Asymptotic Behavior and Observability of Semilinear Evolution Equations

Dissertation

zur Erlangung des akademischen Grades Doktor der Naturwissenschaften

dem Fachbereich C - Mathematik und Naturwissenschaften

der Bergischen Universität Wuppertal

vorgelegt von

Mahmoud Baroun

aus Taroudant in Marokko

Wuppertal, 2012

Die Dissertation kann wie folgt zitiert werden:

urn:nbn:de:hbz:468-20121022-151509-7
[http://nbn-resolving.de/urn/resolver.pl?urn=urn%3Anbn%3Ade%3Ahbz%3A468-20121022-151509-7]

Erstgutachter :	Prof. Dr. B. Jacob
Zweitgutachter:	Prof. Dr. H. Zwart
Mitglieder der Prüfungskommission:	Prof. Dr. B. Jacob
	Prof. Dr. H. Pecher
	Prof. Dr. K. Klamroth
	Dr. C. Wyss
Tag der mündlichen Prüfung:	07 Februar 2012

Acknowledgement

After my thanks to Allah, The Almighty, for granting me the energy and power to continue my efforts to prepare this research work, it is a great pleasure for me to express my most profound gratitude to all people without whom this dissertation most probably would not have been finished. First of all, of course, I would like to thank my supervisor Prof. Dr. B. Jacob for the time and her encouragement, guidance, patience, (not only financial) support and suggestions during the preparation of this thesis.

Special appreciation and thanks is also expressed to Prof. L. Maniar for his perseverance and time in evaluating this dissertation. He not only gave me the scientific support, but he also encouraged me in the last few years.

During this work I have collaborated with many researchers for whom I have great regard, and I wish to extend my warmest thanks to all those who have helped me with my work. I wish to express my warm and sincere thanks to Prof. Dr. R. Schnaubelt, Prof. S. Boulite, Prof. G. M. N'Gurkata and Prof T. Diagana who made an important impact on my research during our continuous collaboration.

Thanks are also due to Prof. Dr. H. Pecher , Prof. Dr. K. Klamroth and Dr. C. Wyss for accepting to be a member of the examination committee and my gratitude also goes to Prof. Dr. H. Zwart for useful discussions concerning my work and for accepting to be reviewer of this thesis.

I would like to express my special thanks to the staff and my colleagues of the functional analysis groups of the university of Wuppertal for their friendship and all of their help over the past years. A special word of gratitude to all my friends in Marrakesh, Delft, Paderborn and Wuppertal, who helped and encouraged me.

Many thanks to everyone who has taught me maths and made me interested in the subject over the years: in particular, my family. I would like to express my deepest appreciation to my parents, sisters and brothers for their monumental support and encouragement. They have truly always been there for me.

Finally, I am most thankful the CICAT (Centre for International Cooperation and Appropriate Technology) for their financial support when I was study in University of Delft for one year and the DFG (German Research Foundation) and the CNRST (National Centre of Coordination and Planning of Technical Scientific Research) for supporting my research visits in Karlsruhe and in Marrakesh.

Wuppertal, February 2012 *Mahmoud Baroun*

Contents

List of Figures

Chapter 1

Introduction

The theory of evolution equations in infinite dimensional spaces plays an important role in mathematics. In fact, frequently a partial differential equation, like the Schrödinger equation, wave equation and heat equation can be transformed into an evolution equation

$$u'(t) = A(t)u(t) + f(t, u(t)), \quad t \in \mathbb{R}, \tag{1.1}$$

in a Banach space X, where $A(t)$ are some unbounded linear operators.

The asymptotic behavior of this equation was studied by several authors. The most extensively studied cases are the autonomous case $A(t) = A$ and the periodic case $A(t + T) = A(t)$, see [8, 12, 21, 58, 59, 100, 116] for the almost periodicity and [32, 47, 56, 63, 93, 94, 95] for the almost automorphy. In the almost periodic case, the authors of [86] proved that the unique bounded mild solution of

$$u'(t) = A(t)u(t) + g(t),\ t \in \mathbb{R}, \tag{1.2}$$

given by

$$u(t) = \int_{\mathbb{R}} \Gamma(t, \tau)g(\tau)\, d\tau, \qquad t \in \mathbb{R}, \tag{1.3}$$

is almost periodic in X if some resolvent $R(\omega, A(\cdot))$ of $A(\cdot)$ and g are almost periodic, and the evolution family $U(t, s)$, solution of the homogeneous equation $g = 0$, has an exponential dichotomy. In Chapter 3, we consider the semilinear equation (1.1). Since, in general the semilinear term f is defined only on some small spaces Y of X, e.g. the interpolation spaces $X_\alpha^t := X_\alpha^{A(t)}$ of $A(t)$, we show first that the bounded mild solution of (1.2) is also almost periodic in some time-invariant interpolation space X_α. Finally, if the function $f : \mathbb{R} \times X_\alpha \longrightarrow X$ is continuous, almost periodic and globally Lipschitz, by the fixed point principle we obtain the existence of a unique almost periodic mild solution to the semilinear evolution equation (1.1) in the interpolation space X_α. To illustrate these results, we study the existence and uniqueness of an almost periodic solution to the thermoelastic plate systems

$$\begin{cases} u_{tt}(t,x) + \Delta^2 u(t,x) + a(t)\Delta\theta(t,x) & = f_1(t, \nabla u(t,x), \nabla\theta(t,x)), \\ \theta_t(t,x) - b(t)\Delta\theta(t,x) - a(t)\Delta u_t(t,x) & = f_2(t, \nabla u(t,x), \nabla\theta(t,x)), \\ \theta = u = \Delta u & = 0, \end{cases}$$

where $t \in \mathbb{R}$, $x \in \Omega$ (open set of $\mathbb{R}^n$); a, b are positive functions on $\mathbb{R}$, and u, θ denote the vertical deflection and the variation of temperature of the plate respectively; the function f_i, $i = 1, 2$ are continuous and globally Lipschitz. Assuming that the coefficients a, b and the nonlinear functions f_1, f_2 are almost periodic, we get the claim.

In the second part of Chapter 3, we study the almost automorphy of solutions of (1.1). Following the arguments of [86], by assuming the exponential dichotomy of U and the almost automorphy of the functions $t \mapsto R(\omega, A(\cdot))$, g and f, we show the almost automorphy of the Green's function corresponding to U. This yields the almost automorphy in X of the unique bounded mild solution of (1.2). Using an interpolation argument, we show the almost automorphy of u in every time-invariant interpolation space X_α. The aim now will be obtained through a fixed-point theorem.

In many systems, the boundary conditions are inhomogeneous, e.g. dynamic population equations, boundary control systems and delay differential equations. These systems can be abstractly written as the following boundary evolution equation

$$\begin{cases} u'(t) & = A_m(t)u(t) + g(t, u(t)), \qquad t \in \mathbb{R}, \\ B(t)u(t) & = h(t, u(t)), \qquad t \in \mathbb{R}, \end{cases} \tag{1.4}$$

for linear operators $A_m(t) : Z \to X$ and $B(t) : Z \to Y$ on Banach spaces $Z \hookrightarrow X$ and Y. Typically, $A_m(t)$ is an elliptic partial differential operator acting in, say, $X = L^p(\Omega)$, and $B(t)$ is a boundary operator mapping $Z = W_p^2(\Omega)$ into a 'boundary space' like $W_p^{1-1/p}(\partial\Omega)$, where $p \in (1, \infty)$, see Example 5.3.6.

If $h = 0$, the boundary evolution equation (1.4) is just the evolution equation (1.1). In the general case, $h \neq 0$, to study the wellposedness and the asymptotic behavior of the equation (1.4), the standard way is to write it as an evolution equation

$$u'(t) = A_{\alpha-1}(t)u(t) + f(t, u(t)), \qquad t \in \mathbb{R}, \tag{1.5}$$

in the continuous extrapolation spaces $X_{\alpha-1}^t$, $\alpha \in (0,1)$, for the operators $A(t) := A_m(t)|\ker B(t)$, where

$$f(t, u(t)) = g(t, u(t)) + (\omega I - A_{\alpha-1}(t))D(t)h(t, u(t))$$

for the solution operator $D(t) : \varphi \mapsto v$ of the corresponding abstract Dirichlet problem $(\omega I - A_m(t))v = 0$ and $B(t)v = \varphi$, where $\omega \in \mathbb{R}$ is large enough. Then (1.4) and (1.5) have the same classical solutions, see e.g. [87].

It is shown in [87] that the evolution family $U(t,s)$ generated by $A(t), t \in \mathbb{R}$, can be extended to operators $U_{\alpha-1}(t,s) : X_{\alpha-1}^s \to X$, see Chapter 2. So we can define mild solutions of (1.5) as the functions $u \in C(\mathbb{R}, X)$ satisfying

$$u(t) = U(t,s)u(s) + \int_s^t U_{\alpha-1}(t,\tau)f(\tau, u(\tau))\, d\tau \tag{1.6}$$

for all $t \geq s$, where $f(\tau, u(\tau))$ belongs to $X^{\tau}_{\alpha-1}$.

In Chapter 4, we study the almost periodicity and automorphy of the semilinear autonomous parabolic boundary evolution equation

$$\begin{cases} u'(t) &= A_m u(t) + g(t, u(t)), \quad t \in \mathbb{R}, \\ Bu(t) &= h(t, u(t)), \quad t \in \mathbb{R}, \end{cases} \tag{1.7}$$

by the ones of its corresponding extrapolated evolution equation

$$u'(t) = A_{\alpha-1} u(t) + f(t, u(t)), \qquad t \in \mathbb{R}. \tag{1.8}$$

In [32], the authors considered this question for a hyperbolic differential equation, when g and the boundary term h are defined on the whole space X. In this chapter, we continue this study in the general case where g and h are defined only on some interpolation space X_β, $\quad 0 \leq \beta < 1$, with respect to the sectorial operator $A := A_m | \ker B$.

In Chapter 5, these results are generalized to the nonautonomous boundary evolution equations (1.4) with inhomogeneous terms, i.e. $g(t, u(t)) = g(t)$ and $h(t, u(t)) = h(t)$. We show that the solutions $u : \mathbb{R} \to X$ of (1.4) inherit the (asymptotic) almost periodicity of the inhomogeneities $g : \mathbb{R} \to X$ and $h : \mathbb{R} \to Y$. Our basic assumptions say that $A_m(\cdot)$ and $B(\cdot)$ are (asymptotically) almost periodic in time and that $A(t)$ satisfy 'Acquistapace-Terreni' conditions. In particular, the operators $A(t)$ are sectorial and they generate a parabolic evolution family $U(t, s)$, $t \geq s$, which solves the homogeneous problem (1.4) with $g = h = 0$. If U has an exponential dichotomy on $\mathbb{R}$, then we show that for each almost periodic g and h there is a unique almost periodic solution of (1.4), see Proposition 5.3.2.

Our main results in this chapter concern the more complicated case where the evolution family U has exponential dichotomies on (possibly disjoint) time intervals $(-\infty, -T]$ and $[T, +\infty)$. Theorem 5.3.5 then gives a Fredholm alternative for (mild) solutions u of (1.4) with inhomogeneous terms in the space $AAP^{\pm}(\mathbb{R}, X)$ of continuous functions $u : \mathbb{R} \to X$ being asymptotically almost periodic on $\mathbb{R}_+$ and on $\mathbb{R}_-$, separately. In fact we prove more detailed results on the Fredholm properties of (5.1), see Theorem 5.2.7, and we also treat the corresponding inhomogeneous initial/final value problems on $\mathbb{R}_{\pm}$, see Propositions 5.3.3 and 5.3.4.

When treating (1.5), it is crucial to identify suitable function spaces for the inhomogeneity f. To that purpose we consider the multiplication operator $A(\cdot)$ in the space $AAP^{\pm}(\mathbb{R}, X)$ endowed with the sup–norm. This space possesses the extrapolation spaces $AAP^{\pm}_{\alpha-1}$ corresponding to $A(\cdot)$. It is shown that the functions in these spaces can be characterized as limits of functions in $AAP^{\pm}(\mathbb{R}, X)$. Moreover, if the operators $A(t)$ possess constant extrapolation spaces $X^t_{\alpha-1} \cong X_{\alpha-1}$, we have $AAP^{\pm}_{\alpha-1} = AAP^{\pm}(\mathbb{R}, X_{\alpha-1})$.

One obtains exponential dichotomies on intervals $(-\infty, -T]$ and $[T, +\infty)$ in the asymptotically hyperbolic case where the operators $A_m(t)$ and $B(t)$ converge as $t \to \pm\infty$ and the resulting limit operators $A_{\pm\infty}$ have no spectrum on $i\mathbb{R}$, see [20], [107], [109]. It should be noted that if the limits at $+\infty$ and $-\infty$ differ, then the operators in (5.1) are asymptotically almost periodic only on

$\mathbb{R}_+$ and $\mathbb{R}_-$ separately, so that the space $AAP^{\pm}(\mathbb{R}, X)$ seems to be a natural setting for our investigations. The asymptotically hyperbolic case can occur if one linearizes a nonlinear problem along an orbit connecting two hyperbolic equilibria, see e.g. [105], and also the references in [76], [87].

For $f \in AAP^{\pm}_{\alpha-1}$ we then set $G_{\alpha-1}u = f$ if $u \in AAP^{\pm}(\mathbb{R}, X)$ satisfies (1.6), thus defining a closed operator $G_{\alpha-1}$ in $AAP^{\pm}_{\alpha-1}$. Its Fredholm properties yield the desired Fredholm alternative for the mild solutions to (1.5) described in Theorems 5.2.7 and 5.2.9.

The second part of this thesis treats the semilinear observation system

$$\dot{u}(t) = Au(t) + F(u(t)), \quad u(0) = x \in X, \ t \geq 0, \tag{1.9}$$

$$y(t) = C(u(t)), \tag{1.10}$$

where A is assumed to be the generator of a linear C_0-semigroup T on a Banach space X, C is a linear (resp. nonlinear) unbounded operator from a domain $D(C)$ to another Banach space Y and F is a globally Lipschitz continuous nonlinear operator from X into itself or locally Lipschitz continuous and has a linear growth.

It is well known, see e.g. [97], that the state equation (1.9) has a global unique mild solution given by $u(\cdot; x)$ for every $x \in X$. Moreover, by $S(t)x = u(t; x)$ one defines a semigroup S of globally (resp. locally) Lipschitz continuous operators. One now looks for sufficient conditions for the admissibility of C for S.

The theory of admissible observation operators and abstract observation systems is well developed for linear systems, see [39], [66], [104] and [118]. In case where C is an admissible linear output operator for T and F is globally Lipschitz, we extend the definition of admissibility of the observation operator C to semilinear systems or with respect to the nonlinear semigroup S. We develop conditions guaranteeing that the set of admissible observation operators for the semilinear problem coincides with the set of admissible observation operators for the linearized system. In another case, where C is a nonlinear unbounded operator and F is locally Lipschitz, we extend the successful linear theory to general nonlinear locally Lipschitz semigroups $S = (S(t))_{t \geq 0}$ (see Definition 7.1.3) and densely defined nonlinear output operators C. In particular, for such semigroups S we define locally Lipschitz observation systems Ψ and locally Lipschitz admissible observation operators. We further prove that such observation systems Ψ can be represented by $\Psi x = \widetilde{C}(S(\cdot)x)$ for a (possibly nonlinear) admissible observation operator $\widetilde{C}$, see Theorem 7.2.6.

As an important special case, we assume that C is an admissible linear output operator for T. In this situation one can in fact construct a nonlinear observation system (S, Ψ_F) given by (7.15), which is the integrated version of (1.9)–(1.10). Moreover, the system (S, Ψ_F) is represented by the Lebesgue extension C_L of C with respect to T, see Theorem 7.2.7.

We also define and study global (resp. local) exact observability of globally (resp. locally) Lipschitz observation systems. Again, in the case of the semilinear system (1.9)–(1.10) with a linear admissible operator C, it is desirable to have criteria of the observability of the system in terms of the linear system given by T and C.

If the system is linear and X is a reflexive Banach space (e.g. Hilbert space), then the concept of controllability is dual to the concept of observability. For semilinear systems the situation is more involved. Consequently, most publications study exact controllability and exact observability separately. There are various publications in the literature on the controllability of specific semilinear systems. We refer the reader to [13, 30, 37, 73, 129] and the references therein. On the other side, to our knowledge, there are only few results on observability of semilinear systems with linear (or nonlinear) observation operators. In particular, Mangnusson established in [84] a robustness result for exact observability near an equilibrium. He allowed for a larger class of nonlinearities in (1.9), but considered only (nonlinear) observation operators defined on X. In contrast, we focus on observation operators defined only on dense subspaces.

Overview of thesis

This thesis is arranged as follows:

In **Chapter 1** we give a general introduction of this work.

Chapter 2 introduces mathematical concepts used in this thesis. We will give basic definitions of the strongly continuous semigroups of linear operators, sectorial operators and hyperbolic semigroups and introduce their most important properties. We will also introduce interpolation and extrapolation spaces and some basic notions of parabolic evolution operators. In the last part of the chapter, we recall some definitions and facts concerning the concept of almost periodicity and almost automorphy. Some new results are also given here with their proofs.

Chapter 3 studies the existence and uniqueness of almost periodic and almost automorphic solutions to semilinear parabolic evolution equations. Under some reasonable assumptions and an interpolation argument we show the existence of a unique almost periodic (almost automorphic) solution in real interpolation spaces of the homogeneous problem. These results will be obtained through studying the inhomogeneous evolution equations and a fixed-point argument. This chapter gave two publications [14, 15].

Chapter 4 investigates the existence and uniqueness of almost periodic and almost automorphic solutions to the semilinear parabolic boundary evolution equations. The idea to achieve this aim is to transform the boundary equation into an equivalent semilinear evolution equation. We show first that the inhomogeneous evolution equation has a unique almost periodic and automorphic mild solution on a real interpolation space for each almost periodic and automorphic inhomogeneous function. The contraction fixed point theorem yields then the unique almost periodic and automorphic mild solution for the semilinear evolution equation and then to the semilinear parabolic boundary evolution equations. The fruits of this chapter are published in [18]

In **Chapter 5**, we show the existence and uniqueness of the (asymptotically) almost periodic solution to parabolic evolution equations with inhomogeneous boundary values on $\mathbb{R}$ and $\mathbb{R}_{\pm}$, if the data are (asymptotically) almost periodic. We assume that the underlying homogeneous problem satisfies the

Acquistapace-Terreni conditions and has an exponential dichotomy on $\mathbb{R}$. If there is an exponential dichotomy only on half intervals $(-\infty, -T]$ and $[T, \infty)$, then we obtain a Fredholm alternative of the equation on $\mathbb{R}$ in the space of functions being asymptotically almost periodic on $\mathbb{R}_+$ and $\mathbb{R}_-$. These results are published in [19].

Chapter 6 deals with semilinear evolution equations with unbounded observation operators. The first part of the chapter introduces the definition of admissible observation operators for semilinear systems and develops conditions on the nonlinearity guaranteeing that the set of admissible observation operators for the semilinear problem coincide with the set of admissible observation operators the linearized system. In the second part, we study the invariance of the Lebesgue extension under globally Lipschitz continuous perturbations of the original generator. The rest of the chapter is used to study the concept of exact observability for semilinear systems and to prove that the exact observability is not changed under small Lipschitz perturbations. The results of this chapter are published in [16].

Chapter 7 introduces locally Lipschitz observation systems for nonlinear semigroups and show that they can be represented by an admissible nonlinear output operator defined on a suitable subspace. In the semilinear case, this concept fits well to the Lebesgue extension known from linear system theory. Also in the semilinear case, we show robustness of exact observability near equilibria under locally small Lipschitz perturbations. These results are submitted, see [17].

Chapter 2

Mathematical background

In this preliminary chapter, we present some notations, basic definitions and results needed for the next chapters. We also give some new results with their proofs produced in this work.

2.1 Some notations

The symbols $\mathbb{N}$, $\mathbb{R}$, $\mathbb{C}$ shall denote the sets of natural, real and complex numbers, respectively, and $\mathbb{R}_+ := [0, \infty)$. Throughout this thesis, X, Y shall be Banach spaces with norms $\|\cdot\|_X$, $\|\cdot\|_Y$ and the Banach space of bounded linear operators between Banach spaces X and Y, shall be denoted by $\mathcal{L}(X, Y)$ and $\mathcal{L}(X) := \mathcal{L}(X, X)$. We denote by $D(A)$, $N(A)$, $R(A)$, $\sigma(A)$, $\rho(A)$ the domain, kernel, range, spectrum and resolvent set of a linear operator A. Moreover, we set $R(\lambda, A) := (\lambda I - A)^{-1} = (\lambda - A)^{-1}$ for $\lambda \in \rho(A)$. We say that the Banach space X is continuously embedded in the Banach space Y if $X \subset Y$ and $||\cdot||_Y \leq C||\cdot||_X$ and write $X \hookrightarrow Y$.

Let $J =]a, b[$, where $-\infty \leq a < b \leq +\infty$, and $1 \leq p < \infty$. Then $L^p(J; X)$ denotes the space of all Bochner-measurable functions $f : J \to X$, such that $\|f(t)\|_X^p$ is integrable for $t \in J$. It is a Banach space when normed by

$$\|f\|_{L^p(J;X)} := \left(\int_J \|f(s)\|_X^p ds \right)^{1/p}.$$

If $p = \infty$ the space $L^p(J; X)$ consists of all a measurable functions with a finite norm

$$\|f\|_{L^\infty(J;X)} := ess \sup_{t \in J} \|f(t)\|_X.$$

Let Ω be an open set of $\mathbb{R}^n$ and $X = \mathbb{R}$ or $\mathbb{C}$. The Sobolev spaces $W^{k,p}(\Omega)$, where k is any positive integer and $1 \leq p < \infty$, consist of all the functions f in $L^p(\Omega)$ which admit weak derivatives $D^\alpha f$ for $|\alpha| \leq k$ belonging to $L^p(\Omega)$. They are endowed with the norm

$$\|f\|_{W^{k,p}(\Omega)} = \sum_{|\alpha| \leq k} \|D^\alpha f\|_p.$$

If $p = 2$, we write $H^k(\Omega)$ for $W^{k,2}(\Omega)$.

We denote by $C(J;X)$, resp. $C^k(J;X)$, $k \in \mathbb{N}$, the space of functions $f : J \to X$, which are continuous, resp. k-times continuously differentiable. For a unbounded closed interval J, the space of bounded continuous functions $f : J \to X$ (vanishing at $\pm\infty$) is denoted by $BC(J,X)$ (by $C_0(J,X)$). Note that $BC(J,X)$ is a Banach space equipped with the supremum norm:

$$\|f\|_{BC(J,X)} := \sup_{t\in J} \|f(t)\|_X.$$

Similarly, $BC(J \times X, Y)$ denotes the space of all bounded continuous functions $f : J \times X \to Y$.

2.2 Semigroups of linear operators

Let X be a Banach space and $A : D(A) \subset X \to X$ be a closed linear densely defined operator in X. In the sequel we suppose that $D(A)$ is equipped with the graph norm of A, i.e. $\|x\|_{D(A)} := \|x\|_X + \|Ax\|_X$; since A is closed, $D(A)$ is a Banach space, continuously and densely embedded into X.

2.2.1 Strongly continuous semigroups

In this subsection we will define strongly continuous semigroups and their generators and introduce their most important properties. For more theory about strongly continuous semigroups see for example the monographs of Engel and Nagel [52] and [89], van Neerven [92] and Pazy [99].

Definition 2.2.1. *The family $T = (T(t))_{t\geq 0}$ of bounded linear operators on X is said to be a* strongly continuous semigroups *if*

(i) $T(t+s) = T(t)T(s)$ *for all* $t,\ s \geq 0$ *(the semigroup property),*

(ii) $T(0) = I$, *(I is the identity operator on X),*

(iii) $\lim_{t\to 0^+} \|T(t)x - x\|_X = 0$ *for every* $x \in X$.

The term strongly continuous semigroup is often abbreviated as *C_0-semigroup.*

Definition 2.2.2. *The* infinitesimal generator *(or generator in short) A of a C_0-semigroup T on a Banach space X is defined by*

$$\begin{aligned} Ax &= \lim_{t\to 0^+} \frac{T(t)x - x}{t} \\ D(A) &= \left\{ x \in X : \lim_{t\to 0^+} \frac{T(t)x - x}{t} \ \text{exists} \right\}. \end{aligned}$$

Example 2.2.3. *If A is a bounded operator, then A is the generator of the semigroup*

$$T(t) = e^{At} := \sum_{n=0}^{\infty} \frac{t^n}{n!} A^n, \quad t \geq 0.$$

We will now give some fundamental properties of C_0-semigroups and their infinitesimal generators.

Proposition 2.2.4. *Let $T = (T(t))_{t\geq 0}$ be a C_0-semigroup on a Banach space X with generator A. The following results hold:*

(i) $\|T(t)\|_{\mathcal{L}(X)}$ *is bounded on every finite subinterval of* $[0, \infty)$.

(ii) *A is a closed linear operator and its domain $D(A)$ is dense in X.*

(iii) *For all $x \in D(A)$ and $t \geq 0$, $T(t)x \in D(A)$, $t \mapsto T(t)x$ is continuously differentiable in X, and*

$$\frac{d}{dt}(T(t)x) = AT(t)x = T(t)Ax, \quad t \geq 0.$$

Exponential stability will be of use in this thesis.

Proposition 2.2.5. *[52, Prop. I.5.5] If $T = (T(t))_{t\geq 0}$ is a C_0-semigroup, then it is* exponentially bounded*, this means, that there exist real constants $M > 0$ and ω such that*

$$||T(t)||_{\mathcal{L}(X)} \leq Me^{\omega t}, \text{ for all } t \geq 0.$$

If $\omega < 0$, we say that T is exponentially stable*.*

The importance of C_0-semigroups is that they provide solutions to the abstract Cauchy problem

$$x'(t) = Ax(t),\ t \geq 0;\ \ x(0) = x_0 \in X$$

Indeed, if $x_0 \in D(A)$ and A generates a C_0-semigroup $(T(t))_{t\geq 0}$, then the map $t \mapsto T(t)x_0 \in C^1(\mathbb{R}_+; X)$ and the solution $x(t) := T(t)x_0$ satisfies $x(t) \in D(A)$ and $x'(t) = Ax(t)$ for all $t \geq 0$. However, for $x_0 \notin D(A)$, the map $t \mapsto T(t)x_0$ is not continuously differentiable and in order to define solutions for these initial values, a weaker notion of solution is required. A mild solution $x(t)$ of the Cauchy problem is a function $x(t) \in C(\mathbb{R}_+; X)$ satisfying

$$\int_0^t x(s)ds \in D(A) \text{ and } x(t) = x_0 + A\int_0^t x(s)ds$$

for each $t \geq 0$. Moreover, if A generates a C_0-semigroup $(T(t))_{t\geq 0}$, then $t \mapsto T(t)x_0 \in C(\mathbb{R}_+; X)$ is the unique mild solution of the Cauchy problem.

2.2.2 Sectorial operators and analytic semigroups

We will start with the study of a special kind of closed, linear operators so called sectorial operators. All the results of this section can you founded in Engel-Nagel [52], D. Henry [61], A. Lunardi [83] and Pazy [99].

For this purpose, let us denote by $S_{\omega,\,\theta}$, where $\omega \in \mathbb{R}$ and $\theta \in]\frac{\pi}{2}, \pi[$, an open sector of the complex plane given by the relation

$$S_{\omega,\,\theta} := \{\lambda \in \mathbb{C} : \lambda \neq \omega, |\arg(\lambda - \omega)| < \theta\},$$

and $S_\theta := S_{0,\,\theta}$ for short. We recall the definition of a sectorial operator in a Banach space X, see Figure 2.1.

Definition 2.2.6. *Let $A : D(A) \subseteq X \to X$ be a linear, closed and densely defined operator on a Banach space X. Then A is a* sectorial operator *in X if and only if there exist the constants $\omega \in \mathbb{R}$, $\theta \in]\frac{\pi}{2}, \pi[$ and $M > 0$ such that*

$$\begin{cases} (a) & \text{the resolvent set } \rho(A) \text{ contains the sector } S_{\omega,\,\theta}, \\ (b) & ||R(\lambda, A)||_{\mathcal{L}(X)} \leq \frac{M}{|\lambda-\omega|},\ \forall\, \lambda \in S_{\omega,\,\theta}. \end{cases} \tag{2.1}$$

where $\rho\,(A)$ is the resolvent set of A.

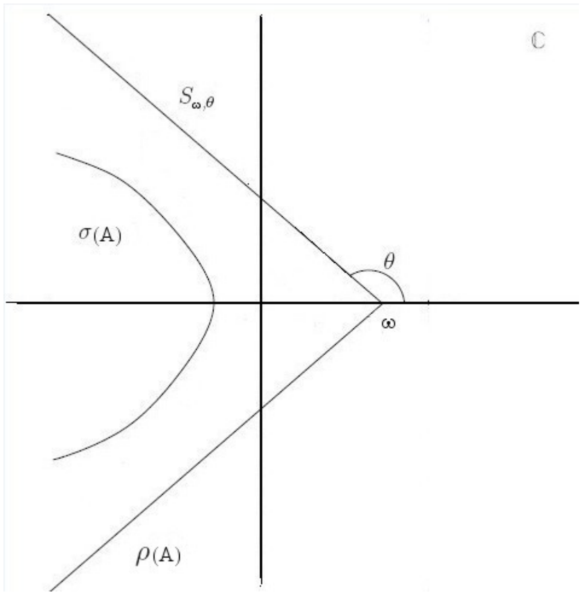

Figure 2.1: *Sectorial operator A with sector $S_{\omega,\,\theta}$.*

Example 2.2.7. *Let $p \geq 1$ and let $\Omega \subset \mathbb{R}^n$ be an open bounded subset such that its boundary $\partial\Omega$ is of class C^2. Let $X := L^p(\Omega)$ be the Lebesgue space equipped with the norm $\|\cdot\|_p$. Define the operator A as follows:*

$$D(A) = W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega),\ A(\varphi) = \Delta\varphi,\ \forall\varphi \in D(A),$$

where $\Delta = \sum_{k=1}^{d} \frac{\partial^2}{\partial x_k^2}$ is the Laplace operator. The operator A is sectorial on X, with $\theta \in]\frac{\pi}{2}, \pi[$.

In the case of a sectorial operator, it is possible to define for every $t > 0$ a linear bounded operator e^{tA} in X, by the mean of the Dunford integral

$$e^{tA} := \frac{1}{2\pi i} \int_{\omega+\gamma_{r,\eta}} e^{t\lambda} R(\lambda, A) d\lambda, \quad t > 0, \qquad e^{0A} := I,$$

where $r > 0$, $\eta \in]\frac{\pi}{2},\ \theta[$, are properly chosen, and $\gamma_{r,\eta}$ is the curve

$$\{\lambda \in \mathbb{C} : |arg\lambda| = \eta, |\lambda| \geq r\} \cup \{\lambda \in \mathbb{C} : |arg\lambda| \leq \eta, |\lambda| = r\},$$

oriented counterclockwise, for more precisions see [83, 126].
Since the operator A is sectorial, this integral converges in $\mathcal{L}(X)$. By Cauchy's Theorem, the definition of $(T(t))_{t\geq 0}$ is independent of the choice of η and r. Moreover, we will see that the operator families $(T(t))_{t\geq 0}$ are analytic semigroups in the following sense

Definition 2.2.8. *([52, Definition II.4.5]) A family of bounded linear operators $(T(z))_{z\in S_\delta\cup 0}$ is called* analytic *semigroup of angle $\delta \in]0, \frac{\pi}{2}]$ if*

(i) $T(0) = I$ and $T(z_1 + z_2) = T(z_1)T(z_2)$ for $z_1,\ z_2 \in S_\delta$.

(ii) The map $z \mapsto T(z)$ is analytic in S_δ.

(iii) For all $x \in X$ and $0 < \delta' < \delta$

$$\lim_{S_{\delta'}\ni z\to 0} T(z)x = x.$$

A semigroup $(T(t))_{t\geq 0}$ of bounded linear operators on X is called analytic, if the mapping

$$]0; \infty) \to X : \quad t \mapsto T(t)x$$

has an analytic extension to a sector S_δ for some $\delta > 0$ and for all $x \in X$.
Remark that the semigroup property holds then automatically in the whole sector S_δ and that $(T(z))_{z\in S_\delta\cup 0}$ is strongly continuous. If $z \mapsto T(z)$ is also strongly continuous in $S_\delta \cup 0$ and $\lim_{z\to 0} T(z)x = x$ for all $x \in X$ then $(T(t))_{t\geq 0}$ is an analytic C_0-semigroup.

Then we obtain the following result.

Theorem 2.2.9. *Let X be a Banach space. Then a densely defined linear operator A is a generator of an analytic semigroup $(T(t))_{t\geq 0}$ of bounded linear operators $T(t) : X \to X, t > 0$, if and only if A is a sectorial operator in X.*

Remark 2.2.10. *We point out that in most of the above results the density of the domain of A is not needed. However, some authors consider sectorial operators without assumption that A is densely defined. In this case semigroups are known which are analytic but not strongly continuous. This is treated in detail in [83].*

In the following, we state some interesting properties concerning analytic operators (see [83]):

Theorem 2.2.11. *Let A be a sectorial operator in a Banach space X and let $(T(t))_{t\geq 0}$ be its analytic semigroup. Then, the following statements hold.*

(i) $T(t)x \in D(A)$ for all $t > 0,\ x \in X$. If $x \in D(A)$, then

$$AT(t)x = T(t)Ax,\ t \geq 0.$$

(ii) There are positive constants M_0, M_1, such that

$$\|T(t)\|_{\mathcal{L}(X)} < M_0\, e^{\omega t}, \qquad t \geq 0,$$
$$\|AT(t)\|_{\mathcal{L}(X)} < \frac{M_1}{t}\, e^{\omega t}, \quad t > 0,$$

where ω is the number in (2.1).

2.2.3 Hyperbolic semigroups

Next, we introduce and study hyperbolic semigroups.

Definition 2.2.12. *A semigroup $(T(t))_{t\geq 0}$ on a Banach space X is said to be hyperbolic if it satisfies the condition* **(H)***:*

(i) There exist two subspaces X_s (the stable space) and X_u (the unstable space) of X such that $X = X_s \oplus X_u$;

(ii) $T(t)X_u \subset X_u$, and $T(t)X_s \subset X_s$ for all $t \geq 0$;

(iii) There exist constants M, $\delta > 0$ such that

$$\|T(t)P_s\| \leq Me^{-\delta t}, \quad t \geq 0, \quad \|T(t)P_u\| \leq Me^{\delta t}, \quad t \leq 0, \tag{2.2}$$

where P_s and P_u are, respectively, the projections onto X_s and X_u.

The most important example of hyperbolic semigroups are the exponentially stable semigroups. In the parabolic case, one obtains regularity properties of the exponential dichotomy, see [8]. For instance, $A|P_u : P_u(X) \longrightarrow P_u(X)$ is bounded, it follows that $||AP_u|| \leq c$.

Recall that an analytic semigroup $(T(t))_{t\geq 0}$ associated with the linear operator A is hyperbolic if and only if

$$\sigma(A) \cap i\mathbb{R} = \emptyset.$$

For details, see, e.g. [52, Prop 1.15, p. 305].

2.3 Interpolation and extrapolation spaces

We begin in this section, by fixing some notations and recalling a few basic results on interpolation and extrapolation spaces of generators. For more details, we refer the reader to [5, 52, 83, 90]. Let A be sectorial operator on X (i.e., (2.1) is satisfied) and $\alpha \in (0, 1)$. We introduce the real interpolation spaces

$$X^A_{\alpha,\infty} := \{x \in X : \sup_{\lambda>0} \|\lambda^\alpha (A - \omega)R(\lambda, A - \omega)x\| < \infty\}, \quad X^A_\alpha := \overline{D(A)}^{||\cdot||_\alpha},$$

with

$$||x||_\alpha := \sup_{\lambda>0} ||\lambda^\alpha (A - \omega)R(\lambda, A - \omega)x||.$$

They are Banach spaces when endowed with the norm $|| \cdot ||_\alpha$.

For convenience we further write $X^A_0 := X$, $X^A_1 := D(A)$ and $||x||_0 = ||x||, ||x||_1 = ||(A - \omega)x||$. We also define on the closed subspace $\widehat{X}^A := \overline{D(A)}$ of X a new norm by

$$\|x\|_{-1} = \left\|(\omega - A)^{-1}x\right\|, \quad x \in X.$$

The completion of $\left(\widehat{X}^A, \|\cdot\|_{-1}\right)$ is called the *extrapolation space* of X associated to A and will be denoted by X^A_{-1}. Then A has a unique continuous extension $A_{-1} : \widehat{X}^A \longrightarrow X^A_{-1}$. Since $T(t)$ commutes with the operator resolvent $R(\omega, A) := (\omega I - A)^{-1}$, the extension of $T(t)$ to X^A_{-1} exists and defines an analytic semigroup $(T_{-1}(t))_{t\geq 0}$ which is generated by A_{-1} with $D(A_{-1}) = \widehat{X}$.

As above, we can then define the space

$$X^A_{\alpha-1} := (X_{-1})^{A_{-1}}_{\alpha} = \overline{\widehat{X}^A}^{||\cdot||_{\alpha-1}}$$

with

$$||x||_{\alpha-1} = \sup_{\lambda>0} ||\lambda^\alpha R(\lambda, A_{-1} - \omega)x||.$$

The restriction $A_{\alpha-1} : X^A_\alpha \longrightarrow X^A_{\alpha-1}$ of A_{-1} generates the analytic semigroup $T_{\alpha-1}(t))_{t\geq 0}$ on $X^A_{\alpha-1}$ which is the extension of $T(t)$ to $X^A_{\alpha-1}$. Observe that $\omega - A_{\alpha-1} : X^A_\alpha \longrightarrow X^A_{\alpha-1}$ is an isometric isomorphism. We will frequently use the continuous embedding

$$\begin{aligned} &D(A) \hookrightarrow X^A_\beta \hookrightarrow D((\omega - A)^\alpha) \hookrightarrow X^A_\alpha \hookrightarrow \widehat{X}^A \subset X, \\ &X \hookrightarrow X^A_{\beta-1} \hookrightarrow D((\omega - A_{-1})^\alpha) \hookrightarrow X^A_{\alpha-1} \hookrightarrow X^A_{-1} \end{aligned} \tag{2.3}$$

for all $0 < \alpha < \beta < 1,$ where the fractional powers are defined as usually.

The real and continuous interpolation and fractional power spaces are in the class of spaces Y satisfying $D(A) \hookrightarrow Y \hookrightarrow X$, and there is a constant $c > 0$ such that

$$||x||_Y \leq c||x||^{1-\alpha}||x||^\alpha_A, \;\; x \in D(A),$$

called *intermediate spaces between* $D(A)$ *and* X or of class $\mathcal{J}_\alpha$. For more details about intermediate spaces, see [52, Chap. II, Section 5.b] and [83].

We give an embedding result of extrapolation spaces needed for Chapter 5. Here, we give the proof in the more general C_0-semigroups context, see [19].

Lemma 2.3.1. *Let A be the generator of a C_0-semigroup $T(\cdot)$ on a Banach space Z. Let Y be an $T(\cdot)$-invariant closed subspace of Z. Endow Y with the norm of Z and consider the restriction A_Y of A to Y. Then the space $Y^{A_Y}_{-1}$ is canonically embedded into Z^A_{-1} as a closed subspace.*

Proof. The operator A_Y generates the semigroup of the restrictions $T_Y(t) \in \mathcal{L}(Y)$ of $T(t)$. By rescaling we may assume that $\|T_Y(t)\| \leq \|T(t)\| \leq ce^{-\epsilon t}$ for some $\epsilon > 0$ and all $t \geq 0$. Observe that then A and A_Y are invertible and that

$$A^{-1}_Y = \int_0^\infty T_Y(t)y\,dt = \int_0^\infty T(t)y\,dt = A^{-1}y$$

for each $y \in Y$. We mostly write A instead of A_Y, and we endow the extrapolation spaces of A and A_Y with the norm $\|x\|_{-1} = \|A^{-1}_{-1}x\|$. By definition, it holds

$$Y^A_{-1} = \{y = (y_n) + N_Y : (y_n) = (y_n)_{n\in\mathbb{N}} \subset Y \;\text{ is Cauchy for }\; \|\cdot\|_{-1}\},$$

where $N_Y = \{(y_n) \subset Y : y_n \to 0 \;\text{ for }\; \|\cdot\|_{-1}\}$. We identify $y \in Y$ with the element $(y)_{n\in\mathbb{N}} + N_Y$ of Y^A_{-1}, thus considering Y as a dense subspace of Y^A_{-1}. We define the operator

$$\Phi : Y^A_{-1} \longrightarrow Z^A_{-1}, \quad \Phi y = (y_n) + N_Z, \qquad \text{where }\; y_n \in Y,\; y_n \to y \text{ in } Y^A_{-1}.$$

If $(y_n), (\tilde{y}_n) \subset Y$ converge to y in Y_{-1}^A, then $y_n - \tilde{y}_n \to 0$ as $n \to \infty$ for $\|\cdot\|_{-1}$. Hence, $(y_n - \tilde{y}_n) \in N_Z$, and so Φ is well defined. Let $y \in Y_{-1}^A$ such that $\Phi y = 0$. This means that $(y_n) \in N_Z$, and hence $y_n \to 0$ in $\|\cdot\|_{-1}$. Therefore $(y_n) \in N_Y$, and thus $y = 0$. It is clear that Φ is linear. It is also bounded since

$$\|\Phi y\|_{Z_{-1}^A} = \inf_{(z_n)\in N_Z} \|(y_n - z_n)\|_\infty \le \inf_{(z_n)\in N_Y} \|(y_n - z_n)\|_\infty = \|y\|_{Y_{-1}^A}.$$

We have shown that $Y_{-1}^A \hookrightarrow Z_{-1}^A$ with the canonical embedding Φ. To prove that the range $R(\Phi)$ is closed in Z_{-1}^A, we take $z_j = \Phi y_j \in R(\Phi) \subseteq Z_{-1}^A$ such that $z_j \to z$ in Z_{-1}^A as $j \to \infty$. Then $A_{-1}^{-1} z_j =: w_j$ converges in Z to $w := A_{-1}^{-1} z$. We further claim that

$$A_{-1}^{-1}\Phi = (A_Y)_{-1}^{-1}. \tag{2.4}$$

Indeed, for $x \in Y$ one has $A_{-1}^{-1}\Phi x = A^{-1}x = A_Y^{-1}x = (A_Y)_{-1}^{-1}x$. So assertion (2.4) follows from the density of Y in Y_{-1}^A. Equation (2.4) then yields

$$(A_Y)_{-1}^{-1} y_j = A_{-1}^{-1} z_j \to w \qquad (\text{in } Z).$$

Since Y is closed in Z and $(A_Y)_{-1}^{-1} y_j \in Y$, we obtain $(A_Y)_{-1}^{-1} y_j \to w$ in Y. As a consequence, y_j converges in Y_{-1}^A to $y := (A_Y)_{-1} w$. We conclude that $z_j = \Phi y_j \to \Phi y$ in Z_{-1}^A which means that $R(\Phi)$ is closed. □

In the sequel of this thesis, we omit the exponent A in the definition of the interpolation and extrapolation spaces.

In the following proposition, we give some estimates of C_0-semigroups on interpolation and extrapolation spaces, needed to obtain results of Chapter 4, see [18].

Proposition 2.3.2. *Assume that* $0 < \alpha \le 1$ *and* $0 \le \beta \le 1$. *Then the following assertions hold for* $0 < t \le t_0$, $t_0 > 0$ *and* $\widetilde{\varepsilon} > 0$ *such that* $0 < \alpha - \widetilde{\varepsilon} < 1$ *with constants possibly depending on* t_0.

(i) The operator $T(t)$ *has continuous extensions* $T_{\alpha-1}(t) : X_{\alpha-1} \longrightarrow X$ *satisfying*

$$||T_{\alpha-1}(t)||_{\mathcal{L}(X_{\alpha-1},X)} \le c\, t^{\alpha-1-\widetilde{\varepsilon}}. \tag{2.5}$$

(ii) For $x \in X_{\alpha-1}$ *we have*

$$||T_{\alpha-1}(t)x||_\beta \le c\, t^{\alpha-\beta-1-\widetilde{\varepsilon}}||x||_{\alpha-1}. \tag{2.6}$$

Proof. Let $0 < t \le t_0$, $0 < \alpha - \widetilde{\varepsilon} < 1$ and $x \in X_{\alpha-1} \hookrightarrow D((\omega - A_{-1})^{\alpha-\widetilde{\varepsilon}})$. We have

$$\begin{aligned}
||T_{\alpha-1}(t)x|| &= ||T_{-1}(t)(\omega - A_{-1})^{-\alpha+\widetilde{\varepsilon}}(\omega - A_{-1})^{\alpha-\widetilde{\varepsilon}}x|| \\
&= ||(\omega - A)^{-\alpha+\widetilde{\varepsilon}}T(\frac{t}{2})T_{-1}(\frac{t}{2})(\omega - A_{-1})^{\alpha-\widetilde{\varepsilon}}x|| \\
&= ||(\omega - A)^{-\alpha+\widetilde{\varepsilon}+1}T(\frac{t}{2})A_{-1}^{-1}T_{-1}(\frac{t}{2})(\omega - A_{-1})^{\alpha-\widetilde{\varepsilon}}x|| \\
&\le (\frac{t}{2})^{\alpha-\widetilde{\varepsilon}-1}||T_{-1}(\frac{t}{2})(\omega - A_{-1})^{\alpha-\widetilde{\varepsilon}}x||_{-1} \\
&\le (\frac{t}{2})^{\alpha-1-\widetilde{\varepsilon}}||T_{-1}(\frac{t}{2})||_{\mathcal{L}(X_{-1})}\ ||(\omega - A_{-1})^{\alpha-\widetilde{\varepsilon}}x||_{-1} \\
&\le 2^{1-\alpha+\widetilde{\varepsilon}} \sup_{0\le s\le t_0} ||T_{-1}(s)||_{\mathcal{L}(X_{-1})}t^{\alpha-1-\widetilde{\varepsilon}}||x||_{D((\omega-A_{-1})^{\alpha-\widetilde{\varepsilon}})}.
\end{aligned}$$

Hence by (2.3) we obtain

$$||T_{\alpha-1}(t)x|| \le c\, t^{\alpha-1-\widetilde{\varepsilon}}||x||_{\alpha-1}.$$

Finally, by (2.5) we have

$$\begin{aligned}
||T_{\alpha-1}(t)x||_{\beta} &\le c\, ||T_{\alpha-1}(t)x||^{1-\beta}||AT(\frac{t}{2})T_{\alpha-1}(\frac{t}{2})x||^{\beta} \\
&\le c\, t^{(\alpha-1-\widetilde{\varepsilon})(1-\beta)}(\frac{t}{2})^{-\beta}(\frac{t}{2})^{(\alpha-1-\widetilde{\varepsilon})\beta}||x||_{\alpha-1} \\
&\le c\, t^{\alpha-1-\widetilde{\varepsilon}-\beta}\|x\|_{\alpha-1}.
\end{aligned}$$

□

Remark 2.3.3. *We can remove $\widetilde{\varepsilon}$ in Proposition 2.3.2 by extending $T(t)$ to operators from $D((\omega - A_{-1})^{\alpha\pm\widetilde{\varepsilon}})$ to X, with norms bounded by $t^{\alpha-1\pm\widetilde{\varepsilon}}$, where $0 < \alpha \pm \widetilde{\varepsilon} < 1$, and therefore, by employing the reiteration theorem and the interpolation property, the inequality in the assertion (i) can be obtained without $\widetilde{\varepsilon}$.*

2.4 Parabolic evolution operators

We investigate a family of linear operators $A(t)$, $t \in \mathbb{R}$, on a Banach space X subject to the following hypotheses:
(H1) There are constants $\omega \in \mathbb{R}$, $\theta \in (\pi/2, \pi)$, $K > 0$ and $\mu, \nu \in (0, 1]$ such that $\mu + \nu > 1$ and

$$\lambda \in \rho(A(t) - \omega), \qquad \|R(\lambda, A(t) - \omega)\| \le \frac{K}{1 + |\lambda|}\,, \tag{2.7}$$

$$\|(A(t) - \omega)R(\lambda, A(t) - \omega)\,[R(\omega, A(t)) - R(\omega, A(s))]\| \le K\frac{|t - s|^{\mu}}{|\lambda|^{\nu}} \tag{2.8}$$

for all $t, s \in \mathbb{R}$ and $\lambda \in \Sigma_{\theta} := \{\lambda \in \mathbb{C}\backslash\{0\} \text{ with} \,|\arg(\lambda)| \le \theta\}$. (Observe that the domains $D(A(t))$ are not required to be dense.)

Remark 2.4.1. *In the case of a constant domain $D(A(t))$, one can replace assumption* (2.8) *(see e.g. [5, 97]) with the following*
(H1)' *There exist constants L and $0 < \mu \leq 1$ such that*

$$\|(A(t) - A(s))A(r)^{-1}\| \leq L|t - s|^{\mu}, \; s, t, r \in \mathbb{R}.$$

Let us mention that assumption **(H1)** was introduced in the literature by P. Acquistapace and B. Terreni in [3, 2] for $\omega = 0$. Among other things, it ensures that there exists a unique evolution family U on X such that:

(a) $U(t, s)U(s, r) = U(t, r)$ and $U(t, t) = I$ for $t \geq s \geq r$;

(b) $(t, s) \mapsto U(t, s) \in \mathcal{L}(X)$ is continuous for $t > s$;

(c) $U(\cdot, s) \in C^1((s, \infty), \mathcal{L}(X))$, $\dfrac{\partial U}{\partial t}(t, s) = A(t)U(t, s)$ and

$$\|A(t)^k U(t, s)\| \leq C\,(t - s)^{-k} \tag{2.9}$$

for $0 < t - s \leq 1$, $k = 0, 1$, $x \in D((\omega - A(s))^{\alpha})$, and a constant C depending only on the constants appearing in **(H1)**;

(d) $\partial_s^+ U(t, s)x = -U(t, s)A(s)x$ for $t > s$ and $x \in D(A(s))$ with $A(s)x \in \overline{D(A(s))}$.

We define the following interpolation and extrapolation spaces as above

$$X_\alpha^t := X_\alpha^{A(t)}, \quad \hat{X}^t := \hat{X}^{A(t)}, \;\; X_{\alpha-1}^t := X_{\alpha-1}^{A(t)}$$

for $0 \leq \alpha \leq 1$ and $t \in \mathbb{R}$, with the corresponding norms. Then the embedding in (2.3) hold with constants independent of $t \in \mathbb{R}$, and there is a constant $c(\alpha)$ such that

$$\|y\|_\alpha^t \leq c(\alpha)\|y\|^{1-\alpha}\|(A(t) - \omega)y\|^{\alpha}, \quad y \in D(A(t)), \quad t \in \mathbb{R}. \tag{2.10}$$

For a closed interval J, we define on $E = E(J) := BC(J, X)$, the multiplication operator $A(\cdot)$ by

$$\begin{aligned} (A(\cdot)f)(t) &:= A(t)f(t) \qquad \text{for all } \; t \in J, \\ D(A(\cdot)) &:= \{f \in E : f(t) \in D(A(t)) \text{ for all } t \in J, \; A(\cdot)f \in E\}. \end{aligned}$$

We can thus introduce the spaces

$$E_\alpha := E_\alpha^{A(\cdot)}, \quad E_{\alpha-1} := E_{\alpha-1}^{A(\cdot)}, \quad \text{and} \quad \hat{E} := \overline{D(A(\cdot))}$$

for $\alpha \in [0, 1]$, where $E_0 := E$ and $E_1 := D(A(\cdot))$. We observe that $E_{-1} \subseteq \prod_{t \in J} X_{-1}^t$ and that the extrapolated operator $A(\cdot)_{-1} : \hat{E} \longrightarrow E_{-1}$ is given by $(A(\cdot)_{-1}f)(t) := A_{-1}(t)f(t)$ for $t \in J$ and $f \in E$. Further, $E_{\alpha-1}$ has the norm

$$\|f\|_{\alpha-1} := \sup_{r>0} \sup_{s \in J} \|r^{\alpha} R(r, A_{-1}(s) - \omega)f(s)\|,$$

and we have $f(t) \in X^t_{\alpha-1}$ for each $t \in J$ if $f \in E_{\alpha-1}$. Since $R(n, A_{\alpha-1}(\cdot))$ is the resolvent of the densely defined sectorial operator $A_{\alpha-1}(\cdot)$, we have $nR(n, A_{\alpha-1}(\cdot))f \to f$ in $E_{\alpha-1}$ as $n \to \infty$, for each $f \in E_{\alpha-1}$ and $0 \le \alpha < 1$.

The next lemma allows to extend the evolution family $U(t,s)$ to the extrapolated spaces $X^t_{\alpha-1}$, see Proposition 2.1 and Remark 3.12 of [87] for the proof.

Lemma 2.4.2. *Assume that* (2.7) *and* (2.8) *hold and let* $1-\mu < \alpha < 1$. *Then the following assertions hold for* $s < t \le s + t_0$ *and* $t_0 > 0$.
(a) The operators $U(t,s)$ *have continuous extensions* $U_{\alpha-1}(t,s) : X^s_{\alpha-1} \to X\beta^t$ *satisfying*

$$\|U_{\alpha-1}(t,s)\|_{\mathcal{L}(X^s_{\alpha-1},X^t_\beta)} \le c(\alpha,t_0)(t-s)^{\alpha-\beta-1}\,, \tag{2.11}$$

and $U_{\alpha-1}(t,s)x = U_{\gamma-1}(t,s)x$ *for* $1-\mu < \gamma < \alpha < 1$, $\beta \in [0,1]$, *and* $x \in X^s_{\alpha-1}$.
(b) The map $\{(t,s) : t > s\} \ni (t,s) \longmapsto U_{\alpha-1}(t,s)f(s) \in X$ *is continuous for* $f \in E_{\alpha-1}$.

Exponential dichotomy is another important tool in our study, cf. [34], [83], [109]. We recall that an evolution family $U(\cdot,\cdot)$ has an *exponential dichotomy* on an interval J if there exists a family of projections $P(t) \in \mathcal{L}(X)$, $t \in J$, being strongly continuous with respect to t, and constants $\delta, N > 0$ such that

(a) $U(t,s)P(s) = P(t)U(t,s)$,

(b) $U(t,s) : Q(s)(X) \to Q(t)(X)$ is invertible with the inverse $\widetilde{U}(s,t)$,

(c) $\|U(t,s)P(s)\| \le Ne^{-\delta(t-s)}$ and $\|\widetilde{U}(s,t)Q(t)\| \le Ne^{-\delta(t-s)}$

for all $s,t \in J$ with $s \le t$, where $Q(t) := I - P(t)$ is the 'unstable projection'. One further defines *Green's function* by

$$\Gamma(t,s) = \begin{cases} U(t,s)P(s), & t \ge s,\ t,s \in J, \\ -\widetilde{U}(t,s)Q(s), & t < s,\ t,s \in J. \end{cases}$$

In the parabolic case one easily obtains regularity results for Green's function and the dichotomy projections, see e.g. [109, Proposition 3.18]. For instance, if J is bounded from below, then we have $\|A(t)Q(t)\| \le c(\eta)$ for all $t > \eta + \inf J$ and each $\eta > 0$ since $A(t)Q(t) = A(t)U(t,t-\eta)\tilde{U}(t-\eta,t)Q(t)$. Similarly, it holds $\|A(t)Q(t)\| \le c$ for all $t \in J$ if J is unbounded from below. As a consequence $P(t) = I - Q(t)$ leaves invariant $\hat{X}^t$ and X^t_α for each $\alpha \in [0,1]$ and $t \in J\backslash\{\inf J\}$. In the next proposition (shown in Proposition 2.2 and Remark 3.12 of [87]) we state some properties of $\Gamma(t,s)$ and $Q(t)$ in extrapolation spaces. We use the convention $\pm\infty + r = \pm\infty$ for $r \in \mathbb{R}$, and we set $J' = J\backslash\{\sup J\}$, i.e., $J = J'$ if J is unbounded from above. Moreover, we write $U_0(t,s) := U(t,s)$, $P_0(t) := P(t)$, and $Q_0(t) := Q(t)$, where $X^t_0 = X$ by definition.

In the following proposition, we state a result from [87] concerning the exponential dichotomy of the extrapolated evolution family $U_{\alpha-1}$.

Proposition 2.4.3. *Assume that* (2.7) *and* (2.8) *hold and that* $U(t,s)$ *has an exponential dichotomy on an interval* J. *Let* $\eta > 0$ *and* $1-\mu < \alpha \le 1$. *Then the* *operators $P(t)$ and $Q(t)$ have continuous extensions $P_{\alpha-1}(t) : X^t_{\alpha-1} \to X^t_{\alpha-1}$ and $Q_{\alpha-1}(t) : X^t_{\alpha-1} \to X$, respectively, for every $t \in J'$; which are uniformly bounded for $t < \sup J - \eta$. Moreover, the following assertions hold for $t, s \in J'$ with $t \geq s$.*

(a) $Q_{\alpha-1}(t)X^t_{\alpha-1} = Q(t)X$;

(b) $U_{\alpha-1}(t,s)P_{\alpha-1}(s) = P_{\alpha-1}(t)U_{\alpha-1}(t,s)$;

(c) $U_{\alpha-1}(t,s) : Q_{\alpha-1}(s)(X^s_{\alpha-1}) \to Q_{\alpha-1}(t)(X^t_{\alpha-1})$ *is invertible with the inverse* $\widetilde{U}_{\alpha-1}(s,t)$;

(d) $\|U_{\alpha-1}(t,s)P_{\alpha-1}(s)x\| \leq N(\alpha,\eta)\max\{(t-s)^{\alpha-1}, 1\}e^{-\delta(t-s)}\|x\|^s_{\alpha-1}$ *for* $x \in X^s_{\alpha-1}$ *and* $s < t < \sup J - \eta$;

(e) $\|\widetilde{U}_{\alpha-1}(s,t)Q_{\alpha-1}(t)x\| \leq N(\alpha,\eta)e^{-\delta(t-s)}\|x\|^t_{\alpha-1}$ *for* $x \in X^t_{\alpha-1}$ *and* $s \leq t < \sup J - \eta$;

(f) let $J_0 \subset J'$ *be a closed interval and* $f \in E_{\alpha-1}(J_0)$*. Then* $P(\cdot)f \in E_{\alpha-1}(J_0)$ *and* $Q(\cdot)f \in BC(J_0, X)$.

Using this proposition, we define

$$\Gamma_{\alpha-1}(t,s) = \begin{cases} U_{\alpha-1}(t,s)P_{\alpha-1}(s), & t \geq s,\ t,s \in J, \\ -\widetilde{U}(t,s)Q_{\alpha-1}(s), & t < s,\ t,s \in J. \end{cases}$$

2.5 Almost periodic and almost automorphic functions

In this section, we recall some definitions and properties concerning almost periodic and almost automorphic functions that we will use later in this thesis.

2.5.1 Almost periodic functions

Next, we give the definition of almost periodic functions due to H. Bohr [28]. The theory of almost periodicity has been generalized in various directions especially by Favard [53, 54], Bochner [26, 27], Levitan [79], Besicovitc [25], Fink [55], and Corduneanu [36]. Recently, motivated by applications, important extensions have been given to the study of almost periodic functions (partial differential equations) see [6, 80, 97] and references therein.

A set $P \subset \mathbb{R}$ is said to be relatively dense in $\mathbb{R}$ if there exists a number $l > 0$ such that any interval $[a, a+l]$, $a \in \mathbb{R}$ of length l contains at least one number from P.

Definition 2.5.1. *A continuous function $f : \mathbb{R} \to X$ is called* almost periodic *if for every $\varepsilon > 0$ there exist a relative dense set $P(\varepsilon) \subseteq \mathbb{R}$, that is, if there is a number $\ell(\varepsilon) > 0$ such that each interval $(a, a+\ell(\varepsilon))$, $a \in \mathbb{R}$, contains an* almost period *$\tau = \tau_\varepsilon \in P(\varepsilon)$ and the estimate*

$$\|g(t+\tau) - g(t)\| \leq \varepsilon$$

holds for all $t \in \mathbb{R}$ and $\tau \in P(\varepsilon)$. The space of almost periodic functions is denoted by $AP(\mathbb{R}, X)$.

Every periodic function is also almost periodic. On the other hand, the inverse is not true, we cite as counterexample the function $f(t) = \cos(t) + \cos(t\sqrt{2}); t \in \mathbb{R}$. (see Figure: 5.1.3).

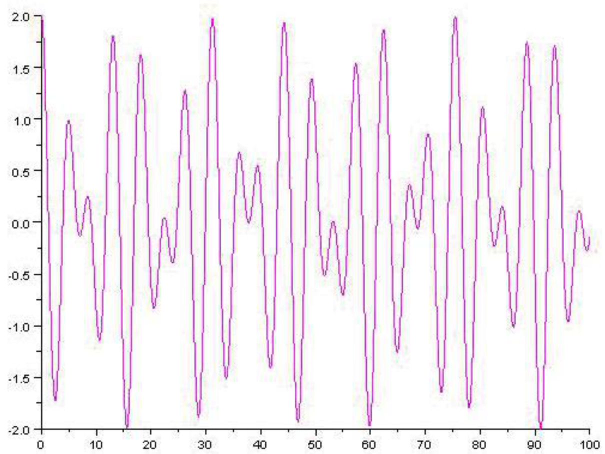

Figure 2.2: *An example of almost periodic function.*

Let us recall that $AP(\mathbb{R}, X)$ is a closed subspace of $BC(\mathbb{R}, X)$ and hence it is itself a Banach space, see [80, Chapter 1].

For a closed unbounded interval J, we also define the space

$$AP(J, X) := \{g : J \to X : \exists \tilde{g} \in AP(\mathbb{R}, X) \text{ s.t. } \tilde{g}|J = g\}$$

of almost periodic functions on J. We remark that the function $\tilde{g}$ in the above definition is uniquely determined, cf. [12, Proposition 4.7.1]. The following notion is important for our investigations.

Definition 2.5.2. *Let $J = [t_0, \infty)$, $t_0 \in \mathbb{R}$. A continuous function $g : J \to X$ is called* asymptotically almost periodic *if for every $\varepsilon > 0$ there exists a set $P(\varepsilon) \subseteq J$ and numbers $s(\varepsilon), \ell(\varepsilon) > 0$ such that each interval $(a, a+\ell(\varepsilon))$, $a \geq 0$, contains an* almost period *$\tau = \tau_\varepsilon \in P(\varepsilon)$ and the estimate $\|g(t+\tau) - g(t)\| \leq \varepsilon$ holds for all $t \geq s(\varepsilon)$ and $\tau \in P(\varepsilon)$. The space of asymptotically almost periodic functions is denoted by $AAP(J, Y)$.*

Due to [12, Theorem 4.7.5], these spaces are related by the equality

$$AAP([t_0, +\infty), X) = AP([t_0, +\infty), X) \oplus C_0([t_0, +\infty), X). \tag{2.12}$$

Analogously, we define the asymptotic almost periodicity on $J = (-\infty, t_0]$, and one also has

$$AAP((-\infty, t_0], X) = AP((-\infty, t_0], X) \oplus C_0((-\infty, t_0], X). \tag{2.13}$$

We recall that $M(\cdot)f \in (A)AP(J,X)$ if $f \in (A)AP(J,X)$ and $M(\cdot) \in (A)AP(J,\mathcal{L}(X))$. This follows from the above definitions if one takes into account that we can find common almost periods for f and M, cf. [80, p.6].

Definition 2.5.3. *A function $f \in BC(\mathbb{R} \times X, Y)$ is called almost periodic if for every $\varepsilon > 0$ and every compact set $K \subset X$ there exists $l(\varepsilon, K) > 0$ such that every interval I of length $l(\varepsilon, K)$ contains a number τ and for $t \in \mathbb{R}$, $x \in K$*

$$\|f(t+\tau, x) - f(t,x)\|_Y < \varepsilon$$

Finally, we mention the following important result.

Lemma 2.5.4. *[55] Let $f : \mathbb{R} \times X \mapsto Y$ be (uniformly) almost periodic, globally Lipschitzian and $y : \mathbb{R} \mapsto X$ be an almost periodic function, then the function $t \mapsto f(t, y(t))$ is also almost periodic.*

2.5.2 Almost automorphic functions

In this subsection, we introduce a class of functions which are more general than the almost periodic ones. Named almost automorphic functions, they were first introduced by S. Bochner [26] in 1955. For more information on almost automorphic functions, we refer the reader to W. A. Veech [113, 114, 115] and others [93, 95, 111, 128].

Definition 2.5.5. *(S. Bochner) A continuous function $f : \mathbb{R} \to X$ is called almost automorphic if for every sequence $(\sigma_n)_{n\in\mathbb{N}}$ there exists a subsequence $(s_n)_{n\in\mathbb{N}} \subset (\sigma_n)_{n\in\mathbb{N}}$ such that*

$$\lim_{n,m\to+\infty} f(t + s_n - s_m) = f(t) \quad \text{for each } t \in \mathbb{R}.$$

This is equivalent to the fact that the limits

$$g(t) := \lim_{n\to+\infty} f(t+s_n) \quad \text{and} \quad f(t) = \lim_{n\to+\infty} g(t-s_n)$$

exist for each $t \in \mathbb{R}$.

The set of all almost automorphic functions with values in X is denoted by $AA(X)$. With the supremum norm

$$\|f\|_{AA(X)} = \sup_{t\in\mathbb{R}} \|f(t)\|,$$

this space turns out to be a Banach space (see [93], page 20).

Remark 2.5.6. *1) It is easy to see that an almost automorphic function is always bounded.*

2) By the pointwise convergence, the function g in Definition 2.5.5 is just measurable, but not necessarily continuous (need not be continuous in general). Moreover, if g is continuous, then f is uniformly continuous (cf. [96] Theorem 2.6).

3) If the convergence in both limits in the definition above is uniform in $t \in \mathbb{R}$, then f is almost periodic (in the sense of Bochner [27]).

4) Clearly from the definition above follows that every almost periodic function (in the sense of Bochner) is necessarily almost automorphic. Thus we have

$$AP(X) \subset AA(X) \subset BC(X).$$

The converse of the last assertion in Remark 2.5.6 is not true, as shown in the following example due to Levitan (see also [23, Example 3.3]).

Example 2.5.7. *Let $p(t) = 2 + \cos t + \cos \sqrt{2}t$ and $f : \mathbb{R} \to \mathbb{R}$ such that $f = \sin \frac{1}{p}$. Then f is almost automorphic, but f is not uniformly continuous on $\mathbb{R}$. It follows that f is not almost periodic.*

Definition 2.5.8. *A function $f : \mathbb{R} \times X \to Y$ is said to be almost automorphic if $f(\cdot, x)$ is almost automorphic for every $x \in X$ and f is continuous jointly in (t, x). We note $f \in AA(\mathbb{R} \times X, Y)$.*

Moreover, we refer to [47, 93, 94, 111] for some new and significant developments on the study of almost automorphic problems.

Chapter 3

Asymptotic behavior of semilinear evolution equations

The aim of this chapter is to study the almost periodicity and the almost automorphicity of solutions of the parabolic semilinear evolution equations

$$u'(t) = A(t)u(t) + f(t, u(t)), \quad t \in \mathbb{R}, \tag{3.1}$$

in a Banach space X, where the linear operators $A(t)$ satisfy the Acquistapace-Terreni conditions, the evolution family U generated by $A(\cdot)$ has an exponential dichotomy and $f : \mathbb{R} \times X_\alpha \to X$. We apply these results to thermoelastic plate systems and the reaction diffusion equation with time-varying coefficients. We show that, if the coefficients and the semilinear term f are almost periodic resp. automorphic, then the solutions are almost periodic resp. automorphic.

3.1 Assumptions and preliminary results

Let $(X, \|\cdot\|)$ be a Banach space and let $A(t)$ for $t \in \mathbb{R}$ be closed linear operators on X with domain $D(A(t))$ (possibly not densely defined). Throughout this chapter, we assume that $A(t)$ satisfies the Aquistapace-Terreni conditions **(H1)**, see Chapter 2, and

(H2) The evolution family U generated by $A(\cdot)$ has an exponential dichotomy with constants $N, \delta > 0$ and dichotomy projections $P(t)$ for $t \in \mathbb{R}$ and Green's function Γ.

(H3) There exist $0 \le \alpha < \beta < 1$ such that

$$X_\alpha^t = X_\alpha \quad \text{and} \quad X_\beta^t = X_\beta$$

for all $t \in \mathbb{R}$, with uniformly equivalent norms.

For the sequel, we need the following fundamental estimates for the evolution family $U := U(t, s)$ generated by $A(\cdot)$.

Proposition 3.1.1. *For $x \in X$, $0 \le \alpha \le 1$ and $t > s$, the following hold:*

(i) There is a constant $c(\alpha)$, such that

$$\|U(t,s)P(s)x\|_\alpha^t \le c(\alpha)e^{-\frac{\delta}{2}(t-s)}(t-s)^{-\alpha}\|x\|. \tag{3.2}$$

(ii) There is a constant $m(\alpha)$, such that

$$\|\widetilde{U}_Q(s,t)Q(t)x\|_\alpha^s \le m(\alpha)e^{-\delta(t-s)}\|x\|. \tag{3.3}$$

Proof. (i) Using (2.10) we obtain

$$\begin{aligned}
\|U(t,s)P(s)x\|_\alpha^t &\le c(\alpha)\|U(t,s)P(s)x\|^{1-\alpha}\|A(t)U(t,s)P(s)x\|^\alpha \\
&\le c(\alpha)\|U(t,s)P(s)x\|^{1-\alpha}\|A(t)U(t,t-1)U(t-1,s)P(s)x\|^\alpha \\
&\le c(\alpha)\|U(t,s)P(s)x\|^{1-\alpha}\|A(t)U(t,t-1)\|^\alpha\|U(t-1,s)P(s)x\|^\alpha \\
&\le c(\alpha)Nc\, e^{-\delta 2(t-s)(1-\alpha)}e^{-\delta(t-s-1)\alpha}\|x\| \\
&\le c(\alpha)(t-s)^{-\alpha}e^{-\frac{\delta}{2}(t-s)}(t-s)^\alpha e^{-\frac{\delta}{2}(t-s)}\|x\|
\end{aligned}$$

for $t-s \ge 1$ and $x \in X$. Since $(t-s)^\alpha e^{-\frac{\delta}{2}(t-s)} \to 0$ as $t \to +\infty$ it easily follows that

$$\|U(t,s)P(s)x\|_\alpha^t \le c(\alpha)(t-s)^{-\alpha}e^{-\frac{\delta}{2}(t-s)}\|x\|.$$

If $0 < t-s \le 1$, we have

$$\begin{aligned}
\|U(t,s)P(s)x\|_\alpha^t &\le c(\alpha)\|U(t,s)P(s)x\|^{1-\alpha}\|A(t)U(t,s)P(s)x\|^\alpha \\
&\le c(\alpha)\|U(t,s)P(s)x\|^{1-\alpha}\|A(t)U(t,\frac{t+s}{2})U(\frac{t+s}{2},s)P(s)x\|^\alpha \\
&\le c(\alpha)\|U(t,s)P(s)x\|^{1-\alpha}\|A(t)U(t,\frac{t+s}{2})\|^\alpha\|U(\frac{t+s}{2},s)P(s)x\|^\alpha \\
&\le c(\alpha)Ne^{-\delta(t-s)(1-\alpha)}2^\alpha(t-s)^{-\alpha}e^{-\frac{\delta\alpha}{2}(t-s)}\|x\| \\
&\le c(\alpha)Ne^{-\frac{\delta}{2}(t-s)(1-\alpha)}2^\alpha(t-s)^{-\alpha}e^{-\frac{\delta\alpha}{2}(t-s)}\|x\| \\
&\le c(\alpha)e^{-\frac{\delta}{2}(t-s)}(t-s)^{-\alpha}\|x\|,
\end{aligned}$$

and hence

$$\|U(t,s)P(s)x\|_\alpha^t \le c(\alpha)(t-s)^{-\alpha}e^{-\frac{\delta}{2}(t-s)}\|x\| \text{ for } t > s.$$

(ii)

$$\begin{aligned}
\|\widetilde{U}_Q(s,t)Q(t)x\|_\alpha^s &\le c(\alpha)\|\widetilde{U}_Q(s,t)Q(t)x\|^{1-\alpha}\|A(s)\widetilde{U}_Q(s,t)Q(t)x\|^\alpha \\
&\le c(\alpha)\|\widetilde{U}_Q(s,t)Q(t)x\|^{1-\alpha}\|A(s)Q(s)\widetilde{U}_Q(s,t)Q(t)x\|^\alpha \\
&\le c(\alpha)\|\widetilde{U}_Q(s,t)Q(t)x\|^{1-\alpha}\|A(s)Q(s)\|^\alpha\|\widetilde{U}_Q(s,t)Q(t)x\|^\alpha \\
&\le c(\alpha)Ne^{-\delta(t-s)(1-\alpha)}\|A(s)Q(s)\|^\alpha e^{-\delta(t-s)\alpha}\|x\| \\
&\le m(\alpha)e^{-\delta(t-s)}\|x\|.
\end{aligned}$$

□

3.2 The almost periodicity

Consider the semilinear evolution equation

$$x'(t) = A(t)x(t) + f(t, x(t)), \qquad t \in \mathbb{R}, \tag{3.4}$$

where the function $f : \mathbb{R} \times X_\alpha \mapsto X$ is continuous and globally Lipschitzian, i.e., there is $k > 0$ such that

$$\|f(t,x) - f(t,y)\| \le k \|x - y\|_\alpha \text{ for all } t \in \mathbb{R} \text{ and } x, y \in X_\alpha. \tag{3.5}$$

To study the almost periodicity of the solutions of (3.4), we assume again the following :

(H4) $R(\omega, A(\cdot)) \in AP(\mathbb{R}, \mathcal{L}(X))$ with pseudo periods $\tau = \tau_\epsilon$ belonging to sets $\mathcal{P}(\epsilon, A)$.

(H5) $f \in AP(\mathbb{R} \times X_\alpha, X)$.

By a mild solution of (3.4) we mean every continuous function $x : \mathbb{R} \mapsto X_\alpha$, which satisfies the following variation of constants formula

$$x(t) = U(t,s)x(s) + \int_s^t U(t,\sigma)f(\sigma, x(\sigma))d\sigma \quad \text{for all } t \ge s,\ t, s \in \mathbb{R}. \tag{3.6}$$

We first study the existence of a unique almost periodic mild solution for the inhomogeneous evolution equation

$$x'(t) = A(t)x(t) + g(t), \quad t \in \mathbb{R}. \tag{3.7}$$

We have the following main result.

Theorem 3.2.1. *Assume that assumptions* **(H1)-(H4)** *hold. Let* $g \in BC(\mathbb{R}, X)$. *Then the following properties hold.*

(i) The equation (3.7) *has a unique bounded mild solution* $x : \mathbb{R} \mapsto X_\alpha$ *given by*

$$x(t) = \int_{-\infty}^t U(t,s)P(s)g(s)ds - \int_t^{+\infty} U_Q(t,s)Q(s)g(s)ds. \tag{3.8}$$

(ii) If $g \in AP(\mathbb{R}, X)$, *then* $x \in AP(\mathbb{R}, X_\alpha)$.

Proof. Since g is bounded, we know from [34] that the function x given by (3.8) is the unique bounded mild solution in X. For the boundedness in X_α, using Proposition 3.1.1, we have

$$\begin{aligned}
\|x(t)\|_\alpha &\le c\|x(t)\|_\beta \\
&\le c\int_{-\infty}^t \|U(t,s)P(s)g(s)\|_\beta \, ds + c\int_t^{+\infty} \|U_Q(t,s)Q(s)g(s)\|_\beta \, ds \\
&\le cc(\beta)\int_{-\infty}^t e^{-\frac{\delta}{2}(t-s)}(t-s)^{-\beta}\|g(s)\| \, ds + cm(\beta)\int_t^{+\infty} e^{-\delta(s-t)}\|g(s)\| \, ds \\
&\le cc(\beta)\|g\|_\infty \int_0^{+\infty} e^{-\sigma}\left(\frac{2\sigma}{\delta}\right)^{-\beta}\frac{2d\sigma}{\delta} + cm(\beta)\|g\|_\infty \int_0^{+\infty} e^{-\delta\sigma}d\sigma \\
&\le cc(\beta)\delta^\alpha\Gamma(1-\beta)\|g\|_\infty + cm(\beta)\delta^{-1}\|g\|_\infty,
\end{aligned}$$

and hence

$$\|x(t)\|_\alpha \leq c\,\|x(t)\|_\beta \leq c[c(\beta)\delta^\beta\Gamma(1-\beta)+m(\beta)\delta^{-1}]\,\|g\|_\infty . \qquad (3.9)$$

For (ii), let $\epsilon > 0$ and $\mathcal{P}(\epsilon, A, f)$ be the set of pseudo periods for the almost periodic function $t \mapsto (f(t), R(\omega, A(t)))$, see details in [80, p.6]. We know, from [86, Theorem 4.5] that x, as an X-valued function is almost periodic. Hence, there exists a number $\tau \in \mathcal{P}((\frac{\varepsilon}{c'})^{\frac{\beta}{\beta-\alpha}}, A, f)$ such that

$$\|x(t+\tau)-x(t)\| \leq \left(\frac{\varepsilon}{c'}\right)^{\frac{\beta}{\beta-\alpha}} \quad \text{for all } t \in \mathbb{R}.$$

For $\theta = \frac{\alpha}{\beta}$, the reiteration theorem implies that $X_\alpha = (X, X_\beta)_{\theta,\infty}$. Using the property of interpolation and (3.9), we obtain

$$\begin{aligned}
\|x(t+\tau)-x(t)\|_\alpha &\leq c(\alpha,\beta)\|x(t+\tau)-x(t)\|^{\frac{\beta-\alpha}{\beta}}\|x(t+\tau)-x(t)\|_\beta^{\frac{\alpha}{\beta}} \\
&\leq c(\alpha,\beta)2^{\frac{\alpha}{\beta}}\left(c[c(\beta)\delta^\beta\Gamma(1-\beta)+m(\beta)\delta^{-1}]\,\|g\|_\infty\right)^{\frac{\alpha}{\beta}} \\
&\quad \|x(t+\tau)-x(t)\|^{\frac{\beta-\alpha}{\beta}} \\
&:= c'\|x(t+\tau)-x(t)\|^{\frac{\beta-\alpha}{\beta}},
\end{aligned}$$

and hence

$$\|x(t+\tau)-x(t)\|_\alpha \leq \varepsilon$$

for $t \in \mathbb{R}$. □

To show the existence of almost periodic solutions for the semilinear evolution equation (3.4), let $y \in AP(\mathbb{R}, X_\alpha)$. By **(H5)** and Lemma 2.5.4, the function $g(\cdot) := f(\cdot, y(\cdot)) \in AP(\mathbb{R}, X)$, and from Theorem 3.2.1, the inhomogeneous equation (3.7) has a unique mild solution $x \in AP\,(\mathbb{R}, X_\alpha)$ given by

$$x(t) = \int_{-\infty}^{t} U(t,s)P(s)f(s,y(s))ds - \int_{t}^{+\infty} U_Q(t,s)Q(s)f(s,y(s))ds, \quad t \in \mathbb{R}.$$

Define the nonlinear operator $F : AP(\mathbb{R}, X_\alpha) \mapsto AP(\mathbb{R}, X_\alpha)$ by

$$(Fy)(t) := \int_{-\infty}^{t} U(t,s)P(s)f(s,y(s))ds - \int_{t}^{+\infty} U_Q(t,s)Q(s)f(s,y(s))ds, \quad t \in \mathbb{R}.$$

For x, $y \in AP\,(\mathbb{R}, X_\alpha)$, one has

$$\begin{aligned}
\|Fx(t)-Fy(t)\|_\alpha &\leq c(\alpha)\int_{-\infty}^{t} e^{-\delta(t-s)}(t-s)^{-\alpha}\,\|f(s,y(s))-f(s,x(s))\|\,ds \\
&\quad + c(\alpha)\int_{t}^{+\infty} e^{-\delta(t-s)}\,\|f(s,y(s))-f(s,x(s))\|\,ds. \\
&\leq k[c(\alpha)\delta^{-\alpha}\Gamma(1-\alpha)+m(\alpha)\delta^{-1}]\,\|x-y\|_\infty \quad \text{for all } t \in \mathbb{R}.
\end{aligned}$$

By taking k small enough, more precisely $k < (c(\alpha)\delta^{\alpha}\Gamma(1-\alpha) + m(\alpha)\delta^{-1})^{-1}$, the operator F becomes a contraction on $AP\,(\mathbb{R}, X_\alpha)$ and hence has a unique fixed point in $AP\,(\mathbb{R}, X_\alpha)$, which obviously is the unique X_α-valued almost periodic solution to (3.4).

The previous discussion can be formulated as follows:

Theorem 3.2.2. *Let $\alpha \in (0,1)$. Suppose that assumptions* **(H1)-(H5)** *hold and $k < (c(\alpha)\delta^{-\alpha}\Gamma(1-\alpha)+m(\alpha)\delta^{-1})^{-1}$. Then* (3.4) *has a unique mild solution x in $AP\,(\mathbb{R}, X_\alpha)$.*

3.3 Application : thermoelastic plate systems

Let a, b be positive functions and let $\Omega \subset \mathbb{R}^N$ $(N \geq 1)$ be a bounded subset, which is sufficiently regular. In this section we study the existence and uniqueness of almost periodic solutions to the thermoelastic plate systems

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} + \Delta^2 u + a(t)\Delta\theta = f_1(t, \nabla u, \nabla\theta), \;\; t \in \mathbb{R}, \; x \in \Omega, \\[2ex] \dfrac{\partial \theta}{\partial t} - b(t)\Delta\theta - a(t)\Delta\dfrac{\partial u}{\partial t} = f_2(t, \nabla u, \nabla\theta), \;\; t \in \mathbb{R}, \; x \in \Omega, \\[2ex] \theta = u = \Delta u = 0, \quad \text{on } \mathbb{R} \times \partial\Omega, \end{cases} \tag{3.10}$$

where u, θ are the vertical deflection and the variation of temperature of the plate and the functions f_1, f_2 are continuous and (globally) Lipschitz.

Assuming the almost periodicity of the functions a, b, f_1, f_2, we show that (3.10) has a unique almost periodic solution. It is worth mentioning that this question was recently studied by H. Leiva et al. [78] in the case when not only the coefficients a, b were constant but also there was no gradient terms in the semilinear terms f_1 and f_2.

To study almost periodic solutions to (3.10), our strategy consists of seeing such a system as an abstract evolution equation. For that, let $H = L^2(\Omega)$ and take A to be the (unbounded) linear operator

$$D(A) = H^2(\Omega) \cap H_0^1(\Omega) \;\text{ and }\; A\varphi = -\Delta\varphi \;\text{ for each }\; \varphi \in D(A).$$

Setting $x := \begin{pmatrix} u \\ \dfrac{\partial u}{\partial t} \\ \theta \end{pmatrix}$, the problem (3.10) can be rewritten in $X := D(A) \times H \times H$ in the following form

$$x'(t) = A(t)x(t) + f(t, x(t)), \; t \in \mathbb{R}, \tag{3.11}$$

where $A(t)$ is the linear operator defined by

$$A(t) = \begin{pmatrix} 0 & I_H & 0 \\ -A^2 & 0 & a(t)A \\ 0 & -a(t)A & -b(t)A \end{pmatrix} \tag{3.12}$$

and whose domain is

$$D(A(t)) = D(A^2) \times D(A) \times D(A), \ t \in \mathbb{R}.$$

Moreover, the semilinear term f is defined only on $\mathbb{R}\times X_\alpha$ for some $\frac{1}{2} < \alpha < 1$ by $f(t,u,v,\theta) = \begin{pmatrix} 0 \\ f_1(t,\nabla u,\nabla\theta) \\ f_2(t,\nabla u,\nabla\theta) \end{pmatrix}$, where X_α is the real interpolation space between X and $D(A(t))$ given by $X_\alpha = H_{1+\alpha} \times H_\alpha \times H_\alpha$, with $H_\alpha = L^2(\Omega)^A_{\alpha,\infty}$, and $H_{1+\alpha}$ is the domain of the part of A in H_α, see Section 2.2 for definitions and properties of these spaces.

We shall assume that the positive real functions a, b are bounded undervalued respectively by a_0, b_0 and a, $b \in C_b^\mu(\mathbb{R}) \cap AP(\mathbb{R})$ (C_b^μ is the space of bounded, globally Hölder continuous functions) and the functions f_1, f_2 : $\mathbb{R} \times H_0^1(\Omega) \times H_0^1(\Omega) \to L^2(\Omega)$ are defined by

$$f_i(t,u,\theta)(x) = f_i(t,\nabla u(x),\nabla\theta(x)) = \frac{Kd_i(t)}{1+|\nabla u(x)|+|\nabla\theta(x)|}$$

for $x \in \Omega$, $t \in \mathbb{R}$, $i = 1,2$, where d_i are almost periodic real functions.

It is not hard to check that the functions f_i $(i = 1,2)$ are continuous in $\mathbb{R} \times H_0^1(\Omega) \times H_0^1(\Omega)$ and globally Lipschitz functions, with Lipschitz constant $L > 0$ i.e.,

$$\|f_i(t,u,\theta) - f_i(t,v,\eta)\| \leq L(\|u-v\|^2_{\mathbb{H}_0^1(\Omega)} + \|\theta-\eta\|^2_{H_0^1(\Omega)})^{\frac{1}{2}}$$

for all $t \in \mathbb{R}, u, v, \eta$ and $\theta \in H_0^1(\Omega)$.

In order to apply the results of Section 2, we need to check that assumptions **(H1)**, **(H1')**, **(H2)** and **(H4)** hold.

To show (2.7) appearing in **(H1)**, we follow along the same lines as in [78]. For that, let $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n \to \infty$ be the eigenvalues of A with the finite multiplicity γ_n equal to the dimension of the corresponding eigenspace and $\{\phi_{n,k}\}$ is a complete orthonormal set of eigenvectors for A. For all $x \in D(A)$ we have

$$Ax = \sum_{n=1}^{\infty} \lambda_n \sum_{k=1}^{\gamma_n} \langle x, \phi_{n,k}\rangle \phi_{n,k} := \sum_{n=1}^{\infty} \lambda_n E_n x,$$

with $\langle\cdot,\cdot\rangle$ being the inner product in H. So, E_n is a complete family of orthogonal projections in H and so each $x \in H$ can be written as

$$x = \sum_{n=1}^{\infty}\sum_{k=1}^{\gamma_n} \langle x, \phi_{n,k}\rangle \phi_{n,k} = \sum_{n=1}^{\infty} E_n x.$$

Hence, for $z := \begin{pmatrix} w \\ v \\ \theta \end{pmatrix} \in D(A(t))$, we have

$$
\begin{aligned}
A(t)z &= \begin{pmatrix} 0 & I_H & 0 \\ -A^2 & 0 & a(t)A \\ 0 & -a(t)A & -b(t)A \end{pmatrix} \begin{pmatrix} w \\ v \\ \theta \end{pmatrix} \\
&= \begin{pmatrix} v \\ -A^2 w + a(t)A\theta \\ -a(t)Av - b(t)A\theta \end{pmatrix} = \begin{pmatrix} \sum_{n=1}^{\infty} E_n v \\ -\sum_{n=1}^{\infty} \lambda_n^2 E_n w + a(t) \sum_{n=1}^{\infty} \lambda_n E_n \theta \\ -a(t) \sum_{n=1}^{\infty} \lambda_n E_n v - b(t) \sum_{n=1}^{\infty} \lambda_n E_n \theta \end{pmatrix} \\
&= \sum_{n=1}^{\infty} \begin{pmatrix} 0 & 1 & 0 \\ -\lambda_n^2 & 0 & a(t)\lambda_n \\ 0 & -a(t)\lambda_n & -b(t)\lambda_n \end{pmatrix} \begin{pmatrix} E_n & 0 & 0 \\ 0 & E_n & 0 \\ 0 & 0 & E_n \end{pmatrix} \begin{pmatrix} w \\ v \\ \theta \end{pmatrix} \\
&= \sum_{n=1}^{\infty} A_n(t) P_n z,
\end{aligned}
$$

where

$$
P_n := \begin{pmatrix} E_n & 0 & 0 \\ 0 & E_n & 0 \\ 0 & 0 & E_n \end{pmatrix}, \ n \geq 1,
$$

and

$$
A_n(t) := \begin{pmatrix} 0 & 1 & 0 \\ -\lambda_n^2 & 0 & a(t)\lambda_n \\ 0 & -a(t)\lambda_n & -b(t)\lambda_n \end{pmatrix}, \ n \geq 1. \tag{3.13}
$$

It is clear that the characteristic equation of the matrix $A_n(t)$ is

$$
\lambda^3 + b(t)\lambda_n \lambda^2 + (1 + a(t)^2)\lambda_n^2 \lambda + b(t)\lambda_n^3 = 0. \tag{3.14}
$$

Setting $\lambda/\lambda_n = -\rho$, this equation takes the form

$$
\rho^3 - b(t)\rho^2 + (1 + a(t)^2)\rho - b(t) = 0. \tag{3.15}
$$

From Routh-Hurwitz theorem we obtain that the real part of the roots $\rho_1(t)$, $\rho_2(t)$, $\rho_3(t)$ of (3.15) are positive. Hence the eigenvalues of $A_n(t)$ are simple and given by $\sigma_i(t) = -\lambda_n \rho_i(t), i = 1, 2, 3$. Therefore, the matrix $A_n(t)$ is diagonalizable and then can be written as

$$
A_n(t) = K_n(t)^{-1} J_n(t) K_n(t), n \geq 1,
$$

with

$$
K_n(t) = \begin{pmatrix} 1 & 1 & 1 \\ \lambda_n \rho_1(t) & \lambda_n \rho_2(t) & \lambda_n \rho_3(t) \\ \frac{a(t)\rho_1(t)}{\rho_1(t) - b(t)} \lambda_n & \frac{a(t)\rho_2(t)}{\rho_2(t) - b(t)} \lambda_n & \frac{a(t)\rho_3(t)}{\rho_3(t) - b(t)} \lambda_n \end{pmatrix},
$$

$$J_n(t) = \begin{pmatrix} -\lambda_n\rho_1(t) & 0 & 0 \\ 0 & -\lambda_n\rho_2(t) & 0 \\ 0 & 0 & -\lambda_n\rho_3(t) \end{pmatrix}$$

and

$$K_n(t)^{-1} = \frac{1}{a(a(t),b(t))\lambda_n} \begin{pmatrix} a_{11}(t) & -a_{12}(t) & a_{13}(t) \\ -a_{21}(t) & a_{22}(t) & -a_{23}(t) \\ a_{31}(t) & -a_{32}(t) & a_{33}(t) \end{pmatrix},$$

where

$$a_{11}(t) = \frac{a(t)\rho_3(t)\rho_2(t)(\rho_2(t)-\rho_3(t))}{(\rho_3(t)-b(t))(\rho_2(t)-b(t))}, \qquad a_{12}(t) = \frac{a(t)\rho_3(t)\rho_1(t)(\rho_1(t)-\rho_3(t))}{(\rho_3(t)-b(t))(\rho_1(t)-b(t))},$$
$$a_{13}(t) = \frac{a(t)\rho_2(t)\rho_1(t)(\rho_1(t)-\rho_2(t))}{(\rho_2(t)-b(t))(\rho_1(t)-b(t))}, \qquad a_{21}(t) = \frac{a(t)b(t)(\rho_2(t)-\rho_3(t))}{(\rho_3(t)-b(t))(\rho_2(t)-b(t))},$$
$$a_{22}(t) = \frac{a(t)b(t)(\rho_1(t)-\rho_3(t))}{(\rho_3(t)-b(t))(\rho_1(t)-b(t))}, \qquad a_{23}(t) = \frac{a(t)b(t)(\rho_1(t)-\rho_2(t))}{(\rho_2(t)-b(t))(\rho_1(t)-b(t))},$$
$$a_{31} = (\rho_3(t)-\rho_2(t)), \qquad a_{32} = (\rho_3(t)-\rho_1(t)),$$
$$a_{33} = (\rho_2(t)-\rho_1(t)),$$

$$\begin{aligned} a(a(t),b(t)) =& \frac{a(t)\rho_3(t)\rho_2(t)}{(\rho_3(t)-b(t))} + \frac{a(t)\rho_1(t)\rho_3(t)}{(\rho_1(t)-b(t))} + \frac{a(t)\rho_2(t)\rho_1(t)}{(\rho_2(t)-b(t))} \\ & - \frac{a(t)\rho_1(t)\rho_2(t)}{(\rho_1(t)-b(t))} - \frac{a(t)\rho_3(t)\rho_1(t)}{(\rho_3(t)-b(t))} - \frac{a(t)\rho_2(t)\rho_3(t)}{(\rho_2(t)-b(t))}. \end{aligned}$$

Since $b(\cdot)$ is not a solution of (3.15), one can show that the matrix operators $K_n(t)$ and $K_n^{-1}(t)$ are well defined and $K_n(t)P_n(t) : Z := H \times H \times H \mapsto X$, $K_n^{-1}(t)P_n(t) : X \mapsto Z$.

The roots $\rho_i(t), i = 1,2,3$, of (3.15) are bounded. Indeed, setting $l(t) = \rho(t) - \frac{b(t)}{3}$, then (3.15) becomes

$$l(t)^3 + p(t)l(t) + q(t) = 0,$$

where $p(t) := (1 + a(t)^2) - \frac{b(t)^2}{3}$, $q(t) := -\frac{2}{27}b(t)^3 + (2 - a(t)^2)\frac{b(t)}{3}$.
Since q is bounded and

$$|q(t)| = |l(t)||l(t)^2 + p(t)| \geq |l(t)||l(t)|^2 - |p(t)||,$$

then l is also bounded. Thus the boundedness of b yields the claim.

Now, define the sector S_θ as

$$S_\theta = \{\lambda \in \mathbb{C} : |\arg(\lambda)| \leq \theta,\ \lambda \neq 0\},$$

where $0 \leq \sup_{t\in\mathbb{R}} |\arg(\rho_i(t))| < \dfrac{\pi}{2}$, $i = 1,2,3$ and $\dfrac{\pi}{2} < \theta < \pi - \max_{i=1,2,3} \sup_{t\in\mathbb{R}}\{|\arg(\rho_i(t))|\}$.

For $\lambda \in S_\theta$ and $z \in X$, one has

$$\begin{aligned} R(\lambda, A(t))z &= \sum_{n=1}^{\infty} (\lambda - A_n(t))^{-1} P_n z \\ &= \sum_{n=1}^{\infty} K_n(t)(\lambda - J_n(t)P_n)^{-1} K_n^{-1}(t) P_n z. \end{aligned}$$

Hence,

$$
\begin{aligned}
&\|R(\lambda, A(t))z\|^2 \\
&\quad \le \sum_{n=1}^{\infty} \|K_n(t)P_n(\lambda - J_n(t)P_n)^{-1}K_n^{-1}(t)P_n\|^2_{\mathcal{L}(X)}\|P_n z\|^2 \\
&\quad \le \sum_{n=1}^{\infty} \|K_n(t)P_n\|^2_{\mathcal{L}(Z,X)}\|(\lambda - J_n(t)P_n)^{-1}\|^2_{\mathcal{L}(Z)}\|K_n^{-1}(t)P_n\|^2_{\mathcal{L}(X,Z)}\|P_n z\|^2.
\end{aligned}
$$

Now, from (3.3) and $b > b_0$, we have

$$
|\rho(t) - b(t)| \ge \frac{a(t)^2|\rho(t)|}{1 + |\rho(t)|^2}, \qquad \inf_{t\in\mathbb{R}} |\rho(t)| > 0. \tag{3.16}
$$

Therefore from $a(t) > a_0$ it follows that

$$
\inf_{t\in\mathbb{R}} |\rho(t) - b(t)| > 0. \tag{3.17}
$$

Moreover, for $z := \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} \in Z$, we have

$$
\begin{aligned}
\|K_n(t)P_n z\|^2 =& \lambda_n^2\|E_n z_1 + E_n z_2 + E_n z_3\|^2 + \lambda_n^2\|\rho_1(t)E_n z_1 + \rho_2(t)E_n z_2 + \rho_3(t)E_n z_3\|^2 \\
&+ \lambda_n^2 \left\| \frac{a(t)\rho_1(t)}{\rho_1(t) - b(t)}E_n z_1 + \frac{a(t)\rho_2(t)}{\rho_2(t) - b(t)}E_n z_2 + \frac{a(t)\rho_3(t)}{\rho_3(t) - b(t)}E_n z_3 \right\|^2.
\end{aligned}
$$

Thus, there is $C_1 > 0$ such that

$$
\|K_n(t)P_n z\|_H \le C_1\lambda_n\|z\|_Z \quad \text{for all} \quad n \ge 1 \text{ and } t \in \mathbb{R}.
$$

Similarly, for $z := \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} \in X$, one can show

$$
\|K_n^{-1}(t)P_n z\| \le \frac{C_2}{\lambda_n}\|z\| \quad \text{for all} \quad n \ge 1 \text{ and } t \in \mathbb{R}.
$$

Now, for $z \in Z$, we have

$$
\begin{aligned}
\|(\lambda - J_n P_n)^{-1}z\|^2_Z &= \left\| \begin{pmatrix} \frac{1}{\lambda+\lambda_n\rho_1(t)} & 0 & 0 \\ 0 & \frac{1}{\lambda+\lambda_n\rho_2(t)} & 0 \\ 0 & 0 & \frac{1}{\lambda+\lambda_n\rho_3(t)} \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} \right\|^2_Z \\
&\le \frac{1}{(\lambda+\lambda_n\rho_1(t))^2}\|z_1\|^2 + \frac{1}{(\lambda+\lambda_n\rho_2(t))^2}\|z_2\|^2 + \frac{1}{(\lambda+\lambda_n\rho_3(t))^2}\|z_3\|^2.
\end{aligned}
$$

Let $\lambda_0 > 0$. The function $\eta(\lambda) := \frac{1+|\lambda|}{|\lambda+\lambda_n\rho_i(t)|}$ is continuous and bounded on the closed set $\Sigma := \{\lambda \in \mathbb{C}/|\lambda| \le \lambda_0,\ |\arg\lambda| \le \theta\}$. On the other hand, it is clear that η is bounded for $|\lambda| > \lambda_0$. Thus η is bounded on S_θ. If one takes

$$
N = \sup\left\{ \frac{1+|\lambda|}{|\lambda+\lambda_n\rho_i(t)|} \ :\ \lambda \in S_\theta, n \ge 1\,;\ i = 1,2,3,\ t \in \mathbb{R} \right\}.
$$

Therefore,

$$\|(\lambda - J_n P_n)^{-1} z\|_Z \leq \frac{N}{1+|\lambda|}\|z\|_Z, \quad \lambda \in S_\theta.$$

Consequently,

$$\|R(\lambda, A(t))\| \leq \frac{K}{1+|\lambda|}$$

for all $\lambda \in S_\theta$ and $t \in \mathbb{R}$.

Since the domain $D(A(t))$ is independent of t, we have only to check **(H1')**. The operator $A(t)$ is invertible and

$$A(t)^{-1} = \begin{pmatrix} -a(t)^2 b(t)^{-1} A^{-1} & -A^{-2} & -a(t)b(t)^{-1}A^{-2} \\ I_X & 0 & 0 \\ -a(t)b(t)^{-1} & 0 & -b(t)^{-1}A^{-1} \end{pmatrix}, \quad t \in \mathbb{R}.$$

Hence, for $t, s, r \in \mathbb{R}$, one has

$$\begin{aligned} &(A(t)-A(s))A(r)^{-1} \\ &= \begin{pmatrix} 0 & 0 & 0 \\ -a(r)b(r)^{-1}(a(t)-a(s))A & 0 & -b(r)^{-1}(a(t)-a(s)) \\ -(a(t)-a(s))A + a(r)b(r)^{-1}(b(t)-b(s))A & 0 & -b(r)^{-1}(b(t)-b(s)) \end{pmatrix}, \end{aligned}$$

and hence

$$\begin{aligned} &\|(A(t)-A(s))A(r)^{-1}z\| \\ &\quad \leq \sqrt{3}(\|a(r)b(r)^{-1}(a(t)-a(s))Az_1\| + \|b(r)^{-1}(a(t)-a(s))z_3\| \\ &\quad + \|(a(t)-a(s))Az_1\| + \|a(r)b(r)^{-1}(b(t)-b(s))Az_1\| + \|b(r)^{-1}(b(t)-b(s))z_3\|) \\ &\quad \leq \sqrt{3}(|a(r)b(r)^{-1}\||t-s|^\mu\|Az_1\| + |b(r)^{-1}|\||t-s|^\mu\|z_3\| + |t-s|^\mu\|Az_1\| \\ &\quad + \|a(r)b(r)^{-1}||t-s|^\mu\|Az_1\| + |b(r)^{-1}\||t-s|^\mu\|z_3\|) \\ &\quad \leq (2\sqrt{3}|a(r)b(r)^{-1}| + 1)|t-s|^\mu\|Az_1\| + 2\sqrt{3}|a(r)b(r)^{-1}|\||t-s|^\mu\|z_3\|. \end{aligned}$$

Consequently,

$$\|(A(t)-A(s))A(r)^{-1}z\| \leq C|t-s|^\mu\|z\|.$$

Let us now check assumption **(H2)**. For every $t \in \mathbb{R}$, $A(t)$ generates an analytic semigroup $(e^{\tau A(t)})_{\tau \geq 0}$ on X given by

$$e^{\tau A(t)}z = \sum_{n=0}^{\infty} K_n(t)^{-1} P_n e^{\tau J_n} P_n K_n(t) P_n z, \; z \in X.$$

On the other hand, we have

$$\|e^{\tau A(t)}z\| = \sum_{n=0}^{\infty} \|K_n(t)^{-1}P_n\|_{\mathcal{L}(X,Z)} \|e^{\tau J_n}P_n\|_{\mathcal{L}(Z)} \|K_n(t)P_n\|_{\mathcal{L}(Z,X)} \|P_n z\|,$$

with for each $z = \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} \in Z$

$$\begin{aligned}
\|e^{\tau J_n} P_n z\|_Z^2 &= \left\| \begin{pmatrix} e^{-\lambda_n \rho_1(t)\tau} E_n & 0 & 0 \\ 0 & e^{-\lambda_n \rho_2(t)\tau} E_n & 0 \\ 0 & 0 & e^{-\lambda_n \rho_3(t)\tau} E_n \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} \right\|_Z^2 \\
&\leq \|e^{-\lambda_n \rho_1(t)\tau} E_n z_1\|^2 + \|e^{-\lambda_n \rho_2(t)\tau} E_n z_2\|^2 + \|e^{-\lambda_n \rho_3(t)\tau} E_n z_3\|^2 \\
&\leq e^{-2\delta\tau} \|z\|_Z^2,
\end{aligned}$$

where $\delta = \lambda_1 \inf_{t\in\mathbb{R}} \{Re(\rho_1(t)), Re(\rho_2(t)), Re(\rho_3(t))\}$.

Therefore

$$\|e^{\tau A(t)}\| \leq C e^{-\delta\tau}, \quad \tau \geq 0. \tag{3.18}$$

Using the continuity of a, b and the equality

$$R(\lambda, A(t)) - R(\lambda, A(s)) = R(\lambda, A(t))(A(t) - A(s))R(\lambda, A(s)),$$

it follows that the mapping $J \ni t \longmapsto R(\lambda, A(t))$ is strongly continuous for $\lambda \in S_\theta$ where $J \subset \mathbb{R}$ is an arbitrary compact interval. Therefore, $A(t)$ satisfies the assumptions of [106, Corollary 2.3], and thus the evolution family $(U(t,s))_{t\geq s}$ is exponentially stable.

Finally, to check **(H4)**, we show that $(A(\cdot))^{-1} \in AP(\mathbb{R}, \mathcal{L}(X))$. Let $\varepsilon > 0$, and $\tau = \tau_\varepsilon \in P(\varepsilon, a, b)$. We have

$$A(t+\tau)^{-1} - A(t)^{-1} = A(t+\tau)^{-1}(A(t+\tau) - A(t))A(t)^{-1}, \tag{3.19}$$

and,

$$A(t+\tau) - A(t) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & (a(t+\tau) - a(t))A \\ 0 & -(a(t+\tau) - a(t))A & -(b(t+\tau) - b(t))A \end{pmatrix}.$$

Therefore, for $z := \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} \in D$, one has

$$\begin{aligned}
\|(A(t+\tau) - A(t))z\| &\leq \|(a(t+\tau) - a(t))Az_3\| + \|(a(t+\tau) - a(t))Az_2\| \\
&\quad + \|(b(t+\tau) - b(t))Az_3\| \\
&\leq \varepsilon \|Az_2\| + \varepsilon \|Az_3\| \\
&\leq \varepsilon \|z\|_D,
\end{aligned}$$

and using (3.19), we obtain

$$\begin{aligned}
\|A(t+\tau)^{-1}y - A(t)^{-1}y\| &\leq \|A(t+\tau)^{-1}(A(t+\tau) - A(t))A(t)^{-1}y\| \\
&\leq \|A(t+\tau)^{-1}\|_{\mathcal{L}(X)} \\
&+ \|(A(t+\tau) - A(t))\|_{\mathcal{L}(D,X)} \|A(t)^{-1}y\|_D, \quad y \in X.
\end{aligned}$$

Since $\|A(t)^{-1}y\|_D \leq c\|y\|$, then

$$\|A(t+\tau)^{-1}y - A(t)^{-1}y\| \leq c'\varepsilon\|y\|.$$

Consequently, $A(t)^{-1}$ is almost periodic.

Finally, for a small constant K, all assumptions of Theorem 3.2.2 are satisfied and thus the thermoelastic system (3.10) has a unique almost periodic mild solution $\binom{u}{\theta}$ with values in $H_{1+\alpha} \times H_\alpha$.

3.4 The almost automorphy

In this section, we study the existence of almost automorphic solutions of the semilinear evolution equations

$$u'(t) = A(t)u(t) + f(t, u(t)), \qquad t \in \mathbb{R}, \tag{3.20}$$

where $A(t), t \in \mathbb{R}$, satisfy **(H1)** and **(H2)**.

To this purpose, define the Yosida approximations $A_n(t) = nA(t)R(n, A(t))$ of $A(t)$ for $n > \omega$ and $t \in \mathbb{R}$. These operators generate an evolution family U_n on X. It has been shown in [86, Lemma 3.1, Proposition 3.3, Corollary 3.4] that assumptions **(H1)** and **(H2)** are satisfied by $A_n(\cdot)$ with the same constants for $n \geq n_0$.

We assume also that

(H4)' $R(\omega, A(\cdot)) \in AA(\mathbb{R}, \mathcal{L}(X))$,

(H5)' the function $f : \mathbb{R} \times X_\alpha \longrightarrow X$ is continuous and globally small Lipschitzian, i.e., there is a small $K_f > 0$ such that

$$\|f(t, u) - f(t, v)\| \leq K_f \|u - v\|_\alpha \text{ for all } t \in \mathbb{R} \text{ and } u, v \in X_\alpha,$$

and $f \in AA(\mathbb{R} \times X_\alpha, X)$.

The Yosida approximations $A_n(\cdot)$ satisfy also this last assumption. More precisely, the following lemma follows. We adopt the same proof of [86] in the almost periodic case.

Lemma 3.4.1. *If* **(H1)** *and* **(H4)'** *hold, then there is a number $n_1 \geq n_0$ such that $R(\omega, A_n(\cdot)) \in AA(\mathbb{R}, \mathcal{L}(X))$ for $n \geq n_1$.*

Proof. Let $t \in \mathbb{R}$ and a sequence $(s'_l)_{l\in\mathbb{N}}$ of real numbers, as $R(w, A(\cdot))$ is almost automorphic, there is a subsequence $(s_l)_{n\in\mathbb{N}}$ such that

$$\lim_{l,\, k\to+\infty} \|R(w, A(t + s_l - s_k)) - R(w, A(t))\| = 0.$$

If $n \geq n_0$ and $|\arg(\lambda - \omega)| \leq \phi$ we have that

$$\begin{aligned}
&R(\omega, A_n(t + s_l - s_k)) - R(\omega, A_n(t)) \qquad (3.21)\\
&\quad= \frac{n^2}{(\omega+n)^2}\Big(R\Big(\frac{\omega n}{\omega+n}, A(t + s_l - s_k)\Big) - R\Big(\frac{\omega n}{\omega+n}, A(t)\Big)\Big)\\
&\quad= \frac{n^2}{(\omega+n)^2} R(\omega, A(t + s_l - s_k))\Big[1 - \frac{\omega^2}{\omega+n} R(\omega, A(t + s_l - s_k))\Big]^{-1}\\
&\qquad - \frac{n^2}{(\omega+n)^2} R(\omega, A(t))\Big[1 - \frac{\omega^2}{\omega+n} R(\omega, A(t))\Big]^{-1}. \qquad (3.22)
\end{aligned}$$

We can also see that

$$\Big\|\frac{\omega^2}{\omega+n}R(\omega,A(s))\Big\| \le \frac{\omega^2}{\omega+n}\frac{K}{1+\omega} \le \frac{\omega K}{n} \le \frac{1}{2}$$

for $n \ge n_1 := \max\{n_0, 2\omega K\}$ and $s \in \mathbb{R}$. In particular,

$$\Big\|\Big[1-\frac{\omega^2}{\omega+n}R(\omega,A(s))\Big]^{-1}\Big\| \le 2. \tag{3.23}$$

Hence, (3.22) implies

$$\begin{aligned}
&\|R(\omega,A_n(t+s_l-s_k)) - R(\omega,A_n(t))\| \\
&\quad \le 2\,\|R(\omega,A(t+s_l-s_k)) - R(\omega,A(t))\| \\
&\quad + \frac{K}{1+\omega}\,\Big\|\Big[1-\frac{\omega^2}{\omega+n}R(\omega,A(t+s_l-s_k))\Big]^{-1} - \Big[1-\frac{\omega^2}{(\omega+n)^2}R(\omega,A(t))\Big]^{-1}\Big\|.
\end{aligned}$$

Employing (3.23) again, we obtain

$$\begin{aligned}
&\Big\|\Big[1-\frac{\omega^2}{\omega+n}R(\omega,A(t+s_l-s_k))\Big]^{-1} - \Big[1-\frac{\omega^2}{\omega+n}R(\omega,A(t))\Big]^{-1}\Big\| \\
&\quad \le 4\Big\|\Big[1-\frac{\omega^2}{\omega+n}R(\omega,A(t+s_l-s_k))\Big] - \Big[1-\frac{\omega^2}{\omega+n}R(\omega,A(t))\Big]\Big\| \\
&\quad \le 4\omega\,\|R(\omega,A(t+s_l-s_k)) - R(\omega,A(t))\|.
\end{aligned}$$

Therefore,

$$\|R(\omega,A_n(t+s_l-s_k)) - R(\omega,A_n(t))\| \le (2+4K)\|R(\omega,A(t+s_l-s_k)) - R(\omega,A(t))\| \tag{3.24}$$

for $n \ge n_1$ and $t \in \mathbb{R}$. The assertion thus follows from **(H4)'**. □

To obtain the aim of this section, we need the following technical lemma.

Lemma 3.4.2. *Assume that* **(H1)**,**(H2)** *and* **(H4)'** *hold. For every sequence* $(s'_l)_{l\in\mathbb{N}} \in \mathbb{R}$, *there is a subsequence* $(s_l)_{l\in\mathbb{N}}$ *such that for every* $\eta > 0$, *and* $t,\ s \in \mathbb{R}$ *there is* $l(\eta,t,s) > 0$ *such that*

$$\|\Gamma_n(t+s_l-s_k, s+s_l-s_k) - \Gamma_n(t,s)\| \le cn^2\eta \tag{3.25}$$

for a large n *and* $l,\ k \ge l(\eta,t,s)$.

Proof. Let a sequence $(s'_l)_{l\in\mathbb{N}} \in \mathbb{R}$. Since $R(\omega,A(\cdot)) \in AA(\mathbb{R},X)$ then we can extract a subsequence (s_l) such that

$$\|R(\omega,A(\sigma+s_l-s_k)) - R(\omega,A(\sigma))\| \to 0, \quad k,l \to \infty. \tag{3.26}$$

From [86], we have

$$\begin{aligned}
&\Gamma_n(t+s_l-s_k, s+s_l-s_k) - \Gamma_n(t,s) \\
&= \int_{\mathbb{R}} \Gamma_n(t,\sigma)(A_n(\sigma)-\omega)\,[R(\omega,A_n(\sigma+s_l-s_k)) - R(\omega,A_n(\sigma))] \\
&\qquad \cdot (A_n(\sigma+s_l-s_k)-\omega)\Gamma_n(\sigma+s_l-s_k, s+s_l-s_k)\,d\sigma
\end{aligned}$$

for $s,t\in\mathbb{R}$ and $l,k,\in\mathbb{N}$ and large n. This formula with the estimate (3.24) and [86, Corollary 3.4] imply that

$$\begin{aligned}
&\|\Gamma_n(t+s_l-s_k,s+s_l-s_k)-\Gamma_n(t,s)\|\\
&\qquad\le cn^2\int_{\mathbb{R}} e^{-\frac{3\delta}{4}|t-\sigma|}\,e^{-\frac{3\delta}{4}|\sigma-s|}\|R(\omega,A_n(\sigma+s_l-s_k))-R(\omega,A_n(\sigma))\|\,d\sigma\\
&\qquad\le cn^2(2+4K)\int_{\mathbb{R}} e^{-\frac{3\delta}{4}|t-\sigma|}\,e^{-\frac{3\delta}{4}|\sigma-s|}\|R(\omega,A(\sigma+s_l-s_k)) \qquad (3.27)\\
&\qquad\quad -R(\omega,A(\sigma))\|\,d\sigma\|\to 0,\ k,l\to\infty,
\end{aligned}$$

by (3.26) and Lebesgue's convergence dominated theorem. Hence, for $\eta>0$ there is $l(\eta,t,s)>0$

$$\|\Gamma_n(t+s_l-s_k,s+s_l-s_k)-\Gamma_n(t,s)\|<cn^2\eta$$

for large n and $l,\ k\ge l(\eta,t,s)$. ☐

We need also this fundamental result. An analogous result for the almost periodicity is shown in [86].

Proposition 3.4.3. *Assume that* **(H1)**, **(H2)** *and* **(H4)'** *hold. Let a sequence* $(s'_l)_{l\in\mathbb{N}}\in\mathbb{R}$ *there is a subsequence* $(s_l)_{l\in\mathbb{N}}$ *such that for every* $h>0$

$$\|\Gamma(t+s_l-s_k,s+s_l-s_k)-\Gamma(t,s)\|\longrightarrow 0,\quad k,l\to\infty$$

for $|t-s|\ge h$.

Proof. Let a sequence $(s'_l)_{l\in\mathbb{N}}\in\mathbb{R}$, and consider the subsequence (s_l) given by Lemma 3.4.2. Let $\varepsilon>0$ and $h>0$. There is $t_\varepsilon>h$ such that

$$\|\Gamma(t+s_l-s_k,s+s_l-s_k)-\Gamma(t,s)\|\le\varepsilon$$

for $|t-s|\ge t_\varepsilon$ and $l,\ k\in\mathbb{N}$. For $h\le|t-s|\le t_\varepsilon$, by [86, Lemma 4.2] we have

$$\|\Gamma(t+s_l-s_k,s+s_l-s_k)-\Gamma_n(t+s_l-s_k,s+s_l-s_k)\|\le c(t_\varepsilon)n^{-\theta},\quad (3.28)$$

$$\|\Gamma(t,s)-\Gamma_n(t,s)\|\le c(t_\varepsilon)n^{-\theta}\quad (3.29)$$

for all k,l and large n. Let $n_\varepsilon>0$ large enough such that $n^{-\theta}<\frac{\varepsilon}{4c(t_\varepsilon)}$ for $n\ge n_\varepsilon$. Take $0<\eta<\frac{\varepsilon}{2cn_\varepsilon^2}$. Hence, by (3.28), (3.29) and Lemma 3.4.2, one has

$$\|\Gamma(t+s_l-s_k,s+s_l-s_k)-\Gamma(t,s)\|\le 2c(t_\varepsilon)n_\varepsilon^{-\theta}+cn_\varepsilon^2\eta\le\varepsilon$$

for all $k,l\ge l(\varepsilon,t,s)$. Consequently, $\|\Gamma(t+s_l-s_k,s+s_l-s_k)-\Gamma(t,s)\|\to 0$ as $l,\ k\to+\infty$ for $|t-s|>h>0$. ☐

These preliminary results will serve to prove the existence of a unique almost automorphic solution of the semilinear evolution equation (3.20).

For this purpose, we show first the existence of a unique almost automorphic mild solution to the inhomogeneous evolution equation

$$u'(t)=A(t)u(t)+g(t),\quad t\in\mathbb{R}.\qquad (3.30)$$

More precisely, we state the following main result.

Theorem 3.4.4. *Assume that* **(H1)-(H3)** *and* **(H4)'** *hold. Then, for every* $g \in AA(\mathbb{R}, X)$*, the unique bounded mild solution* $u(\cdot) = \int_{\mathbb{R}} \Gamma(\cdot, s) g(s)\, ds$ *of* (3.30) *belongs to* $AA(\mathbb{R}, X_\alpha)$.

Proof. First we prove that the mild solution u is almost automorphic in X. Let a sequence $(s'_l)_{l\in\mathbb{N}}$ and $h > 0$. As $g \in AA(\mathbb{R}, X)$ there exists a subsequence $(s_l)_{l\in\mathbb{N}}$ such that $\lim_{l,\, k\to+\infty} \|g(t + s_l - s_k) - g(t)\| \to 0$. Now, we write

$$\begin{aligned}
&u(t + s_l - s_k) - u(t)\\
&\quad = \int_{\mathbb{R}} \Gamma(t + s_l - s_k, s + s_l - s_k) g(s + s_l - s_k)\, ds - \int_{\mathbb{R}} \Gamma(t, s) g(s)\, ds\\
&\quad = \int_{\mathbb{R}} \Gamma(t + s_l - s_k, s + s_l - s_k)(g(s + s_l - s_k) - g(s))\, ds\\
&\quad + \int_{|t-s|\geq h} (\Gamma(t + s_l - s_k, s + s_l - s_k) - \Gamma(t, s)) g(s)\, ds\\
&\quad + \int_{|t-s|\leq h} (\Gamma(t + s_l - s_k, s + s_l - s_k) - \Gamma(t, s)) g(s)\, ds.
\end{aligned}$$

For $\varepsilon' > 0$, we deduce from Proposition 3.4.3 and **(H2)** that

$$\begin{aligned}
&\|u(t + s_l - s_k) - u(t)\|\\
&\leq 2N \int_{\mathbb{R}} e^{-\delta|t-s|} \|g(s + s_l - s_k) - g(s)\|\, ds\ + (\tfrac{4}{\delta}\, \varepsilon' + 4Nh)\|g\|_\infty
\end{aligned}$$

for $t \in \mathbb{R}$ and $l,\ k > l(\varepsilon,\ h) > 0$. Now, for $\varepsilon > 0$, take h small and then $\varepsilon' > 0$ small such that

$$\|u(t + s_l - s_k) - u(t)\| \leq 2N \int_{\mathbb{R}} e^{-\delta|t-s|} \|g(s + s_l - s_k) - g(s)\|\, ds\ + \tfrac{\varepsilon}{2}$$

for $t \in \mathbb{R}$ and $l,\ k > l(\varepsilon) > 0$. Finally, by Lebesgue dominated convergence theorem, u is almost automorphic in X.

Using the reiteration theorem and the interpolation property, we have

$$||u(t + s_l - s_k) - u(t)||_\alpha \leq c(\alpha, \beta)||u(t + s_l - s_k) - u(t)||^{\frac{\beta-\alpha}{\beta}} ||u(t + s_l - s_k) - u(t)||_\beta^{\frac{\alpha}{\beta}}.$$

Using estimates in Proposition 4.1.2, we can show that u is bounded in X_β. Hence,

$$\begin{aligned}
||u(t + s_l - s_k) - u(t)||_\alpha &\leq c(\alpha, \beta) c^{\frac{\beta}{\alpha}} ||u(t + s_l - s_k) - u(t)||^{\frac{\beta-\alpha}{\beta}}\\
&\leq c'||u(t + s_l - s_k) - u(t)||^{\frac{\beta-\alpha}{\beta}}. \qquad (3.31)
\end{aligned}$$

Since u is almost automorphic in X, $u(t + s_l - s_k) \longrightarrow u(t)$, as $l, k \to \infty$, for $t \in \mathbb{R}$, and thus $u \in AA(\mathbb{R}, X_\alpha)$. □

As a consequence of Theorem 3.4.4, we obtain the aim of this section.

Theorem 3.4.5. *Assume that* **(H1)-(H3)**, **(H4)'** *and* **(H5)'** *hold. Then, for small* K_f*,* (3.20) *admits a unique mild solution* u *in* $AA(\mathbb{R}, X_\alpha)$.

Proof. Consider $v \in AA(\mathbb{R}, X_\alpha)$ and $f \in AA(\mathbb{R} \times X_\alpha, X)$. Then, by [95, Theorem 2.2.4, p. 21], the function $g(\cdot) := f(\cdot, v(\cdot)) \in AA(\mathbb{R}, X)$, and from Theorem 3.4.4, the inhomogeneous evolution equation

$$u'(t) = A(t)u(t) + g(t), \quad t \in \mathbb{R},$$

admits a unique mild solution $u \in AA(\mathbb{R}, X)$ given by

$$u(t) = \int_{\mathbb{R}} \Gamma(t,s) f(s, v(s)) ds, \quad t \in \mathbb{R}.$$

Let the operator $F : AA(\mathbb{R}, X_\alpha) \longrightarrow AA(\mathbb{R}, X_\alpha)$ be defined by

$$(Fv)(t) := \int_{\mathbb{R}} \Gamma(t,s) f(s, v(s)) ds \quad \text{for all } t \in \mathbb{R}.$$

Now we prove that F has a unique fixed point. For any x, $y \in AA(\mathbb{R}, X_\alpha)$, we have

$$\begin{aligned} \|Fx(t) - Fy(t)\|_\alpha \le c(\alpha) &\int_{-\infty}^{t} e^{-\delta(t-s)} (t-s)^{-\alpha} \|f(s, y(s)) - f(s, x(s))\| \, ds \\ &+ c(\alpha) \int_{t}^{+\infty} e^{-\delta(t-s)} \|f(s, y(s)) - f(s, x(s))\| \, ds. \\ &\le K_f c'(\alpha) \|x - y\|_\infty \quad \text{for all } t \in \mathbb{R}. \end{aligned}$$

If we assume that $K_f c'(\alpha) < 1$, then F has a unique fixed poind $u \in AA(\mathbb{R}, X_\alpha)$. Thus u is the unique almost automorphic solution to equation (3.20). □

Example 3.4.6. *Consider the parabolic problem*

$$\begin{aligned} \partial_t u(t,x) &= A(t,x,D)u(t,x) + h(t, \nabla u(t,x)), \quad t \in \mathbb{R},\ x \in \Omega, \\ B(x,D)u(t,x) &= 0, \quad t \in \mathbb{R},\ x \in \partial\Omega, \end{aligned} \tag{3.32}$$

on a bounded domain $\Omega \subseteq \mathbb{R}^n$ with boundary $\partial\Omega$ of class C^2 and outer unit normal vector $\nu(x)$, employing the differential expressions

$$\begin{aligned} A(t,x,D) &= \sum\nolimits_{k,l} a_{kl}(t,x) \partial_k \partial_l + \sum\nolimits_{k} a_k(t,x)\, \partial_k + a_0(t,x), \\ B(t,x,D) &= \sum\nolimits_{k} b_k(x)\, \partial_k + b_0(x). \end{aligned}$$

We require that $a_{kl} = a_{lk}$ and b_k are real–valued, $a_{kl}, a_k, a_0 \in C_b^\mu(\mathbb{R}, C(\overline{\Omega}))$, $b_k, b_0 \in C^1(\partial\Omega))$,

$$\sum_{k,l=1}^{n} a_{kl}(t,x)\, X_{i_k} X_{i_l} \ge \eta \left| X_i \right|^2, \quad \text{and} \quad \sum_{k=1}^{n} b_k(x) \nu_k(x) \ge \beta$$

for constants $\mu \in (1/2, 1)$, $\beta, \eta > 0$ and all $X_i \in \mathbb{R}^n$, $k, l = 1, \cdots, n$, $t \in \mathbb{R}$, $x \in \overline{\Omega}$ resp. $x \in \partial\Omega$. We set $X = C(\overline{\Omega})$,

$$D(A(t)) = \{u \in \bigcap\nolimits_{p>1} W_p^2(\Omega) : A(t,\cdot,D)u \in C(\overline{\Omega}),\ B(t,\cdot,D)u = 0 \text{ on } \partial\Omega\},$$

for $t \in \mathbb{R}$. It is known that the operators $A(t)$, $t \in \mathbb{R}$, satisfy **(H1)**, *see [1, 83], or [107, Exa.2.9]. Thus $A(\cdot)$ generates an evolution family $U(\cdot,\cdot)$ on X. Let us fix numbers $\alpha \in (1/2, 1)$ and $p > \frac{n}{2(1-\alpha)}$. Then*

$$X_\alpha^t = X_\alpha = \{f \in C^{2\alpha}(\overline{\Omega}) : B(\cdot, D)u = 0\}$$

with uniformly equivalent constants due to Theorem 3.1.30 in [83], and $X_\alpha \hookrightarrow W_p^2(\Omega)$. It is clear that the function $f(t,u)(x) := h(t, \nabla u(x))$, $x \in \Omega$, is continuous from $\mathbb{R} \times X_\alpha$ to X if h is continuous from $\mathbb{R} \times \mathbb{R}^n$, and if h is small Lipschitzian and almost automorphic then f is. Under the exponential dichotomy of $U(\cdot,\cdot)$ and almost automorphy of $R(\omega, A(\cdot))$, the parabolic equation (3.32) *has a unique almost automorphic solution.*

Chapter 4

Asymptotic behavior of semilinear autonomous boundary evolution equations

In this chapter, we study the existence and uniqueness of the almost periodic and almost automorphic solutions of the semilinear boundary evolution equation

$$\begin{cases} u'(t) = A_m u(t) + h(t, u(t)), & t \in \mathbb{R}, \\ Lu(t) = \phi(t, u(t)), & t \in \mathbb{R}, \end{cases} \tag{4.1}$$

where $A_m \in \mathcal{L}(D(A_m), X)$, $L \in \mathcal{L}(D(A_m), \partial X)$, $t \in \mathbb{R}$, where $D(A_m), X$, and ∂X are Banach spaces such that $D(A_m)$ is dense and continuously embedded in X. The function h is defined from $\mathbb{R} \times X$ into X, and ϕ is defined from $\mathbb{R} \times X$ into ∂X.

In Section 4.3, we show how to transform the semilinear boundary evolution equation to a semilinear evolution equation in an extrapolated space

$$u'(t) = A_{\alpha-1} u(t) + f(t, u(t)), \quad t \in \mathbb{R}, \tag{4.2}$$

where $A_{\alpha-1}$ is the extrapolated extension of the generator A of a hyperbolic analytic semigroup $(T(t))_{t\geq 0}$ on a Banach space X. The semilinear term f is defined on $\mathbb{R} \times X_\beta$ with values in the extrapolated spaces $X_{\alpha-1}$ for $0 \leq \beta < \alpha < 1$.

In Section 4.1, we prove that the exponential dichotomy is inherited by the extrapolated semigroup generated by $A_{\alpha-1}$ in $X_{\alpha-1}$.

In Section 4.2, we show the almost periodicity and automorphy of (4.2). As usual, by a fixed point argument, it is enough to show that the inhomogeneous evolution equation

$$u'(t) = A_{\alpha-1} u(t) + g(t), \quad t \in \mathbb{R}, \tag{4.3}$$

has a unique almost periodic (resp. almost automorphic) mild solution on X_α for each almost periodic (resp. almost automorphic) function $g : \mathbb{R} \longrightarrow X_{\alpha-1}$.

4.1 Hyperbolicity of an extrapolated semigroup

We consider a sectorial operator A on a Banach space X such that $\sigma(A)\cap i\mathbb{R}=\emptyset$, which is equivalent to the fact that A generates a hyperbolic analytic semigroup $(T(t))_{t\geq 0}$ on X. Let $(T_{\alpha-1}(t))_{t\geq 0}$ be its extrapolated semigroup with generator $A_{\alpha-1}$. To show the main result of this chapter, we need the following results.

Proposition 4.1.1. *Assume that $0<\alpha\leq 1$ and that $T(\cdot)$ is hyperbolic. Then the operators P_u and P_s admit continuous extensions $P_{u,\alpha-1}: X_{\alpha-1}\longrightarrow X$ and $P_{s,\alpha-1}: X_{\alpha-1}\longrightarrow X_{\alpha-1}$ respectively. Moreover we have the following assertions.*
(i) $P_{u,\alpha-1}X_{\alpha-1}=P_uX$;
(ii) $T_{\alpha-1}(t)P_{s,\alpha-1}=P_{s,\alpha-1}T_{\alpha-1}(t)$;
(iii) $T_{\alpha-1}(t): P_{u,\alpha-1}(X_{\alpha-1})\longrightarrow P_{u,\alpha-1}(X_{\alpha-1})$ *is invertible with inverse* $T_{\alpha-1}(-t)$;
(iv) for $0<\alpha-\widetilde{\varepsilon}<1$, *we have*

$$||T_{\alpha-1}(t)P_{s,\alpha-1}x||\leq mt^{\alpha-1-\widetilde{\varepsilon}}e^{-\gamma t}||x||_{\alpha-1} \text{ for } x\in X_{\alpha-1} \text{ and } t\geq 0, \tag{4.4}$$

$$||T_{\alpha-1}(t)P_{u,\alpha-1}x||\leq c\, e^{\delta t}||x||_{\alpha-1} \text{ for } x\in X_{\alpha-1} \text{ and } t\leq 0. \tag{4.5}$$

Proof. By applying (i) and (ii) of Condition **(H)** in Definition 2.2.12, we show that $T(t)$ and $R(\omega,A)$ commute with P_s and P_u, and hence

$$\begin{aligned} ||P_ux|| &= \frac{1}{\omega^\alpha}||(2\omega-A)\omega^\alpha R(\omega,A-\omega)P_ux|| \\ &= \frac{1}{\omega^\alpha}||(2\omega-A_{-1})P_u\omega^\alpha R(\omega,A-\omega)x|| \\ &\leq \frac{1}{\omega^\alpha}||(2\omega-A_{-1})||_{\mathcal{L}(\widehat{X},X_{-1})}||P_u||_{\mathcal{L}(X)}||\omega^\alpha R(\omega,A-\omega)x|| \\ &\leq c||x||_{\alpha-1} \end{aligned} \tag{4.6}$$

for all $x\in X$. Hence P_u can be extended to a bounded operator $P_{u,\alpha-1}\in\mathcal{L}(X_{\alpha-1},X)$. Then the operator $P_{s,\alpha-1}=I-P_{u,\alpha-1}\in\mathcal{L}(X_{\alpha-1})$ is the bounded extension of P_s.

Assertion (i) is a consequence of the fact that $P_{u,\alpha-1}$ has values in X and that it is a projection. Since P_s commute with $T(t)$ and by approximation using (2.5), we can see immediately the assertion (ii). To show (iii), we use the fact that $T(t): P_uX\longrightarrow P_uX$ is invertible with inverse $T(-t)$ and (i). To show (iv), let $t\geq 1$ and $x\in X_{\alpha-1}$. Using the estimates (2.2) and (2.5), we obtain

$$\begin{aligned} ||T_{\alpha-1}(t)P_{s,\alpha-1}x|| &= ||T(t-1)P_sT_{\alpha-1}(1)x|| \\ &\leq c\, e^{-\delta t}||x||_{\alpha-1} \\ &\leq c\, t^{\alpha-1-\widetilde{\varepsilon}}.e^{-\delta\frac{t}{2}}t^{-\alpha+1+\widetilde{\varepsilon}}e^{-\delta\frac{t}{2}}||x||_{\alpha-1}. \end{aligned}$$

Since $t^{-\alpha+1+\widetilde{\varepsilon}}e^{-\delta\frac{t}{2}}\to 0$ as $t\to+\infty$, one obtains

$$||T_{\alpha-1}(t)P_{s,\alpha-1}x||=c\, t^{\alpha-1}e^{-\dfrac{t}{2}}||x||_{\alpha-1}.$$

If $0 \le t \le 1$, by assertion (ii) and (2.5), we have

$$\begin{aligned}||T_{\alpha-1}(t)P_{s,\alpha-1}x|| = ||P_s T_{\alpha-1}(t)x|| &\le c\, t^{\alpha-1-a}||x||_{\alpha-1} \\ &\le c\, t^{\alpha-1-a} e^{-\frac{t}{2}} e^{\frac{t}{2}} ||x||_{\alpha-1}.\end{aligned}$$

Hence, there exist constants $m > 0$ and $\gamma := \delta/2$ such that

$$||T_{\alpha-1}(t)P_{s,\alpha-1}x|| \le m\, t^{\alpha-1-\widetilde{\varepsilon}} e^{-\gamma t}||x||_{\alpha-1}.$$

Let $t \le 0$ and $x \in X_{\alpha-1}$. From the equality (i), (2.2) and (4.6), we have

$$\begin{aligned}||T_{\alpha-1}(t)P_{u,\alpha-1}x|| = ||T(t)P_{u,\alpha-1}x|| &\le M||P_{u,\alpha-1}||_{\mathcal{L}(X_{\alpha-1},X)} e^{-\delta t}||x||_{\alpha-1} \\ &= c\, e^{-\delta t}||x||_{\alpha-1}.\end{aligned}$$

□

The following exponential dichotomy estimates in the interpolation and extrapolation spaces are needed.

Proposition 4.1.2. *For $x \in X_{\alpha-1}$ and $0 \le \beta \le 1,\ 0 < \alpha < 1$, we have the following assertions:*
(i) there is a constant $c(\alpha, \beta)$, such that

$$||T_{\alpha-1}(t)P_{u,\alpha-1}x||_\beta \le c(\alpha,\beta)e^{\delta t}||x||_{\alpha-1} \ \textit{for}\ t \le 0, \tag{4.7}$$

(ii) there is a constant $m(\alpha, \beta)$, such that for $t \ge 0$ and $0 < \alpha - \widetilde{\varepsilon} < 1$.

$$||T_{\alpha-1}(t)P_{s,\alpha-1}x||_\beta \le m(\alpha,\beta)e^{-\gamma t}t^{\alpha-\beta-\widetilde{\varepsilon}-1}||x||_{\alpha-1}. \tag{4.8}$$

Proof. (i) As X_β is a space of class $\mathcal{J}_\beta$, see [83, Definition 1.1.1], there is a constant $c(\beta)$ such that

$$||x||_\beta \le c(\beta)||x||^{1-\beta}||Ax||^\beta, \quad x \in D(A).$$

As the part of A in P_u is a bounded operator, from (4.5) one obtains

$$\begin{aligned}||T_{\alpha-1}(t)P_{u,\alpha-1}x||_\beta &\le c(\beta)||T(t)P_{u,\alpha-1}x||^{1-\beta}||AT(t)P_{u,\alpha-1}x||^\beta \\ &\le c(\beta)||T(t)P_{u,\alpha-1}x||^{1-\beta}||AP_u||^\beta||T(t)P_{u,\alpha-1}x||^\beta \\ &\le c(\beta)||AP_u||^\beta||T(t)P_{u,\alpha-1}x|| \\ &\le c(\beta)||AP_u||^\beta c\, e^{\delta t}||x||_{\alpha-1} \\ &\le c(\alpha,\beta)e^{\delta t}||x||_{\alpha-1}.\end{aligned}$$

(ii) For $t \ge 1$, we have

$$||T_{\alpha-1}(t)P_{s,\alpha-1}x||_\beta \le ||T(1)||_{\mathcal{L}(X,X_\beta)}||T_{\alpha-1}(t-1)P_{s,\alpha-1}x||,$$

and hence from (4.4), one obtains

$$\begin{aligned}||T_{\alpha-1}(t)P_{s,\alpha-1}x||_\beta &\leq m(\alpha,\beta)t^{\alpha-1-\widetilde{\varepsilon}}e^{-\delta t}||x||_{\alpha-1}\\ &\leq m(\alpha,\beta)t^{\alpha-1-\beta-\widetilde{\varepsilon}}e^{-\delta\frac{t}{2}}t^\beta e^{-\delta\frac{t}{2}}||x||_{\alpha-1}.\\ &\leq m(\alpha,\beta)t^{\alpha-1-\beta-\widetilde{\varepsilon}}e^{-\delta\frac{t}{2}}||x||_{\alpha-1}.\end{aligned}$$

For $t \in [0,\ 1]$, it follows from (2.6) that

$$\begin{aligned}||T_{\alpha-1}(t)P_{s,\alpha-1}x||_\beta &\leq c\, t^{\alpha-1-\beta-\widetilde{\varepsilon}}||P_{s,\alpha-1}x||_{\alpha-1}\\ &\leq c\, t^{\alpha-1-\beta-\widetilde{\varepsilon}}e^{-\delta\frac{t}{2}}e^{\delta\frac{t}{2}}||x||_{\alpha-1}.\end{aligned}$$

Hence

$$||T_{\alpha-1}(t)P_{s,\alpha-1}x||_\beta \leq m(\alpha,\beta)t^{\alpha-1-\beta-\widetilde{\varepsilon}}e^{-\gamma t}||x||_{\alpha-1} \text{ for } t \geq 0.$$

□

4.2 Semilinear evolution equations

Consider the semilinear evolution equation

$$u'(t) = A_{\alpha-1}u(t) + f(t,u(t)), \qquad t \in \mathbb{R}, \tag{4.9}$$

where the function $f : \mathbb{R}\times X_\beta \longrightarrow X_{\alpha-1}$ is continuous and globally Lipschitzian, i.e., there is $k > 0$ such that

$$\|f(t,x) - f(t,y)\|_{\alpha-1} \leq k\,\|x-y\|_\beta \text{ for all } t \in \mathbb{R} \text{ and } x,y \in X_\beta. \tag{4.10}$$

By a mild solution of (4.9) we will understand a continuous function $x : \mathbb{R} \longrightarrow X_\beta$, which satisfies the following variation of constants formula

$$u(t) = T(t-s)u(s) + \int_s^t T_{\alpha-1}(t-\sigma)f(\sigma,u(\sigma))d\sigma \quad \text{for all } t \geq s,\ t,s \in \mathbb{R}. \tag{4.11}$$

We study first the existence of almost periodic and almost automorphic mild solutions for the inhomogeneous evolution equation

$$u'(t) = A_{\alpha-1}u(t) + g(t), \quad t \in \mathbb{R}. \tag{4.12}$$

We have the following main result.

Theorem 4.2.1. *Let $g \in BC(\mathbb{R}, X_{\alpha-1})$ and $0 \leq \beta < \alpha \leq 1$. Then, the following properties hold.*
(i) The equation (4.12) admits a unique bounded mild solution $u : \mathbb{R} \longrightarrow X_\beta$ given by

$$u(t) = \int_{-\infty}^{t} T_{\alpha-1}(t-\sigma)P_{s,\alpha-1}g(\sigma)d\sigma - \int_{t}^{+\infty} T_{\alpha-1}(t-\sigma)P_{u,\alpha-1}g(\sigma)d\sigma,\ t \in \mathbb{R}. \tag{4.13}$$

(*ii*) *If* $g \in \mathcal{F}(\mathbb{R}, X_{\alpha-1})$, *where* $\mathcal{F}$ *is one of the following abbreviations: AP, AA then* $u \in \mathcal{F}(\mathbb{R}, X_\beta)$.

Proof. (*i*) Since g is bounded, one can show as in [32] that $u(\cdot)$ given by (4.13) is well defined in X for all $t \in \mathbb{R}$. Moreover, one can see easily that $u(\cdot)$ satisfies the variation of constants formula

$$u(t) = T(t-s)u(s) + \int_{s}^{t} T_{\alpha-1}(t-\sigma)g(\sigma)d\sigma \quad \text{for all } t \geq s,\ t,s \in \mathbb{R}.$$

Using Proposition 4.1.2 and a characterization of the continuous interpolation spaces X_β, see [83, Proposition 2.2.8], we show that the function u is continuous from $\mathbb{R}$ to X_β. Hence, u is a mild solution of (4.12). The uniqueness can be shown as in [32]. For the boundedness, let $0 < \widetilde{\varepsilon} + \beta < \alpha$ and $0 < \alpha - \widetilde{\varepsilon} < 1$. By Proposition 4.1.2, we have

$$\begin{aligned}
&\|u(t)\|_\beta \\
&\leq \left\| \int_{-\infty}^{t} T_{\alpha-1}(t-\sigma)P_{s,\alpha-1}g(\sigma)d\sigma \right\|_\beta + \left\| \int_{t}^{+\infty} T_{\alpha-1}(t-\sigma)P_{u,\alpha-1}g(\sigma)d\sigma \right\|_\beta \\
&\leq \int_{-\infty}^{t} \|T_{\alpha-1}(t-\sigma)P_{s,\alpha-1}g(\sigma)\|_\beta \, d\sigma + \int_{t}^{+\infty} \|T_{\alpha-1}(t-\sigma)P_{u,\alpha-1}g(\sigma)\|_\beta \, d\sigma \\
&\leq m(\alpha,\beta) \int_{-\infty}^{t} e^{-\gamma(t-\sigma)}(t-\sigma)^{-(\beta-\alpha+\widetilde{\varepsilon}+1)} \|g(\sigma)\|_{\alpha-1} \, d\sigma \\
&+ c(\alpha,\beta) \int_{t}^{+\infty} e^{-\delta(t-\sigma)} \|g(\sigma)\|_{\alpha-1} \, d\sigma \\
&\leq m(\alpha,\beta) \int_{0}^{+\infty} e^{-\sigma} \left(\frac{\sigma}{\gamma}\right)^{-(\beta-\alpha+\widetilde{\varepsilon}+1)} \frac{d\sigma}{\gamma} \|g\|_\infty + c(\alpha,\beta) \int_{t}^{+\infty} e^{-\delta(t-s)} d\sigma \, \|g\|_\infty \\
&\leq m(\alpha,\beta)\gamma^{\beta-\alpha+\widetilde{\varepsilon}}\Gamma(\alpha-\beta-\widetilde{\varepsilon}) \|g\|_\infty + c(\alpha,\beta)\delta^{-1} \|g\|_\infty ,
\end{aligned}$$

where $\Gamma(\alpha) = \int_0^{+\infty} t^{\alpha-1}e^{-t}dt$ is the function gamma.

(*ii*) To show that the mild solution u is almost periodic, let $g \in AP(X_{\alpha-1})$. From Definition 2.5.1, for each $\varepsilon > 0$ there exists $l(\varepsilon) > 0$ such that for every $a \in \mathbb{R}$, there exists a number $\tau \in [a, a + l(\varepsilon)]$ satisfy $\|g(t+\tau) - g(t)\|_{\alpha-1} \leq \eta\varepsilon$ for all $t \in \mathbb{R}$, where $\eta^{-1} = m(\alpha,\beta)\gamma^{\beta-\alpha+\widetilde{\varepsilon}}\Gamma(\alpha-\beta-\widetilde{\varepsilon}) + c(\alpha,\beta)\delta^{-1}$.

Then,

$$
\begin{aligned}
&\|u(t+\tau)-u(t)\|_\beta \\
&\quad\le \int_{-\infty}^{t} \|T_{\alpha-1}(t-\sigma)P_{s,\alpha-1}[g(\sigma+\tau)-g(\sigma)]\|_\beta\, d\sigma \\
&\quad+ \int_{t}^{+\infty} \|T_{\alpha-1}(t-\sigma)P_{u,\alpha-1}[g(\sigma+\tau)-g(\sigma)]\|_\beta \\
&\quad\le m(\alpha,\beta)\int_{-\infty}^{t} e^{-\gamma(t-\sigma)}(t-\sigma)^{-(\beta-\alpha+\widetilde{\varepsilon}+1)}\,\|g(\sigma+\tau)-g(\sigma)\|_{\alpha-1}\, d\sigma \\
&\quad+ c(\alpha,\beta)\int_{t}^{+\infty} e^{-\delta(t-\sigma)}\,\|g(\sigma+\tau)-g(\sigma)\|_{\alpha-1}\, d\sigma \\
&\quad\le [m(\alpha,\beta)\gamma^{\beta-\alpha+\widetilde{\varepsilon}}\Gamma(\alpha-\beta-\widetilde{\varepsilon})+c(\alpha,\beta)\delta^{-1}]\eta\varepsilon=\varepsilon.
\end{aligned}
$$

Thus, $u\in AP(X_\beta)$.

To prove that the mild solution u is almost automorphic, let us take a sequence (s'_n) of real numbers. As $g\in AA(X_{\alpha-1})$, there is a subsequence $(s_n)_{n\in\mathbb{N}}$ such that

$$
\lim_{n,m\to\infty}\|g(t+s_n-s_m)-g(t)\|_{\alpha-1}=0, \tag{4.14}
$$

for every $t\in\mathbb{R}$. Then,

$$
\begin{aligned}
&u(t+s_n-s_m)-u(t) \\
&=\int_{-\infty}^{t+s_n-s_m} T_{\alpha-1}(t+s_n-s_m-\sigma)P_{s,\alpha-1}g(\sigma)d\sigma-\int_{-\infty}^{t} T_{\alpha-1}(t-\sigma)P_{s,\alpha-1}g(\sigma)d\sigma \\
&-\int_{t+s_n-s_m}^{+\infty} T_{\alpha-1}(t+s_n-s_m-\sigma)P_{u,\alpha-1}g(\sigma)d\sigma+\int_{t}^{+\infty} T_{\alpha-1}(t-\sigma)P_{u,\alpha-1}g(\sigma)d\sigma \\
&=\int_{-\infty}^{t} T_{\alpha-1}(t-\sigma)P_{s,\alpha-1}[g(\sigma+s_n-s_m)-g(\sigma)]d\sigma \\
&-\int_{t}^{+\infty} T_{\alpha-1}(t-\sigma)P_{u,\alpha-1}[g(\sigma+s_n-s_m)-g(\sigma)]d\sigma.
\end{aligned}
$$

Hence, from Proposition 4.1.2, we have

$$
\begin{aligned}
&\|u(t+s_n-s_m)-u(t)\|_\beta \\
&\quad\le m(\alpha,\beta)\int_{-\infty}^{t} e^{-\gamma(t-\sigma)}(t-\sigma)^{-(\beta-\alpha+\widetilde{\varepsilon}+1)}\,\|g(\sigma+s_n-s_m)-g(\sigma)\|_{\alpha-1}\, d\sigma \\
&\quad+ (\alpha,\beta)\int_{t}^{+\infty} e^{-\delta(t-\sigma)}\,\|g(\sigma+s_n-s_m)-g(\sigma)\|_{\alpha-1}\, d\sigma.
\end{aligned}
$$

Finally, the equation (4.14) and the Lebesgue's dominated convergence theorem, yield $\lim_{n,m\to\infty}\|u(t+s_n-s_m)-u(t)\|_\beta=0$ for each $t\in\mathbb{R}$. □

To obtain the same results for the semilinear evolution equation, consider $y\in\mathcal{F}(X_\beta)$ and $f\in\mathcal{F}(\mathbb{R}\times X_\beta,X_{\alpha-1})$. Then, by Lemma 2.5.4, the function

$g(\cdot) := f(\cdot, y(\cdot)) \in \mathcal{F}(X_{\alpha-1})$. Thus, from Theorem 4.2.1, the inhomogeneous evolution equation

$$u'(t) = A_{\alpha-1}u(t) + g(t), \quad t \in \mathbb{R},$$

admits a unique mild solution $u \in \mathcal{F}(X_\beta)$ given by

$$u(t) = \int_{-\infty}^{t} T_{\alpha-1}(t-\sigma)P_{s,\alpha-1}f(\sigma, y(\sigma))d\sigma - \int_{t}^{+\infty} T_{\alpha-1}(t-\sigma)P_{u,\alpha-1}f(\sigma, y(\sigma))d\sigma,$$

for all $t \in \mathbb{R}$. Let the operator $F : \mathcal{F}(X_\beta) \longrightarrow \mathcal{F}(X_\beta)$ be defined by

$$(Fy)(t) : \int_{-\infty}^{t} T_{\alpha-1}(t-\sigma)P_{s,\alpha-1}f(\sigma, y(\sigma))d\sigma - \int_{t}^{+\infty} T_{\alpha-1}(t-\sigma)P_{u,\alpha-1}f(\sigma, y(\sigma))d\sigma$$

for all $t \in \mathbb{R}$ and assume that $k < \dfrac{1}{m(\alpha,\beta)\gamma^{\beta-\alpha+\widetilde{\varepsilon}}\Gamma(\alpha-\beta-\widetilde{\varepsilon}) + c(\alpha,\beta)\delta^{-1}}$, where $\widetilde{\varepsilon}$ is any constant such that $0 < \alpha - \widetilde{\varepsilon} < 1, \ \ 0 < \beta + \widetilde{\varepsilon} < \alpha$. Then, we have for any $x, y \in \mathcal{F}(X_\beta)$

$$\begin{aligned}
&\|Fx(t) - Fy(t)\|_\beta \\
&\qquad \leq m(\alpha,\beta)\int_{-\infty}^{t} e^{-\gamma(t-\sigma)}(t-\sigma)^{-(\beta-\alpha+\widetilde{\varepsilon}+1)} \|f(\sigma, y(\sigma)) - f(\sigma, x(\sigma))\|_{\alpha-1} d\sigma \\
&\qquad + c(\alpha,\beta)\int_{t}^{+\infty} e^{-\delta(t-\sigma)} \|f(\sigma, y(\sigma)) - f(\sigma, x(\sigma))\|_{\alpha-1} d\sigma. \\
&\qquad \leq K[m(\alpha,\beta)\gamma^{\beta-\alpha+\widetilde{\varepsilon}}\Gamma(\alpha-\beta-\widetilde{\varepsilon}) + c(\alpha,\beta)\delta^{-1}] \|x-y\|_\infty \ \text{ for all } t \in \mathbb{R}.
\end{aligned}$$

This shows that F has a unique fixed point in $\mathcal{F}(X_\beta)$, and consequently we have the following theorem.

Theorem 4.2.2. *Let $0 \leq \beta < \alpha$ and $\widetilde{\varepsilon} > 0$ such that $0 < \alpha - \widetilde{\varepsilon} < 1$ and $0 < \beta + \widetilde{\varepsilon} < \alpha$. Assume that $k < (m(\alpha,\beta)\gamma^{\beta-\alpha+\widetilde{\varepsilon}}\Gamma(\alpha-\beta-\widetilde{\varepsilon}) + c(\alpha,\beta)\delta^{-1})^{-1}$ and $f \in \mathcal{F}(\mathbb{R} \times X_\beta, X_{\alpha-1})$.*

Then (4.9) admits a unique mild solution u in $\mathcal{F}(X_\beta)$, which satisfies the variation of constants formula for $t \in \mathbb{R}$

$$u(t) = \int_{-\infty}^{t} T_{\alpha-1}(t-\sigma)P_{s,\alpha-1}f(\sigma, u(\sigma))d\sigma - \int_{t}^{+\infty} T_{\alpha-1}(t-\sigma)P_{u,\alpha-1}f(\sigma, u(\sigma))d\sigma.$$

4.3 Semilinear boundary evolution equations

Consider the semilinear autonomous boundary evolution equation

$$\begin{cases} u'(t) &= A_m x(t) + h(t, u(t)), \ \ t \in \mathbb{R}, \\ Lu(t) &= \phi(t, u(t)), \ \ t \in \mathbb{R}. \end{cases} \tag{4.15}$$

Here $(A_m, D(A_m))$ is a densely defined linear operator on a Banach space X, $L : D(A_m) \longrightarrow \partial X$, the boundary Banach space and the functions $h : \mathbb{R} \times X_m \longrightarrow X$, $\phi : \mathbb{R} \times X_m \longrightarrow \partial X$ are continuous.

Throughout this section, we assume that the following hypotheses hold.

(**A1**) There exists a norm $|\cdot|$ on $D(A_m)$ such that $X_m := (D(A_m), |\cdot|)$ is complete. The space X_m is continuously embedded in X and $A_m \in \mathcal{L}(X_m, X)$.

(**A2**) The restriction $A := A_m \mid \ker(L)$ is a sectorial operator such that $\sigma(A) \cap i\mathbb{R} = \emptyset$.

(**A3**) The operator $L : X_m \longrightarrow \partial X$ is bounded and surjective.

(**A4**) $X_m \hookrightarrow X_\alpha$, for some $0 < \alpha < 1$.

(**A5**) $h : \mathbb{R} \times X_\beta \longrightarrow X$ and $\phi : \mathbb{R} \times X_\beta \longrightarrow \partial X$ are continuous for $0 \leq \beta < \alpha$.

Under the assumptions **(A1)-(A3)** the following properties have been shown by G. Greiner [57, Lemma 1.2, 1.3].

Lemma 4.3.1. *For some $\lambda \in \rho(A)$, the following assertions are true:*

(i) $X_m = D(A) \oplus ker(\lambda - A_m)$.

(ii) The restriction $L : \ker(\lambda - A_m) \to \partial X$ *is invertible and its inverse is the so-called Dirichlet operator* $L_\lambda \in \mathcal{L}(\partial X, X)$.

(iii) $P_\lambda := L_\lambda L$ *is a projection from* $D(A) = \ker L$ *onto* $\ker(\lambda - A_m)$.

(iv) $R(\mu, A)L_\lambda = R(\lambda, A)L_\mu$ *for all* $\lambda, \mu \in \rho(A)$.

(v) $(\lambda - A_m)L_\lambda = LR(\lambda, A) = 0,\ LL_\lambda = Id_{\partial X}$.

We know that the assumption **(A4)** is equivalent to the fact that the operator

$$L_\lambda : \partial X \longrightarrow X_\alpha \quad \text{is bounded for all } \lambda > \lambda_0. \tag{4.16}$$

Recall here that $u : \mathbb{R} \longrightarrow X_\beta$ is a mild solution of (4.15) if for all $t \geq s, t, s \in \mathbb{R}$, we have :

$$(i) \int_s^t u(\tau)d\tau \in X_m, \quad (ii)\, u(t) - u(s) = A_m \int_s^t u(\tau)d\tau + \int_s^t h(\tau, u(\tau))d\tau,$$
$$(iii)\, L \int_s^t u(\tau)d\tau = \int_s^t \phi(\tau, u(\tau))d\tau.$$

In the following lemma we show the equivalence between the boundary equation (4.15) and a semilinear evolution equation

Lemma 4.3.2. *Assume that (***A1***)-(***A3***) are satisfied. A function u is a mild solution of the boundary equation* (4.15) *if and only if u is a mild solution of the semilinear evolution equation on X_β*

$$u'(t) = A_{\alpha-1}u(t) + h(t, u(t)) - A_{\alpha-1}L_0\phi(t, u(t)), \quad t \in \mathbb{R}, \tag{4.17}$$

where $L_0 := (L|Ker(A_m))^{-1}$.

Proof. Let u be a mild solution of (4.15). Then, since $\text{Range}(L_0L) \subset \ker(A_m)$, $\text{Range}(I - L_0L) = D(A)$ and from Lemma 4.3.1(i), we can decompose

$$\int_s^t u(\tau)d\tau = (Id - L_0L)\int_s^t u(\tau)d\tau + L_0L\int_s^t u(\tau)d\tau$$

Then by $(ii) - (iii)$ and (**A4**) we have

$$\begin{aligned} u(t) - u(s) &= A_m\int_s^t u(\tau)d\tau - A_mL_0L\int_s^t u(\tau)d\tau + \int_s^t h(\tau, u(\tau))d\tau \\ &= A\left(\int_s^t u(\tau)d\tau - L_0L\int_s^t u(\tau)d\tau\right) + \int_s^t h(\tau, u(\tau))d\tau \\ &= A_{\alpha-1}\int_s^t u(\tau)d\tau + \int_s^t h(\tau, u(\tau))d\tau - A_{\alpha-1}L_0\int_s^t \phi(\tau, u(\tau))d\tau, \end{aligned}$$

for all $t \geq s,\ t, s \in \mathbb{R}$. The last equation is equivalent to the fact that u satisfies the variation of constants formula (4.11), and thus it is a mild solution of (4.17). Let now u be a mild solution of (4.17), that is, u satisfies

$$u(t) = T(t-s)u(s) + \int_s^t T(t-\tau)h(\tau, u(\tau))d\tau - \int_s^t T_{\alpha-1}(t-\tau)A_{\alpha-1}L_0\phi(\tau, u(\tau))d\tau$$

for all $t \geq s, t, s \in \mathbb{R}$. Since u is a X-valued function, then $\int_s^t T_{\alpha-1}(t - \tau)A_{\alpha-1}L_0\phi(\tau, u(\tau))d\tau \in X$, and then $\int_s^t T(t-\tau)L_0\phi(\tau, u(\tau))d\tau \in D(A)$, and

$$u(t) = T(t-s)u(s) + \int_s^t T(t-\tau)h(\tau, u(\tau))d\tau - A\int_s^t T(t-\tau)L_0\phi(\tau, u(\tau))d\tau.$$

Hence,

$$\begin{aligned} &\int_s^t u(\tau)d\tau \\ &\quad = \int_s^t T(\tau - s)u(s)d\tau + \int_s^t\int_s^\tau T(\tau-\sigma)h(\sigma, u(\sigma))d\sigma d\tau \\ &\quad - \int_s^t\int_s^\tau T_{\alpha-1}(\tau-\sigma)A_{\alpha-1}L_0\phi(\sigma, u(\sigma))d\sigma d\tau \\ &\quad = A^{-1}[T(t-s)u(s) - u(s)] + A^{-1}\int_s^t T(t-\sigma)h(\sigma, u(\sigma))d\sigma \\ &\quad + A^{-1}\int_s^t h(\sigma, u(\sigma))d\sigma - \int_s^t T(t-\sigma)L_0\phi(\sigma, u(\sigma))d\sigma - L_0\int_s^t \phi(\sigma, u(\sigma))d\sigma. \end{aligned}$$

This implies easily that u satisfies (i)–(iii) above. This completes the proof. □

We can now announce the main result of this section.

Theorem 4.3.3. *Assume that* **(A1)-(A5)** *are satisfied, and that the functions* $\phi \in \mathcal{F}(\mathbb{R} \times X_\beta, \partial X), h \in \mathcal{F}(\mathbb{R} \times X_\beta, X)$ *are globally Lipschitzian with small constants. Then, the semilinear boundary evolution equation* (4.15) *has a unique mild solution* $u \in \mathcal{F}(X_\beta)$ *satisfying, for all* $t \in \mathbb{R}$,

$$u(t) = \int_{-\infty}^{t} T(t-s)P_s h(s,u(s))ds - \int_{t}^{+\infty} T(t-s)P_u h(s,u(s))ds \\ - A\left[\int_{-\infty}^{t} T(t-s)P_s L_0\phi(s,u(s))ds - \int_{t}^{+\infty} T(t-s)P_u L_0\phi(s,u(s))ds\right]. \tag{4.18}$$

Proof. One knows that $A_{\alpha-1}L_0$ is a bounded operator from ∂X to $X_{\alpha-1}$. Hence, since $\phi \in \mathcal{F}(\mathbb{R} \times X_\beta, \partial X)$ and $h \in \mathcal{F}(\mathbb{R} \times X_\beta, X)$ and from the injection $X \hookrightarrow X_{\alpha-1}$, the function $f(t,u) := h(t,u) - A_{\alpha-1}L_0\phi(t,u)$ belongs to $\mathcal{F}(\mathbb{R} \times X_\beta, X_{\alpha-1})$. This function is also globally Lipschitzian with a small constant. Hence, by Theorem 4.2.2 there is a unique mild solution $u \in \mathcal{F}(X_\beta)$ of the equation (4.17), satisfying

$$u(t) = \int_{-\infty}^{t} P_{s,\alpha-1}T_{\alpha-1}(t-s)f(s,u(s))ds - \int_{t}^{+\infty} P_{u,\alpha-1}T_{\alpha-1}(t-s)f(s,u(s))ds,$$

from which we deduce the variation of constants formula (4.18) and that $u \in \mathcal{F}(X_\beta)$ is the unique mild solution. □

We conclude this section with the following example.

Example 4.3.4.

Consider the following partial differential equation

$$\begin{cases} \frac{\partial}{\partial t}u(t,x) &= \Delta u(t,x) + au(t,x),\ t \in \mathbb{R},\ x \in \Omega \\ \frac{\partial}{\partial n}u(t,x) &= \Gamma(t, m(x)u(t,x)),\ t \in \mathbb{R},\ x \in \partial\Omega, \end{cases} \tag{4.19}$$

where $a \in \mathbb{R}_+$ and m is a $\mathcal{C}^1$-function. We assume that Ω is a bounded open subset of $\mathbb{R}^n$ with smooth boundary $\partial\Omega$. Let $X = L^2(\Omega)$, $X_m = H^2(\Omega)$ and the boundary space $\partial X = H^{\frac{1}{2}}(\partial\Omega)$. Consider the operator $A_m : X_m \to X,\ A_m\varphi = \Delta\varphi + a\varphi$ and
$L : X_m \to \partial X, \quad L\varphi = \frac{\partial \varphi}{\partial n}$. From [112, Section 4.7.1], the operator L is bounded and surjective. It is known also that the operator $A = A_m \mid \ker(L)$ generates an analytic semigroup. It follows also from [112, Sections 4.3.3, 4.6.1] that for $\alpha < \frac{3}{4}$, $X_m \subset X_\alpha$. The eigenvalues of Neumann Laplacian A form a decreasing sequence (λ_n) with $\lambda_0 = 0$ and $\lambda_1 < 0$. If one takes $a = -\frac{1}{2}\lambda_1$ then $\sigma(A) \cap i\mathbb{R} = \emptyset$. Hence, the analytic semigroup generated

by A is hyperbolic. Consider the function $\phi(t,\varphi)(x) = \Gamma(t, m(x)\varphi(x)) = \dfrac{k\,b(t)}{1+|m(x)\varphi(x)|}$, $t \in \mathbb{R}$, $x \in \partial\Omega$ and $b(t)$ is an almost automorphic (resp. almost periodic) real function. One can see that ϕ is continuous on $\mathbb{R} \times H^{2\beta'}(\Omega)$ for some $\dfrac{1}{2} < \beta < \beta' < \alpha < \dfrac{3}{4}$, which is embedded in $\mathbb{R} \times X_\beta$ (see e.g [112]). By using the definition of fractional Sobolev space, we have $\phi(t,\varphi)(\cdot) \in H^{\frac{1}{2}}(\partial\Omega)$ for all $\varphi \in H^{2\beta'}(\Omega) \hookrightarrow H^1(\Omega)$. Moreover φ is almost automorphic (resp. almost periodic) in $t \in \mathbb{R}$ for each $u \in X_\beta$, and globally Lipshitizian. Now for a small constant k, all assumptions of Theorem 4.3.3 are satisfied and hence (4.19) admits a unique almost automorphic (resp. almost periodic) mild solution u with values in X_β.

Chapter 5

Asymptotic behavior of inhomogeneous non-autonomous boundary evolution equations

The principal aim of this chapter is to study the almost periodicity of the solutions to the parabolic inhomogeneous boundary value problem

$$\begin{cases} u'(t) & = A_m(t)u(t) + g(t), \qquad t \in \mathbb{R}, \\ B(t)u(t) & = h(t), \qquad t \in \mathbb{R}, \end{cases} \tag{5.1}$$

for linear operators $A_m(t) : Z \to X$ and $B(t) : Z \to Y$ on Banach spaces $Z \hookrightarrow X$ and Y. We want to show that the solutions $u : \mathbb{R} \to X$ of (5.1) inherit the (asymptotic) almost periodicity of the inhomogeneities $g : \mathbb{R} \to X$ and $h : \mathbb{R} \to Y$.

As in the previous chapter, we transform our boundary evolution equation to the inhomogeneous evolution equation

$$u'(t) = A_{\alpha-1}(t)u(t) + f(t), \qquad t \in J,$$

in $X^t_{\alpha-1}$. We study first the asymptotic behavior of this last equation and deduce at the end the same result for (5.1).

5.1 Almost periodicity of evolution equations

Consider a family of linear operators $A(t)$, $t \in \mathbb{R}$, on a Banach space X satisfying the Acquistapace-Terreni hypothesis **(H1)**.

In this section, we study the parabolic evolution equation

$$u'(t) = A_{\alpha-1}(t)u(t) + f(t), \qquad t \in J, \tag{5.2}$$

where J is an unbounded closed interval, $f \in E_{\alpha-1}(J)$.

Let $U(t,s)$, $t \geq s$, be the evolution family generated by $A(t), t \in \mathbb{R}$, and be $U_{\alpha-1}(t,s)$, $t \geq s$, its extrapolated evolution family defined in Proposition

2.4.2 for each $\alpha \in (1-\mu, 1]$. A *mild solution* of (5.2) is a function $u \in C(J, X)$ satisfying

$$u(t) = U(t,s)u(s) + \int_s^t U_{\alpha-1}(t,\tau)f(\tau)\,d\tau, \qquad \forall\, t \geq s \ \text{ in } J. \tag{5.3}$$

In Proposition 2.6 of [87], we showed that a mild solution actually satisfies (5.2) pointwise in $X_{\beta-1}^t$ for each $\beta \in [0, \min\{\nu, \alpha\})$ and $t \in J$. Conversely, if $u \in C^1(J, X)$ solves (5.2) (and thus $u \in E_\alpha(J)$), then Proposition 2.1(iv) of [88] implies that

$$\partial_\tau^+ U(t,\tau)u(\tau) = -U_{\alpha-1}(t,\tau)A_{\alpha-1}(\tau)u(\tau) + U(t,\tau)u'(\tau) = U_{\alpha-1}(t,\tau)f(\tau)$$

in X for all $t > \tau$. As a result,

$$U(t, t-\varepsilon)u(t-\varepsilon) - U(t,s)u(s) = \int_s^{t-\varepsilon} U_{\alpha-1}(t,\tau)f(\tau)\,d\tau$$

for $t > t-\varepsilon > s$. Letting $\varepsilon \to 0$, we conclude that u is a mild solution of (5.2).

5.1.1 Evolution equations on $\mathbb{R}$

In this subsection we study the almost periodicity of the solutions to (5.2) on $J = \mathbb{R}$ under the following assumptions.

(H1) The operators $A(t)$, $t \in \mathbb{R}$, satisfy the assumptions (2.7) and (2.8).

(H2) The evolution family U generated by $A(\cdot)$ has an exponential dichotomy on $\mathbb{R}$ with constants $N, \delta > 0$, projections $P(t)$, $t \in \mathbb{R}$, and Green's function Γ.

(H3) $R(\omega, A(\cdot)) \in AP(J, \mathcal{L}(X))$.

It is not difficult to verify that then $R(\lambda, A(\cdot)) \in AP(J, \mathcal{L}(X))$ for $\lambda \in \omega + \Sigma_\theta \cup \{0\}$. We want to solve (5.3) for f belonging to the space $AP_{\alpha-1}(\mathbb{R})$ which is defined by

$$\begin{aligned} AP_{\alpha-1}(\mathbb{R}) &:= \{f \in E_{\alpha-1}(\mathbb{R}) : \exists\,(f_n) \in AP(\mathbb{R}, X) \text{ converging to } f \text{ in } E_{\alpha-1}(\mathbb{R})\} \\ &= \{f \in E_{-1}(\mathbb{R}) : \exists\,(f_n) \in AP(\mathbb{R}, X) \text{ converging to } f \text{ in } E_{\alpha-1}(\mathbb{R})\} \end{aligned}$$

for $\alpha \in [0, 1]$. This space is endowed with the norm of $E_{\alpha-1}(\mathbb{R})$. Note that $AP_0(\mathbb{R}) = AP(\mathbb{R}, X)$.

We first characterize the space $AP_{\alpha-1}(\mathbb{R})$. On $F := AP(\mathbb{R}, X)$, we define the multiplication operator

$$\begin{aligned} (A(\cdot)v)(t) &:= A(t)v(t), \quad t \in \mathbb{R}, \\ D(A(\cdot)) &:= \{v \in F : f(t) \in D(A(t)) \text{ for all } t \in \mathbb{R},\ A(\cdot)v \in F\}. \end{aligned}$$

Assumptions **(H3)** and (2.7) imply that the function $R(\lambda, A(\cdot))v$ belongs to F for every $v \in F$ and $\lambda \in \omega + \Sigma_\theta \cup \{0\}$. Therefore, the operator $A(\cdot)$ is sectorial on F with the resolvent $R(\lambda, A(\cdot))$. We can thus introduce the spaces $F_{\alpha-1} := F_{\alpha-1}^{A(\cdot)}$ for each $\alpha \in [0, 1)$, where we set $F_0 := F$ and $F_1 := D(A(\cdot))$.

Proposition 5.1.1. *Let* (2.7) *and* **(H3)** *hold. We then have* $F_{\alpha-1} \cong AP_{\alpha-1}(\mathbb{R})$ *for each* $\alpha \in [0,1]$.

Proof. We first note that

$$\|f\|_{F_{\alpha-1}} = \|f\|_{E_{\alpha-1}} \quad \text{for all} \quad f \in F \text{ and } \alpha \in [0,1]. \tag{5.4}$$

The embedding $F_{-1} \hookrightarrow E_{-1}$ holds due to Lemma 2.3.1. Therefore we obtain

$$\begin{aligned} F_{\alpha-1} &= \{f \in F_{-1} : \exists\, f_n \in AP(\mathbb{R}, X),\ f_n \to f \ \text{ in } \ \|\cdot\|_{F_{\alpha-1}} = \|\cdot\|_{E_{\alpha-1}}\} \\ &\hookrightarrow \{f \in E_{-1} : \exists\, f_n \in AP(\mathbb{R}, X),\ f_n \to f \ \text{ in } \ \|\cdot\|_{F_{\alpha-1}} = \|\cdot\|_{E_{\alpha-1}}\} \\ &= AP_{\alpha-1}(\mathbb{R}). \end{aligned}$$

The asserted isomorphy now follows from (5.4). □

These spaces are much simpler in the case of constant extrapolation spaces.

Proposition 5.1.2. *Let* (2.7) *and* **(H3)** *hold. Assume that* $X^t_{\alpha-1} \cong X^0_{\alpha-1} =: X_{\alpha-1}$ *for some* $\alpha \in [0,1]$ *and every* $t \in \mathbb{R}$ *with uniformly equivalent norms. Then it holds* $F_{\alpha-1} \cong AP_{\alpha-1}(\mathbb{R}) \cong AP(\mathbb{R}, X_{\alpha-1})$.

Proof. Due to the assumptions, the norms of $E_{\alpha-1}$ and of $C_b(\mathbb{R}, X_{\alpha-1})$ are equivalent on E, so that $E_{\alpha-1} \cong C_b(\mathbb{R}, X_{\alpha-1})$. Take $f \in AP(\mathbb{R}, X_{\alpha-1}) \hookrightarrow E_{\alpha-1}$ and the sequence $f_n := nR(n, A_{\alpha-1}(\cdot))f$ for $n > \omega$. We first show that $f_n \in AP(\mathbb{R}, X)$. For that purpose, let $x \in X_{\alpha-1}$ and take $x_k \in X$ converging to x in $X_{\alpha-1}$. Due to **(H3)**, we have $nR(n, A(\cdot))x_k \in AP(\mathbb{R}, X)$. Since $R(n, A_{\alpha-1}(t))$ is bounded from $X^t_{\alpha-1}$ to X uniformly in t (see e.g. [88, (2.8)], we derive that $nR(n, A_{\alpha-1}(\cdot))x \in AP(\mathbb{R}, X)$. The same is true for functions $f = \phi(\cdot)x$, with scalar almost periodic function ϕ and $x \in X_{\alpha-1}$. Since the span of those functions is dense in $AP(\mathbb{R}, X_{\alpha-1})$ by [12, Theorem 4.5.7], it follows that $f_n \in AP(\mathbb{R}, X)$. Observing that $f_n \to f$ in $E_{\alpha-1}$, we conclude that $f \in AP_{\alpha-1}(\mathbb{R})$. For the converse, let $f \in AP_{\alpha-1}(\mathbb{R})$ and $AP(\mathbb{R}, X) \ni f_n \to f$ in $E_{\alpha-1} \cong C_b(\mathbb{R}, X_{\alpha-1})$. The continuous embedding $X \hookrightarrow X_{\alpha-1}$ implies that $f_n \in AP(\mathbb{R}, X_{\alpha-1})$, and hence $f \in AP(\mathbb{R}, X_{\alpha-1})$. □

We state the main result of this subsection.

Theorem 5.1.3. *Assume that* **(H1)**, **(H2)** *and* **(H3)** *hold. Let* $f \in AP_{\alpha-1}(\mathbb{R})$ *for some* $\alpha \in (1-\mu, 1]$. *Then the evolution equation* (5.2) *has a unique mild solution* $u \in AP(\mathbb{R}, X)$ *given by*

$$u(t) = \int_{\mathbb{R}} \Gamma_{\alpha-1}(t,\tau) f(\tau)\, d\tau, \qquad t \in \mathbb{R}. \tag{5.5}$$

Proof. For $f \in E_{\alpha-1}$, one can show that the function u given by (5.5) is a bounded mild solution of (5.2), and that every bounded mild solution is given by (5.5). (See e.g. the remarks after Theorem 3.10 in [87].) This fact shows the uniqueness of bounded mild solutions to (5.2). Take a sequence $(f_n) \subset AP(\mathbb{R}, X)$ converging to f in $E_{\alpha-1}$. In Theorem 4.5 of [86] we have shown that the functions

$$u_n(t) = \int_{\mathbb{R}} \Gamma(t,\tau) f_n(\tau)\, d\tau, \qquad t \in \mathbb{R}, \tag{5.6}$$

belongs to $AP(\mathbb{R}, X)$. Proposition 2.4.3 further yields

$$\begin{aligned} \|u(t) - u_n(t)\| &\le \int_{\mathbb{R}} \|\Gamma_{\alpha-1}(t,\tau)\|_{\mathcal{L}(X^{\tau}_{\alpha-1},X)} \|f_n(\tau) - f(\tau)\|^{\tau}_{\alpha-1}\, d\tau \\ &\le c\, \|f_n - f\|_{E_{\alpha-1}}, \qquad t \in \mathbb{R}. \end{aligned}$$

Therefore $u_n \to u$ in $C_b(\mathbb{R}, X)$ as $n \to \infty$, and so $u \in AP(\mathbb{R}, X)$. □

5.1.2 Forward evolution equations

We investigate the parabolic initial value problem

$$\begin{aligned} u'(t) &= A_{\alpha-1}(t)u(t) + f(t), \qquad t \ge t_0, \\ u(t_0) &= x, \end{aligned} \tag{5.7}$$

under the following assumptions.

(H1') The operators $A(t)$, $t > a$, satisfy the assumptions (2.7) and (2.8) for $t, s > a$.

(H2') The evolution family U generated by $A(\cdot)$ has an exponential dichotomy on (a, ∞) with projections $P(t)$, $t > a$, constants $N, \delta > 0$, and Green's function Γ.

(H3') $R(\omega, A(\cdot)) \in AAP([t_0, \infty), \mathcal{L}(X))$ for some $t_0 > a$.

Let now $t_0 > a$, $1 - \mu < \alpha \le 1$, $x \in \overline{D(A(t_0))}$ and $f \in E_{\alpha-1}([t_0, \infty))$. Assume that **(H1')** and **(H2')** hold. Then a mild solution of (5.7) is a function $u \in C([t_0, \infty), X)$ being a mild solution of the evolution equation in the first line of (5.7) and satisfying $u(t_0) = x$. We have shown in [87, Proposition 2.7] that there is a bounded mild function u of (5.7) if and only if

$$Q(t_0)x = -\int_{t_0}^{\infty} \tilde{U}(t_0, s) Q_{\alpha-1}(s) f(s)\, ds. \tag{5.8}$$

In this case the mild solution of (5.7) is uniquely given by

$$\begin{aligned} u(t) &= U(t,t_0)P(t_0)x + \int_{t_0}^{t} U_{\alpha-1}(t,s)P_{\alpha-1}(s)f(s)\, ds \\ &\quad - \int_{t}^{+\infty} \tilde{U}_{\alpha-1}(t,s)Q_{\alpha-1}(s)f(s)\, ds \\ &= U(t,t_0)P(t_0)x + \int_{t_0}^{+\infty} \Gamma_{\alpha-1}(t,s)f(s)\, ds, \qquad t \ge t_0. \end{aligned} \tag{5.9}$$

We want to study the asymptotic almost periodicity of this solution in the case of an asymptotically almost periodic f. For a close unbounded interval $J \ne \mathbb{R}$, we introduce the space

$$AAP_{\alpha-1}(J) := \{f \in E_{\alpha-1}(J) : \exists\, (f_n) \subseteq AAP(J,X),\ \ f_n \to f \text{ in } E_{\alpha-1}(J)\},$$

endowed with the norm of $E_{\alpha-1}(J)$.

We define the multiplication operator $A(\cdot)$ on $AAP(J,X)$ by

$$(A(\cdot)v)(t) := A(t)v(t), \qquad t \in J,$$
$$D(A(\cdot)) := \{v \in AAP(J,X) : v(t) \in D(A(t))\ \forall\, t \in J,\ A(\cdot)v \in AAP(J,X)\}.$$

Assumption **(H3')** and (2.7) imply that the function $R(\lambda, A(\cdot))v$ belongs to $AAP([t_0,\infty),X)$ for every $v \in AAP([t_0,\infty),X)$ and $\lambda \in \omega + \Sigma_\theta \cup \{0\}$. Therefore, the operator $A(\cdot)$ is sectorial on $AAP([t_0,\infty),X)$. We can thus introduce also the spaces $AAP([t_0,\infty),X)_{\alpha-1}^{A(\cdot)}$ for $\alpha \in [0,1]$. These spaces can be characterized as in the previous subsection.

Proposition 5.1.4. *Let* (2.7) *and* **(H3')** *hold. Then we have*

$$AAP_{\alpha-1}([t_0,\infty)) \cong AAP([t_0,\infty),X)_{\alpha-1}^{A(\cdot)}.$$

for each $\alpha \in [0,1]$. *If, in addition,* $X_{\alpha-1}^t \cong X_{\alpha-1}$ *with uniform equivalent norms for some* $1-\mu < \alpha \le 1$ *and a Banach space* $X_{\alpha-1}$*, then we obtain*

$$AAP_{\alpha-1}([t_0,\infty)) \cong AAP([t_0,\infty),X_{\alpha-1}).$$

We can now prove the main result of this subsection.

Theorem 5.1.5. *Let* $1-\mu < \alpha \le 1$. *Assume that* **(H1')**, **(H2')**, *and* **(H3')** *hold and that* $x \in \overline{D(A(t_0))}$ *and* $f \in AAP_{\alpha-1}([t_0,\infty))$ *satisfy* (5.8). *Then the unique bounded mild solution* u *of* (5.7) *is asymptotically almost periodic.*

Proof. Let $f \in AAP_{\alpha-1}([t_0,\infty))$ and $x \in X$ satisfy (5.8). Take a sequence $(f_n) \subset AAP([t_0,\infty),X)$ converging to f in $E_{\alpha-1}([t_0,\infty))$. Due to [86, Theorem 5.4], the functions

$$u_n(t) = U(t,t_0)P(t_0)x + \int_{t_0}^{\infty} \Gamma(t,s)f_n(s)\,ds, \qquad t \ge t_0,\ n \in \mathbb{N}$$

are asymptotically almost periodic in X (and they are mild solutions of (5.7) for the inhomogeneities f_n and the initial values $x_n = u_n(t_0)$). As in the proof of Theorem 5.1.3, we see that $u_n \to u$ in $C_b([t_0,\infty),X)$. So we conclude that $u \in AAP([t_0,\infty),X)$. □

5.1.3 Backward evolution equations

As a counterpart to the previous subsection, we now study the parabolic final value problem

$$\begin{aligned} u'(t) &= A_{\alpha-1}(t)u(t) + f(t), \qquad t \le t_0, \\ u(t_0) &= x. \end{aligned} \tag{5.10}$$

Mild solutions of (5.10) are defined as in the forward case. We make the following assumptions.

(H1") The operators $A(t)$, $t < b$, satisfy (2.7) and (2.8) for $t,s < b$.

(H2") The evolution family U has an exponential dichotomy on $(-\infty, b)$ with projections $P(t)$, $t < b$, constants $N, \delta > 0$, and Green's function Γ.

(H3") $R(\omega, A(\cdot)) \in AAP((-\infty, t_0], \mathcal{L}(X))$ for some $t_0 < b$.

Let $1 - \mu < \alpha \leq 1$, $x \in X$, and $f \in E_{\alpha-1}((-\infty, t_0])$. We have shown in [87, Proposition 2.8] that there is a unique bounded mild solution $u \in C((-\infty, t_0], X)$ of (5.10) on $(-\infty, t_0]$ if and only if

$$P(t_0)x = \int_{-\infty}^{t_0} U_{\alpha-1}(t_0, s)P_{\alpha-1}(s)f(s)ds, \tag{5.11}$$

in which case u is given by

$$\begin{aligned} u(t) = \widetilde{U}(t, t_0)Q(t_0)x &+ \int_{-\infty}^{t} U_{\alpha-1}(t, s)P_{\alpha-1}(s)f(s)\, ds \\ &- \int_{t}^{t_0} \widetilde{U}_{\alpha-1}(t, s)Q_{\alpha-1}(s)f(s)\, ds \end{aligned} \tag{5.12}$$

for $t \leq t_0$.

As before, we obtain the asymptotic almost periodicity of this function if f belongs to $AAP_{\alpha-1}((-\infty, t_0])$. We note that the space $AAP_{\alpha-1}((-\infty, t_0])$ can de described as in Proposition 5.1.4.

Theorem 5.1.6. *Assume that* **(H1")**, **(H2")**, *and* **(H3")** *hold. Let* $x \in X$ *and* $f \in AAP_{\alpha-1}((-\infty, t_0])$ *satisfy* (5.11). *Then the unique bounded mild solution* u *of* (5.10) *given by* (5.12) *belongs to* $AAP((-\infty, t_0], X)$.

Proof. Let x and f be as in the assertion. Take a sequence (f_n) in $AAP((-\infty, t_0], X)$ converging to f in $E_{\alpha-1}((-\infty, t_0])$. Define the function

$$u_n(t) = \widetilde{U}(t, t_0)Q(t_0)x + \int_{-\infty}^{t_0} \Gamma_{\alpha-1}(t, s)Q_{\alpha-1}(s)f_n(s)\, ds$$

for $t \leq t_0$ and $n \in \mathbb{N}$. Using the same arguments as in [86, Theorem 5.4], we can show that $u_n \in AAP((-\infty, t_0], X)$ for all $n \in \mathbb{N}$. Finally, as in Theorem 5.1.3 we see that $u_n \to u$ in $C_b((-\infty, t_0], X)$, so that $u \in AAP((-\infty, t_0], X)$. □

5.2 Fredholm properties of almost periodic evolution equations on $\mathbb{R}$

Consider a family of operators $A(t)$, $t \in \mathbb{R}$, on X satisfying the hypothesis **(H1)**. We assume that

$$\begin{aligned} &U(\cdot, \cdot) \text{ has exponential dichotomies on } [T', +\infty) \text{ and } (-\infty, -T'] \\ &\text{for some } T' \in \mathbb{R}. \text{ We fix a number } T \geq 0 \text{ such that } T > T'. \end{aligned} \tag{5.13}$$

In some results we shall assume that $A(\cdot)$ is *asymptotically hyperbolic*, i.e., there are two operators $A_{-\infty} : D(A_{-\infty}) \to X$ and $A_{+\infty} : D(A_{+\infty}) \to X$ which satisfy (2.7) and

$$\lim_{t\to\pm\infty} R(\omega, A(t)) = R(\omega, A_{\pm\infty}) \qquad (\text{in } \mathcal{L}(X)); \tag{5.14}$$

$$\sigma(A_{+\infty}) \cap i\mathbb{R} = \sigma(A_{-\infty}) \cap i\mathbb{R} = \emptyset. \tag{5.15}$$

Under assumptions **(H1)**, (5.14), (5.15), implies that the condition (5.13) is satisfied, see [109, Theorem 2.3], as well as [20] and [107] for earlier results under additional assumptions.

We further assume that

(H4) Let T be the number T from (5.13). Then we assume that $R(\omega, A(\cdot))|[T,\infty) \in AAP([T,\infty), \mathcal{L}(X))$ and $R(\omega, A(\cdot))|(-\infty,-T] \in AAP((-\infty,-T], \mathcal{L}(X))$.

We will work on the space

$$AAP^{\pm} = AAP^{\pm}(\mathbb{R}, X) := \{f \in C_b(\mathbb{R}, X) : f|\mathbb{R}_{\pm} \in AAP(\mathbb{R}_{\pm}, X)\}.$$

of functions being asymptotically almost periodic on $\mathbb{R}_-$ and $\mathbb{R}_+$, separately. This space is endowed with the sup–norm. The following description of this space turns out to be crucial for our work.

Lemma 5.2.1. *Let* (2.7) *and* **(H4)** *hold. We then have* $AAP^{\pm} = \{f \in C_b(\mathbb{R}, X) :\ f|(-\infty,-a] \in AAP((-\infty,-a], X), f|[a,\infty) \in AAP([a,\infty), X)\} =: F^a$ *for each* $a \geq 0$.

Proof. Let $a \geq 0$ and $f \in C_b(\mathbb{R}, X)$ such that

$$f^+ := f|[a,\infty) = g^+ + h^+ \in C_0([a,\infty), X) \oplus AP([a,\infty), X);$$

$$f^- := f|(-\infty,-a] = g^- + h^- \in C_0((-\infty,-a], X) \oplus AP((-\infty,-a], X).$$

It is clear that h^+ and h^- can be extended to functions in $AP(\mathbb{R}_+, X)$ and $AP(\mathbb{R}_-, X)$ respectively. The functions $\tilde{g}^{\pm} := f|\mathbb{R}_{\pm} - h^{\pm}$ then belong to $C_0(\mathbb{R}_{\pm}, X)$, i.e, $f|\mathbb{R}_{\pm} = \tilde{g}^{\pm} + h^{\pm} \in AAP(\mathbb{R}_{\pm}, X)$. So we have shown the inclusion $F^a \subset AAP^{\pm}$. The other inclusion is clear. □

As in the previous sections we define the multiplication operator $A(\cdot)$ on $AAP^{\pm}(\mathbb{R}, X)$ by

$$(A(\cdot)v)(t) := A(t)v(t), \qquad t \in \mathbb{R},$$
$$D(A(\cdot)) := \{v \in AAP^{\pm}(\mathbb{R}, X) : f(t) \in D(A(t))\ \forall\, t \in \mathbb{R},\ A(\cdot)v \in AAP^{\pm}\}.$$

Assumption (H4) shows that function $R(\lambda, A(\cdot))f$ belongs to $AAP^{\pm}$ for every $f \in AAP^{\pm}$ and $\lambda \in \omega + \Sigma_\theta \cup \{0\}$, and thus the operator $A(\cdot)$ is sectorial in $AAP^{\pm}$ with the resolvent $R(\lambda, A(\cdot))$. So we can define the extrapolation spaces

$$AAP^{\pm}_{\alpha-1} = AAP^{\pm}_{\alpha-1}(\mathbb{R}) := (AAP^{\pm}(\mathbb{R}, X))^{A(\cdot)}_{\alpha-1} \qquad \text{for } \alpha \in [0,1],$$

which are characterized in the following proposition.

Proposition 5.2.2. *Let* (2.7) *and* **(H4)** *hold, and let* $\alpha \in [0,1]$. *Then we have*

$$AAP^{\pm}_{\alpha-1} \cong \{f \in E_{\alpha-1}(\mathbb{R}) : f|[T,\infty) \in AAP_{\alpha-1}([T,\infty)), \\ f|(-\infty,-T] \in AAP_{\alpha-1}((-\infty,-T])\}.$$

Assume that, in addition, $X^t_{\alpha-1} \cong X_{\alpha-1}$ *with uniformly equivalent norms for some Banach space* $X_{\alpha-1}$ *and some* $\alpha \in [0,1]$. *Then we have*

$$AAP^{\pm}_{\alpha-1} \cong \{f \in C_b(\mathbb{R}, X_{\alpha-1}) : f|[T,\infty) \in AAP([T,\infty), X_{\alpha-1}), \\ f|(-\infty,-T] \in AAP((-\infty,-T], X_{\alpha-1})\}.$$

Proof. Due to Lemma 5.2.1 the space $AAP^{\pm}_{-1}$ is embedded into $E_{-1}(\mathbb{R})$. Let $f \in AAP^{\pm}_{\alpha-1}$. Then there are $f_n \in AAP^{\pm}$ converging to f in $E_{\alpha-1}$. The restrictions of f_n to $(-\infty,-T]$ and to $[T,+\infty)$ converge to the corresponding restrictions of f in $E_{\alpha-1}((-\infty,-T])$ and $E_{\alpha-1}([T,+\infty))$, respectively. Therefore the restrictions of f belong to $AAP_{\alpha-1}((-\infty,-T])$ and to $AAP_{\alpha-1}([T,+\infty))$, respectively, which shows the inclusion '$\subset$'. Let f belong to the space on the right side in the first assertion. The functions $f_n = nR(n, A_{\alpha-1}(\cdot))f$ then belong to $BC(\mathbb{R}, X)$ for $n \geq \omega$, and their restrictions belong to $AAP((-\infty,-T], X)$ and to $AAP([T,+\infty), X)$ (since $R(n, A_{\alpha-1}(\cdot))$ is the resolvent of the respective multiplication operator $A_{\alpha-1}(\cdot)$). Lemma 5.2.1 thus yields $f_n \in AAP^{\pm}$. Since $f_n \to f$ in $E_{\alpha-1}$ as $n \to \infty$, the first assertion holds. The second assertion now follows from the results of the previous section. □

As in [87], we define the operator $G_{\alpha-1}$ on $AAP^{\pm}_{\alpha-1}(\mathbb{R}, X)$ in the following way. A function $u \in AAP^{\pm}(\mathbb{R}, X)$ belongs to $D(G_{\alpha-1})$ and $G_{\alpha-1}u = f$ if there is a function $f \in AAP^{\pm}_{\alpha-1}$ such that (5.3) holds; i.e.,

$$u(t) = U(t,s)u(s) + \int_s^t U_{\alpha-1}(t,\tau)f(\tau)\,d\tau$$

for all $t, s \in \mathbb{R}$ with $t \geq s$. In particular, G_0 is defined on $AAP^{\pm}(\mathbb{R}, X)$ by (5.3), replacing $U_{\alpha-1}$ by U.

To study the operator $G_{\alpha-1}$, we introduce the stable and unstable subspaces of $U_{\alpha-1}(\cdot,\cdot)$.

Definition 5.2.3. *Let* $t_0 \in \mathbb{R}$. *We define the* stable space at t_0 *by*

$$X_s(t_0) := \{x \in X^{t_0}_{\alpha-1} : \lim_{t\to+\infty} \|U_{\alpha-1}(t,t_0)x\| = 0\},$$

and the unstable space at t_0 *by*

$$X_u(t_0) := \{x \in X : \exists \text{ a mild solution } u \in C_0((-\infty,t_0],X) \text{ of } (5.10) \text{ with } f = 0\}.$$

Observe that the function u in the definition of $X_u(t_0)$ satisfies $u(t) = U(t,s)u(s)$ for $s \leq t \leq t_0$ and $u(t_0) = x$, so that $X_u(t_0) \subset D(A(t_0))$. The following result was shown in [87, Lemma 3.2].

Lemma 5.2.4. *Assume that the assumptions* (2.7), (2.8), *and* (5.13) *are satisfied and that* $1-\mu < \alpha \leq 1$. *Then the following assertions hold.*

(a) $X_s(t_0) = P_{\alpha-1}(t_0)X^{t_0}_{\alpha-1}$ *for* $t_0 \geq T$;

(b) $X_u(t_0) = Q(t_0)X$ *for* $t_0 \leq -T$;

(c) $U_{\alpha-1}(t,t_0)X_s(t_0) \subseteq X_s(t)$ *for* $t \geq t_0$ *in* $\mathbb{R}$;

(d) $U(t,t_0)X_u(t_0) = X_u(t)$ *for* $t \geq t_0$ *in* $\mathbb{R}$;

(e) $X_s(t_0)$ *is closed in* $X^{t_0}_{\alpha-1}$ *for* $t_0 \in \mathbb{R}$.

Finally, for technical purposes we introduce the space

$$F^T := \{f : C_b((-\infty,T],X) : f|(-\infty,-T] \in AAP((-\infty,-T],X)\}$$

and endow it with the sup norm. The corresponding extrapolation spaces $F^T_{\alpha-1}$ for $A(\cdot)$ are defined as above for $\alpha \in [0,1]$.

The restrictions $G^+_{\alpha-1}$ and $G^-_{\alpha-1}$ of $G_{\alpha-1}$ to the halflines $[T,+\infty)$ and $(-\infty,T]$ are given in a similar way: A function $u \in AAP([T,+\infty),X)$ (resp., $u \in F^T$) belongs to $D(G^+_{\alpha-1})$ (resp., $D(G^-_{\alpha-1})$) if there is a function $f \in AAP_{\alpha-1}([T,+\infty))$ (resp., $f \in F^T_{\alpha-1}$) such that

$$u(t) = U(t,s)u(s) + \int_s^t U_{\alpha-1}(t,\sigma)f(\sigma)d\sigma$$

holds for all $t \geq s \geq T$ (resp., for all $s \leq t \leq T$). Then we set $G^+_{\alpha-1}u = f$ and $G^-_{\alpha-1}u = f$, respectively. The operators $G_{\alpha-1}$ and $G^\pm_{\alpha-1}$ are single valued and closed due to Remarks 2.5 and 3.12 of [87]. As in [50], [51] and [87], we obtain right inverses $R^+_{\alpha-1}$ and $R^-_{\alpha-1}$ on $AAP([T,+\infty),X)$ and on F^T for $G^+_{\alpha-1}$ and $G^-_{\alpha-1}$, respectively, by setting

$$(R^+_{\alpha-1}h)(t) = -\int_t^\infty \widetilde{U}_{\alpha-1}(t,s)Q_{\alpha-1}(s)h(s)\,ds + \int_T^t U_{\alpha-1}(t,s)P_{\alpha-1}(s)h(s)\,ds$$

for $h \in AAP_{\alpha-1}([T,+\infty),X)$ and $t \geq T$, and

$$(R^-_{\alpha-1}h)(t) = \begin{cases} \displaystyle\int_{-\infty}^T \Gamma_{\alpha-1}(t,s)h(s)\,ds, & t \leq -T, \\ \displaystyle\int_{-\infty}^{-T} U_{\alpha-1}(t,s)P_{\alpha-1}(s)h(s)\,ds + \int_{-T}^t U_{\alpha-1}(t,s)h(s)\,ds, & |t| \leq T, \end{cases}$$

for $h \in F^T_{\alpha-1}$.

Proposition 5.2.5. *Assume that the assumptions* (2.7), (2.8), (5.13) *and* **(H4)** *are satisfied and that* $1-\mu < \alpha \leq 1$. *Then the following assertions hold.*

(a) *The operator* $R^+_{\alpha-1} : AAP_{\alpha-1}([T,+\infty)) \to AAP([T,+\infty),X)$ *is bounded and* $G^+_{\alpha-1}R^+_{\alpha-1}h = h$ *for each* $h \in AAP_{\alpha-1}([T,+\infty))$.

(b) *The operator* $R^-_{\alpha-1} : F^T_{\alpha-1} \to F^T$ *is bounded and* $G^-_{\alpha-1}R^-_{\alpha-1}h = h$ *for each* $h \in F^T_{\alpha-1}$.

(c) *We have* $R^\pm_{\alpha-1}h(T) \in X^T_\varepsilon$ *for all* $0 \leq \varepsilon < \alpha$.

Proof. Let $h \in AAP_{\alpha-1}([T,+\infty))$. In Proposition 3.3 and Remark 3.12 of [87] it was shown that $R^+_{\alpha-1}h$ is a mild solution of the equation (5.7) for the inhomogeneity h and the initial value $x := -\int_T^\infty \tilde{U}(T,s)Q_{\alpha-1}(s)h(s)\,ds$ at $t_0 = T$. Since (5.8) holds for h and x, Theorem 5.1.5 gives the asymptotic almost periodicity of $R^+_{\alpha-1}h$. So the operator $R^+_{\alpha-1}$ maps $AAP_{\alpha-1}([T,+\infty))$ into $AAP([T,+\infty),X)$, and its boundedness follows from Proposition 2.4.3 d), e) as in the proof of [87, Proposition 3.3]. Assertion (a) is thus established. To show (b), let $h \in F^T_{\alpha-1}((-\infty,T])$. Proposition 3.3 and Remark 3.12 of [87] also yield that $R^-_{\alpha-1}h$ is a mild solution of the equation (5.10) with $t_0 = T$ and the inhomogeneity h. It is clear that $h|(-\infty,-T]$ satisfies (5.12) for $x := \int_{-\infty}^{-T} U_{\alpha-1}(-T,s)P_{\alpha-1}(s)f(s)\,ds$. Theorem 5.1.5 then implies that $R^-_{\alpha-1}h|(-\infty,-T] \in AAP((-\infty,-T],X)$ and consequently $R^-_{\alpha-1}$ maps $F^T_{\alpha-1}$ into F^T. The boundedness of $R^-_{\alpha-1}$ follows again from Proposition 2.4.3 d), e). The last assertion is a consequence of Propositions 2.4.2 a) and 2.4.3 d), e). □

We can now describe the range and the kernel of $G_{\alpha-1}$.

Proposition 5.2.6. *Assume that* (2.7), (2.8), (5.13) *and* **(H4)** *are satisfied and that* $1-\mu < \alpha \le 1$. *For* $f \in AAP^{\pm}_{\alpha-1}$ *we set* $f^+ = f|[T,+\infty)$ *and* $f^- = f|(-\infty,T]$. *Then the following assertions hold for* $G_{\alpha-1}$.

(a) $N(G^+_{\alpha-1}) = \{u \in C_0([T,+\infty),X)) : u(t) = U(t,T)x\ (\forall\, t \ge T),\ x \in P(T)\hat{X}^T\}$;

(b) $N(G^-_{\alpha-1}) = \{u \in C_0((-\infty,T],X) : u(t) = U(t,s)u(s)\ (\forall\, s \le t \le T),\ u(T) \in X_u(T)\}$;

(c) $N(G_{\alpha-1}) = \{u \in C_0(\mathbb{R},X) : u(t) = U(t,s)u(s)\ (\forall\, t \ge s),\ u(T) \in P(T)X \cap X_u(T)\}$;

(d) $R(G_{\alpha-1}) = \{f \in AAP^{\pm}_{\alpha-1} : R^+_{\alpha-1}f^+(T) - R^-_{\alpha-1}f^-(T) \in P(T)X + X_u(T)\}$, *where for* $f \in R(G_{\alpha-1})$ *a function* $u \in D(G_{\alpha-1})$ *with* $G_{\alpha-1}u = f$ *is given by* (5.16) *below;*

(e) $\overline{R(G_{\alpha-1})} = \{f \in AAP^{\pm}_{\alpha-1} : R^+_{\alpha-1}f^+(T) - R^-_{\alpha-1}f^-(T) \in \overline{P(T)X + X_u(T)}\}$, *where the closure on the left (right) side is taken in* $AAP^{\pm}_{\alpha-1}$ *(in* X*).*

Proof. The assertions (a), (b) and (c) follow from Proposition 3.5 and Remark 3.12 of [87]. We note that $P(T)X \cap X_u(T) = P(T)\hat{X}^T \cap X_u(T)$ since $X_u(T) \subseteq D(A(T))$. To show (d), let $G_{\alpha-1}u = f \in AAP^{\pm}_{\alpha-1}(\mathbb{R})$ for some $u \in D(G_{\alpha-1})$. Then the functions $f^{\pm}$ belong to $R(G^+_{\alpha-1})$ and to $R(G^-_{\alpha-1})$, respectively, because of Proposition 5.2.2 and (5.3). Proposition 5.2.5 shows that the functions

$$v_+ = u|[T,+\infty) - R^+_{\alpha-1}f^+ \qquad \text{and} \qquad v_- = u|(-\infty,T] - R^-_{\alpha-1}f^-$$

are contained in the kernels of $G^+_{\alpha-1}$ and of $G^-_{\alpha-1}$, respectively. So we obtain

$$(R^+_{\alpha-1}f^+)(T) - (R^-_{\alpha-1}f^-)(T) = v_-(T) - v_+(T) \in X_u(T) + P(T)X$$

by (a) and (b). Conversely, let $f \in AAP^{\pm}_{\alpha-1}(\mathbb{R})$ with

$$(R^{+}_{\alpha-1}f^{+})(T) - (R^{-}_{\alpha-1}f^{-})(T) = y_s + y_u \in P(T)X + X_u(T).$$

Set $x_0 := (R^{+}_{\alpha-1}f^{+})(T) - y_s = y_u + (R^{-}_{\alpha-1}f^{-})(T)$ and

$$u(t) := \begin{cases} u_+(t) := -U(t,T)y_s + (R^{+}_{\alpha-1}f^{+})(t), & t \geq T, \\ u_-(t) := \tilde{v}(t) + (R^{-}_{\alpha-1}f^{-})(t), & t \leq T, \end{cases} \tag{5.16}$$

where $\tilde{v} \in N(G^{-}_{\alpha-1})$ such that $\tilde{v}(T) = y_u$. Observe that $u_+(T) = u_-(T)$. From Proposition 5.2.5(c) we deduce $y_s \in P(T)\hat{X}^T$, so that $U(\cdot,T)y_s \in C_0([T,\infty),X)$. Proposition 5.2.5 shows that $R^{+}_{\alpha-1}f^{+} \in AAP([T,\infty),X)$, and hence $u|[T,\infty) \in AAP([T,\infty),X)$. We also know from assertion (c) that $\tilde{v} \in C_0((-\infty,T],X)$ and from Proposition 5.2.5 that $R^{-}_{\alpha-1}f^{-} \in F^T$. Using also Lemma 5.2.1, we deduce that u belongs to $AAP^{\pm}(\mathbb{R},X)$. Finally, one can check as in the proof of Proposition 3.5 of [87] that $G_{\alpha-1}u = f$. The last assertion can be shown exactly as Proposition 3.5(e) of [87]. □

Using the above results, we are able to describe the Fredholm properties of the operator $G_{\alpha-1}$ in terms of properties of the subspaces $X_s(T)$ and $X_u(T)$. The proofs are similar to ones of Theorems 3.6 and 3.10 and Proposition 3.8 of [87] and therefore omitted. Recall that subspaces V and W of a Banach space E are called a *semi-Fredholm couple* if $V+W$ is closed and if at least one of the dimensions $\dim(V\cap W)$ and $\operatorname{codim}(V+W)$ is finite. The *index* of (V,W) is defined by $\operatorname{ind}(V,W) := \dim(V\cap W) - \operatorname{codim}(V+W)$. If the index is finite, then (V,W) is a *Fredholm couple.*

Theorem 5.2.7. *Assume that* (2.7), (2.8), *and* (5.13) *are satisfied and that* $1-\mu < \alpha \leq 1$. *Then the following assertions hold for* $G_{\alpha-1}$ *defined on* $AAP^{\pm}_{\alpha-1}(\mathbb{R})$.

(a) $R(G_{\alpha-1})$ *is closed in* $AAP^{\pm}_{\alpha-1}$ *if and only if* $P(T)X + X_u(T)$ *is closed in* X.

(b) If $G_{\alpha-1}$ *is injective, then* $P(T)X \cap X_u(T) = \{0\}$. *The converse is true if* $U(T,-T)_{|Q(-T)(X)}$ *is injective, in addition.*

(c) If $G_{\alpha-1}$ *is invertible, then* $P(T)X \oplus X_u(T) = X$. *The converse is true if* $U(T,-T)_{|Q(-T)(X)}$ *is injective in addition.*

(d) $\dim N(G_{\alpha-1}) = \dim(P(T)X \cap X_u(T)) + \dim N(U(T,-T)_{|Q(-T)(X)})$.
We have $\operatorname{codim}(P(T)X + X_u(T)) = \operatorname{codim} R(G_{\alpha-1})$, *if* $R(G_{\alpha-1})$ *is closed in* $AAP^{\pm}_{\alpha-1}$. *In particular,* $G_{\alpha-1}$ *is surjective if and only if* $P(T)X + X_u(T) = X$.

(e) If $G_{\alpha-1}$ *is a semi-Fredholm operator, then* $(P(T)X, X_u(T))$ *is a semi-Fredholm couple, and* $\operatorname{ind}(P(T)X, X_u(T)) \leq \operatorname{ind} G_{\alpha-1}$. *If in addition the kernel of* $U(T,-T)_{|Q(-T)(X)}$ *is finite dimensional, then*

$$\operatorname{ind}(P(T)X, X_u(T)) = \operatorname{ind} G_{\alpha-1} - \dim N(U(T,-T)_{|Q(-T)(X)}). \tag{5.17}$$

Conversely, if $(P(T)X, X_u(T))$ is a semi-Fredholm couple and the kernel of $U(T,-T)_{|Q(-T)(X)}$ is finite dimensional, then $G_{\alpha-1}$ is a semi-Fredholm operator and (5.17) holds.

Proposition 5.2.8. *Assume that (2.7), (2.8), and (5.13) hold and that $1-\mu < \alpha \le 1$. Then the closure of $R(G_{\alpha-1})$ is equal to the space*

$$\mathcal{F} := \{f \in AAP^{\pm}_{\alpha-1} : \int_{\mathbb{R}} \langle f(s), v(s)\rangle_{X^s_{\alpha-1}}\, ds = 0 \quad \text{for all } v \in \mathcal{V}\},$$

where $\mathcal{V}$ is the space of those $v \in L^1(\mathbb{R}, X^*)$ such that $v(s) = U_{\alpha-1}(t,s)^* v(t)$ for all $t \ge s$ in $\mathbb{R}$.

In the following Fredholm alternative, we restrict ourselves to the asymptotically hyperbolic case. The projections $Q_{\pm\infty}$ have finite rank if, for instance, the domains $D(A_{\pm\infty})$ are compactly embedded in X.

Theorem 5.2.9. *Assume that (2.7), (2.8), (5.14) and (5.15) are true, that $\dim Q_{\pm\infty}X < \infty$, and that $1-\mu < \alpha \le 1$. Let $f \in AAP^{\pm}_{\alpha-1}$. Then there is a mild solution $u \in AAP^{\pm}(\mathbb{R}, X)$ of (5.2) if and only if*

$$\int_{\mathbb{R}} \langle f(s), w(s)\rangle_{X^s_{\alpha-1}}\, ds = 0$$

for each $w \in L^1(\mathbb{R}, X^*)$ with $w(s) = U_{\alpha-1}(t,s)^* w(t)$ for all $t \ge s$ in $\mathbb{R}$. The mild solutions u are given by

$$\begin{aligned} u(t) &= v(t) - U(t,T)y_s + (R^{+}_{\alpha-1}f)(t), && t \ge T,\\ u(t) &= v(t) + \tilde{v}(t) + (R^{-}_{\alpha-1}f)(t), && t \le T, \end{aligned}$$

where $R^{\pm}_{\alpha-1}$ was defined before Proposition 5.2.5, $(R^{+}_{\alpha-1}f)(T) - (R^{-}_{\alpha-1}f)(T) = y_s + y_u \in P(T)X + X_u(T)$, $\tilde{v} \in C_0((-\infty,T],X)$ with $\tilde{v}(T) = y_u$ and $\tilde{v}(t) = U(t,s)\tilde{v}(s)$ for all $T \ge t \ge s$, and $v \in C_0(\mathbb{R},X)$ with $v(t) = U(t,s)v(s)$ for all $t \ge s$.

5.3 Almost periodicity of boundary evolution equations

In this section we study the non–autonomous forward (resp. backward) parabolic boundary evolution equation

$$\begin{cases} u'(t) &= A_m(t)u(t) + g(t), \qquad t \ge t_0 \quad (\text{resp. } t \le t_0),\\ B(t)u(t) &= h(t), \qquad t \ge t_0 \quad (\text{resp. } t \le t_0),\\ u(t_0) &= u_0, \end{cases}$$

and their variant on the line

$$\begin{cases} u'(t) &= A_m(t)u(t) + g(t), \qquad t \in \mathbb{R},\\ B(t)u(t) &= h(t), \qquad t \in \mathbb{R}. \end{cases} \tag{5.18}$$

Here $t_0 \in \mathbb{R}$, $u_0 \in X$, and the inhomogeneities g and h take values in Banach spaces X and Y, respectively.

We assume that the following conditions hold.

(B1) There are Banach spaces $Z \hookrightarrow X$ and Y such that the operators $A_m(t) \in \mathcal{L}(Z,X)$ and $B(t) \in \mathcal{L}(Z,Y)$ are uniformly bounded for $t \in \mathbb{R}$ and that $B(t) : Z \to Y$ is surjective for each $t \in \mathbb{R}$.

(B2) The operators $A(t)u := A_m(t)u$ with domains $D(A(t)) := \{u \in Z : B(t)u = 0\}$, $t \in \mathbb{R}$, satisfy (2.7) and (2.8) with constants $\omega, \theta, K, L, \mu, \nu$. Moreover, the graph norm of $A(t)$ and the norm of Z are equivalent with constants being uniform in $t \in \mathbb{R}$.

In the typical applications $A_m(t)$ is a differential operator with 'maximal' domain not containing boundary conditions and $B(t)$ are boundary operators. Under the hypotheses **(B1)** and **(B2)**, there is an evolution family $(U(t,s))_{t\geq s}$ solving the problem with homogeneous conditions $g = h = 0$. Moreover, by [57, Lemma 1.2] there exists the *Dirichlet map* $D(t)$ for $\omega - A_m(t)$; i.e., $v = D(t)y$ is the unique solution of the abstract boundary value problem

$$(\omega - A_m(t))v = 0, \qquad B(t)v = y,$$

for each $y \in Y$. (In [57] the density of Z in X was assumed, but this does not play a role in the cited Lemma 1.2.) Let $x \in X$ and $y \in Y$ be given. The problem

$$(\omega - A_m(t))v = x, \qquad B(t)v = y,$$

has the solution $v = R(\omega, A(t))x + D(t)y$. This solution is unique in Z since $\omega - A_m(t)$ is injective on $D(A(t)) = N(B(t))$. We further assume that

(B3) there is a $\beta \in (1-\mu, 1]$ such that $Z \hookrightarrow X_\beta^t$ for $t \in \mathbb{R}$ with uniformly bounded embedding constants and $\sup_{t\in\mathbb{R}} \|D(t)\|_{\mathcal{L}(Y,Z)} < \infty$.

Lemma 5.3.1. *Assume that assumptions* **(B1)**, **(B2)** *and* **(B3)** *without* (2.8) *hold. For a closed unbounded interval J, let $A_m(\cdot) \in AP(J, \mathcal{L}(Z,X))$ and $B(\cdot) \in AP(J, \mathcal{L}(Z,Y))$. Then we have*
(a) $D(\cdot) \in AP(J, \mathcal{L}(Y,Z))$,
(b) $R(\omega, A(\cdot)) \in AP(J, \mathcal{L}(X,Z))$,
(c) $(\omega - A_{-1}(\cdot))D(\cdot)h \in AP_{\alpha-1}(J)$ for every $h \in AP(J,Y)$ and $\alpha \in (1-\mu, \beta)$.
The same results hold if one replaces throughout AP by AAP (if $J \neq \mathbb{R}$) or by $AAP^\pm$ (if $J = \mathbb{R}$).

Proof. (a) Let $y \in Y$ and $t, t+\tau \in J$. By the definition of $D(t)$, we have

$$\begin{aligned}(\omega - A_m(t))(D(t+\tau)y - D(t)y) &= (A_m(t+\tau) - A_m(t))D(t+\tau)y =: \varphi(t),\\ B(t)(D(t+\tau)y - D(t)y) &= -(B(t+\tau) - B(t))D(t+\tau)y =: \psi(t),\end{aligned}$$

and thus

$$D(t+\tau)y - D(t)y = R(\omega, A(t))\varphi(t) + D(t)\psi(t).$$

The assumptions now imply that

$$\begin{aligned}\|D(t+\tau)y - D(t)y\|_Z &\leq c\,(\|\varphi(t)\|_X + \|\psi(t)\|_Y)\\ &\leq c\,(\|A_m(t+\tau) - A_m(t)\|_{\mathcal{L}(Z,X)} + \|B(t+\tau) - B(t)\|_{\mathcal{L}(Z,Y)})\,\|y\|_Y.\end{aligned}$$

So the almost periodicity of $D(\cdot)$ follows from that of $A_m(\cdot)$ and $B(\cdot)$.

(b) For $x \in X$ and $t, t+\tau \in J$, set $y = R(\omega, A(t+\tau))x - R(\omega, A(t))x \in Z$. Then we obtain

$$\begin{aligned}(\omega - A_m(t))y &= (A_m(t+\tau) - A_m(t))R(\omega, A(t+\tau))x =: \varphi_1(t),\\ B(t)y &= (B(t) - B(t+\tau))R(\omega, A(t+\tau))x =: \psi_1(t).\end{aligned}$$

Hence $y = R(\omega, A(t))\varphi_1(t) + D(t)\psi_1(t)$, and assertion (b) can now be shown as in (a).

(c) Due to (a) and (b), the functions $D(\cdot)h$ and $f_n := nR(n, A(\cdot))D(\cdot)h$ are almost periodic in Z, and hence in X, for $n > \omega$. Then $A(\cdot)f_n = (n^2R(n, A(\cdot)) - n)D(\cdot)h$ belongs to $AP(J, X)$. Assumptions (2.7) and (A3) imply that f_n is uniformly bounded in the norm of E_β. Since $f_n \to D(\cdot)h$ in $C_b(J, X)$, we conclude by interpolation that $f_n \to D(\cdot)h$ in E_α. As a consequence, $(\omega - A(\cdot))f_n \to (\omega - A_{\alpha-1}(\cdot))D(\cdot)h$ in $E_{\alpha-1}$, whence (c) follows.

Similarly one establishes the assertions concerning AAP and $AAP^\pm$. □

In order to apply the results from the previous sections to the boundary forward (resp. backward) evolution equation (5.18), we write it as the inhomogeneous Cauchy problem

$$\begin{aligned}u'(t) &= A_{-1}(t)u(t) + f(t), \qquad t \geq t_0 \quad (\text{resp. } t \leq t_0),\\ u(t_0) &= u_0,\end{aligned} \tag{5.19}$$

setting $f := g + (\omega - A_{-1}(\cdot))D(\cdot)h$. We also consider the evolution equation

$$u'(t) = A_{-1}(t)u(t) + f(t), \qquad t \in \mathbb{R}. \tag{5.20}$$

In the following we will have $f \in E_{\alpha-1}(J)$, where we fix the number $\alpha \in (1-\mu, \beta)$ from Lemma 5.3.1. We note that a function $u \in C^1(J, X)$ with $u(t) \in Z$ satisfies (5.18), resp. (5.18), if and only if it satisfies (5.19), resp. (5.20). These facts can be shown as in Proposition 4.2 of [44]. This motivates the following definition. We call a function $u \in C(J, X)$ a *mild solution* of (5.18) and (5.20) on J if the equation

$$u(t) = U(t,s)u(s) + \int_s^t U_{\alpha-1}(t,\sigma)[g(\sigma) + (\omega - A_{-1}(\sigma))D(\sigma)h(\sigma)]\,d\sigma \tag{5.21}$$

holds for all $t \geq s$ in J. The function u is called a *mild solution* of (5.18) (resp. (5.19) if in addition $u(t_0) = u_0$ and $J = [t_0, \infty)$ (resp. $J = (-\infty, t_0]$).

Theorems 5.1.3, 5.1.5 and 5.1.6 and Lemma 5.3.1 immediately imply three results on the existence of almost periodic mild solutions for (5.18) and (5.18).

Proposition 5.3.2. *Assume that* **(B1–(B3)** *hold, that* $A_m(\cdot) \in AP(\mathbb{R}, \mathcal{L}(Z, X))$ *and* $B(\cdot) \in AP(\mathbb{R}, \mathcal{L}(Z, Y))$*, and that* $U(t,s)$ *has an exponential dichotomy on* $\mathbb{R}$*. Let* $g \in AP(\mathbb{R}, X)$ *and* $h \in AP(\mathbb{R}, Y)$*. Then there is a unique mild solution* $u \in AP(\mathbb{R}, X)$ *of the boundary equation* (5.18) *given by*

$$u(t) = \int_{\mathbb{R}} \Gamma_{\alpha-1}(t,s)[g(s) + (\omega - A_{-1}(s))D(s)h(s)]\,ds, \quad t \in \mathbb{R}.$$

Proposition 5.3.3. *Assume that* **(B1)**–**(B3)** *hold, that* $A_m(\cdot) \in AAP([a,\infty),\mathcal{L}(Z,X))$*, and* $B(\cdot) \in AAP([a,\infty),\mathcal{L}(Z,Y))$*, and that* $U(t,s)$ *has an exponential dichotomy on* $[a,\infty)$*. Let* $t_0 > a$*,* $g \in AAP([a,\infty),X)$*,* $h \in AAP([a,\infty),Y)$*, and* $u_0 \in \overline{D(A(t_0))}$*. Then the mild solution* u *of the equation* (5.18) *belongs to* $AAP([t_0,+\infty),X)$ *if and only if*

$$Q(t_0)u_0 = -\int_{t_0}^{+\infty} \widetilde{U}_{\alpha-1}(t_0,s)Q_{\alpha-1}(s)[g(s) + (\omega - A_{-1}(s))D(s)h(s)]\,ds.$$

In this case u *is given by*

$$\begin{aligned} u(t) = U(t,t_0)P(t_0)u_0 &+ \int_{t_0}^{t} U_{\alpha-1}(t,s)P_{\alpha-1}(s)[g(s) + (\omega - A_{-1}(s))D(s)h(s)]\,ds \\ &- \int_{t}^{\infty} \widetilde{U}_{\alpha-1}(t,s)Q_{\alpha-1}(s)[g(s) + (\omega - A_{-1}(s))D(s)h(s)]\,ds, \quad t \geq t_0. \end{aligned}$$

Proposition 5.3.4. *Let* **(B1)**–**(B3)** *hold,* $A_m(\cdot) \in AAP((-\infty,b],\mathcal{L}(Z,X))$ $B(\cdot) \in AAP((-\infty,b],\mathcal{L}(Z,Y))$*, and assume that* $U(t,s)$ *has an exponential dichotomy on* $(-\infty,b]$*. Let* $t_0 < b$*,* $g \in AAP((-\infty,b],X)$*,* $h \in AAP((-\infty,b],Y)$*, and* $u_0 \in X$*. Then there is a mild solution* $u \in AAP((-\infty,t_0],X)$ *of the equation* (5.18) *if and only if*

$$P(t_0)u_0 = \int_{-\infty}^{t_0} U_{\alpha-1}(t_0,s)P_{\alpha-1}(s)[g(s) + (\omega - A_{-1}(s))D(s)h(s)]\,ds.$$

In this case u *is given by*

$$\begin{aligned} u(t) = \widetilde{U}(t,t_0)Q(t_0)u_0 &- \int_{t}^{t_0} \widetilde{U}_{\alpha-1}(t,s)Q_{\alpha-1}(s)[g(s) + (\omega - A_{-1}(s))D(s)h(s)]\,ds \\ &+ \int_{-\infty}^{t} U_{\alpha-1}(t,s)P_{\alpha-1}(s)[g(s) + (\omega - A_{-1}(s))D(s)h(s)]\,ds, \quad t \leq t_0. \end{aligned}$$

Moreover, Theorem 5.2.9 implies the following Fredholm alternative for the mild solutions of (5.18), where we focus on the asymptotically hyperbolic case.

Theorem 5.3.5. *Assume that assumptions* **(B1)**–**(B3)** *hold and that* $A_m(t) \to A_m(\pm\infty)$ *in* $\mathcal{L}(Z,X)$ *and* $B(t) \to B(\pm\infty)$ *in* $\mathcal{L}(Z,Y)$ *as* $t \to \pm\infty$*. Set* $A_{\pm\infty} := A_m(\pm\infty)|N(B(\pm\infty))$*. We suppose that* $\sigma(A_{\pm\infty}) \cap i\mathbb{R} = \emptyset$ *and that the corresponding unstable projections* $Q_{\pm\infty}X$ *have finite rank. Let* $g \in AAP^{\pm}(\mathbb{R},X)$ *and* $h \in AAP^{\pm}(\mathbb{R},Y)$*. Then there is a mild solution* $u \in AAP^{\pm}(\mathbb{R},X)$ *of* (5.18) *if and only if*

$$\int_{\mathbb{R}} \langle f(s), w(s)\rangle_{X^s_{\alpha-1}}\,ds = 0$$

for $f := g + (\omega - A_{-1}(\cdot))D(\cdot)h$ *and all* $w \in L^1(\mathbb{R},X^*)$ *with* $w(s) = U_{\alpha-1}(t,s)^*w(t)$ *for all* $t \geq s$ *in* $\mathbb{R}$*. The mild solutions* u *are given by*

$$\begin{aligned} u(t) &= v(t) - U(t,T)y_s + (R^+_{\alpha-1}f^+)(t), \qquad t \geq T, \\ u(t) &= v(t) + \tilde{v}(t) + (R^-_{\alpha-1}f^-)(t), \qquad t \leq T, \end{aligned}$$

where $R^{\pm}_{\alpha-1}$ was defined before Proposition 5.2.5, $f^+ = f|[T,+\infty)$, $f^- = f|(-\infty,-T]$, $(R^+_{\alpha-1}f^+)(T) - (R^-_{\alpha-1}f^-)(T) = y_s + y_u \in P(T)X + X_u(T)$, $\tilde{v} \in C_0((-\infty,T],X)$ with $\tilde{v}(T) = y_u$ and $\tilde{v}(t) = U(t,s)\tilde{v}(s)$ for all $T \geq t \geq s$, and $v \in C_0(\mathbb{R},X)$ with $v(t) = U(t,s)v(s)$ for all $t \geq s$.

Proof. Observe that functions converging at $\pm\infty$ belong to $AAP^{\pm}$. So it remains to show that $R(\omega, A(t)) \to R(\omega, A_{\pm\infty})$ in $\mathcal{L}(X)$ as $t \to \pm\infty$. This can be established as Lemma 5.3.1(b). □

We conclude with a pde example. One could treat more general problems, in particular systems, cf. [51], and one could weaken the regularity assumptions; but we prefer to keep the example simple.

Example 5.3.6. *We study the boundary value problem*

$$\begin{aligned} \partial_t u(t,x) &= A(t,x,D)u(t,x) + g(t,x), \quad t \in \mathbb{R},\ x \in \Omega, \\ B(t,x,D)u(t,x) &= h(t,x), \quad t \in \mathbb{R},\ x \in \partial\Omega, \end{aligned} \tag{5.22}$$

on a bounded domain $\Omega \subseteq \mathbb{R}^n$ with boundary $\partial\Omega$ of class C^2, employing the differential expressions

$$\begin{aligned} A(t,x,D) &= \sum\nolimits_{k,l} a_{kl}(t,x)\partial_k\partial_l + \sum\nolimits_k a_k(t,x)\,\partial_k + a_0(t,x), \\ B(t,x,D) &= \sum\nolimits_k b_k(t,x)\,\partial_k + b_0(t,x), \end{aligned}$$

where $B(t)$ is understood in the sense of traces. We require that $a_{kl} = a_{lk}$ and b_k are real–valued, $a_{kl}, a_k, a_0 \in C^{\mu}_b(\mathbb{R}, C(\overline{\Omega}))$, $b_k, b_0 \in C^{\mu}_b(\mathbb{R}, C^1(\partial\Omega))$,

$$\sum_{k,l=1}^{n} a_{kl}(t,x)\,\xi_k\,\xi_l \geq \eta\,|\xi|^2, \qquad \text{and} \qquad \sum_{k=1}^{n} b_k(t,x)\nu_k(x) \geq \beta$$

for constants $\mu \in (1/2,1)$, $\beta,\eta > 0$ and all $\xi \in \mathbb{R}^n$, $k,l = 1,\cdots,n$, $t \in \mathbb{R}$, $x \in \overline{\Omega}$ resp. $x \in \partial\Omega$. (C^{μ}_b is the space of bounded, globally Hölder continuous functions.) Let $p \in (1,\infty)$. We set $X = L_p(\Omega)$, $Z = W^2_p(\Omega)$, $Y = W^{1-1/p}_p(\Omega)$ (a Slobodeckij space), $A_m(t)u = A(t,\cdot,D)u$ and $B(t)u = B(t,\cdot,D)u$ for $u \in Z$ (in the sense of traces), and $A(t) = A_m(t)|N(B(t))$. The operators $A(t)$, $t \in \mathbb{R}$, satisfy (2.7) and (2.8), see [1], [5], [83], or [109, Example 2.9]. Thus $A(\cdot)$ generates an evolution family $U(\cdot,\cdot)$ on X. It is known that the graph norm of $A(t)$ is uniformly equivalent to the norm of Z, that $B(t) : Z \to Y$ is surjective, that $X^t_\alpha = W^{2\alpha}_p(\Omega)$ with uniformly equivalent norms for $\alpha \in (1-\mu, 1/2)$, and that the Dirichlet map $D(t) : Z \to Y$ is uniformly bounded for $t \in \mathbb{R}$, see e.g. [5, Example IV.2.6.3].

Further let $g \in AAP^{\pm}(\mathbb{R},X)$ and $h \in AAP^{\pm}(\mathbb{R},Y)$. We define mild solutions of (5.22) again by (5.21). We further assume that

$$a_\alpha(t,\cdot) \to a_\alpha(\pm\infty,\cdot) \quad \text{in } C(\overline{\Omega}) \qquad \text{and} \qquad b_j(t,\cdot) \to b_j(\pm\infty,\cdot) \quad \text{in } C^1(\partial\Omega)$$

as $t \to \pm\infty$, where $\alpha = (k,l)$ or $\alpha = j$ for $k,l = 1,\cdots,n$ and $j = 0,\cdots,n$. As a result, $A_m(\cdot) \in AAP^{\pm}(\mathbb{R},\mathcal{L}(Z,X))$ and $B(\cdot) \in AAP^{\pm}(\mathbb{R},\mathcal{L}(Z,Y))$. We define *the sectorial operators* $A_{\pm\infty}$ *in the same way as* $A(t)$. *As in [51, Example 5.1] one can check that* (5.14) *holds.*

Finally we assume that $i\mathbb{R} \subset \rho(A_{\pm\infty})$. *Then the Fredholm alternative stated in Theorem 5.3.5 holds for mild solutions of* (5.22) *on* $X = L^p(\Omega)$ *for* $g \in AAP^{\pm}(\mathbb{R}, X)$ *and* $h \in AAP^{\pm}(\mathbb{R}, Y)$.

Chapter 6

Admissibility and observability of observation operators for semilinear problems

In this chapter we are concerned with abstract semilinear evolution equation with output equation

$$\begin{cases} u'(t) &= Au(t) + F(u(t)), \quad u(0) = x,\ t \geq 0, \quad x \in X, \\ y(t) &= Cu(t), \end{cases} \tag{6.1}$$

where A is assumed to be the infinitesimal generator of a C_0-semigroup $(T(t))_{t\geq 0}$ in a Banach space X and F is a nonlinear continuous function on X. Further, it is assumed that C, the observation operator, is a linear bounded operator from $D(A)$, the domain of A, to another Banach space Y.

It is well-known that global Lipschitz continuity of the nonlinearity F implies that the problem (6.1) admits a unique mild solution given by the variation of parameters formula

$$u(t,x) = T(t)x + \int_0^t T(t-\sigma)F(u(\sigma,x))d\sigma, \;\; t \geq 0,\ x \in X.$$

We define a nonlinear semigroup $(S(t))_{t\geq 0}$ associated to the solution of (6.1) by

$$S(t)x = u(t,x).$$

Hence the output function is formally given by

$$y(t) = CS(t)x.$$

The output function is only well-defined if C is bounded, i.e. if the operator C can be extended to a linear bounded operator from X to Y. However, in case of unbounded observation operators, even if $x \in D(A)$ it might happen that $u(t,x)$ is not in $D(A)$, so that $Cu(t,x)$ is not defined. We call the operator

C admissible for the nonlinear semigroup $(S(t))_{t\geq 0}$ if the output function y is well-defined as locally square integrable function with values in Y.

The problem of admissibility has been studied by many authors, e.g., [38, 39, 66, 118], but in their works they are interested in linear systems only. In this chapter we extend the definition of admissibility of observation operator C for semilinear systems and we develop conditions guaranteeing that the set of admissible observation operators for the semilinear problem coincide with the set of admissible observation operators for the linearized system.

In applications, it is often required that the system is exactly observable, that is, the initial state $x \in X$ can be recovered from the output function y by a bounded operator. This problem is well studied for linear systems, see e.g. [67, 68, 98, 103, 122]. Last results of this chapter is to generalize the concept of exact observability to semilinear problems and we develop conditions guaranteeing that the semilinear system is exact observable if and only if the linearized system has this property.

6.1 Nonlinear semigroups

Throughout this paper, we suppose that

(L) $F : X \to X$ is globally Lipschitz continuous, i.e,

$$||F(x) - F(y)|| \leq L||x - y||, \text{ for all } x, y \in X,$$

where L is a positive constant and $F(0) = 0$.

Under the assumption **(L)**, Equation (6.1) admits an unique mild solution $u(\cdot, x)$ given by the variation of parameters formula

$$u(t;x) = T(t)x + \int_0^t T(t-\sigma)F(u(\sigma;x))d\sigma, \quad t \geq 0 \tag{6.2}$$

$$y(t) = Cu(t;x). \tag{6.3}$$

Let $(S(t))_{t\geq 0}$ be the family of nonlinear operators defined in X by

$$S(t)x = u(t;x), \quad \text{for } t \geq 0, \ x \in X. \tag{6.4}$$

The operators $S(t)$ map X into itself and they satisfy the two properties below:

(P1) $S(0)x = x$, $S(t+s)x = S(t)S(s)x$ for $s, t \geq 0$ and $x \in X$.

(P2) For each $x \in X$, the X-valued function $S(\cdot)x$ is continuous over $[0, +\infty)$.

The first property is obtained through the uniqueness of mild solutions, and the second property follows from the fact that the solution $u(t;x)$ to (6.2) is continuous.

By a nonlinear semigroup on X we mean a family $(S(t))_{t\geq 0}$ of nonlinear operators on X with the above mentioned properties **(P1)** and **(P2)**. If in particular a semigroup on X provides mild solutions of (6.1) in the sense of

(6.4), we call it the nonlinear semigroup on X associated with the semilinear evolution equation (6.1) and we have

$$S(t)x = T(t)x + \int_0^t T(t-\sigma)F(S(\sigma)x)d\sigma, \quad t \geq 0, \ x \in X. \tag{6.5}$$

Since $(T(t))_{t\geq 0}$ is a C_0-semigroup, there exists the constants $M \geq 1, \ \omega \in \mathbb{R}$, such that

$$||T(t)|| \leq Me^{\omega t} \text{ for all } t \geq 0.$$

Moreover, we have the following property

Proposition 6.1.1. *For every x, $y \in X$ and $t \geq 0$, we have*

$$||S(t)x|| \leq Me^{(\omega+ML)t}||x||, \tag{6.6}$$

$$||S(t)x - S(t)y|| \leq Me^{(\omega+ML)t}||x-y||. \tag{6.7}$$

Proof. Let $x, \ y \in X$. Since F is globally Lipschitz continuous, it follows that for $t \geq 0$,

$$\begin{aligned} ||S(t)x - S(t)y|| &\leq ||T(t)x - T(t)y|| + \int_0^t ||T(t-\sigma)[F(S(\sigma)x) - F(S(\sigma)y)]||d\sigma \\ &\leq Me^{\omega t}||x-y|| + \int_0^t MLe^{\omega(t-\sigma)}||S(\sigma)x - S(\sigma)y||d\sigma. \end{aligned}$$

By Gronwall's lemma, we obtain the assertion (6.7). Writing $y = 0$ in (6.7), we get the assertion (6.6). □

Corollary 6.1.2. *If $\omega < -ML$, then $(T(t))_{t\geq 0}$ and $(S(t))_{t\geq 0}$ are exponentially stable.*

6.2 Admissibility of observation operators for semilinear systems

We start this section with the definition of finite-time (resp. infinite-time) admissibility of output operators C for linear semigroups.

Definition 6.2.1. *Let $C \in \mathcal{L}(D(A), Y)$. We say that C is a* finite-time admissible observation operator *for $(T(t))_{t\geq 0}$, if for every $t_0 > 0$, there is some $K_{t_0} > 0$ such that*

$$\int_0^{t_0} ||CT(t)x||_Y^2 dt \leq K_{t_0}||x||^2, \tag{6.8}$$

for any $x \in D(A)$.

Definition 6.2.2. *Let $C \in \mathcal{L}(D(A), Y)$. Then C is called an* infinite-time admissible observation operator *for $(T(t))_{t\geq 0}$, if there is some $K > 0$ such that*

$$\int_0^{\infty} ||CT(t)x||_Y^2 dt \leq K||x||^2, \tag{6.9}$$

for any $x \in D(A)$.

Note that the admissibility of C guarantees that we can extend the mapping $x \mapsto CT(\cdot)x$ to a bounded linear operator from X to $L^2([0,t_0];Y)$ for every $t_0 > 0$. Similarly, if C is an infinite-time admissible observation operator, we can extend this mapping to a bounded linear operator from X to $L^2([0,\infty);Y)$. The reader is referred to see [66, 118, 119, 120] for more details on this concept of admissibility. Next, we introduce the concept of finite-time (resp. infinite-time) admissibility of output operators C for the nonlinear semigroup $(S(t))_{t\geq 0}$ given by (6.5) as follows:

Definition 6.2.3. *Let $C \in \mathcal{L}(D(A),Y)$ with $S(t)D(A) \subset D(A)$ for every $t \geq 0$. We say that C is a* finite-time admissible observation operator for $(S(t))_{t\geq 0}$, *if for every $t_0 > 0$, there is some $K_{t_0} > 0$ such that*

$$\int_0^{t_0} ||CS(t)x - CS(t)y||_Y^2 dt \leq K_{t_0}||x-y||^2, \tag{6.10}$$

for any $x,\ y \in D(A)$.

Definition 6.2.4. *Let $C \in \mathcal{L}(D(A),Y)$ with $S(t)D(A) \subset D(A)$ for every $t \geq 0$. Then C is called an* infinite-time admissible observation operator for $(S(t))_{t\geq 0}$, *if there is some $K > 0$ such that*

$$\int_0^{\infty} ||CS(t)x - CS(t)y||_Y^2 dt \leq K||x-y||^2, \tag{6.11}$$

for any $x,\ y \in D(A)$.

Equation (6.10) (resp. (6.11)) implies that the mapping $x \mapsto CS(\cdot)x$ has a continuous extension from X to $L^2([0,t_0];Y)$ for every $t_0 > 0$ (resp. $L^2([0,\infty);Y)$).

Remark 6.2.5. *(i) It is immediately clear that for linear semigroup equation (6.10) (resp.(6.11)) is equivalent to equation (6.8) (resp.(6.9)).*

(ii) It is not difficult to verify that C is a finite-time admissible observation operator for $(T(t))_{t\geq 0}$ (resp. $(S(t))_{t\geq 0}$) if (6.8) (resp.(6.10)) holds for one $t_0 > 0$.

(iii) If $(T(t))_{t\geq 0}$ (resp. $(S(t))_{t\geq 0}$) is exponentially stable, then the notion of finite-time admissibility and infinite-time admissibility are equivalent.

The objective of this section is to find sufficient conditions guaranteeing that the output function y of the system (6.1) is in $L^2([0,t_0];Y)$.
To begin with, we introduce another Banach space that contains the range of F and has the following properties:

Definition 6.2.6. *(Desch, Schappacher [45, Definition 4]) Let A be the infinitesimal generator of a linear C_0-semigroup $(T(t))_{t\geq 0}$ on X. A Banach space $(Z, |\cdot|_Z)$ is said to satisfy* assumption (**AS**) *with respect to A if and only if*

(Z1) *Z is continuously embedded in X.*

(Z2) *For all continuous functions* $\varphi : [0,\infty) \to Z$ *we have*

$$\int_0^t T(t-s)\varphi(s)ds \in D(A) \ \textit{for all} \ \ t>0,$$

and there exists a continuous nondecreasing function $\gamma : [0,\infty) \to [0,\infty)$ *such that* $\gamma(0)=0$ *and*

$$\left\| A\int_0^t T(t-s)\varphi(s)ds \right\| \le \gamma(t) \sup_{0\le s\le t} |\varphi(s)|_Z.$$

Important examples of Banach spaces that satisfy assumption (**AS**) with respect to A are provided by :

(1) $X_A = (D(A), ||\cdot||_A)$ with $||\cdot||_A$ the graph norm of A.

(2) The Favard class of A (see [89]), given by

$$Z = F_A = \left\{ x\in X| \sup_{0<t\le 1} \frac{1}{t}\|T(t)x - x\| < \infty \right\},$$

$$|x|_Z = ||x|| + \sup_{0<t\le 1} \frac{1}{t}\|T(t)x-x\|.$$

(3) If A generates an analytic semigroup we may take either $Z = D((-A)^\alpha)$, $Z = X^A_{\alpha,\infty}$ or $Z = X^A_\alpha$, $\alpha\in(0,1)$ (see [83, 89]).

One main result concerning admissibility is

Theorem 6.2.7. *Let* $(Z, |\cdot|_Z)$ *satisfy assumption* (**AS**) *with respect to* A *and* $C\in\mathcal{L}(D(A),Y)$. *We assume additionally that* F *maps* X *to* Z *and that* $F : X\to Z$ *is globally Lipschitz continuous. Then the following assertions are equivalent:*

(i) C *is finite-time admissible for* $(T(t))_{t\ge0}$.

(ii) C *is finite-time admissible for* $(S(t))_{t\ge0}$.

Proof. To begin with, we show that (i) implies (ii). Let x, $y\in D(A)$ and $t_0\ge0$. We have, for $0\le t\le t_0$,

$$\begin{aligned}
&\left\|\int_0^t T(t-s)[F(S(s)x - F(S(s)y)]ds\right\|_{X_A}\\
&\qquad = \left\|A\int_0^t T(t-s)[F(S(s)x - F(S(s)y)]ds\right\| + \\
&\qquad\quad \left\|\int_0^t T(t-s)[F(S(s)x - F(S(s)y)]ds\right\|\\
&\le \gamma(t)\sup_{0\le s\le t}|F(S(s)x - F(S(s)y)|_Z + MR\int_0^t e^{\omega(t-s)}|F(S(s)x - F(S(s)y)|_Z ds\\
&\le \gamma(t)L\sup_{0\le s\le t}||S(s)x - S(s)y|| + MLR\int_0^t e^{\omega(t-s)}||S(s)x-S(s)y||ds\\
&\le \gamma(t)LM\max\{1, e^{(\omega+ML)t_0}\}||x-y|| + M^2LR\int_0^t e^{\omega(t-s)}e^{(\omega+ML)s}||x-y||ds\\
&\le \left(\gamma(t)LM\max\{1,e^{(\omega+ML)t_0}\} + MRe^{(\omega+ML)t}\right)||x-y||.
\end{aligned}$$

Since γ is nondecreasing and positive, we obtain

$$\begin{aligned}
&\left\|\int_0^t T(t-s)[F(S(s)x - F(S(s)y)]ds\right\|_{X_A}^2 \\
&\quad \leq 2\left(\gamma(t_0)^2 L^2 M^2 \max\{1, e^{2(\omega+ML)t_0}\} + M^2R^2 e^{2(\omega+ML)t}\right)||x-y||^2 \\
&\quad \leq 2M^2 \max\{\gamma(t_0)^2 L^2, R^2\}\ \max\{1, e^{2(\omega+ML)t_0}\}||x-y||^2.
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
&\int_0^{t_0} ||CS(t)x - CS(t)y||_Y^2 dt \\
&\leq 2\int_0^{t_0} ||CT(t)x - CT(t)y||_Y^2 dt + \\
&\quad 2\int_0^{t_0} \left\|C\int_0^t T(t-s)[F(S(s)x - F(S(s)y)]ds\right\|_Y^2 dt \\
&\leq 2\int_0^{t_0} ||CT(t)x - CT(t)y||_Y^2 dt + \\
&\quad 2\int_0^{t_0} ||C||_{\mathcal{L}(X_A,Y)}^2 \left\|\int_0^t T(t-s)[F(S(s)x - F(S(s)y)]ds\right\|_{X_A}^2 dt \\
&\leq 2K_{t_0}||x-y||^2 + \\
&\quad 4M^2||C||_{\mathcal{L}(X_A,Y)}^2 \max\{\gamma(t_0)^2L^2, R^2\}\ \max\{1, e^{2(\omega+ML)t_0}\}t_0||x-y||^2.
\end{aligned}$$

Defining $K'_{t_0} := 2K_{t_0} + 4M^2||C||_{\mathcal{L}(X_A,Y)}^2 \max\{\gamma(t_0)^2L^2, R^2\}\ \max\{1, e^{2(\omega+ML)t_0}\}t_0$, this implies that

$$\int_0^{t_0} ||CS(t)x - CS(t)y||_Y^2 dt \leq K'_{t_0}\|x-y\|^2.$$

Conversely, suppose that (ii) holds. Using the formula,

$$CT(t)x = CS(t)x - C\int_0^t T(t-s)F(S(s)x)ds, \quad x \in D(A),$$

and by similar calculations as above, we have

$$\int_0^{t_0} ||CT(t)x - CT(t)y||_Y^2 dt \leq K_{t_0}||x-y||^2 \;\; x,\ y \in D(A). \tag{6.12}$$

Therefore C is finite-time admissible for $(T(t))_{t\geq 0}$ by Remark 6.2.5 (i). □

The same result holds for infinite-time admissibility. The proof follows immediately from Remark 6.2.5 (iii).

Theorem 6.2.8. *Suppose that the assumptions of Theorem 6.2.7 are satisfied. If $(T(t))_{t\geq 0}$ and $(S(t))_{t\geq 0}$ are exponentially stable, then the following statement are equivalent*

(i) C is infinite-time admissible for $(T(t))_{t\geq 0}$.

(ii) C is infinite-time admissible for $(S(t))_{t\geq 0}$.

Note that by Corollary 6.1.2 the exponential stability of both semigroup is for example satisfied if $\omega < -ML$.

Another main result of this section is the following theorem.

Theorem 6.2.9. *Assume that A generates an analytic semigroup $(T(t))_{t\geq 0}$ and that $C \in \mathcal{L}(Z_1, Y)$, where Z_1 is any space of class $\mathcal{J}_\alpha$, $0 < \alpha < 1$. Then the following assertions are equivalent:*

(i) C is finite-time admissible for $(T(t))_{t\geq 0}$.

(ii) C is finite-time admissible for $(S(t))_{t\geq 0}$.

Proof. We may assume without loss of generality that $0 \in \rho(A)$. Then the graph norm on $D(A)$ is equivalent to the norm $x \mapsto ||Ax||$, which will be used here. Let $t_0 \geq 0$. For $x \in Z_1$ and $0 \leq t \leq t_0$, we have $T(t)x \in D(A) \hookrightarrow Z_1$ and $||T(t-s)||_{\mathcal{L}(X,Z_1)} \leq cM^{1-\alpha} \max\{1, e^{\omega(1-\alpha)t_0}\}\ (t-s)^{-\alpha}$. We set

$$\sup_{0\leq t\leq t_0} ||S(t)|| \leq M_1 := M \max\{1, e^{(\omega+ML)t_0}\},$$

and

$$c_1 := cM^{1-\alpha} \max\{1, e^{\omega(1-\alpha)t_0}\}.$$

So that

$$\begin{aligned} ||v(t)||_{Z_1} = \left\| \int_0^t T(t-s)F(S(s)x)ds \right\|_{Z_1} &\leq \int_0^t ||T(t-s)F(S(s)x)||_{Z_1} ds \\ &\leq \int_0^t ||T(t-s)||_{\mathcal{L}(X,Z_1)} ||F(S(s)x)||\ ds \\ &\leq c_1 L \int_0^t (t-s)^{-\alpha} ||S(s)x|| ds. \end{aligned}$$

Therefore

$$||v(t)||_{Z_1} = \left\| \int_0^t T(t-s)F(S(s)x)ds \right\|_{Z_1} \leq c_1 L M_1 \frac{t_0^{1-\alpha}}{1-\alpha} ||x||, \quad 0 \leq t \leq t_0. \tag{6.13}$$

Then $S(t)x \in Z_1$. Moreover we have for every $x,\ y \in Z_1$

$$\begin{aligned} &\int_0^{t_0} ||CS(t)x - CS(t)y||_Y^2 dt \\ &\quad \leq 2 \int_0^{t_0} ||CT(t)x - CT(t)y||_Y^2 dt + \\ &\quad 2 \int_0^{t_0} ||C||_{\mathcal{L}(Z_1,Y)}^2 \left\| \int_0^t T(t-s)[F(S(s)x) - F(S(s)y)]ds \right\|_{Z_1}^2 dt \\ &\quad \leq 2K_{t_0} ||x-y||^2 + 2||C||_{\mathcal{L}(Z_1,Y)}^2 \int_0^{t_0} \left(c_1 L M_1 \frac{t_0^{1-\alpha}}{1-\alpha} \right)^2 dt ||x-y||^2 \\ &\quad \leq \left[2K_{t_0} + ||C||_{\mathcal{L}(Z_1,Y)}^2 \left(c_1 L M_1 \frac{t_0^{1-\alpha}}{1-\alpha} \right)^2 t_0 \right] ||x-y||^2. \end{aligned}$$

The converse can be obtain by the same procedure as above and the same way as in the second part of the proof of Theorem 6.2.7 . □

Theorem 6.2.10. *Suppose that the assumptions of Theorem 6.2.9 are satisfied. If $(T(t))_{t\geq 0}$ and $(S(t))_{t\geq 0}$ are exponentially stable, then the following statement are equivalent:*

(i) C is infinite-time admissible for $(T(t))_{t\geq 0}$.

(ii) C is infinite-time admissible for $(S(t))_{t\geq 0}$.

Proof.
The proof follows immediately from Remark 6.2.5 (iii). □

We conclude this section by two examples to illustrate our theory.

Example 6.2.11. *Let Ω be a bounded domain with smooth boundary $\partial\Omega$ in $\mathbb{R}^2$ and let Γ be an open subset of $\partial\Omega$. Consider the following nonlinear initial and boundary value problem*

$$\begin{cases} w_{tt}(x,t) & = -\Delta^2 w(x,t) + f(\int_\Omega \mid \nabla w(x,t) \mid^2 dx)g(x), \quad t \geq 0, \quad x \in \Omega, \\ w(x,t) & = \Delta w(x,t) = 0, \quad t \geq 0, \quad x \in \partial\Omega \\ w(x,0) & = w_0(x), \;\; w_t(x,0) = w_1(x), \quad x \in \Omega, \end{cases} \tag{6.14}$$

with the output function

$$y(t) = \frac{\partial w_t(x,t)}{\partial \nu}|_\Gamma. \tag{6.15}$$

We take $H = L^2(\Omega)$ and $A : D(A) \subset H \to H$ the linear unbounded operator defined by $A\varphi = \Delta^2\varphi$, where $D(A) = \{\phi \in H^4(\Omega) \cap H_0^1(\Omega) |\ \Delta\phi = 0 \text{ on } \partial\Omega\}$,

$$D(A^{\frac{1}{2}}) = H^2(\Omega) \cap H_0^1(\Omega).$$

Setting $W := (w, w_t)^\perp$, the problem (6.14) can be rewritten as an abstract semilinear equation in the Hilbert space $X = D(A^{\frac{1}{2}}) \times H$ of the form

$$W_t(t) = \mathcal{A}W(t) + F(W(t)),$$

where $\mathcal{A} := \begin{pmatrix} 0 & I_H \\ -A & 0 \end{pmatrix}$ defined on a domain $D(\mathcal{A}) = D(A) \times D(A^{\frac{1}{2}})$. Then $\mathcal{A}$ is the generator of a C_0 group on X. If we assume that $g \in H^2(\Omega) \cap H_0^1(\Omega)$ and $f : [0,\infty) \to \mathbb{R}$ is globally Lipschitz continuous. Then the nonlinear mapping $F(W) = (0, f(\int_\Omega \mid \nabla w \mid^2 dx)g)^\perp$ maps X into $D(\mathcal{A})$ and is globally Lipschitz continuous in $D(\mathcal{A})$.
Next, we define the output space $Y = L^2(\Gamma)$ and we can rewrite (6.15) as following

$$y(t) = Cx(t),$$

where

$$C = (0 \;\; C_0), \;\; C_0 w = \frac{\partial w}{\partial \nu}|_\Gamma \;\; \forall w \in D(A^{\frac{1}{2}}).$$

In [81, p. 287], the author proved that $C_0 \in \mathcal{L}(D(A^{\frac{1}{2}}), Y)$ *is an admissible observation operator, for the linear problem, i.e for all* $T \geq 0$ *there exist a constant* $K_T > 0$ *such that*

$$\int_0^T \int_\Gamma ||y(t)||^2 d\Gamma dt \leq K_T^2(||w_0||^2_{H^2(\Omega)} + ||w_1||^2_{L^2(\Omega)}),$$

for all $(w_0, w_1) \in D(A) \times D(A^{\frac{1}{2}})$.
Moreover, one deduced from Theorem 6.2.7 that $C \in \mathcal{L}(D(\mathcal{A}), Y)$ *is an admissible observation operator for the problem (6.14).*

Example 6.2.12. *Let* Ω *be a bounded domain with smooth boundary* $\partial\Omega$ *in* $\mathbb{R}^n$. *We consider the following nonlinear initial value problem*

$$\begin{cases} \dot{w}(x,t) & = \Delta w(x,t) + sin(w(x,t)), \quad x \in \Omega, \quad t \geq 0, \\ w(x,t) & = 0, \quad x \in \partial\Omega, \quad t \geq 0, \\ w(x,0) & = w_0(x), \quad x \in \Omega, \end{cases} \tag{6.16}$$

with the output function

$$y(t) = \frac{\partial w(x,t)}{\partial \nu}|_{\partial\Omega}. \tag{6.17}$$

Let $X = L^2(\Omega)$, $Y = L^2(\partial\Omega)$. *Consider the operator* $A : D(A) \to X$, $A\varphi = \Delta\varphi$, *with* $D(A) = H^2(\Omega) \cap H_0^1(\Omega)$. *Recall that* A *generates an analytic semigroup* $(T(t))_{t\geq 0}$. *Consider* $C\varphi = \dfrac{\partial\varphi}{\partial\nu}|_{\partial\Omega} \in Y$. *Since* $C : X_{\varepsilon+\frac{3}{4}} := D((-A)^{\varepsilon+\frac{3}{4}}) \to Y$, *for every* $\varepsilon > 0$, *is bounded, see [75, Section 3.1], and by Theorem 2.6.13 of [97] we have*

$$||C(-A)^{1-\gamma}T(t)||^2 = ||C(-A)^{-\varepsilon-\frac{3}{4}}(-A)^{-\gamma+\varepsilon+\frac{7}{4}}T(t)|| \leq ct^{2\gamma-2\varepsilon-\frac{7}{2}}$$

is integrable near 0 *for every* $\gamma > \frac{5}{4}$. *This means that* $C \in \mathcal{L}(X_{1-\gamma}, Y)$ *is admissible for* $\gamma > \dfrac{5}{4}$. *Consider the function* $F : X \longrightarrow X$, $F(x) = \sin(x)$, *it is easy to see that* F *is globally Lipschitz. Now Theorem 6.2.9 guarantees that* C *is an admissible observation operator for the problem (6.16)-(6.17).*

6.3 Invariance of admissibility of observations under perturbations

In this section we show that the Lebesgue extension of C is invariant under Lipschitz perturbation and we give relations between the Λ-extension of admissible operators with respect to the semigroup $(T(t))_{t\geq 0}$ and the nonlinear semigroup $(S(t))_{t\geq 0}$.

Definition 6.3.1. *Let* X, Y *be Banach spaces,* $(T(t))_{t\geq 0}$ *a* C_0*-semigroup on* X *with generator* A *and* $C \in \mathcal{L}(D(A), Y)$. *We define the operator* $C_L : D(C_L) \to Y$, *the Lebesgue extension of* C *with respect to* $(T(t))_{t\geq 0}$ *by*

$$C_L x = \lim_{\tau \downarrow 0} C \frac{1}{\tau} \int_0^\tau T(t)x dt, \tag{6.18}$$

where $D(C_L) = \{x \in X |$ *the limit in (6.18) exists*$\}$.

On the domain $D(C_L)$ we define the norm

$$||x||_{D(C_L)} = ||x|| + \sup_{0<\tau\leq 1} \left\| \frac{1}{\tau} \int_0^\tau T(t)xdt \right\|.$$

Then $(D(C_L), ||\cdot||_{D(C_L)})$ is a Banach space. We then have

$$D(A) \subset D(C_L) \subset X$$

with continuous injections, and $C_L \in \mathcal{L}(D(C_L), Y)$. For this definition and further properties we refer the reader to [118, Section 4]. In a similar manner we define the Lebesgue extension of C with respect to a nonlinear semigroup $(S(t))_{t\geq 0}$

Definition 6.3.2. *Let X, Y be Banach spaces, $(S(t))_{t\geq 0}$ a nonlinear semigroup on X given by* (6.5) *and $C \in \mathcal{L}(D(A), Y)$. We define the operator $C'_L : D(C'_L) \to Y$, the* Lebesgue extension of C with respect to $(S(t))_{t\geq 0}$ *by*

$$C'_L x = \lim_{\tau\downarrow 0} C \frac{1}{\tau} \int_0^\tau S(t)xdt, \tag{6.19}$$

where $D(C'_L) = \{x \in X |$ *the limit in (6.19) exists*$\}$.

Theorem 6.3.3. *Let $(T(t))_{t\geq 0}$ be a C_0-semigroup with generator A. Let $(S(t))_{t\geq 0}$ be the nonlinear semigroup given by* (6.5) *and $C \in \mathcal{L}(D(A), Y)$. Then, the Lebesgue extensions C_L and C'_L coincide.*

Proof. Let $x \in X$, $\tau > 0$. The Lebesgue extension C'_L satisfies

$$C'_L x = \lim_{\tau\downarrow 0} \left\{ C\frac{1}{\tau} \int_0^\tau T(t)xdt + C\frac{1}{\tau} \int_0^\tau \int_0^t T(t-\sigma)F(S(\sigma)x)d\sigma dt \right\}, \tag{6.20}$$

if this limit exists. If we can prove that

$$\lim_{\tau\downarrow 0} \left\| \frac{1}{\tau} \int_0^\tau \int_0^t T(t-\sigma)F(S(\sigma)x)d\sigma dt \right\|_A = 0, \tag{6.21}$$

then, the second term on the right-hand side of (6.20) tends to 0. Therefore, the limit in (6.20) exists if and only if the limit in (6.18) exists, and the two limits are equal. Now, we have to show (6.21).
By Fubini's theorem we have

$$\int_0^\tau \int_0^t T(t-\sigma)F(S(\sigma)x)d\sigma dt = \int_0^\tau \int_\sigma^\tau T(t-\sigma)F(S(\sigma)x)dtd\sigma.$$

The integral $\int_\sigma^\tau T(t-\sigma)F(S(\sigma)x)dt = \int_0^{\tau-\sigma} T(t)F(S(\sigma)x)dt$ belongs to $D(A)$ and $A\int_\sigma^\tau T(t-\sigma)F(S(\sigma)x)dt = (T(\tau-\sigma)-I)F(S(\sigma)x)$. It follows that

$$\int_0^\tau \int_0^t T(t-\sigma)F(S(\sigma)x)d\sigma dt \in D(A),$$

and

$$A\int_0^\tau\int_0^t T(t-\sigma)F(S(\sigma)x)d\sigma dt \quad = \int_0^\tau A\int_0^{\tau-\sigma} T(t)F(S(\sigma)x)dtd\sigma$$
$$= \int_0^\tau (T(\tau-\sigma)-I)F(S(\sigma)x)d\sigma.$$

Hence,

$$\frac{1}{\tau}A\int_0^\tau\int_0^t T(t-\sigma)F(S(\sigma)x)d\sigma dt = \frac{1}{\tau}\int_0^\tau (T(\tau-\sigma)-I)F(S(\sigma)x)d\sigma.$$

We decompose

$$T(\tau-\sigma)F(S(\sigma)x) - F(S(\sigma)x)$$
$$= T(\tau-\sigma)\left[F(S(\sigma)x)-F(x)\right] + (T(\tau-\sigma)-I)F(x) - \left[F(S(\sigma)x)-F(x)\right],$$

and we denote

$$M := \max_{t\in[0,1]} ||T(t)||.$$

Fix $x \in X$ and let $\varepsilon > 0$. Then there exists $\delta_\varepsilon \in (0,1]$, such that for $t \in [0,\delta_\varepsilon]$

$$||S(t)x - x|| \le \frac{\varepsilon}{3ML},\ ||(T(t)-I)F(x)|| \le \frac{\varepsilon}{3} \text{ and } ||F(S(t)x)-F(x)|| \le \frac{\varepsilon}{3}.$$

Then for $\tau \in (0,\delta_\varepsilon]$ and $\sigma \in [0,\tau]$ we obtain

$$||T(\tau-\sigma)F(S(\sigma)x) - F(S(\sigma)x)|| \le \varepsilon,$$

which implies

$$\left\|\frac{1}{\tau}A\int_0^\tau\int_0^t T(t-\sigma)F(S(\sigma)x)d\sigma dt\right\| \le \varepsilon.$$

On the other hand, it not difficult to verify that

$$\lim_{\tau\downarrow 0}\frac{1}{\tau}\int_0^\tau\int_0^t T(t-\sigma)F(S(\sigma)x)d\sigma dt = 0.$$

Consequently

$$\lim_{\tau\downarrow 0}\left\|\frac{1}{\tau}\int_0^\tau\int_0^t T(t-\sigma)F(S(\sigma)x)d\sigma dt\right\|_A = 0.$$

□

Remark 6.3.4. *This result coincides with Weiss' result (see [118, Theorem 5.2]), if one considers $F \in \mathcal{L}(X)$.*

In [119, 120], Weiss introduced another extension of C, the Λ-extension.

Definition 6.3.5. *Let $(T(t))_{t\geq 0}$ a C_0-semigroup with generator A and $C \in \mathcal{L}(D(A), Y)$. We define the Λ-extension C_Λ of C by*

$$\begin{aligned} D(C_\Lambda) &:= \left\{ x \in X | \lim_{\lambda \to +\infty} C\lambda R(\lambda, A)x \text{ exists} \right\}, \\ C_\Lambda x &:= \lim_{\lambda \to +\infty} C\lambda R(\lambda, A)x, \quad x \in D(C_\Lambda). \end{aligned} \tag{6.22}$$

$D(C_\Lambda)$ endowed with the norm

$$||x||_{D(C_\Lambda)} = ||x|| + \sup_{\lambda \geq \lambda_0} \|C\lambda R(\lambda, A)x\|_Y \,,$$

for $\lambda_0 \in \mathbb{C}$ such that $[\lambda_0, +\infty) \subset \rho(A)$, is a Banach space satisfying the continuous embedding

$$D(A) \hookrightarrow D(C_\Lambda) \hookrightarrow X,$$

and $C_\Lambda \in \mathcal{L}(D(C_\Lambda), Y)$.

The following result is due to Weiss. The proof was given for Lebesgue extension C_L of C, see [118, Theorem 4.5]. Since $D(C_\Lambda)$ contains $D(C_L)$ (see [120, Remark 5.7]) we obtain the following.

Theorem 6.3.6. *Let $x \in X$. Assume that C is an admissible observation operator for $(T(t))_{t\geq 0}$. Then, $T(t)x \in D(C_\Lambda)$ for all $t \geq 0$, and $C_\Lambda T(\cdot)x \in L^2([0, \tau], Y)$ for all $\tau > 0$.*

The following proposition is proved in [60, Proposition 3.3]

Proposition 6.3.7. *Let $f \in L^2_{loc}(\mathbb{R}_+, X)$. Suppose that C is an admissible observation operator for $(T(t))_{t\geq 0}$. Then, $(T * f)(t) := \int_0^t T(t-s)f(s)ds \in D(C_\Lambda)$ for all $t \geq 0$ and*

$$||C_\Lambda(T * f)||_{L^2([0,\tau],Y)} \leq c(\tau)||f||_{L^2([0,\tau],Y)},$$

for all $\tau > 0$ with $c(\tau) > 0$ is independent of f. Moreover, $\lim_{\tau \downarrow 0} c(\tau) = 0$.

Theorem 6.3.8. *Let C be an admissible observation operator for $(T(t))_{t\geq 0}$ and let $(S(t))_{t\geq 0}$ be the nonlinear semigroup given by (6.5). Then $S(t)x,\ S(t)y \in D(C_\Lambda)$ for all $x,\ y \in X$ and*

$$||C_\Lambda S(t)x - C_\Lambda S(t)y||_{L^2([0,\tau],Y)} \leq K_\tau ||x - y||$$

for $\tau,\ K_\tau > 0$.

Proof. Let $x,\ y \in X$. From Theorem 6.3.6 and Proposition 6.3.7, we deduced that $S(t)x,\ S(t)y \in D(C_\Lambda)$ and

$$\begin{aligned} &||C_\Lambda(T * F(S(\cdot)x)) - C_\Lambda(T * F(S(\cdot)y))||_{L^2([0,\tau],Y)} \\ &\qquad \leq c(\tau)||F(S(\cdot)x) - F(S(\cdot)y)||_{L^2([0,\tau],X)} \end{aligned}$$

for $\tau > 0$. On the other hand, we have

$$\begin{aligned} &||F(S(\cdot)x) - F(S(\cdot)y)||^2_{L^2([0,\tau],X)} \\ &= \int_0^\tau ||F(S(t)x) - F(S(y)x)||^2 dt \\ &\leq \int_0^\tau L^2||S(t)x - S(t)y||^2 dt \\ &\leq L^2M^2 \int_0^\tau e^{2(\omega+ML)t} dt||x-y||^2 =: \eta(\tau)||x-y||^2. \end{aligned}$$

Using formula (6.5), we can write

$$\begin{aligned} &||C_\Lambda S(t)x - C_\Lambda S(t)y||_{L^2([0,\tau],Y)} \\ &\leq ||C_\Lambda T(t)x - C_\Lambda T(t)y||_{L^2([0,\tau],Y)} + \\ &\quad ||C_\Lambda(T * F(S(\cdot)x)) - C_\Lambda(T * F(S(\cdot)y))||_{L^2([0,\tau],Y)} \\ &\leq c'(\tau)||x-y|| + c(\tau)\eta(\tau)||x-y|| \\ &\leq K_\tau||x-y||. \end{aligned}$$

□

6.4 Exact observability of semilinear systems

The object of this section is to prove that exact observability is not changed under small Lipschitz perturbations. We start by giving the definition of exact observability of linear systems described by the equation

$$u'(t) = Au(t), \quad u(0) = x, \quad y(t) = Cu(t),\ t \geq 0, \tag{6.23}$$

and of the semilinear system (6.1), respectively.

Definition 6.4.1. *Let $C \in \mathcal{L}(D(A), Y)$ an admissible observation operator for the linear C_0- semigroup $(T(t))_{t\geq 0}$ and let $\tau > 0$. Then, the system (6.23) is* exactly observable *if there is some $K > 0$ such that*

$$||CT(\cdot)x||_{L^2([0,\,\infty);Y)} \geq K||x||, \quad x \in D(A), \tag{6.24}$$

*and (6.23) is τ-*exactly observable *if there is some $K_\tau > 0$ such that*

$$||CT(\cdot)x||_{L^2([0,\,\tau];Y)} \geq K_\tau||x||, \quad x \in D(A). \tag{6.25}$$

Definition 6.4.2. *Let $C \in \mathcal{L}(D(A), Y)$ an admissible observation operator for the nonlinear semigroup $(S(t))_{t\geq 0}$ given by* (6.5) *and let $\tau > 0$. Then, the system (6.1) is* exactly observable *if there is some $K' > 0$ such that*

$$||CS(\cdot)x - CS(\cdot)y||_{L^2([0,\,\infty);Y)} \geq K'||x-y||, \quad x,\ y \in D(A), \tag{6.26}$$

*and (6.1) is τ-*exactly observable *if there is some $K'_\tau > 0$ such that*

$$||CS(\cdot)x - CS(\cdot)y||_{L^2([0,\,\tau];Y)} \geq K'_\tau||x-y||, \quad x,\ y \in D(A). \tag{6.27}$$

Remark 6.4.3. *i) It is well-known that the notion of τ-exactly observable may depend on τ, see [98, section 5].*

ii) If $(T(t))_{t\geq 0}$ (resp. $(S(t))_{t\geq 0}$) is exponentially stable then for the system (6.23) (resp. (6.1)) there exist equivalence between exact observability and τ-exact observability for some $\tau > 0$, see [103].

Throughout this section, we suppose that we have the following condition **(D)**:

(i) For all $\tau > 0$ and $\varphi \in \mathcal{C}([0,\tau];X)$

$$\int_0^{\tau} T(\tau - s)\varphi(s)ds \in D(A) \tag{6.28}$$

(ii) There exists $\tau_0 > 0$ and a constant $\alpha > 0$ such that

$$\left\| A\int_0^{\tau_0} T(\tau_0 - s)\varphi(s)ds \right\| \leq \alpha \sup_{s\in[0,\,\tau_0]} ||\varphi(s)||, \tag{6.29}$$

for all $\varphi \in \mathcal{C}([0,\tau];X)$.

Remark 6.4.4. *(a) It is easy to see that* **(D)** *holds if X satisfies assumption* **(AS)**.

(b) If there exists $\tau > 0$ and $p \in [1,\infty)$ such (6.28) holds for all $\varphi \in L^p([0,\tau];X)$, then, **(D)** *is satisfied. Indeed, by [45, Proposition 8], one can see that X satisfies the assumption* **(AS)** *and as a consequence of (a), we obtain our result.*

Here we give a useful exponential estimate of the inequality (6.29).

Lemma 6.4.5. *Let φ satisfied* **(D)**, *let $\tau_0 > 0$ and $M \geq 1$, $\omega \in \mathbb{R}$ such that $||T(t)|| \leq Me^{\omega t}$ for all $t \geq 0$. Then, for all $\tau > 0$, we have*

$$\left\| A\int_0^{\tau} T(\tau - s)\varphi(s)ds \right\| \leq N(\omega,\tau,\tau_0)\alpha \sup_{s\in[0,\,t]} ||\varphi(s)||, \tag{6.30}$$

where

$$N(\omega,\tau,\tau_0) := \begin{cases} M\dfrac{e^{|\omega|\tau_0}}{|e^{\omega\tau_0}-1|}e^{\omega\tau}, & \omega > 0, \\ M\left(1+\dfrac{\tau}{\tau_0}\right), & \omega = 0, \\ \dfrac{M}{|e^{\omega\tau_0}-1|} & \omega < 0. \end{cases} \tag{6.31}$$

Proof. We set $(V^A\varphi)(t) := A\displaystyle\int_0^t T(t-s)\varphi(s)ds$ for all $\varphi \in \mathcal{C}([0,\infty);X)$ and let $t \geq \tau \geq 0$. One first has to verify the following equality

$$(V^A\varphi)(t) = T(t-\tau)(V^A\varphi)(\tau) + (V^A\varphi_\tau)(t-\tau), \tag{6.32}$$

where $\varphi_\tau := \varphi(\cdot + \tau)$. Indeed, using integration by part of $(V^A\varphi)(t)$ and (6.28) we obtain

$$\begin{aligned}
(V^A\varphi)(t) &= A\int_0^\tau T(t-s)\varphi(s)ds + A\int_\tau^t T(t-s)\varphi(s)ds \\
&= AT(t-\tau)\int_0^\tau T(\tau-s)\varphi(s)ds + A\int_0^{t-\tau} T(t-\tau-s)\varphi(\tau+s)ds \\
&= T(t-\tau)A\int_0^\tau T(\tau-s)\varphi(s)ds + A\int_0^{t-\tau} T(t-\tau-s)\varphi(\tau+s)ds \\
&= T(t-\tau)(V^A\varphi)(\tau) + (V^A\varphi_\tau)(t-\tau).
\end{aligned}$$

The remaining of the proof follows the proof of Boulite et al. [30, Proposition 4]. □

Now, we can state the main result of this section as follows.

Theorem 6.4.6. *Let L be the Lipschitz constant of F and $\tau > 0$. Then we have:*

(a) There exists a constant $L_0 > 0$ such that:
If $L < L_0$ and system (6.23) is τ-exactly observable, then the system (6.1) is τ-exactly observable.

(b) There exists a constant $L_1 > 0$ such that:
If $L < L_1$ and system (6.1) is τ-exactly observable, then the system (6.23) is τ-exactly observable.

Proof. (a) We assume that system (6.23) is exactly observable on $[0,\tau]$ for $\tau > 0$. Let $x,\ y \in D(A)$, we have

$$CT(\tau)x - CT(\tau)y = CS(\tau)x - CS(\tau)y - C\int_0^\tau T(\tau-\sigma)[F(S(\sigma)x) - F(S(\sigma)y)]d\sigma.$$

Using the hypotheses **(D)**, we obtain

$$\begin{aligned}
&||CT(\tau)x - CT(\tau)y||_Y^2 \\
&\quad\le 2||CS(\tau)x - CS(\tau)y||_Y^2 + 2\left\|C\int_0^\tau T(\tau-\sigma)[F(S(\sigma)x) - F(S(\sigma)y)]d\sigma\right\|_Y^2 \\
&\quad\le 2||CS(\tau)x - CS(\tau)y||_Y^2 + \\
&\qquad 2||C||^2_{\mathcal{L}(D(A),Y)}\left\|\int_0^\tau T(\tau-\sigma)[F(S(\sigma)x) - F(S(\sigma)y)]d\sigma\right\|_A^2 \\
&\quad\le 2||CS(\tau)x - CS(\tau)y||_Y^2 + \\
&\qquad 2||C||^2_{\mathcal{L}(D(A),Y)}\left(\left\|A\int_0^\tau T(\tau-\sigma)[F(S(\sigma)x) - F(S(\sigma)y)]d\sigma\right\| + \right. \\
&\qquad \left.\left\|\int_0^\tau T(\tau-\sigma)[F(S(\sigma)x) - F(S(\sigma)y)]d\sigma\right\|\right)^2.
\end{aligned}$$

Hence,

$$||CT(\tau)x - CT(\tau)y||_Y^2 \leq 2||CS(\tau)x - CS(\tau)y||_Y^2$$
$$+ 2||C||^2_{\mathcal{L}(D(A),Y)}$$
$$\left(\alpha\, N(\omega,\tau,\tau_0) \sup_{0\leq\sigma\leq\tau} ||F(S(\sigma)x) - F(S(\sigma)y)|| + M^2Le^{\omega\tau}\int_0^\tau e^{MLs}||x-y||ds\right)^2$$
$$\leq 2||CS(\tau)x - CS(\tau)y||_Y^2 + 4||C||^2_{\mathcal{L}(D(A),Y)}$$
$$\left(\alpha\, N(\omega,\tau,\tau_0)ML\max\{1, e^{(\omega+ML)\tau}\}\right)^2 ||x-y||^2 +$$
$$4||C||^2_{\mathcal{L}(D(A),Y)}M^4L^2\tau^2e^{2(\omega+ML)\tau}||x-y||^2$$
$$\leq 2||CS(\tau)x - CS(\tau)y||_Y^2 +$$
$$8||C||^2_{\mathcal{L}(D(A),Y)}\left(\alpha\, N(\omega,\tau,\tau_0)M^2L\tau\right)^2 \max\{1, e^{2(\omega+ML)\tau}\}||x-y||^2.$$

Set $M_2 := ||C||^2_{\mathcal{L}(D(A),Y)}M^4\tau^2$, Therefore,

$$\int_0^\tau ||CS(r)x - CS(r)x||_Y^2 dr$$
$$\geq \frac{1}{2}\int_0^\tau ||CT(r)x - CT(r)x||_Y^2 dr -$$
$$4\alpha^2 M_2 L^2 \int_0^\tau N^2(\omega, r,\tau_0)\max\{1, e^{2(\omega+ML)r}\}||x-y||^2 dr$$
$$\geq \frac{1}{2}\int_0^\tau ||CT(r)x - CT(r)y||_Y^2 dr -$$
$$4\alpha^2 M_2 L^2 \max\{1, e^{2(\omega+ML)\tau}\}\int_0^\tau N^2(\omega, r,\tau_0)dr||x-y||^2.$$

Consequently,

$$\int_0^\tau ||CS(r)x - CS(r)y||_Y^2 dr \geq \left(\frac{1}{2}K_\tau - 4\eta(L)\alpha^2L^2\right)||x-y||^2,$$

where $\eta(L) := M_2 \max\{1, e^{2(\omega+ML)\tau}\}\int_0^\tau N^2(\omega,r,\tau_0)dr.$

Set $f(L) = \frac{1}{2}K_\tau - 4\eta(L)\alpha^2L^2$. The function f is continuous from $[0,+\infty)$ to $(-\infty, \frac{1}{2}K_\tau]$ and strictly decreasing, hence it is bijective. Then there exists a unique $L_0 > 0$ such that $f(L_0) = 0$. The parenthesis above becomes positive for $L < L_0$, which implies that system (6.1) is τ-exact observable. The proof of (b) is easy since we use same procedure as above. □

Corollary 6.4.7. *Let L be the Lipschitz constant of F. If the semigroups $(T(t))_{t\geq 0}$ and $(S(t))_{t\geq 0}$ are exponentially stable then we have:*

(a) There exists a constant $L_0 > 0$ such that:
If $L < L_0$ and System (6.23) is exactly observable, then the system (6.1) is exactly observable.

(b) *There exists a constant $L_1 > 0$ such that:*
If $L < L_1$ and System (6.1) is exactly observable, then the system (6.23) is exactly observable.

The statements of the Theorem 6.4.6 and the Corollary 6.4.7 still hold if we drop the assumption **(D)** and instead it is just assumed that A generates an analytic semigroup $(T(t))_{t\geq 0}$, $F : X \to X$ is globally Lipschitz and $C \in \mathcal{L}(X_\alpha, Y)$. The proof is similar to the Theorem 6.4.6 using (6.13).

Example 6.4.8. *Let $\Omega = (0,\pi) \times (0,\pi)$ and let $\Gamma = ([0,\pi] \times 0) \cup (0 \times [0,\pi])$ be a subset of $\partial\Omega$. We consider the following semilinear problem for the wave equation with Neumann boundary observation:*

$$\begin{cases} \ddot{w}(x,t) & = \Delta w(x,t) + \dfrac{\lambda}{1+|w(x,t)|}, \quad x \in \Omega, \quad t \geq 0, \\ w(x,t) & = 0, \quad x \in \partial\Omega, \quad t \geq 0, \\ w(x,0) & = w_0(x), \;\; \dot{w}(x,0) = w_1(x) \quad x \in \Omega, \end{cases} \tag{6.33}$$

with the output function

$$y(t) = \frac{\partial w(x,t)}{\partial \nu}|_\Gamma, \tag{6.34}$$

where $\lambda > 0$. Let $X = L^2(\Omega)$, $Y = L^2(\Gamma)$, we set $x := (w, \dot{w})$. System (6.33)-(6.34) can be written in form (7.3)-(7.4) in the Hilbert space $\mathbb{H} = D(A_0^{\frac{1}{2}}) \times X$, where

$$\mathcal{A} := \begin{pmatrix} 0 & I_X \\ -A_0 & 0 \end{pmatrix}, \quad D(\mathcal{A}) = D(A_0) \times D(A_0^{\frac{1}{2}}),$$

$$A_0\phi = -\Delta\phi \;\; \forall \phi \in D(A_0), \;\; D(A_0) = H^2(\Omega) \cap H_0^1(\Omega),$$

$$D(A_0^{\frac{1}{2}}) = H_0^1(\Omega), \;\; C = (C_0, 0), \;\; C_0\phi = \frac{\partial \phi}{\partial \nu}|_\Gamma \;\; \forall \phi \in D(A_0).$$

It is know that the operator $\mathcal{A}$ generates a C_0 group on $\mathbb{H}$ and the nonlinear mapping $F(x) = (0, \dfrac{\lambda}{1+|w|})$ is globally Lipschitz continuous from $\mathbb{H}$ to $D(\mathcal{A})$ as in Example 6.2.11. From [81, p. 44], it follows that $C \in \mathcal{L}(D(\mathcal{A}), Y)$ is an admissible observation operator for the linearized problem (6.33)-(6.34) and in [101, Theorem 6.2] it is shown that the linearized system of (6.33)-(6.34) is exactly observable in some time τ. Now for a small constant λ, all assumptions of Theorem 6.4.6 are satisfied and hence the semilinear problem (6.33)-(6.34) is exactly observable in some time τ.

Chapter 7

Semilinear observation systems

The concepts of admissible observation operators and of observation systems have been introduced in the linear case by Salamon and Weiss in [104] and [118]. An operator $C \in \mathcal{L}(D(A), Y)$ is called admissible for a C_0-semigroup $T = (T(t))_{t\geq 0}$ with generator A if the output map $x \mapsto C(T(\cdot)x)$, initially defined on $D(A)$, can be extended to a continuous map Ψ from X to $L^2_{loc}(\mathbb{R}_+, Y)$. The pair (T, ψ) is then an observation system; i.e., it holds $(\Psi x)(\cdot+\tau) = \Psi T(\tau)x$ for all $x \in X$ and $\tau \geq 0$. Conversely, for any observation system (T, Ψ) there is an admissible output operator $C \in \mathcal{L}(D(A), Y)$ such that $\Psi x = CT(\cdot)x$ for every $x \in D(A)$. Moreover, there exists the 'Lebesgue extension' C_L of C satisfying $T(t)x \in D(C_L)$ for a.e. $t \geq 0$ and $\Psi x = C_L T(\cdot)x$ for all $x \in X$, see [118] and also [39, 66, 104].

In this chapter we extend this successful linear theory to general nonlinear locally Lipschitz semigroups $S = (S(t))_{t\geq 0}$ (see Definition 7.1.3) and densely defined nonlinear output operators C. In particular, for such semigroups S we define locally Lipschitz observation systems Ψ and locally Lipschitz admissible observation operators in Section 3. We further prove that such observation systems Ψ can be represented by $\Psi x = \widetilde{C}(S(\cdot)x)$ for a (possibly nonlinear) admissible observation operator $\widetilde{C}$, see Theorem 7.2.6.

We consider the linear observation system

$$\begin{aligned} \dot{u}(t) &= Au(t), \quad u(0) = x \in X,\ t \geq 0, && (7.1)\\ y(t) &= C(u(t)), && (7.2) \end{aligned}$$

and focus on the semilinear observation system

$$\begin{aligned} \dot{u}(t) &= Au(t) + F(u(t)), \quad u(0) = x \in X,\ t \geq 0, && (7.3)\\ y(t) &= C(u(t)), && (7.4) \end{aligned}$$

where A is assumed to be the generator of a linear C_0-semigroup T on a Banach space X, C is a nonlinear unbounded operator from a domain $D(C)$ to another Banach space Y and F is a locally Lipschitz continuous nonlinear operator from X into itself. Throughout we assume that F has linear growth.

It is well known, see e.g. [97], that the state equation (7.3) has a global unique mild solution given by $u(\cdot; x)$ for every $x \in X$. Moreover, by $S(t)x =$

$u(t;x)$ one defines a semigroup S of locally Lipschitz continuous operators. One now looks for sufficient conditions for the admissibility of C for S. As an important special case, we assume that C is an admissible linear output operators for T. In this situation one can in fact construct a nonlinear observation system (S, Ψ_F) given by (7.15), which is the integrated version of (7.3)–(7.4). Moreover, the system is (S, Ψ_F) represented by the Lebesgue extension C_L of C with respect to T, see Theorem 7.2.7.

Similar robustness results for admissibility and exact observability were shown for globally Lipschitz F in Chapter 6. In this chapter also additional regularity properties of F or T were assumed which were needed to treat the variation of constants formula related to (7.3). In the present we could discard these extra assumptions by using an estimate for the convolution $f \mapsto C_L T * f$ established in [108] for admissible C, see (7.16).

7.1 Background

In this section we give some results about semilinear evolution equations and linear observation systems. Let X and Y be Banach spaces (the state and the observation space, respectively) and the family $T = (T(t))_{t\geq 0}$ of linear operators be a C_0-semigroup on X with generator $(A, D(A))$. We can fix constants $M, \omega > 0$ such that

$$\|T(t)\| \leq Me^{\omega t} \tag{7.5}$$

holds for all $t \geq 0$. We denote by $\mathcal{L}(X, Y)$ the space of bounded linear operators between two Banach spaces X and Y. Moreover, the (nonlinear) operator $F : X \longrightarrow X$ is always assumed to be locally Lipschitz continuous; that is, for each $r > 0$ there exists a constant $L(r) \geq 0$ such that

$$\|F(x) - F(y)\| \leq L(r)\|x - y\|,$$

for all x, $y \in X$ with $\|x\| \leq r$ and $\|y\| \leq r$.

It is well-known (see e.g. Theorem 6.1.4 in [97]) that, under the above assumptions, for every $x \in X$ there is a maximal $t(x) \in (0, \infty]$ such that the problem (7.3) admits a unique mild solution $u = u(\cdot; x) \in C([0, t(x)), X)$ given by the variation of constant formula

$$u(t) = T(t)x + \int_0^t T(t-\sigma)F(u(\sigma))d\sigma. \tag{7.6}$$

Moreover, if $t(x) < \infty$ then $\lim_{t\to t(x)} \|u(t)\| = \infty$. For our investigations it suffices to consider mild solutions. The question whether they are in fact classical solutions of (7.3) is discussed, e.g., in [97, Chapter 6]. In this chapter we work in the situation of global solvability assuming that

(G) $\|F(x)\| \leq a\|x\| + b$ holds for all $x \in X$ and some constants $a, b \geq 0$.

Under this condition of linear growth, the formula (7.6) and Gronwall's inequality easily yield the next result.

Proposition 7.1.1. *Let A generate a C_0-semigroup T satisfying (7.5) and $F : X \to X$ be locally Lipschitz such that* **(G)** *holds. Then the problem (7.3) has a unique global mild solution in $C([0,\infty),X)$ for each $x \in X$. Moreover, u is exponentially bounded in the sense that*

$$\|u(t)\| \le \frac{Mb}{\omega}e^{\omega t} + Me^{(\omega+aM)t}\|x\| \qquad \text{for all } \; t \ge 0. \tag{7.7}$$

Remark 7.1.2. *If we assume that F is globally Lipschitz continuous, then it has linear growth and thus (7.3) has a unique global mild solution for each $x \in X$.*

Definition 7.1.3. *A family $S = (S(t))_{t\ge 0}$ of locally Lipschitz operators from X into itself is called a* semigroup of locally Lipschitz operators *on X if it satisfies the following conditions:*

(a) $S(t+s)x = S(t)S(s)x$ *and* $S(0)x = 0$ *for all* $t, s \ge 0$ *and* $x \in X$.

(b) *For each* $x \in X$, *the* X*-valued function* $S(\cdot)x$ *is continuous on* $[0,\infty)$.

(c) *For every* $r > 0$ *and* $t_0 > 0$ *there exists a constant* $L(t_0, r) > 0$ *such that for all* $x, y \in X$ *with* $\|x\|, \|y\| \le r$ *we have*

$$\|S(t)x - S(t)y\| \le L(t_0, r)\|x - y\| \qquad \text{for all} \quad t \in [0, t_0]. \tag{7.8}$$

Let $u(\cdot; x)$ be the solution of (7.3) for a given $x \in X$, where we assume that **(G)** holds. We define $S(t)x := u(t;x)$ for all $x \in X$ and $t \ge 0$. The operators $S(t)$ then map X into itself and satisfy the properties stated in Definition 7.1.3. In fact, the first property follows from the uniqueness of mild solutions. The second one is an immediate consequence of the continuity of $t \mapsto u(t;x)$. The last property can be shown using (7.6), (7.7), the local Lipschitz continuity of F and Gronwall's inequality. Hence, the output function in (7.4) is formally given by

$$y(t) = C(S(t)x).$$

Of course, this expression only makes sense if $S(t)x$ belongs to the domain $D(C)$ of C. We note that, in general, $D(C)$ is not invariant under $S(t)$. Such problems already occur in the linear case.

7.2 Locally Lipschitz observation systems

We start with our basic definitions.

Definition 7.2.1. *A* locally Lipschitz observation system *on the Banach spaces X and Y is a pair (S, Ψ) (resp. (T, Ψ)), where $S := (S(t))_{t\ge 0}$ (resp. $T := (T(t))_{t\ge 0}$) is a semigroup of locally Lipschitz operators (resp. a linear C_0-semigroup) on X and Ψ is a family of (possibly nonlinear) operators from X to $L^2_{loc}([0,\infty), Y)$ such that for every t_0, $r > 0$ there exists a constant $k(r, t_0) > 0$ such that*

$$\begin{aligned} &(\Psi x)(\cdot+\tau) = \Psi S(\tau)x \quad (\textit{resp. } (\Psi x)(\cdot+\tau) = \Psi T(\tau)x) \quad \textit{on } \mathbb{R}_+, \\ &\|\Psi x - \Psi y\|_{L^2([0,t_0],Y)} \le k(r, t_0)\|x - y\|, \end{aligned} \tag{7.9}$$

for all $\tau \ge 0$ and $x, y \in X$ with $\|x\|, \|y\| \le r$.

Definition 7.2.2. *Let S (resp. T) be a semigroup of locally Lipschitz operators (resp. a linear C_0-semigroup) on X and let $C : D(C) \to Y$ be a (possibly nonlinear) operator with dense domain $D(C)$ in X. We say that C is a* locally Lipschitz admissible observation operator *for S (resp. T) if, for every $x \in D(C)$, it holds $S(t)x \in D(C)$ (resp. $T(t)x \in D(C)$) for a.e. $t \geq 0$, the function $C(S(\cdot)x) : \mathbb{R}_+ \to Y$ (resp. $CT(\cdot)x : \mathbb{R}_+ \to Y$) is strongly measurable and if for every $t_0 > 0$ and every $r > 0$ there is a constant $\gamma(r, t_0) > 0$ such that*

$$\int_0^{t_0} \|CS(t)x - CS(t)y)\|_Y^2 dt \leq \gamma(r, t_0)^2 \|x - y\|^2, \tag{7.10}$$

$$(resp. \int_0^{t_0} \|CT(t)x - CT(t)y)\|_Y^2 dt \leq \gamma(r, t_0)^2 \|x - y\|^2) \tag{7.11}$$

for all x, $y \in D(C)$ with $\|x\|$, $\|y\| < r$.

Remark 7.2.3. *In case of a linear operator C with $D(C) = D(A)$, and a global Lipshitz F, the above concepts coincide with those of the the previous chapter.*

Let C be locally Lipschitz admissible for S (resp. T). Then the map $\Psi : D(C) \to L^2_{loc}(\mathbb{R}_+, Y)$, $x \mapsto CS(\cdot)x$ (resp. $x \mapsto CT(\cdot)x$), possesses a locally Lipschitz continuous extension from X to $L^2_{loc}(\mathbb{R}_+, Y)$. In fact, let $x \in X$ and $t_0 > 0$. Since $D(C)$ is dense, there exist $x_n \in D(C)$ converging to x in X as $n \to \infty$. Estimate (7.10) (resp. (7.11)) implies that Ψx_n is a Cauchy sequence which therefore converges to some z in the complete metric space $L^2_{loc}(\mathbb{R}_+, Y)$. If $x'_n \in D(C)$ converges to x in X, then $\Psi x'_n$ also converges to z in $L^2_{loc}(\mathbb{R}_+, Y)$ thanks to (7.10) and (7.11). So we can extend Ψ to a map from X to $L^2_{loc}(\mathbb{R}_+, X)$ denoted by the same symbol. Let $t_0, r > 0$ and x, $y \in X$ with $\|x\|$, $\|y\| < r$. There are $x_n \in D(C)$ and $y_n \in D(C)$ converging to x and y, respectively. Using (7.10) and (7.11) we can then estimate

$$\begin{aligned} \|\Psi x - \Psi y\|_{L^2([0,t_0],Y)} &= \lim_{n\to\infty} \|\Psi x_n - \Psi y_n\|_{L^2([0,t_0],Y)} && (7.12) \\ &\leq \gamma(r, t_0) \lim_{n\to\infty} \|x_n - y_n\| = \gamma(r, t_0)\|x - y\|. && (7.13) \end{aligned}$$

Hence, Ψ is locally Lipschitz continuous on X. We further obtain

$$\Psi x(\tau + \cdot) = \lim_{n\to\infty} \Psi x_n(\tau + \cdot) = \lim_{n\to\infty} \Psi S(\tau)x_n = \Psi S(\tau)x$$

in $L^2_{loc}(\mathbb{R}_+, X)$. We state this result in the following lemma.

Lemma 7.2.4. *Let C be a locally Lipschitz admissible observation operator for S (resp. T). There exists a locally Lipschitz continuous extension $\Psi : X \longrightarrow L^2_{loc}([0, \infty), Y)$ of the map $x \mapsto CS(\cdot)x$ (resp. $x \mapsto CT(\cdot)x$) defined on $D(C)$. Moreover, (S, Ψ) (resp. (T, Ψ)) is a locally Lipschitz observation system.*

For a given locally Lipschitz observation system we can now construct a pointwise representation in terms of an observation operator.

Definition 7.2.5. *For a locally Lipschitz observation system* (S, Ψ) *(resp.* (T, Ψ)*) we define*

$$\widetilde{C}x = \lim_{\tau \downarrow 0} \frac{1}{\tau} \int_0^\tau (\Psi x)(t) dt, \tag{7.14}$$

for $x \in D(\widetilde{C}) := \{x \in X : \text{the limit in (7.14) exists in } Y\}$.

The next representation result extends Theorem 4.5 of [118] to locally Lipschitz observation systems.

Theorem 7.2.6. *Let* (S, Ψ) *(resp.* (T, Ψ)*) be a locally Lipschitz observation system, and let* $\widetilde{C} : D(\widetilde{C}) \to Y$ *be the nonlinear operator defined by* (7.14). *Then, for all* $x \in X$ *and* $t \geq 0$ *we have* $S(t)x \in D(\widetilde{C})$ *(resp.* $T(t)x \in D(\widetilde{C})$*) if and only if*

$$\frac{1}{\tau} \int_0^\tau (\Psi x)(t+s)\, ds \quad \text{converges as} \quad \tau \searrow 0.$$

If this is the case, then the limit equals $\widetilde{C}S(t)x$ *(resp.* $\widetilde{C}T(t)x$*). We thus obtain* $(\Psi x)(t) = \widetilde{C}S(t)x$ *(resp.* $(\Psi x)(t) = \widetilde{C}T(t)x$*) for almost every* $t \geq 0$, *namely for all Lebesgue points* $t \geq 0$ *of* Ψx.

Proof. The theorem follows from the identity

$$\frac{1}{\tau} \int_0^\tau (\Psi S(t)x)(r)\, dr = \frac{1}{\tau} \int_0^\tau (\Psi x)(t+r) dr$$

and the fact that this limit exists for almost every $t \geq 0$ since Ψx is locally integrable. □

In particular, $\widetilde{C}$ is an locally Lipschitz admissible observation operator for S (resp. T). According to Lemma 7.2.4, $\widetilde{C}$ and S (resp. T) generate an observation system $(S, \widetilde{\Psi})$ (resp. $(T, \widetilde{\Psi})$). It is easy to see that, in fact, $\Psi = \widetilde{\Psi}$. We say that the operator $\widetilde{C}$ *represents* the observation system (S, Ψ) (resp. (T, Ψ)).

In a second step we now consider the special case of the semilinear system (7.3) and (7.4), and assume that C is linear. So let (T, Ψ) be a linear observation system with observation operator C and Lebesgue extension C_L and $(S(t))_{t \geq 0}$ the semigroup of locally Lipschitz operators solving (7.3) in the mild sense. Recall from Section 2 that $\Psi x = C_L T(\cdot)x$.

In order to describe the output of (7.3) and (7.4), we define

$$\Psi_F x = \Psi x + C_L \mathbb{K} F(S(\cdot)x) \tag{7.15}$$

for all $x \in X$, where $\mathbb{K}f(t) := \int_0^t T(t-s)f(s)ds$ for $f \in L^1_{loc}(\mathbb{R}_+, X)$ and $t \geq 0$. Observe that $F(S(\cdot)x)$ is locally bounded due to **(G)** and (7.7). We recall from Proposition 2.11 in [108] (and its proof) that $\mathbb{K}f(t) \in D(C_L)$ for a.e. $t \geq 0$, $C_L \mathbb{K} f : \mathbb{R}_+ \to Y$ is strongly measurable and

$$\|C_L \mathbb{K} f\|_{L^2([0,t_0],Y)} \leq c(t_0) t_0^{\frac{1}{2}} \|f\|_{L^2([0,t_0],X)} \tag{7.16}$$

for all $f \in L^2_{loc}(\mathbb{R}_+, X)$ and $t_0 > 0$, where $c(t_0) = \gamma(t_0 + 1)$ and γ is given by (7.11). (Hence, $c : \mathbb{R}_+ \to \mathbb{R}_+$ is locally bounded.) We now show that (Ψ_F, S) is a locally Lipschitz observation system represented by C_L.

Theorem 7.2.7. *Let (T, Ψ) be a linear observation system with observation operator $C \in \mathcal{L}(D(A), Y)$, $F : X \to X$ be locally Lipschitz, and $S(\cdot)$ solve (7.3). Assume that (G) holds. Define Ψ_F as in (7.15). Then, (Ψ_F, S) is a locally Lipschitz observation system represented by the Lebesgue extension C_L.*

Proof. Let $t_0 > 0$ and $r > 0$, and take $x, y \in X$ with $\|x\|, \|y\| \le r$. Using the assumptions, (7.16) and (7.8), we can estimate

$$\begin{aligned}\|\Psi_F x - \Psi_F y\|_{L^2([0,t_0],Y)} &\le \|\Psi(x-y)\|_{L^2([0,t_0],Y)} \\ &+\|C_L\mathbb{K}[F(S(\cdot)x) - F(S(\cdot)y)]\|_{L^2([0,t_0],X)} \\ &\le c\|x-y\| + c(t_0)t_0^{\frac{1}{2}}\|F(S(\cdot)x) - F(S(\cdot)y)\|_{L^2([0,t_0],X)} \\ &\le c\|x-y\| + c(r,t_0)t_0^{\frac{1}{2}}\|S(\cdot)x - S(\cdot)y\|_{L^2([0,t_0],X)} \\ &\le c(r,t_0)\|x-y\|.\end{aligned}$$

Let $t \ge 0$. For a.e. $\tau \ge 0$, the formulas (7.15) and (7.6) lead to

$$\begin{aligned}(\Psi_F x)(t+\tau) &= C_L T(\tau)T(t)x + C_L \textstyle\int_t^{t+\tau} T(t+\tau-s)F(S(s)x))\,ds \\ &+C_L T(\tau)\textstyle\int_0^t T(t-s)F(S(s)x))\,ds \\ &= C_L T(\tau)S(t)x + C_L \textstyle\int_0^\tau T(\tau-s)F(S(s)S(t)x))\,ds \\ &= (\Psi_F(S(t)x))(\tau).\end{aligned}$$

So we have shown that (Ψ_F, S) is a locally Lipschitz observation system. For the second assertion, let $x \in X$ and $t \in (0,1]$. Equation (7.15) yields

$$\frac{1}{t}\int_0^t (\Psi_F x)(s)\,ds = \frac{1}{t}\int_0^t (\Psi x)(s)\,ds + \frac{1}{t}\int_0^t C_L\mathbb{K}F(S(\cdot)x)(s)\,ds.$$

The second integral on the right hand side is denoted by $J(t)$. From Hölder's inequality and estimate (7.16) we deduce that

$$\|J(t)\| \le t^{-\frac{1}{2}}\|C_L\mathbb{K}F(S(\cdot)x)\|_{L^2([0,t],Y)} \le c\,\|F(S(\cdot)x)\|_{L^2([0,t],X)} \longrightarrow 0$$

as $t \to 0$. We then conclude that $D(\tilde{C}) = D(C_L)$ and $\tilde{C}x = C_L x$, where $\tilde{C}$ represents Ψ_F. □

7.3 Local exact observability

As the previous chapter, we give the following definitions in the linear and the nonlinear systems.

Definition 7.3.1. *Let $C \in \mathcal{L}(D(A), Y)$ be an admissible observation operator for the linear C_0-semigroup $(T(t))_{t\ge 0}$ with generator A. The system (7.2) is called* exactly observable in time $\tau > 0$ *if there is a constant $\kappa > 0$ such that*

$$\|CT(\cdot)x\|_{L^2([0,\,\tau],Y)} \ge \kappa\|x\| \qquad \textit{for all } x \in D(A). \tag{7.17}$$

Definition 7.3.2. *Let $C : D(C) \to Y$ be an locally Lipschitz admissible observation operator for the semigroup S solving* (7.3)*. The system (7.3) and (7.4)* *is called* locally exact observable in time $\tau > 0$ at $x_0 \in D(C)$ *(or on* $B(x_0, r_0)$*) if there are numbers* $r_0, \kappa > 0$ *such that*

$$\|CS(\cdot)x - CS(\cdot)y\|_{L^2([0,\tau],Y)} \geq \kappa \|x - y\| \tag{7.18}$$

for all $x, y \in D(C)$ *with* $\|x_0 - x\| \leq r_0$ *and* $\|x_0 - y\| \leq r_0$*.*

Remark 7.3.3. *One can see that the linear system (7.2) is exactly observable if and only if is locally exact observable at some* x_0*, see the proof of Theorem 7.3.4 below.*

We now establish a robustness result for exact observability in the semilinear case. Observe that x_0 is fixed point for the semilinear problem (7.3), i.e., $S(t)x_0 = x_0$ holds for all ≥ 0, if and only if $x_0 \in D(A_0)$ and $Ax_0 = -F(x_0)$. In particular, $x_0 = 0$ is a fixed point for (7.3) if and only if $F(0) = 0$.

Theorem 7.3.4. *Let* $C \in \mathcal{L}(D(A), Y)$ *be an admissible linear observation operator for the* C_0*-semigroup* T *with generator* A*. Let* $F : X \to X$ *be locally Lipschitz and* S *be the nonlinear semigroup solving (7.3).Let* $x_0 \in D(A)$ *satisfy* $Ax_0 = -F(x_0)$ *and denote by* $L_0(r)$ *the Lipschitz constant of* F *on the ball* $B(x_0, r)$ *in* X*. Then there are constants* $L_1, L_2 > 0$ *such that the following assertions hold.*

(a) *If the linear system (7.2) is exactly observable in time* $\tau > 0$ *and if there is an* $\tilde{r} > 0$ *with* $L_0(\tilde{r}) < L_1$*, then the nonlinear system (7.3) and (7.4) is locally exact observable in time* τ*.*

(b) *If the nonlinear system (7.3) and (7.4) is locally exact observable in time* $\tau > 0$ *on the ball* $B(x_0, r_0)$ *and there is an* $\tilde{r} \in (0, r_0)$ *with* $L_0(\tilde{r}) < L_2$*, then the linear system (7.2) is exactly observable in time* τ*.*

Proof. We first establish certain Lipschitz estimates for S near x_0. Fix an $R > 0$ and take any $r \in (0, R)$. Let $\rho \in (0, r)$, x, $y \in B(x_0, \rho)$, and $t \in [0, \tau]$. Let $t_1 > 0$ be the supremum of $t \in [0, \tau]$ such that $\|S(s)x - x_0\| < r$ for all $s \in [0, t]$. The formula (7.6) and estimate (7.5) then imply the inequality

$$\|S(t)x - x_0\| = \|S(t)x - S(t)x_0\| \leq Me^{\omega\tau}\|x - x_0\| + Me^{\omega\tau} \int_0^t L_0(r)\|S(s)x - x_0\|\, ds$$

for all $0 \leq t < t_1$. From Gronwall's inequality it follows that

$$\|S(t)x - x_0\| \leq Me^{\omega\tau} \exp(Me^{\omega\tau} L_0(r)\tau)\rho$$

for all $0 \leq t < t_1$. Choosing a sufficiently small $\rho = \rho(r) > 0$ we thus obtain $\|S(t_1)x - x_0\| < r$ so that $t_1 = \tau$ and $S(t)x \in B(x_0, r)$ for all $t \in [0, \tau]$. Using again (7.6), we can now deduce the Lipschitz estimate

$$\|S(t)x - S(t)y\| \leq Me^{\omega\tau} \exp(Me^{\omega\tau} L_0(r)\tau)\|x - y\| =: k(R)\|x - y\|$$

if $\|x - x_0\|$, $\|y - x_0\| \leq \rho(r) < r$ and $t \in [0, \tau]$.

We now assume that the system (7.2) is exactly observable in time $\tau > 0$ with constant $\kappa > 0$. Formula (7.15) yields

$$C_L T(t)(x-y) = C_L S(t)x - C_L S(t)y - C_L \int_0^t T(t-\sigma)[F(S(\sigma)x) - F(S(\sigma)y)]\, d\sigma.$$

Using (7.16) and the above estimates, we then deduce that

$$\begin{aligned} \|C_L \quad & T(\cdot)x - C_L T(\cdot)y\|_{L^2([0,\tau],Y)} \\ \leq \quad & \|C_L S(\cdot)x - C_L S(\cdot)y\|_{L^2([0,\tau],Y)} + c(\tau)\, \|F(S(\cdot)x) - F(S(\cdot)y)\|_{L^2([0,\tau],X)} \\ \leq \quad & \|C_L S(\cdot)x - C_L S(\cdot)y\|_{L^2([0,\tau],Y)} + c(\tau) L_0(r)\, \|S(\cdot)x - S(\cdot)y\|_{L^2([0,\tau],X)} \\ \leq \quad & \|C_L S(\cdot)x - C_L S(\cdot)y\|_{L^2([0,\tau],Y)} + L_0(r) c_1(\tau) k(R) \|x-y\|_X \end{aligned}$$

for $x, y \in B(x_0, \rho(r))$ and $t \in [0,\tau]$. Thus, if $L_0(\tilde{r}) c_1(\tau) k(R) \leq \kappa/2$ for some $\tilde{r} > 0$, the observability of C and T yields

$$\begin{aligned} \|C_L \quad & S(\cdot)x - C_L S(\cdot)y\|_{L^2([0,\tau],Y)} \\ \geq \quad & \|C_L T(\cdot)x - C_L T(\cdot)y\|_{L^2([0,\tau],Y)} - c_1(\tau) k(R) L_0(\tilde{r}) \|x-y\| \geq \tfrac{\kappa}{2} \|x-y\| \end{aligned}$$

for all $x, y \in X$ with $\|x - x_0\|$, $\|y - x_0\| \leq \rho(\tilde{r})$.

To prove part (b) we proceed in the same way, but we require in addition that $0 < \rho < r_0$ and take $y = x_0$. We thus obtain

$$\|C_L T(\cdot)(x - x_0)\|_{L^2([0,\tau],Y)} \geq \tfrac{\kappa}{2} \|x - x_0\|$$

for all x in a ball around x_0. By linearity, this estimate implies the exact observability of the linear system (7.2). □

7.4 Applications

In this section we give examples for the main theorems of this chapter.

Example 7.4.1. *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with boundary $\partial\Omega \in C^4$ and let Γ be an open subset of $\partial\Omega$. Consider the damped nonlinear beam equation*

$$\begin{cases} u_{tt} + \Delta^2 u - 2\beta \Delta u_t - f\left(\int_\Omega |\nabla u|^2 dx\right) \Delta u = 0, \ x \in \Omega, t > 0, \\ u(t,x) = 0, \ x \in \partial\Omega, \ t \geq 0, \\ \Delta u(t,x) = 0, \ x \in \partial\Omega, \ t \geq 0, \\ u(0,x) = u_0(x), \ u_t(0,x) = u_1(x), \ x \in \Omega \end{cases} \tag{7.19}$$

with $\beta > 0$ and the output function

$$y(t) = u_t|_\Gamma. \tag{7.20}$$

Equation (7.19) arise in the mathematical study of structural damped nonlinear vibrations of a string or a beam and was considered in [42, 110] and references therein.
Let $H = L^2(\Omega)$ and $A\phi = \Delta^2\phi$ with $D(A) = H^4(\Omega) : u = \Delta u = 0$ on $\partial\Omega\}$. It is known that A is a self adjoint, positive, boundedly invertible operator and that

$$H_{\frac{1}{2}} := D(A^{\frac{1}{2}}) = H^2(\Omega) \cap H_0^1(\Omega).$$

Let $H_{-\frac{1}{2}}$ *be the dual space of* $H_{\frac{1}{2}}$ *for the pivot space* H.

Set $v = u_t$ *and* $Z(t) = \begin{pmatrix} u(t) \\ v(t) \end{pmatrix}$. *We can then rewrite the problem (7.19)–(7.20) as the abstract first order ordinary differential equation in the Hilbert space* $X = H_{\frac{1}{2}} \times H$

$$\begin{cases} \dfrac{d}{dt}Z(t) & = \mathcal{A}Z(t) + F(Z(t)),\ Z(0) = Z_0, \\ y(t) & = \mathcal{C}Z(t). \end{cases} \tag{7.21}$$

Here the linear operator

$$\mathcal{A} : \mathcal{D}(\mathcal{A}) \subset H_{\frac{1}{2}} \times H \to H_{\frac{1}{2}} \times H$$

is given by

$$\mathcal{A} = \begin{pmatrix} 0 & I \\ -A & -D \end{pmatrix},\ \mathcal{D}(\mathcal{A}) = \mathcal{D}(A) \times \mathcal{D}(A^{\frac{1}{2}}),$$

where the damping operator $D : H_{\frac{1}{2}} \to H$ *defined by* $D = 2\beta A^{\frac{1}{2}}$ *is bounded and positive. Furthermore, we set*

$$C\phi = \phi|_\Gamma \quad \text{for} \quad \phi \in H_{\frac{1}{2}} \quad \text{and} \quad \mathcal{C} = (0, C)$$

and define $F : H_{\frac{1}{2}} \times H \to H_{\frac{1}{2}} \times H$ *by*

$$F\left(\begin{pmatrix} u \\ v \end{pmatrix}\right) = \begin{pmatrix} 0 \\ f\left(\int_\Omega |\nabla u|^2 dx\right) \Delta u \end{pmatrix}.$$

For $z \in H_{\frac{1}{2}}$, *we have*

$$\langle Dz, z\rangle_{H_{-\frac{1}{2}} \times H_{\frac{1}{2}}} = \langle 2\beta A^{\frac{1}{2}} z, z\rangle_H = 2\beta\|z\|_{H_0^1(\Omega)} \geq \frac{2\beta}{c}\|z\|_{L^2(\Gamma)},$$

for some $c > 0$ *by the trace theorem (see e.g. Theorem 2.5.4 in [82]). Hence, the assumptions (A1)-(A3) of [65, Proposition 4.1] are satisfied, and thus the observation operator* $\mathcal{C}$ *is infinite-time admissible for the semigroup generated by* $\mathcal{A}$.

Assuming $f : [0, \infty) \to \mathbb{R}$ *locally Lipschitz and bounded, the mapping* F *is locally Lipschitz continuous on* $H_{\frac{1}{2}} \times H$ *and satisfies the condition of linear growth. Theorem 7.2.7 now implies that the Lebesgue extension of* $\mathcal{C}$ *with respect to the semigroup generated by* $\mathcal{A}$ *is an admissible observation operator for the problem (7.19)-(7.20).*

Example 7.4.2. *Let* Ω *be a bounded open subset of* $\mathbb{R}^N$ *with boundary* $\partial\Omega \in C^4$. *We consider the following semilinear thermo-elastic system*

$$\begin{cases} w_{tt} + \Delta^2 w + \alpha\Delta\theta = f\left(\int_\Omega |\nabla w|^2 dx\right) \Delta w,\ x \in \Omega, t > 0, \\ \theta_t - \Delta\theta + \sigma\theta - \alpha\Delta w_t = 0,\ x \in \Omega, t > 0, \end{cases} \tag{7.22}$$

with the boundary and initial conditions

$$\begin{cases} \theta(t,x) = w(t,x) = \dfrac{\partial w}{\partial \nu}(t,x) = 0, \ x \in \partial\Omega, \ t \geq 0 \\ w(0,x) = w_0(x), \ w_t(0,x) = w_1(x), \ \theta(0,x) = \theta_1(x), \ \ x \in \Omega \end{cases} \tag{7.23}$$

and the output function

$$y(t,x) = -\nabla\theta(t,x), \quad t \geq 0, \ x \in \Omega. \tag{7.24}$$

Here, the coupling parameter α is positive and the constant σ is non negative. Controllability of corresponding linear system of (7.22)–(7.23) with various boundary conditions and controls are well studied, see [11, 43, 74].

We define the linear operators $A_0 = \Delta^2$ and $A_D = -\Delta$ on $L^2(\Omega) \to L^2(\Omega)$ with the domains

$$D(A_0) = H^4(\Omega) \cap H_0^2(\Omega) \qquad \text{and} \qquad D(A_D) = H^2(\Omega) \cap H_0^1(\Omega).$$

It is well known that A_0 and A_D are selfadjoint positive operators and that

$$D(A_0{}^{\frac{1}{2}}) = H_0^2(\Omega) \quad \text{and} \quad D(A_D^{\frac{1}{2}}) = H_0^1(\Omega).$$

We introduce the Hilbert space $\mathbb{H} := D(A_0{}^{\frac{1}{2}}) \times L^2(\Omega) \times L^2(\Omega)$, equipped with its natural inner product. Set $v = w_t$ and

$$z(t) = \begin{pmatrix} w(t) \\ v(t) \\ \theta(t) \end{pmatrix}, \quad z_0 = \begin{pmatrix} w_0 \\ v_0 \\ \theta_0 \end{pmatrix}.$$

The system (7.22)-(7.23) can be rewritten as an abstract semilinear evolution equation in $\mathbb{H}$ of the form

$$z_t = Az + F(z), \ \ z(0) = z_0 \in \mathbb{H},$$

with the output function

$$y(t) = Cz(t),$$

where A is the linear operator defined by

$$A = \begin{pmatrix} 0 & I & 0 \\ -A_0 & 0 & \alpha A_D \\ 0 & -\alpha A_D & -A_D - \sigma I \end{pmatrix}$$

with domain $D(A) = D(A_0) \times D(A_0{}^{\frac{1}{2}}) \times D(A_D)$, and the observation operator $C : D(A) \to Y = 0 \times 0 \times (L^2(\Omega))^N$, $C = (0, 0, -\nabla)$. Further $F : \mathbb{H} \to \mathbb{H}$ is the nonlinear operator given by

$$F\left(\begin{pmatrix} w \\ v \\ \theta \end{pmatrix}\right) = \begin{pmatrix} 0 \\ f\left(\int_\Omega |\nabla w|^2 dx\right) \Delta w \\ 0 \end{pmatrix}.$$

In Proposition 2.1 of [11], it was shown that A *generates a* C_0 *semigroup of contractions on the Hilbert space* $\mathbb{H}$*. Proposition 2.7 of [11] also implies that* C *is admissible with respect to* A*. Finally, in Section 3 of [13] the pair* (A, C) *was proved to be exactly observable. If we assume that* $f : [0, +\infty) \to \mathbb{R}$ *is bounded and locally Lipschitz continuous, then* F *is locally Lipschitz on* $\mathbb{H}$ *and satisfies assumption (G). Moreover,* $F(0) = 0$*. Using Theorem 7.3.4 we deduce that the problem (7.22)–(7.24) is locally exactly observable at* $w_0 = \theta_0 = 0$*.*

Bibliography

[1] P. Acquistapace, Evolution operators and strong solutions of abstract linear parabolic equations, *Differential Integral Equations* **1** (1988), 433-457.

[2] P. Acquistapace, B. Terreni, A unified approach to abstract linear nonautonomous parabolic equations, *Rend. Sem. Mat. Univ. Padova* **78** (1987), 47-107.

[3] P. Acquistapace, F. Flandoli, B. Terreni, Initial boundary value problems and optimal control for nonautonomous parabolic systems, *SIAM J. Control Optim.* **29** (1991), 89-118.

[4] R. A. Adams, Sobolev spaces,*Pure and Applied Mathematics* **65**, Academic Press [A subsidiary of Harcourt Brace Jovanovich, Publishers], New York-London, 1975.

[5] H. Amann, Linear and quasilinear parabolic problems, Birkhäuser, Berlin, 1995.

[6] L. Amerio and G. Prouse, Almost-periodic functions and functional equations, Van Nostrand Reinhold Company, 1971.

[7] F. Ammar-Khodja, A. Bader and A. Benabdallah, Dynamic stabilization of systems via decoupling techniques, *ESAIM Control Optim. Calc. Var.* **4** (1999), 577-593.

[8] B. Amir and L. Maniar, Existence and some asymptotic behaviors of solutions to semilinear Cauchy problems with non dense domain via extrapolation spaces, *Rend. Circ. Mat. Palermo* **4** (2000), no. 3, 481-496.

[9] B. Amir and L. Maniar, Composition of pseudo almost periodic functions and Cauchy problems for Hille-Yosida operators, *Ann. Math. Blaise Pascal* **6** (1999), 1-11.

[10] G. Avalos and I. Lasiecka, Boundary controllability of thermoelastic plates with free boundary conditions, *SIAM J. Control Optim.* **38** (1998), 337-383.

[11] G. Avalos, Exact controllability of thermoelastic system with control in the thermal component only, *Differential Integral Equations* **13** (2000), 613-630.

[12] W. Arendt, C. J. K. Batty, M. Hieber and F. Neubrander, Vector valued Laplace transforms and Cauchy problems, Birkhäuser-Verlag, 2001.

[13] K. Balachandran and J. P. Dauer, Controllability of nonlinear systems in Banach spaces: a survey, *J. Optim. Theory Appl.* **115** (2002), 7-28.

[14] M. Baroun, S. Boulite,T. Diagana, L. Maniar, Almost periodic solutions to some semilinear non-autonomous thermoelastic plate equations, *J. Math. Anal. Appl.* **349** (2009), 74-84.

[15] M. Baroun, S. Boulite, G. M. N'Guérékata, L. Maniar, Almost automorphy of semilinear parabolic evolution equations, *Electron. J. Differential Equations* (2008), 1-9.

[16] M. Baroun and B. Jacob, Admissibility and observability of observation operators for semilinear problems, *Integral Equations Operator Theory* **64** (2009), 1-20.

[17] M. Baroun, B. Jacob, L. Maniar and R. Schnaubelt, Semilinear observation systems, submitted.

[18] M. Baroun, L. Maniar, G.M N'Guérékata, Almost periodic and almost automorphic solutions to semilinear parabolic boundary differential equations, *Nonlinear Anal.* **69** (2008), 2114-2124.

[19] M. Baroun, L. Maniar, R. Schnaubelt, Almost periodicity of parabolic evolution equations with inhomogeneous boundary values, *Integral Equations Operator Theory* **65** (2009), 169-193.

[20] C.J.K. Batty, R. Chill, Approximation and asymptotic behaviour of evolution families, *Differential Integral Equations* **15** (2002), 477-512.

[21] C. J. K. Batty, W. Hutter and F. Räbiger, Almost periodicity of mild solutions of inhomogeneous Cauchy problems, *J. Differential Equations* **156** (1999), 309-327.

[22] A. Bátkai, L. Maniar and A. Rhandi, Regularity properties of perturbed Hille-Yosida operators and retarded differential equations, *Semigroup Forum* **64** (2002), 55-70.

[23] B. Basit, H. Güunzler, Difference property for perturbations of vector-valued Levitan almost periodic functions and their analogs, *Russian J. Math. Physics*, **12** (2005), 424-438.

[24] A. Benabdallah, M. G. Naso, Null controllability of a thermoelastic plate, *Abstr. Appl. Anal.* **7** (2002), 585-599.

[25] A. S., Besicovitch, Almost-periodic functions, Dover Publications, 1954.

[26] S. Bochner, Abstrakte fastperiodische Funktionen, *Acta Math.* **61** (1933), 149-184.

[27] S. Bochner, A new approach to almost periodicity, *Proc. Nat. Acad. Sci. USA* **48** (1962), 2039-2043.

[28] H. Bohr, Zur Theorie der Fastperiodischen Funktionen. (German) II. Zusammenhang der fastperiodischen Funktionen mit Funktionen von unendlich vielen Variabeln; gleichmässige Approximation durch trigonometrische Summen, *Acta Math.* **46** (1925), 101-214.

[29] S. Boulite, S. Hadd, L. Maniar, Critical spectrum and stability for population equation with diffusion in an bounded domains, *Discrete Contin. Dyn. Syst. Ser.* **B 5** (2005), 265-276.

[30] S. Boulite, A. Idrissi, L. Maniar, Robustness of controllability under some unbounded perturbations, *J. Math. Anal. Appl.* **304** (2005), no. 1, 409-421.

[31] S. Boulite, L. Maniar and M. Moussi, Wellposedness and asymptotic behaviour of non-autonomous boundary Cauchy problems, *Forum Math.* **18** (2006), 611-638.

[32] S. Boulite, L. Maniar and G. M. N'Guérékata, Almost automorphic solutions for hyperbolic semilinear evolution equations, *Semigroup Forum* **71** (2005), 231-240.

[33] S. Boulite, L. Maniar, G. M. N'Guérékata, Almost automorphic solutions for semilinear boundary differential equation, *Proc. Amer. Math. Soc.* **134** (2006), 3613-3624.

[34] C. Chicone, Y. Latushkin, Evolution semigroups in dynamical dystems and differential dquations, *Mathematical Surveys and Monographs* **70**, American Math. Society, Providence, 1999.

[35] W. A. Coppel, Dichotomies in stability theory, *Lecture Notes in Mathematics* **629**, Springer-Verlag, 1978.

[36] C. Corduneanu, Almost periodic functions, *Interscience Publishers*, New York-London-Sydney, 1968.

[37] J.-M. Coron, Control and nonlinearity, *American Math. Society*, Providence, 2007.

[38] R. F. Curtain and A. J. Pritchard, Infinite dimensional linear theory, *Lecture Notes in Control and Information Sciences* **8**, Springer-Verlag, 1978.

[39] R. F. Curtain and G. Weiss, Well posedness of triples of operators (in the sense of linear systems theory), *International Series of Numerical Mathematics 91 Birkhäuser-Verlag* (1989), 41-58.

[40] C. M. Dafermos, On the existence and the asymptotic stability of solutions to the equations of linear thermoelasticity, *Arch. Rational Mech. Anal.* **29** (1968), 241-271.

[41] G. Da Prato and P. Grisvard, On extrapolation spaces,*Rend. Accad. Naz. Lincei.* **72** (1982), 330-332.

[42] Luiz Augusto F. de Oliveira, On reaction-diffusion systems, *Electron. J. Differential Equations* **24** (1998),1-10.

[43] L. de Teresa and E. Zuazua, Controllability for the linear system of thermoelastic plates, *Advances in Differential Equations* **1** (1996), 369-402.

[44] W. Desch, J. Milota and Schappacher, Least square control problems in nonreflexive spaces, *Semigroup Forum* **62** (2001), 337-357.

[45] W. Desch and W. Schappacher, Some generation results for pertubed semigroup, Semigroup Theory and Applications (Clémnet, Invernizzi, Mitidieri, and Vrabie, eds.), *Lect. Notes Pure Appl. Math.* **116** (1989), 125-152.

[46] W. Desch, W. Schappacher and K.P. Zhang, Semilinear evolution equations, *Houston J. Math.* **15** (1989), 527-552.

[47] T. Diagana, G. N'Guérékata and N. Van Minh, Almost automorphic solutions of evolution equations, *Proc. Amer. Math. Soc.* **132** (2004), 3289-3298.

[48] T. Diagana, G.M. NGuerekata, Almost automorphic solutions to some classes of partial evolution equations, *Appl. Math. Lett.* **20** (2007), 462-466.

[49] H.S. Ding, J. Liang, T.J. Xiao, Asymptotically almost automorphic solutions for some integrodifferential equations with nonlocal initial conditions, *J. Math. Anal. Appl.* **338** (2008), 141-151.

[50] D. Di Giorgio and A. Lunardi, On Fredholm properties of $\mathcal{L}u = u' - A(t)u$ for paths of sectorial operators, *Proc. Roy. Soc. Edinburgh Sect. A* **135** (2005), 39-59.

[51] D. Di Giorgio, A. Lunardi and R. Schnaubelt, Fredholm properties of abstract parabolic operators in L^p spaces on the real line, *Proc. London Math. Soc.* **91** (2005), 703-737.

[52] K. J. Engel and R. Nagel, One parameter semigroups for linear evolution equations, Graduate Texts in Mathematics, Springer Verlag , New York, 2000.

[53] J. Favard, Sur les équations différentielles einéaires á coefficients presque-périodiques, *Acta Math.* **51** (1927), no. 1, 31-81.

[54] J. Favard, Lecons sur les fonctions presque-périodiques, Gauthier-Villars, Paris, 1933.

[55] A. M. Fink, Almost periodic differential equations, Springer-Verlag, Berlin, Heidelberg, New York, 1974.

[56] J. A. Goldstein, G. M. N'Guérékata, Almost automorphic solutions of semilinear evolution equations, *Proc. Amer. Math. Soc.* **133** (2005), 2401-2408.

[57] G. Greiner: Perturbing the boundary conditions of a generator,*Houston J. Math.* **13** (1987), 213-229.

[58] G. Gühring and F. Räbiger, Asymptotic properties of mild solutions of nonautonomous evolution equations with applications to retarded differential equations, *Abstr. Appl. Anal.* **4** (1999), 169-194.

[59] G. Gühring, F. Räbiger and R. Schnaubelt, A characteristic equation for nonautonomous partial functional differential equations, *J. Differential Equations* **181** (2002), 439-462.

[60] S. Hadd, Unbounded perturbations of C_0-semigroups on Banach spaces and applications, *Semigroup Forum* **70** (2005), 451-465.

[61] D. Henry, Geometric Theory of Semilinear Parabolic Equations, Lecture Notes in Mathematics, Springer-Verlag, Berlin-New York, 1981.

[62] E. Hernández, M. Pelicer and J. dos Santos, Asymptotically almost periodic and almost periodic solutions for a class of evolution equations, *Electron. J. Differential Equations* (2004), 1-15.

[63] Y. Hino, S. Murakami, Almost automorphic solutions for abstract functional differential equations, *J. Math. Math. Analysis and Appl.* **286** (2003), 741-752.

[64] W. Hutter und F. Räbiger, Spectral mapping theorems for evolution semigroups on spaces of almost periodic functions, *Quaest. Math.* **26** (2003), 191-211.

[65] B. Jacob, K. Morris and C. Trunk, Minimum-phase infinite-dimensional second-order systems,*IEEE Trans. Automat. Control* **52** (2007), 1654-1665.

[66] B. Jacob, J.R. Partington, Admissibility of control and observation operators for semigroups: a survey, Current Trends in Operator Theory and its Applications, Proceedings of IWOTA 2002, 199-221, *Operator Theory: Advances and Applications* bf 149, Birkhäuser, Basel, 2004.

[67] B. Jacob, H. Zwart, Exact observability of diagonal systems with a one-dimensional output operator, Infinite-dimensional systems theory and operator theory (Perpignan, 2000), *Int. J. Appl. Math. Comput. Sci.* **11** (2001), 1277-1283.

[68] B. Jacob, H. Zwart, Exact observability of diagonal systems with a finite-dimensional output operator, *Systems Control Lett.* **43** (2001), 101-109.

[69] R. A. Johnson, On a Floquet theory for almost periodic two-dimensional linear systems, *J. Diff. Eqs.* **37** (1980).

[70] R. A. Johnson, A linear almost periodic equation with an almost automorphic solution, *Proc. Amer. Math. Soc.* **82** (1981), 199-205.

[71] R. A. Johnson, Bounded Solutions of Scalar Almost periodic linear equations, *Illinois J. Math.* **25** (1981), 632-643.

[72] R. A. Johnson, A review of recent work on almost periodic differential and difference operators, *Acta Appl. Math.* **1** (1983), 241-261.

[73] J. Klamka, Controllability of dynamical systems, Kluwer Academic, Dordrecht, 1991.

[74] J. Lagnese, The reachability problem for thermoelastic plates, *Arch. Rational Mech.* **112** (1990), 223-267.

[75] I. Lasiecka and R. Triggiani, Control theory for partial differential equations: continuous and approximation theories, Volume I, Cambridge University Press, Cambridge, 2000.

[76] Y. Latushkin, A. Pogan and R. Schnaubelt, Dichotomy and Fredholm properties of evolution equations, *J. Operator Theory* **58** (2007), 387-414.

[77] G. Lebeau and E. Zuazua, Decay rates for the three-dimensional linear system of thermoelasticity, *Arch. Rational Mech. Anal.* **148** (1999), 179-231.

[78] H. Leiva, Z. Sivoli, Existence, stability and smoothness of a bounded solution for nonlinear time-varying thermoelastic plate equations, *J. Math. Anal. Appl.* **285** (2003), 191-211.

[79] B. M. Levitan, A new generalization of the almost periodic functions of H. Bohr, *(Russian) Comm. Inst. Sci. Math. Méc. Univ. Kharkoff [Zapiski Inst. Mat. Mech.] (4)* **15** (1938), 334.

[80] B. Levitan and V. Zhikov, Almost periodic functions and differential equations, Cambridge University Press, 1982.

[81] J.-L. Lions, Contrôlabilité exacte, perturbations et stabilisation de systèmes distribués. Tome 1, Recherches en Mathématiques Appliquées, Vol. 8, Masson, Paris, 1988.

[82] J.-L. Lions and E. Magenes, Non-homogeneous boundary value problems and applications, Vol. I, Springer-Verlag, Berlin, 1972.

[83] A. Lunardi, Analytic semigroups and optimal regularity in parabolic problems, Birkhäuser, Basel-Boston-Berlin, 1995.

[84] K. G. Magnusson, Observability of nonlinear systems, *IMA J. Math. Control Information* **1** (1984), 339-358.

[85] L. Maniar and A. Rhandi, Inhomogeneous retarded differential equation in infinite dimensional Banach space via extrapolation spaces,*Rend. Circ. Mat. Palermo* **47** (1998), 331-346.

[86] L. Maniar, R. Schnaubelt, Almost periodicity of inhomogeneous parabolic evolution equations, *Lecture Notes in Pure and Appl. Math.* **234**, Dekker, New York (2003), 299-318.

[87] L. Maniar and R. Schnaubelt, The Fredholm alternative for parabolic evolution equations with inhomogeneous boundary conditions, *J. Differential Equations* **235** (2007), 308-339.

[88] L. Maniar and R. Schnaubelt, Robustness of Fredholm properties of parabolic evolution equations under boundary perturbations, *J. London Math. Soc.* **77** (2008), 558-580.

[89] R. Nagel, One-parameter semigroups of positive operators. Lecture Notes in Mathematics **1184**, Springer-Verlag, 1986.

[90] R. Nagel and E. Sinestrari, Inhomogeneous Volterra integrodifferential equations for Hille-Yosida operators, Marcel Dekker, *Lecture Notes Pure Appl. Math.* **150**, Dekker, New York, (1994).

[91] M. G. Naso, A. Benabdallah, Thermoelastic plate with thermal interior control, Mathematical models and methods for smart materials (Cortona, 2001), 247-250, *Ser. Adv. Math. Appl. Sci.* **62**, World Sci. Publ., River Edge, NJ, (2002).

[92] Neerven, J. M. A. M. van: The asymptotic behaviour of semigroups of linear operators. Birkhäuser, Basel, 1996.

[93] G. M. N'Guérékata, Almost automorphic functions and almost Periodic functions in abstract spaces, Kluwer Academic/Plenum Publishers, New York-London-Moscow, 2001.

[94] G. M. N'Guérékata, Existence and uniqueness of almost automorphic mild solution to some semilinear abstract differential equations, Semigroup Forum **69** (2004), 80-86.

[95] G. M. N'Guérékata, Topics in almost automorphy, Springer, New York, 2005.

[96] G.M. N'Guérékata, Comments on almost automorphic and almost periodic functions in Banach spaces, *Far East J. Math. Sci. (FJMS)* **17** (2005), 337-344.

[97] A. Pankov, Bounded and almost periodic solutions of nonlinear operator differential equations, Kluwer Academic Publishers, 1985.

[98] J.R. Partington, S. Pott, Admissibility and exact observability of observation operators for semigroups, *Irish Math. Soc. Bull.* **55** (2005), 19-39.

[99] A. Pazy, Semigroups of linear operators and applications to partial differential equations, Springer-Verlag, Berlin, 1983.

[100] J. Prüss, Evolutionary integral equations and applications, Birkhäuser, 1993.

[101] K. Ramdani, T. Takahashi, G. Tenenbaum, M. Tucsnak, A spectral approach for the exact observability of infinite-dimensional systems with skew-adjoint generator, *J. Funct. Anal.* **226** (2005), 193-229.

[102] W.M. Ruess and W.H. Summers, Weak almost periodicity and the strong ergodic limit theorem for periodic evolution systems, *J. Funct. Anal.* **94** (1990), 177-195.

[103] D. L. Russell, G. Weiss, A general necessary condition for exact observability, *SIAM J. Control Optimization* **32** (1994), 1-23.

[104] D. Salamon, Infinite-dimensional linear systems with unbounded control and observation: a functional analytic approach, *Trans. Amer. Math. Soc.* **300** (1987), 383-431.

[105] B. Sandstede, Stability of travelling waves,*Handbook of dynamical systems* **2**, 9831055, North-Holland, Amsterdam, 2002.

[106] R. Schnaubelt, Sufficient conditions for exponential stability and dichotomy of evolution equations, *Forum Math.* **11** (1999), 543-566.

[107] R. Schnaubelt, Asymptotically autonomous parabolic evolution equations, *J. Evol. Equ.* **1** (2001), 19-37.

[108] R. Schnaubelt, Feedbacks for nonautonomous regular linear systems, *SIAM J. Control Optim.* **41** (2002), 1141-1165.

[109] R. Schnaubelt, Asymptotic behaviour of parabolic nonautonomous evolution equations, Functional Analytic Methods for Evolution Equations, *Lecture Notes in Mathematics* **1855**, Springer-Verlag, Berlin, 2004.

[110] D. Sevicovic, Existence and limiting behaviour for damped nonlinear evolution equations with nonlocal term, *Comment. Math. Univ. Carolinae* bf 31 (1990), 283-293.

[111] W. Shen and Y. Yi, Almost automorphic and almsot periodic dynamics in skew-product semiflows, *Memoirs Amer. Math. Soc.* **647** (1998), no. 647.

[112] H. Triebel, Interpolation theory, function spaces, differential operators, North-Holland, Amsterdam, 1978.

[113] W. A. Veech, Almost Automorphic Functions, Thesis, Princetoni, March 1963.

[114] W. A. Veech, Almost automorphic functions, *Proceedings of the National Academy of Sciences, U.S.A.* **49** (1963), 462-464.

[115] W.A. Veech, Almost automorphic functions on groups, *Amer. J. Math.* **87** (1965), 719-751.

[116] Q-P. Vũ, Stability and almost periodicity of trajectories of periodic processes, *J. Differential Equations* **115** (1995), 402-415.

[117] G.F. Webb, Theory of nonlinear age-dependent population dynamics, Marcel Dekker, 1985.

[118] G. Weiss, Admissible observation operators for linear semigroups, *Israel J. Math.* **65** (1989), 17-43.

[119] G. Weiss, Regular linear systems with feedback, *Math. Control Signals Systems* **7** (1994), 23-57.

[120] G. Weiss, Transfer functions of regular linear systems. Part I: Characterization of regularity, *Trans. Amer. Math. Soc.* **342** (1994), 827-854.

[121] J. Wu, Theory and applications of partial functional differential equations, *App. Math. Sc.* **119**, Springer-Verlag, New York, 1996.

[122] G. Q. Xu, C. Liu, S. P. Yung, Necessary conditions for the exact observability of systems on Hilbert spaces, *Systems Control Lett.* **57** (2008), 222-227.

[123] A. Yagi, Parabolic equations in which the coefficients are generators of infinitely differentiable semigroups II, *Funkcial. Ekvac.* **33** (1990), 139-150.

[124] A. Yagi, Abstract quasilinear evolution equations of parabolic type in Banach spaces, Boll. Un. Mat. Ital. B (7) **5** (1991), 341-368.

[125] T. Yoshizawa, Stability theory and the existence of periodic solutions and almost periodic solutions, Springer-Verlag, New York, Heidelberg, Berlin, 1975.

[126] K. Yosida, Functional analysis, volume 123 of Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Springer-Verlag, Berlin, 1980.

[127] S. Zaidman, Almost periodic functions in abstract spaces, Boston, London, Melbourne, 1985.

[128] M. Zaki, Almost automorphic solutions of certain abstract differential equations, *Ann. Mat. Pura Appl.* **101** (1974), 91-114.

[129] E. Zuazua, Exact controllability for the semilinear wave equation, *J. Math. Pures Appl.* **69** (1990), 1-31.