Splitting of power series spaces of (PLS) - type

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Introduction

In the late 60's and early 70's, Retakh and Palamodov published a number of articles which contained applications of methods from Homological Algebra to the theory of locally convex spaces (See [P1], [P2], [R]). Various properties as e.g. topological properties of projective and inductive limits, stability properties of subspaces and extension as well as lifing properties were described using homological notions and characterized by the vanishing of certain derived Functors like Ext^1 and Proj^1 in the considered categories.
The theory of the vanishing of Ext^1 in the category of Fréchet spaces was developed by Vogt in [V1] and [V2]. [V3] and [V4] contain characterizations of the vanishing of the first derived projective limit functor in the class of (LB)-spaces . In [D], [DV1], [DV2] and [DV3], Domański respectively Domański and Vogt developed splitting theories for the space of distributions and the space of smooth functions, in this way e.g. characterizing subspaces and quotients of the space of smooth functions. For further analytical applications we refer to the mentioned articles.
Other fields of study in which the application of the aforementioned homological methods is very useful are existence theorems for fundamental solutions on classes of ultradifferential functions as well as surjectivety problems of convolution operators and partial differential operators on these classes, which was done in [BMV1] and [BMV2].
In this thesis we will try to establish an approach to the splitting problem in the category of (PLB)-spaces, which are reduced projective limits of countable spectra of complete (LB) - spaces, focusing on the subcategory of (PLS)-spaces. The idea is to study spectra of (LB)-spaces and to approach the splitting problem (i.e. the problem of the vanishing of Ext^1) in this category in the way which was developed in [V1]. We will need to give generalizations of some central theorems known so far and we will have to study the settings in this category as opposed to the situation in the Fréchet case.
As an application, we will consider power series spaces of (PLB)-type and show that in some cases our general theorems can be used to characterize the splitting behaviour of these spaces in the category of (PLS)-spaces.
The motivation of studying these spaces may be that quite some classes of analytic functions are isomorphic to sequence spaces of this type (see [BMV1] and [BMV2]).
Chapter 1 contains the fundamental notions with which we will be concerned. We introduce projective spectra and the functors Proj^0, Proj^1 and Ext^1 in the category of (PLB)-spaces, and state the theorems of Vogt, Retakh-Palamodov and Frerick-Wengenroth, of which we will later on try to give suitable generalisations useful for the spaces we will be considering.
Chapter 2 describes the connection between the functors Ext^1 and Proj^1 in the category of (PLB)-spaces, giving us the possibility to describe the vanishing of Ext^1 by the vanishing of Proj^1, a tool most useful in the theory of the functor Ext^1 as developed e.g. for Fréchet spaces by Vogt in [V1]. It also describes the setting in some special cases used later on.
For the vanishing of Proj^1 we will give a characterisation in chapter 3 in the spirit of the theorem of Retakh and Palamodov, which will be generalised to the class of webbed spaces.
In chapter 4 we furtheron give sufficient and necessary conditions for the fulfillment of the condition in the Retakh-Palamodov theorem, the motivation being the wish to have easier calculable conditions which may be used in applications.
We apply in chapter 5 the theory developed so far to (PLB)-sequence spaces defined by certain matrices and show the vanishing of Ext^1 for some of those spaces; we finally char-

acterise the splitting behaviour of sequence spaces E and F under the assumption that one of them is a stable Fréchet space and give an overview concerning the solved problems and open questions.

I wish to thank my supervisor Prof. Dr. D. Vogt for his support during the research for this thesis. I also wish to thank Prof. Dr. P. Domański of the university of Poznań who has been a great help to me during my two months stay at the university. A great part of the results in the last chapter were obtained during this stay. Further thanks I wish to give to Dr. Leonard Frerick, for his invaluable support and lots of fruitful discussions on the subject.

Contents

1 Preliminaries

1.1 Spectra of locally convex spaces and (PLB)-spaces

In this section we introduce (PLB)-spaces and (PLS)-spaces. We assume familiarity with the notions of (LB),(LS) and (LN)-spaces, which we will always consider as given by a countable union $E = \bigcup_{n=1}^{\infty} E_n$, where the E_n, $n \in I\!N$ are Banach spaces and $E_n \hookrightarrow E_{n+1}$, $n \in I\!N$, with a continuous (resp. in the (LS)-case compact and in the (LN)-case nuclear) embedding. Material on these spaces can be found in [FWL]. We start with the notion of a countable spectrum :

Definition 1.1 *A countable projective spectrum of locally convex spaces is a family* $\mathcal{X} = \left(X_N, \iota_N^{N+1}\right)_{N\in I\!N}$ *of locally convex spaces* $(X_N)_{N\in I\!N}$ *and continuous linear operators*

$$\iota_N^{N+1} : X_{N+1} \to X_N \ , N \in I\!N,$$

called connecting maps. For a given spectrum $\mathcal{X}$*, we consider the linear mapping*

$$\begin{array}{rccc} \psi & : & \prod_{N=1}^{\infty} X_N & \to & \prod_{N=1}^{\infty} X_N \\ & & (x_N)_{N\in I\!N} & \to & \left(\iota_N^{N+1} x_{N+1} - x_N\right)_{N\in I\!N} \end{array}$$

ψ *is continuous with respect to the product topology; the space*

$$\ker \psi = \{(x_N)_{N\in I\!N} \in \prod_{N=1}^{\infty} X_N : \iota_N^{N+1} x_{N+1} = x_N, N \in I\!N\}$$

equipped with the subspace topology of the product, will be denoted by $\mathrm{Proj}^0\mathcal{X}$*. If for all* $N \in I\!N$ *the image of* $\left(\mathrm{Proj}^0\mathcal{X}\right)$ *under the canonical mapping*

$$\begin{array}{rccc} \iota^N & : & \mathrm{Proj}^0\mathcal{X} & \to & X_N \\ & & (x_J)_{J\in I\!N} & \to & x_N \end{array}$$

is dense in X_N*, then we will call the projective limit reduced. The cokernel of* ψ*, i.e. the space*

$$\prod_{N=1}^{\infty} X_N \Big/ \ Im\ \psi$$

where $\mathrm{Im}\psi$ *denotes the space*

$$\left\{(y_N)_{N\in I\!N} \in \prod_{N=1}^{\infty} X_N : \exists (x_N)_{N\in I\!N} \in \prod_{N=1}^{\infty} X_N : \iota_N^{N+1} x_{N+1} - x_N = y_N \ , N \in I\!N\right\}$$

will be denoted by $\mathrm{Proj}^1\mathcal{X}$*.*

Remark:

i) We will not need any topological considerations concerning the space $\mathrm{Proj}^1\mathcal{X}$, because for a given spectrum $\mathcal{X}$ we will only need to know, when $\mathrm{Proj}^1\mathcal{X} = 0$, which is a question of a purely algebraic nature.

ii) As we will only be concerned with countable spectra, we will sometimes use merely the notion spectrum meaning in detail a countable one.

The notions of projective spectra and projective limits arise naturally in the theory of locally convex spaces, as e.g. every Fréchet space can be represented as the projective limit of a suitable projective spectrum of Banach spaces, and properties as e.g. being Schwartz or nuclear have their correspondence in the connecting maps being compact, respectively nuclear. Projective Spectra of (DF)-spaces have been studied in [V3] and [V4]; in [BMV1] it is shown that the surjectivity of convolution operators on Gevrey classes in $\mathbb{R}^{\mathbb{N}}$ is characterized by the vanishing of $\mathrm{Proj}^1\mathcal{X}$ for a certain spectrum $\mathcal{X}$ arising out of the properties of the operators studied. Further applications concerning the vanishing of Proj^1 for spectra of (DF) - spaces can be found in [BMV2] and [MTV].

It will be important to know under which conditions two different spectra have the same projective limit. A property ensuring this which we will often need uses the following notion:

Definition 1.2 *Two countable spectra*

$$\mathcal{X} = \quad X_1 \xleftarrow{\iota_1^2} X_2 \xleftarrow{\iota_2^3} \cdots$$

and

$$\mathcal{Y} = \quad Y_1 \xleftarrow{j_1^2} Y_2 \xleftarrow{j_2^3} \cdots$$

are called equivalent, if there are sequences of natural numbers $(K(N))_{N\in\mathbb{N}}$ and $(L(N))_{N\in\mathbb{N}}$ $\in \mathbb{N}^{\mathbb{N}}$ which are increasing and tend to infinity and continuous linear mappings

$$S_N : Y_{K(N)} \to X_N\,,\ N \in \mathbb{N}$$

and

$$T_N : X_{L(N)} \to Y_N\,,\ N \in \mathbb{N}$$

such that for all $N \in \mathbb{N}$

$$T_N \circ S_{L(N)} = \iota_N^{K(L(N))}$$

and

$$S_N \circ T_{K(N)} = j_N^{L(K(N))}.$$

The following proposition is easily verified:

Proposition 1.3 *The projective limits of two equivalent spectra are topologically isomorphic.*

(PLB)-spaces arise out of spectra of complete (LB)-spaces by taking the projective limit:

Definition 1.4 *A (PLB)-space is a locally convex space E which can be represented as the reduced projective limit of a countable spectrum $\left(E_N, \iota_N^{N+1}\right)_{N\in\mathbb{N}}$ of complete (LB)-spaces E_N, $N \in \mathbb{N}$. If the spaces E_N,$N \in \mathbb{N}$ are (LS) (resp. (LN)), we will call E a (PLS) (resp. (PLN)) -space.*

Examples of (PLB)-spaces are e.g. the space of real-analytic funcions on the real line or the spaces $\mathcal{D}'(\Omega)$ of C^∞-functions on an open set Ω in $I\!R^{I\!N}$. Of course every Fréchet space is a (PLB)-space, as it is a projective limit of a countable spectrum of Banach spaces.
The following Proposition, which was proved e.g. in [DV2], shows that for certain (PLB)-spaces the converse of 1.3 holds as well:

Proposition 1.5 *Two spectra of complete (LB)-spaces, having the same projective limit are equivalent.*

The proof is obtained by the following lemma on factorization of mappings between (PLB)-spaces. As we will use it in subsequent chapters, we will give a proof for the sake of completeness:

Lemma 1.6 *For a (PLB) space E, a complete (LB)-space F and $T \in L(E,F)$ the following holds:*

i) For a fixed $M_0 \in I\!N$ T factorizes through E_{M_0} (i.e. there is $\tilde{T} \in L(E_{M_0}, F)$ such that $T = \tilde{T} \circ \iota^{M_0}$) iff for all neighbourhoods of zero $V \subset F$ there is a neighbourhood of zero $U \subset E_{M_0}$, such that

$$T\left(\left(\iota^{M_0}\right)^{-1} U\right) \subset V$$

ii) There is $M_0 \in I\!N$, such that T factorises through E_{M_0}.

Proof: i) If there is a factorization, the inclusion follows from continuity . If the inclusion with the given quantifiers is satisfied, we get a factorisation by putting

$$\begin{array}{cccc} \tilde{T} & : & \iota^{M_0}(E) & \to & F \\ & & \iota^{M_0}x & \to & Tx \end{array}$$

The assumption gives that $\tilde{T}$ is well defined and continuous, and as the spectrum is reduced and F complete, we can extend $\tilde{T}$ onto E_{M_0}.
ii) Assume that there is no factorization. Then for every $M \in I\!N$ there is a neighborhood of zero $V_M \subset F$ such that for all neighbourhoods of zero $U_M \subset E_M$

$$T\left(\left(\iota^{M_0}\right)^{-1} U_M\right) \not\subset V_M.$$

As F is an (LB)-space, there is a neighborhood of zero V, and a sequence $(\lambda_M)_{M\in I\!N}$, such that

$$V \subset \bigcap_{M=1}^{\infty} \lambda_M V_M$$

(cf. [PCB, Proposition 8.3.5]). Furthermore there is a neighbourhood of zero $U \subset E$ such that $T(U) \subset V$. Keeping in mind that E is a closed subspace of $\prod_{M=1}^{\infty} E_M$, and that the ι^M, $M \in I\!N$ are the canonical projections, we find an $M_0 \in I\!N$ and a neighbourhood of zero $U_{M_0} \subset E_{M_0}$, such that $\left(\iota^{M_0}\right)^{-1} U_{M_0} \subset U$, giving the contradiction

$$T\left(\left(\iota^{M_0}\right)^{-1} U_{M_0}\right) \subset \lambda_{M_0} V_{M_0},$$

thus the proof is finished.

As mentioned, every Fréchet space is a (PLB)-space. We will need to know that analogously for every Fréchet-Schwartz space we can find an equivalent spectrum of (LS)-spaces:

Proposition 1.7 *Every Fréchet-Schwartz space is a (PLS)-space.*

Proof: For the proof we will use the following two facts on compact sets in Banach spaces: First, the theorem of Banach-Dieudonne gives that a subset K of a Banach space X is relatively compact iff it is contained in the closure of the absolutely convex hull of a sequence $(x_\nu)_{\nu\in\mathbb{N}}$ tending to zero in X. Second, a subset K of l_1 is relatively compact iff it is bounded and

$$\lim_{N\to\infty} \sup\left\{\sum_{\nu=N}^{\infty} |\lambda_\nu| : (\lambda_\nu)_{\nu\in\mathbb{N}} \in K\right\} = 0.$$

Let now $E = \text{Proj}^0\left(E_N, \iota_N^{N+1}\right)_{N\in\mathbb{N}}$ be a Fréchet-Schwartz space, i.e. the $E_N, N \in \mathbb{N}$ are Banach spaces and the mappings $\iota_N^{N+1} : E_{N+1} \to E_N$, $N \in \mathbb{N}$ are compact. We have to find an equivalent spectrum $\left(X_N, j_N^{N+1}\right)_{N\in\mathbb{N}}$ of (LS)-spaces such that for which we have $E = \text{Proj}^0\left(X_N, j_N^{N+1}\right)_{N\in\mathbb{N}}$. Fix $N \in \mathbb{N}$. The set $\iota_N^{N+1}B$, where B denotes the closed unit ball in E_{N+1}, is relatively compact in E_N, thus there is a sequence $(x_\nu)_{\nu\in\mathbb{N}}$ in E_N tending to zero such that

$$\iota_N^{N+1}B \subset \overline{\Gamma\left(\{x_\nu : \nu \in \mathbb{N}\}\right)}^{E_N}$$

For every $n \in \mathbb{N}$ we can find an increasing sequence $(\gamma_\nu^n)_{\nu\in\mathbb{N}}$ of real numbers such that

i) $\gamma_\nu^n > 1 \quad n, \nu \in \mathbb{N}$

ii) $\lim_{\nu\to\infty} \frac{\gamma_\nu^n}{\gamma_\nu^{n+1}} = 0 \quad n, \nu \in \mathbb{N}$

iii) $\lim_{\nu\to\infty} \gamma_\nu^n = \infty$ for all $n \in \mathbb{N}$

iv) $\lim_{\nu\to\infty} \gamma_\nu^n x_\nu = 0$ for all $n \in \mathbb{N}$

We put for all $n \in \mathbb{N}$

$$B_{N,n} := \overline{\Gamma\left(\{\gamma_\nu^n x_\nu : \nu \in \mathbb{N}\}\right)}^{E_N} = \left\{\sum_{\nu=1}^{\infty} \lambda_\nu \gamma_\nu^n x_\nu : (\lambda_\nu)_{\nu\in\mathbb{N}} \in B_{l_1}\right\}$$

and

$$X_{N,n} := \text{span } B_{N,n}.$$

Due to the theorem of Banach-Dieudonne, the $B_{N,n}$, $n \in \mathbb{N}$ are compact and thus Banach balls, so the $X_{N,n}$, $n \in \mathbb{N}$ equipped with the topology induced by the $B_{N,n}$ as closed unit balls are Banach spaces. For all $n \in \mathbb{N}$ we have $X_{N,n} \hookrightarrow X_{N,n+1}$ with a compact embedding, as $B_{N,n}$ is the closure of the absolutely convex hull of a sequence which tends to zero in $X_{N,n+1}$, thus the space

$$X_N := \bigcup_{n=1}^{\infty} X_{N,n}$$

is an (LS)- space. We will now show that the mapping $\iota_N^{N+1} : E_{N+1} \to E_N$ factorises through X_N. We have

$$\iota_N^{N+1} B \subset \overline{\Gamma\left(\{x_\nu : \nu \in I\!N\}\right)} \subset B_{N,1},$$

so $\iota_N^{N+1} : E_{N+1} \to X_N$ is continuous. $X_{N,n}$ being compact in E_N, we get

$$X_{N,n} \hookrightarrow E_N$$

for all $n \in I\!N$, so

$$X_N \hookrightarrow E_N,$$

which gives the factorisation. We have found (LS)-spaces X_N, $N \in I\!N$ and continuous linear mappings $j_N^{N+1} : X_{N+1} \to X_N$, $N \in I\!N$, such that the spectrum $\left(X_N, j_N^{N+1}\right)_{N \in I\!N}$ is equivalent to $\left(E_N, \iota_N^{N+1}\right)_{N \in I\!N}$, so according to 1.5

$$E = \operatorname{Proj}^0 \left(X_N, j_N^{N+1}\right)_{N \in I\!N}$$

and E is a (PLS)-space.

Finally we will use the theorem of Banach-Dieudonne to find for a given (PLS)-space E another equivalent spectrum of an easier form:

Proposition 1.8 *If*

$$\mathcal{E} = \; E_1 \xleftarrow{\iota_1^2} E_2 \xleftarrow{\iota_2^3} \cdots$$

is a countable spectrum of (LS)-spaces and E its projective limit, then there is an equivalent spectrum

$$\tilde{\mathcal{E}} = \; \tilde{E}_1 \xleftarrow{\tilde{\iota}_1^2} \tilde{E}_2 \xleftarrow{\tilde{\iota}_2^3} \cdots$$

that satisfies

$$\tilde{\iota}_N^{N+1} : \tilde{E}_{N+1,n} \to \tilde{E}_{N,n}$$

continuously for all $n, N \in I\!N$.

For the proof we shall need the following

Lemma 1.9 *If $X_1 \hookrightarrow X_2 \hookrightarrow X_3$ and $X \hookrightarrow Z$ are sequences of Banach spaces with compact embeddings, and if $\iota : Z \to X_3$ is a continuous linear mapping that maps X continuously into X_1, then there is a Banach space Y such that $X \hookrightarrow Y \hookrightarrow Z$ with compact embeddings and ι maps Y continuously into X_2, i.e. the following diagram commutes:*

$$\begin{array}{ccc} X_3 & \xleftarrow{\iota} & Z \\ \uparrow & & \uparrow \\ X_2 & \xleftarrow{\iota} & Y \\ \uparrow & & \uparrow \\ X_1 & \xleftarrow{\iota} & X \end{array}$$

Proof: According to the theorem of Banach-Dieudonne, there is a sequence $(x_n)_{n\in I\!N} \in Z^{I\!N}$ tending to zero in Z, such that

$$B_X \subset \overline{\Gamma\{x_n : n \in I\!N\}}^Z =: M.$$

We put

$$N := \overline{\Gamma\{\mu_n\, x_n : n \in I\!N\}}^Z$$

where $(\mu_n)_{n\in I\!N} \in (1,\infty)^{I\!N}$ is an increasing sequence tending to infinity such that $(\mu_n\, x_n)_{n\in I\!N}$ still tends to zero in Z. Then M and N satisfy the following properties:

- $M \subset N$ as

  $$M = \left\{ \sum_{n=1}^{\infty} \lambda_n\, x_n : (\lambda_n)_{n\in I\!N} \in B_{l_1} \right\},$$

  $$N = \left\{ \sum_{n=1}^{\infty} \nu_n\, \mu_n\, x_n : (\nu_n)_{n\in I\!N} \in B_{l_1} \right\},$$

  and as for $(\lambda_n)_{n\in I\!N} \in B_{l_1}$ we have

  $$\sum_{n=1}^{\infty} \lambda_n\, x_n = \sum_{n=1}^{\infty} \tfrac{\lambda_n}{\mu_n}\, \mu_n\, x_n$$

  with $\left(\frac{\lambda_n}{\mu_n}\right)_{n\in I\!N} \in B_{l_1}$.

- Due to the theorem of Banach-Dieudonne, N is compact in Z, so if $[N]$ denotes the span of N then $([N], \|\cdot\|_N)$, where $\|\cdot\|_N$ denotes the Minkowski functional of N, is a Banach space, and the embedding

  $$([N], \|\cdot\|_N) \hookrightarrow Z$$

  is compact.

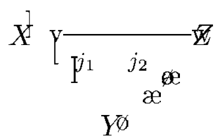
- The sequence $(x_n)_{n\in I\!N}$ tends to zero in $([N], \|\cdot\|_N)$, because for every $m \in I\!N$ we have $x_m \in \frac{1}{\mu_m} N$. Thus M is the closure of the absolutely convex hull of a sequence tending to zero in $([N], \|\cdot\|_N)$, so M is compact in $([N], \|\cdot\|_N)$.

We define the space Y as

$$Y := \{(x,z) \in X_2 \times [N] : x = \iota\, z\}$$

which is a closed subspace of $X_2 \times [N]$ and thus a Banach space. To finish the proof we will show that

1. The embedding $X \hookrightarrow Z$ factorizes over Y with compact embeddings, i.e. there is a commutative diagram

   where j_1 and j_2 are compact.

2. $\iota \circ j_2 : Y \to X_3$ acts continuously into X_2.

To show 1.), we define

$$\begin{array}{rcccl} j_1 & : & X & \to & Y \\ & & x & \to & (\iota\, x, x) \end{array}$$

and

$$\begin{array}{rcccl} j_2 & : & Y & \to & Z \\ & & (x,z) & \to & z. \end{array}$$

j_1 is compact as $\iota\,(X) \subset X_1$ and X_1 is embedded into X_2 by a compact map. Furthermore $X \subset [M]$ and M is relatively compact in [N]; obviously j_1 is injective.
j_2 is compact as $B_Y \subset B_{X_2} \times N$ and thus $j_2\, B_Y \subset N$ and N is compact in Z.
Furthermore it is obvious that $j_2 \circ j_1$ is the embedding of X into Z.
To show 2.), we observe that for $(x, z) \in Y$ we have

$$\iota j_2(x,z) = \iota z = x$$

according to the definition of Y, so $\iota \circ j_2\,(Y) \subset X_2$. That $\iota \circ j_2$ acts continuously into X_2 is implied by the form of the topology on Y. Thus the lemma is proved.

Proof of 1.8: The proof of 1.8 is now reached by inductively applying 1.9. We will show, how this is done for $\iota_1^2 : E_2 \to E_1$. First Grothendieck's factorisation theorem provides $(\gamma(n))_{n\in I\!N} \in I\!N^{I\!N}$ such that for all $n \in I\!N$ $\gamma(n) \geq n$ and ι_1^2 acts continuously from $E_{2,n}$ into $E_{1,\gamma(n)}$. Now using 1.9, we may fill in spaces between $E_{2,n}$ and $E_{2,n+1}$, such that we get the desired spectrum.

1.2 Fundamental known splitting results

This section will be devoted to providing a short introduction to those parts of the theory of the functor Ext^1 in the category of (PLB)-spaces which will be relevant in the following chapters. It will contain the algebraic notions that are used to describe the splitting of short exact sequences of (PLB)-spaces as well as a look on the results already known for the category of Fréchet spaces. These results will be the motivation of our investigation in the next chapters, which will contain some generalizations to the class of webbed spaces.

Definition 1.10 *If we are given (PLB)-spaces E and F, then we will write*

$$\mathrm{Ext}^1_{(PLB)}\,(E,F) = 0,$$

whenever every short exact sequence of (PLB)-spaces

$$0 \to F \to X \to E \to 0\;,$$

where F is a subspace and E a quotient of X, splits. (Here the arrows represent the canonical embedding, respectively the quotient map.) The corresponding meaning will be given to $\mathrm{Ext}^1_{(PLS)}\,(E,F) = 0$ *in the category of (PLS)-spaces and to* $\mathrm{Ext}^1_{(F)}\,(E,F) = 0$ *in the category of Fréchet spaces.*

Remark: Actually it would not be necessary to specify the category of Fréchet spaces, as being a Fréchet space is a three space property, meaning that in a short exact sequence with the subspace and the quotient being Fréchet, the middle space must be Fréchet; this is done merely to avoid misunderstanding.

Recently (in [DV1]) the subspaces and quotients of the space $\mathcal{D}'(\Omega)$ of distributions have been characterized by means of their splitting behaviour: For a (PLN)-space F the property $\mathrm{Ext}^1(F, \mathcal{D}'(\Omega)) = 0$, is equivalent to F being isomorpic to a subspace of $\mathcal{D}'(\Omega)$ and for F being ultrabornological (or equivalently, $\mathrm{Proj}^1\mathcal{F} = 0$ for the spectrum $\mathcal{F}$ defining F), being a quotient of $\mathcal{D}'(\Omega)$ is equivalent to $\mathrm{Ext}^1(\mathcal{D}'(\Omega), F) = 0$. Similiar results for the Fréchet space (s) of rapidly decreasing sequences have been obtained earlier by Vogt in [V2]. Thus the vanishing of Ext^1 has topological consequences for the spaces concerned, it can also help to solve the question of right inverses for partial differential operators (see e.g. [V2]). A thorough investigation on the Functor Ext^1 for Fréchet spaces can be found in [V1] and [V2].

When approaching the problem of the vanishing of Ext^1, a crucial observation is the connection to the spaces $\mathrm{Proj}^1\mathcal{X}$: If one is given (PLB)-spaces $E = \mathrm{Proj}^0\mathcal{E}$ and $F = \mathrm{Proj}^0\mathcal{F}$, with spectra $\mathcal{E} = \left(E_N, j_N^{N+1}\right)_{N\in I\!N}$ and $\mathcal{F} = \left(F_N, \iota_N^{N+1}\right)_{N\in I\!N}$, then we get a spectrum $\mathcal{L}(E,F) = \left(L(E,F_N), \iota^{*N+1}_N\right)_{N\in I\!N}$ by defining

$$\begin{array}{cccc} \iota^{*N+1}_N & : & L(E,F_N) & \to & L(E,F_{N+1}) \\ & & A & \to & \iota_N^{N+1}\circ A \end{array},$$

of which the projective limit will be the space of continuous linear maps L(E,F). Now in the case of E and F being Fréchet spaces there is the following description of $\mathrm{Ext}^1_{(F)}(E,F) = 0$ ([V1], Theorem 1.2):

> **Theorem A:** (Vogt) If $E = \mathrm{Proj}^0\mathcal{E}$ and $F = \mathrm{Proj}^0\mathcal{F}$ are reduced projective limits of spectra $\mathcal{E} = \left(E_N, j_N^{N+1}\right)_{N\in I\!N}$ and $\mathcal{F} = \left(F_N, \iota_N^{N+1}\right)_{N\in I\!N}$ of Banach spaces and if $\mathrm{Ext}^1_{(F)}(E, F_N) = 0$ holds for all $N \in I\!N$, then $\mathrm{Ext}^1_{(F)}(E,F) = 0$ iff $\mathrm{Proj}^1\mathcal{L}(E,F) = 0$.

Remark: In most applications the assumptions of the theorem are satisfied in a very natural way, for instance nuclear spaces F are given as projective limits of l_∞, for details see [V1].

In view of the last theorem it is natural to consider the splitting problem by seeking conditions for $\mathrm{Proj}^1\mathcal{L}(E,F) = 0$. A characterisation for the vanishing of $\mathrm{Proj}^1\mathcal{X}$ was given by Retakh and Palamodov in [R] and ([P2], theorem 5.4) in the following way:

> **Theorem B:** (Retakh - Palamodov) Let $\mathcal{X} = \left(X_N, \iota_N^{N+1}\right)_{N\in I\!N}$ be a countable projective spectrum and let each X_N, $N \in I\!N$ be covered by an increasing sequence of Banach balls such that for every $N \in I\!N$ ι_N^{N+1} maps one of the balls covering X_N into one of those covering X_{N+1}. Then $\mathrm{Proj}^1\mathcal{X} = 0$ iff for every $N \in I\!N$ there is a Banach ball $B_N \subset X_N$ such that

i) $\iota_N^{N+1} B_{N+1} \subset B_N$, $N \in I\!N$

ii) For all $N \in I\!N$ exists $M \in I\!N$ with $M \geq N$ such that for all $K \in I\!N$ with $K \geq M$ we have
$$\iota_N^M X_M \subset \iota_N^K X_K + B_N$$

Although this theorem provides a full characterisation, it is in general difficult to get calculable conditions for $\mathrm{Ext}^1(E, F) = 0$ in terms of the given spaces by it. In ([V4], Theorem 2.5), Vogt obtained conditions on the Banach balls for the vanishing of $\mathrm{Proj}^1 \mathcal{X}$, which could be translated into calculable conditions.

Theorem C: (Vogt) Let $\mathcal{X}$ be a countable spectrum of (LS)-spaces.

- $\mathrm{Proj}^1 \mathcal{X} = 0$ if the following holds:
  There is $n \in I\!N$ such that for all $N \in I\!N$ exist $M \in I\!N$ with $M \geq N$ such that for all $K \in I\!N$ with $K \geq M$ and all $m \in I\!N$ exist $k \in I\!N$ and $S > 0$ such that
$$\iota_N^M B_{M,m} \subset S \left(\iota_N^K B_{K,k} + B_{N,n} \right)$$
- $\mathrm{Proj}^1 \mathcal{X} = 0$ implies the following:
  For all $N \in I\!N$ exist $M \in I\!N$ with $M \geq N$ and $n \in I\!N$ such that for all $K \in I\!N$ with $K \geq M$ and all $m \in I\!N$ exist $k \in I\!N$ and $S > 0$ such that
$$\iota_N^M B_{M,m} \subset S \left(\iota_N^K B_{K,k} + B_{N,n} \right)$$

For Fréchet spaces there are the following sufficient and necessary conditions for the vanishing of $\mathrm{Proj}^1 \mathcal{L}(E, F)$ ([V1], Theorem 2.5):

Theorem D: (Vogt) Let E and F be Fréchet spaces.

i) $\mathrm{Proj}^1 \mathcal{L}(E, F) = 0$ if the following holds:
There is $n \in I\!N$ such that for all $N \in I\!N$ exists $M \in I\!N$ with $M \geq N$ such that for all $K \in I\!N$ with $K \geq M$, all $m \in I\!N$ and all $\varepsilon > 0$ there are $k \in I\!N$ and $S > 0$ such that
$$\iota^{*M}_{\ N} B_{L(E_m, F_M)} \subset S\, \iota^{*K}_{\ N} B_{L(E_k, F_K)} + \varepsilon\, B_{L(E_n, F_N)}$$

ii) $\mathrm{Proj}^1 \mathcal{L}(E, F) = 0$ implies the following: For all $N \in I\!N$ exists $M \in I\!N$ with $M \geq N$ and $n \in I\!N$ such that for all $K \in I\!N$ with $K \geq M$ and all $m \in I\!N$ there are $k \in I\!N$ and $S > 0$ such that
$$\iota^{*M}_{\ N} B_{L(E_m, F_M)} \subset S \left(\iota^* K_N B_{L(E_k, F_K)} + B_{L(E_n, F_N)} \right)$$

In [V1] these conditions are translated into ones containing an inequality between dual norms and thus calculable conditions $(S_1{}^*)$ for sufficiency and $(S_2{}^*)$ for neccessity for $\mathrm{Proj}^1 \mathcal{L}(E, F) = 0$ are obtained. Using these conditions leads to solutions for the splitting problem for Fréchet power series spaces ([V1], section 4) and to results on right inverses for elliptic differential operators with constant coefficients ([V1], section 7).

It was conjectured that the condition $(S_2{}^*)$ might also be sufficient, at least in cases important for applications. At first, this was shown by Krone and Vogt in [KV] for the case of E and F being Koethe sequence spaces. In [FW] the following theorem was proved, and using the means introduced in [V1], this leads to the result, that in the cases of E or F being nuclear as well as in the cases of E or F being Koethe spaces of the types $E = \lambda^1 (A)$ or $F = \lambda^\infty (B)$ the condition $(S_2{}^*)$ will also be sufficient. (For the precise definition of the spaces $\lambda^1 (A)$ or $\lambda^\infty (B)$, see [MV]).

> **Theorem E:** (Frerick-Wengenroth) Let $\mathcal{X} = \left(X_N, \iota_N^{N+1}\right)_{N \in I\!N}$ be a countable projective spectrum and let each X_N, $N \in I\!N$ be covered by a system $\mathcal{B}_N$ of Banach balls which is directed by inclusion and stable under multiplication with scalars. Further assume that for every $N \in I\!N$ ι_N^{N+1} maps each member of $\mathcal{B}_{N+1}$ into a member of $\mathcal{B}_N$. Then the following is sufficient for $\mathrm{Proj}^1 \mathcal{X} = 0$:
> For all $N \in I\!N$ exists $M \in I\!N$ with $M \geq N$ and $B_N \in \mathcal{B}_N$ such that for all $K \in I\!N$ with $K \geq M$ and all $D \in \mathcal{B}_M$ exists $C \in \mathcal{B}_K$ such that
>
> $$\iota_N^M (D) \subset \iota_N^K (C) + B_N$$

Summarizing, the problem of $\mathrm{Ext}^1_{(F)} (E, F)$ can under conditions which are often fulfilled in relevant cases be translated into the problem of the vanishing of $\mathrm{Proj}^1 \mathcal{L} (E, F)$, for which the theorem of Retakh and Palamodov gives a full characterisation. Using Theorem D and the theorem E from [FW] as well as the methods developed in [V1], it is possible to reach a characterisation for Fréchet spaces in the cases important for applications which is accessible for calculation. We will furtheron be concerned with generalisations of the methods described so far to the problem of $\mathrm{Ext}^1_{(PLS)} (E, F) = 0$.

2 Connections between splitting in the category of (PLS) -spaces and the vanishing of the first derived projective limit functor for spectra of spaces of continuous linear maps

In view of Theorem A in section 1.2, we will describe the connection between the problems of $\mathrm{Ext}^1_{(PLS)} (E, F) = 0$ and $\mathrm{Proj}^1 \mathcal{L} (E, F) = 0$ for (PLS)-spaces E and F. For this, we have to understand how the splitting of every short exact sequence

$$0 \to F \to G \to E \to 0$$

is related to the question of the surjectivity of

$$\begin{aligned} \psi^* \quad : \quad \prod_{N=1}^{\infty} L(E, F_N) \quad &\to \quad \prod_{N=1}^{\infty} L(E, F_N) \\ (A_N)_{N \in I\!N} \quad &\to \quad \left(\iota_N^{N+1} \circ A_{N+1} - A_N\right)_{N \in I\!N} \end{aligned}$$

the latter meaning $\mathrm{Proj}^1 \mathcal{L} (E, F) = 0$. For this we need a representation of the short exact sequence as a projective limit of short exact sequences of (LS)-spaces:

2.1 Local sequences and local splitting

Definition 2.1 *Let a subcategory $\mathcal{C}$ of the category of locally convex spaces be given. If for a given short exact sequence*

$$0 \to F \xrightarrow{j} G \xrightarrow{q} E \to 0$$

of locally convex spaces we have for every $N \in I\!N$ short exact sequences

$$0 \to F_N \xrightarrow{j_N} G_N \xrightarrow{q_N} E_N \to 0$$

of locally convex spaces such that $F = \mathrm{Proj}^0 F_N$, $G = \mathrm{Proj}^0 G_N$ and $E = \mathrm{Proj}^0 E_N$ (the projective limits being reduced), if the spaces F_N, G_N and E_N, $N \in I\!N$, belong to the category $\mathcal{C}$, and if for every $N \in I\!N$ we can set up a commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & \longrightarrow & F & \xrightarrow{j} & G & \xrightarrow{q} & E \longrightarrow 0 \\
 & & & \nearrow^{\iota^N,\ \iota^{N+1}} & & \nearrow^{\kappa^N,\ \kappa^{N+1}} & & \nearrow^{\sigma^N,\ \sigma^{N+1}} & \\
 & & \vdots & & \vdots & & \vdots & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & F_N & \xrightarrow{j_N} & G_N & \xrightarrow{q_N} & E_N & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & F_{N+1} & \xrightarrow{j_{N+1}} & G_{N+1} & \xrightarrow{q_{N+1}} & E_{N+1} & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}$$

(where ι^N, ι^{N+1}, κ^N, κ^{N+1}, σ^N and σ^{N+1} denote the canonical projections), we will say that we have local sequences for the sequence

$$0 \to F \xrightarrow{j} G \xrightarrow{q} E \to 0$$

in the category $\mathcal{C}$.

The following proposition from [DV1] ensures the existence of local sequences in the category of (PLS)-spaces:

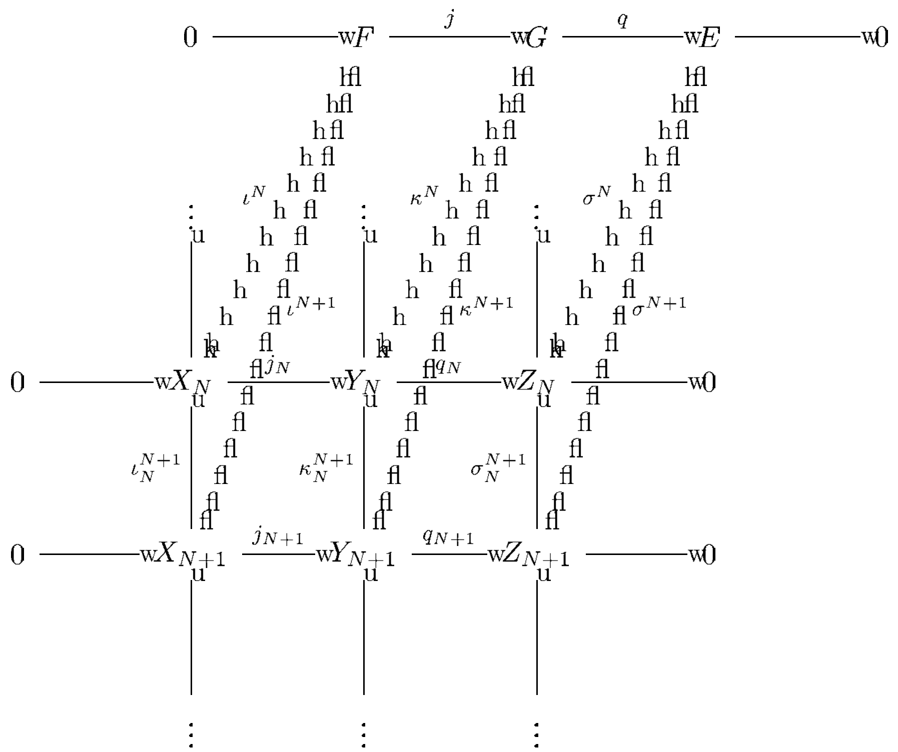
Proposition 2.2 *Let a short exact sequence*

$$0 \to F \xrightarrow{j} G \xrightarrow{q} E \to 0$$

of (PLS)-spaces $E = \mathrm{Proj}^0 E_N$, $G = \mathrm{Proj}^0 G_N$ *and* $F = \mathrm{Proj}^0 F_N$ *be given. Then there exist spectra* $\mathcal{X} = \left(X_N, \iota_N^{N+1}\right)_{N\in I\!N}$, $\mathcal{Y} = \left(Y_N, \kappa_N^{N+1}\right)_{N\in I\!N}$ *and* $\mathcal{Z} = \left(Z_N, \sigma_N^{N+1}\right)_{N\in I\!N}$ *of (LS)-spaces such that* $E = \mathrm{Proj}^0 X_N$, $G = \mathrm{Proj}^0 Y_N$ *and* $F = \mathrm{Proj}^0 Z_N$ *as well as short exact sequences*

$$0 \to X_N \stackrel{j_N}{\to} Y_N \stackrel{q_N}{\to} Z_N \to 0,\ N \in I\!N$$

such that the diagram



commutes for every $N \in I\!N$. *Either the sequence* $(Y_N)_{N\in I\!N}$ *can be chosen as a subsequence of* $(G_N)_{N\in I\!N}$ *or the sequences* $(X_N)_{N\in I\!N}$ *and* $(Z_N)_{N\in I\!N}$ *can be chosen as subsequences of* $(F_N)_{N\in I\!N}$ *and* $(E_N)_{N\in I\!N}$.

For the proof of this proposition we refer to [DV1, Remark after 1.7].
For a characterization of $\mathrm{Ext}^1_{(PLS)}(E, F) = 0$, we will need local sequences which split. This can be described in the following way:

Proposition 2.3 *Let a short exact sequence*

$$0 \to F \stackrel{j}{\to} G \stackrel{q}{\to} E \to 0$$

of (PLS) - spaces be given.Then the following are equivalent:

i) There are local sequences of (LS)-spaces

$$0 \to F_N \stackrel{j_N}{\to} G_N \stackrel{q_N}{\to} E_N \to 0$$

which split for every $N \in I\!N$.

ii) F can be represented by a spectrum $\left(F_N, \iota_N^{N+1}\right)_{N \in I\!N}$ of (LS) - spaces such that for every $N \in I\!N$ the canonical mapping $\iota^N : F \to F_N$ has an extension onto G, i.e. there exist mappings $I_N : G \to F_N$, $N \in I\!N$ such that for all $N \in I\!N$ we have $\iota^N = I_N \circ j$.

Proof: We will only show that ii) implies i) as the other direction is a straightforward application of the splitting of the local sequences. By 2.2 we get local sequences

$$0 \to \tilde{F}_N \overset{j_N}{\to} \tilde{G}_N \overset{q_N}{\to} E_N \to 0$$

with a spectrum $\left(\tilde{F}_N, \tilde{\iota}_N^{N+1}\right)_{N \in I\!N}$ representing F. As we already know that the two spectra $\left(F_N, \iota_N^{N+1}\right)_{N \in I\!N}$ and $\left(\tilde{F}_N, \tilde{\iota}_N^{N+1}\right)_{N \in I\!N}$ are equivalent, by using the push-out procedure (see Appendix), we get local sequences

$$0 \to F_N \overset{j_N}{\to} G_N \overset{q_N}{\to} E_N \to 0$$

(after maybe having to change the numeration). Consider $\iota^1 : F \to F_1$ and its lifting $I_1 : G \to F_1$. According to 1.6, there exists $M \in I\!N$ such that I_1 factorizes through G_M. We obtain a mapping $i_1 : G_M \to F_1$ and a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & F_1 & \overset{j_1}{\longrightarrow} & G_1 & \overset{q_1}{\longrightarrow} & E_1 & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle \iota_1^M} & \nwarrow{\scriptstyle i_1} & \uparrow{\scriptstyle \kappa_1^M} & & \uparrow{\scriptstyle \sigma_1^M} & & \\
0 & \longrightarrow & F_M & \overset{j_M}{\longrightarrow} & G_M & \overset{q_M}{\longrightarrow} & E_M & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}$$

using the pushout-construction (see Appendix), we obtain a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & F_1 & \overset{j_1}{\longrightarrow} & G_1 & \overset{q_1}{\longrightarrow} & E_1 & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle id} & & \uparrow{\scriptstyle S_1} & & \uparrow{\scriptstyle \sigma_1^M} & & \\
0 & \longrightarrow & F_1 & \overset{\tilde{j}_1}{\longrightarrow} & \tilde{G}_1 & \overset{\tilde{q}_1}{\longrightarrow} & E_M & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle \iota_1^M} & \nwarrow{\scriptstyle i_1} & \uparrow{\scriptstyle T_1} & & \uparrow{\scriptstyle id} & & \\
0 & \longrightarrow & F_M & \overset{j_M}{\longrightarrow} & G_M & \overset{q_M}{\longrightarrow} & E_M & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}$$

such that $S_1 \circ T_1 = \kappa_1^M$. The mapping i_1 provides a mapping $r_1 : E_M \to \tilde{G}_1$ which lifts the identity (see Appendix), so the middle row splits. We put $\tilde{F}_1 := F_1$ and $\tilde{E}_1 := E_M$. Proceeding inductively, we find local sequences which split.

Definition 2.4 *If for a given short exact sequence*

$$0 \to F \stackrel{j}{\to} G \stackrel{q}{\to} E \to 0$$

of locally conves spaces one of the two equivalent conditions of 2.3 are fulfilled, we will refer to this situation as local splitting of the sequence

$$0 \to F \stackrel{j}{\to} G \stackrel{q}{\to} E \to 0$$

in the category $\mathcal{C}$.

In subsequent paragraphs we will investigate the splitting of certain (PLS)-sequence spaces. In the course of this we will need the following

Lemma 2.5 *Let E be a Fréchet-Schwartz space which is locally l_1, i.e. there is a countable reduced spectrum*

$$\mathcal{X} = l_1 \stackrel{\iota_1^2}{\longleftarrow} l_1 \stackrel{\iota_2^3}{\longleftarrow} l_1 \longleftarrow \ldots$$

such that $E = \mathrm{Proj}^0 \mathcal{X}$. Let

$$0 \to F \stackrel{j}{\to} G \stackrel{q}{\to} E \to 0$$

be a short exact sequence with (PLS)-spaces $F = \mathrm{Proj}^0 F_N$ and $G = \mathrm{Proj}^0 G_N$. Then the following holds:

i) There exist a reduced spectrum

$$X_1 \longleftarrow X_2 \longleftarrow \ldots$$

of locally convex spaces that satisfies $G = \mathrm{Proj}^0 X_N$ and sequences

$$0 \to F_N \stackrel{w_N}{\to} X_N \stackrel{p_N}{\to} l_1 \to 0, \quad N \in I\!N$$

which are local sequences in the category of locally convex spaces and which split. The space X_N is a subspace of $\tilde{X}_N \times l_1$ with a suitable (LS)-space $\tilde{X}_N$.

ii) The sequence

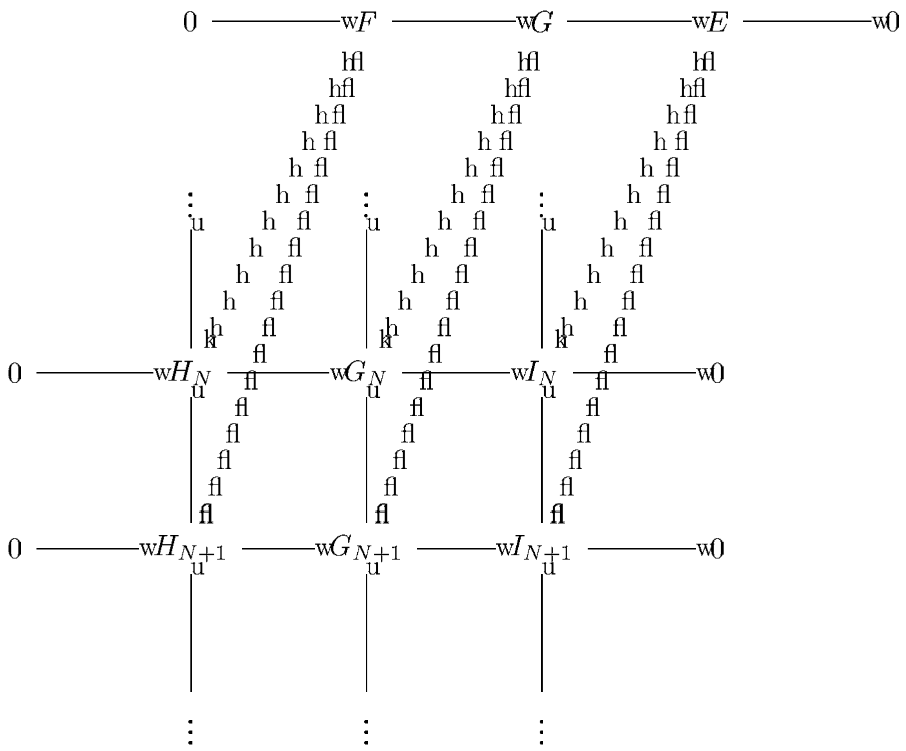
$$0 \to F \stackrel{j}{\to} G \stackrel{q}{\to} E \to 0$$

splits locally in the category of (PLS) -spaces.

Proof: We first prove i). Putting for every $N \in I\!N$

$$H_N := \overline{F}^{G_N} \quad \text{and} \quad I_N := G_N / H_N \ ,$$

we get as in [DV1] a commutative diagram



where the rows are exact and the columns are reduced spectra of (LS)-spaces such that $F = \mathrm{Proj}^0 H_N$ and $E = \mathrm{Proj}^0 I_N$. We know by 1.6 that any two defining spectra of complete (LB)-spaces for a (PLB)-space are equivalent, so the spectra

$$I_1 \longleftarrow I_2 \longleftarrow I_3 \longleftarrow \ldots$$

and

$$l_1 \longleftarrow l_1 \longleftarrow l_1 \longleftarrow \ldots$$

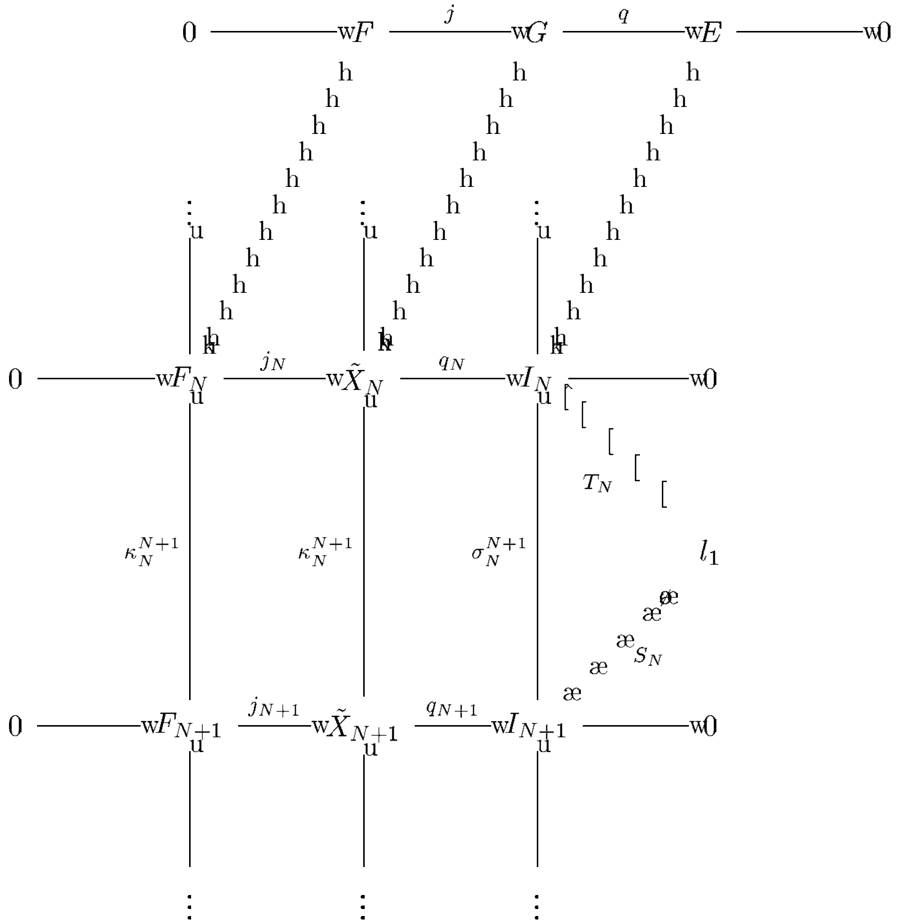
are equivalent, as well as the spectra

$$H_1 \longleftarrow H_2 \longleftarrow H_3 \longleftarrow \ldots$$

and

$$F_1 \longleftarrow F_2 \longleftarrow F_3 \longleftarrow \ldots$$

Inserting the F_N, $N \in I\!N$ into the diagram above by push-out constructions (see Appendix) we get (after maybe having to change the numeration) a diagram



where the $\tilde{X}_N$, $N \in I\!N$, are (LS)-spaces (this category is closed with respect to finite products, closed subspaces and quotients). Now the pullback construction gives for every $N \in I\!N$ a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & F_N & \xrightarrow{j_N} & \tilde{X}_N & \xrightarrow{q_N} & I_N & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle id} & & \uparrow{\scriptstyle u_N} & & \uparrow{\scriptstyle T_N} & & \\
0 & \longrightarrow & F_N & \xrightarrow{w_N} & X_N & \xrightarrow{p_N} & l_1 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \kappa_N^{N+1}} & & \downarrow{\scriptstyle v_N} & & \downarrow{\scriptstyle S_N} & & \\
0 & \longrightarrow & F_{N+1} & \xrightarrow{j_{N+1}} & \tilde{X}_{N+1} & \xrightarrow{q_{N+1}} & I_{N+1} & \longrightarrow & 0
\end{array}$$

where

$$X_N := \Big\{(x,u) \in \tilde{X}_N \times l_1 : q_N x = T_N u\Big\}$$

and the mappings u_N, p_N, w_N and v_N are defined as follows:

$$u_N(x,u) := x, \quad p_N(x,u) := u, \quad w_N(f) := (j_N f, 0)$$

and

$$v_N(\tilde{x}) := \left(\iota_N^{N+1}\tilde{x}, S_N q_{N+1}\tilde{x}\right).$$

The spectrum

$$\mathcal{X} := \quad X_1 \xleftarrow{v_1\circ u_2} X_2 \xleftarrow{v_2\circ u_3} X_3 \longleftarrow \cdots$$

is equivalent to the spectrum

$$\tilde{X}_1 \xleftarrow{\kappa_1^2} \tilde{X}_2 \xleftarrow{\kappa_2^3} \tilde{X}_3 \longleftarrow \cdots,$$

so we have by 1.3:

$$X \simeq \mathrm{Proj}^0 \mathcal{X}.$$

It remains to find right inverses for the mappings p_N, $N \in I\!N$. The idea when showing their existence is that a quotient map onto l_1 has a right inverse iff it lifts bounded sets. Fix $N \in I\!N$. From [DV1, Lemma 1.5] (see also [D, Theorem 2.3]), we get a compact Banach disk $C \subset \tilde{X_N}$, such that

$$B_{l_1} \subset p_N(C)$$

where B_{l_1} is the unit ball in l_1, and by finding a sequence $(c_\nu)_{\nu\in I\!N} \in C^{I\!N}$ such that for all $\nu \in I\!N$

$$e_\nu = p_N(c_\nu)$$

(e_ν, $\nu \in I\!N$, denoting the canonical unit vectors in l_1), we get a lifting $R_N : l_1 \to X_N$ for T_N by putting

$$R_N\left((t_\lambda)_{\lambda\in I\!N}\right) := \sum_{\lambda=1}^{\infty} t_\lambda c_\lambda.$$

The proof of ii) is now just one more application of the push-out construction: Consider for a fixed $N \in I\!N$ the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & F_N & \xrightarrow{w_N} & X_N & \xrightarrow{p_N} & l_1 & \longrightarrow & 0 \\
 & & \Big\uparrow \kappa_N^{N+1} & & \Big\uparrow v_N & & \Big\uparrow S_N & & \\
0 & \longrightarrow & F_{N+1} & \xrightarrow{j_{N+1}} & \tilde{X}_{N+1} & \xrightarrow{q_{N+1}} & l_{N+1} & \longrightarrow & 0
\end{array}$$

the push-out procedure gives a diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & F_N & \xrightarrow{w_N} & X_N & \xrightarrow{p_N} & l_1 & \longrightarrow & 0 \\
 & & \uparrow id & & \uparrow A_N & & \uparrow S_N & & \\
0 & \longrightarrow & F_N & \xrightarrow{\hat{j}_N} & \hat{X}_N & \xrightarrow{\hat{q}_N} & I_{N+1} & \longrightarrow & 0 \\
 & & \uparrow \kappa_N^{N+1} & & \uparrow B_N & & \uparrow id & & \\
0 & \longrightarrow & F_{N+1} & \xrightarrow{j_{N+1}} & \tilde{X}_{N+1} & \xrightarrow{q_{N+1}} & I_{N+1} & \longrightarrow & 0
\end{array}
$$

where $\hat{X}_N$ is an (LS) - space , because it is a quotient of $F_N \times \tilde{X}_{N+1}$. As the upper row splits, there is a left inverse $l_n : X_N \to F_N$ for w_N and we get a left inverse for $\hat{j}_N$ by putting $\hat{l}_N := l_N \circ A_N$.

2.2 The vanishing of $\mathrm{Ext}^1_{(PLS)}(E, F)$ in the case of local splitting

We want to generalize Vogt's Theorem A (see section 1.2) to the category of (PLS)-spaces. For this we first need to describe $\mathrm{Ext}^1_{(PLS)}(E, F) = 0$ in terms of lifting continuous linear maps $A : E \to \prod_{N=1}^{\infty} F_N$:

Theorem 2.6 *If for given (PLS)-spaces $E = \mathrm{Proj}^0 E_N$ and $F = \mathrm{Proj}^0 F_N$ we have local splitting for every short exact sequence*

$$0 \to F \xrightarrow{j} G \xrightarrow{q} H \to 0$$

then the following are equivalent:

i) $\mathrm{Ext}^1_{(PLS)}(E, F) = 0$

ii) For every diagram

$$
\begin{array}{ccccccc}
0 & \to & F & \xrightarrow{j} & G & \xrightarrow{q} & H \\
 & & & & & & \uparrow A \\
 & & & & & & E
\end{array}
$$

with (PLS)-spaces G and H and continuous linear maps q and A such that the upper row is exact and $\mathrm{Im}\, A \subset \mathrm{Im}\, q$, *there is a continuous linear map $\tilde{A} : E \to G$ such that $q \circ \tilde{A} = A$.*

iii) For every diagram

$$
\begin{array}{ccccccc}
0 & \to & F & \overset{\iota}{\hookrightarrow} & \prod_{N=1}^{\infty} F_N & \xrightarrow{\psi} & \prod_{N=1}^{\infty} F_N \\
 & & & & & & \uparrow A \\
 & & & & & & E
\end{array}
$$

with a continuous linear map A such that Im $A \subset$ Im ψ, *where* ψ *as in section 1.1 denotes the mapping*

$$\begin{array}{rcll}\psi & : & \prod_{N=1}^{\infty} F_N & \to \prod_{N=1}^{\infty} F_N \\ & & (x_N)_{N\in\mathbb{N}} & \to \left(\iota_N^{N+1} x_{N+1} - x_N\right)_{N\in\mathbb{N}}\end{array}$$

there is a continuous linear map $\tilde{A} : E \to \prod_{N=1}^{\infty} F_N$ *such that* $\psi \circ \tilde{A} = A$.

iv) For every commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & F & \overset{\iota}{\hookrightarrow} & \prod_{N=1}^{\infty} F_N & \overset{\psi}{\to} & \prod_{N=1}^{\infty} F_N & & \\ & & \uparrow \mathrm{id} & & \uparrow B & & \uparrow A & & \\ 0 & \to & F & \overset{j}{\to} & G & \overset{q}{\to} & E & \to & 0 \end{array}$$

where G is a (PLS)-space, A and B are continuous linear maps and both rows are exact, there is a continuous linear map $\tilde{A} : E \to \prod_{N=1}^{\infty} F_N$ *such that* $\psi \circ \tilde{A} = A$. *(Observe that commutativity and exactnes of the rows imply that* Im $A \subset$ Im ψ*).*

Remark: In ([V1], Theorem 1.8) a similiar version of *i)* $\Leftrightarrow$ *ii)* has been proved for Fréchet spaces, there it suffices to demand that the upper row be exact. Local splitting is needed for the implication of i) by ii),iii) or iv) whereas the other direction always holds. For Fréchet spaces we always have that ψ is surjective, this need not be the case for (PLS)-spaces, though a large class of spaces satisfy that property (in fact, it is equivalent to F being ultrabornological).

Proof of Theorem 2.6 : We only need to show that i) implies ii) and that iv) implies i), as the other implications are obvious. So assume $\mathrm{Ext}^1_{(PLS)}(E,F) = 0$, and let a diagram

$$\begin{array}{ccccccc} 0 & \to & F & \overset{j}{\to} & G & \overset{q}{\to} & H \\ & & & & & & \uparrow A \\ & & & & & & E \end{array}$$

as in ii) be given. Using the pullback construction (see Appendix), we get a commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & F & \overset{j}{\to} & G & \overset{q}{\to} & H & & \\ & & \uparrow \mathrm{id} & & \uparrow P_1 & & \uparrow A & & \\ 0 & \to & F & \to & X & \overset{P_2}{\to} & E & \to & 0 \end{array}$$

with a suitable (PLS)-space X such that the lower row is a short exact sequence of (PLS)-spaces (we have to keep in mind that finite products and closed subspaces of (PLS)-spaces are again (PLS)-spaces, this is shown in [DV1, Proposition 1.2]). Our assumption provides the existence of a right inverse $R : E \to X$ for P_2 and we put $\tilde{A} := P_1 \circ R$. For every $x \in X$ we get from commutativity

$$q\tilde{A}P_2x = qP_1RP_2x = AP_2RP_2x = AP_2x$$

and thus the surjectivety of P_2 gives $q\tilde{A} = A$.
Now assume iv) and consider a short exact sequence

$$0 \to F \overset{j}{\to} G \overset{q}{\to} E \to 0$$

with local sequences

$$0 \to F_N \overset{j_N}{\to} Y_N \overset{q_N}{\to} E_N \to 0 \quad , N \in I\!N.$$

Lifting for all $N \in I\!N$ the canonical maps $\iota^N : F \to F_N$ to $I_N : G \to F_N$ we get a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & F & \overset{\iota}{\longrightarrow} & \prod_{N=1}^{\infty} F_N & \overset{\psi}{\longrightarrow} & \prod_{N=1}^{\infty} F_N & & \\
 & & \uparrow id & & \uparrow B & \nwarrow \tilde{A} & \uparrow A & & \\
0 & \longrightarrow & F & \overset{j}{\longrightarrow} & G & \overset{q}{\longrightarrow} & E & \longrightarrow & 0
\end{array}$$

where B is defined by $B\,x := (I_N\,x)_{N \in I\!N}$, $x \in G$, A is the mapping between the quotients induced by B and $\tilde{A}$ is the lifting which exists due to our assumption. We get a left inverse for j by putting $L := \iota^{-1} \circ \left(B - \tilde{A} \circ q\right)$, which finishes the proof. Notice that in the proof that iv) implies i) we have shown the following

Lemma 2.7 *If for given (PLS)-spaces $E = \text{Proj}^0 E_N$ and $F = \text{Proj}^0 F_N$ we have local splitting for every exact sequence*

$$0 \to F \to G \to E \to 0,$$

then we can set up a diagram

$$\begin{array}{cccccccc}
0 & \to & F & \overset{\iota}{\hookrightarrow} & \prod_{N=1}^{\infty} F_N & \overset{\psi}{\to} & \prod_{N=1}^{\infty} F_N & \\
 & & \uparrow \text{id} & & \uparrow B & & \uparrow A & \\
0 & \to & F & \overset{j}{\to} & G & \overset{q}{\to} & E & \to \; 0
\end{array}$$

and we get a right inverse for q iff A has a lifting $\tilde{A}$ such that $\psi \circ \tilde{A} = A$.

With 2.6 we can describe the connection between the problem of $\text{Ext}^1_{(PLS)}(E, F) = 0$ and $\text{Proj}^1 \mathcal{L}(E, F) = 0$ as follows: Since $\text{Proj}^1 \mathcal{L}(E, F) = 0$ means that every map $A : E \to \prod_{N=1}^{\infty} F_N$ can be lifted to $\tilde{A} : E \to \prod_{N=1}^{\infty} F_N$ such that $\psi \circ \tilde{A} = A$, we get immediately from 2.6:

Corollary 2.8 *If for given (PLS)-spaces* $E = \mathrm{Proj}^0 E_N$ *and* $F = \mathrm{Proj}^0 F_N$ *we have local splitting for every exact sequence*

$$0 \to F \to G \to E \to 0,$$

then $\mathrm{Proj}^1 \mathcal{L}(E, F) = 0$ *implies* $\mathrm{Ext}^1_{(PLS)}(E, F) = 0$.

For the spectrum $\mathcal{F} = \left(F_N, \iota_N^{N+1}\right)_{N \in I\!N}$ $\mathrm{Proj}^1 \mathcal{F} = 0$ just means the surjectivity of the mapping ψ in iii) of theorem 2.6, so the condition $\mathrm{Im}\ A \subset \mathrm{Im}\ \psi$ is satisfied for every A. Thus we get as an analogon to Vogt's Theorem A of section 1.2:

Corollary 2.9 *Assume that for given (PLS)-spaces* $E = \mathrm{Proj}^0 E_N$ *and* $F = \mathrm{Proj}^0 F_N$ *we have local splitting for every exact sequence*

$$0 \to F \to G \to E \to 0,$$

and if $\mathrm{Proj}^1 \mathcal{F} = 0$ *for the spectrum* $\mathcal{F}$ *defining* F. *Then* $\mathrm{Ext}^1_{(PLS)}(E, F) = 0$ *is equivalent to* $\mathrm{Proj}^1 \mathcal{L}(E, F) = 0$

Remark: As was indicated earlier, the additional assumption $\mathrm{Proj}^1 \mathcal{F} = 0$ is in the category of (PLS) -spaces needed for the equivalence of the vanishing of Proj^1 and Ext^1. In fact it is obviously crucial because if ψ is not surjective, then this cannot be expected for ψ^*. In Theorem A this assumption does not appear as it is always fulfilled in the category of Fréchet spaces (cf. [V1, Lemma 1.1]).

Thus for the investigation of the splitting problem it will be very helpful to find conditions for $\mathrm{Proj}^1 \mathcal{L}(E, F) = 0$ in the spirit of the theorems B, C and D that have been introduced in section 1.2.
A necessary condition for local splitting is the following:

Proposition 2.10 *If* E *and* $F = \mathrm{Proj}^0 \left(F_N, \iota_N^{N+1}\right)_{N \in I\!N}$ *are (PLS)-spaces such that*

$$\mathrm{Ext}^1_{(PLS)}(E, F_N) = 0, \; N \in I\!N,$$

then we have local splitting for every exact sequence

$$0 \to F \to G \to E \to 0,$$

of (PLS) -spaces.

Proof: If we consider for every $N \in I\!N$ the diagram

$$\begin{array}{ccccccccc} & & F_N & & & & & & \\ & & \Big\uparrow{\scriptstyle \iota^N} & & & & & & \\ 0 & \longrightarrow & F & \xrightarrow{j} & G & \xrightarrow{q} & E & \longrightarrow & 0 \end{array}$$

we get by the pushout - construction a diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & F_N & \longrightarrow & \tilde{G} & \longrightarrow & E & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle \iota^N} & & \uparrow & & \uparrow{\scriptstyle id} & & \\
0 & \longrightarrow & F & \xrightarrow{j} & G & \xrightarrow{q} & E & \longrightarrow & 0
\end{array}$$

with a (PLS) - space $\tilde{G}$ (the class of (PLS) - spaces is closed with respect to products, closed subspaces and quotients), in which the upper row splits, thus we get an extension of ι^N onto G.

Often the assumption

$$\mathrm{Ext}^1_{(PLS)}(E, F_N) = 0, \; N \in I\!N,$$

is fulfilled in the way that

$$\mathrm{Ext}^1_{(LS)}(E_M, F_N) = 0, \; M, N \in I\!N,$$

which is a question for which the answer is accessible by dualisation through the splitting theory for Fréchet spaces. For the sake of completeness we give the following

Proposition 2.11 *If for given (PLS)-spaces* $E = \mathrm{Proj}^0 E_N$ *and* $F = \mathrm{Proj}^0 F_N$

$$\mathrm{Ext}^1_{(LS)}(E_M, F_N) = 0, \; M, N \in I\!N,$$

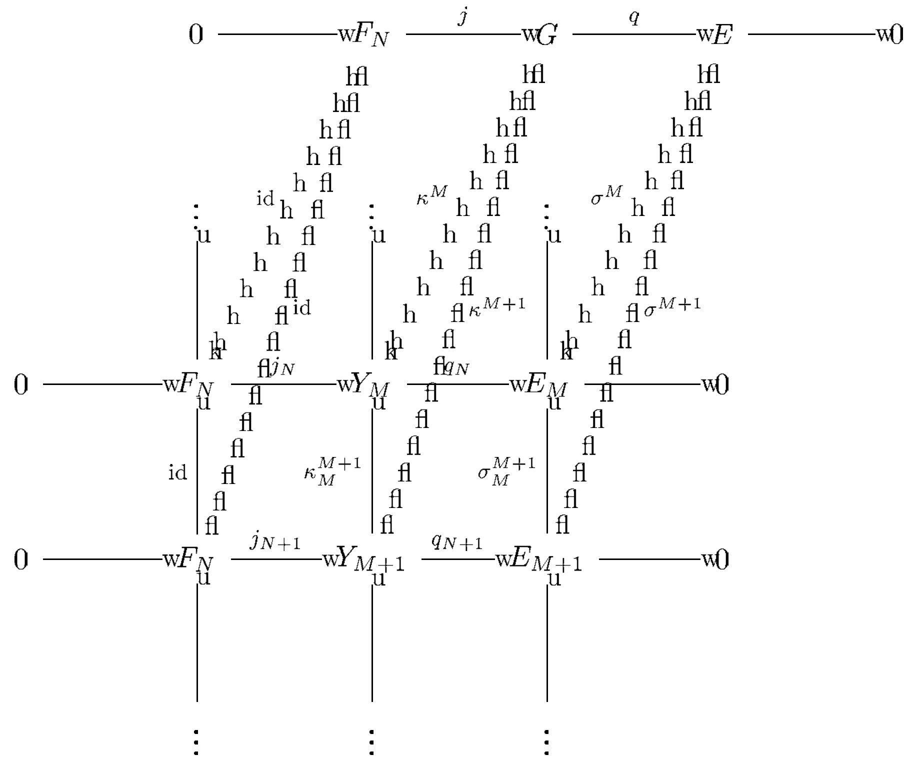
then we have

$$\mathrm{Ext}^1_{(PLS)}(E, F_N) = 0, \; N \in I\!N.$$

Proof: Let for a fixed $N \in I\!N$ a short exact sequence

$$0 \longrightarrow F_N \xrightarrow{j} G \xrightarrow{q} E \longrightarrow 0$$

with a (PLS)-space G be given. 2.2 gives us a commutative diagram



in which the local sequences split because of our assumption. If $l : Y_M \to F_N$ denotes the left inverse for j_N, we get a left inverse for j by putting $L := l \circ \kappa^M$.

2.3 The vanishing of $\mathrm{Ext}^1_{(PLS)}(E, F)$ for E being a Fréchet space

If E is locally l_1 , we can find a stricter version of 2.6. We will use the following notation:

Definition 2.12 *For a given spectrum*

$$\mathcal{X} \;=\; X_1 \xleftarrow{\iota_1^2} X_2 \xleftarrow{\iota_2^3} \cdots$$

we consider for all $K \in I\!N$ the canonical projections

$$p^K \;:\; \prod_{N=1}^{\infty} X_N \;\to\; \prod_{N=1}^{K} X_N$$

$$(x_N)_{N \in I\!N} \;\to\; (x_N)_{N=1}^{K}$$

and

$$p_K^{K+1} \;:\; \prod_{N=1}^{K+1} X_N \;\to\; \prod_{N=1}^{K} X_N$$

$$(x_N)_{N=1}^{K+1} \;\to\; (x_N)_{N=1}^{K}$$

giving a projective spectrum

$$\prod \mathcal{X} \;:=\; X_1 \xleftarrow{\;p_1^2\;} \prod_{N=1}^{2} X_N \xleftarrow{\;p_2^3\;} \prod_{N=1}^{3} X_N \xleftarrow{\;p_3^4\;} \cdots$$

of which $\prod\limits_{N=1}^{\infty} X_N$ *is the projective limit. With*

$$\begin{array}{rcccl} \psi & : & \prod\limits_{N=1}^{\infty} X_N & \to & \prod\limits_{N=1}^{\infty} X_N \\ & & (x_N)_{N\in \mathbb{N}} & \to & \left(\iota_N^{N+1} x_{N+1} - x_N\right)_{N\in \mathbb{N}} \end{array}$$

as in 1.1, we define for all $K \in \mathbb{N}$

$$\begin{array}{rcccl} \psi_K & : & \prod\limits_{N=1}^{K+1} X_N & \to & \prod\limits_{N=1}^{K} X_N \\ & & (x_N)_{N=1}^{K+1} & \to & \left(\iota_N^{N+1} x_{N+1} - x_N\right)_{N=1}^{K} \end{array}$$

The mapping corresponding to ψ *in the spectrum* $\prod \mathcal{X}$ *is*

$$\begin{array}{rcccl} p & : & \prod\limits_{K=1}^{\infty} \prod\limits_{N=1}^{K} X_N & \to & \prod\limits_{K=1}^{\infty} \prod\limits_{N=1}^{K} X_N \\ & & \left(x^K\right)_{K\in \mathbb{N}} & \to & \left(p_K^{K+1} x^{K+1} - x^K\right)_{K\in \mathbb{N}} \end{array}$$

Remark: It is easily verified and will be used in the subsequent calculations that for all $K \in \mathbb{N}$ we have $\psi_K p_{K+1}^{K+2} = p_K^{K+1} \psi_{K+1}$.

We begin with the following

Lemma 2.13 *Let* $F = \mathrm{Proj}^0\left(F_N, \iota_N^{N+1}\right)_{N\in \mathbb{N}}$ *be a (PLS)-space and let* E *be a Fréchet-Schwartz space which is locally* l_1*. Then the following are equivalent:*

i) $\mathrm{Ext}^1_{(PLS)}(E, F) = 0$

ii) For every increasing sequence $(\gamma_N)_{N\in \mathbb{N}} \in \mathbb{N}^{\mathbb{N}}$ *and every continuous linear map*

$$C : E \to F_{1,\gamma_1} \times \prod_{K=1}^{\infty} \left(\ker \psi_K \cap \prod_{N=1}^{K+1} F_{N,\gamma_N} \right)$$

for which there is a commutative diagram

$$(*) \qquad \begin{array}{ccccccccc} & & & & F_1 \times \prod\limits_{K=1}^{\infty} \ker \psi_K & \xrightarrow{\;p\;} & F_1 \times \prod\limits_{K=1}^{\infty} \ker \psi_K & & \\ & & & & \uparrow{\scriptstyle \tilde{C}} & & \uparrow{\scriptstyle C} & & \\ 0 & \longrightarrow & F & \longrightarrow & X & \longrightarrow & E & \longrightarrow & 0 \end{array}$$

where the lower row is a short exact sequence of (PLS) - spaces, there exists a lifting for C, i.e. a continuous linear mapping

$$D : E \to F_1 \times \prod_{K=1}^{\infty} \ker \psi_K$$

such that $p \circ D = C$.

Remark: As $\psi_K p_{K+1}^{K+2} = p_K^{K+1} \psi_{K+1}$, the Mapping p actually maps $F_1 \times \prod_{K=1}^{\infty} \ker \psi_K$ into itself.

Proof: If $\mathrm{Ext}^1_{(PLS)}(E, F) = 0$ and a diagram (*) as above is given, then we get a right inverse $R : E \to X$ in the lower row and we can define a lifting by $T := \tilde{C} \circ R$.
Assume now that ii) holds and let a short exact sequence

$$0 \longrightarrow F \longrightarrow X \stackrel{q}{\longrightarrow} E \longrightarrow 0$$

of (PLS) - spaces be given. We assume without loss of generality that we have $F_N = \bigcup_{n=1}^{\infty} B_{N,n}$. By 2.5 we have local splitting, thus according to 2.7 we can set up a diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & F & \longrightarrow & \prod_{N=1}^{\infty} F_N & \stackrel{\psi}{\longrightarrow} & \prod_{N=1}^{\infty} F_N & & \\
 & & \uparrow \mathrm{id} & & \uparrow \tilde{A} & & \uparrow A & & \\
0 & \longrightarrow & F & \longrightarrow & X & \stackrel{q}{\longrightarrow} & E & \longrightarrow & 0
\end{array}$$

for which we have to show that A has a lifting. We put for all $K \in I\!N$

$$A^K := p^K \circ A : E \to \prod_{N=1}^{K} F_N$$

For all $K \in I\!N$ we have a commutative diagram

$$\begin{array}{ccccc}
\prod_{N=1}^{K+1} F_N & & \stackrel{\psi_K}{\longrightarrow} & & \prod_{N=1}^{K} F_N \\
\uparrow p^{K+1} & & & \nearrow p^K & \\
\prod_{N=1}^{\infty} F_N & \stackrel{\psi}{\longrightarrow} & \prod_{N=1}^{\infty} F_N & & \uparrow A^K \\
 & & & \nwarrow A & \\
 & & & & E
\end{array}$$

and the first step will be to show that we can find a lifting for A^K, i.e. for every $K \in I\!N$ there is a continuous linear map

$$B^{K+1} : E \to \prod_{N=1}^{K+1} F_N$$

such that for all $K \in I\!N$ we have

$$\psi_K \circ B^{K+1} = A^K.$$

We know that the space $\prod_{N=1}^{\infty} F_N$ is strictly webbed [1] by

$$C_{\alpha_{|K}} := \prod_{N=1}^{K} B_{N,\alpha_N} \times \prod_{N=K+1}^{\infty} F_N \,,\; (\alpha_N)_{N \in I\!N} \in I\!N^{I\!N}, K \in I\!N.$$

If $(U_K)_{K \in I\!N}$ denotes a decreasing basis of neighbourhoods of zero in E, then de Wilde's localization theorem gives us strictly increasing sequences $(\gamma_N)_{N \in I\!N}$ and $(L_N)_{N \in I\!N} \in I\!N^{I\!N}$ such that for all $K \in I\!N$

$$A\left(U_{L_K}\right) \subset \psi\left(C_{\gamma_{|K+1}}\right).$$

As E is a Fréchet space, we get a strictly increasing sequence $(M_K)_{K \in I\!N} \in I\!N^{I\!N}$ such that for all $K \in I\!N$ A^K factorizes over E_{M_K}. We assume without loss of generality $M_K = L_K = K$, $K \in I\!N$. So we have for all $K \in I\!N$

$$\begin{aligned} A^K\left(U_K\right) &\subset p^K \psi\left(C_{\gamma_{|K+1}}\right) \\ &= \psi_K p^{K+1}\left(C_{\gamma_{|K+1}}\right) \\ &= \psi_K \left(\prod_{N=1}^{K+1} B_{N,\gamma_N}\right) \end{aligned}$$

and thus by lifting the canonical unit vectors, we can lift A^K to

$$B^{K+1} : E \to \prod_{N=1}^{K+1} F_{N,\gamma_N}$$

such that for all $K \in I\!N$

$$\begin{aligned} B^{K+1}\left(U_K\right) &\subset \prod_{N=1}^{K+1} B_{N,\gamma_N} \\ \text{and}\quad \psi_K B^{K+1} &= A^K. \end{aligned}$$

We define $B^1 := 0$, and use the $\left(B^K\right)_{K \in I\!N}$ to get a continuous linear mapping

$$C : E \to F_{1,\gamma_1} \times \prod_{K=1}^{\infty} \left(\ker \psi_K \cap \prod_{N=1}^{K} F_{N,\gamma_N}\right)$$

[1] For the definition of webbed spaces we refer to the section 3.1. The notion of a web is often used in section 3 wheras before it appears only at this point. It therefore seemed more consistent to state the definition and the elementary properties in section 3.

in the following way: We put

$$C^K := p_K^{K+1} B^{K+1} - B^K \, , \; K \in I\!N.$$

and we observe that for all $K \in I\!N$ we have

$$C^K (U_K) = p_K^{K+1} B^{K+1} (U_K) - B^K (U_K) \subset 2 \prod_{N=1}^{K} B_{N,\gamma_N}$$

and

$$\begin{aligned} \psi_K C^{K+1} &= \psi_K p_{K+1}^{K+2} B^{K+2} - \psi_K B^{K+1} \\ &= p_K^{K+1} \psi_{K+1} B^{K+2} - \psi_K B^{K+1} \\ &= p_K^{K+1} A^{K+1} - A^K \\ &= 0 \end{aligned}$$

and so by defining $C := \left(C^K\right)_{K \in I\!N}$, we get a continuous linear mapping as above. To complete (*), we need to construct $\tilde{C}$. Analogously as before, we define for all $K \in I\!N$

$$\tilde{A}^K := p^K \circ \tilde{A} : X \to \prod_{N=1}^{K} F_N$$

and get for all $K \in I\!N$ the following properties:

a) $p_K^{K+1} \tilde{A}^{K+1} = p^K \tilde{A} = \tilde{A}^K$

b) $\begin{aligned} \psi_K \tilde{A}^{K+1} &= \psi_K p^{K+1} \tilde{A} &= p^K \psi \tilde{A} \\ &= p^K A q &= A^K q \end{aligned}$

c) $\tilde{A}^{K+1} - B^{K+1} q \in \ker \psi_K$ because using b) and the lifting property of B^{K+1}, we get

$$\psi_K B^{K+1} q = A^K q = \psi_K \tilde{A}^{K+1}$$

We then define

$$\tilde{C} := \left(-\tilde{A}^1, \left(B^K q - \tilde{A}^K \right)_{K \geq 2} \right) : X \to F_1 \times \prod_{K=1}^{\infty} \ker \psi_K$$

and get $Cq = p\tilde{C}$ because of

$$\begin{aligned} C^K q &= p_K^{K+1} B^{K+1} q - B^K q \\ &= p_K^{K+1} \left(B^{K+1} q - \tilde{A}^{K+1} \right) - \left(B^K q - \tilde{A}^K \right) \\ &= p_K^{K+1} \tilde{C}^{K+1} - \tilde{C}^K \, , K \in I\!N \end{aligned}$$

where in the second equality we have used a) from above. Thus we have constructed a diagram (*) and the assumption that ii) holds, gives us a continuous linear map

$$D : E \to F_1 \times \prod_{K=1}^{\infty} \ker \psi_K$$

such that $p \circ D = C$, i.e. the mappings $D^K := p^K \circ D$, $K \in I\!N$ fulfill

$$p_K^{K+1} D^{K+1} - D^K = C^K = p_K^{K+1} B^{K+1} - B^K \,, K \in I\!N.$$

Now we can finally define the lifting for A: The continuous linear mappings

$$T^K := B^K - D^K : E \to \prod_{N=1}^{K} F_N \,, K \in I\!N$$

fulfill

$$p_K^{K+1} T^{K+1} = T^K \,,\ K \in I\!N$$

and can be written in the form

$$T^K = \left(T_1^K, \ldots, T_K^K\right).$$

With this notation we put

$$T := \left(T_K^K\right)_{K \in I\!N} : E \to \prod_{K=1}^{\infty} F_K.$$

and get $\psi \circ T = A$ because for all $K \in I\!N$

$$\begin{aligned} p^K \psi T &= \psi_K p^{K+1} T \\ &= \psi_K T^{K+1} \\ &= \psi_K B^{K+1} - \psi_K D^{K+1} \\ &= \psi_K B^{K+1} \\ &= A^K \\ &= p^K A. \end{aligned}$$

Thus the proof is finished.
We will now look for another description of the spaces appearing in condition ii) of 2.13. For this we use the following notions:

Definition 2.14 *Let a spectrum*

$$\mathcal{F} \;=\; F_1 \overset{\iota_1^2}{\hookleftarrow} F_2 \overset{\iota_2^3}{\hookleftarrow} \cdots$$

of (LB)-spaces,

$$F_N = \bigcup_{n=1}^{\infty} F_{N,n} \,,\ N \in I\!N,$$

a sequence $(\gamma_N)_{N \in I\!N} \in I\!N^{I\!N}$ and $K \in I\!N$ be given. We put

$$F_K^\gamma := \bigcap_{N=1}^{K} \left(\iota_N^K\right)^{-1} F_{N,\gamma_N}$$

and

$$B_K^\gamma := \bigcap_{N=1}^{K} \left(\iota_N^K\right)^{-1} B_{N,\gamma_N}$$

where $B_{N,n}$ denotes the unit ball in $F_{N,n}$, $n, N \in I\!N$.

The properties of the spaces F_K^γ which we will need are contained in the following

Proposition 2.15 *Let a spectrum $\mathcal{F}$ as in 2.14, $(\gamma_N)_{N\in\mathbb{N}} \in \mathbb{N}^{\mathbb{N}}$ and $K \in \mathbb{N}$ be given.*

i) *F_K^γ is a Banach space with unit ball B_K^γ and $F_K^\gamma \hookrightarrow F_{K,\gamma_K}$ with a continuous embedding.*

ii) *We have*

$$F_K = \bigcup_{\sigma\in\mathbb{N}^K} F_K^\sigma$$

and the topology of F_K is the inductive limit topology with respect to the embeddings $F_K^\sigma \hookrightarrow F_K$, $\sigma \in \mathbb{N}^{\mathbb{N}}$.

iii) *The continuous linear mappings*

$$\begin{array}{rcccl} s_K & : & F_{K+1}^\gamma & \to & \ker\psi_K \cap \prod_{N=1}^{K+1} F_{N,\gamma_N} \\ & & x & \to & \left(\iota_N^{K+1}x\right)_N^{K+1} \end{array}$$

and

$$\begin{array}{rcccl} S_K & : & F_{K+1} & \to & \ker\psi_K \\ & & x & \to & \left(\iota_N^{K+1}x\right)_N^{K+1} \end{array}$$

are isomorphisms.

iv) *The continuous linear mapping*

$$\begin{array}{rcccl} S & : & \prod_{K=1}^{\infty} F_K & \to & F_1 \times \prod_{K=1}^{\infty} \ker\psi_K \\ & & (x_K)_{K\in\mathbb{N}} & \to & (x_1, (S_K x_{K+1})_{K\in\mathbb{N}}) \end{array}$$

is an isomorphism and the diagram

$$\begin{array}{ccc} F_1 \times \prod_{K=1}^{\infty} \ker\psi_K & \xrightarrow{\ p\ } & F_1 \times \prod_{K=1}^{\infty} \ker\psi_K \\ \uparrow S & & \uparrow S \\ \prod_{K=1}^{\infty} F_K & \xrightarrow{\ \psi\ } & \prod_{K=1}^{\infty} F_K \end{array}$$

commutes.

Proof: i) is elementary as the B_{N,γ_N}, $N = 1,\dots,K$ are Banach balls and thus B_K^γ is a Banach ball and moreover the span of B_K^γ obviously equals F_K^γ. The continuous embedding then follows from $B_K^\gamma \subset B_{K,\gamma_K}$.
To prove ii), let $x \in F_K$ be given. Then there exists $\sigma_K \in I\!N$ such that $x \in F_{K,\sigma_K}$. According to Grothendiecks factorization theorem, there are natural numbers $\sigma_1, \dots, \sigma_{K-1}$, such that for $N = 1,\dots,K$ we have $\iota_N^K F_{K,\sigma_K} \subset F_{N,\sigma_N}$ and thus for a suitable $\sigma \in I\!N^{I\!N}$ $x \in F_K^\sigma$. As to the inductive limit topology, the claim follows from the fact that $F_K^\gamma \hookrightarrow F_K$ is continuous and that for every $n \in I\!N$ there exists $\sigma \in I\!N^{I\!N}$ and $\lambda > 0$ such that $B_{K,n} \subset \lambda B_K^\sigma$.
iii) follows from ii) and from the fact that the mapping $(x_N)_{N=1}^{K+1} \to x_{K+1}$ is a continuous inverse for s_K and S_K respectively.
For iv) we finally we have to check the commutativety of the given diagram. So let $(x_K)_{K\in I\!N} \in \prod_{K=1}^{\infty} F_K$ be given. Then

$$\begin{aligned}
pS(x_K)_{K\in I\!N} &= p\Big(\big(\iota_N^K x_K\big)_{N=1}^{K}\Big)_{K\in I\!N} \\
&= \Big(p_K^{K+1}\big(\iota_N^{K+1} x_{K+1}\big)_{N=1}^{K+1} - \big(\iota_N^K x_K\big)_{N=1}^{K+1}\Big)_{K\in I\!N} \\
&= \Big(\big(\iota_N^{K+1} x_{K+1} - \iota_N^K x_K\big)_{N=1}^{K}\Big)_{K\in I\!N} \\
&= S\big(\iota_K^{K+1} x_{K+1} - x_K\big)_{K\in I\!N} \\
&= S\psi(x_K)_{K\in I\!N}
\end{aligned}$$

which proves iv).

If we collect 2.13 and 2.15 we get the following condition for the vanishing of $\mathrm{Ext}^1(E,F)$:

Theorem 2.16 *Let $F = \mathrm{Proj}^0\big(F_N, \iota_N^{N+1}\big)_{N\in I\!N}$ be a (PLS)-space and let E be a Fréchet space which is locally l_1. Then the following are equivalent:*

i) $\mathrm{Ext}^1_{(PLS)}(E,F) = 0$

ii) For every increasing sequence $(\gamma_N)_{N\in I\!N} \in I\!N^{I\!N}$ and every continuous linear map

$$C : E \to \prod_{K=1}^{\infty} F_K^\gamma$$

for which there is a commutative diagram

$$\begin{array}{ccccccccc}
 & & & & \prod_{K=1}^{\infty} F_K & \xrightarrow{\psi} & \prod_{K=1}^{\infty} F_K & & \\
 & & & & \uparrow \tilde{C} & & \uparrow C & & \\
0 & \longrightarrow & F & \longrightarrow & X & \longrightarrow & E & \longrightarrow & 0
\end{array}$$

where the lower row is a short exact sequence of (PLS) - spaces, there exists a lifting for C.

3 The vanishing of the first derived projective limit functor for spectra of webbed spaces

The appropriate setting for a general formulation of the theorem of Retakh and Palamodov as needed here is the class of webbed spaces. Webbed spaces enjoy nice stability properties as this class is closed with respect to closed subspaces, quotients, and projective as well as inductive limits. Thus all the spaces considered here are webbed, as Banach and Fréchet spaces are webbed in quite an easy way. Also the spaces of continuous linear operators which already appeared in our investigations in the last section will be shown to be webbed. As for the theorem of Retakh and Palamodov, the methods used for the proof (especially the abstract Mittag-Leffler method used by Palamodov in [P2]) demand just the properties being fulfilled by webbed spaces. Thus it is natural to prove the theorem in this abstract setting, we will use it later on the spaces $L(E, F_N)$, $N, \in I\!N$ where E is a (PLS)-space and $\mathcal{F} = \left(F_N, \iota_N^{N+1}\right)_{N \in I\!N}$ a spectrum of (LS)-spaces. First we will introduce the notion of a web in a locally convex space together with some preparatory lemmas needed later on.

3.1 Webs in locally convex spaces

Definition 3.1 *Let X be a locally convex space. A web in X is a family $(C_{\alpha_1,\ldots,\alpha_k})_{k,\alpha_1,\ldots,\alpha_k \in I\!N}$ of absolutely convex sets with the following properties:*

i) $\bigcup\limits_{\alpha=1}^{\infty} C_\alpha = X$

ii) $\bigcup\limits_{\alpha=1}^{\infty} C_{\alpha_1,\ldots,\alpha_k,\alpha} = C_{\alpha_1,\ldots,\alpha_k}$ *for all* $k, \alpha_1, \ldots, \alpha_k \in I\!N$

iii) For every sequence $(\alpha_k)_{k \in I\!N} \in I\!N^{I\!N}$ there is a sequence $(\lambda_k)_{k \in I\!N}$ of positive real numbers such that for every sequence $(x_k)_{k \in I\!N}$ in X with $x_k \in C_{\alpha_1,\ldots,\alpha_k}$, $k \in I\!N$ the series $\sum\limits_{k=1}^{\infty} \lambda_k x_k$ converges in X.

The web is called strict, if in iii) we additionally have for every $k_0 \in I\!N$

$$\sum_{k=k_0}^{\infty} \lambda_k x_k \in C_{\alpha_1,\ldots,\alpha_{k_0}}$$

The web is called ordered, if for all sequences $(\alpha_k)_{k \in I\!N}$ and $(\beta_k)_{k \in I\!N}$ of natural numbers that satisfy $\alpha_k \leq \beta_k$, $k \in I\!N$, the inclusion $C_{\alpha_1,\ldots,\alpha_k} \subset C_{\beta_1,\ldots,\beta_k}$ holds for all $k \in I\!N$.

For abbreviation we will use the following notation: For a sequence $(\alpha_k)_{k \in I\!N}$ and $j, k \in I\!N$ we set $\alpha_j, \ldots \alpha_k := \alpha_{|j,k}$ and $\alpha_1, \ldots \alpha_k := \alpha_{|k}$, thus $C_{\alpha_j,\ldots,\alpha_k}$ will be denoted by $C_{\alpha_{|j,k}}$ and $C_{\alpha_1,\ldots,\alpha_k}$ by $C_{\alpha_{|k}}$.

Examples for webbed spaces are all Fréchet spaces: For a Fréchet space E we can define a strict ordered web by

$$C_{\alpha_{|k}} := \bigcap_{j=1}^{k} \alpha_j U_j$$

where $U_1 \supset U_2 \supset \ldots$ is a fundamental system of neighbourhoods of zero for E. Closed subspaces and quotients of a webbed space are webbed by taking the intersections with the sets $C_{\alpha_{|k}}$, $k, \alpha_1, \ldots, \alpha_k \in \mathbb{N}$ respectively their images under the quotient map. If the original web is strict respectively ordered, these webs will be strict respectively ordered as well. For a countable family $(X_N)_{N \in \mathbb{N}}$ of webbed spaces with strict ordered webs $\mathcal{C}^N = \left(C^N_{\alpha_{|k}}\right)_{k,\alpha_1,\ldots,\alpha_k \in \mathbb{N}}$, $N \in \mathbb{N}$ one gets a strict ordered web on $\prod_{N=1}^{\infty} X_N$ by defining

$$C_{\alpha_{|k}} := \prod_{j=1}^{k} C^j_{\alpha_{|j,k}} \times \prod_{j=k+1}^{\infty} X_j$$

In general one gets in a slightly more complicated way product webs of spaces that are simply webbed, but as we will only work with strict ordered webs, it will be easier to use the form of the product web given above. For a thorough treatment of webbed spaces see e.g. [K].

The condition iii) in definition 3.1 will be important for using the Mittag-Leffler method in the proof of our general version of the Retakh-Palamodov theorem. We will refer to this property using the following notion:

Definition 3.2 *Let X be a locally convex space. A decreasing sequence $(A_k)_{k \in \mathbb{N}}$ of absolutely convex subsets of X is called completing, if there is a sequence $(\lambda_k)_{k \in \mathbb{N}}$ of nonnegative numbers such that for all $(x_k)_{k \in \mathbb{N}}$ in X satisfying $x_k \in A_k$, $k \in \mathbb{N}$ the series*

$$\sum_{k=1}^{\infty} \lambda_k x_k$$

converges. A completing sequence $(A_k)_{k \in \mathbb{N}}$ is called strict, if for all $k_0 \in \mathbb{N}$

$$\sum_{k=k_0}^{\infty} \lambda_k x_k \in A_{k_0}$$

Remark: Examples for completing sequences which we will need come naturally from a given web $\left(C_{\alpha_{|k}}\right)_{k,\alpha_{|k} \in \mathbb{N}}$ on a locally convex space: If we fix a sequence $(\alpha_k)_{k \in \mathbb{N}}$ of natural numbers, then property ii) and iii) in the definition of webs give that the sequence $(A_k)_{k \in \mathbb{N}}$ defined by $A_k = C_{\alpha_{|k}}$ is completing, if the web is strict, then $(A_k)_{k \in \mathbb{N}}$ also is strict. The following proposition is easy to prove, we state it for later reference:

Proposition 3.3 *Let $f : X \to Y$ and $g : Y \to Z$ be continuous maps between locally convex spaces X, Y and Z with completing sequences $\left(A^X_k\right)_{k \in \mathbb{N}}$, $\left(A^Y_k\right)_{k \in \mathbb{N}}$ and $\left(A^Z_k\right)_{k \in \mathbb{N}}$. Then the sequence $(A_k)_{k \in \mathbb{N}}$ of subsets of Y defined by*

$$\left(g^{-1} A^Z_k\right) \cap \left(f\, A^X_k + A^Y_k\right)$$

is completing. If the sequences $\left(A^X_k\right)_{k \in \mathbb{N}}$, $\left(A^Y_k\right)_{k \in \mathbb{N}}$ and $\left(A^Z_k\right)_{k \in \mathbb{N}}$ are strict, then $(A_k)_{k \in \mathbb{N}}$ is strict.

In order to find conditions for $\mathrm{Proj}^1 \mathcal{X} = 0$ similiar to the ones in the theorems B, C, D and E, it will be necessary to construct complete metrisable group topologies from a given completing sequence on a locally convex space. In order to show that these topologies exist (proposition 3.5), we need to know that the sequence $(\lambda_k)_{k \in \mathbb{N}}$ in Definition 3.2 can be substituted by any absolutely convergent sequence:

Lemma 3.4 *If $(A_k)_{k\in\mathbb{N}}$ is a completing sequence in the locally convex space (X,σ), then the following holds:*

i) For all sequences $(\mu_k)_{k\in\mathbb{N}} \in l_1$ and $(x_k)_{k\in\mathbb{N}} \in \prod_{k=1}^{\infty} A_k$ the series $\sum_{k=1}^{\infty} \mu_k x_k$ converges in X.

ii) If $(A_k)_{k\in\mathbb{N}}$ is strict, then for all $(x_k)_{k\in\mathbb{N}} \in \prod_{k=1}^{\infty} A_k$ and $K \in \mathbb{N}$ we also have for all $\varepsilon > 0$

$$\sum_{k=K}^{\infty} \mu_k x_k \in \left(\|(\mu_k)_{k\geq K}\|_{l_1} + \varepsilon\right) A_K.$$

Proof: Let $(\lambda_k)_{k\in\mathbb{N}}$ be a sequence as in the definition of completing sequences. We first claim that for every neighborhood of zero $V \subset X$ there is a $t_0 \in \mathbb{N}$ such that for all $t \geq t_0$

$$\lambda_t A_t \subset V. \tag{1}$$

Assuming the contrary, we find a neighbourhood of zero V_0 and sequences $(t_k)_{k\in\mathbb{N}} \in \mathbb{N}^{\mathbb{N}}$ tending to infinity and $(x_k)_{k\in\mathbb{N}}$ satisfying $x_k \in A_{t_k}$, $k \in \mathbb{N}$, such that $\lambda_{t_k} x_k$ is not a member of V_0 for all $k \in \mathbb{N}$. We reach a contradiction by putting $\tilde{x}_t := x_k$ for $t = t_k$, $k \in \mathbb{N}$ and $\tilde{x}_t := 0$ for all other $t \in \mathbb{N}$; because the series

$$\sum_{t=1}^{\infty} \lambda_t \tilde{x}_t = \sum_{k=1}^{\infty} \lambda_{t_k} x_k$$

converges, so $(\lambda_{t_k} x_k)_{k\in\mathbb{N}}$ must be a null sequence and almost all $\lambda_{t_k} x_k$ must be contained in V_0.
Proceeding to the proof of i), we find a strictly increasing sequence $(k_t)_{t\in\mathbb{N}} \in \mathbb{N}^{\mathbb{N}}$ tending to infinity, such that for all $t \in \mathbb{N}$ we have $k_t \geq t$ and

$$\sum_{k=k_t}^{\infty} |\mu_k| < |\lambda_t|, \tag{2}$$

which implies for all $t \in \mathbb{N}$ and $S, T \geq k_t$

$$\sum_{k=S}^{T} \mu_k x_k \in \lambda_t A_{k_t}\ , \tag{3}$$

because A_k is absolutely convex for all $k \in \mathbb{N}$ and $x_k \in A_k \subset A_{k_t}$ for $k_t \leq\leq S \leq k \leq T$. From this we get first that

$$\sum_{k=k_t}^{k_{t+1}-1} \mu_k x_k \in \lambda_t A_{k_t} \quad \text{for all} \quad t \in \mathbb{N}. \tag{4}$$

and second (also using (1)) that $\sum_{k=k_1}^{\infty} \mu_k x_k$ is a cauchy sequence in X.
By our assumption on $(\lambda_k)_{k\in\mathbb{N}}$ the series

$$\sum_{t=1}^{\infty} \sum_{k=k_t}^{k_{t+1}-1} \mu_k x_k \tag{5}$$

converges, the limit of which we denote by x. Now $\sum_{k=k_1}^{\infty} \mu_k x_k$ also converges to x, because it is cauchy and the series in (5) is a convergent subsequence.
Remark: Notice that we have also proved that

$$\sum_{k=k_1}^{\infty} \mu_k x_k = \sum_{t=1}^{\infty} \sum_{k=k_t}^{k_{t+1}-1} \mu_k x_k$$

In full analogy to the previous we can also show that for every $N \in I\!N$

$$\sum_{k=k_N}^{\infty} \mu_k x_k = \sum_{t=N}^{\infty} \sum_{k=k_t}^{k_{t+1}-1} \mu_k x_k$$

Let us now consider the claim in ii): We fix $K \in I\!N$ and $\varepsilon > 0$ and for abbreviation we put $M_K := \|(\mu_k)_{k\geq K}\|_{l_1}$. We shall proceed analogously to the proof of i): We can find a sequence $(k_t)_{t\in I\!N}$ with $k_1 \geq K$ such that

$$\sum_{k=k_t}^{\infty} |\mu_k| < \varepsilon\, |\lambda_t|,\ t \in I\!N \tag{6}$$

and absolute convexity gives

$$\sum_{k=k_t}^{k_{t+1}-1} \mu_k x_k \in \varepsilon\, \lambda_t\, A_{k_t},\ t \in I\!N. \tag{7}$$

As before we can show

$$\sum_{k=k_1}^{\infty} \mu_k x_k = \sum_{t=1}^{\infty} \sum_{k=k_t}^{k_{t+1}-1} \mu_k x_k \tag{8}$$

and by our assumption on $(\lambda_k)_{k\in I\!N}$ together with (11) we get

$$\sum_{k=k_1}^{\infty} \mu_k x_k \in \varepsilon\, A_{k_1} \subset \varepsilon\, A_K \tag{9}$$

Moreover we know that $\sum_{k=K}^{k_1} |\mu_k| \leq M_K$, so absolute convexity gives

$$\sum_{k=K}^{k_1} \mu_k x_k \in M_K\, A_K, \tag{10}$$

which finally gives

$$\sum_{k=K}^{\infty} \mu_k x_k \in (M_K + \varepsilon)\ A_K,$$

completing our proof.

Proposition 3.5 *Let $(A_k)_{k\in I\!N}$ be a completing sequence in the separated locally convex space (X, σ). Then the sets $U_k := 2^{-(k+1)} A_k$,$k \in I\!N$ define a basis of neighborhoods of zero for a metrizable group topology τ under addition on X. If $(A_k)_{k\in I\!N}$ is strict, then (X, τ) is complete.*

Proof: Notice first that we get from the definition

$$U_{k+1} \subset \frac{1}{2}\, U_k, \quad \text{so} \quad U_{k+1} + U_{k+1} \subset U_k,\ k \in I\!N,$$

which implies that $(U_k)_{k\in I\!N}$ is a neighborhood basis of zero for a topology τ on X and addition is continuous on (X,τ). As metrizability is concerned, the fact that we have a countable basis for τ leaves us to show that (X,τ) is separated and regular. As to the haussdorf property, take an element $x \in X$, which is a member of U_k for all $k \in I\!N$. To show that $x = 0$, we write $x = 2^{-(k+1)}c_k$ with $c_k \in A_k$, $k \in I\!N$ and using Lemma 3.4 we get the convergence of

$$\sum_{k=1}^{\infty} 2^{-(k+1)} c_k = \sum_{k=1}^{\infty} x,$$

thus the constant sequence $(x)_{k\in I\!N}$ is a null sequence. To show regularity, we observe that each point has a neighborhood basis consisting of τ-closed sets, due to the fact that for all $k \in I\!N$

$$\overline{U_{k+1}}^{\tau} \subset U_{k+1} + U_{k+1} \subset U_k.$$

Now we assume that $(A_k)_{k\in I\!N}$ is strict and to prove completeness of (X,τ) we take a τ - Cauchy sequence $(x_k)_{k\in I\!N}$. Then we can find a strictly increasing sequence $(k_j)_{j\in I\!N}$ such that for all $j \in I\!N$ and $k \geq k_j$

$$x_k - x_{k_j} \in U_{2j} = 2^{-(2j+1)}\ A_{2j} \subset 2^{-(2j+1)}\ A_j = \frac{1}{2^j} U_j. \tag{1}$$

We write the x_{k_j}, $j \in I\!N$, as

$$x_{k_j} = \sum_{\nu=2}^{j} (x_{k_\nu} - x_{k_{\nu-1}}) + x_{k_1}.$$

From (1) we get

$$x_{k_\nu} - x_{k_{\nu-1}} \in 2^{-(2\nu-1)}\ A_{\nu-1},\ \nu \geq 2, \tag{2}$$

so by Lemma 3.4 i) the series $\sum_{\nu=2}^{\infty} x_{k_\nu} - x_{k_{\nu-1}}$ converges in (X,σ), and thus also $\left(x_{k_j}\right)_{j\in I\!N}$ has a limit point

$$x = \sum_{\nu=2}^{\infty} (x_{k_\nu} - x_{k_{\nu-1}}) + x_1.$$

in (X,σ). Now the web is strict, so by Lemma 3.4 ii) we may substitute the sequence $\left(2^{-(2\nu+1)}\right)_{\nu\in I\!N}$ for the arbitrary sequence $(\lambda_\nu)_{\nu\in I\!N}$ in the definition of a strict web. As for every $N \in I\!N$

$$\|\left(2^{-(2\nu+1)}\right)_{\nu\geq N}\|_{l_1} \leq 2^{-(N+1)},$$

(2) and Lemma 3.4 ii) imply for all $N \geq 2$

$$\sum_{\nu=N}^{\infty} x_{k_\nu} - x_{k_{\nu-1}} \in 2\,\|\left(2^{-(2\nu+1)}\right)_{\nu\geq N-1}\|_{l_1}\ A_{N-1}, \subset 2^{-(N-1)} A_{N-1} \tag{3}$$

This shows that $\left(x_{k_j}\right)_{j\in I\!N}$ also converges in (X,τ) to the limit x. Thus also $(x_k)_{k\in I\!N}$ converges to x in (X,τ) and the proof is finished.

3.2 The theorem of Retakh and Palamodov in the class of webbed spaces

We are now able to prove the theorem of Retakh and Palamodov in a more general setting than in the one given in Theorem B in 1.2.

Theorem 3.6 *If a projective spectrum* $\mathcal{X} = \left(X_N, \iota_N^{N+1}\right)_{N\in\mathbb{N}}$ *of webbed vector spaces with ordered webs* $\left(C^N_{\alpha_{|k}}\right)_{k,\alpha_{|k}\in\mathbb{N}}$ *is given, then the following are equivalent:*

1) $\mathrm{Proj}^1\mathcal{X} = 0$

2) For every $N \in \mathbb{N}$ *there is a completing sequence* $\left(A_k^N\right)_{k\in\mathbb{N}}$ *in* X_N *satisfying the conditions*

i) $\iota_N^K A_K^K \subset A_K^N$ *for all* $K, N \in \mathbb{N}$ *with* $K \geq N$.

ii) For every $N \in \mathbb{N}$ *there is* $M \in \mathbb{N}$ *with* $M \geq N$ *such that for all* $K \in \mathbb{N}$ *with* $K \geq M$ *we have*

$$\iota_N^M X_M \subset \iota_N^K X_K + A_N^N$$

3) There is a sequence $(\gamma_k)_{k\in\mathbb{N}}$ *and for every* $N \in \mathbb{N}$ *there is* $M \in \mathbb{N}$ *with* $M \geq N$ *such that for all* $K \in \mathbb{N}$ *with* $K \geq M$ *we have*

$$\iota_N^M X_M \subset \iota_N^K X_K + \bigcap_{J=1}^{N} \left(\iota_J^N\right)^{-1} C^J_{\gamma_{|J,N}}$$

If the spectrum is strictly webbed, these conditions are equivalent to

4) For every $N \in \mathbb{N}$ *there is a strict sequence* $\left(A_k^N\right)_{k\in\mathbb{N}}$ *satisfying*

i) $\iota_N^K A_K^K \subset A_K^N$ *for all* $K, N \in \mathbb{N}$ *with* $K \geq N$.

ii) For every $N \in \mathbb{N}$ *there is* $M \in \mathbb{N}$ *with* $M \geq N$ *such that*

$$\iota_N^M X_M \subset \iota^N X + A_N^N$$

5) There is a sequence $(\gamma_k)_{k\in\mathbb{N}}$ *and for every* $N \in \mathbb{N}$ *there is* $M \in \mathbb{N}$ *with* $M \geq N$ *such that for all* $K \in \mathbb{N}$ *with* $K \geq M$ *we have*

$$\iota_N^M X_M \subset \iota^N X + \bigcap_{J=1}^{N} \left(\iota_J^N\right)^{-1} C^J_{\gamma_{|J,N}}$$

Proof: a): First we show that 1) implies 3) by following the ideas of the original proof of Retakh (cf. [R, Theorem 3]). $\Pi = \prod_{N=1}^{\infty} X_N$ has an ordered web of the form

$$C_{\alpha_{|K}} := \prod_{J=1}^{K} C^J_{\alpha_{|J,K}} \times \prod_{J=K+1}^{\infty} X_J.$$

We define a complete metrizable group topology τ under addition on $\Pi = \prod_{N=1}^{\infty} X_N$ by defining as a basis of neighbourhoods of zero

$$U_0 := \Pi \text{ and } U_N := \prod_{J=1}^{N} \{0\} \times \prod_{J=N+1}^{\infty} X_j,\ N \in I\!N.$$

Now as $\mathrm{Proj}^1 \mathcal{X} = 0$, the mapping

$$\begin{array}{rlll} \psi & : & \Pi & \to \Pi \\ & & (x_N)_{N\in I\!N} & \to \left(\iota_N^{N+1} x_{N+1} - x_N\right)_{N\in I\!N} \end{array}$$

is surjective, so

$$\Pi = \bigcup_{\gamma=1}^{\infty} \psi\left(C_\gamma\right)$$

and as (Π, τ) is a Baire space, we get $\gamma_1 \in I\!N$, such that $\psi\left(C_{\gamma_1}\right)$ is of second category in Π, i.e. the closure has an interior point. Now

$$\psi\left(C_{\gamma_1}\right) = \bigcup_{\gamma=1}^{\infty} \psi\left(C_{\gamma_1,\gamma}\right),$$

so we get γ_2, such that $\psi\left(C_{\gamma_1,\gamma_2}\right)$ is of second category in Π. Inductively we can find a sequence $(\gamma_k)_{k\in I\!N}$ such that for all $N \in I\!N$

$$\psi\left(C_{\gamma_{|N}}\right) = \psi\left(\prod_{J=1}^{N} C^J_{\gamma_{|J,N}} \times \prod_{J=N+1}^{\infty} X_J\right)$$

is of second category in (Π, τ). By a simple convexity argument, we can show that for every $N \in I\!N$ the origin must also be an interior point of $\overline{\psi\left(C_{\gamma_{|N}}\right)}$: Fix $N \in I\!N$ and let x_0 be an interior point of $C := \overline{\psi\left(C_{\gamma_{|N}}\right)}$, then there exist a neighbourhood of zero U and another such one V such that $x_0 + U \subset C$ and $V + V \subset U$. Due to absolute convexity also

$$-x_0 + U = -(x_0 + U) \subset -C = C$$

and

$$C \supset \frac{1}{2}\ (x_0 + U) + \frac{1}{2}\ (-x_0 + U) \supset 0 + \frac{1}{2}\ U \supset 0 + \frac{1}{2}\ (V + V) \supset 0 + V.$$

This means that for all $N \in I\!N$ there is an $M \in I\!N$ with $M \geq N$ such that

$$\overline{\psi\left(\prod_{J=1}^{N} C^J_{\gamma_{|J,N}} \times \prod_{J=N+1}^{\infty} X_J\right)}^{\tau} \supset \prod_{J=1}^{M-1} \{0\} \times \prod_{J=M}^{\infty} X_J \tag{1}$$

which implies that for all $K \in I\!N$ with $K \geq M$ we have

$$\begin{aligned} \psi\left(\prod_{J=1}^{N} C^J_{\gamma_{|J,N}} \times \prod_{J=N+1}^{\infty} X_J\right) \quad &+ \quad \prod_{J=1}^{K} \{0\} \times \prod_{J=K+1}^{\infty} X_J \\ &\supset \quad \prod_{J=1}^{M-1} \{0\} \times \prod_{J=M}^{\infty} X_J \end{aligned} \tag{2}$$

Let now $x \in X_M$ be given, then define $\tilde{x} \in \prod_{J=1}^{M-1} \{0\} \times \prod_{J=M}^{\infty} X_J$ by

$$\tilde{x}_J := \begin{cases} 0 & : \quad J \leq M-1 \\ \text{x} & : \quad J = M \\ 0 & : \quad J \geq M+1 \end{cases} \tag{3}$$

Then the inclusion (2) provides the existence of $(y_J)_{J\in I\!N}$ such that $y_J \in C^J_{\gamma|J,N}$ for $J = 1, \ldots, N$ and of $(z_J)_{J\in I\!N}$ such that $z_J = 0$ for $J = 1, \ldots, K$ satisfying

$$(\tilde{x}_J)_{J\in I\!N} = \psi(y_J)_{J\in I\!N} + (z_J)_{J\in I\!N}$$

i.e. for all $J \in I\!N$

$$\tilde{x}_J = \iota_J^{J+1} y_{J+1} - y_J + z_J.$$

From this and from (3) we get the following:

$$\begin{aligned} x &= \iota_M^{M+1} y_{M+1} - y_M \\ \iota_J^{J+1} y_{J+1} &= y_J \qquad J = 1, \ldots, M-1,\; J = M+1, \ldots, K \\ y_J &\in C^J_{\gamma|J,N} \qquad J = 1, \ldots, N \end{aligned} \tag{4}$$

and thus finally

$$\begin{aligned} \iota_N^M x &= \iota_N^{M+1} y_{M+1} - \iota_N^M y_M \\ &= \iota_N^K y_K - y_N \end{aligned}$$

with $\iota_N^K y_K \in \iota_N^K X_K$ and $y_N \in \bigcap_{J=1}^{N} (\iota_J^N)^{-1} C^J_{\gamma|J,N}$. This now implies 3).

b): That 3) implies 2) is trivial if one puts for every $N \in I\!N$

$$A_k^N := \bigcap_{J=1}^{N} (\iota_J^N)^{-1} C^J_{\gamma|J,k},\; k \in I\!N.$$

c): The next step is to show that 2) implies 4); the implication 3) implies 5) is done in exactly the same way. To simplify our notation we will in the following assume that $M(N) = N+1$ so that we have

$$\iota_N^{N+1} X_{N+1} \subset \iota_N^{N+2} X_{N+2} + 2^{-(N+1)} A_N^N,\; N \in I\!N \tag{5}$$

and we will show that

$$\iota_N^{N+1} X_{N+1} \subset \iota^N X + A_N^N,\; N \in I\!N \tag{6}$$

(The proof of (6) for general sequences $(M(N))_{N\in I\!N}$ is exactly the same. Observe that as in the assumption two of the sets mentioned are linear spaces, we may multiply the inclusion with arbitrary scalars). So fix $N_0 \in I\!N$ and let $x \in X_{N_0+1}$ be given. We set $x_{N_0+1} := x$, and using (5), we inductively find sequences $(x_J)_{J\geq N_0+1}$ and $(v_J)_{J\geq N_0}$, such that we have $x_J \in X_J$, $J \geq N_0+1$, $v_J \in 2^{-(J+1)}A_J^J$, $J \geq N_0$ and

$$\iota_J^{J+1}x_{J+1} = \iota_J^{J+2}x_{J+2} + v_J,\ J \geq N_0. \tag{7}$$

Define for all $N \in I\!N$

$$y_N := \lim_{K\to\infty} \iota_N^K x_K$$

To show that the limit exists for all $N \in I\!N$, we write for $K > N > N_0$

$$\begin{aligned} \iota_N^{K+1}x_{K+1} &= \sum_{J=N}^{K-1}\left(\iota_N^{J+2}x_{J+2} - \iota_N^{J+1}x_{J+1}\right) + \iota_N^{N+1}x_{N+1} \\ &= \sum_{J=N}^{K-1}\iota_N^J\left(\iota_J^{J+2}x_{J+2} - \iota_J^{J+1}x_{J+1}\right) + \iota_N^{N+1}x_N \end{aligned} \tag{8}$$

Now from (7) we know that

$$\iota_J^{J+2}x_{J+2} - \iota_J^{J+1}x_{J+1} \in 2^{-(J+1)}A_J^J,\ J \geq N$$

so using the assumption in 2) ii) we get

$$\iota_N^{J+2}x_{J+2} - \iota_N^{J+1}x_{J+1} = \iota_N^J\left(\iota_J^{J+2}x_{J+2} - \iota_J^{J+1}x_{J+1}\right) \in 2^{-(J+1)}A_J^N,\ J \geq N.$$

As the $\left(A_J^N\right)_{J\in I\!N}$ form a strict sequence, and as $\|\left(2^{-(J+1)}\right)_{J\geq N}\|_{l_1} = 2^{-N}$, we conclude with the help of Lemma 3.4 that

$$\sum_{J=N}^{\infty}\iota_N^{J+2}x_{J+2} - \iota_N^{J+1}x_{J+1} \in 2^{-(N-1)}A_N^N \tag{9}$$

and

$$y_N = \lim_{J\to\infty}\iota_N^J x_J = \sum_{J=N}^{\infty}\iota_N^{J+2}x_{J+2} - \iota_N^{J+1}x_{J+1} + \iota_N^{N+1}x_{N+1}. \tag{10}$$

therefore exists. The continuity of the maps ι_N^{N+1} gives that $(y_\nu)_{\nu\in I\!N}$ is an element of $X = \mathrm{Proj}^0\mathcal{X}$, so if we show that

$$\iota_{N_0}^{N_0+1}x - \iota^{N_0}\left((y_N)_{N\in I\!N}\right) \in A_{N_0}^{N_0}$$

we have concluded that 4) holds. Indeed, by (9) and (10) we have

$$\begin{aligned} \iota_{N_0}^{N_0+1}x - y_{N_0} &= -\sum_{J=N_0}^{\infty}\iota_{N_0}^{J+2}x_{J+2} - \iota_{N_0}^{J+1}x_{J+1} \in -2^{-(N_0-1)}A_{N_0}^{N_0} \\ &= 2^{-(N_0-1)}A_{N_0}^{N_0} \subset A_{N_0}^{N_0} \end{aligned}$$

d): It remains to show that 4) implies 1) (the argument that 5) implies 4) is again trivial and the implication that 2) implies 1) is exactly the same, as no strictness is needed). By passing to an equivalent spectrum, and considering that two of the terms in the inclusion are linear spaces, we may assume that for all $N \in I\!N$

$$\iota_N^{N+1}X_{N+1} \subset \iota^N X + 2^{-(N+1)}A_N^N \tag{11}$$

We want to show that for given $(x_N)_{N\in\mathbb{N}} \in \prod_{N=1}^{\infty} X_N$, we can construct $(z_N)_{N\in\mathbb{N}} \in \prod_{N=1}^{\infty} X_N$, such that

$$\left(\iota_N^{N+1} z_{N+1} - z_N\right)_{N\in\mathbb{N}} = (x_N)_{N\in\mathbb{N}}$$

for this aim we construct inductively sequences $\left(y^N\right)_{N\in\mathbb{N}} \in X^{\mathbb{N}}$ and $(v_N)_{N\in\mathbb{N}}$ with $v_N \in 2^{-(N+1)} A_N^N$, $N \in \mathbb{N}$ in the following way: We set $y^1 := 0$. According to (11) there are $y^2 \in X$ and $v_1 \in 2^{-2} A_1^1$ such that

$$\iota_2^1 x_2 = \iota^1 y^2 + v_1.$$

In the next step there are $y^3 \in X$ and $v_2 \in 2^{-3} A_2^2$, such that

$$\iota_3^2 x_3 + \iota^2 y^2 = \iota^2 y^3 + v_2.$$

Proceeding inductively we get the aforementioned sequences fulfilling

$$\iota_N^{N+1} x_{N+1} + \iota^N y^N = \iota^N y^{N+1} + v_N.$$

We define

$$z_N := -x_N + \iota^N y^N - \sum_{J=N}^{\infty} \iota_N^J v_J,\ N \in \mathbb{N}$$

then we get for all $N \in \mathbb{N}$

$$\begin{aligned} \iota_N^{N+1} z_{N+1} - z_N &= -\iota_N^{N+1} x_{N+1} + x_N + \iota_N^{N+1} \iota^{N+1} y^{N+1} - \iota^N y^N \\ &\quad - \sum_{J=N+1}^{\infty} \iota_N^J v_J + \sum_{J=N}^{\infty} \iota_N^J v_J \\ &= -\iota_N^{N+1} x_{N+1} + x_N + \iota^{N+1} y^{N+1} - \iota^N y^N + v_N \\ &= x_N, \end{aligned}$$

thus the proof is complete.

From Theorem 3.6 one can easily set up a sufficient condition for $\mathrm{Proj}^1 \mathcal{X} = 0$ which we will use in section 5:

Corollary 3.7 *Assume that a given spectrum* $\left(X_N, \iota_N^{N+1}\right)_{N\in\mathbb{N}}$ *of webbed spaces with ordered webs* $\left(C_{\alpha_{|k}}^N\right)_{k,\alpha_{|k}\in\mathbb{N}}$ *fulfills*

$$\iota_N^{N+1} C_{\alpha_{|N+1}}^{N+1} \subset C_{\alpha_{|N+1}}^N \quad N \in \mathbb{N}, (\alpha_J)_{J\in\mathbb{N}} \in \mathbb{N}^{\mathbb{N}}.$$

Them the following is sufficient for $\mathrm{Proj}^1 \mathcal{X} = 0$*:*

$$(L)$$

There is a sequence $(\gamma_k)_{k\in\mathbb{N}}$ *of natural numbers such that for every* $N \in \mathbb{N}$ *there is* $M \in \mathbb{N}$ *with* $M \geq N$ *such that for all* $K \in \mathbb{N}$ *with* $K \geq M$ *and all sequences* $(\alpha_k)_{k\in\mathbb{N}}$ *of natural numbers there is a sequence* $(\beta_k)_{k\in\mathbb{N}}$ *and an* $s \in \mathbb{N}$ *such that we have*

$$\iota_N^M C_{\alpha_{|s}}^M \subset \iota_N^K C_{\beta_{|1}}^K + C_{\gamma_{|N}}^N$$

Proof: If we define $A_k^N := C_{\gamma_{|k}}^N$, $N, k \in I\!N$ then for every $N \in I\!N$ the sequence $\left(A_k^N\right)_{k\in I\!N}$ is completing and fulfills the assumption of theorem 3.6. Thus $\mathrm{Proj}^1 \mathcal{X} = 0$.

Another version of the theorem of Retakh and Palamodov for webbed spaces has been proved in [D, Theorem 4.1]:

Theorem: *If a projective spectrum* $\mathcal{X} = \left(X_N, \iota_N^{N+1}\right)_{N\in I\!N}$ *of strictly webbed vector spaces with ordered webs* $\left(C_{\alpha_{|k}}^N\right)_{k,\alpha_{|k}\in I\!N}$ *and subspaces* $Y_N \subset X_N$, $N \in I\!N$ *are given, then the following are equivalent:*

i) For the mapping

$$\begin{aligned} \psi \ : \ \prod_{N=1}^{\infty} X_N \quad &\to \quad \prod_{N=1}^{\infty} X_N \\ (x_N)_{N\in I\!N} \quad &\to \quad \left(\iota_N^{N+1} x_{N+1} - x_N\right)_{N\in I\!N} \end{aligned}$$

we have

$$\prod_{N=1}^{\infty} Y_N \subset \mathrm{Im}\psi.$$

ii) There is a sequence $(\sigma_N)_{N\in I\!N} \in I\!N^{I\!N}$ *such that for all* $N \in I\!N$ *exists* $M \in I\!N$ *with* $M \geq N$ *such that for all* $K \in I\!N$ *with* $K \geq M$ *we have*

$$\iota_N^K Y_K \subset \iota^N X + \bigcap_{J=1}^{N} \left(\iota_J^N\right)^{-1} C_{\sigma_{|J,N}}^J$$

Where $X = \mathrm{Proj}^0 \mathcal{X}$ *denotes the projective limit of the spectrum* $\mathcal{X}$.

3.3 Webs on spaces of continuous linear maps

In view of 2.8 and 2.9, the kind of webbed spaces that will concern us are the spaces $L(E, F)$ where E is a (PLB)-space and F a complete (LB)-space. In this section we will always use the following notation: $E = \left(E_M, \iota_M^{M+1}\right)_{M\in I\!N}$ denotes a (PLB)-space. The E_M, $M \in I\!N$ we will write as $E_M = \bigcup_{m=1}^{\infty} E_{M,m}$, $M \in I\!N$, endowed with the inductive limit topology with respect to the embeddings $I_m^M : E_{M,m} \hookrightarrow E_M$, $M, m \in I\!N$. Let further $B_{M,m}$, $M, m \in I\!N$ denote the corresponding closed unit balls in $E_{M,m}$.

$$F = \bigcup_{n=1}^{\infty} F_n\,,$$

will denote a complete (LB)-space with embeddings $J_n : F_n \hookrightarrow F$, $n \in I\!N$ and closed unit balls $D_n \subset F_n$, $n \in I\!N$.
We assume that $F = \bigcup_{n=1}^{\infty} D_n$ and that for every subset D which is bounded in one of the F_n, $n \in I\!N$, there is $\tilde{n} \in I\!N$, such that $D \subset D_{\tilde{n}}$ (which can always be achieved by multiplying every D_n with a suitable $R_n > 0$). We are now ready to define the web on $L(E, F)$: Let

us recall that for two locally convex spaces X and Y we get a locally convex topology on $L(X, Y)$ by defining as a basis of neighbourhoods of zero the sets

$$W(B, U) := \{T \in L(X, Y) : T(B) \subset U\},$$

where B is a bounded set in X and U a neighborhood of zero in Y. (A thorough treatment of this subject can e.g. be found in [J, Chapter 2.10]

Proposition 3.8 *If E is a (PLB)-space and* $F = \bigcup_{n=1}^{\infty} F_n$ *is a complete (LB) space such that every bounded set in F is contained in one of the unit balls* D_n *of* F_n *for a suitable* $n \in I\!N$*, then we get a strict ordered web on* $L(E, F)$ *by defining*

$$C_\alpha := \{T \in L(E, F) : T \text{ factorizes through } E_\alpha\}, \; \alpha \in I\!N$$

and

$$C_{\alpha_{|k+1}} := \left\{T \in C_{\alpha_{|k}} : \tilde{T}(B_{\alpha_1,k}) \subset D_{\alpha_{k+1}}\right\}, \; k, \alpha_{|k} \in I\!N$$

(here $\tilde{T}$ *denotes the factorization of* T*).*

Remark: Thus for fixed $k \in I\!N$ and $(\alpha_N)_{N \in I\!N}$, $C_{\alpha_{|k}}$ is the set of all linear mappings in L(E,F) that factorize through E_{α_1} and for $j = 1, \ldots, k-1$ map $B_{\alpha_1,j}$ into $D_{\alpha_{j+1}}$.

Proof: To show that $\left(C_{\alpha_{|k}}\right)_{k, \alpha_{|k} \in I\!N}$ is a web, we first observe that because of Lemma 1.6 every element of $L(E, F)$ factorises through a certain E_{α_0}, so

$$L(E, F) = \bigcup_{\alpha=1}^{\infty} C_\alpha$$

Secondly, for for fixed $k, \alpha_{|k} \in I\!N$ and $T \in C_{\alpha_{|k}}$ the mapping $\tilde{T} \circ I_k^{\alpha_1}$ acts continuously from the Banach space $E_{\alpha_1,k}$ into a countable union of Banach Spaces, so Grothendieck's factorization theorem gives us an $n_0 \in I\!N$ such that $\tilde{T} \circ I_k^{\alpha_1}$ acts continuously into F_{n_0}, thus $\tilde{T}(B_{\alpha_1,k})$ is contained in F_{n_0} and bounded there. Due to our assumption on the D_n, $n \in I\!N$, we get an $\alpha_{k+1} \in I\!N$ such that $\tilde{T}(B_{\alpha_1,\alpha_k}) \subset D_{\alpha_{k+1}}$. As a consequence

$$C_{\alpha_{|k}} = \bigcup_{\alpha=1}^{\infty} C_{\alpha_{|k},\alpha}.$$

Obviously our web is ordered. To show that we have obtained a $\mathcal{C}$-web, we claim that for a fixed sequence $(\alpha_k)_{k \in I\!N}$ of natural numbers and every sequence $(T_k)_{k \in I\!N}$ in L(E,F) that satisfies $T_k \in C_{\alpha_{|k}}$, $k \in I\!N$ the series $\sum_{k=1}^{\infty} 2^{-k} T_k$ converges in L(E,F). For all $k \in I\!N$ we will denote the factorization of T_k through E_{α_1} by $\tilde{T}_k$. For $k > \nu$ we have

$$T_k \in C_{\alpha_{|k}} \subset C_{\alpha_{|\nu+1}},$$

thus

$$2^{-k} \tilde{T}_k(B_{\alpha_1,\nu}) \subset 2^{-k} D_{\alpha_{\nu+1}}, \; k > \nu \tag{1}$$

which implies the convergence of

$$\sum_{k=\nu+1}^{\infty} 2^{-k} \, \tilde{T}_k \, I_\nu^{\alpha_1}$$

in L $(E_{\alpha_1,\nu}, F_{\alpha_{\nu+1}})$, so the limit

$$S_\nu := \sum_{k=1}^{\infty} 2^{-k}\, \tilde{T}_k\, I_\nu^{\alpha_1}$$

exists in L $(E_{\alpha_1,\nu}, F)$. In order to see that we are able to put the S_ν, $\nu \in I\!N$ together to a mapping on E_{α_1}, we consider the continuous embeddings

$$I^{\alpha_1}_{\nu,\nu+1} : E_{\alpha_1,\nu} \hookrightarrow E_{\alpha_1,\nu+1}.$$

We have $S_{\nu+1} \circ I^{\alpha_1}_{\nu,\nu+1} = S_\nu$ for all $\nu \in I\!N$, as

$$\begin{aligned} \left(\sum_{k=1}^{\infty} 2^{-k}\, \tilde{T}_k\, I^{\alpha_1}_{\nu+1}\right) I^{\alpha_1}_{\nu,\nu+1} &= \sum_{k=1}^{\infty} 2^{-k}\, \tilde{T}_k\, I^{\alpha_1}_{\nu+1}\, I^{\alpha_1}_{\nu,\nu+1} \\ &= \sum_{k=1}^{\infty} 2^{-k}\, \tilde{T}_k\, I^{\alpha_1}_{\nu}, \end{aligned}$$

therefore we can define a linear operator $S : E_{\alpha_1} \to F$ by putting

$$Sx := S_\nu x,\ x \in E_{\alpha_1,\nu}.$$

E_{α_1} being ultrabornological and S_ν, $\nu \in I\!N$ continuous, S also is continuous, so it remains to show that $S = \sum_{k=1}^{\infty} 2^{-k}\, T_k$ in L (E_{α_1}, F). For this, let a bounded set B in E_{α_1} and a closed neighborhood of the origin U in F be given. We have to show that there is an $N \in I\!N$ such that

$$S - \sum_{k=1}^{M} 2^{-k}\, T_k \in W(B, U)$$

for $M \geq N$. Now there is ν_0 and $\lambda > 0$ such that $B \subset \lambda\, \overline{B_{\alpha_1,\nu_0}}^{\sigma}$, where σ denotes the inductive limit topology of E_{α_1}, (in an (LB)-space the system of the unit balls, closed with respect to the inductive limit topology is a fundamental system of bounded sets, see e.g. [MV, Satz 25.16]), and a sequence $(\mu_l)_{l \in I\!N}$ of positive numbers such that $\Gamma\left(\bigcup_{l=1}^{\infty} \mu_l D_l\right) \subset \lambda^{-1} U$. Using (1) we get $N \geq \nu_0$ satisfying

$$\sum_{k=N+1}^{\infty} 2^{-k} \leq \mu_{\alpha_{\nu_0+1}}\ ,$$

so that we have

$$\sum_{k=N+1}^{\infty} 2^{-k}\, \tilde{T}_k(B_{\alpha_1,\nu_0}) \subset \left(\sum_{k=N+1}^{\infty} 2^{-k}\right) D_{\alpha_{\nu_0+1}} \subset \mu_{\alpha_{\nu_0+1}} D_{\alpha_{\nu_0+1}} \subset \lambda^{-1} U.$$

This gives for $M \geq N$

$$\left(S - \sum_{k=1}^{M} 2^{-k}\, \tilde{T}_k\right) (B_{\alpha_1,\nu_0}) = \sum_{k=M+1}^{\infty} 2^{-k}\, T_k(B_{\alpha_1,\nu_0}) \subset \lambda^{-1} U,$$

and because of continuity and U being closed

$$\left(S - \sum_{k=1}^{M} \lambda_k T_k\right) \left(\overline{B_{n_1,\nu_0}}^{\sigma}\right) \subset \lambda^{-1} U,\ M \geq N$$

so

$$\left(S-\sum_{k=1}^{M}\lambda_k T_k\right)(B)\subset U,\ M\geq N.$$

To show that our web is strict, observe that for all $k_0\in I\!N$ we obtain from (1)

$$\sum_{k=k_0}^{\infty}2^{-k}\,\tilde{T}_k\left(B_{\alpha_1,k_0-1}\right)\in\left(\sum_{k=k_0}^{\infty}2^{-k}\right)D_{\alpha_{k_0}},\subset D_{\alpha_{k_0}}$$

thus

$$\sum_{k=k_0}^{\infty}2^{-k}\,\tilde{T}_k\in C_{\alpha_{|k_0}},$$

and the proof is complete.

4 Further conditions for the vanishing of the first derived projective limit functor for spectra of webbed spaces

In section 2 we characterized the connection between the vanishing of $\mathrm{Ext}^1_{(PLS)}(E,F)$ and $\mathrm{Proj}^1\mathcal{L}(E,F)$ and section 3 contained a generalization of the Retakh-Palamodov theorem, which gives a characterization of the latter problem in terms of the webs on the spaces X_N, $N\in I\!N$. This was done in order to give an analogy to the theorems A and B of Vogt and Retakh-Palamodov cited in section 1.2 . As already indicated there, it is in general very difficult to verify the inclusion

$$\iota_N^M X_M\subset\iota_N^K X_K+\bigcap_{J=1}^{N}\left(\iota_J^N\right)^{-1}C^J_{\gamma_{|J,N}},$$

so we now seek to find conditions similiar to the theorems C and E of Vogt and Frerick-Wengenroth. The advantage of these conditions is that the inclusions between the spaces X_N, $N\in I\!N$, are substituted by inclusions between Banach balls. The analogy will be that we try to substitute the spaces X_N, $N\in I\!N$, and the intersection $\bigcap_{J=1}^{N}\left(\iota_J^N\right)^{-1}C^J_{\gamma_{|J,N}}$ by websets $C^N_{\alpha_{|r}}$ (An example of that kind is the condition in corollary 3.7, we will try to find similiar conditions with general $r\in I\!N$ instead of $r=N$).

4.1 Sufficient conditions

The first step will be to give a "graded" version of a sufficient condition, i.e. we set up a condition in which the intersection $\bigcap_{J=1}^{N}\left(\iota_J^N\right)^{-1}C^J_{\gamma_{|J,N}}$ is substituted by the webset $C^N_{\gamma_{|r}}$ with an arbitrary $r\in I\!N$ and the space X_M is substituted by the webset $C^M_{\alpha_{|r}}$.

Proposition 4.1 *Let a spectrum $\mathcal{X}=\left(X_N,\iota_N^{N+1}\right)_{N\in I\!N}$ of strictly webbed spaces with ordered webs $\left(C^N_{\alpha_{|k}}\right)_{k,\alpha_{|k}\in I\!N}$, $N\in I\!N$ be given. Suppose that we have for all $N,k,\alpha_{|k}\in I\!N$*

$$2\,C^N_{\alpha_{|k}}\subset C^N_{2\alpha_{|k}}.$$

Then the following is sufficient for $Proj^1\mathcal{X} = 0$: For all $N \in I\!N$ there is a sequence $\left(\gamma_\nu^N\right)_{\nu\in I\!N}$ and an $M \in I\!N$, such that for all $K \in I\!N$ with $K \geq M$ and all sequences $(\alpha_\nu)_{\nu\in I\!N}$ there is a sequence $(\beta_\nu)_{\nu\in I\!N}$, such that for every $r \in I\!N$ we have

$$\iota_N^M C^M_{\alpha_{|r}} \subset \iota_N^K C^K_{\beta_{|r+1}} + C^N_{{\gamma^N}_{|r}}$$

Remark:

i) As the proof will show, we have to demand that the substitution for X_K must have the index $r+1$, the same condition with r instead of $r+1$ merely leads to the conclusion that the spectrum is reduced, which is already one of our general assumptions.

ii) The assumption $2\ C^N_{\alpha_{|k}} \subset C^N_{2\alpha_{|k}}$ can always be fulfilled when dealing with spectra $\mathcal{L}(E, F)$ by renorming the Banach spaces constituting the spaces $(F_N)_{N\in I\!N}$.

Proof: As a first step, by replacing $(\alpha_\nu)_{\nu\in I\!N}$ by $(4\,\alpha_\nu)_{\nu\in I\!N}$, we may write in our assumption

$$\iota_N^M C^M_{\alpha_{|r}} \subset \frac{1}{4}\,\iota_N^K C^K_{\beta_{|r+1}} + \frac{1}{4}\,C^N_{\gamma_{|r}}$$

Second, it is sufficient to show $Proj^1\tilde{\mathcal{X}} = 0$ for an equivalent spectrum $\tilde{\mathcal{X}}$, so we may assume that in the assumption we have $M(N) = N+1$ for all $N \in I\!N$. For every $N \in I\!N$ we will construct strict sequences $\left(A_k^N\right)_{k\in I\!N}$ of subsets of X_N, satisfying the following conditions:

i) For every $K, N \in I\!N$ with $K \geq N$ we have

$$\iota_N^K A_K^K \subset A_K^N$$

ii) For every $N, K \in I\!N$ and every sequence $(\alpha_\nu)_{\nu\in I\!N}$ there is a sequence $(\delta_\nu)_{\nu\in I\!N}$ such that for all $r \in I\!N$ we have

$$\iota_N^{N+1} C^{N+1}_{\alpha_{|r}} \subset \iota_N^K C^K_{\tau_{|r+1}} + A_r^N$$

For $N = 1$ we set $A_k^1 := C^1_{\gamma^1{}_{|k}}$ and see that ii) is fulfilled by the assumption of the theorem. For the induction process we show for simplicity of notation the step from $N = 1$ to $N = 2$ (The general induction step from N to $N+1$ we can handle analogously). We use ii) for $N = 1$ with $(\alpha_\nu)_{\nu\in I\!N} := (\gamma_\nu^2)_{\nu\in I\!N}$, $K := 3$ and get a sequence $(\beta_\nu)_{\nu\in I\!N}$ such that for all $r \in I\!N$

$$\iota_1^2 C^2_{\gamma^2{}_{|r}} \subset \iota_1^3 C^3_{\beta_{|r+1}} + A_r^1$$

If we take $x \in C^2_{\gamma^2{}_{|r}}$, we find $y \in C^3_{\beta_{|r+1}}$ and $z \in A_r^1$ such that $\iota_1^2 x = \iota_1^3 y + z$. This implies

$$x - \iota_2^3 y \in \left(\iota_1^2\right)^{-1} A_r^1,$$

furthermore we have

$$x - \iota_2^3 y \in C^2_{\gamma^2{}_{|r}} + \iota_2^3 C^3_{\beta_{|r+1}},$$

so that

$$x = x - \iota_2^3 y + \iota_2^3 y \quad \in \quad \iota_2^3 C^3_{\beta_{|r+1}} + \left\{ \left(\iota_1^2\right)^{-1} A_r^1 \cap \left(\iota_3^2 C^3_{\beta_{|r+1}} + C^2_{\gamma^2{}_{|r}} \right) \right\} \tag{1}$$

We know by 3.3 that

$$\left(\left(\iota_1^2\right)^{-1} A_r^1 \cap \left(\iota_2^3 C^3_{\beta_{|r}} + C^2_{\gamma^2{}_{|r}} \right) \right)_{r \in I\!N}$$

defines a strict sequence on X_2, so we can define for $r \in I\!N$

$$A_r^2 := \left(\iota_1^2\right)^{-1} A_r^1 \cap \left(\iota_2^3 C^3_{\beta_{|r}} + C^2_{\gamma^2{}_{|r}} \right).$$

By this definition we have obviously fulfilled property i) for $N = 2$, so we proceed to the proof of ii). Let for this $K \in I\!N$ and $(\alpha_\nu)_{\nu \in I\!N}$ be arbitrary but fixed. Equation (1) gives us for all $r \in I\!N$

$$C^2_{\gamma^2{}_{|r}} \subset \iota_2^3 C^3_{\beta_{|r}} + A_r^2. \tag{2}$$

We now use our initial assumption in the formulation of the proposition to obtain $(\delta_\nu)_{\nu \in I\!N}$ such that for all $r \in I\!N$

$$\iota_2^3 C^3_{\beta_{|r}} \subset \frac{1}{4}\, \iota_2^K C^K_{\delta_{|r+1}} + \frac{1}{4}\, C^2_{\gamma^2{}_{|r}}.$$

Inserting equation (2), we get for all $r \in I\!N$

$$\begin{aligned} \iota_2^3 C^3_{\beta_{|r}} &\subset \frac{1}{4}\, \iota_2^K C^K_{\delta_{|r+1}} + \frac{1}{4}\, \iota_2^3 C^3_{\beta_{|r+1}} + \frac{1}{4}\, A_r^2. \\ &\subset \frac{1}{4}\, \iota_2^K C^K_{\delta_{|r+1}} + \frac{1}{2}\, \iota_2^3 C^3_{\beta_{|r+1}} + \frac{1}{4}\, A_r^2. \end{aligned}$$

Let $r \in I\!N$ be fixed, and take $x \in C^3_{\beta_{|r}}$; using the last equation inductively, we get sequences $(x_k)_{k \in I\!N}$, $(y_k)_{k \in I\!N}$ and $(z_k)_{k \in I\!N}$ which satisfy for all $k \in I\!N$

$$\begin{aligned} x_k &\in C^K_{\delta_{|r+k}}, \\ y_k &\in C^3_{\beta_{|r+k}}, \\ \text{and} \quad z_k &\in A^2_{r+k-1}, \end{aligned}$$

such that for all $k \in I\!N$

$$\iota_2^3 x = \sum_{j=1}^{k} 2^{-(j+1)} \iota_2^K x_j + \sum_{j=1}^{k} 2^{-(j+1)} z_j + 2^{-k}\, \iota_2^3 y_k. \tag{3}$$

Due to strictness, we know from 3.4 that for every $\varepsilon > 0$

$$\sum_{j=1}^{\infty} 2^{-(j+1)} x_j = 2^{r-1} \sum_{j=r+1}^{\infty} 2^{-j} x_{r+2-j} \in 2^{r-1} \left(\| \left(2^{-j}\right)_{j \geq r+1} \|_{l_1} + \varepsilon \right) C^K_{\delta_{|r+1}}$$

so (putting $\varepsilon := 2^{-r}$)

$$\sum_{j=1}^{\infty} 2^{-j+1} x_j \in C^K_{\delta_{|r+1}}.$$

Analogously we get

$$\sum_{j=1}^{\infty} 2^{-(j+1)} z_j \in A_r^2.$$

The sequence $\left(2^{-k} y_k\right)_{k\in I\!N}$ converges to zero, because the series

$$\sum_{k=1}^{\infty} 2^{-k} y_k$$

converges. So we get

$$\iota_2^3 x \in \iota_2^K C^K_{\delta_{|r+1}} + A_r^2.$$

Thus we get

$$\iota_2^3 C^3_{\beta_{|r}} \subset \iota_2^K C^K_{\delta_{|r+1}} + A_r^2. \tag{4}$$

which is condition ii) for $N = 2$, thus we have finished the induction step. Now we easily get $\operatorname{Proj}^1 \mathcal{X} = 0$ using Theorem 3.6: For every $N \in I\!N$ we set $M(N) := N + 1$. let $K \in I\!N$ with $K \geq M$ and $x \in X_{N+1}$ be given. We can find a sequence $(\alpha_\nu)_{\nu\in I\!N}$ of natural numbers such that $x \in C^{N+1}_{\alpha_{|N}}$; for this sequence we find another one $(\tau_\nu)_{\nu\in I\!N}$ such that the inclusion in ii) with $r = N$ is fulfilled. This shows

$$\iota_N^{N+1} x \in \iota_N^K X_K + A_N^N$$

Form this and from i) we conclude that condition ii) in theorem 3.6 is fulfilled, so $\operatorname{Proj}^1 \mathcal{X} = 0$.

If we want to weaken the condition in the sense that the substitution for X_M does not have to contain the given index r but may have a maybe considerably bigger index $s \in I\!N$ (of course depending on r), thus making the term on the left hand side of the inclusion smaller, we have to place stronger conditions on the substitution for X_K:

Proposition 4.2 *Let a spectrum $\mathcal{X} = \left(X_N, \iota_N^{N+1}\right)_{N\in I\!N}$ of strictly webbed spaces with ordered webs $\left(C^N_{\alpha_{|k}}\right)_{k,\alpha_{|k}\in I\!N}$ be given. Suppose that for all $N, k, \alpha_{|k} \in I\!N$*

$$2\, C^N_{\alpha_{|k}} \subset C^N_{2\alpha_{|k}}.$$

Then $\operatorname{Proj}^1 \mathcal{X} = 0$, if we have the following: For all $N \in I\!N$ there is a sequence $(\gamma_\nu)_{\nu\in I\!N}$ and an $M \in I\!N$, such that for all $K \geq M$ and all sequences $(\alpha_\nu)_{\nu\in I\!N}$ there is a sequence $(\beta_\nu)_{\nu\in I\!N}$, such that for every $r \in I\!N$ there is $s \in I\!N$ such that for all $t \in I\!N$ we have

$$\iota_N^M C^M_{\alpha_{|s}} \subset \iota_N^K C^K_{\beta_{|t}} + C^N_{\gamma_{|r}}$$

Proof: We proceed quite analogously to the preceding: Defining for all $N \in I\!N$ the strict sequences $\left(A_k^N\right)_{k\in I\!N}$ as before, and using the same arguments as before, we get sequences $(\beta_\nu)_{\nu\in I\!N}$ and $(s(r))_{r\in I\!N}$ such that for all $t, r \in I\!N$

$$C^2_{\gamma^2{}_{|s(r)}} \subset \iota_2^3 C^3_{\beta_{|t}} + A_r^2. \tag{1}$$

Applying the assumption to $(\beta_\nu)_{\nu\in I\!N}$, we obtain $(\delta_\nu)_{\nu\in I\!N}$ and $(\tilde{s}(\tilde{r}))_{\tilde{r}\in I\!N}$ such that for all $\tilde{r},\tilde{t}\in I\!N$

$$\iota_2^3 C^3_{\beta_{|\tilde{s}(\tilde{r})}} \subset \frac{1}{4}\, iota_2^K C^K_{\delta_{|\tilde{t}}} + \frac{1}{4}\, C^2_{\gamma^2{}_{|\tilde{r}}}.$$

Setting now $\tilde{r} := s(r), r \in I\!N$ and using (1), we get for all $r,t,\tilde{t}\in I\!N$

$$\iota_2^3 C^3_{\beta_{|\tilde{s}(s(r))}} \subset \frac{1}{4}\, \iota_2^K C^K_{\delta_{|\tilde{t}}} + \frac{1}{2}\, \iota_2^3 C^3_{\beta_{|t}} + \frac{1}{4}\, A^2_r.$$

Here we set $t := \tilde{t} := \tilde{s}\,(s(r+1))$, and get for all $r\in I\!N$(denoting $\tilde{s}\,(s(r))$ by $\bar{s}(r)$)

$$\iota_2^3 C^3_{\beta_{|\bar{s}(r)}} \subset \frac{1}{4}\, \iota_2^K C^K_{\delta_{|\bar{s}(r+1)}} + \frac{1}{2}\, \iota_2^3 C^3_{\beta_{|\bar{s}(r+1)}} + \frac{1}{4}\, A^2_r.$$

Thus the same process as before gives us for all $r\in I\!N$

$$\iota_2^3 C^3_{\beta_{|\bar{s}(r)}} \subset \iota_2^K C^K_{\delta_{|\bar{s}(r+1)}} + A^2_r. \tag{2}$$

The rest is the same as before.

Remark: It should be noted that neither one of the two sufficient conditions implies the other one in any standard way.

4.2 Necessary conditions

We now try to find a similiar condition to Vogt's Theorem C of section 1.2, by finding for a given spectrum $\mathcal{X}$ of srictly webbed spaces a necessary condition for $\mathrm{Proj}^1\mathcal{X}=0$, in which the inclusion relations between the whole spaces as in Theorem 3.6 are substituted by ones betwen the websets $\left(C^N_{\alpha_{|k}}\right)_{k,\alpha_{|k}\in I\!N}$.

Proposition 4.3 *Let a spectrum* $\mathcal{X}=\left(X_N,\iota_N^{N+1}\right)_{N\in I\!N}$ *of strictly webbed spaces with ordered webs* $\left(C^N_{\alpha_{|k}}\right)_{k,\alpha_{|k}\in I\!N}$ *be given. Let for* $k,\alpha_{|k}\in I\!N$ *denote*

$$E_{\alpha_{|k}} := X\cap\left(\prod_{j=1}^{k} C^j_{\alpha_{|k}} \times \prod_{j=k+1}^{\infty} X_j\right)$$

Then $\mathrm{Proj}^1\mathcal{X}=0$ *implies that there is a sequence* $(\gamma_j)_{j\in I\!N}$*, such that for every* $N\in I\!N$ *exists* $M\in I\!N$ *satisfying the following: For all sequences* $(\alpha_j)_{j\in I\!N}$ *there are* $(\beta_j)_{j\in I\!N}$ *and* $(R_j)_{j\in I\!N}$ *such that for all* $r\in I\!N$ *we have* $s\in I\!N$ *satisfying*

$$\iota_N^M C^M_{\alpha_{|s}} \subset R_r\left(\iota^N E_{\beta_{|r}} + \bigcap_{j=1}^{N}\left(\iota_j^N\right)^{-1} C^j_{\gamma_{|j,N},\beta_{|r}}\right)$$

Here R_r *can be chosen as* $2^{s(r)+2}$.

Proof: Let us assume $\operatorname{Proj}^1 \mathcal{X} = 0$ for a spectrum $\mathcal{X}$ of strictly webbed spaces. Then we know from Theorem 3.6, that there is a sequence $(\gamma_j)_{j\in\mathbb{N}}$ of natural numbers such that for all $N \in \mathbb{N}$ there is a $M \in \mathbb{N}$ such that

$$\iota_N^M X_M \subset \iota^N X + \bigcap_{j=1}^{N} \left(\iota_j^N\right)^{-1} C^j_{\gamma_{|j,N}}. \tag{1}$$

We take an arbitrary sequence $(\alpha_j)_{j\in\mathbb{N}}$ of natural numbers, and put

$$A_k := C^M_{\alpha_{|k}},\ k \in \mathbb{N}.$$

Then we know from Definition 3.1 that $(A_k)_{k\in\mathbb{N}}$ forms a strict sequence of subsets of X_N and thus according to Proposition 3.5 induce a complete metrizable group topology under addition on X_N and thus $\left(\iota_N^M A_k\right)_{k\in\mathbb{N}}$ induce such a topology on $\iota_N^M X_M$, which we call τ. The inclusion in (1) enables us to use a Baire argument: Due to the properties i) and ii) of webs, we get a sequence $(\beta_j)_{j\in\mathbb{N}}$ such that for every $r \in \mathbb{N}$ the set

$$\iota^N E_{\beta_{|r}} + \bigcap_{j=1}^{N} \left(\iota_j^N\right)^{-1} C^j_{\gamma_{|j,N},\beta_{|r}}$$

is of second category in $\left(\iota_N^M X_M, \tau\right)$ (the closure has an interior point, which can be considered to be the origin with the same argument as in the proof of Theorem 3.6). In the following, let us use the notation

$$\bigcap_{j=1}^{N} \left(\iota_j^N\right)^{-1} C^j_{\gamma_{|j,N},\beta_{|r}} := D_{\beta_{|r}},$$

so that for every $r \in \mathbb{N}$ $\iota^N E_{\beta_{|r}} + D_{\beta_{|r}}$ is of second category in $\left(\iota_N^M X_M, \tau\right)$. Remembering from proposition 3.5 that the zero neighbourhoods for τ are given by

$$U_k := 2^{-(k+1)} A_k,$$

we get for every $r \in \mathbb{N}$ numbers $s(r) \in \mathbb{N}$ and $R_r = 2^{s(r)+1}$ such that

$$\iota_N^M A_{s(r)} \subset R_r \overline{\iota^N E_{\beta_{|r}} + D_{\beta_{|r}}}^{\tau}. \tag{2}$$

Without loss of generality we assume that for all $r \in \mathbb{N}$ we have $s(r+1) \geq s(r)$ and thus $R_{r+1} \geq R_r$. Our next step is to show that the closure in (2) is redundant, and in the proof of this claim we assume for simplicity of notation $s(r) = r$ for all $r \in \mathbb{N}$. We fix an $r_0 \in \mathbb{N}$ and observe that for all $j \in \mathbb{N}_0$ and any $\varrho > 0$ we have because of (2)

$$\iota_N^M A_{r_0+j} \subset R_{r_0+j} \left(\iota^N E_{\beta_{|r_0+j}} + D_{\beta_{|r_0+j}}\right) + \varrho\, \iota_N^M A_{r_0+j+1} \tag{3}$$

We now take an $x \in A_{r_0}$ and we will show that there is an $\tilde{R}_{r_0} > 0$ which does not depend on x (only on A_{r_0}) and satisfies

$$\iota_N^M x \in \tilde{R}_{r_0} \left(\iota^N E_{\beta_{|N}} + D_{\beta_{|N}}\right). \tag{4}$$

To do this, we set up an induction process: We want to find $(e_n)_{n\in I\!N_0}$, $(d_n)_{n\in I\!N_0}$ and $(c_n)_{n\in I\!N}$ such that

$$\begin{aligned} e_n &\in E_{\beta_{|r_0+n}}, \quad n\in I\!N_0 \\ d_n &\in D_{\beta_{|r_0+n}}, \quad n\in I\!N_0 \\ c_n &\in A_{r_0+n}, \quad n\in I\!N \end{aligned} \tag{5}$$

such that for all $n\in I\!N_0$

$$\begin{aligned} \iota_N^M x &= R_{r_0}\left(\iota^N e_0 + d_0 + \sum_{l=1}^{n} 2^{-(r_0+l)}\iota^N e_l + \sum_{l=1}^{n} 2^{-(r_0+l)} d_l\right) \\ &+ R_{r_0} 2^{-(r_0+n+1)} R_{r_0+n+1}^{-1}\, \iota_N^M c_{n+1}. \ (*) \end{aligned}$$

Using (3) with $j=0$ and $\varrho = R_{r_0}2^{-(r_0+1)}R_{r_0+1}^{-1}$, we obtain $e_0\in E_{\beta_{|r_0}}$, $d_0\in D_{\beta_{|r_0}}$ and $c_1\in A_{r_0+1}$ satisfying

$$\iota_N^M x = R_{r_0}\left(\iota^N e_0 + d_0\right) + R_{r_0}2^{-(r_0+1)}R_{r_0+1}^{-1}\, \iota_N^M c_1.$$

As to the induction step, let for an arbitrary $n\in I\!N$ $e_0,\dots,e_n$, $d_0,\dots,d_n$ and $c_1,\dots,c_{n+1}$ be given, such that for $j=0,\dots,n$ we have $e_j\in E_{\beta_{|r_0+j}}$, $d_j\in D_{\beta_{|r_0+j}}$, $c_{j+1}\in A_{r_0+j+1}$ and the following holds:

$$\begin{aligned} \iota_N^M x &= R_{r_0}\left(\iota^N e_0 + d_0 + \sum_{l=1}^{n} 2^{-(r_0+l)}\iota^N e_l + \sum_{l=1}^{n} 2^{-(r_0+l)} d_l\right) \\ &+ R_{r_0}\, 2^{-(r_0+n+1)} R_{r_0+n+1}^{-1}\, \iota_N^M c_{n+1}. \ (**) \end{aligned}$$

We apply (3) with $j=n+1$ and $\varrho = R_{r_0+n+1}2^{-1}R_{r_0+n+2}^{-1}$ and as $c_{n+1}\in A_{r_0+n+1}$, we find $e_{n+1}\in E_{\beta_{|r_0+n+1}}$, $d_{n+1}\in D_{\beta_{|r_0+n+1}}$ and $c_{n+2}\in A_{r_0+n+2}$ fulfilling

$$\iota_N^M c_{n+1} = R_{r_0+n+1}\left(\iota^N e_{n+1} + d_{n+1}\right) + R_{r_0+n+1}2^{-1}R_{r_0+n+2}^{-1}\iota_N^M c_{n+2}$$

Inserting this equation in (**), we get

$$\begin{aligned} \iota_N^M x &= R_{r_0}\left(\iota^N e_0 + d_0 + \sum_{l=1}^{n+1} 2^{-(r_0+l)}\iota^N e_l + \sum_{l=1}^{n+1} 2^{-(r_0+l)} d_l\right) \\ &+ R_{r_0}\, 2^{-(r_0+n+2)} R_{r_0+n+2}^{-1}\, \iota_N^M c_{n+2}, \ (***) \end{aligned}$$

thus finishing the induction step. The properties of the $e_n, n\in I\!N_0$, $d_n, n\in I\!N_0$ and $c_n, n\in I\!N$ in (5) together with strictness and Lemma 3.4 imply that

$$\sum_{l=1}^{\infty} 2^{-(r_0+l)} e_l \in E_{\beta_{|r_0+1}},$$

$$\sum_{l=1}^{\infty} 2^{-(r_0+l)} d_l \in D_{\beta_{|r_0+1}}$$

and

$$\sum_{l=1}^{\infty} 2^{-(r_0+l+2)} c_l$$

converges in X_M, the latter meaning that

$$R_{r_0} 2^{-(r_0+n+2)} R_{r_0+n+2}^{-1} \iota_N^M c_{n+2}$$

tends to zero as n tends to infinity. (Keep in mind that $R_{r_0} \leq R_{r_0+n+2}$ for all $n \in I\!N_0$). Thus if in (***) we let n tend to infinity we obtain

$$\begin{aligned} \iota_N^M x &= R_{r_0} \left(\iota^N e_0 + d_0 + \sum_{l=1}^{\infty} 2^{-(r_0+l)} \iota^N e_l + \sum_{l=1}^{\infty} 2^{-(r_0+l)} d_l \right) \\ &\in R_{r_0} \left(\iota^N E_{\beta_{|r_0}} + D_{\beta_{|r_0}} + \iota^N E_{\beta_{|r_0+1}} + D_{\beta_{|r_0+1}} \right) \\ &\subset 2R_{r_0} \left(\iota^N E_{\beta_{|r_0}} + D_{\beta_{|r_0}} \right) \end{aligned}$$

which proves (4) with $\tilde{R}_r := 2R_r$, $r \in I\!N$ and thus our proposition.
The same proof gives:

Proposition 4.4 $\mathrm{Proj}^1 \mathcal{X} = 0$ *implies that there is a sequence* $(\gamma_j)_{j \in I\!N}$, *such that for every* $N \in I\!N$ *exists* $M \in I\!N$ *with* $M \geq N$ *such that for all* $K \in I\!N$ *with* $K \geq M$ *we have the following: For all sequences* $(\alpha_j)_{j \in I\!N}$ *there are* $(\beta_j)_{j \in I\!N}$ *and* $(R_j)_{j \in I\!N}$ *such that for all* $r \in I\!N$ *we have* $s \in I\!N$ *satisfying*

$$\iota_N^M C_{\alpha_{|s}}^M \subset R_r \left(\iota_N^K C_{\beta_{|r}}^K + \bigcap_{j=1}^{N} \left(\iota_j^N \right)^{-1} C_{\gamma_{|j,N}, \beta_{|r}}^j \right)$$

Remark:

i) In this proposition the webs need not be ordered (in the last proposition we needed this fact for the easy form of the product web).

ii) One cannot expect the condtitions given here to be sufficient, because the sequence $(R_r)_{r \in I\!N}$ appearing here makes it impossible to reach an inclusion as in the theorem of Retakh and Palamodov. On the other hand, the appearance of this sequence seems natural, because in the proof we used a Baire argument for which it was crucial to have a topology constructed out of the websets belonging to the sequence $(\alpha_j)_{j \in I\!N}$. To construct this topology we have to use weights 2^{r+1}, $r \in I\!N$, which appear in the condition as the numbers R_r, $r \in I\!N$.

5 Weighted sequence spaces

We will now apply the results of the previous sections to power series spaces of (PLS) - type and obtain some splitting results for these kind of spaces. In the first two subsections we state the definitions and elementary properties and state the general assumptions on these spaces which will be valid thoughout the whole section. 5.3 contains a quick look at local splitting for these spaces. The subsections 5.4 - 5.7 contain splitting results in the category of (PLS) - spaces as well as results for the vanishing of $\mathrm{Proj}^1 \mathcal{L}(E, F)$ for special pairs E and F of power series spaces of (PLS) - type. In 5.8 we gather the results we have obtained and the connections we could prove between the vanishing of $\mathrm{Ext}^1_(PLS)(E, F)$ and $\mathrm{Proj}^1 \mathcal{L}(E, F)$.

5.1 Projective limits of special weighted sequence spaces

We explain the kind of spaces we will consider throughout this section. We will investigate (PLB)-spaces of the following kind: Consider an infinite matrix $(a_{\lambda,K,m})_{\lambda,K,m\in I\!N}$ that satisfies the following properties:

$a_{\lambda,K,m} > 0$ for all $\lambda, K, m \in I\!N$.

$a_{\lambda,K,m} \leq a_{\lambda,K+1,m}$ for all $\lambda, K, m \in I\!N$.

$a_{\lambda,K,m} \geq a_{\lambda,K,m+1}$ for all $\lambda, K, m \in I\!N$.

We take a fixed $p \in I\!R$ with $1 \leq p < \infty$ and for $K, m \in I\!N$ we put

$$X^p_{K,m} := l_p\left(a_{K,m}\right) := \{(x_\lambda)_{\lambda\in I\!N} : \|(x_\lambda)_{\lambda\in I\!N}\|^p_{K,m} := \sum_{\lambda=1}^{\infty} a_{\lambda,K,m}{}^p |x_\lambda|^p < \infty\}.$$

For $p = \infty$ we put

$$X^\infty_{K,m} := l_\infty\left(a_{K,m}\right) := \{(x_\lambda)_{\lambda\in I\!N} : \|(x_\lambda)_{\lambda\in I\!N}\|^\infty_{K,m} := \sup_{\lambda\in I\!N} a_{\lambda,K,m}|x_\lambda| < \infty\},$$

or

$$X^{c_0}_{K,m} := c_0\left(a_{K,m}\right) := \{(x_\lambda)_{\lambda\in I\!N} : \lim_{\lambda\to\infty} a_{\lambda,K,m}|x_\lambda| = 0\},$$

where $c_0\left(a_{K,m}\right)$ is endowed with the same norm as $l_\infty\left(a_{K,m}\right)$.
In all cases the $X^p_{K,m}$ and $X^{c_0}_{K,m}$ are Banach spaces and because of the assumed estimates, we get for all $K, m \in I\!N$ the continuous embeddings

$$X^p_{K,m} \hookrightarrow X^p_{K,m+1} \quad \text{respectively} \quad X^{c_0}_{K,m} \hookrightarrow X^{c_0}_{K,m+1}$$

and

$$X^p_{K+1,m} \hookrightarrow X^p_{K,m} \quad \text{respectively} \quad X^{c_0}_{K,m} \hookrightarrow X^{c_0}_{K,m+1}.$$

We put for all $K \in I\!N$ $X^p_K := \bigcup_{m=1}^{\infty} X^p_{K,m}$, and get a projective spectrum of (LB)-spaces with the continuous inclusions $X^p_{K+1} \hookrightarrow X^p_K$ as connecting maps (We deal analagously with the c_0 - case).

Definition 5.1 *For a matrix $A = (a_{\lambda,K,m})_{\lambda,K,m\in I\!N}$ as described before we will denote the (PLB)-spaces arising out of the construction above by $\Lambda^{l_p}(A)$, $1 \leq p \leq \infty$ and $\Lambda^{c_0}(A)$ respectively.*

Using this construction for matrices $(a_{\lambda,K,m})_{\lambda,K,m\in I\!N}$ and $(b_{\nu,L,n})_{\nu,L,n\in I\!N}$, we can consider (PLB)-spaces $X^p = \mathrm{Proj}^0 X^p_K$ and $Y^p = \mathrm{Proj}^0 Y^p_L$, and set up a projective spectrum $L(X^p, Y^p_{L+1}) \hookrightarrow L(X^p, Y^p_L)$, $L \in I\!N$ with again the inclusions as connecting maps (Analogously in the c_0 - case).
In our applications of our general results to these kind of spaces we will consider the following special kind of matrices: Let $x = (x_\lambda)_{\lambda\in I\!N}$ and $y = (y_\lambda)_{\lambda\in I\!N}$ be sequences of positive real numbers tending to infinity. Let $(r_K)_{K\in I\!N}$ and $(s_m)_{m\in I\!N}$ be increasing sequences of real numbers and set $r := \lim_{K\to\infty} r_K$ and $s := \lim_{m\to\infty} s_m$. Then the matrix $(a_{\lambda,K,m})_{\lambda,K,m\in I\!N}$ defined by

$$a_{\lambda,K,m} := exp(r_K x_\lambda - s_m y_\lambda),\ \lambda, K, m \in I\!N$$

satisfies the conditions demanded above.

Definition 5.2 *The (PLB)-space constructed with this special matrix will be denoted by* $\Lambda^{l_p}_{r,s}(x,y)$.

Remark:

i) As in the special case of Fréchet power series spaces one can show that for $s < \infty$ we have $\Lambda^{l_p}_{r,s}(x,y) \tilde{=} \Lambda^{l_p}_{r,0}(x,y)$ and the same holds for $r < \infty$. As we are only interested in the vanishing of the Functors Ext^1 and Proj^1, which is a property independent of isomorphism classes, we will restrict ourselves to the case $r = 0$ respectively $s = 0$ in the finite case.

ii) Moreover we will consider the sequences $(r_k)_{k\in I\!N}$ and $(s_k)_{k\in I\!N}$, to be either $(k)_{k\in I\!N}$ for $r, s = \infty$ or $(-k^{-1})_{k\in I\!N}$ for $r, s < \infty$, in order to avoid too much complicated notation when formulating our conditions. The methods for general $(r_k)_{k\in I\!N}$ and $(s_k)_{k\in I\!N}$ are not different from the ones used here.

First we can quite easily show that the spaces defined in 5.2 are (PLS)-spaces:

Proposition 5.3 *Let an infinte matrix* $A = (a_{\lambda,K,m})_{\lambda,K,m\in I\!N}$ *be given and fix* $K, m \in I\!N$.

i) For (PLB)-spaces X given as $X = \Lambda^{l_p}(A)$, $1 \le p < \infty$ *or* $X = \Lambda^{c_0}(A)$ *the inclusion* $J : X_{K,m} \hookrightarrow X_{K,m+1}$ *is a compact operator iff*

$$\lim_{\lambda\to\infty} \frac{a_{\lambda,K,m+1}}{a_{\lambda,K,m}} = 0$$

ii) If $A = (\exp(r_K x_\lambda - s_m y_\lambda))_{\lambda,K,m\in I\!N}$ *with the sequences* $x = (x_\lambda)_{\lambda\in I\!N}$, $y = (y_\lambda)_{\lambda\in I\!N}$, $(r_K)_{K\in I\!N}$ *and* $(s_m)_{m\in I\!N}$ *as above, then the spaces* $\Lambda^{l_p}_{r,s}(x,y)$ *and* $\Lambda^{c_0}_{r,s}(x,y)$ *are (PLS)-spaces.*

Proof: i) follows from the following lemma ([MV, Lemma 27.8]):

> **Lemma:** *Let* $L = l_p$, $1 \le p < \infty$ or $L \supset c_0$ *be a closed subspace of* l_∞ *such that* $(l_k x_k)_{k\in I\!N} \in L$ *for all* $l = (l_k)_{k\in I\!N} \in l_\infty$ and $x = (x_k)_{k\in I\!N} \in L$. *Then for a given* $d = (d_k)_{k\in I\!N} \in l_\infty$ *the mapping* $D : L \to L$, $D(x_k)_{k\in I\!N} := (d_k x_k)_{k\in I\!N}$ *is continuous and linear and the following are equivalent:*
>
> *i) D is compact*
>
> *ii)* $d \in c_0$

In our case we set $L := l_p$, $1 \le p < \infty$ or $L := c_0$. and define D by means of the sequence $d := \left(\frac{a_{\lambda,K,m+1}}{a_{\lambda,K,m}}\right)_{\lambda\in I\!N}$. Furthermore we use the isometric isomorphisms $T_{K,m}$ and $T_{K,m+1}$ defined by

$$\begin{array}{rcll} T_{K,m} & : & X_{K,m} & \to & L \\ & & (t_\lambda)_{\lambda\in I\!N} & \to & (t_\lambda a_{\lambda,K,m})_{\lambda\in I\!N} \end{array}$$

and analogously for $T_{K,m+1}$. We have a commutative diagram

$$\begin{array}{ccc} X_{K,m} & \overset{J}{\to} & X_{K,m+1} \\ \downarrow T_{K,m} & & \downarrow T_{K,m+1} \\ L & \overset{D}{\to} & L \end{array}$$

which implies that D is compact iff J is compact (keep in mind that $T_{K,m}$ and $T_{K,m+1}$ are isomorphisms and that compactness is an ideal property), thus the claim follows immediately.
To show ii), take an $m_0 \in I\!N$ such that s_{m_0+1} is strictly greater than s_{m_0} and observe that the quotient in i) can be written as

$$\frac{a_{\lambda,K,m_0+1}}{a_{\lambda,K,m_0}} = \exp\left((s_{m_0} - s_{m_0+1})y_\lambda\right).$$

This expression converges to zero as λ tends to infinity iff $\lim\limits_{\lambda\to\infty} y_\lambda = \infty$, so ii) follows.

We now want to describe the webs in the spaces $\mathrm{L}\,(X, Y_N)$, $N \in I\!N$. For this let infinite matrices $(a_{\lambda,K,m})_{\lambda,K,m\in I\!N}$ and $(b_{\nu,N,n})_{\nu,N,n\in I\!N}$ be given. We will restrict ourselves to the case where

$$X_{K,m} = l_1\,(a_{K,m})\,,\ K, m \in I\!N$$

and

$$Y_{N,n} = c_0\,(b_{N,n})\,,\ N, n \in I\!N$$

as in this case it is possible to give a good description of the continuous linear operators $T : X_{K,m} \to Y_{N,n}$. In the following theorems we will always refer to the spaces $X_{K,m}$, $K, m \in I\!N$ and $Y_{N,n}$ $N, n \in I\!N$ in this meaning.

Lemma 5.4 *Let $K, N \in I\!N$ be given.*

i) For every $m, n \in I\!N$ the continuous linear operators $T : X_{K,m} \to Y_{N,n}$ are given by matrix representations $(T_{\nu,\lambda})_{\nu,\lambda\in I\!N}$ such that for every $t = (t_\lambda)_{\lambda\in I\!N} \in X_{K,m}$ we have

$$Tt = \left(\sum_{\lambda=1}^{\infty} T_{\nu,\lambda}\, t_\lambda\right)_{\nu\in I\!N}$$

The operator norm of T is given by

$$\|T\| = \sup_{\nu,\lambda\,\in I\!N}\left\{\frac{b_{\nu,N,n}}{a_{\lambda,K,m}}\,|T_{\nu.\lambda}|\right\}$$

ii) The continuous linear operators $T : X_K \to Y_N$ are given by matrix representations $(T_{\nu,\lambda})_{\nu\in I\!N}$ that fulfill the following: For every $m \in I\!N$ there is an $n \in I\!N$ such that

$$\sup_{\nu,\lambda\,\in I\!N}\left\{\frac{b_{\nu,N,n}}{a_{\lambda,K,m}}\,|T_{\nu.\lambda}|\right\} < \infty$$

Remark: From i) it is obvious that for an operator $T : X_K \to Y_N$ the numbers

$$C_m := \sup_{\nu,\lambda\in I\!N} \left\{ \frac{b_{\nu,N,n(m)}}{a_{\lambda,K,m}} |T_{\nu.\lambda}| \right\},\ m \in I\!N$$

represent the operator norm of $T_{|X_{K,m}} : X_{K,m} \to Y_{N,n(m)}$ $m \in I\!N$

Proof: i) For $t = (t_\lambda)_{\lambda\in I\!N} \in X_{K,m}$ we have

$$t = \sum_{\lambda=1}^{\infty} t_\lambda e_\lambda$$

where $(e_\lambda)_{\lambda\in I\!N}$ denote the canonical unit vectors. We write Tt as $(T_\nu t)_{\nu\in I\!N}$ and get

$$Tt = (T_\nu t)_{\nu\in I\!N} = \left(\sum_{\lambda=1}^{\infty} t_\lambda T_\nu e_\lambda \right)_{\nu\in I\!N},$$

so we can put $T_{\nu,\lambda} := T_\nu e_\lambda$. The formula for the operator norm we get by the estimates

$$\begin{aligned}
\|Tt\|_{Y_{N,n}} &= \sup_{\nu\in I\!N} \left\{ b_{\nu,N,n} \Big| \sum_{\lambda=1}^{\infty} T_{\nu,\lambda} t_\lambda \Big| \right\} \\
&\le \sup_{\nu\in I\!N} \left\{ b_{\nu,N,n} \sup_{\lambda\in I\!N} \left\{ |T_{\nu,\lambda}| a_{\lambda,K,m}^{-1} \right\} \sum_{\lambda=1}^{\infty} a_{\lambda,K,m} |t_\lambda| \right\} \\
&= \sup_{\nu,\lambda\in I\!N} \left\{ b_{\nu,N,n} a_{\lambda,K,m}^{-1} |T_{\nu,\lambda}| \right\} \|t\|_{X_{K,m}}
\end{aligned}$$

and

$$\begin{aligned}
\|T\| &\ge \sup_{\lambda\in I\!N} \left\| T\left(a_{\lambda,K,m}^{-1} e_\lambda\right) \right\|_{Y_{N,n}} \\
&= \sup_{\nu,\lambda\in I\!N} \left\{ b_{\nu,N,n} a_{\lambda,K,m}^{-1} |T_{\nu,\lambda}| \right\}
\end{aligned}$$

The proof of ii) is easily obtained by observing that by Grothendieck's factorisation theorem the continuous linear operators $T : X_K \to Y_N$ are just sequences of operators $T^m : X_{K,m} \to Y_{N,n(m)}$, $m \in I\!N$, such that $T^{m+1}_{|X_{K_m}} = T^m$, $m \in I\!N$. Inserting the vectors e_λ, $\lambda \in I\!N$, which are contained in all $X_{K,m}$, $m \in I\!N$, yields that the matrix representations for all T^m are the same.

We will use the matrix representations to describe the websets $\left(C^N_{\alpha_{|k}}\right)_{k,\alpha_{|k}\in I\!N}$ on the spaces $L(X, Y_N)$, $N \in I\!N$. For this we have to keep in mind that in section 3.3 we made the assumption that for every Y_N, $N \in I\!N$, we know that every set which is bounded in one of the $Y_{N,n}$, $n \in I\!N$, is contained in one of the unit balls $D_{N,n} \subset Y_{N,n}$, $n, N \in I\!N$. As we are interested in the spaces $\Lambda^{l_1}_{r,s}(x, y)$ and $\Lambda^{c_0}_{r,s}(x, y)$ and this assumption is not a priori fulfilled (e.g. if $(s_m)_{m\in I\!N} = \left(-\frac{1}{m}\right)_{m\in I\!N}$), we will use the following convention: If a matrix $(b_{\nu,N,n})_{\nu,N,n\in I\!N}$ does not have the property that for every $n, N \in I\!N$ every bounded set in $Y_{N,n}$ is contained in the unit ball of some Y_{N,n_0}, we put

$$\tilde{b}_{\nu,N,n} := n\, b_{\nu,N,n},\ \nu, N, n \in I\!N,$$

thus ensuring that for every $N, n \in I\!N$ and every bounded set $B \subset Y_{N,n}$ contained in $R\, D_{N,n}$ for an $R > 0$ we have for $n_0 \in I\!N$ with $n_0 > max\{R, n\}$

$$B \subset n_0\, D_{N,n} \subset n_0\, D_{N,n_0}$$

and so our assumption is fulfilled with the new unit balls

$$\tilde{D}_{N,n} := n\, D_{N,n},\ N, n \in I\!N.$$

If the given Matrix has this property, (e.g. in the case $(s_m)_{m\in I\!N} = (m)_{m\in I\!N}$), we do not have to change the norms and simply put

$$\tilde{b}_{\nu,N,n} := b_{\nu,N,n},\ \nu, N, n \in I\!N.$$

With this convention, we can formulate the following description of the sets $\left(C^N_{\alpha_{|k}}\right)_{k,\alpha_{|k}\in I\!N}$:

Proposition 5.5 *Let $K, N \in I\!N$ be fixed. For a continuous linear operator $T : X_K \to Y_N$, a sequence $(\alpha_k)_{k\in I\!N}$ of natural numbers and $r \in I\!N$ the following are equivalent:*

i) T is a member of $C^N_{\alpha_{|r}}$

ii) There are sequences $(n_m)_{m\in I\!N}$ and $(C_m)_{m\in I\!N}$ with $n_m = \alpha_{m+1}$ and $C_m = 1$ for $1 \le m \le r-1$ such that for all $\nu, \lambda \in I\!N$

$$|T_{\nu,\lambda}| \le \inf_{m\in I\!N} \left\{ C_m \frac{a_{\lambda,\alpha_1,m}}{\tilde{b}_{\nu,N,n_m}} \right\}$$

(where $(T_{\nu,\lambda})_{\nu,\lambda\in I\!N}$ denotes the matrix representation of T)

Proof: T being a member of $C^N_{\alpha_{|r}}$ means that T has a factorisation $\tilde{T} : X_{\alpha_1} \to Y_N$ and that for $j = 1, \ldots, r-1$ $\tilde{T}_{X_{\alpha_1,j}}$ acts continuously into $Y_{N,\alpha_{j+1}}$ with operator norm less or equal to 1. Its matrix representation is the same as the one for T, which can be seen by observing that T and $\tilde{T}$ coincide on the system of the canonical unit vectors. From Lemma 5.4 with $K = \alpha_1$ and the remark after it, we see that this can be described by the existence of $(n_m)_{m\in I\!N}$ and $(C_m)_{m\in I\!N}$ such that $n_m = \alpha_{m+1}$ and $C_m = 1$ for $m = 1, \ldots, r-1$ and

$$\sup_{\nu,\lambda\, \in I\!N} \left\{ \frac{\tilde{b}_{\nu,N,n(m)}}{a_{\lambda,\alpha_1,m}} |T_{\nu,\lambda}| \right\} \le C_m,\ m \in I\!N.$$

This finally is equivalent to

$$|T_{\nu,\lambda}| \le \inf_{m\in I\!N} \left\{ C_m \frac{a_{\lambda,\alpha_1,m}}{\tilde{b}_{\nu,N,n_m}} \right\},\ \nu, \lambda \in I\!N.$$

So the websets in $L(X, Y_N)$, $N \in I\!N$ can be described as matrices for which the components satisfy the estimates given in the last proposition. We can use this fact to transform the inclusions in our conditions on $\mathrm{Proj}^1 \mathcal{X} = 0$ into inequalities containing the weights $(\alpha_{\lambda,K,m})_{\lambda,K,m\in I\!N}$ and $\left(\tilde{b}_{\nu,N,n}\right)_{\nu,N,n\in I\!N}$:

Lemma 5.6 *For $N, M, K, r, s, t \in I\!N$, sequences $(\alpha_\nu)_{\nu \in I\!N}$, $(\beta_\nu)_{\nu \in I\!N}$ $(\gamma_\nu)_{\nu \in I\!N}$ of natural numbers the following are equivalent:*
i) There exist $R, S > 0$ such that

$$C^M_{\alpha_{|s}} \subset R\, C^K_{\beta_{|t}} + S\, C^N_{\gamma_{|r}}$$

ii) There exist $R, S > 0$ such that for all sequences $(n_m)_{m \in I\!N}$ and $(C_m)_{m \in I\!N}$ with $n_m = \alpha_{m+1}$ and $C_m = 1$ for $m = 1, \ldots, s-1$ there are sequences $(p_m)_{m \in I\!N}$ and $(D_m)_{m \in I\!N}$ with $p_m = \beta_{m+1}$ and $D_m = 1$ for $m = 1, \ldots, t-1$ and $(q_m)_{m \in I\!N}$ and $(E_m)_{m \in I\!N}$ with $q_m = \gamma_{m+1}$ and $E_m = 1$ for $m = 1, \ldots, r-1$ such that for all $\nu, \lambda \in I\!N$ the following inequality (which we will call (*)) holds:

$$\inf_{m \in I\!N} \left\{ C_m \frac{a_{\lambda,\alpha_1,m}}{\tilde{b}_{\nu,M,n_m}} \right\} \leq max \left\{ R \inf_{m \in I\!N} \left\{ D_m \frac{a_{\lambda,\beta_1,m}}{\tilde{b}_{\nu,K,p_m}} \right\}, S \inf_{m \in I\!N} \left\{ E_m \frac{a_{\lambda,\gamma_1,m}}{\tilde{b}_{\nu,N,q_m}} \right\} \right\}$$

Proof: We first show that i) implies ii). Let $(n_m)_{m \in I\!N}$ and $(C_m)_{m \in I\!N}$ with $n_m = \alpha_{m+1}$ and $C_m = 1$ for $m = 1, \ldots, s-1$ be given. We set for all $\nu, \lambda \in I\!N$

$$T_{\nu,\lambda} := \inf_{m \in I\!N} \left\{ C_m \frac{a_{\lambda,\alpha_1,m}}{\tilde{b}_{\nu,M,n_m}} \right\}$$

Using the last proposition, we see that $(T_{\nu,\lambda})_{\nu,\lambda \in I\!N}$ is an element of $C^M_{\alpha_{|s}}$, so we find matrices $(V_{\nu,\lambda})_{\nu,\lambda \in I\!N}$ and $(W_{\nu,\lambda})_{\nu,\lambda \in I\!N}$ such that for all $\nu, \lambda \in I\!N$

$$T_{\nu,\lambda} = R\, V_{\nu,\lambda} + S\, W_{\nu,\lambda}. \tag{1}$$

These matrices belong to $C^K_{\beta_{|t}}$ and $C^N_{\gamma_{|r}}$ respectively, so there are sequences $(p_m)_{m \in I\!N}$ and $(D_m)_{m \in I\!N}$ with $p_m = \beta_{m+1}$ and $D_m = 1$ for $m = 1, \ldots, t-1$ and $(q_m)_{m \in I\!N}$ and $(E_m)_{m \in I\!N}$ with $q_m = \gamma_{m+1}$ and $E_m = 1$ for $m = 1, \ldots, r-1$ fulfilling for all $\nu, \lambda \in I\!N$ the continuity estimates

$$|V_{\nu,\lambda}| \leq \inf_{m \in I\!N} \left\{ D_m \frac{a_{\lambda,\beta_1,m}}{\tilde{b}_{\nu,K,p_m}} \right\}$$

$$\text{and } |W_{\nu,\lambda}| \leq \inf_{m \in I\!N} \left\{ E_m \frac{a_{\lambda,\gamma_1,m}}{\tilde{b}_{\nu,N,q_m}} \right\},$$

thus we immediately get from (1) for all $\nu, \lambda \in I\!N$

$$\inf_{m \in I\!N} \left\{ C_m \frac{a_{\lambda,\alpha_1,m}}{\tilde{b}_{\nu,M,n_m}} \right\} \leq 2\, max \left\{ R \inf_{m \in I\!N} \left\{ D_m \frac{a_{\lambda,\beta_1,m}}{\tilde{b}_{\nu,K,p_m}} \right\}, S \inf_{m \in I\!N} \left\{ E_m \frac{a_{\lambda,\gamma_1,m}}{\tilde{b}_{\nu,N,q_m}} \right\} \right\}$$

Assuming now ii), we take an operator $(T_{\nu,\lambda})_{\nu,\lambda \in I\!N} \in C^M_{\alpha_{|s}}$. Then lemma 5.4 supplies us with sequences $(n_m)_{m \in I\!N}$ and $(C_m)_{m \in I\!N}$ with $n_m = \alpha_{m+1}$ and $C_m = 1$ for $m = 1, \ldots, s-1$ such that

$$\sup_{\nu,\lambda,m \in I\!N} \left\{ \frac{|T_{\nu,\lambda}|}{C_m} \frac{\tilde{b}_{\nu,M,n_m}}{a_{\lambda,\alpha_1,m}} \right\} \leq 1$$

Now ii) gives us $(p_m)_{m\in I\!N}$ and $(D_m)_{m\in I\!N}$ with $p_m = \beta_{m+1}$ and $D_m = 1$ for $m = 1, \ldots, t-1$ and $(q_m)_{m\in I\!N}$ and $(E_m)_{m\in I\!N}$ with $q_m = \beta_m$ and $E_m = 1$ for $m = 1, \ldots, r-1$ such that for all $\nu, \lambda \in I\!N$ (*) holds, the latter being equivalent to

$$\sup_{m\in I\!N}\left\{\frac{\tilde{b}_{\nu,M,n_m}}{C_m\, a_{\lambda,\alpha_1,m}}\right\} \geq min\left\{\frac{1}{R}\sup_{m\in I\!N}\left\{\frac{\tilde{b}_{\nu,K,p_m}}{D_m\, a_{\lambda,\beta_1,m}}\right\}, \frac{1}{S}\sup_{m\in I\!N}\left\{\frac{\tilde{b}_{\nu,N,q_m}}{E_m\, a_{\lambda,\gamma_1,m}}\right\}\right\} \tag{2}$$

Let us construct the decomposition of $(T_{\nu,\lambda})_{\nu,\lambda\in I\!N}$: Let

$$J_1 := \left\{\nu,\lambda \in I\!N : \sup_{m\in I\!N}\left\{\frac{\tilde{b}_{\nu,M,n_m}}{C_m\, a_{\lambda,\alpha_1,m}}\right\} \geq \frac{1}{R}\sup_{m\in I\!N}\left\{\frac{\tilde{b}_{\nu,K,p_m}}{D_m\, a_{\lambda,\beta_1,m}}\right\}\right\}$$

and

$$J_2 := I\!N \times I\!N - J_1$$

We set for $\nu, \lambda \in I\!N$

$$V_{\nu,\lambda} := \begin{cases} \frac{1}{R} T_{\nu,\lambda} & : (\nu,\lambda) \in J_1 \\ 0 & : (\nu,\lambda) \in J_2 \end{cases} \qquad W_{\nu,\lambda} := \begin{cases} 0 & : (\nu,\lambda) \in J_1 \\ \frac{1}{S} T_{\nu,\lambda} & : (\nu,\lambda) \in J_2 \end{cases}$$

Then we have $T_{\nu,\lambda} = R\, V_{\nu,\lambda} + S\, W_{\nu,\lambda}$ for all $\nu, \lambda \in I\!N$ and to prove i) we have to verify that $(V_{\nu,\lambda})_{\nu,\lambda\in I\!N}$ and $(W_{\nu,\lambda})_{\nu,\lambda\in I\!N}$ belong to $C^K_{\beta_{|t}}$ and $C^N_{\gamma_{|r}}$ respectively. For this, we observe that (2) implies for all $\nu, \lambda \in I\!N$

$$\begin{aligned} \sup_{m\in I\!N}\left\{\frac{|V_{\nu,\lambda}|}{D_m}\frac{\tilde{b}_{\nu,K,p_m}}{a_{\lambda,\beta_1,m}}\right\} &\leq \frac{1}{R}\sup_{m\in I\!N}\left\{\frac{|T_{\nu,\lambda}|}{D_m}\frac{\tilde{b}_{\nu,K,p_m}}{a_{\lambda,\beta_1,m}}\right\} \\ &\leq \sup_{m\in I\!N}\left\{\frac{|T_{\nu,\lambda}|}{C_m}\frac{\tilde{b}_{\nu,M,n_m}}{a_{\lambda,\alpha_1,m}}\right\} \\ &\leq 1, \end{aligned}$$

the last inequality coming from Proposition 5.5, because $(T_{\nu,\lambda})_{\nu,\lambda\in I\!N}$ is a member of $C^M_{\alpha_{|s}}$. Analogously

$$\sup_{m\in I\!N}\left\{\frac{|W_{\nu,\lambda}|}{E_m}\frac{\tilde{b}_{\nu,N,q_m}}{a_{\lambda,\gamma_1,m}}\right\} \leq 1.$$

This completes the proof.

5.2 Power series spaces of (PLS) -type

We will now take a closer look at the spaces $\Lambda_{r,s}(x,y)$, fixing some notation concerning the (LS) - steps and determining under which conditions these spaces are Fréchet - Schwartz respectively (LS) -spaces.

Definition 5.7 *Let a matrix of weights*

$$A = (a_{\lambda,K,m})_{\lambda,K,m\in I\!N} = (\exp(r_K x_\lambda - s_m y_\lambda))_{\lambda,K,m\in I\!N}$$

with sequences $(r_K)_{K\in I\!N}$ and $(s_m)_{m\in I\!N}$ as in section 5.1 be given.

i) For every $K \in I\!N$ and $1 \leq p \leq \infty$ we put

$$\Lambda_s^{l_p *}(r_K, y) := \bigcup_{m=1}^{\infty} l_p(\exp(r_K x_\lambda - s_m y_\lambda))_{\lambda \in I\!N}$$

where the union is equipped with the inductive topology with respect to the inclusions. If for a $K \in I\!N$ we have $r_K = 0$, we put

$$\Lambda_s^{l_p *}(y) := \Lambda_s^{l_p *}(0, y)$$

ii) For every $m \in I\!N$ we define the Frechet space

$$\Lambda_r^{l_p}(x, s_m) := \lim_{\leftarrow K} l_p(\exp(r_K x_\lambda - s_m y_\lambda))_{\lambda \in I\!N}$$

and for any $m \in I\!N$ with $s_m = 0$

$$\Lambda_r^{l_p}(x) := \Lambda_r^{l_p}(x, 0)$$

which are the well known Fréchet power series spaces (cf. [MV, Chapter 29]).

Analogously we define the spaces $\Lambda_s^{c_0 *}(r_K, y)$, $\Lambda_s^{c_0 *}(y)$, $\Lambda_r^{c_0}(x, s_m)$ and $\Lambda_r^{c_0}(x)$.

Remark:

i) Using the notation of definition 5.1, we may write the space $\Lambda_{r,s}^{l_1}(x, y)$ as

$$\Lambda_{r,s}^{l_1}(x, y) = \lim_{\leftarrow K} \Lambda_s^{l_1 *}(r_K, y)$$

ii) Observe that for the space $\Lambda_r^{c_0}(x)$ the theorem of Dieudonne and Gomes (cf. [MV], 27.9) implies that $\Lambda_r^{c_0}(x) = \Lambda_r^{l_\infty}(x)$ iff $\lim\limits_{\lambda \in I\!N} x_\lambda = 0$.

In some of the cases we will consider, it will be important to know, under which conditions the space $\Lambda_{r,s}^{l_1}(x, y)$ will be a Fréchet- resp. an (LB) space:

Proposition 5.8 *For the space $\Lambda_{r,s}^{l_p}(x, y)$ with $1 \leq p \leq \infty$ we have*

i) $\Lambda_{r,0}^{l_p}(x, y) = \Lambda_r^{l_p}(x)$ iff $\inf\limits_{\lambda \in I\!N} \frac{x_\lambda}{y_\lambda} > 0$

ii) $\Lambda_{r,0}^{l_p}(x, y) = \Lambda_0^{l_p *}(y)$ iff $\lim\limits_{\lambda \to \infty} \frac{x_\lambda}{y_\lambda} = 0$

iii) $\Lambda_{r,\infty}^{l_p}(x, y) = \Lambda_r^{l_p}(x)$ iff $\lim\limits_{\lambda \to \infty} \frac{x_\lambda}{y_\lambda} = \infty$

iv) $\Lambda_{r,\infty}^{l_p}(x, y) = \Lambda_\infty^{l_p *}(y)$ iff $\sup\limits_{\lambda \in I\!N} \frac{x_\lambda}{y_\lambda} < \infty$

The analogous characterisation holds for the space $\Lambda_{r,s}^{c_0}(x, y)$

Proof: We will prove the theorem for $1 \leq p < \infty$, as the other two cases are proved in a completely analogous way. Recall that the space $\Lambda_{r,s}^{l_p}(x,y)$ has the form

$$\Lambda_{r,s}^{l_p}(x,y) = \bigcap_{K=1}^{\infty} \Lambda_s^{l_p *}(r_K, y)$$

where

$$\Lambda_s^{l_p *}(r_K, y) = \bigcup_{m=1}^{\infty} l_p\left((\exp(r_K x_\lambda - s_m y_\lambda))_{\lambda \in I\!N}\right)$$

In i) we have $(s_m)_{m \in I\!N} = (-m^{-1})_{m \in I\!N}$, thus for all $K, m, \lambda \in I\!N$

$$\exp(r_K x_\lambda - s_m y_\lambda) \geq \exp(r_K x_\lambda)$$

which implies for all $K, m \in I\!N$

$$l_p\left((\exp(r_K x_\lambda - s_m y_\lambda))_{\lambda \in I\!N}\right) \hookrightarrow \left(l_p(\exp(r_K x_\lambda))_{\lambda \in I\!N}\right)$$

and thus, as $\Lambda_s^{l_p *}(r_K, y)$ is equipped with the inductive limit topology of the union,

$$\Lambda_s^{l_p *}(r_K, y) \hookrightarrow \left(l_p(\exp(r_K x_\lambda))_{\lambda \in I\!N}\right)$$

for all $K \in I\!N$. From the definition of the projective limit we get

$$\Lambda_{r,s}^{l_p}(x,y) \hookrightarrow \Lambda_s^{l_p *}(r_K, y)\,,\; K \in I\!N, \tag{1}$$

so together with the last inclusion we get for all $K \in I\!N$

$$\Lambda_{r,s}^{l_p}(x,y) \hookrightarrow l_p\left((\exp(r_K x_\lambda))_{\lambda \in I\!N}\right)$$

and thus

$$\Lambda_{r,s}^{l_p}(x,y) \hookrightarrow \bigcap_{K=1}^{\infty} l_p\left((\exp(r_K x_\lambda))_{\lambda \in I\!N}\right) = \Lambda_r^{l_p}(x)$$

So for $s = 0$ $\Lambda_r^{l_p}(x) = \Lambda_{r,s}^{l_p}(x,y)$ is equivalent to

$$\Lambda_r^{l_p}(x) \hookrightarrow \Lambda_{r,s}^{l_p}(x,y).$$

Now because of (1) and due to the properties of the projective limit, this is equivalent to the fact that for all $K \in I\!N$

$$\Lambda_r^{l_p}(x) \hookrightarrow \Lambda_s^{l_p *}(r_K, y)\,,$$

which by Grothendieck's factorization theorem is equivalent to the existence of $(m(K))_{K \in I\!N}$ such that for all $K \in I\!N$

$$\Lambda_r^{l_p}(x) \hookrightarrow l_p\left(\left(\exp\left(r_K x_\lambda - s_{m(K)} y_\lambda\right)\right)_{\lambda \in I\!N}.\right)$$

The last continuous inclusion means that there are sequences $(C_K)_{K \in I\!N}$ and $(k(K))_{K \in I\!N}$ such that $k(K) > K$, $K \in I\!N$, and for all $(t_\lambda)_{\lambda \in I\!N} \in \Lambda_r^{l_p}(x)$ and $K \in I\!N$

$$\sum_{\lambda=1}^{\infty} |t_\lambda|^p \exp\left(p\,\left(r_K x_\lambda - s_{m(K)} y_\lambda\right)\right) \leq C_K \sum_{\lambda=1}^{\infty} |t_\lambda|^p \exp\left(p\,\left(r_{k(K)} x_\lambda\right)\right).$$

Inserting the canonical unit vectors, we see that this is equivalent to

$$\exp\left(y_\lambda\left(\left(r_K - r_{k(K)}\right)\frac{x_\lambda}{y_\lambda} - s_{m(K)}\right)\right) \leq (C_K)^{\frac{1}{p}}, \; K,\lambda \in I\!N \tag{2}$$

Consequently, to show i), we have to show that the existence of sequences $(C_K)_{K\in I\!N}$, $(m(K))_{K\in I\!N}$ and $(k(K))_{K\in I\!N}$ such that (2) holds is equivalent to $\inf\limits_{\lambda\in I\!N}\frac{x_\lambda}{y_\lambda} > 0$. Let such sequences be given and assume the latter is not true, then there is an infinite subset $I \subset I\!N$ such that for all $\lambda \in I$

$$\frac{x_\lambda}{y_\lambda} < \delta := \frac{1}{r_1 - r_{k(1)}}\frac{s_{m(1)}}{2}$$

Then for all $\lambda \in I$

$$\begin{aligned} C_1 &\geq \exp\left(y_\lambda\left(\left(r_1 - r_{k(1)}\right)\tfrac{x_\lambda}{y_\lambda} - s_{m(1)}\right)\right) \\ &\geq \exp\left(y_\lambda\left(\left(r_1 - r_{k(1)}\right)\delta - s_{m(1)}\right)\right) \quad = \quad \exp\left(-y_\lambda\tfrac{s_{m(1)}}{2}\right) \end{aligned}$$

As

$$\lim_{\substack{\lambda\to\infty \\ \lambda\in I}} \exp\left(-y_\lambda\frac{s_{m(1)}}{2}\right) = \infty,$$

we reach a contradiction.
If $\delta := \inf\limits_{\lambda\in I\!N}\frac{x_\lambda}{y_\lambda} > 0$, we set $C_K := 1$ and $k(K) := K+1$ and find $(m(K))_{K\in I\!N}$ such that for all $K \in I\!N$ $\frac{s_{m(K)}}{r_K - r_{K+1}} < \delta$. Then we get for all $K,\lambda \in I\!N$

$$\begin{aligned} \exp\left(y_\lambda\left(\left(r_K - r_{k(K)}\right)\tfrac{x_\lambda}{y_\lambda} - s_{m(K)}\right)\right) &\leq \exp\left(y_\lambda\left(\left(r_K - r_{K+1}\right)\delta - s_{m(K)}\right)\right) \\ &\leq \exp(0) = C_K \end{aligned}$$

and the proof of i) is finished.
In ii) we also have $s = 0$, i.e. $(s_m)_{m\in I\!N} = (-m^{-1})_{m\in I\!N}$. We first show the claim for $r = \infty$, i.e. $(r_K)_{K\in I\!N} = (K)_{K\in I\!N}$. We have for all $K, m, \lambda \in I\!N$

$$\exp\left(r_K x_\lambda - s_m y_\lambda\right) \geq \exp\left(-s_m y_\lambda\right),$$

and therefore for all $K, m \in I\!N$

$$l_p\left(\left(\exp\left(r_K x_\lambda - s_m y_\lambda\right)\right)_{\lambda\in I\!N}\right) \hookrightarrow l_p\left(\left(\exp\left(-s_m y_\lambda\right)\right)_{\lambda\in I\!N}\right)$$

which implies

$$\Lambda^{l_p}_{r,0}(x,y) \hookrightarrow \Lambda^{l_p *}_0(y).$$

Now

$$\Lambda^{l_p *}_0(y) \hookrightarrow \Lambda^{l_p}_{r,0}(x,y)$$

iff for all $K \in I\!N$

$$\Lambda^{l_p *}_0(y) \hookrightarrow \Lambda^{l_p *}_0(r_K, y)$$

and that is equivalent to the fact that for all $K, m \in I\!N$ there is $n(K,m) \in I\!N$ with $n > m$ such that

$$l_p(\exp(-s_m y_\lambda))_{\lambda \in I\!N} \hookrightarrow l_p\Big(\exp\Big(r_K x_\lambda - s_{n(K,m)} y_\lambda\Big)\Big)_{\lambda \in I\!N}$$

or equivalently for all $K, m \in I\!N$ there are $n(K,m) \in I\!N$ and $C(K,m) > 0$ such that for all $\lambda \in I\!N$

$$C(K,m) \geq \exp\left(y_\lambda \left(r_K \frac{x_\lambda}{y_\lambda} + s_m - s_{n(K,m)}\right)\right) \tag{3}$$

To show ii) for $r = \infty$, we must therefore show, that (3) holds iff $\lim\limits_{\lambda \to \infty} \frac{x_\lambda}{y_\lambda} = 0$. If (3) holds and we assume that there is an infinite subset $I \subset I\!N$ such that $\delta := \inf\limits_{\lambda \in I} \frac{x_\lambda}{y_\lambda} > 0$, we find an $m_1 \in I\!N$ such that $r_1 \delta + s_{m_1} > 0$ and get for $\lambda \in I$ the contradiction

$$C(1, m_1) \geq \exp\left(y_\lambda \left(r_1 \frac{x_\lambda}{y_\lambda} + s_{m_1} - s_{n(1,m_1)}\right)\right) \geq \exp\left(y_\lambda \left(r_1 \delta + s_{m_1}\right)\right)$$

because the last term tends to infinity as $\lambda \in I$ tends to infinity. If on the other hand $\lim\limits_{\lambda \to \infty} \frac{x_\lambda}{y_\lambda} = 0$, we can find for every $K, m \in I\!N$ a λ_0 such that for $\lambda \geq \lambda_0$

$$r_K \frac{x_\lambda}{y_\lambda} + s_m - s_{m+1} < 0,$$

thus there is $C(K,m) > 0$ such that (3) holds for all $\lambda \in I\!N$ (with $n(K,m) = m+1$, $m \in I\!N$), and we have shown ii) for $r = \infty$.
If $r = 0$, i.e. $(r_K)_{K \in I\!N} = \left(-K^{-1}\right)_{K \in I\!N}$, we get by the estimate

$$\exp(r_K\, x_\lambda + s_m\, y_\lambda) \leq \exp(s_m\, y_\lambda) \;\; K, m, \lambda \in I\!N$$

the inclusion

$$\Lambda_0^{l_p *}(y) \hookrightarrow \Lambda_{0,0}^{l_p}(x,y)$$

and we have to show that $\lim\limits_{\lambda \to \infty} \frac{x_\lambda}{y_\lambda} = 0$ iff for all $K, m \in I\!N$ exist $n(K,m) \in I\!N$ and $C(K,m) > 0$ such that

$$\exp\Big(-r_K x_\lambda + (s_m - s_{n(K,m)}) y_\lambda\Big) \leq C(K,m)$$

which follows by exactly the same arguments as in (3) for $r = \infty$.
The proofs of iii) and iv) run along the same lines as i) and ii): We have $(s_m)_{m \in I\!N} = (m)_{m \in I\!N}$ and thus in iii) the estimate

$$\exp(r_K x_\lambda - s_m y_\lambda) \leq \exp(r_K x_\lambda), \;\; K, m, \lambda \in I\!N$$

shows

$$\Lambda_r^{l_p}(x) \hookrightarrow \Lambda_{r,\infty}^{l_p}(x,y)$$

and the opposite inclusion is equivalent to the fact that for all $N \in I\!N$ there exists $K \in I\!N$ such that for all $m \in I\!N$

$$\Lambda_{r,\infty}^{l_p}(x,y) \cap l_p(\exp(r_K x_\lambda - s_m y_\lambda))_{\lambda \in I\!N} \hookrightarrow l_p(\exp(r_N x_\lambda))_{\lambda \in I\!N}$$

the last continuous inclusion being equivalent to the existence of $C_m > 0$ such that

$$\exp\left(\left(r_N - r_K\right) x_\lambda + s_m y_\lambda\right) \leq C_m$$

Analogous arguments as before give that this is equivalent to $\lim\limits_{\lambda \to \infty} \frac{x_\lambda}{y_\lambda} = \infty$.
In iv) we have for $r = \infty$

$$\Lambda^{l_p}_{r,s}(x,y) \hookrightarrow \Lambda^{l_p *}_s(y)$$

and the opposite inclusion is equivalent to the fact that for all $K, m \in I\!N$ there are $n(K,m)$ and $C(K,m) > 0$ such that

$$\exp\left(r_K x_\lambda + \left(s_m - s_{n(K,m)}\right) y_\lambda\right) \leq C(K,m)$$

which is equivalent to $\sup\limits_{\lambda \in I\!N} \frac{x_\lambda}{y_\lambda} < \infty$. For $r = 0$ we have

$$\Lambda^{l_p *}_s(y) \hookrightarrow \Lambda^{l_p}_{0,\infty}(x,y)$$

and the opposite unclusion is equivalent to the existence of sequences $(n(K,m))_{K,m \in I\!N}$ and $(C(K,m))_{K,m \in I\!N}$ such that

$$\exp\left(-r_K x_l a + \left(s_m - s_{n(K,m)}\right) y_\lambda\right) \leq C(K,m)$$

which is equivalent to $\sup\limits_{\lambda \in I\!N} \frac{x_\lambda}{y_\lambda} < \infty$. Thus the proof is finished.

5.3 Local splitting for (PLS) - power series spaces

In the following we will consider spaces $E = \Lambda^{l_1}_{r,s}(x,y)$ with r, s, x, y as before and $F = \Lambda^{c_0}_{p,q}(v,w)$, where F is defined by a matrix

$$B = \left(b_{\nu,N,n}\right)_{\nu,N,n \in I\!N} = \left(\exp\left(p_N v_\nu - q_n w_\nu\right)\right)_{\nu,N,n \in I\!N},$$

so

$$F = \Lambda^{c_0}_{p,q}(v,w) = \lim_{\leftarrow N} \Lambda^{c_0 *}_s(p_N, w)$$

with sequences $v = (v_k)_{k \in I\!N}$, $w = (w_k)_{k \in I\!N}$ such that $\lim\limits_{k \to \infty} w_k = \infty$, and $(p_k)_{k \in I\!N}$ and $(q_k)_{k \in I\!N}$, putting $p := \lim\limits_{k \to \infty} p_k$ and $q := \lim\limits_{k \to \infty} q_k$. We will treat the case where $r = q = \infty$, as in this case we have local splitting. To see this, we need the following proposition, which contains only well known facts (cf. [MV, Theorem 25.19, Theorem 26.4]):

Proposition 5.9 *If we are given a short exact sequence*

$$0 \to X \overset{S}{\to} Y \overset{T}{\to} Z \to 0 \tag{1}$$

of (LS)-spaces, then the dual sequence

$$0 \to Z' \overset{T'}{\to} Y' \overset{S'}{\to} X' \to 0 \tag{2}$$

is a short exact sequence of Fréchet-Schwartz spaces. The sequence (1) splits iff the sequence (2) splits.

The following follows from ([V1, Corollary 4.4]):

Theorem 5.10 *For any $s \in \{0, \infty\}$ we have*

$$\mathrm{Ext}^1_{(F)}\left(\Lambda^{l_1}_{\infty}(x), \Lambda^{l_\infty}_{s}(v)\right) = 0$$

Thus we see that for $q = \infty$ we have local splitting:

Proposition 5.11 *For every short exact sequence*

$$0 \to \Lambda^{c_0}_{p,\infty}(v, w) \to X \to \Lambda^{l_1}_{r,s}(x, y) \to 0$$

of (PLS)-spaces we have local splitting.

Proof: Given a short exact sequence

$$0 \to \Lambda^{c_0}_{p,\infty}(v, w) \to X \to \Lambda^{l_1}_{r,s}(x, y) \to 0$$

with a (PLS)-space X, we get for all $N \in I\!N$ local sequences

$$0 \to \Lambda^{c_0 *}_{\infty}\left(p_{K(N)}, w\right) \to X_N \to \Lambda^{l_1 *}_{s}\left(r_{L(N)}, y\right) \to 0.$$

with suitable sequences $(K(N))_{N \in I\!N}$ and $(L(N))_{N \in I\!N}$. The dual sequences

$$0 \to \Lambda^{l_\infty}_{s}\left(r_{L(N)}, y\right) \to X'_N \to \Lambda^{l_1}_{\infty}\left(p_{K(N)}, w\right) \to 0$$

split for every $N \in I\!N$ because of Theorem 5.10, so we get local splitting by 5.9.

5.4 Splitting when the quotient is of infinite projective type and the subspace has infinite (LS) - steps

In order to show

$$\mathrm{Ext}^1\left(\Lambda^{l_1}_{\infty,s}(x, y), \Lambda^{c_0}_{p,\infty}(v, w)\right) = 0$$

we must show that we have $\mathrm{Proj}^1 \mathcal{L}(E, F) = 0$. By Corollary 3.7 and Lemma 5.6 it will be sufficient to show the following condition :

$$(L)$$

There is a sequence $(\gamma_k)_{k \in I\!N}$ of natural numbers such that for every $N \in I\!N$ there is an $M \in I\!N$ with $M \geq N$ such that for all $K \in I\!N$ with $K \geq M$ and all sequences $(\alpha_k)_{k \in I\!N}$ of natural numbers we $\beta \in I\!N$ and an $s \in I\!N$ such that for all $(C_k)_{k \in I\!N}$ and $(l_k)_{k \in I\!N}$ with $C_k = 1$ and $l_k = \alpha_{k+1}$ for $1 \leq k \leq s-1$ there are $(D_k)_{k \in I\!N}$ and $(m_k)_{k \in I\!N}$ and $(E_k)_{k \in I\!N}$ and $(n_k)_{k \in I\!N}$ with $E_k = 1$ and $n_k = \gamma_{k+1}$ for $1 \leq k \leq N-1$ such that for all $\nu, \lambda \in I\!N$ we have

$$\inf_{k \in I\!N}\left\{C_k \frac{a_{\lambda,\alpha_1,k}}{b_{\nu,M,l_k}}\right\} \leq max\left\{\inf_{k \in I\!N}\left\{D_k \frac{a_{\lambda,\beta,k}}{b_{\nu,K,m_k}}\right\}, \inf_{k \in I\!N}\left\{E_k \frac{a_{\lambda,\gamma_1,k}}{b_{\nu,N,n_k}}\right\}\right\}$$

Observe that we do not have to renorm the given spaces as we are in the case where the second space consists of (LS) - steps of infinite type.

5.4.1 The case of the quotient having (LS) - steps of infinite type

We assume that we have

$$A = (\exp(Kx_\lambda - ky_\lambda))_{\lambda,K,k\in I\!N}$$

and

$$B = (\exp(p_N v_\nu - nw_\nu))_{\nu,N,n\in I\!N}.$$

where $(p_N)_{N\in I\!N} = (N)_{N\in I\!N}$ or $(-N^{-1})_{N\in I\!N}$. Inserting the definition of the matrices A and B and taking logarithms on both sides the inequality in (L) reads thus:

$$(L)_{\infty,\infty,p,\infty}$$

$$\alpha_1 x_\lambda - p_M v_\nu + \inf_{k\in I\!N}\{\ln C_k + l_k w_\nu - ky_\lambda\} \leq max\{$$

$$\beta_1 x_\lambda - p_K v_\nu + \inf_{k\in I\!N}\{\ln D_k + m_k w_\nu - ky_\lambda\},$$

$$\gamma_1 x_\lambda - p_N v_\nu + \inf_{k\in I\!N}\{\ln E_k + n_k w_\nu - ky_\lambda\} \qquad \}$$

To show this inequality with the quantifiers demanded in (L), we first need the following lemma:

Lemma 5.12 *For $N, M, K \in I\!N$ with $N \leq M \leq K$, sequences $(\alpha_k)_{k\in I\!N}$, $(\beta_k)_{k\in I\!N}$ and $(\gamma_k)_{k\in I\!N}$ of natural numbers and $r, s \in I\!N$ we have for all $\nu, \lambda \in I\!N$*

$$(\tilde{L})_{\infty,\infty,p,\infty}$$

$$\alpha_1 x_\lambda - p_M v_\nu + \inf_{k=1}^{s-1}\{\alpha_{k+1} w_\nu - ky_\lambda\} \leq max\{$$

$$\beta_1 x_\lambda - p_K v_\nu + \inf_{k=1}^{r-1}\{\beta_{k+1} w_\nu - ky_\lambda\},$$

$$\gamma_1 x_\lambda - p_N v_\nu + \inf_{k=1}^{N-1}\{\gamma_{k+1} w_\nu - ky_\lambda\} \qquad \}$$

iff for all $(C_k)_{k\in I\!N}$ and $(l_k)_{k\in I\!N}$ with $C_k = 1$ and $l_k = \alpha_{k+1}$ for $1 \leq k \leq s-1$ there are $(D_k)_{k\in I\!N}$ and $(m_k)_{k\in I\!N}$ and $(E_k)_{k\in I\!N}$ and $(n_k)_{k\in I\!N}$ with $E_k = 1$ and $n_k = \gamma_{k+1}$ for $1 \leq k \leq N-1$ such that for all $\nu, \lambda \in I\!N$ $(L)_{\infty,\infty,p,\infty}$ holds.

Proof: We assume $(\tilde{L})_{\infty,\infty,p,\infty}$. Let us use the following abbreviations: For all $\nu, \lambda \in I\!N$ we put

$$O_{\nu,\lambda} := \alpha_1 x_\lambda - p_M v_\nu,$$

$$P_{\nu,\lambda} := \beta_1 x_\lambda - p_K v_\nu,$$

$$Q_{\nu,\lambda} := \gamma_1 x_\lambda - p_N v_\nu.$$

Let $k_O \in \{1, \ldots, s-1\}$, $k_P \in \{1, \ldots, r-1\}$ and $k_Q \in \{1, \ldots, N-1\}$ be the indices where

$$\alpha_{k_O+1} w_\nu - k_O y_\lambda = \inf_{k=1}^{s-1} \{\alpha_{k+1} w_\nu - k y_\lambda\}$$

$$\beta_{k_P+1} w_\nu - k_P y_\lambda = \inf_{k=1}^{r-1} \{\beta_{k+1} w_\nu - k y_\lambda\}$$

$$\alpha_{k_Q+1} w_\nu - k_Q y_\lambda = \inf_{k=1}^{N-1} \{\gamma_{k+1} w_\nu - k y_\lambda\}$$

Let

$$J_1 := \Big\{(\nu, \lambda) \in I\!N^2 : O_{\nu,\lambda} + \alpha_{k_O+1} w_\nu - k_O y_\lambda \le P_{\nu,\lambda} + \beta_{k_P+1} w_\nu - k_P y_\lambda\Big\}$$

and

$$J_2 := I\!N^2 - J_1.$$

Let $(C_k)_{k \in I\!N}$ and $(l_k)_{k \in I\!N}$ be given. We will show that

i) For every $k \in I\!N$ with $k \ge r$ there is $\hat{k} \in I\!N$ with $\hat{k} \ge s$ and D_k and m_k such that for all $(\nu, \lambda) \in J_1$

$$O_{\nu,\lambda} + \ln C_{\hat{k}} + l_{\hat{k}} w_\nu - \hat{k} y_\lambda \le P_{\nu,\lambda} + \ln D_k + m_k w_\nu - k y_\lambda \tag{1}$$

ii) For every $k \in I\!N$ with $k \ge N$ there is $\tilde{k} \in I\!N$ with $\tilde{k} \ge s$ and E_k as well as n_k such that for all $(\nu, \lambda) \in J_2$

$$O_{\nu,\lambda} + \ln C_{\tilde{k}} + l_{\tilde{k}} w_\nu - \tilde{k} y_\lambda \le P_{\nu,\lambda} + \ln E_k + n_k w_\nu - k y_\lambda. \tag{2}$$

Then setting $D_k = 1$ and $m_k = \beta_{k+1}$ for $k = 1, \ldots, r-1$ and $E_k = 1$ and $n_k = \gamma_{k+1}$ for $k = 1, \ldots, N-1$ we will get from (1) for $(\nu, \lambda) \in J_1$

$$O_{\nu,\lambda} + \inf_{k \in I\!N} \{\ln C_k + l_k w_\nu - k y_\lambda\} \le P_{\nu,\lambda} + \inf_{k \in I\!N} \{\ln D_k + m_k w_\nu - k y_\lambda\}$$

and from (2) for $(\nu, \lambda) \in J_2$

$$O_{\nu,\lambda} + \inf_{k \in I\!N} \{\ln C_k + l_k w_\nu - k y_\lambda\} \le Q_{\nu,\lambda} + \inf_{k \in I\!N} \{\ln E_k + n_k w_\nu - k y_\lambda\}$$

which will imply $(L)_{\infty,\infty,p,\infty}$. To show i), let $k \ge r$ be given. According to the definition of J_1, we know that for $(\nu, \lambda) \in J_1$

$$O_{\nu,\lambda} + \alpha_{k_O+1} w_\nu - k_O y_\lambda \le P_{\nu,\lambda} + \beta_{k_P+1} w_\nu - k_P y_\lambda \tag{3}$$

For all $\hat{k} \ge s$

$$\begin{aligned} O_{\nu,\lambda} + ln\, C_{\hat{k}} + l_{\hat{k}} w_\nu - \hat{k} y_\lambda &= O_{\nu,\lambda} + \alpha_{k_O+1} w_\nu - k_O y_\lambda \\ &+ \Big(\ln C_{\hat{k}} + l_{\hat{k}} w_\nu - \hat{k} y_\lambda - \alpha_{k_O+1} w_\nu + k_0 y_\lambda\Big) \end{aligned}$$

and for $k \geq r$

$$\begin{aligned} P_{\nu,\lambda} + \ln D_k + m_k w_\nu - k y_\lambda &= P_{\nu,\lambda} + \beta_{k_P+1} w_\nu - k_P y_\lambda \\ &+ (\ln D_k + m_k w_\nu - k y_\lambda - \beta_{k_P+1} w_\nu + k_P y_\lambda) \end{aligned}$$

Thus (3) will give us (1) if we can find $\hat{k} \in I\!N$, D_k and m_k such that

$$\ln C_{\hat{k}} + l_{\hat{k}} w_\nu - \hat{k} y_\lambda - \alpha_{k_O+1} w_\nu + k_0 y_\lambda \leq \ln D_k + m_k w_\nu - k y_\lambda - \beta_{k_P+1} w_\nu + k_P y_\lambda$$

or equivalently

$$\ln D_k - \ln C_{\hat{k}} + (\alpha_{k_O+1} - \beta_{k_P+1} + m_k - l_{\hat{k}})\, w_\nu + \left(k_P - k_O + \hat{k} - k\right) y_\lambda \geq 0$$

Now $k_O \in \{1, \ldots, s-1\}$ and $k_P \in \{1, \ldots, r-1\}$, so $\hat{k}$, m_k and D_k can be chosen to fulfill the last inequality, which shows i). The Proof of ii) runs along the same lines, the same calculation gives that one for a given $k \geq N$ has to find $\tilde{k} \geq s$ as well as E_k and n_k that fulfill

$$\ln E_k - \ln C_{\tilde{k}} + \left(\alpha_{k_O+1} - \gamma_{k_Q+1} + n_k - l_{\tilde{k}}\right) w_\nu + \left(k_Q - k_O + \tilde{k} - k\right) y_\lambda \geq 0$$

which can always be done.

Now assume $(L)_{\infty,\infty,p,\infty}$. Fix $\nu_0, \lambda_0 \in I\!N$ and observe that without loss of generality we can assume that for given $(C_k)_{k \in I\!N}$ and $(l_k)_{k \in I\!N}$ the $(D_k)_{k \in I\!N}$, $(m_k)_{k \in I\!N}$, $(E_k)_{k \in I\!N}$ and $(n_k)_{k \in I\!N}$ fulfill $D_k, E_k \geq C_k$, $k \in I\!N$ and $m_k, n_k \geq l_k$ and that the $(C_k)_{k \in I\!N}$ and $(l_k)_{k \in I\!N}$ are increasing. Now use $(L)_{\infty,\infty,p,\infty}$ on sequences $\left(C_k^j\right)_{k \in I\!N}$ and $\left(l_k^j\right)_{k \in I\!N}$, $j \in I\!N$ that fulfill

$$\begin{aligned} &C_k^j = 1,\ l_k^j = \alpha_{k+1}, \qquad j \in I\!N, k = 1, \ldots, s-1 \\ &\lim_{j \to \infty} C_k^j = \lim_{j \to \infty} l_k^j = \infty \quad k \geq s. \end{aligned} \tag{4}$$

We then find $\left(D_k^j\right)_{k \in I\!N}, \left(m_k^j\right)_{k \in I\!N}, \left(E_k^j\right)_{k \in I\!N}$ and $\left(n_k^j\right)_{k \in I\!N}$ for every $j \in I\!N$ fulfilling

$$\begin{aligned} &D_k^j = 1,\ m_k^j = \beta_{k+1}, \qquad j \in I\!N, k = 1, \ldots, r-1 \\ &\lim_{j \to \infty} D_k^j = \lim_{j \to \infty} m_k^j = \infty \quad k \geq r \end{aligned} \tag{5}$$

and

$$\begin{aligned} &E_k^j = 1,\ n_k^j = \gamma_{k+1}, \qquad j \in I\!N, k = 1, \ldots, N-1 \\ &\lim_{j \to \infty} E_k^j = \lim_{j \to \infty} n_k^j = \infty \quad k \geq N \end{aligned} \tag{6}$$

such that for all $j \in I\!N$

$$\begin{aligned} \alpha_1 x_{\lambda_0} - p_M v_{\nu_0} + \inf_{k \in I\!N} \left\{\ln C_k^j + l_k^j w_{\nu_0} - k y_{\lambda_0}\right\} \leq max\{ \\ \beta_1 x_{\lambda_0} - p_K v_{\nu_0} + \inf_{k \in I\!N} \left\{\ln D_k^j + m_k^j w_{\nu_0} - k y_{\lambda_0}\right\}, \\ \gamma_1 x_{\lambda_0} - p_N v_{\nu_0} + \inf_{k \in I\!N} \left\{\ln E_k^j + n_k^j w_{\nu_0} - k y_{\lambda_0}\right\} \quad \} \end{aligned}$$

Due to (4), (5) and (6) and due to our assumption that the $\left(C_k^j\right)_{k\in\mathbb{N}}$ $\left(l_k^j\right)_{k\in\mathbb{N}}$, $j\in\mathbb{N}$ are increasing, there is a $J\in\mathbb{N}$ such that for $j\geq J$

$$\begin{aligned}
\inf_{k\in\mathbb{N}}\left\{\ln C_k^j+l_k^j w_{\nu_0}-ky_{\lambda_0}\right\} &= \inf_{k=1}^{s-1}\left\{\alpha_{k+1}w_{\nu_0}-ky_{\lambda_0}\right\},\\
\inf_{k\in\mathbb{N}}\left\{\ln D_k^j+m_k^j w_{\nu_0}-ky_{\lambda_0}\right\} &= \inf_{k=1}^{r-1}\left\{\beta_{k+1}w_{\nu_0}-ky_{\lambda_0}\right\},\\
\text{and}\quad \inf_{k\in\mathbb{N}}\left\{\ln E_k^j+n_k^j w_{\nu_0}-ky_{\lambda_0}\right\} &= \inf_{k=1}^{N-1}\left\{\gamma_{k+1}w_{\nu_0}-ky_{\lambda_0}\right\}.
\end{aligned}$$

giving

$$\begin{aligned}
\alpha_1 x_{\lambda_0}-p_M v_{\nu_0}+\inf_{k=1}^{s-1}\left\{\alpha_{k+1}w_{\nu_0}-ky_{\lambda_0}\right\}\leq max\{\\
\beta_1 x_{\lambda_0}-p_K v_{\nu_0}+\inf_{k=1}^{r-1}\left\{\beta_{k+1}w_{\nu_0}-ky_{\lambda_0}\right\},\\
\gamma_1 x_{\lambda_0}-p_N v_{\nu_0}+\inf_{k=1}^{N-1}\left\{\gamma_{k+1}w_{\nu_0}-ky_{\lambda_0}\right\}\quad\}.
\end{aligned}$$

As $\nu_0,\lambda_0\in\mathbb{N}$ were chosen arbitrary, this shows $(\tilde{L})_{\infty,\infty,p,\infty}$.

That $(\tilde{L})_{\infty,\infty,p,\infty}$ is fulfilled in quite an easy way will follow from the following

Lemma 5.13 *For all sequences $(\gamma_k)_{k\in\mathbb{N}}$ and $(\alpha_k)_{k\in\mathbb{N}}$ of natural numbers and all numbers $N,M,K\in\mathbb{N}$ with $N<M<K$ there is a sequence $(\beta_k)_{k\in\mathbb{N}}$ such that for all $\nu,\lambda\in\mathbb{N}$*

$$\begin{aligned}
\alpha_1 x_{\lambda}-p_M v_{\nu}+\inf_{k=1}^{N-1}\left\{\alpha_{k+1}w_{\nu}-ky_{\lambda}\right\}\leq max\{\\
\beta_1 x_{\lambda}-p_K v_{\nu}+\inf_{k=1}^{N-1}\left\{\beta_{k+1}w_{\nu}-ky_{\lambda}\right\},\\
\gamma_1 x_{\lambda}-p_N v_{\nu}+\inf_{k=1}^{N-1}\left\{\gamma_{k+1}w_{\nu}-ky_{\lambda}\right\}\quad\}.
\end{aligned}$$

Proof: Let $(\gamma_k)_{k\in\mathbb{N}}$, $(\alpha_k)_{k\in\mathbb{N}}$, and $N,M,K\in\mathbb{N}$ with $N<M<K$ be given. Choose

$$\begin{aligned}
\beta_1 &> \alpha_1+\frac{p_K-p_M}{p_M-p_N}(\alpha_1-\gamma_1),\\
\beta_{k+1} &> \frac{p_K-p_N}{p_M-p_N}\alpha_N, \qquad k=1,\ldots,N-1,
\end{aligned}$$

and β_k arbitrary for $k\geq N+1$. We define as before

$$\begin{aligned}
J_1:=\Big\{(\nu,\lambda)\in\mathbb{N}^2:\quad &\alpha_1 x_\lambda-p_M v_\nu+\inf_{k=1}^{N-1}\left\{\alpha_{k+1}w_\nu-ky_\lambda\right\}\\
\leq\; &\beta_1 x_\lambda-p_K v_\nu+\inf_{k=1}^{N-1}\left\{\beta_{k+1}w_\nu-ky_\lambda\right\}\Big\}
\end{aligned}$$

and

$$J_2 := I\!N^2 - J_1.$$

If we show that for all $\nu, \lambda \in I\!N \in J_2$ we have

$$\alpha_1 x_\lambda - p_M v_\nu + \inf_{k=1}^{N-1} \{\alpha_{k+1} w_\nu - k y_\lambda\} \le \gamma_1 x_\lambda - p_N v_\nu + \inf_{k=1}^{N-1} \{\gamma_{k+1} w_\nu - k y_\lambda\} \qquad (1)$$

we will have shown the inequality in the claim. So take $\nu, \lambda \in J_2$. From the definition of J_1 we get

$$\alpha_1 x_\lambda - p_M v_\nu + \inf_{k=1}^{N-1} \{\alpha_{k+1} w_\nu - k y_\lambda\} > \beta_1 x_\lambda - p_K v_\nu + \inf_{k=1}^{N-1} \{\beta_{k+1} w_\nu - k y_\lambda\}$$

or equivalently

$$v_\nu > \frac{\beta_1 - \alpha_1}{p_K - p_M} x_\lambda + \frac{1}{p_K - p_M} \left(\inf_{k=1}^{N-1} \{\beta_{k+1} w_\nu - k y_\lambda\} - \inf_{k=1}^{N-1} \{\alpha_{k+1} w_\nu - k y_\lambda\} \right) \qquad (2)$$

(observe that $K > M$ and thus $p_K > p_M$).

(1) is equivalent to

$$(\gamma_1 - \alpha_1)\, x_\lambda + (p_M - p_N)\, v_\nu + \inf_{k=1}^{N-1} \{\gamma_{k+1} w_\nu - k y_\lambda\} - \inf_{k=1}^{N-1} \{\alpha_{k+1} w_\nu - k y_\lambda\} \ge 0$$

As $p_M > p_N$, we can insert (2) into this inequality and see that it will be sufficient to prove

$$\begin{aligned} & (\gamma_1 - \alpha_1)\, x_\lambda + \frac{p_M - p_N}{p_K - p_M} (\beta_1 - \alpha_1)\, x_\lambda \\ + \;& \frac{p_M - p_N}{p_K - p_M} \left(\inf_{k=1}^{N-1} \{\beta_{k+1} w_\nu - k y_\lambda\} - \inf_{k=1}^{N-1} \{\alpha_{k+1} w_\nu - k y_\lambda\} \right) \qquad (3) \\ + \;& \inf_{k=1}^{N-1} \{\gamma_{k+1} w_\nu - k y_\lambda\} - \inf_{k=1}^{N-1} \{\alpha_{k+1} w_\nu - k y_\lambda\} \ge 0 \end{aligned}$$

or equivalently

$$\begin{aligned} & \left(\frac{p_M - p_N}{p_K - p_M} (\beta_1 - \alpha_1) - (\alpha_1 - \gamma_1) \right) x_\lambda \\ + \;& \frac{p_M - p_N}{p_K - p_M} \inf_{k=1}^{N-1} \{\beta_{k+1} w_\nu - k y_\lambda\} - \frac{p_K - p_N}{p_K - p_M} \inf_{k=1}^{N-1} \{\alpha_{k+1} w_\nu - k y_\lambda\} \qquad (4) \\ + \;& \inf_{k=1}^{N-1} \{\gamma_{k+1} w_\nu - k y_\lambda\} \ge 0 \end{aligned}$$

The first term is nonnegative due to the choice of β_1. Observe further that

$$\begin{aligned} \inf_{k=1}^{N-1} \{\gamma_{k+1} w_\nu - k y_\lambda\} &\ge -(N-1) y_\lambda \\ \text{and} \quad - \inf_{k=1}^{N-1} \{\alpha_{k+1} w_\nu - k y_\lambda\} &\ge -\left(\alpha_N w_\nu - (N-1) y_\lambda\right), \end{aligned}$$

therefore to show (4), it suffices to show that for every $k_0 \in \{1, \ldots, N-1\}$ we have

$$\frac{p_M - p_N}{p_K - p_M}\left(\beta_{k_0+1} w_\nu - k_0 y_\lambda\right) - \frac{p_K - p_N}{p_K - p_M}\left(\alpha_N w_\nu - (N-1) y_\lambda\right) - (N-1)\, y_\lambda \geq 0 \qquad (5)$$

For such a k_0 the estimates

$$\begin{aligned} -\frac{p_M - p_N}{p_K - p_M} k_0 y_\lambda + \frac{p_K - p_N}{p_K - p_M}(N-1) y_\lambda - (N-1) y_\lambda &\geq \\ \left(-\frac{p_M - p_N}{p_K - p_M}(N-1) + \frac{p_K - p_N}{p_K - p_M}(N-1) - (N-1)\right) y_\lambda &= 0 \end{aligned}$$

and

$$\begin{aligned} \frac{p_M - p_N}{p_K - p_M}\beta_{k_0+1} w_\nu - \frac{p_K - p_N}{p_K - p_M}\alpha_N w_\nu &\geq \\ \frac{p_K - p_N}{p_K - p_M}\alpha_N w_\nu - \frac{p_K - p_N}{p_K - p_M}\alpha_N w_\nu &= 0 \end{aligned}$$

show (5), thus the lemma is proved.

Collecting the previous results, we get

Theorem 5.14 $\mathrm{Ext}^1_{(PLS)}\left(\Lambda^{l_1}_{\infty,\infty}(x,y), \Lambda^{c_0}_{p,\infty}(v,w)\right) = 0$

Proof: As $q = \infty$, we have local splitting, so 2.8 shows that we have to show

$$\mathrm{Proj}^1 \mathcal{L}\left(\Lambda^{l_1}_{\infty,\infty}(x,y), \Lambda^{c_0}_{p,\infty}(v,w)\right) = 0.$$

For this we have according to 3.7 and 5.12 to show that $(\tilde{L})_{\infty,\infty,p,\infty}$ holds, which follows from 5.13.

5.4.2 The case of the quotient having (LS) - steps of finite type

In this case the given matrices are

$$A = \left(\exp\left(K x_\lambda + \tfrac{1}{k} y_\lambda\right)\right)_{\lambda,K,k \in I\!N}$$

and

$$B = \left(\exp\left(p_N v_\nu - n\, w_\nu\right)\right)_{\nu,N,n \in I\!N}.$$

where $(p_N)_{N \in I\!N} = (N)_{N \in I\!N}$ or $\left(-N^{-1}\right)_{N \in I\!N}$. In this case the inequality in (L) reads thus:

$$(L)_{\infty,0,p,\infty}$$

$$\begin{aligned} \alpha_1 x_\lambda - p_M v_\nu + \inf_{k \in I\!N}\left\{\ln C_k + l_k w_\nu + \tfrac{1}{k} y_\lambda\right\} \leq max\{ \\ \beta_1 x_\lambda - p_K v_\nu + \inf_{k \in I\!N}\left\{\ln D_k + m_k w_\nu + \tfrac{1}{k} y_\lambda\right\}, \\ \gamma_1 x_\lambda - p_N v_\nu + \inf_{k \in I\!N}\left\{\ln E_k + n_k w_\nu + \tfrac{1}{k} y_\lambda\right\} \quad \} \end{aligned}$$

We claim that we can show a similiar result as in the last case :

Lemma 5.15 *For all sequences $(\gamma_k)_{k\in I\!N}$ and $(\alpha_k)_{k\in I\!N}$ of natural numbers and all numbers $N, M, K \in I\!N$ with $N < M < K$ there is a sequence $(\beta_k)_{k\in I\!N}$ such that for all $(C_k)_{k\in I\!N}$ and $(l_k)_{k\in I\!N}$ satisfying $C_k = 1$ and $l_k = \alpha_{k+1}$ for $k = 1,\dots,N$ we get $(D_k)_{k\in I\!N}$, $(E_k)_{k\in I\!N}$, $(m_k)_{k\in I\!N}$ and $(n_k)_{k\in I\!N}$ satisfying $E_k = 1$ and $n_k = \gamma_{k+1}$ for $k = 1,\dots,N-1$ such that for all $\nu, \lambda \in I\!N$ we have*

$$\alpha_1 x_\lambda - p_M v_\nu + \inf_{k\in I\!N}\left\{\ln C_k + l_k w_\nu + \tfrac{1}{k} y_\lambda\right\} \le max\{$$

$$\beta_1 x_\lambda - p_K v_\nu + \inf_{k\in I\!N}\left\{\ln D_k + m_k w_{\nu_0} + \tfrac{1}{k} y_\lambda\right\},$$

$$\gamma_1 x_\lambda - p_N v_\nu + \inf_{k\in I\!N}\left\{\ln E_k + n_k w_\nu + \tfrac{1}{k} y_\lambda\right\} \qquad \}.$$

Remark: In the formulation with the websets $\left(C^J_{\alpha_{|k}}\right)_{k,\alpha_{|k}\in I\!N}$, $J \in I\!N$ the inequality means that

$$C^M_{\alpha_{|N+1}} \subset C^K_{\beta_{|1}} + C^N_{\gamma_{|N}}.$$

Proof of the lemma: Let $N, M, K \in I\!N$ with $N < M < K$ and $(\alpha_k)_{k\in I\!N}$ be given. We choose

$$\beta_1 > \alpha_1 + \frac{p_K - p_M}{p_M - p_N}(\alpha_1 - \gamma_1).$$

Let further $(C_k)_{k\in I\!N}$ and $(l_k)_{k\in I\!N}$ be given such that $C_k = 1$ and $l_k = \alpha_{k+1}$ for $k = 1,\dots,N$. We find a sequence $\left(\tilde{k}(k)\right)_{k\in I\!N}$ such that for all $k \in I\!N$

$$\tilde{k}(k) \ge \frac{p_K - p_N}{p_M - p_N} k \tag{1}$$

and choose $(D_k)_{k\in I\!N}$ and $(m_k)_{k\in I\!N}$ such that for all $k \in I\!N$

$$\begin{aligned} \ln D_k &> \frac{p_K - p_N}{p_M - p_N} \ln C_{\tilde{k}(k)} \\ m_k &> \frac{p_K - p_N}{p_M - p_N} l_{\tilde{k}(k)}. \end{aligned} \tag{2}$$

Furthermore choose $(E_k)_{k\in I\!N}$ and $(n_k)_{k\in I\!N}$ such that $E_k = 1$ and $n_k = \gamma_{k+1}$ for $k = 1,\dots,N-1$ and for $k \ge N$ $E_k \ge 1$ and $n_k \in I\!N$ arbitrary. Analogously to the proof of 5.13 we set

$$J_1 := \Big\{(\nu,\lambda) \in I\!N^2 : \quad \alpha_1 x_\lambda - p_M v_\nu + \inf_{k\in I\!N}\left\{\ln C_k + l_k w_\nu + \tfrac{1}{k} y_\lambda\right\}$$

$$\le \quad \beta_1 x_\lambda - p_K v_\nu + \inf_{k\in I\!N}\left\{\ln D_k + m_k w_\nu + \tfrac{1}{k} y_\lambda\right\}\Big\}$$

and

$$J_2 := I\!N^2 - J_1.$$

Now for $(\nu, \lambda) \in J_2$ we get

$$\begin{aligned} v_\nu &> \frac{\beta_1 - \alpha_1}{p_K - p_M} x_\lambda + \\ & \frac{1}{p_K - p_M} \left(\inf_{k \in I\!N} \left\{ \ln D_k + m_k w_\nu + \tfrac{1}{k} y_\lambda \right\} - \inf_{k \in I\!N} \left\{ \ln C_k + l_k w_\nu + \tfrac{1}{k} y_\lambda \right\} \right) \end{aligned} \tag{3}$$

We have to show

$$\begin{aligned} \alpha_1 x_\lambda - p_M v_\nu + \inf_{k \in I\!N} \left\{ \ln C_k + l_k w_\nu + \tfrac{1}{k} y_\lambda \right\} \quad \leq \\ \gamma_1 x_\lambda - p_N v_\nu + \inf_{k \in I\!N} \left\{ \ln E_k + n_k w_\nu + \tfrac{1}{k} y_\lambda \right\} \end{aligned}$$

Inserting (3) gives as in the proof of 5.13, that it will be sufficient to show

$$\begin{aligned} & \left(\frac{p_M - p_N}{p_K - p_M} (\beta_1 - \alpha_1) - (\alpha_1 - \gamma_1) \right) x_\lambda \\ + \; & \frac{p_M - p_N}{p_K - p_M} \inf_{k \in I\!N} \left\{ \ln D_k + m_k w_\nu + \tfrac{1}{k} y_\lambda \right\} - \frac{p_K - p_N}{p_K - p_M} \inf_{k \in I\!N} \left\{ \ln C_k + l_k w_\nu + \tfrac{1}{k} y_\lambda \right\} \\ + \; & \inf_{k \in I\!N} \left\{ \ln E_k + n_k w_\nu + \tfrac{1}{k} y_\lambda \right\} \qquad \geq 0 \end{aligned} \tag{4}$$

The first and the last term are nonnegative, so we have to show that

$$\inf_{k \in I\!N} \left\{ \ln D_k + m_k w_\nu + \frac{1}{k} y_\lambda \right\} \geq \frac{p_K - p_N}{p_M - p_N} \inf_{k \in I\!N} \left\{ \ln C_k + l_k w_\nu + \frac{1}{k} y_\lambda \right\}. \tag{5}$$

Due to the choice of $\left(\tilde{k}(k)\right)_{k \in I\!N}$, $(D_k)_{k \in I\!N}$ and $(m_k)_{k \in I\!N}$, (see (1) and (2)) we have for all $k \in I\!N$

$$\ln D_k + m_k w_\nu + \frac{1}{k} y_\lambda \geq \frac{p_K - p_N}{p_M - p_N} \left(\ln C_{\tilde{k}(k)} + l_{\tilde{k}(k)} w_\nu + \frac{1}{\tilde{k}(k)} y_\lambda \right)$$

which shows (5) and therefore the lemma is proved.

Theorem 5.16 $\mathrm{Ext}^1_{(PLS)} \left(\Lambda^{l_1}_{\infty,0}(x, y), \Lambda^{c_0}_{p,\infty}(v, w) \right) = 0$

5.5 The case of the subspace being a stable Fréchet space

We will solve the splitting problem for spaces $E = \Lambda^{l_1}_{r,s}(x, y)$ $F = \Lambda^{c_0}_p(v)$ under the assumption that F is shift-stable, i.e. $\sup_{\nu \in I\!N} \frac{v_{\nu+1}}{v_\nu} < \infty$. The general method will be to analize the splitting behaviour of F with complemented coordinate Fréchet-subspaces of E.

Definition 5.17 *If $I \subset I\!N$ is an infinte set of indices, and $1 \leq p \leq \infty$, we define for a sequence $(t_\lambda)_{\lambda \in I}$ the new sequence t^I by*

$$\begin{aligned} t^I_\lambda &:= t_\lambda, \quad \lambda \in I \\ t^I_\lambda &:= 0, \quad \lambda \notin I \end{aligned}$$

and put

$$\Lambda^{l_p}_{r,s}(x,y,I) := \left\{t = (t_\lambda)_{\lambda\in I} \in I\!N^I : t^I \in \Lambda^{l_p}_{r,s}(x,y)\right\},$$

$$\Lambda^{l_p}_r(x,I) := \left\{t = (t_\lambda)_{\lambda\in I} \in I\!N^I : t^I \in \Lambda^{l_p}_r(x)\right\}$$

and

$$\Lambda^{l_p*}_s(y,I) := \left\{t = (t_\lambda)_{\lambda\in I} \in I\!N^I : t^I \in \Lambda^{l_p*}_s(y)\right\}$$

From proposition 5.8 we immediately get

Corollary 5.18 *i) If $\Lambda^{l_1}_{r,s}(x,y)$ does not contain a Fréchet subspace of the kind $\Lambda^{l_1}_r(x,I)$, then $\Lambda^{l_1}_{r,s}(x,y) = \Lambda^{l_1*}_s(y)$.*

ii) If $\Lambda^{l_1}_{r,s}(x,y)$ does not contain an (LS)- subspace of the kind $\Lambda^{l_1*}_s(y,I)$, then $\Lambda^{l_1}_{r,s}(x,y) = \Lambda^{l_1}_r(x)$.

Proof: We will show only i) for $s = 0$ as the rest is completely analogous. If there is no infinite subset $I \subset I\!N$ such that $\Lambda^{l_1}_{r,s}(x,y,I) = \Lambda^{l_1}_r(x,I)$, i.e. $\inf_{\lambda\in I} \frac{x_\lambda}{y_\lambda} > 0$, then $\lim_{\lambda\to\infty} \frac{x_\lambda}{y_\lambda} = 0$, i.e. $\Lambda^{l_1}_{r,s}(x,y) = \Lambda^{l_1*}_s(y)$

With this knowledge we can solve the problem of $\mathrm{Ext}^1_{(PLS)}\left(\Lambda^{l_1}_{r,s}(x,y), \Lambda^{c_0}_p(v)\right)$ in the following way:

Theorem 5.19 *If $(v_\nu)_{\nu\in I\!N}$ satisfies*

$$\lim_{\lambda\to\infty} y_\lambda = \lim_{\nu\to\infty} v_\nu = \infty \quad \text{and} \quad \sup_{\nu\in I\!N} \frac{v_{\nu+1}}{v_\nu} < \infty$$

then $\mathrm{Ext}^1_{(PLS)}\left(\Lambda^{l_1}_{r,s}(x,y), \Lambda^{c_0}_p(v)\right) = 0$ iff $r = \infty$ or $E = \Lambda^{l_1*}_s(y)$.

Proof: For $r = \infty$ we set $w_\nu := v_\nu^{\frac{1}{2}}$. Then

$$\lim_{\nu\to\infty} \frac{v_\nu}{w_\nu} = \lim_{\nu\to\infty} v_\nu^{\frac{1}{2}} = \infty,$$

so according to 5.8

$$\Lambda^{c_0}_p(v) = \Lambda^{c_0}_{p,\infty}(v,w)$$

and the theorems 5.14 and 5.16 give $\mathrm{Ext}^1_{(PLS)}\left(\Lambda^{l_1}_{r,s}(x,y), \Lambda^{c_0}_{p,\infty}(v,w)\right) = 0$.
If $r = 0$, and $\Lambda^{l_1}_{0,s}(x,y)$ has a Fréchet subspace of the form $\Lambda^{l_1}_0(x,I)$, we know from ([V1], Corollary 4.4) that $\mathrm{Ext}^1_{(F)}\left(\Lambda^{l_1}_0(x,I), \Lambda^{c_0}_p(v)\right) \neq 0$, so there is a short exact sequence

$$0 \to \Lambda^{c_0}_p(v) \to X \to \Lambda^{l_1}_0(x,I) \to 0$$

with a Fréchet space X which does not split. Let $\tilde{E}$ be a (PLS) -space such that

$$\Lambda^{l_1}_{0,s}(x,y) = \Lambda^{l_1}_0(x,I) \times \tilde{E}$$

(remember that closed subspaces of (PLS)-spaces are again (PLS)-spaces, see [DV1], Proposition 1.2). Adding the short exact sequence

$$0 \to 0 \to \tilde{E} \overset{id}{\to} \tilde{E} \to 0$$

we get a short exact sequence

$$0 \to \Lambda_p^{c_0}(v) \to X \times \tilde{E} \to \Lambda_0^{l_1}(x, I) \to 0$$

which does not split, and as X is a Fréchet-Schwartz space (being a Schwartz space is a three space property) and every Fréchet-Schwartz space is a (PLS)-space, we have $\operatorname{Ext}^1_{(PLS)}\left(\Lambda_{0,s}^{l_1}(x,y), \Lambda_p^{c_0}(v)\right) \neq 0$.
In the case that $\Lambda_{0,s}^{l_1}(x,y)$ does not have a Fréchet subspace of the form $\Lambda_0^{l_1}(x,I)$, then by corollary 5.18 we have $\Lambda_{0,s}^{l_1}(x,y) = \Lambda_s^{l_1 *}(y)$. Setting $\tilde{x}_\lambda := y_\lambda^{\frac{1}{2}}$, $\lambda \in I\!N$ and $w_\nu := v_\nu^{\frac{1}{2}}$, $\nu \in I\!N$, we get

$$\Lambda_s^{l_1 *}(y) = \Lambda_{\infty,0}^{l_1}(\tilde{x}, y)$$

and

$$\Lambda_p^{c_0}(v) = \Lambda_{p,\infty}^{c_0}(v, w),$$

so theorem 5.16 gives

$$\operatorname{Ext}^1_{(PLS)}\left(\Lambda_s^{l_1 *}(y), \Lambda_p^{c_0}(v)\right) = 0,$$

which finishes the proof.

5.6 The case of the quotient being a stable Fréchet space

We use the same method as in the previous section to prove an analogous theorem. Throughout this section we will make the assumption that $(x_\lambda)_{\lambda \in I\!N}$, $(v_\nu)_{\nu \in I\!N}$ and $(w_\nu)_{\nu \in I\!N}$ tend to infinity, and

$$\lim_{\nu \to \infty} \frac{x_{\lambda+1}}{x_\lambda} = 1.$$

Theorem 5.20

$$\operatorname{Ext}^1_{(PLS)}\left(\Lambda_r^{l_1}(x), \Lambda_{p,q}^{c_0}(v, w)\right) = 0$$

iff $r = \infty$ *or* $\Lambda_{p,q}^{c_0}(v,w) = \Lambda_q^{c_0 *}(w)$.

5.6.1 The finite case

To prove 5.20, we will first treat the case $r = 0$.

Lemma 5.21

$$\operatorname{Ext}^1_{(PLS)}\left(\Lambda_0^{l_1}(x), \Lambda_{p,q}^{c_0}(v, w)\right) = 0$$

iff $\Lambda_{p,q}^{c_0}(v,w) = \Lambda_q^{c_0 *}(w)$.

Proof: The first case to consider is that of $\Lambda^{c_0}_{p,q}(v,w)$ having a Fréchet subspace of the form

$$\Lambda^{c_0}_{p}(v,I).$$

Then from ([V1, Theorem 4.2]) it follows that

$$\mathrm{Ext}^1_{(F)}\left(\Lambda^{l_1}_0(x),\Lambda^{c_0}_p(v,I)\right)\neq 0$$

and analogously to the proof of 5.19 we get

$$\mathrm{Ext}^1_{(PLS)}\left(\Lambda^{l_1}_0(x),\Lambda^{c_0}_{p,q}(v,w)\right)\neq 0.$$

In the second case we have $\Lambda^{c_0}_{p,q}(v,w)=\Lambda^{c_0*}_q(w)$, which is (PLS)-space in quite an easy way, namely as the projective limit of the constant spectrum

$$\Lambda^{c_0*}_q(w) \xleftarrow{\mathrm{id}} \Lambda^{c_0*}_q(w) \xleftarrow{\mathrm{id}} \cdots$$

Then we reach the conclusion

$$\mathrm{Ext}^1_{(PLS)}\left(\Lambda^{l_1}_0(x),\Lambda^{c_0*}_q(w)\right)=0.$$

in the following way: From 2.5 we get local splitting, so 2.8 shows that it is necessary to show

$$\mathrm{Proj}^1\mathcal{L}\left(\Lambda^{l_1}_0(x),\Lambda^{c_0*}_q(w)\right)=0$$

for the spectrum

$$L\left(\Lambda^{l_1}_0(x),\Lambda^{c_0*}_q(w)\right)\xleftarrow{\mathrm{id}} L\left(\Lambda^{l_1}_0(x),\Lambda^{c_0*}_q(w)\right)\xleftarrow{\mathrm{id}}\ldots$$

which is obviously fulfilled.

5.6.2 The infinite case

For the case $r=\infty$ we begin with $q=\infty$

Lemma 5.22

$$\mathrm{Ext}^1_{(PLS)}\left(\Lambda^{l_1}_\infty(x),\Lambda^{c_0}_{p,\infty}(v,w)\right)=0.$$

Proof: We set for all $\lambda\in I\!N$

$$y_\lambda:=(x_\lambda)^{\frac{1}{2}}$$

and get from 5.8 iii)

$$\Lambda^{l_1}_\infty(x)=\Lambda^{l_1}_{\infty,\infty}(x,y),$$

so 5.14 gives

$$\mathrm{Ext}^1_{(PLS)}\left(\Lambda^{l_1}_\infty(x),\Lambda^{c_0}_{p,\infty}(v,w)\right)=0.$$

It remains to show that

$$\mathrm{Ext}^1_{(PLS)}\left(\Lambda^{l_1}_\infty(x),\Lambda^{c_0}_{p,0}(v,w)\right)=0,$$

which will need some more consideration; namely we have to use the facts we established in section 2 in 2.16 on Fréchet spaces which are locally l_1. It is trivial that the space $E = \Lambda_\infty^{l_1}(x)$ is locally l_1. To interpret the spaces F_K^γ appearing in 2.16, we observe that according to the definition of the matrix defining F we have

$$B_{N+1,n} \subset B_{N,n},\ n, N \in I\!N,$$

and the connecting maps of the spectrum defining F are simply the inclusions

$$F_{N+1} \hookrightarrow F_N,\ N \in I\!N.$$

For a sequence $(\gamma_N)_{N\in I\!N} \in I\!N^{I\!N}$ the spaces F_K^γ, $K \in I\!N$, are thus given as

$$F_K^\gamma = \bigcap_{N=1}^{K} F_{N,\gamma_N}.$$

Furtheron we observe the following: For $(\gamma_N)_{N\in I\!N} \in I\!N^{I\!N}$ the spectrum

$$\mathcal{F}^\gamma = F_{1,\gamma_1} \hookleftarrow \bigcap_{N=1}^{2} F_{N,\gamma_N} \hookleftarrow \bigcap_{N=1}^{3} F_{N,\gamma_N} \hookleftarrow \dots$$

is a reduced spectrum of Banach spaces, as the canonical unit vectors are dense in every one of the spaces constituting $\mathcal{F}^\gamma$, thus the mapping

$$\begin{aligned} \psi \ : \ \prod_{K=1}^{\infty} F_K^\gamma \quad &\to \quad \prod_{K=1}^{\infty} F_K^\gamma \\ (x_N)_{N\in I\!N} \quad &\to \quad (x_{N+1} - x_N)_{N\in I\!N} \end{aligned}$$

is surjective. This implies that for every $(\gamma_N)_{N\in I\!N} \in I\!N^{I\!N}$ the image of the mapping

$$\begin{aligned} \psi \ : \ \prod_{K=1}^{\infty} F_K \quad &\to \quad \prod_{K=1}^{\infty} F_K \\ (x_N)_{N\in I\!N} \quad &\to \quad (x_{N+1} - x_N)_{N\in I\!N} \end{aligned}$$

contains the space $\prod_{K=1}^{\infty} F_K^\gamma$. With these observations we can in our present case set up the following version of 2.16:

Corollary 5.23 *For $E = \Lambda_\infty^{l_1}(x)$ and $F = \Lambda_{p,0}^{c_0}(v, w)$ the following are equivalent:*

i)

$$\mathrm{Ext}^1{}_{(PLS)}(E, F) = 0$$

ii) For every $(\gamma_N)_{N\in I\!N} \in I\!N^{I\!N}$ and for every continuous linear mapping

$$C : E \to \prod_{K=1}^{\infty} F_K^\gamma$$

there is a lifting in the diagram

$$\begin{array}{ccc} \prod\limits_{K=1}^{\infty} F_K & \xrightarrow{\ \psi\ } & \prod\limits_{K=1}^{\infty} F_K \\ & & \uparrow{\scriptstyle C} \\ & & E \end{array}$$

iii) For the mapping

$$\begin{array}{rccl} \psi^* & : & \prod\limits_{K=1}^{\infty} L(E, F_K) & \rightarrow \prod\limits_{K=1}^{\infty} L(E, F_K) \\ & & (A_K)_{K \in \mathbb{N}} & \rightarrow (A_{K+1} - A_K)_{K \in \mathbb{N}} \end{array}$$

we have

$$\prod_{K=1}^{\infty} L\left(E, \bigcap_{N=1}^{K} F_{N,\gamma_N}\right) \subset \operatorname{Im}\psi^*$$

iv) For every $(\gamma_N)_{N \in \mathbb{N}} \in \mathbb{N}^{\mathbb{N}}$ there exists a sequence $(\sigma_N)_{N \in \mathbb{N}} \in \mathbb{N}^{\mathbb{N}}$ such that for all $N \in \mathbb{N}$ exists $M \in \mathbb{N}$ with $M \geq N$ such that

$$L(E, F_M^{\sigma}) \subset L(E, F) + \bigcap_{J=1}^{N} C^J_{\gamma_{|J,N}}$$

where $\left(C^N_{\alpha_{|k}}\right)_{k, \alpha_{|k} \in \mathbb{N}}$, $N \in \mathbb{N}$ are the ordered webs on $L(E, F_N)$, $N \in \mathbb{N}$ constructed in 3.8.

Proof: If i) holds then we know for a given C that $\operatorname{Im} C \subset \operatorname{Im} \psi$, so we can use the pullback construction to get a lifting. If ii) holds and if we are given a diagram as in 2.16, then ii) gives a desired lifting, so 2.16 ii) holds and we get

$$\operatorname{Ext}^1{}_{(PLS)}(E, F) = 0.$$

iii) is merely a reformulation of ii), and the equivalence of iii) and iv) is exactly the statement of the Retakh-Palamodov theorem for webbed spaces as formulated in ([D], theorem 4.1) (cf. section 3.2).

To show the condition 5.23 iv), we will first consider the spaces and the webs appearing in the inclusion. Recall that in 3.8 we showed that for every $N \in \mathbb{N}$ we get an ordered strict web on $L(E, F_N)$ by

$$C^N_{\alpha} := \{T \in L(E, F_N) : T \text{ factorizes through } E_{\alpha}\}, \ \alpha \in \mathbb{N}$$

and

$$C^N_{\alpha_{|k+1}} := \left\{T \in C^N_{\alpha_{|k}} : \tilde{T}(B_{\alpha_1,k}) \subset D_{N,\alpha_{k+1}}\right\}, \ k, \alpha_{|k} \in \mathbb{N}$$

where $\tilde{T}$ denotes the factorisation of T. For this definition we assumed that every bounded set in F_N is contained in one of the unit balls $D_{N,n}$, $n \in I\!N$. We can fulfill the latter assumption for $F = \Lambda^{c_0}_{p,0}(v,w)$ by putting

$$D_{N,n} := n\, B_{F_{N,n}}$$

as every (LS)-space is regular and as we have $B_{F_{N,n}} \subset B_{F_{N,n+1}}$, $n, N \in I\!N$. Now as E is a Fréchet space, there is the following easier description of the websets $C^N_{\alpha_{|k}}$:

Proposition 5.24

i) For every $N \in I\!N$ and $k \geq 2$ the web on $L(E, F_N)$ has the form

$$\begin{aligned} C^N_{\alpha_{|k}} &= C^N_{\alpha_1,\alpha_2} \\ &= \left\{ T : E \to F_N : \tilde{T} \in \alpha_2\, B_{L\left(E_{\alpha_1}, F_{N,\alpha_2}\right)} \right\} \end{aligned}$$

ii) The web on $L(E,F) = \bigcap_{N=1}^{\infty} L(E, F_N)$ has the form

$$\begin{aligned} C_{\alpha_{|K}} &= \left(\prod_{N=1}^{K} C^N_{\alpha_{|N,K}} \times \prod_{N=K+1}^{\infty} L(E,F_N) \right) \cap L(E,F) \\ &= \left(\prod_{N=1}^{K-1} C^N_{\alpha_N,\alpha_{N+1}} \times C^K_{\alpha_K} \times \prod_{N=K+1}^{\infty} L(E,F_N) \right) \cap L(E,F) \\ &= \Big\{ T : E \to F : \tilde{T} \in \alpha_{J+1}\, B_{L\left(E_{\alpha_J}, F_{N,\alpha_{J+1}}\right)},\ J = 1, \ldots, K-1, \\ &\qquad \text{there is } \beta \in I\!N \text{ such that } \tilde{T} \in \beta\, B_{L\left(E_{\alpha_K}, F_{K,\beta}\right)} \Big\} \end{aligned}$$

From this we may set up a sufficient condition for condition iv) in 5.23:

Corollary 5.25

$$\mathrm{Ext}^1_{(PLS)}\left(\Lambda^{l_1}_{\infty}(x), \Lambda^{c_0}_{p,0}(v,w) \right) = 0$$

is implied by the following:
For every sequence $(\gamma_N)_{N \in I\!N} \in I\!N^{I\!N}$ exists a sequence $(\sigma_N)_{N \in I\!N} \in I\!N^{I\!N}$ such that for all $N \in I\!N$ there is an $M \in I\!N$ with $M \geq N$ such that for all $\alpha \in I\!N$ exists a sequence $(\beta_N)_{N \in I\!N} \in I\!N^{I\!N}$ and $K \in I\!N$ with $K \geq M$ such that

$$\begin{aligned} \alpha\, B_{L\left(E_\alpha, F^{\sigma}_M\right)} &\subset \left(\prod_{J=1}^{K-1} C^J_{\beta_J,\beta_{J+1}} \times C^K_{\beta_K} \times \prod_{J=K+1}^{\infty} L(E,F_J) \right) \cap L(E,F) \\ &+ \bigcap_{J=1}^{N-1} C^J_{\sigma_J,\sigma_{J+1}} \cap C^N_{\sigma_N,\beta_1} \end{aligned}$$

To prove the inclusion in 5.25, we need to describe the sets of operators appearing therein by their matrix representations analogously to 5.5:

Proposition 5.26

i)

$$C^N_{\alpha|K} = \left\{ (T_{\nu,\lambda})_{\nu,\lambda\in I\!N} : |T_{\nu,\lambda}| \le \alpha_2 \, \frac{a_{\lambda,\alpha_1}}{b_{\nu,N,\alpha_2}} \;\; \nu\lambda \in I\!N \right\}$$

ii)

$$\alpha\, B_{L(E_\alpha, F^\gamma_M)} = \left\{ (T_{\nu,\lambda})_{\nu,\lambda\in I\!N} : |T_{\nu,\lambda}| \le \alpha\, a_{\lambda,\alpha} \inf_{J=1}^{M} \left\{ \frac{1}{b_{\nu,J,\gamma_J}} \right\} \;\; \nu,\lambda \in I\!N \right\}$$

iii)

$$\begin{aligned} L(E,F) &= \bigcap_{J=1}^{\infty} L(E,F_J) \\ &= \bigcap_{J=1}^{\infty} \Big\{ (T_{\nu,\lambda})_{\nu,\lambda\in I\!N} : \text{there exist } \delta,\beta \in I\!N \text{ such that} \\ &\qquad\qquad |T_{\nu,\lambda}| \le \beta \, \frac{a_{\lambda,\delta}}{b_{\nu,J,\beta}} \;\; \nu,\lambda \in I\!N \Big\} \\ &= \Big\{ (T_{\nu,\lambda})_{\nu,\lambda\in I\!N} : \text{there exist } (\delta_J)_{J\in I\!N}, (\beta_J)_{J\in I\!N} \in I\!N^{I\!N} \\ &\qquad \text{such that } |T_{\nu,\lambda}| \le \inf_{J=1}^{\infty} \left\{ \beta_J \, \frac{a_{\lambda,\delta_J}}{b_{\nu,J,\beta_J}} \right\} \;\; \nu,\lambda \in I\!N \Big\} \end{aligned}$$

iv)

$$\begin{aligned} &\left(\prod_{J=1}^{K-1} C^J_{\beta_J,\beta_{J+1}} \times C^K_{\beta_K} \times \prod_{N=K+1}^{\infty} L(E,F_N) \right) \cap L(E,F) \\ = \;& \Big\{ (T_{\nu,\lambda})_{\nu,\lambda\in I\!N} : \text{there exist } (\tau_J)_{J\in I\!N}, (\pi_J)_{J\in I\!N} \in I\!N^{I\!N} \\ & \quad \text{such that } \begin{array}{ll} \tau_J = \beta_J & J = 1,\ldots,K \\ \pi_J = \beta_{J+1} & J = 1,\ldots,K-1 \end{array} \\ & \quad \text{and } \; |T_{\nu,\lambda}| \le \inf_{J=1}^{\infty} \left\{ \pi_J \, \frac{a_{\lambda,\tau_J}}{b_{\nu,J,\pi_J}} \right\} \;\; \nu,\lambda \in I\!N \Big\} \end{aligned}$$

v)

$$\begin{aligned} &\bigcap_{J=1}^{N-1} C^J_{\sigma_J,\sigma_{J+1}} \cap C^N_{\sigma_N,\beta_1} \\ = \;& \left\{ (T_{\nu,\lambda})_{\nu,\lambda\in I\!N} : |T_{\nu,\lambda}| \le min \left\{ \inf_{J=1}^{N-1} \left\{ \sigma_{J+1} \, \frac{a_{\lambda,\sigma_J}}{b_{\nu,J,\sigma_{J+1}}} \right\}, \beta_1 \, \frac{a_{\lambda,\sigma_N}}{b_{\nu,J,\beta_1}} \right\} \right\} \end{aligned}$$

With these descriptions the technique of decomposing matrices as used in 5.6 shows

Corollary 5.27

$$\text{Ext}^1_{(PLS)}\left(\Lambda^{l_1}_\infty(x), \Lambda^{c_0}_{p,0}(v,w)\right) = 0$$

if the following holds:

$$(L)_{\infty,p,0}$$

For all $(\gamma_J)_{J\in\mathbb{N}} \in \mathbb{N}^{\mathbb{N}}$ exists a sequence $(\sigma_J)_{J\in\mathbb{N}} \in \mathbb{N}^{\mathbb{N}}$ such that for all $N \in \mathbb{N}$ there is an $M \in \mathbb{N}$ with $M \geq N$ such that for all $\alpha \in \mathbb{N}$ there are sequences $(\beta_J)_{J\in\mathbb{N}}$, $(\delta_J)_{J\in\mathbb{N}}$, $(\pi_J)_{J\in\mathbb{N}} \in \mathbb{N}^{\mathbb{N}}$ and $K \in \mathbb{N}$ with $K \geq M$ such that $\delta_J = \beta_J$ for $J = 1,\ldots,K$ and $\pi_J = \beta_{J+1}$ for $J = 1,\ldots,K-1$ and for all $\nu,\lambda \in \mathbb{N}$ we have

$$\alpha\, a_{\lambda,\alpha} \inf_{J=1}^{M}\left\{\frac{1}{b_{\nu,J,\gamma_J}}\right\} \leq \max\{ \inf_{J=1}^{\infty}\left\{\pi_J\,\frac{a_{\lambda,\delta_J}}{b_{\nu,J,\pi_J}}\right\}, \min\left\{\inf_{J=1}^{N-1}\left\{\sigma_{J+1}\,\frac{a_{\lambda,\sigma_J}}{b_{\nu,J,\sigma_{J+1}}}\right\}, \beta_1\,\frac{a_{\lambda,\sigma_N}}{b_{\nu,N,\beta_1}}\right\}\}$$

It remains to show the condition $(L)_{\infty,p,0}$:

Lemma 5.28 *The spaces $E = \Lambda^{l_1}_\infty(x)$ and $F = \Lambda^{c_0}_{p,0}(v,w)$ fulfill the condition $(L)_{\infty,p,0}$.*

Proof: Let us first recall that E is defined by the matrix

$$(a_{K,\lambda})_{K,\lambda\in\mathbb{N}} := (exp(K\,x_\lambda))_{K,\lambda\in\mathbb{N}}$$

and F by

$$(b_{\nu,l,n})_{\nu,L,n\in\mathbb{N}} := \left(exp\left(p_L\,v_\nu + \tfrac{1}{n}\,w_\nu\right)\right)_{\nu,L,n\in\mathbb{N}}$$

where

$$(p_L)_{L\in\mathbb{N}} = \begin{cases} (L)_{L\in\mathbb{N}}, & p=\infty \\ \left(-\frac{1}{L}\right)_{L\in\mathbb{N}}, & p=0 \end{cases}$$

Inserting these weights into the inequality in $(L)_{\infty,p,0}$ and taking logarithms, we see that we have to show the following inequality with the corresponding quantifiers:

$$(*)$$

$$\ln\alpha + \alpha x_\lambda + \inf_{J=1}^{M}\left\{-p_J\,v_\nu - \tfrac{1}{\gamma_J}\,w_\nu\right\} \leq \max\{ \inf_{J=1}^{\infty}\left\{\ln\pi_J + \delta_J\,x_\lambda - p_J\,v_\nu - \tfrac{1}{\pi_J}\,w_\nu\right\}, \min\left\{\inf_{J=1}^{N-1}\left\{\ln\sigma_{J+1} + \sigma_J\,x_\lambda - p_J\,v_\nu - \tfrac{1}{\sigma_{J+1}}\,w_\nu\right\}, \left\{\ln\beta_1 + \sigma_N\,x_N - p_N\,v_\nu - \tfrac{1}{\beta_1}\,w_\nu\right\}\right\}$$

So let $(\gamma_J)_{J\in I\!N} \in I\!N^{I\!N}$ be given. We assume without loss of generality that $(\gamma_J)_{J\in I\!N} \in I\!N^{I\!N}$ is strictly increasing. We define $(\sigma_J)_{J\in I\!N} \in I\!N^{I\!N}$ by $\sigma_J := \gamma_J$, $J \in I\!N$. For given $N \in I\!N$ we put $M := N+1$. Let $\alpha \in I\!N$ be given. Set $K := N+1$. To find suitable $(\beta_J)_{J\in I\!N}$, $(\delta_J)_{J\in I\!N}$ and $(\pi_J)_{J\in I\!N}$, let $\tilde{J} \in I\!N$ denote a number such that

$$\ln \sigma_{J+1} - \ln \alpha \geq 0,\ J \geq \tilde{J}$$

which we can find as $(\sigma_J)_{J\in I\!N}$ is strictly increasing. Now we can find $(\beta_J)_{J\in I\!N}$, $(\delta_J)_{J\in I\!N}$ and $(\pi_J)_{J\in I\!N} \in I\!N^{I\!N}$ such that

i) For all $J \in I\!N$

$$\beta_J, \pi_J, \delta_J \geq \max\left\{\alpha, \gamma_{N+1}, \alpha + \sup_{L=1}^{N}\left\{\frac{p_J - p_1}{p_{L+1} - p_L}\,|\sigma_L - \alpha|\right\}\right\}$$

ii) For all $J \in I\!N$

$$\ln \pi_J \geq \ln \alpha + \sup_{L=1}^{\tilde{J}}\left\{\frac{p_J - p_1}{p_{L+1} - p_L}\,|\ln \sigma_{L+1} - \ln \alpha|\right\}$$

Let now $\nu, \lambda \in I\!N$ be fixed. To show that (*) holds, we consider two cases: In the first case we have for all $J \in I\!N$

$$\ln \alpha + \alpha\, x_\lambda - p_1\, v_\nu - \frac{1}{\gamma_1}\, w_\nu \leq \ln \pi_J + \delta_J\, x_\lambda - p_J\, v_\nu - \frac{1}{\pi_J}\, w_\nu$$

in which case (*) holds.
In the second case there is $J_0 \in I\!N$ such that

$$\ln \alpha + \alpha\, x_\lambda - p_1\, v_\nu - \frac{1}{\gamma_1}\, w_\nu > \ln \pi_{J_0} + \delta_{J_0}\, x_\lambda - p_{J_0}\, v_\nu - \frac{1}{\pi_{J_0}}\, w_\nu$$

or equivalently

$$v_\nu > \frac{\ln \pi_{J_0} - \ln \alpha}{p_{J_0} - p_1} + \frac{\delta_{J_0} - \alpha}{p_{J_0} - p_1}\, x_\lambda + \frac{\frac{1}{\gamma_1} - \frac{1}{\pi_{J_0}}}{p_{J_0} - p_1}\, w_\nu$$

(observe that due to our choice of the δ_J and π_J, $J \in I\!N$ we know that $J_0 \geq 2$ and that $(p_J)_{J\in I\!N}$ is increasing). To show (*), we will show that

a) For every $J \in \{1, \ldots, N-1\}$ we have

$$\ln \alpha + \alpha\, x_\lambda - p_{J+1}\, v_\nu - \frac{1}{\gamma_{J+1}}\, w_\nu \leq \ln \sigma_{J+1} + \sigma_J\, x_\lambda - p_J\, v_\nu - \frac{1}{\sigma_{J+1}}\, w_\nu$$

b) and

$$\ln \alpha + \alpha\, x_\lambda - p_{N+1}\, v_\nu - \frac{1}{\gamma_{N+1}}\, w_\nu \leq \ln \beta_1 + \sigma_N\, x_\lambda - p_N\, v_\nu - \frac{1}{\beta_1}\, w_\nu$$

or equivalently

a) For every $J \in \{1, \dots, N-1\}$ we have

$$\ln \sigma_{J+1} - \ln \alpha + (\sigma_J - \alpha)\ x_\lambda + (p_{J+1} - p_J)\ v_\nu \geq 0$$

b) and

$$\ln \beta_1 - \ln \alpha + (\sigma_N - \alpha)\ x_\lambda + (p_{N+1} - p_N)\ v_\nu + \left(\frac{1}{\gamma_{N+1}} - \frac{1}{\beta_1} \right) w_\nu \geq 0$$

As the factors in front of w_ν are nonnegative, and as we know that

$$v_\nu > \frac{\ln \pi_{J_0} - \ln \alpha}{p_{J_0} - p_1} + \frac{\delta_{J_0} - \alpha}{p_{J_0} - p_1}\, x_\lambda,$$

we can insert the last inequality into a) and b) and see that it is sufficient to show

a) For every $J \in \{1, \dots, N-1\}$ we have

$$\begin{aligned} & \ln \sigma_{J+1} - \ln \alpha + \frac{p_{J+1} - p_J}{p_{J_0} - p_1}\ (\ln \pi_{J_0} - \ln \alpha) \\ + & \left(\frac{p_{J+1} - p_J}{p_{J_0} - p_1}\ (\delta_{J_0} - \alpha) + (\sigma_J - \alpha) \right) x_\lambda \qquad \geq \ 0 \end{aligned}$$

b)

$$\begin{aligned} & \ln \beta_1 - \ln \alpha + \frac{p_{N+1} - p_N}{p_{J_0} - p_1}\ (\ln \pi_{J_0} - \ln \alpha) \\ + & \left(\frac{p_{N+1} - p_N}{p_{J_0} - p_1}\ (\delta_{J_0} - \alpha) + (\sigma_N - \alpha) \right) x_\lambda \quad \geq \ 0 \end{aligned}$$

a) follows in the following way: Let $J \in \{1, \dots, N-1\}$ be fixed. If $J \geq \tilde{J}$, then we have

$$\ln \sigma_{J+1} - \ln \alpha \geq 0$$

according to the definition of $\tilde{J}$ and

$$\frac{p_{J+1} - p_J}{p_{J_0} - p_1}\ (\ln \pi_{J_0} - \ln \alpha) \geq 0$$

because $\pi_J \geq \alpha$, $J \in I\!N$ and $(p_J)_{J \in I\!N}$ is strictly increasing. Thus

$$\ln \sigma_{J+1} - \ln \alpha + \frac{p_{J+1} - p_J}{p_{J_0} - p_1}\ (\ln \pi_{J_0} - \ln \alpha) \geq 0.$$

If $J < \tilde{J}$ we get from condition ii) on $(\pi_J)_{J \in I\!N}$

$$\begin{aligned} & \ln \sigma_{J+1} - \ln \alpha + \frac{p_{J+1} - p_J}{p_{J_0} - p_1}\ (\ln \pi_{J_0} - \ln \alpha) \\ \geq\ & \ln \sigma_{J+1} - \ln \alpha + \frac{p_{J+1} - p_J}{p_{J_0} - p_1} \sup_{L=1}^{\tilde{J}} \left\{ \frac{p_{J_0} - p_1}{p_{L+1} - p_L} \left| \ln \sigma_{L+1} - \ln \alpha \right| \right\} \\ \geq\ & 0 \end{aligned}$$

In any case we get from the condition i) on $(\delta_J)_{J\in I\!N}$

$$\begin{aligned} & \frac{p_{J+1}-p_J}{p_{J_0}-p_1}\,(\delta_{J_0}-\alpha)+(\sigma_J-\alpha) \\ \geq\ & \frac{p_{J+1}-p_J}{p_{J_0}-p_1}\,\sup_{L=1}^{N}\left\{\frac{p_{J_0}-p_1}{p_{L+1}-p_L}\,|\sigma_L-\alpha|\right\}+(\sigma_J-\alpha) \\ \geq\ & 0. \end{aligned}$$

and a) follows. b) is proved in exactly the same way to finish the proof of the lemma.

So we finally have established th condition $(L)_{\infty,p,0}$, which shows that

$$\mathrm{Ext}^1_{(PLS)}\left(\Lambda^{l_1}_{\infty}(x),\Lambda^{c_0}_{p,0}(v,w)\right)=0$$

and so 5.20 is established.

5.7 Proj$^1\mathcal{L}$ (E,F) = 0 for F consisting of (LS) - steps of finite type

We first state some facts which we will need . The first one comes from [Z, Theorem 1]:

Theorem 5.29 *Let $X=X_1\times X_2$ and $Y=Y_1\times Y_2$ be locally convex spaces such that the continuous linear maps from X_1 to Y_2 and from Y_1 to X_2 are compact. Then $X\tilde{=}Y$ iff there exists $s\in I\!N$ such that one of the following two properties holds:*

i) $Y_1\tilde{=}X_1\times I\!K^s$ and $Y_2\times I\!K^s\tilde{=}X_2$

ii) $Y_1\times I\!K^s\tilde{=}X_1$ and $Y_2\tilde{=}X_2\times I\!K^s$

To apply this theorem in our case, we need the following

Proposition 5.30 *If X is a Fréchet space and $Y=\bigcup\limits_{n=1}^{\infty}Y_n$ is an (LS)-space, then every continuous linear map from X to Y or vice versa is compact.*

Proof: If we have a continuous linear map T from X to Y, then, according to Grothendieck's factorization theorem, it must act continuously into one of the Y_n. As the embeddings into Y are compact, T is compact. If T acts from Y to X, and if for all $n\in I\!N$ B_n denotes the unit ball in Y_n, then $T(B_n)$ is for every $n\in I\!N$ a precompact subset of X. It is easy to check that there is a sequence $(\lambda_n)_{n\in I\!N}$ such that $\bigcup\limits_{n=1}^{\infty}\lambda_n\,T(B_n)$ is precompact in X (Just take for a given basis $(U_n)_{n\in I\!N}$ of neighbourhoods of zero a corresponding sequence $(\lambda_n)_{n\in I\!N}$ such that $\lambda_n\,T(B_n)\subset U_n$, $n\in I\!N$). Then $V:=\Gamma\left(\bigcup\limits_{n=1}^{\infty}\lambda_n\,B_n\right)$ is a neighbourhood of zero in Y whose image under T is precompact in X.

A central problem regarding isomorphisms between power series spaces is the question of when a given space X is shift-stable, i.e. $E\tilde{=}I\!K\times E$ and when it is stable, i.e. $E\tilde{=}E\times E$. The following characterizations are well known, we give a short proof for the sake of completeness:

Theorem 5.31 *For a Fréchet space $E = \Lambda_r(x)$ with a monotonous sequence $(x_\lambda)_{\lambda \in I\!N}$ the following are equivalent:*

i) $\sup\limits_{\lambda \in I\!N} \frac{x_{\lambda+1}}{x_\lambda} < \infty$

ii) $E \cong I\!K \times E$

iii) For all $n \in I\!N$ $E \cong I\!K^n \times E$.

iv) There exists $n \in I\!N$ *such that* $E \cong I\!K^n \times E$.

Proof: That i) implies ii) follows immediately from continuity estimates, we only have to show that iv) implis i) as the other implications are obvious. Now $I\!K^n \times \Lambda_r(x)$ is a power series space $\Lambda_r(\tilde{x})$ defined by the sequence

$$\tilde{x}_\lambda = \begin{cases} 1 & 1 \leq \lambda \leq n \\ x_{\lambda-n} & \lambda > n \end{cases}$$

The isomorphism implies that there exists $C > 0$ such that $x_\lambda \leq C\,\tilde{x}_\lambda$ (cf. [MV, Theorem 29.1]). This together with the monotonicity of $(x_\lambda)_{\lambda \in I\!N}$ implies i).

An anlogous characterization holds for stability:

Theorem 5.32 *For a Fréchet space $E = \Lambda_r(x)$ with a monotonous sequence $(x_\lambda)_{\lambda \in I\!N}$ the following are equivalent:*

i) $\sup\limits_{\lambda \in I\!N} \frac{x_{2\lambda}}{x_\lambda} < \infty$

ii) $E \cong E \times E$

iii) For all $n \in I\!N$ $E \cong E^n$.

iv) There exists $n \in I\!N$ *such that* $E \cong E^n$.

Proof: The same technique as in the proof of 5.31 applied to the spaces $\Lambda_r(x)$ and $\Lambda_r(\tilde{x})$ with

$$\tilde{x}_\lambda = x_\mu,\ n\,(\mu - 1) \leq \lambda < n\,\mu,\ \lambda, \mu \in I\!N,$$

yields the result.

We will need to know that the stability of certain product spaces is equivalent to the stability of the factors:

Proposition 5.33 *The (PLS)-space $E = \Lambda_p(x) \times \Lambda_q^*(y)$, where x and y are monotonous positive sequences tending to infinity, is stable iff every factor is stable.*

Proof: If every factor is stable, the stability of E is trivial. Let $E \cong E \times E$. For abbreviation we put $F := \Lambda_p(x)$ and $G := \Lambda_q^*(y)$. We apply 5.29 to the isomorphism

$$F \times G \cong E \cong E \times E \cong F \times F \times G \times G,$$

getting an $s_1 \in I\!N$ such that

i) $F \cong F \times F \times I\!K^{s_1}$ and $G \times I\!K^{s_1} \cong G \times G$ or

ii) $F \times I\!K^{s_1} \cong F \times F$ and $G \cong G \times G \times I\!K^{s_1}$.

In both cases the same technique as in 5.31 and 5.32 yields the result.

A product of the kind appearing in the last proposition will concern us in connection with the following result ([V4, Corollary 4.4]):

Theorem 5.34 *Let $F = \Lambda^{l_p}_{r,0}(x,y)$, $1 \leq p < \infty$ or $F = \Lambda^{c_0}_{r,0}(x,y)$be a (PLS)-space. Then for the spectrum $\mathcal{F}$ generating F we have* $\mathrm{Proj}^1\mathcal{F} = 0$ *iff there are subsets $I, J \subset I\!N$ such that*

i) $I \cap J = \emptyset$

ii) $I\!N = I \cup J$

iii) $F = \Lambda^{l_p}_r(x,I) \times \Lambda^{l_p*}_0(y,J)$

(resp. in iii) $F = \Lambda^{c_0}_r(x,I) \times \Lambda^{c_0*}_0(y,J)$*).*

The facts which we have collected enable us to give the following characterization:

Theorem 5.35 *If $E = \Lambda^{l_1}_{r,s}(x,y)$ is shift-stable, if $F = \Lambda^{c_0}_{p,0}(v,w)$ is stable, and if*

$$\lim_{\lambda\to\infty} y_\lambda = \lim_{\nu\to\infty} v_\nu = \infty,$$

then

$$\mathrm{Proj}^1\mathcal{L}(E,F) = 0$$

iff $\mathrm{Proj}^1\mathcal{F} = 0$ *and one of the following holds:*

i) $r = \infty$

ii) $E = \Lambda^{l_1*}_s(y)$

iii) $F = \Lambda^{c_0*}_0(w)$

Proof: Assume that $\mathrm{Proj}^1\mathcal{L}(E,F) = 0$ and neither ii) nor iii) hold, then obviously $\mathrm{Proj}^1\mathcal{F} = 0$, so according to 5.34 we have a decomposition $F = \Lambda^{c_0}_p(v,I) \times \Lambda^{c_0*}_0(w,J)$, in which I cannot be finite because iii) does not hold and F is stable. 5.33 shows that $F_1 := \Lambda^{c_0}_p(v,I)$ must be stable and it is easy to see that necessarily we must have

$$\mathrm{Proj}^1\mathcal{L}(E,F_1) = 0.$$

As ii) does not hold and as E is shift-stable, E must contain a complemented coordinate subspace $E_1 := \Lambda^{l_1}_r(x,M)$ where M cannot be finite. Again we must have

$$\mathrm{Proj}^1\mathcal{L}(E_1,F_1) = 0.$$

This implies that we must have

$$\mathrm{Ext}^1_{(F)}(E_1,F_1) = 0.$$

According to [V1, Corollary 4.4] this implies $r = \infty$.
If on the other hand we have a decomposition, then it is obvious that

$$\mathrm{Proj}^1 \mathcal{L}\left(E, \Lambda_0^{co*}(w, J)\right) = 0,$$

and

$$\mathrm{Proj}^1 \mathcal{L}\left(E, \Lambda_p^{co}(v, I)\right) = 0,$$

is in the case i) implied by 2.9 and 5.19 and in the case ii) by writing

$$E = \Lambda_s^{l_1 *}(y) = \Lambda_{\infty,s}^{l_1}(\tilde{x}, y)$$

and

$$\Lambda_p^{co}(v, I) = \Lambda_{p,\infty}^{co}(v, \tilde{w})$$

for suitably chosen $\tilde{x}$ and $\tilde{w}$ and applying 5.14 (for $s = \infty$) respectively 5.16 (for $s = 0$) and 2.9.

5.8 Solved cases and open questions

We want to point out again the connection between the vanishing of $\mathrm{Proj}^1 \mathcal{L}(E, F)$ and the vanishing of $\mathrm{Ext}^1(E, F)$ for $E = \Lambda_{r,s}^{l_1}(x, y)$ and $F = \Lambda_{p,q}^{co}(v, w)$ and which of the problems we solved and which are open. We always assume that $(x_\lambda)_{\lambda \in \mathbb{N}}$, $(y_\lambda)_{\lambda \in \mathbb{N}}$, $(v_\nu)_{\nu \in \mathbb{N}}$ and $(w_\nu)_{\nu \in \mathbb{N}}$ are sequences of positive numbers tending to infinity.

1) $\underline{q = \infty}$ (the case of local splitting)

   1.1 $\underline{r = \infty}$
   We have $\mathrm{Proj}^1 \mathcal{F} = 0$. In section 5.4 we showed that

   $$\mathrm{Proj}^1 \mathcal{L}(E, F) = 0$$

   and thus

   $$\mathrm{Ext}^1_{(PLS)}(E, F) = 0.$$

   1.2 $\underline{r = 0}$
   The vanishing of $\mathrm{Proj}^1 \mathcal{L}(E, F)$ as well as that of $\mathrm{Ext}^1_{(PLS)}(E, F)$ are open in this case. One cannot expect a result like in the case 1.1 to hold, because even in the special case of $E = \Lambda_0^{l_1}(x)$ and $F = \Lambda_p^{co}(v)$ with v being shift-stable neither $\mathrm{Ext}^1_{(PLS)}(E, F)$ nor $\mathrm{Proj}^1 \mathcal{L}(E, F)$ vanish (cf. [V1, Theorem 1.2, Corollary 4.4]).

2) $\underline{q = 0}$
   In section 5.7 we showed that

   $$\mathrm{Proj}^1 \mathcal{L}(E, F) = 0$$

   iff

   $$\mathrm{Proj}^1 \mathcal{F} = 0$$

   and one of the following holds:

   $$r = \infty \quad \text{or} \quad E = \Lambda_s^{l_1 *}(y) \quad \text{or} \quad F = \Lambda_0^{co*}(w).$$

   The question of $\mathrm{Ext}^1_{(PLS)}(E, F) = 0$ remains open, because we do not know, if the vanishing of $\mathrm{Ext}^1_{(PLS)}(E, F)$ also implies the vanishing of $\mathrm{Proj}^1 \mathcal{L}(E, F)$ (remember that equivalence only holds, if we a priori have $\mathrm{Proj}^1 \mathcal{F} = 0$, see 2.8 and 2.9).

6 Appendix

6.1 The Push-Out Construction

Let a commutative diagram

$$\begin{array}{ccccccccc} & & A & & & & & & \\ & & \Big\uparrow{\scriptstyle \iota} & & & & & & \\ 0 & \longrightarrow & X & \xrightarrow{j} & Y & \xrightarrow{q} & Z & \longrightarrow & 0 \end{array}$$

of locally convex spaces with continuous linear operators be given, where the lower row is exact. Then there is a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{u} & \tilde{A} & \xrightarrow{v} & Z & \longrightarrow & 0 \\ & & \Big\uparrow{\scriptstyle \iota} & & \Big\uparrow{\scriptstyle S} & & \Big\uparrow{\scriptstyle id} & & \\ 0 & \longrightarrow & X & \xrightarrow{j} & Y & \xrightarrow{q} & Z & \longrightarrow & 0 \end{array}$$

with the space $\tilde{A}$, and the maps u, v and S defined by

$$\tilde{A} := (A \times Y) \Big/ M$$

where

$$M := \{(\iota\, x, -j\, x) : x \in X\}\,,$$

$$u(a) := (a, 0) + M$$

for all $a \in A$,

$$v\left((a, y) + M\right) := q\, y$$

for all $a, y \in A \times Y$ and

$$S(y) := (0, y) + M$$

for all $y \in Y$.
Let a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X_1 & \xrightarrow{j_1} & Y_1 & \xrightarrow{q_1} & Z_1 & \longrightarrow & 0 \\ & & \Big\uparrow{\scriptstyle \iota_1^2} & & \Big\uparrow{\scriptstyle \sigma_1^2} & & \Big\uparrow{\scriptstyle \kappa_1^2} & & \\ 0 & \longrightarrow & X_2 & \xrightarrow{j_2} & Y_2 & \xrightarrow{q_2} & Z_2 & \longrightarrow & 0 \end{array}$$

of locally convex spaces and continuous linear maps be given, where the lower row is exact. Then there is a commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & X_1 & \xrightarrow{j_1} & Y_1 & \xrightarrow{q_1} & Z_1 & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle id} & & \uparrow{\scriptstyle T} & & \uparrow{\scriptstyle \kappa_1^2} & & \\
0 & \longrightarrow & X_1 & \xrightarrow{u} & \tilde{Y} & \xrightarrow{v} & Z_2 & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle \iota_1^2} & & \uparrow{\scriptstyle S} & & \uparrow{\scriptstyle id} & & \\
0 & \longrightarrow & X_2 & \xrightarrow{j_2} & Y_2 & \xrightarrow{q_2} & Z_2 & \longrightarrow & 0
\end{array}$$

such that $T \circ S = \sigma_1^2$. Here $\tilde{Y}$, u, v and S are defined as before and

$$T\left((x, y) + M\right) := \sigma_1^2\, y + j_1\, x$$

for all $x, y \in X_1 \times Y_2$.

6.2 The Pullback - Construction

Let a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & X & \xrightarrow{j} & Y & \xrightarrow{q} & Z & \longrightarrow & 0 \\
 & & & & & & \uparrow{\scriptstyle \iota} & & \\
 & & & & & & A & &
\end{array}$$

of locally convex spaces with continuous linear operators be given, where the lower row is exact. Then there is a commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & X & \xrightarrow{j} & Y & \xrightarrow{q} & Z & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle id} & & {\scriptstyle S}\uparrow & & {\scriptstyle \iota}\uparrow & & \\
0 & \longrightarrow & X & \xrightarrow{u} & \tilde{A} & \xrightarrow{v} & A & \longrightarrow & 0
\end{array}$$

where

$$\tilde{A} := \{(y, a) \in Y \times A : q\, y = \iota\, a\},$$

$$u(x) := (j\, x, 0)$$

for all $x \in X$,

$$v\left((y, a)\right) := a$$

and

$$S\left((y, a)\right) := y$$

for all $(y, a) \in \tilde{A}$. Let a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X_1 & \xrightarrow{j_1} & Y_1 & \xrightarrow{q_1} & Z_1 & \longrightarrow & 0 \\ & & \uparrow{\scriptstyle \iota_1^2} & & \uparrow{\scriptstyle \sigma_1^2} & & \uparrow{\scriptstyle \kappa_1^2} & & \\ 0 & \longrightarrow & X_2 & \xrightarrow{j_2} & Y_2 & \xrightarrow{q_2} & Z_2 & \longrightarrow & 0 \end{array}$$

of locally convex spaces and continuous linear maps be given, where the lower row is exact. Then there is a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X_1 & \xrightarrow{j_1} & Y_1 & \xrightarrow{q_1} & Z_1 & \longrightarrow & 0 \\ & & \uparrow{\scriptstyle id} & & \uparrow{\scriptstyle S} & & \uparrow{\scriptstyle \kappa_1^2} & & \\ 0 & \longrightarrow & X_1 & \xrightarrow{u} & \tilde{Y} & \xrightarrow{v} & Z_2 & \longrightarrow & 0 \\ & & \uparrow{\scriptstyle \iota_1^2} & & \uparrow{\scriptstyle T} & & \uparrow{\scriptstyle id} & & \\ 0 & \longrightarrow & X_2 & \xrightarrow{j_2} & Y_2 & \xrightarrow{q_2} & Z_2 & \longrightarrow & 0 \end{array}$$

such that $S \circ T = \sigma_1^2$. Here $\tilde{Y}$, u, v and T are defined as before and

$$S\left((x, y) + M\right) := \sigma_1^2\, y + q_2\, y$$

for all $y \in Y_2$.

6.3 A lemma on extending and lifting continuos linear maps

Let a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X_1 & \xrightarrow{j_1} & Y_1 & \xrightarrow{q_1} & Z_1 & \longrightarrow & 0 \\ & & \uparrow{\scriptstyle \iota} & & \uparrow{\scriptstyle \sigma} & & \uparrow{\scriptstyle \kappa} & & \\ 0 & \longrightarrow & X_2 & \xrightarrow{j_2} & Y_2 & \xrightarrow{q_2} & Z_2 & \longrightarrow & 0 \end{array}$$

of locally convex spaces with continuous linear operators and exact rows be given. There exists an extension for ι onto Y_2 (i.e. a map $T : Y_2 \to X_1$ such that $T \circ j_2 = \iota$) iff there exists a lifting for κ to Y_1 (i.e. a map $S : Z_2 \to Y_1$ such that $q_1 \circ S = \kappa$.

Proof: If T exists, then $\tilde{S} := j_1 \circ T - \sigma$ is a mapping from Y_2 to Y_1 which factorizes through Z_2. The factorization is a mapping which lifts κ. If S exists, then $T := \sigma - S \circ q_2$ is a mapping from Y_2 to Y_1 such that $q_1 \circ T = 0$, so T actually has its image in X_1.

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