Conjugation on varieties of nilpotent matrices

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For my mum

Contents

Introduction

The study of algebraic group actions on affine varieties, especially the "vertical" study of orbits and their closures, and the "horizontal" study of parametric families of orbits and quotients, are a common topic in algebraic Lie theory.

A well-known example is the study of the adjoint action of a reductive algebraic group on its Lie algebra and numerous variants thereof, in particular the conjugacy classes of complex (nilpotent) square matrices.

In 1870, the classification of the orbits by so-called Jordan normal forms was described by M. Jordan [Jordan, 1989, Jordan, 1871]. Their closures were described by M. Gerstenhaber [Gerstenhaber, 1959] and W. Hesselink [Hesselink, 1976] in the second half of the twentieth century in terms of partitions and visualized by combinatorial objects named Young Diagrams.

Algebraic group actions of reductive groups have particularly been discussed elaborately in connection with orbit spaces and more generally algebraic quotients, even though their application to concrete examples is far from being trivial. In case of a non-reductive group, even most of these results fail to hold true immediately.

For example, Hilbert's theorem [Hilbert, 1890] yields that for reductive groups, the invariant ring is finitely generated; and a criterion for algebraic quotients is valid [Kraft, 1984]. In 1958, though, M. Nagata [Nagata, 1960, Nagata, 1959] constructed a counterexample of a not finitely generated invariant ring corresponding to a non-reductive algebraic group action, which answered Hilbert's fourteenth problem in the negative.

One exception are algebraic actions of unipotent subgroups that are induced by reductive groups, since the corresponding invariant ring is always finitely generated [Kraft, 1984].

Our main attention in this work is turned towards algebraic non-reductive group actions that are induced by the conjugation action of the general linear group GL_n over $\mathbf{C}$. For example, the standard parabolic subgroups P (and, therefore, the Borel subgroup B) and the unipotent subgroup U of GL_n are not reductive. It suggests itself to consider their action on the variety $\mathcal{N}_n^{(x)}$ of x-nilpotent matrices of square size n via conjugation. We discuss this setup in detail and, thereby, generalize certain known results.

The examination of a group action of P on $\mathcal{N}_n^{(x)}$ can be refined if we consider the P-action on a single nilpotent GL_n-orbit $\mathcal{O}$. This setup generalizes to arbitrary reductive groups G, where the classification of the orbits is equally interesting. Our considerations are, however, restricted to the group GL_n.

Recent studies

A recent development in this field is A. Melnikov's study of the B-action on the variety of upper-triangular 2-nilpotent matrices via conjugation [Melnikov, 2000, Melnikov, 2006, Melnikov, 2007] motivated by Springer Theory. The detailed description of the orbits and their closures is given in terms of so-called link patterns; these are combinatorial objects visualizing the set of involutions in the symmetric group S_n.

The Ph.D. thesis of B. D. Rothbach yields a description of the B-orbits and equations for their closures in the variety of 2-nilpotent matrices. There seems to be a desingularization of the B-orbit closures as well. However, this work was not available to the author of the present thesis, except for parts of it (see [Rothbach, 2009]).

In her Bachelor thesis [Halbach, 2009], B. Halbach describes the B-action on $\mathcal{N}_3^{(3)}$ in all detail. The B-orbits as well as their closures, their minimal degenerations and their singularities are explicity given and a generic normal form is introduced. She generalizes the latter, obtaining a large set of pairwise non-B-conjugate matrices. These yield a generic normal form in the nilpotent cone $\mathcal{N} := \mathcal{N}_n^{(n)}$ for arbitrary n which is proven in [Boos and Reineke, 2011].

Another recent outcome is L. Hille's and G. Röhrle's study of an arbitrary parabolic action on the variety $\mathfrak{n}$ of upper-triangular nilpotent matrices [Hille and Röhrle, 1999]. They obtain a criterion which determines whether the group action admits a finite or infinite number of orbits.

Content of this work

We translate the group action of P to a certain group action in the representation theory of finite dimensional algebras via an associated fibre product. In more detail, we "reduce" the classification of orbits of the action to the knowledge of certain isomorphism classes of representations of a quiver Q given by a linearly oriented quiver of Dynkin type A_n with a loop at the sink and an admissible ideal I given by just one relation. The quiver Q and the ideal I depend on the parabolic subgroup P and on the nilpotency degree x.

Although in general, the classification of these isomorphism classes is far from well-known, there are several cases in which the algebra KQ/I is representation-finite and the representations can be classified using the decomposition theorem of W. Krull, R. Remak and O. Schmidt.

Our first aim is to classify those cases in which a finite group action arises. The most obvious case is an arbitrary parabolic action on the variety of 2-nilpotent matrices; the covering quiver of Q is of Dynkin type A_{2n}, then.

We classify the orbits in terms of "(enhanced) oriented link patterns", a natural generalization of A. Melnikov's link patterns, and provide the concrete structure of the orbit closures: By translating a description of orbit closures given by G. Zwara [Zwara, 1999], a natural generalization of the description A. Melnikov obtains in [Melnikov, 2006] is deduced.

By calculating the dimension of the stabilizer of a 2-nilpotent matrix (with respect to the chosen parabolic subgroup), we are able to compute the dimension of every orbit (closure); the description of the open orbit follows naturally.

In case of the B-action, we give an explicit description of the minimal degenerations. Therefore, we describe all minimal, disjoint degenerations by using several results of K. Bongartz [Bongartz, 1994] and generalize them to arbitrary minimal degenerations afterwards. In this way, the precise degeneration diagram is obtained; this yields a concrete algorithm to derive a set-theoretic description of the orbit closure of a given matrix by turning around the arrows in the corresponding oriented link pattern. We prove that each minimal degeneration is of codimension 1. Since the B-variety $\mathcal{N}_n^{(2)}$ is spherical, this result can be obtained from the theory of sperical varieties as introduced in [Brion, 1989] as well.

In order to state a criterion for the group action of P on $\mathcal{N}_n^{(x)}$ to be finite, we discuss the action of a maximal parabolic subgroup on the variety of 3-nilpotent matrices. This particular action is finite and we classify the orbits and their closures by enhanced oriented link patterns as well. We give an explicit description of the open orbit, which depends on the maximal parabolic subgroup, as well as a detailed example which illustrates the general considerations.

To get an overview of all finite cases, we consider fixed matrix sizes and prove the following theorem by using general techniques from linear algebra.

Theorem 4.2.1
The P-action on $\mathcal{N}_n^{(x)}$ is finite if and only if $x \leq 2$, or P is maximal and $x = 3$.

Every quiver Q not considered up to this point is of wild type and we arrive at a classification problem of wild type which yields explicit 2-parameter families of non-P-conjugate matrices for certain parabolic subgroups P.

We consider "the most difficult case", namely an arbitrary parabolic action on the nilpotent cone $\mathcal{N}$, and initiate a study of the generic classification by specifying a generic normal form. Following [Boos and Reineke, 2011], we construct a large class of determinantal semi-invariants $f^{\mathcal{P}}$; these are used to prove the following lemma using results of A. Schofield and M. van den Bergh [Schofield and van den Bergh, 2001]:

Lemma 5.3.1
The B-semi-invariant ring is generated by determinantal semi-invariants $f^{\mathcal{P}}$.

By observing that all B-semi-invariants are U-invariant functions, we obtain an (infinite) set of generators of the U-invariant ring. We modify a well-known quotient criterion for reductive group actions (see [Kraft, 1984]) and are, thereby, able to state a quotient criterion for unipotent group actions. Algebraic quotients are provided explicitly in the cases $n = 2$ and $n = 3$ by making use of this criterion.

In the general case, we map $\mathcal{N} /\!\!/ U$ to an explicitly described toric variety X, such that $\mathcal{N} /\!\!/ U$ is generically an affine space fibration with fibres isomorphic to $\mathbf{A}^D$.

In addition to calculating the variety X in all detail, we show how the U-orbits are separated generically by certain invariant functions and discuss the interrelation of the varieties $\mathcal{N}/\!\!/U$ and $\mathbf{A}^D \times X$.

If $n = 2$ or $n = 3$, the desription of the U-invariant ring can be used to verify a GIT-quotient of the nilpotent cone by B. In the general case, we separate the B-orbits generically by semi-invariants of the same weight.

We translate this setup into the language of moduli spaces of representations of finite-dimensional algebras and, thereby, initiate further studies.

Structure of this work

In *chapter 1*, we give a brief summary of methods summing up the fundamentals we make use of later on.

We begin by reproducing basic knowledge about algebraic geometry in *section 1.1*, more precisely about algebraic group actions in *subsection 1.1.1* and, thereby, discussing (semi-) invariants and quotients corresponding to them in *subsection 1.1.2* and *subsection 1.1.3*. A brief recapitulation of the concept of an affine toric variety is included in *subsection 1.1.4*.

In *section 1.2*, we introduce basic notions of the representation theory of finite-dimensional algebras using the quiver approach. Basic principles of covering theory are recapitulated in *subsection 1.2.1* (see [Bongartz and Gabriel, 8182]). The notion of tame and wild algebras as well as the classification of tame and wild path algebras by (extended) Dynkin quivers is sketched in *subsection 1.2.2*. In *subsection 1.2.3*, we state general facts about degenerations in the sense of G. Zwara [Zwara, 1999, Zwara, 2000] .

In *chapter 2*, we provide an introduction of the concrete setup of this work, that is, introducing the notations and certain combinatorial objects, repeating basic facts and recent results and realizing an important translation.

Generalizations of A. Melnikov's link patterns, namely "oriented link patterns", "enhanced oriented link patterns" and "labelled oriented link patterns" are considered in *section 2.1* as they represent the combinatorial objects which solve several upcoming classification problems.

We recapitulate known work on the subject in *section 2.2*, starting in *subsection 2.2.1* with the classification of GL_n-orbits in varieties of nilpotent matrices, first given by M. Jordan and M. Gerstenhaber. Recent results of A. Melnikov are considered in *subsection 2.2.2* where the notion of a link pattern comes up. Subsequently, we examine results on parabolic group actions given by L. Hille and G. Röhrle in *subsection 2.2.3*.

In *section 2.3*, we show how the problem of studying the group action of P on $\mathcal{N}_n^{(x)}$ can be translated to a setup in the representation theory of a finite-dimensional algebra.

The conjugation action of an arbitrary parabolic subgroup P on the variety of 2-nilpotent matrices is discussed in *chapter 3*. In addition, the action of the unipotent subgroup is considered.

We start by classifying the orbits in *section 3.1*. More precisely, we discuss and classify the parabolic orbits in terms of enhanced oriented link patterns in *subsection 3.1.1* and deduce representing labelled oriented link patterns of the unipotent orbits in *subsection 3.1.2*. The section ends with two examples in subsection 3.1.3 where in the cases $n = 3$ and $n = 4$ every parabolic action is explicitly described.

In order to depict the orbit closures, we calculate the dimensions of homomorphism spaces and spaces of extensions of certain representations in *section 3.2*.

By making use of this knowledge, we describe the orbit closures concerning the Borel action in *section 3.3*. We start by describing the minimal, disjoint degenerations of representations in *subsection 3.3.1*, which leads to a classification of all minimal degenerations in *subsection 3.3.2*, divided into minimal degenerations obtained from an indecomposable representation and minimal degenerations obtained from extensions. The examination of Borel orbits in *subsection 3.3.3* is concluded by giving an overview of the dimensions of orbits and by calculating the open orbit. Minimal singularities are briefly examined in *subsection 3.3.4*.

In *section 3.4*, we have a closer look at arbitrary parabolic conjugation and describe the orbit closures as well. The calculation of the minimal, disjoint degenerations of certain representations in *subsection 3.4.1* leads to a description of most minimal degenerations. We include the dimensions of the orbits and the description of the open orbit in *subsection 3.4.2*.

Parabolic actions on varieties of x-nilpotent matrices for $2 < x < n$ are considered in *chapter 4*. The considerations are restricted to the examination (and classification) of all finite cases that arise.

The first finite case, discussed in *section 4.1*, is the action of an arbitrary maximal parabolic subgroup P on the variety of 3-nilpotent matrices. The orbits are classified in *subsection 4.1.1* and we give an algorithm on how to calculate the orbit closures in case the parabolic is fixed in *subsection 4.1.2*. After calculating the dimensions of the orbits as well as the open orbit for every parabolic action in *subsection 4.1.3*, we end the section by examining the action of the parabolic subgroup of block sizes $(2, 2)$ on the variety of 3-nilpotent 4×4-matrices in *subsection 4.1.4*.

By observing that every finite action has been examined up to this point, we obtain a complete list of the finite actions in *section 4.2*.

In *section 4.3*, the corresponding wild algebras are considered, that is, in *subsection 4.3.1*, we prove that every non-finite algebra is of wild type before giving explicit 2-parameter families of pairwise non-P-conjugate matrices for certain parabolic subgroups P in *subsection 4.3.2*.

Since every finite action has been classified, we consider a generic approach in *chapter 5* and, therefore, classify the parabolic orbits in an open subset of the nilpotent cone.

A generic normal form for an arbitrary parabolic action is provided in *section 5.1*, which can be used to describe a generic normal form for the unipotent action.

In *section 5.2*, following [Boos and Reineke, 2011], we introduce a large class of B-semi-invariants, thereby defining a large class of unipotent invariants as well.

In case of the Borel action, *section 5.3* shows that the semi-invariant ring is in fact generated by these semi-invariants.

As it is of great interest to describe the U-invariant ring of the nilpotent cone, we start to examine it in *chapter 6*.

In *section 6.1*, by proving a quotient criterion for U-actions, which is directly translated from the reductive setup, we are able to describe the invariant rings for $n = 2$ and $n = 3$ in *subsection 6.1.1* explicitly.

These cases yield the existence of a subring of so-called toric invariants in the U-invariant ring which we discuss in *section 6.2*. More precisely, *subsection 6.2.1* proves that the toric invariants are generated by a finite set containing so-called sum-free toric invariants. In *subsection 6.2.2*, a general description of toric invariants is obtained combinatorially.

We show how the U-orbits can generically be separated by certain invariants in *section 6.3*.

Section 6.4 provides an explicit description of the toric variety X which is induced by the toric invariants. We discuss two toric operations in *subsection 6.4.1* and show how they are related. By translating these operations, we obtain an explicit description of all sum-free toric invariants in *subsection 6.4.2*. *Subsection 6.4.3* discusses the results we obtained so far about the interrelation of the varieties $\mathcal{N}/\!\!/U$ and $\mathbf{A}^D \times X$.

We end the chapter by working through the case $n = 4$ in *section 6.5*.

Since the description of the B-semi-invariant ring of a certain weight is a self-evident aim, we start its discussion in *chapter 7*.

By application of the results on U-invariants in *section 7.1*, the semi-invariant rings for $n = 2$ and $n = 3$ are described explicitly for certain weights.

In *section 7.2*, we then show how the B-orbits can be separated generically by semi-invariants of the same weight.

We initiate the study of moduli spaces in *section 7.3* and introduce a particular stability which comes up naturally by the generic separation.

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1 Theoretical background

In the following, we will fix some notation and give a brief overview about the theories that will be made use of. The reader is referred to [Borel, 1991] for more details on algebraic group actions; questions about algebraic geometry in general, for example about singularities, are answered in [Kraft, 1984] and [Hartshorne, 1977]. Quotients of algebraic group actions, (semi-) invariants and everything closely related are described in [Mukai, 2003] and background information concerning the representation theory of finite-dimensional algebras can, for example, be found in [Assem et al., 2006], [Auslander et al., 1997] and [Ringel, 1984].

1.1 Methods from Algebraic geometry and Invariant theory

Let K be an algebraically closed field of characteristic $\mathrm{char}\, K = 0$; we denote by e_i the i-th coordinate vector of K^n.

If not stated differently, we consider a variety X to be an irreducible quasi-projective variety, we denote its structure sheaf by $\mathcal{O}_X$ and its Krull dimension by $\dim X$. For each point $x \in X$, we denote by $\dim_x(X) := \dim \mathcal{O}_{X,x}$ the local dimension at x, that is, the maximal dimension of those irreducible components containing x. For every closed subvariety X' of X, we define by $\mathrm{codim}_X(X') := \dim X - \dim X'$ the codimension of X' in X. Given an irreducible affine variety X, we denote its associated coordinate ring by $\mathcal{O}_X(X) =: K[X]$ and by $K(X)$ the field of fractions.

An integral domain R with field of fractions $\mathbf{K}$ is called integrally closed if every element of $\mathbf{K}$ which is a root of a monic polynomial in $R[x]$ is an element of R. We obtain the notion of a normal variety.

Definition 1.1.1. *The variety X is called normal if every local ring $\mathcal{O}_{X,x}$ is integrally closed.*

For example, every affine space $\mathbf{A}^n$ is normal and if two varieties X and Y are normal, then their product $X \times Y$ is normal as well. A large class of normal varieties is described by Serre's criterion, which we discuss next in the affine case [Kraft, 1984, AI.6.2] (see, for the general case, [Hartshorne, 1977, Theorem 8.22A]).

A point $x \in X$ is called a singularity in X if $\dim T_x(X) > \dim X$, here T_xX is the tangent space in x; otherwise x is called a regular point. We denote by $\mathrm{Sing}(X)$ the singular locus of X, that is, the closed subvariety of all singular points of X.

The variety X is called regular in codimension 1 if $\mathrm{codim}_X(\mathrm{Sing}(X)) \geq 2$.

Theorem 1.1.2. *(Serre's criterion)*
Let X be an irreducible affine variety. If X is regular in codimension 1, then X is normal.

Let X and Y be two affine varieties. A morphism $\mu : X \to Y$ is called birational if there is an open subset $X' \subseteq X$ on which μ induces an isomorphism $X' \cong \mu(X')$. The following lemma can be found in [Kraft, 1984, AI.3.7].

Lemma 1.1.3. *(Birational morphisms)*
A dominant morphism $\mu : X \to Y$ is birational if and only if there is an open subset $Y' \subseteq Y$, such that for every $y \in Y'$, the fibre $\mu^{-1}(y)$ contains exactly one element.

We include the following lemma of R. Richardson (see [Kraft, 1984, II.3.4]).

Lemma 1.1.4. *(Lemma of Richardson)*
Let $\mu : X \to Y$ be a birational morphism for which $\operatorname{codim}_Y(\overline{Y \setminus \mu(X)}) \geq 2$. If Y is normal, then μ is an isomorphism.

Let (X, x) and (Y, y) be two pointed varieties. They are called smoothly equivalent if there are a pointed variety (Z, z) and smooth morphisms

$$X \xleftarrow{\ \lambda\ } Z \xrightarrow{\ \rho\ } Y$$

fulfilling $\lambda(z) = x$ and $\rho(z) = y$. This definition gives an equivalence relation $\sim$ on the class of pointed varieties; we call the equivalence classes "types of singularities" and denote them by $\operatorname{Sing}(X, x)$. Note that if $(X, x) \sim (Y, y)$, then $x \in \operatorname{Sing}(X)$ if and only if $y \in \operatorname{Sing}(Y)$.

1.1.1 Algebraic group actions

Denote by $\mathrm{GL}_n := \mathrm{GL}_n(K)$ the general linear group for a fixed integer $n \in \mathbf{N}$ regarded as an affine variety and let G be a linear algebraic group, that is, a closed subgroup of some GL_n.

An algebraic group action of G on X is given by a morphism $\_.\_ : G \times X \to X$ which fulfills $(gh).x = g.(h.x)$ and $1_G.x = x$ for all $g, h \in G$ and $x \in X$.

We call X a G-variety and have a closer look at the concept of such group actions.

If possible...

... one would like to classify the orbits by some system of representatives.

... one would like to "understand" each orbit and its closure geometrically.

... one would like to find a set of generators of the invariant ring of the action.

... one would like to find a morphism that separates as many orbits as possible.

We call an algebraic group action finite if the number of orbits is finite and infinite otherwise.

Since an orbit $G.x$ is locally closed but in general not closed, we can consider the smallest Zariski-closed superset in X that contains $G.x$, namely the orbit closure $\overline{G.x}$.

Every orbit closure is a union of orbits, more precisely if $\overline{G.x} = \bigcup_{x'} G.x'$, then for all $x' \in \overline{G.x} \backslash G.x$ the inequality $\dim G.x' < \dim G.x$ holds true.

The calculation of the orbit closure of some orbit, thus, leads to calculating degenerations in X, in more detail we denote $x \leq_{\deg} x'$ if $G.x' \subseteq \overline{G.x}$ and say that "x degenerates to x'".

The Krull dimension of an orbit $G.x$ is given by

$$\dim G.x = \dim \overline{G.x} = \dim G - \dim \mathrm{Iso}_G(x)$$

and given a degeneration $x <_{\deg} x'$, we set

$$\mathrm{codim}(x, x') := \mathrm{codim}_{\overline{G.x}}(\overline{G.x'}).$$

1.1.2 Invariants and algebraic quotients

In order to understand an algebraic group action, we are first of all interested in describing a complete system of representatives, for example, in a combinatorial way. There are, though, several further techniques which yield knowledge about the algebraic group action.

On the one hand, we can classify the orbits generically, that is, in a natural open subset $X_0 \subset X$ (typically by continuous parameters).

On the other hand, there are different kinds of quotients which we will have a closer look at in what follows.

Let $\pi\colon X \to Y$ be a G-invariant morphism of G-varieties, then π is called an algebraic G-quotient of X (sometimes also called a categorical quotient) if it fulfills the universal property that for every G-invariant morphism $f\colon X \to Z$, there exists a unique morphism $\hat{f}\colon Y \to Z$, such that $f = \hat{f} \circ \pi$; we denote $X/\!\!/G := Y$.

Let X be an irreducible affine G-variety. A global section $f \in K[X]$ is called G-invariant if $f(g.x) = f(x)$ holds true for all $g \in G$ and $x \in X$. The G-invariant ring

$$K[X]^G := \{f \in K[X] \mid f \text{ is } G-\text{invariant}\}$$

is a (not necessarily finitely generated) K-algebra which we assume to be finitely generated, though, in the following. In this case, a G-invariant morphism $\pi : X \to Y$ is an algebraic quotient if and only if $K[Y]$ and $K[X]^G$ are isomorphic as K-algebras; the variety Y is then affine as well. We immediately understand that a candidate for an algebraic G-quotient is given by $X/\!\!/G := \mathrm{Spec}\, K[X]^G$.

If the group G is reductive, that is, if every linear representation of G can be decomposed into a direct sum of irreducible representations, D. Hilbert showed that the invariant ring is finitely generated at all times (see [Hilbert, 1890]), even though it can be a problem of large difficulty to find generating invariants.

In order to calculate an algebraic G-quotient of an irreducible affine variety, the criterion what follows (see [Kraft, 1984, II.3.4]) can be of great help.

Theorem 1.1.5. *(Quotient criterion for reductive group actions)*
Let G be a reductive group and X be an affine G-variety. Let Y be an affine variety and $\pi\colon X \to Y$ be a G-invariant morphism of varieties. If

1. *Y is normal,*

2. *π is surjective or* $\operatorname{codim}_Y(\overline{Y\backslash\pi(X)}) \geq 2$ *and*

3. *on an non-empty open subset $Y_0 \subseteq Y$ the fibre $\pi^{-1}(y)$ contains exactly one closed orbit for each $y \in Y_0$,*

then π is an algebraic G-quotient of X.

Note that the condition $\operatorname{codim}_Y(\overline{Y\backslash\pi(X)}) \geq 2$ is weaker than surjectivity if $\dim Y > 1$. If $\dim Y = 1$, though, we need to include surjectivity in order to obtain the criterion for such special situation later on. In case G is not reductive, however, there are counterexamples of only infinitely generated invariant rings (see [Nagata, 1959, Nagata, 1960]). One exception are actions of unipotent subgroups induced by reductive group actions, discussed in [Kraft, 1984, III.3.2].

Lemma 1.1.6. *(Finite generation of U-invariant rings)*
Let U be a unipotent subgroup of G and X be an affine G-variety; the action of G restricts to an action of U on X. Then the invariant ring $K[X]^U$ is finitely generated as a K-algebra.

One can prove that each fibre of π contains exactly one closed orbit. Therefore, an algebraic G-quotient of X parametrizes the closed G-orbits in X.

Let X be a G-variety. We call a G-invariant morphism $\pi : X \to Y =: X/G$ of varieties a geometric quotient if

1. its fibres coincide with the G-orbits in X,

2. a subset $U \subseteq Y$ is open in Y if and only if $\pi^{-1}(U) \subseteq X$ is open in X and

3. the morphism π induces an isomorphism $\pi^* : O_Y(U) \to O_X(\pi^{-1}(U))^G$ for every open subset $U \subseteq Y$ (where $O_X(\pi^{-1}(U))^G \subseteq O_X(\pi^{-1})$ is the subring of G-invariant functions induced by the G-action on X).

Note that if G is reductive, an algebraic quotient as above restricts to a geometric quotient $\pi|_{X^s}\colon X^s \to X^s/G$ if we define X^s to be the closed subset of X of stable points, that is, of points $x \in X$ with finite stabilizers, such that $G.x \subseteq X$ is closed.

We introduce the notion of an associated fibre bundle (see, for example, [Bongartz, 1998] and [Zwara, 2011]) briefly. Given a connected linear algebraic group G and a closed subgroup $H \subseteq G$ with the induced action $h.g := g \cdot h^{-1}$, there exists a geometric quotient $G \to G/H$.

Let X be an H-variety and consider the induced diagonal action of H on $G \times X$.

Then a geometric quotient

$$\pi : G \times X \to (G \times X)/H =: G \times^H X$$

exists, together with an induced G-equivariant fibre bundle $G \times^H X \to G/H$ with typical fibre X.

The following fact on associated fibre bundles sometimes makes it possible to translate an algebraic group action into another algebraic group action that is easier to understand (see, for example, [Serre, 1995] or [Slodowy, 1980]; and [Bongartz, 1994]).

Theorem 1.1.7. *(Translation of algebraic group actions)*
Let G be a linear algebraic group, let X and Y be $G-$varieties, and let $\pi\colon X \to Y$ be a G-equivariant morphism. Assume that Y is a single G-orbit, $Y = G.y_0$. Let H be the stabilizer of y_0 and set $F := \pi^{-1}(y_0)$. Then X is isomorphic to the associated fibre bundle $G \times^H F$, and the embedding $\phi\colon F \hookrightarrow X$ induces a bijection Φ between the H-orbits in F and the G-orbits in X preserving orbit closures and types of singularities.

In the setting of theorem 1.1.7, we deduce the following corollary.

Corollary 1.1.8. *(Preservation of codimensions)*
For each $x \in X$ we obtain

$$\dim G - \dim G.x = \dim H - \dim(G.x \cap F).$$

In more detail, the bijection Φ in theorem 1.1.7 preserves dimensions of stabilizers (of single points) and codimensions.

Proof. Let $x \in X \cong G \times^H F$ and denote the corresponding H-orbit by $H.x \subseteq G \times F$.
Then $G.x \cong G \times^H (G.x \cap F)$ and

$$\dim G.x = \dim(G \times^H (G.x \cap F)) = \dim G + \dim(G.x \cap F) - \dim H$$

yields the above equality.
Since $\dim \mathrm{Iso}_G(s) = \dim G - \dim G.x$ for each $s \in G.x$ and
$\dim \mathrm{Iso}_H(s) = \dim H - \dim(G.x \cap F) = \dim H - \dim H.x$ for all $s \in G.x \cap F$, the bijection Φ obviously preserves the dimensions of stabilizers of single points.
Then $\mathrm{codim}(x, x') = \dim G.x' - \dim G.x = \dim \mathrm{Iso}_G(x) - \dim \mathrm{Iso}_G(x')$ and, thus, Φ also preserves codimensions. □

1.1.3 Semi-invariants and GIT-quotients

Let $\mathbf{P}^n$ be the n-dimensional projective space which is obtained by gluing together certain affine spaces. Given a graded commutative ring $R = \bigoplus_{m \geq 0} R_m$, such that $R_0 = K$ and such that R_1 is a finitely generated K-vector space that generates R as a K-algebra, we define $R_+ := \bigoplus_{m>0} R_m$. We sketch the notion of the projective spectrum $\mathrm{Proj}(R)$ briefly (see [Mukai, 2003, 6.1(a)]).

The underlying set of Proj(R) is given by the maximal homogeneous ideals $R_+ \not\subseteq \mathfrak{m} \subseteq R$. We consider the sets $U_a := \{\mathfrak{m} \mid a \notin \mathfrak{m}\}$ for all homogeneous elements $a \in R$ as a basis of open subsets which induce a Zariski topology on Proj(R). The function field is given by

$$K_0 := \left\{ \frac{a}{b} \mid \deg a = \deg b,\ a, b \in R \right\} \cup \{0\}.$$

Furthermore, the structure sheaf $\mathcal{O}_{\text{Proj}(R)}$ is defined on the above open sets to be the localization $\mathcal{O}_{\text{Proj}(R)}(U_a) := R_{(a)}$. Due to these considerations, the space Proj(R) has a structure of a projective variety.

We define a G-character to be a morphism $\chi \colon G \to \mathbf{G}_m$ of algebraic groups, where $\mathbf{G}_m$ is the multiplicative group $(\mathrm{GL}_1, \cdot)$. The set of G-characters is denoted by $X(G)$ and has a natural structure of an abstract group by setting $(\chi + \chi')(g) := \chi(g) \cdot \chi'(g)$.

Let X be an irreducible affine variety and χ be a G-character. A global section $f \in K[X]$ is called a semi-invariant of weight χ if $f(g.x) = \chi(g) \cdot f(x)$ holds true for all elements $g \in G$ and $x \in X$. We denote the χ-semi-invariant ring by

$$K[X]^G_\chi := \bigoplus_{n \geq 0} K[X]^{G, n\chi},$$

which is a subring of $K[X]$ and naturally $\mathbf{N}$-graded by the sets $K[X]^{G,n\chi}$, that is, by the semi-invariants of weight $n\chi$ (and of degree n). The G-invariant ring is contained in $K[X]^G_\chi$ as a subring, in more detail, as the component of degree 0.

We define the semi-invariant ring corresponding to all characters by

$$K[X]^G_* := \bigoplus_{\chi \in X(G)} K[X]^G_\chi.$$

Given functions $f_0, \ldots, f_s \in K[X]^G_\chi$, such that all ratios $\frac{f_i}{f_j}$ are G-invariant rational functions, the map

$$\begin{aligned} \pi\colon\ X &- - - - > \mathbf{P}^s \\ x &\mapsto (f_0(x) : \ldots : f_s(x)) \end{aligned}$$

is not defined on the common zeros of $f_1, \ldots, f_s$. If we extend the number of functions f_i it is possible that the set of common zeros is diminished even though they in general do not vanish completely.

These thoughts suggests the definition of the so-called unstable locus. Let $\chi \in X(G)$ be a G-character, then we define the unstable locus of χ to be the subset of unstable points $x \in X$, that is, $f(x) = 0$ for every $f \in K[X]^{G,n\chi}$ and for every integer $n > 0$.

We, furthermore, define the semi-stable locus of χ to be the set of χ-semi-stable points in X, that is, of points $x \in X$ for which a χ-semi-invariant $f \in K[X]^{G,n\chi}$ for an integer $n > 0$ exists, such that $f(x) \neq 0$.

We define the so-called GIT-quotient of X by G in direction χ to be

$$X/\!\!/_\chi G := \mathrm{Proj}(K[X]^G_\chi)$$

together with the induced morphism $\pi\colon X^{\chi-\mathrm{sst}} \to X/\!\!/_\chi G$.

If the linear algebraic group G is reductive, the ring $K[X]^G_\chi$ is finitely generated (see [Mukai, 2003, 6.1(b)] or [Reineke, 2008] for more information on the subject) and a morphism

$$\begin{array}{rcll} \pi|_\chi\colon X^{\chi-\mathrm{sst}} & \to & X/\!\!/_\chi G & \subseteq \mathrm{Proj}\, K[x_0,\ldots,x_s] \\ x & \mapsto & (f_0(x):\ldots:f_s(x)). & \end{array}$$

is obtained, where $f_0,\ldots,f_s \in K[X]^G_\chi$ are generating semi-invariants of degrees $a_0,\ldots,a_s$ and x_i is of weight a_i for all $i \in \{0,\ldots,s\}$. We call $\pi|_\chi$ a GIT-quotient map of X by G in direction χ.

Note that in case G is not reductive, the ring $K[X]^G_\chi$ is not necessarily finitely generated.

1.1.4 Toric varieties

Since our considerations will involve the notion of a toric variety, we discuss it briefly in what follows.

A toric variety is an irreducible variety X which containes $(K^*)^n$ as an open subset, such that the action of $(K^*)^n$ on itself extends to an action of $(K^*)^n$ on X. We show in the following how toric varieties can be constructed from cones; for more information on the subject, the reader is referred to [Fulton, 1993] or [Cox et al., 2011].

Let N be a lattice, that is, a free abelian group N of finite rank. By $M := \mathrm{Hom}_{\mathbf{Z}}(N,\mathbf{Z})$ we denote the dual lattice, together with the induced dual pairing $\langle\_,\_\rangle$. Consider the vector space $N_{\mathbf{R}} := N \otimes_{\mathbf{Z}} \mathbf{R} \cong \mathbf{R}^n$.

A subset $\sigma \subseteq N_{\mathbf{R}}$ is called a convex rational polyhedral cone if there is a finite set $S \subseteq N$ that generates σ, that is,

$$\sigma = \mathrm{Cone}(S) := \left\{ \sum_{s\in S} \lambda_s \cdot s \mid \lambda_s \geq 0 \right\}.$$

The cone σ is called strongly convex if $\sigma \cap (-\sigma) = \{0\}$.

Given a strongly convex rational polyhedral cone σ, we define its dual by

$$\sigma^\vee := \{m \in \mathrm{Hom}_{\mathbf{R}}(\mathbf{R}^n,\mathbf{R}) \mid \langle m,v\rangle \geq 0 \text{ for all } v \in \sigma\}$$

and its corresponding additive semigroup by $S_\sigma := \sigma^\vee \cap M$, which is finitely generated due to Gordon's lemma (see [Fulton, 1993]). We associate to it the semigroup algebra KS_σ and obtain an affine toric variety $\mathrm{Spec}\, KS_\sigma$.

Lemma 1.1.9. *(Toric varietys from cones)*
An affine toric variety X is isomorphic to $\mathrm{Spec}\, KS_\sigma$ *for some strongly convex rational polyhedral cone σ if and only if X is normal.*

1.2 Representation theory of finite-dimensional algebras

As we make key use of results from the representation theory of finite-dimensional algebras, we now recall the basic setup of this theory and refer to [Assem et al., 2006] and [Auslander et al., 1997] for a thorough treatment.

Let Q be a finite quiver, that is, a directed graph $Q = (Q_0, Q_1, s, t)$ consisting of a finite set of vertices Q_0 and a finite set of arrows Q_1, whose elements are written as $\alpha\colon s(\alpha) \to t(\alpha)$; the vertices $s(\alpha)$ and $t(\alpha)$ are called the source and the target of α, respectively. A path in Q is a sequence of arrows $\omega = \alpha_s \ldots \alpha_1$ such that $t(\alpha_k) = s(\alpha_{k+1})$ for all $k \in \{1, \ldots, s-1\}$; we formally include a path ε_i of length zero for each $i \in Q_0$ starting and ending in i. We have an obvious notion of concatenation $\omega\omega'$ of paths $\omega = \alpha_s \ldots \alpha_1$ and $\omega' = \beta_t \ldots \beta_1$ such that $t(\beta_t) = s(\alpha_1)$.

The path algebra KQ is defined as the K-vector space with a basis consisting of all paths in Q, and with multiplication

$$\omega \cdot \omega' = \begin{cases} \omega\omega', & \text{if } t(\beta_t) = s(\alpha_1); \\ 0, & \text{otherwise.} \end{cases}$$

The radical $\mathrm{rad}(KQ)$ of KQ is defined to be the (two-sided) ideal generated by all paths of positive length; an ideal I of KQ is called admissible if there is some integer s, such that $\mathrm{rad}(KQ)^s \subset I \subset \mathrm{rad}(KQ)^2$.

The key feature of such pairs (Q, I), consisting of a quiver Q and an admissible ideal $I \subset KQ$, is the following: every finite-dimensional K-algebra $\mathcal{A}$ is Morita-equivalent to an algebra of the form KQ/I, in the sense that their categories of finite-dimensional K-representations are (K-linearly) equivalent.

A finite-dimensional K-representation M of Q consists of a tuple of K-vector spaces M_i for $i \in Q_0$, and a tuple of K-linear maps $M_\alpha\colon M_i \to M_j$ indexed by the arrows $\alpha\colon i \to j$ in Q_1. A morphism of representations $M = ((M_i)_{i\in Q_0}, (M_\alpha)_{\alpha\in Q_1})$ and $M' = ((M'_i)_{i\in Q_0}, (M'_\alpha)_{\alpha\in Q_1})$ consists of a tuple of K-linear maps $(f_i\colon M_i \to M'_i)_{i\in Q_0}$, such that $f_j M_\alpha = M'_\alpha f_i$ for every arrow $\alpha\colon i \to j$ in Q_1.

For a representation M and a path ω in Q as above, we denote $M_\omega = M_{\alpha_s} \cdot \ldots \cdot M_{\alpha_1}$. We call M bound by I if $\sum_\omega \lambda_\omega M_\omega = 0$ whenever $\sum_\omega \lambda_\omega \omega \in I$.

The abelian K-linear category of all representations of Q is denoted by $\mathrm{rep}_K(Q)$, the category of representations of Q bound by I by $\mathrm{rep}_K(Q, I)$; it is equivalent to the category of finite-dimensional representations of the algebra KQ/I. We have, thus, found a "linear algebra model" for the category of finite-dimensional representations of an arbitrary finite-dimensional K-algebra $\mathcal{A}$.

We define the dimension vector $\underline{\dim} M \in \mathbf{N}Q_0$ of M by $(\underline{\dim} M)_i = \dim_k M_i$ for $i \in Q_0$. For a fixed dimension vector $\underline{d} \in \mathbf{N}Q_0$, we define $\mathrm{rep}_K(Q, I)(\underline{d})$ to be the full subcategory of $\mathrm{rep}_K(Q, I)$, consisting of representations of dimension vector $\underline{d}$. We consider the affine space $R_{\underline{d}}(Q) = \bigoplus_{\alpha\colon i\to j} \mathrm{Hom}_K(K^{d_i}, K^{d_j})$; its points m naturally correspond to representations $M \in \mathrm{rep}_K(Q)(\underline{d})$ with $M_i = K^{d_i}$ for $i \in Q_0$.

Via this correspondence, the set of such representations bound by I corresponds to a closed subvariety $R_{\underline{d}}(Q, I) \subset R_{\underline{d}}(Q)$. It is obvious that the algebraic group $\mathrm{GL}_{\underline{d}} = \prod_{i \in Q_0} \mathrm{GL}_{d_i}$ acts on $R_{\underline{d}}(Q)$ and on $R_{\underline{d}}(Q, I)$ via base change $(g_i)_i \cdot (M_\alpha)_\alpha = (g_j M_\alpha g_i^{-1})_{\alpha\colon i \to j}$. By definition, the $\mathrm{GL}_{\underline{d}}$-orbits O_M of this action naturally correspond to the isomorphism classes of representations M in $\mathrm{rep}_K(Q, I)(\underline{d})$.

In order to find generators of certain (semi-) invariant rings later on, we will use a theorem of A. Schofield which we will explain in the following.

There is an induced $\mathrm{GL}_{\underline{d}}$-action on $K[R_{\underline{d}}(Q)]$ which yields the natural notion of a semi-invariant.

Denote by $\mathrm{add}\, Q$ the additive category of Q with objects $O(i)$ corresponding to the vertices $i \in Q_0$ and morphisms induced by the paths in Q. Since every representation $M \in \mathrm{rep}_K(Q)$ can naturally be seen as a functor from $\mathrm{add}\, Q$ to $\mathrm{Mod}\, K$, we denote this functor by M as well. Let $\phi\colon \bigoplus_{i=1}^n O(i)^{x_i} \to \bigoplus_{i=1}^n O(i)^{y_i}$ be an arbitrary morphism in $\mathrm{add}\, Q$ and let $\underline{d}$ be a dimension vector of Q, such that $\sum_{i \in Q_0} x_i \cdot \underline{d}_i = \sum_{i \in Q_0} y_i \cdot \underline{d}_i$.
Then we can define an induced so-called determinantal semi-invariant by

$$\begin{aligned} f_\phi\colon\ R_{\underline{d}}(Q) &\to K \\ m &\mapsto \det(M(\phi)) \end{aligned}$$

where $m \in R_{\underline{d}}(Q)$ and $M \in \mathrm{rep}_K(Q)(\underline{d})$ are related via the above mentioned correspondence.

The following theorem (see [Schofield and van den Bergh, 2001]) is due to A. Schofield and M. van den Bergh.

Theorem 1.2.1. *(Generation of semi-invariant rings)*
The semi-invariants in $K[R_{\underline{d}}(Q)]_*^{\mathrm{GL}_{\underline{d}}}$ *are spanned by the determinantal semi-invariants* f_ϕ.

Coming back to algebraic aspects of the algebra KQ/I, we discuss certain facts about the theory of KQ/I-representations. The theorem of W. Krull, R. Remak and O. Schmidt helps to classify the isomorphism classes of KQ/I-representations; it states that every representation in $\mathrm{rep}_K(Q, I)$ is isomorphic to a direct sum of indecomposables, unique up to isomorphism and permutation.

We call KQ/I representation-finite if it admits only a finite number of isomorphism classes of indecomposable representations. It is called locally representation-finite if for each vertex $i \in Q_0$, the number of isomorphism classes of indecomposable representations M with $M_i \neq 0$ is finite.

For certain classes of finite-dimensional algebras, a convenient tool for the classification of the indecomposable representations is the Auslander-Reiten quiver $\Gamma(Q, I)$ of KQ/I. Its vertices $[M]$ are given by the isomorphism classes of indecomposable representations of KQ/I; the arrows between two such vertices $[M]$ and $[M']$ are parametrized by a basis of the space of so-called irreducible maps $f\colon M \to M'$.

One standard technique to calculate the Auslander-Reiten quiver is the knitting process (see, for example, [Assem et al., 2006, IV.4]). If, for example, the quiver has oriented cycles and the knitting process does not work, in some cases the Auslander-Reiten quiver $\Gamma(Q, I)$ can be calculated using covering techniques (see [Gabriel, 1981]), which we discuss briefly in subsection 1.2.1. In [Bongartz and Gabriel, 8182], a more thorough approach is given.

1.2.1 Covering theory of quiver algebras

Let Q and Q' be connected quivers, of which Q is finite. We set $\mathcal{A} := KQ/I$ for an admissible ideal $I \subseteq KQ$ and $\mathcal{A}' := KQ'/I'$ for $I' \subseteq KQ'$ and denote $C := \operatorname{rep}_K(Q, I)$.

Assume for every vertex $x \in Q'_0$ the number of arrows starting or ending in x is finite and there is a bound $N_x \in \mathbf{N}$, such that all paths of length greater or equal than N_x that start or end in x are contained in I'. Then $C' := \operatorname{rep}_K(Q', I')$ is called locally bounded.

We define the corresponding category of covariant K-linear functors $m : C \to K\text{-}\operatorname{Mod}$ (or $m : C' \to K\text{-}\operatorname{Mod}$, respectively) by $C\text{-}\operatorname{MOD}$ (or $C'\text{-}\operatorname{MOD}$, respectively), where $K\text{-}\operatorname{Mod}$ is the category of K-modules. Define $C\text{-}\operatorname{mod}$ (or $C'\text{-}\operatorname{mod}$, respectively) to be the full subcategory of functors m, such that $\sum_{x \in C} \dim m(x) < \infty$ (or $\sum_{x \in C'} \dim m(x) < \infty$, respectively).

Now let $F\colon C' \to C$ be a K-linear functor. F is called a covering functor if F is surjective on objects and if for all objects x in C' and y in C:

$$\bigoplus_{y'\colon Fy'=y} C'(x, y') \xrightarrow{\sim} C(Fx, y) \quad \text{and} \quad \bigoplus_{y\colon Fy'=y} C'(y', x) \xrightarrow{\sim} C(y, Fx).$$

Then the restriction $F_\bullet\colon C\text{-}\operatorname{MOD} \to C'\text{-}\operatorname{MOD}$ given by $F_\bullet(m) = m \circ F$ has a left and right adjoint functor F_λ, which is uniquely determined (up to isomorphism).

Let G be a torsionfree subgroup of $\operatorname{Aut}_K C'$. We assume G to act freely on C', that is, $g.x \neq x$ for all $e \neq g \in G$ and $x \in C'$.

Theorem 1.2.2. *(Existence of Galois-coverings)*
There is a K-linear category C'/G and a G-invariant covering functor $F\colon C' \to C'/G$, such that for every G-invariant functor $H\colon C' \to C$ there exists a unique $\hat{H}\colon C'/G \to C$ with $H = \hat{H} \circ F$, that is, the following diagram commutes:

$$\begin{array}{ccc} C' & \xrightarrow{\;F\;} & C'/G \\ & \searrow^{H} & \vdots\,\hat{H} \\ & & C \end{array}$$

We call F a Galois-covering with group G.

In case we have a Galois-covering $F\colon C' \to C$, certain structural properties are being preserved.

Lemma 1.2.3. *(Properties of Galois-coverings)*
The following statements hold true:

1. *If m is indecomposable in C'*- mod, *then $F_\lambda m$ is indecomposable in C*- MOD.
2. *F_λ commutes with the Auslander-Reiten translate and preserves Auslander-Reiten sequences.*
3. $C(F_\lambda m, F_\lambda m) \cong C'(m, \bigoplus_{g\in G} m \circ g^{-1})$ *for all $m \in C'$*- mod.
4. *C' is locally representation-finite if and only if C is locally representation-finite.*
   In this case, the Auslander-Reiten quiver $\Gamma_{C'}$ has precisely one connected component and $\Gamma_C = \Gamma_{C'}/G$.
   A bijection $\operatorname{ind} C'/G \leftrightarrow \operatorname{ind} C$ *between the categories of indecomposables is induced.*

In the above setup, we are able to construct a Galois-covering for C:

For every arrow $\alpha\colon x \to y$ in Q_1 we define an inverse $\alpha^{-1}\colon y \to x$. A walk w from x to y is a path $w = \alpha_n \cdots \alpha_1$ of arrows or inverses of arrows. We have a natural composition of walks and an equivalence relation on the set of all walks induced by $w_1 x x^{-1} w_2 \sim w_1 w_2$ for every $x \in Q_1$ or $x^{-1} \in Q_1$.

A walk is called reduced, if it is not equivalent to a walk which arises by cancelling out minimal relations as above. Let $[w]$ be the equivalence class of w, clearly the composition of walks is well-defined on these classes.

Define the fundamental groupoid FQ of Q to be the category with objects Q_0 and morphisms $FQ(x, y)$ given by the equivalence classes of walks from x to y. Then every morphism of FQ is an isomorphism and $FQ(x, x) \cong FQ(y, y)$ for all $x, y \in Q_0$. We fix an element $x \in Q_0$ and define the fundamental group $\Pi_1(Q) := FQ(x, x)$ of Q, which is free in finitely many generators (and, therefore, countable).

Definition 1.2.4. *(Universal covering)*
For $x \in Q_0$, the universal covering $\pi\colon \widehat{Q} \to Q$ is defined by

$$\widehat{Q}_0 := \{[w] \mid w \text{ a walk which starts in } x\}$$

and $\widehat{Q}_1$ as follows:
Every arrow $\alpha \in Q_1$ with $s(\alpha) = y$ induces an arrow $w \mapsto [\alpha^{-1} w]$ for a reduced walk $w \in [w]$ in $\widehat{Q}_1$ and every arrow $\alpha \in Q_1$ with $t(\alpha) = y$ induces an arrow $w \mapsto [\alpha w]$ for a reduced walk $w \in [w]$ in $\widehat{Q}_1$.

The universal covering induces a Galois-covering of the representation categories of the algebras $\mathcal{A}$ and $\widehat{\mathcal{A}} := K\widehat{Q}/\widehat{I}$ with induced relations. Since I is generated by paths of length greater or equal than 2, the following can be proven.

Theorem 1.2.5. *(Interrelation between $\mathcal{A}$ and $\widehat{\mathcal{A}}$)*
The algebra $\mathcal{A}$ is representation-finite if and only if the algebra $\widehat{\mathcal{A}}$ is locally representation-finite.

1.2.2 Tame and wild algebras

Let $\mathcal{A} := KQ/I$ be a finite-dimensional K-algebra, such that $\operatorname{rep}_K(Q, I)$ is locally bounded.

We call $\mathcal{A}$ of tame representation type (or simply "tame") if for every integer d there is an integer m_d and there are finitely generated $K[x]$-$\mathcal{A}$-bimodules $M_1, \dots, M_{m_d}$ that are free over $K[x]$, such that for all but finitely many isomorphism classes of indecomposable right $\mathcal{A}$-modules M of dimension d, there are elements $i \in \{1, \dots, m\}$ and $\lambda \in K$, such that $M \cong K[x]/(x - \lambda) \otimes_{K[x]} M_i$.

The algebra $\mathcal{A}$ is called of wild representation type (or simply "wild") if there is a finitely generated $K\langle X, Y\rangle$-$\mathcal{A}$-bimodule that is free over $K\langle X, Y\rangle$, such that the functor $\_ \otimes_{K\langle X,Y\rangle} M$ sends non-isomorphic finite-dimensional $K\langle X, Y\rangle$-modules to non-isomorphic $\mathcal{A}$-modules.

In 1979, J. A. Drozd proved the following theorem (see [Drozd, 1980]).

Theorem 1.2.6. *(Tame-wild theorem)*
Every finite-dimensional algebra is either tame or wild.

The notion of a tame algebra $\mathcal{A}$ yields that there are at most 1-parameter families of pairwise non-isomorphic indecomposable $\mathcal{A}$-modules; in the wild case there are parameter families of arbitrary many parameters of pairwise non-isomorphic indecomposable $\mathcal{A}$-modules. In order to show that an algebra is wild, it, thus, suffices to describe one particular such 2-parameter family.

Let Q be a quiver of n vertices and consider the path algebra KQ without relations together with the Euler form on $\mathbf{Z}^n \times \mathbf{Z}^n$, defined by

$$\Big\langle (d_1, \dots, d_n), (d'_1, \dots, d'_n) \Big\rangle := \sum_{i=1}^{n} d_i \cdot d'_i - \sum_{(\alpha: i \to j) \in Q_1} d_i \cdot d'_j.$$

We define the Tits form of Q to be the corresponding quadratic form

$$q: \ \mathbf{Z}^n \to \mathbf{Z}; \quad \underline{d} \mapsto \frac{1}{2} \cdot \langle \underline{d}, \underline{d} \rangle$$

A vector $0 \neq \underline{d} \in \mathbf{N}^n$ is called a root if there is an indecomposable representation of Q of this dimension vector. We call $\underline{d}$ a real root if $q(\underline{d}) = 1$ and an imaginary root if $q(\underline{d}) < 1$.

The theorem of P. Gabriel (see [Gabriel, 1972]) shows that KQ is of finite type if and only if the underlying unoriented graph of Q is a disjoint union of Dynkin graphs A, D, E_6, E_7 or E_8; equivalently the corresponding Tits form q is positive definite.

The algebra KQ is representation-infinite and tame if and only if the underlying unoriented graph is a disjoint union of at least one extended Dynkin graph $\widetilde{A}$, $\widetilde{D}$, $\widetilde{E_6}$, $\widetilde{E_7}$ or $\widetilde{E_8}$ and Dynkin graphs; this is equivalent to the associated Tits form being positive semi-definite (see, for example, [Bourbaki, 1972, Donovan and Freislich, 1973, Nazarova, 1973]).

Given a wild algebra KQ, there is an indecomposable representation M with $q(\underline{\dim} M) < 0$.

1.2.3 Degenerations

In order to describe the closure of an orbit $O_M := \mathrm{GL}_{\underline{d}}.m$, where the representation M naturally corresponds to $m \in R_{\underline{d}}(Q, I)$, by degenerations, there are some results of G. Zwara that are extremly powerful - they are quoted from [Zwara, 1999] and [Zwara, 2000] where more information can be found.

Definition 1.2.7. *(Partial orderings on* $\mathrm{rep}_K(Q, I)(\underline{d})$*)*
Let $M, M' \in \mathrm{rep}_K(Q, I)(\underline{d})$*. We denote*

- $M \leq_{\mathrm{deg}} M'$ *if* $O_{M'} \subset \overline{O_M}$ *in* $R_{\underline{d}}(Q, I)$*.*
- $M \leq_{\mathrm{ext}} M'$ *if there exists some integer n, representations* $M_i, M_i', M_i'' \in \mathrm{rep}_K(Q, I)$ *and exact sequences* $0 \to M_i' \to M_i \to M_i'' \to 0$ *for* $i \in \{1, \ldots, n\}$*, such that* $M_i' \oplus M_i'' \cong M_{i+1}$ *for* $i \in \{1, \ldots, n-1\}$*,* $M \cong M_1$ *and* $M' \cong M_n' \oplus M_n''$*.*
- $M \leq_{\mathrm{hom}} M'$ *if* $\dim_K \mathrm{Hom}(V, M) \leq \dim_K \mathrm{Hom}(V, M')$ *for all* $V \in \mathrm{rep}_K(Q, I)$*.*

To simplify the notation, we set $[M, M'] := \dim_K \mathrm{Hom}(M, M')$ for two representations M and M'. In general, the $\leq_{\mathrm{ext}}$-ordering is stronger than the $\leq_{\mathrm{deg}}$-ordering which is stronger than the $\leq_{\mathrm{hom}}$-ordering. For some algebras both $\leq_{\mathrm{deg}}$ and $\leq_{\mathrm{hom}}$ are equivalent as the following result of G. Zwara shows (see [Zwara, 1999]).

Theorem 1.2.8. *(Interrelation between* $\leq_{\mathrm{deg}}$ *and* $\leq_{\mathrm{hom}}$*)*
Suppose an algebra KQ/I *is representation-finite, that is,* KQ/I *admits only finitely many isomorphism classes of indecomposable representations. Let* M *and* M' *be two finite-dimensional representations of* KQ/I *of the same dimension vector. Then* $M \leq_{\mathrm{deg}} M'$ *if and only if* $M \leq_{\mathrm{hom}} M'$*.*

Since the dimension of a homomorphism space is additive with respect to direct sums, one only has to consider the inequality $[V, M] \leq [V, M']$ for indecomposable representations V to characterize a degeneration $M \leq_{\mathrm{deg}} M'$.

To calculate the orbit closure of a given orbit, one needs to calculate every degeneration. Of course, it is sufficient to calculate all minimal degenerations, that is, degenerations $M <_{\mathrm{deg}} M'$ such that if $M \leq_{\mathrm{deg}} L \leq_{\mathrm{deg}} M'$, then $M \cong L$ or $M' \cong L$. We denote a minimal degeneration by $M <_{\mathrm{mdeg}} M'$ and cite the next important result from [Zwara, 2000].

Theorem 1.2.9. *(Types of minimal degenerations)*
Let M *and* M' *be two finite-dimensional representations of* KQ/I*. If* $M <_{\mathrm{mdeg}} M'$*, then one of the following holds true:*

1. $M <_{\mathrm{ext}} M'$ *or*
2. *there are representations* W*,* $\widetilde{M}$*,* $\widetilde{M'}$ *of* KQ/I*, such that*
   a) $M \cong W \oplus \widetilde{M}$
   b) $M' \cong W \oplus \widetilde{M'}$
   c) $\widetilde{M} <_{mdeg} \widetilde{M'}$
   d) $\widetilde{M'}$ *is indecomposable.*

Of course, the converse does not hold true.

2 The concrete setup

We will consider certain subgroups of GL_n, namely

- the Borel subgroup B of upper triangular matrices,
- the parabolic subgroup P of upper-block matrices (that is, $B \subseteq P$) of block sizes $(b_1, \ldots, b_p)$,
- the unipotent subgroup $U \subset B$, where all diagonal entries equal 1, and
- the torus T of diagonal matrices.

Each of these groups has a natural structure of a linear algebraic group. It is a well-known fact that $B = U \cdot T = T \cdot U$ is given as a semi-direct product.

For an integer $x \in \{1, \ldots, n\}$, we denote by $\mathcal{N}^{(x)} \subset K^{n\times n} \cong \mathbf{A}^{n^2}$ the closed subvariety of x-nilpotent matrices N, that is, $N^x = 0$. If x equals n, we obtain an important special case, namely, the nilpotent cone $\mathcal{N} := \mathcal{N}^{(n)}$. If the matrix size n is not clear from the context, we denote it by the index $B_n, P_n, \mathcal{N}_n^{(x)}$, etc.

Obviously, GL_n acts on each variety $\mathcal{N}^{(x)}$ via conjugation; an action that restricts to actions of all the above mentioned algebraic subgroups. The aim of this work is to examine the action of an arbitrary parabolic (which clearly includes the Borel subgroup) on any variety of nilpotent matrices in detail. Where possible, we include results about the unipotent group action.

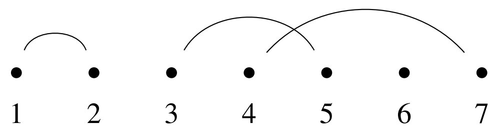
2.1 (Oriented) Link patterns

We will use combinatorial methods to give explicit descriptions of systems of representatives of certain algebraic group actions. In more detail, we will make use of the combinatorial concept of a link pattern (see [Melnikov, 2007]) and its generalizations.

Let $S_n^{(2)}$ be the set of involutions in the symmetric group S_n in n letters.

Definition 2.1.1. *(Link pattern)*
An element σ of $S_n^{(2)}$ is represented by a so-called link pattern, an unoriented graph with vertices $\{1, \ldots, n\}$ and an edge between i and j if $\sigma(i) = j$.

For example, the involution $(1,2)(3,5)(4,7) \in S_7$ corresponds to the link pattern



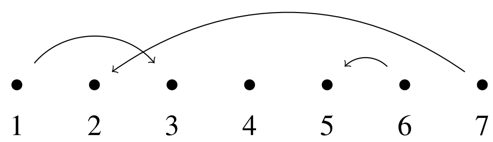
.

As we will see in subsection 2.2.2, these link patterns are used by A. Melnikov to describe the orbits of a certain group action in all detail. Furthermore, we will see that the choice of these link patterns is exactly the right one. In generalizing the results, we obtain different kinds of link patterns that will be described in the following. The starting point is the notion of an oriented link pattern.

Definition 2.1.2. *(Oriented link pattern)*
An oriented link pattern olp *of size n is an oriented graph on $\{1, \ldots, n\}$ together with a set of arrows $\{j_1 \to i_1, \ldots, j_s \to i_s\}$ between vertices $i_k \neq j_k$ for some $0 \leq s < n$, such that every vertex is incident with at most one arrow.*

We call $\mathrm{sh}(\mathrm{olp}) := ((i_1, j_1), \ldots, (i_s, j_s)) \in (\boldsymbol{N} \times \boldsymbol{N})^s$ *the shape of* olp *if $j_1 < j_2 < \ldots < j_s$.*

An example of an oriented link pattern of size 7 and shape $\mathrm{sh}(\mathrm{olp}) = ((3, 1), (5, 6), (2, 7))$ is:

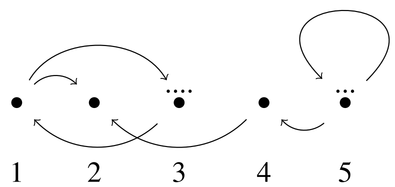


.

Furthermore, we can generalize a link pattern to the following notion of an "enhanced oriented link pattern".

Definition 2.1.3. *(Enhanced oriented link pattern)*
An enhanced oriented link pattern of type $(b_1, \ldots, b_p)$ is a diagram of vertices $\{1, \ldots, p\}$ together with a set of arrows $i \to j$ between vertices i and j and a set of dots at the vertices, such that the sum of the numbers of sources, targets and dots at the i-th vertex equals b_i.

For example, an enhanced oriented link pattern of type $(3, 2, 6, 2, 5)$ is given by

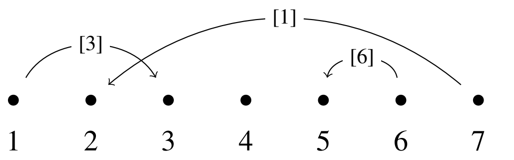


.

A different kind of generalization is given if we label the arrows of an oriented link pattern.

Definition 2.1.4. *(Labelled oriented link pattern)*
A labelled oriented link pattern of size n is defined to be a tuple $\mathrm{olp}_\lambda := (\mathrm{olp}, \lambda)$ *where* olp *is an oriented link pattern of size n and $\lambda \in (K^*)^s$, such that the arrow $j_k \to i_k$ is labelled by λ_k; here s equals the number of arrows in* olp.

We can illustrate the labelled oriented link pattern given by $\lambda = (3, 6, 1)$ which is obtained from the oriented link pattern above as follows:



Note that the shape of the labelled oriented link pattern $\mathrm{olp}_\lambda = (\mathrm{olp}, \lambda)$ is defined to be $\mathrm{sh}(\mathrm{olp}_\lambda) := \mathrm{sh}(\mathrm{olp})$.

2.2 Known results

In case of the action of GL_n on $\mathcal{N}^{(x)}$, the classical theory of M. Jordan and M. Gerstenhaber gives a complete classification of the orbits and their closures in terms of partitions. We will describe the results briefly in 2.2.1 for completeness.

We can also consider $\mathfrak{n} \subseteq \mathcal{N}$, namely the space of all strictly upper triangular matrices. The classification of the orbits $B.N$ of 2-nilpotent matrices $N \in \mathfrak{n}^{(2)} := \mathfrak{n} \cap \mathcal{N}^{(2)}$ under the action of B has been given by A. Melnikov in [Melnikov, 2000, Melnikov, 2006, Melnikov, 2007]. We will sketch the results briefly in 2.2.2 since they yield the starting point of our analysis.

Actions of arbitrary parabolics P on $\mathfrak{n}$ have been coinsidered in [Hille and Röhrle, 1999]; they give a concrete list of the actions that admit a finite number of orbits. We will summarize the results in 2.2.3 and generalize them later on to the action of P on $\mathcal{N}$.

2.2.1 Results of M. Jordan and M. Gerstenhaber

We denote $\lambda \vdash n$ if and only if λ is a partition of n.

For an integer $m \in \mathbf{N}$, we define the Jordan block $J_m \in K^{m\times m}$ (of size m) to be the matrix given by $(J_m)_{i+1,i} = 1$ and $(J_m)_{j,i} = 0$ if $j \neq i+1$.

Given a partition $\lambda := (\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_k) \vdash n$, we denote by $J_\lambda \in K^{n\times n}$ the "Jordan matrix" with Jordan blocks $J_{\lambda_1}, \ldots, J_{\lambda_k}$ on the diagonal and zeros everywhere else. This matrix is unique up to permutation of Jordan blocks and we order the blocks decreasingly by the partition λ.

Theorem 2.2.1. *(Classification of* GL_n*-orbits in* $\mathcal{N}^{(x)}$*)*
Let $N \in \mathcal{N}^{(x)}$, then there exists a unique partition $\lambda \vdash n$ as above, such that $x \geq \lambda_1$ and N is GL_n*-conjugate to J_λ.*

To classify the orbit closures, we briefly sketch the theory of M. Jordan and M. Gerstenhaber which describes them in terms of partitions. Let $\lambda := (\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_k \geq 0)$ and $\lambda' := (\lambda'_1 \geq \lambda'_2 \geq \ldots \geq \lambda'_{k'} \geq 0)$ be two partitions.

We set $\lambda' \lhd \lambda$ if and only if $k \leq k' \leq k+1$ and if there is an index i, such that$\lambda'_x = \lambda_x$ for $x \notin \{i, i+1\}$, $\lambda'_{i+1} = \lambda_{i+1} + 1$ and $\lambda'_i = \lambda_i - 1$.

Note that in terms of Young Diagrams, this relation is given by "lifting a box".

We define $\blacktriangleleft$ to be the transitive closure of $\lhd$.

Theorem 2.2.2. *(GL_n-orbit closures in $\mathcal{N}^{(x)}$)*
Let $\lambda \vdash n$ be a partition as before, then

$$\overline{\mathrm{GL}_n .J_\lambda} = \bigcup_{\substack{\lambda' \vdash n \\ \lambda' \blacktriangleleft \lambda}} \mathrm{GL}_n .J_{\lambda'}$$

The minimal degenerations correspond to the relation $\lhd$.

The only closed orbit is the orbit containing the zero-matrix.

2.2.2 Results of A. Melnikov

For $\sigma \in S_n^{(2)}$ and $i, j \in \{1, \dots, n\}$, define

$$(N_\sigma)_{i,j} := \begin{cases} 1, & \text{if } i < j \text{ and } \sigma(i) = j; \\ 0, & \text{otherwise.} \end{cases}$$

Then the upper-triangular matrix $N_\sigma = ((N_\sigma)_{i,j})_{1 \leq i,j \leq n}$ is 2-nilpotent and, thus, $N_\sigma \in \mathfrak{n}^{(2)}$.

The following theorem is due to [Melnikov, 2000].

Theorem 2.2.3. *(Classification of B-orbits in $\mathfrak{n}^{(2)}$)*
Every B-orbit in $\mathfrak{n}^{(2)}$ is of the form $B.N_\sigma$ for a unique $\sigma \in S_n^{(2)}$.

If $\sigma = (i_1, j_1) \dots (i_k, j_k) \in S_n^{(2)}$, then

$$\dim B_{N_\sigma} = kn + \sum_{s=1}^{k} (i_s - j_s) - \sum_{s=2}^{k} r_s,$$

where $r_s := \sharp\{j_p \mid p < s, j_p < j_s\} + \sharp\{j_p \mid j_p < i_s\}$ (see [Melnikov, 2006]).

For $1 \leq i < j \leq n$, consider the canonical projection $\pi_{i,j} \colon \mathfrak{n}^{(2)} \to \mathfrak{n}^{(2)}_{(j-i+1 \times j-i+1)}$ deleting the first $i - 1$ and the last $n - j$ columns and rows of a matrix in $\mathfrak{n}^{(2)}$.

Define the matrix R_N of $N \in \mathfrak{n}^{(2)}$ by

$$(R_N)_{i,j} = \begin{cases} \operatorname{rank}(\pi_{i,j}(N)), & \text{if } i < j; \\ 0, & \text{otherwise.} \end{cases}$$

This rank matrix R_N is B-invariant, and we denote $R_\sigma := R_{N_\sigma}$ for $\sigma \in S_n^{(2)}$.

We define a partial ordering on the set of rank matrices by

$$R_{\sigma'} \preccurlyeq R_\sigma \text{ if } (R_{\sigma'})_{i,j} \leq (R_\sigma)_{i,j} \text{ for all } i \text{ and } j,$$

inducing a partial ordering on $S_n^{(2)}$ by

$$\sigma' \preccurlyeq \sigma \text{ if } R_{\sigma'} \preccurlyeq R_\sigma.$$

In [Melnikov, 2006], these orderings are used to describe the B-orbit closures in all detail:

Theorem 2.2.4. *(B-orbit closures in $\mathfrak{n}^{(2)}$)*
Let $\sigma \in S_n^{(2)}$ be an involution. Then

$$\overline{B.N_\sigma} = \bigcup_{\sigma' \preccurlyeq \sigma} B.N_{\sigma'}.$$

Moreover, the entry $(R_\sigma)_{i,j}$ of the rank matrix equals the number of edges with end points e_1 and e_2 such that $i \leq e_1, e_2 \leq j$ in the link pattern of σ.

The minimal steps of the partial ordering $\preccurlyeq$ are described explicitly by link patterns in [Melnikov, 2007].

Thus, a combinatorial characterization of the B-orbits in $\mathfrak{n}^{(2)}$ and their orbit closures is given in terms of link patterns.

2.2.3 Results of L. Hille and G. Röhrle

Let P be an arbitrary parabolic subgroup in GL_n of block sizes $(b_1, \ldots, b_p)$, such that $B \subseteq P$.

Denote by P_u the unipotent radical of P, that is, the space of strictly upper block matrices of block sizes $(b_1, \ldots, b_p)$ where the blocks on the diagonal equal zero. We denote by $\mathfrak{p}_u$ its Lie algebra. Then P acts on P_u via conjugation and on $\mathfrak{p}_u$ via the adjoint action.

In order to obtain a criterion as to whether the classification problem is of finite type, L. Hille and G. Röhrle (see [Hille and Röhrle, 1999]) consider the quiver

$$\mathcal{Q}(p): \quad \bullet \underset{\beta_1}{\overset{\alpha_1}{\rightleftarrows}} \bullet \underset{\beta_2}{\overset{\alpha_2}{\rightleftarrows}} \bullet \quad \cdots \quad \bullet \underset{\beta_{p-2}}{\overset{\alpha_{p-2}}{\rightleftarrows}} \bullet \underset{\beta_{p-1}}{\overset{\alpha_{p-1}}{\rightleftarrows}} \bullet$$

together with the relations $\beta_1\alpha_1 = 0$ and $\beta_i\alpha_i = \alpha_{i-1}\beta_{i-1}$ for $i \in \{2, \ldots, p-1\}$ that induce an ideal $I(p)$. They define the full subcategory $\mathcal{M}(p)$ of rep $K\mathcal{Q}(p)/I(p)$, given by the condition that the linear maps at the arrows α_i are injective for all $i \in \{1, \ldots, p-1\}$, and prove that the classification problem explained above is of finite type if and only if the category $\mathcal{M}(p)$ admits a finite number of isomorphism classes of indecomposable representations.

In order to prove a finiteness criterion, they state that the category $\mathcal{M}(p)$ equals the category $\mathcal{F}(\Delta)$ of Δ-filtered modules over the Auslander algebra of the representation-finite algebra $K[x]/(x^p)$. For the precise definitions, see the article [Dlab and Ringel, 1992] by V. Dlab and C. M. Ringel, in which the following theorem is proved.

Theorem 2.2.5. *(The representation type of $\mathcal{M}(p)$)*
Let $p \in \mathbf{N}$. Then the representation type of $\mathcal{M}(p)$ is finite precisely if $p \leq 5$. It is tame if $p = 6$ and wild otherwise.

L. Hille and G. Röhrle deduce the following theorem which answers the question about actions of finite types.

Theorem 2.2.6. *(Classification of finite actions)*
The number of P-orbits on P_u, or $\mathfrak{p}_u$, is finite if and only if $p \leq 5$.

An explicit description of the number of P-orbits for all finite types is obtained and the tame case of theorem 2.2.5 is discussed in more detail.

2.3 Representation-theoretic approach

Let P be a parabolic subgroup of GL_n of block sizes $(b_1, \dots, b_p)$. We are interested in a classification of the P-orbits in the variety $\mathcal{N}^{(x)}$ for some fixed integer $x \in \{1, \dots, n\}$. The starting point is the following translation, which makes use of theorem 1.1.7.

Define $Q(p, x)$ to be the quiver

$$Q(p,x): \quad \underset{1}{\bullet} \xrightarrow{\alpha_1} \underset{2}{\bullet} \xrightarrow{\alpha_2} \underset{3}{\bullet} \quad \cdots \quad \underset{\mathrm{p}-2}{\bullet} \xrightarrow{\alpha_{p-2}} \underset{\mathrm{p}-1}{\bullet} \xrightarrow{\alpha_{p-1}} \underset{\mathrm{p}}{\bullet} \circlearrowleft \alpha$$

and $\mathcal{A}(p, x) = KQ(p, x)/I$ to be the finite-dimensional algebra with a unique element 1, where $I = (\alpha^x)$ is an admissible ideal. We will for now denote $Q_p := Q(p, x)$ and $\mathcal{A} := KQ(p, x)$ even though the reader has to keep in mind that both strongly depend on the choices of p and x.

We fix the dimension vector $\underline{d}_P := (d_1, \dots, d_p) := (b_1, b_1 + b_2 \dots, \sum_{i=1}^{p} b_i)$.

As explained in section 1.2, the algebraic group $\mathrm{GL}_{\underline{d}_P}$ acts on $R_{\underline{d}_P}(Q_p, I)$; the orbits of this action are in bijection with the isomorphism classes of representations in $\mathrm{rep}_K(Q_p, I)(\underline{d}_P)$.

We define $\mathrm{rep}_K^{\mathrm{inj}}(Q_p, I)(\underline{d}_P)$ to be the subcategory of $\mathrm{rep}_K(Q_p, I)(\underline{d}_P)$ consisting of representations $(V_i, M_\rho)_{\substack{1 \leq i \leq p \\ \rho \in Q_1}}$ such that M_ρ is injective if $\rho = \alpha_i$ for $i \in \{1, \dots, p-1\}$. Corresponding to this subcategory, there is an open subset $R_{\underline{d}_P}^{\mathrm{inj}}(Q_p, I) \subset R_{\underline{d}_P}(Q_p, I)$, which is stable under the $\mathrm{GL}_{\underline{d}_P}$-action.

Theorem 1.1.7 yields a correspondence between the P-orbits in $\mathcal{N}^{(x)}$ and the isomorphism classes of representations in $\mathrm{rep}_K^{\mathrm{inj}}(Q_p, I)(\underline{d}_P)$.

Lemma 2.3.1. *(Translation of the P-action on $\mathcal{N}^{(x)}$)*
The variety $R_{\underline{d}_P}^{\mathrm{inj}}(Q_p, I)$ is isomorphic to the associated fibre bundle $\mathrm{GL}_{\underline{d}_P} \times^P \mathcal{N}^{(x)}$. Thus, there exists a closure-preserving bijection Φ between the set of P-orbits in $\mathcal{N}^{(x)}$ and the set of $\mathrm{GL}_{\underline{d}_P}$-orbits in $R_{\underline{d}_P}^{\mathrm{inj}}(Q_p, I)$, which also preserves types of singularities.

Proof. Consider the subquiver $\widetilde{Q_p}$ of Q_p with $(\widetilde{Q_p})_0 = (Q_p)_0$ and $(\widetilde{Q_p})_1 = (Q_p)_1 \setminus \{\alpha\}$. We have a natural $\mathrm{GL}_{\underline{d}_P}$-equivariant projection $\pi\colon R_{\underline{d}_P}^{\mathrm{inj}}(Q_p, I) \to R_{\underline{d}_P}^{\mathrm{inj}}(\widetilde{Q_p})$. The variety $R_{\underline{d}_P}^{\mathrm{inj}}(\widetilde{Q_p})$ consists of tuples of injective maps, thus, the action of $\mathrm{GL}_{\underline{d}_P}$ on $R_{\underline{d}_P}^{\mathrm{inj}}(\widetilde{Q_p})$ is easily seen to be transitive.

Namely, $R^{\mathrm{inj}}_{\underline{d}_P}(\widetilde{Q_p})$ is the orbit of the representation

$$y_0 := K^{d_1} \xrightarrow{\epsilon_1} K^{d_2} \xrightarrow{\epsilon_2} \cdots \xrightarrow{\epsilon_{p-2}} K^{d_{p-1}} \xrightarrow{\epsilon_{p-1}} K^n,$$

with ϵ_j being the canonical embedding of K^{d_j} into $K^{d_{j+1}}$.

The stabilizer H of y_0 is isomorphic to P and the fibre of π over y_0 is isomorphic to $\mathcal{N}^{(x)}$. Thus, $R^{\mathrm{inj}}_{\underline{d}_P}(Q_p, I)$ is isomorphic to the associated fibre bundle $\mathrm{GL}_{\underline{d}_P} \times^P \mathcal{N}^{(x)}$, yielding the claimed bijection Φ. □

We denote $O_M := \mathrm{GL}\,.m$ if $m \in R^{\mathrm{inj}}_{\underline{d}_P}(Q_p, I)$ corresponds to $M \in \mathrm{rep}^{\mathrm{inj}}(Q_p, I)(\underline{d}_P)$ as in section 1.2 and equivalently use the notation $\mathrm{Iso}_{\mathrm{GL}_{\underline{d}_P}}(M)$ and $\mathrm{Iso}_{\mathrm{GL}_{\underline{d}_P}}(m)$ for the isotropy group of m in $R^{\mathrm{inj}}_{\underline{d}_P}(Q_p, I)$. Then

$$\begin{aligned} \dim O_M &= \dim \mathrm{GL}_{\underline{d}_P} - \dim \mathrm{Iso}_{\mathrm{GL}_{\underline{d}_P}}(M) \\ &= \dim \mathrm{GL}_{\underline{d}_P} - [M, M]. \end{aligned}$$

Due to considerations of different parabolic subgroups and nilpotency degrees, the classification of the corresponding quiver representations differs wildly. Considering the related covering quiver, we obtain conditions as to whether the classification is a finite, tame or wild problem in the representation theory of finite-dimensional algebras.

3 Nilpotency degree 2

We will consider the action of an arbitrary parabolic subgroup P of GL_n, in particular of the Borel subgroup B, and of the unipotent subgroup U on the variety of 2-nilpotent matrices by conjugation.

3.1 Classification of the orbits

Fortunately, in the above cases we are able to classify the orbits in detail and to give explicit sets of representatives, that is, normal forms for the orbits.

3.1.1 Parabolic orbits

Let P be the parabolic subgroup of block sizes $(b_1, \ldots, b_p)$ and formally set $b_0 = 0$. Furthermore, define $\underline{d}_P := (d_1, \ldots, d_p) := (b_1, b_1 + b_2, \ldots, \sum_{i=1}^{p} b_i)$.

Given the quiver

$$Q_p := Q(p,2): \quad \begin{array}{ccccccccccc} \bullet & \xrightarrow{\alpha_1} & \bullet & \xrightarrow{\alpha_2} & \bullet & \cdots & \bullet & \xrightarrow{\alpha_{p-2}} & \bullet & \xrightarrow{\alpha_{p-1}} & \bullet \circlearrowleft \alpha, \\ 1 & & 2 & & 3 & & \mathrm{p}-2 & & \mathrm{p}-1 & & \mathrm{p} \end{array}$$

the admissible ideal $I = (\alpha^2)$ and the algebra $\mathcal{A} = KQ_p/I$, in order to classify the P-orbits in $\mathcal{N}^{(2)}$, it suffices to classify the $\mathrm{GL}_{\underline{d}_P}$-orbits in $R^{\mathrm{inj}}_{\underline{d}_P}(Q_p, I)$, see theorem 2.3.1.

As the theorem of W. Krull, R. Remak and O. Schmidt states, every representation in $\mathrm{rep}_K(Q_p, I)$ can be decomposed into a direct sum of indecomposables, which is unique up to permutations and isomorphisms. By [Boos, 2008] and [Boos and Reineke, 2011], the following lemma classifies the indecomposables in $\mathrm{rep}_K(Q_p, I)$.

Lemma 3.1.1. *(Indecomposables in* $\mathrm{rep}_K(Q_p, I)$*)*
Up to isomorphisms, the indecomposable representations in $\mathrm{rep}_K(Q_p, I)$ *are the following (graphically represented by dots for basis elements and arrows for maps sending one basis element to another):*

$\mathcal{U}_{i,j}$ *for* $1 \leq j \leq i \leq p$*:*

$$\begin{array}{ccccccccccccccccc} 0 & \xrightarrow{0} & \cdots & \xrightarrow{0} & 0 & \xrightarrow{0} & K & \xrightarrow{id} & \cdots & \xrightarrow{id} & K & \xrightarrow{e_1} & K^2 & \xrightarrow{id} & \cdots & \xrightarrow{id} & K^2 \circlearrowleft \alpha \\ & & & & & & \bullet & \rightarrow & \cdots & \rightarrow & \bullet & \rightarrow & \bullet & \rightarrow & \cdots & \rightarrow & \bullet \\ & & & & & & j & & & & & & i & & & & p \;\big\downarrow \\ & & & & & & & & & & & & \bullet & \rightarrow & \cdots & \rightarrow & \bullet \end{array}$$

$\mathcal{U}_{i,j}$ for $1 \leq i < j \leq p$:

$$\begin{array}{ccccccccccccccc} 0 & \xrightarrow{0} & \cdots & \xrightarrow{0} & 0 & \xrightarrow{0} & K & \xrightarrow{id} & \cdots & \xrightarrow{id} & K & \xrightarrow{e_2} & K^2 & \xrightarrow{id} & \cdots & \xrightarrow{id} & K^2 \circlearrowleft \alpha \\ & & & & & & \bullet & \rightarrow & \cdots & \rightarrow & \bullet & \rightarrow & \bullet & \rightarrow & \cdots & \rightarrow & \bullet \\ & & & & & & i & & & & & & j & & & & p \\ & & & & & & & & & & & & \bullet & \rightarrow & \cdots & \rightarrow & \bullet \end{array}$$

$\mathcal{V}_i$ for $1 \leq i \leq p$:

$$\begin{array}{ccccccccccc} 0 & \xrightarrow{0} & \cdots & \xrightarrow{0} & 0 & \xrightarrow{0} & K & \xrightarrow{id} & \cdots & \xrightarrow{id} & K \circlearrowleft 0 \\ & & & & & & i & & & & p \\ & & & & & & \bullet & \rightarrow & \cdots & \rightarrow & \bullet \end{array}$$

$\mathcal{W}_{i,j}$ for $1 \leq i \leq j < p$:

$$\begin{array}{ccccccccccccccc} 0 & \xrightarrow{0} & \cdots & \xrightarrow{0} & 0 & \xrightarrow{0} & K & \xrightarrow{id} & \cdots & \xrightarrow{id} & K & \xrightarrow{0} & 0 & \xrightarrow{0} & \cdots & \xrightarrow{0} & 0 \circlearrowleft 0 \\ & & & & & & i & & & & j & & & & & & p \\ & & & & & & \bullet & \rightarrow & \cdots & \rightarrow & \bullet \end{array}$$

Here, e_1 and e_2 are the standard coordinate vectors of K^2 and $\alpha = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$.

Proof. To calculate a system of representatives of the indecomposable representations, we make use of the Auslander-Reiten Theory for finite-dimensional algebras. In more detail, we first calculate the Auslander-Reiten quiver and define representatives for each upcoming indecomposable afterwards.

The universal covering of the quiver Q_p at the vertex p is the (infinite) quiver $\widehat{Q_p}$ given by

$$\widehat{Q_p}: \quad \begin{array}{ccccccccccc} & & & & & \vdots & & & & & \\ & & & & & & & & & & \downarrow \\ \bullet & \longrightarrow & \bullet & \longrightarrow & \bullet & \cdots & \bullet & \longrightarrow & \bullet & \longrightarrow & \bullet \\ & & & & & & & & & & \downarrow \alpha_i \\ \bullet & \longrightarrow & \bullet & \longrightarrow & \bullet & \cdots & \bullet & \longrightarrow & \bullet & \longrightarrow & \bullet \\ & & & & & & & & & & \downarrow \alpha_{i+1} \\ \bullet & \longrightarrow & \bullet & \longrightarrow & \bullet & \cdots & \bullet & \longrightarrow & \bullet & \longrightarrow & \bullet \\ & & & & & & & & & & \downarrow \\ & & & & & \vdots & & & & & \\ 1 & & 2 & & 3 & & p-2 & & p-1 & & p \end{array}$$

together with the induced ideal $\widehat{I}$, generated by all paths $\alpha_{i+1}\alpha_i$ and the fundamental group $\mathbf{Z}$. The natural free action of the group $\mathbf{Z}$ on $\widehat{Q_p}$ is given by shifting the rows.

The algebra $\widehat{\mathcal{A}} := K\widehat{Q_p}/\widehat{I}$ is locally representation-finite, since for each vertex $x \in \widehat{Q_p}$, the number of indecomposables M (up to isomorphism) with $M_i \neq 0$ is finite.

Therefore, due to lemma 1.2.3, we have a bijection between the indecomposables in $\mathcal{A}$ and the indecomposables in $\widehat{\mathcal{A}}/\mathbf{Z}$.

It is easy to see that it suffices to calculate the indecomposable representations of the quiver Q'_p given by

$$
Q'_p: \quad \begin{array}{ccccccccccc}
\bullet & \longrightarrow & \bullet & \longrightarrow & \bullet & \cdots & \bullet & \longrightarrow & \bullet & \longrightarrow & \bullet \\
 & & & & & & & & & & \downarrow \alpha. \\
\bullet & \longrightarrow & \bullet & \longrightarrow & \bullet & \cdots & \bullet & \longrightarrow & \bullet & \longrightarrow & \bullet \\
 & & & & & & & & & & \\
1 & & 2 & & 3 & & \mathrm{p}-2 & & \mathrm{p}-1 & & \mathrm{p}
\end{array}
$$

In more detail, the quiver Q'_p naturally embeds into $\widehat{Q_p}$, such that the composition of this inclusion with the projection $\widehat{Q_p} \to Q_p$ is surjective. We have corresponding maps of the Auslander-Reiten quivers, namely an embedding $\Gamma(Q'_p) \to \Gamma(\widehat{Q_p}, \widehat{I})$ and a quotient $\Gamma(\widehat{Q_p}, \widehat{I}) \to \Gamma(Q_p, I)$ whose composition is also surjective.

Since Q'_p is nothing else than a Dynkin quiver of type A_{2p}, it is routine to calculate its Auslander-Reiten quiver (see [Assem et al., 2006, IV.4]), and we derive the Auslander-Reiten quiver $\Gamma = \Gamma(Q_p, I)$ just by making the identifications resulting from the action of $\mathbf{Z}$, which can be read off from the dimension vectors of indecomposable representations.
We finally arrive at the picture (the marked regions have to be identified) given in figure 3.1.1. As one can see, the only dimension vectors corresponding to indecomposable representations are

$$(0\ldots01\ldots12\ldots2),\ (0\ldots\ldots01\ldots\ldots1) \text{ and } (0\ldots01\ldots10\ldots0).$$

By defining $\mathcal{U}_{i,j}, \mathcal{V}_i$ and $\mathcal{W}_{i,j}$ as above, we can easily compute the endomorphism rings of these representations:

$\mathrm{End}(\mathcal{U}_{i,j}) \cong K$ for $i > j$,

$\mathrm{End}(\mathcal{U}_{i,j}) \cong K[x]/(x^2)$ for $i \leq j$,

$\mathrm{End}(\mathcal{V}_i) \cong K$ for all i and

$\mathrm{End}(\mathcal{W}_{i,j}) \cong K$ for all i, j.

These are local, thus, the defined representations are indeed indecomposable.
Since the vertices of Γ correspond to a system of representatives of the isomorphism classes of indecomposables in $\mathrm{rep}_K(Q_p, I)$ and we have found one representative for each of them, we know that no further indecomposable can exist (up to isomorphism). □

Of course, the representations $\mathcal{U}_{i,j}$ and $\mathcal{V}_i$ form a representative system of the indecomposables in $\mathrm{rep}_K^{\mathrm{inj}}(Q_p, I)$.

First, we classify the B-orbits since the Borel is the finest parabolic subgroup and the classification can easily be generalized to an arbitrary parabolic action.

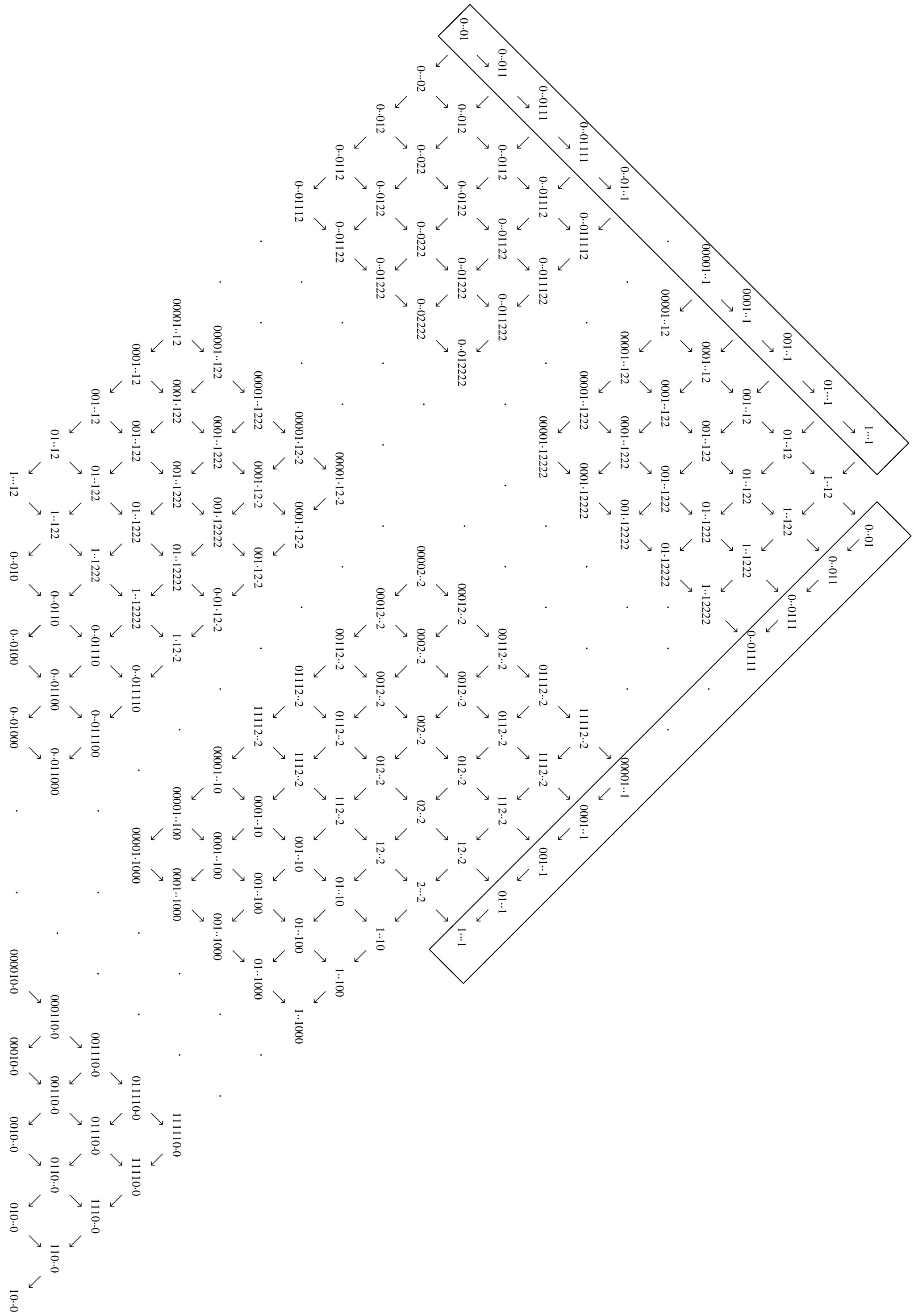


Figure 3.1: The Auslander-Reiten quiver $\Gamma(Q, I)$

Theorem 3.1.2. *(Classification of B-orbits in $\mathcal{N}^{(2)}$)*
There are natural bijections between

1. *B-orbits in $\mathcal{N}^{(2)}$,*
2. *isomorphism classes M in $\mathrm{rep}_K^{\mathrm{inj}}(Q_n, I)$ of dimension vector $\underline{d}_B = (1, 2, \ldots, n)$,*
3. *$n \times n$-matrices $N = (m_{i,j})_{i,j}$ with entries 0 or 1, such that for $i \in \{1, \ldots, n\}$:*

$$\sum_{j=1}^{n} m_{i,j} + \sum_{j=1}^{n} m_{j,i} \leq 1$$

4. *and oriented link patterns of size n.*

Moreover, if an isomorphism class M corresponds to a matrix N under this bijection, the orbit $\mathcal{O}_M \subset R_{\underline{d}_B}^{\mathrm{inj}}(Q_n, I)$ and the orbit $B.N \subset \mathcal{N}^{(2)}$ correspond to each other via the bijection Φ of lemma 2.3.1.

Proof. The bijection between 1. and 2. directly follows from lemma 2.3.1.
Let M be a representation in $\mathrm{rep}_K^{\mathrm{inj}}(Q_n, I)$ of dimension vector $\underline{d}_B$. Then

$$M = \bigoplus_{i,j=1}^{n} \mathcal{U}_{i,j}^{m_{i,j}} \oplus \bigoplus_{i=1}^{n} \mathcal{V}_i^{n_i}$$

for some multiplicities $m_{i,j}, n_i \in \mathbf{N}$, since every representation can be decomposed into indecomposables. Since $\underline{\dim}\, M = (1, 2, \ldots, n)$, we simply need to calculate all tuples $(m_{i,j}, n_i)$ that fulfill

$$\sum_{i,j=1}^{n} m_{i,j} \cdot \underline{\dim}\, \mathcal{U}_{i,j} \ + \ \sum_{i=1}^{n} n_i \cdot \underline{\dim}\, \mathcal{V}_i \ = \ \underline{d}_B.$$

Applying the automorphism δ of $\mathbf{Z}^n$ defined by

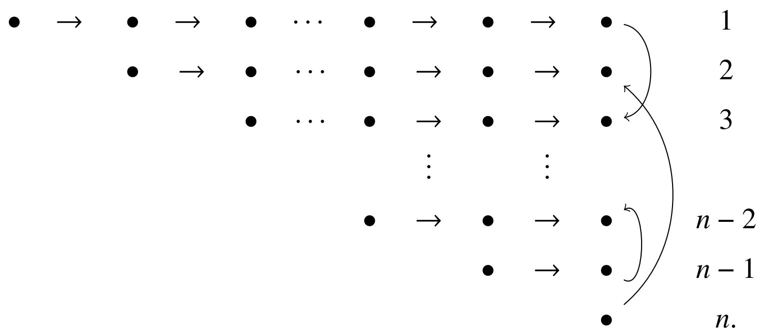
$$\delta(d_1, d_2, \ldots, d_n) = (d_1, d_2 - d_1, d_3 - d_2, \ldots, d_n - d_{n-1}),$$

this condition is equivalent to

$$\sum_{i,j=1}^{n} m_{i,j} \cdot \delta(\underline{\dim}\, \mathcal{U}_{i,j}) + \sum_{i=1}^{n} n_i \cdot \delta(\underline{\dim}\, \mathcal{V}_i) = (1, 1, \ldots, 1, 1).$$

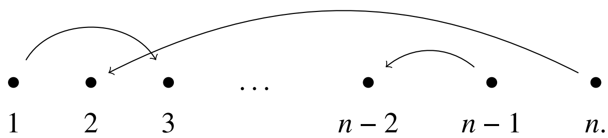
$$1 \ = \ \sum_{j=1}^{n} m_{i,j} \ + \ \sum_{j=1}^{n} m_{j,i} \ + \ n_i.$$

The decomposition of M into indecomposables can be visualized as follows:



The arrows in the rightmost column of the diagram allow us to read off the indecomposable direct summands of M. Namely, $\mathcal{U}_{i,j}$ is a direct summand of M if and only if there is an arrow $j \to i$. If there is no arrow at k, the indecomposable $\mathcal{V}_k$ is a direct summand of M.

Shortening the above picture to the rightmost column, M corresponds to an oriented link pattern:



The conditions on $(m_{i,j})_{i,j}$ ensure that this graph in fact is an oriented link pattern. The matrix $N := (m_{i,j})_{i,j}$ is obviously 2-nilpotent. □

Remark 3.1.3. *We can also rederive theorem 2.2.3 of A. Melnikov.*
Every $B-$orbit of an upper-triangular 2-nilpotent matrix corresponds to the orbit of a representation in $\mathrm{rep}_K^{\mathrm{inj}}(Q_n, I)$ of dimension vector $\underline{d}_B$ which does not contain a representation $\mathcal{U}_{i,j}$ as a direct summand for $i \geq j$. In this case, the corresponding link pattern consists of arrows pointing in the same direction. We can, thus, delete the orientation and arrive at a link pattern as in [Melnikov, 2000].

Define $\mathrm{olp}(X)$ to be the oriented link pattern corresponding to both the isomorphism class of $X \in \mathrm{rep}_K^{\mathrm{inj}}(Q_n, I)(\underline{d}_B)$ and the B-orbit of $X \in \mathcal{N}^{(2)}$.

We denote the set of representatives of B-orbits obtained from theorem 3.1.2 by

$$R_B := \left\{ N = (m_{i,j})_{i,j} \in \{0,1\}^{n \times n} \mid \forall i \in \{1, \ldots, n\} : \sum_{j=1}^{n} m_{i,j} + \sum_{j=1}^{n} m_{j,i} \leq 1 \right\}.$$

Next, we prove a criterion to decide whether two matrices are contained in the same orbit.

Let us define $V_i = \langle e_1, \ldots, e_i \rangle$ to be the span of the first i coordinate vectors in K^n. Given a matrix $N \in \mathcal{N}^{(2)}$, we set $D^N = (d_{i,j}^N)_{i,j}$ with $d_{i,j}^N := \dim(V_i \cap N(V_j))$; we formally define $d_{i,j}^N = 0$ for $i = 0$ or $j = 0$.

The matrix D^N is obviously an invariant for the B-action on $\mathcal{N}^{(2)}$. It is easy to extract an oriented link pattern from D^N as follows:

Proposition 3.1.4. *(Identification of representatives)*
The matrix $N \in \mathcal{N}^{(2)}$ belongs to the B-orbit of a matrix $(m_{i,j})_{1\leq i,j\leq n} \in R_B$ if and only if $d^N_{i,j} = \sum_{i'\leq i;\, j'\leq j} m_{i',j'}$ or, conversely, $m_{i,j} = d^N_{i,j} - d^N_{i-1,j} - d^N_{i,j-1} + d^N_{i-1,j-1}$ for $i, j \in \{1, \ldots, n\}$.

Of course, the number $\mu_B(\mathcal{N}^{(2)})$ of B-orbits in $\mathcal{N}^{(2)}$ is finite. We can moreover give an explicit description.

Proposition 3.1.5. *(Number of B-orbits in $\mathcal{N}^{(2)}$)*

$$\mu_B(\mathcal{N}^{(2)}) = \begin{cases} \sum\limits_{i=1}^{\frac{n-2}{2}} \binom{n}{2i}(n-2i)(n-2i-1) + n^2 - n + 1, & \text{if } n \text{ is even;} \\ \sum\limits_{i=0}^{\frac{n-3}{2}} \binom{n}{2i+1}(n-2i-1)(n-2i-2) + 1, & \text{otherwise.} \end{cases}$$

Proof. We define $\phi^v(x)$ to denote the number of oriented link patterns of size x, such that each vertex is incident with an arrow.

Of course,

$$\mu_B(\mathcal{N}^{(2)}) = \sum_{i=1}^{n-1} \binom{n}{i} \phi^v(n-i) + \phi^v(n) + 1$$

and

$$\phi^v(x) = \begin{cases} 0, & \text{if } x \text{ is odd;} \\ x(x-1), & \text{if } x \text{ is even.} \end{cases}$$

The claimed equalities follow. □

Example 3.1.6. *(Number of B-orbits)*
If...

- *...$n = 2$, then $\mu_B(\mathcal{N}^{(2)}) = 3$,*
- *...$n = 3$, then $\mu_B(\mathcal{N}^{(2)}) = 7$,*
- *...$n = 4$, then $\mu_B(\mathcal{N}^{(2)}) = 25$,*
- *...$n = 5$, then $\mu_B(\mathcal{N}^{(2)}) = 81$.*

Applying the analysis of the B-orbits analogously to 3.1.1, we can classify the P-orbits in $\mathcal{N}^{(2)}$.

We start with the coarsest parabolic subgroup, namely, GL_n itself. The GL_n-orbits in the variety of 2-nilpotent matrices are in bijection with those Jordan normal forms with Jordan blocks of sizes at most 2 (up to permutation of blocks), thus, it is perfectly clear how the B-orbits are related to them. As we can easily describe a basis $\mathbf{B}$ of K^n, for which the representing matrix of the B-orbit representative equals the Jordan normal form, we will describe the connection in detail. Let $N \in \mathcal{N}^{(2)}$ and define $\lambda_x := (\underbrace{2, \ldots, 2}_{x \text{ times}}, 1, \ldots, 1)$.

Proposition 3.1.7. *(Interrelation between B-orbits and* GL_n*-orbits in* $\mathcal{N}^{(2)}$*)*
The B-orbit of N is contained in the GL_n*-orbit of* J_{λ_x} *if and only if* $\mathrm{olp}(N)$ *has exactly x arrows.*

Proof. If $B.N \subseteq \mathrm{GL}_n .J_{\lambda_x}$, then the oriented link pattern of N has x arrows, since the matrices have the same rank.

If $N \in \mathcal{N}^{(2)}$, such that $\mathrm{olp}(N)$ has x arrows, we can without loss of generality assume $N \in R_B$. We define the set of arrows in $\mathrm{olp}(N)$ to be $N_{\mathrm{arr}} := \{j_1 \to i_1, \dots, j_x \to i_x\}$ and the set of fixed vertices to be $N_{\mathrm{fix}} = \{f_1, \dots, f_{n-2x}\}$.
By defining the basis $\mathbf{B} = (e_{j_1}, e_{i_1}, \dots, e_{j_x}, e_{i_x}, e_{f_1}, \dots, e_{f_{n-2x}})$, the induced representing matrix of the linear map l_N (corresponding to N) equals J_{λ_x}. □

Let P be the parabolic subgroup of GL_n of block sizes $(b_1, \dots, b_p)$ and define the vector $\underline{d} := (d_1, \dots, d_p)$ by

$$d_i := \begin{cases} 0, & \text{if } i = 0; \\ d_{i-1} + b_i, & \text{if } i \in \{1, \dots, p\} \end{cases}$$

As an immediate consequence of theorem 3.1.2, we deduce the following corollary.

Corollary 3.1.8. *(Classification of P-orbits in* $\mathcal{N}^{(2)}$*)*
There are natural bijections between

1. *P-orbits in* $\mathcal{N}^{(2)}$*,*
2. *isomorphism classes M in* $\mathrm{rep}_K^{\mathrm{inj}}(Q_p, I)$ *of dimension vector* $\underline{d}_P$*,*
3. *matrices* $N = (p_{i,j})_{i,j} \in \mathbf{N}^{p\times p}$*, such that for all* $i \in \{1, \dots, p\}$*:*

$$\sum_j p_{i,j} + \sum_j p_{j,i} \leq b_i$$

4. *and enhanced oriented link patterns of type* $(b_1, \dots, b_p)$*.*

Note that the multiplicity of the indecomposable $\mathcal{V}_i$ is obtained as the number of dots at the vertex i, due to the missing of a natural notion of fixed vertices. Therefore, we call these dots "fixed vertices", too.

Define $\mathrm{eolp}(X)$ to be the enhanced oriented link pattern corresponding to both the isomorphism class of $X \in \mathrm{rep}_K^{\mathrm{inj}}(Q_p, I)(\underline{d}_P)$ and the P-orbit of $X \in \mathcal{N}^{(2)}$.

Of course, if P is the parabolic subgroup of block sizes $(1, \dots, 1)$, we derive theorem 3.1.2 and every enhanced oriented link pattern of this type is an oriented link pattern. If $P = \mathrm{GL}_n$, we obtain the classification given by M. Jordan and M. Gerstenhaber, see proposition 2.2.1.

We denote the set of P-orbit representatives by

$$R_P := \left\{ N = (p_{i,j})_{i,j} \in \mathrm{N}^{p\times p} \mid \forall i \in \{1, \dots, p\} : \sum_j p_{i,j} + \sum_j p_{j,i} \leq b_i \right\}.$$

Our aim is to verify an algorithm in order to determine each B-orbit contained in a given P-orbit corresponding to a $p \times p$-matrix as in corollary 3.1.8.

The idea is the following: Since each P-orbit is represented by a $p \times p$-matrix N, we can show that all B-orbits contained in this P-orbit are (as B-orbits) represented by matrices, which are obtained by extending N to $n{\times}n$-matrices and thereby translating and interpreting the entries of N. In this way, we obtain the above mentioned algorithm and a precise classification.

Definition 3.1.9. *(Inner sum)*
Let $N = (n_{i,j})_{i,j} \in K^{n\times n}$, then define

$$sum_{i,j}(N) := \sum_{\substack{d_{i-1}<x\leq d_i \\ d_{j-1}<y\leq d_j}} n_{x,y}.$$

Let $\mathbf{B} := (e_1, \ldots, e_n)$ be the basis of coordinate vectors of K^n.

Proposition 3.1.10. *(Interrelation between B-orbits and P-orbits in $\mathcal{N}^{(2)}$)*
Two matrices N and N' in R_B are P-conjugate if and only if $sum_{i,j}(N) = sum_{i,j}(N')$ for $i, j \in \{1, \ldots, p\}$.

Proof. A matrix $S \in P$ with $S^{-1} \cdot N \cdot S = N'$ is induced by a permutation of $\mathbf{B}$, say

$$\sigma\mathbf{B} := (e_{\sigma(1)}, \ldots, e_{\sigma(n)}),$$

such that if $d_{i-1} < x \leq d_i$, then $d_{i-1} < \sigma(x) \leq d_i$ for all $i \in \{1, \ldots, p\}$.

Let i and j be two indices, such that $x := \text{sum}_{i,j}(N) > \text{sum}_{i,j}(N')$ and assume there is a matrix $S \in P$ with $S^{-1} \cdot N \cdot S = N'$. Denote the corresponding non-zero entries of N by (i_s, j_s) for $1 \leq s \leq x$; they fulfill $d_{i-1} < i_s \leq d_i$ and $d_{j-1} < j_s \leq d_j$. Of course, $N \cdot e_{j_s} = e_{i_s}$ and due to $d_{i-1} < \sigma(i_s) \leq d_i$ and $d_{j-1} < \sigma(j_s) \leq d_j$, we obtain $x \leq \text{sum}_{i,j}(N')$, a contradiction.

Given N and N' in R_B fulfilling $\text{sum}_{i,j}(N) = \text{sum}_{i,j}(N')$ for $i, j \in \{1, \ldots, p\}$, we have to define a matrix $S \in P$ such that $S^{-1} \cdot N \cdot S = N'$. We, therefore, define a permutation $\sigma \in S_n$, such that the i-th column $S_{\cdot,i}$ of S equals $e_{\sigma(i)}$. Without loss of generality we assume the oriented link patterns of N and N' to have x arrows.

First, we define σ on fixed vertices.
Let $\mathcal{F}_i$ be the set of fixed vertices f with $d_{i-1} < f \leq d_i$ in olp(N) and $\mathcal{F}'_i$ be the set of fixed vertices f' with $d_{i-1} < f' \leq d_i$ in olp(N'). Of course, the number of elements in $\mathcal{F}_i$ and $\mathcal{F}'_i$ coincides for all i. Given $\mathcal{F}_i = \{f_1, \ldots, f_{l_i}\}$ and $\mathcal{F}'_i = \{f'_1, \ldots, f'_{l_i}\}$, we define $\sigma(f'_k) = f_k$ for all $1 \leq k \leq l_i$.

Next, we define σ on the source vertices of olp(N').
Let $\mathcal{S}_i$ be the set of source vertices s with $d_{i-1} < s \leq d_i$ in olp(N) and $\mathcal{S}'_i$ be the set of source vertices s' with $d_{i-1} < s' \leq d_i$ in olp(N'). We order them in the following way:

Let $(\mathcal{S}_i)_j$ be the set of source vertices of arrows with targets t, such that $d_{j-1} < t \leq d_j$ in olp(N) and let $(\mathcal{S}'_i)_j$ be the set of source vertices of arrows with targets t', such that $d_{j-1} < t' \leq d_j$ in olp(N').

Of course, the number of elements in $(\mathcal{S}_i)_j$ and $(\mathcal{S}'_i)_j$ coincides for $i, j \in \{1, \ldots, p\}$.
Given $(\mathcal{S}_i)_j = \{s_1, \ldots, s_l\}$ and $(\mathcal{S}'_i)_j = \{s'_1, \ldots, s'_l\}$, define $\sigma(s'_k) = s_k$ for all $k \in \{1, \ldots, l\}$.
Finally, we define σ on target vertices.
Let $y' \in (\mathcal{S}'_i)_j$ be mapped to $y \in (\mathcal{S}_i)_j$ by σ. Let x be the target of the arrow $y \to x$ in olp(N) and x' be the target of the arrow $y' \to x'$ in olp(N'). Then we define $\sigma(x') = x$. We have, thus, defined σ on each vertex of the oriented link pattern and, therefore, on S_n. In the same way, we have defined the aforementioned basis $\sigma\mathbf{B} = (e_{\sigma(i)})_{1\leq i\leq n}$.

It now suffices to show $S^{-1} \cdot N \cdot S = N'$, that is, the representing matrix $M^{\sigma\mathbf{B}}_{\sigma\mathbf{B}}(l_N)$ equals N', here we denote by l_N and $l_{N'}$ the induced linear maps.

If i is a fixed vertex in olp(N'), then $\sigma(i)$ is a fixed vertex in olp(N) and $Ne_{\sigma(i)} = 0$. Then the $i-$th column of $M^{\sigma\mathbf{B}}_{\sigma\mathbf{B}}(l_N)$ as well as of N' equals 0.

If i is a source vertex of an arrow in olp(N') with a target j', then $\sigma(i)$ is a source vertex of an arrow in olp(N) with a target j. Thus, $N \cdot e_{\sigma(i)} = e_j$ and since $\sigma(j') = j$, the $i-$th column of $M^{\sigma\mathbf{B}}_{\sigma\mathbf{B}}(l_N)$ and of N' coincide.

If i is a target vertex of an arrow in olp(N') with a source j, then $\sigma(i)$ is a target vertex of an arrow in olp(N) with a source i'. Thus, $Ne_{\sigma(i)} = 0$ and the $i-$th column of N' equals 0 as well. □

Note that the description of the P-orbits can also be deduced directly from the bijection given in 2.3.1. The proof of the theorem however gives an explicit conjugation matrix and therefore presents more details about the connection.

We have proven an explicit description of the P-orbits and derive a natural algorithm to obtain each B-orbit contained in a given P-orbit. The interpretation in terms of oriented link patterns is quite easy.

Given an enhanced oriented link pattern of k vertices, we construct oriented link patterns belonging to the P-orbit as follows:
We draw n vertices numbered by 1, 2 up to n, such that we mark the first b_1 vertices, then the vertices $b_1 + 1$ up to $b_1 + b_2$ and so on. In this way, we obtain n numbered vertices which are ordered in p sets by the block sizes of the parabolic. Now all oriented link patterns have to be constructed, such that the number of arrows from the j-th tuple of vertices to the i-th tuple of vertices equals the number of arrows from j to i in the enhanced oriented link pattern. In this way, it becomes obvious why it is necessarily allowed to draw loops in an enhanced oriented link pattern.

Example 3.1.11. *(Parabolic orbits)*

Consider $n = 4$ *and the parabolic* P *of block sizes* $(3, 1)$.
Then $p = 2$ *and*

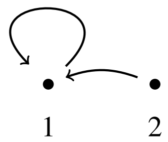
$$R_P = \left\{ \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\}.$$

We fix

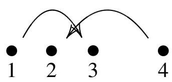
$$A := \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$

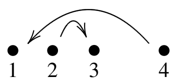
and express a system of representatives of the Borel-orbits contained in the P-orbit of A in the following.

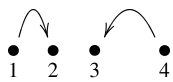
The matrices representing these B-orbits can be obtained from its enhanced oriented link pattern

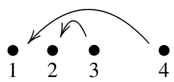
eolp(A) = 

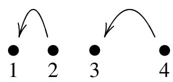
as follows:

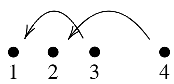
$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} :$$ 



$$: \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} :$$ 



$$: \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} :$$ 



$$: \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The reader can find exactly this picture in theorem 3.3.5.

3.1.2 Unipotent orbits

In case the unipotent subgroup U acts on $\mathcal{N}^{(2)}$, we can make use of the above given classification of the B-orbits in order to classify the orbits in detail, even though the action is of infinite type.

Given a labelled oriented link pattern olp_λ of shape $\mathrm{sh}(\mathrm{olp}) = ((i_1, j_1), \ldots, (i_s, j_s))$ and of size n, we can define the matrix $N(\mathrm{olp}_\lambda) \in \mathcal{N}^{(2)}$ by

$$N(\mathrm{olp}_\lambda)_{i,j} = \begin{cases} \lambda_k, & \text{if } i = i_k \text{ and } j = j_k; \\ 0, & \text{otherwise.} \end{cases}$$

Denote furthermore $N(\mathrm{olp}) := N(\mathrm{olp}_{(1,\dots,1)})$.

Lemma 3.1.12. *(Classification of U-orbits in $\mathcal{N}^{(2)}$)*
There are natural bijections between

1. *U-orbits in $\mathcal{N}^{(2)}$,*
2. *matrices $N(\mathrm{olp}_\lambda)$ where olp_λ is a labelled oriented link pattern of size n and*
3. *labelled oriented link patterns of size n.*

Proof. The bijection between 2. and 3. is immediately clear. We, thus, have to show the following two claims:

Claim 1: For each matrix $N \in \mathcal{N}^{(2)}$ there is a labelled oriented link pattern olp_λ, such that N is U-conjugate to $N(\mathrm{olp}_\lambda)$.

Proof of Claim 1. Due to theorem 3.1.2, the matrix N is B-conjugate to a unique matrix $N_B = (m_{i,j})_{i,j} \in R_B$, say $b \cdot N \cdot b^{-1} = N_B$ with $b \in B$. Then $b = t \cdot u$ for a torus element $t \in T$ and a unipotent element $u \in U$:

$$t \cdot u \cdot N \cdot u^{-1} \cdot t^{-1} \;=\; b \cdot N \cdot b^{-1} \;=\; N_B.$$

Let $((i_1, j_1), \dots, (i_s, j_s))$ be the shape of $\mathrm{olp}(N_B)$.
By setting $\lambda := \left(\frac{t_{j_1,j_1}}{t_{i_1,i_1}}, \dots, \frac{t_{j_s,j_s}}{t_{i_s,i_s}}\right)$ and $\mathrm{olp}_\lambda := (\mathrm{olp}(N_B), \lambda)$, we obtain

$$u \cdot N \cdot u^{-1} = t^{-1} \cdot N_B \cdot t = N(\mathrm{olp}_\lambda).$$

□

Claim 2: If $N(\mathrm{olp}_\lambda)$ and $N(\mathrm{olp}'_\mu)$ are U-conjugate, then $\mathrm{olp} = \mathrm{olp}'$ and $\lambda = \mu$.

Proof of Claim 2. Let $\mathrm{olp}_\lambda = (\mathrm{olp}, \lambda)$ and $\mathrm{olp}'_\mu = (\mathrm{olp}', \mu)$ be labelled oriented link patterns of size n and shapes $\mathrm{sh}(\mathrm{olp}) = ((i_1, j_1), \dots, (i_s, j_s))$ and $\mathrm{sh}(\mathrm{olp}') = ((i'_1, j'_1), \dots, (i'_t, j'_t))$. Assume $N(\mathrm{olp}_\lambda)$ and $N(\mathrm{olp}'_\mu)$ are U-conjugate.

Then $N(\mathrm{olp})$ is T-conjugate to $N(\mathrm{olp}_\lambda)$ by conjugation with a torus element given by $t_{i_k,i_k} = 1$ and $t_{j_k,j_k} = \lambda_k$ for all k. With the same reasoning, $N(\mathrm{olp}')$ is T-conjugate to $N(\mathrm{olp}'_\mu)$. The matrices $N(\mathrm{olp})$ and $N(\mathrm{olp}')$ are, therefore, B-conjugate and $\mathrm{olp} = \mathrm{olp}'$ follows from theorem 3.1.2.

There is a matrix $u \in U$ such that

$$u \cdot N(\mathrm{olp}_\lambda) = N(\mathrm{olp}_\mu) \cdot u.$$

Since

$$(u \cdot N(\mathrm{olp}_\lambda))_{i,j} = \begin{cases} u_{i,i_k} \cdot \lambda_k, & \text{if } j = j_k \text{ and } i \leq i_k; \\ 0, & \text{otherwise,} \end{cases}$$

and

$$(N(\mathrm{olp}_\mu) \cdot u)_{i,j} = \begin{cases} u_{j_k,j} \cdot \mu_k, & \text{if } i = i_k \text{ and } j \geq j_k; \\ 0, & \text{otherwise,} \end{cases}$$

we derive $\lambda_k = \mu_k$ for all k. □

The proof follows from claim 1 and claim 2. □

Each U-orbit is closed itself, see for example [Kraft, 1984].

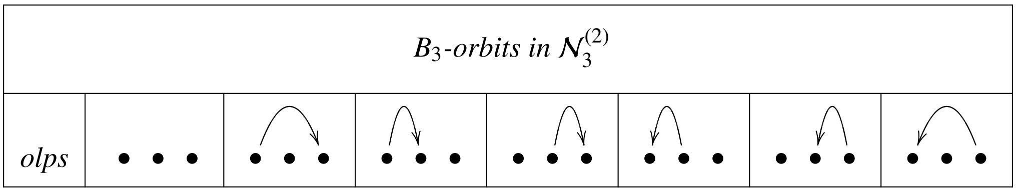
3.1.3 The examples $n = 3$ and $n = 4$

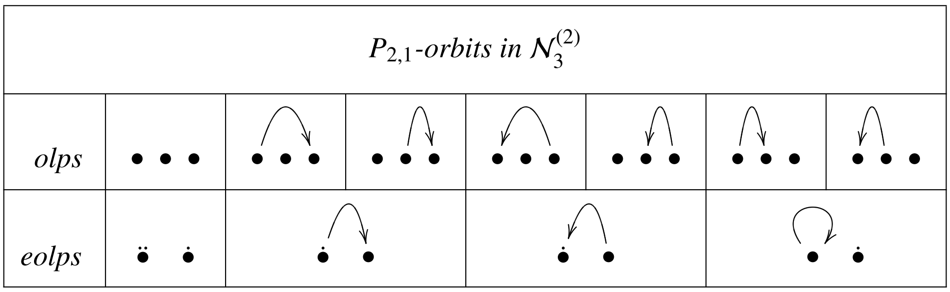
We present how the parabolic orbits in $\mathcal{N}^{(2)}$ are obtained from the Borel orbits if either $n = 3$ or $n = 4$ in the following; in more detail we show how the enhanced oriented link patterns (eolps) are built by the oriented link patterns (olps), depending on the parabolic subgroup. We leave out the numbering of the vertices and denote the parabolic subgroup of block sizes $(i_1, \dots, i_k)$ by $P_{i_1,\dots,i_k}$.

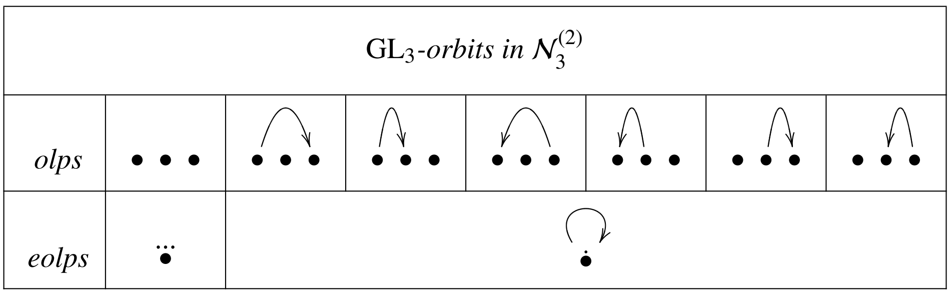
Example 3.1.13. *(The case $n = 3$)*

We consider the actions of the Borel subgroup B_3, the parabolic subgroups $P_{2,1}$ and $P_{1,2}$ and the general linear group GL_3 on the variety $\mathcal{N}_3^{(2)}$ of 2-nilpotent matrices.

Clearly, the classifications of the $P_{2,1}$-orbits and the $P_{1,2}$-orbits are symmetric; hence we restrict our considerations to the parabolic subgroup of block sizes $(2, 1)$.



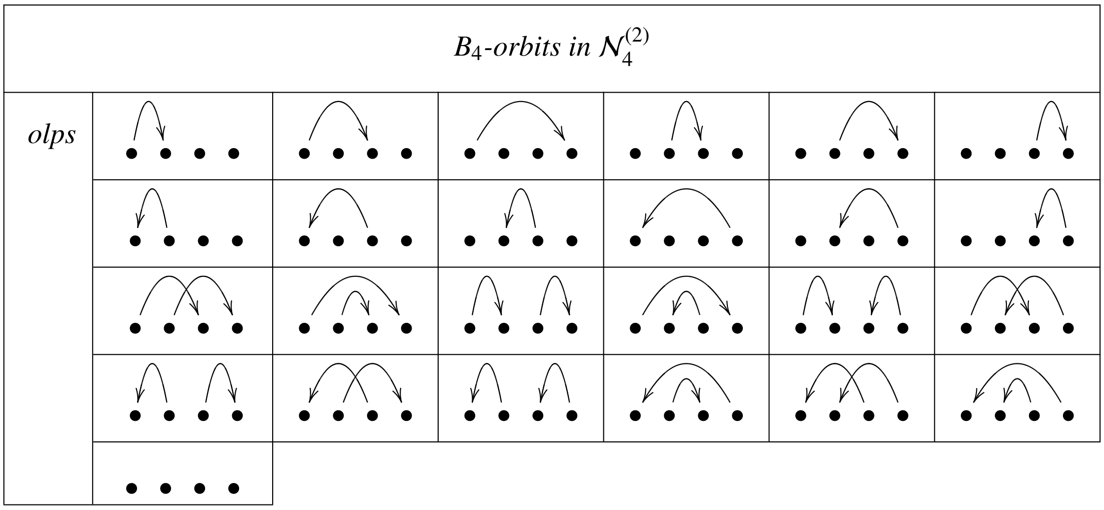


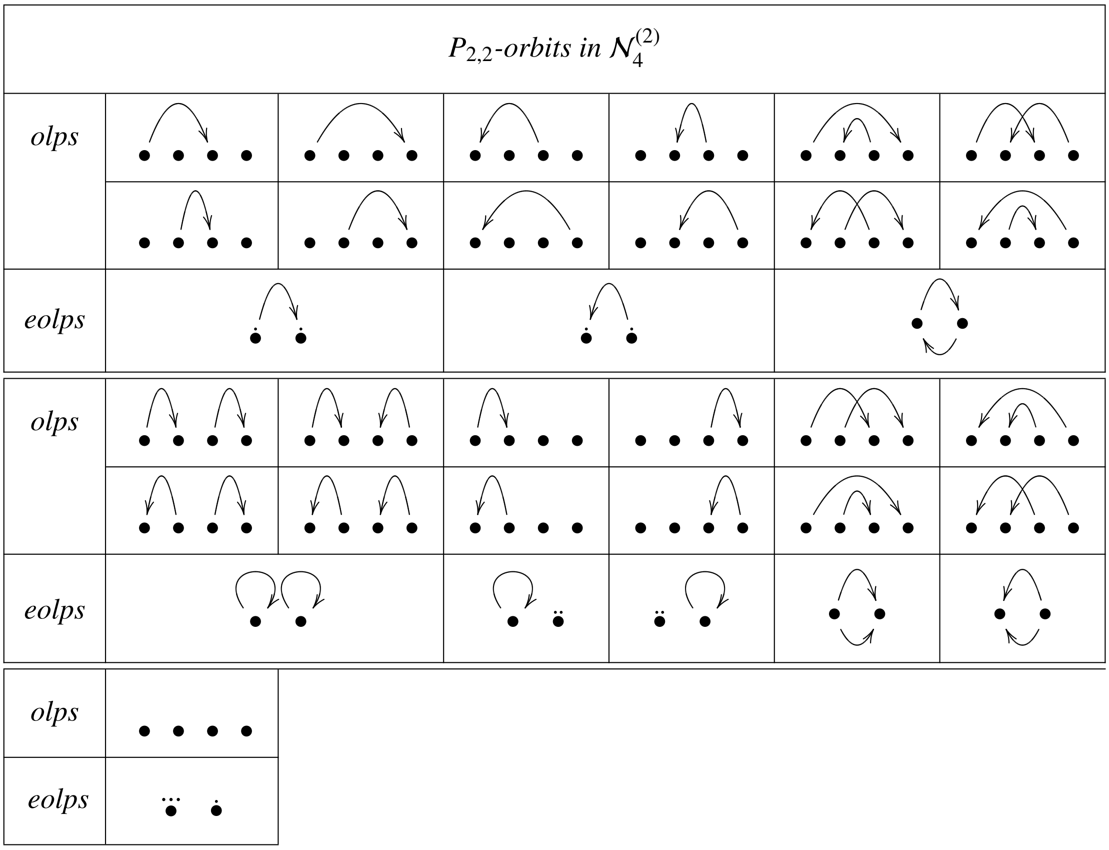


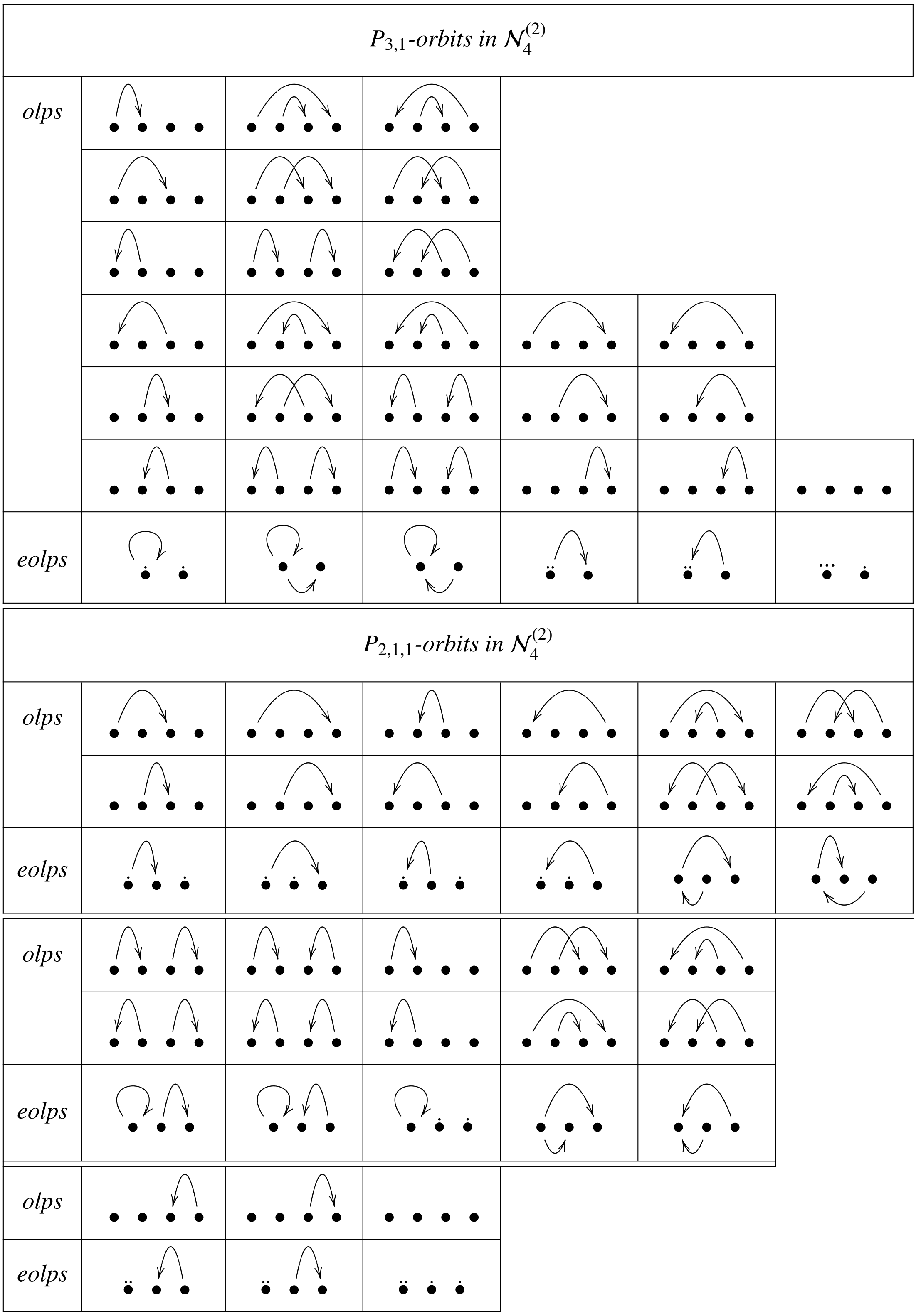
Example 3.1.14. *(The case $n = 4$)*

We consider the Borel action and the actions of the parabolics $P_{2,2}$, $P_{3,1}$ and $P_{2,1,1}$ on the variety $\mathcal{N}_4^{(2)}$ of 2-nilpotent matrices.

We include the parabolic action of $P_{1,2,1}$ even though it can easily be derived from the action of $P_{2,1,1}$. We leave out the actions of $P_{1,3}$ and $P_{1,1,2}$, since they are symmetric to the actions of $P_{3,1}$ and $P_{2,1,1}$, respectively.






$P_{3,1}$-orbits in $\mathcal{N}_4^{(2)}$
olps
eolps
$P_{2,1,1}$-orbits in $\mathcal{N}_4^{(2)}$
olps
eolps
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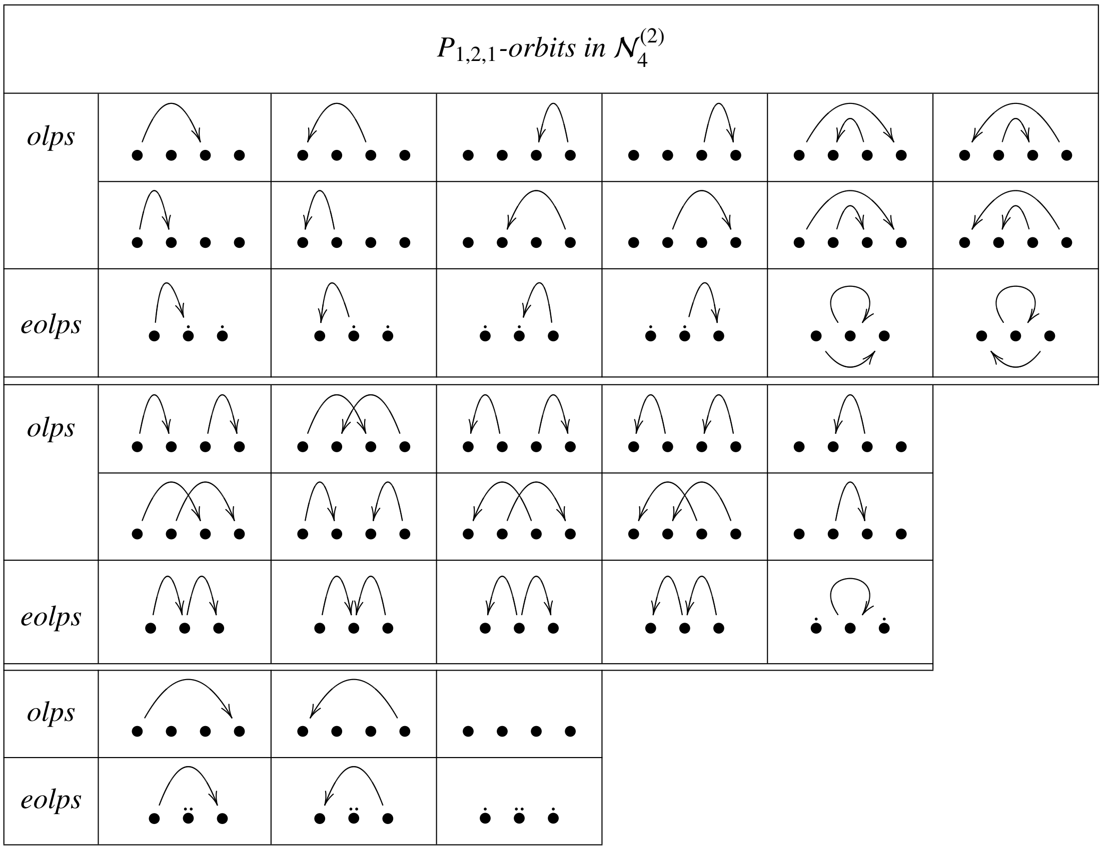
olps
eolps



We have seen how the parabolic orbits can be deduced from the Borel orbits if we make use of their descriptions in terms of link patterns. Of course, in this way, every parabolic action on every nilpotent variety $\mathcal{N}_n^{(2)}$ can be described, even though the number of Borel orbits increases rapidly.

In order to give an explicit system of representatives concerning a parabolic action, it suffices to pick one oriented link pattern out of each orbit; the set of corresponding matrices is a system of representatives.

3.2 Homomorphisms, extensions and their combinatorial interpretation

We can calculate the dimensions of the spaces of homomorphisms between indecomposable representations in $\mathrm{rep}_K(Q_p, I)$, see [Boos, 2008] and [Boos and Reineke, 2011].

Proposition 3.2.1. *(Homomorphisms between indecomposables)*
Let $i, j, k, l \in \{1, \ldots, p\}$. Then

1. $[\mathcal{V}_k, \mathcal{V}_i] = [\mathcal{V}_k, \mathcal{U}_{i,j}] = \delta_{i\leq k}$,
2. $[\mathcal{V}_k, \mathcal{W}_{i,j}] = \delta_{i\leq k\leq j}$,
3. $[\mathcal{U}_{k,l}, \mathcal{V}_i] = \delta_{i\leq l}$,
4. $[\mathcal{U}_{k,l}, \mathcal{U}_{i,j}] = \delta_{i\leq l} + \delta_{j\leq l} \cdot \delta_{i\leq k}$,
5. $[\mathcal{U}_{k,l}, \mathcal{W}_{i,j}] = \delta_{j\geq k} \cdot \delta_{i\leq k} + \delta_{j\geq l} \cdot \delta_{i\leq l}$
6. $[\mathcal{W}_{k,l}, \mathcal{V}_i] = [\mathcal{W}_{k,l}, \mathcal{U}_{i,j}] = 0$,
7. $[\mathcal{W}_{k,l}, \mathcal{W}_{i,j}] = \delta_{i\leq k\leq j\leq l}$,

where $\delta_{x\leq y} := \begin{cases} 1, & \text{if } x \leq y; \\ 0, & \text{otherwise.} \end{cases}$

We also consider the spaces of extensions $\mathrm{Ext}^1_{\mathcal{A}}(M, M')$ between indecomposables M and M' in $\mathrm{rep}^{\mathrm{inj}}_K(Q_p, I)$. For short notation, set $[M, M']^1 := \dim_k \mathrm{Ext}^1_{\mathcal{A}}(M, M')$.

Proposition 3.2.2. *(Extensions between indecomposables)*
Let $i, j, k, l \in \{1, \ldots, p\}$. Then

1. $[\mathcal{V}_k, \mathcal{V}_i]^1 = 1$,
2. $[\mathcal{V}_k, \mathcal{U}_{i,j}]^1 = \delta_{k<j}$,
3. $[\mathcal{U}_{k,l}, \mathcal{V}_i]^1 = \delta_{k<i}$,
4. $[\mathcal{U}_{k,l}, \mathcal{U}_{i,j}]^1 = \delta_{k<j} + \cdot\delta_{l<j} \cdot \delta_{k<i}$,

Proof. We make use of the Auslander-Reiten formula (see [Assem et al., 2006, IV.2])

$$\mathrm{Ext}^1_{\mathcal{A}}(M, M') \cong D\overline{\mathrm{Hom}}_{\mathcal{A}}(M', \tau M).$$

If at all, a homomorphism can factor through the injective I_p at the vertex p, thus, the cases can be computed quickly by using the inequation

$$\dim_K D\overline{\mathrm{Hom}}_{\mathcal{A}}(M', \tau M) \leq \dim_K \mathrm{Hom}_{\mathcal{A}}(M', \tau M) :$$

1. $\tau\mathcal{V}_k = \mathcal{U}_{1,k+1}$ if $k < p$ and $\tau\mathcal{V}_p = \mathcal{V}_1$.
   In both cases $[\mathcal{V}_i, \tau\mathcal{V}_k] = \delta_{1\leq i} = 1$. An easy calculation shows that neither of these homomorphisms factors through I_p.
2. $\tau\mathcal{V}_k = \mathcal{U}_{1,k+1}$ if $k < p$ and $\tau\mathcal{V}_p = \mathcal{V}_1$.
   Then in the first case $[\mathcal{U}_{i,j}, \tau\mathcal{V}_k] = \delta_{1\leq j} + \delta_{k+1\leq j}\delta_{1\leq i} = 1 + \delta_{k+1\leq j}$ and in the second case $[\mathcal{U}_{i,j}, \tau\mathcal{V}_k] = \delta_{1\leq j}$. An easy calculation shows that the dimension is reduced by 1 when taking the quotient modulo homomorphisms that factor through I_p.

3. $\tau\mathcal{U}_{k,l} = \mathcal{U}_{k+1,l+1}$ if $k, l < p$, $\tau\mathcal{U}_{p,l} = 0$ and $\tau\mathcal{U}_{k,p} = \mathcal{V}_{k+1}$ if $k < p$.
   Then $[\mathcal{U}_{k,l}, \mathcal{V}_i]^1 = 0$ if $k = p$ and $[\mathcal{V}_i, \tau\mathcal{U}_{k,l}] = \delta_{k+1\leq i}$ otherwise. If $k \neq p$, then $[I_p, \tau\mathcal{U}_{k,l}] = 0$ follows, therefore, no homomorphism factors through I_p.

4. $\tau\mathcal{U}_{k,l} = \mathcal{U}_{k+1,l+1}$ if $k, l < p$, $\tau\mathcal{U}_{p,l} = 0$ and $\tau\mathcal{U}_{k,p} = \mathcal{V}_{k+1}$ if $k < p$.
   Then $[\mathcal{U}_{k,l}, \mathcal{U}_{i,j}]^1 = 0$ if $k = p$, $[\mathcal{U}_{i,j}, \mathcal{U}_{k+1,l+1}] = \delta_{k+1\leq i} + \delta_{l+1\leq j}\delta_{k+1\leq i}$ if $k, l < p$ and $[\mathcal{U}_{i,j}, \mathcal{V}_{k+1}] = \delta_{k+1\leq i}$ if $k < p$ and $l = p$. Since $[I_p, \tau\mathcal{U}_{k,l}] = 0$ in all cases, no homomorphism factors through I_p.

□

In order to prove an easy description of the parabolic orbit closures in $\mathcal{N}^{(2)}$ in terms of (enhanced) oriented link patterns, we discuss how the dimensions of the homomorphism spaces are linked with these.

Definition 3.2.3. *(The values a_k, $b_{k,l}$, $\overline{a_k}$ and $\overline{b_{i,j}}$)*
Let M be a representation in $\mathrm{rep}_K(Q_p, I)$. For $i, j, k, l \in \{1, \ldots, p\}$ we define

$$a_k(M) := [\mathcal{V}_k, M],\ b_{k,l}(M) := [\mathcal{U}_{k,l}, M],\ \overline{a_i}(M) := [M, \mathcal{V}_i] \text{ and } \overline{b_{i,j}}(M) := [M, \mathcal{U}_{i,j}].$$

If M is a representation in $\mathrm{rep}_K^{\mathrm{inj}}(Q_p, I)(\underline{d}_P)$, then an enhanced oriented link pattern $\mathrm{eolp}(M)$ and a 2-nilpotent matrix N correspond to M via corollary 3.1.8. We use the notations $a_k(M)$, $a_k(\mathrm{eolp}(M))$ and $a_k(N)$ analogously. The same holds true for $b_{k,l}$, $\overline{a_i}$ and $\overline{b_{i,j}}$.

Proposition 3.2.4. *(Combinatorial interpretation)*
Let $M \in \mathrm{rep}_K^{\mathrm{inj}}(Q_p, I)(\underline{d}_P)$ and let $X = \mathrm{eolp}(M)$. Then for $i, j, k, l \in \{1, \ldots, p\}$:

$$a_k(X) = \sharp\{\text{fixed vertices } \leq k\} + \sharp\{\text{targets of arrows } \leq k\},$$

$$b_{k,l}(X) = a_l(X) + \sharp\{\text{arrows with source } \leq l \text{ and target } \leq k\},$$

$$\overline{a_i}(X) = \sharp\{\text{fixed vertices } \geq i\} + \sharp\{\text{sources of arrows } \geq i\},$$

$$\text{and } \overline{b_{i,j}}(X) = \overline{a_i}(X) + \sharp\{\text{arrows with source } \geq j \text{ and target } \geq i\}.$$

Proof. Let $M = \bigoplus_{i,j=1}^{p} \mathcal{U}_{i,j}^{m_{i,j}} \oplus \bigoplus_{i=1}^{p} \mathcal{V}_i^{n_i}$, then

$$a_k(X) = [\mathcal{V}_k, M] = \sum_{i,j=1}^{p} m_{i,j}[\mathcal{V}_k, \mathcal{U}_{i,j}] + \sum_{i=1}^{p} n_i[\mathcal{V}_k, \mathcal{V}_i] = \sum_{\substack{1\leq i\leq k\\ 1\leq j\leq p}} m_{i,j} + \sum_{i\leq k} n_i,$$

$$b_{k,l}(X) = [\mathcal{U}_{k,l}, M] = \sum_{i,j=1}^{p} m_{i,j}[\mathcal{U}_{k,l}, \mathcal{U}_{i,j}] + \sum_{i=1}^{p} n_i[\mathcal{U}_{k,l}, \mathcal{V}_i] = a_l(X) + \sum_{\substack{1\leq i\leq k\\ 1\leq j\leq l}} m_{i,j},$$

$$\overline{a_i}(X) = [M, \mathcal{V}_i] = \sum_{k,l=1}^{p} m_{k,l}[\mathcal{U}_{k,l}, \mathcal{V}_i] + \sum_{k=1}^{p} n_k[\mathcal{V}_k, \mathcal{V}_i] = \sum_{\substack{1\leq k\leq p\\ i\leq l\leq p}} m_{k,l} + \sum_{i\leq k} n_k,$$

and

$$\overline{b_{i,j}}(X) = [M, \mathcal{U}_{i,j}] = \sum_{k,l=1}^{p} m_{k,l}[\mathcal{U}_{k,l}, \mathcal{U}_{i,j}] + \sum_{k=1}^{p} n_k[\mathcal{V}_k, \mathcal{U}_{i,j}] = \overline{a_i}(X) + \sum_{\substack{i \le k \le p \\ j \le l \le p}} m_{k,l}.$$

□

We obtain an explicit combinatorial description of the dimension of the homomorphism space between two arbitrary representations $M, M' \in \mathrm{rep}_K^{\mathrm{inj}}(Q_p, I)$:

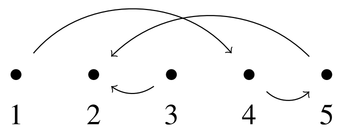
Corollary 3.2.5. *(Dimension of homomorphism spaces)*
Let $M, M' \in \mathrm{rep}_K^{\mathrm{inj}}(Q_p, I)$ with multiplicities $(m_{i,j}, n_k)_{i,j,k}$ and $(m'_{i,j}, n'_k)_{i,j,k}$. Then

$$[M, M'] = \sum_{i,j=1}^{p} m_{i,j} b_{i,j}(M') + \sum_{k=1}^{p} n_k a_k(M') = \sum_{i,j=1}^{p} m'_{i,j} \overline{b_{i,j}}(M) + \sum_{k=1}^{p} n'_k \overline{a_k}(M).$$

If two representations correspond to oriented link patterns as in theorem 3.1.2, an elegant way to calculate these dimensions in concrete terms, is to form an unoriented (generalized) meander of the two oriented link patterns and to count the numbers accordingly (see [Melnikov, 2007] for the analogous definition of a (generalized) meander of two link patterns).

In order to form an oriented (generalized) meander of two oriented link patterns of the same size, we draw both of them on the same n vertices, such that the arrows of the first oriented link pattern are drown upward and the arrows of the second one are drown downward.

Example 3.2.6. *(A meander)*
We consider the representations $M = \mathcal{U}_{4,1} \oplus \mathcal{U}_{2,5} \oplus \mathcal{V}_3$ and $M' = \mathcal{U}_{2,3} \oplus \mathcal{U}_{5,4} \oplus \mathcal{V}_1$. Then in order to calculate $[M, M']$ we can form the meander



.

For every arrow $j \to i$ (and for every fixed vertex k, respectively) in the above oriented link pattern, we count the numbers $b_{i,j}$ (and a_k, respectively) in the below one:

- *for* $1 \to 4$ *we count*
  $b_{4,1}(M') = a_1(M') + \sharp\{$*arrows with source* ≤ 1 *and target* $\le 4\} = 1 + 1 = 2,$
- *for* $5 \to 2$ *we count*
  $b_{2,5}(M') = a_5(M') + \sharp\{$*arrows with source* ≤ 5 *and target* $\le 2\} = 3 + 1 = 4$ *and*
- *for the fixed vertex* 3 *we count*
  $a_3(M') = \sharp\{$*fixed vertices* $\le 3\}$ + $\sharp\{$*targets of arrows* $\le 3\} = 2.$

Thus, $[M, M'] = 8.$

3.3 Closures of Borel orbits

We will give a combinatorial description of the degenerations $N \leq_{\deg} N'$ in $\mathcal{N}^{(2)}$, followed by a generalization of the description of the orbit closures given in [Melnikov, 2006]. Then we present the minimal, disjoint pieces of such degenerations before explicitly calculating all minimal degenerations.

As stated in theorem 1.1.7, the bijection Φ in lemma 2.3.1 is closure-preserving. This leads to calculating the orbit closures of the orbits in $R^{\text{inj}}_{\underline{d}_B}(Q_n, I)$ instead of calculating them directly in $\mathcal{N}^{(2)}$.

We make use of theorem 1.2.8 which states that the partial orderings $\leq_{\deg}$ and $\leq_{\hom}$ are equivalent in $\text{rep}_K(Q_n, I)$.

In more detail, given a dimension vector $\underline{d}$, a representation $M \in \text{rep}^{\text{inj}}_K(Q_n, I)(\underline{d})$ degenerates to another representation $M' \in \text{rep}^{\text{inj}}_K(Q_n, I)(\underline{d})$ if and only if $[V, M] \leq [V, M']$ for every indecomposable representation $V \in \text{rep}_K(Q_n, I)$.

Since $[W_{k,l}, M] = 0$ for every representation $M \in \text{rep}^{\text{inj}}_K(Q_n, I)$ and every pair of integers $k, l \in \{1, \ldots, n\}$ by proposition 3.2.1, we can restrict these indecomposables V to those of type $\mathcal{U}_{k,l}$ and $\mathcal{V}_k$.

Proposition 3.3.1. *(Combinatorial description of degenerations)*
Let $M, M' \in \text{rep}^{\text{inj}}_K(Q_n, I)$ be two representations of the same dimension vector. Then $M \leq_{\deg} M'$ if and only if $a_k(M) \leq a_k(M')$ and $b_{k,l}(M) \leq b_{k,l}(M)$ for all $k, l \in \{1, \ldots, n\}$.

Proof. Given two representations M and M' in $\text{rep}^{\text{inj}}_K(Q_n, I)$ of the same dimension vector, theorem 1.2.8 states that $M \leq_{\deg} M'$ holds true if and only if for all $k, l \in \{1, \ldots, n\}$ the inequalities $[\mathcal{V}_k, M] \leq [\mathcal{V}_k, M']$ and $[\mathcal{U}_{k,l}, M] \leq [\mathcal{U}_{k,l}, M']$ hold true.
Thus, the definition of a_k and $b_{k,l}$ directly yields the claim. □

For $k, l \in \{1, \ldots, n\}$, reconsider the canonical projection $\pi_{i,j} \colon \mathfrak{n}^{(2)} \to \mathfrak{n}^{(2)}_{(j-i+1 \times j-i+1)}$ corresponding to deletion of the first $i-1$ and the last $n-j$ columns and rows of a matrix in $\mathfrak{n}^{(2)}$.

We define the generalized projection $\overline{\pi}_{i,j} \colon \mathcal{N}^{(2)} \to K^{n-i+1 \times j}$ corresponding to deletion of the first $i-1$ rows and the last $n-j$ columns of a matrix in $\mathcal{N}^{(2)}$. Now we can define the generalized rank matrix $\overline{R}(N)$ of $N \in \mathcal{N}^{(2)}_n$ by

$$\overline{R}(N)_{i,j} = \text{rank}\,(\overline{\pi}_{i,j}(N)).$$

Then the definitions in 2.2.2 yield $\overline{R}(N) = R_N$ if $N \in \mathfrak{n}^{(2)}$ and the rank matrix $\overline{R}(N)$ is B-invariant which can be shown easily on the normal forms in R_B.

We define a partial ordering on the set of generalized rank matrices by setting $\overline{R}(N') \preccurlyeq \overline{R}(N)$ if $(\overline{R}(N'))_{i,j} \leq (\overline{R}(N))_{i,j}$ for all i and j, which induces a partial ordering on the B-orbits in $\mathcal{N}^{(2)}$ by $B.N' \preccurlyeq B.N$ if $\overline{R}(N') \preccurlyeq \overline{R}(M)$ for $N, N' \in \mathcal{N}^{(2)}$.

Define $V_{\geq i} := \langle e_i, \ldots, e_n \rangle$ to be the vector space which is spanned by the last $n-i+1$ coordinate vectors of K^n.

Lemma 3.3.2. *(Generalized rank matrices via dimensions of vector spaces)*
Let $N, N' \in \mathcal{N}^{(2)}$, then

$$\overline{R}(N') \preccurlyeq \overline{R}(N) \text{ if and only if } \dim_K(N' \cdot V_j \cap V_{\geq i}) \leq \dim_K(N \cdot V_j \cap V_{\geq i}) \text{ for all } i, j \in \{1, \ldots, n\}.$$

Proof. Given two matrices N and N' in $\mathcal{N}^{(2)}$,

$$\overline{R}(N')_{i,j} = \operatorname{rank}(\overline{\pi}_{i,j}(N).$$

The deletion of the last $n - j$ columns of a matrix N can be translated to multiplying $N \cdot N'$ where $N' \in \{0, 1\}^{n \times j}$ is given by $N'_{k,k} = 1$ and $N'_{k,k'} = 0$ if $k \neq k'$. Then clearly $\operatorname{rank}(N \cdot N') = \dim_K(N \cdot V_j)$.

The deletion of the first $i - 1$ rows of a matrix N can be translated analogously by multiplying $N' \cdot N$ where $N' \in \{0, 1\}^{n-i+1 \times n}$ is given by $N'_{k,k'} = 1$ if $k' = k + i - 1$ and $N'_{k,k'} = 0$ otherwise. Then clearly $\operatorname{rank}(N' \cdot N) = \dim_K(N \cap V_{\geq i})$.

By combining both cases, the claim follows. □

We can reformulate the description and obtain a generalization of theorem 2.2.4.

Corollary 3.3.3. *(Generalization of theorem 2.2.4)*
Let $N, N' \in \mathcal{N}^{(2)}$, then $N \leq_{\deg} N'$ if and only if $\overline{R}(N') \preccurlyeq \overline{R}(N)$.

Proof. Given two matrices N and N' in $\mathcal{N}^{(2)}$, then $N \leq_{\deg} N'$ if and only if

$$a_i(N) \leq a_i(N') \text{ and } b_{i,j}(N) \leq b_{i,j}(N')$$

for all $i, j \in \{1, \ldots, n\}$.

The inequality $a_i(N) \leq a_i(N')$ translates to $\dim_K(N' \cdot V_i) \leq \dim_K(N \cdot V_i)$ and the inequality $b_{i,j}(N) \leq b_{i,j}(N')$ translates to $\dim_K(N' \cdot V_j \cap V_{>i}) \leq \dim_K(N \cdot V_j \cap V_{>i})$.

The claim follows. □

3.3.1 Minimal, disjoint pieces of degenerations

We develop a combinatorial method to produce all degenerations of a given representation $M \in \operatorname{rep}_K^{\text{inj}}(Q_n, I)(\underline{d}_B)$ from its oriented link pattern. Therefore, we construct all minimal degenerations in $R_{\underline{d}_B}^{\text{inj}}(Q_n, I)$ in terms of oriented link patterns. Given the bijection in 2.3.1, we can easily translate them to minimal degenerations in $\mathcal{N}^{(2)}$.

In order to calculate the minimal, disjoint pieces of degenerations in $R_{\underline{d}_B}^{\text{inj}}(Q_n, I)$, we consider degenerations that do not correspond to points in $R_{\underline{d}_B}^{\text{inj}}(Q_n, I)$, that is, representations $M \in \operatorname{rep}_K^{\text{inj}}(Q_n, I)$, such that possibly $\underline{\dim}(M) \neq \underline{d}_B$. They do not correspond to matrices in $\mathcal{N}^{(2)}$, but we will use them to find all minimal degenerations in $\mathcal{N}^{(2)}$ afterwards.

The next proposition gives a characterization of the minimal, disjoint degenerations in $\operatorname{rep}_K^{\text{inj}}(Q_n, I)$, that is, minimal degenerations $M <_{\text{mdeg}} M'$, such that M and M' do not share a common direct summand.

Proposition 3.3.4. *(Types of minimal, disjoint degenerations)*
Let $D <_{\mathrm{mdeg}} D'$ be a minimal, disjoint degeneration in $\mathrm{rep}_K^{\mathrm{inj}}(Q_n, I)$. Then either D' is indecomposable or $D' \cong U \oplus V$, where U and V are indecomposables and there exists an exact sequence $0 \to U \to D \to V \to 0$ or $0 \to V \to D \to U \to 0$.

Proof. We combine theorem 1.2.9 with the technique of [Bongartz, 1994, Theorem 4].

Let $D <_{\mathrm{deg}} D'$ be a minimal, disjoint degeneration in $\mathrm{rep}_K^{\mathrm{inj}}(Q_n, I)$. Then theorem 1.2.9 states that either $D <_{\mathrm{ext}} D'$ or there are representations W, X, X', such that $D \cong W \oplus X$ and $D' \cong W \oplus X'$ where $X <_{\mathrm{deg}} X'$ for an indecomposable representation X'.

Clearly, if the second case arises, the representation D' is indecomposable, since D and D' are disjoint and, therefore, $W = 0$.

Assume $D <_{\mathrm{ext}} D'$. Since $D <_{\mathrm{mdeg}} D'$ is minimal, the $<_{\mathrm{ext}}$-relation has to be "minimal" as well. Thus, there exists a decomposition $D' \cong U \oplus V$ and an exact sequence

$$E = 0 \to U \to D \to V \to 0.$$

Assume U is not indecomposable. Then there is an indecomposable direct summand of U, we call it U', and a retraction $r\colon U \twoheadrightarrow U'$. We denote the kernel of this map by $\mathrm{Ker}\,(r)$ and consider the section $s\colon U' \to U$ such that $r \circ s = id_{U'}$. Looking at the pushout of E by r, we obtain a representation X and the following commutative diagram of exact sequences.

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & \mathrm{Ker}\,(r) & \longrightarrow & \mathrm{Ker}\,(r) & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & U & \xrightarrow{\ \alpha\ } & D & \longrightarrow & V & \longrightarrow & 0 \\
 & & r\downarrow\ \uparrow s & & \downarrow\beta & & \downarrow id_V & & \\
0 & \longrightarrow & U' & \xrightarrow{\ \gamma\ } & X & \longrightarrow & V & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

Given an exact sequence $0 \to A \to B \to C \to 0$ of representations in $\mathrm{rep}_K^{\mathrm{inj}}(Q_n, I)$, we know that $A \oplus C \leq_{\mathrm{hom}} B$.

Thus,
$$D \leq_{\mathrm{hom}} \mathrm{Ker}\,(r) \oplus X \leq_{\mathrm{hom}} \mathrm{Ker}\,(r) \oplus U' \oplus V = D'.$$

If $\mathrm{Ker}\,(r) \oplus U' \oplus V = D' \leq_{\mathrm{hom}} \mathrm{Ker}\,(r) \oplus X$ as well, then $[I, D'] = [I, \mathrm{Ker}\,(r) \oplus X]$ for every indecomposable representation I. The sequence $0 \to U' \xrightarrow{\gamma} X \to V \to 0$ then splits, such that there is a retraction $\rho\colon X \to U'$ fulfilling $\rho \circ \gamma = id_{U'}$.
Thus, $\rho \circ (\beta \circ \alpha) \circ s = \rho \circ (\gamma \circ r) \circ s = id_{U'}$ and U' is a direct summand of both D and D'. A contradiction.

We obtain
$$D \leq_{\mathrm{hom}} \mathrm{Ker}\,(r) \oplus X <_{\mathrm{hom}} \mathrm{Ker}\,(r) \oplus U' \oplus V = D'$$

and since $\leq_{\text{hom}}$ and $\leq_{\text{deg}}$ are equivalent, $D \cong \text{Ker}(r) \oplus X$ due to the minimality of the degeneration. Thus, $\text{Ker}(r)$ is a direct summand of both D and D', a contradiction.

Therefore, the representation U is indecomposable and an analogous proof shows that V is indecomposable, too. □

As we have seen above, all minimal degenerations in $\text{rep}_K^{\text{inj}}(Q_n, I)(\underline{d}_B)$ are of the form $W \oplus D <_{\text{mdeg}} W \oplus D'$, such that D' involves at most two indecomposable direct summands. We have, therefore, "localized" the problem to the consideration of at most four vertices in the corresponding oriented link patterns.

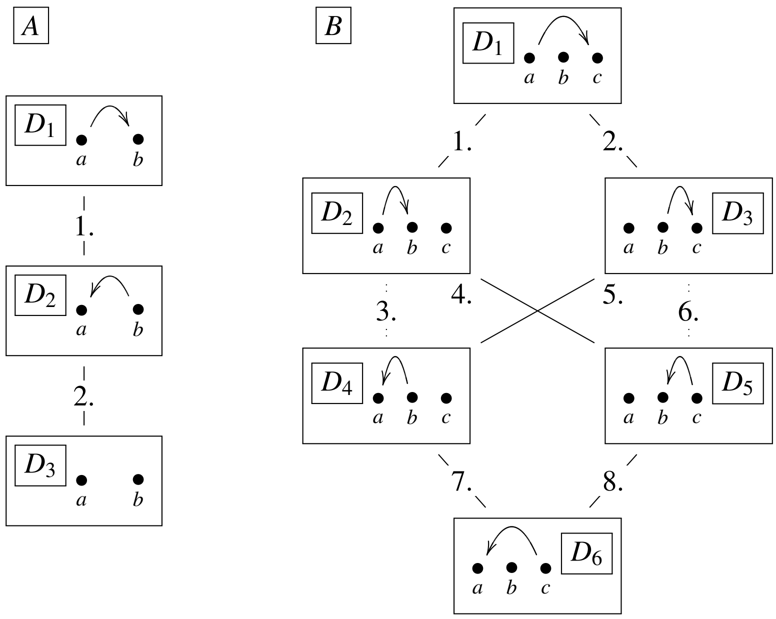
As mentioned before, we consider degenerations between representations in $\text{rep}_K^{\text{inj}}(Q_n, I)$ whose dimension vectors differ from $\underline{d}_B$. We, therefore, define a set of representations $\text{rep}_K^{\text{inj}}(Q_n, I)(\underline{d}_B)_{\text{part}} \subset \text{rep}_K^{\text{inj}}(Q_n, I)$ such that $D \in \text{rep}_K^{\text{inj}}(Q_n, I)(\underline{d}_B)_{\text{part}}$ if and only if there exists a representation $W \in \text{rep}_K^{\text{inj}}(Q_n, I)$ such that $D \oplus W$ is a representation in $\text{rep}_K^{\text{inj}}(Q_n, I)(\underline{d}_B)$.

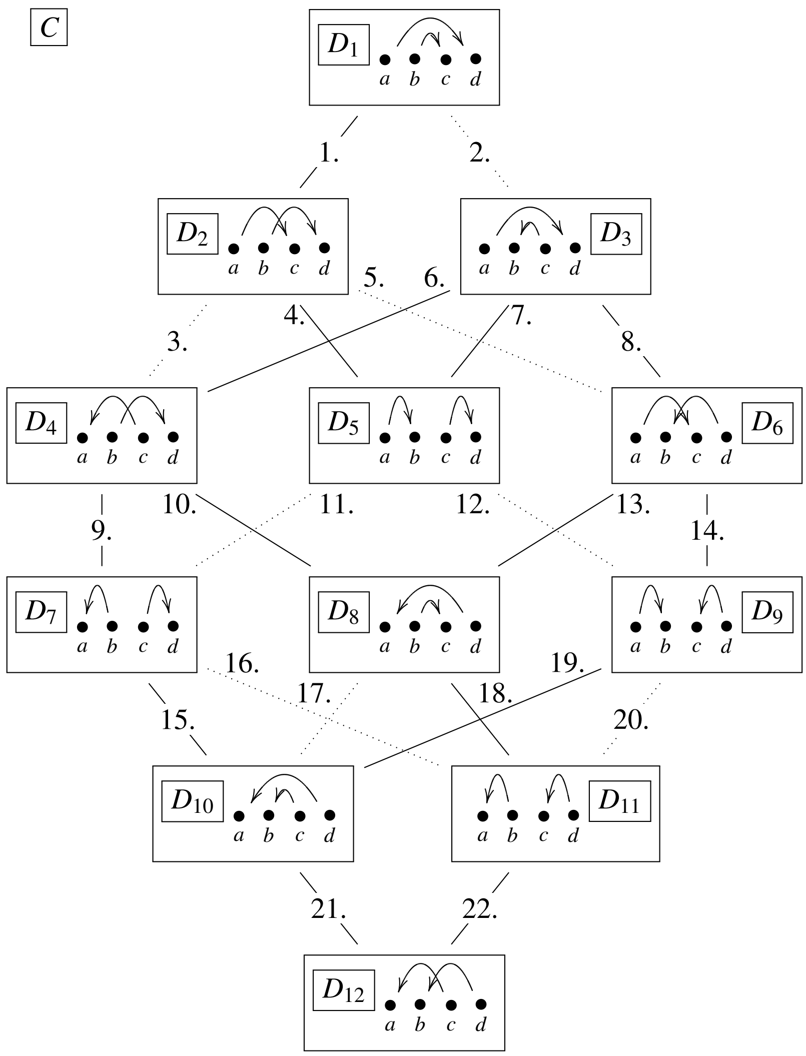
In the local case, that is, in $\text{rep}_K^{\text{inj}}(Q_n, I)(\underline{d}_B)_{\text{part}}$, we can apply proposition 3.3.1 and work out all minimal degenerations.

Given a minimal degeneration $D <_{\text{mdeg}} D'$ in $\text{rep}_K^{\text{inj}}(Q_n, I)(\underline{d}_B)_{\text{part}}$, in order to describe all minimal degenerations, it then suffices to calculate all representations $W \in \text{rep}_K^{\text{inj}}(Q_n, I)$, such that $D \oplus W <_{\text{mdeg}} D' \oplus W$ is a minimal degeneration in $\text{rep}_K^{\text{inj}}(Q_n, I)(\underline{d}_B)$.

Theorem 3.3.5. *(Minimal, disjoint pieces of degenerations)*
Let D and D' be two representations in $\text{rep}_K^{\text{inj}}(Q_n, I)(\underline{d}_B)_{part}$ of the same dimension vector such that $D <_{\text{mdeg}} D'$ is a minimal, disjoint degeneration in $\text{rep}_K^{\text{inj}}(Q_n, I)$.

Considering an arbitrary oriented link pattern, let $a < b$ (respectively $a < b < c$, respectively $a < b < c < d$) be the vertices that D and D' are incident with. Then the degeneration $D <_{\text{mdeg}} D'$ appears in one of the following diagrams.





The diagrams (and the corresponding representations) are named in the diagram as well as the degenerations for the proof later on.
Note: The degenerations marked with dotted lines are not disjoint, but minimal.

Proof. Let $D <_{\text{mdeg}} D'$ be a minimal, disjoint degeneration in $\text{rep}_K^{\text{inj}}(Q_n, I)(\underline{d}_B)_{\text{part}}$.

1. Assume the representation D' is incident with one vertex a.

   Then $D = D' = \mathcal{V}_a$, thus, there is no minimal, disjoint degeneration $D <_{\text{mdeg}} D'$.

2. Assume the representation D' is incident with two vertices $a < b$.

   Then $D' \cong \mathcal{U}_{b,a} = D_1$ or $D' \cong \mathcal{U}_{a,b} = D_2$ or $D' \cong \mathcal{V}_a \oplus \mathcal{V}_b = D_3$.

For $k, l \in \{1, \dots, n\}$ we calculate

$$a_k(\mathcal{U}_{b,a}) = \delta_{b\le k} \le a_k(\mathcal{U}_{a,b}) = \delta_{a\le k} \le a_k(\mathcal{V}_b \oplus \mathcal{V}_a) = \delta_{a\le k} + \delta_{b\le k}$$

$$b_{k,l}(\mathcal{U}_{b,a}) = \delta_{b\le l}+\delta_{a\le l}\delta_{b\le k} \le b_{k,l}(\mathcal{U}_{a,b}) = \delta_{a\le l}+\delta_{b\le l}\delta_{a\le k} \le b_{k,l}(\mathcal{V}_b\oplus\mathcal{V}_a) = \delta_{a\le l}+\delta_{b\le l}.$$

Thus, proposition 3.3.1 yields $\mathcal{U}_{b,a} <^{1.}_{\text{mdeg}} \mathcal{U}_{a,b} <^{2.}_{\text{mdeg}} \mathcal{V}_b \oplus \mathcal{V}_a$ as claimed in diagram A.

3. Assume the representation D' is incident with three vertices $a < b < c$.

Then D' is isomorphic to one of the representations defined in diagram B, that is, $D_1 = \mathcal{U}_{c,a}\oplus\mathcal{V}_b$, $D_2 = \mathcal{U}_{b,a}\oplus\mathcal{V}_c$, $D_3 = \mathcal{U}_{c,b}\oplus\mathcal{V}_a$, $D_4 = \mathcal{U}_{a,b}\oplus\mathcal{V}_c$, $D_5 = \mathcal{U}_{b,c}\oplus\mathcal{V}_a$ or $D_6 = \mathcal{U}_{a,c}\oplus\mathcal{V}_b$, since for each choice of D' the representation $D_7 := \mathcal{V}_a\oplus\mathcal{V}_b\oplus\mathcal{V}_c$ has a direct summand in common with D'.

For $k \in \{1, \dots, n\}$, proposition 3.2.1 yields

$$a_k(D_1) = \delta_{b\le k} + \delta_{c\le k} = a_k(D_2),$$
$$a_k(D_3) = \delta_{a\le k} + \delta_{c\le k} = a_k(D_4),$$
$$a_k(D_5) = \delta_{a\le k} + \delta_{b\le k} = a_k(D_6),$$
$$a_k(D_7) = \delta_{a\le k} + \delta_{b\le k} + \delta_{c\le k}$$

and $a_k(D_1) = a_k(D_2) \le a_k(D_3) = a_k(D_4) \le a_k(D_5) = a_k(D_6) \le a_k(D_7)$.

For $k, l \in \{1, \dots, n\}$, we obtain

$$b_{k,l}(D_1) = \delta_{c\le l} + \delta_{a\le l}\delta_{c\le k} + \delta_{b\le l},\ b_{k,l}(D_2) = \delta_{b\le l} + \delta_{a\le l}\delta_{b\le k} + \delta_{c\le l},$$
$$b_{k,l}(D_3) = \delta_{c\le l} + \delta_{b\le l}\delta_{c\le k} + \delta_{a\le l},\ b_{k,l}(D_4) = \delta_{a\le l} + \delta_{b\le l}\delta_{a\le k} + \delta_{c\le l},$$
$$b_{k,l}(D_5) = \delta_{b\le l} + \delta_{c\le l}\delta_{b\le k} + \delta_{a\le l},\ b_{k,l}(D_6) = \delta_{a\le l} + \delta_{c\le l}\delta_{a\le k} + \delta_{b\le l}.$$

First, let D and D' be representations in $\text{rep}^{\text{inj}}_K(Q_n, I)(\underline{d}_B)_{\text{part}}$, such that $a_k(D) = a_k(D')$ for all k.

- $\mathbf{D_1} <^{\mathbf{1.}}_{\text{deg}} \mathbf{D_2}$, since $b_{k,l}(D_1) \le b_{k,l}(D_2)$ translates to $\delta_{c\le k} \le \delta_{b\le k}$ and $b < c$.
  Assume there is a representation $L \in \text{rep}^{\text{inj}}_K(Q_n, I)(\underline{d}_B)_{\text{part}}$ fulfilling $D_1 <_{\text{deg}} L <_{\text{deg}} D_2$. Then $a_k(D_1) = a_k(L) = a_k(D_2)$, so $L \cong D_1$ or $L \cong D_2$, since D_1 and D_2 are up to isomorphism the only representations fulfilling this condition. Thus, the degeneration is minimal.
- $\mathbf{D_3} <^{\mathbf{5.}}_{\text{deg}} \mathbf{D_4}$, since $b_{k,l}(D_3) \le b_{k,l}(D_4)$ translates to $\delta_{c\le k} \le \delta_{a\le k}$ and $a < c$. Minimality follows in the same way as before.
- $\mathbf{D_5} <^{\mathbf{8.}}_{\text{deg}} \mathbf{D_6}$, since $b_{k,l}(D_5) \le b_{k,l}(D_6)$ translates to $\delta_{b\le k} \le \delta_{a\le k}$ and $a < b$. Minimality follows in the same way as before.

Thus, the degenerations $D_1 <_{\text{mdeg}} D_2$, $D_3 <_{\text{mdeg}} D_4$ and $D_5 <_{\text{mdeg}} D_6$ are minimal.

We consider the remaining cases:

- $\mathbf{D_2} \not< \mathbf{D_3}$ since $b_{k,l}(D_2) \leq b_{k,l}(D_3)$ translates to $\delta_{b\leq l} + \delta_{a\leq l}\delta_{b\leq k} \leq \delta_{b\leq l}\delta_{c\leq k} + \delta_{a\leq l}$. Thus, for $b \leq k < c$ and $l > b$ we have $b_{k,l}(D_2) > b_{k,l}(D_3)$, so $D_2 \nleq D_3$.
- $\mathbf{D_1} <^{\mathbf{2.}}_{\text{mdeg}} \mathbf{D_3}$, since $b_{k,l}(D_1) \leq b_{k,l}(D_3)$ translates to $\delta_{b\leq l} \leq \delta_{a\leq l}$ if $k < c$ and is obvious if $k \geq c$.
- $\mathbf{D_1} \not<_{\text{deg}} \mathbf{D_4}$ is not minimal, since $D_1 <_{\text{mdeg}} D_3 <_{\text{mdeg}} D_4$.
- $\mathbf{D_2} <^{\mathbf{3.}}_{\text{mdeg}} \mathbf{D_4}$, since $b_{k,l}(D_2) \leq b_{k,l}(D_4)$ translates to $\delta_{b\leq l} \leq \delta_{a\leq l}$ if $k < a$, to $0 \leq \delta_{a\leq l}$ if $a \leq k < b$ and is obvious if $k \geq b$.
- $\mathbf{D_4} \not< \mathbf{D_5}$ since $b_{k,l}(D_4) \leq b_{k,l}(D_5)$ translates to $\delta_{b\leq l}\delta_{a\leq k} + \delta_{c\leq l} \leq \delta_{b\leq l} + \delta_{c\leq l}\delta_{b\leq k}$. Thus, for $a \leq k < b$ and $l \geq c$ we have $b_{k,l}(D_4) > b_{k,l}(D_5)$, so $D_4 \nleq D_5$.
- $\mathbf{D_3} <^{\mathbf{6.}}_{\text{mdeg}} \mathbf{D_5}$, since $b_{k,l}(D_3) \leq b_{k,l}(D_5)$ translates to $\delta_{c\leq l} \leq \delta_{b\leq l}$ if $k < b$, to $0 \leq \delta_{b\leq l}$ if $b \leq k < c$ and is obvious if $k \geq c$.
- $\mathbf{D_2} <^{\mathbf{4.}}_{\text{mdeg}} \mathbf{D_5}$, since $b_{k,l}(D_2) \leq b_{k,l}(D_5)$ translates to $\delta_{c\leq l} \leq \delta_{a\leq l}$ if $k < b$ and is obvious if $k \geq b$.
- $\mathbf{D_4} <^{\mathbf{7.}}_{\text{mdeg}} \mathbf{D_6}$, since $b_{k,l}(D_4) \leq b_{k,l}(D_6)$ translates to $\delta_{c\leq l} \leq \delta_{b\leq l}$ if $k < a$ and is obvious if $k \geq a$.

We obtain diagram B.

4. Assume the representation D' is incident with four vertices $a < b < c < d$.

   Then D' is isomorphic to one of the representations defined above in diagram C, that is,

$$D_1 = \mathcal{U}_{d,a} \oplus \mathcal{U}_{c,b},\ D_2 = \mathcal{U}_{c,a} \oplus \mathcal{U}_{d,b},\ D_3 = \mathcal{U}_{d,a} \oplus \mathcal{U}_{b,c},\ D_4 = \mathcal{U}_{a,c} \oplus \mathcal{U}_{d,b},$$
$$D_5 = \mathcal{U}_{b,a} \oplus \mathcal{U}_{d,c},\ D_6 = \mathcal{U}_{c,a} \oplus \mathcal{U}_{b,d},\ D_7 = \mathcal{U}_{a,b} \oplus \mathcal{U}_{d,c},\ D_8 = \mathcal{U}_{a,d} \oplus \mathcal{U}_{c,b},$$
$$D_9 = \mathcal{U}_{b,a} \oplus \mathcal{U}_{c,d},\ D_{10} = \mathcal{U}_{a,d} \oplus \mathcal{U}_{b,c},\ D_{11} = \mathcal{U}_{a,b} \oplus \mathcal{U}_{c,d} \text{ or } D_{12} = \mathcal{U}_{a,c} \oplus \mathcal{U}_{b,d}.$$

   Let $D' = \mathcal{U}_{v,w} \oplus \mathcal{U}_{x,y}$ be a representation. Then $a_k(D) \leq a_k(D') \leq 2$ for all k, thus, D is the sum of at most two indecomposables as well since $a_n(D)$ equals the number of indecomposable direct summands of D up to isomorphisms. The representation D, thus, is isomorphic to some D_i with $i \in \{1, \dots, 12\}$.
   For $k, l \in \{1, \dots, n\}$ we calculate

$$a_k(D_1) = \delta_{c\leq k} + \delta_{d\leq k} = a_k(D_2),$$
$$a_k(D_3) = \delta_{b\leq k} + \delta_{d\leq k} = a_k(D_5),$$
$$a_k(D_4) = \delta_{a\leq k} + \delta_{d\leq k} = a_k(D_7) \mid a_k(D_6) = \delta_{b\leq k} + \delta_{c\leq k} = a_k(D_9),$$
$$a_k(D_8) = \delta_{a\leq k} + \delta_{c\leq k} = a_k(D_{11}),$$
$$a_k(D_{10}) = \delta_{a\leq k} + \delta_{b\leq k} = a_k(D_{12}).$$

Furthermore,

$$
\begin{aligned}
b_{k,l}(D_1) &= \delta_{d\leq l} + \delta_{a\leq l}\delta_{d\leq k} + \delta_{c\leq l} + \delta_{b\leq l}\delta_{c\leq k},\\
b_{k,l}(D_2) &= \delta_{c\leq l} + \delta_{a\leq l}\delta_{c\leq k} + \delta_{d\leq l} + \delta_{b\leq l}\delta_{d\leq k},\\
b_{k,l}(D_3) &= \delta_{d\leq l} + \delta_{a\leq l}\delta_{d\leq k} + \delta_{b\leq l} + \delta_{c\leq l}\delta_{b\leq k},\\
b_{k,l}(D_4) &= \delta_{a\leq l} + \delta_{c\leq l}\delta_{a\leq k} + \delta_{d\leq l} + \delta_{b\leq l}\delta_{d\leq k},\\
b_{k,l}(D_5) &= \delta_{b\leq l} + \delta_{a\leq l}\delta_{b\leq k} + \delta_{d\leq l} + \delta_{c\leq l}\delta_{d\leq k},\\
b_{k,l}(D_6) &= \delta_{c\leq l} + \delta_{a\leq l}\delta_{c\leq k} + \delta_{b\leq l} + \delta_{d\leq l}\delta_{b\leq k},\\
b_{k,l}(D_7) &= \delta_{a\leq l} + \delta_{b\leq l}\delta_{a\leq k} + \delta_{d\leq l} + \delta_{c\leq l}\delta_{d\leq k},\\
b_{k,l}(D_8) &= \delta_{a\leq l} + \delta_{d\leq l}\delta_{a\leq k} + \delta_{c\leq l} + \delta_{b\leq l}\delta_{c\leq k},\\
b_{k,l}(D_9) &= \delta_{b\leq l} + \delta_{a\leq l}\delta_{b\leq k} + \delta_{c\leq l} + \delta_{d\leq l}\delta_{c\leq k},\\
b_{k,l}(D_{10}) &= \delta_{a\leq l} + \delta_{d\leq l}\delta_{a\leq k} + \delta_{b\leq l} + \delta_{c\leq l}\delta_{b\leq k},\\
b_{k,l}(D_{11}) &= \delta_{a\leq l} + \delta_{b\leq l}\delta_{a\leq k} + \delta_{c\leq l} + \delta_{d\leq l}\delta_{c\leq k},\\
b_{k,l}(D_{12}) &= \delta_{a\leq l} + \delta_{c\leq l}\delta_{a\leq k} + \delta_{b\leq l} + \delta_{d\leq l}\delta_{b\leq k}.
\end{aligned}
$$

First, let D and D' be representations in $\mathrm{rep}_K^{\mathrm{inj}}(Q_n, I)(\underline{d}_B)_{\mathrm{part}}$ with $a_k(D) = a_k(D')$ for all k.

- $\mathbf{D_1} <_{\mathrm{deg}}^{\mathbf{1.}} \mathbf{D_2}$, since $b_{k,l}(D_1) \leq b_{k,l}(D_2)$ translates to $\delta_{d\leq k} \leq \delta_{c\leq k}$ if $a \leq l < b$ and is obvious if $l < a$ or $b \leq l$.
  Assume there is a representation $L \in \mathrm{rep}_K^{\mathrm{inj}}(Q_n, I)(\underline{d}_B)_{\mathrm{part}}$ that fulfills $D_1 <_{\mathrm{deg}} L <_{\mathrm{deg}} D_2$. Then $a_k(D_1) = a_k(L) = a_k(D_2)$, so $L \cong D_1$ or $L \cong D_2$, because D_1 and D_2 are up to isomorphisms the only representations fulfilling this condition.
  The same proof shows minimality in the next cases.
- $\mathbf{D_3} <_{\mathrm{deg}}^{\mathbf{7.}} \mathbf{D_5}$ since $b_{k,l}(D_3) \leq b_{k,l}(D_5)$ translates to $\delta_{d\leq k} \leq \delta_{b\leq k}$ if $a \leq l < c$ and is obvious if $l < a$ or $c \leq l$.
- $\mathbf{D_4} <_{\mathrm{deg}}^{\mathbf{9.}} \mathbf{D_7}$ since $b_{k,l}(D_4) \leq b_{k,l}(D_7)$ translates to $\delta_{d\leq k} \leq \delta_{a\leq k}$ if $b \leq l < c$ and is obvious if $l < b$ or $d \leq l$.
- $\mathbf{D_6} <_{\mathrm{deg}}^{\mathbf{14.}} \mathbf{D_9}$ since $b_{k,l}(D_6) \leq b_{k,l}(D_9)$ translates to $\delta_{c\leq k} \leq \delta_{b\leq k}$ if $a \leq l < d$ and is obvious if $l < a$ or $d \leq l$.
- $\mathbf{D_8} <_{\mathrm{deg}}^{\mathbf{18.}} \mathbf{D_{11}}$ since $b_{k,l}(D_8) \leq b_{k,l}(D_{11})$ translates to $\delta_{c\leq k} \leq \delta_{a\leq k}$ if $b \leq l < d$ and is obvious if $l < b$ or $d \leq l$.
- $\mathbf{D_{10}} <_{\mathrm{deg}}^{\mathbf{21.}} \mathbf{D_{12}}$ since $b_{k,l}(D_{10}) \leq b_{k,l}(D_{12})$ translates to $\delta_{b\leq k} \leq \delta_{a\leq k}$ if $c \leq l < d$ and is obvious if $l < c$ or $d \leq l$.

We obtain the diagram

$$
\begin{gathered}
D_1 <_{\mathrm{mdeg}} D_2\\
D_3 <_{\mathrm{mdeg}} D_5\\
D_4 <_{\mathrm{mdeg}} D_7 \mid D_6 <_{\mathrm{mdeg}} D_9\\
D_8 <_{\mathrm{mdeg}} D_{11}\\
D_{10} <_{\mathrm{mdeg}} D_{12}.
\end{gathered}
$$

For two representations D_i and D_j in some cases neither $D_i <_{\deg} D_j$ nor $D_j <_{\deg} D_i$ hold true. We will look at these cases now.

- $\mathbf{D_2} \not<_{\deg} \mathbf{D_3}$, since $b_{k,l}(D_2) \leq b_{k,l}(D_3)$ translates to
$$\delta_{c\leq l} + \delta_{a\leq l}\delta_{c\leq k} + \delta_{b\leq l}\delta_{d\leq k} \leq \delta_{a\leq l}\delta_{d\leq k} + \delta_{b\leq l} + \delta_{c\leq l}\delta_{b\leq k}.$$
We, therefore, obtain $b_{c,a}(D_2) > b_{c,a}(D_3)$.
- $\mathbf{D_5} \not<_{\deg} \mathbf{D_4}$, since $b_{k,l}(D_5) \leq b_{k,l}(D_4)$ translates to
$$\delta_{b\leq l} + \delta_{a\leq l}\delta_{b\leq k} + \delta_{c\leq l}\delta_{d\leq k} \leq \delta_{a\leq l} + \delta_{c\leq l}\delta_{a\leq k} + \delta_{b\leq l}\delta_{d\leq k}.$$
Hence, $b_{b,b}(D_5) > b_{b,b}(D_4)$.
- $\mathbf{D_5} \not<_{\deg} \mathbf{D_6}$, since $b_{k,l}(D_5) \leq b_{k,l}(D_6)$ translates to
$$\delta_{a\leq l}\delta_{b\leq k} + \delta_{d\leq l} + \delta_{c\leq l}\delta_{d\leq k} \leq \delta_{c\leq l} + \delta_{a\leq l}\delta_{c\leq k} + \delta_{d\leq l}\delta_{b\leq k},$$
such that the inequality $b_{b,a}(D_5) > b_{b,a}(D_6)$ follows.
- $\mathbf{D_5} \not<_{\deg} \mathbf{D_8}$, since $b_{k,l}(D_5) \leq b_{k,l}(D_8)$ translates to
$$\delta_{b\leq l} + \delta_{a\leq l}\delta_{b\leq k} + \delta_{d\leq l} + \delta_{c\leq l}\delta_{d\leq k} \leq \delta_{a\leq l} + \delta_{d\leq l}\delta_{a\leq k} + \delta_{c\leq l} + \delta_{b\leq l}\delta_{c\leq k}.$$
Then, clearly, $b_{b,b}(D_5) > b_{b,b}(D_8)$.
- $\mathbf{D_7} \not<_{\deg} \mathbf{D_8}$, since $b_{k,l}(D_7) \leq b_{k,l}(D_8)$ translates to
$$\delta_{b\leq l}\delta_{a\leq k} + \delta_{d\leq l} + \delta_{c\leq l}\delta_{d\leq k} \leq \delta_{d\leq l}\delta_{a\leq k} + \delta_{c\leq l} + \delta_{b\leq l}\delta_{c\leq k}.$$
Therefore, $b_{a,b}(D_7) > b_{a,b}(D_8)$ holds true.
- $\mathbf{D_9} \not<_{\deg} \mathbf{D_8}$, since $b_{k,l}(D_9) \leq b_{k,l}(D_8)$ translates to
$$\delta_{b\leq l} + \delta_{a\leq l}\delta_{b\leq k} + \delta_{d\leq l}\delta_{c\leq k} \leq \delta_{a\leq l} + \delta_{d\leq l}\delta_{a\leq k} + \delta_{b\leq l}\delta_{c\leq k}.$$
We, therefore, obtain $b_{b,b}(D_9) > b_{b,b}(D_8)$.
- $\mathbf{D_{11}} \not<_{\deg} \mathbf{D_{10}}$, since $b_{k,l}(D_{11}) \leq b_{k,l}(D_{10})$ translates to
$$\delta_{b\leq l}\delta_{a\leq k} + \delta_{c\leq l} + \delta_{d\leq l}\delta_{c\leq k} \leq \delta_{d\leq l}\delta_{a\leq k} + \delta_{b\leq l} + \delta_{c\leq l}\delta_{b\leq k}.$$
Thus, $b_{a,c}(D_{11}) > b_{a,c}(D_{10})$ holds true.

Several degenerations, more precisely the degenerations 2., 3., 5., 11., 12., 16., 17. and 20., are not disjoint. We have proven that they, in fact, are degenerations and will prove minimality in the following.

- $\mathbf{D_1} <_{\deg}^{\mathbf{2.}} \mathbf{D_3}$ is minimal, since any representation $L \in \mathrm{rep}_K^{\mathrm{inj}}(Q_n, I)(\underline{d}_B)_{\mathrm{part}}$ fulfilling $D_1 <_{\deg} L <_{\deg} D_3$ has to be isomorphic to D_2. Since we have shown $D_1 \not<_{\deg} D_2$, minimality follows.
- $\mathbf{D_2} <_{\deg}^{\mathbf{3.}} \mathbf{D_4}$ is minimal, since any representation $L \in \mathrm{rep}_K^{\mathrm{inj}}(Q_n, I)(\underline{d}_B)_{\mathrm{part}}$ which fulfills $D_2 <_{\deg} L <_{\deg} D_4$ has to be isomorphic to D_3 or D_5. By our considerations above, $D_2 \not<_{\deg} D_3$ and $D_5 \not<_{\deg} D_4$, thus, minimality follows.
- $\mathbf{D_2} <_{\deg}^{\mathbf{5.}} \mathbf{D_6}$ is minimal, since any representation $L \in \mathrm{rep}_K^{\mathrm{inj}}(Q_n, I)(\underline{d}_B)_{\mathrm{part}}$ that fulfills $D_2 <_{\deg} L <_{\deg} D_6$ has to be isomorphic to D_3 or D_5. Minimality follows from $D_2 \not<_{\deg} D_3$ and $D_5 \not<_{\deg} D_6$.
- $\mathbf{D_5} <_{\deg}^{\mathbf{11.}} \mathbf{D_7}$ is minimal, since any representation $L \in \mathrm{rep}_K^{\mathrm{inj}}(Q_n, I)(\underline{d}_B)_{\mathrm{part}}$ for which $D_5 <_{\deg} L <_{\deg} D_7$ holds true has to be isomorphic to D_4. We have proved $D_5 \not<_{\deg} D_4$, therefore, minimality follows.

- $\mathbf{D_5} <^{\mathbf{12.}}_{\text{deg}} \mathbf{D_9}$ is minimal, since any representation $L \in \text{rep}^{\text{inj}}_K(Q_n, I)(\underline{d}_B)_{\text{part}}$ such that $D_5 <_{\text{deg}} L <_{\text{deg}} D_9$ has to be isomorphic to D_6. Our considerations show $D_5 \not<_{\text{deg}} D_6$ and minimality follows.
- $\mathbf{D_7} <^{\mathbf{16.}}_{\text{deg}} \mathbf{D_{11}}$ is minimal, since any representation $L \in \text{rep}^{\text{inj}}_K(Q_n, I)(\underline{d}_B)_{\text{part}}$ for which $D_7 <_{\text{deg}} L <_{\text{deg}} D_{11}$ holds true has to be isomorphic to D_8. We have shown $D_7 \not<_{\text{deg}} D_8$, thus, minimality follows.
- $\mathbf{D_8} <^{\mathbf{17.}}_{\text{mdeg}} \mathbf{D_{10}}$ is minimal, since any representation $L \in \text{rep}^{\text{inj}}_K(Q_n, I)(\underline{d}_B)_{\text{part}}$ which fulfills $D_8 <_{\text{deg}} L <_{\text{deg}} D_{10}$ has to be isomorphic to D_{11}. Since we have proved $D_{11} \not<_{\text{deg}} D_{10}$, minimality follows.
- $\mathbf{D_9} <^{\mathbf{20.}}_{\text{deg}} \mathbf{D_{11}}$ is minimal, since any representation $L \in \text{rep}^{\text{inj}}_K(Q_n, I)(\underline{d}_B)_{\text{part}}$ fulfilling $D_9 <_{\text{deg}} L <_{\text{deg}} D_{11}$ has to be isomorphic to D_8. By our considerations above, $D_9 \not<_{\text{deg}} D_8$, and minimality follows.

As a last step, we calculate all minimal degenerations that have not been considered yet, but are still possible at this point.

- $\mathbf{D_2} <^{\mathbf{4.}}_{\text{mdeg}} \mathbf{D_5}$:
  The inequality $b_{k,l}(D_2) \leq b_{k,l}(D_5)$ translates to $\delta_{c\leq k} \leq \delta_{b\leq k}$ if $a \leq l < b$, to $\delta_{c\leq k} + \delta_{d\leq k} \leq 1 + \delta_{b\leq k}$ if $b \leq l < c$ and to $\delta_{c\leq k} \leq \delta_{b\leq k}$ if $c \leq l$. It obviously holds true for $l < a$, thus, minimality follows from $D_2 \not<_{\text{deg}} D_3$.
- $\mathbf{D_3} <^{\mathbf{6.}}_{\text{mdeg}} \mathbf{D_4}$:
  The inequality $b_{k,l}(D_3) \leq b_{k,l}(D_4)$ translates to $\delta_{d\leq k} \leq 1$ if $a \leq l < b$ and to $\delta_{b\leq k} \leq \delta_{a\leq k}$ if $c \leq l$. It obviously holds true for $l < a$ or $b \leq l < c$, thus, minimality follows from $D_5 \not<_{\text{deg}} D_4$.
- $\mathbf{D_3} <^{\mathbf{8.}}_{\text{mdeg}} \mathbf{D_6}$:
  The inequality $b_{k,l}(D_3) \leq b_{k,l}(D_6)$ translates to $\delta_{d\leq k} \leq \delta_{c\leq k}$ if $a \leq l < c$, to $\delta_{d\leq k} + \delta_{b\leq k} \leq 1 + \delta_{c\leq k}$ if $c \leq l < d$ and to $\delta_{d\leq k} \leq \delta_{c\leq k}$ if $d \leq l$. The inequality is obviously true for $l < a$ and, since we have shown $D_5 \not<_{\text{deg}} D_6$, minimality follows.
- $\mathbf{D_4} <^{\mathbf{10.}}_{\text{mdeg}} \mathbf{D_8}$:
  The inequality $b_{k,l}(D_4) \leq b_{k,l}(D_8)$ translates to $\delta_{d\leq k} \leq \delta_{c\leq k}$ if $b \leq l < c$, to $\delta_{a\leq k} + \delta_{d\leq k} \leq 1 + \delta_{c\leq k}$ if $c \leq l < d$ and to $\delta_{d\leq k} \leq \delta_{c\leq k}$ if $d \leq l$. For $l < b$, it clearly holds true as well. The degeneration is minimal, since we have shown $D_7 \not<_{\text{deg}} D_8$.
- $\mathbf{D_6} <^{\mathbf{13.}}_{\text{mdeg}} \mathbf{D_8}$:
  The inequality $b_{k,l}(D_6) \leq b_{k,l}(D_8)$ translates to $\delta_{c\leq k} \leq 1$ if $a \leq l < b$ and to $\delta_{b\leq k} \leq \delta_{a\leq k}$ if $d \leq l$; it obviously holds true for $l < a$ or $b \leq l < d$. The minimality of the degeneration follows from $D_9 \not<_{\text{deg}} D_8$.
- $\mathbf{D_7} <^{\mathbf{15.}}_{\text{mdeg}} \mathbf{D_{10}}$:
  The inequality $b_{k,l}(D_7) \leq b_{k,l}(D_{10})$ translates to $\delta_{a\leq k} \leq 1$ if $b \leq l < c$, to $\delta_{a\leq k} + \delta_{d\leq k} \leq 1 + \delta_{b\leq k}$ if $c \leq l < d$ and to $\delta_{d\leq k} \leq \delta_{b\leq k}$ if $d \leq l$; it obviously holds true for $l < b$. Because we have shown $D_7 \not<_{\text{deg}} D_8$ and $D_{11} \not<_{\text{deg}} D_{10}$, minimality follows.

- $\mathbf{D_9} <^{\mathbf{19.}}_{\text{deg}} \mathbf{D_{10}}$:
  The inequality $b_{k,l}(D_9) \leq b_{k,l}(D_{10})$ translates to $\delta_{b\leq k} \leq 1$ if $a \leq l < c$ and to $\delta_{c\leq k} \leq \delta_{a\leq k}$ if $d \leq l$. For $l < a$ or $c \leq l < d$ it obviously holds true as well. The degeneration is minimal, since we have shown $D_9 \not<_{\text{deg}} D_8$ and $D_{11} \not<_{\text{deg}} D_{10}$.
- $\mathbf{D_{11}} <^{\mathbf{22.}}_{\text{deg}} \mathbf{D_{12}}$:
  The inequality $b_{k,l}(D_{11}) \leq b_{k,l}(D_{12})$ translates to $\delta_{a\leq k} \leq 1$ if $b \leq l < c$ and to $\delta_{c\leq k} \leq \delta_{b\leq k}$ if $d \leq l$. It obviously holds true if $l < b$ or if $c \leq l < d$, therefore, it suffices to show minimality which follows from $D_{11} \not<_{\text{deg}} D_{10}$.

We obtain diagram C. □

We can prove the following result about the codimensions of these minimal, disjoint degenerations.

Corollary 3.3.6. *(Codimension of minimal, disjoint degenerations)*
Each minimal, disjoint degeneration $D <_{\text{mdeg}} D'$ in theorem 3.3.5 has codimension 1.

Proof. Let $D <_{\text{mdeg}} D'$ be a minimal, disjoint degeneration in $\text{rep}_K^{\text{inj}}(Q_n, I)(\underline{d}_B)_{\text{part}}$. Then $\text{codim}(D, D') = \dim \overline{O_D} - \dim \overline{O_{D'}} = [D', D'] - [D, D]$.
For every two integers $x, y \in \{1, \ldots, n\}$, proposition 3.2.1 yields

$$[\mathcal{V}_x, \mathcal{V}_x] = 1 \text{ and } [\mathcal{U}_{x,y}, \mathcal{U}_{x,y}] = \begin{cases} 2, & \text{if } x \leq y; \\ 1, & \text{otherwise.} \end{cases}$$

We make use of the notation of the representations given in the proof of theorem 3.3.5.

1. Let $D <_{\text{mdeg}} D'$ be a degeneration in diagram A of theorem 3.3.5.

$$[\mathcal{U}_{b,a}, \mathcal{U}_{b,a}] = 1,$$
$$[\mathcal{U}_{a,b}, \mathcal{U}_{a,b}] = 2,$$
$$[\mathcal{V}_a \oplus \mathcal{V}_b, \mathcal{V}_a \oplus \mathcal{V}_b] = 2 + \delta_{a\leq b} + \delta_{b\leq a} = 3.$$

   Then $\text{codim}(D, D') = 1$ follows for both degenerations.

2. Let $D <_{\text{mdeg}} D'$ be a degeneration in diagram B of theorem 3.3.5. Then

$$[D_1, D_1] = 1 + \delta_{b\leq a} + \delta_{c\leq b} + 1 = 2$$
$$[D_2, D_2] = 1 + \delta_{c\leq a} + \delta_{b\leq c} + 1 = 3$$
$$[D_3, D_3] = 1 + \delta_{a\leq b} + \delta_{c\leq a} + 1 = 3$$
$$[D_4, D_4] = 2 + \delta_{c\leq b} + \delta_{a\leq c} + 1 = 4$$
$$[D_5, D_5] = 2 + \delta_{a\leq c} + \delta_{b\leq a} + 1 = 4$$
$$[D_6, D_6] = 2 + \delta_{b\leq c} + \delta_{a\leq b} + 1 = 5.$$

   The equality $\text{codim}(D, D') = 1$ holds true for every degeneration in diagram B.

3. Let $D <_{\text{mdeg}} D'$ be a degeneration in diagram C of theorem 3.3.5. Then

$$\begin{aligned}
[D_1, D_1] &= 2 + \delta_{c\le a} + \delta_{b\le a}\delta_{c\le d} + \delta_{d\le b} + \delta_{a\le b}\delta_{d\le c} = 2\\
[D_2, D_2] &= 2 + \delta_{d\le a} + \delta_{b\le a}\delta_{d\le c} + \delta_{c\le b} + \delta_{a\le b}\delta_{c\le d} = 3\\
[D_3, D_3] &= 3 + \delta_{b\le a} + \delta_{c\le a}\delta_{b\le d} + \delta_{d\le c} + \delta_{a\le c}\delta_{d\le b} = 3\\
[D_4, D_4] &= 3 + \delta_{d\le c} + \delta_{b\le c}\delta_{d\le a} + \delta_{a\le b} + \delta_{c\le b}\delta_{a\le d} = 4\\
[D_5, D_5] &= 2 + \delta_{d\le a} + \delta_{c\le a}\delta_{d\le b} + \delta_{b\le c} + \delta_{a\le c}\delta_{b\le d} = 4\\
[D_6, D_6] &= 3 + \delta_{b\le a} + \delta_{d\le a}\delta_{b\le c} + \delta_{c\le d} + \delta_{a\le d}\delta_{c\le b} = 4\\
[D_7, D_7] &= 3 + \delta_{d\le b} + \delta_{c\le b}\delta_{d\le a} + \delta_{a\le c} + \delta_{b\le c}\delta_{a\le d} = 5\\
[D_8, D_8] &= 3 + \delta_{c\le d} + \delta_{b\le d}\delta_{c\le a} + \delta_{a\le b} + \delta_{d\le b}\delta_{a\le c} = 5\\
[D_9, D_9] &= 3 + \delta_{c\le a} + \delta_{d\le a}\delta_{c\le b} + \delta_{b\le d} + \delta_{a\le d}\delta_{b\le c} = 5\\
[D_{10}, D_{10}] &= 4 + \delta_{b\le d} + \delta_{c\le d}\delta_{b\le a} + \delta_{a\le c} + \delta_{d\le c}\delta_{a\le b} = 6\\
[D_{11}, D_{11}] &= 4 + \delta_{c\le b} + \delta_{d\le b}\delta_{c\le a} + \delta_{a\le d} + \delta_{b\le d}\delta_{a\le c} = 6\\
[D_{12}, D_{12}] &= 4 + \delta_{b\le c} + \delta_{d\le c}\delta_{b\le a} + \delta_{a\le d} + \delta_{c\le d}\delta_{a\le b} = 7.
\end{aligned}$$

Thus, $\operatorname{codim}(D, D') = 1$ holds true for every minimal, disjoint degeneration. □

Note that the codimension 1 property is obtained from the theory of spherical varieties (see [Brion, 1989, Brion, 1995]) as well (a proof is, for example, given in [Timashev, 1994]).

3.3.2 Minimal degenerations in general

Every minimal, disjoint degeneration $D <_{\text{mdeg}} D'$ in $\text{rep}_K^{\text{inj}}(Q_n, I)(\underline{d}_B)_{\text{part}}$ is described in section 3.3.1. In order to classify the minimal degenerations in $\text{rep}_K^{\text{inj}}(Q_n, I)(\underline{d}_B)$, we need to consider every such minimal, disjoint degeneration $D <_{\text{mdeg}} D'$ and all representations W, such that $D \oplus W <_{\text{deg}} D' \oplus W$ is a degeneration in $\text{rep}_K^{\text{inj}}(Q_n, I)(\underline{d}_B)$. We give an explicit description of those representations W for which the aforementioned degeneration is minimal.

The following proposition is derived directly from proposition 3.2.1.

Proposition 3.3.7. *(Differences of dimensions of homomorphism spaces)*
Let $k, l \in \{1, \dots, n\}$ and consider the minimal degeneration

1. *$D = \mathcal{U}_{t,s} <_{\text{mdeg}} \mathcal{U}_{s,t} = D'$, such that $s < t$.*
   Then $a_k(D') - a_k(D) = \delta_{s\le k<t}$ and $b_{k,l}(D') - b_{k,l}(D) = \delta_{k<t} \cdot \delta_{s\le l<t} + \delta_{s\le k<t} \cdot \delta_{t\le l}$.

2. *$D = \mathcal{U}_{s,t} <_{\text{mdeg}} \mathcal{V}_s \oplus \mathcal{V}_t = D'$, such that $s < t$.*
   Then $a_k(D') - a_k(D) = \delta_{t\le k}$ and $b_{k,l}(D') - b_{k,l}(D) = \delta_{k<s} \cdot \delta_{t\le l}$.

3. *$D = \mathcal{U}_{r,t} \oplus \mathcal{V}_s <_{\text{mdeg}} \mathcal{U}_{s,t} \oplus \mathcal{V}_r = D'$, such that $s < r$.*
   Then $a_k(D') - a_k(D) = 0$ and $b_{k,l}(D') - b_{k,l}(D) = \delta_{s\le k<r} \cdot \delta_{t\le l}$.

4. *$D = \mathcal{U}_{r,s} \oplus \mathcal{V}_t <_{\text{mdeg}} \mathcal{U}_{r,t} \oplus \mathcal{V}_s = D'$, such that $s < t$.*
   $a_k(D') - a_k(D) = \delta_{s\le k<t}$ and $b_{k,l}(D') - b_{k,l}(D) = \delta_{k<r} \cdot \delta_{s\le l<t}$.

5. *$D = \mathcal{U}_{u,t} \oplus \mathcal{U}_{s,r} <_{\text{mdeg}} \mathcal{U}_{s,t} \oplus \mathcal{U}_{u,r} = D'$, such that $r < t$ and $u < s$.*
   Then $a_k(D') - a_k(D) = 0$ and $b_{k,l}(D') - b_{k,l}(D) = \delta_{u\le k<s} \cdot \delta_{r\le l<t}$.

6. *$D = \mathcal{U}_{t,s} \oplus \mathcal{U}_{r,u} <_{\mathrm{mdeg}} \mathcal{U}_{t,r} \oplus \mathcal{U}_{s,u} = D'$, such that $u < t < s < r$, $t < u < s < r$, $t < s < u < r$, $s < t < r < u$ or $t < s < r < u$.*
   Then $a_k(D') - a_k(D) = \delta_{s \leq k < r}$ and $b_{k,l}(D') - b_{k,l}(D) = \delta_{k<t} \cdot \delta_{s \leq l < r} + \delta_{s \leq k < r} \cdot \delta_{u \leq l}$.

7. *$D = \mathcal{U}_{t,s} \oplus \mathcal{U}_{r,u} <_{\mathrm{mdeg}} \mathcal{U}_{r,t} \oplus \mathcal{U}_{u,s} = D'$, such that $s < u < t < r$, $u < t < s < r$ or $u < t < r < s$.*
   Then $a_k(D') - a_k(D) = \delta_{u \leq k < t}$ and $b_{k,l}(D') - b_{k,l}(D) = \delta_{k<r} \cdot \delta_{u \leq l < t} + \delta_{u \leq k < t} \cdot \delta_{s \leq l}$.

Here, $\delta_{x \leq y \leq z} := 1$ if $x \leq y \leq z$ and $\delta_{x \leq y \leq z} := 0$ otherwise.

In order to apply these descriptions in the next proof, it is useful to depict them as follows: We consider matrices $(a_k(D') - a_k(D))_{1 \leq k \leq n} \in K^{1 \times n}$ and $(b_{k,l}(D') - b_{k,l}(D))_{1 \leq k,l \leq n} \in K^{n \times n}$. They can, for example in case of the first degeneration, be visualized as in figure 3.2. The light blue parts yield that the entries equal 1, in the dark blue parts they equal 2 (which is only possible in the case 7.) and otherwise they equal 0.

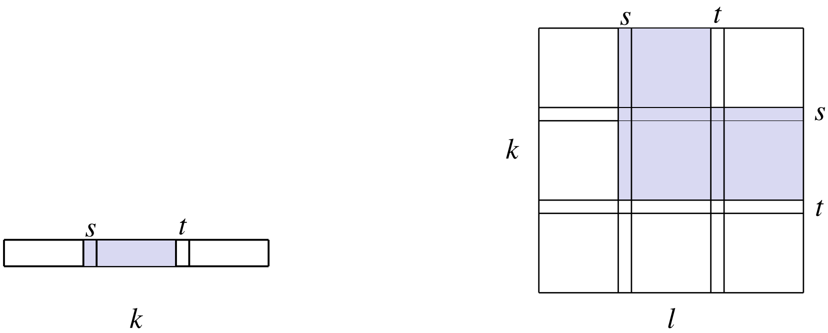


Figure 3.2: The matrices $(a_k(D') - a_k(D))_k$ and $(b_{k,l}(D') - b_{k,l}(D))_{k,l}$ in the case 1.

In the remaining cases of proposition 3.3.7, we depict the matrices $(a_k(D') - a_k(D))_k$ in figure 3.3 and the matrices $(b_{k,l}(D') - b_{k,l}(D))_{k,l}$ exemplary in figure 3.4.

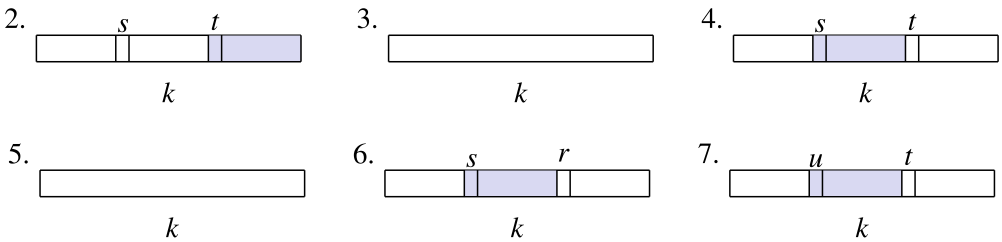


Figure 3.3: The matrices $(a_k(D') - a_k(D))_k$

We divide the calculation of the minimal degenerations in $\mathrm{rep}_K^{\mathrm{inj}}(Q_n, I)(\underline{d}_B)$ into two parts, starting with the degeneration $\mathcal{U}_{t,s} <_{\mathrm{mdeg}} \mathcal{U}_{s,t}$ since it is the only disjoint degeneration $D <_{\mathrm{mdeg}} D'$, such that D' is indecomposable in $\mathrm{rep}_K^{\mathrm{inj}}(Q_n, I)$.

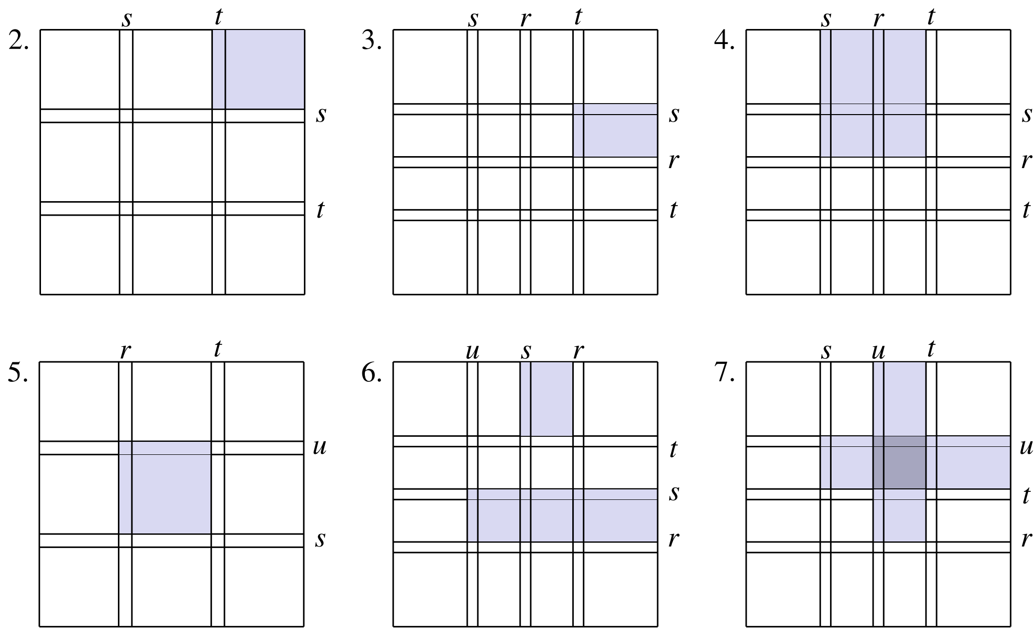


Figure 3.4: The matrices $(b_{k,l}(D') - b_{k,l}(D))_{k,l}$

In this particular setup, the visualizations can be helpful in order to understand the different cases given in the proof of theorem 3.3.8.

Afterwards, we consider every degeneration $D <_{\text{mdeg}} D'$ left. We already know from proposition 3.3.4 that in the second case $D' \cong U \oplus V$, such that U and V are indecomposable in $\text{rep}_K^{\text{inj}}(Q_n, I)$ and that there exists either an exact sequence $0 \to U \to D \to V \to 0$ or $0 \to V \to D \to U \to 0$.

Minimal degenerations obtained from minimal, disjoint degenerations $D <_{\text{mdeg}} D'$ with D' being indecomposable

Consider integers $s, t \in \{1, \ldots, n\}$ for which $s < t$ holds true and the minimal, disjoint degeneration $\mathcal{U}_{t,s} <_{\text{mdeg}} \mathcal{U}_{s,t}$ in $\text{rep}_K^{\text{inj}}(Q_n, I)(\underline{d}_B)_{\text{part}}$.

Let W be a representation in $\text{rep}_K^{\text{inj}}(Q_n, I)$, such that $\mathcal{U}_{s,t} \oplus W \in \text{rep}_K^{\text{inj}}(Q_n, I)(\underline{d}_B)$.

Theorem 3.3.8. *(Minimal degenerations $\mathcal{U}_{t,s} \oplus W <_{\text{deg}} \mathcal{U}_{s,t} \oplus W$)*
The degeneration $\mathcal{U}_{t,s} \oplus W <_{\text{deg}} \mathcal{U}_{s,t} \oplus W$ is minimal if and only if every indecomposable direct summand X of W fulfills $[X, \mathcal{U}_{s,t}] - [X, \mathcal{U}_{t,s}] = 0$.

Lemma 3.3.7, thus, yields the following corollary.

Corollary 3.3.9. *(Concrete description of theorem 3.3.8)*
$\mathcal{U}_{t,s} \oplus W <_{\text{deg}} \mathcal{U}_{s,t} \oplus W$ is minimal if and only if every direct summand $\mathcal{V}_k$ of W fulfills $k < s$ or $k > t$ and every direct summand $\mathcal{U}_{k,l}$ of W fulfills $l < s$ or $k > t$, or $k < s$ and $l > t$.

Proof of corollary 3.3.9. Lemma 3.3.7 and figure 3.2 state

$$[\mathcal{V}_k, D'] - [\mathcal{V}_k, D] = a_k(D') - a_k(D) = \delta_{s\le k<t} \text{ and}$$

$$[\mathcal{U}_{k,l}, D'] - [\mathcal{U}_{k,l}, D] = b_{k,l}(D') - b_{k,l}(D) = \delta_{k<t} \cdot \delta_{s\le l<t} + \delta_{s\le k<t} \cdot \delta_{t\le l}.$$

Thus, $[X, D'] - [X, D] = 0$ translates to $k < s$ or $k > t$ if $X \cong \mathcal{V}_k$ and to $l < s$ or $k > t$, or $k < s$ and $l > t$ if $X \cong \mathcal{U}_{k,l}$. □

Proof of theorem 3.3.8.

<u>The only-if part</u>

Let $\mathcal{V}_k$ be a representation with $s < k < t$. Then the degeneration $\mathcal{U}_{t,s} \oplus \mathcal{V}_k <_{\text{mdeg}} \mathcal{U}_{s,t} \oplus \mathcal{V}_k$ in $\text{rep}_K^{\text{inj}}(Q_n, I)(\underline{d}_B)_{\text{part}}$ is not minimal since

$$\mathcal{U}_{t,s} \oplus \mathcal{V}_k <_{\text{deg}} \mathcal{U}_{k,s} \oplus \mathcal{V}_t <_{\text{deg}} \mathcal{U}_{s,t} \oplus \mathcal{V}_k$$

are proper degenerations.

Let $\mathcal{U}_{k,l}$ be a representation with $s \neq k < t$ and $s < l < t$ (or $s < k < t$ and $l > t$, respectively). Then the degeneration $\mathcal{U}_{t,s} \oplus \mathcal{U}_{k,l} <_{\text{deg}} \mathcal{U}_{s,t} \oplus \mathcal{U}_{k,l}$ in $\text{rep}_K^{\text{inj}}(Q_n, I)(\underline{d}_B)_{\text{part}}$ is not minimal, since

$$\mathcal{U}_{t,s} \oplus \mathcal{U}_{k,l} <_{\text{deg}} \mathcal{U}_{k,s} \oplus \mathcal{U}_{t,l} <_{\text{deg}} \mathcal{U}_{s,t} \oplus \mathcal{U}_{k,l}$$

$$(\mathcal{U}_{t,s} \oplus \mathcal{U}_{k,l} <_{\text{deg}} \mathcal{U}_{k,s} \oplus \mathcal{U}_{t,l} <_{\text{deg}} \mathcal{U}_{s,t} \oplus \mathcal{U}_{k,l}, \text{ respectively})$$

are proper degenerations.

<u>The if-part</u>

Let $W := \bigoplus_{x=1}^{c} \mathcal{U}_{k_x,l_x} \oplus \bigoplus_{x=1}^{c'} \mathcal{V}_{k'_x}$ be a representation in $\text{rep}_K^{\text{inj}}(Q_n, I)(\underline{d}_B)_{\text{part}}$, such that the representation $M := \mathcal{U}_{t,s} \oplus W$ degenerates to $M' := \mathcal{U}_{s,t} \oplus W$ in $\text{rep}_K^{\text{inj}}(Q_n, I)(\underline{d}_B)$.

Furthermore, for $x \in \{1, \ldots, c'\}$ let either $k'_x < s$ or $k'_x > t$ and for $x \in \{1, \ldots, c\}$ let either $k_x > t$, $l_x < s$ or ($l_x > t$ and $k_x < s$).

Assume, the degeneration $M <_{\text{deg}} M'$ is not minimal.

Then there exists a representation $L = \bigoplus_{x=1}^{a} \mathcal{U}_{o_x,p_x} \bigoplus_{x=1}^{b} \mathcal{V}_{o'_x} \in \text{rep}_K^{\text{inj}}(Q_n, I)(\underline{d}_B)$ that fulfills $M <_{\text{deg}} L <_{\text{deg}} M$. Without loss of generality, we can assume $M <_{\text{mdeg}} L$.

Proposition 3.3.1 states $[\mathcal{V}_k, M] \le [\mathcal{V}_k, L] \le [\mathcal{V}_k, M']$ for all k, in more detail, if $k < s$ or $k \ge t$, then $[\mathcal{V}_k, M] = [\mathcal{V}_k, L] = [\mathcal{V}_k, M']$ (see lemma 3.3.7) and we can translate the statement as follows: The source vertices to the left of $s-1$ and to the right of t coincide in $\text{olp}(M)$, $\text{olp}(L)$ and $\text{olp}(M')$. Also, the number of arrows coincides in all three link patterns since $[\mathcal{V}_n, M] = [\mathcal{V}_n, L] = [\mathcal{V}_n, M']$, therefore, $L = \bigoplus_{x=1}^{c+1} \mathcal{U}_{o_x,p_x} \bigoplus_{x=1}^{c'} \mathcal{V}_{o'_x}$.

Claim 1: Let $\mathcal{U}_{k,l}$ be a direct summand of M, L or M'. If $l < s$ or ($k < s$ and $l > t$) or ($k > t$ and $l > t$), then $\mathcal{U}_{k,l}$ is a direct summand of M, L and M'.

Proof of Claim 1. As proposition 3.3.1 states, given a representation $X \in \text{rep}_K^{\text{inj}}(Q_n, I)(\underline{d}_B)$ the values $a_k(X) = [\mathcal{V}_k, X]$ can be translated to counting the fixed and target vertices to the left of k, the values $b_{k,l}(X) = [\mathcal{U}_{k,l}, X]$ can be translated to counting the number of arrows with a source vertex to the left of l and a target vertex to the left of k in $\text{olp}(X)$ and adding $a_l(X)$.

Lemma 3.3.7 immediately yields

- $a_l(M) = a_l(L) = a_l(M')$ if $l < s$ or $l \geq t$, thus, all source vertices $< s$ and $> t$ coincide in the corresponding three oriented link patterns and
- $b_{k,l}(M) = b_{k,l}(L) = b_{k,l}(M')$ if $l < s$ or ($k < s$ and $l \geq t$) or ($k \geq t$ and $l \geq t$), such that the claimed arrows coincide in all three oriented link patterns.

The translation from the oriented link patterns to the direct summands of M, L and M' proves the claim. □

Claim 2: Let $\mathcal{U}_{k,l}$ be a direct summand of M, L or M'. If $t < k$ and $s < l < t$, then $\mathcal{U}_{k,l}$ is a direct summand of M, L and M'.

Proof of Claim 2. Let $t < k$ and $s < l < t$ for two integers k and l.

First, we assume that $U_{k,l}$ is a direct summand of M, but not a direct summand of L.

Since $M <_{\text{mdeg}} L$, the indecomposable $\mathcal{U}_{k,l}$ must be changed by some minimal, disjoint part of the degeneration. The only possibilities for a change like that are the following:

1st case: The indecomposable $\mathcal{U}_{k',l}$ is a direct summand of L, such that $k \neq k'$.

1.1. The minimal, disjoint part is $\mathcal{U}_{k,l} \oplus \mathcal{V}_{k'} <_{\text{mdeg}} \mathcal{U}_{k',l} \oplus \mathcal{V}_k$, such that $k' < k$:

The indecomposable $\mathcal{V}_{k'}$ can only be a direct summand of M if $k' < s$ or $k' > t$.
If $k' < s$, we obtain $[\mathcal{U}_{k',t}, M] < [\mathcal{U}_{k',t}, L]$ and if $k' > t$, we obtain $[\mathcal{U}_{k',l}, M] < [\mathcal{U}_{k',l}, L]$, a contradiction.

1.2. The minimal, disjoint part is $\mathcal{U}_{k,l} \oplus \mathcal{U}_{k',l'} <_{\text{mdeg}} \mathcal{U}_{k',l} \oplus \mathcal{U}_{k,l'}$, such that $k < k'$ and $l' < l$, or such that $k' < k$ and $l < l'$:

The indecomposable $\mathcal{U}_{k',l'}$ can only be a direct summand of M if $k' > t$ or $l' < s$, or if $k' < s$ and $l' > t$. As has been shown in claim 1, every indecomposable $\mathcal{U}_{i,j}$ with $j < s$, or with $j > t$ and $i < s$ is either a direct summand of M, L and M' or a direct summand of none of them. Thus, $k' > t$ and if $k < k'$ and $l' < l$, we obtain $[\mathcal{U}_{k,l'}, M] < [\mathcal{U}_{k,l'}, L]$. If $k' < k$ and $l < l'$, we obtain $[\mathcal{U}_{k',l}, M] < [\mathcal{U}_{k',l}, L]$, a contradiction.

1.3. The minimal, disjoint part is $\mathcal{U}_{k,l} \oplus \mathcal{U}_{l',k'} <_{\text{mdeg}} \mathcal{U}_{l',k} \oplus \mathcal{U}_{k',l}$:

The indecomposable $\mathcal{U}_{l',k'}$ can only be a direct summand of M if $l' > t$ or $k' < s$, or if $l' < s$ and $k' > t$. As has been shown in claim 1, every indecomposable $\mathcal{U}_{i,j}$ with $j < s$, or with $j > t$ and $i < s$ is either a direct summand of M, L and M' or a direct summand of none of them.

Thus, $l' > t$ and the only cases possible are $l < l' < k' < k$ and $l < k' < k < l'$. We immediately obtain $[\mathcal{U}_{k',l}, M] < [\mathcal{U}_{k',l}, L]$, a contradiction.

2nd case: The indecomposable $\mathcal{U}_{k,l'}$ is a direct summand of L, such that $l \neq l'$.

2.1. The minimal, disjoint part is $\mathcal{U}_{k,l} \oplus \mathcal{V}_{l'} <_{\text{mdeg}} \mathcal{U}_{k,l'} \oplus \mathcal{V}_l$, such that $l < l'$:

The indecomposable $\mathcal{V}_{l'}$ can only be a direct summand of M if $l' < s$ or $l' > t$.
Thus, $l' > t$ and we obtain $[\mathcal{U}_{t,l}, M] < [\mathcal{U}_{t,l}, L]$, a contradiction.

2.2. The minimal, disjoint part is $\mathcal{U}_{k,l} \oplus \mathcal{U}_{l',k'} <_{\text{mdeg}} \mathcal{U}_{k,l'} \oplus \mathcal{U}_{l,k'}$:

The indecomposable $\mathcal{U}_{l',k'}$ can only be a direct summand of M if $l' > t$ or $k' < s$, or if $l' < s$ and $k' > t$. As has been shown in claim 1, every indecomposable $\mathcal{U}_{i,j}$ with $j < s$, or with $j > t$ and $i < s$ is either a direct summand of M, L and M' or a direct summand of none of them, thus, $l' > t$. But then we obtain $[\mathcal{U}_{t,l}, M] < [\mathcal{U}_{t,l}, L]$, a contradiction.

3rd case: The indecomposable $\mathcal{U}_{l,k}$ is a direct summand of L.

Then $[\mathcal{U}_{1,t}, M] < [\mathcal{U}_{1,t}, L]$ if $s > 1$ and $[\mathcal{U}_{t,n}, M] < [\mathcal{U}_{t,n}, L]$ if $t < n$. Of course, if $s = 1$ and $t = n > 2$, no representation W as given in the assumption can exist at all, a contradiction.

Now we assume that $U_{k,l}$ is a direct summand of L, but not a direct summand of M.

As before, the indecomposable $\mathcal{U}_{k,l}$ must have been changed by some minimal, disjoint part of the degeneration. The only possibilities for this change are the following:

1st case: The indecomposable $\mathcal{U}_{k',l}$ is a direct summand of M, such that $k \neq k'$.

1.1. The minimal, disjoint part is $\mathcal{U}_{k',l} \oplus \mathcal{V}_k <_{\text{mdeg}} \mathcal{U}_{k,l} \oplus \mathcal{V}_{k'}$, such that $k < k'$:

The indecomposable $\mathcal{V}_k$ can only be a direct summand of M if $k < s$ or $k > t$, thus, $k > t$ and we obtain $[\mathcal{U}_{k,l}, M] < [\mathcal{U}_{k,l}, L]$, a contradiction.

1.2. The minimal, disjoint part is $\mathcal{U}_{k',l} \oplus \mathcal{U}_{k,l'} <_{\text{mdeg}} \mathcal{U}_{k,l} \oplus \mathcal{U}_{k',l'}$, such that $k' < k$ and $l' < l$, or such that $k < k'$ and $l < l'$:

The indecomposable $\mathcal{U}_{k,l'}$ can only be a direct summand of M if $k > t$ or $l' < s$, or if $k < s$ and $l' > t$ and the indecomposable $\mathcal{U}_{k',l}$ can only be a direct summand of M if $k' > t$ or $l < s$, or if $k' < s$ and $l > t$.
As has been shown in claim 1, every indecomposable $\mathcal{U}_{i,j}$ with $j < s$, or with $j > t$ and $i < s$ is either a direct summand of M,L and M' or a direct summand of none of them. Thus, $k > t$ and $k' > t$ and if $k' < k$ and $l' < l$, we obtain $[\mathcal{U}_{k',l'}, M] < [\mathcal{U}_{k',l'}, L]$; if $k < k'$ and $l < l'$, we obtain $[\mathcal{U}_{k,l}, M] < [\mathcal{U}_{k,l}, L]$, a contradiction.

1.3. The minimal, disjoint part is $\mathcal{U}_{k',l} \oplus \mathcal{U}_{l',k} <_{\text{mdeg}} \mathcal{U}_{l',k'} \oplus \mathcal{U}_{k,l}$:

The indecomposable $\mathcal{U}_{k',l}$ can only be a direct summand of M if $k' > t$ or $l < s$, or if $k' < s$ and $l > t$ and the indecomposable $\mathcal{U}_{l',k}$ can only be a direct summand of M if $l' > t$ or $k < s$, or if $l' < s$ and $k > t$.

As has been shown in claim 1, every indecomposable $\mathcal{U}_{i,j}$ with $j < s$, or with $j > t$ and $i < s$ is either a direct summand of M, L and M' or a direct summand of none of them. Thus, $k' > t$ and $l' > t$ and the only cases possible are $l < l' < k < k'$ and $l < k < k' < l'$, we immediately obtain $[\mathcal{U}_{k,l}, M] < [\mathcal{U}_{k,l}, L]$, a contradiction.

2nd case: The indecomposable $\mathcal{U}_{k,l'}$ is a direct summand of M, such that $l \neq l'$.

2.1. The minimal, disjoint part is $\mathcal{U}_{k,l'} \oplus \mathcal{V}_l <_{\text{mdeg}} \mathcal{U}_{k,l} \oplus \mathcal{V}_{l'}$, such that $l' < l$:

The indecomposable $\mathcal{V}_l$ can only be a direct summand of M if $l < s$ or $l > t$, an immediate contradiction, since $s < l < t$.

2.2. The minimal, disjoint part is $\mathcal{U}_{k,l'} \oplus \mathcal{U}_{l,k'} <_{\text{mdeg}} \mathcal{U}_{k,l} \oplus \mathcal{U}_{l',k'}$:

The indecomposable $\mathcal{U}_{l,k'}$ can only be a direct summand of M if $l > t$ or $k' < s$, or if $l < s$ and $k' > t$. As has been shown in claim 1, every indecomposable $\mathcal{U}_{i,j}$ with $j < s$, or with $j > t$ and $i < s$ is either a direct summand of M, L and M' or a direct summand of none of them, thus, $l > t$ yields an immediate contradiction.

3rd case: The indecomposable $\mathcal{U}_{l,k}$ is a direct summand of M.

The indecomposable $\mathcal{U}_{l,k}$ can only be a direct summand of M if $l > t$ or $k < s$, or if $l < s$ and $k > t$, a contradiction.

Clearly, the indecomposable $\mathcal{U}_{k,l}$ with $t < k$ and $s < l < t$ is a direct summand of M if and only if it is a direct summand of M' due to the definitions $M = \mathcal{U}_{t,s} \oplus W$ and $M' = \mathcal{U}_{s,t} \oplus W$ and we have shown that the direct summands $\mathcal{U}_{k,l}$ with $t < k$ and $s < l < t$ coincide in all three representations M, L and M'. □

Claim 1 and claim 2 show that all arrows $l \to k$ with $b_{k,l}(M) = b_{k,l}(M')$ and $k, l \notin \{s, t\}$ coincide in $\text{olp}(M)$, $\text{olp}(L)$ and $\text{olp}(M')$. More precisely, either there is an arrow $l \to k$ in $\text{olp}(M)$, $\text{olp}(L)$ and $\text{olp}(M')$ or in none of them at all.

The minimal degeneration $M <_{\text{mdeg}} L$, therefore, equals

$$M = \mathcal{U}_{t,s} \oplus \bigoplus_{x=1}^{c} \mathcal{U}_{k_x,l_x} \oplus \bigoplus_{x=1}^{c'} \mathcal{V}_{k'_x} <_{\text{mdeg}} \mathcal{U}_{o,p} \oplus \bigoplus_{x=1}^{c} \mathcal{U}_{k_x,l_x} \bigoplus_{x=1}^{c'} \mathcal{V}_{o'_x} = L.$$

The minimal, disjoint piece of the degeneration has to be one of the following three, since every other option is excluded due to the considerations so far.

- $\mathcal{U}_{t,s} <_{\text{mdeg}} \mathcal{U}_{s,t}$
  Then $L \cong M'$, a contradiction to the assumption $L <_{\text{deg}} M'$;
- $\mathcal{U}_{t,s} \oplus \mathcal{V}_{k'} <_{\text{mdeg}} \mathcal{U}_{t,k'} \oplus \mathcal{V}_s$ with $k' > t$
  In this case $\mathcal{U}_{t,k'} \oplus \mathcal{V}_s \not<_{\text{deg}} \mathcal{U}_{s,t} \oplus \mathcal{V}_{k'}$ due to theorem 3.3.5 and therefore $L \not<_{\text{deg}} M'$, a contradiction;
- $\mathcal{U}_{t,s} \oplus \mathcal{V}_{k'} <_{\text{mdeg}} \mathcal{U}_{k',s} \oplus \mathcal{V}_t$ with $k' < s$
  In this case $\mathcal{U}_{k',s} \oplus \mathcal{V}_t \not<_{\text{deg}} \mathcal{U}_{s,t} \oplus \mathcal{V}_{k'}$ due to theorem 3.3.5 and therefore $L \not<_{\text{deg}} M'$, a contradiction.

Since we obtain a contradiction in each case, the degeneration $M <_{\text{deg}} M'$ is minimal. □

Minimal degenerations obtained from extensions

Consider a minimal, disjoint degeneration $D <_{\mathrm{mdeg}} D'$, such that $D' \cong U \oplus V$ and U and V are indecomposable representations in $\mathrm{rep}_K^{\mathrm{inj}}(Q_n, I)$. Then, without loss of generality, there exists an exact sequence $0 \to U \to D \to V \to 0$. Let W be a representation in $\mathrm{rep}_K^{\mathrm{inj}}(Q_n, I)$, such that $D \oplus W \in \mathrm{rep}_K^{\mathrm{inj}}(Q_n, I)(\underline{d}_B)$.

The aim of this subsection is to give an explicit description of those W for which the degeneration $D \oplus W <_{\mathrm{deg}} D' \oplus W$ is minimal. A great help for proving this condition will be the following two theorems. They have both been proven in [Bongartz, 1994].

For two representations $M, M' \in \mathrm{rep}_K(Q_n, I)$, let us define

$$\langle M, M' \rangle := \{L \in \mathrm{rep}_K(Q_n, I) \mid M \leq_{\mathrm{deg}} L \leq_{\mathrm{deg}} M'\}.$$

Theorem 3.3.10. *(Cancellation of direct summands)*
If $\mathrm{codim}(D, D') = \mathrm{codim}(D \oplus W, D' \oplus W)$, *then the map* $L \mapsto L \oplus W$ *induces an isomorphism between the partially ordered sets* $\langle D, D' \rangle$ *and* $\langle D \oplus W, D' \oplus W \rangle$.

A general description of the above considered minimal degenerations follows.

Theorem 3.3.11. *(Minimal degenerations obtained from extensions)*
The degeneration $D \oplus W <_{\mathrm{deg}} D' \oplus W$ *in* $\mathrm{rep}_K^{\mathrm{inj}}(Q_n, I)(\underline{d}_B)$ *is minimal if and only if the equalities* $[X, D] = [X, D']$ *and* $[D, X] = [D', X]$ *hold true for every direct summand* X *of* W.

Proof. We extract the argumentation from [Bongartz, 1994, Theorem 4].

Let $D \oplus W <_{\mathrm{mdeg}} U \oplus V \oplus W$ be a minimal degeneration in $\mathrm{rep}_K^{\mathrm{inj}}(Q_n, I)(\underline{d}_B)$, such that U and V are indecomposables and there exists an exact sequence $0 \to U \to D \to V \to 0$.

Let X be a direct summand of W, such that $[X, D'] > [X, D]$. Then the exact sequence

$$0 \to U \to D \to V \to 0$$

yields the existence of an exact sequence

$$0 \to \mathrm{Hom}(X, U) \to \mathrm{Hom}(X, D) \to \mathrm{Hom}(X, V) \to \mathrm{Ext}^1(X, U) \to \mathrm{Ext}^1(X, D),$$

such that the last map is not injective.

Thus, there exists a representation Y and an exact sequence $0 \to U \to Y \to X \to 0$, such that the pushout sequence splits and we obtain the commutative diagram

$$\begin{array}{ccccccccc} & & 0 & & 0 & & & & \\ & & \downarrow & & \downarrow & & & & \\ 0 & \to & U & \to & Y & \to & X & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & D & \to & D \oplus X & \to & X & \to & 0 \\ & & \downarrow & & \downarrow & & & & \\ & & V & \to & V & & & & \\ & & \downarrow & & \downarrow & & & & \\ & & 0 & & 0 & & & & \end{array}$$

with $D \oplus X <_{\mathrm{deg}} V \oplus Y <_{\mathrm{deg}} V \oplus U \oplus X$.

We denote by Z the representation that fulfills $W = X \oplus Z$ and obtain

$$D \oplus W <_{\deg} V \oplus Y \oplus Z <_{\deg} D' \oplus W,$$

a contradiction.
A dual argument contradicts the assumption $[D', X] > [D, X]$ for a direct summand X of W.

Assume $[X, D] = [X, D']$ and $[D, X] = [D', X]$ holds true for every direct summand X of W. Then $\mathrm{codim}(D, D') = \mathrm{codim}(D \oplus W, D' \oplus W)$ and theorem 3.3.10 yields that the degeneration $D \oplus W <_{\deg} D' \oplus W$ is minimal if and only if $D <_{\deg} D'$ is. □

In the following theorem we are able to give all minimal degenerations obtained from extensions.

Lemma 3.3.12. *(Minimal degenerations obtained from extensions in detail)*
Let $D <_{\mathrm{mdeg}} D'$ be a minimal, disjoint degeneration in $\mathrm{rep}_K^{\mathrm{inj}}(Q_n, I)(\underline{d}_B)_{part}$ and let

$$M := D \oplus W <_{\deg} D' \oplus W =: M'$$

be a degeneration in $\mathrm{rep}_K^{\mathrm{inj}}(Q_n, I)(\underline{d}_B)$.
Then $M <_{\mathrm{mdeg}} M'$ if and only if (in the numbering of proposition 3.3.7)

2. $[X, D'] - [X, D] = 0$ *for every direct summand $X \cong \mathcal{U}_{k,l}$ of W and $s < k < t$ for every direct summand $\mathcal{V}_k$ of W (if $D \cong \mathcal{U}_{s,t}$) and*

3. $[X, D'] - [X, D] = 0$ *for every direct summand $X \cong \mathcal{U}_{k,l}$ of W and $k < s$ or $k > r$ for every direct summand $\mathcal{V}_k$ of W (if $D \cong \mathcal{U}_{r,t} \oplus \mathcal{V}_s$) and*

4.-7. $[X, D'] - [X, D] = 0$ *for every direct summand X of W (if $D \ncong \mathcal{U}_{s,t}, \mathcal{U}_{r,t} \oplus \mathcal{V}_s$).*

Proof. We consider every minimal, disjoint degeneration $D <_{\mathrm{mdeg}} D'$ separately, numbered referring to proposition 3.3.7. Given one such degeneration and an indecomposable $X \in \mathrm{rep}_K^{\mathrm{inj}}(Q_n, I)$, proposition 3.3.7 states the values of $[X, D'] - [X, D]$. We, thus, calculate $[D', X] - [D, X]$ and via theorem 3.3.11 obtain a condition stating in which cases X can be a direct summand of W in order to $D \oplus W <_{\deg} D' \oplus W$ being minimal.

Let $k, l \notin \{s, t, u, r\}$.

2. $D = \mathcal{U}_{s,t} <_{\mathrm{mdeg}} \mathcal{V}_s \oplus \mathcal{V}_t = D'$, such that $s < t$.

   $[\mathcal{V}_k, D'] - [\mathcal{V}_k, D] = \delta_{t<k} = 0$ if and only if $k < t$.
   Fix $k < t$, then $[D', \mathcal{V}_k] - [D, \mathcal{V}_k] = (\delta_{k<s} + \delta_{k<t}) - \delta_{k<t} = \delta_{k<s}$.
   Thus, $[D', \mathcal{V}_k] - [D, \mathcal{V}_k] = 0$ if $s < k$.

   $[\mathcal{U}_{k,l}, D'] - [\mathcal{U}_{k,l}, D] = \delta_{k<s} \cdot \delta_{t<l} = 0$ if and only if $k > s$ or $l < t$.
   Fix $k > s$ and $l < t$, then $[D', \mathcal{U}_{k,l}] - [D, \mathcal{U}_{k,l}] = (\delta_{k<s} + \delta_{k<t}) - (\delta_{k<t} + \delta_{l<t} \cdot \delta_{k<s}) = 0$.

3. $D = \mathcal{U}_{r,t} \oplus \mathcal{V}_s <_{\text{mdeg}} \mathcal{U}_{s,t} \oplus \mathcal{V}_r = D'$, such that $s < r$.

   $[\mathcal{V}_k, D'] - [\mathcal{V}_k, D] = 0$ and $[D', \mathcal{V}_k] - [D, \mathcal{V}_k] = (\delta_{k<t} + \delta_{k<r}) - (\delta_{k<t} + \delta_{k<s}) = 0$ if and only if $k < s$ or $k > r$.

   $[\mathcal{U}_{k,l}, D'] - [\mathcal{U}_{k,l}, D] = \delta_{s<k<r} \cdot \delta_{t<l} = 0$ if and only if $k < s$, $k > r$ or $l < t$.
   Fix k and l such that $k < s$, $k > r$ or $l < t$, then
   $[D', \mathcal{U}_{k,l}] - [D, \mathcal{U}_{k,l}] = (\delta_{k<t} + \delta_{l<t} \cdot \delta_{k<s} + \delta_{k<r}) - (\delta_{k<t} + \delta_{l<t} \cdot \delta_{k<r} + \delta_{k<s}) = 0.$

4. $D = \mathcal{U}_{r,s} \oplus \mathcal{V}_t <_{\text{mdeg}} \mathcal{U}_{r,t} \oplus \mathcal{V}_s = D'$, such that $s < t$.

   $[\mathcal{V}_k, D'] - [\mathcal{V}_k, D] = \delta_{s<k<t} = 0$ if and only if $k < s$ or $k > t$.
   Fix $k < s$ or $k > t$, then $[D', \mathcal{V}_k] - [D, \mathcal{V}_k] = (\delta_{k<t} + \delta_{k<s}) - (\delta_{k<s} + \delta_{k<t}) = 0$.

   $[\mathcal{U}_{k,l}, D'] - [\mathcal{U}_{k,l}, D] = \delta_{k<r} \cdot \delta_{s<l<t} = 0$ if and only if $k > r$, $l < s$ or $l > t$.
   Fix k and l such that $k > r$, $l < s$ or $l > t$, then
   $[D', \mathcal{U}_{k,l}] - [D, \mathcal{U}_{k,l}] = (\delta_{k<t} + \delta_{l<t} \cdot \delta_{k<r} + \delta_{k<s}) - (\delta_{k<s} + \delta_{l<s} \cdot \delta_{k<r} + \delta_{k<t}) = 0.$

5. $D = \mathcal{U}_{u,t} \oplus \mathcal{U}_{s,r} <_{\text{mdeg}} \mathcal{U}_{s,t} \oplus \mathcal{U}_{u,r} = D'$, such that $r < t$ and $u < s$.

   $[\mathcal{V}_k, D'] - [\mathcal{V}_k, D] = 0$ and $[D', \mathcal{V}_k] - [D, \mathcal{V}_k] = (\delta_{k<t} + \delta_{k<r}) - (\delta_{k<r} + \delta_{k<t}) = 0$.

   $[\mathcal{U}_{k,l}, D'] - [\mathcal{U}_{k,l}, D] = \delta_{u\le k<s} \cdot \delta_{r\le l<t} = 0$ if and only if $k < u$, $k > s$, $l < r$ or $l > t$.
   Fix k and l such that $k < u$, $k > s$, $l < r$ or $l > t$, then
   $$\begin{aligned}[D', \mathcal{U}_{k,l}] - [D, \mathcal{U}_{k,l}] &= (\delta_{k<t} + \delta_{l<t} \cdot \delta_{k<s} + \delta_{k<r} + \delta_{l<r} \cdot \delta_{k<u}) \\ &\quad - (\delta_{k<t} + \delta_{l<t} \cdot \delta_{k<u} + \delta_{k<r} + \delta_{l<r} \cdot \delta_{k<s}) = 0.\end{aligned}$$

6. $D = \mathcal{U}_{t,s} \oplus \mathcal{U}_{r,u} <_{\text{mdeg}} \mathcal{U}_{t,r} \oplus \mathcal{U}_{s,u} = D'$, such that $u < t < s < r$, $t < u < s < r$, $t < s < u < r$, $s < t < r < u$ or $t < s < r < u$.

   $[\mathcal{V}_k, D'] - [\mathcal{V}_k, D] = \delta_{s\le k<r} = 0$ if and only if $k < s$ or $k > r$.
   Fix $k < s$ or $k > r$, then $[D', \mathcal{V}_k] - [D, \mathcal{V}_k] = (\delta_{k<r} + \delta_{k<u}) - (\delta_{k<s} + \delta_{k<u}) = 0$.

   $[\mathcal{U}_{k,l}, D'] - [\mathcal{U}_{k,l}, D] = \delta_{k<t} \cdot \delta_{s\le l<r} + \delta_{s\le k<r} \cdot \delta_{u\le l} = 0$ if and only if
   $k > t$, $l < s$ or $l > r$ holds true and $k < s$, $k > r$ or $l < u$ holds true.
   Fix k and l such that ($k > t$, $l < s$ or $l > r$) and ($k < s$, $k > r$ or $l < u$), then
   $$\begin{aligned}[D', \mathcal{U}_{k,l}] - [D, \mathcal{U}_{k,l}] &= (\delta_{k<r} + \delta_{l<r} \cdot \delta_{k<t} + \delta_{k<u} + \delta_{l<u} \cdot \delta_{k<s}) \\ &\quad - (\delta_{k<s} + \delta_{l<s} \cdot \delta_{k<t} + \delta_{k<u} + \delta_{l<u} \cdot \delta_{k<r}) = 0\end{aligned}$$

7. $D = \mathcal{U}_{t,s} \oplus \mathcal{U}_{r,u} <_{\text{mdeg}} \mathcal{U}_{r,t} \oplus \mathcal{U}_{u,s} = D'$, such that $s < u < t < r$, $u < t < s < r$ or $u < t < r < s$.

   $[\mathcal{V}_k, D'] - [\mathcal{V}_k, D] = \delta_{u\le k<t} = 0$ if and only if $k < u$ or $k > t$.
   Fix $k < u$ or $k > t$, then $[D', \mathcal{V}_k] - [D, \mathcal{V}_k] = (\delta_{k<t} + \delta_{k<s}) - (\delta_{k<s} + \delta_{k<u}) = 0$.

   $[\mathcal{U}_{k,l}, D'] - [\mathcal{U}_{k,l}, D] = \delta_{k<r} \cdot \delta_{u\le l<t} + \delta_{u\le k<t} \cdot \delta_{s\le l} = 0$ if and only if
   $k > r$, $l < u$ or $l > t$ holds true and $k < u$, $k > t$ or $l < s$ holds true.
   Fix k and l such that ($k > r$, $l < u$ or $l > t$) and ($k < u$, $k > t$ or $l < s$), then
   $$\begin{aligned}[D', \mathcal{U}_{k,l}] - [D, \mathcal{U}_{k,l}] &= (\delta_{k<t} + \delta_{l<t} \cdot \delta_{k<r} + \delta_{k<s} + \delta_{l<s} \cdot \delta_{k<u}) \\ &\quad - (\delta_{k<s} + \delta_{l<s} \cdot \delta_{k<t} + \delta_{k<u} + \delta_{l<u} \cdot \delta_{k<r}) = 0.\end{aligned}$$

In all cases, theorem 3.3.11 yields the claim. □

Corollary 3.3.13. *(Explicit minimal degenerations obtained from extensions)*
Lemma 3.3.12 translates as follows: $D \oplus W <_{\mathrm{deg}} D' \oplus W$ is a minimal degeneration in $\mathrm{rep}_K^{\mathrm{inj}}(Q_n, I)(\underline{d}_B)$ if and only if for every direct summand X of W, the following holds true.

2. $D = \mathcal{U}_{s,t} <_{\mathrm{mdeg}} \mathcal{V}_s \oplus \mathcal{V}_t = D'$:
   If $X = \mathcal{V}_k$, then $s < k < t$ and if $X = \mathcal{U}_{k,l}$, then $k > s$ or $l < t$.
3. $D = \mathcal{U}_{r,t} \oplus \mathcal{V}_s <_{\mathrm{mdeg}} \mathcal{U}_{s,t} \oplus \mathcal{V}_r = D'$:
   If $X = \mathcal{V}_k$, then $k < s$ or $k > r$ and if $X = \mathcal{U}_{k,l}$, then $k < s$, $k > r$ or $l < t$.
4. $D = \mathcal{U}_{r,s} \oplus \mathcal{V}_t <_{\mathrm{mdeg}} \mathcal{U}_{r,t} \oplus \mathcal{V}_s = D'$:
   If $X = \mathcal{V}_k$, then $k < s$ or $k > t$ and if $X = \mathcal{U}_{k,l}$, then $k > r$, $l < s$ or $l > t$.
5. $D = \mathcal{U}_{u,t} \oplus \mathcal{U}_{s,r} <_{\mathrm{mdeg}} \mathcal{U}_{s,t} \oplus \mathcal{U}_{u,r} = D'$:
   If $X = \mathcal{U}_{k,l}$, then $k < u$, $k > s$, $l < r$ or $l > t$.
6. $D = \mathcal{U}_{t,s} \oplus \mathcal{U}_{r,u} <_{\mathrm{mdeg}} \mathcal{U}_{t,r} \oplus \mathcal{U}_{s,u} = D'$:
   If $X = \mathcal{V}_k$ then $k < s$ or $k > r$ and if $X = \mathcal{U}_{k,l}$, then $k > t$, $l < s$ or $l > r$ holds true and $k < s$, $k > r$ or $l < u$.
7. $D = \mathcal{U}_{t,s} \oplus \mathcal{U}_{r,u} <_{\mathrm{mdeg}} \mathcal{U}_{r,t} \oplus \mathcal{U}_{u,s} = D'$:
   If $X = \mathcal{V}_k$ then $k < u$ or $k > t$ and if $X = \mathcal{U}_{k,l}$, then $k > r$, $l < u$ or $l > t$ holds true and $k < u$, $k > t$ or $l < s$.

Proof. Follows directly from the proof of lemma 3.3.12. □

Of course, the possible direct summands of W can be read off the values of the matrices $(b_{k,l}(D') - b_{k,l}(D))_{k,l} \in K^{n\times n}$. For whatever k, l the entry $(b_{k,l}(D') - b_{k,l}(D))_{k,l}$ equals zero, the indecomposable $\mathcal{U}_{k,l}$ can be a direct summand of W. The same holds true for $X = \mathcal{V}_k$ except in the two differing cases mentioned in theorem 3.3.11 and of course concerning the matrix $(a_k(D') - a_k(D))_k \in K^{1\times n}$.

Lemma 3.3.14. *(Codimension of minimal degenerations)*
Given $M, M' \in \mathrm{rep}_K^{\mathrm{inj}}(Q_n, I)(\underline{d}_B)$ with $M \leq_{\mathrm{deg}} M'$, we have $M <_{\mathrm{mdeg}} M'$ if and only if $\mathrm{codim}(M, M') = 1$.

Proof. If $M <_{\mathrm{mdeg}} M'$ comes up from the minimal, disjoint degeneration $\mathcal{U}_{t,s} <_{\mathrm{mdeg}} \mathcal{U}_{s,t}$ by adding W, then $[\mathcal{U}_{t,s}, X] = [\mathcal{U}_{s,t}, X]$ whenever X is a direct summand of W follows from theorem 3.3.8 and $[X, \mathcal{U}_{t,s}] = [X, \mathcal{U}_{s,t}]$ follows by a direct calculation.

If the degeneration is obtained by extensions, the results of corollary 3.3.6, theorem 3.3.8 and lemma 3.3.12 yield $\mathrm{codim}(M, M') = 1$.
Let $\mathrm{codim}(M, M') = 1$ and assume that $M \leq_{\mathrm{deg}} M'$ is not minimal. Since M and M' cannot be isomorphic, the degeneration is a chain of at least two minimal degenerations, each of which has at least codimension 1. Thus, $\mathrm{codim}(M, M') > 1$, a contradiction. □

3.3.3 Dimensions and the open orbit

We will now calculate the dimensions of the isotropy groups, the orbits and their closures. Note that $\dim \mathcal{O}_M = \dim \overline{\mathcal{O}_M}$ in $R^{\mathrm{inj}}_{\underline{d}_B}(Q_n, I)$ for each representation $M \in \mathrm{rep}^{\mathrm{inj}}_K(Q_n, I)(\underline{d}_B)$ and $\dim B.N = \dim \overline{B.N}$ in $\mathcal{N}^{(2)}$ for every matrix $N \in \mathcal{N}^{(2)}_n$.

Although the dimensions of the orbits are changed by the bijection Φ in theorem 2.3.1, we can nevertheless obtain from proposition 1.1.8 that codimensions and the dimensions of isotropy groups are being preserved.

Let $N \in \mathcal{N}^{(2)}$ be a 2-nilpotent matrix that corresponds to the representation

$$M = \bigoplus_{i,j=1}^{n} \mathcal{U}_{i,j}^{m_{i,j}} \oplus \bigoplus_{i=1}^{n} \mathcal{V}_i^{n_i}$$

in $\mathrm{rep}^{\mathrm{inj}}_K(Q_n, I)(\underline{d}_B)$ via the bijection of lemma 2.3.1.

Proposition 3.3.15. *(Dimension of* $\mathrm{GL}_{\underline{d}_B}$*-orbits in* $R^{\mathrm{inj}}_{\underline{d}_B}(Q_n, I)$*)*
The equalities

$$\dim \mathrm{Iso}_{\mathrm{GL}_{\underline{d}_B}}(M) = \sum_{i,j=1}^{n} m_{i,j} b_{i,j}(M) + \sum_{i=1}^{n} n_i a_i(M)$$

and

$$\dim \mathcal{O}_M = \sum_{i=1}^{n} i^2 - \sum_{i,j=1}^{n} m_{i,j} b_{i,j}(M) - \sum_{i=1}^{n} n_i a_i(M)$$

hold true.

Proof. Let m be the point in $R^{\mathrm{inj}}_{\underline{d}_B}(Q_n, I)$ corresponding to M.
Since $\dim \mathrm{Iso}_{\mathrm{GL}_{\underline{d}_B}}(m) = [M, M]$, the first equality follows from corollary 3.2.5.
Then $\dim \mathcal{O}_M = \dim \mathrm{GL}_{\underline{d}_B} - \dim \mathrm{Iso}_{\mathrm{GL}_{\underline{d}_B}}(M) = \sum_{i=1}^{n} i^2 - [M, M]$ yields the claim. □

The interpretation in terms of oriented link patterns is as follows:
In order to calculate the dimension of the isotropy group, we add up the invariants $b_{i,j}(M)$ for each arrow $j \to i$ in the oriented link pattern, which were defined in 3.3. Then for each fixed vertex i, we add the invariant $a_i(M)$.

Proposition 3.3.16. *(Minimal* $\mathrm{GL}_{\underline{d}_B}$*-orbit in* $R^{\mathrm{inj}}_{\underline{d}_B}(Q_n, I)$*)*
There is one unique orbit of minimal dimension in $R^{\mathrm{inj}}_{\underline{d}_B}(Q_n, I)$*, which is represented by* $M_0 = \bigoplus_{i=1}^{n} \mathcal{V}_i$ *of dimension*

$$\dim \mathcal{O}_{M_0} = \sum_{i=1}^{n} i^2 - \frac{n(n+1)}{2}.$$

It corresponds naturally to the zero-matrix N_0 *in* $\mathcal{N}^{(2)}$*, that is,* $(N_0)_{i,j} = 0$ *for all* i, j*.*
Of course, $\dim B.N_0 = 0$ *in* $\mathcal{N}^{(2)}$.

Proof. The B-orbit of the zero-matrix N_0 has dimension 0. The bijection Φ of theorem 3.1.2 yields the correspondence to the representation M_0. Since codimensions are being preserved, clearly the orbit corresponding to the isomorphism class of M_0 is the orbit of minimal dimension in $R^{\text{inj}}_{\underline{d}_B}(Q_n, I)$. Its actual dimension can be computed by using proposition 3.3.15. □

Proposition 3.3.17. *(Open* $\mathrm{GL}_{\underline{d}_B}$*-orbit in* $R^{\text{inj}}_{\underline{d}_B}(Q_n, I)$*)*
There is one unique orbit of maximal dimension in $R^{\text{inj}}_{\underline{d}_B}(Q_n, I)$*, namely* $\mathcal{O}_{M_{\max}}$*, such that*

$$M_{\max} = \bigoplus_{k=1}^{n/2} \mathcal{U}_{n-k+1,k} \ \text{ and } \ \dim \mathcal{O}_{M_{\max}} = \sum_{i=1}^{n} i^2 - \frac{n}{2}$$

if n is even and

$$M_{\max} = \bigoplus_{k=1}^{n-1/2} \mathcal{U}_{n-k+1,k} \oplus \mathcal{V}_{\frac{n-1}{2}+1} \ \text{ and } \ \dim \mathcal{O}_{M_{\max}} = \sum_{i=1}^{n} i^2 - \frac{n+1}{2}$$

if n is odd.

Proof. Regardless of n being even or odd, theorem 3.3.5 yields that every oriented link pattern corresponding to an arbitrary representation $M \in \mathrm{rep}^{\text{inj}}_K(Q_n, I)(\underline{d}_B)$ is obtained by applying "decreasing minimal changes" to the oriented link pattern of $M_{\max}$. Thus, for each representation $M_{\max} \not\cong M \in \mathrm{rep}^{\text{inj}}_K(Q_n, I)(\underline{d}_B)$, the degeneration $M_{\max} <_{\deg} M$ is a proper chain of minimal degenerations and $\dim \mathcal{O}_{M_{\max}} - \dim \mathcal{O}_M \geq 1$by lemma 3.3.14. □

We have, thus, found the open orbit in $R^{\text{inj}}_{\underline{d}_B}(Q_n, I)$. Note that there are no extensions between direct summands of the representations in the open orbit if n is even. If n is odd, however, this is not the case. More precisely, $\mathrm{Ext}^1_{\mathcal{A}}(V_{(n+1)/2}, V_{(n+1)/2}) = 1$ holds true.

Of course, we deduce

$$\dim R_{\underline{d}_B}(Q_n, I) = \begin{cases} \sum\limits_{i=1}^{n} i^2 - \frac{n}{2}, & \text{if } n \text{ is even;} \\ \sum\limits_{i=1}^{n} i^2 - \frac{n+1}{2}, & \text{if } n \text{ is odd.} \end{cases}$$

Lemma 3.3.18. *(Dimension of B-orbits in* $\mathcal{N}^{(2)}$*)*
Let $N \in \mathcal{N}^{(2)}$*, then*

$$\dim B.N = \frac{n(n+1)}{2} - \sum_{i,j=1}^{n} m_{i,j} b_{i,j}(N) - \sum_{i=1}^{n} n_i a_i(N).$$

Proof. Let $N \in \mathcal{N}^{(2)}$, then $\dim \mathrm{Iso}_B(N) = \sum\limits_{i,j=1}^{n} m_{i,j} b_{i,j}(N) + \sum\limits_{i=1}^{n} n_i a_i(N)$.

Since $\dim B = \frac{n(n+1)}{2}$ and $\dim B.N = \dim B - \dim \mathrm{Iso}_B(N)$, we obtain the claimed description of $\dim B.N$. □

We obtain the orbits of maximal dimension from proposition 3.3.17 and lemma 3.3.18.

Corollary 3.3.19. *(Open B-orbit in $\mathcal{N}^{(2)}$)*
There is one unique orbit of maximal dimension in $\mathcal{N}^{(2)}$, namely $B.N_{\max}$ with

$$(N_{\max})_{i,j} = \begin{cases} 1, & \text{if } i = n-j+1 \text{ and } 1 \leq j \leq \frac{n}{2}; \\ 0, & \text{otherwise}; \end{cases} \quad \text{and} \quad \dim B.N_{\max} = \frac{n^2}{2}$$

if n is even and

$$(N_{\max})_{i,j} = \begin{cases} 1, & \text{if } i = n-j+1 \text{ and } 1 \leq j \leq \frac{n-1}{2}; \\ 0, & \text{otherwise}; \end{cases} \quad \text{and} \quad \dim B.N_{\max} = \frac{n^2-1}{2}$$

if n is odd.

Proof. Translating proposition 3.3.17 to $\mathcal{N}^{(2)}$ yields the description of $N_{\max}$. Lemma 3.3.18 then provides the claimed dimensions. □

Of course,

$$\dim \mathcal{N}^{(2)} = \begin{cases} \frac{n^2}{2}, & \text{if } n \text{ is even;} \\ \frac{n^2-1}{2}, & \text{if } n \text{ is odd.} \end{cases}$$

3.3.4 Minimal singularities

Since the bijection Φ of theorem 2.3.1 preserves types of singularities, we consider singularities in $R^{\text{inj}}_{\underline{d}_B}(Q,I)$ in order to examine singularities in the B-orbit closures in $\mathcal{N}^{(2)}$.

We denote a representation in $\text{rep}^{\text{inj}}_K(Q_n,I)$ by a capital letter and the corresponding point in $R^{\text{inj}}_{\underline{d}_B}(Q,I)$ by the same small letter.

In the following, minimal singularities are discussed, that is, given a minimal degeneration $M <_{\text{mdeg}} M'$, we examine if m' is a singularity in $\overline{O_M}$, where $M \in \text{rep}^{\text{inj}}_K(Q_n,I)(\underline{d}_B)$. Since the bijection Φ of lemma 2.3.1 preserves types of singularities, the translation to the B-orbit closures follows right away.

Note that if a point m' is contained in the singular locus, then every $\text{GL}_{\underline{d}_B}$-conjugate of m' is contained as well. Therefore, it suffices to consider representations in normal form.

Given a minimal degeneration $M <_{\text{mdeg}} M'$ in $\text{rep}^{\text{inj}}_K(Q_n,I)(\underline{d}_B)$, we know that $M = D \oplus W$ and $M' = D' \oplus W$, such that D and D' are disjoint and $D <_{\text{mdeg}} D'$ is a minimal, disjoint degeneration as in theorem 3.3.5 and $\text{codim}(M,M') = \text{codim}(D,D') = 1$.

Furthermore, $[X,D] = [X,D']$ and $[D,X] = [D',X]$ for every indecomposable direct summand X of W. Of course, then $[X,M] = [X,M']$ and $[M,X] = [M',X]$ holds true as well.

The following theorem is due to K. Bongartz (see [Bongartz, 1994]) and yields the reduction to minimal, disjoint degenerations; we formulate it for the setup given above.

Theorem 3.3.20. *(Cancellation theorem)*
Let $D <_{\mathrm{deg}} D'$ and $M = D \oplus U <_{\mathrm{mdeg}} D' \oplus U = M'$ be degenerations of the same codimension. Then the two pointed varieties $(\overline{O_{D\oplus U}}, d' \oplus u)$ and $(\overline{O_D}, d')$ are (very) smoothly equivalent.

Thus, the pointed varieties $(\overline{O_D}, d')$ and $(\overline{O_M}, m')$ are (very) smoothly equivalent. Therefore, in order to classify the minimal singularities, it suffices to describe singularities arising from the minimal, disjoint degenerations in theorem 3.3.5.

If the minimal, disjoint degeneration $D <_{\mathrm{mdeg}} D'$ is given by extensions, K. Bongartz proves the following theorem (see [Bongartz, 1994]) which can easily be applied in the setup above.

Theorem 3.3.21. *(Smoothness for certain degenerations with two direct summands)*
Let $M <_{\mathrm{mdeg}} M' = U \oplus V$ be a minimal, disjoint degeneration of codimension one.
Then $\overline{O_M}$ is smooth at m'.

As an easy consequence we see that no singularity arises if the minimal, disjoint degeneration is given by extensions.

Corollary 3.3.22. *(Smoothness of minimal, disjoint degenerations from extensions)*
For each minimal, disjoint degeneration $D <_{\mathrm{mdeg}} D'$ given in theorem 3.3.5 by extensions, the point d' is smooth in $\overline{O_D}$.

We aim to describe the minimal singularities arising from the minimal, disjoint degeneration $\mathcal{U}_{t,s} <_{\mathrm{mdeg}} \mathcal{U}_{s,t}$ for $s < t$ and, therefore, examine the corresponding B-orbits.

Proposition 3.3.23. *(Description of B-orbits by equations)*
Let $N \in \mathcal{N}^{(2)}$, then $B.N$ is given by matrices X fulfilling the equations $X^2 = 0$ and $\dim(X \cdot V_j \cap V_{\geq i}) = \dim(N \cdot V_j \cap V_{\geq i})$ for all $i, j \in \{1, \ldots, n\}$.

Proof. The datum $\dim(N \cdot V_j \cap V_{\geq i})$ is B-invariant, it therefore suffices to consider $N \in R_B$. Furthermore, given $N, N' \in R_B$ the equality $\dim(N \cdot V_j \cap V_{\geq i}) = \dim(N' \cdot V_j \cap V_{\geq i})$ holds true for all $i, j \in \{1, \ldots, n\}$ if and only if $N = N'$. □

We denote by $E_{i,j}$ the $n \times n$-matrix given by $(E_{i,j})_{i,j} = 1$ and $(E_{i,j})_{k,l} = 0$ otherwise.

Example 3.3.24. *(The case $n = 2$)*
We show that the point $E_{1,2}$ is smooth in the closure of $B.E_{2,1} \subseteq \mathcal{N}_2^{(2)}$:

Theorem 3.3.5 yields $\overline{B.E_{2,1}} = B.E_{2,1} \cup B.E_{1,2} \cup \{0\}$. The explicit structure of the orbits is obtained from proposition 3.3.23.

- $B.E_{2,1} = \left\{ \begin{pmatrix} n_{1,1} & n_{1,2} \\ n_{2,1} & n_{2,2} \end{pmatrix} \mid n_{2,1} \neq 0;\ n_{1,1} + n_{2,2} = 0;\ n_{1,1}n_{2,2} - n_{1,2}n_{2,1} = 0 \right\}$
- $B.E_{1,2} = \left\{ \begin{pmatrix} 0 & n_{1,2} \\ 0 & 0 \end{pmatrix} \mid n_{1,2} \neq 0 \right\}$

- $B.0 = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\}$

Of course,

$$\overline{B.E_{2,1}} = \left\{ \begin{pmatrix} n_{1,1} & n_{1,2} \\ n_{2,1} & n_{2,2} \end{pmatrix} \mid n_{1,1} + n_{2,2} = 0;\ n_{1,1}n_{2,2} - n_{1,2}n_{2,1} = 0 \right\} = \mathcal{N}_2^{(2)}.$$

The ideal

$$\langle n_{1,1} + n_{2,2}, n_{1,1}n_{2,2} - n_{1,2}n_{2,1} \rangle \subset k[n_{1,1}, n_{1,2}, n_{2,1}, n_{2,2}]$$

is reduced, thus, the associated Jacobian matrix is given by

$$J = \begin{pmatrix} 1 & 0 & 0 & 1 \\ n_{2,2} & -n_{2,1} & -n_{1,2} & n_{1,1} \end{pmatrix}$$

and we can read off the smoothness of every point contained in $\overline{B.E_{2,1}}$, except the zero-matrix.

In the example $n = 3$, minimal singularities arise.

Example 3.3.25. *(The case $n = 3$)*
The orbits can due to proposition 3.3.23 be described by equations as follows.

- $B.E_{1,3} = \left\{ \begin{pmatrix} 0 & 0 & n_{1,3} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \mid n_{1,3} \neq 0 \right\}$

- $B.E_{1,2} = \left\{ \begin{pmatrix} 0 & n_{1,2} & n_{1,3} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \mid n_{1,2} \neq 0 \right\}$

- $B.E_{2,3} = \left\{ \begin{pmatrix} 0 & 0 & n_{1,3} \\ 0 & 0 & n_{2,3} \\ 0 & 0 & 0 \end{pmatrix} \mid n_{2,3} \neq 0 \right\}$

- $B.E_{2,1} = \left\{ \begin{pmatrix} n_{1,1} & n_{1,2} & n_{1,3} \\ n_{2,1} & n_{2,2} & n_{2,3} \\ 0 & 0 & 0 \end{pmatrix} \mid \begin{matrix} n_{2,1} \neq 0;\ n_{1,1}n_{2,2} - n_{1,2}n_{2,1} = 0; \\ n_{1,1}n_{2,3} - n_{1,3}n_{2,1} = 0;\ n_{1,1} + n_{2,2} = 0 \end{matrix} \right\}$

- $B.E_{3,2} = \left\{ \begin{pmatrix} 0 & n_{1,2} & n_{1,3} \\ 0 & n_{2,2} & n_{2,3} \\ 0 & n_{3,2} & n_{3,3} \end{pmatrix} \mid \begin{matrix} n_{3,2} \neq 0;\ n_{2,2} + n_{3,3} = 0;\ n_{1,2}n_{3,3} - n_{1,3}n_{3,2} = 0; \\ n_{2,2}n_{3,3} - n_{2,3}n_{3,2} = 0 \end{matrix} \right\}$

- $B.E_{3,1} = \left\{ \begin{pmatrix} n_{1,1} & n_{1,2} & n_{1,3} \\ n_{2,1} & n_{2,2} & n_{2,3} \\ n_{3,1} & n_{3,2} & n_{3,3} \end{pmatrix} \mid \begin{matrix} n_{3,1} \neq 0;\ n_{1,1} + n_{2,2} + n_{3,3} = 0; \\ n_{1,1}n_{3,2} - n_{1,2}n_{3,1} = 0;\ n_{2,1}n_{3,2} - n_{2,2}n_{3,1} = 0; \\ n_{1,1}n_{3,3} - n_{1,3}n_{3,1} = 0;\ n_{2,1}n_{3,3} - n_{2,3}n_{3,1} = 0 \end{matrix} \right\}$

Thus, the orbit closures are given by the following equations:

- $\overline{B.E_{2,1}} = \left\{ \begin{pmatrix} n_{1,1} & n_{1,2} & n_{1,3} \\ n_{2,1} & n_{2,2} & n_{2,3} \\ 0 & 0 & 0 \end{pmatrix} \middle| \begin{array}{c} n_{1,1}+n_{2,2}=0;\ n_{1,1}n_{2,2}-n_{1,2}n_{2,1}=0; \\ n_{1,1}n_{2,3}-n_{1,3}n_{2,1}=0 \end{array} \right\}$

- $\overline{B.E_{3,2}} = \left\{ \begin{pmatrix} 0 & n_{1,2} & n_{1,3} \\ 0 & n_{2,2} & n_{2,3} \\ 0 & n_{3,2} & n_{3,3} \end{pmatrix} \middle| \begin{array}{c} n_{2,2}+n_{3,3}=0;\ n_{1,2}n_{2,3}-n_{1,3}n_{2,2}=0; \\ n_{1,2}n_{3,3}-n_{1,3}n_{3,2}=0;\ n_{2,2}n_{3,3}-n_{2,3}n_{3,2}=0 \end{array} \right\}$

- $\overline{B.E_{3,1}} = \left\{ \begin{pmatrix} n_{1,1} & n_{1,2} & n_{1,3} \\ n_{2,1} & n_{2,2} & n_{2,3} \\ n_{3,1} & n_{3,2} & n_{3,3} \end{pmatrix} \middle| \begin{array}{c} n_{1,1}+n_{2,2}+n_{3,3}=0;\ n_{1,1}n_{2,2}-n_{1,2}n_{2,1}=0; \\ n_{2,1}n_{3,2}-n_{2,2}n_{3,1}=0;\ n_{1,1}n_{3,3}-n_{1,3}n_{3,1}=0; \\ n_{2,1}n_{3,3}-n_{2,3}n_{3,1}=0;\ n_{1,1}n_{2,3}-n_{1,3}n_{2,1}=0; \\ n_{1,2}n_{3,3}-n_{1,3}n_{3,2}=0;\ n_{2,2}n_{3,3}-n_{2,3}n_{3,2}=0 \end{array} \right\}$

By using the computer algebra system "Singular" (the computation is attached to the Appendix A), we can show that the induced ideals

$$I_{2,1} := \langle n_{1,1}+n_{2,2},\ n_{1,1}n_{2,2}-n_{1,2}n_{2,1},\ n_{1,1}n_{2,3}-n_{1,3}n_{2,1},\ n_{3,1},\ n_{3,2},\ n_{3,3} \rangle$$

$$I_{3,2} := \left\langle \begin{array}{c} n_{2,2}+n_{3,3},\ n_{1,2}n_{2,3}-n_{1,3}n_{2,2},\ n_{1,2}n_{3,3}-n_{1,3}n_{3,2}, \\ n_{2,2}n_{3,3}-n_{2,3}n_{3,2},\ n_{1,1},\ n_{2,1},\ n_{3,1} \end{array} \right\rangle$$

and

$$I_{3,1} := \left\langle \begin{array}{c} n_{1,1}+n_{2,2}+n_{3,3},\ n_{1,1}n_{2,2}-n_{1,2}n_{2,1},\ n_{2,1}n_{3,2}-n_{2,2}n_{3,1}, \\ n_{1,1}n_{3,3}-n_{1,3}n_{3,1},\ n_{2,1}n_{3,3}-n_{2,3}n_{3,1},\ n_{1,1}n_{2,3}-n_{1,3}n_{2,1}, \\ n_{1,2}n_{3,3}-n_{1,3}n_{3,2},\ n_{2,2}n_{3,3}-n_{2,3}n_{3,2} \end{array} \right\rangle$$

are reduced in $k[n_{1,1}, n_{1,2}, n_{1,3}, n_{2,1}, n_{2,2}, n_{2,3}, n_{3,1}, n_{3,2}, n_{3,3}]$.

Thus, the associated Jacobian matrices can be computed directly. Without loss of generality, we consider the shortened ideal, deleting zero-variables.

The associated Jacobian matrix of $\overline{B.E_{2,1}}$ *is*

$$J = \begin{pmatrix} n_{2,2} & -n_{2,1} & 0 & -n_{1,2} & n_{1,1} & 0 \\ n_{2,3} & 0 & n_{2,1} & -n_{1,3} & 0 & n_{1,1} \\ 1 & 0 & 0 & 0 & 1 & 0 \end{pmatrix},$$

we directly see that $E_{1,2}$, $E_{1,3}$ *and* $E_{2,3}$ *are singular points in* $B.E_{2,1}$.

The associated Jacobian matrix of $\overline{B.E_{3,2}}$ *is*

$$J = \begin{pmatrix} n_{3,3} & -n_{3,2} & 0 & 0 & -n_{1,3} & n_{1,2} \\ 0 & 0 & n_{3,3} & -n_{3,2} & -n_{2,3} & n_{2,2} \\ 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}.$$

and $E_{2,3}$, $E_{1,2}$ *and* $E_{1,3}$ *are singular in* $\overline{B.E_{3,2}}$.

The final orbit closure to consider is $\overline{B.E_{3,1}}$, in which case we obtain the Jacobian matrix

$$J = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ n_{2,2} & -n_{2,1} & 0 & -n_{1,2} & n_{1,1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & n_{3,2} & -n_{3,1} & 0 & -n_{2,2} & n_{2,1} & 0 \\ n_{3,3} & 0 & -n_{3,1} & 0 & 0 & 0 & -n_{1,3} & 0 & n_{1,1} \\ 0 & 0 & 0 & n_{3,3} & 0 & -n_{3,1} & -n_{2,3} & 0 & n_{2,1} \\ n_{2,3} & 0 & -n_{2,1} & -n_{1,3} & 0 & 0 & 0 & 0 & n_{1,1} \\ 0 & n_{3,3} & -n_{3,2} & 0 & 0 & 0 & 0 & -n_{1,3} & n_{1,2} \\ 0 & 0 & 0 & 0 & n_{3,3} & -n_{3,2} & 0 & -n_{2,3} & n_{2,2} \end{pmatrix}.$$

The points $E_{1,3}$, $E_{1,2}$, $E_{2,1}$, $E_{2,3}$ and $E_{3,2}$, thus, are singular in $\overline{B.E_{3,1}}$.

As we have seen, minimal degenerations in general do not correspond to smooth points in the orbit closures. A conjecture for the general case suggests itself.

Conjecture 3.3.26. *(Conjecture for minimal singularities)*
The point $E_{s,t}$ is smooth in $\overline{B.E_{t,s}}$ if and only if $n = 2$, $s = 1$ and $t = 2$.

3.4 Closures of parabolic orbits

In view of section 3.3, we generalize the results on B-orbit closures to arbitrary parabolic actions in what follows.

Let P be the parabolic subgroup of block sizes $(b_1, \dots, b_p)$. The closures of the orbits $P.N$ with $N \in \mathcal{N}^{(2)}$ can be classified easily.

Lemma 3.4.1. *(Parabolic orbit closures in $\mathcal{N}^{(2)}$)*
Let $N \in \mathcal{N}^{(2)}$, then

$$\overline{P.N} = \bigcup \{N' \in \mathcal{N}^{(2)} \mid a_i(N) \leq a_i(N') \text{ and } b_{i,j}(N) \leq b_{i,j}(N') \text{ for all } 1 \leq i, j \leq p\}.$$

Proof. Follows directly from section 3.3. □

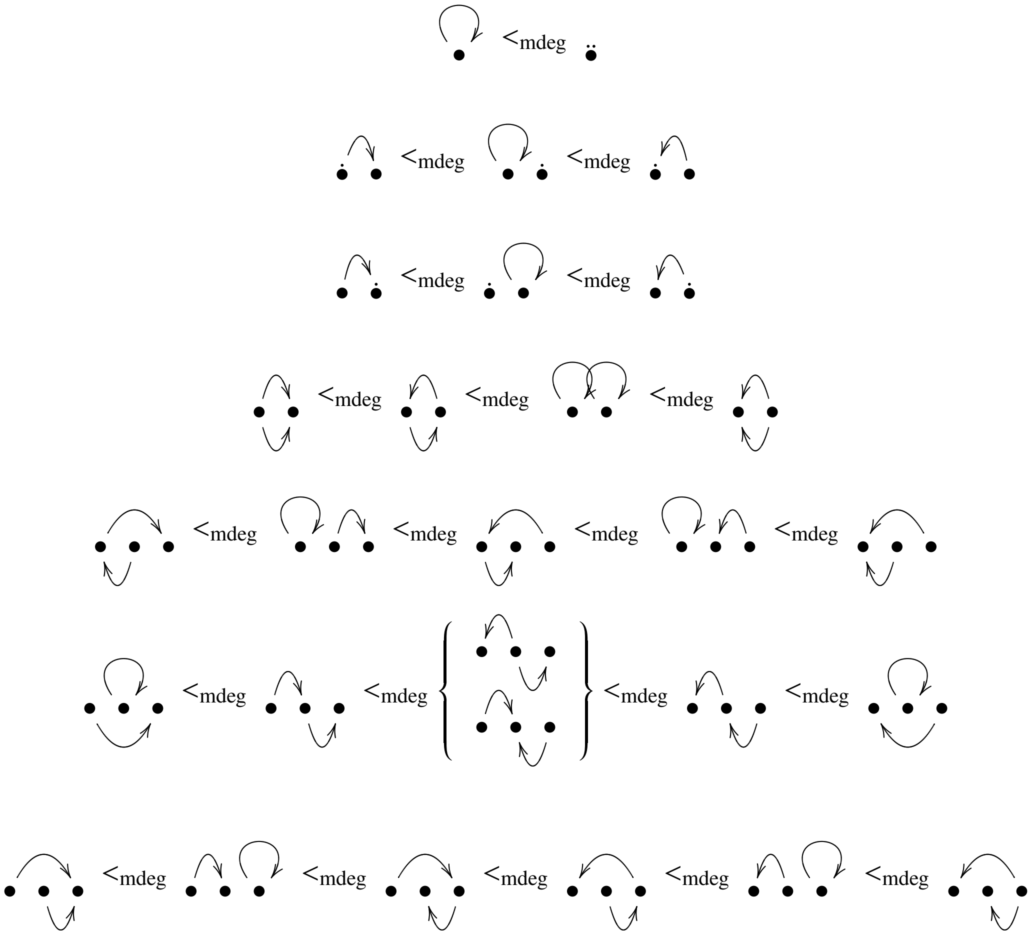
3.4.1 Minimal, disjoint degenerations

We consider the description of the orbits as enhanced oriented link patterns. The minimal, disjoint degenerations are given by exactly the same proof as in theorem 3.3.5.

Of course, in case we are looking at $\mathrm{rep}_K^{\mathrm{inj}}(Q_n, I)(\underline{d}_B)_{\mathrm{part}}$ some degenerations are missing, since the link patterns then cannot have loops or double arrows.

An indecomposable $\mathcal{V}_i$ is denoted, as before, by a dot and an indecomposable $\mathcal{U}_{i,i}$ by a loop at the vertex i.

Theorem 3.4.2. *(Minimal, disjoint pieces of degenerations)*
Let $D <_{\text{mdeg}} D'$ be a minimal, disjoint degeneration in $\text{rep}_K^{\text{inj}}(Q_p, I)$. Then it either appears in theorem 3.3.5 or in one of the following chains.



Proof. Due to proposition 3.3.4, either D' is indecomposable or D' is the direct sum of two indecomposables. In the same way as in the proof of theorem 3.3.5, the so far missing minimal, disjoint degenerations in $\text{rep}_K^{\text{inj}}(Q_p, I)$ are obtained. □

Each minimal P-degeneration in $\mathcal{N}^{(2)}$ is given by some degeneration $D \oplus W <_{\text{mdeg}} D' \oplus W$, such that the degeneration $D <_{\text{mdeg}} D'$ is one of theorem 3.4.2. Due to this fact, we can construct the P-orbit closures for a given P-orbit by adjusting minimal, disjoint moves to the enhanced oriented link pattern corresponding to this orbit.

We will calculate the representations W which lead to minimal degenerations for parabolic orbit closures, too. Given a minimal, disjoint degeneration $D <_{\text{mdeg}} D'$ due to the parabolic action, we have seen that D' decomposes into two indecomposables. Thus, we are in the case where the degenerations are obtained by extensions and can prove the minimality by calculating the dimensions of the homomorphism spaces.

Lemma 3.4.3. *(Minimal degenerations)*
Let $D <_{\mathrm{mdeg}} D'$ be a disjoint, minimal degeneration in $\mathrm{rep}_K^{\mathrm{inj}}(Q_p, I)$ with $D \ncong \mathcal{U}_{t,s}$.
A degeneration $M := D \oplus W <_{\mathrm{deg}} D' \oplus W =: M'$ in $\mathrm{rep}_K^{\mathrm{inj}}(Q_p, I)(\underline{d}_P)$ is minimal if and only if

$$a_i(M) = a_i(M') \text{ and } \overline{a_i}(M) = \overline{a_i}(M')$$

for every direct summand $\mathcal{V}_i$ of W and

$$b_{i,j}(M) = b_{i,j}(M') \text{ and } \overline{b_{i,j}}(M) = \overline{b_{i,j}}(M')$$

for every direct summand $U_{i,j}$ of W.

Proof. Since every degeneration is obtained by extensions, the claim follows directly from proposition 3.2.4 and the proof of theorem 3.3.11. □

3.4.2 Dimensions of orbits

The same reasoning as in the previous section yields the following results about the dimensions of the P-orbits. Let $N \in \mathcal{N}^{(2)}$ be a 2-nilpotent matrix that corresponds to the representation

$$M = \bigoplus_{i,j=1}^{p} \mathcal{U}_{i,j}^{m_{i,j}} \oplus \bigoplus_{i=1}^{p} \mathcal{V}_i^{n_i}$$

in $\mathrm{rep}_K^{\mathrm{inj}}(Q_p, I)(\underline{d}_P)$ via the bijection of lemma 2.3.1.

Proposition 3.4.4. *(Dimension of $\mathrm{GL}_{\underline{d}_P}$-orbits in $R_{\underline{d}_P}^{\mathrm{inj}}(Q_p, I)$)*

$$\dim \mathcal{O}_M = \sum_{i=1}^{p} \left(\sum_{x=1}^{i} b_i \right)^2 - \sum_{i,j=1}^{p} m_{i,j} b_{i,j}(M) - \sum_{i=1}^{p} n_i a_i(M).$$

There is a unique $\mathrm{GL}_{\underline{d}_P}$-orbit of minimal dimension in $R_{\underline{d}_P}(Q_p, I)$, represented by $M_0 := \bigoplus_{i=1}^{p} \mathcal{V}_i^{b_i}$ of dimension

$$\dim \mathcal{O}_{M_0} = \sum_{i=1}^{p} b_i^2 - \sum_{i=1}^{p} \sum_{x=1}^{i} (b_i \cdot b_x).$$

It corresponds naturally to the P-orbit of minimal dimension in $\mathcal{N}^{(2)}$, which is represented by the zero-matrix and has dimension 0.

Corollary 3.4.5. *(Dimension of P-orbits in $\mathcal{N}^{(2)}$)*

$$\dim P.N = \sum_{i=1}^{p} \sum_{x=1}^{i} (b_i \cdot b_x) - \sum_{i,j=1}^{p} m_{i,j} b_{i,j}(N) - \sum_{i=1}^{p} n_i a_i(N).$$

Assume $N \in \mathcal{N}^{(2)}$ to be a matrix of rank $N = x$. Considering the GL_n-action, the results above yield

$$\begin{aligned}\dim \mathrm{GL}_n .N &= n^2 - m_{1,1}b_{1,1}(N) - n_1 a_1(N)\\ &= n^2 - x \cdot n - (n-x)(n-x)\\ &= x \cdot (n-x).\end{aligned}$$

Furthermore, we now know $\dim \mathrm{Iso}_{\mathrm{GL}_n}(N) = n^2 - x \cdot (n-x) = n^2 - xn + x^2$.

We end the section with the description of the open orbits for these group actions. In case GL_n acts, the open orbit is clearly given by the highest rank matrices. In case of a parabolic action, the description is slightly more difficult, though.

Let M be a representation in $R^{\mathrm{inj}}_{\underline{d}_P}(Q_p, I)$ and consider the enhanced oriented link pattern corresponding to M. As has been seen in proposition 3.1.10, this enhanced oriented link pattern can be extended to an oriented link pattern by splitting each vertex k into b_k vertices $k^{(1)}, \ldots, k^{(b_k)}$ and drawing arrows accordingly. Without loss of generality, we denote the vertices by $1_P, \ldots, n_P$ and can read off the open orbit directly.

We define $\mathcal{U}^P_{i_P, j_P} := \mathcal{U}_{x,y}$ if there exist $1 \leq s \leq b_x$ and $1 \leq t \leq b_y$, such that $i_P = b_x^{(s)}$ and $j_P = b_y^{(t)}$. Furthermore, set $\mathcal{V}^P_{i_P} := \mathcal{V}_x$ if there exists an integer $1 \leq s \leq b_x$, such that $i_P = b_x^{(s)}$.

Proposition 3.4.6. *(Open $\mathrm{GL}_{\underline{d}_P}$-orbit in $R^{\mathrm{inj}}_{\underline{d}_P}(Q_p, I)$)*
The open orbit is represented by

$$M_{\max} = \bigoplus_{k=1}^{n/2} \mathcal{U}^P_{(n-k+1)_P, k_P}$$

if n is even and by

$$M_{\max} = \bigoplus_{k=1}^{(n-1)/2} \mathcal{U}^P_{(n-k+1)_P, k_P} \oplus \mathcal{V}^P_{(\frac{n-1}{2}+1)_P}$$

if n is odd.

4 Finite classifications in higher nilpotency degrees

We will next take matrices of nilpotency degree greater than 2 into consideration.

If B acts on $\mathcal{N}^{(x)}$ and $x > 2$, the number of orbits is infinite as we will see in section 4.2. Thinking in detail about the classification of B-orbits in $\mathcal{N}^{(x)}$ via the associated algebra of theorem 2.3.1, one realizes that the corresponding quiver as well as the associated classification problem are of wild type. The same holds true for arbitrary non-maximal parabolic actions, which we will prove in section 4.3.1.

Considering arbitrary parabolic actions, infinite classifications arise in most cases, but there is one particular exception: the action of a maximal parabolic subgroup on matrices of nilpotency degree 3.

4.1 Maximal parabolic action for $x = 3$

The only case where the algebra associated to the action of P on $\mathcal{N}^{(x)}$ is representation-finite comes up for $x = 3$ and a maximal parabolic subgroup P_m of arbitrary block-sizes (b_1, b_2). We classify this case in the following before proving that it is the only finite case in section 4.2.

4.1.1 Classification of the orbits

Let us define the quiver

$$Q := Q(2,3): \qquad \underset{1}{\bullet} \xrightarrow{\alpha_1} \underset{2}{\bullet} \circlearrowleft \alpha \quad .$$

Section 2.3 proposes to consider representations of the algebra $\mathcal{A} := KQ/I$ where I is the admissible ideal $I = (\alpha^3)$ in order to classify the P_m-orbits in $\mathcal{N}^{(3)}$.

Define $E_{i,j}^{(s)}$ to be the elementary $s \times s$-matrix with $(E_{i,j}^{(s)})_{i,j} = 1$ and $(E_{i,j}^{(s)})_{i',j'} = 0$ for $(i, j) \neq (i', j')$.

If $i \leq j$, we furthermore define $e_{i,j}$ to be the natural embedding of K^i into K^j.

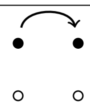
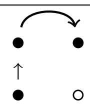
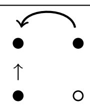
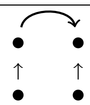
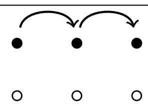
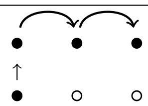
Theorem 4.1.1. *(Indecomposable representations in* $\mathrm{rep}_K(Q,I)$*)*
All indecomposable representations in $\mathrm{rep}_K(Q,I)$ *are (up to isomorphism) of the form*

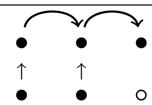
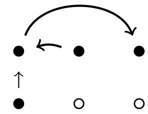
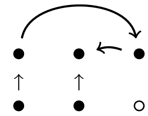
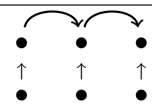
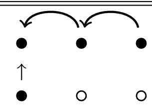
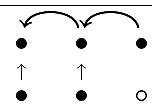
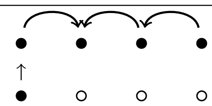
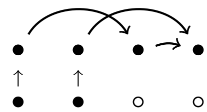
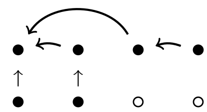
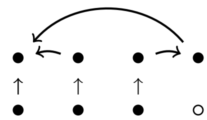
$$U = \qquad K^i \xrightarrow{e_{i,j}} K^j \circlearrowleft N$$

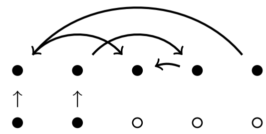
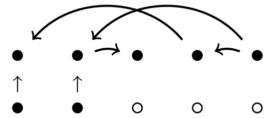
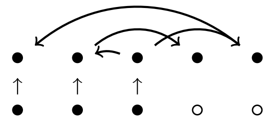
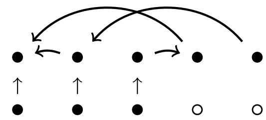
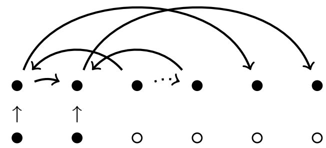
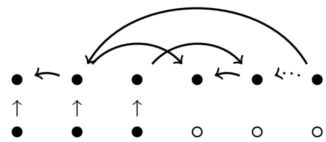
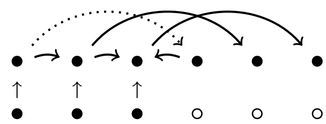
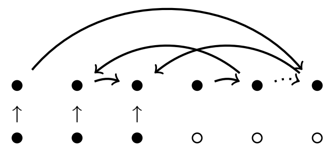
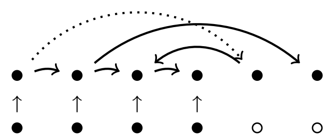
for certain integers i, j *and nilpotent matrices* N *which are explicitly listed in the table below.*

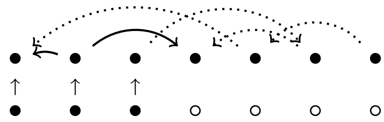
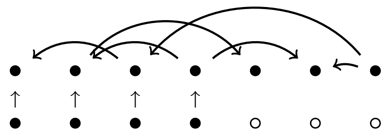
We thereby name the indecomposables (the indeces coincide with the above mentioned integers i, j*) and describe their dimension vectors which are due to the corresponding covering quiver and will be understood from the proof.*

We again visualize each indecomposable graphically as in theorem 3.1.1 by a diagram of dots for basis elements and arrows for maps sending one basis element to another; a dotted arrow marks a map that sends a basis element to the negative of another basis element.

Indecom–posable $\mathcal{U}$	*Dimension–vector*	*Matrix N*	*Diagram* $D(\mathcal{U})$
$\mathcal{U}_{0,1}$	01	0	
$\mathcal{U}_{1,1}$	11	0	
$\mathcal{U}_{0,2}$	01 01	$E^{(2)}_{2,1}$	
$\mathcal{U}^{(1)}_{1,2}$	11 01	$E^{(2)}_{2,1}$	
$\mathcal{U}^{(2)}_{1,2}$	01 11	$E^{(2)}_{1,2}$	
$\mathcal{U}_{2,2}$	11 11	$E^{(2)}_{2,1}$	
Projectives:			
$\mathcal{U}_{0,3}$	01 01 01	$E^{(3)}_{2,1} + E^{(3)}_{3,2}$	
$\mathcal{U}^{(1)}_{1,3}$	11 01 01	$E^{(3)}_{2,1} + E^{(3)}_{3,2}$	

Indecom-posable $\mathcal{U}$	*Dimension-vector*	*Matrix N*	*Diagram D($\mathcal{U}$)*
$\mathcal{U}^{(1)}_{2,3}$	11 11 01	$E^{(3)}_{2,1} + E^{(3)}_{3,2}$	
$\mathcal{U}^{(2)}_{1,3}$	01 11 01	$E^{(3)}_{3,1} + E^{(3)}_{1,2}$	
$\mathcal{U}^{(2)}_{2,3}$	11 01 11	$E^{(3)}_{3,1} + E^{(3)}_{2,3}$	
Injectives:			
$\mathcal{U}_{1,0}$	10	0	
$\mathcal{U}_{3,3}$	11 11 11	$E^{(3)}_{2,1} + E^{(3)}_{3,2}$	
$\mathcal{U}^{(3)}_{1,3}$	01 01 11	$E^{(3)}_{1,2} + E^{(3)}_{2,3}$	
$\mathcal{U}^{(3)}_{2,3}$	01 11 11	$E^{(3)}_{1,2} + E^{(3)}_{2,3}$	
$\mathcal{U}_{1,4}$	01 12 01	$E^{(4)}_{2,1} + E^{(4)}_{2,3}$ $+E^{(4)}_{3,4}$	
$\mathcal{U}^{(1)}_{2,4}$	11 12 01	$E^{(4)}_{3,1} + E^{(4)}_{4,2}$ $+E^{(4)}_{4,3}$	
$\mathcal{U}^{(2)}_{2,4}$	01 12 11	$E^{(4)}_{1,2} + E^{(4)}_{1,3}$ $+E^{(4)}_{3,4}$	
$\mathcal{U}_{3,4}$	11 12 11	$E^{(4)}_{1,2} + E^{(4)}_{1,4}$ $+E^{(4)}_{4,3}$	

Indecom–posable $\mathcal{U}$	*Dimension–vector*	*Matrix N*	*Diagram D*($\mathcal{U}$)
$\mathcal{U}_{2,5}^{(1)}$	12 12 01	$E_{3,1}^{(5)} + E_{4,2}^{(5)}$ $+E_{3,4}^{(5)} + E_{1,5}^{(5)}$	
$\mathcal{U}_{2,5}^{(2)}$	01 12 12	$E_{3,2}^{(5)} + E_{1,4}^{(5)}$ $+E_{2,5}^{(5)} + E_{4,5}^{(5)}$	
$\mathcal{U}_{3,5}^{(1)}$	12 12 11	$E_{4,2}^{(5)} + E_{2,3}^{(5)}$ $+E_{5,3}^{(5)} + E_{1,5}^{(5)}$	
$\mathcal{U}_{3,5}^{(2)}$	11 12 12	$E_{1,2}^{(5)} + E_{4,3}^{(5)}$ $+E_{1,4}^{(5)} + E_{2,5}^{(5)}$	
$\mathcal{U}_{2,6}$	01 12 12 01	$E_{1,3}^{(6)} + E_{2,1}^{(6)}$ $+E_{2,4}^{(6)} - E_{4,3}^{(6)}$ $+E_{5,1}^{(6)} + E_{6,2}^{(6)}$	
$\mathcal{U}_{3,6}$	12 12 12	$E_{1,2}^{(6)} + E_{4,2}^{(6)}$ $+E_{5,3}^{(6)} + E_{4,5}^{(6)}$ $+E_{2,6}^{(6)} - E_{5,6}^{(6)}$	
$\mathcal{U}_{3,6}^{(1)}$	11 12 12 01	$E_{2,1}^{(6)} + E_{3,2}^{(6)}$ $+E_{3,4}^{(6)} - E_{4,1}^{(6)}$ $+E_{5,2}^{(6)} + E_{6,3}^{(6)}$	
$\mathcal{U}_{3,6}^{(2)}$	01 12 12 11	$E_{2,5}^{(6)} + E_{3,2}^{(6)}$ $+E_{3,6}^{(6)} + E_{5,4}^{(6)}$ $+E_{6,1}^{(6)} - E_{6,5}^{(6)}$	
$\mathcal{U}_{4,6}$	11 12 12 11	$E_{2,1}^{(6)} + E_{3,2}^{(6)}$ $+E_{3,5}^{(6)} + E_{4,3}^{(6)}$ $-E_{5,1}^{(6)} + E_{6,2}^{(6)}$	

Indecom-posable $\mathcal{U}$	*Dimension-vector*	*Matrix N*	*Diagram D*($\mathcal{U}$)
$\mathcal{U}_{3,7}$	12 13 12	$E^{(7)}_{1,2} + E^{(7)}_{4,2}$ $-E^{(7)}_{6,3} - E^{(7)}_{1,5}$ $-E^{(7)}_{4,6} - E^{(7)}_{5,7}$	
$\mathcal{U}_{4,7}$	12 23 12	$E^{(7)}_{5,2} + E^{(7)}_{1,3}$ $+E^{(7)}_{2,4} + E^{(7)}_{6,4}$ $+E^{(7)}_{3,7} + E^{(7)}_{6,7}$	

The only indecomposable not contained in $\mathrm{rep}^{\mathrm{inj}}_K(Q, I)$ *is the indecomposable* $\mathcal{U}_{1,0}$.

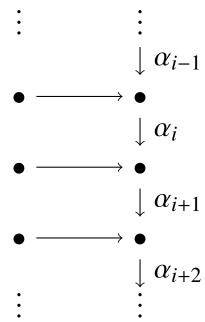
Proof. In order to calculate representatives of the isomorphism classes of indecomposable representations of the quiver

$$Q: \quad \underset{1}{\bullet} \xrightarrow{\alpha_1} \underset{2}{\bullet} \circlearrowleft \alpha$$

we make use of covering theory which was briefly recapitulated in subsection 1.2.1.

The universal covering quiver of Q at the vertex 2 is the (infinite) quiver $\widehat{Q}$ given by

$\widehat{Q}$:



together with the induced ideal $\widehat{I}$, generated by all paths $\alpha_{i+1}\alpha_i\alpha_{i-1}$, and the fundamental group $\mathbf{Z}$. The natural free action of the group $\mathbf{Z}$ on $\widehat{Q}$ is given by shifting the rows.

The algebra $\widehat{\mathcal{A}} = K\widehat{Q}/\widehat{I}$ is locally representation-finite since for each vertex $x \in \widehat{Q}$, the number of indecomposables M (up to isomorphism) with $M_x \neq 0$ is finite as we will see in the following. Therefore, due to lemma 1.2.3, we have a bijection between the indecomposables in $\mathcal{A}$ and the indecomposables in $\hat{\mathcal{A}}/\mathbf{Z}$.

For every integer k, we consider the finite subquiver

$$Q(k)\colon \quad \begin{array}{ccc} \bullet & \longrightarrow & \bullet \\ & & \downarrow \alpha_1 \\ \bullet & \longrightarrow & \bullet \\ \vdots & & \vdots \\ \bullet & \longrightarrow & \bullet \\ & & \downarrow \alpha_{k-1} \\ \bullet & \longrightarrow & \bullet \end{array}$$

together with the ideal $I(k)$ generated by the paths $\alpha_{i+1}\alpha_i\alpha_{i-1}$ for $i \in \{2, \dots, k-2\}$.

The Auslander-Reiten quivers $\Gamma(Q(k), I(k))$ can be calculated with elementary methods (see [Assem et al., 2006, IV.4]).

By calculating the Auslander-Reiten quivers $\Gamma(Q(4), I(4))$ and $\Gamma(Q(5), I(5))$, we realize that all isomorphism classes of indecomposables in $KQ(5)/I(5)$ already appear (up to the action of $\mathbf{Z}$) in the quiver $\Gamma(Q(4), I(4))$. The translation of the indecomposables between the algebras is deduced directly from the action of $\mathbf{Z}$.

It, therefore, suffices to calculate the indecomposable representations of the quiver

$$Q(4)\colon \quad \begin{array}{ccc} \bullet & \longrightarrow & \bullet \\ & & \downarrow \alpha_1 \\ \bullet & \longrightarrow & \bullet \\ & & \downarrow \alpha_2 \\ \bullet & \longrightarrow & \bullet \\ & & \downarrow \alpha_3 \\ \bullet & \longrightarrow & \bullet \end{array}$$

with the associated ideal $I(4)$ generated by the path $\alpha_3\alpha_2\alpha_1$, since all indecomposables arise up to the action of $\mathbf{Z}$ and up to isomorphism.

The Auslander-Reitem quiver $\Gamma(Q(4), I(4))$ is sketched in figure 5.1, where we denote the indecomposables by their dimension vectors. Note that we directly delete zero rows in the dimension vectors, such that the identifications by the action of $\mathbf{Z}$ can be seen right away.

We derive the Auslander-Reiten quiver $\Gamma = \Gamma(Q, I)$ just by making the identifications resulting from the action of $\mathbf{Z}$ as in the proof of lemma 3.1.1. Figure 5.2 shows $\Gamma(Q, I)$, the dotted lines mark the mentioned identifications.

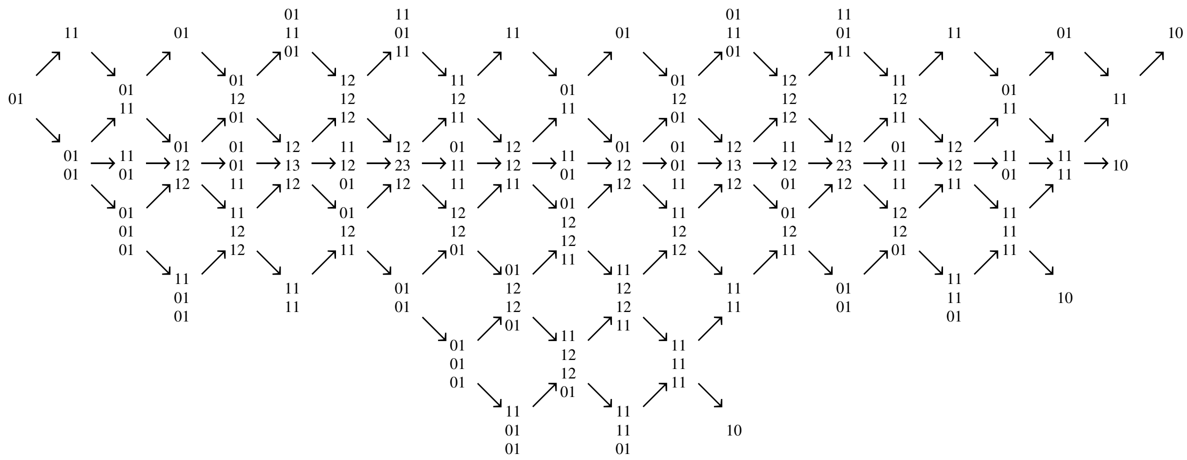


Figure 4.1: The Auslander-Reiten quiver $\Gamma(Q(4), I(4))$

The representations given in the table are all indecomposable, which can, for example, be proved by showing that the corresponding endomorphism rings are local. Either the representations have been considered in the 2-nilpotent case or the number of given representations coincides with the number of corresponding indecomposables with the same dimension vectors in the Auslander Reiten quiver $\Gamma(Q, I)$.

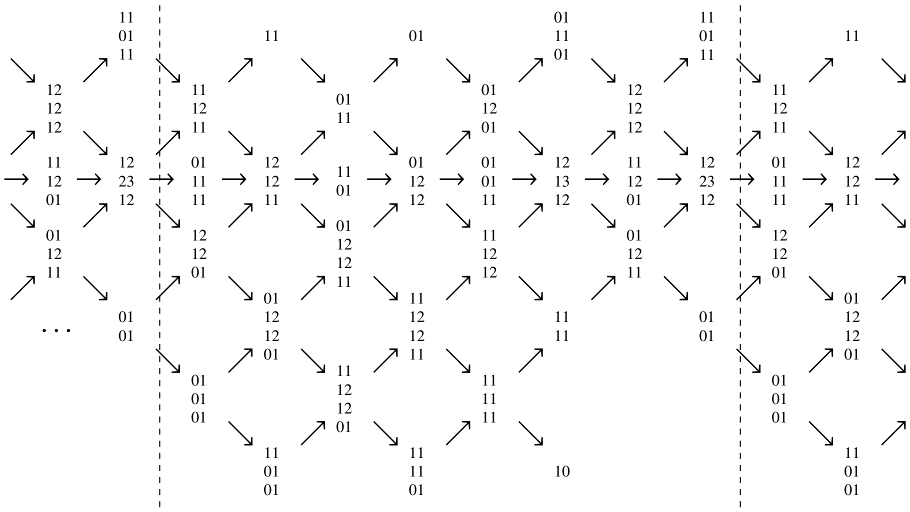


Figure 4.2: The Auslander-Reiten quiver $\Gamma(Q, I)$

Thus, for each isomorphism class of indecomposables, a representative has been found. □

We have found a normal form for each isomorphism class of indecomposable representations in $\text{rep}_K^{\text{inj}}(Q, I)$ and can, without loss of generality, assume each upcoming indecomposable to be in normal form.

Denote by $\underline{d} := (b_1, n)$ the dimension vector of theorem 2.3.1 and set $\mathcal{U}_{i,j}^{(0)} := \mathcal{U}_{i,j}$ for reasons of formality.

Corollary 4.1.2. *($\text{GL}_{\underline{d}}$-orbits in $R_{\underline{d}}^{\text{inj}}(Q, I)$)*
Each isomorphism class in $\text{rep}_K^{\text{inj}}(Q, I)(\underline{d})$ *contains a unique representation*

$$M := \bigoplus_{i,j,x} U_{i,j}^{(x)m_{i,j}^{(x)}},$$

such that $\sum_{i,j,x} m_{i,j}^{(x)} \cdot i = b_1$ *and* $\sum_{i,j,x} m_{i,j}^{(x)} \cdot j = n$.

Following lemma 2.3.1, the P_m-orbits of 3-nilpotent matrices are in bijection to the isomorphism classes of representations in $\text{rep}_K^{\text{inj}}(Q, I)$ of dimension vector $\underline{d}$. We translate the description of these representatives to the P_m-orbits in $\mathcal{N}^{(3)}$:

Let $\mathcal{U}$ and $\mathcal{V}$ be two indecomposable representations in $\text{rep}_K^{\text{inj}}(Q, I)$. We denote $\mathcal{U} \leq_{\text{tab}} \mathcal{V}$ if $\mathcal{U}$ comes up before $\mathcal{V}$ in the table of theorem 4.1.1, or if $\mathcal{U} = \mathcal{V}$.

Let $\mathcal{U}$ be an indecomposable representation in $\text{rep}_K^{\text{inj}}(Q, I)$ with $\underline{\dim}\mathcal{U} = (i, j)$. We denote $i(\mathcal{U}) := i$ and $j(\mathcal{U}) := j$. Theorem 4.1.1 yields a diagram $D(\mathcal{U})$ of $i(\mathcal{U})$ bullets and $j(\mathcal{U}) - i(\mathcal{U})$ circles in the bottom row and $j(\mathcal{U})$ bullets in the top row.

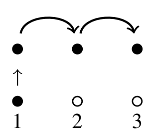
Assume $M = \mathcal{U}_1 \oplus \ldots \oplus \mathcal{U}_s$ is a representation in $\text{rep}_K^{\text{inj}}(Q, I)$ of dimension vector $\underline{d}$, that is, $\sum_{k=1}^{s} i(\mathcal{U}_k) = b_1$ and $\sum_{k=1}^{s} j(\mathcal{U}_k) = n$. Assume furthermore $\mathcal{U}_k \leq_{\text{tab}} \mathcal{U}_{k+1}$ for all k and denote the columns of the diagram $D(\mathcal{U}_k)$ from left to right by $(\mathcal{U}_k)_1$ up to $(\mathcal{U}_k)_{j(\mathcal{U}_k)}$.

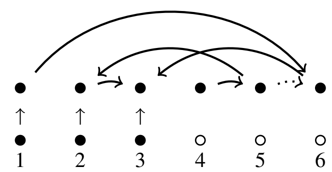
If we regard the sequence of diagrams $D(\mathcal{U}_1), \ldots, D(\mathcal{U}_s)$ as one diagram of n columns and s disjoint subdiagrams, we obtain a diagram called $D(M)$ naturally. By changing the positions of the columns of $D(M)$ to

$$(\mathcal{U}_1)_1,.., (\mathcal{U}_1)_{i(\mathcal{U}_1)},.., (\mathcal{U}_s)_1,.., (\mathcal{U}_s)_{i(\mathcal{U}_s)}, (\mathcal{U}_1)_{i(\mathcal{U}_1)+1},.., (\mathcal{U}_1)_{j(\mathcal{U}_1)},.., (\mathcal{U}_s)_{i(\mathcal{U}_s)+1},.., (\mathcal{U}_s)_{j(\mathcal{U}_s)}$$

and by adapting the arrows accordingly, we obtain a diagram which we denote by $\widetilde{D}(M)$.

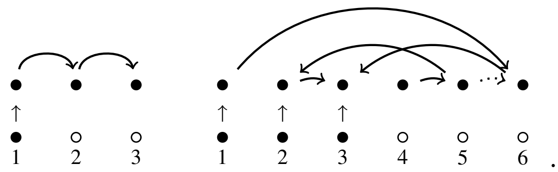
Example 4.1.3. *(The diagrams $D(M)$ and $\widetilde{D}(M)$)*
We consider the maximal parabolic subgroup P_m of block sizes $(4, 9)$ *which acts on $\mathcal{N}_9^{(3)}$ and define the representation $M := \mathcal{U}_{1,3}^{(1)} \oplus \mathcal{U}_{3,6}^{(2)}$. The corresponding diagrams (see theorem 4.1.1) which show how basis elements are mapped to each other are given by*

$D(\mathcal{U}_{1,3}^{(1)}) =$ 

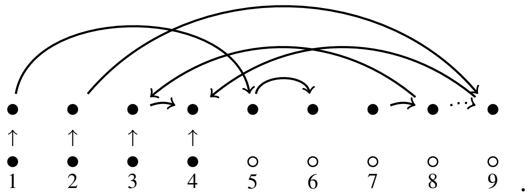
and $D(\mathcal{U}_{3,6}^{(2)}) =$ 

.

We obtain the diagrams

$D(M) =$ 

and

$\widetilde{D}(M) :=$ 

Corresponding to the diagram $\widetilde{D}(M)$, we define the 3-nilpotent matrix $N(M) \in K^{n\times n}$ as follows:

$$(N(M))_{i,j} := \begin{cases} 1, & \text{if there is a permanent arrow } j \to i \text{ in } \widetilde{D}(M); \\ -1, & \text{if there is a dotted arrow } j \to i \text{ in } \widetilde{D}(M); \\ 0, & \text{otherwise.} \end{cases}$$

Example 4.1.4. *(The matrix $N(M)$)*
In the setting of example 4.1.3, we can read off the matrix

$$N(M) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}$$

which is a representative of the corresponding P_m-orbit.

The matrix $N(M)$ corresponds to the representation M naturally via the bijection Φ of lemma 2.3.1. We have, therefore, proved the following corollary of theorem 4.1.1.

Corollary 4.1.5. *(Classification of P_m-orbits in $\mathcal{N}^{(3)}$)*
Let $N \in \mathcal{N}^{(3)}$, then the orbit $P_m.N$ contains a unique matrix $N(M)$, where M is a representation in $\text{rep}_K^{\text{inj}}(Q, I)(\underline{d})$.

We have found a normal form for each P_m-orbit.

4.1.2 Orbit closures

As in section 3.3, by making use of theorem 1.2.8, the orbit closures can be calculated.

Lemma 4.1.6. *(P_m-orbit closures in $\mathcal{N}^{(3)}$)*
The P_m-orbit of a matrix $N \in \mathcal{N}^{(3)}$ is contained in the P_m-orbit closure of $N' \in \mathcal{N}^{(3)}$ if and only if $[V, \Phi(N')] \leq [V, \Phi(N)]$ for all indecomposables $V \in \operatorname{rep}_K(Q_p, I)$.

The dimensions are given in the following tables. With these, all degenerations can be obtained in case the numbers n, b_1 and b_2 are fixed.

$\nearrow$	$\mathcal{U}_{0,1}$	$\mathcal{U}_{0,2}$	$\mathcal{U}_{0,3}$	$\mathcal{U}_{1,0}$	$\mathcal{U}_{1,1}$	$\mathcal{U}^{(1)}_{1,2}$	$\mathcal{U}^{(2)}_{1,2}$	$\mathcal{U}^{(1)}_{1,3}$	$\mathcal{U}^{(2)}_{1,3}$	$\mathcal{U}^{(3)}_{1,3}$	$\mathcal{U}_{1,4}$	$\mathcal{U}_{2,2}$	$\mathcal{U}^{(1)}_{2,3}$	$\mathcal{U}^{(2)}_{2,3}$	$\mathcal{U}^{(3)}_{2,3}$
$U_{0,1}$	1	1	1	0	1	1	1	1	1	1	2	1	1	1	1
$U_{0,2}$	1	2	2	0	1	2	2	2	2	2	3	2	2	2	2
$U_{0,3}$	1	2	3	0	1	2	2	3	3	3	4	2	3	3	3
$U_{1,0}$	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0
$U_{1,1}$	0	0	0	1	1	0	1	0	0	1	0	1	0	1	1
$U^{(1)}_{1,2}$	0	0	0	1	1	1	1	0	1	1	1	2	1	1	2
$U^{(2)}_{1,2}$	1	1	1	1	1	1	2	1	1	2	2	2	1	2	2
$U^{(1)}_{1,3}$	0	0	0	1	1	1	1	1	1	1	1	2	2	2	2
$U^{(2)}_{1,3}$	1	1	1	1	1	1	2	1	2	2	2	2	2	2	2
$U^{(3)}_{1,3}$	1	2	2	1	1	2	2	2	2	3	3	2	2	2	3
$U_{1,4}$	1	2	2	1	2	2	3	2	3	3	4	2	3	3	4
$U_{2,2}$	0	0	0	2	1	0	1	0	0	1	0	2	0	1	2
$U^{(1)}_{2,3}$	0	0	0	2	1	0	1	0	0	1	0	2	1	1	2
$U^{(2)}_{2,3}$	0	0	0	2	1	1	1	0	1	1	1	2	1	2	2
$U^{(3)}_{2,3}$	1	1	1	2	1	1	2	1	1	2	2	2	1	2	3
$U^{(1)}_{2,4}$	0	0	0	2	2	1	2	0	1	2	1	3	2	2	3
$U^{(2)}_{2,4}$	1	2	2	2	2	2	3	2	2	3	3	3	2	3	4
$U^{(1)}_{2,5}$	1	1	1	2	2	2	3	1	2	3	3	4	3	3	4
$U^{(2)}_{2,5}$	1	2	2	2	2	2	3	2	3	4	4	4	3	4	5
$U_{2,6}$	1	2	2	2	2	2	3	2	3	4	4	4	4	4	5
$U_{3,3}$	0	0	0	3	1	0	1	0	0	1	0	2	0	1	2
$U_{3,4}$	0	0	0	3	2	1	2	0	1	2	1	3	2	2	4
$U^{(1)}_{3,5}$	0	0	0	3	2	1	2	0	1	2	1	4	2	3	4
$U^{(2)}_{3,5}$	1	1	1	3	2	2	3	1	2	3	3	4	2	3	4
$U_{3,6}$	1	1	1	3	2	2	3	1	2	3	3	4	3	4	5
$U^{(1)}_{3,6}$	0	0	0	3	2	1	2	0	1	2	2	4	3	3	4
$U^{(2)}_{3,6}$	1	2	2	3	2	2	3	1	2	3	3	4	3	4	5
$U_{3,7}$	1	2	2	3	3	3	4	2	3	4	4	5	4	5	6

↗	$U_{0,1}$	$U_{0,2}$	$U_{0,3}$	$U_{1,0}$	$U_{1,1}$	$U^{(1)}_{1,2}$	$U^{(2)}_{1,2}$	$U^{(1)}_{1,3}$	$U^{(2)}_{1,3}$	$U^{(3)}_{1,3}$	$U_{1,4}$	$U_{2,2}$	$U^{(1)}_{2,3}$	$U^{(2)}_{2,3}$	$U^{(3)}_{2,3}$
$U_{4,6}$	0	0	0	4	2	2	2	0	1	2	2	4	2	3	4
$U_{4,7}$	1	1	1	4	3	2	3	1	2	4	3	5	3	4	6

↗	$U^{(1)}_{2,4}$	$U^{(2)}_{2,4}$	$U^{(1)}_{2,5}$	$U^{(2)}_{2,5}$	$U_{2,6}$	$U_{3,3}$	$U_{3,4}$	$U^{(1)}_{3,5}$	$U^{(2)}_{3,5}$	$U_{3,6}$	$U^{(1)}_{3,6}$	$U^{(2)}_{3,6}$	$U_{3,7}$	$U_{4,6}$	$U_{4,7}$
$U_{0,1}$	2	2	2	2	2	1	2	2	2	2	2	2	3	2	3
$U_{0,2}$	3	3	4	4	4	2	3	4	4	4	4	4	5	4	5
$U_{0,3}$	4	4	5	5	6	3	4	5	5	6	6	6	7	6	7
$U_{1,0}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$U_{1,1}$	0	1	0	1	0	1	1	1	1	1	0	1	1	1	1
$U^{(1)}_{1,2}$	1	2	1	2	1	2	2	2	2	2	1	2	2	3	3
$U^{(2)}_{1,2}$	2	3	2	3	2	2	2	3	3	3	2	2	4	3	4
$U^{(1)}_{1,3}$	2	2	2	2	2	3	3	3	3	3	3	3	3	4	4
$U^{(2)}_{1,3}$	2	3	3	3	3	3	3	3	4	4	3	4	4	4	5
$U^{(3)}_{1,3}$	3	4	4	4	4	3	4	4	5	5	4	5	6	5	6
$U_{1,4}$	4	5	5	6	6	4	5	5	6	5	5	6	7	5	8
$U_{2,2}$	0	2	0	1	0	2	2	1	2	1	0	2	1	2	2
$U^{(1)}_{2,3}$	0	2	0	1	0	3	2	1	2	1	0	2	1	2	2
$U^{(2)}_{2,3}$	1	2	1	2	1	3	3	2	3	2	1	3	2	3	3
$U^{(3)}_{2,3}$	2	3	2	3	2	3	3	3	4	3	2	4	4	4	4
$U^{(1)}_{2,4}$	2	3	2	3	2	4	4	3	4	3	2	4	3	4	5
$U^{(2)}_{2,4}$	3	5	4	5	4	4	5	5	6	5	4	6	6	6	7
$U^{(1)}_{2,5}$	3	5	4	5	4	5	5	5	6	5	4	6	6	6	8
$U^{(2)}_{2,5}$	4	6	5	6	3	5	6	6	7	7	5	7	8	6	9
$U_{2,6}$	4	6	5	6	6	6	6	6	8	7	6	8	8	8	9
$U_{3,3}$	0	2	0	1	0	3	2	1	2	1	0	2	1	2	2
$U_{3,4}$	1	3	1	3	1	4	4	3	4	3	1	4	3	4	4
$U^{(1)}_{3,5}$	2	4	2	3	2	5	5	4	5	4	2	5	4	5	6
$U^{(2)}_{3,5}$	3	5	3	5	3	5	5	5	6	5	3	6	6	6	7
$U_{3,6}$	4	5	4	5	4	6	6	5	7	6	4	7	6	7	8
$U^{(1)}_{3,6}$	2	4	2	3	2	6	5	4	5	4	3	4	5	6	6
$U^{(2)}_{3,6}$	5	6	5	6	5	6	6	6	7	7	5	8	8	8	8
$U_{3,7}$	5	7	5	7	6	7	8	7	9	8	6	9	9	8	11
$U_{4,6}$	2	4	2	3	3	6	5	4	5	4	2	5	4	3	6
$U_{4,7}$	3	6	4	6	4	7	8	6	8	6	4	8	8	7	9

4.1.3 Dimensions and the open orbit

As in subsection 4.1.1, in order to calculate the dimensions of the P_m-orbits and to describe the open P_m-orbit in $\mathcal{N}^{(3)}$ for a maximal parablic subgroup P_m of block sizes (b_1, b_2), we make use of the translation of lemma 2.3.1 and corollary 1.1.8.

Dimensions of orbits

Let $M \in \mathrm{rep}_K^{\mathrm{inj}}(Q, I)(\underline{d})$ be a representation, then

$$\begin{aligned}\dim \mathcal{O}_M = \dim \overline{\mathcal{O}_M} &= \dim \mathrm{GL}_{\underline{d}} - \dim \mathrm{Iso}_{\mathrm{GL}_{\underline{d}}}(M) \\ &= \dim \mathrm{GL}_{\underline{d}} - [M, M] \\ &= b_1^2 + n^2 - [M, M]\end{aligned}$$

due to section 2.3. The dimensions, therefore, can be calculated by using the tables of subsection 4.1.2.

Note that the zero matrix in $\mathcal{N}^{(3)}$ corresponds to the representation $M_0 = \mathcal{U}_{1,1}^{b_1} \oplus \mathcal{U}_{0,1}^{b_2}$ via the bijection of lemma 2.3.1 and the orbit $\mathcal{O}_{M_0}$ fulfills

$$\begin{aligned}\dim \mathcal{O}_{M_0} &= \dim \mathrm{GL}_{\underline{d}} - [\mathcal{U}_{1,1}^{b_1} \oplus \mathcal{U}_{0,1}^{b_2}, \mathcal{U}_{1,1}^{b_1} \oplus \mathcal{U}_{0,1}^{b_2}] \\ &= b_1^2 + n^2 - b_1^2 - b_1 b_2 - b_2^2 \\ &= n \cdot b_1.\end{aligned}$$

Proposition 4.1.7. *(Dimensions of P_m-orbits in $\mathcal{N}^{(3)}$)*
If a matrix $N \in \mathcal{N}^{(3)}$ and a representation $M \in \mathrm{rep}_K^{\mathrm{inj}}(Q, I)(\underline{d})$ correspond to each other via the bijection Φ of lemma 2.3.1, then

$$\dim P_m.N = n^2 - b_1 b_2 - [M, M].$$

The open orbit

We denote the matrix in normal form in the open P_m-orbit in $\mathcal{N}^{(3)}$ by N_{open} and the representation in normal form in the open $\mathrm{GL}_{\underline{d}}$-orbit in $R_{\underline{d}}^{\mathrm{inj}}(Q, I)$ by M_{open}.

Since the open orbit is the orbit of maximal dimension, we have $\dim P_m.N_{\mathrm{open}} = \dim \mathcal{N}^{(3)}$ and $\dim \mathcal{O}_{M_{\mathrm{open}}} = \dim R_{\underline{d}}^{\mathrm{inj}}(Q, I)$.

Proposition 4.1.8. *(Dimension of the variety $\mathcal{N}^{(3)}$)*

$$\dim \mathcal{N}^{(3)} = \begin{cases} n^2 - 3r^2, & \text{if } n = 3r; \\ n^2 - 3r^2 - 2r - 1, & \text{if } n = 3r + 1; \\ n^2 - 3r^2 - 4r - 2, & \text{if } n = 3r + 2. \end{cases}$$

Proof. In case of the GL_n-action, the open orbit is represented by the matrix N of maximal Jordan blocks.

Since $\dim \mathcal{N}^{(3)} = \dim \mathrm{GL}_n .N = \dim \mathrm{GL}_n - \dim \mathrm{Iso}_{\mathrm{GL}_n}(N) = n^2 - \dim \mathrm{Iso}_{\mathrm{GL}_n}(N)$ it suffices to show

$$\dim \mathrm{Iso}_{\mathrm{GL}_n}(N) = \begin{cases} 3r^2, & \text{if } n = 3r; \\ 3r^2 + 2r + 1, & \text{if } n = 3r + 1; \\ 3r^2 + 4r + 2, & \text{if } n = 3r + 2. \end{cases}$$

Let $n = 3r$ and let $g \in \mathrm{GL}_n$. Then $g \in \mathrm{Iso}_{\mathrm{GL}_n}(N)$ if and only if $g = (G_{i,j})_{1 \leq i,j \leq r}$ where

$$G_{i,j} = \begin{pmatrix} a_{i,j} & 0 & 0 \\ b_{i,j} & a_{i,j} & 0 \\ c_{i,j} & b_{i,j} & a_{i,j} \end{pmatrix} \in K^{3\times 3}$$

for certain elements $a_{i,j}, b_{i,j}, c_{i,j} \in K$.

Let $n = 3r + 1$ and let $g \in \mathrm{GL}_n$. Then $g \in \mathrm{Iso}_{\mathrm{GL}_n}(N)$ if and only if $g = \begin{pmatrix} G_{i,j} & B_i \\ C_j & D \end{pmatrix}_{1 \leq i,j \leq r}$ where $G_{i,j}$ is a matrix as above,

$$B_i = \begin{pmatrix} b_i \\ 0 \\ 0 \end{pmatrix} \text{ and } C_j = \begin{pmatrix} c_j & 0 & 0 \end{pmatrix} \text{ and } D = (d)$$

for certain elements $b_i, c_j, d \in K$.

Let $n = 3r + 2$ and let $g \in \mathrm{GL}_n$. Then $g \in \mathrm{Iso}_{\mathrm{GL}_n}(N)$ if and only if $g = \begin{pmatrix} G_{i,j} & B_i \\ C_j & D \end{pmatrix}_{1 \leq i,j \leq r}$ where $G_{i,j}$ is a matrix as above,

$$B_i = \begin{pmatrix} b_i & 0 \\ b'_i & b_i \\ 0 & 0 \end{pmatrix} \text{ and } C_j = \begin{pmatrix} c_j & 0 & 0 \\ c'_j & c_j & 0 \end{pmatrix} \text{ and } D = \begin{pmatrix} d & 0 \\ d' & d \end{pmatrix}$$

for certain elements $b_i, b'_i, c_j, c'_i, d, d' \in K$.

The claim follows. □

Lemma 4.1.9. *(The representation M_{open})*
The representation M_{open} is the unique representation in normal form that fulfills

$$[M_{open}, M_{open}] = \begin{cases} 3r^2 - b_1 b_2, & \textit{if } n = 3r; \\ 3r^2 + 2r + 1 - b_1 b_2, & \textit{if } n = 3r + 1; \\ 3r^2 + 4r + 2 - b_1 b_2, & \textit{if } n = 3r + 2. \end{cases}$$

Proof. Since

$$\dim \mathcal{N}^{(3)} = \dim P_m.N_{\text{open}} = \dim O_{M_{\text{open}}} - n \cdot b_1 = b_1^2 + n^2 - [M_{\text{open}}, M_{\text{open}}] - n \cdot b_1,$$

the claim follows from proposition 4.1.8. □

Making use of lemma 4.1.9, we are able to give an explicit description of the representation M_{open}.

Corollary 4.1.10. *(Explicit description of the open orbit)*
We consider the following cases:

1.1 If $b_1 \leq b_2$, such that $b_1 \leq r$, then

$$M_{open} = (\mathcal{U}_{1,3}^{(1)})^{b_1} \oplus (\mathcal{U}_{0,3})^{r-b_1} \oplus \begin{cases} 0, & \text{if } n = 3r; \\ \mathcal{U}_{0,1}, & \text{if } n = 3r+1; \\ \mathcal{U}_{0,2}, & \text{if } n = 3r+2; \end{cases}$$

1.2 If $b_1 \leq b_2$, such that $b_1 > r$, then

$$M_{open} = (\mathcal{U}_{3,6}^{(1)})^{b_1-r-1} \oplus (\mathcal{U}_{1,3}^{(1)})^{n-2b_1} \oplus \begin{cases} \mathcal{U}_{3,6}^{(1)}, & \text{if } n = 3r; \\ \mathcal{U}_{2,4}^{(1)}, & \text{if } n = 3r+1; \\ \mathcal{U}_{1,2}^{(1)}, & \text{if } n = 3r+2; \end{cases}$$

2.1 If $b_1 \geq b_2$, such that $b_2 \leq r$, then

$$M_{open} = (\mathcal{U}_{2,3}^{(1)})^{b_2} \oplus (\mathcal{U}_{3,3})^{r-b_2} \oplus \begin{cases} 0, & \text{if } n = 3r; \\ \mathcal{U}_{1,1}, & \text{if } n = 3r+1; \\ \mathcal{U}_{2,2}, & \text{if } n = 3r+2; \end{cases}$$

2.2 If $b_1 \geq b_2$, such that $b_2 > r$, then

$$M_{open} = (\mathcal{U}_{3,6}^{(1)})^{b_2-r-1} \oplus (\mathcal{U}_{2,3}^{(1)})^{n-2b_2} \oplus \begin{cases} \mathcal{U}_{3,6}^{(1)}, & \text{if } n = 3r; \\ \mathcal{U}_{2,4}^{(1)}, & \text{if } n = 3r+1; \\ \mathcal{U}_{1,2}^{(1)}, & \text{if } n = 3r+2; \end{cases}$$

Proof. We only consider the first two cases since the case 1.1 is symmetric to the case 2.1 and the case 1.2 is symmetric to the case 2.2.

Let $n = 3r$. If $b_1 \leq b_2$, such that $b_1 \leq r$, then

$$\begin{aligned} [M_{\text{open}}, M_{\text{open}}] &= [(\mathcal{U}_{1,3}^{(1)})^{b_1} \oplus (\mathcal{U}_{0,3})^{r-b_1}, (\mathcal{U}_{1,3}^{(1)})^{b_1} \oplus (\mathcal{U}_{0,3})^{r-b_1}] \\ &= b_1^2 + 3b_1(r-b_1) + 3(r-b_1)^2 \\ &= 3r^2 - b_1 b_2. \end{aligned}$$

If $b_1 \leq b_2$, such that $b_1 > r$, then

$$\begin{aligned} [M_{\text{open}}, M_{\text{open}}] &= [(\mathcal{U}_{3,6}^{(1)})^{b_1-r} \oplus (\mathcal{U}_{1,3}^{(1)})^{n-2b_1}, (\mathcal{U}_{3,6}^{(1)})^{b_1-r} \oplus (\mathcal{U}_{1,3}^{(1)})^{n-2b_1}] \\ &= 3(b_1-r)^2 + 3(b_1-r)(3r-2b_1) + (3r-2b_1)^2 \\ &= 3r^2 - b_1 b_2. \end{aligned}$$

Lemma 4.1.9 yields the claim for the case $n = 3r$.

Let $n = 3r + 1$. If $b_1 \leq b_2$, such that $b_1 \leq r$, then

$$\begin{aligned}[M_{\text{open}}, M_{\text{open}}] &= [(\mathcal{U}^{(1)}_{1,3})^{b_1} \oplus (\mathcal{U}_{0,3})^{r-b_1} \oplus \mathcal{U}_{0,1}, (\mathcal{U}^{(1)}_{1,3})^{b_1} \oplus (\mathcal{U}_{0,3})^{r-b_1} \oplus \mathcal{U}_{0,1}] \\ &= b_1^2 + 3b_1(r - b_1) + 3(r - b_1)^2 + 2(r - b_1) + b_1 + 1 \\ &= 3r^2 + 2r + 1 - b_1 b_2.\end{aligned}$$

If $b_1 \leq b_2$, such that $b_1 > r$, then

$$\begin{aligned}[M_{\text{open}}, M_{\text{open}}] &= [(\mathcal{U}^{(1)}_{3,6})^{b_1-r-1} \oplus (\mathcal{U}^{(1)}_{1,3})^{n-2b_1} \oplus \mathcal{U}^{(1)}_{2,4}, (\mathcal{U}^{(1)}_{3,6})^{b_1-r-1} \oplus (\mathcal{U}^{(1)}_{1,3})^{n-2b_1} \oplus \mathcal{U}^{(1)}_{2,4}] \\ &= 3(b_1 - r - 1)^2 + 3(b_1 - r - 1)(n - 2b_1) + (n - 2b_1)^2 \\ &\quad + 4(b_1 - r - 1) + 2(n - 2b_1) + 2 \\ &= 3r^2 + 2r + 1 - b_1 b_2.\end{aligned}$$

Lemma 4.1.9 yields the claim for the case $n = 3r + 1$.

Let $n = 3r + 2$. If $b_1 \leq b_2$, such that $b_1 \leq r$, then

$$\begin{aligned}[M_{\text{open}}, M_{\text{open}}] &= [(\mathcal{U}^{(1)}_{1,3})^{b_1} \oplus (\mathcal{U}_{0,3})^{r-b_1} \oplus \mathcal{U}_{0,2}, (\mathcal{U}^{(1)}_{1,3})^{b_1} \oplus (\mathcal{U}_{0,3})^{r-b_1} \oplus \mathcal{U}_{0,2}] \\ &= b_1^2 + 3b_1(r - b_1) + 3(r - b_1)^2 + 4(r - b_1) + 2b_1 + 2 \\ &= 3r^2 + 4r + 2 - b_1 b_2.\end{aligned}$$

If $b_1 \leq b_2$, such that $b_1 > r$, then

$$\begin{aligned}[M_{\text{open}}, M_{\text{open}}] &= [(\mathcal{U}^{(1)}_{3,6})^{b_1-r-1} \oplus (\mathcal{U}^{(1)}_{1,3})^{n-2b_1} \oplus \mathcal{U}^{(1)}_{1,2}, (\mathcal{U}^{(1)}_{3,6})^{b_1-r-1} \oplus (\mathcal{U}^{(1)}_{1,3})^{n-2b_1} \oplus \mathcal{U}^{(1)}_{1,2}] \\ &= 3(b_1 - r - 1)^2 + 3(b_1 - r - 1)(n - 2b_1) + (n - 2b_1)^2 \\ &\quad + 2(b_1 - r - 1) + (n - 2b_1) + 1 \\ &= 3r^2 + 4r + 2 - b_1 b_2.\end{aligned}$$

□

The translation to the normal form $N_{\text{open}} \in \mathcal{N}^{(3)}$ is obtained from subsection 4.1.1.

Example 4.1.11. *(The case $n = 3$)*
If P_m is a parabolic of block sizes $(1, 2)$, then $M_{open} = \mathcal{U}^{(1)}_{1,3}$ and if P_m is a parabolic of block sizes $(2, 1)$, then $M_{open} = \mathcal{U}^{(1)}_{2,3}$. In both cases, the matrix in normal form that represents the open orbit is given by

$$N_{open} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Example 4.1.12. *(The case $n = 4$)*
If P_m is a parabolic of block sizes $(1, 3)$, then $M_{open} = \mathcal{U}^{(1)}_{1,3} \oplus \mathcal{U}_{0,1}$ and

$$N_{open} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

If P_m is a parabolic of block sizes $(2,2)$, then $M_{open} = \mathcal{U}^{(1)}_{2,4}$ and

$$N_{open} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix}.$$

We will examine this example in more detail in subsection 4.1.4.

If P_m is a parabolic of block sizes $(3,1)$, then $M_{open} = \mathcal{U}^{(1)}_{1,3} \oplus \mathcal{U}_{0,1}$ and

$$N_{open} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

4.1.4 The parabolic of block sizes $(2,2)$

We exemplify the results of the previous subsections by calculating a system of representatives for the orbits of the action of the maximal parabolic P_m of block sizes $(2,2)$ on $\mathcal{N}_4^{(3)}$. We furthermore describe all minimal degenerations that arise in detail.

As an example which shows how easily even a large number of orbits can be classified, we include the parabolic subgroup of block sizes $(3,4)$ in the appendix A.2.

The action of P_m provides 14 orbits which are obtained combinatorially by considering every direct sum of indecomposables of dimension vector $(2,4)$ up to isomorphism.

Given a representation M in normal form, we can calculate the dimension of $O(M)$ by

$$\begin{aligned} \dim O(M) &= \dim \mathrm{GL}_{\underline{d}} - \dim \mathrm{Iso}_{\mathrm{GL}_{\underline{d}}}(M) \\ &= 20 - [M,M]. \end{aligned}$$

The normal forms classifying the orbits are listed in the table in figure 5.3 as well as the dimension of their orbits.

Representation M	$\dim O(M)$	Representation M	$\dim O(M)$
$M_1 := U^{(1)}_{2,4}$	18	$M_8 := U^{(1)}_{1,3} \oplus \mathcal{U}_{1,1}$	17
$M_2 := U^{(2)}_{2,4}$	15	$M_9 := U^{(2)}_{1,3} \oplus \mathcal{U}_{1,1}$	16
$M_3 := U^{(1)}_{2,3} \oplus \mathcal{U}_{0,1}$	17	$M_{10} := U^{(3)}_{1,3} \oplus \mathcal{U}_{1,1}$	14
$M_4 := U^{(2)}_{2,3} \oplus \mathcal{U}_{0,1}$	16	$M_{11} := U^{(1)}_{1,2} \oplus \mathcal{U}^{(2)}_{1,2}$	15
$M_5 := U^{(3)}_{2,3} \oplus \mathcal{U}_{0,1}$	14	$M_{12} := U^{(1)}_{1,2} \oplus \mathcal{U}_{1,1} \oplus \mathcal{U}_{0,1}$	14
$M_6 := U_{2,2} \oplus \mathcal{U}_{0,2}$	14	$M_{13} := U^{(2)}_{1,2} \oplus \mathcal{U}_{1,1} \oplus \mathcal{U}_{0,1}$	11
$M_7 := U_{2,2} \oplus \mathcal{U}^2_{0,1}$	12	$M_{14} := U^2_{1,1} \oplus \mathcal{U}^2_{0,1}$	10

Figure 4.3: Normal forms in the P_m-orbits

We can read off a necessary criterion for degenerations, since the orbits contained in an orbit closure are of smaller or equal dimension.

In order to calculate the exact degeneration diagram, we calculate the dimensions of the homomorphism spaces, namely $[V, M_i]$ for every indecomposable representation $V \in \mathrm{rep}_K(Q, I)$ following lemma 1.2.8. The dimensions are listed in the tables in figure 5.4.

We compare every two representation that fulfill the necessary condition given by their orbit dimensions and arrive at the degeneration diagram in figure 5.4 which sketches how the diagrams and the representing matrices of the orbits degenerate to each other.

$\swarrow$	$U_{0,1}$	$U_{0,2}$	$U_{0,3}$	$U_{1,0}$	$U_{1,1}$	$U^{(1)}_{1,2}$	$U^{(2)}_{1,2}$	$U^{(1)}_{1,3}$	$U^{(2)}_{1,3}$	$U^{(3)}_{1,3}$	$U_{1,4}$	$U_{2,2}$	$U^{(1)}_{2,3}$	$U^{(2)}_{2,3}$	$U^{(3)}_{2,3}$
M_1	2	3	4	0	0	1	2	2	2	3	4	0	0	1	2
M_3	2	3	4	0	0	1	2	2	3	3	4	0	1	1	2
M_8	2	3	4	0	1	1	2	2	2	3	4	1	1	1	2
M_4	2	3	4	0	1	1	3	2	3	3	4	1	1	2	3
M_9	2	3	4	0	1	2	2	2	3	3	5	1	1	2	2
M_2	2	3	4	0	1	2	3	2	3	4	5	2	2	2	3
M_{11}	2	4	4	0	1	2	3	2	3	4	5	1	1	2	3
M_5	2	3	4	0	1	2	3	2	3	4	5	2	2	2	4
M_6	2	4	4	0	1	2	3	2	3	4	4	2	2	2	3
M_{10}	2	3	4	0	2	2	3	2	3	4	5	2	2	2	3
M_{12}	3	4	4	0	1	2	3	2	3	4	5	1	1	2	3
M_7	3	4	4	0	1	2	4	2	4	4	4	2	2	2	4
M_{13}	3	4	4	0	2	2	4	2	4	4	6	2	2	2	4
M_{14}	4	4	4	0	2	2	4	2	4	4	6	2	2	2	4

$\swarrow$	$U^{(1)}_{2,4}$	$U^{(2)}_{2,4}$	$U^{(1)}_{2,5}$	$U^{(2)}_{2,5}$	$U_{2,6}$	$U_{3,3}$	$U_{3,4}$	$U^{(1)}_{3,5}$	$U^{(2)}_{3,5}$	$U_{3,6}$	$U^{(1)}_{3,6}$	$U^{(2)}_{3,6}$	$U_{3,7}$	$U_{4,6}$	$U_{4,7}$
M_1	2	3	3	4	4	0	1	2	3	4	2	4	5	2	3
M_3	2	3	4	4	5	0	2	2	3	4	3	4	5	2	4
M_8	2	4	3	4	4	1	2	2	3	4	2	4	5	2	4
M_4	2	4	4	5	5	1	2	3	4	5	3	5	6	3	5
M_9	3	4	4	5	5	1	3	3	4	5	3	4	6	3	5
M_2	3	5	5	6	6	2	3	4	5	5	4	6	7	4	6
M_{11}	3	5	5	5	5	1	3	3	5	5	3	5	7	4	5
M_5	3	5	5	6	6	2	4	4	5	6	4	6	7	4	7
M_6	3	5	5	6	6	2	3	4	5	5	4	6	7	4	6
M_{10}	4	5	5	6	6	2	4	4	5	6	4	5	7	4	7
M_{12}	3	5	5	5	5	1	3	3	5	6	3	5	7	4	6
M_7	3	5	6	6	6	2	3	4	6	6	4	6	7	4	7
M_{13}	4	6	6	6	6	2	4	4	6	7	4	6	8	4	7
M_{14}	4	6	6	6	6	2	4	4	6	8	4	6	8	4	8

Figure 4.4: Dimensions of homomorphism spaces

Note that we delete the bottom row of the diagrams corresponding to the orbits, since they coincide for each orbit.

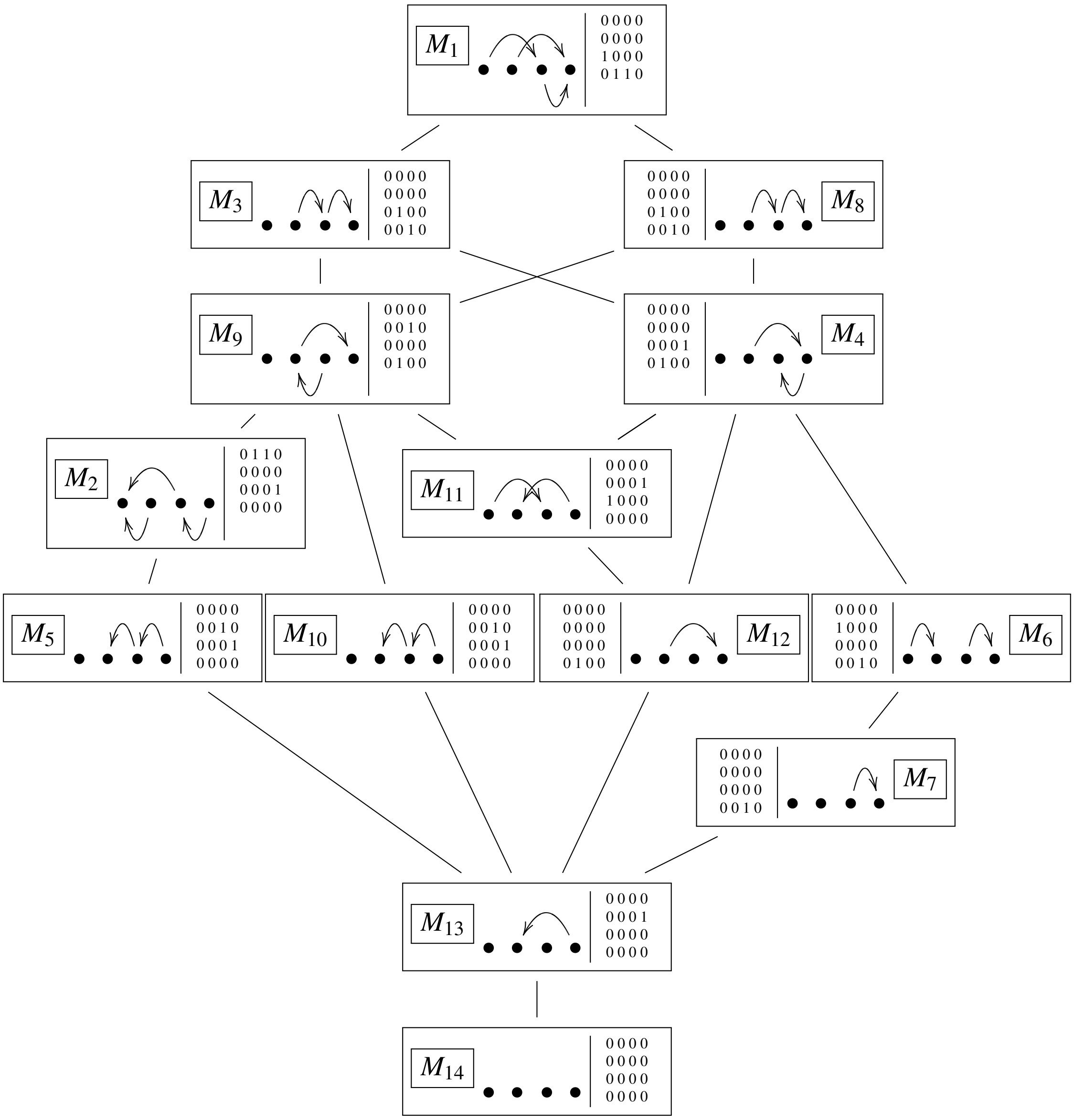


Figure 4.5: Degeneration-diagram for the parabolic subgroup of block sizes $(2,2)$

4.2 A finiteness criterion

Next, we consider fixed integers n and give an explicit description of all finite types of the action of a parabolic P on any variety of nilpotent $n \times n$-matrices that appear.

It is redundant to consider 2-nilpotent matrices, because it has been shown in chapter 3 that all parabolic actions on 2-nilpotent matrices are finite. In the following, we look at small examples, thereby discussing the methods and fixing the notations.

Let **n=2**. All matrices are either 2-nilpotent or equal to the zero-matrix.

Let **n=3**. The Borel subgroup B_3 acts infinitely on $\mathcal{N}_3^{(3)}$, since

$$\{D_1(\lambda) := \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ \lambda & 1 & 0 \end{pmatrix} \mid \lambda \in K^*\}$$

is a 1-parameter family of pairwise non-conjugate matrices (this has been worked out in [Halbach, 2009]). The only proper parabolic subgroups left in GL_3 are those of block-sizes either $(1,2)$ or $(2,1)$. Both are maximal and have been discussed in section 4.1.

Note that B_n, thus, acts infinitely on $\mathcal{N}_n^{(x)}$ if $x > 2$, so that we do not have to consider the action of the Borel subgroup anymore.

Let **n=4**.

- Let $x = 3$.
  The maximal parabolic subgroups of block-sizes $(1,3)$, $(3,1)$ and $(2,2)$ have been considered in 4.1. The parabolic subgroup P of block-sizes $(1,1,2)$ acts infinitely, since
  $$\{D_1^1(\lambda) := \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \lambda & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \mid \lambda \in K^*\},$$
  is a 1-parameter family of pairwise non-P-conjugate matrices which is clear by our considerations in the case of the B_3-action on $\mathcal{N}_3^{(3)}$. Due to symmetry, the parabolic subgroup of block-sizes $(2,1,1)$ acts infinitely, too.
  The parabolic subgroup P of block-sizes $(1,2,1)$ acts infinitely because
  $$\{D_2(\lambda) := \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \lambda & 1 & 1 & 0 \end{pmatrix} \mid \lambda \in K^*\}$$
  is a 1-parameter family of pairwise non-P-conjugate matrices.

- Let $x = 4$.
  The maximal parabolic subgroup P of block-sizes $(2,2)$ acts infinitely, since
  $$\{E(\lambda) := \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ \lambda & 1 & 1 & 0 \end{pmatrix} \mid \lambda \in K^*\}$$

is a 1-parameter family of pairwise non-P-conjugate matrices. The maximal parabolic subgroup P of block-sizes $(1,3)$ acts infinitely, because

$$\{F(\lambda) := \begin{pmatrix} 1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ \lambda-1 & \lambda & -1 & 1 \\ \lambda & \lambda-1 & -1 & 1 \end{pmatrix} \mid \lambda \in K^*\}$$

is a 1-parameter family of pairwise non-P-conjugate matrices. Due to symmetry, the maximal parabolic subgroup of block-sizes $(3,1)$ acts infinitely, too.

Let **n=5**.

- Let $x = 3$.
  The maximal parabolic subgroups of block-sizes $(1,4)$, $(4,1)$, $(3,2)$ and $(2,3)$ have been considered in section 4.1. The parabolic subgroup of block-sizes $(1,1,3)$, $(3,1,1)$, $(1,1,2,1)$, $(1,2,1,1)$, $(1,1,1,2)$ and $(2,1,1,1)$ act infinitely, since the 1-parameter family of the B_3-action on $\mathcal{N}_3^{(3)}$ yields a 1-parameter family of pairwise non-conjugate matrices for each of these cases.
  The parabolic subgroup P of block-sizes $(1,3,1)$ does not act finitely because

$$\{D_3(\lambda) := \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ \lambda & 1 & 1 & 1 & 0 \end{pmatrix} \mid \lambda \in K^*\}$$

  is a 1-parameter family of pairwise non-P-conjugate matrices.

- Let $x = 4$.
  The parabolic subgroup P of block-sizes $(2,3)$ acts infinitely because

$$\{E^1(\lambda) := \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ \lambda & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \mid \lambda \in K^*\}$$

  is a 1-parameter family of pairwise non-P-conjugate matrices.

  The parabolic subgroup P of block-sizes $(1,4)$ acts infinitely because

$$\{F^1(\lambda) := \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 \\ \lambda-1 & \lambda & -1 & 1 & 0 \\ \lambda & \lambda-1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \mid \lambda \in K^*\}$$

  is a 1-parameter family of pairwise non-P-conjugate matrices. Due to symmetry, the maximal parabolic subgroups of block-sizes $(3,2)$ and $(4,1)$ act infinitely, too.

- Let $x = 5$.
  Since all parabolic actions already act infinitely on $\mathcal{N}_5^{(4)}$, there is nothing left to consider.

One can see that for $n = 4$, only the maximal parabolic subgroups act finitely on the variety of 3-nilpotent matrices and each action becomes infinite if we swich to considering all 4-nilpotent matrices. In the case $n = 5$, every parabolic action on $\mathcal{N}_5^{(4)}$ is infinite. We generalize these observations in the following.

Theorem 4.2.1. *(Classification of finite parabolic actions)*
The action of a parabolic subgroup P in GL_n on the variety $\mathcal{N}_n^{(x)}$ is finite if and only if $x \leq 2$, or P is maximal and $x = 3$.

The proof of theorem 4.2.1 follows from lemma 4.2.2 and lemma 4.2.3.

Lemma 4.2.2. *(Infiniteness of non-maximal parabolic actions on $\mathcal{N}_n^{(3)}$)*
Each conjugation action of a non-maximal parabolic $P \subsetneq \mathrm{GL}_n$ on $\mathcal{N}_n^{(x)}$ is infinite if $x \geq 3$.

Proof. Let $P \subsetneq \mathrm{GL}_n$ be a parabolic subgroup of block-sizes $(b_1, \ldots, b_p)$.

We divide the proof into two parts. First, we show that each parabolic subgroup $P(x)$ of block-sizes $(1, x, 1)$ acts infinitely and deduce the claim afterwards. As has been seen above, it is of utility to define 3-nilpotent matrices $D_x(\lambda)$ for $\lambda \in K^*$ as follows:

$$(D_x(\lambda))_{i,j} = \begin{cases} \lambda, & \text{if } i = n \text{ and } j = 1; \\ 1, & \text{if } (1 \leq i < n \text{ and } j = 1) \text{ or } (i = n \text{ and } 1 \leq j < n); \\ 0, & \text{otherwise.} \end{cases}$$

Claim: A 1-parameter family of pairwise non-$P(x)$-conjugate matrices is given by

$$\{D_x(\lambda) \mid \lambda \in K^*\}.$$

Let $\lambda, \mu \in K^*$, such that $\lambda \neq \mu$. Let us assume there is a matrix $P \in P(x)$ with

$$P \cdot D_x(\lambda) = D_x(\mu) \cdot P.$$

Then $P_{1,i} = P_{i-1,n} = 0$ for all $2 \leq i \leq n$ and

$$\sum_{i,j=2}^{n-1} P_{i,j} = (n-2) \cdot P_{1,1} = (n-2) \cdot \frac{\lambda}{\mu} \cdot P_{1,1},$$

an immediate contradiction since $\lambda \neq \mu$.

Let us consider a parabolic subgroup P of block-sizes $(b_1, \ldots, b_p)$, where $p \geq 3$ and $b_i \geq 1$ for all i, say $b_1 = s + 1$ and $b_3 = 1 + t$.

We define matrices $D_{b_2}^s(n, \lambda)$ for $\lambda \in K^*$ as follows:

$$(D_{b_2}^s(n, \lambda))_{i,j} = \begin{cases} (D_{b_2}(\lambda))_{i-s,j-s} & \text{if } s + 1 \leq i, j \leq s + b_2 + 2; \\ 0 & \text{otherwise.} \end{cases}$$

Claim: A 1-parameter family of pairwise non-P-conjugate matrices is given by

$$\{D^s_{b_2}(n,\lambda) \mid \lambda \in K^*\}.$$

We proceed inductively on the matrix size n.

The beginning of the induction is given by the case $n = 3$. Clearly $(b_1, b_2, b_3) = (1,1,1)$ and we arrive at the introductory example given above. In more detail, the 1-parameter family $\{D_1(\lambda) \mid \lambda \in K^*\}$ consists of pairwise non-B_3-conjugate matrices.

Assume that, for fixed $k < n$ and each parabolic subgroup $P' \subset \mathrm{GL}_k$ of block-sizes $(b'_1, \ldots, b'_m)$, where $b'_1 = s' + 1$ and $b'_3 = t' + 1$, all matrices in the 1-parameter family

$$\{D^{s'}_{b'_2}(k,\lambda) \mid \lambda \in K^*\}$$

are non-P'-conjugate.

Let $\lambda, \mu \in K^*$, such that $\lambda \neq \mu$ and assume there is a matrix $A \in P$, such that

$$A \cdot D^s_{b_2}(n,\lambda) = D^s_{b_2}(n,\mu) \cdot A.$$

First case: $t \neq 0$ or $p \geq 4$

The equality is independent of the entries $A_{i,n}$ if $i < s$ or $i > s + b_2 + 2$ and of the entries $A_{n,j}$ if $j \in \{1, \ldots, n\}$. Furthermore, $A_{i,*} \cdot (D^s_{b_2}(n,\lambda))_{*,n} = 0$ and $(D^s_{b_2}(n,\mu))_{n,*} \cdot A_{*,i} = 0$ for all i; therefore, without loss of generality, we can set $A_{n,n} = 1$ and $A_{i,n} = A_{n,i} = 0$ for $i \in \{1, \ldots, n-1\}$. It is easily examined that the existence of a matrix $A \in P$ solving

$$A \cdot D^s_{b_2}(n,\lambda) = D^s_{b_2}(n,\mu) \cdot A$$

is equivalent to the existence of a matrix A' that is obtained from A by deleting the n-th column and row, solving

$$A' \cdot D^s_{b_2}(n-1,\lambda) = D^s_{b_2}(n-1,\mu) \cdot A'.$$

A' is a matrix of block-sizes $(b_1, \ldots, b_p - 1)$ and the induction yields the claim.

Second case: $t = 0$ and $p = 3$

Without loss of generality we assume $s > 0$, since otherwise we derive at a parabolic subgroup of block-sizes $(1, b_2, 1)$, which has already been considered.
The equality does not depend on the entries $A_{1,j}$ if $j \leq s$ and on the entries $A_{i,1}$ if $1 \neq s+1$. Additionally, $A_{i,*} \cdot (D^s_{b_2}(n,\lambda))_{*,1} = 0$ and $(D^s_{b_2}(n,\mu))_{1,*} \cdot A_{*,i} = 0$ for all i; without loss of generality we set $A_{1,1} = 1$ and $A_{i,1} = A_{1,i} = 0$ for all $i \neq 1$. It can be verified quickly that the existence of a matrix $A \in P$ which solves

$$A \cdot D^s_{b_2}(n,\lambda) = D^s_{b_2}(n,\mu) \cdot A$$

is equivalent to the existence of a matrix A' that arises from A by deleting the first column and row which solves

$$A' \cdot D^{s-1}_{b_2}(n-1,\lambda) = D^{s-1}_{b_2}(n-1,\mu) \cdot A'.$$

A' is a matrix of block-sizes $(b_1 - 1, b_2, 1)$ and the induction yields the claim. □

Lemma 4.2.3. *(Infiniteness of maximal parabolic actions on $\mathcal{N}_n^{(4)}$)*
Each conjugation action of a maximal parabolic subgroup $P \subset \mathrm{GL}_n$ on $\mathcal{N}_n^{(4)}$ is infinite.

Proof. Let $P \subsetneq \mathrm{GL}_n$ be a parabolic subgroup of block-sizes (x, y).

First case: $x \geq 2$ and $y \geq 2$

Say $x = s + 2$ and $y = t + 2$, where it is possible that $s = 0$ or $t = 0$.

Claim: A 1-parameter family of pairwise non-P-conjugate 4-nilpotent matrices is given by

$$\{E^s(n, \lambda) \mid \lambda \in K^*\},$$

where $E^s(n, \lambda)$ is defined by

$$(E^s(n,\lambda))_{i,j} := \begin{cases} (E(\lambda))_{i-s,j-s}, & \text{if } s+1 \leq i, j \leq s+4; \\ 0, & \text{otherwise.} \end{cases}$$

The beginning of the induction is given by the case $n = 4$. Then $(x, y) = (2, 2)$ and the introductory example given above yields the claim. More precisely, the 1-parameter family $\{E(\lambda) \mid \lambda \in K^*\}$ consists of pairwise non-P-conjugate matrices.

Assume that, for fixed $k < n$ and for each parabolic subgroup $P' \subset \mathrm{GL}_k$ of block-sizes (x', y'), where $x' = s' + 2 \geq 2$ and $y' = t' + 2 \geq 2$, all matrices in the 1-parameter family

$$\{E^{s'}(k, \lambda) \mid \lambda \in K^*\}$$

are pairwise non-P'-conjugate.

Without loss of generality, we can assume $s > 0$ (if $s = 0$, the claim follows for $t > 0$ due to symmetry).

Let $\lambda, \mu \in K^*$, such that $\lambda \neq \mu$, and assume there is a matrix $A \in P$, such that

$$A \cdot E^s(n, \lambda) = E^s(n, \mu) \cdot A.$$

The product $A \cdot E^s(n, \lambda)$ of the equation is independent of the entries $A_{i,1}$ if $i \in \{1, \ldots, n\}$ and of the entries $A_{1,j}$ if $j \leq s$ or $j > s + 4$. Also, the product $E^s(n, \mu) \cdot A$ is independent of the entries $A_{1,j}$ if $j \in \{1, \ldots, n\}$ and of the entries $A_{i,1}$ if $i \leq s$ or $i > s + 4$. Moreover, $A_{i,*} \cdot E^s(n, \lambda)_{*,1} = 0$ and $E^s(n, \mu)_{1,*} \cdot A_{*,i} = 0$ for all i, thus, without loss of generality, we set $A_{1,1} = 1$ and $A_{i,1} = A_{1,i} = 0$ for $i \in \{2, \ldots, n\}$.

Then there is a matrix $A \in P$ solving

$$A \cdot E^s(n, \lambda) = E^s(n, \mu) \cdot A$$

if and only if there is a matrix A' that is obtained from A by deletion of the first column and row, such that

$$A' \cdot E^{s-1}(n-1, \lambda) = E^{s-1}(n-1, \mu) \cdot A'.$$

A' is a matrix of block-sizes $(x - 1, y)$ and the induction yields the claim.

Second case: $x = 1$ or $y = 1$

Let without loss of generality $x = 1$, thus, $y = n - 1$. The opposite case follows due to symmetry.

For $\lambda \in K^*$, let us define

$$(F(n,\lambda))_{i,j} = \begin{cases} (F(\lambda))_{i,j}, & \text{if } 1 \le i,j \le 4; \\ 0, & \text{otherwise.} \end{cases}$$

Claim: A 1-parameter family of pairwise non-P-conjugate 4-nilpotent matrices is given by

$$\{F(n,\lambda) \mid \lambda \in K^*\}.$$

We proceed inductively on the matrix size n as in the first case.

The beginning of the induction is given by the case $n = 4$. Then the block sizes of P are $(x, y) = (1, 3)$ and we arrive at the introductory example given above where a 1-parameter family $\{F(\lambda) \mid \lambda \in K^*\}$ of pairwise non-P-conjugate matrices was given.

Let us assume that, for fixed $k < n$ and each parabolic subgroup $P' \subset \mathrm{GL}_k$ of block-sizes $(1, y')$, all matrices in the 1-parameter family $\{F(k,\lambda) \mid \lambda \in K^*\}$ are pairwise non-P'-conjugate.

Let $\lambda, \mu \in K^*$, such that $\lambda \neq \mu$ and assume, there is a matrix $A \in P$, such that

$$A \cdot F(n,\lambda) = F(n,\mu) \cdot A.$$

Then, as before, $A \cdot F(n,\lambda)$ is independent of the entries $A_{i,n}$ if $i \in \{1,\dots,n\}$ and of the entries $A_{n,j}$ if $j \le 5$; and $F(n,\mu) \cdot A$ independent of the entries $A_{i,n}$ if $j \ge 5$ and of the entries $A_{n,j}$ if $j \in \{1,\dots,n\}$. The equalities $(A_{n,j})_{1\le j\le 4} \cdot F(\lambda) = 0$ and $F(\mu) \cdot (A_{i,n})_{1\le i\le 4} = 0$, therefore, yield that we can, without loss of generality, set $A_{n,n} = 1$ and $A_{i,n} = A_{n,i} = 0$ for $i \in \{1,\dots,n-1\}$.
Then the existence of a matrix $A \in P$ for which

$$A \cdot F(n,\lambda) = F(n,\mu) \cdot A$$

is equivalent to the existence of a matrix A' which is deduced from A by deletion of the n-th column and row for which

$$A' \cdot F(n-1,\lambda) = F(n-1,\mu) \cdot A'$$

holds true. A' is a matrix of block-sizes $(1, y - 1)$ and the induction yields the claim. □

Remark 4.2.4. *The proofs of lemma 4.2.2 and lemma 4.2.3 provide a concrete* 1*-parameter family of non-conjugate nilpotent matrices for each parabolic subgroup P that acts infinitely.*

Given an arbitrary parabolic subgroup P of GL_n, the action on $\mathcal{N}_n^{(x)}$ translates to an infinite classification problem if either $x = 3$ and P is not maximal or $x > 3$. In all these cases, we consider the generic approach in chapter 5 which is a common tool to analyze infinite classification problems.

4.3 A wildness criterion

Let us fix $p > 1$. Section 4.2 shows that the algebra $\mathcal{A}(p, x)$ is of finite representation type if and only if $x \in \{1, 2\}$, or $p = 2$ and $x = 3$.

4.3.1 The representation type of the corresponding algebras

In this subsection, it will be shown that each remaining algebra is of wild representation type.

Proposition 4.3.1. *(Representation type of $\mathcal{A}(p, x)$)*
The algebra $\mathcal{A}(p, x)$ is of wild representation type if and only if $\mathcal{A}(p, x)$ is not of finite representation type.

Proof. If $\mathcal{A}(p, x)$ is not of finite representation type, then either $x = 3$ and $p > 2$, or $x \geq 4$.

If $x = 3$ and $p > 2$, then the covering quiver of $\mathcal{A}(p, x)$ at the vertex p contains the subquiver

$$Q': \qquad \begin{array}{ccccc} \bullet & \longrightarrow & \bullet & \longrightarrow & \bullet \\ & & & & \downarrow \alpha_1 \\ \bullet & \longrightarrow & \bullet & \longrightarrow & \bullet \\ & & & & \downarrow \alpha_2 \\ \bullet & \longrightarrow & \bullet & \longrightarrow & \bullet \end{array}$$

without any relations.

If $x \geq 4$, then the covering quiver of $\mathcal{A}(p, x)$ at the vertex p contains the subquiver

$$Q': \qquad \begin{array}{ccc} \bullet & \longrightarrow & \bullet \\ & & \downarrow \alpha_1 \\ \bullet & \longrightarrow & \bullet \\ & & \downarrow \alpha_2 \\ \bullet & \longrightarrow & \bullet \\ & & \downarrow \alpha_3 \\ \bullet & \longrightarrow & \bullet \end{array}$$

without any relations.

These subquivers are not quivers of extended Dynkin types, therefore, the algebra $\mathcal{A}(p, x)$ is of wild representation type. □

Of course, we have shown that $\mathcal{A}(p, x)$ is never of infinite tame representation type.

Note that we cannot conclude that each parabolic action admits 2-parameter families of non-conjugate matrices. It is possible that certain parabolic actions admit at most 1-parameter families of pairwise non-conjugate matrices. One example is the Borel-action on the nilpotent cone for $n = 3$ as has been seen in section 4.2.

4.3.2 Concrete 2-parameter families via tree modules

Since the algebra $\mathcal{A}(p,x)$ is of wild representation type if either $x = 3$ and $p > 2$, or $x \geq 4$, it is natural to try to exhibit a 2-parameter family of pairwise non-conjugate matrices for at least one parabolic action corresponding to $\mathcal{A}(p,x)$.

We follow the method for constructing indecomposable modules T. Weist describes in [Weist, 2010] of which we describe the general idea first.

Let U and U' be two indecomposable representations of a finite-dimensional path algebra $\mathcal{A} = KQ$, such that $\underline{\dim}U$ and $\underline{\dim}U'$ are real roots and such that the root $\underline{\dim}U + \underline{\dim}U'$ is an imaginary root of Q. Assume that $[U', U] = 0 = [U, U']^1$ and $[U', U]^1 = 3$, then the representatives X of the middleterms of the classes of extensions

$$[0 \to U \to X \to U' \to 0]$$

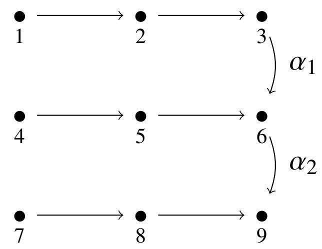
yield a 2-parameter family of pairwise non-isomorphic indecomposable $\mathcal{A}$-representations.

We consider the two cases that come up in the proof of proposition 4.3.1.

First case:

Let us define the quiver

Q':



and the root

$$\underline{d} = (1,2,3,1,3,4,1,2,3) = \underbrace{(0,1,1,0,1,2,0,1,1)}_{\underline{e}:=} + \underbrace{(1,1,2,1,2,2,1,1,2)}_{\underline{e}':=}.$$

Then $q(\underline{d}) = 54 - 55 = -1$ and, thus, $\underline{d}$ is an imaginary root of Q'; furthermore the equalities $q(\underline{e}) = 9 - 8 = 1$ and $q(\underline{e}') = 21 - 20 = 1$ yield that $\underline{e}$ and $\underline{e}'$ are positive real roots.

There are indecomposable KQ-representations $U_{\underline{e}}$ and $U_{\underline{e}'}$, such that $\underline{\dim}U_{\underline{e}} = \underline{e}$ and $\underline{\dim}U_{\underline{e}'} = \underline{e}'$:

$$U_{\underline{e}}: \quad \begin{array}{ccccc} 0 & \xrightarrow{0} & K & \xrightarrow{\mathrm{id}} & K \\ & & & & \downarrow{\scriptstyle e_1} \\ 0 & \xrightarrow{0} & K & \xrightarrow{e_2} & K^2 \\ & & & & \downarrow{\scriptstyle (e_1+e_2)^t} \\ 0 & \xrightarrow{0} & K & \xrightarrow{\mathrm{id}} & K \end{array} \qquad U_{\underline{e}'}: \quad \begin{array}{ccccc} K & \xrightarrow{\mathrm{id}} & K & \xrightarrow{-e_1+e_2} & K^2 \\ & & & & \downarrow{\scriptstyle \mathrm{id}} \\ K & \xrightarrow{e_1} & K^2 & \xrightarrow{\mathrm{id}} & K^2 \\ & & & & \downarrow{\scriptstyle \mathrm{id}} \\ K & \xrightarrow{\mathrm{id}} & K & \xrightarrow{e_2} & K^2 \end{array}$$

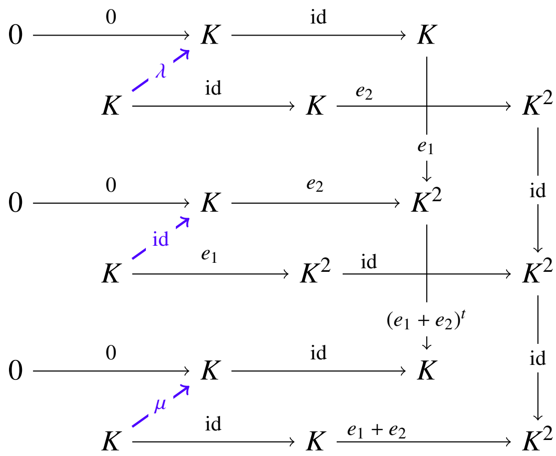
We calculate the Euler forms

$$\langle \underline{e}, \underline{e}' \rangle = 0 \quad \text{and} \quad \langle \underline{e}', \underline{e} \rangle = -3.$$

Furthermore, $[\mathcal{U}_{\underline{e}'}, \mathcal{U}_{\underline{e}}] = 0$. In general, for two KQ'-representations M and M', the equality

$$\left\langle \underline{\dim} M, \underline{\dim} M' \right\rangle = [M, M'] - [M, M']^1$$

holds true (see, for example, [Ringel, 1976]). We obtain $[\mathcal{U}_{\underline{e}'}, \mathcal{U}_{\underline{e}}]^1 = 3$ and use these extensions to glue the two representations together in order to obtain the sought representations, here $\lambda, \mu \in K$:



We obtain the representation

$$K^3 \xrightarrow{a_{\lambda,\mu}} K^7 \xrightarrow{b} K^{10} \circlearrowleft A$$

with

$$a_{\lambda,\mu} = \begin{pmatrix} 1 & 0 & 0 \\ \lambda & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & \mu \end{pmatrix}; \quad b = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}; \quad A = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

For fixed parameters $\lambda, \mu \in K^*$, this representation is isomorphic to a unique representation of the form

$$K^3 \xrightarrow{e_{3,7}} K^7 \xrightarrow{e_{7,10}} K^{10} \circlearrowleft N_{\lambda,\mu}\ .$$

Let P be the parabolic subgroup P of block sizes $(3,4,3)$. Our considerations prove the following proposition (see [Weist, 2010]).

Proposition 4.3.2. *(2-parameter family for the first quiver)*
A 2-parameter family of pairwise non-P-conjugate matrices in $\mathcal{N}^{(3)}$ is induced by the matrices

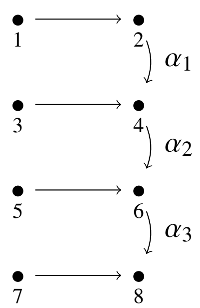
$$N_{\lambda,\mu} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & -\mu & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \lambda & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

for $\lambda, \mu \in K^*$.

Second case:

We consider the quiver

Q':



and the root

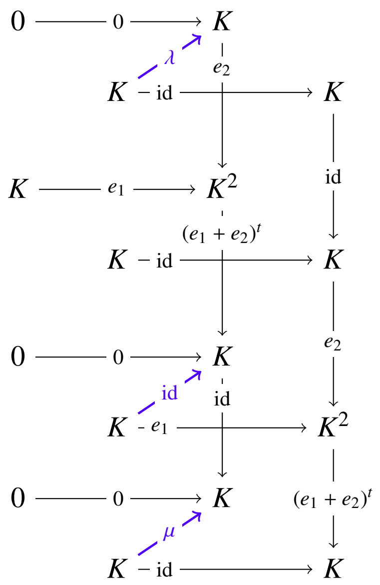
$$\underline{d} = (1,2,2,3,1,3,1,2) = \underbrace{(0,1,1,2,0,1,0,1)}_{\underline{e}:=} + \underbrace{(1,1,1,1,1,2,1,1)}_{\underline{e}':=}.$$

Then $q(\underline{d}) = 33-34 = -1$ and, thus, $\underline{d}$ is an imaginary root of Q'; furthermore the equalities $q(\underline{e}) = 8-7 = 1$ and $q(\underline{e}') = 11-10 = 1$ yield that $\underline{e}$ and $\underline{e}'$ are positive real roots as before.

We proceed as in the first case and define indecomposable KQ-representations $U_{\underline{e}}$ and $U_{\underline{e}'}$, such that $\underline{\dim} U_{\underline{e}} = \underline{e}$ and $\underline{\dim} U_{\underline{e}'} = \underline{e}'$.

$$U_{\underline{e}}: \quad \begin{array}{ccc} 0 & \xrightarrow{0} & K \\ & & \downarrow e_2 \\ K & \xrightarrow{e_1} & K^2 \\ & & \downarrow (e_1+e_2)^t \\ 0 & \xrightarrow{0} & K \\ & & \downarrow \mathrm{id} \\ 0 & \xrightarrow{0} & K \end{array} \qquad\qquad U_{\underline{e}'}: \quad \begin{array}{ccc} K & \xrightarrow{\mathrm{id}} & K \\ & & \downarrow \mathrm{id} \\ K & \xrightarrow{\mathrm{id}} & K \\ & & \downarrow e_2 \\ K & \xrightarrow{e_1} & K^2 \\ & & \downarrow (e_1+e_2)^t \\ K & \xrightarrow{\mathrm{id}} & K \end{array}$$

The equality $[U_{\underline{e}'}, U_{\underline{e}}]^1 = 3$ holds true, thus, we can again glue the representations $U_{\underline{e}}$ and $U_{\underline{e}'}$ together and obtain the sought representations, here $\lambda, \mu \in K$:



We obtain the representations

$$K^5 \xrightarrow{a_{\lambda,\mu}} K^{10} \circlearrowleft A$$

for certain matrices $a_{\lambda,\mu}$ and A, which can be read off the diagram. For fixed parameters λ, μ, this representation is isomorphic to a unique representation of the form

$$K^5 \xrightarrow{e_{5,10}} K^{10} \circlearrowleft N_{\lambda,\mu} \quad .$$

Let P be the parabolic subgroup P of block sizes $(5, 5)$; we obtain the following proposition as in the first case:

Proposition 4.3.3. *(2-parameter family for the second quiver)*
A 2-parameter family of pairwise non-P-conjugate matrices in $\mathcal{N}^{(3)}$ *is induced by the matrices*

$$N_{\lambda,\mu} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \lambda & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1-\mu & 0 & 0 & 0 & 1 & -\mu & 0 \end{pmatrix}$$

for $\lambda, \mu \in K^*$.

5 Generic classification in the nilpotent cone

In the following, we verify a parabolic normal form for the orbits in an open subset of the nilpotent cone $\mathcal{N}$ for an arbitrary parabolic subgroup P of GL_n.

5.1 Generic normal forms

In [Halbach, 2009] and [Boos and Reineke, 2011], a generic normal form for the orbits of the Borel-action is given, generalizing the case $n = 3$, which has been described in all detail in [Halbach, 2009]. We will generalize the results to arbitrary parabolic actions.

As in 3.1.1, let P be a proper parabolic subgroup of GL_n of block sizes $\underline{b} := (b_1, \ldots, b_p)$. Define

$$d_i := \begin{cases} 0, & \text{if } i = 0; \\ d_{i-1} + b_i, & \text{if } 1 \leq i \leq p \end{cases}$$

and $\underline{d} := (d_1, \ldots, d_p)$. Let V be an n-dimensional K-vector space and denote the space of partial p-step flags of dimensions $\underline{d}$ by $\mathcal{F}_{\underline{d}}(V)$, that is, $\mathcal{F}_{\underline{d}}(V)$ contains flags

$$(0 = F_0 \subset F_1 \subset \ldots \subset F_{p-2} \subset F_{p-1} \subset F_p = V),$$

such that $\dim_K F_i = d_i$. Let φ be a nilpotent endomorphism of V and consider pairs of a nilpotent endomorphism and a p-step flag up to base change in V, that is, up to the $\mathrm{GL}(V)$-action via $g.(F_*, \varphi) = (gF_*, g\varphi g^{-1})$.

Let us fix a partial flag $F_* \in \mathcal{F}_{\underline{d}}(V)$ and a nilpotent endomorphism φ of V.

Lemma 5.1.1. *(Interrelation between partial flags and nilpotent operators)*
The following properties of the pair (F_*, φ) are equivalent:

1. $\dim_K \varphi^{n-d_k}(F_k) = d_k$ *for every* $k \in \{0, \ldots, p\}$,

2. *there exists a basis* $w_1, \ldots, w_n$ *of* V, *such that for all* $k \in \{1, \ldots, p\}$*:*

   (a_k) $F_k = \langle w_1, \ldots, w_{d_k} \rangle$

   and for every $k \in \{2, \ldots, p\}$*:*

   (b) $$\varphi(w_x) = \begin{cases} w_{x+1} \mod \langle w_{d_1+2}, \ldots, w_n \rangle, & \text{if } x < d_1; \\ w_{d_{k-1}+1} \mod \langle w_{d_k+1}, \ldots, w_n \rangle, & \text{if } x = d_{k-1}; \\ w_{x+1} \mod \langle w_{d_k+1}, \ldots, w_n \rangle, & \text{if } d_{k-1} < x < d_k; \\ 0, & \text{if } x = n. \end{cases}$$

Proof. If 2. *holds true, then* 1. *follows:*

Let $w_1, \dots, w_n$ be a basis of V that fulfills (a) and (b).
An easy induction shows

$$\varphi^i(w_x) = \begin{cases} w_{x+i} \mod \left\langle w_j \mid j > x+i \right\rangle, & \text{if } x+i \le n; \\ 0, & \text{if } x+i > n. \end{cases}$$

Thus,

$$\varphi^{n-d_k}(F_k) = \left\langle \varphi^{n-d_k}(w_1), \dots, \varphi^{n-d_k}(w_{d_k}) \right\rangle = \langle w_{n-d_k+1}, \dots, w_n \rangle$$

and $\dim_K \varphi^{n-d_k}(F_k) = d_k$ for all $k \in \{0, \dots, p\}$.

If 1. *holds true, then* 2. *follows:*

We will start discussing permanence properties of the rank of φ^i for arbitrary i:

Let $\dim_K \varphi^{n-d_k}(F_k) = d_k$ for all $k \in \{0, \dots, p\}$, then $\dim_K \varphi^{n-d_k}(V) = d_k$ holds true and, therefore, $\operatorname{rank} \varphi^{n-d_k} = d_k$.

If for an integer i the condition $\operatorname{rank} \varphi^i \neq 0$ holds true, then $\operatorname{rank} \varphi^i > \operatorname{rank} \varphi^{i+1}$ follows.
Furthermore, $\operatorname{rank} \varphi^{n-d_{p-1}} = d_{p-1}$ and $\operatorname{rank} \varphi^i \neq 0$ for $i \in \{1, \dots, n-d_{p-1}\}$, we, therefore, immediately see $\operatorname{rank} \varphi^{n-i} = i$ for $i \in \{d_{p-1}, \dots, n-1\}$.

Since $\operatorname{rank} \varphi = n-1$ and, thus, $\dim_K \ker \varphi = 1$, we arrive at $\operatorname{rank} \varphi^{n-i} = i$ for $i \in \{1, \dots, d_1\}$.
We have proven that the following equation holds true for every integer $i \in \{1, \dots, n\}$:

$$\operatorname{rank} \varphi^{n-i} = i.$$

Let $u_1, \dots, u_n$ be a basis of V that is adapted to F_*, that is, $F_k = \langle u_1, \dots, u_{d_k} \rangle$ for every integer $k \in \{1, \dots, p\}$.

Let $x \in \{1, \dots, n\}$, then without loss of generality, due to the rank conditions examined above, we can assume

$$\dim_K \varphi^{n-x}(\langle u_1, \dots, u_x \rangle) = x.$$

In particular $\ker \varphi = \left\langle \varphi^{n-1}(u_1) \right\rangle \subseteq \varphi^{n-x}(\langle u_1, \dots, u_x \rangle)$ and there exist elements $b_{i,x} \in K$, such that

$$\varphi^{n-1}(u_1) = \sum_{i=1}^{x} b_{i,x} \cdot \varphi^{n-x}(u_i).$$

We define

$$u'_x \coloneqq \sum_{i=1}^{x} b_{i,x} \cdot u_i$$

and note that $u'_1, \dots, u'_n$ form a basis of V, which is adapted to F_*:

Assume that $b_{x,x} = 0$, then

$$\varphi^{n-1}(u_1) = \sum_{i=1}^{x-1} b_{i,x} \cdot \varphi^{n-x}(u_i).$$

Application of φ yields $0 = \sum_{i=1}^{x-1} b_{i,x} \cdot \varphi^{n-(x-1)}(u_i)$. But then $b_{i,x} = 0$ for all $i < x$, since

$$\varphi^{n-(x-1)}(u_1), \ldots, \varphi^{n-(x-1)}(u_{x-1})$$

are linearly independent. We derive $\varphi^{n-1}(u_1) = 0$, a contradiction.

Since the basis $u_1, \ldots, u_n$ is adapted to F_*, the basis $u'_1, \ldots, u'_n$ is adapted to F_* as well; it furthermore fulfills

$$\dim_K \left\langle \varphi^{n-x}\left(u'_1\right), \ldots, \varphi^{n-x}\left(u'_x\right) \right\rangle = x.$$

Let $x \in \{1, \ldots, n\}$, then in particular $\varphi^{n-x}\left(u'_x\right) = u'_n = \varphi^{n-1}\left(u'_1\right)$ per definition and

$$\varphi^{x}\left(u'_i\right) = \varphi^{x+i-n}\left(\varphi^{n-i}\left(u'_i\right)\right) = \varphi^{x+i-n}\left(\varphi^{n-1}\left(u'_1\right)\right) = 0$$

if $x + i > n$.

Given $x \in \{1, \ldots, n\}$, we define $c_{x,i} \in K$ to be elements, such that $\varphi\left(u'_x\right) = \sum_{i=1}^{n} c_{x,i} \cdot u'_i$. Then $\varphi\left(u'_x\right) = u'_{x+1} \mod \left\langle u'_{x+2}, \ldots, u'_n \right\rangle$ holds true, since

$$\begin{aligned}
\varphi^{n-x-1}\left(u'_{x+1}\right) &= u'_n \\
&= \varphi^{n-x}\left(u'_x\right) \\
&= \varphi^{n-x-1} \circ \varphi\left(u'_x\right) \\
&= \sum_{i=1}^{n} c_{x,i} \cdot \varphi^{n-x-1}\left(u'_i\right) \\
&= c_{x,x+1} \cdot \varphi^{n-x-1}\left(u'_{x+1}\right) + \sum_{i<x+1} c_{x,i} \cdot \varphi^{n-x-1}\left(u'_i\right).
\end{aligned}$$

Thus, $c_{x,x+1} = 1$ and $c_{x,i} = 0$ if $i < x + 1$ and, therefore,

$$\varphi\left(u'_x\right) = u'_{x+1} \mod \left\langle u'_{x+2}, \ldots, u'_n \right\rangle.$$

Since every endomorphism of a K-vector space V' has a representing matrix in Jordan normal form, we can choose a basis $v_1, \ldots, v_n$, such that for every integer $k \in \{1, \ldots, p\}$ the following two conditions hold true:

- $F_k = \langle v_1, \ldots, v_{d_k} \rangle$,
- $\varphi(v_x) = \begin{cases} v_{x+1} \mod \langle v_{d_k+1}, \ldots, v_n \rangle, & \text{if } d_{k-1} < x < d_k; \\ 0 \mod \langle v_{d_k+1}, \ldots, v_n \rangle, & \text{if } x = d_k. \end{cases}$

In more detail:

If we pass to $\varphi' \colon K^n \to K^n$ by base change with respect to the basis of coordinate vectors in K^n, we see that the representing matrix A has rank $n - 1$; the partial flag translates to $F_k = \langle e_1, \ldots, e_{d_k} \rangle$.

Since the adaption to the flag does not depend on the choice of basis vectors of each space $\langle e_{d_{k-1}+1}, \dots, e_{d_k} \rangle$, the base change "inside the blocks" can be brought into Jordan normal form and we arrive at the above mentioned block-Jordan basis $v_1, \dots, v_n$, such that for every integer $k \in \{2, \dots, p\}$ and every $d_{k-1} < x \le d_k$ there are elements $d_{i,x} \in K$ for which

$$v_x = \sum_{i=d_{k-1}+1}^{d_k} d_{i,x} \cdot u'_i.$$

Let $k \in \{2, \dots, p\}$. The next aim is to define a basis $(v'_1, \dots, v'_n)$, such that

$$\varphi(v'_x) = \begin{cases} v'_{d_{k-1}+1} \mod \left\langle v'_{d_k+1}, \dots, v'_n \right\rangle, & \text{if } x = d_{k-1}; \\ v'_{x+1} \mod \left\langle v'_{d_k+1}, \dots, v'_n \right\rangle, & \text{if } d_{k-1} < x < d_k; \\ 0, & \text{if } x = n. \end{cases}$$

So far, we know $\varphi(v_{d_{k-1}}) = v_{d_{k-1}+1} \mod \langle v_{d_{k-1}+2}, \dots, v_n \rangle$, thus, there are elements $\eta_i \in K$, such that $\varphi(v_{d_{k-1}}) = v_{d_{k-1}+1} + \sum_{i=d_{k-1}+2}^{d_k} \eta_i \cdot v_i \mod \langle v_{d_k+1}, \dots, v_n \rangle$.

We define

$$v'_x := \begin{cases} v_x, & \text{if } x = d_{k-1}; \\ v_x + \sum_{i=x+1}^{d_k} \eta_{d_{k-1}-x+1+i} \cdot v_i, & \text{if } d_{k-1} < x < d_k; \\ v_n, & \text{if } x = n. \end{cases}$$

Then clearly $v'_1, \dots, v'_n$ build a basis of V that is adapted to F_*.

Given $d_{k-1} < x < d_k$, we calculate

$$\begin{aligned} \varphi(v'_x) &= \varphi\left(v_x + \sum_{i=x+1}^{d_k} \eta_{d_{k-1}-x+1+i} \cdot v_i\right) \\ &= v_{x+1} + \sum_{i=x+1}^{d_k} \eta_{d_{k-1}-x+1+i} \cdot v_{i+1} \mod \langle v_{d_k+1}, \dots, v_n \rangle \\ &= v_{x+1} + \sum_{i=x+2}^{d_k} \eta_{d_{k-1}-(x+1)+1+i} \cdot v_i \mod \langle v_{d_k+1}, \dots, v_n \rangle \\ &= v'_{x+1} \mod \left\langle v'_{d_k+1}, \dots, v'_n \right\rangle. \end{aligned}$$

Let $x = d_{k-1}$, then $v'_x = v_x$ and

$$\begin{aligned} \varphi(v'_x) &= v_{x+1} + \sum_{i=x+2}^{d_k} \eta_i \cdot v_i \mod \langle v_{d_k+1}, \dots, v_n \rangle \\ &= v_{x+1} + \sum_{i=x+2}^{d_k} \eta_{d_{k-1}-(x+1)+1+i} \cdot v_i \mod \langle v_{d_k+1}, \dots, v_n \rangle \\ &= v'_{x+1} \mod \langle v_{d_k+1}, \dots, v_n \rangle \\ &= v'_{x+1} \mod \left\langle v'_{d_k+1}, \dots, v'_n \right\rangle. \end{aligned}$$

Since $\varphi(v'_n) = \varphi(v_n) = 0$, the basis $v'_1, \ldots, v'_n$ has the desired properties.

The last step is to define a basis $w_1, \ldots, w_n$, such that

$$\varphi(w_x) = \begin{cases} w_{x+1} \mod \langle w_{d_1+2}, \ldots, w_n \rangle, & \text{if } x < d_1; \\ w_{d_{k-1}+1} \mod \langle w_{d_k+1}, \ldots, w_n \rangle, & \text{if } x = d_{k-1}; \\ w_{x+1} \mod \langle w_{d_k+1}, \ldots, w_n \rangle, & \text{if } d_{k-1} < x < d_k; \\ 0, & \text{if } x = n. \end{cases}$$

We fix elements $\lambda_x \in K$, such that for $1 \leq x < d_1$:

$$\varphi(w_x) = w_{x+1} + \lambda_x \cdot w_{d_1+1} \mod \langle w_{d_1+2}, \ldots, w_n \rangle .$$

By defining

$$\mu_{x-1} := \begin{cases} 1, & \text{if } x = 1; \\ -\lambda_{d_1-1}, & \text{if } x = 2; \\ -\sum_{i=0}^{x-1} \mu_i \cdot \lambda_{d_1-x+i}, & \text{if } 2 < x < d_1, \end{cases}$$

we are able to introduce vectors w_x as described above by setting

$$w_x := \begin{cases} \sum_{i=0}^{d_1-x} \mu_i \cdot v'_{x+i}, & \text{if } x < d_1; \\ v'_x, & \text{if } x \geq d_1. \end{cases}$$

Then $w_1, \ldots, w_n$ is a basis of V that is obviously adapted to F_* since $v'_1, \ldots, v'_n$ is adapted to F_*. We calculate

$$\begin{aligned} \varphi(w_x) &= \sum_{i=0}^{d_1-x-1} \mu_i \cdot \varphi\left(v'_{x+i}\right) + \mu_{d_1-x} \cdot \varphi\left(v'_{d_1}\right) \\ &= \sum_{i=0}^{d_1-x-1} \mu_i \cdot \left(v'_{x+i+1} + \lambda_{x+i} v'_{d_1+1}\right) + \mu_{d_1-x} \cdot v'_{d_1+1} \mod \left\langle v'_{d_1+2}, \ldots, v'_n \right\rangle \\ &= w_{x+1} + \left(\sum_{i=0}^{d_1-x-1} \mu_i \cdot \lambda_{x+i} + \mu_{d_1-x} \right) \cdot v'_{d_1+1} \mod \langle w_{d_1+2}, \ldots, w_n \rangle \\ &= w_{x+1} \mod \langle w_{d_1+2}, \ldots, w_n \rangle . \end{aligned}$$

Since $w_x = v'_x$ for $d_1 \leq x \leq n$, the basis $w_1, \ldots, w_n$ fulfills the conditions of 2. □

We make use of theorem 5.1.1 in order to find a generic normal form in the variety of nilpotent matrices over K. The following definition will be the key to this translation. Note that it will play a significant role in section 5.2 as well.

Definition 5.1.2. *(The submatrix $N_{(a,b)}$)*
Let $a, b \in \{0, \ldots, n\}$ and let $N \in \mathcal{N}$ be a nilpotent matrix.
We define $N_{(a,b)}$ to be the submatrix formed by the last a rows and the first b columns of N.

Corollary 5.1.3. *(Generic P-normal form in $\mathcal{N}$)*
The following conditions on a matrix $N \in \mathcal{N}$ are equivalent:

1. *The first d_k columns of N^{n-d_k} are linearly independent for each $k \in \{1, \ldots, p-1\}$,*
2. *the minor $\det((N^{n-d_k})_{(d_k,d_k)})$ is non-zero for each $k \in \{1, \ldots, p-1\}$,*
3. *N is P-conjugate to a unique matrix H, such that for all $k \in \{1, \ldots, p\}$:*

$$H_{i,j} = \begin{cases} 0, & \text{if } i \leq j; \\ 0, & \text{if } i = d_1 + 1 \text{ and } j < d_1; \\ 0, & \text{if } d_{k-1} + 3 \leq i \leq d_k \text{ and } d_{k-1} + 1 \leq j \leq d_k - 2, \text{ such that } i > j + 1; \\ 0, & \text{if } d_{k-1} + 2 \leq i \leq d_k \text{ and } j = d_{k-1}; \\ 1, & \text{if } i = j + 1. \end{cases}$$

The normal form is sketched in figure 5.1; the block sizes are those of the parabolic subgroup P.

$$H = \left(\begin{array}{c:c:c:c:c}
\begin{matrix} 0 & \cdots & 0 \\ 1 & & \\ & \ddots & \vdots \\ 0 & 1 & 0 \end{matrix} & 0 & 0 & 0 & 0 \\ \hdashline
\begin{matrix} 0 & \cdots & 0 & 1 \\ * & \cdots & * & 0 \\ \vdots & & \vdots & \vdots \\ * & \cdots & * & 0 \end{matrix} & \begin{matrix} 0 & \cdots & 0 \\ 1 & & \\ & \ddots & \vdots \\ 0 & 1 & 0 \end{matrix} & 0 & 0 & 0 \\ \hdashline
\begin{matrix} * & \cdots & * \\ \vdots & & \vdots \\ * & \cdots & * \end{matrix} & \begin{matrix} * & \cdots & * & 1 \\ * & \cdots & * & 0 \\ \vdots & & \vdots & \vdots \\ * & \cdots & * & 0 \end{matrix} & \ddots & 0 & 0 \\ \hdashline
\begin{matrix} * & \cdots & * \\ \vdots & & \vdots \\ * & \cdots & * \end{matrix} & \begin{matrix} * & \cdots & * \\ \vdots & & \vdots \\ * & \cdots & * \end{matrix} & \ddots & \begin{matrix} 0 & \cdots & 0 \\ 1 & & \\ & \ddots & \vdots \\ 0 & 1 & 0 \end{matrix} & 0 \\ \hdashline
\begin{matrix} * & \cdots & * \\ \vdots & & \vdots \\ * & \cdots & * \end{matrix} & \begin{matrix} * & \cdots & * \\ \vdots & & \vdots \\ * & \cdots & * \end{matrix} & \begin{matrix} * & \cdots & * \\ \vdots & & \vdots \\ * & \cdots & * \end{matrix} & \begin{matrix} * & \cdots & * & 1 \\ * & \cdots & * & 0 \\ \vdots & & \vdots & \vdots \\ * & \cdots & * & 0 \end{matrix} & \begin{matrix} 0 & \cdots & 0 \\ 1 & & \\ & \ddots & \vdots \\ 0 & 1 & 0 \end{matrix}
\end{array}\right)$$

Figure 5.1: The generic parabolic normal form

Proof of corollary 5.1.3. The equivalence between 1. and 3. follows directly from theorem 5.1.1 when defining the partial flag F_* by $F_k = \langle e_1, \ldots, e_{d_k} \rangle$ for $k \in \{1, \ldots, p\}$.

If 3. holds true, then 2. obviously holds true as well. Assume that the first d_k columns of N^{n-d_k} are linearly dependent for some $k \in \{1, \ldots, p-1\}$, then we find $\lambda_i \in K$, such that

$$\sum_{i=1}^{d_k} \lambda_i \cdot N_{*,i}^{n-d_k} = 0$$

and at least one of the λ_i is non-zero. But then $\det\left(\left(N^{n-d_k}\right)_{(d_k,d_k)}\right) = 0$. □

What comes to mind when looking at the normal form given in figure 5.1 is a missing symmetry: The entries in the first row of the very left second upper block equal zero. As the proof of lemma 5.1.1 suggests, there are some possibilities for a normal form; we choose this particular normal form, since it facilitates the clearest and most concise proof.

The conditions of Corollary 5.1.3 define an open subset of $\mathcal{N}$; we have, thus, found a generic normal form for nilpotent matrices up to P-conjugacy. In more detail, we can define a locally closed variety $\mathcal{H}_P$ containing all these P-normal forms, which is isomorphic to the affine space $\mathbf{A}^r$ of dimension $r = \sum_{i<j}\left(b_i \cdot b_j\right) - n + 1$.

The set $P.\mathcal{H}_P$ of all P-conjugates of normal forms in $\mathcal{H}_P$ is an open subset of $\mathcal{N}$ and will be denoted by $\mathcal{N}_P := P.\mathcal{H}_P$.

Note that all diagonal matrices with the same non-zero entries on the diagonal act trivially on $\mathcal{N}$. Then the calculation

$$\begin{aligned}
\dim\left(\mathcal{N}_P/P\right) &= \dim\mathcal{N} - \dim\left(P/K^*\right) \\
&= n\left(n-1\right) + \sum_{i\leq j}\left(b_i \cdot b_j\right) + 1 \\
&= \sum_{i=1}^{l} b_i^2 + 2 \cdot \sum_{i<j}\left(b_i \cdot b_j\right) - n - \sum_{i\leq j}\left(b_i \cdot b_j\right) + 1 \\
&= \sum_{i<j}\left(b_i \cdot b_j\right) - n + 1 \\
&= \dim\mathcal{H}_P
\end{aligned}$$

yields that the normal form is as fine as possible and separates the P-orbits in $\mathcal{N}$ generically.

One important special case is given by the Borel normal form. Since it will be examined in more detail in chapter 7, we include a visualization of the Borel normal form in figure 5.2.

$$\begin{pmatrix} 0 & & & \cdots & & 0 \\ 1 & 0 & & & & \\ * & 1 & 0 & & 0 & \\ & & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & 0 & \\ & & & & 1 & 0 \\ * & & \cdots & & * & 1 & 0 \end{pmatrix}$$

Figure 5.2: The generic B- normal form

We conclude the subsection by giving a normal form for the U-action on $\mathcal{N}$, that is, the conjugation action by the unipotent subgroup.

Corollary 5.1.4. *(Generic U-normal form in $\mathcal{N}$)*
The following conditions on a matrix $N \in \mathcal{N}$ are equivalent:

1. *For $k \in \{1, \ldots, n-1\}$, the first k columns of N^{n-k} are linearly independent,*
2. *for $k \in \{1, \ldots, n-1\}$, the minor $\det\left(\left(N^{n-k}\right)_{(k,k)}\right)$ is non-zero,*
3. *N is U-conjugate to a unique matrix H', such that $H'_{i,j} = 0$ for $i \leq j$ and $H'_{i+1,i} \neq 0$ for $i \in \{1, \ldots, n-1\}$.*

Proof. Let $N \in \mathcal{N}$. Then there is some matrix $b \in B$, such that $b \cdot N \cdot b^{-1} = H \in \mathcal{H}_B$ is a matrix in the above B-normal form. We have a unique decomposition $b = u \cdot t$ with $u \in U$ and $t \in T$. Thus, $H = u \cdot t \cdot N \cdot t^{-1} \cdot u^{-1}$ and the U-normal forms are, thus, given by

$$\begin{aligned} \mathcal{H}_U &:= \{t \cdot H \cdot t^{-1} \mid t \in T, H \in \mathcal{H}_B\} \\ &= \{H' \mid H'_{i,j} = 0 \text{ if } i \leq j \text{ and } H'_{i+1,i} \neq 0 \text{ for } i \in \{1, \ldots, n-1\}\}. \end{aligned}$$ □

We define $\mathcal{N}_U := U.\mathcal{H}_U$.

$$\begin{pmatrix} 0 & & & \cdots & & & 0 \\ x_1 & 0 & & & & & \\ * & x_2 & 0 & & 0 & & \\ & & \ddots & \ddots & & \vdots & \\ \vdots & & \ddots & \ddots & 0 & & \\ & & & & x_{n-2} & 0 & \\ * & & \cdots & & * & x_{n-1} & 0 \end{pmatrix}$$

Figure 5.3: The generic U- normal form

5.2 (Semi-) Invariants

We start calculating a class of B-semi-invariants on $\mathcal{N}$ (see [Boos and Reineke, 2011]).

For $i \in \{1, \ldots, n\}$, we denote by $\omega_i \colon B \to \mathbf{G}_m$ the character defined by $\omega_i(g) = g_{i,i}$; the ω_i form a basis for the group of characters of B.

Fix integers $a_1, \ldots, a_s, a'_1, \ldots, a'_t \in \{1, \ldots, n\}$, such that $a_1 + \ldots + a_s = a'_1 + \ldots + a'_t =: r$.

Moreover, for $i \in \{1, \ldots, s\}$ and $j \in \{1, \ldots, t\}$, fix polynomials $\mathcal{P}_{i,j}(x) \in K[x]$, and denote the datum $\left((a_i)_i, \left(a'_j\right)_j, \left(\mathcal{P}_{i,j}\right)_{i,j}\right)$ by $\mathcal{P}$.

Let $N \in \mathcal{N}$, then for all such i and j we consider the submatrices

$$\mathcal{P}_{i,j}(N)_{\left(a_i, a'_j\right)} \in K^{a_i \times a'_j}$$

as defined in definition 5.1.2, and form the $r \times r$-block matrix

$$N^{\mathcal{P}} := \left(\mathcal{P}_{i,j}(N)_{\left(a_i, a'_j\right)}\right)_{i,j}.$$

These considerations provide a class of B-semi-invariants on the nilpotent cone:

Proposition 5.2.1. *(B-semi-invariants on $\mathcal{N}$)*
For every datum $\mathcal{P}$ as above, the function

$$\begin{array}{rcl} f^{\mathcal{P}} \colon \mathcal{N} & \to & K \\ N & \mapsto & \det\left(N^{\mathcal{P}}\right) \end{array}$$

defines a B-semi-invariant regular function on $\mathcal{N}$ of weight

$$\sum_{i=1}^{s} \left(\omega_{n-a_i+1} + \ldots + \omega_n\right) - \sum_{j=1}^{t} \left(\omega_1 + \ldots + \omega_{a'_j}\right).$$

Proof. For $g \in B_n$ and $1 \leq a, a' \leq n$, denote by $g_{(\geq a)} \in B_a$ (respectively by $g_{(\leq (a')} \in B_{a'}$) the submatrix formed by the last a rows and columns (respectively by the first a' rows and columns) of g. With these definitions, we verify immediately that

$$\left(g \cdot N \cdot g^{-1}\right)_{(a,a')} = g_{(\geq a)} \cdot N_{(a,a')} \cdot g^{-1}_{(\leq a')}.$$

This yields the following equalities of block matrices

$$\begin{aligned} \left(g \cdot N \cdot g^{-1}\right)^{\mathcal{P}} &= \left(\mathcal{P}_{i,j}\left(g \cdot N \cdot g^{-1}\right)_{\left(a_i, a'_j\right)}\right)_{i,j} \\ &= \left(\left(g \cdot \mathcal{P}_{i,j}(N) \cdot g^{-1}\right)_{\left(a_i, a'_j\right)}\right)_{i,j} \\ &= \left(g_{(\geq a_i)} \cdot \mathcal{P}_{i,j}(N)_{\left(a_i, a'_j\right)} \cdot g_{\left(\leq a'_j\right)}^{-1}\right)_{i,j} \\ &= \left(\delta_{i,j} \cdot g_{(\geq a_i)}\right)_{i,j} \cdot N^{\mathcal{P}} \cdot \left(\delta_{i,j} \cdot g_{\left(\leq a'_j\right)}^{-1}\right)_{i,j}. \end{aligned}$$

Therefore,

$$f^{\mathcal{P}}\left(g\cdot N\cdot g^{-1}\right)=\det\left(\left(g\cdot N\cdot g^{-1}\right)^{\mathcal{P}}\right)=\prod_i\det\left(g_{(\geq a_i)}\right)\prod_j\det\left(g_{\left(\leq a'_j\right)}\right)^{-1}f^{\mathcal{P}}\left(N\right). \quad \square$$

We call r the size of $f^{\mathcal{P}}$.

Corollary 5.2.2. *(U-invariants on $\mathcal{N}$)*
All B-semi-invariants $f^{\mathcal{P}}$ of proposition 5.2.1 are U-invariants.

Proof. Of course, $\det\left(g_{(\geq a_i)}\right)=1$ and $\det\left(g_{\left(\leq a'_j\right)}\right)=1$ for all i, j. $\square$

Next, we will generalize the above proven semi-invariants to arbitrary parabolic subgroups of GL_n.

We consider the character group $X\left(\mathrm{GL}_n\right)$, which is freely generated by the determinant det.

Definition 5.2.3. *(The submatrix $N_{a\to b}$)*
Given a matrix $N\in K^{n\times n}$ and two integers $1\leq a<b\leq n$, we define $N_{a\to b}$ to be the submatrix of N which is obtained by deletion of the first a and the last $n-b$ columns and rows.

Given the parabolic subgroup P of GL_n of block sizes $\left(b_1,\ldots,b_p\right)$, we define $\omega_i\colon P\to\mathbf{G}_m$ by $\omega_i\left(g\right)=\det\left(g_{d_{i-1}\to d_i}\right)$, where d_i is defined as before. Then the ω_i form a basis for the group $X(P)$ of characters of P.

We construct a class of determinantal P-(semi-) invariant functions on $\mathcal{N}$ in the following.

Define $a_i:=n-d_{i-1}$ and $a'_i:=d_i$ for all $i\in\{1,\ldots,p\}$.

Fix integers $x_1,\ldots,x_s,y_1,\ldots,y_t\in\{1,\ldots,p\}$, such that $a_{x_1}+\ldots+a_{x_s}=a'_{y_1}+\ldots+a'_{y_t}=:r$. Moreover, fix polynomials $P_{i,j}\left(x\right)\in K[x]$ for $i\in\{1,\ldots,s\}$ and $j\in\{1,\ldots,t\}$, and denote the datum $\left((x_i)_i,\left(y_j\right)_j,\left(P_{i,j}\right)_{i,j}\right)$ by $\mathcal{P}$ as before.

Let $N\in\mathcal{N}$, then for all such i and j consider the submatrices $\mathcal{P}_{i,j}\left(N\right)_{\left(a_{x_i},a'_{y_j}\right)}\in K^{a_{x_i}\times a'_{y_j}}$ and form the $r\times r$-block matrix $N^{\mathcal{P}}:=\left(\mathcal{P}_{i,j}\left(N\right)_{\left(a_{x_i},a'_{y_j}\right)}\right)_{i,j}$.

Proposition 5.2.4. *(P-semi-invariants on $\mathcal{N}$)*
For every datum $\mathcal{P}$ as above, the function

$$\begin{aligned}f^{\mathcal{P}}\colon\ \mathcal{N}&\to K\\ N&\mapsto\det\left(N^{\mathcal{P}}\right)\end{aligned}$$

defines a P-semi-invariant regular function on $\mathcal{N}$ of weight

$$\sum_{1=i}^{s}\left(\omega_{x_i}+\ldots+\omega_p\right)-\sum_{j=1}^{t}\left(\omega_1+\ldots+\omega_{y_j}\right).$$

Proof. With the definitions of proposition 5.2.1 for $1 \leq x, y \leq p$, it follows immediately that

$$\left(g \cdot N \cdot g^{-1}\right)_{\left(a_x, a'_y\right)} = g_{(\geq n-d_{x-1})} \cdot N_{\left(a_x, a'_y\right)} \cdot g^{-1}_{\left(\leq d_y\right)}.$$

This yields the following equalities of block matrices:

$$\begin{aligned} \left(g \cdot N \cdot g^{-1}\right)^{\mathcal{P}} &= \left(\mathcal{P}_{i,j}\left(g \cdot N \cdot g^{-1}\right)_{\left(a_{x_i}, a'_{y_j}\right)}\right)_{i,j} \\ &= \left(\left(g \cdot \mathcal{P}_{i,j}\left(N\right) \cdot g^{-1}\right)_{\left(a_{x_i}, a'_{y_j}\right)}\right)_{i,j} \\ &= \left(g_{\left(\geq n-d_{x_i-1}\right)} \cdot \mathcal{P}_{i,j}\left(N\right)_{\left(a_{x_i}, a'_{y_j}\right)} \cdot g_{\left(\leq d_{y_j}\right)}{}^{-1}\right)_{i,j} \\ &= \left(\delta_{i,j} \cdot g_{\left(\geq n-d_{x_i-1}\right)}\right)_{i,j} \cdot N^{\mathcal{P}} \cdot \left(\delta_{i,j} \cdot g_{\left(\leq d_{y_j}\right)}{}^{-1}\right)_{i,j}. \end{aligned}$$

Therefore,

$$f^{\mathcal{P}}\left(g \cdot N \cdot g^{-1}\right) = \det\left(\left(g \cdot N \cdot g^{-1}\right)^{\mathcal{P}}\right) = \prod_i \det\left(g_{\left(\geq n-d_{x_i-1}\right)}\right) \prod_j \det\left(g_{\left(\leq d_{y_j}\right)}\right)^{-1} f^{\mathcal{P}}\left(N\right). \qquad \square$$

5.3 Generation of semi-invariant rings

In order to show that the above defined semi-invariants in fact generate the ring of semi-invariants

$$R := \bigoplus_{\chi \in X(B)} K[\mathcal{N}]^B_\chi,$$

we make use of theorem 1.2.1 and the translation of lemma 2.3.1:

To calculate generating B-semi-invariants of $\mathcal{N}$, we can also translate generating determinantal semi-invariants of the $\mathrm{GL}_{\underline{d}_B}$-action on $R^{\mathrm{inj}}_{\underline{d}_B}(\mathcal{Q}_n)$ where

$$\mathcal{Q}_n := \quad \underset{1}{\bullet} \xrightarrow{\alpha_1} \underset{2}{\bullet} \xrightarrow{\alpha_2} \underset{3}{\bullet} \quad \cdots \quad \underset{n-2}{\bullet} \xrightarrow{\alpha_{n-2}} \underset{n-1}{\bullet} \xrightarrow{\alpha_{n-1}} \underset{n}{\bullet} \circlearrowleft \alpha.$$

Thus, we fix the dimension vector $\underline{d}_B$ and an arbitrary morphism in $\operatorname{add}\mathcal{Q}$, say

$$\phi\colon \bigoplus_{j=1}^{n} O(j)^{x_j} \to \bigoplus_{i=1}^{n} O(i)^{y_i}$$

with

$$\sum_{j \in \mathcal{Q}_0} x_j \cdot j = \sum_{i \in \mathcal{Q}_0} y_i \cdot i.$$

The homomorphism spaces $P(j,i)$ between two objects $O(j)$ and $O(i)$ in add $\mathcal{Q}$ are generated as K-vector spaces by

$$P(j,i) = \begin{cases} 0, & \text{if } j > i; \\ \left\langle \rho_{j,i} := \alpha_{i-1} \cdots \alpha_j \right\rangle, & \text{if } j \leq i < n; \\ \left\langle \rho_{j,n}^{(k)} := \alpha^k \alpha_{n-1} \cdots \alpha_j \mid k \in \mathbf{N} \cup \{0\} \right\rangle, & \text{if } i = n. \end{cases}$$

Then we obtain a determinantal semi-invariant

$$\begin{array}{rccl} f_\phi\colon & R^{\mathrm{inj}}_{\underline{d}_B}(Q_n) & \to & K \\ & m & \mapsto & \det M(\phi), \end{array}$$

as defined in [Schofield and van den Bergh, 2001]; the definition has also been recapitulated briefly in section 1.2.

Given an arbitrary matrix $N \in K^{n\times n}$, define M^N to be the representation

$$M^N = \qquad K \xrightarrow{\epsilon_1} K^2 \xrightarrow{\epsilon_2} K^3 \longrightarrow \cdots \longrightarrow K^{n-2} \xrightarrow{\epsilon_{n-2}} K^{n-1} \xrightarrow{\epsilon_{n-1}} K^n \circlearrowleft N$$

in $\mathrm{rep}^{\mathrm{inj}}_K(Q_n)(\underline{d}_B)$ with natural embeddings $\epsilon_i\colon K^i \hookrightarrow K^{i+1}$.

Since $\mathrm{GL}_{\underline{d}_B}$ acts transitively on $R^{\mathrm{inj}}_{\underline{d}_B}(Q')$ with Q' being the linearly oriented quiver of Dynkin type A_n, we can examine the semi-invariants on these representations M^N.

If the determinantal $\mathrm{GL}_{\underline{d}_B}$-semi-invariant f_ϕ of $R^{\mathrm{inj}}_{\underline{d}_B}(Q_n)$, interpreted as a B-semi-invariant, can be expressed by the semi-invariants from proposition 5.2.1, we have found generating semi-invariants of R, therefore we formulate the following lemma.

Lemma 5.3.1. *(Generation of the B-semi-invariant ring of $\mathcal{N}$)*
The determinantal semi-invariant f_ϕ corresponds to one of the B-semi invariants constructed in proposition 5.2.1.

Proof. The morphism ϕ is given by a $\sum_{i=1}^{n} y_i \times \sum_{j=1}^{n} x_j$-matrix H with entries being morphisms between objects in add $\mathcal{Q}$.

We can view the matrix H as an $n \times n$ block matrix $H = (H_{i,j})_{1\leq i,j\leq n}$ with $H_{i,j} \in K^{y_i \times x_j}$ for $i, j \in \{1, \ldots, n\}$. Then

$$\left(H_{i,j}\right)_{k,l} = \begin{cases} 0, & \text{if } i < j; \\ \lambda_{i,j}^{k,l} \cdot \rho_{j,i}, & \text{for some } \lambda_{i,j}^{k,l} \in K \text{ if } j \leq i < n; \\ \sum_{h=0}^{\infty} \left(\lambda_{n,j}^{k,l}\right)_h \cdot \rho_{j,n}^{(h)}, & \text{for some } \left(\lambda_{n,j}^{k,l}\right)_h \in K \text{ if } j \leq i = n. \end{cases}$$

Let us denote by f^ϕ the B-semi-invariant of $\mathcal{N}$ associated to f_ϕ via the translation of lemma 2.3.1,

$$\begin{array}{rccl} f^\phi\colon & \mathcal{N} & \to & K \\ & N & \mapsto & \det M^N(\phi). \end{array}$$

The matrix

$$M^N(\phi) = \left(M^N_{i,j}\right)_{1\le i,j\le n} \in K^{\sum_{i\in Q_0} iy_i \times \sum_{j\in Q_0} jx_j}$$

is given as a block matrix where each block

$$M^N_{i,j} = \left(\left(M^N_{i,j}\right)_{k,l}\right)_{\substack{1\le k\le y_i\\ 1\le l\le x_j}} \in K^{iy_i\times jx_j}$$

is again a block matrix. The blocks of $M^N_{i,j}$ are given by

$$K^{i\times j} \ni \left(M^N_{i,j}\right)_{k,l} = \begin{cases} 0, & \text{if } i < j;\\ \lambda^{k,l}_{i,j}\cdot E^{(i)}_{(i,j)}, & \text{if } j \le i < n;\\ \sum\limits_{h=0}^{\infty} \left(\lambda^{k,l}_{n,j}\right)_h \cdot \left(N^h\right)_{(n,j)}, & \text{if } j \le i = n; \end{cases}$$

where $E^{(i)} \in K^{i\times i}$ is the identity matrix and the notation $N_{(i,j)}$ is the same as in 5.1, that is, the submatrix of the last i rows and the first j columns of N.

Note that if $i, j \in \{1,\ldots,n\}$, then $M^N_{i,j} = M^{N'}_{i,j} =: M_{i,j}$ for every pair of matrices $N, N' \in \mathcal{N}$.

Reduction

We prove in the following that, without loss of generality, $y_1 = \ldots = y_{n-1} = 0$ and proceed by induction on the index i of y_i.

The beginning of the induction is given by the case $i = 1$, we claim that we can, without loss of generality, assume $y_1 = 0$.

Clearly

$$M^N(\phi) = \begin{pmatrix} M_{1,1} & 0 \\ * & \left(M^N_{i,j}\right)_{1<i,j\le n} \end{pmatrix}.$$

If $y_1 > x_1$, the determinant $\det M^N(\phi)$ equals 0 for every matrix N due to rank considerations.

If $y_1 = x_1$, then

$$\det M^N(\phi) = \det M_{1,1} \cdot \det\left(M^N_{i,j}\right)_{1<i,j\le n} = \lambda \cdot \det\left(M^N_{i,j}\right)_{1<i,j\le n}$$

for some $\lambda \in K$ independent of N and without loss of generality $y_1 = 0$.

Let us assume $y_1 < x_1$. We can apply elementary row operations to the first y_1 rows and elementary column operations to the first x_1 columns of $M^N(\phi)$ to obtain the equality

$$\det M^N(\phi) = \lambda \cdot \det \begin{pmatrix} E^{(y_1)} & 0 & 0 \\ * & S & \left(M^N_{i,j}\right)_{1<i,j\le n} \end{pmatrix}$$

for some $\lambda \in K$ independent of N and a naturally occurring matrix S.
In more detail, we can exchange two rows or columns and multiply a row or column with an element $\mu \in K^*$ because the same operations can be applied to H and the semi-invariant stays the same up to scalar multiplication.

Given elements $\mu, \mu' \in K^*$, we can add the μ-th multiple of one row (or column) to the μ'-th multiple of another row (or column, respectively) and for the same reasoning, the semi-invariant does not change. Since the operations applied above are independent of N, we obtain the given equality.
Then

$$\det M^N(\phi) = \lambda \cdot \det \begin{pmatrix} S & \left(M^N_{i,j}\right)_{1<i,j\leq n} \end{pmatrix},$$

thus, without loss of generality, we can assume $y_1 = 0$; this proves the beginning of the induction.

Now let $i \in \{2, \ldots, n-1\}$ and assume $y_1 = \ldots = y_{i-1} = 0$. Then

$$M^N(\phi) = \begin{pmatrix} M_{i,1} & M_{i,2} & \ldots & M_{i,i} & 0 \\ \left(M^N_{x,1}\right)_{i<x} & \left(M^N_{x,2}\right)_{i<x} & \ldots & \left(M^N_{x,i}\right)_{i<x} & \left(M^N_{x,y}\right)_{i<x,y} \end{pmatrix}$$

and we distinguish between the following three cases.

First case: $i \cdot y_i > x_1 + 2 \cdot x_2 + \ldots + i \cdot x_i$.

Then $\det M^N(\phi) = 0$ for every matrix N due to rank considerations.

Second case: $i \cdot y_i = x_1 + 2 \cdot x_2 + \ldots + i \cdot x_i$.

Then

$$\det M^N(\phi) = \det \begin{pmatrix} M_{i,1} & M_{i,2} & \ldots & M_{i,i} \end{pmatrix} \cdot \det \left(M^N_{x,y}\right)_{i<x,y} = \lambda \cdot \det \left(M^N_{x,y}\right)_{i<x,y}$$

for some $\lambda \in K$ independent of N and without loss of generality $y_i = 0$.

Third case: $i \cdot y_i < x_1 + 2 \cdot x_2 + \ldots + i \cdot x_i$.

Then $i \cdot y_i \leq i \cdot x_i$, since otherwise $\det M^N(\phi) = 0$ for every matrix N.
Define

$$\left(M^N_{s,.}\right)_y := \left(\left(M^N_{s,j}\right)_{y,l}\right)_{\substack{1\leq j\leq n \\ 1\leq l\leq x_j}} \quad \text{for } 1 \leq y \leq y_s$$

$$\text{and } \left(M^N_{.,t}\right)_x := \left(\left(M^N_{a,t}\right)_{k,x}\right)_{\substack{i\leq a\leq n \\ 1\leq k\leq y_a}} \quad \text{for } 1 \leq x \leq x_t.$$

Remember that $M_{i,x} = 0$ if $i < x$ and $\left(M^N_{i,x}\right)_{k,l} = \lambda^{k,l}_{i,x} \cdot E^{(i)}_{(i,x)} \in K^{i\times x}$ if $x \leq i$.
With the same reasoning as in the case $i = 1$, by application of appropriate elementary row and column operations, we obtain

$$\det M^N(\phi) = \lambda \cdot \det \begin{pmatrix} M'_{i,1} & M'_{i,2} & \ldots & M'_{i,i-1} & E^{(iy_i)} & 0 & 0 \\ \left(M^N_{x,1}\right)_{i<x} & \left(M^N_{x,2}\right)_{i<x} & \ldots & \left(M^N_{x,i-1}\right)_{i<x} & S' & S & \left(M^N_{x,y}\right)_{i<x,y} \end{pmatrix}$$

for some $\lambda \in K$ independent of N and naturally occurring matrices $M'_{i,x}$ for $x \in \{1, \ldots, i-1\}$, S and S'.
In more detail, given $y, y' \in \{1, \ldots, y_i\}$ and $x, x' \in \{1, \ldots, x_i\}$, we can exchange $\left(M^N_{i,.}\right)_y$ and $\left(M^N_{i,.}\right)_{y'}$ and we can exchange $\left(M^N_{.,i}\right)_x$ and $\left(M^N_{.,i}\right)_{x'}$, because the same operations can be

applied to H and the semi-invariant stays the same up to scalar multiplication.
We can multiply $\left(M^N_{i,\cdot}\right)_y$ or $\left(M^N_{\cdot,i}\right)_x$ with an element $\mu \in K^*$ for the same reasoning.

Furthermore, given elements $\mu, \mu' \in K^*$, we can add the μ-th multiple of $\left(M^N_{i,\cdot}\right)_y$ (or $\left(M^N_{\cdot,i}\right)_x$) to the μ'-th multiple of $\left(M^N_{i,\cdot}\right)_{y'}$ (or $\left(M^N_{\cdot,i}\right)_{x'}$, respectively) as well, the semi-invariant does not change. Since the operations applied above are independent of N, we obtain the given equality.

The above matrix can be simplified since, given $\mu, \mu' \in K$, we can without loss of generality add the μ-fold of $\left(\left(M^N_{\cdot,i}\right)_k\right)_{(i,l)}$ to the μ'-fold of $\left(M^N_{\cdot,j}\right)_l$. The translation to H is easily done and the semi-invariant is only changed by multiplication of a scalar independent of N. We are, therefore, able to show

$$\begin{aligned}\det M^N(\phi) &= \lambda' \det \begin{pmatrix} 0 & 0 & \dots & 0 & E^{(iy_i)} & 0 & 0 \\ R'_1 & R'_2 & \dots & R'_{i-1} & S' & S & \left(M^N_{x,y}\right)_{i<x,y} \end{pmatrix} \\ &= \pm\, \lambda' \det \begin{pmatrix} R'_1 & R'_2 & \dots & R'_{i-1} & S & \left(M^N_{x,y}\right)_{i<x,y} \end{pmatrix}\end{aligned}$$

for some $\lambda' \in K$ and naturally occurring matrices R'_x for $1 \leq x \leq i-1$.
Thus, in all cases, we can without loss of generality assume $y_i = 0$.

Now assume $y_1 = \ldots = y_{n-1} = 0$, then we are able to extract a semi-invariant as in 5.2. Define

$$a := (\underbrace{n, \ldots, n}_{=:a_1,\ldots,a_{y_n}}) \quad \text{and} \quad a' := (\underbrace{1, \ldots, 1}_{=:a'_{1,1},\ldots,a'_{1,x_1}}, \underbrace{2, \ldots, 2}_{=:a'_{2,1},\ldots,a'_{2,x_2}}, \ldots, \underbrace{n, \ldots, n}_{=:a'_{n,1},\ldots,a'_{n,x_n}}).$$

Furthermore, define for $j \in \{1, \ldots, n\}$ and for each pair of integers $k \in \{1, \ldots, y_n\}$ and $l \in \{1, \ldots, x_j\}$ a polynomial

$$\mathcal{P}^{(k,l)}_j := \sum_{h=0}^{\infty} \left(\lambda^{k,l}_{n,j}\right)_h \cdot X^h.$$

Let us denote $\mathcal{P} := \left(a, a', \left(P^{(k,l)}_j\right)_{j,k,l}\right)$ and consider the B-semi-invariant

$$\begin{aligned} f^{\mathcal{P}}: \quad \mathcal{N} &\to K \\ N &\mapsto \det N^{\mathcal{P}}. \end{aligned}$$

We claim that $f^{\mathcal{P}}(N) = f^{\phi}(N)$ holds true for all $N \in \mathcal{N}$.

Let $N \in \mathcal{N}$. Then

$$\begin{aligned}
f^{\phi}(N) \;=\; \det M^N(\phi) &= \det\left(M^N_{n,j}\right)_{1\leq j\leq n} \\
&= \det\left(\left(\left(M^N_{n,j}\right)_{k,l}\right)_{\substack{1\leq k\leq y_n\\ 1\leq l\leq x_j}}\right)_{1\leq j\leq n} \\
&= \det\left(\left(\sum_{h=0}^{\infty}\left(\lambda^{k,l}_{n,j}\right)_h \cdot \left(N^h\right)_{(n,j)}\right)_{\substack{1\leq k\leq y_n\\ 1\leq l\leq x_j}}\right)_{1\leq j\leq n} \\
&= \det\left(\left(\left(\sum_{h=0}^{\infty}\left(\lambda^{k,l}_{n,j}\right)_h \cdot N^h\right)_{(n,j)}\right)_{\substack{1\leq k\leq y_n\\ 1\leq l\leq x_j}}\right)_{1\leq j\leq n} \\
&= \det\left(\left(\mathcal{P}^{(k,l)}_j(N)_{(n,j)}\right)_{\substack{1\leq k\leq y_n\\ 1\leq l\leq x_j}}\right)_{1\leq j\leq n} \;=\; \det N^{\mathcal{P}} \;=\; f^{\mathcal{P}}(N). \qquad \square
\end{aligned}$$

Corollary 5.3.2. *(Generation of the U-invariant ring of $\mathcal{N}$)*
The U-invariant ring $K[\mathcal{N}]^U$ is generated by the U-invariants given in corollary 5.2.2.

6 Towards an algebraic U-quotient of the nilpotent cone

Of course, GL_n is a reductive group but neither U, B nor P are. Therefore, to calculate quotients of the nilpotent cone by these groups, we cannot rely on those results which assume the acting group to be reductive. In case of the U-action, however, there is a translation to a reductive action.

6.1 A quotient criterion for unipotent actions

We will prove the quotient criterion given in theorem 1.1.5 in a more general way than necessary for our aim, that is, for arbitrary unipotent actions that are induced by a reductive group.

Let G be a reductive algebraic group and U be a unipotent subgroup. Then U acts on G by right multiplication and lemma 1.1.6 states that the U-invariant ring $K[G]^U$ is finitely generated as a K-algebra. Thus, an algebraic U-quotient of G, namely $G/\!\!/U := \operatorname{Spec} K[G]^U$, exists together with a dominant morphism $\pi_{G/\!\!/U} \colon G \to G/\!\!/U := \operatorname{Spec} K[G]^U$ which is in general not surjective. Note that there is an element $\overline{e} \in G/\!\!/U$, such that $\pi_{G/\!\!/U}(g) = g\overline{e}$ for all $g \in G$.

The group G acts on $G/\!\!/U$ by left multiplication. Let X be an affine G-variety and consider the diagonal operation of G on the affine variety $G/\!\!/U \times X$.
Let $\pi' \colon G/\!\!/U \times X \to (G/\!\!/U \times X)/\!\!/G := \operatorname{Spec} K[G/\!\!/U \times X]^G$ be the associated algebraic G-quotient, then we obtain the following commutative diagram:

$$\begin{array}{ccc} X & \xrightarrow[x \,\mapsto\, (\overline{e}, x)]{\iota} & G/\!\!/U \times X \\ & \searrow^{\rho = \pi' \circ \iota} & \downarrow \pi' \\ & & (G/\!\!/U \times X)/\!\!/G \end{array}$$

By definition, the map ι^* is given by

$$\begin{aligned} \iota^* \colon \; K[G]^U \otimes K[X] &\to K[X] \\ \sum_i h_i \otimes f_i &\mapsto \sum_i h_i(\overline{e}) f_i. \end{aligned}$$

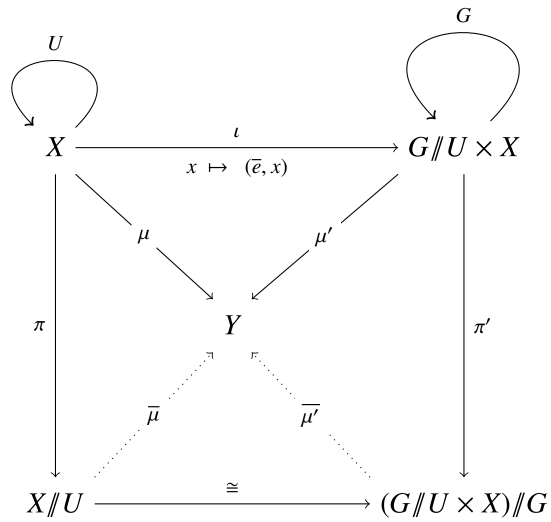
The morphism ρ induces an isomorphism

$$\rho^* \colon (K[G]^U \otimes K[X])^G \to K[X]^U.$$

Thus, $X /\!\!/ U \cong (G /\!\!/ U \times X) /\!\!/ G$ and

$$K[X]^U \cong (K[G /\!\!/ U \times X])^G \cong (K[G /\!\!/ U] \otimes K[X])^G \cong (K[G]^U \otimes K[X])^G.$$

Let Y be an affine G-variety and consider the following commutative diagram



where μ' is assumed to be G-invariant and

$$\begin{aligned} \mu\colon\ & X \to Y \\ & x \mapsto (f_1(x), \ldots, f_s(x)) \end{aligned}$$

is assumed to be a dominant U-invariant morphism of affine varieties.

In this setting, we obtain the following criterion for μ to be an algebraic U-quotient.

Lemma 6.1.1. *(Quotient criterion for unipotent actions)*
Assume that

(1.) *Y is normal,*

(2.) *μ separates the U-orbits generically, that is, there is an open subset $X_U \subseteq X$, such that $\mu(x) \neq \mu(x')$ for all $x, x' \in X_U$ and $\mu(X_U)$ is an open subvariety of Y, and*

(3.) $\operatorname{codim}_Y(\overline{Y \backslash \mu(X)}) \geq 2$ *or μ is surjective.*

Then μ is an algebraic U-quotient of X, that is, $Y \cong X /\!\!/ U$.

Proof. Let $g_1, \ldots, g_s \in K[G /\!\!/ U \times X]^G$, such that $\rho^*(g_i) = f_i$ for all i.

Y is assumed to be a normal variety.

If $Z \subset Z'$, then $\overline{Z} \subseteq \overline{Z'}$. Thus,

$$\overline{Y\backslash\mu(X)} = \overline{Y\backslash\mu' \circ \iota(X)} \supseteq \overline{Y\backslash\mu'(G/\!\!/U \times X)}$$

and, therefore,

$$2 \leq \operatorname{codim}_Y(\overline{Y\backslash\mu(X)}) \leq \operatorname{codim}_Y(\overline{Y\backslash\mu'(G/\!\!/U \times X)}).$$

The orbit $G.(\{\overline{e}\} \times X_U)$ is an open subvariety of $G/\!\!/U \times X$, since X_U is an open subvariety of X.

Then π separates the G-orbits in $G/\!\!/U \times X$ generically, namely on $G.(\{\overline{e}\} \times X_U)$:

Let $y_i := g_i((\overline{e}, x)) = g_i((\overline{e}, x'))$ for all i and two non-U-conjugate $x, x' \in X_U$. Since the diagram

$$\begin{array}{ccc} X & \xrightarrow{\quad \iota \quad} & G/\!\!/U \times X \\ & \searrow^{\mu} & \vdots\, \mu' \\ & & Y \end{array}$$

commutes, we derive

$$\mu(x) = \mu' \circ \iota(x) = \mu'((\overline{e}, x)) = \mu'((\overline{e}, x')) = \mu' \circ \iota(x') = \mu(x'),$$

a contradiction to assumption (3.). We have shown that generically each fibre of μ' contains a unique G-orbit. Since the fibres of a morphism are closed, each of the fibres $\mu'^{-1}(y)$ for $y \in \mu'(G.(\{\overline{e}\} \times X_U))$ contains one unique closed orbit.

Of course, the assumptions yield that $\mu'(G.(\{\overline{e}\} \times X_U)) = \mu(X_U)$ is open in Y.

Thus, theorem 1.1.5 yields that $\pi \colon G/\!\!/U \times X \to Y$ is an algebraic G-quotient. Since f_i and g_i correspond to each other via the isomorphism $\rho^* \colon (K[G]^U \otimes K[X])^G \to K[X]^U$, the morphism $\mu \colon X \to Y$ is an algebraic U-quotient of X. □

6.1.1 The examples $n = 2$ and $n = 3$

We are now able to give explicit descriptions of algebraic U-quotients of the nilpotent cone in case n equals 2 or 3.

Example 6.1.2. *(An algebraic U-quotient in $\mathcal{N}_2$)*

We consider $\mathcal{N} = \mathcal{N}_2$. In this case, the U-normal form of corollary 5.1.4 is given by matrices H_x for $x \in K^*$, where

$$H_x := \begin{pmatrix} 0 & 0 \\ x & 0 \end{pmatrix}.$$

Then we define a morphism by

$$f_{2,1} \colon \mathcal{N} \to K; \quad N \mapsto f_{2,1}(N) := N_{2,1},$$

which is a U-invariant due to corollary 5.2.2.

Claim: *The morphism*

$$\begin{aligned} \mu\colon\ \mathcal{N} &\to \boldsymbol{A}^1 = \operatorname{Spec} K[f_{2,1}] \\ N &\mapsto f_{2,1}(N). \end{aligned}$$

is an algebraic U-quotient of $\mathcal{N}$.

Proof. To make use of theorem 6.1.1, we show that

(1.) $\mathbf{A}^1$ is normal:

This is a well-known fact from algebraic geometry (see also section 1.1).

(2.) μ separates the U-orbits generically, that is, there is an open subset $\mathcal{N}_U \subseteq \mathcal{N}$, such that $\mu(N) \neq \mu(N')$ for all $N, N' \in \mathcal{N}_U$ and $\mu(\mathcal{N}_U)$ is an open subvariety of $\mathbf{A}^1$:

Let $\mathcal{N}_U$ be defined as in 5.1 as the set of U-conjugates of normal forms. Then $\mu(\mathcal{N}_U) = \mathbf{A}^1 \backslash \{0\}$ is open in $\mathbf{A}^1$ and $\mu(N) = \mu(N')$ for two normal forms directly yields $N = N'$.

(3.) Since μ is surjective, there is nothing left to show. □

We have, therefore, proven

$$\mathcal{N} /\!\!/ U = \boldsymbol{A}^1$$

and

$$K[\mathcal{N}]^U = K[f_{2,1}].$$

The case $n = 3$ is slightly more complex, but can still be proven by making use of theorem 6.1.1.

Example 6.1.3. *(An algebraic U-quotient for $\mathcal{N}_3$)*
We consider $\mathcal{N} = \mathcal{N}_3$, in this case the U-normal forms are given by matrices

$$H = \begin{pmatrix} 0 & 0 & 0 \\ x_1 & 0 & 0 \\ x & x_2 & 0 \end{pmatrix}$$

for $x_1, x_2 \in K^*$.
Consider the following U-invariants (as in corollary 5.2.2), given by

$$\begin{aligned} f_{3,1}\colon\ \mathcal{N} \to K;\quad N &\mapsto N_{3,1}, \\ \det_1\colon\ \mathcal{N} \to K;\quad N &\mapsto N_{2,1}N_{3,2} - N_{2,2}N_{3,1}, \\ \det_2\colon\ \mathcal{N} \to K;\quad N &\mapsto N_{1,1}N_{3,1} + N_{2,1}N_{3,2} + N_{3,1}N_{3,3}, \end{aligned}$$

The equality $\det_1(N) = \det_2(N)$ *holds true for all $N \in \mathcal{N}$ due to the nilpotency conditions.*

Let $f_1(N)$ be the U-invariant given by $a_1 = 2, b_1 = 1, b_2 = 1, \mathcal{P}_{1,1} = x$ and $\mathcal{P}_{1,2} = x^2$. Then

$$f_1\colon\ \mathcal{N} \to K;\quad N \mapsto N_{2,1} \cdot \det_1 + N_{3,1} \cdot (N_{2,1}N_{3,3} - N_{3,1}N_{2,3}).$$

Let $f_2(N)$ be the U-invariant given by $a_1 = 1, a_2 = 1, b_1 = 2, \mathcal{P}_{1,1} = x^2$ and $\mathcal{P}_{1,2} = x$. Then

$$f_2\colon\ \mathcal{N} \to K;\quad N \mapsto N_{3,2} \cdot \det_1 + N_{3,1} \cdot (N_{1,1}N_{3,2} - N_{1,2}N_{3,1}).$$

Claim: *The morphism*

$$\begin{aligned} \mu\colon\ \mathcal{N} &\to \boldsymbol{A}^1 \times \operatorname{Spec} \frac{K[X_1, X_2, Z]}{(X_1X_2 = Z^3)} \\ N &\mapsto (f_{3,1}(N), f_1(N), f_2(N), \det_1(N)) \end{aligned}$$

is an algebraic U-quotient of $\mathcal{N}$.

Proof. Given a matrix H in normal form as above, the equalities

$$f_1(H) \cdot f_2(H) = (x_1^2 \cdot x_2) \cdot (x_1 \cdot x_2^2) = x_1^3 \cdot x_2^3 = (x_1 \cdot x_2)^3 = \det_1^3(H)$$

yield $f_1 \cdot f_2 = \det_1$ on H_U. Since the morphisms f_1, f_2 and $\det_1$ are U-invariant, this equality holds true on $\mathcal{N}_U$ as well, and, since $\mathcal{N}_U \subset_{\text{open}} \mathcal{N}$, on the whole nilpotent cone.

Furthermore,

$$Y := \mathbf{A}^1 \times \underbrace{\operatorname{Spec} \frac{K[X_1, X_2, Z]}{(X_1X_2 = Z^3)}}_{X:=} = \operatorname{Spec} \frac{K[f_{3,1}, f_1, f_2, \det_1]}{\left(f_1 \cdot f_2 = \det_1^3\right)}.$$

To make use of theorem 6.1.1, we show that

(1.) the affine variety Y is normal:

The normality of the variety X follows immediately from Serre's criterion 1.1.2 or from the fact that X is a toric affine variety (see subsection 1.1.4) induced by the strongly convex rational polyhedral cone

$$\sigma := \operatorname{Cone}\left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix}\right).$$

Since products of normal varieties are normal, the variety Y is normal as well.

(2.) the morphism μ separates the U-orbits generically, that is, there is an open subset $\mathcal{N}_{\text{open}} \subseteq \mathcal{N}$, such that $\mu(N) \neq \mu(N')$ for all $N, N' \in \mathcal{N}_{\text{open}}$ and $\mu(\mathcal{N}_{\text{open}})$ is an open subvariety of Y:

Let $\mathcal{N}_{\text{open}} := \mathcal{N}_U$ be defined as in 5.1 as the set of U-conjugates of U-normal forms in $\mathcal{H}_U$ and set $X' \subset X$ to contain those tuples of which all entries are being non-zero. Then $\mu(\mathcal{N}_{\text{open}}) = \mathbf{A}^1 \times X'$ is open in Y and $f_1 \cdot f_2 = \det_1^3$, thus, if $\mu(N) = \mu(N') \in \mathbf{A}^1 \times X'$, then $\det_1(N) \neq 0 \neq \det_1(N')$ and N and N' are contained in $\mathcal{N}_{\text{open}}$. We can, thus, assume them to be two normal forms and clearly derive $N = N'$.

(3.) $\operatorname{codim}_Y(\overline{Y \backslash \mu(\mathcal{N})}) \geq 2$:

Since $\mathbf{A}^1 \times X' \subset \mu(\mathcal{N})$, it suffices to show that $(s,t,u,v) \in \mu(\mathcal{N})$ whenever either s, t or u equals zero and $v^3 = ut$.

We are left with the three cases

(3.1) $\mu(\begin{pmatrix} 0 & 0 & 0 \\ t/v & 0 & 0 \\ 0 & u/v & 0 \end{pmatrix}) = (0,t,u,v)$ for arbitrary $t,u \in K^*$.

(3.2) $\mu(\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -t/s^2 \\ s & 0 & 0 \end{pmatrix}) = (s,t,0,0)$ for arbitrary $s,t \in K^*$.

(3.3) $\mu(\begin{pmatrix} 0 & -u/s^2 & 0 \\ 0 & 0 & 0 \\ s & 0 & 0 \end{pmatrix}) = (s,0,u,0)$ for arbitrary $s,u \in K^*$.

Therefore, $\operatorname{codim}_Y(\overline{Y \backslash \mu(\mathcal{N})}) \geq 2$. □

We have proven

$$\mathcal{N} /\!\!/ U = \boldsymbol{A}^1 \times \operatorname{Spec} \frac{K[X_1, X_2, Z]}{(X_1 X_2 = Z^3)}$$

and

$$K[\mathcal{N}]^U = \frac{K[f_{3,1}, f_1, f_2, \det_1]}{\left(f_1 \cdot f_2 = \det_1^3\right)}.$$

6.2 Toric invariants

As the case $n = 3$ suggests, there is a toric variety closely related to $\mathcal{N} /\!\!/ U$.

The idea of a generalization is the following: By considering a special type of U-invariants, so-called toric invariants, we define a toric variety X together with a dominant morphism $\mathcal{N} /\!\!/ U \to X$, such that the generic fibres are affine spaces of the same dimension.

Given a matrix $H = (x_{i,j})_{i,j} \in \mathcal{H}_U$, we denote $x_i := x_{i+1,i}$ and define its “toric part” $H_{\mathrm{tor}} \in K^{n \times n}$ by

$$(H_{\mathrm{tor}})_{i,j} := \begin{cases} x_i, & \text{if } i = j+1; \\ 0, & \text{otherwise.} \end{cases}$$

We call an invariant f “toric” if $f(H) = f(H_{\mathrm{tor}})$ for every matrix $H \in \mathcal{H}_U$.

There exists a minimal, finite set $\{f_1, \ldots, f_s\}$ of toric invariants that generates all toric invariants, such that for each $i \in \{1, \ldots, s\}$, there are integers $h_1, \ldots, h_{n-1}$ with

$$f(H) = x_1^{h_1} \cdot \ldots \cdot x_{n-1}^{h_{n-1}}.$$

The set S of these tuples $(h_1, \ldots, h_{n-1})$ yields a cone $\sigma = \operatorname{Cone}(S)$, such that the variety $X := \operatorname{Spec} KS_\sigma$ is the aforementioned toric variety.

Our aim, thus, is to characterize all toric invariants in the invariant ring $K[\mathcal{N}]^U$ by describing a finite set of generators of these invariants.

6.2.1 Reductions

Let $f \neq 0$ be a toric invariant of size r, given by the data

$$\mathcal{P} = ((a_i)_{1\le i\le s}, (a'_j)_{1\le j\le t}, (\mathcal{P}_{i,j})_{\substack{1\le i\le s\\1\le j\le t}}).$$

Proposition 6.2.1. *(First reduction of toric invariants)*
Let $\sigma \in S_r$ be a permutation, such that $\prod_{i=1}^{r}(H^{\mathcal{P}})_{i,\sigma(i)} \neq 0$ for every $H \in \mathcal{H}_U$. Then there is an element $\lambda \in K^*$, such that for every $H \in \mathcal{H}_U$

$$f(H) = \lambda \cdot \prod_{i=1}^{r}(H^{\mathcal{P}})_{i,\sigma(i)}.$$

Proof. Since

$$f(H) = \det H^{\mathcal{P}} = \sum_{\tau\in S_r} \operatorname{sgn}(\tau)\cdot\prod_{i=1}^{r}(H^{\mathcal{P}})_{i,\tau(i)},$$

it is adequate to show that for every choice $\sigma, \tau \in S_r$, such that

$$\prod_{i=1}^{r}(H^{\mathcal{P}})_{i,\sigma(i)} \neq 0 \neq \prod_{i=1}^{r}(H^{\mathcal{P}})_{i,\tau(i)},$$

there exists an element $\lambda \in K^*$ fulfilling

$$\prod_{i=1}^{r}(H^{\mathcal{P}})_{i,\sigma(i)} = \lambda\cdot\prod_{i=1}^{r}(H^{\mathcal{P}})_{i,\tau(i)}.$$

Every permutation equals a product of transpositions, thus, it suffices to show that for every choice $1 \le i, i', j, j' \le r$ with $H^{\mathcal{P}}_{i,j}\cdot H^{\mathcal{P}}_{i',j'} \neq 0 \neq H^{\mathcal{P}}_{i,j'}\cdot H^{\mathcal{P}}_{i',j}$, there is an element $\lambda \in K^*$, such that

$$H^{\mathcal{P}}_{i,j}\cdot H^{\mathcal{P}}_{i',j'} = \lambda\cdot H^{\mathcal{P}}_{i,j'}\cdot H^{\mathcal{P}}_{i',j}.$$

We consider single entries first:

Let $H \in \mathcal{H}_U$ be an arbitrary matrix in normal form and denote the entries on the second diagonal by $H_{m+1,m} =: x_m$ for all m.
Given $k \in \{1,\dots,s\}$ and $l \in \{1,\dots,t\}$ and elements $x \in \{1,\dots,a_k\}$ and $y \in \{1,\dots,a'_t\}$, there is an element $\mu \in K$, such that

$$(\mathcal{P}_{k,l}(H))_{(a_k,a'_l)})_{x,y} = \mu\cdot\prod_{h=y}^{n-a_k+x-1} x_h.$$

Since $H^{\mathcal{P}} = ((\mathcal{P}_{i,j}(H))_{(a_i,a'_j)})_{\substack{1\le i\le s\\1\le j\le t}}$, there are integers $s', s'' \in \{1,\dots,s\}$ and $t', t'' \in \{1,\dots,t\}$ and integers $x' \in \{1,\dots,a_{s'}\}$ and $x'' \in \{1,\dots,a_{s''}\}$, as well as integers $y' \in \{1,\dots,a'_{t'}\}$ and $y'' \in \{1,\dots,a'_{t''}\}$, such that

$$H^{\mathcal{P}}_{i,j} = (\mathcal{P}_{s',t'}(H))_{(a_{s'},a'_{t'})})_{x',y'},$$

$$H^{\mathcal{P}}_{i',j'} = (\mathcal{P}_{s'',t''}(H))_{(a_{s''},a'_{t''})})_{x'',y''},$$
$$H^{\mathcal{P}}_{i,j'} = (\mathcal{P}_{s',t''}(H))_{(a_{s'},a'_{t''})})_{x',y''} \text{ and}$$
$$H^{\mathcal{P}}_{i',j} = (\mathcal{P}_{s'',t'}(H))_{(a_{s''},a'_{t'})})_{x'',y'}.$$

Following the above considerations, there are elements $\mu_1, \mu_2 \in K^*$, such that

$$\begin{aligned} H^{\mathcal{P}}_{i,j} \cdot H^{\mathcal{P}}_{i',j'} &= (\mathcal{P}_{s',t'}(H))_{(a_{s'},a'_{t'})})_{x',y'} \cdot (\mathcal{P}_{s'',t''}(H))_{(a_{s''},a'_{t''})})_{x'',y''} \\ &= \left(\mu_1 \cdot \prod_{k=y'}^{n-a_{s'}+x'-1} x_k\right) \cdot \left(\mu_2 \cdot \prod_{k=y''}^{n-a_{s''}+x''-1} x_k\right) \end{aligned}$$

and elements $\mu_3, \mu_4 \in K^*$ that fulfill

$$\begin{aligned} H^{\mathcal{P}}_{i,j'} \cdot H^{\mathcal{P}}_{i',j} &= (\mathcal{P}_{s',t''}(H))_{(a_{s'},a'_{t''})})_{x',y''} \cdot (\mathcal{P}_{s'',t'}(H))_{(a_{s''},a'_{t'})})_{x'',y'} \\ &= \left(\mu_3 \cdot \prod_{k=y''}^{n-a_{s'}+x'-1} x_k\right) \cdot \left(\mu_4 \cdot \prod_{k=y'}^{n-a_{s''}+x''-1} x_k\right). \end{aligned}$$

Then

$$H^{\mathcal{P}}_{i,j} \cdot H^{\mathcal{P}}_{i',j'} = \frac{\mu_1 \cdot \mu_2}{\mu_3 \cdot \mu_4} \cdot H^{\mathcal{P}}_{i,j'} \cdot H^{\mathcal{P}}_{i',j}.$$

yields the claim. □

In order to calculate a set of minimal generators, we can without loss of generality assume $a_i, a'_j \leq n-1$ for all $i \in \{1, \ldots, s\}$ and $j \in \{1, \ldots, t\}$, since otherwise the corresponding semi-invariant f fulfills $f(H) = 0$ for every $H \in \mathcal{H}_U$ or deletion of these blocks leads to changing f by a scalar.

We call f a sum-free toric invariant, if its block sizes $a_1, \ldots, a_s$ and $a'_1, \ldots, a'_t$ do not share any partial sums, that is,

$$\sum_{i \in I} a_i \neq \sum_{i' \in I'} a'_{i'}$$

for all $I \subsetneq \{1, \ldots, s\}$ and $I' \subsetneq \{1, \ldots, t\}$.

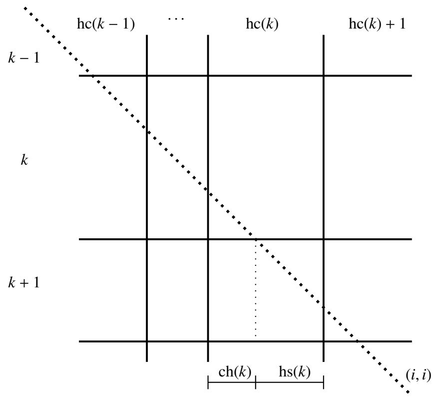
Given such sum-free toric invariant, we define some corresponding combinatorial data that depend on the block sizes $\underline{a} := (a_1, \ldots, a_s)$ and $\underline{a}' := (a'_1, \ldots, a'_t)$. Note, however, that they do not depend on the polynomials defining an invariant of these block sizes.

Definition 6.2.2. *(Combinatorial data of toric invariants)*
For $k \in \{1, \ldots, s\}$ we denote the "horizontal change" of k by $\mathrm{hc}(k)$*, that is, the minimal integer, such that there is an integer* $\mathrm{hs}(k) > 0$ *(the "horizontal split" of k) with*

$$\sum_{j=1}^{k} a_j = \sum_{j=1}^{\mathrm{hc}(k)} a'_j - \mathrm{hs}(k).$$

We denote the "complement of $\mathrm{hs}(k)$*" by* $\mathrm{ch}(k) := a'_{\mathrm{hc}(k)} - \mathrm{hs}(k)$*; for formal reasons, we define* $\mathrm{hc}(0) := 0$.

These data can be visualized as follows:

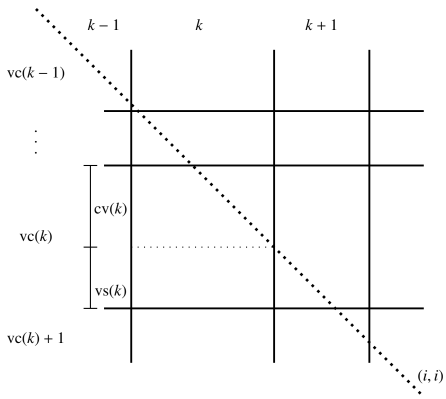


For $k \in \{1, \ldots, t\}$ *denote the "vertical change" by* $\mathrm{vc}(k)$*, that is, the minimal integer, such that there is an integer* $\mathrm{vs}(k) > 0$ *(the "vertical split") with*

$$\sum_{j=1}^{k} a'_j = \sum_{j=1}^{\mathrm{vc}(k)} a_j - \mathrm{vs}(k).$$

We denote the "complement of vs*" by* $\mathrm{cv}(k) := a'_{\mathrm{vc}(k)} - \mathrm{vs}(k)$*; for formal reasons we define* $\mathrm{vc}(0) := 0$ *as above.*

These data can be visualized by:



For every $i \in \{1, \ldots, r\}$*, we define*

the "horizontal block" $\mathrm{hb}(i)$*, that is, the maximal integer with* $i = \sum_{j=1}^{\mathrm{hb}(i)-1} a_j + \mathrm{hd}(i)$ *for a positive integer* $\mathrm{hd}(i)$ *(which we call the "horizontal datum"),*

the "vertical block" $\mathrm{vb}(i)$*, that is, the maximal integer with* $i = \sum_{j=1}^{\mathrm{vb}(i)-1} a'_j + \mathrm{vd}(i)$ *for a positive integer* $\mathrm{vd}(i)$ *(which we call the "vertical datum").*

We call an entry $(i, j) \in \{1, \dots, r\}^2$ *acceptable for* $(\underline{a}, \underline{a}')$ *if* $\mathrm{vd}(j) < \mathrm{hd}(i) + n - a_{\mathrm{hb}(i)}$ *and unacceptable otherwise.*

A permutation $\sigma \in S_r$ *is called acceptable for* $(\underline{a}, \underline{a}')$ *if every entry* $(i, \sigma(i))$ *is acceptable for* f.

Given a toric invariant, following lemma 6.2.1 it suffices to find one acceptable permutation in order to calculate f on $\mathcal{N}_U$.

Lemma 6.2.3. *(Second reduction of toric invariants)*
The toric invariants are generated by sum-free toric invariants.

Proof. Let f be a toric U-invariant. Due to proposition 6.2.1, to see of which form f is on $\mathcal{H}_U$, we can without loss of generality order $\underline{a} := (a_1, \dots, a_s)$ and $\underline{a} := (a'_1, \dots, a'_t)$ as we like and adapt the permutation accordingly.

It, therefore, suffices to consider an arbitrary $r \times r$-matrix of sum-free block sizes $\underline{a}$ and $\underline{a}'$, that is, for every pair of subsets $I \subsetneq \{1, \dots, s\}$ and $I' \subsetneq \{1, \dots, t\}$ the partial sums do not coincide:

$$\sum_{i \in I} a_i \neq \sum_{i' \in I'} a'_{i'}.$$

If we find an acceptable permutation σ for $(\underline{a}, \underline{a}')$, following proposition 6.2.1 there exists an element $\mu \in K^*$ and a datum $\mathcal{P}$ which fulfills $f(H) = \mu \cdot f^{\mathcal{P}}(H) = \mu \cdot \prod_{i=1}^{r} (H^{\mathcal{P}})_{i,\sigma(i)} \neq 0$ for every $H \in \mathcal{H}_U$.

We define a permutation $\sigma \in S_r$, such that every $(i, \sigma(i))$ is acceptable for $(\underline{a}, \underline{a}')$ by double induction on s and t.

Let $s = 1$ and $t = 1$, then every entry (i, i) is acceptable for $(\underline{a}, \underline{a}')$, since $a_{\mathrm{hb}(i)} = a_1 \leq n - 1$ and, therefore,

$$\mathrm{vd}(i) = i < i + n - a_1 = \mathrm{hd}(i) + n - a_1.$$

Let $t = 1$ and assume that for every $k \leq s$, the above claim holds true. Consider the block sizes $\underline{a} := (a_1, \dots, a_{s+1})$ and $\underline{a}' := a'_1$, then every entry (i, i) is acceptable for $(\underline{a}, \underline{a}')$, since

$$\mathrm{vd}(i) = i < i + n - a_{\mathrm{hb}(i)} \leq \mathrm{hd}(i) + n - a_{\mathrm{hb}(i)}.$$

Let $s = 1$ and assume for every $k \leq t$, the above claim holds true. Consider the block sizes $\underline{a} := (a_1)$ and $\underline{a}' := (a'_1, \dots, a'_{t+1})$, then every (i, i) is acceptable for $(\underline{a}, \underline{a}')$ in the same way:

$$\mathrm{vd}(i) = i < i + n - a_1 = \mathrm{hd}(i) + n - a_1.$$

We can set $\sigma = id$ in every of these cases.

Let us fix an arbitrary integer t and let us assume that for $s' \leq s$ and for every choice of block sizes $a_1, \dots, a_{s'}$ and $a'_1, \dots, a'_t$ with $\sum_{j=1}^{s'} a_j = \sum_{j=1}^{t} a'_j$, there is a permutation σ as claimed.
We consider block sizes $\underline{a} := (a_1, \dots, a_{s+1})$ and $\underline{a}' := (a'_1, \dots, a'_t)$ with $\sum_{j=1}^{s+1} a_j = \sum_{j=1}^{t} a'_j = r$ and show in the following that we can find a permutation as wished for.

First case: We can order the block sizes $a'_1, \dots, a'_t$, such that $a'_t \geq a_{s+1}$.

We can apply the premise of the induction to the $r - a_{s+1} \times r - a_{s+1}$-upper-left submatrix of block sizes $\underline{a(s)} := (a_1, \dots, a_s)$ and $\underline{a(s)}' := (a'_1, \dots, a'_{t-1}, a'_t - a_{s+1})$ and obtain a permutation $\sigma' \in S_{r-a_{s+1}}$, such that $(i, \sigma'(i))$ is acceptable for $(\underline{a(s)}, \underline{a(s)}')$ for every $i \leq r - a_{s+1}$.

We define $\sigma \in S_r$ by

$$\sigma(i) := \begin{cases} \sigma'(i), & \text{if } i \leq r - a_{s+1}; \\ i, & \text{otherwise.} \end{cases}$$

Then every entry $(i, \sigma(i))$, where $i \leq r - a_{s+1}$, is acceptable for $(\underline{a}, \underline{a}')$, since it is acceptable for $(\underline{a(s)}, \underline{a(s)}')$.

Every entry (i, i), where $i > r - a_{s+1}$, is acceptable for $(\underline{a}, \underline{a}')$, since

$$\mathrm{vd}(i) \;=\; i - \sum_{j=1}^{t-1} a'_j \;<\; i - \sum_{j=1}^{t} a'_j + n \;=\; i - \sum_{j=1}^{s} a_j + n - a_{s+1} \;=\; \mathrm{hd}(i) + n - a_{s+1}.$$

Second case: The inequality $a'_i < a_j$ holds true for every $i \in \{1, \dots, s+1\}$ and $j \in \{1, \dots, t\}$.

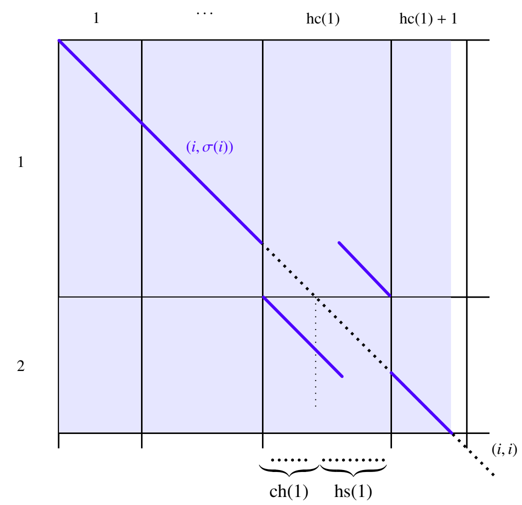
Claim: For every $k \in \{1, \dots, s\}$, there is a permutation $\sigma \in S_{a_1+\ldots+a_{k+1}}$, such that every entry $(i, \sigma(i))$ is acceptable for $(\underline{a}, \underline{a}')$. Furthermore, the entry (i, i) is acceptable for $(\underline{a}, \underline{a}')$ for every integer $a_1 + \ldots + a_k + \mathrm{hs}(k) < i \leq a_1 + \ldots + a_{k+1}$.

We prove the claim by induction on k.

Let $k = 1$.
Define

$$\sigma(i) := \begin{cases} i, & \text{if } i \leq a_1 - \mathrm{ch}(1); \\ i + \mathrm{hs}(1), & \text{if } a_1 - \mathrm{ch}(1) < i \leq a_1; \\ i - \mathrm{ch}(1), & \text{if } a_1 < i \leq a_1 + \mathrm{hs}(1); \\ i, & \text{otherwise.} \end{cases}$$

The permutation σ can be vizualized as follows:



For $i \leq a_1 - \mathrm{ch}(1)$, the entry (i, i) is acceptable for $(\underline{a}, \underline{a}')$ due to the considerations in the case $s = 1$.

For $a_1 - \mathrm{ch}(1) < i \leq a_1$, the entry $(i, \sigma(i))$ is acceptable for $(\underline{a}, \underline{a}')$, since

$$\mathrm{vd}(\sigma(i)) = i + \mathrm{hs}(1) - \sum_{j=1}^{\mathrm{hc}(1)-1} a'_j < i + n - a_1 = \mathrm{hd}(i) + n - a_1.$$

For $a_1 < i \leq a_1 + \mathrm{hs}(1)$, the entry $(i, \sigma(i))$ is acceptable for $(\underline{a}, \underline{a}')$, since

$$\mathrm{vd}(\sigma(i)) = i + \mathrm{hs}(1) - \sum_{j=1}^{\mathrm{hc}(1)} a'_j = i - a_1 < i - a_1 + n - a_2 = \mathrm{hd}(i) + n - a_2.$$

For $i > a_1 + \mathrm{hs}(1)$, the entry $(i, \sigma(i))$ is acceptable for $(\underline{a}, \underline{a}')$, since

$$\mathrm{vd}(\sigma(i)) = i - \sum_{j=1}^{\mathrm{vb}(\sigma(i))-1} a'_j < i - a_1 + n - a_2 = \mathrm{hd}(i) + n - a_2.$$

Now let $k + 1 > 1$.

Assume the claim holds true for k, that is, there is a permutation $\sigma' \in S_{a_1+\ldots+a_{k+1}}$, such that every entry $(i, \sigma'(i))$ is acceptable for $(\underline{a}, \underline{a}')$ and such that $\sigma'(i) = i$ for every integer $a_1 + \ldots + a_k + \mathrm{hs}(k) < i \leq a_1 + \ldots + a_{k+1}$.

Then we set

$$\sigma(i) := \begin{cases} \sigma'(i), & \text{if } i \leq \sum_{j=1}^{k+1} a_j - \mathrm{ch}(k+1); \\ i + \mathrm{hs}(k+1), & \text{if } \sum_{j=1}^{k+1} a_j - \mathrm{ch}(k+1) < i \leq \sum\limits_{j=1}^{k+1} a_j; \\ i - \mathrm{ch}(k+1), & \text{if } \sum\limits_{j=1}^{k+1} a_j < i \leq \sum\limits_{j=1}^{k+1} a_j + \mathrm{hs}(k+1); \\ i, & \text{otherwise.} \end{cases}$$

For $i \leq \sum\limits_{j=1}^{k+1} a_j - \mathrm{ch}(k+1)$, the entry $(i, \sigma'(i))$ is acceptable for $(\underline{a}, \underline{a}')$ due to the assumption of the induction.

For $\sum\limits_{j=1}^{k+1} a_j - \mathrm{ch}(k+1) < i \leq \sum\limits_{j=1}^{k+1} a_j$, the entry $(i, \sigma(i))$ is acceptable for $(\underline{a}, \underline{a}')$, since

$$\begin{aligned} \mathrm{vd}(\sigma(i)) &= i + \mathrm{hs}(k+1) - \sum_{j=1}^{\mathrm{hc}(k+1)-1} a'_j = i - (\sum_{j=1}^{\mathrm{hc}(k+1)} a'_j - \mathrm{hs}(k+1)) + a'_{\mathrm{hc}(k+1)} \\ &= i - \sum_{j=1}^{k+1} a_j + a'_{\mathrm{hc}(k+1)} < i + n - \sum_{j=1}^{k+1} a_j = \mathrm{hd}(i) + n - a_{k+1}. \end{aligned}$$

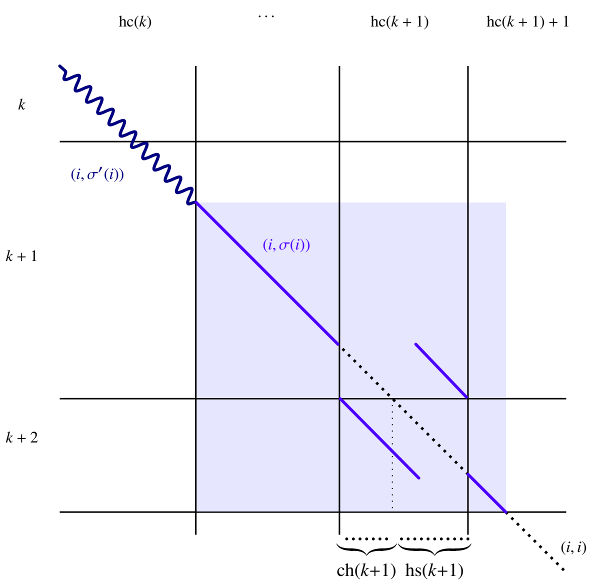
For $\sum_{j=1}^{k+1} a_j < i \leq \sum_{j=1}^{k+1} a_j + \mathrm{hs}(k+1)$, the entry $(i, \sigma(i))$ is acceptable for $(\underline{a}, \underline{a}')$, since

$$\begin{aligned} \mathrm{vd}(\sigma(i)) &= i - \mathrm{ch}(k+1) - \sum_{j=1}^{\mathrm{hc}(k+1)-1} a'_j = i - a_{k+1} - \sum_{j=1}^{k} a_j \\ &< i - \sum_{j=1}^{k+1} a_j + n - a_{k+2} = \mathrm{hd}(i) + n - a_{k+2}. \end{aligned}$$

For $i > \sum_{j=1}^{k+1} a_j + \mathrm{hs}(k+1)$, the entry (i, i) is acceptable for $(\underline{a}, \underline{a}')$, since

$$\mathrm{vd}(\sigma(i)) = i - \sum_{j=1}^{\mathrm{vb}(\sigma(i))-1} a'_j < i - \sum_{j=1}^{k+1} a_j + n - a_{k+2} = \mathrm{hd}(i) + n - a_{k+2}.$$

As in the case $k = 1$, the permutation σ can be vizualized by



If s is fixed and the assumption holds true for every $k \leq t$, then it also holds true for $t + 1$ by an argumentation symmetric to the above one.

Therefore, we have found a permutation as wished for in every case.

We can define the polynomials

$$\mathcal{P}_{k,l} := \begin{cases} x^{n-a_k+\mathrm{hd}(i_{\min})-\mathrm{vd}(i_{\min})}, & \text{if there is a minimal element } i_{\min} \text{ with } \mathrm{hb}(i_{\min}) = k \text{ and } \mathrm{vb}(\sigma(i_{\min})) = l; \\ 0, & \text{otherwise.} \end{cases}$$

Then, corresponding to the datum $\mathcal{P} = ((a_i)_{1\leq i\leq s}, (a'_j)_{1\leq j\leq t}, (\mathcal{P}_{i,j})_{\substack{1\leq i\leq s\\ 1\leq j\leq t}})$, proposition 6.2.1 yields the existence of an element $\mu \in K$, such that

$$f(H) = \mu \cdot \prod_{i=1}^{r} (H^{\mathcal{P}})_{i,\sigma(i)}$$

for every $H \in \mathcal{H}_U$. □

6.2.2 General description of toric invariants

We fix a sum-free toric invariant f of block sizes $\underline{a} := (a_1, \ldots, a_s)$ and $\underline{a}' := (a'_1, \ldots, a'_t)$ and assume, without loss of generality, $a_1 \leq \ldots \leq a_s$ and $a'_1 \leq \ldots \leq a'_t$.

Given an integer $i \in \{1, \ldots, s\}$, we define $s_i := \sum_{l=1}^{i} a_l + 1$.

Definition 6.2.4. *(Block crossings)*
Let us define so-called block crossings of $(\underline{a}, \underline{a}')$ *, that is, tuples of integers* (i_k, j_k)*, such that "the diagonal crosses the* $(i_k - 1) \times j_k$*-th block in the upper right corner", recursively as follows:*

- $i_0 = j_0 := 0$

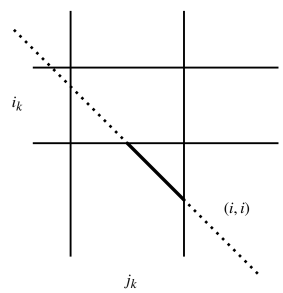
Let $k \geq 0$*. Then we define*

- $i'_k := \min\left\{i_{k-1} < i \leq s \mid \mathrm{vd}(s_i + \mathrm{hs}(i)) = a'_{\mathrm{vb}(s_i+\mathrm{hs}(i))} \text{ and } \mathrm{vd}(s_i) \neq 1\right\}$ *and*

$$i_k := \max\left\{i \mid \mathrm{hc}(i) = \mathrm{hc}(i'_k)\right\}$$

- $j_k := \mathrm{vb}(s_{i_k})$.

Note that the condition $\mathrm{vd}(s_i) \neq 1$ *is required to exclude a block crossing in the* $(1, 1)$*-th block. If the tuple* $(\underline{a}, \underline{a}')$ *is sum-free, the condition is redundant for* $k > 1$.



There is a minimal integer x, such that the set

$$\left\{i_{x-1} < i \leq s \mid \mathrm{vd}(s_i + \mathrm{hs}(i)) = a'_{\mathrm{vb}(s_i+\mathrm{hs}(i))}\right\} = \emptyset.$$

Finally, we set

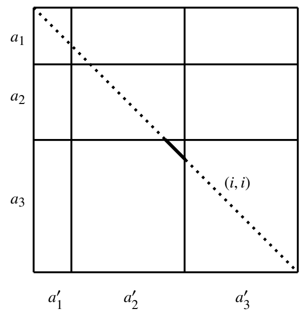
- $i_x := s$ *and* $j_x := t$.

We define $\mathcal{BC} := \{(i_k, j_k) \mid 0 \leq k \leq x\}$ to be the set of block crossings of $(\underline{a}, \underline{a}')$.

Example 6.2.5. *(Block crossings)*

If $r = 14$*, consider the block sizes* $\underline{a} = (3, 4, 7)$ *and* $\underline{a}' = (2, 6, 6)$.

The block matrix can be depicted as follows; we mark the (potentially) unacceptable entries by coloring them.



Then the block crossings are given in the following table:

k	0	1	2
i_k	0	2	3
j_k	0	2	3

Proposition 6.2.6. *(Acceptable entries and block crossings)*
If

$$i \notin \{s_{i_k} + h \mid h \in \{0, \ldots, \mathrm{hs}(i_k)\} \text{ and } k \in \{1, \ldots, x\}\},$$

then the entry (i, i) *is acceptable for* $(\underline{a}, \underline{a}')$.

Proof. Follows from definition 6.2.4 and since $a_i, a'_j \leq n - 1$ for all integers $i \in \{1, \ldots, s\}$ and $j \in \{1, \ldots, t\}$. □

Following from the definition of block crossings, we obtain a general description of an acceptable permutation of f.

Corollary 6.2.7. *(An acceptable permutation)*
The permutation $\sigma \in S_r$ *defined by*

$$\sigma(i) := \begin{cases} i + \mathrm{hs}(i_k), & \text{if } \sum_{j=1}^{j_k - 1} a'_j < i \leq \sum_{j=1}^{i_k} a_j; \\ i - \mathrm{ch}(i_k), & \text{if } \sum_{j=1}^{i_k} a_j < i \leq \sum_{j=1}^{j_k} a'_j; \\ i, & \text{otherwise}. \end{cases}$$

for $k \in \{1, \ldots, x\}$ *is acceptable for* $(\underline{a}, \underline{a}')$.

Proof. We distinguish between two cases:

Let $x = 1$, then the permutation $\sigma = id$ is acceptable in all cases.

Let $x > 1$; we look at certain blocks that are induced by the crossings (i_k, j_k). In more detail, for $k \in \{1, \ldots, x\}$, we show that every entry $(i, \sigma(i))$ is acceptable for $(\underline{a}, \underline{a}')$ if

$$\sum_{j=1}^{j_{k-1}} a'_j < i \leq \sum_{j=1}^{j_k} a'_j.$$

The first block:

The permutation σ is given by

$$\sigma(i) := \begin{cases} i, & \text{if } 0 < i \leq \sum_{j=1}^{j_1-1} a'_j; \\ i + \mathrm{hs}(i_1), & \text{if } \sum_{j=1}^{j_1-1} a'_j < i \leq \sum_{j=1}^{i_1} a_j; \\ i - \mathrm{ch}(i_1), & \text{if } \sum_{j=1}^{i_1} a_j < i \leq \sum_{j=1}^{j_1} a'_j. \end{cases}$$

We visualize these entries $(i, \sigma(i))$ in figure 6.1. Each of them is acceptable for $(\underline{a}, \underline{a}')$ which can be proven with the help of proposition 6.2.6 analogously to the proof of lemma 6.2.3.

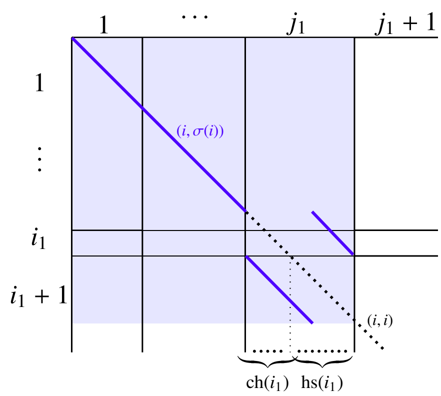


Figure 6.1: The first block

The k-th block for $k \in \{2, \ldots, x-1\}$:

The permutation σ is given by

$$\sigma(i) := \begin{cases} i, & \text{if } \sum_{j=1}^{j_{k-1}} a'_j < i \leq \sum_{j=1}^{j_k-1} a'_j; \\ i + \mathrm{hs}(i_k), & \text{if } \sum_{j=1}^{j_k-1} a'_j < i \leq \sum_{j=1}^{i_k} a_j; \\ i - \mathrm{ch}(i_k), & \text{if } \sum_{j=1}^{i_k} a_j < i \leq \sum_{j=1}^{j_k} a'_j. \end{cases}$$

and is depicted in figure 6.2.

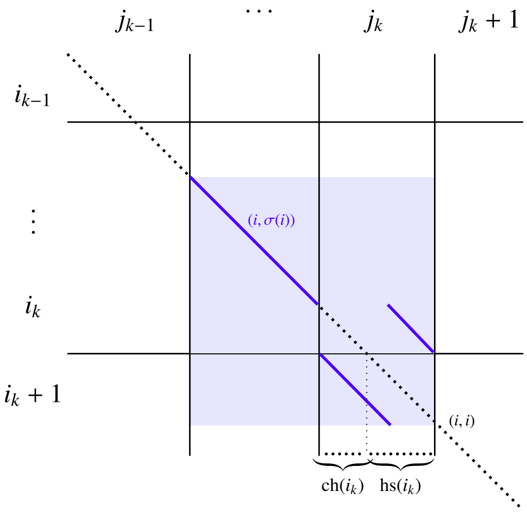


Figure 6.2: The k-th block

Every entry $(i, \sigma(i))$ for

$$\sum_{j=1}^{j_{k-1}} a'_j < i \leq \sum_{j=1}^{j_k} a'_j$$

is acceptable for $(\underline{a}, \underline{a}')$ which can be proven with the help of proposition 6.2.6 analogously to the proof of lemma 6.2.3.

<u>The last block:</u>

The permutation σ is given by $\sigma(i) := i$ for every integer $\sum_{j=1}^{j_{x-1}} a'_j \leq r$.

The fact that each entry (i, i) is acceptable for $(\underline{a}, \underline{a}')$ follows from proposition 6.2.6 and the condition $a_s \leq n - 1$. Figure 6.3 visualizes the permutation σ in the last block.

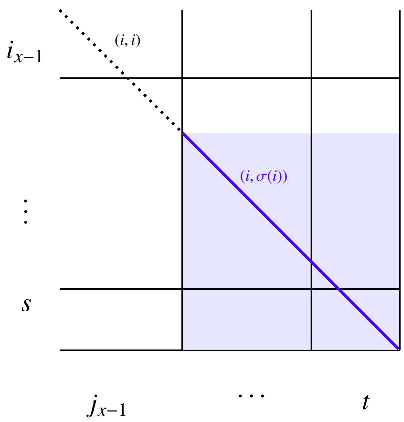


Figure 6.3: The last block

Every entry $(i, \sigma(i))$ for
$$\sum_{j=1}^{j_{x-1}} a'_j \leq r$$
is acceptable for $(\underline{a}, \underline{a}')$ which can be proven as in the proof of lemma 6.2.3.

Therefore, for every integer $i \in \{1, \ldots, r\}$, the entry $(i, \sigma(i))$ is acceptable for $(\underline{a}, \underline{a}')$ and the defined permutation σ is, thus, acceptable for $(\underline{a}, \underline{a}')$ as well. □

We fix the acceptable permutation $\sigma \in S_r$ and the induced datum $\mathcal{P}$ (as in the proof of lemma 6.2.3).

Proposition 6.2.8. *(Description of acceptable entries)*
Let $H = H_{\text{tor}} \in \mathcal{H}_U$ be a matrix with entries $H_{k+1,k} =: x_k$. Then
$$H^{\mathcal{P}}_{(i,\sigma(i))} = (H^{n-\text{vd}(\sigma(i))-a_{\text{hb}(i)}+\text{hd}(i)})_{\text{hd}(i),\text{vd}(\sigma(i))} = \prod_{k=\text{vd}(\sigma(i))}^{n-a_{\text{hb}(i)}+\text{hd}(i)-1} x_k.$$

Proof. Clearly,
$$H^{\mathcal{P}}_{(i,\sigma(i))} = (H^{n-\text{vd}(\sigma(i))-a_{\text{hb}(i)}+\text{hd}(i)})_{(\text{hd}(i),\text{vd}(\sigma(i))}$$
due to the definition of hd and vd. The remaining equality follows from the consideration of potencies of H (for example in the proof of proposition 6.2.1). □

For $i, j \in \{1, \ldots, n-1\}$, we consider so-called part-diagonal determinants, that is, determinants along the blue entries, depicted in figure 6.4.
We say the determinant is of type (1) or (2), of height h and of width w.

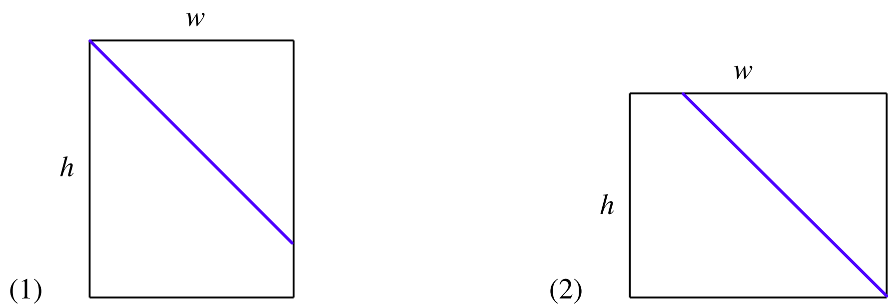


Figure 6.4: Part-diagonal determinants

Corollary 6.2.9. *(Description of part-diagonal determinants)*
If the part-diagonal determinant of height h and width w is given by acceptable entries $(i, \sigma(i))$ for $i \in I$, then it is given by
$$\prod_{i \in I} H^{\mathcal{P}}_{i,\sigma(i)} = \prod_{k=1}^{w} \prod_{l=0}^{n-h-1} x_{k+l} \quad \textit{if it is of type } (1)$$
and by
$$\prod_{i \in I} H^{\mathcal{P}}_{i,\sigma(i)} = \prod_{k=w-h+1}^{w} \prod_{l=0}^{n-w-1} x_{k+l} \quad \textit{if it is of type } (2).$$

Proof. Follows directly from proposition 6.2.8. □

We define data in order to describe the block diagonal determinants that are induced by the permutation σ.

Definition 6.2.10. *(The data $s(\sigma)$ and $\operatorname{diag}(f)$)*
We start by defining a sequence $(s(\sigma)_k)_{1\leq k\leq h}$ recursively as follows:

- $s(\sigma)_1 = 1$ *and*
- *for $k > 1$, the integer $s(\sigma)_k > s(\sigma)_{k-1}$ is the minimal integer, such that*

$$\operatorname{hb}(s(\sigma)_k) > \operatorname{hb}(s(\sigma)_{k-1}) \text{ or } \operatorname{vb}(s(\sigma)_k) > \operatorname{vb}(s(\sigma)_{k-1}).$$

  If no such integer exists, we set $k - 1 =: h$ and determine the sequence at $s(\sigma)_h$.

Then a tuple

$$\operatorname{diag}(f) = (\operatorname{diag}(f)_1, \ldots, \operatorname{diag}(f)_h) \in (\boldsymbol{N} \times \boldsymbol{N} \times \{(1), (2)\})^h$$

is obtained as follows:
Let $l \in \{1, \ldots, h\}$ and denote $i := \operatorname{hb}(s(\sigma)_l)$ as well as $j := \operatorname{vb}(s(\sigma)_l)$.

- $\operatorname{diag}(f)_l := (d_l, d'_l, (1))$ *if the permutation σ yields a block determinant as in the figure 6.5.*

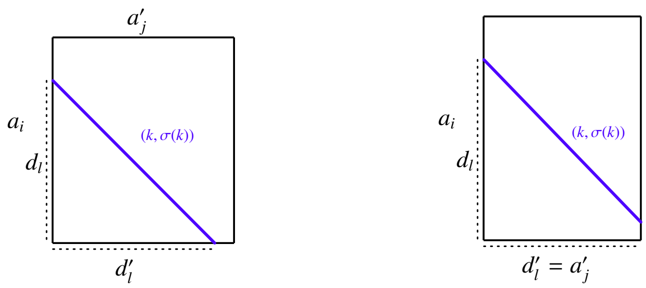


Figure 6.5 : Part − diagonal determinants; first case

- $\operatorname{diag}(f)_l := (d_l, d'_l, (2))$ *if the permutation σ yields a block determinant as in figure 6.6.*

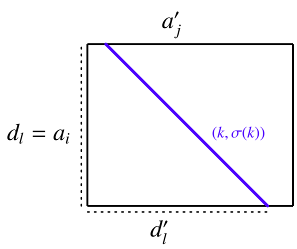
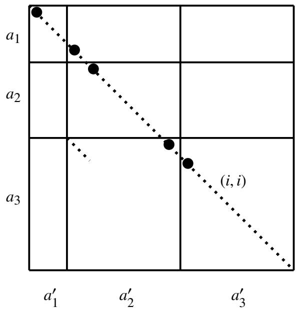


Figure 6.6 : Part − diagonal determinants; second case

Example 6.2.11. *(The data $s(\sigma)$ and* $\mathrm{diag}(f)_l$*)*
We consider the example 6.2.5. The sequence $s(\sigma)$ is given by

$$s(\sigma) = (1, 3, 4, 8, 9)$$



and $\mathrm{diag}(f) = ((3, 2, (1)); (1, 1, (1)); (4, 6, (2)); (7, 1, (1)); (6, 6, (1)))$.

We describe the tuple $\mathrm{diag}(f)_l$ in detail in order to verify a general description of all toric invariants.

Proposition 6.2.12. *(Explicit description of* $\mathrm{diag}(f)$*)*
Let $l \in \{1, \ldots, h\}$ and denote $i := \mathrm{hb}(s(\sigma)_l)$ as well as $j := \mathrm{vb}(s(\sigma)_l)$.

1. *Assume* $\mathrm{vd}(s(\sigma)_l) = 1$.
   - *Let $i \neq i_k$ and $j \neq j_k$ for all $k \in \{1, \ldots, x\}$.*
     - *If* $\mathrm{hb}(s(\sigma)_{l+1}) = a_i + 1$, *then*

       $$\mathrm{diag}(f)_l = (a_i - \mathrm{hd}(s(\sigma)_l) + 1, \mathrm{vd}(s(\sigma)_{l+1}) - 1, (1))$$

       and
     - *if* $\mathrm{hb}(s(\sigma)_{l+1}) = a_i$, *then*

       $$\mathrm{diag}(f)_l = (a_i - \mathrm{hd}(s(\sigma)_l) + 1, a'_j, (1)).$$

   - *If $j = j_k$ for an integer $k \in \{1, \ldots, x\}$, then*

     $$\mathrm{diag}(f)_l = (a_{i_k+1}, \mathrm{hs}(k), (1)).$$

2. *Assume* $\mathrm{vd}(s(\sigma)_l) = 1$.
   - *If $i \neq i_k$ and $j \neq j_k$ for all $k \in \{1, \ldots, x\}$, then*

     $$\mathrm{diag}(f)_l = (a_i, \mathrm{vd}(s(\sigma)_{l+1} - 1, (2)).$$

   - *If $i = i_k$ for an integer $k \in \{1, \ldots, x\}$, then*

     $$\mathrm{diag}(f)_l = (a_i - \mathrm{hd}(s(\sigma)_l) + 1, a'_j, (2)).$$

We are now able to give an explicit description of the sum-free toric invariants.

Define

$$\delta(l) := \begin{cases} 1 & \text{if } \mathrm{diag}(f)_l \in \mathbf{N}\times\mathbf{N}\times\{(1))\}; \\ 0 & \text{otherwise;} \end{cases}$$

and

$$\delta'(l) := \begin{cases} 1 & \text{if } \mathrm{diag}(f)_l \in \mathbf{N}\times\mathbf{N}\times\{(2))\}; \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 6.2.13. *(Description of sum-free toric invariants)*
Let f be a sum-free toric invariant of block sizes $\underline{a} := (a_1, \dots, a_s)$ and $\underline{a}' := (a'_1, \dots, a'_t)$. Then

$$f = \prod_{l=1}^{h}\left(\prod_{k=1}^{\delta(l)\cdot d'_l}\left(\prod_{y=0}^{n-d_l-1} x_{k+y}\right)\right)\cdot\prod_{l=1}^{h}\left(\prod_{k=d'_l-d_l+1}^{\delta'(l)\cdot d'_l}\left(\prod_{y=0}^{n-d'_l-1} x_{k+y}\right)\right)$$

Proof. Lemma 6.2.9 gives the description of the partial diagonal block matrices. It, therefore, suffices to combine the datum $\mathrm{diag}(f)$ with these descriptions. □

We define $f_{(\underline{a},\underline{a}')}$ to be the (unique) toric U-invariant given by the datum $(\underline{a}, \underline{a}', (\mathcal{P}_{i,j})_{i,j})$ where $\mathcal{P}_{i,j}$ are polynomials induced by an acceptable permutation σ for $(\underline{a}, \underline{a}')$ as in the proof of lemma 6.2.3.

6.3 Generic separation of the U-orbits

We define a set of U-invariants and a morphism $\mu : \mathcal{N} \to \mathbf{A}^D \times \mathbf{A}^{n-1}$ for an integer D, such that μ separates the U-orbits in $\mathcal{N}_U$. More explicitly, if $H, H' \in \mathcal{H}_U$ and $\mu(H) = \mu(H')$, then $H = H'$.

Let $H = (x_{i,j})_{i,j}$ be a matrix in U-normal form in the following and define $x_i := x_{i+1,i}$ for $i \in \{1, \dots, n-1\}$.

Definition 6.3.1. *(Separating invariants)*
We define the toric invariants $\det_i := f_{((n-i),(n-i))}$ and $f_i := f_{((i),(1,\dots,1))}$ for every integer $i \in \{1, \dots, n-1\}$.

Furthermore, for integers $i, j \in \{1, \dots, n\}$, such that $j < i-1$, we define the datum

$$\mathcal{P} = \left((j-1, n-i+1), (j, n-i), \begin{pmatrix} x^{n-j+1} & 0 \\ x & x^i \end{pmatrix}\right).$$

Then $f_{i,j} := f^{\mathcal{P}}$ is a U-invariant due to corollary 5.2.2.

Clearly, for $i \in \{1, \dots, \lfloor \frac{n+1}{2} \rfloor\}$, the equalities

$$\det_i(H) = \det_{n-i}(H) = \det\left(H^i_{(n-i,n-i)}\right) = \prod_{k=1}^{i-1} x_k^k \cdot \prod_{k=i}^{n-i} x_k^i \cdot \prod_{k=n-i+1}^{n-1} x_k^{n-k}$$

hold true.

Furthermore,

$$f_{i,j}(H) = \det\begin{pmatrix} H^{n-j+1}_{(j-1,j)} & 0 \\ H_{(n-i+1,j)} & H^{i}_{(n-i+1,n-i)} \end{pmatrix} = \det_{j-1}(H)\cdot\det_{n-i}(H)\cdot x_{i,j}$$

and

$$f_i(H) = \det\begin{pmatrix} H^{n-i}_{(i,1)} & \dots & H^{n-1}_{(i,1)} \end{pmatrix} = \prod_{k=1}^{n-i} x_k^i \cdot \prod_{k=n-i+1}^{n-1} x_k^{n-k}.$$

We set $D := \frac{(n-1)(n-2)}{2}$ and can separate the U-orbits in the open subset $\mathcal{N}_U \subset \mathcal{N}$.

Lemma 6.3.2. *(Generic separation of the U-orbits)*
The morphism

$$\begin{aligned} \mu\colon\ \mathcal{N} &\to \mathbf{A}^D \times \mathbf{A}^{n-1} \\ N &\mapsto ((f_{i,j}(N))_{1\le j<i-1\le n-1}, (f_i(N))_{1\le i\le n-1}) \end{aligned}$$

separates the U-orbits in $\mathcal{N}_U$.

Proof. Let $a_{i,j} \in K^*$ and $a_i \in K^*$ be elements fulfilling

$$\pi(H) = ((f_{i,j}(H))_{1\le j<i-1\le n-1}, (f_i(H))_{1\le i\le n-1}) = ((a_{i,j})_{i,j}, (a_i)_i).$$

We verify that the matrix H in normal form as above with these properties is uniquely determined.

First, we show that the toric invariants f_i separate the second diagonal, that is, the entries $x_1, \dots, x_{n-1}$.

If $i = 1$, then

$$x_i = \frac{f_{n-1}(H)}{f_{n-2}(H)} = \frac{a_{n-1}}{a_{n-2}} \in K^*;$$

if $1 < i < n-1$, then

$$x_i = \frac{f_{n-i}(H)^2}{f_{n-i-1}(H) f_{n-i+1}(H)} = \frac{a_{n-i}^2}{a_{n-i-1}a_{n-i+1}} \in K^*$$

and if $i = n-1$, then

$$x_i = \frac{f_1(H)^2}{f_2(H)} = \frac{a_1^2}{a_2} \in K^*.$$

Every entry x_i is uniquely determined, therefore, the entries $x_{i,j}$, where $j+1 \neq i$, are also uniquely determined, since

$$a_{i,j} = f_{i,j}(H) = x_{i,j}\cdot\det_{j-1}(H)\cdot\det_{n-i}(H)$$

and because of the above description of $\det_{j-1}$ and $\det_{n-i}$. □

The U-invariant ring of $\mathcal{N}_U$, thus, is given by

$$K[\mathcal{N}_U]^U = K[\mathcal{H}_U] \cong K[\mathbf{A}^D \times (K^*)^{n-1}].$$

6.4 The toric variety X

We denote the toric part of the invariant ring $K[\mathcal{N}]^U$, that is, the subring which is generated by the toric invariants given in section 6.2, by $K[\mathcal{N}]^U_{\text{tor}}$.

Corresponding to $K[\mathcal{N}]^U_{\text{tor}}$, there is a variety $X := \operatorname{Spec} K[\mathcal{N}]^U_{\text{tor}}$ which is a toric variety as follows:

As before, let $H = (x_{i,j})_{i,j}$ be an arbitrary matrix in U-normal form and define $x_i := x_{i+1,i}$ for $i \in \{1, \dots, n-1\}$.

Given a sum-free tuple $(\underline{a}, \underline{a}')$, there are integers $h_1(\underline{a}, \underline{a}'), \dots, h_{n-1}(\underline{a}, \underline{a}')$, such that

$$f_{(\underline{a},\underline{a}')}(H) = x_1^{h_1(\underline{a},\underline{a}')} \cdot \ldots \cdot x_{n-1}^{h_{n-1}(\underline{a},\underline{a}')}.$$

Denote by S the set of tuples $(h_1(\underline{a}, \underline{a}'), \dots, h_{n-1}(\underline{a}, \underline{a}')) \in \mathbf{N}^{n-1}$ for arbitrary sum-free tuples that induce a minimal set of generating toric invariants and denote $\sigma := \operatorname{Cone}(S)$.

Let N be the lattice $\mathbf{Z}^{n-1}$, then σ is generated by the finite set $S \subset \mathbf{Z}^{n-1}$ and, therefore, is a convex rational polyhedral cone. It is strongly convex, since $\sigma \cap (-\sigma) = \{0\}$.

Therefore, $X = \operatorname{Spec} K[\mathcal{N}]^U_{\text{tor}} \cong \operatorname{Spec} K[S_\sigma]$ is a normal toric variety by lemma 1.1.9.

6.4.1 Toric operation(s)

Let $T \subset \mathrm{GL}_n$ be the torus of diagonal matrices. There is a natural action τ of T on the U-invariant ring of $\mathcal{N}$ as follows:

$$\begin{aligned} \tau\colon\ T \times K[\mathcal{N}]^U &\to K[\mathcal{N}]^U \\ (t, f) &\mapsto \left(\begin{array}{ll} f\colon & \mathcal{N} \to K \\ & N \mapsto f(tNt^{-1}) \end{array} \right). \end{aligned}$$

We examine τ on the separating toric invariants f_i of section 6.3. Let $t \in T$ be a matrix with diagonal entries $t_1, \dots, t_n \in K^*$. Then

$$\tau(t, f_i)(H) = \frac{t_{n-i+1} \ldots t_n}{t_1^i} \cdot \prod_{k=1}^{n-i} x_k^i \cdot \prod_{k=n-i+1}^{n-1} x_k^{n-k}$$

can be verified directly, or by considering the character ω, such that f_i is a B-semi-invariant of weight ω as in proposition 5.2.1.

Lemma 6.4.1. *(The operation τ)*
Let f be a toric invariant, such that $f(H) = x_1^{h_1} \ldots x_{n-1}^{h_{n-1}}$. Then

$$\tau(t, f)(H) = \prod_{k=1}^{n} \left(\frac{t_{k+1}}{t_k} \right)^{h_k} \cdot f(H).$$

Proof. Following section 6.3, we have the equality

$$f(H) = \left(\left(\frac{f_{n-1}}{f_{n-2}} \right)^{h_1} \cdot \prod_{k=2}^{n-2} \left(\frac{f_{n-k}^2}{f_{n-k-1} \cdot f_{n-k+1}} \right)^{h_k} \cdot \left(\frac{f_1^2}{f_2} \right)^{h_{n-1}} \right) (H).$$

The claim follows from our considerations above. □

Another operation is given, since the variety $X = \operatorname{Spec} K[\mathcal{N}]^U_{\mathrm{tor}}$ is a toric variety, that is,

$$\tau' \colon \ (K^*)^{n-1} \times K[\mathcal{N}]^U_{\mathrm{tor}} \to K[\mathcal{N}]^U_{\mathrm{tor}}.$$

Let f be a toric invariant, such that $f(H) = x_1^{h_1} \dots x_{n-1}^{h_{n-1}}$, and $c := (c_1, \dots, c_{n-1}) \in (K^*)^{n-1}$, the operation τ' is given by

$$\tau'(c, f)(H) = f(H) \cdot c_1^{h_1} \dots c_{n-1}^{h_{n-1}}.$$

Consider the morphism

$$\begin{aligned} \rho \colon \ T &\to (K^*)^{n-1} \\ (t_1, \dots, t_n) &\mapsto \left(\frac{t_2}{t_1}, \dots, \frac{t_n}{t_{n-1}} \right). \end{aligned}$$

Corollary 6.4.2. *(Interrelation of the operations τ and τ')*
The operation τ is induced by the operation τ' via the morphism ρ.

Proof. Follows from the description of τ and τ' as well as of lemma 6.4.1. □

6.4.2 Explicit description of toric invariants

We make use of the interrelation of the operations τ and τ' in order to describe the toric invariants in detail.

Lemma 6.4.3. *(Explicit description of toric invariants)*
Let f be a sum-free toric invariant of block sizes $\underline{a} := (a_1, \dots, a_s)$ and $\underline{a}' := (a'_1, \dots, a'_t)$ and let $f(H) = x_1^{h_1} \dots x_{n-1}^{h_{n-1}}$. Then, for $l \in \{1, \dots, n-2\}$,

$$h_l = t + \sum_{k=2}^{l} \sharp\{j \in \{1, \dots, t\} \mid a'_j \geq k\} - \sum_{k=1}^{l-1} \sharp\{i \in \{1, \dots, s\} \mid a_i \geq n-k\}$$

and

$$h_{n-1} = s.$$

Proof. The invariant f is a B-semi-invariant of weight

$$\omega := \sum_{i=1}^{s} (\omega_{n-a_i+1} + \dots + \omega_n) - \sum_{j=1}^{t} \left(\omega_1 + \dots + \omega_{a'_j} \right).$$

Therefore, if $t \in T$ is a matrix with diagonal entries $t_1, \dots, t_n$, then the weight ω and corollary 6.4.2 yield

$$\frac{\prod_{i=1}^{s} (t_{n-a_i+1} \cdot \ldots \cdot t_n)}{\prod_{j=1}^{t} \left(t_1 \cdot \ldots \cdot t_{a'_j}\right)} = t.f(H) = \prod_{i=1}^{n-1} \left(\frac{t_{i+1}}{t_i}\right)^{h_i}.$$

The claim follows by counting the corresponding factors. □

6.4.3 Interrelation between $\mathcal{N}/\!\!/U$ and X

We summarize the results, we have proven so far about the affine variety $\mathcal{N}/\!\!/U$.

Let $\pi : \mathcal{N} \to \mathcal{N}/\!\!/U$ be an algebraic U-quotient of $\mathcal{N}$ which exists, since $K[\mathcal{N}]^U$ is finitely generated.

The space of U-normal forms is given by $\mathcal{H}_U \cong \mathbf{A}^D \times (K^*)^{n-1}$ and the map π restricts to a morphism $i : \mathcal{H}_U \to \mathcal{N}/\!\!/U$.

Consider the toric variety X described above by its cone σ which is induced by the sum-free toric invariants of subsection 6.2 and let $X' \cong (K^*)^{n-1}$ be the dense orbit in X.

The generic separation in section 6.3 yields that the morphism $i : \mathcal{H}_U \to i(\mathcal{H}_U)$ is injective and that we can construct an explicit morphism $i' : i(\mathcal{H}_U) \to \mathcal{H}_U$, such that $i \circ i' = \mathrm{id}_{i(\mathcal{H}_U)}$ and $i' \circ i = \mathrm{id}_{\mathcal{H}_U}$. Thus,

$$\mathbf{A}^D \times (K^*)^{n-1} \cong i(\mathcal{H}_U) \subseteq \mathcal{N}/\!\!/U$$

and the morphism i is birational.

Lemma 6.4.4. *(The morphism $\mathcal{N}/\!\!/U \to X$)*
The natural embedding $K[\mathcal{N}]^U_{\mathrm{tor}} \to K[\mathcal{N}]^U$ induces a dominant, T-equivariant morphism

$$p : \mathcal{N}/\!\!/U \to X,$$

such that for each point $x' \in X'$, its fibre fulfills $p^{-1}(x) \cong \mathbf{A}^D$.

Proof. The morphism p is T-equivariant due to our considerations in subsection 6.4.1 and dominant, since it is induced by the inclusion $K[\mathcal{N}]^U_{\mathrm{tor}} \to K[\mathcal{N}]^U$. More explicitly, $X' = p(\mathcal{H}_U) \subseteq \operatorname{im} p$.

Let $x' \in X'$, then $p^{-1}(x) \subseteq i(\mathcal{H}_U)$, since every determinant $\det_i$ for $i \in \{1, \dots, n-1\}$ is a toric invariant. If $x' \in X'$, none of these determinants vanishes on x' and corollary 5.1.4, therefore, yields $p^{-1}(x') \subseteq i(\mathcal{H}_U)$. Since the orbits in $\mathcal{N}_U$ are separated by the U-invariants of section 6.3 and since $\mathcal{H}_U \cong \mathbf{A}^D \times X'$, the claim $p^{-1}(x) \cong \mathbf{A}^D$ follows. □

The morphism

$$\mathcal{H}_U \xrightarrow{i} \mathcal{N}/\!\!/U \xrightarrow{p} X$$

can be thought of as the projection on the first diagonal, that is, $p \circ i(H) = (x_i)_{1 \le i \le n-1} \in X'$.

There is a morphism $q : \mathcal{N}/\!\!/U \to \mathbf{A}^D$ as well, such that the composition

$$\mathcal{H}_U \xrightarrow{i} \mathcal{N}/\!\!/U \xrightarrow{q} \mathbf{A}^D$$

yields $q \circ i(H) = (x_{i,j})_{1<j+1\leq i-1<n} \in \mathbf{A}^D$.

Lemma 6.4.5. *(The morphism (p,q))*
The morphism

$$(q,p) : \mathcal{N}/\!\!/U \to \mathbf{A}^D \times X$$

is dominant and birational.

Proof. The morphism (p,q) is dominant, since (following our considerations above) $\mathbf{A}^D \times X' \subseteq \operatorname{im}(p,q) \subseteq \mathbf{A}^D \times X$.

The morphism (p,q) is birational following lemma 1.1.3, since (p,q) is dominant and for every $y \in \mathbf{A}^D \times X' \subseteq \mathbf{A}^D \times X$, the fibre $(p,q)^{-1}(y)$ contains exactly one element by our considerations in section 6.3. More straight forward, (p,q) restricts to an isomorphism $i(\mathcal{H}_U) \cong \mathbf{A}^D \times X'$. □

Remark 6.4.6. *The morphism (p,q) is not surjective for $n \geq 4$:*

Let us assume the morphism (p,q) to be surjective. Since $\mathbf{A}^D \times X$ is a normal variety and (p,q) is birational, the lemma of Richardson 1.1.4 yields that (p,q) is an isomorphism.

We obtain a contradiction, since $K[\mathcal{N}]^U \not\cong K[\mathcal{H}_U]$:

Define a U-invariant g by the data

$$\mathcal{P} = \begin{cases} ((2),(2),(x)), & \text{if } n = 4; \\ ((n-2),(2,n-4),(x,x^4)) & \text{otherwise}. \end{cases}$$

Let $H \in \mathcal{H}_U$ be a matrix in U-normal form as before. Then

$$g(H) = (x_{3,1} \cdot x_{4,2} - x_2 \cdot x_{4,1}) \cdot \det_{n-4}(H)$$

and the relation

$$g \cdot \underbrace{\det_{n-3} \cdot \det_1 \cdot f_{n-3} \cdot f_{n-1}}_{:=F} = \underbrace{f_{3,1} \cdot f_{4,2} \cdot f_{n-3} \cdot f_{n-1} - f_{4,1} \cdot f_{n-2}^2 \cdot \det_{n-3} \cdot \det_1}_{:=F'}$$

holds true in $K[\mathcal{N}]^U$. The inequality $K[\mathcal{N}]^U \not\cong K[\mathcal{H}_U] \cong K[\mathbf{A}^D \times X]$ follows.

Furthermore, since $F \neq F'$, the set $M := \{\underline{x} \in \mathbf{A}^D \times X \mid F(\underline{x}) \neq 0; F'(\underline{x}) = 0\}$ is non-empty. Then the inclusion $M \subseteq (\mathbf{A}^D \times X) \setminus \operatorname{im}(p,q)$ directly yields that the morphism (p,q) is not surjective.

By the same reasoning, we obtain the stronger result

$$\operatorname{codim}_{\mathbf{A}^D \times X}(\overline{(\mathbf{A}^D \times X) \setminus \operatorname{im}(p,q)}) \leq 1.$$

6.5 The case $n = 4$

We work through the case $n = 4$ in all detail.

Toric invariants

Let us consider a matrix

$$H = \begin{pmatrix} 0 & 0 & 0 & 0 \\ x_1 & 0 & 0 & 0 \\ x_{3,1} & x_2 & 0 & 0 \\ x_{4,1} & x_{4,2} & x_3 & 0 \end{pmatrix}$$

in U-normalform.

We deduce minimal toric generators following section 6.2 and describe them by their monomials, that is, if f is a toric invariant, we describe the tuple (a, b, c), such that

$$f(H) = x_1^a \cdot x_2^b \cdot x_3^c.$$

We start by describing all sum-free tuples $(\underline{a}, \underline{a}')$ with $\underline{a} = (a_1, \ldots, a_s)$ and $\underline{a}' = (a_1', \ldots, a_t')$ with increasingly ordered entries, such that $a_i \leq 3$ and $a_j' \leq 3$ for all i, j.

There are integers $s_1, s_2 \in \{1, \ldots, s\}$ and $t_1, t_2 \in \{1, \ldots, t\}$ with

$$1 = a_1 = \ldots = a_{s_1} < a_{s_1+1} = \ldots = a_{s_2} < a_{s_2+1} = \ldots = a_s = 3$$

and

$$1 = a_1' = \ldots = a_{t_1}' < a_{t_1+1}' = \ldots = a_{t_2}' < a_{t_2+1}' = \ldots = a_t' = 3.$$

The following are the only sum-free tuples:

1. $(\underline{a}, \underline{a}') = ((1), (1))$ 2. $(\underline{a}, \underline{a}') = ((2), (2))$ 3. $(\underline{a}, \underline{a}') = ((3), (3))$

4. $(\underline{a}, \underline{a}') = ((1, 1), (2))$ 5. $(\underline{a}, \underline{a}') = ((2), (1, 1))$

6. $(\underline{a}, \underline{a}') = ((1, 1, 1), (3))$ 7. $(\underline{a}, \underline{a}') = ((3), (1, 1, 1))$

8. $(\underline{a}, \underline{a}') = ((1, 2), (3))$ 9. $(\underline{a}, \underline{a}') = ((3), (1, 2))$

10. $(\underline{a}, \underline{a}') = ((2, 2), (1, 3))$ 11. $(\underline{a}, \underline{a}') = ((1, 3), (2, 2))$

Of course, given the toric normal form H,

$$H^2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ x_1x_2 & 0 & 0 & 0 \\ 0 & x_2x_3 & 0 & 0 \end{pmatrix} \text{ and } H^3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ x_1x_2x_3 & 0 & 0 & 0 \end{pmatrix}.$$

The above sum-free tuples yield generating sum-free toric invariants as follows:

1. $\det_1 = f_1 = f_{((1),(1))}$: $(a, b, c) = (1, 1, 1)$, since

$$\det\left((H^3)_{(1,1)}\right) = \det(x_1x_2x_3) = x_1x_2x_3;$$

2. $\det_2 = f_{((2),(2))}$: $(a,b,c) = (1,2,1)$, since

$$\det\left((H^2)_{(2,2)}\right) = \det\begin{pmatrix} x_1x_2 & 0 \\ 0 & x_2x_3 \end{pmatrix} = x_1x_2^2x_3;$$

3. $\det_3 = f_{((3),(3))}$: $(a,b,c) = (1,1,1)$, since

$$\det\left((H)_{(3,3)}\right) = \det\begin{pmatrix} x_1 & 0 & 0 \\ 0 & x_2 & 0 \\ 0 & 0 & x_3 \end{pmatrix} = x_1x_2x_3;$$

4. $g_2 \coloneqq f_{((1,1),(2))}$: $(a,b,c) = (1,2,2)$, since

$$\det\begin{pmatrix} H^3_{(1,2)} \\ \hline H^2_{(1,2)} \end{pmatrix} = \det\begin{pmatrix} x_1x_2x_3 & 0 \\ \hline 0 & x_2x_3 \end{pmatrix} = x_1x_2^2x_3^2;$$

5. $f_2 = f_{((2),(1,1))}$: $(a,b,c) = (2,2,1)$, since

$$\det\left(\begin{array}{c|c} H^2_{(2,1)} & H^3_{(2,1)} \end{array}\right) = \det\left(\begin{array}{c|c} x_1x_2 & 0 \\ 0 & x_1x_2x_3 \end{array}\right) = x_1^2x_2^2x_3;$$

6. $g_3 \coloneqq f_{((1,1,1),(3))}$: $(a,b,c) = (1,2,3)$, since

$$\det\begin{pmatrix} H^3_{(1,3)} \\ \hline H^2_{(1,3)} \\ \hline H_{(1,3)} \end{pmatrix} = \det\begin{pmatrix} x_1x_2x_3 & 0 & 0 \\ \hline 0 & x_2x_3 & 0 \\ \hline 0 & 0 & x_3 \end{pmatrix} = x_1x_2^2x_3^3;$$

7. $f_3 = f_{((3),(1,1,1))}$: $(a,b,c) = (3,2,1)$, since

$$\det\left(\begin{array}{c|c|c} H_{(3,1)} & H^2_{(3,1)} & H^3_{(3,1)} \end{array}\right) = \det\left(\begin{array}{c|c|c} x_1 & 0 & 0 \\ 0 & x_1x_2 & 0 \\ 0 & 0 & x_1x_2x_3 \end{array}\right) = x_1^3x_2^2x_3;$$

8. $f_{((1,2),(3))}$: $(a,b,c) = (1,2,2)$, since

$$\det\begin{pmatrix} H^3_{(1,3)} \\ \hline H_{(2,3)} \end{pmatrix} = \det\begin{pmatrix} x_1x_2x_3 & 0 & 0 \\ \hline 0 & x_2 & 0 \\ 0 & 0 & x_3 \end{pmatrix} = x_1x_2^2x_3^2;$$

9. $f_{((3),(1,2))}$: $(a,b,c) = (2,2,1)$, since

$$\det\left(\begin{array}{c|c} H_{(3,1)} & H^2_{(3,2)} \end{array}\right) = \det\left(\begin{array}{c|cc} x_1 & 0 & 0 \\ 0 & x_1x_2 & 0 \\ 0 & 0 & x_2x_3 \end{array}\right) = x_1^2x_2^2x_3;$$

10. $f_{((2,2),(1,3))}$: $(a, b, c) = (2, 3, 2)$, since

$$\det\left(\begin{array}{c|c} H^2_{(2,1)} & H^3_{(2,3)} \\ \hline 0 & H_{(2,3)} \end{array}\right) = \det\left(\begin{array}{c|ccc} x_1x_2 & 0 & 0 & 0 \\ 0 & x_1x_2x_3 & 0 & 0 \\ \hline 0 & 0 & x_2 & 0 \\ 0 & 0 & 0 & x_3 \end{array}\right) = x_1^2x_2^3x_3^2;$$

11. $f_{((1,3),(2,2))}$: $(a, b, c) = (2, 3, 2)$, since

$$\det\left(\begin{array}{c|c} H^2_{(1,2)} & H^3_{(1,2)} \\ \hline H_{(3,2)} & H_{(3,2)} \end{array}\right) = \det\left(\begin{array}{cc|cc} 0 & x_2x_3 & 0 & 0 \\ \hline x_1 & 0 & 0 & 0 \\ 0 & 0 & x_1x_2 & 0 \\ 0 & 0 & 0 & x_2x_3 \end{array}\right) = (-1) \cdot x_1^2x_2^3x_3^2.$$

The sum-free invariants are listed in the table in figure 6.7.

a	a'	a	b	c	a	a'	a	b	c
(1)	(1)	1	1	1	(3)	(1,1,1)	3	2	1
(2)	(2)	1	2	1	(1,2)	(3)	2	2	1
(3)	(3)	1	1	1	(3)	(1,2)	1	2	2
(1,1)	(2)	2	2	1	(2,2)	(1,3)	2	3	2
(2)	(1,1)	1	2	2	(1,3)	(2,2)	2	3	2
(1,1,1)	(3)	1	2	3					

Figure 6.7: Generating toric invariants for $n = 4$

Define the ideal

$$R := \left(g_2^2 = g_3\det_2, f_2^2 = f_3\det_2, f_3g_3 = \det_1^4, f_3g_2 = \det_1^2 f_2, g_3f_2 = \det_1^2 g_2, f_2g_2 = \det_1^2\det_2\right)$$

in the subring of toric invariants $K[\mathcal{N}]^U_{\text{tor}} \subset K[\mathcal{N}]^U$.

Then

$$X := \operatorname{Spec} K[\mathcal{N}]^U_{\text{tor}} = \operatorname{Spec} \frac{K[\det_1, \det_2, f_1, f_2, g_2, g_3]}{R}$$

due to our considerations above.

Claim: X is a normal affine toric variety.

There is an isomorphism $X \cong \operatorname{Spec} K[S_\sigma]$ of affine varieties where σ is the convex polyhedral cone generated by

$$S = \left\{ \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} \right\}.$$

The variety X is a normal toric variety if and only if σ is a strongly convex rational polyhedral cone. Let N be the lattice $\mathbf{Z}^3$, then σ is generated by the finite set $S \subset \mathbf{Z}^3$ and, therefore, is a convex rational polyhedral cone. It is strongly convex, since $\sigma \cap (-\sigma) = \{0\}$.

Toric operations

Let $T \subset \mathrm{GL}_n$ be the torus of diagonal matrices. Consider the operation

$$\begin{aligned} \tau\colon\ T \times K[\mathcal{N}]^U &\to K[\mathcal{N}]^U \\ (t,f) &\mapsto \begin{pmatrix} f\colon & \mathcal{N} \to K \\ & N \mapsto f(tNt^{-1}) \end{pmatrix} \end{aligned}$$

which we examine the operation τ on the generating invariants defined in section 6.3 in the following.

Let $t \in T \subset K^{4\times 4}$ be a diagonal matrix with diagonal entries $t_1, t_2, t_3, t_4 \in K^*$. Then

$$\begin{array}{lll} \det_1(tHt^{-1}) = \frac{t_4}{t_1}; & f_2(tHt^{-1}) = \frac{t_3t_4}{t_1^2}; & g_2(tHt^{-1}) = \frac{t_4^2}{t_1t_2}; \\ \det_2(tHt^{-1}) = \frac{t_3t_4}{t_1t_2}; & f_3(tHt^{-1}) = \frac{t_2t_3t_4}{t_1^3}; & g_3(tHt^{-1}) = \frac{t_4^3}{t_1t_2t_3}. \end{array}$$

These equalities can be verified directly, or by calculating the character ω, such that the U-invariant is a B-semi-invariant of weight ω. The operation is fixed by those matrices for which $t_1 = t_2 = t_3 = t_4$.

The variety X is a toric variety via the induced operation of

$$\tau'\colon (\mathbf{C}^*)^3 \times K[\mathcal{N}]^U_{\mathrm{tor}} \to K[\mathcal{N}]^U_{\mathrm{tor}},$$

such that $c := (c_1, c_2, c_3) \in (\mathbf{C}^*)^3$ acts on the generating invariants as follows:

$$\begin{array}{lll} c.\det_1(H) = \det_1(H)c_1c_2c_3; & c.f_2(H) = f_2(H)c_1^2c_2^2c_3; & c.g_2(H) = g_2(H)c_1c_2^2c_3^2; \\ c.\det_2(H) = \det_2(H)c_1c_2^2c_3; & c.f_3(H) = f_3(H)c_1^3c_2^2c_3; & c.g_3(H) = g_3(H)c_1c_2^2c_3^3. \end{array}$$

We immediately understand that the operation τ is induced by the operation τ' via the morphism

$$\begin{aligned} \rho\colon\ T &\to (\mathbf{C}^*)^3 \\ t &\mapsto \left(\frac{t_2}{t_1}, \frac{t_3}{t_2}, \frac{t_4}{t_3}\right). \end{aligned}$$

Generic separation

Define

$$\begin{aligned} \mu\colon\ \mathcal{N} &\to \mathbf{A}^6 \\ N &\mapsto (f_{3,1}(N), f_{2,1}(N), f_{3,2}(N), f_1(N), f_2(N), f_3(N)) \end{aligned}$$

where the U-invariants $f_{i,j}$ and f_i are defined as in definition 6.3.1.

Claim: The morphism μ separates the U-orbits generically, that is, on the open subset $\mathcal{N}_U \subseteq \mathcal{N}$, the inequality $\mu(N) \neq \mu(N')$ holds true for all $N, N' \in \mathcal{N}_U$.

Proof. Let $H = (x_{i,j})_{i,j} \in \mathcal{H}_U$ and $H' = (x'_{i,j})_{i,j} \in \mathcal{H}_U$ be two matrices in normal form; we set $x_i := x'_{i+1,i}$ and $x'_i := x'_{i+1,i}$.

It is sufficient to show $x_i = x'_i$ for all i.

Let $(a, b, c, s, t, u) \in \mathbf{A}^6$ and assume $\mu(H) = \mu(H') = (a, b, c, s, t, u)$.

$$\begin{aligned}
x_{4,1} &= f_{4,1}(H) = a = f_{4,1}(H') = x'_{4,1} \\
x_{3,1} \cdot x_1 x_2 x_3 &= f_{3,1}(H) = b = f_{3,1}(H') = x'_{3,1} \cdot x'_1 x'_2 x'_3 \\
x_{4,2} \cdot x_1 x_2 x_3 &= f_{4,2}(H) = c = f_{4,2}(H') = x'_{4,2} \cdot x'_1 x'_2 x'_3 \\
x_1 x_2 x_3 &= f_1(H) = s = f_1(H') = x'_1 x'_2 x'_3 \\
x_1^2 x_2^2 x_3 &= f_2(H) = t = f_2(H') = x_1'^2 x_2'^2 x'_3 \\
x_1^3 x_2^2 x_3 &= f_3(H) = u = f_3(H') = x_1'^3 x_2'^2 x'_3
\end{aligned}$$

We calculate $x_1 = x'_1 = u/t$ and $x_2 = t^2/us = x'_2$ and $x_3 = s^2/t = x'_3$.

Therefore, $H = H'$ which proves the claim. □

The morphism (p, q)

We define the dominant morphism

$$(p, q) : \mathcal{N} /\!\!/ U \to \mathbf{A}^3 \times X$$

as in subsection 6.4.3 which separates the U-orbits in $\mathbf{A}^3 \times X'$ as has been shown above.

Consider the following three invariants:

- the (non-toric) invariant g given by the datum $\mathcal{P} = ((2), (2), (x))$ for which
$$g(H) = \det\left(H_{(2,2)}\right) = x_{3,1} \cdot x_{4,2} - x_2 \cdot x_{4,1}$$
holds true,
- the (non-toric) invariant g' given by the datum $\mathcal{P} = ((1), (1), (x^2))$ for which
$$g'(H) = \det\left(H^2_{(1,1)}\right) = x_1 \cdot x_{4,2} + x_3 \cdot x_{3,1}$$
holds true and
- the (non-toric) invariant G_2 by the datum $\mathcal{P} = ((2), (1, 1), (x, x^2))$ for which
$$g''(H) = \det\left(\; H_{(2,1)} \;\middle|\; H^2_{(2,1} \; \right) = x_{3,1} \cdot g'(H) - x_{4,1} \cdot x_1 \cdot x_2$$
holds true.

Then, for example, the relations

$$g \cdot \det_1^2 = f_{3,1} f_{4,2} - \det_1 \det_2 f_{4,1} \quad \text{and} \quad g'' \cdot \det_1^2 = f_{3,1} \det_1 g' - f_2 f_{4,1}$$

yield $K[\mathcal{N}]^U \not\cong K[\mathbf{A}^3] \otimes K[\mathcal{N}]^U_{\mathrm{tor}}$ and $\mathcal{N} /\!\!/ U \not\cong \mathbf{A}^3 \times X$.

7 Towards a GIT-quotient for the Borel action

We initiate the study of a GIT-quotient for the Borel action on the nilpotent cone $\mathcal{N}$ in the following.

7.1 The examples $n = 2$ and $n = 3$

We start by discussing $n = 2$ and consider $\mathcal{N} = \mathcal{N}_2$.

Example 7.1.1. *(A GIT-quotient in $\mathcal{N}_2$ for the Borel action)*

Example 6.1.2 proves

$$K[\mathcal{N}]^U = K[f_{2,1}].$$

The U-invariant morphism $f_{2,1}$ is a B-semi-invariant of weight $\chi_0 := \omega_2 - \omega_1$. Therefore,

$$\bigoplus_{\chi \in X(B)} \bigoplus_{n \geq 0} K[\mathcal{N}]^{B,n\chi} = \bigoplus_{n \geq 0} K[\mathcal{N}]^{B,n\chi_0}.$$

Of course, $N \in \mathcal{N}_B$ if and only if $f_{2,1}(N) \neq 0$ and therefore $\mathcal{N}^{\chi_0-\mathrm{sst}} = \mathcal{N}_B$.

The morphism

$$\begin{aligned} \mu\colon\ \mathcal{N}^{\chi_0-\mathrm{sst}} &\to \{1\} = \operatorname{Proj} K[f_{2,1}] \\ N &\mapsto f_{2,1}(N) = 1, \end{aligned}$$

thus, is a GIT-quotient.

Example 7.1.2. *(A GIT-quotient in $\mathcal{N}_3$ for the Borel action)*

Let us consider $\mathcal{N} = \mathcal{N}_3$. Example 6.1.3 proves

$$K[\mathcal{N}]^U = \frac{K[f_{3,1}, f_1, f_2, \det_1]}{\left(f_1 \cdot f_2 = \det_1^3\right)}.$$

We consider these U-invariants:

1. *$f_{3,1}$ and $\det_1$ are B-semi-invariants of weight $\chi_{3,1} := \omega_3 - \omega_1$,*

2. *f_1 is a B-semi-invariant of weight $\chi_1 := -2\omega_1 + \omega_2 + \omega_3$ and*

3. *f_2 is a B-semi-invariant of weight $\chi_2 := -\omega_1 - \omega_2 + 2\omega_3$.*

The equality $\det_1 = \det_2$ *holds true on* $\mathcal{N}$, *therefore* $\mathcal{N}^{\chi_{3,1}-\mathrm{sst}} = \mathcal{N}_B \cup \{N \in \mathcal{N} \mid N_{3,1} \neq 0\}$. *Thus, the morphism*

$$\begin{aligned} \mu\colon\ \mathcal{N}^{\chi_{3,1}-\mathrm{sst}} &\to \mathbf{P}^1 = \operatorname{Proj} K[f_{3,1}, \det_1] \\ N &\mapsto (f_{3,1}(N) : \det_1(N)) \end{aligned}$$

is a GIT-quotient.

7.2 Generic separation of the same weight

We define the character

$$\chi := \sum_{i=1}^{n-1}(\omega_{n-i+1} + \ldots + \omega_n) - \sum_{i=1}^{n-1}(\omega_1 + \ldots + \omega_i).$$

There is one particular B-semi-invariant f_B of weight χ which extracts the matrices in $\mathcal{N}_B$ from $\mathcal{N}$.

Definition 7.2.1. *(The B-semi-invariant f_B)*
Let $a_i := a'_i := i$ *for* $i \in \{1, \ldots, n-1\}$ *and let*

$$P_{i,j} := \begin{cases} x^{n-i}, & \text{if } i = j; \\ 0, & \text{otherwise.} \end{cases}$$

Define $\mathcal{P}_B := ((a_i)_i, (a'_i)_i, (P_{i,j})_{i,j})$ *and*

$$\begin{aligned} f_B\colon\ \mathcal{N} &\to K \\ N &\mapsto \det(N^{\mathcal{P}_B}) = \prod_{i=1}^{n-1} \det(N^{n-i})_{(i,i)}. \end{aligned}$$

We directly see $f_B(H) = 1$ for all $H \in \mathcal{H}_B$ and that f_B is a B-semi-invariant of character χ.

Proposition 7.2.2. *(Extraction of $\mathcal{N}_B$)*
The B-semi-invariant f_B *fulfills*

$$f_B(N) \neq 0 \text{ if and only if } N \in \mathcal{N}_B.$$

Proof. Let $N \in \mathcal{N}$. Clearly $f_B(N) \neq 0$ if and only if $\prod_{i=1}^{n-1} \det(N^{n-i})_{(i,i)} \neq 0$, thus, if and only if $\det N^{n-i}_{(i,i)} \neq 0$ for all i. Corollary 5.1.3 yields $N \in \mathcal{N}_B$. □

We show how to extract the entries of the normal forms H in the affine space $\mathcal{H}_B \cong \mathbf{A}^D$ of dimension $D := \frac{(n-1)(n-2)}{2}$ with the generating semi-invariants from proposition 5.2.1. In particular, we are able to separate them with semi-invariants of the same weight χ.

Lemma 7.2.3. *(Separating B-semi-invariants)*
For each i and j, such that $2 < j + 2 \leq i \leq n$, there is a semi-invariant $g_{i,j}$ of weight χ which fulfills

$$g_{i,j}(H) = H_{i,j}$$

for every normal form $H \in \mathcal{H}_B$.

Proof. We consider three cases.

1. Let $n - i + 1 \notin \{j - 1, j\}$.

   Define the datum $\mathcal{P} := ((a_k)_k, (a'_k)_k, (P_{k,l})_{k,l})$ as follows:

   - $(a_k)_{1\leq k\leq n-1} := (j-1, n-i+1, j, 1, \ldots, j-2, j+1, \ldots, n-i, n-i+2, \ldots, n-1)$,
   - $(a'_k)_{1\leq k\leq n-1} := (j, n-i+1, j-1, 1, \ldots, j-2, j+1, \ldots, n-i, n-i+2, \ldots, n-1)$ and
   - $P_{k,l} := \begin{cases} x^{n-j+1}, & \text{if } k = l \in \{1,3\}; \\ x & \text{if } k = 2 \text{ and } l = 1; \\ x^{i} & \text{if } k = l = 2; \\ x^{i-j} & \text{if } k = 3 \text{ and } l = 2; \\ x^{n-a_k} & \text{if } k = l > 3; \\ 0 & \text{otherwise.} \end{cases}$

   Let us denote $g_{i,j} := f^{\mathcal{P}}$ and let $H \in \mathcal{H}_B$, then

$$\begin{aligned} g_{i,j}(H) &= \det(H^{\mathcal{P}}) \\ &= \det((P_{k,l}(H)_{(a_k,a'_l)})_{1\leq k,l\leq 3}) \cdot \det((P_{k,l}(H)_{(a_k,a'_l)})_{4\leq k,l\leq n-1}) \\ &= \det((P_{k,l}(H)_{(a_k,a'_l)})_{1\leq k,l\leq 3}) \cdot \prod_{k=4}^{n-1} \det(P_{k,k}(H)_{(a_k,a'_k)}) \\ &= \det((P_{k,l}(H)_{(a_k,a'_l)})_{1\leq k,l\leq 3}). \end{aligned}$$

   Of course,

$$(P_{k,l}(H)_{(a_k,a'_l)})_{1\leq k,l\leq 3} = \left(\begin{array}{cccc|cccc|ccc} 1 & & 0 & 0 & & & & & & & \\ & \ddots & & \vdots & & 0 & & & & 0 & \\ * & & 1 & 0 & & & & & & & \\ \hline & & & & 0 & & & 0 & & & \\ & & & & 1 & & & & & & \\ & H_{(n-i+1,j)} & & & & \ddots & & & & 0 & \\ & & & & * & & 1 & 0 & & & \\ \hline & & & & * & \cdots & * & 1 & 0 & \cdots & 0 \\ & & & & * & \cdots & * & * & 1 & & 0 \\ & 0 & & & \vdots & & \vdots & \vdots & & \ddots & \\ & & & & * & \cdots & * & * & * & & 1 \end{array}\right)$$

This is a lower triangular matrix, all diagonal entries being 1 except the (j, j)-entry, which equals $H_{i,j}$.

Thus,

$$g_{i,j}(H) = \det((P_{k,l}(H)_{(a_k,a'_l)})_{1\le k,l\le 3}) = H_{i,j}.$$

2. Let $n - i + 1 = j$.

We define the datum $\mathcal{P} := ((a_k)_k, (a'_k)_k, (P_{k,l})_{k,l})$ by

- $(a_k)_{1\le k\le n-1} := (j-1, j, 1, \dots, j-2, j+1, \dots, n-1)$,
- $(a'_k)_{1\le k\le n-1} := (j, j-1, 1, \dots, j-2, j+1, \dots, n-1)$ and
- $P_{k,l} := \begin{cases} x^{n-j+1}, & \text{if } k = l \in \{1,2\}; \\ x & \text{if } k = 2 \text{ and } l = 1; \\ x^{n-a_k} & \text{if } k = l > 2; \\ 0 & \text{otherwise.} \end{cases}$

Denote $g_{i,j} := f^{\mathcal{P}}$ and let $H \in \mathcal{H}_B$, then

$$\begin{aligned} g_{i,j}(H) &= \det(H^{\mathcal{P}}) \\ &= \det((P_{k,l}(H)_{(a_k,a'_l)})_{1\le k,l\le 2}) \cdot \det((P_{k,l}(H)_{(a_k,a'_l)})_{3\le k,l\le n-1}) \\ &= \det((P_{k,l}(H)_{(a_k,a'_l)})_{1\le k,l\le 2}) \cdot \prod_{k=3}^{n-1} \det(P_{k,k}(H)_{(a_k,a'_k)}) \\ &= \det((P_{k,l}(H)_{(a_k,a'_l)})_{1\le k,l\le 2}). \end{aligned}$$

Of course,

$$(P_{k,l}(H)_{(a_k,a'_l)})_{1\le k,l\le 2} = \left(\begin{array}{ccc|ccc} 1 & & 0 \quad 0 & & & \\ & \ddots & \vdots & & 0 & \\ * & & 1 \quad 0 & & & \\ \hline & & & 0 & \dots & 0 \\ & & & 1 & & 0 \\ & H_{(n-i+1,j)} & & & \ddots & \\ & & & * & & 1 \end{array}\right)$$

This is a lower triangular matrix, all diagonal entries being 1 except the (j, j)-entry, which equals $H_{i,j}$.

Thus,

$$g_{i,j}(H) = \det((P_{k,l}(H)_{(a_k,a'_l)})_{1\le k,l\le 2}) = H_{i,j}.$$

3. Let $n-i+1 = j-1$.

- Let $j = 2$ first, then $i = n$:

  We define the datum $\mathcal{P} = ((a_k)_k, (a'_k)_k, (P_{k,l})_{k,l})$ as follows:

  · $(a_k)_{1\leq k\leq n-1} := (2, 1, 3, \ldots, n-1)$,

  · $(a'_k)_{1\leq k\leq n-1} := (1, 2, 3, \ldots, n-1)$ and

  · $P_{k,l} := \begin{cases} x^{n-2}, & \text{if } k = l = 1; \\ x^{n-1} & \text{if } k = 1 \text{ and } l = 2; \\ x & \text{if } k = l = 2; \\ x^{n-k} & \text{if } k = l > 2; \\ 0 & \text{otherwise.} \end{cases}$

  Consider $g_{i,j} := f^{\mathcal{P}}$ and let $H \in \mathcal{H}_B$ be a matrix in normal form, then

$$\begin{aligned} g_{i,j}(H) &= \det(H^{\mathcal{P}}) \\ &= \det((P_{k,l}(H)_{(a_k,a'_l)})_{1\leq k,l\leq 2}) \cdot \det((P_{k,l}(H)_{(a_k,a'_l)})_{3\leq k,l\leq n-1}) \\ &= \det((P_{k,l}(H)_{(a_k,a'_l)})_{1\leq k,l\leq 2}) \cdot \prod_{k=3}^{n-1} \det(P_{k,k}(H)_{(a_k,a'_k)}) \\ &= \det((P_{k,l}(H)_{(a_k,a'_l)})_{1\leq k,l\leq 2}). \end{aligned}$$

  Since

$$(P_{k,l}(H)_{(a_k,a'_l)})_{1\leq k,l\leq 2} = \left(\begin{array}{c|cc} 1 & 0 & 0 \\ * & 1 & 0 \\ \hline 0 & * & H_{n,2} \end{array}\right),$$

  we arrive at

$$g_{i,j}(H) = \det((P_{k,l}(H)_{(a_k,a'_l)})_{1\leq k,l\leq 2}) = H_{i,j}.$$

- Let us assume $j \geq 3$.

  We consider the datum $\mathcal{P} = ((a_k)_k, (a'_k)_k, (P_{k,l})_{k,l})$ defined by

  · $(a_k)_{1\leq k\leq n-1} := (j, j-1, 1, \ldots, j-2, j+1, \ldots, n-1)$,

  · $(a'_k)_{1\leq k\leq n-1} := (1, j, j-1, 2 \ldots, j-2, j+1, \ldots, n-1)$ and

  · $P_{k,l} := \begin{cases} x^{n-j+1} & \text{if } (k = 1 \text{ and } l = 2) \text{ or if } k = l = 3; \\ x & \text{if } k = l = 2; \\ x^{n-j+2} & \text{if } k = 2 \text{ and } l = 3; \\ x^{n-a_k} & \text{if } k = l = 1 \text{ or if } k = l > 3; \\ 0 & \text{otherwise.} \end{cases}$

Let $g_{i,j} \coloneqq f^{\mathcal{P}}$ and let $H \in \mathcal{H}_B$, then

$$\begin{aligned} g_{i,j}(H) &= \det(H^{\mathcal{P}}) \\ &= \det((P_{k,l}(H)_{(a_k,a'_l)})_{1\le k,l\le 3}) \cdot \det((P_{k,l}(H)_{(a_k,a'_l)})_{4\le k,l\le n-1}) \\ &= \det((P_{k,l}(H)_{(a_k,a'_l)})_{1\le k,l\le 3}) \cdot \prod_{k=4}^{n-1} \det(P_{k,k}(H)_{(a_k,a'_k)}) \\ &= \det((P_{k,l}(H)_{(a_k,a'_l)})_{1\le k,l\le 3}). \end{aligned}$$

Of course,

$$(P_{k,l}(H)_{(a_k,a'_l)})_{1\le k,l\le 3}) = \left(\begin{array}{c|ccc|ccc} 1 & 0 & & 0 & & & \\ * & 1 & & & & & \\ \vdots & & \ddots & & & 0 & \\ * & * & & 1 & 0 & & \\ \hline & & & & 0 & & 0 \\ & & & & 1 & & \\ 0 & & H_{(n-i+1,j)} & & & \ddots & \\ & & & & * & 1 & 0 \\ \hline 0 & & 0 & & * & \dots\ * & 1 \end{array}\right)$$

This is a lower triangular matrix, all diagonal entries being 1 except the $(j+1, j+1)$-entry, which equals $H_{i,j}$.

Thus,

$$g_{i,j}(H) = \det((P_{k,l}(H)_{(a_k,a'_l)})_{1\le k,l\le 3}) = H_{i,j}.$$

It follows from proposition 5.2.1 that in all cases, the given semi-invariant $g_{i,j}$ is of weight χ. □

We have, thus, found semi-invariants of the same character that extract the coordinates of $\mathcal{H}_B \cong \mathbf{A}^D$.

Proposition 7.2.4. *(Description of χ)*
The character χ can be described as a linear combination $\chi = \sum_{i=1}^{n} \lambda_i \omega_i$ of the basis $\omega_1, \dots, \omega_n$ of $X(B)$ with

$$\lambda_i = -n + 2i - 1.$$

Proof. The character χ is defined by

$$\chi \coloneqq \sum_{k=1}^{n-1} (\omega_{n-k+1} + \ldots + \omega_n) - \sum_{k=1}^{n-1} (\omega_1 + \ldots + \omega_k),$$

where ω_i appears $(i-1)$-times in the first sum and $(n-i)$-times in the second sum. □

7.3 Translation to the language of quiver moduli

The notion of a GIT-quotient of B in $\mathcal{N}$ can be translated to the language of moduli spaces. We give a brief overview of this translation in the following; the reader is referred to [Reineke, 2008] for a thorough treatment of the subject.

Let $Q = Q(n, n)$ be the quiver defined in section 2.3, that is,

$$Q: \quad \underset{1}{\bullet} \xrightarrow{\alpha_1} \underset{2}{\bullet} \xrightarrow{\alpha_2} \underset{3}{\bullet} \quad \cdots \quad \underset{n-2}{\bullet} \xrightarrow{\alpha_{n-2}} \underset{n-1}{\bullet} \xrightarrow{\alpha_{n-1}} \underset{n.}{\bullet} \circlearrowleft \alpha$$

Fix $\theta \in (\mathbf{Z}Q_0)^*$, that is, a linear map $\theta : \mathbf{Z}^n \to \mathbf{Z}$ and assume $\theta(\underline{d}) = \sum_{i=1}^{n} \theta_i \cdot d_i$.

We define the slope corresponding to θ by

$$\mu : \mathbf{N}^n \backslash \{0\} \to K^*; \ \underline{d} \mapsto \frac{\theta(\underline{d})}{\dim \underline{d}}$$

where $\dim \underline{d} = \sum_{i=1}^{n} d_i$ is the total dimension of $\underline{d}$.

Let $M \in \mathrm{rep}_K(Q)$ be a representation that naturally corresponds to the point $m \in R_{\underline{d}}(Q)$ via section 1.2 and denote $\underline{d} := \underline{\dim} M$. We define

$$\hat{\theta} : \mathbf{N}^n \to K; \ \underline{d}' \mapsto \theta(\underline{d}) \cdot \dim \underline{d}' - \dim(\underline{d}) \cdot \theta(\underline{d}').$$

Then $\hat{\theta}(\underline{d}) = 0$ and

$$\mu(\underline{\dim} U) \leq \mu(\underline{d}) \text{ if and only if } \hat{\theta}(\underline{\dim} U) \geq 0.$$

Given $\hat{\theta}$, such that $\hat{\theta}(\underline{d}) = \sum_{i=1}^{n} \hat{\theta}_i \cdot d_i$ we define a $G_{\underline{d}}$-character by

$$\chi_{\hat{\theta}} : \ G_{\underline{d}} \to K^*; \ (g_i)_{i \in Q_0} \mapsto \prod_{i \in Q_0} \det(g_i)^{\theta_i}.$$

A. King proves the following theorem in [King, 1994].

Theorem 7.3.1. *(Theorem of King)*
The representation $M \in \mathrm{rep}_K(Q)$ is semi-stable of weight $\chi_{\hat{\theta}}$ if and only if for every subrepresentation $0 \neq M' \subsetneq M$ the inequality $\mu(\underline{\dim} M') \leq \mu(\underline{d})$ (or, equivalently, $\hat{\theta}(\underline{\dim} M') \geq 0$) holds true.

Let $N_\bullet = (N_1, \ldots, N_n) \in \mathbf{N}^n$ and define a B-character $\chi_{N_\bullet}$ by

$$\chi_{N_\bullet} : \ B \to K^*; \ b \mapsto \prod_{i=1}^{n} b_{i,i}^{N_i}.$$

We translate the characters $\chi_{\hat{\theta}}$ and χ_{N_*} via the bijection of theorem 2.3.1 now.

If we restrict $\chi_{\hat{\theta}}$ to the isotropy group $\mathrm{Iso}_{G_{\underline{d}}}(R_{\underline{d}}(Q)) \cong B$, we obtain

$$\chi_{\hat{\theta}}: \quad \mathrm{Iso}_{G_{\underline{d}}}(R_{\underline{d}}(Q)) \to K^*; \quad (b_i)_{i\in Q_0} \ \mapsto \prod_{i\in Q_0} \det(b_i)^{\hat{\theta}_i},$$

such that

$$b_i = \begin{pmatrix} b_{1,1} & \cdots & b_{1,i} \\ 0 & \ddots & \vdots \\ 0 & 0 & b_{i,i} \end{pmatrix} \in K^{i\times i}.$$

Therefore,

$$\prod_{i\in Q_0} \det(b_i)^{\hat{\theta}_i} = \prod_{k=1}^{n} b_{1,1}^{\hat{\theta}_k} \cdot \ldots \cdot b_{k,k}^{\hat{\theta}_k}$$

and, thus, the characters can be translated via $N_i = \sum_{j=i}^{n} \hat{\theta}_j$ and $\hat{\theta}_i = N_i - N_{i+1}$ if we set $N_{n+1} = 0$.

We call a representation M (and the corresponding point m) θ-semi-stable if they fulfill the equivalent conditions of theorem 7.3.1.

Furthermore, they are called θ-stable if for every subrepresentation $0 \neq M' \subsetneq M$, the inequality $\mu(\underline{\dim} M') < \mu(\underline{\dim} M)$ holds true.
Let us define

$$R_{\underline{d}}^{\theta-\mathrm{sst}}(Q) := \{m \in R_{\underline{d}}(Q) \mid m \text{ is } \theta - \text{semi} - \text{stable}\}$$

and

$$R_{\underline{d}}^{\theta-\mathrm{st}}(Q) := \{m \in R_{\underline{d}}(Q) \mid m \text{ is } \theta - \text{stable}\}.$$

Then we obtain the corresponding quotient varieties which will be denoted by

$$M_{\underline{d}}^{\theta-\mathrm{sst}}(Q) := R_{\underline{d}}^{\theta-\mathrm{sst}}(Q) /\!\!/ G_{\underline{d}}$$

and

$$M_{\underline{d}}^{\theta-\mathrm{st}}(Q) := R_{\underline{d}}^{\theta-\mathrm{st}}(Q) / G_{\underline{d}}.$$

We denote by $PG_{\underline{d}}$ the factor group $G_{\underline{d}}/K^*$ and by $M_{\underline{d}}^{\mathrm{ssimp}}(Q)$ the quotient variety

$$M_{\underline{d}}^{\mathrm{ssimp}}(Q) := R_{\underline{d}}(Q) /\!\!/ PG_d$$

which parametrizes the semi-simple representations of Q of dimension vector $\underline{d}$, we call this variety the "moduli space of semi-simple representations".

There is a projective morphism $\pi : M_{\underline{d}}^{\theta-\mathrm{sst}}(Q) \to M_{\underline{d}}^{\mathrm{ssimp}}(Q)$ and we define

$$M_{\underline{d}}^{\theta-\mathrm{sst,nilp}}(Q) := \pi^{-1}(0),$$

where $0 := \bigoplus_{i\in Q_0} S_i^{d_i} \in M_{\underline{d}}^{\mathrm{ssimp}}(Q)$ is the point corresponding to the canonical semi-simple representation of dimension vector $\underline{d}$; the representation S_i denotes the one-dimensional representation at the vertex i.

If χ and θ can be translated as above. we have an isomorphism

$$\mathcal{N}/\!\!/_{\chi}B \cong M_{\underline{d}_B}^{\theta-\mathrm{sst,nilp}}(Q_n).$$

In order to prove semi-stability for a representation M of dimension vector $\underline{\dim} M = \underline{d}$, for each subrepresentation $M' \subseteq M$ we have to verify

$$\sum_{i=1}^{n}(N_i - N_{i+1}) \dim_K M'_i \leq 0.$$

Without loss of generality, we assume $N_i - N_{i+1} > 0$ for all $i < n$.

Proposition 7.3.2. *(Translation of semi-stability)*
M is θ-semi-stable if and only if for every subspace $0 \neq M'_n \subsetneq M_n$, such that $M_\alpha(M'_n) \subseteq M'_n$, the inequality

$$\sum_{i=1}^{n-1}(N_i - N_{i+1}) \dim_K M_{\alpha_i}^{-1} \cdot \ldots \cdot M_{\alpha_{n-1}}^{-1}(M'_n) + N_n \dim_K M'_n \leq 0$$

holds true.

Proof. Follows from the translation of $\hat{\theta}$ and $N_\bullet$ given above and from theorem 7.3.1. □

Corollary 7.3.3. *(Injectivity of certain maps)*
If M is semi-stable, then M_{α_i} is injective for every integer $i \in \{1, \ldots, n-1\}$.

Proof. Assume, M_{α_i} is not injective for an integer $i \in \{1, \ldots, n-1\}$. Then consider the representation

$$M' = 0 \to \ldots \to 0 \to \langle v \rangle \to 0 \to \ldots \to 0$$

where $\langle v \rangle$ is the space at the i-th vertex of Q for a vector $v \in \mathrm{Ker}\, M_{\alpha_i}$. Clearly,

$$\sum_{i=1}^{n-1}(N_i - N_{i+1}) \dim_K M_{\alpha_i}^{-1} \cdot \ldots \cdot M_{\alpha_{n-1}}^{-1}(M'_n) + N_n \dim_K M'_n = N_i - N_{i+1} > 0,$$

a contradiction. □

Since we were able to separate the B-orbits generically by B-semi-invariants of the same weight χ in section 7.2, it is natural to consider the weight

$$\chi = \sum_{i=1}^{n-1}(\omega_{n-i+1} + \ldots + \omega_n) - \sum_{i=1}^{n-1}(\omega_1 + \ldots + \omega_i)$$

which can be described as a linear combination $\chi = \sum_{i=1}^{n} \lambda_i \omega_i$ with $\lambda_i = -n + 2i - 1$ by proposition 7.2.4.

The translation above yields a stability

$$\theta = (2, \ldots, 2, 1-n)$$

and a semi-stability criterion for θ is obtained.

Proposition 7.3.4. *(Concrete semi-stability)*
M is θ-semi-stable if and only if for every subspace $0 \neq M'_n \subsetneq M_n$, such that $M_\alpha(M'_n) \subseteq M'_n$, the inequality

$$\sum_{i=1}^{n-1} \dim_K M_{\alpha_i}^{-1} \cdot \ldots \cdot M_{\alpha_{n-1}}^{-1}(M'_n) \leq \frac{n-1}{2} \dim_K M'_n.$$

holds true.

Due to proposition 7.2.2,

$$\mathcal{N}_B \subseteq R_{\underline{d}}^{\theta-\mathrm{sst}}(Q)$$

follows immediately.

As the translation to the representation theory of the algebra KQ/I provides an insight into the classification of finite parabolic actions in case the algebra is representation-finite, the translation to the language of moduli spaces may provide further knowledge about quotients if the algebra is of wild representation type.

A Appendix

A.1 Singular computations

1. > ring $R = 0,(a, b, c, d, e, f, g, h, i)$,dp;
> ideal $I = (a + e, a * e - b * d, a * f - c * d, g, h, i)$;
> LIB "primdec.lib";
// ** loaded /opt/Singular/3-1-1/LIB/primdec.lib (12962,2010-07-09)
// ** loaded /opt/Singular/3-1-1/LIB/ring.lib (12231,2009-11-02)
// ** loaded /opt/Singular/3-1-1/LIB/absfact.lib (12231,2009-11-02)
// ** loaded /opt/Singular/3-1-1/LIB/triang.lib (12231,2009-11-02)
// ** loaded /opt/Singular/3-1-1/LIB/matrix.lib (12898,2010-06-23)
// ** loaded /opt/Singular/3-1-1/LIB/nctools.lib (12790,2010-05-14)
// ** loaded /opt/Singular/3-1-1/LIB/inout.lib (12541,2010-02-09)
// ** loaded /opt/Singular/3-1-1/LIB/random.lib (12827,2010-05-28)
// ** loaded /opt/Singular/3-1-1/LIB/poly.lib (12443,2010-01-19)
// ** loaded /opt/Singular/3-1-1/LIB/elim.lib (12231,2009-11-02)
// ** loaded /opt/Singular/3-1-1/LIB/general.lib (12904,2010-06-24)
> quotient(I, $radical(I)$);
_[1] = 1

2. > ring $R = 0,(a, b, c, d, e, f, g, h, i)$,dp;
> ideal $I = (e + i, b * f - c * e, b * i - c * h, e * i - f * h, a, d, g)$;
> LIB "primdec.lib";
// ** loaded /opt/Singular/3-1-1/LIB/primdec.lib (12962,2010-07-09)
// ** loaded /opt/Singular/3-1-1/LIB/ring.lib (12231,2009-11-02)
// ** loaded /opt/Singular/3-1-1/LIB/absfact.lib (12231,2009-11-02)
// ** loaded /opt/Singular/3-1-1/LIB/triang.lib (12231,2009-11-02)
// ** loaded /opt/Singular/3-1-1/LIB/matrix.lib (12898,2010-06-23)
// ** loaded /opt/Singular/3-1-1/LIB/nctools.lib (12790,2010-05-14)
// ** loaded /opt/Singular/3-1-1/LIB/inout.lib (12541,2010-02-09)
// ** loaded /opt/Singular/3-1-1/LIB/random.lib (12827,2010-05-28)
// ** loaded /opt/Singular/3-1-1/LIB/poly.lib (12443,2010-01-19)
// ** loaded /opt/Singular/3-1-1/LIB/elim.lib (12231,2009-11-02)
// ** loaded /opt/Singular/3-1-1/LIB/general.lib (12904,2010-06-24)
> quotient(I, $radical(I)$);
_[1] = 1

3.
```
> ring R = 0, (a, b, c, d, e, f, g, h, i),dp;
> ideal I = (a + e + i, a * e - b * d, d * h - e * g, a * i - c * g, d * i - f * g,
a * f - c * d, b * i - c * h, e * i - f * h);
> LIB "primdec.lib";
// ** loaded /opt/Singular/3-1-1/LIB/primdec.lib (12962,2010-07-09)
// ** loaded /opt/Singular/3-1-1/LIB/ring.lib (12231,2009-11-02)
// ** loaded /opt/Singular/3-1-1/LIB/absfact.lib (12231,2009-11-02)
// ** loaded /opt/Singular/3-1-1/LIB/triang.lib (12231,2009-11-02)
// ** loaded /opt/Singular/3-1-1/LIB/matrix.lib (12898,2010-06-23)
// ** loaded /opt/Singular/3-1-1/LIB/nctools.lib (12790,2010-05-14)
// ** loaded /opt/Singular/3-1-1/LIB/inout.lib (12541,2010-02-09)
// ** loaded /opt/Singular/3-1-1/LIB/random.lib (12827,2010-05-28)
// ** loaded /opt/Singular/3-1-1/LIB/poly.lib (12443,2010-01-19)
// ** loaded /opt/Singular/3-1-1/LIB/elim.lib (12231,2009-11-02)
// ** loaded /opt/Singular/3-1-1/LIB/general.lib (12904,2010-06-24)
> quotient(I, radical(I));
_[1] = 1
```

A.2 The parabolic subgroup of block sizes $(4, 3)$

Let us consider the action of the parabolic subgroup of block sizes $(4, 3)$ on $\mathcal{N}_7^{(3)}$. There are 136 orbits of which the normal forms are listed in the following table; one can see how easily they are obtained by combinatorially considering each representation M (as a direct sum of indecomposables), such that $\underline{\dim} M = (4, 7)$.

$\mathcal{U}_{4,7}$	$\mathcal{U}^{(2)}_{2,4} \oplus \mathcal{U}^{2}_{1,1} \oplus \mathcal{U}_{0,1}$	$\mathcal{U}_{2,2} \oplus \mathcal{U}_{1,4} \oplus \mathcal{U}_{1,1}$
$\mathcal{U}_{4,6} \oplus \mathcal{U}_{0,1}$	$\mathcal{U}^{(1)}_{2,3} \oplus \mathcal{U}^{(1)}_{2,3} \oplus \mathcal{U}_{0,1}$	$\mathcal{U}_{2,2} \oplus \mathcal{U}^{(1)}_{1,3} \oplus \mathcal{U}^{(1)}_{1,2}$
$\mathcal{U}_{3,6} \oplus \mathcal{U}_{1,1}$	$\mathcal{U}^{(1)}_{2,3} \oplus \mathcal{U}^{(2)}_{2,3} \oplus \mathcal{U}_{0,1}$	$\mathcal{U}_{2,2} \oplus \mathcal{U}^{(2)}_{1,3} \oplus \mathcal{U}^{(1)}_{1,2}$
$\mathcal{U}^{(1)}_{3,6} \oplus \mathcal{U}_{1,1}$	$\mathcal{U}^{(1)}_{2,3} \oplus \mathcal{U}^{(3)}_{2,3} \oplus \mathcal{U}_{0,1}$	$\mathcal{U}_{2,2} \oplus \mathcal{U}^{(3)}_{1,3} \oplus \mathcal{U}^{(1)}_{1,2}$
$\mathcal{U}^{(2)}_{3,6} \oplus \mathcal{U}_{1,1}$	$\mathcal{U}^{(2)}_{2,3} \oplus \mathcal{U}^{(2)}_{2,3} \oplus \mathcal{U}_{0,1}$	$\mathcal{U}_{2,2} \oplus \mathcal{U}^{(1)}_{1,3} \oplus \mathcal{U}^{(2)}_{1,2}$
$\mathcal{U}^{(1)}_{3,5} \oplus \mathcal{U}^{(1)}_{1,2}$	$\mathcal{U}^{(2)}_{2,3} \oplus \mathcal{U}^{(3)}_{2,3} \oplus \mathcal{U}_{0,1}$	$\mathcal{U}_{2,2} \oplus \mathcal{U}^{(2)}_{1,3} \oplus \mathcal{U}^{(2)}_{1,2}$
$\mathcal{U}^{(1)}_{3,5} \oplus \mathcal{U}^{(2)}_{1,2}$	$\mathcal{U}^{(3)}_{2,3} \oplus \mathcal{U}^{(3)}_{2,3} \oplus \mathcal{U}_{0,1}$	$\mathcal{U}_{2,2} \oplus \mathcal{U}^{(3)}_{1,3} \oplus \mathcal{U}^{(2)}_{1,2}$
$\mathcal{U}^{(1)}_{3,5} \oplus \mathcal{U}_{1,1} \oplus \mathcal{U}_{0,1}$	$\mathcal{U}^{(1)}_{2,3} \oplus \mathcal{U}_{2,2} \oplus \mathcal{U}_{0,2}$	$\mathcal{U}_{2,2} \oplus \mathcal{U}^{(1)}_{1,3} \oplus \mathcal{U}_{1,1} \oplus \mathcal{U}_{0,1}$
$\mathcal{U}^{(2)}_{3,5} \oplus \mathcal{U}^{(1)}_{1,2}$	$\mathcal{U}^{(2)}_{2,3} \oplus \mathcal{U}_{2,2} \oplus \mathcal{U}_{0,2}$	$\mathcal{U}_{2,2} \oplus \mathcal{U}^{(2)}_{1,3} \oplus \mathcal{U}_{1,1} \oplus \mathcal{U}_{0,1}$
$\mathcal{U}^{(2)}_{3,5} \oplus \mathcal{U}^{(2)}_{1,2}$	$\mathcal{U}^{(3)}_{2,3} \oplus \mathcal{U}_{2,2} \oplus \mathcal{U}_{0,2}$	$\mathcal{U}_{2,2} \oplus \mathcal{U}^{(3)}_{1,3} \oplus \mathcal{U}_{1,1} \oplus \mathcal{U}_{0,1}$
$\mathcal{U}^{(2)}_{3,5} \oplus \mathcal{U}_{1,1} \oplus \mathcal{U}_{0,1}$	$\mathcal{U}^{(1)}_{2,3} \oplus \mathcal{U}_{2,2} \oplus \mathcal{U}^{2}_{0,1}$	$\mathcal{U}_{2,2} \oplus \mathcal{U}^{(1)}_{1,2} \oplus \mathcal{U}^{(1)}_{1,2} \oplus \mathcal{U}_{0,1}$
$\mathcal{U}_{3,4} \oplus \mathcal{U}^{(1)}_{1,3}$	$\mathcal{U}^{(2)}_{2,3} \oplus \mathcal{U}_{2,2} \oplus \mathcal{U}^{2}_{0,1}$	$\mathcal{U}_{2,2} \oplus \mathcal{U}^{(1)}_{1,2} \oplus \mathcal{U}^{(2)}_{1,2} \oplus \mathcal{U}_{0,1}$
$\mathcal{U}_{3,4} \oplus \mathcal{U}^{(2)}_{1,3}$	$\mathcal{U}^{(3)}_{2,3} \oplus \mathcal{U}_{2,2} \oplus \mathcal{U}^{2}_{0,1}$	$\mathcal{U}_{2,2} \oplus \mathcal{U}^{(2)}_{1,2} \oplus \mathcal{U}^{(2)}_{1,2} \oplus \mathcal{U}_{0,1}$
$\mathcal{U}_{3,4} \oplus \mathcal{U}^{(3)}_{1,3}$	$\mathcal{U}^{(1)}_{2,3} \oplus \mathcal{U}^{(1)}_{1,3} \oplus \mathcal{U}_{1,1}$	$\mathcal{U}_{2,2} \oplus \mathcal{U}^{(1)}_{1,2} \oplus \mathcal{U}_{1,1} \oplus \mathcal{U}_{0,2}$
$\mathcal{U}_{3,4} \oplus \mathcal{U}^{(1)}_{1,2} \oplus \mathcal{U}_{0,1}$	$\mathcal{U}^{(1)}_{2,3} \oplus \mathcal{U}^{(2)}_{1,3} \oplus \mathcal{U}_{1,1}$	$\mathcal{U}_{2,2} \oplus \mathcal{U}^{(2)}_{1,2} \oplus \mathcal{U}_{1,1} \oplus \mathcal{U}_{0,2}$
$\mathcal{U}_{3,4} \oplus \mathcal{U}^{(2)}_{1,2} \oplus \mathcal{U}_{0,1}$	$\mathcal{U}^{(1)}_{2,3} \oplus \mathcal{U}^{(3)}_{1,3} \oplus \mathcal{U}_{1,1}$	$\mathcal{U}_{2,2} \oplus \mathcal{U}^{(1)}_{1,2} \oplus \mathcal{U}_{1,1} \oplus \mathcal{U}^{2}_{0,1}$
$\mathcal{U}_{3,4} \oplus \mathcal{U}_{1,1} \oplus \mathcal{U}_{0,2}$	$\mathcal{U}^{(2)}_{2,3} \oplus \mathcal{U}^{(1)}_{1,3} \oplus \mathcal{U}_{1,1}$	$\mathcal{U}_{2,2} \oplus \mathcal{U}^{(2)}_{1,2} \oplus \mathcal{U}_{1,1} \oplus \mathcal{U}^{2}_{0,1}$
$\mathcal{U}_{3,4} \oplus \mathcal{U}_{1,1} \oplus \mathcal{U}^{2}_{0,1}$	$\mathcal{U}^{(2)}_{2,3} \oplus \mathcal{U}^{(2)}_{1,3} \oplus \mathcal{U}_{1,1}$	$\mathcal{U}_{2,2} \oplus \mathcal{U}^{2}_{1,1} \oplus \mathcal{U}_{0,3}$
$\mathcal{U}_{3,3} \oplus \mathcal{U}_{1,4}$	$\mathcal{U}^{(2)}_{2,3} \oplus \mathcal{U}^{(3)}_{1,3} \oplus \mathcal{U}_{1,1}$	$\mathcal{U}_{2,2} \oplus \mathcal{U}^{2}_{1,1} \oplus \mathcal{U}_{0,2} \oplus \mathcal{U}_{0,1}$
$\mathcal{U}_{3,3} \oplus \mathcal{U}^{(1)}_{1,3} \oplus \mathcal{U}_{0,1}$	$\mathcal{U}^{(3)}_{2,3} \oplus \mathcal{U}^{(1)}_{1,3} \oplus \mathcal{U}_{1,1}$	$\mathcal{U}_{2,2} \oplus \mathcal{U}^{2}_{1,1} \oplus \mathcal{U}^{3}_{0,1}$
$\mathcal{U}_{3,3} \oplus \mathcal{U}^{(2)}_{1,3} \oplus \mathcal{U}_{0,1}$	$\mathcal{U}^{(3)}_{2,3} \oplus \mathcal{U}^{(3)}_{1,3} \oplus \mathcal{U}_{1,1}$	$\mathcal{U}_{1,4} \oplus \mathcal{U}^{3}_{1,1}$
$\mathcal{U}_{3,3} \oplus \mathcal{U}^{(3)}_{1,3} \oplus \mathcal{U}_{0,1}$	$\mathcal{U}^{(3)}_{2,3} \oplus \mathcal{U}^{(2)}_{1,3} \oplus \mathcal{U}_{1,1}$	$\mathcal{U}^{(1)}_{1,3} \oplus \mathcal{U}^{(1)}_{1,2} \oplus \mathcal{U}^{2}_{1,1}$
$\mathcal{U}_{3,3} \oplus \mathcal{U}^{(1)}_{1,2} \oplus \mathcal{U}_{0,2}$	$\mathcal{U}^{(1)}_{2,3} \oplus \mathcal{U}^{(1)}_{1,2} \oplus \mathcal{U}^{(1)}_{1,2}$	$\mathcal{U}^{(1)}_{1,3} \oplus \mathcal{U}^{(2)}_{1,2} \oplus \mathcal{U}^{2}_{1,1}$
$\mathcal{U}_{3,3} \oplus \mathcal{U}^{(2)}_{1,2} \oplus \mathcal{U}_{0,2}$	$\mathcal{U}^{(1)}_{2,3} \oplus \mathcal{U}^{(1)}_{1,2} \oplus \mathcal{U}^{(2)}_{1,2}$	$\mathcal{U}^{(2)}_{1,3} \oplus \mathcal{U}^{(1)}_{1,2} \oplus \mathcal{U}^{2}_{1,1}$
$\mathcal{U}_{3,3} \oplus \mathcal{U}^{(1)}_{1,2} \oplus \mathcal{U}^{2}_{0,1}$	$\mathcal{U}^{(1)}_{2,3} \oplus \mathcal{U}^{(2)}_{1,2} \oplus \mathcal{U}^{(2)}_{1,2}$	$\mathcal{U}^{(2)}_{1,3} \oplus \mathcal{U}^{(2)}_{1,2} \oplus \mathcal{U}^{2}_{1,1}$
$\mathcal{U}_{3,3} \oplus \mathcal{U}^{(2)}_{1,2} \oplus \mathcal{U}^{2}_{0,1}$	$\mathcal{U}^{(2)}_{2,3} \oplus \mathcal{U}^{(1)}_{1,2} \oplus \mathcal{U}^{(1)}_{1,2}$	$\mathcal{U}^{(3)}_{1,3} \oplus \mathcal{U}^{(1)}_{1,2} \oplus \mathcal{U}^{2}_{1,1}$
$\mathcal{U}_{3,3} \oplus \mathcal{U}_{1,1} \oplus \mathcal{U}_{0,3}$	$\mathcal{U}^{(2)}_{2,3} \oplus \mathcal{U}^{(1)}_{1,2} \oplus \mathcal{U}^{(2)}_{1,2}$	$\mathcal{U}^{(3)}_{1,3} \oplus \mathcal{U}^{(2)}_{1,2} \oplus \mathcal{U}^{2}_{1,1}$
$\mathcal{U}_{3,3} \oplus \mathcal{U}_{1,1} \oplus \mathcal{U}_{0,2} \oplus \mathcal{U}_{0,1}$	$\mathcal{U}^{(2)}_{2,3} \oplus \mathcal{U}^{(2)}_{1,2} \oplus \mathcal{U}^{(2)}_{1,2}$	$\mathcal{U}^{(1)}_{1,3} \oplus \mathcal{U}^{3}_{1,1} \oplus \mathcal{U}_{0,1}$
$\mathcal{U}_{3,3} \oplus \mathcal{U}_{1,1} \oplus \mathcal{U}^{3}_{0,1}$	$\mathcal{U}^{(3)}_{2,3} \oplus \mathcal{U}^{(1)}_{1,2} \oplus \mathcal{U}^{(1)}_{1,2}$	$\mathcal{U}^{(2)}_{1,3} \oplus \mathcal{U}^{3}_{1,1} \oplus \mathcal{U}_{0,1}$
$\mathcal{U}^{(1)}_{2,5} \oplus \mathcal{U}_{2,2}$	$\mathcal{U}^{(3)}_{2,3} \oplus \mathcal{U}^{(1)}_{1,2} \oplus \mathcal{U}^{(2)}_{1,2}$	$\mathcal{U}^{(3)}_{1,3} \oplus \mathcal{U}^{3}_{1,1} \oplus \mathcal{U}_{0,1}$
$\mathcal{U}^{(2)}_{2,5} \oplus \mathcal{U}_{2,2}$	$\mathcal{U}^{(3)}_{2,3} \oplus \mathcal{U}^{(2)}_{1,2} \oplus \mathcal{U}^{(2)}_{1,2}$	$(\mathcal{U}^{(1)}_{1,2})^{3} \oplus \mathcal{U}_{0,1}$

$\mathcal{U}^{(1)}_{2,5} \oplus \mathcal{U}^{2}_{1,1}$	$\mathcal{U}^{(1)}_{2,3} \oplus \mathcal{U}^{(1)}_{1,2} \oplus \mathcal{U}_{1,1} \oplus \mathcal{U}_{0,1}$	$(\mathcal{U}^{(1)}_{1,2})^2 \oplus \mathcal{U}^{(2)}_{1,2} \oplus \mathcal{U}_{0,1}$
$\mathcal{U}^{(2)}_{2,5} \oplus \mathcal{U}^{2}_{1,1}$	$\mathcal{U}^{(1)}_{2,3} \oplus \mathcal{U}^{(2)}_{1,2} \oplus \mathcal{U}_{1,1} \oplus \mathcal{U}_{0,1}$	$\mathcal{U}^{(1)}_{1,2} \oplus (\mathcal{U}^{(2)}_{1,2})^2 \oplus \mathcal{U}_{0,1}$
$\mathcal{U}^{(1)}_{2,4} \oplus \mathcal{U}^{(1)}_{2,3}$	$\mathcal{U}^{(2)}_{2,3} \oplus \mathcal{U}^{(1)}_{1,2} \oplus \mathcal{U}_{1,1} \oplus \mathcal{U}_{0,1}$	$(\mathcal{U}^{(2)}_{1,2})^3 \oplus \mathcal{U}_{0,1}$
$\mathcal{U}^{(1)}_{2,4} \oplus \mathcal{U}^{(2)}_{2,3}$	$\mathcal{U}^{(2)}_{2,3} \oplus \mathcal{U}^{(2)}_{1,2} \oplus \mathcal{U}_{1,1} \oplus \mathcal{U}_{0,1}$	$\mathcal{U}^{(1)}_{1,2} \oplus \mathcal{U}^{3}_{1,1} \oplus \mathcal{U}_{0,2}$
$\mathcal{U}^{(1)}_{2,4} \oplus \mathcal{U}^{(3)}_{2,3}$	$\mathcal{U}^{(3)}_{2,3} \oplus \mathcal{U}^{(1)}_{1,2} \oplus \mathcal{U}_{1,1} \oplus \mathcal{U}_{0,1}$	$\mathcal{U}^{(2)}_{1,2} \oplus \mathcal{U}^{3}_{1,1} \oplus \mathcal{U}_{0,2}$
$\mathcal{U}^{(2)}_{2,4} \oplus \mathcal{U}^{(1)}_{2,3}$	$\mathcal{U}^{(3)}_{2,3} \oplus \mathcal{U}^{(2)}_{1,2} \oplus \mathcal{U}_{1,1} \oplus \mathcal{U}_{0,1}$	$\mathcal{U}^{(1)}_{1,2} \oplus \mathcal{U}^{3}_{1,1} \oplus \mathcal{U}^{2}_{0,1}$
$\mathcal{U}^{(2)}_{2,4} \oplus \mathcal{U}^{(2)}_{2,3}$	$\mathcal{U}^{(1)}_{2,3} \oplus \mathcal{U}^{2}_{1,1} \oplus \mathcal{U}_{0,2}$	$\mathcal{U}^{(2)}_{1,2} \oplus \mathcal{U}^{3}_{1,1} \oplus \mathcal{U}^{2}_{0,1}$
$\mathcal{U}^{(2)}_{2,4} \oplus \mathcal{U}^{(3)}_{2,3}$	$\mathcal{U}^{(2)}_{2,3} \oplus \mathcal{U}^{2}_{1,1} \oplus \mathcal{U}_{0,2}$	$(\mathcal{U}^{(1)}_{1,2})^2 \oplus \mathcal{U}^{2}_{1,1} \oplus \mathcal{U}_{0,1}$
$\mathcal{U}^{(1)}_{2,4} \oplus \mathcal{U}_{2,2} \oplus \mathcal{U}_{0,1}$	$\mathcal{U}^{(3)}_{2,3} \oplus \mathcal{U}^{2}_{1,1} \oplus \mathcal{U}_{0,2}$	$\mathcal{U}^{(1)}_{1,2} \oplus \mathcal{U}^{(2)}_{1,2} \oplus \mathcal{U}^{2}_{1,1} \oplus \mathcal{U}_{0,1}$
$\mathcal{U}^{(2)}_{2,4} \oplus \mathcal{U}_{2,2} \oplus \mathcal{U}_{0,1}$	$\mathcal{U}^{(1)}_{2,3} \oplus \mathcal{U}^{2}_{1,1} \oplus \mathcal{U}^{2}_{0,1}$	$(\mathcal{U}^{(2)}_{1,2})^2 \oplus \mathcal{U}^{2}_{1,1} \oplus \mathcal{U}_{0,1}$
$\mathcal{U}^{(1)}_{2,4} \oplus \mathcal{U}^{(1)}_{1,2} \oplus \mathcal{U}_{1,1}$	$\mathcal{U}^{(2)}_{2,3} \oplus \mathcal{U}^{2}_{1,1} \oplus \mathcal{U}^{2}_{0,1}$	$\mathcal{U}^{4}_{1,1} \oplus \mathcal{U}_{0,3}$
$\mathcal{U}^{(1)}_{2,4} \oplus \mathcal{U}^{(2)}_{1,2} \oplus \mathcal{U}_{1,1}$	$\mathcal{U}^{(3)}_{2,3} \oplus \mathcal{U}^{2}_{1,1} \oplus \mathcal{U}^{2}_{0,1}$	$\mathcal{U}^{4}_{1,1} \oplus \mathcal{U}_{0,2} \oplus \mathcal{U}_{0,1}$
$\mathcal{U}^{(2)}_{2,4} \oplus \mathcal{U}^{(1)}_{1,2} \oplus \mathcal{U}_{1,1}$	$\mathcal{U}^{2}_{2,2} \oplus \mathcal{U}_{0,3}$	$\mathcal{U}^{4}_{1,1} \oplus \mathcal{U}^{3}_{0,1}$
$\mathcal{U}^{(2)}_{2,4} \oplus \mathcal{U}^{(2)}_{1,2} \oplus \mathcal{U}_{1,1}$	$\mathcal{U}^{2}_{2,2} \oplus \mathcal{U}_{0,2} \oplus \mathcal{U}_{0,1}$	
$\mathcal{U}^{(1)}_{2,4} \oplus \mathcal{U}^{2}_{1,1} \oplus \mathcal{U}_{0,1}$	$\mathcal{U}^{2}_{2,2} \oplus \mathcal{U}^{3}_{0,1}$	

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