On some homological conjectures for finite dimensional algebras

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Denis Skorodumov

aus Mažejkjaj
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Contents

Introduction

Free resolutions and syzygies occur already in the fundamental article "Über die Theorie der algebraischen Formen" published by Hilbert in 1890. Later on during the formalization of homological algebra in the 1950's one studied the slightly more general projective resolutions. This lead to striking results in commutative algebra like the characterization of regular local noetherian rings by Auslander, Buchsbaum and Serre.

For finite dimensional algebras and their representations the homological point of view motivated the invention of almost split sequences - which are of central importance nowadays - but it also produced some interesting homological conjectures. Two of these, the finitistic dimension conjectures and the various no loop conjectures, are the theme of this dissertation.

For simplicity, we will always work with a finite dimensional associative algebra Λ over an algebraically closed field $\mathbf{k}$. We will deal with finite dimensional right Λ-modules except otherwise stated. By Morita-equivalence and by an observation of Gabriel we can assume that Λ is isomorphic to $\mathbf{k}\,\mathcal{Q}\,/I$ for some quiver $\mathcal{Q}$ with path algebra $\mathbf{k}\,\mathcal{Q}$ and some two-sided ideal I generated by linear combinations of paths of length at least two.

The (little) **finitistic dimension** $\operatorname{findim}\Lambda$ of Λ is defined to be the supremum of the projective dimensions of all finitely generated modules of finite projective dimension. Similarly, the big finitistic dimension $\operatorname{Findim}\Lambda$ is defined allowing arbitrary right Λ-modules. In 1960, Bass [2] formulated two so-called **finitistic dimension conjectures**. The first one asserts that

$$\operatorname{findim}\Lambda = \operatorname{Findim}\Lambda\ ,$$

while the second one claims that

$$\operatorname{findim}\Lambda < \infty\ .$$

A counterexample to the first conjecture was first given by Zimmerman-Huisgen in [14]. Later on Smalø [22] gave another example showing in addition that the difference $\operatorname{Findim}\Lambda - \operatorname{findim}\Lambda$ can be arbitrarily large. We reproduce Smalø's example in chapter 6 thereby making some of his arguments more transparent.

The second finitistic dimension conjecture is still open and we refer to it as the finitistic dimension conjecture. It is known for:

- algebras Λ where the subcategory of all modules with finite projective dimension is contravariantly finite in mod-Λ (by Auslander, Reiten [1]),
- monomial algebras (by Green, Kirkman, Kuzmanovich [10]),
- algebras where the cube of the radical is zero (by Green, Huisgen-Zimmermann [12]),
- algebras of representation dimension at most three (by Igusa, Todorov [16]),
- some special kinds of algebras (by Wang [23] and Xi [24, 25, 26, 27]).

This conjecture is also related to many other homological conjectures (e.g., the Gorenstein symmetry conjecture, the Wakamatsu tilting conjecture and the generalized Nakayama conjecture).

As our main result in that direction we construct in chapter 5 from a given algebra $\Lambda = \mathbf{k}\,\mathcal{Q}/I$ a new algebra $\widetilde{\Lambda} = \mathbf{k}\,\widetilde{\mathcal{Q}}/\widetilde{I}$ with $\operatorname{findim}\Lambda \le \operatorname{findim}\tilde{\Lambda} \le \operatorname{findim}\Lambda + 2$ such that $\widetilde{\mathcal{Q}}$ has neither multiple arrows nor loops. $\tilde{\tilde{\Lambda}}$ is then called **single-arrowed** and it suffices to prove the finitistic dimension conjecture for single-arrowed algebras.

Let $\Lambda = \mathbf{k}\,\mathcal{Q}/I$ be given with a point x in $\mathcal{Q}$. Denote the corresponding simple by S_x. The **strong no loop conjecture** is due to Zacharia [15]. It says that there is no loop at x provided the projective dimension $\operatorname{pdim}_\Lambda S_x$ is finite.

The conjecture is known for

- monomial algebras (by Igusa [15]),
- truncated extensions of semi-simple rings (by Marmaridis, Papistas [20]),
- bound quiver algebras $k\,\mathcal{Q}/I$ such that for each loop $\alpha \in \mathcal{Q}$ there exists an $n \in \mathbb{N}$ with $\alpha^n \in I \setminus (IJ + JI)$, where J denotes the ideal generated by the arrows (by Green, Solberg, Zacharia [11]),
- special biserial algebras (by Liu, Morin [19]),
- two point algebras with radical cube zero (by Jensen [17]).

The **stronger no loop conjecture** says that $\operatorname{Ext}^i(S_x, S_x) \neq 0$ for infinitely many indices i. In [11] and [19] this stronger assertion was proven for the cases where the loop behaves as in the third case above.

Note that the existence of a loop at x just means $\operatorname{Ext}^1(S_x, S_x) \neq 0$. This implies easily $\operatorname{Ext}^2(S_x, S_x) \neq 0$, but $\operatorname{Ext}^3(S_x, S_x) = 0$ occurs for a representation-finite example of Happel stated in [11, Section 4].

Observe that the finitistic dimension conjecture is obviously true for representation-finite algebras, whereas the truth of the strong no loop conjecture is not clear for these algebras. In chapter 3 we will prove it for an even bigger class of algebras containing all representation-finite algebras. The author has published this in [21].

To state the main result precisely we introduce for any point x in $\mathcal{Q}$ its **neighborhood** $\Lambda(x) = e\,\Lambda\,e$. Here e is the sum of all primitive idempotents $e_z \in \Lambda$ such that z belongs to the support of the projective $P_x := e_x\,\Lambda$ or such that there is an arrow $z \to x$ in $\mathcal{Q}$ or a configuration $y' \leftarrow x \rightleftarrows y \leftarrow z$ with 4 pairwise distinct points x, y, y' and z.

Recall that an algebra Λ is called **distributive** if it has a distributive lattice of two-sided ideals and **mild** if it is distributive and any proper quotient $\Lambda\,/J$ is representation-finite.

Our main result reads as follows:

Theorem 1
Let $\Lambda = \mathbf{k}\,\mathcal{Q}\,/I$ be a finite dimensional algebra over an algebraically closed field $\mathbf{k}$. Let x be a point in $\mathcal{Q}$ such that the corresponding simple Λ-module S_x has finite projective dimension. If $\Lambda(x)$ is mild, then there is no loop at x.

Of course, it follows immediately that the strong no loop conjecture holds for all mild algebras, in particular for all representation-finite algebras.

Corollary 2
Let Λ be a mild algebra over an algebraically closed field. Let S be a simple Λ-module. If the projective dimension of S is finite, then $\mathrm{Ext}^1_\Lambda(S,S) = 0$.

To prove the theorem we do not look at projective resolutions. Instead, in chapter 2, we slightly refine the K-theoretic arguments of Lenzing [18, Satz 5], also used by Igusa in his proof of the strong no loop conjecture for monomial algebras [15, Corollary 6.2].

Chapter 4 is devoted to show that for a representation-finite algebra Λ the stronger no loop conjecture is invariant under passing to the standard form $\overline{\Lambda}$ of Λ. Moreover we prove that the extensions of the simple Λ-modules coincide with the extensions of the corresponding simple $\overline{\Lambda}$-modules.

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Chapter 1

Basic notations and facts

1.1 Preliminaries

Throughout this paper, $\mathbf{k}$ stands for an algebraically closed field. Let $\mathcal{Q}$ be a finite quiver with $\mathcal{Q}_0 = \{1, \dots, n\}$ the set of points and $\mathcal{Q}_1 = \{\alpha_1, \dots, \alpha_r\}$ the set of arrows in $\mathcal{Q}$. The starting point of an arrow α_i is denoted by $s(\alpha_i)$ and its ending point is $e(\alpha_i)$. Let $\mathbf{k}\,\mathcal{Q}$ be the path algebra of $\mathcal{Q}$ over $\mathbf{k}$, this is the vector space over $\mathbf{k}$ having as a basis W the set of all directed paths in $\mathcal{Q}$. For any point i in $\mathcal{Q}$ there is a path e_i of length 0 such that $\sum_{i \in \mathcal{Q}_0} e_i = 1_{\mathbf{k}\,\mathcal{Q}}$.

The multiplication in $\mathbf{k}\,\mathcal{Q}$ is the concatenation of paths if the starting and ending points match. Since we work with right modules, the product of two paths v and w with $e(v) = s(w)$ is vw and zero otherwise. An element $w \in \mathbf{k}\,\mathcal{Q}$ is called uniform if $w = e_i w e_j$ for some suitable idempotents e_i, e_j.

An ideal $I \subset \mathbf{k}\,\mathcal{Q}$ is admissible if $\mathbf{k}\,\mathcal{Q}^{+m} \subset I \subset \mathbf{k}\,\mathcal{Q}^{+2}$ for some $m \geq 2$. Here $\mathbf{k}\,\mathcal{Q}^{+m}$ is the $\mathbf{k}$-vector space with basis W^m, the set of all paths of length at least m. The algebra $\mathbf{k}\,\mathcal{Q}\,/I$ is finite dimensional and associative.

Let $w_1, \dots, w_r$ be pairwise distinct paths of $\mathcal{Q}$ from a vertex x to a vertex y, and let $\lambda_1, \dots, \lambda_r \in \mathbf{k}$ be non-zero scalars. We call

$$\rho = \sum_{i=1}^{r} \lambda_i \, w_i$$

a relation on $\mathcal{Q}$ if $\rho \in I$ while $\sum_{i \in N}^{r} \lambda_i \, w_i \notin I$ for all proper subsets $N \subset \{1, \dots, r\}$. In this case $w_1, \dots, w_r$ are called the components of ρ. Moreover, ρ is called monomial or polynomial if $r = 1$ or $r \geq 2$ respectively.

The quotient $\Lambda = \mathbf{k}\,\mathcal{Q}\,/I$ is called the algebra of the bound quiver $(\mathcal{Q}, I)$. Let mod-Λ denote the category of finite dimensional right Λ-modules. It is well known that mod-Λ is equivalent to the category $\mathrm{rep}_{\mathbf{k}}(\mathcal{Q}, I)$ of $\mathbf{k}$-linear representations of the bound quiver $(\mathcal{Q}, I)$.

The radical, the top and the socle of an Λ-module M will be denoted by $\operatorname{rad} M, \operatorname{top} M$ and $\operatorname{soc} M$. We call $m \in M$ a top element of M if $m \notin \operatorname{rad} M$.

The indecomposable projective Λ-modules are $P_i = e_i \Lambda$ for $i \in \mathcal{Q}_0$. The simple modules are $S_i = P_i / \operatorname{rad} P_i$. For every module $M \in \text{mod-}\Lambda$ there exists a projective module P_M and an epimorphism $\pi : P_M \to M$ with kernel contained in the radical of P_M; this module is called the projective cover of M. It is unique up to isomorphism.

A projective resolution of an Λ-module M is an exact sequence of projective Λ-modules together with an epimorphism $d_0 : P_0 \to M$:

$$\ldots \to P_n \stackrel{d_n}{\to} P_{n-1} \stackrel{d_{n-1}}{\to} \ldots \stackrel{d_3}{\to} P_1 \stackrel{d_1}{\to} P_0 \stackrel{d_0}{\to} M \to 0 \ .$$

A projective resolution is called minimal if $d_j : P_j \to \operatorname{Im} d_j$ is a projective cover for all $j \geq 0$. By $\Omega(M)$ we denote the kernel of a projective cover $d_0 : P_0 \to M$ and call it the first syzygy of M. The higher syzygies are defined inductively; $\Omega^{i+1}(M) = \ker d_i$ if d_i is the i'th morphisms in a minimal projective resolution of M.

Every $\mathbf{k}$-algebra Λ we consider as a $\mathbf{k}$-category which has the idempotents $e_i \in \Lambda$ as objects and the $e_i \Lambda e_j$ as the morphism spaces. Then a finite dimensional Λ-module M is a covariant $\mathbf{k}$-linear functor from Λ to the category of $\mathbf{k}$-vectorspaces such that the sum of the dimensions of all $M(e_i)$ is finite.

The modules arising in our examples can be represented by directed graphs of a format which is intuitively suggestive. We use the conventions from [13]. For the convenience of the reader we briefly review the construction on a simple example.

Example 1.1.1
Let $\Lambda = \mathbf{k}\, \mathcal{Q} / I$ be an algebra of a bound quiver $(\mathcal{Q}, I)$ as follows.

$$\mathcal{Q} = \begin{tikzcd} x \arrow[loop left, "\alpha"] \arrow[r, bend left, "\beta"] & y \arrow[l, bend left, "\gamma"] \end{tikzcd}$$

A generating set for the relation ideal I can be communicated by way of graphs of the indecomposable projective right Λ-modules. Presenting $M = e_x \Lambda$ by way of the directed graph:

$$\begin{tikzcd} e_x \arrow[d] \arrow[dr] & \\ \alpha \arrow[d] \arrow[dr] & \beta \arrow[dl, "\gamma"'] \\ \alpha^2 & \alpha\beta \end{tikzcd}$$

holds the following information. The right ideal $e_x I \subseteq \mathbf{k}\,\mathcal{Q}$ is generated by $\alpha^3, \alpha^2\beta$ and $\lambda\alpha^2 + \beta\gamma$ for some $\lambda \in \mathbf{k}\setminus\{0\}$. Moreover, $\operatorname{top} M = M/\operatorname{rad} M \cong S_x$, $\operatorname{rad} M/\operatorname{rad}^2 M \cong S_x \oplus S_y \cong \operatorname{rad}^2 M$ and $\operatorname{rad}^3 M = 0$.

1.2 Calculating syzygies is worthless

Some obvious method to prove the strong no loop conjecture is detecting repetitions in the sequence $\Omega(S)$,$\Omega^2(S)$,$\Omega^3(S), \ldots$ of syzygies of a simple Λ-module S. The objective of this section is to give, for each natural number n, an example of a finite dimensional $\mathbf{k}$-algebra Λ_n such that the sequence of syzygies of a simple Λ_n-module with non-trivial self extensions becomes periodic at the $2n+2$'th step for the first time. Since the algebras Λ_n are of finite representation type with radical cube zero it appears to have no prospect of success to prove the strong no loop conjecture by this method even for such 'simple' algebras. This example emerged at the investigation of projective resolutions for representation-finite algebras with radical cube zero. As a result which we won't prove here it came out that the stronger no loop conjecture is true for such algebras.

The algebra Λ_n is given as a bound quiver algebra. Namely, let $\mathcal{Q}_n$ be the following quiver:

$$\begin{tikzcd}
1 \arrow[loop left, "\alpha"] \arrow[r, bend left, "\beta_2"] & 2 \arrow[l, bend left, "\gamma_1"] \arrow[r, bend left, "\beta_3"] & 3 \arrow[l, bend left, "\gamma_2"] \arrow[r, bend left, "\beta_4"] & 4 \arrow[l, bend left, "\gamma_3"] \cdots\cdots n \arrow[r, bend left, "\beta_{n+1}"] & n+1 \arrow[l, bend left, "\gamma_n"]
\end{tikzcd}$$

let $\mathbf{k}$ be any field, and $\Lambda_n = \mathbf{k}\,\mathcal{Q}_n / I_n$ with relation ideal I_n such that the indecomposable projective Λ_n-modules are given by the following graphs:

$$P_1: \begin{tikzcd}
e_1 \arrow[d] \arrow[dr] & \\
\alpha \arrow[d] & \beta_2 \arrow[dl] \\
\alpha^2 &
\end{tikzcd}
\qquad
P_{n+1}: \begin{tikzcd}
& e_{n+1} \arrow[dl] \\
\gamma_n \arrow[dr] & \\
& \gamma_n\beta_{n+1}
\end{tikzcd}
\qquad
P_i: \begin{tikzcd}
& e_i \arrow[dl] \arrow[dr] & \\
\gamma_{i-1} \arrow[dr] & & \beta_{i+1} \arrow[dl] \\
& \beta_{i+1}\gamma_i &
\end{tikzcd}
\qquad i = 2, \ldots, n$$

Consider the Λ_n-modules V_i, W_j, X given by the graphs:

$$V_0 = W_0: \ \alpha^2 \qquad
V_1: \begin{tikzcd}
\alpha \arrow[d] & \beta_2 \arrow[dl] \\
\alpha^2 &
\end{tikzcd}
\qquad
W_1: \begin{tikzcd}
\alpha+\gamma_1 \arrow[d] \arrow[dr] & \\
\alpha^2 & \gamma_1\beta_2
\end{tikzcd}$$

$$V_i: \begin{tikzcd}
\alpha+\gamma_1 \arrow[d] \arrow[dr] & \beta_2+\gamma_2 \arrow[dl] \arrow[dr] & \beta_3+\gamma_3 \arrow[dl] \cdots\cdots & \beta_{i-1}+\gamma_{i-1} \arrow[dr] & \beta_i \arrow[dl] & \beta_{i+1} \arrow[dll] \\
\alpha^2 & \gamma_1\beta_2 & \gamma_2\beta_3 \ \cdots\cdots & \gamma_{i-2}\beta_{i-1} & \gamma_{i-1}\beta_i &
\end{tikzcd}$$

$$
\begin{tikzcd}[column sep=small]
W_j: \ \alpha+\gamma_1 \arrow[d] \arrow[dr] & \beta_2+\gamma_2 \arrow[dl] \arrow[dr] & \beta_3+\gamma_3 \arrow[dl] & \cdots\cdots\cdot & \beta_{j-1}+\gamma_{j-1} \arrow[dr] & \beta_j+\gamma_j \arrow[dl] \arrow[dr] & \\
\alpha^2 & \gamma_1\beta_2 & \gamma_2\beta_3 & \cdots\cdots\cdot & \gamma_{j-2}\beta_{j-1} & \gamma_{j-1}\beta_j & \gamma_j\beta_{j+1}
\end{tikzcd}
$$

for $i, j = 2, \ldots, n$.

$$
\begin{tikzcd}[column sep=small]
X: \ \alpha+\gamma_1 \arrow[d] \arrow[dr] & \beta_2+\gamma_2 \arrow[dl] \arrow[dr] & \beta_3+\gamma_3 \arrow[dl] & \cdots\cdots\cdot & \beta_{n-1}+\gamma_{n-1} \arrow[dr] & \beta_n+\gamma_n \arrow[dl] \arrow[dr] & \beta_{n+1} \arrow[dl] \\
\alpha^2 & \gamma_1\beta_2 & \gamma_2\beta_3 & \cdots\cdots\cdot & \gamma_{n-2}\beta_{n-1} & \gamma_{n-1}\beta_n & \gamma_n\beta_{n+1}
\end{tikzcd}
$$

These modules are indecomposable. They are pairwise not isomorphic except $W_0 = V_0$. An easy computation shows that $\Omega(V_i) = V_{i+1}$ for $i = 0, \ldots, n-1$; $\Omega(W_j) = W_{j-1}$ for $j = 1, \ldots, n$; $\Omega(V_n) = X$, $\Omega(X) = W_n$. Hence $\Omega^{2n+2}(S_1) \cong S_1$ since $S_1 \cong V_0 = W_0$.

Chapter 2

Lenzing's result

Let R be a ring with 1_R and mod- R the category of all finitely generated right R-modules. Let $\mathcal{P}(R)$ be the full subcategory of projective R-modules and denote by $\mathcal{P}^{\infty}(R)$ the full subcategory in mod- R consisting of modules of finite projective dimension.

In this chapter we present some slight generalization of a result of Lenzing [18] giving us a tool at hand to prove the strong no loop conjecture for some special types of algebras. The great advantage of this tool is that we don't have to calculate any projective resolutions in order to make assertions about the projective dimension of the simple modules in view.

Following Igusa [15] we introduce the relative K-theory groups $K_1(R)$ resp. $K_1'(R)$ first.

2.1 The relative K-theory group $K_1(R)$

Definition 2.1.1
Let $K_1(R)$ resp. $K_1'(R)$ be the additive group given by generators and relations as follows. The generators are pairs (M, f) where $M \in \mathcal{P}(R)$ resp. $M \in \mathcal{P}^{\infty}(R)$ and $f \in \operatorname{End}_R(M)$. The relations are:

a) $(M, f) + (M, g) = (M, f + g)$,

b) $(M, f) + (N, g) = (L, h)$ *for every commutative diagram with exact rows:*

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r] \arrow[d, "f"] & L \arrow[r] \arrow[d, "h"] & N \arrow[r] \arrow[d, "g"] & 0\, , \\
0 \arrow[r] & M \arrow[r] & L \arrow[r] & N \arrow[r] & 0
\end{tikzcd}$$

c) $(M, fg) = (N, gf)$ *for every sequence* $M \xrightarrow{f} N \xrightarrow{g} M \xrightarrow{f} N$.

As a main result it will turn out that in fact $K_1(R)$ and $K_1'(R)$ are both isomorphic to the zeroth Hochschild homology group $H_0(R)$ of R.

Lemma 2.1.2 *a)* $(M \oplus N, \begin{bmatrix} 0 & f \\ g & 0 \end{bmatrix}) = 0$ *in* $K_1(R)$ *resp.* $K_1'(R)$.

b) $(P, f) + (Q, g) = (P \oplus Q, \begin{bmatrix} f & 0 \\ 0 & g \end{bmatrix})$ *in* $K_1(R)$.

c) For $(P, f) \in K_1(R)$ *there exists* $r_f \in R$ *such that* $(P, f) = (R, \lambda_{r_f})$*. Here* $\lambda_r : R \to R$ *is the left multiplication with* r.

d) $(R, \lambda_r \lambda_{r'}) = (R, \lambda_{r'} \lambda_r)$ *for all* $r, r' \in R$*. Moreover* $(R, \lambda_{(rr'-r'r)}) = 0$ *in* $K_1(R)$.

Proof. a) Since

$$\begin{bmatrix} 0 & f \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & \mathrm{id}_N \end{bmatrix} = \begin{bmatrix} 0 & f \\ 0 & 0 \end{bmatrix}$$

and

$$\begin{bmatrix} 0 & 0 \\ 0 & \mathrm{id}_N \end{bmatrix} \begin{bmatrix} 0 & f \\ 0 & 0 \end{bmatrix} = 0 \ ,$$

we have by 2.1.1 c) :

$$(M \oplus N, \begin{bmatrix} 0 & f \\ 0 & 0 \end{bmatrix}) = 0 \ .$$

Hence $(M \oplus N, \begin{bmatrix} 0 & f \\ g & 0 \end{bmatrix}) = (M \oplus N, \begin{bmatrix} 0 & f \\ 0 & 0 \end{bmatrix}) + (M \oplus N, \begin{bmatrix} 0 & 0 \\ g & 0 \end{bmatrix}) = 0.$

b) The claim holds by 2.1.1 b) since every exact sequence ending in a projective module splits.

c) Since P is a projective R-module, there is a complement Q and an isomorphism $\alpha : P \oplus Q \to R^n$. Now we set

$$f \oplus 0_Q := \begin{bmatrix} f & \\ & 0_Q \end{bmatrix} : P \oplus Q \to P \oplus Q \ .$$

Hence $\alpha^{-1}(f \oplus 0_Q)\alpha : R^n \to R^n$ is represented by a left multiplication with an $n \times n$ matrix $[r_{ij}]_{i,j}$ having entries in R. Using 2.1.1 a,b,c) and 2.1.2 a) we derive

$$(P, f) = (P \oplus Q, f \oplus 0_Q) = (R^n, \alpha^{-1}(f \oplus 0_Q)\alpha) = (R^n, [\lambda_{r_{ij}}]_{i,j}) = (R, \sum_{i=1}^{n} \lambda_{r_{ii}}) \ .$$

Thus $r_f = \sum_{i=1}^{n} r_{ii}$ does the job.

d) The claim is trivial.

□

By $H_0(R)$ we denote the zeroth Hochschild homology group of R. It is well known that $H_0(R) = R/[R,R]$ is the quotient of R by the additive subgroup $[R,R]$ generated by all elements of the form $rs - sr$ where $r, s \in R$.

Definition 2.1.3
Let the trace map $\mathrm{Tr} : K_1(R) \to H_0(R)$ *be defined as follows:*

i) For $f : R^n \to R^n$, $f = (f_{ij})$ *we define* $\mathrm{tr}(f) := \sum_{i=1}^n f_{ii}(1_R)$. *Then* $\mathrm{Tr}(R^n, f) := \overline{\mathrm{tr}(f)}$ *is the residue class of* $\mathrm{tr}(f)$ *in* $H_0(R)$.

ii) For $f : F \to F$, $\alpha : F \xrightarrow{\sim} R^n$ *we define* $\mathrm{Tr}(F, f) := \mathrm{Tr}(R^n, \alpha^{-1} f \alpha)$.

iii) For $f : P \to P$, $P \oplus Q \simeq R^n$ *we define* $\mathrm{Tr}(P, f) := \mathrm{Tr}(P \oplus Q, f \oplus 0_Q)$.

Lemma 2.1.4
$Tr : K_1(R) \to H_0(R)$ *is a well defined homomorphism.*

Proof. Let $\alpha : P \oplus Q \to R^n$ and $\beta : P \oplus Q' \to R^m$ be isomorphisms. Without loos of generality we can assume that $m = n$ and $Q = Q'$ since

$$\begin{aligned} \mathrm{tr}(\alpha^{-1}(f \oplus 0_Q)\alpha) &= \mathrm{tr}(\alpha^{-1}(f \oplus 0_Q)\alpha \oplus 0_{R^k}) \\ &= \mathrm{tr}((\alpha^{-1} \oplus \mathrm{id}_{R^k})(f \oplus 0_Q \oplus 0_{R^k})(\alpha \oplus \mathrm{id}_{R^k}))\ . \end{aligned}$$

It is well known that for matrices $A, B \in R^{n\times n}$ one has $\mathrm{tr}(AB) = \mathrm{tr}(BA)$ modulo $[R,R]$. Hence

$$\mathrm{tr}(\beta^{-1}(f \oplus 0_Q)\beta) = \mathrm{tr}((\alpha^{-1}\beta)(\beta^{-1}(f \oplus 0_Q)\beta)(\beta^{-1}\alpha)) = \mathrm{tr}(\alpha^{-1}(f \oplus 0_Q)\alpha)$$

modulo $[R,R]$.

i) $\mathrm{Tr}(P, f) + \mathrm{Tr}(P, g) = \mathrm{Tr}(P, f + g)$ holds since tr is additive.

ii) If we have a commutative diagram with exact rows:

$$\begin{tikzcd} 0 \arrow[r] & P \arrow[r] \arrow[d, "f"] & T \arrow[r] \arrow[d, "h"] & Q \arrow[r] \arrow[d, "g"] & 0\ , \\ 0 \arrow[r] & P \arrow[r] & T \arrow[r] & Q \arrow[r] & 0 \end{tikzcd}$$

then $T = P \oplus Q$ and $h = f \oplus g$. Moreover there are complements P' and Q' with $\alpha : P \oplus P' \oplus Q \oplus Q' \xrightarrow{\sim} R^n$. For

$$\gamma := \begin{bmatrix} \mathrm{id}_P & & & \\ & 0 & \mathrm{id}_Q & \\ & \mathrm{id}_{P'} & 0 & \\ & & & \mathrm{id}_{Q'} \end{bmatrix} : P \oplus Q \oplus P' \oplus Q' \xrightarrow{\sim} P \oplus P' \oplus Q \oplus Q'$$

we have

$$\gamma^{-1}(f \oplus g \oplus 0_{P'} \oplus 0_{Q'})\gamma = (f \oplus 0_{P'} \oplus g \oplus 0_{Q'}) \ .$$

Now we derive

$$\begin{array}{rcl} \alpha^{-1}(f \oplus 0_{P'} \oplus 0_Q \oplus 0_{Q'})\alpha & & \\ +\alpha^{-1}(0_P \oplus 0_{P'} \oplus g \oplus 0_{Q'})\alpha & = & \alpha^{-1}(f \oplus 0_{P'} \oplus g \oplus 0_{Q'})\alpha \\ & = & \alpha^{-1}\gamma^{-1}(f \oplus g \oplus 0_{P'} \oplus 0_{Q'})\gamma\alpha \\ & = & (\gamma\alpha)^{-1}(h \oplus 0_{P'\oplus Q'})(\gamma\alpha) \ . \end{array}$$

Thus $\mathrm{Tr}(P,f) + \mathrm{Tr}(Q,g) = \mathrm{Tr}(T,h)$.

iii) Let $f : P \to Q$, $g : Q \to P$, $\alpha : P \oplus P' \xrightarrow{\sim} R^n$ and $\beta : Q \oplus Q' \xrightarrow{\sim} R^m$ be homomorphisms. Then modulo $[R,R]$ we have:

$$\begin{array}{rcl} \mathrm{tr}(\alpha^{-1}(fg \oplus 0_{P'})\alpha) & = & \mathrm{tr}(\alpha^{-1}(f \oplus 0)\beta\beta^{-1}(g \oplus 0)\alpha) \\ & = & \mathrm{tr}(\beta^{-1}(g \oplus 0)\alpha\alpha^{-1}(f \oplus 0)\beta) \\ & = & \mathrm{tr}(\beta^{-1}(gf \oplus 0_{Q'})\beta) \end{array}$$

Thus $\mathrm{Tr}(P,fg) = \mathrm{Tr}(Q,gf)$.

□

Theorem 2.1.5
For any ring R we have an isomorphism $Tr : K_1(R) \to H_0(R)$.

Proof. Tr is surjective since $\mathrm{Tr}(R,\lambda_r) = \overline{r} \in H_0(R)$ for all $r \in R$.
To show that Tr is injective let (P,f) satisfy $\mathrm{Tr}(P,f) = 0$. By Lemma 2.1.2 c) $(P,f) = (R,\lambda_{r_f})$, hence $\mathrm{tr}(\lambda_{r_f}) = r_f \in [R,R]$. Thus $(P,f) = 0$ in $K_1(R)$ holds by 2.1.2 d). □

2.2 Lenzing's Theorem

Theorem 2.2.1
For any ring R the inclusion functor $\mathcal{P}(R) \to \mathcal{P}^\infty(R)$ induces an isomorphism $K_1(R) \to K_1'(R)$.

Proof. Let $\phi : K_1(R) \to K_1'(R)$ be the homomorphism induced by the inclusion $\mathcal{P}(R) \to \mathcal{P}^\infty(R)$. We construct the inverse map $\psi : K_1'(R) \to K_1(R)$. First of all we define ψ as a map from the free additive group given by generators (M,f) with $\operatorname{pdim} M < \infty$ to $K_1(R)$. Let

$$P_* : 0 \to P_n \xrightarrow{d_n} P_{n-1} \to \ldots \to P_1 \xrightarrow{d_1} P_0 (\to M \to 0)$$

be a projective resolution of M. Given a morphism $f : M \to M$ we choose a lifting of f to a chain map $f_* : P_* \to P_*$. We define $\psi(M, f) := \sum_{i=0}^{n}(-1)^i(P_i, f_i)$. First we show that ψ is well defined.

i) If $g_* : P_* \to P_*$ is another lifting of f, then there are maps $h_i : P_i \to P_{i+1}$ such that
$$g_i = f_i + h_i d_{i+1} + d_i h_{i-1}.$$
Thus
$$\begin{aligned}
\sum_{i=0}^{n}(-1)^i(P_i, g_i) &= \sum_{i=0}^{n}(-1)^i((P_i, f_i) + (P_i, h_i d_{i+1}) + (P_i, d_i h_{i-1})) \\
&= \sum_{i=0}^{n}(-1)^i(P_i, f_i) + \sum_{i=0}^{n}(-1)^i(P_i, h_i d_{i+1}) - \sum_{i=-1}^{n-1}(-1)^i(P_{i+1}, d_{i+1} h_i) \\
&= \sum_{i=0}^{n}(-1)^i(P_i, f_i) + \sum_{i=0}^{n-1}(-1)^i((P_i, h_i d_{i+1}) - (P_{i+1}, d_{i+1} h_i)) \\
&\quad + (-1)^n(P_n, h_n d_{n+1}) + (P_0, d_0 h_{-1}) \\
&= \sum_{i=0}^{n}(-1)^i(P_i, f_i)
\end{aligned}$$

ii) If Q_* is another projective resolution of M let $\alpha_* : P_* \to Q_*$ be a lifting of id_M and $f_* : Q_* \to P_*$ a lifting of f. Then $\alpha_* f_*$ and $f_* \alpha_*$ are liftings of f. Actually we have $P_i \overset{\alpha_i}{\to} Q_i \overset{f_i}{\to} P_i \overset{\alpha_i}{\to} Q_i$ thus $(P_i, \alpha_i f_i) = (Q_i, f_i \alpha_i)$ holds for all $i = 0, \ldots, n$ and
$$\sum_{i=0}^{n}(-1)^i(P_i, \alpha_i f_i) = \sum_{i=0}^{n}(-1)^i(Q_i, f_i \alpha_i)\ .$$

Now we have to check that the relations given in definition of $K_1'(R)$ are in the kernel of ψ.

a) Obviously $f_* + g_*$ is a lifting of $f + g$ if $f_*,\ g_* : P_* \to P_*$ are liftings of $f,\ g$.

b) For a commutative diagram with exact rows:
$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r] \arrow[d, "f"] & L \arrow[r] \arrow[d, "h"] & N \arrow[r] \arrow[d, "g"] & 0 \\
0 \arrow[r] & M \arrow[r] & L \arrow[r] & N \arrow[r] & 0
\end{tikzcd}$$

let $f_* : P_* \to P_*$ resp. $g_* : Q_* \to Q_*$ be a lifting of f resp. g. It is well known [8, see V.2.3] that there exist $\gamma_i : Q_i \to P_i$ such that

$$h_* = \begin{bmatrix} f_* & 0 \\ \gamma_* & g_* \end{bmatrix} : P_* \oplus Q_* \to P_* \oplus Q_*$$

is a lifting of h. Thus $\psi(L,h) = \psi(M,f) + \psi(N,g)$ by Lemma 2.1.2.

c) Since a lifting of a composition $fg : M \to M$ is the composition f_*g_* of liftings $f_* : P_* \to P_*$, $g_* : Q_* \to Q_*$ the equality

$$\psi(M,fg) = \sum(-1)^i(P_i, f_ig_i) = \sum(-1)^i(Q_i, g_if_i) = \psi(N,gf)$$

holds.

Therefore ψ induces a morphism $\psi : K_1'(R) \to K_1(R)$. Trivially $\psi \circ \phi = \mathrm{id}_{K_1(R)}$ holds, thus it remains to verify that $\phi \circ \psi = \mathrm{id}_{K_1'(R)}$. Let

$$0 \to P_n \stackrel{d_n}{\to} P_{n-1} \to \ldots \to P_1 \stackrel{d_1}{\to} P_0 \to M \to 0$$

be a projective resolution of M. Then

$$(\phi \circ \psi)(M,f) = \phi(\sum_{i=0}^{n}(-1)^i(P_i,f_i)) = \sum_{i=0}^{n}(-1)^i(P_i,f_i).$$

We proceed by induction on n. For $n = 0$ the claim is trivial. Now let U be the kernel of the projective cover $P_0 \to M$ then

$$0 \to P_n \stackrel{d_n}{\to} P_{n-1} \to \ldots \to P_1 \stackrel{d_1}{\to} U \to 0$$

is a projective resolution of U and there is a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & U \arrow[r] \arrow[d, "f'"] & P_0 \arrow[r] \arrow[d, "f_0"] & M \arrow[r] \arrow[d, "f"] & 0\,. \\ 0 \arrow[r] & U \arrow[r] & P_0 \arrow[r] & M \arrow[r] & 0 \end{tikzcd}$$

By the induction hypothesis we have $(U,f') = \sum_{i=1}^{n}(-1)^i(P_i,f_i)$ and $(P_0,f_0) = (U,f') + (M,f)$ by 2.1.1 b). Thus

$$(M,f) = (P_0,f_0) + \sum_{i=1}^{n}(-1)^i(P_i,f_i) = (\phi \circ \psi)(M,f)$$

holds. $\square$

Definition 2.2.2
*Let $M \in \text{mod-}R$, $f : M \to M$. An **f-filtration** of M is a finite filtration*

$$M = M_0 \supset M_1 \supset \ldots \supset M_n = 0$$

by submodules with

$$f(M_i) \subset M_{i+1} \ \forall\ i = 0, \ldots, n-1.$$

The f-filtration has finite projective dimension if $\operatorname{pdim}_R M_i < \infty$ holds for all $i = 0, \ldots, n-1$.

Proposition 2.2.3
Suppose that $M \in \mathcal{P}^\infty(R)$ has an f-filtration of finite projective dimension. Then $(M, f) = 0$ in $K_1'(R)$.

Proof. We proceed by induction on n. When $n = 0$ the claim is trivial. If $n \geq 1$ we consider the map $f_1 : M_1 \to M_1$ induced by the restriction of f, then $(M_1, f_1) = 0$ in $K_1'(R)$ by induction hypothesis. Since $f(M) \subset M_1$ we have the following commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & M_1 \arrow[r] \arrow[d, "f_1"] & M \arrow[r] \arrow[d, "f"] & M/M_1 \arrow[r] \arrow[d, "0"] & 0 . \\
0 \arrow[r] & M_1 \arrow[r] & M \arrow[r] & M/M_1 \arrow[r] & 0
\end{tikzcd}$$

Thus $(M, f) = (M_1, f_1) + (M/M_1, 0) = 0$. □

Theorem 2.2.4
Let R be a ring with 1_R and $e \in R$ a primitive idempotent. Let $\alpha \in eRe$ be a nilpotent element and denote by $\lambda_\alpha : eR \to eR$ the left multiplication with α. If eR has a λ_α-filtration of finite projective dimension, then α is in $[R, R]$.

Proof. By Proposition 2.2.3 $(eR, \lambda_\alpha) = 0$ in $K_1'(R)$; hence by Theorem 2.2.1 $(eR, \lambda_\alpha) = 0$ in $K_1(R)$ and $0 = \operatorname{Tr}(eR, \lambda_\alpha) = \overline{\alpha}$ in $H_0(R)$. That means $\alpha \in [R, R]$. □

Definition 2.2.5
In the sequel we shortly say α-filtration and mean a λ_α-filtration as in theorem 2.2.4.

Corollary 2.2.6
Let $\Lambda = \mathbf{k}\,\mathcal{Q}/I$ be a finite dimensional algebra, x a point in $\mathcal{Q}$ and α an oriented cycle at x. If P_x has an α-filtration of finite projective dimension, then α is not a loop.

Proof. Consider the following commutative diagram of $\mathbb{Z}$-modules with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & I' \arrow[r] \arrow[d, "\varepsilon_|"] & {[\mathbf{k}\,\mathcal{Q}, \mathbf{k}\,\mathcal{Q}]} \arrow[r, "\pi_|"] \arrow[d, "\varepsilon"] & {[\Lambda, \Lambda]} \arrow[r] \arrow[d, "\overline{\varepsilon}"] & 0\, . \\
0 \arrow[r] & I \arrow[r] & \mathbf{k}\,\mathcal{Q} \arrow[r, "\pi"] & \Lambda \arrow[r] & 0
\end{tikzcd}$$

Since ε and $\overline{\varepsilon}$ are injective, $\varepsilon_|$ is injective too. By Theorem 2.2.4 $\pi(\alpha) \in [\Lambda, \Lambda]$ holds. Hence there is $w \in [\mathbf{k}\,\mathcal{Q}, \mathbf{k}\,\mathcal{Q}]$ such that $\pi(w) = \pi(\alpha)$. That means $\alpha - w \in I$. Since I is generated by paths of length at least two, either α has length ≥ 2 or $\alpha = w \in [\mathbf{k}\,\mathcal{Q}, \mathbf{k}\,\mathcal{Q}]$. But the loops have length one and the elements of length one in $[\mathbf{k}\,\mathcal{Q}, \mathbf{k}\,\mathcal{Q}]$ are linear combinations of arrows which are not loops. Therefore α can't be a loop in both cases. $\square$

Chapter 3

The strong no loop conjecture is true for mild algebras

In this chapter we will use Corollary 2.2.6 as an essential tool to prove the strong no loop conjecture for the algebras considered in Theorem 1. Our strategy is as follows: We consider the point x with $\operatorname{pdim}_\Lambda S_x < \infty$ and its mild neighborhood $A := \Lambda(x)$. We assume in addition that there is a loop α in x. Then we deduce a contradiction either by showing that $\operatorname{pdim}_\Lambda S_x = \infty$ or by constructing a certain α-filtration $\mathcal{F}$ of P_x having finite projective dimension in mod-Λ and implying that α is not a loop by Proposition 2.2.6. Since $\Lambda(x)$ contains the support of P_x, these filtrations coincide for P_x as a Λ-module and as a $\Lambda(x)$-module. Thus we are dealing with a mild algebra, and we use in an essential way the deep structure theorems about such algebras given in [3] and [5] to obtain the wanted α-filtrations. In particular, we show that we always work in the ray-category attached to $\Lambda(x)$. This makes it much easier to use cleaving diagrams. But still the construction of the appropriate α-filtrations depends on the study of several cases and it remains a difficult technical problem. The α-filtrations are always built in such a way that they have finite projective dimension in mod-Λ provided $\operatorname{pdim}_\Lambda S_x < \infty$.

To illustrate the method by two examples we define $\langle w_1, \ldots, w_k \rangle$ as the submodule of P_x generated by elements $w_1, \ldots, w_k \in P_x$.

Example 3.0.7
Let Λ be an algebra such that $\Lambda(x)$ is given by the quiver

$$\mathcal{Q} = \begin{tikzcd} & x \arrow[loop above, "\alpha"] \arrow[dl, "\gamma_1"'] \arrow[d, "\beta_1"] & \\ z \arrow[r, "\gamma_2"] & y_1 \arrow[r, "\beta_2"] & y_2 \arrow[ul, "\beta_3"'] \end{tikzcd}$$

and a relation ideal I such that the projective module P_x is described by the following graph:

$$\begin{tikzcd}
& e_x \arrow[dl] \arrow[d] \arrow[dr] & \\
\gamma_1 \arrow[d, "\gamma_2"'] & \alpha \arrow[dl] \arrow[d] & \beta_1 \arrow[d] \\
\alpha\beta_1 & \alpha^2 & \beta_1\beta_2 \arrow[l]
\end{tikzcd} .$$

Notice that the picture means that there are relations $\alpha^2 - \lambda_1\, \beta_1\beta_2\beta_3,\ \alpha\beta_1 - \lambda_2\, \gamma_1\gamma_2 \in I$ for some $\lambda_i \in \mathbf{k}\setminus\{0\}$. From the obvious exact sequences

$$0 \to \operatorname{rad} P_x \to P_x \to S_x \to 0$$

$$0 \to \langle \beta_1, \gamma_1 \rangle \to \operatorname{rad} P_x \to S_x \to 0$$

$$0 \to \langle \alpha^2, \gamma_1 \rangle \to \langle \alpha, \gamma_1 \rangle \to S_x \to 0$$

we see that $\operatorname{pdim}_\Lambda S_x < \infty$ leads to $\operatorname{pdim}_\Lambda \operatorname{rad} P_x < \infty$ and $\operatorname{pdim}_\Lambda \langle \beta_1, \gamma_1 \rangle < \infty$. Since $\langle \beta_1, \gamma_1 \rangle = \langle \beta_1 \rangle \oplus \langle \gamma_1 \rangle$ and $\langle \alpha^2, \gamma_1 \rangle = \langle \alpha^2 \rangle \oplus \langle \gamma_1 \rangle$ in this example, both $\operatorname{pdim}_\Lambda \langle \gamma_1 \rangle$ and $\operatorname{pdim}_\Lambda \langle \alpha, \gamma_1 \rangle$ are finite. Then the following α-filtration $\mathcal{F}$: $P_x \supset \langle \alpha, \gamma_1 \rangle \supset \langle \alpha^2 \rangle \supset 0$ has finite projective dimension in mod-Λ.

In the next example we see that this method may not work if the neighborhood $\Lambda(x)$ is not mild, even if the support of P_x is mild.

Example 3.0.8
Let $\Lambda(x) = \mathbf{k}\,\mathcal{Q}/I$ be given by the quiver

$$\mathcal{Q} = \begin{tikzcd}
x \arrow[loop left, "\alpha"] \arrow[r, bend left, "\beta_1"] \arrow[d, "\gamma"] & y \arrow[l, bend left, "\beta_2"] \arrow[d, bend left] \\
z \arrow[loop left, "\delta"] \arrow[r, bend left] & z' \arrow[l, bend left] \arrow[u, bend left]
\end{tikzcd}$$

and by a relation ideal I such that P_x is represented by

$$\begin{tikzcd}
& e_x \arrow[dl] \arrow[d] \arrow[dr] & \\
\gamma \arrow[d, "\delta"'] & \alpha \arrow[dl] \arrow[d] & \beta_1 \arrow[dl, "\beta_2"] \\
\alpha\gamma & \alpha^2 &
\end{tikzcd} .$$

Here we get stuck because the uniserial module with basis $\{\gamma, \alpha\gamma\}$ allows only the composition series as an α-filtration. Since we do not know $\operatorname{pdim}_\Lambda S_z$, which depends on Λ and not only on $\Lambda(x)$, our method does not apply.

In the next section we recall some facts about ray-categories and we show how to reduce the proof to standard algebras without penny-farthings. This case is then analyzed in the last section.

3.1 The reduction to standard algebras

3.1.1 Ray-categories and standard algebras

We recall some well-known facts from [3], [9].

Let $A := \Lambda(x) = \mathbf{k}\,\mathcal{Q}_A\,/I_A$ be a basic distributive $\mathbf{k}$-algebra. Then every space e_xAe_y is a cyclic module over e_xAe_x or e_yAe_y and we can associate to A its **ray-category** $\overrightarrow{A}$. Its objects are the points of $\mathcal{Q}_A$. The morphisms in $\overrightarrow{A}$ are called **rays** and $\overrightarrow{A}(x,y)$ consists of the orbits $\overrightarrow{\mu}$ in e_xAe_y under the obvious action of the groups of units in e_xAe_x and e_yAe_y. The composition of two morphisms $\overrightarrow{\mu}$ and $\overrightarrow{\nu}$ is either the orbit of the composition $\mu\nu$, in case this is independent of the choice of representatives in $\overrightarrow{\mu}$ and $\overrightarrow{\nu}$, or else 0. We call a non-zero morphism $\eta \in \overrightarrow{A}$ **long** if it is non-irreducible and satisfies $\nu\eta = 0 = \eta\nu'$ for all non-isomorphisms $\nu, \nu' \in \overrightarrow{A}$. One crucial fact about ray-categories frequently used in this paper is that A is mild iff $\overrightarrow{A}$ is so [9, see Theorem 13.17].

The ray-category is a finite category characterized by some nice properties. For instance, given $\lambda\,\mu\kappa = \lambda\,\nu\kappa \neq 0$ in $\overrightarrow{A}$, $\mu = \nu$ holds. We shall refer to this property as the **cancellation law**.

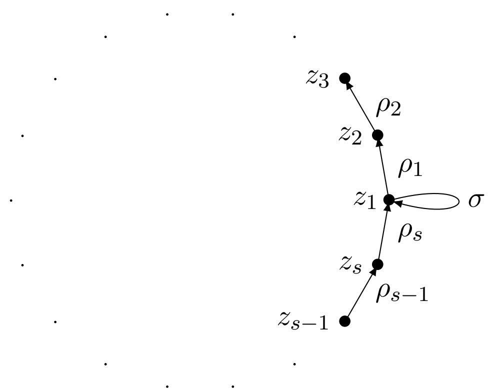
Given $\overrightarrow{A}$, we construct in a natural way its linearization $\mathbf{k}(\overrightarrow{A})$ and obtain a finite dimensional algebra

$$\overline{A} = \bigoplus_{x,y\in\mathcal{Q}_A} \mathbf{k}(\overrightarrow{A})(x,y),$$

the **standard form** of A. In general, A and $\overline{A}$ are not isomorphic, but they are if either A is minimal representation-infinite [5, Theorem 2] or representation-finite with $\operatorname{char}\mathbf{k} \neq 2$ [9, Theorem 13.17].

Similar to A, the ray-category $\overrightarrow{A}$ admits a description by quiver and relations. Namely, there is a canonical full functor $\vec{\ } : \mathcal{P}\,\mathcal{Q}_A \to \overrightarrow{A}$ from the path category of $\mathcal{Q}_A$ to $\overrightarrow{A}$. Two paths in $\mathcal{Q}_A$ are **interlaced** if they belong to the transitive closure of the relation given by $v \sim w$ iff $v = pv'q,\ w = pw'q$ and $\overrightarrow{v'} = \overrightarrow{w'} \neq 0$, where p and q are not both identities.

A **contour** of $\overrightarrow{A}$ is a pair (v,w) of non-interlaced paths with $\overrightarrow{v} = \overrightarrow{w} \neq 0$. Note that these contours are called essential contours in [3, 2.7]. Throughout this paper we will need a special kind of contours called penny-farthings. A **penny-farthing** P in $\overrightarrow{A}$ is a contour $(\sigma^2, \rho_1 \ldots \rho_s)$ such that the full subquiver $\mathcal{Q}_P$ of $\mathcal{Q}_A$ that supports the arrows of P has the following shape:



Moreover, we ask the full subcategory $A_P \subset A$ living on $\mathcal{Q}_P$ to be defined by $\mathcal{Q}_P$ and one of the following two systems of relations

$$\begin{aligned} (1)\ 0 = \sigma^2 - \rho_1 \dots \rho_s &= \rho_s \rho_1 = \rho_{i+1} \dots \rho_s \sigma \rho_1 \dots \rho_{f(i)}, \\ (2)\ 0 = \sigma^2 - \rho_1 \dots \rho_s &= \rho_s \rho_1 - \rho_s \sigma \rho_1 = \rho_{i+1} \dots \rho_s \sigma \rho_1 \dots \rho_{f(i)}, \end{aligned}$$

where $f : \{1, 2, \dots, s-1\} \to \{1, 2, \dots, s\}$ is some non-decreasing function (see [3, 2.7]. For penny-farthings of type (1) A_P is standard, for that of type (2) A_P is not standard in case the characteristic is two.

A functor $F : D \to \overrightarrow{A}$ between ray categories is **cleaving** ([9, 13.8]) iff it satisfies the following two conditions and their duals:

a) $F(\mu) = 0$ iff $\mu = 0$.

b) If $\eta \in D(y, z)$ is irreducible and $F(\mu) : F(y) \to F(z')$ factors through $F(\eta)$ then μ factors already through η.

The key fact about cleaving functors is that $\overrightarrow{A}$ is not representation-finite if D is not. In this paper D will always be given by its quiver $\mathcal{Q}_D$, that has no oriented cycles and some relations. Two paths between the same points give always the same morphism, and zero relations are indicated by a dotted line. As in [9, section 13], the cleaving functor is then defined by drawing the quiver of D with relations and by writing the morphism $F(\mu)$ in $\overrightarrow{A}$ close to each arrow μ.

By abuse of notation, we denote the irreducible rays of $\overrightarrow{A}$ and the corresponding arrows of $\mathcal{Q}_A$ by the same letter.

3.1.2 Getting rid of penny-farthings

Using the above notations let $P = (\sigma^2, \rho_1 \dots \rho_s)$ be a penny-farthing in $\overrightarrow{A}$. We shall show now that $x = z_1$. Therefore $\sigma = \alpha$ and P is the only penny-farthing in $\overrightarrow{A}$ by [9, Theorem 13.12].

Lemma 3.1.1
If there is a penny-farthing $P = (\sigma^2, \rho_1 \dots \rho_s)$ in $\overrightarrow{A}$, then $z_1 = x$.

Proof. We consider two cases:

i) $x \in \mathcal{Q}_P$: Hence $\mathcal{Q}_P$ has the following shape:

$$\begin{tikzcd} z_1 \arrow[loop left, "\sigma"] \arrow[rrr, bend left, "\rho_1 \cdots \rho_l"] & & & x \arrow[loop right, "\alpha"] \arrow[lll, bend left, "\rho_{l+1} \cdots \rho_s"] \end{tikzcd}$$

But this can be the quiver of a penny-farthing only for $z_1 = x$.

ii) $x \notin \mathcal{Q}_P$: Since A is the neighborhood of x, only the following cases are possible:

a) $e_x A e_z \neq 0$: Since $x \notin \mathcal{Q}_P$ we can apply the dual of [4, Theorem 1] or [9, Lemma 13.15] to $\overrightarrow{A}$ and we see that the following quivers occur as subquivers of $\mathcal{Q}_A$:

$$\begin{tikzcd} z_1 \arrow[loop left, "\sigma"] \arrow[r, bend left, "\rho_1"] & z_2 \arrow[l, bend left, "\rho_2"] \\ x \arrow[loop left, "\alpha"] \arrow[u] & \end{tikzcd} \quad resp. \quad \begin{tikzcd} z_1 \arrow[loop left, "\sigma"] \arrow[r, bend left, "\rho_1"] & z_2 \arrow[l, bend left, "\rho_2"] \\ & x \arrow[loop left, "\alpha"] \arrow[u] \end{tikzcd} .$$

Moreover, there can be only one arrow starting in x. This is a contradiction to the actual setting.

b) $\exists\ z_1 \to x$: By applying [4, Theorem 1] or the dual of [9, Lemma 13.15] we deduce that the following quiver occurs as a subquiver of $\mathcal{Q}_A$:

$$\begin{tikzcd} z_1 \arrow[loop left, "\sigma"] \arrow[r, bend left, "\rho_1"] \arrow[d] & z_2 \arrow[l, bend left, "\rho_2"] \\ x \arrow[loop left, "\alpha"] & \end{tikzcd}$$

and there can be only one arrow ending in x contradicting the present case.

c) $\exists\ y' \leftarrow x \rightleftarrows y \leftarrow z_1$: If $y \notin \mathcal{Q}_P$, then

$$\begin{tikzcd} z_1 \arrow[loop left, "\sigma"] \arrow[r, bend left, "\rho_1"] \arrow[d] & z_2 \arrow[l, bend left, "\rho_2"] \\ y \arrow[r, bend left] & x \arrow[l, bend left] \end{tikzcd}$$

is a subquiver of $\mathcal{Q}_A$ leading to the same contradiction as in b).
If $y \in \mathcal{Q}_P$, then $y = z_2$ and the quiver

$$\begin{tikzcd}
x \arrow[loop left, "\alpha"] \arrow[rr, bend left, "\beta_1"] \arrow[d, "\gamma"'] & & z_2 \arrow[ll, bend left, "\beta_2"] & z_1 \arrow[l, "\rho_1"'] \arrow[loop right, "\sigma"] \\
y' & & &
\end{tikzcd}$$

is a subquiver of $\mathcal{Q}_A$. Since $x \notin \mathcal{Q}_P$, all morphisms occurring in the following diagram

$$D := \begin{tikzcd}
\bullet & \bullet \arrow[l, "\rho_2"'] \arrow[r, "\beta_2"] & \bullet & \bullet \arrow[l, "\alpha"'] \arrow[r, "\beta_1"] \arrow[d, "\gamma"'] & \bullet & \bullet \arrow[l, "\rho_1"'] \arrow[r, "\sigma"] & \bullet \\
& & & \bullet & & &
\end{tikzcd}$$

are irreducible and pairwise distinct. Therefore D is a cleaving diagram in $\overrightarrow{A}$. Moreover, some long morphism $\eta = \nu\sigma^3\nu'$ does not occur in D; hence D is still cleaving in $\overrightarrow{A}/\eta$ by [5, Lemma 3]. Since D is of representation-infinite Euclidean type $\widetilde{E}_7$, $\overrightarrow{A}/\eta$ is representation-infinite contradicting the mildness of A.

$\square$

Now, we show that, provided the existence of a penny-farthing in $\overrightarrow{A}$, there exists an α-filtration of P_x having finite projective dimension.

Lemma 3.1.2
Let $A = \Lambda(x)$ be mild and standard. If there is a penny-farthing in $\overrightarrow{A}$, then there exists an α-filtration $\mathcal{F}$ of P_x having finite projective dimension.

Proof. If there is a penny-farthing P in $\overrightarrow{A}$, then $P = (\alpha^2, \rho_1 \dots \rho_s)$ is the only penny-farthing in $\overrightarrow{A}$ by the last lemma. Since A is standard and mild, there are three cases for the graph of P_x which can occur by [4, Theorem 1] or the dual of [9, Lemma 13.15].

i) There exists an arrow $\gamma : x \to z,\ \gamma \neq \rho_1$. Then $s = 2$, the quiver

$$\begin{tikzcd}
x \arrow[loop left, "\alpha"] \arrow[r, bend left, "\rho_1"] \arrow[d, "\gamma"] & y \arrow[l, bend left, "\rho_2"] \\
z &
\end{tikzcd}$$

is a subquiver of $\mathcal{Q}_A$, and P_x is represented by the following graph:

$$\begin{tikzcd}[column sep=small]
 & e_x \arrow[dl] \arrow[d] \arrow[dr] & \\
\gamma & \alpha \arrow[d] \arrow[dr] & \rho_1 \arrow[dl] \\
 & \alpha^2 \arrow[d] & \alpha\rho_1 \arrow[dl] \\
 & \alpha^3 &
\end{tikzcd} .$$

Let M be a quotient of P_x defined by the following exact sequence:

$$0 \to \langle\gamma\rangle \oplus \langle\rho_1, \alpha\rho_1\rangle \to P_x \to M \to 0.$$

Then M has S_x as the only composition factor. Hence $\operatorname{pdim}_\Lambda M < \infty$ and $\operatorname{pdim}_\Lambda \langle\rho_1, \alpha\rho_1\rangle < \infty$. Now, we consider the exact sequence

$$0 \to \langle\alpha^3\rangle \to \langle\rho_1, \alpha\rho_1\rangle \to \langle\rho_1\rangle/\langle\alpha^3\rangle \oplus \langle\alpha\rho_1\rangle/\langle\alpha^3\rangle \to 0.$$

But $\langle\alpha^3\rangle \cong S_x$ and $\operatorname{pdim}_\Lambda S_x < \infty$, hence $\langle\alpha\rho_1\rangle/\langle\alpha^3\rangle \cong S_y$ has finite projective dimension in mod-Λ. Finally, the α-filtration $P_x \supset \langle\alpha\rangle \supset \langle\alpha^2\rangle \supset \langle\alpha^3\rangle \supset 0$ has finite projective dimension since all filtration modules $\ncong P_x$ have S_x and S_y as the only composition factors.

ii) In the second case there exists a point $z \notin \mathcal{Q}_P$ such that $A(x, z) \neq 0$. Then $s = 2$, the quiver

$$\begin{tikzcd}
x \arrow[loop left, "\alpha"] \arrow[r, bend left, "\rho_1"] & y \arrow[l, bend left, "\rho_2"] \arrow[d, "\delta"] \\
 & z
\end{tikzcd}$$

is a subquiver of $\mathcal{Q}_A$, and P_x is represented by:

$$\begin{tikzcd}[column sep=small]
e_x \arrow[d] \arrow[dr] & & \\
\alpha \arrow[d] \arrow[dr] & \rho_1 \arrow[dl] \arrow[dr] & \\
\alpha^2 \arrow[d] & \alpha\rho_1 \arrow[dl] & \rho_1\delta \\
\alpha^3 & &
\end{tikzcd} .$$

With similar considerations as in I) we obtain that the same filtration fits.

iii) In the last possible case we have $A(x,z) = 0$ for all points $z \notin \mathcal{Q}_P$. Hence P_x is represented by:

$$
\begin{tikzcd}
 & e_x \arrow[r] \arrow[d] & \rho_1 \arrow[d] \\
\alpha\rho_1 \arrow[d] & \alpha \arrow[l] \arrow[d] & \rho_1\rho_2 \arrow[d, squiggly] \\
\alpha\rho_1\rho_2 \arrow[d, squiggly] & \alpha^2 \arrow[d] & \rho_1\rho_2\cdots\rho_{s-1} \arrow[l] \\
\alpha\rho_1\rho_2\cdots\rho_{s-1} \arrow[r] & \alpha^3 &
\end{tikzcd}
.
$$

As a Λ-module, $M := P_x/\langle\alpha^2\rangle$ has finite projective dimension since $\langle\alpha^2\rangle$ has S_x as the only composition factor. Let K be the kernel of the epimorphism $M \to \langle\alpha^2\rangle,\ e_x \mapsto \alpha^2$, then $K = \langle\rho_1\rangle/\langle\alpha^2\rangle \oplus \langle\alpha\rho_1\rangle/\langle\alpha^3\rangle$ has finite projective dimension. Moreover, $\mathrm{pdim}_\Lambda\langle\rho_1\rangle, \mathrm{pdim}_\Lambda\langle\alpha\rho_1\rangle < \infty$. Since

$$0 \to \langle\alpha\rho_1\rangle \to \langle\alpha\rangle \overset{\lambda_\alpha}{\to} \langle\alpha^2\rangle \to 0$$

is exact, $\mathrm{pdim}_\Lambda\langle\alpha\rangle < \infty$. Thus the same filtration as in the first two cases fits again.

□

Lemma 3.1.3
With above notations let $A = \Lambda(x)$ be mild and non-standard. There exists an α-filtration $\mathcal{F}$ of P_x having finite projective dimension.

Proof. If A is non-standard, then A is representation-finite by [5], $\mathrm{char}\,\mathbf{k} = 2$ and there is a penny-farthing in $\overrightarrow{A}$ by [9, Theorem 13.17]. Since Lemma 3.1.1 remains valid, the penny-farthing $(\alpha^2, \rho_1 \dots \rho_s)$, $\rho_i : z_i \to z_{i+1}$, $z_1 = z_{s+1} = x$, is unique. By [9, 13.14, 13.17] the difference between A and $\overline{A}$ in the composition of the arrows shows up in the graphs of the projectives to $z_2, \dots, z_s$ only. Thus the graph of P_x remains the same in all three cases of the proof of Lemma 3.1.2 and the filtrations constructed there still do the job. □

3.2 The proof for standard algebras without penny-farthings

3.2.1 Some preliminaries

If there is no penny-farthing in $\overrightarrow{A}$, then $A = \overline{A}$ is standard by Gabriel, Roiter [9, Theorem 13.17] and Bongartz [5, Theorem 2]. By a result of Liu, Morin [19, Corollary 1.3], deduced from a proposition of Green, Solberg, Zacharia [11], a power of α is a summand of a polynomial relation in $I = I_\Lambda$. Otherwise $\operatorname{pdim}_\Lambda S_x$ would be infinite contradicting the choice of x. Furthermore, α is a summand of a polynomial relation in I_A by definition of A. But I_A is generated by paths and differences of paths in $\mathcal{Q}_A$. Hence we can assume without loss of generality that there is a relation $\alpha^t - \beta_1\beta_2 \ldots \beta_r$ in I_A for some $t \in \mathbb{N}$ and arrows $\beta_1, \beta_2, \ldots, \beta_r$. Among all relations of this type we choose one with minimal t. Hence $(\alpha^t, \beta_1\beta_2 \ldots \beta_r)$ is a contour in $\overrightarrow{A}$ with $t, r \geq 2$. Let $y = e(\beta_1)$ be the ending point of β_1 and $\tilde{\beta} = \beta_2 \ldots \beta_r$.

By the structure theorem for non-deep contours in [3, 6.4] the contour $(\alpha^t, \beta_1\beta_2 \ldots \beta_r)$ is deep, i.e. we have $\alpha^{t+1} = 0$ in A. Since A is mild, the cardinality of the set x^+ of all arrows starting in x is bounded by three. Before we consider the cases $|x^+| = 2$ and $|x^+| = 3$ separately we shall prove some useful general facts.

The following trivial fact about standard algebras will be essential hereafter.

Lemma 3.2.1
Let $A = \overline{A}$ be a standard $\mathbf{k}$-algebra. Consider rays $v_i, w_j \in \overrightarrow{A} \setminus \{0\}$ for $i = 1, \ldots, n$ and $j = 1, \ldots, m$ such that $v_l \neq v_k$ and $w_l \neq w_k$ for $l \neq k$. If there are $\lambda_i, \mu_j \in \mathbf{k} \setminus \{0\}$ such that $\sum_{i=1}^n \lambda_i v_i = \sum_{j=1}^m \mu_j w_j$, then $n = m$ and there exists a permutation $\pi \in S(n)$ such that $v_i = w_{\pi(i)}$ and $\lambda_i = \mu_{\pi(i)}$ for $i = 1, \ldots, n$.

Proof. Since the set of non-zero rays in $\overrightarrow{A}$ forms a basis of A, it is linearly independent and the claim follows. □

In what follows we denote by $\mathcal{L}$ the set of all long morphisms in $\overrightarrow{A}$. By μ we denote some long morphism $\nu\alpha^t\nu'$ which exists since $\alpha^t \neq 0$.

Lemma 3.2.2
Using the above notations we have:

$$\langle \beta_1 \rangle \cap \langle \alpha\beta_1 \rangle = 0$$

Proof. We assume to the contrary that $\langle \beta_1 \rangle \cap \langle \alpha\beta_1 \rangle \neq 0$. Then, by Lemma 3.2.1, there are rays $v, w \in \overrightarrow{A}$ such that $\beta_1 v = \alpha\beta_1 w \neq 0$. We claim that

$$D := \begin{tikzcd} \bullet \arrow[d, "\alpha^{t-1}"'] \arrow[dr, "\beta_1 w" near start] & \bullet \arrow[dl, "\tilde{\beta}"' near start] \arrow[d, "v"] \\ \bullet & \bullet \end{tikzcd}$$

is a cleaving diagram in $\overrightarrow{A}$. It is of representation-infinite, Euclidean type $\widetilde{A}_3$. Since all morphisms occurring in D are not long, the long morphism $\mu = \nu\alpha^t\nu'$ does not occur in D and D is still cleaving in $\overrightarrow{A}/\mu$ by [5, Lemma 3]. Thus $\overrightarrow{A}/\mu$ is representation-infinite contradicting the mildness of A.

Now we show in detail, using [5, Lemma 3 d)], that D is cleaving. First of all we assume that there is a ray ρ with $\rho\tilde{\beta} = \alpha^{t-1}$. Then we get $0 \neq \alpha^t = \alpha\rho\tilde{\beta} = \beta_1\tilde{\beta}$, whence $\alpha\rho = \beta_1$ by the cancellation law. This contradicts the fact that β_1 is an arrow. In a similar way it can be shown that $\rho\alpha^{t-1} = \tilde{\beta},\ \rho v = \beta_1 w$ and $\rho\beta_1 w = v$ are impossible.
The following four cases are left to exclude.

i) $\alpha^{t-1}\rho = \beta_1 w$: Left multiplication with α gives us $\alpha^t\rho = \alpha\beta_1 w \neq 0$. Hence there is a non-deep contour $(\alpha^{t-1}\rho_1 \ldots \rho_k, \beta_1 w_1 \ldots w_l)$ in $\overrightarrow{A}$. Here $\rho = \rho_1 \ldots \rho_k$ resp. $w = w_1 \ldots w_l$ is a product of irreducible rays (arrows). Since the arrow β_1 is in the contour, the cycle $\beta_1\tilde{\beta}$ and the loop α belong to the contour. Hence it can only be a penny-farthing by the structure theorem for non-deep contours [3, 6.4]. But this case is excluded in the current section.

ii) $\tilde{\beta}\rho = v$: We argue as before and deduce $\beta_1\tilde{\beta}\rho = \beta_1 v = \alpha^t\rho = \alpha\beta_1 w \neq 0$. Hence there is a non-deep contour $(\alpha^{t-1}\rho_1 \ldots \rho_k, \beta_1 w_1 \ldots w_l)$ leading again to a contradiction.

iii) $\beta_1 w\rho = \alpha^{t-1}$: Since $t-1 < t$ we have a contradiction to the minimality of t.

iv) $v\rho = \tilde{\beta}$: Then $\beta_1 v\rho = \beta_1\tilde{\beta} = \alpha^t = \alpha\beta_1 v\rho \neq 0$. Using the cancellation law we get $\alpha^{t-1} = \beta_1 v\rho$ a contradiction as before.

$\square$

Lemma 3.2.3
*If $t \geq 3$ **and** $\mathcal{L} \not\subseteq \{\alpha^3, \alpha^2\beta_1\}$, then $\alpha^2\beta_1 = 0$.*

Proof. If $\alpha^2\beta_1 \neq 0$, then

$$D := \begin{tikzcd} \bullet \arrow[r, "\alpha"] & \bullet \arrow[r, "\beta_1"] \arrow[d, "\alpha"] & \bullet \\ \bullet & \bullet \arrow[l, "\alpha"'] \arrow[r, "\beta_1"] & \bullet \end{tikzcd}$$

is a cleaving diagram of Euclidian type $\widetilde{D}_5$ in $\overrightarrow{A}$. It is cleaving since:

i) $\alpha^2 = \beta_1\rho \neq 0$ contradicts the choice of $t \geq 3$.

ii) $\alpha\beta_1 = \beta_1\rho \neq 0$ contradicts Lemma 3.2.2.

It is also cleaving in $\overrightarrow{A}/\eta$ for $\eta \in \mathcal{L} \setminus \{\alpha^3, \alpha^2\beta_1\} \neq \emptyset$ contradicting the mildness of A. $\square$

Lemma 3.2.4
If $\langle\alpha^2\rangle \cap \langle\alpha\beta_1\rangle = 0 = \langle\beta_1\rangle \cap \langle\alpha\beta_1\rangle$, then $\langle\alpha^2, \beta_1\rangle \cap \langle\alpha\beta_1\rangle = 0$.

Proof. Let $\alpha^2 u + \beta_1 v = \alpha\beta_1 w \neq 0$ be an element in $\langle \alpha^2, \beta_1 \rangle \cap \langle \alpha\beta_1 \rangle$. By Lemma 3.2.1 we can assume that u, v, w are rays and the following two cases might occur:

i) $\beta_1 v = \alpha\beta_1 w \neq 0$: This is a contradiction since $\langle \beta_1 \rangle \cap \langle \alpha\beta_1 \rangle = 0$.

ii) $\alpha^2 u = \alpha\beta_1 w \neq 0$: This is impossible because $\langle \alpha^2 \rangle \cap \langle \alpha\beta_1 \rangle = 0$.

□

3.2.2 The case $|x^+| = 2$

Lemma 3.2.5
If $x^+ = \{\alpha, \beta_1\}$ and $\mathcal{L} \subseteq \{\alpha^3, \alpha^2\beta_1\}$, then there exists an α-filtration $\mathcal{F}$ of P_x having finite projective dimension.

Proof. We treat two cases:

i) $\alpha\beta_1 = 0$: Then for $\langle \alpha^k \rangle$ with $k \geq 1$ only S_x is possible as a composition factor; hence $\mathrm{pdim}_\Lambda \langle \alpha^k \rangle < \infty$. Thus $P_x \supset \langle \alpha \rangle \supset \langle \alpha^2 \rangle \supset \langle \alpha^3 \rangle \supset 0$ is the wanted α-filtration.

ii) $\alpha\beta_1 \neq 0$: Since α^3 and $\alpha^2\beta_1$ are the only morphisms in $\overrightarrow{A}$ which can be long, we have $t = 3$, $0 \neq \alpha^3 \in \mathcal{L}$, $\langle \alpha\beta_1 \rangle = \mathbf{k}\,\alpha\beta_1 \cong S_y$ and $\langle \alpha^2\beta_1 \rangle \in \{\mathbf{k}\,\alpha^2\beta_1,\ 0\}$. Now we show that $\langle \alpha^2 \rangle \cap \langle \alpha\beta_1 \rangle = 0$. If there are rays $v = v_1 \ldots v_s,\ w \in \overrightarrow{A}$ with irreducible $v_i,\ i = 1 \ldots, s$ such that $\alpha^2 v = \alpha\beta_1 w \neq 0$, then $s > 0$ because $s = 0$ would contradict the irreducibility of α. Therefore $v_1 = \alpha$ or $v_1 = \beta_1$.

- If $v_1 = \alpha$, then $v' = v_2 \ldots v_s = \mathrm{id}$ since α^3 is long and $0 \neq \alpha^2 v = \alpha^3 v'$. Hence $0 \neq \alpha^3 = \alpha^2 v = \alpha\beta_1 w$ and $\alpha^2 = \beta_1 w$ contradicts the minimality of t.
- If $v_1 = \beta_1$, then $0 \neq \alpha^2 v = \alpha^2\beta_1 v' = \alpha\beta_1 w$; hence $0 \neq \alpha\beta_1 v' = \beta_1 w \in \langle \beta_1 \rangle \cap \langle \alpha\beta_1 \rangle = 0$.

Since $\langle \beta_1 \rangle \cap \langle \alpha\beta_1 \rangle = 0 = \langle \alpha^2 \rangle \cap \langle \alpha\beta_1 \rangle$, we deduce $\langle \beta_1, \alpha^2, \alpha\beta_1 \rangle = \langle \beta_1, \alpha^2 \rangle \oplus \langle \alpha\beta_1 \rangle$ by Lemma 3.2.4. Therefore the graph of P_x has the following shape:

$$\begin{tikzcd}
e_x \arrow[d] \arrow[dr] & & . \\
\alpha \arrow[d] \arrow[dr] & \langle \beta_1 \rangle \arrow[ddl, squiggly] & \\
\alpha^2 \arrow[d] \arrow[dr] & \alpha\beta_1 & \\
\alpha^3 & \alpha^2\beta_1 &
\end{tikzcd}$$

Here $\langle\beta_1\rangle$ stands for the graph of the submodule $\langle\beta_1\rangle$ which is not known explicitly. Consider the module M defined by the following exact sequence:

$$0 \to \langle\beta_1, \alpha^2, \alpha\beta_1\rangle \to P_x \to M \to 0$$

Then $\mathrm{pdim}_\Lambda M < \infty$ since M is filtered by S_x and $\mathrm{pdim}_\Lambda(\langle\beta_1, \alpha^2\rangle \oplus \langle\alpha\beta_1\rangle) = \mathrm{pdim}_\Lambda\langle\beta_1, \alpha^2, \alpha\beta_1\rangle < \infty$. Thus $\mathrm{pdim}_\Lambda(\langle\alpha\beta_1\rangle \cong S_y)$ is finite too and the wanted α-filtration is $P_x \supset \langle\alpha\rangle \supset \langle\alpha^2\rangle \supset \langle\alpha^3\rangle \supset 0$.

□

Lemma 3.2.6
If $x^+ = \{\alpha, \beta_1\}$, $t \geq 3$ ***and*** *$\mathcal{L} \nsubseteq \{\alpha^3, \alpha^2\beta_1\}$, then $\alpha^2\rho = 0$ for all rays $\rho \notin \{e_x, \alpha, \ldots, \alpha^{t-2}\}$. Moreover, $\langle\alpha^2\rangle \cap \langle\alpha\beta_1\rangle = 0$.*

Proof. Let $\rho \in \overrightarrow{A}$ with $\alpha^2\rho \neq 0$ be written as a composition of irreducible rays $\rho = \rho_1 \ldots \rho_s$. Then the following two cases are possible:

i) $\rho = \alpha^s$: Since $0 \neq \alpha^2\rho = \alpha^{2+s}$ and $\alpha^{t+1} = 0$ we have $s \leq t-2$ and $\rho = \alpha^s \in \{e_x, \alpha, \ldots, \alpha^{t-2}\}$.

ii) There exists a minimal $1 \leq i \leq s$ such that $\rho_i \neq \alpha$. Since $x^+ = \{\alpha, \beta_1\}$, we have $\rho_i = \beta_1$ and $0 \neq \alpha^2\rho = \alpha^{2+i-1}\beta_1\rho_{i+1}\ldots\rho_s = 0$ by Lemma 3.2.3.

If $0 \neq \alpha^2 v = \alpha\beta_1 w$, then $v = \alpha^s$ with $0 \leq s \leq t-2$. Hence $0 = \alpha^2 v = \alpha^{s+2} = \alpha\beta_1 w$ and $\alpha^{s+1} = \beta_1 w$ by cancellation law. This contradicts the minimality of t. □

Corollary 3.2.7
If $x^+ = \{\alpha, \beta_1\}$, $t \geq 3$ ***and*** *$\mathcal{L} \nsubseteq \{\alpha^3, \alpha^2\beta_1\}$, then $\langle\alpha^2, \beta_1\rangle \cap \langle\alpha\beta_1\rangle = 0$.*

Proof. The claim is trivial using Lemmas 3.2.2, 3.2.4 and 3.2.6. □

Proposition 3.2.8
If $x^+ = \{\alpha, \beta_1\}$, then there exists an α-filtration $\mathcal{F}$ of P_x having finite projective dimension.

Proof. If $\mathcal{L} \subseteq \{\alpha^3, \alpha^2\beta_1\}$, then the claim is the statement of Lemma 3.2.5. If $\mathcal{L} \nsubseteq \{\alpha^3, \alpha^2\beta_1\}$, then we consider the value of t:

i) $t = 2$: Then the graph of P_x has the following shape:

$$\begin{tikzcd}
e_x \arrow[d] \arrow[dr] & & . \\
\alpha \arrow[d] \arrow[dr] & \langle \beta_1 \rangle \arrow[dl] & \\
\alpha^2 \arrow[dr] & \langle \alpha\beta_1 \rangle & \\
 & \langle \alpha^2\beta_1 \rangle &
\end{tikzcd}$$

Let a subquotient M of P_x be defined by the following exact sequence:

$$0 \to \langle \beta_1, \alpha\beta_1 \rangle \to P_x \to M \to 0$$

Then M and $\langle \beta_1, \alpha\beta_1 \rangle$ have finite projective dimension in mod-Λ. By Lemma 3.2.2 we have $\langle \beta_1, \alpha\beta_1 \rangle = \langle \beta_1 \rangle \oplus \langle \alpha\beta_1 \rangle$; hence $\mathrm{pdim}_\Lambda \langle \beta_1 \rangle$ and $\mathrm{pdim}_\Lambda \langle \alpha\beta_1 \rangle$ are both finite.
Let K be the kernel of the epimorphism $\lambda_\alpha : \langle \beta_1 \rangle \to \langle \alpha\beta_1 \rangle$, $\lambda_\alpha(\rho) = \alpha\rho$. Then $\mathrm{pdim}_\Lambda K < \infty$ and for the α-filtration $\mathcal{F}$ we take the following: $P_x \supset \langle \alpha, \beta_1 \rangle \supset \langle \beta_1 \rangle \oplus \langle \alpha\beta_1 \rangle \supset \langle \alpha\beta_1 \rangle \oplus K \supset K \supset 0$.

ii) $t \geq 3$: Consider the following exact sequences:

$$0 \to \langle \alpha, \beta_1 \rangle \to P_x \to S_x \to 0$$

$$0 \to \langle \alpha^2, \beta_1, \alpha\beta_1 \rangle \to \langle \alpha, \beta_1 \rangle \to S_x \to 0$$

Hence $\mathrm{pdim}_\Lambda \langle \alpha, \beta_1 \rangle$ and $\mathrm{pdim}_\Lambda \langle \alpha^2, \beta_1, \alpha\beta_1 \rangle$ are finite. By Corollary 3.2.7 $\langle \alpha^2, \beta_1, \alpha\beta_1 \rangle = \langle \alpha^2, \beta_1 \rangle \oplus \langle \alpha\beta_1 \rangle$, that means $\mathrm{pdim}_\Lambda \langle \alpha\beta_1 \rangle$ is finite too. With Lemma 3.2.6 it is easily seen that for $2 \leq k \leq t$ the module $\langle \alpha^k \rangle$ is a uniserial module with S_x as the only composition factor. Hence $\mathrm{pdim}_\Lambda \langle \alpha^k \rangle$ is finite for $2 \leq k \leq t$. Thereby we have the wanted α-filtration

$$P_x \supset \langle \alpha, \beta_1 \rangle \supset \langle \alpha^2 \rangle \oplus \langle \alpha\beta_1 \rangle \supset \langle \alpha^3 \rangle \supset \langle \alpha^4 \rangle \supset \ldots \supset \langle \alpha^t \rangle \supset 0.$$

$\square$

3.2.3 The case $|x^+| = 3$

With previous notations $x^+ = \{\alpha, \beta_1, \gamma\}$, $(\alpha^t, \beta_1\beta_2 \ldots \beta_r)$ is a contour in $\overrightarrow{A}$, $t \geq 2$, $\alpha^{t+1} = 0$, $\tilde{\beta} := \beta_2 \ldots \beta_r$ and $\mu = \nu\alpha^t\nu'$ is a long morphism in $\overrightarrow{A}$.

The α-filtrations will be constructed depending on the set $\mathcal{L}$ of long morphisms in $\overrightarrow{A}$. The case $\mathcal{L} \subseteq \{\alpha^2, \alpha\beta_1, \alpha\gamma\}$ is treated in Lemma 3.2.17, the case $\mathcal{L} \subseteq \{\alpha^t, \alpha^2\beta_1\}$ in 3.2.18 and the remaining case in 3.2.19.

But first, we derive some technical results.

The following well known result is straight forward and holds for arbitrary Λ.

Lemma 3.2.9
Let B be a full convex subcategory of Λ i.e. any path in Λ with source and target in B lies entirely in B. The canonical restriction functor maps a projective resolution of a simple Λ-module with support in B to a projective resolution in mod-B.

Lemma 3.2.10
If $r = 2$ and $\delta : z' \to z$ is an arrow in $\mathcal{Q}_A$ ending in $z = e(\gamma)$, then $\delta = \gamma$.

Proof. Assume to the contrary that $\gamma \neq \delta : z' \to z$, then there is no arrow $\beta_1 \neq \varepsilon : y' \to y$ in $\mathcal{Q}_\Lambda$. If there is such an arrow, then by the definition of a neighborhood ε belongs to $\mathcal{Q}_A$. This arrow induces an irreducible ray $\beta_1 \neq \varepsilon : y' \to y$ in $\overrightarrow{A}$ and

$$D := \begin{tikzcd} \bullet \arrow[r, "\delta"] & \bullet & \bullet \arrow[l, "\gamma"'] \arrow[r, "\beta_1"] \arrow[d, "\alpha"] & \bullet & \bullet \arrow[l, "\varepsilon"'] \\ & & \bullet & & \\ & & \bullet \arrow[u, "\beta_2"'] & & \end{tikzcd}$$

is a cleaving diagram in $\overrightarrow{A}/\mu$ of Euclidian type $\widetilde{E}_6$.
In a similar way an arrow $\alpha, \beta_2 \neq \varepsilon : x' \to x$ in $\mathcal{Q}_\Lambda$ leads to a cleaving diagram of type $\widetilde{D}_5$ in $\overrightarrow{A}/\mu$. Hence the full subcategory B of Λ supported by the points x, y is a convex subcategory of Λ. Therefore the projective dimensions of S_x is finite in mod-B since it is finite in mod-Λ. But in B we have $x^+ = \{\alpha, \beta_1\}$, whence we can apply Proposition 3.2.8 together with 2.2.6 to get the contradiction that α is not a loop. $\square$

Lemma 3.2.11
If $\alpha\gamma \neq 0$, then $\beta_1 v \neq \alpha\gamma \neq \gamma w$ for all rays $v, w \in \overrightarrow{A}$.

Proof. i) Assume that there exists a ray $v \in \overrightarrow{A}$ such that $\beta_1 v = \alpha\gamma \neq 0$. Then

$$D := \begin{tikzcd} \bullet \arrow[d, "\gamma"'] \arrow[dr, "\alpha^{t-1}" near start] & \bullet \arrow[d, "\tilde{\beta}"] \arrow[dl, "v"' near start] \\ \bullet & \bullet \end{tikzcd}$$

is a cleaving diagram of Euclidian type $\widetilde{A}_3$ in $\overrightarrow{A}/\mu$.

- For $\gamma\rho = \alpha^{t-1}$ or $v\rho = \tilde{\beta}$ we have $\alpha\gamma\rho = \beta_1 v\rho = \beta_1\tilde{\beta} = \alpha^t \neq 0$. Thus $\alpha^{t-1} = \gamma\rho$ contradicts the choice of t.
- If $\alpha^{t-1}\rho = \gamma$ or $\tilde{\beta}\rho = v$, then $\alpha^t\rho = \beta_1\tilde{\beta}\rho = \beta_1 v = \alpha\gamma \neq 0$. Then $\alpha^{t-1}\rho = \gamma$ contradicts the irreducibility of γ.

ii) Assume that there exists a ray $w = w_1 \dots w_s : z \rightsquigarrow z \in \overrightarrow{A}$ with irreducible w_i such that $\gamma w = \alpha\gamma \neq 0$.

$r = 2$: Since w_s is an irreducible ray ending in z, $w_s = \gamma$ by Lemma 3.2.10. Thus we get a contradiction $\gamma w_1 \dots w_{s-1} = \alpha$.

$r \geq 3$: We look at the value of s. If $s = 1$, then $w = w_1$ is a loop and

$$D := \begin{tikzcd}
\bullet \arrow[r, "w_1=w"] & \bullet \arrow[ddr, dotted, "(1)"'] & \bullet \arrow[l, "\gamma"'] \arrow[r, "\alpha"] \arrow[d, "\beta_1"] & \bullet \arrow[ddl, dotted, "(2)"] & \bullet \arrow[l, "\beta_r"'] \\
& & \bullet \arrow[d, "\beta_2"] & & \\
& & \bullet & &
\end{tikzcd}$$

is a cleaving diagram in $\overrightarrow{A}/\mu$.
If $s \geq 2$, then

$$D := \begin{tikzcd}
\bullet \arrow[r, "w_{s-1}"] \arrow[rrr, bend right, dotted, "(3)"'] & \bullet \arrow[r, "w_s"] & \bullet & \bullet \arrow[l, "\gamma"'] \arrow[r, "\alpha"] \arrow[d, "\beta_1"] & \bullet & \bullet \arrow[l, "\beta_r"'] & \bullet \arrow[l, "\beta_{r-1}"'] \arrow[lll, bend left, dotted, "(4)"] \\
& & & \bullet & & &
\end{tikzcd}$$

is cleaving in $\overrightarrow{A}/\mu$.
We still have to show that not any morphisms indicated by the dotted lines make the diagrams commute.

(1): $\gamma\rho = \beta_1\beta_2$, with $\rho = \rho_1 \dots \rho_l$. If $\rho = w_1^l = w^l$, then $\beta_1\beta_2 = \gamma\rho = \gamma w^l = \alpha\gamma w^{l-1}$ and $\beta_1\beta_2 \dots \beta_r = \alpha^t = \alpha\gamma w^{l-1}\beta_3 \dots \beta_r \neq 0$. Therefore $\alpha^{t-1} = \gamma w^{l-1}\beta_3 \dots \beta_r$ is a contradiction. If $\rho \neq w_1^l$, then one of the irreducible rays $\rho_i \neq w_1$ starts in z and

$$D := \begin{tikzcd}
\bullet & \bullet \arrow[l, "\rho_i"'] \arrow[r, "w_1"] & \bullet & \bullet \arrow[l, "\gamma"'] \arrow[r, "\alpha"] \arrow[d, "\beta_1"] & \bullet & \bullet \arrow[l, "\beta_r"'] & \bullet \arrow[l, "\beta_{r-1}"'] \arrow[lll, bend left, dotted, "(4)"] \\
& & & \bullet & & &
\end{tikzcd}$$

is cleaving in $\overrightarrow{A}/\mu$.

(2): If $\alpha\rho = \beta_1\beta_2$, then $\alpha\rho\beta_3 \dots \beta_r = \beta_1\beta_2 \dots \beta_r = \alpha^t \neq 0$ and $\alpha^{t-1} = \rho\beta_3 \dots \beta_r$ contradicts the minimality of t.

(3): If $\rho\gamma = w_{s-1}w_s$, then $\gamma w_1 \dots w_{s-2}\rho\gamma = \gamma w = \alpha\gamma \neq 0$ and $\alpha = \gamma w_1 \dots w_{s-2}\rho$ contradicts the irreducibility of α.

(4): If $\rho\alpha = \beta_{r-1}\beta_r$, then $\beta_1\beta_2 \dots \beta_{r-2}\rho\alpha = \beta_1\beta_2 \dots \beta_r = \alpha^t \neq 0$ and $\alpha^{t-1} = \beta_1\beta_2 \dots \beta_{r-2}\rho$ contradicts the minimality of t.

□

Lemma 3.2.12
If $t \geq 3$, then $\alpha\gamma = 0$.

Proof. Assume that $\alpha\gamma \neq 0$, then

$$D := \begin{tikzcd} \bullet & \bullet \arrow[l, "\gamma"'] \arrow[r, "\beta_1"] \arrow[d, "\alpha"] & \bullet \\ \bullet & \bullet \arrow[l, "\gamma"'] \arrow[r, "\alpha"] & \bullet \end{tikzcd}$$

is a cleaving diagram of Euclidian type in $\overrightarrow{A}/\mu$. It is cleaving since:

i) $\gamma\rho = \alpha\gamma$ or $\beta_1\rho = \alpha\gamma$ contradicts Lemma 3.2.11,

ii) $\gamma\rho = \alpha^2$ or $\beta_1\rho = \alpha^2$ contradicts the minimality of $t \geq 3$.

□

Lemma 3.2.13

a) If $\mathcal{L} \nsubseteq \{\alpha^2, \alpha\beta_1, \alpha\gamma\}$, then $\alpha\beta_1 = 0$ or $\alpha\gamma = 0$.

b) If $\alpha^2\beta_1 \neq 0$, then $\gamma w \neq \alpha\beta_1$ for all $w \in \overrightarrow{A}$.

Proof. a) If $\alpha\beta_1 \neq 0$ and $\alpha\gamma \neq 0$, then

$$D := \begin{tikzcd} & \bullet \arrow[d, "\alpha"] & \\ \bullet & \bullet \arrow[l, "\gamma"] \arrow[r, "\beta_1"] \arrow[d, "\alpha"] & \bullet \\ & \bullet & \end{tikzcd}$$

is a cleaving diagram of Euclidian type $\widetilde{D}_4$ in $\overrightarrow{A}$. It is still cleaving in $\overrightarrow{A}/\eta$ for $\eta \in \mathcal{L} \setminus \{\alpha^2, \alpha\beta_1, \alpha\gamma\} \neq \emptyset$.

b) Since $\alpha^2\beta_1 \neq 0$, we have $\alpha\gamma = 0$ by *a*). But $\gamma w = \alpha\beta_1$ leads to the contradiction $0 \neq \alpha^2\beta_1 = \alpha\gamma w = 0$.

□

Lemma 3.2.14
If $t = 2$ ***or*** *$\mathcal{L} \not\subseteq \{\alpha^t, \alpha^2\beta_1\}$, then:*

a) $\alpha^2\beta_1 = 0 = \alpha^2\gamma$, $\alpha^2\rho = 0$ *for all rays* $\rho \notin \{e_x, \alpha, \dots, \alpha^{t-2}\}$.

b) $\langle\beta_1\rangle \cap \langle\alpha\gamma\rangle = 0$.

c) If $\langle\gamma\rangle \cap \langle\beta_1\rangle = 0$, *then* $\langle\gamma\rangle \cap \langle\alpha^2\rangle = 0$.

d) $\langle\gamma\rangle \cap \langle\alpha^t\rangle = 0$ *or* $\langle\gamma\rangle \cap \langle\alpha\beta_1\rangle = 0$.

e) $\langle\gamma\rangle \cap \langle\alpha\beta_1\rangle = 0$ *or* $\langle\gamma\rangle \cap \langle\beta_1\rangle = 0$.

f) $\langle\alpha\beta_1\rangle \cap \langle\alpha^2\rangle = 0$ *and* $\langle\alpha\gamma\rangle \cap \langle\alpha^2\rangle = 0$.

Proof. a) Consider the case $t = 2$.

i) If $\alpha^2\beta_1 \neq 0$, then $\beta_r\beta_1 \neq 0$ and

$$\begin{tikzcd}
\bullet & \bullet \arrow[l, "\gamma"'] \arrow[r, "\beta_1"] \arrow[d, "\alpha"] & \bullet \\
\bullet & \bullet \arrow[l, "\beta_1"'] & \bullet \arrow[l, "\beta_r"']
\end{tikzcd}$$

is a cleaving diagram of Euclidian type $\widetilde{D}_5$ in $\overrightarrow{A}/\mu$. The diagram is cleaving because:

- $\beta_1\rho = \alpha\beta_1 \neq 0$ is a contradiction of Lemma 3.2.2,
- $\gamma\rho = \alpha\beta_1 \neq 0$ contradicts Lemma 3.2.13 b).

ii) If $\alpha^2\gamma \neq 0$, then $\beta_r\gamma \neq 0$ and

$$\begin{tikzcd}
\bullet & \bullet \arrow[l, "\gamma"'] \arrow[r, "\beta_1"] \arrow[d, "\alpha"] & \bullet \\
\bullet & \bullet \arrow[l, "\gamma"'] & \bullet \arrow[l, "\beta_r"']
\end{tikzcd}$$

is a cleaving diagram in $\overrightarrow{A}/\mu$. It is cleaving since $\beta_1\rho = \alpha\gamma$ resp. $\gamma\rho = \alpha\gamma$ contradicts Lemma 3.2.11.

In the case $t \geq 3$, $\alpha^2\gamma = 0$ by Lemma 3.2.12. If $t = 3$, then $\mathcal{L} \not\subseteq \{\alpha^3, \alpha^2\beta_1\}$ by assumption. If $t > 3$, then $\mu = \nu\alpha^t\nu' \in \mathcal{L} \setminus \{\alpha^3, \alpha^2\beta_1\}$. Hence $\alpha^2\beta_1 = 0$ by Lemma 3.2.3 in both cases.

b) If v, w are rays in $\overrightarrow{A}$ such that $\beta_1 v = \alpha\gamma w \neq 0$, then the diagram

$$D := \begin{tikzcd} \bullet \arrow[d, "\gamma w"'] \arrow[dr, "\alpha^{t-1}" near start] & \bullet \arrow[d, "\tilde{\beta}"] \arrow[dl, "v"' near start] \\ \bullet & \bullet \end{tikzcd}$$

is a cleaving diagram in $\overrightarrow{A}/\mu$.

i) If $\gamma w\rho = \alpha^{t-1}$ or $v\rho = \tilde{\beta}$, then $\beta_1 v\rho = \beta_1\tilde{\beta} = \alpha^t = \alpha\gamma w\rho \neq 0$. Hence $\gamma w\rho = \alpha^{t-1}$ contradicts the minimality of t.

ii) If $\alpha^{t-1}\rho = \gamma w$ or $\tilde{\beta}\rho = v$, then $0 \neq \beta_1 v = \beta_1\tilde{\beta}\rho = \alpha\gamma w = \alpha^t\rho = 0$ by a).

c) Let v, w be rays such that $\gamma v = \alpha^2 w \neq 0$. By *a)* we have $w = \alpha^k$ with $0 \leq k \leq t-2$, that means $\gamma v = \alpha^{2+k}$. Since t is minimal, we have $t = 2+k$ and $0 \neq \gamma v = \alpha^t = \beta_1\tilde{\beta} \in \langle\gamma\rangle \cap \langle\beta_1\rangle = 0$.

d) Let v, w, v', w' be rays in $\overrightarrow{A}$ such that $\gamma w = \alpha^t v \neq 0$ and $\gamma w' = \alpha\beta_1 v' \neq 0$. Then

$$D := \begin{tikzcd} \bullet \arrow[d, "w"'] \arrow[dr, "w'" near start] & \bullet \arrow[d, "\beta_1 v'"] \arrow[dl, "\alpha^{t-1}v"' near start] \\ \bullet & \bullet \end{tikzcd}$$

is a cleaving diagram in $\overrightarrow{A}/\mu$.

i) If $w\rho = w'$ or $\alpha^{t-1}v\rho = \beta_1 v'$, then $\gamma w\rho = \gamma w' = \alpha^t v\rho = \alpha\beta_1 v' \neq 0$. Hence there is a non-deep contour $(\alpha^{t-1}v_1 \dots v_k\rho_1 \dots \rho_l, \beta_1 v'_1 \dots v'_s)$ in $\overrightarrow{A}$ which can only be a penny-farthing by the structure theorem for non-deep contours. But this case is excluded in the current section.

ii) If $w'\rho = w$ or $\beta_1 v'\rho = \alpha^{t-1}v$, then $\gamma w'\rho = \gamma w = \alpha\beta_1 v'\rho = \alpha^t v \neq 0$. Again, we have a non-deep contour $(\alpha^{t-1}v_1 \dots v_k, \beta_1 v'_1 \dots v'_l\rho_1 \dots \rho_s)$ which leads to a contradiction as before.

e) Let v, w, v', w' be rays such that $\beta_1 v = \gamma w \neq 0$ and $\alpha\beta_1 v' = \gamma w' \neq 0$. Then

$$\begin{tikzcd} \bullet \arrow[d, "w"'] \arrow[dr, "w'" near start] & \bullet \arrow[dl, "v"' near start] \arrow[dr, "\tilde{\beta}" near start] & \bullet \arrow[d, "\alpha^{t-1}"] \arrow[dl, "\beta_1 v'"' near start] \\ \bullet & \bullet & \bullet \end{tikzcd}$$

is a cleaving diagram in $\overrightarrow{A}/\mu$.

i) If $w\rho = w'$, we get the contradiction $0 \neq \gamma w\rho = \gamma w' = \beta_1 v\rho = \alpha\beta_1 v' \in \langle\beta_1\rangle \cap \langle\alpha\beta_1\rangle = 0$.

ii) If $w'\rho = w$, then $0 \neq \gamma w'\rho = \gamma w = \alpha\beta_1 v'\rho = \beta_1 v \in \langle\beta_1\rangle \cap \langle\alpha\beta_1\rangle = 0$.

iii) If $v\rho = \tilde{\beta}$, then $0 \neq \beta_1 v\rho = \beta_1\tilde{\beta} = \gamma w\rho = \alpha^t \in \langle\gamma\rangle \cap \langle\alpha^t\rangle = 0$ by d).

iv) If $\tilde{\beta}\rho = v$, then $0 \neq \beta_1\tilde{\beta}\rho = \beta_1 v = \alpha^t\rho = \gamma w \in \langle\gamma\rangle \cap \langle\alpha^t\rangle = 0$ by d).

v) If $\alpha^{t-1}\rho = \beta_1 v'$, then $0 \neq \alpha^t\rho = \alpha\beta_1 v' = \gamma w' \in \langle\gamma\rangle \cap \langle\alpha^t\rangle = 0$ by d).

vi) The case $\beta_1 v'\rho = \alpha^{t-1}$ contradicts the minimality of t.

f) If v, w are rays in $\overrightarrow{A}$ such that $\alpha\beta_1 v = \alpha^2 w \neq 0$ resp. $\alpha\gamma v = \alpha^2 w \neq 0$, then $w = \alpha^k$ with $0 \leq k \leq t-2$ and $\beta_1 v = \alpha^{1+k}$ resp. $\gamma v = \alpha^{1+k}$. Since t is minimal, we get the contradiction $t = 1 + k < t$.

$\square$

Lemma 3.2.15
If $\mathcal{L} \nsubseteq \{\alpha^2, \alpha\beta_1, \alpha\gamma\}$, then $\langle\gamma\rangle \cap \langle\alpha\gamma\rangle = 0$.

Proof. In the case $t \geq 3$, the claim is trivial since $\alpha\gamma = 0$ by 3.2.12.

Consider the case $t = 2$. Assume that there exist rays v, w in $\overrightarrow{A}$ such that $\gamma v = \alpha\gamma w \neq 0$. First of all, we deduce that $w \neq \mathrm{id}$ by Lemma 3.2.11 and $v \neq \mathrm{id}$ since γ is an arrow. Therefore we can write $v = v_1 \ldots v_s$, $w = w_1 \ldots, w_q$ with irreducible rays $v_i, w_j \in \overrightarrow{A}$. Consider the value of q:

a) If $q = 1$, then the diagram

$$\begin{tikzcd}
\bullet \arrow[r, "v_s"] & \bullet & \bullet \arrow[l, "w_1=w"'] & \bullet \arrow[l, "\gamma"'] \arrow[r, "\beta_1"] \arrow[d, "\alpha"'] & \bullet \\
 & & & \bullet \arrow[d, "\gamma"'] & \bullet \arrow[l, "\beta_r"'] \arrow[dl, dash, dashed] \\
 & & & \bullet &
\end{tikzcd}$$

is a cleaving diagram of Euclidian type $\widetilde{E}_7$ in $\overrightarrow{A}/\mu$ (see [9, 10.7]).

b) If $q \geq 2$, then the diagram

$$\begin{tikzcd}
\bullet & \bullet \arrow[l, "w_2 \ldots w_q"'] & \bullet \arrow[l, "w_1"'] & \bullet \arrow[l, "\gamma"'] \arrow[r, "\beta_1"] \arrow[d, "\alpha"'] & \bullet \\
 & & & \bullet \arrow[d, "\gamma"'] & \bullet \arrow[l, "\beta_r"'] \arrow[dl, dash, dashed] \\
 & & & \bullet &
\end{tikzcd}$$

is cleaving in $\overrightarrow{A}/\mu$.

The diagrams are cleaving because:

i) $\alpha\rho = \gamma w \neq 0$: Then $0 \neq \alpha\gamma w = \alpha^2\rho = 0$ by Lemma 3.2.14 a).

ii) $\gamma\rho = \alpha\gamma \neq 0$ contradicts Lemma 3.2.11.

iii) $\beta_1\rho = \gamma w \neq 0$: Then $0 \neq \alpha\gamma w = \alpha\beta_1\rho = 0$ since $\alpha\beta_1 = 0$ by Lemma 3.2.13.

iv) $\rho v_s = \gamma w \neq 0$: Then $\alpha\rho v_s = \alpha\gamma w \neq 0$. If $\rho = \beta_1\rho'$, then $0 = \alpha\beta_1\rho' v_s = \alpha\gamma w \neq 0$. If $\rho = \gamma\rho'$, then $\alpha\gamma\rho' v_s = \alpha\gamma w$ and $w_1 = w = \rho' v_s$. Hence $\rho' = \mathrm{id}$ and $v_s = w_1$. Therefore $0 \neq \gamma v = \gamma v_1 \dots v_{s-1} w_1 = \alpha\gamma w_1$ and $\gamma v_1 \dots v_{s-1} = \alpha\gamma$ contradicting Lemma 3.2.11. If $\rho = \alpha\rho'$, then $0 \neq \alpha\gamma w = \alpha^2\rho' v_s = 0$ by Lemma 3.2.14 a).

v) $\beta_1\rho = \alpha\gamma \neq 0$ contradicts Lemma 3.2.11.

□

Lemma 3.2.16
Let $\mathcal{L} \nsubseteq \{\alpha^t, \alpha^2\beta_1\}$ and $\mathcal{L} \nsubseteq \{\alpha^2, \alpha\beta_1, \alpha\gamma\}$.

a) If $\langle\alpha\gamma\rangle = 0 = \langle\gamma\rangle \cap \langle\alpha\beta_1\rangle$, then $\langle\beta_1, \gamma, \alpha^2\rangle \cap \langle\alpha\beta_1\rangle = 0$.

b) If $\langle\alpha\gamma\rangle = 0 = \langle\gamma\rangle \cap \langle\beta_1\rangle$, then $\langle\beta_1, \alpha^2\rangle \cap \langle\gamma, \alpha\beta_1\rangle = 0$.

c) If $\langle\alpha\beta_1\rangle = 0$, then $\langle\beta_1, \gamma, \alpha^2\rangle \cap \langle\alpha\gamma\rangle = 0$.

Proof. We only prove b); the other cases are proven analogously. Let $v, v', w, w' \in A$ be such that $\beta_1 v + \alpha^2 v' = \gamma w + \alpha\beta_1 w' \neq 0$. That means we have rays $v_i, w_j \in \overrightarrow{A}$, numbers $\lambda_i, \mu_j \in \mathbf{k}$ and integers $s_1, s_2 \geq 0,\ n_1, n_2 \geq 1$ such that

$$\sum_{i=1}^{s_1} \lambda_i\, \beta_1 v_i + \sum_{i=s_1+1}^{n_1} \lambda_i\, \alpha^2 v_i = \sum_{j=1}^{s_2} \mu_j \gamma w_j + \sum_{j=s_2+1}^{n_2} \mu_j \alpha\beta_1 w_j$$

and $\beta_1 v_i \neq \beta_1 v_j,\ \alpha^2 v_i \neq \alpha^2 v_j,\ \gamma w_i \neq \gamma w_j,\ \alpha\beta_1 w_i \neq \alpha\beta_1 w_j$ for $i \neq j$. Without loss of generality we can assume that all λ_i, μ_j are non-zero, that $\beta_1 v_i \neq \alpha^2 v_j$ for $i = 1, \dots, s_1,\ j = s_1 + 1, \dots, n_1$ and $\gamma w_i \neq \alpha\beta_1 w_j$ for $i = 1, \dots, s_2,\ j = s_2 + 1, \dots, n_2$. Then by Lemma 3.2.1 we have $n_1 = n_2$ and there exists a permutation π such that $\beta_1 v_i = \gamma w_{\pi(i)} \in \langle\beta_1\rangle \cap \langle\gamma\rangle = 0$ or $\beta_1 v_i = \alpha\beta_1 w_{\pi(i)} \in \langle\beta_1\rangle \cap \langle\alpha\beta_1\rangle = 0$ by Lemma 3.2.2. Hence $s_1 = 0$. Moreover, by Lemma 3.2.14 we have $\alpha^2 v_i = \gamma w_{\pi(i)} \in \langle\alpha^2\rangle \cap \langle\gamma\rangle = 0$ or $\alpha^2 v_i = \alpha\beta_1 w_{\pi(i)} \in \langle\alpha^2\rangle \cap \langle\alpha\beta_1\rangle = 0$; this is possible for $n_1 - s_1 = 0$ only. Hence $n_1 = 0$, contradicting the choice of n_1. □

Lemma 3.2.17
If $\mathcal{L} \subseteq \{\alpha^2, \alpha\beta_1, \alpha\gamma\}$, then there exists an α-filtration $\mathcal{F}$ of P_x having finite projective dimension.

Proof. Since $\mathcal{L} \subseteq \{\alpha^2, \alpha\beta_1, \alpha\gamma\}$, $\mu = \alpha^2$ is long and $t = 2$. Now it is easily seen that $\langle\alpha^2\rangle = \mathbf{k}\,\alpha^2 \cong S_x$, $\langle\alpha\gamma\rangle = \mathbf{k}\,\alpha\gamma$, $\langle\alpha\beta_1\rangle = \mathbf{k}\,\alpha\beta_1$ and $\langle\alpha\rangle$ has a $\mathbf{k}$-basis $\{\alpha, \alpha^2, \alpha\beta_1, \alpha\gamma\}$. Using Lemma 3.2.2 and 3.2.11 we conclude $\langle\beta_1\rangle \cap \langle\alpha\beta_1\rangle = 0$ and $\langle\gamma\rangle \cap \langle\alpha\gamma\rangle = 0 = \langle\beta_1\rangle \cap \langle\alpha\gamma\rangle$.
By Lemma 3.2.14 d) $\langle\gamma\rangle \cap \langle\alpha^2\rangle = 0$ or $\langle\gamma\rangle \cap \langle\alpha\beta_1\rangle = 0$. Thus the graph of P_x has one of the following shapes:

$$\begin{tikzcd} & e_x \arrow[dl] \arrow[d] \arrow[dr] & \\ \langle\gamma\rangle \arrow[dr, bend right] & \alpha \arrow[dl] \arrow[d] \arrow[dr] & \langle\beta_1\rangle \arrow[dl, bend left] \\ \alpha\gamma & \alpha^2 & \alpha\beta_1 \end{tikzcd} \quad \textit{or} \quad \begin{tikzcd} & e_x \arrow[dl] \arrow[d] \arrow[dr] & \\ \langle\gamma\rangle \arrow[ddrr, bend left] & \alpha \arrow[dl] \arrow[d] \arrow[dr] & \langle\beta_1\rangle \arrow[dl, bend left] \\ \alpha\gamma & \alpha^2 & \alpha\beta_1 \end{tikzcd} \quad .$$

In the first case we consider the following exact sequence:

$$0 \to \langle\alpha^2\rangle \to \langle\alpha, \beta_1, \gamma\rangle \to \langle\alpha, \beta_1, \gamma\rangle/\langle\alpha^2\rangle \to 0$$

Since $\langle\alpha\rangle$ has $\mathbf{k}$-basis $\{\alpha, \alpha^2, \alpha\beta_1, \alpha\gamma\rangle$ and $\mathcal{L} \subseteq \{\alpha^2, \alpha\beta_1, \alpha\gamma\}$ we have $\langle\alpha, \beta_1, \gamma\rangle/\langle\alpha^2\rangle = \langle\alpha\rangle/\langle\alpha^2\rangle \oplus \langle\beta_1, \gamma\rangle/\langle\alpha^2\rangle$. Hence $\operatorname{pdim}_\Lambda\langle\alpha\rangle < \infty$ and $P_x \supset \langle\alpha\rangle \supset \langle\alpha^2\rangle \supset 0$ is the wanted filtration.

In the second case we have $\langle\alpha, \beta_1, \gamma\rangle/\langle\alpha^2\rangle = \langle\alpha, \gamma\rangle/\langle\alpha^2\rangle \oplus \langle\beta_1\rangle/\langle\alpha^2\rangle$. Thus $\operatorname{pdim}_\Lambda\langle\alpha, \gamma\rangle < \infty$. Now we consider

$$0 \to \langle\beta_1, \gamma, \alpha\gamma\rangle \to \langle\alpha, \beta_1, \gamma\rangle \to S_x \to 0.$$

Since $\langle\beta_1, \gamma, \alpha\gamma\rangle = \langle\beta_1, \gamma\rangle \oplus \langle\alpha\gamma\rangle$, we have $\operatorname{pdim}_\Lambda\langle\alpha\gamma\rangle < \infty$ and $P_x \supset \langle\alpha, \gamma\rangle \supset \langle\alpha^2, \alpha\gamma\rangle \supset 0$ is a suitable filtration. $\square$

Lemma 3.2.18
If $\mathcal{L} \subseteq \{\alpha^t, \alpha^2\beta_1\}$, then there exists an α-filtration $\mathcal{F}$ of P_x having finite projective dimension.

Proof. If $t = 2$, then $\alpha^2\beta_1 = 0$ by Lemma 3.2.14 a). Hence $\mathcal{L} \subseteq \{\alpha^2\}$ and the filtration exists by Lemma 3.2.17.
If $t \geq 3$, then $\alpha\gamma = 0$ by Lemma 3.2.12. From the assumption $\mathcal{L} \subseteq \{\alpha^t, \alpha^2\beta_1\}$ it is easily seen that $\langle\alpha\beta_1\rangle = \mathbf{k}\,\alpha\beta_1$ and $\langle\alpha^2\beta_1\rangle = \mathbf{k}\,\alpha^2\beta_1$.

i) If $\alpha^2\beta_1 = 0$, then α^t is the only long morphism in $\overrightarrow{A}$; hence $\alpha\beta_1 = 0$ and $\langle\alpha^k\rangle$, $k \geq 1$, is uniserial of finite projective dimension. Thus $P_x \supset \langle\alpha\rangle \supset \langle\alpha^2\rangle \supset \ldots \supset \langle\alpha^t\rangle \supset 0$ is a suitable α-filtration.

ii) If $\alpha^2\beta_1 \neq 0$, then $\langle\alpha\beta_1\rangle = \mathbf{k}\,\alpha\beta_1 \cong S_y \cong \langle\alpha^2\beta_1\rangle$. By 3.2.2 and 3.2.13 b) $\langle\beta_1\rangle \cap \langle\alpha\beta_1\rangle = 0 = \langle\gamma\rangle \cap \langle\alpha\beta_1\rangle$. Therefore the graph of P_x has the following shape:

$$\begin{tikzcd}
 & e_x \arrow[dl] \arrow[d] \arrow[dr] & \\
\langle\gamma\rangle \arrow[dddr, bend right] \arrow[ddrr, bend right=15] & \alpha \arrow[d] \arrow[dr] & \langle\beta_1\rangle \arrow[dddl, bend left] \\
 & \alpha^2 \arrow[d] \arrow[dr] & \alpha\beta_1 \\
 & \alpha^3 \arrow[d, dotted] & \alpha^2\beta_1 \\
 & \alpha^t &
\end{tikzcd} .$$

Moreover, $\langle\alpha\beta_1\rangle \cong S_y$ is a direct summand of the module $\langle\alpha^2, \beta_1, \gamma, \alpha\beta_1\rangle$, which has finite projective dimension. Since the modules $\langle\alpha\rangle, \langle\alpha^2\rangle, \ldots, \langle\alpha^t\rangle$ have S_x and S_y as the only composition factors, they are of finite projective dimension. Thus $P_x \supset \langle\alpha\rangle \supset \langle\alpha^2\rangle \supset \ldots \supset \langle\alpha^t\rangle \supset 0$ is a suitable α-filtration.

□

Proposition 3.2.19
If $x^+ = \{\alpha, \beta_1, \gamma\}$, then there exists an α-filtration $\mathcal{F}$ of P_x having finite projective dimension.

Proof. By lemmata 3.2.17 and 3.2.18 we can assume that $\mathcal{L} \not\subseteq \{\alpha^t, \alpha^2\beta_1\}$ and $\mathcal{L} \not\subseteq \{\alpha^2, \alpha\beta_1, \alpha\gamma\}$. Then $\operatorname{pdim}_\Lambda\langle\alpha^k\rangle < \infty$ for $2 \leq k \leq t$ since $\langle\alpha^k\rangle$ has only S_x as a composition factor by 3.2.14 a). Moreover, $\operatorname{pdim}_\Lambda\langle\alpha, \beta_1, \gamma\rangle < \infty$ since it is the left hand term of the following exact sequence:

$$0 \to \langle\alpha, \beta_1, \gamma\rangle \to P_x \to S_x \to 0.$$

By Lemma 3.2.13 a) only the following two cases are possible:

i) $\alpha\beta_1 = 0$: Consider the following exact sequence:

$$0 \to \langle\beta_1, \gamma, \alpha^2, \alpha\gamma\rangle \to \langle\alpha, \beta_1, \gamma\rangle \to S_x \to 0.$$

Then $\operatorname{pdim}_\Lambda\langle\beta_1, \gamma, \alpha^2, \alpha\gamma\rangle < \infty$. By 3.2.16 c) we have $\langle\beta_1, \gamma, \alpha^2, \alpha\gamma\rangle = \langle\beta_1, \gamma, \alpha^2\rangle \oplus \langle\alpha\gamma\rangle$; hence $\operatorname{pdim}_\Lambda\langle\alpha\gamma\rangle < \infty$. Therefore $P_x \supset \langle\alpha, \beta_1, \gamma\rangle \supset \langle\alpha^2\rangle \oplus \langle\alpha\gamma\rangle \supset \langle\alpha^3\rangle \supset \ldots \supset \langle\alpha^t\rangle \supset 0$ is a suitable α-filtration.

ii) $\alpha\gamma = 0$: Then $\operatorname{pdim}_\Lambda\langle\beta_1, \gamma, \alpha^2, \alpha\beta_1\rangle < \infty$ since we have the exact sequence

$$0 \to \langle\beta_1, \gamma, \alpha^2, \alpha\beta_1\rangle \to \langle\alpha, \beta_1, \gamma\rangle \to S_x \to 0.$$

If $\langle\gamma\rangle \cap \langle\alpha\beta_1\rangle = 0$, then by 3.2.16 a) we have $\langle\beta_1, \gamma, \alpha^2, \alpha\beta_1\rangle = \langle\beta_1, \gamma, \alpha^2\rangle \oplus \langle\alpha\beta_1\rangle$; hence $\mathrm{pdim}_\Lambda\langle\alpha\beta_1\rangle < \infty$. Therefore $P_x \supset \langle\alpha, \beta_1, \gamma\rangle \supset \langle\alpha^2\rangle \oplus \langle\alpha\beta_1\rangle \supset \langle\alpha^3\rangle \supset \ldots \supset \langle\alpha^t\rangle \supset 0$ is a suitable α-filtration.
By Lemma 3.2.14 e) it remains to consider the case $\langle\gamma\rangle \cap \langle\beta_1\rangle = 0$: Then $\langle\beta_1, \gamma, \alpha^2, \alpha\beta_1\rangle = \langle\beta_1, \alpha^2\rangle \oplus \langle\gamma, \alpha\beta_1\rangle$ by 3.2.16 b). Thus $\mathrm{pdim}_\Lambda\langle\gamma, \alpha\beta_1\rangle < \infty$. Now $P_x \supset \langle\alpha, \beta_1, \gamma\rangle \supset \langle\alpha^2\rangle \oplus \langle\gamma, \alpha\beta_1\rangle \supset \langle\alpha^3\rangle \supset \ldots \supset \langle\alpha^t\rangle \supset 0$ is a suitable α-filtration.

$\square$

Chapter 4

Reduction to standard algebras

Let Λ be a representation-finite, connected $\mathbf{k}$-algebra with standard form $\overline{\Lambda}$. In this chapter we prove that the extensions of the simple Λ-modules coincide with the extensions of the corresponding simple $\overline{\Lambda}$-modules.

4.1 Coverings of k-categories

Let Λ be a representation-finite, connected $\mathbf{k}$-algebra and let Γ be the Auslander-Reiten quiver of Λ. By Bretscher, Gabriel [7, 3.1] the standard form $\overline{\Lambda}$ of Λ is isomorphic as $\mathbf{k}$-category to the full subcategory of the projective objects in the mesh-category $k(\Gamma)$ [6, see 2.2, 5.1].

A functor $F : A \to B$ between $\mathbf{k}$-categories is called a covering functor if the induced maps

$$\coprod_{F(y)=b} A(a, y) \to B(F(a), b) \text{ and } \coprod_{F(y)=b} A(y, a) \to B(b, F(a))$$

are bijective for all $a \in A$ and $b \in B$.

Let $\pi : \widetilde{\Gamma} \to \Gamma$ be the universal cover of Γ as defined in [6, 1.3], then π induces a universal covering of mesh-categories $k(\pi) : k(\widetilde{\Gamma}) \to k(\Gamma)$ [6, 2.5]. Let $\widetilde{\Lambda} \subset k(\widetilde{\Gamma})$ be the full subcategory of projectives, then the restriction of $k(\pi)$ to $\widetilde{\Lambda}$ gives a covering $G : \widetilde{\Lambda} \to \overline{\Lambda}$ [6, 3.1]. Moreover there is another covering $k(\widetilde{\Gamma}) \to$ ind- Λ which maps $y \in \widetilde{\Gamma}$ onto $\pi(y)$ and by restriction induces a covering $F : \widetilde{\Lambda} \to \Lambda$, identifying Λ with the full subcategory of the projective modules in ind- Λ. Applying [6, 3.2] we get well behaved exact functors $F_\lambda :$ mod- $\widetilde{\Lambda} \to$ mod- Λ and $G_\lambda :$ mod- $\widetilde{\Lambda} \to$ mod- $\overline{\Lambda}$ which 'coincide' on the simple modules. In which sense these functors are well behaved is summarized in the next proposition.

Proposition 4.1.1 ([6] 3.2)
Let $F : A \to B$ be a covering of $\mathbf{k}$-categories. Then there exists an exact functor $F_\lambda :$ mod- $A \to$ mod- B with the following properties:

a) $\dim M = \dim F_\lambda(M)$ *for all* $M \in \text{mod-}A$.

b) $F_\lambda(P)$ *is projective iff* P *is projective.*
$F_\lambda(S)$ *is (semi-)simple iff* S *is (semi-)simple.*

c) $F_\lambda(\operatorname{rad} M) = \operatorname{rad} F_\lambda(M)$ *and* $F_\lambda(\operatorname{top} M) = \operatorname{top} F_\lambda(M)$ *for all* $M \in \text{mod-}A$.

d) F_λ *is dense on the (semi-)simple modules.*

As a trivial consequence of this proposition we get the following.

Corollary 4.1.2
Let $F : A \to B$ *be a covering of* $\mathbf{k}$*-categories. Then there exists an exact functor* $F_\lambda : \text{mod-}A \to \text{mod-}B$ *which preserves minimal projective resolutions.*

4.2 Extensions of simples are invariant under passing to the standard form

Using the above notations we are now able to prove the main result of this chapter.

Proposition 4.2.1
Let $\widetilde{S}, \widetilde{T}$ *be simple* $\widetilde{\Lambda}$*-modules, then we have:*

$$\dim \operatorname{Ext}^i_\Lambda(F_\lambda \widetilde{S}, F_\lambda \widetilde{T}) = \dim \operatorname{Ext}^i_{\overline{\Lambda}}(G_\lambda \widetilde{S}, G_\lambda \widetilde{T}) \ \forall i \in \mathbb{N}$$

Proof. Let $\ldots \to \widetilde{Q}_i \to \widetilde{Q}_{i-1} \to \ldots \to \widetilde{Q}_1 \to \widetilde{Q}_0 \to \widetilde{S} \to 0$ be a minimal projective resolution in mod-$\tilde{\Lambda}$. Then by the above corollary we get minimal projective resolutions

$$\ldots \to F_\lambda \widetilde{Q}_i \to F_\lambda \widetilde{Q}_{i-1} \to \ldots \to F_\lambda \widetilde{Q}_1 \to F_\lambda \widetilde{Q}_0 \to F_\lambda \widetilde{S} \to 0$$

$$\ldots \to G_\lambda \widetilde{Q}_i \to G_\lambda \widetilde{Q}_{i-1} \to \ldots \to G_\lambda \widetilde{Q}_1 \to G_\lambda \widetilde{Q}_0 \to G_\lambda \widetilde{S} \to 0$$

of the simple Λ resp. $\overline{\Lambda}$-modules $S := F_\lambda \widetilde{S}$ resp. $\overline{S} := G_\lambda \widetilde{S}$. Set $T := F_\lambda \widetilde{T}$ resp. $\overline{T} := G_\lambda \widetilde{T}$. It is well known that $m = \dim \operatorname{Ext}^i_\Lambda(S, T)$ is maximal with the property that T^m is a direct summand of $\operatorname{top} F_\lambda \widetilde{Q}_i$. Let $\operatorname{top} \widetilde{Q}_i = \bigoplus_{j=1}^s \widetilde{T}_j$ be a decomposition in simple modules. Since F_λ commutes with top, m is maximal such that T^m is a direct summand of $F_\lambda \bigoplus_j \widetilde{T}_j = \bigoplus_j F_\lambda \widetilde{T}_j$. Hence we can assume $F_\lambda \widetilde{T} = T \cong F_\lambda \widetilde{T}_j$ for $j = 1, \ldots, m$ and $F_\lambda \widetilde{T}_j \ncong T$ for $j = m+1, \ldots, s$. Since F_λ and G_λ coincide on the simple modules we get $G_\lambda \widetilde{T}_j \cong G_\lambda \widetilde{T} = \overline{T}$ for $j = 1, \ldots, m$ and $G_\lambda \widetilde{T}_j \ncong \overline{T}$ for $j = m+1, \ldots, s$, hence $\overline{T}^m$ is a direct summand of $\operatorname{top} G_\lambda \widetilde{Q}_i$ and m is maximal with that property. This proves that

$$\dim \operatorname{Ext}^i_{\overline{\Lambda}}(G_\lambda \widetilde{S}, G_\lambda \widetilde{T}) = m = \dim \operatorname{Ext}^i_\Lambda(F_\lambda \widetilde{S}, F_\lambda \widetilde{T}) \ .$$

□

As an application of this proposition we get:

Corollary 4.2.2
Let Λ be a representation-finite $\mathbf{k}$*-algebra. Then the stronger no loop conjecture holds for Λ iff it holds for the standard form $\overline{\Lambda}$ of Λ.*

Remark 4.2.3
Since $\overline{\Lambda}$ is a degeneration of Λ in the sense of algebraic geometry it would be interesting to know how the strong(-er) no loop conjecture behaves under arbitrary degenerations.

Chapter 5

Reduction to single-arrowed algebras

In this chapter we introduce some technique allowing us to reduce the proof of the finitistic dimension conjecture for arbitrary algebras to single-arrowed algebras i.e. algebras having neither multiple arrows nor loops in their quiver. It will be discussed as well how this is useful for the stronger no loop conjecture.

5.1 Getting rid of double arrows and loops

Definition 5.1.1
Let $\Lambda = \mathbf{k}\mathcal{Q}\,/I$ be a finite dimensional algebra. I an admissible ideal. For $x, y \in \mathcal{Q}_0$ with α an arrow or a loop from x to y, we set

$$\tilde{\mathcal{Q}} = (\mathcal{Q}\setminus\{\alpha\}) \cup \{\alpha_1 : x \to x'\ , \alpha_2 : x' \to y\}$$

and define an injective non-unital algebra homomorphism $f : \mathbf{k}\mathcal{Q} \to \mathbf{k}\tilde{\mathcal{Q}}$ by

$$\begin{aligned} f(e_z) &= e_z,\ \forall\ z \in \mathcal{Q}_0\ ; \\ f(\alpha) &= \alpha_1\alpha_2\ ; \\ f(\beta) &= \beta,\ \forall \beta \neq \alpha \in \mathcal{Q}_1\ . \end{aligned}$$

Now we can define $\tilde{I}$ to be the two sided ideal in $\mathbf{k}\tilde{\mathcal{Q}}$ generated by $f(I)$ and set $\tilde{\Lambda} := \mathbf{k}\tilde{\mathcal{Q}}\,/\,\tilde{I}$.

Lemma 5.1.2 *a) $\tilde{I}$ is an admissible ideal in $\mathbf{k}\tilde{\mathcal{Q}}$.*

b) For $\rho \in \tilde{I}$ uniform there exists $\xi \in I,\ i, j \in \{0, 1\}$ such that $\rho = \alpha_2^i f(\xi)\alpha_1^j$.

Proof. a) Let J resp. $\tilde{J}$ be the ideal generated by the arrows in $\mathbf{k}\mathcal{Q}$ resp. $\mathbf{k}\tilde{\mathcal{Q}}$ then there exists $t \geq 2$ such that $J^t \subset I \subset J^2$. From the definition of f it is clear that $f(I) \subset \tilde{J}^2$. Now let $w \neq 0$ be a path in $\tilde{J}^{2t+2}$, we can write $w = \alpha_2^i w' \alpha_1^j$ for some $i, j \in \{0,1\}$ and $w' \in \tilde{J}^{2t}$. By the construction we know that each α_1 in w' must be followed by α_2, otherwise w would be zero. Now we can replace each $\alpha_1\alpha_2$ in w' by α to get a path $v \in \mathbf{k}\mathcal{Q}$ such that $f(v) = w'$. Since the length of w' decreases by 1 for each replacement and there are maximal t replacements possible the least possible length of v is $2t - t = t$. Hence we have $v \in J^t \subset I$, $w = \alpha_2^i w' \alpha_1^j = \alpha_2^i f(v) \alpha_1^j \in \tilde{I}$ and $\tilde{J}^{2t+2} \subset \tilde{I}$.

b) Let $\rho \in \tilde{I}$ be uniform. Then there exists a set W of paths from some z to some z' in $\tilde{\mathcal{Q}}$ such that $\rho = \sum_{w \in W} \lambda_w w$ with $\lambda_w \neq 0$ for all $w \in W$. If α_1 is a right divisor of some w then it is a right divisor of all $w \in W$ since $z' = x'$ in this case and α_1 is the single one arrow to x'. α_2 can be a left divisor in the same way. Hence we can assume that there are $i, j \in \{0,1\}$ and for each $w \in W$ paths $w' \in \tilde{\mathcal{Q}}$ with $w = \alpha_2^i w' \alpha_1^j$. Replacing $\alpha_1\alpha_2$ in each w' by α gives paths v_w in $\mathcal{Q}$ such that $f(v_w) = w'$ $\forall w \in W$. Set $\xi = \sum_{w \in W} \lambda_w v_w$ then we get $\alpha_2^i f(\xi) \alpha_1^j = \alpha_2^i f(\sum_{w \in W} \lambda_w v_w) \alpha_1^j = \sum_{w \in W} \lambda_w \alpha_2^i f(v_w) \alpha_1^j = \sum_{w \in W} \lambda_w \alpha_2^i w' \alpha_1^j = \sum_{w \in W} \lambda_w w = \rho$.

□

Definition 5.1.3
Define a covariant functor $F : \text{mod-}\Lambda \to \text{mod-}\tilde{\Lambda}$. *On objects* $M \in \text{mod-}\Lambda$ *set:*

$$F(M)(z) := \begin{cases} M(x), & z = x'; \\ M(z), & z \neq x'. \end{cases} \qquad F(M)(\beta) := \begin{cases} \mathrm{id}_{M(x)}, & \beta = \alpha_1; \\ M(\alpha), & \beta = \alpha_2; \\ M(\beta), & \beta \neq \alpha_1, \alpha_2 \end{cases},$$

on morphisms $\varphi : M \to N$*:*

$$F(\varphi)_z := \begin{cases} \varphi_x, & z = x'; \\ \varphi_z, & z \neq x'. \end{cases}$$

Let $\mathcal{C} := \{M \in \text{mod-}\tilde{\Lambda} \mid M(\alpha_1)$ *is bijective*$\}$ *be a full subcategory in* $\text{mod-}\tilde{\Lambda}$.

Lemma 5.1.4 *a) For* $\xi \in \mathbf{k}\mathcal{Q}$, $M \in \text{mod-}\Lambda$ *we have* $F(M)(f(\xi)) = M(\xi)$.

b) $\mathrm{Im}(F) = \{N \in \text{mod-}\tilde{\Lambda} \mid N(\alpha_1) = \mathrm{id}_{N(x)}\} \subset \mathcal{C}$.

c) $\mathcal{C}$ *is closed under direct summands and extensions.*

Proof. a) $F(M)(f(\beta)) = F(M)(\beta) = M(\beta)$ $\forall \beta \neq \alpha$,

$$F(M)(f(\alpha)) = F(M)(\alpha_1\alpha_2) = F(M)(\alpha_1)F(M)(\alpha_2) = \mathrm{id}_{M(x)} M(\alpha) = M(\alpha).$$

b) Let $\rho \in \tilde{I}$ be uniform, then we can write $\rho = af(\xi)b$ with $a, b \in \mathbf{k}\tilde{\mathcal{Q}},\ \xi \in I$. Since $F(M)(f(\xi)) = M(\xi)$ and $M(\xi) = 0$ for $\xi \in I$ we compute:

$$F(M)(\rho) = F(M)(af(\xi)b) = (F(M)(a))(F(M)(f(\xi)))(F(M)(b)) = 0\ .$$

Thus $F(M)$ is a $\tilde{\Lambda}$-module.
Let $N \in \text{mod-}\tilde{\Lambda}$ satisfy $N(\alpha_1) = \mathrm{id}_{N(x)}$. We define $M \in \text{mod-}\Lambda$ by

$$\begin{aligned} M(z) &:= N(z)\ for\ z \in \mathcal{Q}_0\ and \\ M(\alpha) &:= N(\alpha_2), \\ M(\beta) &:= N(\beta). \end{aligned}$$

Then $F(M) = N$ and M is a Λ-module since for $\xi \in I$ we have

$$0 = N(f(\xi)) = M(\xi)\ .$$

The last equality it suffices to check on arrows:

$$N(f(\alpha)) = N(\alpha_1\alpha_2) = N(\alpha_1)N(\alpha_2) = N(\alpha_2) = M(\alpha)\ .$$

c) For $N \in \mathcal{C}$ let $N = X \oplus Y$ be a direct sum of $\tilde{\Lambda}$-modules, that means

$$N(\alpha_1) = \begin{bmatrix} X(\alpha_1) & 0 \\ 0 & Y(\alpha_1) \end{bmatrix}.$$

Since $N(\alpha_1) : N(x) \to N(x')$ is an isomorphism by definition of $\mathcal{C}$, $X(\alpha_1)$ and $Y(\alpha_1)$ have to be isomorphisms too. Hence $X, Y \in \mathcal{C}$.
Let $0 \to X \stackrel{f}{\to} N \stackrel{g}{\to} Y \to 0$ be an exact sequence of $\tilde{\Lambda}$-modules with $X, Y \in \mathcal{C}$. Since f, g are $\tilde{\Lambda}$-module homomorphism we get the following commutative diagram of vector spaces:

$$\begin{tikzcd}
 & 0 \arrow[d] & & 0 \arrow[d] & \\
0 \arrow[r] & X(x) \arrow[r, "f_x"] \arrow[d, "X(\alpha_1)"] & N(x) \arrow[r, "g_x"] \arrow[d, "N(\alpha_1)"] & Y(x) \arrow[r] \arrow[d, "Y(\alpha_1)"] & 0 \\
0 \arrow[r] & X(x') \arrow[r, "f_{x'}"] \arrow[d] & N(x') \arrow[r, "g_{x'}"] & Y(x') \arrow[r] \arrow[d] & 0 \\
 & 0 & & 0 &
\end{tikzcd}$$

By the Snake-Lemma $N(\alpha_1)$ has to be an isomorphism.

□

Lemma 5.1.5
$F : \text{mod-}\Lambda \to \text{mod-}\tilde{\Lambda}$ is a full, faithful and exact functor. F induces an equivalence of categories $\text{mod-}\Lambda \to \mathcal{C}$.

Proof. a) For a morphism $\varphi : M \to N$ we have

$$F(M)(\alpha_1)F(\varphi)_{x'} = \mathrm{id}_{M(x)}\varphi_x = \varphi_x \mathrm{id}_{N(x)} = F(\varphi)_x F(N)(\alpha_1),$$

$$F(M)(\alpha_2)F(\varphi)_y = M(\alpha)\varphi_y = \varphi_x N(\alpha) = F(\varphi)_{x'} F(N)(\alpha_2).$$

For an arrow $\beta : z \to z'$ in $\tilde{\mathcal{Q}}$ not equal to α_1, α_2 we have $z, z' \neq x'$. Thus

$$F(M)(\beta)F(\varphi)_{z'} = M(\beta)\varphi_{z'} = \varphi_z N(\beta) = F(\varphi)_z F(N)(\beta)$$

and $F(\varphi) : F(M) \to F(N)$ is a morphism of $\tilde{\Lambda}$-modules.
Let $\varphi' : N \to K$ be another morphism of Λ-modules. Then

$$F(\varphi\varphi')_{x'} = (\varphi\varphi')_x = \varphi_x\varphi'_x = F(\varphi)_{x'}F(\varphi')_{x'},$$

$$F(\varphi\varphi')_z = (\varphi\varphi')_z = \varphi_z\varphi'_z = F(\varphi)_z F(\varphi')_z \;\forall\; z \neq x'.$$

b) F is full: Let $\tilde{\varphi} : F(M) \to F(N)$ be a morphism in $\text{mod-}\tilde{\Lambda}$. Then

$$\mathrm{id}_{M(x)}\tilde{\varphi}_{x'} = F(M)(\alpha_1)\tilde{\varphi}_{x'} = \tilde{\varphi}_x F(N)(\alpha_1) = \tilde{\varphi}_x \mathrm{id}_{N(x)}$$

and $\tilde{\varphi}_{x'} = \tilde{\varphi}_x$ holds. Define $\varphi : M \to N$ by $\varphi_z = \tilde{\varphi}_z$ for all $z \neq x'$, then $F(\varphi) = \tilde{\varphi}$.
F is faithful: Let $\varphi : M \to N$ be such that $F(\varphi) = 0$, then $F(\varphi)_z = 0$ for all $z \in \tilde{\mathcal{Q}}_0$. Hence $0 = F(\varphi)_z = \varphi_z$ for all $z \neq x'$ and $\varphi = 0$.

c) F is dense in $\mathcal{C}$: For $M \in \mathcal{C}$ $M(\alpha_1) : M(x) \to M(x')$ is an isomorphism. Define $N \in \text{mod-}\tilde{\Lambda}$ by $N(x) := M(x'),\; N(z) := M(z)$ for $z \neq x$.

$$x \neq y: \;\; N(\beta) = \begin{cases} \mathrm{id}_{M(x')}, & \beta = \alpha_1; \\ M(\alpha_1)^{-1}M(\beta), & \beta : x \to y; \\ M(\beta)M(\alpha_1), & \beta : z \to x; \\ M(\beta), & \textit{else}. \end{cases}$$

$$x = y: \;\; N(\beta) = \begin{cases} \mathrm{id}_{M(x')}, & \beta = \alpha_1; \\ M(\alpha_1)^{-1}M(\beta)M(\alpha_1), & \beta : x \to y; \\ M(\beta)M(\alpha_1), & \beta : z \to x,\; z \neq x; \\ M(\alpha_1)^{-1}M(\beta), & \beta : x \to z,\; z \neq x; \\ M(\beta)\;, & \textit{else}. \end{cases}$$

Then $M \overset{\varphi}{\cong} N$ with $\varphi_x := M(\alpha_1),\; \varphi_z := \mathrm{id}_{M(z)}$ for $z \neq x$. And $N \in \mathrm{Im}(F)$ since $N(\alpha_1) = \mathrm{id}_{N(x)}$.

e) F is exact: Let $0 \to M \overset{\varphi}{\to} N \overset{\psi}{\to} K \to 0$ be an exact sequence in mod-Λ, that means $0 \to M(z) \overset{\varphi_z}{\to} N(z) \overset{\psi_z}{\to} K(z) \to 0$ is exact for all $z \in \mathcal{Q}_0$. Hence $0 \to F(M)(z) \overset{F(\varphi)_z}{\to} F(N)(z) \overset{F(\psi)_z}{\to} F(K)(z) \to 0$ is exact for all $z \in \tilde{\mathcal{Q}}_0$.

$\square$

Lemma 5.1.6 *a) For $M := P_{z\tilde{\Lambda}} = e_z\,\tilde{\Lambda},\ z \in \tilde{\mathcal{Q}}_0,\ M(\alpha_1) : M(x) \to M(x')$ is injective.*

b) The indecomposable projective $\tilde{\Lambda}$-modules $P_{z\tilde{\Lambda}},\ z \neq x'$ form a complete representative system of the indecomposable projective objects in $\mathcal{C}$.

c) $F(P_{z\Lambda}) \cong P_{z\tilde{\Lambda}}$ for all $z \neq x'$.

Proof. a) For $M := P_{z\tilde{\Lambda}}$ by definition $M(\alpha_1) : M(x) \xrightarrow{-\cdot\alpha_1} M(x')$ is the right multiplication with α_1 in $\tilde{\Lambda}$ and $M(x)$ resp. $M(x')$ has the residue classes of paths from z to x resp. x' as a basis. Let $\rho \in \mathbf{k}\tilde{\mathcal{Q}}$ be uniform such that $\overline{\rho} \in \ker M(\alpha_1)$, then $\rho\alpha_1 \in \tilde{I}$ implies $\rho\alpha_1 = \alpha_2^i f(\xi)\alpha_1$ for suitable $i \in \{0, 1\}$ and $\xi \in I$. Thus $\rho = \alpha_2^i f(\xi) \in \tilde{I}$ holds implying $\overline{\rho} = 0$.

b) Since α_1 is the single arrow which ends in x', all paths from $z \neq x'$ to x' have to end by α_1. Hence $M(\alpha_1)$ is surjective and an isomorphism since it is injective by a).
Since mod-Λ and $\mathcal{C}$ are equivalent categories, they have the number of $|\,\mathcal{Q}_0\,|$ indecomposable projective objects, hence the $P_{z\tilde{\Lambda}}$ for $z \neq x'$ are the right number of indecomposable projectives in mod-$\tilde{\Lambda}$ which are indecomposable projective in $\mathcal{C}$ too.

c) For $z \neq x'$ we have $M := F(P_{z\Lambda})$ is indecomposable projective in $\mathcal{C}$. Since $\mathcal{C}$ is closed under direct summands M is indecomposable projective in mod-$\tilde{\Lambda}$ too. For $z \neq x, x'$ we have $F(S_{z\Lambda}) \cong S_{z\tilde{\Lambda}}$ by definition and since F is exact $S_{z\tilde{\Lambda}}$ is the top of M. For $z = x$ $F(S_{x\Lambda})$ is not the simple $S_{x\tilde{\Lambda}}$ but it is easily seen that it has $S_{x\tilde{\Lambda}}$ in the top, hence in this case $S_{x\tilde{\Lambda}}$ is the top of M.

$\square$

Theorem 5.1.7
With above notations we have $\Omega_{\tilde{\Lambda}}(F(M)) \cong F(\Omega_\Lambda(M))$ for all Λ-modules M. In particular: $\operatorname{pdim}_\Lambda M = \operatorname{pdim}_{\tilde{\Lambda}} F(M)$ and $\operatorname{findim}\Lambda \leq \operatorname{findim}\tilde{\Lambda}$.

Proof. Let $0 \to \Omega(M) \to P \to M \to 0$ be exact with P a projective cover of M. Since F is exact $0 \to F(\Omega(M)) \to F(P) \to F(M) \to 0$ is exact with projective middle term. If $F(P)$ is not the projective cover of $F(M)$ we get a projective direct summand $Q \neq 0$ of $F(\Omega(M)) = X \oplus Q \in$ mod-$\tilde{\Lambda}$. As $\mathcal{C} \subset$ mod-$\tilde{\Lambda}$ is closed under direct summands and F is an equivalence on $\mathcal{C}$, Q induces a projective direct summand $F^{-1}(Q)$ of $\Omega(M)$. Hence $F^{-1}(Q) = 0$ and $Q = 0$ a contradiction. $\square$

Iteration of the previous construction leads to a new algebra $\tilde{\Lambda}$ which arises from Λ by replacing all multiple arrows and loops $x_i \stackrel{\alpha_i}{\to} y_i$ by $x_i \stackrel{\alpha_{i1}}{\to} x_i' \stackrel{\alpha_{i2}}{\to} y_i$ for $i = 1, \ldots, t$. Generalization of the above results to this situation provides a subcategory $\mathcal{C} = \{M \in \text{mod-}\tilde{\Lambda} \mid M(\alpha_{i1}) \text{ bijective } \forall\ i = 1, \ldots, t\} \subset \text{mod-}\tilde{\Lambda}$, an equivalence $F : \text{mod-}\tilde{\Lambda} \to \mathcal{C}$ and the following result:

Theorem 5.1.8
For Λ and $\tilde{\Lambda}$ as above we have:

a) $\operatorname{findim} \Lambda \leq \operatorname{findim} \tilde{\Lambda} \leq \operatorname{findim} \Lambda + 2$, *and these bounds are sharp.*

b) $\operatorname{gldim} \tilde{\Lambda} = \begin{cases} \operatorname{gldim} \Lambda + 1, & \mid \mathcal{Q}_0 \mid = 1; \\ \operatorname{gldim} \Lambda, & \mid \mathcal{Q}_0 \mid > 1. \end{cases}$

Proof. a) Let M be a submodule of the radical of a projective $\tilde{\Lambda}$-module Q. As $Q(\alpha_{i1})$ is injective and M is a submodule of Q, $M(\alpha_{i1})$ has to be injective too. Now we look at the following exact sequence:

$$0 \to N \stackrel{\varphi}{\to} P \stackrel{\psi}{\to} M \to 0$$

with P a $\tilde{\Lambda}$-projective cover of M. Since the only indecomposable projective $\tilde{\Lambda}$-modules which are not in $\mathcal{C}$ are $P_{x_i'}$ and $\operatorname{rad} P_{x_i'} = P_{y_i} \in \mathcal{C}$ for $i = 1, \ldots, t$ there exists a module $V \in \mathcal{C}$ such that

$$N \subset \operatorname{rad} P \subset V \subset P$$

and for $i \in \{1, \ldots, t\}$ $V(\alpha_{i1}) : V(x_i) \to V(x_i')$ is bijective. Moreover $V(\alpha_{i1})$ is the restriction of $P(\alpha_{i1})$ to $V(x_i) = Ve_{x_i}$. Let $v = ve_{x_i'} \in N$ be right uniform, then

$$\varphi(v) = \varphi(v)e_{x_i'} \in Ve_{x'} = \operatorname{Im} V(\alpha_{i1}).$$

Hence there exists $p \in Ve_{x_i} \subset Pe_{x_i}$ such that

$$\varphi(v) = V(\alpha_{i1})(p) = P(\alpha_{i1})(p).$$

Now we derive

$$0 = \psi(\varphi(v)) = \psi(P(\alpha_{i1})(p)) = M(\alpha_{i1})(\psi(p))$$

and deduce that $p \in \ker \psi = \operatorname{Im} \varphi$ since $M(\alpha_{i1})$ is injective. Hence there exists $w = we_{x_i} \in Ne_{x_i}$ such that $\varphi(w) = p$ and

$$\varphi(N(\alpha_{i1})(w)) = P(\alpha_{i1})(\varphi(w)) = V(\alpha_{i1})(\varphi(w)) = \varphi(v).$$

As φ is injective we have $N(\alpha_{i1})(w) = v$. This shows that $N(\alpha_{i1})$ is surjective. Trivially $N(\alpha_{i1})$ is injective since N is a submodule of P and $P(\alpha_{i1})$ is injective. Hence the second syzygy of every $\tilde{\Lambda}$-module is in $\mathcal{C}$ and the upper bound follows.

The sharpness of the lower bound we see for hereditary algebras. For the upper bound set $\Lambda = \mathbf{k}\left(\begin{tikzcd} x \arrow[loop above, "\alpha"] \end{tikzcd} \right)/(\alpha^2)$. Then $\operatorname{findim} \Lambda = 0$. For $\tilde{\Lambda}$ we get the quiver $\tilde{\mathcal{Q}} = \begin{tikzcd} x \arrow[r, bend left, "\alpha_1"] & x' \arrow[l, bend left, "\alpha_2"] \end{tikzcd}$ and the minimal projective resolution of the indecomposable injective $I_{x'}$ is $0 \to P_x \to P_{x'} \to P_{x'} \to I_{x'} \to 0$ hence $\operatorname{findim} \tilde{\Lambda} = 2$.

b) Since $0 \to P_{y_i} \to P_{x'_i} \to S_{x'_i} \to 0$ is exact, we have $\operatorname{pdim} S_{x'_i} = 1$ for all $i = 1, \ldots, t$.

$\square$

Corollary 5.1.9
The finitistic dimension conjecture holds in general if it holds for all finite dimensional single-arrowed algebras.

5.2 Use for the stronger no loop conjecture

Now we would like to get some analogous results for the stronger no loop conjecture. But the problem is that for the simple module $S_{x\Lambda} \in \operatorname{mod-}\Lambda$ we have $F(S_{x\Lambda}) \ncong S_{x\tilde{\Lambda}}$. Hence we have to know something about the self-extensions of $F(S_{x\Lambda})$ to get information about those of $S_{x\Lambda}$. Nevertheless for all other simple Λ-modules $S_{z\Lambda}$, $z \neq x$, we have by Theorem 5.1.7:

$$\dim \operatorname{Ext}^i_\Lambda(S_{z\Lambda}, S_{z\Lambda}) = \dim \operatorname{Ext}^i_{\tilde{\Lambda}}(S_{z\tilde{\Lambda}}, S_{z\tilde{\Lambda}}).$$

The equality holds since $F(S_{z\Lambda}) \cong S_{z\tilde{\Lambda}}$ and $F(P_{z\Lambda}) \cong P_{z\tilde{\Lambda}}$.

To solve the problem we define an analogous functor $G : \operatorname{mod-}\Lambda \to \operatorname{mod-}\tilde{\Lambda}$. On objects $M \in \operatorname{mod-}\Lambda$ we set:

$$G(M)(z) := \begin{cases} M(y), & z = x'; \\ M(z), & z \neq x'. \end{cases} \qquad G(M)(\beta) := \begin{cases} \mathrm{id}_{M(y)}, & \beta = \alpha_2; \\ M(\alpha), & \beta = \alpha_1; \\ M(\beta), & \beta \neq \alpha_1, \alpha_2. \end{cases}$$

On morphisms $\varphi : M \to N$:

$$G(\varphi)_z := \begin{cases} \varphi_y, & z = x'; \\ \varphi_z, & z \neq x'. \end{cases}$$

Define $\mathcal{D} := \{M \in \text{mod-}\tilde{\Lambda} \mid M(\alpha_2) \textit{ is bijective}\}$. Then we get analogous results:

Lemma 5.2.1 *a) $\mathcal{D}$ is closed under direct summands and extensions.*

b) The indecomposable injective $\tilde{\Lambda}$-modules $I_{z\tilde{\Lambda}}$, $z \neq x'$ form a complete representative system of the indecomposable injective objects in $\mathcal{D}$.

c) $G : \text{mod-}\Lambda \to \text{mod-}\tilde{\Lambda}$ is a full, faithful and exact functor. G induces an equivalence of categories $\text{mod-}\Lambda \to \mathcal{D}$.

d) $G(I_{z\Lambda}) = I_{z\tilde{\Lambda}}$ for all $z \neq x'$.

Now the advantage is that for $x \neq y$ i.e. α is not a loop, the image of $S_{x\Lambda}$ under G is $S_{x\tilde{\Lambda}}$. Hence $\dim \operatorname{Ext}^i_\Lambda(S_{x\Lambda}, S_{x\Lambda}) = \dim \operatorname{Ext}^i_{\tilde{\Lambda}}(S_{x\tilde{\Lambda}}, S_{x\tilde{\Lambda}})$ for all $i \geq 0$. And we can formulate the desired result.

Corollary 5.2.2
The stronger no loop conjecture holds in general if it holds for all algebras without multiple arrows, multiple loops are allowed.

5.3 Generalization

The following generalization is due to Xi [27].

Let A be an artin algebra and e an idempotent element of A. Then eA is an (eAe, A)-bimodule and we get functors $F := \_ \otimes_{eAe} eA : \text{mod-}eAe \to \text{mod-}A$, $H := \operatorname{Hom}_A(eA, \_) : \text{mod-}A \to \text{mod-}eAe$.

Lemma 5.3.1 *a) F is left adjoint to H and $H \circ F \cong \mathrm{id}_{\text{mod-}eAe}$*

b) H is exact and F maps projective eAe-modules to projective A-modules.

c) If eA is projective in eAe-mod, then F is exact and H maps projective A-modules to projective eAe-modules.

Proof. a) Well known.

b) Since $eA \in \text{mod-}A$ is projective H is exact. For $e' = e'e$ an idempotent element of A let e'' be the complement of e' in e i.e. $e = e' + e''$, $e'e'' = 0$. Then we have $F(e'Ae) = e'Ae \otimes_{eAe} eA \cong e'A$ as A-module, since $e'A = e'(eA) = e'(eAe \otimes_{eAe} eA) = e'(e'Ae \otimes_{eAe} eA \oplus e''Ae \otimes_{eAe} eA) = e'Ae \otimes_{eAe} eA$.

c) F is exact since $eA \in \text{mod-}A$ is projective. For $P \in \text{mod-}A$ projective we have $\operatorname{Hom}_{eAe}(-, H(P)) \cong \operatorname{Hom}_A(F(-), P)$ is an exact functor, hence $H(P)$ is projective eAe-module.

□

Proposition 5.3.2
Let e be such that eA is projective in eAe-mod. Then: $\operatorname{findim} eAe \leq \operatorname{findim} A$.

Proof. Set $m = \operatorname{findim} A$ and take $M \in \text{mod-}eAe$ of finite projective dimension. If $s := \operatorname{pdim}_{eAe} M \leq m$ we have nothing to show hence we can assume $s > m$. As the projective eAe-modules are of the form $H(P) = Pe$ for projective $P \in \text{mod-}A$ we get

$$0 \to P_s e \to P_{s-1} e \to \cdots \to P_1 e \to P_0 e \to M \to 0$$

as minimal projective resolution of M. Since F is exact

$$0 \to F(P_s e) \to F(P_{s-1} e) \to \cdots \to F(P_1 e) \to F(P_0 e) \to F(M) \to 0$$

is exact as well and a projective resolution of $F(M)$ in mod- A. Since $\operatorname{findim} A = m$ the kernel N of the morphism $F(P_{m-1}) \to F(P_{m-2})$ has to be projective in mod- A and leads to the exact sequence:

$$0 \to N \to F(P_{m-1} e) \to \cdots \to F(P_1 e) \to F(P_0 e) \to F(M) \to 0 \ .$$

By applying H to the last sequence we get

$$0 \to H(N) \to P_{m-1} e \to \cdots \to P_1 e \to P_0 e \to M \to 0$$

with $H(N)$ projective in mod- eAe. Hence $s = \operatorname{pdim}_{eAe} M \leq m$ a contradiction. $\square$

Chapter 6

The first finitistic dimension conjecture fails

In the first section of this chapter we present the example of Smalø [22] showing that the first finitistic dimension conjecture fails. In the second section the finitistic dimension of tensor algebras will be computed following the arguments of Rickard.

But first we state some easy fact:

Definition 6.0.3
For a Λ-module M we call a subspace $U \subset M$ ***characteristic*** *if $f(U) \subset U$ holds for all $f \in \mathrm{End}_\Lambda(M)$.*

Remark 6.0.4
Let $U \subset M$ be characteristic and $a \in \Lambda$, then

$$Ua := \{ua | u \in U\}$$

and

$$Ua^{-1} := \{m \in M | ma \in U\}$$

are characteristic in M.

6.1 The Smalø example

Definition 6.1.1
Let $\mathcal{Q}_n$ be the following quiver:

$$\begin{tikzcd} n \arrow[r, bend left, "\rho_n"] \arrow[r, "\sigma_n"] \arrow[r, bend right, "\tau_n"'] & n-1 \arrow[r, bend left, "\rho_{n-1}"] \arrow[r, "\sigma_{n-1}"] \arrow[r, bend right, "\tau_{n-1}"'] & n-2 \;\cdots\cdots\; 2 \arrow[r, bend left, "\rho_2"] \arrow[r, "\sigma_2"] \arrow[r, bend right, "\tau_2"'] & 1 \arrow[r, bend left, "\rho_1"] \arrow[r, "\sigma_1"] \arrow[r, bend right, "\tau_1"'] & 0 \arrow[loop above right, "\beta"] \arrow[loop below right, "\alpha"'] \end{tikzcd}$$

and let I_n be the ideal in $\mathbf{k}\mathcal{Q}$ generated by the following elements:

$$\begin{aligned} &\alpha^2,\ \beta^2,\ \alpha\beta,\ \beta\alpha,\ \sigma_1\alpha,\ \rho_1\alpha,\ \tau_1\beta\ , \\ &x_{i+1}y_i \ for\ x \neq y\ x,y \in \{\rho,\sigma,\tau\}, i = 1,\dots,n\ , \\ &x_{i+1}x_i - y_{i+1}y_i\ for\ x,y \in \{\rho,\sigma,\tau\},\ i = 1,\dots,n\ . \end{aligned}$$

Theorem 6.1.2
For $\Lambda_n := \mathbf{k}\mathcal{Q}_n/I_n$, $n \geq 1$ we have

$$\operatorname{findim}\Lambda_n = 1 \ \textit{and}\ \operatorname{Findim}\Lambda_n = n.$$

Proof. First of all we look at the graphs of the indecomposable projective Λ_n-modules.

$$\begin{tikzcd} & e_0 \arrow[dl] \arrow[dr] & \\ \alpha & & \beta \end{tikzcd} \qquad \begin{tikzcd} & e_1 \arrow[dl] \arrow[d] \arrow[dr] & \\ \rho_1 \arrow[d] & \sigma_1 \arrow[d] & \tau_1 \arrow[d] \\ \rho_1\beta & \sigma_1\beta & \tau_1\alpha \end{tikzcd} \qquad \begin{tikzcd} & e_2 \arrow[dl] \arrow[d] \arrow[dr] & \\ \rho_2 \arrow[dr] & \sigma_2 \arrow[d] & \tau_2 \arrow[dl] \\ & \rho_2\rho_1 & \end{tikzcd}$$

$$\begin{tikzcd} & e_{i+1} \arrow[dl] \arrow[d] \arrow[dr] & \\ \rho_{i+1} \arrow[dr] & \sigma_{i+1} \arrow[d] & \tau_{i+1} \arrow[dl] \\ & \rho_{i+1}\rho_i & \end{tikzcd} \qquad \forall i = 1,\dots,n-1.$$

Hence the radical lengths $\operatorname{rl} P_i$ of the projective modules $P_i = e_i\,\Lambda_n$ are 3 except for P_0, which has radical length 2. Remark that for $i = 0,\dots,n$ Λ_i-Mod embeds into Λ_n-Mod.

a) First we show that $\operatorname{findim}\Lambda_n = 1$. Let M be a Λ_n-module of finite projective dimension. Then the last two terms

$$0 \to Q_m \to Q_{m-1} \to \dots$$

of a minimal projective resolution of M induce an inclusion f of a projective module $P = Q_m \in \text{mod-}\Lambda_n$ into the radical of another projective Λ_n-module

$Q = Q_{m-1}$. Since the radical length of the radical of a projective Λ_n-module is less than 3, P has to be a finite product of copies of P_0, namely $P = P_0^m$. As P_0 has non-trivial morphisms only to P_0, P_1 and P_2 we can assume that $Q = P_0^{m_0} \oplus P_1^{m_1} \oplus P_2^{m_2}$. Hence

$$f = (f_0, f_1, f_2) : P_0^m \to P_0^{m_0} \oplus P_1^{m_1} \oplus P_2^{m_2}$$

with $\mathrm{Im}(f_i) \subset \operatorname{rad} P_i^{m_i}$ for $i = 0, 1, 2$. Inspection of the graphs of the indecomposable projectives reveals that

$$f_0(P_0^m) \subset \operatorname{rad} P_0^{m_0} = \operatorname{soc} P_0^{m_0} \ ,$$

$$f_2(P_0^m) \subset \operatorname{rad}^2 P_2^{m_2} = \operatorname{soc} P_2^{m_2}$$

thus

$$f_i(\operatorname{soc} P_0^m) = f_i(\operatorname{rad} P_0^m) \subset \operatorname{rad} \operatorname{soc} P_i^{m_i} = 0 \ for\ i = 0, 2.$$

Now let $u \neq 0$ be an element in the socle of P_0^m, then

$$0 \neq f(u) = (f_0(u), f_1(u), f_2(u)) = (0, f_1(u), 0) \ ,$$

that means f_1 is injective on the socle of P_0^m, hence $f_1 : P_0^m \to P_1^{m_1}$ is an inclusion and induces an inclusion $\alpha P_0^m \to \alpha P_1^{m_1}$. This is possible only for

$$m = \dim \alpha P_0^m \leq \dim \alpha P_1^{m_1} = m_1.$$

Since

$$\dim \operatorname{rad}^2 P_1^{m_1} \cap \mathrm{Im} f = 2m < 3m_1 = \dim \operatorname{rad}^2 P_1^{m_1}$$

there exists $u \in \operatorname{rad}^2 P_1^{m_1} \setminus \mathrm{Im} f$. Hence $0 \neq u \in \operatorname{rad}^2 \operatorname{coker} f$ and $\operatorname{rl} \operatorname{coker} f = 3$. That means that the cokernel of any inclusion $P \to Q$ of finitely generated projective Λ_n-modules P and Q with the image in the radical of Q has radical length three and can't embed in the radical of any projective Λ_n-module. Therefore a minimal projective resolution of finitely generated Λ_n-modules has maximally the length 1, which shows that $\operatorname{findim} \Lambda_n = 1$ for $n \geq 1$.

b) Now we show that $\operatorname{Findim} \Lambda_n = n$ holds. An easy observation is that for $i = 1, \ldots, n$, $M \in \text{mod-}\Lambda_i$ the first syzygy $N := \Omega_{\Lambda_i}(M)$ of M is a Λ_{i-1}-module and the projective resolutions of N as Λ_i- or Λ_{i-1}-module coincide. Therefore $\Omega^n_{\Lambda_n}(M)$ is an Λ_0-module and hence projective or of infinite projective dimension as Λ_0-module. Therefore $\operatorname{pdim}_{\Lambda_n} M \leq n$ or $\operatorname{pdim}_{\Lambda_n} M = \infty$. That means $\operatorname{Findim} \Lambda_n \leq n$.
To show that $\operatorname{Findim} \Lambda_n \geq n$ let for $i \in \mathbb{N}$, $j \in \{0, 1\}$, $e_{j,i}$ be the coordinates of $P_j^{(\mathbb{N})}$ with e_j in the i'th place and zero otherwise. Now let $\varphi : P_0^{(\mathbb{N})} \to P_1^{(\mathbb{N})}$ be given by

$$\varphi(e_{0,2i-1}) = e_{1,2i-1}\tau_1 + e_{1,i}\sigma_1 \ ,$$

$$\varphi(e_{0,2i}) = e_{1,2i}\tau_1 + e_{1,i}\rho_1 \ ,$$

where τ_1, σ_1, ρ_1 denote the related residue classes in P_1. To show that φ is an inclusion it suffices to verify that it is injective on the socle. But this is clear since the coordinates $e_{0,i}$ form a basis of the socle of $P_0^{(\mathbb{N})}$ and there images under φ are linearly independent. An easy calculation provides that

$$\left\langle \begin{array}{ll} e_{1,i}\tau_1\alpha & \\ e_{1,i}\rho_1\beta & \mid i \in \mathbb{N} \\ e_{1,i}\sigma_1\beta & \end{array} \right\rangle \subset \operatorname{Im}\varphi$$

hence $\operatorname{coker}\varphi$ is annihilated by the residue classes of α and β and we get:

$$X_1 := \operatorname{coker}\varphi = \left\langle \begin{array}{ll} e_{1,i} & \\ e_{1,i}\rho_1 & \\ e_{1,i}\sigma_1 & \mid i \in \mathbb{N} \\ e_{1,i}\tau_1 & \end{array} \right\rangle \Big/ \left\langle \begin{array}{ll} e_{1,2i-1}\tau_1 + e_{1,i}\sigma_1 & \\ e_{1,2i}\tau_1 + e_{1,i}\rho_1 & \mid i \in \mathbb{N} \end{array} \right\rangle .$$

Therefore

i) $\operatorname{rl} X_1 = 2$,

ii) $\operatorname{soc} X_1 = S_0^{(\mathbb{N})}$.

Moreover X_1 is a $A := \Lambda_1 / (\alpha, \beta)$-module and we can embed X_1 in his minimal injective envelope $I_{0A}^{(\mathbb{N})}$. But since $I_{0A} = \operatorname{rad} P_{2\Lambda_n}$ we get an embedding $\psi : X_1 \to P_2^{(\mathbb{N})}$ as Λ_n-module such that $\operatorname{soc} P_2^{(\mathbb{N})} \subset \operatorname{Im}\psi \subset \operatorname{rad} P_2^{(\mathbb{N})}$. This leads to a quotient module $X_2 := \operatorname{coker}\psi$ which is annihilated by α, β and, under the assumption

iii) X_1 indecomposable

we deduce that

i) $\operatorname{rl} X_2 = 2$,

ii) $\operatorname{soc} X_2 = S_1^{(\mathbb{N})}$ and

iii) X_2 is indecomposable.

Since $\operatorname{soc} P_2^{(\mathbb{N})} \subset \operatorname{Im}\psi$ we have $\operatorname{rl} X_2 \leq \operatorname{rl}(P_2^{(\mathbb{N})} / \operatorname{soc} P_2^{(\mathbb{N})}) = 2$. $\operatorname{Im}\psi \subset \operatorname{rad} P_2^{(\mathbb{N})}$ hence X_1 is the first syzygy of X_2 as Λ_n-module. If we assume $X_2 = Y \oplus Y'$ as Λ_n-module then $X_1 = \Omega(X_2) = \Omega(Y) \oplus \Omega(Y')$ and since X_1 is indecomposable we can assume $\Omega(Y) = 0$. That means Y is a projective direct summand of X_2. Since $\operatorname{rl} X_2 \leq 2$ and all indecomposable projectives which are annihilated by α, β have radical length 3, Y has to be zero.

As an infinite dimensional indecomposable module X_2 can't have radical length 1 hence $\operatorname{rl} X_2 = 2$. By construction $\operatorname{soc} X_2 \in \operatorname{add}(S_1 \oplus S_2)$, assume $S_2 \overset{\iota}{\hookrightarrow} X_2$. Inspection of $S_2 \overset{\iota}{\hookrightarrow} X_2$ on the level of representations provides that S_2 has to be a direct summand of X_2. This is impossible since X_2 is indecomposable of radical length 2. Therefore $\operatorname{soc} X_2 = S_1^{(\mathbb{N})}$. Proceeding by induction we construct an exact sequence

$$0 \to P_0^{(\mathbb{N})} \xrightarrow{f_1:=\varphi} P_1^{(\mathbb{N})} \xrightarrow{f_2:=\psi} \dots \xrightarrow{f_n} P_n^{(\mathbb{N})} \to X_n \to 0$$

with $X_i = \operatorname{coker} f_i$ such that

i) $\operatorname{rl} X_i = 2$,

ii) $\operatorname{soc} X_i = S_{i-1}^{(\mathbb{N})}$ and

iii) X_i is indecomposable.

Therefore this is a minimal projective resolution of X_n and $\operatorname{pdim} X_n = n$ which proves the claim.

To complete the proof we have to verify that X_1 is indecomposable. First of all we derive

$$0(\tau_1 + \sigma_1)^{-1} = \ker(x \mapsto x\tau_1 + x\sigma_1) = \mathbf{k}\, e_{1,1} \subset X_1.$$

Hence

$$\mathbf{k}\, e_{1,1} \subset X_1,\ (\mathbf{k}\, e_{1,1})\tau_1 = \mathbf{k}(e_{1,1}\tau_1),\ (\mathbf{k}\, e_{1,1})\rho_1 = \mathbf{k}(e_{1,2}\tau_1) \subset X_1\ ,$$
$$\mathbf{k}\, e_{1,2} = (\mathbf{k}\, e_{1,2}\tau_1)\tau_1^{-1} \subset X_1$$

are characteristic in X_1. Proceeding by induction we get that

$$\mathbf{k}\, e_{1,i},\ \mathbf{k}\, e_{1,i}\tau_1 \subset X_1$$

are characteristic in X_1 for all $i \in \mathbb{N}$. Now let f be an endomorphism of X_1, then

$$f(\mathbf{k}\, e_{1,i}) \subset \mathbf{k}\, e_{1,i}\ ,$$
$$f(\mathbf{k}\, e_{1,i}\tau_1) \subset \mathbf{k}\, e_{1,i}\tau_1\ for\ all\ i \in \mathbb{N}\ .$$

Hence

$$f(e_{1,i}) = \lambda_i e_{1,i}\ ,$$
$$f(e_{1,i}\tau_1) = \lambda_i e_{1,i}\tau_1$$

for $\lambda_i \in \mathbf{k}$ and since

$$0 = f(e_{1,2i-1}\tau_1 + e_{1,i}\sigma_1) = \lambda_{2i-1}e_{1,2i-1}\tau_1 + \lambda_i \underbrace{e_{1,i}\sigma_1}_{-e_{1,2i-1}\tau_1} = (\lambda_{2i-1} - \lambda_i)e_{1,2i-1}\tau_1$$

we have $\lambda_{2i-1} = \lambda_i =: \lambda$ for all $i \in \mathbb{N}^{>0}$, therefore $f = \lambda \cdot \operatorname{id}$ and X_1 is indecomposable.

$\square$

6.2 Finitistic dimension of tensor algebras

In this section we show that the big resp. little finitistic dimension of the tensor algebra $A \otimes_{\mathbf{k}} B$ equals the sum of the corresponding finitistic dimensions of A and B. Using this and the first example of Huisgen-Zimmermann [14] of a finite dimensional algebra where the two dimensions do not coincide one can construct an algebra with an arbitrary large difference between these dimensions.

Remark 6.2.1
Consider the following commutative diagram of Λ-modules:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & A_1 \arrow[r, "\alpha_1"] \arrow[d, "\varphi_1"] & A_2 \arrow[r, "\alpha_2"] \arrow[d, "\varphi_2"] & A_3 \arrow[r] \arrow[d, "\varphi_3"] & 0 \\
0 \arrow[r] & B_1 \arrow[r, "\beta_1"] \arrow[d, "\psi_1"] & B_2 \arrow[r, "\beta_2"] \arrow[d, "\psi_2"] & B_3 \arrow[r] \arrow[d, "\psi_3"] & 0 \\
0 \arrow[r] & C_1 \arrow[r, "\gamma_1"] \arrow[d] & C_2 \arrow[r, "\gamma_2"] \arrow[d] & C_3 \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Then we get the following exact sequences:

a) $0 \to A_1 \xrightarrow{\begin{bmatrix} \alpha_1 \\ \varphi_1 \end{bmatrix}} A_2 \oplus B_1 \xrightarrow{\begin{bmatrix} \varphi_2, & -\beta_1 \end{bmatrix}} B_2 \xrightarrow{\psi_3 \circ \beta_2} C_3 \to 0,$

b) $0 \to A_1 \xrightarrow{\varphi_2 \circ \alpha_1} B_2 \xrightarrow{\begin{bmatrix} \beta_2 \\ \psi_2 \end{bmatrix}} B_3 \oplus C_2 \xrightarrow{\begin{bmatrix} \psi_3, & -\gamma_2 \end{bmatrix}} C_3 \to 0.$

Using this easy remark we can prove the following:

Lemma 6.2.2
For $M \in \text{Mod-}A$, $N \in \text{Mod-}B$ we have

$$\operatorname{pdim}_{A\otimes B} M \otimes N = \operatorname{pdim}_A M + \operatorname{pdim}_B N \ .$$

Proof. If one of the projective dimensions $\operatorname{pdim} M$, $\operatorname{pdim} N$ is infinite, then the equation holds, since an $A \otimes B$ projective resolution of $M \otimes N$ restricts to an A resp. B projective resolution. Now we can assume that $n = \operatorname{pdim} N \leq m =$

$\operatorname{pdim} M < \infty$.
For $n = 0$, let

$$0 \to P_m \to P_{m-1} \to \cdots \to P_1 \to P_0 \to M \to 0$$

be a minimal projective resolution in Mod- A. Then

$$0 \to P_m \otimes N \to P_{m-1} \otimes N \to \cdots \to P_1 \otimes N \to P_0 \otimes N \to M \otimes N \to 0$$

is exact with $P_i \otimes N$ projective in Mod- $A \otimes B$ for all $i = 0, \ldots, m$. Since

$$P_m \otimes N \subset (\operatorname{rad} P_{m-1}) \otimes N \subset \operatorname{rad}(P_{m-1} \otimes N)$$

the last sequence is a minimal projective resolution, hence $\operatorname{pdim} M \otimes N = m$. Now we proceed by induction on $d = m + n$.

i) If $d \leq 2$ the only non-trivial case is $(m, n) = (1, 1)$. Let

$$0 \to P_1 \to P_0 \to M \to 0$$

resp.

$$0 \to Q_1 \to Q_0 \to N \to 0$$

be a minimal projective resolution in Mod- A resp. Mod- B. We tensor these two sequences to the following commutative diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & P_1 \otimes Q_1 \arrow[r] \arrow[d] & P_0 \otimes Q_1 \arrow[r] \arrow[d] & M \otimes Q_1 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & P_1 \otimes Q_0 \arrow[r] \arrow[d] & P_0 \otimes Q_0 \arrow[r] \arrow[d] & M \otimes Q_0 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & P_1 \otimes N \arrow[r] \arrow[d] & P_0 \otimes N \arrow[r] \arrow[d] & M \otimes N \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Using the above remark we get the exact sequence

$$0 \to P_1 \otimes Q_1 \to (P_0 \otimes Q_1) \oplus (P_1 \otimes Q_0) \to P_0 \otimes Q_0 \to M \otimes N \to 0$$

which is a minimal projective resolution since

$$P_1 \otimes Q_1 \subset ((\operatorname{rad} P_0) \otimes Q_1) \oplus (P_1 \otimes (\operatorname{rad} Q_0)) \subset \operatorname{rad}((P_0 \otimes Q_1) \oplus (P_1 \otimes Q_0)).$$

ii) Now we conclude from d to $d+1$ for $d \geq 2$. Consider the following exact sequence

$$0 \to L \to P \to M \to 0$$

with P projective in Mod- A and $\operatorname{pdim} L = m-1$. Then

$$0 \to L \otimes N \to P \otimes N \to M \otimes N \to 0$$

is exact and we have

$$\operatorname{pdim} M \otimes N \leq \max\{\operatorname{pdim} P \otimes N, \operatorname{pdim} L \otimes N + 1\} \leq m + n = d + 1$$

by induction. Now we look at the related homological sequence

$$\operatorname{Ext}^d(P \otimes N, -) \to \operatorname{Ext}^d(L \otimes N, -) \to \operatorname{Ext}^d(M \otimes N, -).$$

Since $m + n = d + 1 > 2$ we can assume $m \geq 2$. Then by induction we get

$$\operatorname{pdim} P \otimes N = n < m - 1 + n = d = \operatorname{pdim} L \otimes N \ .$$

Hence

$$\operatorname{Ext}^d(P \otimes N, -) = 0 \neq \operatorname{Ext}^d(L \otimes N, -)$$

and $\operatorname{pdim} M \otimes N = d + 1$.

□

Lemma 6.2.3
Let $X \in \text{Mod-}\, A \otimes B$ *with* $\operatorname{pdim}_A X \leq m < \infty$ *and* $M \in A\text{-Mod}, N \in B\text{-Mod}$. *Then*

$$\operatorname{Tor}_i^{A \otimes_{\mathbf{k}} B}(\Omega^m_{A \otimes_{\mathbf{k}} B}(X), M \otimes_{\mathbf{k}} N) \cong \operatorname{Tor}_i^B(\Omega^m_{A \otimes_{\mathbf{k}} B}(X) \otimes_A M, N)$$

holds for all $i \in \mathbb{N}$.

Proof. The claim is a special case of [8, Theorem IX.2.8], but we will use some arguments of the proof in the proof of the next theorem. Set $\Lambda = A \oplus_{\mathbf{k}} B$. For projective $P \in \text{Mod-}\, \Lambda$ we have the well known isomorphism

$$(P \otimes_A M) \otimes_B N \cong P \otimes_\Lambda (M \otimes_{\mathbf{k}} N)$$

which is functorial in P, M, N [8, see IX.2.1]. Thus $(P \otimes_A M) \otimes_B -$ is an exact functor and $P \otimes_A M$ is projective in Mod- B. Let

$$\ldots \to P_j \to P_{j-1} \to \ldots \to P_0 \to X \to 0$$

be a projective resolution in Mod- Λ. By restriction it becomes a projective resolution in Mod- A hence $\Omega^m_\Lambda(X)$ is projective in Mod- A and

$$\ldots \to P_j \to P_{j-1} \to \ldots \to P_m \to \Omega^m_\Lambda(X) \to 0$$

is a split-exact sequence in Mod- A. Thus

$$\ldots \to P_j \otimes_A M \to P_{j-1} \otimes_A M \to \ldots \to P_m \otimes_A M \to \Omega^m_\Lambda(X) \otimes_A M \to 0 \quad (6.1)$$

is a projective resolution in Mod- B. Finally using the above isomorphism we get the following commutative diagram of complexes:

$$\begin{tikzcd}
\cdots \arrow[r] & P_j \otimes_\Lambda (M \otimes_{\mathbf{k}} N) \arrow[r] \arrow[d, "\wr"] & P_{j-1} \otimes_\Lambda (M \otimes_{\mathbf{k}} N) \arrow[r] \arrow[d, "\wr"] & \cdots \arrow[r] & P_m \otimes_\Lambda (M \otimes_{\mathbf{k}} N) \arrow[r] \arrow[d, "\wr"] & 0 \\
\cdots \arrow[r] & (P_j \otimes_A M) \otimes_B N \arrow[r] & (P_{j-1} \otimes_A M) \otimes_B N \arrow[r] & \cdots \arrow[r] & (P_m \otimes_A M) \otimes_B N \arrow[r] & 0
\end{tikzcd}$$

Since $\mathrm{Tor}^\Lambda_i(\Omega^m_\Lambda(X), M \otimes_{\mathbf{k}} N)$ is the i'th homology group of the upper complex and $\mathrm{Tor}^B_i(\Omega^m_\Lambda(X) \otimes_A M, N)$ of the lower one the claim follows. □

Theorem 6.2.4
Let A and B be finite dimensional algebras over an algebraically closed field $\mathbf{k}$. Then

a) $\mathrm{Findim}\, A \otimes_{\mathbf{k}} B = \mathrm{Findim}\, A + \mathrm{Findim}\, B$.

b) $\mathrm{findim}\, A \otimes_{\mathbf{k}} B = \mathrm{findim}\, A + \mathrm{findim}\, B$.

Proof. By Lemma 6.2.2 we have $\mathrm{Findim}\, A \otimes_{\mathbf{k}} B \geq \mathrm{Findim}\, A + \mathrm{Findim}\, B$. Set $\Lambda = A \oplus_{\mathbf{k}} B, m = \mathrm{Findim}\, A$ and $n = \mathrm{Findim}\, B$. Let $X \in$ Mod- Λ be of finite projective dimension. Since every Λ-projective resolution of X restricts to an A-projective resolution, $\mathrm{pdim}_A X \leq m$ holds. Since $\mathrm{pdim}_\Lambda X$ is finite the B-projective resolution (6.1) is finite too. Thus $\Omega^m_\Lambda(X) \otimes_A M$ has finite projective dimension in Mod- B. By Lemma 6.2.3 we have:

$$\mathrm{Tor}^\Lambda_i(\Omega^m_\Lambda(X), M \otimes_{\mathbf{k}} N) \cong \mathrm{Tor}^B_i(\Omega^m_\Lambda(X) \otimes_A M, N) = 0 \ \forall\ i \geq n.$$

Since the simple Λ-modules are of type $S \otimes_{\mathbf{k}} T$ with simple modules $S \in A$-Mod, $T \in B$-Mod, $\mathrm{Tor}^\Lambda_i(\Omega^m_\Lambda(X), V) = 0$ for all $i \geq n$ and $V \in \Lambda$-Mod. Hence

$$\mathrm{pdim}_\Lambda X = m + \mathrm{pdim}_\Lambda(\Omega^m_\Lambda(X) \leq m + n\ .$$

The proof of part b) is analogous. □

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