Hilbert Schemes of Quiver Algebras

Thesis submitted in partial fulfillment of the requirements of the degree "Doktor rer. nat."

Johannes Christoph Norbert Engel

13th July 2010

Bergische Universität Wuppertal
Fachbereich C – Mathematik

Die Dissertation kann wie folgt zitiert werden:

urn:nbn:de:hbz:468-20100375
[http://nbn-resolving.de/urn/resolver.pl?urn=urn%3Anbn%3Ade%3Ahbz%3A468-20100375]

Introduction

Und jedem Anfang
wohnt ein Zauber inne

(Hermann Hesse)

Representation theory of algebras

Representation theory in general deals with abstract algebraic structures by studying their representations. These representations are translations into the language of vector spaces and linear maps inheriting essential properties of the original object. By this approach, it becomes possible to study rather difficult algebraic objects using methods of linear algebra, an area considered well-understood. The structures which can be treated in this way include groups, Lie algebras and algebras. The latter ones, especially algebras associated to quivers, are in the central focus of interest of this work.

One of the main problems arising in that context is the classification of representations of finite dimensional algebras up to isomorphism, in particular finite dimensional representations. In general, this is far from being solved, because the set of these isomorphism classes is too big. To make that statement precise consider a theorem of Victor Kac, stating that for an algebra associated to a so-called wild quiver the problem of classifying the indecomposable representations depends on arbitrarily many continuous parameters (c. f. [12], [13]). For the so-called wild algebras the problem is essentially of the same quality, which implies that the standard toolbox of representation theory[1] cannot be applied here.

[1] c. f. for example [10]

A different approach though is to handle these continuous phenomena using tools from algebraic geometry, that means considering the isomorphism classes of representations as points in a suitable space and trying to analyse the geometric properties of the resulting objects called *moduli spaces*. A special class of moduli spaces is formed by so called *non-commutative Hilbert schemes* explained in chapter 1.

This work investigates some geometric properties of the non-commutative Hilbert schemes assigned to quiver algebras and uses the results to construct operators on the Borel-Moore homology groups of these. Some applications are presented linking the results to earlier works by Markus Reineke [17] and Yakuo Teranishi [22].

Structure of this work

In chapter 1 a short collection of facts and notations of the representation theory of quivers resp. quiver algebras is given and the corresponding moduli spaces will be introduced. In order to obtain some very desirable geometric properties of the moduli spaces, there will be a slight modification in the formulation of the moduli problem: Instead of parametrising just isomorphism classes of representations, a framing datum will be added which basically consists of some additional linear maps and a stability condition. The advantage of this approach is that one obtains, for example, that the resulting moduli spaces are always smooth. In general, that does not hold for moduli spaces of polystable representations for coprime dimension vectors, see [5]. Hence, this special class of moduli spaces is called "smooth models", details of this construction as well as the rest of this chapter have been published in a joint work with Markus Reineke in [5].

As a main result, we construct an algebraic cell decomposition in section 1.2 for a special class of smooth models, extending [17] where this result is proved in the special case of the m-loop quiver. These moduli spaces will

be called *non-commutative Hilbert schemes*, since, similar to Hilbert schemes of points in an affine variety, they parametrise the left ideals of finite codimension in a quiver algebra.
The main instrument used in the construction of the cell decomposition is a combinatorial calculus of trees and forests. This is derived from a covering quiver for any given quiver resolving oriented cycles and multiple edges, which will be introduced in detail in Definition 1.2.2.
Besides, we obtain a combinatorial formula for the Betti numbers of the non-commutative Hilbert schemes, and therefore also for the corresponding Poincaré polynomial, see Corollary 1.2.18. As a third result, functional equations for the generating functions of the Betti numbers are obtained in this chapter.

Chapter 2 presents the facts and notations for Borel-Moore homology used in this work. This special homology theory is a very powerful tool to handle even singular varieties. For smooth compact manifolds it coincides with singular homology, but also works here for nilpotent non-commutative Hilbert schemes in chapter 3, which are not necessarily smooth. Instead, they are projective which makes them well suited for the construction of the operators in Borel-Moore homology later.
The content of that chapter is mostly following [4], where the essential facts about Borel-Moore homology are listed in a comprehensive form. Most of the skipped proofs in this section can be found in [11]. For the original introduction the reader should refer to [3].

In chapter 3, the tools developed in chapter 2 will be applied to the setting from chapter 1. Here we make a slight modification restricting to the nilpotent non-commutative Hilbert schemes. They have a cell decomposition analogous to the non-commutative Hilbert schemes; the cells are even parametrised by the same objects. Therefore, most of the properties stated in chapter 1 also hold in this case.
The main reason to use Borel-Moore homology in this context is the existence of fundamental classes in Borel-Moore homology corresponding to the

closures of the cells arising from the cell decomposition constructed earlier. These are generators of the respective homology groups.

The main result of this chapter is the definition of generation operators in Borel-Moore homology using a technique called *correspondences* which is applied in the context of Hilbert schemes of points on surfaces in [16] to construct a Heisenberg algebra. In our case, they turn the direct sum of all homology groups of the non-commutative Hilbert schemes into a geometric model for the Fock space. In the special case of the m-loop quiver one can even derive the structure of a bigraded algebra here which is conjectured to work in a more general setting, c. f. Conjecture 3.3.3. This method is explained for example in [4, section 2.7] and makes use of the properties of Borel-Moore homology introduced in chapter 2. In this situation the fundamental classes come in quite handy, since they allow to calculate the operators explicitely in terms of fundamental classes associated to the closures of the cells. In chapter 1, a combinatorial parametrisation for them is given which is used in the following to derive a combinatorial way of calculating this convolution product in Borel-Moore homology using trees and forests. Secondly, we present a description of the cell closures in the case of the nilpotent non-commutative Hilbert schemes.

Finally, in the last chapter we give an algebraic realisation of the functional equations from chapter 1. In the context of chapter 3, this amounts to giving some complements and analogies to shed some light onto various aspects of the convolution algebra conjectured there. For that purpose section 4.1 presents an algebra isomorphism to a so-called *cable algebra* generated by cable diagrams. This construction belongs to the class of diagram algebras. Another very well known example of this class are *Temperley-Lieb algebras* introduced in [21]. Although there are certain similarities in the diagrams, the composition of diagrams used in this case is a different one.

In section 4.2, we take a functional equation for the Hilbert series of the non-commutative Hilbert schemes as a motivation to find an isomorphism corresponding to this between two algebras one of which is the convolution algebra conjectured in chapter 3. Together these two sections can give an idea of how the Betti numbers of the non-commutative Hilbert schemes behave, and how this behaviour is related to the structure of the convolution algebra. The second main result of this chapter is a link between the algebra structure on a vector space with a basis parametrised by trees and non-commutative Invariant Theory: The topic discussed in section 4.3 relates such an algebra – which can be thought of as the convolution algebra from chapter 3 – to the algebra of invariants under the operation of $\mathrm{SL}(V)$ on the tensor algebra of a vector space V. In the general situation, there is a monomorphism of graded algebras, which specialises to an isomorphism for $Q = L_m$ with $m = 1$ or $m = 2$. The construction of the SL_m-invariants used here is due to [22].

Acknowledgements

I want to thank everyone who has contributed to the completion of this work, first of all my advisor Markus Reineke for his ideas, for his patience when I kept bothering him with emails and for always having an open door.

Second I want to name Alastair D King and the geometry group at the University of Bath who gave me a warm welcome, 6 months of interesting discussions and talks and new perspectives to some of the aspects of this work.

Also a big thank you goes to my colleagues at the Bergische Universität Wuppertal for interesting discussions and for cheering me up during the periods of doubts; especially here to Guido Frank and Roland Olbricht who thoroughly read the preliminary version of this text and made many valuable suggestions for improvements.

Not only for spotting several almost untraceable mistakes (that even includes the month of a publication in the bibliography), but also for hosting me during my time in Bath and many other things I am deeply grateful to Yoonhee Tina Chang.

Last but not least I owe the Research Training Group 1052 “Representation Theory and its Applications in Mathematics and Physics” a debt of gratitude for funding me during my PhD studies. The same holds for the German Academic Exchange Service for a scholarship during my stay in Bath. I have also benefitted from the various opportunities of a scholarship of the German National Academic Foundation, especially I appreciated very much the doctoral candidates’ meetings.

Contents

1 Geometry of non-commutative Hilbert schemes 10

1.1 Quiver representations 12

1.2 Cell decomposition 19

1.2.1 Trees and Forests 20

1.2.2 Multipartitions 34

1.3 An explicit formula for the Euler characteristic 38

2 Borel-Moore Homology 46

2.1 Basic facts and properties 48

2.2 Convolution products 53

3 Operators on Hilbert Schemes 57

3.1 Nilpotent Hilbert Schemes 58

3.2 Convolution Operators 66

3.3 Open Question 74

4 Applications 77

4.1 The Cable Algebra 78

4.1.1 Cable Diagrams 78

4.1.2 Tree Diagrams 79

4.2 Functional equation 81

4.3 SL_m-Invariants 84

Bibliography 90

Chapter 1

1 Geometry of non-commutative Hilbert schemes

Geometry is to open up my mind
so I may see
what has always been behind
the illusions that time
and space construct.

(David Henderson, [9])

In this chapter, we give a short introduction to the notations of quiver representations and the corresponding moduli spaces. Furthermore, this chapter contains the details of the construction of non-commutative Hilbert schemes for quiver algebras. A close investigation of their geometric properties reveals amongst others an algebraic cell decomposition generalising a result by Markus Reineke for the case of the m-loop quiver.
Using a combinatorial model of trees and forests, we give an explicit formula for the Euler characteristic of the non-commutative Hilbert schemes and functional equations for the generating function.
Even though the first definitions also hold for general fields, we shall always assume $\mathbb{F} = \mathbb{C}$. The combinatorial setting of this cell decomposition will be used later in chapter 3 and chapter 4 for explicit computations.

1.1 Quiver representations

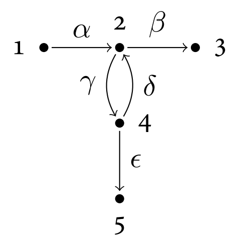
Definition 1.1.1 (Quiver) A quiver Q consists of two sets Q_0 and Q_1 and maps $t, h: Q_1 \to Q_0$.
One can think of Q_0 as a set of vertices and Q_1 as a set of arrows α with tail $t(\alpha)$ and head $h(\alpha)$. Having that in mind, elements of Q_1 are sometimes written as $\alpha: t(\alpha) \to h(\alpha)$.

A quiver Q is called *finite*, if Q_0 and Q_1 are finite sets. We will always assume any quiver to be finite in the context of this work.

A subquiver S of a quiver Q is a quiver S such that $S_0 \subseteq Q_0$ and $S_1 \subseteq Q_1$. The maps t and h corresponding to S are the restrictions of the corresponding maps of Q to S_1.
A subquiver is called *full*, if for all $q, r \in S_0$ the following condition holds:

$$\{\alpha: q \to r \mid \alpha \in Q_1\} \subset S_1.$$

Example 1.1.2 From now on we will recur to the following quiver Q as a standard example:



Definition 1.1.3 (Quiver representation) Let Q be a quiver and $\mathbb{C}$ a field. A *representation* M of Q over $\mathbb{C}$ consists of

- $\mathbb{C}$-vector spaces $(M_i)_{i \in Q_0}$ and

- linear maps $\left(M_\alpha\colon M_{t(\alpha)} \to M_{h(\alpha)}\right)_{\alpha\in Q_1}$.

The vector $d = (\dim M_i)_{i\in Q_0} \in \mathbb{N}\,Q_0$ is called the dimension vector of M.

A map between quiver representations

$$\varphi\colon \left((M_i)_{i\in Q_0}, (M_\alpha)_{\alpha\in Q_1}\right) \to \left((N_i)_{i\in Q_0}, (N_\alpha)_{\alpha\in Q_1}\right)$$

is a tuple of linear maps $(\varphi_i\colon M_i \to N_i)_{i\in Q_0}$, such that for all arrows $\alpha \in Q_1$ the following holds: $\varphi_{h(\alpha)} \circ M_\alpha = N_\alpha \circ \varphi_{t(\alpha)}$. This allows us to consider the Abelian category formed by the representations of Q and morphisms as above; this category is equivalent to the category of left modules of the path algebra kQ. Under this equivalence the notions of isomorphisms and subrepresentations translate as follows:

A map as above is called an *isomorphism* if all maps $(\varphi_i)_{i\in Q_0}$ are isomorphisms.

A *subrepresentation* W of V is a representation W of Q with a map of representations $\iota\colon W \to V$ such that all $\iota_i\colon W_i \to V_i$ for $i \in Q_0$ are injective.

Definition 1.1.4 (Euler form) Let Q be a quiver. Define the so called *Euler form* on $\mathbb{N}\,Q_0$ by

$$\langle a, b\rangle := \sum_{q\in Q_0} a_q b_q - \sum_{\alpha\in Q_1} a_{t(\alpha)} b_{h(\alpha)} \qquad (a, b \in \mathbb{N}\,Q_0)\,.$$

This form is sometimes also called *Ringel form*.

Consider the variety

$$R_d(Q) := \bigoplus_{\alpha\in Q_1} \operatorname{Hom}\left(\mathbb{C}^{d_{t(\alpha)}}, \mathbb{C}^{d_{h(\alpha)}}\right);$$

$R_d(Q)$ is called *representation space*. It carries a natural operation of the group

$$\mathrm{GL}_d := \prod_{i\in Q_0} \mathrm{GL}_{d_i}(\mathbb{C})$$

via

$$(g_i)_{i\in Q_0} \cdot (M_\alpha)_{\alpha\in Q_1} = \left(g_{h(\alpha)} \circ M_\alpha \circ g_{t(\alpha)}^{-1}\right)_{\alpha\in Q_1}.$$

It is not hard to see that the orbits of this operation correspond in a natural way to the isomorphism classes of representations of Q of dimension vector d.

Definition 1.1.5 Let Q be a quiver. A path in Q is a sequence of arrows $\alpha_1, \ldots, \alpha_r \in Q_1$ with $h\alpha_i = t\alpha_{i+1}$ for all i. Consider the *path algebra* $\mathbb{C}Q$ generated by all paths in Q where the product is given by

$$w \times w' = \begin{cases} w.w' & \text{if } hw = tw' \\ 0 & \text{else} \end{cases}$$

where $w.w'$ denotes the concatenation of paths. This defines an associative $\mathbb{C}$-algebra.
Note that in each vertex $i \in Q_0$ one has the empty path e_i in that vertex.

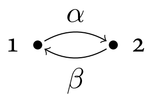
For a representation M of Q and a path $w = \alpha_1 \ldots \alpha_r$ consisting of arrows $\alpha_1, \ldots, \alpha_r \in Q_1$ let $M_w := M_{\alpha_1} \circ \cdots \circ M_{\alpha_r}$ be the composition of linear maps.
Fix the following notions: $h(w) = h(\alpha_r)$ and $t(w) = t(\alpha_1)$.

The subset of $R_d(Q)$ consisting of all simple representations will be denoted by $R_d^{\text{simp}}(Q)$, this is an open, but possibly empty, subset of $R_d(Q)$. The same holds for the subset $R_d^{\text{ssimp}}(Q)$ of all semisimple representations of Q.

Example 1.1.6

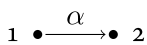
1. Let Q be a quiver without oriented cycles. Then the simple representations are precisely the representations E_i with dimension vector $(\delta_{i,j})_{j\in Q_0}$ for $i \in Q_0$.

2. Let Q be the quiver



Then the representation given by $M_1 = M_2 = \mathbb{C}$ and $M_\alpha = M_\beta = \mathrm{id}$ is simple.

3. Let Q be the quiver



Then the representation given by $M_1 = M_2 = \mathbb{C}$ and $M_\alpha(x) = ax$ for some $a \in \mathbb{C}$ is not simple, since there exists a non-trivial subrepresentation given by $M'_1 = 0$, $M'_2 = \mathbb{C}$ and $M'_\alpha = 0$.

4. Let Q be the quiver with two loops, that is $Q_0 = \{i\}$ and $Q_1 = \{\alpha, \beta\}$. In this situation the simple representations of dimension 2 are the pairs of 2×2 matrices without simultaneous eigenspaces.

Michael Artin showed in [1] the existence of a complex algebraic variety (not necessarily smooth, though) $M_d^{\mathrm{ssimp}}(Q)$ parametrising the isomorphism classes of semisimple representations of Q of dimension vector d. This $M_d^{\mathrm{ssimp}}(Q)$ is an affine variety.
Furthermore, [1] proves that there is a smooth complex algebraic variety $M_d^{\mathrm{simp}}(Q)$ parametrising the isomorphism classes of simple representations of Q with dimension vector d; the variety $M_d^{\mathrm{simp}}(Q)$ can be obtained as the geometric quotient of $R_d^{\mathrm{simp}}(Q)$ by the action of GL_d.

Definition 1.1.7 (Extended quiver representation) Let Q be a quiver and assume given dimension vectors $d, n \in \mathbb{N}Q_0$; we will denote such a triple (Q, d, n) as a *quiver datum* . An *extended representation* of Q of dimension vectors d, n consists of

- $\mathbb{C}$-vector spaces $(V_i)_{i\in Q_0}$ and $(M_i)_{i\in Q_0}$ of dimensions $\dim V_i = n_i$ and $\dim M_i = d_i$ for all $i \in Q_0$ as well as
- linear maps $(f_i : V_i \to M_i)_{i\in Q_0}$ and $\left(M_\alpha : M_{t(\alpha)} \to M_{h(\alpha)}\right)_{\alpha\in Q_1}$.

The corresponding extended representation space is given by

$$R_{d,n}(Q) := \bigoplus_{\alpha\in Q_1} \operatorname{Hom}\left(\mathbb{C}^{d_{t(\alpha)}}, \mathbb{C}^{d_{h(\alpha)}}\right) \oplus \bigoplus_{i\in Q_0} \operatorname{Hom}\left(\mathbb{C}^{n_i}, \mathbb{C}^{d_i}\right).$$

On $R_{d,n}(Q)$, we have an operation of the group GL_d via

$$\begin{aligned}&(g_i)_{i\in Q_0} \cdot \left((M_\alpha)_{\alpha\in Q_1}, (f_i)_{i\in Q_0}\right)\\ &= \left(\left(g_{h(\alpha)} \circ M_\alpha \circ g_{t(\alpha)}^{-1}\right)_{\alpha\in Q_1}, (g_i \circ f_i)_{i\in Q_0}\right).\end{aligned}$$

Two extended representations

$$\left((M_\alpha)_{\alpha\in Q_1}, (f_i)_{i\in Q_0}\right) \quad \text{and} \quad \left((M'_\alpha)_{\alpha\in Q_1}, (f'_i)_{i\in Q_0}\right)$$

are equivalent if there exists an isomorphism of representations $\varphi : M \to M'$ such that $f'_i = \varphi_i \circ f_i$ for all $i \in Q_0$. Then the orbits of the group operation of GL_d on $R_{d,n}(Q)$ correspond naturally to equivalence classes of extended representations.

Alternatively, the following description of extended representations of Q is also possible (c. f. [5, chapter 3]): Let d, n be two dimension vectors for Q and denote by $\hat{Q}^n$ the extended quiver with

$$\begin{aligned}\hat{Q}_0^n &= Q_0 \cup \{\infty\} \quad \text{and}\\ \hat{Q}_1^n &= Q_1 \cup \left\{\zeta_j^i : \infty \to i \mid i \in Q_0,\ j = 1, \ldots, n_i\right\}.\end{aligned}$$

Then the extended representations of Q for dimension vectors $d, n \in \mathbb{N}Q_0$ correspond to representations of $\hat{Q}^n$ for the dimension vector $\hat{d}$ where $\hat{d}_i = d_i$

for $i \in Q_0$ and $\hat{d}_\infty = 1$. This is obvious by the following: Choose bases $v_1, \dots, v_{n_i}$ of V_i for $i \in Q_0$ and set

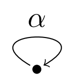
$$\zeta_j^i (1) = f_i \big| (v_j) \qquad (j = 1, \dots, n_i,\ i \in Q_0)\,.$$

.

Definition 1.1.8 (Stability) An extended representation (M, f) of Q for dimension vectors $d, n \in \mathbb{N} Q_0$ is called stable if the images of the f_i for $i \in Q_0$ generate the representation M.
The open subset of $R_{d,n}(Q)$ of all stable representations of Q of dimension vectors $d, n \in \mathbb{N} Q_0$ is denoted by $R_{d,n}^{\text{st}}(Q)$. One can easily see that the operation of $\mathrm{GL}_d(Q)$ on $R_{d,n}(Q)$ restricts to an operation on $R_{d,n}^{\text{st}}(Q)$.

Remark 1.1.9 M is generated by the images of the f_i for $i \in Q_0$ if there are bases $(v_{i,j})_{j=1,\dots,n_i}$ of the vector spaces V_i such that each M_i for $i \in Q_0$ is spanned by elements of the form $\left(M_w f_\ell \left(v_{\ell,k}\right)\right)_{\ell,k,w}$, where $w = \alpha_1 \dots \alpha_r$ is a word in the alphabet Q_1 with $h\alpha_s = t\alpha_{s+1}$ for all s. Note that this implies $h(w) = i$.

Example 1.1.10 Let Q be the quiver



Then the extended representation given by $V = \mathbb{C}$, $M = \langle m_1, m_2 \rangle \simeq \mathbb{C}^2$, $f(x) = axm_1 + bxm_2$ for some $a, b, x \in \mathbb{C}$ and $M_\alpha = \mathrm{id}$ is not stable, since M is not generated by the image of V.
If in the same setting M_α is given by $\begin{pmatrix} M_{1,1} & M_{1,2} \\ M_{2,1} & M_{2,2} \end{pmatrix}$ with

$$(aM_{1,1} + bM_{2,1})\, b \neq (aM_{2,1} + bM_{2,2})\, a,$$

then (M, f) is stable, since $(f(v), M_\alpha f(v))$ for some $v \in V \setminus \{0\}$ forms a basis of M.

Alastair King showed in [14] that there exists a smooth complex algebraic variety $M_{d,n}^{\text{st}}(Q)$ parametrising the isomorphism classes of stable representations. Via the notion of stability introduced in [5], this also holds for extended representations. This variety is given as the geometric quotient of $R_{d,n}^{\text{st}}(Q)$ by the action of GL_d.

Definition 1.1.11 (Non-commutative Hilbert Scheme) Denote by

$$\mathrm{Hilb}_{d,n}(Q) := M_{d,n}^{\text{st}}(Q)$$

the non-commutative Hilbert scheme for the quiver datum (Q, d, n).

In the special case of the m-loop quiver with $n = 1$, the elements of the Hilbert scheme $\mathrm{Hilb}_{d,1}(Q)$ correspond to left ideals in the path algebra $\mathbb{C}Q$: Let $(M, f) \in \mathrm{Hilb}_{d,1}(Q)$. Then M is a $\mathbb{C}Q$-module and thus corresponds to an ideal $I(M, f)$ by the following

$$0 \to I(M, f) \to \mathbb{C}Q \to M \to 0,$$

where the morphism $\mathbb{C}Q \to M$ is given by $1 \mapsto f(v)$. Under this construction factor representations correspond to subideals as one can see from the following diagram of exact sequences:

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
 & & I(L,h) & \hookrightarrow & I\left(L/M, \bar{h}\right) & & \\
 & & \downarrow & & \downarrow & & \\
 & & \mathbb{C}Q & = & \mathbb{C}Q & & \\
 & & \downarrow & & \downarrow & & \\
0 \longrightarrow M & \longrightarrow & L & \longrightarrow & L/M, \bar{h} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & &
\end{array}$$

In coordinates this can be written as follows: Assume $(M, f) \in U_{S_*}$ is given by

$$M_w f(v) = \sum_{S_* \ni w' < w} \mu_{w,w'} M_{w'} f(v) \qquad (w \in C(S_*)).$$

Then (M, f) corresponds to the ideal

$$I(M, f) := \left\langle w - \sum_{S_* \ni w' < w} \mu_{w,w'} w' \right\rangle_{w \in C(S_*)} \subset \mathbb{C}Q$$

with $\operatorname{codim}_{\mathbb{C}Q} I = \dim M$.

1.2 Cell decomposition

In this section, we construct an algebraic cell decomposition for the non-commutative Hilbert schemes of a quiver Q. In the following, this construction will be used to compute the Betti numbers of the non-commutative Hilbert schemes and to give a formula for the Poincaré polynomial.

Definition 1.2.1 (Algebraic cell decomposition) For a variety X an algebraic cell decomposition is a filtration by closed algebraic subsets

$$X = X_0 \supseteq X_1 \supseteq \cdots \supseteq X_{s-1} \supseteq X_s = \varnothing,$$

such that $X_i \setminus X_{i+1}$ is isomorphic to an affine space for $i = 0, \ldots, s-1$. The varieties $X_i \setminus X_{i+1}$ are called cells.

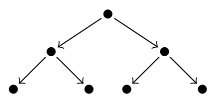
If a variety X has an algebraic cell decomposition, then all odd homology groups vanish, see for example [6, Appendix B].

For the construction of a cell decomposition, we apply a method from [17]: 1. We will construct a covering of $\mathrm{Hilb}_{d,n}(Q)$ by open affine subsets U_S, which 2. will then be modified to a stratification into disjoint subsets Z_S by cutting out intersections in a suitable way. 3. The crucial point in the whole proof is now to show that these strata are isomorphic to affine spaces. To achieve this we will make use of a combinatorial technique to enumerate the cells by certain forests. This tool was developed in [17] and will be generalised to arbitrary quivers in subsection 1.2.1.

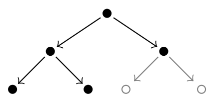
1.2.1 Trees and Forests

Definition 1.2.2 (Trees and Forests) A *tree* T is a connected quiver without cycles, such that for all $q \in T_0$ there exists at most one $\alpha \in T_1$ with $h(\alpha) = q$. This implies for example that every tree T has a unique root, that is a vertex $t_0 \in T_0$ such that there exists no $\alpha \in T_1$ with $h(\alpha) = t_0$.
A *subtree* of a tree T is a full connected subquiver S of T, which is closed under predecessors, i. e. for all $s \in S_0$ and all $T_1 \ni \alpha\colon r \to s$ we have $r \in S_0$.
A quiver $\mathcal{F}$, whose connected components are trees, is called a *forest*; a *sub-forest* $\mathcal{F}'$ of $\mathcal{F}$ is a forest $\mathcal{F}'$ such that the components of $\mathcal{F}'$ are subtrees of the trees in $\mathcal{F}$.

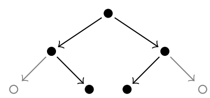
Example 1.2.3 Take following tree:



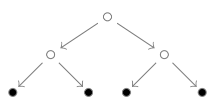
Among its subtrees are the following:



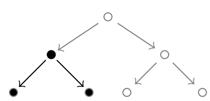
and



but not



or



the first one is not a tree, the second one is not closed under predecessors.

Let (Q, d, n) be a quiver datum. For $q \in Q_0$ define a tree T_q as follows: the vertices of T_q are paths in Q starting in q. There is an arrow $\alpha\colon w \to w'$ in T_q, if there is an arrow $\bar{\alpha} \in Q_1$ such that $w' = w\bar{\alpha}$. Here $w\bar{\alpha}$ means the path which can be obtained by appending $\bar{\alpha}$ at w in $h(w)$.

Definition 1.2.4 Denote by $\mathcal{F}_n(Q)$ the forest consisting of n_q copies of T_q for all $q \in Q_0$.
The vertices in $\mathcal{F}_n(Q)$ can be written as (q, i, w) for $q \in Q_0$, $i \in \{1, \ldots, n_q\}$ and w a path in Q starting in q. This denotes the vertex w in the i-th copy of T_q in $\mathcal{F}_n(Q)$.

Remark 1.2.5 All T_q for $q \in Q_0$ are trees, since 1. they are connected by construction (vertices correspond to paths starting in q, thus are connected to the root corresponding to the empty path in q) and 2. every vertex – apart from the root given by the empty path in q – has exactly one predecessor, which can be obtained by cutting off the last arrow of the corresponding path in Q. 3. If Q does not have oriented cycles, all covering quivers are finite.

Example 1.2.6 For the quiver Q from Example 1.1.2 one obtains the trees depicted in Figure 1.1.

Using these trees it is now possible to construct the index set which will later be used to parametrise the cells of a non-commutative Hilbert scheme $\mathrm{Hilb}_{d,n}(Q)$. This set will consist of certain subforests of $\mathcal{F}_n(Q)$.

Example 1.2.7 Let Q be the quiver

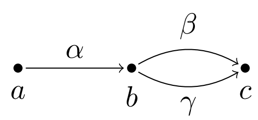
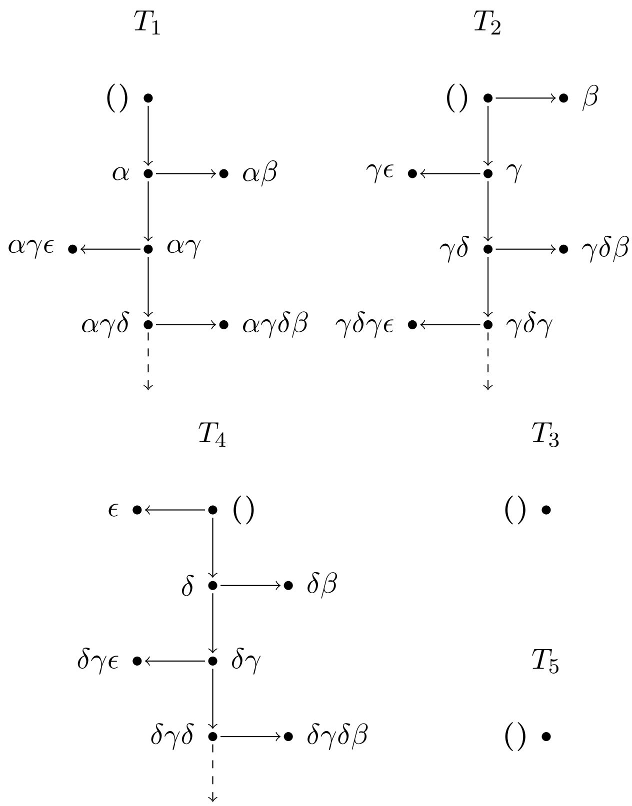
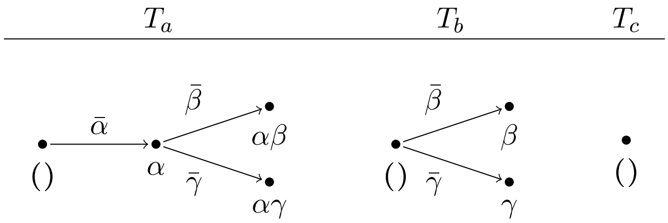


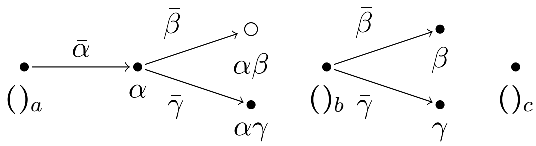
Figure 1.1: Covering trees T_q for Q from Example 1.1.2



Then one obtains the trees

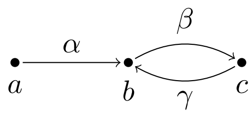


Choosing for example $n = (1,1,1) \in \mathbb{N}Q_0$ as a dimension vector implies that $\mathcal{F}_n(Q)$ consists of one copy of each of these trees T_a, T_b and T_c:

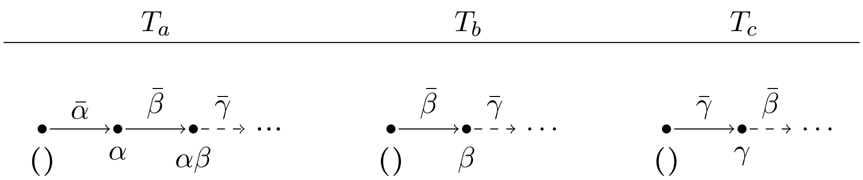


The marked vertex has the description $(a, 1, \alpha\beta)$.

Example 1.2.8 Let Q be the quiver



which looks very similar to the last one, but has very different properties as we will see. This change introduces an oriented cycle to Q which makes the covering trees infinitely long:



Choosing $n = (1, 0, 1)$ determines $\mathcal{F}_n(Q)$ as consisting of one copy of T_a and one copy of T_c.

Example 1.2.9 For the m-loop quiver all covering trees are infinite m-ary trees. Thus, $\mathcal{F}_n(Q)$ consists of n copies of the infinite m-ary tree.

Ordering vertices and forests

The aim is to construct a total ordering of the subtrees of $\mathcal{F}_n(Q)$. After constructing an affine covering for the non-commutative Hilbert schemes whose sheets are parametrised by the subtrees of $\mathcal{F}_n(Q)$, we can use this total ordering to make the covering disjoint by cutting off from each sheet the intersection with sheets of smaller order. Several steps will be needed to construct this order which implies the construction of preliminary orders on related sets. The whole process of constructing this order is almost canonical apart from two choices one has to make right in the beginning.

step 1 Choose an arbitrary total ordering on the vertices of Q and for each pair $(i, j) \in Q_0$ a total ordering of the arrows from i to j in Q_1.

step 2 This gives a total ordering on the set of arrows of Q as follows: For two arrows $\alpha: i \to j$ and $\beta: k \to \ell$ in Q_1 with $i, j, k, \ell \in Q_0$ define $\alpha < \beta$ if one of the following conditions holds:

a) $i = k$ and $j = \ell$ and $\alpha < \beta$ in the chosen order on the arrows between i and j,

b) $i = k$ and $j < \ell$ in the chosen order of Q_0,

c) $i < k$ in the chosen order of Q_0.

step 3 By means of the lexicographic order, this allows us to compare paths in Q as they are words in the alphabet Q_1. Thus we have an order on the set of vertices of the trees T_q for $q \in Q_0$: Let $w = \alpha_1 \ldots \alpha_r$ and

$w' = \beta_1 \dots \beta_s$ be two vertices in T_q, that is paths in Q. Define $w < w'$ if one of the following conditions holds:

a) w is a proper subword of w' or

b) $\alpha_k < \beta_k$ in the order on the arrows of Q where $k \in \mathbb{N}$ is minimal such that $\alpha_k \neq \beta_k$.

step 4 This can now be used to define an order on the vertices in $\mathcal{F}_n(Q)$ by setting $(q, i, w) < (q', i', w')$ if one of the following conditions holds:

a) $q < q'$ in the order on Q_0 defined in item 1,

b) $q = q'$ and $i < i'$ in $\mathbb{N}$,

c) $q = q'$, $i = i'$ and $w < w'$ in the order of the vertices of T_q defined in item 3.

step 5 Furthermore, define an order on the subforests of $\mathcal{F}_n(Q)$. For that purpose we will use again the lexicographic order.
For two such subforests $S = \{x_1 < \dots < x_k\}$ and $S' = \{x'_1 < \dots < x'_\ell\}$ define $S < S'$ if one of the following two conditions holds:

a) $k > \ell$ or

b) $k = \ell$ and $x_r < x'_r$ in the order on the vertices of $\mathcal{F}_n(Q)$ defined in item 4, where $r \in \{1, \dots, k\}$ is chosen minimal with respect to the property $x_r \neq x'_r$.

step 6 The order on Q_0 allows us to enumerate the trees in $\mathcal{F}_n(Q)$ by setting $(q, i) < (q', i')$ for $q, q' \in Q_0$, $i \in \{1, \dots, n_q\}$ and $i' \in \{1, \dots, n_{q'}\}$ if

a) $q < q'$ in the order defined in item 1 or

b) $q = q'$ and $i < i'$ in $\mathbb{N}$.

For an index (q, i) denote by $\operatorname{succ}(q, i)$ the unique smallest index greater than (q, i), which is

$$\operatorname{succ}(q, i) := \begin{cases} (q, i+1), & \text{if } i < n_q \\ (q', 1), & \text{if } i = n_q, \end{cases}$$

where $q' = \min \{\bar{q} \in Q_0 \mid \bar{q} > q,\ n_{\bar{q}} > 0\}$.

An affine covering

Let (Q, d, n) be a quiver datum and (M, f) an extended representation. Fix bases $(v_{q,i})_{i=1,\ldots,n_q}$ of V_q for $q \in Q_0$. For a finite subforest S_* of $\mathcal{F}_n(Q)$ let

$$U_{S_*} := \left\{ \overline{\left((M_\alpha)_{\alpha \in Q_1}, (f_q)_{q \in Q_0} \right)} \in \operatorname{Hilb}_{d,n}(Q) \mid (1.1) \right\}$$

where

$$\left((M_w \circ f_q)(v_{q,i}) \right)_{\substack{(q,i,w) \in S_* \\ h(w) = r}} \text{ is a basis of } M_r \text{ for all } r \in Q_0. \qquad (1.1)$$

Definition 1.2.10 (Corona) For a subforest S_* of $\mathcal{F}_n(Q)$ let $C(S_*)$ be the set of all vertices $(q, i, w) \in \mathcal{F}_n(Q)$ subject to one of the following conditions:

a) $w = \alpha_1 \ldots \alpha_\ell \notin S_{q,i}$ and $w' := \alpha_1 \ldots \alpha_{\ell-1} \in S_{q,i}$, where $S_{q,i} \neq \varnothing$, or

b) $w = ()$ and $S_{q,i} = \varnothing$.

$C(S_*)$ is called the *corona* of S_*.

Lemma 1.2.11 All U_{S_*} for subforests S_* of $\mathcal{F}_n(Q)$ are isomorphic to affine spaces.

Proof For $(M,f) \in U_{S_*}$ and $(q,i,w) \in C(S_*)$ one can write uniquely

$$M_w f_q(v_{q,i}) = \sum_{(q',i',w') \in S_*} \lambda_{(q,i,w),(q',i',w')} M_{w'} f_{q'}(v_{q',i'})$$

and assigning to (M,f) the coefficient $\lambda_{(q,i,w),(q',i',w')}$ induces an algebraic function $\Lambda_{(q,i,w),(q',i',w')}: U_{S_*} \to \mathbb{C}$. Thus, U_{S_*} is isomorphic to an affine space under the isomorphism given by the functions $\left(\Lambda_{(q,i,w),(q',i',w')}\right)$ for $(q,i,w) \in C(S_*)$ and $(q,i,w) \in S_*$. □

Thus the sets U_{S_*} are open subsets of $\mathrm{Hilb}_{d,n}(Q)$, since the defining condition is equivalent to the non-vanishing of the following determinant:

$$\det\left(M_w\left(f_q\left(v_{q,i}\right)\right)\right)_{(q,i,w) \in S_*} \neq 0.$$

Furthermore, it is easy to see, that these sets U_{S_*} cover $\mathrm{Hilb}_{d,n}(Q)$, because the stability condition asserts that every point in $\mathrm{Hilb}_{d,n}(Q)$ corresponds to a representation which is generated by $\mathrm{im}\, f$.

In the future, the notation $(M,f) \in U_{S_*}$ will be used dropping indices if the respective quiver is obvious. (M,f) will denote a class in $\mathrm{Hilb}_{d,n}(Q)$ as well as an extended representation where the actual meaning is obvious from the context.

The set of all subforests S_* of $\mathcal{F}_n(Q)$ such that $\mathrm{Hilb}_{d,n}(Q) \supseteq U_{S_*} \neq \varnothing$ is denoted by $\Phi_{d,n}(Q)$. For $S_* \in \Phi_{d,n}$ this implies

$$|S_*| := \left(\#\left\{(q,i,w) \in S_* \mid h(w) = j\right\}\right)_{j \in Q_0} = d$$

and each such S_* appears in $\Phi_{d,n}(Q)$, thus we have

$$\Phi_{d,n}(Q) = \left\{S_* \subset \mathcal{F}_n(Q) \mid |S_*| = d\right\}.$$

Lemma 1.2.12 Let $(M,f) \in \mathrm{Hilb}_{d,n}(Q)$ and $\bar{S}_*$ be a subforest of $\mathcal{F}_n(Q)$ such that the vectors $\left(M_w \circ f_q\left(v_{q,i}\right)\right)_{(q,i,w) \in \bar{S}_*}$ are linearly independent. Then there is a subforest S'_* of $\mathcal{F}_n(Q)$ with $\bar{S}_* \subseteq S'_*$ such that $(M,f) \in U_{S'_*}$.

Proof Use a downward induction on $\left|\bar{S}_*\right|$. For $\left|\bar{S}_*\right| = d$ there is nothing to prove.
Let $S_* = \bar{S}_* \cup C\left(\bar{S}_*\right)$ and

$$\begin{aligned} U_r &:= \langle M_w f_q\left(v_{q,i}\right)\rangle_{\substack{(q,i,w)\in S_*\\ h(w)=r}}, \\ \bar{U}_r &:= \langle M_w f_q\left(v_{q,i}\right)\rangle_{\substack{(q,i,w)\in \bar{S}_*\\ h(w)=r}} \qquad (r \in Q_0). \end{aligned}$$

Here one obviously has, due to $S_* \supseteq \bar{S}_*$

$$\sum_{r\in Q_0} \dim U_r \geq \sum_{r\in Q_0} \dim \bar{U}_r.$$

Note that both sides are equal, if and only if $U_r = M_r$ for all $r \in Q_0$, because $\mathrm{Hilb}_{d,n}(Q)$ is generated as a representation by $\operatorname{im} f$ and $(\operatorname{im} f)_r \subseteq U_r$ for all $r \in Q_0$.
If this is a proper inequality on the other hand, there exists a vertex $(q,i,w) \in S_* \setminus \bar{S}_*$ such that $M_w f_q\left(v_{q,i}\right) \notin \bar{U}_{h(w)}$. Replacing $\bar{S}_*$ by $\bar{S}_* \cup \{(q,i,w)\}$ yields $\left|\bar{S}_* \cup \{(q,i,w)\}\right| > \left|\bar{S}_*\right|$ and thus completes the induction. □

For a quiver datum (Q,d,n) and a finite subforest S_* of $\mathcal{F}_n(Q)$ define $Z_{S_*} \subseteq U_{S_*}$ as

$$Z_{S_*} := \left\{\overline{\left(\left(M_\alpha\right)_{\alpha\in Q_1}, \left(f_q\right)_{q\in Q_0}\right)} \in \mathrm{Hilb}_{d,n}(Q) \mid (1.1) \text{ and } (1.2)\right\},$$

where

$$M_w f_q\left(v_{q,i}\right) \in \langle M_{w'} f_{q'}\left(v_{q',i'}\right)\rangle_{\substack{(q,i,w)>(q',i',w')\in S_*\\ h(w)=h(w')}} \tag{1.2}$$

for all $(q,i,w) \in C(S_*)$.
These sets are obviously isomorphic to affine spaces, since they arise from the affine spaces U_{S_*} by setting a fixed set of coordinates to zero. In order to see

that they are the desired cells, one has to show that they actually form a decomposition of $\mathrm{Hilb}_{d,n}(Q)$. This follows immediately from Theorem 1.2.13, which is the main theorem of this chapter.

Theorem 1.2.13 Let (Q, d, n) be a quiver datum and S_* a finite subforest of $\mathcal{F}_n(Q)$. Then

$$Z_{S_*} = U_{S_*} \smallsetminus \bigcup_{S'_* < S_*} U_{S'_*}. \tag{1.3}$$

Corollary 1.2.14 For a quiver datum (Q, d, n) there is a cell decomposition of $\mathrm{Hilb}_{d,n}(Q)$, the cells of which are parametrised by $\Phi_{d,n}(Q)$.

Proof Enumerate the forests in $\Phi_{d,n}(Q)$ as $S_*^1 < \cdots < S_*^\ell$ according to the order defined above, and let

$$A_k := A_{S_*^k} := \mathrm{Hilb}_{d,n}(Q) \smallsetminus \bigcup_{S'_* < S_*^k} Z_{S'_*}.$$

All A_k are closed subvarieties of $\mathrm{Hilb}_{d,n}(Q)$, and we have a filtration

$$\mathrm{Hilb}_{d,n}(Q) = A_1 \supset \cdots \supset A_\ell \supset A_{\ell+1} := \varnothing,$$

where $A_i \smallsetminus A_{i+1} \simeq Z_{S_*^i}$ is isomorphic to an affine space for $i = 1, \ldots, \ell$. □

To prove Theorem 1.2.13 we will show both inclusions of formula (1.3) separately.

Lemma 1.2.15

$$Z_{S_*} \subseteq U_{S_*} \smallsetminus \bigcup_{S'_* < S_*} U_{S'_*}.$$

Proof The inclusion $Z_{S_*} \subseteq U_{S_*}$ is clear by definition.
Assume $Z_{S_*} \cap \bigcup_{S'_* < S_*} U_{S'_*} \neq \varnothing$, for instance $Z_{S_*} \cap U_{S'_*} \neq \varnothing$ for a $S'_* < S_*$, and choose some representative (M, f) for a class in this intersection.

Let (q,i) with $q \in Q_0$ and $i \in \{1,\dots,n_q\}$ be maximal with respect to the property that $S_{q',i'} = S'_{q',i'}$ for all $(q',i') \le (q,i)$. Because of $S'_* < S_*$ it follows that $S'_{\operatorname{succ}(q,i)} < S_{\operatorname{succ}(q,i)}$ holds and therefore one of the following cases:

$\left|S'_{\operatorname{succ}(q,i)}\right| > \left|S_{\operatorname{succ}(q,i)}\right|$**:** Since (M,f) represents a class in $U_{S'_*}$, the elements $M_w f_{q'}(v_{q',i'})$ for $(q',i',w) \in S'_*$ are linearly independent. Thus it follows that

$$\sum_{\substack{(q',i',w)\in\mathcal{F}_n(Q)\\(q',i')\le\operatorname{succ}(q,i)\\s\in Q_0}} \dim\left\langle M_w f_{q'}(v_{q',i'})\right\rangle_{h(w)=s} \ge \sum_{(q',i')\le\operatorname{succ}(q,i)} \left|S'_{q',i'}\right|.$$

By assumption one has

$$\sum_{(q',i')\le\operatorname{succ}(q,i)} \left|S'_{q',i'}\right| > \sum_{(q',i')\le\operatorname{succ}(q,i)} \left|S_{q',i'}\right|.$$

Since (M,f) represents a class in Z_{S_*}, it follows from formula (1.2) that

$$\sum_{(q',i')\le\operatorname{succ}(q,i)} \left|S_{q',i'}\right| = \sum_{\substack{(q',i',w)\in\mathcal{F}_n(Q)\\(q',i')\le\operatorname{succ}(q,i)\\s\in Q_0}} \dim\left\langle M_w f_{q'}(v_{q',i'})\right\rangle_{h(w)=s}.$$

Altogether this is a contradiction, so this case cannot occur.

$\left|S'_{\operatorname{succ}(q,i)}\right| = \left|S_{\operatorname{succ}(q,i)}\right|$**:** Let

$$\begin{aligned} S_{\operatorname{succ}(q,i)} &= \{w_1 < \dots < w_\ell\}, \\ S'_{\operatorname{succ}(q,i)} &= \{w'_1 < \dots < w'_\ell\} \end{aligned}$$

and k be minimal with respect to the property $w'_k < w_k$. That includes two cases:

1. If w'_k is not a proper subword of w_k, then it is in $S_{\mathrm{succ}(q,i)}$, since trees are closed under taking predecessors. But that contradicts the minimality of k.
2. Otherwise let $w_k = (\alpha_1 \dots \alpha_r)$ and $w'_k = (\alpha'_1 \dots \alpha'_{r'})$ and s be chosen minimal such that $\alpha_{s+1} \neq \alpha'_{s+1}$ and thus $\alpha'_{s+1} < \alpha_{s+1}$. Let furthermore $(\bar{q}, \bar{i}) := \mathrm{succ}\,(q, i)$.
Then $\bar{w} := (\alpha_1 \dots \alpha_s \alpha'_{s+1})$ satisfies
$$(\bar{q}, \bar{i}, \bar{w}) \in S'_{\mathrm{succ}(q,i)} \cap C\,(S_*)\,.$$
Together with formula (1.2) this is a contradiction as follows: Because (M, f) represents a class in Z_{S_*}, the following holds:
$$M_{\bar{w}} f_{\bar{q}} \left(v_{\bar{q},\bar{i}}\right) \in \left\langle M_{w'} f_{q'}\left(v_{q',i'}\right)\right\rangle_{\substack{S_* \ni (q',i',w') < (\bar{q},\bar{i},\bar{w}) \\ h(w) = h(\bar{w})}}\,.$$
The indices on the right side are contained in S'_* because of the assumed minimality of s, so the left hand side has to be linearly independent from the right hand side, since (M, f) represents a class in $U_{S'_*}$. □

Lemma 1.2.16
$$Z_{S_*} \supseteq U_{S_*} \setminus \bigcup_{S'_* < S_*} U_{S'_*}\,.$$

Proof Again we prove this lemma by contradiction. For that purpose, assume there exists a class in $\left(U_{S_*} \setminus \bigcup_{S'_* < S_*} U_{S'_*}\right) \setminus Z_{S_*}$ for a subforest S_* of $\mathcal{F}_n\,(Q)$. Let (M, f) represent such a class. For (M, f) this violates formula (1.2) at least in one point; let $(q, i, w) \in C\,(S_*)$ be minimal with respect to that property, i. e.
$$M_w f_q \left(v_{q,i}\right) \notin \left\langle M_{w'} f_{q'}\left(v_{q',i'}\right)\right\rangle_{\substack{S_* \ni (q',i',w') < (q,i,w) \\ h(w') = h(w)}}\,.$$

Let $S_{q,i} = \{w_1 < \dots < w_p < \dots\}$, where w_p is maximal with $w_p < w$. Note that this implies

$$(q, i, w_p) \neq \max\left\{\left(q', i', w'\right) \in S_*\right\}.$$

Define another forest $\bar{S}_*$ by setting $\bar{S}_{q',i'} = S_{q',i'}$ for all $(q', i') < (q, i)$. Furthermore, let $\bar{S}_{q,i} := \{w_1 < \dots < w_p < w\}$. Because of the minimality of (q, i, w) the requirements of Lemma 1.2.12 are satisfied for $\bar{S}_*$, so one can extend $\bar{S}_*$ to a forest S'_* such that (M, f) represents a class in $U_{S'_*}$.
Then one of the following cases is true:

1. $S'_{q',i'} = S_{q',i'}$ for all $(q', i') < (q, i)$. In this case we have $S'_{q,i} < S_{q,i}$ because $S'_{q,i}$ is an extension of $\bar{S}_{q,i}$ which contains all $w_i < \dots < w_p < w$. For the smallest vertex $(\bar{q}, \bar{i}, \bar{w}) \in S_*$ with $(\bar{q}, \bar{i}, \bar{w}) > (q, i, w_p)$ one has $(q, i, w) < (\bar{q}, \bar{i}, \bar{w})$.

2. $\left|S'_{q',i'}\right| > |S_{q',i'}|$ for some $(q', i') < (q, i)$.

In both cases we obtain $S'_* < S_*$. Hence $(M, f) \in U_{S'_*}$ contradicts the initial assumption. □

Remark 1.2.17 For a subforest S_* of $\mathcal{F}_n(Q)$ define

$$D(S_*) := \left\{\left((q, i, w), \left(q', i', w'\right)\right) \in C(S_*) \times S_* \mid \left(q', i', w'\right) < (q, i, w)\right\}$$

and let $d(S_*) := |D(S_*)|$. Then one can easily see that

$$\dim Z_{S_*} = d(S_*). \tag{1.4}$$

As already stated, we know that given an algebraic cell decomposition of the non-commutative Hilbert scheme, the number of n-dimensional cells is just the $2n$-th Betti number. The i-th Betti number $b_i\left(\mathrm{Hilb}_{d,n}(Q)\right)$ is defined as

$$b_i\left(\mathrm{Hilb}_{d,n}(Q)\right) := \dim_{\mathbb{Q}} \mathrm{H}_i\left(\mathrm{Hilb}_{d,n}(Q), \mathbb{Q}\right),$$

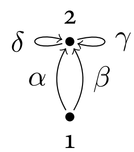
where $\mathrm{H}_i\left(\mathrm{Hilb}_{d,n}(Q), \mathbb{Q}\right)$ denotes the i-th singular homology group with rational coefficients. The Betti numbers appear in the so called *Poincaré polynomial*, which is their generating function:

$$P_{\mathrm{Hilb}_{d,n}(Q)}(q) = \sum_{i=0}^{\infty} b_i\left(\mathrm{Hilb}_{d,n}(Q)\right) q^i.$$

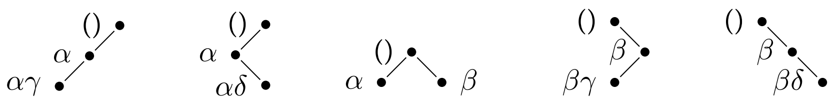
Corollary 1.2.18 One obtains the following formula for the Poincaré polynomial of $\mathrm{Hilb}_{d,n}(Q)$:

$$P_{\mathrm{Hilb}_{d,n}(Q)}(q) = \sum_{S_* \in \Phi_{d,n}(Q)} q^{2 \dim Z_{S_*}}.$$

Example 1.2.19 Consider the following quiver Q:



with dimension vectors $d = (1,2)$ and $n = (1,0)$. Then the cells of the corresponding non-commutative Hilbert scheme $\mathrm{Hilb}_{d,n}(Q)$ are parametrised by the trees



The corresponding cells have dimensions 3,2,2,1 and 0, respectively.

1.2.2 Multipartitions

The previous section contains the construction of a cell decomposition for the non-commutative Hilbert schemes. For this purpose certain subforests of $\mathcal{F}_n(Q)$ were used to parametrise the cells. From this we obtained a description of the Betti numbers in terms of these forests.
A different combinatorial description of the Betti numbers can be derived from [5, Theorem 6.2], where so called *multipartitions* appear as exponents in the Poincaré polynomial:

$$P_{\mathrm{Hilb}_{d,n}(Q)}(t) = t^{n\cdot d - \langle d,d\rangle} \sum_{\lambda\in\Lambda_{d,n}} t^{|\lambda|}$$

where $\Lambda_{d,n}$ is the set of admissible multipartitions that will be defined in Definition 1.2.20. This identity comes from counting rational points of the Hilbert scheme over finite fields and reordering by exponents of t^{-1}. Comparing coefficients, it is natural to ask for the relation between these two classes of objects, resp. for an explicit bijection. Following this approach, in this section we will proove a generalisation of [20, Proposition 6.2.1].

Definition 1.2.20 (Multipartitions) A *multipartition* λ for a quiver datum (Q, d, n) is a tuple $(\lambda^q)_{q\in Q_0}$ of partitions

$$\lambda^q = \left(\lambda^q_1 \geq \cdots \geq \lambda^q_{d_q} \geq 0\right).$$

Such a multipartition is called *admissible* if the $\lambda^q_i \in \mathbb{N}$ are subject to the following condition:

$$\text{For all } 0 \leq e < d \text{ there is a } q \in Q_0 \text{ such that } \lambda^q_{d_q - e_q} < n_q - \langle e, 1^q\rangle. \quad (1.5)$$

Here 1^q denotes the vector $(\delta_{q,r})_{r\in Q_0} \in \mathbb{N}Q_0$. The set of all multipartitions satisfying this condition will be denoted by $\Lambda_{d,n}$. In [5] this set is called $S_{d,n}$, this is changed here to avoid any confusion with the subforests of $\mathcal{F}_n(Q)$.
Let $|\lambda| := \sum_{q\in Q_0} \sum_{i=1}^{d_q} \lambda^q_i$ be the weight of a multipartition λ.

Construct a map $\varphi\colon \Phi_{d,n}(Q) \to \Lambda_{d,n}$ as follows: For $S_* \in \Phi_{d,n}(Q)$ define $\varphi(S_*) := (\lambda^q)_{q \in Q_0}$ with

$$\lambda_i^q := \#\left\{\left(q', i', w'\right) \in C(S_*) \mid h\left(w'\right) = q \text{ and } \#M\left(q', i', w'\right) \geq i\right\},$$

where

$$M\left(q', i', w'\right) := \left\{(q, i, w) \in S_* \mid h(w) = h\left(w'\right),\ (q, i, w) > \left(q', i', w'\right)\right\}.$$

This will be the bijection mentioned above.

Lemma 1.2.21 This map $\varphi\colon \Phi_{d,n}(Q) \to \Lambda_{d,n}$ is well defined, i. e. $\varphi(S_*)$ satisfies formula (1.5) for all $S_* \in \Phi_{d,n}(Q)$.

Proof Let $0 \leq e < d$, and for $(q, i, w) \in S_*$ define the set $M^-(q', i', w')$ as

$$\left\{(q, i, w) \in S_* \mid h(w) = h\left(w'\right) \wedge (q, i, w) < \left(q', i', w'\right)\right\}.$$

Choose $(\bar{q}, \bar{i}, \bar{w}) \in S_*$ minimal with respect to the property $|M^-(\bar{q}, \bar{i}, \bar{w})| = e_{h(\bar{w})} + 1$ and let $q := h(\bar{w})$.

By definition $\lambda_{d_q - e_q}^q$ counts those vertices in $C(S_*)$ which point towards q and for which there exist at least $d_q - e_q$ greater vertices in S_* with the same property. Here we mean the order of the vertices in $\mathcal{F}_n(Q)$ as defined on page 25. Since S_* contains exactly d_q vertices pointing towards q, this condition is equivalent to the following: there exist at most e_q smaller vertices in S_* pointing towards q.
Therefore, we count the possibilities for a vertex pointing towards q to appear in S_* or $C(S_*)$, which is smaller than the $(e_q + 1)$-th such "q-vertex" in S_*. Vertices with this property either do not have a predecessor, then they are precisely one of the roots of the n_q trees with root q. Or they have a predecessor $r \in Q_0$, and because of the minimality of q there can be at most e_r smaller r-vertices.

If $\bar{w} = ()$, then one of the roots does not appear, since it acts already as the $e_q + 1$-th q-vertex in S_*. Otherwise, one of the predecessors does not appear, since it is the predecessor of the $(e_q + 1)$-th q-vertex.
In each case one still has to substract the q-vertices in S_*; there are precisely e_q of them.
Altogether, $\lambda^q_{d_q - e_q}$ has an upper bound:

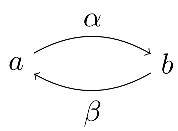
$$\lambda^q_{d_q - e_q} \leq n_q + \sum_{Q_1 \ni r \to q} e_r - e_q - 1 < n_q - \langle e, 1^q \rangle . \qquad \square$$

It remains to show that this map φ is bijective. Since both $\Phi_{d,n}(Q)$ and $\Lambda_{d,n}(Q)$ are finite and of the same cardinality (c. f. Corollary 1.2.18 and [5, Theorem 6.2]), it suffices to prove that φ is injective.

Lemma 1.2.22 The map $\varphi \colon \Phi_{d,n}(Q) \to \Lambda_{d,n}$ defined above is injective.

Proof Let $S_* > S'_*$ be two subforests of $\mathcal{F}_n(Q)$ of dimension type d and choose $(q, i, w) \in S_* \cap S'_*$ maximal with respect to the property, that for all $(q', i', w') \in S_* \cup S'_*$ with $(q', i', w') \leq (q, i, w)$ the condition $(q', i', w') \in S_* \cap S'_*$ holds. Because of $S_* > S'_*$ the immediate successor $(\bar{q}, \bar{i}, \bar{w}) \in S_* \cup S'_*$ bigger than (q, i, w) is contained in $S'_* \cap C(S_*)$. Let $q_0 := h(\bar{w})$ and $(\lambda) := \varphi(S_*)$ as well as $(\lambda') := \varphi(S'_*)$ and $(\bar{q}, \bar{i}, \bar{w})$ be the $(m+1)$-th vertex in S'_* pointing towards q_0.
Since the number of vertices of S_* and S'_* pointing towards q_0 is equal, we can deduce, that there must be at least one more vertex in $C(S_*)$ smaller than the $(m+1)$-th vertex pointing towards q_0 than there are vertices in $C(S'_*)$ subject to this condition with respect to S'_* instead of S_*. This is due to the fact, that we know all these vertices for S'_*, and for S_* all those and additionally $(\bar{q}, \bar{i}, \bar{w})$ satisfy this condition. But this means $\lambda^{q_0}_m > \lambda'^{q_0}_m$ and hence in particular $\lambda \neq \lambda'$. $\square$

Example 1.2.23 As an example take the quiver



together with dimension vectors $d = n = (2,2)$. Then T_a is an infinite line, and so is T_b. Therefore $\mathcal{F}_n(Q)$ consists of two copies of each of them. The subforests of dimension type d which parametrize the cells are listed in Table 1.1 together with the corresponding multipartitions.

Table 1.1: List of forests/ multipartitions for example 1.2.23

	Forest	Multipartition
1	$((( ), \alpha, \alpha\beta, \alpha\beta\alpha), \varnothing, \varnothing, \varnothing)$	$(0,0 \mid 0,0)$
2	$((( ), \alpha, \alpha\beta), \varnothing, ( ), \varnothing)$	$(0,0 \mid 1,0)$
3	$((( ), \alpha, \alpha\beta), \varnothing, \varnothing, ( ))$	$(0,0 \mid 2,0)$
4	$((( ), \alpha), (( ), \alpha), \varnothing, \varnothing)$	$(1,0 \mid 0,0)$
5	$((( ), \alpha), ( ), ( ), \varnothing)$	$(1,0 \mid 1,0)$
6	$((( ), \alpha), ( ), \varnothing, ( ))$	$(1,0 \mid 2,0)$
7	$((( ), \alpha), \varnothing, (( ), \beta), \varnothing)$	$(2,0 \mid 0,0)$
8	$((( ), \alpha), \varnothing, \varnothing, (( ), \beta))$	$(2,0 \mid 1,0)$
9	$(( ), (( ), \alpha), ( ), \varnothing)$	$(0,0 \mid 1,1)$
10	$(( ), (( ), \alpha), \varnothing, ( ))$	$(0,0 \mid 2,1)$
11	$(( ), ( ), ( ), ( ))$	$(0,0 \mid 2,2)$
12	$(( ), \varnothing, (( ), \beta, \beta\alpha), \varnothing)$	$(1,0 \mid 1,1)$
13	$(( ), \varnothing, (( ), \beta), ( ))$	$(1,0 \mid 2,1)$
14	$(( ), \varnothing, ( ), (( ), \beta))$	$(2,0 \mid 1,1)$
15	$(( ), \varnothing, \varnothing, (( ), \beta, \beta\alpha))$	$(1,0 \mid 2,2)$
16	$(\varnothing, (( ), \alpha, \alpha\beta, \alpha\beta\alpha), \varnothing, \varnothing)$	$(1,1 \mid 0,0)$
17	$(\varnothing, ( ), (\alpha, \alpha\beta), ( ), \varnothing)$	$(1,1 \mid 1,0)$
18	$(\varnothing, (( ), \alpha, \alpha\beta), ( ), \varnothing)$	$(1,1 \mid 2,0)$

Table 1.1: List of forests/ multipartitions for example 1.2.23 (cont.)

	Forest	Multipartition
19	$(\varnothing,((),\alpha),((),\beta),\varnothing)$	$(2,1\mid 0,0)$
20	$(\varnothing,((),\alpha),\varnothing,((),\beta))$	$(2,1\mid 1,0)$
21	$(\varnothing,(),((),\beta,\beta\alpha),\varnothing)$	$(1,1\mid 1,1)$
22	$(\varnothing,(),((),\beta),())$	$(1,1\mid 2,1)$
23	$(\varnothing,(),(),((),\beta))$	$(2,1\mid 1,1)$
24	$(\varnothing,(),\varnothing((),\beta,\beta\alpha))$	$(1,1\mid 2,2)$
25	$(\varnothing,\varnothing,((),\beta,\beta\alpha,\beta\alpha\beta),\varnothing)$	$(2,2\mid 0,0)$
26	$(\varnothing,\varnothing,((),\beta),((),\beta))$	$(2,2\mid 1,0)$
27	$(\varnothing,\varnothing,\varnothing,((),\beta,\beta\alpha,\beta\alpha\beta))$	$(2,2\mid 1,1)$

1.3 An explicit formula for the Euler characteristic

Taking the formula for the Poincaré polynomial from Corollary 1.2.18, one obtains the Euler characteristic of the non-commutative Hilbert schemes as

$$P_{\mathrm{Hilb}_{d,n}(Q)}(1) = \sum_{S_* \in \Phi_{d,n}(Q)} 1^{2\dim Z_{S_*}} = \left|\Phi_{d,n}\right|,$$

since all odd homology groups vanish as mentioned earlier. In [17, Corollary 4.5] the author derives a formula for the Euler characteristic from this by using a formular for the enumeration of "plain forests" from [20, Theorem 5.3.10] in the case of the m-loop quiver. However, in the general case this method does not apply anymore in the same way. Therefore in the following an explicit formula for the Euler characteristics of the non-commutative Hilbert schemes is given extending the m-loop case in [17, Corollary 4.5].

Definition 1.3.1 For $d \in \mathbb{N}$ and variables $x = (x_1, \ldots, x_d)$ and $k \in \mathbb{N}^d$ define powers

$$x^d := \prod_{i=1}^{d} x_i^{d_i}.$$

For $k, n \in \mathbb{N}^d$ define

$$\binom{n}{k} := \prod_{i=1}^{d} \binom{n_i}{k_i}.$$

An easy consequence of that definition is the following generalisation of the usual binomial formula for variables x, y as in Definition 1.3.1 and $n \in \mathbb{N}^d$:

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}. \tag{1.6}$$

Here the sum runs over all $k \in \mathbb{N}^d$ with $0 \leq k_i \leq n_i$ for $i = 1, \ldots, d$ and $x + y$ denotes the component-wise summation.

Let $\chi\left(\mathrm{Hilb}_{d,n}(Q)\right)$ denote the Euler characteristic of $\mathrm{Hilb}_{d,n}(Q)$ and write

$$F_Q^n(t) := \sum_{d \in \mathbb{N}_0 Q_0} \chi\left(\mathrm{Hilb}_{d,n}(Q)\right) t^d$$

for the generating functions of these Euler characteristics. These are considered as formal power series in $|Q_0|$ variables with coefficients in $\mathbb{Q}$. In [17, Corollary 5.8] a fundamental equation for these generating functions in the case of $Q = L_m$ is given, which we can now extend to the general case:

Proposition 1.3.2 For $n \in \mathbb{N}_0 Q_0$ the formal power series $F_Q^n(t)$ are the uniquely determined elements of $\mathbb{Q}[[\mathbb{Z} Q_0]]$ satisfying the following functional equations:

$$F_Q^n(t) = \prod_{i \in Q_0} F_Q^i(t)^{n_i} \qquad (n \in \mathbb{N}_0 Q_0) \tag{1.7}$$

$$F_Q^i(t) = 1 + t_i \prod_{j \in Q_0} F_Q^j(t)^{r_{i,j}} \qquad (i \in Q_0). \tag{1.8}$$

where

$$r_{i,j} := |\{\alpha \in Q_1 \mid t\alpha = i,\ h\alpha = j\}| \qquad (i, j \in Q_0)$$

denotes the number of arrows in Q from i to j and $i = (\delta_{i,j})_{j \in Q_0}$ as a dimension vector.

Proof Comparing coefficients in (1.7), this equation is equivalent to the following:

$$\chi\left(\mathrm{Hilb}_{d,n}(Q)\right) = \prod_{i \in Q_0} \prod_{j=1}^{n_i} \sum_{e} \chi\left(\mathrm{Hilb}_{e_{i,j},i}\right)$$

such that $\sum_{i,j} e_{i,j} = d$. This corresponds directly to the decomposition of each subforest of $\mathcal{F}_n(Q)$ of dimension type d into its components.

The second equation follows similarly from the decomposition of a tree starting in i into the root i and the trees with one of the successors of i:

$$\sum_{d} \chi\left(\mathrm{Hilb}_{d,i}(Q)\right) = \sum_{d} \left|\Phi_{d,i}\right| = \underbrace{\left|\Phi_{i,i}\right|}_{=1} + \sum_{d>i} \left|\Phi_{d,i}\right|,$$

and for $d > i$ we can decompose any forest of dimension d with root i into the root and $r_{i,j}$ possibly empty subforests with root j for $j \in Q_0$. Comparing coefficients yields the desired equation. □

To derive a formula for the coefficients of the functions satisfying these functional equations the *multivariate Lagrange inversion* is used following [2].

Definition 1.3.3 (Tree Derivation) Let $\mathcal{D}$ be a graph with $\ell := |\mathcal{D}_0|$, $x = (x_1, \ldots, x_\ell)$ a vector of variables and $f = (f_1, \ldots, f_\ell) \in \mathbb{Q}[[X_1, \ldots, X_\ell]]^\ell$ a vector of formal power series.

Then the derivation of f by $\mathcal{D}$ is defined as

$$\frac{\partial f}{\partial \mathcal{D}} := \prod_{j \in \mathcal{D}_0} \left(\prod_{\substack{\alpha \in \mathcal{D}_1 \\ t\alpha = j}} \frac{\partial}{\partial x_{h\alpha}} \right) f_j(x).$$

In the situation of Definition 1.3.3 let $[t^d] f$ denote the coefficient of t^d in f.

Theorem 1.3.4 ([2, Theorem 2]) Let $g, f_1, \ldots, f_\ell$ be formal power series in $x = (x_1, \ldots, x_\ell)$ with $f_i(0) \neq 0$ for $i = 1, \ldots, \ell$. Then the set of functional equations

$$w_i = t_i f_i(w(t)) \qquad (i = 1, \ldots, \ell) \tag{1.9}$$

uniquely determines w_i as formal power series in t, and one has the following equation for the coefficients of $w = (w_1, \ldots, w_\ell)$:

$$[t^d] g(w(t)) = \frac{1}{\prod_{i=1}^{\ell} d_i} [t^{d-1}] \sum_{\mathcal{T}} \frac{\partial \left(g, f_1^{d_1}, \ldots, f_\ell^{d_\ell} \right)(t)}{\partial \mathcal{T}}, \tag{1.10}$$

where the sum runs over all trees $\mathcal{T}$ with $\mathcal{T}_0 = \{0, 1, \ldots, \ell\}$ and all edges directed away from 0.

Let $\ell = |Q_0|$. Comparing formula (1.9) and formula (1.8), one observes that F^i satisfies formula (1.8) if and only if $w_i := F^i - 1$ satisfies formula (1.9) for

$$f_i(x) := \prod_{j \in Q_0} (1 + x_j)^{r_{i,j}}.$$

Using formula (1.7) one has

$$F^n(t) = (w(t) + 1)^n \overset{(1.6)}{=} \sum_{k=0}^{n} \binom{n}{k} w^k.$$

Thus, in order to obtain a formula for the Euler characteristics, that is for the coefficients of F^n, one has to set $g(x) := x^n$. Then $f(0) = 1$ and thus Theorem 1.3.4 can be applied to the situation and gives:

$$\begin{aligned}\chi\left(\mathrm{Hilb}_{d,n}(Q)\right) &= \sum_{k=0}^{n}\binom{n}{k}\left[t^d\right]w^k(t)\\ &= \sum_{k=0}^{n}\binom{n}{k}\frac{1}{\prod_{i=1}^{\ell}d_i}\left[t^{d-1}\right]\sum_{\mathcal{T}}\frac{\partial\left(t^k, f_1^{d_1}(t),\ldots,f_\ell^{d_\ell}(t)\right)}{\partial\mathcal{T}}.\end{aligned}$$

For the computation of the right hand side some calculations are necessary. One has

$$\begin{aligned}\left(\prod_{b\in Q_0}\frac{\partial}{\partial t_b}\right)f_j^{d_j}(t) &= \left(\prod_{b\in Q_0}\frac{d_j r_{j,b}}{1+t_b}\right)\prod_{c\in Q_0}(1+x_c)^{d_j r_{j,c}}\\ \left(\prod_{b\in Q_0}\frac{\partial}{\partial t_b}\right)t^k &= \left(\prod_{b\in Q_0}\frac{k_b}{t_b}\right)\prod_{c\in Q_0}t_c^{k_c}.\end{aligned}$$

To avoid confusion, in the following we will write $a \underset{\mathcal{T}}{\rightarrow} b$ for arrows in $\mathcal{T}$ and $a \rightarrow b$ for those in Q. One has

$$\begin{aligned}&\sum_{\mathcal{T}}\frac{\partial\left(t^k, f_1^{d_1}(t),\ldots,f_\ell^{d_\ell}(t)\right)}{\partial\mathcal{T}}\\ &= \sum_{\mathcal{T}}\prod_{a\in\mathcal{T}_0}\left(\left(\prod_{a\underset{\mathcal{T}}{\rightarrow}b}\frac{\partial}{\partial t_b}\right)\left(t^k, f_1^{d_1}(t),\ldots,f_\ell^{d_\ell}(t)\right)\right)\\ &= \sum_{\mathcal{T}}\left(\prod_{0\underset{\mathcal{T}}{\rightarrow}b}\frac{\partial}{\partial t_b}\right)t^k\cdot\prod_{a\in Q_0}\left(\prod_{a\underset{\mathcal{T}}{\rightarrow}b}\frac{\partial}{\partial t_b}\right)f_a^{d_a}(t)\end{aligned}$$

$$
\begin{aligned}
&= \sum_{\mathcal{T}} \left(\prod_{0 \underset{\mathcal{T}}{\longrightarrow} b} k_b \right) t^{k - \sum_{0 \underset{\mathcal{T}}{\longrightarrow} b} b} \cdot \left(\prod_{a \in Q_0} \prod_{a \underset{\mathcal{T}}{\longrightarrow} b} d_a r_{a,b} \right) (t+1)^{\sum_{a \in Q_0} \left(d_a r_a - \sum_{a \underset{\mathcal{T}}{\longrightarrow} b} b \right)} \\
&= \sum_{\mathcal{T}} \left(\prod_{0 \underset{\mathcal{T}}{\longrightarrow} b} k_b \right) t^{k - \sum_{0 \underset{\mathcal{T}}{\longrightarrow} b} b} \cdot \left(\prod_{a \in Q_0} \prod_{a \underset{\mathcal{T}}{\longrightarrow} b} d_a r_{a,b} \right) \cdot \\
&\qquad \cdot \sum_{p=0}^{\sum_{a \in Q_0} \left(d_a r_a - \sum_{a \underset{\mathcal{T}}{\longrightarrow} b} b \right)} \binom{\sum_{a \in Q_0} \left(d_a r_a - \sum_{a \underset{\mathcal{T}}{\longrightarrow} b} b \right)}{p} t^p \\
&= \sum_{\mathcal{T}} \left(\prod_{0 \underset{\mathcal{T}}{\longrightarrow} b} k_b \right) \left(\prod_{a \in Q_0} \prod_{a \underset{\mathcal{T}}{\longrightarrow} b} d_a r_{a,b} \right) \cdot \\
&\qquad \cdot \sum_{p=0}^{\sum_{a \in Q_0} \left(d_a r_a - \sum_{a \underset{\mathcal{T}}{\longrightarrow} b} b \right)} \binom{\sum_{a \in Q_0} \left(d_a r_a - \sum_{a \underset{\mathcal{T}}{\longrightarrow} b} b \right)}{p} t^{p + k - \sum_{0 \underset{\mathcal{T}}{\longrightarrow} b} b}.
\end{aligned}
$$

From here it is obvious that all summands will vanish where

- $\mathcal{T}$ has an arrow $0 \to b$ with $k_b = 0$ (note: $0 \le k \le n$) or
- $\mathcal{T}$ contains an arrow $a \to b$ that cannot be found in Q_1, since in that case $r_{a,b}$ and hence the whole product will vanish.

Therefore it is sufficient to sum over the spanning subtrees of Q.

Taking the coefficient of t^{d-1} here means choosing p such that

$$
p + k - \sum_{0 \underset{\mathcal{T}}{\longrightarrow} b} b = d - 1 \quad \iff \quad p = d - k - \sum_{0 \underset{\mathcal{T}}{\not\longrightarrow} b} b.
$$

Since $\mathcal{T}$ can assumed to be a spanning tree of Q, one sees that the condition $0 \underset{\mathcal{T}}{\not\to} b$ is equivalent to $a \underset{\mathcal{T}}{\to} b$ for some $a \in Q_0$. Hence this can be reformulated as

$$p = d - k - \sum_{\substack{a \xrightarrow[\mathcal{T}]{} b \\ a \in Q_0}} b.$$

In the above formula the coefficient of t^{d-1} has the form

$$\sum_{k=0}^{n} \binom{n}{k} \frac{1}{\prod_{i \in Q_0} d_i} \cdot$$
$$\cdot \sum_{\mathcal{T}} \left(\prod_{0 \xrightarrow[\mathcal{T}]{} b} k_b \right) \left(\prod_{\substack{a \xrightarrow[\mathcal{T}]{} b \\ a \in Q_0}} d_a r_{a,b} \right) \prod_{i \in Q_0} \binom{\sum_{a \in Q_0} \left(d_a r_a - \sum_{a \xrightarrow[\mathcal{T}]{} b} b \right)}{d - k - \sum_{\substack{a \xrightarrow[\mathcal{T}]{} b \\ a \in Q_0}} b},$$

with $r_a = \left(r_{a,b} \right)_b$. This can be written as follows:

$$\frac{1}{d^1} \sum_{\mathcal{T}} \sum_{k=0}^{n} \left(\prod_{0 \to b} n_b \binom{n_b - 1}{k_b - 1} \binom{\sum_{a \in Q_0} \left(d_a r_{a,b} - \sum_{a \xrightarrow[\mathcal{T}]{} c} \delta_{b,c} \right)}{d_b - k_b - \sum_{\substack{a \xrightarrow[\mathcal{T}]{} c \\ a \in Q_0}} \delta_{b,c}} \right) \cdot$$
$$\cdot \left(\prod_{\substack{a \xrightarrow[\mathcal{T}]{} b \\ a \in Q_0}} d_a r_{a,b} \binom{n_b}{k_b} \binom{\sum_{c \in Q_0} \left(d_c r_{c,b} - \sum_{\substack{c \xrightarrow[\mathcal{T}]{} e \\ c \in Q_0}} \delta_{b,e} \right)}{d_b - k_b - \sum_{\substack{c \xrightarrow[\mathcal{T}]{} e \\ c \in Q_0}} \delta_{b,e}} \right)$$

Using

$$\sum_{k=0}^{n} \binom{a}{k} \binom{b}{c-k} = \binom{a+b}{c}$$

and setting formally $r_{0,b} := n_b$ for $b \in Q_0$ and $d_0 := 1$ one obtains

Proposition 1.3.5 The Euler characteristic $\chi\left(\mathrm{Hilb}_{d,n}(Q)\right)$ can be written as

$$\sum_{\mathcal{T}} \frac{\prod_{a \xrightarrow[\mathcal{T}]{} b} d_a r_{a,b}}{\prod_{i \in Q_0} d_i} \binom{n + \sum_{a \in Q_0} d_a r_a - 1}{d-1}. \tag{1.11}$$

where $\mathcal{T}$ is running over all spanning trees of Q.

Remark 1.3.6 In the case of the m-loop quiver one has only one vertex and thus exactly one spanning tree $\mathcal{T} : 0 \to 1$. Therefore the above formula simplyfies to

$$\frac{n}{d}\binom{md+n-1}{d-1} = \frac{n}{(m-1)\,d+n}\binom{md+n-1}{d},$$

which is exactly the formula given in [17, Corollary 4.5].

Chapter 2

2 Borel-Moore Homology

This chapter contains a brief overview of a special homology theory which is called *Borel-Moore Homology*. It is named after Armand Borel and James Moore who developed it in [3].The facts presented here are based mostly on [4, chapters 2.6 and 2.7] and sometimes also on [6, Appendix B]. Facts about homology used in this chapter can be found in [11] and [8].
If not mentioned otherwise all homology groups are with rational coefficients. All topological spaces in this chapter are assumed to be locally compact spaces X with the homotopy type of a finite CW-complex. They should admit a closed embedding into a countable at infinity C^∞-manifold and there should always be an open neighbourhood $U \supset X$ such that X is a homotopy retract of U. These properties are always satisfied for real and complex algebraic varieties which will be our main target of interest.
In section 2.2 the general setting of a prominent application of this homology theory is explained. Starting from the usual convolution product on sets and the convolution by integration on smooth compact manifolds this approach is translated and extended to the situation of varieties using Borel-Moore homology. This will later be used in chapter 3 to construct operators in Borel-Moore homology of non-commutative Hilbert schemes.

2.1 Basic facts and properties

If not explicitly mentioned otherwise, we are using the usual $\mathbb{C}$-topology on manifolds in this section; nonetheless, all relevant statements can be expressed in the context of algebraic varieties using the Zariski topology instead due to [19].

Definition 2.1.1 (Borel-Moore Homology) For a topological space X, let $C_*^{\mathrm{BM}}(X)$ be the chain complex of infinite singular chains $\sum_{i=0}^{\infty} a_i\sigma_i$ where $a_i \in \mathbb{C}$ and σ_i is a singular simplex, and the sum is finite in the following sense: For any compact set $D \subset X$ there are only finitely many non-zero coefficients a_i such that $D \cap \operatorname{supp} \sigma_i \neq \varnothing$. The usual boundary map ∂ on singular chains is well-defined on this complex, too, because the faces of any simplex are still subject to the same finiteness condition.

Define the *Borel-Moore homology* as the homology groups of this complex:

$$\mathrm{H}_*^{\mathrm{BM}}(X) := \mathrm{H}_\bullet\left(C_*^{\mathrm{BM}}(X), \partial\right).$$

We see from this definition, that for compact spaces the notions of singular homology and Borel-Moore homology coincide.

Remark 2.1.2 The following alternative definitions are equivalent to Definition 2.1.1:

1. Set $\mathrm{H}_*^{\mathrm{BM}}(X) := \mathrm{H}_*\left(\hat{X}, \infty\right)$, where $\hat{X} = X \cup \{\infty\}$ is the one-point compactification of X.
2. For an arbitrary compactification $\bar{X}$ of X such that $\left(\bar{X}, \bar{X} \smallsetminus X\right)$ is a CW-pair, there is an isomorphism $\mathrm{H}_*^{\mathrm{BM}}(X) \simeq \mathrm{H}_*\left(\bar{X}, \bar{X} \smallsetminus X\right)$.

Also the following lemma can serve as an equivalent definition for Borel-Moore homology groups:

Lemma 2.1.3 (Poincaré Duality) If X can be embededded into a smooth, oriented manifold M of real dimension m as a closed subset which is a proper deformation retract of another closed neighbourhood, then we have a canonical isomorphism

$$\mathrm{H}_i^{\mathrm{BM}}(X) \simeq \mathrm{H}^{m-i}(M, M \smallsetminus X).$$

The conditions are satisfied for example if X is a complex algebraic variety and M is a smooth complex algebraic variety. In the light of this lemma we can understand the relation between Borel-Moore homology and singular cohomology to be the same as the relation between singular homology and cohomology with compact support.

Lemma 2.1.4 (Fundamental Class) A complex algebraic variety X determines a fundamental class $[X]$ in Borel-Moore homology.

If X has irreducible components $X_1, \ldots, X_n$, $[X]$ is the sum of fundamental classes $\sum_{i=1}^{n} [X_i]$.

Corollary 2.1.5 Let X be a complex algebraic variety and $U \subset X$ a closed subvariety. Then there is a fundamental class $[U] \in \mathrm{H}^{\mathrm{BM}}_{\dim_{\mathbb{R}} U}(X)$.

This fundamental class is defined to be the image of $[U] \in \mathrm{H}^{\mathrm{BM}}_{\dim_{\mathbb{R}} U}(U)$ under the pushforward map coming from the inclusion $\iota: U \hookrightarrow X$.

Lemma 2.1.6 If an algebraic variety X has a filtration $X = X_s \supset \cdots \supset X_0 = \varnothing$ by closed algebraic subsets, such that $X_i \smallsetminus X_{i+1}$ is the disjoint union of affine pieces $U_{i,j} \simeq \mathbb{C}^{n(i,j)}$ for all i, then the fundamental classes of the closures $\left[\overline{U_{i,j}}\right]$ form an additive base of the vector spaces $\mathrm{H}_*^{\mathrm{BM}}(X)$.

In particular, if X has an algebraic cell decomposition as in Definition 1.2.1, the fundamental classes of the cell closures form a basis of $\mathrm{H}_*^{\mathrm{BM}}(X)$.

In contrast to singular homology, Borel-Moore homology is not a covariant functor with respect to regular maps. This can be seen by considering the alternate definition given above: One has to ensure that for a map $f\colon X \to Y$ also the induced map $\hat{f}\colon \hat{X} \to \hat{Y}$ with $\hat{f}(\infty) = \infty$ is continuous. This can be guaranteed by restricting to proper maps:

Lemma 2.1.7 (Proper Pushforward) Borel-Moore homology is a covariant functor with respect to proper maps, that is for any proper map $f\colon X \to Y$ there is a morphism in Borel-Moore homology

$$f_*\colon \mathrm{H}_*^{\mathrm{BM}}(X) \to \mathrm{H}_*^{\mathrm{BM}}(Y).$$

For a closed subvariety $Z \subset X$ this satisfies

$$f^*\colon [Z] \mapsto \left[\overline{f(Z)}\right].$$

Since any proper algebraic morphism is also a proper map in $\mathbb{C}$-topology, this lemma also holds in the context of algebraic varieties.

Lemma 2.1.8 (Restriction) Let $U \subset X$ be an open subset with inclusion map $\iota\colon U \to X$. Then we have a natural restriction morphism

$$\iota^*\colon \mathrm{H}_*^{\mathrm{BM}}(X) \to \mathrm{H}_*^{\mathrm{BM}}(U).$$

Details about this and an alternative construction can be found in [11].

Lemma 2.1.9 (Long exact sequence) Let $F \subset X$ be a closed subset with the complement $U := X \setminus F$; write $i\colon F \to X$ and $j\colon U \to X$ for the corresponding embeddings. Being a closed embedding i is in particular proper, so we have maps i_* and j^* as above. They give rise to a long exact sequence:

$$\cdots \to \mathrm{H}_p^{\mathrm{BM}}(F) \xrightarrow{i_p} \mathrm{H}_p^{\mathrm{BM}}(X) \xrightarrow{j^p} \mathrm{H}_p^{\mathrm{BM}}(U) \to \mathrm{H}_{p-1}^{\mathrm{BM}}(F) \to \dots .$$

Lemma 2.1.10 (Smooth Pullback) Let X be a locally compact space and

$$p: \tilde{X} \to X$$

a locally trivial fibration with smooth oriented fiber F, such that all transition functions of the fibration preserve the orientation of the fiber. In particular, these conditions are satisfied if p is Zariski locally trivial. Then we have a natural pullback morphism

$$p^*: \mathrm{H}_\bullet^{\mathrm{BM}}(X) \to \mathrm{H}_{\bullet+\dim_{\mathbb{R}} F}^{\mathrm{BM}}(\tilde{X}).$$

This has the property that it restricts to the map $c \mapsto c \times [F]$ where the product comes from the Künneth formula in Borel Moore homology on any open subset $U \subset X$ such that $p: p^{-1}(U) \to U$ is trivial. That means for any cartesian square

$$\begin{array}{ccc} U \times F & \xhookrightarrow{\tilde{i}} & \tilde{X} \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle p} \\ U & \xhookrightarrow{i} & X \end{array}$$

the induced square commutes:

$$\begin{array}{ccc}
H_{\bullet+\dim_{\mathbb{R}} F}(U \times F) & \xleftarrow{\tilde{i}^*} & H_{\bullet+\dim_{\mathbb{R}} F}(\tilde{X}) \\
{\scriptstyle c \mapsto c \times [F]} \big\uparrow & & \big\uparrow {\scriptstyle p^*} \\
H_\bullet(U) & \xleftarrow{i^*} & H_\bullet(X)
\end{array}$$

Assume given spaces $Z, S, \tilde{S}$, a morphism $f\colon Z \to S$ and a Zariski locally trivial morphism $\varphi\colon \tilde{S} \to S$ with smooth fiber. Set $\tilde{Z} := Z \times_S \tilde{S}$ and form the natural cartesian diagram

$$\begin{array}{ccc}
\tilde{Z} & \xrightarrow{\tilde{\varphi}} & Z \\
{\scriptstyle \tilde{f}} \big\downarrow & & \big\downarrow {\scriptstyle f} \\
\tilde{S} & \xrightarrow{\varphi} & S
\end{array}$$

Then one has a well-defined pullback homomorphism

$$\tilde{\varphi}^*\colon \mathrm{H}_*^{\mathrm{BM}}(Z) \to \mathrm{H}_*^{\mathrm{BM}}(\tilde{Z})$$

in Borel-Moore homology given by the smooth pullback.

Proposition 2.1.11 (Base change) If furthermore $f\colon Z \to S$ is proper, then also the following diagram commutes:

$$\begin{array}{ccc}
\mathrm{H}_*^{\mathrm{BM}}(\tilde{Z}) & \xleftarrow{\tilde{\varphi}^*} & \mathrm{H}_*^{\mathrm{BM}}(Z) \\
{\scriptstyle \tilde{f}_*} \big\downarrow & & \big\downarrow {\scriptstyle f_*} \\
\mathrm{H}_*^{\mathrm{BM}}(\tilde{S}) & \xleftarrow{\varphi^*} & \mathrm{H}_*^{\mathrm{BM}}(S)
\end{array}$$

The proof of this can be found in [11].

2.2 Convolution products

Assume there are given three finite sets M_1, M_2 and M_3 and let $C(M)$ be the finite dimensional vector space of $\mathbb{C}$-valued functions on M. Define then a convolution product:

$$\begin{aligned} f_{1,3}(m_1, m_3) &= (f_{1,2} * f_{2,3})(m_1, m_3) \\ := &\sum_{m_2 \in M_2} f_{1,2}(m_1, m_2) \cdot f_{2,3}(m_2, m_3) \end{aligned}$$

for $f_{i,j} \in \mathbb{C}(M_i \times M_j)$ and $m_i \in M_i$. Switching from finite sets to compact oriented manifolds with C^∞-functions the sum becomes an integral.

The goal of this section is to make a similar approach in Borel-Moore homology, that is to define a map

$$\mathrm{H}_\bullet^{\mathrm{BM}}(M_1 \times M_2) \to \mathrm{H}_{\bullet+k}^{\mathrm{BM}}(N_{1,2})$$

for some $k \in \mathbb{N}$. By setting $P_{1,2} = M_1 \times M_2 \times N_{1,2}$ one can then obtain the setting above as a special case. For this purpose assume a variety $P_{1,2}$ such that $p_{1,2}\colon P_{1,2} \to M_1 \times M_2$ is a Zariski locally trivial morphism with fiber F and $\bar{p}_{1,2}\colon P_{1,2} \to N_{1,2}$ a proper morphism. Let $Z_1 \subset M_1$ and $Z_2 \subset M_2$ be two closed subvarieties. They induce fundamental classes $[Z_1] \in \mathrm{H}^{\mathrm{BM}}_{\dim_{\mathbb{R}} Z_1}(M_1)$ resp. $[Z_2] \in \mathrm{H}^{\mathrm{BM}}_{\dim_{\mathbb{R}} Z_2}(M_2)$ according to Lemma 2.1.4. Then for U open with $M_1 \times M_2 \subset U$ such that $p_{1,2}|_U$ is trivial, Lemma 2.1.10 gives that $p_{1,2}^*$ satisfies:

$$p_{1,2}^* : [Z_1] \times [Z_2] = [Z_1 \times Z_2] \mapsto [Z_1 \times Z_2] \times [F] = [Z_1 \times Z_2 \times F].$$

Since $\bar{p}_{1,2}$ was assumed to be proper, according to Lemma 2.1.7 there is a natural morphism

$$(\bar{p}_{1,2})_* : \mathrm{H}_\bullet^{\mathrm{BM}}(P_{1,2}) \to \mathrm{H}_\bullet^{\mathrm{BM}}(N_{1,2})$$

satisfying

$$(\bar{p}_{1,2})_* : [Z_1 \times Z_2 \times F] \mapsto \left[\overline{\bar{p}_{1,2}(Z_1 \times Z_2 \times F)}\right].$$

To show that the so defined convolution product is associative we have to prove that

1. in diagram in Figure 2.1 there exists an isomorphism from the top to the bottom making this diagram commutative, and that

2. also the induced diagram in Borel-Moore homology commutes.

Here the top square and the bottom square are fibre products.

The first claim can be proved in a straightforward way by diagram chase. All one has to do is to give an isomorphism

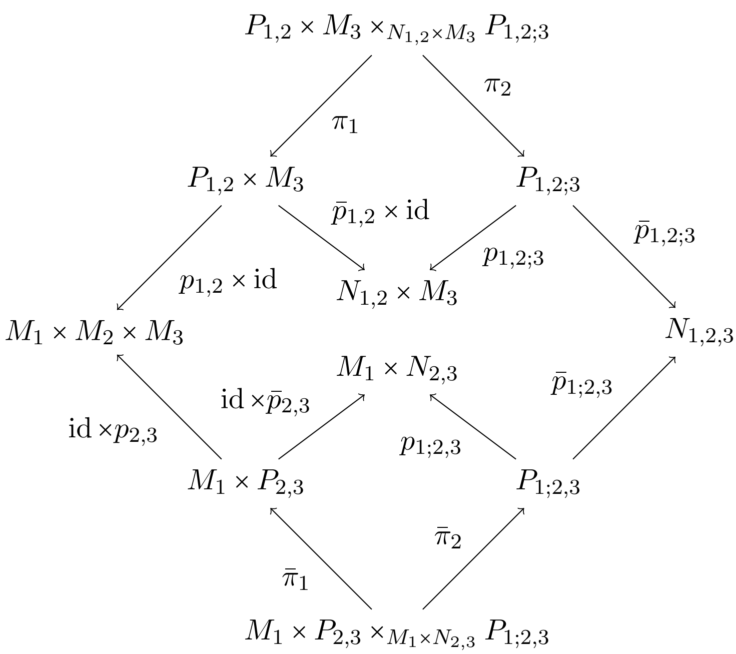
$$p: P_{1,2} \times M_3 \times_{N_{1,2} \times M_3} P_{1,2;3} \to M_1 \times P_{2,3} \times_{M_1 \times N_{2,3}} P_{1;2,3}$$

such that

$$\bar{p}_{1;2,3} \circ \bar{\pi}_2 \circ p = \bar{p}_{1,2;3} \circ \pi_2 \qquad \text{and}$$
$$\mathrm{id} \times p_{2,3} \circ \bar{\pi}_1 \circ ps = p_{1,2} \times \mathrm{id} \circ \pi_1 .$$

Given that this isomorphism p exists, the second item on the list can be

Figure 2.1: Associativity of a convolution product in Borel-Moore homology



proved as follows:

$$
\begin{aligned}
&\left(\bar{p}_{1,2;3}\right)_* \left(p_{1,2;3}\right)^* \left(\bar{p}_{1,2} \times \mathrm{id}\right)_* \left(p_{1,2} \times \mathrm{id}\right)^* \\
&= \left(\bar{p}_{1,2;3}\right)_* \left(\pi_2\right)_* \left(\pi_1\right)^* \left(p_{1,2} \times \mathrm{id}\right)^* \\
&= \left(\bar{p}_{1,2;3}\pi_2\right)_* \left(\left(p_{1,2} \times \mathrm{id}\right)\pi_1\right)^* \\
&= \left(\bar{p}_{1;2,3}\bar{\pi}_2 p\right)_* \left(\left(\mathrm{id} \times p_{2,3}\right)\bar{\pi}_1 p\right)^* \\
&= \left(\bar{p}_{1;2,3}\right)_* \left(\bar{\pi}_2\right)_* p_* p^* \left(\bar{\pi}_1\right)^* \left(\mathrm{id} \times p_{2,3}\right)^* \\
&= \left(\bar{p}_{1;2,3}\right)_* \left(\bar{\pi}_2\right)_* \left(\bar{\pi}_1\right)^* \left(\mathrm{id} \times p_{2,3}\right)^* \\
&= \left(\bar{p}_{1;2,3}\right)_* \left(p_{1;2,3}\right)^* \left(\mathrm{id} \times \bar{p}_{2,3}\right)_* \left(\mathrm{id} \times p_{2,3}\right)^*
\end{aligned}
$$

using $p_* p^* = \mathrm{id}$, since p is an isomorphism, given π_1 and $p_{1,2} \times \mathrm{id}$ are Zariski locally trivial and π_2 and $\bar{p}_{1,2;3}$ proper. The respective conditions have to be satisfied for the lower half of the diagram.

Chapter 3

3 Operators on Hilbert Schemes

Mathematical science is in my opinion an indivisible whole, an organism whose vitality is conditioned upon the connection of its parts.

(David Hilbert)

This chapter gives an application of the techniques presented in chapter 2 to the non-commutative Hilbert schemes constructed in chapter 1 in the special case of the m-loop quiver L_m. The operators defined here on the nilpotent non-commutative Hilbert schemes can be used for an implementation of a Fock space. This is in analogy of the construction by Hiraku Nakajima in [16] for $\mathrm{Hilb}^n\left(\mathbb{A}^2\right)$ where the result is a Heisenberg algebra.

We will restrict to the nilpotent Hilbert schemes which will be introduced first. The reason for this restriction can be seen in the proof of Lemma 3.2.5. Furthermore, we give a description of the cell closures for the nilpotent non-commutative Hilbert schemes.

3.1 Nilpotent Hilbert Schemes

Recall the definition of the non-commutative Hilbert schemes from Definition 1.1.11.

Definition 3.1.1 (Nilpotent Hilbert Scheme) Let (Q, d, n) be some quiver datum. The *nilpotent Hilbert scheme* $\mathrm{Hilb}^0_{d,n}(Q)$ consists of those points $(M, f) \in \mathrm{Hilb}_{d,n}(Q)$ such that M_w is nilpotent for all cycles w in Q.

Equivalently, $\mathrm{Hilb}^0_{d,n}(Q)$ can be defined as the zero fibre of the *Hilbert-Chow morphism*

$$\pi\colon \mathrm{Hilb}_{d,n}(Q) \to M^{\mathrm{sst}}_d(Q), \quad (M, f) \mapsto M$$

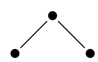
which is projective (c. f. [5]). Here $M^{\mathrm{sst}}_d(Q) := R^{\mathrm{sst}}_d(Q) / \mathrm{GL}_d$, where we denote by $R^{\mathrm{sst}}_d(Q)$ the variety of semistable representations of dimension type d of Q. Therefore the nilpotent Hilbert schemes are projective, but not necessarily smooth.

The nilpotent Hilbert schemes $\mathrm{Hilb}^0_{d,n}(Q)$ are covered by open subsets $U^0_{S_*}$ for a subforest S_* of $\mathcal{F}_n(Q)$ with

$$U^0_{S_*} := \left\{ (M, f) \in \mathrm{Hilb}^0_{d,n}(Q) \mid (1.1) \right\}.$$

Unlike the situation for the usual non-commutative Hilbert schemes which is described in Lemma 1.2.11, in the nilpotent case the sets $U^0_{S_*}$ are not necessarily isomorphic to affine spaces as the following example illustrates:

Example 3.1.2 Let $Q := L_2$ be the 2-loop quiver and S_* be the tree



Then $M \in U^0_{S_*}$ has the form

$$M_\alpha = \begin{pmatrix} 0 & 0 & 0 \\ 1 & \lambda a & \lambda b \\ 0 & \lambda c & \lambda d \end{pmatrix}, \qquad M_\beta = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \mu a & \mu b \\ 1 & \mu c & \mu d \end{pmatrix}$$

where $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ is nilpotent and $\lambda, \mu \in \mathbb{C}$.

This can be seen as follows: For $(M, f) \in U^0_{S_*}$ we can assume M to be of the form

$$M_\alpha = \begin{pmatrix} 0 & a & d \\ 1 & b & e \\ 0 & c & f \end{pmatrix} \qquad M_\beta = \begin{pmatrix} 0 & g & j \\ 0 & h & k \\ 1 & i & \ell \end{pmatrix}.$$

Since all cycles have to be nilpotent, this implies especially that both of these matrices have to be nilpotent. But that is equivalent to the condition that the characteristic polynomial is of the form

$$\det(\lambda E - M_\alpha) = \lambda^3 = \det\left(\lambda E - M_\beta\right).$$

From this condition we can deduce immediately

$$\begin{aligned} b &= -f, \qquad & -b^2 &= ce + a, \qquad & cd + af &= 0, \\ h &= -\ell, \qquad & -h^2 &= ik + g, \qquad & ij + g\ell &= 0. \end{aligned}$$

Furthermore, the second or third column vector of any of these matrices have to vanish under M_α and M_β, since they can be written as a word of length two applied to $f(1)$. That yields

$$0 = M_\alpha \cdot \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} ab + cd \\ a + b^2 + ce \\ bc + cf \end{pmatrix}, \tag{3.1}$$

$$0 = M_\alpha \cdot \begin{pmatrix} d \\ e \\ f \end{pmatrix} = \begin{pmatrix} ae + df \\ d + be + ef \\ ce + f^2 \end{pmatrix} \tag{3.2}$$

and analogously for M_β instead of M_α or the column vectors of M_β instead of those of M_α.
The second row in formula (3.2) implies $0 = d + be + ef = d + be - be = d$ using $b = -f$ from above. Similarly the first row of formula (3.1) implies $0 = ab + cd = ab$ and thus $a = 0$ or $b = 0$, but $b = 0$ implies $c = 0$ or $e = 0$ using the third row of formula (3.2) and hence also $a = 0$ using $-b^2 = ce + a$. Analogously one obtains $g = j = 0$. It is now clear that $\begin{smallmatrix} b & e \\ c & f \end{smallmatrix}$ has to be a nilpotent matrix so as $\begin{smallmatrix} h & k \\ i & \ell \end{smallmatrix}$, since now we know that $M_\alpha f$ and $M_\beta f$ span a subrepresentation.

Since all cycles have to be nilpotent, the two nilpotent submatrices obviously have to be proportional.

However, the following is easy to see:

Lemma 3.1.3 The nilpotent Hilbert schemes have a cell decomposition and the cells are given as

$$Z^0_{S_*} := \left\{ (M, f) \in U^0_{S_*} \mid (3.3) \right\}$$

with

$$M_w f_q \left(v_{q,i} \right) \in \langle M_{w'} f_{q'} \left(v_{q',i'} \right) \rangle_{\substack{(q,i,w) > (q',i',w') \in S_* \\ (q',i',w') \not\prec (q,i,w) \\ h(w) = h(w')}} \qquad \left((q,i,w) \in C\left(S_* \right) \right). \tag{3.3}$$

Proof Assume this condition does not hold. Let $(M, f) \in Z^0_{S_*}$ and choose $(\bar{q}, \bar{i}, \bar{w}) \in C\left(S_*\right)$ minimal with

$$M_{(\bar{q},\bar{i},\bar{w})} f = \sum_{(q,i,w) \in S_*} \mu_{(\bar{q},\bar{i},\bar{w}),(q,i,w)} M_{(q,i,w)} f$$

such that $\mu_{(\bar{q},\bar{i},\bar{w}),(q,i,w)} \neq 0$ for some predecessor $w << \bar{w}$ of $\bar{w}$ with $h(w) = h(\bar{w})$ and $\bar{q} = q, \bar{i} = i$.
For such (q,i,w) let $j((\bar{q},\bar{i},\bar{w}),(q,i,w))$ denote the point given by

$$(q,i,wj((\bar{q},\bar{i},\bar{w}),(q,i,w))) = (\bar{q},\bar{i},\bar{w}).$$

Let $p(\bar{w})$ denote the maximal predecessor of $\bar{w}$, that is $\bar{w} = p(\bar{w})\alpha$ for some $\alpha \in Q_1$. Choose w' such that

$$\begin{aligned} j' :=& j\left((\bar{q},\bar{i},\bar{w}),\left(\bar{q},\bar{i},w'\right)\right) \\ =& \min\left\{j((\bar{q},\bar{i},\bar{w}),(\bar{q},\bar{i},w)) \mid w << \bar{w},\ \mu_{(\bar{q},\bar{i},\bar{w}),(\bar{q},\bar{i},w)} \neq 0\right\}. \end{aligned}$$

Then by the minimality condition of w' and $(\bar{q},\bar{i},\bar{w})$ and the defining condition of Z_{S_*} one has

$$M_{j'}M_{(\bar{q},\bar{i},\bar{w})}f = \sum_{S_* \ni (q,i,w) < (\bar{q},\bar{i},\bar{w})} \mu'_{(\bar{q},\bar{i},\bar{w}),(q,i,w)} M_{(q,i,w)} f$$

where $\mu'_{(\bar{q},\bar{i},\bar{w}),(\bar{q},\bar{i},\tilde{w}')} \neq 0$ for

$$\tilde{w}' := \min\left\{w << \bar{w} \mid \mu_{\bar{w},w} \neq 0\right\}.$$

By assumption such a $\tilde{w}'$ does always exist.
Choose w'' such that

$$\begin{aligned} j'' :=& j\left((\bar{q},\bar{i},\bar{w}),\left(\bar{q},\bar{i},w''\right)\right) \\ =& \min\left\{j((\bar{q},\bar{i},\bar{w}),(\bar{q},\bar{i},w)) \mid w << \bar{w},\ \mu'_{(\bar{q},\bar{i},\bar{w}),(\bar{q},\bar{i},w)} \neq 0\right\}. \end{aligned}$$

and obtain coefficients μ'' and $\tilde{w}''$ from replacing $M_{j'}$ by $M_{j'j''}$ such that

$$\mu''_{(\bar{q},\bar{i},\bar{w}),(\bar{q},\bar{i},\tilde{w}'')} \neq 0.$$

Continuning inductively gives a cycle $j = j'j'' \dots$ such that

$$M_j^\ell M_{(\bar{q},\bar{i},p(\bar{w}))} f \neq 0 \qquad (\ell \in \mathbb{N}) .$$

But that contradicts the nilpotency of (M, f) we assumed. Note that this cycle is finite due to the defining condition of Z_{S_*} and since $(\bar{q}, \bar{i}, \bar{w})$ has only finitely many predecessors. Therefore the assumption was wrong and $Z^0_{S_*}$ has the form claimed above. □

The sets $Z^0_{S_*}$ are isomorphic to affine spaces as in the case mentioned before; this follows because $Z^0_{S_*}$ arises from Z_{S_*} by setting a fixed set of coordinates to zero, which can be seen by comparing formula (1.2) and formula (3.3). Also in this case the formula

$$Z^0_{S_*} = U^0_{S_*} \setminus \bigcup_{S'_* < S_*} U^0_{S'_*}$$

holds: $Z_{S_*} = U_{S_*} \setminus \bigcup_{S'_* < S_*} U_{S'_*}$ implies

$$\begin{aligned} Z^0_{S_*} =& Z_{S_*} \cap \mathrm{Hilb}^0_{d,n}(Q) \\ =& \left(U_{S_*} \setminus \bigcup_{S'_* < S_*} U_{S'_*} \right) \cap \mathrm{Hilb}^0_{d,n}(Q) = U^0_{S_*} \setminus \bigcup_{S'_* < S_*} U^0_{S'_*} . \end{aligned}$$

This means that the combinatorics of the cells is the same as in the general case described in chapter 1.

For a subforest $S_* \in \Phi_{d,n}(Q)$ define a quiver $Q(S_*)$ consisting of

- vertices $Q(S_*)_0 := (S_*)_0$ and
- arrows $\alpha_{(q,i,w),(q',i',w')} \colon (q,i,w) \to (q',i',w')$ if
  - $q = q'$, $i = i'$ and $w' = w\alpha$ for some $\alpha \in Q_1$,

– $w\alpha \in C(S_*)$ for some $\alpha \in Q_1$ with $h(w) = h(w')$ and

$$\left(q', i', w'\right) < (q, i, w), \quad \left(q', i', w'\right) \nless (q, i, w).$$

For dimension vectors $d, n \in \mathbb{N}Q$ define dimension vectors $\tilde{d}$ and $\tilde{n}$ for $Q(S_*)$ by setting

$$\begin{aligned} \tilde{d}_{(q,i,w)} &:= 1 \qquad ((q, i, w) \in S_*), \\ \tilde{n}_{(q,i,w)} &:= \delta_{w,()} \qquad ((q, i, w) \in S_*). \end{aligned}$$

$Q(S_*)$ contains no oriented cycles because arrows to predecessors are explicitely excluded, hence the nilpotency condition is empty, therefore all representations of $Q(S_*)$ are nilpotent.

The set $R_{\tilde{d},\tilde{n}}(Q(S_*))$ consists of tuples of scalars

$$\left(\lambda_{\alpha,(q,i,w),(q',i',w')}, \phi_{(q,i,())}\right),$$

where the first tuple runs over all arrows in $Q(S_*)$ and the second one over those tuples (q, i) where $S_{(q,i)} \neq \varnothing$. Formally we set $\phi_{(q,i,())} = 0$ if $S_{q,i} = \varnothing$.

On $R_{\tilde{d},\tilde{n}}(Q(S_*))$ one has an operation of the group

$$\mathrm{GL}_{\tilde{d}}(Q(S_*)) = \prod_{(q,i,w) \in S_*} \mathrm{GL}_1.$$

Let M_j for $j \in Q_0$ be the vector space with basis vectors

$$\left(m_{(q,i,w)}\right)_{\substack{(q,i,w) \in S_* \\ h(w) = j}},$$

and V_j the vector space given by basis vectors

$$(v_{j,i})_{i=1,\ldots,n_j}.$$

Define a map

$$\sigma\colon R_{\tilde{d},\tilde{n}}\left(Q\left(S_*\right)\right) \to R_{d,n}\left(Q\right), \quad \sigma\left(p\right) := \left(M, f\right)$$

with

$$M_\alpha\left(m_{(q,i,w)}\right) := \sum_{Q(S_*)_1 \ni \alpha_{(q,i,w),(q',i',w')}} \lambda_{(q,i,w),(q',i',w')} m_{(q',i',w')}$$

and

$$f_q\left(v_{q,i}\right) := \phi_{q,i} m_{(q,i,())}.$$

It is easy to see that σ is injective.
$\mathrm{GL}_{\tilde{d}}\left(Q\left(S_*\right)\right)$ can be embedded into $\mathrm{GL}_d\left(Q\right)$ by setting

$$g \cdot m_{(q,i,w)} = g_{(q,i,w)} m_{(q,i,w)} \qquad \left(g = \left(g_{(q,i,w)}\right)_{(q,i,w)\in S_*}\right).$$

Here g is invertible, since $g_{(q,i,w)} \in \mathrm{GL}_1$.

Lemma 3.1.4 The map σ defined above is $\mathrm{GL}_{\tilde{d}}\left(Q\left(S_*\right)\right)$-invariant.

Proof Let $p \in R_{\tilde{d},\tilde{n}}\left(Q\left(S_*\right)\right)$ be given by

$$\left(\left(\lambda_{(q,i,w),(q',i',w')}\right), \left(\phi_{(q,i)}\right)\right).$$

Then for $g \in \mathrm{GL}_{\tilde{d}}\left(Q\left(S_*\right)\right)$ one has

$$g \cdot p = \left(\left(g_{(q',i',w')} \lambda_{(q,i,w),(q',i',w')} g^{-1}_{(q,i,w)}\right), \left(g_{(q,i,())} \phi_{(q,i)}\right)\right).$$

Hence $\left(\tilde{M}, \tilde{f}\right) := \sigma\left(gp\right)$ is given by

$$\tilde{f}_q\left(v_{q,i}\right) = g_{q,i,()} \phi_{q,i} m_{(q,i,())} = g \cdot f_q\left(v_{q,i}\right)$$

and

$$\begin{aligned}
&\tilde{M}_\alpha\left(m_{(q,i,w)}\right)\\
&= \sum_{\substack{Q(S_*)_1 \ni \alpha_{(q,i,w),(q',i',w')}\\ h(\alpha)=h(w')}} g_{(q',i',w')} \lambda_{(q,i,w),(q',i',w')} g^{-1}_{(q,i,w)} m_{(q',i',w')}\\
&= \sum_{\substack{Q(S_*)_1 \ni \alpha_{(q,i,w),(q',i',w')}\\ h(\alpha)=h(w')}} g_{(q',i',w')} \lambda_{(q,i,w),(q',i',w')} m_{(q',i',w')} g^{-1}_{(q,i,w)}\\
&= g \sum_{\substack{Q(S_*)_1 \ni \alpha_{(q,i,w),(q',i',w')}\\ h(\alpha)=h(w')}} \lambda_{(q,i,w),(q',i',w')} m_{(q',i',w')} g^{-1}_{(q,i,w)}\\
&= g M_\alpha g^{-1}\left(m_{(q,i,w)}\right). \qquad \square
\end{aligned}$$

Let $\bar{w} = \alpha_1 \dots \alpha_k$ be an oriented cycle in Q and $(M, f) = \sigma(p)$ as above. Then one has

$$M_{\bar{w}}\left(m_{(r,j,x)}\right) = \sum_{\bar{w}'} \prod_{\ell=1}^{k} \lambda_{\alpha_\ell,(q_\ell,i_\ell,w_\ell),(q_{\ell+1},i_{\ell+1},w_{\ell+1})} m_{(q_{k+1},i_{k+1},w_{k+1})},$$

where the sum runs over all words

$$\bar{w}' = (\alpha_1)_{(q_1,i_1,w_1),(q_2,i_2,w_2)} \dots (\alpha_k)_{(q_k,i_k,w_k),(q_{k+1},i_{k+1},w_{k+1})}$$

in $Q(S_*)$ with $(q_1, i_1, w_1) = (r, j, x)$. By definition of $Q(S_*)$ one has

$$\begin{gathered}
(r,j,x) \prec (q_{k+1}, i_{k+1}, w_{k+1}) \qquad \text{or}\\
(q_{k+1}, i_{k+1}, w_{k+1}) < (r,j,x), \quad \text{and} \quad (q_{k+1}, i_{k+1}, w_{k+1}) \nprec (r,j,x)
\end{gathered}$$

for all $m_{(q_{k+1},i_{k+1},w_{k+1})}$ with non-vanishing coefficient. Hence by reordering the basis vectors according to this condition, one sees that the matrix representing M_w is similar to an upper triangular matrix and thus M_w is nilpotent.

Thus the composition of σ with the quotient map

$$R^{\mathrm{st}}_{d,n}(Q) \to \mathrm{Hilb}_{d,n}(Q)$$

is $\mathrm{GL}_{\tilde{d}}(Q(S_*))$-invariant and therefore induces a map

$$\varphi\colon \mathrm{Hilb}^0_{\tilde{d},\tilde{n}}(Q(S_*)) \hookrightarrow \mathrm{Hilb}^0_{d,n}(Q)\,.$$

This gives a description of the closures of the cells as follows:

Proposition 3.1.5 Let $d, n \in \mathbb{N}Q$ and $S_* \subset \mathcal{F}_n(Q)$ be a subforest. Then we have

$$\overline{Z^0_{S_*}} \simeq \mathrm{Hilb}^0_{d(S_*),n(S_*)}(Q(S_*))\,.$$

Proof Both $\mathrm{Hilb}^0_{d,n}(Q)$ as well as $\mathrm{Hilb}^0_{d(S_*),n(S_*)}(Q(S_*))$ contain a cell $Z^0_{S_*}$ and σ maps these cells onto each other. The isomorphism of cells extends to an embedding of projective varieties as mentioned above

$$\mathrm{Hilb}^0_{d(S_*),n(S_*)}(Q(S_*)) \hookrightarrow \mathrm{Hilb}^0_{d,n}(Q)\,.$$

By construction, $\mathrm{Hilb}^0_{\tilde{d},\tilde{n}}(Q(S_*))$ contains $Z^0_{S_*}$ as the generic cell, hence the closure is $\mathrm{Hilb}^0_{\tilde{d},\tilde{n}}(Q(S_*))$. Thus, the image of this projective morphism is closed and irreducible due to [15] and contains $Z^0_{S_*}$. Therefore this is the closure of $Z^0_{S_*}$. □

3.2 Convolution Operators

From now on we shall restrict to the special class of quivers $Q = L_m$ and $n = 1$; in this case f amounts to the choice of a vector in the representation. To implement the convolution operators using the setup presented in chapter 2 one has to define a variety in the products of Hilbert schemes with morphisms to each component. That is

$$P_{d,e} \subseteq \mathrm{Hilb}_d^0(Q) \times \mathrm{Hilb}_{d+e}^0(Q)$$

such that in

$$\begin{array}{ccc} & P_{d,e} & \\ \swarrow^{\pi_1} & & \searrow^{\pi_2} \\ \mathrm{Hilb}_d^0(Q) & & \mathrm{Hilb}_{d+e}^0(Q) \end{array}$$

the projection onto the second component $\pi_2|_{P_{d,e}}$ is proper and the projection onto the first component $\pi_1|_{P_{d,e}}$ is Zariski locally trivial. At this point one of the reasons for the restriction to nilpotent non-commutative Hilbert schemes becomes obvious: This guarantees that π_2 is projective and therefore proper, which is the statement of Lemma 3.2.4.

Let

$$P_{d,e} := \left\{ \begin{pmatrix} (M,f) \\ (L,h) \end{pmatrix} \mid (3.5) \right\} \subseteq \mathrm{Hilb}_d^0(Q) \times \mathrm{Hilb}_{d+e}^0(Q) \tag{3.4}$$

subject to the condition

$$I(L) \subset I(M). \tag{3.5}$$

One can immediately see that in this setting L has some subrepresentation N of dimension e, such that $L/N \simeq M$. This shows that for $M \in U_{S_*}^0$ the elements $(L_w h)_{w \in S_*}$ are linearly independent, so according to Lemma 1.2.12 there exists a tree extension $\bar{S}_*$ of S_* such that $L \in U_{\bar{S}_*}^0$.

Lemma 3.2.1 The set $P_{d,e}$ defined above is closed.

Proof Consider the set $\bar{P}_{d,e}$ of representations of Q of dimension vector d resp. $d+e$ satisfying formula (3.5). This is closed in $R_d(Q) \times R_{d+e}(Q)$ according to [18, Lemma 2.2]. Adding an additional vector to each representation and reducing to the case where these vectors are stable for the respective representations gives a closed set

$$\tilde{P}_{d,e} := \{((M,f),(L,h)) \mid I(L,h) \subseteq I(M,f)\} \subseteq R_d^{\mathrm{st}} \times R_{d+e}^{\mathrm{st}}.$$

This is stable under the action of $\mathrm{GL}_d \times \mathrm{GL}_{d+e}$, hence the quotient $P_{d,e}$ is closed. □

Example 3.2.2 Let $Q = L_2$ and

$$M_\alpha = M_\beta = (0),$$

Then $P_{1,2} = \mathrm{Hilb}_1^0(Q) \times \mathrm{Hilb}_3^0(Q)$. Note however, in general it is not true that inclusion from (3.4) is an equation.

Theorem 3.2.3 The projection

$$\pi_1 \colon P_{d,e} \to \mathrm{Hilb}_d^0(Q)$$

is a Zariski locally trivial morphism.

Proof Let $(M,f),(M',f') \in U_{S_*}^0$ and $(L,h) \in F_M$. Denote by $F_M := \pi_1^{-1}(M,f)$ the fibre of π_1 over (M,f). By definition $I(L,h)$ is a subideal of $I(M,f)$, thus there is a subrepresentation N of L such that L can be written as

$$L = \begin{pmatrix} M & 0 \\ X & N \end{pmatrix}.$$

Define $L' \in \mathrm{Hilb}_{d+e}^0$ by

$$L' := \begin{pmatrix} M' & 0 \\ X & N \end{pmatrix}.$$

This implies $I(L',h) < I(M',f')$ and $I(L',h)$ has codimension e. Since L is nilpotent, the same holds for N. By definition also M' is nilpotent and therefore any matrix consisting of the columns of L' as one can easily see by multiplication of block matrices. Thus one obtains $(L',h) \in \mathrm{Hilb}^0_{d+e}(Q)$.

Also the stability of L' follows from the stability of L with an argument that is to some extent similar to Lemma 1.2.12. The difference is that here we do not a priori have the stability condition, in fact, as we prove the stability here it turns out to be the same argument. Starting with $(M,f),(M',f') \in U^0_{S_*}$, one has a linearly independent system $(L_w h)_{h\in S_*}$. Assume $(L,h) \in U^0_{R_*}$ for some R_* with subtree S_* and define

$$V_w := \left\{w' \in R_* \mid w' < w\right\} \cup S_* \qquad (w \in R_*)$$

and

$$\tilde{w} := \min\left\{w' \in R_* \mid L'_{w'}h \in \left\langle L'_{w''}h\right\rangle_{w''\in V_{w'}}\right\}.$$

If no such $\tilde{w}$ exists, then also L' has a basis of type R_*, so we have $(L',h) \in U^0_{R_*}$.
Otherwise $(L_{\tilde{w}}h - L'_{\tilde{w}}h) \in \langle L_w h\rangle_{w\in S_*}$ by definition of L', and thus

$$\left\langle L'_w h\right\rangle_{w\in V_{\tilde{w}}\cup\{\tilde{w}\}} = \langle L_w h\rangle_{w\in V_{\tilde{w}}\cup\{\tilde{w}\}}.$$

But this gives a contradiction between the definition of $\tilde{w}$ and $(L,h) \in U^0_{R_*}$, since the latter requires the left hand side to be linearly independent, whereas the first one requires the opposite for the right hand side. But both sides contain the same number of vectors.

Thus $((M'.f'),(L',h)) \in P_{d,e}$. This gives a map

$$\varphi_M^{M'}: F_M \to F_{M'}, \qquad ((M,f),(L,h)) \mapsto \left(\left(M',f'\right),\left(L',h\right)\right).$$

Obviously this is an isomorphism, too, since the inverse map is given by $\varphi_{\bar{M}}^{M}$.

Because $\mathrm{Hilb}^0_d(Q)$ is covered by open sets $U^0_{S_*}$ we have a commutative diagram

$$\begin{array}{ccc} \pi_1^{-1}\left(U^0_{S_*}\right) & \xrightarrow{\quad\sim\quad} & U^0_{S_*} \times F \\ \downarrow{\scriptstyle \pi_1} & \swarrow{\scriptstyle p_{1,2}} & \\ U^0_{S_*} & & \end{array}$$

This proves that π_1 is Zariski locally trivial. □

Lemma 3.2.4 The projection

$$\pi_2 \colon P_{d,e} \to \mathrm{Hilb}^0_{d+e}(Q)$$

onto the third factor is proper.

Proof This is evident from the fact, that π_2 is a morphism between projective varieties, and hence projective. But projective morphisms are always proper (c. f. [7, Theorem II.4.9]). □

Lemma 3.2.5 The morphism $\pi_2 \colon P_{d,e} \to \mathrm{Hilb}^0_{d+e}(Q)$ is generically a bijection.

Proof Assume $L \in Z^0_{T_*}$, then L contains an e-dimensional subrepresentation N spanned by the basis vectors corresponding to the e vertices constructed as follows:
Start with the minimal vertex $w \in T_*$ such that $w\alpha \in C(T_*)$ for all $\alpha \in Q_1$. If $e = 1$, the corresponding base vector spans a one-dimensional subrepresentation. Otherwise assume $w = \bar{w}\bar{\alpha}$ for some $\bar{\alpha} \in Q_1$. Let β be minimal with respect to the property $\bar{w}\beta \in T_*$, $\beta > \bar{\alpha}$. Add $\bar{w}\beta w'$ where w' is minimal with respect to the property that $\bar{w}\beta w'\gamma \in C(T_*)$ for all $\gamma \in T_*$, or

if no such β exists add $\bar{w}$, and obtain a two-dimensional subrepresentation spanned by the corresponding base vectors. Continuing analogously for e steps. By the defining condition of $Z^0_{T_*}$, this yields a subrepresentation N of L. Note that this holds only for the nilpotent case, otherwise the existence of a subrepresentation cannot be guaranteed!

Let $M := L / N$ with $f(v) := h(v) + N$. That shows $(M, L) \in P_{d,e}$.

This proves that π_2 is generically surjective. On the other hand we can see that π_2 is generically injective, since in the generic case there exists at most one subrepresentation of dimension e. □

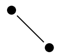
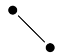
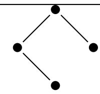
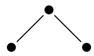
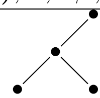
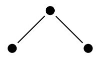
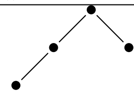
Definition 3.2.6 (Tree Grafting) Let $S_*, S'_* \in \mathcal{F}_1(L_m)$ be two trees. Define a tree $S_* \curlyvee S'_* \in \mathcal{F}_1(L_m)$ by the following: For $(S_*)_0 = \{w_1, \ldots, w_\ell\}$, $(S'_*)_0 = \{w'_1, \ldots, w'_{\ell'}\}$ and $\bar{w} := \min\{w \in C(S_*)\}$ set

$$\left(S_* \curlyvee S'_*\right)_0 := (S_*)_0 \cup \left\{\bar{w}w'_1, \ldots, \bar{w}w'_{\ell'}\right\}$$

and define arrows

1. $\alpha\colon w \to w'$ for $\alpha \in (L_m)_1$, $w, w' \in S_*$ and $w' = w\alpha$,
2. $\alpha\colon \bar{w}w \to \bar{w}w'$ for $\alpha \in (L_m)_1$, $w, w' \in S'_*$ and $w' = w\alpha$,
3. $\alpha\colon w \to \bar{w}$ if $\bar{w} = w\alpha$.

Example 3.2.7 The following table illustrates the grafting of trees:

S_*	S'_*	$S_* \leftthreetimes S'_*$
 $\{(), \beta\}$	 $\{(), \beta\}$	 $\{(), \alpha, \alpha\beta, \beta\}$
$\{()\}$	 $\{(), \alpha, \beta\}$	 $\{(), \alpha, \alpha^2, \alpha\beta\}$
 $\{(), \alpha, \beta\}$	$\{()\}$	 $\{(), \alpha, \alpha^2, \beta\}$

Denote by $T_*^e := \left((), \alpha_1, \ldots, \alpha_1^{e-1}\right)$ the minimal subtree of $\mathcal{F}_1(L_m)$ with e vertices. If $L \in Z^0_{S_* \leftthreetimes T_*^e}$, then the vertices in $(S_* \leftthreetimes T_*^e) \smallsetminus S_*$ span a subrepresentation as mentioned in the proof of Lemma 3.2.5. Let $\bar{w}$ be the minimal vertex in this set, then $\bar{w} \in C(S_*)$ and because of the defining condition of $Z^0_{S_*}$ and the definition of $S_* \leftthreetimes T_*^e$ one has $M_{\bar{w}} = 0$ in the quotient as above. Furthermore, for $w \in C(S_* \leftthreetimes T_*^e) \smallsetminus C(S_*)$ with $L_w = \sum_{w'} \lambda_{w,w'} w'$ it follows that $\bar{w} \prec w'$ if $\lambda_{w,w'} \neq 0$ because of the defining condition of $Z^0_{S_* \leftthreetimes T_*^e}$. Moreover, it is clear that $\bar{w} \prec w'$ for all $w' \in (S_* \leftthreetimes T_*^e) \smallsetminus S_*$ and $w \in C(S_* \leftthreetimes T_*^e) \cap C(S_*)$ with $\lambda_{w,w'} \neq 0$. Hence $I(L)$ is a subideal of $I(M)$ and $M \in Z^0_{S_*}$.

Lemma 3.2.8 Under the composition $\pi_2 \circ \pi_1^{-1}$ the closure of the cell $\overline{Z^0_{S_*}}$ is mapped to the closure of the cell associated to the tree $S_* \leftthreetimes T_*$ which is obtained by identifying $() \in T_* = \left\{(), \alpha_1, \ldots, \alpha_1^{e-1}\right\}$ with $\min\{w \in C(S_*)\}$.

Proof For $L \in Z^0_{S_* \leftthreetimes T_*^e}$ there exists $M \in Z^0_{S_*}$ with $(M, L) \in P_{d,e}$ as shown above.

Since π_2 is proper, it follows that $\pi_2 \circ \pi_1^{-1}\left(\overline{Z^0_{S_*}}\right)$ is closed. Therefore using the above it contains $\overline{Z^0_{S_* \leftthreetimes T_*^e}}$.

On the other hand for $M \in \overline{Z^0_{S_*}} \simeq \mathrm{Hilb}^0_1(Q(S_*))$ any $L \in \mathrm{Hilb}^0_{d+e}(Q)$ with $I(L) < I(M)$ is a representation of $Q(S_* \rightthreetimes T^e_*)$ of dimension type $\widetilde{d+e}$ and thus an element of $\mathrm{Hilb}^0_{\widetilde{d+e}}(Q(S_* \rightthreetimes T^e_*))$, since this contains all equivalence classes (M, f) of dimension $d+e$ where $(M_w f)_{w \in S_*}$ are linearly independent. Using the embedding σ this corresponds to $Z^0_{S_* \rightthreetimes T^e_*}$. Hence $\pi_2 \circ \pi_1^{-1}\left(\overline{Z^0_{S_*}}\right) \subseteq \overline{Z^0_{S_* \rightthreetimes T^e_*}}$ and therefore

$$\pi_2 \circ \pi_1^{-1}\left(\overline{Z^0_{S_*}}\right) = \overline{Z^0_{S_* \rightthreetimes T^e_*}}. \qquad \square$$

From this setting one obtains operators G_e for $e \in \mathbb{N}$ on

$$\bigoplus_{d \in \mathbb{N}} \mathrm{H}^{\mathrm{BM}}_*\left(\mathrm{Hilb}^0_d(L_m)\right) \quad \text{by means of}$$

$$G_e \colon \left[\overline{Z^0_{S_*}}\right] \mapsto \left[\overline{\pi_2 \circ \pi_1^{-1}\left(Z^0_{S_*}\right)}\right] = \left[\overline{Z^0_{S_* \rightthreetimes T^e_*}}\right].$$

These will be called operators of degree e.

The dual operators G_{-e} can be defined as

$$G_{-e}\left[\overline{Z^0_{S_*}}\right] = \begin{cases} \left[\overline{Z^0_{S'_*}}\right] & \text{if } S_* = S'_* \rightthreetimes T^e_* \\ 0 & \text{otherwise.} \end{cases}$$

They satisfy

$$G_a \circ G_b = \begin{cases} G_{a+b} & \text{for } \operatorname{sgn} a = \operatorname{sgn} b \text{ or } b \geq 0, \\ G_{a+b} \circ F_b & \text{for } a \geq 0, b < 0. \end{cases}$$

where F_b is defined as follows: Let $Q_1 = \{\alpha_1 < \cdots < \alpha_\ell\}$ and for $S_* \in \mathcal{F}_1(Q)$ let $p(S_*) := \max\left\{p \in \mathbb{N} \mid \alpha_1^p \in S_*\right\}$. Then

$$F_b\left(\left[\overline{Z_{S_*}}\right]\right) := \begin{cases} \left[\overline{Z_{S_*}}\right] & \text{if } \{\alpha_1^r \alpha_s \mid s > 1,\ p(S_*) - |b| < r \leq p(S_*)\} \subseteq C(S_*) \\ 0 & \text{otherwise} \end{cases}$$

Hence one also has

$$(G_a \circ G_b) \circ G_c = G_{a+b} \circ G_c = G_{a+b+c} = G_a \circ (G_b \circ G_c)$$

for all $a, b, c \in \mathbb{Z}$. Thus they form a monoid with the neutral element G_0.

3.3 Open Question

The open question in this setting is the following: Is it possible to modify $P_{d,e}$ to a Zariski-closed set

$$P_{d,e} \subseteq \mathrm{Hilb}_d^0(Q) \times \mathrm{Hilb}_e^0(Q) \times \mathrm{Hilb}_{d+e}^0(Q)$$

such that for the projections

$$\begin{array}{ccc} & P_{d,e} & \\ {\scriptstyle \pi_{1,2}} \swarrow & & \searrow {\scriptstyle \pi_3} \\ \mathrm{Hilb}_d^0(Q) \times \mathrm{Hilb}_e^0(Q) & & \mathrm{Hilb}_{d+e}^0(Q) \end{array}$$

$\pi_{1,2}$ is Zariski locally trivial, π_3 is proper and

$$\pi_3 \circ \pi_{1,2}^{-1}\left(\overline{Z_{S_*}} \times \overline{Z_{T_*}}\right) = \overline{Z_{S_* \times T_*}}? \tag{3.6}$$

Under this assumption one would obtain using the results from chapter 2

Proposition 3.3.1 There is a natural convolution product in Borel-Moore homology:

$$\mathrm{H}_*^{\mathrm{BM}}\left(\mathrm{Hilb}_d^0(Q)\right) \times \mathrm{H}_*^{\mathrm{BM}}\left(\mathrm{Hilb}_e^0(Q)\right) \to \mathrm{H}_*^{\mathrm{BM}}\left(\mathrm{Hilb}_{d+e}^0(Q)\right).$$

Applying formula (3.6) to this product gives

Proposition 3.3.2 The convolution product from Proposition 3.3.1 can be calculated in terms of fundamental classes associated to the cell closures:

$$\left[\overline{Z^0_{S_*}}\right] \times \left[\overline{Z^0_{T_*}}\right] = \left[\overline{Z^0_{S_* \rtimes T_*}}\right].$$

In this case the result would be a geometric realisation of an algebra with a convolution product whose combinatorial data arises from the tree structure of the cells. Some purely combinatorial aspects of this algebra can be found in chapter 4.

Conjecture 3.3.3 Let $Q = L_m$ be the m-loop quiver. Then

$$C^{(m)} := \bigoplus_{d \in \mathbb{N}} \mathrm{H}_*^{\mathrm{BM}}\left(\mathrm{Hilb}_d^0(Q)\right)$$

is an associative graded algebra where the product is given by the convolution product defined in Proposition 3.3.1. It will be called the *convolution algebra* of the nilpotent Hilbert scheme.

Assuming this, the following is easy to obtain:

Corollary 3.3.4 Let $Q_1 = \{\alpha_1 < \cdots < \alpha_m\}$. As an algebra $C^{(m)}$ is generated by the fundamental classes corresponding to trees S_* such that $\alpha_1 \notin S_*$.

Proof This is obvious from the definition of $S_* \rtimes T_*$. □

$C^{(m)}$ also has another grading given by the dimensions of the cells; this makes $C^{(m)}$ a bigraded algebra as follows:

$$C^{(m)} = \bigoplus_{d \in \mathbb{N}} C_d^{(m)} = \bigoplus_{d \in \mathbb{N}} \bigoplus_{i=0}^{d(d-1)} C_{d,i}^{(m)}$$

where

$$C_{d,i}^{(m)} = \left\langle \left[\overline{Z_{S_*}} \right] \mid \dim Z_{S_*} = i \right\rangle .$$

The convolution is compatible with that grading, that is it can be written as

$$*\colon C_{d,i}^{(m)} \times C_{e,j}^{(m)} \to C_{d+e,i+j+k}^{(m)}$$

for some $k \in \mathbb{N}$.

In Theorem 4.2.2, even a third grading on $C^{(m)}$ is derived from an isomorphism between $C^{(m)}$ and the tensor algebra of a shifted $(m-1)$-fold product of $C^{(m)}$.

Chapter 4

4 Applications

The greatest mathematicians, as Archimedes, Newton and Gauß, always united theory and applications in equal measure.

(Felix Klein)

In this chapter we give an algebraic realisation of the functional equations for the Euler characteristic presented in chapter 1. Using the conjecture from chapter 3, section 4.2 becomes a statement about the internal structure of the convolution algebra of non-commutative Hilbert schemes.
Furthermore, we present a link between the algebra structure on a vector space with a basis given by trees and non-commutative Invariant Theory. In particular, this applies to the convolution algebra conjectured in chapter 3: In section 4.1 it is proved that the convolution algebras are isomorphic to certain cable algebras. This comes from a combinatorial bijection between m-ary trees and cable diagrams which can be found in [20], where both classes of objects are stated to parametrise the Catalan numbers.

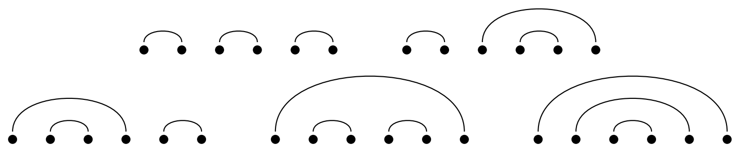
4.1 The Cable Algebra

4.1.1 Cable Diagrams

Definition 4.1.1 (m-ary arc) An m-ary arc is a line connecting m vertices, such that each vertex is passed exactly once.

Definition 4.1.2 (Cable Diagram) Let $d, m \in \mathbb{N}$ and define an m-ary cable diagram of weight d to consist of dm vertices connected by d m-ary arcs without intersections such that each vertex is hit by exactly one arc.
Denote by Δ_d^m the sets of all m-ary cable diagrams of weight d.

Example 4.1.3 Let $m = 2$ and $d = 3$. Then there are the following cable diagrams:



Definition 4.1.4 Define the so called *cable algebra* D^m as a vector space by

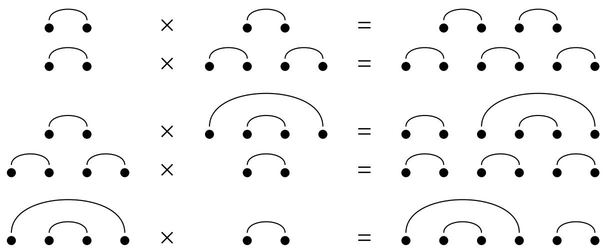
$$D^m := \bigoplus_{d=0}^{\infty} \mathbb{C}\, \Delta_d^m,$$

where $\mathbb{C}\, M$ denotes the vector space spanned by basis elements $m \in M$. This becomes an algebra by means of the non-commutative multiplication

$$*\colon D_d^m \times D_e^m \to D_{d+e}^m$$

which can be defined using the following operation on the basis elements: For $c_d \in \Delta_d^m$ and $c_e \in \Delta_e^m$ let $c_d * c_e$ be the diagram in Δ_{d+e}^m obtained by concatenating c_d and c_e to a new diagram $c_d c_e$.

Example 4.1.5 There are the following convolution products in degree 2 and 3:



Remark 4.1.6 It is easy to see that D^m is generated as an algebra by those diagrams with an m-ary arc including the first and the last vertex.

4.1.2 Tree Diagrams

Let T_d^m be the set of m-ary trees with d vertices for $d, m \in \mathbb{N}$. As in Definition 1.2.10 denote by $C(T)$ the corona of a tree T. In an m-ary tree each vertex x has m potential successors, the set $C(x)$ of which will be denoted as the *corona* of this vertex.

The following is easy to see:

$$C(T) = \bigcup_{x \in T_0} C(x) \setminus T_0.$$

Definition 4.1.7 Define a product

$$T_d^m \times T_e^m \to T_{d+e}^m \qquad (S, T) \mapsto S \leftthreetimes T$$

where $S \leftthreetimes T$ arises from S by attaching T identifying the root of T with the smallest vertex in $C(S)$.

Remark 4.1.8 This definition turns

$$T^m := \bigoplus_{d=0}^{\infty} \mathbb{C}\, T_d^m$$

into a non-commutative algebra. It is generated by those trees which do not contain the smallest possible edge starting in the root.

Assuming Conjecture 3.3.3, the algebra T^m is isomorphic to the convolution algebra in Borel-Moore homology for the nilpotent Hilbert schemes of the quiver L_m conjectured in chapter 3.

Theorem 4.1.9 For each $m \in \mathbb{N}$ there is an isomorphism of graded algebras

$$\psi\colon D^m \xrightarrow{\sim} T^m.$$

Proof Take a cable diagram $\Gamma \in \Delta_d^m$ for $d \in \mathbb{N}$ and construct a tree in T_d^m from Γ as follows: Because the arcs in Γ are free of intersections, for every m-ary arc $\gamma \in \Gamma$ there are exactly two possibilities:

1. The leftmost leg of γ is the leftmost leg of Γ,

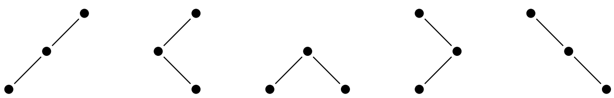
2. the first leg in Γ lying left of γ is the k-th leg of some arc γ'.

In the first case $\psi(\gamma)$ is the root of the tree to construct. In the second case $\psi(\gamma)$ is attached to the $m-k+1$-th vertex in the corona of $\psi(\gamma')$.

This map is bijective, since it can be reversed as follows: for any vertex ν, which is attached as the k-th vertex in the corona of some other vertex ν', insert an m-ary arc immediately right of the $m-k+1$-th leg of the arc corresponding to ν'. The root of the tree which has no predecessor is mapped to the arc with the leftmost leg.

Under this bijection, the appending of cable diagrams corresponds to the attaching of trees at the smallest vertex in the corona. Therefore the two convolution structures are compatible, hence this gives an isomorphism of algebras.
Furthermore, ψ is obviously compatible with the grading. □

Example 4.1.10 Under this bijection the trees corresponding to the cable diagrams from Example 4.1.3 are the following:



4.2 Functional equation

Let V be an $\mathbb{N}$-graded vector space. Then the tensor space $V^{\otimes n}$ is a graded vector space via

$$\left(V^{\otimes n}\right)_\ell = \bigcup_{\ell_1+\cdots+\ell_n=\ell} V_{\ell_1} \otimes \cdots \otimes V_{\ell_n}.$$

We denote by $V[k]$ the graded vector space obtained from V by shifting the graduation by k. Let

$$T(V) := \bigoplus_{n\in\mathbb{N}_0} V^{\otimes n}$$

the *tensor algebra* of V with the grading inherited from $V^{\otimes n}$:

$$(T(V))_\ell = \bigcup_{\substack{\ell_1+\cdots+\ell_s=\ell \\ s\in\mathbb{N}}} V_{\ell_1} \otimes \cdots \otimes V_{\ell_s}.$$

Denote by

$$H_V(t) := \sum_{n\geq 0} \dim V_n t^n$$

the *Hilbert series* of V.

Remark 4.2.1 One has the functional equation

$$H_{T(V)}(t) = \frac{1}{1 - H_V(t)}.$$

Proof Using the geometric series one obtains

$$\begin{aligned}\frac{1}{1 - H_V(t)} &= \sum_{\ell=0}^{\infty} H_V(t)^{\ell} \\ &= \sum_{\ell=0}^{\infty} \left(\sum_{j=0}^{\infty} \dim V_j t^j \right)^{\ell} \\ &= \sum_{\ell=0}^{\infty} \sum_{\substack{\ell_1 + \cdots + \ell_s = \ell \\ s \in \mathbb{N}}} \left(\prod_{i=1}^{s} \dim V_{\ell_i} \right) t^{\ell} \\ &= H_{T(V)}(t)\end{aligned}$$

which is the claimed identity. □

Recall the functional equation for the generating function of the Poincaré polynomials of the non-commutative Hilbert schemes from formula (1.8). Here the left hand side is the Hilbert series of $C^{(m)}$. This motivates the question for a morphism of algebras reflecting this functional equation, that is we expect an algebra $X^{(m)}$ such that the right hand side of formula (1.8) is the Hilbert series of $X^{(m)}$ and an isomorphism of algebras $\Psi: C^{(m)} \to X^{(m)}$.

This question is answered in

Theorem 4.2.2 There is an isomorphism of graded algebras

$$\Psi: C^{(m)} \xrightarrow{\sim} T\left(\left(C^{(m)} \right)^{\otimes m-1} [1] \right).$$

The Hilbert series of the algebras here are precisely those in formula (1.8).

Proof Define

$$\Psi\left(\left[\overline{Z^0_{S_*}}\right]\right) := \left(\left[\overline{Z^0_{S^1_1}}\right]\otimes\cdots\otimes\left[\overline{Z^0_{S^1_{m-1}}}\right]\right)\otimes\cdots\otimes\left(\left[\overline{Z^0_{S^p_1}}\right]\otimes\cdots\otimes\left[\overline{Z^0_{S^p_{m-1}}}\right]\right).$$

Here for S_* define

$$S^i_* := \left\{w \in \mathcal{F}_1\left(L_m\right) \mid \alpha_1^{i-1} w \in S_*,\ \alpha_1^i \notin w\right\} \qquad (i \in \mathbb{N})$$

and

$$S^i_j := \left\{w \in \mathcal{F}_1\left(Q\right) \mid \alpha_{j+1} w \in S^i\right\} \qquad (j = 1, \dots, m-1).$$

The fact that this defines an isomorphism is evident from the inverse map which makes the following assignment:

$$\left(\left[\overline{Z^0_{S^1_1}}\right]\otimes\cdots\otimes\left[\overline{Z^0_{S^1_{m-1}}}\right]\right)\otimes\cdots\otimes\left(\left[\overline{Z^0_{S^p_1}}\right]\otimes\cdots\otimes\left[\overline{Z^0_{S^p_{m-1}}}\right]\right)$$

$$\Big\downarrow$$

$$\left[\overline{Z^0_{S^1}}\right]\cdots\left[\overline{Z^0_{S^p}}\right]$$

where for trees S^i_j in $\mathcal{F}_n\left(Q\right)$ a tree S^i is constructed by an operation called *grafting*, which is defined in [17, Definition 5.1]. The product is the convolution product assumed in chapter 3.

Since this isomorphism is compatible with the convolution product by definition, it induces a grading on the tensor algebra that makes it a graded isomorphism. This can be obtained by the following procedure: Consider $\left(C^{(m)}\right)^{\otimes m-1}$ as a graded vector space via

$$\left(\left(C^{(m)}\right)^{\otimes m-1}\right)_\ell := \bigcup_{\ell_1+\cdots+\ell_{m-1}=\ell}\left(C^{(m)}\right)_{\ell_1}\otimes\cdots\otimes\left(C^{(m)}\right)_{m-1}.$$

Then the induced grading on $T\left(\left(C^{(m)}\right)^{\otimes m-1}[1]\right)$ is the one corresponding to the primary grading on C_m given by

$$C_d^{(m)} = \left\langle \left[\overline{Z^0_{S_*}}\right] \mid |S_*| = d \right\rangle$$

under the isomorphism Ψ defined above. □

4.3 SL_m-Invariants

In subsection 4.1.2 we saw that m-ary trees used for parametrising the cells in the non-commutative Hilbert scheme of the m-loop quiver are in bijection to m-ary cable diagrams. This is a generalisation of [20, Exercise 6.19] where this can be found for $m = 2$. The author gives a list of 66 different classes of objects which are enumerated by the *Catalan numbers*; amongst others that list includes standard Young tableaux with 2 rows. Here we use the following definition from [6]:

Definition 4.3.1 (Young diagram) A *Young diagram* is a collection of rows of boxes, with a non-increasing number of boxes in each row. The number of boxes of a Young diagram Y will be denoted by $|Y|$.
A Young diagram is called *rectangular* if each row has the same number of boxes.

Definition 4.3.2 (Standard Young tableau) A *standard Young tableau* consists of a Young diagram Y and a filling of the boxes in Y with the numbers from 1 to $|Y|$, which is increasing along each row (from the left to the right) and each column from top to bottom.
For $i \in \{1, \dots, |Y|\}$ let $c_Y(i)$ denote the column in which i appears in Y and analogously $r_Y(i)$ the row in Y.

Let $V \simeq \mathbb{C}^m$ be a vector space. Then one has the standard operation of the group $\mathrm{SL}_m := \mathrm{SL}_m(\mathbb{C})$ on V. This operation extends to an operation of

SL_m on the tensor algebra $T(V)$ via

$$g \cdot v_1 \otimes \cdots \otimes v_r := gv_1 \otimes \cdots \otimes gv_r \qquad (g \in \mathrm{SL}_m,\ v_i \in V).$$

In this situation one is interested in the structure of the invariant ring of this operation, which will be denoted $T(V)^{\mathrm{SL}_m}$. This problem was solved in [22]. The author uses a special class of standard Young tableaux, namely those which are rectangular, to parametrise the generators of the invariant ring of the operation of SL_m on the tensor algebra $T(V)$. In fact, Teranishi deals with a more general setting, which is only interesting here in the special case mentioned above.

Let $e_1, \ldots, e_m$ denote the standard basis of V and let Y be a rectangular standard Young tableau with m rows and d columns. Because of the special form of Y the map

$$\{1, \ldots, md\} \to \{1, \ldots, m\} \times \{1, \ldots, d\}, \qquad i \mapsto (r_Y(i), c_Y(i))$$

is a bijection. To each Y we can associate a generator of $T(V)^{\mathrm{SL}_m}$ which is given as:

$$\sum_{(\sigma_j)_j \in (S_m)^d} \operatorname{sgn}(\sigma_j)_j \bigotimes_{i=1}^{md} e_{\sigma_{c_Y(i)}(r_Y(i))},$$

where $\operatorname{sgn}(\sigma_j)_j := \prod_j \operatorname{sgn} \sigma_j$. The multiplication of these generators corresponds to the following operation on the Young tableaux: Let Y_1, Y_2 be two rectangular standard Young diagrams with m rows and define $Y_1 \hat{\oplus} Y_2$ as the diagram obtained by appending $\hat{Y}_2$ at the right end of Y_1. Here $\hat{Y}_2$ denotes the diagram arising from Y_2 by replacing each entry i with $i + |Y_1|$.

Definition 4.3.3 (Indecomposable Young Tableau) A rectangular standard Young tableau Y is called *indecomposable* if there are no two rectangular standard Young tableaux Y_1, Y_2 such that $Y = Y_1 \hat{\oplus} Y_2$.

Example 4.3.4 Let $d = m = 2$. Then the Young tableau

$$Y_1 := \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline\end{array}$$

corresponds to the generator

$$e_1 \otimes e_1 \otimes e_2 \otimes e_2 + e_1 \otimes e_2 \otimes e_2 \otimes e_1 + e_2 \otimes e_1 \otimes e_1 \otimes e_2 + e_2 \otimes e_2 \otimes e_1 \otimes e_1.$$

Setting

$$Y_2 := \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline\end{array}$$

one obtains

$$Y_1 \hat{\oplus} Y_2 = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 5 & 7 \\ \hline 3 & 4 & 6 & 8 \\ \hline\end{array}$$

It is also clear that Y_1 is indecomposable whereas Y_2 can be written as

$$Y_2 = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline\end{array} = \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline\end{array} \hat{\oplus} \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline\end{array}.$$

Theorem 4.3.5 The set of generators associated to indecomposable rectangular Young tableaux forms a free system of generators of $T(V)^{\mathrm{SL}_m}$.

Proof This is [22, Theorem 3.3]. □

Definition 4.3.6 (Admissible Young Tableaux) Take a rectangular standard Young tableau Y with m rows. Y is called *admissible* if one of the following conditions holds for all $i \in \{1, \ldots, |Y|\}$:

1. $r_Y(i) = 1$,
2. $r_Y(i) = r_Y(i-1) + 1$,

3. $r_Y(i) = r_Y(\max\{k < i \mid r_Y(k+1) = 1 \wedge r_Y(k) \neq m\}) + 1$.

Extending the bijection between cable diagrams resp. binary trees and rectangular standard Young tableaux with two rows from [20, Exercise 6.19], it is possible to give an injective map from the set of m-ary cable diagrams into the set of rectangular standard Young tableaux with m rows. The number of arcs corresponds to the number of columns under this injection. This works as follows: Write the number i into the field $(\ell(i), a(i))$, where $\ell(i)$ denotes the number of the leg of the m-ary arc hitting i and

$$a(i) = \begin{cases} \max\{j < i,\ \ell(j) = \ell(i)\} + 1, & \text{if the maximum exists,} \\ 1, & \text{otherwise.} \end{cases}$$

The Young tableaux in the image of this injection are precisely those which are admissible in the notion of Definition 4.3.6. This can be seen as follows: For a rectangular standard Young tableau, we can construct the preimage: Let k be the number of columns and $\{1, \dots, km\}$ the set of vertices of the cable diagram to be constructed. Construct a graph with these vertices connecting the vertex i to the vertex

$$\max\{j < i \mid \ell(j) = \ell(i) - 1\} \qquad (\ell(i) > 1).$$

The resulting graph consists of m-ary arcs and the conditions for the Young tableau to be admissible guarantee that the resulting graph has no intersecting arcs and thus meets our definition of an m-ary cable diagram. It is easy to see that this diagram is a preimage of the original Young tableau under the map given above.
Furthermore, one can easily see that appending of cable diagrams corresponds to the special appending operation on Young tableaux defined above. Therefore, the product of two admissible Young tableaux is admissible, too.

Example 4.3.7 Let $m = 3$ and $d = 3$. Then one has the pairs of cable diagrams and admissible Young tableaux depicted in Figure 4.1.

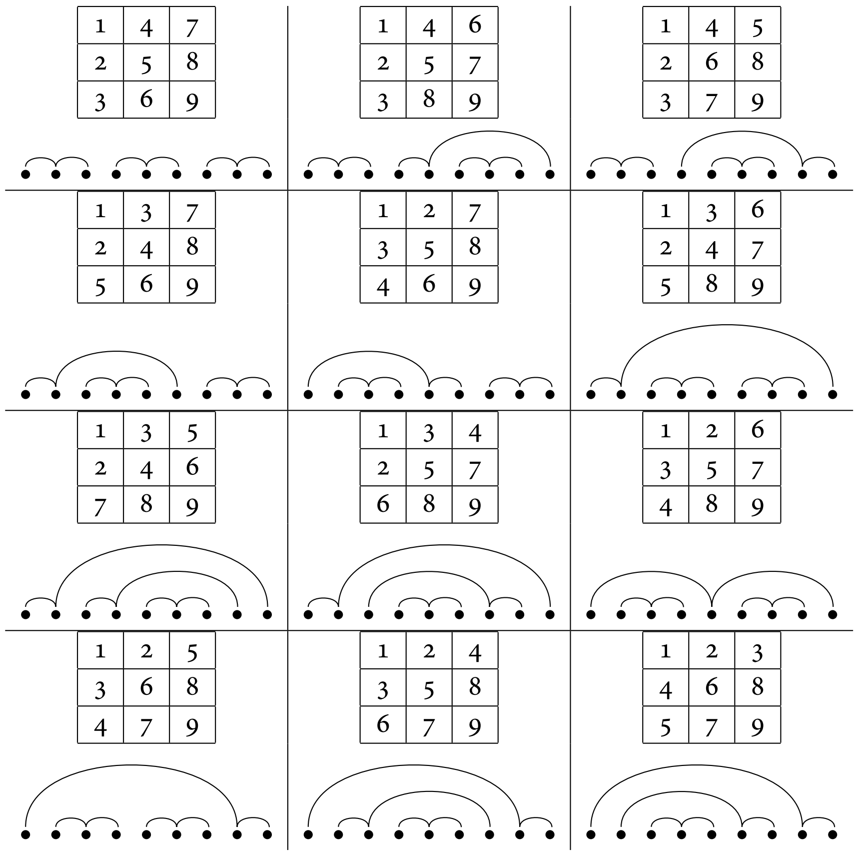


Figure 4.1: Cable diagrams and corresponding Young tableaux for $m = d = 3$

Using the results of section 4.1 this gives

Theorem 4.3.8 Let V be a complex vector space of dimension $m \in \mathbb{N}$. Then there is a monomorphism of graded algebras

$$C^{(m)} \hookrightarrow T(V)^{\mathrm{SL}_m}. \tag{4.1}$$

In the special cases $m = 1$ or $m = 2$ this monomorphism is an isomorphism.

Example 4.3.9 Consider once more the case $d = 2$ and $m = 3$. It is easy to see that the two Young tableaux

1	2
3	4
5	6

1	3
2	5
4	6

are irreducible, hence in this setting $C^{(3)}$ is isomorphic to a proper subalgebra of $T(\mathbb{C}^3)^{\mathrm{SL}_3}$.

Proof One obtains a monomorphism by mapping the generators corresponding to the cable diagrams to those corresponding to the associated admissible Young tableaux as constructed earlier. This respects the product given by concatenation of cable diagrams.

The graded subalgebra of $T(V)$ given as the image of this monomorphism can be described as follows: It is spanned as a vector space by all admissible rectangular standard Young tableaux with the product from appending diagrams as defined above. The fact that these generate a subalgebra is clear from the definition of the product. As proved earlier in this section, this algebra is isomorphic to the cable algebra $C^{(m)}$ and hence proves the claim.

It is also clear from Definition 4.3.6 that for $m \leq 2$ all standard Young tableaux are admissible. Hence the monomorphism from Theorem 4.3.8 is an isomorphism of algebras. □

Bibliography

[1] Michael Artin. "On Azumaya Algebras and finite dimensional representations of rings". In: *Journal of Algebra* 11.4 (Apr. 1969), pp. 532–563. DOI: 10.1016/0021-8693(69)90091-X.

[2] Edward A. Bender and L. Bruce Richmond. "A Multivariate Lagrange Inversion Formula for Asymptotic Calculations". In: *The Electronic Journal of Combinatorics* 5 (1998). DOI: 10.1.1.46.6593. URL: http://www.emis.de/journals/EJC/Volume_5/PDF/v5i1r33.pdf.

[3] Armand Borel and James C. Moore. "Homology theory for locally compact spaces". In: *Michigan Math. J.* 7.2 (1960), pp. 137–159. ISSN: 0026-2285. DOI: 10.1307/mmj/1028998385.

[4] Neil Chriss and Victor Ginzburg. *Representation Theory and Complex Geometry*. Boston: Birkhäuser, 1997.

[5] Johannes Engel and Markus Reineke. "Smooth models of quiver moduli". In: *Mathematische Zeitschrift* 262 (2009), pp. 817–848. DOI: 10.1007/s00209-008-0401-y.

[6] William Fulton. *Young Tableaux: With Applications to Representation Theory and Geometry*. London Mathematical Society Student Texts 35. Cambridge: Cambridge University Press, 1997. ISBN: 0-521-56724-6.

[7] Robin Hartshorne. *Algebraic geometry*. 15th ed. Graduate texts in mathematics 52. New York: Springer, 1977. ISBN: 0-387-90244-9.

[8] Allen Hatcher. *Algebraic Topology*. Cambridge: Cambridge University Press, 2002. URL: http://www.math.cornell.edu/~hatcher/AT/ATpage.html.

[9] David Henderson. *Experiencing Geometry: On Plane and Space*. 1st ed. New York: Prentice Hall College Division, 1995. ISBN: 978-0133737707.

[10] James E. Humphreys. *Introduction to Lie Algebras and Representation Theory*. Graduate Texts in Mathematics 9. New York: Springer, 2001. ISBN: 0-387-90053-5.

[11] Birger Iversen. *Cohomology of Sheaves*. Berlin: Springer, 1986.

[12] Victor G. Kac. "Infinite root systems, representations of graphs and invariant theory". In: *Inventiones Mathematicae* 56 (1980), pp. 57–92.

[13] Victor G. Kac. "Root systems, representations of quivers and invariant theory". In: *Invariant Theory*. Montecatini. Vol. 996. Lecture Notes in Mathematics. Berlin, Heidelberg: Springer, 1982, pp. 74–108. ISBN: 978-3-540-12319-4. DOI: 10.1007/BFb0063236.

[14] Alastair D King. "Moduli of representations of finite dimensional algebras". In: *Quarterly J. Math. Oxford* 2.45 (1994), pp. 515–530. URL: http://qjmath.oxfordjournals.org/cgi/reprint/45/4/515.

[15] Lieven Le Bruyn and George Seelinger. "Fibers of generic Brauer-Severi schemes". In: *Journal of Algebra* 214.1 (1999), pp. 222–234. DOI: 10.1006/jabr.1998.7656.

[16] Hiraku Nakajima. *Lectures on Hilbert schemes of points on surfaces*. Vol. 18. University Lecture Series. Providence, RI: American Mathematical Society, 1999. xii+132. ISBN: 0-8218-1956-9.

[17] Markus Reineke. "Cohomology of non-commutative Hilbert schemes". In: *Algebras and Representation Theory* 8 (Sept. 2003), pp. 541–561. ISSN: 1386-923X. arXiv: math/0306185.

[18] Markus Reineke. “The monoid of families of quiver representations”. In: *Proc. London Math. Soc.* 84.3 (2002), pp. 663–685. DOI: 10.1112/S0024611502013497. URL: http://plms.oxfordjournals.org/cgi/content/abstract/84/3/663.

[19] Jean-Pierre Serre. “Géometrie Algébrique et Géométrie Analytique”. In: *Annales de l'Insitut Fourier* 6 (1955), pp. 1–42.

[20] Richard P. Stanley. *Enumerative Combinatorics*. Vol. 2. Cambridge Studies in Advanced Mathematics. Cambridge: Cambridge University Press, 1997. 337 pp. ISBN: 0521663512.

[21] Harold Neville Vazeille Temperley and Elliott H. Lieb. “Relations between the percolation and colouring problem and other graph-theoretical problems associated with regular planar lattices. some exact results for the percolation problem”. In: *Proceedings of the Royal Society London Series A* 322.1549 (Apr. 1971), pp. 251–280. DOI: 10.1098/rspa.1971.0067.

[22] Yasuo Teranishi. “Noncommutative classical invariant theory”. In: *Nagoya Mathematical Journal* 112 (1988), pp. 153–169.

List of Symbols

$\hat{\oplus}$	Composition of Young tableaux	85
$\langle \cdot, \cdot \rangle$	Euler form, also known as Ringel form	13
$\leftthreetimes$	Composition of trees	72
1^q	Unit vector for $q \in Q_0$	34
$C(S_*)$	Corona of a forest S_*	26
χ	Euler characteristic	39
C^m	Cable algebra for m-ary cable diagrams	78
$C^{(m)}$	Convolution algebra for the quiver L_m	75
$C_d^{(m)}$	Grade d part of $C^{(m)}$ in the scheme dimension grading	76
$C_{d,i}^{(m)}$	Grade i part of $C_d^{(m)}$ in the cell dimension grading	76
$c_Y(m)$	Column of m in the standard Young tableau Y	84
$\mathcal{F}_n(Q)$	Covering forest for a quiver Q	21
GL_d	General linear group associated to a dimension vector d	14
h	Map assigning to an arrow its head	12
$\mathrm{H}_*^{\mathrm{BM}}()$	Borel-Moore homology group	48
$\mathrm{Hilb}_{d,n}^0(Q)$	Nilpotent non-commutative Hilbert scheme of a quiver Q for dimension vectors $d, n \in Q_0$	58
$\mathrm{Hilb}_{d,n}(Q)$	Non-commutative Hilbert scheme of dimension vectors d, n for a quiver Q	18
$\mathrm{H}^*(X)$	Singular cohomology groups of X	49

$\mathrm{H}_*(X)$	Singular homology groups of X	33
$(\lambda^q)_{q\in Q_0}$	Multipartition for Q consisting of a tuple of partitions	34
$\Lambda_{d,n}$	Set of all admissible multipartitions	34
L_m	Quiver with 1 vertex and m loops	58
$\mathbb{C}Q$	Path algebra for a quiver Q	14
$P_{d,e}$	Correspondence set for the convolution product	67
$\Phi_{d,n}$	Set of all subforests in $\mathcal{F}_n(Q)$ parametrising the cells of $\mathrm{Hilb}_{d,n}(Q)$	27
$P_X(q)$	Poincaré polynomial of X in q	33
Q_0	Set of vertices of a quiver Q	12
Q_1	Set of edges of a quiver Q	12
(Q,d,n)	Quiver datum consisting of a quiver Q and corresponding dimension vectors $d,n\in\mathbb{N}Q_0$	15
(q,i,w)	Vertex in $\mathcal{F}_n(Q)$ corresponding to the vertex w in the i-th copy of T_q	21
$\hat{Q}^n$	Extended quiver for Q containing one extra vertex with n arrows	16
$R_{d,n}(Q)$	Extended representation space of a quiver Q and dimension vectors d,n	16
$R_d(Q)$	Representation space of dimension d for a quiver Q	13
$r_{i,j}$	Number of arrows in Q from i to j	40
$r_Y(m)$	Row of m in the standard Young tableau Y	84
sgn	Signum function for (tuples of) permutations	85
SL_m	Special linear group of $\mathbb{C}^m$	84
$\mathrm{succ}(q,i)$	Successor of (q,i), i. e. smallest amongst the indices of the trees in $\mathcal{F}_n(Q)$ that is greater than (q,i)	26

t	Map assigning to an arrow its tail	12
T^m	Tree algebra for m-ary trees	80
T_d^m	Set of m-ary trees with d vertices	79
T_q	Covering tree for a quiver Q based in $q \in Q_0$	21
$T(V)$	Tensor algebra of V	85
$T(V)^{\mathrm{SL}_m}$	Ring of the SL_m-invariants in $T(V)$	85
$U_{S_*}^0$	Open set for a forest S_*, part of the covering of a nilpotent non-commutative Hilbert scheme	58
U_{S_*}	Open affine set, part of the covering of the non-commutative Hilbert scheme	26
w	Path in the quiver Q	14
$[X]$	Fundamental class of X in Borel-Moore homology	49
$\hat{Y}$	Shifted Young diagram	85
$Z_{S_*}^0$	Affine cell of a nilpotent non-commutative Hilbert scheme corresponding to the forest S_*	60
Z_{S_*}	Cell of a non-commutative Hilbert scheme corresponding to a forest S_*	28

Index

A

algebra
- graded 75

arc
- m-ary 78

B

Betti number 33

C

cable algebra 6, 78
Catalan numbers 77, 84
cell 19
cell decomposition 4, 19
convolution algebra 75
convolution product 74
corona 26, 79
correspondence 6

D

dimension vector 13

E

Euler characteristic 38
Euler form 13

F

forest 20
fundamental class 49

G

grading 75f
grafting 83

H

Hilbert scheme
- nilpotent 58
- non-commutative 4f, 18

Hilbert-Chow morphism 58
homology
- Borel-Moore 47f

L

Lagrange Inversion
- multivariate 40

M

moduli space 4

multipartition 34
 admissible 34

O

order
 lexicographic 24

P

path 14
Poincaré duality 49
Poincaré polynomial 33

Q

quiver 12
 wild 3
quiver datum 15
quiver representation 12

R

representation space 13
Ringel form 13
root 20

S

stability 17
subforest 20
subquiver 12
subrepresentation 13
subtree 20

T

Temperley-Lieb algebra 6
tensor algebra 76
tree 20

Y

Young diagram 84
Young tableau 84
 admissible 86
 indecomposable 85
 standard 84