

On Chow Rings of Fine Quiver Moduli and Modules over the Cohomological Hall Algebra



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Hans Franzen

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Meiner Familie.

Introduction

Many interesting problems in mathematics deal with the classification of certain objects up to isomorphism. One of the most famous problems of this type is the study of square matrices (say, over the complex numbers) up to conjugation. A solution was provided by Jordan in 1871. It states that every conjugacy class contains precisely one matrix which is in Jordan canonical form. Viewing a square matrix as a finite-dimensional module over the polynomial algebra in one variable, the parametrization of $(d \times d)$ -matrices up to conjugation amounts to classifying d -dimensional $\mathbb{C}[t]$ -modules up to isomorphism.

Generalizing the above example, we might be interested in classifying finite-dimensional modules over a finitely generated (not necessarily commutative) algebra A up to isomorphism. When studying A -modules, we may assume that A is the quotient of a path algebra by an admissible ideal (cf. [ASS06]). We are particularly interested in the case of a path algebra of a quiver (i.e. the admissible ideal is zero). In general, we cannot expect a canonical form like the Jordan canonical form as a result of Kac asserts (cf. [Kac80]). He proves that, for the path algebra of a wild quiver, there are arbitrarily many continuous families of indecomposable modules.

Taking into account Kac's observation, we focus on another strategy to gain more information on these continuous phenomena. We interpret isomorphism classes of modules over (path) algebras as points of a suitable space and try to use techniques from algebraic geometry to analyze its properties. These algebro-geometric objects whose points correspond to isomorphism classes of modules over the path algebra are called quiver moduli spaces. They will be the objects of our primary interest.

When dealing with moduli spaces, we have to decide which kind of space we would like to work with. In the course of the text (see Example 1.3.1), we will see that, in general, a variety or a scheme whose points (over the ground field k) are in bijection to all isomorphism classes of representations of a given quiver (and of a fixed dimension vector of this quiver) cannot exist. Basically, the reason for this is that there are representations which would have to be contained in the closure of another representation. This phenomenon cannot be captured by a scheme as its k -valued points are always closed. There are two ways of circumventing this problem, both of which will play a role in this thesis. We can, on the one hand, allow a weaker type of space, not requiring the moduli space to be a variety or a scheme, but to be just a stack (cf. [LMB00]). This allows us to capture the whole moduli problem but at the cost of obtaining an algebro-geometric object which has less desirable properties. On the other hand, we might restrict to a certain subclass of representations, the (semi-)stable ones (with respect to a certain stability condition). Mumford's geometric invariant theory [MFK94] then guarantees the existence of a moduli space *as a variety*. Most of the time, we will deal with moduli spaces of (semi-)stable representations, the stacks point of view will be taken implicitly in the last chapter.

In this work, we investigate properties of Chow rings of fine quiver moduli, these are moduli spaces, where stability and semi-stability coincide, which assures that the moduli space is a non-singular variety. Our first main result states that the Chow ring of a fine, projective quiver moduli can be described explicitly in terms of generators and relations which are encoded in the quiver setting, i.e. the underlying quiver, the dimension vector and the stability condition. Studying non-projective fine quiver moduli, we focus on non-commutative Hilbert schemes. It turns out that a basis of the Chow group arising from a cell decomposition constructed by Reineke can be transformed into a basis of monomials in Chern classes of the universal bundle. This is the second main observation of this thesis. Our third central result asserts that the (direct sum of the) Chow groups of non-commutative Hilbert schemes, viewed as a module over Kontsevich's and Soibelman's Cohomological Hall algebra, is in fact a quotient of this very algebra, whose ideal we are able to describe explicitly.

Structure of this Work

In Chapter 1, we give an introduction to moduli spaces of quivers. We recall the definition of a quiver representation. After giving a general definition of (fine and coarse) moduli spaces, which, by the way, resembles the stacks point of view, we turn our attention to the construction of quiver moduli and to their properties, using techniques of Mumford's geometric invariant theory.

The second chapter gives a short introduction to intersection theory. We start by giving a definition of the Chow group and its two basic functorial properties - the proper push-forward and the flat pull-back. Afterwards, we sketch the construction of the Gysin homomorphism and explain how the Chow group of a non-singular variety becomes a commutative ring. We describe how Chern classes of bundles can be constructed and also provide a localized version of Chern classes, which will be used in the subsequent chapters. Finally, we introduce Edidin's and Graham's equivariant intersection theory.

Chapter 3 is devoted to the first main result of this thesis (Theorem 3.2.1). For an acyclic quiver, a coprime dimension vector, and a stability condition for which stability and semi-stability coincide, we give an explicit presentation of the Chow ring of the resulting quiver moduli space. In this situation, it is already known, by a theorem of King and Walter [KW95], that the Chow ring is generated by the Chern classes of the universal family. We start by constructing relations between these generators. Passing to the complete flag bundle over the universal family, we translate the notion of stability to a statement on some localized top Chern classes. By a theorem of Grothendieck (Theorem 2.3.2 taken from [Gro58b]), there is a close connection between the Chow ring of the complete flag bundle of the universal family and the Chow ring of the moduli space itself. We are thus provided the tautological relations, which (almost) yield a presentation. Theorem 3.2.1 states that, together with a linear relation, the tautological relations do, indeed, present the Chow ring of the moduli space. The proof proceeds in two parts. As a first step, we prove the statement when the moduli space is a toric variety (this is Proposition 3.2.2). Afterwards, we link the general statement to the toric case by using a certain covering quiver. We illustrate the theorem in several examples. All of these examples are bipartite quivers and use the canonical stability condition, which reduces the computational effort. We recover a result of Kirwan about the Poincaré series of the moduli space of points on the projective line (cf. Example 3.3.5). Moreover, we describe the Chow ring of a six-dimensional Kronecker moduli space of which a description in terms of generators and relations was not previously known (Example 3.3.7).

In the fourth chapter, we deal with non-commutative Hilbert schemes of the multi-loop quiver. These are special types of quiver moduli spaces, so-called framed quiver moduli. The non-commutative Hilbert scheme of the m -loop quiver parametrizes left-ideals of the free non-commutative algebra in m generators having a fixed codimension. The framing datum makes these varieties particularly well-behaved. For example, Reineke showed in [Rei05] that these non-commutative Hilbert schemes possess a cell decomposition. We show in Theorem 4.4.1, the main statement of this chapter, that the cycles of the cell closures can be displayed as a linear combination of monomials in Chern classes of the universal bundle. The coefficients are given by intersection multiplicities, equaling lengths of certain artinian local rings. Putting them in a reasonable order, these coefficients provide a transformation matrix which is upper unitriangular. As a consequence, these monomials also provide a basis of the Chow group (Corollary 4.4.5). Our guiding example will be the non-commutative Hilbert scheme of left-ideals of $k\langle x, y \rangle$ of codimension three.

Finally, the last chapter deals with modules over the Cohomological Hall algebra (short: CoHa). We give Kontsevich's and Soibelman's definition from [KS11] of the CoHa of a quiver. As Kontsevich and Soibelman work entirely over the complex numbers, they use equivariant cohomology of the representation varieties. However, their construction works equally well in arbitrary characteristic when replacing equivariant cohomology with equivariant Chow groups (see Section 5.1 for a justification). Fixing a quiver Q and a framing for it, we realize the direct sum \mathcal{A} of the Chow groups of the non-commutative Hilbert schemes of Q as a module over the CoHa \mathcal{H} of Q . We find a map $\mathcal{H} \rightarrow \mathcal{A}$ which is \mathcal{H} -linear and surjective and whose kernel we can describe explicitly using the Harder-Narasimhan stratification. This is Theorem 5.2.1, the main result of Chapter 5. An obvious resemblance between the generators of the kernel and the tautological relations of the framed quiver shows that the Chow ring of a non-commutative Hilbert scheme is also tautologically (generated and) presented, although it does not meet the requirements of Theorem 3.2.1. We conclude by giving a detailed description of \mathcal{A} in the cases of the 0- and the 1-loop quiver, the two cases where the CoHa is understood best (cf. Corollaries 5.3.3 and 5.3.4).

Conventions

Throughout the text, we fix an algebraically closed field k of characteristic $\text{char } k = 0$. All vector spaces will be k -vector spaces and linear maps will be k -linear maps. Unless stated otherwise, a ring is always commutative and unital. Morphisms of rings are required to respect the unit element. An algebra is a commutative k -algebra, i.e. a homomorphism $k \rightarrow A$ of rings. Just like for rings, if an algebra is not necessarily commutative, we will say so explicitly.

We assume that the reader is familiar with the basics of algebraic geometry, say at the level of the first two chapters of [Har77]. At some points, we use the technique of faithfully flat descent which can be found in [Gro63]. A scheme is always assumed to be an algebraic k -scheme, this means it is a separated scheme of finite type over $\text{Spec } k$. Hence, a morphism of schemes is always separated. Using this terminology, a variety - which is used short hand for k -variety - is an integral scheme. This amounts to the same thing as to require the scheme to be reduced and irreducible. The structure sheaf of a scheme X is denoted by \mathcal{O}_X . For $x \in X$, not necessarily a closed point, the local ring of X at x is denoted by $\mathcal{O}_{X,x}$. If V is a subvariety of X , we write $\mathcal{O}_{X,V}$ for the local ring \mathcal{O}_{X,η_V} of X at the generic point η_V of V . In this terminology, if X is itself a variety, the function field $\kappa(X)$ is just $\mathcal{O}_{X,X}$.

When we talk about a point x of a variety X , we mean a k -valued point of X , apart from obvious exceptions like, for example, generic points. If X is an affine scheme, denote by $k[X] := \Gamma(X, \mathcal{O}_X)$ its ring of global sections. This is a finitely generated k -algebra.

Furthermore, we suppose that the reader is familiar with the notion of an abelian category and the basic definitions in this context. This includes the notion of a subobject, a (semi-)simple and an indecomposable object, as well as the Grothendieck group of an abelian category. The reader may, for example, refer to [\[Wei94\]](#).

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Chapter 1

Moduli Spaces of Quiver Representations

1.1 Representations of Quivers

As a gentle start, we introduce the notion of a representation of a quiver. Classical references for this are [ASS06] or [ARS97]. In representation theory, a finite directed graph is usually called a quiver. Loosely speaking, a quiver is a bunch of points and a bunch of arrows such that every arrow has a distinguished starting- and endpoint. More precisely:

Definition. A **quiver** Q consists of two finite sets Q_0 , the set of **vertices** of Q , and Q_1 , the set of **arrows** of Q , together with two mappings $s, t : Q_1 \rightrightarrows Q_0$. The vertex $s(\alpha)$ is called the **source** of α , whereas we call $t(\alpha)$ the **target** of α . An arrow α with source i and target j is often symbolized $\alpha : i \rightarrow j$.

Example 1.1.1. Here are a few prominent examples:

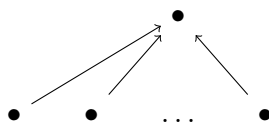
- (i) The r -loop quiver consists of a single vertex and r distinct loops. Pictorially,



- (ii) The r -arrow Kronecker quiver has two vertices and r parallel arrows, i.e. all from one point to the other.



- (iii) The r -subspace quiver has $r + 1$ vertices, say q_1, \dots, q_r and s , and r arrows $\alpha_i : q_i \rightarrow s$. In a picture:



The idea of a representation of a quiver is the following: Given a quiver Q , a representation of Q is a diagram - in some fixed category - that has the shape of the quiver Q . A reasonable category to work in is the category of finitely generated vector spaces. Hence, the precise definition reads as follows:

Definition. Let Q be a quiver. A **representation** M of Q (over k) consists of finitely generated vector spaces M_i for every vertex i of Q and linear maps $M_\alpha : M_i \rightarrow M_j$ for each arrow $\alpha : i \rightarrow j$ of Q .

Example 1.1.2. Consider the three quivers from above.

- (i) A representation of the r -loop quiver consists of a finite-dimensional vector space V and an r -tuple $(\varphi_1, \dots, \varphi_r)$ of endomorphisms of this vector space.
- (ii) For the r -arrow Kronecker quiver, a representation consists of finitely generated vector spaces V and W and r linear maps $\varphi_1, \dots, \varphi_r$ from V to W .
- (iii) A representation of the r -subspace quiver has $r + 1$ vector spaces V_1, \dots, V_r and W and linear maps $\varphi_\nu : V_\nu \rightarrow W$ with $1 \leq \nu \leq r$.

The notion of a morphism of representations comes naturally: Let M and N be representations of Q . A **morphism of representations** $f : M \rightarrow N$ is a tuple of linear maps $f_i : M_i \rightarrow N_i$ for all $i \in Q_0$ such that $N_\alpha f_i = f_j M_\alpha$ holds for every arrow $\alpha : i \rightarrow j$. The composition given in the obvious way, we obtain a category $\mathbf{rep}_k(Q) =: \mathbf{rep}(Q)$ of representations of Q .

It is a well known fact that the category of representations of a quiver Q is equivalent to the category of finitely generated left- kQ -modules, where kQ denotes the path algebra of the quiver. In particular, the category of representations of Q is abelian. The path algebra is finitely generated (over k) and, moreover, it is finite dimensional if and only if Q is **acyclic**, i.e. contains no oriented cycles. As a consequence, we obtain that the category $\mathbf{rep}(Q)$ is always noetherian and, additionally, it is artinian if and only if Q is acyclic. We refer the reader to [ASS06] for the definition of the path algebra and for proofs of the above facts.

We would like to classify all representations of a given quiver Q up to isomorphism. As with every (reasonable) classification problem, a first reduction is to fix the class in the Grothendieck group. A basic fact on quiver representations (cf. [ASS06] for details) tells us that the set of isomorphism classes of simple representations of Q is in bijection to the set Q_0 of vertices of Q . Hence, the Grothendieck group $K_0(\mathbf{rep}(Q))$ is isomorphic to the free abelian group with basis Q_0 . The isomorphism is induced by sending a representation M to the tuple $\underline{\dim} M := (\dim_k M_i \mid i)$ which we call the **dimension vector** of M . Hence, the problem of our main concern is the following:

Classification Problem (First version). Classify all representations of Q of a fixed dimension vector d up to isomorphism.

Still, this is a rather vague request. We are not quite sure, what a classification is supposed to be. In order to get an idea of how precise a classification can be obtained, let us have a look at an example:

Example 1.1.3. Consider the loop quiver Q with r loops from above. A representation of Q is a finite-dimensional vector space V equipped with r endomorphisms $\varphi_1, \dots, \varphi_r$ of V . This representation is isomorphic to another, say, consisting of W and endomorphisms ψ_1, \dots, ψ_r , if and only if there exists an isomorphism $g : V \rightarrow W$ of vector spaces such that $\varphi_\nu g = g\psi_\nu$, or equivalently,

$$\varphi_\nu = g\psi_\nu g^{-1}$$

for all $\nu = 1, \dots, r$. Choosing a particular basis of V , we obtain an isomorphic representation whose vector space is \mathbf{k}^n and consequently, the endomorphisms are $(n \times n)$ -matrices A_1, \dots, A_r . Fixing a dimension vector amounts to fixing the size of the matrix. The Classification Problem thus reads as follows:

Classify r -tuples of $(n \times n)$ -matrices up to simultaneous conjugation.

The case $r = 1$ is well known to us. A solution to the Classification Problem is provided by the Jordan canonical form. We see that in the simplest case already, the problem has a discrete parameter, the Jordan type of the matrix, and n continuous parameters, the eigenvalues of the matrix. For $r > 1$, the problem seems totally out of control, a canonical form is not known.

In other words, the one-loop quiver admits a canonical form for its representations, while the r -loop quiver (with $r > 1$) does not. This has a general reason:

An immediate reduction to the Classification Problem is provided by the Krull-Schmidt theorem which tells us that it suffices to classify all *indecomposable* representations of Q . For simplicity, assume further that Q is connected, as the connected components may be treated individually.

If the underlying unoriented graph of Q is one of the Dynkin diagrams A_n, D_n, E_6, E_7, E_8 , Gabriel's theorem (cf. [Gab72]) states that there exists at most one indecomposable of a fixed dimension vector. For example, the 2-subspace quiver is a D_3 . If Q is of extended Dynkin type $\tilde{A}_n, \tilde{D}_n, \tilde{E}_6, \tilde{E}_7, \tilde{E}_8$, then, for a fixed dimension vector d , the indecomposable representations of Q of dimension vector d depend on at most one continuous parameter (see [DF73] and [Naz73]). This includes the one-loop quiver (which is of type \tilde{A}_0) and the 2-arrow Kronecker quiver (type \tilde{A}_1). Quivers which are neither of Dynkin type ADE nor of extended Dynkin type $\tilde{A}\tilde{D}\tilde{E}$ are called wild. Kac's theorem (cf. [Kac80]) asserts that for these quivers, the Classification Problem of indecomposable representations of dimension vector d depends on $1 - \langle d, d \rangle$ continuous parameters if d is an imaginary root. The r -loop quiver is wild for $r \geq 2$, the r -arrow Kronecker quiver and the r -subspace quiver are both wild for $r \geq 3$.

As, in general, a canonical form cannot be obtained, we follow a different strategy: We try to find a reasonable space, for us, this will be a variety, whose points are in bijection to isomorphism classes of representations of Q . This leads us to the notion of a moduli space.

1.2 Fine and Coarse Moduli Spaces

We loosely follow the treatment of [New78]. Let us start with the definition of a moduli problem.

Definition. A **moduli problem** is a contravariant functor from the category of varieties to the category of sets.

Given a moduli problem t , we can look at the set $t(\text{Spec } k)$ which we abbreviate $t(k)$. A moduli space for t is a variety whose (k -valued) points are in bijection to $t(k)$ *in a reasonable way*. Usually, we distinguish two interpretations of what a “reasonable way” is supposed to be. The first is the following:

Definition. Let t be a moduli problem. A **fine moduli space** for t is a representing object of t , i.e. a pair consisting of a variety M and an isomorphism of functors $t \rightarrow \underline{M}$.

In the above definition, \underline{M} is the contravariant set-valued functor that sends a variety X to the set of X -valued points of M . In every book on category theory, e.g. [Mac71], the reader will find the statement that a representing object of a functor is uniquely determined up to (natural) isomorphism.

Representability is a strong property. Often, we have an interesting functor which is not representable but which allows some parametrization by a variety thus making it similar to a Hom-functor.

Definition. Let t be a moduli problem. A **coarse moduli space** for t is a pair consisting of a variety M and a natural transformation $\varphi : t \rightarrow \underline{M}$ satisfying the following:

- (M.1) The map $\varphi_k : t(k) \rightarrow M(k)$ is bijective.
- (M.2) For every variety N and every natural transformation $\psi : t \rightarrow \underline{N}$, there exists a unique morphism $a : M \rightarrow N$ such that $\underline{a} \circ \varphi = \psi$.

In the above context, \underline{a} is the natural transformation which is given by composition with a . More precisely, this means that $\underline{a}(f) = a \circ f$ for an X -valued point f of M . Using axiom (M.2), we conclude that for two coarse moduli spaces (M, φ) and (M', φ') , there exists a unique isomorphism $a : M \rightarrow M'$ such that $\underline{a} \circ \varphi = \varphi'$. This means that a coarse moduli space for a moduli problem t is unique up to a natural isomorphism.

The terminology suggests that the notion of a fine moduli space is stronger than the notion of a coarse one. A routine verification shows that, indeed, a fine moduli space for t is also a coarse moduli space for t .

In practice, moduli problems often arise from certain equivalence relations on families. This concept can also be found in Newstead’s book (c.f. [New78]). We give a brief introduction which concentrates on representations of quivers.

Definition. Let X be a variety and let Q be a quiver. A **family E of representations of Q over X** consists of vector bundles E_i over X associated to every vertex i of Q and bundle maps $E_\alpha : E_i \rightarrow E_j$ for every arrow $\alpha : i \rightarrow j$.

By a bundle map, we mean a morphism of varieties over the basis which induces a linear map on the fibers. We might interpret a family of representations of Q over X as a representation of Q in the category of vector bundles and bundle maps. Using this interpretation, a **morphism of families of representations of Q over X** is defined in the same vein as a morphism of (ordinary) representations of Q . In this way, we obtain the category of families of representations of Q over X which is easily seen to be an additive category. It is also possible to define a dimension vector of a family. It is given by the ranks of the vector bundles E_i . Note that our definition of a family coincides with the usage in [Kin94].

The definition of a family over X (which will always be understood as a family of representations of Q) behaves functorially in X . For a morphism $f : Y \rightarrow X$ of varieties and a family E over X , we

get a family f^*E over Y by pulling back every vector bundle E_i and taking the induced maps on the pull-backs. In particular, for every point x of X , which is nothing but a morphism $x : \text{Spec } k \rightarrow X$, we may define the **fiber** of E in x by $E_x := x^*E$. This is a representation of Q .

Let \mathbf{P} be a property of representations of Q which is stable under isomorphisms. We extend this property to a family E over X by requiring every fiber E_x to have the property \mathbf{P} . Moreover, we define an equivalence relation for families as follows: Two families E' and E'' parametrized by X are defined to be equivalent, short $E' \sim E''$, if their fibers E'_x and E''_x are isomorphic for every point x of X . The equivalence class of E is denoted $[E]$. Fix a quiver Q and a dimension vector d for Q , i.e. a tuple $d = (d_i \mid i \in Q_0)$ of non-negative integers. Define a moduli problem $t = \text{Rep}^{\mathbf{P}}(Q, d)$ by letting $t(X)$ be the set of equivalence classes of families of representations of Q over X of dimension vector d having property \mathbf{P} . On morphisms, t is defined by the pull-back of families. We call $\text{Rep}^{\mathbf{P}}(Q, d)$ the **moduli problem of \mathbf{P} -representations of Q of dimension vector d** . For convenience, a representation (or a family of representations) of Q with dimension vector d will sometimes be called a representation (or a family of representations) of (Q, d) .

Definition. A family U of representations of Q with dimension vector d over M is called **universal** for $\text{Rep}^{\mathbf{P}}(Q, d)$ if it satisfies \mathbf{P} and if there exists a unique morphism $u_E : X \rightarrow M$ such that $E \sim u_E^*U$ for every variety X and every family E of representations of (Q, d) over X which has \mathbf{P} .

For the rest of this section, let us assume Q , d and \mathbf{P} to be fixed. Denote $t := \text{Rep}^{\mathbf{P}}(Q, d)$. For brevity, we frequently drop the dependencies of these data in the terminology when there is no confusion about to arise. The next result shows that a universal family provides a fine moduli space for the associated moduli problem and vice versa. The proof can be found in [New78].

Lemma 1.2.1. If U is a universal family over M , then the pair consisting of M and the natural transformation $\varphi : t \rightarrow \underline{M}$ defined by

$$\varphi[E] := u_E$$

for every family E over X is a fine moduli space for t . Here, u_E is the unique morphism $X \rightarrow M$ such that $E \sim u_E^*U$. Conversely, if M together with φ is a fine moduli space for t , then any representative U of the equivalence class $\varphi^{-1}(\text{id}_M)$ is a universal family.

There is also a possibility to describe coarse moduli spaces of t in terms of families. It will turn out that a coarse moduli space is the same as a weakly universal pair (defined below). The proof of this result (Lemma 1.2.2) can also be found in [New78].

Definition. A pair consisting of a variety M and a bijection $\alpha : t(k) \rightarrow M(k)$ is called **weakly universal** (for \mathbf{P}) if the following hold:

- (W.1) For all varieties X and all families E over X having \mathbf{P} , there exists a morphism $f : X \rightarrow M$ such that $fx = \alpha[E_x]$ for every point x of X .
- (W.2) For every variety X and every natural transformation $\psi : t \rightarrow \underline{X}$, there exists a morphism $g : M \rightarrow X$ with $gm = \psi(\alpha^{-1}(m))$ for all points m of M .

We recall that, by Hilbert's Nullstellensatz, a morphism of varieties is uniquely determined by its values on (k -valued) points. Therefore, the two morphisms f and g , whose existence we postulated in (W.1) and (W.2), respectively, are unique. Given a weakly universal pair as above, we define a

natural transformation $\widehat{\alpha} : t \rightarrow \underline{M}$ by $\widehat{\alpha}[E] = f$ for every family E over X , where f is as in (W.1). This is well defined, as we have just explained. It is easy to see that $\widehat{\alpha}$ is a natural transformation.

Lemma 1.2.2. If M together with φ is a coarse moduli space for t , then the pair (M, φ_k) is weakly universal. On the other hand, let (M, α) be a weakly universal pair. Then $\widehat{\alpha}$ makes M into a coarse moduli space for t .

Finally, we state a criterion for a coarse moduli space of t to be a fine moduli space. In other words, given a weakly universal pair (M, φ) , we give a necessary and sufficient condition for a family U over M to be universal.

Proposition 1.2.3. Let M together with φ be a coarse moduli space for t . Then, M and φ form a fine moduli space if and only if there exists a family U over M such that $\varphi[U_m] = m$ for every point m of M .

Proof. We assume that (M, φ) is a fine moduli space. By Lemma 1.2.1, a representative U of $\varphi^{-1}(\text{id}_M)$ is a universal family. For every point m of M , we obtain

$$\varphi[U_m] = \underline{m}\varphi[U] = m.$$

This proves the “only if” part.

Conversely, suppose that there exists a family U such that $\varphi[U_m] = m$ for all points m . We show that the family U is universal. By Lemma 1.2.2, the bijection $\varphi_k : t(k) \rightarrow M(k)$ is weakly universal. Let E be a family over X . By axiom (W.1), there exists a (unique) morphism $f : X \rightarrow M$ with $fx = \varphi[E_x]$ for all points x of X . Thus

$$\varphi[E_x] = fx = \varphi[U_{fx}] = \varphi[(f^*U)_x].$$

As φ_k is injective, we obtain $E_x \cong (f^*U)_x$. This holds for every point x of X , whence f^*U is equivalent to E . Every morphism $f : X \rightarrow M$ satisfying $f^*U \sim E$ must obey $fx = \varphi[E_x]$ for all points x of X . As f is uniquely determined by its values on points, the uniqueness of f is proved. \square

The theory of moduli spaces allows us to give a more precise formulation of the Classification Problem from section 1. Fix a quiver Q and a dimension vector d for Q . The Classification Problem from section 1 amounts to finding a coarse (or maybe even a fine) moduli space for $\text{Rep}(Q, d)$. However, by interpreting this moduli problem geometrically (cf. Section 1.3), we will see that such a moduli space cannot exist in general. Therefore, we are forced to restrict ourselves to the following task:

Classification Problem (Second version). Find a “reasonable” property \mathbf{P} of representations of Q such that the moduli problem $\text{Rep}^{\mathbf{P}}(Q, d)$ possesses a coarse moduli space.

1.3 Quiver Moduli as GIT-Quotients

As outlined in the previous section, we wish to construct moduli spaces for moduli problems of representations of quivers. We will do this by interpreting these moduli problems in terms of actions

of algebraic groups on varieties. For us, an algebraic group is a group object in the category of reduced algebraic schemes over k (i.e. it is “almost” a variety, but not necessarily irreducible). By regarding a scheme as a representable contravariant functor from the category of schemes to the category of sets, we may interpret an algebraic group as a contravariant functor from schemes to the category of *groups* which is representable by a reduced algebraic scheme. We assume that the reader is familiar with the basic results on algebraic groups (see for example [Bor91], [Hum75], or [Spr98]; see also [DG70] and [Jan03] for an emphasis on the functorial point of view). An action of an algebraic group on a variety (or a scheme) is an action in the categorical sense.

Let Q be a quiver and let d be a dimension vector for Q . For every vertex i , fix vector spaces M_i of dimension d_i and define

$$R(Q, d) := \bigoplus_{\alpha: i \rightarrow j} \text{Hom}_k(M_i, M_j),$$

which we regard as an affine space (over k). Its points are obviously in bijection to representations of Q on the given vector spaces M_i . On $R(Q, d)$, we define an action of the linear (i.e. affine) algebraic group

$$G(Q, d) := \prod_{i \in Q_0} \text{Gl}(M_i)$$

by base change. This means that an element $g = (g_i \mid i)$ acts on a representation $(M_\alpha \mid \alpha)$ via $g \cdot M := (g_j M_\alpha g_i^{-1} \mid \alpha : i \rightarrow j)$. By construction, the set of $G(Q, d)$ -orbits corresponds bijectively to the set of isomorphism classes of representations of Q with dimension vector d .

The quiver Q and the dimension vector d being fixed, we abbreviate $R := R(Q, d)$ and $G := G(Q, d)$. There is a natural family E of representations of (Q, d) over R which arises as follows: For every vertex i , let $E_i := R \times M_i$ be the trivial vector bundle. Let $E_\alpha : E_i \rightarrow E_j$ be the bundle map which maps a point (M, v) of $R \times M_i$ to

$$E_\alpha(M, v) := (M, M_\alpha v),$$

where $\alpha : i \rightarrow j$ is an arrow. For a point M of R , the fiber E_M is precisely M regarded as a representation of Q .

We want to analyze which properties P for representations do have a chance to allow a coarse moduli space for $\text{Rep}^P(Q, d)$. We require P to be reasonable in the geometric sense, i.e. we assume the subset of those points of R which have the property P to be locally closed. Let R^P be the reduced (algebraic) scheme whose points have the property P . As P is stable under isomorphisms, R^P is a union of G -orbits. Suppose that there exists a coarse moduli space M^P for $\text{Rep}^P(Q, d)$ (which comes with a bijection identifying isomorphism classes of representations of (Q, d) having P and points of M^P). Restricting E to R^P defines a family satisfying P , so by the property (W.1) from Lemma 1.2.2, there exists a morphism $\pi : R^P \rightarrow M^P$ that maps $M \in R^P(k)$ to the point of $[M] \in M^P(k)$ which represents the isomorphism class of M . Hence, the fiber of the point $[M]$ is precisely the G -orbit of M in R^P . As fibers of k -valued points of a variety are closed, we see that, in order for a coarse moduli space to exist, the G -orbits of R^P must be closed in R^P .

Using this observation, it is easy to construct examples where a coarse moduli space for representations of Q of dimension vector d (without imposing an additional condition P) cannot exist.

Example 1.3.1. Let Q be the one-loop quiver and let $d > 0$ be an integer. Then, $R = R(Q, d)$ is the space of $(d \times d)$ -matrices and $G = G(Q, d) = \mathrm{Gl}_d$ acts on R by conjugation. It is fairly easy to see that for any $(d \times d)$ -matrix A , the diagonal matrix whose entries are the d eigenvalues of A is contained in the closure of the G -orbit of A .

Remark 1.3.2. A way of dealing with the lack of existence of a moduli space may be to relax the notion of a space. We can always construct the quotient *stack* $[R^{\mathrm{P}}/G]$, especially $[R/G]$ (cf. [LMB00]). This stack plays a role in Chapter 5 where we relate the cohomology of this stack to the cohomology of non-commutative Hilbert schemes (which are not just stacks but actual *varieties*). As the image Γ of the multiplicative group in G under the diagonal embedding acts trivially on R , we obtain an action of $PG := G/\Gamma$ on R . We can also form the moduli stack $[R^{\mathrm{P}}/PG]$ and especially $[R/PG]$. Being stacks, both $[R/G]$ and $[R/PG]$ are categories fibered in groupoids, but we may also interpret them as pseudo-functors, i.e. (contravariant) lax 2-functors from schemes to the 2-category of groupoids (cf. [Vis04]). When composing both of these pseudo-functors with the (2-)functor from groupoids to sets which assigns to a groupoid its set of isomorphism classes, we obtain two (contravariant) functors from schemes to sets. Via this process, $[R/G]$ induces the functor “families of representations of (Q, d) up to *isomorphism*” while $[R/PG]$ yields “families of representations of (Q, d) up to *equivalence*”.

As a next step, we use techniques from Mumford’s geometric invariant theory (GIT, for short) to construct moduli spaces. We review the basic theorems without giving the proofs. The classical reference is [MFK94], however we do not need the full generality of Mumford’s theory. For our purposes, [Muk03] will suffice. We will basically follow the presentation in [Rei08b].

A Reminder on Geometric Invariant Theory

Let us consider a linear algebraic group G which acts linearly on a finite-dimensional vector space V , this means it induces a morphism $G \rightarrow \mathrm{Gl}(V)$ of algebraic groups. Interpreting $f \in \mathbb{k}[V]$ as a regular function $f : V \rightarrow \mathbb{k} = \mathbb{A}^1(\mathbb{k})$, we say that f is **G -invariant** if $f(gv) = f(v)$ holds for all points g of G and all $v \in V$. The subset $\mathbb{k}[V]^G$ is, indeed, a subalgebra of $\mathbb{k}[V]$. A theorem of Hilbert tells us that the ring $\mathbb{k}[V]^G$ is finitely generated if the group G is **linearly reductive** (which means that every linear representation of G decomposes into a direct sum of representations which do not possess a proper subrepresentation). Therefore, the affine scheme

$$V//G := \mathrm{Spec} \mathbb{k}[V]^G$$

is a variety. The embedding $\mathbb{k}[V]^G \rightarrow \mathbb{k}[V]$ gives rise to a morphism $V \rightarrow V//G$. This morphism is a **categorical G -quotient** of V which means that it is universal for being a G -invariant morphism from V . One can show that every fiber contains precisely one closed G -orbit. The above holds also true when V is replaced by any affine variety.

We introduce a version of Mumford’s notion of (semi-)stability. Choose a character χ of G , i.e. a morphism $\chi : G \rightarrow \mathbb{G}_m$ of algebraic groups. A function $f \in \mathbb{k}[V]$ is called **χ -semi-invariant** if $f(gv) = \chi(g)f(v)$ for every point g of G and every $v \in V$. The subspace of $\mathbb{k}[V]$ of all χ -semi-invariant functions on V is denoted by $\mathbb{k}[V]^{G,\chi}$. We define a graded ring

$$\mathbb{k}[V]_{\chi}^G := \bigoplus_{n \geq 0} \mathbb{k}[V]^{G,n\chi}.$$

In the above definition, $n\chi$ denotes the character which maps g to $\chi(g)^n$. The additive notation is common.

Definition. An element $v \in V$ is called χ -**semi-stable** if there exists a positive integer n and an $n\chi$ -semi-invariant function f such that $f(v) \neq 0$.

The subset $V^{\chi\text{-sst}}$ of all χ -semi-stable points is an open subset and a union of G -orbits of V . For an $n\chi$ -semi-invariant f , the ring of G -invariants of the affine open subset $D_V(f)$ of all points not vanishing at f equals $(k[V]_f)^G = (k[V]_\chi^G)_{(f)}$, the degree 0 part of the localization at f of the graded ring $k[V]_\chi^G$. The categorical quotients hence glue together to give a categorical quotient

$$\pi : V^{\chi\text{-sst}} \rightarrow V^{\chi\text{-sst}} // G := \text{Proj}(k[V]_\chi^G).$$

Again, every fiber of π contains exactly one orbit which is closed in $V^{\chi\text{-sst}}$. As the degree 0 component of $k[V]_\chi^G$ is just $k[V]^G$, we obtain a natural map $V^{\chi\text{-sst}} // G \rightarrow V // G$ which is projective.

Definition. A point $v \in V$ is called χ -**stable** if v is χ -semi-stable, the G -orbit of v is closed in $V^{\chi\text{-sst}}$ and the stabilizer of v in G is zero-dimensional (i.e. finite).

The set $V^{\chi\text{-st}}$ of stable points of V is an open subset of $V^{\chi\text{-sst}}$ and its image under π is open in $V^{\chi\text{-sst}} // G$. The categorical quotient $V^{\chi\text{-sst}} \rightarrow V^{\chi\text{-sst}} // G$ restricts to a **geometric quotient** $\pi : V^{\chi\text{-st}} \rightarrow V^{\chi\text{-st}} / G := \pi(V^{\chi\text{-st}})$. Among other properties, this means that its fibers are precisely the G -orbits of $V^{\chi\text{-st}}$ (see [MFK94, Def. 0.6] for a precise definition). The dimension of $V^{\chi\text{-st}} / G$ is $\dim V - \dim G$.

If the action of G on $V^{\chi\text{-st}}$ is free, then the geometric quotient $\pi : V^{\chi\text{-st}} \rightarrow V^{\chi\text{-st}} / G$ is even a **G -principal bundle**, which means that π is flat, surjective, and locally trivial in the fpqc-topology (or, equivalently, in the étale topology) with fiber G . In particular, given a free action, $V^{\chi\text{-st}} / G$ is non-singular, as regularity is a property that descends along faithfully flat morphisms.

In Mumford's theory (cf. [MFK94]), (semi-)stability is defined with respect to a G -linearized ample line bundle. It is fairly easy to see that, if we define a G -linearization of the trivial line bundle on V by having G act on it via the character χ , our notion of (semi-)stability coincides with Mumford's.

The condition of (semi-)stability can be described numerically with the help of one-parameter subgroups. Moreover, the GIT-equivalence relation (which we will introduce below) can be interpreted in numerical terms. This will be essential to identify (semi-)stable points of $R(Q, d)$ with (semi-)stable representations.

A **one-parameter subgroup** of G is a morphism $\lambda : \mathbb{G}_m \rightarrow G$ of algebraic groups. Here and in the following, \mathbb{G}_m denotes the multiplicative group; its points are $\mathbb{G}_m(\mathbf{k}) = \mathbf{k}^\times$. There is an integral pairing between characters χ and one-parameter subgroups λ of G which is defined as the unique integer $n = \langle \chi, \lambda \rangle$ such that $\chi\lambda(t) = t^n$ for every $t \in \mathbb{G}_m(\mathbf{k}) = \mathbf{k}^\times$. When choosing a one-parameter subgroup λ and a vector $v \in V$, we obtain a morphism $\mathbb{G}_m \rightarrow V$ mapping $t \in \mathbf{k}^\times$ to $\lambda(t)v$. By the valuation criterion for separability, there exists at most one extension of this morphism to a morphism $\mathbb{A}^1 \rightarrow V$. If this extension exists, we denote its value at 0 by $\lim_{t \rightarrow 0} \lambda(t)v$.

Theorem 1.3.3 (Hilbert-Mumford criterion for (semi-)stability). A point $v \in V$ is χ -semi-stable (χ -stable) if and only if $\langle \chi, \lambda \rangle \geq 0$ (or $\langle \chi, \lambda \rangle > 0$, respectively) for all non-trivial one-parameter subgroups λ of G for which $\lim_{t \rightarrow 0} \lambda(t)v$ exists.

Moreover, we can tell with the help of one-parameter subgroups which orbits are closed and also whether or not two semi-stable points are mapped to the same point in the quotient $V^{\chi\text{-sst}}//G$. Two such χ -semi-stable points $v_1, v_2 \in V$ have the same image in $V^{\chi\text{-sst}}//G$ if and only if their orbit closures in $V^{\chi\text{-sst}}$ intersect. We call v_1 and v_2 **GIT-equivalent** in this case. Using a theorem of Kempf (cf. [Kem78]), King shows in [Kin94]:

Proposition 1.3.4. A χ -semi-stable point v has a closed orbit in $V^{\chi\text{-sst}}$ if and only if the limit $\lim_{t \rightarrow 0} \lambda(t)v$ does not exist or lies in Gv for every one-parameter subgroup λ of G with $\langle \chi, \lambda \rangle = 0$. Two χ -semi-stable points $v_1, v_2 \in V$ are GIT-equivalent if and only if there exist one-parameter subgroups λ_1, λ_2 of G with $\langle \chi, \lambda_1 \rangle = \langle \chi, \lambda_2 \rangle = 0$ such that $\lim_{t \rightarrow 0} \lambda_1(t)v_1$ and $\lim_{t \rightarrow 0} \lambda_2(t)v_2$ (exist and) lie in the same closed G -orbit.

In order to construct fine moduli spaces, it is crucial to know how to obtain vector bundles on a quotient. Considering the quotient map $\pi : V^{\chi\text{-sst}} \rightarrow V^{\chi\text{-sst}}//G$ and a vector bundle U on $V^{\chi\text{-sst}}//G$, the pull back π^*U possesses a natural action of G making it a G -vector bundle. We say that a G -vector bundle E on $V^{\chi\text{-sst}}$ **descends** to $V^{\chi\text{-sst}}//G$ if there exists a vector bundle U on $V^{\chi\text{-sst}}//G$ such that E is isomorphic to π^*U . In [DN89], Drezet and Narasimhan give a necessary and sufficient condition for a G -vector bundle to descend to $V^{\chi\text{-sst}}//G$. The authors attribute this result to Kempf. For this theorem, $\text{char } k = 0$ is necessary.

Theorem 1.3.5. A G -vector bundle E on $V^{\chi\text{-sst}}$ descends to $V^{\chi\text{-sst}}//G$ if and only if for every point v of $V^{\chi\text{-sst}}$, the stabilizer G_v of v acts trivially on the fiber E_v .

Moreover, if $f : E \rightarrow F$ is a G -equivariant bundle map between bundles that descend to the quotient, then the map also descends.

Application to Representations of Quivers

We wish to apply the above results to the affine space $R(Q, d)$ of representations of Q on some fixed vector spaces M_i of dimension d_i . This was introduced by King in [Kin94] for the first time. Yet, we will follow Reineke's presentation in [Rei08b], which is slightly different. Denote again $R := R(Q, d)$. The action of $G = G(Q, d)$ which we introduced previously is not useful to allow stability: The diagonally embedded rank 1 torus Γ acts trivially on R . Therefore, we pass to the linearly reductive group $PG := PG(Q, d) := G/\Gamma$. A character of PG is a character of G factoring through Γ , whence it has the form

$$\chi(g) = \prod_i \det(g_i)^{\chi_i}$$

for some integers χ_i satisfying $\sum_i d_i \chi_i = 0$. We want to develop a homological interpretation of (semi-)stability with respect to such a character. Let $\theta : \mathbb{Z}^{Q_0} \rightarrow \mathbb{Z}$ be a linear form on the Grothendieck group of the category of representations of Q . Such a linear form is called a **stability condition** for Q . Define the associated **slope function** $\mu = \mu_\theta : \mathbb{Z}_{\geq 0}^{Q_0} - \{0\} \rightarrow \mathbb{Q}$ by

$$\mu(d) := \frac{\theta(d)}{\dim d},$$

where $\dim d := \sum_i d_i$. If M is a representation of Q , we abbreviate $\theta(\underline{\dim} M) =: \theta(M)$, and so forth. We define a notion of stability which strongly resembles the Hilbert-Mumford criterion.

Definition. A representation M of Q is called θ -semi-stable (θ -stable) if $\mu(M') \leq \mu(M)$ (or $\mu(M') < \mu(M)$, respectively) for all non-zero proper subrepresentations M' of M .

A stability condition θ of Q is defined by mapping each unit vector $e_i \in \mathbb{Z}^{Q_0}$ to an integer θ_i . We define a character χ_θ of G by

$$\chi_\theta(g) = \prod_{i \in Q_0} \det(g_i)^{\theta(d) - \theta_i \dim d}$$

for every point g of G . This induces a character of PG , *without requiring* $\theta(d) = 0$. We thus gain the freedom of choosing θ independently of d .

King's key observation is that a one-parameter subgroup λ of PG (induced by one of G) such that $\lim_{t \rightarrow 0} \lambda(t)M$ exists induces a filtration of M by subrepresentations, and vice versa. We thus obtain:

Theorem 1.3.6. A representation of Q of dimension vector d is θ -(semi-)stable if and only if it is a χ_θ -(semi-)stable point of $R(Q, d)$ with respect to the $PG(Q, d)$ -action.

We denote by $R^{\theta\text{-sst}}(Q, d)$ and $R^{\theta\text{-st}}(Q, d)$ the open subsets of θ -(semi-)stable representations of Q of dimension vector d . They hence coincide with $R^{\chi_\theta\text{-sst}}$ and $R^{\chi_\theta\text{-st}}$, respectively.

The property of having a closed orbit and the notion of GIT-equivalence do also readily translate to homological properties. Let us consider the full subcategory $\mathbf{rep}^{\mu_0}(Q)$ of $\mathbf{rep}(Q)$ whose objects are the θ -semi-stable representations M of Q having slope $\mu(M) = \mu(d) =: \mu_0$. This is in fact an abelian category (cf. [Rei08b]). The simple objects of this category are precisely the θ -stable representations of slope μ_0 . A semi-simple object in $\mathbf{rep}^{\mu_0}(Q)$ is a θ -semi-stable representation which decomposes into a direct sum of θ -stable representations of slope μ_0 . Such a representation is called μ_0 -**polystable**. By the Jordan-Hölder theorem, every θ -semi-stable representation of slope μ_0 possesses a filtration with θ -stable subquotients (of slope μ_0). We call the associated graded to this filtration the **associated μ_0 -polystable**.

Proposition 1.3.7. A θ -semi-stable representation of Q of dimension vector d has a closed $PG(Q, d)$ -orbit in $R^{\theta\text{-sst}}(Q, d)$ if and only if it is μ_0 -polystable. Moreover, two θ -semi-stable representations of Q of dimension vector d are GIT-equivalent if and only if their associated μ_0 -polystables coincide.

Denote by $M^{\text{ssimp}}(Q, d) := R // PG$ and $M^{\theta\text{-sst}}(Q, d) := R^{\theta\text{-sst}} // PG$ as well as $M^{\theta\text{-st}}(Q, d) := R^{\theta\text{-st}} // PG$. Summarizing the above, we obtain a diagram

$$\begin{array}{ccccc} R^{\theta\text{-st}}(Q, d) & \subseteq & R^{\theta\text{-sst}}(Q, d) & \subseteq & R(Q, d) \\ \downarrow & & \downarrow & & \downarrow \\ M^{\theta\text{-st}}(Q, d) & \subseteq & M^{\theta\text{-sst}}(Q, d) & \longrightarrow & M^{\text{ssimp}} \end{array}$$

such that the following holds:

Theorem 1.3.8. In the above diagram,

- (i) $R^{\theta\text{-st}}(Q, d) \subseteq R^{\theta\text{-sst}}(Q, d) \subseteq R(Q, d)$ are open inclusions,
- (ii) the maps $R(Q, d) \rightarrow M^{\text{ssimp}}(Q, d)$ and $R^{\theta\text{-sst}}(Q, d) \rightarrow M^{\theta\text{-sst}}(Q, d)$ are categorical quotients and $R^{\theta\text{-st}}(Q, d) \rightarrow M^{\theta\text{-st}}(Q, d)$ is a $PG(Q, d)$ -principal fiber bundle,
- (iii) $M^{\text{ssimp}}(Q, d)$ is an affine variety which is a coarse moduli space for semi-simple representations of Q of dimension vector d ,
- (iv) $M^{\theta\text{-sst}}(Q, d)$ is projective over $M^{\text{ssimp}}(Q, d)$ and is a coarse moduli space parametrizing μ_0 -polystable representations of Q of dimension vector d , and
- (v) $M^{\theta\text{-st}}(Q, d)$ is open in $M^{\theta\text{-sst}}(Q, d)$, it is a non-singular variety of dimension $1 - \langle d, d \rangle$ (if not empty) and it is a coarse moduli space for θ -stable representations of Q of dimension vector d .

If the dimension vector d is **coprime**, which means the greatest common divisor of all d_i is one, then $M^{\theta\text{-st}}(Q, d)$ is even a fine moduli space for θ -stable representations of (Q, d) . The existence of a universal family is also shown in [Kin94]. As we will explicitly make use of it, let us repeat the construction. By Theorem 1.3.5, it suffices to define an action of PG on the vector bundles $E_i | R^{\theta\text{-st}}$ (by abuse of notation, we denote the restriction of E_i to $R^{\theta\text{-st}}$ also by E_i) such that the maps E_α are PG -equivariant. As d is coprime, there exist integers ψ_i such that $\sum_i \psi_i d_i = 1$. The choice of such numbers amounts to choosing a character ψ of G of weight 1 by defining $\psi(g) = \prod_i \det(g_i)^{\psi_i}$. Define a G -action on $E_i = R^{\theta\text{-st}} \times M_i$ by

$$g \cdot (M, v) := (g \cdot M, \psi(g)^{-1} gv).$$

Twisting with ψ assures that the diagonally embedded rank 1 torus Γ acts trivially. This action therefore induces a PG -action on E_i . As the stabilizer of a θ -stable representation in PG is trivial, Theorem 1.3.5 assures that E_i , equipped with this PG -action, descends to a vector bundle U_i on $M^{\theta\text{-st}}$. The bundle maps E_α are PG -equivariant, hence induce bundle maps $U_\alpha : U_i \rightarrow U_j$ for every arrow $\alpha : i \rightarrow j$. We obtain a family U of θ -stable representations of Q of dimension vector d which can easily be seen to be universal. Note that, in general, different choices of the character ψ give rise to *non-isomorphic* universal families. A universal family is uniquely determined only up to *equivalence* of families.

Chapter 2

Intersection Theory and Equivariant Intersection Theory

In this chapter, we first give a brief overview of some results of intersection theory. As a reference, we recommend Fulton's book [Ful98]. Afterwards, we give an introduction to equivariant intersection theory which was invented by Edidin and Graham (cf. [EG98a]). Equivariant intersection theory relates to non-equivariant intersection theory in the same way that equivariant cohomology relates to ordinary cohomology.

Remember our conventions about schemes and varieties: Schemes are algebraic over a fixed algebraically closed field k and varieties are integral (i.e. reduced and irreducible) schemes.

2.1 The Chow Group

Let X be a scheme.

Definition. Let n be a non-negative integer. An n -**cycle** of X is a finite \mathbb{Z} -linear combination $\sum_{\nu} m_{\nu}[V_{\nu}]$ of n -dimensional closed subvarieties V_{ν} of X . There is an obvious notion of addition of n -cycles making the set $Z_n(X)$ of all n -cycles into an abelian group. We denote $Z_*(X) := \bigoplus_{n \geq 0} Z_n(X)$ the **cycle group** of X .

We see at once that the cycle group does not depend on the scheme structure of X . If X is purely n -dimensional, an $(n - 1)$ -cycle is a Weil divisor of X .

Let W be an $(n + 1)$ -dimensional closed subvariety of X . For an n -dimensional subvariety V of W , we consider the local ring $\mathcal{O}_{W,V}$ of W at the generic point of V . Its quotient field coincides with the function field $\kappa(W)$ of W . Therefore, we obtain a homomorphism $\text{ord}_V : \kappa(W)^{\times} \rightarrow \mathbb{Z}$ of groups by assigning to a non-zero element r of $\mathcal{O}_{W,V}$ the integer

$$\text{ord}_V(r) := l_{\mathcal{O}_{W,V}}(\mathcal{O}_{W,V}/r\mathcal{O}_{W,V}).$$

Note that $\text{ord}_V(r)$ is also the length of the ring $\mathcal{O}_{W,V}/r\mathcal{O}_{W,V}$ over itself which is finite by Krull's principal ideal theorem. Let $r \in \kappa(W)^{\times}$. We define

$$[\text{div}(r)] := \sum_V \text{ord}_V(r)[V],$$

regarded as an element of $Z_n(X)$, where the sum ranges over all n -dimensional subvarieties V of W . Note that this sum is indeed a finite linear combination as an artinian ring has only finitely many prime ideals. We denote by $\text{Rat}_n(X)$ the subgroup of $Z_n(X)$ which is generated by cycles $[\text{div}(r)]$, where r is an element of $\kappa(W)^\times$ and W runs through all $(n+1)$ -dimensional subvarieties of X . Two n -cycles of X are called **rationally equivalent** if their difference lies in $\text{Rat}_n(X)$. We define $A_n(X)$ to be the quotient group $Z_n(X)/\text{Rat}_n(X)$. Furthermore, let $A_*(X)$ be defined as $\bigoplus_{n \geq 0} A_n(X)$, the group of cycles up to rational equivalence.

Definition. We call $A_n(X)$ the n -th **Chow group** of X and $A_*(X)$ the **total Chow group** of X .

Just like the group of cycles, the Chow group does not depend on the scheme structure of X either. If X is a purely n -dimensional scheme, we obtain that $Z_n(X) = A_n(X)$ is the free group generated by the cycles belonging to the irreducible components of X .

We define the cycle associated to a scheme X . Let X be a scheme and let X_1, \dots, X_k be its irreducible components. Let $[X]$ be defined as

$$[X] := \sum_{i=1}^k l(\mathcal{O}_{X, X_i})[X_i].$$

If X is purely n -dimensional, $[X]$ is an n -cycle which we call the **cycle associated to X** . This cycle *does* depend on the scheme structure of X . Oftentimes, we regard $[X]$ as an element of the Chow group of X , using the same symbol.

Similar to the above construction, we may associate a cycle to a Cartier divisor of a purely n -dimensional scheme. Defining \mathcal{K}_X to be the sheaf of rings which arises as the sheafification of the presheaf on X whose ring of sections over U is the localization $\text{Frac} \Gamma(U, \mathcal{O}_X)$ of the ring $\Gamma(U, \mathcal{O}_X)$ making all regular elements invertible. Then, a Cartier divisor of X is nothing but a global section of the sheaf $\mathcal{K}_X^\times / \mathcal{O}_X^\times$. Let D be a Cartier divisor of X . For a closed subvariety V of codimension 1, we write the image of D under the map

$$\Gamma(X, \mathcal{K}_X^\times / \mathcal{O}_X^\times) \rightarrow \text{Frac}(\mathcal{O}_{X, V})^\times / \mathcal{O}_{X, V}^\times$$

as a fraction a/b of two regular elements of $\mathcal{O}_{X, V}$. The value $l(\mathcal{O}_{X, V}/a\mathcal{O}_{X, V}) - l(\mathcal{O}_{X, V}/b\mathcal{O}_{X, V})$, which we denote $\text{ord}_V(D)$, is independent of the choice of representatives. The $(n-1)$ -cycle

$$[D] := \sum_V \text{ord}_V(D)[V]$$

is called the **associated Weil divisor** of D . It is supported in $|D|$. If D is effective, then the associated Weil divisor coincides with the associated cycle of D regarded as a closed subscheme.

Let $f: X \rightarrow Y$ be a *proper* morphism of schemes. Let V be a closed subvariety of X of dimension n . Then $f(V) =: W$ is a closed, irreducible subset of Y which we regard as a subvariety by equipping it with the reduced subscheme structure. The surjection $f: V \rightarrow W$ induces a field extension $\kappa(W) \hookrightarrow \kappa(V)$ of transcendence degree $n - \dim W$. If $\dim W = n$, it is a finite extension. We define

$$\deg(V | W) := \begin{cases} [\kappa(V) : \kappa(W)] & \text{if } \dim W = n \text{ and} \\ 0 & \text{if } \dim W < n. \end{cases}$$

Thus, $f_*[V] := \deg(V | W)[W]$ gives an n -cycle of Y . Extending this definition linearly, we obtain a homomorphism $f_* : Z_n(X) \rightarrow Z_n(Y)$ for every n . This construction is evidently functorial, i.e. $(gf)_* = g_*f_*$ for morphisms of schemes which are composable in this way. Using a reasonable amount of algebraic geometry, one can show that if $\alpha \in \text{Rat}_n(X)$, then $f_*\alpha \in \text{Rat}_n(Y)$. Hence, the homomorphism $f_* : Z_n(X) \rightarrow Z_n(Y)$ descends to a homomorphism $f_* : A_n(X) \rightarrow A_n(Y)$. We call it the **proper push-forward** associated to f .

Let $f : X \rightarrow Y$ be a *flat* morphism of schemes. We say that f has **relative dimension** r if for every closed subvariety W of Y , every irreducible component of the inverse image scheme $f^{-1}(W)$ has dimension $\dim W + r$. Throughout this text, we make the convention that a flat morphism should have a relative dimension. Let $f : X \rightarrow Y$ be flat of relative dimension r . For an n -dimensional closed subvariety W of Y , we define $f^*[W]$ to be the $(n+r)$ -cycle $[f^{-1}(W)]$ associated to the inverse image scheme $f^{-1}(W)$. Extending linearly, we obtain a homomorphism $f^* : Z_n(Y) \rightarrow Z_{n+r}(X)$ of abelian groups. If Y' is a closed subscheme of Y , one can show that $f^*[Y'] = [f^{-1}(Y')]$. This is not as evident as it may seem at first sight. However, using this fact, we can easily deduce that $(gf)^* = f^*g^*$ for flat morphisms f and g which can be composed in this way. Just like for the proper push-forward, we would like the homomorphism f^* to descend to the level of Chow groups. Indeed, for a flat morphism $f : X \rightarrow Y$ and $\beta \in \text{Rat}_n(Y)$, we have $f^*\beta \in \text{Rat}_{n+r}(X)$. The induced morphism $f^* : A_n(Y) \rightarrow A_{n+r}(X)$ is called the **flat pull-back** associated to f .

These two constructions interchange in the following situation: Let

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ \downarrow f' & & \downarrow f \\ Y' & \xrightarrow{g} & Y \end{array}$$

be a cartesian diagram of schemes. If g is flat of relative dimension r , then g' is also flat of the same relative dimension and if f is proper then f' is proper, too. In this case

$$f'_*g'^*\alpha = g^*f_*\alpha$$

for every $\alpha \in Z_n(X)$.

Finally, we introduce an effective method for the calculation of Chow groups. Thinking about cohomology theories, we have a very efficient tool at hand that converts excisions into long exact sequences. Unfortunately, we do not get a long exact sequence but just a slightly weaker result.

Proposition 2.1.1 (Localization exact sequence). Let X be a scheme, let Y be a closed subscheme of X with closed embedding $i : Y \rightarrow X$ and denote $U := X - Y$ with open embedding $j : U \rightarrow X$. Then, the sequence

$$A_n(Y) \xrightarrow{i_*} A_n(X) \xrightarrow{j^*} A_n(U) \rightarrow 0$$

is exact.

An immediate consequence of the above exact sequence is the following: Let X be a scheme and let $X = X_0 \supseteq X_1 \supseteq \dots \supseteq X_d$ be closed subsets, such that for every i , the open subset $Z_i := X_{i-1} - X_i$

is isomorphic to an affine space, say of dimension n_i . If this is fulfilled, we say that X possesses a **cell decomposition** and the locally closed subsets Z_i are called the **cells**. In this case, $A_*(X)$ is the free group with basis $[\overline{Z_1}], \dots, [\overline{Z_d}]$, where $\overline{Z_i}$ denotes the closure of Z_i in X .

Let E be a vector bundle of rank r on a scheme X with projection $\pi : E \rightarrow X$. Using the localization exact sequence, one can show that the flat pull-back $\pi^* : A_{i-r}(X) \rightarrow A_i(E)$ is surjective for all i . Applying more sophisticated arguments, we see that π^* is in fact an isomorphism.

2.2 Gysin Maps and the Chow Ring

Let $i : Y \rightarrow X$ be a closed embedding and let \mathcal{I} be the corresponding sheaf of ideals of \mathcal{O}_X . The scheme $C_Y X := \text{Spec}(\bigoplus_{n \geq 0} \mathcal{I}^n / \mathcal{I}^{n+1})$ is a linear scheme over Y which we call the normal cone of Y in X . If $i : Y \rightarrow X$ is a regular embedding of codimension d , i.e. i is a closed embedding whose sheaf of ideals \mathcal{I} is locally given by a regular sequence of length d , then $\mathcal{I} / \mathcal{I}^2$ is a locally free \mathcal{O}_Y -module of rank d . Hence, the normal cone $C_Y X$ is a vector bundle on Y . For a k -dimensional subvariety V of X , the normal cone $C_{V \cap Y} V$ embeds naturally as a purely k -dimensional closed subscheme of $C_Y X$. With a substantial amount of work, we can show that the map $Z_k(X) \rightarrow Z_k(C_Y X)$, sending $[V]$ to $[C_{V \cap Y} V]$, descends to a map $\sigma : A_k(X) \rightarrow A_k(C_Y X)$, called the **specialization to the normal cone**.

Definition. The composition $i^* : A_k(X) \xrightarrow{\sigma} A_k(C_Y X) \xrightarrow{(\pi^*)^{-1}} A_{k-d}(Y)$ is called the **Gysin homomorphism** of the regular embedding i of codimension d .

In the above definition, π is the projection of the normal cone $C_Y X$. For a pure-dimensional closed subscheme V of X , we call $Y \cdot V := Y \cdot_X V := i^*[V]$ the **intersection product** of Y and V in X . If X itself is pure-dimensional then Y is, too, and we obtain $Y \cdot X = i^*[X] = [Y]$, the cycle associated to Y .

There is also a relative version of the Gysin map (constructed in a very similar way, see [Ful98, 6.2]): Consider a cartesian diagram of schemes

$$\begin{array}{ccc} Y' & \xrightarrow{i'} & X' \\ \downarrow & & \downarrow f \\ Y & \xrightarrow{i} & X \end{array}$$

in which i is a regular embedding of codimension d , then there exists a relative (or refined) Gysin homomorphism $i^! := i_f^! : A_k(X') \rightarrow A_{k-d}(Y')$ which provides a generalization of the above Gysin map: If the closed embedding $i' : Y' \rightarrow X'$ is also a regular embedding of the same codimension as i , then $i^!$ coincides with i'^* . The relative Gysin map (just like the absolute) commutes with proper push-forward and flat pull-back under base extension.

The formation of the Gysin homomorphism is also contravariantly functorial, i.e. $(ji)^! = i^!j^!$ for regular embeddings i and j . Using the refined versions of the Gysin homomorphisms, we can formulate commutativity: If

$$\begin{array}{ccccc}
 Z & \longrightarrow & Y'_2 & \longrightarrow & Y_2 \\
 \downarrow & & \downarrow & & \downarrow i_2 \\
 Y'_1 & \longrightarrow & X' & \xrightarrow{f_2} & X_2 \\
 \downarrow & & \downarrow f_1 & & \\
 Y_1 & \xrightarrow{i_1} & X_1 & &
 \end{array}$$

is a diagram in which every square is cartesian and such that i_1 and i_2 are regular embeddings of codimension d_1 and d_2 , respectively, then $i_2^! i_1^! \alpha = i_1^! i_2^! \alpha$ in $A_{k-d_1-d_2}(Z)$ for every $\alpha \in A_k(X')$.

We apply this to the following situation: Let X be purely n -dimensional and let Y_1 and Y_2 be pure-dimensional closed subschemes of X . Then, the above properties yield

$$Y_1 \cdot Y_2 = Y_2 \cdot Y_1 = (Y_1 \times Y_2) \cdot \Delta_X$$

in $A_{n-d_1-d_2}(Y_1 \cap Y_2)$, with Δ_X the diagonal in $X \times X$.

Now, assume that we are given a *non-singular* variety of dimension n . This implies that the diagonal morphism $\delta : X \rightarrow X \times X$ is a regular embedding of codimension n . We define a multiplication on the graded group $A^*(X) := \bigoplus_i A^i(X)$, where $A^i(X) := A_{n-i}(X)$, by letting

$$x \cdot y := \delta^*(x \times y),$$

an element of $A^{i+j}(X)$, for all $x \in A^i(X)$ and $y \in A^j(X)$. The cycle $x \times y \in A^{i+j}(X \times X)$ is defined by extending $[V] \times [W] := [V \times W]$ linearly. Therefore,

$$[V] \cdot [W] = V \cdot W = W \cdot V = [W] \cdot [V],$$

interpreted as elements of $A^*(X)$, for pure-dimensional closed subschemes V and W of X . This multiplication makes $A^*(X)$ a commutative graded ring with unit $[X]$.

More generally, let Y be a scheme and $f : Y \rightarrow X$ be a morphism. Then, the graph $\gamma_f : Y \rightarrow Y \times X$ is a regular embedding. For $x \in A^i(X)$ and $y \in A_j(Y)$, define

$$f^* x \cap y := \gamma_f^*(y \times x)$$

which is an element of $A_{j-i}(Y)$. In this vein, $A_*(Y)$ becomes an $A^*(X)$ -module. If Y is also non-singular, we write $f^* x := f^* x \cap [Y]$ and obtain an additive map $f^* : A^*(X) \rightarrow A^*(Y)$. It is actually a homomorphism of rings which preserves degrees. If $f : Y \rightarrow X$ is a *proper* morphism of non-singular varieties we obtain the **projection formula**, which reads

$$f_*(f^* x \cdot y) = x \cdot f_* y.$$

Let V and W be closed subschemes of pure dimension k and l , respectively, of a non-singular variety X of dimension n . Every component of the (scheme-theoretic) intersection $V \cap W$ has dimension at least $k + l - n$. An irreducible component of $V \cap W$ which has exactly this dimension is called a *proper component*. The intersection product $V \cdot W$ can be displayed as a unique linear combination

$$V \cdot W = \sum_Z i(Z, V \cdot W; X)[Z],$$

the sum ranging over all proper components of $V \cap W$. The integer $i(Z, V \cdot W; X)$ is called the **intersection multiplicity** of Z . Note that it equals $i(Z, (V \times W) \cdot \Delta_X; X \times X)$. One can show:

Proposition 2.2.1. For every proper component Z of $V \cap W$, we have

$$1 \leq i(Z, V \cdot W; X) \leq l(\mathcal{O}_{V \cap W, Z})$$

and if the local ring of $V \times W$ at Z is Cohen-Macaulay, then the right-hand estimate is an equality.

2.3 Chern Classes and Localized Chern Classes

We will now define Chern classes of vector bundles on schemes. We start by defining first Chern classes of line bundles. We will follow Fulton's exposition (cf. [Ful98, Chapter 3]). Let L be a line bundle on a scheme X . Denote $s : X \rightarrow L$ its zero section. For $\alpha \in A_m(X)$, define

$$c_1(L) \cap \alpha := s^* s_* \alpha$$

which induces an operator $c_1(L) \cap _- : A_m(X) \rightarrow A_{m-1}(X)$. We call it the **first Chern class** of L . By properties of the Gysin homomorphism, the first Chern class is compatible with flat pull-backs and is commutative, meaning $c_1(L) \cap (c_1(L') \cap \alpha) = c_1(L') \cap (c_1(L) \cap \alpha)$. Moreover, it fulfills the projection formula which asserts that $f_*(c_1(f^*L) \cap \beta) = c_1(L) \cap f_*\beta$ for a proper morphism $f : Y \rightarrow X$. Finally, if $L = \mathcal{O}(D)$ is the line bundle of a Cartier divisor D on X , then $c_1(\mathcal{O}(D)) \cap [X] = [D]$.

From this, we construct Chern classes of arbitrary vector bundles on schemes. Let E be a vector bundle of rank r on X . The i -th **Segre class** of E is the operator $s_i(E) \cap _- : A_m(X) \rightarrow A_{m-i}(X)$ defined by

$$s_i(E) \cap \alpha := p_* (c_1(\mathcal{O}(1))^{r-1+i} \cap p^* \alpha).$$

In the above context, $p : \mathbb{P}E \rightarrow X$ is the projection from the projectivization of the bundle E and $\mathcal{O}(1)$ is the twisting bundle on $\mathbb{P}E$. We can easily see that $s_0(E) = 1$ and $s_i(E) = 0$ for $i < 0$. Let $s(E) := \sum_{i \geq 0} s_i(E)$ be the **total Segre class**.

Definition. The **total Chern class** $c(E)$ is defined as $s(E)^{-1}$. Writing $c(E) = \sum_{i \geq 0} c_i(E)$, we call $c_i(E)$ the i -th **Chern class** of E .

These Chern classes satisfy certain nice properties. Clearly, $c_0(E) = 1$. More important features are the following:

- **Vanishing:** For $i > \text{rk } E$, we have $c_i(E) = 0$.
- **Functoriality:** If $f : Y \rightarrow X$ is *flat* (of some relative dimension) or a *regular embedding*, then $f^*(c_i(E) \cap \alpha) = c_i(f^*E) \cap f^*\alpha$.
- **Commutativity:** We have $c(E) \cdot c(E') = c(E') \cdot c(E)$.
- **Whitney sum formula:** For a short exact sequence $0 \rightarrow E' \rightarrow E \rightarrow E'' \rightarrow 0$ of vector bundles on X , we obtain $c(E) = c(E') \cdot c(E'')$.
- **Projection formula:** If $f : Y \rightarrow X$ is *proper*, then $f_*(c_i(f^*E) \cap \beta) = c_i(E) \cap f_*\beta$.

- **Normalization:** For a Cartier divisor D on X , we get $c_1(\mathcal{O}(D)) \cap [X] = [D]$.

The normalization property has been mentioned before and functoriality, commutativity and the projection formula are consequences of the corresponding properties of the Segre classes. A key ingredient to proving the vanishing property and the Whitney sum formula is the **splitting principle**. It states that to every vector bundle E on X , there exists a flat morphism $f : F \rightarrow X$ such that the bundle f^*E possesses a complete flag of subbundles and such that $f^* : A_*(X) \rightarrow A_*(F)$ is injective. In order to prove any relation between Chern classes, it hence suffices to show that the relation holds true if the bundle has a complete filtration and that the relation is preserved by flat pull-backs. Denoting by L_i the successive subquotients of the filtration and $\xi_i := c_1(L_i)$ their first Chern classes, both vanishing and Whitney sum formula are consequences of the identity

$$c(f^*E) = \prod_{i=1}^r (1 + \xi_i) \tag{*}$$

of operators on $A_*(F)$. We call ξ_1, \dots, ξ_r the **Chern roots** of E . To prove (*), it is crucial to observe that $\prod_{i=1}^r \xi_i = 0$ if E has a nowhere vanishing section.

Another application of the splitting principle is the following: When putting $\zeta := c_1(\mathcal{O}(1))$, the first Chern class of the twisting bundle on $\mathbb{P}E$, we have the relation

$$\zeta^r + c_1(f^*E)\zeta^{r-1} + \dots + c_{r-1}(f^*E)\zeta + c_r(f^*E) = 0 \tag{**}$$

of operators on $A_*(\mathbb{P}E)$. More sophisticated arguments show that every element $\beta \in A_m(\mathbb{P}E)$ can be expressed as $\beta = \sum_{i=0}^{r-1} \zeta^i \cap p^* \alpha_i$ for unique α_i . Here, $p : \mathbb{P}E \rightarrow X$ denotes the projection.

When dealing with a *non-singular variety* X , we can interpret $c_i(E)$ as an element of $A^i(X)$ by identifying it with $c_i(E) \cap [X]$. This determines the operator $c_i(E) \cap _$ completely since $c_i(E) \cap \alpha = c_i(E) \cdot \alpha$. We reconsider the case of the projectivization $p : \mathbb{P}E \rightarrow X$. Viewing $A^*(\mathbb{P}E)$ as an $A^*(X)$ -module, we have seen that with $\zeta = c_1(\mathcal{O}(1))$, we obtain a basis $1, \zeta, \zeta^2, \dots, \zeta^{r-1}$ of $A^*(\mathbb{P}E)$ over $A^*(X)$. This fact is known as the **projective bundle formula**. A reformulation yields:

Theorem 2.3.1. If E is a vector bundle on a non-singular variety X of rank r , then $A^*(\mathbb{P}E)$ is isomorphic to the $A^*(X)$ -algebra generated by the symbol ζ which satisfies only the relation $\zeta^r + \sum_{i=1}^r c_i(E)\zeta^{r-i} = 0$.

In [Gro58b] and [Gro58a], Grothendieck uses the projective bundle formula as an axiom in an abstract setup to define Chern classes for vector bundles on non-singular varieties. He defines them as the (unique) coefficients in the relation (**). In [Gro58b], he also shows with the help of these axioms that the Chow ring of a flag bundle of a vector bundle E on X is closely related to the Chow ring of the basis X . By the Whitney sum formula, the expression (*) necessarily holds for the complete flag bundle. But analogously to Theorem 2.3.1, the first Chern classes of the subquotients of the universal bundle generate $A^*(\text{Fl}(E))$ and obey *only* relation (*). More generally:

Theorem 2.3.2. Let E be a vector bundle of rank r on a non-singular variety X , let $\underline{r} = (r_1, \dots, r_n)$ be a sequence of positive integers that add up to r , and denote by $\text{Fl}_{\underline{r}}(E)$ the flag bundle of E of nationality \underline{r} . Let

$$0 = U_0 \subseteq U_1 \subseteq \dots \subseteq U_n = E_{\text{Fl}_{\underline{r}}(E)}$$

be the universal flag on $\mathrm{Fl}_r(E)$ and let $F_i := U_i/U_{i-1}$ be the successive subquotients of rank r_i . Then, in the Chow ring $A^*(\mathrm{Fl}(E))$, we have

$$c(E) = \prod_{i=1}^n (1 + c_1(F_i) + \dots + c_{r_i}(F_i)),$$

when identifying $c(E)$ with $c(E_{\mathrm{Fl}_r(E)})$, and, moreover, $A^*(\mathrm{Fl}_r(E))$ is isomorphic to the free $A^*(X)$ -algebra in symbols $c_j(F_i)$, satisfying only those relations comprised in the above expression.

The *complete* flag bundle $\mathrm{Fl}(E)$ therefore qualifies for the splitting principle. The first Chern classes $\xi_i := c_1(L_i)$ of the subquotients of the universal flag are thus Chern roots of E . They may be regarded as formal variables subject only to the relation $c(E) = \prod_{i=1}^r (1 + \xi_i)$. A formal consequence of this is that the i -th Chern class $c_i(E)$ equals the i -th elementary symmetric function in the Chern roots ξ_1, \dots, ξ_r .

The Chern classes of many constructions, forming new vector bundles from given ones, may be described more easily using Chern roots. For example, let F be another vector bundle on X , say of rank s , and let η_1, \dots, η_s be its Chern roots. Then, the tensor product $E \otimes F$ has total Chern class

$$c(E \otimes F) = \prod_{i=1}^r \prod_{j=1}^s (1 + \xi_i + \eta_j),$$

that means the k -th Chern class of $E \otimes F$ is the k -th elementary symmetric function in the $\xi_i + \eta_j$. Another example is the dual of a bundle. Its total Chern class is $c(E^\vee) = \prod_i (1 - \xi_i)$. This implies $c_i(E^\vee) = (-1)^i c_i(E)$.

An important connection between intersection products and Chern classes is given by the **self-intersection formula**. It states that, for a regular embedding $i : Y \rightarrow X$ of relative dimension d , we obtain

$$i^* i_* \beta = c_d(C_Y X) \cap \beta$$

for every $\beta \in A_*(Y)$. In the above formula, $C_Y X$ denotes the normal bundle of the regular embedding i . The self-intersection formula is a formal consequence of the excess intersection formula (cf. [Ful98, 6.3]).

There exists a localized version of Chern classes. Let us start with the localized top Chern class. Let E be a vector bundle of rank r on a purely n -dimensional scheme X . Let s be a section of E and let $Z(s)$ be its zero scheme with the natural closed subscheme structure which comes from the cartesian diagram

$$\begin{array}{ccc} Z(s) & \xrightarrow{i} & X \\ \downarrow i & & \downarrow s \\ X & \xrightarrow{s_0} & E, \end{array}$$

s_0 denoting the zero section of E . As s_0 is a regular embedding of codimension r , we can form its refined Gysin pull-back. The class $\mathbb{Z}(s) := s_0^! [X] \in A_{n-r}(Z(s))$ is called the **localized top Chern class** of E . This terminology is justified by the following:

Proposition 2.3.3. Let s be a section of a rank r bundle E on a purely n -dimensional scheme X .

- (i) In $A_*(X)$, we have $i_*\mathbb{Z}(s) = c_r(E) \cap [X]$.
- (ii) Every irreducible component of $Z(s)$ has codimension at most r . If s is a regular section (i.e. a regular embedding of codimension r), then every component of $Z(s)$ has precisely codimension r and $\mathbb{Z}(s) = [Z(s)]$.
- (iii) Let $f : Y \rightarrow X$ be a morphism, let $s' := f^*s$ the induced section of f^*E and let $f' : Z(s') \rightarrow Z(s)$ be the restriction.
 - If f is flat, then $f'^*\mathbb{Z}(s) = \mathbb{Z}(s')$.
 - If f is a regular embedding, then $f'^*\mathbb{Z}(s) = \mathbb{Z}(s')$.
 - If f is a proper morphism of varieties, then $f'_*\mathbb{Z}(s') = \deg(Y/X)\mathbb{Z}(s)$.

Moreover, $\mathbb{Z}(s)$ is uniquely determined by the properties (ii) and (iii).

Item (i) resembles the self-intersection formula stated above. In fact, it is an easy consequence of it (see [Ful98, 14.1]).

Using localized top Chern classes, we can define a localized version of the lower Chern classes as well. Let again E be a vector bundle of rank r on a purely n -dimensional scheme X . Let $i \leq r$ and let s_1, \dots, s_{r-i+1} be sections of E . Abbreviate $l := r - i + 1$. The subset $D(s_1, \dots, s_l)$ of all $x \in X$ such that $s_1(x), \dots, s_l(x)$ are linearly dependent has a natural structure of a closed subscheme when identifying it with $Z(s_1 \wedge \dots \wedge s_l)$. To realize the localized i -th Chern class, we interpret $D(s_1, \dots, s_l)$ as a degeneracy locus (cf. [Ful98, 14.4]). The sections induce a map $\sigma : \mathcal{O}^{\oplus l} \rightarrow E$ from the trivial rank l bundle to E . Let $P := \mathbb{P}^{l-1} \times X$ be the $(l-1)$ -dimensional projective space over X . The line bundle $\mathcal{O}(-1)$ is a subbundle of the trivial rank l bundle on P . Let f be the composition

$$\mathcal{O}(-1) \hookrightarrow \mathcal{O}^{\oplus l} \xrightarrow{\sigma_P} E_P$$

which we interpret as a section of $\mathcal{O}(1) \otimes E_P$. The projection $\pi : P \rightarrow X$ restricts to a surjective map $\pi' : Z(f) \rightarrow D(s_1, \dots, s_l)$. We call $\mathbb{D}(s_1, \dots, s_l) := \pi'_*\mathbb{Z}(f)$ the **localized i -th Chern class** of E . Assume that X is a non-singular variety. Letting $\xi = c_1(\mathcal{O}(1))$ and $\alpha_1, \dots, \alpha_r$ the Chern roots of E , the image of $\mathbb{Z}(f)$ in $A^*(P)$ is

$$\begin{aligned} c_{\text{top}}(\mathcal{O}(1) \otimes E_P) &= \prod_{\nu=1}^r (\xi + \alpha_\nu) \\ &= \xi^r + c_1(E)\xi^{r-1} + \dots + c_{r-1}(E)\xi + c_r(E) \\ &= c_{r-l+1}(E)\xi^{l-1} + \dots + c_{r-1}(E)\xi + c_r(E) \end{aligned}$$

as $\xi^j = 0$ for $j \geq l$. Note that $r - l + 1 = i$. Applying π_* to this expression, we obtain $c_i(E)$. We have proved:

Proposition 2.3.4. Let s_1, \dots, s_{r-i+1} be sections of a rank r bundle on a purely n -dimensional scheme.

- (i) The image of $\mathbb{D}(s_1, \dots, s_{r-i+1})$ in $A_*(X)$ is $c_i(E) \cap [X]$.
- (ii) Every irreducible component of $D(s_1, \dots, s_{r-i+1})$ has codimension at most i . If X is Cohen-Macaulay and every component of $D(s_1, \dots, s_{r-i+1})$ has codimension i , then $\mathbb{D}(s_1, \dots, s_{r-i+1}) = [D(s_1, \dots, s_{r-i+1})]$.
- (iii) The formation of $\mathbb{D}(s_1, \dots, s_{r-i+1})$ commutes with the formation of flat pull-backs, relative Gysin maps of regular embeddings and proper push-forward (the latter up to degree).

The localized i -th Chern class is uniquely determined by the properties (ii) and (iii).

2.4 Equivariant Intersection Theory

In [EG98a], Edidin and Graham give a definition of equivariant Chow groups which provides a suitable analog of equivariant cohomology. For a topological group G , there exists a classifying space, i.e. a quotient $E_G \rightarrow B_G$, where E_G is a weakly contractible space on which G acts freely. Every principal G -bundle (with a reasonable basis) arises as a pull-back of the principal bundle $E_G \rightarrow B_G$. For a topological space X endowed with an action of G , the G -equivariant cohomology is defined as

$$H_G^*(X) := H^*((X \times E_G)/G),$$

where G acts diagonally on $X \times E_G$.

The goal is to find an intersection theory equivalent of equivariant cohomology preserving all its fundamental properties (such as functorial behavior, the technique of torus localization, or the fact that $H_G^*(X) \cong H^*(X/G)$ in case a reasonable quotient exists). Moreover, there should be an equivariant cycle map, when dealing with varieties over the complex numbers.

The main idea behind Edidin's and Graham's construction is to use a finite-dimensional approximation of $E_G \rightarrow B_G$ for a given X . This approximation was proposed by Totaro [Tot99]. Let G be a linear algebraic group of dimension d which acts on a purely n -dimensional scheme X . For technical reasons, suppose X is quasi-projective and has a linearized G -action (which is sufficient for our purposes). Following Brion (cf. [Bri98]), we call such X a G -scheme. Fix an integer i . We wish to define $A_i^G(X)$. There exists a finite-dimensional representation V of G , which possesses a G -invariant open subset $U \subseteq V$ with the following properties:

- The codimension of $V - U$ in V is greater than $n - i$ and
- a principal G -bundle $U \rightarrow U/G$ exists.

Remember that, for us, a principal G -bundle is a G -invariant map of *schemes* (not algebraic spaces) which is flat, surjective and locally trivial in the étale (or fpqc-) topology with fiber G . Consider the diagonal action of G on $X \times U$. Our technical assumption assures that a principal G -bundle $X \times U \rightarrow (X \times U)/G$ exists. We denote this quotient by $X \times^G U$. By using Proposition 2.1.1 and applying the ‘‘codimension property’’, one can show (cf. [EG98a, Prop.-Def. 1.1]) that, for every i ,

the Chow group $A_{i+\dim V-d}(X \times^G U)$ is independent of the choices of V and U . Therefore, it is reasonable to define the i -th G -equivariant Chow group of X as

$$A_i^G(X) := A_{i+\dim V-d}(X \times^G U).$$

It is clear that $A_i^G(X)$ vanishes if $i > n$, but $A_i^G(X)$ may be non-trivial for $i < 0$.

Let $f : X \rightarrow Y$ be a G -equivariant morphism of G -schemes. Choosing V and U appropriately for both X and Y induces an equivariant morphism $\bar{f} : X \times^G U \rightarrow Y \times^G U$. Note that $X \times^G U \cong X \times_Y (Y \times^G U)$. Hence, if f is a proper morphism, then, by faithfully flat descent, \bar{f} is a proper morphism, too. The same holds true if we replace “proper” by “flat”, “smooth” or “regular embedding”. Therefore, we are able to transfer all functorial properties of (ordinary) Chow groups to equivariant Chow groups: There exist equivariant versions of the proper push-forward, the flat pull-back and the (relative) Gysin homomorphism.

Additionally, if X is a *non-singular* n -dimensional G -variety, we have seen that $X \times^G U$ is smooth over the non-singular U/G , whence $X \times^G U$ is itself a non-singular variety. In this case, define

$$A_G^j(X) := A_{n-j}^G(X) = A_{n-j+\dim V-d}(X \times^G U) = A^j(X \times^G U).$$

For two classes $x \in A_G^i(X)$ and $y \in A_G^j(X)$, choose V and U appropriately for the maximum of i and j and define $x \cdot y$ to be the intersection product in $A_G^*(X \times^G U)$. An inspection of the proof of [EG98a, Prop.-Def. 1.1] shows that the isomorphisms given there are multiplicative. Thus, $A_G^*(X) := \bigoplus_{j \geq 0} A_G^j(X)$ becomes a graded ring. A similar argument shows that a G -equivariant morphism $Y \rightarrow X$, with X a non-singular G -variety, makes $A_G^*(Y)$ into a graded $A_G^*(X)$ -module.

Let E be a G -equivariant vector bundle on a G -scheme X . Fix i and choose V and U appropriately. Then, $E \times^G U$ is a linear scheme over $X \times^G U$ which is locally trivial in the étale topology. Using the equivalence between vector bundles of rank r and GL_r -principal bundles (over both the étale and the Zariski topology), we obtain that $E \times^G U$ is also Zariski locally trivial because GL_r is special. The j -th Chern class $c_j(E \times^G U) \cap _ : A_i^G(X) \rightarrow A_{i-j}^G(X)$ is independent of the choice of V and U . Therefore,

$$c_j^G(E) := c_j(E \times^G U)$$

is well-defined. We call it the j -th G -equivariant Chern class of E . The equivariant Chern classes fulfill the same properties as the “usual” Chern classes that we introduced in the previous section.

The equivariant Chow group also behaves functorially in G : Let H be a closed subgroup of G . If V and U are chosen such that a G -principal bundle $U \rightarrow U/G$ exists, then, also an H -principal bundle $U \rightarrow U/H$ exists. We obtain a smooth morphism $X \times^H U \rightarrow X \times^G U$ whose fiber is G/H . This gives rise to a map

$$A_i^G(X) \rightarrow A_i^H(X)$$

in equivariant Chow groups. If, for example, H is a Levi subgroup, then G/H is an affine space and therefore, the above map is an isomorphism. Choosing H to be the trivial subgroup gives a morphism $A_i^G(X) \rightarrow A_i(X)$. If X is non-singular, we obtain a homomorphism of graded rings $A_G^*(X) \rightarrow A_H^*(X)$. As for equivariant cohomology, there exist results relating equivariant Chow groups with respect to a reductive linear algebraic group to the equivariant Chow group with respect to a maximal torus (remember that we always work over an algebraically closed field). The following result is also due to Edidin and Graham (cf. [EG98a]).

Theorem 2.4.1. Let G be a reductive linear algebraic group and let T be a maximal torus of G with Weyl group W .

- (i) The ring $A_T^*(\text{pt})$ is isomorphic to $S(T)$, the symmetric algebra of the group of characters of the torus.
- (ii) The map $A_G^*(\text{pt}) \rightarrow A_T^*(\text{pt})$ is injective and $A_G^*(\text{pt})$ is isomorphic to $S(T)^W$ as a graded ring.
- (iii) For any G -scheme X , the map $A_*^G(X)_{\mathbb{Q}} \rightarrow A_T^*(X)_{\mathbb{Q}}$ is injective and induces isomorphisms

$$A_*^G(X)_{\mathbb{Q}}^W \cong A_*^T(X)_{\mathbb{Q}} \quad \text{and} \quad A_*^G(X)_{\mathbb{Q}} \otimes_{S(T)_{\mathbb{Q}}^W} S(T)_{\mathbb{Q}} \cong A_*^T(X)_{\mathbb{Q}}$$

of graded $S(T)_{\mathbb{Q}}^W$ -modules and graded $S(T)_{\mathbb{Q}}$ -modules, respectively.

The isomorphism in (i) is obtained by sending a character χ of T to the first T -equivariant Chern class $c_1^T(L(\chi))$ of the line bundle $L(\chi) = \mathbb{k}$ on pt with T acting by χ . In the context of (iii), $M_{\mathbb{Q}}$ is defined as $M \otimes_{\mathbb{Z}} \mathbb{Q}$ for any abelian group M .

Of particular interest for us (see Chapter 5) will be the case when $X = R$ is a finite-dimensional representation of a reductive linear algebraic group G . Then, choosing V and U appropriately, $R \times^G U$ is a vector bundle over U/G , whence $A_G^*(R) \cong A_G^*(\text{pt})$.

An essential tool to calculate in T -equivariant Chow groups is the method of localization. It works analogously to localization in T -equivariant cohomology. Suppose a torus T acts on a scheme X . Denote by X^T the closed subscheme of T -fixed points. In the ring $S(T)$, consider the multiplicative system $S_+(T)$ of homogeneous elements of positive degree. Let $Q(T) := S_+(T)^{-1}S(T)$ be the localization. Edidin and Graham proved in [EG98b]:

Theorem 2.4.2. The closed embedding $X^T \rightarrow X$ induces an isomorphism $A_*^T(X^T) \otimes_{S(T)} Q(T) \rightarrow A_*^T(X) \otimes_{S(T)} Q(T)$ of graded $Q(T)$ -modules.

In case X is a non-singular, complete T -variety, they give an explicit localization formula. In this case, every component F of X^T is itself non-singular and complete. The embedding $i_F : F \rightarrow X$ is regular and the T -equivariant top Chern class of the corresponding normal bundle $C_F X$ is invertible in $A_*^T(F) \otimes_{S(T)} Q(T)$. For an $\alpha \in A_*^T(X) \otimes_{S(T)} Q(T)$, we obtain

$$\alpha = \sum_F i_{F*} \frac{i_F^* \alpha}{c_{\text{top}}^T(C_F X)},$$

from which we conclude the so-called integration formula (cf. [EG98b]):

Theorem 2.4.3. For a non-singular, complete T -variety and an $\alpha \in A_*^T(X) \otimes_{S(T)} Q(T)$, we have

$$\int_X^T \alpha = \sum_F \int_F^T \frac{i_F^* \alpha}{c_{\text{top}}^T(C_F X)},$$

the sum ranging over all irreducible components F of X^T . Here, \int_X^T and \int_F^T denote the T -equivariant proper push-forwards along $X \rightarrow \text{pt}$ and $F \rightarrow \text{pt}$, respectively.

If, in the context of the above theorem, α has a lift to $A_*^T(X)$, then $\int_X^T \alpha$ belongs to $S(T)$, and hence does the right-hand side.

We conclude this section by relating equivariant Chow groups to ordinary Chow groups of the quotient if the latter exists. Suppose $\pi : X \rightarrow X/G$ is a G -principal bundle. Choosing V and U with respect to an integer i , we obtain that $X \times^G V$ exist. The natural morphism $X \times^G V \rightarrow X/G$ is smooth with fiber V , this yields isomorphisms

$$A_i^G(X) = A_{i+\dim V-d}(X \times^G U) \cong A_{i+\dim V-d}(X \times^G V) \cong A_{i-d}(X/G).$$

If X is non-singular, then X/G is, too, and we get an isomorphism $A_G^*(X) \cong A^*(X/G)$ of graded rings. Edidin and Graham proved a substantially stronger result which states that if $\pi : X \rightarrow Y$ is a geometric G -quotient (in the sense of Mumford's GIT, cf. [MFK94]) and G is a reductive (linear) algebraic group, then $A_i^G(X)_{\mathbb{Q}} \cong A_{i-d}(Y)_{\mathbb{Q}}$. Moreover, π induces an isomorphism of rings $A^*(Y) \cong A_G^*(X)$ if X is smooth. Here, $A^*(Y)$ denotes the operational Chow ring of Y (cf. [Ful98, Chapter 17]). However, for us, the weaker statement will suffice.

Chapter 3

The Chow Ring of a Fine Projective Quiver Moduli is Tautologically Presented

Let us fix our setup for this chapter. Let Q be an *acyclic* quiver and let $d = (d_i \mid i \in Q_0)$ be a *coprime* dimension vector. Fix vector spaces M_i of dimension d_i . Abbreviate $R := R(Q, d)$ and $G := G(Q, d)$. Fix integers ψ_i such that $\sum_i \psi_i d_i = 1$ and let ψ be the resulting character of G of weight one. Let $\theta : \mathbb{Z}^{Q_0} \rightarrow \mathbb{Z}$ be a stability condition for Q such that d is θ -*coprime*. This means $\mu(d') \neq \mu(d)$ for every $0 \leq d' \leq d$ with $0 \neq d' \neq d$. If d is θ -coprime, then every θ -semi-stable representation of (Q, d) is already θ -stable. Let $R^\theta := R^{\theta-(s)\text{st}}(Q, d)$, let $M^\theta := M^{\theta-(s)\text{st}}(Q, d)$ and let U be the universal representation of (Q, d) over M^θ constructed by means of ψ . An application of Theorem 1.3.8 yields:

Theorem. The variety M^θ is a non-singular projective variety which, together with U , is a fine moduli space parametrizing θ -stable representations of (Q, d) up to isomorphism. The quotient map $\pi : R^\theta \rightarrow M^\theta$ is a PG -principal bundle.

Our goal in this chapter is to find an explicit description of the Chow ring $A(M^\theta) := A^*(M^\theta)_{\mathbb{Q}}$ with rational coefficients in terms of generators and relations. The prototype is the Grassmannian $\text{Gr}_p(k^r)$ of p -dimensional linear subspaces in k^r (cf. [Gro58b] or [Ful98, Chapter 14]). By Theorem 2.3.2, the Chow ring of $\text{Gr}_p(k^r)$ is the ring generated by variables c_1, \dots, c_p (which correspond to the Chern classes of the universal rank p subbundle of the trivial rank r bundle) subject to the relation that

$$(1 + c_1 t + \dots + c_p t^p)^{-1}$$

is a polynomial of degree at most $r - p$. It is easy to describe $\text{Gr}_p(k^r)$ as a quiver moduli. Let Q be the Kronecker quiver with r arrows (cf. Example 1.1.1(ii)) and put $d = (1, r - p)$. Choosing an appropriate stability condition θ , a representation of (Q, d) , which is nothing but a linear map $k^r \rightarrow k^{r-p}$, is θ -(semi-)stable if and only if it is surjective. Hence, $M^\theta(Q, d)$ identifies with $\text{Gr}_p(k^r)$. A reasonable description of a Chow ring of a quiver moduli should generalize this standard example.

We should point out that Chapter 3 is based on the author's paper [Fra13a].

3.1 Tautological Generators for the Chow Ring of $M^\theta(Q, d)$ and Relations Between them

A first step towards an explicit description of $A(M^\theta)$ is to give generators which arise naturally from the quiver Q and the dimension vector d . This is provided by a theorem of King and Walter (cf. [KW95]).

Theorem 3.1.1. The Chow ring $A(M^\theta)$ is generated by the Chern classes $c_{i,\nu} := c_\nu(U_i)$, with $i \in Q_0$ and $1 \leq \nu \leq d_i$, as a \mathbb{Q} -algebra.

So, generators for the ring in question are known. We want to gather some information about relations between these generators. There is one rather non-canonical relation which comes from the choice of the character ψ . Considering the construction of U , it is easy to see that

$$\bigotimes_{i \in Q_0} (\det U_i)^{\otimes \psi_i} = \mathcal{O},$$

where ψ_i are the integers with $\psi(g) = \prod_i (\det g_i)^{\psi_i}$. This, in turn, yields the so-called **linear relation**

$$\sum_{i \in Q_0} \psi_i c_{i,1} = 0.$$

Next, we will construct certain degeneracy loci in an iterated flag bundle over M^θ that provide relations between the $c_\nu(U_i)$.

Let us have a closer look at the notion of (semi-)stability. Let $M \in R$ and let $d' \neq 0$ be a dimension vector of Q with $d' \leq d$ and $\mu(d') > \mu(d)$. For convenience, such a sub-dimension vector $d' \leq d$ will be called **forbidden** for θ or **θ -forbidden**. If M has a subrepresentation M' of dimension vector d' , then there exists a tuple $(M'_i \mid i \in Q_0)$ of subspaces M'_i of M_i of dimension d'_i such that $M_\alpha M'_i \subseteq M'_j$ holds for every $\alpha : i \rightarrow j$. Conversely, M has no subrepresentation of dimension vector d' if and only if for all such tuples $(M'_i \mid i \in Q_0)$, there exists an arrow $\alpha : i \rightarrow j$ in Q such that

$$M_\alpha M'_i \not\subseteq M'_j.$$

Next, we form for every $i \in Q_0$ the complete flag bundle $\text{Fl}(U_i)$ and denote by $\text{Fl}(U)$ the fiber product of all $\text{Fl}(U_i)$ over M^θ . Let $p : \text{Fl}(U) \rightarrow M^\theta$ be the projection. This variety possesses a “universal flag” $U^\cdot = (U_i^\cdot \mid i \in Q_0)$ consisting of complete flags U_i^\cdot of p^*U_i . The flag U_i^\cdot arises as the pull-back of the universal flag on $\text{Fl}(U_i)$ along the natural morphism $\text{Fl}(U) \rightarrow \text{Fl}(U_i)$. Together with U^\cdot , the variety $\text{Fl}(U)$ is the universal M^θ -variety being equipped with a family of complete flags of (the pull-backs of) all U_i^\cdot 's. Therefore, a point $p \in \text{Fl}(U)$ can be regarded as a pair $p = ([M], W^\cdot)$ consisting of an isomorphism class of a θ -(semi-)stable representation M of (Q, d) and a tuple $W^\cdot = (W_i^\cdot \mid i \in Q_0)$ of complete flags W_i^\cdot of subspaces of $(U_i)_{[M]} = M_i$.

We will see that every forbidden sub-dimension vector $d' \leq d$ induces a relation in $A(\text{Fl}(U))$. Fix a forbidden $d' \leq d$. Let $y = ([M], W^\cdot)$ be a point of $\text{Fl}(U)$. In particular, we are given a d'_i -dimensional subspace of M_i for every $i \in Q_0$. As M is a θ -(semi-)stable representation, we know that there exists an arrow $\alpha : i \rightarrow j$ with

$$M_\alpha W_i^{d'_i} \not\subseteq W_j^{d'_j}$$

or equivalently, the composition of linear maps $W_i^{d'_i} \rightarrow M_i \xrightarrow{M_\alpha} M_j \rightarrow M_j/W_j^{d'_j}$ is not identically zero. We consider the map

$$\varphi_\alpha^{d'} : U_i^{d'_i} \rightarrow p^*U_i \xrightarrow{p^*U_\alpha} p^*U_j \rightarrow p^*U_j/U_j^{d'_j}$$

of vector bundles on $\text{Fl}(U)$. On the fibers of the point $y \in \text{Fl}(U)$, we obtain

$$\begin{array}{ccccccc} (U_i^{d'_i})_y & \longrightarrow & (p^*U_i)_y & \xrightarrow{(p^*U_\alpha)_y} & (p^*U_j)_y & \longrightarrow & (p^*U_j/U_j^{d'_j})_y \\ \parallel & & \parallel & & \parallel & & \parallel \\ W_i^{d'_i} & \longrightarrow & M_i & \xrightarrow{M_\alpha} & M_j & \longrightarrow & M_j/W_j^{d'_j}, \end{array}$$

so $(\varphi_\alpha^{d'})_y$ is not the zero-map. This implies that for every point y of $\text{Fl}(U)$, there exists $\alpha \in Q_1$ such that y is not contained in the zero locus $Z(\varphi_\alpha^{d'})$ (see Section 2.3). Hence, $\bigcap_{\alpha \in Q_1} Z(\varphi_\alpha^{d'}) = \emptyset$ and therefore, we obtain the following result for the localized top Chern classes.

Proposition 3.1.2. If $d' \leq d$ is θ -forbidden then $\prod_{\alpha \in Q_1} \mathbb{Z}(\varphi_\alpha^{d'}) = 0$ in $A(\text{Fl}(U))$.

Let us calculate these products more explicitly. For $\alpha : i \rightarrow j$, the localized top Chern class $\mathbb{Z}(\varphi_\alpha^{d'})$, or more precisely its push-forward to $A(\text{Fl}(U))$, coincides with the top Chern class of the bundle

$$(U_i^{d'_i})^\vee \otimes p^*U_j/U_j^{d'_j}.$$

Let $F_i^\nu := U_i^\nu/U_i^{\nu-1}$ be the successive subquotients of the universal flag U . With $\xi_{i,\nu} := c_1(F_i^\nu)$, we get

$$\begin{aligned} c(U_i^{d'_i}) &= (1 + \xi_{i,1}) \dots (1 + \xi_{i,d'_i}) \\ c(p^*U_j/U_j^{d'_j}) &= (1 + \xi_{j,d'_j+1}) \dots (1 + \xi_{j,d_j}) \end{aligned}$$

and consequently, we obtain

$$\mathbb{Z}(\varphi_\alpha^{d'}) = c_{\text{top}} \left((U_i^{d'_i})^\vee \otimes p^*U_j/U_j^{d'_j} \right) = \prod_{\mu=1}^{d'_i} \prod_{\nu=d'_j+1}^{d_j} (\xi_{j,\nu} - \xi_{i,\mu}).$$

The previous proposition then reads as follows:

Corollary 3.1.3. For every θ -forbidden $d' \leq d$, we have $\prod_{\alpha: i \rightarrow j} \prod_{\mu=1}^{d'_i} \prod_{\nu=d'_j+1}^{d_j} (\xi_{j,\nu} - \xi_{i,\mu}) = 0$ in $A(\text{Fl}(U))$.

We can view $\text{Fl}(U)$ as an iterated formation of flag bundles. Therefore, it is easy to determine the Chow ring of $\text{Fl}(U)$ in terms of the Chow ring of M^θ and the Chern classes $c_1(F_i^\nu)$. Let \tilde{Q}_0 be the set of all pairs (i, ν) with $i \in Q_0$ and $1 \leq \nu \leq d_i$ and let C be the polynomial ring $C := \mathbb{Q}[t_{i,\nu} \mid (i, \nu) \in \tilde{Q}_0]$. We define an action of the group $W := \prod_{i \in Q_0} S_{d_i}$ on C by

$$w \cdot t_{i,\nu} = t_{i,w_i(\nu)},$$

where $w = (w_i \mid i \in Q_0)$. Then, $A := C^W$ is generated by the algebraically independent elements $x_{i,\nu} := \sigma_\nu(t_{i,1}, \dots, t_{i,d_i})$, and σ_ν denotes the ν -th elementary symmetric function (in the suitable number of variables).

Define the ring homomorphism $\Psi : C \rightarrow A(\text{Fl}(U))$ by $\Psi(t_{i,\nu}) := \xi_{i,\nu}$. As $\sigma_\nu(\xi_{i,1}, \dots, \xi_{i,d_i}) = c_\nu(U_i)$, the map Ψ restricts to $\Phi : A \rightarrow A(M^\theta)$, sending $x_{i,\nu}$ to $c_\nu(U_i)$. We get a commuting square

$$\begin{array}{ccc} C & \xrightarrow{\Psi} & A(\text{Fl}(U)) \\ \uparrow & & \uparrow p^* \\ A & \xrightarrow{\Phi} & A(M^\theta). \end{array}$$

Theorem 2.3.2 implies at once the following fact:

Theorem 3.1.4. The homomorphism Ψ induces an isomorphism $C \otimes_A A(M^\theta) \xrightarrow{\cong} A(\text{Fl}(U))$ of (graded) $A(M^\theta)$ -algebras.

We can easily see that C is a free A -module. For example, an A -basis of C is given by $(t^\lambda \mid \lambda \in \Delta)$, where

$$t^\lambda := \prod_{i \in Q_0} \left(t_{i,1}^{\lambda_{i,1}} \cdots t_{i,d_i}^{\lambda_{i,d_i}} \right)$$

and Δ is the set of all tuples $\lambda = (\lambda_{i,\nu} \mid (i, \nu) \in \tilde{Q}_0)$ of non-negative integers with $\lambda_{i,\nu} \leq d_i - \nu$ for all i and ν . This implies that $A(\text{Fl}(U))$ is a free $A(M^\theta)$ -module.

Let $d' \leq d$ be a θ -forbidden sub-dimension vector. For an arrow $\alpha : i \rightarrow j$, let $f_\alpha^{d'} \in C$ be defined as

$$f_\alpha^{d'} := \prod_{\mu=1}^{d'_i} \prod_{\nu=d'_j+1}^{d_j} (t_{j,\nu} - t_{i,\mu}).$$

Definition. For a θ -forbidden sub-dimension vector d' of d , we call $f^{d'} := \prod_{\alpha \in Q_1} f_\alpha^{d'}$ the **forbidden polynomial** associated to d' .

Corollary 3.1.3 shows that $\Psi(f^{d'}) = 0$. Let \mathcal{B} be a basis of C as an A -module. It is of the form $\mathcal{B} = (y_\lambda \mid \lambda \in \Delta)$. There exist uniquely determined $\tau_\lambda(d', \mathcal{B}) \in A$ such that

$$f^{d'} = \sum_{\lambda \in \Delta} \tau_\lambda(d', \mathcal{B}) \cdot y_\lambda.$$

Definition. The elements $\tau_\lambda(d', \mathcal{B})$ are called **tautological relations** for d' with respect to \mathcal{B} .

We will show in the next paragraph that, together with the linear relation $l := \sum_i \psi_i x_{i,1}$ that we figured out earlier, the tautological relations generate the kernel of Φ and hence are a complete system of relations for $A(M^\theta)$.

3.2 The Chow Ring of $M^\theta(Q, d)$ is Tautologically Presented

Let C , A and W be as in the previous section. Fix a basis $\mathcal{B}^{d'} = (y_\lambda(d') \mid \lambda)$ of C as an A -module for every θ -forbidden sub-dimension vector d' , and let $\tau_\lambda(d') := \tau_\lambda(d', \mathcal{B}^{d'})$ be the tautological relations for d' with respect to this basis. Let $l := \sum_i \psi_i x_{i,1}$ be the linear relation corresponding to the character ψ .

This section is devoted to the proof of the following result (which is the main result of Chapter 3 and [Fra13a]):

Theorem 3.2.1. The map $A \rightarrow A(M^\theta)$ sending $x_{i,\nu}$ to $c_\nu(U_i)$ yields an isomorphism $A/\mathfrak{a} \cong A(M^\theta)$ of graded \mathbb{Q} -algebras, where \mathfrak{a} is the ideal generated by the linear relation l and the tautological relations $\tau_\lambda(d')$ for d' running through all θ -forbidden sub-dimension vectors of d and $\lambda \in \Delta$.

The proof proceeds in several steps. We start with some simple reductions. Remember the commuting square from above

$$\begin{array}{ccc} C & \xrightarrow{\Psi} & A(\text{Fl}(U)) \\ \uparrow & & \uparrow p^* \\ A & \xrightarrow{\Phi} & A(M^\theta). \end{array}$$

Evidently, $\Psi(l) = \Phi(l) = 0$. If $f^{d'} = \sum_\lambda \tau_\lambda y_\lambda$, then $0 = \Psi(f^{d'}) = \sum_\lambda \Phi(\tau_\lambda) \Psi(y_\lambda)$, and thus, Theorem 3.1.4 yields $\Phi(\tau_\lambda) = 0$. Therefore, we obtain $\Phi(\mathfrak{a}) = 0$, and consequently, Φ induces $\bar{\Phi} : A/\mathfrak{a} \rightarrow A(M^\theta)$. Theorem 3.1.1 yields that Φ is onto, so we obtain the surjectivity of $\bar{\Phi}$. Hence, it remains to prove that $\bar{\Phi}$ is injective.

Furthermore, we note that there is no loss of generality in assuming $\theta(d) = 0$. This is because neither multiplication of the stability condition with a positive integer, nor adding an integral multiple of \dim changes the set of forbidden sub-dimension vectors.

The following is inspired by the proof of a result due to Ellingsrud and Stromme (cf. [ES89, Thm. 4.4]). Like they do, we first prove the desired result for a torus quotient using methods of toric geometry. In our situation, this amounts to choosing the dimension vector consisting of ones only. Afterwards, we reduce the general case to the toric one. Ellingsrud and Stromme use a symmetrization map ρ to obtain the ideal of relations. This map will also play a role in the following proof: We show that the ideal of tautological relations contains the image via ρ of the ideal generated by the forbidden polynomials. After having done so, we proceed in almost the same way as in [ES89] (cf. Lemmas 3.2.8, 3.2.9 and 3.2.10).

The toric case

We prove that $\bar{\Phi}$ is injective when $d = \mathbf{1}$, the dimension vector that consists of ones entirely. A sub-dimension vector of $\mathbf{1}$ is of the form $\mathbf{1}_{I'}$, the characteristic function on a subset I' of Q_0 . Denote $\theta(I') := \theta(\mathbf{1}_{I'})$. Using this description of sub-dimension vectors, Theorem 3.2.1, which we want to prove for $d = \mathbf{1}$, reads like this:

Proposition 3.2.2. There is an isomorphism $\mathbb{Q}[t_i \mid i \in Q_0]/\mathfrak{a} \cong A(M^\theta)$ of graded \mathbb{Q} -algebras sending t_i to $c_1(U_i)$, where \mathfrak{a} is the ideal generated by functions $l = \sum_i \psi_i t_i$ and

$$f^{I'} = \prod_{\alpha: i \rightarrow j, i \in I', j \notin I'} (t_j - t_i)$$

with I' running through all subsets of Q_0 with $\theta(I') < 0$.

We will prove this by showing that M^θ is a toric variety and giving an explicit description of its toric fan. This enables us to employ a theorem of Danilov which displays the Chow ring of a non-singular projective toric variety in terms of generators and relations.

Let M be a representation of $(Q, \mathbf{1})$. A subrepresentation M' consists of subspaces $M'_i \subseteq M_i \cong \mathbb{k}$ such that $M_\alpha M'_i \subseteq M'_j$ holds for every $\alpha: i \rightarrow j$. This is equivalent to requiring $M'_j \neq 0$ for every $\alpha: i \rightarrow j$ with $M_\alpha \neq 0$ and $M'_i \neq 0$. Define the subset I' (which depends on M') by

$$I' := \{i \in Q_0 \mid M'_i \neq 0\}.$$

This subset satisfies the following condition:

- (1) For every arrow $\alpha: i \rightarrow j$ with $i \in I'$ and $M_\alpha \neq 0$, we have $j \in I'$.

Conversely, every subset $I' \subseteq Q_0$ satisfying condition (1) defines a subrepresentation M' of M . The dimension vector of this subrepresentation M' is $\mathbf{1}_{I'}$. We obtain that a representation M of $(Q, \mathbf{1})$ is (semi-)stable if and only if $\theta(I') > 0$ (or $\theta(I') \geq 0$) for every subset $I' \subseteq Q_0$ that has the property (1).

Let us have a look at property (1) again. It actually does not depend on M , but only on whether or not $M_\alpha = 0$. So, if we define $\text{Supp}(M) := \{\alpha \in Q_1 \mid M_\alpha \neq 0\}$, it is clear that the (semi-)stability of M only depends on the set $\text{Supp}(M)$ of arrows.

Definition. A subset $J \subseteq Q_1$ is called θ -**(semi-)stable** if there exists a θ -(semi-)stable representation $M \in R$ with $J = \text{Supp}(M)$.

We define (J) to be the set of all subsets $I' \subseteq Q_0$ such that $j \in I'$ for every arrow $\alpha: i \rightarrow j$ with $i \in I'$ and $\alpha \in J$. We have seen:

Lemma 3.2.3. A subset $J \subseteq Q_1$ is θ -(semi-)stable if and only if $\theta(I') > 0$ (or $\theta(I') \geq 0$, respectively) for every $I' \in (J)$.

This is the simplest way to describe (semi-)stability of a representation of $(Q, \mathbf{1})$. On the other hand, we can interpret M as an element of the variety $R := R(Q, \mathbf{1}) = \bigoplus_\alpha R_\alpha$ with $R_\alpha = \text{Hom}(M_i, M_j) \cong \mathbb{k}$. Let us work out another characterization of (semi-)stability from this geometric point of view.

Let T^+ be the maximal torus of $\text{Gl}(R)$ that corresponds to the decomposition $R = \bigoplus_\alpha R_\alpha$. Let $T := G(Q, \mathbf{1}) = \prod_i \mathbb{G}_m$. The action of T on R is compatible with the decomposition of R , thus it induces a morphism $r: T \rightarrow T^+$ of tori. The kernel of this morphism r is the image Γ of the diagonal embedding $\mathbb{G}_m \rightarrow T$. This gives an embedding of $PT := PG(Q, \mathbf{1}) = T/\Gamma \rightarrow T^+$. Let $T^- := T^+/PT$. We have an exact sequence of tori

$$1 \rightarrow \mathbb{G}_m \rightarrow T \xrightarrow{r} T^+ \xrightarrow{s} T^- \rightarrow 1.$$

This gives rise to exact sequences

$$0 \rightarrow \mathbb{Z} \rightarrow N := N(T) \xrightarrow{r_*} N^+ := N(T^+) \xrightarrow{s_*} N^- := N(T^-) \rightarrow 0$$

of the lattices of one-parameter subgroups and

$$0 \leftarrow \mathbb{Z} \leftarrow M := M(T) \xleftarrow{r^*} M^+ := M(T^+) \xleftarrow{s^*} M^- := M(T^-) \leftarrow 0$$

of the character lattices (cf. for example [Spr98]). Denote by $\langle \cdot, \cdot \rangle$ the pairings between the characters and the one-parameter subgroups. We have dual bases $M = \bigoplus_i \mathbb{Z}\chi_i$ and $N = \bigoplus_i \mathbb{Z}\lambda_i$, as well as $M^+ = \bigoplus_\alpha \mathbb{Z}\chi_\alpha$ and $N^+ = \bigoplus_\alpha \mathbb{Z}\lambda_\alpha$. Let $\delta := \sum_i \lambda_i \in N$. This is the image of 1 under the map $\mathbb{Z} \rightarrow N$.

Theorem 1.3.3 applied to this particular situation yields the following characterization of (semi-)stability:

Theorem 3.2.4. For a representation $M \in R$ with support $J \subseteq Q_1$, the following are equivalent:

- (i) M is θ -(semi-)stable.
- (ii) For every $\lambda \in N_{\mathbb{R}} - \mathbb{R}\delta$ such that $\langle \chi_\alpha, r_*\lambda \rangle \geq 0$ for all $\alpha \in J$, we have $\langle \theta, \lambda \rangle > 0$ (or $\langle \theta, \lambda \rangle \geq 0$, respectively).

The varieties R^θ and $M^\theta = R^\theta/PT$ are toric: The torus T^+ acts on R^θ with a dense orbit isomorphic to PT and therefore, T^- acts on M^θ with a dense orbit isomorphic to T^- . As references on toric geometry, we recommend Danilov's article [Dan78] and Fulton's book [Ful93]. An application of toric geometry to quiver moduli has been done by Hille [Hil98]. Let Δ^+ be the fan of R^θ in $N_{\mathbb{R}}^+$ and Δ^- be the fan of M^θ in $N_{\mathbb{R}}^-$. We want to give an explicit description of these fans. We need some auxiliary results to finally obtain this description in Proposition 3.2.7. Define

$$\Phi_\theta := \{ J \subseteq Q_1 \mid J^c \text{ } \theta\text{-}(semi\text{-})\text{stable} \},$$

where $J^c := Q_1 - J$. Note that θ -stability and θ -semi-stability coincide as we have already pointed out. As every subset of Q_1 containing a (semi-)stable set is itself (semi-)stable, we obtain that Φ_θ is a simplicial complex. For every $J \subseteq Q_1$, let $\sigma_J^+ := \text{cone}(\lambda_\alpha \mid \alpha \in J) = \{ \sum_{\alpha \in J} a_\alpha \lambda_\alpha \mid a_\alpha \geq 0 \}$ in $N_{\mathbb{R}}^+$ and let $\sigma_J^- := s_*\sigma_J^+ \subseteq N_{\mathbb{R}}^-$.

Lemma 3.2.5. (i) For every $J \in \Phi_\theta$, the cone σ_J^- is a simplex of dimension $\sharp J$.

(ii) For $J_1, J_2 \in \Phi_\theta$, we have $\sigma_{J_1}^- \cap \sigma_{J_2}^- = \sigma_{J_1 \cap J_2}^-$.

(iii) If $J' \notin \Phi_\theta$ such that $\{\alpha\} \in \Phi_\theta$ holds for every $\alpha \in J'$, then $\sigma_{J'}^- \notin \{ \sigma_J^- \mid J \in \Phi_\theta \}$.

(iv) If $J = \{\alpha_0\} \in \Phi_\theta$, then $s_*\lambda_{\alpha_0}$ is the minimal lattice point of σ_J^- .

Proof. (i) We have to show that the elements $s_*\lambda_\alpha$ with $\alpha \in J$ are linearly independent over the reals. Let us assume there were an element $0 \neq \lambda^+ = \sum_{\alpha \in J} b_\alpha \lambda_\alpha \in \sigma_J^+$ with $s_*\lambda^+ = 0$. Then, there would exist $\lambda \in N_{\mathbb{R}} - \mathbb{R}\delta$ with $\lambda^+ = r_*\lambda$. For every $\alpha \notin J$, we would get

$$\langle \chi_\alpha, r_*\lambda \rangle = \langle \chi_\alpha, \lambda^+ \rangle = 0$$

as λ^+ is supported in J . By stability of J^c , we would obtain that $\langle \theta, \lambda \rangle > 0$. But, on the other hand, we would also get $\langle \chi_\alpha, r_*(-\lambda) \rangle = 0$ and thus, $\langle \theta, -\lambda \rangle > 0$. A contradiction.

- (ii) Assume there existed $\lambda' = \sum_{\alpha \in J_1} b'_\alpha \lambda_\alpha \in \sigma_{J_1}^+$ and $\lambda'' = \sum_{\alpha \in J_2} b''_\alpha \lambda_\alpha \in \sigma_{J_2}^+$ with $\lambda' \neq \lambda''$ and $s_*\lambda' = s_*\lambda''$. Then, there would exist an element $\lambda \in N_{\mathbb{R}} - \mathbb{R}\delta$ such that $\lambda' - \lambda'' = r_*\lambda$. For all $\alpha \notin J_2$, we would have

$$\langle \chi_\alpha, r_*\lambda \rangle = b'_\alpha - b''_\alpha = b'_\alpha \geq 0,$$

and as J_2^c is θ -stable, we would obtain that $\langle \theta, \lambda \rangle > 0$. But with the same argument, we would get $\langle \chi_\alpha, r_*(-\lambda) \rangle = b''_\alpha \geq 0$ and therefore, $\langle \theta, -\lambda \rangle > 0$ by stability of J_1^c . Again, this is a contradiction.

- (iii) We suppose there existed $J \in \Phi_\theta$ with $\sigma_J^- = \sigma_{J'}^-$. The set J' could not be contained in J , thus there would exist an arrow $\alpha \in J' - J$. Consider $s_*\lambda_\alpha \in \sigma_{J'}^-$. As $\{\alpha\} \in \Phi_\theta$, part (i) would yield $s_*\lambda_\alpha \neq 0$. But $s_*\lambda_\alpha \in \sigma_J^- \cap \sigma_{\{\alpha\}}^- = \sigma_\emptyset^- = 0$ by (ii).

- (iv) We will show that for every $\alpha_0 \in Q_1$, the generator $s_*\lambda_{\alpha_0}$ is either 0 or primitive (i.e. cannot be displayed as a positive integer multiple of a lattice element, apart from itself). This proves the desired statement as $s_*\lambda_{\alpha_0} \neq 0$ if $\{\alpha_0\} \in \Phi_\theta$. Let $\alpha_0 : i_0 \rightarrow j_0$ with $s_*\lambda_{\alpha_0} \neq 0$ and assume there were an integer $n > 1$ and an element $\lambda^- \in N^-$ such that $s_*\lambda_{\alpha_0} = n\lambda^-$. We would find $\lambda^+ \in N^+$ with $s_*\lambda_{\alpha_0} = n\lambda^- = s_*(n\lambda^+)$. This would imply that there existed $\lambda = \sum_i b_i \lambda_i \in N$ with

$$\sum_{\alpha: i \rightarrow j} (b_j - b_i) \lambda_\alpha = r_*\lambda = n\lambda^+ - \lambda_{\alpha_0},$$

which shows that n would divide $b_j - b_i$ if there exists an arrow $\alpha : i \rightarrow j$ with $\alpha \neq \alpha_0$, and also that $b_{j_0} - b_{i_0}$ would *not* be a multiple of n . Consequently, n would divide $b_j - b_i$ if there exists an unoriented path between i and j that does not involve α_0 , and $b_j - b_i$ would not be a multiple of n if there exists an unoriented path between i and j that passes through α_0 exactly once. We distinguish two cases. If there exist vertices i and j (not necessarily distinct) such that two unoriented paths between i and j exist, one of which does not run through α_0 and the other does exactly once, we get a contradiction. So, we are down to the case where such vertices i and j do not exist. In this situation, removing the arrow α_0 splits the quiver into two disjoint subquivers C^1 and C^2 with $i_0 \in C^1$ and $j_0 \in C^2$. This means that in Q , there is no arrow from C_0^1 to C_0^2 apart from α_0 and no arrow from C_0^2 to C_0^1 . Consider the element $\lambda' := \sum_{i \in C_0^2} \lambda_i$. We would obtain

$$r_*\lambda' = \sum_{\alpha: i \rightarrow j} (\mathbf{1}_{C_0^2}(j) - \mathbf{1}_{C_0^2}(i)) \lambda_\alpha = \lambda_{\alpha_0}.$$

In turn, this would imply that $s_*\lambda_{\alpha_0} = s_*r_*\lambda' = 0$, which contradicts our assumption. \square

We need a simple algebraic result to calculate the invariant ring of the affine toric varieties $X_{\sigma_J^+}$ under the T -action.

Lemma 3.2.6. Let $0 \rightarrow M'' \xrightarrow{\varphi} M \xrightarrow{\psi} M'$ be an exact sequence of lattices (or just abelian groups) and let S be a submonoid of M . Let Λ be a (commutative) ring. Consider the co-action

$$c : \Lambda[S] \rightarrow \Lambda[M'] \otimes_\Lambda \Lambda[S] = \Lambda[M' \times S],$$

defined by $c(x^m) = x^{\psi m} \otimes x^m = x^{(\psi m, m)}$. Then, the co-invariant ring $\Lambda[S]^{\Lambda[M']}$ equals the subring $\Lambda[\varphi^{-1}S]$.

Proof. By definition, $\Lambda[S]^{\Lambda[M']} = \{f \in \Lambda[S] \mid c(f) = 1 \otimes f\}$. As a co-action, c is compatible with multiplication in $\Lambda[S]$, thus, $\Lambda[S]^{\Lambda[M']}$ is a subring of $\Lambda[S]$. As every $x^{\varphi(m)}$ is $\Lambda[M']$ -co-invariant, we obtain that $\Lambda[\varphi^{-1}S]$ is contained in $\Lambda[S]^{\Lambda[M']}$. On the other hand, let $f = \sum_{m \in S} f_m x^m \in \Lambda[S]$ be $\Lambda[M']$ -co-invariant. That means

$$\sum_{m \in S} f_m x^{(\psi m, m)} = c(f) = 1 \otimes f = \sum_{m \in S} f_m x^{(0, m)}.$$

As the elements $x^{(m', m)}$ with $m' \in M'$ and $m \in S$ are linearly independent, we obtain that $\psi m = 0$ for all $m \in S$ with $f_m \neq 0$. But then $m \in \text{im } \varphi$, and therefore, m possesses an inverse image in $\varphi^{-1}S$. \square

Proposition 3.2.7. We obtain $\Delta^+ = \{\sigma_J^+ \mid J \in \Phi_\theta\}$ and $\Delta^- = \{\sigma_J^- \mid J \in \Phi_\theta\}$.

Proof. (i) Let $\Sigma^+ := \{\sigma_J^+ \mid J \in \Phi_\theta\}$ and let $X := X_{\Sigma^+}$ be the toric variety associated to Σ^+ . Then X is an open subset of R which is defined as the union $X = \bigcup_{J \in \Phi_\theta} X_{\sigma_J^+}$. By definition,

$$X_{\sigma_J^+} = \text{Spec}k[M^+ \cap (\sigma_J^+)^{\vee}] = \left\{ \sum_{\alpha \in Q_1} M_\alpha \in R \mid M_\alpha \neq 0 \text{ for all } \alpha \notin J \right\},$$

and thus, it is clear that $R^\theta = X$.

(ii) Put $\Sigma^- := \{\sigma_J^- \mid J \in \Phi_\theta\} = s_* \Sigma^+$. Let $Y := X_{\Sigma^-}$ be the associated toric variety. The homomorphism $s_* : N^+ \rightarrow N^-$ yields a morphism $\eta : R^\theta = X_{\Sigma^+} \rightarrow X_{\Sigma^-} = Y$ of toric varieties. This morphism is T^+ -equivariant via $s : T^+ \rightarrow T^-$ and therefore, PT -invariant with respect to the induced PT -actions via $PT \rightarrow T^+$. We know that a geometric PT -quotient exists. Thus, it suffices to show that Y is a categorical PT -quotient.

Let $f : R^\theta \rightarrow Z$ be a PT -invariant morphism of varieties. Let $X_J := X_{\sigma_J^+}$. This is an affine open subset of R . Lemma 3.2.6 shows that

$$k[M^+ \cap (\sigma_J^+)^{\vee}]^{PT} = k[M^- \cap (s^*)^{-1}((\sigma_J^+)^{\vee})] = k[M^- \cap (\sigma_J^-)^{\vee}],$$

and therefore, $\eta_J : X_J \rightarrow Y_J := X_{\sigma_J^-}$ is a *universal* categorical quotient (cf. [MFK94, Theorem 1.1]). Note that Y_J is an open subset of Y and η_J coincides with the restriction of η . By the quotient property, we obtain that there exists a unique morphism $g_J : Y_J \rightarrow Z$ with $g_J \eta_J = f_J := f|_{X_J}$. Applying Lemma 3.2.5(ii), we get $\sigma_{J_1}^- \cap \sigma_{J_2}^- = \sigma_{J_1 \cap J_2}^-$ for all $J_1, J_2 \in \Phi_\theta$. Let $J := J_1 \cap J_2$. For $\nu = 1, 2$ we have

$$g_J \eta_J = f_J = f_{J_\nu}|_{X_{J_\nu}} = g_{J_\nu} \eta_{J_\nu}|_{X_{J_\nu}} = (g_{J_\nu}|_{Y_{J_\nu}}) \eta_{J_\nu}$$

because η_{J_ν} is a universal categorical quotient. As g_J is uniquely determined by the property $g_J \eta_J = f_J$, we obtain $g_{J_\nu}|_{Y_{J_\nu}} = g_J$. This proves that the maps g_J with $J \in \Phi_\theta$ glue together to a map $g : Y \rightarrow Z$ with $g \eta = f$. Conversely, every such map g has to fulfill $g|_{Y_J} = g_J$. \square

We have obtained an explicit description of the fan of M^θ . This enables us to prove Proposition 3.2.2 with the help of Danilov's theorem (cf. [Dan78, Theorem 10.8]).

Proof of Proposition 3.2.2. Let $\sigma_1^-, \dots, \sigma_r^-$ be the rays of the fan Δ^- . By Lemma 3.2.5 (i), we know that these come from arrows $\alpha_1, \dots, \alpha_r$ with $\{\alpha_i\} \in \Phi_\theta$ and part (iv) of the same lemma tells us that $s_*\lambda_{\alpha_1}, \dots, s_*\lambda_{\alpha_r}$ are their minimal lattice points. Using that Δ^- is the toric fan of M^θ , Danilov's theorem implies

$$A(M^\theta) \cong \mathbb{Q}[x_{\alpha_1}, \dots, x_{\alpha_r}] / (\mathfrak{r}_1 + \mathfrak{r}_2),$$

where \mathfrak{r}_1 is the ideal generated by all monomials $x_{\alpha_{i_1}} \dots x_{\alpha_{i_l}}$, with $J = \{\alpha_{i_1}, \dots, \alpha_{i_l}\}$ such that $\sigma_J^- \notin \Delta^-$, and \mathfrak{r}_2 is spanned by expressions $\sum_i \langle u, s_*\lambda_{\alpha_i} \rangle x_{\alpha_i}$, where u runs through M^- . The above isomorphism is given by sending x_{α_i} to the toric Weil divisor D_{α_i} . Consider the homomorphism $\text{Sym}_{\mathbb{Q}}(M^+) = \mathbb{Q}[\chi_\alpha \mid \alpha \in Q_1] \rightarrow \mathbb{Q}[x_{\alpha_1}, \dots, x_{\alpha_r}]$ mapping χ_{α_i} to x_{α_i} and χ_α to 0 if $\{\alpha\} \notin \Phi_\theta$. We write $S(T^+)$ instead of $\text{Sym}_{\mathbb{Q}}(M^+)$, for brevity. The inverse image of \mathfrak{r}_1 under this map is

$$\left(\chi_{\alpha_{i_1}} \dots \chi_{\alpha_{i_l}} \mid J := \{\alpha_{i_1}, \dots, \alpha_{i_l}\} \text{ s.t. } \sigma_J^- \notin \Delta^- \right) + \left(\chi_\alpha \mid \{\alpha\} \notin \Phi_\theta \right) = \left(\prod_{\alpha \in J} \chi_\alpha \mid J \notin \Phi_\theta \right)$$

by virtue of Lemma 3.2.5 (iii). In the quotient $\mathbb{Q}[\chi_\alpha \mid \alpha \in Q_1] / (\chi_\alpha \mid \{\alpha\} \notin \Phi_\theta)$, the expression $\sum_i \langle u, s_*\lambda_{\alpha_i} \rangle \chi_{\alpha_i}$ can be read as

$$\sum_{\alpha \in Q_1} \langle u, s_*\lambda_\alpha \rangle \chi_\alpha = \sum_{\alpha \in Q_1} \langle s^*u, \lambda_\alpha \rangle \chi_\alpha = s^*u.$$

Thus, we have shown that

$$\begin{aligned} \mathbb{Q}[x_{\alpha_1}, \dots, x_{\alpha_r}] / (\mathfrak{r}_1 + \mathfrak{r}_2) &\cong S(T^+) / \left(\left(\prod_{\alpha \in J} \chi_\alpha \mid J \notin \Phi_\theta \right) + S(T^+) \cdot M^- \right) \\ &\cong S(PT) / \left(\prod_{\alpha \in J} r^* \chi_\alpha \mid J \notin \Phi_\theta \right). \end{aligned}$$

The map $M(\Gamma) \rightarrow M$ sending the generator of $M(\Gamma) \cong \mathbb{Z}$ to $\sum_i \psi_i \chi_i$ is a right inverse of the map $M \rightarrow M(\Gamma)$ that sends every χ_i to the generator. Remember that ψ_i are integers such that $\sum_i \psi_i = 1$. This yields an identification of $M(PT)$ with the quotient group

$$M / \left(\mathbb{Z} \cdot \sum_i \psi_i \chi_i \right).$$

Under this identification, the natural homomorphism $S(T^+) \rightarrow S(PT)$ is the map $\mathbb{Q}[\chi_\alpha \mid \alpha] \rightarrow \mathbb{Q}[\chi_i \mid i] / (\sum_i \psi_i \chi_i)$ that sends χ_α to the coset of $\chi_j - \chi_i$ if $\alpha : i \rightarrow j$. The above calculation shows that we have an isomorphism

$$A(M^\theta) \cong \mathbb{Q}[\chi_i \mid i \in Q_0] / \mathfrak{b},$$

where \mathfrak{b} is the ideal generated by $\sum_i \psi_i \chi_i$ and by the terms $\prod_{\alpha \in J, \alpha: i \rightarrow j} (\chi_j - \chi_i)$ for all $J \notin \Phi_\theta$. This looks already pretty similar to the ideal \mathfrak{a} in Proposition 3.2.2. We show that \mathfrak{a} and \mathfrak{b} are in fact the same after renaming the variables $\chi_i \mapsto t_i$. Both ideals contain $l = \sum_i \psi_i t_i$. Let $J \subseteq Q_1$ with $J \notin \Phi_\theta$. That means J^c is not θ -(semi)-stable and therefore, there exists a θ -forbidden $I' \subseteq Q_0$ with $I' \in (J^c)$ by Lemma 3.2.3. Thus, if $\alpha : i \rightarrow j$ is an arrow with $i \in I'$ and $j \notin I'$, then $\alpha \notin J^c$. This implies

$$f^{I'} = \prod_{\alpha: i \rightarrow j, i \in I', j \notin I'} (t_j - t_i) \text{ divides } \prod_{\alpha: i \rightarrow j, \alpha \in J} (t_j - t_i)$$

and thus $\mathfrak{b} \subseteq \mathfrak{a}$. Conversely, consider the element $f^{I'} \in \mathfrak{a}$ for a θ -forbidden $I' \in Q_0$. If we define $J := \{\alpha : i \rightarrow j \mid i \in I', j \notin I'\}$, then $I' \in (J^c)$ and $f^{I'} = \prod_{\alpha: i \rightarrow j, \alpha \in J} (t_j - t_i)$, so $f^{I'} \in \mathfrak{b}$. \square

The general case

Now, we proceed to the general case. Let \tilde{Q} be a covering quiver of (Q, d) in the sense of Reineke [Rei08b] and Weist [Wei09] defined by

$$\begin{aligned}\tilde{Q}_0 &= \{(i, \nu) \mid i \in Q_0, 1 \leq \nu \leq d_i\} \\ \tilde{Q}_1 &= \{(\alpha, \mu, \nu) \mid (\alpha : i \rightarrow j) \in Q_1, 1 \leq \mu \leq d_i, 1 \leq \nu \leq d_j\},\end{aligned}$$

and for every $\beta := (\alpha, \mu, \nu) \in \tilde{Q}_1$ with $\alpha : i \rightarrow j$ in Q , the arrow β starts at (i, μ) and ends at (j, ν) . Note that this definition of the symbol \tilde{Q}_0 coincides with the one we have used before. We get $R = R(Q, d) \cong R(\tilde{Q}, \mathbf{1})$ and $T := G(\tilde{Q}, \mathbf{1})$ is isomorphic to a maximal torus of $G = G(Q, d)$ in such a way that the action of T on R coincides with the induced T -action by the G -action on R . The same holds for the PT - and PG -action.

For $\tilde{a} \in \mathbb{Z}^{\tilde{Q}_0}$, let $s(\tilde{a})$ be the tuple $(a_i \mid i \in Q_0)$ of integers defined by $a_i := \sum_{\nu=1}^{d_i} \tilde{a}_{(i, \nu)}$. Define

$$\tilde{\theta}(\tilde{a}) := \theta(s(\tilde{a})).$$

This is a stability condition for \tilde{Q} . We analyze its properties.

A sub-dimension vector I' of $\mathbf{1}$ is nothing but a subset $I' \subseteq \tilde{Q}_0$. For a subset I' of \tilde{Q}_0 , the tuple $d' := s(I')$ consists of the integers d'_i defined as the number of $\nu \in \{1, \dots, d_i\}$ with $(i, \nu) \in I'$. Therefore, I' is forbidden for $\tilde{\theta}$ if and only if the induced sub-dimension vector d' of d is θ -forbidden. Furthermore,

$$f^{I'} = \prod_{\substack{(\alpha, \mu, \nu) : (i, \mu) \rightarrow (j, \nu), \\ (i, \mu) \in I', (j, \nu) \notin I'}} (t_{j, \nu} - t_{i, \mu}) = \prod_{\alpha : i \rightarrow j} \prod_{\mu : (i, \mu) \in I'} \prod_{\nu : (j, \nu) \notin I'} (t_{j, \nu} - t_{i, \mu}) = w f^{d'}$$

if we define $w \in W$ such that $w_i \{1, \dots, d'_i\} = \{\mu \mid (i, \mu) \in I'\}$. Remember $W = \prod_i S_{d_i}$ which is, by the way, the Weyl group of T in G and of PT in PG . Conversely, every $w f^{d'}$ is of the form $f^{I'}$, where $I' = \{(i, w_i(\nu)) \mid i \in Q_0, 1 \leq \nu \leq d'_i\}$. With these identifications, the first step of the proof shows that the natural map

$$C/\mathfrak{c} \rightarrow A(\tilde{Y})$$

is an isomorphism, where $\tilde{Y} := M^{\tilde{\theta}}(\tilde{Q}, \mathbf{1})$ and \mathfrak{c} is the ideal generated by l and all elements $w f^{d'}$ for θ -forbidden sub-dimension vectors $d' \leq d$ and $w \in W$.

Definition. Let A be a ring, let M be a free A -module of finite rank and let M' be a sub- A -module of M . Choose a basis $\{m_1, \dots, m_n\}$ of M , and define

$$\text{coeff}_A(M', M) := p_1(M') + \dots + p_n(M'),$$

where $p_i : M \rightarrow A$ is the A -linear map defined by $p_i(\sum_j a_j m_j) = a_i$. We can easily see that $\text{coeff}_A(M', M)$ is an ideal of A that does not depend on the choice of a basis of M .

If $M' = A \cdot m$ with $m = \sum_i a_i m_i$ then $\text{coeff}_A(M', M)$ is the ideal of A generated by a_1, \dots, a_n . This explains the notation.

We get $\mathfrak{a} = \text{coeff}_A(\mathfrak{c}, C)$ with the following argument: By definition, $\mathfrak{a} = Al + \sum_{d'} \text{coeff}_A(f^{d'}, C)$, where the sum runs over all θ -forbidden sub-dimension vectors $d' \leq d$. Let $\mathcal{B} = (y_\lambda \mid \lambda)$ be a basis of C as an A -module and write $f^{d'} = \sum_\lambda \tau_\lambda y_\lambda$. For every $w \in W$, we get

$$wf^{d'} = \sum_\lambda \tau_\lambda wy_\lambda$$

and $(wy_\lambda \mid \lambda)$ is also a basis of C over A . We obtain that $\text{coeff}_A(wf^{d'}, C) = \text{coeff}_A(f^{d'}, C)$ and therefore,

$$\text{coeff}_A(\mathfrak{c}, C) = Al + \sum_{d'} \sum_{w \in W} \text{coeff}_A(wf^{d'}, C) = Al + \sum_{d'} \text{coeff}_A(f^{d'}, C) = \mathfrak{a}.$$

Let us summarize the situation in a picture. With $Y := M^\theta(Q, d)$, we have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathfrak{c} & \longrightarrow & C & \longrightarrow & \mathcal{C} := A(\tilde{Y}) & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & & & \\ 0 & \longrightarrow & \mathfrak{a} = \text{coeff}_A(\mathfrak{c}, C) & \longrightarrow & A & \longrightarrow & \mathcal{A} := A(Y) & \longrightarrow & 0 \end{array}$$

and we know that the upper row is exact. In order to prove the exactness of the lower row, we have remarked earlier that it suffices to check that $A/\mathfrak{a} \rightarrow \mathcal{A}$ is injective.

Define the discriminant of C by

$$\Delta = \prod_{i \in Q_0} \prod_{1 \leq \mu < \nu \leq d_i} (t_{i,\nu} - t_{i,\mu}).$$

This is an anti-invariant, i.e. W acts on Δ by $w\Delta = (\prod_i \text{sign}(w_i)) \Delta$. It is a basic fact that every anti-invariant y of C is of the form $y = a\Delta$ for some $a \in A$. Furthermore, define an A -linear map $\rho : C \rightarrow A$, called the symmetrization map, by

$$\rho(f) = \Delta^{-1} \sum_{w \in W} \text{sign}(w)wf$$

Note that $\rho(\Delta) = \#W$ regarded as an element of A . We show that \mathfrak{a} contains the image $\rho(\mathfrak{c})$. Let $f \in \mathfrak{c}$. Display f as a linear combination $f = \sum_i a_i y_i$ with respect to a basis y_1, \dots, y_n of C as an A -module. As ρ is A -linear, we get $\rho(f) = \sum_i a_i \rho(y_i)$ which lies in $\text{coeff}_A(\mathfrak{c}, C) = \mathfrak{a}$ as every a_i is an element of $\text{coeff}_A(\mathfrak{c}, C)$.

To prove Theorem 3.2.1, it then suffices to show that the induced map $A/\rho(\mathfrak{c}) \rightarrow \mathcal{A}$ is injective. A couple of lemmas will do the trick.

Lemma 3.2.8. There is a natural isomorphism $\mathcal{C}^{\text{a}} \xrightarrow{\cong} A/\rho(\mathfrak{c})$ of $\mathbb{Q}W$ -modules decreasing degrees by $\delta := \deg \Delta$.

In the lemma, \mathcal{C}^{a} denotes the anti-invariant part of \mathcal{C} as a $\mathbb{Q}W$ -module. As the action of W is compatible with the grading of $\mathcal{C} = \bigoplus_{i \geq 0} \mathcal{C}_i$ with $\mathcal{C}_i := A^i(\tilde{Y})$, the anti-invariant part is a graded subspace of \mathcal{C} , i.e. $\mathcal{C}^{\text{a}} = \bigoplus_{i \geq 0} \mathcal{C}_i^{\text{a}}$.

Proof. The composition $C \xrightarrow{\rho} A \rightarrow A/\rho(\mathfrak{c})$ induces an onto map $\bar{\rho} : \mathcal{C} \cong C/\mathfrak{c} \rightarrow A/\rho(\mathfrak{c})$. As taking anti-invariants is an exact functor of $\mathbb{Q}W$ -modules, we obtain

$$\mathcal{C}^a \cong (C/\mathfrak{c})^a = C^a/\mathfrak{c}^a = A \cdot \Delta / (A \cdot \Delta \cap \mathfrak{c}).$$

But $A \cdot \Delta \cap \mathfrak{c} = \rho(\mathfrak{c}) \cdot \Delta$ because of the following: On the one hand, if f is of the form $f = a\Delta$ for some $a \in A$, then $a = \rho(\frac{1}{\#W}a\Delta) \in \rho(\mathfrak{c})$. On the other hand, let $f = \rho(y)\Delta$ for some $y \in \mathfrak{c}$. As the ideal \mathfrak{c} is W -invariant, we get $wy \in \mathfrak{c}$ for every w . This implies $\rho(y) \in \rho(\mathfrak{c})$ and thus, $f \in \mathfrak{c}$. We deduce

$$\mathcal{C}^a \cong A \cdot \Delta / (\rho(\mathfrak{c}) \cdot \Delta) \cong A/\rho(\mathfrak{c})$$

as $\mathbb{Q}W$ -modules. It is clear that this isomorphism decreases degrees by δ . \square

Lemma 3.2.9. For $i < \delta$ or $i > \dim \tilde{Y} - \delta$, we have $\mathcal{C}_i^a = 0$.

Proof. As every anti-invariant of \mathcal{C} is divisible by Δ , there can be no anti-invariant of degree less than $\delta = \deg \Delta$. By Poincaré duality, the multiplication on \mathcal{C} induces a perfect pairing $\mathcal{C}_i \otimes_{\mathbb{Q}} \mathcal{C}_{l-i} \rightarrow \mathcal{C}_l = \mathbb{Q}$ for every i , where $l := \dim \tilde{Y}$. As the W -action on the Chow ring comes from the W -action on the moduli space, this pairing is also W -equivariant. Therefore, we obtain

$$\mathcal{C}_i^a \cong (\mathcal{C}_{l-i}^\vee)^a \cong (\mathcal{C}_{l-i}^a)^\vee$$

as taking anti-invariants commutes with taking duals. If $i > l - \delta$ then $l - i < \delta$, so the assertion is proved. \square

Combining the preceding two lemmas, we obtain that $(A/\rho(\mathfrak{c}))_i = 0$ if $i < 0$ or $i > r := l - 2\delta$. Note also that r is precisely the dimension of Y , because $2\delta + \dim T = \dim G$.

Lemma 3.2.10. We have $(A/\rho(\mathfrak{c}))_r \cong \mathbb{Q}$ and the ring multiplication induces a perfect pairing

$$(A/\rho(\mathfrak{c}))_i \otimes_{\mathbb{Q}} (A/\rho(\mathfrak{c}))_{r-i} \rightarrow (A/\rho(\mathfrak{c}))_r \cong \mathbb{Q}.$$

Proof. We know that

$$(A/\rho(\mathfrak{c}))_r = (A/\rho(\mathfrak{c}))_{l-2\delta} \cong \mathcal{C}_{l-\delta}^a \cong (\mathcal{C}_\delta^a)^\vee \cong (A/\rho(\mathfrak{c}))_0^\vee \cong \mathbb{Q}$$

because $\Delta \notin \mathfrak{c}$. We show that the pairing $(A/\rho(\mathfrak{c}))_i \otimes_{\mathbb{Q}} (A/\rho(\mathfrak{c}))_{r-i} \rightarrow \mathbb{Q}$ is perfect. Let $x \in (A/\rho(\mathfrak{c}))_i$ with $x \neq 0$. There is a unique anti-invariant $y \in \mathcal{C}_{i+\delta}$ with $\bar{\rho}(y) = x$. As the pairing $\mathcal{C}_{i+\delta} \otimes_{\mathbb{Q}} \mathcal{C}_{l-i-\delta} \rightarrow \mathbb{Q}$ is perfect and W equivariant, we obtain that there exists an *anti-invariant* $y' \in \mathcal{C}_{l-i-\delta}$ with $yy' \neq 0$. Choose representatives $\dot{y} \in C_{i+\delta}^a$ and $\dot{y}' \in C_{l-i-\delta}^a$ of y and y' and define $\dot{x} := \rho(\dot{y})$ and $\dot{x}' := \rho(\dot{y}')$. There are unique $a, a' \in A$ such that $a\Delta = \dot{y}$ and $a'\Delta = \dot{y}'$. We get

$$\dot{x}\dot{x}' = \rho(\dot{y})\rho(\dot{y}') = (\#W)^2 \cdot aa' = \#W \cdot \rho(aa'\Delta).$$

As yy' is non-zero, the anti-invariant $aa'\Delta$ is not contained in \mathfrak{c}^a . Hence, $\rho(aa'\Delta) \notin \rho(\mathfrak{c})$ because the isomorphism $\rho : C^a \rightarrow A$ maps \mathfrak{c}^a onto $\rho(\mathfrak{c})$. Therefore, $\dot{x}\dot{x}' \neq 0$. \square

Finally, we are able to show that the map $A/\rho(\mathfrak{c}) \rightarrow \mathcal{A}$ is injective.

Proof of Theorem 3.2.1. Call this map β . We have seen in the above lemmas that $(A/\rho(\mathfrak{c}))_i$ vanishes for $i > r = \dim Y$. In addition, β maps the degree r part of $A/\rho(\mathfrak{c})$ isomorphically onto $\mathcal{A}_r \cong \mathbb{Q}$. Now let $x \in A/\rho(\mathfrak{c})$ with $x \neq 0$. Without loss of generality, we assume x is homogeneous of degree i . By Lemma 3.2.10, there exists $x' \in (A/\rho(\mathfrak{c}))_{r-i}$ with $xx' \neq 0$. Then $0 \neq \beta(xx') = \beta(x)\beta(x')$, which implies $\beta(x) \neq 0$. \square

3.3 Examples

We now turn to some classes of quiver settings where we can compute the Chow ring more explicitly.

The canonical stability condition and the bipartite case

Let θ be the stability condition of Q which comes from the character $G \xrightarrow{\text{act}} \text{Gl}(R) \xrightarrow{\text{det}} \mathbb{G}_m$. Note that θ depends on d . It is of the form

$$\theta(a) = \sum_{i \in Q_0} a_i \cdot \left(\sum_{\alpha: j \rightarrow i} d_j - \sum_{\alpha: i \rightarrow k} d_k \right) = \langle a, d \rangle - \langle d, a \rangle,$$

where $\langle \cdot, \cdot \rangle$ denotes the Euler form of the quiver Q . This is a reasonable stability condition in many examples. In fact, the examples in the following section will all use this stability condition. It is called the **canonical stability condition** of (Q, d) . It is evident that $\theta(d) = 0$. Anyway, d is not necessarily θ -coprime for this choice of θ . Let us assume it is.

We define a partial ordering on the set of dimension vectors of Q . We write $d' \leq d''$ if $d'_i \leq d''_i$ for every source i of Q , $d'_j \geq d''_j$ if j is a sink of Q , and $d'_k = d''_k$ for every vertex k of Q which is neither a source nor a sink.

For two sub-dimension vectors d' and d'' with $d' \leq d''$, the polynomial $f^{d'}$ divides $f^{d''}$. A simple calculation shows that in this case $\text{coeff}_A(f^{d'}, C)$ contains $\text{coeff}_A(f^{d''}, C)$.

Let d' and d'' be two sub-dimension vectors of d with $d' \leq d''$. Then,

$$\begin{aligned} \theta(d'') - \theta(d') &= \sum_i (d''_i - d'_i) \left(\sum_{\alpha: j \rightarrow i} d_j - \sum_{\alpha: i \rightarrow k} d_k \right) \\ &= \sum_{i \text{ source}} (d'_i - d''_i) \sum_{\alpha: i \rightarrow k} d_k + \sum_{i \text{ sink}} (d''_i - d'_i) \sum_{\alpha: j \rightarrow i} d_j \leq 0, \end{aligned}$$

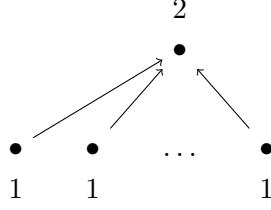
and thus $\theta(d') \geq \theta(d'')$. So, if d' is θ -forbidden, d'' will be “even more forbidden”. This shows that we can restrict ourselves to the *minimal* θ -forbidden sub-dimension vectors d' of d . This substantially reduces the computational effort, in particular if Q is bipartite.

Let Q be a bipartite quiver, let d be a coprime dimension vector and let θ be the canonical stability condition. Assume that d is θ -coprime. Let A be as usual. Under these circumstances, Theorem 3.2.1 reads like this:

Corollary 3.3.1. The Chow ring $A(M^\theta)$ is isomorphic to A/\mathfrak{a} , where \mathfrak{a} is generated by the linear relation l and the tautological relations $\tau_\lambda(d')$. Here, $\lambda \in \Delta$ and d' runs through all *minimal* θ -forbidden sub-dimension vectors of d .

Subspace quivers

Let (Q, d) be the following quiver setting



with sources q_1, \dots, q_m , where $m = 2r + 1$ is an odd number, and sink s . The quiver is the m -subspace quiver from Example 1.1.1. Let the stability condition θ be defined by

$$\theta(a) = ma_s - 2(a_{q_1} + \dots + a_{q_m})$$

for $a \in \mathbb{Z}^{Q_0}$. This is the canonical stability condition for (Q, d) described above. As m is an odd number, it is immediate that d is θ -coprime.

The moduli space $M^\theta := M^\theta(Q, d)$ can easily be identified with the space of m ordered points on the projective line, of which no more than r coincide, up to the natural PGL_2 -action.

We want to calculate the ring $A(M^\theta) = A/\mathfrak{a}$ from Theorem 3.2.1. We have $C = \mathbb{Q}[x_1, \dots, x_m, y_1, y_2]$ if we rename $t_{q_i,1} =: x_i$ and $t_{s,j} =: y_j$, for convenience. Then, A is the subring $\mathbb{Q}[x_1, \dots, x_m, y, z]$, where $y = y_1 + y_2$ and $z = y_1 y_2$. The group W is just S_2 that acts on C by swapping y_1 and y_2 . As in the general setting, we fix some integers a_i and b such that $\sum_{i=1}^m a_i + 2b = 1$, thus

$$l = \sum_{i=1}^m a_i x_i + by \in A.$$

By Corollary 3.3.1, we have to find the minimal forbidden sub-dimension vectors of d' . We distinguish between three cases (the last of which does not appear at all).

- (a) $d'_s = 0$. Then, d' is minimal forbidden if and only if there exists exactly one index $1 \leq i \leq m$ such that $d'_{q_i} = 1$.
- (b) $d'_s = 1$. We obtain $\theta(d') = m - 2\#\{i \mid d'_{q_i} = 1\}$, which is negative if and only if $\#\{i \mid d'_{q_i} = 1\} > r$. Moreover, d' is minimal forbidden if and only if the set $\{i \mid d'_{q_i} = 1\}$ has precisely $r + 1$ elements.
- (c) $d'_s = 2$. In this case, $\theta(d') \geq 0$.

Now we turn to the elements $f^{d'}$ for some fixed minimal forbidden sub-dimension vector $d' \leq d$. Distinguish again between the cases (a) and (b).

- (a) If $d'_s = 0$, then there exists a unique i with $d'_{q_i} = 1$. Then, $f^{d'} = (y_1 - x_i)(y_2 - x_i) = z - yx_i + x_i^2$. In particular, $f^{d'} \in A$.
- (b) In case $d'_s = 1$, the set I of all i with $d'_{q_i} = 1$ has exactly $r + 1$ elements. Let $I = \{i_1, \dots, i_{r+1}\}$. We obtain

$$f^{d'} = \prod_{\nu=1}^{r+1} (y_2 - x_{i_\nu}) = \sum_{j=0}^{r+1} (-1)^j \sigma_j^{(I)} y_2^{r+1-j},$$

where $\sigma_j^{(I)}$ denotes the j -th elementary symmetric function on the variables $x_{i_1}, \dots, x_{i_{r+1}}$.

We know that C decomposes as $C = A \cdot y_2 \oplus A \cdot 1$. By definition, y_1 and y_2 are roots of the polynomial $p(t) = t^2 - yt + z \in A[t]$. We want to give a presentation of y_2^k as a linear combination of y_2 and 1 with coefficients in A . Let

$$y_2^k = \alpha_k y_2 + \beta_k$$

for all $k \geq 0$. Then, we obtain that (α_k) and (β_k) are sequences (a_k) of homogeneous elements in A that fulfill the recursion

$$a_k = ya_{k-1} - za_{k-2}$$

for $k \geq 2$, but with different initial values $(\alpha_0, \alpha_1) = (0, 1)$ and $(\beta_0, \beta_1) = (1, 0)$. The sequences (a_k) satisfying the two-term recursion above can be described with linear algebra methods: Consider the matrix

$$M = \begin{pmatrix} 0 & 1 \\ -z & y \end{pmatrix}.$$

Let $v^{(k)} := (a_k, a_{k+1})^T$ for all $k \geq 0$. Then obviously $v^{(k)} = M^k \cdot v^{(0)}$.

Back to our relations. We get for d' and I as in case (b) above

$$f^{d'} = \left(\sum_{j=0}^{r+1} (-1)^j \sigma_j^{(I)} \alpha_{r+1-j} \right) y_2 + \left(\sum_{j=0}^{r+1} (-1)^j \sigma_j^{(I)} \beta_{r+1-j} \right).$$

As d' is forbidden, we obtain that $f^{d'} = 0$ and this implies

$$0 = \sum_{j=0}^{r+1} (-1)^j \sigma_j^{(I)} \alpha_{r+1-j} \quad \text{and} \quad 0 = \sum_{j=0}^{r+1} (-1)^j \sigma_j^{(I)} \beta_{r+1-j}$$

in A/\mathfrak{a} . Thus, the ideal \mathfrak{a} from 3.2.1 is generated by the expressions

(l) $\sum_{i=1}^m a_i x_i + by,$

(a) $z - yx_i + x_i^2$ for all $1 \leq i \leq m,$

(b1) $\sum_{j=0}^{r+1} (-1)^j \sigma_j^{(I)} \alpha_{r+1-j},$ and

(b2) $\sum_{j=0}^{r+1} (-1)^j \sigma_j^{(I)} \beta_{r+1-j}$ for all $I \subseteq \{1, \dots, m\}$ with $r+1 := \#I > r.$

Here, a_i and b are integers such that $a_1 + \dots + a_m + 2b = 1$. Theorem 3.2.1 states that there exists an isomorphism $\mathbb{Q}[x_1, \dots, x_m, y, z]/\mathfrak{a} \cong A(M^\theta)$ sending $x_i \mapsto c_1(U_{q_i})$, $y \mapsto c_1(U_s)$, and $z \mapsto c_2(U_s)$, where U is the universal representation corresponding to the character of weight one according to the integers a_i and b .

We see that the presentation of the Chow ring depends essentially on the choice of a character of weight one. In this case, there are two “reasonable choices”.

Let $a_i = \delta_{i,m}$ the Kronecker symbol and let $b = 0$. Then, the generator $l = \sum_i a_i x_i + by$ is just $l = x_m$. We consider the image of $z + yx_i + x_i^2$ in the quotient $A/A \cdot l$. For $i = m$, we obtain that $z + yx_i + x_i^2$ maps to z . Therefore, we have an isomorphism of $A(M^\theta)$ to $\mathbb{Q}[x_1, \dots, x_{m-1}, y]/\mathfrak{a}$, where \mathfrak{a} is generated by expressions

- (a) $yx_i - x_i^2$ for all $1 \leq i \leq m - 1$,
- (b1) $\sum_{j=0}^{r+1} (-1)^j \sigma_j^{(I)} \alpha_{r+1-j}$, and
- (b2) $\sum_{j=0}^{r+1} (-1)^j \sigma_j^{(I)} \beta_{r+1-j}$ for all $I \subseteq \{1, \dots, m\}$ with $r + 1$ elements.

Here, we have (formally) $x_m := 0$ and $(\alpha_k), (\beta_k)$ fulfill the recursions $\alpha_k = y\alpha_{k-1}$ and $\beta_k = y\beta_{k-1}$ for all $k \geq 2$. As $\beta_1 = 0$, we obtain that all β_k vanish if $k \geq 1$, so $\beta_k = \delta_{k,0}$. As $\alpha_1 = 1$, we get $\alpha_k = y^{k-1}$ for $k \geq 1$. By definition, we have $\alpha_0 = 0$ and thus, $\alpha_k = (1 - \delta_{k,0})y^{k-1}$. Therefore, the expressions (b1) and (b2) read as follows: For all subsets $I \subseteq \{1, \dots, m\}$ with $r + 1$ elements, we have

- (b1) $\sum_{j=0}^r (-1)^j \sigma_j^{(I)} y^{r-j}$ and
- (b2) $\prod_{i \in I} x_i$

lying in \mathfrak{a} . If $m \in I$, we can reformulate (b1). Let $I' := I - \{m\}$. We obtain $\sigma_j^{(I)} = \sigma_j^{(I')}$ for all $j \leq r$ and thus,

$$\sum_{j=0}^r (-1)^j \sigma_j^{(I)} y^{r-j} = \sum_{j=0}^r (-1)^j \sigma_j^{(I')} y^{r-j} = \prod_{i \in I'} (y - x_i).$$

This implies that for $m \notin I$, the product $\prod_{i \in I} (y - x_i)$ is also contained in the ideal \mathfrak{a} . But we can rewrite this as

$$\prod_{i \in I} (y - x_i) = \sum_{j=0}^{r+1} (-1)^j \sigma_j^{(I)} y^{r+1-j} = (-1)^{r+1} \prod_{i \in I} x_i + y \cdot \left(\sum_{j=0}^r (-1)^j \sigma_j^{(I)} y^{r-j} \right)$$

and thereby, the expressions of type (b2) are already contained in \mathfrak{a} if all of type (b1) are. In summary, we get the following presentation of $A(M^\theta)$:

Corollary 3.3.2. The ring $A(M^\theta)$ is isomorphic to $\mathbb{Q}[x_1, \dots, x_{m-1}, y]/\mathfrak{a}$, where \mathfrak{a} is the ideal generated by the expressions

- (a) $x_i(y - x_i)$ for all $1 \leq i \leq m - 1$,
- (b1') $\prod_{i \in I'} (y - x_i)$, and
- (b1'') $\sum_{j=0}^r (-1)^j \sigma_j^{(I)} y^{r-j}$

for all $I', I \subseteq \{1, \dots, m - 1\}$ with $\#I' = r$ and $\#I = r + 1$.

This presentation is similar to the one in the paper of Hausmann and Knutson (c.f. [HK98]). We describe it briefly: Let \mathbb{S} be the direct product of m copies of the 2-sphere. Consider the diagonal action of SO_3 on \mathbb{S} . Identifying \mathfrak{so}_3^\vee with \mathbb{R}^3 , this action has the moment map $\mu(p) = \sum_{i=1}^m p_i$. The symplectic reduction $\mathrm{Pol} := \mu^{-1}(0)/\mathrm{SO}_3$ is called the polygon space. This space can be identified with $M^\theta(\mathbb{C})$ regarded as a symplectic manifold.

Theorem 3.3.3 (Hausmann-Knutson). The cohomology ring $H^*(\text{Pol})$ is isomorphic to the quotient $\mathbb{Z}[V_1, \dots, V_{m-1}, R]/\mathcal{I}$ of a polynomial ring with generators in degree 2, and the ideal \mathcal{I} is generated by the expressions

$$(R1) \quad V_i^2 + RV_i \text{ for } i = 1, \dots, m-1,$$

$$(R2) \quad \prod_{i \in L} V_i \text{ for all subsets } L \subseteq \{1, \dots, m-1\} \text{ with } \#L \geq r \text{ and}$$

$$(R3) \quad \sum_{S \subseteq L, \#S \leq r-1} R^{\#(L-S)-1} \prod_{i \in S} V_i \text{ for all subsets } L \subseteq \{1, \dots, m-1\} \text{ with } \#L \geq r+1.$$

We should remark that this is the equal weight version of the main result of [HK98]. In the actual theorem, a description of the cohomology ring of the polygon space associated to an m -tuple of positive weights is given. This result is proved by embedding the polygon space into a transverse self-intersection of a toric subvariety in some toric variety.

Although the presentation in Theorem 3.3.3 strongly resembles the presentation from Corollary 3.3.2, it is hard to show that these two rings are isomorphic (over the rationals with a doubling of the grades).

The other “reasonable choice” is the following: We put $a_i = 1$ for all i and $b = -r$. This choice comes from viewing the $(2r+1)$ -subspace quiver as a covering quiver of a generalized Kronecker quiver with dimension vector $(2, 2r+1)$.

In order to make things simpler, we calculate in the quotient modulo \mathfrak{a} . Let $\sigma_j = \sigma_j(x_1, \dots, x_m)$ for $j = 1, 2$. The expression (1) reads $\sigma_1 - ry = 0$. Replacing y with $1/r \cdot \sigma_1$, we get for (a)

$$z - \frac{1}{r}\sigma_1 x_i + x_i^2 = 0,$$

and summing over all i yields

$$0 = mz - \frac{1}{r}\sigma_1^2 + \sigma_1(x_1^2, \dots, x_m^2) = mz - \frac{1}{r}\sigma_1^2 + (\sigma_1^2 - 2\sigma_2),$$

or equivalently $z = \frac{1-r}{mr}\sigma_1^2 + \frac{2}{m}\sigma_2$. We get:

Corollary 3.3.4. The ring $A(M^\theta)$ is isomorphic to $\mathbb{Q}[x_1, \dots, x_m]/\mathfrak{a}$, where \mathfrak{a} is the ideal generated by the expressions

$$(a) \quad (1-r)\sigma_1^2 + 2r\sigma_2 - m\sigma_1 x_i + qm x_i^2 \text{ for all } 1 \leq i \leq m,$$

$$(b1) \quad \sum_{j=0}^{r+1} (-1)^j \sigma_j^{(I)} \alpha_{r+1-j}, \text{ and}$$

$$(b2) \quad \sum_{j=0}^{r+1} (-1)^j \sigma_j^{(I)} \beta_{r+1-j}$$

for all $I \subseteq \{1, \dots, m\}$ with $r+1$ elements and (α_k) and (β_k) are sequences (a_k) that satisfy

$$a_k = \frac{1}{r}\sigma_1 a_{k-1} - \left(\frac{1-r}{mr}\sigma_1^2 + \frac{2}{m}\sigma_2 \right) a_{k-2},$$

and $(\alpha_0, \alpha_1) = (0, 1)$, $(\beta_0, \beta_1) = (1, 0)$.

This might look more frightening than the presentation calculated before, but the latter has the advantage of preserving the natural S_m -action on the moduli space.

The first presentation can be used to read off the Poincaré polynomial of $\mathcal{A} := A(M^\theta)$. We observe that these defining relations are homogeneous of degree 2 for type (a) and of degree $\geq r$ for the other types (b1') and (b1''). We know that

$$\dim M^\theta = \dim R - \dim G + \dim \Gamma = 2m - (m + 4) + 1 = m - 3 = 2(r - 1).$$

By Poincaré-duality, $\mathcal{A}^i \cong \mathcal{A}^{2(r-1)-i}$ for every $0 \leq i \leq r - 1$, so we just have to determine the dimensions of $\mathcal{A}^0, \dots, \mathcal{A}^{r-1}$. But in these degrees, \mathcal{A} coincides with the quotient of the polynomial ring

$$\mathbb{Q}[x_1, \dots, x_{m-1}, y] / (yx_i - x_i^2 \mid 1 \leq i \leq m).$$

Now, we verify almost at once that $yx_1 - x_1^2, \dots, yx_{m-1} - x_{m-1}^2$ is a regular sequence, and thus, the Poincaré series of the latter ring computes as

$$\frac{(1 - t^2)^{m-1}}{(1 - t)^m} = \frac{(1 + t)^{m-1}}{1 - t} = \sum_{j=0}^{\infty} \left(\sum_{\nu=0}^{\min\{j, m-1\}} \binom{m-1}{\nu} \right) t^j = \sum_{j=0}^{\infty} \left(\sum_{\nu=0}^{\min\{j, 2r\}} \binom{2r}{\nu} \right) t^j.$$

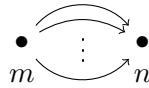
This proves a result of Kirwan (c.f. [Kir84]):

Corollary 3.3.5. The Poincaré polynomial of $A(M^\theta)$ is

$$\sum_{j=0}^{2(r-1)} \left(\sum_{\nu=0}^{\min\{j, 2(r-1)-j\}} \binom{2r}{\nu} \right) t^j.$$

Generalized Kronecker quivers

Let Q be the generalized Kronecker quiver with r arrows $\alpha_1, \dots, \alpha_r$ pointing from the source q to the sink s . Let $d = (m, n)$ with coprime positive integers n and m . In a picture:



Without loss of generality, we may assume that $m < n$ as the moduli space of the quiver with reverted arrows is isomorphic to the original moduli space (with an obvious modification of the stability condition). The canonical stability condition θ is given by

$$\theta(m', n') = r \cdot (mn' - nm').$$

As m and n are coprime, d is θ -coprime. The moduli space M^θ is often denoted $N(r; m, n)$ and called a Kronecker module. Its cohomology has already been studied by Drezet [Dre88] and also by Ellingsrud and Stromme [ES89].

The ring C is, in this case, given by $C = \mathbb{Q}[t_1, \dots, t_m, s_1, \dots, s_n]$, where $t_i := t_{q,i}$ and $s_j := t_{s,j}$. Furthermore, $A = \mathbb{Q}[x_1, \dots, x_m, y_1, \dots, y_n]$, where x_i is the i -th elementary symmetric polynomial in

the variables t_1, \dots, t_m and y_j is the j -th in s_1, \dots, s_n . The Weyl group W is the product $W = S_m \times S_n$ acting separately on the t_i and the s_j in the usual manner. Choose integers a and b with $am + bn = 1$. Then

$$l = ax_1 + by_1.$$

Let us work out the forbidden sub-dimension vectors. We see that a sub-dimension vector $d' = (m', n')$ of d is forbidden if and only if

$$n' < \left\lceil \frac{nm'}{m} \right\rceil.$$

The corresponding $f^{d'} \in C$ computes as

$$f^{d'} = \prod_{i=1}^{m'} \prod_{j=n'+1}^n (s_j - t_i)^r.$$

Moreover, the forbidden sub-dimension vector d' is minimal if and only if $n' = \lceil nm'/m \rceil - 1$. Fix a minimal forbidden sub-dimension vector $d' = (m', n')$. Note that different values for m' yield different values of the expression $\lceil nm'/m \rceil$, as $m < n$ by assumption. There are m different polynomials

$$f^{(m')} := f^{d'} = \prod_{i=1}^{m'} \prod_{j=\lceil nm'/m \rceil}^n (s_j - t_i)^r$$

for $m' = 1, \dots, m$. Letting $\tau_{(\mu, \nu)}(m') := \tau_{(\mu, \nu)}(d')$ be the tautological relations with respect to the basis $\{t^\mu s^\nu \mid (\mu, \nu) \in \Delta\}$, we obtain with Theorem 3.2.1 that $A(M^\theta)$ is isomorphic to A/\mathfrak{a} , and \mathfrak{a} is generated by

- (i) $ax_1 + by_1$ and
- (ii) $\tau_{(\mu, \nu)}(m')$ for $1 \leq m' \leq m$ and $(\mu, \nu) \in \Delta$.

In Drezet's paper [Dre88], a formula for the Betti numbers of Kronecker modules is given by a recursive formula, after showing that the cohomology with integral coefficients is torsion free and finitely generated. Ellingsrud and Stromme (c.f. [ES89]) obtain a presentation of the Chow ring as the quotient of A by the image $\rho(\mathfrak{c})$, where $\rho : C \rightarrow A$ is the symmetrization map (as in the proof of Theorem 3.2.1) and \mathfrak{c} is the ideal generated by l and all $wf^{d'}$. However, as ρ is not multiplicative but just A -linear, generators of \mathfrak{c} do not provide generators of $\rho(\mathfrak{c})$. A set of generators of $\rho(\mathfrak{c})$ is hard to calculate by actually using ρ and its cardinality grows with a factor of $m!n!$.

Let us turn to some special cases.

Example 3.3.6. Let $m = 1$ and $r \geq n$. Then, the moduli space M^θ can easily be identified with the Grassmannian $\text{Gr}_{r-n}(k^r)$. We choose $a = 1$ and $b = 0$ and obtain $x_1 \in \mathfrak{a}$. Therefore, $\mathcal{A} := A(M^\theta)$ is isomorphic to $\mathbb{Q}[y_1, \dots, y_n]/\mathfrak{a}'$ by sending x_1 to 0. The ideal \mathfrak{a}' is generated by the coefficients of $f^{(1)}(0, s_1, \dots, s_n)$ in terms of the basis s^λ with $\lambda_i \leq i - 1$. We get

$$f^{(1)}(0, s_1, \dots, s_n) = s_n^r.$$

Abbreviate $s := s_n$. We have $s^n = \sum_{i=1}^n (-1)^{i-1} y_i s^{n-i}$. Letting $(\beta_j^{(\nu)} \mid \nu)$ be sequences such that $s^\nu = \sum_{i=1}^n \beta_i^{(\nu)} s^{n-i}$ for all $\nu \geq 0$, we obtain that these sequences fulfill the recursion

$$\beta_j^{(\nu)} = \sum_{i=1}^n (-1)^{i-1} y_i \beta_j^{(\nu-i)}$$

for $\nu \geq n$, and that they have the initial values $\beta_j^{(\nu)} = \delta_{j,n-\nu}$ for $0 \leq \nu \leq n-1$. Using this terminology, the ideal \mathfrak{a}' is given as

$$\mathfrak{a}' = (\beta_1^{(r)}, \dots, \beta_n^{(r)}).$$

We can determine these $\beta_j^{(r)}$ using Linear Algebra methods. Let B be the $(n \times n)$ -matrix

$$B = \begin{pmatrix} 0 & 1 & & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ (-1)^{n-1} y_n & \dots & -y_2 & y_1 & \end{pmatrix}.$$

Let $v_j^{(\nu)} := (\beta_j^{(\nu)}, \dots, \beta_j^{(\nu-1+n)})^T$. Then, $v_j^{(\nu)} = B^\nu v_j^{(0)} = B^\nu e_j$, and thus, $\beta_j^{(r)}$ is the last entry of the vector $B^{r-n} e_j$.

The above description of the Chow ring \mathcal{A} is similar to the one we gave at the very beginning of this chapter: Let $c(t) = 1 + y_1 t + \dots + y_n t^n \in A[t]$. We can describe \mathcal{A} as the ring generated by y_1, \dots, y_n modulo the relations contained in the condition that the formal power series $c(t)^{-1} = \sum_i \delta_i t^i \in A[[t]]$ is actually a polynomial of degree at most $r - n$. This means \mathcal{A} is isomorphic to $\mathbb{Q}[y_1, \dots, y_n]/(\delta_{r-n+1}, \dots, \delta_r)$. We get $\delta_0 = 1$ and for $\nu > 0$,

$$\delta_\nu = (-1)^\nu \begin{vmatrix} y_1 & y_2 & \dots & y_\nu \\ 1 & y_1 & y_2 & \dots & y_{\nu-1} \\ 0 & 1 & y_1 & \ddots & \vdots \\ \vdots & & \ddots & \ddots & y_2 \\ 0 & \dots & 0 & 1 & y_1 \end{vmatrix} =: (-1)^{\nu-1} d_\nu$$

(defining $y_i := 0$ for $i > n$). Using Laplace's formula, we see that $d_\nu = \sum_{i=1}^n (-1)^{i-1} y_i d_{\nu-i}$ for $\nu \geq n$. Letting $d_{-n+1} = \dots = d_{-1} = 0$, this formula holds also true for every $\nu < 0$. Let $w^{(\nu)} := (d_{\nu-n+1}, \dots, d_\nu)^T$. We get that d_{r-n+j} is the last entry of $B^{r-n} w^{(j)}$. As $w^{(0)}, \dots, w^{(n)}$ is also a basis of $\mathbb{Q}[y_1, \dots, y_n]^n$, we have shown that

$$(\beta_1^{(r)}, \dots, \beta_n^{(r)}) = (d_{r-n+1}, \dots, d_r) = (\delta_{r-n+1}, \dots, \delta_r).$$

Example 3.3.7. Let $d = (2, 3)$ and $r = 3$. Here, we have $f^{(1)} = (s_3 - t_1)^3 (s_3 - t_2)^3$ and $f^{(2)} = (s_2 - t_1)^3 (s_3 - t_1)^3$. A basis of $C = \mathbb{Q}[t_1, t_2, s_1, s_2, s_3]$ considered as an $A = \mathbb{Q}[x_1, x_2, y_1, y_2, y_3]$ -module is given by the elements

$$t_2^{\lambda_2} s_2^{\mu_2} s_3^{\mu_3},$$

with $0 \leq \lambda_2, \mu_2 \leq 1$ and $0 \leq \mu_3 \leq 2$. Display the polynomials $f^{(1)}$ and $f^{(2)}$ as linear combinations of these basis vectors. We obtain presentations

$$f^{(m')} = \sum_{0 \leq \lambda_2, \mu_2 \leq 1} \sum_{0 \leq \mu_3 \leq 2} \tau_{\lambda_2, \mu_2, \mu_3}(m') \cdot t_2^{\lambda_2} s_2^{\mu_2} s_3^{\mu_3}$$

for $m' = 1, 2$. We choose a linear relation, i.e. we choose integers a and b with $2a + 3b = 1$. For example, let $a = -1$ and $b = 1$. The linear relation thus obtained is $l = y_1 - x_1$. In order to simplify the tautological relations, we calculate in the ring $A/A \cdot l$, meaning we replace x_1 by y_1 . Using SINGULAR (you really do not want to do this by hand), we obtain, after a couple of simplifications, that $A(M^\theta)$ is isomorphic to $\mathbb{Q}[x_2, y_1, y_2, y_3]/\mathfrak{a}$, where \mathfrak{a} is generated by

- $3x_2^2 - 3x_2y_2 + y_2^2 - y_1y_3$,
- $(3x_2 - 2y_2)y_3$,
- $x_2^3 - y_1y_2y_3 + y_3^2$,
- $-4x_2y_1 + y_1^3 + 3y_3$,
- $3x_2^2 - x_2y_1^2$,
- $3x_2^2 + x_2y_2 - y_1^2y_2$,
- $x_2y_1y_2 - 3y_2y_3$,
- $3y_1^2 - 5y_2y_3$, and
- $x_2^3 - x_2y_1y_3$.

Chapter 4

Chow Rings of Non-Commutative Hilbert Schemes

The cohomology of the Hilbert scheme $\text{Hilb}_d(\mathbb{A}^m)$ of d points in an m -dimensional affine space has been studied intensively by various authors (e.g. [Gro96], [Nak97] and [LS01]). The objective of this chapter is to investigate cohomological properties of a certain non-commutative generalization of these Hilbert schemes. The Hilbert scheme $\text{Hilb}_d(\mathbb{A}^m)$ parametrizes ideals of codimension d of the polynomial algebra $k[x_1, \dots, x_m]$. We are interested in the moduli space of *left*-ideals of codimension d in the free *non-commutative* algebra $k\langle x_1, \dots, x_m \rangle$. This is the most prominent example of a non-commutative Hilbert scheme.

The results of Chapter 4 are contained in the author's article [Fra13b], more precisely in the first two sections of it.

4.1 Non-Commutative Hilbert Schemes of a Loop Quiver

We basically keep the notation of [Rei05]. Fix positive integers d, m and n and vector spaces V of dimension n and W of dimension d . Define \widehat{R} as the vector space $\text{Hom}(V, W) \oplus \text{End}(W)^m$ and let G be the algebraic group $\text{Gl}(W)$ which acts on \widehat{R} via

$$g \cdot (f, \varphi_1, \dots, \varphi_m) = (gf, g\varphi_1g^{-1}, \dots, g\varphi_mg^{-1}).$$

An element $(f, \varphi) = (f, \varphi_1, \dots, \varphi_m)$ is called **stable** if $k\langle \varphi_1, \dots, \varphi_m \rangle f(V) = W$, i.e. the image of f generates W regarded as a representation of the free non-commutative algebra $A = k\langle x_1, \dots, x_m \rangle$ in m variables. On the set \widehat{R}^{st} of stable points of \widehat{R} , a geometric G -quotient

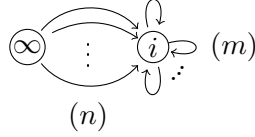
$$\pi : \widehat{R}^{\text{st}} \rightarrow \text{Hilb}_{d,n}^{(m)}$$

exists. It is even a principal G -bundle. The variety $\text{Hilb}_{d,n}^{(m)}$ is called a **non-commutative Hilbert scheme**. As m is fixed throughout this text, we sometimes suppress the dependency on m and write $\text{Hilb}_{d,n}$, for convenience. It is a smooth and irreducible variety of dimension $N := (m-1)d^2 + nd$. Its points parametrize A -subrepresentations of codimension d of the free representation A^n . Denote the image $\pi(f, \varphi)$ by $[f, \varphi]$.

On \widehat{R}^{st} , we consider the G -equivariant vector bundle $\widehat{R}^{\text{st}} \times W \rightarrow \widehat{R}^{\text{st}}$, the trivial bundle, equipped with the G -action $g \cdot ((f, \varphi), v) = ((gf, g\varphi g^{-1}), gv)$. This descends to a vector bundle \mathcal{U} of rank d on $\text{Hilb}_{d,n}$, meaning there exists a vector bundle \mathcal{U} on $\text{Hilb}_{d,n}$ such that $\pi^{-1}\mathcal{U}$ with its canonical G -action equals the G -bundle we have just described. The G -linear endomorphisms $\widehat{R}^{\text{st}} \times W \rightarrow \widehat{R}^{\text{st}} \times W$ mapping a point $((f, \varphi), v)$ to $((f, \varphi), \varphi_i v)$ descend to endomorphisms $\Phi_i : \mathcal{U} \rightarrow \mathcal{U}$. Choosing a basis e_1, \dots, e_n of V gives rise to G -linear sections $\widehat{R}^{\text{st}} \rightarrow \widehat{R}^{\text{st}} \times W$ defined by sending (f, φ) to $((f, \varphi), fe_i)$. These sections, in turn, induce sections s_1, \dots, s_n of \mathcal{U} .

The variety $\text{Hilb}_{d,n}$ is a so-called framed quiver moduli space (cf. [Nak96], [CB03], or [Rei08a]). Consider the m -loop quiver Q consisting of a single vertex i and m loops. A dimension vector for this quiver is just a natural number, say d . A representation of Q of dimension vector d is a d -dimensional vector space W together with m endomorphisms $\varphi_1, \dots, \varphi_m$. In the sense of section 1.3, every representation of Q is stable, when choosing the linear form $\theta : \mathbb{Z} \rightarrow \mathbb{Z}$ to be zero.

Define a quiver \widehat{Q}_n with two vertices ∞ and i and with n arrows pointing from ∞ to i and m loops at i . In a picture



For this quiver, we consider the dimension vector $(1, d)$. A representation of \widehat{Q}_n consists of a representation of Q of dimension vector d and additionally, a linear map f from an n -dimensional space V to W . When choosing the extended stability condition $\widehat{\theta}$ as in [Rei08a] according to $\theta = 0$, we obtain that such a representation (f, φ) is stable if and only if it is $\widehat{\theta}$ -stable.

4.2 Words and Forests

Let $\Omega := \Omega^{(m)}$ be the set of words on the alphabet $\{1, \dots, m\}$. The empty word will be denoted by ε .

Definition. A finite subset S of Ω is called a **(m -ary) tree** if it is closed under taking left subwords, that means, $w \in S$ provided $ww' \in S$ for some $w' \in \Omega$.

A **(m -ary) forest with n roots** is an n -tuple $S_* = (S_1, \dots, S_n)$ of (m -ary) trees.

Let $\mathcal{F}_{d,n} := \mathcal{F}_{d,n}^{(m)}$ be the set of m -ary forests with n roots and d nodes. Here, a forest $S_* = (S_1, \dots, S_n)$ is said to have d nodes if $\#S_1 + \dots + \#S_n = d$. For a word w with $w \in S_k$, we write $(k, w) \in S_*$.

A pair (k', w') consisting of an index $1 \leq k' \leq n$ and a word $w' \in \Omega$ is called **critical** for a forest S_* if either $w' = \varepsilon$ and $S_{k'} = \emptyset$ or if $w' \notin S_{k'}$ but there exists a word $w \in S_{k'}$ and a letter $i \in \{1, \dots, m\}$ with $w' = wi$. We define $C(S_*)$ to be the set of critical pairs of S_* . Its cardinality $c(S_*)$ equals $(m-1)\#S_* + n$.

We introduce an ordering on Ω , the so-called **lexicographic ordering**. For two words $w = i_1 \dots i_s$ and $w' = i'_1 \dots i'_t$, let p be the largest index such that $i_p = i'_p$. Formally, we define $p = 0$ if such an index does not exist. Define $w \leq w'$ if either $p = s$ (note that s is the length of w) or $i_{p+1} < i'_{p+1}$. This ordering can be extended to an ordering on the set of trees. Let S and S' be two distinct

trees. Define $S < S'$ if either $\#S > \#S'$ or $\#S = \#S'$ and, writing $S = \{w_1 < \dots < w_s\}$ and $S' = \{w'_1 < \dots < w'_s\}$, we obtain $w_{p+1} < w'_{p+1}$ for the maximal index p such that $w_p = w'_p$.

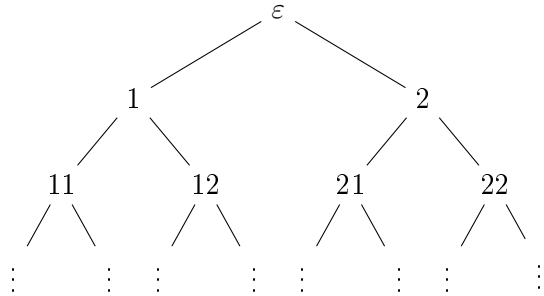
Let us enlarge this to an ordering on the set $\mathcal{F}_{d,n}$. For two distinct forests $S_*, S'_* \in \mathcal{F}_{d,n}$, let p be the largest index with $S_p = S'_p$ (again, $p = 0$ if $S_1 \neq S'_1$). Define $S_* < S'_*$ if $S_{p+1} < S'_{p+1}$.

Let S_* be a forest. Define $D(S_*)$ to be the set of all quadruples (k, w, k', w') consisting of indexes $1 \leq k, k' \leq n$ and words $w, w' \in \Omega$ satisfying

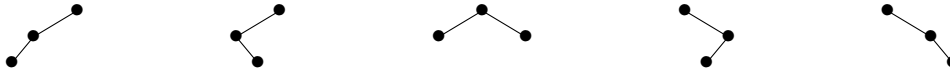
- $(k, w) \in S_*$,
- $(k', w') \in C(S_*)$, and
- $(k, w) < (k', w')$, that means either $k < k'$ or $k = k'$ and $w < w'$.

The cardinality of $D(S_*)$ will be denoted $d(S_*)$.

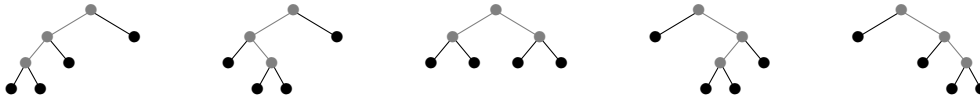
Example. Let us describe all $S_* \in \mathcal{F}_{d,n}^{(m)}$ for $m = 2, d = 3$ and $n = 1$. This example will accompany us throughout the text. When displaying Ω as follows



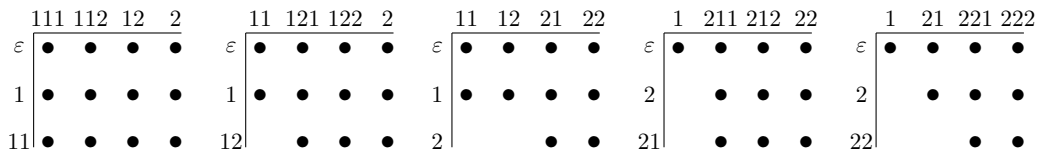
then the 2-ary - let us call them binary - trees with 3 nodes are exactly those:



Compare this to Stanley's list of descriptions of the Catalan numbers (cf. [Sta99, Ex. 6.19]). The above is part (c) of the list. When considering the sets $S_* \sqcup C(S_*)$, we also get a tree (or a forest, in general), more precisely a *plane* binary tree with $(m - 1)d + n = 7$ vertices. This is part (d) of Stanley's list. In the sketch below, the vertices belonging to S_* are displayed in gray:



Now, let us determine the sets $D(S_*)$. They are given by



and when viewing the missing entries as Young diagrams (after turning them upside down) fitting in some triangular shape, we obtain (vv) of [Sta99, Ex. 6.19].

4.3 A Cell Decomposition

For a word $w \in \Omega$, say $w = i_1 \dots i_s$, and a point $(f, \varphi) \in \widehat{R}$, define the endomorphism φ_w of W to be the composition $\varphi_{i_s} \dots \varphi_{i_1}$. In the same vein, define $\Phi_w := \Phi_{i_s} \dots \Phi_{i_1}$ to obtain an endomorphism of the bundle \mathcal{U} . Finally, define the section $s_{(k,w)}$ of \mathcal{U} to be $\Phi_w s_k$.

Definition. Let $S_* \in \mathcal{F}_{d,n}^{(m)}$ be a forest. Define U_{S_*} to be the subset of all $[f, \varphi] \in \text{Hilb}_{d,n}^{(m)}$ such that the vectors $\varphi_w f e_k$ with $(k, w) \in S_*$ form a basis of W .

Reineke shows that for every point $[f, \varphi]$ of $\text{Hilb}_{d,n}$ and every forest S'_* for which the tuple of vectors $(\varphi_w f e_k \mid (k, w) \in S'_*)$ is linearly independent, there exists a forest $S_* \in \mathcal{F}_{d,n}$ containing S'_* such that $[f, \varphi] \in U_{S_*}$. Furthermore, by expressing $[f, \varphi]$ in terms of the basis $\varphi_w f e_k$ with $(k, w) \in S_*$, he shows that U_{S_*} is isomorphic to an affine space. This implies:

Lemma 4.3.1. The variety $\text{Hilb}_{d,n}^{(m)}$ is covered by the open subsets U_{S_*} with $S_* \in \mathcal{F}_{d,n}^{(m)}$, each of which is isomorphic to an affine space of dimension $N = (m-1)d^2 + nd$.

Next, we define certain closed subsets of the U_{S_*} . These subsets will be the cells of the cell decomposition we are about to describe.

Definition. Let $S_* \in \mathcal{F}_{d,n}^{(m)}$ be a forest. Define Z_{S_*} to be the set of all $[f, \varphi] \in U_{S_*}$ such that for all critical pairs $(k', w') \in C(S_*)$, the vector $\varphi_{w'} f e_{k'}$ is contained in the span of all $\varphi_w f e_k$ with $(k, w) \in S$ and $(k, w) < (k', w')$.

In [Rei05], Theorem 3.6 gives a description of Z_{S_*} as a set in terms of the $U_{S'_*}$ for $S'_* < S_*$. It reads as follows:

Theorem 4.3.2. For all forests $S_* \in \mathcal{F}_{d,n}^{(m)}$, we obtain $Z_{S_*} = U_{S_*} \setminus \bigcup_{S'_* < S_*} U_{S'_*}$.

Let us equip Z_{S_*} with the reduced closed subscheme structure of U_{S_*} . Displaying $[f, \varphi]$ in terms of the basis $\varphi_w f e_k$ with $(k, w) \in S_*$ shows that Z_{S_*} is also an affine space. Precisely:

Lemma 4.3.3. The closed subset Z_{S_*} , viewed as a reduced closed subscheme of U_{S_*} , is isomorphic to an affine space of dimension $d(S_*)$.

These results culminate in a main result of [Rei05], the existence of a cell decomposition of $\text{Hilb}_{d,n}$. By definition, a cell decomposition of a variety is a descending sequence of closed subsets such that the successive complements are isomorphic to affine spaces. Define $A_{S_*} := \text{Hilb}_{d,n} - \bigcup_{S'_* < S_*} U_{S'_*}$. Enumerating the forests of $\mathcal{F}_{d,n}$ lexicographically, say $S_*^1 < \dots < S_*^u$, and abbreviating $A_i := A_{S_*^i}$, we obtain a filtration

$$\text{Hilb}_{d,n} = A_1 \supseteq A_2 \supseteq \dots \supseteq A_u \supseteq A_{u+1} := \emptyset$$

by closed subsets satisfying $A_i - A_{i+1} = Z_{S_*^i}$. Cutting a long story short:

Theorem 4.3.4. The variety $\text{Hilb}_{d,n}^{(m)}$ has a cell decomposition parametrized by forests $S_* \in \mathcal{F}_{d,n}^{(m)}$, whose cells Z_{S_*} are of respective dimensions $d(S_*)$.

An immediate application yields a basis of the Chow group of $\text{Hilb}_{d,n}$. Denote by \mathcal{Z}_{S_*} the closure of Z_{S_*} in $\text{Hilb}_{d,n}$ equipped with the reduced closed subscheme structure. As \mathcal{Z}_{S_*} is irreducible, it becomes a closed subvariety of $\text{Hilb}_{d,n}$.

Corollary 4.3.5. The Chow group $A_*(\text{Hilb}_{d,n}^{(m)})$ is the free abelian group with basis $[\mathcal{Z}_{S_*}]$ for $S_* \in \mathcal{F}_{d,n}^{(m)}$.

Example (continued). Again, let $m = 2$, $d = 3$ and $n = 1$. We describe the cells Z_{S_*} belonging to the binary trees S_* with 3 nodes. A point of \widehat{R} may be viewed as a triple (v, A, B) , where $v \in \mathbb{k}^3$ and A and B are (3×3) -matrices. Write $[v, A, B]$ for its image in the non-commutative Hilbert scheme. The cells are

$$\begin{aligned} Z_{\nearrow} &= \{[v, A, B] \mid v, Av, A^2v \text{ basis of } \mathbb{k}^3\}, \\ Z_{\leftarrow} &= \{[v, A, B] \mid v, Av, BAv \text{ basis of } \mathbb{k}^3 \text{ and } A^2v \in \langle v, Av \rangle\}, \\ Z_{\wedge} &= \{[v, A, B] \mid v, Av, Bv \text{ basis of } \mathbb{k}^3 \text{ and } A^2v, BAv \in \langle v, Av \rangle\}, \\ Z_{\succ} &= \{[v, A, B] \mid v, Bv, ABv \text{ basis of } \mathbb{k}^3 \text{ and } Av \in \langle v \rangle\}, \text{ and} \\ Z_{\searrow} &= \{[v, A, B] \mid v, Bv, B^2v \text{ basis of } \mathbb{k}^3, Av \in \langle v \rangle, \text{ and } ABv \in \langle v, Bv \rangle\}. \end{aligned}$$

Their dimensions allow us to determine the Poincaré polynomial $\sum_i \dim_{\mathbb{Q}}(A_i(\text{Hilb}_{d,n}^{(m)}))t^i$ at once. It reads $t^{12} + t^{11} + 2t^{10} + t^9$.

4.4 Another Basis of the Chow Group

We are interested in the ring structure on the Chow group of the non-commutative Hilbert scheme $\text{Hilb}_{d,n}^{(m)}$. It turns out that computing the intersection product of two cell closures is rather difficult. We would therefore like to find another basis that is better adapted to the multiplication. This basis will be provided by Chern classes of the universal bundle \mathcal{U} which we have already introduced in the previous section.

Let $S_* \in \mathcal{F}_{d,n}$ be a forest. Order the words of the trees lexicographically, i.e. $S_k = \{w_{k,1} < \dots < w_{k,d_k}\}$. Consider all pairs $(k, w) \in S_*$ and order them lexicographically, too. This gives

$$(1, w_{1,1}) < \dots < (1, w_{1,d_1}) < \dots < (n, w_{n,1}) < \dots < (n, w_{n,d_n})$$

and we denote these pairs as $x_1 < \dots < x_d$. This means $(k, w_{k,\nu}) = x_{d_1+\dots+d_{k-1}+\nu}$. For a critical pair $x' = (k', w')$ of S_* , let $j = j_{S_*}(x')$ be the maximal index such that $x_j < x'$. Formally, let $j = 0$ if such an index does not exist.

We express j in a slightly different way. If $w' = \varepsilon$, then $j = d_1 + \dots + d_{k'-1}$. Otherwise, it is $j = d_1 + \dots + d_{k'-1} + \nu$, where ν is the maximal index such that $w_\nu < w'$ (and ν is not zero in this case, but possibly $d_{k'}$).

As $D(S_*)$ is clearly in bijection to the disjoint union of the sets $\{(x_1, x'), \dots, (x_{j(x')}, x')\}$, with x' ranging over all critical pairs of S_* , we see that $\sum_{x' \in C(S_*)} j(x') = d(S_*)$. Define $i(x') := i_{S_*}(x')$ to be $d - j(x')$. We will show the following:

Theorem 4.4.1. For all forests $S_* \in \mathcal{F}_{d,n}^{(m)}$, we have

$$\prod_{x' \in C(S_*)} c_{i(x')}(\mathcal{U}) \cap [\text{Hilb}_{d,n}^{(m)}] = [\mathcal{L}_{S_*}] + \sum_{\substack{S'' > S_*, \\ d(S'')=d(S_*)}} n_{S_*, S''} [\mathcal{L}_{S''}]$$

for some positive integers $n_{S_*, S''}$.

Recall the section $s_x = s_{(k,w)}$ associated to any pair $x = (k, w)$ consisting of an index $1 \leq k \leq n$ and a word w . For a forest S_* , define D_{S_*} as the intersection of the degeneracy loci

$$D_{S_*} = \bigcap_{x' \in C(S_*)} D_{S_*}(x'),$$

where $D_{S_*}(x') := D(s_{x_1}, \dots, s_{x_{j(x')}})$ is the degeneracy locus as defined in Section 2.3 (cf. also [Ful98, Chap. 14]). As all degeneracy loci possess a natural structure of a closed subscheme of $\text{Hilb}_{d,n}$, the subset D_{S_*} does, too.

Lemma 4.4.2. The underlying closed subset of D_{S_*} is $A_{S_*} = \text{Hilb}_{d,n}^{(m)} - \bigcup_{S'_* < S_*} U_{S'_*}$.

Proof. Order the pairs of S_* , i.e. $x_1 = (k_1, w_1) < \dots < x_d = (k_d, w_d)$. A point $[f, \varphi]$ lies in D_{S_*} if and only if the vectors

$$\varphi_{w_1} f e_{k_1}, \dots, \varphi_{w_{j(x')}} f e_{k_{j(x')}}, \varphi_{w'} f e_{k'}$$

are linearly dependent for all critical pairs $x' = (k', w')$ of S_* . Let $S'_* \in \mathcal{F}$ with $S'_* < S_*$, say $S'_* = \{x'_1 < \dots < x'_d\}$. Define p to be the maximal index such that $x_p = x'_p$. We have $x'_{p+1} < x_{p+1}$ and therefore $x'_{p+1} \notin S_*$. We write x'_{p+1} as (k', w') , this means $w' \notin S_{k'}$. If $w' = \varepsilon$, then $S_{k'} = \emptyset$ and if w' is not the empty word, we write $w' = wi$ for some $w \in S'_{k'}$. As $(k', w) < x'_{p+1}$, we obtain $w \in S_{k'}$. This means x'_{p+1} is a critical pair for S'_* . Moreover, we get $j(x'_{p+1}) = p$. Therefore, the vectors

$$\varphi_{w_1} f e_{k_1}, \dots, \varphi_{w_p} f e_p, \varphi_{w'} f e_{k'}$$

are linearly dependent and this implies that $[f, \varphi]$ is not contained in $U_{S'_*}$.

Conversely, assume that $[f, \varphi]$ does not belong to the union $\bigcup_{S'_* < S_*} U_{S'_*}$. Let $x' = (k', w')$ be a critical pair for S_* . Let $j := j(x')$ and write $x_j = (k', w_{k', \nu})$. Consider the forest S'_* consisting of $S'_k := S_k$ for all $k < k'$, of

$$S'_{k'} = \{w_{k', 1} < \dots < w_{k', \nu} < w'\}$$

and of $S'_k = \emptyset$ for $k > k'$. Assume that the vectors $\varphi_{w_1} f e_{k_1}, \dots, \varphi_{w_j} f e_{k_j}, \varphi_{w'} f e_{k'}$ were linearly independent. By [Rei05, Lemma 3.2], there would exist a forest S''_* containing S'_* such that $[f, \varphi]$ belongs to $U_{S''_*}$. But this forest would fulfill $S''_* < S_*$. A contradiction. \square

Example (continued). Let us determine the underlying closed subsets A_{S_*} of the D_{S_*} in the - by now well known - case $m = 2$, $d = 3$ and $n = 1$. We have

$$\begin{aligned} A_{\curvearrowright} &= \text{Hilb}_{3,1}^{(2)}, \\ A_{\curvearrowleft} &= \{[v, A, B] \mid v, Av, A^2v \text{ linearly dependent}\}, \\ A_{\curvearrowright\curvearrowleft} &= \{[v, A, B] \mid v, Av, A^2v \text{ and } v, Av, BAv \text{ both linearly dependent}\}, \\ A_{\curvearrowleft\curvearrowright} &= \{[v, A, B] \mid v, Av \text{ linearly dependent}\}, \text{ and} \\ A_{\curvearrowleft\curvearrowleft} &= \{[v, A, B] \mid v, Av \text{ and } v, Bv, ABv \text{ both linearly dependent}\}. \end{aligned}$$

We can easily see that the successive complements are, indeed, the cells Z_{S_*} .

For general reasons (cf. [Ful98, Thm. 14.4]), we know that every irreducible component of $D_{S_*}(x')$ has dimension at least $N - i(x')$. We will show that, in fact, equality holds.

Lemma 4.4.3. Let $T_* = \{x_1 < \dots < x_j < x'\}$ be a forest. Then, the closed subset $D_{T_*} = D(s_{x_1}, \dots, s_{x_j}, s_{x'})$ has pure dimension $N - (d - j)$ (or is empty).

Proof. The proof proceeds by induction on j . In the case $j = 0$, the forest T_* equals $\{(k', \varepsilon)\}$ for an index $1 \leq k' \leq n$. Choose a forest $S_* \in \mathcal{F}_{d,n}$ such that $(k', \varepsilon) \in C(S_*)$. If $n = 1$, such a forest does not exist and D_{T_*} is empty. Otherwise, $D_{T_*} \cap U_{S_*} \neq \emptyset$. We apply the isomorphism $U_{S_*} \cong \mathbb{A}^N$ from [Rei05], provided by the functions $\lambda_{x,x'}$ on U_{S_*} for every $x' = (k', w') \in C(S_*)$ and every $x \in S_*$. By definition, $\lambda_{x,x'}[f, \varphi]$ is the coefficient occurring in the linear combination

$$\varphi_{w'} f e_{k'} = \sum_{x=(k,w) \in S_*} \lambda_{x,x'}[f, \varphi] \cdot \varphi_w f e_k$$

for $x' = (k', w') \in C(S_*)$. The closed subscheme $D_{T_*} \cap U_{S_*}$ is defined by annihilation of all functions $\lambda_{x,(k',\varepsilon)}$. This describes an affine space of dimension $N - d$.

Assume that $j > 0$. Let $T_*' := \{x_1 < \dots < x_j\}$. This is also a forest. By induction hypothesis, the closed subset $D_{T_*'}$, which is contained in D_{T_*} , has pure dimension $N - (d - j) - 1$ (or is empty). But as we know that every irreducible component of D_{T_*} has dimension at least $N - (d - j)$, it suffices to show that $D_{T_*} \cap U_{S_*}$ has pure dimension $N - (d - j)$ for every forest $S_* \in \mathcal{F}_{d,n}$ containing T_*' . For $D_{T_*} \cap U_{S_*}$ not being empty, we require $x' \notin S_*$. This means that x' is a critical pair for S_* . Via the isomorphism $U_{S_*} \cong \mathbb{A}^N$ described above, $D_{T_*} \cap U_{S_*}$ is given by the ideal generated by all functions $\lambda_{x,x'}$ with $x \in S_*$ and $x > x'$. This describes an affine space of dimension $N - (d - j)$. \square

The above lemma implies, using Proposition 2.3.4 (see also [Ful98, Ex. 14.4.2]), that the cycle $[D_{S_*}(x')]$ associated to the closed subscheme $D_{S'}(x')$, regarded as an element of $A_{N-i(x')}(\text{Hilb}_{d,n})$, equals $c_{i(x')}(\mathcal{Z}) \cap [\text{Hilb}_{d,n}]$. We use this observation to prove Theorem 4.4.1.

Proof of Theorem 4.4.1. By the cell decomposition, we obtain that $A_{S_*} = Z_{S_*} \cup \bigcup_{S_*'' > S_*} Z_{S_*}'' = \mathcal{Z}_{S_*} \cup \bigcup_{S_*'' > S_*} \mathcal{Z}_{S_*}''$. The proper components of the intersection of the $D_{S_*}(x')$ with $x' \in C(S_*)$ are

among those $\mathcal{Z}_{S''}$ with $S'' \geq S_*$ and $d(S_*) = d(S'')$. Hence, using [Ful98, Ex. 8.2.1], there are positive integers n_{S_*, S_*} and $n_{S_*, S''}$ such that

$$\prod_{x' \in C(S_*)} c_{i(x')}(\mathcal{U}) \cap [\text{Hilb}_{d,n}] = \prod_{x' \in C(S_*)} [D_{S_*}(x')] = n_{S_*, S_*}[\mathcal{Z}_{S_*}] + \sum_{\substack{S'' > S_*, \\ d(S'')=d(S_*)}} n_{S_*, S''}[\mathcal{Z}_{S''}].$$

It remains to prove that the coefficient n_{S_*, S_*} is 1. In order to do so, it suffices to prove that $D_{S_*} \cap U_{S_*} = Z_{S_*}$ as schemes. As mentioned above, Reineke shows that an isomorphism $U_{S_*} \rightarrow \mathbb{A}^N$ (with $N := (m-1)d^2 + nd$) is given by the functions $\lambda_{x, x'}$ with $x \in S_*$ and $x' \in C(S_*)$ assigning to every point $[f, \varphi]$ of U_{S_*} the coefficient $\lambda_{x, x'}[f, \varphi]$ that occurs displaying $\varphi_{w'} f e_{k'}$ as a linear combination

$$\varphi_{w'} f e_{k'} = \sum_{x=(k, w) \in S_*} \lambda_{x, x'}[f, \varphi] \cdot \varphi_w f e_k$$

where $x' = (k', w')$. Over U_{S_*} , the bundle \mathcal{U} trivializes. Moreover, for every pair $x_0 = (k_0, w_0)$, the sections s_{x_0} of \mathcal{U} correspond to the sections of the trivial rank d bundle on U_{S_*} assigning to $[f, \varphi]$ the matrix $(a_{x, x_0} \mid x \in S_*)$ of coefficients of the linear combination $\varphi_{w_0} f e_{k_0} = \sum_{x=(k, w) \in S_*} a_{x, x_0} \varphi_w f e_k$. Therefore, enumerating $S_* = \{x_1 < \dots < x_d\}$, the section s_{x_i} restricted to \mathbb{A}^N maps a matrix λ to the i -th coordinate vector and $s_{x'}$ maps λ to the vector $(\lambda_{x_1, x'}, \dots, \lambda_{x_d, x'})$. Under the isomorphism $U_{S_*} \rightarrow \mathbb{A}^N$, the degeneracy locus is therefore given by the vanishing of all $j(x')$ -minors of the matrices

$$\begin{pmatrix} 1 & & & \lambda_{x_1, x'} \\ & \ddots & & \vdots \\ & & 1 & \lambda_{x_{j(x')}, x'} \\ & & & \vdots \\ & & & \lambda_{x_d, x'} \end{pmatrix}.$$

Thus, locally on $U_{S_*} \cong \mathbb{A}^N$, the degeneracy locus D_{S_*} is given by the ideal generated by the coordinate functions $\lambda_{x_{j(x')+1}, x'}, \dots, \lambda_{x_d, x'}$ with x' ranging over all critical pairs of S_* . It is therefore an affine space of dimension $d(S_*)$ and thus isomorphic to Z_{S_*} . \square

Remark 4.4.4. Fixing the notation as in Theorem 4.4.1, we are able to determine the numbers $n_{S_*, S''}$ - at least in principle. They are given as intersection multiplicities

$$n_{S_*, S''} = i\left(\mathcal{Z}_{S''}, D_{S_*}(x'_1) \dots D_{S_*}(x'_r); \text{Hilb}_{d,n}^{(m)}\right)$$

as defined in [Ful98, Ex. 8.2.1]. Here, $\{x'_1, \dots, x'_r\} = C(S'')$. As the non-commutative Hilbert scheme is non-singular, it is also Cohen-Macaulay. Applying Lemma 4.4.3 and [Ful98, Ex. 14.4.2], we obtain that every $D_{S_*}(x')$ is Cohen-Macaulay, too. Thus, Proposition 2.2.1 (see also [Ful98, Prop. 8.2]) implies that

$$n_{S_*, S''} = l\left(\mathcal{O}_{D_{S_*}, \mathcal{Z}_{S''}}\right) = l\left(\mathcal{O}_{D_{S_*} \cap U_{S''}, Z_{S''}}\right),$$

D_{S_*} being equipped with its natural scheme structure.

We reformulate Theorem 4.4.1 slightly. Let $S_* \in \mathcal{F}_{d,n}$ be a forest. Enumerate $S_* = \{x_1 < \dots < x_d\}$ and $C(S_*) = \{x'_1, \dots, x'_{(m-1)d+n}\}$. Define j_ν to be the index $j_{S_*(x'_\nu)}$. One can show that this induces a bijection

$$\mathcal{F}_{d,n}^{(m)} \rightarrow \mathcal{J}_{d,n}^{(m)},$$

where the right hand side is the set of tuples $(j_1, \dots, j_{(m-1)d+n})$ of integers $0 \leq j_1 \leq \dots \leq j_{(m-1)d+n} \leq d$ such that $j_\nu \geq l$ for every $1 \leq l \leq d$ and every $(m-1)(l-1) + n \leq \nu \leq (m-1)l + n - 1$.

For a tuple $(j_1, \dots, j_{(m-1)d+n})$, define values b_0, \dots, b_{d-1} by letting b_i be the number of ν with $j_\nu = i$. This, in turn induces a bijection

$$\mathcal{J}_{d,n}^{(m)} \rightarrow \mathcal{B}_{d,n}^{(m)},$$

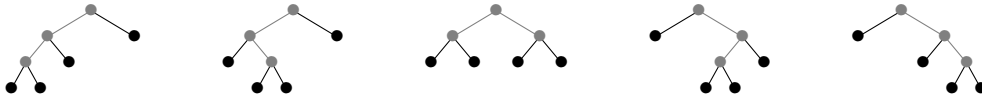
the latter being the set of all tuples (b_0, \dots, b_{d-1}) of non-negative integers such that $\sum_{r=0}^i b_r < (m-1)i + n$ for all $0 \leq i \leq d-1$. This proves the following:

Corollary 4.4.5. The underlying additive group of the Chow ring $A^*(\text{Hilb}_{d,n}^{(m)})$ is a free abelian group with basis

$$c_1(\mathcal{U})^{b_{d-1}} \dots c_d(\mathcal{U})^{b_0},$$

where (b_0, \dots, b_{d-1}) ranges over all tuples of non-negative integers satisfying $b_0 + \dots + b_i < (m-1)i + n$ for every $0 \leq i \leq d-1$.

Example (continued). Let us illustrate Corollary 4.4.5 in our favorite example $m = 2, d = 3$ and $n = 1$. The bijections $\mathcal{F} \rightarrow \mathcal{J} \rightarrow \mathcal{B}$ yield the following result: The trees



again displayed together with the sets $C(S_*)$, give rise to the following sequences of numbers in $\mathcal{J}_{3,1}^{(2)}$

3333 2333 2233 1333 1233

which, in turn, correspond to the following sequences of integers in $\mathcal{B}_{3,1}^{(2)}$

000 001 002 010 011.

By forming the sequences of partial sums and then increasing every entry by one, we end up at Stanley's (s) (cf. [Sta99, Ex. 6.19]).

We have thus obtained a basis of the underlying (free) abelian group of the Chow ring of the non-commutative Hilbert scheme. It reads

$$A^*(\text{Hilb}_{3,1}^{(2)}) = \mathbb{Z} \cdot 1 \oplus \mathbb{Z} \cdot c_1(\mathcal{U}) \oplus \mathbb{Z} \cdot c_1(\mathcal{U})^2 \oplus \mathbb{Z} \cdot c_2(\mathcal{U}) \oplus \mathbb{Z} \cdot c_1(\mathcal{U})c_2(\mathcal{U}).$$

Applying Theorem 4.4.1 also gives us a relation between these (monomials in) Chern classes and the cycles associated to the cell closures. The theorem tells us that

$$\begin{aligned} 1 \cap [\mathrm{Hilb}_{3,1}^{(2)}] &= [\mathcal{L}_{\nearrow}], \\ c_1(\mathcal{U}) \cap [\mathrm{Hilb}_{3,1}^{(2)}] &= [\mathcal{L}_{\nearrow}], \\ c_1(\mathcal{U})^2 \cap [\mathrm{Hilb}_{3,1}^{(2)}] &= [\mathcal{L}_{\nearrow}] + n[\mathcal{L}_{\searrow}], \\ c_2(\mathcal{U}) \cap [\mathrm{Hilb}_{3,1}^{(2)}] &= [\mathcal{L}_{\searrow}], \text{ and} \\ c_1(\mathcal{U})c_2(\mathcal{U}) \cap [\mathrm{Hilb}_{3,1}^{(2)}] &= [\mathcal{L}_{\searrow}] \end{aligned}$$

for some positive integer n .

As we have remarked (cf. Remark 4.4.4), the integer n is precisely the intersection multiplicity

$$i\left(\mathcal{L}_{\searrow}, D_{\nearrow}(11) \cdot D_{\nearrow}(12); \mathrm{Hilb}_{3,1}^{(2)}\right)$$

which equals the length of the local ring $\mathcal{O}_{D_{\nearrow} \cap U_{\searrow}, Z_{\searrow}}$. Employing the isomorphism $U_{\searrow} \cong \mathbb{A}^{12}$ from [Rei05], we regard an element $[v, A, B] \in U_{\searrow}$ as a tuple

$$1 \begin{vmatrix} x_{\varepsilon,1} & 0 & x_{\varepsilon,211} \\ x_{2,1} & 1 & x_{2,211} \\ x_{21,1} & 0 & x_{21,211} \end{vmatrix} \begin{vmatrix} 0 & x_{\varepsilon,22} & x_{\varepsilon,212} \\ 0 & x_{2,22} & x_{2,212} \\ 1 & x_{21,22} & x_{21,212} \end{vmatrix} =: 0 \begin{vmatrix} x_1 & 0 & x_2 \\ y & 1 & x_3 \\ z & 0 & x_4 \end{vmatrix} \begin{vmatrix} 0 & x_5 & x_8 \\ 0 & x_6 & x_9 \\ 1 & x_7 & x_{10} \end{vmatrix}$$

which comes from displaying $[v, A, B]$ in terms of the basis v, Bv, ABv . The closed subscheme $D_{\nearrow} \cap U_{\searrow}$ is defined by the vanishing of the determinants

$$\begin{aligned} \det(v \mid Av \mid A^2v) &= \begin{vmatrix} 1 & x_1 & x_1^2 + x_2z \\ 0 & y & x_1y + x_3z \\ 0 & z & x_1z + y + x_4z \end{vmatrix} = y^2 + x_4yz - x_3z^2 \text{ and} \\ \det(v \mid Av \mid BAv) &= \begin{vmatrix} 1 & x_1 & x_5y + x_8z \\ 0 & y & x_1 + x_6y + x_9z \\ 0 & z & x_7y + x_{10}z \end{vmatrix} = x_7y^2 + (x_{10} - x_6)yz - x_1z - x_9z^2. \end{aligned}$$

On the other hand, the closed subvariety Z_{\searrow} is given by the vanishing of y and z . Therefore, the local ring of $D_{\nearrow} \cap U_{\searrow}$ along the closed subvariety Z_{\searrow} is

$$\mathbb{k}(x_1, \dots, x_{10})[y, z]/(y^2 + x_4yz - x_3z^2, x_7y^2 + (x_{10} - x_6)yz - x_1z - x_9z^2).$$

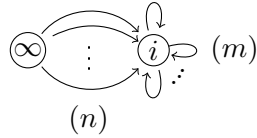
A lengthy computation shows that the length of this (artinian) ring - which equals its dimension over $\mathbb{k}(x_1, \dots, x_{10})$ - is 4. The author has determined this using SINGULAR.

4.5 Tautological Presentation of the Chow Ring of $\text{Hilb}_{d,n}^{(m)}$

Applying Corollary 4.4.5, we see that the Chow ring of $\text{Hilb}_{d,n}^{(m)}$, which we again abbreviate $\text{Hilb}_{d,n}$ is generated by the Chern classes $c_1(\mathcal{U}), \dots, c_d(\mathcal{U})$. An aforementioned result of King and Walter (cf. [KW95]) asserts that the Chow ring of a fine quiver moduli space is generated by the Chern classes of the universal bundle if the quiver has no oriented cycles. This theorem is not applicable here, yet the statement holds nonetheless.

Thus, in view of Chapter 3, we might wonder if Theorem 3.2.1 holds also true for $\text{Hilb}_{d,n}$, in other words: Viewing the non-commutative Hilbert scheme as a fine quiver moduli, do the tautological relations (which do hold, even if the quiver is not acyclic) give a *presentation* of $A(\text{Hilb}_{d,n}) := A^*(\text{Hilb}_{d,n})_{\mathbb{Q}}$? The answer will turn out to be positive.

In Section 4.1, we have figured out that $\text{Hilb}_{d,n}$ is isomorphic to $M^{\widehat{\theta}}(\widehat{Q}_n, \widehat{d})$, where \widehat{Q}_n is the quiver



and $\widehat{d} = (1, d)$. The stability condition $\widehat{\theta}$ is chosen according to $\theta = 0$. When choosing the character ψ according to $(1, 0)$, the universal bundle on the sink coincides with the bundle \mathcal{U} on the Hilbert scheme. The linear relation from Section 3.1 hence tells us that the first Chern class of the universal line bundle on the source is zero.

The $\widehat{\theta}$ -forbidden sub-dimension vectors of \widehat{d} are of the form $(1, p)$ with $0 \leq p < d$. The associated forbidden polynomials are

$$f^{(p)} = x_{p+1}^n \dots x_d^n \prod_{\mu=1}^p \prod_{\nu=p+1}^d (x_\nu - x_\mu)^m$$

which live in $\mathbb{Q}[x_1, \dots, x_p, x_{p+1}, \dots, x_d]^{S_p \times S_q}$ where x_1, \dots, x_d are the Chern roots of the universal bundle of $\text{Hilb}_{d,n}$.

We fix the same notation as in Section 3.1. Let $C = \mathbb{Q}[x_1, \dots, x_d]$ be the polynomial ring in variables that correspond to the Chern roots of \mathcal{U} . Let $A := C^{S_d}$ be the ring of symmetric polynomials in these variables. In order to obtain the ideal of tautological relations, we may compute $\rho(bf^{(p)})$, where b runs through C and $\rho : C \rightarrow A$ is the A -linear symmetrization map defined by

$$\rho(f) = \Delta^{-1} \sum_{w \in S_d} \text{sign}(w) w f.$$

In this context $\Delta = \Delta_d$ is the discriminant $\prod_{i < j} (x_j - x_i)$. We now use the fact that for every $w \in S_d$, there exist a unique $\tau \in S_p \times S_q$ and a unique (p, q) -shuffle permutation σ with $w = \sigma\tau$. Being an (r, s) -shuffle permutation means that σ is of the form

$$\sigma = \begin{pmatrix} 1 & \dots & p & p+1 & \dots & d \\ i_1 & \dots & i_p & j_1 & \dots & j_q \end{pmatrix}$$

for sequences $i_1 < \dots < i_p$ and $j_1 < \dots < j_q$ which are necessarily complementary. For any $b \in C$, we obtain

$$\begin{aligned} \rho(bf^{(p)}) &= \Delta^{-1} \sum_{\sigma \text{ (p,q)-shuffle}} \sum_{\tau \in S_p \times S_q} \text{sign}(\sigma\tau)(\sigma\tau)(bf^{(p)}) \\ &= \Delta^{-1} \sum_{\sigma \text{ (p,q)-shuffle}} \text{sign}(\sigma)\sigma f^{(p)} \sum_{\tau \in S_p \times S_q} \text{sign}(\tau)(\sigma\tau)b \\ &= \Delta^{-1} \sum_{\sigma \text{ (p,q)-shuffle}} \text{sign}(\sigma)\sigma f^{(p)} \cdot \sigma \left(\sum_{\tau \in S_p \times S_q} \text{sign}(\tau)\tau b \right). \end{aligned}$$

As $\sum_{\tau} \text{sign}(\tau)\tau b$ is alternating under the action of $S_p \times S_q$, it is divisible by $\Delta_{p \times q} = \Delta_p \Delta_q$. Putting $\delta := \prod_{i=1}^p \prod_{j=p+1}^d (x_j - x_i)$, we obtain $\Delta = \delta \Delta_{p \times q}$ and therefore

$$\text{sign}(\sigma)\Delta = \sigma\Delta = (\sigma\delta)(\sigma\Delta_{p \times q}).$$

With $\rho_{p \times q}(b) = \Delta_{p \times q}^{-1} \sum_{\tau \in S_p \times S_q} \text{sign}(\tau)\tau b$, we get

$$\rho(bf^{(p)}) = \sum_{\sigma \text{ (p,q)-shuffle}} (\sigma\delta)^{-1} \cdot \sigma f^{(p)} \cdot \sigma \rho_{p \times q}(b).$$

We insert the definition of $f^{(p)}$. We write it as $f^{(p)} = x_{p+1}^n \dots x_d^n \delta^m$. This yields

$$\rho(bf^{(p)}) = \sum_{\substack{1 \leq i_1 < \dots < i_p \leq d \\ 1 \leq j_1 < \dots < j_q \leq d \\ \text{complementary}}} x_{j_1}^n \dots x_{j_q}^n \prod_{\mu=1}^p \prod_{\nu=1}^q (x_{j_\nu} - x_{i_\mu})^{m-1} \cdot (\rho_{p \times q}(b))(x_{i_1}, \dots, x_{i_p}, x_{j_1}, \dots, x_{j_q}).$$

As b runs through C , the image $\rho_{p \times q}(b)$ runs through $\mathbb{Q}[x_1, \dots, x_p, x_{p+1}, \dots, x_d]^{S_p \times S_q}$ which we may identify with the tensor product $\mathbb{Q}[x_1, \dots, x_p]^{S_p} \otimes \mathbb{Q}[x_{p+1}, \dots, x_d]^{S_q}$. Thus, the ideal $\rho(C \cdot f^{(p)})$ is generated by expressions

$$\sum_{\substack{1 \leq i_1 < \dots < i_p \leq d \\ 1 \leq j_1 < \dots < j_q \leq d \\ \text{complementary}}} f(x_{i_1}, \dots, x_{i_p}) x_{j_1}^n \dots x_{j_q}^n g(x_{j_1}, \dots, x_{j_q}) \prod_{\mu=1}^p \prod_{\nu=1}^q (x_{j_\nu} - x_{i_\mu})^{m-1}$$

as a \mathbb{Q} -vector space, with f ranging over all symmetric polynomials in p variables and g over those in q variables. In Chapter 5 (more precisely in Corollary 5.3.2), we will show that these relations provide a presentation for the Chow ring of $\text{Hilb}_{d,n}$.

Corollary 4.5.1. The homomorphism $\mathbb{Q}[x_1, \dots, x_d]^{S_d} \rightarrow A(\text{Hilb}_{d,n}^{(m)})$ sending the i -th elementary symmetric function e_i to the i -th Chern class of \mathcal{U} is surjective and its kernel is generated by expressions of the form

$$\sum_{\substack{1 \leq i_1 < \dots < i_p \leq d \\ 1 \leq j_1 < \dots < j_q \leq d \\ \text{complementary}}} f(x_{i_1}, \dots, x_{i_p}) x_{j_1}^n \dots x_{j_q}^n g(x_{j_1}, \dots, x_{j_q}) \prod_{\mu=1}^p \prod_{\nu=1}^q (x_{j_\nu} - x_{i_\mu})^{m-1}$$

as a \mathbb{Q} -vector space, where $0 \leq p < d$ and f ranges over all symmetric polynomials in p variables and g over those in $q = d - p$ variables.

In Chapter 5, we will give a conceptual proof of this result by interpreting the cohomology of non-commutative Hilbert schemes a module over the CoHa.

Note that, in order to obtain a set of generators of the kernel of $A \rightarrow A(\text{Hilb}_{d,n}^{(m)})$ as an *ideal* of A , it suffices to choose bases $\mathcal{B}_{p,q}$ of $\mathbb{Q}[x_1, \dots, x_p, x_{p+1}, \dots, x_d]^{S_p \times S_q}$ as a module over A whose elements we may require to be of the form $f_{\lambda,p} \otimes g_{\lambda,q}$. We illustrate this in the case $m = 2$, $n = 1$ and $d = 3$.

Example (continued). We have $A = \mathbb{Q}[e_1, e_2, e_3] = \mathbb{Q}[x, y, z]^{S_3}$.

- Let $p = 0$. Inserting $g = 1$ yields the relation $e_3 = 0$.
- For $p = 1$, a basis of $\mathbb{Q}[x][y, z]^{S_2}$ as an A -module is given by $1, x, x^2$. Putting $f(x) = 1$ yields

$$0 = yz(y - x)(z - x) + xz(x - y)(z - y) + xy(x - z)(y - z) \equiv (xy + xz + yz)^2 \equiv e_2^2$$

when employing the relation $xyz = 0$. The other basis vectors result in multiples of xyz .

- Finally, let $p = 2$. Then, a basis of $\mathbb{Q}[x, y]^{S_2}[z]$ over A is $1, z, z^2$. We consider $g(z) = 1$ first and obtain, using $xyz = 0$,

$$0 \equiv e_1^3 - 4e_1e_2.$$

After some lengthy computation, we see that for $g(z) = z$, we obtain the relation $e_1^4 = 0$. The basis element z^2 does not provide a new relation.

All in all, we have computed a presentation for the Chow ring of $\text{Hilb}_{3,1}^{(2)}$. It is isomorphic to

$$\mathbb{Q}[e_1, e_2]/(e_1^3 - 4e_1e_2, e_2^2, e_1^4).$$

Chapter 5

Modules over the Cohomological Hall Algebra

The Cohomological Hall algebra, which we will call CoHa in the future, was invented by Kontsevich and Soibelman in [KS11]. We will consider the CoHa of a quiver Q and define a CoHa-module structure on the Chow rings of non-commutative Hilbert schemes. This has been done in the author's paper [Fra13b] for the special case when Q is the multi-loop quiver.

To motivate the Cohomological Hall algebra, let us have a look at classical Hall algebras. We follow Kapranov's exposition in [Kap97]. Let \mathfrak{A} be an abelian category. The **Hall algebra** $\mathcal{H}(\mathfrak{A})$ of \mathfrak{A} is, by definition, the complex vector space of functions $f : \mathcal{M}(\mathfrak{A}) \rightarrow \mathbb{C}$ with finite support on the set $\mathcal{M}(\mathfrak{A})$ of isomorphism classes of \mathfrak{A} , equipped with the following multiplication: For functions f and g , we define

$$(f \circ g)(M) = \sum_{M' \hookrightarrow M} f(M')g(M/M'),$$

the sum ranging over all subobjects of M , i.e. objects with a *fixed* monomorphism $M' \hookrightarrow M$. It might also make sense to consider a twisted version of the above product. Suppose we are given a non-zero real number $\varepsilon_{M',M''}^M$ for every short exact sequence $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$ which is invariant under isomorphisms of short exact sequences. The formula

$$(f * g)(M) = \sum_{M' \hookrightarrow M} \varepsilon_{M',M''}^M f(M')g(M/M')$$

provides a product under certain requirements on the ε 's. For example, in [Lus90, Lus91], Lusztig considers twists of this multiplication by a bilinear form and Ringel found that, among these, the Euler form is particularly important (see [Rin93]).

Let us apply this construction to the following situation: Fix a quiver Q and a finite field \mathbb{F} . We consider the category $\mathbf{rep}_{\mathbb{F}}(Q)$ of \mathbb{F} -linear representations of Q . Let d be a dimension vector of Q . If we define $R_d(\mathbb{F}) := \bigoplus_{\alpha:i \rightarrow j} \mathbf{Hom}(\mathbb{F}^{d_i}, \mathbb{F}^{d_j})$ and $G_d(\mathbb{F}) := \prod_i \mathbf{GL}_{d_i}(\mathbb{F})$, then $R_d(\mathbb{F})$ is the set of \mathbb{F} -valued points of $R_d := R(Q, d)$ and analogously for $G_d := G(Q, d)$. The $G_d(\mathbb{F})$ -orbits of $R_d(\mathbb{F})$ correspond bijectively to the isomorphism classes of representations of Q over \mathbb{F} with dimension vector d . Thus,

$$\mathcal{M}(\mathbf{rep}_{\mathbb{F}}(Q)) = \bigsqcup_{d \in \mathbb{Z}_{\geq 0}^{Q_0}} R_d(\mathbb{F})/G_d(\mathbb{F}).$$

Note that $\mathbb{R}_d(\mathbb{F})/G_d(\mathbb{F})$ is the set of \mathbb{F} -valued points of the moduli stack $[R_d/G_d]$ (which makes sense as the variety R_d and the group G_d are defined over \mathbb{Z}). The vector space $\mathcal{H}(\mathbf{rep}_{\mathbb{F}}(Q)) =: \mathcal{H}(Q, \mathbb{F})$ decomposes as $\mathcal{H}(Q, \mathbb{F}) = \bigoplus_d \mathcal{H}_d(Q, \mathbb{F})$, where $\mathcal{H}_d(Q, \mathbb{F})$ is the subspace of all $G_d(\mathbb{F})$ -invariant, complex-valued functions on $R_d(\mathbb{F})$. In this particular situation, we give yet another interpretation of the Hall algebra multiplication. If G is an algebraic group acting on a variety X , both defined over \mathbb{F} , denote $\mathbb{C}^G(X)$ to be the set of all $G(\mathbb{F})$ -invariant, complex-valued functions on $X(\mathbb{F})$. There are two functorial constructions: Let $\pi : Y \rightarrow X$ be a G -equivariant morphism. We obtain a pull-back $\pi^* : \mathbb{C}^G(X) \rightarrow \mathbb{C}^G(Y)$ by pre-composition with π . Moreover, we define a push-forward $\pi_* : \mathbb{C}^G(Y) \rightarrow \mathbb{C}^G(X)$ as follows: For a $G(\mathbb{F})$ -invariant function $h : Y(\mathbb{F}) \rightarrow \mathbb{C}$ and an \mathbb{F} -valued point x of X , we define

$$(\pi_* h)(x) := \sum_{y \in (f^{-1}x)(\mathbb{F})} h(y)$$

and obtain a $G(\mathbb{F})$ -invariant function on $Y(\mathbb{F})$. For two dimension vectors p and q with $p + q = d$, consider the maps

$$\begin{array}{ccccccc} \mathbb{C}^{G_p}(R_p) \otimes \mathbb{C}^{G_q}(R_q) & \xrightarrow{\times} & \mathbb{C}^{G_{p_q}}(R_{p_q}) & \dashrightarrow & \mathbb{C}^{G_{p_q^*}}(R_{p_q^*}) & \xrightarrow{i_*} & \mathbb{C}^{G_{p_q^*}}(R_d) \xrightarrow{\int} \mathbb{C}^{G_d}(R_d). \\ & & \downarrow \text{pr}^* & \swarrow & & & \\ & & \mathbb{C}^{G_{p_q}}(R_{p_q^*}) & & & & \end{array}$$

Here, $G_{p_q^*}$ denotes the upper parabolic of G_d to the decomposition of every k^{d_i} into the coordinate space of the first p_i and the last q_i unit vectors. Write $G_{p_q} := G_p \times G_q$. It is the Levi factor of the parabolic $G_{p_q^*}$. The subspace R_{p_q} is just $R_p \times R_q$ and $R_{p_q^*}$ is the subspace of R_d consisting of all $M \in R_d$ such that M_α maps the coordinate space k^{p_i} into k^{p_j} for every arrow $\alpha : i \rightarrow j$.

Some of these maps deserve a little explanation. The map \times assigns to $f \otimes g$ the function $f \times g$ on $R_{p_q}(\mathbb{F})$ defined by $(f \times g)(M' \oplus M'') = f(M')g(M'')$. The map pr^* is the pull-back with respect to the projection of the G_{p_q} -equivariant vector bundle $R_{p_q} \rightarrow R_{p_q}$. We can see at once that $\text{pr}^* h$ is invariant with respect to the parabolic. The map \int arises as follows: For a function h on $R_d(\mathbb{F})$ which is $G_{p_q^*}$ -invariant, define

$$(\int h)(M) := \frac{1}{\sharp_{G_{p_q^*}}} \sum_{g \in G_d} h(g \cdot M).$$

Under the composition of the horizontal maps in the diagram, $f \otimes g$ is mapped to the function $\int i_* \text{pr}^*(f \times g)$ which is defined by

$$(\int i_* \text{pr}^*(f \times g))(M) = \sum_{M' \hookrightarrow M} \varepsilon_{M', M''}^M f(M')g(M/M''),$$

where $\varepsilon_{M', M''}^M$ is the number of representatives $\dot{g} \in G_d(\mathbb{F})/G_{p_q^*}(\mathbb{F})$ such that $g \cdot M$ restricted to the coordinate subspaces k^{d_i} is isomorphic to the representation M' .

We are going to use a similar formalism to develop the Cohomological Hall algebra. The idea is to use the equivariant cohomology groups (or equivariant Chow groups) instead of spaces of invariant functions.

5.1 The Cohomological Hall Algebra of a Quiver

Let Q be a quiver and let $d = (d_i \mid i)$ be a dimension vector for Q . Abbreviate $R_d := R(Q, d)$ and $G_d := G(Q, d)$. We define

$$\mathcal{H}_d := \mathcal{H}_d(Q) := A_{G_d}^*(R_d)\mathbb{Q},$$

the G_d -equivariant Chow ring of R_d with rational coefficients (cf. Section 2.4). In [KS11], Kontsevich and Soibelman use equivariant cohomology (with rational coefficients) instead of equivariant intersection theory, as they work entirely over the complex numbers. However, if $k = \mathbb{C}$, the equivariant cycle map becomes an isomorphism after tensoring with the rationals and all the constructions work in equivariant intersection theory, as well.

We make the following convention: In this section, all (equivariant) Chow rings are extended to the rationals, i.e. write $A_G^*(X)$ instead of $A_G^*(X)\mathbb{Q}$. On the direct sum $\mathcal{H} := \bigoplus_d \mathcal{H}_d$ ranging over all dimension vectors, Kontsevich and Soibelman define a ‘‘convolution like’’ multiplication $\mathcal{H}_p \otimes \mathcal{H}_q \rightarrow \mathcal{H}_{d=p+q}$, assigning $f \otimes g \mapsto f * g$, as the composition

$$\begin{array}{ccccccc} A_{G_p}^i(R_p) \otimes A_{G_q}^j(R_q) & \xrightarrow{\times} & A_{G_{p_q}}^{n=i+j}(R_{p_q}) & \cong & A_{G_{p_q^*}}^n(R_{p_q^*}) & \xrightarrow{i^*} & A_{G_{p_q^*}}^{n+s_1}(R_d) \xrightarrow{\pi^*} A_{G_d}^{n+s_1+s_2}(R_d). \\ & & \downarrow \cong & & \swarrow \cong & & \\ & & A_{G_{p_q}}^n(R_{p_q^*}) & & & & \end{array}$$

As in the introduction, $G_{p_q^*}$ denotes the upper parabolic of G_d to the decomposition of every k^{d_i} into the coordinate space of the first p_i and the last q_i unit vectors. Write $G_{p_q} := G_p \times G_q$. It is the Levi factor of the parabolic $G_{p_q^*}$. The subspace R_{p_q} is just $R_p \times R_q$ and $R_{p_q^*}$ is the subspace of R_d consisting of all $M \in R_d$ such that M_α maps the coordinate space k^{p_i} into k^{q_j} for every arrow $\alpha : i \rightarrow j$. This means that k^p is a sub- Λ -module of ${}_\Lambda M$ which we denote $({}_\Lambda M)|_{k^p}$. In this context, $\Lambda := kQ$ denotes the path algebra of Q . Moreover,

$$\begin{aligned} s_1 &= \dim R_d - \dim R_{p_q^*} = \sum_{\alpha:i \rightarrow j} p_j q_i \\ s_2 &= -\dim G_d/G_{p_q^*} = -\sum_i p_i q_i \\ s_1 + s_2 &= \sum_{\alpha:i \rightarrow j} p_j q_i + \sum_i p_i q_i = -\langle p, q \rangle \end{aligned}$$

where the brackets denote the Euler form of the quiver Q . The above maps arise as follows: The map \times is the equivariant exterior product. For the other maps, choose a G_d -module V such that there exists an open G_p -equivariant subset $U \subseteq V$ on which a fiber bundle quotient $U \rightarrow U/G_d$ exists and with the property that $\dim V - \dim(V - U) > n + s_1$. We have morphisms

$$\begin{array}{ccccc} (R_{p_q}) \times^{G_{p_q}} U & & R_{p_q^*} \times^{G_{p_q^*}} U & \xrightarrow{i} & R_d \times^{G_{p_q^*}} U \xrightarrow{\pi} R_d \times^{G_d} U, \\ & \uparrow & \nearrow & & \\ & R_{p_q^*} \times^{G_{p_q}} U & & & \end{array}$$

the non-horizontal maps being affine space bundles, i is a closed embedding and π is a smooth morphism with fiber $G_d/G_{p_i^*} = \prod_i \text{Gr}_{p_i}(\mathbb{k}^{d_i}) =: \text{Gr}_{p,d}$. In particular, π is proper.

This multiplication makes \mathcal{H} an associative graded algebra with a unit $1 \in \mathcal{H}_0$. Moreover, we can define a $(\mathbb{Z}_{\geq 0}^{Q_0} \times \mathbb{Z})$ -bigrading on \mathcal{H} by putting

$$\mathcal{H}_{d,k} := A_{G_d}^{-(d,d)-k}(R_d).$$

The bigrading is compatible with the multiplication. Note that this bigrading is a little different from the one used in [KS11]. The one we use here was suggested in [Rei12].

In [KS11], an explicit formula for the multiplication is derived. The proof given there is valid in equivariant intersection theory as well. It involves the equivariant self-intersection formula and the integration formula (cf. Theorem 2.4.3). Identifying $\mathcal{H}_d = A_{G_d}^*(R_d)$ with $A_{G_d}^*(\text{pt}) \cong A_{T_d}^*(\text{pt})^{W_d} = \mathbb{Q}[x_{i,\nu} \mid i \in Q_0, 1 \leq \nu \leq d_i]^{W_d}$, where $W_d = \prod_i S_{d_i}$ is the Weyl group, we obtain that $(f * g)(x_{i,\nu} \mid i, 1 \leq \nu \leq d_i)$ is the expression

$$\sum_{\sigma \text{ (p,q)-shuffle}} f(x_{\sigma(i,\mu)} \mid i, 1 \leq \mu \leq p_i) \cdot g(x_{\sigma(i,\nu)} \mid i, p_i + 1 \leq \nu \leq d_i) \cdot \frac{\prod_{\alpha:i \rightarrow j} \prod_{\mu=1}^{p_i} \prod_{\nu=p_j+1}^{d_j} (x_{\sigma(j,\nu)} - x_{\sigma(i,\mu)})}{\prod_{i \in Q_0} \prod_{\mu=1}^{p_i} \prod_{\nu=p_i+1}^{d_i} (x_{\sigma(i,\nu)} - x_{\sigma(i,\mu)})}.$$

A (p, q) -shuffle is a tuple $\sigma = (\sigma_i \mid i) \in W_d$ such that $\sigma_i(1) < \dots < \sigma_i(p_i)$ and $\sigma_i(p_i + 1) < \dots < \sigma_i(d_i)$ for all i . We write $\sigma(i, \nu)$ for the pair $(i, \sigma_i(\nu))$.

5.2 Cohomology of Non-Commutative Hilbert Schemes as a Module over the CoHa

For another dimension vector $n = (n_i \mid i)$, we consider the vector space $F_{d,n} := \prod_{i \in Q_0} M_{d_i \times n_i}$ on which the group G_d acts by

$$g \cdot f = (g_i \cdot f_i \mid i).$$

We define $R_{d,n} := R_d \times F_{d,n}$ on which G_d acts on each component separately.

Let again $\Lambda = \mathbb{k}Q$ be the path algebra of the quiver. Choosing an $M \in R_d$ declares a (left-) Λ -module structure on the vector space $\mathbb{k}^d := \bigoplus_i \mathbb{k}^{d_i}$. We denote this Λ -module by ${}_{\Lambda}M$. Denoting P_i the projective indecomposable Λ -module associated to the vertex i , we define $P^n := \bigoplus_i P_i^{n_i}$. The choice of a pair (M, f) corresponds to defining a Λ -linear map

$$P^n \rightarrow {}_{\Lambda}M.$$

A pair (M, f) is called **stable** if this map is surjective. On the open subset $R_{d,n}^{\text{st}}$ of stable points, a geometric G_d -quotient

$$\pi : R_{d,n}^{\text{st}} \rightarrow \text{Hilb}_{d,n}(Q)$$

exists. The image $\pi(M, f)$ of a stable pair will be denoted $[M, f]$. The variety $\text{Hilb}_{d,n}(Q)$ is called a **non-commutative Hilbert scheme**. Note that this provides a generalization of the non-commutative Hilbert schemes that we considered in Chapter 4. Using our new terminology, $\text{Hilb}_{d,n}^{(m)}$

equals $\text{Hilb}_{d,n}(Q_m)$, where Q_m denotes the m -loop quiver. As Q is fixed throughout the text, we will sometimes suppress the dependency on Q in the notation.

Fix a dimension vector n for Q . Put $\mathcal{A} := \bigoplus_d \mathcal{A}_d$, where $\mathcal{A}_d := A^*(\text{Hilb}_{d,n}(Q))$ is the Chow ring of the non-commutative Hilbert scheme. We are going to define an \mathcal{H} -module structure on \mathcal{A} and show that we can realize it as a quotient of \mathcal{H} itself and describe the kernel of the quotient map.

Note that we may construct a similar diagram as above for the “decorated” representation variety $R_{d,n}$: We have morphisms

$$\begin{array}{ccc} (R_p \times R_{q,n}) \times^{G_{p_q}} U & & R_{p_q^*,n} \times^{G_{p_q^*}} U \xrightarrow{i} R_{d,n} \times^{G_{p_q^*}} U \xrightarrow{\pi} R_{d,n} \times^{G_d} U, \\ \uparrow & \nearrow & \\ R_{p_q^*,n} \times^{G_{p_q}} U & & \end{array}$$

where the arrows without names are again affine space bundles. Here, $R_{p_q^*,n} := R_{p_q^*} \times F_{d,n}$. This induces an \mathcal{H} -module structure on the direct sum $\bigoplus_d A_{G_d}^*(R_{d,n})$. But as $R_{d,n} \rightarrow R_d$ is also a G_d -equivariant affine bundle, the direct sum of these Chow groups coincides with \mathcal{H} , as a vector space. It is not hard to see that the thus induced module structure coincides with the natural \mathcal{H} -module structure on \mathcal{H} itself.

As a next step, we pass to the stable locus of $R_{d,n}$. It consists of the tuples (M, f) such that the image of f generates M as a representation. Consider the open subset $R_{p_q^*,n}^{\text{st}}$ of $R_{p_q^*,n} = R_{p_q^*} \times \prod_i M_{d_i, n_i}$ which is the intersection $R_{p_q^*,n} \cap R_{d,n}^{\text{st}}$. An element (M, f) of $R_{p_q^*,n}^{\text{st}}$ is of the form

$$\left(\left(\begin{array}{c|c} M' & * \\ \hline 0 & M'' \end{array} \right), \left(\begin{array}{c} f' \\ f'' \end{array} \right) \right)$$

We can easily see that (M'', f'') is also a stable point of $R_{q,n}$. By restricting the projection $R_{p_q^*,n} \rightarrow R_p \times R_{q,n}$, we obtain a well defined morphism

$$\varphi : R_{p_q^*,n}^{\text{st}} \rightarrow R_p \times R_{q,n}^{\text{st}}.$$

Being an affine space bundle, the projection $R_{p_q^*,n} \rightarrow R_p \times R_{q,n}$ is flat. This implies at once that φ is a flat morphism, too. We can draw the, by now, well known diagram

$$\begin{array}{ccc} (R_p \times R_{q,n}^{\text{st}}) \times^{G_{p_q}} U & & R_{p_q^*,n}^{\text{st}} \times^{G_{p_q^*}} U \xrightarrow{i} R_{d,n}^{\text{st}} \times^{G_{p_q^*}} U \xrightarrow{\pi} R_{d,n}^{\text{st}} \times^{G_d} U, \\ \uparrow \varphi & \nearrow & \\ R_{p_q^*,n}^{\text{st}} \times^{G_{p_q}} U & & \end{array}$$

which gives us maps in equivariant intersection theory as follows:

$$\begin{array}{ccccccc} A_{G_p}^i(R_p) \otimes A_{G_q}^j(R_{q,n}^{\text{st}}) & \xrightarrow{\times} & A_{G_{p_q}}^n(R_p \times R_{q,n}^{\text{st}}) & \longrightarrow & A_{G_{p_q^*}}^n(R_{p_q^*,n}^{\text{st}}) & \xrightarrow{i_*} & A_{G_{p_q^*}}^{n+s_1}(R_{d,n}^{\text{st}}) \xrightarrow{\pi_*} A_{G_d}^{n+s_1+s_2}(R_{d,n}^{\text{st}}). \\ & & \downarrow \varphi^* & & \swarrow \cong & & \\ & & A_{G_{p_q}}^n(R_{p_q^*,n}^{\text{st}}) & & & & \end{array}$$

Composing these maps, we get $\mathcal{H}_p \otimes \mathcal{A}_q \rightarrow \mathcal{A}_d$. A similar argument to [KS11, 2.3] shows that this map indeed makes \mathcal{A} into an \mathcal{H} -module. Let us write $f \otimes g$ for the image of $f \otimes g$ under this map.

Let us look at the map $j^* : \mathcal{H} \rightarrow \mathcal{A}$ which is induced by the open embeddings $R_{d,n}^{\text{st}} \hookrightarrow R_{d,n}$. It is clearly surjective. It is also \mathcal{H} -linear as the following commutative diagram asserts:

$$\begin{array}{ccccc}
 (R_p \times R_{q,n}^{\text{st}}) \times^{G_{p_q}} U & \xleftarrow{\varphi} & R_{p_q^*,n}^{\text{st}} \times^{G_{p_q}} U & & \\
 \swarrow & & \swarrow & & \downarrow \\
 (R_p \times R_{q,n}) \times^{G_{p_q}} U & \xleftarrow{\quad} & R_{p_q^*,n} \times^{G_{p_q}} U & & R_{p_q^*,n}^{\text{st}} \times^{G_{p_q^*}} U \\
 & & \downarrow & & \downarrow i \\
 & & R_{p_q^*,n} \times^{G_{p_q^*}} U & & R_{d,n}^{\text{st}} \times^{G_{p_q^*}} U \\
 & & \downarrow i & & \downarrow \pi \\
 & & R_{d,n} \times^{G_{p_q^*}} U & & R_{d,n}^{\text{st}} \times^{G_d} U \\
 & & \downarrow \pi & & \swarrow \\
 & & R_{d,n} \times^{G_d} U & &
 \end{array}$$

In this diagram, all maps pointing from north-east to south-west are induced by the open embeddings. Note that every “square”, except for the uppermost, is cartesian. Passing to intersection theory, the outer arrows of the diagram give two ways to go from $A_{G_{p_q}}^*(R_p \times R_{q,n})$ to $A_{G_d}^*(R_{d,n}^{\text{st}})$. One way describes $f \otimes j^*g$, whereas the other represents $j^*(f * g)$. In a picture:

$$\begin{array}{ccc}
 A_{G_{p_q}}^*(R_p \times R_{q,n}) & \xrightarrow{\quad} & A_{G_d}^*(R_{d,n}^{\text{st}}) \\
 \searrow & \searrow & \uparrow \\
 & & j^*(f * g) \\
 & & \uparrow \\
 & & f \otimes j^*g
 \end{array}$$

If we define a bigrading on \mathcal{A} by letting

$$\mathcal{A}_{d,k} := A_{G_d}^{-(d,d)-k}(R_{d,n}^{\text{st}}),$$

the map $j^* : \mathcal{H} \rightarrow \mathcal{A}$ is also homogeneous of bidegree $(0,0)$. We have therefore shown that \mathcal{A} can be written as a quotient of \mathcal{H} . We want to prove the following result about the kernel of the quotient map j^* . Using the above identification of \mathcal{H}_d with a certain ring of symmetric functions, we let $e_d^n \in \mathcal{H}_d$ be $e_d^n(x_{i,\nu} \mid i, \nu) = \prod_i \prod_{\nu=1}^{d_i} x_{i,\nu}^{n_i}$. The product here is the multiplication in \mathcal{H}_d which corresponds to the intersection product and *not* the CoHa-multiplication.

Theorem 5.2.1. The kernel of $j^* : \mathcal{H} \rightarrow \mathcal{A}$ equals $\sum_{p,q \in \mathbb{Z}_{\geq 0}^2, q \neq 0} \mathcal{H}_p * (e_q^n \cdot \mathcal{H}_q)$.

We point out that every graded component \mathcal{I}_d of $\mathcal{S} = \ker j^*$ is an ideal of \mathcal{H}_d with respect to the intersection product. This is true for trivial reasons, as the flat pull-back is multiplicative, but it can also be verified from the explicit formula given in the above theorem.

Proof. Let $s_0 : R_d \rightarrow R_{d,n}$ be the zero section of $R_{d,n}$ considered as a G_d -linear bundle on R_d . Under the identifications $\mathcal{H}_d = A_{G_d}^*(R_d) \cong A_{G_d}^*(R_{d,n}) \cong A_{G_d}^*(\text{pt})$, the map

$$A_{G_d}^k(\text{pt}) \cong A_{G_d}^k(R_d) \xrightarrow{(s_0)^*} A_{G_d}^{k+dn}(R_{d,n}) \cong A_{G_d}^{k+dn}(\text{pt}),$$

is the multiplication with the top G_d -equivariant Chern class of $R_{d,n}$. Here, $dn := \sum_i d_i n_i$. Identifying $A_{G_d}^*(\text{pt})$ with the ring of W_d -invariant functions in variables $x_{i,\nu}$, the top G_d -equivariant Chern class of $R_{d,n}$ is the polynomial $e_d^n(x_{i,\nu} \mid i, \nu)$ above. Taking this into account, the statement to prove is equivalent to showing that the horizontal sequence in the diagram

$$\begin{array}{ccccccc} \bigoplus_{\substack{p+q=d, q \neq 0 \\ i+j=k-\langle p,q \rangle}} \mathcal{H}_p^i \otimes \mathcal{H}_q^{j-qn} & \longrightarrow & \mathcal{H}_d^k & \xrightarrow{j^*} & \mathcal{A}_d^k & \longrightarrow & 0 \\ & & \downarrow (s_0)^* & \nearrow * & & & \\ \bigoplus_{\substack{p+q=d, q \neq 0 \\ i+j=k-\langle p,q \rangle}} \mathcal{H}_p^i \otimes (e_q \cdot \mathcal{H}_q^j) & & & & & & \end{array}$$

is exact. For all $p+q=d$, we have the Künneth isomorphism

$$\bigoplus_{i+j=r} \mathcal{H}_p^i \otimes \mathcal{H}_q^{j-q} \xrightarrow{\cong} A_{G_{p_q}}^{r-q}(R_{p_q}^*).$$

Modulo these isomorphisms, the maps that we are interested in are $A_{G_{p_q}}^{r-q}(R_{p_q}^*) \rightarrow A_{G_d}^{r+(m-1)pq}(R_{d,n})$. These maps arise from the following morphisms

$$\begin{array}{ccccccc} R_{p_q}^* \times^{G_{p_q}} U & \xrightarrow{s_0} & (R_{p_q}^* \times F_{q,n}) \times^{G_{p_q}} U & & R_{p_q^*,n} \times^{G_{p_q^*}} U & \xrightarrow{i} & R_{d,n} \times^{G_{p_q^*}} U \xrightarrow{p} R_{d,n} \times^{G_d} U, \\ & & \uparrow & \nearrow & & & \\ & & R_{p_q^*,n} \times^{G_{p_q}} U & & & & \end{array}$$

the non-horizontal ones being affine space bundles. Considering the cartesian squares

$$\begin{array}{ccc} R_{p_q}^* \times^{G_{p_q}} U & \xrightarrow{s_0} & (R_{p_q}^* \times F_{q,n}) \times^{G_{p_q}} U \\ \uparrow & & \uparrow \\ (R_{p_q}^* \times F_{p,n}) \times^{G_{p_q}} U & \xrightarrow{s_0} & R_{p_q^*,n} \times^{G_{p_q}} U \\ \downarrow & & \downarrow \\ (R_{p_q}^* \times F_{p,n}) \times^{G_{p_q^*}} U & \xrightarrow{s_0} & R_{p_q^*,n} \times^{G_{p_q^*}} U \end{array}$$

and using the commutativity of flat pull-back and proper push-forward, we are bound to show the exactness of

$$\bigoplus_{d=p+q, q \neq 0} A_{G_{p^*_q}}^{k-\langle p,q \rangle - qn}(R_{p^*_q} \times F_{p,n}) \rightarrow A_{G_d}^k(R_{d,n}) \xrightarrow{j^*} A_{G_d}^k(R_{d,n}^{\text{st}}) \rightarrow 0. \quad (*)$$

Let (M, f) be an instable point of $R_{d,n}$. Then, the subrepresentation $L(M, f)$ of M generated by $\text{im}(f)$ is a proper subrepresentation. Let X_p be the closed G_d -subset of all (M, f) where the dimension vector of $L(M, f)$ is at most p (its natural scheme structure is the reduced one). Let X_p^o be the locally closed subset where $\underline{\dim} L(M, f) = p$. We obtain

$$R_{d,n}^{\text{inst}} = \bigsqcup_{p \leq d, p \neq d} X_p^o$$

which is precisely the Harder-Narasimhan stratification, as defined in [Rei03]. Denote $W_p := R_{p^*_q} \times F_{p,n}$ and let W_p^o be the open subset of W_p where $L(M, f) = (\wedge M)|_{k^p}$, i.e. $\underline{\dim} L(M, f) = p$. Evidently, the G_d -saturation of W_p lies in X_p . Therefore, we obtain a morphism

$$\psi_p : W_p \times^{G_{p^*_q}} U \rightarrow X_p \times^{G_{p^*_q}} U \rightarrow X_p \times^{G_d} U.$$

As the first map is a closed immersion and the latter is a $G_d/G_{p^*_q}$ -bundle, ψ_p is proper. We claim that ψ_p induces an isomorphism

$$W_p^o \times^{G_{p^*_q}} U \xrightarrow{\cong} X_p^o \times^{G_d} U.$$

As $W_p^o \times^{G_{p^*_q}} U$ is naturally isomorphic to $(W_p^o \times^{G_{p^*_q}} G_d) \times^{G_d} U$ as a G_d -variety and as $X_p^o \times U \rightarrow X_p^o \times^{G_d} U$ is a G_d -principal fiber bundle, by faithfully flat descent, it suffices to show that

$$W_p^o \times^{G_{p^*_q}} G_d \rightarrow X_p^o$$

is an isomorphism of G_d -varieties. This will be done in Lemma 5.2.2.

Denote by W_p^c the complement of W_p^o in W_p and let X_p^c be defined analogously. Applying [Ful98, Ex. 1.8.1], the cartesian diagram

$$\begin{array}{ccc} W_p^c \times^{G_{p^*_q}} U & \longrightarrow & W_p \times^{G_{p^*_q}} U \\ \downarrow \pi' & & \downarrow \pi \\ X_p^c \times^{G_d} U & \longrightarrow & X_p \times^{G_d} U \end{array}$$

induces an exact sequence

$$A_*^{G_{p^*_q}}(W_p^c) \rightarrow A_*^{G_d}(X_p^c) \oplus A_*^{G_{p^*_q}}(W_p) \rightarrow A_*^{G_d}(X_p) \rightarrow 0,$$

where the first map sends α to $\pi'_* \alpha + (-\alpha)$ and the second maps $\beta + \beta'$ to $\beta + \pi_* \beta'$. As $X_p^c = \bigsqcup_r X_r$, the disjoint union over all $r' \leq p$ with $r' \neq p$ maximal, there exists a surjection

$$A_*^{G_{p^*_q}}(W_p) \oplus \bigoplus_{r \leq p, r \neq p \text{ max.}} A_*^{G_d}(X_r) \rightarrow A_*(X_p)^{G_d} \rightarrow 0.$$

By induction on p , we obtain that the natural map

$$\bigoplus_{r \leq p} A_*^{Gr_{r^*}}(W_r) \rightarrow A_*^{G_d}(X_p)$$

is onto. Inserting for p the maximal dimension vectors $p \leq d$ with $p \neq d$ finally yields the exactness of the sequence (*). \square

Lemma 5.2.2. With the notation as in the proof of Theorem 5.2.1, the natural map $W_p^o \times^{G_{p^*}} G_d \rightarrow X_p^o$ is an isomorphism.

Proof. In order to prove this, consider the morphism $L : X_p^o \rightarrow \text{Gr}_{p,d} := \prod_i \text{Gr}_{p_i}(k^{d_i}) = G_d/G_{p^*}$ assigning to every point $(M, f) \in X_p^o$ the subrepresentation $L(M, f)$. We show that

$$\begin{array}{ccc} G_d \times W_p^o & \xrightarrow{\text{act}} & X_p^o \\ \downarrow \text{pr} & & \downarrow L \\ G_d & \xrightarrow{\text{act}} & \text{Gr}_{p,d} \end{array}$$

is a cartesian diagram of varieties. In fact, $G_d \times_{\text{Gr}_{p,d}} X_p^o$ consists of those pairs $(g, (M, f))$ such that $L(M, f)$ equals the subrepresentation $g \cdot (\wedge M)|_{k^p}$. An isomorphism

$$G_d \times W_p^o \rightarrow G_d \times_{\text{Gr}_{p,d}} X_p^o$$

is therefore given by mapping $(g, (M, f))$ to $(g, g \cdot (M, f))$. \square

5.3 The Case of the m -Loop Quiver

If Q is the m -loop quiver, the Hilbert scheme $\text{Hilb}_{d,n}(Q)$ coincides with $\text{Hilb}_{d,n}^{(m)}$, as we have already remarked.

Cohomological Hall Algebra of a Loop Quiver

The explicit formula for the CoHa-multiplication from [KS11] in the case of the loop quiver reads as follows: Identifying $\mathcal{H}_d = A_{G_d}^*(R_d)$ with $\mathbb{Q}[x_1, \dots, x_d]^{S_d}$, we obtain

$$(f * g)(x_1, \dots, x_d) = \sum_{\substack{1 \leq i_1 < \dots < i_p \leq d \\ 1 \leq j_1 < \dots < j_q \leq d \\ \text{complementary}}} f(x_{i_1}, \dots, x_{i_p}) g(x_{j_1}, \dots, x_{j_q}) \prod_{\mu=1}^p \prod_{\nu=1}^q (x_{j_\nu} - x_{i_\mu})^{m-1},$$

where two sequences $1 \leq i_1 < \dots < i_p \leq d$ and $1 \leq j_1 < \dots < j_q \leq d$ are called complementary if the union of these numbers is $\{1, \dots, d\}$. Using this formula, it is evident that the multiplication $*$ is graded commutative if m is even, and commutative if m is odd.

We have seen that $\mathcal{A} = \bigoplus_{d \geq 0} A^*(\text{Hilb}_{d,n}^{(m)})$ can be realized as an \mathcal{H} -module and that the map $j^* : \mathcal{H} \rightarrow \mathcal{A}$ from Section 2 is \mathcal{H} -linear. Taking into account that \mathcal{H} is either commutative (if m is odd) or graded commutative (if m is even), we obtain:

Corollary 5.3.1. The vector space \mathcal{A} inherits the structure of a bigraded \mathcal{H} -algebra.

Theorem 5.2.1 for the loop quiver reads as follows:

Corollary 5.3.2. The kernel of $j^* : \mathcal{H} \rightarrow \mathcal{A}$ equals $\sum_{p \geq 0, q > 0} \mathcal{H}_p * (e_q^n \cdot \mathcal{H}_q)$.

With the help of Corollary 5.3.2, we can prove Corollary 4.5.1 at once. Using the CoHa-multiplication, the tautological relations from Section 4.5 read as $f * (e_q^n \cdot g)$. In this sense, Corollary 4.5.1 is just a reformulation of the above statement.

Two Examples

For $m = 0$ and $m = 1$, there is an explicit description of the CoHa as an exterior algebra and as a symmetric algebra, respectively, both over a vector space of countably infinite dimension. We would like to describe the ideal $\ker j^*$ under these isomorphisms. For simplicity, we restrict to the case $n = 1$, but the results generalize to arbitrary n .

Let $m = 0$. In this case, the (non-commutative) Hilbert schemes Hilb_0 and Hilb_1 are singletons while Hilb_d is empty for $d > 1$. The multiplication in \mathcal{H} is given by the formula

$$(f * g)(x_1, \dots, x_d) = \sum_{\substack{1 \leq i_1 < \dots < i_p \leq d \\ 1 \leq j_1 < \dots < j_q \leq d \\ \text{complementary}}} f(x_{i_1}, \dots, x_{i_p}) g(x_{j_1}, \dots, x_{j_q}) \prod_{\mu=1}^p \prod_{\nu=1}^q (x_{j_\nu} - x_{i_\mu})^{-1}.$$

It is easy to see that $(f * f)(x, y) = 0$ for all $f \in \mathcal{H}_1$. Therefore, the embedding $\mathcal{H}_1 \hookrightarrow \mathcal{H}$ induces a homomorphism of (graded) algebras $\varphi : \bigwedge^*(\mathcal{H}_1) \rightarrow \mathcal{H}$. We identify the ring \mathcal{H}_1 (equipped with the intersection product \cdot) with the polynomial ring $\mathbb{Q}[x]$. Let $\psi_0, \psi_1, \psi_2, \dots$ be the basis of \mathcal{H}_1 that corresponds to x^0, x^1, x^2, \dots under this isomorphism. Note that ψ_i lives in bidegree $(1, -i)$. A basis of $\bigwedge^d(\mathcal{H}_1)$ is given by expressions $\psi_{k_1} \wedge \dots \wedge \psi_{k_d}$, where $k_1 < \dots < k_d$ is an increasing sequence of d non-negative integers. By induction on d , we can show that

$$(\psi_{k_1} * \dots * \psi_{k_d})(x_1, \dots, x_d) = s_\lambda(x_1, \dots, x_d),$$

where s_λ is the Schur function belonging to the partition $\lambda = (k_d - d + 1, \dots, k_1)$. Hence, it follows that φ is an isomorphism.

Let us determine $\mathcal{I} := \ker j^*$. Denoting $\mathcal{I}_d \subseteq \mathcal{H}_d$ the d -th homogeneous component, Theorem 5.2.1 implies that $\mathcal{I}_d = \sum_{q=1}^d \mathcal{H}_{d-q} * (e_q \cdot \mathcal{H}_q)$. We obtain that $\mathcal{I}_0 = 0$, and $\mathcal{I}_1 \subseteq \mathcal{H}_1$ is $e_1 \cdot \mathcal{H}_1$, which equals the ideal generated by x under the isomorphism $(\mathcal{H}_1, \cdot) \cong \mathbb{Q}[x]$. For $d \geq 2$, the element

$$(\psi_0 * \psi_2 * \dots * \psi_{d-1}) * \psi_1$$

is contained in \mathcal{I}_d . But as \mathcal{H} is graded commutative, this element is - up to a sign - just

$$(\psi_0 * \psi_1 * \dots * \psi_{d-1})(x_1, \dots, x_d) = s_{(0, \dots, 0)}(x_1, \dots, x_d) = 1.$$

Therefore, $\mathcal{I}_d = \mathcal{H}_d$. This shows that $\varphi^{-1}(\mathcal{I}) \subseteq \bigwedge^*(\mathcal{H}_1)$ is the ideal generated by ψ_1, ψ_2, \dots (with respect to the multiplication \wedge). Consequently:

Corollary 5.3.3. In the case $m = 0$, the $\mathcal{H} = \bigwedge^*(\psi_0, \psi_1, \dots)$ -algebra $\mathcal{A} = \bigoplus_d A^*(\text{Hilb}_{d,1}^{(0)})$ equals $\bigwedge^*(\mathbb{Q}\psi_0) \cong \mathbb{Q}[\psi]/(\psi^2)$ with a generator of bidegree $(1, 0)$.

Let us turn to the case $m = 1$. The (non-commutative) Hilbert scheme Hilb_d is an affine space of dimension d . The CoHa-multiplication has the form

$$(f * g)(x_1, \dots, x_d) = \sum_{\substack{1 \leq i_1 < \dots < i_p \leq d \\ 1 \leq j_1 < \dots < j_q \leq d \\ \text{complementary}}} f(x_{i_1}, \dots, x_{i_p}) g(x_{j_1}, \dots, x_{j_q}).$$

Again, the immersion $\mathcal{H}_1 \hookrightarrow \mathcal{H}$ yields a homomorphism $\varphi : \text{Sym}^*(\mathcal{H}_1) \rightarrow \mathcal{H}$ of algebras (which respects the grading). A basis element $\psi_{k_1} \dots \psi_{k_d}$ of $\text{Sym}^d(\mathcal{H}_1)$ with $k_1 \geq \dots \geq k_d$ is mapped to

$$(\psi_{k_1} * \dots * \psi_{k_d})(x_1, \dots, x_d) = c_\lambda m_\lambda(x_1, \dots, x_d).$$

In the above equation, m_λ denotes the monomial symmetric function of $\lambda = (k_1, \dots, k_d)$ and c_λ is some positive integer. This proves that φ is an isomorphism.

We compute \mathcal{I} and its inverse image under φ . We see that $\mathcal{I}_0 = 0$. For $d \geq 1$, the element

$$\left(1_{\mathcal{H}_{d-q}} * (e_q^{(q)} \cdot 1_{\mathcal{H}_q})\right)(x_1, \dots, x_d) = \sum_{1 \leq j_1 < \dots < j_q \leq d} x_{j_1} \dots x_{j_q} = e_q(x_1, \dots, x_d)$$

is contained in \mathcal{I}_d for all $q = 1, \dots, d$. As the unit of \mathcal{H}_d is not contained in \mathcal{I}_d , we obtain that \mathcal{I}_d is generated by the elementary symmetric functions e_1, \dots, e_d in d variables. It follows that $\varphi^{-1}(\mathcal{I}) \subseteq \text{Sym}^*(\mathcal{H}_1)$ is the ideal generated by ψ_1, ψ_2, \dots and thus:

Corollary 5.3.4. For $m = 1$, the $\mathcal{H} = \text{Sym}^*(\psi_0, \psi_1, \dots)$ -algebra $\mathcal{A} = \bigoplus_d A^*(\text{Hilb}_{d,1}^{(1)})$ coincides with the polynomial ring $\text{Sym}^*(\mathbb{Q}\psi_0) \cong \mathbb{Q}[\psi]$ in one variable of bidegree $(1, 0)$.

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