Three Papers in Risk-Factors and Asset Allocations

by

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For my wife

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Acronyms

ADF test	Augmented Dickey-Fuller test
BIS	Bank for International Settlements
CAPM	Capital Asset Pricing Model
CBOE	Chicago Board Options Exchange
CES	Component Expected Shortfall
CPI	Consumer Price Index
CVaR	Conditional Value at Risk
DCC	Dynamic Conditional Correlation
DD pair	Double Description pair
EM algorithm	Expectation Maximisation algorithm
EoD	end-of-day
ERC	Equal Risk to Contribution
ES	Expected Shortfall
ETF	Exchange Traded Fund
FAVAR	Factor Augmented Vector Auto Regression
FRP	Factor Risk Parity
IMF	International Monetary Fund
I(n)	Integrated of order n
IP	Industrial Production
KPSS test	Kwiatkowski–Phillips–Schmidt–Shin test
MES	Marginal Expected Shortfall
(M)RC	(Marginal) Risk Contribution
MV	Minimum Variance
P1	Paper "Analyzing Precious Metals Returns using a Kalman Smoother Approach"
P2	Paper "Factor Risk Parity with Portfolio Weight Constraints"
P3	Paper "Tail Driven Factor Risk Parity with Volatility Investments"
PCA	Principal Component Analysis
PPI	Producer Price Index
PRCC	Performance/Risk Contribution Concentration
RTS smoother	Rauch-Tung-Striebel smoother
SDE	Stochastic Difference Equation
VaR	Value at Risk
VAR-Model	Vector Auto-Regressive Model

1 Introduction

1.1 Overview, Aim and Motivation

Dealing with the underlying risk factors of asset classes is compulsory for understanding what the "real" performance drivers of those assets are. Setting up a framework based on this analysis is a subsequent step that should achieve better risk management/diversification in portfolio construction.

This study wants to shed some light on the topics of underlying risk factors on the one hand and portfolio construction based on risk factors on the other. The former is done in form of a time-sensitive Kalman smoother approach for precious metals. The aim is to understand how different external factors are related to precious metals, how those sensitivities change over time and how the precious metals differ regarding those factor sensitivities. The portfolio construction analysis on the other hand is done by two studies that analyze a modified version of a factor risk parity portfolio construction process. The aim is to understand the solution set and how the solution changes when specific constraints are set. Furthermore, the analysis also covers an exchange of the standard deviation as the risk measure in the third paper. Volatility short positions as a skewed and fat tailed time series are included in this context as well.

1.1.1 Risk factors

Multifactor models can be classified into three different types: macroeconomic, fundamental and statistical factor models. The first uses observable factors such as inflation or industrial production, the second focuses on security specific factors such as book-to-market and the final type derives the factor values directly from the statistics of the asset returns. Connor (1995) compare the three approaches by applying the model to the US equity market.

For a macroeconomic multifactor model, the loadings to each factor such as inflation can be estimated via a simple regression. Chen et al. (1986) describe the popular Chen, Roll and Ross model which explains equity returns mostly via changes in industrial production, the risk premium, twists in the yield curve and to a lesser extent through unexpected inflation.

In a fundamental factor model, observable security specific factors are considered. A classical example is the Fama-French three factor model. Fama and French (1993) identify three factors for equities, the overall market factor as well as the factors related to firm size as well as book-to-market ratio. In their work they also find the two factors of maturity and default risk relating to bond price changes.

Finally, statistical models usually have non-observable factors as well as loadings. A typical approach is to take a statistical method such as the *Principal Component Analysis (PCA)* to determine both. As mentioned above, this approach is chosen in a modified version of a factor risk parity approach to set up the model in the second and third paper.

The models mentioned above, although popular, assume that the factor loadings are constant over time. Although this simplifies the calculations, the assumption can be challenged as dynamics in financial markets constantly change. That is the reason why models such as the *Capital Asset Pricing Model (CAPM)* have been amended to account for time-variation of the factor loading. The conditional CAPM as for example described in Jagannathan and Wang (1996) is an example that allows the factor loading for the market factor to change over time. In general, the question arises on how to properly determine the changing loadings in the model. The first paper will discuss the Kalman smoother approach and apply that concept to the changing sensitivities of specific, macroeconomic factors regarding precious metals returns.

1.1.2 Factor based portfolio construction and Risk Parity

From the classical naive 1/N-diversification approaches to Markowitz and the Fama-French model, portfolio construction has become increasingly sophisticated. Factor based portfolio construction, even though not a new investment style, has seen a significant increase in interest in research literature while a large number of factors offer a lot of opportunities for research analyses. Cazalet and Roncalli (2014) speak of a "zoo of factors" where investors can get easily lost.

A different topic is the surge of risk parity as an investment concept that has become very popular especially after the financial crisis in 2008/09. As described in Chaves et al. (2011), the idea is to not set up an equity/bond portfolio with standard 60%/40% weights as the equity risk would in this scenario dominate the portfolio returns due to higher volatility, but to set up a portfolio where each asset has the same risk contribution to total risk. Especially during the financial crisis, the positive return of the "safe haven bonds" did not sufficiently offset the huge losses on the equity side in many investor portfolios.

Combining factor based portfolio construction on the one hand and risk parity on the other leads to the concept of factor risk parity as for example described in Roncalli and Weisang (2012) or Kind (2013). An intuitive motivation for this approach is given in Bhansali et al. (2012) as they argue: "Having diversification in risk contribution from assets is generally not the same as having diversification in the primitive sources of risk underlying asset returns. An easy way to understand this argument is to think of assets as foods and risk factors as nutrients. While the body consumes foods, it actually needs the underlying nutrients to build bones and muscles. A healthy diet is not necessarily one that contains a diversified basket of foods but a diversified basket of nutrients". Continuing this illustration, the general idea in a classical factor risk parity approach is to diversify over all identified nutrients. The second and third paper will adjust this standard factor risk parity model by focusing on the important nutrients only, thereby adding the freedom on the selection of the food.

1.1.3 Outline and Contribution

This thesis consists of three papers that can each be taken as a separate study: "Analyzing Precious Metals Returns using a Kalman Smoother Approach" (P1), "Factor Risk Parity with Portfolio Weight Constraints" (P2) and "Tail Driven Factor Risk Parity with Volatility Investments" (P3). They are covering the topic of risk factors as well as factor risk parity investment strategies. The order is chosen to describe precious metals returns and their relationship to specific, pre-determined factors in the first paper, showing that sensitivities of asset returns to specific factors can significantly change over time. Additionally, it illustrates similarities but also differences between the returns of gold, silver, platinum and palladium.

A modified version of a risk-parity concept in a multi-asset framework is set up in the second and third paper, which are a joint work with Steffen Möllenhoff. A rolling-window approach for the PCA acknowledges the time sensitivity of asset returns to the main factors. In the case of precious metals, the changing time sensitivity has already been illustrated in the first study. The principal components are taken and portfolios are determined that have an "equal risk contribution" regarding those factors. The classical factor risk parity model, however, is altered in a way that it lets the less important components of the PCA float for the benefit of adding additional portfolio constraints. The analysis is performed by using the standard deviation in the second paper as well as the expected shortfall as a risk measure in the third paper. The following subsections will give a more detailed overview on each study including its contribution to existing literature while adding additional and related research.

1.1.3.1 Analyzing Precious Metal Returns using a Kalman Smoother Approach

The first study analyzes the returns of precious metals in a time-varying context. As a typical asset class in a well-diversified multi-asset portfolio it is analyzed how precious metals returns are affected by specific pre-determined factors such as the *Consumer Price Index (CPI)* or the industrial production. As the analysis of the returns is performed in a time-varying setting, the study relates to some extend to the second and third paper in which a PCA with a rolling-window approach is used to determine the main risk drivers for a multi-asset portfolio.

After some econometric tests, which will already hint at the importance of using a time-varying approach, a Kalman smoother is used to identify sensitivities between the factors of CPI, Producer Price Index (PPI), industrial production, volatility, dollar, real and nominal interest rates as well as the S&P500 to the precious metals of gold, silver, platinum and palladium. The Kalman smoother approach is chosen as it allows to continuously analyze and describe the sensitivities while at the same time using all the available data points to determine a sensitivity at time t. A majority of the research has focused on the relationship between gold and inflation and the question whether gold is a suitable investment to hedge inflation. Supporting the view of Lucey et al. (2017) or Ghosh et al. (2004) who in general find a positive answer to that question, this analysis also finds a positive relationship between gold and the CPI which is taken as an approximation for inflation.

The study, however, deals with further questions such as sensitivities of different factors to also silver, platinum and palladium and not only gold alone. Thereby it for example finds that gold also exhibits a positive sensitivity to equity volatility in the first part of the data set which, however, decreases over time until it is practically non-existent around 2015. Similar to Akram (2009), a negative relationship between the value of the dollar and the price of precious metals are found. Possible economic interpretations are given.

Finally a *Dynamic Time Warping (DTW)* approach is used to compare the normalized sensitivities among each other. A factor is taken and the sensitivities versus two metals compared. The DTW distances are determined whereby lower values mean a higher degree of similarity between the sensitivity of two different precious metals regarding a specific factor. The results indicate that, on average, platinum and palladium react similarly towards most factors. A similar pattern can additionally be observed between gold and silver whereas gold offers a different "sensitivity structure" than for example palladium.

The study adds to existing research on precious metal price movements as for example conducted by Batten et al. (2014), Batten et al. (2010), Golosnoy and Rossen (2014) or Sari et al. (2010). The main benefit is the Kalman smoother combined with a DTW approach applied to the four main precious metals which leads to a direct sensitivity analysis for each metal regarding a specific factor. Furthermore, a direct comparison between two precious metals can be made for specific factors.

Due to the large amount of (macroeconomic) factors and assets, an analysis of sensitivities of other asset classes from a multi-asset setup is not covered in this study. To close the gap to the PCA that is used in a moving time window approach in the second and third paper, a quick overview of other asset classes is given for the rest of this section.

Additionally to the work mentioned directly in the study, Byrne et al. (2013) use a *Factor Augmented Vector Auto Regression (FAVAR)* approach to also stress the importance of interest rates on commodity prices.

For equities, researchers have mainly identified inflation and money supply as having a significant effect on equity prices. Influence of other macroeconomic factors are more difficult to identify. Flannery and Protopapadakis (2002) find the CPI and PPI to affect the market portfolios return, whereas balance of trades, employment and housing starts affect the returns' conditional volatility. Monetary aggregate is supposed to affect the return as well as the conditional volatility. Using a co-integration approach, Nasseh and Strauss (2000) find a co-integrating relationship between stock price levels in some European countries and industrial production, business surveys of manufacturing orders, short- and long-term interest rates as well as foreign stock prices.

Figure 1 illustrates the sensitivity analysis of the percentage changes of the S&P500 to the CPI as well as the *real interest rates* when the Kalman smoother approach is used. The figure contains the development of the sensitivity as well as the 95% upper- and lower-bound confidence interval which are plotted as dotted lines. This procedure is identical to the one in the first paper, with the only difference in using a stock market index in exchange of precious metals. Even though this is just a very brief analysis of the sensitivities of the S&P500, it already indicates how those sensitivities can change over time and how the CPI or the real interest rates are moving relative to the changes in the S&P500. The relationship between stock price movements and inflation, interpreted as changes in the CPI, is negative in the first part of the time period which overlaps with results found in Flannery and Protopapadakis (2002) or Geske and Roll (1983). As mentioned there: "... stock returns are negatively related to contemporaneous changes in expected inflation because they signal a chain of events which results in a higher rate of monetary expansion." The results from Figure 1, however, are mixed when recent equity returns are considered. This might be interpreted to some extend by recent events such as the financial crisis or the "dot.com" crash as those events could overshadow other factors.

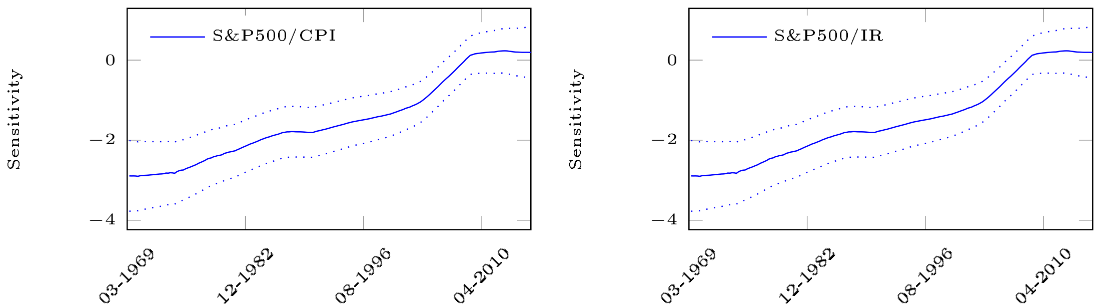

Figure 1: *Sensitivity of S&P500 to the CPI and IR using the Kalman smoother.*

Focusing on bond price movements, Diebold et al. (2006) take macroeconomic variables such as real activity, inflation and stance of monetary policy on the one hand and yield curve factors such as level, slope, and curvature on the other. They find "strong evidence of macroeconomic effects on the future yield curve and somewhat weaker evidence of yield curve effects on future macroeconomic developments". Chen and Maringer (2011) use a multi-factor smooth transition regression model to analyze returns of corporate bond indices during two different regimes of strong and weak economic periods. They find that "the sensitivities of default spread, dividend yield and excess stock index return are significant and more negative (or statistically equal) in strong economic regimes than in weak regimes". The dependence of the results on the different regimes again stress the importance of using a time-sensitive analysis.

1.1.3.2 Factor Risk Parity and Portfolio Weight Constraints Risk Parity is a popular investment topic that weights assets in a portfolio in a way that their contribution to the total portfolio risk is equal for all positions. Factor risk parity enhances this concept by requiring risk contributions for the underlying risk factors to be equal instead of considering the risk of assets itself. This study modifies this concept by focusing on the first two principal components and letting the residual component weights float, thereby having the freedom to further restrict the portfolio weights in the asset space as the solution set increases. The second and third paper take advantage of this benefit to allow only long positions as many investors are prohibited from taking short positions or leverage. The factors are determined by using a PCA while using a rolling window approach on the data set to acknowledge changing sensitivities described for example for the case of precious metals in the first study.

The solution to the above mentioned problem with the added restrictions is not a single portfolio but a set of portfolios which can be described by a polytope. To ensure sufficiently high explanation levels of the first two principal components relative to the total risk, minimum explanation levels are added as a restriction. Given the resulting solution set, the study focuses on two portfolios within that set that minimize the in-the-sample variance and the portfolio that maximizes in-the-sample diversification. A portfolio is constructed thereby that either minimizes the variance or maximizes diversification with the additional benefit that the risk contributions regarding the first two main risk drivers, determined through a PCA, are equal. A data set reaching back to 1998 and 15 different assets are taken to determine those portfolios and back-test them against some naive and heuristic allocations.

The study contributes to existing research as for example conducted by Bhansali et al. (2012), Lohre et al. (2012), Kind (2013), Bernardi et al. (2018), Costa and Kwon (2019) or Deguest et al. (2013). Even though the setup is similar to Meucci (2009), the main advantage of the model proposed here is the flexible consideration of different constraints for a different number of asset classes while considering the risk contributions of the core underlying risks. Meucci (2009), on the contrary, allows constraints which lead to only approximately equal risk contributions without describing the solution as implemented in our model.

1.1.3.3 Tail driven Factor Risk Parity with Volatility Investments The third paper deals with the above-mentioned model while exchanging the risk measure of the standard deviation by the expected shortfall. The standard deviation is often considered a bad choice in the light of fat tails and skewed distributions. When exchanging the risk measure, the solution does not change compared to the results in the second paper P2 if some assumptions on the return distributions of the underlying assets are made. In the general case, however, this might not be true and numerical calculations are needed to determine the optimal portfolios. By using a downside risk measure, the study follows research of Boudt et al. (2008), Mausser and Romanko (2018) or Tasche (2002) who stress the importance of non-symmetrical risk measures that focus on downside risks.

Other enhancements of the model in the third paper P3 include the introduction of unfunded positions as for example future or forward positions. That allows the introduction of volatility short positions, which in recent years have delivered attractive returns but which also exhibit tailed, non-symmetric return profiles. That, however, again motivates the use of the expected shortfall as the risk measure.

2 Analyzing Precious Metals Returns using a Kalman Smoother Approach

2.1 Introduction

2.1.1 Overview

Precious metals such as gold or silver are, next to the use as jewellery or in industrial processes, a popular commodity and investment vehicle. It is therefore important to understand the sensitivities of different (external) factors to the price of those metals. The most commonly analyzed relationship is gold to inflation as investigated among others in Beckmann and Czudaj (2013). Batten et al. (2014) focus on a time varying approach via Kalman filter with an additional analysis of the determinants of that relationship. Another common research topic is the role of precious metals serving as a "safe haven" as for example discussed in Li and Lucey (2017) or the analysis of precious metals price volatilities as done by Batten et al. (2010).

For risk management purposes or setting up an investment strategy that contains precious metals it is important though to understand what drives the price of not only gold but also silver, platinum or palladium. It is also key to understand whether the sensitivities of specific factors such as inflation, industrial production or US-Dollar movements to precious metals are permanent or changing over time and how a specific factor impacts the price of one precious metal compared to another one. A time-varying approach in form of a Kalman smoother is therefore chosen to analyze the sensitivities based on quarterly data since 1969. Finally, DTW is used to better understand how different or similar the impact of a factor to two different precious metals is. A precious metal investor who would like to build up a specific factor exposure thereby gets an indication whether one precious metal can easily be exchanged by another. Examples will be given that illustrate different behavior of two portfolios of two precious metals, one with low and one with high DTW distances, when a factor is "under stress".

The range of factors considered in this chapter is larger than in some other work. Investors are provided with factors that illustrate stable sensitivities for precious metals such as gold to CPI as for example also done in Batten et al. (2014). The results presented below find similar sensitivities when a more robust Kalman smoother approach and a longer data set is used. Additionally, the chapter contributes to other research such as Batten et al. (2010) who conclude that precious metals are too distinct to be considered a single asset class: by focusing on individual factors and their influence over time on precious metal prices, similarities and differences between precious metals can be analyzed in detail. The results for the relationship of precious metals to equity markets also partly overlap with Klein (2017) who use a dynamic correlation approach and thereby compare different precious metals when stock indices such as the S&P500 or DAX crash.

The use of a Kalman smoother, however, contains the problem of describing the model parameters properly, which will briefly be discussed in Section 2.2.1.1. DTW is finally applied to identify and summarize the differences/similarities between the precious metals. The explanatory power is superior to a classical Euclidian distance metric as delays of response or different reaction times of the sensitivities are taken into account. Results indicate that exchanging platinum for palladium, for example, should still leave investors with a similar exposure to different factors.

Chapter 2 is organized as follows: Section 2.1.2 gives a brief overview of the current literature, before Section 2.2 describes the Kalman filter and smoother approach as well as briefly the concept of Dynamic Time Warping. Section 2.3 explains the data for the precious metals as well as the factors including some econometric characteristics. The following Section 2.4 covers the empirical part that illustrates the results for the co-integration, the Kalman smoother and DTW approach as well as consequences that arise for investors.

2.1.2 Literature

The mathematical tools applied in this work mainly consist of basic time series analysis and the Kalman filter/smoother as well as the DTW approach. The former consists of co-integration as for example vividly explained in Murray (1994) and later extended by Smith and Harrison (1995). The basic idea of the Kalman filter has been developed in 1960 and presented in Kalman (1960), whereas the idea of the *Rauch–Tung–Striebel (RTS) smoother* which is later applied in this work has only been pubished 5 years later in Rauch et al. (1965). As the methods turned out to be extremely useful, their use has been extended from the original idea in physics to all different topics such as the explanation of time series movements as done for example in the work of Batten et al. (2014). The application of the DTW approach can for example be found in Ratanamahatana and Keogh (2004) or Müller (2007).

Metal prices have been analyzed among others in Golosnoy and Rossen (2014). Next to the precious metals they also take non-ferrous metals into consideration and try to identify two factors that explain a majority of the return variation. For the 100 year monthly dataset they also try to determine the correlation of the factors to common macroeconomic fundamentals. Equivalently to previous results for the one-factor model, they find that "the full sample correlations of the single common factor with the U.S. IP (industrial production), oil, S&P500 and CPI are significantly positive, whereas they are significantly negative with the interest rates." For the second subsample of their dataset, which approximately overlaps with the data in this work, they find that "the second factor... seems to be a good proxy for the monetary indicators such as CPI."

Batten et al. (2010) try to identify common factors which explain the return volatility in precious metals. They find limited evidence of the same macroeconomic factors jointly influencing the volatility processes of the commodity price time series examined and conclude that gold, silver, platinum and palladium are too distinct to be considered a single asset class. This result will later be supported by the RTS smoother analysis in this study.

Akram (2009) focus on commodities in general and use a Vector Auto-Regressive (VAR) model to conclude that shocks to the real interest rate and the dollar real exchange rate contribute significantly to movements in commodity prices.

Focusing on the popular gold to inflation topic, Lucey et al. (2017) analyze the time-varying relationship between gold prices and inflation in different countries and find a time varying co-integration between gold and inflation in nearly all time series. They also identify a break between gold and official inflation in the US in the mid 1990's which seems to be less clear when the inflation of UK or Japan is used. Especially for Japan they find deviating results, concluding that gold prices are not co-integrated with the inflation in Japan.

Ghosh et al. (2004) also focus on the gold to inflation relationship and in this context distinguish between short-term and long-term movements of the price of gold. Using co-integration regression techniques, they find that gold can be regarded as a long run inflation hedge, however they also find that the nominal price of gold is dominated by short-term influences.

Other research supporting that gold can be seen as a hedge against inflation is found in Murach (2019) or Worthington and Pahlavani (2007). Latter find a stable long-term relationship between the price of gold and the rate of inflation after accounting for structural breaks in the time series. The main structural breaks relate to the gold market moving to purely open market operations as well as the acceleration of inflation in the 1970s. Murach (2019) stress the positive long-run relationship between excess global liquidity and the real gold price, concluding that inflation fears driven by excess liquidity might be the cause for a higher demand of gold.

Sari et al. (2010) take gold, silver, platinum and palladium together with oil and the Euro to USD exchange rate and examine the co-movements and information transmission between those time series. They find some weak long-run equilibrium relationship while, for the short term, they argue that precious metals react strongly to the prices of the other precious metals and exchange rate.

By using different factors and precious metals combined with the Kalman smoother and DTW approach, this work expands the research mentioned above. Using a smoother is similar to the filter approach of Batten et al. (2014) while benefiting from the usage of all data points at every point of time. This approach also covers unique characteristics of price changes of the metals as for example done in Bräuninger et al. (2013), but delivers additional insight by using a time-varying smoother combined with the DTW approach. Finally, it extends the sensitivity analysis of factors to different precious metals and by that mostly supports the results found in Lucey et al. (2017), Ghosh et al. (2004) or Cohen and Qadan (2010). The use of different factors and precious metals, however, expands their results for the benefit of better identifying differences and similarities between price changes in precious metals. DTW distances between the sensitivities further illustrate the findings.

2.2 Mathematical Methods

This section briefly describes the mathematical methods used to analyze the time series. The standard approaches of the Johansen method for co-integration as well as Augmented Dickey-Fuller (ADF) or Kwiatkowski–Phillips–Schmidt–Shin (KPSS) tests are omitted and the reader is referred to standard econometrics literature instead. The Kalman filter and smoother as well as the concept of Dynamic Time Warping will be explained in more detail in the sequel.

2.2.1 The Kalman Filter

The basic concept of the Kalman Filter has been developed some time ago by Kalman (1960). One of the first applications at that time was the trajectory estimation and control problem for the Apollo project at NASA. This is nowadays just one typical application of many while the approach since then has found much appeal among different practical problems (see Grewal and Andrews (2001), Gelb (1974) or Maybeck (1982)). The basic idea of the Kalman Filter is to efficiently estimate the unobservable state of a process by recursively taking new observations into account. Applied to this work, the state of the process is the relationship between the explanatory variables (e.g. inflation) and the variable being explained (asset price movement), whereas the observations are the historical price changes.

After using the Kalman filter, a RTS-smoother as presented in Rauch et al. (1965) is used to take all the data points at time t into account (instead of just using information available up to time t). The next subsection will therefore give a brief overview of the mathematical description of the Kalman filter and smoother before they are implemented in Section 2.4.

2.2.1.1 An introduction to the Kalman Filter As mentioned in Section 2.2.1, the aim is to estimate the state x_k of a discrete-time (controlled) process which is given by a linear stochastic difference equation (SDE)

$$x_t = Ax_{t-1} + Bu_{t-1} + w_{t-1} \tag{1}$$

with $x_t \in \mathbb{R}^n$, matrix $A \in \mathbb{R}^{n\times n}$ and optionally $u_t \in \mathbb{R}^l$ with matrix $B \in \mathbb{R}^{n\times l}$. For this analysis, the term including B and u_t are dropped and the matrix A will later be chosen to be the identity matrix. w_t is the process noise and assumed to be a random variable that is normally distributed with mean zero and fixed covariance matrix Σ. The measurement relates via

$$z_t = H_t x_t + v_t \tag{2}$$

with $z_t \in \mathbb{R}^m$ and matrix $H_t \in \mathbb{R}^{m\times n}$. The relationship between the state x_t and the measurement z_t is therefore assumed to be of linear nature. Similar to equation (1), v_t denotes the measurement noise and is assumed to be a normally distributed random variable with zero mean and covariance Γ. The noise terms v_t and w_t are supposed to be independent.

In our analysis, the state x_t will be the sensitivity in the change of the price of a precious metal, given observations in the change of factors of z_t. The equations (1) and (2) above will therefore be of the form:

$$Sensitivities_t = Sensitivities_{t-1} + noise_{t-1} \tag{3}$$

for the sensitivities and

$$ReturnMetal_t = \sum Sensitivity(i)_t \cdot ReturnFactor(i)_t + c + noise_t \tag{4}$$

with a constant c. The constant appears by setting the last entry of matrix H_t to 1. The price change of the metal is therefore the sum of the sensitivity of the metal to a factor multiplied by the change in the factor plus a constant and noise.

A typical first question to ask is whether the price level of the commodity or its relative change, the first difference, is used in this context. Batten et al. (2014) focus on the price level of the metals, stating that an advantage of the Kalman filter is the robustness to non-stationary data. However, as stationary data, like for example given in the form of the volatility time series, will also be analyzed next to integrated time series of order one ($I(1)$), differenced data instead of price levels will later be used for the Kalman smoother if the order of integration is greater than zero.

As the sensitivities are not directly observable, they have to be estimated. Let therefore $\hat{x}_{t|t-1}$ be the estimation of the sensitivity at time t given the information until $t-1$, meaning $\hat{x}_{t|t-1} = E(x_t|z_1, ..., z_{t-1})$. As these are estimates, $P_{t|t-1}$ and $P_{t|t}$ describe equivalently the covariances, meaning $P_{t|t-1} = E((x_t - \hat{x}_{t|t-1})(x_t - \hat{x}_{t|t-1})^T)$. The algorithm for the Kalman filter tries to determine the state estimates while minimizing the estimation error covariance $P_{t|t}$. It can be described by two simple steps: 1. prediction and 2. measurement update. Algorithm 1 describes all the necessary computations (see for example Simon and Shmaliy (2013)).

Algorithm 1 Classical Kalman Filter algorithm

input : Process noise covariance matrix Σ

input : Measurement covariance matrix Γ

input : Inital sensitivity estimate $\hat{x}_{0|0}$

input : Inital covariance estimate $P_{0|0}$

foreach $t \in \{1, 2, .., r\}$ ▷ *For all r observations*

do

1. $\hat{x}_{t|t-1} = A\hat{x}_{t-1|t-1}$ ▷ prediction of sensitivity
2. $P_{t|t-1} = AP_{t-1|t-1}A^T + \Sigma$ ▷ prediction estimation error covariance
3. $K_t = P_{t|t-1}H_t^T(H_tP_{t|t-1}H_t^T + \Gamma)^{-1}$ ▷ Calculation of Kalman gain
4. $\hat{x}_{t|t} = \hat{x}_{t|t-1} + K_t(z_t - H_t\hat{x}_{t|t-1})$ ▷ update of sensitivity
5. $P_{t|t} = P_{t|t-1}K_tH_tP_{t|t-1}$ ▷ update estimation error covariance

end

Lines 1 and 2 within the 'do'-part of the algorithm represent the prediction step, lines 3 to 5 are the update steps. This is recursively computed for all the r available observations. The first equation in the prediction step is directly determined through equation 1. The second one follows from the definition of $P_{t|t-1} = E((x_t - \hat{x}_{t|t-1})(x_t - \hat{x}_{t|t-1})^T)$ and combining the first prediction step with equation (1). Note that for those calculations, the precondition of uncorrelated noise terms has to hold (see for example Faragher et al. (2012) for the detailed calculations).

Line 4 represents the first update step. The new estimated sensitivity $\hat{x}_{t|t}$ is a mix of the newly predicted sensitivity $\hat{x}_{t|t-1}$, adjusted by the term $K_t(z_t - H_t\hat{x}_{t|t-1})$. $(z_t - H_t\hat{x}_{t|t-1})$ is the so called measurement innovation or residual term (see Welch and Bishop (1995)), calculating how much the actual observation z_t is different from the prediction $H_t\hat{x}_{t|t-1}$. K_t is called the *"Kalman gain"* that determines how much the residual term is to be weighted for the update step. The lower the value, the more the prediction of the sensitivity in line 1 is trusted and vice versa. The Kalman gain is determined by minimizing the a posteriori error covariance $P_{t|t} = E((x_t - \hat{x}_{t|t})(x_t - \hat{x}_{t|t})^T)$. This is done by using the update of the sensitivity equation (line 4 in Algorithm 1), calculating the expectation and setting the derivative to zero (see e.g. Welch and Bishop (1995) or Brown et al. (1992)).

The final question, which from a practical point of view is probably the most important one, is the choice of the input parameters, namely the initial sensitivity and covariance estimate as well as the noise covariances Σ and Γ. Naik et al. (2015) discuss those aspects, including some readily available algorithms such as the *Expectation Maximisation algorithm (EM)*. That and other algorithms, however, are not further discussed in this analysis. It is probably more important that setting those input parameters to "reasonable" values when running the Kalman filter. These include:

1. Inital sensitivity estimate $\hat{x}_{0|0}$ is set to zero in the absence of further information about any sensitivities.

2. The importance of the choice of the initial covariance estimate should not be underestimated (Naik et al. (2015)). Setting it to near zero would mean that the initial sensitivity estimate of zero is "certain". The value should therefore be not too small. This is even more important as the RTS smoother will later be applied.

3. In general, the measurement noise is determined by the physical properties of the measurement device, which in this analysis is not given. Also, the process noise covariance Σ should not be chosen "too big" (in comparison to P), as this would suggest that the model

does not predict the sensitivity well.

What does this mean in this case? In our equation 1 with an identity matrix A, a new sensitivity is calculated as the value from the previous quarter plus noise, as quarterly relative price changes will be taken. The question would therefore be how much we expect the sensitivity to change from one quarter to the next. According to Naik et al. (2015), the basic procedure is to start with values close to zero for both covariances and gradually increase the values. For the case of the analysis of multiple factors, the covariance matrix will be chosen with values on the diagonal only, meaning that the sensitivity parameters are independent from each other. The use of the Kalman Filter to estimate a beta is similar to a rolling window regression approach. The Kalman filter, however, has some advantages though: the choice of the parameters is seen as superior to an arbitrary choice of the window length for the rolling regression method. The filter should also better track a rolling beta as for example illustrated for some numerical examples in Roncalli and Teiletche (2007).

2.2.1.2 Rauch-Tung-Striebel Smoother The basic idea of the Kalman smoother is to take not only information until time t into consideration, but to use the whole spectrum of observations for every point in time. As the purpose is to explain the relationship of precious metals to specific factors, the use of all data points at any time via a smoother is a reasonable approach. If the Kalman filter is used for predicting values, it is crucial to not take future values into consideration as that would distort the prediction. In this context, the smoother is used in a similar way as done for example in Chen and Tindall (2013) or Swinkels and Van Der Sluis (2002).

The RTS smoother discussed here has already been presented in 1965 in Rauch et al. (1965). It is a so-called fixed-interval smoother that starts by using a classical Kalman filter as discussed above and then running through the time series from the end back to the first observation and adjusting the estimates accordingly. The procedure is in detail described in Algorithm 2 (see Simon and Shmaliy (2013)).

Keep in mind that $P_{i+1|i} = AP_{i|i}A^T + \Sigma$ and $\hat{x}_{i+1|i} = A\hat{x}_{i|i}$ are from the prediction step of the Kalman filter. A proof with simple examples can be found in Särkkä (2013).

The following example is an extension to the simulation of Welch and Bishop (1995) to show the effects and benefits of using the RTS smoother. Like in their example, let $x = -0.37727$ be the constant to be estimated. 50 measurements around that constant are drawn with a mean of zero and standard deviation of 0.1. Further, let $\Gamma = 0.01$, $\Sigma = 1e-5$, $r = 50$, the initial values

Algorithm 2 RTS Kalman smoother algorithm

input : Results ($\hat{x}_{t|t}$, $P_{t|t}$) from the execution of Kalman Filter

foreach $t \in \{1, 2, .., r\}$ ▷ *Go through all r observations*

do

1. $i = r - t$ ▷ Successively go back, starting at second last observation
2. $K_i^s = P_{i|i} A^T (P_{i+1|i})^{-1}$ ▷ Kalman smoother gain
3. $P_i^s = P_{i|i} - K_i^s (P_{i+1|i} - P_{i+1}^s)(K_i^s)^T$ ▷ covariance of smoothed sensitivity
4. $\hat{x}_{i|r} = \hat{x}_{i|i} + K_i^s (\hat{x}_{i+1|r} - \hat{x}_{i+1|i})$ ▷ Smoothed sensitivity

end

of $\hat{x}_{0|0} = 0$ and $P_{0|0} = 1$ and transition matrix of $A = 1$ be given. Figure 2 shows the results: the left figure shows the estimation of the error (co)variance which decreases over time for the Kalman filter as more and more estimates are considered. As the initial value is set to zero with $P_{0|0} = 1$, the line decreases rapidly. The estimation error (co)variance however is quite stable for the RTS smoother with small increases at time step 0 and 50 as all estimates are considered at every point in time. The same is true for the actual estimation of the value with estimation of the RTS smoother being more accurate and stable. The example is of course overly simplified for demonstration purposes as the core advantage of the Kalman filter (or smoother) plays out in the case when the signal is changing, however it still illustrates the effects of using a smoother instead of a Kalman filter approach.

As the name already indicates, the Kalman smoother also smoothes the sensitivities. Figure 2 illustrates how the smoothed sensitivity looks much more stable and less volatile than the results from the Kalman filter. When selecting this approach one assumes therefore that sensitivities don't change abruptly over time which is an assumption that should be kept in mind when analyzing the results. Also, it should be noted that only a sensitivity may be identified which excludes statements about causality as in a Granger causality framework (Granger (1969)). This avoids, e.g., discussions such as whether the USD-Dollar influences the price of gold or whether gold influences the USD-Dollar. Finally, the following analyses talk of relationships between factors and the price of a precious metal. By that it is meant that over longer time horizons, the sensitivity as well as the 95% confidence interval in total is either positive or negative.

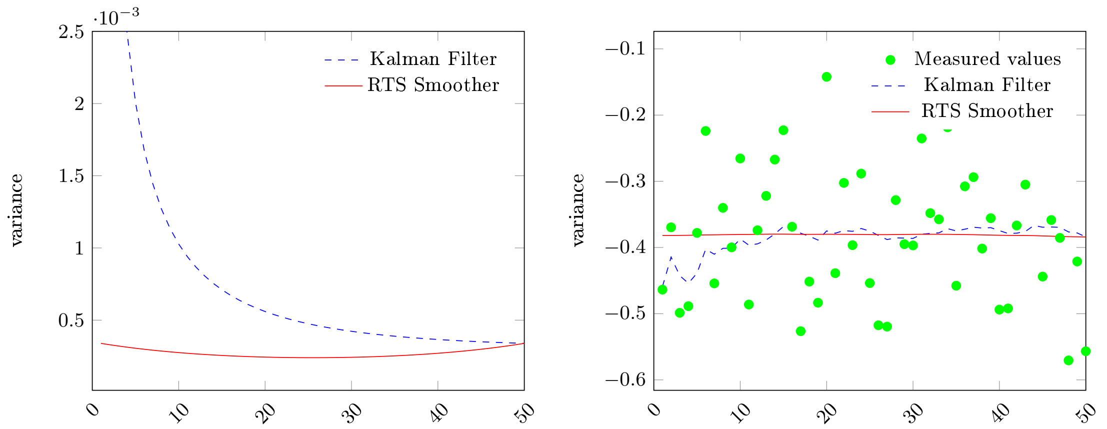

Figure 2: *Left: Estimation error (co)variance over the 50 time steps of the Kalman filter as well as for the RTS smoother, Right: The filter estimate is inprecise at the beginning, the smoother estimate is closer to the true value of x.*

2.2.2 Dynamic Time Warping

In this study, Dynamic Time Warping (DTW) as for example explained in Ratanamahatana and Keogh (2004) is used instead of a simple Euclidian distance metric to compare the time-sensitive dynamics of precious metals to specific factors. After normalizing the sensitivities, the differences in curves is analyzed to identify similar but also different movements of a factor to two different precious metals. Figure 3 illustrates the differences between DTW and Euclidian distances in a simple example: both sensitivities look similar with a major difference that sensitivity 2 reacts with a time lag compared to sensitivity 1. The Euclidian distance metric determines the difference for a each time t between the two curves, whereas the DTW algorithm better finds "suitable" points to connect for the determination of the difference. As well as time lags, the DTW also better copes with stretched and compressed sections in the sensitivities which is the reason for the approach being used for example in speech recognition as it is able to recognize voices spoken at different speeds.

For a more formal description, let $S = \{s_1, s_2, ..., s_r\}$ and $T = \{t_1, t_2, ..., t_r\}$ be the two sensitivities of length r. In a general DTW framework, the length for the two time series are allowed to differ, in this study, however, the sensitivities that are determined via the Kalman smoother are of same length. Let $d_{i,j} = \|s_i - t_j\|$ with $\|x\|$ be the Euclidian norm so that $D = (d_{i,j}) \in \mathbb{R}^{r \times r}$. As described in Müller (2007), a (r,r)-warping path can then be defined as a sequence $p = (p_1, ..., p_L)$ with $p_l = (n_l, m_l) \in [1 : r] \times [1 : r]$ and $l \in 1, 2, .., L$ meeting the conditions

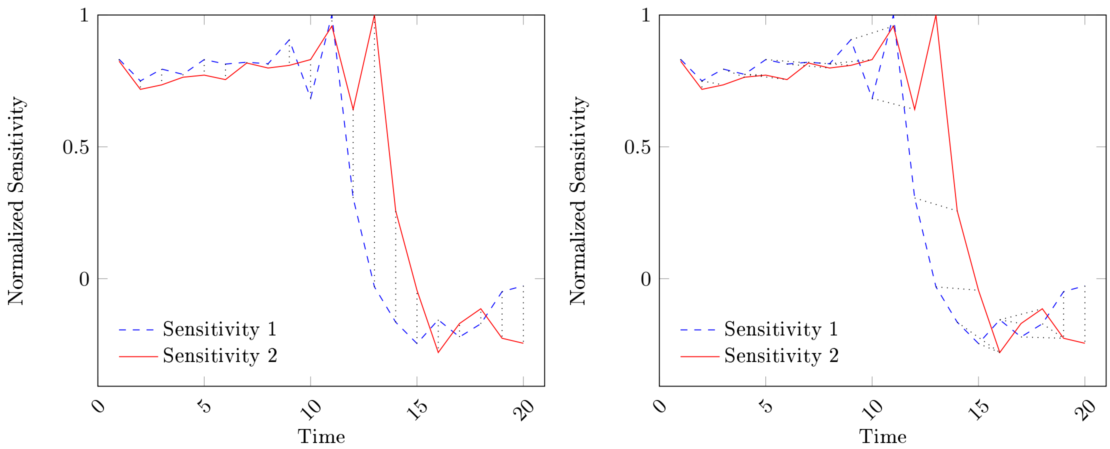

Figure 3: *Example that illustrates the differences between two sensitivities for the normal Euclidian (left) vs the DTW (right) distance. The DTW distance much better copes with the time lag of sensitivity 2, leading to a lower distance than in the Euclidian case.*

- Boundary condition: $p_1 = (1, 1)$ and $p_L = (r, r)$
- Step size condition: $p_{l+1} - p_l \in \{(1, 0), (0, 1), (1, 1)\}$ for all $l \in 1, 2, .., L - 1$

The first condition guarantees that the first and the last sensitivity point for both sensitivity curves match. The second condition describes a restriction on how the path through the matrix elements are to be found and assures that no sensitivity point is omitted. For $r = 5$ for example, $((1, 1), (2, 2), (3, 2), (4, 3), (4, 4), (5, 5))$ would be a proper warping path that determines that s_1 is connected with t_1, s_2 with t_2, s_3 with t_2, s_4 with t_3 and so forth.

The problem is to find a path p_{opt} that minimizes the total cost

$$c_{p_{opt}}(S, T) = \sum_l d_{n_l, m_l}. \tag{5}$$

To find that optimal path, a dynamic programming approach can be used to determine an accumulated cost matrix C, which for $i, j > 1$ is given by

$$C(i, j) = d_{i,j} + \min \Big\{ C(i-1, j-1), C(i-1, j), C(i, j-1) \Big\} \tag{6}$$

Thus $C(i, j)$ determines the total cost for an optimal path from point $(1, 1)$ to (i, j). More detailed information can be found in Müller (2007). Boundary conditions that limit the optimal path as presented for example in Ratanamahatana and Keogh (2004) will not be used in later computations.

2.3 Data Description

2.3.1 Precious Metals

The following section gives an overview of the data that is used as well as some of its characteristics determined through econometric tests. For all the metal prices as well as factors, quarterly data since March 1969 until the end of 2015 for a total of 188 data points is taken either from Bloomberg or directly from the International Monetary Fund (IMF).

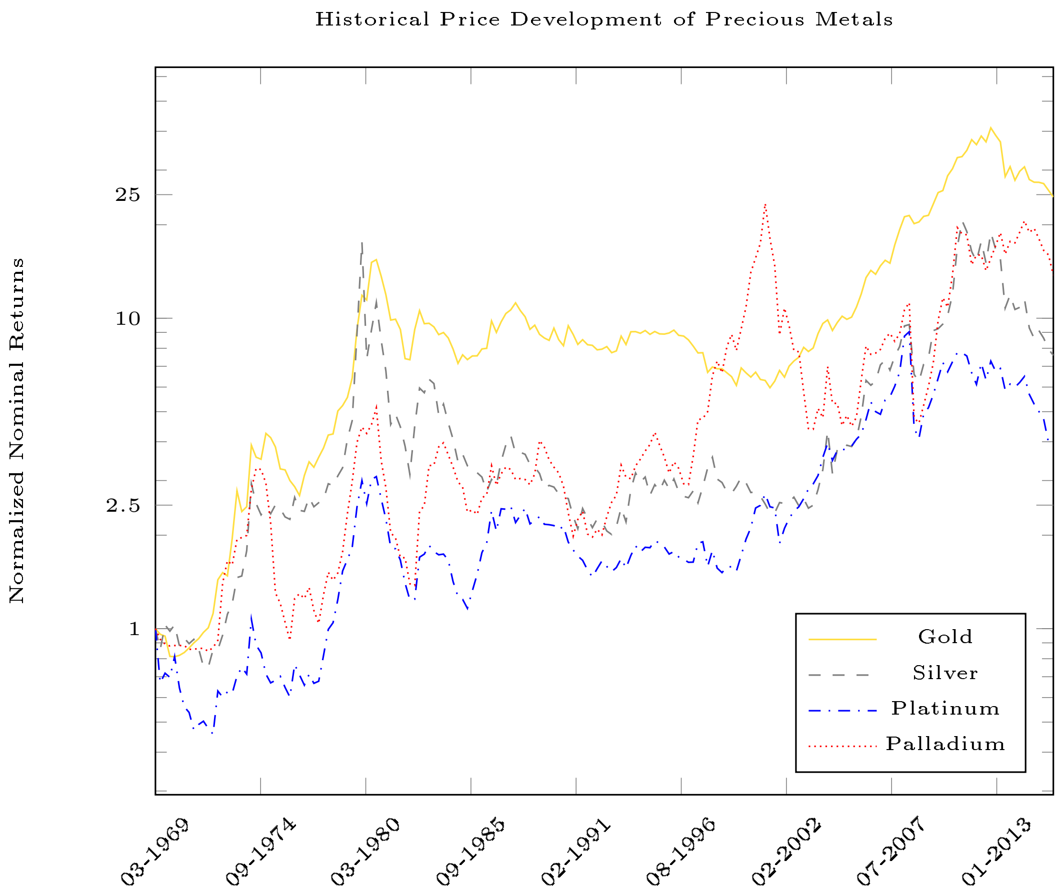

Figure 4: *Historical price development of the four precious metals since March 1969 on a normalized basis with log scaling.*

Table 1 illustrates briefly some statistics of that dataset. The length of the interval is chosen so that a sufficiently long time period is available while at the same time avoiding (half-)yearly data. Furthermore, macroeconomic time series are used in the analysis which usually change only slowly, making daily or weekly intervals, as for example done in Bhatia et al. (2018), less reasonable. For other factors where daily data is readily available, such as equity volatility or the S&P500, results for the sensitivities may of course differ when the time length is changed. The motivation for the analyses here, especially for the ADF as well as KPSS tests and the co-integration test later, is to illustrate the similarities, but also differences, between the time series for the different precious metals for longer time horizons.

	Min	Max	Mean	std	skew	kurt
Gold	-0.258	0.456	0.017	0.097	1.057	6.41
Silver	-0.87	0.684	0.011	0.16	-0.138	9.897
Platinum	-0.715	0.408	0.007	0.125	-0.837	8.816
Palladium	-0.854	0.549	0.014	0.172	-0.674	6.569

Table 1: *Basic statistics of log changes in prices of precious metals, quarterly basis*

2.3.1.1 Gold Gold is probably the most common precious metal in the investment sector. The question on what drives the gold price has therefore been intensively discussed in literature: Baur et al. (2017) discuss the relative valuation of gold to other asset classes and conclude that gold can be interpreted as a "safe haven". The characteristic of protecting during equity market turmoil is also discussed in Hood and Malik (2013), Li and Lucey (2017) or Pan (2018). Both questions will briefly be covered below as well. Other influencing factors that are often discussed are inflation (see Batten et al. (2014), Kumar (2017) or Ghosh et al. (2004)) or the US-Dollar (see (Oxford Economics, 2011)). To first get a better understanding of the price development of gold, Figures 4 and 12 in the Appendix give an overview of the price history in comparison to the other precious metals as well as on a log scale for the price as well as the differences in the log price for gold itself. After the end of the Bretton-Woods system, the gold price started a strong upward move until 1980, before moving down-/sideward for a long period of around 20 years. The second strong upward move started in 2001 and lasted until 2011 when gold reached (in nominal terms) its highest price of around 1920 US-Dollar before falling back again in the following years.

Looking at Table 6, the results of the ADF test support the view that the series is of type $I(1)$, meaning it is integrated of order one. The KPSS test indicates some evidence to suggest that the log of the gold price is unit root nonstationary. The results are, however, not completely clear with a long period of flat prices in gold.

2.3.1.2 Silver Like gold, silver reached a high in 1980 of nearly $50 per troy ounce, mostly driven by the Hunt Brothers who amassed a large amount of silver and silver futures in the 1970s. The following correction in the prices was huge, before another large upward move between 2001 and 2011 led to an increase of nearly 800%. Since then, prices have declined again. The similarity between gold and silver is both visible in the commonly quoted gold/silver ratio, which is tracked by many traders, as well as the correlation in the data of 0.64 (see Table 8), the highest among the correlation pairs of the four precious metals.
The results of the ADF test as well as for the KPSS test hint at similar results as already seen in the case of gold, indicating that the price is of type $I(1)$.

2.3.1.3 Platinum and Palladium Hageluken (2006) provide a detailed overview on platinum as well as palladium. They argue that consumer applications and jewellery have recently become the main drivers for the demand of those metals, making technical applications an important factor when identifying price trends. Even though platinum and palladium are sometimes seen as substitutes for each other, differences between the metals exist. The development of palladium-based catalyst systems, for example, led to a higher demand of palladium and as a consequence its price, relative to that of platinum, increased at that time.

Table 6 illustrates the values from the ADF test with similar results to gold or silver (time series being of form $I(1)$). In contrast to gold and silver, the KPSS test directly fails to reject the null hypothesis that the log prices of platinum as well as palladium are trend stationary.

Although similarities between all four metals are obviously present in the price development in figure 4, differences in the price returns cannot be neglected. As the KPSS test indicates, differences in the kind of upward trend exist. This is for example driven by the huge increase in prices for gold and silver around 1980 with the down-/sideward trend for a longer period after 1980 to around 2000. The trend appears to be more consistent for platinum and palladium.

2.3.2 Factors

Different factors are selected which had been studied in other research before. They include volatility (as done in Cohen and Qadan (2010)), CPI/PPI (Beckmann and Czudaj (2013) among others), industrial production (Christie-David et al. (2000)), the US-dollar (Oxford Economics (2011), Lin et al. (2016)) and other market data such as real interest rates, 10y treasury yield or the S&P500 (Baur (2013), Nguyen et al. (2016) or Bukowski et al. (2016)). The factors, which were taken either from Bloomberg or directly from the IMF, are briefly described in the following section.

2.3.2.1 Consumer Price Index/Producer Price Index As mentioned above, the CPI is the factor that is most often found in analyses related to gold as it is considered a representation for inflation. Research focuses on a relationship between the gold price and the CPI number to find an answer to whether gold can be used as an inflation hedge. The argument behind is that gold is a physical metal that is durable, transportable, universally acceptable and easily authenticated and should therefore reflect inflation expectations. In this context, both the CPI as well as the PPI in the US are taken for this analysis. The use of the PPI is taken for comparative purposes, following the analysis of Beckmann and Czudaj (2013). Both values are taken from the IMF.

Table 6 shows that at common probability levels, the log of the CPI seems to be of order I(2) and PPI of order I(1). The results so far in the literature are mixed with many analyses such as Beckmann and Czudaj (2013) and Batten et al. (2014) finding the CPI to be I(1), some others such as Claus et al. (1997) argue in favor of I(2). As the results are mixed, the CPI in first difference is taken for the co-integration analysis as well as for the general analysis with the Kalman smoother later. Thereby the acceleration or deceleration of the inflation instead of the general price change from year to year is measured which makes interpretation easier than in the case of a CPI in second difference.

2.3.2.2 Industrial Production The data for the industrial production (IP) is taken from the IMF and in this context used as an approximation for general economic activity. Christie-David et al. (2000) analyze the short-term effect of news releases of the industrial production on the prices of gold and silver but do not find any significant effect. IP is nevertheless taken into consideration, especially as platinum or palladium, as mentioned above, are intensively used in selected industrial production processes such as the automobile or mobile industry. A higher IP should therefore lead to a higher demand for platinum or palladium which should result in higher prices of those metals. Tests indicated that the log time series of the industrial production is integrated of order one.

2.3.2.3 Realized equity volatility 90-days realized volatility of the S&P500 is taken as an approximation of investor fears to have a sufficiently large time series. This contrasts to for example using the VIX index as done in Cohen and Qadan (2010). The VIX index, which reaches back until 1990, is an index of implied volatility. It plays an important role in risk prediction and, through its construction, reflects the nature of asymmetric volatility found in capital markets (see Aboura and Wagner (2016) for further details). The insecurity of market participants is often supposed to drive the price of gold as it is seen as a store of value and a currency which cannot be manipulated during times when stocks and other assets are sold. Cohen and Qadan (2010) study the relationship between gold price and fear sentiments of the market participants and find that the Chicago Board Options Exchange (CBOE) Volatility Index VIX is positively related to previous day gold return. Tests show that the time series of the volatility are integrated of order zero, which can be explained by the mean reversion characteristic of volatility. This is the only I(0) time series where the level and not the first difference is used for the Kalman smoother.

2.3.2.4 Dollar The time series is taken from an index which represents the general international value of the USD by averaging the exchange rates between the USD and major world currencies. The value is calculated by the ICE US by using the rates supplied by some 500 banks. Futures on that index can be traded via the exchange as well. The correlation to the nominal effective exchange rate that is issued by the BIS is with around 96% for quarterly data since 1969 very high. The IMF in 2008 in their IMF World Economic Outlook (April 2008) estimated that 40%-50% of the moves in the gold price between 2002 and 2008 were dollar-related. According to a study commissioned by the World Gold Council and conducted by Oxford Economics (Oxford Economics, 2011), the reasoning is that (a) a falling dollar increases the purchasing power of non-dollar area countries which drives up prices of commodities and (b) during periods of dollar weakness, investors look for an alternative store of value, driving up the gold price. The time series itself appears to be of type I(1).

2.3.2.5 Real interest rates/10Y treasury yield (IP)/S&P500 Other market data, such as the 10y real US interest rates, nominal rates in form of the 10 year US treasury yield and the price of the S&P500 are also taken into consideration. Interest rates are taken as many people argue that an increase in rates would make commodities such as gold less attractive as precious metals do not pay any dividend. The S&P500 is taken as a broad US equity index to answer the question as to how far the precious metals move differently from a normal equity index.

The positive correlation between the S&P500 and platinum and palladium for example could be explained similarly to the IP case above as economic activity relates positively to stock prices as well as the demand for those metals. The opposite could be argued in the case of gold: as investors shift their assets into equity, gold as a substitute could become less appealing, leading to price decrease. Table 8 illustrates a low positive correlation between the S&P500 to platinum or palladium and a low negative correlation to gold. The log price of this equity index is I(1).

2.4 Empirical Analysis

2.4.1 Co-integration

Co-integration in this study is analyzed by using a Johansen framework. The results from a likelihood ratio test as illustrated in Sims (1980) indicate that a lag of thirteen should be included for the further analysis of the four precious metals. The results of the test are mixed and depend on the time horizon that is taken. Similar to the results of Bräuninger et al. (2013) for example, the tests indicate that palladium is not co-integrated to the other metals. This, however, seems to be only true when taking the whole time period of 1969 to 2015 into account. As mentioned above, the price of palladium especially around the year 2000 was largely influenced by market speculations. If only the time period of 1969 to 1998 is considered, the analysis does find some strong evidence of co-integration between palladium and gold. Lucey and Tully (2006) focus alone on the co-integration between gold and silver. They find a strong relationship between those two metals but also argue that "there are significant periods when it is weakened or broken". Excluding for example the time period of the silver bubble around 1980 and only focusing on the time period of 1982 to 1998, the Johansen test for the quarterly dataset also strongly supports the view of co-integration of gold and silver.

Focusing on the co-integrating relationship of precious metal prices to selected factors, Batten et al. (2014), among others, focus on the gold-inflation relationship. They do not find co-integration for their time period but argue that the relationship is very time-sensitive. Analyzing the quarterly price levels since 1969, the finding is confirmed in this study as co-integration is not observable for the whole period. Sub-periods such as the time from 1990 to 2015, however, do show a co-integrating relationship at a 99% significance level.

Table 2 illustrates the results of the Johansen test when all precious metals and the whole time period are taken into account. The trace and the eigenvalue test reject the hypothesis that there are no co-integrating relationships among the time series. Both tests also fail to reject the hypothesis that there is at most one co-integrating relationship. We can therefore conclude there is a single co-integrating relationship among the precious metals.

# of co-integrating	trace	critical			eigen	critical		
relationships	statistic	90%	95%	99%	statistic	90%	95%	99%
<=0	49.4	44.5	47.9	54.7	25.9	25.1	27.6	32.7
<=1	23.5	27.1	29.8	35.5	14.2	18.9	21.1	25.9
<=2	9.2	13.4	15.5	19.9	8.2	12.3	14.3	18.5
<=3	1.0	2.7	3.8	6.6	1.0	2.7	3.8	6.6

Table 2: *Tests statistics of Johansen test using 13 lags.*

All results point to separating the analysis for each asset to determine the influence of the factors mentioned in the previous section. Also, the variation in the relationship supports the use of a time-sensitive analysis. Both points motivate the next section, where a Kalman smoother approach is used for the individual analysis of the relationship of a precious metal to specific, pre-determined factors, before comparing those sensitivities via a DTW analysis.

2.4.2 Time-Varying Sensitivity Analysis

The following section describes the results for the Kalman Smoother when only a single factor is used as an input. The smoother is used in a similar way as for example in Chen and Tindall (2013) which determine market sensitivities for hedge funds or Swinkels and Van Der Sluis (2002) who analyze the returns of specific investment funds in a time-varying context. In this study, the Kalman smoother is taken to determine sensitivities for precious metals and (economic) explanations are tried to be given wherever suitable. The figures contain the development of the sensitivity as well as the 95% upper- and lower-bound confidence interval which are plotted as dotted lines. Factors are not further discussed if sensitivities are less clear or of less interest.

The section intentionally uses words such as "relationship" or "sensitivity" and does not focus on causality as for example in a Granger causality framework. Even though some plausible economic interpretations are given which hint at a causality such as "factor A implies a price move of precious metal B", it should be noted that a third parameter influencing simultaneously the factor as well as the precious metal price may exist.

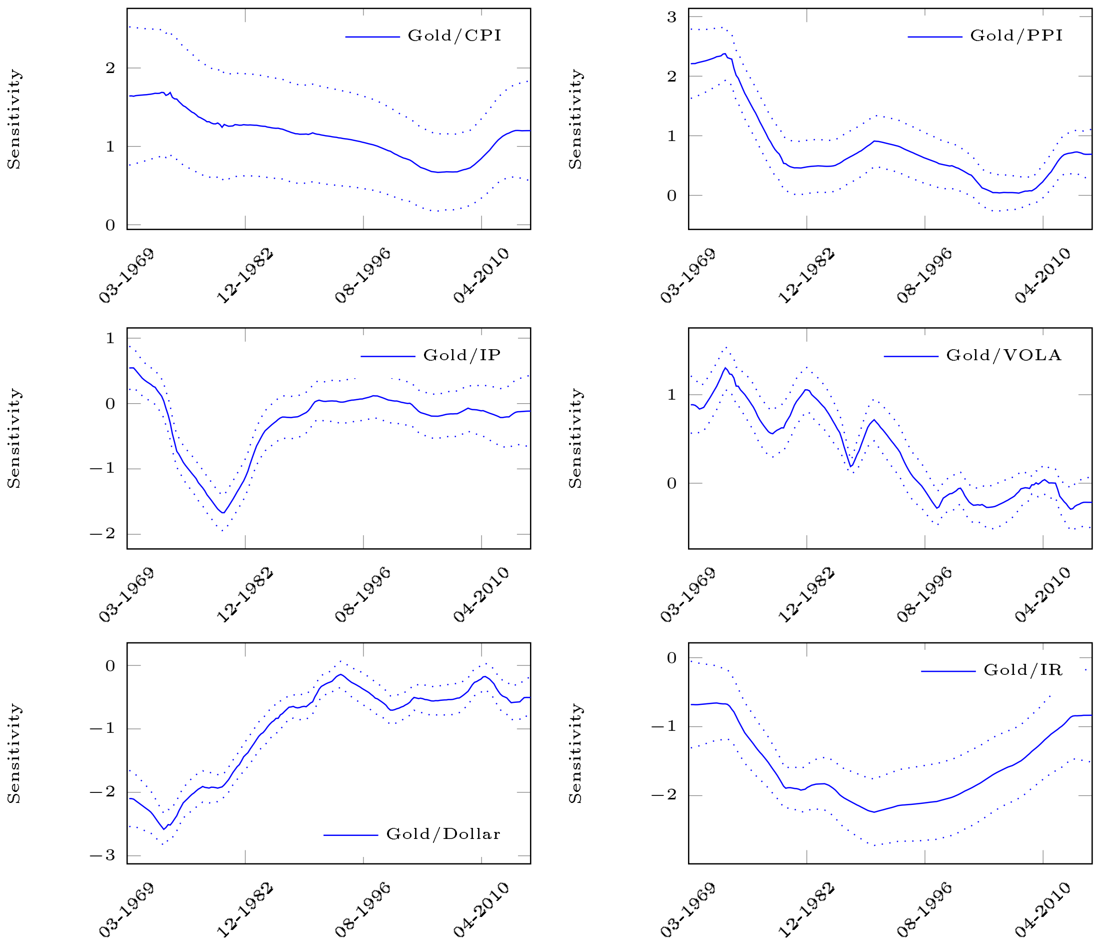

Figure 5: *Sensitivity of gold to selected factors using the Kalman smoother.*

2.4.2.1 Results for Gold Figure 5 illustrates that the sensitivity of gold towards the CPI is positive for the whole time period, although decreasing over time and moving up slightly only in recent years again. The results are similar to those given in Batten et al. (2014) who use a Kalman filter instead of a smoother and whose results are based on the level of the gold price. Although Batten et al. (2014) also show an increase in the sensitivity in recent years, using the smoother and a longer data set in this study indicates that the positive relationship had been even stronger in the high inflation environment of the 1970s. Sensitivities for the PPI are similar, although not significantly different from zero for some periods around 2000.

The sensitivity of the relative price change of gold to the changes in industrial production is not significantly different from zero for the last 25 years. Around 1980, however, we see a negative relationship that arises due to weak industrial production at that time combined with the above mentioned spike in precious metal prices. A general economic explanation is probably difficult to find and is due to the price bubble in precious metals around that time. It has been mentioned already that gold is not used in the same amount as platinum or palladium in industrial production process, which could be a possible explanation for the lack of relationship between those time series.

The sensitivity of gold to realized equity volatility is positive in the first half of the data set and around zero for the second half. At first glance this seems to be surprising, as gold is nowadays often seen as a hedge during market turmoil. Looking at the data more closely, though, explains the absence of a positive relationship: the correlation for gold and equity volatility is positive in the first half of the data set but slightly negative in the second one. Furthermore, looking at specific events such as the stock market crash in 1987 or 2002, no major (upward) moves of gold can be identified in the quarterly data. A possible explanation of the decreasing sensitivity could be the increased usage of other instruments such as options or volatility futures to hedge equity positions in times of market turmoil, decreasing the demand for gold.

The relationship between gold and the dollar index is also plotted in Figure 5. The negative sensitivity illustrated there supports the finding of Akram (2009), although the strength of that link found in this study has been decreasing since the 1970s. The same negative relationship between the price of gold and the real interest rates is also illustrated in the figure above. The results confirm the assumptions from above as well as the results from Akram (2009), who found that negative relationship of interest rates to a broader commodities index.

2.4.2.2 Results for Silver Figure 6 illustrates the results for silver. As mentioned above, it should be kept in mind that the market distortions of the price around 1980 were very high, thereby significantly influencing the results of the analysis. The sensitivities for example of the CPI or PPI around that time were negative and only back in positive territory in later years again, thereby matching the results of the case of gold. The influence of the price moves in 1980 is even more dominant when analyzing the sensitivity to the industrial production or the Dollar. For the former one, for example, a strong negative sensitivity can be observed at that time. This finding is not observable during other time periods. An economic rationale for the negative sensitivity is difficult to find. The negative relationship to the real interest rates as well as nominal bond yields, however, is constantly negative and thereby identical to the case of gold.

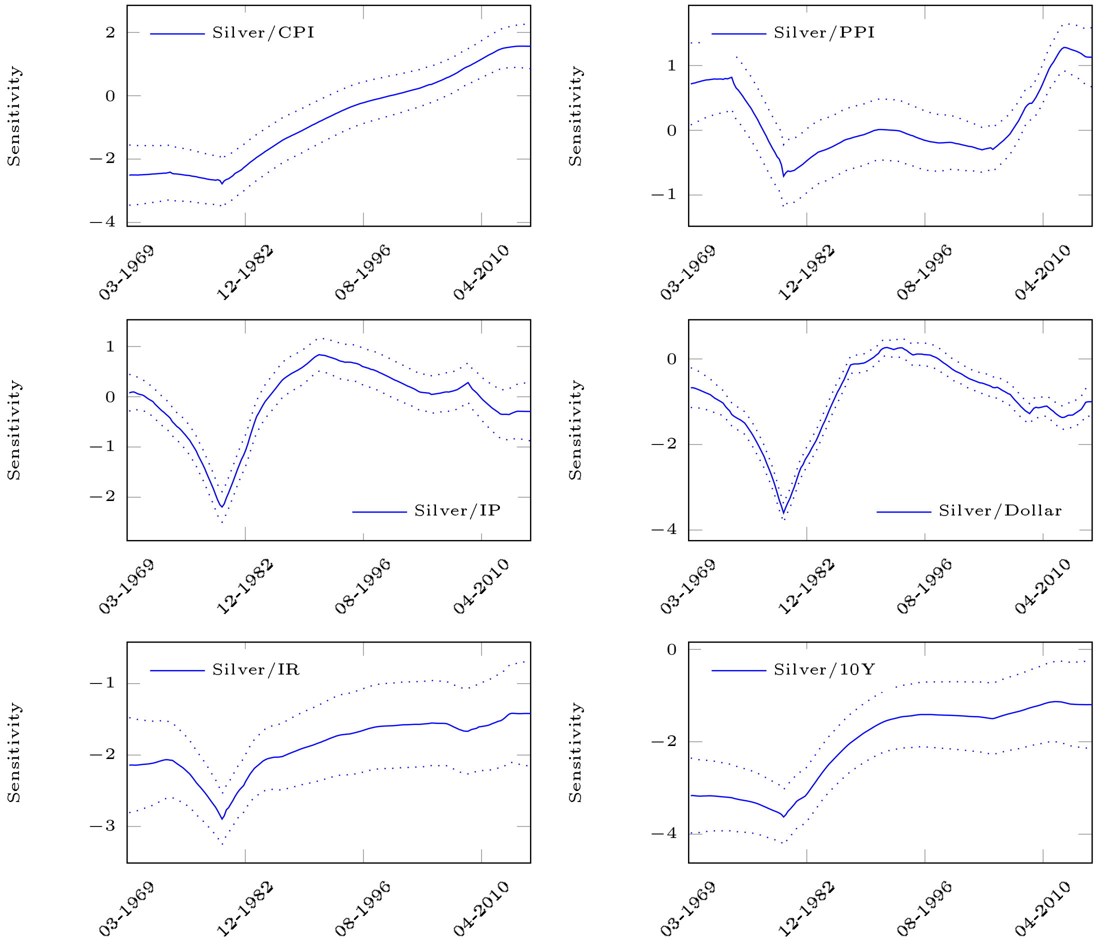

Figure 6: *Sensitivity of silver to selected factors using the Kalman smoother*

2.4.2.3 Results for Platinum Sensitivities for platinum as well as palladium are less distinct compared to the results above. The positive relationship between platinum and the CPI is increasing supporting the results of Bilgin et al. (2018) who find platinum to be the most effective hedge against inflation among white precious metals. The sensitivity to the industrial production has turned positive after being negative around 1980, which might be explained by an increased usage of platinum in different industries such as the mobile or through catalysts in the automobile sector. This led to a higher demand and therefore higher prices followed by the increasing sensitivity of platinum to the industrial production factor. This statement is supported by the results for the other precious metals, as especially palladium, in contrast to gold or silver, offers a very similar sensitivity pattern to the IP. Similar to palladium, platinum has also been used in the automobile industry and is less so used as an investment vehicle. The relationship to the dollar is also negative, although changing significantly over time. Sensitivity

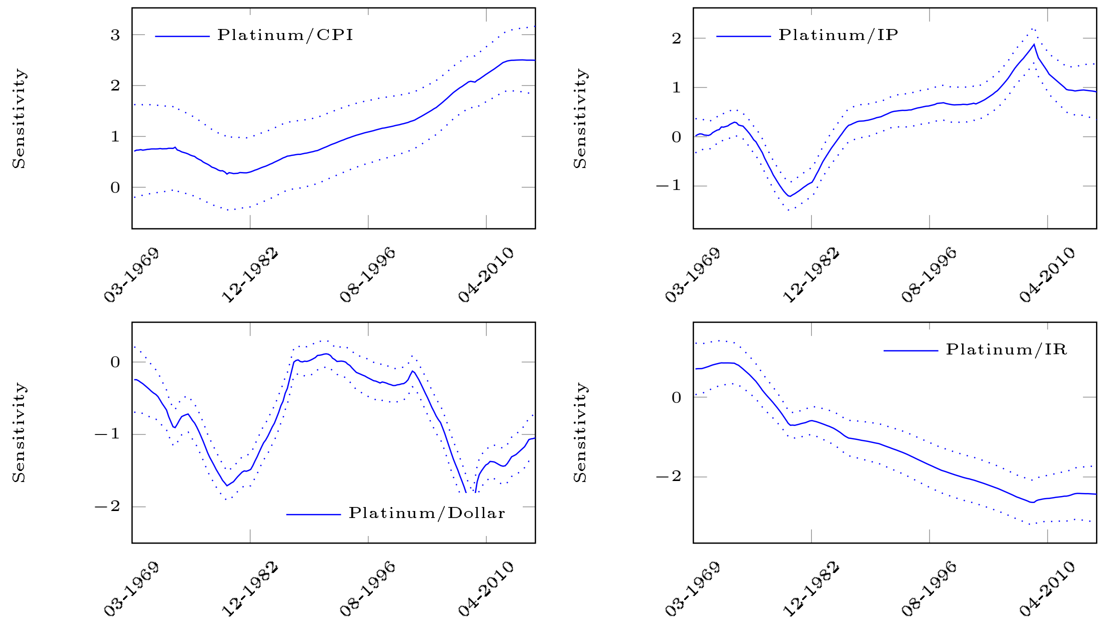

Figure 7: *Sensitivity of platinum to selected factors using the Kalman smoother*

to the interest rates is, equivalently to the other precious metals, negative as well. As mentioned above, the lack of dividends of precious metals might be taken as an explanation.

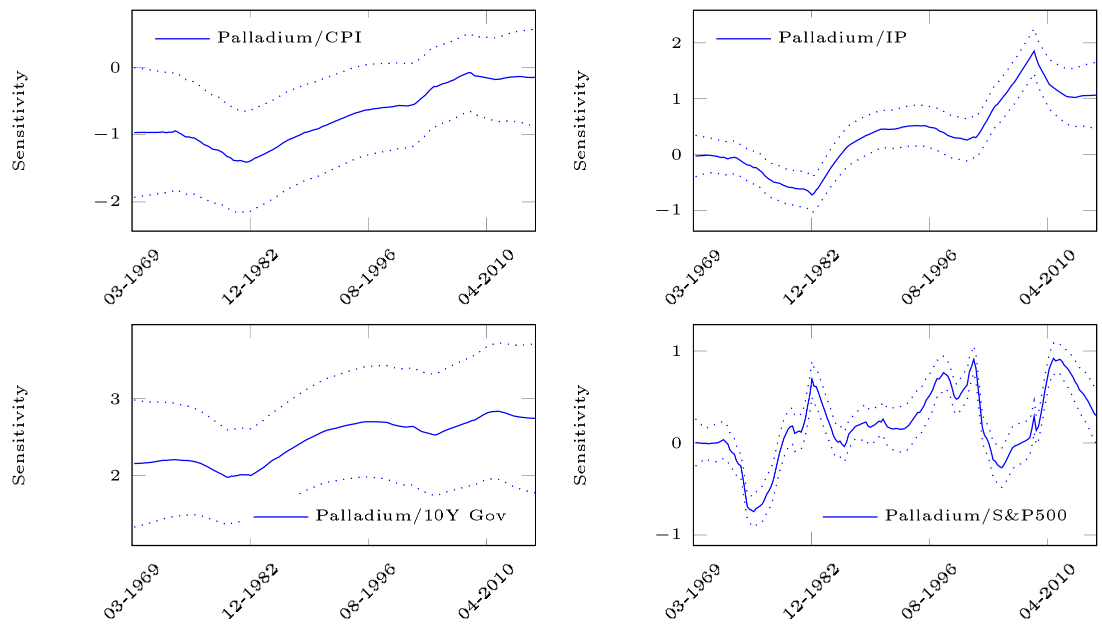

Figure 8: *Sensitivity of palladium to selected factors using the Kalman smoother.*

2.4.2.4 Results for Palladium Bräuninger et al. (2013) argue that palladium follows a different trend than other precious metals as the production of palladium is more flexible. The results found in their work could be taken as an explanation why the sensitivities, calculated via the Kalman smoother approach, also often show different patterns in this study. First, Figure 8 shows that the price of palladium has a negative relationship to the CPI, although this sensitivity has more or less been neutralized in the last years. A possible explanation for the equality in the characteristics of the sensitivities of platinum and palladium to the industrial production has already been given above. The relationship to the nominal 10 year bond yield is even positive and by that different than for other assets. The sensitivity of palladium to the S&P500 is less clear. This, however, is true for the other precious metals as well, as no clear and consistent results can be found.

2.4.2.5 Simultaneous analysis of sensitivities The analysis has so far focused on just including one factor at a time to better identify the influence of each time series. The following brief section wants to apply the Kalman smoother in the multi-factor case. Figure 9 shows the results for gold, the figures for silver, platinum and palladium are also illustrated here. The same factors for gold are taken as those analyzed above (CPI, IP, Vola, Dollar and IR) with the exception of PPI which is excluded due to the high correlation to the CPI. The results exhibit similar patterns as above with the CPI showing a constantly high positive relationship and the dollar a negative, but in absolute value decreasing, sensitivity to gold.

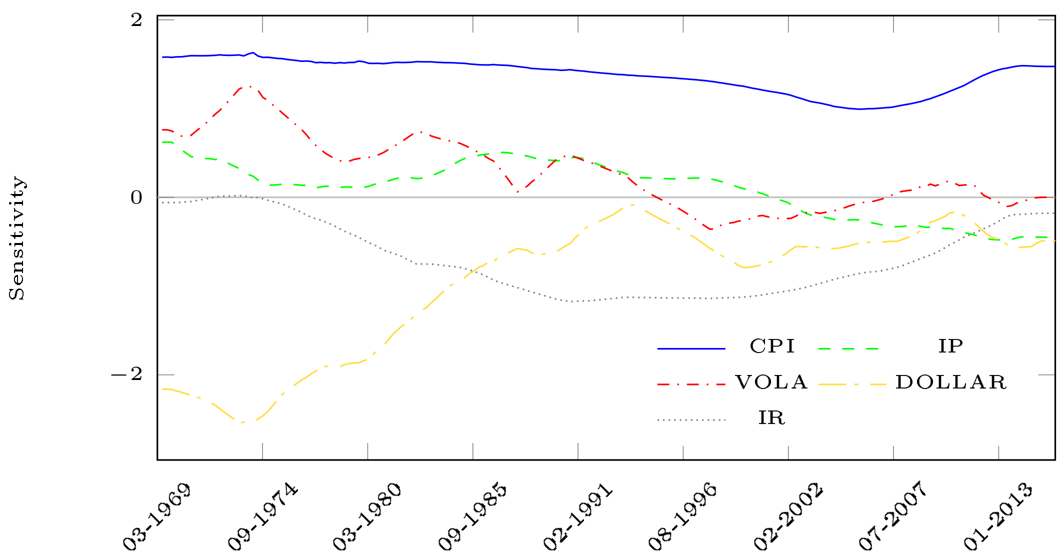

Figure 9: *Simultaneous sensitivity analysis for gold.*

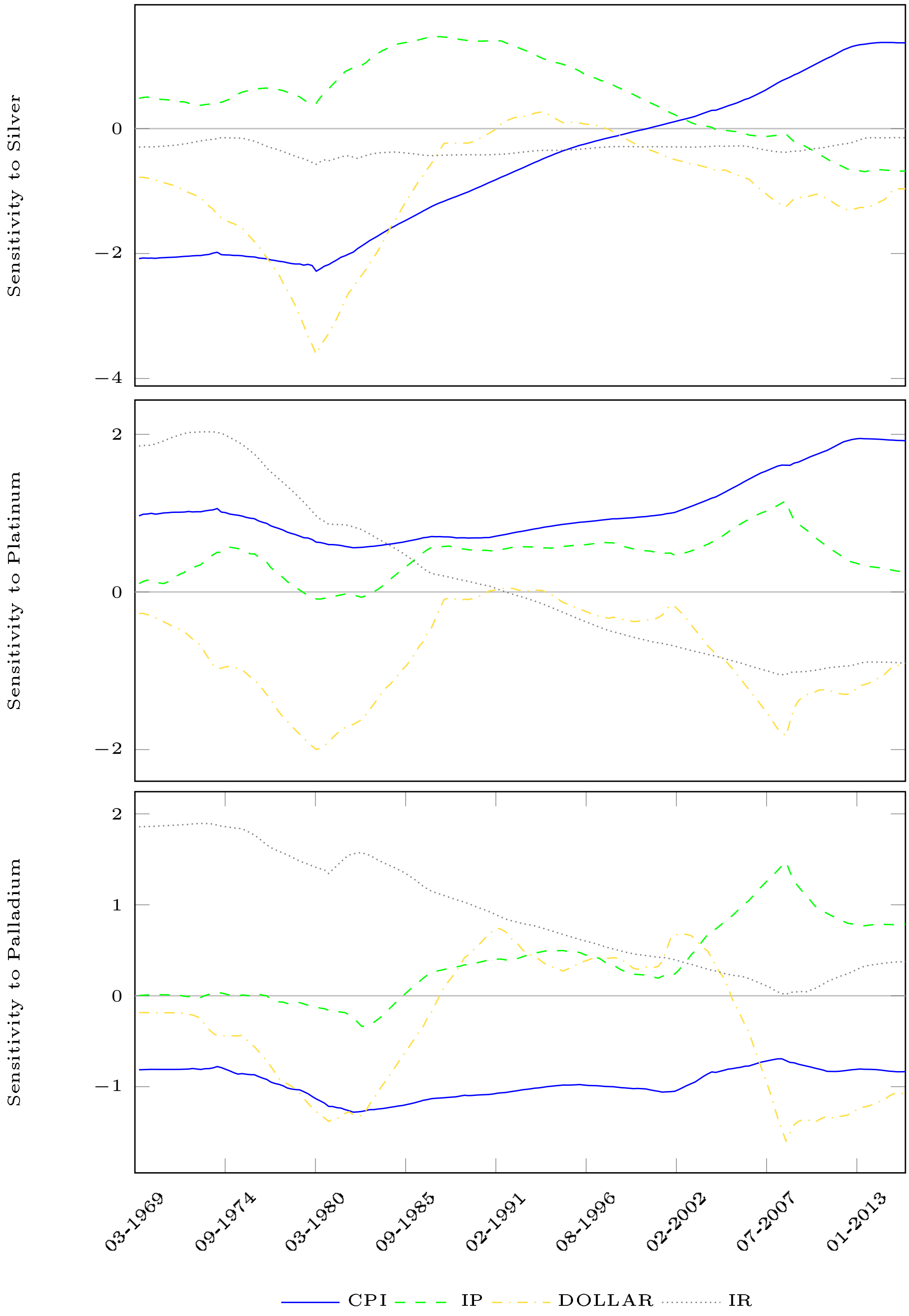

Figure 10: *Simultaneous sensitivity analysis of silver (top), platinum (middle) and palladium (bottom).*

2.4.3 Dynamic Time Warping results

For a deeper analysis of the time-varying sensitivities, Batten et al. (2014) use a linear regression to explain the changes in the sensitivities by other differences in macroeconomic state variables to identify factors that have an influence on the gold to inflation relation. This study also uses the sensitivity results but focuses on the similarities and differences that occur for a factor in relation to each precious metal.

The sensitivity series are first normalized by $x_t/sup_i|x_i|$ where x_t is the sensitivity at time t. By that normalization, which is for example also implemented by Tang and Müller (2009), the sensitivities are compressed to values between -1 and 1. Results are normalized as we are more interested in the changes of the sensitivities. The magnitude is influenced significantly by the input parameters (measurement noise, covariance estimates, etc) for the Kalman smoother and is of lesser importance. The downside, however, is that sensitivities which are low or even non-existent are practically not distinguishable from clearly obvious relationships, making a proper selection of the factors necessary.

Figure 11 illustrates two examples: the left hand side exhibits the normalized sensitivity of gold to the CPI as well as palladium to the CPI. The gap between the two is large, showing especially at the beginning of the time series a completely different behavior of the two metals regarding the CPI. The right hand side shows platinum and palladium in relationship to the industrial production. As mentioned above, both metals show a similar sensitivity pattern, leading to a lower distance between the two sensitivity curves.

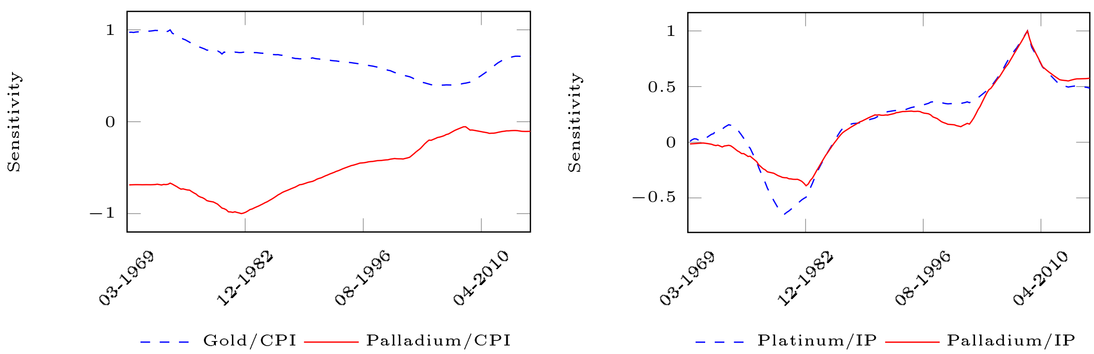

Figure 11: *Normalized sensitivities with huge gap of gold/CPI vs palladium/CPI (left) and similar pattern style of platinum/IP vs palladium/IP (right).*

The results in Table 3 focus mostly on the factors already mentioned above and excludes those where results were not as clear. The values in the table illustrate the DTW distances between the sensitivity of a factor to one precious metal and the same factor to another precious metal. The distance between the normalized sensitivity curve of gold to CPI and that of platinum and the CPI is for example 177. The lower the number in the table, the more the normalized sensitivity for the corresponding precious metals of the pair are therefore alike. The number in brackets illustrate the percentage of the distance relative to the maximum of distance over all pairs for a specific factor. The 79% for the CPI to gold/silver e.g. is calculated by dividing the values of 177 by 225, which corresponds to the maximum distance in the row in form of the gold/palladium to CPI distance. The number is given to easier identify which sensitivity patterns are more alike and which, given a specific factor, are more distinct.

	gold/silver	gold/plat	gold/pall	silver/plat	silver/pall	plat/pall
CPI	177 (79%)	69 (30.9%)	225 (100%)	106 (47.2%)	31 (14%)	182 (81%)
IP	17 (26.4%)	61 (92.5%)	66 (100%)	45 (67.5%)	47 (70.5%)	10 (14.5%)
Vola	53 (77.1%)	56 (81.9%)	69 (100%)	18 (25.7%)	27 (39.4%)	24 (35.5%)
Dollar	27 (37.9%)	46 (63.4%)	72 (100%)	17 (24.1%)	33 (46.3%)	20 (27.9%)
IR	14 (9.6%)	41 (26.8%)	106 (70.2%)	87 (57.3%)	151 (100%)	45 (29.6%)

Table 3: *Resulting distances for each possible pair of precious metals when a dynamic time warping approach is applied to the normalized sensitivities from Section 2.4.2 for a total of 187 data points. The percentages in brackets represent the percent in comparison to the maximum over all pairs for a specific factor.*

The table indicates that the differences between the sensitivities for gold and silver versus the CPI seem to be high, which at first thought might be confusing. Even though one might think that the influence of the CPI on the price of gold and silver might follow the same pattern, the possible explanation of volatile price moves around 1980 has been mentioned above already. Looking at platinum and palladium versus the industrial production, a high similarity can be identified: being substitutes in some industrial production processes has already been given as a possible explanation for that link.

Looking at the other factors, the table illustrates a higher degree of similarity of movement between platinum and palladium. Also, gold and palladium seem to be quite different with having the highest DTW-distances as a pair regarding most factors. Although the values are of limited explanatory power due to the choice of the factors as well as the model parameters of the Kalman smoother, they point, however, in the direction that have been assumed upfront: in tendency a higher similarity between gold and silver on the one hand and platinum and palladium on the other. Different possible explanations for that have already been mentioned above.

2.5 Implications

A first implication for investors that have a specific view for example on the CPI or on the IP could use the analysis to pick a specific precious metal that matches their expectation: investors who expect an increase in the CPI should prefer gold whereas those who expect an increase in the IP should favor platinum.

The DTW results should give an indication to investors on how interchangeable the results are for a given factor. In the case of the industrial production, for example, the investor might choose palladium instead of platinum as both show a positive sensitivity with very low DTW distances. In case the sensitivity for two metals point in the same direction with higher DTW distances, however, the similarities between the two metals might only be temporary.

To illustrate this, the following calculations show the impact on a portfolio consisting of two different precious metals when a specific factor shows a strong upward-/downward move. The aim is therefore to calculate the conditional volatility of a simple precious metal portfolio that consists of two positions with equal weight, given that a factor is "under stress". As a first example, the CPI is taken as a factor and 20 scenarios are identified that exhibit the highest upward moves in the Consumer Price Index. This corresponds to an approximately 10% quantile over all CPI quarterly moves. 19 out of 20 occurrences from the tail are found in the time period of the 1970s or 1980s, a time in which the sensitivities of gold/CPI and palladium/CPI according to figure 11 are quite distinct. The idea is therefore to compare the quarterly volatilities of a portfolio that consists of 50% gold and 50% palladium to a portfolio of 50% silver and 50% palladium, the pair that has the lowest DTW distances to CPI according to Table 3. Table 4 shows the conditional volatilities of both portfolios and compares those to the volatilities for the whole time period. The table illustrates two points: first, the conditional volatility is higher for both portfolios than the volatility that is calculated for the whole period of time. This can be explained by the quarters chosen for the calculations: the time periods that illustrate a significant increase in the CPI seem to be related to general market stress or at least related to higher volatilities in the returns of precious metals in general.

	Volatility all	conditional Volatility
Portfolio1: 50% gold + 50% palladium	11.2%	17.5%
Portfolio2: 50% silver + 50% palladium	14.3%	26.1%

Table 4: *Quartly volatility of the two portfolios when measured over the whole time period as well as only the tail when CPI increases*

The second point related to the table above is the magnitude of the increase: the volatility of portfolio 1 increases less than portfolio 2. This means that when the CPI increases significantly, the portfolio with different sensitivities to the CPI and higher DTW distances regarding the CPI shows a lower increase in the volatility. In this scenario, portfolio 1 seems to offer a better diversification benefit.

Another example is illustrated in the next Table 5: here, the tails for IP decreases are taken. The quarters with high decreases in the IP can be found throughout the time series and of course include for example the time periods around 2008/09 or 2001/02. For this example, gold and palladium is compared to a portfolio of platinum and palladium. Those combinations have again been chosen based on figure 11 and table 3. Table 5 shows again that the volatility of both portfolios in the tail increases and that the increase is again less for portfolio 1 compared to portfolio 2.

	Volatility all	conditional Volatility
Portfolio1: 50% gold + 50% palladium	11.2%	16.0%
Portfolio2: 50% platinum + 50% palladium	13.5%	22.9%

Table 5: *Quartly volatility of the two portfolios when measured over the whole time period as well as only the tail when IP decreases.*

To sum up, it is important for investors to understand how precious metals react and what drives their returns. The two examples illustrate how different portfolios behave when specific factors are "under stress". Nevertheless it should be taken into account that the sensitivities as well as the DTW distances that have been calculated only give a rough estimate and can hardly be taken to determine precise portfolio weights. But if an investor has finally decided to invest into precious metals, it can be easily implemented via Exchange Traded Funds (ETFs). Leung and Ward (2015), for example, analyze the tracking error of leveraged exchange-traded funds on gold, but other standard ETFs for silver, platinum or palladium exist as well.

2.6 Conclusion

Understanding the relationship between external factors and the prices of precious metals is an important as well as challenging task. Important, as the metals are not only used in production or the jewellery industry, but also in finance as an investment vehicle. Challenging, as the nature of the relationship is unclear and possibly changing over time. This paper discusses a linear Kalman smoother approach for gold, silver, platinum and palladium and analyzes their sensitivities to the Consumer and Producer Price Indices, the industrial production, US dollar, equity volatility, real interest rates as well as nominal bond yield and the S&P500 index.

The paper extends the existing research by focusing on factors other than the commonly used inflation. By additionally analyzing factors such as equity volatility, the dollar or the S&P500, it compares the four metals in a time-varying, integrated, approach by using a Kalman smoother. A DTW approach delivers further insight into the different sensitivities. Before that, the time series are analyzed using standard econometric tests including the Johansen test for co-integration between the precious metals.

The first result of the computations are not surprising and motivate the use of a time-varying approach: the sensitivities are not constant as the time for the data set between 1969 and 2015 shows changing characteristics for different time periods. A co-integration between gold and silver, for example, can only be found for a specific time frame and not for the whole period. Sensitivities using the Kalman smoother also exhibit stronger changes in the sensitivity during specific time periods.

Identifying strong relationships between specific precious metals and factors, the results are in accordance to other research as presented for example by Batten et al. (2014) or Akram (2009): there is on average a positive relationship between the CPI (or PPI) and precious metals. Also, a negative relationship between the metals as well as the US-dollar can be identified, meaning that a depreciation of the US-dollar positively relates to a tendency for precious metals to increase in price. The relationship between real interest rates and precious metals tends to be negative, as increasing real rates seem to decrease the price of precious metals. Other connections can also be found: the relationship between equity volatility as an indicator for market fear, for example, has been positively related to the gold price. This relationship, however, has vanished in the last years. Positive relationships can also be identified between platinum and palladium and the industrial production in the last years. Sensitivities to the S&P500 are practically non-existent and those to the nominal bond yields similar to the real bond rates.

The sensitivities of a factor to different precious metals are compared via Dynamic Time Warping. It allows to describe the differences and similarities of a factor to two metals by a single number. Results show a higher degree of similarity between platinum to palladium and gold to silver to selected factors.

Further research could be undertaken in different areas: important for a Kalman filter or smoother approach are the input parameters. Helping to choose the "right" ones are for example described in Naik et al. (2015). Further analyses are also possible by including non-ferrous metals or a general comparison of the results to other, time-varying approaches.

Appendix 2.A Tables and figures

	Level				First Difference			
	lag	model	statistic	p	lag	model	statistic	p
Gold (log)	6	T	-2.76	0.221	5	C	-4.66	<0.001
Silver (log)	14	T	-2.45	0.369	11	C	-3.90	<0.001
Platinum (log)	4	T	-3.10	0.110	7	C	-5.40	<0.001
Palladium (log)	10	T	-3.35	0.062	9	C	-5.77	<0.001
CPI (log)	12	T	-2.33	0.431	11	C	-1.11	0.246
PPI (log)	11	T	-2.92	0.159	10	C	-1.98	0.046
IP (log)	13	T	-1.43	0.850	12	C	-3.34	<0.001
Vola	6	C	-4.42	0.003				
US-Dollar	3	T	-2.98	0.140	8	C	-4.48	<0.001
IR	13	C	-2.35	0.425	12	C	-3.81	<0.001
10Y Yield	14	C	-2.77	0.213	6	C	-6.65	<0.001
S&P500 (log)	0	T	-2.44	0.376	0	C	-9.30	<0.001

Table 6: *Augmented Dickey Fuller test results, lags determined according to Schwert (2002) with maximum number of lags of 14.*

	Level			First Difference		
	trend	statistic	p	trend	statistic	p
Gold(log)	Y	0.17/0.16	0.034/0.040	N	0.16/0.16	>0.1/>0.1
Silver(log)	Y	0.16/0.15	0.040/0.047	N	0.09/0.09	>0.1/>0.1
Platinum(log)	Y	0.11/0.11	>0.1/>0.1	N	0.06/0.06	>0.1/>0.1
Palladium(log)	Y	0.06/0.06	>0.1/>0.1	N	0.03/0.04	>0.1/>0.1
CPI(log)	Y	0.34/0.32	<0.01/<0.01	N	0.93/0.89	<0.01/<0.01
PPI(log)	Y	0.25/0.23	<0.01/<0.01	N	0.51/0.50	0.039/0.042
IP(log)	Y	0.15/0.15	0.043/0.049	N	0.096/0.103	>0.1/>0.1
Vola	N	0.29/0.29	>0.1/>0.1	N	0.034/0.036	>0.1/>0.1
US-Dollar	Y	0.05/0.05	>0.1/>0.1	N	0.07/0.07	>0.1/>0.1
IR	N	0.26/0.25	>0.1/>0.1	N	0.06/0.06	>0.1/>0.1
10Y Yield	N	0.95/0.89	<0.01/<0.01	N	0.19/0.19	>0.1/>0.1
S&P500(log)	Y	0.16/0.15	0.036/0.043	N	0.11/0.11	>0.1/>0.1

Table 7: *KPSS test results, lags determined according to Kwiatkowski et al. (1992).*

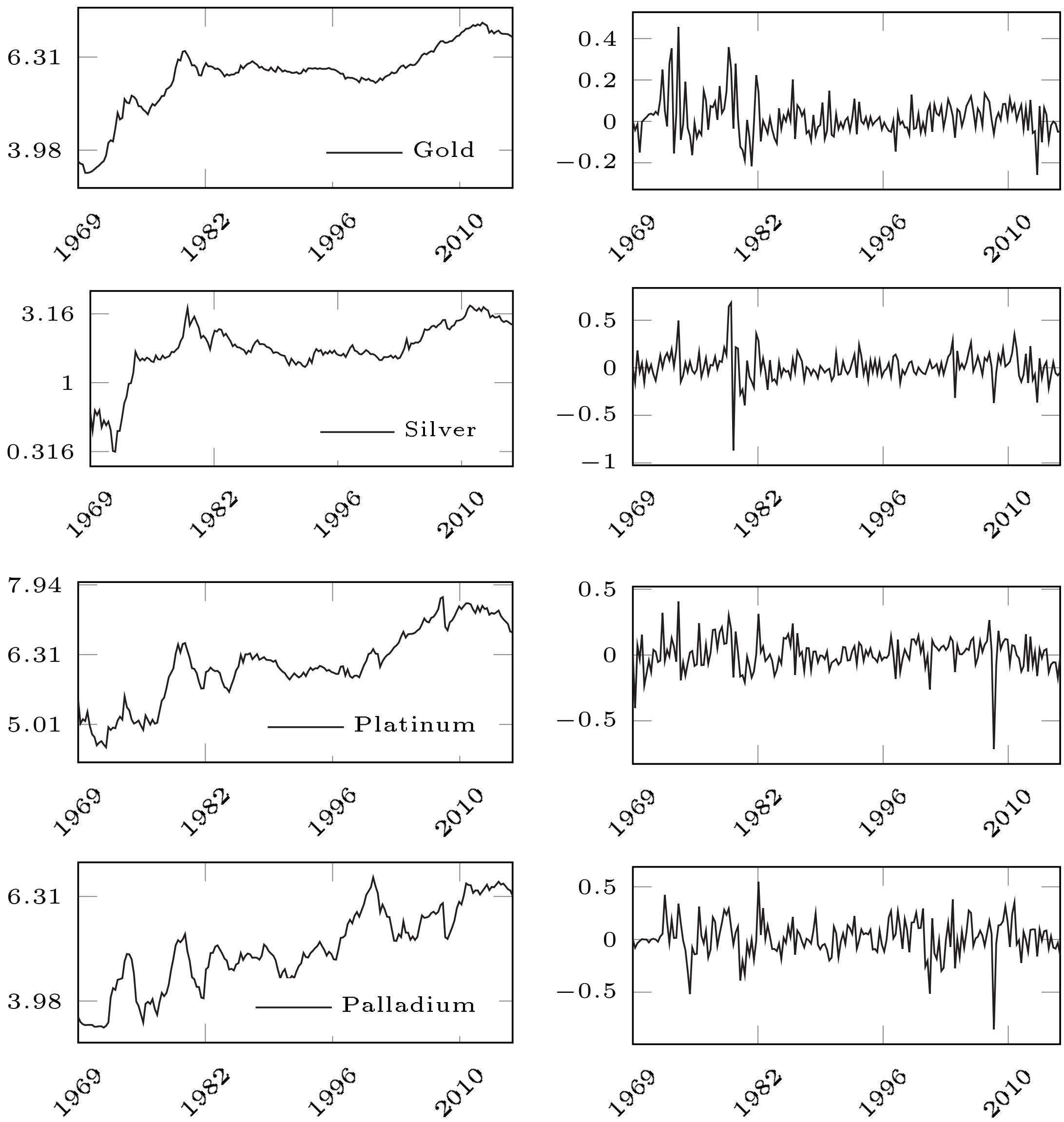

Figure 12: *Log Prices of gold, silver, platinum and palladium (left) and the first difference of the log prices (right) since 1969.*

	Gold	Silver	Platinum	Palladium	CPI	PPI	IP	Vola	US-Dollar	IR	10Y Yield	S&P500
Gold		0.64	0.52	0.33	0.18	0.25	-0.06	0.12	-0.44	-0.28	-0.09	-0.05
Silver	0.64		0.61	0.47	0.08	0.17	0	-0.11	-0.27	-0.19	0.02	0.15
Platinum	0.52	0.61		0.64	0.12	0.24	0.13	-0.14	-0.32	-0.14	0.11	0.19
Palladium	0.33	0.47	0.64		0.02	0.14	0.14	-0.1	-0.09	-0.06	0.12	0.16
CPI	0.18	0.08	0.12	0.02		0.75	0.14	-0.11	0.02	-0.13	0.34	-0.1
PPI	0.25	0.17	0.24	0.14	0.75		0.23	-0.13	-0.15	-0.17	0.38	-0.04
IP	-0.06	0	0.13	0.14	0.14	0.23		-0.06	0.1	0.17	0.18	-0.03
Vola	0.12	-0.11	-0.14	-0.1	-0.11	-0.13	-0.06		-0.14	0.04	-0.33	-0.53
US-Dollar	-0.44	-0.27	-0.32	-0.09	0.02	-0.15	0.1	-0.14		0.27	0.09	-0.06
IR	-0.28	-0.19	-0.14	-0.06	-0.13	-0.17	0.17	0.04	0.27		0.1	-0.08
10Y Yield	-0.09	0.02	0.11	0.12	0.34	0.38	0.18	-0.33	0.09	0.1		0.16
S&P500	-0.05	0.15	0.19	0.16	-0.1	-0.04	-0.03	-0.53	-0.06	-0.08	0.16	

Table 8: *Correlation*

3 Factor Risk Parity with Portfolio Weight Constraints

3.1 Introduction

Recent literature (Poddig and Unger (2012), Roncalli and Weisang (2012), Kind (2013) and Deguest et al. (2013)) focuses on applying the equal risk to contribution approach on risk factors derived through a PCA. Yet, there has been much criticism on the robustness, definiteness and the manageability of such portfolio allocations.

To address the major critical points, we introduce a modified version of a factor risk parity model. The model focuses only on the risk contribution of the main risk factors and, in contrast to the "classical" Factor Risk Parity model, allows the risk contribution of the less important principal components to float to some extent. The key benefits are the opportunity to add asset weight constraints for the portfolio selection while, at the same time, building robust allocations along the main risk factors. By meeting the constraints regarding the asset weights and risk factors, the model describes all possible portfolios as a convex solution set and is thereby able to describe those portfolios efficiently while handling various investors' needs or possible regulatory requirements.

Meucci (2009) is a main contributor to the field of portfolio constructions along principal components. His idea of well diversified portfolio allocations in terms of uncorrelated risk sources is a driving force for this paper. Poddig and Unger (2012) use the approach of Meucci (2009) to compute maximum diversified portfolio allocations by maximizing the entropy based on the risk contributions of all principal components. By that, they determine portfolios with equal risk contributions for all principal components. Using a Monte Carlo simulation and historical data, they show that those portfolios are far from being stable and robust. As the authors find that the strategies perform poorly as well, they argue that practitioners would probably ignore an allocation that aims at equalizing the risk contribution from uncorrelated sources of risk. In particular, the criticism on the robustness of those principal portfolios described in the literature is justified. We will show in this chapter why portfolio allocations are unstable when solving the factor risk parity problem using only a numerical optimization and taking all principal components into account.

Moreover, we address the detailed and technical criticism on portfolios built along principal components formulated by Meucci et al. (2014). First, they argue that principal components with the lowest eigenvalues tend to be unstable. That is something we will not contradict. Our model, however, will focus only on the first m principal components which explain most of the portfolio variance. By not taking the components with lower eigenvalues into account, we bypass the instability in those principal components. This is also an advantage as, in the authors' view, the meaning of those components change more quickly in a dynamic allocation setting. Second, when using principal components, a criticism is that the eigenvectors are not unique. Meucci et al. (2014) conclude that 2^n possible solutions exist. We will show that in the factor risk parity framework in this paper with n assets and m equalized risk factors, only 2^{m-1} solutions exist. Out of that set, only one solution that meets another desired criterion is chosen. Due to all those deficiencies they suggest an approach based on minimum torsion bets whereas our model still sticks to a PCA with modifications in the risk parity approach. The model works with equal risk contributions for an arbitrary number of risk factors, but a focus is put on the first two principal components, thereby following the discussion of Bhansali et al. (2012) which argue that the first two principal components explain a significant amount of variation in a multi-asset setup.

This work contributes to all the other works in the field of factor risk parity or principal portfolio allocation (see Meucci (2009), Bhansali et al. (2012), Lohre et al. (2012), Kind (2013),Bernardi et al. (2018) and Deguest et al. (2013)). To the best of our knowledge, the model described in this paper is not covered in literature so far.

The main advantage of using the model described in this paper is the flexible and simultaneous consideration of different constraints for a different number of asset classes while considering the risk contributions of the core underlying risks. It opens the possibility to test different constraints regarding the asset weights or the level of explanation of the risk contributions of the main risk factors. It thereby forms a development of the well-known equal risk to contribution (ERC) approach. The advantages of the factor risk parity versus a classical risk parity approach, such as the focus on the main risk drivers or avoidance of the duplication invariance problem (see Choueifaty et al. (2013)), stay also valid.

We know that the PCA as a tool of risk source separation is not free of criticism. One major problem that remains and that cannot be eliminated is the interpretation of the principal components. Interpreting risk factors derived by blind source separations is rather a task of macroeconomic research than of quantitative portfolio construction. Another problem is the requirement of a PCA that data is (nearly) normally distributed, which some critics may doubt when handling financial time series. The same issue comes up when using the standard deviation as the risk measure. This may not be the best choice, particularly in the light of possible skewed distributions. Finally, depending on the structure of the historical data as well as the number of equal risk components, solutions might not exist. We think that the model and the way we construct and solve the allocation problem offers the possibility for further research in the field of factor risk parity.

The chapter is organized as follows: Section 3.3 discusses the basic naive and heuristic allocation methods which serves, in the risk parity case, as the theoretical basis. We believe that these allocations directly compete with the factor risk parity allocation and we will take these allocations into account within the empirical section. In Section 3.4, we introduce the factor risk parity framework and provide some examples. After the data description in Section 3.5, some robustness checks are done in the following section before performing empirical analysis on our factor risk parity model. We backtest this model in Section 3.6 as well as the other naive and heuristic models using the same dynamic backtest approach. Some proofs, tables and figures are provided in the Appendix.

3.2 Related literature

The mean-variance framework described first by Markowitz (1952) forms the basis for the Modern Portfolio Theory. Today strategic asset management is sometimes still based on his concept. Yet, as commonly known, implementing the mean-variance optimization is a tough challenge due to estimating the expected returns and covariances of the assets. Additionally, the Markowitz model is highly sensitive to changes in the input data and works only under very special conditions. In his work Michaud (1989) calls this effect "error maximization". Regardless of the problems related to the application of the mean-variance optimization, one of the most important ideas in asset management is strongly associated with this model: Diversification.

Since Markowitz published its fundamental work, much effort has been put into fixing the estimation problems or developing alternative asset allocation models against the backdrop of diversification. The simplest and often discussed alternative is the naive 1/N allocation. Under the assumption of independent and identically distributed returns, the 1/N allocation is optimal in the mean-variance framework.

Recent works compare the 1/N allocation with different mean-variance optimizations. DeMiguel et al. (2009) evaluate the out-of-sample performance of 14 different mean-variance models and the 1/N allocation. For seven real world data sets and one simulated data set, they find that no single model beats the 1/N allocation consistently over time. For diversified equity portfolios and single stocks Brown et al. (2013) give an explanation why mean-variance allocations are not able to outperform a naive 1/N allocation. They argue that the outperformance of 1/N allocation against optimal allocations is paid by increasing tail risks and a reduced upside potential. The recent work by Platen and Rendek (2012) supports the results by DeMuguel et al. and formulates the Naive Diversification Theorem which states that the 1/N allocation is a good proxy for the numéraire portfolio under specific conditions. Of course, a number of other works support the naive 1/N diversification strategy.

Yet, there is a significant number of recent papers neglecting the 1/N strategy in terms of the out-of-sample performance comparing to other strategies. For example, the recent work by Abankwa et al. (2013) claims the 1/N approach is outperformed by sophisticated mean-variance optimization using dynamic conditional correlations (DCC) in out-of-sample test in terms of Sharpe Ratio. Using a simple VAR(1) model for the mean and variance forecast, Caporin and Pelizzon (2012) achieve similar results.

Another well known benchmark allocation strategy is the very special case of Markowitz's mean-variance optimization, the minimum-variance portfolio. Given identical expected returns for each asset class, the minimum-variance portfolio is the only optimal portfolio under the Markowitz framework. Certainly, this strategy will lead to very concentrated portfolio allocations on low risk assets. Even though returns are not the same in the real world, the minimum-variance idea attracted a lot of interest by researchers since Markowitz proposed its mean-variance framework due to the fact that the expected value of future asset returns is irrelevant for the portfolio construction. Particularly in equity portfolio construction, this approach was successful in both research and practice, due to almost identical expected returns on stocks. Nonetheless, allocating multi-asset portfolios along the minimum-variance strategy will lead to heavy fixed-income portfolios.

Much research nowadays is still being done on those portfolio construction approaches, even though they are simple and have been developed a while ago. Jiang et al. (2019) for example try to combine the minimum-variance with the 1/N portfolio and find that it is possible to enhance the Sharpe ratio as well as reduce the risk with this portfolio if short-selling is allowed.

The risk parity allocation is a newer approach designed to fill the gap between equal weighted and minimum-variance allocations. For none of them there is a need to estimate the future expected returns of the asset classes within the risk parity heuristic. Just as the minimum-variance optimization, risk parity only needs a semi-definite covariance matrix. Comparing the risk parity heuristic with Markowitz's optimization, risk parity is optimal under the mean-variance framework if all asset classes have the same correlations and Sharpe Ratios (Chaves et al., 2011).

We use the approach in which the risk contribution from each portfolio component, that can be defined as a stock, another asset or an asset class, is equalized within the portfolio (ERC), with risk defined as the standard deviation of returns. For this approach, we will refer mostly to the works of Qian (2005), Neukirch (2008) and Maillard et al. (2008). The total portfolio risk with variance as the risk measure is partitioned into the risk contribution (RC) of each portfolio component, which means that the sum of all risk contributions is equal to the portfolio variance. In terms of the risk-profile of this heuristic allocation, the risk parity to contribution approach can be positioned between the $1/N$ naive allocation and the minimum-variance portfolio. In recent time, a lot of research focused on the field of ERC portfolios. Maillard et al. (2008) compare the ERC strategy with the minimum-variance portfolio as well as the $1/N$ strategy. They examine the ERC strategy at the theoretic level given different special cases, such as equal correlation or equal variances, compared them and showed the mathematical link between these three asset allocation strategies. Moreover, they compare the three strategies for three different data sets as well as for a numerical example. Chaves et al. (2012) introduce two different computing methods for the ERC problem and took again a closer look at the special cases of the ERC strategy where assets are uncorrelated. Moreover, they show the efficiency of the ERC strategy in terms of risk diversification.

Bai et al. (2016), in their recent work, provide a comprehensive theoretical overview of the general long-short ERC. They show that the ERC problem can be formulated as a convex optimization problem with a unique solution for the long-only case. For the generalized long-short case, they demonstrate that multiple solutions exist and gave advice how investors should deal with those solutions. Additionally, Bai et al. (2016) discuss different numerical optimization methods for solving the ERC optimization problem efficiently. Complementary work was published by Griveau-Billion et al. (2013). They develop a fast algorithm for high-dimensional covariance matrices with $n > 500$.

Cesarone and Colucci (2018) also compare the ERC strategy to other portfolio construction processes, namely the minimum variance as well as the maximum diversification approach. They highlight the strengths and weaknesses when the strategy is applied to seven different investment universes.

Ardia et al. (2018) include the performance contribution in a risk-based portfolio construction process. They design a *"Performance/Risk Contribution Concentration" (PRCC)* measure to set up portfolios that do not deviate much regarding the performance and risk contribution while at the same time staying close to a risk-based portfolio benchmark.

3.3 Naive and heuristic allocation strategies

As part of the theoretical foundation of this paper, we start by defining the benchmark allocations. The $1/N$, minimum-variance and the equally-weighted risk contribution allocation strategies are the subject of numerous recent research papers. Moreover, the risk parity to contribution strategy is also a pillar of the factor risk parity model described in a later section. Hence, the naive $1/N$ and the minimum-variance allocation are solely used as benchmark strategies for the empirical Section 3.6. The risk parity strategy serves, in addition, as an introduction to the factor risk parity model described in Section 3.4.1.

For all allocations in this chapter let $x \in \mathbb{R}^{k \times n}$ with $k > n$ be a real matrix where $x_{i,j}$ describes the percentage price change of an asset $i \in \{1, \ldots, n\}$ at time period $j \in \{1, \ldots, k\}$ [1]. The real covariance matrix $\Sigma \in \mathbb{R}^{n \times n}$ of all asset price changes is symmetric and assumed to be positive semi-definite. The entire paper focuses on the fully invested long-only case

$$\sum_{i=1}^{n} w_i = 1, \tag{7}$$

$$w_i \geq 0, \tag{8}$$

where $w = [w_1, w_2, \ldots, w_n]^T \in \mathbb{R}^n$ defines the vector of asset weights of a portfolio P. Given matrix x and vector w, the percentage price changes of the corresponding portfolio P can be written as

$$P := x\, w \in \mathbb{R}^k. \tag{9}$$

For all following investment strategies described in this section and backtested in Section 3.6.3, a two-step approach for computing the out-of-sample weights for each asset is used. First we take all the time series for one asset class and apply the strategy to those assets only. The

[1] The model will use data in $\mathbb{R}$ for asset price changes as we will also use historical data for the backtest later. Random variables are only used sometimes when we refer to other work or papers.

result is a new time series which belongs to that specific asset class. We repeat the process for all asset classes to finally get a new time series for each asset class. Finally, we use the same strategy again and apply it to all the new time series we calculated for each asset class. The reason for using this approach is in the following simple example: if we use the $1/N$ approach directly with four equity indices and only one bond index, we would get an $\frac{4}{5} = 80\%$ equity weight versus a $\frac{1}{5} = 20\%$ bond weight. With the two-step approach, we first build a new time series for the equities to receive a final 50%/50% weighting for equities and bonds.

3.3.1 Equal-weighted allocations

The $1/N$ allocation is the simplest naive diversification an investor can choose. Under the assumptions of equal expected value E for the returns with $E(x_i)$ the expected return of asset i, equal volatility in the returns σ and no correlation

$$\begin{aligned} E(X_1) &= E(X_i) && \forall i \in \{2, \dots, n\} \\ \sigma_1 &= \sigma_i && \forall i \in \{2, \dots, n\} \\ \rho_{k,i} &= 0 && \forall i, j \in \{1, \dots, n\} \text{ and } i \neq j \end{aligned} \tag{10}$$

the $1/N$ allocation is optimal and equal to the market portfolio.

Accordingly, the two-step approach is applied to the equal-weighted strategy. The aim is to achieve a naive $1/N$ allocation in the space of the asset class as well as within the entire portfolio for a different asset classes. Given k_m assets within an asset class m and a asset classes, the asset weights can be calculated by

$$w_i = \frac{1}{k_m \cdot a} \ \forall i \in \{1, \dots, n\}. \tag{11}$$

3.3.2 Minimum-variance allocations

The assumption for the minimum-variance portfolio is less restrictive than for the equal weighted portfolio:

$$E(X_1) = E(X_i) \ \forall i \in \{2, \dots, n\}. \tag{12}$$

Given that all expected asset returns are equal, the minimum-variance portfolio coincides with the market portfolio. Computing the minimum-variance portfolio for the following risk parity approach as well, the *"marginal risk contribution" (MRC)* is defined as the change in portfolio risk σ when the weight of an asset is marginally changed:

$$MRC_i := \frac{\partial \sigma(w)}{\partial w_i} \tag{13}$$

For all calculations in this paper the standard deviation $\sigma(x) = \sqrt{x^T \Sigma x}$ is used as the risk measure. In this case, the deviation can be calculated directly and expresses the marginal risk contribution as

$$MRC_i = \frac{(\Sigma w)_i}{\sqrt{w^T \Sigma w}}. \tag{14}$$

Solving the problem

$$\frac{\partial \sigma(w)}{\partial w_i} = \frac{\partial \sigma(w)}{\partial w_j} \quad \forall i, j \in \{1, 2, \ldots, n\} \tag{15}$$

one obtains the minimum variance (MV) portfolio for the given set of time series within an asset class and for all asset classes in the second computing step.

Following Maillard et al. (2008) the solution for the minimum-variance problem can be written in the following way:

$$\begin{aligned} w^* &= \left\{ w \in [0,1]^n : \frac{\partial \sigma(w)}{\partial w_i} = \frac{\partial \sigma(w)}{\partial w_j} \quad \forall i, j \in \{1, 2, \ldots, n\} \right\} \\ s.t. & \sum w_i = 1 \end{aligned} \tag{16}$$

Using an interior-point optimization of the Matlab function "fmincon", the above-stated problem can be solved numerically under long-only constraints:

$$\begin{aligned} f^* &= argmin \left(\sum_{i=1}^{n} \sum_{j<i} \left(\frac{\partial \sigma(w)}{\partial w_i} - \frac{\partial \sigma(w)}{\partial w_j} \right)^2 \right) \\ & \sum_{i=1}^{n} w_i = 1 \text{ and } w_i \geq 0. \end{aligned} \tag{17}$$

Since we demand the portfolio weights to be positive, the problem of finding the portfolio must be solved numerically. Otherwise a Lagrange multiplier method can be used to calculate the minimum variance portfolio directly.

3.3.3 Risk parity allocations

A combination of both the naive $1/N$ strategy and the minimum-variance optimization leads to the concept of a simple risk parity. The absence of return expectations and zero correlations are basic assumptions for this naive allocation:

$$\begin{aligned} E(X_1) &= E(X_i) \quad && \forall i \in \{2, \ldots, n\} \\ \rho_{i,j} &= 0 && \forall i, j \in \{1, \ldots, n\} \text{ and } j \neq i. \end{aligned} \tag{18}$$

Given these assumptions, the allocation is optimal and no diversification effect is implied due to zero correlations. Equalizing the risk of all assets in the portfolio or, in our case, the risk within an asset class and later the risk of all asset classes in the portfolio, implies that less risky assets receive a higher weight and vice versa. We apply the most widely used concept of risk parity. This means that the risk contribution of an asset to the entire portfolio, and not only the simple risk of an asset, determines the portfolio allocation and thus the asset weights. Therefore non-zero correlations are taken into account:

$$\rho_{i,j} \neq 0 \ \ \forall i, j \in \{1, \ldots, n\} \text{ and } i \neq j. \tag{19}$$

The risk parity portfolio is based on the concept of risk contribution of a single asset to the portfolio, which is the marginal risk contribution weighted by the corresponding weight

$$RC_i := w_i \ MRC_i. \tag{20}$$

With the standard deviation as the risk measure and Σ as the covariance matrix of x, the risk contribution can be calculated directly as:

$$RC_i = w_i \ \frac{(\Sigma w)_i}{\sqrt{w^T \Sigma w}} \tag{21}$$

With these definitions the actual risk parity problem can be formulated as:

$$\begin{aligned} & w_i \frac{\partial \sigma(w)}{\partial w_i} = w_j \frac{\partial \sigma(w)}{\partial w_j} \ \ \forall i, j \in \{1, 2, .., n\} \\ & s.t. \sum_{i=1}^{n} w_i = 1, \ w_i \geq 0. \end{aligned} \tag{22}$$

The following theorem shows that summing up all the risk contributions leads to the portfolio volatility (see Roncalli and Weisang (2012) for details):

Euler decomposition

For positive homogeneous functions of the degree one, Euler's theorem states that these functions satisfy the following equation:

$$f(x) = \sum_{i=1}^{n} x_i \left(\frac{\partial f(x)}{\partial x_i} \right) \tag{23}$$

As a consequence, it follows for the risk parity problem under a convex risk measure

$$\sigma(w) = \sum_{i=1}^{n} \left(w_i \frac{\partial \sigma(w)}{\partial w_i} \right). \tag{24}$$

Given Eulers' theorem, the portfolio risk can be written as the sum of the assets' risk contribution to the portfolio:

$$\sigma_P = \sum_{i=1}^{n} RC_i. \tag{25}$$

The solution for the risk parity to contribution problem can be written as follows:

$$w^* = \left\{ w \in [0,1]^n : \sum w_i = 1,\ w_i \frac{\partial \sigma(w)}{\partial w_i} = w_j \frac{\partial \sigma(w)}{\partial w_j} \quad \forall i,j \in \{1,2,..,n\} \right\}. \tag{26}$$

The problem of finding the weights w that satisfy the conditions in (22) can be solved numerically using the same Matlab optimization as described for the minimum-variance problem (see Maillard et al. (2008)):

$$\begin{aligned} &f^* = argmin \left(\sum_{i=1}^{n} \sum_{j<i} \left(w_i \frac{\partial \sigma(w)}{\partial w_i} - w_j \frac{\partial \sigma(w)}{\partial w_j} \right)^2 \right), \\ &\sum_{i=1}^{n} w_i = 1 \text{ and } w_i \geq 0. \end{aligned} \tag{27}$$

As to the other two allocations, the two-step approach is adopted for the risk parity allocation in the same way as for the minimum variance optimization.

The risk parity concept described up to this point deals with correlated assets. In the case of uncorrelated time series x, the problem of finding the optimal risk parity portfolio weights is simplified. Covariances do not play a role in calculating the asset weights given zero correlations. For σ_i^2 as the variance of time series i and given uncorrelated time series, we can write the correlation and covariance matrix as

$$\rho_x = diag\,[1,\ 1,\ 1, \ldots,\ 1], \tag{28}$$

$$\Sigma = diag\left[\sigma_1^2,\ \sigma_2^2,\ \sigma_3^2, \ldots,\ \sigma_n^2\right], \tag{29}$$

and the risk contribution from equation (21) simplifies to

$$RC_i = \frac{w_i^2\ \sigma_i^2}{\sigma_P}. \tag{30}$$

Under these assumptions, the portfolio variance can with the use of equation (25) be written as the simple sum of squared asset weights multiplied by the assets' variances:

$$\sigma_P^2 = \sum_{i=1}^{n} \left(w_i^2 \, \sigma_i^2 \right). \tag{31}$$

The risk parity problem under long-only constraint is then described by a system of $(n-1)$ linear equations.

$$\begin{aligned} w_1 \, \sigma_1 &= \sigma_2 \, w_2, \\ w_1 \, \sigma_1 &= \sigma_3 \, w_3, \\ &\cdots \\ w_1 \, \sigma_1 &= \sigma_n \, w_n. \end{aligned} \tag{32}$$

The constraint $\sum_{i=1}^{n} w_i = 1$ leads to the closed form solution:

$$w_i = \frac{\sigma_i^{-1}}{\sum_{j=1}^{n} \sigma_j^{-1}}. \tag{33}$$

Again, the simplified risk parity problem can be written mathematically as follows:

$$w^* = \left\{ w \in [0,1]^n : \sum w_i = 1, \; w_i \, \sigma_i = w_j \, \sigma_j \;\; \forall i,j \in \{1, 2, \ldots, n\} \right\} \tag{34}$$

A comprehensive theoretical overview of the general long-short ERC can be found in Bai et al. (2016). Additionally, they discuss different numerical optimization methods for solving the ERC optimization problem efficiently.

3.4 Factor risk parity allocations

3.4.1 Principal component analysis

The motivation for using a PCA for constructing risk parity portfolios is straightforward: instead of focusing on equal risk contributions concerning each asset, we analyze the underlying factors that drive the performance of the portfolio. The idea is to balance the risk regarding the most important factors which explain the majority of the portfolio variation. Furthermore, well-balanced portfolios usually consist of a larger amount of asset (classes) which are highly correlated. PCA focuses on the underlying risk drivers, thereby determining the relevant dependencies beneath the asset movements, and is independent of the number of asset (classes) selected.

As above, let $x = (x_{i,j}) \in \mathbb{R}^{k \times n}$ with $k > n$ be a matrix with all asset returns and let $C \in \mathbb{R}^{n \times n}$ be a matrix that is still to be determined. n is again the number of assets and k the number of time periods. The goal is to linearly transform the data x by multiplying with that matrix C to receive a new set of variables, the principal components:

$$\widetilde{x} = x\, C. \tag{35}$$

This multiplication can be interpreted in different ways (see Shlens (2005)):

1. A transformation of x into $\widetilde{x}$, consisting of rotation and stretching of x into $\widetilde{x}$
2. The rows of C are a set of new basis vectors for expressing the columns of x.

The question now is how to determine matrix C in a proper way, such that we gain some insight into the structure of the data when transforming it?

As mentioned above, the goal is to find the most relevant factors that drive the returns of the assets. Those factors should be orthogonal to each other. Thus, the covariance matrix of $\widetilde{x}$, $\widetilde{\Sigma}$, should be diagonalized:

$$\widetilde{\Sigma} = diag(\lambda_1, \ldots, \lambda_k) \tag{36}$$

with $\lambda_1 \geq \lambda_2 \geq \ldots \lambda_k$. Thus, the eigenvectors from the covariance matrix from x, Σ, can be placed into the columns of C in a descending order (for a more detailed explanation see Shlens (2005)). This is what happens when using a PCA. The steps in the PCA therefore consist of:

1. Calculating the matrix $\Sigma(X)$
2. Finding all eigenvectors p_i: $\Sigma(X)p_i = \lambda_i p_i$
3. Sorting the eigenvalues λ_i so that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k$
4. Taking the eigenvectors p_i and placing them in the columns of C

Figure 13 describes the working principle of the PCA in a two-dimensional setting.

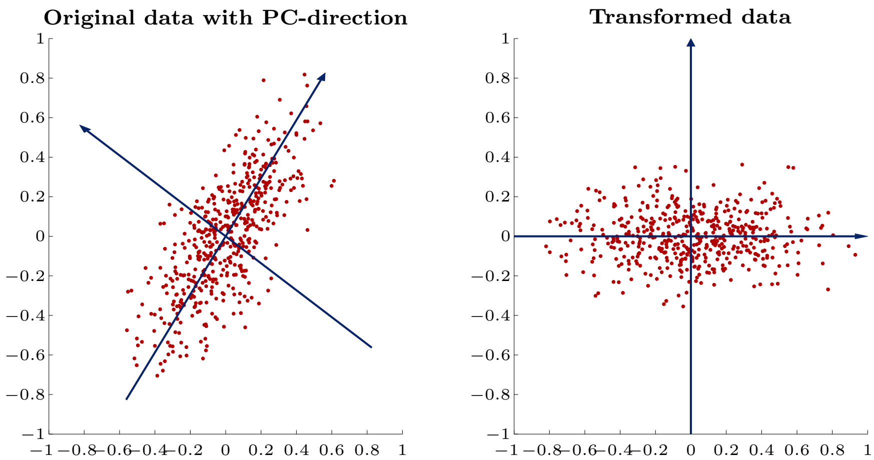

Figure 13: *The PCA tries to find the direction with the highest variance first (left). The next component, in turn, contains the highest variance left, under the constraint that it is orthogonal to the preceding components. The axes are finally rotated (right).*

In the following, we assume that the matrix $C \in \mathbb{R}^{n \times n}$ has full rank and therefore its inverse C^{-1} exists. We will use the matrix C to move from the original data space to the PCA space by simply multiplying the matrix with the original data: $\widetilde{x}_i = C\, x_i$. The change from the PCA space back to the original space is achieved by $x_i = C^{-1}\, \widetilde{x}_i$. However, as C is a matrix consisting of orthogonal eigenvectors in its columns, $C^T\, C$ is the identity matrix and therefore $C^{-1} = C^T$.

3.4.2 Risk parity applied to principal components

We now want to apply the concept of PCA to the Risk Parity problem. The PCA is used here as a blind source separation method as uncorrelated risk factors are later needed to determine the solution set. The idea of factor risk parity that is used is thereby similar to the concept described in Roncalli and Weisang (2012) whereby the specific risk factors in this work are not, as mentioned above, further economically motivated. A core advantage of using a risk parity approach is that return expectations are not needed for the model, especially as setting those expectations are often a point of criticism. Also, historical returns are not put in any relationship to risk or used as an approximation for future returns.

We therefore take the price changes x of the assets described in Section 3.5 and calculate the matrix $C \in \mathbb{R}^{n \times n}$ as described in the subsection above. We assume the matrix to be fully ranked. As mentioned above, we can transform asset price changes into changes along the directions of the principal components and vice versa by multiplying with C or C^T respectively. We therefore have

$$\widetilde{x} = x\, C, \tag{37}$$

with $\widetilde{x} \in \mathbb{R}^{k \times n}$ being the changes in the principal components. In the same way we can transform the weights w into $\widetilde{w}$ by

$$\widetilde{w} = C^T\, w, \tag{38}$$

with $\widetilde{w} \in \mathbb{R}^n$ interpreted as weights in a principal component space. The portfolio return of principal components $\widetilde{P}$ is then calculated as

$$\widetilde{P} := \widetilde{x}\, \widetilde{w}, \tag{39}$$

with $\widetilde{P} \in \mathbb{R}^k$. Given the asset space portfolio returns P as described in Section 3.3, the daily portfolio returns in P are equal to the daily portfolio returns in $\widetilde{P}$.
With a covariance matrix $\widetilde{\Sigma} \in \mathbb{R}^{n \times n}$ of the principal components $\widetilde{x}$, the marginal risk contribution for each component can be calculated by

$$\widetilde{MRC}_i := \frac{\partial \sigma(\widetilde{w})}{\partial \widetilde{w}_i} = \frac{(\widetilde{\Sigma}\widetilde{w})_i}{\sqrt{\widetilde{w}^T \widetilde{\Sigma} \widetilde{w}}}. \tag{40}$$

Through the orthogonal transformation used by the PCA, each component is orthogonal to the preceding components. Therefore, by using $\widetilde{\sigma_i^2}$ as the variance of the i-th principal component, the covariance matrix is given by:

$$\widetilde{\Sigma} = diag(\widetilde{\sigma}_1^2, \ldots, \widetilde{\sigma}_n^2) \tag{41}$$

Hence, the correlation matrix is:

$$\rho_{\widetilde{x}} = diag\,[1,\ 1,\ 1, \ldots,\ 1]\,. \tag{42}$$

The orthogonality in the principal components and equation (40) imply that changing the weight $\widetilde{w}_i$ only changes the i-th marginal risk contribution $\widetilde{MRC}_i$ and leaves $\widetilde{MRC}_j$ with $j \neq i$ unchanged.
As mentioned above, orthogonality considerably simplifies the calculation of the risk contribution:

$$\widetilde{RC}_i = \frac{\widetilde{w}_i^2\, \widetilde{\sigma}_i^2}{\widetilde{\sigma_P}} \;\; \forall i \in \{1, \ldots, n\}\,. \tag{43}$$

We now try to find weights $\widetilde{w} \in \mathbb{R}^n$ which lead to an equal risk contribution of the first m principal components in the factor space of solutions. Those first m principal components should explain most of the variation. In practice, m will be 2.[2]

$$\widetilde{w}_i^2 \, \widetilde{\sigma}_i^2 = \widetilde{w}_j^2 \, \widetilde{\sigma}_j^2 \;\; \forall i, j \in \{1, 2, .., m\} \tag{44}$$

Considering an equality in squared variables, $\beta_i \in \{-1, 1\}$ serves as a long-short indicator related to these factors:

$$\begin{aligned} \beta_1 \, \widetilde{w}_1 \, \widetilde{\sigma}_1 &= \beta_2 \, \widetilde{w}_2 \, \widetilde{\sigma}_2, \\ \beta_1 \, \widetilde{w}_1 \, \widetilde{\sigma}_1 &= \beta_3 \, \widetilde{w}_3 \, \widetilde{\sigma}_3, \\ &\cdots \\ \beta_1 \, \widetilde{w}_1 \, \widetilde{\sigma}_1 &= \beta_m \, \widetilde{w}_m \, \widetilde{\sigma}_m. \end{aligned} \tag{45}$$

β_i equates to the principal components direction so that we receive 2^{m-1} possible solutions. For each permutation of β, the vector $\widetilde{w}$ is a solution in the factor space. The directions of the other $n - m$ principal components are irrelevant. Due to the transformation $w = C \, \widetilde{w}$ we do not need to consider the direction of the other principal components.

The concept presented here is similar to the approach of Meucci (2009) who does not use the term of risk contributions but volatility concentration curves. As the variance $\widetilde{\sigma}_i^2$ of the i-th principal component equals the i-th eigenvalue of the PCA, those terms though largely overlap. Additionally, Meucci (2009) focuses on setting up portfolios by analyzing diversification distributions represented by the (exponential of the) entropy (see equation (73)). There are major differences to this model, however: first, risk parity is independent of any historical returns or return distributions which in general is seen as a key advantage of this approach. Meucci (2009) on the other hand relates the effective number of bets to the expected return. For the case that solely the entropy is maximized, all principal components, excluding those determined through the constraints, are taken into consideration. That is a core difference to the approach discussed in this paper, which lets the residual risk components float, thereby generalizing the classical factor risk parity solution set. As the residual components in a PCA are more unstable and often considered to be noisy, letting the weights float to some extend is in our view a more intuitive approach.

As PCA is not free of criticism, it is sometimes replaced by other methods as for example done by Meucci et al. (2014), who present a model that is based on minimum torsion bets to

[2] For all further considerations, $m < n$ as the model seeks to reduce the number of risk factors.

determine uncorrelated factors to track the original ones. They argue, among others, that principal components are statistically unstable, especially those regarding to the lower eigenvalues. They further mention the problem of uniqueness as well as interpretation issues of the principal components. The model described here, by focussing only on the first principal components, bypasses some of those issues: the focus is placed on the more stable, important principal components and the residual ones are allowed to float to some extend. As the number of equal risk contributions is set to only a few components, the amount of solutions due to different principal component directions does not increase significantly.

3.4.3 Introduction to polyhedra

After describing the allocation problem, we now focus on the solution set if we consider portfolios without leverage or short positions. So far we have been able to arbitrarily choose $\widetilde{w}_{m+1}$ to $\widetilde{w}_n$ but we will later restrict those weights. As a subset of $\mathbb{R}^n$ we do not know though whether a solution in general will exist.

It turns out that by introducing the long-only constraint we get a convex polytope which describes the portfolios with the desired attributes as a set of solutions. We will therefore continue with a short introduction to polyhedra and polytopes before describing our modified factor risk parity model.

A polyhedron Q is a subset of $\mathbb{R}^n$ that can be written as a finite number of linear inequalities, meaning $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$ exist with:

$$Q = \{x \in \mathbb{R}^n \mid Ax \leq b\}. \tag{46}$$

This definition describes the subset of $\mathbb{R}^n$ as a cut out by a finite number of hyperplanes (see Gallier (2008)). The description of a polyhedron via equation (46) is also called the H-representation of a polyhedron. Very simple two-dimensional examples of polyhedra in $\mathbb{R}^2$ include structures such as a square or triangle. However, the inequalities do not necessarily mean that the set is bounded nor that a solution exists. By bounded we mean that Q fits into a cube, so $k \in \mathbb{R}$ exists with

$$Q \subset \{x \in \mathbb{R}^n \mid \quad |x_i| \leq k \quad \forall i \in \{1, ..., n\}\}. \tag{47}$$

Let us next define a corner point (extreme point) of a polyhedron. There are many different ways which are used equivalently in the literature. We define a corner point of a polyhedron Q as a point $x \in \mathbb{R}^n$ so that there are no two $y, z \in \mathbb{R}^n$ such that x is a convex combination of y and z. If the polyhedron is bounded we also speak of a polytope.

According to the Weyl-Minkowski Theorem, polytopes can either be written in the form of equation (46) or equivalently as a convex hull of a finite set of points. Thus, let $q_i \in \mathbb{R}^n$ with $i \in \{1, \dots, r\}$ be a finite number of points, then the polytope can be written as:

$$Q = conv\{q_1, \dots, q_r\} = \{x \in \mathbb{R}^n \mid x = \sum \lambda_i q_i \text{ with } \lambda_i \geq 0, \sum \lambda_i = 1\}. \tag{48}$$

Equation (48) is called the V-representation of a polytope. For a proof of the Weyl-Minkowski Theorem see Gallier (2008).

We will later consider polyhedra which also include equalities. However, the definition above already includes equalities as $Ax = b \Leftrightarrow Ax \leq b$ and $-Ax \leq -b$. In implementing the model, the portfolio weights $w_i \in \mathbb{R}^n$ will be subject to some restrictions which can be formulated by a number of equalities and inequalities. The solution set can therefore be formulated through a H-representation as described in equation (46).

Considering the matrix A and vector b as another example: [3]

$$A = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \\ -1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & -1 \\ -1 & -1 & -1 \end{bmatrix} \quad and\ b = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \tag{49}$$

The resulting three-dimensional object that represents all points of $Ax \leq b$ is called an octahedron and is shown in Figure 14.

[3] Matlab-Code and example can be found at http://www.mathworks.com/matlabcentral/fileexchange/9261-plot-2d-3d-region/content/example4.m.

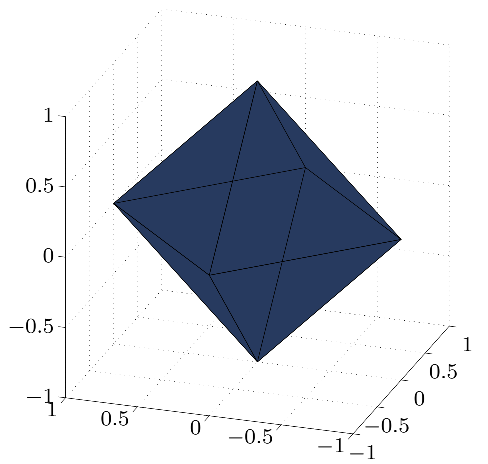

Figure 14: *The figure shows a special form of a polytope, an octahedron according to the example (49) with six corner points and eight hyperplanes in* $\mathbb{R}^3$.

There are many computational issues related to polyhedra or polytopes. One problem is the conversion between the H-representation and the V-representation of a polytope, which is called the "vertex enumeration problem". If we are defining a set of inequalities describing the restrictions for the portfolio weights, how to obtain the corner points of the polytope? Later each corner point will represent a portfolio that is of special interest and will be used for further calcuations. In this case we will compute the extreme points by using the cdd-mex library of Komai Fukuda which is based on the "Double Description Method" of Motzkin et al. (1953). The documentation says: "The C-library cddlib is a C implementation of the Double Description Method of Motzkin et al. for generating all vertices (i.e. extreme points) and extreme rays of a general convex polyhedron in $\mathbb{R}^d$ given by a system of linear inequalities..." The program can be used for the reverse operation (i.e. convex hull computation). This means that one can move back and forth between an inequality representation and a generator (i.e. vertex and ray) representation of a polyhedron with cdd"[4]. See appendix (3.A) for further details.

At a later stage, we will use the fact that a polyhedron stays a polyhedron under affine transformations in Euclidian Spaces. The corresponding proof can be found in Gallier (2008).
Note also that polyhedra are convex sets which means that for $y_1, y_2 \in Q$ and $0 \leq \lambda \leq 1$ s.t. we have $\lambda y_1 + (1-\lambda) y_2 \in Q$. We will later call a portfolio "optimal" if and only if the weights of the portfolio are within a specific polyhedron. As the corner points will later, due to our restrictions, represent portfolios with sum of weights equal to 1, an investor can conveniently choose which affine combination of those portfolios to take. As the set is convex, this affine combination will

[4] ftp://ftp.ifor.math.ethz.ch/pub/fukuda/cdd/README.libcdd

automatically be an optimal portfolio as well. And as each corner point will have weights equal to 1, so will the affine combination.

We will also briefly touch on the topic of whether the inequalities actually do have a solution, meaning that the polyhedron consists of at least one point and whether the polyhedron is bounded or not.

3.4.4 Factor risk parity polytopes

In this section we describe the set of portfolios which meets the following criteria and solves the problem described in Section 3.4.1:

1. **Risk parity condition**
   This condition has already been explained above: the weights in the PCA space need to meet conditions from equation (45).

2. **Positive weights in the asset space**
   We do not want any short selling, meaning $w = C\,\widetilde{w} \geq 0$.

3. **Asset weights sum up to 1**
   Equivalent to $\sum w_i = 1$ as no leverage as well as no cash position is allowed.

If forcing weights $\widetilde{w}_{m+1}, \ldots, \widetilde{w}_n$ in the principal component space to be zero, it is often not possible to find a solution that meets all the above mentioned criteria. It might be of interest, however, to allow those weights to differ from zero for the benefit of avoiding short sales and leverage.

There are now different ways to proceed from here to find the solution set. We will first take a step by step approach to find weights that fulfill conditions 1 and 2. By normalizing the results those criteria are still met and the weights will also sum up to 1. This is not a direct way but will, however, provide some insights on the path. We subsequently focus on an integrated approach which meets all conditions simultaneously and which we will use in further calculations.

First, we focus on weights $\widetilde{w}_1, \ldots, \widetilde{w}_m$ according to equation (45). The condition only refers to the ratio of the weights $\widetilde{w}_1$ to $\widetilde{w}_m$. Without loss of generality, we just describe the case where $\beta_i = 1 \;\; \forall i \in \{1, \ldots, m\}$ to simplify the calculations. From equation (45) we can directly conclude

$$\widetilde{w}_i = \widetilde{w}_1 \frac{\widetilde{\sigma_1}}{\widetilde{\sigma_i}} \;\; \forall\, i \; \in \{2, \ldots, m\}\,. \tag{50}$$

Thereafter, we turn to condition 2 to find $\widetilde{w}_{m+1}, \ldots, \widetilde{w}_n$ and meet condition

$$\begin{pmatrix} c_{1,1} & \cdots & c_{1,n} \\ \vdots & \vdots & \vdots \\ c_{n,1} & \cdots & c_{n,n} \end{pmatrix} \begin{pmatrix} \widetilde{w_1} \\ \cdots \\ \widetilde{w}_m \\ \widetilde{w}_{m+1} \\ \cdots \\ \widetilde{w}_n \end{pmatrix} \geq \begin{pmatrix} 0 \\ \cdots \\ 0 \\ 0 \\ \cdots \\ 0 \end{pmatrix}. \tag{51}$$

Next let the matrix C_1 be the matrix containing the first m columns of the PCA transformation matrix and the matrix C_2 be the matrix with the residual columns:

$$C_1 := \begin{pmatrix} c_{1,1} & \cdots & c_{1,m} \\ \vdots & \ddots & \vdots \\ c_{n,1} & \cdots & c_{n,m} \end{pmatrix} \in \mathbb{R}^{n \times m}, \;\; C_2 := \begin{pmatrix} c_{1,m+1} & \cdots & c_{1,n} \\ \vdots & \ddots & \vdots \\ c_{n,m+1} & \cdots & c_{n,n} \end{pmatrix} \in \mathbb{R}^{n \times (n-m)}. \tag{52}$$

Under the constraint $w = C\ \widetilde{w} \geq 0$ and as $(\widetilde{w}_1, \ldots, \widetilde{w}_m)^T$ is defined by equation (45), $(\widetilde{w}_{m+1}, \ldots, \widetilde{w}_n)^T$ must meet condition

$$\begin{aligned} -C_2 \begin{pmatrix} \widetilde{w}_{m+1} \\ \vdots \\ \widetilde{w}_n \end{pmatrix} &\leq C_1 \begin{pmatrix} \widetilde{w}_1 \\ \vdots \\ \widetilde{w}_m \end{pmatrix} = C_1 \begin{pmatrix} \widetilde{w}_1 \\ \vdots \\ \widetilde{w}_1 \frac{\widetilde{\sigma_1}}{\widetilde{\sigma_m}} \end{pmatrix} \\ \Rightarrow -C_1 \begin{pmatrix} \widetilde{w}_1 \\ \vdots \\ \widetilde{w}_1 \frac{\widetilde{\sigma_1}}{\widetilde{\sigma_m}} \end{pmatrix} &- C_2 \begin{pmatrix} \widetilde{w}_{m+1} \\ \vdots \\ \widetilde{w}_n \end{pmatrix} \leq \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \end{aligned} \tag{53}$$

which is a linear inequality in $\widetilde{w}_1, \widetilde{w}_{m+1}, \ldots, \widetilde{w}_n$. The solution set describes a convex polyhedron. As noted above, convex means, in this case, that for $z_1, z_2 \in Q$ and $0 \leq \lambda \leq 1$, $\lambda z_1 + (1-\lambda) z_2 \in Q$ holds. Thus, all affine combinations of extreme points are feasible solutions. Once we find solutions for $\widetilde{w}_1, \widetilde{w}_{m+1}, \ldots, \widetilde{w}_n$, we can also calculate $\widetilde{w}_2, \ldots, \widetilde{w}_m$.

On the one hand, we have now defined the conditions for $\widetilde{w}_1, \ldots, \widetilde{w}_m$ which relate to the risk parity condition in the first m components and, on the other hand, we define $\widetilde{w}_{m+1}, \ldots, \widetilde{w}_n$ for non-negative weights constraints while the risk parity condition is still met. We can transform the weights using $C\ \widetilde{w}$ in the asset space. Taking into consideration, however, that the polyhedron in this case might not be bounded.

So far we have ignored condition 3 from above, meaning that the weights in the asset space sum up to 1. We could proceed the following way: for every solution $\widetilde{w}$ we can just normalize the weight in the asset space. Knowing that the solution set in the asset space is a polyhedron and, with $c \times W$, still stays a polyhedron with $c > 0$, the solution can be normalized and is still a valid portfolio in the asset space with asset weights sum equal to 1. The complete polyhedron which was calculated above is thereby normalized.

We now focus on the method that is more straightforward, particularly as we compute the extreme points of the polyhedron by using the cdd-mex library. To find the vertices of the polytope, the conditions need to be written in the form of equalities and inequalities. The idea therefore is to describe each of the conditions 1 to 3 above as equations.

Describing the solution set as a combination of equalities and inequalities

As it has been shown, the solutions for the constraints of the weights, being within certain boundaries, can be written in accordance with equation (45). Standard algorithms for determining the polyhedron, however, can usually deal with inequalities and equalities.

The condition $\widetilde{w_1}\widetilde{\sigma_1} = \widetilde{w_i}\widetilde{\sigma_i}$ for $i \in \{1, \ldots, m\}$ regarding the risk equality can be written in matrix form as

$$\begin{pmatrix} \widetilde{\sigma_1} & -\widetilde{\sigma_2} & 0 & \cdots & 0 & 0 \\ \widetilde{\sigma_1} & 0 & -\widetilde{\sigma_3} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \widetilde{\sigma_1} & \cdots & \cdots & -\widetilde{\sigma_m} & \cdots & 0 \end{pmatrix} \begin{pmatrix} \widetilde{w_1} \\ \vdots \\ \vdots \\ \widetilde{w_n} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ \vdots \\ 0 \end{pmatrix}. \tag{54}$$

The condition that the portfolio weights sum up to 1 is equivalent to

$$1 = \sum_i w_i = \sum_i \sum_j (c_{i,j}\widetilde{w_j}) = \sum_j \widetilde{w_j} \sum_i c_{i,j} = \sum_i C_i \widetilde{w_i}, \tag{55}$$

with $c_{i,j}$ being the elements of the transformation matrix from the PCA and $C_j = \sum_i (c_{i,j})$ being the sum of the j-th column in the matrix C.
We can therefore write

$$\begin{pmatrix} C_1 & C_2 & \cdots & C_n \end{pmatrix} \begin{pmatrix} \widetilde{w_1} \\ \vdots \\ \widetilde{w_n} \end{pmatrix} = 1. \tag{56}$$

Finally, the requirement that the weights in the asset space are non-negative is already given above by the pre-condition $w = C\,\widetilde{w} \geq 0$.

Some remarks on the existence of the solutions and the boundary aspects

One of the questions that arises when dealing with inequalities and polyhedra is whether a solution exists. The inequalities and equalities may be set up in a way that no point in $\mathbb{R}^n$ fulfills all conditions. Given our application, the question is whether a portfolio exists which has equal risk contributions in the first two components in the PCA space with positive weights in the asset space that sum up to 1. This problem of finding a solution to $A\,x \leq b$ is often referred to as the *"linear feasibility problem"* which is very closely related to the topic of linear programming. Due to its importance it has been widely discussed in the literature such as Cartis and Gould (2007), which focuses heavily on finding efficient algorithms for solving this problem. As discussed above, we use the cdd-mex library for finding the vertices of the polyhedron. The method returns an empty set when the inequalities do not have any solutions, therefore we will not go into further detail regarding this problem.

Another issue is whether the solution to the equalities and inequalities above is bounded or not, i.e. whether the solution is a polytope or not. The standard cdd-mex code does not deal with unbounded polyhedra and returns an error when calculating the V-representation. In our case, however, boundedness is not a problem: the solution in the form of a polyhedron in the asset space is bounded: the weights are supposed to be positive and the condition that the weights sum up to 1 implies that each weight is lower than or equal to 1. We also know that we can transform the polyhedron back and forth using the matrix multiplication with C or C^T accordingly. As the image of a compact set under a matrix multiplication is compact, we know that the polyhedron in the PCA space must also be a polytope if the asset space polytope exists. Thus, we will subsequently use the expression "polytope" to describe the solution set.

Sufficient explanation of the first few risk contributions and benefits of calulating the polytope

As discussed, the conditions with the corresponding solution above lead to a polytope. This, however, might still not be satisfying, as the risk contributions in the first m components may be much smaller than those in the last components. This would contradict the initial idea of factor risk parity as the majority of the portfolio variance is determined not by the first risk contributions which are equal in size but by other factors which are not taken into consideration at all. Solutions from the polytope should therefore be excluded when the first risk contributions are too small compared to the total risk. With a minimum explanation of e_{min} and the first m risk contributions equal, we require:

$$\begin{aligned} \frac{\sum_{j=1}^{m} \widetilde{RC}_j}{\sum_{i=1}^{n} \widetilde{RC}_i} &\geq e_{min}, \\ \Leftrightarrow (1-e_{min}) \sum_{j=1}^{m} \widetilde{RC}_j - e_{min} \sum_{j=m+1}^{n} \widetilde{RC}_j &\geq 0, \\ \Leftrightarrow (1-e_{min}) \sum_{j=1}^{m} \widetilde{w}_j^2 \sigma_j^2 - e_{min} \sum_{j=m+1}^{n} \widetilde{w}_j^2 \sigma_j^2 &\geq 0. \end{aligned} \tag{57}$$

Equation (57) is of the form

$$c_1 \widetilde{w}_1^2 + \cdots + c_m \widetilde{w}_m^2 - c_{m+1} \widetilde{w}_{m+1}^2 - \cdots - c_n \widetilde{w}_n^2 \geq 0, \tag{58}$$

with $c_i > 0 \; \forall i$. In contrast to the inequalities we had before, this equation is an inequality with sums of squared variables. To find solutions for the factor risk parity problem with equal risk contributions in the first components and a sufficiently high explanation of those components, we have to find the intersection of the polytope with the solution set of equation (45). Usually this intersection, however, is not a polytope anymore and solving this problem is often complex. This is why we define the minimum explanation level as a constraint for each optimization problem within the polytope to determine a solution.

Finding an optimal portfolio later can also be done completely numerically. However, knowing that the solution set can be described by a convex polytope offers some benefits versus just trying to find an optimal portfolio numerically: first, by calculating the polytope, the investor is aware of the structure of the solution set, whether a solution in general exists and if so, how large the set is. Next, the optimization for finding a solution within the polytope is much more efficient compared to a full numerical optimization which would have to incorporate more parameters and constraints simultaneously. The explicit structure of the solution set in the principal component space opens up further opportunities. For example, by using the corner points of the polytope, an investor can individually combine all the corner point portfolios to construct his desired allocation.

Example in a lower dimension case

The following paragraph describes a simple example where we assume there are 4 different assets, where the risk contributions in the first two components are supposed to be equal. The coefficient matrix from the PCA in this example is given by

$$C = \begin{pmatrix} 0.506 & 0.848 & 0.145 & -0.059 \\ -0.110 & -0.107 & 0.987 & -0.056 \\ 0.057 & 0.024 & 0.066 & 0.996 \\ 0.854 & -0.518 & 0.036 & -0.039 \end{pmatrix}. \tag{59}$$

We require asset weights between 0 and 1. Let 0.0169, 0.0113, 0.0047 and 0.0030 be the standard deviations of the principal components. Using the algorithm to determine the polyhedra based on the conditions above, we get two possible solution sets reflecting the two orthogonal solutions, meaning

$$\beta = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ or } \beta = \begin{pmatrix} 1 \\ -1 \end{pmatrix}. \tag{60}$$

The reasons why only two different polytopes can exist if the risk of the first two principal components is equalized is the following: first, from the first polytope constraint (see page 60), which can be written as:

$$\begin{pmatrix} \widetilde{\sigma_1} & -\widetilde{\sigma_2} & 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} \widetilde{w_1} \\ \vdots \\ \widetilde{w_n} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}, \tag{61}$$

one can see the the solution changes when the direction of one principal component is changed (i.e. using $\beta = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$. The relation of the sign of the first two principal is fixed facing one another. Thus, two different solutions are possible.

Second, the reason why not more than two polytopes exists has to do with the second polytope constraint from page 60. From $w = C\ \widetilde{w} \geq 0$ follows $w = C\ \widetilde{w} = -C(-\widetilde{w}) \geq 0$. If one changes, the direction of one or more principal components C_i for $i > 2$, only the sign of the weights $\widetilde{-w_i}$ for $i > 2$ changes. Due to the transformation with C, the solution in the asset space stays the same.

For the example, we are focusing again on the scenario with $\beta = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. If we use the algorithm that reflects all three criteria for the factor risk parity polytopes, we get a polytope with 4 corner points in a four-dimensional PCA space:

$$\widetilde{w_1} = \begin{pmatrix} 0.5228 \\ 0.7819 \\ 0.1397 \\ -0.0580 \end{pmatrix}, \widetilde{w_2} = \begin{pmatrix} 0.2375 \\ 0.3549 \\ 0.0974 \\ 0.5724 \end{pmatrix}, \widetilde{w_3} = \begin{pmatrix} -0.0300 \\ -0.0448 \\ 0.5477 \\ 0.4446 \end{pmatrix}, \widetilde{w_4} = \begin{pmatrix} -0.0801 \\ -0.1198 \\ 0.9574 \\ -0.0560 \end{pmatrix}. \tag{62}$$

or, transformed back into asset space:

$$w_1 = \begin{pmatrix} 0.9513 \\ 0 \\ 0 \\ 0.0487 \end{pmatrix}, w_2 = \begin{pmatrix} 0.4014 \\ 0 \\ 0.5986 \\ 0 \end{pmatrix}, w_3 = \begin{pmatrix} 0 \\ 0.5238 \\ 0.4762 \\ 0 \end{pmatrix}, w_4 = \begin{pmatrix} 0 \\ 0.9697 \\ 0 \\ 0.0303 \end{pmatrix}. \tag{63}$$

We can already see from the results that the weights in the asset space sum up to 1. We also see from the weights in the PCA space that the ratio of second to first coordinate is constant:

$$\frac{0.782}{0.523} \approx \frac{0.355}{0.238} \approx \frac{-0.045}{-0.030} \approx \frac{-0.120}{-0.080} \approx 1.496. \tag{64}$$

Let us now take another look and compare the results to the methods with lengthy computations. We already know that the ratio of the second to first coordinate is 1.49.

Next we turn to the condition from equation (53) which is

$$-\begin{pmatrix} 0.506 & 0.848 \\ -0.110 & -0.107 \\ 0.057 & 0.024 \\ 0.854 & -0.518 \end{pmatrix} \begin{pmatrix} \widetilde{w_1} \\ \frac{0.0169}{0.0113}\widetilde{w_1} \end{pmatrix} - \begin{pmatrix} 0.145 & -0.059 \\ 0.987 & -0.056 \\ 0.066 & 0.996 \\ 0.036 & -0.039 \end{pmatrix} \begin{pmatrix} \widetilde{w_3} \\ \widetilde{w_4} \end{pmatrix} \leq \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \tag{65}$$

This is a linear inequality in $\widetilde{w_1}, \widetilde{w_3}$ and $\widetilde{w_4}$. Using the cdd-mex library again we can determine the polyhedron relating to equation (65). The polyhedron, however, is unbounded, as we can see from the equation. Yet, in this case, this is not a major obstacle[5]. So far, we have not considered the restriction that the weights in the asset space sum up to 1. This criterion will lead to the boundedness of the solution set of the problem. In this case the hyperplane is defined by:

$$\begin{aligned} \sum C_i \widetilde{w_i} &= 1.307 \cdot \widetilde{w_1} + 0.247 \cdot \widetilde{w_2} + 1.234 \cdot \widetilde{w_3} + 0.842\, \widetilde{w_4} \\ &= 1.676 \cdot \widetilde{w_1} + 1.234 \cdot \widetilde{w_3} + 0.842 \cdot \widetilde{w_4} \\ &= 1 \end{aligned} \tag{66}$$

[5] For fixed $\widetilde{w_1}$ and $\widetilde{w_4}$ we can let $\widetilde{w_3} \to \infty$ for example. For this example, we will simply place a box as boundaries around the polyhedron. Figure 15 will show there is no restriction when the space chosen is big enough.

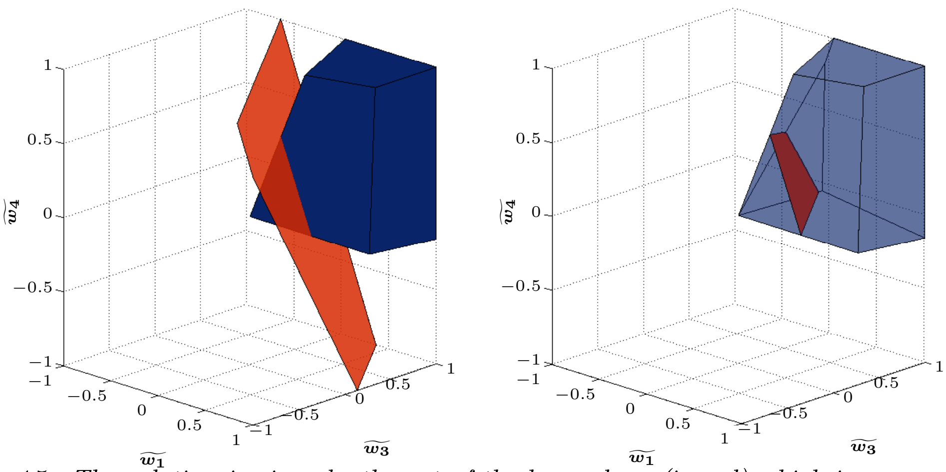

Figure 15: *The solution is given by the cut of the hyperplane (in red) which is responsible for asset weights equal to 1, with the polyhedron in blue which represents the portfolios with positive asset weights and risk contributions equal in first two components. Note that $\widetilde{w_2}$ as a multiple of $\widetilde{w_1}$ is not plotted.*

Figure 15 shows the results: we first of all got the solution for $\widetilde{w_1}$, $\widetilde{w_3}$ and $\widetilde{w_4}$ in the form of the unbounded polyhedron (blue box). That polyhedron represents the weights in the PCA space where the first two risk contributions are equal and the weights are non-negative. Keep in mind that the weights for $\widetilde{w_2}$ are not plotted in that figure. The hyperplane in the form of equation (66) is also plotted in that figure. That hyperplane (red plane) represents the portfolios with weights in the PCA space which, when transferred back into the asset space, sum up to 1. The solution set is therefore restricted to all portfolios on that hyperplane. As we are interested in portfolios that are both within the polyhedron as well as on the hyperplane, we have to take the intersection of those two sets. Please note that the corner points of the intersection match exactly with the corner points that we received above (see equation (62)).

Finally, we address the level of explanation from equation (57). We now want to see whether the first two components explain at least 70% of the total risk contributions. Equation (57) in this example then turns into:

$$\begin{aligned} &0.3 \cdot 0.0169^2 \cdot \widetilde{w_1}^2 + 0.3 \cdot 0.0113^2 \cdot \widetilde{w_2}^2 - 0.7 \cdot 0.0047^2 \cdot \widetilde{w}_3^2 - 0.7 \cdot 0.003^2 \cdot \widetilde{w}_4^2 \geq 0 \\ \Leftrightarrow &1.714 \cdot \widetilde{w_1}^2 \geq 0.155 \cdot \widetilde{w_3}^2 + 0.063 \cdot \widetilde{w_4}^2 \end{aligned} \tag{67}$$

Figure 16 displays the results of equation (67) together with the polytope that we have calculated above. The axes represent the weights $\widetilde{w_1}$, $\widetilde{w_3}$ and $\widetilde{w_4}$. In addition to a minimum explanation level of 70%, we have added the 50% and the 90% levels. It can be observed that the solution in the form of the polytope consists of a wide variety of portfolios, even with a level of explanation of the first two principal components (risk contribution) of over 90% but also with less than 50%.

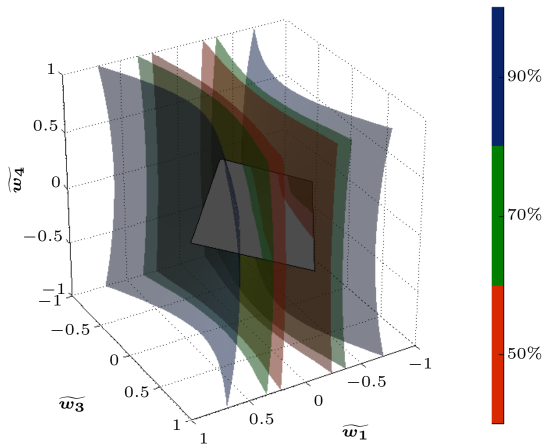

Figure 16: *The rectangle in this figure is the solution from above while the curved surfaces represent the curves of equal explanation ratio. As an example, the blue surface represents all portfolios where the explanation of the first two components is equal to 90%.*

We can also see that the weight $\widetilde{w}_1$ is most relevant for a high explanation: a high absolute weight $\widetilde{w}_1$ implies a high absolute weight $\widetilde{w}_2$. Additionally to the characteristic of sorted variances of the principal components received through the PCA, this increases the risk contributions $\widetilde{RC}_1$ and $\widetilde{RC}_2$ relative to $\widetilde{RC}_3$ or $\widetilde{RC}_4$.

3.5 Data description

The following section describes the historical datasets used for the empirical analysis. We used two US multi-asset datasets with different periods of time and different numbers of time series but with a comparable multi-asset class setup. For both datasets daily end-of-day (EoD) data is used. Dataset 1 focuses on a broad selection within the asset classes and avoids aggregated indices to allow a more granular view. Given this requirement, the time series reaches from 2014 back to 1994. For dataset 2, aggregated indices are used for the backtest to benefit from a longer time period that starts in 1986.

There are different reasons for the focus on US data instead of worldwide assets, one being the problem of different time zones, which is crucial when dealing with daily data. This is even more important in cases of risk-based strategies. Important key statistics, such as correlations, get distorted by lags in trading hours of different exchanges around the globe or different holiday calendars. We excluded other markets, like private equity, hedge funds or real estate, in order to obtain reliable and long period EoD series. Stock indices such as the S&P 500 have a long history and are well understood with reliable data. Finally, the data is not affected by any foreign exchange issues when we completely focus on the US dollar as the currency.

3.5.1 Dataset 1

In dataset 1, we focused on a broadly diversified set of US assets. To be balanced and cover different risk factors, the five most liquid asset classes as well as indices within those classes are used. All the data is taken from Bloomberg. These include:

- Three equity indices
- Treasury bond indices with three different maturity bands
- Corporate bond indices with three different maturity bands
- High-yield bond indices with three different maturity bands
- Three commodity indices

The equity indices include the S&P 500 (SPX), as a broad index of 500 US stocks, the NASDAQ 100 (NDX) as the 100 largest and most active non-financial domestic and international issues listed on the NASDAQ and the Russell 2000 index (RTY), which is an index consisting of 2000 equities with lower market capitalization.

On the fixed income side, we add 3 different treasury bond indices which include the Bank of America (BoA) Merrill Lynch indices with different maturity bands. These enclose indices with maturities of 1-3 years (G1O2), 5-7 years (G3O2) and 10-15 years (G7O2) to cover the entire yield curve. Those indices represent the pure interest rates' development over different maturities excluding credit risk.

To account for credit risk, corporate debt and high-yield indices are added. Corporate debt is represented via BofA Merrill Lynch indices with different maturity bands, too. These comprise the BofA Merrill Lynch 1-3 Year AA US corporate index (C1A2), the ones for maturities 5-7 years (C3A2) and 10-15 years (C7A2). The same maturity buckets are used for high-yield bonds, the BofA Merrill Lynch 1-3 Year B Cash Pay High-Yield Index (J1A2), 5-7 years (J3A2) and

10-15 years (J7A2).

For the commodities, three S&P GSCI sub-indices are included: S&P GSCI Energy Total Return Index (SPGSENTR), S&P GSCI Industrial Metals Total Return Index (SPGSINTR), and S&P GSCI Precious Metals Total Return Index (SPGSPMTR). There are several reasons for choosing exactly these indices. First, these indices are directly investable. Especially with commodities it is necessary to select indices which reflect the real performance that investors achieve when they want to invest into that asset class. Special characteristics such as backwardation or contango can significantly influence the risk and return and, as such, the investors' earnings.

Regarding commodities, we choose three sub-indices that reflect a wide range of the commodity universe. Agricultural investments are excluded due to general concerns related to ethical aspects. We also choose to take the sub-indices instead of a single commodity index as each sub-index reacts differently and contains its own characteristics. This can be even more important in the case of factor risk parity as we try to extract the underlying risk factors varying from index to index. As an example, the precious metals index is driven by different factors than the energy index. Gold, as for example discussed in the first study, is sometimes interpreted as a type of safe haven when markets crash whereas the price of oil may be influenced by completely different factors. We can see from the correlation matrix (see Figure 32 in the appendix) as well as from the optimization results that the sub-indices behave quite differently and therefore justify their inclusion.

For the analysis, bond positions are split into treasuries versus corporates and high-yield, in order to split the credit risk from pure interest rate movements. As the credit spread moves differently for investment grade bonds and high-yield bonds, we include both types. The time buckets cover most maturities, from short-term in the form of the 1-3 years bucket to very long-term in the form of bonds with maturities longer than 10 years. Different maturity buckets are included to account for any shifts or twists that may occur in the yield or spread curves. The chosen buckets are standard and guarantee a long time series in the form of the BoA Merrill Lynch indices.

We deliberately ignore any alternative investment indices such as private equity, hedge funds or real estate. Although one might consider these assets to be key in a well-diversified multi-asset portfolio, we decided to exclude them due to different reasons. The most important one is the lack of liquidity, particularly as we focus on daily data. As the range of indices selected already covers a huge universe of assets and includes the relevant risk factors, other indices such as mid-cap equity are excluded. Particularly focusing on the extraction of the driving risk factors by using the PCA, adding another equity index for example, should not influence the analysis or diversification.

Considering the historical data and their characteristics, it is not surprising that the volatility of commodity and equitiy indices on average exceeds that of the bond indices. Figure 17 illustrates the annualized standard deviation and the annualized return for the data from 1994 to 2014. Tables 21 and 22 provide a more detailed view on the data.

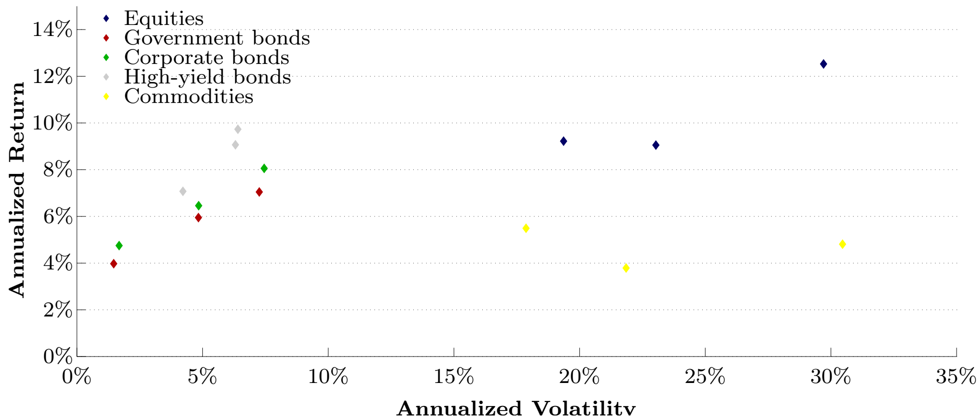

Figure 17: *Annualized asset returns and annualized volatilities for the five asset classes (three different time series each) are plotted in this figure for the entire period from 1994 to 2014.*

On average, the equity time series offer a more or less similar volatility level as the commodities while at the same time offering the highest returns. The Sharpe Ratio for different time periods in Figure 18 reflects the average high volatility and low return in the form of average negative Sharpe Ratios for commodities for different time periods.

Treasury and corporate bonds perform similarly, with corporate bonds offering higher returns at a slightly higher level of volatility. For most time periods in Figure 18, those proportions are reflected by the Sharpe Ratios. US high-yield bonds have performed well in the last 20 years with annual returns between 6% and 12% and manageable volatility levels. It is worth noting that high-yield bonds had the highest Sharpe Ratio in each period we observed.

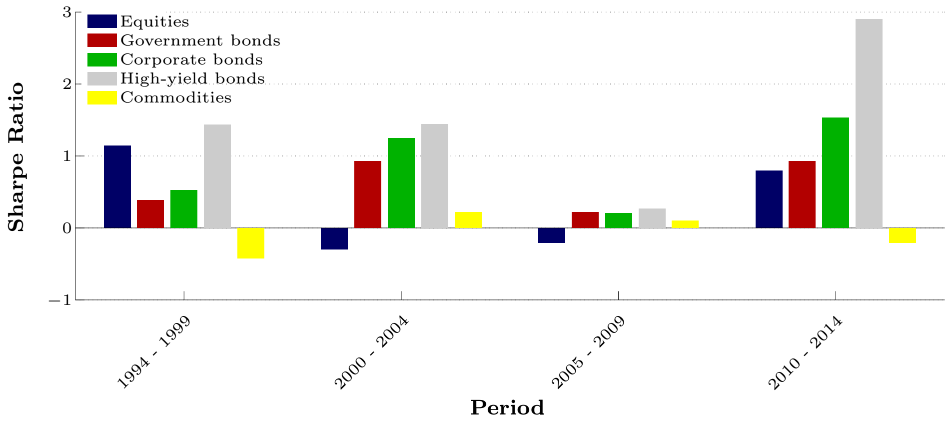

Figure 18: *The asset classes' Sharpe Ratios are plotted for four periods. For calculating the Sharpe Ratios we used the 3M-USD-Libor as the risk-free rate aggregated all time series within an asset class to one time series with equal weights. Negative Sharpe Ratios are shown even though interpretation is difficult in those cases.*

When working with historical data, however, it is crucial to consider the time period. For example, the yield of bonds in the last 20 years has dropped significantly. The 10-year US government bond yield has dropped from around 8% p.a. in 1994 to under 1.5% p.a. in 2012. Despite difficult periods, such as the "dot-com crash" or the financial crisis in 2008/09, equities had an impressive 20-year history with an average annual performance between 9% and 12.5%. The high-yield bonds had a strong performance in the last 20 years, too. Equities showed positive Sharpe Ratios between 1994-1999 and 2009-2014, whereas they were negatively affected by the two crises mentioned above. The expansion of the Federal Reserve Balance sheet after the financial crisis has led to a strong performance with higher Sharpe Ratios in all asset classes except for commodities. To calculate the Sharpe Ratio, we used the 3M-USD-Libor as the risk free-rate.

3.5.2 Dataset 2

The idea behind the composition of dataset 2 is to maximize the time period under analysis while at the same time focusing on a similar asset structure as in the dataset above. We do not include high-yield time series due to their shorter data history. Although it was necessary to use more aggregated indices to cover the whole period from 1986 to 2014, we believe that the dataset is still well diversified in terms of risk factors. We again cover the same four asset risk factors as in dataset 1: equity risk, interest rate risk, credit spread risk and commodity risk. Data are again taken from Bloomberg as EoD time series. These include:

- Two equity indices
- Two treasury bond indices
- Two corporate bond indices
- One commodity Index

Due to their long history, we included the S&P500 (SPX) and the Russell 2000 Index (RTY). Given the shorter data history, the NASDAQ 100 (NDX) could not be used. To cover treasury bonds, we used the 1-3 years (G2O2) and the 15+ (G8O2) years BofA Merrill Lynch treasury bond indices. To account for credit risk in this dataset, we used the 1-3 years (C2A0) und the 15+ years (C8A0) Bank of America Merrill Lynch US Corporate Bond indices. The bond indices are responsible for determining the length of this dataset. Finally, we replace the commodity sub-indices used in dataset 1 by the S&P GSCI commodity index due to the longer data history.

The historical data characteristics are similar to the first dataset. Figure 19 shows annualized return and risk relations for the whole period in the same way as for dataset 1. Equity time series again offer similar volatility as the commodity time series, but at a higher level of return. The volatility of both asset classes are on average between 18% and 22% and thus the highest in dataset 2.

Corporate and government bonds have similar risk/return characteristics. As in dataset 1, corporate bonds again perform better with a slightly lower level of volatility. The reason for this difference lies in the time period from 1986 to 1994. Tables 25 and 26 provide a more detailed view on the data. The heatmap in Figure 33 shows the asset correlations. Sharpe Ratios are very similar to that in dataset 1 for the four periods from 1994 to 2014. For the period from 1989 to 1994 we measured low but positive Sharpe Ratios for all asset classes (see Figure 20).

In sum, both datasets are quite similar in terms of their statistical characteristics. We will use each dataset to cross-validate the results using them independently. Notwithstanding the different length and the different time series, we would expect similar allocations for the comparable time periods. Otherwise the model would be very sensitive to marginal changes in the input time series.

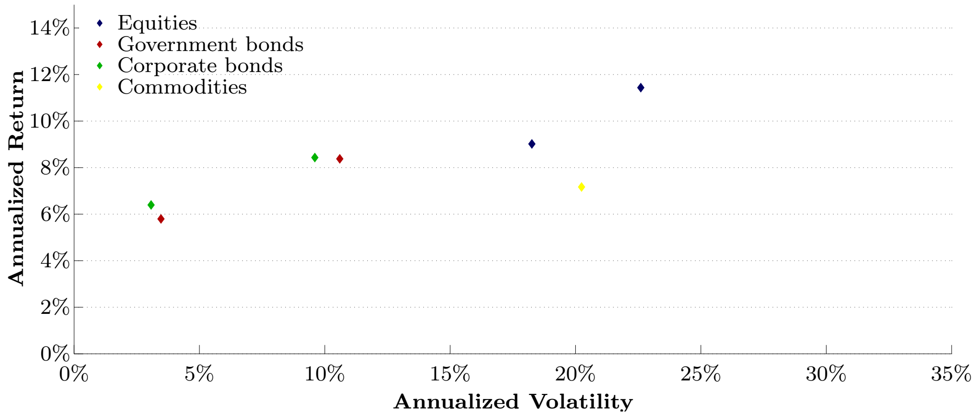

Figure 19: *Annualized asset returns and annualized volatilities for the four asset classes (three different time series each) are plotted in this figure for the entire period from 1986 to 2014.*

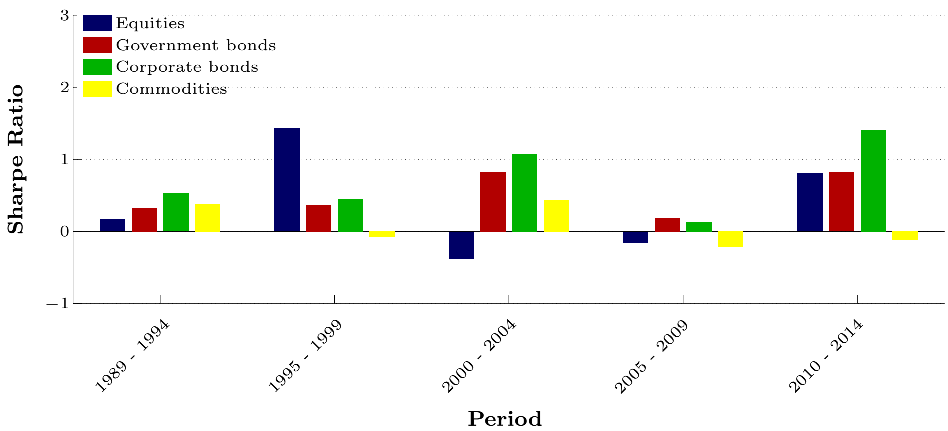

Figure 20: *The asset classes Sharpe Ratios are plotted for five periods. For calculating the Sharpe Ratios we used the 3M-USD-Libor as the risk-free rate and aggregated all time series within an asset class to one time series with equal weights. Negative Sharpe Ratios are shown even though interpretation is difficult in those cases.*

3.6 Empirical analysis

In this section we apply the model described in detail in Section 3.4.1. Before performing the empirical analysis using the model and the strategies described in Section 3.3, the out-of-sample backtest setup will be explained. Some robustness checks regarding the stability of the composition are made and the stability of the risk contribution of the first two principal components will be analyzed.

3.6.1 Backtest setup

For all following empirical analyses and tests, we define $t = 1$ as the first entry in dataset 1 and dataset 2. Despite the different length of the two datasets, the same backtest approach in the form of a rolling window approach for both datasets will be used. The out-of-sample asset weights w are calculated at time t based on the historical data from $t - x$ to $t - 1$. After that, these weights are used to calculate the performance and other statistics for the time from t to $t + (y - 1)$. The asset weights will be rebalanced at $t + y$ accordingly. Using this approach, the length of the in-the-sample window x and the length of the out-of-sample window y until reallocation have to be chosen.

The in-the-sample window length is often a subject of discussion in literature. We will therefore backtest three different in-the-sample window lengths: 500, 1,000 and 1,500 trading days. The length of the out-of-sample window is set to 60 trading days which corresponds to roughly 3 months. The rebalancing period should not be too short to avoid higher trading costs but also not too long so as to be able to adjust to any changes in the market environment.

For computing the solution set corner points, the direct way as explained in Section 3.4.4 will be used only. For each rebalancing period a new polytope is calculated. When two polytopes due to the two beta factors from formula 60 exist, the one with the higher level of explanation is taken.

For the empirical analysis the two datasets as specified in Section 3.5 are used. A special focus will be put on the level of explanation by the first two principal components, which as discussed have to be sufficiently high. For a more detailed insight into the behavior of the risk contribution and the composition of the first two principal components in the context of the models' robustness, the same approach with daily calculated risk contributions is used. In Section 3.6.2 the risk contribution and the composition of the first two principal components for each trading day will be computed. For this, a window of 500, 1,000 and 1,500 days will be used, which is then rolled over on a daily basis.

3.6.2 Robustness

Bhansali et al. (2012) take a closer look at a sample universe of 9 international assets and argue that only 2 factors already explain a majority of the variance of the data. For that, they calculate the PCA and analyze the factor loadings for the assets on the first two factors. They argue that the first factor can be interpreted as a kind of global growth risk factor whereas the second factor would represent the inflation risk.

For dataset 1, the factor loadings of the first two principal components for dataset 1 are illustrated in figure 21. Particularly, the equity and the commodity time series load heavy on the first two principal components. Equities, however, load positively on the first principal component and negatively on the second principal component, whereas all commodities load positively on the first and second principal component.

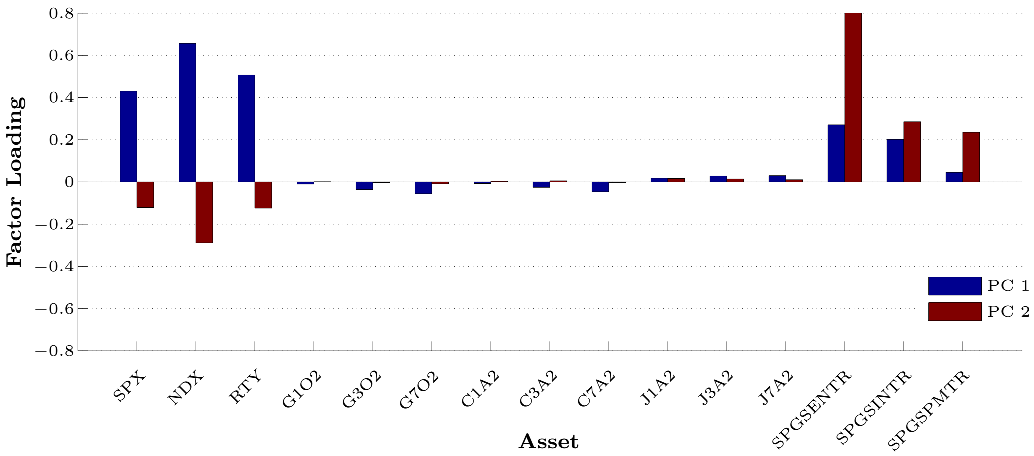

Figure 21: *Factor loadings of all assets for the first two principal components are plotted for the period of dataset 1 from 1994 to 2014.*

Tables 9 and 10 provide a statistical overview of all principal components computed for the entire period of dataset 1 whereas table 23 in the appendix shows the factor loadings of all principal components for the entire period. As shown in the tables the first two principal components explain most of the variance. Moreover, we believe that it makes little sense to use more than two principal components due to the minor contribution of the third and fourth components to the variance. Regardless of the interpretation of the first two principal components, we therefore share the view that the first two components explain the majority of the variance in a multi-asset setting.

Against the backdrop of a robust backtest and a robust portfolio construction, we are interested in the behavior of the level of explanation of the first two principal components. Figure 22 shows the total risk contribution of the first two principal components for each trading day from $t = 501,\ 1,001$ and $1,501$ to $t = 4,992$ using the rolling window setup.

On the one hand, that graph illustrates that the level of explanation becomes smoother the longer the window is chosen. On the other hand, the level of explanation decreases marginally

	$Mean$	$Median$	$Mean_{Ann.}$	Std	$Std_{Ann.}$	$Skew$	$Kurt$
Principal Component 1	0.07%	0.20%	18.55%	2.59%	40.95%	(0.37)	7.94
Principal Component 2	0.00%	0.01%	1.01%	1.97%	31.18%	(0.16)	5.88
Principal Component 3	0.01%	0.02%	2.71%	1.33%	20.97%	(0.15)	4.87
Principal Component 4	0.03%	0.03%	6.55%	0.98%	15.48%	0.11	7.48
Principal Component 5	(0.01%)	0.01%	(2.10%)	0.79%	12.53%	(0.21)	7.11
Principal Component 6	0.07%	0.08%	16.95%	0.73%	11.48%	(0.34)	7.09
Principal Component 7	0.05%	0.06%	11.56%	0.50%	7.98%	(1.67)	70.77
Principal Component 8	(0.01%)	(0.02%)	(3.22%)	0.44%	7.03%	0.38	10.18
Principal Component 9	0.00%	(0.00%)	0.62%	0.30%	4.77%	2.72	91.37
Principal Component 10	0.01%	0.01%	1.35%	0.17%	2.67%	(1.47)	21.95
Principal Component 11	(0.01%)	(0.01%)	(1.69%)	0.14%	2.27%	0.61	61.84
Principal Component 12	(0.01%)	(0.00%)	(1.34%)	0.13%	2.03%	0.14	21.29
Principal Component 13	0.01%	0.01%	2.24%	0.07%	1.09%	(0.46)	49.89
Principal Component 14	0.01%	0.01%	2.38%	0.05%	0.77%	(1.81)	46.43
Principal Component 15	0.00%	0.00%	0.56%	0.02%	0.37%	2.37	60.08

Table 9: *Principal components of dataset 1 - 1994-2014 (1)*

	Min	Max	$Q_{(5\%)}$	$Q_{(1\%)}$	Ø $Q_{(5\%)}$	Ø $Q_{(1\%)}$
Principal Component 1	(18.88%)	15.78%	(4.13%)	(7.37%)	(6.17%)	(10.15%)
Principal Component 2	(15.43%)	10.09%	(3.16%)	(5.40%)	(4.54%)	(6.80%)
Principal Component 3	(8.06%)	6.32%	(2.09%)	(3.50%)	(2.95%)	(4.53%)
Principal Component 4	(7.94%)	7.53%	(1.50%)	(2.48%)	(2.15%)	(3.34%)
Principal Component 5	(5.93%)	4.08%	(1.22%)	(2.16%)	(1.81%)	(2.96%)
Principal Component 6	(6.29%)	6.48%	(1.10%)	(1.98%)	(1.64%)	(2.59%)
Principal Component 7	(11.04%)	6.23%	(0.57%)	(1.42%)	(1.16%)	(2.35%)
Principal Component 8	(4.02%)	3.67%	(0.67%)	(1.16%)	(0.99%)	(1.54%)
Principal Component 9	(3.92%)	6.30%	(0.31%)	(0.80%)	(0.64%)	(1.38%)
Principal Component 10	(2.01%)	1.41%	(0.22%)	(0.56%)	(0.45%)	(0.90%)
Principal Component 11	(1.81%)	2.82%	(0.17%)	(0.39%)	(0.33%)	(0.69%)
Principal Component 12	(1.50%)	1.86%	(0.20%)	(0.33%)	(0.30%)	(0.52%)
Principal Component 13	(1.33%)	0.96%	(0.08%)	(0.16%)	(0.14%)	(0.26%)
Principal Component 14	(0.95%)	0.53%	(0.06%)	(0.12%)	(0.10%)	(0.20%)
Principal Component 15	(0.24%)	0.50%	(0.03%)	(0.06%)	(0.05%)	(0.09%)

Table 10: *Principal components of dataset 1 - 1994-2014 (2)*

the longer the rolling window becomes. Given this tradeoff between the level of explanation and smoothness of explanation, we decided to test different window lengths at the beginning of the empirical analysis. For reasons of simplicity, subsequent detailed analysis will focus on one in-the-sample window length only.

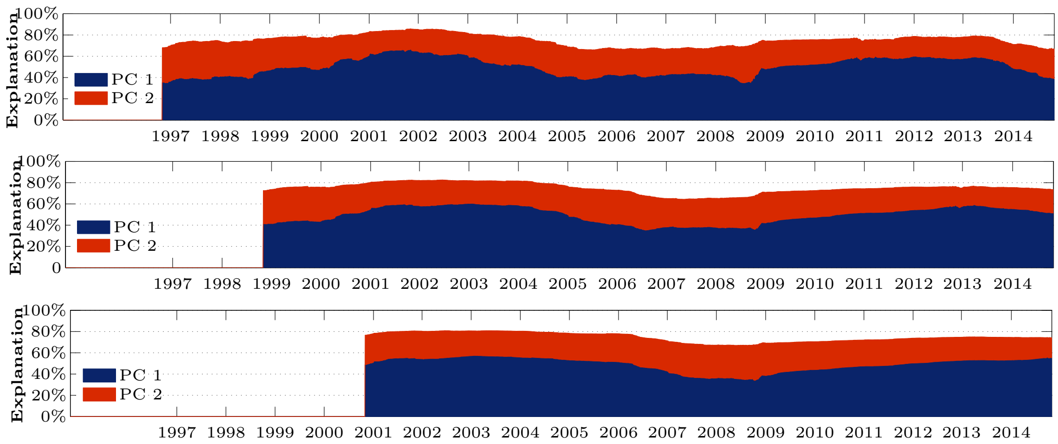

Figure 22: *Explanation of risk contribution of the first two principal components for 500, 1,000 and 1,500 days in-the-sample windows using dataset 1.*

As a last point, Figures 23 to 25 show the principal components' composition for single equity, bond and commodity time series. It can be seen that the 1,000 days window as well as the 1,500 days window are much more stable in terms of the principal component composition. Particularly, the global financial crisis is reflected by the very volatile composition using the 500 days window.

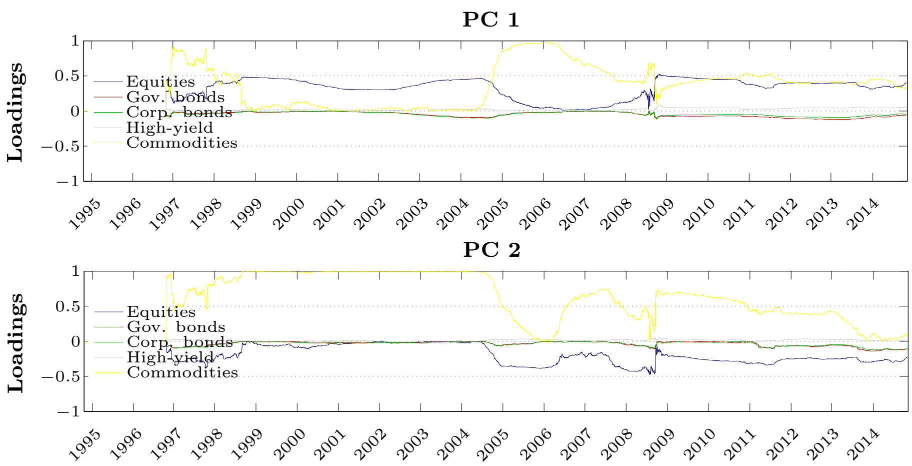

Figure 23: *Principal components' loadings - 500 days in-the-sample window - dataset 1*

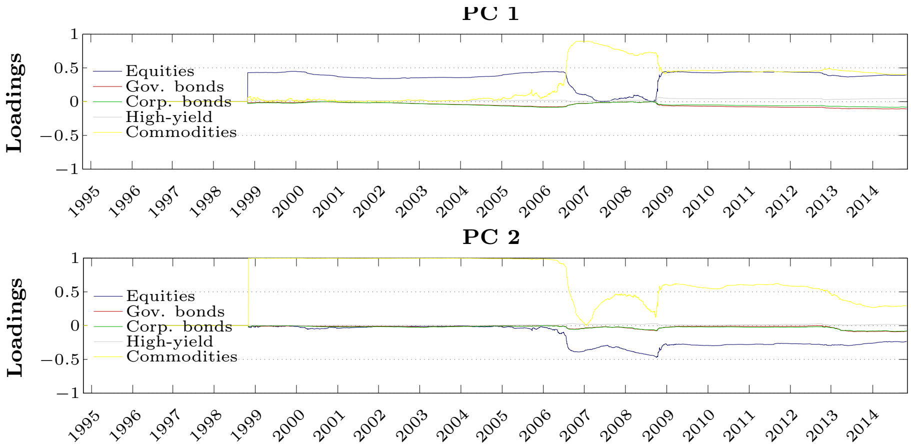

***Figure 24:** Principal components' loadings - 1,000 days in-the-sample window - dataset 1*

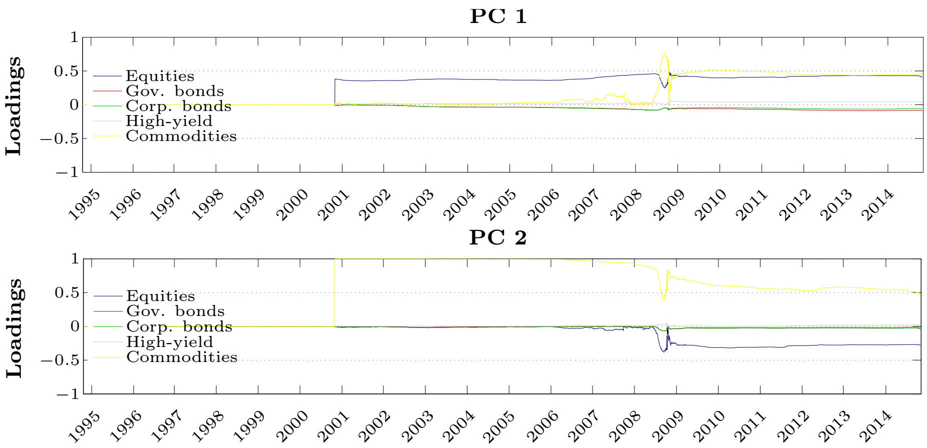

***Figure 25:** Principal components' loadings - 1,500 Days in-the-sample window - dataset 1*

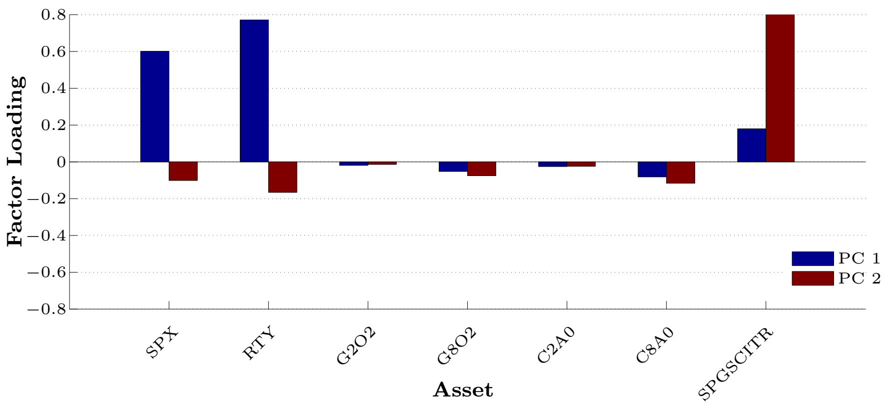

***Figure 26:** Factor loadings on all assets for the first two principal components are plotted for the whole period of dataset 2 from 1986 to 2014.*

After gaining insight into the the principal components' structure of dataset 1, we will now see that dataset 2 has the same characteristics. Figure 26 shows the factor loadings of the first two principal components for dataset 2. The loadings for the equity and bond indices are very similar compared to the loadings from dataset 1. The equity time series loads highly positively on the first principal components and less high negatively on the second principal component. All bond time series load negatively on both principal components. The loading of the commodity index (SPGSCITR) is comparable to that of the energy sub-index with a slightly lower weight on the first principal component.

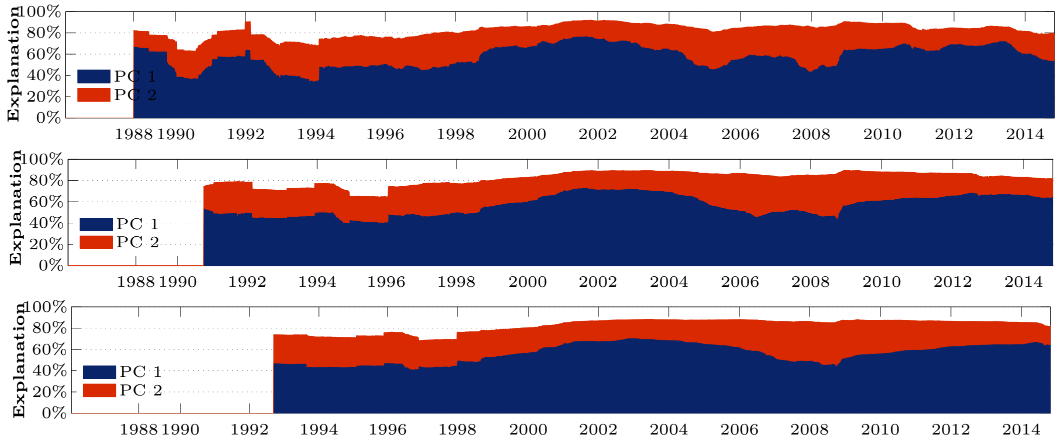

Figure 27: *Explanation of risk contribution of the first two principal components for 500, 1,000 and 1,500 days in-the-sample windows using dataset 2.*

Tables 27 and 28 in the appendix provide a statistical overview of the backtest of all principal components for the entire period of dataset 2 as well.

As for dataset 1, the risk contribution of the first two principal components computed for dataset 2 becomes smoother the longer the in-the-sample window is chosen, but for the disadvantage of a lower explanation level (see figure 27).

Figures 34 to 36 in the appendix show the composition for the first two principal components using dataset 2. Again, the composition for the 1,000 and 1,500 days window is more stable than for the 500 trading days window. The financial crisis in 2008 is again reflected by a very volatile composition.

Despite the higher volatility of the risk contribution and the more volatile composition of the first two principal components using a 500 days window, we will compute the backtest statistics using all three rolling window lengths for dataset 1 and dataset 2 in order to show the impact of this factor on the portfolio allocation. As for the factor risk parity allocations, the rolling window backtest setup for computing the benchmark allocations, namely the ERC, EW and MV strategies, will be used.

3.6.3 Backtest

For calculating the out-of-sample allocation weights, we define $q_i \in \mathbb{R}^n$ for $i = 1, ..., r$ to be the corner points of the polyhedron and λ_i to be the weight that we place on those corner points. We make use of the nature of the convex polytope and formulate all allocations as a function of the corner points of the polytope which we determine numerically. As it is impossible to invest in the entire set of solutions, we decided to backtest two special points within the set of solutions. First, the point where the in-the-sample variance has a minimum and second, the point where the in-the-sample diversification has a maximum. We will find both points using the *fmincon* function in Matlab.

The most important constraint for both optimizations is on the minimum level of explanation of the first two principal components within the set of all solutions,

$$\frac{\sum_{j=1}^{2} \widetilde{RC}_j}{\sum_{i=1}^{n} \widetilde{RC}_i} \geq e_{min}. \tag{68}$$

with e_{min} being the minimum explanation level. Due to the nature of the affine combinations, the constraints regarding the asset weights are always met.

$$\begin{aligned} \sum_{i=1}^{r} \lambda_i &= 1 \\ \lambda_i &\geq 0 \quad \forall i \in \{1...r\} \end{aligned} \tag{69}$$

This means we are trying to determine a portfolio inside the polyhedron by finding an affine combination of the corner points with a sufficiently high explanation of the first two risk contributions that minimizes the portfolio variance (maximizes portfolio diversification). The weights of the corresponding portfolio are given by $\sum \lambda_j q_j$. The variance and return of the resulting portfolios are given by $\sigma^2_{Q(\lambda)}$ and $\mu_{Q(\lambda)}$. Given the global constraints, we are now able to formulate the two optimization problems. The Minimum Variance Optimization can be written as

$$f^*(\lambda) = arg\ min \left(\sigma^2_{Q(\lambda)} \right), \tag{70}$$

where the Maximum Diversification optimization is a problem in the form of:

$$f^*(\lambda) = arg\ max \left(\frac{\sum_{i=1}^{n} (\sum_{j=1}^{r} \lambda_j q_j)_i \sigma_i}{\sigma_{Q(\lambda)}} \right) \tag{71}$$

with $(\sum_{j=1}^{r} \lambda_j p_j)_i$ is the i-th element of the vector in $\mathbb{R}^n$.

In the next section, we will apply the two optimization problems to the polytopes we calculated in accordance with the backtest setup. In a first step, we will compute some standard statistics such as the annualized return, standard deviation, Sharpe Ratio and Maximum Drawdown as well as two diversification measures. We calculate the "Diversification" as

$$\mathcal{D} = \frac{\sum (w_i \cdot \sigma_i)}{\sigma_P} \tag{72}$$

and the entropy in the asset space of weights as

$$\mathcal{E} = \exp\left(- \sum_{i=1}^{N} w_i \ln w_i \right) \tag{73}$$

for all backtests and allocation strategies. In a second step, we take a closer look at the risk contribution of the first two principal components and the asset weights of factor risk parity allocation and try to interpret the results.

Backtest dataset 1

Using dataset 1 and a 500 days in-the-sample window, we observe for both factor risk parity (FRP) allocations with a 66% minimum level of explanation a lower volatility, a lower maximum drawdown and a higher diversification than for the two factor risk parity allocations at an 80% minimum level of explanation. In terms of return and Sharpe Ratio, we found that the ERC and the FRP Maximum Diversification strategy as well as the MV and the FRP Minimum Variance strategy are comparable. Those results are in line with our expectations as the FRP Minimum Variance allocation, like the MV allocation, also tries to minimize the variance, with the only difference on operating on a more restricted set of possible solutions. For all allocation strategies in table 11, the ERC strategy has the highest Sharpe Ratio at a relative high level of diversification. Additionally, it can be seen from all allocation strategies that a high diversification does not necessarily go along with a low volatility.

Table 11 also illustrates the average yearly turnover statistics for those strategies. The ERC and EW allocation exhibit the lowest turnover while the MV allocation has the highest turnover values due to the instability of the minimum variance approach. In contrast to that, using the

FRP Minimum Variance strategy reduces the turnover significantly. This is interesting to note, as a PCA for financial time series is often considered to be unstable over time, leading to less robust strategies with higher turnover figures (see for example Deguest et al. (2013)). However, the FRP Minimum Variance strategy focuses only on the first two principal components, which are more stable than the residual components in a classical PCA, and can therefore explain the lower turnover of that approach. Even when the FRP Maximum Diversification strategy with higher turnover values is used, those numbers stay lower than in the MV approach.

	$Return$	$Mean_{Ann.}$	$Std_{Ann.}$	SR	Max_{DD}	$\mathcal{D}$	$\mathcal{E}$	$Turnover$
ERC	152.40%	5.28%	2.55%	0.95	14.74%	2.44	13.88	28.10%
EW	207.36%	6.44%	6.26%	0.57	28.91%	2.05	7.86	19.05%
MV	91.14%	3.67%	1.36%	0.60	7.66%	2.03	8.30	152.27%
$Max.\ Div._{66\%}$	153.09%	5.30%	3.29%	0.74	12.84%	2.54	10.37	126.03%
$Max.\ Div._{80\%}$	153.00%	5.30%	4.10%	0.60	15.57%	2.30	8.92	135.95%
$Min.\ Var._{66\%}$	97.65%	3.86%	2.02%	0.50	12.27%	2.22	10.79	65.26%
$Min.\ Var._{66\%}$	104.87%	4.07%	2.80%	0.43	14.92%	1.94	9.44	85.67%

Table 11: *Allocation strategies - 500 days in-the-sample window - dataset 1*

Using a 1,000 days in-the-sample window, the results are very similar to those using a 500 days window (see table 12). Due to the shorter backtest period, the total return is a little bit lower. During the first years from 1997 to 1999, all allocation strategies delivered positive returns. The turnover for the strategies usually decrease with the longer in-the-sample window as the principal components become more stable.

	$Return$	$Mean_{Ann.}$	$Std_{Ann.}$	SR	Max_{DD}	$\mathcal{D}$	$\mathcal{E}$	$Turnover$
ERC	132.06%	5.41%	2.72%	1.07	16.24%	2.35	14.20	17.77%
EW	179.08%	6.63%	6.43%	0.64	28.10%	2.04	7.86	19.33%
MV	74.52%	3.55%	1.40%	0.75	8.30%	1.84	7.94	119.21%
$Max.\ Div._{66\%}$	130.34%	5.36%	3.20%	0.89	14.44%	2.47	10.29	70.48%
$Max.\ Div._{80\%}$	126.99%	5.26%	3.86%	0.72	16.58%	2.30	8.60	69.96%
$Min.\ Var._{66\%}$	81.65%	3.81%	2.10%	0.62	12.56%	2.16	10.20	22.39%
$Min.\ Var._{80\%}$	80.71%	3.77%	2.63%	0.48	14.91%	2.00	8.55	24.32%

Table 12: *Allocation strategies - 1,000 days in-the-sample window - dataset 1*

Annualized standard deviation and diversification do not differ much compared to the allocation previously described. Maximum drawdown increases for all allocation strategies, except for EW where weights are continuously reset to constant levels over time. The Sharpe Ratios for all FRP allocations are higher than for the same allocations where we use a 500 days window, while the ERC strategy has again the highest Sharpe Ratio in this set of allocations.

Using a 1,500 days in-the-sample period, annualized returns of all allocations drop significantly (see table 13). The factor risk parity allocation performances deteriorate sharply compared to the ERC and EW strategies. Diversification and maximum drawdown remain nearly unchanged compared to the 500 and 1,000 day window allocations.

	Return	$Mean_{Ann.}$	$Std_{Ann.}$	SR	Max_{DD}	$\mathcal{D}$	$\mathcal{E}$	*Turnover*
ERC	108.15%	5.38%	2.84%	1.19	15.88%	2.31	14.27	13.83%
EW	120.22%	5.81%	6.47%	0.59	28.79%	2.04	7.97	18.38%
MV	63.00%	3.56%	1.35%	1.15	6.71%	1.91	7.99	132.27%
$Max.\ Div._{66\%}$	87.09%	4.58%	3.13%	0.82	14.32%	2.40	9.94	44.85%
$Max.\ Div._{80\%}$	79.12%	4.26%	3.78%	0.60	16.49%	2.25	8.10	45.35%
$Min.\ Var._{66\%}$	55.53%	3.21%	2.04%	0.59	11.58%	2.20	9.34	15.65%
$Min.\ Var._{80\%}$	51.32%	3.01%	2.54%	0.40	13.88%	2.06	7.62	17.68%

Table 13: *Allocation strategies - 1,500 days in-the-sample window - dataset 1*

The in-the-sample window length is a softer factor when dealing with backtests. The results above indicate that the in-the-sample window length does not have a major impact on the portfolio statistics. To reduce the number of backtests we therefore decided to select the 1000 day window for further analysis. This selection has been made as we consider a 500 days time period to be too short, as the volatility of the risk contributions of the first two principal components already indicate. The statistics such as Sharpe Ratio or maximum drawdown for the 1,500 days window are quite similar or only slightly inferior when compared to the 1,000 days window.

We will now take a closer look at the allocations that are of high interest, namely the FRP Maximimum Diversification and Minimum Variance allocation at a minimum explanation level of 66% and 80% using a 1,000 days rolling in-the-sample window.

As the allocation statistics lead one to assume, it can be seen from figure 28 that the ERC and FRP Max. Diversification allocations and the MV and FRP Min. Variance allocations behave similarly. For a detailed view on the allocations shown in Figures 28 and 29, we calculated yearly risk and return data on pages 104 to 105 and plotted the results accordingly.

The behavior of the factor risk parity allocations at different minimum levels of explanation are very similar. Figure 29 shows only slightly changed results for the factor risk parity allocations at a minimum level of explanation of 80% compared to the allocations shown in chart 28

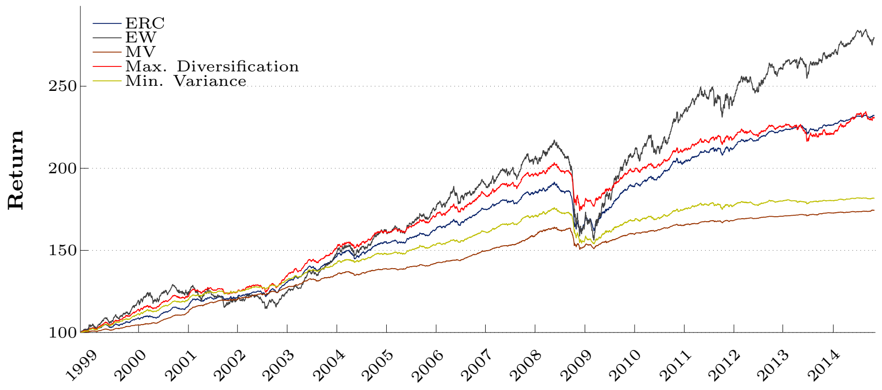

Figure 28: *Allocation strategies - 66% level of explanation - 1,000 days in-the-sample window - dataset 1*

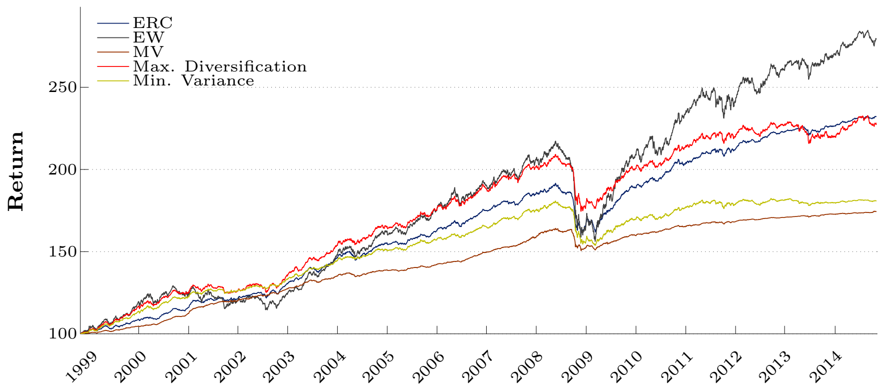

Figure 29: *Allocation strategies - 80% level of explanation - 1,000 Days in-the-sample window - dataset 1*

before.

Nonetheless, figure 38 reveals some interesting aspects. Data points for the factor risk parity strategy at an 80% minimum level of explanation are almost continuously moved to the right compared to the same strategy at a 66% minimum level of explanation, meaning that we see higher volatility at the same return levels. In the case of minimum variance, it is not possible to find lower volatility portfolios at the 80% level as the set we are working on is only a subset of the portfolios in the 66% case.

The following section takes a closer look at the asset weights in the backtest to get a better feeling for the behavior of each strategy. For that, the average weights for each year and strategy are calculated, which can be found in Tables 14 and 15 for the factor risk parity strategies and in the appendix for the MV, EW and ERC strategies. First, let us take a look at the MV strategy with the weights in Table 34: as government bonds offer the lowest volatility, the MV strategy loads heavily on this asset class. Particularly after the financial crises in 2008/09, high-yield positions were mostly replaced by government bonds. Equities and commodities usually only play a minor role in this allocation strategy. The ERC approach seems to be more stable in terms of the asset weights. Government bond positions tended to increase and commodities and high-yield positions tended to decrease in the time period from 1998 to 2014. Equity positions with around 4% to 7% and corporate bond positions with around 23% to 29% tend to be more stable than in other allocations. We omit the EW allocation here as the weights are reset regularly.

Let us next focus on the factor risk parity allocations. For that, the results for the FRP Maximum Diversification allocation are displayed in Table 14. Although government bonds are not as dominant here as in the MV strategy, they still play an important role in the allocation. Especially after the financial crisis, the weights increase from around 40% to 50% to around 70%. Only in 2013/14 does that weight decrease, as on average falling volatility levels favored more "risky" assets such as equities, commodities or high-yield bonds. Surprisingly, corporate bonds have low or even zero weights after the "dot.com" crash around 2003.

	FRP Max. Diversification 66%					*FRP Max. Diversification* 80%				
	EQ	*GOV*	*CORP*	*HY*	*COM*	*EQ*	*GOV*	*CORP*	*HY*	*COM*
1998	6.4%	43.5%	3.7%	29.4%	17.0%	9.4%	41.0%	3.7%	27.8%	18.2%
1999	5.4%	28.1%	20.7%	30.4%	15.4%	8.0%	26.6%	19.7%	29.0%	16.7%
2000	4.7%	28.8%	21.0%	32.5%	13.0%	6.8%	27.5%	20.0%	31.1%	14.5%
2001	4.6%	39.3%	13.0%	30.4%	12.6%	6.4%	37.8%	12.4%	29.3%	14.2%
2002	5.5%	34.5%	15.5%	30.6%	13.9%	7.3%	32.9%	14.7%	29.4%	15.7%
2003	6.8%	52.0%	2.8%	24.3%	14.1%	8.6%	49.6%	2.6%	23.3%	15.9%
2004	8.1%	51.8%	2.5%	24.1%	13.4%	10.3%	49.1%	2.3%	23.0%	15.3%
2005	10.1%	51.4%	1.1%	25.6%	11.8%	12.9%	48.3%	0.9%	24.2%	13.7%
2006	10.9%	47.4%	0.0%	30.5%	11.1%	14.0%	43.8%	0.0%	28.5%	13.6%
2007	8.1%	38.2%	4.8%	40.2%	8.7%	10.3%	35.6%	4.2%	37.8%	12.1%
2008	9.5%	59.6%	0.0%	24.9%	6.0%	11.7%	56.2%	0.0%	23.7%	8.3%
2009	3.1%	73.6%	0.2%	11.4%	11.7%	3.1%	69.9%	0.2%	11.2%	15.6%
2010	2.7%	72.7%	0.0%	12.1%	12.5%	2.4%	69.1%	0.0%	11.8%	16.6%
2011	3.0%	72.5%	0.0%	11.7%	12.8%	2.8%	68.7%	0.0%	11.5%	17.0%
2012	2.3%	72.6%	0.0%	11.7%	13.4%	1.8%	68.9%	0.0%	11.6%	17.8%
2013	2.9%	52.0%	0.0%	28.0%	17.0%	1.5%	49.1%	0.0%	26.3%	23.1%
2014	7.3%	42.3%	0.0%	31.4%	19.1%	6.4%	39.1%	0.0%	29.0%	25.5%

Table 14: *Max. Diversification asset weights - 1,000 days in-the-sample window - dataset 1*

The weights for the FRP Minimum Variance allocations are displayed in Table 15. Not surprisingly they are similar to the results we calculated for the general Minimum Variance approach. Still significant differences exist however: although government bonds play the most important role in the allocation, high-yield bonds have the second-highest weights. These positions were reduced after the financial crisis in 2008/09 but recovered to around 10% in 2014. Equities are only represented by a very low amount, particularly in the last few years. Instead, commodities, on average, are the third most important asset class in this allocation. Like in the case above, the difference in the minimum level of explanation leads to differences in the allocation weights. The general picture though stays the same again.

	FRP Min. Variance 66%					*FRP Min. Variance* 80%				
	EQ	*GOV*	*CORP*	*HY*	*COM*	*EQ*	*GOV*	*CORP*	*HY*	*COM*
1998	5.0%	64.7%	0.0%	22.1%	8.2%	7.4%	62.4%	0.0%	21.0%	9.2%
1999	4.7%	66.3%	0.0%	21.0%	8.0%	6.8%	64.1%	0.0%	20.0%	9.1%
2000	4.0%	68.7%	0.0%	20.7%	6.6%	5.6%	66.6%	0.0%	20.0%	7.7%
2001	4.0%	72.8%	0.0%	17.2%	6.0%	5.2%	70.8%	0.0%	16.7%	7.3%
2002	4.5%	73.7%	0.0%	16.0%	5.9%	5.8%	71.5%	0.0%	15.5%	7.2%
2003	5.1%	75.1%	0.0%	14.2%	5.6%	6.5%	72.9%	0.0%	13.8%	6.9%
2004	6.0%	74.6%	0.0%	14.3%	5.2%	7.5%	72.1%	0.0%	13.8%	6.6%
2005	7.2%	72.5%	0.0%	15.8%	4.5%	9.3%	69.5%	0.0%	15.2%	6.0%
2006	6.9%	72.9%	0.0%	15.9%	4.4%	8.9%	69.8%	0.0%	15.2%	6.0%
2007	5.1%	63.2%	0.0%	26.5%	5.2%	6.7%	60.3%	0.0%	25.5%	7.5%
2008	7.6%	64.7%	0.0%	23.0%	4.6%	9.6%	61.6%	0.0%	22.2%	6.7%
2009	2.1%	80.9%	0.5%	8.9%	7.6%	2.2%	78.4%	0.3%	8.6%	10.5%
2010	1.8%	82.0%	0.1%	8.3%	7.8%	1.6%	79.6%	0.0%	8.0%	10.8%
2011	1.8%	82.4%	0.1%	8.0%	7.6%	1.7%	80.0%	0.0%	7.7%	10.6%
2012	0.9%	84.4%	2.3%	5.6%	6.8%	0.7%	82.7%	1.9%	5.0%	9.7%
2013	0.1%	84.5%	0.0%	9.5%	5.9%	0.0%	85.2%	0.0%	5.8%	9.0%
2014	0.6%	84.8%	0.0%	10.0%	4.5%	0.5%	83.3%	0.0%	9.6%	6.5%

Table 15: *Min. Variance asset weights - 1,000 days in-the-sample window - dataset 1*

Finally, let us take a look at the risk contributions, which are plotted for factor risk parity allocations and provided in the appendix (see figure 41 to 44). The results are not surprising: the floor of the risk contribution of the first and second component and the equality in the first two components can be observed for these allocations. In dataset 1, the risk contribution of the first and second component sum up to the minimum requirement of 66% or 80% accordingly and only seldom does the explanation of the components exceed the minimum level. It seems that there is no dominating other risk contribution with all the other risk contributions floating over time.

To conclude this section, what are the findings and distinctive features from the backtest and how could they be explained or best described?

- **Volatility levels**

  First, the general MV strategy has the lowest volatility whereas the equally weighted strategy has the highest volatility. Usually the ERC or the factor risk parity strategies heavily weigh on low risk assets. As we have seen in the weighting tables, government bonds tend to be overweighted relative to the more volatile asset classes such as equities. This indicates that the ERC and the FRP allocations can, from a volatility point of view, be placed between the general MV and equally weighted allocations. The volatility of the FRP Minimum Variance allocation increases when we increase the minimum level of explanation. As mentioned above, a lower explanation level "leaves more room" to find portfolios with a lower volatility.

- **Diversification**
  Similar to the point above, the diversification increases for the FRP Maximum Diversification approach when we lower the minimum explanation requirement: decreasing the minimum explanation level increases the amount of portfolios we can use for the search of the portfolio with the highest diversification.

- **Commodity weights**
  Higher minimum explanation levels seem on average to increase the commodity weights, as the backtest using dataset 1 indicates. This might be explained by the dominating factor loadings especially on the second component. Additionally, this explains part of the lower performance as commodity as an asset class has, on average, not performed well in the last 20 years, as data from Section 3.5 shows.

- **Influence of historical data**
  As seen above, the historical data used for the in-the-sample time window does have a significant influence on the portfolio allocation. The weights, for example, change significantly once the huge price changes during the financial crisis enter the calculation of the backtest but also when they leave the in-the-sample window again. Among other things, one can see that in the allocation from Table 14 the high yield position drops in 2009 and recovers to around 30% in 2014.

Backtest dataset 2

Using dataset 2, we perform the same backtest as with dataset 1. Although we change the dataset, we expect the behavior and characteristics of all allocation strategies to be somewhat similar. We anticipate the FRP Maximum Diversification strategy and the ERC strategy as well as the FRP Minimum Variance strategy and the general MV strategy to be comparable.

For the 500 days in-the-sample window length in particular, we find that the FRP Maximum Diversification strategies perform slightly better than the ERC strategy on a total return basis. As for dataset 1, ERCs' Sharpe Ratio is still the highest for all strategies using this backtest setup.

As expected, the FRP Minimum Variance strategy outperforms the general MV strategy on a total return basis at a higher level of diversification but also a higher level of volatility.

	Return	$Mean_{Ann.}$	$Std_{Ann.}$	*SR*	Max_{DD}	$\mathcal{D}$	$\mathcal{E}$	*Turnover*
ERC	601.48%	7.47%	4.42%	0.84	16.27%	2.01	6.41	24.05%
EW	703.33%	8.01%	7.38%	0.57	32.33%	1.84	3.81	16.50%
MV	396.51%	6.10%	2.93%	0.80	11.80%	1.49	4.00	103.17%
$Max.\ Div._{66\%}$	623.99%	7.59%	5.47%	0.70	17.50%	1.94	4.86	117.05%
$Max.\ Div._{80\%}$	615.50%	7.55%	6.13%	0.62	22.81%	1.86	4.36	122.33%
$Min.\ Var._{66\%}$	436.62%	6.41%	4.18%	0.63	20.72%	1.67	5.20	79.60%
$Min.\ Var._{80\%}$	451.50%	6.52%	5.04%	0.54	25.09%	1.63	4.63	96.54%

Table 16: *Allocation strategies - 500 days in-the-sample window - dataset 2*

For the 1,000 days rolling window, an almost identical behavior of all allocations compared to the 500 days rolling window can be observed. The FRP Maximum Diversification strategy with a minimum level of explanation of 66% is an exception, as its total return performance is a bit lower than that of the ERC strategy. Compared with the results obtained using dataset 1, in general, however, we cannot observe structural differences.

	Return	$Mean_{Ann.}$	$Std_{Ann.}$	*SR*	Max_{DD}	$\mathcal{D}$	$\mathcal{E}$	*Turnover*
ERC	444.82%	7.00%	4.48%	0.77	16.17%	1.99	6.45	19.56%
EW	476.76%	7.25%	7.55%	0.49	33.24%	1.82	3.76	16.49%
MV	315.76%	5.85%	2.94%	0.79	13.51%	1.48	4.04	82.76%
$Max.\ Div._{66\%}$	429.58%	6.88%	5.25%	0.64	17.46%	1.93	4.76	66.47%
$Max.\ Div._{80\%}$	415.08%	6.76%	5.99%	0.54	22.60%	1.84	4.09	78.01%
$Min.\ Var._{66\%}$	323.25%	5.93%	4.11%	0.58	22.65%	1.73	4.67	43.48%
$Min.\ Var._{80\%}$	325.64%	5.95%	4.97%	0.49	26.70%	1.67	4.03	65.30%

Table 17: *Allocation strategies - 1,000 days in-the-sample window - dataset 2*

An FRP Maximum Diversification allocation at a minimum level of explanation of 66% and a 1,500 days in-the-sample window again performs slightly better than the ERC strategy in terms of total return. Yet, as for the results using a 500 days and a 1,000 days in-the-sample window, we cannot make out any big difference regarding the results obtained using dataset 1.

For all three backtests described above, some characteristics show up again as they have done already in dataset 1. For example, the volatility for the FRP Minimum Variance strategy increases when the minimum level of explanation is increased. Also, the diversification for the FRP Maximum Diversification strategy increases when the minimum level of explanation is decreased.

	Return	$Mean_{Ann.}$	$Std_{Ann.}$	*SR*	Max_{DD}	$\mathcal{D}$	$\mathcal{E}$	*Turnover*
ERC	316.01%	6.38%	4.56%	0.69	17.10%	1.96	6.49	18.04%
EW	364.41%	6.89%	7.59%	0.48	32.94%	1.82	3.77	16.52%
MV	245.03%	5.52%	2.95%	0.77	9.40%	1.49	3.99	98.10%
$Max.\ Div._{66\%}$	320.54%	6.43%	5.43%	0.59	15.54%	1.89	3.67	53.93%
$Max.\ Div._{80\%}$	306.98%	6.28%	6.18%	0.49	20.62%	1.80	3.05	53.83%
$Min.\ Var._{66\%}$	249.47%	5.58%	4.33%	0.54	22.17%	1.78	3.76	34.80%
$Min.\ Var._{80\%}$	250.14%	5.59%	5.14%	0.46	25.89%	1.70	3.18	36.97%

Table 18: *Allocation strategies - 1,500 days in-the-sample window - dataset 2*

The reason has already been explained and is determined by the nature of the restrictions for the portfolio weights. The time window also plays an important role in the calculation of the optimal portfolio as the changes in portfolio weights during and after the financial crisis in 2008/09 indicates. Moreover, we identify again that FRP Maximum Diversification and ERC allocation as well as FRP Minimum Variance and general MV allocation are widely comparable.

As for the backtest using dataset 1, we will continue our analysis using the 1,000 days in-the-sample window only. Figure 30 and figure 31 illustrate the performances using dataset 2.

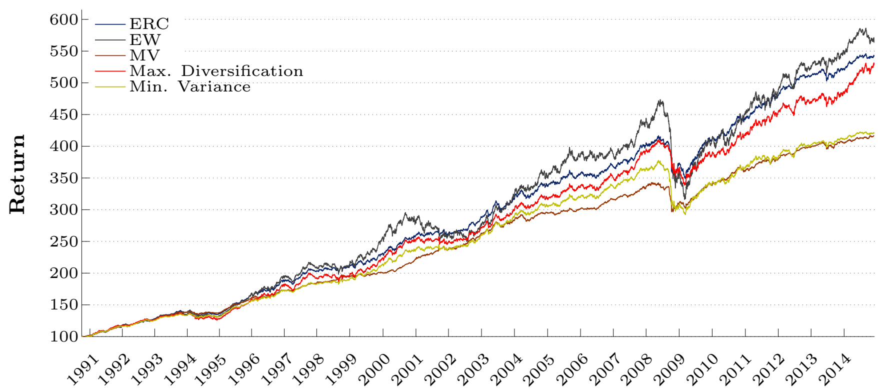

Figure 30: *Allocation strategies - 66% level of explanation - 1,000 days in-the-sample window - dataset 2*

The results are quite similar compared to the case above using dataset 1, mainly: equally weighted allocations deliver the highest returns, the FRP Minimum Variance allocations are similar to the gerneral MV strategy and the FRP Maximum Diversification allocations are somewhere in between. Differences in factor risk parity allocations at different level of explanation are hardly distinguishable.

It is worth noting that the equally weighted strategy for the longer period in this backtest is not able to outperform the ERC and the FRP Maximum Diversification strategies in terms of total return as much as in the first backtest using dataset 1.

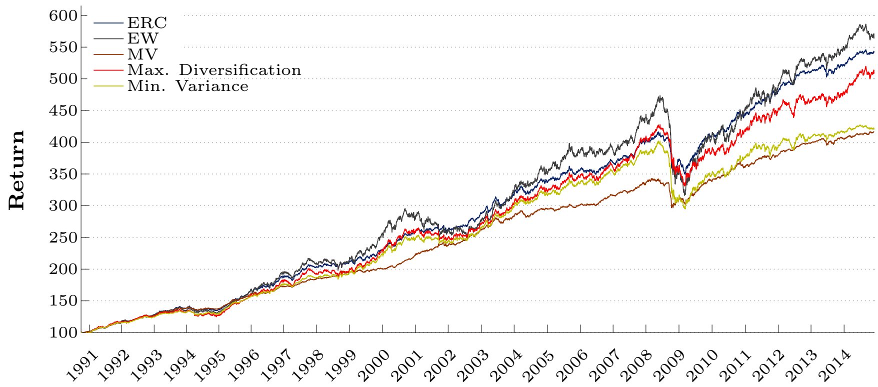

Figure 31: *Allocation strategies - 80% level of explanation - 1,000 days in-the-sample window - dataset 2*

For all allocations using a 1,000 days in-the-sample window, we calculate the detailed yearly risk and return data (see Table 32 and 33) and plot the results in figures 39 and 40. For the ERC, the EW and the general MV strategy, clearly distinct groups of data points in the risk/return plots can be observed again. As for the backtest using dataset 1, risk and return data points of the factor risk parity allocations with a minimum level of explanation of 80% are positioned further to the right compared to those with a minimum level of 66%.

Finally, let us take a look at the weights of the strategies since 1990. The choice of different indices and the exclusion of high-yield bonds already lead to differences in weighting.

On the one hand, the FRP Maximum Diversification strategies (see Table 19) load heavily on corporate bonds and quite significantly on equities for the last 20 years. On the other hand, the ERC strategy weighted equities and corporate bonds considerably too, whereas it reduced government bonds in recent years in return.

If we compare the FRP Minimum Variance allocation with the general MV strategy from Table 35, we recognize again the similarities of these two allocations: the general MV strategy has also increased corporate bond positions for lower government positions when compared to dataset 1. The equity position in both allocations, in the general MV and in the FRP Minimum Variance strategy, are significantly higher than in the same allocations computed with dataset 1.

	FRP Max. Diversification 66%				*FRP Max. Diversification* 80%			
	EQ	*GOV*	*CORP*	*COM*	*EQ*	*GOV*	*CORP*	*COM*
1990	14.2%	40.3%	27.2%	18.3%	18.7%	35.7%	23.7%	21.9%
1991	15.2%	43.9%	28.0%	13.0%	19.9%	39.8%	25.2%	15.1%
1992	0.0%	62.4%	15.1%	22.5%	0.0%	62.2%	15.0%	22.7%
1993	0.0%	58.5%	21.3%	20.2%	1.2%	55.5%	18.5%	24.8%
1994	15.9%	43.6%	23.3%	17.2%	21.8%	39.9%	21.3%	17.0%
1995	7.8%	34.3%	43.5%	14.4%	11.0%	38.5%	41.5%	9.0%
1996	15.2%	16.0%	43.4%	25.3%	15.5%	16.7%	43.2%	24.6%
1997	20.6%	16.2%	39.9%	23.2%	20.6%	16.2%	39.9%	23.2%
1998	18.4%	9.7%	51.6%	20.2%	18.8%	9.7%	50.9%	20.6%
1999	13.5%	20.0%	49.0%	17.6%	16.5%	18.1%	44.6%	20.8%
2000	12.4%	25.7%	43.8%	18.0%	15.8%	22.5%	39.3%	22.3%
2001	12.0%	24.2%	47.2%	16.7%	14.6%	21.3%	43.0%	21.1%
2002	12.6%	15.2%	56.0%	16.2%	15.2%	13.5%	50.6%	20.6%
2003	14.2%	10.7%	60.1%	14.9%	16.7%	9.2%	54.9%	19.2%
2004	17.2%	7.5%	61.3%	14.0%	20.0%	6.5%	55.4%	18.2%
2005	20.7%	2.4%	64.1%	12.8%	24.2%	2.0%	57.2%	16.7%
2006	24.0%	0.0%	65.4%	10.6%	28.4%	0.0%	57.7%	13.9%
2007	20.2%	0.2%	65.8%	13.7%	24.9%	0.2%	57.4%	17.5%
2008	19.2%	2.0%	66.6%	12.2%	22.7%	1.4%	59.9%	16.0%
2009	9.8%	12.3%	59.2%	18.7%	10.7%	10.9%	54.3%	24.1%
2010	8.2%	11.4%	59.4%	21.1%	8.5%	9.8%	54.8%	27.0%
2011	7.8%	8.8%	60.2%	23.2%	7.7%	7.2%	55.5%	29.7%
2012	5.3%	8.6%	57.7%	28.4%	12.8%	5.3%	55.6%	26.3%
2013	25.5%	0.4%	57.3%	16.9%	23.2%	0.3%	52.9%	23.6%
2014	38.3%	0.0%	50.6%	11.1%	36.3%	0.0%	48.2%	15.5%

Table 19: *Max. Diversification asset weights - 1,000 days in-the-sample window - dataset 2*

The risk contribution of the first two principal components of the FRP Minimum Variance allocation at both minimum levels of explanation sums up to the minimum level of 66% or 80% for most of the time with only one notable exception. Risk contribution for the FRP Maximum Diversification strategy is dominated by the minimum level too, but with longer exceptions in the early years of the backtest period (see figure 45 to 48). One point that we cannot observe in the previous backtest using dataset 1 is that the risk contribution of the third principal component seems to play a major role while the other risk contributions float over time.

The most obvious difference in the asset class weighting between dataset 1 and 2 is, however, the lower weight of government bonds and a higher weight of corporate bonds in dataset 2. Although the returns are quite similar, it can be seen that the choice of the asset classes does have a significant influence on the final portfolio allocation. We have already briefly analyzed the factor loadings in Section 3.6.2 and have seen the differences in the factor loadings which might explain some of the different weightings between dataset 1 and 2. Another plausible explanation might be the following: as the more volatile high-yield bonds leave the asset universe, some of those positions together with government bonds are exchanged for corporate bonds.

	FRP Min. Variance 66%				*FRP Min. Variance* 80%			
	EQ	*GOV*	*CORP*	*COM*	*EQ*	*GOV*	*CORP*	*COM*
1990	11.4%	57.7%	16.7%	14.2%	15.2%	53.4%	13.8%	17.5%
1991	11.6%	61.2%	17.9%	9.3%	15.6%	55.8%	17.2%	11.4%
1992	0.6%	77.0%	7.9%	14.5%	2.4%	68.4%	10.7%	18.5%
1993	1.6%	82.7%	1.0%	14.7%	3.3%	76.2%	1.2%	19.2%
1994	10.7%	79.7%	0.0%	9.6%	15.2%	73.5%	1.6%	9.7%
1995	4.3%	91.5%	0.0%	4.3%	8.1%	81.7%	8.1%	2.1%
1996	5.5%	79.8%	0.0%	14.7%	10.5%	72.8%	0.0%	16.7%
1997	3.9%	84.5%	0.0%	11.6%	8.8%	77.3%	0.0%	13.9%
1998	8.9%	79.1%	0.0%	12.0%	13.3%	71.6%	0.0%	15.1%
1999	10.8%	66.9%	8.4%	13.9%	14.3%	61.4%	6.5%	17.8%
2000	10.2%	50.7%	24.3%	14.9%	13.3%	47.1%	20.5%	19.1%
2001	10.1%	37.6%	38.2%	14.0%	12.5%	34.8%	34.5%	18.2%
2002	11.2%	54.1%	21.1%	13.5%	13.6%	50.0%	18.7%	17.7%
2003	12.7%	62.2%	12.2%	13.0%	15.0%	58.1%	9.9%	17.0%
2004	15.2%	72.1%	0.7%	12.0%	17.8%	66.0%	0.3%	15.9%
2005	18.0%	71.0%	0.0%	11.0%	21.5%	63.9%	0.0%	14.7%
2006	19.8%	71.7%	0.0%	8.5%	24.2%	64.3%	0.0%	11.5%
2007	17.2%	72.7%	0.0%	10.1%	21.2%	65.4%	0.0%	13.4%
2008	17.8%	63.1%	8.3%	10.8%	21.3%	56.7%	7.6%	14.5%
2009	8.1%	37.4%	40.5%	14.0%	9.0%	35.0%	37.1%	18.9%
2010	6.7%	37.4%	40.2%	15.7%	7.1%	35.1%	36.6%	21.1%
2011	6.1%	36.9%	40.3%	16.6%	6.3%	34.7%	36.6%	22.5%
2012	3.0%	37.0%	41.8%	18.2%	11.4%	25.2%	45.0%	18.4%
2013	8.3%	35.2%	46.7%	9.8%	7.4%	32.2%	46.4%	14.0%
2014	12.0%	25.7%	56.5%	5.8%	11.0%	22.6%	56.7%	9.7%

Table 20: *Min. Variance asset weights - 1,000 days in-the-sample window - dataset 2*

To sum up this brief analysis, an interpretation of why each strategy weighs some assets specifically in the way it does is difficult to give. The optimal allocation relies heavily on the results from the PCA, from factors such as the choice of time series and the choice of the final, optimal portfolio inside of the polytope. We believe that the backtest with its results, tables, and charts illustrated here, however, gives a good impression of what to expect when implementing the strategy discussed in this chapter.

3.7 Conclusion

Allocation of multi-asset portfolios presents a major challenge in capital market research and practice. A vast amount of different models have been intensively discussed in literature. The subject of this paper is a type of factor risk parity which has been discussed controversially in very recent literature. We extended the classical risk parity model and used PCA to define a factor risk parity model.

The problem, however, that comes up when using the model are the asset weights of the "optimal" portfolio which, in many cases, consist of leveraged or short positions. Due to regulatory constraints or other possible concerns or restrictions, investors are often very restrained when it comes to shorting or leveraging assets. It is, however, often not possible to find a portfolio with equal risk contributions while, at the same time, keeping the constraints that prohibit short or leveraged positions.

We therefore modified the factor risk parity model and presented one that keeps the first two risk contributions equal while, at the same time, allowing the other risk contributions of the principal components to float within special limits. The limits have to be set to keep the explanation of the first two components high relative to the total amount of variation. We therefore added the restriction that the risk contributions of the first two components explained a significant minimum amount of variance. Keeping the strict restrictions for the portfolio weights and first ignoring the minimum explanation levels it turns out that the solution set for all portfolios with those constraints are described by a convex polytope.

A convex solution set of this type is convenient as one can easily take any affine combination points inside the convex body and still stay inside the polytope, meaning the solution stays optimal regarding the goals and restrictions that have been set. Particularly, this is true for the corner points which are calculated and which describe the polytope. We make use of this special attribute to find portfolios with specific characteristics inside the polytope for the backtests. Computing the asset weights in this way has some distinct advantages versus a fully numerical optimization. By determining the polytope, one knows whether a factor risk parity solution exists and, if so, what the solution set looks like.

We selected two types, the portfolio inside the polytope which minimizes the in-the-sample variance and the portfolio that maximizes in-the-sample diversification. We then use two different datasets, one reaching back to 1994 and using 15 different assets and one reaching back to 1986 with 7 assets, to determine those portfolios and backtest them to some naive and heuristic allocations. Those backtests for the strategies and two datasets can be found in detail in Section 3.6.3.

The model presented in this paper is a more systematic approach on how to invest in a risk-approach way based on factors and should therefore integrate in the area of risk-controlled investment allocations. The solution set can conveniently be described by a polytope which is a common algebraic structure. Computer software exists in different programming languages, so it is easy to deal with questions and problems that arise when analyzing the solution set, such as the "Double Description Method" briefly described in this paper. This leaves room for plenty of changes or further development which have not been covered in this paper. Some extensions could include:

- **Flexible weight constraints**
  As mentioned above, it is possible to exchange the weight restrictions of 0 and 1 with any other number. If investors are interested to keep some assets within some individual limits, it is easy to adjust the values in the model and set any arbitrary minimum and maximum asset weight.

- **Unfunded positions**
  Taking into account strategical or tactical future positions or simple portfolio hedging means including unfunded positions in portfolio allocation. Minor model adjustments for this type of transaction are necessary to compute the polytope.

- **Arbitrary optimization for portfolios within the polytope**
  For backtesting purposes, we analyzed mainly two portfolios which, to us, appeared most appealing: the minimum variance as well as the maximum diversification allocation. It is, however, possible to select any numerical optimization approach to find an "optimal" portfolio most suitable for an investor. One option, for example, could be to find a portfolio which minimizes turnover over time, meaning one could leave the portfolio constant at time of turnover as long as the actual portfolio is still inside or nearly inside the polytope.

- **Other portfolio selections inside the polytope**
  As discussed in this paper, the polytope is described by a number of corner points. We also showed that any affine combination is still a valid portfolio in terms of asset weight constraints and risk contribution. If a solution therefore is found, it is also possible to present investors with a list of corner point portfolios and ask the investor directly to set up a portfolio consisting of those corner points. The resulting portfolio would automatically be "optimal" regarding the set goals if aspects of minimum explanation level are excluded.

- **Further research**
  Another application would be to take a fixed portfolio and find a portfolio inside the polytope with minimal distance to the original portfolio. By doing so, the original portfolio would be adjusted to result in a hopefully similar portfolio that fulfills the desired requirements.

As seen above, these are just some examples of how the model might be used or extended to fit the individual needs of investors. The model can of course also be modified in other directions as well which do not focus on the specific shape of the polytope as the solution set. One major improvement could be the change of the standard deviation as the risk measure which has been heavily criticized, especially as it assumes symmetric risk exposure without considering fat tails inherent in financial time series.

Appendix 3.A Double Description method

For the Double Describtion Method, we refer to Fukuda and Prodon (1996).

A pair (A, R) of real matrices A and R is said to be a double description pair (DD pair) if the relationship

$$Ax \geq 0 \text{ if and only if } x = R\lambda \text{ for some } \lambda \geq 0$$

holds. For such pair, the set of solutions $Q(A)$ represented by A as $Q(A) = \{x \in \mathbb{R}^d \ : \ A\,x \geq 0\}$ is simultaneously represented by R as $\{x \in \mathbb{R}^d \ : \ x = R\lambda \text{ for some } \lambda \geq 0\}$.

A subset Q of $\mathbb{R}^d$ is called polyhedral cone if $Q = Q(A)$ for some matrix A. A is called the representation matrix of $Q(A)$ and R ist called the generating matrix for Q. The following two key theorems can also be found in Fukuda and Prodon (1996):

Theorem A.1 - Minikowski's Theorem for Polyhedral Cones

For any $m \times d$ real matrix A, there exists some $d \times n$ real matrix R such that (A, R) is a double description pair, or in other words, the cone $Q(A)$ is generated by R.

Theorem A.2 - Weyl's Theorem for Polyhedral Cones

For any $d \times n$ real matrix R, there exists some $m \times d$ real matrix A such that (A, R) is a double description pair, or in other words, the set generated by R is the cone $Q(A)$.

The DD method used in this paper is an incremental algorithms to construct a $d \times m$ matrix R such that (A, R) is a DD pair. The algorithm for the standard form of DD method which produces a minimal generating set of Q.

Algorithm 3 Standard DD Method

output: R

begin

- Obtain any initial DD pair (A_k, R) such that R is minimal
- **while** $K \neq \{1, 2, ..., m\}$ **do**
  - Select any index i from $\{1, 2, ..., m\} \setminus K$
  - Construct a DD pair (A_{k+i}, R^T) from (A_k, R) /* by using following Lemma */
  - $R := R^T$; $K := K + i$
- **end**

end

Lemma

Let (A_k, R) be a DD pair such that $rank(A_k) = d$ and let i be a row index of A not in k. Then the pair (A_{k+i}, R') is a DD pair, where R' is the $d \times |J'|$ matrix with the column vector r_j $(j \in J')$ defined by

$$J' = J^+ \cup J^0 \cup Adj$$

$$Adj = \{(j, j') \in J^+ \times J^- : r_j \text{ and } r_{j'} \text{ are adjecent in } P(A_k\}$$

$$r = (A_i\, r_j)\, r_{j'} - (A_i\, r_j) r_j \text{ for each } (j, j') \in Adj$$

Furthermore, if R is a minimal generating matrix for $P(A_k)$ then R' is a minimal generating matrix for $P(A_{k+i})$.

Appendix 3.B Random vector for numerical calculations

For numerical tests, we build a uniform distributed random vector $\nu \in 1, 2, ..., n$ with the dimension $\mathbb{R}^n$. The sum of all vector elements is always 1, which is especially important for the affine combinations of the polytopes' extreme points.

Algorithm 4 Algorithm to create a random vector

---

input : Define element $\epsilon > 0$ in vector ν and set $\lambda = \frac{1}{\epsilon}$

input : Define length $\kappa \in \mathbb{N}$ of vector ν

input : Create dummy vector $z = [1, 2, 3, ..., \kappa]$

output: Random vector ν

```
begin
    while sum of all elements in ν ≠ 1 ∧ κ ≥ 1 do
        Choose X ∈ (1, ..., κ) randomly and set V = z(X)
        Choose a whole numbered element Y ∈ (1, ..., λ) for position V in vector ν
        Set λ = λ − Y
        Delete element of position Y in z s.t. κ(ν) = κ − 1
        Set κ = κ − 1
    end
    ν = ν^T ∘ ϵ
end
```

Appendix 3.C Tables and charts

	Mean	*Median*	*Mean*$_{Ann.}$	*Std*	*Std*$_{Ann.}$	*Skew*	*Kurt*
S&P 500 Index	0.04%	0.08%	9.22%	1.22%	19.36%	(0.29)	11.02
Nasdaq Index	0.05%	0.12%	12.53%	1.88%	29.71%	0.06	8.03
Russel 2000 Index	0.04%	0.11%	9.05%	1.46%	23.04%	(0.40)	8.44
1-3 Years Treasury Index	0.02%	0.01%	3.98%	0.09%	1.47%	(0.14)	9.70
5-7 Years Treasury Index	0.02%	0.03%	5.95%	0.31%	4.84%	(0.17)	6.33
10-15 Years Treasury Index	0.03%	0.04%	7.05%	0.46%	7.26%	(0.14)	5.48
1-3 Years AA US Corp. Index	0.02%	0.02%	4.75%	0.11%	1.68%	(1.40)	24.55
5-7 Years AA US Corp. Index	0.03%	0.03%	6.46%	0.31%	4.85%	(0.55)	7.97
10-15 Years AA US Corp. Index	0.03%	0.04%	8.06%	0.47%	7.45%	(0.23)	5.95
1-3 Years B US HY Index	0.04%	0.04%	9.73%	0.41%	6.41%	0.60	209.86
5-7 Years B US HY Index	0.03%	0.05%	7.07%	0.27%	4.22%	(2.02)	34.31
10-15 Years B US HY Index	0.04%	0.05%	9.07%	0.40%	6.31%	(0.45)	30.08
DJ UBS Energy Subindex	0.02%	0.06%	4.81%	1.93%	30.46%	(0.23)	5.67
DJ UBS Ind. Metal Subindex	0.02%	0.02%	3.79%	1.38%	21.85%	(0.23)	6.33
DJ UBS Prec. Metals Subindex	0.02%	0.03%	5.49%	1.13%	17.87%	(0.34)	9.93

Table 21: *Dataset 1 - 1994-2014 (1)*

	Min	*Max*	$Q_{(5\%)}$	$Q_{(1\%)}$	Ø $Q_{(5\%)}$	Ø $Q_{(1\%)}$
S&P 500 Index	(9.46%)	10.43%	(1.87%)	(3.38%)	(2.94%)	(5.03%)
Nasdaq Index	(11.12%)	17.20%	(3.03%)	(5.18%)	(4.42%)	(6.82%)
Russel 2000 Index	(12.61%)	8.17%	(2.31%)	(4.02%)	(3.46%)	(5.79%)
1-3 Years Treasury Index	(0.90%)	0.75%	(0.12%)	(0.24%)	(0.20%)	(0.34%)
5-7 Years Treasury Index	(2.25%)	2.58%	(0.47%)	(0.82%)	(0.69%)	(1.06%)
10-15 Years Treasury Index	(2.71%)	3.57%	(0.73%)	(1.27%)	(1.05%)	(1.56%)
1-3 Years AA US Corp. Index	(1.60%)	0.88%	(0.13%)	(0.27%)	(0.23%)	(0.44%)
5-7 Years AA US Corp. Index	(2.79%)	2.35%	(0.47%)	(0.83%)	(0.71%)	(1.15%)
10-15 Years AA US Corp. Index	(2.73%)	3.47%	(0.73%)	(1.24%)	(1.07%)	(1.68%)
1-3 Years B US HY Index	(9.87%)	9.25%	(0.29%)	(0.94%)	(0.79%)	(1.99%)
5-7 Years B US HY Index	(4.18%)	2.72%	(0.34%)	(0.93%)	(0.71%)	(1.43%)
10-15 Years B US HY Index	(4.79%)	4.27%	(0.51%)	(1.20%)	(1.01%)	(1.99%)
DJ UBS Energy Subindex	(14.38%)	9.81%	(3.14%)	(5.05%)	(4.46%)	(6.84%)
DJ UBS Ind. Metal Subindex	(9.02%)	7.59%	(2.17%)	(3.94%)	(3.28%)	(5.29%)
DJ UBS Prec. Metals Subindex	(10.10%)	8.76%	(1.80%)	(3.40%)	(2.77%)	(4.51%)

Table 22: *Dataset 1 - 1994-2014 (2)*

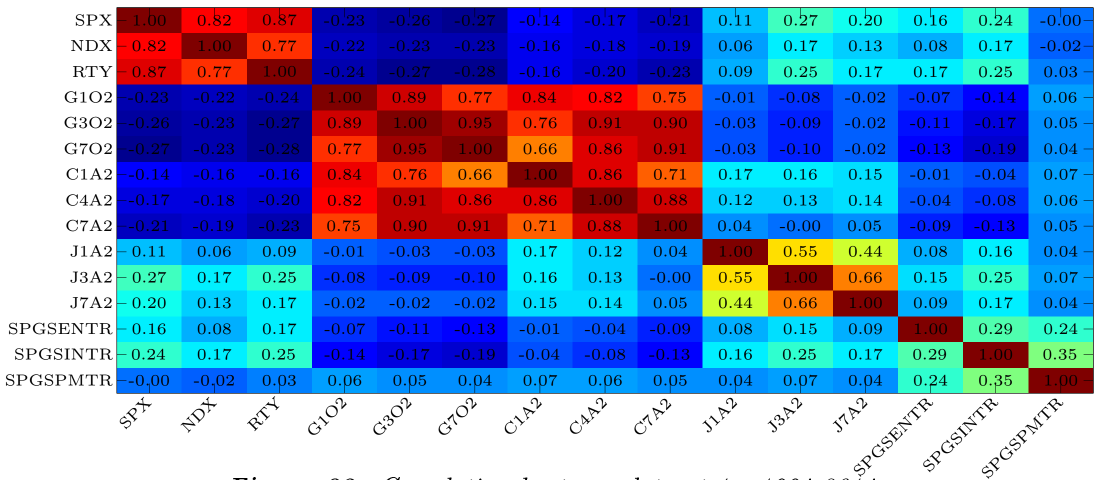

Figure 32: *Correlation heatmap dataset 1 - 1994-2014*

	PC 1	PC 2	PC 3	PC 4	PC 5	PC 6	PC 7	PC 8
SPX	0.43	(0.12)	(0.02)	(0.03)	0.27	0.14	0.21	0.81
NDX	0.66	(0.29)	(0.09)	0.16	(0.64)	(0.12)	(0.01)	(0.15)
RTY	0.51	(0.12)	(0.01)	(0.02)	0.65	0.15	(0.24)	(0.47)
G1O2	(0.01)	0.00	(0.00)	0.02	(0.02)	0.09	(0.02)	0.00
G3O2	(0.04)	(0.00)	(0.01)	0.09	(0.08)	0.36	(0.09)	0.01
G7O2	(0.06)	(0.01)	(0.01)	0.13	(0.13)	0.53	(0.13)	(0.00)
C1A2	(0.01)	0.00	0.00	0.02	(0.02)	0.11	0.02	(0.01)
C4A2	(0.03)	0.01	0.00	0.07	(0.06)	0.37	0.02	(0.01)
C7A2	(0.05)	(0.00)	(0.00)	0.12	(0.12)	0.57	(0.07)	0.00
J1A2	0.02	0.02	0.03	(0.03)	0.02	0.10	0.61	(0.23)
J3A2	0.03	0.01	0.02	(0.03)	0.04	0.07	0.37	(0.10)
J7A2	0.03	0.01	0.03	(0.03)	0.03	0.12	0.59	(0.18)
SPGSENTR	0.27	0.87	(0.42)	(0.02)	(0.06)	0.02	(0.02)	(0.00)
SPGSINTR	0.20	0.29	0.79	(0.47)	(0.16)	0.08	(0.09)	(0.00)
SPGSPMTR	0.05	0.24	0.45	0.84	0.12	(0.15)	0.05	0.03

Table 23: *Principal components loadings dataset 1 - 1994-2014 (1)*

	PC 9	PC 10	PC 11	PC 12	PC 13	PC 14	PC 15
SPX	0.04	(0.03)	(0.01)	0.01	0.00	0.00	0.00
NDX	(0.01)	0.01	(0.00)	(0.01)	0.00	(0.00)	0.00
RTY	(0.00)	(0.01)	0.00	(0.00)	(0.00)	0.00	(0.00)
G1O2	0.01	0.01	(0.17)	(0.19)	0.44	0.29	0.81
G3O2	0.01	(0.04)	(0.37)	(0.12)	0.63	(0.44)	(0.33)
G7O2	0.01	(0.17)	(0.29)	0.66	(0.21)	0.25	0.05
C1A2	0.01	0.10	(0.14)	(0.35)	0.10	0.78	(0.46)
C4A2	(0.01)	0.22	(0.34)	(0.53)	(0.58)	(0.22)	0.15
C7A2	0.02	0.04	0.78	(0.14)	0.08	(0.01)	0.02
J1A2	0.72	(0.21)	(0.01)	(0.00)	0.00	(0.01)	0.00
J3A2	(0.11)	0.86	(0.01)	0.30	0.12	(0.01)	0.00
J7A2	(0.68)	(0.36)	0.00	(0.05)	0.00	0.00	0.00
SPGSENTR	(0.00)	(0.01)	0.00	0.01	0.00	0.00	(0.00)
SPGSINTR	(0.00)	(0.01)	(0.01)	0.01	0.00	0.00	(0.00)
SPGSPMTR	0.00	(0.00)	0.01	(0.00)	(0.00)	(0.00)	0.00

Table 24: *Principal components loadings dataset 1 - 1994-2014 (2)*

	Mean	*Median*	$Mean_{Ann.}$	*Std*	$Std_{Ann.}$	*Skew*	*Kurt*
S&P 500 Index	0.04%	0.04%	9.02%	1.15%	18.26%	(0.85)	25.06
Russel 2000 Index	0.05%	0.08%	11.44%	1.43%	22.60%	(0.02)	11.06
1-3 Years Treasury Index	0.02%	0.02%	5.80%	0.22%	3.47%	0.15	9.81
15+ Years Treasury Index	0.03%	0.04%	8.38%	0.67%	10.60%	(0.01)	6.31
1-3 Years US Corp. Index	0.03%	0.03%	6.40%	0.19%	3.07%	(0.38)	7.72
15+ Years US Corp. Index	0.03%	0.04%	8.43%	0.61%	9.60%	1.02	116.17
GS Commodity Index	0.03%	0.01%	7.17%	1.28%	20.24%	(0.43)	10.90

Table 25: *Dataset 2 - 1986-2014 (1)*

	Min	*Max*	$Q_{(5\%)}$	$Q_{(1\%)}$	$\varnothing\ Q_{(5\%)}$	$\varnothing\ Q_{(1\%)}$
S&P 500 Index	(20.47%)	11.58%	(1.71%)	(3.08%)	(2.73%)	(4.81%)
Russel 2000 Index	(11.35%)	14.17%	(2.28%)	(4.08%)	(3.46%)	(5.63%)
1-3 Years Treasury Index	(1.83%)	2.64%	(0.32%)	(0.59%)	(0.49%)	(0.75%)
15+ Years Treasury Index	(3.60%)	6.12%	(1.06%)	(1.81%)	(1.53%)	(2.25%)
1-3 Years US Corp. Index	(1.73%)	1.50%	(0.29%)	(0.54%)	(0.45%)	(0.72%)
15+ Years US Corp. Index	(12.44%)	14.99%	(0.82%)	(1.44%)	(1.27%)	(2.14%)
GS Commodity Index	(16.83%)	7.89%	(1.99%)	(3.46%)	(3.02%)	(4.92%)

Table 26: *Dataset 2 - 1986-2014 (2)*

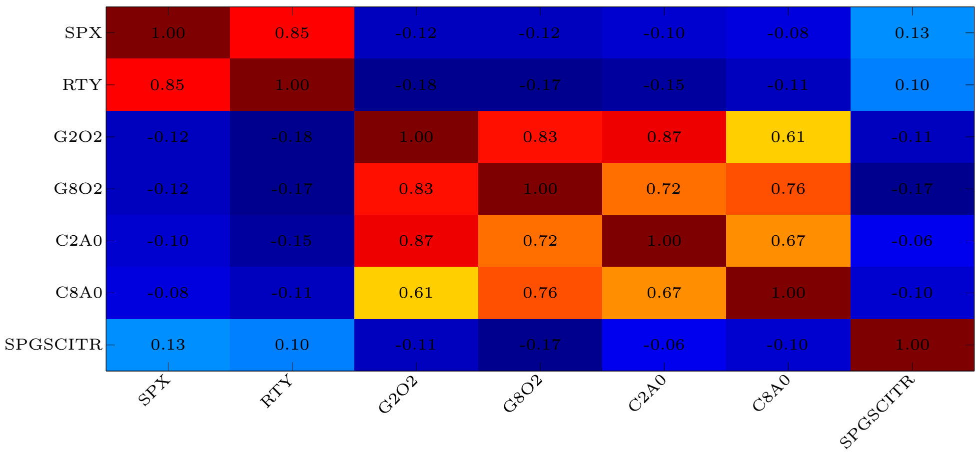

Figure 33: *Correlation heatmap dataset 2 - 1986-2014*

	Mean	*Median*	$Mean_{Ann.}$	*Std*	$Std_{Ann.}$	*Skew*	*Kurt*
Principal Component 1	0.06%	0.11%	14.15%	1.79%	28.31%	(0.33)	13.43
Principal Component 2	0.01%	(0.00%)	2.31%	1.27%	20.06%	(0.39)	11.72
Principal Component 3	0.06%	0.07%	15.64%	0.86%	13.53%	0.03	9.44
Principal Component 4	(0.00%)	(0.01%)	(1.02%)	0.49%	7.79%	0.23	41.80
Principal Component 5	(0.00%)	0.00%	(0.57%)	0.32%	5.01%	2.81	570.01
Principal Component 6	0.02%	0.02%	5.26%	0.16%	2.51%	(0.39)	10.64
Principal Component 7	(0.00%)	(0.00%)	(0.34%)	0.06%	1.00%	2.23	69.92

Table 27: *Principal components of dataset 2 - 1986-2014 (1)*

	Min	*Max*	$Q_{(5\%)}$	$Q_{(1\%)}$	Ø $Q_{(5\%)}$	Ø $Q_{(1\%)}$
Principal Component 1	(21.38%)	16.76%	(2.81%)	(5.11%)	(4.34%)	(7.27%)
Principal Component 2	(17.56%)	9.04%	(2.00%)	(3.49%)	(2.96%)	(4.75%)
Principal Component 3	(7.01%)	9.59%	(1.33%)	(2.24%)	(1.93%)	(2.90%)
Principal Component 4	(9.00%)	9.25%	(0.70%)	(1.33%)	(1.10%)	(1.93%)
Principal Component 5	(10.23%)	11.37%	(0.29%)	(0.60%)	(0.56%)	(1.21%)
Principal Component 6	(1.57%)	1.54%	(0.23%)	(0.43%)	(0.36%)	(0.62%)
Principal Component 7	(0.87%)	1.16%	(0.07%)	(0.16%)	(0.13%)	(0.26%)

Table 28: *Principal components of dataset 2 - 1986-2014 (2)*

	PC 1	PC 2	PC 3	PC 4	PC 5	PC 6	PC 7
SPX	0.60	(0.10)	0.10	0.79	0.02	(0.01)	(0.00)
RTY	0.77	(0.17)	0.01	(0.61)	(0.03)	0.01	0.00
G2O2	(0.02)	(0.01)	0.17	(0.00)	(0.05)	0.72	(0.67)
G8O2	(0.05)	(0.08)	0.63	(0.08)	0.76	(0.03)	0.07
C2A0	(0.03)	(0.02)	0.20	0.01	(0.22)	0.62	0.73
C8A0	(0.08)	(0.12)	0.71	(0.03)	(0.60)	(0.32)	(0.11)
SPGSCITR	0.18	0.97	0.15	(0.03)	(0.02)	(0.01)	0.00

Table 29: *Principal components loadings of dataset 2 - 1986-2014*

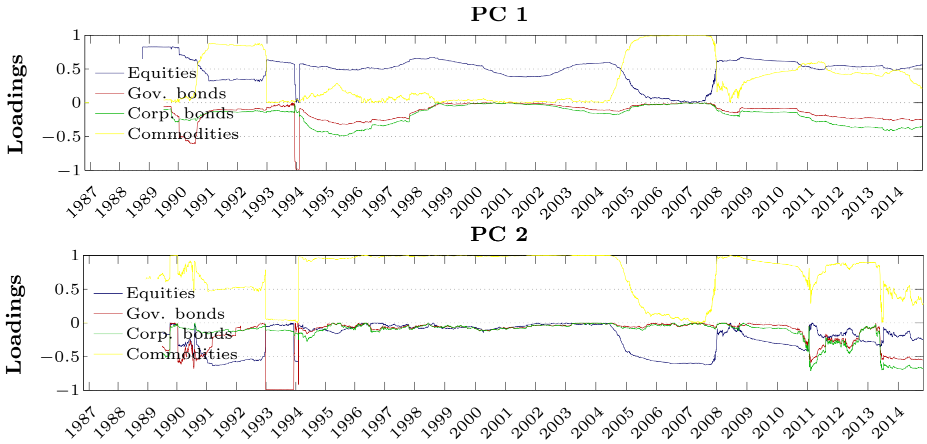

Figure 34: *Principal components' loadings - 500 Days in-the-sample window - dataset 2*

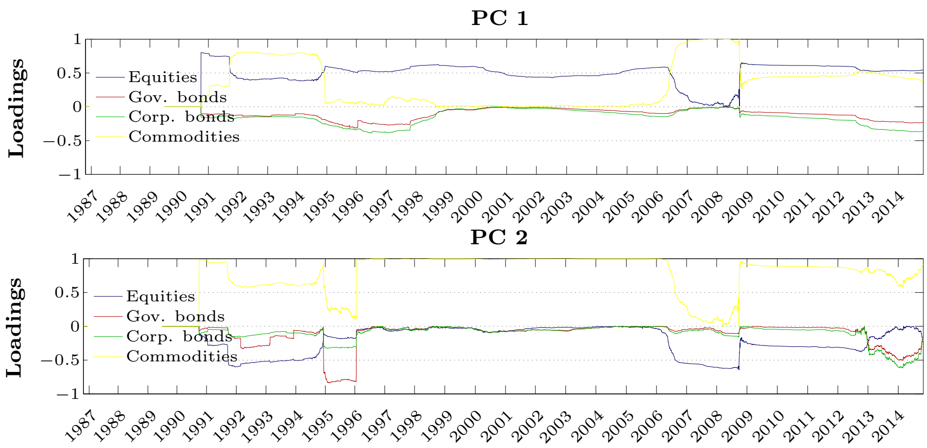

Figure 35: *Principal components' loadings - 1,000 Days in-the-sample window - dataset 2*

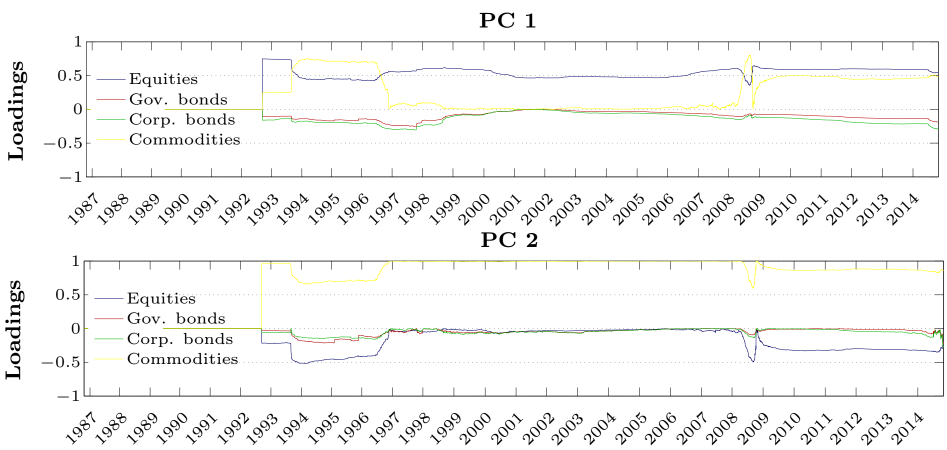

Figure 36: *Principal components' loadings - 1,500 Days in-the-sample window - dataset 2*

	Returns			$Std_{Ann.}$		
	ERC	*EW*	*MV*	*ERC*	*EW*	*MV*
1998	2.01%	2.98%	0.64%	1.87%	4.48%	1.49%
1999	6.62%	15.71%	4.02%	2.28%	5.39%	1.11%
2000	6.73%	4.18%	7.54%	2.26%	7.20%	1.11%
2001	4.29%	(2.42%)	6.43%	2.65%	6.79%	1.69%
2002	7.05%	3.21%	5.88%	2.64%	5.98%	1.50%
2003	11.81%	18.10%	6.03%	2.36%	4.33%	1.33%
2004	5.73%	8.36%	2.67%	3.06%	5.48%	1.45%
2005	5.73%	9.99%	2.67%	2.11%	4.36%	0.95%
2006	7.00%	7.57%	4.97%	2.38%	5.72%	0.98%
2007	6.00%	8.31%	6.23%	2.11%	5.15%	1.30%
2008	(9.80%)	(18.40%)	(4.11%)	5.65%	10.90%	3.28%
2009	12.38%	22.40%	4.54%	3.58%	9.44%	1.53%
2010	7.74%	11.51%	3.47%	2.57%	6.82%	1.00%
2011	3.95%	2.38%	1.63%	2.48%	7.35%	0.95%
2012	4.62%	6.78%	1.42%	1.43%	4.96%	0.42%
2013	1.57%	3.15%	1.13%	1.65%	4.79%	0.54%
2014	2.38%	3.43%	0.87%	1.07%	3.26%	0.61%

Table 30: *Risk/Return data - naive and heuristic strategies - 1,000 days in-the-sample window - dataset 1*

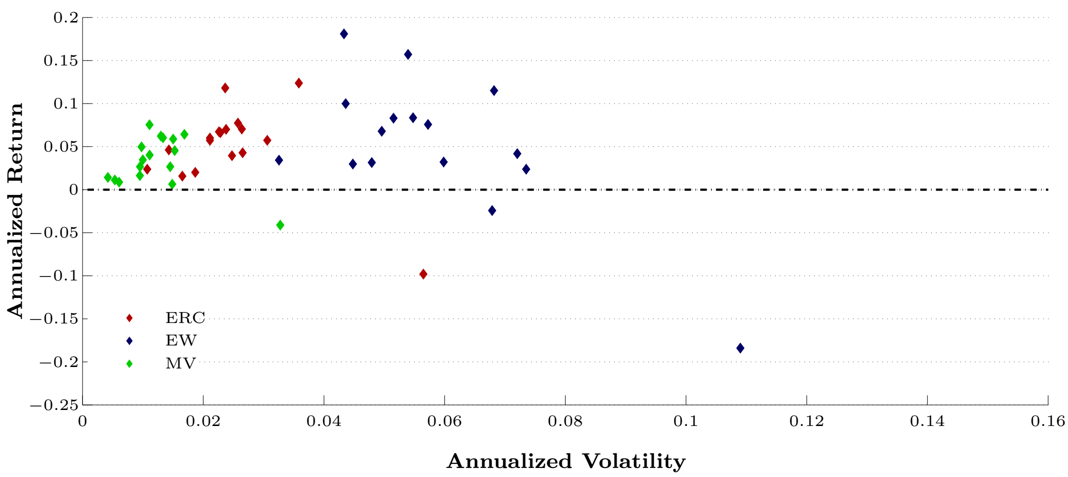

Figure 37: *Risk/Return plot - naive and heuristic strategies - 1,000 days in-the-sample window - dataset 1*

	Returns				$Std_{Ann.}$			
	Max. Div.		*Min. Variance*		*Max. Div.*		*Min. Variance*	
	66%	88%	66%	88%	66%	88%	66%	88%
1998	1.48%	1.65%	1.40%	1.54%	2.76%	3.53%	1.86%	2.45%
1999	12.19%	14.55%	9.73%	11.74%	2.68%	3.36%	2.06%	2.66%
2000	7.08%	7.28%	7.68%	7.88%	2.98%	3.86%	2.34%	3.10%
2001	2.23%	0.98%	4.10%	3.09%	2.92%	3.56%	2.14%	2.61%
2002	7.37%	7.20%	5.58%	5.52%	3.09%	3.69%	1.83%	2.33%
2003	12.15%	12.84%	7.51%	8.06%	2.67%	3.16%	1.75%	2.14%
2004	5.99%	6.39%	3.69%	4.03%	3.40%	3.90%	2.10%	2.50%
2005	6.80%	7.47%	4.20%	4.72%	2.61%	3.18%	1.74%	2.17%
2006	6.84%	6.46%	4.75%	4.56%	3.26%	3.86%	1.68%	2.09%
2007	7.00%	7.17%	6.09%	6.14%	2.65%	3.31%	1.65%	2.08%
2008	(8.66%)	(10.88%)	(8.41%)	(10.36%)	5.63%	6.56%	4.18%	5.06%
2009	10.02%	11.03%	6.82%	7.45%	3.49%	4.46%	2.69%	3.45%
2010	6.33%	6.43%	3.96%	4.04%	2.82%	3.48%	1.92%	2.45%
2011	2.97%	2.49%	1.03%	0.78%	2.99%	3.76%	2.03%	2.59%
2012	2.88%	2.62%	1.31%	1.04%	2.19%	2.79%	1.24%	1.65%
2013	(2.01%)	(3.22%)	0.08%	(0.94%)	3.51%	4.07%	1.04%	1.47%
2014	4.14%	3.18%	0.70%	0.54%	2.65%	2.95%	0.67%	0.79%

Table 31: *Risk/Return data - FRP strategies - 1,000 days in-the-sample window - dataset 1*

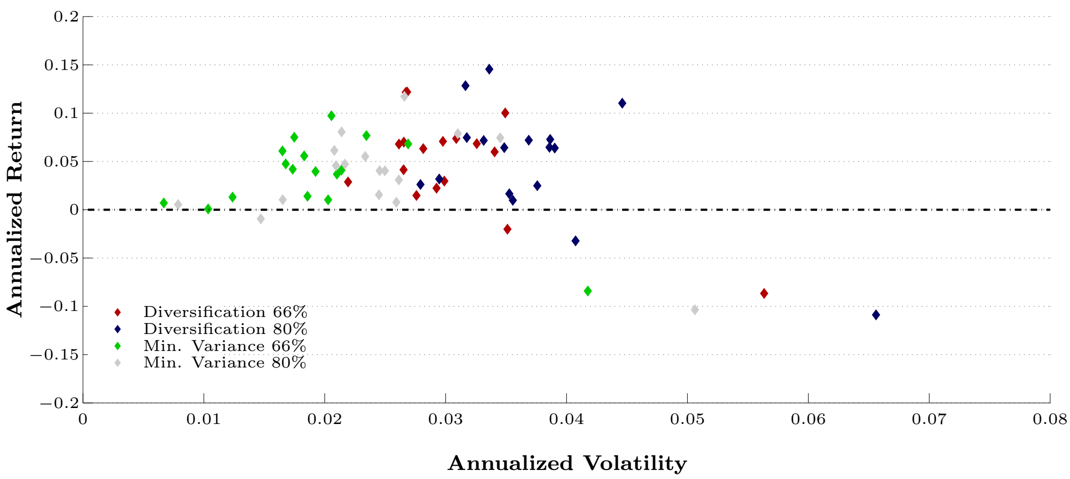

Figure 38: *Risk/Return plot - FRP strategies - 1,000 days in-the-sample window - dataset 1*

	Returns			*$Std_{Ann.}$*		
	ERC	*EW*	*MV*	*ERC*	*EW*	*MV*
1990	2.27%	1.39%	2.43%	5.06%	8.08%	2.49%
1991	16.82%	18.50%	14.88%	3.79%	6.52%	1.95%
1992	8.42%	7.96%	8.44%	3.45%	4.82%	2.59%
1993	7.63%	5.49%	8.61%	3.08%	4.22%	2.22%
1994	(1.84%)	(1.86%)	(0.58%)	4.32%	5.40%	3.04%
1995	23.22%	26.96%	17.37%	3.92%	4.78%	2.98%
1996	10.87%	13.64%	6.20%	5.98%	7.18%	3.67%
1997	8.51%	8.24%	6.88%	4.74%	6.18%	2.57%
1998	3.53%	2.27%	5.55%	4.82%	7.34%	3.09%
1999	9.03%	19.07%	2.61%	5.18%	7.95%	2.92%
2000	13.06%	10.45%	11.43%	4.88%	10.03%	2.57%
2001	0.97%	(8.32%)	6.65%	5.32%	9.55%	3.96%
2002	9.58%	6.11%	10.27%	4.06%	8.43%	3.08%
2003	9.57%	15.25%	7.87%	5.05%	7.14%	3.45%
2004	6.23%	7.80%	4.23%	5.05%	7.79%	3.27%
2005	6.17%	10.65%	2.52%	4.30%	7.30%	2.46%
2006	2.81%	(0.10%)	4.56%	3.87%	6.50%	2.33%
2007	10.37%	15.29%	6.57%	3.88%	6.50%	2.59%
2008	(7.98%)	(18.57%)	(8.20%)	7.32%	14.13%	5.45%
2009	10.33%	14.30%	9.20%	6.17%	12.20%	3.92%
2010	8.04%	9.68%	6.55%	4.40%	8.08%	2.83%
2011	8.15%	8.44%	4.84%	4.01%	8.18%	2.54%
2012	6.50%	7.48%	5.09%	2.57%	5.61%	1.40%
2013	2.01%	4.17%	1.58%	3.21%	5.28%	2.22%
2014	4.08%	4.66%	2.50%	2.41%	4.26%	1.80%

Table 32: *Risk/Return data - naive and heuristic strategies - 1,000 days in-the-sample window - dataset 2*

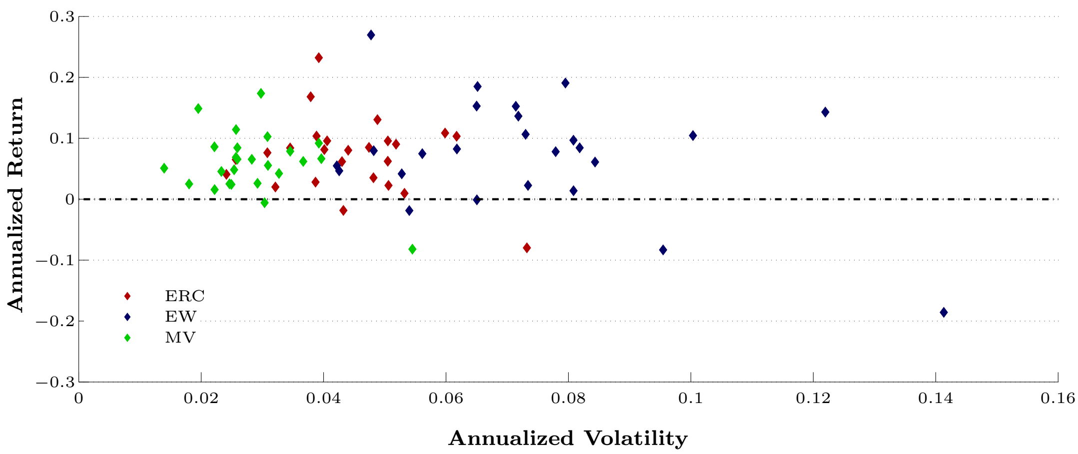

Figure 39: *Risk/Return plot - naive and heuristic strategies - 1,000 days in-the-sample window - dataset 2*

	Returns				$Std_{Ann.}$			
	Max. Div.		*Min. Variance*		*Max. Div.*		*Min. Variance*	
	66%	88%	66%	88%	66%	88%	66%	88%
1990	1.81%	1.30%	1.58%	1.16%	6.03%	7.11%	4.88%	5.84%
1991	16.69%	17.08%	15.68%	15.99%	4.32%	5.00%	3.27%	3.87%
1992	7.86%	7.85%	8.08%	7.74%	3.99%	4.00%	2.63%	2.81%
1993	5.26%	4.73%	6.15%	5.31%	2.93%	3.26%	2.37%	2.59%
1994	(4.34%)	(4.33%)	(1.86%)	(1.93%)	6.13%	6.41%	3.95%	4.30%
1995	24.40%	25.99%	19.36%	21.45%	4.20%	4.71%	3.28%	4.01%
1996	12.10%	11.92%	9.16%	10.59%	6.40%	6.35%	4.32%	4.74%
1997	6.67%	6.67%	6.07%	6.27%	5.50%	5.50%	2.94%	3.33%
1998	2.23%	2.38%	4.20%	3.79%	5.86%	5.95%	3.62%	4.41%
1999	13.25%	16.24%	11.99%	15.40%	5.56%	6.23%	4.41%	5.27%
2000	13.41%	12.88%	11.23%	11.06%	6.06%	7.61%	5.19%	6.70%
2001	(1.83%)	(4.31%)	0.23%	(2.20%)	5.93%	6.94%	5.12%	6.09%
2002	9.17%	8.91%	8.85%	8.72%	4.63%	5.76%	4.05%	5.10%
2003	9.53%	10.82%	10.61%	11.78%	4.95%	5.58%	4.21%	4.87%
2004	5.42%	5.78%	5.22%	5.59%	5.16%	5.93%	4.39%	5.16%
2005	6.71%	7.87%	5.38%	6.57%	4.50%	5.31%	3.92%	4.74%
2006	3.66%	3.56%	4.57%	4.40%	4.60%	5.36%	3.69%	4.46%
2007	13.33%	14.24%	9.23%	10.33%	4.41%	5.25%	3.39%	4.19%
2008	(7.89%)	(11.90%)	(16.19%)	(19.10%)	7.89%	9.76%	8.31%	10.13%
2009	6.35%	7.30%	10.90%	11.40%	7.46%	8.95%	5.73%	7.17%
2010	7.67%	7.76%	7.07%	7.23%	5.27%	6.25%	4.18%	5.16%
2011	8.93%	8.32%	4.85%	4.84%	5.49%	6.74%	4.43%	5.64%
2012	3.25%	2.76%	4.60%	3.40%	4.71%	5.46%	3.24%	4.46%
2013	3.22%	2.63%	2.22%	1.84%	5.04%	5.22%	2.52%	2.72%
2014	8.88%	7.39%	2.44%	1.75%	4.47%	4.51%	1.92%	2.05%

Table 33: *Risk/Return data - FRP strategies - 1,000 days in-the-sample window - dataset 2*

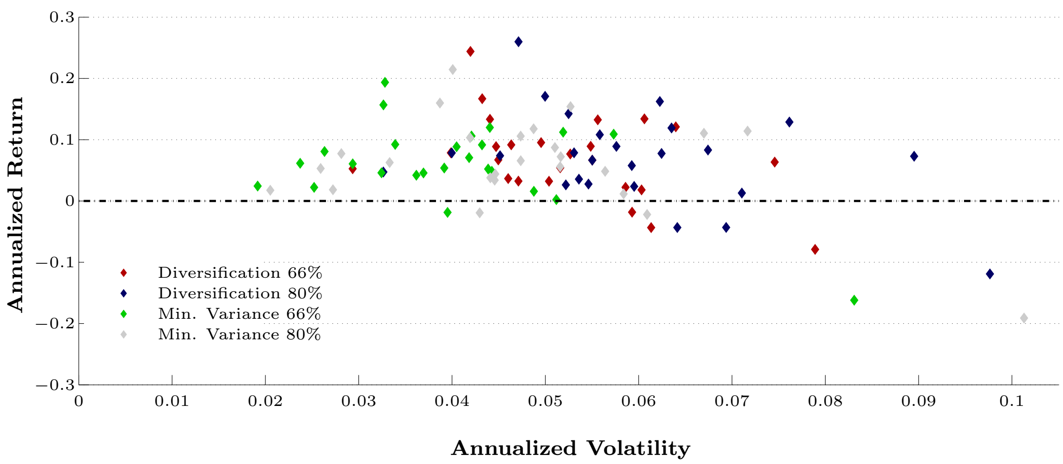

Figure 40: *Risk/Return plot - FRP strategies - 1,000 days in-the-sample window - dataset 2*

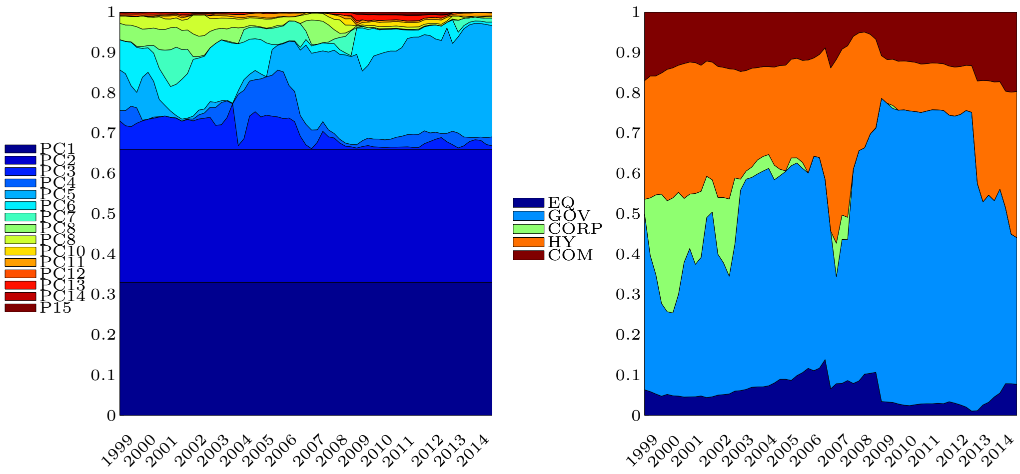

Figure 41: *Risk contribution and asset weights - Max. Diversification strategy 66% - dataset 1*

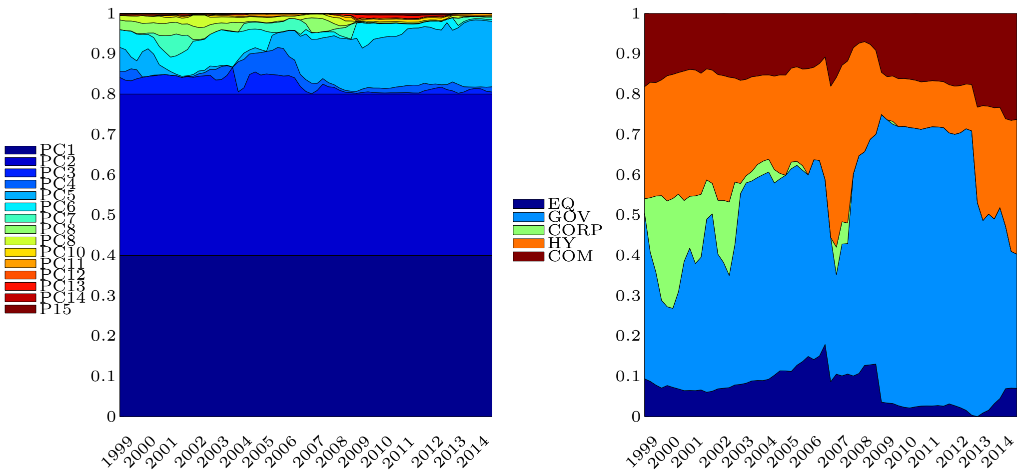

Figure 42: *Risk contribution and asset weights - Max. Diversification strategy 80% - dataset 1*

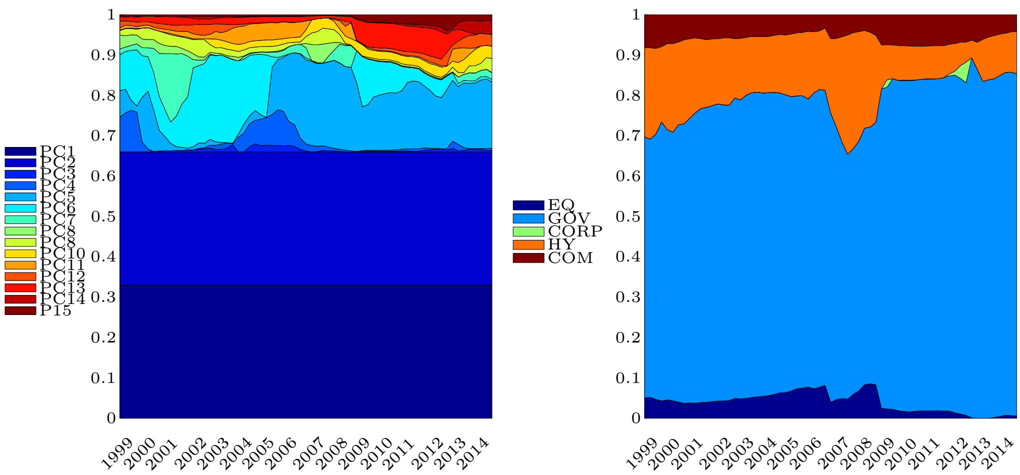

Figure 43: *Risk contribution and asset weights - Min. Variance strategy 66% - dataset 1*

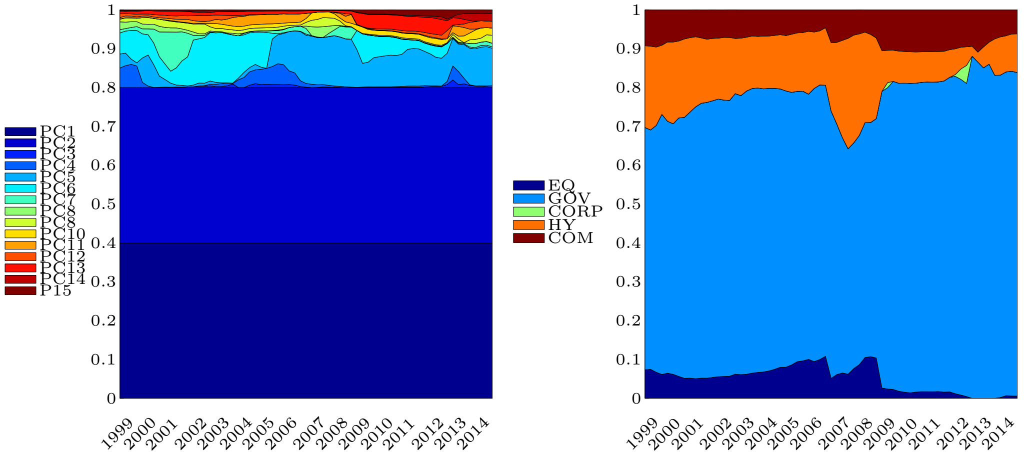

Figure 44: *Risk contribution and asset weights - Min. Variance strategy 80% - dataset 1*

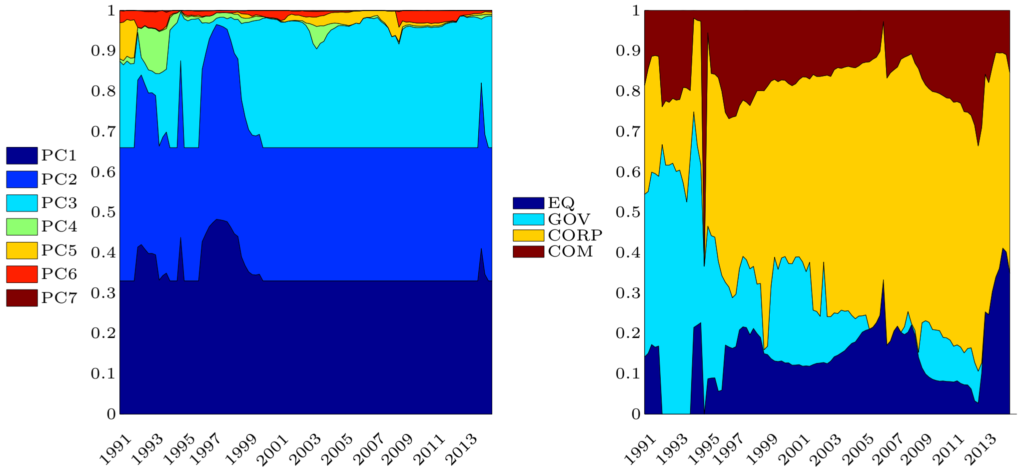

Figure 45: *Risk contribution and asset weights - Max. Diversification strategy 66% - dataset 2*

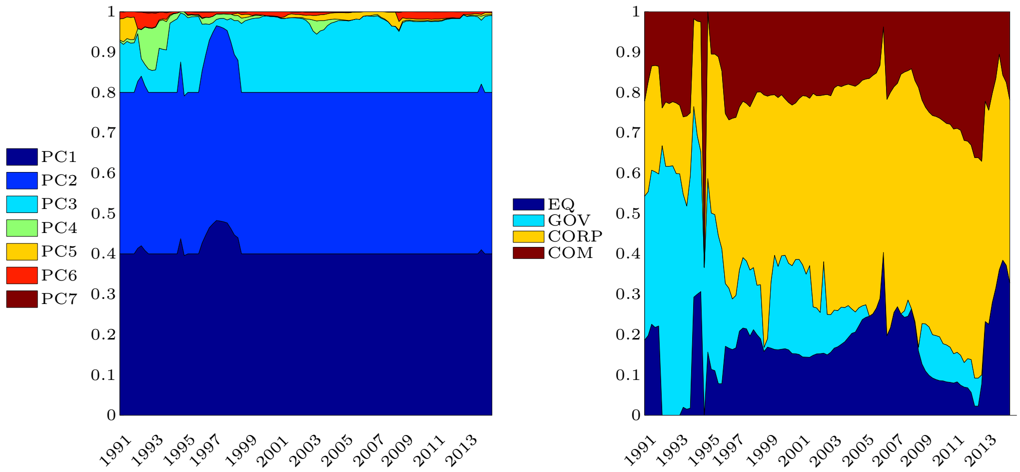

Figure 46: *Risk contribution and asset weights - Max. Diversification strategy 80% - dataset 2*

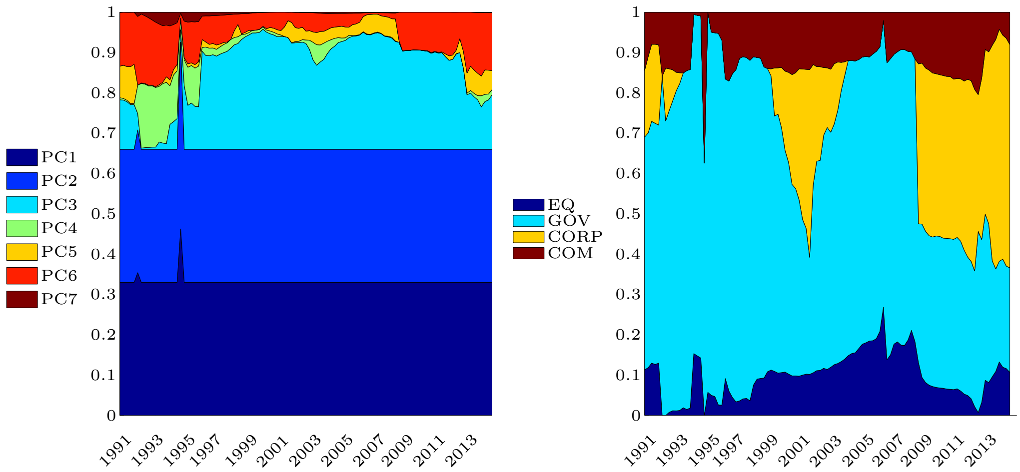

Figure 47: *Risk contribution and asset weights - Min. Variance strategy 66% - dataset 2*

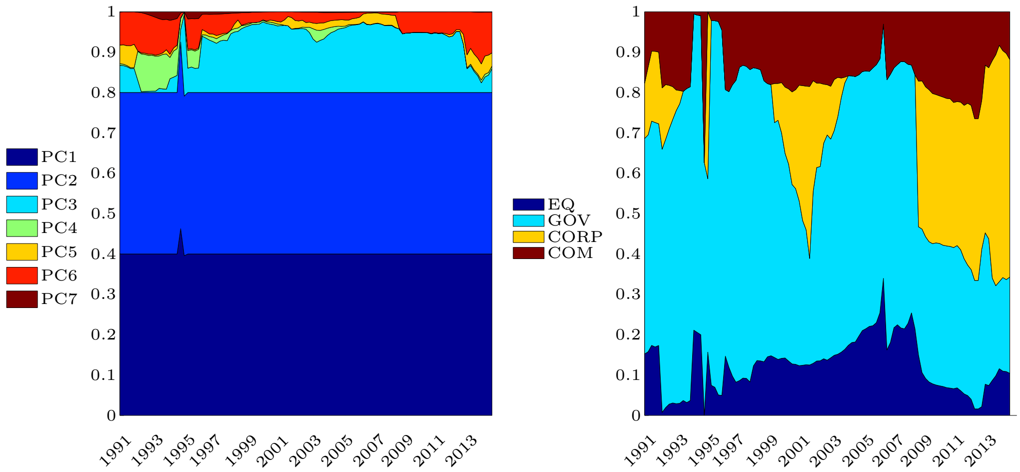

Figure 48: *Risk contribution and asset weights - Min. Variance strategy 80% - dataset 2*

	ERC					*MV*				
	EQ	*GOV*	*CORP*	*HY*	*COM*	*EQ*	*GOV*	*CORP*	*HY*	*COM*
1998	5.6%	25.3%	23.7%	33.6%	11.8%	0.0%	82.2%	0.0%	13.1%	4.7%
1999	5.3%	25.8%	24.1%	33.5%	11.3%	0.0%	77.3%	0.0%	18.6%	4.1%
2000	4.4%	27.6%	24.7%	33.8%	9.5%	0.2%	67.8%	7.6%	21.8%	2.7%
2001	4.1%	30.2%	26.1%	30.8%	8.8%	0.8%	77.8%	0.1%	19.1%	2.1%
2002	4.6%	30.6%	26.8%	28.7%	9.4%	1.2%	69.4%	9.5%	18.0%	1.8%
2003	5.0%	32.3%	27.8%	25.2%	9.7%	1.9%	55.2%	25.4%	16.1%	1.5%
2004	5.8%	32.1%	27.7%	25.1%	9.2%	2.3%	60.7%	20.3%	16.0%	0.6%
2005	6.7%	31.7%	27.6%	25.9%	8.1%	2.5%	59.7%	20.9%	16.9%	0.0%
2006	6.9%	31.8%	27.8%	27.0%	6.5%	2.4%	70.5%	11.6%	15.5%	0.0%
2007	6.7%	27.4%	25.1%	35.9%	4.9%	1.6%	69.6%	9.2%	19.7%	0.0%
2008	7.0%	30.3%	27.1%	30.7%	4.8%	3.0%	62.4%	18.8%	15.8%	0.0%
2009	6.2%	44.2%	28.3%	15.7%	5.6%	2.4%	89.4%	0.3%	7.8%	0.1%
2010	5.9%	45.1%	28.0%	15.4%	5.7%	2.4%	89.8%	0.0%	7.6%	0.2%
2011	5.8%	45.6%	27.8%	15.1%	5.7%	2.4%	90.3%	0.0%	7.2%	0.1%
2012	4.9%	49.4%	27.3%	13.4%	5.0%	1.8%	92.4%	0.9%	4.9%	0.1%
2013	4.2%	44.1%	28.5%	19.6%	3.7%	1.2%	91.8%	0.0%	7.0%	0.0%
2014	3.9%	44.5%	29.4%	18.9%	3.3%	1.1%	98.6%	0.0%	0.3%	0.0%

Table 34: *ERC and MV asset weights - 1,000 days in-the-sample window - dataset 1*

	ERC				*MV*			
	EQ	*GOV*	*CORP*	*COM*	*EQ*	*GOV*	*CORP*	*COM*
1990	10.8%	44.3%	30.1%	14.8%	1.8%	85.1%	8.9%	4.2%
1991	11.2%	44.8%	31.8%	12.2%	1.6%	85.7%	9.7%	3.0%
1992	13.6%	42.9%	32.2%	11.3%	0.6%	83.5%	13.3%	2.6%
1993	13.6%	43.3%	31.7%	11.4%	1.4%	94.2%	1.7%	2.8%
1994	14.5%	42.3%	30.7%	12.5%	0.8%	96.0%	0.0%	3.2%
1995	15.5%	38.0%	28.2%	18.3%	0.6%	92.0%	0.0%	7.4%
1996	16.8%	36.2%	28.2%	18.8%	0.1%	92.6%	0.0%	7.3%
1997	16.7%	35.9%	30.0%	17.5%	0.0%	94.1%	0.0%	5.9%
1998	14.7%	36.7%	32.5%	16.1%	0.1%	69.7%	25.8%	4.3%
1999	12.5%	37.7%	35.3%	14.4%	1.4%	67.7%	27.5%	3.4%
2000	10.5%	38.7%	37.3%	13.4%	1.8%	61.4%	33.3%	3.5%
2001	9.5%	39.8%	38.6%	12.2%	3.5%	48.6%	44.7%	3.1%
2002	10.0%	40.5%	37.8%	11.7%	4.6%	56.0%	36.6%	2.8%
2003	10.6%	40.7%	37.6%	11.1%	6.1%	79.1%	12.0%	2.7%
2004	12.6%	39.9%	36.6%	11.0%	7.7%	90.1%	0.0%	2.2%
2005	14.9%	38.6%	36.1%	10.5%	8.7%	81.8%	7.6%	1.9%
2006	16.9%	37.8%	35.7%	9.5%	9.1%	64.5%	25.2%	1.3%
2007	18.4%	37.5%	34.9%	9.2%	8.0%	74.1%	17.0%	0.9%
2008	16.8%	37.4%	36.8%	8.9%	9.3%	67.0%	22.5%	1.2%
2009	11.7%	38.0%	41.6%	8.6%	6.5%	42.8%	49.3%	1.5%
2010	11.4%	38.0%	41.7%	8.9%	6.4%	42.6%	49.2%	1.9%
2011	11.4%	37.8%	41.7%	9.1%	6.5%	41.8%	49.8%	1.9%
2012	10.3%	38.2%	42.8%	8.8%	5.0%	39.9%	52.7%	2.3%
2013	11.4%	38.1%	41.9%	8.7%	5.7%	40.1%	52.5%	1.7%
2014	11.4%	37.3%	42.1%	9.2%	5.3%	31.6%	61.5%	1.6%

Table 35: *ERC and MV asset weights - 1,000 days in-the-sample window - dataset 2*

Appendix 3.D Matrices and variables

Name	Space	Explanation
n	$\mathbb{N}$	Number of assets
k	$\mathbb{N}$	Number of time periods with $k < n$
x	$\mathbb{R}^{k \times n}$	Daily asset returns
ρ	$\mathbb{R}$	Correlation of two assets with $-1 \leq \rho \leq 1$
σ, σ_i	$\mathbb{R}$	Volatility of P or of asset i
Σ	$\mathbb{R}^{n \times n}$	Covariance matrix of x
w	$\mathbb{R}^n$	Asset weights
P	$\mathbb{R}^k$	Daily portfolio returns $(x \cdot w)$
MRC_i	$\mathbb{R}$	Marginal risk contribution of asset i
RC_i	$\mathbb{R}$	Risk contribution of asset i
m	$\mathbb{N}$	Number of equalized risk factors
C	$\mathbb{R}^{n \times n}$	Principal component mixing matrix
$\widetilde{x}$	$\mathbb{R}^{k \times n}$	Principal components $(x \cdot C)$
$\widetilde{\sigma}$,$\widetilde{\sigma}_i$	$\mathbb{R}$	Volatility of $\widetilde{P}$ or $\widetilde{x}_i$
$\widetilde{\Sigma}$	$\mathbb{R}^{n \times n}$	Covariance matrix of $\widetilde{x}$
$\widetilde{w}$	$\mathbb{R}^n$	Principal component weights $(C^T \cdot w)$
$\widetilde{P}$	$\mathbb{R}^k$	Daily principal component portfolio returns $(\widetilde{x} \cdot \widetilde{w})$
$\widetilde{MRC_i}$	$\mathbb{R}$	Marginal risk contribution of a principal component i
$\widetilde{RC_i}$	$\mathbb{R}$	Risk contribution of a principal component i
e_{min}	$\mathbb{R}$	Minimum level of explanation with $0 \leq e_{min} \leq 1$
A	$\mathbb{R}^{z \times n}$	Matrix used for polytope definition
b	$\mathbb{R}^z$	Inequalities solutions for polytope definition
Q	$\mathbb{R}^n$	Polytope as subset of $\mathbb{R}^n$
q	$\mathbb{R}^{n \times r}$	Finite number of polytope corner points, $Q = conv\{q_1, ..., q_r\}$
β	$\{-1, 1\}^2$	Principal component directions

4 Tail Driven Factor Risk Parity with Volatility Investments

4.1 Introduction

Risk factors have in recent years become a key factor in the development of asset allocation strategies (see, inter alia, Fama and French (1993) and Clarke et al. (2005)). One group of concepts dealing with risk factors are Factor Risk Parity (FRP) allocations (Bhansali et al. (2012), Roncalli and Weisang (2012), Bruder and Roncalli (2012), Lohre et al. (2012), Kind (2013)). Such strategies try to combine the characteristics of risk parity allocations on the one hand and investments made along predefined risk factors on the other. Due to fat tails in the asset return distributions, the use of the standard deviation as the risk measure in the context of these strategies has been heavily criticised. If the standard deviation is replaced and further asset weight constraints added, the computations get complex.

We will contribute to the existing literature in two points: first, the standard deviation as the risk measure is replaced by the expected shortfall in the FRP model. Due to heavily skewed asset return distributions, the expected shortfall seems to be a good choice to capture special properties in the tails of the asset distributions. This risk measure is preferred to the value at risk as we strive for a robust risk estimation. We focus on equal risk contributions in the first few risk contributions only to avoid leverage or short sales. Second, volatility as an asset class is examined against the backdrop of an strategic FRP allocation. A volatility short strategy will be set up due to their attractive return profile during longer market periods of contango in the volatility term structure. It is analyzed to what extend this strategy replaces equity or commodity investments in a multi-asset portfolio while at the same time serving as an assset with a skewed and fat-tailed return distribution.

Using a portfolio allocation along a downside risk measure is not a new concept. Boudt et al. (2013) introduced the minimum Conditional Value at Risk (CVaR) portfolio in their paper. They find that the ex ante application of a downside risk measure is able to determine portfolios with a low portfolio risk, a low risk concentration and a high portfolio diversification. In an unconstrained setup, they find that the minimum CVaR portfolio is similar to the equal risk to contribution portfolio. In a Gaussian world, where asset returns are normally distributed, the risk parity, and therefore the factor risk parity solutions derived with the value at risk or the expected shortfall, do not differ from that using the standard deviation as the risk measure (Bruder and Roncalli, 2012). The VIX futures and their relationship to VIX price changes are described and explained in Zhang et al. (2010). Eraker and Wu (2013) and Asensio (2013) explore the negative yield premium in long VIX futures investments and thereby serve as a motivation for our volatility short strategy.

In contrast to the commonly used application of volatility futures as a hedging against equity losses, this paper focuses on using short volatility positions to profit from possible positive rolling effects when futures are rolled over. Within the factor risk parity framework, the incorporation of such volatility strategies has a positive impact on the return, Sharpe Ratio and Maximum Drawdown properties. Compared to a simple equal risk to contribution strategy, the positive impact of the volatility strategies is greater when using the factor risk parity model. Due to heavily skewed return distributions of the VIX-future strategies, the use of the expected shortfall is particularly appropriate. Although data history of volatility futures is limited to ten years, some conclusions can be drawn from the results we receive.

The structure of this part is as follows: Section 4.2 presents some related literature. Section 4.3 presents the theory for the marginal expected shortfall and component expected shortfall before describing in detail the factor risk parity model with the expected shortfall as the risk measure. In Section 4.4, we introduce the volatility short strategy and its detailed setup. Data used for the backtests are described in Section 4.5. The optimal weight ratio of the first two principal components is analyzed in Section 4.6.1 using the standard deviation versus the expected shortfall as the risk measure before performing the out-of-sample backtest of the factor risk parity model (Section 4.6.2). Section 4.7 concludes. Appendix 4.A provides the theory of the closed-form model for the modified expected shortfall. In appendix 4.B we validate the results computed in Section 4.6.2 using the modified expected shortfall. Appendix 4.C provides additional figures and tables.

4.2 Related literature

Contributing to the existing risk parity and factor risk parity literature, this paper replaces the standard deviation by the expected shortfall within a factor risk parity framework as defined in the second study above. In a Gaussian world, where asset returns are normally distributed, the risk parity, and therefore the factor risk parity solutions derived with the value at risk or the expected shortfall, does not differ from that using the standard deviation as the risk measure Bruder and Roncalli (2012). Baitinger et al. (2017) also focus on higher moments, stressing how incorporating higher risk moment terms can also lead to better performance when the underlying data exhibit tailed distributions.

As the number of observations in the tail of a real world distribution is always limited to a smaller number, the robustness of the derived factor risk parity solutions is a crucial issue. Against the backdrop of an asset allocation along the tails of the relevant distributions, we follow the works of Härdle et al. (2014) and Mercuri and Rroji (2014) and make use of the closed-form model for the (marginal) modified expected shortfall to validate the results using the non-parametric calculation for the expected shortfall.

As the second contribution of the existing risk parity and factor risk parity literature, an equity volatility strategy is set up in a multi-asset allocation context. In contrast to the general terminology of volatility, which has been thoroughly researched in numerous mathematical and financial literature, volatility as an own investment in form of volatility futures or volatility/variance swaps is a relatively young innovation and therefore less covered by research so far. When talking about volatility futures specifically, the research can be grouped into two different topics: one deals with pricing issues, volatility curves, etc. and the other one focuses on embedding the volatility futures in an investment setup. The work of Zhu and Lian (2012) for example belongs to the first group of papers. The authors present a closed-form solution for the pricing of VIX futures under a stochastic volatility model with jumps in asset prices and in the volatility. The implications of using VIX futures in the context of a risk parity approach discussed in this paper, however, belongs to the second group of papers which deal with volatility futures in an investment setup. Signori et al. (2010), for example, present two volatility strategies and a calibrated combination of those two. The first strategy they discuss is taking a long volatility position by a long VIX future position with an adjusted exposure depending on the absolute volatility level. They use this strategy to benefit from the strong negative correlations to equity markets to establish a hedge, particularly, in weak market periods. The other strategy discussed is to capture the volatility risk premium by using variance swaps. A calibrated approach combining these two strategies is then analyzed through a mean/modified Value-at-Risk optimization with the result that the absolute and risk-adjusted return of the portfolio for a nearly 20-year historical dataset is significantly enhanced when the calibrated strategy is added to an equity portfolio.

Avellaneda and Papanicolaou (2018) analyze VIX and VIX futures together with corresponding ETNs/ETFs. As in our analysis they find large volatility and skewness in the financial data of those products. By using a 2-factor lognormal model with mean-reverting factors, they show the profitability of constantly shorting VIX futures based on data between 2011 and 2016. They also argue, however, that due to surges in the VIX during market turmoil the Sharpe ratio is with around 0.5 somewhat modest.

Simon and Campasano (2012) deal with two questions: the first one covers the aspect of predictability of future VIX spot movements based on the VIX basis and the other deals with a long/short VIX future strategy. This strategy consists of short VIX future positions if the basis is in contango and long VIX future positions if the VIX term structure is in backwardation. The underlying market risk inherent in those VIX futures is then hedged by S&P futures. Exit rules for the VIX futures are added in a final chapter of the paper to illustrate how to improve the risk/reward relation of the strategy.

Other papers also deal with backtest and the risk/return characteristics of volatility futures: Stanescu and Tunaru (2012) test the use of those instruments for improving Sharpe Ratios by adding them to an equity portfolio and to a balanced equity/bond portfolio. The results show improved risk and return figures for European and US portfolios, particularly, due to hedging effects of the volatility futures during stress periods at the equity markets. They also present a long/short model for the VIX and VSTOXX futures using GARCH forecasting.

4.3 Factor risk parity model

4.3.1 Expected shortfall as the risk measure

To recap the conditions in the factor risk parity model, using the notation from the second study above, and to introduce unfunded positions, the following restrictions which are formulated here as a set of linear equalities and inequalities have to hold:

1. **Risk parity condition**
   The risk contributions of the first two principal components have to be equal. This condition can be written as

$$\begin{pmatrix} \widetilde{\sigma_1} & -\widetilde{\sigma_2} & 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} \widetilde{w_1} \\ \vdots \\ \widetilde{w}_n \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{74}$$

2. **Positive weights in the asset space**
   Prohibiting short selling means that the condition $w = C\,\widetilde{w} \geq 0$ must be fulfilled.

3. **Funded asset weights sum up to 1**
   Expression $\sum {w_i}_{i=1}^n = 1$ only applies to funded assets. Unfunded assets, such as volatility futures, have to be excluded. To include unfunded portfolio positions to the model, the

constraint defined in the part above has to be modified. We redefine $c_{i,j}$ being the elements of the transformation matrix C from the PCA for funded positions and

$$\hat{c}_{i,j} = \begin{cases} c_{i,j}, & \text{if asset } i \text{ is funded} \\ 0, & \text{if asset } i \text{ is unfunded} \end{cases}, \tag{75}$$

The sum for funded positions are constituted by $C_i = \sum_{j=1}^{n}(\hat{c}_{i,j})$ and exclude unfunded positions from the sum of all elements in the i-th column in the matrix C. One can then write the constraints as:

$$\begin{pmatrix} C_1 & C_2 & \cdots & C_n \end{pmatrix} \begin{pmatrix} \widetilde{w_1} \\ \vdots \\ \widetilde{w}_n \end{pmatrix} = 1. \tag{76}$$

In the second study above, the marginal risk contribution has been calculated with the standard deviation as the risk measure. The standard deviation, however, may not be perfectly suitable as historical return distributions often exhibit fat tails or are not symmetrically distributed. This is even more distinct if asset classes such as volatility are included in the multi-asset portfolio. The expected shortfall as a non-parametric tail-risk measure therefore appears suitable to capture special distributional properties.

Using the quantile function $VaR_\alpha(z) = F_z^{-1}(\alpha)$ for the value at risk to the quantile α, the expected shortfall for a continuously distributed random variable z to the same probability level α is defined as

$$ES_\alpha(z) := \frac{1}{1-\alpha}\int_\alpha^1 VaR_u(z)du. \tag{77}$$

Given the portfolio $P := x\ w \in \mathbb{R}^k$ consisting of empirical time series, the empirical expected shortfall of portfolio P can be written in the form of

$$ES_\alpha(P) = \mu\left[P|P \leq VaR_\alpha(P)\right], \tag{78}$$

with $\mu\left[P|x\right]$ as the conditional expectation of P under x. Rewriting the portfolio using $P = \sum_i x_i\ w_i$, the empirical portfolio expected shortfall can be decomposed into

$$ES_\alpha(P) = \sum_i w_i\ \mu\left[x_i|P \leq VaR_\alpha(P)\right]. \tag{79}$$

As it can be seen below, this decomposition is important for calculating the marginal expected shortfall (MES_i). Following Tasche (2002), the marginal expected shortfall is equal to the first order partial derivative of expression (79). The component expected shortfall (CES_i), which can be interpreted as the risk contribution of the empirical time series x_i to the empirical portfolio expected shortfall $ES_\alpha(P)$, is then the marginal expected shortfall multiplied by the weights of the time series x_i in the portfolio:

$$MES_i = \frac{\partial ES_\alpha}{\partial w_i} = \mu[x_i | P \leq VaR_\alpha(P)], \tag{80}$$

$$CES_i = w_i \frac{\partial ES_\alpha}{\partial w_i} = w_i \; \mu[x_i | P \leq VaR_\alpha(P)]. \tag{81}$$

The marginal expected shortfall is equal to the the average value of asset i if the portfolio loss is bigger than the historical value at risk to the quantile α.

For historical scenarios, this value can be calculated by first taking the change in the portfolio value for each scenario, sorting them to take the cutoff point for determining the value at risk and finally taking the average value of price change for asset i for all those scenarios where the portfolio loss is larger than $VaR_\alpha(P)$.

Given the portfolio $\widetilde{P} := \widetilde{x}\, \widetilde{w}$ and the quantile function $VaR_\alpha(\widetilde{P}) = F^{-1}_{\widetilde{P}}(\alpha)$ for the principal component time series, one can formulate an equation similar to the expression in the standard deviation case for the expected shortfall as the risk measure:

$$\widetilde{w}_1 \; \widetilde{MES}_1(\widetilde{w}) = \widetilde{w}_2 \; \widetilde{MES}_2(\widetilde{w}). \tag{82}$$

So far, the solutions for $\widetilde{w}_1$ and $\widetilde{w}_2$ to this equation is not trivial as $\widetilde{MES}_1$ and $\widetilde{MES}_2$ depend on $\widetilde{w}$. In the case of the standard deviation, the i-th marginal risk contribution only depends on weight $\widetilde{w_i}$ which simplifies the equations significantly. The orthogonal attribute of the principal components is not sufficient to guarantee that $\widetilde{MES}_i$ does not also depend on a different weight j. Rewriting the equation (82) may clarify those dependencies:

$$\widetilde{w}_1 \; \mu[\widetilde{x}_1 | (\widetilde{x}\, \widetilde{w}) \leq VaR_\alpha(\widetilde{x}\, \widetilde{w})] = \widetilde{w}_2 \; \mu[\widetilde{x}_2 | (\widetilde{x}\, \widetilde{w}) \leq VaR_\alpha(\widetilde{x}\, \widetilde{w})]. \tag{83}$$

The solution to the equation depends heavily on the form of the assets' distributions and thereby on the principal components' distributions. Before proceeding with the FRP model, some general statements are first made. Later, closed form solutions for the marginal expected shortfall under different distribution assumptions will be presented.

Nonetheless, a numerical process for computing the FRP model weights $\widetilde{w}$ is needed and will be described in the last part of this section.

General statements

As mentioned above in the case of the standard deviation as the risk measure, the ratio of the weights $\frac{\widetilde{w_1}}{\widetilde{w_2}}$ is equal for all risk parity solutions in the principal component space. This is a direct consequence of the attribute that only the i-th weight $\widetilde{w}_i$ changes the i-th marginal risk contribution $\widetilde{MRC}_i$.

When using the expected shortfall we allow returns to also be non-normally distributed. As financial data are analyzed, we expect distributions e.g. to be fat tailed and skewed. Regardless of the asset or principal component return distributions, the component expected shortfall is homogeneous, i.e.

$$CES_i(c \cdot w) = c \cdot CES_i(w), \tag{84}$$

with $c > 0$. Particularly, it follows that if a portfolio has equal component expected shortfalls, so does a portfolio with weights $c \cdot w$ for $c > 0$. The proof is straightforward as for given w and $c > 0$, due to positive homogeneity of the value at risk,

$$\begin{aligned} CES_i(c \cdot w) &= c \cdot w_i \; E[x_i | c \cdot x \leq VaR_\alpha(c \cdot x)] \\ &= c \cdot w_i \; E[x_i | c \cdot x \leq c \cdot VaR_\alpha(x)] \\ &= c \cdot CES_i(w). \end{aligned} \tag{85}$$

So far, we know that scaling the weights of a factor risk parity portfolio with a factor $c > 0$ means that the new scaled portfolio stays a factor risk parity portfolio when the expected shortfall as the risk measure is used. This, in general, is only true if the scaling factor is positive. It will be shown later that, in the case of the normal or elliptical distribution assumptions when the distributions are symmetrical with mean zero, the solution is still a solution to the problem even if all the optimal portfolio weights are multiplied by minus one, meaning that scaling by a factor of $c < 0$ is a valid operation. Therefore, the characteristic of scaling the weights of a factor risk parity portfolio with the expected shortfall for the normal and elliptical distributions is the same as in the general case with the standard deviation as the risk measure.

With skewed distributions the remarks above are incorrect. The reason for this is that by using skewed distributions and the expected shortfall as a risk measure, a left skewed distribution becomes a right skewed distribution and vice versa when the sign of the weight changes.

Closed form solution - normal case

Let X denote asset return, where X is a normally distributed random variable with mean μ and variance σ^2, $X \sim \mathcal{N}(\mu, \sigma^2)$. Following Boudt et al. (2008), the component expected shortfall in this case can be written as

$$CES_i = w_i \frac{(\Sigma w)_i}{\sqrt{w^T \Sigma w}} \frac{1}{\alpha} \phi[\Phi^{-1}[\alpha]], \tag{86}$$

with α being the corresponding quantile, $\phi()$, $\Phi()$ and $\Phi^{-1}()$ being the standard Gaussian density, distribution and quantile functions. As the time series are normally distributed, it is not surprising that the first part of equation (86) is identical to the equation in the standard deviation case. As the term $\frac{1}{\alpha}\phi[\Phi^{-1}[\alpha]]$ is independent of i, the term crosses out when we set $CES_i = CES_j$. This leads to the same solution set when we compare the standard deviation versus the expected shortfall results with normal distributed returns.

Closed form solution - elliptical case

Landsman and Valdez (2003) extend the calculation of the component expected shortfall to the larger class of elliptical distributions which includes the case of the normal distribution above. Multivariate elliptical distributions are those where the characteristic function can be expressed as

$$\varphi_X(t) = exp\left(it^T \mu\right) \psi\left(\frac{1}{2} t^T \Sigma t\right), \tag{87}$$

for some column vector $\mu \in \mathbb{R}^n$, complex number i, a positive-definite matrix $\Sigma \in \mathbb{R}^{n \times n}$ and some function $\psi(t) : [0, \infty) \to \mathbb{R}$ such that $\psi(\sum_{i=1}^n (t_i^2))$ is an n-dimensional characteristic function. In this case the solution can be given as

$$CES_i = w_i \mu_i + \lambda_P w_i \sigma_i \sigma_P \rho_{X_i,P} \tag{88}$$

for some λ_P as given in Landsman and Valdez (2003) (be aware that $CES_i = w_i \cdot MES_i$). Note that the paper discusses the case $P = X_1 + \cdots + X_n$. However, without loss of generality, X_i can be replaced by $w_i X_i$ as linear combinations of elliptical distributions are elliptical distributions again. If we assume μ_i to be zero and X_i to be independent,

$$\begin{aligned} CES_i &= w_i\mu_i + \lambda_P w_i\sigma_i\sigma_P \frac{Cov(w_iX_i, P)}{w_i\sigma_i\sigma_P} \\ &= \lambda_P Cov\left(w_iX_i, \sum_{k=1}^{n} w_kX_k\right) \\ &= \lambda_P w_i^2\sigma_i^2. \end{aligned} \tag{89}$$

For CES_i to be equal to CES_j,

$$\lambda_P w_i^2\sigma_i^2 = \lambda_P w_j^2\sigma_j^2. \tag{90}$$

Therefore, the solution is the same as in the case using the standard deviation as the risk measure.

For the sake of completeness, the (marginal) modified expected shortfall (see appendix 4.A) is additionally described. The idea is to approximate the tail by using a Cornish-Fisher expansion. This approach will be used for validating the numerical results determined in Section 4.6.1.

Numerical solutions

As discussed above, a numerical process for computing FRP portfolios in the principal component space of solutions under the expected shortfall as the risk measure is required for non-normally distributed returns. The computations now consist of 2 steps: the calculation of the component expected shortfall followed by finding the optimal weights so that the component expected shortfall of the first two components are equal.

Given the weights w, the calculation of the component expected shortfall is straightforward as given directly by equations (81) and (82). We first determine the portfolio in the factor space, determine the tail scenarios for the portfolio, take a component i and average the scenario changes for exactly those tail scenarios.

Being able to determine the CES and MES this way, the optimal portfolio can be found numerically by minimizing the squared difference between the CES of the first 2 principal components. The general optimization problem to achieve equality in the first m principal components can be written as:[6]

$$f^* = argmin\left(\sum_{i=1}^{m}\sum_{j<i}\left(w_i\frac{\partial ES_\alpha}{\partial w_i} - w_j\frac{\partial ES_\alpha}{\partial w_j}\right)^2\right), \tag{91}$$

[6]For all further considerations, m<n as the model seeks to reduce the number of risk factors

or equivalently

$$f^* = argmin \left(\sum_{i=1}^{m} \sum_{j<i} (w_i \; MES_i - w_j \; MES_j)^2 \right). \tag{92}$$

The constraints we define for the optimization problem above are directly taken from the standard deviation case. These include:

- Asset weights sum up to 1
- Non-negative asset weights - short sale prohibited
- Minimum explanation level of the first two risk contributions of at least 66%

As mentioned above, an additional objective function, similar to the case where the standard deviation is used, is needed to narrow the choices down to one portfolio. The allocations we want to analyze are the

- Minimum Variance allocation and
- Maximum Diversification allocation.

Both allocations and the constraints have already been explained in the second study above.

4.4 Volatility investments

4.4.1 Volatility futures - introduction

Volatility futures, such as the VIX futures, have become increasingly important financial instruments in the investment industry since they have been established in 2004. Figure 49 illustrates the increase in open interest in recent years.

One field of application is using VIX futures as an equity hedge instrument through a long position in the volatility futures. However, long positions usually suffer from a contango in the volatility term structure, which is often observable during longer stable market periods. Eraker and Wu (2013) studied the returns of VIX futures and quantified the negative return premium implied in long VIX futures investments. The paper indicates that another application - constantly shorting the VIX future - might be an attractive investment strategy for an investor. The strategy would imply to receive the premium by shorting longer dated VIX futures and rolling over the position closer to maturity. The technique is similar to the concept of "riding the yield curve", where investors profit by a contango in the yield curve through investing in longer dated bonds and, all else being equal, gaining from the passage of time. As mentioned in Bhansali and Harris (2018), this investment strategy has in the meantime become so popular that people start

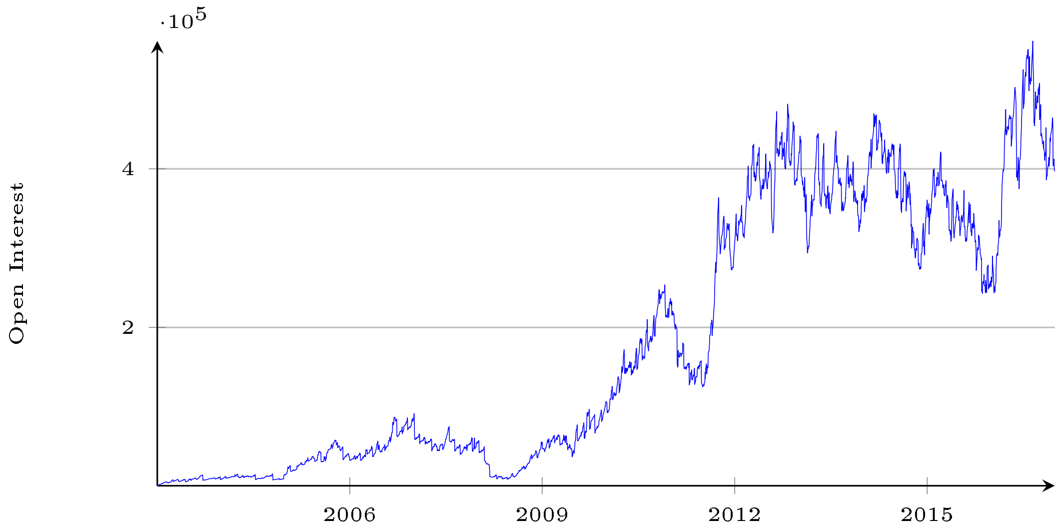

Figure 49: *VIX Future open interest at CBOE from 2004 to 2016. Source: CBOE*

warning that the strategy may trigger the next serious market crash.

Directly related to the question of contango versus backwardation is the question of overpriced VIX futures relative to the VIX itself. Asensio (2013) analyzed this characteristic. Some of his conclusions are:

1. VIX futures are consistently overpriced relative to subsequent moves of the VIX index.
2. This overpricing is more marked for longer term futures.
3. The overpricing is less during periods of extreme volatility.

As confirmed by Avellaneda and Papanicolaou (2018), the next subsection will show that selling VIX futures and continuously rolling over close to expiry should be an attractive investment strategy. The authors, however, also mention the problem of tail events in the VIX futures' distributions which occur in the case of stock market downturns when volatility, and with it volatility futures, spike. Therefore, it can be of interest for an investor to see how this strategy embeds in a risk-based portfolio construction process, especially when the expected shortfall as a risk measure is used.

In the following sections we do not try to approach the investment strategy from an analytical point of view by, for example, explaining the structure of the volatility curve, etc., but from an exclusively empirical point of view to answer the following questions:

1. As long volatility and long equity positions are strongly negatively correlated, how does the strategy of constantly selling volatility futures affect the long equity positions and other asset classes in our factor risk parity allocation?
2. If shorter dated volatility futures are continuously replaced by longer dated ones, what influence does the rolling strategy have?

4.4.2 The volatility short strategy in detail

The question we are facing now is how to construct a time series that represents an investable volatility short strategy. In constrast to VIX futures, the VIX index is not an investable asset.

One point an investor therefore has to decide is which VIX future to sell. He has the choice of longer versus shorter dated futures. For that, we will take a closer look at three-months and one-month futures. The difference is quite big and should not be neglected. Longer dated VIX futures usually are less volatile with a lower sensitivity toward movements of the underlying: Table 37 for example shows that the standard deviation of the volatility short strategy using 1 month futures is much higher than using 3 month futures.

Once the decision on the maturity of the future has been made, the question is when to roll over into the next future. It should be noted that the closer to maturity the future is rolled over, the more volatile the entire volatility strategy becomes. Due to the much higher expected return in the last years though we choose the strategy where we roll the future one day prior to expiration.

The final point to decide is the quantity of the future to be sold when rolling over a position. The first idea that may come to mind is to leave the number of contracts constant. Usually the investor earns a premium by selling the futures which would be ignored and not re-invested using this method. The concept is similar to simply holding a bond which has just matured: the interest earned should be re-invested when the bond matures. The same is true when analyzing equity investments as dividend payments should not be ignored.

To better illustrate the selection of the right future, Table 36 serves as an example on the choice of the instrument: to answer the question when, for example, the February 2015 VIX future is used in the 3-month to expiry scenario, one has to go back to the months of November and December 2014 to determine the expiry dates. In the example, this would be the 17th of

December and 19th of November 2014. As the future is rolled over one day prior to expiration, the period when the future is selected would be from the 18th of November to the 16th of December.

Future used	Future expires	Used between	Future expiry in relevant month
Feb 15	18.02.15	18.11.14 - 16.12.14	17.12.14
Jan 15	21.01.15	20.10.14 - 18.11.14	19.11.14
Dec 14	17.12.14	15.09.14 - 20.10.14	21.10.14

Table 36: *VIX futures example - rollover dates*

The time series is therefore set up in the following way: starting from the end of the time series we will take the price changes of the latest volatility future. Going back in time, the time T on which the future is rolled over will be identified. The price around the rollover day of the current future is denoted by P_{curr} and by P_{prev} for the previous future. At time t, one gets a value in the time series $TS(t)$ in the case where the "notional was kept constant" (re-investment of premium earned) as

$$TS(t) = \begin{cases} TS(t+1)\frac{P_{curr}(t)}{P_{curr}(t+1)}, & \text{if } t > T, \\ TS(t+1)\frac{P_{prev}(t)}{P_{prev}(t+1)}, & \text{if } t \leq T. \end{cases} \tag{93}$$

The interpretation of the constructed time series is straight forward: the geometric changes are taken and, at the rollover date, the price changes of the new future are taken instead of the old contract.

To sum up, the following graph illustrates the performance for the rolling volatility strategy for the one-month future with rollover one day prior to expiration and starting value of 100. The strategy appears quite attractive due to its high return especially during low volatility regimes such as the time from 2004 to 2008. On the downside, however, the strategy itself is quite volatile with sharpe downturns during times of market turmoil.

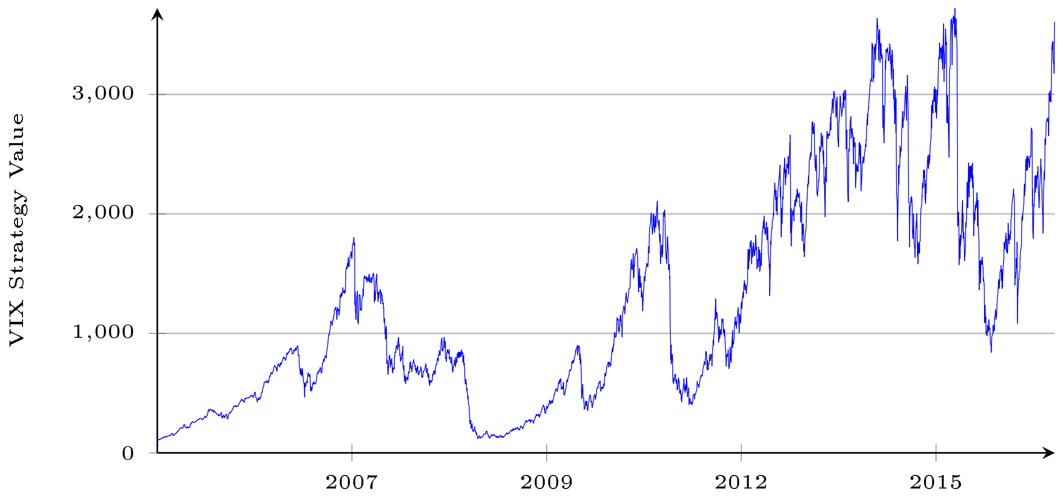

Figure 50: *VIX future short strategies until end of 2016. Source: Bloomberg*

4.5 Data description

For the empirical part of this paper, a dataset of 17 time series is used. The dataset is composed entirely of US end-of-day (EoD) return data for five asset classes and two different equity volatility strategies. Time series reach from May 2004 to the end of 2016 and are taken from Bloomberg. When choosing the data, we took care of a broad diversified multi-asset set of time series. The entire dataset consists of:

- Two equity volatility strategies
- Three equity indices
- Three treasury bond indices
- Three corporate bond indices
- Three high-yield bond indices
- Three commodity indices

The volatility strategies have already been described in detail in section 4.4.2. The basic idea is to use two strategies on the VIX index future that have two different expiration dates. The three equity indices are a selection to cover a broad universe of US equities. For that, the S&P 500 (SPX) is taken as a broad index of 500 US stocks. The small cap equities are represented by the Russell 2000 index (RTY). These indices are complemented by the NASDAQ 100 (NDX) as the 100 largest and most active non-financial domestic and international issues listed on the NASDAQ. We focus on different maturity bands for all bond indices, 1-3 years, 5-7 years and 10-15 years, to account for all changes that occur for the different maturities in the treasury yield and in the corporate spread curves.

The pure interest rate movements are represented by the US treasuries in form of the Bank of America (BoA) Merrill Lynch indices (G1O2, G3O2 and G7O2). The BoA Merrill Lynch indices for the corporate bonds and the abovementioned maturities are C1A2, C3A2 and C7A2. Those indices, together with the high-yield indices of BoA Merrill Lynch (J1A2, J3A2 and J7A2), represent the credit risk in the portfolio.

For the commodities, the directly investable S&P GSCI Energy Total Return Index (SPGSENTR), the S&P GSCI Industrial Metals Total Return Index (SPGSINTR) and the S&P GSCI Precious Metals Total Return Index (SPGSPMTR) are included. The indices are mostly taken from above section. Further reasons on why to avoid other indices, such as ethical concerns in the case of agricultural investments, and why to include specifically these indices are given. Further statistical evaluations and a correlation heatmap are provided in the appendix (see table 43 and 44 and figure 64).

However, as the focus of this paper is particularly on the expected shortfall as the risk measure and the implementation of the volatility strategies in the multi-asset factor risk parity model, the tails of the distributions are of special interest.

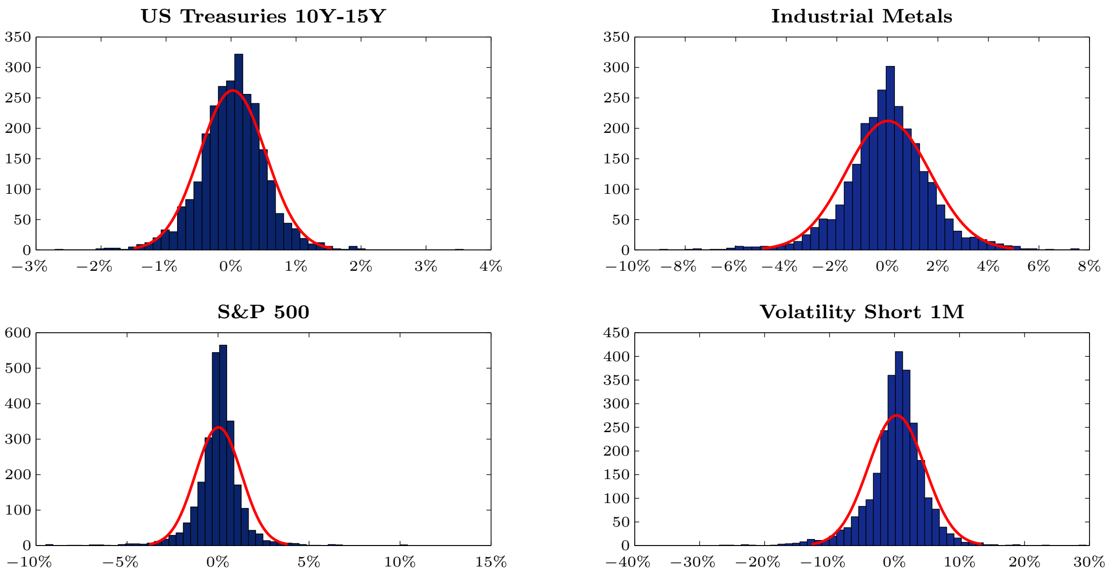

Figure 51: *Data distributions - the red line represents a normal distribution with the same mean and variance as the underlying data.*

The historical distributions in Figure 51 give a first indication of how important the consideration of the left hand tail can be for an allocation using a factor risk parity model. In figure

51, a left hand fat tail is particularly noticeable for the Industrial Metals in the range of -6% to -2%, for the S&P 500 between -5% and -1% and for the 1-month volatility short time series between -20% and -5%. The numbers in table 37 offer further evidence of the existence of the fat tails and the skewness in the historical data. The volatility short, equity, commodity and mid and long maturity high-yield bond time series are all, in part strongly, left skewed. Moreover, the quantile figures show particularly high negative returns for the volatility short, equity and commodity time series (see Tables 43 and 44 for more detailed figures). To sum up, most asset time series exhibit fat tails and are far from being normally distributed.

	Std	*Skew*	*Kurt*	$Q_{(5\%)}$	Ø $Q_{(5\%)}$
Volatility Short 1 Month	4.35%	(0.63)	7.53	(7.44%)	(11.08%)
Volatility Short 3 Months	2.46%	(0.45)	6.85	(4.27%)	(6.21%)
S&P 500 Index	1.27%	(0.43)	14.28	(1.88%)	(3.22%)
Nasdaq Index	1.36%	(0.38)	9.83	(2.19%)	(3.29%)
Russel 2000 Index	1.64%	(0.42)	8.50	(2.54%)	(3.96%)
1-3 Years B US HY Index	0.50%	0.43	164.59	(0.29%)	(0.96%)
5-7 Years B US HY Index	0.31%	(1.84)	31.17	(0.39%)	(0.84%)
10-15 Years B US HY Index	0.40%	(0.51)	30.38	(0.52%)	(0.98%)
DJ UBS Energy Subindex	1.92%	(0.26)	6.07	(3.08%)	(4.63%)
DJ UBS Ind. Metal Subindex	1.66%	(0.23)	5.22	(2.65%)	(3.89%)
DJ UBS Prec. Metals Subindex	1.34%	(0.50)	7.85	(2.21%)	(3.28%)

Table 37: *Higher moments and quantile figures for the skewed and fat tailed asset distributions.*

As the FRP model deals with principal components, the tail behavior of the principal component time series are of particular interest. Figure 52 indicates left hand fat tails for both principal components with and without volatility short positions. However, the effect is significantly reinforced when adding the volatility short time series to the set of return series. This does not come as a surprise, as the principal components are a linear combination of the fat tailed assets described above. Tables 38 and 39 provide the numbers for the skewness and kurtosis for the principal components for the cases with and without volatility short positions. Furthermore, the risk figures increase and skewness and kurtosis decrease, in part, considerably when adding volatility short to the portfolio.

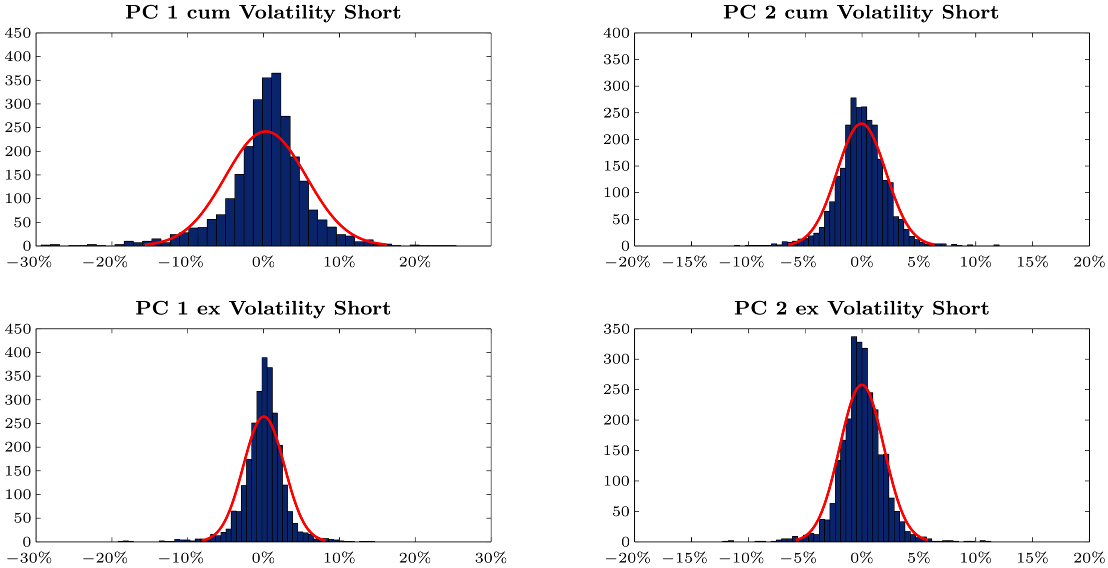

Figure 52: *Principal component distributions with and without volatility short positions - the red line represents a normal distribution with the same mean and variance as the underlying data.*

	Std	*Skew*	*Kurt*	$Q_{(5\%)}$	Ø $Q_{(5\%)}$
Principal Component 1	2.71%	(0.69)	9.21	(4.13%)	(6.83%)
Principal Component 2	1.94%	(0.10)	7.16	(2.98%)	(4.53%)

Table 38: *Higher moments and quantile figures for the first two principal components without volatility short positions.*

As quantile data in both tables show, the first two principal components exhibit strong negative returns on the distributions' left hand side. Again, this is even more severe when adding volatility to the set of time series. More detailed data for all principal components are provided in the appendix (see Tables 45, 46, 49 and 50). The principal component loadings for the two datasets with and without volatility short positions are provided in the appendix as well (see Tables 47 and 51).

	Std	*Skew*	*Kurt*	$Q_{(5\%)}$	Ø $Q_{(5\%)}$
Principal Component 1	5.33%	(0.63)	6.86	(9.06%)	(13.63%)
Principal Component 2	2.15%	(0.06)	5.48	(3.41%)	(5.03%)

Table 39: *Higher moments and quantile figures for the first two principal components with volatility short positions.*

Having described the tail behavior of the single assets and principal components, we will continue to analyze the conditional tail behavior under the constraints in the next section. The conditional tail behavior is of great importance when building a factor risk parity solution under the expected shortfall as the risk measure as it will influence the weight ratio of the first two weights which was key when determining the polytope in the standard deviation case.

4.6 Empirical analysis

The first part of the empirical analysis deals with the issue of the optimal weight ratio of $\widetilde{w_1}$ versus $\widetilde{w_2}$ as discussed in the second study for the standard deviation case. Due to the non-normality distributions of the financial time series and the volatility short time series described in Section 4.5 in particular, we expect a different weight ratio depending on whether the standard deviation or the expected shortfall as the risk measure is used. When using the expected shortfall, the quantile that is chosen should also influence the results. The following subsection illustrates the results for the standard deviation and the expected shortfall. For validation, the appendix 4.A will discuss the results when using the (marginal) modified expected shortfall (a Cornish-Fisher approximation) mentioned in Section 4.3.1.

After the illustration of the differences in Section 4.6.1, Section 4.6.2 will describe the backtesting results in detail with descriptions of performances, risks and other statistics.

4.6.1 Optimal weight ratio of first two principal components

To get a better understanding on the changes that occur when moving from the standard deviation to the expected shortfall, Figure 53 first illustrates the case where the volatility time series are excluded from the multi-asset portfolio. The ratios of $\widetilde{w_1}$ to $\widetilde{w_2}$ for the optimal portfolios are calculated for the standard deviation as the risk measure (red line) and for numerical approximations with the expected shortfall as the risk measure for a larger amount of portfolios (blue dots) with different quantiles (84.1% - equal to the 1. standard deviation - and 95%).

For the calculations with the standard deviation as the risk measure, the ratio of $\widetilde{w_1}$ to $\widetilde{w_2}$ is first determined via equation (74). The minimum and maximum weight for $\widetilde{w_1}$ (and $\widetilde{w_2}$ respectively) are then determined numerically so that the explanation level is between 66.6% and 100%, the weights sum up to 1 and are between 0 and 1. When volatility short positions are excluded, the permissible values for $\widetilde{w_1}$ are between 0.0259 and 0.3502. Figure 53 therefore illustrates the solution in the standard deviation case as a solid line when inside those boundaries and as a dotted line when outside those boundaries.

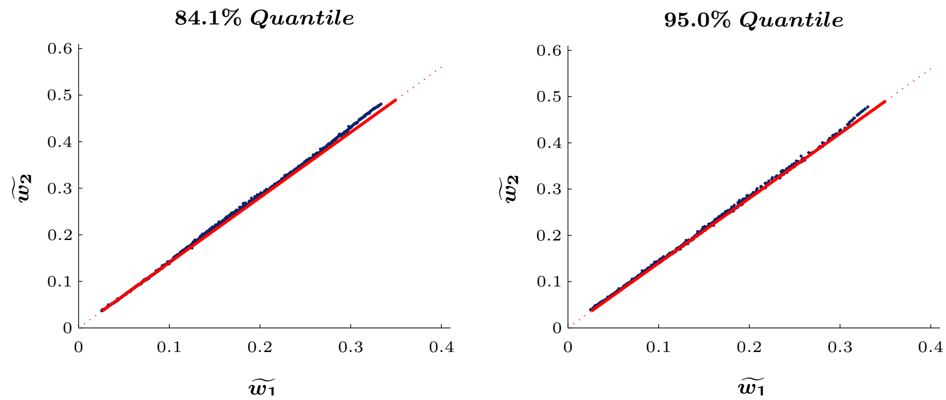

Figure 53: *Expected shortfall quantile simulations for a multi-asset portfolio excluding volatility short time series.*

Keep in mind that the general solution for the problem under the standard deviation as the risk measure is a polyhedron. For a better understanding of the structure and calculations see Section 3.4. For calculating the numerical solutions (blue dots), values for $\widetilde{w_1}$ are taken and the weights $\widetilde{w_2}$ to $\widetilde{w_n}$ are determined so that all constraints are met and where $\widetilde{RC_1}$ is equal to $\widetilde{RC_2}$.

The results are plotted in Figure 53 and 54. The left graph in both figures shows the ratio of the weights in the PCA space for the quantile corresponding to the first standard deviation (84.1%). The figure indicates that the numerical results calculated with the expected shortfall to the quantile 84.1% match closely the results calculated with the standard deviation as the risk measure.

As the quantile for the first standard deviation is not far out in the tail, one could expect that there are no major differences whether the standard deviation or the expected shortfall to the quantile 84.1% is used. The left plot of Figure 53 confirms this assumption initially.

However, the picture changes when adding the volatility short strategy to the multi-asset portfolio. Figure 54 shows the results for the identical quantile simulations but with extended dataset. The permissible values calculated with the standard deviation as the risk measure for the weight $\widetilde{w_1}$ in the case with volatility short time series are now between 0.0132 and 0.2837. The results for the 84.1% quantile (blue dots) again are almost identical with the values cal-

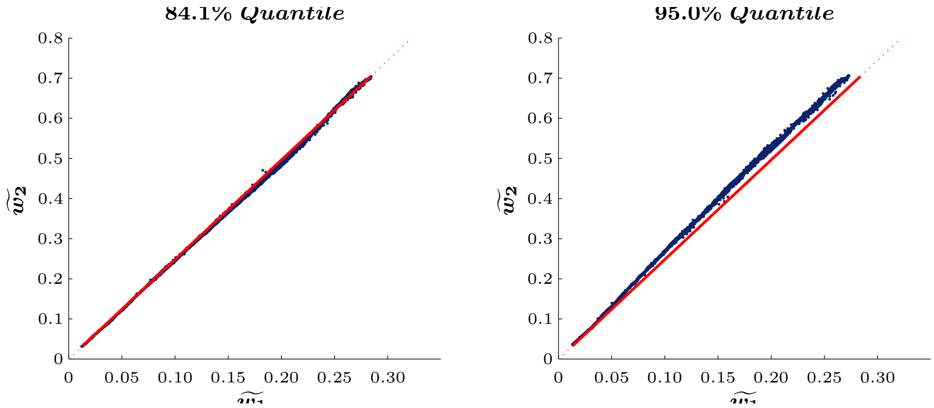

Figure 54: *Expected shortfall quantile simulations for a multi-asset portfolio including volatility short time series.*

culated with the standard deviation (red line). The results change, however, when a smaller quantile (95%) for the numerical calculations is chosen: the weight ratio $\frac{\widetilde{w_1}}{\widetilde{w_2}}$ increases as the plot in Figure 54 on the right hand side indicates.

Keep in mind that the ratio was key when determining the solution in the standard deviation case, particularly when weight restrictions lead to polyhedron solutions. It seems that the differences in the weight ratio for expected shortfall versus standard deviation is higher for heavily skewed time series in a multi-asset portfolio.

As already mentioned, we validated the results described above using a closed form solution for calculating the (marginal) modified expected shortfall with a the Cornish-Fisher approximation. This results can be found in appendix 4.B. The results from the (marginal) modified expected shortfall support the findings described in this section.

In the next section, the impact of changing the risk measure and therefore the change in the weight ratio and the impact of including volatility short positions as described in Section 4.4 on the allocations will be reviewed in a general backtest.

4.6.2 Backtest

For computing the out-of-sample asset weights, a distinction between an allocation along the standard deviation and the expected shortfall as the risk measure in the factor risk parity model (FRP) has to be made. Under the standard deviation, the corner points of the polyhedron are used within a numerical optimization to find the minimum variance and the maximum diversification portfolio. Using the expected shortfall, numerical calculations as described in Section 4.3.1 are chosen to determine the out-of-sample weights. Independently from choosing the way the asset weights are computed, we set a minimum level of explanation of 66% of the first two principal components for all FRP allocations.

As a benchmark allocation, the heuristic equal risk to contribution (ERC) allocation is computed for the time series including and excluding the volatility short strategies. For further details on the computation of the ERC strategy, see Section 3.3.

All backtests and analyses are constructed in the following way: $t = 1$ marks the first entry of the dataset. The out-of-sample weights w are computed at time t based on the historical data from days $t - 1,000$ to $t - 1$. The weights are calculated based on the last 1,000 trading days which are then used for the next 60 trading days from t to $t + 59$. The asset weights are rebalanced at $t + 59$ accordingly. As the in-the-sample window length is often the subject of discussion in literature, the window length for the analysis in this paper is fixed to 1,000 trading days. The reasoning behind using this setup can also be found in the above section.

Under this allocation setup, all strategies are first computed excluding the volatility short positions and, in a second step, volatility short positions are added to the portfolio. In the next part, the main statistics for all strategies will be compared before explaining special properties of single allocations.

Using the dataset described in Section 4.5 and excluding the volatility short strategies from the allocations, the ERC strategy performs best and delivers the highest Sharpe Ratios in the period from 2008 to 2014. Table 40 provides the detailed figures. Despite that, the maximum drawdown for the ERC allocation is the second highest of all allocations backtested.

When including the volatility short strategies, the ERC allocation still achieves the highest Sharpe Ratio (see Table 41). However, the FRP maximum diversification allocation computed with the expected shortfall as the risk measure performs best when including the volatility short strategies. As the volatility short strategy is itself a major source of volatility in all allocations in this setup, the diversification figures decreases compared to the allocations computed excluding the volatility short strategies. At the same time, the entropy increases when adding the volatility short strategy to the portfolio.

	VIX	*Return*	$Mean_{Ann.}$	$Std_{Ann.}$	*SR*	Max_{DD}	$\mathcal{D}$	$\mathcal{E}$
ERC	*exVIX*	21.58%	2.99%	3.00%	0.79	15.19%	2.05	14.42
Max. Div.$_{66\%}$ *Std*	*exVIX*	12.04%	1.73%	3.45%	0.32	12.47%	2.40	9.98
Min. Var.$_{66\%}$ *Std*	*exVIX*	2.84%	0.42%	2.18%	-0.09	11.91%	1.81	8.76
Max. Div.$_{66\%}$ *ES*	*exVIX*	15.36%	2.18%	5.05%	0.31	21.66%	1.75	9.84
Min. Var.$_{66\%}$ *ES*	*exVIX*	0.27%	0.04%	2.15%	-0.27	11.72%	1.75	8.19

Table 40: *Allocation strategies without the volatility short strategies. SR is the Sharpe Ratio, Max_{DD} the maximum drawdown, $\mathcal{D}$ the diversification and $\mathcal{E}$ the entropy.*

	VIX	*Return*	$Mean_{Ann.}$	$Std_{Ann.}$	*SR*	Max_{DD}	$\mathcal{D}$	$\mathcal{E}$
ERC	*cumVIX*	110.25%	11.87%	5.13%	2.19	12.83%	1.32	16.27
Max. Div.$_{66\%}$ *Std*	*cumVIX*	79.62%	9.24%	5.05%	1.71	10.19%	1.45	12.77
Min. Var.$_{66\%}$ *Std*	*cumVIX*	21.25%	2.95%	2.76%	0.84	9.04%	1.28	11.69
Max. Div.$_{66\%}$ *ES*	*cumVIX*	129.50%	13.36%	8.54%	1.49	19.20%	1.18	12.40
Min. Var.$_{66\%}$ *ES*	*cumVIX*	14.50%	2.06%	2.58%	0.56	9.76%	1.25	11.02

Table 41: *Allocation strategies with the volatility short strategies.*

Figure 55 shows the total return chart for the ERC and the FRP allocations using the expected shortfall as the risk measure. Particularly, the return of the FRP maximum diversification allocation is notable for the period from 2008 to 2012. However, the performance of the ERC strategy is not as high as for the FRP maximum diversification strategy in the period from 2008 to 2012, but more steady in the period after 2012.

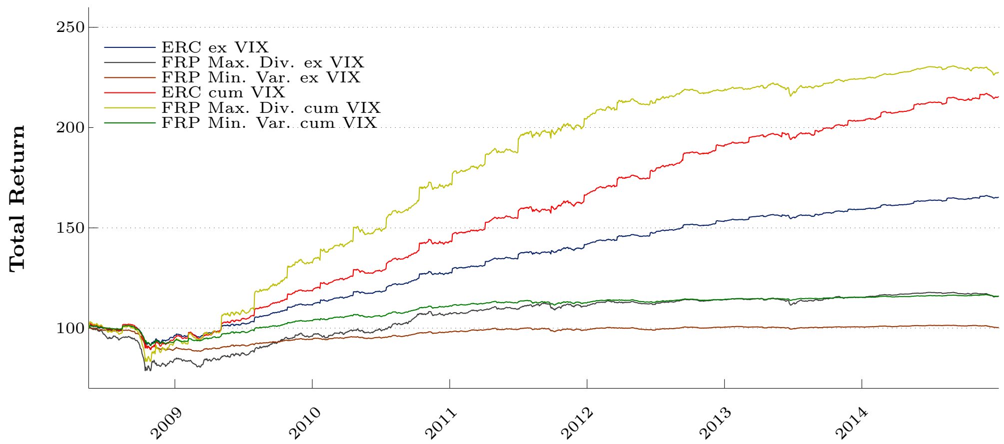

Figure 55: *ERC and FRP allocations including and excluding volatility short strategies using the expected shortfall as the risk measure.*

For the sake of completeness, the total return plot of the FRP allocations using the standard deviation as the risk measure can be found in Figure 56. A very steady return development again can be observed for the ERC strategy. After this general overview, the next section continues to describe the effect of adding volatility to the multi-asset portfolio.

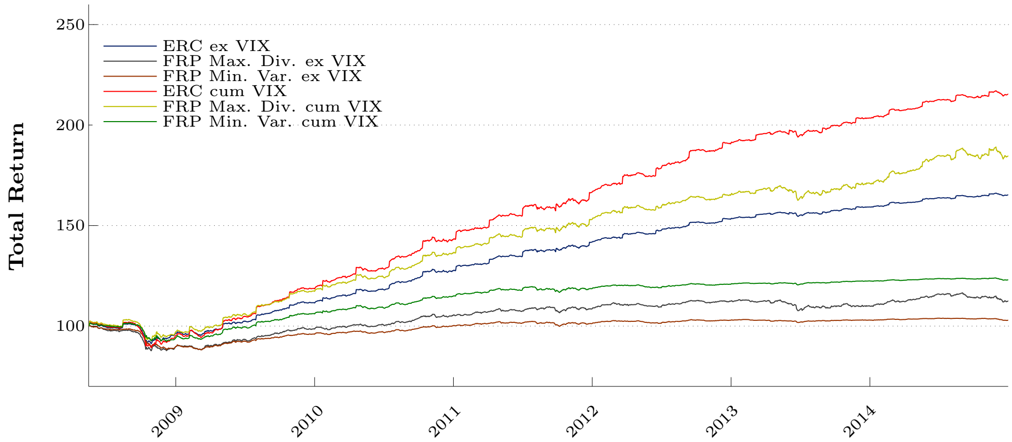

Figure 56: *Ex vs cum volatility short positions for FRP allocations*

Adding volatility short positions

One question already asked in Section 4.4 is the impact of adding volatility short strategies to the portfolio. For that, Figure 57 illustrates the annualized risk-return profile for all the strategies backtested in this chapter, while focusing on the changes that occur when the volatility short positions are added to the portfolio.

The figure shows for all strategies an increase in return and risk when volatility positions are added. Due to the much higher increase in the return though, the Sharpe Ratios increase as Table 40 and Table 41 also illustrate. The change in returns are higher for the ERC and the maximum diversification strategy as for the the minimum variance strategy (red and green dots), which do not exhibit bigger changes in the risk-return profile. As those portfolios try to minimize the variance, adding the volatility short strategy should not really have a major influence on the allocation.

The increase in portfolio volatility has to be considered from another viewpoint: on the one hand, adding volatility short positions to a multi-asset portfolio may increase portfolio volatility, on the other hand, volatility short positions decrease the maximum drawdown in the backtest.

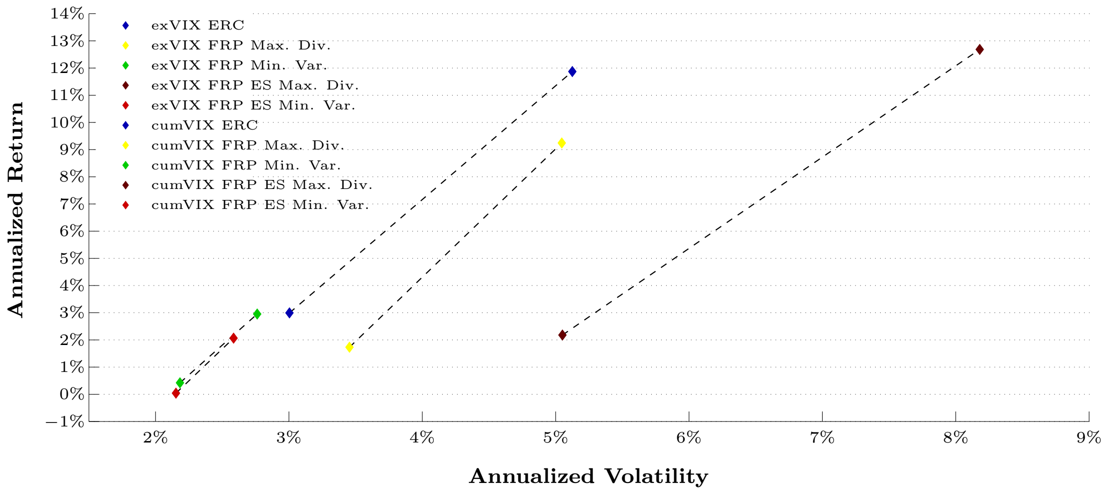

Figure 57: *Risk/return plot for the ERC strategy and the FRP strategies including and excluding volatility short positions to the multi-asset portfolio.*

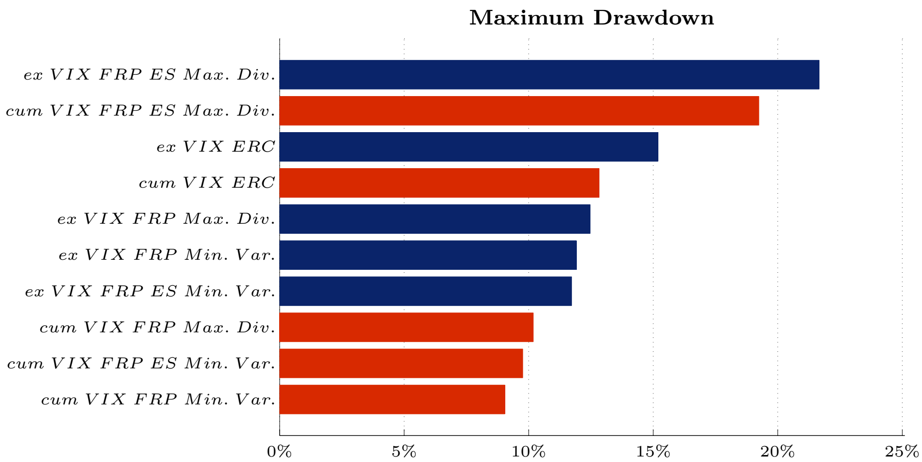

Figure 58: *Portfolio maximum drawdown for all allocation strategies in a descending order.*

Figure 58 shows the maximum drawdown of all strategies for the entire backtesting period. The red bars represent the strategies including the volatility short positions, the blue bars display the maximum drawdown of the allocations excluding volatility strategies. The results are robust and clear: adding volatility short positions to the multi-asset portfolio decreases the maximum drawdown significantly for each allocation strategy in our backtest. Additionally, the two FRP strategies including volatility short positions and computed with the standard deviation as the risk measure are among the three allocations with the lowest maximum drawdown.

There are two possible explanations for this phenomenom: firstly, the contango in the "normal" volatility term structure provides an independent alpha source. This alpha source absorbs slight negative portfolio returns or delivers positive portfolio returns in periods when markets are trending sideways. This was particularly the case in the time before the outbreak of the global financial crisis. During this 4-month period in the backtests, the portfolios including volatility short positions performed better than the portfolios excluding volatility short positions.

Secondly, including volatility short positions changes the portfolios' asset weights. Figure 59 clarifies the impact of the inclusion of volatility short positions on the average asset weights. The asset weights are computed as the average over the entire backtest period.

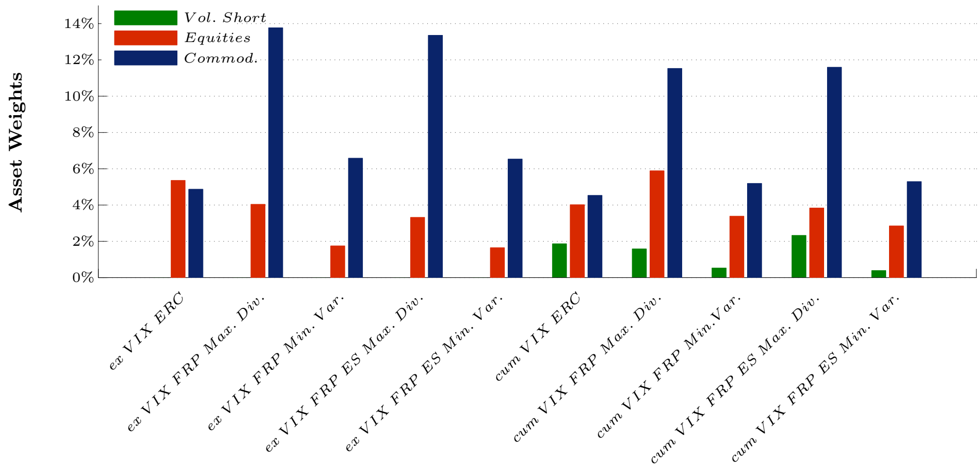

Figure 59: *Aggregated volatility short, equity, bond and commodity weights for all allocations including and excluding volatility short positions.*

The changes in the asset weights might explain some of the effects already described. Due to the higher volatility of the volatility short strategy itself, the weights of the volatility positions are relatively low for all allocation (green bars). It can be observed, that the equity weights are only reduced in the ERC allocation. For all FRP allocations, equity weights increase when adding volatility short positions to the portfolio. Table 42 gives an overview on the asset weights. The numbers indicate that, when adding a volatility short strategy to the portfolio, asset weights are shifted from commodity to equity positions, the asset weights for bond positions stay almost constant.

In particular, the average daily return of the UBS Energy subindex (-0.73%) and the UBS Ind. Metals subindex (-0.57%) are the lowest of all assets in the time period up to the end of 2008 covered in the backtest. A reduction of the commodity time series in this period will therefore have a positive effect on the portfolio return.

On the one hand, the effect that the weights of commodities are reduced can explain one part of the higher returns when volatility short strategies are included as the returns of commodities during the backtesting period has been on average lower than that for other assets. On the other hand, the returns of the volatility short strategy itself explain the other part of the higher return of those strategies. Despite the negative drawdowns during market turmoil described in Avellaneda and Papanicolaou (2018), including the VIX short strategy therefore does have a significant positive impact on the asset allocation.

	VIX Strategy	*Volatility short*	*Equities*	*Bonds*	*Commodities*
ERC	*ex VIX*	0.00%	5.36%	89.77%	4.87%
$Max.\ Div._{66\%}$ *Std*	*ex VIX*	0.00%	4.04%	82.19%	13.77%
$Min.\ Var._{66\%}$ *Std*	*ex VIX*	0.00%	1.75%	91.67%	6.58%
$Max.\ Div._{66\%}$ *ES*	*ex VIX*	0.00%	3.32%	83.32%	13.36%
$Min.\ Var._{66\%}$ *ES*	*ex VIX*	0.00%	1.65%	91.81%	6.54%
ERC	*cum VIX*	1.87%	4.02%	91.45%	4.53%
$Max.\ Div._{66\%}$ *Std*	*cum VIX*	1.59%	5.89%	82.58%	11.53%
$Min.\ Var._{66\%}$ *Std*	*cum VIX*	0.53%	3.39%	91.42%	5.19%
$Max.\ Div._{66\%}$ *ES*	*cum VIX*	2.33%	3.84%	84.56%	11.60%
$Min.\ Var._{66\%}$ *ES*	*cum VIX*	0.39%	2.85%	91.85%	5.29%

Table 42: *Asset class weights for all allocations including and excluding volatility short positions.*

Standard deviation versus expected shortfall as the risk measure

Another core issue of this work is the exchange of the standard deviation for the expected shortfall as the risk measure for computing the optimal weight ratio of the first two weights in the principal component space as well as for the backtest. Section 4.3.1 gives a theoretic overview on this matter and discusses in detail the reasons for the exchange of the risk measure.

Subsection 4.6.1 explains and presents the changes in the weight ratios of the first two principal components that occur when the 95.0% quantile is exchanged for the 84.1% quantile or the standard deviation. These results are validated using the (marginal) modified expected shortfall and are presented in appendix 4.B.

As the differences in the ratios are not very high but significant when adding volatility short positions to the portfolio, Figure 57 shows the differences in the risk and return relation that occur when changing the risk measure. The effect of the change of the risk measure for the FRP minimum variance allocations is slightly negative in terms the total return. For the FRP maximum diversification allocation, the change in the risk measure increases both, the volatility and the return of the allocation, significantly.

Additionally, Table 42 displays the changes in the asset weights that occur when the risk measure is changed. Keep in mind that the weights are averaged over the entire period, which might blur the differences. However, when comparing the changes in the asset weights for the allocations excluding volatility, almost no differences can be determined. These results reflect the effect that the optimal weight ratios do not differ significantly from those computed with the standard deviation when the 95.0% quantile is used for the numerical calculations (see Figure 53). Moreover, the weight ratios computed without volatility short positions using the expected shortfall as the risk measure are also validated in Section 4.6.1. Those weight ratios are almost identical to those computed with the standard deviation and the polytope approach.

In comparing the changes for the allocations including volatility short positions, significant changes for the volatility short and the equity weights can be observed, when the 95.0% quantile instead of the 84.1% quantile or the standard deviation is used. These findings again are in line with the numerical results in Section 4.6.1. Figure 54 shows the effect of a higher weight $\widetilde{w_2}$ when the smaller quantile is used for the calculations.

The differences in the allocation weights become more apparent when the statistics of the allocations are considered (see Tables 40 and 41). The return, for example, increases for the FRP maximum diversification strategy including volatility short positions from around 80% to around 130% when changing the risk measure from the standard deviation to the expected shortfall. However, the maximum drawdown almost doubles, the volatility increases and diversification decreases for this allocation. As bond and commodity weights stay almost constant and equity weights decrease, the significant volatility short weight increase seems to be the suitable explanation for the changes in the properties of the FRP maximum diversification allocation.

To sum up, the exchange of the risk measure leads to reasonable changes in the allocations weights. Particularly, the return of the FRP maximum diversification allocation increases significantly. The increase in risk figures is the other side of the coin when implementing this allocation.

4.7 Conclusion

Constructing robust and well-balanced multi-asset portfolios is a challenge for theorists and practitioners. This paper contributes to the subject of portfolio construction by extending a version of the factor risk parity model discussed in the previous section. The model is able to construct a portfolio with equal risk contributions in the first m components when using the Expected Shortfall as the risk measure while, at the same time, avoiding leverage and short positions.

This chapter extends the research mainly by two points. First, volatility short strategies are constructed and included in the multi-asset portfolio as unfunded positions. Due to their attractive return characteristics, a rolling short volatility future strategy appears to be an interesting addition in a multi-asset context. Second, the standard deviation is exchanged by the expected shortfall as the risk measure. Despite the advantage of simplified calculations, some drawbacks, such as the non-consideration of skewness or fat tails, exist when the standard deviation as the risk measure is used. The expected shortfall focuses on the tails of the resulting distributions, which is, in the case of volatility short positions, an advantageous attribute. At the same time, using the expected shortfall as the risk measure makes calculations more complex as closed-form solutions for arbitrary distributions of the asset returns cannot be determined. Under specific distribution assumptions some closed form solutions are provided which were identical to the standard deviation case.

The results when the standard deviation is exchanged by the expected shortfall are in line with our expectations: when volatility short positions are included in the portfolio, the optimal weight ratio changes when using a smaller quantile for the expected shortfall. The differences in the weight ratios of the first two principal components are not huge but significant. For the optimal weight ratio $\widetilde{w_2}$ to $\widetilde{w_1}$, as described in Section 4.6.1, the simulations show a significant higher weight for the second principal component for our dataset when using the expected shortfall to the 95% quantile as when using the standard deviation or the 84.1% quantile respectively. Data shows that the more this quantile is moved outside to the tail, the more the optimal weight ratio differs from the standard deviation case. The results have been validated by using the (marginal) modified expected shortfall, an extension of the Cornish-Fisher approximation for the value at risk measure. The idea behind this concept is to approximate the expected shortfall in the tail by using a polynomial and then calculate the marginal expected shortfall based on that approximation. The method is described in appendix 4.A. The results confirm our initial results from Section 4.6.1.

Backtesting the models for the period of 2004 to the end of 2014 shows that including volatility short positions and exchanging the risk measure does have an influence on the portfolio weights and thereby on the portfolios' risk and return characteristic. When excluding volatility short positions from the portfolio, the average portfolio weights hardly change and the portfolios' characteristics remain almost unchanged using the expected shortfall. Adding volatility short strategies to the portfolio does not radically change the portfolio weights as the strategy itself is very volatile, but shows an impact, particularly, on equity and commodity positions. Commodity positions are, on average, reduced for an increase in equity positions as the volatility positions are required to be unfunded. Due to the lower long term performance of equities and commodities relative to the volatility short strategy, this shift in weights has a significant positive impact on the portfolio return and Sharpe Ratio. The exchange of the risk measure in this case reinforced these asset weight shifts and thereby the positive effect on the portfolio return and Sharpe Ratio.

Further research could be undertaken in various subjects. One obvious point of criticism is the use of the principal component analysis for the data analysis and data conversion. The PCA is a linear method that assumes more or less a normal distribution. The expected shortfall that is used as a risk measure on the other hand focuses specifically on the distribution on the tails in that transformed factor space. In this context, a linear transformation of the input data, such as a principal component analysis, might not necessarily be the perfect choice. PCA, however, has been described in detail in the above section where solely the standard deviation as the risk measure is used. PCAs' simplicity combined with the intuitive use proves beneficial. It may be useful, however, to take other blind source separation and data analysis tools into consideration.

Appendix 4.A Closed form solution - modified expected shortfall

As seen above, the analytic calculation of the component expected shortfall (CES) or modified expected shortfall (MES) can be troublesome, particularly if the loss distribution is not described by a simple normal distribution. It therefore can be useful to take a closer look at approximations to (a) speed up the calculation and (b) gain some deeper understanding of the analysis of the distributions' tails and the expected shortfall as the risk measure. We therefore want to take a look at the (marginal) modified expected shortfall like it is described by Boudt et al. (2008) which serves as an estimator for the expected shortfall.

First, we can write the exact expected shortfall of the portfolio return as

$$ES(\alpha) = -w'\mu - \sigma_P E_G[x|x \leq G^{-1}(\alpha)] \tag{94}$$

with $G(\cdot)$ being the true distribution function of the portfolio returns.

Boudt, Peterson and Croux define the modified expected shortfall then as

$$ES^{mod}(\alpha) := -w'\mu - \sigma_P E_{G_2}[x|x \leq G_2^{-1}(\alpha)] \tag{95}$$

with G_2^{-1} being the 2nd order Cornish-Fisher expansion of the quantile function $G^{-1}(\cdot)$ around the Gaussian quantile function $\Phi^{-1}(\dot{)}$ as

$$G_2^{-1}(\alpha) = z_\alpha + \sum_{i=1}^{2} P_i^*(z_\alpha) \tag{96}$$

with $z_\alpha = \Phi^{-1}(\alpha)$ and

$$\begin{aligned} P_1^*(z_\alpha) &= \frac{1}{6}(z_\alpha^2 - 1)s_p \\ P_2^*(z_\alpha) &= \frac{1}{24}(z_\alpha^3 - 3z_\alpha)k_p - \frac{1}{36}(2z_\alpha^3 - 5z_\alpha)s_p^2 \end{aligned} \tag{97}$$

with portfolio skewness s_p and the excess kurtosis k_p given below.

Even though second degree in the polynom should be sufficient for the approximation, higher degrees could of course be added. Those polynomes can be found, for example, in Draper and Tierney (1973). Given those formulas the approximated expected shortfall can be calculated as

$$\begin{aligned} E_{G_2} = E_{G_2}[x|x \leq G_2^{-1}(\alpha)] = -\frac{1}{\alpha}\Big\{ &\phi(g_\alpha) + \frac{1}{24}[I^4 - 6I^2 + 3\phi(g_\alpha)]k_p + \frac{1}{6}[I^3 - 3I^1]s_p \\ &+ \frac{1}{72}[I^6 - 15I^4 + 45I^2 - 15\phi(g_\alpha)]s_p^2 \Big\} \end{aligned} \tag{98}$$

with $g_\alpha = G_2^{-1}(\alpha)$ and I^1, I^2, I^3, I^4, I^6 given as

$$I^q = \begin{cases} \sum_{i=1}^{q/2} \left(\frac{\prod_{j=1}^{q/2} 2j}{\prod_{j=1}^{i} 2j} \right) g_\alpha^{2i} \phi(g_\alpha) + \left(\prod_{j=1}^{q/2} 2j \right) \phi(g_\alpha) & \text{for } q \text{ even} \\ \sum_{i=0}^{q*} \left(\frac{\prod_{j=0}^{q*} (2j+1)}{\prod_{j=0}^{i} (2j+1)} \right) g_\alpha^{2i+1} \phi(g_\alpha) - \left(\prod_{j=0}^{q*} (2j+1) \right) \Phi(g_\alpha) & \text{for } q \text{ odd} \end{cases} \tag{99}$$

and $q* = (q-1)/2$. Given the formula for E_{G_2}, we can differentiate formula 95 and as $m_2 = \sigma^2$ get by applying the product rule:

$$\begin{aligned} \frac{\partial ES_\alpha^{mod}}{\partial w_i} =& - \mu_i - \frac{1}{2\sqrt{m_2}} \frac{\partial m_2}{\partial w_i} E_{G_2}[x | x \leq G_2^{-1}(\alpha)] \\ &+ \frac{\sqrt{m_2}}{\alpha} \Big\{ \frac{1}{24} [I^4 - 6I^2 + 3\phi(g_\alpha)] \frac{\partial k_p}{\partial w_i} + \frac{1}{6} [I^3 - 3I^1] \frac{\partial s_p}{\partial w_i} \\ &+ \frac{1}{36} [I^6 - 15I^4 + 45I^2 - 15\phi(g_\alpha)] s_p \frac{\partial s_p}{\partial w_i} \\ &+ \frac{\partial g_\alpha}{\partial w_i} \Big[-g_\alpha \phi(g_\alpha) + \frac{1}{24} [\frac{\partial I^4}{\partial w_i} - 6 \frac{\partial I^2}{\partial w_i} - 3 g_\alpha \phi(g_\alpha)] k_p \\ &+ \frac{1}{6} [\frac{\partial I^3}{\partial w_i} - 3 \frac{\partial I^1}{\partial w_i}] s_p + \frac{1}{72} [\frac{\partial_i}{\partial w_i} I^6 - 15 \frac{\partial I^4}{\partial w_i} + 45 \frac{\partial I^2}{\partial w_i} \\ &+ 15 g_\alpha \phi(g_\alpha)] s_p^2 \Big] \Big\} \end{aligned} \tag{100}$$

with

$$\frac{\partial g_\alpha}{\partial w_i} = \frac{1}{6}(z_\alpha^2 - 1) \frac{\partial s_p}{\partial w_i} + \frac{1}{24}(z_\alpha^3 - 3z_\alpha) \frac{\partial k_p}{\partial w_i} - \frac{1}{18}(2z_\alpha^3 - 5z) s_p \frac{\partial s_p}{\partial w_i} \tag{101}$$

$$\frac{\partial I^q}{\partial w_i} = \begin{cases} \sum_{i=1}^{q/2} \left(\frac{\prod_{j=1}^{q/2} 2j}{\prod_{j=1}^{i} 2j} \right) g_\alpha^{2i-1} (2i - g_\alpha^2) \phi(g_\alpha) - \left(\prod_{j=1}^{q/2} 2j \right) g_\alpha \phi(g_\alpha) & \text{for } q \text{ even} \\ \sum_{i=0}^{q*} \left(\frac{\prod_{j=0}^{q*} (2j+1)}{\prod_{j=0}^{i} (2j+1)} \right) g_\alpha^{2i} (2i + 1 - g_\alpha^2) \phi(g_\alpha) - \left(\prod_{j=0}^{q*} (2j+1) \right) \phi(g_\alpha) & \text{for } q \text{ odd} \end{cases} \tag{102}$$

and $q* = (q-1)/2$.

For the sake of completeness, the $N \times N^2$ co-skewness matrix M_3 and the $N \times N^3$ co-kurtosis matrix M_4 as well as the portfolio moments m_2, m_3 and m_4 and their partial derivative are given by:

$$\begin{aligned} M_3 =& E\left[(r-\mu)(r-\mu)' \otimes (r-\mu)'\right] \\ M_4 =& E\left[(r-\mu)(r-\mu)' \otimes (r-\mu)' \otimes (r-\mu)'\right] \\ m_2 =& w'\Sigma w & \frac{\partial m_2}{\partial w_i} &= 2(\Sigma w)_i \\ m_3 =& w' M_3 (w \otimes w) & \frac{\partial m_3}{\partial w_i} &= 3(M_3(w \otimes w))_i \\ m_4 =& w' M_4 (w \otimes w \otimes w) & \frac{\partial m_4}{\partial w_i} &= 4(M_4(w \otimes w \otimes w))_i \end{aligned} \tag{103}$$

The portfolio skewness s_p and the excess kurtosis k_p and their partial derivative are given by:

$$\begin{aligned} s_p &= m_3/m_2^{3/2} & \frac{\partial s_p}{\partial w_i} &= (2m_2^{3/2}\frac{\partial m_3}{\partial w_i} - 3m_3 m_2^{1/2}\frac{\partial m_2}{\partial w_i})/2m_2^3 \\ k_p &= m_4/m_2^2 - 3 & \frac{\partial k_p}{\partial w_i} &= (m_2\frac{\partial m_4}{\partial w_i} - 2m_4\frac{\partial m_2}{\partial w_i})/m_2^3 \end{aligned} \tag{104}$$

Appendix 4.B Optimal weight ratio validation

The (marginal) modified expected shortfall (see appendix 4.A for further details) is used to validate the results from Section 4.6. Using this approximation for the expected shortfall and the component expected shortfall to calculate the optimal ratio of $\widetilde{w_1}$ to $\widetilde{w_2}$, we expect the results to be similar to that in Section 4.6.1.

At first, the results using the (marginal) modified expected shortfall while excluding volatility short positions from the portfolio will be discussed and compared to those results from Section 4.6.1. Figure 60 therefore illustrates the results when the optimal weight ratio for the first two principal components is calculated via the standard deviation as the risk measure (red line), the expected shortfall as described in Section 4.3.1 (blue dots) and the (marginal) modified expected shortfall (green dots). The results for the (marginal) modified expected shortfall are calculated in the same numerical way as for the expected shortfall in Section 4.6.1 with changes only in the way the risk contribution is calculated. For all numerical calculations, an interior point algorithm in form of the Matlab-Function "fmincon" is used to solve the optimization problem so that the risk contributions of the first two principal components under the (marginal) modified expected shortfall are practically equal.

The results for the (marginal) modified expected shortfall (green dots) plotted in figure 60 differ only slightly from that in Section 4.6.1 (blue dots). Using the 84.1% quantile for both models, the optimal weight $\widetilde{w_2}$ increases slightly more the greater the weight $\widetilde{w_1}$ becomes. However, the results from both models are very similar to that computed with the standard deviation (solid red line).

The optimal weight ratio from the (marginal) modified expected shortfall does not significantly change when using the 95.0% quantile. As the results from the (marginal) modified expected shortfall for both quantiles do not seem to differ much from the results computed with the standard deviation, figure 61 shows that the behavior of the optimal weight ratio is not linear

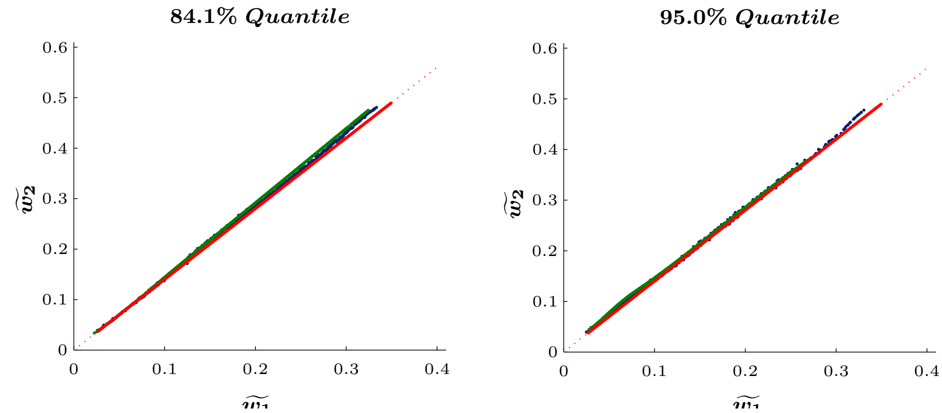

Figure 60: *Expected shortfall quantile simulation using the modified expected shortfall excluding volatility short positions to the multi-asset portfolio.*

as it is in the case with the standard deviation. The values of the weight ratio on the y-axis are plotted against the weight $\widetilde{w_1}$ on the x-axis.

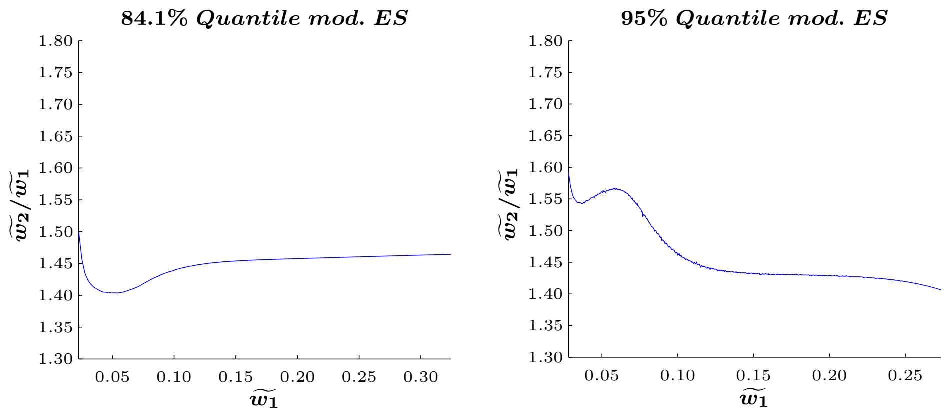

Figure 61: *Weight ratios $\widetilde{w_1}$ to $\widetilde{w_2}$ computed with the (marginal) modified expected shortfall to the quantiles* 84.1% *and* 95.0%*.*

Including volatility short positions in the portfolio changes the results from the (marginal) modified expected shortfall. Those results calculated to the quantile of 84.1% again are quite similar compared to the results computed with the standard deviation and mostly in line with the results from Section 4.6.1. The results computed with the (marginal) modified expected shortfall changes when moving the quantile further to the tail. In this case, the weight ratio change $\frac{\widetilde{w_2}}{\widetilde{w_1}}$ increases slightly compared to the results from Section 4.6.1. The results gained from the (marginal) modified expected shortfall confirm the increase in the optimal weight ratio already described in Section 4.6.1.

The results from the (marginal) modified expected shortfall again indicate that the optimal weight ratio of $\widetilde{w_2}$ to $\widetilde{w_1}$ to the quantile 84.1% and 95% is a straight line, as it is in the case when using the standard deviation as the risk measure. This, however, is not true as figure 63 shows again. Obviously, the optimal weight ratio again is not constant for all $\widetilde{w_1}$ and therefore not a straight line in figure 62.

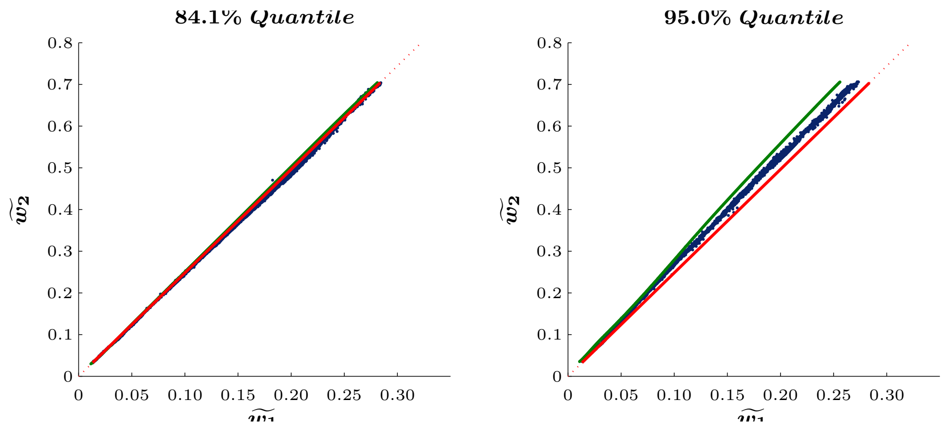

Figure 62: *Expected shortfall quantile simulation using the (marginal) modified expected shortfall including volatility short positions to the multi-asset portfolio.*

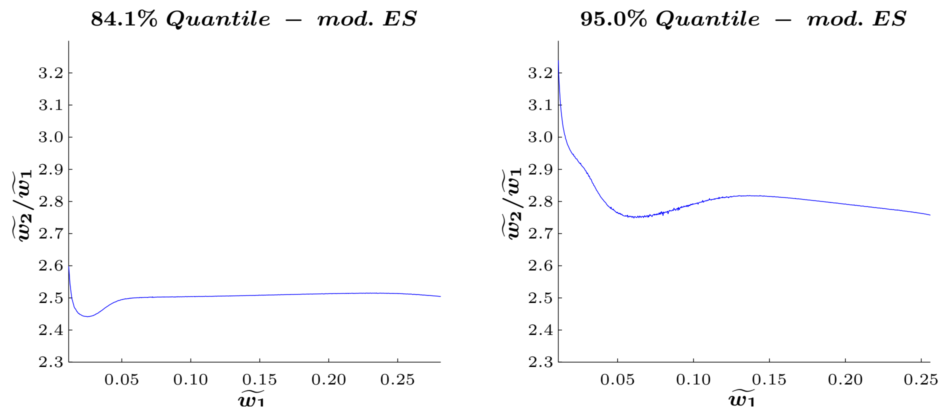

Figure 63: *Weight ratios $\widetilde{w_2}$ to $\widetilde{w_1}$ computed with the (modified) marginal expected shortfall to the quantiles 84.1% and 95.0%.*

Instead, the figure already indicates what has been mentioned above: adding volatility short positions to the portfolio, increasing the quantile and going further to the tails, does not only increase the optimal ratio $\widetilde{w_2}$ to $\widetilde{w_1}$, but let it looks less and less like the optimal ratio computed with the standard deviation as the risk measure.

Appendix 4.C Tables and charts

	$Mean$	$Median$	$Mean_{Ann.}$	Std	$Std_{Ann.}$	$Skew$	$Kurt$
Volatility Short 1M Strategy	0.31%	0.66%	76.67%	4.35%	68.75%	(0.63)	7.53
Volatility Short 3M Strategy	0.08%	0.00%	20.80%	2.46%	38.84%	(0.45)	6.85
S&P 500 Index	0.02%	0.08%	5.97%	1.27%	20.00%	(0.43)	14.28
Nasdaq Index	0.04%	0.11%	10.44%	1.36%	21.52%	(0.38)	9.83
Russel 2000 Index	0.03%	0.10%	7.51%	1.64%	25.87%	(0.42)	8.50
1-3 Years US Treasury Index	0.01%	0.01%	2.46%	0.09%	1.37%	(0.09)	15.33
5-7 Years US Treasury Index	0.02%	0.02%	4.85%	0.31%	4.93%	0.01	7.41
10-15 Years US Treasury Index	0.03%	0.04%	6.52%	0.51%	8.03%	(0.01)	5.22
1-3 Years AA US Corp. Index	0.01%	0.01%	3.14%	0.10%	1.63%	(2.76)	45.36
5-7 Years AA US Corp. Index	0.02%	0.03%	4.99%	0.31%	4.85%	(0.70)	10.71
10-15 Years AA US Corp. Index	0.03%	0.05%	7.46%	0.48%	7.62%	(0.07)	6.73
1-3 Years B US HY Index	0.03%	0.04%	7.98%	0.50%	7.94%	0.43	164.59
5-7 Years B US HY Index	0.03%	0.05%	6.56%	0.31%	4.95%	(1.84)	31.17
10-15 Years B US HY Index	0.04%	0.06%	9.83%	0.40%	6.36%	(0.51)	30.38
DJ UBS Energy Subindex	(0.03%)	0.04%	-7.50%	1.92%	30.39%	(0.26)	6.07
DJ UBS Ind. Metal Subindex	0.02%	0.02%	5.68%	1.66%	26.22%	(0.23)	5.22
DJ UBS Prec. Metals Subindex	0.04%	0.07%	9.79%	1.34%	21.25%	(0.50)	7.85

Table 43: *Dataset - 2004-2014 (1)*

	Min	Max	$Q_{(5\%)}$	$Q_{(1\%)}$	$Ø\ Q_{(5\%)}$	$Ø\ Q_{(1\%)}$
Volatility Short 1M Strategy	(26.99%)	29.48%	(7.44%)	(13.26%)	(11.08%)	(17.16%)
Volatility Short 3M Strategy	(13.54%)	13.99%	(4.27%)	(7.28%)	(6.21%)	(9.70%)
S&P 500 Index	(9.47%)	10.42%	(1.88%)	(3.93%)	(3.22%)	(5.81%)
Nasdaq Index	(11.11%)	10.37%	(2.19%)	(4.00%)	(3.29%)	(5.50%)
Russel 2000 Index	(12.61%)	8.15%	(2.54%)	(4.94%)	(3.96%)	(6.74%)
1-3 Years US Treasury Index	(0.90%)	0.75%	(0.11%)	(0.22%)	(0.19%)	(0.34%)
5-7 Years US Treasury Index	(2.25%)	2.58%	(0.46%)	(0.82%)	(0.69%)	(1.06%)
10-15 Years US Treasury Index	(2.71%)	3.57%	(0.79%)	(1.31%)	(1.12%)	(1.63%)
1-3 Years AA US Corp. Index	(1.60%)	0.88%	(0.12%)	(0.26%)	(0.24%)	(0.52%)
5-7 Years AA US Corp. Index	(2.79%)	2.35%	(0.46%)	(0.82%)	(0.72%)	(1.20%)
10-15 Years AA US Corp. Index	(2.73%)	3.47%	(0.75%)	(1.22%)	(1.08%)	(1.70%)
1-3 Years B US HY Index	(9.87%)	9.25%	(0.29%)	(1.10%)	(0.96%)	(2.72%)
5-7 Years B US HY Index	(4.18%)	2.72%	(0.39%)	(1.04%)	(0.84%)	(1.69%)
10-15 Years B US HY Index	(4.79%)	4.13%	(0.52%)	(1.14%)	(0.98%)	(1.88%)
DJ UBS Energy Subindex	(10.34%)	9.81%	(3.08%)	(5.32%)	(4.63%)	(7.37%)
DJ UBS Ind. Metal Subindex	(9.02%)	7.59%	(2.65%)	(4.95%)	(3.89%)	(6.02%)
DJ UBS Prec. Metals Subindex	(10.10%)	8.76%	(2.21%)	(3.87%)	(3.28%)	(5.23%)

Table 44: *Dataset - 2004-2014 (2)*

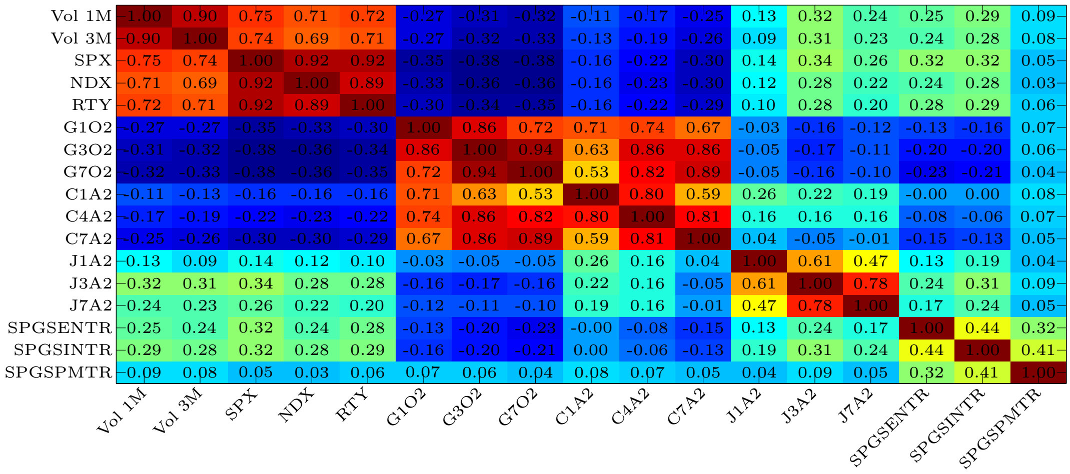

Figure 64: *Correlation heatmap of the dataset*

	Mean	*Median*	$Mean_{Ann.}$	*Std*	$Std_{Ann.}$	*Skew*	*Kurt*
Principal Component 1	0.30%	0.65%	75.53%	5.33%	84.26%	(0.63)	6.86
Principal Component 2	(0.05%)	(0.06%)	(13.52%)	2.15%	33.97%	(0.06)	5.48
Principal Component 3	(0.09%)	(0.13%)	(22.86%)	1.49%	23.48%	0.42	10.20
Principal Component 4	(0.02%)	(0.03%)	(5.28%)	1.35%	21.29%	0.06	4.70
Principal Component 5	0.02%	0.01%	4.90%	1.10%	17.32%	0.04	6.73
Principal Component 6	(0.07%)	(0.07%)	(17.72%)	0.94%	14.91%	(0.24)	9.79
Principal Component 7	0.05%	0.07%	13.24%	0.73%	11.52%	(0.21)	7.50
Principal Component 8	0.04%	0.04%	9.19%	0.58%	9.16%	(1.68)	72.02
Principal Component 9	0.00%	0.01%	0.87%	0.50%	7.86%	(0.23)	8.78
Principal Component 10	(0.01%)	(0.01%)	(1.40%)	0.34%	5.31%	2.37	55.88
Principal Component 11	(0.02%)	(0.02%)	(4.21%)	0.31%	4.94%	0.30	10.89
Principal Component 12	(0.00%)	(0.00%)	(0.08%)	0.17%	2.72%	0.46	44.92
Principal Component 13	(0.00%)	0.01%	(0.23%)	0.16%	2.46%	(1.38)	16.94
Principal Component 14	(0.00%)	0.00%	(0.65%)	0.14%	2.17%	(0.58)	23.09
Principal Component 15	0.01%	0.01%	1.79%	0.08%	1.21%	(0.71)	42.35
Principal Component 16	0.01%	0.01%	1.61%	0.05%	0.87%	(2.18)	55.35
Principal Component 17	0.00%	0.00%	0.40%	0.03%	0.42%	1.23	28.05

Table 45: *Principal components of the dataset - 2004-2014 (1)*

	Min	Max	$Q_{(5\%)}$	$Q_{(1\%)}$	Ø $Q_{(5\%)}$	Ø $Q_{(1\%)}$
Principal Component 1	(29.34%)	31.49%	(9.06%)	(16.15%)	(13.63%)	(20.72%)
Principal Component 2	(11.23%)	12.06%	(3.41%)	(6.15%)	(5.03%)	(7.44%)
Principal Component 3	(8.90%)	12.46%	(2.12%)	(4.80%)	(3.35%)	(5.95%)
Principal Component 4	(6.08%)	7.52%	(2.24%)	(3.55%)	(3.02%)	(4.25%)
Principal Component 5	(7.27%)	6.35%	(1.74%)	(2.82%)	(2.45%)	(3.79%)
Principal Component 6	(9.37%)	5.60%	(1.51%)	(2.42%)	(2.20%)	(3.51%)
Principal Component 7	(5.83%)	6.05%	(1.15%)	(1.94%)	(1.64%)	(2.52%)
Principal Component 8	(10.75%)	7.06%	(0.61%)	(1.54%)	(1.28%)	(2.82%)
Principal Component 9	(4.26%)	3.44%	(0.75%)	(1.42%)	(1.16%)	(1.90%)
Principal Component 10	(2.90%)	5.73%	(0.40%)	(0.75%)	(0.70%)	(1.41%)
Principal Component 11	(1.97%)	2.52%	(0.45%)	(0.83%)	(0.70%)	(1.21%)
Principal Component 12	(1.74%)	2.67%	(0.19%)	(0.46%)	(0.39%)	(0.83%)
Principal Component 13	(1.55%)	0.92%	(0.21%)	(0.52%)	(0.41%)	(0.81%)
Principal Component 14	(1.71%)	1.40%	(0.20%)	(0.37%)	(0.33%)	(0.59%)
Principal Component 15	(1.25%)	0.84%	(0.09%)	(0.18%)	(0.16%)	(0.29%)
Principal Component 16	(0.99%)	0.53%	(0.06%)	(0.14%)	(0.12%)	(0.24%)
Principal Component 17	(0.20%)	0.39%	(0.04%)	(0.07%)	(0.06%)	(0.10%)

Table 46: *Principal components of the dataset - 2004-2014 (2)*

	PC 1	PC 2	PC 3	PC 4	PC 5	PC 6	PC 7	PC 8	PC 9
Vol 1M	0.80	(0.18)	(0.34)	0.13	(0.05)	(0.43)	(0.04)	(0.03)	(0.00)
Vol 3M	0.43	(0.09)	(0.07)	0.05	0.03	0.89	0.09	0.02	0.01
SPX	0.20	0.05	0.40	(0.14)	0.07	(0.03)	0.04	0.09	0.13
NDX	0.20	0.02	0.46	(0.20)	0.10	(0.08)	0.03	0.11	0.71
RTY	0.25	0.05	0.56	(0.22)	0.18	(0.07)	0.06	(0.19)	(0.66)
G1O2	(0.00)	(0.00)	(0.01)	0.00	0.01	(0.01)	0.08	(0.03)	(0.00)
G3O2	(0.02)	(0.01)	(0.05)	0.00	0.06	(0.03)	0.35	(0.11)	0.02
G7O2	(0.03)	(0.03)	(0.07)	(0.00)	0.09	(0.05)	0.57	(0.17)	0.05
C1A2	(0.00)	0.00	(0.01)	0.00	0.01	(0.01)	0.09	0.03	(0.01)
C4A2	(0.01)	(0.00)	(0.04)	(0.00)	0.03	(0.03)	0.36	0.02	0.00
C7A2	(0.03)	(0.01)	(0.07)	(0.00)	0.07	(0.05)	0.57	(0.10)	0.05
J1A2	0.01	0.03	(0.01)	(0.02)	(0.06)	(0.06)	0.16	0.71	(0.18)
J3A2	0.02	0.03	0.00	(0.01)	(0.04)	(0.01)	0.09	0.38	(0.09)
J7A2	0.02	0.03	0.00	(0.02)	(0.05)	(0.01)	0.12	0.47	(0.09)
SPGSENTR	0.12	0.74	0.14	0.64	(0.03)	(0.01)	0.04	(0.04)	0.03
SPGSINTR	0.11	0.53	(0.21)	(0.62)	(0.52)	0.03	0.05	(0.11)	0.00
SPGSPMTR	0.03	0.36	(0.34)	(0.28)	0.81	(0.00)	(0.13)	0.08	0.02

Table 47: *Principal components loadings of the dataset (1)*

	PC 10	PC 11	PC 12	PC 13	PC 14	PC 15	PC 16	PC 17
Vol 1M	(0.01)	(0.00)	0.00	(0.00)	0.00	0.00	(0.00)	0.00
Vol 3M	0.04	(0.01)	(0.00)	(0.01)	(0.00)	(0.00)	0.00	(0.00)
SPX	(0.48)	0.71	(0.03)	(0.04)	(0.00)	0.03	0.01	0.00
NDX	0.26	(0.34)	(0.00)	0.02	0.00	(0.00)	(0.00)	0.00
RTY	0.14	(0.22)	0.02	0.01	(0.01)	(0.01)	(0.00)	(0.00)
G1O2	0.02	(0.01)	(0.14)	0.03	(0.16)	0.42	0.27	0.84
G3O2	0.02	(0.01)	(0.34)	(0.08)	(0.14)	0.70	(0.31)	(0.37)
G7O2	0.02	0.00	(0.23)	(0.40)	0.54	(0.24)	0.22	0.07
C1A2	(0.00)	(0.01)	(0.15)	0.19	(0.29)	0.02	0.84	(0.38)
C4A2	(0.07)	(0.03)	(0.44)	0.35	(0.44)	(0.50)	(0.28)	0.13
C7A2	(0.00)	0.05	0.76	0.23	(0.14)	0.06	(0.02)	0.02
J1A2	0.56	0.30	0.01	(0.15)	(0.05)	(0.01)	(0.02)	0.00
J3A2	(0.23)	(0.17)	(0.10)	0.64	0.56	0.16	0.00	(0.00)
J7A2	(0.55)	(0.45)	0.10	(0.43)	(0.21)	(0.00)	0.01	0.00
SPGSENTR	0.02	(0.02)	0.00	(0.01)	0.01	(0.00)	(0.00)	(0.00)
SPGSINTR	0.02	0.00	(0.01)	(0.01)	0.00	0.00	0.00	(0.00)
SPGSPMTR	(0.01)	0.01	0.01	0.00	(0.01)	(0.00)	(0.00)	0.00

Table 48: *Principal components loadings of the dataset (2)*

	$Mean$	$Median$	$Mean_{Ann.}$	Std	$Std_{Ann.}$	$Skew$	$Kurt$
Principal Component 1	0.04%	0.16%	10.31%	2.71%	42.89%	(0.69)	9.21
Principal Component 2	(0.02%)	(0.05%)	(5.91%)	1.94%	30.62%	(0.10)	7.16
Principal Component 3	(0.06%)	(0.05%)	(13.81%)	1.36%	21.58%	0.11	4.60
Principal Component 4	0.03%	0.04%	7.49%	1.10%	17.36%	(0.01)	7.00
Principal Component 5	0.06%	0.08%	14.93%	0.73%	11.56%	(0.24)	7.96
Principal Component 6	0.04%	0.05%	10.71%	0.58%	9.21%	(1.88)	73.78
Principal Component 7	0.00%	0.02%	1.01%	0.50%	7.86%	(0.22)	8.69
Principal Component 8	(0.00%)	(0.01%)	(1.19%)	0.34%	5.39%	2.43	55.02
Principal Component 9	(0.02%)	(0.02%)	(4.06%)	0.31%	4.95%	0.29	11.44
Principal Component 10	(0.00%)	(0.00%)	(0.17%)	0.17%	2.72%	0.47	44.88
Principal Component 11	(0.00%)	0.01%	(0.09%)	0.16%	2.47%	(1.42)	17.16
Principal Component 12	(0.00%)	0.00%	(0.69%)	0.14%	2.17%	(0.58)	23.14
Principal Component 13	0.01%	0.01%	1.77%	0.08%	1.21%	(0.72)	43.47
Principal Component 14	0.01%	0.01%	1.61%	0.05%	0.87%	(2.19)	55.44
Principal Component 15	0.00%	0.00%	0.40%	0.03%	0.42%	1.23	28.22

Table 49: *Principal components of the dataset - 2004-2014 (1)*

	Min	Max	$Q_{(5\%)}$	$Q_{(1\%)}$	Ø $Q_{(5\%)}$	Ø $Q_{(1\%)}$
Principal Component 1	(19.17%)	14.69%	(4.13%)	(8.68%)	(6.83%)	(11.81%)
Principal Component 2	(12.25%)	11.31%	(2.98%)	(5.74%)	(4.53%)	(7.24%)
Principal Component 3	(5.87%)	7.06%	(2.28%)	(3.76%)	(3.08%)	(4.23%)
Principal Component 4	(8.15%)	6.80%	(1.73%)	(2.79%)	(2.46%)	(3.80%)
Principal Component 5	(6.01%)	6.20%	(1.14%)	(2.02%)	(1.64%)	(2.57%)
Principal Component 6	(11.00%)	6.96%	(0.61%)	(1.53%)	(1.29%)	(2.84%)
Principal Component 7	(4.22%)	3.44%	(0.74%)	(1.42%)	(1.16%)	(1.90%)
Principal Component 8	(2.83%)	5.93%	(0.43%)	(0.83%)	(0.72%)	(1.38%)
Principal Component 9	(2.11%)	2.43%	(0.45%)	(0.82%)	(0.71%)	(1.22%)
Principal Component 10	(1.75%)	2.67%	(0.19%)	(0.46%)	(0.39%)	(0.83%)
Principal Component 11	(1.55%)	0.95%	(0.21%)	(0.52%)	(0.42%)	(0.81%)
Principal Component 12	(1.71%)	1.40%	(0.20%)	(0.37%)	(0.33%)	(0.59%)
Principal Component 13	(1.26%)	0.84%	(0.09%)	(0.18%)	(0.16%)	(0.29%)
Principal Component 14	(0.99%)	0.53%	(0.06%)	(0.14%)	(0.12%)	(0.24%)
Principal Component 15	(0.21%)	0.39%	(0.04%)	(0.07%)	(0.06%)	(0.10%)

Table 50: *Principal components of dataset 1 - 2004-2014 (2)*

	PC 1	PC 2	PC 3	PC 4	PC 5	PC 6	PC 7	PC 8
SPX	0.40	(0.30)	(0.00)	0.05	0.05	0.09	0.13	(0.52)
NDX	0.41	(0.36)	(0.03)	0.06	0.04	0.10	0.71	0.29
RTY	0.51	(0.41)	(0.01)	0.13	0.07	(0.22)	(0.66)	0.17
G1O2	(0.01)	0.01	(0.00)	0.01	0.08	(0.03)	(0.00)	0.02
G3O2	(0.04)	0.02	(0.02)	0.06	0.35	(0.11)	0.02	0.02
G7O2	(0.08)	0.03	(0.03)	0.10	0.57	(0.17)	0.05	0.01
C1A2	(0.00)	0.01	(0.00)	0.01	0.09	0.03	(0.01)	(0.00)
C4A2	(0.02)	0.02	(0.02)	0.04	0.36	0.02	0.00	(0.07)
C7A2	(0.06)	0.04	(0.03)	0.07	0.57	(0.10)	0.05	(0.01)
J1A2	0.03	0.02	(0.02)	(0.06)	0.17	0.70	(0.18)	0.55
J3A2	0.04	0.01	(0.01)	(0.04)	0.09	0.38	(0.10)	(0.22)
J7A2	0.04	0.00	(0.02)	(0.05)	0.12	0.47	(0.09)	(0.51)
SPGSENTR	0.48	0.56	0.67	(0.02)	0.04	(0.03)	0.03	0.02
SPGSINTR	0.39	0.39	(0.63)	(0.54)	0.04	(0.11)	0.00	0.02
SPGSPMTR	0.16	0.38	(0.39)	0.81	(0.13)	0.08	0.02	(0.01)

Table 51: *Principal components loadings of the dataset (1)*

	PC 9	PC 10	PC 11	PC 12	PC 13	PC 14	PC 15
SPX	0.67	(0.03)	(0.05)	(0.00)	0.03	0.01	0.00
NDX	(0.33)	(0.00)	0.02	0.00	(0.00)	(0.00)	0.00
RTY	(0.21)	0.02	0.01	(0.01)	(0.01)	(0.00)	(0.00)
G1O2	(0.01)	(0.14)	0.03	(0.16)	0.42	0.27	0.84
G3O2	(0.01)	(0.34)	(0.08)	(0.14)	0.70	(0.31)	(0.37)
G7O2	0.01	(0.23)	(0.40)	0.54	(0.24)	0.22	0.07
C1A2	(0.01)	(0.15)	0.19	(0.29)	0.02	0.84	(0.38)
C4A2	(0.03)	(0.44)	0.35	(0.45)	(0.50)	(0.28)	0.13
C7A2	0.05	0.76	0.23	(0.14)	0.06	(0.02)	0.02
J1A2	0.34	0.01	(0.15)	(0.05)	(0.00)	(0.02)	0.00
J3A2	(0.19)	(0.10)	0.64	0.55	0.16	0.00	(0.00)
J7A2	(0.50)	0.10	(0.43)	(0.21)	(0.00)	0.01	0.00
SPGSENTR	(0.02)	0.00	(0.01)	0.01	(0.00)	(0.00)	(0.00)
SPGSINTR	0.00	(0.01)	(0.01)	0.00	0.00	0.00	(0.00)
SPGSPMTR	0.01	0.01	0.00	(0.01)	(0.00)	(0.00)	0.00

Table 52: *Principal components loadings of the dataset (2)*

Appendix 4.D Matrices and variables

Name	Space	Explanation
n	$\mathbb{N}$	Number of assets
k	$\mathbb{N}$	Number of time periods with $k < n$
x	$\mathbb{R}^{k \times n}$	Daily asset returns
ρ	$\mathbb{R}$	Correlation of two assets with $-1 \leq \rho \leq 1$
σ, σ_i	$\mathbb{R}$	Volatility of P or of asset i
Σ	$\mathbb{R}^{n \times n}$	Covariance matrix of x
w	$\mathbb{R}^n$	Asset weights
P	$\mathbb{R}^k$	Daily portfolio returns $(x \cdot w)$
MRC_i	$\mathbb{R}$	Marginal risk contribution of asset i
RC_i	$\mathbb{R}$	Risk contribution of asset i
m	$\mathbb{N}$	Number of equalized risk factors
C	$\mathbb{R}^{n \times n}$	Principal component mixing matrix
$\widetilde{x}$	$\mathbb{R}^{k \times n}$	Principal components $(x \cdot C)$
$\widetilde{\sigma}$,$\widetilde{\sigma}_i$	$\mathbb{R}$	Volatility of $\widetilde{P}$ or $\widetilde{x}_i$
$\widetilde{\Sigma}$	$\mathbb{R}^{n \times n}$	Covariance matrix of $\widetilde{x}$
$\widetilde{w}$	$\mathbb{R}^n$	Principal component weights $(C^T \cdot w)$
$\widetilde{P}$	$\mathbb{R}^k$	Daily principal component portfolio returns $(\widetilde{x} \cdot \widetilde{w})$
$\widetilde{MRC}_i$	$\mathbb{R}$	Marginal risk contribution of a principal component i
$\widetilde{RC}_i$	$\mathbb{R}$	Risk contribution of a principal component i
e_{min}	$\mathbb{R}$	Min. level of explanation with $0 \leq e_{min} \leq 1$
A	$\mathbb{R}^{z \times n}$	Matrix used for polytope definition
b	$\mathbb{R}^z$	Inequalities solutions for polytope definition
Q	$\mathbb{R}^n$	Polytope as subset of $\mathbb{R}^n$
q	$\mathbb{R}^{n \times r}$	Finite number of polytope corner points, $Q = conv\{q_1, ..., q_r\}$
β	$\{-1, 1\}^2$	Principal component directions
$VaR_\alpha(P)$	$\mathbb{R}$	Value at risk of portfolio P to probability level α
$ES_\alpha(P)$	$\mathbb{R}$	Expected shortfall of portfolio P to probability level α
$\widetilde{MES}_i$	$\mathbb{R}$	Marginal expected shortfall of a principal component i
$\widetilde{CES}_i$	$\mathbb{R}$	Component expected shortfall of a principal component i

5 Conclusion and final remarks

This thesis consists of three different works, one individual paper and two studies that were jointly conducted with Steffen Moellenhoff. They cover the topic of return analysis of precious metals in a time-varying context, using a Kalman smoother and a Dynamic Time Warping approach, as well as a modified version of the factor risk parity model. This model is designed in a way that it allows to add further restrictions such as portfolio weight constraints by focusing on the main principal components only. The model is described when the standard deviation as the risk measure is taken as well as when the expected shortfall is used as a measure to account for non-symmetrical, fat tailed return distributions.

"Analyzing Precious Metals returns using a Kalman Smoother Approach" is focusing on the analysis of returns of precious metals in a time-varying context. The results include the similarities but also differences between gold, silver, platinum and palladium regarding an external factor. The study illustrates how the sensitivities in the case of precious metals, especially for longer time series, often vary over time. Platinum or palladium, for example, illustrate that finding as the demand in industrial processes in 2015 has completely changed from that 30 or 40 years ago. Other research that tries to identify a fixed and constant relationship between factors and asset returns without taking the factor of time into consideration, should therefore be handled with care.

Additional insight is given on how different/similar the four precious metals are related to movements in specific factors. This also delivers a much more detailed analysis of the relationship as for example a simple correlation coefficient can do. The sensitivity is related to only a specific factor and delivers insight into how stable that sensitivity has been over time. The same approach can easily be applied to other asset classes as well, which has been briefly illustrated in the case of the S&P500 in the introduction of this thesis. This also enables a comparison between the factor analysis done in other research and the approach taken here.

On the downside, the Kalman smoother that is used in the analysis needs input parameters which are not directly observable. Those parameters have to be estimated or calculated using a different approach. Similar to the length in a rolling window approach, the question on how to choose those parameters is not easy to answer and is often up to interpretation. As mentioned in the study, a stringent approach that has to be followed at all cost does not exist.

The second and third study build on existing factor risk parity models but add flexibility by letting "residual" components float. The idea consists of ignoring the less important principal components which leads to higher flexibility as the size of the solution set for the problem increases. This enables the investor to choose from a wider variety of portfolios, thereby allowing to add additional restrictions on the portfolio weights or identifying portfolios that have additional characteristics. In the study, we choose "long only" portfolios with a minimum variance and maximum diversification on the newly determined solution set.

One of the core problems with factor risk parity portfolios consists of the optimal portfolio having short or leveraged positions, which investors often want to avoid or which investors are prohibited from investing in. Using a PCA, there are different ways to identify "nearly" optimal portfolios. One way is to let the risk contributions in the PCA space be "approximately" equal while at the same time adding further restrictions, as for example done by Meucci (2009).

Our approach is similar. By setting the first risk contributions exactly equal and letting residuals float, however, we gain further advantages: first, the residual components are often considered noise and risk contributions do not need to be equal. Second, focusing on first contributions only and defining special portfolio weight constraints let the solution set be described in an efficient way through polyhedrons. Knowing that a solution exists, further operations on this set are easy to implement.

Finally, this version is extended by exchanging the risk measure that is used within the model. For a general distribution assumption of the underlying assets, the solution can only be determined numerically when the expected shortfall is used. Nevertheless, this downside is outweighed by the benefit of the non-symmetric risk measure as financial time series often exhibit a higher skew and fat tails. Results show that a higher skewed/fat tailed time series with a higher confidence level lead to bigger changes to the allocation than in the standard deviation case.

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