Generalized Kähler metrics on complex spaces and a supplement to a Theorem of Fornæss and Narasimhan

Dissertation
zur Erlangung des akademischen Grades
doctor rerum naturalium (Dr. rer. nat.)

eingereicht am Fachbereich Mathematik
der Bergischen Universität-GHS Wuppertal

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geboren am 8. Juli 1971 in Sighişoara/Schäßburg, Rumänien

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Tag der mündlichen Prüfung: 28. November 2000

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Chapter 1

Introduction

Singularities do occur immediately, even if one wants to study complex manifolds only.

For example when considering fibers of holomorphic maps $X \longrightarrow Y$ between complex manifolds, the objects we obtain are in general analytic sets with singularities.

Or if considering quotients of complex manifolds, e. g. for the group

$$G := \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \right\}$$

the orbit space $\mathbb{C}^2/G$ is isomorphic to the affine surface F in $\mathbb{C}^3$ given by $z_3^2 - z_1 z_2$ which is not a topological manifold around $0 \in F$.

These remarks show that complex manifolds cannot be studied successfully without studying more general objects such as complex spaces (with singularities).

In the study of complex spaces *analytic convexity* is of fundamental importance; it establishes a link between geometrical and analytical properties of sets.

Analytic convexity was introduced in complex analysis by E. E. Levi by discovering (around 1910) that the (smooth) boundary of a domain of holomorphy in $\mathbb{C}^n$ is not arbitrary, but satisfies a certain condition of *pseudoconvexity*.

A systematic treatment of complex spaces involves the analysis of *punctual*, *local* and *global* properties.

In passing from the punctual to the local point of view the concept of coherent sheaves plays a decisive role. The coherence of the structure sheaf gives rise to an interplay between algebra and geometry. So, for example, the (punctual) algebraic notion of *analytic algebras* corresponds to the (local) geometric objects *germs of complex spaces*.

For the step from local to global one can apply cohomology theory. The global theory is particularly well developed for two classes of complex spaces that can be characterized in terms of topological or function-theoretic "completeness", namely the compact spaces and the Stein spaces.

Stein spaces are exactly the 1-complete complex spaces and the strongly plurisubharmonic functions, which are in close relation to the Stein spaces, are exactly the 1-convex functions.

q-complete spaces generalize the Stein spaces, as q-convex functions do with the strongly plurisubharmonic ones.

The q-convex with corners functions on a complex space introduced by Diederich and Fornæss play an important role in q-convexity theory.

The concept of q-convexity developed by Andreotti and Grauert in 1962 generalizes the concept of pseudoconvexity.

This dissertation deals, roughly speaking, with analytic convexity and q-convexity on complex spaces.

The results are structured into two parts: one concerning compact spaces, proper modifications and generalized Kähler metrics (chapters 3 and 4) and the other concerning q-complete spaces and q-plurisubharmonic functions (chapter 5).

There is less relation between the two parts, but they are both in close relation to me!

Concerning the first part: on complex spaces there is a kind of surgery called proper modifications. Roughly speaking one replaces a nowhere dense closed complex subspace A of Y by another complex space B in such a way that $X := (Y \setminus A) \cup B$ becomes a complex space endowed with a proper holomorphic map $\pi : X \longrightarrow Y$ which maps $X \setminus B$ biholomorphically onto $Y \setminus A$. Classical is the blowing-up of points: e. g. one replaces $0 \in \mathbb{C}^n$ by the projective space $\mathbb{P}^{n-1}$ of all line directions at 0. This procedure can be generalized: every closed complex subspace A can be blown-up in a natural way along A.

By a theorem of Hironaka and Chow proper modifications are not too far away from blowing-ups: they are always dominated by a locally finite sequence of blowing-ups.

But although proper modifications modify the geometry of the space only along a rare analytic set it is enough to "disturb" important analytic properties.

So, for example, Moishezon [Mo1] proved by a counterexample that for a surjective, proper modification $\pi : X \longrightarrow Y$ between complex spaces such that Y has a Kähler metric it doesn't follow necessarily that X is also Kähler.

Among the compact complex manifolds the Kähler manifolds enjoy a number of remarkable properties. Kähler spaces were first introduced by Grauert [Gr] and their study was continued by Moishezon [Mo1]. It is known that the definition of Moishezon of a Kähler metric coincides with that one of Grauert in the case of normal spaces.

The counterexample of Moishezon gives naturally rise to the question whether one can prove in that context general results about X. Other stated the question is: how far is X from being Kähler.

In chapter 3 we prove that X possesses a so called *generalized Kähler metric* (Theorem 3.1.6) by using a method from the article of M. Coltoiu and M. Mihalache [Co-Mi] and a smooth-glueing technique from Demailly [Dem].

The notion of generalized Kähler metrics that we introduce (Definition 3.1.5) differs only a little from the definition of Moishezon we use for Kähler metrics: we admit $-\infty$ as value for the system of defining functions. More precisely it is:

Definition 3.1.5 *We say that the reduced compact complex space X has a generalized Kähler metric if there exists a covering of X with open sets $(U_i)_i$ such that on each set U_i there exists a function $\varphi_i : U_i \longrightarrow [-\infty, \infty), \varphi_i \not\equiv -\infty$ on each irreducible component of U_i, which is strongly plurisubharmonic, regular of class $\mathcal{C}^\infty$ outside the set $\{\varphi_i = -\infty\}$ and such that on each nonempty intersection $U_i \cap U_j$ we have (locally) the compatibility condition $\varphi_i = \varphi_j + \operatorname{Re} f_{ij}$ for some holomorphic function f_{ij}*

The precise theorem is the following:

Theorem 3.1.6 *Let X and Y be two reduced, compact, complex spaces (with singularities) and $p : X \longrightarrow Y$ a surjective, holomorphic map, which is a proper modification. Suppose that Y is Kähler. Then X has a generalized Kähler metric.*

In chapter 4 we generalize this theorem to proper maps by using Theorem 3.1.6 as a special case in its proof. More exactly, the result is the following:

Theorem 4.1.1 *Let $p : X \longrightarrow Y$ be a holomorphic and surjective map between two reduced, compact, complex spaces with singularities and with the property that p sends each irreducible component C_X of X (surjective) onto an irreducible component C_Y of Y of the same dimension, $\dim C_X = \dim C_Y$.*
If Y is Kähler, then X has a generalized Kähler metric.

Concerning the second part: it contains a generalization of a theorem of Fornæss and Narasimhan ([F-N] Theorem 5.3.1.) which asserts that on each complex space we have equality between the class of plurisubharmonic functions and that one of weakly plurisubharmonic functions (see section 5.1). This result is evident for complex manifolds but totally non-trivial for complex spaces. The proof of Fornæss and Narasimhan uses among other results also a local maximum theorem of H. Rossi (Ann. of Math. 72, 1-11, 1960) which cannot be generalized so easy (if ever) to the q-convex case.

To generalize to complex spaces a certain object in complex analysis already defined by a local condition in open sets in $\mathbb{C}^n$ in a biholomorphic manner one has the choice between the following two possibilities:

(i) to define it as the trace, by means of local embeddings, of the analogue object in open sets in $\mathbb{C}^n$

(ii) to ask that for each holomorphic map $f : U \longrightarrow X$ with $U \subset \mathbb{C}^q$ open set (for some fixed q) the "pull-back" to U of the object by means of the map f is such an object in U.

It is well known that the resulting objects are not necessarily the same in the two methods if X has singularities.

Chapter 5 deals with the equivalence of this two defining methods for the case of the q-plurisubharmonic functions in the sense of Fujita [Fu] on complex spaces of finite dimension. We obtain so a generalization of the result of Fornæss and Narasimhan, but only in the case of continuous functions (Theorem 5.3.14). Our proof is by induction over the dimension of the space and is based on a new proof for the result of Fornæss and Narasimhan (i. e. the case $q = 1$) (see section 5.2).

At the same time we obtain a generalization of a result of Siu, namely we show that every q-complete subspace with corners of a complex space admits a q-complete with corners neighbourhood.

I want to express my warmest thanks to Prof. M. Colţoiu and to Prof. K. Diederich for their very helpful discussions and advices during the whole time of preparing my dissertation.

I thank Prof. Colţoiu for suggesting me the problems treated here. This dissertation was written at the University of Wuppertal. I thank the University of Wuppertal and especially Prof. Diederich for this opportunity and for providing me a nice and proper atmosphere to work.

To all my friends who were near me I am grateful.

Especially to my parents and to Bert Fischer, who answered me all questions concerning LaTeX.

Chapter 2

Preliminaries

2.1 Introduction to q-convexity

Throughout this work all complex spaces are assumed to be reduced and with countable topology, unless it is otherwise stated. The main definitions concerning complex spaces we will use are those from [Ka].

Definition 2.1.1 *A holomorphic map between two complex spaces $\varphi : X \longrightarrow Y$ is called an embedding if there is a closed subspace $i : Y' \hookrightarrow Y$ and a factorisation*

$$\begin{array}{ccc} X & \xrightarrow{\varphi} & Y \\ & \searrow^{\varphi'} & \uparrow i \\ & & Y' \end{array}$$

where φ' is biholomorphic.

φ is called immersion at $x \in X$ if there are open neighbourhoods U of x in X and V of $\varphi(x)$ in Y such that $\varphi_{|_U} : U \longrightarrow V$ is an embedding (i.e. it is locally an embedding).

φ is called an immersion if it is an immersion at every point.

Let X be a complex space (with singularities). A function $\theta : X \longrightarrow \mathbb{R}$ is called *smooth* (of class $\mathcal{C}^\infty$) if we can find for every point $x \in X$ an open neighbourhood U of x and a holomorphic embedding $i : U \hookrightarrow G$, where G is an open subset in some euclidian space $\mathbb{C}^n$ (such a i is called a "local chart"), such that there exists on G a function of class $\mathcal{C}^\infty$, $f \in \mathcal{C}^\infty(G)$, with the property that $f \circ i = \theta_{|_U}$.

We denote by $PSH(X)$ and call them *plurisubharmonic functions* those upper semi-continuous functions $\varphi : X \longrightarrow [-\infty, \infty)$ such that for every $x \in X$ there exists a local embedding $i : U \hookrightarrow \tilde{U} \subset \mathbb{C}^n$, where U is a neighbourhood of x and $\tilde{U}$ is an open subset of $\mathbb{C}^n$, and there exists a plurisubharmonic function $\tilde{\varphi}$ on the euclidian open subset $\tilde{U}$ such that $\tilde{\varphi} \circ i = \varphi$.

$SPSH(X)$ stands for the *strongly plurisubharmonic functions* on X, i.e. those PSH-functions for which we have: for every $\theta \in \mathcal{C}_0^\infty(X, \mathbb{R})$ (i.e. smooth and with compact support), there exists $\varepsilon_0 > 0$ such that $\varphi + \varepsilon\theta \in PSH(X)$, for $0 \le \varepsilon \le \varepsilon_0$.

Example If $f : X \longrightarrow \mathbb{C}$ is holomorphic on X then $|f|$, $|f|^2$, $\log(1+|f|^2)$ and $\log|f|$ are plurisubharmonic functions on X (the last function is indeed pluriharmonic on the set $\{f \neq 0\}$). Furthermore $\log(1+|z|^2)$ is strongly plurisubharmonic on $\mathbb{C}^n$.

Remark 2.1.2

(i) If $f : Z \longrightarrow X$ is a holomorphic map between complex spaces and $\varphi \in PSH(X)$ then $\varphi \circ f \in PSH(Z)$.

(ii) If $\varphi : X \longrightarrow \mathbb{R}$ is plurisubharmonic and $g : \mathbb{R} \longrightarrow \mathbb{R}$ is convex and increasing then $g \circ \varphi$ is plurisubharmonic.

Definition 2.1.3 *For a function $\varphi \in \mathcal{C}^2(U, \mathbb{R})$, where U is an open subset of some euclidian space $\mathbb{C}^n$, we define the Levi form $\mathcal{L}(\varphi)$ of φ as the hermitian form*

$$\mathcal{L}(\varphi) = \sum_{i,j=1}^{n} \frac{\partial^2 \varphi}{\partial z_i \partial \bar{z}_j} dz_i d\bar{z}_j$$

Remark 2.1.4

(i) The Levi form of φ at the point $x \in U$, calculated in $\psi, \eta \in \mathbb{C}^n$ is then the complex number given by the expression

$$\mathcal{L}(\varphi, x)(\psi, \eta) = \sum_{i,j=1}^{n} \frac{\partial^2 \varphi}{\partial z_i \partial \bar{z}_j}(x) \psi_i \bar{\eta}_j$$

(ii) A real-valued $\mathcal{C}^2$-function $\varphi : U \longrightarrow \mathbb{R}$, where U is an open set in $\mathbb{C}^n$, is plurisubharmonic (resp. strongly plurisubharmonic) if and only if its Levi form is positive-semidefinite (resp. positive definite), that is that for each $x \in U$ and for every $\psi \in \mathbb{C}^n$ the inequality $\mathcal{L}(\varphi, x)(\psi, \psi) \ge 0$ (resp. > 0 on $\mathbb{C}^n \setminus \{0\}$) holds. This is equivalent to the fact that the Levi-form of the plurisubharmonic (resp. strongly plurisubharmonic) function φ has exactly n non-negative (resp. n positive) eigenvalues.

If we require the Levi-form of a $\mathcal{C}^2$-function to have at each point a certain number of positive eigenvalues, we get the notions of q-convex functions (see [A-G]). More precisely, we have the following definitions:

Definition 2.1.5 *A function $\varphi \in \mathcal{C}^\infty(U, \mathbb{R})$, where U is an open subset of $\mathbb{C}^n$ is called q-convex ($q \in \mathbb{N}, 1 \le q \le n$) if its Levi form $\mathcal{L}(\varphi)$ has at least $n - q + 1$ positive (> 0) eigenvalues at every point of U.*

Example Let U be an open subset in a complex number space $\mathbb{C}^n$ and φ a smooth strongly plurisubharmonic function on U. The function $\psi \in \mathcal{C}^\infty(U, \mathbb{R})$ defined by $\psi := \varphi \cdot (1 + |f_2|^2 + \ldots + |f_q|^2)^\alpha$ is q-convex, where $\alpha \in \mathbb{R}$ and $f_2, \ldots, f_q$, $(2 \leq q \leq n)$ are holomorphic functions on U.

Remark 2.1.6

(i) The 1-convex functions on open euclidian subsets are exactly the smooth strongly plurisubharmonic ones.

(ii) The above notion of q-convex functions is easily generalized to complex spaces by using local embeddings (see [A-G]). The precise definition is the following:

Definition 2.1.7 *Let X be a complex space and consider a function $\varphi \in \mathcal{C}^\infty(X, \mathbb{R})$. φ is called a q-convex function ($q \in \mathbb{N}$) if for any point $x \in X$ there exists a local chart $i : U \longrightarrow \tilde{U} \subset \mathbb{C}^n$ and a function $\tilde{\varphi} \in \mathcal{C}^\infty(\tilde{U}, \mathbb{R})$ with the property that $\tilde{\varphi} \circ i = \varphi_{|_U}$ and that the Levi form of $\tilde{\varphi}$ has at least $n - q + 1$ positive eigenvalues at any point of $\tilde{U}$.*

Remark 2.1.8 It can be checked that the existence of $\tilde{\varphi}$ does not depend on the local embedding chosen.

A continuous function $\varphi : X \longrightarrow \mathbb{R}$ is called an *exhaustion function* for the space X if the sublevel sets $\{\varphi \leq c\} := \{x \in X \mid \varphi(x) \leq c\}$ are compact for every $c \in \mathbb{R}$.

Definition 2.1.9 *A complex space X is called q-convex if there exists a compact set K in X and a smooth exhaustion function $f : X \longrightarrow \mathbb{R}$ which is q-convex on $X \setminus K$.*
If it is possible to choose $K = \emptyset$, then X is called q-complete.

Remark 2.1.10

(i) The 1-complete spaces are exactly the Stein spaces.

(ii) By a result of Narasimhan (see [RN]) it holds that any 1-convex space X is a proper modification of a Stein space at a finite set, i.e. there exists a Stein space Z, a finite set $A := \{z_1, \ldots, z_p\} \subset Z$ and a proper (i.e. inverse images of compact sets are still compact), holomorphic, surjective map $f : X \longrightarrow Z$ such that $f_{|_{X \setminus f^{-1}(A)}} : X \setminus f^{-1}(A) \longrightarrow Z \setminus A$ is biholomorphic and that $\mathcal{O}_Z \simeq f_* \mathcal{O}_X$. The (analytic) set $f^{-1}(A)$ is called the exceptional set of X; it is the maximal compact analytic set of positive dimension in X. If the exceptional set is empty it means that X is 1-complete.

Examples

(i) If $U \subset \mathbb{C}^n$ is a Stein open subset and if we consider the analytic subset given by $A = \{x \in U \mid f_1(x) = \ldots = f_k(x) = 0\}$ where $f_1, \ldots, f_k \in \mathcal{O}(U)$, then $U \setminus A$ is k-complete.

More generally, if $U \subset \mathbb{C}^n$ is q-complete then $U \setminus A$ is $(q+k-1)$-complete. Indeed, if we fix a q-convex exhaustion function ϱ for U, then the function φ given by

$$\varphi = \varrho + \frac{1}{(\sum_{i=1}^{k} |f_i|^2)^k}$$

is $(q+k-1)$-convex on $U \setminus A$ and exhausts $U \setminus A$.

(ii) Consider a linear subspace $L \subset \mathbb{P}^n$ of codimension q in some complex projective space $\mathbb{P}^n$. Then $\mathbb{P}^n \setminus L$ is q-complete.

Indeed, if L is given by $L = \{(z_0 : \ldots : z_n) \mid z_0 = \ldots = z_{q-1} = 0\}$ then we obtain a q-convex exhaustion function for $\mathbb{P}^n \setminus L$ by putting:

$$\varphi : \mathbb{P}^n \setminus L \longrightarrow \mathbb{R}$$

$$\varphi(z) = \frac{|z_0|^2 + \ldots\ldots + |z_n|^2}{|z_0|^2 + \ldots + |z_{q-1}|^2}$$

An important tool in complex analysis are the q-convex functions with corners, as they were introduced by Diederich and Fornæss in [D-F.1] and [D-F.2].

Definition 2.1.11 *On a complex space X a continuous real valued function f is called q-convex with corners if for any point $x \in X$ there exists an open neighbourhood U of x and finitely many q-convex functions $f_1, \ldots, f_p$ on U such that $f_{|_U} = \max\{f_1, \ldots, f_p\}$.*

The following definition is also given in [D-F.1] and [D-F.2].

Definition 2.1.12 *A complex space X is called q-convex with corners if there is a continuous exhaustion function f on X which is q-convex with corners outside a compact set $K \subset X$.*

If it is possible to choose $K = \emptyset$, then X is called q-complete with corners.

Remark 2.1.13 The class of the q-convex functions with corners on a complex space X will be denoted by $F_q(X)$.

We list in what follows some examples of q-complete spaces which are provided in [MP]: Expl.(i) and [Co-Di] Expl.(iii).

Examples

(i) If X is a Stein manifold and $A \subset X$ is a complex subvariety which is at every point of codimension $\leq q$ in X, then $X' = X \setminus A$ is q-complete with corners.

By using the smoothing result for q-convex functions with corners from [D-F.1] one obtains from the above example the following one:

(ii) Let X and A be as above and suppose that X is of pure dimension n. Then X' is $\tilde{q}$-complete (in the smooth sense, i.e. without corners) with $\tilde{q} = n - [\frac{n}{q}] + 1$ (where $[\cdot]$ means the integral part).

This example is relevant if $q \leq \frac{n}{2}$.

(iii) Let X be a normal Stein space of pure dimension $n \geq 2$ and $A \subset X$ a closed complex analytic subvariety without isolated points. Then $X \setminus A$ is $(n-1)$-complete (being the union of $(n-1)$ open Stein subspaces!).

2.2 The tangent space on complex spaces with singularities

In what follows we refer to the book of G. Fischer [Fi]. We need the underneath notions and results for the proof of our Theorem 4.1.1.

We will denote by $\mathcal{O}(X)$ the set of holomorphic functions (they are indeed functions in the case of reduced spaces!) on X, by $\mathcal{O}_{X,x}$ for a given point $x \in X$ we will denote the set of germs in X of holomorphic functions defined in a neighbourhood of x, by $\mathrm{Sing}(X)$ the (analytic) set of singular points and by $\mathrm{Reg}(X) := X \setminus \mathrm{Sing}(X)$ the set of regular points.

$\mathfrak{m}_{X,x}$ will stand for the maximal ideal of the algebra $\mathcal{O}_{X,x}$.

Let now X be a complex space and $x \in X$ be a point. The following definition is analogue to the complex manifold-case:

Definition 2.2.1 *A $\mathbb{C}$-linear map $\delta_x : \mathcal{O}_{X,x} \longrightarrow \mathbb{C}$ is called a tangent vector to X at x (also called a derivation of $\mathcal{O}_{X,x}$) if it satisfies one of the following equivalent conditions:*

(i) $\delta_x(f \cdot g) = g(x) \cdot \delta_x(f) + f(x) \cdot \delta_x(g)\ ,\ \forall f, g \in \mathcal{O}_{X,x}$

(ii) $\delta_x(\mathfrak{m}^2_{X,x}) = 0 \quad \text{and} \quad \delta_x(\mathbb{C} \cdot 1) = 0$

Remark 2.2.2 These are indeed equivalent conditions:

(i) ⇒ *(ii)* If $f, g \in \mathfrak{m}_{X,x}$ then $f(x) = g(x) = 0$ and by (i) it follows that $\delta_x(f \cdot g) = 0$. Because $\delta_x(1) = \delta_x(1 \cdot 1) = \delta_x(1) + \delta_x(1)$ we have that $\delta_x(1) = 0$ so $\delta_x(\mathbb{C} \cdot 1) = 0$.

(ii) ⇒ *(i)* We use the decomposition $\mathcal{O}_{X,x} = \mathbb{C} \oplus \mathfrak{m}_{X,x}$ which is true for all complex spaces, by definition. So, for an arbitrary germ $f \in \mathcal{O}_{X,x}$ we have that $f = f(x) + \bar{f}$ and now a short computation gives the desired equality:

$$\delta_x(f \cdot g) = \delta_x((f(x) + \bar{f}) \cdot (g(x) + \bar{g})) = \delta_x(f(x) \cdot g(x)) + \delta_x(\bar{f} \cdot \bar{g}) +$$

$$+ g(x)\delta_x(\bar{f}) + f(x)\delta_x(\bar{g}) = g(x) \cdot \delta_x(f) + f(x) \cdot \delta_x(g)$$

Remark 2.2.3 It is obvious that $T_x X$, the set of all tangent vectors to X at x, is a $\mathbb{C}$-vector space. It is called the tangent space of X at x.

Now, if $\varphi : X \longrightarrow Y$ is a holomorphic map, $x \in X$ and $y \in Y$ are points with $\varphi(x) = y$ and if $\tilde{\varphi}_x : \mathcal{O}_{Y,y} \longrightarrow \mathcal{O}_{X,x}$ denotes the canonical morphism (in the case of reduced spaces it is given by composition with φ), then the map

$$T_x\varphi : T_x X \longrightarrow T_y Y$$

given by

$$\delta_x \mapsto \delta_x \circ \tilde{\varphi}_x$$

is a morphism of $\mathbb{C}$-vector spaces, called the *Jacobian map* of φ at x.

Remark 2.2.4 For every tangent vector δ_x to X at x we have the induced $\mathbb{C}$-linear map

$$\mathfrak{m}_{X,x}/\mathfrak{m}_{X,x}^2 \longrightarrow \mathbb{C}$$

Thus for every $x \in X$ we get a map

$$T_xX \longrightarrow (\mathfrak{m}_{X,x}/\mathfrak{m}_{X,x}^2)^*,$$

where $(\cdot)^*$ represents the dual space. This map turns out to be an isomorphism of $\mathbb{C}$-vector spaces.

The description of the tangent space by means of local coordinates is the following:

Consider first an open subset $U \subset \mathbb{C}^n$ and denote by $z_1, \ldots, z_n$ the coordinate functions. We then have (being in an euclidian space!) the following tangent vectors:

$$\begin{array}{rcl} \frac{\partial}{\partial z_i}(x) : \mathcal{O}_{U,x} & \longrightarrow & \mathbb{C} \\ f & \mapsto & \frac{\partial F}{\partial z_i}(x) =: \frac{\partial f}{\partial z_i}(x) \end{array}$$

where F represents the germ $f \in \mathcal{O}_{U,x}$ in some neighbourhood of x.

It is known that the map

$$\mathbb{C}^n \longrightarrow T_xU = T_x\mathbb{C}^n$$

given by

$$(s_1, \ldots, s_n) \mapsto s_1 \frac{\partial}{\partial z_1}(x) + \ldots + s_n \frac{\partial}{\partial z_n}(x)$$

is an isomorphism of $\mathbb{C}$-vector spaces.

Consider now the case when X is a closed complex subspace of U with the canonical injection $i : X \hookrightarrow U$ and defined by a coherent ideal sheaf $\mathcal{J} \subset \mathcal{O}_U$.

We then have at each point $x \in X$ an exact sequence

$$0 \longrightarrow \mathcal{J}_x \longrightarrow \mathcal{O}_{U,x} \longrightarrow \mathcal{O}_{X,x} \longrightarrow 0$$

This implies that $T_xi : T_xX \longrightarrow T_xU$ is injective and that its image is equal to

$$\{\delta_x \in T_xU \mid \delta_x(\mathcal{J}_x) = 0\}$$

This means in local coordinates that T_xX is isomorphic to the set:

$$\{(s_1, \ldots, s_n) \in \mathbb{C}^n \mid s_1 \frac{\partial f}{\partial z_1}(x) + \ldots + s_n \frac{\partial f}{\partial z_n}(x) = 0 \ , \ \forall f \in \mathcal{J}_x\}$$

Remark 2.2.5 The dimension of the tangent space at a point is always finite. But whereas in the case of connected manifolds this number is the same for all points, this is no more true for complex spaces. In the singular points of a complex space this number is always bigger than the dimension of the tangent space at the regular points in a small neighbourhood. More precisely, it holds:

Proposition 2.2.6 *For every point x of a complex space X we have that* $\dim_{\mathbb{C}} T_x X = \operatorname{embdim}_x X$*, where* $\operatorname{embdim}_x X$ *denotes the embedding dimension of X at x, that means the minimal $m \in \mathbb{N}$ such that there exists an open neighbourhood U of x in X together with an embedding $U \longrightarrow \mathbb{C}^m$.*

Remark 2.2.7 It is clear that the embedding dimension at a singular point x is (strictly) greater than the dimension of the regular points in a small neighbourhood of x.

Proposition 2.2.8 *Let $\varphi : X \longrightarrow Y$ be a holomorphic map, $x \in X$ and $y = \varphi(x)$. Then φ is an immersion at x if and only if the Jacobian map $T_x\varphi : T_x X \longrightarrow T_y Y$ is injective.*

We want to point out that we considered so far only tangent vectors at a point of a complex space. But if one wants for instance to define vector fields and differential forms the punctual tangent vector spaces have to be glued together to a space TX. In case of a manifold it is well known that this is a vector bundle over X, $\pi : TX \longrightarrow X$. In the singular case this cannot be a vector bundle (for example because of Proposition 2.2.6). It turns out to be a *linear fiber space over* X, $\pi : TX \longrightarrow X$.

Definition 2.2.9 *A linear space over a complex space S (or simply, a linear fiber space) is a complex space L together with a holomorphic map $\varphi : L \longrightarrow S$ and with compositions*

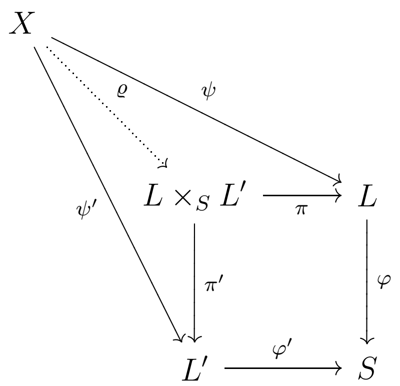
$$+ : L \times_S L \longrightarrow L$$

and

$$\bullet : \mathbb{C} \times L \longrightarrow L$$

such that the module axioms hold.

Remark 2.2.10 $\times_S$ denotes the fiber product in the category of complex spaces over S, i.e. for given holomorphic maps $\varphi : L \longrightarrow S$ and $\varphi' : L' \longrightarrow S$ between complex spaces the fiber product of L and L' over S is a complex space $L \times_S L'$ together with holomorphic maps $\pi : L \times_S L' \longrightarrow L$ and $\pi' : L \times_S L' \longrightarrow L'$ such that $\varphi \circ \pi = \varphi' \circ \pi'$ and such that the following universal property holds:
Given any complex space X together with holomorphic maps $\psi : X \longrightarrow L$ and $\psi' : X \longrightarrow L'$ such that $\varphi \circ \psi = \varphi' \circ \psi'$, there is a unique holomorphic map $\varrho : X \longrightarrow L \times_S L'$ such that the following diagram commutes:



Remark that the universal property implies the uniqueness up to isomorphisms of the fiber product.

Remark 2.2.11 $L \times_S L'$ is a closed subspace of $L \times L'$. As a set it is given by $L \times_S L' = \{(l, l') \in L \times L' \mid \varphi(l) = \varphi'(l')\}$.

Definition 2.2.12 *Let $(L, +, \bullet)$ and $(L', +', \bullet')$ be linear spaces over S. A holomorphic map $\xi : L \longrightarrow L'$ over S (i.e. such that $\varphi' \circ \xi = \varphi$) is called a morphism (of linear spaces over S) if the diagrams*

$$\begin{array}{ccc} L \times_S L & \xrightarrow{\xi \times_S \xi} & L' \times_S L' \\ {\scriptstyle +}\downarrow & & \downarrow{\scriptstyle +'} \\ L & \xrightarrow{\xi} & L' \end{array}$$

and

$$\begin{array}{ccc} \mathbb{C} \times L & \xrightarrow{\mathrm{Id} \times \xi} & \mathbb{C} \times L' \\ {\scriptstyle \bullet}\downarrow & & \downarrow{\scriptstyle \bullet'} \\ L & \xrightarrow{\xi} & L' \end{array}$$

commute.

We shall now define a change of base for linear fiber spaces. If $\psi : S' \longrightarrow S$ is a holomorphic map and L is a linear space over S, then we define

$$\psi^* L := L \times_S S'$$

The compositions in L induce compositions in $\psi^* L$ and $\psi^* L$ becomes a linear space over S'.

Now, for the construction of the space TX one simply takes for the underlying set the disjoint union $\bigcup_{x \in X} T_x X$.

In order to define a complex structure on TX consider first the case when

$$X \hookrightarrow W \subset \mathbb{C}^n$$

is a closed complex subspace of an open subset W of $\mathbb{C}^n$ defined by a coherent ideal sheaf $\mathcal{J} \subset \mathcal{O}_W$ which is generated by $f_1, \ldots, f_r \in \mathcal{O}_W(W)$.

If $(z_1, \ldots, z_n, s_1, \ldots, s_n)$ are coordinates in $W \times \mathbb{C}^n$ one defines the space

$$TX \hookrightarrow W \times \mathbb{C}^n$$

as the closed complex subspace defined by

$$f_1, \ldots, f_r \quad \text{and} \quad s_1 \frac{\partial f_\nu}{\partial z_1} + \ldots + s_n \frac{\partial f_\nu}{\partial z_n} \quad \forall \nu = 1, \ldots, r.$$

Remark that f_ν and $\frac{\partial f_\nu}{\partial z_i}$ are considered here as holomorphic functions on $W \times \mathbb{C}^n$ by means of the canonical projection map $W \times \mathbb{C}^n \longrightarrow W$.

Considering TX as a subspace in $X \times \mathbb{C}^n$ it is then defined by the sheaf of *linear forms* on $X \times \mathbb{C}^n$ given by:

$$f_{\nu,1} \cdot s_1 + \ldots + f_{\nu,n} \cdot s_n \quad \text{for} \quad \nu = 1, \ldots, r$$

where $f_{\nu,\mu} \in \mathcal{O}_X(X)$ denotes here the residue class of $\frac{\partial f_\nu}{\partial z_\mu} \in \mathcal{O}_W(W)$.

The projection of $X \times \mathbb{C}^n$ onto the first component induces the canonical map $\pi : TX \longrightarrow X$. Hence TX is a linear space over X.

The coordinate description of T_xX immediately implies that $T_xX = (TX)_x$ is the fiber of π in x, for all $x \in X$.

Now it is easy to show that this construction does not depend on the choices made. Hence, for an arbitrary complex space X the local pieces (i.e. where X is locally embedded in a complex number space) may be glued together in the usual way of glueing complex spaces (see for instance [Ka] E.31.m) to obtain the so called *tangent space of* X, denoted by TX, which is a linear space over X.

Remark 2.2.13 We want to point out that by this canonical construction of a complex structure on TX one gets a complex structure which is in general not reduced, even if X was reduced!

We want to define also a morphism of tangent spaces,

$$T\varphi : TX \longrightarrow \varphi^* TY$$

for a given holomorphic map $\varphi : X \longrightarrow Y$.

The problem being local with respect to X and Y we may suppose that we have a commutative diagram of the form:

$$\begin{array}{ccc} \mathbb{C}^n & & \mathbb{C}^m \\ \uparrow & & \uparrow \\ U & \xrightarrow{\Phi} & V \\ \uparrow & & \uparrow \\ X & \xrightarrow{\varphi} & Y \end{array}$$

where U and V are open euclidian sets.

We may further assume that the ideal $\mathcal{J} \subset \mathcal{O}_U$ of X is generated by $f_1, \ldots, f_k \in \mathcal{O}_U(U)$ and the ideal $\mathcal{I} \subset \mathcal{O}_V$ of Y is generated by $g_1, \ldots, g_l \in \mathcal{O}_V(V)$.

In order to define $T\varphi$ we also consider the following diagram:

$$\begin{array}{ccccc}
TU & & \Phi^*TV & & TV \\
\| & & \| & & \| \\
U \times \mathbb{C}^n & \xrightarrow{T\Phi} & U \times \mathbb{C}^m & \xrightarrow{\Phi\times\mathrm{Id}} & V \times \mathbb{C}^m \\
\uparrow & & \uparrow & & \uparrow \\
TX & \overset{T\varphi}{\dashrightarrow} & \varphi^*TY & \xrightarrow{\varphi\times\mathrm{Id}} & TY
\end{array}$$

where $T\Phi$ is the classical Jacobian map defined by:

$$(u, s_1 \dots, s_n) \mapsto (u, \sum_{r=1}^{n} \frac{\partial \Phi_1}{\partial z_r} \cdot s_r, \dots, \sum_{r=1}^{n} \frac{\partial \Phi_m}{\partial z_r} \cdot s_r)$$

and we have to show that $T\Phi$ admits a restriction $T\varphi$.

But we now know that $TY \hookrightarrow V \times \mathbb{C}^m$ is the subspace generated by

$$g_1, \dots, g_l \quad \text{and} \quad t_1 \frac{\partial g_\lambda}{\partial w_1} + \dots + t_m \frac{\partial g_\lambda}{\partial w_m} \quad \forall \lambda = 1, \dots, l$$

where $(w_1, \dots, w_m)$ are the coordinates in V and $(t_1, \dots, t_m)$ are the coordinates in $\mathbb{C}^m$.

Consequently $\varphi^*TY \hookrightarrow U \times \mathbb{C}^m$ is the subspace generated by

$$f_1, \dots, f_k \quad \text{and} \quad t_1 \frac{\partial g_\lambda}{\partial w_1} \circ \Phi + \dots + t_m \frac{\partial g_\lambda}{\partial w_m} \circ \Phi \quad \forall \lambda = 1, \dots, l. \tag{2.1}$$

We apply in what follows a criterion for the existence of the restriction of a map $\chi := (|\chi|, \chi^0) : S \longrightarrow T$ between complex spaces (see for instance [Ka] 31.9):

If $A \hookrightarrow S$ and $B \hookrightarrow T$ are complex subspaces generated by the ideal sheafs $\mathcal{J}$ resp. $\mathcal{I}$ then the restriction $\tilde{\chi} : A \longrightarrow B$ of χ exists if and only if $\chi^*\mathcal{I} \subset \mathcal{J}$, where the germ of the inverse image sheaf $\chi^*\mathcal{I}$ is given by $(\chi^*\mathcal{I})_s = \chi^0_s(\mathcal{I}_{\chi(s)}) \cdot \mathcal{O}_{S,s}$.

In our case we have to substitute in (2.1)

$$t_\nu = s_1 \frac{\partial \Phi_\nu}{\partial z_1} + \dots + s_n \frac{\partial \Phi_\nu}{\partial z_n}$$

By computation we obtain then

$$s_1 \frac{\partial (g_\lambda \circ \Phi)}{\partial z_1} + \dots + s_n \frac{\partial (g_\lambda \circ \Phi)}{\partial z_n} \tag{2.2}$$

Since φ is the restriction of Φ it follows that the function $g_\lambda \circ \Phi$ is a section in $\mathcal{J}$ over U. This implies that (2.2) is a section in the ideal sheaf defining $TX \hookrightarrow U \times \mathbb{C}^n$ if we shrink U sufficiently. But this means that there is a restriction $T\varphi : TX \to \varphi^*TY$ of

$T\Phi$. It is called the *Jacobian map* of φ and we have the property that $(T\varphi)_x = T_x\varphi$ for each point $x \in X$.

Let $\varphi : X \to Y$ be a holomorphic map and define

$$T(X/Y) := \ker T\varphi \hookrightarrow TX$$

It is called the *tangent space of* φ (or the *tangent space of* X *over* Y). Geometrically, $T(X/Y)$ consists of all tangent vectors "in the fiber direction".
More precisely, this is justified by the following remark:

Remark 2.2.14 Given a holomorphic map $\varphi : X \to Y$ and $q \in Y$ denote by X_q the fiber of φ over q, that is the inverse image through φ of the simple point $(\{q\}, \mathbb{C})$. Denote by $i : X_q \hookrightarrow X$ the canonical injection. It then holds that $T(X_q) = i^*T(X/Y)$.

Indeed, since the question is local and with the same notations as above it follows from the definition of $T\varphi$ that $T(X/Y) \hookrightarrow U \times \mathbb{C}^n$ is the complex subspace generated by the following holomorphic functions on $U \times \mathbb{C}^n$:

$$\left.\begin{array}{ll} f_1, \dots, f_k & \\ \dfrac{\partial f_\lambda}{\partial z_1} s_1 + \cdots + \dfrac{\partial f_\lambda}{\partial z_n} s_n, & \lambda = 1, \dots, k \\ \dfrac{\partial \Phi_\nu}{\partial z_1} s_1 + \cdots + \dfrac{\partial \Phi_\nu}{\partial z_n} s_n, & \nu = 1, \dots, m \end{array}\right\} \tag{2.3}$$

We may assume without loss of generality that $q = 0 \in V$. It then follows that $X_q \hookrightarrow U$ is the subspace generated by $f_1, \dots, f_k$ and $\Phi_1, \dots, \Phi_m$. Consequently $T(X_q) \hookrightarrow U \times \mathbb{C}^n$ is generated by $f_1, \dots, f_k$, $\Phi_1, \dots, \Phi_m$ and (2.3). The remark is verified by observing that $i^*T(X/Y) \hookrightarrow T(X/Y)$ is just the subspace generated by the residue classes of $\Phi_1, \dots, \Phi_m$.

We have listed the above notions and properties about tangent spaces because we will need them in the proof of our Theorem 4.1.1. More precisely we will need the following lemma:

Lemma 2.2.15 *For any holomorphic map* $\varphi : X \to Y$ *the set*

$$\mathrm{Sing}^0(\varphi) := \{x \in X \mid \operatorname{corank}_x \varphi > 0\}$$

is analytic. Here $\operatorname{corank}_x \varphi := \dim_{\mathbb{C}} T_x(X/Y) = \dim_{\mathbb{C}} T_x X_{\varphi(x)}$.

This lemma follows at once from the next lemma if we recall the defining ideal sheaf of $TX_{\varphi(x)}$. We introduce first the following.
Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module and $x \in X$. Then $\mathcal{F}(x) := \mathcal{F}_x/(\mathfrak{m}_{X,x}\mathcal{F}_x) = \mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} \mathbb{C}$ is a finite dimensional $\mathbb{C}$-vector space.

We call $\operatorname{corank}_x \mathcal{F} := \dim_{\mathbb{C}} \mathcal{F}(x)$ the corank of $\mathcal{F}$ at x and introduce the set $A_0(\mathcal{F}) := \{x \in X \mid \operatorname{corank}_x \mathcal{F} > 0\} = \{x \in X \mid \mathcal{F}_x \neq \mathfrak{m}_{X,x}\mathcal{F}_x\}$. We then have the following lemma:

Lemma 2.2.16 *For any coherent $\mathcal{O}_X$-module $\mathcal{F}$ the subset $A_0(\mathcal{F}) \subset X$ is analytic.*

Proof. It is a local question, so we can assume the existence of an exact sequence

$$\mathcal{O}_X^l \xrightarrow{\alpha} \mathcal{O}_X^k \longrightarrow \mathcal{F} \longrightarrow 0$$

This yields immediately by tensoring with $\otimes_{\mathcal{O}_{X,x}}\mathbb{C}$ the following exact sequence:

$$\mathbb{C}^l \xrightarrow{\alpha(x)} \mathbb{C}^k \longrightarrow \mathcal{F}(x) \longrightarrow 0 \quad \text{for all} \quad x \in X.$$

Consequently $A_0(\mathcal{F}) = \{x \in X \mid \operatorname{rank} \alpha(x) < k\}$ and since α is given by a holomorphic $(k \times l)$-matrix on X, the subset $A_0(\mathcal{F}) \subset X$ is the common set of zeros of all subdeterminants of order k, so it is analytic. □

2.3 The linear space associated to a coherent ideal sheaf. The projective space associated to a cone

2.3.1 The analytic inverse image of a sheaf and the blowing-up

We want to introduce the analytic inverse image of a coherent analytic sheaf. We will need it for the blowing-up in the proof of our Theorem 3.1.6.

Let $\varphi : X \to Y$ be a holomorphic map between two complex spaces and let $\mathcal{F}$ be an $\mathcal{O}_X$-module. First we have to introduce the direct image and the topological inverse image.

The presheaf on Y given by $V \mapsto \mathcal{F}(\varphi^{-1}(V))$ for each V open in Y is a sheaf. We denote it by $\varphi_*\mathcal{F}$ and call it the direct image of $\mathcal{F}$. It is a $\varphi_*\mathcal{O}_X$-module.

Remark 2.3.1 Remember that by definition a holomorphic map φ is a pair $(|\varphi|, \varphi^0)$ where $|\varphi| : |X| \longrightarrow |Y|$ is a continuous map between the underlying topological spaces and $\varphi^0 : \mathcal{O}_Y \longrightarrow |\varphi|_*\mathcal{O}_X$ is a morphism of sheaves of algebras on Y, which induces a morphism $\varphi_p^0 : \mathcal{O}_{Y,\varphi(p)} \longrightarrow \mathcal{O}_{X,p}$ on the stalks. We will denote $|\varphi|$ also by φ.

Now, if $\mathcal{G}$ is an $\mathcal{O}_Y$-module we denote by $\varphi^{-1}\mathcal{G}$ the topological inverse image of $\mathcal{G}$, which is a sheaf on X uniquely determined by the property $(\varphi^{-1}\mathcal{G})_p = \mathcal{G}_{\varphi(p)}$ for all $p \in X$.
$\varphi^{-1}\mathcal{G}$ is a $\varphi^{-1}\mathcal{O}_Y$-module.

We have seen that $\varphi_*\mathcal{F}$ is a $\varphi_*\mathcal{O}_X$-module. But via the morphism $\varphi^0 : \mathcal{O}_Y \longrightarrow \varphi_*\mathcal{O}_X$ it is also an $\mathcal{O}_Y$-module.

The morphism φ^0 induces a morphism $\varphi^{0\#} : \varphi^{-1}\mathcal{O}_Y \longrightarrow \mathcal{O}_X$ with $(\varphi^{0\#})_p = \varphi_p^0$ for $p \in X$. Hence $\mathcal{O}_X$ may be considered as a $\varphi^{-1}\mathcal{O}_Y$-module.

If $\mathcal{G}$ is an $\mathcal{O}_Y$-module, we define the analytic inverse image $\varphi^*\mathcal{G} = \varphi^{-1}\mathcal{G} \otimes_{\varphi^{-1}\mathcal{O}_Y} \mathcal{O}_X$ which is an $\mathcal{O}_X$-module.

Remark 2.3.2 For any $\mathcal{O}_X$-module $\mathcal{F}$ and any $\mathcal{O}_Y$-module $\mathcal{G}$ there is a canonical isomorphism $\operatorname{Hom}_X(\varphi^*\mathcal{G}, \mathcal{F}) \longrightarrow \operatorname{Hom}_Y(\mathcal{G}, \varphi_*\mathcal{F})$.
By substituting $\mathcal{F} = \varphi^*\mathcal{G}$ and $\mathcal{G} = \varphi_*\mathcal{F}$ one obtains canonical morphisms $\mathcal{G} \longrightarrow \varphi_*(\varphi^*\mathcal{G})$ and $\varphi^*(\varphi_*\mathcal{F}) \longrightarrow \mathcal{F}$.

We recall some properties of φ_* and φ^*:

(i) φ_* is left exact

(ii) φ^* is right exact

(iii) $\varphi^*\mathcal{O}_Y^k = \mathcal{O}_X^k$

(iv) In general φ_* is not right exact and φ^* is not left exact.

In Chapter 3 we make use of the blowing-up of a complex space along a certain subspace (called the center of the blowing-up). In complex analysis the method of blowing-up was introduced by Hopf as a "σ-Prozess". The general existence theorem is the following:

Theorem 2.3.3 *Let Y be a complex space together with a closed complex subspace $A \hookrightarrow Y$. Then there exists a complex space Y^* together with a holomorphic map $\pi : Y^* \longrightarrow Y$ with the following properties:*

(i) π is proper

(ii) $A^* := \pi^{-1}(A) \hookrightarrow Y^*$ is a hypersurface (that is, its defining coherent ideal sheaf $\mathcal{J}^*$ has the property that for every $p \in Y^*$ there exists a non-zero divisor $a \in \mathcal{O}_{Y^*,p}$ such that $\mathcal{J}^*_p = a \cdot \mathcal{O}_{Y^*,p}$)

(iii) π is universal with respect to (ii), i.e. for any holomorphic map $\tau : Y' \longrightarrow Y$ such that $A' := \tau^{-1}(A) \hookrightarrow Y'$ is a hypersurface there is a unique holomorphic map $\varphi : Y' \longrightarrow Y^*$ such that the following diagram commutes:

$$\begin{array}{ccc} Y' & \overset{\varphi}{\dashrightarrow} & Y^* \\ & \searrow^{\tau} & \downarrow{\scriptstyle \pi} \\ & & Y \end{array}$$

(iv) The restriction of π

$$\pi_{|Y^*\setminus A^*} Y^* \setminus A^* \longrightarrow Y \setminus A$$

is biholomorphic

(v) If Y is a manifold and A is a submanifold then Y^* is a manifold.

For a complete proof of the above theorem see [Fi], p. 162.

We indicate here the explicite construction of the blowing-up with help of the projective space associated to coherent ideal sheaves, as far as we need it in the proof of our Theorem 3.1.6.

2.3.2 The linear space associated to a coherent sheaf

Let $\mathcal{J}$ be a coherent ideal sheaf on a complex space Y. Then there exists locally an exact sequence of the form

$$\mathcal{O}_W^p \xrightarrow{h} \mathcal{O}_W^q \xrightarrow{\alpha} \mathcal{J}_W \longrightarrow 0$$

where h is given by a holomorphic matrix of type $q \times p$ on W. The transposed matrix gives rise to "dual" morphisms $h^* = h_W^* : W \times \mathbb{C}^q \longrightarrow W \times \mathbb{C}^p$. The linear space $\ker h_W^*$ doesn't depend on the choice of the exact sequence, in the sense that they are isomorphic for different choices, so we can define it globally.

Indeed, if we have two exact sequences on W, this gives rise to a commutative diagram of the form

$$\begin{array}{ccccccc} \mathcal{O}_W^{p'} & \xrightarrow{h'} & \mathcal{O}_W^{q'} & \xrightarrow{\alpha'} & \mathcal{J}_W & \longrightarrow & 0 \\ \downarrow{\scriptstyle \alpha_1} & & \downarrow{\scriptstyle \alpha_2} & & \downarrow{\scriptstyle \mathrm{Id}} & & \\ \mathcal{O}_W^{p} & \xrightarrow{h} & \mathcal{O}_W^{q} & \xrightarrow{\alpha} & \mathcal{J}_W & \longrightarrow & 0 \end{array}$$

that is, one proves easily the existence of the $\mathcal{O}_W$-linear maps α_1 and α_2. For instance let $a \in \mathcal{O}_W^{q'}$ and consider $\mathrm{Id} \circ \alpha'(a)$ which belongs to $\mathcal{J}_W$. Now α being surjective we have at least one point $b \in \mathcal{O}_W^q$ with $\alpha(b) = \mathrm{Id} \circ \alpha'(a)$. This allows us to define a linear map α_2. The map α_2 is of course not necessarily unique.

Having now α_2 one defines similarly a linear map α_1 making the diagram commutative.

Because the construction of α_1 and α_2 is not unique we have to investigate what happens if we have two pairs of linear maps (α_1, α_2) and (α_1', α_2') making the respective above diagram commutative. It is easy to see that the pairs are homotopic, that means that we obtain the existence of a linear map $s : \mathcal{O}^{q'} \longrightarrow \mathcal{O}^p$ such that $h \circ s = \alpha_2 - \alpha_2'$, because of the fact that $\alpha_2(a) - \alpha_2'(a) \in \ker \alpha = \mathrm{Im}\, h$.

It holds that the dual maps α_2^*, ${\alpha'}_2^*$ coincide on $\ker h^*$; this follows at once from $\alpha_2^* - {\alpha'}_2^* = s^* \circ h^*$.

In order to show that $\ker h^* \simeq \ker h'^*$ we first consider the analogous commutative diagram

$$\begin{array}{ccc} \mathcal{O}_W^{p'} & \xrightarrow{h'} & \mathcal{O}_W^{q'} \\ \uparrow{\scriptstyle \beta_1} & & \uparrow{\scriptstyle \beta_2} \\ \mathcal{O}_W^{p} & \xrightarrow{h} & \mathcal{O}_W^{q} \end{array}$$

and also the following commutative diagram

$$\begin{array}{ccc} \mathcal{O}_W^{p} & \xrightarrow{h} & \mathcal{O}_W^{q} \\ \downarrow{\scriptstyle \alpha_1 \circ \beta_1} & & \downarrow{\scriptstyle \alpha_2 \circ \beta_2} \\ \mathcal{O}_W^{p} & \xrightarrow{h} & \mathcal{O}_W^{q} \end{array}$$

But having also the trivial commutative diagram

$$\begin{array}{ccccccc}
\mathcal{O}_W^p & \xrightarrow{h} & \mathcal{O}_W^q & \xrightarrow{\alpha} & \mathcal{J}_W & \longrightarrow & 0 \\
\downarrow \scriptstyle{\mathrm{Id}} & & \downarrow \scriptstyle{\mathrm{Id}} & & \downarrow \scriptstyle{\mathrm{Id}} & & \\
\mathcal{O}_W^p & \xrightarrow{h} & \mathcal{O}_W^q & \xrightarrow{\alpha} & \mathcal{J}_W & \longrightarrow & 0
\end{array}$$

we obtain by the considerations above that

$$(\alpha_2 \circ \beta_2)^* = \beta_2^* \circ \alpha_2^* = \mathrm{Id}^* \quad \text{on} \quad \ker h^*$$

In an analogous way it follows that we have also

$$(\beta_2 \circ \alpha_2)^* = \alpha_2^* \circ \beta_2^* = \mathrm{Id}^* \quad \text{on} \quad \ker h'^*$$

That means that α_2^* is an isomorphism between $\ker h^*$ and $\ker h'^*$.

The construction of the linear space $\ker h^*$ being independent of the choices made, we can glue them together to obtain a linear space over Y, denoted by $V(\mathcal{J})$, defined locally as $V(\mathcal{J})_{|W} := \ker h_W^*$. We have also a canonical projection map $\pi_1 : V(\mathcal{J}) \longrightarrow Y$.

We remark that $V(\mathcal{J})_{|W} \hookrightarrow W \times \mathbb{C}^q$ being a linear subspace with respect to the second component it is in particular a *cone* over W, that means it is a subspace invariant to the scalar multiplication. More precisely it means that we have a commutative diagram

$$\begin{array}{ccc}
\mathbb{C} \times W \times \mathbb{C}^q & \xrightarrow{\mu} & W \times \mathbb{C}^q \\
\uparrow & & \uparrow \\
\mathbb{C} \times V(\mathcal{J})_{|W} & \xrightarrow{\mu'} & V(\mathcal{J})_{|W}
\end{array}$$

where $\mu(t, y, z_1, \ldots, z_q) = (y, tz_1, \ldots, tz_q)$ and $\mu' = \mu_{|\mathbb{C} \times V(\mathcal{J})_{|W}}$

2.3.3 The projective space associated to a cone

Now let $V(\mathcal{J}) \hookrightarrow Y \times \mathbb{C}^q$ be a cone over Y. Then it is given locally (on U) as a complex subspace by the common zeros of homogeneous polynomials

$$H_1, \ldots, H_r \in \mathcal{O}_Y(U)[z_1, \ldots, z_q]$$

where $(z_1, \ldots, z_q)$ are the coordinates on $\mathbb{C}^q$ (see [Fi], p. 45, Prop. 1.2.).

For each $\varrho \in \{1, \ldots, r\}$ and $\nu \in \{1, \ldots, q\}$ there exists on $\{z_\nu \neq 0\}$ a unique polynomial

$$H_{\varrho,\nu}(x_1, \ldots, \hat{x}_\nu, \ldots, x_q) \in \mathcal{O}_Y(U)[x_1, \ldots, \hat{x}_\nu, \ldots, x_q]$$

such that

$$H_\varrho(z_1, \ldots, z_q) = z_\nu^k H_{\varrho,\nu}(\frac{z_1}{z_\nu}, \ldots, \frac{z_{\nu-1}}{z_\nu}, \frac{z_{\nu+1}}{z_\nu}, \ldots, \frac{z_q}{z_\nu})$$

where k is the degree of H_ϱ. Via the biholomorphism

$$\varphi_\nu : \mathbb{C}^{q-1} \longrightarrow \tilde{V}_\nu := \{(z_1 : \ldots : z_q) \mid z_\nu \neq 0\} \subset \mathbb{P}^{q-1}(\mathbb{C})$$

$$(x_1, \ldots, \hat{x}_\nu, \ldots, x_q) \mapsto (x_1 : \ldots : x_{\nu-1} : 1 : x_{\nu+1} : \ldots : x_q)$$

we can consider $H_{1,\nu}, \ldots, H_{r,\nu}$ as holomorphic functions on $U \times \tilde{V}_\nu$. Let then $\mathcal{J}_\nu \subset \mathcal{O}_{U\times\tilde{V}_\nu}$ denote the ideal sheaf generated by $H_{1,\nu}, \ldots, H_{r,\nu}$. One verifies that $\mathcal{J}_\nu = \mathcal{J}_\mu$ on $U \times (\tilde{V}_\nu \cap \tilde{V}_\mu)$ and so it gives rise to an ideal sheaf $\mathcal{J}_U \subset \mathcal{O}_{U\times\mathbb{P}^{q-1}(\mathbb{C})}$.

Indeed, we have on $U \times \tilde{V}_\nu$ that

$$H_{\varrho,\nu} = \frac{1}{z_\nu^k} \cdot H_\varrho$$

and respectively, on $U \times \tilde{V}_\mu$

$$H_{\varrho,\mu} = \frac{1}{z_\mu^k} \cdot H_\varrho$$

But then we have the relations

$$\begin{aligned} H_{\varrho,\nu} &= \left(\frac{z_\mu}{z_\nu}\right)^k \cdot H_{\varrho,\mu} \\ \text{and} \quad H_{\varrho,\mu} &= \left(\frac{z_\nu}{z_\mu}\right)^k \cdot H_{\varrho,\nu} \end{aligned}$$

which hold on $U \times (\tilde{V}_\nu \cap \tilde{V}_\mu)$, where $\frac{z_\nu}{z_\mu} \neq 0$ and $\frac{z_\mu}{z_\nu} \neq 0$. This means actually that the respective generators generate the same ideal sheaf $\mathcal{J}_\nu = \mathcal{J}_\mu$ on $U \times (\tilde{V}_\nu \cap \tilde{V}_\mu)$, so they give rise to a well defined ideal sheaf $\mathcal{J}_U \subset \mathcal{O}_{U\times\mathbb{P}^{q-1}(\mathbb{C})}$.

By covering Y with open sets of this type we get a coherent ideal sheaf $\mathcal{J} \subset \mathcal{O}_{Y\times\mathbb{P}^{q-1}(\mathbb{C})}$ independent of all choices made. The closed subspace (which is in general not reduced) corresponding to this ideal sheaf $\mathcal{J}$,

$$\mathbb{P}(V(\mathcal{J})) \hookrightarrow Y \times \mathbb{P}^{q-1}(\mathbb{C})$$

is called the *projective variety over Y associated to the cone* $V(\mathcal{J})$, or the *projective variety associated to the ideal sheaf* $\mathcal{J} \subset \mathcal{O}_Y$ denoted also by $\mathbb{P}(\mathcal{J})$.

Remark 2.3.4 We have also a canonical projection map $\xi : \mathbb{P}(\mathcal{J}) \longrightarrow Y$.

The blowing-up with center $(A, (\mathcal{O}_Y/\mathcal{J})_{|A})$ is now given by the following closed subspace in $\mathbb{P}(\mathcal{J})$

$$Y^* := \mathrm{clos}_{\mathbb{P}(\mathcal{J})}(\mathbb{P}(\mathcal{J})\backslash\xi^{-1}(A))$$

and the map $\pi : Y^* \longrightarrow Y$ is given by $\pi := \xi_{|Y^*}$.

Chapter 3

Proper modifications and generalized Kähler metrics

3.1 Definitions and main result

First we would like to recall some well-known definitions.

Definition 3.1.1 *An analytic set A in a reduced complex space X is called rare (or thin) if it does not contain any irreducible component of X. In case that X is reduced and irreducible this means that A is nowhere dense.*

Definition 3.1.2 *A holomorphic map p between two complex spaces X and Y is called a proper modification if it is proper and there exists a rare analytic set A in Y such that $p^{-1}(A)$ is rare in X and such that $p_{|_{X\setminus p^{-1}(A)}} : X\setminus p^{-1}(A) \longrightarrow Y\setminus A$ is biholomorphic.*

Definition 3.1.3 *A reduced compact complex space Y is called Kähler (in the sense of Moishezon) if there exists a covering $(V_i)_{i\in I}$ of Y with open sets such that there exists for each index i a strongly plurisubharmonic function $\lambda_i : V_i \longrightarrow \mathbb{R}$ which is regular of class $\mathcal{C}^\infty$ and such that on each nonempty intersection $V_i \cap V_j$ we have the pluriharmonic compatibility condition: $\lambda_i - \lambda_j = \operatorname{Re} g_{ij}$, locally on $V_i \cap V_j$ for some holomorphic function g_{ij}.*

We say that two such collections $(V_i, \lambda_i)_{i\in I}$ and $(W_j, \psi_j)_{j\in J}$ define the same Kähler metric on Y if each $\lambda_i - \psi_j$ is pluriharmonic (i.e. is locally the real part of a holomorphic function) on $V_i \cap W_j \neq \emptyset$.

Remark 3.1.4 In the case of complex manifolds such a collection $(U_i, \lambda_i)_{i\in I}$ defines indeed a metric on Y, by endowing Y with the $(1,1)$-form given locally (on each open set U_i) by $\partial\bar{\partial}\lambda_i$.

Example

(i) It is known that complex projective algebraic spaces are Kähler (in particular the Grassmann manifolds are).

(ii) It can be shown that *Hopf surfaces* are not Kähler.
The Hopf surface is given by the quotient $(\mathbb{C}^2 \setminus \{0\})/G$, where G is the free cyclic group $G := \{z \mapsto 2^j z \mid j \in \mathbb{Z}\}$ of automorphisms of $\mathbb{C}^2 \setminus \{0\}$. It is a compact manifold (actually it is homeomorphic with $S^1 \times S^3$).

We want to generalize the above concept of Kähler metrics.

Definition 3.1.5 *We say that the reduced compact complex space X has a generalized Kähler metric if there exists a covering of X with open sets $(U_i)_i$ such that on each set U_i there exists a function $\varphi_i : U_i \longrightarrow [-\infty, \infty)$, $\varphi_i \not\equiv -\infty$ on each irreducible component of U_i, which is strongly plurisubharmonic, regular of class $\mathcal{C}^\infty$ outside the set $\{\varphi_i = -\infty\}$ and such that on each nonempty intersection $U_i \cap U_j$ we have (locally) the compatibility condition $\varphi_i = \varphi_j + \operatorname{Re} f_{ij}$ for some holomorphic function f_{ij}.*

In this chapter we prove the following theorem:

Theorem 3.1.6 *Let X and Y be two reduced, compact, complex spaces (with singularities) and $p : X \longrightarrow Y$ a surjective, holomorphic map, which is a proper modification. Suppose that Y is Kähler. Then X has a generalized Kähler metric.*

It is known by an example of Moishezon [Mo1] (see also [Bi]) that it does not follow in general from the hypothesis of our theorem that X is then also Kähler. So the question arises natural of how "far" X is from being Kähler, that is: what general assertion one can prove in the above context about X.

In Chapter 4 we will generalize our theorem to the following weaker hypothesis, by using in its proof Theorem 3.1.6 as a special case:

Theorem 4.1.1 *Let $p : X \longrightarrow Y$ be a holomorphic and surjective map between two reduced, compact, complex spaces with singularities and with the property that p sends each irreducible component C_X of X (surjective) onto an irreducible component C_Y of Y of the same dimension, $\dim C_X = \dim C_Y$.*
If Y is Kähler, then X has a generalized Kähler metric.

We start now with the proof of Theorem 3.1.6.

Proof. Consider the covering of Y given by Definition 3.1.3 and the covering of X given by $U_i := p^{-1}(V_i)$, $i \in I$ and on each U_i the function $\tilde{\varphi}_i = \lambda_i \circ p$. Then it follows at once that $\tilde{\varphi}_i \in \mathcal{C}^\infty(U_i)$ and that $\tilde{\varphi}_i$ is plurisubharmonic on U_i, but not necessarily strongly plurisubharmonic. The idea in what follows is to modify in a first step $\tilde{\varphi}_i$ such that they become strongly plurisubharmonic. But then we destroy the "pluriharmonic compatibility condition", that is $\tilde{\varphi}_i - \tilde{\varphi}_j = \operatorname{Re}(g_{ij} \circ p)$ locally on $U_i \cap U_j$. In a second step we get also this condition back.

3.2 First step: Strong plurisubharmonicity

To modify $\tilde{\varphi}$ such that they become strongly plurisubharmonic we use a technique from an article of Colţoiu-Mihalache [Co-Mi]. We look at the following commutative diagram given by Chow's lemma (see for instance [Hi] and [Mo2] or [Fi],p.171):

$$\begin{array}{ccc} Y^* & \xrightarrow{F} & X \\ & \overset{\pi}{\searrow} & \Big\downarrow p \\ & & Y \end{array} \tag{3.1}$$

More precisely, given p and so implicitly the rare analytic set A, the lemma of Chow says that there exists a coherent ideal $\mathcal{J}$ on Y, with $\operatorname{supp}(\mathcal{O}_Y/\mathcal{J}) = A$ and such that denoting by $\pi : Y^* \longrightarrow Y$ the blowing-up of Y with center $(A, (\mathcal{O}_Y/\mathcal{J})_{|A})$ it follows the existence of a holomorphic, proper and surjective map F making the above diagram commutative. The ideal $\mathcal{J}$ is called the ideal of Hironaka, it is not necessary the Nullstellenideal, so the resulting complex space is not necessarily reduced (for more details about the blowing-up see below and/or Section 2.3)

Without loss of generality we can suppose that the open sets of the covering of Y given by the definition of the Kähler metric are all Stein open sets. Fix now for the moment an arbitrary Stein open set V_j of the finite covering $(V_i)_{i\in I}$ of Y. V_j being Stein it follows from Cartan's Theorem A that each fiber $\mathcal{J}_y$, for $y \in V_j$, is generated by global sections in $\mathcal{J}(V_j)$.

But because the sets V_j are relatively compact there exists a finite number of sections $f_{1,j}, \ldots, f_{s,j} \in \mathcal{J}(V_j)$ which generate each fiber $\mathcal{J}_y$ for $y \in V_j$. Then we have the following equality for the zero sets:

$$N(V_j, \mathcal{J}) = \bigcap_{f\in\{f_{1,j},\ldots,f_{s,j}\}} N(V_j, f)$$

In other words there exist sections $f_{1,j}, \ldots, f_{s,j} \in \Gamma(V_j, \mathcal{J})$ generating each fiber of $\mathcal{J}$ such that

$$A \cap V_j = \{x \in V_j \mid f_{1,j}(x) = \ldots = f_{s,j}(x) = 0\}$$

Then it follows for the map

$$f_j := (f_{1,j}, \ldots, f_{s,j}) : V_j \longrightarrow \mathbb{C}^s$$

that we have:

$$f_j^{-1}(0) = (A \cap V_j, (\mathcal{O}_Y/\mathcal{J})|_{A\cap V_j})$$

Now consider the function

$$\psi_j : V_j \longrightarrow [-\infty, \infty)$$

given by

$$\psi_j = \lambda_j + \log(\sum_{k=1}^{s} |f_{k,j}|^2)$$

It is clear that ψ_j is strongly plurisubharmonic on V_j, $\{\psi_j = -\infty\} = A \cap V_j$ and that ${\psi_j}_{|_{V_j \setminus A}} \in \mathcal{C}^\infty(V_j \setminus A)$. Considering now the composed function $\psi_j \circ p$ we have that $\psi_j \circ p$ is plurisubharmonic on $U_j = p^{-1}(V_j)$, $\mathcal{C}^\infty$ on $U_j \setminus p^{-1}(A)$ and $\{\psi_j \circ p = -\infty\} = p^{-1}(A) \cap U_j$. We will see below that $\psi_j \circ p$ are even strongly plurisubharmonic. But unfortunately the pluriharmonic-compatibility condition is no longer satisfied.

With the same proof as in the article of Colţoiu and Mihalache it follows that $\psi_j \circ p$ is strongly plurisubharmonic. We recall the ideas from there. We use the following lemma which is true for all reduced complex spaces (not necessarily compact). For a proof see [Co-Mi], [Bo-Na] or our Appendix, Proposition 6.0.1.

Lemma 3.2.1 *Let X and Y be complex spaces and $p : X \longrightarrow Y$ a proper, holomorphic, surjective map. Let $\Phi : Y \longrightarrow [-\infty, \infty)$ be an upper semicontinuous function such that $\Phi \circ p$ is (strongly) plurisubharmonic on X. Then Φ is (strongly) plurisubharmonic on Y.*

Using the diagram (3.1), we can conclude with help of this lemma that, in order to show that $\psi_j \circ p$ is strongly plurisubharmonic, it is enough to prove that $\psi_j \circ \pi = (\psi_j \circ p) \circ F$ is strongly plurisubharmonic on $\pi^{-1}(V_j)$.

In order to do this we need the explicit description of the analytic blowing-up. Let $\mathfrak{m} \subset \mathcal{O}_{\mathbb{C}^s}$ denote the maximal ideal of the origin in $\mathbb{C}^s$. One has then an exact sequence (the syzygy-theorem) of the form:

$$\mathcal{O}_{\mathbb{C}^s}^{\binom{s}{2}} \xrightarrow{\alpha} \mathcal{O}_{\mathbb{C}^s}^{s} \xrightarrow{\beta} \mathfrak{m} \longrightarrow 0$$

where β is given by multiplication with the coordinates $(z_1, \ldots, z_s)$ of $\mathbb{C}^s$ and α is given by the $s \times \binom{s}{2}$ matrix:

$$\begin{pmatrix}
z_2 & z_3 & z_4 & \cdots & z_s & 0 & 0 & \cdots & 0 & \cdots & \cdots & 0 & \cdots & \cdots & 0 \\
-z_1 & 0 & 0 & \cdots & 0 & z_3 & z_4 & \cdots & z_s & \cdots & \cdots & 0 & \cdots & \cdots & 0 \\
0 & -z_1 & 0 & \cdots & 0 & -z_2 & 0 & \cdots & 0 & \cdots & \cdots & 0 & \cdots & \cdots & 0 \\
0 & 0 & -z_1 & \cdots & 0 & 0 & -z_2 & \cdots & 0 & \cdots & \cdots & 0 & \cdots & \cdots & 0 \\
\vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & & & \vdots & & & \vdots \\
\vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & & & z_j & & & \vdots \\
\vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & & & \vdots & & & \vdots \\
\vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & & & -z_i & & & \vdots \\
\vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & & & \vdots & & & \vdots \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & \cdots & 0 & \cdots & \cdots & z_s \\
0 & 0 & 0 & \cdots & -z_1 & 0 & 0 & \cdots & -z_2 & \cdots & \cdots & 0 & \cdots & \cdots & -z_{s-1}
\end{pmatrix}$$

Via the analytic inverse image (see Section 2.3) this gives rise to an exact sequence on V_j

$$\mathcal{O}_{V_j}^{\binom{s}{2}} \xrightarrow{f_j^*\alpha} \mathcal{O}_{V_j}^{s} \xrightarrow{f_j^*\beta} \mathcal{J}_{|V_j} \longrightarrow 0$$

(recall that $f_j^*\mathfrak{m} = \mathcal{J}_{|V_j}$).

Let $\mathbb{P}(\mathcal{J})$ denote the projective space over Y associated to the coherent ideal sheaf $\mathcal{J}$. We recall the construction of $\mathbb{P}(\mathcal{J})$ in two steps (for more details see Section 2.3).

The linear space associated to a coherent sheaf $\mathcal{J}$ being a coherent sheaf on Y there exists locally an exact sequence of the form

$$\mathcal{O}_W^m \xrightarrow{\eta} \mathcal{O}_W^s \longrightarrow \mathcal{J}_{|W} \longrightarrow 0$$

where η is given by a holomorphic matrix of type $s \times m$ on W. The transposed matrix gives rise to a morphism $\psi_W : W \times \mathbb{C}^s \longrightarrow W \times \mathbb{C}^m$. The linear space $\ker \psi_W$ doesn't depend on the choice of the exact sequence, in the sense that they are isomorphic for different choices, so we can define it globally.

Then $V(\mathcal{J})$ defined locally as $V(\mathcal{J})_{|W} := \ker \psi_W$ is called the linear space over Y associated to $\mathcal{J}$ and we have a canonical projection map $\pi_1 : V(\mathcal{J}) \longrightarrow Y$.

We remark that $V(\mathcal{J})_{|W} \hookrightarrow W \times \mathbb{C}^s$ is a cone over W.

In our case, Y being compact we can cover it with a finite number of such open sets W and by taking the maximum of the dimensions of $s = s_W$, again denoted by s, we can say that $V(\mathcal{J})$ is a cone over Y, $V(\mathcal{J})_{|W} \hookrightarrow W \times \mathbb{C}^s$.

The projective space associated to a cone $V(\mathcal{J})_{|W} \hookrightarrow W \times \mathbb{C}^s$ being a cone it is given as a complex subspace locally (on U) by the common zeros of homogeneous polynomials (see [Fi], p.45). In our case there are $\binom{s}{2}$ homogeneous linear equations given by the matrix $f_j^*\alpha$ on $U = V_j$:

$$G_1, \dots, G_{\binom{s}{2}} \in \mathcal{O}_Y(U)[z_1, \dots, z_s]$$

where $(z_1, \dots, z_s)$ are the coordinates on $\mathbb{C}^s$.

For each $\varrho \in \{1, \dots, \binom{s}{2}\}$ and $\nu \in \{1, \dots, s\}$ there exists on $\{z_\nu \neq 0\}$ a unique polynomial

$$G_{\varrho,\nu}(x_1, \dots, \hat{x}_\nu, \dots x_s) \in \mathcal{O}_Y(U)[x_1, \dots, \hat{x}_\nu, \dots x_s]$$

such that

$$G_\varrho = z_\nu G_{\varrho,\nu}(\frac{z_1}{z_\nu}, \dots, \frac{z_{\nu-1}}{z_\nu}, \frac{z_{\nu+1}}{z_\nu}, \dots, \frac{z_s}{z_\nu})$$

(remember that the degree of G_ϱ is 1). Via the biholomorphism

$$\varphi_\nu : \mathbb{C}^{s-1} \longrightarrow \tilde{V}_\nu := \{(z_1 : \dots : z_s) \mid z_\nu \neq 0\} \subset \mathbb{P}^{s-1}(\mathbb{C})$$

$$(x_1, \dots, \hat{x}_\nu, \dots, x_s) \mapsto (x_1 : \dots : x_{\nu-1} : 1 : x_{\nu+1} : \dots : x_s)$$

we can consider $G_{1,\nu}, \dots, G_{\binom{s}{2},\nu}$ as holomorphic functions on $U \times \tilde{V}_\nu$. Let then $\mathcal{J}_\nu \subset \mathcal{O}_{U \times \tilde{V}_\nu}$ denote the ideal sheaf generated by $G_{1,\nu}, \dots, G_{\binom{s}{2},\nu}$. Because $\mathcal{J}_\nu = \mathcal{J}_\mu$ on $U \times (\tilde{V}_\nu \cap \tilde{V}_\mu)$ it gives rise to an ideal sheaf $\mathcal{J}_U \subset \mathcal{O}_{U \times \mathbb{P}^{s-1}(\mathbb{C})}$.

By covering Y with open sets of this type we get a coherent ideal sheaf $\mathcal{J} \subset \mathcal{O}_{Y \times \mathbb{P}^{s-1}(\mathbb{C})}$, independent of all choices made. The closed subspace corresponding to the ideal sheaf $\mathcal{J}$, denoted $\mathbb{P}(V(\mathcal{J}))$,

$$\mathbb{P}(V(\mathcal{J}))_{|W} \hookrightarrow W \times \mathbb{P}^{s-1}(\mathbb{C})$$

is called the projective variety over Y associated to the cone $V(\mathcal{J})$, or the projective variety associated to the ideal sheaf $\mathcal{J} \subset \mathcal{O}_Y$, denoted also by $\mathbb{P}(\mathcal{J})$.

Remember that we obtained also a canonical projection map $\xi : \mathbb{P}(\mathcal{J}) \longrightarrow Y$.

The blowing-up with center $(A, (\mathcal{O}_Y/\mathcal{J})_{|A})$ is now given by the closed subspace in $\mathbb{P}(\mathcal{J})$,

$$Y^* := \operatorname{clos}_{\mathbb{P}(\mathcal{J})}(\mathbb{P}(\mathcal{J}) \backslash \xi^{-1}(A))$$

together with the map $\pi := \xi_{|Y^*} : Y^* \longrightarrow Y$.

So we have the commutative diagram

$$\begin{array}{ccc} Y^* & \overset{i}{\hookrightarrow} & \mathbb{P}(\mathcal{J}) \\ & {\scriptstyle\pi}\searrow & \downarrow{\scriptstyle\xi} \\ & & Y \end{array}$$

Returning to our problem, $\mathcal{J}_{|V_j}$ being equal with $f_j^*\mathfrak{m}$ on V_j we also have the diagram

$$\begin{array}{ccc} V_j^* & \overset{i}{\hookrightarrow} & \mathbb{P}(f_j^*\mathfrak{m}) \\ & {\scriptstyle\pi_{|V_j^*}}\searrow & \downarrow{\scriptstyle\xi} \\ & & V_j \end{array}$$

(Remark that it holds that $V_j^* = \pi^{-1}(V_j)$).

Therefore because of Lemma 3.2.1 it remains for us to prove that

$$\psi_j \circ \xi : \mathbb{P}(f_j^*\mathfrak{m}) \longrightarrow [-\infty, \infty)$$

is strongly plurisubharmonic.

But in this form the advantage is that for the closed subspace

$$\mathbb{P}(f_j^*\mathfrak{m}) \hookrightarrow V_j \times \mathbb{P}^{s-1}(\mathbb{C})$$

we can give the defining equations explicitly. They are

$$f_{k,j}(y)z_m - f_{m,j}(y)z_k = 0, \ \forall\ 1 \leq m < k \leq s$$

where $(z_1 : \ldots : z_s)$ are the homogeneous coordinates on $\mathbb{P}^{s-1}(\mathbb{C})$.

Let

$$V_j \times \tilde{V}_\nu := \{(y, z) \in V_j \times \mathbb{P}^{s-1}(\mathbb{C}) \mid z_\nu \neq 0\} \quad \text{for} \quad \nu \in \{1, \ldots, s\}$$

and

$$\alpha_\nu : V_j \times \tilde{V}_\nu \longrightarrow V_j \times \mathbb{C}^{s-1}$$

be the biholomorphic map given by

$$\alpha_\nu(y,z) = (y, \frac{z_1}{z_\nu}, \ldots, \frac{z_{\nu-1}}{z_\nu}, \frac{z_{\nu+1}}{z_\nu}, \ldots, \frac{z_s}{z_\nu}) \ , \ (\alpha_\nu = \mathrm{Id} \times \varphi_\nu^{-1})$$

and define

$$\tau_\nu^j : V_j \times \mathbb{C}^{s-1} \longrightarrow [-\infty, \infty)$$

given by

$$\tau_\nu^j(y, t_1, \ldots, t_{s-1}) = \lambda_j(y) + \log|f_{\nu,j}(y)|^2 + \log(1 + \sum_{k=1}^{s-1} |t_k|^2)$$

where $(t_1, \ldots, t_{s-1})$ denote the coordinates on $\mathbb{C}^{s-1}$.

It is then clear that τ_ν^j is strongly plurisubharmonic on $V_j \times \mathbb{C}^{s-1}$ and because α_ν is biholomorphic it follows that $\tau_\nu^j \circ \alpha_\nu$ is strongly plurisubharmonic on $V_j \times \tilde{V}_\nu$. But on $(V_j \times \tilde{V}_\nu) \cap \xi^{-1}(V_j)$ we have that

$$\tau_\nu^j \circ \alpha_\nu = \psi_j \circ \xi$$

so that finally it follows that $\psi_j \circ \xi$ is strongly plurisubharmonic on $\xi^{-1}(V_j)$. So we obtained also that $\psi_j \circ \pi$ is strongly plurisubharmonic on V_j^*. As already seen above this means that $\psi_j \circ p$ is strongly plurisubharmonic on U_j.

As a conclusion of the first step of the proof we obtained the following properties for $\psi_j \circ p$: it is strongly plurisubharmonic on U_j, regular of class $\mathcal{C}^\infty$ on $U_j \backslash p^{-1}(A)$, $\{\psi_j \circ p = -\infty\} = p^{-1}(A) \cap U_j$, but we have destroyed the pluriharmonic-compatibility condition, because now

$$\psi_j \circ p = \lambda_j \circ p + \log(\sum_{k=1}^{s} |f_{k,j} \circ p|^2)$$

the last term being a "perturbation factor".

3.3 Second step: Pluriharmonic compatibility

In order to obtain on X a collection of strongly plurisubharmonic functions with the pluriharmonic compatibility condition we proceed as follows.

Let

$$a_j := |f_{1,j}|^2 + \ldots + |f_{s,j}|^2 \quad \text{on} \quad V_j$$

and

$$a_k := |f_{1,k}|^2 + \ldots + |f_{l,k}|^2 \quad \text{on} \quad V_k.$$

Consider now a relatively compact subcover of Y with open subsets $V'_j \subset V_j$, $\forall j \in I$. Then the quotient

$$\frac{a_j}{a_k} = \frac{|f_{1,j}|^2 + \ldots + |f_{s,j}|^2}{|f_{1,k}|^2 + \ldots + |f_{l,k}|^2}$$

remains bounded (upper and lower) on $(V_j' \cap V_k')\backslash A$. The problem is only in small neighbourhoods of A in $(V_j' \cap V_k')\backslash A$. But we know that on $V_j \cap V_k$ the sections in $\mathcal{J}(V_j \cap V_k)$ are generated by $(f_{1,j}, \ldots, f_{s,j})_{|V_j \cap V_k}$ and also by $(f_{1,k}, \ldots, f_{l,k})_{|V_j \cap V_k}$ because the respective germs generate $\mathcal{J}_y$ for all $y \in V_j \cap V_k$ and $V_j \cap V_k$ is Stein. So the following holds:

$$\forall\ m \in \{1, \ldots, s\}\ ,\ f_{m,j} = \sum_{t=1}^{l} g_{m,t} f_{t,k} \quad \text{with} \quad g_{m,t} \in \mathcal{O}(V_j \cap V_k)$$

So on $V_j' \cap V_k'$ we have the inequalities

$$|f_{m,j}| \leq \sum_{t=1}^{l} |g_{m,t}||f_{t,k}| \leq C \sum_{t=1}^{l} |f_{t,k}|$$

(In the following estimates C denotes a generic constant.) From here we obtain

$$|f_{m,j}|^2 \leq C(\sum_{t=1}^{l} |f_{t,k}|)^2 = C(\sum_{t=1}^{l} |f_{t,k}|^2 + 2 \sum_{t<n} |f_{t,k}||f_{n,k}|)$$

But because of

$$2 \sum_{t<n} |f_{t,k}||f_{n,k}| \leq (l-1) \sum_{t=1}^{l} |f_{t,k}|^2$$

it finally follows that

$$|f_{m,j}|^2 \leq C \sum_{t=1}^{l} |f_{t,k}|^2\ ,\ \forall m = 1, \ldots, s$$

which implies that

$$\frac{a_j}{a_k} \leq C \quad \text{on} \quad (V_j' \cap V_k')\backslash A.$$

The lower boundedness of $\frac{a_j}{a_k}$ follows of course from the analogue upper boundedness of $\frac{a_k}{a_j}$. So now we know that $\log a_j - \log a_k$ is bounded on $(V_j' \cap V_k')\backslash A$.
In what follows we apply a glueing technique of Demailly [Dem] for a collection of certain functions, which has the advantage that the glueing result is of class $\mathcal{C}^\infty$.

More precisely, we can suppose from the begining, without loss of generality, that the open sets V_j' are isomorphic with analytic sets in open balls $B(0, r_j) \subset \mathbb{C}^{N_j}$.

Let $\Phi_j : V_j' \longrightarrow B(0, r_j)$ denote the chart and we can assume that $0 \in \Phi_j(V_j')$. Consider for each j the functions

$$\mathfrak{v}_j : V_j' \longrightarrow [-\infty, \infty)$$

given by

$$\mathfrak{v}_j(z) = \log a_j(z) - \frac{1}{r_j^2 - |\Phi_j(z)|^2} =: \log a_j(z) - \theta_j(z)$$

One sees at once that $\theta_j \in \mathcal{C}^\infty(V_j')$ and $\theta_j(z) \longrightarrow +\infty$ for $z \longrightarrow \partial V_j'$, $z \in V_j'$, so that we have $\mathfrak{v}_j \in \mathcal{C}^\infty(V_j' \backslash A)$ and $\mathfrak{v}_j(z) \longrightarrow -\infty$ for $z \longrightarrow \partial V_j'$, $z \in V_j'$ (we have also that $\mathfrak{v}_j(z) = -\infty$ for $z \in A \cap V_j'$).

In order to get a $\mathcal{C}^\infty$ -glueing of the functions $\mathfrak{v}_i$ on $Y \backslash A$, to overcome the fact that the function $\max(\mathfrak{v}_i)_i$ is only continuous, we proceed as follows:

Let $\varrho : \mathbb{R} \longrightarrow \mathbb{R}$ be a function of class $\mathcal{C}^\infty$ with $\varrho \geq 0$, $\operatorname{supp} \varrho \subset [-\frac{1}{2}, \frac{1}{2}]$ and with $\int_{\mathbb{R}} \varrho(u)du = 1$ and let m denote the function

$$m : \mathbb{R}^q \longrightarrow \mathbb{R}$$

given by

$$m(t_1, \ldots, t_q) = \int_{\mathbb{R}^q} \max\{t_1 + u_1, \ldots, t_q + u_q\} \prod_{1 \leq n \leq q} \varrho(u_n)du_n$$

(in our case q will be the number of open sets of the finite covering $(V_j')_j$ of Y).

It is clear that m is increasing in each variable , that it is convex and of class $\mathcal{C}^\infty$ and that the following property holds:

$$m(t_1, \ldots, \hat{t}_j, \ldots, t_q) = m(t_1, \ldots, t_j, \ldots, t_q) \tag{3.2}$$

whenever

$$t_j < \max\{t_1, \ldots, t_{j-1}, t_{j+1}, \ldots, t_q\} - 2$$

(where $\hat{}$ denotes as usual that the respective variable is missing).

Let now $\mathfrak{v}$ denote the function on Y given by

$$\mathfrak{v}(z) = m(\mathfrak{v}_1(z), \ldots, \mathfrak{v}_q(z))$$

We then have that $\mathfrak{v} \in \mathcal{C}^\infty(Y \backslash A)$. However, written in this form we have to ignore the $\mathfrak{v}_i$'s for which $z \notin V_i'$. This can be done because of the following: the maximum is taken over the $\mathfrak{v}_i$'s with $z \in V_i'$, so for different positions of z we have a different number of functions over which we take the maximum. But we have that $\mathfrak{v}_i(z) \longrightarrow -\infty$, for $z \longrightarrow \partial V_i'$, $z \in V_i'$, i.e. the values of $\mathfrak{v}_i(z)$ with z near the boundary of V_i' doesn't play an effective role in the maximum. This fact together with (3.2) shows that $\mathfrak{v}$ is globally well defined.

At the same time it allows us to choose a covering $(V_j'')_j$ of Y, $V_j'' \subset\subset W_j \subset\subset V_j'$ such that already each $\mathfrak{v}_j(z)$ for $z \in V_j' \setminus V_j''$ does not play an effective role in the maximum, in particular $m(\mathfrak{v}_{1_{|W_1}}(z), \ldots, \mathfrak{v}_{q_{|W_q}}(z)) = m(\mathfrak{v}_1(z), \ldots, \mathfrak{v}_q(z))$ for each $z \in Y$. We will need the covering $(W_j)_j$ in what follows.

Remark first that we have $\{z \in Y \mid \mathfrak{v}(z) = -\infty\} = A$.

The listed properties of the function m imply that $m(\eta_1, \ldots, \eta_q)$ is still plurisubharmonic if $\eta_1, \ldots, \eta_q$ are plurisubharmonic. Because of the special form of m it follows that it also preserves the strongly plurisubharmonicity.

Indeed, we have to check that for any strongly plurisubharmonic functions (such that the composition makes sense) $\eta_1, \ldots, \eta_q$ and for each $\theta \in \mathcal{C}_0^\infty$ there exists $\varepsilon_0 > 0$ such that $m(\eta) + \varepsilon\theta$ is plurisubharmonic $\forall\ 0 \leq \varepsilon \leq \varepsilon_0$.

But this follows at once from:

$$m(\eta)+\varepsilon\theta = m(\eta)+\int_{\mathbb{R}^q}\varepsilon\theta \prod_{1\le n\le q}\varrho(u_n)du_n =$$

$$= \int_{\mathbb{R}^q}\max(\eta_1+u_1+\varepsilon\theta,\dots,\eta_q+u_q+\varepsilon\theta)\prod_{1\le n\le q}\varrho(u_n)du_n = m(\eta+\varepsilon\theta)$$

Now consider on V_j, for each index j, the function

$$M\lambda_j+\mathfrak{v}_{|V_j}$$

We will show that if M is a sufficiently big constant then

$$\varphi_j = (M\lambda_j+\mathfrak{v})\circ p_{|_{p^{-1}(V_j)}}$$

is strongly plurisubharmonic on $p^{-1}(V_j)$.

To do this consider first the function $M\lambda_j-\theta_i$ on $V_j\cap W_i$. Because θ_i and its derivatives are bounded on W_i and λ_j is strongly plurisubharmonic on V_j it follows that there exists a constant M such that $M\lambda_j-\theta_i$ is strongly plurisubharmonic on $V_j\cap W_i$.

Now look on $p^{-1}(V_j)$ at

$$\varphi_j = (M\lambda_j\circ p+\mathfrak{v}\circ p)_{|_{p^{-1}(V_j)}} =$$

$$= M\lambda_j\circ p+\int_{\mathbb{R}^q}\max(\log a_1\circ p-\theta_1\circ p+u_1,\dots,\log a_q\circ p-\theta_q\circ p+u_q)\prod_{1\le n\le q}\varrho(u_n)du_n =$$

$$= \int_{\mathbb{R}^q}\max(M\lambda_j\circ p+\log a_1\circ p-\theta_1\circ p+u_1,\dots,M\lambda_j\circ p+\log a_q\circ p-\theta_q\circ p+u_q)\prod_{1\le n\le q}\varrho(u_n)du_n$$

(where $M\lambda_j\circ p+\log a_i\circ p-\theta_i\circ p$ is defined on $p^{-1}(V_j\cap W_i)$ respectively).

We have shown in the first step that

$$\psi_j\circ p = \lambda_j\circ p+\log(\sum_{k=1}^{s}|f_{k,j}|^2)\circ p = \lambda_j\circ p+\log a_j\circ p$$

is strongly plurisubharmonic on $U_j := p^{-1}(V_j)$

What concerns λ_j it was important only that λ_j was strongly plurisubharmonic on V_j. So we can replace it by any other strongly plurisubharmonic function, for example on $V_j\cap W_i$ by $M\lambda_j-\theta_i$, to obtain by the same type of argumentation the analogue conclusion, that

$$M\lambda_j\circ p+\log a_i\circ p-\theta_i\circ p$$

is strongly plurisubharmonic on $p^{-1}(V_j\cap W_i), \forall\, j, \forall\, i$.

So, it follows finally from the above listed properties of m that the function φ_j is strongly plurisubharmonic on $U_j := p^{-1}(V_j)$.

In conclusion, we obtained a covering

$$(U_j := p^{-1}(V_j))_{j\in I}$$

of X and on each open set U_j a strongly plurisubharmonic function

$$\varphi_j : U_j \longrightarrow [-\infty, \infty)$$

with the property that φ is regular of class $\mathcal{C}^\infty$ outside the rare set $\{x \in U_j \mid \varphi_j(x) = -\infty\} = U_j \cap p^{-1}(A)$.

This collection of functions satisfies also the desired pluriharmonic-compatibility condition, that is, we have on each non-empty intersection $U_i \cap U_j$ locally that:

$$\varphi_i = M\lambda_i \circ p + \mathfrak{v} \circ p_{|_{p^{-1}(V_i)\cap p^{-1}(V_j)}} = M\lambda_j \circ p + M\operatorname{Re}(f_{ij} \circ p) + \mathfrak{v} \circ p_{|_{p^{-1}(V_i)\cap p^{-1}(V_j)}} = \varphi_j + \operatorname{Re} g_{ij}$$

with g_{ij} holomorphic. This completes the proof of our Theorem 3.1.6. $\square$

Chapter 4

Proper maps and generalized Kähler metrics

4.1 General setup

Now we can extend our result to the following more general context:

Theorem 4.1.1 *Let $p : X \longrightarrow Y$ be a holomorphic and surjective map between two reduced, compact, complex spaces with singularities and with the property that p sends each irreducible component C_X of X (surjective) onto an irreducible component C_Y of Y of the same dimension,* $\dim C_X = \dim C_Y$.
If Y is Kähler, then X has a generalized Kähler metric.

Remark 4.1.2

(i) In general we only know that the image of an irreducible component of X is contained in an irreducible component of Y

(ii) In the context of the above theorem it follows that $\dim X = \dim Y$

(iii) X being compact and p continuous it follows that p is automatically proper

(iv) The hypothesis of the above theorem concerning the irreducible components of X and Y is satisfied for example in the following special cases:

   (a) X and Y are irreducible (and therefore pure dimensional) and $\dim X = \dim Y$

   (b) X and Y are pure dimensional with $\dim X = \dim Y$ and they have the same number of irreducible components

The idea of the proof is to reduce this problem to the now known context of a proper modification between compact complex spaces, where the "base" space is Kähler. This is possible with help of the following "Stein factorization theorem" (see for instance [Fi] p.70, Th.1.24).

Theorem 4.1.3 *Let $p : X \longrightarrow Y$ be a proper holomorphic map. Then there is a commutative diagram*

$$\begin{array}{ccc} X & & \\ {\scriptstyle p}\downarrow & \searrow^{\sigma} & \\ Y & \xleftarrow[\tau]{} & Z \end{array}$$

of complex spaces and holomorphic maps with the following properties:

(i) *τ is finite*

(ii) *σ is proper, surjective, has connected fibers and the canonical map $\sigma^0 : \mathcal{O}_Z \longrightarrow \sigma_* \mathcal{O}_X$ is an isomorphism.*

Remark 4.1.4 In our context we have also the following supplementary properties:

(i) Z is compact (because X is compact and σ is continuous and surjective; or because Y is compact and τ is proper)

(ii) τ is surjective (because p and σ are surjective)

(iii) Z is reduced

Indeed, if there would exist an open set $V \subset Z$ such that $\mathcal{O}_Z(V)$ contains a nilpotent element, then because of $\mathcal{O}_Z(V) \simeq \mathcal{O}_X(\sigma^{-1}(V))$ it would follow that $\mathcal{O}_X(\sigma^{-1}(V))$ contains nilpotent elements, which is a contradiction to the fact that X is reduced

(iv) τ being finite and surjective it follows also that $\dim Y = \dim Z$, so that $\dim X = \dim Z$

(v) Y being Kähler and τ being finite it follows that Z is also Kähler (see for instance [Bi] or [VV] or see our Appendix, Proposition 6.0.2).

Our goal is to show that σ is a proper modification. For this we need some additional short lemmas concerning the images and inverse images through σ and τ of rare analytic sets and of the irreducible components.

4.2 Images and inverse images of analytic sets

Lemma 4.2.1 *For every holomorphic, proper and surjective map $f : X \longrightarrow Y$ between two complex spaces it holds that for each irreducible component C_Y of Y there exists an irreducible component C_X of X with $f(C_X) = C_Y$.*

Remark that we will apply this lemma to the maps σ and τ.

Proof. Assume the contrary is true. Because f is surjective there would exist at least 2 components C_{X_1} and C_{X_2} such that $f(C_{X_1}) \cup f(C_{X_2}) = f(C_{X_1} \cup C_{X_2}) = C_Y$ and with the property that $f(C_{X_1}) \neq C_Y \neq f(C_{X_2})$. But f being proper, $f(C_{X_1})$ and $f(C_{X_2})$ are analytic sets different from C_Y with the union equal to C_Y. This would be a contradiction to the irreducibility of C_Y. □

Remark that this lemma also implies that X has at least the same number of irreducible components as Y.

Lemma 4.2.2 *With the notations and with the hypothesis of Theorem 4.1.1 and Theorem 4.1.3 it holds that each irreducible component C_X of X goes (surjective) onto an irreducible component C_Z of Z.*

Proof. By Reductio ad absurdum, we assume the existence of a component C_X such that $\sigma(C_X) =: B \overset{\neq}{\hookrightarrow} C_Z$ is a rare analytic subset in C_Z. Because there exists by our hypothesis on p a component C_Y such that $p(C_X) = C_Y$, that is such that $\tau\sigma(C_X) = C_Y$ i.e. $\tau(B) = C_Y$ it would follow that we have the following two surjective and finite mappings (B is closed in C_Z !):

$$B \twoheadrightarrow C_Y$$

$$C_Z \twoheadrightarrow C_Y$$

from where we can deduce that (see for instance [Ka] E.49.p) $\dim B = \dim C_Y$ and $\dim C_Z = \dim C_Y$ so, it follows that $\dim B = \dim C_Z$. But C_Z is reduced and irreducible and B is a closed, proper analytic subset, so $\dim B < \dim C_Z$, which comes in contradiction to the above deduced fact. So the Lemma 4.2.2 is proved. □

Lemma 4.2.3 *With the notations and with the hypothesis of Theorem 4.1.1 and Theorem 4.1.3 it holds that each irreducible component C_Z of Z goes (surjective) through τ onto an irreducible component C_Y of Y.*

Proof. Again by contradiction, if there exists a component C_Z with $\tau(C_Z) =: C \overset{\neq}{\hookrightarrow} C_Y$ consider an irreducible component C_X of X with $\sigma(C_X) = C_Z$. Then it follows that $p(C_X) = \tau\sigma(C_X) = \tau(C_Z) =: C \overset{\neq}{\hookrightarrow} C_Y$ which comes in contradiction to our hypothesis on p. □

Remark 4.2.4 So σ as well as τ fulfil the same condition as p does by hypothesis.

Lemma 4.2.5 *Let $\varrho : X' \longrightarrow Y'$ be a surjective, holomorphic map between two reduced complex spaces such that each irreducible component of X' is mapped through ϱ (surjective) onto an irreducible component of Y'. Let $A \hookrightarrow Y'$ be a rare analytic subset. Then $\varrho^{-1}(A) \hookrightarrow X'$ is also rare.*

Proof. Suppose for the moment that X' is an irreducible, reduced complex space. If, by contradiction, we suppose that the inverse image of A is not rare we then have that

$\varrho^{-1}(A) = X'$. But this means also that $\varrho(X') \subset A \neq Y'$, which is a contradiction with the fact that ϱ is surjective.

Consider now the general case, i.e. X' is no longer supposed to be irreducible. But $A \hookrightarrow Y'$ is rare if and only if A contains no irreducible component of Y'. This is equivalent with the fact that $A \cap C_{Y'}$ is rare in $C_{Y'}$ for every irreducible component $C_{Y'}$ of Y'. So, we have that for every $C_{Y'}$, $\varrho^{-1}(A \cap C_{Y'})$ is rare in $C_{X'}$, for each $C_{X'}$ with $\varrho(C_{X'}) = C_{Y'}$. But this means that $\varrho^{-1}(A)$ is rare in X'. □

Lemma 4.2.6 *Let $\varrho : X' \longrightarrow Y'$ be a surjective, holomorphic and proper map between two reduced complex spaces of the same finite dimension, $\dim X' = \dim Y'$, such that each irreducible component $C_{X'}$ of X' is mapped through ϱ (surjective) onto an irreducible component $C_{Y'}$ of Y'. Let $A \hookrightarrow X'$ be a rare analytic subset. Then $\varrho(A) \hookrightarrow Y'$ is also rare.*

Proof. Suppose again for the moment that X' and Y' are irreducible (reduced) complex spaces. If, by contradiction, we suppose that the image of A is not rare we then have that $\varrho(A) = Y' = \varrho(X')$ so $\varrho_{|A} : A \longrightarrow Y'$ is surjective, holomorphic and proper and because X' is reduced and A is rare we have $\dim A < \dim X' = \dim Y'$. We want to show that it follows from the surjectivity of $\varrho_{|A}$ that $\dim A \geq \dim Y'$ which will lead to a contradiction.

For this it is enough to consider the case when A is irreducible. Indeed, else applying Lemma 4.2.1 we obtain the existence of an irreducible component A_1 of A with $\varrho(A_1) = Y'$. We work further with A_1 instead of A (we have $\dim A \geq \dim A_1 \geq \dim Y'$).

By repeatedly replacing A by $\mathrm{Sing}(A)$ we can also assume that $\varrho(A) = Y'$ and that $\varrho(\mathrm{Sing}\, A) \neq Y'$. Indeed, this process has to end after finitely many steps because otherwise we would have a contradiction by the existence of a sequence of surjective maps (all being restrictions of ϱ)

$$A \twoheadrightarrow Y'$$
$$\mathrm{Sing}\, A \twoheadrightarrow Y'$$
$$\mathrm{Sing}(\mathrm{Sing}\, A) \twoheadrightarrow Y'$$
$$\mathrm{Sing}(\ldots \mathrm{Sing}(A)) \twoheadrightarrow Y'$$

where at each step the dimension of the domains of definitions is by one lower. But remember that A has finite dimension, so we would get once a surjective map with an empty defining space.

So we are now in the situation of having a surjective map

$$A \twoheadrightarrow Y'$$

with A irreducible and reduced and such that

$$\mathrm{Sing}\, A \not\twoheadrightarrow Y'.$$

Consider now the following map between two manifolds given by the restriction of ϱ:

$$M = A \backslash \varrho^{-1}(\mathrm{Sing}\, Y' \cup \varrho(\mathrm{Sing}\, A)) \twoheadrightarrow Y' \backslash (\mathrm{Sing}\, Y' \cup \varrho(\mathrm{Sing}\, A))$$

Now we use the following theorem of Sard:

Theorem 4.2.7 (Sard) *Let M and N be real manifolds of dimensions m resp. n and let $\varphi \in \mathcal{C}^k(M, N)$, where $k \geq \max(m-n+1, 1)$. Then the set C_φ of critical values of φ is locally of measure zero in N, i.e. $\beta(C_\varphi \cap V)$ is a set of measure zero in $\mathbb{R}^n$ for every local chart $\beta : V \longrightarrow \mathbb{R}^n$ of N.*

Remark 4.2.8 A value $b \in \varphi(M)$ is called *critical value* if the fiber $M_b := \varphi^{-1}(b)$ contains at least one point where φ is not a submersion.

Applying this theorem to our map above we obtain the existence of a point in the connected manifold M (remember that A is irreducible!), say $a \in M$, where ϱ is a submersion, so in particular it is a point where $\dim A = \dim A_a = \dim M \geq \dim Y'$ holds (A_a denotes the germ of A at a). That was our goal to prove in order to get a contradiction, so the special case where X' and Y' are irreducible is proved.

For the general case the argument is now again very easy. Indeed, by using the previous special case we have the following: $A \hookrightarrow X'$ is rare if and only if $A \cap C_{X'}$ is rare for all irreducible components $C_{X'}$ of X'. But then from the previous case we have that $\varrho(A \cap C_{X'}) = \varrho(A) \cap C_{Y'}$ is rare in $C_{Y'}$, this being equivalent to the fact that $\varrho(A)$ is rare in Y'. □

Remark 4.2.9 Another proof for this lemma can be obtained at once with the help of a Corollary from [Fi] ,p.141.

4.3 Proof of Theorem 4.1.1

Now we have all preliminaries to start proving that σ is a proper modification. As we have seen above this is enough in order to prove Theorem 4.1.1.

Using the lemmas from Section 4.2 and the fact that X and Z are reduced we see that

$$\mathrm{Sing}(Z) \hookrightarrow Z$$

$$\mathrm{Sing}(X) \hookrightarrow X$$

$$\sigma(\mathrm{Sing}(X)) \hookrightarrow Z$$

$$\sigma^{-1}(\mathrm{Sing}(Z)) \hookrightarrow X$$

and

$$\sigma^{-1}(\sigma(\mathrm{Sing}(X))) \hookrightarrow X$$

are all rare analytic sets.

Consider now $D = \mathrm{Sing}\, Z \cup \sigma(\mathrm{Sing}\, X)$, $D \hookrightarrow Z$ which is a rare analytic set in Z. Consider also the surjective map $\sigma_{|_{X\setminus\sigma^{-1}(D)}} : X\setminus\sigma^{-1}(D) \longrightarrow Z\setminus D$ given by the restriction of σ to $X \setminus \sigma^{-1}(D)$. For each irreducible component C_X of X we then have a surjective map between two connected manifolds

$$\sigma_{|_{C_X\setminus\sigma^{-1}(D)}} : C_X\setminus\sigma^{-1}(D) \longrightarrow C_Z\setminus D$$

where C_Z is chosen such that $\sigma(C_X) = C_Z$, in particular by our hypothesis we then have that $\dim C_X = \dim C_Z$. Applying Sard's Theorem 4.2.7 it follows that there exists a regular point $a \in C_X \backslash \sigma^{-1}(D)$ where σ is a submersion, that is a point a where the linear tangent map of σ , $T_a(\sigma_{|_{C_X \backslash \sigma^{-1}(D)}})$ is surjective. Because of the same finite dimension of the spaces it follows that the linear tangent map $T_a(\sigma_{|_{C_X \backslash \sigma^{-1}(D)}})$ is in fact bijective, that is the determinant of the Jacobian matrix is nonvanishing, $\det J_{\sigma_{|_{C_X \backslash \sigma^{-1}(D)}}}(a) \neq 0$. But this tells us that the set

$$\{x \in C_X \backslash \sigma^{-1}(D) \mid \det J_\sigma(x) = 0\}$$

is a rare analytic set in $C_X \backslash \sigma^{-1}(D)$. This being true for all irreducible components C_X of X it follows that

$$B := \{x \in X \backslash \sigma^{-1}(D) \mid \det J_\sigma(x) = 0\}$$

is a rare analytic set in $X \backslash \sigma^{-1}(D)$. Note that we do not know whether $\overline{B}$ is analytic in X.

We want to prove the existence of a rare analytic set C in X such that $\overline{B} \subset C$, (where $\overline{B}$ denotes the closure of B in X). In fact we will find C such that $C \cap (X \setminus \sigma^{-1}(D)) = B$. Because $\sigma^{-1}(D)$ is rare it follows at once that C is rare in X.

To find C we make use of some known notions and results about the tangent space and the tangent map for complex spaces with singularities. For a complete description we refer to [Fi]. A short introduction was given in Section 2.2.

It follows from Lemma 2.2.15 that the set

$$C := \mathrm{Sing}^0(\sigma) := \{x \in X \mid \mathrm{corank}_x\, \sigma > 0\}$$

is analytic. That means that the set

$$C = \{x \in X \mid \dim \ker T_x\sigma > 0\} = \{x \in X \mid T_x\sigma \text{ is not injective}\}$$

is analytic in X.

Moreover because $C \cap (X \setminus \sigma^{-1}(D)) = B$, this set is also rare as noticed above.

Let $A := D \cup \sigma(C)$ which is rare in Z and consider the surjective map

$$\sigma_{|_{X \backslash \sigma^{-1}(A)}} : X \setminus \sigma^{-1}(A) \longrightarrow Z \setminus A$$

We have for all $x \in X \setminus \sigma^{-1}(A)$ that $x \in \mathrm{Reg}(X)$ and $x \notin C$. Therefore $x \notin B$, so that $\det J_\sigma(x) \neq 0$ for each $x \in X \setminus \sigma^{-1}(A)$.

But this means that σ is locally biholomorphic in $X \setminus \sigma^{-1}(A)$. Because $\sigma_{|_{X \backslash \sigma^{-1}(A)}}$ has connected fibers it follows that $\sigma_{|_{X \backslash \sigma^{-1}(A)}}$ is injective, so we deduce finally that the map

$$\sigma_{|_{X \backslash \sigma^{-1}(A)}} : X \setminus \sigma^{-1}(A) \longrightarrow Z \setminus A$$

is biholomorphic, which proves the fact that $\sigma : X \longrightarrow Z$ is a proper modification.

As we mentioned above this is enough to conclude as desired that X has a generalized Kähler metric. So the proof of Theorem 4.1.1 is now complete. □

Chapter 5

A supplement to a Theorem of Fornæss and Narasimhan

Fornæss and Narasimhan proved (in [F-N], Theorem 5.3.1) that for any complex space X the identity $WPSH(X) = PSH(X)$ holds, where $WPSH(X)$ denotes the weakly plurisubharmonic functions on X and $PSH(X)$ denotes, as usual, the plurisubharmonic functions on X.

Of course, for the case when X has no singularities, i.e. X is a complex manifold, this identity is trivial. But for the singular case the inclusion $WPSH(X) \subseteq PSH(X)$ is no more trivial.

In this chapter we give another proof for this identity (Theorem 5.2.2) but for continuous functions only, which is shorter and easier and which has the advantage that it can be generalized to q-plurisubharmonic functions (Theorem 5.3.14) as they were introduced by Hunt and Murray in [H-M](see also [Fu]).

In the same time we obtain also a generalization of a theorem of Siu ([Siu]), namely we show (Theorem 5.3.16) that every q-complete subspace with corners of a complex space X admits a q-complete with corners neighbourhood in X. This result will be needed in the proof of our Theorem 5.3.14.

5.1 Some definitions and needed results

We will denote by $WPSH(X)$ the class of *weakly plurisubharmonic functions* on X, as they were introduced in [F-N], i.e. the class of upper semicontinuous functions $\varphi : X \longrightarrow [-\infty, \infty)$ such that for any holomorphic function $f : \Delta \longrightarrow X$, where Δ denotes the unit disc in $\mathbb{C}$, the composition $\varphi \circ f$ is subharmonic on Δ.

$SWPSH(X)$ stands for the *strongly weakly plurisubharmonic functions* on X, i.e. those $WPSH(X)$- functions for which we have: for every $\theta \in \mathcal{C}_0^\infty(X, \mathbb{R})$ (i.e. smooth and with compact support) there exists $\varepsilon_0 > 0$ such that $\varphi + \varepsilon\theta$ is in $WPSH(X)$ for $0 \leq \varepsilon \leq \varepsilon_0$.

In the alternative proof of Fornæss-Narasimhan's Theorem we will use an extension theorem of Richberg, which can be found in [R], Satz 3.3:

Theorem 5.1.1 (Richberg) *Let X be a complex space and Y a closed complex subspace of X. Then for every function ψ on Y which is continuous (resp. smooth) and strongly plurisubharmonic, there exist a neighbourhood V of Y in X and a function $\tilde{\psi}$ on V which is continuous (resp. smooth), strongly plurisubharmonic and such that $\tilde{\psi}_{|Y} = \psi$.*

We shall also need a theorem due to M. Colţoiu in [MC.1].

Theorem 5.1.2 (M. Colţoiu) *Let X be a complex space which admits a strongly plurisubharmonic exhaustion function $\varphi : X \longrightarrow [-\infty, \infty)$. Then X is 1-convex.*

Remark 5.1.3 If in the above theorem φ is supposed to be real-valued, as remarked in [MC.1] it follows easily from the Maximum Principle that the exceptional set of X (i.e. the maximal compact analytic subset) is empty, hence X is Stein. This is a theorem of Fornæss-Narasimhan, Theorem 6.1 in [F-N].

The following result is due to M. Peternell (Lemma 5 in [MP]; see also [Dem]). For this we need first the following:

Definition 5.1.4 *Let X be a manifold. A function $v : X \longrightarrow [-\infty, \infty)$ is called almost plurisubharmonic if it can be written locally as a sum of a plurisubharmonic and a smooth function.*

If X is a complex space we require that v can be locally extended as an almost plurisubharmonic function in the ambient space of an embedding.

Theorem 5.1.5 (M. Peternell) *If Y is a closed analytic subset in a complex space X then there exists an almost plurisubharmonic function v on X such that $v \in \mathcal{C}^\infty(X \setminus Y)$ and $Y = \{x \in X \mid v(x) = -\infty\}$.*

The next needed result is due to Siu ([Siu]):

Theorem 5.1.6 *Let Y be a closed Stein subspace in a complex space X. Then Y has a Stein open neighbourhood in X.*

5.2 Another proof of Fornæss-Narasimhan's Theorem

First of all we prove a lemma which shows the interplay between $SWPSH$- and $SPSH$-functions on a complex space under certain conditions.

Lemma 5.2.1 *Let Ω be an open subset of a reduced Stein space X with $\dim X < +\infty$ and such that Ω admits a $SWPSH$- exhaustion function $\varphi : \Omega \longrightarrow \mathbb{R}$. Then Ω is Stein.*

Proof. We may assume, without loss of generality that $\varphi > 0$.

The proof is by induction on $n = \dim X$.

If $n = 0$ then X has only isolated points and it is therefore a manifold, so there is nothing to prove.

Suppose now that the lemma is true for all complex spaces Y with $\dim Y \leq n-1$ and let $\dim X = n$.
Consider $Y = \mathrm{Sing}(X)$, the singular locus of X. We have $\dim Y \leq n-1$ and by the induction hypothesis, since $\varphi_{|_{Y\cap\Omega}} \in SWPSH(Y\cap\Omega)$ is an exhaustion function for $Y\cap\Omega$, it therefore follows that $Y\cap\Omega$ is Stein. So $Y\cap\Omega$ admits a smooth, SPSH-exhaustion function, which we shall denote by ψ_1 (the smoothness condition, at least the continuity will be necessary for applying Richberg's extension theorem).

Now Theorem 5.1.1 yields a $SPSH$ and smooth extension of ψ_1 to an open neighbourhood V of $Y\cap\Omega$ in Ω, denoted by $\tilde{\psi} : V \longrightarrow \mathbb{R}$. By shrinking V, if necessary, we can suppose that $\tilde{\psi}$ is defined in a neighbourhood of $\overline{V}$ (the closure being in Ω) and that $\{x \in \overline{V} \mid \tilde{\psi}(x) < c\}$ is relatively compact in $\overline{V}$ for all real numbers c.

But Y being a closed analytic subset of a Stein space X there exists $f_1, \ldots, f_m \in \mathcal{O}(X)$ such that $Y = \{x \in X \mid f_1(x) = \ldots = f_m(x) = 0\}$. If we define $p := \log(|f_1|^2 + \ldots + |f_m|^2)$ then we have also that $Y = \{x \in X \mid p(x) = -\infty\}$.

Let now $\chi : (0,\infty) \longrightarrow \mathbb{R}$ be a smooth, convex, rapidly increasing function (to be made precise later) and define:

$$\psi = \begin{cases} \max(\tilde{\psi}, \chi\circ\varphi + p) & \text{on} \quad V \\ \chi\circ\varphi + p & \text{on} \quad \Omega\backslash V \end{cases}$$

We choose χ such that :

(i) $\chi\circ\varphi + p > \tilde{\psi}$ on ∂V (the border being considered in Ω)

(ii) ψ is an exhaustion function of Ω.

These two conditions can be achieved for a suitable choice of χ, for example in the following way:

Consider a sufficiently small open neighbourhood W of $Y\cap\Omega$ in Ω such that $\overline{W} \subset V$ and such that $\tilde{\psi} > \chi\circ\varphi + p$ on $\overline{W}$. This last condition can be fulfilled because p takes the value $-\infty$ exactly on Y and $\tilde{\psi}$ is real and continuous on $V \supset Y\cap\Omega$.

Consider now a strictly increasing sequence of non-negative numbers $(c_n)_n$ with $c_0 = 0$ and $\lim\limits_{n\to\infty} c_n = +\infty$ and consider the relatively compact sets given by

$$A_i := \{x \in \Omega \mid c_i \leq \varphi(x) < c_{i+1}\}, \quad i \in \mathbb{N}$$

We have that $A_i \cap \partial V \cap Y = \emptyset$ so that $\tilde{\psi} - p$ is bounded from above on $A_i \cap \partial V$ by a positive constant M_i. So the condition we impose on χ in order to obtain condition (i) is to have:

$$\chi(x) > M_i \quad \text{for} \quad x \in [c_i, c_{i+1}), \quad i \in \mathbb{N}$$

Now in order to find conditions for χ such that ψ is an exhaustion function for Ω we remark that it is enough to find conditions for χ such that $\chi \circ \varphi + p$ is an exhaustion function on $\Omega \setminus W$ because on $\overline{V} \supset \overline{W}$ the function $\tilde{\psi}$ is already an exhaustion function and we have chosen W such that $\tilde{\psi} > \chi \circ \varphi + p$ on $\overline{W}$. In order to do this it is enough to require to χ to satisfy the following condition (observe that in particular it follows that $\chi(x) \geq x \, \forall x \in \mathbb{R}$):

$$\chi_{|[c_i, c_{i+1})} \geq c_{i+1} + L_i$$

where the constants $L_i > 0$ are chosen such that $|p| < L_i$ on $A_i \setminus W$.

Indeed, with this condition imposed on χ we have that for all i the following inequality holds on $A_i \setminus W$:

$$\chi \circ \varphi + p > c_{i+1} \tag{5.1}$$

because on $A_i \setminus W$ we have $\chi \circ \varphi \geq c_{i+1} + L_i > c_{i+1} - p$. Remark that then (5.1) holds also on $A_k \setminus W$ for all $k \geq i$, because $(c_i)_i$ increases. But this implies that

$$\{x \in \Omega \setminus W \mid \chi \circ \varphi + p < c_{i+1}\} \subset \bigcup_{j=0}^{i-1} (A_j \setminus W)$$

which is relatively compact in Ω (and also in $\Omega \setminus W$). But this means that $\chi \circ \varphi + p$ is an exhaustion function for $\Omega \setminus W$.

So all we need in order to satisfy our conditions (i) and (ii) is the existence of a convex, smooth and strictly increasing function $\chi : (0, \infty) \longrightarrow \mathbb{R}$ which satisfies

$$\chi_{|[c_i, c_{i+1})} \geq k_i := \max(M_i, c_{i+1} + L_i)$$

(in particular we have $\lim_{i \to \infty} k_i = +\infty$). But this is a well known fact.

Now, to finish the proof of our Lemma 5.2.1 we observe that by the definition of ψ and condition (i) one obviously has that $\psi \in PSH(\Omega)$. If now $\tau > 0$ is a smooth strongly plurisubharmonic function on X then $\psi + \tau_{|\Omega} \in SPSH(\Omega)$ and it is exhaustive.

By Theorem 5.1.2, Ω is Stein and the proof of our Lemma 5.2.1 is complete. □

Now we are ready to give a new proof of Fornæss-Narasimhan's Theorem for the case of continuous functions:

Theorem 5.2.2 *On any complex space X it holds: any continuous $WPSH(X)$- function is also a $PSH(X)$- function.*

Proof. The problem being locally we may assume that X is a closed analytic subset in some Stein open subset U of $\mathbb{C}^n$.

Let $\varphi \in WPSH(X)$ be continuous. Consider $\tilde{X} := X \times \mathbb{C}$ which is Stein and

$$\Omega := \{(z, w) \in \tilde{X} \mid \varphi(z) + \log |w| < 0\}.$$

We notice that Ω is itself Stein. Indeed, to see this choose $g > 0$ a smooth, $SPSH$, exhaustion function for $X \times \mathbb{C}$ and define

$$h(z, w) = g(z, w) - \frac{1}{\varphi(z) + \log |w|}$$

which is a $SWPSH(\Omega)$-exhaustion function for Ω. By Lemma 5.2.1 Ω is Stein. We have $\Omega \subset X \times \mathbb{C} \subset U \times \mathbb{C} \subset \mathbb{C}^{n+1}$. Consider now an open set W in $\mathbb{C}^{n+1}$ with the property that $W \cap (X \times \mathbb{C}) = \Omega$. By Siu's Theorem 5.1.6 applied for the situation $\Omega \subset W$, it follows that there exists V an open Stein set in $\mathbb{C}^{n+1}$ with $V \cap (X \times \mathbb{C}) = \Omega$. Since V is Stein it follows that $-\log \delta_w$ is plurisubharmonic on V, where δ_w denotes the boundary distance of V in the w-direction (see [G-R]). From the definition of $\Omega = V \cap (X \times \mathbb{C})$ it follows that $-\log \delta_{w_{|X}} = \varphi$ and so we have the required plurisubharmonic extension of φ. □

5.3 A generalization to the q-convex case

To generalize Fornæss-Narasimhan's Theorem to the q-plurisubharmonic case (but for continuous functions only) we will follow the general ideas of the proof in Section 5.2. But first of all we will give the precise definitions of *q-plurisubharmonic* (in notation q-PSH) and *weakly q-plurisubharmonic* (in notation q-$WPSH$) functions on complex spaces. We recall the definitions for open sets in $\mathbb{C}^n$.

Definition 5.3.1 (*see for instance* [*Fu*]) *An upper semicontinuous function $\varphi : D \longrightarrow [-\infty, \infty)$, where $D \subset \mathbb{C}^n$ is an open subset, is called subpluriharmonic if for every relatively compact subset $G \subset\subset D$ and for every pluriharmonic function u defined on a neighbourhood of $\overline{G}$ such that $\varphi_{|\partial G} \leq u_{|\partial G}$ we have also $\varphi \leq u$ on $\overline{G}$.*

Definition 5.3.2 ([*H-M*]) *A function defined on $D \subset \mathbb{C}^n$ and with values in $[-\infty, \infty)$ is called q-plurisubharmonic $(1 \leq q \leq n)$ in D if it is upper semicontinuous and if it is subpluriharmonic on the intersection of every q-dimensional complex plane with D.*

Remark 5.3.3

(i) n-plurisubharmonic means subpluriharmonic

(ii) 1-plurisubharmonic means plurisubharmonic

(iii) If a function is q-plurisubharmonic, it also is q'-plurisubharmonic, for every $q' \geq q$

(iv) If $\varphi : D \longrightarrow [-\infty, \infty)$ is q-plurisubharmonic, where D is open in $\mathbb{C}^n$, and if $f : D' \longrightarrow D$ is a holomorphic map defined on the open set D' in $\mathbb{C}^m$ then $\varphi \circ f$ is q-plurisubharmonic on D', for every $q \leq \min(n, m)$

(v) A real valued $\mathcal{C}^2$-function defined on an open set D in $\mathbb{C}^n$ is q-plurisubharmonic $(1 \leq q \leq n)$ if and only if the Levi form of φ has at least $n - q + 1$ non-negative eigenvalues at every point of D.

Now we define the q-plurisubharmonic functions on an arbitrary complex space.

Definition 5.3.4 *Let X be a complex space and $\varphi : X \longrightarrow [-\infty, \infty)$ be an upper semicontinuous function on X. Then φ is called q-plurisubharmonic on X if for every point $x \in X$ there exists a local embedding $i : U \hookrightarrow \tilde{U} \subset \mathbb{C}^n$, where U is a neighbourhood of x, $\tilde{U}$ an open subset of $\mathbb{C}^n$, and there exists $\tilde{\varphi}$ a q-plurisubharmonic function on $\tilde{U}$ such that $\tilde{\varphi} \circ i = \varphi$.*

Remark 5.3.5 If we suppose that φ is continuous we don't require in the above definition that $\tilde{\varphi}$ shall be continuous; it is in general assumed to be only upper semicontinuous.

We also define the weakly q-plurisubharmonic functions on complex spaces as follows:

Definition 5.3.6 *Let X be a complex space and let $\varphi : X \longrightarrow [-\infty, \infty)$ be an upper semicontinuous function. Then φ is called weakly q-plurisubharmonic on X if for every holomorphic function $f : G \longrightarrow X$, where G is open in $\mathbb{C}^q$, it follows that $\varphi \circ f$ is subpluriharmonic on G.*

Remark 5.3.7 It is known (see a remark in [KM]) that on a complex manifold the two classes q-$WPSH$ and q-PSH coincide. What is not obvious is the inclusion q-$WPSH \subseteq$ q-PSH for the singular case.

We may define the *q-SPSH* and *q-SWPSH* functions on a complex space X in a similar way as it was done in Section 5.1 or in Section 2.1.

More precisely one has:

Definition 5.3.8 *We will say that a q-PSH (q-WPSHresp.) function $\varphi : X \longrightarrow [-\infty, \infty)$ is a q-SPSH function (q-SWPSH resp.) if for every $\theta \in \mathcal{C}_0^\infty(X, \mathbb{R})$ there exists $\varepsilon_0 > 0$ such that if $0 \leq \varepsilon \leq \varepsilon_0$ then $\varphi + \varepsilon\theta$ is in q-PSH(X) (in q-WPSH(X)resp.).*

We remind that we have denoted (see Section 2.1) by $F_q(X)$ the set of the q-convex functions with corners on X. Theorem 5.1.1 which was needed in Section 5.2 has to be replaced in the q-convex case by the following:

Lemma 5.3.9 ([*MC.3*]) *Let X be a complex space, $A \subset X$ a closed analytic subset, $f \in F_q(A)$ and $\eta > 0$ a continuous function on A. Then there exists an open neighbourhood V of A in X and $\tilde{f} \in F_q(V)$ such that $|\tilde{f} - f| < \eta$ on A.*

We will need also the following approximation result due to Bungart [B] :

Theorem 5.3.10 (Bungart) *Let X be a complex manifold and $\varphi : X \longrightarrow \mathbb{R}$ a continuous q-SPSH(X) function. Then for any continuous function $\eta : X \longrightarrow (0, \infty)$ there exists a function $\tilde{\varphi} \in F_q(X)$ such that $|\tilde{\varphi} - \varphi| < \eta$ on X.*

Remark 5.3.11 In fact Bungart proved this result only when X is an open subset of some euclidian space $\mathbb{C}^n$. But as Matsumoto ([KM]) remarked, this result still holds when X is any complex manifold. For the sake of completeness we give here a proof for the manifold case using Bungart's theorem.

Proof. Fix three locally finite open coverings $(U_i)_{i\in\mathbb{N}}$, $(V_i)_{i\in\mathbb{N}}$, $(W_i)_{i\in\mathbb{N}}$ of X such that $U_i \subset\subset V_i \subset\subset W_i \subset\subset X$ for all $i \in \mathbb{N}$ and such that each W_i is the domain of a biholomorphic map $i : W_i \longrightarrow \tilde{W}_i$, $\tilde{W}_i$ being an open set in $\mathbb{C}^{n_i}$.

Consider for each index $i \in \mathbb{N}$ a function $\theta_i \in \mathcal{C}_0^\infty(X,\mathbb{R})$ such that $\theta_i \equiv -1$ on ∂V_i, $\theta_i \equiv 1$ on $\overline{U}_i$ and $\theta_i \equiv 0$ on $X \setminus W_i$.

Let now $\varepsilon_i > 0$ be small enough such that $2\varepsilon_i\theta_i \le \eta$ and such that $\varphi + \varepsilon_i\theta_i$ is still q-SPSH.

Now because $\overline{V}_i$ is contained in $W_i \simeq \tilde{W}_i$, we can apply Bungart's theorem in order to get for all $i \in \mathbb{N}$ a q-convex with corners function $\varphi_i \in F_q(W_i)$ with the property that

$$|\varphi(x) + \varepsilon_i\theta_i(x) - \varphi_i(x)| < \min(\varepsilon_i, \frac{\eta(x)}{2})$$

on a neighbourhood of $\overline{V}_i$.

It follows that we have $\varphi_i < \varphi$ on ∂V_i and $\varphi_i > \varphi$ on $\overline{U}_i$. Therefore we may define $\tilde{\varphi} : X \longrightarrow \mathbb{R}$ by $\tilde{\varphi}(x) := \max\{\varphi_i(x) \mid x \in V_i\}$. Clearly $\tilde{\varphi} \in F_q(X)$, $\varphi \le \tilde{\varphi}$ and $\tilde{\varphi} < \varphi + \eta$ as desired. □

We shall use also the following result due to Fujita (Theorem 1 in [Fu]):

Theorem 5.3.12 (Fujita) *Let D be an open subset of $\mathbb{C}^n$ which is q-complete with corners, let $w \in \mathbb{C}^n, ||w|| = 1$ and denote by δ_w the boundary distance function of D along the w-direction. Then $-\log\delta_w$ is q-plurisubharmonic on D.*

Remark 5.3.13 In fact Fujita proves this result for "pseudoconvex domains of order $(n-q)$", but by Bungart's approximation result (Theorem 5.3.10 above) this assumption is equivalent to q-complete with corners.

We can state now our main result of this chapter:

Theorem 5.3.14 *If X is a complex space and φ a continuous q-$WPSH$ function on X then it is also a q-PSH function on X.*

(*Remember that the local extensions are not required to be continuous.*)

In order to prove it we give first two other results.

Lemma 5.3.15 *Let X be a reduced complex space of finite dimension for which there exists a continuous exhaustion function $\varphi : X \longrightarrow \mathbb{R}$ which is in q-$SWPSH(X)$. Then there exists a q-convex function with corners $\psi : X \longrightarrow \mathbb{R}$, exhausting X.*

Proof. We may assume that $\varphi > 0$. In the regular case, i.e. if X is a complex manifold this lemma is a direct consequence of Bungart's approximation theorem. In the singular case the proof is by induction on $n = \dim(X)$.

The case $n = 0$ is obvious. Now suppose that Lemma 5.3.15 holds for all complex spaces Y with $\dim Y \le n-1$ and let $\dim X = n$.

Consider $Y = \mathrm{Sing}(X)$, the singular locus of X. Because $\dim Y \leq n-1$ and $\varphi_{|Y}$ satisfies the conditions of our Lemma we conclude that there exists an exhaustion function $\psi_1 : Y \longrightarrow \mathbb{R}$ which is q-convex with corners. By Lemma 5.3.9 we can find a neighbourhood V of Y in X and $\tilde{\psi}_1 \in F_q(V)$ such that $|\tilde{\psi}_1 - \psi_1| < 1$ on Y.

By shrinking V if necessary, we can suppose that $\tilde{\psi}_1$ is defined on a neighbourhood of $\overline{V}$ and and that $\{x \in \overline{V} \mid \tilde{\psi}_1(x) < c\}$ is relatively compact in $\overline{V}$, for all real numbers c.

By Peternell's Theorem there exists an almost plurisubharmonic function $\theta : X \to [-\infty, \infty)$ such that $\theta_{|\mathrm{Reg}(X)}$ is smooth and such that $Y = \mathrm{Sing}(X) = \{x \in X \mid \theta(x) = -\infty\}$.

Now let $\chi : [0, \infty) \longrightarrow \mathbb{R}_+$ be a continuous, convex, increasing function which is linear on segments. This means that there is a division $0 = a_0 < a_1 < \ldots < a_n < \ldots$ of $[0, \infty)$ such that the restriction of χ to each $[a_i, a_{i+1}]$ is linear. Therefore on $[a_i, a_{i+1}]$, $\chi(t) = A_i t + B_i$ with $A_i > 0$ and the convexity of χ gives that $A_{i+1} \geq A_i$.
If χ is as above and it increases rapidly at infinity then clearly $(\chi \circ \varphi + \theta)_{|\mathrm{Reg}(X)}$ is in q-$SPSH(\mathrm{Reg}(X))$. This can be seen as follows: take a locally finite open covering $(U_j)_{j \in \mathbb{N}}$, $U_j \subset\subset X$, of X such that for each j on a neighbourhood of $\overline{U}_j$ one has $\theta = \theta_{1,j} + \theta_{2,j}$ with $\theta_{1,j}$ smooth and $\theta_{2,j}$ plurisubharmonic. Then if the constants $A_i > 0$ in the definition of χ are chosen large enough, we have that for all $j \in \mathbb{N}, \chi \circ \varphi + \theta_{1,j}$ is q-$SWPSH$ on U_j.
Therefore we can find χ as above such that $(\chi \circ \varphi + \theta)_{|\mathrm{Reg}(X)}$ is in q-$SPSH(\mathrm{Reg}(X)) = q$-$SWPSH(\mathrm{Reg}(X))$.
Also if χ increases rapidly we may assume that $(\chi \circ \varphi + \theta)_{|\partial V} > \tilde{\psi}_{1|\partial V}$ and that $(\chi \circ \varphi + \theta)_{|X \setminus V}$ exhausts $X \setminus V$ (see also the proof of our Lemma 5.2.1).

By Bungart's approximation theorem there is a function $u : \mathrm{Reg}(X) \longrightarrow \mathbb{R}$ which is q-convex with corners and such that:

(i) $|u - (\chi \circ \varphi + \theta)| < 1$ on $\mathrm{Reg}(X)$

(ii) $u_{|\partial V} > \tilde{\psi}_{1|\partial V}$

We define now $\psi : X \longrightarrow \mathbb{R}$ as follows:

$$\psi = \begin{cases} \max(\tilde{\psi}_1, u) & \text{on} \quad V \setminus Y \\ \tilde{\psi}_1 & \text{on} \quad Y \\ u & \text{on} \quad X \setminus V \end{cases}$$

Then clearly ψ is an exhaustion function on X and ψ is q-convex with corners. Thus our lemma is proved. □

The next theorem (a generalization of Siu's theorem 5.1.6) will also be needed.

Theorem 5.3.16 *Let S be a closed analytic subset of a complex space X and assume that S is q-complete with corners. Then there exists an open neighbourhood V of S in X such that V is q-complete with corners.*

Proof. Since $S \subset X$ is a closed complex subspace there exists by Peternell's Theorem an almost plurisubharmonic function λ on X such that $S = \{x \in X \mid \lambda(x) = -\infty\}$ and such that $\lambda_{|X\setminus S} \in \mathcal{C}^\infty(X \setminus S)$.

Denote by $\psi : S \longrightarrow \mathbb{R}$ a positive, q-convex exhaustion function with corners. Applying Lemma 5.3.9 we deduce that there exists a q-convex function with corners, $\tilde{\psi}$, in a neighbourhood U of S such that $|\tilde{\psi} - \psi| < 1$ on S. We can assume that $\tilde{\psi} > 0$.

We may suppose again, by eventually shrinking U, that $\tilde{\psi}$ is defined on a neighbourhood of $\overline{U}$ and that $\tilde{\psi}$ exhausts $\overline{U}$.

Consider again χ a continuous, convex, increasing, real function defined on $[0, \infty)$ which is linear on segments such that:

(i) if $V = \{x \in U \mid \chi \circ \tilde{\psi}(x) + \lambda(x) < 0\}$, then $\partial V \cap \partial U = \emptyset$
and that

(ii) the function $\varphi := \max(-\frac{1}{\chi\circ\tilde{\psi}+\lambda}, \tilde{\psi})$ defined on V is q-convex with corners.

The choice of χ satisfying (ii) is possible as in Lemma 5.3.15. We can also realize condition (i) by choosing a sequence of real numbers $(\lambda_n)_n \searrow -\infty$ such that $\{x \in U \mid \tilde{\psi}(x) < n \text{ and } \lambda(x) < \lambda_n\}$ is relatively compact in U and requiring that $\chi : [0, \infty) \longrightarrow \mathbb{R}$ satisfies additionally $\chi_{|[n-1,n)} \geq -\lambda_n \ \forall n \in \mathbb{N}$.

Then it follows that the set $V = \{x \in U \mid \chi\circ\tilde{\psi}(x)+\lambda(x) < 0\}$ is an open q-complete with corners neighbourhood of S (φ being the exhaustion function) and therefore Theorem 5.3.16 is proved. ☐

We are now in a position to prove Theorem 5.3.14.

Proof. The problem being locally, we can assume, without loss of generality, that X is a closed analytic subset in a Stein open subset $U \subset \mathbb{C}^n$. Let $\varphi \in q\text{-}WPSH(X)$ be continuous.

We have $X \times \mathbb{C} \subset U \times \mathbb{C} \subset \mathbb{C}^{n+1}$ and consider $\Omega \subset X \times \mathbb{C}$ the open set given by:

$$\Omega = \{(z, w) \in X \times \mathbb{C} \mid |w| < e^{-\varphi(z)}\}$$

On Ω there exists a continuous q-$SWPSH$ exhaustion function. Indeed, denote by $s : X \times \mathbb{C} \longrightarrow \mathbb{R}$ a smooth, $SPSH(X \times \mathbb{C})$, positive exhaustion function and consider

$$s(z, w) - \frac{1}{\varphi(z) + \log|w|} : \Omega \longrightarrow \mathbb{R}$$

This function has the desired properties, so that for Ω we can apply Lemma 5.3.15 and we get so a q-convex with corners exhaustion function $\psi : \Omega \longrightarrow \mathbb{R}$. But this means that Ω is q-complete with corners.

Consider now an open set W in $\mathbb{C}^{n+1}$ with the property that $W \cap (X \times \mathbb{C}) = \Omega$. Then Theorem 5.3.16 can be applied for the situation $\Omega \subset W$. We conclude the existence of an open set $\tilde{\Omega} \subset \mathbb{C}^{n+1}$ which is q-complete with corners and for which $\tilde{\Omega} \cap (X \times \mathbb{C}) = \Omega$ holds.

Now it is enough to consider δ_w the distance to the boundary of $\tilde{\Omega}$ along the w-direction. By Theorem 5.3.12 this is a q-$PSH(\tilde{\Omega})$ function (not necessary continuous). But by the definition of Ω, it follows that $-\log \delta_{w_{|_X}} = \varphi$ and so we have the desired conclusion that φ is a q-$PSH(X)$ function. $\square$

Chapter 6

Appendix

In the proof of our Theorems 3.1.6 and 4.1.1 we used the following two known results for which we want to give here proofs (see [Co-Mi], Prop. 2.1 resp. [VV] ,Thm. 1).

Proposition 6.0.1 *Let X, Y be complex spaces and $p : X \longrightarrow Y$ a proper, surjective, holomorphic map. Let $\Phi : Y \longrightarrow [-\infty, \infty)$ be an upper semicontinuous function such that $\Phi \circ p$ is plurisubharmonic (respectively strongly plurisubharmonic) on X. Then Φ is plurisubharmonic (respectively strongly plurisubharmonic) on Y.*

Proposition 6.0.2 *Let $f : X \longrightarrow Y$ be a finite, holomorphic map between compact complex spaces. If Y is Kähler, then X is Kähler too.*

Proof of Proposition 6.0.1. Because on each complex space the equality $WPSH(X) = PSH(X)$ between weakly plurisubharmonic and plurisubharmonic functions holds (also for non-continuous functions see [F-N]!) we have to show that the composition $\Phi \circ f$ is subharmonic on W for each holomorphic map $f : W \longrightarrow Y$, where W is an open set in the complex plane $\mathbb{C}$.

In what follows we make some reductions to the problem:

1) We show first that it is enough to consider the case when Y is a domain in $\mathbb{C}$.

For this consider the set

$$Z := \{(w, x) \in W \times X \mid f(w) = p(x)\}$$

It is endowed with a natural structure of a complex space, being the inverse image of the diagonal in $Y \times Y$ under the holomorphic map $f \times p : W \times X \longrightarrow Y \times Y$. The structure on Z is such that the projections onto the two factors induce holomorphic maps $\pi_X : Z \longrightarrow X$ and $\pi_W : Z \longrightarrow W$ verifying $p \circ \pi_X = f \circ \pi_W$, i.e. we have a commutative diagram of the form

$$\begin{array}{ccc}
W \times X & & \\
\uparrow & & \\
Z & \xrightarrow{\pi_X} & X \\
\downarrow{\scriptstyle \pi_W} & & \downarrow{\scriptstyle p} \\
W & \xrightarrow{f} & Y
\end{array}$$

Since p is surjective it follows by the definition of Z that π_W is also surjective. But from our hypothesis that $\Phi \circ p$ is plurisubharmonic (the strongly plurisubharmonic case will be considered at the end) it follows that $(\Phi \circ f) \circ \pi_W = (\Phi \circ p) \circ \pi_X$ is also plurisubharmonic. Remember that we want to show that $\Phi \circ f$ is subharmonic. But $\Phi \circ f$ being upper semicontinuous and π_W being holomorphic, proper and surjective it follows that if $\Phi \circ f$ plays the role of Φ and π_W that one of p it is indeed enough to consider the case when $Y = W$ is a domain in $\mathbb{C}$.

2) Now in this case, p being proper, we have that for any irreducible component C_X of X it follows that $p_{|C_X}$ is constant or that $p(C_X) = Y$, because the only analytic subsets in $\mathbb{C}$ are those of dimension 0 (i.e. the points) or the whole plane.

The map p being surjective, that means that it is enough to assume that X is irreducible; we don't loose generality.

3) The last reduction is to show that it is enough to prove the subharmonicity of Φ outside a discrete set of points in $\mathbb{C}$.

Indeed, the result we will use and which will be proved later is the following (see [Gr-Re])

Lemma 6.0.3 *Let D be a domain in $\mathbb{C}$ and $z_0 \in D$ be an arbitrary point. Suppose we have given a function s defined and subharmonic in $D \setminus \{z_0\}$ and which is bounded from above on a neighbourhood of z_0.*

Then there exists a unique subharmonic extension of s to D, that one given by

$$\tilde{s}(z) = \begin{cases} s(z) & \text{for} \quad z \neq z_0 \\ \limsup\limits_{\substack{w \longrightarrow z_0 \\ (w \neq z_0)}} s(w) & \text{for} \quad z = z_0 \end{cases}$$

Now it remains to prove that for an irreducible complex space X and Y an open subset in $\mathbb{C}$, the function Φ as given in Proposition 6.0.1, is subharmonic outside a discrete set of points in $\mathbb{C}$.

To prove this we use the fact that the fibers of p (which are analytic sets) have all (pure) codimension 1 (see [Ka] 48.3_{geo} and remember that X is irreducible and p surjective). Consider now an arbitrary point $y_0 \in Y$ and $x_0 \in p^{-1}(y_0)$ and a neighbourhood $\tilde{V}$ of x_0 in X which goes surjective through p on a neighbourhood $\tilde{W}$ of y_0 in $\mathbb{C}$. We can also suppose that $\tilde{V}$ is embedded as an analytic subset in some complex euclidian set $\hat{V}$.

We can apply then, for instance, the following (see [Gu], vol II, chap. G, thm. 4, p. 75)

Theorem 6.0.4 *If $\tilde{U}$ is the germ at the origin of an analytic set in $\mathbb{C}^r$ then* $\operatorname{codim} \tilde{U}$ *is the largest integer k for which there is a k-dimensional linear subspace $\tilde{L}$ through the origin in $\mathbb{C}^r$ with $\tilde{U} \cap \tilde{L} = 0$.*

In our situation (eventually shrinking the neighbourhoods) we get the existence of a linear complex subspace $\tilde{L}$ in $\mathbb{C}^r$ of maximal dimension with respect to the property that $\tilde{L} \cap (p_{|_{\tilde{V}}})^{-1}(y_0) = \{x_0\}$.

But then, again because of Theorem 6.0.4, we obtain that the complex subspace $\tilde{L} \cap \tilde{V}$ of $\tilde{V}$ has (at least) dimension 1. In conclusion we got the existence of a subspace L of dimension 1 in $\tilde{V}$ such that $L \cap p^{-1}(y_0) = \{x_0\}$. So $p_{|_L}$ is not constant, so it is an open map between complex spaces of dimension 1.

But this implies that $p_{|_L}$ is a local isomorphism outside a discrete set of points with discrete image. It is then clear that because $\Phi \circ p$ is subharmonic on L we have that Φ is subharmonic in a neighbourhood of y_0 in Y outside a discrete set of points. But y_0 being arbitrary and subharmonicity being a local problem, this is enough to conclude the proof, that is we have that Φ is subharmonic on Y outside a discrete set of points and thus also on Y.

The strongly plurisubharmonic case follows at once from the definition of the strongly plurisubharmonicity and the plurisubharmonic case proved above. □

Proof of Lemma 6.0.3. Because s is supposed to be bounded from above in a neighbourhood of z_0 we have that $\limsup_{z \longrightarrow z_0} s(z) < +\infty$, so $\tilde{s}$ is well defined, i.e. it has values in $[-\infty, \infty)$.

Of course $\tilde{s}$ is upper semicontinuous, by definition.

If by contradiction, we suppose that $\tilde{s}$ is not subharmonic on D this means that we can find an open set W in D, $z_0 \in W$ and a harmonic function h on W such that $\tilde{s} + h$ is not constant and takes its maximum m only in z_0 because on $D \setminus \{z_0\}$ $\tilde{s}$ is subharmonic.

Consider a disc $K_0 := \{z \in \mathbb{C} \mid |z - z_0| \leq d_0\}$ contained in W. We then have on ∂K_0 that $\tilde{s}(z) + h(z) < m$. But $\tilde{s} + h$ being upper semicontinuous on W we can find $\varepsilon > 0$ such that $\tilde{s}(z) + h(z) < m - \varepsilon$ on ∂K_0 .

Now consider the annulus $R_d := \{z \in W \mid d < |z - z_0| < d_0\}$, where $0 < d < d_0$, and the harmonic function given by

$$h_d(z) := \frac{\varepsilon}{\ln \frac{d}{d_0}} \cdot \ln \left(\left(\frac{d}{d_0} \right)^{\frac{m}{\varepsilon}} \cdot \frac{|z - z_0|}{d} \right)$$

It is clear that

$$h_d(z) = \begin{cases} m & \text{for} \quad |z - z_0| = d \\ m - \varepsilon & \text{for} \quad |z - z_0| = d_0 \end{cases}$$

so that $\tilde{s}(z) + h(z) \leq h_d(z)$ on ∂R_d and because $z_0 \notin R_d$ and $\tilde{s} + h$ is subharmonic on $D \setminus \{z_0\}$ it follows that we have the inequality $\tilde{s}(z) + h(z) \leq h_d(z)$ for all $z \in R_d$.

But on $K_0 \setminus \{z_0\}$ the function h_d converges uniformly on compact subsets to $m - \varepsilon$, for $d \longrightarrow 0$, so that we have on $K_0 \setminus \{z_0\}$ that $\tilde{s}(z) + h(z) \leq m - \varepsilon$ holds. This implies that $\tilde{s}(z_0) + h(z_0) \leq m - \varepsilon$. But this last inequality is not compatible with the equality $\tilde{s}(z_0) + h(z_0) = m$. Getting a contradiction it follows that the assumption we made that $\tilde{s}$ is not subharmonic is false.

In order to show that $\tilde{s}$ is the unique subharmonic extension of s we apply the following fact: for each plurisubharmonic function $f : X \longrightarrow [-\infty, \infty)$ on an irreducible complex space X, and for every point $x_0 \in X$ the equality

$$\limsup_{\substack{x \longrightarrow x_0 \\ (x \neq x_0)}} f(x) = f(x_0)$$

holds.

Indeed, we have the following sequence of inequalities:

$$f(x_0) \leq \limsup_{\substack{x \longrightarrow x_0 \\ (x \neq x_0)}} f(x) \leq f(x_0)$$

the first inequality being justified by the Maximum principle for plurisubharmonic functions (remember that we work on an irreducible space) and the last one is given by the upper semicontinuity of f.

With this the lemma is proved. □

Proof of Proposition 6.0.2. In what follows we adapt for our situation the proof of Theorem 1, p. 254 given in [VV].
In our situation – that one of a finite mapping between compact spaces – we can choose two finite coverings $(U_j)_{j \in J}$ of X and $(V_i)_{i \in I}$ of Y with the following properties:

(i) On each open set V_i there exists a strongly plurisubharmonic function $\psi_i : V_i \longrightarrow \mathbb{R}$ such that (V_i, ψ_i) defines a Kähler metric on Y.

(ii) Every set U_j is a connected component of some $f^{-1}(V_i)$.

(iii) On each open set U_j there exists a smooth, strongly plurisubharmonic function $\varphi_j : U_j \longrightarrow (0, \infty)$. (This can be achieved since each complex space has a neighbourhood base of Stein open sets.)

We now define a function

$$\delta : J \longrightarrow I$$

by

$$\delta(j) = i$$

where i is such that U_j is a connected component of $f^{-1}(V_i)$.
Consider a shrinking with compact sets $(K_i)_{i \in I}$ of the covering $(V_i)_{i \in I}$, that is: each set K_i is compact, $K_i \subset V_i$ and $\bigcup_{i \in I} K_i = Y$. We can also suppose that if we denote by $\tilde{K}_j$ the connected component of the compact set $f^{-1}(K_{\delta(j)})$ which is included in U_j,

we have also $\bigcup_{j\in J}\tilde{K}_j = X$. Choose now functions $\varrho_i \in \mathcal{C}_0^\infty(V_i)$, $\varrho_i \geq 0$, $\varrho_i = 1$ on a neighbourhood of K_i.

Then consider for each index $j \in J$ the real function defined on X by

$$\theta_j := \begin{cases} (\varrho_{\delta(j)} \circ f)^2 & \text{on} \quad U_j \\ 0 & \text{on} \quad X \setminus U_j. \end{cases}$$

It follows that the function $\theta_j\varphi_j$ is $\mathcal{C}^\infty$ on X with compact support in U_j.

Define then for any collection $\varepsilon = (\varepsilon_j)_{j\in J}$ of positive numbers the following smooth functions:

$$\varphi_\varepsilon : X \longrightarrow (0, \infty)$$

given by

$$\varphi_\varepsilon(x) := \sum_{j\in J} \varepsilon_j \theta_j(x)\varphi_j(x)$$

Define then on U_k the function

$$p_k(x) := p_{k,\varepsilon}(x) := \psi_{\delta(k)} \circ f(x) + \varphi_\varepsilon(x) \quad \text{for} \quad k \in J.$$

One sees at once that $p_k \in \mathcal{C}^\infty(U_k, \mathbb{R})$ and that $p_l - p_k = (\psi_{\delta(l)} - \psi_{\delta(k)}) \circ f$, so $p_l - p_k$ is pluriharmonic on $U_l \cap U_k$.

We want to prove that each p_k is strongly plurisubharmonic on an open set W_k with the property that $\operatorname{supp}\theta_k \supset W_k \supset \tilde{K}_k$. Because $(W_k)_{k\in J}$ is then a covering of X we will get the collection $(W_k, p_k)_{k\in J}$ which defines a Kähler metric on X.

Because strongly plurisubharmonicity is a local property it is enough to show that p_k is strongly plurisubharmonic in a neighbourhood of each point $x_0 \in \operatorname{supp}\theta_k$.

So, fix an arbitrary point $x_0 \in \operatorname{supp}\theta_k$ and choose sufficiently small neighbourhoods $U'_{x_0} \subset\subset U''_{x_0}$ of x_0, i.e. with the property that $U''_{x_0} \subset \bigcap_{l\in J_{x_0}} U_l$ where $J_{x_0} := \{l \in J \mid x_0 \in \operatorname{supp}\theta_l\}$.

We can also suppose that we have two neighbourhoods $V'_{f(x_0)} \subset\subset V''_{f(x_0)}$ of $f(x_0)$ such that $f(U'_{x_0}) \subset V'_{f(x_0)} \subset \bigcap_{l\in I_{f(x_0)}} V_l$, where $I_{f(x_0)} = \{i \in I \mid f(x_0) \in V_i\}$, and that $\tilde{U}_{x_0} \subset \mathbb{C}^n$, $\tilde{V}_{f(x_0)} \subset \mathbb{C}^m$ are open euclidian sets such that $U''_{x_0} \subset \tilde{U}_{x_0}$ and $V''_{f(x_0)} \subset \tilde{V}_{f(x_0)}$ are embedded closed analytic subsets such that $f_{|U''_{x_0}}$ extends as a holomorphic function to $\tilde{f}_{\tilde{U}_{x_0}}$. Thus we have the following commutative diagram

$$\begin{array}{ccc} \tilde{U}_{x_0} & \overset{\tilde{f}_{\tilde{U}_{x_0}}}{\dashrightarrow} & \tilde{V}_{f(x_0)} \\ \uparrow & & \uparrow \\ U''_{x_0} & \overset{f_{|U''_{x_0}}}{\longrightarrow} & V''_{f(x_0)} \end{array}$$

Without loss of generality we can also assume (by eventually shrinking U''_{x_0} resp. $V''_{f(x_0)}$) that each function $\varphi_{l_{|U''_{x_0}}}$ where $l \in J_{x_0}$ extends as a strongly plurisubharmonic function to the open subset $\tilde{U}_{x_0}$ in $\mathbb{C}^n$, and that $\varrho_{\delta(l)}|_{V''_{f(x_0)}}$ extends as a smooth function resp. $\psi_{\delta(l)}|_{V''_{f(x_0)}}$ extends as a strongly plurisubharmonic function to the open set $\tilde{V}_{f(x_0)}$ in $\mathbb{C}^m$.

These considerations show us that it is enough to consider the case when U''_{x_0} and $V''_{f(x_0)}$ are euclidian open sets of dimension n resp. m. We will suppose this in what follows.

Then we can find for each index $j \in J_{x_0}$ and $k \in J$ positive constants $A_{\delta(k)}$, B_j, C_j, D_j such that for each $x \in U'_{x_0}$, $y = f(x)$, for each $\eta \in \mathbb{C}^n$ and for each vector $\xi = (\xi_1, \ldots, \xi_m)$ given by $\xi_s = \partial f_s(x) \cdot \eta$, for $s = 1, \ldots, m$ (where $\partial f_s(x)$ denotes as usual the vector $\left(\frac{\partial f_s}{\partial z_1}(x), \ldots, \frac{\partial f_s}{\partial z_n}(x)\right)$) the following inequalities hold:

(i) $\mathcal{L}(\psi_{\delta(k)}, y)\xi \geq A_{\delta(k)} \|\xi\|^2$

(ii) $\mathcal{L}(\varphi_j, x)\eta \geq B_j \|\eta\|^2$

(iii) $|\varphi_j(x) \cdot \mathcal{L}(\varrho_{\delta(j)}, y)\xi| \leq C_j \|\xi\|^2$

(iv) $|\operatorname{Re}((\partial \varrho_{\delta(j)}(y)\xi) \cdot \overline{(\partial \varphi_j(x)\eta)})| \leq D_j \|\xi\| \cdot \|\eta\|$

where $\|\cdot\|$ denotes the euclidean norm and $\mathcal{L}(\cdot,\cdot)$ denotes the Levi form.

For the existence of $A_{\delta(k)}$ and B_j one uses the strongly plurisubharmonicity of $\psi_{\delta(k)}$ and of φ_j and the fact that U'_{x_0} and $V'_{f(x_0)}$ are relatively compact.

The Levi form of p_k gives by computation:

$$\begin{aligned}\mathcal{L}(p_k, x)\eta &= \mathcal{L}(\psi_{\delta(k)}, y)\xi \\ &\quad + \sum_{j \in J} 2\varepsilon_j \varphi_j(x) \left(\left| \partial \varrho_{\delta(j)}(y)\xi \right|^2 + \varrho_{\delta(j)}(y) \mathcal{L}(\varrho_{\delta(j)}, y)\xi \right) \\ &\quad + \sum_{j \in J} \varepsilon_j {\varrho_{\delta(j)}}^2(y) \mathcal{L}(\varphi_j, x)\eta + \sum_{j \in J} 4\varepsilon_j \varrho_{\delta(j)}(y) \operatorname{Re}\left((\partial \varrho_{\delta(j)}(y)\xi) \overline{(\partial \varphi_j(x)\eta)} \right)\end{aligned}$$

We can estimate it in the following way:

$$\mathcal{L}(p_k, x)\eta \geq A\|\xi\|^2 + B\|\eta\|^2 - C\|\xi\|^2 - 2D\|\xi\| \cdot \|\eta\|$$

where we have denoted

$$\begin{aligned} A &= A_{\delta(k)} \\ B &= \sum_{j \in J} \varepsilon_j B_j \varrho^2_{\delta(j)}(y) \\ C &= \sum_{j \in J} 2\varepsilon_j C_j \varrho_{\delta(j)}(y) \\ D &= \sum_{j \in J} 2\varepsilon_j D_j \varrho_{\delta(j)}(y) \end{aligned}$$

If we will prove that we have

$$(A - C)B > D^2 \tag{6.1}$$

then it will follow that $\mathcal{L}(p_k, x)\eta$ is positive unless $\eta = 0$, which would complete the proof.

In order to prove (6.1) choose $\delta_1 > 0$ small enough such that $C \leq A/2$ for each $0 \leq \varepsilon_j \leq \delta_1$, which is possible because A does not depend on ε. By using the Schwarz inequality one obtains for $0 \leq \varepsilon_j \leq \delta_1$ that

$$(A - C)B \geq \frac{A}{2} \sum_{j \in J} \varepsilon_j B_j \varrho^2_{\delta(j)}(y) \geq \tilde{A} \left(\sum_{j \in J} \sqrt{\varepsilon_j B_j} \varrho_{\delta(j)}(y) \right)^2$$

where $\tilde{A}$ is a constant. But because $\sqrt{\varepsilon_j B_j} \geq \varepsilon_j \sqrt{B_j/\delta_1}$, by suitable choice of $\delta_1 > 0$ one obtains $\tilde{A}^2 \sqrt{B_j/\delta_1} > 2D_j$ and so the desired inequality (6.1).

Hence the proof is complete. X turns out to be indeed Kähler. $\square$

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