due to Rothstein, see [Rot55]. Observe that the definition of strict $q$-pseudoconvexity that was introduced in the smooth case by Andreotti-Grauert in [AG62] and that we will use in Section 3.1 below is indexed differently with respect to $q$ when compared to the definition of Rothstein. Throughout this thesis $q$-pseudoconcavity will always be understood in the sense of Rothstein.)

Before we formulate the statements of this section, we want to point out a series of results, which indicate that it is reasonable to interpret $q$-pseudoconcavity as a generalization of $q$-dimensional analytic structure: Let $U \subset \mathbb{C}^n$ be open.

RESULT 1. *Every $q$-dimensional analytic variety $A \subset U$ is $q$-pseudoconcave in $U$.*

This follows from Theorem 4.2, Theorem 4.3 and Proposition 5.2 in [Sl86]. (If $A$ is a complete intersection, then an easy way to avoid the reference to [Sl86] is the following: Assume, to get a contradiction, that $A$ is not $q$-pseudoconcave. Then there exists an $(n-q,q)$ Hartogs figure $H = \{(\zeta,\eta) \in \Delta^{n-q} \times \Delta^q : \|\zeta\|_\infty > r_1 \text{ or } \|\eta\|_\infty < r_2\}$ and an injective holomorphic mapping $\Phi\colon \hat{H} \to U$ such that $\Phi(H) \subset U \setminus A$ but $\Phi(\hat{H}) \cap A \neq \varnothing$. Let $P\colon U \to \mathbb{C}^{n-q}$ be holomorphic sucht that $A = \{P = 0\}$. Then for a fixed regular value $c \in \mathbb{C}^{n-q}$ of $P$ close enough to zero, and after possibly shrinking $H$, the complex $q$-dimensional manifold $\mathcal{N} := \{P = c\}$ satisfies $\Phi(\bar{H}) \subset U \setminus \mathcal{N}$ and $\Phi(\hat{H}) \cap \mathcal{N} \neq \varnothing$. Thus, for $\varepsilon > 0$ small enough, the function $\varphi := (-\log\|\eta\| + \varepsilon(\|\zeta\|^2 + \|\eta\|^2)) \circ \Phi^{-1}$ attains a maximum along $\mathcal{N} \cap \Phi(\hat{H})$ which contradicts the fact that the Levi form of $\varphi|_{\mathcal{N} \cap \Phi(\hat{H})}$ has at least one positive eigenvalue at every point of $\mathcal{N} \cap \Phi(\hat{H})$.)

Moreover, let $\{A_\nu\}$ be a sequence of relatively closed subsets $A_\nu \subset U$ that converges in the Hausdorff metric to a relatively closed subset $A \subset U$ (i.e., $\lim_{\nu\to\infty} A_\nu \cap K = A \cap K$ in the Hausdorff metric for every compact set $K \subset U$).

RESULT 2. *Assume that each set $A_\nu$ is a $q$-dimensional analytic variety. Then in general $A$ is not an analytic variety.*

A counterexample is provided by the existence of Wermer type sets. However, if the $2q$-dimensional Hausdorff measure of the sets $A_\nu$, $\nu \in \mathbb{N}$, is locally uniformly bounded, then $A$ is a $q$-dimensional analytic variety, see Theorem 1 in [Bi64].

RESULT 3. *Assume that each set $A_\nu$ is $q$-pseudoconcave. Then $A$ is $q$-pseudoconcave.*

This can be seen, for example, as in the proof of Lemma 2.3.1 below.

We will further stress the viewpoint that $q$-pseudoconcave sets generalize $q$-dimensional analytic varieties in Section 3.3 below. For the moment, we only mention that Theorem 1.3 in [Sl89] also states a partial result on approximation in the Hausdorff metric of $q$-pseudoconcave sets by $q$-dimensional analytic varieties.

We now formulate the results of this section: For every closed set $A \subset \mathbb{C}^n$, the set of all smooth plurisubharmonic functions which are defined near $A$ and which are constant on $A$ will be denoted by $T(A)$. If $A_q \subset \mathbb{C}^n$ is an analytic variety of dimension $q$, then clearly

$$\max\big\{0 \le p \le n : \operatorname{rank}\operatorname{Lev}(\Phi) \le n-p \text{ along } A_q \text{ for every } \Phi \in T(A_q)\big\} = q.$$

We will show that for every $q = 1, \dots, n-1$ there exists a $q$-pseudoconcave set $\mathcal{A}_q \subset \mathbb{C}^n$ such that

$$\max\big\{0 \le p \le n : \operatorname{rank}\operatorname{Lev}(\Phi) \le n-p \text{ along } \mathcal{A}_q \text{ for every } \Phi \in T(\mathcal{A}_q)\big\} = n.$$

The general idea of our construction is as follows: The set $\mathcal{E} \subset \mathbb{C}^{n-k}_z \times \mathbb{C}^k_w$ is $(n-k)$-pseudoconcave but not $(n-k+1)$-pseudoconcave, since it is essentially an $(n-k)$-dimensional object. On the other hand, despite possibly large codimension of $\mathcal{E}$ in $\mathbb{C}^n$, for every $(p,q) \in \mathbb{N}_{n-k} \times \mathbb{N}_k$, there is an everywhere dense sequence of root branches along the $z_p$-axis originating in $w_q$-direction. This geometric property will enforce the Levi form of every $\Phi \in T(\mathcal{E})$ to vanish along all coordinate directions at every point of $\mathcal{E}$. Letting $k$ vary between 1 and $n-1$, this proves our claim. The above considerations are made precise by the following two lemmas.

**Lemma 2.3.1.** *If $\{\varepsilon_l\}$ is decreasing fast enough, then $\mathcal{E} \subset \mathbb{C}^{n-k}_z \times \mathbb{C}^k_w$ is $(n-k)$-pseudoconcave but not $(n-k+1)$-pseudoconcave.*

**Proof.** Assume, to get a contradiction, that $\mathcal{E}$ is not $(n-k)$-pseudoconcave. Then there exists an $(k, n-k)$ Hartogs figure $H = \{(\zeta,\eta) \in \Delta^k \times \Delta^{n-k} : \|\zeta\|_\infty > r_1 \text{ or } \|\eta\|_\infty < r_2\}$ and an injective holomorphic mapping $\Phi\colon \hat{H} \to \mathbb{C}^n$ such that $\Phi(H) \subset \mathbb{C}^n \setminus \mathcal{E}$ but $\Phi(\hat{H}) \cap \mathcal{E} \neq \varnothing$. After possibly shrinking $H$, one can easily see that for $\nu \in \mathbb{N}$ large enough the pure $(n-k)$-dimensional varieties $E_\nu$ will satisfy the conditions $\Phi(\bar{H}) \subset \mathbb{C}^n \setminus E_\nu$ and $\Phi(\hat{H}) \cap E_\nu \neq \varnothing$. Then $V := \Phi(\hat{H})$ is a relatively compact subset of $\mathbb{C}^n$ such that the $(n-k-1)$-plurisubharmonic function $\varphi := -\log\|\eta\| \circ \Phi^{-1}$ satisfies $\max_{E_\nu \cap V} \varphi > \max_{E_\nu \cap bV} \varphi$. This contradicts the local maximum property of $(n-k-1)$-plurisubharmonic functions on $(n-k)$-dimensional analytic varieties, see Corollary 5.3 in [Sl86].

To see that $\mathcal{E}$ is not $(n-k+1)$-pseudoconcave, let $z_0 \in \mathbb{C}^{n-k}$ be an arbitrary fixed point and let $\operatorname{co}\mathcal{E}_{z_0}$ denote the convex hull of $\mathcal{E}_{z_0}$. We claim that the set $X := b(\operatorname{co}\mathcal{E}_{z_0}) \cap \mathcal{E}_{z_0}$ is nonempty. Indeed, by compactness of $\mathcal{E}_{z_0}$, we conclude that $\operatorname{co}\mathcal{E}_{z_0}$ is compact too, and thus it follows easily from Minkowski's theorem that $X$ contains the nonempty set of extreme points of $\operatorname{co}\mathcal{E}_{z_0}$. Hence we can find a supporting real hyperplane $L \subset \mathbb{C}^k_w$ for $\mathcal{E}_{z_0}$ that contains at least one point $w_0 \in \mathcal{E}_{z_0}$. Since $L$ contains a $(k-1)$-dimensional complex subspace, one now constructs easily an $(k-1, n-(k-1))$ Hartogs figure $H = \{(\zeta,\eta) \in \Delta^{k-1} \times \Delta^{n-(k-1)} : \|\zeta\|_\infty > r_1 \text{ or } \|\eta\|_\infty < r_2\}$ and an injective holomorphic mapping $\Phi\colon \hat{H} \to \mathbb{C}^n$ such that $\Phi(H) \subset \mathbb{C}^n \setminus \mathcal{E}$ but $\Phi(\hat{H}) \cap \mathcal{E} \neq \varnothing$. $\square$

**Lemma 2.3.2.** *If $\{\varepsilon_l\}$ is decreasing fast enough, then Lev$(\Phi) \equiv 0$ on $\mathcal{E}$ for every $\Phi \in T(\mathcal{E})$.*

**Proof.** First we observe that it suffices to prove the claim in the case $k = n - 1$. Indeed, for every $p \in \mathbb{N}_{n-k}$ and every $\xi = (\xi', \xi'') \in \mathbb{C}^{p-1} \times \mathbb{C}^{n-k-p}$, the set

$$\mathcal{E}_{p,\xi} := \mathcal{E} \cap \left[(\{\xi'\} \times \mathbb{C}_{z_p} \times \{\xi''\}) \times \mathbb{C}^k_w\right]$$

is, up to inclusion of $\mathbb{C}_{z_p} \times \mathbb{C}^k_w$ into $\mathbb{C}^n$, of the form $\mathcal{E}_{p,\xi} = \bigcup_{w \in W}[\mathcal{F}_p + (0, w)]$ for a Wermer type set $\mathcal{F}_p \subset \mathbb{C}_{z_p} \times \mathbb{C}^k_w$ and a suitable set $W = W(p, \xi) \subset \mathbb{C}^k$. Thus it is enough to choose $\{\varepsilon_l\}$ in such a way that the assertion of the lemma holds true simultaneously for all sets $\mathcal{F}_1, \dots, \mathcal{F}_{n-k}$.

Let $k = n - 1$ and fix $\Phi \in T(\mathcal{E})$. By assumption, there exists a constant $C \in \mathbb{R}$ such that $\Phi \equiv C$ on $\mathcal{E}$. Now let $\{B_\nu\}_{\nu=\nu_0}^\infty$ be an exhaustion of $\mathbb{C}$ by open sets $B_\nu \subset \mathbb{C}$ such that $E_\nu \cap (B_\nu \times \mathbb{C}^{n-1}) \subset \Omega$ for every $\nu \geq \nu_0$, and consider the following sequence of functions $\varphi_\nu \colon B_\nu \to \mathbb{R}$,

$$\varphi_\nu(z) := \frac{1}{2^\nu} \sum_{j=1}^{2^\nu} \Phi\big(z, w_j^{(\nu)}(z)\big).$$

Since on compact subsets of $\mathbb{C}^n$ the sequence $\{E_\nu\}$ converges in the Hausdorff metric to $\mathcal{E}$, one can easily see that $\{\varphi_\nu\}$ converges locally uniformly to the function $\varphi \equiv C$. Now recall that for fixed $z_0 \in \mathbb{C}$ and for $\nu \geq \nu_0$ large enough the Poisson-Jensen formula for $\varphi_\nu$ on $\Delta(z_0, 1)$ states that

$$\frac{1}{2\pi} \int_0^{2\pi} \varphi_\nu(z_0 + e^{i\theta})\, d\theta - \varphi_\nu(z_0) = -\frac{1}{2\pi} \int_{\Delta(z_0,1)} \log|z - z_0| \Delta\varphi_\nu(z)\, d\mu.$$

Assume, to get a contradiction, that we can find $\nu_0 \in \mathbb{N}$, a positive constant $L > 0$ and a subset $M \subset \Delta(z_0, 1)$ of positive Lebesgue measure such that $\Delta\varphi_\nu > L$ on $M$ for every $\nu \geq \nu_0$. Then from the above formula and locally uniform convergence of $\{\varphi_\nu\}$ we get that

$$\begin{aligned} 0 = C - C &= \frac{1}{2\pi} \int_0^{2\pi} \varphi(z_0 + e^{i\theta})\, d\theta - \varphi(z_0) \\ &= \lim_{\nu \to \infty} -\frac{1}{2\pi} \int_{\Delta(z_0,1)} \log|z - z_0| \Delta\varphi_\nu(z)\, d\mu > 0, \end{aligned} \tag{2.1}$$

which is a contradiction. We will use this observation to show that for a suitable choice of $\{\varepsilon_l\}$ the Levi form of every $\Phi \in T(\mathcal{E})$ has to vanish identically on $\mathcal{E}$.

We first specify the choice of the sequence $\{\varepsilon_l\}$. For every $\nu \in \mathbb{N}$, let $\operatorname{Reg} E_\nu \subset$

$\{(z,w)\in E_\nu : z\neq a_l$ for every $l\in\mathbb{N}_\nu\}$ denote the regular part of $E_\nu$ and for every $(z,w)\in\operatorname{Reg}E_\nu$ let $\lambda_\nu(z,w)\subset\mathbb{C}^n$ be the complex 1-dimensional subspace that is tangent to $E_\nu$ at $(z,w)$. Further, for every $q\in\mathbb{N}_{n-1}$ and every $\alpha\geq 1$, let $\Gamma^q(\alpha)\subset\mathbb{C}^n$ denote the closed cone $\Gamma^q(\alpha):=\{(z,w)\in\mathbb{C}^n : |w_q|\geq(1-1/\alpha)\|(z,w)\|\}$. Since $\lambda_\nu(z,w)$ converges in $\mathbb{CP}^{n-1}$ to the $w_{\langle l\rangle}$-axis for $z\to a_l$ (here we use the fact that $a_l\neq a_{l'}$ for $l\neq l'$), it is then easy to see that we can choose inductively the sequence $\{\varepsilon_l\}_{l=1}^\infty$ and a second sequence $\{\delta_l\}_{l=1}^\infty$ of positive numbers both converging to zero so fast that the following assertion is satisfied for every $\nu\in\mathbb{N}$:

$$\begin{aligned}&\text{for each } l\in\mathbb{N}_\nu \text{ one has that } \lambda_\nu(z,w)\subset\Gamma^{\langle l\rangle}\big(l+1-\textstyle\sum_{l'=l+1}^{\nu}1/2^{l'}\big)\\ &\text{for } (z,w)\in\Big[\big(\Delta(a_l,\delta_l)\setminus\textstyle\bigcup_{l'=l+1}^{\nu}\Delta(a_{l'},\delta_l/2^{l'+1})\big)\times\mathbb{C}^{n-1}\Big]\cap\operatorname{Reg}E_\nu.\end{aligned}\qquad (I_\nu)$$

Indeed, for the case $\nu=1$ fix arbitrary $\varepsilon_1>0$ and then choose $\delta_1>0$ so small that $\lambda_1(z,w)\subset\Gamma^{\langle 1\rangle}(2)$ for every $(z,w)\in[\Delta(a_1,\delta_1)\times\mathbb{C}^{n-1}]\cap\operatorname{Reg}E_1$. Assume now that $\varepsilon_1,\ldots,\varepsilon_\nu$ and $\delta_1,\ldots,\delta_\nu$ are already chosen in such a way that $(I_\nu)$ holds true. Since $E_\nu$ and $E_{\nu+1}$, viewed as set-valued functions over $\mathbb{C}_z$, differ only by the term $\varepsilon_{\nu+1}\mathbf{e}_{\nu+1}\sqrt{z-a_{\nu+1}}$, we can now choose $\varepsilon_{\nu+1}>0$ so small that $\lambda_{\nu+1}(z,w)\subset\Gamma^{\langle l\rangle}\big(l+1-\sum_{l'=l+1}^{\nu+1}1/2^{l'}\big)$ for every $l\in\mathbb{N}_\nu$ and $(z,w)\in\Big[\big(\Delta(a_l,\delta_l)\setminus\bigcup_{l'=l+1}^{\nu+1}\Delta(a_{l'},\delta_l/2^{l'+1})\big)\times\mathbb{C}^{n-1}\Big]\cap\operatorname{Reg}E_{\nu+1}$ (observe that $\operatorname{Reg}E_{\nu+1}\subset\{(z,w)\in\mathbb{C}\times\mathbb{C}^{n-1} : \exists(z,w')\in\operatorname{Reg}E_\nu$ such that $w=w'+\varepsilon_{\nu+1}\mathbf{e}_{\nu+1}\sqrt{z-a_{\nu+1}}\}$). With $\varepsilon_{\nu+1}$ now being fixed we can then choose $\delta_{\nu+1}>0$ so small that $\lambda_{\nu+1}(z,w)\subset\Gamma^{\langle\nu+1\rangle}(\nu+2)$ for every $(z,w)\in[\Delta(a_{\nu+1},\delta_{\nu+1})\times\mathbb{C}^{n-1}]\cap\operatorname{Reg}E_{\nu+1}$. But then $(I_{\nu+1})$ is satisfied which completes our induction on $\nu$. Hence for $M'_l:=\Delta(a_l,\delta_l)\setminus\bigcup_{l'=l+1}^{\infty}\Delta(a_{l'},\delta_l/2^{l'+1})$ we now have

$$\lambda_{\nu+\mu}(z,w)\subset\Gamma^{\langle\nu\rangle}(\nu)\text{ for every }\mu,\nu\geq 1\text{ and }(z,w)\in(M'_\nu\times\mathbb{C}^{n-1})\cap\operatorname{Reg}E_{\nu+\mu},\qquad(2.2)$$

and $M'_\nu\subset\mathbb{C}$ has positive Lebesgue measure. Moreover, as above, we choose $\{\varepsilon_l\}$ in such a way that $\varepsilon_l\sqrt{|z-a_l|}<1/2^l$ on $\Delta(0,l)$ for every $l\in\mathbb{N}$.

We now want to show that with the above choice of the sequence $\{\varepsilon_l\}$ the Levi form $\operatorname{Lev}(\Phi)((z_0,w_0),\,\cdot\,)$ vanishes on $\mathbb{C}_{w_q}$ for every $\Phi\in T(\mathcal{E})$, $(z_0,w_0)\in\mathcal{E}$ and $q\in\mathbb{N}_{n-1}$. Indeed, assume, to get a contradiction, that $\operatorname{Lev}(\Phi)((z_0,w_0),\,\cdot\,)>0$ on $\mathbb{C}_{w_q}$ for some fixed data $\Phi\in T(\mathcal{E})$, $(z_0,w_0)\in\mathcal{E}$ and $q\in\mathbb{N}_{n-1}$. By smoothness of $\Phi$, we can then find positive constants $r,\alpha,\tilde{L}>0$ such that $\operatorname{Lev}(\Phi)((z,w),\xi)\geq\tilde{L}\cdot\|\xi\|^2$ for every $(z,w)\in B^n((z_0,w_0),r)$ and every $\xi\in\Gamma^q(\alpha)$. For every $\nu,\mu\in\mathbb{N}$, $j\in\mathbb{N}_{2^\nu}$ and $z\in\mathbb{C}$ let $\{w_1^{(\mu)}(\nu,j;z),\ldots,w_{2^\mu}^{(\mu)}(\nu,j;z)\}=(z,w_j^{(\nu)}(z))+\sum_{l=\nu+1}^{\nu+\mu}\varepsilon_l\mathbf{e}_{\langle l\rangle}\sqrt{z-a_l}$. Since $\{\varepsilon_l\}$ is converging to zero so fast that (2.2) holds true, it is easy to see that we can find $\nu\in\mathbb{N}$, $\langle\nu\rangle=q$, and $j_0\in\mathbb{N}_{2^\nu}$ such that the graphs of the functions $w_1^{(\mu)}(\nu,j_0;\,\cdot\,),\ldots,w_{2^\mu}^{(\mu)}(\nu,j_0;\,\cdot\,)$ over $M'_\nu$ are contained in $B^n((z_0,w_0),r)$ and such that $\lambda_{\nu+\mu}(z,w_k^{(\mu)}(\nu,j_0;z))\subset\Gamma^q(\alpha)$ for every $\mu\in\mathbb{N}$, $k\in\mathbb{N}_{2^\mu}$ and $z\in M'_\nu\setminus\pi(\operatorname{Sing}E_{\nu+\mu})$, where $\pi\colon\mathbb{C}^n\to\mathbb{C}_z$ is the canonical projection. Now if we define the functions $\varphi_\nu$ as before, then we get

$$\begin{aligned}\Delta\varphi_{\nu+\mu}(z) &= \frac{1}{2^{\nu+\mu}}\sum_{j=1}^{2^\nu}\sum_{k=1}^{2^\mu}\Delta_z\big[\Phi(z,w_k^{(\mu)}(\nu,j;z))\big]\\ &\geq \frac{1}{2^{\nu+\mu}}\sum_{k=1}^{2^\mu}\Delta_z\big[\Phi(z,w_k^{(\mu)}(\nu,j_0;z))\big] \geq \frac{\tilde{L}}{2^\nu} =: L\end{aligned} \tag{2.3}$$

for every $\mu \in \mathbb{N}$ and $z \in M'_\nu \setminus \pi(\operatorname{Sing} E_{\nu+\mu})$, since on every convex subset of $\mathbb{C} \setminus \pi(\operatorname{Sing} E_{\nu+\mu})$ we can assume the functions $w_k^{(\mu)}(\nu, j_0;\,\cdot\,)$ to be holomorphic. But it is clear from the construction of the sequence $\{E_{\nu+\mu}\}$ that each of the sets $\pi(\operatorname{Sing} E_{\nu+\mu}) \subset \mathbb{C}$ has Lebesgue measure zero. Hence the Lebesgue measure of $M_\nu := M'_\nu \setminus \bigcup_{\mu=1}^\infty \pi(\operatorname{Sing} E_{\nu+\mu})$ is positive, and we have already seen in (2.1) that this leads to a contradiction.

We already know that for every $\Phi \in T(\mathcal{E})$ and $(z_0, w_0) \in \mathcal{E}$ the Levi form $\operatorname{Lev}(\Phi)((z_0,w_0),\,\cdot\,)$ vanishes on $\mathbb{C}_w^{n-1}$. Assume, to get a contradiction, that there exist $\Phi \in T(\mathcal{E})$ and $(z_0,w_0) \in \mathcal{E}$ such that the Levi form $\operatorname{Lev}(\Phi)((z_0,w_0),\,\cdot\,)$ is not identically zero. Then $\operatorname{Lev}(\Phi)((z_0,w_0),\xi) = \tilde{c} \cdot |\xi_z|^2$ for some constant $\tilde{c}$, where $\xi = (\xi_z, \xi_w) \in \mathbb{C}_z \times \mathbb{C}_w^{n-1}$. Hence, by smoothness of $\Phi$, we can find $r, c > 0$ such that $\operatorname{Lev}(\Phi)((z,w),\xi) \geq c \cdot |\xi_z|^2$ for every $(z,w) \in B^n((z_0,w_0),r)$. Thus whenever $f$ is a holomorphic mapping from an open subset of $\Delta(z_0,r)$ to $\mathbb{C}_w^{n-1}$ such that its graph is completely contained in $B^n((z_0,w_0),r)$, we have $\Delta_z[\Phi(z,f(z))] \geq c$. Then we can argue as in (2.3) to conclude that there exists $\nu \in \mathbb{N}$ such that $\Delta\varphi_{\nu+\mu}(z) \geq c/2^\nu =: L$ for every $\mu \in \mathbb{N}$ and $z \in M_\nu$, where $\varphi_{\nu+\mu}(z) := \frac{1}{2^{\nu+\mu}}\sum_{j=1}^{2^{\nu+\mu}}\Phi(z, w_j^{(\nu+\mu)}(z))$ as before. In view of (2.1), this again leads to a contradiction. □

# Part II

# Core sets of unbounded domains

# 3 Global plurisubharmonic defining functions and the core

We prove the existence of global defining functions for unbounded strictly pseudoconvex domains and we investigate properties of the core.

In Section 3.1 we prove the Main Theorem on existence of global plurisubharmonic defining functions for strictly pseudoconvex domains $\Omega$ with smooth boundary in arbitrary complex manifolds. In the same context we also prove a theorem that guarantees the existence of smooth plurisubharmonic functions defined in a neighbourhood of $\bar{\Omega}$ which are strictly plurisubharmonic near $b\Omega$ and have arbitrary bounded from below and smooth boundary data. Analogous results are shown for strictly $q$-pseudoconvex domains in complex manifolds and strictly hyper-$q$-pseudoconvex domains in complex spaces. Moreover, we show that every strictly pseudoconvex domain in a complex manifold (not necessarily relatively compact or with smooth boundary) admits a neighbourhood basis consisting of strictly pseudoconvex domains with smooth boundary. In Section 3.2 we construct examples of unbounded domains $\Omega \subset \mathbb{C}^n$ such that the core $\mathfrak{c}(\Omega)$ is nonempty, and we investigate for which domains the core can have the special product structure $\mathfrak{c}(\Omega) = E \times \mathbb{C}^k$. In Section 3.3 we show that $\mathfrak{c}(\Omega)$ is always 1-pseudoconcave in $\Omega$, and we construct an example of a strictly pseudoconvex domain $\Omega \subset \mathbb{C}^n$ such that $\mathfrak{c}(\Omega)$ is nonempty and contains no analytic variety of positive dimension. Section 3.4 is devoted to the study of Liouville type properties of the core. Finally, we construct examples of higher order cores with arbitrarily prescribed pseudoconcavity properties in Section 3.5.

## 3.1 Existence of global plurisubharmonic defining functions

### 3.1.1 Existence results in complex manifolds

In this section we prove the existence of global defining functions in the setting of complex manifolds. Our focus will lie on strictly pseudoconvex domains, but when it is possible we formulate the results in the more general context of strictly $q$-pseudoconvex domains. We also discuss to which extent smoothness of $b\Omega$ is needed in our results. We start by recalling some definitions and by fixing our notation.

Let $\mathcal{M}$ be a complex manifold of complex dimension $n := \dim_{\mathbb{C}} \mathcal{M}$. The holomorphic tangent space to $\mathcal{M}$ at $z \in \mathcal{M}$ is denoted by $T_z(\mathcal{M})$, and we write $T(\mathcal{M})$ for the holomorphic tangent bundle of $\mathcal{M}$. If $\varphi\colon \mathcal{M} \to \mathbb{R}$ is a smooth function, then we define $(\partial\varphi)_z, (\bar{\partial}\varphi)_z \colon T_z(\mathcal{M}) \to \mathbb{C}$ and $\mathrm{Lev}(\varphi)(z, \cdot)\colon T_z(\mathcal{M}) \to \mathbb{R}$ in local holomorphic coordinates $h = (z_1, \dots, z_n)$ by

$$(\partial\varphi)_z(\xi) := \sum_{j=1}^{n} \frac{\partial(\varphi \circ h^{-1})}{\partial z_j}(h(z))\xi_j, \quad (\bar{\partial}\varphi)_z(\xi) := \sum_{j=1}^{n} \frac{\partial(\varphi \circ h^{-1})}{\partial \bar{z}_j}(h(z))\bar{\xi}_j,$$
$$\mathrm{Lev}(\varphi)(z,\xi) := \sum_{j,k=1}^{n} \frac{\partial^2(\varphi \circ h^{-1})}{\partial z_j \partial \bar{z}_k}(h(z))\xi_j\bar{\xi}_k,$$

where $\xi = \sum_{j=1}^{n} \xi_j \, (\partial/\partial z_j)$. Moreover, $(d\varphi)_z(\xi) := (\partial\varphi)_z(\xi) + (\bar{\partial}\varphi)_z(\xi)\colon T_z(\mathcal{M}) \to \mathbb{R}$ denotes the real differential of $\varphi$ in $z$. Further, we write $H_z(\varphi)$ for the complex subspace of $T_z(\mathcal{M})$ defined by $H_z(\varphi) := \{\xi \in T_z(\mathcal{M}) : (\partial\varphi)_z(\xi) = 0\}$. If $h$ is a hermitian metric on $\mathcal{M}$, then for every $z \in \mathcal{M}$ we denote by $\|\cdot\|_{h_z}$ and $\|\cdot\|_{h_z^*}$ the induced norms on $T_z(\mathcal{M})$ and on the dual space $T_z^*(\mathcal{M})$, respectively. If the context is clear, then we sometimes omit the index $z$ and simply write $\|\cdot\|_h$ and $\|\cdot\|_{h^*}$. (Throughout this article the term "smooth" always means "$\mathcal{C}^\infty$-smooth". Of course, the above definitions of the various differentials and of the Levi form are possible for $\mathcal{C}^1$-smooth and $\mathcal{C}^2$-smooth functions, respectively.)

An upper semicontinuous function $\varphi\colon \mathcal{M} \to [-\infty, \infty)$ is called plurisubharmonic if for every holomorphic mapping $f\colon G \to \mathcal{M}$ of an open set $G \subset \mathbb{C}$ into $\mathcal{M}$ the composition $\varphi \circ f$ is subharmonic on $G$. It is called strictly plurisubharmonic if for every compactly supported smooth function $\theta\colon \mathcal{M} \to \mathbb{R}$ there exists some number $\varepsilon_0 > 0$ such that $\varphi + \varepsilon\theta$ is plurisubharmonic whenever $|\varepsilon| \le \varepsilon_0$. If the function $\varphi$ is $\mathcal{C}^2$-smooth, then it is (strictly) plurisubharmonic if and only if $\mathrm{Lev}(\varphi)(z, \cdot)$ has precisely $n$ (positive) nonnegative eigenvalues for every $z \in \mathcal{M}$. An open set $\Omega \subset \mathcal{M}$ is called strictly pseudoconvex at $z \in b\Omega$ if there exist an open neighbourhood $U_z \subset \mathcal{M}$ of $z$ and a continuous strictly plurisubharmonic function $\varphi_z\colon U_z \to \mathbb{R}$ such that $\Omega \cap U_z = \{\varphi_z < 0\}$. It is called $\mathcal{C}^s$-smooth at $z \in b\Omega$, $s \ge 1$, if there exist an open neighbourhood $U_z \subset \mathcal{M}$ of $z$ and a $\mathcal{C}^s$-smooth function $\tilde{\varphi}_z\colon U_z \to \mathbb{R}$ such that $\Omega \cap U_z = \{\tilde{\varphi}_z < 0\}$ and $d\tilde{\varphi}_z \neq 0$ on $b\Omega \cap U_z$. The open set $\Omega$ is called strictly pseudoconvex or $\mathcal{C}^s$-smooth if it is strictly pseudoconvex or $\mathcal{C}^s$-smooth at each boundary point, respectively.

Let $q \in \mathbb{N}_0 := \{0\} \cup \mathbb{N}$. An upper semicontinuous function $\varphi\colon \mathcal{M} \to [-\infty, \infty)$ is called $q$-plurisubharmonic if for every holomorphic mapping $f\colon G \to \mathcal{M}$ of an open set $G \subset \mathbb{C}^{q+1}$ into $\mathcal{M}$ the composition $\varphi \circ f$ is subpluriharmonic on $G$. It is called strictly $q$-plurisubharmonic if for every compactly supported smooth function $\theta\colon \mathcal{M} \to \mathbb{R}$ there exists some number $\varepsilon_0 > 0$ such that $\varphi + \varepsilon\theta$ is $q$-plurisubharmonic whenever $|\varepsilon| \leq \varepsilon_0$. Observe that $q$-plurisubharmonic functions are also $(q+1)$-plurisubharmonic for every $q \in \mathbb{N}_0$. Moreover, the 0-plurisubharmonic functions are precisely the plurisubharmonic ones, and a function is $q$-plurisubharmonic for $q \geq n$ if and only if it is upper semicontinuous. If the function $\varphi$ is $\mathcal{C}^2$-smooth, then it is (strictly) $q$-plurisubharmonic if and only if $\mathrm{Lev}(\varphi)(z, \cdot\,)$ has at least $n-q$ (positive) nonnegative eigenvalues for every $z \in \mathcal{M}$. An open set $\Omega \subset \mathcal{M}$ is called strictly $q$-pseudoconvex at $z \in b\Omega$, $q \geq 1$, if there exist an open neighbourhood $U_z \subset \mathcal{M}$ of $z$ and a $\mathcal{C}^2$-smooth strictly $q$-plurisubharmonic function $\varphi_z\colon U_z \to \mathbb{R}$ such that $\Omega \cap U_z = \{\varphi_z < 0\}$. It is called strictly 0-pseudoconvex at $z \in b\Omega$, if it is strictly pseudoconvex at $z$. Observe that $\Omega$ is strictly $q$-pseudoconvex at $z \in b\Omega$ for every $q \geq n-1$, provided that $\Omega$ is $\mathcal{C}^2$-smooth at $z$. The open set $\Omega$ is called strictly $q$-pseudoconvex if it is strictly $q$-pseudoconvex at each boundary point.

Before we start our studies on global defining functions, we want to mention that it is not completely trivial to see that a strictly pseudoconvex domain with $\mathcal{C}^2$-smooth boundary can locally near each boundary point be defined by a $\mathcal{C}^2$-smooth strictly plurisubharmonic function. Since we were not able to find a proof of this fact in the literature, we state here the following proposition.

**Proposition 3.1.1.** *Let $\mathcal{M}$ be a complex manifold, let $\Omega \subset \mathcal{M}$ be open and let $b\Omega$ be $\mathcal{C}^s$-smooth at $z_0 \in b\Omega$, $s \geq 2$. Assume that there exist an open neighbourhood $U \subset \mathcal{M}$ of $z_0$ and a continuous strictly plurisubharmonic function $\psi\colon U \to [-\infty, \infty)$ such that $\Omega \cap U = \{\psi < 0\}$. Then, after possibly shrinking $U$, there exists a $\mathcal{C}^s$-smooth strictly plurisubharmonic function $\varphi\colon U \to \mathbb{R}$ such that $\Omega \cap U = \{\varphi < 0\}$ and $d\varphi \neq 0$ on $b\Omega \cap U$.*

**Proof.** Observe that the statement is trivial in the case $\dim_{\mathbb{C}} \mathcal{M} = 1$. Thus we may assume that $n := \dim_{\mathbb{C}} \mathcal{M} \geq 2$. By assumption, we can find an open neighbourhood $U \subset \mathcal{M}$ of $z_0$, a strictly plurisubharmonic function $\psi\colon U \to [-\infty, \infty)$ such that $\Omega \cap U = \{\psi < 0\}$ and a $\mathcal{C}^s$-smooth function $\tilde{\varphi}\colon U \to \mathbb{R}$ such that $\Omega \cap U = \{\tilde{\varphi} < 0\}$ and $d\tilde{\varphi} \neq 0$ on $b\Omega \cap U$. After possibly shrinking $U$, and after introducing suitable holomorphic coordinates around $z_0$, we can assume that $U \subset\subset \mathbb{C}^n$, $z_0 = 0$, $\psi$ and $\tilde{\varphi}$ are defined in a neighbourhood of $\bar{U}$ and the Taylor expansion of $\tilde{\varphi}$ around 0 has the form

$$\tilde{\varphi}(\xi) = \mathrm{Re}\,\xi_1 + \mathrm{Lev}(\tilde{\varphi})(0, \xi) + o(\|\xi\|^2). \tag{3.1}$$

For every $z \in U$, let $\mathrm{dist}(z, b\Omega) := \inf_{z' \in b\Omega \cap U} \|z - z'\|$ and for $\zeta \in b\Omega \cap U$ let $N_\Omega(\zeta)$ be the outward unit normal vector to $b\Omega$ at $\zeta$.

We claim that $\psi \equiv 0$ on $b\Omega \cap U$. Indeed, for every plurisubharmonic function $u$ defined near some point $\zeta \in \mathbb{C}^n$ and for every $w \in \mathbb{C}^n$ it holds true that $u(\zeta) = \limsup_{t\to 0+} u(\zeta + tw)$, see, for example, Proposition 7.4 in [FSt87]. In particular, $\psi(\zeta) = \limsup_{t\to 0+} \psi(\zeta - tN_\Omega(\zeta)) \leq 0$ for every $\zeta \in b\Omega \cap U$. The fact that $\psi \geq 0$ on $b\Omega \cap U$ is clear by the choice of $\psi$.

As the next step we claim that, after possibly shrinking $U$, there exist numbers $l, L > 0$ such that $\tilde{\varphi}(z) \geq l \operatorname{dist}(z, b\Omega)$ and $\psi(z) \leq L \operatorname{dist}(z, b\Omega)$ for every $z \in U \setminus \Omega$. Clearly, we only need to show the assertion on $\psi$, since the inequality for $\tilde{\varphi}$ follows immediately from the fact that $d\tilde{\varphi} \neq 0$ on $b\Omega \cap U$. The proof is similar to that of the Hopf Lemma: First we can assume, after possibly shrinking $U$, that the orthogonal projection $\pi\colon U \to b\Omega \cap U$ along the normal vectors $N_\Omega(\zeta)$ is well defined. In particular, $z = \pi(z) + \operatorname{dist}(z, b\Omega) N_\Omega(\pi(z))$ for every $z \in U \setminus \Omega$. For every $\zeta \in b\Omega \cap U$ and every $r > 0$, let $\zeta_r := \zeta - rN_\Omega(\zeta)$. By $\mathcal{C}^2$-smoothness of $b\Omega \cap U$, we can then choose $r > 0$ so small that for every $\zeta \in b\Omega \cap B^n(0, r)$ one has

(i) $B^n(\zeta, 4r) \subset U$ and $b\Omega \cap B^n(\zeta, 4r)$ is the graph of a $\mathcal{C}^2$-smooth function over some open subset of $\zeta + T^{\mathbb{R}}_\zeta(b\Omega)$,

(ii) $B^n(\zeta_{2r}, 2r) \subset \Omega \cap U$ and $\bar{B}^n(\zeta_{2r}, 2r) \cap b\Omega = \{\zeta\}$,

where $B^n(a, r) := \{z \in \mathbb{C}^n : \|z - a\| < r\}$ and $T^{\mathbb{R}}_\zeta(b\Omega)$ denotes the real tangent space to $b\Omega$ at $\zeta$. For every $\zeta \in b\Omega \cap B^n(0, r)$ let $G_\zeta := B^n(\zeta, r) \setminus \bar{\Omega}$ and let $h_\zeta\colon \bar{G}_\zeta \to \mathbb{R}$ be the function

$$h_\zeta(z) := \frac{1}{r^{2n-2}} - \frac{1}{\|z - \zeta_r\|^{2n-2}}.$$

Observe that $h_\zeta$ is harmonic on $G_\zeta$ and continuous on $\bar{G}_\zeta$, $h_\zeta(\zeta) = 0$, $h_\zeta > 0$ on $bG_\zeta \setminus \{\zeta\}$ and there exists a constant $c > 0$ such that $h_\zeta > c$ on $bG_\zeta \setminus b\Omega$ for every $\zeta \in b\Omega \cap B^n(0, r)$. Choose $C > 0$ so large that $\psi \leq C$ on $U$ and set $M := C/c$. Then, since $\psi \equiv 0$ on $b\Omega \cap U$, we have $\psi \leq Mh_\zeta$ on $bG_\zeta$ for every $\zeta \in b\Omega \cap B^n(0, r)$. By subharmonicity of $\psi$, it follows that $\psi \leq Mh_\zeta$ on $\bar{G}_\zeta$. In particular,

$$\begin{aligned}\psi(\zeta + tN_\Omega(\zeta)) \leq Mh_\zeta(\zeta + tN_\Omega(\zeta)) &= M\Big(\frac{1}{r^{2n-2}} - \frac{1}{(r+t)^{2n-2}}\Big) \\ &\leq M(2n-2)\frac{1}{r^{2n-1}} t =: Lt\end{aligned}$$

for every $t \in (0, r)$ and $\zeta \in b\Omega \cap B^n(0, r)$. This shows that $\psi(z) \leq L \operatorname{dist}(z, b\Omega)$ for every $z \in \{\zeta + tN_\Omega(\zeta) \in \mathbb{C}^n : \zeta \in b\Omega \cap B^n(0, r), t \in [0, r)\}$.

Now assume, to get a contradiction, that there exists $\xi_0 \in H_0(\tilde{\varphi})$ such that $\operatorname{Lev}(\tilde{\varphi})(0, \xi_0) \leq 0$ and $\|\xi_0\| = 1$. Choose $\varepsilon > 0$ such that $\tilde{\psi} := \psi - \varepsilon \|\cdot\|^2$ is still plurisubharmonic on $U$. We claim that $\tilde{\psi} < 0$ on the punctured complex disc $\Delta_{\xi_0}(0,\delta) \setminus \{0\} := \{\lambda\xi_0 : \lambda \in \mathbb{C}, 0 < |\lambda| < \delta\}$, provided $\delta > 0$ is chosen small enough. Since $\tilde{\psi}$ is subharmonic on $\Delta_{\xi_0}(0,\delta)$, and since $\tilde{\psi}(0) = 0$, this will be a contradiction to the maximum principle. Indeed, if $\lambda \in \mathbb{C} \setminus \{0\}$ is chosen in such a way that $\lambda\xi_0 \in \bar{\Omega}$, then the statement is trivial. But otherwise we can use (3.1) and the estimates on $\psi$ and $\tilde{\varphi}$ which where given above to see that

$$\tilde{\psi}(\lambda\xi_0) = \big(\frac{\psi}{\tilde{\varphi}}\tilde{\varphi} - \varepsilon\|\cdot\|^2\big)(\lambda\xi_0) \leq \frac{L}{l}o(|\lambda|^2) - \varepsilon|\lambda|^2,$$

which is negative if $0 < |\lambda| << 1$. This shows that $\mathrm{Lev}(\tilde{\varphi})(0,\cdot\,)$ is positive definit on $H_0(\tilde{\varphi})$.

To conclude the proof of the proposition, choose a smooth function $\chi\colon \mathbb{R} \to \mathbb{R}$ such that $\chi(0) = 0$, $\chi'(0) = 1$ and $\chi''(0) = k$. It follows then by a standard argument that for $k > 0$ large enough the function $\varphi := \chi \circ \tilde{\varphi}$ is strictly plurisubharmonic near 0 as desired (for a version of this argument see, for example, the proof of Lemma 3.1.1 below). □

**Remarks.** 1) The above definition of strictly pseudoconvex open sets in complex manifolds is the same as the one given in [Na62]. In particular, the strictly plurisubharmonic functions $\varphi_z$ that define $\Omega$ near a given point $z \in b\Omega$ are assumed to be continuous. Observe that by dropping the assumption on continuity of the local defining functions $\varphi_z$ one obtains a class of sets that is strictly larger than the class of strictly pseudoconvex sets. Indeed, the function $u(z) := \sum_{j=1}^{\infty} 2^{-j}\log|z - 1/j|$ is well defined and subharmonic on $\mathbb{C}$ such that $u(0) \neq -\infty$, and thus $\psi(z,w) := u(z) + (|z|^2 + |w|^2) - u(0)$ is a strictly plurisubharmonic function on $\mathbb{C}^2$. Consider the open set $\Omega := \{\psi < 0\}$. Then $L := \{0\} \times \mathbb{C} \subset b\Omega$, since $\{1/j\} \times \mathbb{C} \subset \Omega$ for every $j \in \mathbb{N}$. In particular, since $\psi(z,w) = |w|^2 \not\equiv 0$ on $L$, there exists no neighbourhood of $b\Omega$ on which $\psi$ is continuous. Assume, to get a contradiction, that there exist an open neighbourhood $U \subset\subset \mathbb{C}^2$ of $0 \in b\Omega$ and a continuous strictly plurisubharmonic function $\varphi\colon U \to \mathbb{R}$ such that $\Omega \cap U = \{\varphi < 0\}$. Let then $U' \subset\subset U$ be open such that $0 \in U'$ and let $\lambda\colon \mathbb{C}^2 \to (-\infty, 0]$ be smooth such that $\bar{U}' = \{\lambda = 0\}$. After possibly shrinking $U$, we can find $\varepsilon > 0$ such that $\varphi' := \varphi + \varepsilon\lambda$ is still plurisubharmonic on $U$. Since $\varphi$ is continuous, we have $\varphi \equiv 0$ on $L \subset b\Omega$, and thus there exists $c > 0$ such that $\varphi' < -c$ near $L \cap bU$. It follows that $\varphi'|_{L\cap U}$ is a nonconstant subharmonic function that attains a maximum at $0 \in L \cap U$, which is a contradiction. Observe, in particular, that the boundary of a sublevel set of a not necessarily continuous strictly plurisubharmonic function may contain non trivial analytic sets.

2) The described above problem cannot happen if $b\Omega$ satisfies some mild regularity assumptions. Namely, the following analogue of Proposition 3.1.1 holds true in

the $\mathcal{C}^0$-smooth category:

> *Let $\mathcal{M}$ be a complex manifold, let $\Omega \subset \mathcal{M}$ be open and let $b\Omega$ be $\mathcal{C}^0$-smooth at $z_0 \in b\Omega$. Assume that there exist an open neighbourhood $U \subset \mathcal{M}$ of $z_0$ and a strictly plurisubharmonic function $\psi\colon U \to [-\infty, \infty)$ such that $\Omega \cap U = \{\psi < 0\}$. Then, after possibly shrinking $U$, there exists a continuous strictly plurisubharmonic function $\varphi\colon U \to \mathbb{R}$ such that $\Omega \cap U = \{\varphi < 0\}$.*

Indeed, after possibly shrinking $U$, we can assume that $U \subset \mathbb{C}^n$. By continuity of $b\Omega$ at $z_0$, there exists $w \in \mathbb{C}^n \setminus \{0\}$ such that, after possibly further shrinking $U$, $z + tw \in \Omega$ for every $z \in b\Omega \cap U$ and $t \in (0, 1)$. Then, by the same argument as above, it follows that $\psi(z) = \limsup_{t\to 0+} \psi(z + tw) \leq 0$ for every $z \in b\Omega \cap U$. Thus $\psi \equiv 0$ on $b\Omega \cap U$. The existence of the function $\varphi$ now follows from Theorem 2.5 in [Ric68].

Now we begin to prove the existence of global defining functions. We will formulate our results in the general context of strictly $q$-pseudoconvex domains, since the essential part of our proof will be the same in both cases $q = 0$ and $q > 0$. However, at a certain point of our construction a technical problem will occur in the case $q > 0$, which is not present if $q = 0$. This problem is related to the fact that the sum of two $q$-plurisubharmonic functions $\varphi_1, \varphi_2\colon U \to \mathbb{R}$ on an open set $U \subset \mathcal{M}$ will in general be again $q$-plurisubharmonic only if both $\mathrm{Lev}(\varphi_1)(z, \cdot)$ and $\mathrm{Lev}(\varphi_2)(z, \cdot)$ are positive definit on the same $(n - q)$-dimensional subspaces of $T_z(\mathcal{M})$ for every $z \in U$. Thus in the case $q > 0$ we need to keep track of the directions of positivity of the Levi forms of the $q$-plurisubharmonic functions involved in our construction. That is why before stating our theorems on global defining functions we first prove the following lemma which deals with this particular problem of the case $q > 0$.

**Lemma 3.1.1.** *Let $\mathcal{M}$ be a complex manifold of dimension $n$ equipped with a hermitian metric $h$ and let $\Omega \subset \mathcal{M}$ be a strictly $q$-pseudoconvex domain with smooth boundary for some $q \in \{0, 1, \ldots, n-2\}$. Then for every smooth function $\varphi\colon V \to \mathbb{R}$ defined on an open neighbourhood $V \subset \mathcal{M}$ of $b\Omega$ such that $\Omega \cap V = \{\varphi < 0\}$ and $d\varphi \neq 0$ on $b\Omega$, there exist a neighbourhood $V' \subset V$ of $b\Omega$ and for each $z \in V'$ an $(n - q)$-dimensional complex subspace $L_z \subset T_z(\mathcal{M})$ such that the following assertion holds true: for every open set $U \subset\subset \mathcal{M}$ there exist a strictly increasing and strictly convex smooth function $\mu\colon \mathbb{R} \to \mathbb{R}$ satisfying $\mu(0) = 0$ and a constant $c > 0$ such that $Lev(\mu \circ \varphi)(z, \xi) \geq c\|\xi\|_{h_z}$ for every $z \in V' \cap U$ and $\xi \in L_z$.*

**Proof.** Let $\varphi\colon V \to \mathbb{R}$ be a smooth function defined on an open neighbourhood $V \subset \mathcal{M}$ of $b\Omega$ such that $\Omega \cap V = \{\varphi < 0\}$ and $d\varphi \neq 0$ on $b\Omega$. After possibly shrinking $V$, we can assume that $d\varphi \neq 0$ on $V$, and then for every $z \in V$ we write $N_z$ for the orthogonal complement of $H_z(\varphi)$ in $T_z(\mathcal{M})$ with respect to $h_z$.

For every $l = 1, 2, \ldots, n$, denote by $G_l(\mathcal{M})$ the Grassmann bundle of dimension $l$ over $\mathcal{M}$, i.e., for every $z \in \mathcal{M}$ the fiber $G_l(\mathcal{M})_z = \{L \subset T_z(\mathcal{M}) : (z, L) \in$

$G_l(\mathcal{M})\}$ consists of all complex $l$-dimensional subspaces of $T_z(\mathcal{M})$. Write $\pi = \pi_l\colon G_l(\mathcal{M}) \to \mathcal{M}$ for the canonical projection of $G_l(\mathcal{M})$ onto $\mathcal{M}$. Since $\Omega$ is strictly $q$-pseudoconvex with smooth boundary, there exists a closed in $\mathcal{M}$ neighbourhood $V' \subset V$ of $b\Omega$ and a closed subset $\tilde{\mathcal{L}} \subset G_{n-1-q}(\mathcal{M})$ such that $\pi(\tilde{\mathcal{L}}) = V'$ with the following properties: for every $z \in V'$ and every $\tilde{L} \in \tilde{\mathcal{L}}_z := \{\tilde{L} \subset T_z(\mathcal{M}) : (z, \tilde{L}) \in \tilde{\mathcal{L}}\}$ we have $(\partial\varphi)_z(\,\cdot\,) \equiv 0$ on $\tilde{L}$ and $\mathrm{Lev}(\varphi)(z, \cdot\,) > 0$ on $\tilde{L} \setminus \{0\}$. Set

$$\mathcal{L} := \big\{(z, L) \in G_{n-q}(\mathcal{M})|_{V'} : L = \tilde{L} \oplus N_z \text{ for some } \tilde{L} \in \tilde{\mathcal{L}}_z\big\}.$$

We claim that $V'$ and any choice of $\{L_z\}_{z\in V'}$ such that $L_z \in \mathcal{L}_z$ for every $z \in V'$ are a neighbourhood of $b\Omega$ and a family of complex subspaces as desired.

Indeed, let $U \subset\subset \mathcal{M}$ be open. Define a map $\tau = \tau_l\colon G_l(\mathcal{M}) \to \mathcal{P}(T(\mathcal{M}))$ from $G_l(\mathcal{M})$ to the set of subsets of $T(\mathcal{M})$ by $\tau((z, L)) := \bigcup_{\xi\in L, \|\xi\|_{h_z}=1}(z, \xi)$. Let $\tilde{\mathcal{S}} := \tau(\tilde{\mathcal{L}})$, $\mathcal{S} := \tau(\mathcal{L})$ and $\mathcal{S}_0 := \{(z, \xi) \in \mathcal{S} : \mathrm{Lev}(\varphi)(z, \xi) \le 0\}$. Observe that, by construction, $\tilde{\mathcal{S}} = \{(z, \xi) \in \mathcal{S} : (\partial\varphi)_z(\xi) = 0\} \subset \mathcal{S} \setminus \mathcal{S}_0$, and $\mathcal{S}_0$ is closed in $T(\mathcal{M})$. In particular, for every $z \in V'$ we have that $\delta_0(z) := \min_{\xi\in\mathcal{S}_{0,z}} |(\partial\varphi)_z(\xi)| > 0$, where $S_{0,z} := \{\xi \in T_z(\mathcal{M}) : (z, \xi) \in \mathcal{S}_0\}$. Moreover, since $\mathcal{S} \setminus \mathcal{S}_0$ is an open neighbourhood of $\tilde{\mathcal{S}}$ in $\mathcal{S}$, one sees easily that it is possible to choose a continuous function $\delta\colon V' \to (0, \infty)$ such that $\delta(z) < \delta_0(z)$ for every $z \in V'$. Let $C\colon V' \to \mathbb{R}$ be a continuous function such that $\mathrm{Lev}(\varphi)(z, \xi) > C(z)$ for every $(z, \xi) \in \mathcal{S}$. Now choose $k > 0$ so large that $C(z) + 2k\delta^2(z) > 0$ on $V' \cap \bar{U}$ and define $\mu\colon \mathbb{R} \to \mathbb{R}$ as $\mu(t) := te^{kt}$. Then $\mathrm{Lev}(\mu \circ \varphi)(z, \xi) = \mathrm{Lev}(\varphi)(z, \xi) + 2k|(\partial\varphi)_z(\xi)|^2 > 0$ for every $z \in V' \cap \bar{U}$ and $\xi \in \mathcal{S}_{0,z}$, and clearly $\mathrm{Lev}(\mu \circ \varphi)(z, \xi) > 0$ for every $z \in V' \cap \bar{U}$ and $\xi \in \mathcal{S} \setminus \mathcal{S}_0$. Since $\mathcal{S}$ is closed in $T(\mathcal{M})$, and since $\mathcal{S} = \tau(\mathcal{L})$, it follows that there exists a constant $c > 0$ such that $\mathrm{Lev}(\mu \circ \varphi)(z, \cdot\,) \ge c\|\cdot\|^2$ on $L$ for every $z \in V' \cap \bar{U}$ and $L \in \mathcal{L}_z$. □

After these preparations we can now prove the first two theorems of this section.

**Theorem 3.1.1.** *Let $\mathcal{M}$ be a complex manifold of dimension $n$, let $\Omega \subset \mathcal{M}$ be a strictly $q$-pseudoconvex domain with smooth boundary for some $q \in \{0, 1, \ldots, n-2\}$ and let $f\colon b\Omega \to \mathbb{R}$ be a smooth function that is bounded from below. Then there exists a smooth $q$-plurisubharmonic function $F$ defined on an open neighbourhood of $\bar{\Omega}$ such that $F|_{b\Omega} = f$ and $F$ is strictly $q$-plurisubharmonic near $b\Omega$.*

**Proof.** Since $f$ is bounded from below, we can assume without loss of generality that $f > 0$. Let $\tilde{F}\colon \mathcal{M} \to (0, \infty)$ be a smooth extension of $f$. Choose open sets $U'_j \subset\subset U_j \subset\subset \mathcal{M}$ such that $\{U'_j\}_{j=1}^{\infty}$ covers $b\Omega$ and $\{U_j\}_{j=1}^{\infty}$ is locally finite.

Let $\beta\colon (0, \infty) \to (0, \infty)$ be a strictly increasing and strictly convex smooth function such that $\beta(t) := e^{-1/t}$ for small values of $t$, and let $\tilde{\beta}\colon \mathbb{R} \to [0, \infty)$ be the smooth extension of $\beta$ such that $\tilde{\beta}|_{(-\infty,0]} \equiv 0$. We will construct a family $\{\chi_j\}_{j=1}^{\infty}$ of smooth functions $\chi_j\colon \mathcal{M} \to [0,1]$ such that $\{\chi_j > 0\} = U_j'$, $\sum_{j=1}^{\infty} \chi_j \leq 1$ on $\mathcal{M}$, $\sum_{j=1}^{\infty} \chi_j \equiv 1$ near $b\Omega$ and such that the trivial extension $g_j\colon \mathcal{M} \to [0,\infty)$ of the function $\beta^{-1} \circ (\tilde{F}\chi_j)\colon U_j' \to \mathbb{R}$ by 0 is smooth on $\mathcal{M}$. For this purpose let $\delta_j\colon \mathcal{M} \to \mathbb{R}$ be smooth such that $U_j' = \{\delta_j > 0\}$ and $\mathcal{M} \setminus \bar{U}_j' = \{\delta_j < 0\}$ and let $\psi_j := \tilde{\beta} \circ \tilde{\beta} \circ \delta_j$. Further, let $\theta\colon \mathcal{M} \to [0,\infty)$ be smooth such that $\theta > 0$ on $\mathcal{M} \setminus \bigcup_{j=1}^{\infty} U_j'$ and such that $\theta \equiv 0$ near $b\Omega$. Then choosing $\chi_j := \psi_j/\big(\theta + \sum_{k=1}^{\infty} \psi_k\big)$ and writing $\sigma := \tilde{F}/\big(\theta + \sum_{k=1}^{\infty} \psi_k\big)$ we get for points in $U_j'$ close to $bU_j'$ that $\beta^{-1} \circ (\tilde{F}\chi_j) = \beta^{-1}\big(\sigma \cdot (\beta \circ \beta \circ \delta_j)\big) = \big[-\log\big(\sigma \cdot (\beta \circ \beta \circ \delta_j)\big)\big]^{-1} = \big[-\log \sigma - \log(\beta \circ \beta \circ \delta_j)\big]^{-1} = \big[1/(\beta \circ \delta_j) - \log \sigma\big]^{-1} = (\beta \circ \delta_j)/\big(1 - (\beta \circ \delta_j) \log \sigma\big)$. Hence $\beta^{-1} \circ (\tilde{F}\chi_j)$ extends smoothly to $\mathcal{M}$ by 0, since $\beta$ extends smoothly to $\tilde{\beta}$. The other properties are clear from the construction.

Fix a hermitian metric $h$ on $\mathcal{M}$. Let $\varphi\colon V \to \mathbb{R}$ be a smooth function defined on an open neighbourhood $V \subset \mathcal{M}$ of $b\Omega$ such that $\Omega \cap V = \{\varphi < 0\}$ and $d\varphi \neq 0$ on $b\Omega$. By Lemma 3.1.1, there exist an open neighbourhood $V' \subset V$ of $b\Omega$ and for every $z \in V'$ an $(n-q)$-dimensional complex subspace $L_z \subset T_z(\mathcal{M})$ with the following properties: for each $j \in \mathbb{N}$ there exist a number $c_j > 0$ and a strictly increasing strictly convex smooth function $\mu_j\colon \mathbb{R} \to \mathbb{R}$ satisfying $\mu_j(0) = 0$ such that the function $\varphi_j := \mu_j \circ \varphi$ satisfies $\operatorname{Lev}(\varphi_j)(z,\xi) \geq c_j \|\xi\|_{h_z}$ for every $z \in V' \cap U_j$ and $\xi \in L_z$. Moreover, without loss of generality, we can assume that $U_j \subset\subset V'$ for every $j \in \mathbb{N}$.

Fix $j \in \mathbb{N}$. Let $\lambda_j\colon \mathcal{M} \to (-\infty, 0]$ be smooth such that $\bar{U}_j' = \{\lambda_j = 0\}$. Then choose $\varepsilon_j > 0$ so small and $C_j > 0$ so large that $\operatorname{Lev}\big(g_j + C_j(\varphi_j + \varepsilon_j\lambda_j)\big)(z, \cdot\,)$ is positive definit on $L_z$ for every $z \in U_j$. Observe that, by construction, $g_j + C_j(\varphi_j + \varepsilon_j\lambda_j) < 0$ on $bU_j \cap \bar{\Omega}$, hence the function $\tilde{\beta} \circ (g_j + C_j(\varphi_j + \varepsilon_j\lambda_j))|_{U_j}$ vanishes near this set and thus its trivial extension by 0 to the open neighbourhood $\mathcal{U}_j := \mathcal{M} \setminus \{z \in bU_j : (g_j + C_j(\varphi_j + \varepsilon_j\lambda_j))(z) \geq 0\}$ of $\bar{\Omega}$ defines a smooth $q$-plurisubharmonic function $F_j\colon \mathcal{U}_j \to [0,\infty)$ such that $F_j|_{b\Omega} = f\chi_j$ and $F_j \equiv 0$ outside $U_j$. Moreover, $W_j := \{F_j > 0\} \subset U_j$ is an open neighbourhood of $b\Omega \cap U_j'$ such that $F_j$ is strictly $q$-plurisubharmonic on $W_j$. In particular, $\operatorname{Lev}(F_j)(z, \cdot\,) > 0$ on $L_z \setminus \{0\}$ for every $z \in W_j$ and $\operatorname{Lev}(F_j)(z, \cdot\,) \equiv 0$ if $z \notin W_j$.

Set $F := \sum_{j=1}^{\infty} F_j$. Then $F$ is a well defined smooth function on the open neighbourhood $\mathcal{U} := \bigcap_{j=1}^{\infty} \mathcal{U}_j \subset \mathcal{M}$ of $\bar{\Omega}$. By construction, $F|_{b\Omega} = f$. Moreover, $\operatorname{Lev}(F)(z, \cdot\,) > 0$ on $L_z \setminus \{0\}$ for every $z \in W := \bigcup_{j=1}^{\infty} W_j \supset b\Omega$ and $\operatorname{Lev}(F)(z, \cdot\,) \equiv 0$ if $z \notin W$. Hence $F$ is a function as desired. □

**Theorem 3.1.2.** *Let $\mathcal{M}$ be a complex manifold of dimension $n$ and let $\Omega \subset \mathcal{M}$ be a strictly $q$-pseudoconvex domain with smooth boundary for some $q \in \{0, 1, \ldots, n-2\}$. Then there exists a smooth $q$-plurisubharmonic function $\varphi$ defined on an* *open neighbourhood of* $\bar{\Omega}$ *such that* $\Omega = \{\varphi < 0\}$, $d\varphi \neq 0$ *on* $b\Omega$ *and* $\varphi$ *is strictly* $q$*-plurisubharmonic near* $b\Omega$.

**Proof.** Let $\varphi := F - 1$, where $F$ is the function from Theorem 3.1.1 corresponding to the boundary values $f \equiv 1$. Then $\varphi$ is a smooth $q$-plurisubharmonic function on an open neighbourhood of $\bar{\Omega}$ that vanishes identically on $b\Omega$ and that is strictly $q$-plurisubharmonic near $b\Omega$. Observe that in the construction of $F$ we can choose $\tilde{F}$ such that $\Omega = \{\tilde{F} < 1\}$ and $\mathcal{M} \setminus \bar{\Omega} = \{\tilde{F} > 1\}$. Moreover, after possibly shrinking $\mathcal{U}$, we can assume that $\sum_{j=1}^{\infty} \chi_j \equiv 1$ on $\mathcal{U} \setminus \bar{\Omega}$. For $z \in \mathcal{M}$ let $I(z) := \{j \in \mathbb{N} : z \in W_j\}$ and $J(z) := \{j \in \mathbb{N} : z \in U_j'\}$. Then

$$\begin{aligned} F(z) &= \sum_{j \in I(z)} F_j(z) = \sum_{j \in I(z)} (\tilde{\beta} \circ (g_j + C_j(\varphi_j + \varepsilon_j \lambda_j)))(z) \leq \sum_{j \in I(z)} (\tilde{\beta} \circ g_j)(z) \\ &= \sum_{j \in I(z)} (\tilde{F}\chi_j)(z) \leq \tilde{F}(z) < 1 \quad \text{for } z \in \Omega \end{aligned}$$

(here the sum over the empty index set is understood to be zero), and

$$\begin{aligned} F(z) &\geq \sum_{j \in J(z)} F_j(z) = \sum_{j \in J(z)} (\tilde{\beta} \circ (g_j + C_j(\varphi_j + \varepsilon_j \lambda_j)))(z) \geq \sum_{j \in J(z)} (\tilde{\beta} \circ g_j)(z) \\ &= \sum_{j \in J(z)} (\tilde{F}\chi_j)(z) = \tilde{F}(z) > 1 \quad \text{for } z \in \mathcal{U} \setminus \bar{\Omega}. \end{aligned}$$

This shows that $\Omega = \{F < 1\}$, i.e., $\Omega = \{\varphi < 0\}$. Finally, we have $d\varphi \neq 0$ on $b\Omega$, provided that the numbers $C_j$ which appear in the construction of $F$ are chosen large enough (in fact, since $b\Omega$ is smooth, the non-vanishing of $d\varphi$ along $b\Omega$ is automatically satisfied, see, for example, the proof of Proposition 1.5.16 in [HeL84], which can be adapted easily to the case of $q$-plurisubharmonic functions). □

**Remarks.** 1) The assumption in Theorem 3.1.1 that $f$ is bounded from below is crucial. In fact, it was shown in Example 8.2 of [ShT99] that there exist an unbounded strictly pseudoconvex domain $\Omega \subset \mathbb{C}^2$ with smooth boundary and a smooth function $f: b\Omega \to \mathbb{R}$ that is not bounded from below such that the only plurisubharmonic function $F: \Omega \to [-\infty, \infty)$ satisfying $\limsup_{z \to z_0} F(z) \leq f(z_0)$ for every $z_0 \in b\Omega$ is the function $F \equiv -\infty$.

2) The function $F$ from Theorem 3.1.1 is strictly $q$-plurisubharmonic on the open neighbourhood $W$ of $b\Omega$ and $F$ is a constant on $\Omega \setminus W$. It is clear from the construction that for every open set $\omega \subset \Omega$ such that $\bar{\omega} \subset \Omega$ we can choose $F$ in such a way that $F$ is constant on $\omega$.

3) Let $h$ be a hermitian metric on $\mathcal{M}$ and let $\nu, \mu\colon b\Omega \to (0,\infty)$ be positive continuous functions. Then $F$ can be chosen in such a way that $\|(dF)_z\|_{h_z^*} \geq \nu(z)$ for every $z \in b\Omega$ and $\operatorname{Lev}(F)(z, \cdot\,) \geq \mu(z)\|\cdot\|_{h_z}^2$ on $L_z$ for every $z \in b\Omega$. Indeed, for every $j \in \mathbb{N}$ let $U_j'' \subset\subset U_j'$ be an open set such that $\{U_j''\}_{j=1}^{\infty}$ still covers $b\Omega$. Now in the construction of $F$ we can choose for every $j \in \mathbb{N}$ the numbers $\varepsilon_j > 0$ so small and $C_j > 0$ so large that $(dF_j)_z(N_\Omega(z)) \geq 0$ for every $z \in b\Omega$, $(dF_j)_z(N_\Omega(z)) \geq \nu(z)$ for every $z \in b\Omega \cap U_j''$ and $\operatorname{Lev}(F_j)(z, \cdot\,) \geq \mu(z)\|\cdot\|_{h_z}^2$ on $L_z$ for every $z \in b\Omega \cap U_j''$, where $N_\Omega(z)$ denotes the outward unit normal to $b\Omega$ at $z$ with respect to $h$. Then $F$ is a function as desired.

4) The statements of Theorems 3.1.1 and 3.1.2 as well as the above remarks remain true if $\mathcal{C}^\infty$-smoothness is replaced by $\mathcal{C}^s$-smoothness for $s \geq 2$. If $q = 0$ and if for each point $z \in b\Omega$ there exists an open neighbourhood $U \subset \mathcal{M}$ of $z$ and a $\mathcal{C}^1$-smooth strictly plurisubharmonic function $\varphi\colon U \to \mathbb{R}$ such that $\Omega \cap U = \{\varphi < 0\}$ and $d\varphi \neq 0$ on $b\Omega \cap U$ (note that this is a stronger assumption than $\Omega$ being strictly pseudoconvex with $\mathcal{C}^1$-smooth boundary), and if $f\colon b\Omega \to \mathbb{R}$ is $\mathcal{C}^2$-smooth (i.e., for every $z \in b\Omega$ there exists an open neighbourhood $U_z \subset \mathcal{M}$ of $z$ and a $\mathcal{C}^2$-smooth function $F_z\colon U_z \to \mathbb{R}$ such that $F_z$ coincides with $f$ on $b\Omega \cap U_z$), then a statement analoguous to Theorem 3.1.1 holds true with $\mathcal{C}^1$-smooth $F$. Further, if $\Omega$ is just strictly pseudoconvex (with no smoothness assumptions on $b\Omega$) and if $f\colon b\Omega \to \mathbb{R}$ is $\mathcal{C}^2$-smooth, then there always exists a continuous plurisubharmonic function $F$ as in Theorem 3.1.1. Analoguous generalizations are possible for Theorem 3.1.2 (but of course no assertion on the differential of $\varphi$ is imposed if $s = 0$). Moreover, when considering the case of possibly nonsmooth boundaries, it is also worth mentioning that we do not need connectedness of the set $\Omega$ in the proofs of Theorem 3.1.1 and Theorem 3.1.2. In particular, every strictly pseudoconvex open set in a complex manifold admits a continuous global defining function.

5) Finally, we want to mention without giving the details of the proof that it is possible to weaken the assumptions on smoothness of $b\Omega$ even further. Indeed, in Theorem 3.1.1 it suffices to assume that $\Omega$ can be represented locally near each boundary point as the sublevel set of a $\mathcal{C}^\infty$-smooth strictly $q$-plurisubharmonic function with possibly vanishing differential along $b\Omega$ (or, more general, as the sublevel set of a $\mathcal{C}^s$-smooth strictly $q$-plurisubharmonic function for some $s \geq 2$ or $(q, s) = (0, 1)$, but then the function $F$ from Theorem 3.1.1 will only be $\mathcal{C}^s$-smooth in general). Domains of this type were considered, for example, in [HeL84] and [HeL88]. If $q = 0$, this is clear. In the case $q > 0$ this is a consequence of the following fact: If $\Omega \subset \mathcal{M}$ is open, $z \in b\Omega$, $U \subset \mathcal{M}$ is an open neighbourhood of $z$ and $\varphi_1, \varphi_2\colon U \to \mathbb{R}$ are $\mathcal{C}^2$-smooth functions such that $\Omega \cap U = \{\varphi_1 < 0\} = \{\varphi_2 < 0\}$, then for every $\xi \in H_z(\varphi_1) = H_z(\varphi_2)$ (see again Proposition 1.5.16 in [HeL84] for the fact that $(d\varphi_1)_z = 0$ if and only if $(d\varphi_2)_z = 0$) we have $\operatorname{Lev}(\varphi_1)(z, \xi) \geq 0$ if and only if $\operatorname{Lev}(\varphi_2)(z, \xi) \geq 0$ (see, for example, the proof of Proposition 15 in [AG62]). In particular, the sum $\varphi_1 + \varphi_2$ is strictly $q$-plurisubharmonic near $z$ if

both $\varphi_1$ and $\varphi_2$ are strictly $q$-plurisubharmonic near $z$ and $(d\varphi_1)_z = (d\varphi_2)_z = 0$. Thus, if $\Sigma(b\Omega)$ denotes the set of points $z \in b\Omega$ such that $b\Omega$ is smooth in $z$, then the function $F = \sum_{j=1}^{\infty} F_j$ that appears in the proof of Theorem 3.1.1 will be automatically strictly $q$-plurisubharmonic in a neighbourhood of $b\Omega \setminus \Sigma(b\Omega)$, and strict $q$-plurisubharmonicity near the remaining part of $b\Omega$ can be achieved as before. The same weakening of assumptions is possible in Theorem 3.1.2, but then the constructed function $\varphi$ can be guaranteed to have nonvanishing differential only along $\Sigma(b\Omega)$.

Our next goal is to show that the core is the only obstruction for strict plurisubharmonicity of global defining functions, i.e., we want to construct a global defining function that is strictly plurisubharmonic precisely in the complement of $\mathfrak{c}(\Omega)$ (in particular, we now work in the case $q = 0$). This will give a stronger version of the statement of Theorem 3.1.2, namely, the Main Theorem. For the proof we will use the following notion of smooth maximum: Let $\delta > 0$ and let $\chi_\delta \colon \mathbb{R} \to \mathbb{R}$ be a smooth function such that $\chi$ is strictly convex for $|t| < \delta/2$ and $\chi_\delta(t) = |t|$ for $|t| \geq \delta/2$. Then we define a smooth maximum by

$$\widetilde{\max}_\delta(x, y) := \frac{x + y + \chi_\delta(x - y)}{2}.$$

Observe that the smooth maximum of two smooth (strictly) plurisubharmonic functions is again a smooth (strictly) plurisubharmonic function (see, for example, Corollary 4.14 in [HeL88]). Moreover, $\widetilde{\max}_\delta(x, y) = \max(x, y)$ if $|x - y| \geq \delta$.

**Main Theorem.** Every strictly pseudoconvex domain $\Omega$ with smooth boundary in a complex manifold $\mathcal{M}$ admits a bounded global defining function that is strictly plurisubharmonic outside $\mathfrak{c}(\Omega)$.

**Proof.** For every $p \in \Omega \setminus \mathfrak{c}(\Omega)$, there exists a smooth global defining function $\psi_p$ for $\Omega$ that is strictly plurisubharmonic on an open neighbourhood $V_p \subset\subset \Omega \setminus \mathfrak{c}(\Omega)$ of $p$. Indeed, let $\varphi_1 \colon \Omega \to \mathbb{R}$ be smooth plurisubharmonic and bounded from above such that $\varphi_1$ is strictly plurisubharmonic at $p$, and let $\varphi_2 \colon \bar{\Omega} \to \mathbb{R}$ be a smooth global defining function for $\Omega$. Then we can choose $\varphi_p := \widetilde{\max}_1(\varphi_1 - C_1, C_2\varphi_2)$, where $C_1, C_2 > 0$ are constants such that $\varphi_1 - C_1 < C_2\varphi_2 - 1$ near $b\Omega$ and $C_2\varphi_2(p) < \varphi_1(p) - C_1 - 1$.

Let $\{p_j\}_{j=1}^{\infty}$ be a sequence of points $p_j \in \Omega$ such that $\bigcup_{j=1}^{\infty} V_{p_j} = \Omega \setminus \mathfrak{c}(\Omega)$. Without loss of generality we can assume that each set $V_{p_j}$ is contained in some coordinate patch of $\mathcal{M}$. Choose a sequence $\{\delta_j\}_{j=1}^{\infty}$ of positive numbers $\delta_j$ such that $\delta_j \psi_{p_j} > -1/2$ on $V_{p_j}$ for every $j \in \mathbb{N}$. Moreover, let $\{\varepsilon_j\}_{j=1}^{\infty}$ be a second sequence of suitably chosen positive numbers $\varepsilon_j$ and define $\phi_1 := \sum_{j=1}^{\infty} \varepsilon_j \widetilde{\max}_{1/2}(\delta_j \psi_{p_j}, -1)$. If $\{\varepsilon_j\}_{j=1}^{\infty}$ converges to zero fast enough, then $\phi_1$ is a smooth plurisubharmonic function on $\bar{\Omega}$ such that $\phi_1$ is strictly plurisubharmonic outside $\mathfrak{c}(\Omega)$, $b\Omega = \{\phi_1 = 0\}$ and $0 \geq \phi_1 > -1$. By construction, $\phi_1$ has nonvanishing differential along $b\Omega$, hence a smooth extension $\varphi$ of $\phi_1$ to a small enough open neighbourhood $\mathcal{U} \subset \mathcal{M}$ of $\bar{\Omega}$ will be a global defining function as desired. □

**Remarks.** 1) In the same way as described in the remarks after Theorem 3.1.2, we can prescribe along $b\Omega$ the size of the differential and the Levi form of the global defining function constructed in the Main Theorem.

2) As for the case of $\mathcal{C}^\infty$-smooth functions, we can define the sets

$$\mathfrak{c}^s(\Omega) := \big\{z \in \Omega : \text{every } \mathcal{C}^s\text{-smooth plurisubharmonic function on } \Omega \text{ that is bounded from above fails to be strictly plurisubharmonic in } z\big\}$$

for every $s \in \mathbb{N}_0^\infty := \{0\} \cup \mathbb{N} \cup \{\infty\}$. Then a statement analoguous to the Main Theorem holds for every $s \in \mathbb{N}_0^\infty$. Observe, however, that it is not clear whether in general $\mathfrak{c}^{s_1}(\Omega) = \mathfrak{c}^{s_2}(\Omega)$ for $s_1 \neq s_2$.

3) One can also define yet another version of the core as

$$\tilde{\mathfrak{c}}(\Omega) := \big\{z \in \Omega : \text{every plurisubharmonic function on } \Omega \text{ that is bounded from above and not identically } -\infty \text{ on any connected component of } \Omega \text{ fails to be strictly plurisubharmonic in } z\big\}.$$

Observe that this definition leads to a weaker notion, i.e., in general we have $\tilde{\mathfrak{c}}(\Omega) \subsetneq \mathfrak{c}(\Omega)$. For example, the function $\varphi(z, w) := \log|w - f(z)| + C_1(|z|^2 + |w|^2)$ is strictly plurisubharmonic and bounded from above on the domain $\Omega$ from Example II. Hence in this case we have $\tilde{\mathfrak{c}}(\Omega) = \varnothing$, but $\mathfrak{c}(\Omega) \neq \varnothing$. We do not know if there exists a complex manifold $\mathcal{M}$ and a strictly pseudoconvex domain $\Omega \subset \mathcal{M}$ such that $\tilde{\mathfrak{c}}(\Omega) \neq \varnothing$.

4) A result analoguous to the Main Theorem holds also true if $b\Omega$ is only smooth in the weaker sense as it is described in Remark 5 after Theorem 3.1.2. Indeed, to extend $\phi_1$ from $\bar{\Omega}$ to an open neighbourhood of $\bar{\Omega}$ let then $\phi_2$ be a global defining function for $\Omega$ as constructed in Theorem 3.1.2. In particular, $\phi_2$ is defined on an open neighbourhood $U$ of $\bar{\Omega}$, $\phi_2 \geq -1$ and $\phi_2$ is strictly plurisubharmonic on $\{\phi_2 > -1\}$. Then

$$\varphi(z) := \begin{cases} 2\phi_2(z) & , \quad z \in U \setminus \bar{\Omega} \\ \widetilde{\max}_{1/4}(\phi_1(z) - 1/2, 2\phi_2(z)) & , \quad z \in \bar{\Omega} \cap \{\phi_2 > -1\} \\ \phi_1(z) - 1/2 & , \quad z \in \bar{\Omega} \cap \{\phi_2 = -1\} \end{cases}$$

is a function as desired.

Following [SlT04], we introduce the following notion of minimal functions for a domain $\Omega \subset \mathcal{M}$.

**Definition.** Let $\mathcal{M}$ be a complex manifold and let $\Omega \subset \mathcal{M}$ be a domain. A smooth and bounded from above plurisubharmonic function $\varphi\colon \Omega \to \mathbb{R}$ will be called *minimal* if $\varphi$ is strictly plurisubharmonic outside $\mathfrak{c}(\Omega)$.

Our Main Theorem can then be rephrased as follows: every strictly pseudoconvex domain with smooth boundary in a complex manifold admits a bounded minimal global defining function. Moreover, by using similar arguments as in the proof of the Main Theorem, it also follows that every domain in a complex manifold admits a bounded minimal function.

As in the case of plurisubharmonic functions, it now would also be possible to introduce for every domain $\Omega$ in a complex manifold $\mathcal{M}$ the core $\mathfrak{c}(\Omega, q)$ with respect to the class of $q$-plurisubharmonic functions, namely,

$$\mathfrak{c}(\Omega, q) := \big\{z \in \Omega : \text{every smooth } q\text{-plurisubharmonic function on } \Omega \text{ that is bounded from above fails to be strictly } q\text{-plurisubharmonic in } z\big\}.$$

However, we do not know whether this definition is meaningful, in the sense that we do not have any examples of domains $\Omega \subset \mathcal{M}$ such that $\mathfrak{c}(\Omega, q) \neq \varnothing$ for $q > 0$. Indeed, for domains in Stein manifolds the set $\mathfrak{c}(\Omega, q)$ is always empty for every $q > 0$ as it is shown in the following proposition.

**Proposition 3.1.2.** *Every Stein manifold $\mathcal{M}$ admits a smooth and bounded 1-plurisubharmonic function. In particular, $\mathfrak{c}(\Omega, q) = \varnothing$ for every $\Omega \subset \mathcal{M}$ and every $q > 0$.*

**Proof.** Let $\psi\colon \mathcal{M} \to \mathbb{R}$ be a smooth strictly plurisubharmonic function. After replacing $\psi$ by $e^{\psi}$ if necessary, we can assume without loss of generality that $\psi \geq 0$. Define $\chi\colon (-1, \infty) \to \mathbb{R}$ as $\chi(t) := -1/(1+t)$ and consider the bounded smooth function $\varphi := \chi \circ \psi$. Then

$$\operatorname{Lev}(\varphi)(z, \xi) = \chi''(\psi(z))|(\partial\psi)_z(\xi)|^2 + \chi'(\psi(z))\operatorname{Lev}(\psi)(z, \xi)$$

for every $z \in \mathcal{M}$ and $\xi \in T_z(\mathcal{M})$. In particular, $\operatorname{Lev}(\varphi)(z, \cdot\,) > 0$ on the at least $(\dim_{\mathbb{C}} \mathcal{M} - 1)$-dimensional subspace $H_z(\psi) = \{\xi \in \mathbb{C}^n : (\partial\psi)_z(\xi) = 0\}$. □

One might expect that at least compact analytic subsets $A \subset \Omega$ of pure dimension $q + 1$ are always contained in $\mathfrak{c}(\Omega, q)$. However, this is not necessarily the case as it is shown by the following example.

**Example 1.** As in Example I, let $\mathcal{M} := \{(z,x) \in \mathbb{C}^3 \times \mathbb{CP}^2 : z_i x_j = z_j x_i,\ i,j = 0,1,2\}$ be the blow-up of $\mathbb{C}^3$ at the origin. For every $j = 0,1,2$, define mappings $h_j : U_j \to \mathbb{C}^3$ on the dense open subsets $U_j := \{(z,x) \in \mathcal{M} : x_j \neq 0\}$ as

$$h_j(z,x) := \Big(\frac{x_0}{x_j}, \dots, \frac{x_{j-1}}{x_j}, z_j, \frac{x_{j+1}}{x_j}, \dots, \frac{x_2}{x_j}\Big).$$

Each map $h_j$ is a homeomorphism with inverse

$$h_j^{-1}(w_0,w_1,w_2) := \big((w_j w_0, \dots, w_j w_{j-1}, w_j, w_j w_{j+1}, \dots, w_j w_2), \\ [w_0 : \dots : w_{j-1} : 1 : w_{j+1} : \dots : w_2]\big)$$

and the tupel $\{(U_j, h_j) : j = 0,1,2\}$ defines a complex structure on $\mathcal{M}$. For every $j = 0,1,2$, define a smooth function $\varphi_j : \mathcal{M} \to \mathbb{R}$ as

$$\varphi_j(z,x) := -\frac{|x_j|^2}{|z_j|^2|x_j|^2 + |x_0|^2 + |x_1|^2 + |x_2|^2}.$$

Then

$$(\varphi_j \circ h_j^{-1})(w_0,w_1,w_2) = -\frac{1}{1 + |w_0|^2 + |w_1|^2 + |w_2|^2},$$

and as in the proof of Proposition 3.1.2 we see that this function is strictly 1-plurisubharmonic on $\mathbb{C}^3$. Hence $\varphi_j$ is 1-plurisubharmonic on $\mathcal{M}$ and strictly 1-plurisubharmonic on $U_j$. Now let $\Omega \subset\subset \mathcal{M}$ be the strictly pseudoconvex domain with smooth boundary defined by

$$\Omega := \big\{(z,x) \in \mathcal{M} : \|z\| < 1\big\}.$$

Then the above computations show that for every $(z,x) \in \Omega$ there exists a smooth 1-plurisubharmonic function on $\Omega$ that is bounded from above and that is strictly 1-plurisubharmonic near $(z,x)$, i.e., $\mathfrak{c}(\Omega,1) = \varnothing$. In particular, the pure 2-dimensional compact analytic set $\{0\} \times \mathbb{CP}^2 \subset \Omega$ is not contained in $\mathfrak{c}(\Omega,1)$.

Observe also that in general no analogue of the Main Theorem holds true in the case of strictly $q$-pseudoconvex domains $\Omega$ if $q > 0$, i.e., in general it is not possible to have a global defining function for $\Omega$ as in Theorem 3.1.2 that is strictly $q$-plurisubharmonic outside $\mathfrak{c}(\Omega,q)$. Indeed, the domain $\Omega$ from the last example satisfies $\mathfrak{c}(\Omega,1) = \varnothing$, but there exists no smooth strictly 1-plurisubharmonic function on $\Omega$, since there exists no such function on $\mathbb{CP}^2$.

We now want to give an application of our construction of global defining functions.

**Theorem 3.1.3.** *Let $\mathcal{M}$ be a complex manifold and let $\Omega \subset \mathcal{M}$ be a strictly pseudoconvex open set (not necessarily relatively compact or with smooth boundary). Let $U \subset \mathcal{M}$ be an arbitrary open neighbourhood of $b\Omega$. Then the following assertions hold true:*

*(1) There exists a strictly pseudoconvex open set $\Omega' \subset \mathcal{M}$ with smooth boundary such that $\Omega \setminus U \subset \Omega'$, $\bar{\Omega}' \subset \Omega$ and $\mathfrak{c}(\Omega') = \mathfrak{c}(\Omega)$.*

*(2) There exists a strictly pseudoconvex open set $\Omega'' \subset \mathcal{M}$ with smooth boundary such that $\bar{\Omega} \subset \Omega''$, $\bar{\Omega}'' \subset \Omega \cup U$ and $\mathfrak{c}(\Omega'') = \mathfrak{c}(\Omega)$.*

*In particular, $\bar{\Omega}$ admits a neighbourhood basis consisting of strictly pseudoconvex open sets with smooth boundary. Moreover, if $\Omega$ is a domain, then one can also choose $\Omega'$ and $\Omega''$ to be domains.*

**Proof.** Fix an open neighbourhood $U \subset \mathcal{M}$ of $b\Omega$.

(1) We first show the existence of the strictly pseudoconvex set $\Omega'$. Let $\omega \subset \Omega$ be an arbitrary but fixed open set such that $\Omega \setminus U \subset \omega$ and $\bar{\omega} \subset \Omega$. By Theorem 3.1.2 and the related Remarks 2 and 4, there exists a continuous plurisubharmonic function $\varphi$ defined near $\bar{\Omega}$ such that $\Omega = \{\varphi < 0\}$, $\varphi \geq -1$, $\varphi \equiv -1$ on $\omega$ and $\varphi$ is strictly plurisubharmonic on $\{\varphi > -1\}$. Applying Richberg's smoothing procedure (see, for example, Theorem I.5.21 in [De12]), we can then find a continuous plurisubharmonic function $\tilde{\varphi}$ defined near $\bar{\Omega}$ such that $\tilde{\varphi} \geq \varphi$, $\tilde{\varphi} \equiv -1$ on $\omega$, $\tilde{\varphi}$ is smooth and strictly plurisubharmonic on $\{\tilde{\varphi} > -1\}$, and $|\tilde{\varphi} - \varphi| < 1/2$. Let $c \in (-1, -1/2)$ be a regular value of $\tilde{\varphi}$ and set $\Omega' := \{\tilde{\varphi} < c\}$. Then $\Omega'$ is a strictly pseudoconvex open set such that $\Omega \setminus U \subset \Omega'$ and $\bar{\Omega}' \subset \Omega$.

It remains to show that $\mathfrak{c}(\Omega') = \mathfrak{c}(\Omega)$. Since $\Omega' \subset \Omega$, it follows immediately that $\mathfrak{c}(\Omega') \subset \mathfrak{c}(\Omega)$. On the other hand, observe that for small enough $\delta > 0$ the function $\varphi_2 := \widetilde{\max}_\delta(\tilde{\varphi} - c, -(c+1)/2)$ is smooth plurisubharmonic and bounded from above on $\Omega$, strictly plurisubharmonic near $\Omega \setminus \Omega'$ and $\Omega' = \{\varphi_2 < 0\}$. In particular, this shows that $\mathfrak{c}(\Omega) \subset \Omega'$. By repeating the same arguments as in the proof of the Main Theorem, it now follows easily that $\mathfrak{c}(\Omega) \subset \mathfrak{c}(\Omega')$.

(2) We now show the existence of the strictly pseudoconvex set $\Omega''$. After possibly shrinking $U$, let $\varphi$ be a continuous plurisubharmonic function defined on a neighbourhood of $\overline{\Omega \cup U}$ such that $\Omega = \{\varphi < 0\}$, $\varphi \geq -1$, $\varphi > -1/2$ on $U$ and $\varphi$ is strictly plurisubharmonic on $\{\varphi > -1\}$. Without loss of generality we can assume that $\varphi > 0$ outside $\bar{\Omega}$ (in fact, the function $\varphi$ from Theorem 3.1.2 has this property by construction).

We claim that there exists a strictly pseudoconvex open set $\tilde{\Omega}'' \subset \mathcal{M}$ (not necessarily with smooth boundary) such that $\bar{\Omega} \subset \tilde{\Omega}'' \subset \Omega \cup U$. The proof is essentially the same as in Lemma 2 of [To83]: Choose a locally finite open covering $\{U_j\}_{j=1}^{\infty}$ of $b\Omega$ by open sets $U_j \subset\subset U$. For every $j \in \mathbb{N}$, let $\eta_j \colon \mathcal{M} \to (-\infty, 0]$ be a smooth function such that $\{\eta_j < 0\} = U_j$. Set $\phi := \varphi + \sum_{j=1}^{\infty} \varepsilon_j \eta_j$ with positive constants $\varepsilon_j$, $j \in \mathbb{N}$. Clearly, $\phi > 0$ on $b(\Omega \cup U)$, $\phi < 0$ on $\bar{\Omega}$ and if the numbers $\varepsilon_j$ are chosen small enough, then $\phi$ is still strictly plurisubharmonic on $U$. Set $\tilde{\Omega}'' := \{\phi < 0\}$.

Note that, by a suitable choice of the numbers $\varepsilon_j$, we can also guarantee that $\mathfrak{c}(\tilde{\Omega}'') = \mathfrak{c}(\Omega)$. Indeed, since $\Omega \subset \tilde{\Omega}''$, it is immediately clear that $\mathfrak{c}(\Omega) \subset \mathfrak{c}(\tilde{\Omega}'')$. Further, observe that in the construction of $\phi$ we can choose the numbers $\varepsilon_j$ so small that $\phi > -1/2$ on $U$. Thus we can use the same smoothing procedure as in the proof of part (1) (choosing $c = -1/2$) to obtain a smooth and bounded from above plurisubharmonic function $\phi_2 \colon \tilde{\Omega}'' \to [-1/4, \infty)$ such that $\phi_2 > 0$ on $U$ and $\phi_2$ is strictly plurisubharmonic on $\{\phi_2 > -1/4\}$. Then, as before, the same argument as in the proof of the Main Theorem shows that $\mathfrak{c}(\tilde{\Omega}'') \subset \mathfrak{c}(\Omega)$.

Now we can apply part (1) of the theorem to the strictly pseudoconvex set $\tilde{\Omega}''$ and an open neighbourhood of $b\tilde{\Omega}''$ that does not intersect $\Omega$ to obtain a set $\Omega''$ as desired. This completes the proof of (2).

The last two properties claimed in the theorem are obvious by the construction.□

At the end of this section we want to prove again, but in a different way, the existence of global defining functions for strictly pseudoconvex domains $\Omega$ with $\mathcal{C}^\infty$-smooth boundary. We first prove the existence of defining functions for $\Omega$ that have prescribed differentials along the boundary of $\Omega$. In a next step we use this result to construct a global defining function for $\Omega$.

**Lemma 3.1.2.** *Let $\Omega$ be a strictly pseudoconvex domain with smooth boundary in a complex manifold $\mathcal{M}$. Let $h$ be a hermitian metric on $\mathcal{M}$ and let $f \colon b\Omega \to (0, \infty)$ be a smooth positive function. Then there exists an open neighbourhood $V \subset \mathcal{M}$ of $b\Omega$ and a smooth strictly plurisubharmonic function $\psi \colon V \to \mathbb{R}$ such that $\Omega \cap V = \{\psi < 0\}$ and $\|d\psi\|_{h^*} = f$ on $b\Omega$.*

**Proof.** Let $\rho \colon V \to \mathbb{R}$ be a smooth function on an open neighbourhood $V \subset \mathcal{M}$ of $b\Omega$ such that $\Omega \cap V = \{\rho < 0\}$ and $d\rho \neq 0$ on $b\Omega$. Let $q \colon b\Omega \to (0, \infty)$ be a positive smooth function that we consider to be fixed, but that will be further specified later on. Choose smooth extensions $F \colon V \to (0, \infty)$ of $f / \|d\rho\|_{h^*} \colon b\Omega \to (0, \infty)$ and $Q \colon V \to (0, \infty)$ of $q \colon b\Omega \to (0, \infty)$, respectively, and define $\psi \colon V \to \mathbb{R}$ as

$$\psi(z) := F(z)\rho(z) + Q(z)\rho(z)^2.$$

Then $\psi$ is smooth, $\|d\psi(z)\|_{h^*} = f(z)$ for every $z \in b\Omega$ and, after possibly shrinking $V$, $\Omega \cap V = \{\psi < 0\}$. By smoothness of $\psi$, it only remains to show that $\psi$ is strictly plurisubharmonic at every point $z \in b\Omega$. We claim that this is always the case, provided that the function $q$ is chosen large enough (observe that the Levi form of $F \cdot \rho$ in $z \in b\Omega$ is automatically positive definit on the complex tangent space $T_z^{\mathbb{C}}(b\Omega)$ of $b\Omega$ in $z$, since $\Omega$ is strictly pseudoconvex).

Indeed, a straightforward calculation shows that for every $z \in b\Omega$ and $\xi \in T_z(\mathcal{M})$

$$\mathrm{Lev}(\psi)(z,\xi) = F(z)\mathrm{Lev}(\rho)(z,\xi) + 2\mathrm{Re}\,[(\partial\rho)_z(\xi)\cdot(\bar{\partial}F)_z(\xi)] + 2q(z)|(\partial\rho)_z(\xi)|^2. \tag{3.2}$$

Since $T_z^{\mathbb{C}}(b\Omega) = H_z(\rho)$, we have $\mathrm{Lev}(\psi)(z,\,\cdot\,) = F(z)\mathrm{Lev}(\rho)(z,\,\cdot\,)$ on $T_z^{\mathbb{C}}(b\Omega)$, and by strict pseudoconvexity of $\Omega$ we know that $\mathrm{Lev}(\rho)(z,\,\cdot\,)$ is positive definit on $T_z^{\mathbb{C}}(b\Omega)$ for every $z \in b\Omega$. Let $K := \{(z,\xi) \in T(\mathcal{M})|_{b\Omega} : \|\xi\|_{h_z} = 1\}$ and define $K_0 \subset K$ to be the subset $K_0 := \{(z,\xi) \in K : F(z)\mathrm{Lev}(\rho)(z,\xi) + 2\mathrm{Re}\,[(\partial\rho)_z(\xi)\cdot(\bar{\partial}F)_z(\xi)] \le 0\}$. Since $\rho$ and $F$ are smooth, we can choose a smooth function $C\colon b\Omega \to \mathbb{R}$ such that $F(z)\mathrm{Lev}(\rho)(z,\xi) + 2\mathrm{Re}\,[(\partial\rho)_z(\xi)\cdot(\bar{\partial}F)_z(\xi)] > C(z)$ for every $(z,\xi) \in K$. Moreover, observe that, by construction, $(\partial\rho)_z(\xi) \neq 0$ for every $(z,\xi) \in K_0$. Hence we can further choose a positive smooth function $\varepsilon\colon b\Omega \to (0,\infty)$ such that $|(\partial\rho)_z(\xi)|^2 > \varepsilon(z)$ for every $(z,\xi) \in K_0$. Now assume that $q\colon b\Omega \to (0,\infty)$ is chosen so large that $C + 2q\varepsilon > 0$ on $b\Omega$. Then we conclude from (3.2) and the choice of $C$ that $\mathrm{Lev}(\psi)(z,\xi) > 0$ on $K_0$. But it is clear from the choice of $K_0$ that $\mathrm{Lev}(\psi)(z,\xi) > 0$ on $K \setminus K_0$. Thus $\psi$ is strictly plurisubharmonic at every point $z \in b\Omega$ as claimed. □

**Theorem 3.1.2′.** *Let $\mathcal{M}$ be a complex manifold and let $\Omega \subset \mathcal{M}$ be a strictly pseudoconvex domain with smooth boundary. Then there exists a smooth plurisubharmonic function $\varphi$ defined on an open neighbourhood of $\bar{\Omega}$ such that $\Omega = \{\varphi < 0\}$, $d\varphi \neq 0$ on $b\Omega$ and $\varphi$ is strictly plurisubharmonic near $b\Omega$.*

**Proof.** As in Theorem 5 of [SiT08] we can choose a countable locally finite covering $\{U_j\}_{j=1}^{\infty}$ of $b\Omega$ by open subsets $U_j \subset\subset \mathcal{M}$ such that there exist biholomorphisms $\phi_j\colon U_j \to U_j'$ onto open subsets $U_j' \subset \mathbb{C}^n$, strictly convex bounded domains $G_j' \subset\subset U_j'$ with smooth boundaries and a smooth partition of unity $\{\theta_j\}_{j=1}^{\infty}$ on $b\Omega$ subordinated to $\{b\Omega \cap U_j\}_{j=1}^{\infty}$ such that $G_j' \subset \phi_j(\Omega \cap U_j)$ and $\operatorname{supp}\theta_j' \subset\subset bG_j' \cap \phi_j(b\Omega \cap U_j)$, where $\theta_j' := \theta_j \circ \phi_j^{-1}$ on $\phi_j(b\Omega \cap U_j)$. Moreover, by strict pseudoconvexity of $b\Omega$, we can assume that $\bar{\Omega} \cap \bigcup_{j=1}^{\infty} U_j$ is contained in a one-sided neighbourhood $U \subset \mathcal{M}$ of $b\Omega$ that is filled with analytic discs attached to $b\Omega$. For every $j \in \mathbb{N}$, let $g_j'\colon bG_j' \to [0,1]$ be the smooth extension of $\theta_j'\colon bG_j' \cap \phi_j(b\Omega \cap U_j) \to [0,1]$ by 0, let $S_j' := \operatorname{supp}\theta_j' = \operatorname{supp} g_j' \subset bG_j'$ and let $Z_j' := bG_j' \setminus S_j'$. Let $f_j'\colon \bar{G}_j' \to (-\infty,1]$ be the strictly plurisubharmonic solution of the following Dirichlet problem for the complex Monge-Ampère equation,

$$\begin{cases} f_j'|_{bG_j'} = g_j' \\ \mathrm{MA}[f_j'] \equiv 1 \end{cases}.$$

The existence and uniqueness as well as smoothness of $f_j'$ is guaranteed by Theorem 1.1 in [CKNS85]. Observe that, by strict convexity of $bG_j'$, the set $D_j' := \{z' \in \bar{G}_j'\,:\,$ there exists a complex line $L_{z'} \ni z'$ such that $L_{z'} \cap S_j' = \varnothing\}$ is an open neighbourhood of $Z'_j$ in $\bar{G}'_j$. By the maximum principle, we have $f'_j \leq 0$ on $D'_j$. Hence the function $\tilde{f}'_j := \max(0, f'_j)$ satisfies $\tilde{f}'_j \equiv 0$ on $D'_j$. For every $j \in \mathbb{N}$, let $X_j \subset b\Omega$ be an open set such that $X_j \subset\subset \{\theta_j > 0\}$ and such that $\{X_j\}_{j=1}^{\infty}$ covers $b\Omega$. Further, let $W'_j$, $j \in \mathbb{N}$, be an open neighbourhood of $X'_j := \phi_j(X_j) \subset\subset \{\theta'_j > 0\}$ in $\bar{G}'_j$ such that $f'_j > c_j > 0$ on $W'_j$ for some $c_j > 0$. Then $\tilde{f}'_j$ is strictly plurisubharmonic on a relatively open neighbourhood of $\bar{W}'_j$ in $\bar{G}'_j$.

Fix $j \in \mathbb{N}$. Without loss of generality we can assume that $0 \in G'_j$. In particular, there exists $\varepsilon_{j,1} > 0$ such that $G'_{j,\varepsilon_j} := (1+\varepsilon_j)G_j'^{(-\varepsilon_j^2)}$ satisfies $G'_j \subset\subset G'_{j,\varepsilon_j} \subset\subset U'_j$ for every positive $\varepsilon_j \leq \varepsilon_{j,1}$, where $G_j'^{(-\varepsilon_j^2)} := G'_j \setminus \bigcup_{z' \in bG'_j} B^n(z', \varepsilon_j^2)$. Define a smooth plurisubharmonic function $\tilde{f}'_{j,\varepsilon_j} \colon G'_{j,\varepsilon_j} \to [0,1]$ by $\tilde{f}'_{j,\varepsilon_j}(z') := \big(\tilde{f}'_j * \delta_{\varepsilon_j^2}\big)(z'/(1+\varepsilon_j))$, where for $\gamma > 0$ we denote by $\delta_\gamma$ some fixed smooth nonnegative function depending only on $\|z\|$ such that $\operatorname{supp} \delta_\gamma = \bar{B}^n(0,\gamma)$ and $\int_{\mathbb{C}^n} \delta_\gamma = 1$. It follows from the constructions of $G'_{j,\varepsilon_j}$ and $\tilde{f}'_{j,\varepsilon_j}$ that there exists $\varepsilon_{j,2} > 0$ such that for every positive $\varepsilon_j \leq \varepsilon_{j,2}$ the set $D'_{j,\varepsilon_j} := (1+\varepsilon_j)D_j'^{(-\varepsilon_j^2)} \subset \mathbb{C}^n$ is an open neighbourhood of $bG'_j \setminus \phi_j(b\Omega \cap U_j)$ and $\tilde{f}'_{j,\varepsilon_j} \equiv 0$ on $D'_{j,\varepsilon_j}$. In particular, the trivial extension of $\tilde{f}'_{j,\varepsilon_j} \circ \phi_j \colon \bar{G}_j \to [0,1]$ by 0 defines a smooth plurisubharmonic function $F_{j,\varepsilon_j} \colon \bar{\Omega} \to [0,1]$, where $G_j := \phi_j^{-1}(G'_j)$. Moreover, there exists $\varepsilon_{j,3} > 0$ such that for every $\varepsilon_j \leq \varepsilon_{j,3}$ the function $\tilde{f}'_{j,\varepsilon_j}$ is strictly plurisubharmonic on $W'_j$, and hence the function $F_{j,\varepsilon_j}$ is strictly plurisubharmonic on $W_j := \phi_j^{-1}(W'_j)$. Finally, for $\varepsilon_j \to 0$ the function $\tilde{f}'_{j,\varepsilon_j}|_{bG'_j}$ converges uniformly to $g'_j$, i.e., $F_{j,\varepsilon_j}|_{b\Omega}$ converges uniformly to $\theta_j$.

For every $j \in \mathbb{N}$, let $\varepsilon_{j,0} := \min\{\varepsilon_{j,1}, \varepsilon_{j,2}, \varepsilon_{j,3}\}$. Consider the sets $e$ and $d$ of sequences of nonnegative numbers defined by $e := \big\{\varepsilon = \{\varepsilon_j\}_{j=1}^{\infty} : 0 < \varepsilon_j \leq \varepsilon_{j,0}\big\}$ and $d := \big\{\delta = \{\delta_j\}_{j=1}^{\infty} : 0 < \delta_j \leq 1/2\big\}$. For every $(\varepsilon,\delta) \in e \times d$, define a function $F_{\varepsilon,\delta} \colon \bar{\Omega} \to [0,1]$ as $F_{\varepsilon,\delta} := \sum_{j=1}^{\infty} F_{j,\varepsilon_j}(1-\delta_j)$. Since $\operatorname{supp} F_{j,\varepsilon_j} \subset\subset U_j$ for every $j \in \mathbb{N}$, and since $\{U_j\}_{j=1}^{\infty}$ is locally finite, each of the functions $F_{\varepsilon,\delta}$ is a well defined smooth and plurisubharmonic function such that $\operatorname{supp} F_{\varepsilon,\delta} \subset U$. Moreover, $F_{\varepsilon,\delta}$ is strictly plurisubharmonic on $W := \bigcup_{j=1}^{\infty} W_j$. By construction, each set $W_j$ is an open neighbourhood of $X_j$ in $\bar{\Omega}$, and since $\{X_j\}_{j=1}^{\infty}$ covers $b\Omega$, it follows that $W$ is an open neighbourhood of $b\Omega$ in $\bar{\Omega}$. Moreover, we claim that the following assertion holds true: for every continuous function $k \colon b\Omega \to (0,\infty)$ we can chose $(\varepsilon,\delta) \in e \times d$ such that

$$1 - k < F_{\varepsilon,\delta} < 1 \text{ on } b\Omega. \tag{3.3}$$

Indeed, for every $\delta \in d$ define functions $\delta_{min}, \delta_{max} \colon b\Omega \to (0,1/2]$ as $\delta_{min}(z) :=$

$\min\{\delta_j : z \in U_j\}$ and $\delta_{max} := \max\{\delta_j : z \in U_j\}$, respectively, and for every $\varepsilon \in e$ let $F_\varepsilon := \sum_{j=1}^{\infty} F_{j,\varepsilon_j}$. Since $\{U_j\}_{j=1}^{\infty}$ is locally finite, and since $k$ is continuous, we can choose $\delta \in d$ so small that $1 - k/2 < 1 - \delta_{max}$. Then, since for $\varepsilon_j \to 0$ the function $F_{j,\varepsilon_j}|_{b\Omega \cap U_j}$ converges uniformly to $\theta_j|_{b\Omega\cap U_j}$ for every $j \in \mathbb{N}$, and since $F_{j,\varepsilon_j}|_{b\Omega\setminus U_j} = \theta_j|_{b\Omega\setminus U_j} \equiv 0$ for every $j \in \mathbb{N}$ and $\varepsilon_j > 0$, we can choose $\varepsilon \in e$ so small that $(1-k)/(1-k/2) < F_\varepsilon < 1/(1-\delta_{min})$ on $b\Omega$. Now observe that, by definition of $F_{\varepsilon,\delta}$, we have $(1-\delta_{\max})F_\varepsilon \leq F_{\varepsilon,\delta} \leq (1-\delta_{min})F_\varepsilon$, hence it follows that $1-k < F_{\varepsilon,\delta} < 1$ on $b\Omega$ as claimed. Finally, note that the inequality $F_{\varepsilon,\delta} < 1$ on $b\Omega$ implies that $F_{\varepsilon,\delta} < 1$ on $\Omega$, since $\operatorname{supp} F_{\varepsilon,\delta} \subset U$, $U$ is filled by analytic discs attached to $b\Omega$, and $F_{\varepsilon,\delta}$ is smooth and plurisubharmonic on $\bar{\Omega}$.

Now define a continuous function $\nu\colon \bar{\Omega} \to (0,\infty)$ as $\nu := \sup_{(\varepsilon,\delta)\in e\times d} \|dF_{\varepsilon,\delta}\|_{h^*}$ and observe that indeed $\nu(z) < \infty$ for every $z \in \bar{\Omega}$. Let $\psi\colon V \to \mathbb{R}$ be a smooth strictly plurisubharmonic function on an open neighbourhood $V \subset \mathcal{M}$ of $b\Omega$ such that $\Omega \cap V = \{\psi < 0\}$ and $\|d\psi\|_{h^*} > 1 + \nu$. The existence of such a function $\psi$ follows immediately from Lemma 3.1.2. Let $k\colon b\Omega \to (0,\infty)$ be a sufficiently small continuous function and let $(\varepsilon,\delta) \in e \times d$ be chosen in such a way that (3.3) holds true. Since $\|d\psi\|_{h^*} > 1 + \|dF_{\varepsilon,\delta}\|_{h^*}$ on $b\Omega$, it is easy to see that we have $\psi < F_{\varepsilon,\delta} - 1$ on $b(V\cap W)\cap\Omega$, provided that $k$ is chosen small enough. Hence the function $\tilde{\varphi}\colon \bar{\Omega}\cup V \to \mathbb{R}$ defined by

$$\tilde{\varphi}(z) := \begin{cases} \psi(z) & , \quad z \in V\setminus\Omega \\ \max(\psi(z), F_{\varepsilon,\delta}(z)-1), & \quad z \in (V\cap W)\cap\Omega \\ F_{\varepsilon,\delta}(z) - 1 & , \quad z \in \Omega\setminus(V\cap W) \end{cases}$$

is a continuous plurisubharmonic function such that $\tilde{\varphi} = \psi$ near $b\Omega$ and $\Omega = \{\tilde{\varphi} < 0\}$. That is why $\tilde{\varphi}$ has all the properties that we seek, except, possibly, for smoothness in points of the set $A := \{z \in V\cap W\cap\Omega : \psi(z) = F_{\varepsilon,\delta}(z)\}$. But both $\psi$ and $F_{\varepsilon,\delta}$ are strictly plurisubharmonic on $V\cap W\cap\Omega$. Hence, if $\omega$ is an arbitrary fixed neighbourhood of $A$ such that $\bar{\omega} \subset \Omega$, we can apply Richberg's smoothing method to obtain from $\tilde{\varphi}$ a smooth plurisubharmonic function $\varphi\colon V\cup\bar{\Omega}$ such that $\varphi = \tilde{\varphi}$ outside $\omega$ and such that still $\Omega = \{\varphi < 0\}$ (see, for example, Theorem I.5.21 in [De12] for a version of Richberg's smoothing procedure that is strong enough for our purpose). Then $\varphi$ is a function as desired. □

### 3.1.2 Existence results in complex spaces

In this section we extend the above results to the setting of complex spaces. However, at least at the following two points our results are weaker when compared to the case of complex manifolds. First, we are not able to establish a general existence theorem for global defining functions of smoothly strictly $q$-pseudoconvex domains if $q > 0$. Instead, we have to restrict ourselves to the case of strictly hyper-$q$-pseudoconvex domains. And second, if $\Omega$ is a smoothly strictly pseudoconvex domain (i.e., $q = 0$) in an arbitrary complex space, a subtle technical problem concerning the regularity of the desired function arises, when one tries to construct a smoothly global defining function that is smoothly strictly plurisubharmonic outside $\mathfrak{c}(\Omega)$. We start by gathering the necessary definitions and results.

Let $X = (X, \mathcal{O}_X)$ be a complex space (all complex spaces are assumed to be reduced and paracompact). A holomorphic chart for $X$ is a tupel $(U, \tau, A, G)$ where $U \subset X$ is open, $A$ is an analytic subset of a domain $G \subset \mathbb{C}^n$ and $\tau \colon U \xrightarrow{\sim} A$ is biholomorphic. For every $x \in X$, let $T_x(X)$ denote the Zariski tangent space of $X$ at $x$, i.e., $T_x(X) := (\mathfrak{m}_x/\mathfrak{m}_x^2)^*$ where $\mathfrak{m}_x \subset \mathcal{O}_x$ is the maximal ideal of germs of holomorphic functions that vanish in $x$. If $f \colon X \to Y$ is a holomorphic map between complex spaces $X$ and $Y$, then we write $f_* = f_{*,x} \colon T_x(X) \to T_{f(x)}(Y)$ for the induced differential map. Let $\varphi \colon X \to \mathbb{R}$ be a smooth function, let $(U, \tau, A, G)$ be a holomorphic chart for $X$ around $x$ and let $\hat{\varphi} \colon G \to \mathbb{R}$ be a smooth function such that $\varphi = \hat{\varphi} \circ \tau$ on $U$ (see below for the definition of smooth functions on complex spaces). Then we can define functionals $(d\varphi)_x \colon T_x(X) \to \mathbb{R}$ and $(\partial\varphi)_x, (\bar{\partial}\varphi)_x \colon T_x(X) \to \mathbb{C}$ by setting

$$(\partial\varphi)_x(\xi) := (\partial\hat{\varphi})_{\tau(x)}(\tau_*\xi), \quad (\bar{\partial}\varphi)_x(\xi) := (\bar{\partial}\hat{\varphi})_{\tau(x)}(\tau_*\xi),$$
$$(d\varphi)_x(\xi) := (\partial\varphi)_x(\xi) + (\bar{\partial}\varphi)_x(\xi)$$

for every $\xi \in T_x(X)$. Indeed, by part 1 of the Proposition in [V93], this definition is independent of the smooth extension $\hat{\varphi}$, and by assertion (1) in Section 1 of [Gr62] it is also independent of the holomorphic chart $(U, \tau, A, G)$. In particular, $H_x(\varphi) := \{\xi \in T_x(X) : (\partial\varphi)_x(\xi) = 0\}$ is a well defined subspace of $T_x(X)$. In the same way we want to define $\operatorname{Lev}(\varphi)(x, \cdot\,) \colon T_x(X) \to \mathbb{R}$ as

$$\operatorname{Lev}(\varphi)(x, \xi) := \operatorname{Lev}(\hat{\varphi})\big(\tau(x), \tau_*\xi\big). \tag{3.4}$$

However, as it is shown by Example 1 in [V93], the number $\operatorname{Lev}(\varphi)(x, \xi)$ defined in this way will in general depend on the choice of the smooth extension $\hat{\varphi}$. In fact, in order for (3.4) to be well defined, we need to require that $X$ is locally irreducible at $x$, see part 2 of the Proposition in [V93]. (We do not know if the functionals $(\partial\varphi)_x, (\bar{\partial}\varphi)_x$ and $(d\varphi)_x$ can be well defined in general if $\varphi$ is only assumed to be $\mathcal{C}^1$-smooth. We also do not know whether on locally irreducible complex spaces the Levi form $\operatorname{Lev}(\varphi)(x, \cdot\,)$ can be well defined for arbitrary $\mathcal{C}^2$-smooth functions $\varphi$.)

A function $\varphi \colon X \to \mathbb{R}$ is called smooth or (strictly) plurisubharmonic, if for every $x \in X$ there exist a holomorphic chart $(U, \tau, A, G)$ around $x$ and a smooth or (strictly) plurisubharmonic function $\hat{\varphi} \colon G \to \mathbb{R}$ such that $\varphi|_U = \hat{\varphi} \circ \tau$, respectively.

Observe that it is not clear from the definition whether a smooth and (strictly) plurisubharmonic function $\varphi\colon X \to \mathbb{R}$ does admit local extensions $\hat{\varphi}$ as above that are both smooth and (strictly) plurisubharmonic at the same time. In fact, this is not true in general, see, for example, Warning 1.5 in [Sm86] and Example 2 in [V93]. If around each point $x \in X$ the function $\varphi$ admits holomorphic charts and local extensions $\hat{\varphi}$ that are smooth and (strictly) plurisubharmonic, then $\varphi$ will be called smoothly (strictly) plurisubharmonic. A domain $\Omega \subset X$ is called strictly pseudoconvex if for every $x \in b\Omega$ there exist an open neighbourhood $U_x \subset X$ of $x$ and a continuous strictly plurisubharmonic function $\varphi_x\colon U_x \to \mathbb{R}$ such that $\Omega \cap U_x = \{\varphi_x < 0\}$. The domain $\Omega$ will be called smoothly strictly pseudoconvex if for every $x \in b\Omega$ the function $\varphi_x\colon U_x \to \mathbb{R}$ can be chosen to be smoothly strictly plurisubharmonic. (In the same way we can define the notions of $\mathcal{C}^s$-smoothly (strictly) plurisubharmonic functions and $\mathcal{C}^s$-smoothly strictly pseudoconvex domains for every $s \in \mathbb{N}_0^\infty$. Note that a function $\varphi\colon X \to \mathbb{R}$ is $\mathcal{C}^0$-smooth and (strictly) plurisubharmonic if and only if it is $\mathcal{C}^0$-smoothly (strictly) plurisubharmonic, see Theorem 2.4 in [Ric68].)

Let $q \in \mathbb{N}_0$. A function $\varphi\colon X \to \mathbb{R}$ is called (strictly) $q$-plurisubharmonic, if for every $x \in X$ there exist a holomorphic chart $(U, \tau, A, G)$ around $x$ and a (strictly) $q$-plurisubharmonic function $\hat{\varphi}\colon G \to \mathbb{R}$ such that $\varphi|_U = \hat{\varphi} \circ \tau$. It is called smoothly (strictly) plurisubharmonic if around each point $x \in X$ the function $\varphi$ admits holomorphic charts and local extensions $\hat{\varphi}$ that are smooth and (strictly) $q$-plurisubharmonic. A domain $\Omega \subset X$ is called (smoothly) strictly $q$-pseudoconvex if for every $x \in b\Omega$ there exist an open neighbourhood $U_x \subset X$ of $x$ and a (smoothly) strictly $q$-plurisubharmonic function $\varphi_x\colon U_x \to \mathbb{R}$ such that $\Omega \cap U_x = \{\varphi_x < 0\}$. (Analoguous definitions are possible for $\mathcal{C}^s$-smoothly (strictly) $q$-plurisubharmonic functions, $s \in \mathbb{N}_0^\infty$, and $\mathcal{C}^s$-smoothly strictly $q$-pseudoconvex domains, $s \geq 2$; in the latter case, the restriction to $s \geq 2$ is a matter of convention.)

Finally, we will say that the boundary $b\Omega$ is smooth in $x \in b\Omega$, if there exists a smooth function $\varphi\colon U \to \mathbb{R}$ defined on an open neighbourhood $U \subset X$ of $x$ such that $\Omega \cap U = \{\varphi < 0\}$ and $(d\varphi)_x \neq 0$. Observe that $b\Omega$ is smooth in $x \in b\Omega$ if and only if in every small enough minimal holomorphic chart around $x$ (i.e., every small enough chart $(U, \tau, A, G)$ around $x$ such that $G \subset \mathbb{C}^{\operatorname{ebdim}_x X}$, where $\operatorname{ebdim}_x X = \dim_{\mathbb{C}} T_x(X)$ denotes the embedding dimension of $X$ at $x$) $\Omega$ is the intersection of $X$ with a smoothly bounded subdomain of the ambient $\mathbb{C}^n$. A function $f\colon b\Omega \to \mathbb{R}$ will be called smooth if $f$ is the restriction of a smooth function defined on an open neighbourhood $U \subset X$ of $b\Omega$. In the case when $X$ is a manifold and $b\Omega$ is smooth this definition coincides with the usual one. (Again, analoguous definitions of $\mathcal{C}^s$-smooth boundaries and $\mathcal{C}^s$-smooth functions can be given for every $s \in \mathbb{N}_0^\infty$.)

Now we can formulate the main results of this section which generalize Theorem

3.1.1 and Theorem 3.1.2 to the case of smoothly strictly pseudoconvex domains in complex spaces.

**Theorem 3.1.4.** *Let $X$ be a complex space, let $\Omega \subset X$ be a smoothly strictly pseudoconvex domain and let $f\colon b\Omega \to \mathbb{R}$ be a smooth function that is bounded from below. Then there exists a smoothly plurisubharmonic function $F$ defined on an open neighbourhood of $\bar{\Omega}$ such that $F|_{b\Omega} = f$ and $F$ is smoothly strictly plurisubharmonic near $b\Omega$.*

**Theorem 3.1.5.** *Let $X$ be a complex space and let $\Omega \subset X$ be a smoothly strictly pseudoconvex domain. Then there exists a smoothly plurisubharmonic function $\varphi$ defined on an open neighbourhood of $\bar{\Omega}$ such that $\Omega = \{\varphi < 0\}$ and $\varphi$ is smoothly strictly plurisubharmonic near $b\Omega$.*

We would also like to prove results analoguous to Theorem 3.1.1 and Theorem 3.1.2 for the case of smoothly strictly $q$-pseudoconvex domains in complex spaces. However, we do not know whether this is possible in general if $q > 0$. The problem is essentially the following: If $\Omega$ is a domain in a complex manifold $\mathcal{M}$, $z \in b\Omega$, $U \subset \mathcal{M}$ is an open neighbourhood of $z$ and $\varphi_1, \varphi_2\colon U \to \mathbb{R}$ are smooth functions such that $\Omega \cap U = \{\varphi_1 < 0\} = \{\varphi_2 < 0\}$, then for every $\xi \in H_z(\varphi_1)$ we have $\operatorname{Lev}(\varphi_1)(z, \xi) \geq 0$ if and only if $\operatorname{Lev}(\varphi_2)(z, \xi) \geq 0$ (see Remark 5 after Theorem 3.1.2). Thus when adding $\varphi_2$ to $\varphi_1$ we do not loose positivity of the Levi form on $H_z(\varphi_1)$, and if $H_z(\varphi_1) \neq T_z(\mathcal{M})$, then a possible loss of positivity in the direction normal to $H_z(\varphi_1)$ (with respect to some hermitian metric $h$ on $\mathcal{M}$) can be reacquired by composing the sum $\varphi_1 + \varphi_2$ with a smooth strictly increasing and strictly convex function $\chi\colon \mathbb{R} \to \mathbb{R}$. However, this is not longer true in general in the setting of complex spaces as it is shown by the following example.

**Example 2.** Let $X := \{(z, w) \in \mathbb{C}^2 : z^3 = w^2\}$ and let $\Omega := X \setminus \{0\}$. Consider the smoothly 1-plurisubharmonic functions $\varphi_1, \varphi_2\colon X \to \mathbb{R}$ which are defined as the restrictions to $X$ of the functions $\hat{\varphi}_1(z, w) := |z + w|^2 - 2|z - w|^2$ and $\hat{\varphi}_2(z, w) := |z - w|^2 - 2|z + w|^2$ on $\mathbb{C}^2$, respectively. One easily verifies that in a small open neighbourhood $U \subset X$ of $0 \in X$ it holds true that $\Omega \cap U = \{x \in U : \varphi_1(x) < 0\} = \{x \in U : \varphi_2(x) < 0\}$ and hence $\Omega$ is a smoothly strictly 1-pseudoconvex domain. Since $T_0(X) \simeq \mathbb{C}^2$, and since $X$ is locally irreducible, the Levi form at the origin of every smooth extension of $\varphi_1 + \varphi_2$ to an open neighbourhood of $0 \in \mathbb{C}^2$ coincides with the Levi form of $\hat{\varphi}_1 + \hat{\varphi}_2$ in 0. However, $\operatorname{Lev}(\hat{\varphi}_1 + \hat{\varphi}_2)(0, \cdot\,)$ is negative definit on $\mathbb{C}^2 \simeq H_0(\varphi_1)$.

One might argue that the above example is of a rather pathological nature. On the other hand, observe that typically no assumptions about the smoothness of $b\Omega$ are made in the definition of smoothly strictly $q$-pseudoconvex domains in complex spaces (see, for example, [AG62]). Anyway, even if $b\Omega$ is assumed to be smooth at $x \in b\Omega$ we do not know whether the Levi forms at $x$ of two local defining functions for $\Omega$ around $x$ can be compared as it is done in the manifold case.

As a consequence of the described above problem, in the case $q > 0$ we prove generalizations of Theorem 3.1.1 and Theorem 3.1.2 only for hyper-$q$-pseudoconvex domains instead of smoothly strictly $q$-pseudoconvex domains. Before stating the precise results we collect all necessary definitions.

First we remind the definition of hermitian metrics on complex spaces. Let $\pi\colon T(X) \to X$ be the Zariski tangent linear space, i.e., the underlying set of $T(X)$ is simply the disjoint union $\bigcup_{x\in X} T_x(X)$ (see, for example, [Fi76] for more details). A hermitian metric $h$ on $X$ is a smooth mapping $h\colon T(X) \times_\pi T(X) \to \mathbb{C}$ such that $h|_{T_x(X)\times T_x(X)}$ is a hermitian metric on $T_x(X)$ for every $x \in X$. If $h$ is a hermitian metric on $X$, then for every $x \in X$ we denote by $\|\cdot\|_{h_x}$ and $\|\cdot\|_{h_x^*}$ the induced norms on $T_x(X)$ and $T_x^*(X)$, respectively. If the context is clear, then we sometimes omit the index $x$ and simply write $\|\cdot\|_h$ or $\|\cdot\|_{h^*}$.

Let $X$ be a complex space endowed with a hermitian metric $h$. A smooth function $\varphi\colon X \to \mathbb{R}$ is called hyper-$q$-plurisubharmonic (respectively strictly hyper-$q$-plurisubharmonic) if for every complex subspace $Y \subset X$ and every $y \in Y$, every holomorphic chart $(U, \tau, A, G)$ for $Y$ around $y$ and every hermitian metric $\hat{h}$ on $G \subset \mathbb{C}^n$ which satisfies $h|_U = \tau^*\hat{h}$ there exist an open neighbourhood $G' \subset G$ of $\tau(y)$ and a smooth function $\hat{\varphi}\colon G' \to \mathbb{R}$ such that $\varphi = \hat{\varphi}\circ\tau$ on $U' := \tau^{-1}(G') \subset U$ with the following property: for every $z \in G'$ the trace with respect to $\hat{h}$ of the restriction of the Levi form $\mathrm{Lev}(\hat{\varphi})(z,\cdot)$ to any $(q+1)$-dimensional subspace of $\mathbb{C}^n$ is nonnegative (respectively positive), i.e., for every $\hat{h}$-orthonormal collection of vectors $e_1, e_2, \ldots, e_{q+1} \subset \mathbb{C}^n$ we have that $\sum_{j=1}^{q+1} \mathrm{Lev}(\hat{\varphi})(z, e_j) \geq 0$ (respectively $> 0$). Observe that these definitions depend on the given hermitian metric $h$ and that in general the Levi form of $\hat{\varphi}$ is not uniquely determined by $\varphi$ (it is if $X$ is locally irreducible). Moreover, it is clear from the definition that every (strictly) hyper-$q$-plurisubharmonic function is (strictly) $q$-plurisubharmonic. The main advantage of the set of (strictly) hyper-$q$-plurisubharmonic functions over the set of all (strictly) $q$-plurisubharmonic functions is that the former set is closed under addition. (In the case of not necessarily strictly hyper-$q$-plurisubharmonic functions we need here that $\varphi$ has an extension $\hat{\varphi}$ as described above with respect to every extension $\hat{h}$ of $h$. Then the additional assumption on the complex subspaces $Y \subset X$ is imposed in order to guarantee that restrictions of hyper-$q$-plurisubharmonic functions to complex subspaces are again hyper-$q$-plurisubharmonic; we will not need this property in our constructions. Moreover, for not necessarily strictly hyper-$q$-plurisubharmonic functions it is also not clear if requiring the existence of the extension $\hat{\varphi}$ only with respect to one fixed chart of $Y$, instead of requiring it with respect to every chart of $Y$, yields an equivalent definition. All these complications do not arise in the case of strictly hyper-$q$-plurisubharmonic functions, and thus there is a less technical but equivalent definition in this situation, see, for example, Proposition 2.2 in [FrN10]. In particular, it follows easily from the above remarks that for every strictly hyper-$q$-plurisubharmonic function $\varphi\colon X \to \mathbb{R}$ and every compactly supported smooth function $\theta\colon X \to \mathbb{R}$ there exists $\varepsilon_0 > 0$ such that $\varphi + \varepsilon\theta$ is strictly hyper-$q$-plurisubharmonic for every $|\varepsilon| \leq \varepsilon_0$.) Finally, note that (strictly) hyper-0-plurisubharmonic just means smoothly (strictly) plurisubharmonic. The notion of hyper-q-plurisubharmonicity was first introduced in the context of complex manifolds by Grauert and Riemenschneider in [GR70], the above definition for complex spaces is taken from [FrN10] (actually Grauert and Riemenschneider use the term hyper-$(q+1)$-convex functions instead of strictly hyper-$q$-plurisubharmonic functions, but since we prefered the term of strict $q$-plurisubharmonicity over $(q+1)$-convexity before, we stick to this convention).

A domain $\Omega \subset X = (X, h)$ will be called strictly hyper-$q$-pseudoconvex if for every $x \in b\Omega$ there exist an open neighbourhood $U_x \subset X$ of $x$ and a strictly hyper-$q$-plurisubharmonic function $\varphi_x\colon U_x \to \mathbb{R}$ such that $\Omega \cap U_x = \{\varphi_x < 0\}$. Observe that $\Omega$ is strictly hyper-0-pseudoconvex if and only if it is smoothly strictly pseudoconvex. (Analoguous definitions would be possible in the $\mathcal{C}^s$-smooth categories for every $s \geq 2$.)

Now we turn to the generalizations of Theorem 3.1.1 and Theorem 3.1.2 to hyper-$q$-pseudoconvex domains in complex spaces. Note that for $q = 0$ this will include the case of smoothly strictly pseudoconvex domains. Hence Theorems 3.1.4 and 3.1.5 are special cases of the next two theorems, and thus it suffices to prove only these more general results.

**Theorem 3.1.4′.** *Let $X$ be a complex space, let $\Omega \subset X$ be a strictly hyper-$q$-pseudoconvex domain and let $f\colon b\Omega \to \mathbb{R}$ be a smooth function that is bounded from below. Then there exists a hyper-$q$-plurisubharmonic function $F$ defined on an open neighbourhood of $\bar{\Omega}$ such that $F|_{b\Omega} = f$ and $F$ is strictly hyper-$q$-plurisubharmonic near $b\Omega$.*

**Proof.** Since $f$ is bounded from below, we can assume without loss of generality that $f > 0$. Let $\tilde{F}\colon X \to (0, \infty)$ be a smooth extension of $f$. Let $\{U''_j\}_{j=1}^{\infty}$ be a locally finite covering of $b\Omega$ by open sets $U''_j \subset\subset X$ such that for every $j \in \mathbb{N}$ there exists a strictly hyper-$q$-plurisubharmonic function $\varphi_j\colon U''_j \to \mathbb{R}$ such that $\Omega \cap U''_j = \{\varphi_j < 0\}$. Moreover, let $U'_j \subset\subset U_j \subset\subset U''_j$ be open sets such that $\{U'_j\}_{j=1}^{\infty}$ still covers $b\Omega$.

Let $\{\chi_j\}_{j=1}^{\infty}$ be a family of smooth functions $\chi_j\colon X \to [0, \infty)$ such that $\{\chi_j > 0\} = U'_j$ for every $j \in \mathbb{N}$, $\sum_{j=1}^{\infty} \chi_j \leq 1$ on $X$ and $\sum_{j=1}^{\infty} \chi_j \equiv 1$ near $b\Omega$. Let $\beta\colon (0, \infty) \to (0, \infty)$ be a strictly increasing and strictly convex smooth function such that $\beta(t) = e^{-1/t}$ for small values of $t$, and let $\tilde{\beta}\colon \mathbb{R} \to [0,\infty)$ be the smooth extension of $\beta$ such that $\tilde{\beta}|_{(-\infty,0]} \equiv 0$. As in the proof of Theorem 3.1.1, by a proper choice of $\{\chi_j\}_{j=1}^{\infty}$, we can guarantee that for every $j \in \mathbb{N}$ the trivial extension $g_j\colon X \to [0,\infty)$ of the function $\beta^{-1} \circ (\tilde{F}\chi_j)\colon U_j' \to (0,\infty)$ by 0 is smooth on $X$.

Let $\lambda_j\colon X \to (-\infty, 0]$ be smooth such that $\bar{U}_j' = \{\lambda_j = 0\}$. Then choose $\varepsilon_j > 0$ so small and $C_j > 0$ so large that $g_j + C_j(\varphi_j + \varepsilon_j\lambda_j)$ is still strictly hyper-$q$-plurisubharmonic on $U_j$. Observe that, by construction, $g_j + C_j(\varphi_j + \varepsilon_j\lambda_j) < 0$ on $bU_j \cap \bar{\Omega}$, hence the function $\tilde{\beta} \circ (g_j + C_j(\varphi_j + \varepsilon_j\lambda_j))|_{U_j}$ vanishes near this set and thus its trivial extension by 0 to the open neighbourhood $\mathcal{U}_j := X \setminus \{x \in bU_j : (g_j + C_j(\varphi_j + \varepsilon_j\lambda_j))(x) \geq 0\}$ of $\bar{\Omega}$ defines a hyper-$q$-plurisubharmonic function $F_j\colon \mathcal{U}_j \to [0,\infty)$ such that $F_j|_{b\Omega} = f\chi_j$ and $F_j \equiv 0$ outside $U_j$. Moreover, $W_j := \{F_j > 0\} \subset U_j$ is an open neighbourhood of $b\Omega \cap U_j'$ such that $F_j$ is strictly hyper-$q$-plurisubharmonic on $W_j$. Hence $F := \sum_{j=1}^{\infty} F_j$ is hyper-$q$-plurisubharmonic on the open neighbourhood $\mathcal{U} := \bigcap_{j=1}^{\infty} \mathcal{U}_j \subset X$ of $\bar{\Omega}$ such that $F|_{b\Omega} = f$ and $F$ is strictly hyper-$q$-plurisubharmonic on $W := \bigcup_{j=1}^{\infty} W_j \supset b\Omega$. □

**Theorem 3.1.5′.** *Let $X$ be a complex space and let $\Omega \subset X$ be a strictly hyper-$q$-pseudoconvex domain. Then there exists a hyper-$q$-plurisubharmonic function $\varphi$ defined on an open neighbourhood of $\bar{\Omega}$ such that $\Omega = \{\varphi < 0\}$ and $\varphi$ is strictly hyper-$q$-plurisubharmonic near $b\Omega$.*

**Proof.** Let $\varphi := F - 1$ where $F$ is the function from Theorem 3.1.4 ′ corresponding to the boundary values $f \equiv 1$. Then $\varphi$ is a hyper-$q$-plurisubharmonic function on an open neighbourhood of $\bar{\Omega}$ that vanishes identically on $b\Omega$ and that is strictly hyper-$q$-plurisubharmonic near $b\Omega$. As before, in the construction of $F$ we can choose $\tilde{F}$ such that $\Omega = \{\tilde{F} < 1\}$, and this choice implies that $\Omega = \{F < 1\}$, i.e., $\Omega = \{\varphi < 0\}$; see the proof of Theorem 3.1.2 for more details. □

**Remarks.** 1) In the last two theorems strict hyper-$q$-convexity is understood with respect to an arbitrary but fixed hermitian metric on $X$.

2) Similar to what we had above, the function $F$ from Theorem 3.1.4′ is strictly hyper-$q$-plurisubharmonic on the open neighbourhood $W$ of $b\Omega$ and it is constant on $\Omega \setminus W$. One sees immediately from our construction that for every open set $\omega \subset \Omega$ such that $\bar{\omega} \subset \Omega$ we can choose $F$ in such a way that it will be constant on $\omega$.

3) The statements of Theorems 3.1.4′ and 3.1.5′ remain true if $\mathcal{C}^\infty$-smoothness is replaced by $\mathcal{C}^s$-smoothness for any $s \geq 2$. Also for $\mathcal{C}^s$-smoothly strictly pseudoconvex domains with $s \in \{0, 1\}$ the proofs still work, but in Theorem 3.1.4′ we have to assume that the function $f\colon b\Omega \to \mathbb{R}$ is at least $\mathcal{C}^2$-smooth. In particular,

every strictly pseudoconvex domain in a complex space admits a continuous global defining function.

It is also possible to generalize Theorem 3.1.3 to the case of complex spaces. The precise statement is contained in the following theorem.

**Theorem 3.1.6.** *Let $X$ be a complex space and let $\Omega \subset X$ be a strictly pseudoconvex domain. Let $U \subset X$ be an arbitrary open neighbourhood of $b\Omega$. Then the following assertions hold true:*

*(1) There exists a smoothly strictly pseudoconvex domain $\Omega' \subset X$ such that $\Omega \setminus U \subset \Omega'$ and $\bar{\Omega}' \subset \Omega$.*

*(2) There exists a smoothly strictly pseudoconvex domain $\Omega'' \subset X$ such that $\bar{\Omega} \subset \Omega''$ and $\bar{\Omega}'' \subset \Omega \cup U$.*

*In particular, $\bar{\Omega}$ admits a neighbourhood basis consisting of smoothly strictly pseudoconvex domains.*

**Proof.** In view of Theorem 2.4 in [Ric68], it is easy to see that the version of Richberg's smoothing procedure which is formulated in Theorem I.5.21 in [De12], remains true in the setting of complex spaces. Thus the theorem can be proved in the same way as Theorem 3.1.3 above. □

As in the case of manifolds, we can now introduce the notion of the core of a smoothly strictly pseudoconvex domain in an arbitrary complex space.

**Definition.** Let $X$ be a complex space and let $\Omega \subset X$ be a domain. Then the set

$$\mathfrak{c}(\Omega) := \big\{z \in \Omega : \text{every smoothly plurisubharmonic function on } \Omega \text{ that is bounded from above fails to be smoothly strictly plurisubharmonic in } z\big\}$$

will be called the *core* of $\Omega$.

However, observe that a subtle technical problem concerning the regularity of the global defining function occurs if one tries to extend the Main Theorem to the setting of complex spaces. In fact, in the proof of the Main Theorem we construct the smooth plurisubharmonic function $\phi_1$ as a limit of a series of smooth plurisubharmonic functions. Correspondingly, in the case of complex spaces the function $\phi_1$ would be a limit of a series of smoothly plurisubharmonic functions. Observe though that when taking limits in the class of smoothly plurisubharmonic functions defined on a complex space, it is not clear in general in which cases the limit function will again be smoothly plurisubharmonic. Indeed, if we try to repeat the proof of the Main Theorem for complex spaces, then one can easily choose the sequence $\{\varepsilon_j\}_{j=1}^{\infty}$ in such a way that the function $\phi_1$ is smooth and plurisubharmonic (here we need the equivalence of weakly plurisubharmonic and plurisubharmonic functions on complex spaces, see Theorem 5.3.1 in [FoN80]), but it is not clear whether $\phi_1$ will also be smoothly plurisubharmonic. The problem is that given a point $x \in X$ and a sequence $\{\Psi_j\}$ of smoothly plurisubharmonic functions on $X$, then after a local embedding of $X$ into some $\mathbb{C}^n$ each $\Psi_j$ extends as a smooth plurisubharmonic function onto some neighbourhood $\hat{U}_j \subset \mathbb{C}^n$ of $x$, but it is not clear whether one can guarantee that also $\bigcap_{j=1}^{\infty} \hat{U}_j$ will contain some neighbourhood of $x$. For the function $\phi_1$, we only know how to avoid this problem away from $\mathfrak{c}(\Omega)$, namely, we can at least show that $\phi_1$ is smoothly strictly plurisubharmonic outside $\mathfrak{c}(\Omega)$. We sketch briefly the corresponding argument: By construction, $\phi_1 = \sum_{j=1}^{\infty} \varepsilon_j \Psi_j$ for some smoothly plurisubharmonic functions $\Psi_j$ on $\Omega$. Fix arbitrary $x \in \Omega \setminus \mathfrak{c}(\Omega)$. After a local embedding of the complex space X, we can find smooth extensions $\hat{\Psi}_j$ of the functions $\Psi_j$ to a uniformly large neighbourhood of $x$ in the ambient $\mathbb{C}^n$ such that each function $\hat{\Psi}_j$ has nonnegative Levi form in $x$. Moreover, since $x \notin \mathfrak{c}(\Omega)$, and by the choice of the functions $\Psi_j$, at least one of the functions $\hat{\Psi}_j$ has positive Levi form in $x$. Thus if $\{\varepsilon_j\}$ is chosen suitably, then the function $\hat{\phi}_1 := \sum_{j=1}^{\infty} \varepsilon_j \hat{\Psi}_j$ is a smooth extension of $\phi_1$ which has a positive Levi form in $x$, i.e., $\phi_1$ is smoothly strictly plurisubharmonic in $x$. (Analoguous results hold in the $\mathcal{C}^s$-smooth categories for every $s \geq 2$. Moreover, in view of Theorem 2.4 from [Ric68], a full analogue of the Main Theorem holds true for strictly pseudoconvex domains in complex spaces in the $\mathcal{C}^0$-smooth category.)

Finally, an analogue of Proposition 3.1.2 holds true for arbitrary Stein spaces (for the existence of a smoothly strictly plurisubharmonic function on a Stein space see, for example, the Lemma in Section 3 of [Na61]).

## 3.2 Examples of unbounded domains with nonempty core

Let $\Omega$ be a domain in a complex manifold $\mathcal{M}$. It follows immediately from the definition of the core, that $\mathfrak{c}(\Omega)$ is always relatively closed in $\Omega$. If $\Omega$ is strictly pseudoconvex with smooth boundary, then Theorem 3.1.2$'$ implies that $\mathfrak{c}(\Omega)$ is also closed in $\mathcal{M}$. Moreover, $\mathfrak{c}(\Omega) = \varnothing$ if $\mathcal{M}$ is Stein and $\Omega$ is relatively compact in $\mathcal{M}$. As remarked above, every domain $\Omega \subset \mathcal{M}$ admits a smooth and bounded plurisubharmonic function $\varphi\colon \Omega \to \mathbb{R}$ that is strictly plurisubharmonic precisely in $\Omega \setminus \mathfrak{c}(\Omega)$ (see the remarks following the definition of minimal functions on page 79).

In order to get a better understanding of properties of the core, we construct in this section several examples of unbounded domains $\Omega \subset \mathbb{C}^n$ such that $\mathfrak{c}(\Omega) \neq \varnothing$. Before we start with these constructions, we want to make the following observation: If $\Omega \subset \mathcal{M}$ is a domain, and $\omega \subset \Omega$ is a subdomain such that $\mathfrak{c}(\Omega) \subset \omega$, then clearly $\mathfrak{c}(\omega) \subset \mathfrak{c}(\Omega)$. We do not know, however, whether the reverse inclusion holds also true here, i.e., we do not know if in general $\mathfrak{c}(\omega) = \mathfrak{c}(\Omega)$. In fact, in all examples of domains $\Omega \subset \mathcal{M}$ with nonempty core that we will construct here (and also in the Examples I and II that have already been given in the Introduction), the above equality does indeed hold true for every subdomain $\omega \subset \Omega$ that contains $\mathfrak{c}(\Omega)$. Thus, loosely speaking, in all examples that we are able to construct, the presence of the core $\mathfrak{c}(\Omega) \subset \Omega$ is only related to intrinsic properties of $\mathfrak{c}(\Omega)$, but not to properties of $\Omega$.

These observations will lead us in Section 3.4 to define the notion of sets of *core type*. For the moment, we just want to point out, that in view of a possible dependence of $\mathfrak{c}(\Omega)$ on $\Omega$, it is desirable to construct not only examples of arbitrary domains with nonempty core, but also of domains with additional properties, like, for example, pseudoconvexity. This concern is further illustrated by the following theorem.

**Theorem 3.2.1.** *The following assertions hold true for domains $\Omega \subset \mathbb{C}^n$, $n \geq 2$:*

*(1) There exists a domain $\Omega \subset \mathbb{C}^n$ such that $\mathfrak{c}(\Omega) = E \times \mathbb{C}^{n-1}$, where $E \subset \mathbb{C}$ is the set $E = [0,1] \times \mathbb{R}$.*

*(2) Let $\Omega \subset \mathbb{C}^n$ be a pseudoconvex domain such that $\mathfrak{c}(\Omega) = E \times \mathbb{C}^k$ for some $k \in \mathbb{N}_{n-1}$ and some set $E \subset \mathbb{C}^{n-k}$. Then either $E$ is locally complete pluripolar or $E$ is open. In the latter case $\Omega = E \times \mathbb{C}^k$.*

*(3) Let $k \in \mathbb{N}_{n-1}$ be arbitrary but fixed. Then there exists a strictly pseudoconvex domain $\Omega \subset \mathbb{C}^n$ such that $\mathfrak{c}(\Omega) = E \times \mathbb{C}^k$ for a set $E \subset \mathbb{C}^{n-k}$ if and only if $E$ is closed and complete pluripolar.*

**Remarks.** 1) Statement (1) shows that the core of an arbitrary domain $\Omega \subset \mathbb{C}^n$ may divide $\Omega$ into several connected components, and, moreover, $\mathfrak{c}(\Omega)$ may have nonemtpy interior. However, it is not clear to us at the moment, whether a pseudoconvex domain $\Omega \subset \mathbb{C}^n$ can have nonempty and disconnected complement $\Omega \setminus \mathfrak{c}(\Omega)$, or if the core of a strictly pseudoconvex domain $\Omega \subset \mathbb{C}^n$ can have nonempty interior.

2) Statement (3) shows that if the core of a strictly pseudoconvex domain is assumed to have a product structure as described above, then it is always complete pluripolar. In view of this result, and also of the examples that will be given below, one can raise the following question: Is it always true that the core of a

strictly pseudoconvex domain is complete pluripolar? At the moment we are not able to give an answer to this question.

3) Some part of the arguments which we will use to prove the statements (2) and (3) of the theorem are similar to the ones which appear in the proof of Theorem 1.11 in [Au83]. In fact, to some extent, the corresponding statements are already implicitly contained in the above mentioned result of Aupetit. Another partial result, which is slightly different from the one proven by Aupetit, can also be found in Theorem 1 in [Kaz83].

**Proof.** (i) We start with proving statement (1) of the theorem. Let $(z_1, z_2, \ldots, z_n)$, $z_j = x_j + iy_j$, denote the coordinates in $\mathbb{C}^n$. For every $j \in \mathbb{N}$, let $\psi_j \colon B^n(0,j) \to \mathbb{R}$ be the smooth and strictly plurisubharmonic function defined by

$$\psi_j(z_1, \ldots, z_n) := x_1 - \frac{1}{2^{j-2}} + \frac{1}{j^2 2^{j-1}}\big(y_1^2 + |z_2|^2 + \cdots + |z_n|^2\big).$$

Choose a smooth function $\chi_j \colon \mathbb{R} \to [0, \infty)$ such that $\chi_j \equiv 0$ on $(-\infty, -1/2^j]$ and such that $\chi_j$ is strictly increasing and strictly convex on $(-1/2^j, \infty)$. Set $\tilde{\varphi}_j := \chi_j \circ \psi_j$. Then $\tilde{\varphi}_j$ is a smooth plurisubharmonic function on $B^n(0,j)$ such that $\tilde{\varphi}_j \equiv 0$ on $\{\psi_j \leq -1/2^j\} \supset B^n(0,j) \cap \{x_1 \leq 1/2^j\}$ and such that $\tilde{\varphi}_j$ is strictly plurisubharmonic and positive on $\{\psi_j > -1/2^j\} \supset B^n(0,j) \cap \{x_1 > 3/2^j\}$. Thus

$$\varphi_j(z) := \begin{cases} \tilde{\varphi}_j(z)\,, & z \in B^n(0,j) \cap \{x_1 \geq 1/2^j\} \\ 0 \quad , & z \in \{x_1 < 1/2^j\} \end{cases}$$

is a smooth plurisubharmonic function on $W_j := B^n(0,j) \cup \{x_1 < 1/2^j\}$ such that $\varphi_j$ is strictly plurisubharmonic and positive on $B^n(0,j) \cap \{x_1 > 3/2^j\}$. Observe that $W := \bigcap_{j=1}^{\infty} W_j$ is a connected open neighbourhood of $\{x_1 \leq 0\}$. Then one easily sees that for a sequence $\{\varepsilon_j\}_{j=1}^{\infty}$ of positive numbers that converges to zero fast enough, the function $\varphi := \sum_{j=1}^{\infty} \varepsilon_j \varphi_j$ is smooth and plurisubharmonic on $W$ such that $\varphi \equiv 0$ on $\{x_1 \leq 0\}$ and such that $\varphi$ is strictly plurisubharmonic and positive on $W \cap \{x_1 > 0\}$.

Now define a domain $\Omega \subset \mathbb{C}^n$ as

$$\Omega := [W + (1, 0, \ldots, 0)] \cap [-W] = \big\{z \in \mathbb{C}^n : (z_1 - 1, z_2, \ldots, z_n) \in W \text{ and } -z \in W\big\}.$$

Then $E \times \mathbb{C}^{n-1} \subset \Omega$, where $E := [0,1] \times \mathbb{R}_{y_1} \subset \mathbb{C}$. By the Liouville theorem, every plurisubharmonic function $u$ on $\Omega$ that is bounded from above has to be constant on $\{z\} \times \mathbb{C}^{n-1}$ for every $z \in E$. Hence $u$ fails to be strictly plurisubharmonic at every point of $E \times \mathbb{C}^{n-1}$, i.e., $E \times \mathbb{C}^{n-1} \subset \mathfrak{c}(\Omega)$. On the other hand, $\Phi(z) := \varphi(z_1 - 1, z_2, \ldots, z_n) + \varphi(-z)$ is a smooth plurisubharmonic function on $\Omega$ such that $\Phi$ is strictly plurisubharmonic on $\Omega \setminus (E \times \mathbb{C}^{n-1})$, i.e., $\mathfrak{c}(\Omega) \subset E \times \mathbb{C}^{n-1}$. Thus $\mathfrak{c}(\Omega) = E \times \mathbb{C}^{n-1}$, which completes the proof of part 1 of the theorem.

(ii) We now prove the statements (2) and (3) of the theorem. At first, let $\Omega \subset \mathbb{C}^n$ be an arbitrary domain such that $\mathfrak{c}(\Omega) = E \times \mathbb{C}^k$ for some set $E \subset \mathbb{C}^{n-k}$. Then, in view of Liouville's theorem, one can easily see that $E = \{z' \in \mathbb{C}^{n-k} : \{z'\} \times \mathbb{C}^k \subset \Omega\}$. For every $z'' \in \mathbb{C}^k$ let $V_{z''} := \{z' \in \mathbb{C}^{n-k} : (z', z'') \in \Omega\}$ and define $\psi_{z''} \colon V_{z''} \to [-\infty, \infty)$ as $\psi_{z''}(z') := -\log \mathcal{R}(z', z'')$, where $\mathcal{R}(z) := \sup\{r > 0 : \{z'\} \times B^k(z'', r) \subset \Omega\}$, $z = (z', z'') \in \mathbb{C}^{n-k} \times \mathbb{C}^k$. By definition, $\psi_0(z') = -\infty$ if and only if $\{z'\} \times \mathbb{C}^k \subset \Omega$. Thus $E = \{\psi_0 = -\infty\}$.

Assume now that $\Omega$ is pseudoconvex. Then $\psi_0$ is plurisubharmonic on $V_0$, since $\psi_0(z') = \sup_{w'' \in \mathbb{C}^k, \|w''\|=1}[-\log \mathcal{R}_{(0,w'')}(z', 0)]$, where for every $w \in \mathbb{C}^n$ the function $\mathcal{R}_w(z) := \sup\{r > 0 : z + \zeta w \in \Omega \text{ for every } \zeta \in \Delta(0, r)\}$ denotes the Hartogs radius of $\Omega$ in the $w$-direction; here $\Delta(a, r) := \{z \in \mathbb{C} : |z - a| < r\}$. Thus $E$ is locally complete pluripolar if $\psi_0 \not\equiv -\infty$ on every connected component of $V_0$. On the other hand, suppose that $\psi_0 \equiv -\infty$ on some connected component $U$ of $V_0$, i.e. $U \times \mathbb{C}^k \subset \Omega$. Assume, to get a contradiction, that $\Omega \neq U \times \mathbb{C}^k$. Then there exists $z'' \in \mathbb{C}^k$ such that $U$ is a proper subset of the connected component $V'_{z''}$ of $V_{z''}$ containing $U$. Since $\psi_{z''} \equiv -\infty$ on the open set $U$, it follows that $\psi_{z''} \equiv -\infty$ on $V'_{z''}$. Thus $V'_{z''} \times \mathbb{C} \subset \Omega$ and hence, by definition of $U$, we have $V'_{z''} \subset U$. This contradicts the fact that $U \subsetneq V'_{z''}$ and thus proves that $\Omega = U \times \mathbb{C}^k$. Another application of Liouville's theorem then shows that $E = U$, which completes the proof of statement (2).

Now assume that $\Omega$ is even strictly pseudoconvex. Then, by what we have already proven, it follows that $E$ is locally complete pluripolar. Assume, to get a contradiction, that $E$ is not closed. Then there exist $p \in \mathbb{C}^{n-k} \setminus E$ and a sequence $\{p_j\}_{j=1}^{\infty} \subset E$ such that $\lim_{j \to \infty} p_j = p$. Since $E \times \mathbb{C}^k \subset \Omega$, it follows that $L := \{p\} \times \mathbb{C}^k \subset \bar{\Omega}$. By Theorem 3.1.2 and the related Remark 4, there exists a continuous plurisubharmonic function $\varphi$ on an open neighbourhood of $\bar{\Omega}$ such that $\Omega = \{\varphi < 0\}$. In particular, $\varphi \leq 0$ on $\bar{\Omega}$. Thus, by Liouville's theorem, $\varphi \equiv c$ on $L$ for some constant $c \leq 0$. If $c < 0$, then $L \subset \Omega$ and hence also $L \subset \mathfrak{c}(\Omega)$. This implies that $p \in E$, which contradicts the assumption on $p$. On the other hand, if $c = 0$, then $L \subset b\Omega$, which is not possible by strict pseudoconvexity of $b\Omega$. This shows that $E$ is closed in $\mathbb{C}^{n-k}$. By Corollary 1 in [Co90], it follows that $E$ is complete pluripolar.

Finally, let $E \subset \mathbb{C}^{n-k}$ be a closed complete pluripolar set. Then, by Corollary 1 in [Co90], there exists a plurisubharmonic function $u$ on $\mathbb{C}^{n-k}$ such that $u$ is smooth on $\mathbb{C}^{n-k} \setminus E$ and $E = \{u = -\infty\}$. Define

$$\Omega' := \big\{(z', z'') \in \mathbb{C}^n : u(z') + \|z\|^2 < C\big\}.$$

Then for generic $C \in \mathbb{R}$, $\Omega'$ is a strictly pseudoconvex open set with smooth boundary such that $E \times \mathbb{C}^k \subset \Omega'$. By Liouville's theorem, $E \times \mathbb{C}^k \subset \mathfrak{c}(\Omega')$.

Moreover, let $v\colon \mathbb{C}^n \to [-\infty, \infty)$ be defined as $v(z) = u(z') + \|z\|^2$. It is easy to see that if $\{\eta_j\}_{j=1}^{\infty}$ is a sequence of positive numbers that converges to zero fast enough, then $\tilde{v} := \sum_{j=1}^{\infty} \eta_j \widetilde{\max}_1(v - C, -j)$ is a smooth global defining function for $\Omega'$ which is strictly plurisubharmonic outside $E \times \mathbb{C}^k$. Thus we also have that $\mathfrak{c}(\Omega') \subset E \times \mathbb{C}^k$. This shows that $\mathfrak{c}(\Omega') = E \times \mathbb{C}^k$. The assertion of statement (3) then follows from the following lemma. ☐

**Lemma 3.2.1.** *Let $\mathcal{M}$ be a connected complex manifold and let $\Omega' \subset \mathcal{M}$ be a strictly pseudoconvex open set (not necessarily connected or with smooth boundary). Then there exists a strictly pseudoconvex domain $\Omega \subset \mathcal{M}$ with smooth boundary such that $\bar{\Omega}' \subset \Omega$ and $\mathfrak{c}(\Omega) = \mathfrak{c}(\Omega')$.*

**Proof.** We proceed in three steps.

Step 1. *Let $G_0, G_1 \subset \mathbb{C}^n$ be two strictly pseudoconvex domains with smooth boundary such that $\bar{G}_0 \cap \bar{G}_1 = \varnothing$. Let $\gamma\colon [0,1] \to \mathbb{C}^n$ be a smooth embedding such that $z_0 := \gamma(0) \in bG_0$, $z_1 := \gamma(1) \in bG_1$ and $\gamma(t) \in \mathbb{C}^n \setminus (\bar{G}_0 \cup \bar{G}_1)$ for $t \in (0,1)$. Let $\psi$ be a smooth plurisubharmonic function on an open neighbourhood $V \subset \mathbb{C}^n$ of $\bar{G}_0 \cup \bar{G}_1$ such that for $j = 0,1$ we have $\psi(z_j) \le 0$ and $\psi$ is strictly plurisubharmonic near $z_j$. Then for every open neighbourhood $\Gamma \subset \mathbb{C}^n$ of $\gamma([0,1])$ there exist a strictly pseudoconvex domain $\Omega \subset \mathbb{C}^n$ with smooth boundary and a smooth plurisubharmonic function $\varphi$ on an open neighbourhood $U \subset \mathbb{C}^n$ of $\bar{\Omega}$ such that the following assertions hold true:*

*(i)* $U = V' \cup \Gamma'$ *for some open neighbourhood* $V' \subset V$ *of* $\bar{G}_0 \cup \bar{G}_1$ *and some open neighbourhood* $\Gamma' \subset \Gamma$ *of* $\gamma([0,1])$,

*(ii)* $\Omega \setminus \Gamma = (G_0 \cup G_1) \setminus \Gamma$,

*(iii)* $\varphi = \psi$ *on* $V'$, *while* $\varphi$ *is strictly plurisubharmonic and less than* $1$ *on* $\Gamma'$.

Proof. Fix constants $\varepsilon_0, \delta_0 > 0$ such that $\bar{B}^n(z_0, \varepsilon_0) \cap \bar{B}^n(z_1, \varepsilon_0) = \varnothing$, $\psi$ is strictly plurisubharmonic and less than $1/2$ on $B^n(z_0, \varepsilon_0) \cup B^n(z_1, \varepsilon_0) \subset V \cap \Gamma$ and such that $\gamma([0,1])^{(\delta_0)} \subset \Gamma$, where for $K \subset \mathbb{C}^n$ and $d > 0$ we let $K^{(d)} := \bigcup_{z \in K} B^n(z, d)$.

Choose $s > 0$ so small that $\gamma([0,s])$ and $\gamma([1-s,1])$ are contained in $B^n(z_0, \varepsilon_0) \cup B^n(z_1, \varepsilon_0)$. Let $f\colon \gamma([0,1]) \to (-\infty, 1/2)$ be a smooth function such that for some constant $c \in (0,1)$ one has $f + c < \psi$ in $\gamma(0)$ and $\gamma(1)$, and $f > \psi + c$ in $\gamma(s)$ and $\gamma(1-s)$. Let $F\colon \mathbb{C}^n \to \mathbb{R}$ be a smooth extension of $f$. Since one can see easily that $\gamma([0,1])$ is contained in a closed embedded smooth real 1-dimensional submanifold $M \subset \mathbb{C}^n$, it follows from Lemma 1 in [Ch69] that there exists a smooth strictly plurisubharmonic function $\theta\colon W \to \mathbb{R}$ on an open neighbourhood $W \subset \mathbb{C}^n$ of $\gamma([0,1])$ such that $\theta \equiv 0$ on $\gamma([0,1])$. Thus for $C > 0$ large enough, and after possibly shrinking $W$, the function $\rho := F + C\theta$ is smooth and strictly plurisubharmonic on $W$ such that $\rho < 1/2$ and $\rho|_{\gamma([0,1])} = f$.

Choose $\varepsilon \in (0, \varepsilon_0)$ so small that

- $B^n(z_0, \varepsilon) \cup B^n(z_1, \varepsilon) \subset W$,
- $\gamma([s, 1-s]) \cap (\bar{B}^n(z_0, \varepsilon) \cup \bar{B}^n(z_1, \varepsilon)) = \varnothing$, and
- $\rho + c < \psi$ on $B^n(z_0, \varepsilon) \cup B^n(z_1, \varepsilon)$.

Moreover, let $\delta \in (0, \min(\delta_0, \varepsilon)/2)$ be so small that

- $\gamma([0,1])^{(2\delta)} \subset W$,
- $\gamma([0,1])^{(2\delta)} \cap (G_0 \cup G_1)^{(\delta)} \subset B^n(z_0, \varepsilon) \cup B^n(z_1, \varepsilon)$,
- the orthogonal projection $\pi\colon \gamma([0,1])^{(2\delta)} \to M$ along the normal directions of the manifold $M$ is well defined, and
- there exists a constant $a \in (0, s)$ such that $\pi^{-1}(\gamma([0, s+a) \cup (1-s-a, 1])) \subset B^n(z_0, \varepsilon_0) \cup B^n(z_1, \varepsilon_0)$, $\pi^{-1}(\gamma((s-a, 1-s+a))) \cap (B^n(z_0, \varepsilon) \cup B^n(z_1, \varepsilon)) = \varnothing$ and $\rho > \psi + c$ on $\pi^{-1}(\gamma((s-a, s+a) \cup (1-s-a, 1-s+a)))$.

Let $V' := V \cap (G_0 \cup G_1)^{(\delta)}$, let $\Gamma' := B^n(z_0, \varepsilon) \cup \gamma([0,1])^{(2\delta)} \cup B^n(z_1, \varepsilon)$ and set $U := V' \cup \Gamma'$. Then $\varphi\colon U \to \mathbb{R}$ defined as

$$\varphi := \begin{cases} \psi & \text{on } V \cap (G_0 \cup G_1)^{(\delta)} \\ \widetilde{\max}_c(\psi, \rho) & \text{on } B^n(z_0, \varepsilon) \cup \pi^{-1}(\gamma([0, s+a) \cup (1-s-a, 1])) \cup B^n(z_1, \varepsilon) \\ \rho & \text{on } \pi^{-1}(\gamma((s-a, 1-s+a))) \end{cases}$$

is a smooth plurisubharmonic function on $U$ such that $\varphi = \psi$ on $V'$, $\varphi < 1$ on $\Gamma'$ and $\varphi$ is strictly plurisubharmonic on $\Gamma'$.

It only remains to construct a strictly pseudoconvex domain $\Omega \subset \mathbb{C}^n$ with smooth boundary such that $\Omega \setminus \Gamma = (G_0 \cup G_1) \setminus \Gamma$ and $\bar{\Omega} \subset U$. To do so, fix $\tilde{\varepsilon} > 0$ so small that $\bar{B}^n(z_0, \tilde{\varepsilon}) \cup \bar{B}^n(z_1, \tilde{\varepsilon}) \subset U \cap \Gamma$. Then for $j = 0, 1$ choose strictly pseudoconvex domains $\tilde{G}_j \subset \mathbb{C}^n$ with smooth boundary such that $\tilde{G}_j \subset G_j$, $\tilde{G}_j \setminus B^n(z_j, \tilde{\varepsilon}) = G_j \setminus B^n(z_j, \tilde{\varepsilon})$ and such that near $z_j$ the domain $\tilde{G}_j$ looks like a ball with $z_j$ as a boundary point. (The existence of the domains $\tilde{G}_j$ is essentially an observation by H. Boas, which is based on a result due to Y. Eliashberg. A detailed proof of this fact, together with references to the results of Boas and Eliashberg, can be found, for example, in Corollary 4.1.46 of [JP00]. Observe that our assertions on the domains $\tilde{G}_j$ are slightly stronger than the ones formulated in the statement of the mentioned above corollary. However, the fact that $\tilde{G}_j$ can be assumed to be strictly pseudoconvex with smooth boundary follows, for example, from the remark after Corollary 4.1.46 in [JP00], or from the construction of smooth maximum as described in Section 3.1.1 of this article.) Moreover, let $\tilde{\gamma}\colon [0,1] \to \mathbb{C}^n$ be a smooth embedding such that $\tilde{\gamma}(0) = z_0$, $\tilde{\gamma}(1) = z_1$, $\tilde{\gamma}([0,1]) \setminus (B^n(z_0, \tilde{\varepsilon}/2) \cup B^n(z_1, \tilde{\varepsilon}/2)) = \gamma([0,1]) \setminus (B^n(z_0, \tilde{\varepsilon}/2) \cup B^n(z_1, \tilde{\varepsilon}/2))$, and such that for some $r > 0$ the curves $\tilde{\gamma}([0,r])$ and $\tilde{\gamma}([1-r,1])$ are segments of lines orthogonal to $b\tilde{G}_0$ and $b\tilde{G}_1$, respectively. Finally, choose $\tilde{\delta} \in (0, \min(\tilde{\varepsilon}/2, \delta))$. Then, by the corollary in Section 1 of [Sh83], and after possibly further shrinking $\tilde{\delta}$, there exists a strictly pseudoconvex domain $\Omega \subset \mathbb{C}^n$ with smooth boundary such that

$$\Omega\setminus(B^n(z_0,\tilde{\varepsilon}/2)\cup B^n(z_1,\tilde{\varepsilon}/2)) = (\tilde{G}_0\cup\tilde{\gamma}([0,1])^{(\tilde{\delta})}\cup\tilde{G}_1)\setminus(B^n(z_0,\tilde{\varepsilon}/2)\cup B^n(z_1,\tilde{\varepsilon}/2)).$$

(The corollary quoted above is only formulated for domains in $\mathbb{C}^2$, but the statement and the proof remain true also in the case of $\mathbb{C}^n$.) In particular,

$$\Omega \setminus (B^n(z_0,\tilde{\varepsilon}) \cup B^n(z_1,\tilde{\varepsilon})) = (G_0 \cup \gamma([0,1])^{(\tilde{\delta})} \cup G_1) \setminus (B^n(z_0,\tilde{\varepsilon}) \cup B^n(z_1,\tilde{\varepsilon})).$$

It now follows easily from the constructions that $\Omega$ is a domain as desired. This completes the proof of Step 1.

Step 2. *The statement of Step 1 remains true if $\mathbb{C}^n$ is replaced by an arbitrary complex manifold $\mathcal{M}$.*

Proof. Let $D_1, \ldots, D_N \subset \mathcal{M}$ be open coordinate patches such that $\gamma([0,1]) \subset \bigcup_{j=1}^N D_j$, $z_0 \in D_1$, $z_1 \in D_N$, $D_j \cap \gamma([0,1])$ is connected, $1 \le j \le N$, $D_j \cap D_{j+1} \cap \gamma([0,1]) \neq \varnothing$, $1 \le j \le N-1$, and $D_j \cap D_k = \varnothing$ if $|j-k| > 1$. For every $j = 1, \ldots, N-1$, let $\tilde{G}_j \subset\subset \Gamma \cap (D_j \cap D_{j+1})$ be a strictly pseudoconvex domain with smooth boundary such that the sets $G_0 =: \tilde{G}_0, \tilde{G}_1, \ldots, \tilde{G}_{N-1}, \tilde{G}_N := G_1$ have pairwise disjoint closures and there exists numbers $0 =: t_0^1 < t_1^1 < t_0^2 < t_1^2 < \cdots < t_0^N < t_1^N := 1$ such that $\bigcup_{j=1}^N \gamma((t_0^j, t_1^j)) \subset \mathcal{M} \setminus \bigcup_{j=0}^N \bar{\tilde{G}}_j$ and $\gamma((t_1^j, t_0^{j+1})) \subset \tilde{G}_j$, $1 \le j \le N-1$. Define $\tilde{\gamma}_j \colon [0,1] \to \mathcal{M}$ as $\tilde{\gamma}_j(t) := \gamma(t_0^j + t(t_1^j - t_0^j))$. Choose open neighbourhoods $\tilde{V}_j \subset \mathcal{M}$ of $\bar{\tilde{G}}_j$ and smooth functions $\tilde{\psi}_j \colon \tilde{V}_j \to \mathbb{R}$, $0 \le j \le N$, such that the sets $\tilde{V}_0, \ldots, \tilde{V}_N$ are pairwise disjoint, $\tilde{V}_j \subset V$ and $\tilde{\psi}_j \equiv \psi$ if $j \in \{0, N\}$, and $\tilde{\psi}_j$ is a strictly plurisubharmonic global defining function for $\tilde{G}_j$ such that $\tilde{\psi}_j < 1$ on $\tilde{V}_j$ if $j \in \{1, \ldots, N-1\}$. Finally, let $\tilde{\Gamma}_j \subset \Gamma$, $1 \le j \le N$, be open neighbourhoods of $\tilde{\gamma}_j([0,1])$ with pairwise disjoint closures. Then application of Step 1 to the tupel $(\tilde{G}_{j-1}, \tilde{G}_j, \tilde{V}_{j-1}, \tilde{V}_j, \tilde{\gamma}_j, \tilde{\Gamma}_j, \tilde{\psi}_{j-1}, \tilde{\psi}_j)$ for every $j = 1, \ldots, N$ gives the desired result.

Step 3. *The assertion of the lemma holds true.*

Proof. By Theorem 3.1.3, there exists a strictly pseudoconvex open set $G \subset \mathcal{M}$ with smooth boundary such that $\bar{\Omega}' \subset G$ and $\mathfrak{c}(G) = \mathfrak{c}(\Omega')$. Let $\{G_j\}_{j=1}^N$ be the different connected components of $G$, $N \in \mathbb{N} \cup \{\infty\}$. Fix an arbitrary increasing sequence $\{D_R\}_{R=1}^\infty$ of relatively compact domains $D_R \subset \mathcal{M}$ such that $\bigcup_{R=1}^\infty D_R = \mathcal{M}$. Since $bG$ is smooth, it is easy to see that there exist a family $\{\gamma_j\}_{j=1}^{N-1}$ of smooth embeddings $\gamma_j \colon [0,1] \to \mathcal{M}$ and natural numbers $\nu(j), \mu(j)$,

$1 \leq j < N$, such that $\gamma_j(0) \in bG_{\nu(j)}$, $\gamma_j(1) \in bG_{\mu(j)}$, $\gamma_j(t) \in \mathcal{M} \setminus \bar{G}$ for $t \in (0,1)$, $\gamma_j([0,1]) \cap \gamma_k([0,1]) = \varnothing$ if $j \neq k$, $\#\{1 \leq j < N : D_R \cap \gamma_j([0,1]) \neq \varnothing\}$ is finite for every $R > 0$, and $G \cup \bigcup_{j=1}^{N-1} \gamma_j([0,1])$ is connected. Let $\psi$ be a minimal global defining function for $G$.

Choose open neighbourhoods $\Gamma_j \subset\subset \mathcal{M}$ of $\gamma_j([0,1])$, $1 \leq j < N$, such that

- $\bar{\Gamma}_j \cap \bar{\Gamma}_k = \varnothing$ if $j \neq k$,
- $\bar{\Gamma}_j \cap \bar{G} \subset \bar{G}_{\nu(j)} \cup \bar{G}_{\mu(j)}$, and
- $\Gamma_j \cap \Omega' = \varnothing$.

Then for every $1 \leq j < N$ we can apply Step 2 to obtain a strictly pseudoconvex domain $\Omega_j \subset \mathcal{M}$ with smooth boundary, an open set $\Gamma_j' \subset \Gamma_j$ and a smooth plurisubharmonic function $\varphi_j$ on $\Omega_j$ such that

- $\Omega_j \setminus \Gamma_j' = (G_{\nu(j)} \cup G_{\mu(j)}) \setminus \Gamma_j'$,
- $\varphi_j = \psi$ on $\Omega_j \setminus \Gamma_j'$, while $\varphi_j$ is strictly plurisubharmonic and less than 1 on $\Omega_j \cap \Gamma_j'$.

Then define a strictly pseudoconvex domain $\Omega \subset \mathbb{C}^n$ with smooth boundary as

$$\Omega := \Big[G \setminus \bigcup_{j=1}^{N-1} \Gamma_j'\Big] \cup \bigcup_{j=1}^{N-1} \left[\Omega_j \cap \Gamma_j'\right]$$

and a smooth plurisubharmonic function $\varphi\colon \Omega \to \mathbb{R}$ as

$$\varphi := \begin{cases} \psi & \text{on } \Omega \setminus \bigcup_{j=1}^{N-1} \Gamma_j' \\ \varphi_j & \text{on } \Omega \cap \Gamma_j' \end{cases} .$$

By construction, $\varphi < 1$ on $\Omega$ and $\varphi$ is strictly plurisubharmonic outside $\mathfrak{c}(G) = \mathfrak{c}(\Omega')$. Thus $\mathfrak{c}(\Omega) \subset \mathfrak{c}(\Omega')$. Moreover, observe that, by construction of $\Omega$, one has $\bar{\Omega}' \subset \Omega$, hence $\mathfrak{c}(\Omega') \subset \mathfrak{c}(\Omega)$. It follows that $\mathfrak{c}(\Omega) = \mathfrak{c}(\Omega')$, which completes the proof the lemma. □

In order to get a better understanding of properties of the core we now consider some examples.

**Example 3.** Fix $n \geq 2$ and let $1 \leq q \leq n-1$. Then for generic $C \in \mathbb{R}$

$$\Omega := \left\{(z,w) \in \mathbb{C}^{n-q} \times \mathbb{C}^q : \log\|z\| + \left(\|z\|^2 + \|w\|^2\right) < C\right\}$$

is an unbounded strictly pseudoconvex domain with smooth boundary. By the Liouville theorem, every plurisubharmonic function $\varphi$ on $\Omega$ that is bounded from above has to be constant on $\{0\}\times\mathbb{C}^q$. Hence $\varphi$ fails to be strictly plurisubharmonic at every point of $\{0\}\times\mathbb{C}^q$, i.e., $\{0\}\times\mathbb{C}^q\subset\mathfrak{c}(\Omega)$. On the other hand, let $\psi\colon\mathbb{C}^n\to\mathbb{R}$ be defined as $\psi(z,w)=\log\|z\|+(\|z\|^2+\|w\|^2)$. As before, if $\{\eta_j\}_{j=1}^\infty$ is a sequence of positive numbers that converges to zero fast enough, then $\varphi:=\sum_{j=1}^\infty\eta_j\,\widetilde{\max}_1(\psi-C,-j)$ is a smooth global defining function for $\Omega$ that is strictly plurisubharmonic outside $\{0\}\times\mathbb{C}^q$. This shows that $\mathfrak{c}(\Omega)\subset\{0\}\times\mathbb{C}^q$, and hence $\mathfrak{c}(\Omega)=\{0\}\times\mathbb{C}^q$. In particular, $\mathfrak{c}(\Omega)$ is $q$-pseudoconcave.

**Example 4.** Fix $n\geq 2$ and let $1\leq q\leq n-1$. Further, fix pairwise distinct points $a_1,a_2,\ldots,a_N\in\mathbb{C}^{n-q}$, $N\geq 2$. Then for generic and large enough $C\in\mathbb{R}$

$$\Omega:=\Big\{(z,w)\in\mathbb{C}^{n-q}\times\mathbb{C}^q:\sum_{j=1}^N\log\|z-a_j\|+\big(\|z\|^2+\|w\|^2\big)<C\Big\}$$

is an unbounded strictly pseudoconvex domain with smooth boundary. Using the same argument as before, it can be shown that $\mathfrak{c}(\Omega)=\bigcup_{j=1}^N\{a_j\}\times\mathbb{C}^q$. In particular, the core $\mathfrak{c}(\Omega)$ is not connected.

**Example 5.** Let $\Omega'\subset\mathbb{C}^2_{z,w}$ be a Fatou-Bieberbach domain such that $\varnothing\neq\bar{\Omega}'\cap\{w=0\}\subset\Delta(0,1)\times\{0\}$ and $\overline{\Omega'\cap\{w=0\}}=\bar{\Omega}'\cap\{w=0\}$ (the existence of such a domain is guaranteed by Corollary 1.1 in [Gl98]). Let $\varepsilon>0$ and let $\psi\colon\bar{\Delta}(0,1+\varepsilon)\to(-\infty,-C)$ be a smooth superharmonic function, where $C>0$ is chosen so large that $\{(z,w)\in\mathbb{C}^2:|z|=1+\varepsilon,|w|\leq e^{\psi(z)}\}\subset\mathbb{C}^2\setminus\bar{\Omega}'$. Let $\Phi\colon\mathbb{C}^2\to\Omega'$ be a biholomorphism and define $\Omega\subset\mathbb{C}^2$ as

$$\Omega:=\Phi^{-1}\big(\Omega'\cap\big\{(z,w)\in\mathbb{C}^2:|z|<1+\varepsilon,|w|<e^{\psi(z)}\big\}\big).$$

After possibly replacing $\Omega$ by one of its connected components, $\Omega$ is an unbounded strictly pseudoconvex domain with smooth boundary. Since $\varphi:=\|\cdot\|^2\circ\Phi\colon\Omega\to\mathbb{R}$ is a smooth strictly plurisubharmonic function on $\Omega$ that is bounded from above, we see that $\mathfrak{c}(\Omega)=\varnothing$.

**Example 6.** Let $\Omega\subset\mathbb{C}^n$ be a Fatou-Bieberbach domain or a domain of the form $\Omega=D\times\mathbb{C}^k$ for some domain $D\subset\mathbb{C}^{n-k}$. Then $\mathfrak{c}(\Omega)=\Omega$. It follows easily from our construction of global defining functions that the situation $\mathfrak{c}(\Omega)=\Omega$ cannot happen if $b\Omega$ has points of strict pseudoconvexity.

**Example 7.** Let $X$ be a compact Riemann surface and let $E\subset X$ be a polar subset. Then $X\setminus E$ is a Stein manifold of dimension 1, hence there exists a proper holomorphic embedding $F\colon X\setminus E\to\mathbb{C}^3$. Let $g_1,g_2\colon\mathbb{C}^3\to\mathbb{C}$ be holomorphic functions such that $F(X\setminus E)=\{z\in\mathbb{C}^3:g_1(z)=g_2(z)=0\}$ (see [FR68]). Define

$$\Omega:=\big\{z\in\mathbb{C}^3:\log\big(|g_1(z)|^2+|g_2(z)|^2\big)+\|z\|^2<C\big\}$$

for generic $C \in \mathbb{R}$. Then, after possibly replacing $\Omega$ by a suitable connected component, $\Omega$ is a strictly pseudoconvex domain with smooth boundary such that $F(X \setminus E) \subset \Omega$. We claim that $\mathfrak{c}(\Omega) = F(X \setminus E)$. Indeed, if $\{\eta_j\}_{j=1}^{\infty}$ is a sequence of positive numbers that converges to zero fast enough, then the function $\varphi\colon \mathbb{C}^3 \to \mathbb{R}$ defined by $\varphi(z) := \sum_{j=1}^{\infty} \eta_j \widetilde{\max}_1(\log(|g_1(z)|^2 + |g_2(z)|^2) + \|z\|^2, -j)$ is a smooth plurisubharmonic function that is strictly plurisubharmonic in the complement of $F(X \setminus E)$ and that is bounded from above on $\Omega$, hence $\mathfrak{c}(\Omega) \subset F(X \setminus E)$. On the other hand, if $\varphi\colon \Omega \to \mathbb{R}$ is a plurisubharmonic function that is bounded from above, then $\psi := \varphi \circ F|_{X \setminus E}$ extends to a bounded subharmonic function $\hat{\psi}$ on $X$, and since $X$ is compact we conclude that $\hat{\psi}$ is constant. This means that $\varphi$ is constant on $F(X \setminus E)$, hence $\varphi$ cannnot be strictly plurisubharmonic at any point of $F(X \setminus E)$. This proves that $F(X \setminus E) \subset \mathfrak{c}(\Omega)$, and hence $\mathfrak{c}(\Omega) = F(X \setminus E)$ as claimed.

**Example 8.** Let $H$ be a complex hypersurface in the complex projective space $\mathbb{CP}^n$. Then $\mathbb{CP}^n \setminus H$ is a Stein manifold (see, for example, Corollary V.3.4 in [FG02]), hence there exists a proper holomorphic embedding $F\colon \mathbb{CP}^n \setminus H \to \mathbb{C}^N$ for some $N \in \mathbb{N}$. Let $g_1, g_2, \ldots, g_k\colon \mathbb{C}^N \to \mathbb{C}$ be holomorphic functions such that $F(\mathbb{CP}^n \setminus H) = \{z \in \mathbb{C}^N : g_1(z) = g_2(z) = \cdots = g_k(z) = 0\}$. Define

$$\Omega := \left\{z \in \mathbb{C}^N : \log\left(|g_1(z)|^2 + \cdots + |g_k(z)|^2\right) + \|z\|^2 < C\right\}$$

for generic $C \in \mathbb{R}$. Then, after possibly replacing $\Omega$ by a suitable connected component, $\Omega$ is a strictly pseudoconvex domain with smooth boundary such that $F(\mathbb{CP}^n \setminus H) \subset \Omega$. As before we see that $\mathfrak{c}(\Omega) = F(\mathbb{CP}^n \setminus H)$.

## 3.3 1-pseudoconcavity of the core

Observe that in each example of a domain $\Omega \subset \mathbb{C}^n$ such that $\mathfrak{c}(\Omega) \neq \varnothing$ that we have constructed so far, the core $\mathfrak{c}(\Omega)$ is a (possibly infinite) union of nontrivial analytic subsets of $\Omega$. In this section we investigate the question whether this is a general phenomenon, i.e., whether $\mathfrak{c}(\Omega)$ always carries an analytic structure. We first show that this is not the case by proving the following theorem.

**Theorem 3.3.1.** *For every $n \geq 2$, there exists an unbounded strictly pseudoconvex domain $\Omega \subset \mathbb{C}^n$ with smooth boundary such that $\mathfrak{c}(\Omega)$ is nonempty and contains no analytic variety of positive dimension.*

**Proof.** Let $\mathcal{E}$ be the Wermer type set constructed in Chapter 1 and let $\Phi\colon \mathbb{C}^n \to [0, \infty)$ be the smooth plurisubharmonic function from Corollary 1.4.1. In particular, $\mathcal{E} = \{\Phi = 0\}$ and $\Phi$ is strictly plurisubharmonic outside $\mathcal{E}$. For generic $C > 0$, define $\Omega := \{z \in \mathbb{C}^n : \Phi(z) < C\}$. Then, after possibly replacing $\Omega$ by one of its connected components, $\Omega$ is an unbounded strictly pseudoconvex domain with smooth boundary such that $\mathcal{E} \subset \Omega$. We will show that $\mathfrak{c}(\Omega) = \mathcal{E}$. In view of Lemma 1.3.6, this completes the proof.

Indeed, from the properties of $\Phi$ we immediately see that $\mathfrak{c}(\Omega) \subset \mathcal{E}$. On the other hand, let $\varphi\colon \Omega \to \mathbb{R}$ be a smooth plurisubharmonic function which is bounded from above. By Theorem 1.5.1, there exists a constant $C \in \mathbb{R}$ such that $\varphi \equiv C$ on $\mathcal{E}$. Moreover, since, in view of Lemma 1.2.2, the set $\mathcal{E}$ is pseudoconcave, it follows from Lemma 3.3.2 below that $\varphi$ fails to be strictly plurisubharmonic at every point of $\mathcal{E}$. This shows that $\mathcal{E} \subset \mathfrak{c}(\Omega)$. □

However, we will now prove that the core $\mathfrak{c}(\Omega)$ is always 1-pseudoconcave in $\Omega$, and we will explain that 1-pseudoconcavity can be interpreted as a generalized notion of analytic structure. The main step of this proof is contained in the following lemma.

**Lemma 3.3.1.** *Let $\mathcal{M}$ be a complex manifold and let $\Omega$ be a domain in $\mathcal{M}$. Then it is not possible "to touch" $\mathfrak{c}(\Omega)$ by a strictly pseudoconvex hypersurface contained in $\Omega$. More precisely, one cannot find a domain $U \subset \Omega$ and a smooth real hypersurface $M \subset U$ such that $U \setminus M$ consists of two connected components $U_1$ and $U_2$, $M \cap \mathfrak{c}(\Omega) \neq \varnothing$, $U \cap \mathfrak{c}(\Omega) \subset \overline{U}_1$ and $U_1$ is strictly pseudoconvex at every point $p \in M$.*

**Proof.** Assume, to get a contradiction, that there exist $U$ and $M$ as above. Fix $p \in M \cap \mathfrak{c}(\Omega)$. After possibly shrinking $U$ and performing a local biholomorphic change of variables, we can assume that $U \subset \mathbb{C}^n$ and that $U_1$ is strictly convex at every point of $M$. By slightly enlarging $U_1$, we can choose a smooth real hypersurface $M' \subset U$ such that $U \setminus M'$ consists of two connected components $U_1'$ and $U_2'$, $U_1 \subset U_1'$, $M' \cap M = \{p\}$ and $U_1'$ is strictly convex at every point of $M'$. Moreover, we may assume without loss of generality that $p = 0$ and that the outward unit normal vector to $U_1'$ at 0 equals $e_{y_n} := (0, \dots, 0, i) \in \mathbb{C}^n$, $z_j = x_j + iy_j$, $j = 1, 2, \dots, n$. Let $\tilde{G} \subset U$ be the domain bounded by $M'' := M' + \varepsilon_1 e_{y_n}$ and $\{y_n = \varepsilon_2(|z_1|^2 + \cdots + |z_{n-1}|^2 + x_n^2) - \varepsilon_3\}$, where $\varepsilon_1, \varepsilon_2, \varepsilon_3$ are small positive constants, and let $G \subset \tilde{G}$ be a domain obtained by smoothing the wedge of $\tilde{G}$. Then for suitably choosen $\varepsilon_1, \varepsilon_2, \varepsilon_3$ and a good enough smoothing of $\tilde{G}$ the domain $G$ is a strictly convex smoothly bounded domain in $U$ such that $bG \cap \{y_n > -\frac{\varepsilon_3}{2}\} \subset M'' \subset U \setminus \mathfrak{c}(\Omega)$. (A suitable smoothing of $\tilde{G}$ is obtained as follows: Since the outward unit normal to $U_1'$ is $e_{y_n}$, there exists a smooth strictly concave function $f\colon \mathbb{C}^{n-1}_{z_1,\dots,z_{n-1}} \times \mathbb{R}_{x_n} \to \mathbb{R}_{y_n}$ such that $M''$ is contained in the graph of $f$. Then for $\delta > 0$ small enough let $u := \widetilde{\max}_\delta\big(y_n - f(z_1, \dots, z_{n-1}, x_n), \varepsilon_2(|z_1|^2 + \cdots + |z_{n-1}|^2 + x_n^2) - \varepsilon_3 - y_n\big)$ and set $G := \{u < 0\}$.)

Now let $\varphi$ be a smooth and bounded from above plurisubharmonic function on $\Omega$ that is strictly plurisubharmonic on $\Omega \setminus \mathfrak{c}(\Omega)$ (see the definition of minimal functions, which was stated after the Main Theorem, and the related remarks). Let $\tilde{\varphi}\colon \bar{G} \to \mathbb{R}$ be the maximal plurisubharmonic function such that $\tilde{\varphi}|_{bG} = \varphi$. Since $M'' \cap \mathfrak{c}(\Omega) = \varnothing$, $\varphi$ is strictly plurisubharmonic near $M''$ and hence $\varphi < \tilde{\varphi}$ in a one-sided neighbourhood $W \subset G$ of $M'' \cap bG$ (indeed, for $z_0 \in G \setminus \mathfrak{c}(\Omega)$ the function $\psi(z) := \varphi(z) + \max(\delta_1 - \delta_2\|z - z_0\|^2, 0)$ with $0 < \delta_1 << \delta_2 << 1$ is plurisubharmonic on $G$ with $\psi|_{bG} = \varphi$, hence $\varphi(z_0) < \psi(z_0) \leq \tilde{\varphi}(z_0)$ by maximality of $\tilde{\varphi}$). We want to show that $\varphi < \tilde{\varphi}$ holds not only on $W$, but in fact on the whole set $G \cap \{y_n > -\frac{\varepsilon_3}{3}\} \ni 0$. If we have done so, then $\psi := \delta_{\gamma_2} * (\tilde{\varphi} - \gamma_1) + \gamma_3\|\cdot\|^2$, where $\gamma_1 := (\tilde{\varphi}(0) - \varphi(0))/2$, $\gamma_2$ and $\gamma_3$ are small enough positive constants and $\delta_{\gamma_2}$ is a smooth nonnegative function depending only on $\|z\|$ such that $\operatorname{supp}\delta_{\gamma_2} = \bar{B}^n(0, \gamma_2)$ and $\int_{\mathbb{C}^n} \delta_{\gamma_2} = 1$, is a smooth strictly plurisubharmonic function on $\bar{G}_{\gamma_2} := \{z \in G : \operatorname{dist}(z, bG) \geq \gamma_2\}$ such that $\psi + \delta < \varphi$ on $bG_{\gamma_2}$ and $\psi(0) > \varphi(0) + \delta$ for some $\delta > 0$. In particular, $\widetilde{\max}_\delta(\varphi, \psi)$ is a smooth and bounded from above plurisubharmonic function on $\Omega$ that is strictly plurisubharmonic in 0. This contradicts the fact that $0 \in \mathfrak{c}(\Omega)$.

In order to show that $\varphi < \tilde{\varphi}$ on $G \cap \{y_n > -\frac{\varepsilon_3}{3}\}$ let $G' \subset\subset G$ be a smoothly bounded strictly convex domain such that $bG' \cap \{y_n \geq -\frac{\varepsilon_3}{3}\} \subset W$ and $G \cap \{y_n \geq -\frac{\varepsilon_3}{3}\} \setminus W \subset G'$. Since $\varphi < \tilde{\varphi}$ on $W$, the function $h\colon [-\frac{\varepsilon_3}{3}, \varepsilon] \to \mathbb{R}$ defined by $h(t) := \min_{bG' \cap \{y_n = t\}}(\tilde{\varphi} - \varphi)$ is strictly positive, where $\varepsilon := \sup_{G'} y_n > 0$. In particular, we can choose a smooth function $\chi\colon (-\infty, \varepsilon] \to \mathbb{R}$ such that $\chi|_{(-\varepsilon_3/3, \varepsilon]}$ is strictly convex, $\chi(t) = 0$ for $-\infty < t \leq -\frac{\varepsilon_3}{3}$ and $0 < \chi(t) < h(t)$ for $-\frac{\varepsilon_3}{3} < t \leq \varepsilon$. Let $\rho\colon \bar{G}' \to \mathbb{R}$ be defined as $\rho(z) := \chi(y_n)$ and observe that $\rho$ is plurisubharmonic. Then, by construction of $\rho$, one has $\varphi + \rho \leq \tilde{\varphi}$ on $b\big(G' \cap \{y_n > -\frac{\varepsilon_3}{3}\}\big)$, and hence $\varphi + \rho \leq \tilde{\varphi}$ on $G' \cap \{y_n > -\frac{\varepsilon_3}{3}\}$ by maximality of $\tilde{\varphi}$. Since $\rho > 0$ on $\{y_n > -\frac{\varepsilon_3}{3}\}$, this proves our claim. ☐

As the first consequence of Lemma 3.3.1, we obtain the following property of the core.

**Proposition 3.3.1.** *Let $\mathcal{M}$ be a Stein manifold and let $\Omega \subset \mathcal{M}$ be a domain such that $\mathfrak{c}(\Omega) \neq \varnothing$. Then $\mathfrak{c}(\Omega)$ cannot be relatively compact in $\Omega$.*

**Proof.** Assume, to get a contradiction, that $\mathfrak{c}(\Omega)$ is relatively compact in $\Omega$. Let $\varphi$ be a smooth strictly plurisubharmonic exhaustion function for $\mathcal{M}$ and let $C \in \mathbb{R}$ be the minimal value such that $\mathfrak{c}(\Omega) \subset \{\varphi \leq C\}$. It may happen that $C$ is not a regular value of $\varphi$. In this case choose $p \in \mathfrak{c}(\Omega) \cap \{\varphi = C\}$ and let $U \subset \mathcal{M}$ be an open coordinate patch around $p$ with corresponding chart $h\colon U \to \mathbb{C}^n$. Choose $\delta > 0$ so small that $h(U)_\delta := \{z \in h(U) : \operatorname{dist}(z, b(h(U))) > \delta\}$ still contains $h(p)$, and let $U' := h^{-1}(h(U)_\delta)$. For each $v \in \mathbb{C}^n$, let $\tau_v\colon \mathbb{C}^n \to \mathbb{C}^n$ be the translation $\tau_v(z) :=$

$z-v$ and define a map $g\colon B^n(0,\delta)\to\mathbb{R}$ by $g(v):=\max_{z\in\mathfrak{c}(\Omega)\cap U'}(\varphi\circ h^{-1}\circ\tau_v\circ h)(z)$. Then the image of $g$ contains an open intervall $I\subset\mathbb{R}$ and, by Sard's theorem, there exists a regular value $C'\in I$ of $\varphi$. Let $v'\in B^n(0,\delta)$ such that $g(v')=C'$. Then $\varphi'\colon U'\to\mathbb{R}$ defined by $\varphi':=\varphi\circ h^{-1}\circ\tau_{v'}\circ h$ is a smooth strictly plurisubharmonic function and $\max_{\mathfrak{c}(\Omega)\cap U'}\varphi'=C'$. In particular, $\{\varphi'=C'\}$ is a smooth strictly pseudoconvex hypersurface that touches $\mathfrak{c}(\Omega)$ as described in Lemma 3.3.1. □

**Remark.** As Example I from the Introduction shows, the core $\mathfrak{c}(\Omega)$ can be relatively compact in $\Omega$ if $\mathcal{M}$ is not Stein.

Now we use Lemma 3.3.1 to prove that $\mathfrak{c}(\Omega)$ is always 1-pseudoconcave in $\Omega$.

**Theorem 3.3.2.** *Let $\mathcal{M}$ be a complex manifold and let $\Omega\subset\mathcal{M}$ be a domain. Then $\mathfrak{c}(\Omega)$ is 1-pseudoconcave in $\Omega$. In particular, $\mathfrak{c}(\Omega)$ is pseudoconcave in $\Omega$ if $\dim_{\mathbb{C}}\mathcal{M}=2$.*

**Proof.** Assume, to get a contradiction, that $\mathfrak{c}(\Omega)$ is not 1-pseudoconcave in $\Omega$. Then there exists an $(n-1,1)$ Hartogs figure $H=\big\{(z,w)\in\Delta^{n-1}\times\Delta : \|z\|_\infty > r_1 \text{ or } |w|<r_2\big\}$ and an injective holomorphic mapping $\Phi\colon\hat{H}\to\Omega$ such that $\Phi(H)\subset\Omega\setminus\mathfrak{c}(\Omega)$ but $\Phi(\hat{H})\cap\mathfrak{c}(\Omega)\neq\varnothing$. For small $\varepsilon>0$ let $\varphi\colon\mathbb{C}^{n-1}_z\times\mathbb{C}^*_w\to\mathbb{R}$ be the smooth strictly plurisubharmonic function defined by $\varphi(z,w):=-\log|w|+\varepsilon\big(\|z\|^2+|w|^2\big)$, and for each $C\in\mathbb{R}$ let $G_C$ denote the domain $G_C:=\big\{p\in\Phi(\hat{H}) : (\varphi\circ\Phi^{-1})(p)<C\big\}$. Since for $C$ large enough the set $\hat{H}\cap\{(z,w)\in\mathbb{C}^n : \varphi<C\}$ contains $\hat{H}\setminus H$, and since $\Phi(\hat{H})\cap\mathfrak{c}(\Omega)\subset\Phi(\hat{H}\setminus H)$, we know that for $C$ large enough $\Phi(\hat{H})\cap\mathfrak{c}(\Omega)\subset G_C$. Let $C_0:=\inf\{C\in\mathbb{R} : \Phi(\hat{H})\cap\mathfrak{c}(\Omega)\subset G_C\}$. Then $M:=bG_{C_0}\cap\Phi(\hat{H})$ is a strictly pseudoconvex hypersurface that touches $\mathfrak{c}(\Omega)$ as described in Lemma 3.3.1 (observe that $\{\varphi=C_0\}\cap b\hat{H}\subset b\Delta^{n-1}_z\times\Delta_w$ if $\varepsilon<<1$). Since the lemma states that such hypersurfaces cannot exist, we arrived at a contradiction. □

**Remarks.** 1) Observe that it follows from Example 3 above that in general $\mathfrak{c}(\Omega)$ is not $q$-pseudoconcave in $\Omega$ for any $q>1$.

2) A different proof of Theorem 3.3.2 can also be obtained by modifying the arguments of the proof of Theorem 3.6 in [SlT04] and adapting them to our setting.

Recall that, by Theorems 4.2 and 5.1 in [Sl86], a nonempty relatively closed subset $A$ of an open set $U\subset\mathbb{C}^n$ is $(q+1)$-pseudoconcave in $U$ if and only if $q$-plurisubharmonic functions have the local maximum property on $A$. In particular, $A$ is 1-pseudoconcave if and only if plurisubharmonic functions have the local maximum property on $A$. An analoguous statement is also true in the setting of complex manifolds. Since we were not able to find this statement in the literature, and since in the more general setting the precise formulation of the local maximum property needs a little bit of caution, we state here the following proposition for the convenience of reading.

**Proposition 3.3.2.** *Let $\mathcal{M}$ be a complex manifold of dimension $n$, let $A \subset \mathcal{M}$ be a closed set and let $q \in \{0, 1, \dots, n-2\}$. Then the following assertions are equivalent:*

(1) *For every $p \in A$, there exists an open neighbourhood $U \subset \mathcal{M}$ of $p$ such that $A \cap U$ is $(q+1)$-pseudoconcave in $U$.*

(1′) *$A$ is $(q+1)$-pseudoconcave in $\mathcal{M}$.*

(2) *For every $p \in A$, there exists an open neighbourhood $U \subset \mathcal{M}$ of $p$ such that for every compact set $K \subset U$ and every $q$-plurisubharmonic function $\varphi$ defined in a neighbourhood of $K$ one has $\max_{A\cap K} \varphi \leq \max_{A\cap bK} \varphi$.*

*If $\mathcal{M}$ is Stein, then the above statements are also equivalent to the following one:*

(2′) *For every compact set $K \subset \mathcal{M}$ and every $q$-plurisubharmonic function $\varphi$ defined in a neighbourhood of $K$, one has $\max_{A\cap K} \varphi \leq \max_{A\cap bK} \varphi$.*

*Here $\max_{A\cap bK} \varphi$ is meant to be $-\infty$ if $A \cap bK = \varnothing$.*

**Remark.** If $\mathcal{M}$ is not Stein, then in general the assertion (2′) does not follow from (2), as it is shown by the following simple examples:

i) $\mathcal{M} = \mathbb{CP}^1_z \times \mathbb{C}^{n-1}_w$, $A = \mathbb{CP}^1_z \times \{0\}$, $K = \mathbb{CP}^1_z \times \bar{B}^{n-1}(0,1)$ and $\varphi(z,w) = \|w\|^2$.

ii) $\mathcal{M}_q = \mathbb{CP}^{n-q}_z \times \mathbb{C}^q_w$, $A_q = \mathbb{CP}^{n-q}_z \times \bar{B}^q(0,1)$, $K_q = A_q$ and $\varphi_q(z,w) = -\|w\|^2$, where $q \in \{1, 2, \dots, n-1\}$.

**Proof.** The implication (1′) ⇒ (1) is clear and the implication (1) ⇒ (2) follows from Theorems 4.2 and 5.1 of [Sl86]. We will show that also (2) ⇒ (1′). Indeed, let $A$ have the properties from (2) and assume, to get a contradiction, that $A$ is not $(q+1)$-pseudoconcave in $\mathcal{M}$. Then, by the same kind of arguments as in the proof of Theorem 3.3.2, we can find an open set $V \subset \mathcal{M}$ and a smooth real hypersurface $M \subset V$ such that $V \setminus M$ consists of two connected components $V_1$ and $V_2$, $M \cap A \neq \varnothing$, $V \cap A \subset \bar{V}_1$ and $V_1$ is strictly $q$-pseudoconvex at every point of $M$. After possibly shrinking $V$ and perturbing $M$, we can assume that $M \cap A = \{p\}$ for some $p \in V$. Let $U \subset V$ be an arbitrary neighbourhood of $p$. Let $W \subset\subset U$ be another open neighbourhood of $p$ and let $\varphi$ be a smooth strictly $q$-plurisubharmonic function defined near $\bar{W}$ such that $V_1 \cap W = \{\varphi|_W < 0\}$. Then for $K := \bar{W}$ we have $\max_{A\cap K}\varphi > \max_{A\cap bK}\varphi$. This contradicts the assumptions in (1).

It remains to consider statement (2′). Clearly, one always has that (2′) ⇒ (2). Now let $\mathcal{M}$ be Stein and let $A$ satisfy the properties from (2). Assume, to get a contradiction, that there exists a compact set $K \subset \mathcal{M}$ and a $q$-plurisubharmonic function $\varphi$ defined in a neighbourhood of $K$ such that $\max_{A\cap K}\varphi > \max_{A\cap bK}\varphi$. Let $m := \max_{A\cap K}\varphi$ and consider the set $L := \{z \in A\cap K : \varphi(z) = m\}$. Since $\mathcal{M}$ is Stein, we can use the same arguments as in Proposition 3.3.1 to obtain an open set $V \subset \mathcal{M}$ and a smooth real hypersurface $M \subset V$ such that $V \setminus M$ consists of two connected components $V_1$ and $V_2$, $M \cap L \neq \varnothing$, $V \cap L \subset \bar{V}_1$, and $V_1$ is strictly pseudoconvex at every point of $M$. After possibly shrinking $V$, and after introducing suitable holomorphic coordinates, we can assume that $V \subset \mathbb{C}^n$ and that $V_1$ is strictly convex at every point of $M$. Fix arbitrary $p \in L \cap M$ and let $U \subset\subset V$ be an open neighbourhood of $p$ as described in (2). Without loss of generality we can assume that $p = 0$. By strict convexity of $M$, we can then choose an $\mathbb{R}$-linear functional $\lambda\colon V \to \mathbb{R}$ such that $\lambda \leq 0$ on $V \cap L$ and $\{\lambda = 0\} \cap L = \{p\}$. Let $W \subset\subset U$ be another open neighbourhood of $p$. Then one sees easily that for $\tilde{K} := \bar{W}$ and for $\varepsilon > 0$ small enough the $q$-plurisubharmonic function $\tilde{\varphi} := \varphi + \varepsilon\lambda\colon V \to \mathbb{R}$ satisfies $\max_{A\cap\tilde{K}}\tilde{\varphi} > \max_{A\cap b\tilde{K}}\tilde{\varphi}$. But this contradicts the choice of $U$. □

We conclude this section by a brief discussion on the role of 1-pseudoconcavity of $\mathfrak{c}(\Omega)$. Namely, we want to point out that for our purpose it is reasonable to interpret 1-pseudoconcavity as a generalized notion of analytic structure (see also the discussion at the beginning of Section 2.3). This viewpoint is motivated by the following simple lemma, which was already used in the proof of Theorem 3.3.1, and which is an easy consequence of the above mentioned results of Słodkowski.

**Lemma 3.3.2.** *Let $\mathcal{M}$ be a complex manifold and let $A \subset \mathcal{M}$ be closed and* 1*-pseudoconcave in $\mathcal{M}$. Then every plurisubharmonic function $\varphi$ which is defined on an open neighbourhood of $A$ and which is constant on $A$ fails to be strictly plurisubharmonic at every point of $A$.*

**Proof.** Let $\varphi$ be a plurisubharmonic function defined on an open neighbourhood of $A$ such that $\varphi$ is constant on $A$. Assume, to get a contradiction, that there exists $z \in A$ such that $\varphi$ is strictly plurisubharmonic on a small open coordinate neighbourhood $U \subset \mathcal{M}$ of $z$. Let $\theta\colon \mathcal{M} \to [0,\infty)$ be a smooth function with compact support in $U$ such that $\theta|_A$ is not constant, and choose $\varepsilon > 0$ so small that $\psi := \varphi + \varepsilon\theta$ is still plurisubharmonic. Then $\psi$ attains a local maximum along the 1-pseudoconcave set $A$. But this is not possible, since plurisubharmonic functions have the local maximum property on $A$, see [Sl86]. □

To further support our interpretation of 1-pseudoconcavity, we also want to formulate the following version of Rossi's local maximum modulus principle. It is easily achieved from Rossi's original result by applying Słodkowski's characterization of 1-pseudoconcave sets as local maximum sets for absolute values of holomorphic functions (the original theorem of Rossi is contained in [Ros60]; a formulation of this result which is better suited for our purpose can be found, for example, in Theorem 2.1.8 of [St07]). This version most likely was known to some people before, therefore we do not claim any originality for its proof.

**Proposition 3.3.3.** *Let $K \subset \mathbb{C}^n$ be a compact set and let $z_0$ in $\mathbb{C}^n$. Let $\hat{K}$ denote the polynomial hull of $K$. Then the following assertions are equivalent:*

*(1)* $z_0 \in \hat{K} \setminus K$.

*(2) There exists a connected bounded locally closed set $\lambda \subset \mathbb{C}^n \setminus K$ with the following properties:*

*(i)* $\lambda$ *is* 1*-pseudoconcave in* $\mathbb{C}^n \setminus K$.

*(ii)* $\bar{\lambda} \setminus \lambda \neq \varnothing$ *and* $\bar{\lambda} \setminus \lambda \subset K$.

*(iii)* $z_0 \in \lambda$.

**Proof.** Let first $z_0 \in \hat{K} \setminus K$ be an arbitrary fixed point. Define $\lambda \subset \mathbb{C}^n$ to be the connected component of $\hat{K} \setminus K$ that contains $z_0$. Then, by definition, $\lambda$ is closed in $\mathbb{C}^n \setminus K$, but, by the Shilov idempotent theorem (see, for example, Corollary 6.5 in [Ga69]), $\lambda$ is not closed in $\mathbb{C}^n$. Thus $\bar{\lambda} \setminus \lambda \neq \varnothing$ and $\bar{\lambda} \setminus \lambda \subset K$. Moreover, Rossi's local maximum modulus principle states that absolute values of holomorphic polynomials have the local maximum property on $\lambda$, i.e., $\lambda$ is 1-pseudoconcave in $\mathbb{C}^n \setminus K$, see [Sl86].

For the other direction, fix a set $\lambda \subset \mathbb{C}^n$ such that $\lambda$ satisfies all the properties (i)-(iii) above. Assume, to get a contradiction, that $z_0 \notin \hat{K}$. Then there exists a holomorphic polynomial $p$ on $\mathbb{C}^n$ such that $|p(z_0)| > \max_{z \in K}|p(z)|$. Hence, slightly shrinking $\lambda$, one will find a compact set $L \subset \lambda \subset \hat{K} \setminus K$ such that $z_0 \in L$ and $|p(z_0)| > \max_{z \in b_\lambda L}|p(z)|$, where $b_\lambda L$ denotes the relative boundary of $L$ in $\lambda$. But, in view of the results from see [Sl86], this contradicts 1-pseudoconcavity of $\lambda$. □

Finally, we want to mention the following result due to Fornaess-Sibony (see Corollary 2.6 in [FSi95]): Let $T$ be a positive closed current of bidimension $(p, p)$ on $\mathbb{C}^n$, $1 \leq p \leq n-1$. Then the support of $T$ is $p$-pseudoconcave in $\mathbb{C}^n$ (hence, in particular, it is 1-pseudoconcave in $\mathbb{C}^n$).

## 3.4 Liouville type properties of the core

In all examples of strictly pseudoconvex domains $\Omega \subset \mathbb{C}^n$ such that $\mathfrak{c}(\Omega) \neq \varnothing$, which we have constructed so far, the core has the following Liouville type property: if $\varphi$ is a smooth and bounded from above plurisubharmonic function on $\Omega$, then $\varphi$ is constant on every connected component of $\mathfrak{c}(\Omega)$. Thus it is natural to ask whether this is a general property of the core, i.e., we want to investigate whether every connected component of the core of a strictly pseudoconvex domain $\Omega \subset \mathbb{C}^n$ satisfies a Liouville type theorem.

Further interest to this question is derived from 1-pseudoconcavity of $\mathfrak{c}(\Omega)$, see Theorem 3.3.2 above, and the fact that the following easy lemma holds true.

**Lemma 3.4.1.** *Let $\mathcal{M}$ be a complex manifold and let $L \subset \mathcal{M}$ be a closed set consisting of more than one point such that every smooth and bounded from above plurisubharmonic function defined near $L$ is constant on $L$. Then $L$ is 1-pseudoconcave in $\mathcal{M}$.*

**Proof.** Assume, to get a contradiction, that $L$ is not 1-pseudoconcave in $\mathcal{M}$. Then, using the same argument as in the proof of Theorem 3.3.2, we can find an open set $U \subset \mathcal{M}$ and a smooth real hypersurface $M \subset U$ such that $U \setminus M$ consists of two connected components $U_1$ and $U_2$, $M \cap L \neq \varnothing$, $U \cap L \subset \bar{U}_1$ and $U_1$ is strictly pseudoconvex at every point of $M$. After possibly shrinking $U$ and perturbing $M$, we can assume that $M \cap L = \{p\}$ for some $p \in U$ and that there exists a smooth strictly plurisubharmonic function $\tilde{\varphi}\colon U \to (-\infty, 1]$ defined on an open neighbourhood of $\bar{U}$ such that $U_1 = \{\tilde{\varphi}|_U < 0\}$. Let $\tilde{c} := \max_{L \cap bU} \tilde{\varphi}$ and set $c := \max(\tilde{c}, -1)$. Then for $\delta > 0$ small enough the trivial extension of $\widetilde{\max}_\delta(\tilde{\varphi}, c/2)\colon U \to \mathbb{R}$ by $c/2$ defines a smooth and bounded from above plurisubharmonic function $\varphi$ on a suitable neighbourhood of $L$ such that $\varphi$ is not constant on $L$. This is a contradiction. $\square$

We will prove in this section that a Liouville type theorem holds true for the core of highest order, i.e., for the set $\mathfrak{c}_n(\Omega) \subset \mathfrak{c}(\Omega)$ of all points $z \in \Omega$ where every smooth and bounded from above plurisubharmonic function $\varphi\colon \Omega \to \mathbb{R}$ satisfies $\operatorname{Lev}(\varphi)(z, \cdot\,) \equiv 0$ (for the general definition of cores of higher order see Section 3.5). More precisely, we will prove the following theorem.

**Theorem 3.4.1.** *Let $\mathcal{M}$ be a complex manifold of complex dimension $n$ and let $\Omega \subset \mathcal{M}$ be a domain. Then every smooth and bounded from above plurisubharmonic function on $\Omega$ is constant on each connected component of $\mathfrak{c}_n(\Omega)$.*

However, we will show that in general no analogue of Theorem 3.4.1 holds true for $\mathfrak{c}(\Omega)$, even if $\Omega$ is strictly pseudoconvex. In particular, we will construct strictly pseudoconvex domains $\Omega \subset \mathbb{C}^2_z \times \mathbb{C}^{n-2}_w$ that are bounded in the $z$-directions such that the core has the form $\mathfrak{c}(\Omega) = A \times L$ for some connected set $A \subset \mathbb{C}^2$ that consists of more than one point and some connected set $L \subset \mathbb{C}^{n-2}$ with the property that every smooth and bounded from above plurisubharmonic function defined near $L$ is constant on $L$. Then for $z_0 \in \mathbb{C}^2$ the function $\varphi(z,w) := \|z - z_0\|^2$ is a smooth and bounded from above plurisubharmonic function on $\Omega$, but for almost every choice of $z_0$ the function $\varphi$ is not constant on $\mathfrak{c}(\Omega)$.

These examples show that the connected components of $\mathfrak{c}(\Omega)$ in general do not satisfy a Liouville type theorem. However, observe that here $\mathfrak{c}(\Omega) = \dot{\bigcup}_{\alpha \in A} \{\alpha\} \times L$, where each set $L_\alpha := \{\alpha\} \times L$ has the property that smooth and bounded from above plurisubharmonic functions are constant on $L_\alpha$ (here the symbol $\dot{\bigcup}$ is used in order to indicate that the union is disjoint). Thus the question arises whether one can still formulate a Liouville type theorem for suitably defined "irreducible components" of $\mathfrak{c}(\Omega)$ instead of connected components. At the moment we do not know whether this is possible or not. However, at least in the 2-dimensional case we are able to give a partial answer. It is contained in the Theorem 3.4.2 and the subsequent remarks below. (Note that a local version of Theorem 3.4.2 in the different setting of exhaustion functions was given earlier in Lemma 4.1 of [SlT04].)

Before stating the result, we recall some definitions: Let $M$ be a smooth manifold. An immersed submanifold of $M$ is a subset $S \subset M$ endowed with a topology with respect to which it is a topological manifold, and a smooth structure with respect to which the inclusion map $i\colon S \hookrightarrow M$ is a smooth immersion. An immersed submanifold $S \subset M$ is called weakly embedded in $M$ if every smooth map $f\colon N \to M$ from a smooth manifold $N$ to $M$ that satisfies $f(N) \subset S$ is smooth as a map from $N$ to $S$. An immersed submanifold $S \subset M$ is called complete if for every complete Riemannian metric $g$ on $M$ the induced metric $i^*g$ on $S$ is complete (a Riemannian metric $g$ on $M$ is called complete if the metric on $M$ that is induced by $g$ turns $M$ into a complete metric space). Now let $\mathcal{M}$ be a complex manifold. An immersed complex submanifold of $\mathcal{M}$ is a subset $\mathcal{S} \subset \mathcal{M}$ endowed with a topology with respect to which it is a topological manifold, and a complex structure with respect to which the inclusion map $i\colon \mathcal{S} \hookrightarrow \mathcal{M}$ is a holomorphic immersion. An immersed complex submanifold $\mathcal{S} \subset \mathcal{M}$ will be called weakly embedded or complete if the underlying smooth manifold is weakly embedded or complete, respectively. By a complex curve $\gamma \subset \mathcal{M}$ we will mean a 1-dimensional immersed complex submanifold of $\mathcal{M}$.

**Theorem 3.4.2.** *Let $\mathcal{M}$ be a 2-dimensional complex manifold and let $\Omega \subset \mathcal{M}$ be a domain. Then the following assertions hold true:*

*(1) Let $\varphi\colon \Omega \to \mathbb{R}$ be a minimal function for $\Omega$. Then for every regular value $t \in \mathbb{R}$ of $\varphi$ there exists a family $\{\gamma_\alpha\}_{\alpha \in \mathcal{A}}$ (possibly empty) of weakly embedded*

*complete connected complex curves* $\gamma_\alpha \subset \mathcal{M}$ *such that*

$$\mathfrak{c}_t := \mathfrak{c}(\Omega) \cap \{\varphi = t\} = \bigcup_{\alpha \in \mathcal{A}}^{\cdot} \gamma_\alpha.$$

*Moreover, there is a decomposition* $\mathcal{A} = \mathcal{A}' \cup \mathcal{A}''$ *such that*

$$\mathring{\mathfrak{c}}_t = \bigcup_{\alpha \in \mathcal{A}'}^{\cdot} \gamma_\alpha \quad \textit{and} \quad b\mathfrak{c}_t = \bigcup_{\alpha \in \mathcal{A}''}^{\cdot} \gamma_\alpha,$$

*where* $\mathring{\mathfrak{c}}_t$ *and* $b\mathfrak{c}_t$ *denote the interior and the boundary of* $\mathfrak{c}_t$ *in the relative topology of* $\{\varphi = t\}$*, respectively.*

*(2) Let* $\varphi\colon \Omega \to \mathbb{R}$ *be a smooth and bounded from above plurisubharmonic function, let* $t \in \mathbb{R}$ *be a regular value of* $\varphi$ *and let* $\gamma \subset \mathfrak{c}(\Omega) \cap \{\varphi = t\}$ *be a connected complex curve. Then every smooth and bounded from above plurisubharmonic function* $\tilde{\varphi}\colon \Omega \to \mathbb{R}$ *is constant on* $\gamma$*.*

**Remark.** We do not have an example of a domain $\Omega$ in 2-dimensional complex manifold such that for some minimal function $\varphi\colon \Omega \to \mathbb{R}$ and some regular value $t$ of $\varphi$ the set $\mathfrak{c}(\Omega) \cap \{\varphi = t\}$ has nonempty interior in $\{\varphi = t\}$. Moreover, we do not know if it can happen that the sets $\gamma_\alpha$ from part (1) of the theorem are only immersed but not embedded submanifolds of $\Omega$.

We discuss briefly some consequences of the results in Theorem 3.4.2. Let $\Omega$ be a domain in a 2-dimensional complex manifold and consider the set

$$\mathfrak{c}_{\text{reg}}(\Omega) := \{z \in \mathfrak{c}(\Omega) : \text{there exists a minimal function } \varphi\colon \Omega \to \mathbb{R} \\ \text{such that } \varphi(z) \text{ is a regular value of } \varphi\}.$$

It follows from part (1) of Theorem 3.4.2 that for every $z \in \mathfrak{c}_{\text{reg}}(\Omega)$ there exists a minimal function $\varphi_z : \Omega \to \mathbb{R}$ and a weakly embedded complete complex curve $\gamma_z \subset \mathfrak{c}_{\text{reg}}(\Omega) \cap H_z$, where $H_z := \{\zeta \in \Omega : \varphi_z(\zeta) = \varphi_z(z)\}$, such that $z \in \gamma_z$. Fix arbitrary $z \in \mathfrak{c}_{\text{reg}}(\Omega)$ and assume that $\gamma_z \cap \gamma_{z'} \neq \varnothing$ for some $z' \in \mathfrak{c}_{\text{reg}}(\Omega)$. Then, by part (2) of Theorem 3.4.2, we conclude that $\gamma_z' \subset H_z$. Since for every $p \in H_z$ there exists at most one germ of a complex curve $\gamma \subset H_z$ through $p$, it will follow from Step 2 in the proof of part (1) below and, in particular, from maximality of the curves $\gamma_z, \gamma_{z'}$ that $\gamma_z = \gamma_{z'}$. This shows that $\mathfrak{c}_{\text{reg}}(\Omega) = \dot{\bigcup}_{\alpha \in A} \gamma_\alpha$ for a suitable set $A \subset \mathfrak{c}_{\text{reg}}(\Omega)$. In view of part (2) of the theorem, every smooth and bounded from above plurisubharmonic function on $\Omega$ is constant on each curve $\gamma_\alpha$, $\alpha \in A$. Now consider the set

$$\mathfrak{c}_{\text{sing}}(\Omega) := \mathfrak{c}(\Omega) \setminus \mathfrak{c}_{\text{reg}}(\Omega)$$

and let $\mathfrak{c}_{\mathrm{sing}}(\Omega) = \bigcup_{\beta \in B} \sigma_\beta$ be the decomposition of $\mathfrak{c}_{\mathrm{sing}}(\Omega)$ into its connected components $\sigma_\beta$, $\beta \in B$. We claim that every smooth and bounded from above plurisubharmonic function $\varphi\colon \Omega \to \mathbb{R}$ is constant on each set $\sigma_\beta$, $\beta \in B$. Indeed, assume, to get a contradiction, that there exist $\varphi$ and $\sigma_\beta$ as above such that $\varphi$ is not constant on $\sigma_\beta$. Then let $\psi$ be a minimal function for $\Omega$ and observe that for $\varepsilon > 0$ small enough $\tilde{\varphi} := \varphi + \varepsilon\psi$ is a minimal function for $\Omega$ which is not constant on $\sigma_\beta$. Hence there exist points $p, q \in \sigma_\beta$ such that $\tilde{\varphi}(p) < \tilde{\varphi}(q)$. By connectedness of $\sigma_\beta$, every hypersurface $\{\tilde{\varphi} = t\}$ for $\tilde{\varphi}(p) < t < \tilde{\varphi}(q)$ has nonempty intersection with $\sigma_\beta$, which, in view of Sard's theorem, contradicts the definition of $\mathfrak{c}_{\mathrm{sing}}(\Omega)$. Thus we have shown that in the 2-dimensional case there exists a decomposition

$$\mathfrak{c}(\Omega) = \mathfrak{c}_{\mathrm{reg}}(\Omega) \cup \mathfrak{c}_{\mathrm{sing}}(\Omega) = \Big( \dot{\bigcup_{\alpha \in A}} \gamma_\alpha \Big) \dot{\cup} \Big( \dot{\bigcup_{\beta \in B}} \sigma_\beta \Big)$$

such that each of the sets $\gamma_\alpha$, $\alpha \in A$, and $\sigma_\beta$, $\beta \in B$, is connected and satisfies a Liouville type theorem for smooth plurisubharmonic functions on $\Omega$. Observe, however, that the described above decomposition of $\mathfrak{c}(\Omega)$ is not completely satisfactory, since we do not have much information on the sets $\sigma_\beta$ so far. In particular, we do not know if it can happen that some of the sets $\sigma_\beta$ consist of only one point.

We conclude the discussion of Theorem 3.4.2 by introducing the following two definitions.

**Definition.** Let $\Omega$ be a domain in a complex manifold $\mathcal{M}$. Let $L$ be the family of all subsets $\lambda \subset \mathfrak{c}(\Omega)$ with the property that every smooth and bounded from above plurisubharmonic function on $\Omega$ is constant on $\lambda$, and define a partial ordering on $L$ by setting $\lambda \leq \lambda'$ if and only if $\lambda \subset \lambda'$. Then $\lambda \subset \mathfrak{c}(\Omega)$ is called a *maximal component* of $\mathfrak{c}(\Omega)$ if $\lambda$ is a maximal element in $L$.

One might expect that the curves $\gamma_\alpha$ from Theorem 3.4.2 or the sets $\sigma_\beta$ from the above remarks are maximal components of $\mathfrak{c}(\Omega)$ according to the previous definition. However, at the moment we do not know whether this is true or not. We also do not know if it can happen that a maximal component of $\mathfrak{c}(\Omega)$ consists of only one point.

**Definition.** Let $\mathcal{M}$ be a complex manifold and let $E \subset \mathcal{M}$ be a closed set. We say that $E$ is of *core type* in $\mathcal{M}$ if for every open set $\Omega \subset \mathcal{M}$ such that $E \subset \Omega$ one has $E \subset \mathfrak{c}(\Omega)$.

Observe that in all examples of domains $\Omega \subset \mathcal{M}$ such that $\mathfrak{c}(\Omega) \neq \varnothing$, which we have constructed so far, the set $\mathfrak{c}(\Omega)$ was always of core type in $\mathcal{M}$. We do not know whether this holds true in general (see also the remarks at the beginning of Section 3.2). In particular, we do not know if the assertions of Theorem 3.4.2 hold true for functions which are not defined on $\Omega$ but only in a neighbourhood of $\mathfrak{c}(\Omega)$. Moreover, it is not clear if the curves $\gamma_\alpha$ from Theorem 3.4.2 or the sets $\sigma_\beta$ from the above remarks are of core type in $\Omega$. Note, however, that in the special case of an irreducible closed subvariety $E \subset \mathbb{C}^n$ of pure dimension, some sufficient conditions for $E$ to be of core type in $\mathbb{C}^n$ are given, for example, by Corollary 1 in [Ta93] and Theorem 4 in [Kan96]. Note also that if $E \subset \mathcal{M}$ is of core type in $\mathcal{M}$, then $E$ is 1-pseudoconcave in $\mathcal{M}$. (Indeed, assume, to get a contradiction, that $E$ is not 1-pseudoconcave in $\mathcal{M}$. Then, by the same kind of arguments as in Proposition 3.3.2, we can find $p \in E$, an open neighbourhood $W \subset\subset \mathcal{M}$ of $p$ and a smooth strictly plurisubharmonic function $\varphi$ defined near $\bar{W}$ such that $\varphi(p) = 0$ and $\varphi < 0$ on $(E \cap \bar{W}) \setminus \{p\}$. Let $C := \max_{bW \cap E} \varphi < 0$ if $bW \cap E \neq \varnothing$, and let $C := -1$ otherwise. Then for $\delta > 0$ small enough the trivial extension of $\widetilde{\max}_\delta(\varphi, C/2)\colon W \to \mathbb{R}$ to a suitable open neighbourhood $\Omega \subset \mathcal{M}$ of $E$ defines a smooth and bounded from above plurisubharmonic function on $\Omega$ which is strictly plurisubharmonic near $p$. This contradicts the fact that $E$ is of core type in $\mathcal{M}$.)

We now begin to prove the statements of this section. We start by showing that for every $n \geq 3$ there exists an unbounded strictly pseudoconvex domain $\Omega \subset \mathbb{C}^n$ with smooth boundary and a smooth plurisubharmonic function $\varphi\colon \mathbb{C}^n \to \mathbb{R}$ which is bounded from above on $\Omega$ such that $\mathfrak{c}(\Omega)$ is nonempty and connected but $\varphi$ is not constant on $\mathfrak{c}(\Omega)$. For this we first mention two results on complete pluripolar subsets of $\mathbb{C}^2$.

Result 1. *There exist compact connected complete pluripolar subsets $A \subset \mathbb{C}^2$ that consist of more than one point.*

The first construction of a bounded connected complete pluripolar set $A \subset \mathbb{C}^2$ that consists of more than one point is contained in Example 2.4 in [Sa79]; the fact that this set is complete pluripolar follows from Proposition 2.4 in [LMP92]. Here $A$ is the graph of a certain holomorphic function $f \in \mathcal{O}(\Delta)$ on the unit disc $\Delta \subset \mathbb{C}$, which is not analytically continuable across any point of $b\Delta$. By slightly improving the construction from [Sa79], it is possible to choose $A$ as the graph of a function $f \in \mathcal{O}(\Delta) \cap \mathcal{C}^\infty(\bar{\Delta})$, see Example 2.17 and Proposition 2.15 in [LMP92], or as the graph of a function $f \in \mathcal{C}^\infty(b\Delta)$, see Theorem 1 in [Ed04]. In particular, the last two examples show that $A$ can be assumed to be compact. More examples of compact complete pluripolar sets in $\mathbb{C}^2$ can be found, for example, in [Ed04] and [El].

Result 2. *If $A \subset \mathbb{C}^n$ is a closed complete pluripolar set, then there exists a strictly plurisubharmonic function $\psi\colon \mathbb{C}^n \to [-\infty, \infty)$ such that $A = \{\psi = -\infty\}$ and $\psi$ is smooth on $\mathbb{C}^n \setminus A$.*

The existence of such a function follows from Corollary 1 in [Co90]. It is also a consequence of the earlier Proposition II.2 in [El] and the smoothing procedure of Richberg as formulated, for example, in Theorem I.5.21 from [De12].

**Example 9.** Let $n \geq 3$. Let $A$ be a compact connected complete pluripolar subset of $\mathbb{C}^2$ that consists of more than one point, and let $L$ be a connected complete pluripolar subset of $\mathbb{C}^{n-2}$ such that every smooth and bounded from above plurisubharmonic function defined near $L$ is constant on $L$. (Possible choices for $A$ are described in Result 1 above. If $n = 3$, then for $L$ we can take, for example, $L = \mathbb{C}$; if $n \geq 3$ is large enough, then for $L$ we can also take, for example, unions of positive-dimensional complex subspaces of $\mathbb{C}^{n-2}$, the Wermer type sets $\mathcal{E}$ from Part I or any of the sets from Examples 7 and 8.) Moreover, let $\psi_1 \colon \mathbb{C}^2 \to [-\infty, \infty)$ and $\psi_2 \colon \mathbb{C}^{n-2} \to [-\infty, \infty)$ be strictly plurisubharmonic functions such that $\psi_1$ is smooth on $\mathbb{C}^2 \setminus A$ and $A = \{\psi_1 = -\infty\}$ and such that $\psi_2$ is smooth on $\mathbb{C}^{n-2} \setminus L$ and $L = \{\psi_2 = -\infty\}$. Then the function $\tilde{\psi}(z,w) := \max(\psi_1(z), \psi_2(w))$ is strictly plurisubharmonic and continuous outside $A \times L = \{\tilde{\psi} = -\infty\}$. Hence, by Richberg, we can smooth it up to get a strictly plurisubharmonic function $\psi \colon \mathbb{C}^2 \times \mathbb{C}^{n-2} \to [-\infty, \infty)$ such that $\psi$ is smooth on $\mathbb{C}^2 \times \mathbb{C}^{n-2} \setminus A \times L$, $|\psi - \tilde{\psi}| < 1$ on $\mathbb{C}^2 \times \mathbb{C}^{n-2} \setminus A \times L$ and $A \times L = \{\psi = -\infty\}$. Choose a strictly increasing and strictly convex smooth function $\chi \colon \mathbb{R} \to \mathbb{R}$ such that $\lim_{z \to \infty} \left[\psi_1(z) + \chi(\|z\|^2)\right] = \infty$ (if for each $N \in \mathbb{N}$ such that $A \subset B^2(0,N)$ we set $C_N := \max\{|\psi_1(z)| : z \in \bar{B}^2(0, N+1) \setminus B^2(0,N)\}$, then every strictly increasing and strictly convex smooth function $\chi \colon \mathbb{R} \to \mathbb{R}$ satisfying $\chi(N^2) > C_N + N$ has the required property). Then for generic $C \in \mathbb{R}$, define an unbounded strictly pseudoconvex open set with smooth boundary $\tilde{\Omega} \subset \mathbb{C}^n$ as

$$\tilde{\Omega} := \left\{(z,w) \in \mathbb{C}^2 \times \mathbb{C}^{n-2} : \psi(z,w) + \chi(\|z\|^2) + \|w\|^2 < C\right\}$$

and denote by $\Omega$ the connected component of $\tilde{\Omega}$ that contains the set $A \times L$. Observe that, by the choice of $L$, and by Lemmas 3.3.2 and 3.4.1 above, every smooth and bounded from above plurisubharmonic function on $\Omega$ fails to be strictly plurisubharmonic on $A \times L$. On the other hand, for small enough constants $\eta_j > 0$, $j \in \mathbb{N}$, the function $\Psi(z,w) := \sum_{j=1}^{\infty} \eta_j \widetilde{\max}_1(\psi(z,w) + \chi(\|z\|^2) + \|w\|^2 - C, -j)$ is a smooth global defining function for $\Omega$ that is strictly plurisubharmonic outside $A \times L$. This shows that $\mathfrak{c}(\Omega) = A \times L$ and, in particular, that $\mathfrak{c}(\Omega)$ is connected. Observe now that, by the choice of the functions $\psi$ and $\chi$, the domain $\Omega$ is bounded in the $z$-directions. Hence for every $z_0 \in \mathbb{C}^2$ the smooth plurisubharmonic function $\varphi(z,w) := \|z - z_0\|^2$ is bounded from above on $\Omega$. But for almost every choice of $z_0$ it is not constant on $\mathfrak{c}(\Omega)$.

This gives us, for every $n \geq 3$, an example of a strictly pseudoconvex domain with smooth boundary $\Omega \subset \mathbb{C}^n$ and a connected component of $\mathfrak{c}(\Omega)$ (actually here it is the whole of $\mathfrak{c}(\Omega)$) without Liouville type property for plurisubharmonic functions. By slightly changing the above constructions, we can also show that for every $n \geq 4$, we can additionally assume that $\mathfrak{c}(\Omega)$ contains no analytic variety of positive dimension.

**Example 10.** Let $n \geq 4$. Let $A$ be a compact and connected complete pluripolar subset of $\mathbb{C}^2$ that consists of more than one point such that the projections $\pi_{z_1}(A)$ and $\pi_{z_2}(A)$ of $A$ onto the coordinate axes $\mathbb{C} \times \{0\}$ and $\{0\} \times \mathbb{C}$, respectively, have no interior points (for example, take $A$ to be the graph of a suitable smooth function $f\colon b\Delta \to \mathbb{C}$, see [Ed04]). Moreover, let $L := \mathcal{E} \subset \mathbb{C}^{n-2}$ be the Wermer type set as in Theorem 1.1 of [HST12]. Then repeat the construction of the previous example to obtain an unbounded strictly pseudoconvex domain with smooth boundary $\Omega \subset \mathbb{C}^n$ such that $\mathfrak{c}(\Omega) = A \times \mathcal{E}$. In particular, $\mathfrak{c}(\Omega)$ is connected and contains no analytic variety of positive dimension. For the last assertion observe that every holomorphic function $f = (f_z, f_w)\colon \Delta \to \mathbb{C}^2_z \times \mathbb{C}^{n-2}_w$ has to have constant $f_z$ component, since, by choice of $A$, the holomorphic images $(\pi_{z_1} \circ f_z)(A), (\pi_{z_2} \circ f_z)(A) \subset \mathbb{C}$ have no interior points, and also constant $f_w$ component, since $\mathcal{E}$ contains no analytic variety of positive dimension, which implies that $f$ is constant. Finally, observe that, as before, for almost every choice of $z_0 \in \mathbb{C}^2$ the function $\varphi\colon \mathbb{C}^2 \times \mathbb{C}^{n-2} \to \mathbb{R}$ defined as $\varphi(z, w) := \|z - z_0\|^2$ is a smooth and bounded from above plurisubharmonic function on $\Omega$ which is not constant on $\mathfrak{c}(\Omega)$.

We now begin to prove the theorems of this section. First we prove the Liouville type property of the highest order core, as formulated in Theorem 3.4.1.

**Proof of Theorem 3.4.1.** Let $Z$ be a connected component of $\mathfrak{c}_n(\Omega)$ and let $\varphi\colon \Omega \to \mathbb{R}$ be a smooth plurisubharmonic function which is bounded from above. Assume, to get a contradiction, that there exist points $z_1, z_2 \in Z$ such that $\varphi(z_1) \neq \varphi(z_2)$. Then, by Sard's theorem and by connectedness of $Z$, there exists a regular value $t \in \mathbb{R}$ of $\varphi$ such that $\{\varphi = t\} \cap \mathfrak{c}_n(\Omega) \neq \varnothing$. Choose a strictly increasing and strictly convex smooth function $\chi\colon \mathbb{R} \to \mathbb{R}$ such that $\chi \circ \varphi$ is still bounded from above on $\mathcal{M}$. Then $\chi \circ \varphi$ is a smooth plurisubharmonic function on $\Omega$ that is bounded from above, but $\mathrm{Lev}(\chi \circ \varphi)(z, \cdot\,) = \chi''(\varphi(z))|(\partial\varphi)_z(\,\cdot\,)|^2 + \chi'(\varphi(z))\mathrm{Lev}(\varphi)(z, \cdot\,) \not\equiv 0$ for every $z \in \{\varphi = t\}$, which contradicts the fact that $\{\varphi = t\} \cap \mathfrak{c}_n(\Omega) \neq \varnothing$. ☐

We now turn to the proof of Theorem 3.4.2. For a domain $\Omega \subset \mathbb{C}^n$ denote by $\mathcal{A}(\Omega) := \mathcal{O}(\Omega) \cap \mathcal{C}(\bar{\Omega})$ the algebra of functions holomorphic on $\Omega$ and continuous on $\bar{\Omega}$ and write $\mathcal{P}(\Omega) := \mathcal{PSH}(\Omega) \cap \mathcal{USC}(\bar{\Omega})$ for the functions plurisubharmonic on $\Omega$ and upper semicontinuous on $\bar{\Omega}$. Recall that a plurisubharmonic function $\varphi\colon \Omega \to [-\infty, \infty)$ is called maximal if for every relatively compact open set $G \subset \Omega$

and for each $\psi \in \mathcal{P}(G)$ such that $\psi \le \varphi$ on $bG$ we have $\psi \le \varphi$ on $G$. If $\Omega$ is a bounded strictly pseudoconvex domain in $\mathbb{C}^n$ and $f\colon b\Omega \to \mathbb{R}$ is a continuous function, then by Theorem 4.1 in [Br59] and Theorem 1 in [Wa68] there exists a unique continuous function $F\colon \bar{\Omega} \to \mathbb{R}$ such that $F$ is maximal plurisubharmonic on $\Omega$ and $F|_{b\Omega} = f$ (note that both mentioned above theorems are stated for $\mathcal{C}^2$-smooth strictly pseudoconvex domains, but actually no assumption on smoothness of $b\Omega$ is needed in the proof of Theorem 4.1 in [Br59] and, hence, also in the proof of Theorem 1 in [Wa68]). Moreover,

$$F(z) = \sup\big\{\varphi(z) : \varphi \in \mathcal{U}(\Omega, f)\big\}, \tag{3.5}$$

where $\mathcal{U}(\Omega, f)$ denotes the family of all $\varphi \in \mathcal{P}(\Omega)$ such that $\varphi \le f$ on $b\Omega$. If $f \ge 0$, then one can easily see that $\{F = 0\} = \widehat{\{f = 0\}}^{\mathcal{P}(\Omega)}$, where $\widehat{\{f = 0\}}^{\mathcal{P}(\Omega)} := \{z \in \bar{\Omega} : \varphi(z) \le \sup_{w \in \{f=0\}} \varphi(w) \text{ for every } \varphi \in \mathcal{P}(\Omega)\}$. In fact, more is true as it is shown in the following lemma (the statement of the lemma seems to be well known, but since we were not able to find a reference in the literature, we include here its proof for the convenience of reading).

**Lemma 3.4.2.** *Let $\Omega \subset \mathbb{C}^n$ be a bounded strictly pseudoconvex domain (not necessarily with smooth boundary). Let $f\colon b\Omega \to [0, \infty)$ be a continuous function and let $F\colon \bar{\Omega} \to [0, \infty)$ be the maximal plurisubharmonic function on $\Omega$ such that $F|_{b\Omega} = f$. Then $\{F = 0\} = \widehat{\{f = 0\}}^{\mathcal{A}(\Omega)}$. If $\bar{\Omega}$ is polynomially convex, then $\{F = 0\} = \widehat{\{f = 0\}}$, where $\widehat{\{f = 0\}}$ denotes the polynomially hull of $\{f = 0\}$.*

**Proof.** Set $K := \{f = 0\}$. Let first $z_0 \in \bar{\Omega} \setminus \hat{K}^{\mathcal{A}(\Omega)}$, i.e., there exists $h \in \mathcal{A}(\Omega)$ such that $|h(z_0)| > \max_{z \in K} |h(z)|$. Then for suitably chosen constants $C, \varepsilon > 0$ the function $\varphi := \varepsilon(|h| - C) \in \mathcal{P}(\Omega)$ satisfies $\varphi|_{b\Omega} \le f$ and $\varphi(z_0) > 0$. By (3.5), this implies that $F(z_0) > 0$, i.e., $z_0 \in \bar{\Omega} \setminus \{F = 0\}$. On the other hand, let now $z_0 \in \bar{\Omega} \setminus \{F = 0\}$, i.e., there exists $\varphi \in \mathcal{P}(\Omega)$ such that $\varphi|_{b\Omega} \le f$ but $\varphi(z_0) > 0$. Let $g\colon \mathbb{C}^n \to \mathbb{R}$ be a smooth function such that $g|_{b\Omega} > \varphi|_{b\Omega}$ and $g|_K < \varphi(z_0) < g(z_0)$, and let $\psi\colon U \to \mathbb{R}$ be a strictly plurisubharmonic function on an open neighbourhood $U \subset \mathbb{C}^n$ of $\bar{\Omega}$ such that $\Omega = \{\psi < 0\}$. Moreover, choose $C > 0$ so large that, after possibly shrinking $U$, the function $C\psi + g$ is plurisubharmonic on $U$, and, in the case when $z_0 \in \Omega$, one also has $(C\psi + g)(z_0) < \varphi(z_0)$. Then $\phi := \max(\varphi, C\psi + g)\colon U \to [-\infty, \infty)$ is a plurisubharmonic function such that $\phi(z_0) > \max_{z \in K} \phi(z)$. Since $\Omega$ has a Stein neighbourhood basis, we can assume that $U$ is pseudoconvex. By the equality of holomorphic and plurisubharmonic convex hulls of compact sets in pseudoconvex domains (see, for example, Theorem 4.3.4 in [Hö90]), we then can find a holomorphic function $h \in \mathcal{O}(U) \subset \mathcal{A}(\Omega)$ such that $|h(z_0)| > \max_{z \in K} |h(z)|$, i.e., $z_0 \in \bar{\Omega} \setminus \hat{K}^{\mathcal{A}(\Omega)}$. If $\bar{\Omega}$ is polynomially convex, then, by the Oka-Weil theorem, $h$ can be chosen to be a holomorphic polynomial.$\square$

**Proof of Theorem 3.4.2.** (1) We start with proving the first part of the theorem. In order to do so, we proceed in three steps.

STEP 1. *For every $p \in \mathfrak{c}_t$, there exist local holomorphic coordinates on an open neighbourhood $U \subset \mathcal{M}$ of $p$ and numbers $\varepsilon, \delta, c > 0$ such that $p$ is the origin in $\mathbb{C}^2_{z,w}$ and the following assertions hold true:*

*(i)* $U = U' \times (-c, c) \subset \mathbb{C}^2$ *for some domain* $U' \subset \mathbb{C}_z \times \mathbb{R}_u$, *where* $w = u + iv$,

*(ii)* *there exist a smooth function* $\Phi\colon U' \to (-c, c)$ *and a continuous function* $\Psi\colon U' \to (-c, c)$, $\Psi \leq \Phi$, *such that* $\Gamma_\Phi = U \cap \{\varphi = t\}$, $\Gamma_\Psi$ *is a Levi-flat hypersurface and* $\mathfrak{c}_t \cap U = \Gamma_\Phi \cap \Gamma_\Psi$, *where* $\Gamma_\Phi$ *and* $\Gamma_\Psi$ *denote the graphs of* $\Phi$ *and* $\Psi$, *respectively,*

*(iii)* *there exists a continuous one-parameter family* $\{f_u\}_{-\delta<u<\delta}$ *of holomorphic functions* $f_u\colon \Delta_\varepsilon \to \mathbb{C}_w$ *satisfying* $Re\, f_u(0) = u$ *for every* $u \in (-\delta, \delta)$ *such that* $U' = \dot{\bigcup}_{-\delta<u<\delta}\{(z, Re\, f_u(z)) \in \mathbb{C} \times \mathbb{R} : z \in \Delta_\varepsilon\} =: \dot{\bigcup}_{-\delta<u<\delta} D'_u$ *and* $\Gamma_\Psi = \dot{\bigcup}_{-\delta<u<\delta}\{(z, f_u(z)) \in \mathbb{C}^2 : z \in \Delta_\varepsilon\} =: \dot{\bigcup}_{-\delta<u<\delta} D_u$, *where* $\Delta_\varepsilon := \{z \in \mathbb{C} : |z| < \varepsilon\}$,

*(iv)* *there exists a subset* $d \subset (-\delta, \delta)$ *such that* $\mathfrak{c}_t \cap U = \dot{\bigcup}_{u\in d} D_u$.

PROOF. Without loss of generality we can assume that $t = 0$. Moreover, by introducing suitable local coordinates around $p$, we can also assume that $p$ is the origin in $\mathbb{C}^2_{z,w}$ and that $T_p(\{\varphi = 0\}) = \mathbb{C}_z \times \mathbb{R}_u$, where $w = u+iv$. Choose constants $r, R > 0$ such that for the convex domain $G := \{|z|^2 + u^2 < r\} \cap \{|z|^2 + |w|^2 < R\} \subset \mathbb{C}^2$ there exists a smooth function

$$\Phi\colon \{(z, u) \in \mathbb{C} \times \mathbb{R} : |z|^2 + u^2 \leq r\} \to \mathbb{R}_v \quad \text{such that} \quad \Gamma_\Phi = \{\varphi = 0\} \cap \bar{G},$$

and such that $\operatorname{int} \Gamma_\Phi := \Gamma_\Phi \cap \{(z, w) \in \mathbb{C}^2 : |z|^2 + u^2 < r\}$ satisfies $\operatorname{int} \Gamma_\Phi = \Gamma_\Phi \cap G$. After smoothing the wedges of $G$, we can assume without loss of generality that $bG$ is smooth. Since $\{\varphi = 0\} \cap bG$ is the graph of a smooth function over $\{(z, u) \in \mathbb{C} \times \mathbb{R} : |z|^2 + u^2 = r\}$, it follows from Theorem 3 in [BK91] that there exists a continuous function

$$\Psi\colon \{(z, u) \in \mathbb{C} \times \mathbb{R} : |z|^2 + u^2 \leq r\} \to \mathbb{R}_v \quad \text{such that} \quad \Gamma_\Psi = \widehat{\{\varphi = 0\} \cap bG},$$

and $\operatorname{int} \Gamma_\Psi$ is a Levi-flat hypersurface. Let $f := \max(\varphi, 0)|_{bG}$ and let $F\colon \bar{G} \to \mathbb{R}$ be the continuous function such that $F$ is maximal plurisubharmonic on $G$ and $F|_{bG} = f$. Finally, set $K := \{f = 0\} \subset bG$ and observe that $\hat{K} = \{F = 0\} \subset \bar{G}$ by Lemma 3.4.2.

We claim that $\Phi \geq \Psi$ and that $\hat{K} = \{(z, w) \in \bar{G} : v \leq \Psi(z, u)\} =: \Gamma_\Psi^-$. Indeed, by the choice of $f$, we have $\Gamma_\Psi \subset \hat{K}$ and hence $\hat{K} = \widehat{K \cup \Gamma_\Psi} \supset \Gamma_\Psi^-$, since

$K \cup \Gamma_\Psi = b\Gamma_\Psi^-$. For the other direction, note first that, by strict pseudoconvexity of $bG$, for every $a \in \hat{K} \cap bG$, there exists a holomorphic polynomial $P$ such that $|P|$ attains a strict local maximum at $a$ along $\hat{K}$. (Indeed, we can choose for $P$ a finite part of the Taylor expansion of $1/L_G(a, z - \varepsilon N_G(a))$, where $L_G(a, \cdot\,)$ denotes the Levi polynomial at $a$ of a strictly plurisubharmonic defining function for $G$, $N_G(a)$ is the outward unit normal vector to $G$ at $a$ and $\varepsilon > 0$ is small enough.) Thus, by Rossi's local maximum modulus principle, it follows that $\hat{K} \cap bG = K$. In particular, $\hat{K} \cap b\Gamma_\Psi^+ = \hat{K} \cap \Gamma_\Psi$, where $\Gamma_\Psi^+ := \{(z,w) \in G : v \geq \Psi(z,u)\}$. Another application of Rossi's local maximum modulus principle now shows that $\hat{K} \cap \Gamma_\Psi^+ = \widehat{\hat{K} \cap b\Gamma_\Psi^+}$, and, in view of polynomial convexity of $\Gamma_\Psi$, we get that $\widehat{\hat{K} \cap \Gamma_\Psi} \subset \Gamma_\Psi$. Hence $\hat{K} \cap \Gamma_\Psi^+ \subset \Gamma_\Psi$, i.e., $\hat{K} \subset \Gamma_\Psi^-$. The proof of the second claim is now complete. For the first claim observe that $\varphi \in \mathcal{U}(G, f)$, i.e., $\varphi \leq F$ in view of (3.5) and hence $\{F = 0\} \subset \{\varphi \leq 0\}$. In particular, $\Gamma_\Psi \subset \{\varphi \leq 0\}$ and thus $\Phi \geq \Psi$.

Next we want to show that $\mathfrak{c}(\Omega) \cap \Gamma_\Phi \subset \Gamma_\Psi$. Indeed, in view of the assertions that we have just proven, it suffices to show that $\mathfrak{c}(\Omega) \cap \Gamma_\Phi \subset \hat{K}$. Thus let $q \in \mathfrak{c}(\Omega) \cap \bar{G}$ such that $\varphi(q) = 0$ and assume, to get a contradiction, that $q \notin \hat{K}$ and hence, in view of Lemma 3.4.2, that $F(q) > 0$. Since $\Gamma_\Phi \cap bG \subset K$, it follows that $q \in G$. Then for $\gamma_1 := F(q)/2 > 0$ define $\varphi^* \colon \Omega \to \mathbb{R}$ as $\varphi^* := \widetilde{\max}_{\gamma_1}(\varphi, 0)$ and observe that $\max(\varphi, 0) \leq \varphi^* \leq \max(\varphi, 0) + \gamma_1/2$ by definition of the smooth maximum. Hence $\varphi^*(q) \leq \max(\varphi(q), 0) + \gamma_1/2 = \gamma_1/2 < F(q) - \gamma_1$ while on $bG$ we have $\varphi^* \geq \max(\varphi, 0) = F > F - \gamma_1$. Then $F^* := \delta_{\gamma_2} * (F - \gamma_1) + \gamma_3 \|\cdot\|^2$, where $\gamma_2$ and $\gamma_3$ are small enough positive constants and $\delta_{\gamma_2}$ is a smooth nonnegative function depending only on $\|(z,w)\|$ such that $\operatorname{supp} \delta_{\gamma_2} \subset \bar{B}^2(0, \gamma_2)$ and $\int_{\mathbb{C}^2} \delta_{\gamma_2} = 1$, is a smooth strictly plurisubharmonic function on $\bar{G}_{\gamma_2} := \{(z,w) \in G : \operatorname{dist}((z,w), bG) \geq \gamma_2\}$ such that $\varphi^*(q) < F^*(q)$ and $\varphi^* > F^*$ on $bG_{\gamma_2}$. In particular, for $\delta > 0$ small enough $\widetilde{\max}_\delta(\varphi^*, F^*)$ is a smooth and bounded from above plurisubharmonic function on $\Omega$ that is strictly plurisubharmonic in $q$. This contradicts the fact that $q \in \mathfrak{c}(\Omega)$.

From the Main Theorem in [Sh93] we know that $\Gamma_\Psi$ is the disjoint union of a family of complex discs. Moreover, it follows from the Main Lemma in [Sh93] that there exist positive constants $\varepsilon, \delta > 0$ and a continuous one-parameter family $\{f_u\}_{-\delta<u<\delta}$ of holomorphic functions $f_u \colon \Delta_\varepsilon \to \mathbb{C}_w$ satisfying $\operatorname{Re} f_u(0) = u$ for every $u \in (-\delta, \delta)$ such that $U' := \dot{\bigcup}_{-\delta<u<\delta}\{(z, \operatorname{Re} f_u(z)) \in \mathbb{C} \times \mathbb{R} : z \in \Delta_\varepsilon\}$ is an open neighbourhood of $p$ in $\mathbb{C} \times \mathbb{R}$ contained in $\{(z,u) \in \mathbb{C} \times \mathbb{R} : |z|^2 + u^2 < r\}$ and $\Gamma_\Psi \cap (U' \times \mathbb{R}_v) = \dot{\bigcup}_{-\delta<u<\delta}\{(z, f_u(z)) \in \mathbb{C}^2 : z \in \Delta_\varepsilon\}$. Choose $c > 0$ such that $\Psi(U') \subset (-c, c)$.

Now fix some $u \in (-\delta, \delta)$ and assume that $D_u \cap \Gamma_\Phi \neq \varnothing$, i.e, there exists $p \in D_u$ such that $\varphi(p) = 0$. Since $\Gamma_\Psi \subset \hat{K} \subset \{\varphi \leq 0\}$, it follows that $\varphi|_{D_u}$ is a subharmonic function that attains a maximum at $p$. Hence $D_u \subset \{\varphi = 0\} \cap U = \Gamma_\Phi$. This shows that there exists $d \subset (-\delta, \delta)$ such that $\Gamma_\Phi \cap \Gamma_\Psi = \dot{\bigcup}_{u \in d} D_u$. In particular, for every $p \in \Gamma_\Phi \cap \Gamma_\Psi$ there exists $u \in d$ such that $p \in D_u \subset \Gamma_\Phi \cap \Gamma_\Psi$. But then $\varphi$ is constant on $D_u$ and hence not strictly plurisubharmonic at $p$, which implies $p \in \mathfrak{c}(\Omega)$ by minimality of $\varphi$. This shows that $\Gamma_\Phi \cap \Gamma_\Psi \subset \mathfrak{c}_0 \cap U$. On the other hand, we already know that $\mathfrak{c}_0 \cap U = \mathfrak{c}(\Omega) \cap \Gamma_\Phi \subset \Gamma_\Phi \cap \Gamma_\Psi$. Hence $\Gamma_\Phi \cap \Gamma_\Psi = \mathfrak{c}_0 \cap U$. The proof of Step 1 is now complete.

STEP 2. *Let $H \subset \mathcal{M}$ be a smooth real hypersurface in the 2-dimensional complex manifold $\mathcal{M}$. For every $p \in H$, let $\{\gamma_{p,j}\}_{j \in J_p}$ be the family of all connected complex curves $\gamma_{p,j} \subset H$ such that $p \in \gamma_{p,j}$. Then $\gamma_p := \bigcup_{j \in J_p} \gamma_{p,j}$ is a connected complex curve in $H$ and each $\gamma_{p,j}$ is an open complex submanifold of $\gamma_p$.*

PROOF. This statement surely is well known, but for the convenience of reading we sketch its proof (observe that $J_p$ here might be empty).

First we note that for every $p \in H$ there exists at most one germ $\delta_p$ of an embedded 1-dimensional complex submanifold $\delta \subset H$ of $\mathcal{M}$. Indeed, assume, to get a contradiction, that there exist two submanifolds $\delta_1, \delta_2 \subset H$ such that $\delta_{1,p} \neq \delta_{2,p}$. After possibly shrinking $\delta_1$ and $\delta_2$ we can assume that $\delta_1 \cap \delta_2 = \{p\}$. Choose an open coordinate neighbourhood $U \subset \mathcal{M}$ of $p$ and local holomorphic coordinates on $U$ such that there exist $U' = \Delta_\varepsilon \times (-a, a) \subset \mathbb{C}_z \times \mathbb{R}_u$ and a smooth function $h \colon U' \to \mathbb{R}_v$ satisfying $U \cap H = \Gamma_h$, and, moreover, holomorphic functions $f_1, f_2 \colon \Delta_\varepsilon \to \mathbb{C}_w$ such that $\Gamma_{f_1} = \delta_1 \cap U$ and $\Gamma_{f_2} = \delta_2 \cap U$. It follows then from Rouché's Theorem that for $\delta > 0$ small enough the two functions $g_0, g_\delta \colon \Delta_\varepsilon \to \mathbb{C}$, $g_0 = f_1 - f_2$ and $g_\delta := f_1 - (f_2 + i\delta)$ have the same number of zeros, which contradicts the facts that $\Gamma_{f_1} \cap \Gamma_{f_2} \neq \varnothing$ but $\Gamma_{f_1} \cap \Gamma_{f_2 + i\delta} \subset \Gamma_h \cap \Gamma_{h+\delta} = \varnothing$.

Define $V \subset \gamma_p$ to be open in $\gamma_p$ if $V \cap \gamma_{p,j}$ is open in $\gamma_{p,j}$ for every $j \in J_p$. By the unicity of germs of complex manifolds in $H$ described above, we conclude that $\gamma_{p,j_1} \cap \gamma_{p,j_2}$ is open in $\gamma_{p,j_1}$ and $\gamma_{p,j_2}$. Thus the open sets in $\gamma_p$ define a topology on $\gamma_p$ and each $\gamma_{p,j}$ is open in $\gamma_p$. Since each $\gamma_{p,j}$ is locally an embedded submanifold of $\mathcal{M}$, and since each $\gamma_{p,j}$ is open in $\gamma_p$, it follows that there exists a unique complex structure on $\gamma_p$ such that the inclusion $\gamma_p \hookrightarrow \mathcal{M}$ is a holomorphic immersion (and with respect to this complex structure the inclusions $\gamma_{p,j} \hookrightarrow \gamma_p$ are holomorphic for every $j \in J_p$). From the continuity of the inclusion we conclude that the topology on $\gamma_p$ is Hausdorff. Then Radó's Theorem on second countability of Riemann surfaces shows that the topology of $\gamma_p$ has a countable basis.

STEP 3. *The set $\mathfrak{c}_t$ is the disjoint union of a family of weakly embedded complete connected complex curves. Moreover, the sets $\mathring{\mathfrak{c}}_t$ and $b\mathfrak{c}_t$ have the structure as described above.*

PROOF. It follows immediately from Step 1 that for every $p \in \mathfrak{c}_t$ there exists a complex disc $\delta_p \subset \mathcal{M}$ such that $p \in \delta_p \subset \mathfrak{c}_t$. Applying Step 2 in the case $H :=$

$\{\varphi = t\}$, we conclude that each complex disc $\delta_p$ extends to a maximal connected complex curve $\gamma_p$ in $H$, and that there exists $\mathcal{A} \subset \mathfrak{c}_t$ such that $\mathfrak{c}_t = \dot{\bigcup}_{\alpha\in\mathcal{A}} \gamma_\alpha$.

We claim that each $\gamma_\alpha$ is weakly embedded in $\mathcal{M}$. Indeed, let $N$ be a smooth manifold and let $f\colon N \to \mathcal{M}$ be a smooth map such that $f(N) \subset \gamma_\alpha$, where $\alpha \in \mathcal{A}$ is fixed. Choose arbitrary $q \in N$ and local holomorphic coordinates on $U \subset \mathcal{M}$ around $f(q) \in \gamma_\alpha \cap U \subset \mathfrak{c}_t \cap U = \dot{\bigcup}_{u\in d} D_u$ as described in Step 1. Since $\gamma_\alpha \cap U$ is open in $\gamma_\alpha$, it has at most countably many connected components $\{\gamma_\alpha^k\}_{k\in K_\alpha}$. Moreover, by the unicity of germs of complex curves in $H$ (see the proof of Step 2), and by the identity theorem applied to the function $g_u := w - f_u(z)$, we see that $\gamma_\alpha^k \subset D_u$ whenever $\gamma_\alpha^k \cap D_u \neq \varnothing$. In fact, we even get that $\gamma_\alpha^k = D_u$, since $\gamma_\alpha$ is maximal. This shows that there exists an at most countable set $d_\alpha \subset d$ such that $\gamma_\alpha \cap U = \dot{\bigcup}_{u\in d_\alpha} D_u$. Now let $W \subset N$ be a connected neighbourhood of $q$ such that $f(W) \subset U$. Observe that the function $e\colon \dot{\bigcup}_{u\in d} D_u \to \mathbb{R}$ defined as $e(z,w) = u$ if and only if $f_u(z) = w$ is continuous. Hence $e \circ f\colon W \to \mathbb{R}$ is a continuous function that takes at most countably many values. By connectedness of $W$, we conclude that $f(W) \subset D_{u_W}$ for some $u_W \in d_\alpha$. Since $D_{u_W}$ is an embedded submanifold of $\mathcal{M}$, it follows that $f\colon W \to D_{u_W}$ is smooth. Moreover, $D_{u_W}$ is open in $\gamma_\alpha$, hence the inclusion $D_{u_W} \hookrightarrow \gamma_\alpha$ is smooth too. This shows that $f\colon W \to \gamma_\alpha$ is a smooth map, and thus that $\gamma_\alpha$ is weakly embedded.

Next we want to show that each $\gamma_\alpha \subset \mathcal{M}$ is complete. Indeed, fix $\alpha \in \mathcal{A}$ and let $g$ be a complete Riemannian metric on $\mathcal{M}$. Let $\{p_j\}_{j=1}^\infty \subset \gamma_\alpha$ be a Cauchy sequence with respect to $i^*g$ and let $p := \lim_{j\to\infty} p_j \in \mathcal{M}$. Since $\gamma_\alpha \subset \mathfrak{c}_t$, and since $\mathfrak{c}_t$ is closed in $\mathcal{M}$, it follows that $p \in \mathfrak{c}_t$. Choose local holomorphic coordinates on $U \subset \mathcal{M}$ around $p \in \mathfrak{c}_t \cap U = \dot{\bigcup}_{u\in d} D_u$ as described in Step 1. Then $p \in D_{u_p}$ for some $u_p \in d$. Observe that, after possibly shrinking $U$, there exists a constant $C > 0$ such that for every $u_1, u_2 \in d$, $u_1 \neq u_2$, and every $q_1 \in \gamma_\alpha \cap D_{u_1}$, $q_2 \in \gamma_\alpha \cap D_{u_2}$ one has $\mathrm{dist}_{i^*g}(q_1, q_2) > C$, where $\mathrm{dist}_{i^*g}$ denotes the metric on $\gamma_\alpha$ induced by $i^*g$. Since $\{p_j\}_{j=1}^\infty$ is a Cauchy sequence with respect to $\mathrm{dist}_{i^*g}$, it follows that there exists $j_0 \in \mathbb{N}$ such that $p_j \in D_{u_p}$ for every $j \geq j_0$. Hence $D_{u_p} \cap \gamma_\alpha \neq \varnothing$. By Step 2 and by maximality of the set $\gamma_\alpha$, we conclude that $D_{u_p} \subset \gamma_\alpha$, i.e., $p \in \gamma_\alpha$.

Finally, observe that from property $(iv)$ of the local holomorphic coordinates in Step 1 it follows immediately that $\mathcal{A}$ has a decomposition $\mathcal{A} = \mathcal{A}' \mathbin{\dot\cup} \mathcal{A}''$ such that $\mathring{\mathfrak{c}}_t = \dot{\bigcup}_{\alpha\in\mathcal{A}'} \gamma_\alpha$ and $b\mathfrak{c}_t = \dot{\bigcup}_{\alpha\in\mathcal{A}''} \gamma_\alpha$. This concludes the proof of Step 3 and hence also of part (1) of the theorem.

(2) We now prove the second part of the theorem. Assume, to get a contradiction, that there exists a function $\tilde{\varphi}$ as above that is not constant on $\gamma$. After possibly replacing $\tilde{\varphi}$ by $\tilde{\varphi} + \varepsilon\psi$, where $\psi\colon \Omega \to \mathbb{R}$ is a minimal function for $\Omega$ and $\varepsilon > 0$ is small enough, we can assume without loss of generality that $\tilde{\varphi}$ is minimal. Applying Sard's theorem to the functions $\tilde{\varphi}$ and $\tilde{\varphi}|_\gamma$ simultaneously, we see that

we can choose a regular value $\tilde{t} \in \mathbb{R}$ for $\tilde{\varphi}$ such that $\gamma$ and $\{\tilde{\varphi} = \tilde{t}\}$ intersect transversally. Let $p \in \gamma \cap \{\tilde{\varphi} = \tilde{t}\}$. From part (1) we know that there exists a complex curve $\tilde{\gamma} \subset \{\tilde{\varphi} = \tilde{t}\}$ such that $p \in \tilde{\gamma}$. Observe that, by transversality of $\gamma$ and $\{\tilde{\varphi} = \tilde{t}\}$ at $p$, we have $T_p\gamma \cap T_p\tilde{\gamma} = \{0\}$. Now let $\chi \colon \mathbb{R} \to \mathbb{R}$ be a smooth strictly increasing and strictly convex function such that the smooth plurisubharmonic function $\Phi := \chi \circ \varphi + \chi \circ \tilde{\varphi}$ is still bounded from above on $\Omega$. We claim that $\Phi$ is strictly plurisubharmonic in $p$ which contradicts the fact that $p \in \mathfrak{c}(\Omega)$. Indeed, observe that $\operatorname{Lev}(\chi \circ \varphi)(p, \xi) = \chi''(\varphi(p))|(\partial\varphi)_p(\xi)|^2 + \chi'(\varphi(p))\operatorname{Lev}(\varphi)(p, \xi) > 0$ for every $\xi \in T_p\mathcal{M} \setminus \operatorname{Ker}[(\partial\varphi)_p] = T_p\mathcal{M} \setminus T_p\gamma$. In the same way we conclude that $\operatorname{Lev}(\chi \circ \tilde{\varphi})(p, \xi) > 0$ for every $\xi \in T_p\mathcal{M} \setminus T_p\tilde{\gamma}$. Since $T_p\gamma \cap T_p\tilde{\gamma} = \{0\}$, this proves our claim. □

## 3.5 Pseudoconcavity of higher order cores

Recall the following definition, which was already given before in the Introduction.

**Definition.** Let $\mathcal{M}$ be a complex manifold of complex dimension $n$ and let $\Omega \subset \mathcal{M}$ be a domain. For every $q = 1, \dots, n$, we call the set

$$\mathfrak{c}_q(\Omega) := \big\{ z \in \Omega : \operatorname{rank} \operatorname{Lev}(\varphi)(z, \cdot\,) \leq n - q \text{ for every smooth plurisub-harmonic function } \varphi \colon \Omega \to \mathbb{R} \text{ that is bounded from above} \big\}$$

the *core of order* $q$ of $\Omega$.

It follows immediately from the definition that $\mathfrak{c}_1(\Omega) \supset \mathfrak{c}_2(\Omega) \supset \cdots \supset \mathfrak{c}_n(\Omega)$ and that $\mathfrak{c}(\Omega) = \mathfrak{c}_1(\Omega)$. Let us now illustrate this notion with the following example.

**Example 11.** For generic $C \in \mathbb{R}$, let

$$\Omega := \big\{ (z_1, z_2, z_3) \in \mathbb{C}^3 : \log|z_1| + \log(|z_2| + |z_3|) + \big(|z_1|^2 + |z_2|^2 + |z_3|^2\big) < C \big\}.$$

Then $\Omega$ is strictly pseudoconvex with smooth boundary and, in view of Liouville's theorem, $\mathfrak{c}(\Omega) = l \cup \Pi$, where $l = \{(z_1, 0, 0) \in \mathbb{C}^3 : z_1 \in \mathbb{C}\}$ and $\Pi = \{(0, z_2, z_3) \in \mathbb{C}^3 : z_2, z_3 \in \mathbb{C}\}$.

In the above example one can easily see that $\mathfrak{c}_1(\Omega) = l \cup \Pi$ is 1-pseudoconcave and $\mathfrak{c}_2(\Omega) = \Pi$ is 2-pseudoconcave. We know from Theorem 3.3.2 that $\mathfrak{c}_1(\Omega)$ is always 1-pseudoconcave in $\Omega$ for every domain $\Omega \subset \mathcal{M}$. Moreover, in view of the discussion on Liouville type properties of the core in Section 3.4, observe that the following generalization of Lemma 3.3.2 holds true for every $q = 1, \dots, n$: every smooth plurisubharmonic function $\varphi$ which is defined on an open neighbourhood of a closed $q$-pseudoconcave set $A \subset \mathcal{M}$ and which is constant on $A$ satisfies $\operatorname{rank} \operatorname{Lev}(\varphi)(z, \cdot\,) \leq n - q$ for every $z \in A$ (by the results from [Sl86], $(q-1)$-plurisubharmonic functions have the local maximum property on $q$-pseudoconcave sets for every $q = 1, \dots, n$; thus the statement follows by the same argument as in the proof of Lemma 3.3.2). These observations lead us to the following question: Is it always true that $\mathfrak{c}_q(\Omega)$ is $q$-pseudoconcave in $\Omega$ for $q > 1$?

We will show that in general the answer to the raised above question is negative, by proving the following theorem.

**Theorem 3.5.1.** *For every $n \geq 2$ and every $q = 1, \dots, n$, $q' = 0, \dots, n-1$ such that $(q, q') \neq (1, 0)$, there exists a strictly pseudoconvex domain $\Omega \subset \mathbb{C}^n$ with smooth boundary such that $\mathfrak{c}_q(\Omega)$ is $q'$-pseudoconcave but not $(q'+1)$-pseudoconcave.*

**Proof.** Consider first the case $q' = 0$ (recall that a set $A \subset \mathbb{C}^n$ is 0-pseudoconcave if and only if it is closed). Indeed, fix arbitrary $q \in \{2, \dots, n\}$ and for generic $C \in \mathbb{R}$ consider the set

$$\Omega := \Big\{ z \in \mathbb{C}^n : \|z\|^2 + \sum_{j=1}^{q} \log(\|z\|^2 - |z_j|^2) < C \Big\}.$$

After possibly passing to a suitable connected component, $\Omega$ is a strictly pseudoconvex domain with smooth boundary such that $L := \bigcup_{j=1}^{q} \{ z \in \mathbb{C}^n : z_k = 0 \text{ for every } k \neq j \} \subset \Omega$. By Liouville's theorem, every smooth and bounded from above plurisubharmonic function on $\Omega$ has to be constant on $L$. In particular, $L \subset \mathfrak{c}(\Omega)$ and $0 \in \mathfrak{c}_q(\Omega)$. Moreover, a straightforward computation shows that $\varphi(z) := \exp(\|z\|^2 + \sum_{j=1}^{q} \log(\|z\|^2 - |z_j|^2))$ is a smooth and bounded from above plurisubharmonic function on $\Omega$ such that $\varphi$ is strictly plurisubharmonic on $\Omega \setminus L$ and such that $\operatorname{rank} \operatorname{Lev}(\varphi)(z, \cdot\,) = n - 1$ for every $z \in L \setminus \{0\}$. Hence $\mathfrak{c}(\Omega) \subset L$ and $\mathfrak{c}_q(\Omega) \subset \{0\}$. It follows that $\mathfrak{c}(\Omega) = L$ and $\mathfrak{c}_q(\Omega) = \{0\}$. In particular, $\mathfrak{c}_q(\Omega)$ is not 1-pseudoconcave.

Now fix $q' \in \{1, 2, \dots, n-1\}$. Let $\mathcal{E} \subset \mathbb{C}^{q'}_z \times \mathbb{C}^{n-q'}_w$ be the Wermer type set which was constructed in Chapter 2. Observe that, by Lemma 2.3.1, the set $\mathcal{E}$ is $q'$-pseudoconcave but not $(q'+1)$-pseudoconcave. Now let $\Phi \colon \mathbb{C}^n \to [0, \infty)$ be the smooth plurisubharmonic function from Section 2.2. In particular, $\mathcal{E} = \{\Phi = 0\}$ and $\Phi$ is strictly plurisubharmonic outside $\mathcal{E}$. For generic $C > 0$, define

$$\Omega := \big\{ z \in \mathbb{C}^n : \Phi(z) < C \big\}.$$

Then, after possibly replacing $\Omega$ by one of its connected components, $\Omega$ is an unbounded strictly pseudoconvex domain with smooth boundary such that $\mathcal{E} \subset \Omega$.

We will show that $\mathfrak{c}_1(\Omega) = \mathfrak{c}_2(\Omega) = \cdots = \mathfrak{c}_n(\Omega) = \mathcal{E}$. Note that this completes the proof of the theorem. Indeed, from the properties of $\Phi$ we immediately see that $\mathfrak{c}_1(\Omega) \subset \mathcal{E}$. On the other hand, let $\varphi\colon \Omega \to \mathbb{R}$ be a smooth plurisubharmonic function which is bounded from above. By Theorem 2.2.1, there exists a constant $C \in \mathbb{R}$ such that $\varphi \equiv C$ on $\mathcal{E}$. Thus, in view of Lemma 2.3.2, it follows that $\mathrm{Lev}(\varphi) \equiv 0$ along $\mathcal{E}$. This shows that $\mathcal{E} \subset \mathfrak{c}_n(\Omega)$. The claim now follows from the fact that $\mathfrak{c}_n(\Omega) \subset \mathfrak{c}_{n-1}(\Omega) \subset \cdots \subset \mathfrak{c}_1(\Omega)$. □

At the end of this section, we also state the following generalization of the Main Theorem.

**Theorem 3.5.2.** *Let $\Omega$ be a strictly pseudoconvex domain with smooth boundary in a complex manifold $\mathcal{M}$. Then there exists a bounded global defining function $\varphi$ for $\Omega$ such that $\varphi$ is strictly plurisubharmonic in the complement of $\mathfrak{c}_1(\Omega)$ and* rank *$Lev(\varphi)(z, \cdot\,) = n - q$ for every $z \in \mathfrak{c}_q(\Omega) \setminus \mathfrak{c}_{q+1}(\Omega)$ and $q = 1, 2, \ldots, n$.*

**Proof.** We know from the Main Theorem in Section 3.1.1 that there exists a smooth global defining function $\varphi_1$ for $\Omega$ such that $\operatorname{rank} \mathrm{Lev}(\varphi_1)(z, \cdot\,) = n$ for every $z \notin \mathfrak{c}_1(\Omega)$. Observe that by repeating the same arguments as in the proof of the Main Theorem we can also construct for each $q = 2, 3, \ldots, n$ a smooth global defining function $\varphi_q$ for $\Omega$ such that $\operatorname{rank} \mathrm{Lev}(\varphi_q)(z, \cdot\,) \geq n - q + 1$ for every $z \notin \mathfrak{c}_q(\Omega)$. Then $\varphi := \sum_{q=1}^{n} \varphi_q$ is a function as desired. □

We would like to point out here that the most essential achievement of the Main Theorem and Theorem 3.5.2 is the proof of existence of global defining functions (the construction of these functions is carried out in Theorem 3.1.2). The proof of the additional properties of these functions, namely, of being strictly plurisubharmonic outside the core $\mathfrak{c}(\Omega)$ or having the corresponding rank of the Levi form outside the core $\mathfrak{c}_q(\Omega)$ of order $q$ for every $q = 1, 2, \ldots, n$, is simple and rather standard. Note also that a version of the last argument as well as the definition of a notion similar to our notion of the core $\mathfrak{c}_q(\Omega)$, $q = 1, 2, \ldots, n$, in the different setting of exhaustion functions was given earlier in Lemma 3.1 of [SlT04].

# 4 Holomorphic extension of CR functions and the CR-core

We study the $CR$-core of unbounded strictly pseudoconvex domains $\Omega \subset \mathbb{C}^n$.

In Section 4.1 we construct an example of a strictly pseudoconvex domain $\Omega \subset \mathbb{C}^n$ such that $\mathfrak{c}_{CR}(\Omega)$ is nonempty and contains no analytic variety of positive dimension. Moreover, we show in Section 4.2 that in general $\mathfrak{c}(\Omega) \neq \mathfrak{c}_{CR}(\Omega)$, even for strictly pseudoconvex domains $\Omega \subset \mathbb{C}^2$.

## 4.1 A CR-core with no analytic structure

In this section we study the problem of holomorphic extension of $CR$ functions defined on the boundary of an unbounded strictly pseudoconvex domain $\Omega \subset \mathbb{C}^n$. In particular, we are interested in the question whether the $CR$-core $\mathfrak{c}_{CR}(\Omega)$ of $\Omega$ always carries an analytic structure. We will show that this is not the case, by proving the following theorem.

**Theorem 4.1.1.** *For each $n \in \mathbb{N}$, $n \geq 2$, there exist an unbounded strictly pseudoconvex domain $\Omega \subset \mathbb{C}^n$ and a smooth $CR$ function $f$ on $b\Omega$ such that:*

*(1) The envelope of holomorphy $E(b\Omega)$ of the set $b\Omega$ is one-sheeted.*

*(2) $\mathfrak{c}_{CR}(\Omega)$ is nonempty and contains no analytic variety of positive dimension.*

*(3) $f$ has a single-valued holomorphic extension exactly to $\Omega \setminus \mathfrak{c}_{CR}(\Omega)$.*

**Proof.** Let $\mathcal{E}$ be the Wermer type set constructed in Chapter 1 and let $\varphi\colon \mathbb{C}^n \to [-\infty, \infty)$ be the plurisubharmonic function from Lemma 1.4.3 such that $\mathcal{E} = \{\varphi = 0\}$. For generic $C > 0$, define

$$\Omega := \big\{ z \in \mathbb{C}^n : \varphi(z) + \|z\|^2 < C \big\}.$$

Then, after possibly replacing $\Omega$ by one of its connected components, $\Omega$ is an unbounded strictly pseudoconvex domain with smooth boundary such that $\mathcal{E} \subset \Omega$. By Lemma 1.3.6, the set $\mathcal{E}$ contains no analytic variety of positive dimension.

Moreover, we know from Lemma 1.2.2 that $\Omega \setminus \mathcal{E}$ is pseudoconvex and hence the projection $\pi_n(E(b\Omega))$ of the envelope of holomorphy $E(b\Omega)$ of $b\Omega$ onto $\mathbb{C}^n$ is contained in $\bar{\Omega} \setminus \mathcal{E}$. Thus, in order to show (1) and (2), it only remains to show that $E(b\Omega)$ is one-sheeted and coincides with $\bar{\Omega} \setminus \mathcal{E}$.

Recall first that, by Lemma 1.4.3, there exists a sequence $\{\varphi_\nu\}$ of plurisubharmonic functions $\varphi_\nu \colon \mathbb{C}^n \to [-\infty, \infty)$ such that every function $\varphi_\nu$ is of the form $\varphi_\nu = (1/2^\nu)\log|P_\nu|$ for some holomorphic polynomial $P_\nu$ and such that $\{\varphi_\nu\}$ converges uniformly on compact subsets of $\mathbb{C}^n \setminus \mathcal{E}$ to $\varphi$. Observe then that for every $a \in \mathbb{R}$ the set $\bar{\Omega} \cap \{\varphi \geq a\}$ is compact and hence, since $\varphi_\nu \to \varphi$ uniformly on compact subsets of $\mathbb{C}^n \setminus \mathcal{E}$, for each $a \in \mathbb{R}$ we can choose a natural number $N(a) \in \mathbb{N}$ such that

$$\Omega\cap\{\varphi > a\} \subset \Omega\cap\{\varphi_{N(a)} > a-1\} = \Omega\cap\big\{|P_{N(a)}| > e^{2^{N(a)}(a-1)}\big\} \subset \Omega\cap\{\varphi > a-2\}.$$

Fix some $a \in \mathbb{R}$ and let $N := N(a)$. Observe that $P_N$, being a polynomial, has only finitely many singular values $c_1, c_2, \ldots, c_k$ and let $S_N := \bigcup_{j=1}^k \{P_N = c_j\}$ (indeed, using the explicit formula for $P_N$ stated after Lemma 1.1.1, one can even see that $k = 1$ and $c_1 = 0$). Let now $f \in CR(b\Omega)$. Since $\Omega$ is strictly pseudoconvex, $f$ extends to a holomorphic function on some one-sided neighbourhood $U \subset \bar{\Omega}$ of $b\Omega$, which will be denoted by $f$ as well.

Let $H \subset \mathbb{C}^n$ denote a complex two-dimensional affine subspace of $\mathbb{C}^n$ which is obtained by fixing $n-2$ of the coordinates $z_1, z_2, \ldots, z_{n-1}, w$ (for $n = 2$ the only possible choice is $H = \mathbb{C}^2$). Then $\Omega \cap H = \bigcup_\alpha \Gamma_\alpha$ is the disjoint union of a family $\{\Gamma_\alpha\}$ of strictly pseudoconvex domains $\Gamma_\alpha \subset H \cong \mathbb{C}^2$, and $b_H\Gamma_\alpha \subset b\Omega \cap H$ for each $\alpha$, where $b_H\Gamma_\alpha$ denotes the boundary of $\Gamma_\alpha$ with respect to the relative topology on $H$. In particular we can view each $\Gamma_\alpha$ as a strictly pseudoconvex domain in $\mathbb{C}^2$ and for each $\alpha$ the restriction of $f$ to $U \cap H$ defines a holomorphic function in a one-sided neighbourhood of $b_H\Gamma_\alpha$. With the situation reduced to a two-dimensional case we can now argue as in the example from introduction and conclude from Theorem A in [J95] that $E(b_H\Gamma_\alpha)$ is single-sheeted (of course here $E(b_H\Gamma_\alpha)$ denotes the envelope of holomorphy of $b_H\Gamma_\alpha$ with respect to functions holomorphic in $H \cong \mathbb{C}^2$). On the other hand, since for each $\nu \in \mathbb{N}$ the restriction $P_\nu|_H$ is again a polynomial and we can assume it to be nonconstant (for $\nu \geq \nu_0$ big enough this clearly is satisfied), for each $a' \in \mathbb{R}$ the sets $\{P_{N(a')} = c\}$ with $c \in \mathbb{C}$, $|c| > e^{2^{N(a')}(a'-1)}$, constitute a continuous family of analytic curves in $H \cong \mathbb{C}^2$ that fills $(\Omega \cap H) \cap \{\varphi > a'\}$. Using the Kontinuitätssatz we thus conclude that $E(b_H\Gamma_\alpha) = \bar{\Gamma}_\alpha \cap \{\varphi > -\infty\} = \bar{\Gamma}_\alpha \setminus \mathcal{E}$ for each $\alpha$. Hence, since the domains $\Gamma_\alpha$ are disjoint and pseudoconvex, we get that $E(\bigcup_\alpha b_H\Gamma_\alpha)$ is single-sheeted and $(\Omega \cap H) \setminus \mathcal{E} \subset E(\bigcup_\alpha b_H\Gamma_\alpha)$. In particular $f|_{U\cap H}$ extends to a holomorphic function

$$f_H \colon (\Omega \setminus \mathcal{E}) \cap H \to \mathbb{C}, \qquad f_H = f \text{ near } b\Omega.$$

Observe that this already proves our claim in the case $n = 2$.

Assume now that $n \geq 3$. For each $c \in \mathbb{C} \setminus \{c_1, c_2, \dots, c_k\}$ the hypersurface $\{P_N = c\}$ is a Stein manifold of dimension at least 2, and if $|c| > e^{2^N(a-1)}$, then each connected component of $\Omega_c := \Omega \cap \{P_N = c\}$ is a bounded strictly pseudoconvex domain in $\{P_N = c\}$. Further, $f$ restricts to a holomorphic function on $\Omega_c \setminus K$, where $K \subset \Omega_c$ is a compact set of the form $K = \Omega_c \setminus \tilde{U}$ for some one-sided neighbourhood $\tilde{U} \subset U$ of $b\Omega$. Since each connected component $\Gamma$ of $\Omega_c$ is bounded and strictly pseudoconvex, the boundary of $\Gamma$ in $\{P_N = c\}$ is connected and hence we can assume $\Gamma \setminus K = \Gamma \cap \tilde{U}$ to be connected. Thus we can apply Hartogs theorem on removability of compact singularities to extend $f|_{\Omega_c \setminus K}$ to a holomorphic function $\tilde{f}_c$ on $\Omega_c$ (for a version of the classical Hartogs theorem in the setting of Stein manifolds see [AH72]). In this way we can define a function

$$f_a \colon \big[\Omega \cap \{P_N > e^{2^N(a-1)}\}\big] \setminus S \to \mathbb{C}, \qquad f_a = f \text{ near } b\Omega,$$

by letting $f_a(z,w) = \tilde{f}_c(z,w)$ if $P_N(z,w) = c$. We claim that for every two-dimensional subspace $H \subset \mathbb{C}^n$ described above the functions $f_a$ and $f_H$ coincide on their common domain of definition, namely on the set $\big[\Omega \cap H \cap \{P_N > e^{2^N(a-1)}\}\big] \setminus S$. Indeed, let $c \in \mathbb{C} \setminus \{c_1, c_2, \dots, c_k\}$, $|c| > e^{2^N(a-1)}$. Since the restriction $P_N|_H$ is again a (nonconstant) polynomial, the set $\gamma_c := \Omega \cap H \cap \{P_N = c\}$ is an analytic curve in $\Omega \cap H \cap \{P_N > e^{2^N(a-1)}\}$. Observe that the boundary of $\gamma_c$ is contained in $b\Omega$ and recall that $f_a$ and $f_H$ are holomorphic on $\gamma_c$ and coincide near $b\Omega$. Thus it follows from the uniqueness theorem that $f_a = f_H$ on $\gamma_c$. Hence, since $c \in \mathbb{C} \setminus \{c_1, c_2, \dots, c_k\}$ with $|c| > e^{2^N(a-1)}$ was arbitrary, we conclude that

$$f_a = f_H \quad \text{on} \quad \big[\Omega \cap H \cap \{P_N > e^{2^N(a-1)}\}\big] \setminus S. \tag{4.1}$$

In particular this shows that $f_a$ is holomorphic in each variable separately (recall the definition of $H$). Thus by Hartogs separate analyticity theorem $f_a$ is a holomorphic function on $\big[\Omega \cap \{P_N > e^{2^N(a-1)}\}\big] \setminus S$. Moreover we see from (4.1) and the holomorphicity of $f_H$ on $(\Omega \cap H) \setminus \mathcal{E} \supset \Omega \cap H \cap \{P_N > e^{2^N(a-1)}\}$ that $f_a$ remains bounded near $S$. It follows then from Riemann's removable singularities theorem that $f_a$ extends to a holomorphic function $\tilde{f}_a$ on $\Omega \cap \{P_N > e^{2^N(a-1)}\} \supset \Omega \cap \{\varphi > a\}$. Since $a \in \mathbb{R}$ was arbitrary, and since $\Omega \setminus \mathcal{E} = \bigcup_{a \in \mathbb{R}} \Omega \cap \{\varphi > a\}$, we conclude that $f$ has a single-valued holomorphic extension to $\bar{\Omega} \setminus \mathcal{E}$. Hence $E(b\Omega)$ is single-sheeted and $E(b\Omega) = \bar{\Omega} \setminus \mathcal{E}$.

It only remains to construct a $CR$ function $f$ on $b\Omega$ which extends exactly to $\bar{\Omega} \setminus \mathcal{E}$. In order to do so, let

$$\tilde{\Omega} := \big\{(z,w) \in \mathbb{C}^n : \varphi(z,w) + \big(\|z\|^2 + |w|^2\big) < C_2\big\},$$

where the constant $C_2 > C_1$. Then the domain $\tilde{\Omega}$ is also pseudoconvex and $\bar{\Omega} \subset \tilde{\Omega}$. As before we see that $\tilde{\Omega} \setminus \mathcal{E}$ is pseudoconvex, hence there exists a holomorphic function $f \in \mathcal{O}(\tilde{\Omega} \setminus \mathcal{E})$ which does not extend to $\mathcal{E}$. Then $f|_{b\Omega}$ is a function as required. □

Although the previous theorem shows that in general $\mathfrak{c}_{CR}(\Omega)$ does not possess any analytic structure, observe that it follows immediately from the Kontinuitätssatz and the definition of the $CR$-core, that $\mathfrak{c}_{CR}(\Omega)$ is always pseudoconcave in $\Omega$. Note also that, in view of the discussions at the beginning of Section 2.3 and at the end of Section 3.3, for our purpose it is reasonable to interpret pseudoconcavity of $\mathfrak{c}_{CR}(\Omega)$ as a generalized notion of $(n-1)$-dimensional analytic structure.

## 4.2 Comparison of the core and the CR-core

Let $\Omega \subset \mathbb{C}^n$ be a strictly pseudoconvex domain with smooth boundary such that the envelope of holomorphy $E(b\Omega)$ of $b\Omega$ is single-sheeted. In some simple cases (for example the domain $\Omega$ from Example II in the Introduction) one can observe that $\mathfrak{c}_{CR}(\Omega) = \mathfrak{c}(\Omega)$. The same equality holds true for the strictly pseudoconvex neighbourhoods of Wermer type sets $\mathcal{E} \subset \mathbb{C}^n$, which were constructed in Theorem 3.3.1 and Theorem 4.1.1 above. This is why one can be tempted to think that the equality $\mathfrak{c}_{CR}(\Omega) = \mathfrak{c}(\Omega)$ holds true for every domain $\Omega$ as above. However, we claim that this is false and, moreover, in general the sets $\mathfrak{c}_{CR}(\Omega)$ and $\mathfrak{c}(\Omega)$ are not related at all.

**Proposition 4.2.1.** *(1) There exists an unbounded strictly pseudoconvex domain $\Omega \subset \mathbb{C}^3$ with smooth boundary such that $\mathfrak{c}(\Omega) \neq \varnothing$ but $\mathfrak{c}_{CR}(\Omega) = \varnothing$.*

*(2) There exists an unbounded strictly pseudoconvex domain $\Omega \subset \mathbb{C}^2$ with smooth boundary such that $\mathfrak{c}(\Omega) = \varnothing$ but $\mathfrak{c}_{CR}(\Omega) \neq \varnothing$.*

**Proof.** Let first $\Omega \subset \mathbb{C}^3$ be the domain from Example 3 such that $\mathfrak{c}(\Omega) = \{0\} \times \mathbb{C} \neq \varnothing$. We claim that $\mathfrak{c}_{CR}(\Omega) = \varnothing$. Indeed, by strict pseudoconvexity of $b\Omega$, every CR function $f$ on $b\Omega$ extends to a holomorphic function $\tilde{f}$ on a one-sided neighbourhood $U \subset \Omega$ of $b\Omega$. Further, every slice $S_c := \Omega \cap \{w = c\}$ is a ball in $\mathbb{C}^2_z$, hence by Hartogs theorem on removability of compact singularities each function $\tilde{f}|_{S_c \cap U}$ extends to a holomorphic function $F_c \colon S_c \to \mathbb{C}$. An easy investigation of the proof of Hartogs theorem shows that the function $F \colon \Omega \to \mathbb{R}$ defined by $F(z, w) := F_w(z)$ is holomorphic in the $w$-variable. But it is clear from the construction that $F$ is also holomorphic in the $z$-variables. By Hartogs theorem on separate analyticity, it follows that $F$ is a holomorphic extension of $\tilde{f}$. Since here $f$ was arbitrary, it follows that $\mathfrak{c}_{CR}(\Omega) = \varnothing$.

On the other hand, let now $\Omega$ be the domain from Example 5. We have already seen that $\mathfrak{c}(\Omega) = \varnothing$, and we claim that $\mathfrak{c}_{CR}(\Omega) \neq \varnothing$. Indeed, let $\Omega^* \subset \mathbb{C}^2$ be a strictly pseudoconvex domain with smooth boundary such that $\bar{\Omega} \subset \Omega^*$ (the existence of such a domain $\Omega^*$ follows, for example, from Theorem 3.1.3; the other direct way to see this is by repeating the construction of $\Omega$ with $\psi$ replaced by $\psi + \delta$ for a some small enough constant $\delta > 0$). Moreover, let $h\colon \Delta(0, 1+\varepsilon) \to \mathbb{R}$ be a harmonic function such that $h < \psi$. Then $V := \Phi^{-1}(\Omega' \cap \{(z,w) \in \mathbb{C}^2 : |z| < 1+\varepsilon, |w| < e^{h(z)}\})$ is an unbounded open set with smooth Levi-flat boundary such that, after possibly replacing $V$ by a suitable connected component, $\bar{V} \subset \Omega$. In particular, $\Omega^* \setminus V$ is a pseudoconvex open set, hence there exists a holomorphic function $F\colon (\Omega^* \setminus V) \to \mathbb{C}$ that does not extend holomorphically to any larger domain. But, by construction, $b\Omega \subset \Omega^* \setminus V$, hence $F|_{b\Omega}$ is a CR function on $b\Omega$ that does not extend holomorphically to any point of $V$. Thus $V \subset \mathfrak{c}_{CR}(\Omega)$. $\square$

# Open Questions

We conclude the present thesis by stating some open questions related to the content of this work.

## 1. Existence of global defining functions

**Question 1.** Let $X$ be a complex space and let $\Omega \subset X$ be a smoothly strictly pseudoconvex domain. Does there exist a minimal global defining function for $\Omega$, i.e., does there exist a smoothly plurisubharmonic function $\varphi\colon U \to \mathbb{R}$ defined on an open neighbourhood $U \subset X$ of $\bar{\Omega}$ such that $\Omega = \{\varphi < 0\}$ and such that $\varphi$ is strictly smoothly plurisubharmonic outside $\mathfrak{c}(\Omega)$? (For the definition of $\mathfrak{c}(\Omega)$ in the setting of complex spaces see p. 92.)

**Question 2.** Let $X$ be a complex space and let $\Omega \subset X$ be a smoothly strictly $q$-pseudoconvex domain. Does there exist a smoothly $q$-plurisubharmonic function $\varphi\colon U \to \mathbb{R}$ defined on an open neighbourhood $U \subset X$ of $\bar{\Omega}$ such that $\Omega = \{\varphi < 0\}$ and such that $\varphi$ is smoothly strictly $q$-plurisubharmonic near $b\Omega$?

## 2. The core of a domain

**Question 3.** Let $\Omega \subset \mathbb{C}^n$ be a domain and let $\omega \subset \Omega$ be a domain such that $\mathfrak{c}(\Omega) \subset \omega$. Does it follow that $\mathfrak{c}(\omega) = \mathfrak{c}(\Omega)$?

**Question 4.** Let $\mathcal{M}$ be a complex manifold. Is it possible to characterize the core type subsets $E \subset \mathcal{M}$? (For the definition of core type sets see p. 112.)

**Question 5.** Let $\Omega \subset \mathbb{C}^n$ be a strictly pseudoconvex domain with smooth boundary. Can it happen that the set $\mathfrak{c}(\Omega)$ is not pluripolar? Or, even more, can it happen that $\mathfrak{c}(\Omega)$ has a nonempty interior? And finally the strongest version of this question: Does there exist a strictly pseudoconvex domain $\Omega \subset \mathbb{C}^n$ containing a Fatou-Bieberbach domain?

**Question 6.** Let $\Omega \subset \mathbb{C}^n$ be a domain. Is it true that $\widehat{bB^n(0,R) \cap \mathfrak{c}(\Omega)} = \bar{B}^n(0,R) \cap \mathfrak{c}(\Omega)$ for every $R > 0$? Here $\widehat{bB^n(0,R) \cap \mathfrak{c}(\Omega)}$ denotes the polynomial hull of the set $bB^n(0,R) \cap \mathfrak{c}(\Omega)$.

**Question 7.** Is it true that $\mathfrak{c}^{s_1}(\Omega) = \mathfrak{c}^{s_2}(\Omega)$ for every domain $\Omega \subset \mathbb{C}^n$ and every $s_1, s_2$ such that the corresponding cores $\mathfrak{c}^{s_1}(\Omega)$ and $\mathfrak{c}^{s_2}(\Omega)$ are defined? (For the definition of $\mathfrak{c}^s(\Omega)$ see p. 78.)

**Question 8.** Let $\Omega \subset \mathbb{C}^2$ be a strictly pseudoconvex domain with smooth boundary. Is it true that every smooth and bounded from above plurisubharmonic function on $\Omega$ is constant on each connected component of $\mathfrak{c}(\Omega)$?

**Question 9.** Let $\mathcal{M}$ be a complex manifold and let $\Omega \subset \mathcal{M}$ be a domain. Can it happen that a maximal component of $\mathfrak{c}(\Omega)$ consists of only one point? Moreover, in the case when $\mathcal{M}$ is Stein, is it true that no maximal component of $\mathfrak{c}(\Omega)$ is relatively compact in $\mathcal{M}$? (For the definition of maximal components of $\mathfrak{c}(\Omega)$ see p. 112.)

**Question 10.** Let $\Omega \subset \mathbb{C}^n$ be a strictly pseudoconvex domain with smooth boundary. Is it always true that $\Omega \setminus \mathfrak{c}(\Omega)$ is connected?

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*"... aaaand it's gone!"*