From Anomalous Deterministic Diffusion to the Continuous-Time Random Walk

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Abstract

Anomalous diffusion is commonly observed in nature. There are several stochastic processes which model this phenomenon, e.g., the fractional Brownian motion and the continuous-time random walk (CTRW). In general, one differentiates between these models by statistical indicators or by using a phenomenological approach. This thesis depicts a method of obtaining a continuous-time random walk as the asymptotic description of a deterministic system showing anomalous diffusion. It also describes how to obtain the parameters of the asymptotic continuous-time random walk from the dynamical system.

At first, a diagrammatical method is introduced which allows to write down the joint probability distributions of a CTRW. This approach is easily adopted to other similar settings. It can be used to show that the scaling limit of a CTRW with finite mean waiting time is a Markovian process. Using this method one can derive the limit behavior for several important classes of CTRWs. It can also be adopted to nonindependent CTRWs, i.e., to CTRWs which allow a coupling between different steps which is described by an internal state space of the random walker. The parts of the diagrams correspond then to operators on the space of probability distributions on the internal state space instead of being simple factors. If this state space is finite, it is shown that the scaling limit is described by an independent CTRW and it is demonstrated how the parameters of this limit can be determined.

The diagrammatic approach can also be used in the case of a deterministic system where the method of inducing is applicable. Therefore, it provides a unifying framework for the stochastic description of these systems. Under certain assumptions one can give a good justification that the scaling limit in this case is also an independent CTRW. The parameters of this limit can be identified from the dynamical system. For two examples of maps of Manneville-Pomeau type the analytical predictions are confirmed by numerical simulations.

Zusammenfassung

Anomale Diffusion wird häufig in der Natur beobachtet. Es existieren verschiedene stochastische Prozesse, die dieses Phänomen beschreiben, z.B. die fraktionale Brownsche Bewegung oder der Continuous-Time Random Walk (CTRW). Die Unterscheidung zwischen diesen Modellen wird im Allgemeinen mit Hilfe von statistischen Indikatoren oder phänomenologischen Ansätzen getroffen. Diese Arbeit beschreibt eine Methode, mit der man einen Continuous-Time Random Walk als asymptotische Beschreibung eines deterministischen Prozesses mit anomaler Diffusion erhält. Dieser Ansatz beinhaltet auch, wie man die Parameter des asymptotischen Continuous-Time Random Walks aus dem dynamischen System bestimmt.

Zuerst wird eine diagrammatische Methode vorgestellt, mit deren Hilfe man die Mehrpunktverteilungen eines CTRWs bestimmen kann. Dieser Ansatz kann einfach auf ähnliche Situationen übertragen werden. Mit ihr kann man zeigen, dass der Skalierungslimes für CTRWs mit einer endlichen mittleren Wartezeit ein Markovscher Prozess ist. Weiterhin kann man für mehrere wichtige Klassen von CTRWs das Verhalten im Skalierungslimes bestimmen. Die Methode kann auch auf abhängige CTRWs ausgedehnt werden, dies sind CTRWs, die eine stochastische Abhängigkeit zwischen verschiedenen Schritten zulassen. Diese Abhängigkeit wird durch einen inneren Zustandsraum beschrieben. Die einzelnen Komponenten eines Diagramms entsprechen dann nicht mehr einfachen Faktoren sondern Operatoren auf den Wahrscheinlichkeitsverteilungen des Zustandsraumes. Wenn dieser Zustandsraum endlich ist, ist der Skalierungslimes durch einen unabhängigen CTRW gegeben. Weiterhin wird gezeigt, wie die Parameter dieses Grenzwertes bestimmt werden können.

Der diagrammatische Ansatz läßt sich auch auf den Fall eines deterministischen Systems, das mit der Methode der induzierten Abbildung behandelt werden kann, übertragen. Somit stellt dieser Ansatz einen Rahmen dar, in dem die stochastische Beschreibung dieser Systeme effektiv behandelt werden kann. Unter bestimmten Bedingungen kann man motivieren, dass der Skalierungslimes auch in diesem Fall ein unabhängiger CTRW ist. Die Parameter dieses Grenzprozesses können aus dem dynamischen System bestimmt werden. Für zwei Beispiele aus der Klasse der Manneville-Pomeau Abbildungen werden die analytischen Ergebnisse durch numerische Simulationen bestätigt.

Contents

1. Introduction

Stochastic descriptions of physical systems have a long history. One of the earliest examples is the explanation of the Brownian motion of a particle suspended in a fluid by Einstein and von Smoluchowski [Ein05, Ein06, vS06]. The interaction of the fluid with the particle is modeled by a stochastic process. This already contains key points why stochastic modeling is still an important issue:

- The system is much too complex to describe all degrees of freedom. Therefore one searches for an effective description.
- The reduction to a few observables of the system can give rise to a description which shows the main properties of the system in a much more direct way.

In the given example, the motion of all fluid particles is much too complicated to be described (today one can do numerical simulations up to a system size determined by the computing power available). But on the other hand, the details of the motion of the fluid particles is not interesting in most cases, therefore the description with a Brownian motion with only one parameter (the diffusion constant) gives a better understanding of the system. Of course these models have a finite range of applicability. If one would resolve every interaction of the observed particle with the embedding fluid in space and time, then a more detailed model would be necessary.

A modern example of a stochastic description is stochastic climate modeling [Has76, IvS01]. The weather has a big impact on the long term behavior of the climate, but it takes too much computational power to compute current weather models over the time scales interesting for climate (the weather models themselves are already a reduction of the degrees of freedom, but this does not disturb the argument). Therefore, one tries to reduce the weather to simpler processes which capture its statistical behavior on longer time scales but which fail to be as accurate on the time scales of a few days which are important for weather prediction.

This is a typical example for a time scale separation. We have a model which works well on the time scale of a few days and we want to extract a much simpler model which only needs to work on the time scales of months or years. The question is, how one gets the relevant parameters from the detailed dynamics. Taking the Brownian motion as an example: the long time dynamics is described by one parameter, the diffusion constant. This quantity has to be extracted from the description of the fluid (which Einstein and Smoluchowski did based on the atomic hypothesis). Later, by using concepts from chaos theory, more general methods were developed under the name of "elimination of fast degrees of freedom" [VK85, JGB$^+$03, JKRH01]. Often, the exponential decay of correlations of the fast system is assumed, such that it is driven by white Gaussian noise and the resulting stochastic trajectories are continuous.

This ubiquitous appearance of the white Gaussian noise (respectively its integrated form, the Brownian motion) can be understood from a purely deterministic point of view. If

the system is sufficiently chaotic, one can show its convergence to a Brownian motion (see [Bec90, Bec95, BR87b], or from a more mathematical point of view [Kel09]). As the Gaussian noise always leads to continuous trajectories, its coupling with the system can be described by local parameters, the drift and the diffusion. This corresponds to the fact that the Fokker-Planck operator is a regular differential operator.

But not all processes fall into this category, some of them show long memory effects. Harold Edwin Hurst was the first to observe such a behavior. He was analyzing the water heights of the river Nile and found that its long-time behavior was incompatible with the assumption of a fast decay of correlations [Hur51]. For this, he developed a method which was later called R/S statistics (and was mathematically analyzed by Mandelbrot and Taqqu [Man72, MT79]). The parameter which describes the type of memory is now called Hurst coefficient H (where $H = \frac{1}{2}$ corresponds to a memoryless process).

His results stimulated a lot of research. By analyzing real world time series many more examples of processes with long memory were found. By now there are several different measures of long memory which nevertheless also use the name of Hurst coefficient. Many of these definitions are collected in [Gué05]. Perhaps the most used method today is the "detrended fluctuation analysis" (DFA) [PHSG95]. Examples of long memory found with this method are in heart rates [BHK$^+$00], in continent and oceanic temperatures [ARAD$^+$08] and in DNA sequences [BDG$^+$98].

In general, a diffusive process driven by noise with long memory will show anomalous diffusion, i.e., the variance will not increase linearly, but with a power law. This thesis is mainly concerned with the description of such integrated systems. I refer to the reviews [MK00, MK04] for a collection of many experimental examples and some models.

There exist several stochastic models which all describe anomalous diffusion, e.g.: the FARIMA model [Ber94, Rob03] which is in discrete time and is mostly used in an economic setting, the fractional Brownian motion introduced by Mandelbrot and Van Ness [MVN68] and the continuous-time random walk (CTRW) introduced by Montroll and Weiss [MW65]. The fractional Brownian motion was introduced as an example of a process with Gaussian joint probability distribution and any given Hurst coefficient H ($0 < H < 1$). It is singled out by being the only self-similar process (see section 2.2) with these properties, i.e., it appears as limit process for systems with these properties.

The continuous-time random walk was introduced to describe stickiness phenomena in solids, i.e., the random walker does not hop at uniformly distributed time steps, but the time it stays at one position is itself random. It was directly applied by Scher and Montroll to explain the anomalous transport properties in amorphous solids [SM75]. Consequently, the CTRW was successfully applied to model transport in geological formations [BCDS06, DCSB04], wind fields [KFGS06] and blinking quantum dots [MB05]. It is also used in non-physical contexts such as finance [Sca06] or human travel [BHG06]. Hamiltonian systems often show stickiness properties at KAM islands (dynamical traps) which are similar to the behavior of a CTRW [Zas02, Zas94, AMK06].

Even though there is a lot of experimental evidence in favor of the continuous-time random walk, it is not as universal as the Brownian motion, e.g., for the dielectric response in a glassy medium [Goy07], the fractional Brownian motion is a much better model.

When one considers the coupling to a potential the options how to do introduce it increase drastically. As mentioned above, when the driving noise is Gaussian, one has continuous trajectories and the coupling with the potential is local. Mainly, one only has to choose the interpretation to use (e.g., Itō, Stratonovich, ...). In contrast to this, the CTRW

is a jump-type process, i.e., the time evolution of a probability density is not local any more (asymptotically, it is often expressible by a Fokker-Planck equation with fractional derivatives [BMK00, BF07a]). This leaves much more freedom in the possible ways how one can insert a potential [EFJS08, MKS98].

The question emerges how to choose the correct model. A typical way is to look at experimental or simulation data and apply statistical indicators which discriminate between the different models. Nevertheless this implies that one is aware of the complete class of models which can emerge in such a situation. A common alternative approach is the use of phenomenological models.

This thesis provides a third way to this problem by starting from the underlying deterministic system. It shows a way how to construct a continuous-time random walk from a dynamical system and identifying the parameters of the asymptotic process from the original system. But it will also turn out that one has to consider different types of models for the CTRW (see section 2.3) to cover all possible limits.

The developed theory is applied to a standard example for intermittency, the Manneville-Pomeau map which is known to lead to anomalous diffusion [GT84, KCK$^+$05, AC03]. While this thesis considers only the potential free case, I expect the developed methods to be helpful in determining the way of coupling the noise with a potential.

About this thesis

This thesis was written under the supervision of Prof. Holger Kantz at the Max-Planck-Institut für Physik komplexer Systeme in Dresden, Germany.

In chapter 2 some mathematical concepts and notations are introduced which are needed for this thesis. This is mainly the description of jump processes and their use in the theory of continuous-time random walks.

Chapter 3 introduces a diagrammatic method which allows to write down the joint probability distributions and correlation functions of a CTRW. The decomposition according to the diagrams will find its justification in the following chapters as the scaling limits can be taken for each diagram separately.

Chapter 4 starts with a look at CTRWs with a finite mean waiting time (section 4.2). While it is known that the transport of a biased CTRW with finite mean but infinite variance waiting time is not Fickian, the methods of chapter 3 help elucidate the origin of this. It will turn out that it is not due to a memory in the limit process (it is Markovian), but because of the non Gaussianity of the driving noise. The rest of the chapter focuses on the case of an infinite mean waiting time. It constructs conditions how to determine the parameters of the limit process.

Chapter 5 focuses on non-independent CTRWs, i.e., continuous-time random walks which can have correlations between successive steps. The memory can be generally modeled by an internal state space with Markovian dynamics. This chapter considers in detail the case of a finite internal state space where the line of argument can serve as a guideline in more complex situations. The scaling limit of these CTRWs are simple independent CTRWs. How to get the parameters of the asymptotics is worked out in detail.

In chapter 6 the focus shifts to deterministic systems which can be treated by the method of inducing. By using the diagrammatic method of chapter 3 and adopting the discussions of chapters 4 and 5, there are good arguments supporting the hypothesis that the scaling

limit of this deterministic dynamics is described by an independent CTRW. Moreover, one can identify the asymptotic parameters from the dynamics. These results are applied to two maps of Manneville-Pomeau type (one unbiased and one biased) and checked numerically.

Chapter 7 summarizes the results and gives an outlook.

2. Preliminaries

In this chapter, I want to introduce the basic mathematical concepts and notations.

2.1. Probability Distributions

Probability distributions on $\mathbb{R}^d$ will play a significant role in this work. I will summarize important aspects of them which can be found in standard references (e.g., [Bau96, Fel68, Fel71, Kal01]). Using the standard constructions, one obtains the Borel σ-algebra $\mathcal{B}_d$ on $\mathbb{R}^d$ [Els02]. A probability distribution on $\mathbb{R}^d$ is then given by a probability measure

$$\mu : \mathcal{B}_d \to \mathbb{R}_{\geq 0} \tag{2.1}$$

with $\mu(\mathbb{R}^d) = 1$.

The *characteristic function* $\tilde{\mu}(k)$ of a probability distribution is the Fourier transform of μ

$$\tilde{\mu}(\mathbf{k}) = \int e^{i(k_1 x_1 + \cdots + k_q x_q)} \, \mu(d^q x). \tag{2.2}$$

The characteristic function defines uniquely the probability distribution.

Assume we have two random variables X and Y with values in $\mathbb{R}^d$ and measures μ_X and μ_Y. The distribution of the sum $X + Y$ is given by the convolution

$$\begin{aligned} \mu_{X+Y} &= \mu_X \star \mu_Y \\ (\mu_X \star \mu_Y)(A) &= \int \mu_X(A - y) \, \mu_Y(dy) = \int \mu_Y(A - x) \, \mu_X(dx) \quad \text{for } A \in \mathcal{B}_d. \end{aligned} \tag{2.3}$$

The convolution of the measures is represented in Fourier space by a simple multiplication of the characteristic functions

$$\tilde{\mu}_{X+Y}(\mathbf{k}) = \tilde{\mu}_X(\mathbf{k}) \tilde{\mu}_Y(\mathbf{k}). \tag{2.4}$$

If every set A of zero Lebesgue measure is also a set of zero measure $\mu(A) = 0$, then μ is called *absolutely continuous* with respect to the Lebesgue measure. The theorem of Radon-Nikodým says that we can write μ with help of a density function $\rho : \mathbb{R}^d \to \mathbb{R}_{\geq 0}$

$$\mu(A) = \int_{\mathbf{x} \in A} \rho(\mathbf{x}) \, d^q x. \tag{2.5}$$

The density $\rho(\mathbf{x})$ is defined up to sets of zero measure of μ. The characteristic function $\tilde{\rho}(\mathbf{k})$ is then

$$\tilde{\rho}(\mathbf{k}) = \int e^{i\mathbf{k}\cdot\mathbf{x}} \rho(\mathbf{x}) \, d^q x \tag{2.6}$$

which allows to obtain the density as inverse Fourier transform

$$\rho(\mathbf{x}) = \frac{1}{(2\pi)^d} \int e^{-i\mathbf{k}\cdot\mathbf{x}} \tilde{\rho}(\mathbf{k}) \, d^q k. \tag{2.7}$$

The convolution is defined as

$$(\rho_X \star \rho_Y)(\mathbf{x}) = \int_{-\infty<y_i<\infty} \rho_X(\mathbf{x}-\mathbf{y})\rho_Y(\mathbf{y})\, d^q y = \int_{-\infty<y_i<\infty} \rho_Y(\mathbf{x}-\mathbf{y})\rho_X(\mathbf{x})\, d^q y. \quad (2.8)$$

The connection between the characteristic functions under convolution stays the same with densities

$$\widetilde{(\rho_X \star \rho_Y)}(\mathbf{k}) = \tilde{\rho}_X(\mathbf{k})\tilde{\rho}_X(\mathbf{k}). \quad (2.9)$$

Where no ambiguity is possible, I will distinguish between the density and the characteristic function by the nomenclature of the arguments.

When the measure μ is concentrated on the positive quadrant (i.e., its support is contained in $\mathbb{R}^d_{\geq 0}$), it is often more convenient to consider the Laplace transform

$$\hat{\mu}(\boldsymbol{\lambda}) = \int e^{-(\lambda_1 x_1 + \cdots + \lambda_q x_q)}\, \mu(d^q x). \quad (2.10)$$

For probability distribution the Laplace transform is defined for $\boldsymbol{\lambda} \in \mathbb{C}^d$ with $\operatorname{Re}\lambda_i \geq 0$ $(i = 1, \ldots, d)$. It is analytic for $\boldsymbol{\lambda} \in \mathbb{C}^d$ with $\operatorname{Re}\lambda_i > 0$ and continuous for $\operatorname{Re}\lambda_i \geq 0$. Therefore, one can construct the characteristic function of μ by analytic continuation

$$\tilde{\mu}(\mathbf{k}) = \hat{\mu}(-i\mathbf{k} + 0), \quad (2.11)$$

where the $+0$ indicates that one approaches the imaginary axis with a positive real part. Not surprisingly, the Laplace transform has also the convolution property

$$\widehat{(\mu_X \star \mu_Y)}(\boldsymbol{\lambda}) = \hat{\mu}_X(\boldsymbol{\lambda})\hat{\mu}_Y(\boldsymbol{\lambda}). \quad (2.12)$$

The Laplace transform can also be used with densities

$$\hat{\rho}(\boldsymbol{\lambda}) = \int_0^\infty e^{-\boldsymbol{\lambda}\cdot\mathbf{x}}\rho(\mathbf{x})\, d^q x. \quad (2.13)$$

The convolution for densities with support in $\mathbb{R}^d_{\geq 0}$ is

$$(\rho_X \star \rho_Y)(\mathbf{x}) = \int_{0\leq y_i\leq x_i} \rho_X(\mathbf{x}-\mathbf{y})\rho_Y(\mathbf{y})\, d^q y = \int_{0\leq y_i\leq x_i} \rho_Y(\mathbf{x}-\mathbf{y})\rho_X(\mathbf{x})\, d^q y. \quad (2.14)$$

For each $\mathbf{x}$, the integration domain is bounded. The convolution (2.14) is sometimes called Laplace convolution. As in the case of the characteristic function, I will distinguish between a density and its Laplace transform by the arguments.

For the rest of the section, I will only consider probability distributions in one dimension (i.e., on $\mathbb{R}$). One important class are the *infinitely divisible* distributions [[Fel71], chapter XVII]: a probability distribution μ is called infinitely divisible, if for every $n \in \mathbb{N}$, there is a probability distribution μ_n such that $\mu = \star^n \mu_n$ (i.e., μ_n convoluted n times with itself). In other words, a random variable X with infinitely divisible distribution μ can be written as sum of n independent, identical distributed (i.i.d.) random variables for each n. An important application are random processes with stationary and independent increments in continuous time (see section 2.2): it is clear by construction, that the one point probability distributions appearing in these processes have to be infinitely divisible.

The condition for infinite divisibility is easily paraphrased in terms of the characteristic functions

$$\tilde{\mu}(k) = \tilde{\mu}_n(k)^n \text{ for all } n. \tag{2.15}$$

Following [[Fel71], theorem XVII.1.1], we know that the limit

$$\omega(k) = \lim_{n\to\infty} n(1 - \tilde{\mu}_n(k)) \tag{2.16}$$

exists and that

$$\begin{aligned} \tilde{\mu}(k) &= e^{-\omega(k)} \\ \text{and } \tilde{\mu}_n(k) &= e^{-\frac{1}{n}\omega(k)} \text{ for all } n \in \mathbb{N}. \end{aligned} \tag{2.17}$$

The possibility to assume the form (2.17) for infinitely divisible distributions eases many calculations.

One special kind of infinite divisible distributions are the *stable* distributions [[Fel71], chapter VI.1] (they are also called Lévy stable distributions in honor of Paul Pierre Lévy) They are defined by the property that the distributions μ and μ_n differ only by an offset and scaling: assume that X and $X_1, X_2, \ldots$ are i.i.d. random variables with distribution μ; if μ is stable, then there are constant a_n and b_n such that for every n

$$X = \frac{X_1 + \cdots + X_n}{a_n} - b_n \quad \text{in distribution.} \tag{2.18}$$

If $b_n = 0$, μ is called *strictly stable*. The condition can be expressed in terms of the characteristic function:

$$\tilde{\mu}\left(\frac{k}{a_n}\right)^n e^{-ikb_n} = \tilde{\mu}(k). \tag{2.19}$$

The constants a_n must be of the form $a_n = n^{\frac{1}{\alpha}}$ with $0 < \alpha \leq 2$ [[Fel71], theorem VI.1.1]. The exponent α is called the *characteristic exponent*. Using the form (2.17), we have the following parameterization of the stable distributions by $0 < \alpha \leq 2$,$-1 \leq \beta \leq 1$,$\sigma \geq 0$ and m [[ST94], definition 1.1.6]

$$\omega(k) = \begin{cases} \sigma^2 k^2 - imk & \text{for } \alpha = 2 \\ \sigma^\alpha |k|^\alpha (1 - i\beta \operatorname{sgn}(k) \tan\left(\frac{\pi\alpha}{2}\right)) - imk & \text{for } 0 < \alpha < 2 \text{ and } \alpha \neq 1 \\ \sigma|k|(1 + i\beta\frac{2}{\pi}\operatorname{sgn}(k)\ln|k|) - imk & \text{for } \alpha = 1. \end{cases} \tag{2.20}$$

The case $\alpha = 2$ corresponds to the normal (Gaussian) distribution. Here β is irrelevant. Except for this case, the parameterization is unique. The parameter β is often called *skewness* as it measures the asymmetry of the distribution (however, one should notice that this use does not coincide with the more common definition of skewness as the normalized third moment — this does not exist for $\alpha < 2$). The parameter σ gives the overall scale and m shifts the distribution.

The case $\alpha = 1$ is special due to the appearance of the logarithm. This has the effect that any distribution with $\alpha = 1$ and $\beta \neq 0$ is *not* strictly stable (on the other hand, for $\alpha \neq 1$ every distribution can be made strictly stable by shifting it to $m = 0$). Therefore, this case has normally to be treated separately.

Stable distributions are important as limit distributions of scaled sums: assume that $X_1, X_2, \ldots$ are i.i.d.. If there are constants $a_n > 0$ and b_n such that

$$\frac{X_1 + \cdots + X_n}{a_n} - b_n \to X \text{ in distribution as } n \to \infty, \tag{2.21}$$

than X is a stable distribution. The values for α and β of the stable limit distribution are fixed by the distributions of the X_i. By an overall scaling of the a_n, one can change the value σ, and by an overall additive constant to the b_n one can change the value m. One can proceed by fixing the values for σ and m and then determine a_n and b_n accordingly (e.g., [[Fel71], equation (XVII.5.23)]). This is the most general approach. However, in a physical setting an alternative way is often more convenient: one considers only distributions of the X_i which are in the *domain of normal attraction* of a stable distribution X, i.e., equation (2.21) is valid with the constants $a_n = n^{\frac{1}{\alpha}}$ [[Fel71], page 581]. The constants b_n are fixed by the requirement that the stable limit distribution is strictly stable, i.e., $m = 0$. This excludes the case $\alpha = 1$ and $\beta \neq 0$. A distribution μ is in the domain of normal attraction with exponent α, if the following limits exist [[Fel71], page 581]:

$$\begin{aligned} \lim_{x\to\infty} x^\alpha \mu([x, \infty[) &= c_1 \\ \lim_{x\to-\infty} |x|^\alpha \mu(]-\infty, x]) &= c_2. \end{aligned} \tag{2.22}$$

For $\alpha = 2$, we are in the domain of attraction of a normal distribution and the value of σ is determined by the standard deviation of μ. For $\alpha < 2$ and $\alpha \neq 1$, we can put equation [[Fel71], (XVII.5.11)] into [[Fel71], (XVII.2.9)] resulting in the limit distribution (up to a shifting constant b):

$$\begin{aligned} \omega(k) - ibk =& c_2(2-\alpha) \int_{-\infty}^{0} \frac{1 - e^{ikx} - ik\sin(x)}{|x|^{1+\alpha}}\, dx + c_1(2-\alpha) \int_0^{\infty} \frac{1 - e^{ikx} - ik\sin(x)}{x^{1+\alpha}}\, dx \\ =& -\Gamma(-\alpha)(c_1 + c_2)(2-\alpha)\cos\left(\frac{\pi\alpha}{2}\right) |k|^\alpha \left[1 - i\frac{c_1 - c_2}{c_1 + c_2}\operatorname{sgn}(k)\tan\left(\frac{\pi\alpha}{2}\right)\right] \\ &+ i\Gamma(-\alpha)\sin\left(\frac{\pi\alpha}{2}\right) k. \end{aligned} \tag{2.23}$$

Therefore,

$$\begin{aligned} \sigma^\alpha &= -\Gamma(-\alpha)(c_1 + c_2)(2-\alpha)\cos\left(\frac{\pi\alpha}{2}\right) \\ \text{and } \beta &= \frac{c_1 - c_2}{c_1 + c_2}. \end{aligned} \tag{2.24}$$

As can be seen from equation (2.22), a distribution μ in the domain of normal attraction with exponent $\alpha < 2$ possesses all absolute moments of order $< \alpha$ while all absolute moments of order $> \alpha$ are infinite (this is actually true for the complete domain of attraction [[Fel71], lemma on page 578]). If $\alpha > 1$, μ has a mean value

$$\overline{m} = \int x\, \mu(dx). \tag{2.25}$$

Following [[Fel71], theorem XVII.5.3], we can set $b_n = 0$ for $0 < \alpha < 1$ and $b_n = \overline{m} n^{1-\frac{1}{\alpha}}$ for $1 < \alpha < 2$ in equation (2.21) to obtain a strictly stable limit. For convenience, I set $\overline{m} = 0$ in the case $0 < \alpha < 1$, such that we can define $b_n = \overline{m} n^{1-\frac{1}{\alpha}}$ in general.

It is instructive to express these statements in terms of the characteristic functions. The basis for this is the following theorem [[Fel71], theorem XV.3.2]: a sequence of probability distributions converges in distribution if and only if the sequence of characteristic functions converges pointwise to a function which is continuous at the origin. This function is the characteristic function of the limit. The convergence is uniform in every bounded interval.

From this follows

$$\lim_{n\to\infty} \tilde{\mu}\left(\frac{k}{n^{\frac{1}{\alpha}}}\right)^n e^{-ik\overline{m}n^{1-\frac{1}{\alpha}}} = \exp\left(-\sigma^\alpha |k|^\alpha (1 - i\beta \operatorname{sgn}(k) \tan\left(\frac{\pi\alpha}{2}\right))\right) \tag{2.26}$$

and with [[Fel71], theorem XVII.1.1]

$$\lim_{n\to\infty} n(1 - \tilde{\mu}\left(\frac{k}{n^{\frac{1}{\alpha}}}\right) e^{-in^{-\frac{1}{\alpha}}k\overline{m}}) = \sigma^\alpha |k|^\alpha (1 - i\beta \operatorname{sgn}(k) \tan\left(\frac{\pi\alpha}{2}\right)). \tag{2.27}$$

Using the uniform convergence, this can be rewritten with the continuous parameter ζ ($\zeta = n^{-\frac{1}{\alpha}}$)

$$\tilde{\mu}(\zeta k) = 1 + i\zeta\overline{m}k - \zeta^\alpha \sigma^\alpha |k|^\alpha (1 - i\beta \operatorname{sgn}(k) \tan\left(\frac{\pi\alpha}{2}\right)) + o(\zeta^\alpha) \text{ as } \zeta \searrow 0. \tag{2.28}$$

On the other hand, equation (2.26) follows from equation (2.28). Therefore, a probability distribution μ is in the domain of normal attraction of a stable distribution ($\alpha \neq 1$), if and only if its characteristic function $\tilde{\mu}(k)$ behaves as equation (2.28) near the origin.

One important subclass of the stable distributions remains to be discussed: the one-sided Lévy stable distributions. Expect for the parameters $0 < \alpha < 1$ with $\beta = \pm 1$, all stable distributions are supported on the whole real axis (except the trivial case $\sigma = 0$). When we have $0 < \alpha < 1$ and $\beta = 1$, the distribution is concentrated on the positive half-axis ($m = 0$). For $\beta = -1$ we have the same situation on the negative axis. Unless mentioned otherwise, I will always refer to the positive case.

If μ is a one-sided stable distribution, its Laplace transform is defined. They can be parameterized by the two coefficients $0 < \alpha < 1$ and $s > 0$ as

$$\hat{\mu}(\lambda) = \exp(-s^\alpha \lambda^\alpha). \tag{2.29}$$

The connection to the previous parameterization can be made by analytic continuation (equation (2.11))

$$\begin{aligned} \tilde{\mu}(k) &= \hat{\mu}(-ik + 0) \\ &= \exp\left(-s^\alpha \cos\left(\frac{\pi\alpha}{2}\right) |k|^\alpha (1 - i \operatorname{sgn}(k) \tan\left(\frac{\pi\alpha}{2}\right))\right). \end{aligned} \tag{2.30}$$

Similar to the two-sided case, the domain of normal attraction is given by the distributions whose Laplace transform has the behavior

$$\hat{\mu}(\lambda) = 1 - s^\alpha \lambda^\alpha + o(\lambda^\alpha) \text{ as } \lambda \searrow 0. \tag{2.31}$$

2.2. Stochastic Processes

Stochastic processes play a very prominent role in this thesis. At this point, I want to informally introduce the concepts which are relevant later. I refer to [[Øks03], chapter 2], [[Kal01], chapter 3] and [Bau96] for more details. A stochastic process can be seen as a set of random variables $\{X(t)\}_t$ indexed by a time t. Here, I always assume that these random variables take values in $\mathbb{R}^d$. The time can be continuous, then t takes the values $t \in \mathbb{R}_{\geq 0}$ or $t \in \mathbb{R}$. The time can also be discrete with t taking values $t \in \mathbb{N}$ of $t \in \mathbb{Z}$. For any finite collection of times $\{t_1, \ldots, t_n\}$, we can look at the probability distribution of the random vector $(X(t_1), \ldots, X(t_n))$ (called joint probability distribution). If this distribution can be written as a density, we have for the n-point joint probability distribution

$$p_n(x_1, \ldots, x_n; t_1, \ldots, t_n) = \langle \delta(x_1 - X(t_1)) \cdots \delta(x_n - X(t_n)) \rangle \tag{2.32}$$

When all finite joint probability distributions are given (and are consistent), this defines a stochastic process by Kolmogorov's extension theorem [[Øks03], theorem 2.1.5].

A stochastic process is called *Markov*, if the process has no memory in the sense that the future behavior depends only on the current state. This is commonly expressed with the help of conditional probabilities ($B \in \mathcal{B}_d$)

$$\Pr(X(t) \in B | X(t_1), X(t_2), \ldots, X(t_n)) = \Pr(X(t) \in B | X(t_1)) \text{ for any } t > t_1 > t_2 > \cdots > t_n. \tag{2.33}$$

The density factorizes for a Markov process ($t_1 > t_2 > \cdots > t_n$)

$$p_n(x_1, \ldots, x_n; t_1, \ldots, t_n) = p_c(x_1, t_1 | x_2, t_2) \cdots p_c(x_{n-1}, t_{n-1} | x_n, t_n) p_1(x_n; t_n) \tag{2.34}$$

where $p_c(x_1, t_1 | x_2, t_2)$ is the conditional probability density of $X(t_1)$ conditioned on $X(t_2)$.

A special class are the processes with stationary, independent increments. If the random variables

$$X(t_1) - X(s_1), X(t_2) - X(s_2), \ldots, X(t_n) - X(s_n) \tag{2.35}$$

with $t_1 > s_1 \geq t_2 > s_2 \geq \cdots \geq t_n > s_n$ are independent, than the process $X(t)$ is said to have *independent increments* [[Fel71], section VI.4]. A process with independent increments is clearly Markovian. If for a fixed δt all distributions

$$X(t + \delta t) - X(t) \text{ with } t \text{ arbitrary} \tag{2.36}$$

coincide, than this process is said to have *stationary increments*. For a process with stationary, independent increments, the conditional probability density can be reduced to a function

$$p_c(x_1, t_1 | x_2, t_2) = g(x_1 - x_2, t_1 - t_2). \tag{2.37}$$

The most famous example of a stochastic process in continuous time with stationary, independent increments is of course the Brownian motion $B(t)$ in d dimensions. It is defined by the initial condition $X(0) = 0$ and the conditional distribution [[Øks03], page 12]

$$p_c(x_1, t_1 | x_2, t_2) = \frac{1}{(2\pi(t_1 - t_2))^{d/2}} \exp\left(-\frac{1}{2(t_1 - t_2)} \|x_1 - x_2\|^2 \right). \tag{2.38}$$

To simplify the notation, I will now concentrate on stochastic processes in one dimension and continuous time. I suppose that the processes start at $t = 0$ with $X(0) = 0$ (i.e., $t \in$

$\mathbf{R}_{\geq 0}$). A process with stationary, independent increments is then defined by the probability distributions $X(t_1) - X(t_2)$ which is given by the density $g(x,t)$ (with $t = t_1 - t_2$)[1]. These processes are called *Lévy processes* (mathematically, one additionally needs the technical assumption that $X(t)$ is right continuous and has left limits [[Kal01], page 290]). As all time step are defined, we can write $g(x,t)$ for every $n \in \mathbb{N}$ as convolution (in x):

$$g(x,t) = \star^n g(x, \frac{t}{n}) \tag{2.39}$$

which corresponds to expressing

$$\begin{aligned} X(t_1) - X(t_2) = &\left[X(t_1) - X(t_1 - \frac{t_1 - t_2}{n})\right] + \left[X(t_1 - \frac{t_1 - t_2}{n}) - X(t_1 - 2\frac{t_1 - t_2}{n})\right] \\ &+ \cdots + \left[X(t_2 + \frac{t_1 - t_2}{n}) - X(t_2)\right] \end{aligned} \tag{2.40}$$

and noting that the expression in the square brackets are independent with density $g(x, t/n)$. But this is exactly the definition of infinite divisibility. Therefore, we have a function $\omega(k)$ with

$$\begin{gathered} \int e^{ikx} g(x,1)\, dx = e^{-\omega(k)} \\ \text{and more generally} \quad g(k,t) = e^{-t\omega(k)} \end{gathered} \tag{2.41}$$

Here, I use the convention to distinguish the density $g(x,t)$ from its characteristic function $g(k,t)$ by use of the arguments. In accordance with [BKMS04], I call $\omega(k)$ the *log-Fourier transform* of $g(x,1)$. The Brownian motion corresponds to $\omega(k) = \frac{1}{2}k^2$.

In general, the Fourier transform of the multi-point density can be expressed with help of $\omega(k)$ (here: $t_1 \geq t_2 \geq \cdots \geq t_n$)

$$\begin{aligned} p_n(\mathbf{k}, \mathbf{t}) &= \int e^{i\mathbf{k}\cdot\mathbf{x}} g(x_1 - x_2, t_1 - t_2) \cdots g(x_{n-1} - x_n, t_{n-1} - t_n) g(x_n, t_n)\, d^n x \\ &= \exp\left(-(t_1 - t_2)\omega(k_1) - (t_2 - t_3)\omega(k_1 + k_2) - \cdots - t_n \omega(k_1 + \cdots k_n)\right). \end{aligned} \tag{2.42}$$

The log-Fourier transform can be used to decompose stochastic processes, based on a additivity principle: assume $X(t)$ and $Y(t)$ are independent Lévy processes with log-Fourier transforms $\omega_X(k)$ and $\omega_Y(k)$. Then the log-Fourier transform of the sum $X(t) + Y(t)$ is given by

$$\omega_{X+Y}(k) = \omega_X(k) + \omega_Y(k). \tag{2.43}$$

For Lévy processes, there exists a general representation theorem [[Kal01], theorem 15.4 and corollary 15.7]: any Lévy process $X(t)$ can be decomposed into three independent processes

$$X(t) = mt + \sigma B(t) + J(t). \tag{2.44}$$

The first term is a deterministic linear drift (with $\omega_{\text{drift}}(k) = ikm$), the second is Brownian motion (with scale $\sigma \geq 0$ and $\omega_{\text{BM}}(k) = \frac{\sigma^2}{2}k^2$). $J(t)$ is a *pure jump-type* process. While

[1] In agreement with the usual notation in the physics literature, I write the probability distributions as densities. The arguments can be adopted to the more general notation of probability measures.

the first two terms are continuous stochastic processes, the last term is responsible for all discontinuities of $X(t)$.

Most of the processes we will encounter in this work are of jump-type. A mathematically comprehensive introduction can be found in [[Kal01], chapter 12]. Here, I concentrate on $J(t)$. The *Poisson process* is a special case of a jump-type process which I want to introduce first.

An important tool for the definition of a Poisson process is the notion of a counting measure: a *counting measure* is a measure ν on some space which takes only values in $\mathbb{N}_0 \cup \{\infty\}$ (i.e., it counts some occurrences). The behavior of $J(t)$ is completely determined by its jumps. For a fixed realization of $J(t)$, we can describe the jumps by a counting measure ν on $\mathbb{R} \times \mathbb{R}_{\geq 0}$ by

$$\nu([x_1, x_2] \times [t_1, t_2]) = \text{number of jumps of } J(t) \text{ of size } x \in [x_1, x_2] \text{ at some time } t \in [t_1, t_2]. \tag{2.45}$$

Therefore, we can write

$$J(t) = \int_{\tau \in [0,t]} x \, \nu(dx, d\tau) \quad \text{for a fixed realization.} \tag{2.46}$$

The idea is to turn ν into a random quantity. For any two Borel sets A_1 and A_2 which do not intersect ($A_1 \cap A_2 = \{\}$), the distributions of $\nu(A_1)$ and $\nu(A_2)$ are supposed to be independent. Additionally, one assumes that the distribution of $\nu(A)$ is Poissonian with rate $\overline{\nu}(A)$, i.e.,

$$\Pr(\nu(A) = n) = e^{-\overline{\nu}(A)} \frac{\overline{\nu}(A)^n}{n!}. \tag{2.47}$$

This notation stems from the fact, that $\overline{\nu}(A)$ is the expectation value of $\nu(A)$. By the properties of the Poisson distribution, $\overline{\nu}$ is a (nonrandom) measure on $\mathbb{R} \times \mathbb{R}_{\geq 0}$, the *intensity measure* (or *jump density*). Together with the above assumptions, it completely defines the stochastic properties of the random measure ν. The random process $J(t)$ is then defined by equation (2.46).

For a Lévy process, the intensity measure $\overline{\mu}$ can be factorized with the Lebesgue measure and an intensity measure κ on $\mathbb{R}$ by

$$\overline{\nu}(A \times [t_1, t_2]) = (t_2 - t_1)\kappa(A) \quad \text{for } t_2 \geq t_1. \tag{2.48}$$

How does the log-Fourier transform for this process look like? To calculate the characteristic function of $J(1)$, one approximates the measure κ on a partition using the intervals $A_j = [j\Delta x, (j+1)\Delta x[$ with $j \in \mathbb{Z}$:

$$\begin{aligned} \langle e^{ikJ(1)} \rangle &\simeq \prod_{j=-\infty}^{\infty} \sum_{n=0}^{\infty} e^{-\kappa(A_j)} \frac{\kappa(A)^n}{n!} e^{iknj\Delta x} \\ &= \prod_{j=-\infty}^{\infty} \exp\left(-(1 - e^{ikj\Delta x})\kappa(A_j)\right) \\ &= \exp\left(-\sum_{j=-\infty}^{\infty} (1 - e^{ikj\Delta x})\kappa(A_j)\right). \end{aligned} \tag{2.49}$$

Letting $\Delta x \searrow 0$ gives

$$\langle e^{ikJ(1)} \rangle = \exp\left(- \int (1 - e^{ikx})\, \kappa(dx) \right) \tag{2.50}$$

and therefore for the log-Fourier transform

$$\omega_J(k) = \int (1 - e^{ikx})\, \kappa(dx). \tag{2.51}$$

Unfortunately, it turns out that the Poisson processes (equation (2.51)) do not describe all possible jump-type processes [[Kal01], theorem 15.4]. Equation (2.51) has to be extended by a regularization function

$$\omega_J(k) = \int (1 - e^{ikx} + ikx\Xi(x))\, \kappa(dx). \tag{2.52}$$

This is a different process as the one described by equation (2.51) – it can be reduced to a Poisson process only if the integral $\int x\Xi(x)\, \kappa(dx)$ exists and is finite. Typical forms for the regularization function $\Xi(x)$ which are generally valid are [[MS01], theorem 3.1.14]

$$\begin{aligned} \Xi_a(x) &= \frac{1}{1+x^2} \\ \text{or } \Xi_b(x) &= \begin{cases} 1 & \text{for } |x| \leq 1 \\ 0 & \text{for } |x| > 1. \end{cases} \end{aligned} \tag{2.53}$$

To be well defined, the jump density κ has to fulfill

$$\int \max(|x|^2, 1)\, \kappa(dx) < \infty. \tag{2.54}$$

The process $J(t)$ is then described by extending equation (2.46)

$$J(t) = \int_{\tau \in [0,t]} x\, (\nu - \Xi(x)\overline{\nu})(dx, d\tau). \tag{2.55}$$

In general it is not possible to split the integral into separate integrations over the measures ν and $\Xi(x)\overline{\nu}$.

At this point, it is perhaps helpful to consider an example: assume that $J(1)$ (and therefore all $J(t)$) follows a stable distribution with exponent $0 < \alpha < 2$. The representation with the jump density is [[Kal01], proposition 15.9]

$$\kappa_{\mathrm{s}}(dx) = \begin{cases} c_+ \frac{dx}{|x|^{1+\alpha}} & \text{for } x > 0 \\ c_- \frac{dx}{|x|^{1+\alpha}} & \text{for } x < 0 \end{cases} \tag{2.56}$$

with some $c_\pm \geq 0$. The scaling behavior is directly reflected in the jump density. Using the regularization function $\Xi_b(x)$, this gives

$$\omega_J(k) = \begin{cases} i\frac{(c_+ - c_-)}{1-\alpha} k - \Gamma(-\alpha)(c_+ + c_-) \cos\left(\frac{\pi\alpha}{2}\right) |k|^\alpha (1 - i\frac{c_+ - c_-}{c_+ + c_-} \operatorname{sgn}(k) \tan\left(\frac{\pi\alpha}{2}\right)) & \text{for } \alpha \neq 1 \\ -i(c_+ - c_-)(1-\gamma)k + (c_+ + c_-)\frac{\pi}{2}|k|(1 + i\frac{c_+ - c_-}{c_+ + c_-}\frac{2}{\pi} \operatorname{sgn}(k) \ln|k|) & \text{for } \alpha = 1 \end{cases} \tag{2.57}$$

where $\gamma = 0.5772\ldots$ is the Euler-Mascheroni constant. One sees that the result corresponds to a stable, but not to a strictly stable distribution. This is due to the fact that the regularization function $\Xi_b(x)$ is not linear which leads to additional linear terms in k.

This observation is related to a more general fact: The decomposition of a Lévy process into the drift and the jump process is only determined for a fixed regularization function. It can change when one uses a different function, e.g., a change from $\Xi_a(x)$ to $\Xi_b(x)$:

$$\omega_J^{\text{using } \Xi_a(x)}(k) - \omega_J^{\text{using } \Xi_b(x)}(k) = ik \int x(\Xi_a(x) - \Xi_b(x))\, \kappa(dx). \tag{2.58}$$

This can become inconvenient, when one tries to determine the scaling behavior of the distributions. One way is to use special regularization functions, which behave better in this respect but which have the disadvantage of not being applicable to all cases. In general, one can introduce a new regularization function $\Xi'(x)$, if the following integral is defined

$$\int x(\Xi_a(x) - \Xi'(x))\, \kappa(dx) \tag{2.59}$$

(this condition is independent under exchange of $\Xi_a(x)$ and $\Xi_b(x)$). The following possibilities emerge [[MS01], theorem 3.1.14]

1. If
$$\int \max(|x|, 1)\, \kappa(dx) < \infty \tag{2.60}$$
then we can use $\Xi_0(x) = 0$, i.e., we have the Poisson process without regularization described above.

2. If
$$\int \min(|x|^2, |x|)\, \kappa(dx) < \infty \tag{2.61}$$
then we can use $\Xi_1(x) = 1$, i.e., the log-Fourier transform is given by
$$\omega_J(k) = \int (1 - e^{ikx} + ikx)\, \kappa(dx). \tag{2.62}$$

In general, one needs to restrict the behavior of $\kappa(dx)$ near the origin to be able to use $\Xi_0(x)$ while one needs to restrict the asymptotic behavior to be able to use $\Xi_1(x)$. When κ is a probability measure, $\Xi_0(x)$ can always be used, while $\Xi_1(x)$ can be used when the first absolute moment is finite.

If we look again at the densities corresponding to stable distributions:

$$\kappa_{\text{s}}(dx) = \begin{cases} c_+ \frac{dx}{|x|^{1+\alpha}} & \text{for } x > 0 \\ c_- \frac{dx}{|x|^{1+\alpha}} & \text{for } x < 0, \end{cases} \tag{2.63}$$

we can use $\Xi_0(x)$ in the domain $0 < \alpha < 1$ and $\Xi_1(x)$ in the domain $1 < \alpha < 2$. The case $\alpha = 1$ is again special, as it cannot be treated with either of these functions. Using these functions, the scaling behavior can be directly inferred from a linear rescaling of the integration variable without even having to calculate the integral:

$$\int (1 - e^{ikx})\, \kappa_{\text{s}}(dx) = |k|^\alpha \int (1 - e^{i \operatorname{sgn}(k) x})\, \kappa_{\text{s}}(dx) \qquad \text{for } 0 < \alpha < 1 \tag{2.64}$$

$$\int (1 - e^{ikx} + ikx)\, \kappa_{\text{s}}(dx) = |k|^\alpha \int (1 - e^{i \operatorname{sgn}(k) x} + i \operatorname{sgn}(k) x)\, \kappa_{\text{s}}(dx) \quad \text{for } 1 < \alpha < 2. \tag{2.65}$$

Especially, the log-Fourier transforms correspond to strictly stable distributions. We do not have to consider the linear terms in contrast in equation (2.57).

At this point it is convenient to look at a special case of Lévy processes: a process $X(t)$ which only takes nonnegative values. These processes play an important role as subordinators and are motivated in section 2.3 (one typical examples are random processes describing time which is monotonically increasing). By the stationarity property, these processes have only positive increments and $X(t)$ is for each realization a (not necessarily strictly) monotonic function. As $X(t)$ takes only nonnegative values, it is convenient to look at the Laplace transform, and define a *log-Laplace* transform $\omega'(\lambda)$ via

$$\langle e^{-\lambda X(1)} \rangle = e^{-\omega'(\lambda)}. \tag{2.66}$$

As before, one can obtain the log-Fourier transform as $\omega(k) = \omega'(-ik + 0)$. The positivity condition restricts the possible processes: in the decomposition (2.44), the Brownian component has to vanish; the jump density κ which support lies in $[0, \infty[$, has to fulfill [[Kal01], theorem 15.4]

$$\int \min(x, 1)\, \kappa(dx) < \infty \tag{2.67}$$

such that we can always use $\Xi_0(x)$ (i.e., we do not need a regularizing function). In general, the log-Laplace transform has the form

$$\omega'(\lambda) = m\lambda + \int (1 - e^{-\lambda x})\, \kappa(dx) \tag{2.68}$$

with $m \geq 0$.

Many parts of this thesis are concerned with the asymptotic behavior of processes. In the next paragraphs, I will define the *scaling limit* of a stochastic process. Let us leave for a moment the context of Lévy processes and consider a general stochastic process $X(t)$. Following Lamperti [Lam62], the process $X(t)$ is called *semi-stable* (or *self-similar* [[Kal01], page 291] — the notation is not unified yet), if for every $\zeta > 0$ there is a $s(\zeta)$ such that

$$\zeta^{\beta} X(\zeta^{-1} t) \stackrel{d}{=} X(t). \tag{2.69}$$

The notation $\stackrel{d}{=}$ denotes that all finite joint probability distributions coincide. The parameter β is the *scaling exponent* or *Hurst exponent*[2]. Similar to the stable distributions the semistable processes appear as limits of stochastic processes. There are many different versions how to define the limit of a stochastic process, in this thesis I am only considering the notion of the finite joint probabilities converging in distribution to the joint probability distributions of the limit process. I refer to [Whi02] for more details on other definitions of convergence and the interplay between them. In this thesis, I call the process $X^{\lim}(t)$ *scaling limit* of $X(t)$ if there exist $s(\zeta)$, $a(\zeta)$ and $b(\zeta)$ such that

$$X^{\lim}(t) \stackrel{d}{=} \lim_{\zeta \searrow 0} s(\zeta)(X(\zeta^{-1} t) - a(\zeta) - b(\zeta)\zeta^{-1} t). \tag{2.70}$$

[2] as mentioned in the introduction, there are many definitions of the Hurst exponent – therefore I prefer the name of scaling exponent in this context

The functions $a(\zeta)$ and $b(\zeta)$ correct a possible offset and a deterministic drift, respectively (this is the definition [[Lam62], equation (5)] extended by the drift component). In most cases, $a(\zeta)$ and $b(\zeta)$ will simply vanish. Especially for Lévy processes $a(\zeta) = 0$ as we assume the initial condition $X(0) = 0$. In analogy to the case of probability distributions, I will use the name of *domain of normal attraction*, if equation (2.70) applies with $s(\zeta) = \zeta^\beta$.

Processes with $\beta < \frac{1}{2}$ are called *subdiffusive*, with $\beta > \frac{1}{2}$ *superdiffusive*. Additionally, one uses the names *ballistic* for processes with $\beta = 1$ and correspondingly *subballistic* for $\beta < 1$ and *superballistic* for $\beta > 1$.

How does this look for Lévy processes: assume $X(t)$ is a Lévy process with log-Fourier transform $\omega(k)$. From the condition $X(\zeta) \stackrel{d}{=} \zeta^\beta X(1)$ follows

$$\begin{aligned} \langle e^{ikX(\zeta^{-1})} \rangle &= \langle e^{i\zeta^{-\beta}kX(1)} \rangle \\ \Rightarrow \quad e^{-\zeta^{-1}\omega(k)} &= e^{-\omega(\zeta^{-\beta}k)} \\ \Rightarrow \quad \zeta^{-1}\omega(k) &= \omega(\zeta^{-\beta}k). \end{aligned} \tag{2.71}$$

Therefore, $X(1)$ is strictly stable with exponent $\alpha = \frac{1}{\beta}$ (note that here β is the scaling exponent of the process and not related to the skewness parameter of the stable distributions). If $X(t)$ converges to a semi-stable process, then the limit process can be determined by looking at the stable limit distribution corresponding to the one-point distribution $X(1)$. The notion of the domain of normal attraction are the same for the limit of processes and the limit of distributions.

It is instructive to look at the convergence in terms of the jump density. The log-Fourier transform behaves under the limit (2.70) as

$$\begin{aligned} e^{-\omega^{\lim}(k)} &= \langle e^{ikX^{\lim}(1)} \rangle \\ &= \lim_{\zeta \searrow 0} \langle e^{ik\zeta^\beta (X(\zeta^{-1}) - b(\zeta)\zeta^{-1})} \rangle \\ &= \lim_{\zeta \searrow 0} e^{-\zeta^{-1}(\omega(\zeta^\beta k) - i\zeta^\beta b(\zeta)k)} \end{aligned} \tag{2.72}$$

and therefore

$$\omega^{\lim}(k) = \lim_{\zeta \searrow 0} \left(\zeta^{-1}(\omega(\zeta^\beta k) + i\zeta^\beta b(\zeta)k) \right). \tag{2.73}$$

According to the splitting (2.44), we can write

$$\omega(k) = -ikm + \frac{\sigma^2}{2}k^2 + \int (1 - e^{ikx} + ikx\Xi(x))\, \kappa(dx). \tag{2.74}$$

If $X(1)$ is in the domain of normal attraction of a stable distribution with exponent $0 < \alpha < 2$, then we know (equation (2.22))

$$\begin{aligned} \lim_{x \to \infty} x^\alpha \Pr(X(1) \geq x) &= c_+ \\ \lim_{x \to \infty} x^\alpha \Pr(X(1) \leq -x) &= c_- \end{aligned} \tag{2.75}$$

with $c_+, c_- \geq 0$ and at least one of them > 0. This behavior is reflected in the jump density

$$\begin{aligned} \lim_{x \to \infty} x^\alpha \kappa([x, \infty[) &= c_+ \\ \lim_{x \to \infty} x^\alpha \kappa(]-\infty, -x]) &= c_-. \end{aligned} \tag{2.76}$$

First, assume that $0 < \alpha < 1$, i.e., we can choose $\Xi(x) = 0$ such that

$$\omega(k) = -ikm_0 + \frac{\sigma^2}{2}k^2 + \int (1 - e^{ikx})\,\kappa(dx). \tag{2.77}$$

We can choose $b(\zeta) = 0$ and get for the limit (with $\beta = \frac{1}{\alpha} > 1$)

$$\begin{aligned}\omega^{\lim}(k) &= \lim_{\zeta \searrow 0} \zeta^{-1} \left(-i\zeta^{\beta} k m_0 + \zeta^{2\beta} \frac{\sigma^2}{2} k^2 + \int (1 - e^{i\zeta^{\beta} kx})\,\kappa(dx) \right) \\ &= \lim_{\zeta \searrow 0} \int (1 - e^{ikx})\,\zeta^{-1} \kappa(\zeta^{-\beta} dx).\end{aligned} \tag{2.78}$$

It boils down to the determination of the limit behavior for the measure $\zeta^{-1}\kappa(\zeta^{-\beta}\cdot)$. As $\zeta \searrow 0$, the asymptotic behavior of κ determines the limit, therefore using equation (2.76)

$$\begin{aligned}\kappa^{\lim}(dx) &= \lim_{\zeta \searrow 0} \zeta^{-1} \kappa(\zeta^{-\beta} dx) \\ &= \begin{cases} c_+ \frac{dx}{x^{1+\alpha}} & \text{for } x > 0 \\ c_- \frac{dx}{|x|^{1+\alpha}} & \text{for } x < 0. \end{cases}\end{aligned} \tag{2.79}$$

(A mathematically more precise formulation is given in [[Kal01], theorem 15.14]). With

$$\omega^{\lim}(k) = \int (1 - e^{ikx})\,\kappa^{\lim}(dx) \tag{2.80}$$

we are back to the strictly stable distributions.

For $1 < \alpha < 2$ we can use $\Xi(x) = 1$, i.e.,

$$\omega(k) = -ikm_1 + \frac{\sigma^2}{2}k^2 + \int (1 - e^{ikx} + ikx)\,\kappa(dx). \tag{2.81}$$

The constant m_1 is the mean of $X(1)$. To compensate the drift, we can choose $b(\zeta) = m_1$ (with $\beta = \frac{1}{\alpha} < 1$). The limit becomes

$$\begin{aligned}\omega^{\lim}(k) &= \lim_{\zeta \searrow 0} \zeta^{-1} \left(-i\zeta^{\beta} k m_1 + \zeta^{2\beta} \frac{\sigma^2}{2} k^2 + \int (1 - e^{i\zeta^{\beta} kx} + i\zeta^{\beta} kx)\,\kappa(dx) \right) + im_1 \zeta^{\beta - 1} k \\ &= \lim_{\zeta \searrow 0} \int (1 - e^{ikx} + ikx)\,\zeta^{-1} \kappa(\zeta^{-\beta} dx).\end{aligned} \tag{2.82}$$

Using again equation (2.79), we get

$$\omega^{\lim}(k) = \int (1 - e^{ikx} + ikx)\,\kappa^{\lim}(dx) \tag{2.83}$$

and the stable distribution emerges.

One sees that the convergence to a non-Gaussian stable process differs significantly from the convergence to Brownian motion. Only the asymptotic behavior of the large discontinuous jumps survive, the rest is 'scaled away'.

2.3. The Continuous-Time Random Walk

The *continuous-time random walk* was introduced by Montroll and Weiss in 1965 [MW65]. The concept in its simplest form describes a random walker (a random process) which rests at a site for a time which is determined by a random distribution and then makes a step which size is given by another random distribution. More formally, one uses a *operational time* s (alternative names in use are *virtual time* or *internal time*) which parameterizes the steps (at this point the operational time is considered to be discrete: $s \in \mathbb{N}_0$). We consider two random walks[3]

$$\begin{aligned} Y(s) &= \sum_{j=0}^{s-1} \mathcal{Y}_j \\ T(s) &= \sum_{j=0}^{s-1} \mathcal{T}_j. \end{aligned} \tag{2.84}$$

The $\mathcal{Y}_j$ and $\mathcal{T}_j$ are i.i.d. random variables with the additional constraint $\mathcal{T}_j \geq 0$ as the time should increase. The resulting process $X(t)$ in real (or physical) time is defined by

$$X(t) = Y(\sup\{s : T(s) < t\}). \tag{2.85}$$

While $Y(s)$ and $T(s)$ are Markovian processes, the resulting process $X(t)$ is in general not Markovian any more. The procedure equation (2.85) of using one stochastic process to define the parameter of an other stochastic process is called *subordination*. At this point it is worth mentioning that there are different notions of subordination. In its original form which was introduced by Bochner in 1949 (which is nowadays often termed "subordination in the sense of Bochner") [[Fel71], section X.7], two Markovian processes $Y(t)$ and $T(s)$ are considered and the subordinated process is given by $Y(T(s))$. This process is Markovian and clearly different from the process (2.85) which uses the inverse of $T(s)$. In the context of CTRWs, the term subordination is only used for the construction (2.85) and I adopt this usage for this thesis.

I will consider only processes $Y(s)$ which live in one dimension, but many results generalize readily to more dimensions. For a given j, the random variables $\mathcal{Y}_j$ and $\mathcal{T}_j$ need not to be independent. Therefore, their distribution is described by a two-dimensional density

$$\psi(x,t) = \langle \delta(x - \mathcal{Y}_j)\delta(t - \mathcal{T}_j)\rangle. \tag{2.86}$$

Again, I adopt the notion commonly used in the literature of writing this distribution as density, the result are unchanged by using the more general probability measure. As x can take all values in $\mathbb{R}$ while t can only take positive values, one takes the Fourier-Laplace transform of $\psi(x,t)$ (Fourier transform in x, Laplace transform in t):

$$\psi(k,\lambda) = \int e^{-\lambda t + ikx}\psi(x,t)\,dx\,dt. \tag{2.87}$$

[3] The original paper by Montroll and Weiss was set in the context of solid state physics and they considered the random walk on a lattice. In this thesis, I will work with a setup commonly used today. A good reference is the review paper [MK00].

Similarly, one introduces the Fourier-Laplace transform of the one-point density

$$\begin{aligned} p_1(k,\lambda) &= \int e^{-\lambda t}\langle e^{ikX(t)}\rangle \, dt \\ &= \int e^{-\lambda t + ikx} p_1(x,t)\, dx\, dt. \end{aligned} \tag{2.88}$$

As seen in the first line, $p_1(x,t)$ is only a probability density in x while t is a parameter. In Fourier-Laplace space, one can determine $p_1(k,\lambda)$ which is given by the *Montroll-Weiss equation* [[MK00], equation (25)]

$$p_1(k,\lambda) = \frac{1}{\lambda}\frac{1-\psi(0,\lambda)}{1-\psi(k,\lambda)}. \tag{2.89}$$

How does this behave under the scaling limit (2.70) (as the processes start at the origin, we can set $a(\zeta) = 0$)

$$X^{\lim}(t) = \lim_{\zeta \searrow 0} \zeta^{\beta}\left(X(\zeta^{-1}t) - \zeta^{-1}b(\zeta)t\right)? \tag{2.90}$$

The one-point density becomes

$$p_1^{\lim}(k,\lambda) = \lim_{\zeta\searrow 0} \zeta p_1(\zeta^{\beta}k, \zeta\lambda + i\zeta^{\beta}b(\zeta)k) \tag{2.91}$$

using the analyticity of the Laplace transform. Assume the simplest example, namely that the waiting time distribution and the spatial step distribution are independent. Further assume the waiting time distribution to be a one-sided Lévy stable distribution (2.29) and the step size distribution to be Gaussian. Then

$$\psi(k,\lambda) = \exp\left(-c^{\alpha}\lambda^{\alpha} - \frac{\sigma^2}{2}k^2\right). \tag{2.92}$$

We get a nontrivial scaling limit with $\beta = \frac{\alpha}{2}$ and $b(\zeta) = 0$

$$\begin{aligned} p_1(k,\lambda) &= \lim_{\zeta\searrow 0} \zeta p_1(\zeta^{\beta}k, \zeta\lambda) \\ &= \frac{\lambda^{\alpha-1}}{\lambda^{\alpha} + \frac{\sigma^2}{2c^{\alpha}}k^2}. \end{aligned} \tag{2.93}$$

As $\beta = \frac{\alpha}{2} < \frac{1}{2}$ the process is clearly subdiffusive.

Before I look at CTRWs in continuous operational time s, I want to introduce some notations. Unfortunately, the names for the different concepts are far from being unique — one has to check in every paper which definitions are used. I will stick to the notations in [MM07]. If for a step in operational time j the waiting time $\mathcal{T}_j$ and the spatial step size $\mathcal{Y}_j$ are independent, then the process is called *uncoupled CTRW*. This implies that $\psi(x,t)$ and $\psi(k,\lambda)$ can be factored in functions depending only on x and t (k and λ, respectively). If this is not the case then the process is called *coupled CTRW*. Till now, we have only looked at CTRWs where the $\mathcal{T}_j$ and $\mathcal{Y}_j$ are i.i.d. for different j. These CTRWs are called *independent CTRWs*. Later, a coupling between different steps will be considered. This process is then called *nonindependent CTRW*.

The other notation is related to the model of CTRW. In the definition (2.85) the random process stays at one position during the waiting time, then jumps instantaneously to its new

position. Following [[Hug95], page 286], this is called *leaper* model of the CTRW. An example of an alternative model is the *creeper* [[Hug95], page 287]: the random process moves with constant velocity during the waiting time and reaches the new position continuously. In general, looking at the scaling limit with $b(\zeta) \neq 0$ will change the model of the CTRW. Unless indicated otherwise, I will always refer to the leaper model in this work.

For the notion of the scaling limit, it is necessary to have a description of continuous-time random walks on a continuous operational time $s \in \mathbb{R}_{\geq 0}$. The first approach introduced by Fogedby [Fog94] who used two independent stochastic differential equations

$$\begin{aligned} \frac{dY}{ds} &= \xi(s) \\ \frac{dT}{ds} &= \eta(s) \end{aligned} \tag{2.94}$$

with $\xi(s)$ and $\eta(s) \geq 0$ being two noise sources. The processes $Y(s)$ and $T(s)$ are Lévy processes. These two equations were combined in a manner similar to (2.85) to give a process $X(t)$. This has been studied extensively (e.g., [BF05, BF07a, KF07]).

The general form (for the leaper model) was then introduced by Becker-Kern, Benson, Meerschaert and Scheffler [BKMS04, MBSBK02]. They consider the two-dimensional Lévy process

$$\begin{pmatrix} Y(s) \\ T(s) \end{pmatrix}. \tag{2.95}$$

(In their work, they allow $y(s)$ to be multidimensional. However, I will restrict myself to the one-dimensional case.) Similar to the situation in section 2.2, one has a *log-Fourier-Laplace transform* $\omega(k, \lambda)$ such that

$$\langle e^{-\lambda T(s) + ikY(s)} \rangle = e^{-s\omega(k,\lambda)}. \tag{2.96}$$

This log-Fourier-Laplace transform can also be decomposed into drift, normal and jump contributions [[BKMS04], lemma 2.1]. Here, I use a slightly different but equivalent approach.

The approach by Fogedby [Fog94] can be extended by introducing a coupled noise via

$$\begin{aligned} \frac{dY}{ds} &= \xi_{\mathrm{uc}}(s) + \xi_{\mathrm{c}}(s) \\ \frac{dT}{ds} &= \eta(s). \end{aligned} \tag{2.97}$$

The term $\xi_{\mathrm{uc}}(s)$ takes here the function of the uncoupled noise (it corresponds to $\xi(s)$ in equation (2.94)). The term $\xi_{\mathrm{c}}(s)$ corresponds to a noise source which is coupled to $\eta(s)$ but is independent of $\xi_{\mathrm{uc}}(s)$. Correspondingly, we can decompose

$$Y(s) = Y_{\mathrm{uc}}(s) + Y_{\mathrm{c}}(s) \tag{2.98}$$

with

$$\frac{dY_{\mathrm{uc}}}{ds} = \xi_{\mathrm{uc}}(s) \quad \text{and} \quad \frac{dY_{\mathrm{c}}}{ds} = \xi_{\mathrm{c}}(s). \tag{2.99}$$

The process $Y_{\mathrm{uc}}(s)$ is again a Lévy process with a log-Fourier transform $\theta(k)$

$$\langle e^{ikY_{\mathrm{uc}}(s)} \rangle = e^{-s\theta(k)}. \tag{2.100}$$

The process $T(s)$ is an increasing Lévy process with jump density κ and log-Laplace transform $\omega(\lambda)$ (see equation (2.68))

$$\begin{aligned} \langle e^{-\lambda T(s)} \rangle &= e^{-s\omega'(\lambda)} \\ \omega'(\lambda) &= m\lambda + \int (1 - e^{-\lambda t})\, \kappa(dt). \end{aligned} \tag{2.101}$$

For each realization we have a counting measure $\nu(dt, ds)$ such that

$$T(s) = \int_{\tau \in [0,s]} t\, \nu(dt, d\tau). \tag{2.102}$$

As described in section 2.2 the distribution of ν is determined by κ. The coupling between $Y_c(s)$ and $T(s)$ is introduced as follows: for each jump of $T(s)$ with size τ there is a jump in $Y_c(s)$ which follows a probability distribution η_τ (this corresponds to the notion of a *ν-randomization* in [[Kal01], page 226]). Using essentially the same argument leading to equation (2.51) one obtains

$$\langle e^{-\lambda T(s) + ikY_c(s)} \rangle = \exp\left(-sm\lambda - s \int (1 - e^{-\lambda t + ikx})\, \eta_t(dx)\, \kappa(dt) \right). \tag{2.103}$$

The general case needs again the introduction of a regulization function $\Xi(x)$ in x such that the complete log-Fourier-Laplace transform reads (including the uncoupled contribution)

$$\begin{aligned} \langle e^{-\lambda T(s) + ikY(s)} \rangle &= e^{-s\omega(k,\lambda)} \\ \omega(k, \lambda) &= m\lambda + \theta(k) + \int (1 - e^{-\lambda t + ikx} + ikx\Xi(x))\, \eta_t(dx)\, \kappa(dt). \end{aligned} \tag{2.104}$$

The function $\Xi(x)$ can be chosen as in the one-dimensional case. This form is equivalent to [[BKMS04], lemma 2.1] (From [[Bau96], theorem 44.3] follows that the factorization of the measure is always possible). The condition on the measures η_t and κ is

$$\int \min(x^2 + t, 1)\, \eta_t(dx)\, \kappa(dt) < \infty. \tag{2.105}$$

Let us return one moment to the original definition of the CTRW with discrete steps (equation (2.84)). In [BKMS04] the authors look at the scaling limit of the process in two dimensions where different scalings are allowed for $T(s)$ and $Y(s)$

$$\begin{pmatrix} Y^{\lim}(s) \\ T^{\lim}(s) \end{pmatrix} = \lim_{\zeta \searrow 0} \begin{pmatrix} r_Y(\zeta)(Y(\zeta^{-1}s) - b(\zeta)\zeta^{-1}s) \\ r_T(\zeta) T(\zeta^{-1}s) \end{pmatrix}. \tag{2.106}$$

Again, I focus on the case of the domain of normal attraction, i.e., I put $r_X(\zeta) = \zeta^{\frac{\beta}{\alpha}}$, $r_T(\zeta) = \zeta^{\frac{1}{\alpha}}$ and $b(\zeta) = b$. The limit process is given in the density formalism with the log-Fourier-Laplace transform $\omega^{\lim}(k, \lambda)$

$$\langle e^{-\lambda T^{\lim}(s) + ikY^{\lim}(s)} \rangle = e^{-s\omega^{\lim}(k,\lambda)}. \tag{2.107}$$

By using [[BKMS04], theorem 2.2], this log-Fourier-Laplace transform can be determined by (with the jump distribution $\psi(x, t)$ and arguing in the spirit of [[Fel71], theorem XVII.1.1])

$$\omega^{\lim}(k, \lambda) = \lim_{\zeta \searrow 0} \zeta^{-\alpha}(1 - \psi(\zeta^\beta k, \zeta\lambda) + i\zeta^\beta kb). \tag{2.108}$$

We will see in chapter 4 that for infinite mean waiting time, physically one can always set $b = 0$ (because only $X^{\lim}(t)$ is relevant in a physical setting). The resulting $\omega^{\lim}(k, \lambda)$ is scale invariant, i.e.,

$$\omega^{\lim}(\zeta^\beta k, \zeta\lambda) = \zeta^\alpha \omega^{\lim}(k, \lambda) \quad \text{for all } \zeta > 0. \tag{2.109}$$

This scaling behavior with different exponents is a special case of *operator scaling* laws (see, e.g., [MBB01, MS01]). The most general form with infinite mean waiting time (i.e., $0 < \alpha < 1$) is [[BKMS04], theorem 2.2]

$$\omega^{\lim}(k, \lambda) = \theta(k) + ik\mu + c \int (1 - e^{-\lambda t + ikx} + ikx\Xi(x))\, \eta^{\lim}(t^{-\beta} dx)\, \frac{dt}{t^{1+\alpha}} \tag{2.110}$$

where $c > 0$, and μ depends on the regularization function $\Xi(x)$ by the requirement (2.109) (e.g., for $\Xi(x) = 0$ or $\Xi(x) = 1$ we have $\mu = 0$). In general, β needs to fulfill $\beta \geq \frac{\alpha}{2}$ and $\theta(k)$ is the log-Fourier transform of a strictly stable distribution with characteristic exponent $\frac{\alpha}{\beta}$ (equation (2.20))

$$\theta(k) = \sigma^{\frac{\alpha}{\beta}} |k|^{\frac{\alpha}{\beta}} \left(1 - i\gamma \operatorname{sgn}(k) \tan\left(\frac{\pi\alpha}{2\beta}\right)\right) \quad \text{with} \quad -1 \leq \gamma \leq 1. \tag{2.111}$$

For a strictly stable distribution, the case $\alpha = \beta$ implies $\gamma = 0$ (i.e., no logarithmic term). The term $\eta^{\lim}$ describes an arbitrary probability distribution which leads to an $\omega^{\lim}(k, \lambda)$ fulfilling condition (2.105). For $\beta = \frac{\alpha}{2}$ this enforces $\eta^{\lim}(dx) = \delta(x)\, dx$, therefore this case is only possible by an uncoupled waiting time and step size.

Assume $\beta > \frac{\alpha}{2}$. If $\eta^{\lim}$ has a second moment σ^2, then condition (2.105) is fulfilled. More generally, for a probability distribution $\eta^{\lim}$ with a regularly varying tail, the application of [[Fel71], theorem VIII.9.2] gives: it is necessary for condition (2.105) that $\eta^{\lim}$ is in the domain of attraction of a stable distribution with characteristic exponent $\geq \frac{\alpha}{\beta}$; the property of $\eta^{\lim}$ being in the domain of attraction of a stable distribution with characteristic exponent $> \frac{\alpha}{\beta}$ is sufficient for condition (2.105).

2.4. The Manneville-Pomeau Map

Manneville and Pomeau [MP79, PM80] introduced a one-dimensional map to illustrate intermittent behavior in the Lorenz system. Today, one considers a one-dimensional map $f(x)$ of an interval to itself to be of Manneville-Pomeau type, if it has an indifferent instable fixed point and is expanding otherwise. This indifferent unstable fixed point is of the form (if located at the origin)

$$f(x) = x + ax^z + o(x^z) \quad \text{for } x \to 0 \tag{2.112}$$

with $z > 1$ (the original version was with $z = 2$). (A set of suitable conditions on the map away from the fixed point can be found in [Aar97, Tha01].) A typical example is plotted in figure 2.1.

This map defines a dynamic system by the recursion relation

$$x_{n+1} = f(x_n). \tag{2.113}$$

This type has become a standard example to study intermittent behavior and anomalous diffusion (e.g., [GW88, TG02, GT84, ZK93b, ZK93a, KCK$^+$05, KKC$^+$07, AC03]). They

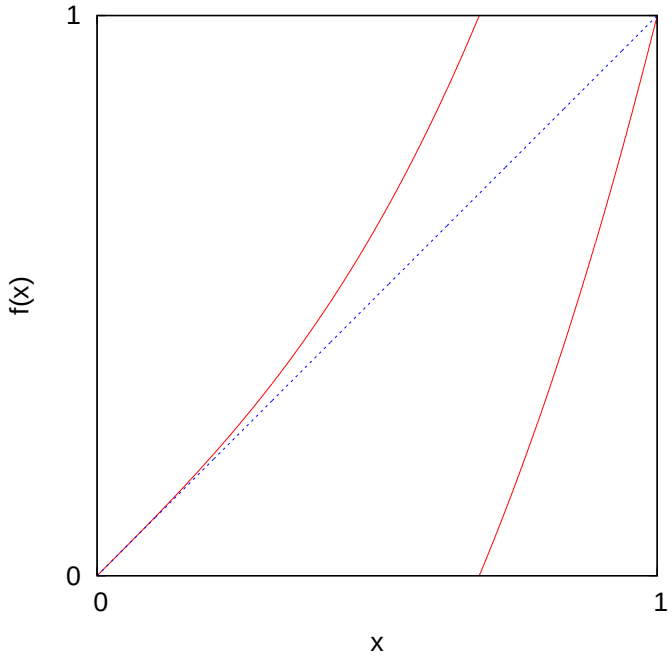

Figure 2.1.: Example of an map of Manneville-Pomeau type

are also used as examples for a non-exponential decay of correlations (e.g., [[Bal00], section 3.5], [LSV00, LSV93]).

In general, these maps admit an invariant measure μ which is absolutely continuous with respect to the Lebesgue measure, i.e., $\mu(dx) = g(x)\,dx$. These maps show the interesting property that the invariant measure μ forms a singularity at the origin of the form

$$g(x) \sim \frac{1}{x^{1-z}} \text{ as } x \searrow 0. \tag{2.114}$$

For $z \geq 2$ this singularity leads to an invariant measure which is not normalizable (i.e., it is infinite) [Tha83, Tha95]. One can proof that this measure is still ergodic if one adopts the definition of ergodicity in the sense that for any invariant set A either the set A or its complement A^{c} has zero measure. Not all properties of standard ergodic theory translate into this new setting, e.g., the Birkhoff ergodic theorem does not hold for an infinite ergodic measure. Therefore new phenomena arise such as the fact that the time average stays a probability distribution in the limit and does not converge almost surely to the ergodic average. Depending on the point of view, this a called *distributional limit theorem* in the mathematical literature [Tha01] or *weak ergodicity breaking* in the physics literature [Bou92]. In mathematics this line of research is known under the name of *infinite ergodic theory*. I refer to the standard text book by Aaronson [Aar97] for details and references to the original mathematical paper, and to the very readable introduction by Thaler [Tha01].

An important concept for the treatment of these system is the method of *inducing*. To ease the notation assume that f is a map $f : [0,1] \to [0,1]$ with one indifferent fixed point at $x = 0$ which behaves as equation (2.112). A suitable set of technical conditions for the structure of the map is given in, e.g., [Tha01]. For any given $\epsilon > 0$ one can define a first

return map

$$\phi(x) = \min\{n \in \mathbb{N} : f^n(x) \geq \epsilon\} \tag{2.115}$$

where $f^n(x)$ is the n-times iteration of $f(x)$. This defines an *induced map*

$$f_{\text{ind}} : [\epsilon, 1] \to [\epsilon, 1], \quad f_{\text{ind}}(x) = f^{\phi(x)}(x). \tag{2.116}$$

This induced map $f_{\text{ind}}(x)$ admits an invariant measure $\mu\big|_{[\epsilon,1]}$ by restricting μ to $[\epsilon, 1]$. This restricted measure is finite and $f_{\text{ind}}(x)$ is ergodic with respect to this measure [[Aar97], proposition 1.5.2]. Therefore the normal ergodic theory can be applied and we can calculate averages with respect to this measure. The expectation value of the first return map is infinite when the original measure μ is infinite [[Aar97], theorem (Kac's formula) 1.5.5], i.e., the non-existence of a mean return time is deeply linked with the non-normalizable ergodic measure of the original map $f(x)$.

The distribution of the values of $\phi(x)$ where x is a random variable drawn from the ergodic measure

$$\mu([\epsilon, 1])^{-1}\mu\big|_{[\epsilon,1]}(dx) = \frac{g(x)}{\int_{[\epsilon,1]} g(y)\, dy}\, dx \tag{2.117}$$

is important as it reflects how long the dynamical system (2.113) stays in the vicinity of the indifferent fixed point. The tail behavior of $\phi(x)$ is interesting for asymptotic considerations as it determines the characteristic exponent α of the one-sided Lévy-stable distribution it converges to. Following equation (2.22) α is defined by the condition (for the domain of normal attraction)

$$\lim_{t\to\infty} t^\alpha \mu(\phi^{-1}([t, \infty[)) \in \mathbb{R}_{>0}. \tag{2.118}$$

There are essentially two ways to determine the tail behavior. One is to approximate the dynamics x_n near the origin by a differential equation [HHS82]

$$\frac{dx}{dn} \simeq ax^z. \tag{2.119}$$

Alternatively, one can use the renormalization group formalism [HNS82, HR82]. The renormalization operator T has the form

$$T\{f(x)\} = \rho f(f(\frac{x}{\rho})). \tag{2.120}$$

Using equation (2.112) and expanding $T\{f(x)\}$ gives

$$T\{f(x)\} = x + 2\rho^{1-z}ax^z + o(x^z). \tag{2.121}$$

To stay consistent with condition (2.112) one has to choose ρ as (see [[HR82], equation (10)])

$$\rho = 2^{\frac{1}{z-1}}. \tag{2.122}$$

It can be shown that the operator T has a fixed point which is attracting [HNS82, HR82]. Therefore, this fixed point describes the behavior of $f(x)$ as the number of iterations goes to infinity. Hu and Rudnick were able to construct this attracting fixed point $T\{f_*(x)\} = f_*(x)$ [[HR82], equation (12)]:

$$f_*(x) = \left[x^{-(z-1)} - a(z-1)\right]^{-\frac{1}{z-1}}. \tag{2.123}$$

This formula has the advantage that its iterates can be given explicitely

$$f_*^n(x) = \left[x^{-(z-1)} - na(z-1)\right]^{-\frac{1}{z-1}}. \tag{2.124}$$

For points which stay many iterations in the domain $[0, \epsilon[$ one can approximate $f(x)$ by $f_*(x)$. Therefore, for large t and $x \in [\epsilon, 1]$ (and ϵ small enough)

$$\begin{aligned} \phi(x) \geq t \quad &\Leftrightarrow \quad f(x) < \epsilon \text{ and } \left[f(x)^{-(z-1)} - (t-2)a(z-1)\right]^{-\frac{1}{z-1}} < \epsilon \\ &\Leftrightarrow \quad f(x) < \left[\epsilon^{-(z-1)} + (t-2)a(z-1)\right]^{-\frac{1}{z-1}}. \end{aligned} \tag{2.125}$$

If the insertion near $x \simeq 0$ is approximately linear, i.e., $\mu(f^{-1}([0,\delta]) \cap [\epsilon, 1]) \propto \delta$ as $\delta \searrow 0$, condition (2.118) holds with the parameter

$$\alpha = \frac{1}{z-1}. \tag{2.126}$$

In other words, the distribution of $\phi(x)$ is in the domain of normal attraction of a one-sided stable distribution with characteristic exponent $\alpha = \frac{1}{z-1}$ [[BMPR$^+$85], equation (3.13)].

3. Multi point properties of continuous-time random walks

3.1. Introduction

In general, a CTRW is not Markovian in real time. Therefore the joint probability distributions do not factorize. We need to determine all joint probability distributions to fully characterize the process [ŠM05, BM04]. The Montroll-Weiss equation [MW65] gives the Fourier-Laplace transform of the one-point distribution. This equation is one of the corner stones of the subsequent application. Barkai and Sokolov extended it to a description of the two-point distributions [BS07]. Another extension to multi point correlation by Baule and Friedrich [BF05, BF07b] starts from the description by Fogedby [Fog94] with two independent Langevin equations in the scaling limit. Their method is based on the fact that the multi-point correlations of Brownian motion are polynomials in the observation times (these are operational times). The correlations of the CTRW can then be determined by knowing the moments of the inverted time process. Unfortunately, this method relies substantially on the independence of the two processes and on the Gaussianity of the spatial process and does not provide a generalization to other processes. Šanda and Mukamel [ŠM05] look at a CTRW on a spatial lattice where the transition probability to another lattice site depends on the current position while the spatial motion is independent of the waiting times. They are mainly considering the case of an external potential such that the transition matrix has a stationary ensemble while aforementioned papers and I consider the situation that temporal and spatial step size is independent of the current position.

In this chapter I want to present a different method which is essentially an extension of the argument of Montroll and Weiss and which allows to directly write down the joint probability distribution of a possibly space-time coupled CTRW in Fourier-Laplace space. It is a diagrammatic method, i.e., for each joint probability distribution, one has to write a number of diagrams and can associate terms with the different parts of the diagram. This chapter is focused on the basic method using a CTRW with discrete jumps with densities $\psi(x,t)$ and independence between successive jumps. In later chapters this will be extended to accommodate the asymptotic density formulation and correlations between successive jumps. Additionally, I introduce a method to determine the Laplace transforms of the multi point correlations without having to determine the joint probability distributions first which can become quite complicated when considering several points.

In this paragraph, I want to introduce some notation. I use the standard convention of writing the probability distribution of the waiting time/step size as density $\psi(x,t)$. The usage of a probability measure does not change the arguments. I am assuming here the leaper [Hug95] type of CTRW, i.e., the spatial movement is done in one leap after waiting while the walker rests during the waiting time, the adaptation to other models is exemplified in section 3.4.1 on the creeper [Hug95] model. I denote the distribution of the waiting time by $\phi(t) = \int dx\, \psi(x,t)$. The Fourier- or Laplace transform of a function is distinguished by

the naming of the arguments, i.e., the Fourier-Laplace transform of $\psi(x,t)$ is written as

$$\psi(k,\lambda) = \int dt \int dx\, e^{-\lambda t + ikx} \psi(x,t). \tag{3.1}$$

The n-point joint probability density function will be referred to as $p_n(x_1,\ldots,x_n,t_1,\ldots,t_n)$ or short as $p_n(\mathbf{x},\mathbf{t})$. The Fourier-Laplace transform is then

$$p_n(\mathbf{k},\boldsymbol{\lambda}) = \int d^n t \int d^n x\, e^{-\boldsymbol{\lambda}\cdot\mathbf{t} + i\mathbf{k}\cdot\mathbf{x}} p_n(\mathbf{x},\mathbf{t}) \tag{3.2}$$

with $\boldsymbol{\lambda}\cdot\mathbf{t} = \lambda_1 t_1 + \cdots + \lambda_n t_n$ and $\mathbf{k}\cdot\mathbf{x} = k_1 x_1 + \cdots + k_n x_n$. Section 3.2 is mainly concerned with the determination of this Fourier-Laplace transform. Especially for the leaper model, it will be handy to have the following definition ready

$$\breve{p}_n(\mathbf{k},\boldsymbol{\lambda}) = \lambda_1 \cdots \lambda_n p_n(\mathbf{k},\boldsymbol{\lambda}) \tag{3.3}$$

which implies the following relation in the untransformed variables:

$$p_n(\mathbf{x},\mathbf{t}) = \int_{0-}^{t_1} dt_1' \cdots \int_{0-}^{t_n} dt_n'\, \breve{p}_n(\mathbf{x},\mathbf{t}'). \tag{3.4}$$

3.2. Joint Probability Distributions

First, I start with splitting up the probability $p_n(\mathbf{x},\mathbf{t})$ into the contributions of the different steps. For non negative integers $q_1,\ldots,q_n$ I call $p_n[q_1,\ldots,q_n](\mathbf{x},\mathbf{t}) = p_n[\mathbf{q}](\mathbf{x},\mathbf{t})$ the density $p_n(\mathbf{x},\mathbf{t})$ intersected with the event that t_i is in the waiting time of the (q_i+1)th step (for $i = 1,\ldots,n$). Therefore one gets

$$p_n(\mathbf{x},\mathbf{t}) = \sum_{q_1,\ldots,q_n=0}^{\infty} p_n[\mathbf{q}](\mathbf{x},\mathbf{t}). \tag{3.5}$$

Now, this probability distribution can be factorized into terms describing only one step. I denote by χ_i the random variable describing the step size of the ith step and τ_i the corresponding waiting time. The probability density of the pair (χ_i,τ_i) is given by $\psi(x,t)$ and all (χ_i,τ_i) $(i = 1,\ldots)$ are independent random variables. With this notation one can write[1]

$$\begin{aligned} p_n[\mathbf{q}](\mathbf{x},\mathbf{t}) = \Big\langle \prod_{i=1}^{n} & \delta\left(x_i - \chi_1 - \cdots - \chi_{q_i-1}\right) \\ & \times \theta\left(t_i - \tau_1 - \cdots - \tau_{q_i-1}\right) \\ & \times \theta\left(\tau_1 + \cdots + \tau_{q_i} - t_i\right) \Big\rangle, \end{aligned} \tag{3.6}$$

[1] Being more precise, the two Heaviside θ-functions in equation (3.6) should be replaced with a function which is one for $\tau_1+\cdots+\tau_{q_i-1} \le t_i < \tau_1+\cdots+\tau_{q_i}$ and zero otherwise. This would only change countably many points and there will not be any difference in the Fourier-Laplace transform.

where δ is the Dirac delta function, θ the Heaviside step function and $\langle \dots \rangle$ denotes the expectation value. Now, I introduce the auxiliary function ($j = 1, \dots$)

$$\begin{aligned}\Delta_n^{(j)}[\mathbf{q}](\mathbf{x},\mathbf{t}) = \prod_{i:q_i<j} \delta(x_i)\delta(t_i) \prod_{i:q_i\geq j} & \delta\left(x_i - \chi_j - \cdots - \chi_{q_i-1}\right) \\ & \times \theta\left(t_i - \tau_j - \cdots - \tau_{q_i-1}\right) \\ & \times \theta\left(\tau_j + \cdots + \tau_{q_i} - t_i\right).\end{aligned} \tag{3.7}$$

For $j = 1$ we regain equation (3.6) by taking the expectation value

$$p_n[\mathbf{q}](\mathbf{x},\mathbf{t}) = \langle \Delta_n^{(1)}[\mathbf{q}](\mathbf{x},\mathbf{t}) \rangle \tag{3.8}$$

while for j larger than every q_i, $\Delta_n^{(j)}[\mathbf{q}](\mathbf{x},\mathbf{t})$ collapses to

$$\Delta_n^{(j)}[\mathbf{q}](\mathbf{x},\mathbf{t}) = \prod_{i=1}^{n} \delta(x_i)\delta(t_i) \tag{3.9}$$

and does not depend any more on the (χ_i, τ_i). In general $\Delta_n^{(j)}[\mathbf{q}](\mathbf{x},\mathbf{t})$ does only depend on the (χ_i, τ_i) with $i \geq j$.

The idea is now to find a function $\eta_n^{(j)}[\mathbf{q}](\mathbf{x},\mathbf{t})$ which contains only the pair (χ_j, τ_j) of random variables and fulfills the recursion relation

$$\Delta_n^{(j)}[\mathbf{q}](\mathbf{x},\mathbf{t}) = \eta_n^{(j)}[\mathbf{q}](\mathbf{x},\mathbf{t}) \star \Delta_n^{(j+1)}[\mathbf{q}](\mathbf{x},\mathbf{t}) \tag{3.10}$$

where $\star$ denotes the convolution with respect to $x_1, \dots, x_n$ and $t_1, \dots, t_n$ (a Fourier convolution for the x_i and a Laplace convolution for the t_i, i.e. we always assume $t_i \geq 0$). The following function will do as can easily be checked by inserting it in equation (3.10)

$$\begin{aligned}\eta_n^{(j)}[\mathbf{q}](\mathbf{x},\mathbf{t}) = & \prod_{i:q_i=j} \delta(x_i)\theta(\tau_j - t_i) \\ & \times \prod_{i:q_i<j} \delta(x_i)\delta(t_i) \\ & \times \prod_{i:q_i>j} \delta(x_i - \chi_j)\delta(t_i - \tau_j).\end{aligned} \tag{3.11}$$

Putting together equations (3.8) and (3.10), we get

$$p_n[\mathbf{q}](\mathbf{x},\mathbf{t}) = \Big\langle \bigstar_{j=1}^{m} \eta_n^{(j)}[\mathbf{q}](\mathbf{x},\mathbf{t}) \Big\rangle, \tag{3.12}$$

where m is any natural number greater than every q_i. In Fourier-Laplace space, this reads

$$\begin{aligned}p_n[\mathbf{q}](\mathbf{k},\boldsymbol{\lambda}) &= \left\langle \prod_{j=1}^{\infty} \eta_n^{(j)}[\mathbf{q}](\mathbf{k},\boldsymbol{\lambda}) \right\rangle \\ &= \prod_{j=1}^{\infty} \left\langle \eta_n^{(j)}[\mathbf{q}](\mathbf{k},\boldsymbol{\lambda}) \right\rangle\end{aligned} \tag{3.13}$$

where I have used the fact that the (χ_i, τ_i) are independent and the $\eta_n^{(j)}[\mathbf{q}](\mathbf{k}, \boldsymbol{\lambda})$ depend only on the random variable (χ_j, τ_j). Since $\eta_n^{(m)}[\mathbf{q}](\mathbf{k}, \boldsymbol{\lambda}) = 1$ for every m greater than every q_i we can use any of these m as the upper limit for j in the product. The function $\eta_n^{(j)}[\mathbf{q}](\mathbf{k}, \boldsymbol{\lambda})$ can be interpreted as the contribution of the jth step. As a side note, equation (3.11) can easily be adapted to other models. I exemplify this for the creeper model in section 3.4.1.

Before I proceed to use equation (3.13) to get an explicit expression for $p_n[\mathbf{q}](\mathbf{k}, \boldsymbol{\lambda})$, I take a step back to equation (3.5). Though one can calculate $p_n[\mathbf{q}](\mathbf{k}, \boldsymbol{\lambda})$ one still has to sum over infinitely many of these terms. In the proceeding paragraphs I split this sum (equation (3.5)) into finitely many subsums which are easy to calculate. The idea is to group the "partial" probabilities (i.e., the $p_n[\mathbf{q}](\mathbf{x}, \mathbf{t})$) according to the relative ordering of the steps. E.g., for two coefficients we get the orderings $q_1 < q_2$, $q_1 = q_2$ and $q_2 < q_1$; for three we get $q_1 < q_2 < q_3$, $q_1 = q_2 < q_3$, $q_1 < q_2 = q_3$ and $q_1 = q_2 = q_3$ plus the permutations which result in a different relation (in total 13). I extend the notation $p_n[\dots](\mathbf{x}, \mathbf{t})$ such that when I write the relation instead of the coefficients between the square brackets, I mean that we sum over all q_i which fulfill this relation. In other words, we intersect the probability with the event that this relation is fulfilled. E.g.,

$$p_2[q_1 < q_2](\mathbf{x}, \mathbf{t}) = \sum_{\substack{q_1, q_2 = 0 \\ q_1 < q_2}}^{\infty} p_2[q_1, q_2](\mathbf{x}, \mathbf{t}) \tag{3.14}$$

and

$$p_2[q_1 = q_2](\mathbf{x}, \mathbf{t}) = \sum_{q_1 = 0}^{\infty} p_2[q_1, q_1](\mathbf{x}, \mathbf{t}). \tag{3.15}$$

This splits the sum into finitely many different parts. As an example $p_2(\mathbf{x}, \mathbf{t})$ can be written as

$$p_2(\mathbf{x}, \mathbf{t}) = p_2[q_1 < q_2](\mathbf{x}, \mathbf{t}) + p_2[q_1 = q_2](\mathbf{x}, \mathbf{t}) + p_2[q_2 < q_1](\mathbf{x}, \mathbf{t}), \tag{3.16}$$

respectively for $p_2(\mathbf{k}, \boldsymbol{\lambda})$

$$p_2(\mathbf{k}, \boldsymbol{\lambda}) = p_2[q_1 < q_2](\mathbf{k}, \boldsymbol{\lambda}) + p_2[q_1 = q_2](\mathbf{k}, \boldsymbol{\lambda}) + p_2[q_2 < q_1](\mathbf{k}, \boldsymbol{\lambda}). \tag{3.17}$$

For each of these contributions, the sum can easily be evaluated. An alternative way to represent these relative orderings is the use of diagrams. One example for a five point function can be seen in figure 3.1a. The number of steps increases from left to right. A vertex corresponds to a single step in whose waiting time the time parameters of the indices indicated by the outgoing arrows lie. A horizontal line corresponds to any finite number of steps (including zero) taken by the walker without having any time parameter in the step. The example represents $q_1 = q_4 < q_3 < q_2 = q_5$, or using the terminology of the continuous-time random walk, that t_1 and t_4 are in the same step, t_3 in some later while t_2 and t_5 are together again in an even later step. We do not yet impose a corresponding ordering on the times $t_1, \dots, t_5$ as this will lead to more complicated terms. Later, a more efficient way of using the symmetry under exchange of indices will be introduced.

Now, fix a diagram (respectively a given ordering). I start with a vertex which corresponds always to a single step of the continuous-time random walk. While in this general frame

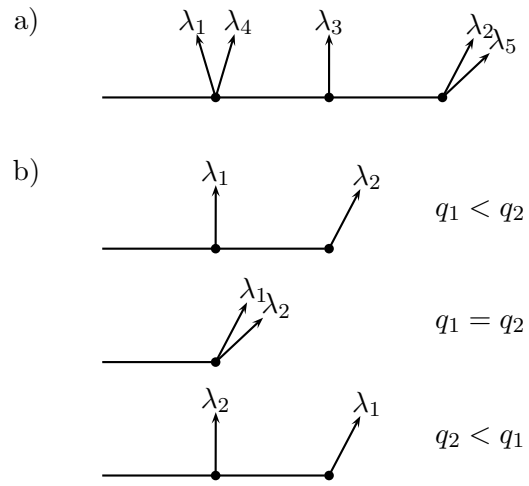

Figure 3.1.: (a) Example of a diagram appearing for a five point density. Explanations are given in the text. (b) The three diagrams for the two point density. The corresponding relations of the step numbers are given in the right column.

the exact step number j of the vertex in a diagram can vary, the three types of indices in equation (3.11) are always the same for a given ordering. I use the following notation: the set of indices belonging to the times in a later step than the current one will be denoted by $\mathcal{L}$ ("later", $\mathcal{L} = \{i : q_i > j\}$). The set of of indices belonging to the times in an earlier step will be denoted by $\mathcal{E}$ ("earlier", $\mathcal{E} = \{i : q_i < j\}$). Finally, the set of indices belonging to the times which lie in the waiting time of the current time step will be denoted by $\mathcal{V}$ ("vertex", $\mathcal{V} = \{i : q_i = j\}$). In the example figure 3.1a we would have for the first vertex $\mathcal{E} = \{\}$, $\mathcal{V} = \{1, 4\}$ and $\mathcal{L} = \{2, 3, 5\}$; for the second vertex $\mathcal{E} = \{1, 4\}$, $\mathcal{V} = \{3\}$ and $\mathcal{L} = \{2, 5\}$; and for the last $\mathcal{E} = \{1, 3, 4\}$, $\mathcal{V} = \{2, 5\}$ and $\mathcal{L} = \{\}$. For working with sets of indices the following definitions are helpful

$$\begin{aligned} \Lambda_{\mathcal{I}} &= \sum_{i \in \mathcal{I}} \lambda_i \text{ and} \\ K_{\mathcal{I}} &= \sum_{i \in \mathcal{I}} k_i. \end{aligned} \tag{3.18}$$

With this notation we have for a vertex

$$\begin{aligned} \left\langle \eta_n^{(j)}[\mathbf{q}](\mathbf{x}, \mathbf{t}) \right\rangle = \int d\chi \int d\tau \, \psi(\chi, \tau) \prod_{e \in \mathcal{E}} \delta(x_e)\delta(t_e) \prod_{v \in \mathcal{V}} \delta(x_v)\theta(\tau - t_v) \\ \times \prod_{l \in \mathcal{L}} \delta(x_l - \chi)\delta(t_l - \tau). \end{aligned} \tag{3.19}$$

which is easy to calculate in Fourier-Laplace space:

$$\begin{aligned}\rho_{\text{vertex}}(\mathbf{k},\boldsymbol{\lambda}) &= \left\langle \eta_n^{(j)}[\mathbf{q}](\mathbf{k},\boldsymbol{\lambda}) \right\rangle \\ &= \frac{1}{\prod_{v\in\mathcal{V}}\lambda_v}\int d\chi \int d\tau\, \psi(\chi,\tau) e^{-\Lambda_{\mathcal{L}}\tau + iK_{\mathcal{L}}\chi} \prod_{v\in\mathcal{V}}\left(1-e^{-\lambda_v\tau}\right) \\ &= \frac{1}{\prod_{v\in\mathcal{V}}\lambda_v}\sum_{\mathcal{J}\in\mathcal{P}(\mathcal{V})}(-1)^{|\mathcal{J}|}\psi(K_{\mathcal{L}},\Lambda_{\mathcal{L}}+\Lambda_{\mathcal{J}})\end{aligned} \tag{3.20}$$

where $\mathcal{P}(\mathcal{V})$ is the power set of $\mathcal{V}$ and $|\mathcal{J}|$ denotes the number of elements in $\mathcal{J}$ (the cardinality). The sum in the last line has in the λ-argument of ψ the term $\Lambda_{\mathcal{L}}$ plus every combination of λ_j with $j\in\mathcal{J}$ with a positive sign in front of ψ if it is an even number of elements and a negative sign if it is an odd number.

Now, I proceed to calculate the contribution of a horizontal line. For a given horizontal line all steps have the same types of indices in the sets $\mathcal{E}$ and $\mathcal{L}$ while always $\mathcal{V}=\{\}$. Looking again at the example figure 3.1a: the line before the first vertex has $\mathcal{E}=\{\}$ and $\mathcal{L}=\{1,2,3,4,5\}$, the line between the first and second vertex $\mathcal{E}=\{1,4\}$ and $\mathcal{L}=\{2,3,5\}$ while the last line between the second and third vertex has $\mathcal{E}=\{1,3,4\}$ and $\mathcal{L}=\{2,5\}$. By using $\mathcal{V}=\{\}$ in equation (3.20), we see that the contribution of a single step inside a horizontal line in the diagram is $\psi(K_{\mathcal{L}},\Lambda_{\mathcal{L}})$. When we calculate the probability distribution corresponding to a given diagram (respectively ordering), every nonnegative number of steps in a horizontal line is possible and we have to add them up. This gives the contribution of a horizontal line:

$$\begin{aligned}\rho_{\text{line}}(\mathbf{k},\boldsymbol{\lambda}) &= \sum_{r=0}^{\infty}\psi^r(K_{\mathcal{L}},\Lambda_{\mathcal{L}}) \\ &= \frac{1}{1-\psi(K_{\mathcal{L}},\Lambda_{\mathcal{L}})}.\end{aligned} \tag{3.21}$$

Now we can put the contributions together. When summing the $p_n[\mathbf{q}](\mathbf{k},\boldsymbol{\lambda})$ belonging to a fixed diagram, we only have to sum over all possible numbers of steps in the horizontal lines. Comparing with equation (3.13) one sees that one gets this sum by multiplying the contribution equation (3.20) for each vertex and equation (3.21) for each line in the diagram. This result can alternatively be derived by using a renewal equation [Fel71]. I show this approach in appendix A.

The calculation of a joint probability distribution $p_n(\mathbf{k},\boldsymbol{\lambda})$ in Fourier-Laplace space of a continuous-time random walk reduces therefore to the following steps: draw all possible diagrams with n indices (or equivalently, determine all possible orderings of the n indices). For each diagram multiply the finitely many contributions given by equations (3.20) and (3.21). Adding up these products gives $p_n(\mathbf{k},\boldsymbol{\lambda})$.

The simplest case is of course the one point density which only has one diagram with one line and one vertex. Using the described procedure directly gives the Montroll-Weiss equation [MW65]

$$p_1(k,\lambda) = \frac{1}{\lambda}\frac{1-\phi(\lambda)}{1-\psi(k,\lambda)}. \tag{3.22}$$

The next example is the two point density for which the corresponding diagrams are

shown in figure 3.1b

$$\begin{aligned}
p_2[q_1 < q_2](\mathbf{k},\boldsymbol{\lambda}) =& \rho_{\text{line},\mathcal{L}=\{1,2\}}(\mathbf{k},\boldsymbol{\lambda})\rho_{\text{vertex},\mathcal{V}=\{1\},\mathcal{L}=\{2\}}(\mathbf{k},\boldsymbol{\lambda}) \\
&\times \rho_{\text{line},\mathcal{L}=\{2\}}(\mathbf{k},\boldsymbol{\lambda})\rho_{\text{vertex},\mathcal{V}=\{2\},\mathcal{L}=\{\}}(\mathbf{k},\boldsymbol{\lambda}) \\
=& \frac{1}{\lambda_1}\frac{\psi(k_2,\lambda_2)-\psi(k_2,\lambda_1+\lambda_2)}{1-\psi(k_1+k_2,\lambda_1+\lambda_2)}\frac{1}{\lambda_2}\frac{1-\phi(\lambda_2)}{1-\psi(k_2,\lambda_2)} \\
p_2[q_1 = q_2](\mathbf{k},\boldsymbol{\lambda}) =& \rho_{\text{line},\mathcal{L}=\{1,2\}}(\mathbf{k},\boldsymbol{\lambda})\rho_{\text{vertex},\mathcal{V}=\{1,2\},\mathcal{L}=\{\}}(\mathbf{k},\boldsymbol{\lambda}) \\
=& \frac{1}{\lambda_1\lambda_2}\frac{1-\phi(\lambda_1)-\phi(\lambda_2)+\phi(\lambda_1+\lambda_2)}{1-\psi(k_1+k_2,\lambda_1+\lambda_2)} \\
p_2[q_2 < q_1](\mathbf{k},\boldsymbol{\lambda}) =& \rho_{\text{line},\mathcal{L}=\{1,2\}}(\mathbf{k},\boldsymbol{\lambda})\rho_{\text{vertex},\mathcal{V}=\{2\},\mathcal{L}=\{1\}}(\mathbf{k},\boldsymbol{\lambda}) \\
&\times \rho_{\text{line},\mathcal{L}=\{1\}}(\mathbf{k},\boldsymbol{\lambda})\rho_{\text{vertex},\mathcal{V}=\{1\},\mathcal{L}=\{\}}(\mathbf{k},\boldsymbol{\lambda}) \\
=& \frac{1}{\lambda_2}\frac{\psi(k_1,\lambda_1)-\psi(k_1,\lambda_1+\lambda_2)}{1-\psi(k_1+k_2,\lambda_1+\lambda_2)}\frac{1}{\lambda_1}\frac{1-\phi(\lambda_1)}{1-\psi(k_1,\lambda_1)}
\end{aligned} \tag{3.23}$$

where I have used $\phi(\lambda) = \psi(0,\lambda)$. In combination with equation (3.17) this reproduces a result obtained by Barkai and Sokolov [BS07].

Simplifications

Till now, I have introduced a method which allows to write down directly the Fourier-Laplace transform of the n-point joint probability distribution. It is clear that these terms become very large, e.g., already for $p_3(\mathbf{k},\boldsymbol{\lambda})$ we would have to consider 13 diagrams. But the joint probability distributions are symmetric under exchange of the indices. If we look at equation (3.23), it is easy to see that the situations $q_1 < q_2$ and $q_2 < q_1$ emerge from each other by permutation of the indices while the case $q_1 = q_2$ is symmetric in the indices. In the rest of this section, I will show how to use this symmetry to ease the calculation.

In more detail, one is looking for a $\omega_n(\mathbf{k},\boldsymbol{\lambda})$ which gives $p_n(\mathbf{k},\boldsymbol{\lambda})$ via

$$p_n(\mathbf{k},\boldsymbol{\lambda}) = \sum_{\pi\in S_n} \omega_n(k_{\pi(1)},\ldots,k_{\pi(n)},\lambda_{\pi(1)},\ldots,\lambda_{\pi(n)}), \tag{3.24}$$

where S_n is the symmetric group of n elements, i.e., the sum runs over all permutations π of the indices. Ideally, $\omega_n(\mathbf{k},\boldsymbol{\lambda})$ will contain every type of diagram only once (e.g., for the two-point probability $q_1 < q_2$ and $q_2 < q_1$ are different contributions, but the diagrams have the same type or form). To achieve this, one draws all different structures of diagrams that may appear only once and numbers the outgoing arrows from n down to 1 (the λ_i are not put at the arrows any more to emphasize that it is not connected to a specific index but an abstract numbering). For the case $n = 3$ this is shown in figure 3.2. In principle, we can now apply the same rules to construct the $p_n(\mathbf{k},\boldsymbol{\lambda})$ to get the $\omega_n(\mathbf{k},\boldsymbol{\lambda})$ by summing over the different types of diagrams. One only has to bear in mind that in equation (3.24) one sums over diagrams several times when they have vertices with more than one emerging arrow – namely taken the factorial of the number of arrows for each vertex. Therefore one has to divide by this symmetry factor. In figure 3.2 these numbers are given in the right column. Therefore one possible way to describe the contributions for $\omega_n(\mathbf{k},\boldsymbol{\lambda})$ is

$$\rho^{\omega}_{\text{vertex}}(\mathbf{k},\boldsymbol{\lambda}) = \frac{1}{|\mathcal{V}|!}\frac{1}{\prod_{v\in\mathcal{V}}\lambda_v}\sum_{\mathcal{J}\in\mathcal{P}(\mathcal{V})}(-1)^{|\mathcal{J}|}\psi(K_{\mathcal{L}},\Lambda_{\mathcal{L}}+\Lambda_{\mathcal{J}}) \tag{3.25}$$

for each vertex while the contribution of a line stays the same. This form has the advantage that one can directly generalize it to other models then the leaper model and is useful in some limit considerations shown in section 4.2.

If one uses the leaper model of the continuous-time random walk, I consider in general an alternative form as being better suited which uses specifics of this model. Similarly to equation (3.3), one can factor out the $\frac{1}{\lambda_i}$ which gives rise to the definition

$$\breve{\omega}_n(\mathbf{k}, \boldsymbol{\lambda}) = \lambda_1 \cdots \lambda_n \omega_n(\mathbf{k}, \boldsymbol{\lambda}), \tag{3.26}$$

and one has

$$\breve{p}_n(\mathbf{k}, \boldsymbol{\lambda}) = \sum_{\pi \in S_n} \breve{\omega}_n(k_{\pi(1)}, \ldots, k_{\pi(n)}, \lambda_{\pi(1)}, \ldots, \lambda_{\pi(n)}). \tag{3.27}$$

The form which is derived here will additionally allow a direct interpretation of $\breve{\omega}_n(\mathbf{k}, \boldsymbol{\lambda})$ and determines it unambiguously.

The first change is that I now consider $\breve{\omega}_n(\mathbf{k}, \boldsymbol{\lambda})$ instead of $\omega_n(\mathbf{k}, \boldsymbol{\lambda})$. This is achieved by removing the prefactor $\left(\prod_{v \in \mathcal{V}} \lambda_v\right)^{-1}$ from the definition of a vertex. Second, one notices that — for a given diagram — we can apply any permutation on the indices $\mathcal{V}$ of any vertex without changing anything in the factors coming from horizontal lines of other vertices. I use this freedom to apply a permutation on every element of the sum in equation (3.25) such that it can be written solely with the combinations $K_q = K_{\{1,\ldots,q\}}$ and $\Lambda_q = \Lambda_{\{1,\ldots,q\}}$. This is possible because of the decreasing numbering and gives an alternative form for the vertex (now as a contribution to $\breve{\omega}_n(\mathbf{k}, \boldsymbol{\lambda})$)

$$\nu(d, h) = \frac{1}{d!} \sum_{j=0}^{d} \binom{d}{j} (-1)^j \psi(K_{h-d}, \Lambda_{h-d+j}) \tag{3.28}$$

which takes as arguments the degree of the vertex d (i.e., the number of arrows leaving) and the first (and therefore highest) index h. Now, applying any permutation to the indices either leaves the term invariant or transforms it to one which uses another combination of arguments other than K_q or Λ_q ($q = 1, \ldots$). Therefore it is not possible that different summands in equation (3.27) cancel in whole or in part which is useful in connection with computer algebra systems.

At a first glance, the restriction to the arguments K_q and Λ_q can be seen as a method simply to reduce the number of possible arguments. But it is possible to give an interpretation of this representation. To see this, start with a function $f(t_1, \ldots, t_n)$ in the variables $t_1, \ldots, t_n \geq 0$. The corresponding parameters of the Laplace transform are denoted as usual by $\lambda_1, \ldots, \lambda_n$. Now, if $f(\mathbf{t})$ has support contained in the domain $t_1 \geq t_2 \geq \cdots \geq t_n$, it is possible to change variables to $(t_1 - t_2), \ldots, (t_n - t_{n-1}), t_n \geq 0$. Carrying out this variable transform in the Laplace transform, one sees from the identity

$$\lambda_1 t_1 + \cdots + \lambda_n t_n = \Lambda_1(t_1 - t_2) + \cdots + \Lambda_{n-1}(t_{n-1} - t_n) + \Lambda_n t_n \tag{3.29}$$

that the Laplace parameters to these variables are just the $\Lambda_1, \ldots, \Lambda_n$. Conversely, if we write the Laplace transform $f(\boldsymbol{\lambda})$ in the variables $\Lambda_1, \ldots, \Lambda_n$ and if we know that it is a Laplace transform in these variables then we can conclude that $f(\mathbf{t})$ has support in the domain $t_1 \geq \cdots \geq t_n$. Now, if one builds $\breve{\omega}_n(\mathbf{k}, \boldsymbol{\lambda})$ from the contributions in equation (3.28), one can expand the products and end with summands which consist of factors depending

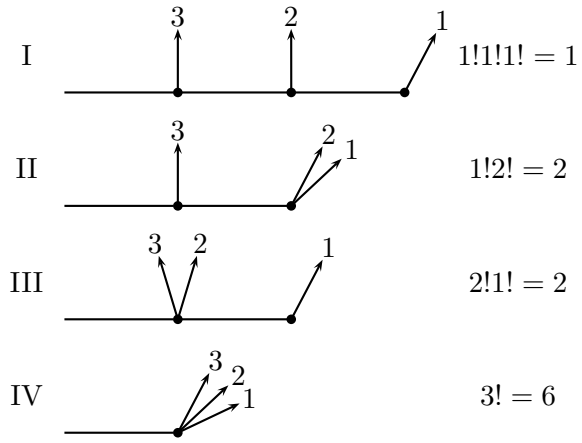

Figure 3.2.: The diagrams relevant for the calculation of the three point probability density. To indicate that these are only the ones with sorted indices, the ends of the arrows are not attached to variables but just labeled with decreasing numbers. The right column gives the multiplicity with which this diagram is counted.

only on one Λ_i. These are either of the form $\psi(K_j, \Lambda_i)$ or $\psi(K_j, \Lambda_i)/(1 - \psi(K_i, \Lambda_i))$. For both it is then clear from construction that they constitute a Laplace transform. Therefore, we know that $\breve{\omega}_n(\mathbf{x}, \mathbf{t})$ vanishes outside of $t_1 \geq \cdots \geq t_n$ if we use equation (3.28) for the vertex. In reverse, putting this condition on $\breve{\omega}_n(\mathbf{x}, \mathbf{t})$ would have fixed $\breve{\omega}_n(\mathbf{x}, \mathbf{t})$ under all functions which satisfy equation (3.27).

Applying this to the three point function with the diagrams depicted in figure 3.2 one gets the contributions

$$\begin{aligned}
\breve{\omega}_3^{\mathrm{I}}(\mathbf{k}, \boldsymbol{\lambda}) &= \frac{\nu(1,3)}{1-\psi(K_3,\Lambda_3)} \frac{\nu(1,2)}{1-\psi(K_2,\Lambda_2)} \frac{\nu(1,1)}{1-\psi(K_1,\Lambda_1)} \\
&= \frac{\psi(K_2,\Lambda_2) - \psi(K_2,\Lambda_3)}{1-\psi(K_3,\Lambda_3)} \frac{\psi(K_1,\Lambda_1) - \psi(K_1,\Lambda_2)}{1-\psi(K_2,\Lambda_2)} \frac{1-\phi(\Lambda_1)}{1-\psi(K_1,\Lambda_1)} \\
\breve{\omega}_3^{\mathrm{II}}(\mathbf{k}, \boldsymbol{\lambda}) &= \frac{\nu(1,3)}{1-\psi(K_3,\Lambda_3)} \frac{\nu(2,1)}{1-\psi(K_2,\Lambda_2)} \\
&= \frac{\psi(K_2,\Lambda_2) - \psi(K_2,\Lambda_3)}{1-\psi(K_3,\Lambda_3)} \frac{1}{2} \frac{1-2\phi(\Lambda_1)+\phi(\Lambda_2)}{1-\psi(K_2,\Lambda_2)} \\
\breve{\omega}_3^{\mathrm{III}}(\mathbf{k}, \boldsymbol{\lambda}) &= \frac{\nu(2,3)}{1-\psi(K_3,\Lambda_3)} \frac{\nu(1,1)}{1-\psi(K_1,\Lambda_1)} \\
&= \frac{1}{2} \frac{\psi(K_1,\Lambda_1) - 2\psi(K_1,\Lambda_2) + \psi(K_1,\Lambda_3)}{1-\psi(K_3,\Lambda_3)} \frac{1-\phi(\Lambda_1)}{1-\psi(K_1,\Lambda_1)} \\
\breve{\omega}_3^{\mathrm{IV}}(\mathbf{k}, \boldsymbol{\lambda}) &= \frac{\nu(3,3)}{1-\psi(K_3,\Lambda_3)} \\
&= \frac{1}{6} \frac{1-3\phi(\Lambda_1)+3\phi(\Lambda_2)-\phi(\Lambda_3)}{1-\psi(K_3,\Lambda_3)} \\
\breve{\omega}_3(\mathbf{k}, \boldsymbol{\lambda}) &= \breve{\omega}_3^{\mathrm{I}}(\mathbf{k}, \boldsymbol{\lambda}) + \breve{\omega}_3^{\mathrm{II}}(\mathbf{k}, \boldsymbol{\lambda}) + \breve{\omega}_3^{\mathrm{III}}(\mathbf{k}, \boldsymbol{\lambda}) + \breve{\omega}_3^{\mathrm{IV}}(\mathbf{k}, \boldsymbol{\lambda}).
\end{aligned} \tag{3.30}$$

It is possible to formulate the contributions to $\breve{\omega}_n$ in a recursive way. For this one notices that for each diagram, leaving out the factor stemming from the first line and the first vertex gives exactly the same contribution as the diagram (with less indices) with this line and vertex removed. Putting this into formulas, gives

$$\breve{\omega}_n(\mathbf{k}, \boldsymbol{\lambda}) = \sum_{d=1}^{n} \frac{\nu(d,n)}{1-\psi(\mathbf{k}, \boldsymbol{\lambda})} \breve{\omega}_{n-d}(\mathbf{k}, \boldsymbol{\lambda}). \tag{3.31}$$

The $\breve{\omega}_{n-d}$ takes of course only $(n-d)$ k and λ arguments, but with the decreasing numbering introduced above, these are the first $(n-d)$ ones.

3.3. Multi Point Correlations

When one wants to compare a stochastic model with measured or numerical data, one often does not look at the full probability distributions. This can have several reasons: estimating probability distributions normally needs a lot of data which is not always available, or in the case of the continuous-time random walk the analytical expressions for the joint probability distributions become large quite fast (e.g., see equations (3.23) and (3.30)). Therefore one is often interested in the multi point correlations which are defined by

$$C_n(\mathbf{t}) = \langle X(t_1) \cdots X(t_n) \rangle, \tag{3.32}$$

or in Laplace space

$$C_n(\boldsymbol{\lambda}) = \frac{\partial}{i\partial k_1} \cdots \frac{\partial}{i\partial k_n} p_n(\mathbf{k}, \boldsymbol{\lambda}) \bigg|_{\mathbf{k}=\mathbf{0}}. \tag{3.33}$$

equation (3.33) can be applied to any joint probability distribution determined with the method introduced in section 3.2. This works with any model of a continuous-time random walk but it has the disadvantage of having to determine the joint probability distribution first. In this section I want to focus on the leaper model and introduce a method which allows to write down the Laplace transform of the multi point correlations without having to determine the joint probabilities first.

Similar to the case of probability densities, it is convenient to have the definitions

$$\breve{C}_n(\boldsymbol{\lambda}) = \lambda_1 \cdots \lambda_n C_n(\boldsymbol{\lambda}) \tag{3.34}$$

which gives in the untransformed variables

$$C_n(\mathbf{t}) = \int_{0-}^{t_1} dt_1' \cdots \int_{0-}^{t_n} dt_n' \, \breve{C}_n(\mathbf{t}'). \tag{3.35}$$

Additionally, I define

$$\breve{\gamma}_n(\boldsymbol{\lambda}) = \frac{\partial}{i\partial k_1} \cdots \frac{\partial}{i\partial k_n} \breve{\omega}_n(\mathbf{k}, \boldsymbol{\lambda}) \bigg|_{\mathbf{k}=\mathbf{0}}, \tag{3.36}$$

such that

$$\breve{C}_n(\boldsymbol{\lambda}) = \sum_{\pi \in S_n} \breve{\gamma}_n(\lambda_{\pi(1)}, \ldots, \lambda_{\pi(n)}). \tag{3.37}$$

In the remainder of this chapter I will use $\breve{\omega}_n(\mathbf{k}, \boldsymbol{\lambda})$ by constructing it with the definition for the vertex given in equation (3.28), i.e., the support of $\breve{\omega}_n(\mathbf{x}, \mathbf{t})$ is contained in the domain

$t_n \leq \cdots \leq t_2 \leq t_1$ (i.e., we can use the Λ_i as the natural variables of the Laplace transform). This directly gives that the support of $\breve{\gamma}_n(\mathbf{t})$ is also contained in $t_n \leq \cdots \leq t_2 \leq t_1$.

We can see from equation (3.33) that the multi point correlations do not depend on the full spatial-temporal probability distribution of the steps but only the first n derivatives with respect to k. This suggests the definition

$$\phi_q(\lambda) = \left(\frac{\partial}{i\partial k}\right)^q \psi(k,\lambda)\bigg|_{k=0}. \tag{3.38}$$

We have $\phi_0(\lambda) = \phi(\lambda)$ as the marginal distribution of the waiting times, while $\phi_q(\lambda)$ can be interpreted as the Laplace transform of the qth moment of the spatial jump distribution depending on the waiting time.

Now, I proceed to calculate $\breve{\gamma}_n(\boldsymbol{\lambda})$. One notices that for any diagram contributing to $\breve{\gamma}_n(\boldsymbol{\lambda})$ the variable k_n does only appear in the contribution of the first horizontal line. This can also be seen from equation (3.31) by noting that $K_n = k_1 + \cdots + k_n$ is the only sum of the $K_i = k_1 + \cdots + k_i$ which contains k_n. Therefore multiplying equation (3.31) with $(1-\psi(K_n,\Lambda_n))$ leaves us with an equation for which the right hand side is now independent of k_n. This gives

$$\frac{\partial}{i\partial k_n}(1-\psi(K_n,\Lambda_n))\breve{\omega}_n(\mathbf{k},\boldsymbol{\lambda}) = 0. \tag{3.39}$$

Taking additionally the derivatives with respect to $k_1,\ldots,k_{n-1}$ therefore leads to

$$\frac{\partial^n}{i^n\partial k_{\{1,\ldots,n\}}}(1-\psi(K_n,\Lambda_n))\breve{\omega}_n(\mathbf{k},\boldsymbol{\lambda})\bigg|_{\mathbf{k}=\mathbf{0}} = 0 \tag{3.40}$$

where I introduced the notation

$$\frac{\partial^{|\mathcal{J}|}}{\partial k_{\mathcal{J}}} \tag{3.41}$$

to describe the differentiation with respect to all k_i for which the index i is in the set $\mathcal{J}$. Applying the product rule of differentiation gives

$$(1-\phi_0(\Lambda_n))\breve{\gamma}_n(\boldsymbol{\lambda}) = \sum_{d=1}^{n}\phi_d(\Lambda_n)\sum_{\substack{\mathcal{J}\in\mathcal{P}(\{1,\ldots,n\})\\|\mathcal{J}|=n-d}}\frac{\partial^{n-d}}{i^{n-d}\partial k_{\mathcal{J}}}\breve{\omega}_n(\mathbf{k},\boldsymbol{\lambda})\bigg|_{\mathbf{k}=\mathbf{0}} \tag{3.42}$$

where the second sum runs over all subsets of $\{1,\ldots,n\}$ with a given cardinality. In other words, it is a sum over all differential operators of a given order which can be built by the differentiation with respect to $k_1,\ldots,k_n$ with no second or higher order differentiation with respect to a single k_i.

I will show in section 3.3.3 that we can evaluate these operators with the equality (actually, there I derive a more general form, allowing for higher derivatives)

$$\sum_{\substack{\mathcal{J}\in\mathcal{P}(\{1,\ldots,n\})\\|\mathcal{J}|=n-d}}\frac{\partial^{n-d}}{i^{n-d}\partial k_{\mathcal{J}}}\breve{\omega}_n(\mathbf{k},\boldsymbol{\lambda})\bigg|_{\mathbf{k}=\mathbf{0}} = \frac{1}{d!}\breve{\gamma}_{n-d}(\boldsymbol{\lambda}). \tag{3.43}$$

Since the derivation of equation (3.43) is a simple but purely technical handling of indices, I will only give some ideas which make equation (3.43) plausible, but leave the details for a later separate section.

Assume we would have one of these operators of order $(n-d)$ acting on $\breve{p}_n(\mathbf{k},\boldsymbol{\lambda})$, i.e., given $\mathcal{J} \in \mathcal{P}(\{1,\ldots,n\}$ with $|\mathcal{J}| = (n-d)$ we look at

$$\frac{\partial^{n-d}}{i^{n-d}\partial k_{\mathcal{J}}}\breve{p}_n(\mathbf{k},\boldsymbol{\lambda})\bigg|_{\mathbf{k}=\mathbf{0}}. \tag{3.44}$$

By construction this corresponds to a $(n-d)$-point correlation, i.e., it can be expressed by $\breve{C}_{n-d}(\ldots)$ (dropping the arguments with indices which are not contained in $\mathcal{J}$). Now, symmetrizing $\breve{\omega}_n(\mathbf{k},\boldsymbol{\lambda})$ gives $\breve{p}_n(\mathbf{k},\boldsymbol{\lambda})$, while in equation (3.42) we have the reversed situation: consider $\breve{\omega}_n(\mathbf{k},\boldsymbol{\lambda})$ with a symmetrized differential operator acting on it. Since one can distinguish the different permutations of $\breve{\omega}_n(\ldots)$ by their support, it is likely that the result can by identified as a multiple of $\breve{\gamma}_{n-d}(\boldsymbol{\lambda})$.

Inserting equation (3.43) into equation (3.42) becomes

$$\breve{\gamma}_n(\boldsymbol{\lambda}) = \sum_{d=1}^{n} \frac{1}{d!}\frac{\phi_d(\Lambda_n)}{1-\phi(\Lambda_n)}\breve{\gamma}_{n-d}(\boldsymbol{\lambda}) \tag{3.45}$$

with the natural definition $\breve{\gamma}_0 = 1$. Again I take up the convention that the $(n-d)$ arguments of $\breve{\gamma}_{n-d}(\boldsymbol{\lambda})$ are the first λ_is, i.e., $\breve{\gamma}_{n-d}(\boldsymbol{\lambda}) = \breve{\gamma}_{n-d}(\lambda_1,\ldots,\lambda_{n-d})$.

Equation (3.45) is the main result of this section. Applying it gives

$$\breve{\gamma}_1(\boldsymbol{\lambda}) = \frac{\phi_1(\Lambda_1)}{1-\phi(\Lambda_1)} \tag{3.46}$$

$$\breve{\gamma}_2(\boldsymbol{\lambda}) = \frac{\phi_1(\Lambda_2)}{1-\phi(\Lambda_2)}\frac{\phi_1(\Lambda_1)}{1-\phi(\Lambda_1)} + \frac{1}{2}\frac{\phi_2(\Lambda_2)}{1-\phi(\Lambda_2)} \tag{3.47}$$

$$\begin{aligned}\breve{\gamma}_3(\boldsymbol{\lambda}) = {} & \frac{\phi_1(\Lambda_3)}{1-\phi(\Lambda_3)}\frac{\phi_1(\Lambda_2)}{1-\phi(\Lambda_2)}\frac{\phi_1(\Lambda_1)}{1-\phi(\Lambda_1)} + \frac{1}{2}\frac{\phi_1(\Lambda_3)}{1-\phi(\Lambda_3)}\frac{\phi_2(\Lambda_2)}{1-\phi(\Lambda_2)} \\ & + \frac{1}{2}\frac{\phi_2(\Lambda_3)}{1-\phi(\Lambda_3)}\frac{\phi_1(\Lambda_1)}{1-\phi(\Lambda_1)} + \frac{1}{6}\frac{\phi_3(\Lambda_3)}{1-\phi(\Lambda_3)}.\end{aligned} \tag{3.48}$$

I now proceed to translate the results back from Laplace space to the real time. For this I define $g_q(t)$ as the inverse Laplace transform of

$$\frac{\phi_q(\lambda)}{1-\phi(\lambda)} \tag{3.49}$$

for $t \geq 0$ and as zero otherwise. The first two examples (equations (3.46) and (3.47)) are in real time

$$\begin{aligned}\breve{\gamma}_1(t_1) &= g_1(t_1) \\ \breve{\gamma}_2(t_1,t_2) &= g_1(t_2)g_1(t_1-t_2) + \frac{1}{2}g_2(t_2)\delta(t_1-t_2).\end{aligned} \tag{3.50}$$

To finally get the n-point correlation function $C_n(\mathbf{t})$, one still has to perform an integration. In accordance with the notation used before, I denote this integration of $\breve{\gamma}_n(\mathbf{t})$ by $\gamma_n(\mathbf{t})$ and one has

$$\gamma_n(\boldsymbol{\lambda}) = \frac{1}{\lambda_1\cdots\lambda_n}\breve{\gamma}_n(\boldsymbol{\lambda}) \tag{3.51}$$

where one gets $C_n(\mathbf{t})$ just by summing over all permutations of the indices of $\gamma_n(\mathbf{t})$ (similar to equation (3.37)). The evaluation of $\gamma_n(\mathbf{t})$ can always be reduced to arguments in the domain $t_1 \geq t_2 \geq \cdots \geq t_n$ via

$$\gamma_n(\mathbf{t}) = \gamma_n(\mathbf{t}') \tag{3.52}$$

with $t'_j = \min\{t_1, t_2, \ldots, t_j\}$. These two terms are equal since the difference is an integration over a domain in which $\breve{\gamma}_n(\mathbf{t})$ vanishes. As example, for $t_1 \geq t_2$ we have

$$\begin{aligned} C_2(t_1, t_2) &= \gamma_2(t_1, t_2) + \gamma_2(t_2, t_1) \\ &= \gamma_2(t_1, t_2) + \gamma_2(t_2, t_2). \end{aligned} \tag{3.53}$$

In the domain $t_1 \geq t_2 \geq \cdots \geq t_n$ the evaluation of $\gamma_n(\mathbf{t})$ leads to the integral

$$\gamma_n(\mathbf{t}) = \int_{0-}^{t_n} d\tau_n \int_{\tau_n-}^{t_{n-1}} d\tau_{n-1} \cdots \int_{\tau_2-}^{t_1} d\tau_1 \, \breve{\gamma}_n(\boldsymbol{\tau}) \tag{3.54}$$

The minus sign behind the lower bounds expresses that a δ-function sitting at the boundary (e.g., a $\delta(\tau_{i+1} - \tau_i)$) is fully evaluated (which is in the definition of the Laplace transform and which can be seen in the fact that the Laplace transform of $\delta(t)$ is simply 1). Before I get to some examples on how to evaluate this expression in the long time limit, I still want to give two small remarks.

Evaluating the integral in equation (3.54) with two or more t_i being equal, we have a convolution which we can also express directly in Laplace space. Taking additionally the δ function into account, equation (3.53) in Laplace space is

$$\begin{aligned} \tilde{C}_2(\lambda_1, \lambda_2) = &\frac{1}{\lambda_1 \lambda_2} \frac{\phi_1(\lambda_1 + \lambda_2)}{1 - \phi(\lambda_1 + \lambda_2)} \frac{\phi_1(\lambda_1)}{1 - \phi(\lambda_1)} \\ &+ \frac{1}{\lambda_1 \lambda_2} \frac{\phi_2(\lambda_2)}{1 - \phi(\lambda_2)} + \frac{1}{\lambda_1 \lambda_2} \left(\frac{\phi_1(\lambda_2)}{1 - \phi(\lambda_2)} \right)^2 . \end{aligned} \tag{3.55}$$

This is not the complete two point correlation, but $C_2(t_1, t_2)$ and $\tilde{C}_2(t_1, t_2)$ coincide for $t_1 \geq t_2$.

When the correlation function is not symmetric under exchange of the times (e.g., we take different powers of the $X(t_i)$) the corresponding differential operator in Fourier-space is also not symmetric under permutations of the indices and it is disadvantageous to work directly with the $\breve{\gamma}_n(\boldsymbol{\lambda})$. In this case it is easier to use $\breve{p}_n(\mathbf{k}, \boldsymbol{\lambda})$ by noting that $(1 - \psi(K_n, \Lambda_n))\breve{p}_n(\mathbf{k}, \boldsymbol{\lambda})$ is a sum of terms which are independent of at least one k_i and therefore differentiating with respect to each k_i $(i = 1, \ldots, n)$ at least once will yield a zero.

3.3.1. The uncoupled case

I assume here that the step size distribution is independent from the waiting time distribution, i.e., $\phi_q(\lambda) = \mu_q \phi(\lambda)$ where μ_q is the qth moment of the step size distribution. I further assume that the waiting time distribution is in the domain of normal attraction of an one-sided Lévy-distribution with exponent β $(0 < \beta < 1)$, i.e., $\phi(\lambda) = 1 - (\tau_0 \lambda)^\beta + o(\lambda^\beta)$ where τ_0 is the time constant (see equation (2.31). The asymptotic results in this subsection for the uncoupled case can also be obtained by applying the method by Baule and Friedrich [BF05] which I therefore use as a consistency check.

As an example I assume an uncoupled CTRW with a step size distribution with vanishing mean (i.e., $\mu_1 = 0$). The four-point correlation is then given by

$$\begin{aligned}\breve{\gamma}_4(\boldsymbol{\lambda}) &= \frac{\mu_2^2}{4}\frac{\phi(\Lambda_4)}{1-\phi(\Lambda_4)}\frac{\phi(\Lambda_2)}{1-\phi(\Lambda_2)} + \frac{\mu_4}{24}\frac{\phi(\Lambda_4)}{1-\phi(\Lambda_4)} \\ &\simeq \frac{1}{4}\frac{\mu_2^2}{\tau_0^{2\beta}}\frac{1}{\Lambda_4^\beta}\frac{1}{\Lambda_2^\beta}\end{aligned} \tag{3.56}$$

where I use "$\simeq$" for the long time behavior (corresponding to small λ).

For $t_1 \geq t_2 \geq t_3 \geq t_4$ this leads to

$$\begin{aligned}\gamma_4(\mathbf{t}) &\simeq \frac{1}{4}\frac{\mu_2^2}{\tau_0^{2\beta}}\frac{1}{\Gamma(\beta)^2}\int_0^{t_4} ds_4 \int_{s_4}^{t_2} ds_2\, s_4^{\beta-1}(s_2-s_4)^{\beta-1} \\ &= \frac{1}{4}\frac{\mu_2^2}{\tau_0^{2\beta}}\frac{t_2^\beta t_4^\beta}{\Gamma(\beta+1)^2}\beta\int_0^1 ds\, s^{\beta-1}\left(1-\frac{t_4}{t_2}s\right)^\beta.\end{aligned} \tag{3.57}$$

The last integral (with the prefactor β) is an integral representation of the hypergeometric function $F(a,b;c;z)$ [[AS72], equation (15.3.1)]. Therefore, we can write

$$\gamma_4(\mathbf{t}) \simeq \frac{1}{4}\frac{\mu_2^2}{\tau_0^{2\beta}}\frac{t_2^\beta t_4^\beta}{\Gamma(\beta+1)^2}F(\beta,-\beta;1+\beta;\frac{t_4}{t_2}). \tag{3.58}$$

Summing over the permutations gives

$$\begin{aligned}C_4(\mathbf{t}) \simeq \frac{\mu_2^2}{\tau_0^{2\beta}}\Bigg(&\frac{t_4^\beta t_2^\beta}{\Gamma(\beta+1)^2}F(\beta,-\beta;1+\beta;\frac{t_4}{t_2}) \\ &+ 2\frac{t_4^\beta t_3^\beta}{\Gamma(\beta+1)^2}F(\beta,-\beta;1+\beta;\frac{t_4}{t_3}) \\ &+ 3\frac{t_4^{2\beta}}{\Gamma(2\beta+1)}\Bigg)\end{aligned} \tag{3.59}$$

for $t_1 \geq t_2 \geq t_3 \geq t_4$. The prefactors arise from counting how often t_2, t_3 and t_4 arise as the smallest of the first two elements of any permutation of t_1, t_2, t_3, t_4. For the last summand, I additionally used the identity [[AS72], equation (15.1.20)]

$$F(\beta_1,-\beta_2;1+\beta_1;1) = \frac{\Gamma(\beta_1+1)\Gamma(\beta_2+1)}{\Gamma(\beta_1+\beta_2+1)}. \tag{3.60}$$

As a consistency check, I put $t_1 = t_2$ and $t_3 = t_4$ to get

$$\langle X^2(t_1)X^2(t_3)\rangle \simeq \frac{\mu_2^2}{\tau_0^{2\beta}}\left(5\frac{t_3^{2\beta}}{\Gamma(2\beta+1)} + \frac{t_3^\beta t_1^\beta}{\Gamma(\beta+1)^2}F(\beta,-\beta;1+\beta;\frac{t_3}{t_1})\right). \tag{3.61}$$

This can also be calculated by applying the introduced method to give

$$\begin{aligned}\frac{\partial^2}{\partial k_1^2}\frac{\partial^2}{\partial k_2^2}\breve{\omega}_2(\mathbf{k},\boldsymbol{\lambda})\Big|_{\mathbf{k}=\mathbf{0}} &= \mu_2^2\frac{\phi(\Lambda_2)}{1-\phi(\Lambda_2)}\frac{\phi(\Lambda_1)}{1-\phi(\Lambda_1)} + 2\mu_2^2\left(\frac{\phi(\Lambda_2)}{1-\phi(\Lambda_2)}\right)^2 + \frac{\mu_4}{2}\frac{\phi(\Lambda_2)}{1-\phi(\Lambda_2)} \\ &\simeq \frac{\mu_2^2}{\tau_0^{2\beta}}\left(\frac{1}{\Lambda_2^\beta}\frac{1}{\Lambda_1^\beta} + \frac{2}{\Lambda_2^{2\beta}}\right)\end{aligned} \tag{3.62}$$

which subsequently reproduces equation (3.61) (by substituting t_3 for t_2). For equation (3.62) I used the generalized form of equation (3.43) shown in section 3.3.3 (equation (3.79)).

I want to close this subsection on the uncoupled case with the calculation of the long term behavior of $\langle X^2(t_1)X(t_2)\rangle$ as an example of an unsymmetric correlation. If $\mu_1 = 0$ this term vanishes. Therefore I consider a biased CTRW with $\mu_1 \neq 0$. Since the correlation function is not symmetric with respect to index permutation, one has to work with $\breve{p}_2(\mathbf{k}, \boldsymbol{\lambda})$ directly. In the long time limit, this leads to

$$\begin{aligned} i\frac{\partial^2}{\partial k_1^2}\frac{\partial}{\partial k_2}\breve{p}_2(\mathbf{k},\boldsymbol{\lambda})\Big|_{\mathbf{k}=\mathbf{0}} &\simeq \frac{\mu_1^3\,\phi(\Lambda_{\{1,2\}})}{1-\phi(\Lambda_{\{1,2\}})}\left(\frac{\phi(\Lambda_{\{1\}})}{1-\phi(\Lambda_{\{1\}})}\right)^2 \\ &\quad + 2\mu_1^3\left(\frac{\phi(\Lambda_{\{1,2\}})}{1-\phi(\Lambda_{\{1,2\}})}\right)^2\frac{\phi(\Lambda_{\{1\}})}{1-\phi(\Lambda_{\{1\}})} \\ &\quad + 2\mu_1^3\left(\frac{\phi(\Lambda_{\{1,2\}})}{1-\phi(\Lambda_{\{1,2\}})}\right)^2\frac{\phi(\Lambda_{\{2\}})}{1-\phi(\Lambda_{\{2\}})} \\ &\simeq \frac{\mu_1^3}{\tau_0^3}\left(\frac{1}{\Lambda^{\beta}_{\{1,2\}}}\frac{1}{\Lambda^{2\beta}_{\{1\}}} + 2\frac{1}{\Lambda^{2\beta}_{\{1,2\}}}\frac{1}{\Lambda^{\beta}_{\{1\}}} + 2\frac{1}{\Lambda^{2\beta}_{\{1,2\}}}\frac{1}{\Lambda^{\beta}_{\{2\}}}\right). \end{aligned} \tag{3.63}$$

The determination of equation (3.63) can be shortened by convincing oneself in advance that only the terms with $\phi_1(\cdot)$ are relevant for the long time behavior. The first two terms in the last line of equation (3.63) are the Laplace transforms of a function with support in $t_1 \geq t_2$ while the last term is the Laplace transform of a function supported in $t_2 \geq t_1$. The individual summands can be determined analogously to equations (3.57) and (3.58). Putting these together with equation (3.60) gives the result for $t_1 \geq t_2$

$$\begin{aligned} \langle X^2(t_1)X(t_2)\rangle \simeq \frac{\mu_1^3}{\tau_0^3}\Bigg(&2\frac{t_2^{3\beta}}{\Gamma(3\beta+1)} \\ &+ 2\frac{t_1^{\beta}}{\Gamma(\beta+1)}\frac{t_2^{2\beta}}{\Gamma(2\beta+1)}F(2\beta,-\beta;1+2\beta;\frac{t_2}{t_1}) \\ &+ \frac{t_1^{2\beta}}{\Gamma(2\beta+1)}\frac{t_2^{\beta}}{\Gamma(\beta+1)}F(\beta,-2\beta;1+\beta;\frac{t_2}{t_1})\Bigg) \end{aligned} \tag{3.64}$$

and for $t_2 \geq t_1$

$$\langle X^2(t_1)X(t_2)\rangle \simeq \frac{\mu_1^3}{\tau_0^3}\left(3\frac{t_2^{3\beta}}{\Gamma(3\beta+1)} + 2\frac{t_1^{2\beta}}{\Gamma(2\beta+1)}\frac{t_2^{\beta}}{\Gamma(\beta+1)}F(2\beta,-\beta;1+2\beta;\frac{t_1}{t_2})\right). \tag{3.65}$$

3.3.2. The coupled case

In this subsection I allow a coupling between the step distribution and the waiting time. When looking at the n-point correlation and assuming that the marginal distribution of the step size has all first n moments μ_q, one has

$$\lim_{\lambda\to 0}\phi_q(\lambda) = \mu_q. \tag{3.66}$$

Therefore, in the long time limit one ends with the same terms as in the uncoupled case, except that one has to take the average (or marginal) moments.

Therefore I will concentrate here on the situation where not all marginal moments exist, but nevertheless for every bounded interval of waiting times, the step size distribution has all necessary moments (i.e., up to the nth moment if we look at the n-point correlation). To be able to calculate the long time limit, the $\phi_q(\lambda)$ have to exist for $\lambda > 0$, while they are allowed to diverge to $+\infty$ as $\lambda \to 0+$. One way to ensure this is to put the condition that the increase of the moments with the waiting time is bounded by a polynomial.

Let us look at the Lévy walk [KBS87]

$$\psi(x,t) = \phi(t)\frac{\delta(|x| - vt^\kappa)}{2} \tag{3.67}$$

with $\phi(t)$ being for simplicity an one-sided Lévy stable distribution, which implies

$$\phi(t) \simeq \frac{\tau_0^\beta}{-\Gamma(-\beta)}\frac{1}{t^{1+\beta}} \text{ for } t \to \infty. \tag{3.68}$$

I should remark that I use an other parameterization as in [KBS87]. The long time behavior of the second and forth moment is

$$\begin{aligned} \phi_2(t) &\simeq \frac{\tau_0^\beta v^2}{-\Gamma(-\beta)}\frac{t^{2\kappa}}{t^{1+\beta}} \\ \phi_4(t) &\simeq \frac{\tau_0^\beta v^4}{-\Gamma(-\beta)}\frac{t^{4\kappa}}{t^{1+\beta}}. \end{aligned} \tag{3.69}$$

I concentrate here on the case $\kappa > \frac{\beta}{2}$. Then the Tauberian theorems [[Fel71], section XIII.5] give

$$\begin{aligned} \phi_2(\lambda) &\simeq \frac{\tau_0^\beta v^2 \Gamma(2\kappa-\beta)}{-\Gamma(-\beta)}\frac{1}{\lambda^{2\kappa-\beta}} \\ \phi_4(\lambda) &\simeq \frac{\tau_0^\beta v^4 \Gamma(4\kappa-\beta)}{-\Gamma(-\beta)}\frac{1}{\lambda^{4\kappa-\beta}} \end{aligned} \tag{3.70}$$

for $\lambda \to 0+$. Similar to equation (3.62) we get

$$\begin{aligned} \frac{\partial^2}{\partial k_1^2}\frac{\partial^2}{\partial k_2^2}\breve{\omega}_2(\mathbf{k},\boldsymbol{\lambda})\Big|_{\mathbf{k}=\mathbf{0}} &= \frac{\phi_2(\Lambda_2)}{1-\phi(\Lambda_2)}\frac{\phi_2(\Lambda_1)}{1-\phi(\Lambda_1)} + 2\left(\frac{\phi_2(\Lambda_2)}{1-\phi(\Lambda_2)}\right)^2 + \frac{1}{2}\frac{\phi_4(\Lambda_2)}{1-\phi(\Lambda_2)} \\ &\simeq v^4\left(\frac{\Gamma(2\kappa-\beta)}{-\Gamma(-\beta)}\right)^2\left(\frac{1}{\Lambda_2^{2\kappa}}\frac{1}{\Lambda_1^{2\kappa}} + \frac{2}{\Lambda_2^{4\kappa}}\right) + \frac{v^4}{2}\frac{\Gamma(4\kappa-\beta)}{-\Gamma(-\beta)}\frac{1}{\Lambda_2^{4\kappa}}. \end{aligned} \tag{3.71}$$

In this case the term $\phi_4(\Lambda_2)/(1-\phi(\Lambda_2))$ does not vanish in the long time limit. We get for $t_1 \geq t_2$

$$\langle X^2(t_1)X^2(t_2)\rangle \simeq D_1 v^4 \frac{t_2^{4\kappa}}{\Gamma(4\kappa+1)} + D_2 v^4 \frac{t_1^{2\kappa}t_2^{2\kappa}}{\Gamma(2\kappa+1)^2}F(2\kappa,-2\kappa;1+2\kappa;\frac{t_2}{t_1}) \tag{3.72}$$

with

$$\begin{aligned} D_1 &= 5\left(\frac{\Gamma(2\kappa-\beta)}{-\Gamma(-\beta)}\right)^2 + \frac{\Gamma(4\kappa-\beta)}{-\Gamma(-\beta)} \\ D_2 &= \left(\frac{\Gamma(2\kappa-\beta)}{-\Gamma(-\beta)}\right)^2. \end{aligned} \tag{3.73}$$

Since the last example showed that unlike to the uncoupled case the term $\phi_4(\Lambda_2)/(1-\phi(\Lambda_2))$ need not to be negligible for large times, one can ask if it is possible that it dominates in this limit. For this, let us look at the following example ($\phi(t)$ as before)

$$\psi(x,t) = \phi(t)\left(\frac{t^{2\kappa}}{1+t^{2\kappa}}\frac{\delta(|x|-v_1t^\kappa)}{2} + \frac{1}{1+t^{2\kappa}}\frac{\delta(|x|-v_2t^{2\kappa})}{2}\right). \tag{3.74}$$

This corresponds to a CTRW where the walker makes after a waiting time t, either a step with modulus v_1t^κ or with $v_2t^{2\kappa}$. Which of the step sizes is taken is chosen randomly with a probability of $t^{2\kappa}/(1+t^{2\kappa})$ for the first and a probability of $1/(1+t^{2\kappa})$ for the second step size. For $\kappa > \frac{\beta}{2}$ one gets

$$\begin{aligned}\phi_2(\lambda) &\simeq \tau_0^\beta(v_1^2+v_2^2)\frac{\Gamma(2\kappa-\beta)}{-\Gamma(-\beta)}\frac{1}{\lambda^{2\kappa-\beta}}\\ \phi_4(\lambda) &\simeq \tau_0^\beta v_2^4\frac{\Gamma(6\kappa-\beta)}{-\Gamma(-\beta)}\frac{1}{\lambda^{6\kappa-\beta}}.\end{aligned} \tag{3.75}$$

For the four-point correlation the term with $\phi_4(\cdot)$ dominates in the long time limit and one gets (with $\phi_1(\lambda)=0$)

$$C_4(\mathbf{t}) \simeq v^4\frac{\Gamma(6\kappa-\beta)}{-\Gamma(-\beta)}\frac{t_4^{6\kappa}}{\Gamma(6\kappa+1)} \tag{3.76}$$

for $t_1 \geq t_2 \geq t_3 \geq t_4$. The value is therefore determined by the earliest time only.

3.3.3. Derivation of equation (3.43)

In this subsection, I derive a more general version of equation (3.43). For this I introduce the operator $\mathcal{D}^m[f(\mathbf{k},\boldsymbol{\lambda})]$ for a function $f(\mathbf{k},\boldsymbol{\lambda})$ as

$$\mathcal{D}^m[f(\mathbf{k},\boldsymbol{\lambda})] = \left(\frac{\partial}{i\partial k_1}\right)^{q_1}\cdots\left(\frac{\partial}{i\partial k_m}\right)^{q_m} f(\mathbf{k},\boldsymbol{\lambda})\Big|_{\mathbf{k}=\mathbf{0}}. \tag{3.77}$$

With this notation $\mathcal{D}^m[p_n(\mathbf{k},\boldsymbol{\lambda})]$ can be written for the Laplace transform of $\langle X^{q_1}(t_1)\cdots X^{q_m}(t_m)\rangle$ (I suppress the $\mathbf{q}$-dependence for simplicity). I additionally define a permuted version of $\mathcal{D}^m[f(\mathbf{k},\boldsymbol{\lambda})]$ for any $\sigma \in S_n$ (S_n is the symmetric group of n elements):

$$\mathcal{D}^m_\sigma[f(\mathbf{k},\boldsymbol{\lambda})] = \left(\frac{\partial}{i\partial k_{\sigma(1)}}\right)^{q_1}\cdots\left(\frac{\partial}{i\partial k_{\sigma(m)}}\right)^{q_m} f(\mathbf{k},\boldsymbol{\lambda})\Big|_{\mathbf{k}=\mathbf{0}}. \tag{3.78}$$

When calculating correlation functions with the method introduced in section 3.3 one encounters terms like $\mathcal{D}^m[\breve{\omega}_n(\mathbf{k},\boldsymbol{\lambda})]$ with $m \leq n$. For $m < n$ the problem arises that for the recursive procedure of section 3.3 one would need a differentiation of $\breve{\omega}_n(\mathbf{k},\boldsymbol{\lambda})$ with respect to k_n. In the case of symmetric differential operators it is possible to circumvent the problem by use of the equation

$$\sum_{\sigma\in S_n}\mathcal{D}^m_\sigma[\breve{\omega}_n(\mathbf{k},\boldsymbol{\lambda})] = \sum_{\theta\in S_m}\mathcal{D}^m_\theta[\breve{\omega}_m(\mathbf{k},\boldsymbol{\lambda})] \tag{3.79}$$

where I identify S_m as a subgroup of S_n in the standard way by acting on the first m elements. In case of unsymmetric differential operators, it is better to work with $\breve{p}_n(\mathbf{k},\boldsymbol{\lambda})$

instead of $\breve{\omega}_n(\mathbf{k},\boldsymbol{\lambda})$ since then the reduction is simply equation (3.84) (in fact, the derivation of equation (3.79) is done by tracing back the problem to equation (3.84)). Equation (3.43) follows then from equation (3.79) by the following identities (here $q_1 = \cdots = q_{n-d} = 1$):

$$\sum_{\substack{\mathcal{J}\in\mathcal{P}(\{1,\ldots,n\})\\|\mathcal{J}|=n-d}} \frac{\partial^{n-d}}{i^{n-d}\partial k_{\mathcal{J}}}\breve{\omega}_n(\mathbf{k},\boldsymbol{\lambda})\bigg|_{\mathbf{k}=\mathbf{0}} = \frac{1}{d!(n-d)!}\sum_{\sigma\in S_n}\mathcal{D}_\sigma^{n-d}[\breve{\omega}_n(\mathbf{k},\boldsymbol{\lambda})] \tag{3.80}$$

and

$$\breve{\gamma}_{n-d}(\boldsymbol{\lambda}) = \frac{1}{(n-d)!}\sum_{\theta\in S_{n-d}}\mathcal{D}_\theta^{n-d}[\breve{\omega}_n(\mathbf{k},\boldsymbol{\lambda})]. \tag{3.81}$$

For the derivation of equation (3.79), we use the notation $\pi\mathbf{k}$ with $\pi\in S_n$ for the permutation of the different elements of the vector (i.e., π acts on the indices). This allows to write equation (3.27) as

$$\breve{p}_n(\boldsymbol{\lambda}) = \sum_{\pi\in S_n}\breve{\omega}_n(\pi\mathbf{k},\pi\boldsymbol{\lambda}). \tag{3.82}$$

Symmetrizing equation (3.79) by summing the left hand side over all permutations $\pi\in S_n$ of the arguments, we get

$$\begin{aligned}\sum_{\pi\in S_n}\sum_{\sigma\in S_n}\mathcal{D}_{\pi\sigma}^m[\breve{\omega}_n(\pi\mathbf{k},\pi\boldsymbol{\lambda})] &= \sum_{\pi,\eta\in S_n}\mathcal{D}_\eta^m[\breve{\omega}_n(\pi\mathbf{k},\pi\boldsymbol{\lambda})]\\ &= \sum_{\eta\in S_n}\mathcal{D}_\eta^m[\breve{p}_n(\mathbf{k},\boldsymbol{\lambda})]\\ &= \sum_{\eta\in S_n}\mathcal{D}_\eta^m[\breve{p}_n(\eta\mathbf{k},\eta\boldsymbol{\lambda})].\end{aligned} \tag{3.83}$$

Here I have used the fact that $\breve{p}_n(\mathbf{k},\boldsymbol{\lambda})$ is symmetric in the arguments. The term $\breve{p}_n(\mathbf{k},\boldsymbol{\lambda})$ can be simplified by noting that integrating one parameter in a joint probability distribution gives a joint probability distribution with lesser points, i.e.,

$$\breve{p}_n(k_1,\ldots,k_{n-1},k_n=0,\lambda_1,\ldots,\lambda_n) = \breve{p}_{n-1}(k_1,\ldots,k_{n-1},\lambda_1,\ldots,\lambda_{n-1}) \tag{3.84}$$

and similar relations for other $k_i = 0$. Continuing equation (3.83) gives

$$\begin{aligned}\sum_{\pi\in S_n}\sum_{\sigma\in S_n}\mathcal{D}_{\pi\sigma}^m[\breve{\omega}_n(\pi\mathbf{k},\pi\boldsymbol{\lambda})] &= \sum_{\eta\in S_n}\mathcal{D}_\eta^m[\breve{p}_m(\pi'\mathbf{k},\eta\boldsymbol{\lambda})]\\ &= \sum_{\eta\in S_n}\sum_{\theta\in S_m}\mathcal{D}_\eta^m[\breve{\omega}_m(\eta\theta^{-1}\mathbf{k},\eta\theta^{-1}\boldsymbol{\lambda})]\\ &= \sum_{\pi\in S_n}\sum_{\theta\in S_m}\mathcal{D}_{\pi\theta}^m[\breve{\omega}_m(\pi\mathbf{k},\pi\boldsymbol{\lambda})].\end{aligned} \tag{3.85}$$

equation (3.79) follows now from equation (3.85) by noting that the right hand side of equation (3.79) has support in the domain $t_1\geq\cdots\geq t_n$ and the only term in the sum in the last line of equation (3.85) which shares this property is $\breve{\gamma}_l(\boldsymbol{\lambda})$ (one can assure oneself that this argument is not invalidated by the fact that these domains overlap when some t_i are equal since then the values are still fixed for symmetry reasons).

3.4. Applications to other CTRW models

This section is mainly concerned with the leaper type of continuous-time random walk. However, the methods introduced in section 3.2 generalize directly to other models. In the remainder of this chapter, I will show two such generalizations, one to the creeper model and one to a model for weak-ergodicity breaking introduced by Rebenshtok and Barkai [RB07, RB08]. The second example is not a diffusive one, but the process takes only values on a bounded set. Nevertheless, the methods can be applied to yield information about the spectrum.

3.4.1. The Creeper model

The first model I want to consider is the creeper model of the CTRW [[Hug95], section 5.4.3]. While for the leaper model the random walker stayed at the same x-position during the waiting time and then leapt to its new position, the random walker in the creeper model moves during the waiting time with constant velocity to its new position.

The objective is now to find the equivalent expression for the contribution of one step. For this we start with the expression for $\eta_n^{(j)}(\mathbf{x}, \mathbf{t})$ (the equivalent to equation (3.11)). For the creeper model we have

$$\begin{aligned}\eta_n^{(j)}(\mathbf{x}, \mathbf{t}) = &\prod_{i:q_i=j} \delta(x_i - \frac{\chi_j}{\tau_j} t_i)\theta(\tau_j - t_i) \\ &\times \prod_{i:q_i<j} \delta(x_i)\delta(t_i) \\ &\times \prod_{i:q_i>j} \delta(x_i - \chi_j)\delta(t_i - \tau_j).\end{aligned} \tag{3.86}$$

I want to motivate this expression: the change of model does only affect the behavior during the waiting time, but not the step sizes and waiting time of a completed step. Therefore we can expect, that we have a change in $\eta_n^{(j)}(\mathbf{x}, \mathbf{t})$ only in the indices i with $q_i = j$ (that are the times which are in the waiting time of the jth step – the step we are looking at). The difference for these is that they do not stay at the same position, but move with constant velocity $\frac{\chi_j}{\tau_j}$ where χ_j and τ_j are the step size and the waiting time respectively for the current step. This motivates why we have to replace $\delta(x_i)$ with $\delta(x_i - \frac{\chi_j}{\tau_j} t_i)$.

We can now use the same procedure as in section 3.2. The contribution of a horizontal line does not change, since they consist of steps with no indices i fulfilling $q_i = j$. The contribution of a vertex becomes

$$\rho_{\text{vertex}}(\mathbf{k}, \boldsymbol{\lambda}) = \int d\chi\, d\tau\, \psi(\chi, \tau) e^{-\Lambda_{\mathcal{L}}\tau + iK_{\mathcal{L}}\chi} \prod_{v \in \mathcal{V}} \frac{1 - e^{-\lambda_v \tau + ik_v \chi}}{\lambda_v - ik_v \frac{\chi}{\tau}}. \tag{3.87}$$

The one point probability distribution can again be calculated from the simple diagram having only one horizontal line and one vertex:

$$p_1(k, \lambda) = \frac{1}{1 - \psi(k, \lambda)} \int d\chi\, d\tau\, \psi(\chi, \tau) \frac{1 - e^{-\lambda\tau + ik\chi}}{\lambda - ik\frac{\chi}{\tau}}. \tag{3.88}$$

This expression coincides with the one given by Hughes in [[Hug95], page 287f] where he also shows how to obtain the asymptotic behavior.

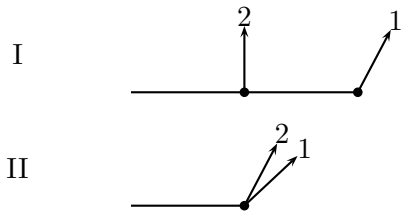

Figure 3.3.: The diagrams relevant for the calculation of equation (3.89)

In equivalence to equation (3.25) one can also define the contributions to $\omega_n(\mathbf{k}, \boldsymbol{\lambda})$. With the diagrams given in figure 3.3 we get

$$\begin{aligned}
\omega_2^{\mathrm{I}}(\mathbf{k}, \boldsymbol{\lambda}) &= \frac{1}{1 - \psi(k_1 + k_2, \lambda_1 + \lambda_2)} \int d\chi \, d\tau \, \psi(\chi, \tau) \frac{(1 - e^{-\lambda_2 \tau + i k_2 \chi}) e^{-\lambda_1 \tau + i k_1 \chi}}{\lambda_2 - i k_2 \frac{\chi}{\tau}} \\
&\quad \times \frac{1}{1 - \psi(k_1, \lambda_1)} \int d\chi \, d\tau \, \psi(\chi, \tau) \frac{1 - e^{-\lambda_1 \tau + i k_1 \chi}}{\lambda_1 - i k_1 \frac{\chi}{\tau}} \\
\omega_2^{\mathrm{II}}(\mathbf{k}, \boldsymbol{\lambda}) &= \frac{1}{2} \frac{1}{1 - \psi(k_1 + k_2, \lambda_1 + \lambda_2)} \int d\chi \, d\tau \, \psi(\chi, \tau) \frac{1 - e^{-\lambda_1 \tau + i k_1 \chi}}{\lambda_1 - i k_1 \frac{\chi}{\tau}} \frac{1 - e^{-\lambda_2 \tau + i k_2 \chi}}{\lambda_2 - i k_2 \frac{\chi}{\tau}} \\
\omega_2(\mathbf{k}, \boldsymbol{\lambda}) &= \omega_2^{\mathrm{I}}(\mathbf{k}, \boldsymbol{\lambda}) + \omega_2^{\mathrm{II}}(\mathbf{k}, \boldsymbol{\lambda}).
\end{aligned} \tag{3.89}$$

The two point probability distribution is then given by symmetrization

$$p_2(k_1, k_2, \lambda_1, \lambda_2) = \omega_2(k_1, k_2, \lambda_1, \lambda_2) + \omega_2(k_2, k_1, \lambda_2, \lambda_1). \tag{3.90}$$

3.4.2. Spectral properties of a model for weak ergodicity breaking

Rebenshtok and Barkai introduced the following model for a thermodynamic system showing weak ergodicity breaking [RB07, RB08]: In its simplest form it is characterized by two distributions, one waiting time distribution with density $\phi(t)$ and one distribution of an observable $x \in \mathbb{R}$ according to the probability density $\kappa(x)$. In the model let $\chi_0, \chi_1, \ldots$ be i.i.d. random variables distributed according to $\kappa(x)$ and let $\tau_0, \tau_1, \ldots$ be i.i.d. waiting times distributed according to $\phi(t)$. The process $x(t)$ takes the value χ_0 in the time $0 \leq t < t_0$ and χ_1 for the next time interval of length t_1, in general

$$x(t) = \chi_i \text{ for } \overline{T}_{i-1} \leq t < \overline{T}_i \tag{3.91}$$

with

$$\overline{T}_i = \sum_{j=0}^{i} \tau_j \tag{3.92}$$

As in equation (3.6) one has for the "partial" joint probabilities

$$p_n[\mathbf{q}](\mathbf{x}, \mathbf{t}) = \left\langle \prod_{k=1}^{n} \delta(x_k - \chi_{q_k}) \theta(t_k - \overline{T}_{q_k - 1}) \theta(\overline{T}_{q_k} - t_k) \right\rangle. \tag{3.93}$$

In this situation I only take the Laplace transform of the time coordinates and do not transform the spatial coordinates, i.e., I am considering

$$p_n[\mathbf{q}](\mathbf{x}, \boldsymbol{\lambda}) = \int d^n t \, e^{-\boldsymbol{\lambda}\mathbf{t}} p_n[\mathbf{q}](\mathbf{x}, \mathbf{t}). \tag{3.94}$$

Here the contribution of one step (equation (3.11)) is

$$\eta_n^{(j)}[\mathbf{q}](\mathbf{x}, \mathbf{t}) = \prod_{i:q_i=j} \delta(x_i - \chi_j)\theta(\tau_j - t_i) \prod_{i:q_i<j} \delta(t_i - \tau_j) \prod_{i:q_i>j} \delta(t_i) \tag{3.95}$$

which gives

$$p_n[\mathbf{q}](\mathbf{x}, \mathbf{t}) = \left\langle \bigstar_j \eta_n^{(j)}[\mathbf{q}](\mathbf{x}, \mathbf{t}) \right\rangle \tag{3.96}$$

where $\star$ denotes the Laplace convolution in the time variables. In Laplace space we have

$$p_n[\mathbf{q}](\mathbf{x}, \boldsymbol{\lambda}) = \prod_j \langle \eta_n^{(j)}[\mathbf{q}](\mathbf{x}, \boldsymbol{\lambda}) \rangle. \tag{3.97}$$

Now, using the result from the main section gives the following contributions

$$\begin{aligned} \rho_{\text{vertex}}(\mathbf{x}, \boldsymbol{\lambda}) &= \frac{1}{\prod_{v\in\mathcal{V}} \lambda_v} \left(\sum_{\mathcal{J}\in\mathcal{P}(\mathcal{V})} (-1)^{|\mathcal{J}|} \phi(\Lambda_{\mathcal{J}\cup\mathcal{L}}) \right) \left(\int d\chi \, \kappa(\chi) \prod_{v\in\mathcal{V}} \delta(\chi - x_v) \right) \\ \rho_{\text{line}}(\mathbf{x}, \boldsymbol{\lambda}) &= \frac{1}{1 - \phi(\Lambda_{\mathcal{L}})}. \end{aligned} \tag{3.98}$$

In the following I will mainly be interested in the correlation functions

$$\begin{aligned} C_n(\mathbf{t}) &= \langle x(t_1) \cdots x(t_n) \rangle \\ &= \int d^n x \, x_1 \cdots x_n p_n(\mathbf{x}, \mathbf{t}) \end{aligned} \tag{3.99}$$

or in Laplace space

$$C_n(\boldsymbol{\lambda}) = \int d^n x \, x_1 \cdots x_n p_n(\mathbf{x}, \boldsymbol{\lambda}). \tag{3.100}$$

With the notion of the moments of the distribution $\kappa(x)$

$$\mu_n = \int dx \, x^n \kappa(x) \tag{3.101}$$

the contributions for $C_n(\boldsymbol{\lambda})$ are given by

$$\begin{aligned} \gamma_{\text{vertex}} &= \mu_{|\mathcal{V}|} \frac{1}{\prod_{v\in\mathcal{V}} \lambda_v} \sum_{\mathcal{J}\in\mathcal{P}(\mathcal{V})} (-1)^{|\mathcal{J}|} \phi(\Lambda_{\mathcal{J}\cup\mathcal{L}}) \\ \gamma_{\text{line}} &= \frac{1}{1 - \phi(\Lambda_{\mathcal{L}})}. \end{aligned} \tag{3.102}$$

Equipped with these tools let us have a look at the spectral properties. Assume that the distribution $\phi(t)$ is in the domain of normal attraction of the one-sided Lévy-stable

distribution with exponent α ($0 < \alpha < 1$). I define the Fourier-transform of the process $x(t)$ up to time T as

$$F_T(\omega) = \frac{1}{T^{\alpha/2}} \int_0^T dt\, e^{i\omega t} x(t). \tag{3.103}$$

I want to draw attention to the non standard normalization which is employed here. For a fixed T this definition (apart from the normalization) corresponds to the Fourier transform one encounters in practice: if we have measured a time series of length T and one does a Fourier decomposition with periodic continuation the above definition coincides for the frequency values

$$\omega_n = \frac{2\pi}{nT} \text{ with } n \in \mathbb{N} \tag{3.104}$$

for which the Fourier decomposition is defined. If one would have measured only one length T and looks only at the spectrum one would not see the normalization constant. The spectrum is now given by

$$\begin{aligned} S_T(\omega) &= F_T(\omega) F_T(-\omega) \\ S(\omega) &= \lim_{T\to\infty} S_T(\omega). \end{aligned} \tag{3.105}$$

At first I am interested in the expectation value of the spectrum $\langle S_T(\omega)\rangle$. To be able to apply the multidimensional Tauberian theorems [Dro83, DZ80] I will consider

$$\langle F_{T_1}(\omega) F_{T_2}(-\omega)\rangle = \frac{1}{(T_1 T_2)^{\alpha/2}} \int_0^{T_1} dt_1 \int_0^{T_2} dt_2\, e^{i\omega(t_1-t_2)} \langle x(t_1) x(t_2)\rangle. \tag{3.106}$$

The Laplace transform of $(T_1T_2)^{\alpha/2}\langle F_{T_1}(\omega)F_{T_2}(-\omega)\rangle$ is then given by (with λ_1 corresponding to T_1 and λ_2 to T_2 respectively)

$$L[(T_1T_2)^{\alpha/2}\langle F_{T_1}(\omega)F_{T_2}(-\omega)\rangle] = \frac{1}{\lambda_1\lambda_2} C_2(\lambda_1 - i\omega, \lambda_2 + i\omega). \tag{3.107}$$

Assume for simplicity that $\kappa(x)$ is centered (i.e., $\mu_1 = 0$) this leads to

$$C_2(\lambda_1, \lambda_2) = \frac{\mu_2}{\lambda_1\lambda_2} \frac{1 - \phi(\lambda_1) - \phi(\lambda_2) + \phi(\lambda_1 + \lambda_2)}{1 - \phi(\lambda_1 + \lambda_2)} \tag{3.108}$$

and therefore

$$\begin{aligned} L[(T_1T_2)^{\alpha/2}\langle F_{T_1}(\omega)F_{T_2}(-\omega)\rangle] =& \frac{\mu_2}{\lambda_1\lambda_2} \frac{1}{(\lambda_1 - i\omega)(\lambda_2 + i\omega)} \\ & \times \frac{1 - \phi(\lambda_1 - i\omega) - \phi(\lambda_2 + i\omega) + \phi(\lambda_1 + \lambda_2)}{1 - \phi(\lambda_1 + \lambda_2)}. \end{aligned} \tag{3.109}$$

According to the multidimensional Tauberian theorem by Drozhzhinov [Dro83]

$$\lim_{r\to\infty} \frac{1}{r^\alpha} L[(rT_1 rT_2)^{\alpha/2}\langle F_{rT_1}(\omega)F_{rT_2}(-\omega)\rangle] = \lim_{r\to\infty} L[(T_1T_2)^{\alpha/2}\langle F_{rT_1}(\omega)F_{rT_2}(-\omega)\rangle] \tag{3.110}$$

can be calculated in Laplace space as

$$\lim_{\zeta\to 0} \zeta^{2+\alpha} \frac{1}{\zeta^2\lambda_1\lambda_2} C_2(\zeta\lambda_1 - i\omega, \zeta\lambda_2 + i\omega) = \frac{1}{\lambda_1\lambda_2} \lim_{\zeta\to 0} \zeta^\alpha C_2(\zeta\lambda_1 - i\omega, \zeta\lambda_2 + i\omega). \tag{3.111}$$

For $\omega \neq 0$ and $\phi(\lambda) = 1 - (\tau_0\lambda)^\alpha + o(\lambda^\alpha)$ one gets

$$\lim_{\zeta\to 0} \zeta^\alpha C_2(\zeta\lambda_1 - i\omega, \zeta\lambda_2 + i\omega) = \frac{\mu_2}{\tau_0^\alpha(\lambda_1+\lambda_2)^\alpha} \frac{2-\phi(i\omega)-\phi(-i\omega)}{\omega^2}. \tag{3.112}$$

From the inverse Laplace transform

$$L^{-1}\left[\frac{1}{\lambda_1\lambda_2}\frac{1}{(\lambda_1+\lambda_2)^\alpha}\right] = \frac{\min(T1,T2)^\alpha}{\Gamma(1+\alpha)} \tag{3.113}$$

(which can simply be checked by applying the Laplace transform) one obtains

$$\lim_{r\to\infty} \langle F_{rT_1}(\omega) F_{rT_2}(-\omega)\rangle = \frac{\mu_2}{\tau_0^\alpha\Gamma(1+\alpha)} \frac{1}{\max(T_1,T_2)^\alpha} \frac{2-\phi(i\omega)-\phi(-i\omega)}{\omega^2}. \tag{3.114}$$

The expectation value for the spectrum is then (by setting ($T_1 = T_2 = 1$)

$$\begin{aligned} \langle S(\omega)\rangle &= \frac{\mu_2}{\tau_0^\alpha\Gamma(1+\alpha)} \frac{2-\phi(i\omega)-\phi(-i\omega)}{\omega^2} \\ &= \frac{2\mu_2\cos\left(\frac{\pi}{2}\alpha\right)}{\tau_0^\alpha\Gamma(1+\alpha)} \frac{1}{|\omega|^{2-\alpha}} + o\left(\frac{1}{|\omega|^{2-\alpha}}\right) \qquad \text{for } \omega\to 0. \end{aligned} \tag{3.115}$$

Therefore we see in this model a typical $1/f^\beta$ noise (as long as we have this non standard normalization; with standard normalization we could not have a singularity at zero in the frequency spectrum since as soon as μ_2 exists, the process has finite power).

But the result is more general, it describes the whole spectrum. Assume that $\phi(t)$ is not only in the domain of attraction, but is actually a one-sided Lévy stable distribution, i.e.,

$$\phi(\lambda) = e^{-(\tau_0\lambda)^\alpha}. \tag{3.116}$$

The exact expression for the expectation value of the spectrum becomes

$$\begin{aligned} \langle S(\omega)\rangle &= \frac{\mu_2}{\tau_0^\alpha\Gamma(1+\alpha)} \frac{2-\exp\left[-|\tau_0\omega|^\alpha\exp\left(i\frac{\pi}{2}\alpha\right)\right]-\exp\left[-|\tau_0\omega|^\alpha\exp\left(-i\frac{\pi}{2}\alpha\right)\right]}{\omega^2} \\ &= \frac{2\mu_2}{\tau_0^\alpha\Gamma(1+\alpha)} \frac{1-\cos\left[\sin\left(\frac{\pi}{2}\alpha\right)|\tau_0\omega|^\alpha\right]\exp\left[-\cos\left(\frac{\pi}{2}\alpha\right)|\tau_0\omega|^\alpha\right]}{\omega^2}. \end{aligned} \tag{3.117}$$

As worked out by Rebenshtok and Barkai [RB07, RB08] the weak ergodicity breaking in this model manifests itself in the property that the time average of the observable does not converge towards its expectation value. The reason for this is that the relaxation of this process is so slow due to the waiting times that the time average stays a proper probability distribution in the long time limit. Since the spectrum is also a time average one can ask the question if the distribution of the limit $S(\omega)$ collapses to its expectation value $\langle S(\omega)\rangle$ or will stay a nontrivial probability distribution. Unfortunately, the complete distribution does not seem to be accessible, but it is possible to calculate the variance $\mathrm{Var}[S(\omega)] = \langle S^2(\omega)\rangle - \langle S(\omega)\rangle^2$ of the process which would vanish if the spectrum would converge to its expectation value.

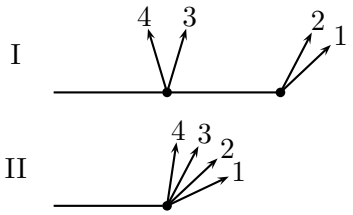

Figure 3.4.: The diagrams relevant for the calculation of the variance of $S(\omega)$

Therefore the value of $\langle S^2(\omega)\rangle$ needs to be determined. As for the value of $\langle S(\omega)\rangle$ I look first at a slightly more general situation to be able to apply the generalized Tauberian theorem:

$$\begin{aligned}(T_1T_2T_3T_4)^{\alpha/2}&F_{T_1}(\omega)F_{T_2}(\omega)F_{T_3}(-\omega)F_{T_4}(-\omega)\\ &= \int_0^{T_1} dt_1 \int_0^{T_2} dt_2 \int_0^{T_3} dt_3 \int_0^{T_4} dt_4 e^{i\omega(t_1+t_2-t_3-t_4)} C_4(t_1,t_2,t_3,t_4)\end{aligned} \tag{3.118}$$

of which the Laplace transform reads

$$\begin{aligned}L[(T_1T_2T_3T_4)^{\alpha/2}&F_{T_1}(\omega)F_{T_2}(\omega)F_{T_3}(-\omega)F_{T_4}(-\omega)]\\ &= \frac{1}{\lambda_1\lambda_2\lambda_3\lambda_4} C_4(\lambda_1-i\omega,\lambda_2-i\omega,\lambda_3+i\omega,\lambda_4+i\omega).\end{aligned} \tag{3.119}$$

The limit is given by the generalized Tauberian theorem [Dro83]:

$$\begin{aligned}L[(T_1T_2T_3T_4)^{\alpha/2}&\lim_{r\to\infty} F_{rT_1}(\omega)F_{rT_2}(\omega)F_{rT_3}(-\omega)F_{rT_4}(-\omega)]\\ &= \frac{1}{\lambda_1\lambda_2\lambda_3\lambda_4} \lim_{\zeta\to 0} \zeta^{2\alpha} C_4(\zeta\lambda_1-i\omega,\zeta\lambda_2-i\omega,\zeta\lambda_3+i\omega,\zeta\lambda_4+i\omega).\end{aligned} \tag{3.120}$$

Next, I am determining $C_4(\lambda_1',\lambda_2',\lambda_3',\lambda_4')$, respectively its unsymmetric version $\gamma_4(\lambda_1',\lambda_2',\lambda_3',\lambda_4')$ with

$$C_4(\lambda_1',\lambda_2',\lambda_3',\lambda_4') = \sum_{\eta\in S_4} \gamma_4(\lambda_{\eta(1)}',\lambda_{\eta(2)}',\lambda_{\eta(3)}',\lambda_{\eta(4)}'). \tag{3.121}$$

Since I am assuming that the mean of $\rho(x)$ is vanishing (i.e., $\mu_1 = 0$) there will not be any contribution from diagrams that have vertices with one leaving line. For the four point correlation this leaves only the two diagrams shown in figure 3.4. This gives

$$\begin{aligned}\gamma^{\mathrm{I}}(\boldsymbol{\lambda}') &= \frac{1}{4}\frac{\mu_2^2}{\lambda_1'\lambda_2'\lambda_3'\lambda_4'}\frac{\phi(\Lambda_2')-2\phi(\Lambda_3')+\phi(\Lambda_4')}{1-\phi(\Lambda_4')}\frac{1-2\phi(\Lambda_1')+\phi(\Lambda_2')}{1-\phi(\Lambda_2')}\\ \gamma^{\mathrm{II}}(\boldsymbol{\lambda}') &= \frac{1}{24}\frac{\mu_4}{\lambda_1'\lambda_2'\lambda_3'\lambda_4'}\frac{1-4\phi(\Lambda_1')+6\phi(\Lambda_2')-4\phi(\Lambda_3')+\phi(\Lambda_4')}{1-\phi(\Lambda_4')}\\ \gamma(\boldsymbol{\lambda}') &= \gamma^{\mathrm{I}}(\boldsymbol{\lambda}')+\gamma^{\mathrm{II}}(\boldsymbol{\lambda}')\end{aligned} \tag{3.122}$$

where I have used again the notation $\Lambda_n' = \lambda_1' + \cdots + \lambda_n'$. If we now plug in the parameters of equation (3.120)

$$\lambda_1' = \zeta\lambda_1 - i\omega, \quad \lambda_2' = \zeta\lambda_2 - i\omega, \quad \lambda_3' = \zeta\lambda_3 + i\omega, \quad \lambda_4' = \zeta\lambda_4 + i\omega \tag{3.123}$$

we have for every permutation $\eta \in S_4$

$$\lim_{\zeta\to 0} \zeta^{2\alpha}\gamma^{\mathrm{II}}(\eta\boldsymbol{\lambda}') = 0 \tag{3.124}$$

as the numerator of $\gamma^{\mathrm{II}}(\boldsymbol{\lambda}')$ is not singular and the denominator behaves around $\zeta \to 0+$ as

$$\lambda_1'\lambda_2'\lambda_3'\lambda_4'(1-\phi(\Lambda_4')) = \omega^4\tau_0^\alpha(\lambda_1+\lambda_2+\lambda_3+\lambda_4)^\alpha\zeta^\alpha + o(\zeta^\alpha) \tag{3.125}$$

independently of the permutation η. Therefore diagram II does not contribute in the long time limit.

Likewise we have to determine the permutations $\eta \in S_4$ such that

$$\lim_{\zeta\to 0} \zeta^{2\alpha}\gamma^{\mathrm{I}}(\eta\boldsymbol{\lambda}') \neq 0. \tag{3.126}$$

The numerators of $\gamma^{\mathrm{I}}(\eta\boldsymbol{\lambda}')$ are always nonsingular while the first part of the denominators behaves as described by equation (3.125). To have a non-vanishing limit the remaining denominator $1-\phi(\Lambda_2')$ should behave as ζ^α. This is exactly the case when the $i\omega$ parts in $\lambda'_{\eta(1)}$ and $\lambda'_{\eta(2)}$ cancel. In total the following possibilities arise

$$\lambda_1' = \zeta\lambda_1 - i\omega, \quad \lambda_2' = \zeta\lambda_3 + i\omega, \quad \lambda_3' = \zeta\lambda_2 - i\omega, \quad \lambda_4' = \zeta\lambda_4 + i\omega \tag{3.127}$$
$$\lambda_1' = \zeta\lambda_2 - i\omega, \quad \lambda_2' = \zeta\lambda_3 + i\omega, \quad \lambda_3' = \zeta\lambda_1 - i\omega, \quad \lambda_4' = \zeta\lambda_4 + i\omega \tag{3.128}$$
$$\lambda_1' = \zeta\lambda_1 - i\omega, \quad \lambda_2' = \zeta\lambda_4 + i\omega, \quad \lambda_3' = \zeta\lambda_2 - i\omega, \quad \lambda_4' = \zeta\lambda_3 + i\omega \tag{3.129}$$
$$\lambda_1' = \zeta\lambda_2 - i\omega, \quad \lambda_2' = \zeta\lambda_4 + i\omega, \quad \lambda_3' = \zeta\lambda_1 - i\omega, \quad \lambda_4' = \zeta\lambda_3 + i\omega \tag{3.130}$$

plus the ones that appear by exchanging λ_1' and λ_2', or λ_3' and λ_4'. Adding these up gives

$$\begin{aligned}
&\lim_{\zeta\to 0}\zeta^{2\alpha}C_4(\zeta\lambda_1 - i\omega, \zeta\lambda_2 - i\omega, \zeta\lambda_3 + i\omega, \zeta\lambda_4 + i\omega) \\
&\quad= \left(\frac{\mu_2}{\tau_0^\alpha}\right)^2 \frac{(2-\phi(i\omega)-\phi(-i\omega))^2}{\omega^4}\frac{1}{(\lambda_1+\lambda_2+\lambda_3+\lambda_4)^\alpha} \\
&\qquad\times\left(\frac{1}{(\lambda_1+\lambda_3)^\alpha} + \frac{1}{(\lambda_1+\lambda_4)^\alpha} + \frac{1}{(\lambda_2+\lambda_3)^\alpha} + \frac{1}{(\lambda_2+\lambda_4)^\alpha}\right) \\
&\quad= \Gamma(1+\alpha)^2\langle S(\omega)\rangle^2\frac{1}{(\lambda_1+\lambda_2+\lambda_3+\lambda_4)^\alpha} \\
&\qquad\times\left(\frac{1}{(\lambda_1+\lambda_3)^\alpha} + \frac{1}{(\lambda_1+\lambda_4)^\alpha} + \frac{1}{(\lambda_2+\lambda_3)^\alpha} + \frac{1}{(\lambda_2+\lambda_4)^\alpha}\right).
\end{aligned} \tag{3.131}$$

It remains to invert the Laplace transform

$$\frac{1}{\lambda_1\lambda_2\lambda_3\lambda_4}\frac{1}{(\lambda_1+\lambda_2+\lambda_3+\lambda_4)^\alpha}\left(\frac{1}{(\lambda_1+\lambda_3)^\alpha} + \frac{1}{(\lambda_1+\lambda_4)^\alpha} + \frac{1}{(\lambda_2+\lambda_3)^\alpha} + \frac{1}{(\lambda_2+\lambda_4)^\alpha}\right). \tag{3.132}$$

As in section 3.3.1 (equation (3.58)) we can invert the Laplace transform of each summand as

$$\begin{aligned}
&L^{-1}\left[\frac{1}{\lambda_1\lambda_2\lambda_3\lambda_4}\frac{1}{(\lambda_1+\lambda_2+\lambda_3+\lambda_4)^\alpha}\frac{1}{(\lambda_1+\lambda_3)^\alpha}\right] \\
&\quad= \frac{\min(T1,T2,T3,T4)^\alpha\min(T1,T3)^\alpha}{\Gamma(1+\alpha)^2}F\left(\alpha,-\alpha;1+\alpha;\frac{\min(T1,T2,T3,T4)}{\min(T1,T3)}\right).
\end{aligned} \tag{3.133}$$

Evaluating at $T_1 = T_2 = T_3 = T_4 = 1$ we get finally

$$\langle S^2(\omega)\rangle = 4\frac{\Gamma(1+\alpha)^2}{\Gamma(1+2\alpha)}\langle S(\omega)\rangle^2, \tag{3.134}$$

which gives as variance

$$\mathrm{Var}[S(\omega)] = \left(4\frac{\Gamma(1+\alpha)^2}{\Gamma(1+2\alpha)} - 1\right)\langle S(\omega)\rangle^2. \tag{3.135}$$

As the prefactor is between 1 and 3 the standard deviation of $S(\omega)$ is of the order of the expected value for $S(\omega)$.

Since the values of $S(\omega)$ stay random variables in the long time limit, one can ask about the correlations between the determined values of the spectrum at different frequencies. One measure for this is the covariance between $S(\omega_1)$ and $S(\omega_2)$

$$\mathrm{Cov}[S(\omega_1), S(\omega_2)] = \langle S(\omega_1)S(\omega_2)\rangle - \langle S(\omega_1)\rangle\langle S(\omega_2)\rangle. \tag{3.136}$$

For simplicity I will assume $\omega_1 \neq \omega_2$. The quantity which I have not yet determined is $\langle S(\omega_1)S(\omega_2)\rangle$. Doing the steps as before leads to

$$\begin{aligned} &L[(T_1T_2T_3T_4)^{\alpha/2}F_{T_1}(\omega_1)F_{T_2}(-\omega_1)F_{T_3}(\omega_2)F_{T_4}(-\omega_2)] \\ &\qquad = \frac{1}{\lambda_1\lambda_2\lambda_3\lambda_4}C_4(\lambda_1 - i\omega_1, \lambda_2 + i\omega_1, \lambda_3 - i\omega_2, \lambda_4 + i\omega_2). \end{aligned} \tag{3.137}$$

For $\omega_1 \neq \omega_2$ only half of the possible permutations (3.127)–(3.130) survive:

$$\lambda_1' = \zeta\lambda_1 - i\omega_1, \qquad \lambda_2' = \zeta\lambda_2 + i\omega_1, \qquad \lambda_3' = \zeta\lambda_3 - i\omega_2, \qquad \lambda_4' = \zeta\lambda_4 + i\omega_2 \tag{3.138}$$

$$\lambda_1' = \zeta\lambda_3 - i\omega_2, \qquad \lambda_2' = \zeta\lambda_4 + i\omega_2, \qquad \lambda_3' = \zeta\lambda_1 - i\omega_1, \qquad \lambda_4' = \zeta\lambda_2 + i\omega_1 \tag{3.139}$$

(plus of course the ones that appear by exchanging λ_1' and λ_2', or λ_3' and λ_4'). This reduces the limit to

$$\begin{aligned} &\lim_{\zeta\to 0}\zeta^{2\alpha}C_4(\zeta\lambda_1 - i\omega_1, \zeta\lambda_2 + i\omega_1, \zeta\lambda_3 - i\omega_2, \zeta\lambda_4 + i\omega_2) \\ &\quad = \left(\frac{\mu_2}{\tau_0^\alpha}\right)^2 \frac{2 - \phi(i\omega_1) - \phi(-i\omega_1)}{\omega^2}\frac{2 - \phi(i\omega_2) - \phi(-i\omega_2)}{\omega^2} \\ &\qquad \times \frac{1}{(\lambda_1+\lambda_2+\lambda_3+\lambda_4)^\alpha}\left(\frac{1}{(\lambda_1+\lambda_2)^\alpha} + \frac{1}{(\lambda_3+\lambda_4)^\alpha}\right) \\ &\quad = \Gamma(1+\alpha)^2\langle S(\omega_1)\rangle\langle S(\omega_2)\rangle\frac{1}{(\lambda_1+\lambda_2+\lambda_3+\lambda_4)^\alpha}\left(\frac{1}{(\lambda_1+\lambda_2)^\alpha} + \frac{1}{(\lambda_3+\lambda_4)^\alpha}\right). \end{aligned} \tag{3.140}$$

Using again equation (3.133) gives

$$\langle S(\omega_1)S(\omega_2)\rangle = 2\frac{\Gamma(1+\alpha)^2}{\Gamma(1+2\alpha)}\langle S(\omega_1)\rangle\langle S(\omega_2)\rangle \tag{3.141}$$

and therefore the covariance

$$\mathrm{Cov}[S(\omega_1), S(\omega_2)] = \left(2\frac{\Gamma(1+\alpha)^2}{\Gamma(1+2\alpha)} - 1\right)\langle S(\omega_1)\rangle\langle S(\omega_2)\rangle \tag{3.142}$$

and the correlation coefficient

$$\begin{aligned}\rho[S(\omega_1), S(\omega_2)] &= \frac{\mathrm{Cov}[S(\omega_1), S(\omega_2)]}{\sqrt{\mathrm{Var}[S(\omega_1)]\,\mathrm{Var}[S(\omega_2)]}} \\ &= \frac{2\Gamma(1+\alpha)^2 - \Gamma(1+2\alpha)}{4\Gamma(1+\alpha)^2 - \Gamma(1+2\alpha)}.\end{aligned} \tag{3.143}$$

Interestingly, the correlation coefficient depends only on the parameter α describing the waiting time distribution and is independent of the two frequencies ω_1 and ω_2 involved. Furthermore, it is always in the domain $\rho[S(\omega_1), S(\omega_2)] \in [0, \frac{1}{3}]$, i.e., two values in the spectrum are always positively correlated, but this correlation is always weak.

In general, the observed spectrum at a specific frequency stays a fluctuating quantity even for standard processes, e.g., white Gaussian noise. Therefore one usually applies techniques as binning for spectral estimates. It remains to be investigated if there are cases of weak ergodicity breaking for such observables. Nevertheless, the observed variance (equation (3.135)) exceeds the usual spreading.

4. Asymptotic behavior of continuous-time random walks

4.1. Introduction

The purpose of this chapter is to derive conditions which allow to determine the jump density of the scaling limit of a continuous-time random walk from the probability distribution describing a single step. Both descriptions have been introduced in section 2.3. For simplicity, I will restrict the derivations to the case of the leaper model of the CTRW and to the domains of normal attraction. In a first step, I will consider the situation where the waiting time has a finite mean. The resulting random walks are Markovian. The more interesting situation emerges for a diverging mean waiting time. After this the diagrammatic approach to the joint probability distributions is extended to the usage of a jump density. With this, one can derive the limit description. As these derivations are quite technical and not very illuminating, I am going to summarize the results without the derivations at the end.

It will be convenient to have the formulation of an *essential limit* in the spirit of other definitions as, e.g., essential supremum. Assume, we have a measure κ on $\mathbb{R}_{\geq 0}$ which is not bounded, i.e., we have $\kappa([C, \infty[) > 0$ for all $C > 0$. Further, let $f : \mathbb{R}_{\geq 0} \to \mathbb{C}$ a function (which needs only to be defined up to sets of measure 0). The value $c \in \mathbb{C}$ is the essential limit of $f(t)$, written

$$\operatorname*{ess\,lim}_{t\to\infty} f(t) = c, \tag{4.1}$$

if

$$\lim_{t\to\infty} \operatorname{ess\,sup}\{|f(\tau) - c| : \tau > t\} = 0. \tag{4.2}$$

This corresponds to the normal limit definition if we neglect sets of measure 0.

4.2. Scaling Limit with Finite Mean Waiting Time

In this Section I want to give an example of how to use the method introduced in the last chapter to determine limits. I want to look at the scaling limit in case the waiting time distribution has a finite mean (or additionally a finite second moment, depending on the definition used when the mean of the spatial step size distribution does not vanish). I am going to show that under these conditions all finite point joint probability distributions become Markovian in the limit. The result in the case of spatial step size with zero mean or of a finite first and second moment of the waiting time distribution is not new (e.g., see Dentz and Berkowitz [DB03]), but I think that it is nevertheless instructive to look at the argument.

I recall the definition of the scaling limit in the domain of normal attraction (introduced in section 2.2) as the limit (of all finite joint probability distributions)

$$X^{\lim}(t) = \zeta^{\beta} X(\frac{t}{\zeta}) \text{ for } \zeta \to 0 \tag{4.3}$$

with the scaling exponent β. Writing the convergence conditions for the probability density, yields the limit as

$$p_n^{\lim}(\mathbf{x}, \mathbf{t}) = \lim_{\zeta \to 0} \frac{1}{\zeta^{n\beta}} p_n(\frac{1}{\zeta^\beta}\mathbf{x}, \frac{1}{\zeta}\mathbf{t}) \tag{4.4}$$

(more precisely, this should be a limit of measures). Going to Fourier-Laplace space and the more convenient $\breve{p}_n$, gives

$$p_n^{\lim}(\mathbf{k}, \boldsymbol{\lambda}) = \lim_{\zeta \to 0} \zeta^n p_n(\zeta^\beta \mathbf{k}, \zeta \boldsymbol{\lambda}) \tag{4.5}$$

as the limit we are interested in. The scaling exponent β is chosen to be the largest value with this object still depending non-trivially on $\mathbf{k}$.

Now, I focus on the case that the waiting time distribution has the finite mean value τ and spatial step distribution has zero mean. To avoid distracting technicalities, we will restrict ourselves to the space-time independent case where the spatial step has variance σ^2 which will lead to Brownian motion. The argument generalizes straightforwardly to other settings. The assumptions give rise to the following asymptotic behavior of the Fourier-Laplace transform around $\lambda, k \to 0$

$$\phi(\lambda) = 1 - \tau\lambda + o(\lambda) \tag{4.6}$$

and

$$\psi(k, \lambda) = \phi(\lambda)\left(1 - \frac{\sigma^2}{2}k^2 + o(k^2)\right). \tag{4.7}$$

where I use the Landau notation. The scaling exponent is $\beta = 1/2$. Let us first look at the denominators stemming from the lines. Its contributions are

$$\begin{aligned} \rho_{\text{line},\mathcal{L}}^{-1}(\zeta^{1/2}\mathbf{k}, \zeta\boldsymbol{\lambda}) &= 1 - \psi(\zeta^{1/2}K_{\mathcal{L}}, \zeta\Lambda_{\mathcal{L}}) \\ &= \zeta\left(\tau\Lambda_{\mathcal{L}} + \frac{\sigma^2}{2}K_{\mathcal{L}}^2\right) + o(\zeta). \end{aligned} \tag{4.8}$$

Now I move on to the contribution of a vertex. The $\mathbf{k}$-contribution factors out by our assumptions and becomes unity in the scaling limit (which in retrospect justifies the scaling exponent because it is therefore determined purely by the behavior of the denominators). A degree one vertex gives the contribution

$$\begin{aligned} \rho_{\text{vertex},\mathcal{V}=\{v\},\mathcal{L}}(\zeta^{1/2}\mathbf{k}, \zeta\boldsymbol{\lambda}) &= \frac{1}{\zeta\lambda_v}\left(\phi(\zeta\Lambda_{\mathcal{L}}) - \phi(\zeta\Lambda_{\mathcal{L}\cup\{v\}}) + o(\zeta)\right) \\ &= \tau + o(1). \end{aligned} \tag{4.9}$$

For the vertices of higher degree I apply the definition used in equation (3.25)

$$\begin{aligned} &\rho_{\substack{\text{vertex},\mathcal{V},\mathcal{L} \\ |\mathcal{V}|\geq 2}}(\zeta^{1/2}\mathbf{k}, \zeta\boldsymbol{\lambda}) \\ &\quad= \frac{1}{\zeta^{|\mathcal{V}|}\prod_{v\in\mathcal{V}}\lambda_v}\left(\sum_{\mathcal{J}\in\mathcal{P}(\mathcal{V})}(-1)^{|\mathcal{J}|}\phi\left(\zeta(\Lambda_{\mathcal{L}} + \Lambda_{\mathcal{J}})\right) + o(\zeta)\right) \\ &\quad= \frac{1}{\zeta^{|\mathcal{V}|}\prod_{v\in\mathcal{V}}\lambda_v}\left(\zeta\tau\sum_{\mathcal{J}\in\mathcal{P}(\mathcal{V})}(-1)^{|\mathcal{J}|}\Lambda_{\mathcal{J}} + o(\zeta)\right) \\ &\quad= o(\zeta^{1-|\mathcal{V}|}). \end{aligned} \tag{4.10}$$

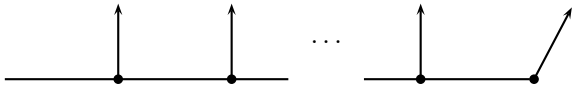

Figure 4.1.: This diagrams represents the Markovian contribution of the random walk. It is the only one that survives in the scaling limit with finite mean waiting time.

Since we have the same number of lines and vertices, all terms with vertices of degree two or higher vanish in the scaling limit. Therefore only the diagram in figure 4.1 survives. The interpretation is that we can neglect the case with two or more times in the same step. The scaling limit gives

$$\breve{p}_n^{\lim}(\mathbf{k},\boldsymbol{\lambda}) = \frac{\tau}{\tau\Lambda_n + \frac{\sigma^2}{2}K_n}\cdots\frac{\tau}{\tau\Lambda_1 + \frac{\sigma^2}{2}K_1} \tag{4.11}$$

and by symmetrizing

$$p_n^{\lim}(\mathbf{k},\boldsymbol{\lambda}) = \sum_{\sigma\in S_n}\breve{p}_n^{\lim}(\sigma\mathbf{k},\sigma\boldsymbol{\lambda}). \tag{4.12}$$

With

$$p_1^{\lim}(x,t) = \sqrt{\frac{\tau}{2\pi\sigma^2}\frac{1}{t}}\exp\left(-\frac{\tau}{2\sigma^2}\frac{x^2}{t}\right) \tag{4.13}$$

being the inverse Fourier-Laplace transform of

$$p_1^{\lim}(k,\lambda) = \frac{1}{\lambda + \frac{\sigma^2}{2\tau}k^2} \tag{4.14}$$

one gets for $p_n^{\lim}(\mathbf{x},\mathbf{t})$ with $t_1 \geq t_2 \geq \cdots \geq t_{n-1} \geq t_n$

$$p_n^{\lim}(\mathbf{x},\mathbf{t}) = p_1^{\lim}(x_n,t_n)p_1^{\lim}(x_{n-1}-x_n,t_{n-1}-t_n)\cdots p_1^{\lim}(x_1-x_2,t_1-t_2) \tag{4.15}$$

which is exactly the multi point density of Brownian motion.

Let us now have a look at the case where the spatial step distribution has a finite mean $\mu \neq 0$. Using the definition equation (4.3) with $\alpha = 1$ and the same arguments gives then

$$p_n^{\lim}(\mathbf{x},\mathbf{t}) = \prod_{i=1}^{n}\delta\left(x_i - \frac{\mu}{\tau}t_i\right), \tag{4.16}$$

which corresponds to a uniform deterministic motion with velocity $\frac{\mu}{\tau}$. Therefore, we have to use definition (2.70) with $b(\zeta) = \frac{\mu}{\tau}$. The scaling limit is then

$$p_n^{\lim}(\mathbf{k},\boldsymbol{\lambda}) = \lim_{\zeta\to 0}\zeta^n p_n(\zeta^\beta\mathbf{k},\zeta\boldsymbol{\lambda} + i\frac{\mu}{\tau}\zeta^\beta\mathbf{k}). \tag{4.17}$$

If we want this to converge to a normal diffusion (with $\beta = \frac{1}{2}$), we additionally need that the second moment of the waiting time distribution is finite to account for the different orders of ζ appearing in the second argument. Then the line of argument is essentially the same as above.

The question remains what happens if the waiting time has a finite mean but an infinite variance. It is known that the resulting process describes non-Fickian transport [DB03].

But as Fickian transport needs Markovianity and Gaussianity, it is not clear which of this conditions is violated. It will turn out, that the limit process is Markovian, but not Gaussian.

Assume that the waiting time distribution has the asymptotic behavior

$$\phi(t) \propto \frac{1}{t^{1+\alpha}} \quad \text{for } t \to \infty \tag{4.18}$$

with $1 < \alpha < 2$. This corresponds to the asymptotic behavior of the Laplace transform around $\lambda \searrow 0$

$$\phi(\lambda) = 1 - \tau\lambda + \gamma\lambda^\alpha + o(\lambda^\alpha). \tag{4.19}$$

The jump size distribution is assumed to have finite mean μ, such that

$$\psi(k,\lambda) = \phi(\lambda)\left(1 + i\mu k + o(k)\right). \tag{4.20}$$

For simplicity, I assume $\mu > 0$ (the case $\mu < 0$ will only change some signs). The scaling exponent is $\beta = \frac{1}{\alpha}$ (that this is the correct choice is shown by the calculations below). Now, using the same approach

$$\begin{aligned}\rho^{-1}_{\text{line},\mathcal{L}}(\zeta^{1/2}\mathbf{k}, \zeta\boldsymbol{\lambda} + i\frac{\mu}{\tau}\zeta^\beta\mathbf{k}) &= 1 - \psi(\zeta^\beta K_\mathcal{L}, \zeta\Lambda_\mathcal{L} + i\frac{\mu}{\tau}\zeta^\beta K_\mathcal{L}) \\ &= \zeta\left(\tau\lambda - \gamma\left|\frac{\mu}{\tau}\right| e^{i\frac{\pi}{2}\beta\,\mathrm{sgn}(k)}|k|^\beta\right) + o(\zeta).\end{aligned} \tag{4.21}$$

The vertices give

$$\begin{aligned}&\rho_{\text{vertex},\mathcal{V}=\{v\},\mathcal{L}}(\zeta^\beta\mathbf{k}, \zeta\boldsymbol{\lambda} + i\frac{\mu}{\tau}\zeta^\beta\mathbf{k}) \\ &\quad= \frac{1}{\zeta\lambda_v + i\frac{\mu}{\tau}\zeta^\beta k_v}\left(\phi\left(\zeta\Lambda_\mathcal{L} + i\frac{\mu}{\tau}\zeta^\beta K_\mathcal{L}\right) - \phi\left(\zeta\Lambda_{\mathcal{L}\cup\{v\}} + i\frac{\mu}{\tau}\zeta^\beta K_{\mathcal{L}\cup\{v\}}\right)\right) \\ &\qquad\times\left(1 + O(\zeta^\beta)\right) \\ &\quad= \tau + o(1)\end{aligned} \tag{4.22}$$

and (for $k_v \neq 0$)

$$\begin{aligned}&\rho_{\substack{\text{vertex},\mathcal{V},\mathcal{L}\\|\mathcal{V}|\geq 2}}(\zeta^\beta\mathbf{k}, \zeta\boldsymbol{\lambda} + i\frac{\mu}{\tau}\zeta^\beta\mathbf{k}) \\ &\quad= \frac{1}{\prod_{v\in\mathcal{V}}(\zeta\lambda_v + i\frac{\mu}{\tau}\zeta^\beta k_v)}\left(\sum_{\mathcal{J}\in\mathcal{P}(\mathcal{V})}(-1)^{|\mathcal{J}|}\phi\left(\zeta(\Lambda_{\mathcal{L}\cup\mathcal{J}} + i\frac{\mu}{\tau}\zeta^\beta K_{\mathcal{L}\cup\mathcal{J}}\right)\right) \\ &\qquad\times\left(1 + O(\zeta^\beta)\right) \\ &\quad= O(\zeta^{1-|\mathcal{V}|}).\end{aligned} \tag{4.23}$$

Therefore, if v_1 is the number of vertices of order one, and $v_{\geq 2}$ the number of vertices of order two or larger in a diagram D, we get the behavior

$$\begin{aligned}\zeta^n p_n[D](\zeta^\beta\mathbf{k}, \zeta\boldsymbol{\lambda} + i\frac{\mu}{\tau}\zeta^\beta\mathbf{k}) &= O\left(\zeta^n\frac{1}{\zeta^{v_1+v_{\geq 2}}}\zeta^{\beta(v_{\geq 2}-(n-v_1))}\right) \\ &= O\left(\zeta^{(n-v_1-v_{\geq 2})\beta}\right).\end{aligned} \tag{4.24}$$

The only possibility of getting a nontrivial contribution is $v_{\geq 2} = 0$, and therefore the diagrams of form 4.1 are the only ones to survive in the scaling limit. This means again, that the limit process is Markovian.

The scaling limit is

$$\breve{p}_n^{\lim}(\mathbf{k}, \boldsymbol{\lambda}) = p_1^{\lim}(K_n, \Lambda_n) \cdots p_1^{\lim}(K_1, \Lambda_1) \tag{4.25}$$

which gives the complete result by symmetrization:

$$p_n^{\lim}(\mathbf{k}, \boldsymbol{\lambda}) = \sum_{\sigma \in S_n} \breve{p}_n^{\lim}(\sigma\mathbf{k}, \sigma\boldsymbol{\lambda}). \tag{4.26}$$

This implies for $p_n^{\lim}(\mathbf{x}, \mathbf{t})$ with $t_1 \geq t_2 \geq \cdots \geq t_{n-1} \geq t_n$

$$p_n^{\lim}(\mathbf{x}, \mathbf{t}) = p_1^{\lim}(x_n, t_n) p_1^{\lim}(x_{n-1} - x_n, t_{n-1} - t_n) \cdots p_1^{\lim}(x_1 - x_2, t_1 - t_2). \tag{4.27}$$

This is clearly Markovian with a propagator

$$p_1^{\lim}(k, \lambda) = \frac{1}{\lambda - \frac{\gamma}{\tau} \left|\frac{\mu}{\tau}\right|^\alpha \exp\left(i\frac{\pi}{2}\alpha \operatorname{sgn}(k)\right) |k|^\alpha}. \tag{4.28}$$

Inverting the Laplace transform gives

$$p_1^{\lim}(k, t) = \exp\left(t\frac{\gamma}{\tau}\left|\frac{\mu}{\tau}\right|^\alpha \cos\left(\frac{\pi}{2}\alpha\right) |k|^\alpha \left(1 + i\tan\left(\frac{\pi}{2}\alpha\right)\operatorname{sgn}(k)\right)\right) \tag{4.29}$$

which corresponds to a maximally (negatively) skewed Lévy-stable distribution with exponent α (see equation (2.20), note that the cosine is negative for $1 < \alpha < 2$). This limit process is know as a Lévy-flight [[MK00], section 3.5]. This shows that the non-Fickianity is due to the non-Gaussianity of the process while it is still Markovian.

One should however note, that while the limit process does not posses second moments, the original process has. Especially, the variance has a definite asymptotic scaling behavior [DB03] which cannot be inferred from the limit process.

4.3. Joint probability distributions with jump densities

In chapter 3 a diagrammatic approach for the determination of the joint probability distributions of a CTRW was introduced. The derivation applied to classical case where there are discrete steps in the CTRW which are described by a probability distribution for the step size/waiting time. The limit behavior is formulated in the jump density formalism (see section 2.3). It is necessary to extend the diagrammatic formalism to densities to formulate the limits later in this chapter. Recalling equation (2.96), in continuous operational time we have the two processes $T(s)$ and $Y(s)$ which jointly form a Lévy process with the log-Fourier-Laplace transform $\rho_{\text{step}}(k, \lambda)$ such that

$$\langle e^{-\lambda T(s) + ikY(s)} \rangle = e^{-s\rho_{\text{step}}(k,\lambda)} \tag{4.30}$$

(the notation $\rho_{\text{step}}(k, \lambda)$ will become clear soon). The most general form for $\rho_{\text{step}}(k, \lambda)$ was given by equation (2.104).

The transition can be done by a limit consideration. For this, one can discretize the operational time in steps of length Δs and consider what happens during this time as a single step. Thus, I take as Fourier-Laplace transform of one step

$$\hat{\psi}^{(\Delta s)}(k,\lambda) = \exp\left(-\Delta s \rho_{\text{step}}(k,\lambda)\right) \tag{4.31}$$

and let $\Delta s \to 0$ at the end. Now, one can apply the results from the last chapter 3 for the contributions of a diagram (using the notation from chapter 3):

$$\begin{aligned}
\hat{\rho}^{(\Delta s)}_{\text{vertex}}(\mathbf{k},\boldsymbol{\lambda}) &= \frac{1}{\prod_{v\in\mathcal{V}}\lambda_v}\sum_{\mathcal{J}\in\mathcal{P}(\mathcal{V})}(-1)^{|\mathcal{J}|}\hat{\psi}^{(\Delta s)}(K_{\mathcal{L}},\Lambda_{\mathcal{J}\cup\mathcal{L}}) \\
&= \Delta s\frac{1}{\prod_{v\in\mathcal{V}}\lambda_v}\sum_{\mathcal{J}\in\mathcal{P}(\mathcal{V})}(-1)^{|\mathcal{J}|+1}\rho_{\text{step}}(K_{\mathcal{L}},\Lambda_{\mathcal{J}\cup\mathcal{L}}) + O(\Delta s^2) \\
\hat{\rho}^{(\Delta s)}_{\text{line}}(\mathbf{k},\boldsymbol{\lambda}) &= \frac{1}{1-\hat{\psi}^{(\Delta s)}(K_{\mathcal{L}},\Lambda_{\mathcal{L}})} \\
&= \frac{1}{\Delta s}\frac{1}{\rho_{\text{step}}(K_{\mathcal{L}},\Lambda_{\mathcal{L}})} + O(1).
\end{aligned} \tag{4.32}$$

As we have always the same number of lines and vertices, the following definitions give the same result as the limit $\Delta s \to 0$ with the rules (4.32)

$$\begin{aligned}
\rho_{\text{vertex}}(\mathbf{k},\boldsymbol{\lambda}) &= \frac{1}{\prod_{v\in\mathcal{V}}\lambda_v}\sum_{\mathcal{J}\in\mathcal{P}(\mathcal{V})}(-1)^{|\mathcal{J}|+1}\rho_{\text{step}}(K_{\mathcal{L}},\Lambda_{\mathcal{J}\cup\mathcal{L}}) \\
\rho_{\text{line}}(\mathbf{k},\boldsymbol{\lambda}) &= \frac{1}{\rho_{\text{step}}(K_{\mathcal{L}},\Lambda_{\mathcal{L}})}
\end{aligned} \tag{4.33}$$

By comparing the rules (4.33) with the ones for discrete steps (equations (3.20) and (3.21)) the previous formulation with the probability distribution $\psi(x,t)$ is reproduced by setting

$$\begin{aligned}
\rho_{\text{step}}(k,\lambda) &= 1-\psi(k,\lambda) \\
&= \int (1-e^{-\lambda t+ikx})\psi(x,t)\,dx\,dt.
\end{aligned} \tag{4.34}$$

Therefore, we can use the probability distribution as jump density (without regularization function) and obtain the same process $X(t)$. This becomes clear when one considers that this transition to the densities randomizes only the operational times at which these jumps happen. But as the operational time is only an auxiliary construct which is not visible in the resulting process, this does not effect the process itself.

Nevertheless, the formulation (4.33) is a real extension of the previous stepwise formulation of the CTRW since as soon as the jump density is not normalizable any more, there is no step probability distribution $\psi(x,t)$ corresponding to this process.

4.4. Asymptotic behavior of an independent CTRW

Before I start to look at continuous-time random walks with a possible coupling between successive steps, I want to introduce some criteria which allow in many cases to determine the limit distribution. To keep things simple, I will concentrate on domains of normal

attraction (i.e., I am disregarding the case of slowly varying corrections to the scaling law). As before one is interested in the scaling limit

$$X^{\lim}(t) = \lim_{\zeta \searrow 0} \zeta^{\beta} X(\frac{t}{\zeta}) \tag{4.35}$$

with β being the scaling exponent. The limit for the stochastic process is meant as the convergence of all finite multi-point distributions. The limit should be *full*, i.e., for $t > 0$ the distribution of $X^{\lim}(t)$ is not a δ-distribution.

The result of Becker-Kern, Meerschaert and Scheffler who have identified all possible limit forms [BKMS04, MBSBK02] was presented at the end of section 2.3. Their setting consists of looking at the random walk $(Y(s), T(s))$ depending on the operational time and then looking at the limit behavior where they allow for different scaling in spatial and time direction by using operator stable laws. The general form is given by [[BKMS04], theorem 2.2], respectively equation (2.110).

$$\rho^{\lim}_{\text{step}}(k,\lambda) = -ik\mu + \theta(k) + c \int (1 - e^{-\lambda t + ikx} + ikx\Xi(x)) \eta^{\lim}\left(\frac{dx}{t^{\beta}}\right) \frac{dt}{t^{1+\alpha}} \tag{4.36}$$

with the scaling exponent α of $t(s)$ $(0 < \alpha < 1)$ and the additional condition $\beta \geq \frac{\alpha}{2}$. I will mainly use the regularization functions $\Xi(x) = 0$ or $\Xi(x) = 1$, so I can set $\mu = 0$. Since the requirement is that the process $(Y(s), T(s))$ is full in both variables, this contains more information than actually is seen in the scaling limit (4.35). Let me give an example to illustrate this point: let $x(s)$ to be a purely deterministic drift with constant velocity $v > 0$: $x(s) = vs$. The process $T(s)$ is assumed be be a stable subordinator with exponent α. Viewed as the two-dimensional process $(Y(s), T(s))$, this is obviously not full (no randomness in x-direction). But as a CTRW, we have for the one-point distributions

$$p_1(k,\lambda) = \frac{\lambda^{\alpha-1}}{\lambda^{\alpha} - ivk}. \tag{4.37}$$

While it is hard (or impossible) to analytically invert the Fourier-Laplace transform, it is possible to determine the first two moments

$$\begin{aligned} \frac{\partial}{i\partial k} p_1(k,\lambda)\Big|_{k=0} &= \frac{v}{\lambda^{1+\alpha}} \quad \Rightarrow \quad \int dx\, x p_1(x,t) = \frac{v}{\Gamma(1+\alpha)} t^{\alpha} \\ -\frac{\partial^2}{\partial k^2} p_1(k,\lambda)\Big|_{k=0} &= \frac{2v}{\lambda^{1+2\alpha}} \quad \Rightarrow \quad \int dx\, x^2 p_1(x,t) = \frac{2v^2}{\Gamma(1+2\alpha)} t^{2\alpha}. \end{aligned} \tag{4.38}$$

Thus, $X(t)$ is a proper stochastic process for $0 < \alpha < 1$.

Therefore, if we would have started with $Y(s)$ being a Brownian motion on top of a constant drift, the limit theorem for $(Y(s), T(s))$ would still keep the Brownian motion visible, while it is scaled away for the process $X(t)$. In this thesis, I am only considering the second type of limit, because the operational time is just an auxiliary construct and not a physical parameter.

In the region of interest where $T(s)$ is a proper stochastic process, we can therefore assume the scaling behavior

$$\zeta^n p_n(\zeta^{\beta}\mathbf{k}, \zeta\boldsymbol{\lambda}) = p_n(\mathbf{k}, \boldsymbol{\lambda}) \quad \text{for all } \zeta > 0. \tag{4.39}$$

By looking at $p_1(k,\lambda)$ one sees that this implies a scaling behavior

$$\zeta^{-\alpha}\rho_{\text{step}}(\zeta^\beta k, \zeta\lambda) = \rho_{\text{step}}(k,\lambda) \quad \text{for all } \zeta > 0. \tag{4.40}$$

Respectively, equation (4.39) follows from equation (4.40) which can be seen from the diagrammatic rules.

In the following, I will derive conditions on how the limit form (4.36) looks when one starts from a probability distribution $\psi(x,t)$. The proofs will always be of the form, that I show the convergence

$$\lim_{\zeta \searrow 0} \zeta^{-\alpha}\rho_{\text{step}}(\zeta^\beta k, \zeta\lambda) = \rho^{\lim}_{\text{step}}(k,\lambda) \tag{4.41}$$

pointwise in k and λ. This implies directly the pointwise convergence of the Fourier-Laplace transform of the finite multi-point distribution by using the diagrammatic rules. One can sharpen the type of convergence to uniform convergence in any bounded intervals in k and bounded intervals in λ which are also bounded away from $\lambda = 0$. The proofs are essentially unmodified, one only has to choose the corresponding constants in the inequalities.

I use the form (4.36) as a guideline, but the proofs are actually independent from the results in [BKMS04].

4.4.1. Convergence to an uncoupled CTRW

This refers to the case, that spatial step size is independent of the waiting time. This means that in the limit $\eta(dx) = \delta(x)\,dx$. Equation (4.36) is then

$$\begin{aligned}\rho_{\text{step}}(k,\lambda) &= \theta(k) + c\int (1-e^{-\lambda t})\frac{dt}{t^{1+\alpha}} \\ &= \theta(k) + c(-\Gamma(-\alpha))\lambda^\alpha.\end{aligned} \tag{4.42}$$

We have assumed that the waiting time distribution lies in the domain of normal attraction of the one-sided Lévy stable distribution with exponent α. This means that we have the weak convergence of measure [[Kal01], proposition 15.9 and lemma 15.15]

$$\zeta^{-\alpha}(1-e^{-\lambda t})\kappa\left(\frac{dt}{\zeta}\right) \to c(1-e^{-\lambda t})\frac{dt}{t^{1+\alpha}} \text{ as } \zeta \searrow 0. \tag{4.43}$$

An important tool is the limit taken at an fixed time point t:

$$\theta_t(k) = \lim_{\zeta\searrow 0}\frac{1}{\zeta^\alpha}\int (1 - e^{i\zeta^\beta kx} + i\zeta^\beta kx\Xi(x))\,\eta_t(dx). \tag{4.44}$$

These functions $\theta_t(k)$ are of the form (2.111). We have $\theta_t(\zeta^\beta k) = \zeta^\alpha\theta_t(k)$. But since α and β are fixed, it is possible that $\theta_t(k) = 0$ for some t even when η_t represents a nontrivial probability distribution. We will see that $\theta(k)$ is simply given as the integral over the different ts.

$$\theta(k) = \int \theta_t(k)\,\kappa(dt) \tag{4.45}$$

Now, we need to distinguish which regularization function $\Xi(x)$ to use.

The case $\beta > \alpha$. In this case we can use $\Xi(x) = 0$, i.e.,

$$\begin{aligned}\theta_t(k) &= \lim_{\zeta \searrow 0} \frac{1}{\zeta^\alpha} \int (1 - e^{i\zeta^\beta kx})\, \eta_t(dx) \\ &= \lim_{\zeta \searrow 0} \int (1 - e^{ikx})\, \frac{1}{\zeta\alpha} \eta_t \left(\frac{dx}{\zeta^\beta} \right).\end{aligned} \tag{4.46}$$

If the convergence

$$\int (1 - e^{ikx})\, \frac{1}{\zeta\alpha} \eta_t \left(\frac{dx}{\zeta^\beta} \right) \to \theta_t(k) \tag{4.47}$$

is in the $L^1(\kappa)$ sense (as a function of t), we have

$$\begin{aligned}&\left| \zeta^{-\alpha} \rho_{\text{step}}(\zeta^\beta k, \zeta\lambda) - \rho^{\lim}_{\text{step}}(k, \lambda) \right| \\ &\quad = \left| \zeta^{-\alpha} \int (1 - e^{-\zeta\lambda t + i\zeta^\beta kx}) \eta_t(dx) \kappa(dt) - \rho^{\lim}_{\text{step}}(k, \lambda) \right| \\ &\quad \le \left| \int e^{-\lambda t} \left(\zeta^{-\alpha} \int (1 - e^{i\zeta^\beta kx}) \eta_t(dx) - \theta_t(k) \right) \kappa(dt) \right| \\ &\qquad + \left| \int e^{-\zeta\lambda t} \theta_t(k) \kappa(dt) - \theta(k) \right| + \left| \zeta^{-\alpha} \int (1 - e^{-\zeta\lambda t}) \kappa(dt) - c(-\Gamma(-\alpha))\lambda^\alpha \right| \\ &\quad \le \int \left| \zeta^{-\alpha} \int (1 - e^{i\zeta^\beta kx}) \eta_t(dx) - \theta_t(k) \right| \kappa(dt) \\ &\qquad + \left| \int e^{-\zeta\lambda t} \theta_t(k) \kappa(dt) - \theta(k) \right| + \left| \zeta^{-\alpha} \int (1 - e^{-\zeta\lambda t}) \kappa(dt) - c(-\Gamma(-\alpha))\lambda^\alpha \right| \\ &\quad \to 0.\end{aligned} \tag{4.48}$$

The case $\frac{\alpha}{2} \le \beta < \alpha$. In this case we are using $\Xi(x) = 1$, i.e.,

$$\begin{aligned}\theta_t(k) &= \lim_{\zeta \searrow 0} \frac{1}{\zeta^\alpha} \int (1 - e^{i\zeta^\beta kx} + i\zeta^\beta kx)\, \eta_t(dx) \\ &= \lim_{\zeta \searrow 0} \int (1 - e^{ikx} + ikx)\, \frac{1}{\zeta^\alpha} \eta_t \left(\frac{dx}{\zeta^\beta} \right).\end{aligned} \tag{4.49}$$

This time the distribution η_t have a mean value, which I denote

$$\mu_t = \int x\, \eta_t(dx). \tag{4.50}$$

In addition to the $L^1(\kappa)$-convergence

$$\zeta^{-\alpha} \int (1 - e^{i\zeta^\beta kx} + i\zeta^\beta k\mu_t)\, \eta_t(dx) \to \theta_t(k) \tag{4.51}$$

we need

$$\int \mu_t\, \kappa(dt) = 0. \tag{4.52}$$

(The case $\int \mu_t \, \kappa(dt) \neq 0$ will be treated later). The argument goes analogously to equation (4.48)

$$\begin{aligned}
&\left|\zeta^{-\alpha}\rho_{\text{step}}(\zeta^\beta k, \zeta\lambda) - \rho^{\lim}_{\text{step}}(k,\lambda)\right| \\
&\quad= \left|\zeta^{-\alpha}\int (1 - e^{-\zeta\lambda t + i\zeta^\beta kx} + i\zeta^\beta k\mu_t)\eta_t(dx)\kappa(dt) - \rho^{\lim}_{\text{step}}(k,\lambda) - i\zeta^{\beta-\alpha}k\int \mu_t\,\kappa(dt)\right| \\
&\quad\leq \left|\int e^{-\lambda t}\left(\zeta^{-\alpha}\int (1 - e^{i\zeta^\beta kx} + i\zeta^\beta k\mu_t)\eta_t(dx) - \theta_t(k)\right)\kappa(dt)\right| \\
&\qquad + \left|\int e^{-\zeta\lambda t}\theta_t(k)\kappa(dt) - \theta(k)\right| + \left|\zeta^{-\alpha}\int (1 - e^{-\zeta\lambda t})\kappa(dt) - c(-\Gamma(-\alpha))\lambda^\alpha\right| \\
&\quad\leq \int \left|\zeta^{-\alpha}\int (1 - e^{i\zeta^\beta kx} + i\zeta^\beta k\mu_t)\eta_t(dx) - \theta_t(k)\right|\kappa(dt) \\
&\qquad + \left|\int e^{-\zeta\lambda t}\theta_t(k)\kappa(dt) - \theta(k)\right| + \left|\zeta^{-\alpha}\int (1 - e^{-\zeta\lambda t})\kappa(dt) - c(-\Gamma(-\alpha))\lambda^\alpha\right| \\
&\quad\to 0.
\end{aligned} \tag{4.53}$$

The convergence in the $L^1(\kappa)$ can still be hard to determine, but fortunately most processes have second moments which implies in the uncoupled case $\beta = \frac{\alpha}{2}$. This case allows further simplification by the use of higher moments.

Specialties of the case $\beta = \frac{\alpha}{2}$. While the case of $\beta = \frac{\alpha}{2}$ was already included in the last section, it is special in the respect that the generator $\theta(k) = \frac{1}{2}\sigma^2 k^2$ corresponds to Gaussian noise. This allows simplifications which I will discuss in this paragraph.

We have already defined the mean μ_t of η_t. Similarly, we get for the variance σ_t^2 of η_t

$$\sigma_t^2 = \int (x - \mu_t)^2 \, \eta_t(dx). \tag{4.54}$$

With these definitions we have the limit

$$\lim_{\zeta\searrow 0} \zeta^{-\alpha}\int (1 - e^{i\zeta^\beta kx} + i\zeta^\beta kx)\,\eta_t(dx) = \frac{1}{2}(\sigma_t^2 + \mu_t^2)k^2. \tag{4.55}$$

The limit is

$$\theta(k) = \frac{1}{2}\sigma^2 k^2 \quad \text{with} \quad \sigma^2 = \int (\sigma_t^2 + \mu_t^2)\,\kappa(dt). \tag{4.56}$$

Here the limit is directly expressible in terms of the first and second moment.

In many cases, the proof of the convergence can by simplified by use of an absolute moment of order $\epsilon > 2$. Define

$$m_t^{|\epsilon|} = \int |x|^\epsilon \, \eta_t(dx). \tag{4.57}$$

If the integral over times is finite

$$C_{|\epsilon|} = \int m_t^{|\epsilon|}\,\kappa(dt) < \infty, \tag{4.58}$$

then we know that the convergence (4.51) is in the $L^1(\kappa)$ sense. To see this, we can use the inequality (C.6) (derived in appendix C)

$$\left|1 - iy - \frac{1}{2}y^2 - e^{iy}\right| \leq 2|y|^\epsilon. \tag{4.59}$$

Substituting $y = \zeta^\beta kx$ we see

$$\left|\zeta^{-\alpha}\int (1 - e^{i\zeta^\beta kx} - i\zeta^\beta kx)\,\eta_t(dx) - \frac{1}{2}(\sigma_t^2 + \mu_t^2)k^2\right| \leq 2\zeta^{\epsilon\beta-\alpha}|k|^\epsilon m_t^{|\epsilon|} \tag{4.60}$$

where I have used $\beta = 2\alpha$. Integrating over t gives the $L^1(\kappa)$ convergence.

The case of non-vanishing mean ($\alpha = \beta$). Assume the condition (4.52) would be violated, i.e., we would have a mean

$$\mu = \int \mu_t\,\kappa(dt) \neq 0. \tag{4.61}$$

If we additionally have the existence of absolute moments

$$m_t^{|\epsilon|} = \int |x|^\epsilon\,\eta_t(dx) \tag{4.62}$$

for some $\epsilon > 1$ and

$$C_\epsilon = \int m_t^{|\epsilon|}\kappa(dt) < \infty \tag{4.63}$$

then the limit will be a biased CTRW (4.37), i.e.,

$$\rho_{\text{step}}^{\lim}(k,\lambda) = -i\mu k + c(-\Gamma(-\alpha))\lambda^\alpha. \tag{4.64}$$

For simplicity, I will restrict myself to $1 < \epsilon < 2$ and show how to generalize this later. Using the inequality (C.5)

$$\left|1 + iy - e^{iy}\right| \leq 2|y|^\epsilon. \tag{4.65}$$

This gives

$$\begin{aligned}
&\left|\zeta^{-\alpha}\rho_{\text{step}}(\zeta^\alpha k, \zeta\lambda) - \rho_{\text{step}}^{\lim}(k,\lambda)\right| \\
&\quad\leq \zeta^{-\alpha}\int \left|1 - e^{i\zeta^\alpha kx} + i\zeta^\alpha k\mu_t\right|\,\eta_t(dx)\,\kappa(dt) \\
&\qquad + |k|\left|\int e^{-\zeta\lambda t}\mu_t\kappa(dt) - \mu\right| + \left|\zeta^{-\alpha}\int (1 - e^{-\zeta\lambda t}) - c(-\Gamma(-\alpha))\lambda^\alpha\right| \\
&\quad\leq 2\zeta^{-\alpha}C_\epsilon|\zeta^\alpha k|^\epsilon + |k|\left|\int e^{-\zeta\lambda t}\mu_t\kappa(dt) - \mu\right| + \left|\zeta^{-\alpha}\int (1 - e^{-\zeta\lambda t}) - c(-\Gamma(-\alpha))\lambda^\alpha\right| \\
&\quad\to 0.
\end{aligned} \tag{4.66}$$

The above conditions on the moments hold for a $\epsilon \geq 2$, we can simply choose any $1 < \tilde{\epsilon} < 2$. Since $|x|^{\frac{\epsilon}{\tilde{\epsilon}}}$ is convex, we can apply Jensen's inequality [[Kal01], lemma 3.5] twice

$$\left(m^{|\tilde{\epsilon}|}\right)^{\frac{\epsilon}{\tilde{\epsilon}}} \leq m^{|\epsilon|} \quad\Rightarrow\quad (C_{\tilde{\epsilon}})^{\frac{\epsilon}{\tilde{\epsilon}}} \leq C_\epsilon \tag{4.67}$$

and therefore we can use the above argument.

4.4.2. Convergence to a coupled CTRW

In this case the uncoupled contribution, represented by $\theta(k) = 0$ vanishes. The general limit form is therefore (4.36)

$$\rho_{\text{step}}^{\lim}(k,\lambda) = -i\mu k + c\int (1 - e^{-\lambda t + ikx} + ikx\Xi(x))\eta^{\lim}\left(\frac{dx}{t^\beta}\right)\frac{dt}{t^{1+\alpha}} \tag{4.68}$$

where for the important cases $\Xi(x) = 0$ and $\Xi(x) = 1$, we have $\mu = 0$. The distribution $\eta_{\lim}(dx)$ is best described using the characteristic functions

$$\tilde{\eta}^{\lim}(k) = \int e^{ikx}\,\eta_{\lim}(dx) \quad \text{and} \quad \tilde{\eta}_t(k) = \int e^{ikx}\,\eta_t(dx). \tag{4.69}$$

The limit distribution is obtained by the limit (for fixed k)

$$\tilde{\eta}^{\lim}(k) = \operatorname*{ess\,lim}_{t\to\infty} \tilde{\eta}_t(t^{-\beta}k) \tag{4.70}$$

where β is chosen such that the limit corresponds to a probability distribution and is not $\tilde{\eta}^{\lim}(k) = 1$. Using [[Fel71], theorem 2 on page 508] we know that this limit is uniform in every finite interval of k (except for a set of measure zero in t). The necessary conditions are easily expressed by use of the absolute moments $m^{|\epsilon|}$. Therefore, I will use their definitions from above. It is necessary to distinguish the different regularization functions $\Xi(x)$.

The case $\beta > \alpha$. As in the uncoupled case, we can choose $\Xi(x) = 0$ as regularization function. Additionally, I demand the existence of $\delta > \frac{\alpha}{\beta}$ and $0 < A < \infty$ such that

$$\int_{\tau\in[0,A]} m_\tau^{|\delta|}\,\kappa(d\tau) = D_1 < \infty \quad \text{and} \quad \operatorname*{ess\,sup}_{t\in[A,\infty[} \frac{m_t^{|\delta|}}{t^{\beta\delta}} = D_2 < \infty. \tag{4.71}$$

As above, we can restrict the argument to $\frac{\alpha}{\beta} < \delta < 1$. One has to split the integration domain. Take $0 < a < b < \infty$:

$$\begin{aligned}\left|\zeta^{-\alpha}\rho_{\text{step}}(\zeta^\alpha k, \zeta\lambda) - \rho_{\text{step}}^{\lim}(k,\lambda)\right| \le{}& \left|\zeta^{-\alpha}\int_{t\in\zeta^{-1}[a,b]}\left(1 - e^{-\zeta\lambda t}\tilde{\eta}_t(\zeta^\beta k)\right)\,\kappa(dt)\right.\\ &\left. - c\int_{t\in[a,b]}\left(1 - e^{-\lambda t}\tilde{\eta}^{\lim}(t^\beta k)\right)\frac{dt}{t^{1+\alpha}}\right|\\ &+ \left|\zeta^{-\alpha}\int_{t\in\mathbb{R}_+\setminus\zeta^{-1}[a,b]}\left(1 - e^{-\zeta\lambda t}\tilde{\eta}_t(\zeta^\beta k)\right)\,\kappa(dt)\right.\\ &\left. - c\int_{t\in\mathbb{R}_+\setminus[a,b]}\left(1 - e^{-\lambda t}\tilde{\eta}^{\lim}(t^\beta k)\right)\frac{dt}{t^{1+\alpha}}\right|.\end{aligned} \tag{4.72}$$

The first term is

$$\begin{aligned}
&\left|\zeta^{-\alpha}\int_{t\in\zeta^{-1}[a,b]}\left(1-e^{-\zeta\lambda t}\tilde{\eta}_t(\zeta^\beta k)\right)\,\kappa(dt)-c\int_{t\in[a,b]}\left(1-e^{-\lambda t}\tilde{\eta}^{\lim}(t^\beta k)\right)\frac{dt}{t^{1+\alpha}}\right| \\
&\quad\leq\zeta^{-\alpha}\left|\int_{t\in\zeta^{-1}[a,b]}e^{-\zeta\lambda t}\left(\tilde{\eta}^{\lim}(\zeta^\beta t^\beta k)-\tilde{\eta}_t(\zeta^\beta k)\right)\,\kappa(dt)\right| \\
&\qquad+\left|\zeta^{-\alpha}\int_{t\in\zeta^{-1}[a,b]}\left(1-e^{-\zeta\lambda t}\tilde{\eta}^{\lim}(\zeta^\beta t^\beta k)\right)\,\kappa(dt)-c\int_{t\in[a,b]}\left(1-e^{-\lambda t}\tilde{\eta}^{\lim}(t^\beta k)\right)\frac{dt}{t^{1+\alpha}}\right| \\
&\quad\leq\operatorname*{ess\,sup}_{\substack{x\in[a,b]\\ t=\zeta^{-1}x}}\left|\tilde{\eta}^{\lim}(x^\beta k)-\tilde{\eta}_t(t^{-\beta}x^\beta k)\right|\int_{t\in[a,b]}\zeta^{-\alpha}\,\kappa\left(\frac{dt}{\zeta}\right) \\
&\qquad+\left|\int_{t\in[a,b]}\left(1-e^{-\lambda t}\tilde{\eta}^{\lim}(t^\beta k)\right)\zeta^{-\alpha}\left(\frac{dt}{\zeta}\right)-c\int_{t\in[a,b]}\left(1-e^{-\lambda t}\tilde{\eta}^{\lim}(t^\beta k)\right)\frac{dt}{t^{1+\alpha}}\right| \\
&\quad\to 0
\end{aligned}\tag{4.73}$$

where I have used the fact that

$$\operatorname*{ess\,sup}_{\substack{x\in[a,b]\\ t=\zeta^{-1}x}}\left|\tilde{\eta}^{\lim}(x^\beta k)-\tilde{\eta}_t(t^{-\beta}x^\beta k)\right|\to 0\quad\text{for }\zeta\searrow 0\tag{4.74}$$

following from the uniform convergence of $\tilde{\eta}_t(t^{-\beta}k)$ on bounded intervals in k. From the weak convergence of the measure (with $\lambda>0$)

$$(1-e^{-\lambda t})\zeta^{-\alpha}\kappa\left(\frac{dt}{\zeta}\right)\to c(1-e^{-\lambda t})\frac{dt}{t^{1+\alpha}}\tag{4.75}$$

follow the convergences

$$\int_{t\in[a,b]}\left(1-e^{-\lambda t}\tilde{\eta}^{\lim}(t^\beta k)\right)\zeta^{-\alpha}\left(\frac{dt}{\zeta}\right)\to c\int_{t\in[a,b]}\left(1-e^{-\lambda t}\tilde{\eta}^{\lim}(t^\beta k)\right)\frac{dt}{t^{1+\alpha}}\tag{4.76}$$

and

$$\int_{t\in[a,b]}\left(1-e^{-\lambda t}\tilde{\eta}^{\lim}(t^\beta k)\right)\zeta^{-\alpha}\left(\frac{dt}{\zeta}\right)\to c\int_{t\in[a,b]}\left(1-e^{-\lambda t}\tilde{\eta}^{\lim}(t^\beta k)\right)\frac{dt}{t^{1+\alpha}}.\tag{4.77}$$

Putting equation (4.73) in equation (4.72) leaves us with

$$\begin{aligned}
&\limsup_{\zeta\searrow 0}\left|\zeta^{-\alpha}\rho_{\text{step}}(\zeta^\alpha k,\zeta\lambda)-\rho^{\lim}_{\text{step}}(k,\lambda)\right| \\
&\quad\leq\limsup_{\zeta\searrow 0}\left|\zeta^{-\alpha}\int_{t\in\mathbb{R}_+\setminus\zeta^{-1}[a,b]}\left(1-e^{-\zeta\lambda t}\tilde{\eta}_t(\zeta^\beta k)\right)\,\kappa(dt)\right. \\
&\qquad\qquad\left.-c\int_{t\in\mathbb{R}_+\setminus[a,b]}\left(1-e^{-\lambda t}\tilde{\eta}^{\lim}(t^\beta k)\right)\frac{dt}{t^{1+\alpha}}\right| \\
&\quad\leq\limsup_{\zeta\searrow 0}\left|\zeta^{-\alpha}\int_{t\in\zeta^{-1}[0,a[}\left(1-e^{-\zeta\lambda t}\tilde{\eta}_t(\zeta^\beta k)\right)\,\kappa(dt)\right| \\
&\qquad+2\limsup_{\zeta\searrow 0}\left|\zeta^{-\alpha}\int_{t\in\zeta^{-1}]b,\infty[}\kappa(dt)\right|+\left|c\int_{t\in\mathbb{R}_+\setminus[a,b]}\left(1-e^{-\lambda t}\tilde{\eta}^{\lim}(t^\beta k)\right)\frac{dt}{t^{1+\alpha}}\right|.
\end{aligned}\tag{4.78}$$

In the following, I will show that one can choose a and b such that the right hand side of equation (4.78) is arbitrarily small. One has for the first term

$$\begin{aligned}
&\limsup_{\zeta\searrow 0}\left|\zeta^{-\alpha}\int_{t\in\zeta^{-1}[0,a[}\left(1-e^{-\zeta\lambda t}\tilde{\eta}_t(\zeta^\beta k)\right)\,\kappa(dt)\right| \\
&\quad\leq \limsup_{\zeta\searrow 0}\zeta^{-\alpha}\int_{t\in\zeta^{-1}[0,a[}\left(1-e^{-\zeta\lambda t}\right)\,\kappa(dt) \\
&\quad\quad+\limsup_{\zeta\searrow 0}\zeta^{-\alpha}\int_{t\in\zeta^{-1}[0,a[}\left|1-\tilde{\eta}_t(\zeta^\beta k)\right|\,\kappa(dt) \\
&\quad\leq c\int_{t\in[0,a[}\left(1-e^{-\lambda t}\right)\frac{dt}{t^{1+\alpha}}+2|k|^\delta\limsup_{\zeta\searrow 0}\zeta^{\beta\delta-\alpha}\int_{t\in\zeta^{-1}[0,a[}m_t^{|\delta|}\,\kappa(dt).
\end{aligned}\tag{4.79}$$

The last step used

$$\begin{aligned}
\left|1-\tilde{\eta}_t(\zeta^\beta k)\right| &\leq \int\left|1-e^{i\zeta^\beta kx}\right|\,\eta_t(dx) \\
&\leq 2\int\left|\zeta^\beta kx\right|^\delta\,\eta_t(dx) \\
&= \zeta^{\beta\delta}|k|^\delta m_t^{|\delta|}.
\end{aligned}\tag{4.80}$$

By choosing a small enough the term

$$c\int_{t\in[0,a[}\left(1-e^{-\lambda t}\right)\frac{dt}{t^{1+\alpha}}\tag{4.81}$$

in equation (4.79) can be made arbitrary small. For the second term, I will use the conditions (4.71)

$$\begin{aligned}
&\limsup_{\zeta\searrow 0}\zeta^{\beta\delta-\alpha}\int_{t\in\zeta^{-1}[0,a[}m_t^{|\delta|}\,\kappa(dt) \\
&\quad\leq D_1\limsup_{\zeta\searrow 0}\zeta^{\beta\delta-\alpha}+D_2\limsup_{\zeta\searrow 0}\zeta^{\beta\delta-\alpha}\int_{t\in\zeta^{-1}[0,a[}t^{\beta\delta}\,\kappa(dt) \\
&\quad= D_2\limsup_{\zeta\searrow 0}\zeta^{\beta\delta-\alpha}\int_{t\in\zeta^{-1}[0,a[}t^{\beta\delta}\,\kappa(dt).
\end{aligned}\tag{4.82}$$

We have assumed that the distribution $\kappa(dt)$ is in the domain of normal attraction of the one-sided Lévy stable distribution with exponent α, i.e., [[Fel71], remark on page 581]

$$\lim_{\tau\to\infty}\tau^\alpha\int_{t\in[\tau,\infty[}\kappa(dt)=c_1\quad\text{with } 0<c_1<\infty.\tag{4.83}$$

Using [[Fel71], equation (XVII.5.22)], we have the existence of an other constant $0<c_2<\infty$ with

$$\lim_{\tau\to\infty}\frac{\int_{t\in[0,\tau[}t^{\beta\delta}\,\kappa(dt)}{\tau^{\beta\delta}\int_{t\in[\tau,\infty[}\kappa(dt)}=c_2.\tag{4.84}$$

This gives

$$\lim_{\zeta\searrow 0}\zeta^{\beta\delta-\alpha}\int_{t\in\zeta^{-1}[0,a[}t^{\beta\delta}\,\kappa(dt)=c_1c_2a^{\beta\delta-\alpha}.\tag{4.85}$$

Therefore equation (4.79) can be made arbitrary small by choosing a small enough.

For the second term of equation (4.78)

$$\begin{aligned}\limsup_{\zeta \searrow 0} \left| \zeta^{-\alpha} \int_{t \in \zeta^{-1}]b,\infty[} \kappa(dt) \right| &= c \int_{t \in]b,\infty[} \frac{dt}{t^{1+\alpha}} \\ &= \frac{c}{\alpha} b^{-\alpha}.\end{aligned} \tag{4.86}$$

Again, this can be made arbitrary small by choosing b large enough.

The third term of equation (4.78) is

$$\left| c \int_{t \in \mathbb{R}_+ \setminus [a,b]} \left(1 - e^{-\lambda t} \tilde{\eta}^{\lim}(t^\beta k)\right) \frac{dt}{t^{1+\alpha}} \right| \leq c \left| \int_{t \in [0,a[} \left(1 - e^{-\lambda t} \tilde{\eta}^{\lim}(t^\beta k)\right) \frac{dt}{t^{1+\alpha}} \right| + \frac{2c}{\alpha} b^{-\alpha}. \tag{4.87}$$

Next note that for any $C > 0$

$$\begin{aligned}\int \max(C,|x|)^\delta \eta^{\lim}(dx) &= \operatorname*{ess\,lim}_{t \to \infty} \int \max(C,|x|)^\delta \eta_t(t^\beta dx) \\ &\leq \lim_{\tau \to \infty} \operatorname*{ess\,sup}_{t \in]\tau,\infty[} \int |x|^\delta \eta_t(t^\beta dx) \\ &= \lim_{\tau \to \infty} \operatorname*{ess\,sup}_{t \in]\tau,\infty[} t^{-\beta\delta} m_t^{|\delta|} \\ &\leq D_2.\end{aligned} \tag{4.88}$$

By letting $C \to \infty$

$$\int |x|^\delta \eta^{\lim}(dx) \leq D_2. \tag{4.89}$$

Since the δth fractional moment is finite, we know

$$1 - \tilde{\eta}^{\lim}(k) = O(|k|^\delta) \quad \text{for } k \to 0 \tag{4.90}$$

and therefore (λ, k fixed)

$$\left| \int_{t \in [0,a[} \left(1 - e^{-\lambda t} \tilde{\eta}^{\lim}(t^\beta k)\right) \frac{dt}{t^{1+\alpha}} \right| = O(a^{\beta\delta - \alpha}) \quad \text{for } a \to 0. \tag{4.91}$$

Finally, putting these results together, gives the convergence

$$\lim_{\zeta \searrow 0} \zeta^{-\alpha} \rho_{\text{step}}(\zeta^\alpha k, \zeta\lambda) = \rho_{\text{step}}^{\lim}(k,\lambda). \tag{4.92}$$

The case $\frac{\alpha}{2} < \beta < \alpha$. As in the uncoupled case, we can use $\Xi(x) = 1$ as regularization function. The limit is described by

$$\tilde{\eta}^{\lim}(k) = \operatorname*{ess\,lim}_{t \to \infty} \tilde{\eta}_t(t^{-\beta} k) \tag{4.93}$$

and

$$\begin{aligned}\rho_{\text{step}}^{\lim}(k,\lambda) &= c \int (1 - e^{-\lambda t + ikx} + ikx) \eta^{\lim}\left(\frac{dx}{t^\beta}\right) \frac{dt}{t^{1+\alpha}} \\ &= c \int \left(1 - e^{-\lambda t} \tilde{\eta}^{\lim}(t^\beta k) + i t^\beta k \mu^{\lim}\right) \frac{dt}{t^{1+\alpha}}\end{aligned} \tag{4.94}$$

with

$$\mu^{\lim} = \int x\, \eta^{\lim}(dx) = -i\frac{d}{dk}\tilde{\eta}^{\lim}(k)\bigg|_{k=0}. \tag{4.95}$$

To show the convergence one needs the existence of $\delta > \frac{\alpha}{\beta}$ and $0 < A < \infty$ such that

$$\int_{\tau\in[0,A]} m_\tau^{|\delta|}\, \kappa(d\tau) = D_1 < \infty \quad \text{and} \quad \operatorname*{ess\,sup}_{t\in[A,\infty[} \frac{m_t^{|\delta|}}{t^{\beta\delta}} = D_2 < \infty. \tag{4.96}$$

Additionally, one needs the condition

$$\int \mu_t\, \kappa(dt) = 0. \tag{4.97}$$

This leads to

$$\begin{aligned} \rho_{\text{step}}(k,\lambda) &= \int (1 - e^{-\lambda t + ikx})\eta_t(dx)\kappa(dt) \\ &= \int (1 - e^{-\lambda t + ikx} + ikx)\eta_t(dx)\kappa(dt). \end{aligned} \tag{4.98}$$

The proof of the convergence

$$\lim_{\zeta\searrow 0} \zeta^{-\alpha}\rho_{\text{step}}(\zeta^\alpha k, \zeta\lambda) = \rho_{\text{step}}^{\lim}(k,\lambda) \tag{4.99}$$

is similar to the case $\beta > \alpha$, but there are some changes due to the regularization function. For completeness, I will go through the argument, but skip the parts which are completely analogous. Again, the first part is to show the convergence

$$\begin{aligned} \Bigg|\zeta^{-\alpha} \int_{t\in\zeta^{-1}[a,b]} &\left(1 - e^{-\zeta\lambda t}\tilde{\eta}_t(\zeta^\beta k) + i\zeta^\beta k\mu_t\right)\kappa(dt) \\ &- c\int_{t\in[a,b]} \left(1 - e^{-\lambda t}\tilde{\eta}^{\lim}(t^\beta k) + ikt^\beta\mu^{\lim}\right)\frac{dt}{t^{1+\alpha}}\Bigg| \to 0. \end{aligned} \tag{4.100}$$

This time, in addition to the convergence

$$\operatorname*{ess\,sup}_{\substack{x\in[a,b]\\ t=\zeta^{-1}x}} \left|\tilde{\eta}^{\lim}(x^\beta k) - \tilde{\eta}_t(t^{-\beta}x^\beta k)\right| \to 0 \quad \text{for } \zeta \searrow 0 \tag{4.101}$$

we need

$$\operatorname*{ess\,sup}_{\substack{x\in[a,b]\\ t=\zeta^{-1}x}} \left|\mu^{\lim} - t^{-\beta}\mu_t\right| \to 0 \quad \text{for } \zeta \searrow 0, \tag{4.102}$$

which follows from the limit

$$\operatorname*{ess\,lim}_{t\to\infty} t^{-\beta}\mu_t = \mu^{\lim}. \tag{4.103}$$

The second step is to show that

$$\begin{aligned}\limsup_{\zeta\searrow 0}\Bigg| \zeta^{-\alpha}\int_{t\in\mathbb{R}_+\setminus\zeta^{-1}[a,b]}\left(1-e^{-\zeta\lambda t}\tilde{\eta}_t(\zeta^\beta k)+i\zeta^\beta k\mu_t\right)\kappa(dt)\\ -c\int_{t\in\mathbb{R}_+\setminus[a,b]}\left(1-e^{-\lambda t}\tilde{\eta}^{\lim}(t^\beta k)+it^\beta k\mu^{\lim}\right)\frac{dt}{t^{1+\alpha}}\Bigg|\\ \leq \limsup_{\zeta\searrow 0}\left|\zeta^{-\alpha}\int_{t\in\zeta^{-1}[0,a[}\left(1-e^{-\zeta\lambda t}\tilde{\eta}_t(\zeta^\beta k)+i\zeta^\beta k\mu_t\right)\kappa(dt)\right|\\ +2\limsup_{\zeta\searrow 0}\left|\zeta^{-\alpha}\int_{t\in\zeta^{-1}]b,\infty[}\kappa(dt)\right|+\limsup_{\zeta\searrow 0}\left|\zeta^{\beta-\alpha}\int_{t\in\zeta^{-1}]b,\infty[}|\mu_t|\,\kappa(dt)\right|\\ +\left|c\int_{t\in\mathbb{R}_+\setminus[a,b]}\left(1-e^{-\lambda t}\tilde{\eta}^{\lim}(t^\beta k)+it^\beta k\mu^{\lim}\right)\frac{dt}{t^{1+\alpha}}\right|.\end{aligned} \tag{4.104}$$

can be made arbitrary small by choosing a and b accordingly. The argument is again essentially the same by using equation (C.5) ($1\leq\delta\leq 2$)

$$\begin{aligned}\left|1-\tilde{\eta}_t(\zeta^\beta k)+i\zeta^\beta k\mu_t\right| &\leq 2\int\left|\zeta^\beta kx\right|^\delta\,\eta_t(dx)\\ &= \zeta^{\beta\delta}|k|^\delta m_t^{|\delta|}.\end{aligned} \tag{4.105}$$

We are left with the term

$$\limsup_{\zeta\searrow 0}\left|\zeta^{\beta-\alpha}\int_{t\in\zeta^{-1}]b,\infty[}|\mu_t|\,\kappa(dt)\right|. \tag{4.106}$$

Jensen's inequality gives

$$\operatorname*{ess\,sup}_{t\in[A,\infty[}\frac{|\mu_t|}{t^\beta}\leq\operatorname*{ess\,sup}_{t\in[A,\infty[}\left(\frac{m_t^{|\delta|}}{t^{\beta\delta}}\right)^{\frac{1}{\delta}}\leq D_2^{\frac{1}{\delta}}, \tag{4.107}$$

therefore

$$\limsup_{\zeta\searrow 0}\left|\zeta^{\beta-\alpha}\int_{t\in\zeta^{-1}]b,\infty[}|\mu_t|\,\kappa(dt)\right|\leq D_2^{\frac{1}{\delta}}\limsup_{\zeta\searrow 0}\left|\zeta^{\beta-\alpha}\int_{t\in\zeta^{-1}]b,\infty[}t^\beta\,\kappa(dt)\right|. \tag{4.108}$$

Using again [[Fel71], equation (XVII.5.22)] gives the existence of $0<c_3<\infty$ with

$$\lim_{\tau\to\infty}\frac{\int_{t\in[0,\tau]}t^2\,\kappa(dt)}{\tau^2\int_{t\in[\tau,\infty[}\kappa(dt)}=c_3 \tag{4.109}$$

and [[Fel71], equation (XVII.5.21)] with another constant $0<c_4<\infty$

$$\lim_{\tau\to\infty}\frac{\tau^{2-\beta}\int_{t\in[\tau,\infty[}t^\beta\,\kappa(dt)}{\int_{t\in[0,\tau]}t^2\,\kappa(dt)}=c_4. \tag{4.110}$$

Combining this gives

$$\limsup_{\zeta\searrow 0}\left|\zeta^{\beta-\alpha}\int_{t\in\zeta^{-1}]b,\infty[}|\mu_t|\,\kappa(dt)\right|\leq D_2^{\frac{1}{\delta}}c_1c_3c_4b^{\beta-\alpha}. \tag{4.111}$$

4.5. Summary of Limits

While deriving the limit distributions, it is easily possible to get lost in the details. Therefore, I want to finish this chapter by summarizing the conditions and the limits they imply in the cases which are relevant in later chapters.

For all cases I assume that the waiting time distribution κ converges as

$$\lim_{\zeta \searrow 0} (1 - e^{-\lambda t}) \zeta^\alpha \kappa(\frac{dt}{\zeta}) = c(1 - e^{-\lambda t}) \frac{dt}{t^{1+\alpha}}. \tag{4.112}$$

1. If an absolute moment of order $\epsilon > 2$ of the spatial step size distribution exists

$$\int |x|^\epsilon \, \eta_t(dx) \, \kappa(dt) < \infty \tag{4.113}$$

and the mean vanishes

$$\int x \, \eta_t(dx) \, \kappa(dt) = 0, \tag{4.114}$$

then we have convergence to an uncoupled CTRW with log-Fourier-Laplace transform

$$\rho^{\lim}_{\text{step}}(k, \lambda) = c(-\Gamma(-\alpha))\lambda^\alpha + \frac{1}{2}\sigma^2 k^2 \tag{4.115}$$

with

$$\sigma^2 = \int x^2 \, \eta_t(dx) \, \kappa(dt) < \infty. \tag{4.116}$$

The scaling exponent is $\beta = \frac{\alpha}{2}$.

2. If an absolute moment of order $\epsilon > 1$ of the spatial step size distribution exists

$$\int |x|^\epsilon \, \eta_t(dx) \, \kappa(dt) < \infty \tag{4.117}$$

and the mean

$$\mu = \int x \, \eta_t(dx) \, \kappa(dt) \tag{4.118}$$

does not vanish ($\mu \neq 0$), then we have convergence to an uncoupled CTRW with bias

$$\rho^{\lim}_{\text{step}}(k, \lambda) = c(-\Gamma(-\alpha))\lambda^\alpha - i\mu k. \tag{4.119}$$

The scaling exponent is $\beta = \alpha$.

3. If there exists a $\beta > \alpha$ such that

$$\tilde{\eta}^{\lim}(k) = \operatorname*{ess\,lim}_{t \to \infty} \tilde{\eta}_t(t^{-\beta} k) \tag{4.120}$$

is a non-trivial characteristic function of a probability distribution and we have a constant A and $\epsilon > \frac{\alpha}{\beta}$ such that

$$\int_{\tau \in [0,A], x \in \mathbb{R}} |x|^\epsilon \, \eta_t(dx) \, \kappa(d\tau) < \infty \quad \text{and} \quad \operatorname*{ess\,sup}_{t \in [A, \infty[} \frac{1}{t^{\beta\epsilon}} \int |x|^\epsilon \, \eta_t(dx) < \infty, \tag{4.121}$$

then we have convergence to a coupled CTRW with scaling exponent β and

$$\rho^{\lim}_{\text{step}}(k, \lambda) = c \int (1 - e^{-\lambda t} \tilde{\eta}^{\lim}(t^{-\beta} k)) \, \frac{dt}{t^{1+\alpha}}. \tag{4.122}$$

4. If there exists a β with $\frac{\alpha}{2} < \beta < \alpha$ such that

$$\tilde{\eta}^{\lim}(k) = \operatorname*{ess\,lim}_{t\to\infty} \tilde{\eta}_t(t^{-\beta}k) \tag{4.123}$$

is a non-trivial characteristic function of a probability distribution, we have a constant A and $\epsilon > \frac{\alpha}{\beta}$ such that

$$\int_{\tau\in[0,A],x\in\mathbb{R}} |x|^\epsilon \, \eta_t(dx)\, \kappa(d\tau) < \infty \quad \text{and} \quad \operatorname*{ess\,sup}_{t\in[A,\infty[} \frac{1}{t^{\beta\epsilon}} \int |x|^\epsilon \, \eta_t(dx) < \infty \tag{4.124}$$

and the mean step size vanishes

$$\int x \, \eta_t(dx)\, \kappa(dt) = 0, \tag{4.125}$$

then we have convergence to a coupled CTRW with scaling exponent β and

$$\rho^{\lim}_{\text{step}}(k,\lambda) = c \int (1 - e^{-\lambda t}\tilde{\eta}^{\lim}(t^\beta k) + i t^\beta k \mu^{\lim}) \, \frac{dt}{t^{1+\alpha}} \tag{4.126}$$

with

$$\mu^{\lim} = -i \frac{\partial}{\partial k} \tilde{\eta}^{\lim}(k)\Big|_{k=0}. \tag{4.127}$$

5. Nonindependent continuous-time random walks

5.1. Introduction

In this section, I want to have a look at CTRWs where the step size distribution can depend on the previous steps. This can of course always appear when approximating a deterministic dynamics with such a process. The concept of coupled steps in a CTRW was first introduced by Landman, Montroll and Shlesinger [LMS77] with a finite set of internal states. The internal state represents the memory of the process. In contrast to the approach in this chapter they worked on a lattice with periodic boundary conditions while I consider the case of a random walker on $\mathbb{R}$. The concept of the nonindependent CTRW was again introduced thirty years later by Montero and Masoliver [MM07] (who seem to be unaware of the earlier work). They make a Markov assumption which states that the probability distribution of one step depends only on the realization of the previous one. They set up an integral equation [[MM07], equation (11)] which describes the one point density. To provide a solvable example they restrict themself to the case that the sign of the current step depends only on the sign of the previous step which corresponds then to the case of a finite internal state space.

As before, I will concentrate on the case of the leaper model (an example for an other model can be seen in chapter 6). The different states are conveniently organized in vectors. I adopt the quantum Hamilton formalism (as it is used in, e.g., [HS07]) to write column vectors by Kets $|\cdot\rangle$ and and row vectors by Bras $\langle\cdot|$.

I adopt the setting of [LMS77]: The idea is to have a set M (equipped with a σ-algebra) which is the internal state of the CTRW and which describes all information the process has about its history (in [LMS77] the authors restrict themselves to finite sets M). This internal state evolves according to a Markov kernel [[Bau96], definition 36.1], i.e., it is a Markov process on M in discrete time parameterized by the number of steps. For every internal state $m \in M$, we define a step distribution $\psi_{(m)}(x,t)$. The process works now such that at the beginning of a step in operational time, the internal state is propagated by a random process described by the Markov kernel to a new state $m \in M$, then the systems makes a spatio-temporal step described by the probability distribution $\psi_{(m)}(x,t)$. In other words, we have a skew system where the process on M drives the random walk, but there is no feedback from the random walk to M. The treatment of Montero and Masoliver [MM07] can be incorporated by taking $M = \mathbb{R}\times\mathbb{R}^+$ corresponding to realization of the step size x and waiting time t of the current step. The Markov kernel describes the conditional probability distribution describing the step depending on the realization of the last step. Here, the functions $\psi_{(x,t)}(\chi,\tau)$ are trivial, as they simply map the internal state to its realization in the random walk:

$$\psi_{(x,t)}(\chi,\tau) = \delta(x-\chi)\delta(t-\tau). \tag{5.1}$$

For their second example, it is sufficient to take M as a two element set, corresponding to

the sign of the current step.

5.2. Nonindependent CTRWs with finite memory

In this chapter, I want to concentrate on the simplest example, namely a finite set M, i.e., the internal memory is described by a discrete Markov chain with finite state space. In the spirit of chapter 3 I will set up the diagrammatic rules, which allow to write down the finite multi-point functions of this process. But the main point is that the asymptotic behavior of these processes which can be described in the scaling limit with a simple CTRW without internal state space. This relies on the short memory of a finite state Markov chain. In general, it is natural to assume that the memory of the internal state is short, as all long memory should be described by the $\psi_{(m)}(x,t)$. Therefore, I assume that the basic ideas presented here generalize to more complex situations. But it is most probably a nontrivial task to establish conditions when one would consider on a given process as having short memory and to show its asymptotic behavior. In an other formulation one can account for the fact that in a physical system the splitting in an internal memory and waiting time distributions is somehow arbitrary and therefore one could consider the internal state space to have short memory, if it is possible to obtain these asymptotic results. This is illustrated in chapter 6. It is not even possible to generalize these results to Markov chains with countable infinite state space without further assumptions, as it is possible to construct examples of such chains with a recurrent state which has infinite mean recurrence time [GW88].

In the next paragraph, I want to introduce the notation I will use for the Markov chains. For the mathematical details, I want to refer to standard texts on probability theory (e.g., [[Fel68], chapter XV] or [[GS97], chapter 11]). The number of states is m, and the states will be numbered consecutively, i.e., $M = \{1, \ldots, m\}$. Since we have a finite state space, the transfer operator is described by a stochastic matrix F. I adopt the convention of a right-stochastic matrix in which the element F_{ij} denotes the conditional probability to go to state j if the process is in state i. Therefore, the rows sum up to 1:

$$\sum_{j=1}^{m} F_{ij} = 1 \text{ for all } 1 \leq i \leq m. \tag{5.2}$$

A probability distribution of a state is described by a row vector $\langle\pi| = (\pi_1, \ldots, \pi_m)$ which evolves in one step to $\langle\pi|F$. Therefore, the condition on $\langle\pi|$ to be a stationary distribution is

$$\langle\pi|F = \langle\pi|. \tag{5.3}$$

I look now at the CTRWs with this type of internal memory. One needs one additional datum: the probability distribution of the internal states prior to the first step[1] which is denoted by $\langle\pi_0|$. Additionally, we have a collection of spatiotemporal step size distributions $\psi_{(i)}(x,t)$ which depend on the current internal state i. It is useful to define a matrix $\Psi(x,t)$

[1] It might seem inconvenient to give the distribution prior to the first step instead of the distribution of the first step, but this is just a matter of interchanging matrices and becomes irrelevant asymptotically.

which has these densities on the diagonal

$$\Psi(x,t) = \begin{pmatrix} \psi_{(1)}(x,t) & & & & \\ & \psi_{(2)}(x,t) & & & \\ & & \ddots & & \\ & & & \psi_{(m-1)}(x,t) & \\ & & & & \psi_{(m)}(x,t) \end{pmatrix}. \tag{5.4}$$

Further, I assume, that none of these distributions $\psi_{(i)}(x,t)$ has almost surely a zero waiting time. Then, we have for the Fourier-Laplace transform

$$\|\Psi(k,\lambda)\|_1 = \max_{i=1,\ldots,m} |\psi_{(i)}(k,\lambda)| < 1 \text{ for } \operatorname{Re}\lambda > 0. \tag{5.5}$$

To apply the methods of section 3.2, we need to determine here the equivalent of equation (3.11), i.e., the contribution of the jth step $\eta_n^{(j)}[\mathbf{q}](\mathbf{x},\mathbf{t})$. First, fix a realization of the internal states $s_1, s_2, \ldots \in M$. This means that the process is in state s_1 for the first step, in s_2 for the second step, etc. Then the different random variables (χ_j, τ_j), describing the spatial jump and the waiting time are independent for different j and (χ_j, τ_j) is distributed according to the density $\psi_{(s_j)}(x,t)$. This means we can take equations (3.13) and (3.20) to get

$$p_n[\mathbf{q}](\mathbf{k},\boldsymbol{\lambda})\Big|_{s_1,s_2,\ldots \text{ fixed}} = \prod_{j=1}^{\infty} \left\langle \eta_n^{(j)}[\mathbf{q}](\mathbf{k},\boldsymbol{\lambda}) \right\rangle_{s_1,s_2,\ldots \text{ fixed}} \tag{5.6}$$

with

$$\left\langle \eta_n^{(j)}[\mathbf{q}](\mathbf{k},\boldsymbol{\lambda}) \right\rangle_{s_1,s_2,\ldots \text{ fixed}} = \frac{1}{\prod_{v\in\mathcal{V}} \lambda_v} \sum_{\mathcal{J}\in\mathcal{P}(\mathcal{V})} (-1)^{|\mathcal{J}|} \psi_{(s_j)}(K_{\mathcal{L}}, \Lambda_{\mathcal{L}\cup\mathcal{J}}). \tag{5.7}$$

Second, we now weight this result with the probability of the realization $s_1, s_2, \ldots$. As soon as $j > \max(q_1, \ldots, q_n)$, we have

$$\left\langle \eta_n^{(j)}[\mathbf{q}](\mathbf{k},\boldsymbol{\lambda}) \right\rangle_{s_1,s_2,\ldots \text{ fixed}} = 1 \tag{5.8}$$

independent of $s_1, s_2, \ldots$. Therefore we only need to look at the probability of the finite string $s_1, s_2, \ldots, s_q$ for any $q > \max(q_1, \ldots, q_n)$. The probability for this string is

$$\Pr[s_1, s_2, \ldots, s_q] = \sum_{s_0=1}^{q} \pi_0(s_0) F_{s_0 s_1} F_{s_1 s_2} \cdots F_{s_{q-1} s_q}. \tag{5.9}$$

Defining the matrix $H_n^{(j)}[\mathbf{q}](\mathbf{k},\boldsymbol{\lambda})$ as

$$H_n^{(j)}[\mathbf{q}](\mathbf{k},\boldsymbol{\lambda}) = \frac{1}{\prod_{v\in\mathcal{V}} \lambda_v} \sum_{\mathcal{J}\in\mathcal{P}(\mathcal{V})} (-1)^{|\mathcal{J}|} F\Psi(K_{\mathcal{L}}, \Lambda_{\mathcal{L}\cup\mathcal{J}}) \tag{5.10}$$

and the column vector

$$|1\rangle = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}, \tag{5.11}$$

we can combine equations (5.6), (5.7) and (5.9) in matrix notation to give

$$\begin{aligned} p_n[\mathbf{q}](\mathbf{k},\boldsymbol{\lambda}) &= \sum_{s_1,s_2,\ldots,s_q=1}^{m} p_n[\mathbf{q}](\mathbf{k},\boldsymbol{\lambda})\Big|_{s_1,s_2,\ldots \text{ fixed}} \Pr[s_1,s_2,\ldots,s_q] \\ &= \langle \pi_0 | H_n^{(1)}[\mathbf{q}](\mathbf{k},\boldsymbol{\lambda}) H_n^{(2)}[\mathbf{q}](\mathbf{k},\boldsymbol{\lambda}) \cdots H_n^{(q)}[\mathbf{q}](\mathbf{k},\boldsymbol{\lambda}) | 1 \rangle. \end{aligned} \tag{5.12}$$

Now, we can proceed to sum over the contributions for a fixed diagram. The contributions of a vertex and a line are now matrices and therefore it is important to consider their relative ordering. We see in equation (5.12) that the number of the step increases from left to right and therefore we have to put the contributions in the same order as the diagram is drawn. The contribution of a vertex is given by one step, therefore we get for the matrix $R_{\text{vertex}}(\mathbf{k},\boldsymbol{\lambda})$ (the equivalent to $\rho_{\text{vertex}}(\mathbf{k},\boldsymbol{\lambda})$) from equation (5.10)

$$R_{\text{vertex}}(\mathbf{k},\boldsymbol{\lambda}) = \frac{1}{\prod_{v\in\mathcal{V}} \lambda_v} F \sum_{\mathcal{J}\in\mathcal{P}(\mathcal{V})} (-1)^{|\mathcal{J}|} \Psi(K_{\mathcal{L}}, \Lambda_{\mathcal{L}\cup\mathcal{J}}). \tag{5.13}$$

The lines consists of steps with $\mathcal{V} = \{\}$, i.e., $H_n^{(j)}[\mathbf{q}](\mathbf{k},\boldsymbol{\lambda}) = F\Psi(K_{\mathcal{L}}, \Lambda_{\mathcal{L}})$. The contribution of a line is then

$$\begin{aligned} R_{\text{line}}(\mathbf{k},\boldsymbol{\lambda}) &= \sum_{i=0}^{\infty} \left(F\Psi(K_{\mathcal{L}}, \Lambda_{\mathcal{L}})\right)^i \\ &= \left[\mathbb{I} - F\Psi(K_{\mathcal{L}}, \Lambda_{\mathcal{L}})\right]^{-1} \end{aligned} \tag{5.14}$$

where the geometric series of matrices converges since

$$\|F\Psi(k,\lambda)\|_1 \leq \|F\|_1 \|\Psi(k,\lambda)\|_1 = \|\Psi(k,\lambda)\|_1 < 1 \text{ for } \operatorname{Re}\lambda > 0. \tag{5.15}$$

One could go through all simplifications presented in chapter 3 by using the permutation symmetry of the indices. But this does not offer anything new, nor do I need it for the following argumentation.

I want to give two examples to show how this machinery works. The first one is the one point density, which is given by

$$\begin{aligned} p_1(k,\lambda) &= \langle \pi_0 | R_{\text{line},\mathcal{L}=\{1\}}(k,\lambda) R_{\text{vertex},\mathcal{V}=\{1\},\mathcal{L}=\{\}}(k,\lambda) | 1 \rangle \\ &= \frac{1}{\lambda} \langle \pi_0 | \left[\mathbb{I} - F\Psi(k,\lambda)\right]^{-1} F \left[\mathbb{I} - \Psi(0,\lambda)\right] | 1 \rangle. \end{aligned} \tag{5.16}$$

As a check, it is possible to plug in the functions given for the solvable example in [[MM07] section IV]. They consider a CTRW where the waiting time is independent from the internal state and the jump size. Its distribution is given by $\psi^{\text{MM}}(t)$ (I will use the subscript "MM" as indication that this is the notation in [MM07]). The internal state describes the sign of the spatial jump. The probability that the next step has the same direction as the current one is parameterized by $(1+\epsilon)/2$. This gives the Markov matrix

$$F = \begin{pmatrix} \frac{1+\epsilon}{2} & \frac{1-\epsilon}{2} \\ \frac{1-\epsilon}{2} & \frac{1+\epsilon}{2} \end{pmatrix} \tag{5.17}$$

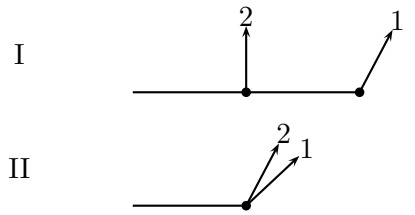

Figure 5.1.: The diagrams relevant for the calculation of a two-point correlation

(with $-1 < \epsilon < 1$). The probability distribution for the spatial steps is given by $h^{\mathrm{MM}}(x)$ which is assumed to be symmetric. The Fourier transform of the positive side is

$$H^{\mathrm{MM}}(k) = \int_0^\infty dx\, e^{ikx} h^{\mathrm{MM}}(x). \tag{5.18}$$

Therefore, $2H^{\mathrm{MM}}(k)$ corresponds to the Fourier transform of the probability distribution of a positive step. This gives

$$\Psi(k,\lambda) = \begin{pmatrix} 2H^{\mathrm{MM}}(k)\psi^{\mathrm{MM}}(\lambda) & 0 \\ 0 & 2H^{\mathrm{MM}}(-k)\psi^{\mathrm{MM}}(\lambda) \end{pmatrix}. \tag{5.19}$$

Plugging equations (5.17) and (5.19) with the initial distribution $\langle \pi_0| = (1/2, 1/2)$ into equation (5.16) gives

$$p_1(k,\lambda) = \frac{1-\psi^{\mathrm{MM}}(\lambda)}{\lambda} \frac{1-\epsilon\psi^{\mathrm{MM}}(\lambda)h^{\mathrm{MM}}(k)}{1-(1+\epsilon)\psi^{\mathrm{MM}}(\lambda)h^{\mathrm{MM}}(k) + 4\epsilon\psi^{\mathrm{MM}}(\lambda)^2|H^{\mathrm{MM}}(k)|^2} \tag{5.20}$$

with $h^{\mathrm{MM}}(k) = H^{\mathrm{MM}}(k) + H^{\mathrm{MM}}(-k)$. This is exactly the result [[MM07] equation (25)].

For the two point density, there are two distinct diagrams, given in figure 5.1. Their contributions are

$$\begin{aligned} p_2^{\mathrm{I}}(\mathbf{k},\boldsymbol{\lambda}) &= \langle \pi_0|\, R_{\mathrm{line},\mathcal{L}=\{1,2\}}(\mathbf{k},\boldsymbol{\lambda}) R_{\mathrm{vertex},\mathcal{V}=\{2\},\mathcal{L}=\{1\}}(\mathbf{k},\boldsymbol{\lambda}) \\ &\qquad \cdot R_{\mathrm{line},\mathcal{L}=\{1\}}(\mathbf{k},\boldsymbol{\lambda}) R_{\mathrm{vertex},\mathcal{V}=\{1\},\mathcal{L}=\{\}}(\mathbf{k},\boldsymbol{\lambda}) |1\rangle \\ &= \frac{1}{\lambda_1\lambda_2} \langle \pi_0|\, \left[\mathbb{I} - F\Psi(k_1+k_2, \lambda_1+\lambda_2)\right]^{-1} F\left[\Psi(k_1,\lambda_1) - \Psi(k_1,\lambda_1+\lambda_2)\right] \\ &\qquad \cdot \left[\mathbb{I} - F\Psi(k_1,\lambda_1)\right]^{-1} F\left[\mathbb{I} - \Psi(0,\lambda_1)\right] |1\rangle \\ p_2^{\mathrm{II}}(\mathbf{k},\boldsymbol{\lambda}) &= \langle \pi_0| R_{\mathrm{line},\mathcal{L}=\{1,2\}}(\mathbf{k},\boldsymbol{\lambda}) R_{\mathrm{vertex},\mathcal{V}=\{1,2\},\mathcal{L}=\{\}}(\mathbf{k},\boldsymbol{\lambda}) |1\rangle \\ &= \frac{1}{\lambda_1\lambda_2} \langle \pi_0|\, \left[\mathbb{I} - F\Psi(k_1+k_2, \lambda_1+\lambda_2)\right]^{-1} \\ &\qquad \cdot F\left[1 - \Psi(0,\lambda_1) - \Psi(0,\lambda_2) + \Psi(0,\lambda_1+\lambda_2)\right] |1\rangle. \end{aligned} \tag{5.21}$$

The complete two point density is then given by

$$p_2(\mathbf{k},\boldsymbol{\lambda}) = p_2^{\mathrm{I}}(k_1,k_2,\lambda_1,\lambda_2) + p_2^{\mathrm{I}}(k_2,k_1,\lambda_2,\lambda_1) + p_2^{\mathrm{II}}(k_1,k_2,\lambda_1,\lambda_2). \tag{5.22}$$

Fortunately, these formulas simplify a lot when one is interested in the asymptotic behavior. As a Markov chain with finite state space can always be decomposed into different irreducible blocks and a set of transient states, I will start with the case that the whole Markov chain is irreducible and then use the results to determine the general case.

5.3. Asymptotic behavior for irreducible Markov chains

A Markov chain is called *irreducible* (or *ergodic*) if it is possible to reach every state from every state, i.e., for each pair $i, j \in M$ there is a $n \geq 1$ such that

$$(F^n)_{ij} > 0. \tag{5.23}$$

Irreducible Markov chains possess exactly one stationary distribution, which I denote by $\langle\overline{\pi}|$. That they are also called ergodic is justified by the following variant of the ergodic theorem [[GS97], theorem 11.11]

$$\lim_{n\to\infty} \frac{\mathbb{I} + F + \cdots + F^n}{n+1} = |1\rangle\langle\overline{\pi}|. \tag{5.24}$$

By multiplying the left hand side with an initial distribution, the limit gives the average time the process spends in the different states, i.e., this corresponds to the temporal average. The right hand side corresponds to the ensemble average by telling us that, independent from the initial distribution, the limit it determined by the stationary distribution $\langle\overline{\pi}|$.

The uniqueness of the stationary distribution is connected with the fact that 1 is a simple eigenvalue of F. Equation (5.24) additionally tells us that 1 is simple also as generalized eigenvalue (in other words, the characteristic polynomial of F has a simple zero at 1 [[MM92], 5.5.1(i)]). Therefore, $\mathbb{R}^m$ is a direct sum of the kernel and the image of $(\mathbb{I} - F)$, where $|1\rangle\langle\overline{\pi}|$ is a projector on the kernel and $\mathbb{I} - |1\rangle\langle\overline{\pi}|$ a projector on the image. This gives

$$\big(\mathbb{I} - F\big)|1\rangle\langle\overline{\pi}| = |1\rangle\langle\overline{\pi}|\big(\mathbb{I} - F\big) = 0, \tag{5.25}$$

$$\big(\mathbb{I} - F + |1\rangle\langle\overline{\pi}|\big)^{-1}|1\rangle\langle\overline{\pi}| = |1\rangle\langle\overline{\pi}|\big(\mathbb{I} - F + |1\rangle\langle\overline{\pi}|\big)^{-1} = |1\rangle\langle\overline{\pi}|, \tag{5.26}$$

$$\mathbb{I} - F = \big(\mathbb{I} - F + |1\rangle\langle\overline{\pi}|\big)\big(\mathbb{I} - |1\rangle\langle\overline{\pi}|\big) \tag{5.27}$$

which correspond to the conditions (B.18)–(B.20). This will come in handy for the power series expansion of the determinant equation (B.31) which is derived in appendix B.

From the result [BKMS04] (only domain of normal attraction) we know that for each $\phi_{(i)}(k,\lambda)$ we have constants α_i and β_i such that

$$\psi_{(i)}(\zeta^{\beta_i}k, \zeta\lambda) = 1 - i\mu_{(i)}\zeta^{\beta_i}k - \zeta^{\alpha_i}\phi_{(i)}(k,\lambda) + o(\zeta^{\alpha_i}) \tag{5.28}$$

with the property

$$\phi_{(i)}(\zeta^{\beta_i}k, \zeta\lambda) = \zeta^{\alpha_i}\phi_{(i)}(k,\lambda). \tag{5.29}$$

The $\phi_{(i)}(k,\lambda)$ correspond to the full limit, i.e., its projection on the spatial component $\phi_{(i)}(k,0)$ has positive real part:

$$\operatorname{Re}\phi_{(i)}(k,0) = d_i|k|^{\frac{\alpha_i}{\beta_i}} \quad \text{with } d_i > 0. \tag{5.30}$$

Similarly for the projection on the temporal component

$$\phi_{(i)}(0,\lambda) = d_i'\lambda^{\alpha_i} \quad \text{with } d_i' > 0. \tag{5.31}$$

The mean $\mu_{(i)}$ is only defined for $\beta_i < \alpha_i$. For $\beta_i \geq \alpha_i$, I set $\mu_{(i)} = 0$ to avoid the treatment of different cases.

Define

$$\hat{\alpha} = \min(\alpha_1, \ldots, \alpha_m) \quad \text{and} \quad \hat{\beta} = \hat{\alpha} \max\left(\frac{\beta_1}{\alpha_1}, \ldots, \frac{\beta_m}{\alpha_m}\right). \tag{5.32}$$

Further define

$$\hat{\phi}_{(i)}(k,\lambda) = \begin{cases} \phi_{(i)}(k,\lambda) & \alpha_i = \hat{\alpha} \text{ and } \beta_i = \hat{\beta} \\ \phi_{(i)}(0,\lambda) & \alpha_i = \hat{\alpha} \text{ and } \beta_i < \hat{\beta} \\ \phi_{(i)}(k,0) & \alpha_i > \hat{\alpha} \text{ and } \frac{\beta_i}{\alpha_i} = \frac{\hat{\beta}}{\hat{\alpha}} \\ 0 & \alpha_i > \hat{\alpha} \text{ and } \frac{\beta_i}{\alpha_i} < \frac{\hat{\beta}}{\hat{\alpha}}. \end{cases} \tag{5.33}$$

In matrix form:

$$\hat{\Phi}(k,\lambda) = \begin{pmatrix} \hat{\phi}_{(1)}(k,\lambda) & & & & \\ & \hat{\phi}_{(2)}(k,\lambda) & & & \\ & & \ddots & & \\ & & & \hat{\phi}_{(m-1)}(k,\lambda) & \\ & & & & \hat{\phi}_{(m)}(k,\lambda) \end{pmatrix} \tag{5.34}$$

and for $\hat{\beta} \leq \hat{\alpha}$

$$\hat{M} = \begin{pmatrix} \mu_{(1)} & & & & \\ & \mu_{(2)} & & & \\ & & \ddots & & \\ & & & \mu_{(m-1)} & \\ & & & & \mu_{(m)} \end{pmatrix}, \tag{5.35}$$

or $\hat{M} = \mathbf{0}$ for $\hat{\beta} > \hat{\alpha}$, respectively. With these definition, one can expand $\Psi(\zeta^{\hat{\beta}}k, \zeta\lambda)$ as

$$\Psi(\zeta^{\hat{\beta}}k, \zeta\lambda) = \mathbb{I} - i\hat{M}\zeta^{\hat{\beta}}k - \zeta^{\hat{\alpha}}\hat{\Phi}(k,\lambda) + o(\zeta^{\hat{\alpha}}). \tag{5.36}$$

Defining the mean with respect to the ergodic measure $\overline{\pi}$:

$$\begin{aligned} \overline{\psi}(k,\lambda) &= \langle\overline{\pi}|\Psi(k,\lambda)|1\rangle \\ \overline{\mu} &= \langle\overline{\pi}|\hat{M}|1\rangle \\ \overline{\phi}(k,\lambda) &= \langle\overline{\pi}|\hat{\Phi}(k,\lambda)|1\rangle \end{aligned} \tag{5.37}$$

gives the expansion

$$\overline{\psi}(\zeta^{\hat{\beta}}k, \zeta\lambda) = 1 - i\overline{\mu}\zeta^{\hat{\beta}}k - \zeta^{\hat{\alpha}}\overline{\phi}(k,\lambda) + o(\zeta^{\hat{\alpha}}). \tag{5.38}$$

Since $\pi_i > 0$ for all $i = 1, \ldots, m$ and both $\alpha_i = \hat{\alpha}$ and $\frac{\beta_j}{\alpha_j} = \frac{\hat{\beta}}{\hat{\alpha}}$ for some i, j, one knows that $\overline{\phi}(k,\lambda)$ is full in the sense that

$$\begin{aligned} \operatorname{Re}\overline{\phi}(k,0) &= \overline{d}|k|^{\frac{\hat{\alpha}}{\hat{\beta}}} \quad \text{with } \overline{d} > 0 \\ \text{and} \quad \overline{\phi}(0,\lambda) &= \overline{d'}\lambda^{\hat{\alpha}} \quad \text{with } \overline{d'} > 0. \end{aligned} \tag{5.39}$$

Similarly to the last chapter, one has to distinguish the case $\overline{\mu} = 0$ from the case $\overline{\mu} \neq 0$.

The case $\overline{\mu} = 0$ and $\hat{\beta} > \frac{\hat{\alpha}}{2}$. I start with the expansion of a line:

$$\begin{aligned} R_{\text{line}}(\zeta^{\hat{\beta}}\mathbf{k}, \zeta\boldsymbol{\lambda}) &= \left[\mathbb{I} - F\Psi(\zeta^{\hat{\beta}}K_{\mathcal{L}}, \zeta\Lambda_{\mathcal{L}})\right]^{-1} \\ &= \frac{1}{\det\left(\mathbb{I} - F\Psi(\zeta^{\hat{\beta}}K_{\mathcal{L}}, \zeta\Lambda_{\mathcal{L}})\right)} \operatorname{adj}\left(\mathbb{I} - F + O(\zeta^{\hat{\beta}})\right) \\ &= \frac{\det\left(\mathbb{I} - F + |1\rangle\langle\overline{\pi}|\right)}{\det\left(\mathbb{I} - F\Psi(\zeta^{\hat{\beta}}K_{\mathcal{L}}, \zeta\Lambda_{\mathcal{L}})\right)} \left(|1\rangle\langle\overline{\pi}| + O(\zeta^{\hat{\beta}})\right) \end{aligned} \tag{5.40}$$

since since the adjugate matrix $\operatorname{adj}(\cdot)$ (see appendix B) is a polynomial in the entries of the argument and (equation (B.27))

$$\operatorname{adj}\left(\mathbb{I} - F\right) = \det\left(\mathbb{I} - F + |1\rangle\langle\overline{\pi}|\right)|1\rangle\langle\overline{\pi}|. \tag{5.41}$$

Using equation (B.14) we can expand the determinant

$$\begin{aligned} &\det\Big(\mathbb{I} - F\Psi(\zeta^{\hat{\beta}}K_{\mathcal{L}}, \zeta\Lambda_{\mathcal{L}})\Big) \\ &\quad= \det\left(\mathbb{I} - F + iF\hat{M}\zeta^{\hat{\beta}}K_{\mathcal{L}} + F\zeta^{\hat{\alpha}}\hat{\Phi}(K_{\mathcal{L}}, \Lambda_{\mathcal{L}}) + o(\zeta^{\hat{\alpha}})\right) \\ &\quad= \det\left(\mathbb{I} - F\right) + \det\left(\mathbb{I} - F + |1\rangle\langle\overline{\pi}|\right) \operatorname{Tr}\left[|1\rangle\langle\overline{\pi}|\left(iF\hat{M}\zeta^{\hat{\beta}}K_{\mathcal{L}} + F\zeta^{\hat{\alpha}}\hat{\Phi}(K_{\mathcal{L}}, \Lambda_{\mathcal{L}})\right)\right] \\ &\qquad+ O(\zeta^{2\min(\hat{\alpha},\hat{\beta})}) \\ &\quad=\zeta^{\hat{\alpha}} \det\left(\mathbb{I} - F + |1\rangle\langle\overline{\pi}|\right) \overline{\phi}(K_{\mathcal{L}}, \Lambda_{\mathcal{L}}) + o(\zeta^{\hat{\alpha}}) \end{aligned} \tag{5.42}$$

which gives finally

$$R_{\text{line}}(\zeta^{\hat{\beta}}\mathbf{k}, \zeta\boldsymbol{\lambda}) = \frac{\zeta^{-\hat{\alpha}}}{\overline{\phi}(K_{\mathcal{L}}, \Lambda_{\mathcal{L}}) + o(1)} \left(|1\rangle\langle\overline{\pi}| + O(\zeta^{\hat{\beta}})\right). \tag{5.43}$$

Combining the line with the following vertex gives

$$\begin{aligned} &R_{\text{line},\mathcal{L}'=\mathcal{L}\cup\mathcal{V}}(\zeta^{\hat{\beta}}\mathbf{k}, \zeta\boldsymbol{\lambda}) R_{\text{vertex},\mathcal{L},\mathcal{V}}(\zeta^{\hat{\beta}}\mathbf{k}, \zeta\boldsymbol{\lambda}) \\ &=\frac{\zeta^{-|\mathcal{V}|}}{\prod_{v\in\mathcal{V}}\lambda_v} \frac{\zeta^{-\hat{\alpha}}}{\overline{\phi}(K_{\mathcal{L}'}, \Lambda_{\mathcal{L}'}) + o(1)} \left(|1\rangle\langle\overline{\pi}| + O(\zeta^{\hat{\beta}})\right) F \sum_{\mathcal{J}\in\mathcal{P}(\mathcal{V})} (-1)^{|\mathcal{J}|}\Psi(K_{\mathcal{L}}, \Lambda_{\mathcal{L}\cup\mathcal{J}}) \\ &=\frac{\zeta^{-|\mathcal{V}|}}{\prod_{v\in\mathcal{V}}\lambda_v} \frac{1}{\overline{\phi}(K_{\mathcal{L}'}, \Lambda_{\mathcal{L}'}) + o(1)} \left(|1\rangle\langle\overline{\pi}| + O(\zeta^{\hat{\beta}})\right) \\ &\quad\times F \sum_{\mathcal{J}\in\mathcal{P}(\mathcal{V})} (-1)^{|\mathcal{J}|+1} \left(i\hat{M}\zeta^{\hat{\beta}-\hat{\alpha}}K_{\mathcal{L}} + \hat{\Phi}(K_{\mathcal{L}}, \Lambda_{\mathcal{L}\cup\mathcal{J}}) + o(1)\right) \\ &=\frac{\zeta^{-|\mathcal{V}|}}{\prod_{v\in\mathcal{V}}\lambda_v} \frac{1}{\overline{\phi}(K_{\mathcal{L}'}, \Lambda_{\mathcal{L}'})} \sum_{\mathcal{J}\in\mathcal{P}(\mathcal{V})} (-1)^{|\mathcal{J}|+1} \left(i\zeta^{\hat{\beta}-\hat{\alpha}}|1\rangle\langle\overline{\pi}|\hat{M}K_{\mathcal{L}} + |1\rangle\langle\overline{\pi}|\hat{\Phi}(K_{\mathcal{L}}, \Lambda_{\mathcal{L}\cup\mathcal{J}}) + o(1)\right). \end{aligned} \tag{5.44}$$

Next, I want to combine this formula to the n-point distribution corresponding to a general Diagram D with q vertices. I use the following notation which is illustrated in figure 5.2: the sets of the line indices are $\mathcal{L}_0, \mathcal{L}_1, \ldots, \mathcal{L}_{q-1}$ with the definition $\mathcal{L}_0 = \{1, \ldots, n\}$.

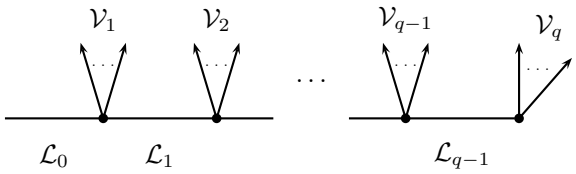

Figure 5.2.: Illustration of the notation of the indices for a general diagram

Correspondingly the vertex indices are the sets $\mathcal{V}_1, \dots, \mathcal{V}_q$. The sets of indices fulfill the consistency equation $\mathcal{L}_i = \mathcal{V}_{i+1} \dot\cup \mathcal{L}_{i+1}$. Therefore with the initial distribution π_0 and the fact $\langle\overline{\pi}|\hat{M}|1\rangle = 0$, we get

$$\begin{aligned}
\zeta^n p_n[D](\zeta^{\hat\beta}\mathbf{k}, \zeta\boldsymbol{\lambda}) =& \zeta^n \langle\pi_0|\, R_{\text{line},\mathcal{L}_0}(\zeta^{\hat\beta}\mathbf{k}, \zeta\boldsymbol{\lambda}) R_{\text{vertex},\mathcal{L}_1,\mathcal{V}_1}(\zeta^{\hat\beta}\mathbf{k}, \zeta\boldsymbol{\lambda}) \cdots \\
& \cdots R_{\text{line},\mathcal{L}_{q-1}}(\zeta^{\hat\beta}\mathbf{k}, \zeta\boldsymbol{\lambda}) R_{\text{vertex},\mathcal{L}_{q-1},\mathcal{V}_{q-1}}(\zeta^{\hat\beta}\mathbf{k}, \zeta\boldsymbol{\lambda})|1\rangle \\
=& \frac{1}{\lambda_1 \cdots \lambda_n} \frac{1}{\overline{\phi}(K_{\mathcal{L}_0}, \Lambda_{\mathcal{L}_0})} \sum_{\mathcal{J}_1 \mathcal{P}(\mathcal{V}_1)} (-1)^{|\mathcal{J}_1|+1} \overline{\phi}(K_{\mathcal{L}_1}, \Lambda_{\mathcal{L}_1 \cup \mathcal{J}_1}) \cdots \\
& \cdots \frac{1}{\overline{\phi}(K_{\mathcal{L}_{q-1}}, \Lambda_{\mathcal{L}_{q-1}})} \sum_{\mathcal{J}_q \mathcal{P}(\mathcal{V}_q)} (-1)^{|\mathcal{J}_q|+1} \overline{\phi}(0, \Lambda_{\mathcal{J}_q}) + o(1).
\end{aligned} \tag{5.45}$$

Except for the term $o(1)$, the last line can be expressed as the n-point distribution of an independent CTRW. Using the density formalism with

$$\rho^{\lim}_{\text{step}}(k, \lambda) = \overline{\phi}(k, \lambda) \tag{5.46}$$

we see from the rules (4.33) that the density $p_n^{\lim}[D](\mathbf{k}, \boldsymbol{\lambda})$ corresponding to the diagram D can be written as

$$\begin{aligned}
p_n^{\lim}[D](\mathbf{k}, \boldsymbol{\lambda}) =& \rho^{\lim}_{\text{line},\mathcal{L}_0}(\zeta^{\hat\beta}\mathbf{k}, \zeta\boldsymbol{\lambda}) \rho^{\lim}_{\text{vertex},\mathcal{L}_1,\mathcal{V}_1}(\zeta^{\hat\beta}\mathbf{k}, \zeta\boldsymbol{\lambda}) \cdots \\
& \cdots \rho^{\lim}_{\text{line},\mathcal{L}_{q-1}}(\zeta^{\hat\beta}\mathbf{k}, \zeta\boldsymbol{\lambda}) \rho^{\lim}_{\text{vertex},\mathcal{L}_{q-1},\mathcal{V}_{q-1}}(\zeta^{\hat\beta}\mathbf{k}, \zeta\boldsymbol{\lambda}) \\
=& \frac{1}{\lambda_1 \cdots \lambda_n} \frac{1}{\overline{\phi}(K_{\mathcal{L}_0}, \Lambda_{\mathcal{L}_0})} \sum_{\mathcal{J}_1 \mathcal{P}(\mathcal{V}_1)} (-1)^{|\mathcal{J}_1|+1} \overline{\phi}(K_{\mathcal{L}_1}, \Lambda_{\mathcal{L}_1 \cup \mathcal{J}_1}) \cdots \\
& \cdots \frac{1}{\overline{\phi}(K_{\mathcal{L}_{q-1}}, \Lambda_{\mathcal{L}_{q-1}})} \sum_{\mathcal{J}_q \mathcal{P}(\mathcal{V}_q)} (-1)^{|\mathcal{J}_q|+1} \overline{\phi}(0, \Lambda_{\mathcal{J}_q}) \\
=& \zeta^n p_n[D](\zeta^{\hat\beta}\mathbf{k}, \zeta\boldsymbol{\lambda}) + o(1).
\end{aligned} \tag{5.47}$$

The scaling limit of the joint distributions is obtained by a sum over all diagrams D

$$\begin{aligned}
\lim_{\zeta \searrow 0} \zeta^n p_n(\zeta^{\hat\beta}\mathbf{k}, \zeta\boldsymbol{\lambda}) &= \sum_D \lim_{\zeta \searrow 0} \zeta^n p_n[D](\zeta^{\hat\beta}\mathbf{k}, \zeta\boldsymbol{\lambda}) \\
&= \sum_D p_n^{\lim}[D](\mathbf{k}, \boldsymbol{\lambda}) \\
&= p_n^{\lim}(\mathbf{k}, \boldsymbol{\lambda}).
\end{aligned} \tag{5.48}$$

The case $\overline{\mu}=0$ and $\hat{\beta}=\frac{\hat{\alpha}}{2}$. The case $\hat{\beta}=\frac{\hat{\alpha}}{2}$ is special in the respect that the term $i\zeta^{\hat{\beta}}\hat{M}k$ has to be expanded to second order to get all terms up to order $\zeta^{\hat{\alpha}}$. On the other hand the expressions are easier, since $\phi_{(i)}(k,\lambda)$ cannot contain a coupled contribution (see end of section 2.3). In general, we have the expansion

$$\psi_{(i)}(\zeta^{\hat{\beta}}k,\zeta\lambda)=1-i\zeta^{\hat{\beta}}\mu_{(i)}k-\frac{1}{2}\zeta^{\hat{\alpha}}(\mu_{(i)}^2+\sigma_{(i)}^2)k^2-\beth_{(i)}\lambda^{\hat{\alpha}}+o(\zeta^{\hat{\alpha}}) \tag{5.49}$$

with $\beth_{(i)}\geq 0$ (and $\beth_{(i)}>0$ for at least one i) and $\sigma_{(i)}^2$ being the variance of spatial step distribution. This gives rise to the variance matrix

$$\hat{V}=\begin{pmatrix}\sigma_{(1)}^2 & & & & \\ & \sigma_{(2)}^2 & & & \\ & & \ddots & & \\ & & & \sigma_{(m-1)}^2 & \\ & & & & \sigma_{(m)}^2\end{pmatrix} \tag{5.50}$$

and its mean $\overline{\sigma^2}=\langle\overline{\pi}|\hat{V}|1\rangle$. Additionally,

$$\hat{G}=\begin{pmatrix}\beth_{(1)} & & & & \\ & \beth_{(2)} & & & \\ & & \ddots & & \\ & & & \beth_{(m-1)} & \\ & & & & \beth_{(m)}\end{pmatrix} \tag{5.51}$$

with $\overline{\beth}=\langle\overline{\pi}|\hat{G}|1\rangle>0$. Therefore, we have the expansions

$$\Psi(\zeta^{\hat{\beta}}k,\zeta\lambda)=1-i\zeta^{\hat{\beta}}\hat{M}k-\frac{1}{2}\zeta^{\hat{\alpha}}(\hat{M}^2+\hat{V})k^2-\hat{G}\lambda^{\hat{\alpha}}+o(\zeta^{\hat{\alpha}}) \tag{5.52}$$

$$\text{and}\quad \langle\overline{\pi}|\Psi(\zeta^{\hat{\beta}}k,\zeta\lambda)|1\rangle=1-\frac{1}{2}\zeta^{\hat{\alpha}}(\langle\overline{\pi}|\hat{M}^2|1\rangle+\overline{\sigma^2})-\overline{\beth}\lambda^{\hat{\alpha}}+o(\zeta^{\hat{\alpha}}). \tag{5.53}$$

Similarly to the last case, I will expand the contribution of a line

$$\begin{aligned}R_{\text{line},\mathcal{L}}(\zeta^{\hat{\beta}}\mathbf{k},\zeta\boldsymbol{\lambda})&=\left[\mathbb{I}-F\Psi(\zeta^{\hat{\beta}}K_{\mathcal{L}},\zeta\Lambda_{\mathcal{L}})\right]^{-1}\\&=\frac{\det\left(\mathbb{I}-F+|1\rangle\langle\overline{\pi}|\right)}{\det\left(\mathbb{I}-F\Psi(\zeta^{\hat{\beta}}K_{\mathcal{L}},\zeta\Lambda_{\mathcal{L}})\right)}\left(|1\rangle\langle\overline{\pi}|+O(\zeta^{\hat{\beta}})\right)\end{aligned} \tag{5.54}$$

where I need for the determinant the expansion up to second order (B.31):

$$\begin{aligned}
&\det\Big(\mathbb{I} - F\Psi(\zeta^{\hat\beta}K_{\mathcal{L}}, \zeta\Lambda_{\mathcal{L}})\Big) \\
&\quad= \det\left[\mathbb{I} - F + F\left(i\zeta^{\hat\beta}\hat{M}k + \frac{1}{2}\zeta^{\hat\alpha}(\hat{M}^2+\hat{V})k^2 + \hat{G}\zeta^{\hat\alpha}\lambda^{\hat\alpha}\right)\right] + o(\zeta^{\hat\alpha}) \\
&\quad=\zeta^{\hat\alpha}\det\left(\mathbb{I} - F + |1\rangle\langle\overline{\pi}|\right)\left[\mathrm{Tr}\left(|1\rangle\langle\overline{\pi}|F\left(\frac{1}{2}(\hat{M}^2+\hat{V})k^2 + \hat{G}\lambda^{\hat\alpha}\right)\right)\right. \\
&\qquad\left. + \mathrm{Tr}\left(|1\rangle\langle\overline{\pi}|F\hat{M}\Big(\mathbb{I} - F + |1\rangle\langle\overline{\pi}|\Big)^{-1}F\hat{M}\right)k^2\right] + o(\zeta^{\hat\alpha}) \\
&\quad=\zeta^{\hat\alpha}\det\left(\mathbb{I} - F + |1\rangle\langle\overline{\pi}|\right)\left[\frac{1}{2}\left(D_{\mathrm{MC}} + \overline{\sigma^2}\right)k^2 + \overline{\mathfrak{J}}\lambda^{\hat\alpha}\right] + o(\zeta^{\hat\alpha}) \\
&\quad=\zeta^{\hat\alpha}\det\left(\mathbb{I} - F + |1\rangle\langle\overline{\pi}|\right)\left[\overline{\phi}(k,\lambda) + \frac{1}{2}D_{\mathrm{MC}}k^2\right] + o(\zeta^{\hat\alpha})
\end{aligned} \tag{5.55}$$

with

$$\begin{aligned}
D_{\mathrm{MC}} &= \langle\overline{\pi}|\hat{M}^2|1\rangle + 2\langle\overline{\pi}|\hat{M}\Big(\mathbb{I} - F + |1\rangle\langle\overline{\pi}|\Big)^{-1}F\hat{M}|1\rangle \\
&= \langle\overline{\pi}|\hat{M}\Big(\mathbb{I} - F + |1\rangle\langle\overline{\pi}|\Big)^{-1}\Big(\mathbb{I} + F\Big)\hat{M}|1\rangle.
\end{aligned} \tag{5.56}$$

Therefore the contribution of a line is

$$R_{\mathrm{line},\mathcal{L}}(\zeta^{\hat\beta}\mathbf{k}, \zeta\boldsymbol{\lambda}) = \frac{\zeta^{-\hat\alpha}}{\overline{\phi}(k,\lambda) + \frac{1}{2}D_{\mathrm{MC}}k^2 + o(1)}\left(|1\rangle\langle\overline{\pi}| + O(\zeta^{\hat\beta})\right). \tag{5.57}$$

The contributions of a vertex is

$$\begin{aligned}
R_{\mathrm{vertex},\mathcal{L},\mathcal{V}}(\zeta^{\hat\beta}\mathbf{k}, \zeta\boldsymbol{\lambda}) =& \frac{\zeta^{-|\mathcal{V}|}}{\prod_{v\in\mathcal{V}}\lambda_v}F\sum_{\mathcal{J}\in\mathcal{P}(\mathcal{V})}(-1)^{|\mathcal{J}|}\Psi(\zeta^{\hat\beta}K_{\mathcal{L}}, \zeta\Lambda_{\mathcal{L}\cup\mathcal{J}}) \\
=& \frac{\zeta^{\hat\alpha-|\mathcal{V}|}}{\prod_{v\in\mathcal{V}}\lambda_v}F\sum_{\mathcal{J}\in\mathcal{P}(\mathcal{V})}(-1)^{|\mathcal{J}|+1}\left(\hat{G}\Lambda^{\hat\alpha}_{\mathcal{L}\cup\mathcal{J}} + o(1)\right).
\end{aligned} \tag{5.58}$$

Using the setup as in the last case, one gets the correspondence to equation (5.45)

$$\begin{aligned}
&\zeta^n p_n[D](\zeta^{\hat\beta}\mathbf{k}, \zeta\boldsymbol{\lambda}) \\
&\quad=\zeta^n\langle\pi_0|\, R_{\mathrm{line},\mathcal{L}_0}(\zeta^{\hat\beta}\mathbf{k}, \zeta\boldsymbol{\lambda})R_{\mathrm{vertex},\mathcal{L}_1,\mathcal{V}_1}(\zeta^{\hat\beta}\mathbf{k}, \zeta\boldsymbol{\lambda})\cdots \\
&\qquad\cdots R_{\mathrm{line},\mathcal{L}_{q-1}}(\zeta^{\hat\beta}\mathbf{k}, \zeta\boldsymbol{\lambda})R_{\mathrm{vertex},\mathcal{L}_{q-1},\mathcal{V}_{q-1}}(\zeta^{\hat\beta}\mathbf{k}, \zeta\boldsymbol{\lambda})|1\rangle \\
&\quad=\frac{1}{\lambda_1\cdots\lambda_n}\frac{1}{\overline{\phi}(K_{\mathcal{L}_0}, \Lambda_{\mathcal{L}_0}) + \frac{1}{2}D_{\mathrm{MC}}k^2}\sum_{\mathcal{J}_1\mathcal{P}(\mathcal{V}_1)}(-1)^{|\mathcal{J}_1|+1}\overline{\phi}(K_{\mathcal{L}_1}, \Lambda_{\mathcal{L}_1\cup\mathcal{J}_1})\cdots \\
&\qquad\cdots\frac{1}{\overline{\phi}(K_{\mathcal{L}_{q-1}}, \Lambda_{\mathcal{L}_{q-1}}) + \frac{1}{2}D_{\mathrm{MC}}k^2}\sum_{\mathcal{J}_q\mathcal{P}(\mathcal{V}_q)}(-1)^{|\mathcal{J}_q|+1}\overline{\phi}(0, \Lambda_{\mathcal{J}_q}) + o(1).
\end{aligned} \tag{5.59}$$

With the limit density

$$\rho^{\lim}_{\mathrm{step}}(k,\lambda) = \overline{\phi}(k,\lambda) + \frac{1}{2}D_{\mathrm{MC}}k^2 \tag{5.60}$$

of an independent CTRW, one sees by following the steps of equations (5.47) and (5.48) that one has again

$$\lim_{\zeta \searrow 0} \zeta^n p_n(\zeta^{\hat{\beta}}\mathbf{k}, \zeta\boldsymbol{\lambda}) = p_n^{\lim}(\mathbf{k}, \boldsymbol{\lambda}). \tag{5.61}$$

It is possible to give a simple interpretation for the value D_{MC}: while $\overline{\sigma^2}$ is the average variance of the spatial distributions, D_{MC} is the diffusion due to the fluctuations of the mean spatial step driven by the Markov Chain. More precisely: consider a random walk $X_{\mathrm{MC}}(t)$ which is driven by the Markov Chain as before, but this time the realization of the step is deterministic with temporal step size 1 and spatial step size $\mu_{(i)}$, i.e.,

$$\psi_{(i)}^{\mathrm{MC}}(x,t) = \delta(x-\mu_{(i)})\delta(t-1) \quad \text{and therefore} \quad \psi_{(i)}^{\mathrm{MC}}(k,\lambda) = e^{-\lambda + ik\mu_{(i)}} \tag{5.62}$$

which is in matrix notation

$$\Psi^{\mathrm{MC}}(k,\lambda) = e^{-\lambda} \exp\left(ik\hat{M}\right). \tag{5.63}$$

The one-point density is (I assume for simplicity that we take the equilibrium distribution $\overline{\pi}$ as initial distribution)

$$p_1^{\mathrm{MC}}(k,\lambda) = \frac{1-e^{-\lambda}}{\lambda} \langle\overline{\pi}| \left[\mathbb{I} - F\Psi^{\mathrm{MC}}(k,\lambda)\right]^{-1} |1\rangle. \tag{5.64}$$

The Laplace transform of the second moment

$$c_2^{\mathrm{MC}}(t) = \langle X_{\mathrm{MC}}^2(t)\rangle \tag{5.65}$$

is given by

$$\begin{aligned}
c_2^{\mathrm{MC}}(\lambda) =& -\frac{\partial^2}{\partial k^2} p_1^{\mathrm{MC}}(k,\lambda)\Big|_{k=0} \\
=& \frac{1-e^{-\lambda}}{\lambda}\Big(2\langle\overline{\pi}|\left[\mathbb{I}-e^{-\lambda}F\right]^{-1}F\hat{M}\left[\mathbb{I}-e^{-\lambda}F\right]^{-1}F\hat{M}\left[\mathbb{I}-e^{-\lambda}F\right]^{-1}|1\rangle \\
&\qquad + \langle\overline{\pi}|\left[\mathbb{I}-e^{-\lambda}F\right]^{-1}F\hat{M}^2\left[\mathbb{I}-e^{-\lambda}F\right]^{-1}|1\rangle\Big) \\
=& \frac{1}{\lambda(1-e^{-\lambda})}\Big(2\langle\overline{\pi}|\hat{M}\left[\mathbb{I}-e^{-\lambda}|1\rangle\langle\overline{\pi}|\right]\left[\mathbb{I}-e^{-\lambda}F\right]^{-1}F\hat{M}|1\rangle + \langle\overline{\pi}|\hat{M}^2|1\rangle\Big) \\
=& \frac{1}{\lambda(1-e^{-\lambda})}\langle\overline{\pi}|\hat{M}\left[\mathbb{I}-e^{-\lambda}(F-|1\rangle\langle\overline{\pi}|)\right]^{-1}\left[\mathbb{I}+(2-e^{-\lambda})F\right]\hat{M}|1\rangle
\end{aligned} \tag{5.66}$$

where I used the following equalities (which can be checked by multiplication)

$$\begin{aligned}
\langle\overline{\pi}|\left[\mathbb{I}-e^{-\lambda}F\right]^{-1} &= \frac{1}{1-e^{-\lambda}}\langle\overline{\pi}| \\
\left[\mathbb{I}-e^{-\lambda}F\right]^{-1}|1\rangle &= \frac{1}{1-e^{-\lambda}}|1\rangle \\
\left[\mathbb{I}-e^{-\lambda}|1\rangle\langle\overline{\pi}|\right]\left[\mathbb{I}-e^{-\lambda}F\right]^{-1} &= \left[\mathbb{I}-e^{-\lambda}(F-|1\rangle\langle\overline{\pi}|)\right]^{-1}.
\end{aligned} \tag{5.67}$$

The diffusion constant is established by use of the Tauberian theorems [[Fel71], section XIII.5]

$$\begin{aligned}\lim_{t\to\infty}\frac{\langle X^2_{\mathrm{MC}}(t)\rangle}{t} &= \lim_{\lambda\searrow 0}\lambda^2 c_2^{\mathrm{MC}}(\lambda)\\ &= \langle\overline{\pi}|\hat{M}\big[\mathbb{I}-F+|1\rangle\langle\overline{\pi}|\big]^{-1}\big[\mathbb{I}+F\big]\hat{M}|1\rangle\\ &= D_{\mathrm{MC}}.\end{aligned} \tag{5.68}$$

The case $\overline{\mu}\neq 0$. In this case we have by definition $\hat{\beta}>\hat{\alpha}$, i.e., we have the expansion (note that the scaling factor for k is $\zeta^{\hat{\alpha}}$ and not $\zeta^{\hat{\beta}}$ as before)

$$\Psi(\zeta^{\hat{\alpha}}k,\zeta\lambda) = \mathbb{I} - i\zeta^{\hat{\alpha}}\hat{M}k - \zeta^{\hat{\alpha}}\hat{G}\lambda^{\hat{\alpha}} + o(\zeta^{\hat{\alpha}}). \tag{5.69}$$

Similar to the independent case, the term containing $\hat{\phi}(k,\lambda)$ will not contribute.

The derivation is an analogy to the other cases, therefore I will reduce it to the main results. The contribution of a line is

$$\begin{aligned}R_{\mathrm{line},\mathcal{L}}(\zeta^{\hat{\alpha}}\mathbf{k},\zeta\boldsymbol{\lambda}) &= \Big[\mathbb{I}-F\Psi(\zeta^{\hat{\alpha}}K_{\mathcal{L}},\zeta\Lambda_{\mathcal{L}})\Big]^{-1}\\ &= \frac{\det\left(\mathbb{I}-F+|1\rangle\langle\overline{\pi}|\right)}{\det\left(\mathbb{I}-F\Psi(\zeta^{\hat{\alpha}}K_{\mathcal{L}},\zeta\Lambda_{\mathcal{L}})\right)}\Big(|1\rangle\langle\overline{\pi}|+O(\zeta^{\hat{\alpha}})\Big)\end{aligned} \tag{5.70}$$

with the determinant

$$\det\Big(\mathbb{I}-F\Psi(\zeta^{\hat{\alpha}}K_{\mathcal{L}},\zeta\Lambda_{\mathcal{L}})\Big) = \zeta^{\hat{\alpha}}\det\left(\mathbb{I}-F+|1\rangle\langle\overline{\pi}|\right)\Big(\overline{\mathfrak{J}}\Lambda^{\hat{\alpha}}_{\mathcal{L}}+i\overline{\mu}K_{\mathcal{L}}\Big)+o(\zeta^{\hat{\alpha}}). \tag{5.71}$$

The vertex expands to

$$\begin{aligned}R_{\mathrm{vertex},\mathcal{L},\mathcal{V}}(\zeta^{\hat{\alpha}}\mathbf{k},\zeta\boldsymbol{\lambda}) =& \frac{\zeta^{-|\mathcal{V}|}}{\prod_{v\in\mathcal{V}}\lambda_v}F\sum_{\mathcal{J}\in\mathcal{P}(\mathcal{V})}(-1)^{|\mathcal{J}|}\Psi(\zeta^{\hat{\alpha}}K_{\mathcal{L}},\zeta\Lambda_{\mathcal{L}\cup\mathcal{J}})\\ =& \frac{\zeta^{\hat{\alpha}-|\mathcal{V}|}}{\prod_{v\in\mathcal{V}}\lambda_v}F\sum_{\mathcal{J}\in\mathcal{P}(\mathcal{V})}(-1)^{|\mathcal{J}|+1}\Big(\hat{G}\Lambda^{\hat{\alpha}}_{\mathcal{L}\cup\mathcal{J}}+o(1)\Big).\end{aligned} \tag{5.72}$$

Asymptotically, this can be described by an independent CTRW with jump density

$$\rho^{\lim}_{\mathrm{step}}(k,\lambda) = \overline{\mathfrak{J}}\lambda^{\hat{\alpha}}+i\overline{\mu}k. \tag{5.73}$$

Summary. Let me summarize the limits obtained:

$$\rho^{\lim,\mathrm{nonindependent}}_{\mathrm{step}}(k,\lambda) = \begin{cases}\overline{\phi}(k,\lambda) & \text{for } \overline{\mu}=0 \text{ and } \hat{\beta}>\frac{\hat{\alpha}}{2}\\ \overline{\phi}(k,\lambda)+\frac{1}{2}D_{\mathrm{MC}}k^2 & \text{for } \overline{\mu}=0 \text{ and } \hat{\beta}=\frac{\hat{\alpha}}{2}\\ \overline{\mathfrak{J}}\lambda^{\hat{\alpha}}+i\overline{\mu}k & \text{for } \overline{\mu}\neq 0.\end{cases} \tag{5.74}$$

For comparison, consider the independent CTRW with probability distribution

$$\overline{\psi}(k,\lambda) = \langle\overline{\pi}|\Psi(k,\lambda)|1\rangle. \tag{5.75}$$

Its limit densities are

$$\rho_{\text{step}}^{\text{lim,independent}}(k,\lambda) = \begin{cases} \overline{\phi}(k,\lambda) & \text{for } \overline{\mu}=0 \text{ and } \hat{\beta} > \frac{\hat{\alpha}}{2} \\ \overline{\phi}(k,\lambda) & \text{for } \overline{\mu}=0 \text{ and } \hat{\beta} = \frac{\hat{\alpha}}{2} \\ \overline{\mathbb{J}}\lambda^{\hat{\alpha}} + i\overline{\mu}k & \text{for } \overline{\mu} \neq 0. \end{cases} \tag{5.76}$$

Therefore we have

$$\rho_{\text{step}}^{\text{lim,independent}}(k,\lambda) = \rho_{\text{step}}^{\text{lim,nonindependent}}(k,\lambda) \qquad \text{for } \hat{\beta} > \frac{\hat{\alpha}}{2} \tag{5.77}$$

and

$$\rho_{\text{step}}^{\text{lim,independent}}(k,\lambda) = \rho_{\text{step}}^{\text{lim,nonindependent}}(k,\lambda) + \frac{1}{2}D_{\text{MC}}k^2 \qquad \text{for } \hat{\beta} = \frac{\hat{\alpha}}{2}. \tag{5.78}$$

This gives the following algorithm to determine the scaling limit of a nonindependent CTRW driven by a ergodic Markov Chain with finite state space and transition matrix F.

1. Determine the stationary state distribution $\langle\overline{\pi}|$ of the Markov Chain

   $$\langle\overline{\pi}|F = \langle\overline{\pi}|. \tag{5.79}$$

   Determine the average jump probability distribution

   $$\overline{\psi}(k,\lambda) = \langle\overline{\pi}|\Psi(k,\lambda)|1\rangle. \tag{5.80}$$

2. Determine the jump density and the scaling exponent α and β for the independent CTRW with jump probability distribution $\overline{\psi}(k,\lambda)$

   $$\rho_{\text{step}}^{\text{lim,independent}}(k,\lambda) = \lim_{\zeta \searrow 0} \zeta^{\alpha}\Big(1 - \overline{\psi}(\zeta^{\beta}k, \zeta\lambda)\Big). \tag{5.81}$$

   The results from chapter 4 can be used for this.

3. If $\beta > \frac{\alpha}{2}$ then the limit of the nonindependent CTRW is given by the jump density of the independent CTRW

   $$\rho_{\text{step}}^{\text{lim,independent}}(k,\lambda) = \rho_{\text{step}}^{\text{lim,nonindependent}}(k,\lambda). \tag{5.82}$$

4. If $\beta = \frac{\alpha}{2}$ then one has to calculate

   $$D_{\text{MC}} = \langle\overline{\pi}|\hat{M}\Big(\mathbb{I} - F + |1\rangle\langle\overline{\pi}|\Big)^{-1}\Big(\mathbb{I} + F\Big)\hat{M}|1\rangle \tag{5.83}$$

   with $\hat{M}$ being the diagonal matrix containing the means of the spatial jump probability distributions. The limit of the nonindependent CTRW is then given by

   $$\rho_{\text{step}}^{\text{lim,independent}}(k,\lambda) = \rho_{\text{step}}^{\text{lim,nonindependent}}(k,\lambda) + \frac{1}{2}D_{\text{MC}}k^2. \tag{5.84}$$

I want to finish this section with one remark: in all cases, I have assumed that $\Psi(k,\lambda)$ has an expansion (5.36) and therefore $\overline{\psi}(k,\lambda)$ has the expansion (5.38). On the other hand, the conditions for the convergence of $\overline{\psi}(k,\lambda)$ for the case $\mu \neq 0$ in chapter 4 implies only the expansion

$$\overline{\psi}(\zeta^{\alpha}k, \zeta\lambda) = 1 - i\zeta^{\alpha}\overline{\mu}k - \zeta^{\alpha}\overline{\mathbb{J}}\lambda^{\alpha} + o(\zeta^{\alpha}). \tag{5.85}$$

As one can see from equation (5.69), this is sufficient to handle the case $\overline{\mu} \neq 0$.

5.4. The reducible Markov Chain

The case of a reducible (i.e., non-ergodic) Markov Chain with finite state space does not behave significantly differently from the case of the ergodic chain. Using [[Fel68], theorem XV.6.3] one can decompose the states into disjoint sets $T, C_1, C_2, \ldots, C_q$. The set T contains the transient states, i.e., the states that have a recurrence probability smaller than 1. The sets C_i are irreducible closed sets, i.e., a state in C_i will never leave this set and the Markov Chain restricted to this set is irreducible. There exists at least one of these sets [[Fel68], theorem XV.6.4]. Since these C_i are irreducible, each has a unique stationary distribution $\langle\overline{\pi}_{C_i}|$.

Assume that we start with a probability distribution $\langle\pi_0|$ which is completely contained in a C_i. Then the internal state of the CTRW will never leave C_i and we can reduce the whole process to the states C_i. As this reduced Markov Chain is ergodic, we can use the results from the last section with the average jump distribution

$$\overline{\psi}_{C_i}(k,\lambda) = \langle\overline{\pi}_{C_i}|\Psi(k,\lambda)|1\rangle. \tag{5.86}$$

To simplify the following discussing, I introduce the linear projection operators Π_{C_i} onto the states supported by C_i, i.e.,

$$\langle j|\Pi_{C_i} = \begin{cases} \langle j| & \text{for } j \in C_i \\ 0 & \text{for } j \notin C_i \end{cases} \tag{5.87}$$

$$\text{with} \quad \langle j| = (0,\ldots,0,\underset{j}{1},0,\ldots,0). \tag{5.88}$$

As corresponding definition applies to Π_T and we have $\Pi_T + \Pi_{C_1} + \cdots + \Pi_{C_q} = \mathbb{I}$.

If $T = \{\}$ (or more generally $\langle\pi_0|\Pi_T = 0$), the concrete realization of the initial state determines which of the CTRWs corresponding to the irreducible subsets $C_1, \ldots, C_q$ we are going to see. In other words, we have a set of q different CTRWs where the initial condition of the internal state determines which is realized. The random walk corresponding to C_i is thereby chosen with a probability of $\langle\pi_0|\Pi_{C_i}|1\rangle$.

In the case that the set of transient state is not empty, we have the problem that any random walk starting with a transient state will almost surely fall after a finite number of steps in one of the C_is and then follow the corresponding CTRW. One has to answer the question if this transient period is asymptotically relevant or not. For this, look at the first line of any diagram with $\mathcal{L}_0 = \{1,\ldots,n\}$

$$\begin{aligned} R_{\text{line},\mathcal{L}_0}(\zeta^\beta K_{\mathcal{L}_0}, \zeta\Lambda_{\mathcal{L}_0}) =& \Big[\mathbb{I} - F\Psi(\zeta^\beta K_{\mathcal{L}_0}, \zeta\Lambda_{\mathcal{L}_0})\Big]^{-1} \\ =& \left(\mathbb{I} - \Pi_T + \Pi_T\Big[\mathbb{I} - \Pi_T F\Psi(\zeta^\beta K_{\mathcal{L}_0}, \zeta\Lambda_{\mathcal{L}_0})\Big]^{-1}\right) \\ & \times \left(\Pi_T + \sum_{i=1}^{q} \Pi_{C_i}\Big[\mathbb{I} - F\Psi(\zeta^\beta K_{\mathcal{L}_0}, \zeta\Lambda_{\mathcal{L}_0})\Big]^{-1}\right) \end{aligned} \tag{5.89}$$

which can be seen by multiplying with $[\mathbb{I} - F\Psi(\zeta^\beta K_{\mathcal{L}_0}, \zeta\Lambda_{\mathcal{L}_0})]$ from the right. Since there is by definition no stationary distribution of F which is supported in T, one knows

$$\lim_{\zeta\searrow 0}\Big[\mathbb{I} - \Pi_T F\Psi(\zeta^\beta K_{\mathcal{L}_0}, \zeta\Lambda_{\mathcal{L}_0})\Big]^{-1} = \Big[\mathbb{I} - \Pi_T F\Big]^{-1}. \tag{5.90}$$

Therefore we have for this line as $\zeta \searrow 0$

$$\begin{aligned} R_{\mathrm{line},\mathcal{L}_0}(\zeta^\beta K_{\mathcal{L}_0}, \zeta\Lambda_{\mathcal{L}_0}) =& \left(\mathbb{I} - \Pi_T + \Pi_T\left[\mathbb{I} - \Pi_T F\right]^{-1}\right) \\ & \times \left(\sum_{i=1}^{q} \Pi_{C_i}\left[\mathbb{I} - F\Psi(\zeta^\beta K_{\mathcal{L}_0}, \zeta\Lambda_{\mathcal{L}_0})\right]^{-1}\right) + O(1) \\ =& \left[\mathbb{I} - \Pi_T F\right]^{-1}(\mathbb{I} - \Pi_T)\, R_{\mathrm{line},\mathcal{L}_0}(\zeta^\beta K_{\mathcal{L}_0}, \zeta\Lambda_{\mathcal{L}_0}) + O(1). \end{aligned} \tag{5.91}$$

The term of order $O(1)$ will be canceled by the following vertex. The projector $\mathbb{I} - \Pi_T = \sum_{i=1}^{q} \Pi_{C_i}$ indicates that no transient states are left after this point. Therefore, the transient states do not contribute to the asymptotic behavior of the CTRW. They only occur as probability which of the q different CTRWs will be realized. More precisely, the probability that the CTRW corresponding to the irreducible set C_i is going to be realized for an initial distribution $\langle\pi_0|$ is

$$\Pr[\text{states will end in } C_i] = \langle\pi_0|\left[\mathbb{I} - \Pi_T F\right]^{-1}\Pi_{C_i}|1\rangle. \tag{5.92}$$

Not surprisingly, these are the absorption probabilities of the Markov chain (which can be inferred from [[Fel68], section XVI.4] by noting that [[Fel68], equation (XVI.4.4)] corresponds to $[(\Pi_{C_i} + \Pi_T)F]^n$ and that $\lim_{n\to\infty}\langle\pi_0|\left[(\Pi_{C_i} + \Pi_T)F\right]^n|1\rangle = \langle\pi_0|\left[\mathbb{I} - \Pi_T F\right]^{-1}\Pi_{C_i}|1\rangle$).

Summarizing, the case of the reducible Markov chain can be traced back to irreducible Markov chains. For this one has to calculate by the standard techniques the absorption probabilities which of the different irreducible sets is reached. The resulting asymptotic random walk is a superposition of the asymptotic random walks for the irreducible sets which are chosen by the absorption probabilities.

6. Anomalous Deterministic Diffusion

6.1. Introduction

In this chapter the methods of the last chapter are applied to anomalous deterministic diffusion. The notions of the nonindependent CTRWs can be adopted to accommodate a deterministic map which can be treated by the method of inducing. The role of the Markov matrix is then taken by the Frobenius-Perron operator of the induced map. Under certain assumptions on this operator, it is possible to give good justifications that the asymptotic behavior is analogous to the case of a finite state space. In other words: under these assumptions the scaling limit of the deterministic dynamics converges to a continuous-time random walk in all joint probability distributions, i.e., we have convergence as a stochastic process. Therefore, the CTRW is the correct asymptotic description of the deterministic diffusion and the parameters of this CTRW (e.g., the scaling exponents α and β) can be inferred from the dynamical system. Additionally, it turns out that the leaper model of a CTRW is not enough, sometimes more refined models are needed.

As an application, I look at two examples based on the Manneville-Pomeau map (though it is not known if these examples verify all assumptions). The analytic description from the stochastic properties of the asymptotic CTRW are tested against numerical simulations of the deterministic dynamics which supports the derivation.

In section 6.2 I introduce the type of deterministic diffusion $S(t)$ which is considered. When making use of the method of inducing, it turns out that it is more convenient to consider first a slightly modified dynamics $S_{\mathrm{mod}}(t)$. In section 6.3 the dynamics $S_{\mathrm{mod}}(t)$ is put in the context of nonindependent CTRWs. The main argument is similar to chapter 5: I give a justification that the contribution of a line becomes asymptotically a projection operator on the ergodic measure of the induced map, i.e., the limit is an independent CTRW. In section 6.4 the difference between $S(t)$ and $S_{\mathrm{mod}}(t)$ is exposed as the difference between different models of CTRWs. An application to the two examples is worked out in section 6.5.

6.2. The basic setup

Assume we have a deterministic dynamics in discrete time on a state space $\mathcal{S}$, which is given by a function $g : \mathcal{S} \to \mathcal{S}$. The states are given by $s_1, s_2, \ldots$ given by the law

$$s_{t+1} = g(s_t). \tag{6.1}$$

Further, we have an observable $\mathcal{X}$ on $\mathcal{S}$, i.e., a function

$$\mathcal{X} : \mathcal{S} \to \mathbb{R}. \tag{6.2}$$

In a setup similarly to the one used by Beck and Roepstorff [BR87b] for sufficiently mixing maps, I am interested in the scaling limit of the process

$$S(t) = \sum_{i=1}^{t} \mathcal{X}(s_i). \tag{6.3}$$

The interpretation in terms of a stochastic process enters by choosing the initial condition s_1 according to a probability distribution. For a sufficiently mixing law g, it has been shown [BR87b] that this process converges to a Brownian motion. In this work, I am focusing on the case when the dynamics belongs to the class of weakly broken ergodic dynamics. While the methods is not restricted to these examples, I will apply them in two Manneville-Pomeau like settings. As described in section 2.4 a common way to treat systems which support an infinite ergodic measure is the method of inducing. One chooses a subset

$$\mathcal{S}_{\text{ind}} \subset \mathcal{S} \tag{6.4}$$

a first return map

$$\mathcal{T} : \mathcal{S}_{\text{ind}} \to \mathbb{N}; \quad \mathcal{T}(s) = \min\{n \in \mathbb{N} : g^n(s) \in \mathcal{S}_{\text{ind}}\} \tag{6.5}$$

and the induced dynamics

$$g_{\text{ind}}(s) = g^{\mathcal{T}(s)}(s). \tag{6.6}$$

It is useful in the case, when $g(s)$ on $\mathcal{S}$ does not have a finite ergodic measure (i.e., an infinite ergodic measure), but $g_{\text{ind}}(s)$ on $\mathcal{S}_{\text{ind}}$ does have an ergodic probability measure $\overline{\pi}$ [Aar97, Tha01].

Therefore, using the dynamics of $g_{\text{ind}}(s)$ we can hope to extract stochastic information. Of course, the dynamics $g_{\text{ind}}(s)$ is not parameterized by the real time T as above but by a virtual time. This virtual time drives both the real time and the space coordinate. Define

$$T(m) = \sum_{i=0}^{m-1} \mathcal{T}(g_{\text{ind}}^i(s_1)) \tag{6.7}$$

and consider as inverse

$$T^{-1}(t) = \max\{m \in \mathbb{N} : T(m) \leq t\}. \tag{6.8}$$

We have then $T^{-1}(T(m)) = m$ but in general *not* $T(T^{-1}(t)) = t$. Similarly for the spatial variable

$$Y(m) = \sum_{i=0}^{m-1} \mathcal{Y}(g_{\text{ind}}^i(s_1)) \qquad \text{with} \qquad \mathcal{Y}(s) = \sum_{i=0}^{\mathcal{T}(s)-1} \mathcal{X}(g^i(s)). \tag{6.9}$$

Let me introduce a modified dynamics, given by

$$S_{\text{mod}}(t) = Y(T^{-1}(t)). \tag{6.10}$$

This coincides with $S(t)$ for $t = T(m)$:

$$S_{\text{mod}}(T(m)) = S(T(m)). \tag{6.11}$$

In the times $T(m) \leq t < T(m+1)$ the value of $S_{\text{mod}}(t)$ is constant which is usually not true for $S(t)$. The reason to look at $S_{\text{mod}}(t)$ is of course that it corresponds to the leaper model

of a CTRW which we considered in the last chapter. Therefore, I will concentrate first on $S_{\mathrm{mod}}(t)$ and come back later to the problem of adapting the results to $S(t)$.

Now, one can write the process $S_{\mathrm{mod}}(t)$ in the formalism of chapter 5 for nonindependent CTRWs. The internal state space (or memory states) are given by $\mathcal{S}_{\mathrm{ind}}$. They evolve by the deterministic map $g_{\mathrm{ind}}(s)$ (which is Markovian). To each of the states $s \in \mathcal{S}_{\mathrm{ind}}$ belongs a jump of $\mathcal{T}(s)$ in temporal direction and $\mathcal{Y}(s)$ in spatial direction, i.e., we have the distribution

$$\psi_{(s)}(x,t) = \delta(x - \mathcal{Y}(s))\delta(t - \mathcal{T}(s)). \tag{6.12}$$

There is a difference with respect to chapter 5: in the present case the propagation according to $\psi_{(s)}(x,t)$ is done first and only after this, the internal state is evolved according to $g_{\mathrm{ind}}(s)$.

6.3. Asymptotics for the dynamics of $S_{\mathrm{mod}}(t)$

While the state space in chapter 5 was finite, here we have to deal with an infinite state space $\mathcal{S}_{\mathrm{ind}}$. By assumption, we have a distinct probability measure on $\mathcal{S}_{\mathrm{ind}}$, the ergodic measure $\overline{\pi}$ (implicitly assuming the existence of a σ-algebra which turns $\mathcal{S}_{\mathrm{ind}}$ into a measure space). The probability distributions on $\mathcal{S}_{\mathrm{ind}}$ which are absolutely continuous with respect to the ergodic measure $\overline{\pi}$ can be written as densities which are represented as positive elements of $L^1(\overline{\pi})$ [[LM96], theorem 2.2.1]. In the following I will always assume that the function spaces are over the field $\mathbb{C}$. The ergodic distribution is represented by the function $f(x) = 1$. In this setting, the Markov matrix F of the last chapter becomes the Frobenius-Perron operator (or transfer operator) $\mathcal{F} : L^1(\overline{\pi}) \to L^1(\overline{\pi})$ which can be defined by the property [[LM96], equation (3.2.2)]

$$\int_A \mathcal{F}f(s)\,\overline{\pi}(ds) = \int_{g_{\mathrm{ind}}^{-1}(A)} f(s)\,\overline{\pi}(ds) \quad \text{for all measurable } A \subseteq \mathcal{S}_{\mathrm{ind}} \text{ and } f \in L^1(\overline{\pi}). \tag{6.13}$$

More generally, we have [[LM96], equation (3.3.4)]

$$\int \mathcal{F}f(s)h(s)\,\overline{\pi}(ds) = \int f(s)h(g_{\mathrm{ind}}(s))\,\overline{\pi}(ds) \quad f \in L^1(\overline{\pi}) \text{ and } h \in L^\infty(\overline{\pi}). \tag{6.14}$$

From its property as a Markov operator [[LM96], section 3.1], the L^1 operator norm $\|\cdot\|_1$ of $\mathcal{F}$ is $\|\mathcal{F}\|_1 = 1$. Using further [[LM96], proposition 3.1.1(M3)] shows that the Frobenius-Perron operator also maps $L^\infty(\overline{\pi})$-functions to $L^\infty(\overline{\pi})$-functions, i.e., $\mathcal{F}(L^\infty(\overline{\pi})) \subseteq L^\infty(\overline{\pi})$. The corresponding norm is also $\|\mathcal{F}\|_\infty = 1$.

The diagonal matrix $\Psi(k,\lambda)$ is replaced by the operator which multiplies a function with $\psi_{(s)}(k,\lambda)$, i.e.,

$$\Psi(k,\lambda) : L^1(\overline{\pi}) \to L^1(\overline{\pi}), \quad f(s) \mapsto \exp\left(-\lambda\mathcal{T}(s) + ik\mathcal{Y}(s)\right) f(s). \tag{6.15}$$

Since the exponential is bounded:

$$|\exp\left(-\lambda\mathcal{T}(s) + ik\mathcal{Y}(s)\right)| \leq e^{-\lambda} \tag{6.16}$$

this operator also maps $L^\infty(\overline{\pi})$ to $L^\infty(\overline{\pi})$ functions and the two operator norms can be estimated by

$$\|\Psi(k,\lambda)\|_1 \leq e^{-\lambda} \quad \text{and} \quad \|\Psi(k,\lambda)\|_\infty \leq e^{-\lambda}. \tag{6.17}$$

I will also use the Bra-Ket notation for vectors (here functions) and their duals (here linear functionals). We have a notational difference to the last chapter due to the fact that operators conventionally act to the right while the propagation of a probability vector with a Markov matrix is denoted by a left multiplication of a vector. Therefore, we have to reverse the direction of notation.

The probability density with respect to $\overline{\pi}$ of a distribution π_0 is denoted by $|\pi_0\rangle$. The stationary density $|\overline{\pi}\rangle$ corresponds of course to the constant function $f(s) = 1$.

The functional of integration with respect to the measure $\overline{\pi}$ is denoted by $\langle 1|$:

$$\langle 1| : L^1(\overline{\pi}) \to \mathbb{C}, \quad f(s) \mapsto \int f(s)\, \overline{\pi}(ds). \tag{6.18}$$

Now, we can proceed analogously to section 5.2: In a first step, fix s_1 which determines the complete series $s_1, s_2, \ldots$ as we have a deterministic evolution equation. In analogy to equations (5.6) and (5.7) we obtain

$$p_n[\mathbf{q}](\mathbf{k}, \boldsymbol{\lambda})\Big|_{s_1 \text{ fixed}} = \prod_{j=1}^{\infty} \left\langle \eta_n^{(j)}[\mathbf{q}](\mathbf{k}, \boldsymbol{\lambda}) \right\rangle_{s_1 \text{ fixed}} \tag{6.19}$$

with

$$\left\langle \eta_n^{(j)}[\mathbf{q}](\mathbf{k}, \boldsymbol{\lambda}) \right\rangle_{s_1 \text{ fixed}} = \frac{1}{\prod_{v \in \mathcal{V}_|} \lambda_v} \sum_{\mathcal{J} \in \mathcal{P}(\mathcal{V}_j)} (-1)^{|\mathcal{J}|} \psi_{(s_j)}(K_{\mathcal{L}_j}, \Lambda_{\mathcal{L}_j \cup \mathcal{J}}). \tag{6.20}$$

For an given initial density π_0 for s_1 we have ($q > \max(q_1, \ldots, q_n)$)

$$p_n[\mathbf{q}](\mathbf{k}, \boldsymbol{\lambda}) = \int \prod_{j=1}^{q} \left\langle \eta_n^{(j)}[\mathbf{q}](\mathbf{k}, \boldsymbol{\lambda}) \right\rangle_{s_1 \text{ fixed}} \pi_0(s_1)\, \overline{\pi}(ds_1) \tag{6.21}$$

where the $\eta_n^{(j)}[\mathbf{q}](\mathbf{k}, \boldsymbol{\lambda})$ depend on s_1 through s_j. In this case one gets an operator $H_n^{(j)}[\mathbf{q}](\mathbf{k}, \boldsymbol{\lambda})$ (see (5.10)), defined as

$$H_n^{(j)}[\mathbf{q}](\mathbf{k}, \boldsymbol{\lambda}) = \frac{1}{\prod_{v \in \mathcal{V}_j} \lambda_v} \sum_{\mathcal{J} \in \mathcal{P}(\mathcal{V}_j)} (-1)^{|\mathcal{J}|} \mathcal{F}\Psi(K_{\mathcal{L}_j}, \Lambda_{\mathcal{L}_j \cup \mathcal{J}}). \tag{6.22}$$

From equation (6.14) we get with $f \in L^1(\overline{\pi})$ and $h \in L^\infty(\overline{\pi})$

$$\int h(s_{j+1}) \left\langle \eta_n^{(j)}[\mathbf{q}](\mathbf{k}, \boldsymbol{\lambda}) \right\rangle_{s_j \text{ fixed}} f(s_j)\, \overline{\pi}(ds_j) = \int f(s_{j+1}) \left(H_n^{(j)}[\mathbf{q}](\mathbf{k}, \boldsymbol{\lambda}) h \right)(s_{j+1})\, \overline{\pi}(ds_{j+1}). \tag{6.23}$$

By induction, equation (6.21) can be rewritten as

$$p_n[\mathbf{q}](\mathbf{k}, \boldsymbol{\lambda}) = \langle 1 | H_n^{(q)}[\mathbf{q}](\mathbf{k}, \boldsymbol{\lambda}) H_n^{(q-1)}[\mathbf{q}](\mathbf{k}, \boldsymbol{\lambda}) \cdots H_n^{(1)}[\mathbf{q}](\mathbf{k}, \boldsymbol{\lambda}) | \pi_0 \rangle. \tag{6.24}$$

In comparison to equation (5.12) the order of the operators is reversed. This means also, that the order of the components will be reversed with respect to the ordering in a diagram. The rules are

$$R_{\text{vertex}}(\mathbf{k}, \boldsymbol{\lambda}) = \frac{1}{\prod_{v \in \mathcal{V}} \lambda_v} \mathcal{F} \sum_{\mathcal{J} \in \mathcal{P}(\mathcal{V})} (-1)^{|\mathcal{J}|} \Psi(K_{\mathcal{L}}, \Lambda_{\mathcal{L} \cup \mathcal{J}}). \tag{6.25}$$

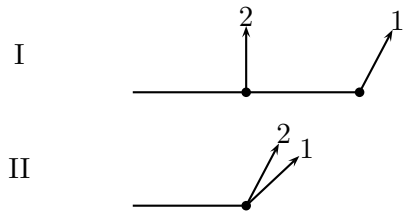

Figure 6.1.: The diagrams relevant for the calculation of a two-point correlation

and

$$\begin{aligned} R_{\text{line}}(\mathbf{k},\boldsymbol{\lambda}) &= \sum_{i=0}^{\infty} \left(\mathcal{F}\Psi(K_{\mathcal{L}},\Lambda_{\mathcal{L}})\right)^i \\ &= \left[\mathbb{I} - \mathcal{F}\Psi(K_{\mathcal{L}},\Lambda_{\mathcal{L}})\right]^{-1}. \end{aligned} \tag{6.26}$$

The last equation is a Neumann series, which converges for $\Lambda_{\mathcal{L}} > 0$ both as an operator $L^1(\overline{\pi}) \to L^1(\overline{\pi})$ and $L^\infty(\overline{\pi}) \to L^\infty(\overline{\pi})$ since

$$\begin{aligned} &\|\mathcal{F}\Psi(K_{\mathcal{L}},\Lambda_{\mathcal{L}})\|_1 \leq \|\mathcal{F}\|_1 \, \|\Psi(K_{\mathcal{L}},\Lambda_{\mathcal{L}})\|_1 \leq e^{-\Lambda_{\mathcal{L}}} \\ \text{and} \qquad &\|\mathcal{F}\Psi(K_{\mathcal{L}},\Lambda_{\mathcal{L}})\|_\infty \leq \|\mathcal{F}\|_\infty \, \|\Psi(K_{\mathcal{L}},\Lambda_{\mathcal{L}})\|_\infty \leq e^{-\Lambda_{\mathcal{L}}}. \end{aligned} \tag{6.27}$$

Using these rules, the Fourier-Laplace transform of the one-point density is given by

$$\begin{aligned} p_1(k,\lambda) &= \langle 1 | R_{\text{vertex},\mathcal{V}=\{1\},\mathcal{L}=\{\}}(k,\lambda) R_{\text{line},\mathcal{L}=\{1\}}(k,\lambda) | \pi_0 \rangle \\ &= \frac{1}{\lambda} \langle 1 | \mathcal{F} \left[\mathbb{I} - \Psi(0,\lambda)\right] \left[\mathbb{I} - \mathcal{F}\Psi(k,\lambda)\right]^{-1} | \pi_0 \rangle \\ &= \frac{1}{\lambda} \langle 1 | \left[\mathbb{I} - \Psi(0,\lambda)\right] \left[\mathbb{I} - \mathcal{F}\Psi(k,\lambda)\right]^{-1} | \pi_0 \rangle. \end{aligned} \tag{6.28}$$

For the two point density, there are two distinct diagrams, given in figure 6.1. Their contributions are

$$\begin{aligned} p_2^{\text{I}}(\mathbf{k},\boldsymbol{\lambda}) &= \langle 1 | \, R_{\text{vertex},\mathcal{V}=\{1\},\mathcal{L}=\{\}}(\mathbf{k},\boldsymbol{\lambda}) R_{\text{line},\mathcal{L}=\{1\}}(\mathbf{k},\boldsymbol{\lambda}) \\ &\qquad \cdot R_{\text{vertex},\mathcal{V}=\{2\},\mathcal{L}=\{1\}}(\mathbf{k},\boldsymbol{\lambda}) R_{\text{line},\mathcal{L}=\{1,2\}}(\mathbf{k},\boldsymbol{\lambda}) | \pi_0 \rangle \\ &= \frac{1}{\lambda_1 \lambda_2} \langle 1 | \, \left[\mathbb{I} - \Psi(0,\lambda_1)\right] \left[\mathbb{I} - \mathcal{F}\Psi(k_1,\lambda_1)\right]^{-1} \\ &\qquad \cdot \mathcal{F} \left[\Psi(k_1,\lambda_1) - \Psi(k_1,\lambda_1+\lambda_2)\right] \left[\mathbb{I} - \mathcal{F}\Psi(k_1+k_2,\lambda_1+\lambda_2)\right]^{-1} | \pi_0 \rangle, \\ p_2^{\text{II}}(\mathbf{k},\boldsymbol{\lambda}) &= \langle 1 | R_{\text{vertex},\mathcal{V}=\{1,2\},\mathcal{L}=\{\}}(\mathbf{k},\boldsymbol{\lambda}) R_{\text{line},\mathcal{L}=\{1,2\}}(\mathbf{k},\boldsymbol{\lambda}) | \pi_0 \rangle \\ &= \frac{1}{\lambda_1 \lambda_2} \langle 1 | \, \left[1 - \Psi(0,\lambda_1) - \Psi(0,\lambda_2) + \Psi(0,\lambda_1+\lambda_2)\right] \\ &\qquad \cdot \left[\mathbb{I} - \mathcal{F}\Psi(k_1+k_2,\lambda_1+\lambda_2)\right]^{-1} | \pi_0 \rangle. \end{aligned} \tag{6.29}$$

The complete two point density is then given by

$$p_2(\mathbf{k},\boldsymbol{\lambda}) = p_2^{\text{I}}(k_1,k_2,\lambda_1,\lambda_2) + p_2^{\text{I}}(k_2,k_1,\lambda_2,\lambda_1) + p_2^{\text{II}}(k_1,k_2,\lambda_1,\lambda_2). \tag{6.30}$$

So far, the derivation has been completely analogously to the ones in chapter 5, except for the reversed ordering and more formal overhead due to the need of working on function spaces. But the results of chapter 5 can not be generalized without further assumptions. Already for the simplest infinite state space $\mathbb{N}$, one can construct a Markov chain for which the derivation does not work any more (in general, a recurrent state with infinite mean recurrence time will void the argument. One example of such a Markov chain can be found in [GW88, Iso99]).

In general, allowing all densities in $L^1(\overline{\pi})$ for the initial distribution $|\pi_0\rangle$ can lead to very different behaviors. The following argument should illustrate this fact (I concentrate only on the spatial direction): assume we had a process where the ergodic distribution is given by a normal distribution. Still, all stable distributions can be described by densities in $L^1(\overline{\pi})$, i.e., it is possible that at least the first step has a power law tail. In general, this power law will still be visible at later times and we end with a dynamic very different from the one suggested by the ergodic average.

A possibility to avoid a "fattening of tails" is the restriction of the densities to $L^\infty(\overline{\pi})$. As mentioned while deriving, all operators considered will also map $L^\infty(\overline{\pi})$ to $L^\infty(\overline{\pi})$. Therefore, if not mentioned otherwise, I will work in the following on the space $L^\infty(\overline{\pi})$.

Analogously to the last chapter, I will consider the behavior of

$$\mathbb{I} - \mathcal{F}\Psi(\zeta^\beta k, \zeta\lambda) = \mathbb{I} - \mathcal{F} + \mathcal{F}(\mathbb{I} - \Psi(\zeta^\beta k, \zeta\lambda)) \tag{6.31}$$

where the last term is treated as a perturbation. As it seems that making the results below rigorously is quite demanding with help of the perturbation theory for operators [Kat80], I can only motivate them in an informal manner. A rigorous treatment has to be postponed for a future work which should also provide details on the necessary conditions. As example, when one treats simple systems which converge to a Brownian motion [BG97, Kel09], one needs still to restrict the space $L^\infty(\overline{\pi})$ to functions of bounded variation or Lipschitz functions. It is not clear at the moment, if a similar restriction is necessary in this context.

The behavior

$$R_{\text{line}}(\zeta^\beta \mathbf{k}, \zeta\boldsymbol{\lambda}) = \Big[\mathbb{I} - \mathcal{F}\Psi(\zeta^\beta K_{\mathcal{L}}, \zeta\Lambda_{\mathcal{L}})\Big]^{-1} \tag{6.32}$$

as $\zeta \searrow 0$ is essential for the asymptotics. From equation (6.27), we know

$$\|\mathcal{F}\Psi(\zeta^\beta K_{\mathcal{L}}, \zeta\Lambda_{\mathcal{L}})\|_\infty \leq e^{-\zeta\Lambda_{\mathcal{L}}} \tag{6.33}$$

and for $\zeta = 0$, we have the eigenvector $|\overline{\pi}\rangle$ to the eigenvalue 1. Therefore, the dominating part will be determined by the spectrum of $\mathcal{F}$ near the values of modulus 1. A helpful concept from the theory of dynamical systems is the notion of a *quasi-compact* operator [[Bal00] page 31 and [HH01] definition II.1]: the spectrum quasicompact Frobenius-Perron operator $\mathcal{F}$ can be decomposed into isolated eigenvalues on the unit sphere while the rest of the spectrum is contained in a disc of radius strictly smaller than one[1]. Here, I will concentrate on a slightly simpler setting also used in [[Kel09] section 6]: I assume that the ergodic measure is the only eigenvector of modulus one, i.e.,

$$\text{spectral radius}\big(\mathcal{F} - |\overline{\pi}\rangle\langle 1|\big) < 1. \tag{6.34}$$

[1] Tasaki and Gaspard [TG02] have worked out the spectrum of the Frobenius-Perron operator of a piecewise linear map similar to a Manneville-Pomeau map (without inducing). They find a continuous spectrum on the unit interval $[0, 1]$, so this Frobenius-Perron operator is clearly not quasi-compact.

I will shortly describe the results of [[Kel09] theorem 6.1] (which is a good summary of [[HH01] theorem III.8 and corollary III.11]): Assume that we perturb the operator $\mathbb{I} - \mathcal{F}$ by an operator $Q(\zeta)$ such that $\mathbb{I} - \mathcal{F} + Q(\zeta)$ is quasi-compact for $|\zeta| < \epsilon$ and $Q(0) = 0$. Informally, we can Taylor expand $Q(\zeta)$ around $\zeta \to 0$

$$Q(\zeta) = \zeta Q'(0) + \frac{1}{2}\zeta^2 Q''(0) + \text{h.o.t.} \tag{6.35}$$

where h.o.t. denotes higher order terms. The theorem [[Kel09] theorem 6.1] states the existence of densities $|\overline{\pi}(\zeta)\rangle$, dual densities $\langle 1(\zeta)|$, an operator $N(\zeta)$ and a function $\lambda(\zeta)$ such that

$$\mathbb{I} - \mathcal{F} + Q(\zeta) = \mathbb{I} - |\overline{\pi}(\zeta)\rangle\langle 1(\zeta)| + N(\zeta) \tag{6.36}$$

with

$$\begin{aligned} \big(\mathbb{I} - \mathcal{F} + Q(\zeta)\big)|\overline{\pi}(\zeta)\rangle &= \lambda(\zeta)|\overline{\pi}(\zeta)\rangle \\ \langle 1(\zeta)|\big(\mathbb{I} - \mathcal{F} + Q(\zeta)\big) &= \lambda(\zeta)\langle 1(\zeta)| \\ \langle 1(\zeta)|\overline{\pi}(\zeta)\rangle &= 1 \\ \text{spectral radius}(N(\zeta)) &< r < 1. \end{aligned} \tag{6.37}$$

The terms $|\overline{\pi}(\zeta)\rangle$, $\langle 1(\zeta)|$, $N(\zeta)$ and $\lambda(\zeta)$ have at least the same order of differentiability as $Q(\zeta)$.

Similarly to equation (6.35) one can do an expansion for $\lambda(\zeta)$ up to second order where I use the result [[Kel09] equations (6.1) and (6.2)] for $\lambda'(0)$ and $\lambda''(0)$

$$\begin{aligned} \lambda(\zeta) =& \lambda(0) + \zeta\lambda'(0) + \frac{1}{2}\zeta^2\lambda''(0) + \text{h.o.t.} \\ =& 1 + \zeta\langle 1|Q'(0)|\overline{\pi}\rangle + \frac{1}{2}\zeta^2\langle 1|Q''(0)|\overline{\pi}\rangle \\ &+ \zeta^2\langle 1|Q'(0)\Big[\mathbb{I} - \mathcal{F} + |\overline{\pi}\rangle\langle 1|\Big]^{-1}\Big(1 - |\overline{\pi}\rangle\langle 1|\Big)Q'(0)|\overline{\pi}\rangle + \text{h.o.t.} \\ =& 1 + \langle 1|Q(\zeta)|\overline{\pi}\rangle - \langle 1|Q(\zeta)|\overline{\pi}\rangle^2 + \langle 1|Q(\zeta)\Big[\mathbb{I} - \mathcal{F} + |\overline{\pi}\rangle\langle 1|\Big]^{-1}Q(\zeta)|\overline{\pi}\rangle + \text{h.o.t.} \end{aligned} \tag{6.38}$$

Now, I will switch back to the situation

$$Q(\zeta) = \mathcal{F}(\mathbb{I} - \Psi(\zeta^\beta K_{\mathcal{L}}, \zeta\Lambda_{\mathcal{L}})) \tag{6.39}$$

and assume that we can still use the expansion (6.38). The contribution of a line is in this formalism

$$\begin{aligned} R_{\text{line}}(\zeta^\beta \mathbf{k}, \zeta\boldsymbol{\lambda}) &= \Big[\mathbb{I} - \mathcal{F}\Psi(\zeta^\beta K_{\mathcal{L}}, \zeta\Lambda_{\mathcal{L}})\Big]^{-1} \\ &= \left[\mathbb{I} - |\overline{\pi}(\zeta)\rangle\langle 1(\zeta)| + N(\zeta)\right]^{-1}. \end{aligned} \tag{6.40}$$

Here, I introduce the auxiliary variable z:

$$[z\mathbb{I} - |\overline{\pi}(\zeta)\rangle\langle 1(\zeta)| + N(\zeta)]^{-1} \tag{6.41}$$

to apply the perturbation theory for linear operators [[Kat80], chapter III §6.5] around $z \to \lambda(\zeta)$

$$[z\mathbb{I} - |\overline{\pi}(\zeta)\rangle\langle 1(\zeta)| + N(\zeta)]^{-1} = \frac{1}{z - \lambda(\zeta)}|\overline{\pi}(\zeta)\rangle\langle 1(\zeta)| + O(1). \tag{6.42}$$

For small enough ζ, one can put $z = 1$ and get

$$R_{\text{line}}(\zeta^\beta \mathbf{k}, \zeta \boldsymbol{\lambda}) = \frac{1}{1-\lambda(\zeta)} |\overline{\pi}\rangle\langle 1| + \text{h.o.t.}. \tag{6.43}$$

Now, we are again in the situation of the last chapter and the same argument shows that we end up with an independent CTRW with

$$\rho^{\lim}_{\text{step}}(k,\lambda) = \lim_{\zeta \searrow 0} \zeta^{-\alpha}(1-\lambda(\zeta)). \tag{6.44}$$

With the average one-step distribution

$$\overline{\psi}(k,\lambda) = \langle 1|\Psi(k,\lambda)|\overline{\pi}\rangle \tag{6.45}$$

we have for $\beta > \frac{\alpha}{2}$

$$\rho^{\lim}_{\text{step}}(k,\lambda) = \lim_{\zeta \searrow 0} \zeta^{-\alpha}(1-\overline{\psi}(\zeta^\beta k, \zeta\lambda)) \tag{6.46}$$

which coincides with the limit of the independent CTRW with distribution $\overline{\psi}(k,\lambda)$. For $\beta = \frac{\alpha}{2}$ we can have again second order contributions

$$\begin{aligned} \rho^{\lim}_{\text{step}}(k,\lambda) = &\lim_{\zeta \searrow 0} \zeta^{-\alpha}(1-\overline{\psi}(\zeta^\beta k, \zeta\lambda)) \\ &- \lim_{\zeta \searrow 0} \zeta^{-\alpha} \langle 1|(1-\Psi(\zeta^\beta k,\zeta\lambda))\Big[\mathbb{I} - \mathcal{F} + |\overline{\pi}\rangle\langle 1|\Big]^{-1} \mathcal{F}(1-\Psi(\zeta^\beta k,\zeta\lambda))|\overline{\pi}\rangle. \end{aligned} \tag{6.47}$$

It seems likely that the last term can be identified with

$$-k^2 \langle 1|\hat{x}\Big[\mathbb{I} - \mathcal{F} + |\overline{\pi}\rangle\langle 1|\Big]^{-1} \mathcal{F}\hat{x}|\overline{\pi}\rangle \tag{6.48}$$

where the operator $\hat{x}$ is the multiplication with x, but I have not found an argument in this operator setting.

6.4. Discussion on the dynamic $S(t)$

The question remains what we can infer from the dynamics of $S_{\text{mod}}(t)$ about the original dynamics $S(t)$. We have already seen from equation (6.11)

$$S_{\text{mod}}(T(m)) = S(T(m))$$

that these two systems have points at which they are necessarily equal (of course this set $\{T(m)\}$ differs for each realization). Therefore, if the dynamics during the waiting times is of the order of the rest of the dynamics, the anomalous scaling exponents β of $S(t)$ and $S_{\text{mod}}(t)$ should be the same. Of course, the diffusion constant of $S_{\text{mod}}(t)$ will be in general smaller.

But does the description of $S(t)$ require a completely different formalism than the description of $S_{\text{mod}}(t)$? Of course $S_{\text{mod}}(t)$ is connected with the leaper model of a CTRW. But in section 3.4.1 we have already seen the creeper model in which the observable moves with constant velocity during the waiting time. It is possible to apply the same formalism; only the description of the vertices has to be changed. The same applies in this more general case:

the vertices correspond to the steps (or waiting times) in which the observation times lie. Respectively, what happens during the steps which are represented by a line, does not have any influence on $S(t)$. Therefore, comparing the rules for $S(t)$ and $S_{\mathrm{mod}}(t)$: the line will have the same rule, while the vertices will be different in general. Unfortunately, to derive a unified treatment seems not to be easy. But it can be worked out on specific examples that will be presented in the next section.

6.5. Application to Manneville-Pomeau maps

A typical guinea pig for the study of anomalous diffusion are maps of Manneville-Pomeau type (see section 2.4 for an introduction). I will use two different types of these maps: a Manneville-Pomeau map which is point symmetric about the origin and an asymmetric one taking only positive values. The invariant measure of the first one will by symmetric to the origin (therefore, if the mean exists it is necessarily zero, i.e., $\overline{\mu} = 0$) and the resulting asymptotic stable distributions will show no skewness. The second one will clearly be skewed and will have a non-vanishing mean (when the mean exists).

In this section, I will also present data obtained by means of numerical calculations and compare them with the analytical results. Therefore, I describe here the techniques which were used. Numerical simulations of chaotic systems face the problem that the round-off effects due to the use of finite-sized floating-point numbers will force all orbits to be periodic. The typical length of these orbits for a sufficiently chaotic system can be estimated from the relative precision of the floating-point number [BR87a, GOY88]. Without going into details, I did some tests which indicate that the situation becomes worse for maps as the Manneville-Pomeau map. In general, the precision of the standard floating-point numbers led to periodic orbits which were shorter than the time needed for a good estimate of the observables (e.g., diffusion exponents). Therefore, I resorted to an arbitrary precision library, the MPFR Library [FHL+07, MPF]. All simulations were done with a precision of 100 significant bits with the rounding mode set to 'nearest value'.

6.5.1. A symmetrized Manneville-Pomeau map

The first map I want to consider can be constructed from a one-sided map $g_+(x)$

$$g_+(x) : [0,1] \to [-1,1], \quad g_+(x) = \begin{cases} x + 2x^z & \text{for } x + 2x^z \leq 1 \\ x + 2x^z - 2 & \text{for } x + 2x^z > 1 \end{cases} \tag{6.49}$$

by anti-symmetrization to give a map $g(x)$ which is point-symmetric about the origin

$$g(x) : [-1,1] \to [-1,1], \quad g(x) = \begin{cases} g_+(x) & \text{for } x \geq 0 \\ -g_+(-x) & \text{for } x < 0. \end{cases} \tag{6.50}$$

This map is plotted in figure 6.2 for the parameter value $z = 3$.

This map has a singularity of the invariant measure at the origin $x = 0$ (equation (2.114)). Therefore, we choose a sufficiently small $\epsilon > 0$ and cut out a symmetric interval around the origin

$$\mathcal{S}_{\mathrm{ind}} = [-1,1] \setminus [-\epsilon, \epsilon]. \tag{6.51}$$

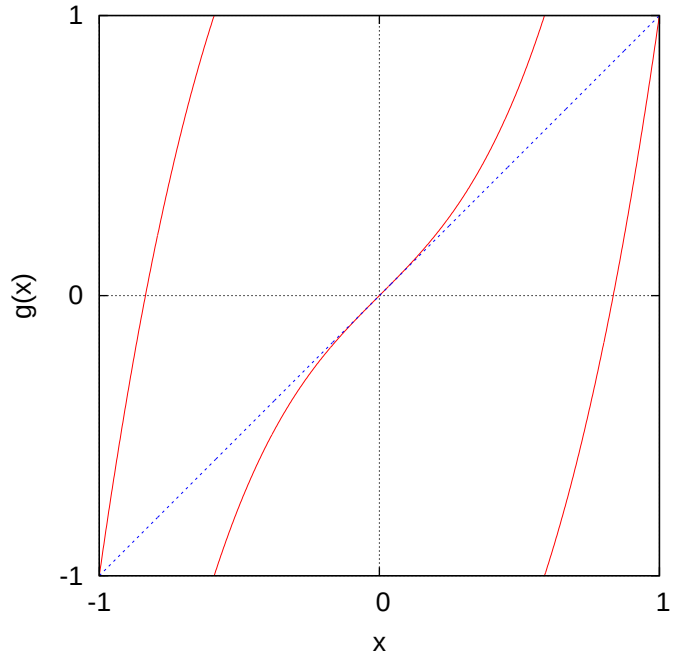

Figure 6.2.: Plot of the symmetrized Manneville-Pomeau map ($z = 3$)

with the first return map (i.e., the waiting time)

$$\mathcal{T}(x) = \min\{n \in \mathbb{N} : g^n(x) \in \mathcal{S}_{\text{ind}}\} \tag{6.52}$$

and the spatial jump

$$\mathcal{Y}(x) = \sum_{i=0}^{\mathcal{T}(x)-1} g^i(x) \tag{6.53}$$

which are driven by the induced dynamics

$$g_{\text{ind}}(x) = g^{\mathcal{T}(s)}(x). \tag{6.54}$$

We know that $g_{\text{ind}}(x)$ is ergodic (see section 2.4). I denote the ergodic measure by $\overline{\pi}$. In the following I will assume that the methods of section 6.3 can be applied to $g_{\text{ind}}(x)$. The dynamics of $S_{\text{mod}}(t)$ is mostly given by

$$\begin{aligned} \overline{\psi}(k,\lambda) &= \langle 1|\Psi(k,\lambda)|\overline{\pi}\rangle \\ &= \int e^{-\lambda\mathcal{T}(x)+ik\mathcal{Y}(x)}\,\overline{\pi}(dx). \end{aligned} \tag{6.55}$$

For $z < 2$ we know (see section 2.4) that the waiting time distribution has finite mean value. By symmetry, it is clear that the spatial step size distribution does not have a bias. Section 4.2 tells us that $S_{\text{mod}}(t)$ converges to a Brownian motion and we have a scaling exponent of $\beta = \frac{1}{2}$.

For $z > 2$ the waiting time distribution has an infinite mean. More precisely, it is in the domain of attraction of a one-sided Lévy-stable distribution with exponent $\alpha = \frac{1}{z-1}$ (equation (2.126)).

We can check if the conditions for convergence to an uncoupled CTRW are fulfilled with $\beta = \frac{\alpha}{2}$ (point 1 in section 4.5). To do so, we have to look at the absolute moment of order $\delta > 2$ (equation (4.58))

$$C_\delta = \int |\mathcal{Y}(x)|^\delta \, \overline{\pi}(dx). \tag{6.56}$$

By choosing a $t > 0$ this can be split into two parts

$$\begin{aligned} C_\delta &= \int_{\mathcal{T}(x)\leq t} |\mathcal{Y}(x)|^\delta \, \overline{\pi}(dx) + \int_{\mathcal{T}(x)>t} |\mathcal{Y}(x)|^\delta \, \overline{\pi}(dx) \\ &\leq t + \int_{\mathcal{T}(x)>t} |\mathcal{Y}(x)|^\delta \, \overline{\pi}(dx) \end{aligned} \tag{6.57}$$

where the last estimate follows from the fact

$$|\mathcal{Y}(x)| \leq \sum_{i=0}^{\mathcal{T}(x)-1} |g^i(x)| \leq \mathcal{T}(x). \tag{6.58}$$

When we choose t large enough, we can assume that $\mathcal{Y}(x)$ with $\mathcal{T}(x) > t$ can be calculated by using the renormalization group limit $g_*(x)$ instead of $g(x)$. This function $g_*(x)$ is an adaptation of equation (2.123) [HNS82, HR82] to the point-symmetric case[2]

$$\begin{aligned} g_*(x) &= \operatorname{sgn}(x) \left[|x|^{-(z-1)} - 2(z-1) \right]^{-\frac{1}{z-1}} \\ \text{and therefore } g_*^j(x) &= \operatorname{sgn}(x) \left[|x|^{-(z-1)} - 2(z-1)j \right]^{-\frac{1}{z-1}}. \end{aligned} \tag{6.59}$$

Using the fact that $g_*(x)$ has the same sign as x we see

$$\begin{aligned} |\mathcal{Y}(x)| &= \sum_{j=0}^{\mathcal{T}(x)-1} \left[|x|^{-(z-1)} - 2(z-1)j \right]^{-\frac{1}{z-1}} \\ &\leq \int_{r\in[0,\mathcal{T}(x)]} \left[|x|^{-(z-1)} - 2(z-1)r \right]^{-\frac{1}{z-1}} dr \\ &= \frac{1}{2(z-2)} \left(\left[|x|^{-(z-1)} \right]^{\frac{z-2}{z-1}} - \left[|x|^{-(z-1)} - 2(z-1)\mathcal{T}(x) \right]^{\frac{z-2}{z-1}} \right) \\ &\leq \frac{1}{2(z-2)} \left(\left[\epsilon^{-(z-1)} + 2(z-1)\mathcal{T}(x) \right]^{\frac{z-2}{z-1}} - \left[\epsilon^{-(z-1)} \right]^{\frac{z-2}{z-1}} \right) \end{aligned} \tag{6.60}$$

where the last line used the fact that by definition of $\mathcal{T}(x)$

$$\left[|x|^{-(z-1)} - 2(z-1)\mathcal{T}(x) \right]^{\frac{z-2}{z-1}} > \epsilon. \tag{6.61}$$

A further simplification gives

$$|\mathcal{Y}(x)| \leq \left(\frac{1}{2(z-2)} \right)^{\frac{1}{z-1}} \mathcal{T}(x)^{\frac{z-2}{z-1}}. \tag{6.62}$$

[2] In the articles [HNS82, HR82] it is shown that $g_*(x)$ is an attracting fixed point of the renormalization group. A priori, one would need to extend their calculation to the convergence of $\mathcal{Y}(x)$ which I assume here. There is also an alternative way by approximating $g(x)$ near $x = 0$ by a differential equation, e.g. [[KKC+07], equation (10)], which gives directly this result.

Plugging this in equation (6.57) gives the estimate for the absolute moment of order δ

$$C_\delta \lesssim t + \left(\frac{1}{2(z-2)}\right)^{\frac{\delta}{z-1}} \int_{\mathcal{T}(x)>t} |\mathcal{T}(x)|^{\delta\frac{z-2}{z-1}}\, \overline{\pi}(dx) \tag{6.63}$$

(the symbol "$\lesssim$" indicates that this inequality is in general not exactly true, but only approximately since it already uses the function $g_*(x)$ obtained as the limit of the renormalization group). Since the waiting time distribution is in the domain of attraction of a stable law with exponent $\alpha = \frac{1}{z-1}$, its absolute moments of order less than $\alpha = \frac{1}{z-1}$ exist. Therefore, if

$$\delta\frac{z-2}{z-1} < \frac{1}{z-1} \quad \Leftrightarrow \quad z < 2 + \frac{1}{\delta}, \tag{6.64}$$

we know that C_δ is finite. Especially, for $2 < z < \frac{5}{2}$ there is a $\delta > 2$ such that C_δ is finite which implies the convergence to an uncoupled CTRW with a scaling exponent

$$\beta = \frac{\alpha}{2} = \frac{1}{2}\frac{1}{z-1}. \tag{6.65}$$

The asymptotic jump density for $S_{\mathrm{mod}}(t)$ is given by

$$\rho_{\mathrm{step}}(k,\lambda) = \overline{\mathbb{J}}\lambda^\alpha + \frac{1}{2}Dk^2 \tag{6.66}$$

where the constants $\overline{\mathbb{J}}, D > 0$ could be inferred from equation (6.47) (which is non-trivial).

Let us have a look at the remaining case $z > \frac{5}{2}$. In this domain the diffusion is converging towards a coupled CTRW, therefore I am using the results of points 3 and 4 of section 4.5. By symmetry reasons, we know as long as the integral exists, that

$$\overline{\mu} = \int \mathcal{Y}(x)\, \overline{\pi}(dx) = 0. \tag{6.67}$$

We can extend the estimate (6.60) to a lower bound by

$$\begin{aligned} |\mathcal{Y}(x)| &= \sum_{j=0}^{\mathcal{T}(x)-1} \left[|x|^{-(z-1)} - 2(z-1)j\right]^{-\frac{1}{z-1}} \\ &\geq \int_{r\in[0,\mathcal{T}(x)-1]} \left[|x|^{-(z-1)} - 2(z-1)r\right]^{-\frac{1}{z-1}}\, dr \\ &= \frac{1}{2(z-2)}\left(\left[|x|^{-(z-1)}\right]^{\frac{z-2}{z-1}} - \left[|x|^{-(z-1)} - 2(z-1)(\mathcal{T}(x)-1)\right]^{\frac{z-2}{z-1}}\right). \end{aligned} \tag{6.68}$$

Combining these estimates gives

$$\lim_{t\to\infty} \operatorname*{ess\,sup}_{x:\mathcal{T}(x)>t} \left|\frac{|\mathcal{Y}(x)|}{\mathcal{T}(x)^{\frac{z-2}{z-1}}} - \frac{1}{2(z-2)}\right| = 0. \tag{6.69}$$

This gives in the notation of section 4.4.2 (equation (4.70)) with $\beta = \frac{z-2}{z-1}$

$$\begin{aligned} \tilde{\eta}^{\lim}(k) &= \operatorname*{ess\,lim}_{t\to\infty} \tilde{\eta}_t(t^{-\beta}k) \\ &= \cos\left(\frac{k}{2(z-2)}\right). \end{aligned} \tag{6.70}$$

The rest of the conditions is easily checked (equation (4.71), because of the symmetry, we do not get additional conditions for $\beta < \alpha$): The fact $|\mathcal{Y}(x)| \leq \mathcal{T}(x)$ implies $m_t^{|\delta|} \leq t^\delta$ and therefore

$$\begin{aligned} D_1 &= \int_{t\in[0,A]} m_t^{|\delta|}\, \kappa(dt) \\ &\leq A^\delta \\ &< \infty. \end{aligned} \tag{6.71}$$

As used above, for A sufficiently large, we can approximate $m_t^{|\delta|}$ by

$$\begin{aligned} m_t^{|\delta|} &\lesssim |\mathcal{Y}(x)|^\delta \Big|_{\mathcal{T}(x)\simeq t} \\ &\lesssim \frac{1}{2(z-2)} t^{\delta\frac{z-2}{z-1}} \end{aligned} \tag{6.72}$$

and therefore

$$D_2 = \operatorname*{ess\,sup}_{t\in[A,\infty[} \frac{m_t^{|\delta|}}{t^{\beta\delta}} \lesssim \frac{1}{2(z-2)} < \infty. \tag{6.73}$$

Summarizing, we have convergence to a coupled CTRW with scaling exponent $\beta = \frac{z-2}{z-1}$ and waiting time exponent $\alpha = \frac{1}{z-1}$. The asymptotic jump density for $S_{\mathrm{mod}}(t)$ is given by

$$\rho_{\mathrm{step}}(k,\lambda) = c \int \left[1 - e^{-\lambda t} \cos\left(\frac{t^\beta k}{2(z-2)}\right)\right] \frac{dt}{t^{1+\alpha}} \tag{6.74}$$

with one (irrelevant) constant $c > 0$.

As discussed in section 6.4, the scaling exponent β should be the same for the asymptotic dynamics of $S(t)$ and $S_{\mathrm{mod}}(t)$. As a function of the parameter z

$$\beta(z) = \begin{cases} \frac{1}{2} & \text{for } z < 2 \\ \frac{1}{2(z-1)} & \text{for } 2 < z < \frac{5}{2} \\ \frac{z-2}{z-1} & \text{for } z > \frac{5}{2}. \end{cases} \tag{6.75}$$

The behavior directly at the transition points $z = 2$ and $z = \frac{5}{2}$ is not discussed in this work. There seem to be logarithmic corrections to the pure power law at these points.

In a next step, I want to check these analytic results with simulations. These numerical simulation were carried out as follows: 1 000 points were uniformly distributed on the unit interval $[0, 1]$. For better convergence, the induced map was applied 100 times to each of these points. These were taken as the initial conditions for the dynamics (6.3). For the statistical evaluations, these 1 000 points were mirrored on the origin which also results in a trajectory for $S(t)$ mirrored at the origin (of course, these trajectories were not calculated separately). For the scaling exponents $\beta(z)$, the trajectories were calculated over $t = 10^9$ time steps. The empirical scaling exponents were determined by a linear fit in the double-logarithmic plot of $\sqrt{\langle S^2(t)\rangle}$ against t in the domain $t = 10^7 \ldots 10^9$ (400 points for each increase of t by a factor of 10). The results together with the theoretical curve are plotted in figure 6.3. Except near the boundary points $z = 2$ and $z = \frac{5}{2}$ there is a good agreement between theory and simulation. Close to these transition points, the scaling exponents of

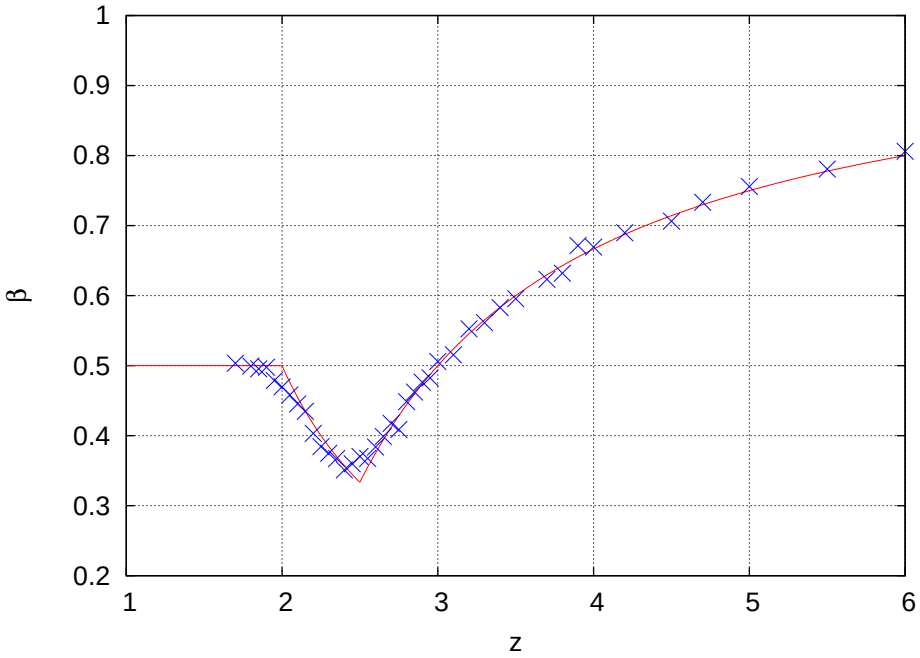

Figure 6.3.: Scaling exponents of the dynamics $S(t)$ driven by the symmetrized Manneville-Pomeau map. The line gives the theoretical result, the crosses correspond to the numerical simulations.

the two contributions are close to each other, therefore one can expect a slow convergence. If this is the reason for the departure of the simulations from the theory at these points, remains to be checked in a future work.

As the theory implies the convergence of $S(t)$ (respectively, $S_{\mathrm{mod}}(t)$) as a stochastic process, it is natural to compare also multi-point correlations with the numerical simulations. To do this, it is necessary to look at the details of the process $S(t)$, especially during the waiting time. We have seen that in the parameter domain $2 < z < \frac{5}{2}$, the scaling exponents are $\alpha = \frac{1}{z-1}$ for the time and $\frac{\beta}{\alpha} = \frac{1}{2}$ for the space (with respect to the operational time). In a waiting time interval, we have with $0 \leq m \leq \mathcal{T}(x)$

$$\left| \sum_{i=0}^{m} g^i(x) \right| \leq |\mathcal{Y}(x)|. \tag{6.76}$$

When we look at the asymptotic behavior for a given waiting time $t > 0$, we have to look

at the x with $\mathcal{T}(x) \simeq \zeta^{-\frac{1}{\alpha}} t$. For $\zeta \searrow 0$, this gives

$$\begin{aligned}
\zeta^{\frac{1}{2}} \left| \sum_{i=0}^{m} g^i(x) \right| &\leq \zeta^{\frac{1}{2}} |\mathcal{Y}(x)| \\
&\lesssim \zeta^{\frac{1}{2}} \left(\frac{1}{2(z-2)} \right)^{\frac{1}{z-1}} \mathcal{T}(x)^{\frac{z-2}{z-1}} \\
&\simeq \zeta^{\frac{1}{2}} \left(\frac{1}{2(z-2)} \right)^{\frac{1}{z-1}} (\zeta^{-\alpha} t)^{\frac{z-2}{z-1}} \\
&\simeq \zeta^{\frac{5}{2}-z} \left(\frac{1}{2(z-2)} \right)^{\frac{1}{z-1}} t^{\frac{z-2}{z-1}} \\
&\to 0 \text{ for } \zeta \searrow 0.
\end{aligned} \tag{6.77}$$

Therefore, the process stays at one position during the waiting time which corresponds directly to the leaper model of the CTRW. An interpretation is that the bursts between the intermittent phases dominate over the contributions of the intermittent phases. We can further deduce that the scaling limits of the two processes $S(t)$ and $S_{\mathrm{mod}}(t)$ coincide and we can use the results from section 3.3.1. A nontrivial expression is obtained for the two-point correlation $\langle S^2(t_1) S^2(t_2) \rangle$. In the scaling limit, this is described for $t_1 \leq t_2$ by equation (3.62) (see also [[BF05] equation (67)])

$$\langle S^2(t_1) S^2(t_2) \rangle^{\lim} = \frac{D^2}{\overline{\mathfrak{J}}^2} \left(5 \frac{t_1^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{t_1^{\alpha} t_2^{\alpha}}{\Gamma(\alpha+1)^2} F(\alpha, -\alpha; 1+\alpha; \frac{t_1}{t_2}) \right) \tag{6.78}$$

with the hypergeometric function $F(\dots)$. This expression still depends on the unknown constants D and $\overline{\mathfrak{J}}$. But we also have in the limit

$$\langle S^2(t) \rangle^{\lim} = \frac{D}{\overline{\mathfrak{J}}} \frac{t^{\alpha}}{\Gamma(\alpha+1)}. \tag{6.79}$$

The following normalized two-point correlation function is therefore helpful ($t_1 \leq t_2$)

$$\begin{aligned}
C^{(2)}(t_1, t_2) &= \frac{\langle S^2(t_1) S^2(t_2) \rangle^{\lim}}{\langle S^2(t_1) \rangle^{\lim} \langle S^2(t_2) \rangle^{\lim}} \\
&= 5 \frac{\Gamma(\alpha+1)^2}{\Gamma(2\alpha+1)} \left(\frac{t_1}{t_2} \right)^{\alpha} + F(\alpha, -\alpha; 1+\alpha; \frac{t_1}{t_2}).
\end{aligned} \tag{6.80}$$

This is a function of the ratio $\frac{t_1}{t_2}$ only and it does not contain any free parameter (i.e., there is no parameter which has to be fitted to the numerical simulations).

To check this analytic expression with numerical simulations, I used the same setting as above with the parameter $z = 2.2$. For better statistics, the data for $S(t)$ was generated up to the time step $t = 10^{10}$. All data in the domain $t = 10^4 \dots 10^{10}$ was used. For a given ratio of $\frac{t_1}{t_2}$ all available combinations for $C^{(2)}(t_1, t_2)$ were considered (again 400 data points for an increase of t by a factor of 10). The theoretical curve together with the mean and standard deviation of these sets are plotted in figure 6.4. There is a good agreement between the numerical calculations with the analytical form. While the analytical form is in the range of the statistical errors, it is noticeable that the numerics is systematically below the theoretical curve. If one uses shorter time series, this effect increases in the sense

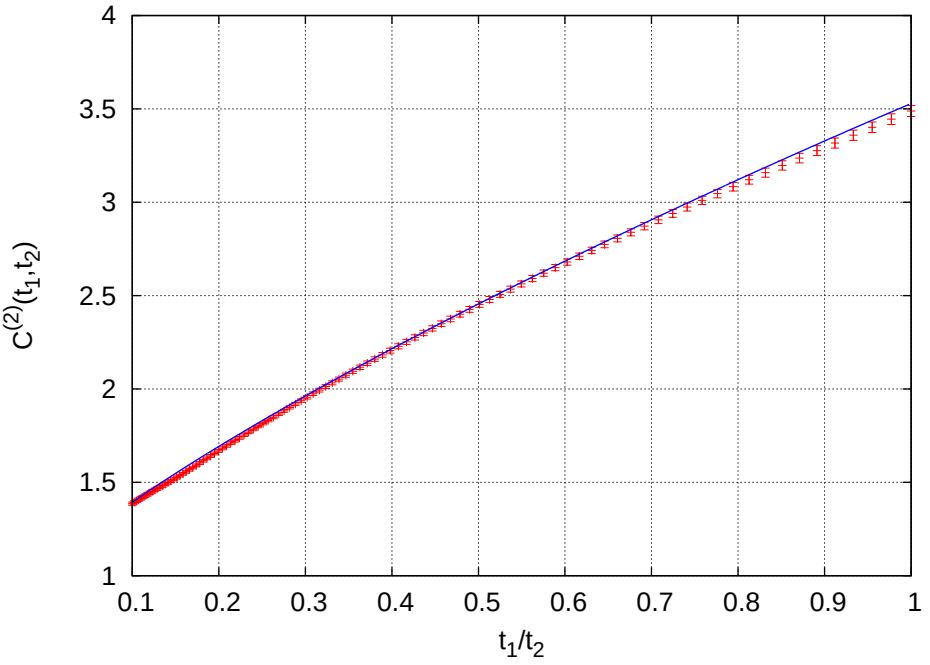

Figure 6.4.: Normalized two-point correlations of the dynamics $S(t)$ driven by the symmetrized Manneville-Pomeau map ($z = 2.2$)

that the mean becomes even lower, but the standard deviations of the data sets increase accordingly. This indicates an effect which causes a convergence from below.

In the case $z > \frac{5}{2}$, we have a coupling between the waiting time and the spatial jump size. For simplicity, I will only work with asymptotic expressions. For large times, we get from equation (6.59)

$$\begin{aligned} \left| g_*^{\mathcal{T}(x)}(x) \right| &= \left[|x|^{-(z-1)} - 2(z-1)\mathcal{T}(x) \right]^{-\frac{1}{z-1}} \\ &\simeq \epsilon \end{aligned} \tag{6.81}$$

which gives

$$|x| \simeq \left[\epsilon^{-(z-1)} + 2(z-1)\mathcal{T}(x) \right]^{-\frac{1}{z-1}} . \tag{6.82}$$

The x dependence can therefore be removed for large waiting times

$$g_*^j(x) \simeq \mathrm{sgn}(x) \left[\epsilon^{-(z-1)} + 2(z-1)(\mathcal{T}(x) - j) \right]^{-\frac{1}{z-1}} . \tag{6.83}$$

One obtains the spatial position of the walker at time t in a waiting time $\mathcal{T}(x)$ $(0 \leq t \leq \mathcal{T}(x))$

$$\begin{aligned}
\sum_{i=0}^{t-1} g^i(x) &\simeq \mathrm{sgn}(x) \int_{r\in[0,t]} \left[\epsilon^{-(z-1)} + 2(z-1)(\mathcal{T}(x)-r)\right]^{-\frac{1}{z-1}} dr \\
&= \frac{\mathrm{sgn}(x)}{2(z-1)} \left(\left[\epsilon^{-(z-1)} + 2(z-1)\mathcal{T}(x)\right]^{\frac{z-2}{z-1}} - \left[\epsilon^{-(z-1)} + 2(z-1)(\mathcal{T}(x)-t)\right]^{\frac{z-2}{z-1}} \right) \\
&\simeq \mathcal{Y}(x) \frac{\left[\epsilon^{-(z-1)} + 2(z-1)\mathcal{T}(x)\right]^{\frac{z-2}{z-1}} - \left[\epsilon^{-(z-1)} + 2(z-1)(\mathcal{T}(x)-t)\right]^{\frac{z-2}{z-1}}}{\left[\epsilon^{-(z-1)} + 2(z-1)\mathcal{T}(x)\right]^{\frac{z-2}{z-1}} - \epsilon^{-(z-2)}} \\
&\simeq \mathcal{Y}(x) \left(1 - \left[1 - \frac{t}{\mathcal{T}(x)} \right]^{\frac{z-2}{z-1}} \right).
\end{aligned} \tag{6.84}$$

This describes the motion of the walker during a waiting time. One can see that the resulting process is continuous as $\mathcal{Y}(x)$ accumulates during the waiting time. Therefore, one can say that the intermittent phases dominate over the chaotic bursts between the long waiting times. Adapting the steps from section 3.4.1, one see that the contribution of a vertex has to be modified to be

$$\begin{aligned}
\rho_{\mathrm{vertex}}(\mathbf{k},\boldsymbol{\lambda}) \simeq & \int \prod_{v\in\mathcal{V}} \left[\int_{t_v\in[0,\mathcal{T}(x)]} \exp\left(-\lambda_v t_v + ik_v \mathcal{Y}(x) \left(1 - \left[1 - \frac{t_v}{\mathcal{T}(x)} \right]^{\frac{z-2}{z-1}} \right) \right) dt_v \right] \\
& \times \exp\left(-\Lambda_{\mathcal{L}} \mathcal{T}(x) + iK_{\mathcal{L}} \mathcal{Y}(x) \right) \overline{\pi}(dx) \\
= & \int \prod_{v\in\mathcal{V}} \left[\mathcal{T}(x) \int_{r_v\in[0,1]} \exp\left(-\lambda_v \mathcal{T}(x) r_v + ik_v \mathcal{Y}(x) \left(1 - (1-r_v)^\beta \right) \right) dr_v \right] \\
& \times \exp\left(-\Lambda_{\mathcal{L}} \mathcal{T}(x) + iK_{\mathcal{L}} \mathcal{Y}(x) \right) \overline{\pi}(dx). \\
\simeq & \frac{1}{2} \sum_{\sigma=\pm 1} \int \prod_{v\in\mathcal{V}} \left[\int_{r_v\in[0,1]} \exp\left(-\lambda_v \mathcal{T}(x) r_v + ik_v \frac{\sigma \mathcal{T}(x)^\beta}{2(z-1)} \left(1 - (1-r_v)^\beta \right) \right) dr_v \right] \\
& \times \mathcal{T}(x)^{|\mathcal{V}|} \exp\left(-\Lambda_{\mathcal{L}} \mathcal{T}(x) + iK_{\mathcal{L}} \frac{\sigma \mathcal{T}(x)^\beta}{2(z-1)} \right) \overline{\pi}(dx).
\end{aligned} \tag{6.85}$$

Let $\kappa(dt)$ be the distribution of the waiting time $\mathcal{T}(x)$ with respect to the ergodic measure

$\overline{\pi}(dx)$. Then we have for the limit

$$\begin{aligned}
&\rho^{\lim}_{\text{vertex}}(\mathbf{k},\boldsymbol{\lambda}) = \lim_{\zeta\searrow 0} \zeta^{|\mathcal{V}|-\alpha}(\zeta^\beta \mathbf{k}, \zeta\boldsymbol{\lambda}) \\
&= \lim_{\zeta\searrow 0} \zeta^{|\mathcal{V}|-\alpha} \frac{1}{2} \sum_{\sigma=\pm 1} \int \prod_{v\in\mathcal{V}} \left[\int_{r_v\in[0,1]} \exp\left(-\zeta\lambda_v t r_v + i\zeta^\beta k_v \frac{\sigma t^\beta}{2(z-1)} \left(1-(1-r_v)^\beta\right)\right) dr_v \right] \\
&\qquad\qquad \times t^{|\mathcal{V}|} \exp\left(-\zeta\Lambda_{\mathcal{L}} t + i\zeta^\beta K_{\mathcal{L}} \frac{\sigma t^\beta}{2(z-1)} \right) \kappa(dt) \\
&= \frac{c}{2} \int \sum_{\sigma=\pm 1} \prod_{v\in\mathcal{V}} \left[\int_{r_v\in[0,1]} \exp\left(-\lambda_v t r_v + i k_v \frac{\sigma t^\beta}{2(z-1)} \left(1-(1-r_v)^\beta\right)\right) dr_v \right] \\
&\qquad\qquad \times t^{|\mathcal{V}|} \exp\left(-\Lambda_{\mathcal{L}} t + i K_{\mathcal{L}} \frac{\sigma t^\beta}{2(z-1)} \right) \frac{dt}{t^{1+\alpha}}.
\end{aligned} \tag{6.86}$$

Together with the rule (6.74), equation (6.86) allows to write down the Fourier-Laplace transforms of the joint probability distributions for the scaling limit of $S(t)$. Unfortunately, it is hard to work with equation (6.86). If one is only interested in the correlation function, one can simplify the relevant expressions, but one still encounters the problem of inverting a double Laplace transform. This seems not to be possible analytically and so far the numerical routines were not stable enough to provide reliable estimates for $C^{(2)}(t_1,t_2)$. Therefore, a precise numerical test of this prediction remains to be done in a future work.

6.5.2. An asymmetric Manneville-Pomeau map

As a second example, I consider the function

$$g(x) : [0,1] \to [0,1], \quad g(x) = \begin{cases} x + x^z & \text{for } x + x^z \leq 1 \\ x + x^z - 1 & \text{for } x + x^z > 1. \end{cases} \tag{6.87}$$

A plot of this function is given in figure 6.5 with the parameter value $z = 3$. The function $g(x)$ gives only positive values, so it is clearly asymmetric.

The singularity of the invariant measure again sits at the origin $x = 0$. Therefore, we choose a $\epsilon > 0$ (small enough) and define

$$\mathcal{S}_{\text{ind}} = [0,1] \setminus [0,\epsilon]. \tag{6.88}$$

The functions $\mathcal{T}(x)$ and $\mathcal{Y}(x)$ are defined analogously to equations (6.52) and (6.53). In the parameter range of z where the mean value

$$\overline{\mu} = \int \mathcal{Y}(x)\, \overline{\pi}(dx) \tag{6.89}$$

is finite, this value is clearly positive, $\overline{\mu} > 0$.

In the domain $\frac{3}{2} < z < 2$, the distribution of the waiting time has a finite mean, but an infinite variance. We saw in section 4.2 that the one-point distributions of the limit process do not have second moments, i.e.,

$$\left\langle (S(t) - \langle S(t)\rangle^{\lim})^2 \right\rangle^{\lim} = \infty. \tag{6.90}$$

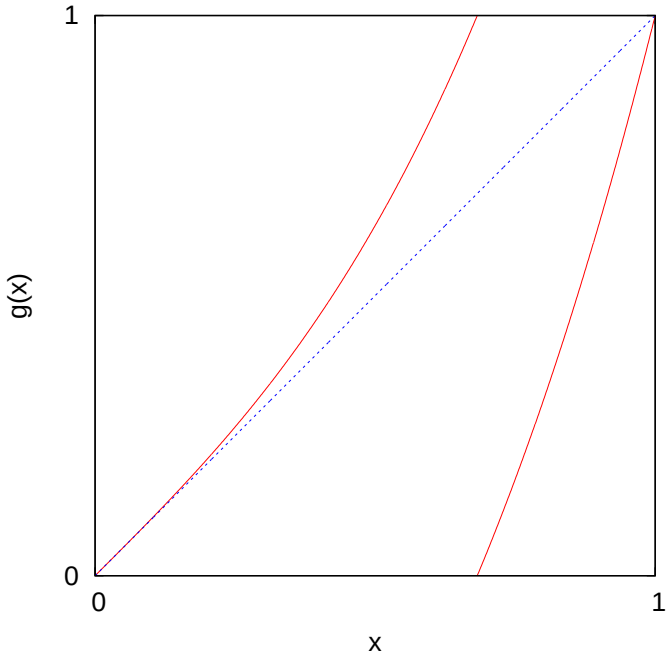

Figure 6.5.: Plot of the asymmetric Manneville-Pomeau map ($z = 3$)

To avoid these problems, I will simply determine the scaling exponents β without subtracting the mean, i.e., via

$$\sqrt{\langle S(t)^2 \rangle} \propto e^{\beta t} \text{ for } t \to \infty. \tag{6.91}$$

Since for $z < 2$ we have a waiting time with finite mean, only the ballistic component is visible and we will get $\beta = 1$.

Let me proceed to the more interesting case $z > 2$ with infinite mean of the waiting time distribution. Following point 2 in section 4.5, we have convergence to an uncoupled CTRW with $\alpha = \beta = \frac{1}{z-1}$ if we can show that an absolute moment δ with $\delta > 1$ is finite:

$$C_\delta = \int \mathcal{Y}(x)^\delta \, \overline{\pi}(dx) < \infty. \tag{6.92}$$

The argument from the last section using the renormalization group works completely analogously. Therefore, we know that C_δ is finite when (equation (6.64))

$$z < 2 + \frac{1}{\delta}. \tag{6.93}$$

Accordingly, for $2 < z < 3$ one obtains convergence to an uncoupled CTRW with scaling exponent $\beta = \frac{1}{z-1}$.

For $z > 3$ the argument is the same as in the last section, i.e., we have convergence to a coupled CTRW with scaling exponent $\beta = \frac{z-2}{z-1}$. Summarizing, we have

$$\beta(z) = \begin{cases} 1 & \text{for } z < 2 \\ \frac{1}{z-1} & \text{for } 2 < z < 3 \\ \frac{z-2}{z-1} & \text{for } z > 3. \end{cases} \tag{6.94}$$

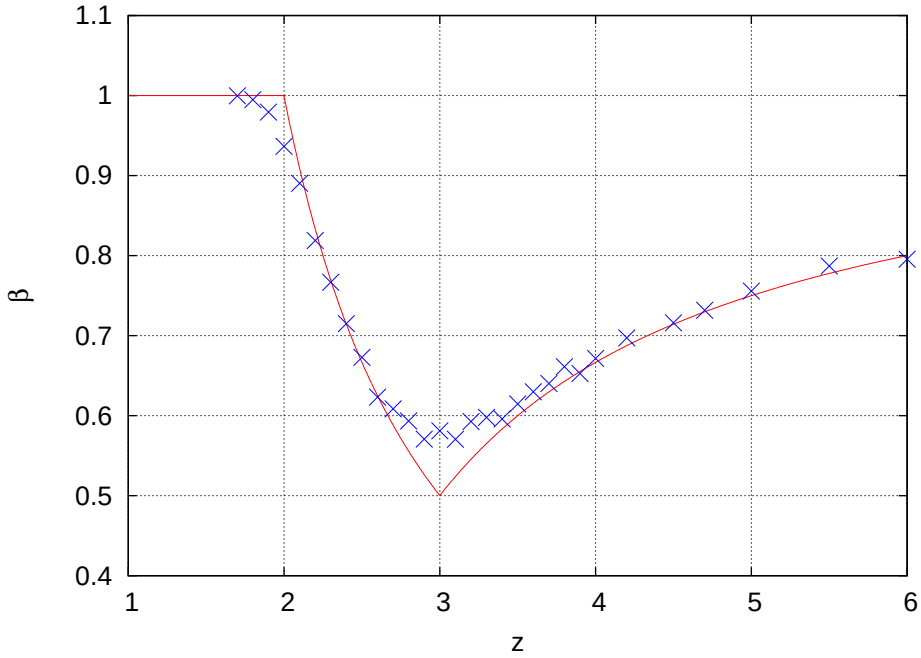

Figure 6.6.: Scaling exponents of the dynamics $S(t)$ driven by the asymmetric Manneville-Pomeau map. The line gives the theoretical result, the crosses are the numerical simulations.

Again, the transition points $z = 2$ and $z = 3$ have not been considered. The numerical simulations were done as in the last section (except for the mirroring of the trajectories, therefore only 1 000 trajectories were used). The numerical results for the scaling limit and the analytical form (6.94) are plotted in figure 6.6. Again, the agreement is good, except near the transition points $z = 2$ and $z = 3$.

7. Summary and Outlook

Anomalous diffusion and long memory are phenomena which are commonly observed in nature. Several stochastic models have been introduced which to model this property. Frequently, one encounters the continuous-time random walk and the fractional Brownian motion. Both models can give any asymptotic scaling exponent β with $0 < \beta < 1$. But both models are clearly different, e.g., the fractional Brownian motion has Gaussian joint probability distributions while the CTRW does not. Therefore, one can construct statistical indicators to discriminate between these models. Also phenomenological approaches are available. The question remains, how one can discriminate these models from the point of view of dynamical systems.

In this thesis the continuous-time random walk and its use as asymptotic description for anomalous deterministic diffusion has been studied. The CTRW approach is useful for systems which contain parts of their phase space where the decay of correlations slows down significantly. This leads to the introduction of an operational time parameter which is driven by the decay of correlation (e.g., parameterization of the reentries into the domains of fast decay of correlations). More generally, the idea is applicable to systems where one can parameterize the rate of mixing via the operational time. This work provides a unifying setup in which both the deterministic and the stochastic system can be treated. For simple systems, the details are worked out in this thesis.

In general, one can not even asymptotically neglect the possibility that two arbitrarily distant time points are trapped in the same region of slow decay. This observation is a central theme in this thesis. It is captured by the diagrammatic approach to the joint probability distributions, introduced in its basic form in chapter 3. This approach can be adapted to several other models of CTRWs (section 3.4.1) or non-diffusive processes (section 3.4.2).

For the description of asymptotic properties of CTRWs it is useful to switch from a probability distribution of a jump to a jump density (as introduced by Becker-Kern, Meerschaert and Scheffler, see section 2.3). In combination with the diagrammatic approach, this allows a direct method of proving asymptotic properties of CTRWs. Several important cases are worked out in chapter 4.

These asymptotic CTRWs with independence of different jumps form the basis for the limit behavior. In general, one needs to allow correlations between jumps. This can be done quite generally by use of an internal state space. The diagrammatic approach is also valid in this case, but the contributions are no longer simple factors but operators on the space of probability distributions of the internal states. The litmus test for the quality of the parameterization with the operational time is a good one, is whether it is possible to show that the contribution of a line converges asymptotically (with a specific speed of convergence) to a projection operator on the ergodic measure (or measures) of the internal state space. Then the asymptotic joint probability distributions factorize and are described by a CTRW with independent jumps.

For a finite internal state space this is always true. The argument uses only elementary

methods of finite Markov chains (chapter 5). The derivation can be used as scheme for more complex cases. The case of a deterministic map was also considered (chapter 6). The main ingredient is the spectral assumption of quasi-compactness of the Frobenius-Perron operator (with respect to the operational time). This assumption controls the eigenvalues which approach unit modulus and captures therefore the long-time behavior (again with respect to the operational time). There are good indications that this setting gives rise to the limit behavior mentioned above (section 6.3). It also turned out that in general the restriction to the leaper model is not sufficient. Other models appear naturally.

These ideas have been tested in a Manneville-Pomeau setting. For certain parameter ranges a map of Manneville-Pomeau type has an infinite ergodic measure. The induced map – which is commonly used for these cases – implies naturally a specific operational time. From this the asymptotic stochastic process describing the deterministic dynamics has been calculated and compared with numerical simulations (section 6.5) in two different settings. There is good agreement between the analytical and the numerical calculations.

7.1. List of specific results

The specific results of this thesis can be summarized as follows:

- Chapter 3:
  - A diagrammatic method has been developed which is usable in a wide range of circumstances which allows a simple determination of the joint probability distributions.
  - In the case of the leaper model there is an analogous method for the multi point correlations.
  - The method allows also to effectively treat non-diffusive models, exemplified by determining the spectral properties of a model for weak ergodicity breaking.
- Chapter 4:
  - The scaling limit of an independent CTRW with finite mean waiting time is Markovian — even in the case of an infinite variance. In the case of a spatial jump distribution with non-vanishing mean and infinite variance of the waiting time, the limit is a maximally skewed stable process. However, the scaling limit does not capture all asymptotic information.
  - For several important cases, the scaling limits of a CTRW have been derived (in combination with appropriate necessary conditions).
- Chapter 5:
  - The diagrammatic approach of chapter 3 has been extended to a non-independent CTRW with internal states. The contributions of the diagrams become operators on the space of probability distributions on the internal states.
  - When the internal states form a finite and ergodic Markov chain the scaling limit is an independent CTRW. The limit can be determined from the ergodic average of the jump size/waiting time distribution – only in one special case, there can be a second order contribution.

  - The case of a finite and reducible Markov chain can be split into two parts: first: determining the probabilities of the different irreducible sets (using only the theory of Markov chains); second: treating the different irreducible components with the methods for irreducible Markov chains.

- Chapter 6:

  - The deterministic diffusion of maps which can be treated by the method of inducing can be reformulated in terms of a non-independent CTRW. The internal state space dynamics is ergodic.
  - The spectral assumption of quasi-compactness of the induced Frobenius-Perron operator (together with a suitable choice of the basis spaces) leads to the well supported hypothesis that the scaling limit is described by an independent CTRW. Under these assumptions the parameters of the asymptotic process can be determined from the deterministic map (e.g., the scaling exponents α, β).
  - The theoretical predictions were tested by numerical simulations using an unbiased and a biased map of Manneville-Pomeau type. The results of the simulations agree well with the analytical predictions.
  - In general, the leaper model of the CTRW is not enough for general limit considerations.

7.2. Outlook

This thesis leaves several open questions.

- The analytical form of the two point correlation in the coupled case (equation (6.86)) remains to be compared with the results of the numerical simulations.

- A mathematically more rigorous derivation of the scaling limit for the anomalous deterministic diffusion (section 6.3) would be favorable. Especially, the exact conditions for the validity and the spaces to work on is an open question.

- The derivations always focused on the leaper model of the CTRW. Other models were incorporated afterwards. A more natural description of a general class of models is desirable.

- In this thesis, the theory has been applied to simple one-dimensional maps. The use in higher dimensional and more complex situations has to be tested (both analytically and numerically).

- It was mentioned in the introduction that Hamiltonian systems can also show stickiness phenomena and intermittent behavior. However, one generally does not have specific points which account for the stickiness of the system, but the concept of a hierarchical phase space. It would be interesting to work out if the methods used in this thesis can be transferred to this situation. Especially the motion during a period of stickiness near to an integrable torus can be highly non-trivial. Accordingly, the behavior during the waiting time (i.e., the model of CTRW) will be complex.

- The calculation of the spectral properties of a model for weak ergodicity breaking (section 3.4.2) did only consider the spectrum at specific frequencies. In general, these do fluctuate and stay a probability distribution – also for standard processes; therefore one normally looks at averaged quantities like a binned spectrum. It would be rewarding to extend the calculations to these observables.

- This thesis focuses on processes obtained as scaling limits. This is relevant for many observations (e.g., scaling exponents), but there are also observables which are not covered with these limits (e.g., correlations of a fixed time distance).

A. Renewal Equation Approach to Joint Probability Distributions

In this appendix, I want to show an alternative derivation for the joint probability distributions of a continuous-time random walk. I concentrate on the probability corresponding to a given diagram and use a renewal equation approach to get a recursion relation on the number of vertices. Figure A.1 shows the relevant diagrams for this calculation. The sets $\mathcal{V}_j$ describe the indices leaving the jth vertex (counting from the right). The diagram D_r in the first line contains all r vertices while the diagram D_{r-1} in the second line follows from this by removing the leftmost vertex. The corresponding joint probability distributions are denoted $p_{D_r}(\mathbf{x}, \mathbf{t})$ and $p_{D_{r-1}}(\mathbf{x}, \mathbf{t})$ respectively. The renewal equation is set up by splitting the process in the first step (the first renewal) and the rest. This step can either contain all times t_v ($v \in \mathcal{V}_r$) and then continue with the shifted probability for the diagram D_{r-1}, or all times t_v ($v \in \mathcal{V}_r$) are in a later step which means that after the first step the situation is still described by the diagram D_r. The renewal equation then reads

$$\begin{aligned} p_{D_r}(\mathbf{x}, \mathbf{t}) = &\int d\chi \int d\tau\, \psi(\chi, \tau) \prod_{v \in \mathcal{V}_r} \Big(\theta(\tau - t_v) \delta(x_v) \Big) p_{D_{r-1}}(\mathbf{x} - \chi, \mathbf{t} - \tau) \\ &+ \int d\chi \int d\tau\, \psi(\chi, \tau)\, p_{D_r}(\mathbf{x} - \chi, \mathbf{t} - \tau) \end{aligned} \tag{A.1}$$

where I write $\mathbf{x} - \chi$ and $\mathbf{t} - \tau$ for $(x_1 - \chi, x_2 - \chi, \dots)$ and $(t_1 - \tau, t_2 - \tau, \dots)$ respectively. Taking the Fourier-Laplace transform of equation. (A.1) and solving for $p_{D_r}(\mathbf{k}, \boldsymbol{\lambda})$ gives

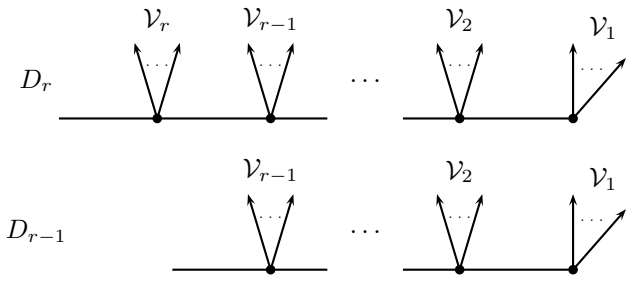

Figure A.1.: The two diagrams which appear in the renewal equation in appendix A. The sets $\mathcal{V}_j$ contain the indices of the jth vertex (from the right). The second diagram stems from the first one by removing the leftmost vertex.

(with $\mathcal{L}_r = \mathcal{V}_1 \cup \cdots \cup \mathcal{V}_{r-1}$)

$$p_{D_r}(\mathbf{k}, \boldsymbol{\lambda}) = \frac{\sum_{\mathcal{J} \in \mathcal{P}(\mathcal{V}_r)} (-1)^{|\mathcal{J}|} \psi(K_{\mathcal{L}_r}, \Lambda_{\mathcal{L}_r} + \Lambda_{\mathcal{J}})}{(1 - \psi(K_{\mathcal{V}_r \cup \mathcal{L}_r})) \prod_{v \in \mathcal{V}_r} \lambda_v} \times p_{D_{r-1}}(\mathbf{k}, \boldsymbol{\lambda}). \tag{A.2}$$

The factor in front of $p_{D_{r-1}}(\mathbf{k}, \boldsymbol{\lambda})$ is identical to the contribution of the leftmost line and vertex as calculated in section 3.2. Iterating equation (A.2) therefore reproduces the result of section 3.2 for $p_{D_r}(\mathbf{k}, \boldsymbol{\lambda})$.

B. Power Series Expansion of a Determinant

The power series expansion of a determinant is best expressed in the language of the exterior algebra. While assuming the basic definitions to be known, I introduce here some notations and properties from [[Bou89], III §7 and §8]. I assume that we work on a n-dimensional $\mathbb{R}$-vector space V.

- The exterior algebra of this vector space is called $\Lambda = \bigoplus_j \Lambda^j(V)$.
- Any endomorphism $A : V \to V$ induces a $\mathbb{R}$-algebra homomorphism $\Lambda(A) : \Lambda \to \Lambda$ via

$$\Lambda(A)(x_1 \wedge \cdots \wedge x_p) = A(x_1) \wedge \cdots \wedge A(x_p). \tag{B.1}$$

- The determinant of an endomorphism $A : V \to V$ is defined to be the number $\det(A)$ such that

$$\Lambda^n(A)(x) = \det(A)x \text{ for all } x \in \Lambda^n(V). \tag{B.2}$$

- Let $(e_i)_{i\in\{1,\ldots,n\}}$ be a basis of V with its dual basis $(e_i^*)_{i\in\{1,\ldots,n\}}$ of V^*.
- For any $H = \{j_1, \ldots, j_p\} \subseteq \{1, \ldots, n\}$ with $j_1 < j_2 < \cdots < j_p$ I use the definition

$$e_H = e_{j_1} \wedge \cdots \wedge e_{j_p}. \tag{B.3}$$

  The complementary set of indices is denoted by $H' = \{1, \ldots, n\} \setminus H$.

- For two sets $J, H \subseteq \{1, \ldots, n\}$ one can define a number $\rho_{J,H} \in \{-1, 0, 1\}$ such that

$$e_J \wedge e_H = \rho_{J,H} e_{J\cup H}. \tag{B.4}$$

  For the details how to determine $\rho_{J,H}$ I refer to [[Bou89], III §7.8 equations (19) and (20)].

- By a slight abuse of notation I identify the endomorphism A with its representation with respect to the basis $(e_i)_{i\in\{1,\ldots,n\}}$. For two subsets of indices of same size $J, H \subseteq \{1, \ldots, n\}$, $|J| = |H|$ the symbol $A_{J,H}$ denotes the $(n - |J|) \times (n - |H|)$ matrix which has the rows (resp. columns) of A given by the indices J (resp. H). In other words, $A_{J,H}$ is the matrix A where one has deleted the lines J' and the rows H'. The minors of A are the determinants $\det(A_{J,H})$.
- With help of the minors one has [[Bou89], III §8.6]

$$\Lambda^q(A)(e_H) = \sum_{\substack{R\subseteq\{1,\ldots,n\} \\ |R|=q}} \det(A_{R,H}) e_R. \tag{B.5}$$

In the following I need a generalization of the *adjugate matrix*[1] (it is a fairly straightforward generalization, so I guess it has been defined before, but I was not able to locate any reference).

Definition B.1 *Let A be an endomorphism $A : V \to V$, then the* generalized adjugate $\mathrm{Adj}(A) : \Lambda \to \Lambda$ *is defined by*

$$\mathrm{Adj}(A)(x) \wedge y = x \wedge \Lambda(A)(y) \quad \textit{for all } p \in \{0, \dots, n\} \textit{ and } x \in \Lambda^p(V), y \in \Lambda^{n-p}(V). \tag{B.6}$$

In the following, I list some properties of $\mathrm{Adj}(A)$ which follow directly from the definition:

- $\mathrm{Adj}(A)$ is a graded endomorphism, i.e.,

$$\mathrm{Adj}(A)\left(\Lambda^p(V)\right) \subseteq \Lambda^p(V). \tag{B.7}$$

  The restriction of $\mathrm{Adj}(A)$ to $\Lambda^p(V)$ is denoted by $\mathrm{Adj}_p(A)$.

- It follows from [[Bou89], p. 532] that $\mathrm{Adj}_1(A) : V \to V$ is the standard adjugate $\mathrm{Adj}_1(A) = \mathrm{adj}(A)$.

- For a given basis $(e_i)_{i\in\{1,\dots,n\}}$ the matrix representation of $\mathrm{Adj}(A)$ can be expressed in term of the minors of A (using equation (B.5))

$$\begin{aligned} e^*_H\left(\mathrm{Adj}(A)(e_R)\right) &= \rho_{H,H'} e^*_{\{1,\dots,n\}}\left(\mathrm{Adj}(A)(e_R) \wedge e_{H'}\right) \\ &= \rho_{H,H'} e^*_{\{1,\dots,n\}}\left(e_R \wedge \Lambda(A)(e_{H'})\right) \\ &= \sum_{\substack{S\subseteq\{1,\dots,n\} \\ |S|=|H'|}} \rho_{H,H'} \det(A_{S,H'}) e^*_{\{1,\dots,n\}}\left(e_R \wedge e_S\right) \\ &= \rho_{H,H'}\rho_{R,R'} \det(A_{R',H'}). \end{aligned} \tag{B.8}$$

- We have $\mathrm{Adj}(A)\Lambda(A) = \det(A)\mathbb{I}_\Lambda$ which is essentially the Laplace expansion of the determinant. This follows directly from

$$\begin{aligned} \mathrm{Adj}(A) \circ \Lambda(A)(x) \wedge y &= \Lambda(A)(x) \wedge \Lambda(A)(y) \\ &= \det(A) \cdot (x \wedge y). \end{aligned} \tag{B.9}$$

  Especially for A invertible,

$$\mathrm{Adj}(A) = \det(A)\Lambda(A^{-1}). \tag{B.10}$$

- We have

$$\mathrm{Adj}(AB) = \mathrm{Adj}(B) \circ \mathrm{Adj}(A). \tag{B.11}$$

Lemma B.2 *Let A, B be two endomorphisms of V. Then*

$$\det(A + B) = \mathrm{Tr}_\Lambda\left(\mathrm{Adj}(A) \circ \Lambda(B)\right). \tag{B.12}$$

[1] sometimes also called *classical adjoint* or simply *adjoint* in older texts

This follows from

$$\begin{aligned}\det(A+B) &= e^*_{\{1,\dots,n\}}\left(\Lambda(A+B)(e_{\{1,\dots,n\}})\right)\\ &= \sum_{S\subseteq\{1,\dots,n\}} (e_S\wedge e_{S'})^*\left(\Lambda(B)(e_S)\wedge\Lambda(A)(e_{S'})\right)\\ &= \sum_{S\subseteq\{1,\dots,n\}} (e_S\wedge e_{S'})^*\left(\mathrm{Adj}(A)\circ\Lambda(B)(e_S)\wedge e_{S'}\right)\\ &= \sum_{S\subseteq\{1,\dots,n\}} e^*_S\left(\mathrm{Adj}(A)\circ\Lambda(B)(e_S)\right)\\ &= \mathrm{Tr}_\Lambda\left(\mathrm{Adj}(A)\circ\Lambda(B)\right)\quad\square\end{aligned} \tag{B.13}$$

This Lemma can be used to determine power expansions of determinants. For this let A be the fixed matrix which is the base point of the expansion and $B(t)$ contains the higher order terms. The trace of Λ can often be simplified by noting that $\Lambda^q(B)=O(\|B\|^q)$. We get

$$\det(A+B)=\det(A)+\mathrm{Tr}_V\left(\mathrm{adj}(A)\circ B\right)+O(\|B\|^2) \tag{B.14}$$

which corresponds to Jacobi's formula [[MN99], III.8.3 equation (1)]. The next order is given by

$$\det(A+B)=\det(A)+\mathrm{Tr}_V\left(\mathrm{adj}(A)\circ B\right)+\mathrm{Tr}_{\Lambda^2(V)}\left(\mathrm{Adj}_2(A)\circ\Lambda^2(B)\right)+O(\|B\|^3). \tag{B.15}$$

The problem boils down to an efficient determination of $\mathrm{Adj}(A)$. When A is invertible, the answer is given by equation (B.10). In general, one needs to be able to control the generalized eigenspace of A corresponding to the eigenvector 0. Here, I will focus only on the case which is important for the ergodic Markov chains, namely, the kernel of A is one-dimensional and we have the direct sum

$$V=\ker(A)\oplus\mathrm{im}(A). \tag{B.16}$$

This corresponds to the fact, that we only have a trivial Jordan block for the eigenvalue 0. Now, define a linear projection M (i.e., $M^2=M$) on the kernel of A, i.e.,

$$M\big|_{\ker(A)}=\mathbb{I}_{\ker(A)}\text{ and }M\big|_{\mathrm{im}(A)}=0. \tag{B.17}$$

This implies that $A+M$ is invertible and we have the following properties

$$A\circ M=M\circ A=0, \tag{B.18}$$

$$(A+M)^{-1}\circ M=M\circ(A+M)^{-1}=M, \tag{B.19}$$

$$A=(A+M)\circ(\mathbb{I}-M). \tag{B.20}$$

Therefore,

$$\begin{aligned}\mathrm{Adj}(A) &= \mathrm{Adj}(\mathbb{I}-M)\circ\mathrm{Adj}(A+M)\\ &= \det(A+M)\,\mathrm{Adj}(\mathbb{I}-M)\circ\Lambda(A+M)^{-1}.\end{aligned} \tag{B.21}$$

Lemma B.3 *Let $M : V \to V$ be a projector with the properties described above. Then*

$$\mathrm{Adj}_1(\mathbb{I} - M) : V \to V, \qquad x \mapsto M(x) \tag{B.22}$$

$$\mathrm{Adj}_2(\mathbb{I} - M) : \Lambda^2(V) \to \Lambda^2(V), \qquad x \wedge y \mapsto M(x) \wedge y + x \wedge M(y). \tag{B.23}$$

Since the dimension of $\mathrm{im}(M)$ is one, we have $M(x) \wedge M(y) = 0$ for $x, y \in V$. By induction this gives for $x \in V$ and $z \in \Lambda$:

$$M(x) \wedge \Lambda(\mathbb{I} - M)(z) = M(x) \wedge z. \tag{B.24}$$

The first identity follows from ($x \in V$, $z \in \Lambda^{n-1}(V)$):

$$\begin{aligned} 0 &= \Lambda(\mathbb{I} - M)(x \wedge z) \\ &= x \wedge \Lambda(\mathbb{I} - M)(z) - M(x) \wedge \Lambda(\mathbb{I} - M)(z) \\ &= \mathrm{Adj}_1(\mathbb{I} - M)(x) \wedge z - M(x) \wedge z. \end{aligned} \tag{B.25}$$

Similarly for the second ($x, y \in V$, $z \in \Lambda^{n-2}(V)$):

$$\begin{aligned} 0 &= \Lambda(\mathbb{I} - M)(x \wedge y \wedge z) \\ &= x \wedge y \wedge \Lambda(\mathbb{I} - M)(z) - M(x) \wedge y \wedge \Lambda(\mathbb{I} - M)(z) - x \wedge M(y) \wedge \Lambda(\mathbb{I} - M)(z) \\ &= \mathrm{Adj}_2(\mathbb{I} - M)(x \wedge y) \wedge z - (M(x) \wedge y + x \wedge M(y)) \wedge z. \quad \square \end{aligned} \tag{B.26}$$

This result gives immediately

$$\begin{aligned} \mathrm{adj}(A) &= \det(A + M)\, \mathrm{adj}(\mathbb{I} - M) \circ (A + M)^{-1} \\ &= \det(A + M) M. \end{aligned} \tag{B.27}$$

Furthermore, one can simplify the traces in equation (B.15):

$$\mathrm{Tr}_V \left(\mathrm{Adj}_1(\mathbb{I} - M) \circ (A + M)^{-1} \circ B \right) = \mathrm{Tr}_V \left(M \circ B \right) \tag{B.28}$$

and

$$\begin{aligned} &\mathrm{Tr}_{\Lambda^2(V)} \left(\mathrm{Adj}_2(\mathbb{I} - M) \circ \Lambda(A + M)^{-1} \circ \Lambda(B) \right) \\ &\quad = \sum_{\substack{i,j=1 \\ i<j}}^{n} (e_i \wedge e_j)^* \Big(M \circ (A + M)^{-1} \circ B(e_i) \wedge (A + M)^{-1} \circ B(e_j) \\ &\qquad\qquad + (A + M)^{-1} \circ B(e_i) \wedge M \circ (A + M)^{-1} \circ B(e_j) \Big) \\ &\quad = \sum_{i,j=1}^{n} (e_i \wedge e_j)^* \left(M \circ B(e_i) \wedge (A + M)^{-1} \circ B(e_j) \right) \\ &\quad = \sum_{i,j=1}^{n} e_i^* \left(M \circ B(e_i) \right) e_j^* \left((A + M)^{-1} \circ B(e_j) \right) \\ &\qquad\qquad - e_i^* \left(M \circ B(e_j) \right) e_j^* \left((A + M)^{-1} \circ B(e_i) \right) \\ &\quad = \mathrm{Tr}_V \left(M \circ B \right) \mathrm{Tr}_V \left((A + M)^{-1} \circ B \right) - \mathrm{Tr}_V \left(M \circ B \circ (A + M)^{-1} \circ B \right). \end{aligned} \tag{B.29}$$

Where I have used the general equality for two endomorphisms $W_1, W_2 : V \to V$:

$$\sum_{i,j=1}^{n} e_i^* \left(W_1(e_j) \right) e_j^* \left(W_2(e_i) \right) = \mathrm{Tr}_V \left(W_1 \circ W_2 \right). \tag{B.30}$$

By using these traces in equation (B.15), one gets the main result of this appendix

$$\begin{aligned}\det(A+B) = \det(A+M)\Big[& \operatorname{Tr}_V (M \circ B) + \operatorname{Tr}_V (M \circ B) \operatorname{Tr}_V \left((A+M)^{-1} \circ B\right) \\ & - \operatorname{Tr}_V \left(M \circ B \circ (A+M)^{-1} \circ B\right) \Big] + O(\|B\|^3).\end{aligned} \tag{B.31}$$

C. Some Estimates for the Exponential Function

In this appendix, I want to derive some simple estimates for the exponential function.[1] Let $p_n(x)$ be the nth Taylor polynomial of the exponential function, i.e.,

$$p_n(x) = \sum_{k=0}^{n} \frac{1}{k!} x^k. \tag{C.1}$$

The following inequality is valid for all non-negative integers n and $x \in \mathbb{R}$ [[Kal01], lemma 5.13]

$$|p_n(ix) - e^{ix}| \leq \min\left(\frac{2|x|^n}{n!}, \frac{|x|^{n+1}}{(n+1)!}\right). \tag{C.2}$$

Therefore, one obtains for all α with $n \leq \alpha \leq n+1$ and all $x \in \mathbb{R}$

$$\begin{aligned} |p_n(ix) - e^{ix}| &\leq \frac{2}{n!} \min\left(|x|^n, |x|^{n+1}\right) \\ &\leq \frac{2}{n!} |x|^\alpha. \end{aligned} \tag{C.3}$$

In particular, the following inequalities are valid for all $x \in \mathbb{R}$:

$$|1 - e^{ix}| \leq 2|x|^\alpha \qquad \text{for } 0 \leq \alpha \leq 1 \tag{C.4}$$

$$|1 + ix - e^{ix}| \leq 2|x|^\alpha \qquad \text{for } 1 \leq \alpha \leq 2 \tag{C.5}$$

$$|1 + ix - \frac{1}{2}x^2 - e^{ix}| \leq 2|x|^\alpha \qquad \text{for } 2 \leq \alpha \leq 3 \tag{C.6}$$

[1] This appendix contained an error in the submitted version. This error has been corrected in this version.

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