Bergische Universität Wuppertal
Fachberichte C

Doctoral thesis

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Non-perturbative approach to calculation of correlation functions in 1D Fermi gases

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Wuppertal, 2019

The PhD thesis can be quoted as follows:

urn:nbn:de:hbz:468-20200103-091211-1
[http://nbn-resolving.de/urn/resolver.pl?urn=urn%3Anbn%3Ade%3Ahbz%3A468-20200103-091211-1]

DOI: 10.25926/zwc5-b017
[https://doi.org/10.25926/zwc5-b017]

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0.1 Introduction

0.1.1 Integrable systems and Bethe ansatz

Quantum low dimensional systems attract huge interest since they provide a test area for the investigation of the general behaviour of complicated nonlinear systems. Indeed, a lot of interesting phenomena in condensed matter, high-energy physics and quantum field theory emerge due to the presence of complicated non inear interactions. The study of these systems is often an extremely complicated task that is out of the reach of perturbative approaches. However, it is known that in many cases interesting nontrivial many body effects prevail even if the system is restricted to 1D. Moreover, the simplified but still nontrivial cases of 1D system allows one to concentrate on key properties of a system while avoiding bulky technical problems. Sometimes this allows to make crucial simplifications because the system will exhibit integrability, i.e. can be solved exactly. We also see that a lot of specific, interesting properties emerge in 1D systems, especially in integrable models, thus making 1D integrable models interesting in their own right.

Among these interesting systems it is worth noting t-J [16, 73–79] and Hubbard [6, 7, 71, 72] models, which describe lattice gases of strongly-correlated electrons and are expected to exhibit the high-T superconductivity behaviour. While this phenomenon was intensively studied over the last 50 years, still most of the mechanisms of high-T superconductivity are unknown. One of the main open questions of the field is the mechanism of formation of electron pairs. The derivation of exact solutions of the t-J and Hubbard models can help to answer such a question and thus to understand the nature of the high temperature superconductivity. These are important models from the perspective of studying the entanglement entropy and Kondo model [9] which describes an anomalous low-T behaviour of conductivity of doped metals (Kondo problem).

Some other widely studied integrable 1D systems are models of ions with exchange interaction (Heisenberg magnets or *quantum spin chains*) [8] used for description of ferromagnets or lattice models of ferroelectricity [10]. These models were intensively studied both theoretically (see sort review [17]) and experimentally [1–5].

The various 1D Fermi [24] and Bose [29] gases generate significant interest as their investigation could allow for the creation of stable qubits, a crucial ingredient for the development of a quantum computers. Also, motivations arise from the recent successes in the realisation of 1D optical traps that open up new possibilities for experiments with 1D systems. In the 2000s series of such experiments were performed [11, 12, 19, 136].

In field theories, integrable models appear in the context of quantization of the gravitational interaction in two dimensions, where Einsteins equations of general relativity are reduced to the Liouville equation whose quantization leads to the Liouville field theory. Non-perturbative solution of this theory could be an important step toward understanding quantum gravity. One more integrable system is the Gross-Neveu model that describes a simple 2D version of the quantum chromodynamics [31]. The key property of this model is an existence of the asymptotic freedom, while in contrast to the 4D analog called Nambo-Jona-Lasinio model the Gross-Neveu model is renormalisable. While the understanding of asymptotic freedom is one of the most complicated and important questions in the theory of the fundamental interaction, the understanding of this property on the simplest example can help to get an insight into the 4D problem.

Despite study of quantum integrable systems track down to work of H. Bethe published in 1931 [13] and fascinating success gained during last 80 years, a wide range of questions still open up to now especially in field of multicomponent and nonequilibrium systems. Nontrivial questions remaining are also system behaviour dependence on temperature, external fields, interaction strength, etc. The reason lies in the extreme complexity of integrability. The obtaining of the non-perturbative results requires a lot of specific and complicated mathematical techniques. Since the publication by H. Bethe in his celebrated work on the Heisenberg spin chains, the method, called *the Bethe ansatz*, became one of the most powerful and common instrument of study of integrable systems. The method allowed to exactly calculate the spectrum and the eigenvectors of a wide range of models. Thus, this method was used for solving the problem of 1D Bose and Fermi gases (e.g. mixtures of gases with different particle types and higher spins), Kondo problem, various 1D field and lattice statistical systems. Later the approach was also generalised to the case of non-zero temperature.

From the technical sight the Bethe ansatz is roughly nothing but expansion the eigenvectors of the periodical system in the basis of planar waves, where each wave describes a single (quasi)particle of the system (say, boson in 1D Bose gas or spinons in the Heisenberg magnet) while the equation for the spectrum is nothing but a periodicity condition for the wave functions of (quasi)particles. Consider, for example, the Heisenberg spin chain

$$H = J\sum_{j=1}^{L} \sigma_j^x\sigma_{j+1}^x + \sigma_j^y\sigma_{j+1}^y + \Delta\sigma_j^z\sigma_{j+1}^z. \tag{0.1.1}$$

Here L is the chain length, operators σ_j (Pauli matrices) are spin operators on site j. This Hamiltonian describes (quasi)one-dimensional electrons connected to heavy ions, on a periodic lattice. A typical example of such system is $KCuF_3$ ($\Delta = 1$ in this case). The proposed method, called also *coordinate Bethe ansatz* can be applied to periodical system (so $\sigma_{L+1}^k = \sigma_1^k$) and based on presenting of system of eigenvectors in the form of a linear combination

$$|\Psi\rangle = \sum_{k_1\ldots k_j} a(p_1,\ldots,p_j|k_1,\ldots,k_j)\sigma_-^{(k_1)}\cdots\sigma_-^{(k_j)}\Omega, \tag{0.1.2}$$

where $a(p_1, \dots, p_j | k_1, \dots, k_j)$ are j-particles wavefunctions of corresponding (quasi)particles excitations with rapidities $\{p_1, \dots, p_j\}$ at positions $\{k_1, \dots, k_j\}$, Ω is the state with all spins ordered up, and $\sigma_-^{(k_1)}$, $\sigma_-^{(k_2)}$, ... are spin-flip operators that create excitations ($j = L/2$ corresponds to case where half of spins are flipped). Such linear combinations are called Bethe vectors. The periodicity condition imposes restriction on these rapidities (called *Bethe parameters* or *spectral parameters*). For the isotropic (or XXX) spin chain ($\Delta = 1$ in (0.1.1)) the system of Bethe equations (BAE) is

$$\left(\frac{p_k + i/2}{p_k - i/2}\right)^L = \prod_{\ell \neq k}^{j} \frac{p_k - p_\ell + i}{p_k - p_\ell - i}, \qquad j \leq L/2. \tag{0.1.3}$$

Since the set $\{p_1, \dots, p_j\}$ univocal defines the energy, equation (0.1.3) can be considered as equation for the spectrum of the system. Thereby a linear combination of the secondary quantised multiparticle wavefunctions span eigenvector of the system if their spectral parameters satisfy the BAE. Such Bethe vectors are called *on-shell*. In the case where no restrictions are imposed on spectral parameters Bethe vectors are called *off-shell* and they are no more eigenfunctions of the system.

Further, that approach was generalised for an anisotriopic (or XXZ) chain [14, 15], Bose and Fermi gases [18, 20–24], multicomponent systems [25, 27] (Nested Bethe ansatz, NBA), finite temperature [30] (thermodynamic Bethe ansatz, TBA) and field systems [31]. Bethe ansatz became one of most universally used approach for calculations in quantum integrable systems.

As a very simple and an intuitive approach Bethe ansatz is widely used in different fields and proved itself a fruitful and powerful method. It turns out, however, that the method is very complicated technically to apply to the calculation of the correlation functions of operators. Thus, the method allows to build eigenfunctions of Hamiltonian of integrable systems, but even the calculation of the norm of these eigenfunctions in case of many (quasi)particles is a complicated problem [32], while the calculation of matrix elements of physical operators was completely out of reach.

In the 70th a new method, called an *Algebraic Bethe ansatz* (ABA) was developed by the Leningrad school of physics (L. D. Faddeev et al.) [34–42]. The main components of this approach are the *monodromy matrix* [43–45], whose matrix entries are (quasi)particles creation/annihilation operators, the R-matrix (introduced by C. N. Yang [25] and R. Baxter [26]), that defines 2-particle scattering processes, and the $RTT-$ and Yang-Baxter relations on these operators and scattering matrix entries. The new approach allowed to establish a more unified way to construct the eigenvectors (that now are certain polynomials on the monodromy matrix entries) and the spectrum of the system. It allows explicitly build *transfer matrix* of a system (operator of evolution on the lattice), that produces in a certain way all integrals of motion. New formulation of the Bethe ansatz allows deeply understand the nature of integrability that now can be seen as a condition of factorisation of scattering processes into 2-particle scatterings.

Further, ABA was generalised to the case of multicomponent systems, higher spins, etc. The method gives more simple structure of the eigenvectors, that no more contains multiple summations on the coordinates of the (quasi)particles of the system. Calculation of the matrix elements of operators (*form factors*) however still was complicated problem: although it was possible to calculate the form factors explicitly (Izergin-Korepin formula), the final result was given by quite bulky expression and was not suitable even for the numerical analysis.

The most important results in this direction are work [46], where the norm of eigenvectors was calculated and work [47], where the scalar product of on-shell and off-shell Bethe vectors was obtained in a compact form (determinant representation of a scalar product). There was shown, that the scalar product can be rewritten as a determinant

$$S(\{p_i\}|\{k_j\}) \sim \det\left(\mathcal{N}(k_i, p_j|\{p\}, \{k\})\right), \tag{0.1.4}$$

where $\mathcal{N}$ is special matrix, $\{k\}$ are spectral parameters of the off-shell and $\{p_j\}$ are spectral parameters of the on-shell Bethe vectors, $\#\{p\}$ and $\#\{k\}$ are n. The fact that one of the Bethe vectors is still off-shell (no restriction on spectral parameters) is very important, since form factors $\mathcal{F}^O_{mn}$ of arbitrary physical operator O between eigenvectors $|m\rangle$, $|n\rangle$ can be rewritten as

$$\mathcal{F}^O_{mn} = \langle m|O|n\rangle = \langle n|\Psi\rangle, \tag{0.1.5}$$

where $|\Psi\rangle \equiv O|n\rangle$, as it can be shown, is a linear combination of the off-shell Bethe vectors.

In the same work [47] a determinant representation for the form factors of T-matrix entries was calculated in determinant form

$$\mathcal{F}(\{\lambda\}, \{\mu\}) \sim \det\left(\mathcal{M}(\mu_i, \lambda_j|\{\lambda\}, \{\mu\})\right), \tag{0.1.6}$$

where $\mathcal{F}$ is the form factor, the matrix $\mathcal{M}(\mu_i, \lambda_j|\{\lambda\}, \{\mu\})$ depends on particular operator, the sets $\{\lambda\}$ and $\{\mu\}$ are rapidities of the (quasi)particles in the left and the right eigenstates.

In work [97] one-point (ultralocal) physical operators were expressed via the matrix entries of monodromy matrix T in quantum spin chain solving a quantum inverse problem. Later, for arbitrary models form factors of local operator were expressed via form factors of matrix entries T_{ij} in work [110] using *composite model* approach.

The discovery of the determinant representations for the form factors of ultralocal[1] operators allowed to make a breakthrough in the study of correlation functions both numerically and analytically. Such determinant representations are very important, because they are the only known compact form for the representation of the form factors. The two point dynamical correlation functions of arbitrary local operators O can be presented then as the form factor series

$$\begin{aligned} \langle O(x,t)O^\dagger(0,0)\rangle = Z^{-1}\sum_{m,n} &\frac{\langle n|O(0,0)|m\rangle\langle m|O^\dagger(0,0)|n\rangle e^{i\omega_{mn}t-\beta\omega_n-ixp_{mn}}}{\langle m|m\rangle\langle n|n\rangle} \\ &\equiv Z^{-1}\sum_{mn}\left|\mathcal{F}^O_{mn}\right|^2 e^{i\omega_{mn}t-\beta\omega_n-ixp_{mn}}, \end{aligned} \tag{0.1.7}$$

where $\left|\mathcal{F}^O_{m,n}\right|$ is the form factor of the operator O

$$\mathcal{F}^O_{mn} = \frac{\langle n|O|m\rangle}{\sqrt{\langle n|n\rangle}\sqrt{\langle m|m\rangle}} \tag{0.1.8}$$

between the eigenstates $|m\rangle$ and $|n\rangle$, $\omega_{mn} = \omega_n - \omega_m$, $p_{mn} = p_m - p_n$ are energy and momentum of states and β is an inverse temperature and Z is a statistical sum

$$Z = \sum_n \frac{\langle n|e^{-\beta\omega_n}|n\rangle}{\langle n|n\rangle}. \tag{0.1.9}$$

[1]Below we often use term form factor of local operators implying, in fact, the form factors of ultralocal (i.e. acting in one point, not on interval) operators.

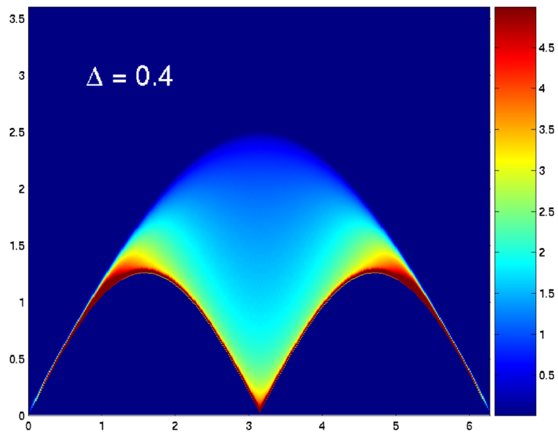

Figure 1: The image of zero-temperature correlator $\langle \sigma_1^z(t)\sigma_{m+1}^z(0)\rangle$ in the Heisenberg spin chain is given in the energy–momentum coordinates (by J.-S. Caux *et al.* [60, 61])

The method can be easy applied for many point function too.

Finally, in a linear response theory, the transport coefficients σ can be expressed via the correlators of correspondent currents (Kubo formula)

$$\sigma(q,\omega) = \frac{1}{\omega V}\int_0^\infty dt e^{i\omega t}\langle [J(q,t), J(q,0)]\rangle + i\frac{ne^2}{\omega}, \tag{0.1.10}$$

where V is a system volume (in 1D length), n is density and e^2 eigenvalue of a charge conjugated to current.

The summation of form factor series is a complicated problem. A huge contribution to the study of the analytical properties of low-temperature Bose gases, critical spin chains, etc. was done by the Lyon group [50–52]. This method allowed finally to move to consideration of the correlation functions, that allow to calculate, using Kubo formula, all transport coefficients in a model under consideration. Important progress was made by Wuppertal-Shizuoka group that allows to calculate correlation functions at non-zero temperatures [53–56] and in the dynamical case [58, 59]. The large part of progress connected with the method of the *quantum transfer matrix* [111–113], that allows to reduce double summation to just a single in (0.1.7).

The numerical algorithms for the form factor summation were developed in works of Wuppertal-Providence [159–162] and Amsterdam groups [60–62, 104, 135, 163]. With these algorithms Fourier images of the form factors of were produced (see (1)).

These results show a perfect coincidence with experimental measurements [3–5] (neutron scattering experiments) despite the real Heisenberg magnet is not pure 1D but rather quasi 1D that demonstrates that restriction of real system to toy 1D model conserves the general behavior and justify results of the long time development of the mathematical physics of 1D integrable systems.

The key moment here become *basis scanning* algorithm. Performing the summation over all states is nontrivial (and in the case of very large system an unsolvable problem), since the number of terms grows exponentially with the system size and degrees of freedom. But in fact only a very limited number of form factors give a significant contribution to the form factor series. The proper choice of such form factors is a necessary part of the numerical algorithm. Up to now such a problem was solved for wide range of models at zero-temperature but still unsolved for quantum transfer matrix.

One more interesting application of the ABA appears from the connection of the Bethe vectors scalar products with the super Yang-Mills theories [118]. Thus, it was shown that the scalar products of off-shell and on-shell Bethe vectors define many point-correlation functions in the super-Yang-Mills theories.

0.1.2 Nested Bethe ansatz

Since all information about a physical system is encoded in the monodrom matrix, it is natural that systems with a more complicated structure of the monodromy matrix have more rich physics. The generating function of the integrals of motion is the transfer-matrix that is nothing but trace of monodromy matrix. Original ABA allows to deal only with the systems whose monodromy matrices posses algebra symmetry $\mathfrak{gl}(2)$. Therefore, it is interesting to approach a theory that allows to work with the monodromy matrices associated with algebra symmetry $\mathfrak{gl}(N)$ or graded algebra symmetry $\mathfrak{gl}(m|n)$. Systems with such monodromy matrices have more degrees of freedom. For instance, both Gross-Neveu field model and 1D Fermi gas are fermionic, so that their analysis requires to take into account the spin degree of freedom. Another important example are t-J and Hubbard lattice models or many-component Bose and Fermi gases with (iso)spins, discussed earlier.

ABA that allows to work with such multicomponent systems is called Nested ABA (NABA). In this approach Bethe vectors have a much more complicated structure, than in the case of ordinary ABA. The first method of derivation of such vectors for algebra symmetry $\mathfrak{gl}(3)$ was developed in [63, 64] and later generalised for algebra symmetry $\mathfrak{gl}(N)$. It was shown that in this case Bethe vectors are polynomials of very special form in the monodromy matrix entries contrary to ordinary ABA, where they are just monomials. The calculation of scalar products and form factors of operators in this case is much more involved. The first results for algebra symmetry $\mathfrak{gl}(3)$ were obtained in [65–68]. Generalisation of such results for algebra symmetry $\mathfrak{gl}(4)$ and higher is still an open problem. An important point here is the fact, that monodromy matrices associated with certain symmetry still have many degrees of freedom, so physically different systems can have monodromy matrices associated with the same algebra symmetry. If form factors of monodromy matrix entries that possess certain algebra symmetry are found, the problem of calculation of one-point correlation functions is solved for all these models. For instance, both Heisenberg spin chain and 1D spinless one-component Bose gas have monodromy matrices associated with algebra symmetry $\mathfrak{gl}(2)$.

This thesis is devoted to the application of NABA to the calculation of form factors and correlation functions in case of algebra symmetry $\mathfrak{gl}(2|1)$. It is the first non-trivial example of such calculations in a graded algebra symmetry. Such calculation is first step in generalisation of ABA for higher-rank algebra symmetry case, that is important, since huge number physically interesting systems posses algebra symmetry $\mathfrak{gl}(N)$ or $\mathfrak{gl}(m|n)$. This case is interesting in itself, since it has physical applications. In particular, $t-J$ model at t=J/2 (see (1.1.31)) and 1D one-component Fermi gas (Gaudin-Yang model) have monodromy matrices with algebra symmetry $\mathfrak{gl}(2|1)$. t-J model was already studied using NABA by [73–76] (see also [80, 82]). Gaudin-Yang model was studied in [123–130, 150] (see also review [122]). Other interesting examples with such symmetry are models of 1D lattice gases [28, 83].

The goals of this thesis are calculations of one-point (also we call them ultralocal) form factors and correlation functions in such model. The first step consists in the calculation (like in usual ABA) of the determinant representation of scalar product. Such representation is among the main results of this work. The second problem is the calculation of the determinant representations for form factors. The last step is the numerical calculation of the correlation function using form factor series summation.

Summarising above, we can calculate correlation functions in 1D integrable systems associated with algebra symmetry $\mathfrak{gl}(2|1)$ using the following approach.

1. Initially the systems with finite (but still arbitrary) number of (quasi)particles are considered. In the ABA terminology it means that there is a finite number of spectral parameters and Bethe vectors consist of a finite number of (quasi)particles creation operators. Bethe vectors of the systems with $\mathfrak{gl}(2|1)$ algebra symmetry were obtained in work [84]. As first step the action rules of the (quasi)particles creation/annihilation operators on the Bethe vectors should be found. Such action rules can be found relatively easy using recursion relations for the Bethe vectors and commutation rules between the creation/annihilation operators, that are known from the definition of monodromy matrix of system. Creation(annihilation) operators act trivially on the left(right) vacuum, so despite complicated commutation relations between these creation and annihilation operator the computation is, in fact, nothing but normal ordering of the operators similar to ordinary quantum field theory. It can be shown that actions of these creation/annihilation operators on Bethe vector $|\psi\rangle$ spawn a linear combination of the off–shell Bethe vectors $|psi^{(n)}\rangle$

$$O|\psi\rangle = \sum_n C_n|\psi^{(n)}\rangle. \tag{0.1.11}$$

2. Using rules, established above, and a fact that a (dual)Bethe vector is special polynomial on creation/annihilation operators that acts on (dual) vacuum Ω, it is possible to examine scalar product of ordinary Bethe and dual Bethe vector. This technically cumbersome procedure and result has pretty bulky form. In particular, the final expression is given by the bilinear combination of two scalar products of special form, so called *highest coefficient* (details are given in chapter 3). This form is an analog of a representation for the scalar product found in [85] for the $\mathfrak{gl}(3)$ algebra symmetry case. The calculated expression at this stage contains multiple summations and barely can be applied to calculation of the correlation functions.

3. At the next step the multiple sum expression for the scalar product should be reduced to compact form (namely, to determinant of special matrix). This is the most nontrivial phase since there is no general approach for the calculation of such sums for arbitrary algebra symmetry and method should be developed independently for algebra symmetry $\mathfrak{gl}(2|1)$. In particular, the details of calculation and result for the graded algebra symmetry are quite different form algebra symmetry $\mathfrak{gl}(3)$ systems.

4. Further, using already obtained scalar product and action rules of *zero modes* (see [86]), the form factors of T-matrix are calculated. Since all form factors of ultralocal physical operators can be expressed via form factors of monodromy matrix the problem of one-point correlation function calculation is solved.

5. Finally, the form factor series should be calculated using the explicit determinant representation of the form factors and the numerical summation algorithm. At this step the “scanning algorithm”, that allows to make a scan of the basis for choosing the most contributing form factor, is required.

Results, given in this thesis, were published in works [87–90].

The thesis contains 6 chapters and two appendices:

In chapter 1 the basics of ABA are introduced, the preliminary facts given and necessary notation are defined.

Chapters 2-3 are devoted to steps 1-2 correspondingly of the plan above. Some cumbersome but technically simple calculations here are not shown, but they can be found in the Appendices.

Chapter 4 devoted to the calculation of scalar product in the determinant form, and, as particular case, the norm of Bethe vectors.

In chapter 5 form factor of the monodromy matrix entries are derived.

In chapter 6 numerical algorithm for the form factor series summation is developed and correlation functions are calculated.

Appendix 1 devoted to details of calculation of the multiple actions from chapter 2.

Appendix 2 devoted to description of some properties of the highest coefficient from chapter 3.

0.2 Acknowledgments

I appreciate to my colleagues A. Liashyk, N. A. Slavnov, S. Z. Pakuliak and E. Ragoucy with whom a large part of this work was done. I am appreciate to N. A. Slavnov, A. Liashyk, F. Göhmann, A. Klümper and M. Karbach for the numerous discussions on the Bethe ansatz computations. I am thankful to F. Göhmann, R. Kleinemühl and C. Babenko for the careful reading and useful commentaries on this work.

Chapter 1

Algebraic Bethe ansatz

1.1 Algebraic Bethe ansatz

Here the necessary tools of integrable systems are given. Nested algebraic Bethe ansatz is briefly described. Specific terminology introduced.

1.1.1 Monodromy matrix and RTT-relation

We deal with 1D quantum integrable systmes on the lattice. In case of Heisenberg magnets, or lattice gas models the systems are defined on a lattice, in case of Gaudin-Yang or Lieb-Liniger type models we can define the model on a lattice for the technical convenience and consider a continuous limit later.

The main component of integrable systems theory is the *L-operator*, that defines the particular physical system. L-operator satisfies the relation

$$R(u,v)\big(L(u)\otimes\mathbb{I}\big)\big(\mathbb{I}\otimes L(v)\big)=\big(\mathbb{I}\otimes L(v)\big)\big(L(u)\otimes\mathbb{I}\big)R(u,v), \tag{1.1.1}$$

called *RLL-(or RTT-)relation*.

Both sides of (1.1.1) act in a tensor product of three spaces $\mathbb{C}^N\otimes\mathbb{C}^N\otimes\mathcal{H}$, where $\mathcal{H}$ is a Hilbert space of a system under consideration, called also *quantum space*, while spaces $\mathbb{C}^N$ are called *auxiliary spaces*. Variables u, v traditionally are called *spectral parameters*, in case they satisfy a special system of equations (see (1.1.20)) we will also call them *Bethe parameters*. Each L-operator acts in the tensor product of auxiliary and quantum spaces (further we will write it as a matrix in the auxiliary space which matrix elements are operators that act in the quantum space), R-matrix also acts in two auxiliary spaces and itself satisfies the *Yang-Baxter equation*

$$R_{12}(x,y)R_{13}(x,z)R_{23}(y,z)=R_{23}(y,z)R_{13}(x,z)R_{12}(x,y). \tag{1.1.2}$$

Both sides act on a tensor product of 3 spaces $\mathbb{C}^N$, subscripts denote the space number where certain matrix acts nontrivially. There exist many solutions of equation (1.1.2) — rational, trigonometric, elliptic. Further we will use only the simplest case of the rational R-matrix

$$R(x,y)=\mathbb{I}+g(x,y)P,\qquad g(x,y)=\frac{c}{x-y}, \tag{1.1.3}$$

that posses $\mathfrak{gl}(N)$ symmetry. Here c is an arbitrary constant, $\mathbb{I}$ is the unit operator and P is operator of spaces permutation,

$$P = \sum_{ij} E_{ij} \otimes E_{ji}, \qquad \text{where } N \times N \text{ matrices } E_{ij} \text{ are defined as } (E_{ij})_{kl} = \delta_{ik}\delta_{jl} \tag{1.1.4}$$

Further we consider the case of $\mathfrak{gl}(N)$ algebra symmetry, implying the use of a rational R-matrix and auxiliary space $\mathbb{C}^N$.

In terms of lattice model L-operator defines lattice systems consisting of one site. In order to have many site system the product of operators L_{0j}, $j = 1, \ldots, n$ should be taken. Subscripts 0 here denote an auxiliary space (it is common for all L-operators), while subscripts j index quantum spaces, each of them corresponds to one site of system

$$T(u) = L_n(u) \ldots L_1(u), \qquad L_j \equiv L_{0j}. \tag{1.1.5}$$

Correspondingly, the resulting operator will be matrix acting on the auxiliary space and an operator on all quantum spaces, i.e. in all sites of the lattice. It can be easily shown, that if L-operator satisfies (1.1.1), their product also satisfies same equation, where $\mathcal{H}$ will denote tensor product of all quantum spaces $j = 1, \ldots, n$. Operator (1.1.5) is called *monodromy matrix*.

From relation (1.1.1) and explicit expression (1.1.3) follow commutation relation for matrix elements of T (here and further indecies numerate matrix element, not the quantum spaces, if opposite is not stated explicitly):

$$[T_{ij}(u), T_{kl}(v)] = g(u,v) \left[T_{kj}(v)T_{il}(u) - T_{kj}(u)T_{il}(v)\right]. \tag{1.1.6}$$

Transfer matrix is defined as trace of the monodromy matrix

$$\mathcal{T}(u) = \operatorname{tr} T(u) = \sum_{i=1}^{N} T_{ii}(u). \tag{1.1.7}$$

It is easy to show, that from (1.1.1)-relation follows

$$[\mathcal{T}(u)\,,\, \mathcal{T}(v)] = 0. \tag{1.1.8}$$

Expanding (1.1.8) in Taylor series with respect to u and v at point u_0, we obtain commutation property of all coefficients of the expansion $\mathcal{T} = sum I_j u^j$

$$[I_j, I_k] = 0, \qquad \forall j, k. \tag{1.1.9}$$

These coefficients are some operators that act in Hilbert space $\mathcal{H}$. Identifying one of I_j with Hamiltonian of some system we obtain the model with a set of mutually commuted integrals of motion I_j.

It follows from (1.1.31), that eigenvectors of the transfer-matrix will also be eigenvectors of the Hamiltonian and all integrals of motion.

Since operator $L_i(u - \xi)$ also satisfies RTT-relation, it is possible to introduce monodromy matrix

$$T\left(u|\{\xi_1, \ldots, \xi_n\}\right) = L_n(u|\xi_n) \ldots L_1(u|\xi_1). \tag{1.1.10}$$

We call the model, defined by this monodromy matrix, *the inhomogeneous model*, and the set of parameters $\{\xi_1, \ldots, \xi_n\}$ — *inhomogenities.* Transfer-matrix, integrals of motion, etc. are defined for inhomogeneous model in similar way.

1.1.2 Graded algebra case

RLL-relation and Yang-Baxter equation, introduced in (1.1.1), can be extended for the case of graded algebras $\mathfrak{gl}(m|n)$. Space grading $\mathbb{C}^{m|n}$ can be defined in following way : $[i] = 0$ for $i = 1 \ldots m$, $[i] = 1$, for $i = m+1, \ldots, m+n$, square brackets denote parity. Matrices E_{ij} have grading $[E_{ij}] = [i] + [j]$. Tensor product is defined according to rule

$$(E_{ij} \otimes E_{kl}) \cdot (E_{pq} \otimes E_{rs}) = (-1)^{([k]+[l])([p]+[q])} E_{ij} E_{pq} \otimes E_{kl} E_{rs}. \tag{1.1.11}$$

The graded unity matrix and permutation operator P, acting on the tensor product of spaces $\mathbb{C}^{m|n} \otimes \mathbb{C}^{m|n}$ are defined as

$$P = \sum_{ij} (-1)^{[j]} E_{ij} \otimes E_{ji}, \tag{1.1.12}$$

$$\mathbb{I} = \sum_{i=1}^{m+n} E_{ii}. \tag{1.1.13}$$

Taking into account the grading of a tensor product and using graded R-matrix commutation relation for matrix elements T can be written as:

$$[T_{ij}(u), T_{kl}(v)\} = (-1)^{[i]([k]+[l])+[k][l]} g(u,v) \left[T_{kj}(v) T_{il}(u) - T_{kj}(u) T_{il}(v)\right], \tag{1.1.14}$$

where *the graded commutator* is used

$$[T_{ij}(u), T_{kl}(v)\} \equiv T_{ij}(u) T_{kl}(v) - (-1)^{([i]+[j])([k]+[l])} T_{kl}(v) T_{ij}(u). \tag{1.1.15}$$

Graded transfer-matrix is defined as graded supertrace of monodromy matrix

$$\mathcal{T}(u) = \operatorname{str} T(u) = \sum_{i=1}^{m+n} (-1)^{[i]}\, T_{ii}(u). \tag{1.1.16}$$

All terminology introduced above conserves in case of graded algebras up to replacement $\mathfrak{gl}(N) \to \mathfrak{gl}(m|n)$, introducing the graded tensor product, trace and spaces $\mathbb{C}^{m|n}$ and replacement $\mathbb{I}, P$ in definition of R-matrix (1.1.3) by graded.

Further we will consider graded algebra case $\mathfrak{gl}(2|1)$.

1.1.3 Eigenvectors of transfer-matrix and Bethe equations

ABA allows to build eigenvectors of the algebra $\mathfrak{gl}(m|n)$ case in explicit form. In particular, these eigenvectors are polynomial of special form on matrix elements of the monodromy matrix $T_{ij}(u)$ with $i \le j$ acting on vector Ω, that define pseudovacuum. Vector Ω satisfies following properties:

$$\begin{aligned} &T_{ii}(u)\Omega = \lambda_i(u)\Omega, \quad i = 1, \ldots, N, \\ &T_{ij}(u)\Omega = 0, \qquad N \ge i > j \ge 1. \end{aligned} \tag{1.1.17}$$

Here $\lambda_i(u)$ are scalar functions, that depend on the particular physical model. Further we also use ratios of these functions

$$r_1(u) = \frac{\lambda_1(u)}{\lambda_2(u)}, \qquad r_3(u) = \frac{\lambda_3(u)}{\lambda_2(u)}. \tag{1.1.18}$$

The particular form of polynomial, that defines Bethe vectors depends on the algebra symmetry. For instance, in the simplest case algebra $\mathfrak{gl}(2)$, Bethe vectors are monomials with respect to matrix element $T_{12}(u)$

$$\mathbb{B}_a(\bar{u}) = T_{12}(u_1)\dots T_{12}(u_a)\Omega. \tag{1.1.19}$$

$T_{12}(u)$ can be considered as a creation operator of (quasi)particles with rapidity u (correspondingly T_{21} can be considered as an annihilation operator). Further we will use the following notation for the subsets: $\bar{u} = \{u_1, \dots, u_n\}$, $\bar{v} = \{v_1, \dots, v_n\}$, $\bar{\eta} = \{\eta_1, \dots, \eta_n\}$, etc. Bethe parameters here are arbitrary complex numbers. In special case they satisfy system of equations, called *Bethe equations*.

$$\frac{\lambda_1(u_i)}{\lambda_2(u_i)} = \prod_{j\neq i}^{a} \frac{u_i - u_j + ic}{u_i - u_j - ic}, \qquad i = 1, \dots, a, \qquad a = \#\bar{u}. \tag{1.1.20}$$

Definition 1.1.1. *In the case when Bethe parameters $\bar{u}$ satisfy the system of Bethe equations, Bethe vectors become eigenvectors of the transfer-matrix. In this case we call them on-shell Bethe vectors. Otherwise they are called off-shell Bethe vectors.*

In addition to right, the left (or dual) Bethe vectors are used. In the algebra symmetry $\mathfrak{gl}(2)$ case they are

$$\mathbb{C}_a(\bar{v}) = \Omega^\dagger T_{21}(v_1)\dots T_{21}(v_a). \tag{1.1.21}$$

Vector $\Omega^\dagger$ here is a dual vacuum, normalised in such way that $\Omega^\dagger\Omega = 1$.

Similarly, dual Bethe vectors become eigenvectors of the transfer matrix if the set of $\bar{v}$ satisfies the same Bethe equation system (1.1.20).

In case of algebra $\mathfrak{gl}(N)$ the procedure that allows to build Bethe vectors is based on the modification of ABA called "nested" ABA (NABA). Consider the monodromy matrix in case $\mathfrak{gl}(2)$. It can be represented as a square matrix of size 2×2. Matrix elements of this matrix traditionally are called as $T_{11}(u) = A(u)$, $T_{12}(u) = B(u)$, $T_{21}(u) = C(u)$ and $T_{22}(u) = D(u)$. The monodromy matrix in case of algebra symmetry $\mathfrak{gl}(3)$ can be represented as matrix of size 3×3. It can be presented as a block matrix of size 2×2, whose matrix elements $\mathbb{B}$, $\mathbb{C}$ are vectors, and element $\mathbb{D}$ is a 2×2 matrix :

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \longrightarrow \begin{pmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{pmatrix} = \begin{pmatrix} A & \mathbb{B} \\ \mathbb{C} & \mathbb{D} \end{pmatrix}, \qquad \mathbb{D} = \begin{pmatrix} T_{22} & T_{23} \\ T_{32} & T_{33} \end{pmatrix}, \tag{1.1.22}$$
$$\mathbb{B} = \{B_1, B_2\}, \qquad \mathbb{C} = \{C_1, C_2\}^T.$$

Obviously, in the case of $\mathfrak{gl}(N)$ algebra symmetry a matrix of size N can be rewritten as a matrix of smaller sizer. In this case block $\mathbb{D}$ of size $N-1$ also can be rewritten as a monodromy matrix with with matrix element $\mathbb{D}'$ of size $N-2$, and i. g. In such a way consequence of embedding ("nesting") can be build: $\mathfrak{gl}(2) \subset \mathfrak{gl}(3) \subset \dots \subset \mathfrak{gl}(N)$, that give the name to nested ABA.

Such consequence of nesting allows to establish explicit forms of the arbitrary (off-shell) Bethe vectors for algebra symmetry $\mathfrak{gl}(N)$. In work [63] using the consequence of embedding an explicit form of the polynomials that define Bethe vector was found for an arbitrary N in case of fundamental representation of the monodromy matrix. The following procedure was used there.

Consider the case of the *fundamental representation* of algebra symmetry $gl(3)$. In this case the action of T_{23} on the vacuum Ω is trivial, so the only possibility is to use an ansatz for the Bethe vectors that contains only $B_1(u_i)$ and $B_2(u_j)$

$$\begin{aligned}\mathbb{B}_a(\bar{u}) = \mathbb{F}_\beta(\bar{u}) B^\beta(\bar{u})\Omega, \\ B^\beta(\bar{u}) \equiv B_{\beta_1}(u_1)\dots B_{\beta_a}(u_a),\end{aligned} \tag{1.1.23}$$

where the sum is taken over repeated (multi)index $\beta = \{\beta_1, \dots, \beta_a\}$, $\beta_i = 1, 2$. Coefficients $F_{\beta_1,\dots,\beta_a}$ here is convenient to present as components of some vector $\mathbb{F}$ of size a, that acts in space $\mathcal{H}^{(2)} = \mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2$ (tensor product of a spaces). The structure of vector $\mathbb{F}$ should be fixed from the condition that vector $\mathbb{B}_a(\bar{u})$ is an eigenvector of the transfer-matrix $\mathcal{T} = A(w) + D_{11}(w) + D_{22}(w)$. It can be shown that this leads to the following condition on $\mathbb{F}$

$$\mathcal{T}^{(2)}_{a,a}(w)\mathbb{F} = \tau^{(2)}(w|\bar{u})\mathbb{F}, \tag{1.1.24}$$

here $\mathcal{T}^{(2)}(w)$ and $\tau^{(2)}(w|\bar{u})$ are the transfer-matrix of the spin chain with the algebra symmetry $\mathfrak{gl}(2)$ ($\mathcal{T}^{(2)}(w) = D_{11}(w) + D_{22}(w)$) and its eigenvalue, $\bar{u}$ here satisfy the system of Bethe equation for $\mathfrak{gl}(2)$ spin chain. In such way vector $\mathbb{F}$ is an eigenvector that acts in space $\mathcal{H}^{(2)}$, that is Hilbert space of inhomogeneous spin chain of length $a = \#\bar{u}$, with vacuum $\Omega^{(2)}$. Eigenvectors of this model are already known

$$\mathbb{F} = \mathcal{B}(v_1)\cdots\mathcal{B}(v_b)\Omega^{(2)}, \qquad b = 0, \dots, a, \tag{1.1.25}$$

where $\mathcal{B}$ are B operators in case of algebra symmetry $\mathfrak{gl}(2)$ and $\Omega^{(2)}$ is a corresponding vacuum. Parameters $\bar{v}$ should satisfy Bethe equations of inhomogeneous spin chain

$$\prod_{j=1}^{a} \frac{v_i - u_j + c/2}{v_i - u_j - c/2} = (-1)^{b-1} \prod_{k\neq i}^{b} \frac{v_i - v_k + c}{v_k - v_i + c}, \qquad i = 1, \dots, b, \tag{1.1.26}$$

or using shorthand notation

$$\frac{h(v_i, \bar{u})}{h(\bar{u}, v_i)} = (-1)^{b-1} \frac{h(v_i, \bar{v})}{h(\bar{v}, v_i)}, \qquad i = 1, \dots, b, \tag{1.1.27}$$

$\#\bar{v} = b \le a$, $h(x, y)$ is defined in (1.1.36). We come to the conclusion, that eigenvectors of the model with algebra symmetry $\mathfrak{gl}(3)$ should depend already on two sets of Bethe parameters $\bar{v}$ and $\bar{u}$, that satisfy coupled system of Bethe equations.

Generalisation of this approach to another model (nonfundamental representation of the algebra) is also possible.

Remark. It is clear, that embedding can be organised in such way that matrix element A will be itself matrix 2×2 while D just a scalar

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \longrightarrow \begin{pmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{pmatrix} = \begin{pmatrix} \mathbb{A} & \mathbb{B} \\ \mathbb{C} & D \end{pmatrix}. \tag{1.1.28}$$

Such embedding leads to the same Bethe vectors, written, however, in different form.

The approach introduced here can be generalised for the graded algebras $\mathfrak{gl}(m|n)$, differences appear only the details of calculation.

In the later works [91–93], the method of off-shell Bethe vectors calculation based on the Drinfeld's current realisation of the affine algebra was developed. This approach leads to the same result and gives the hierarchy of embedding of Bethe vectors in the same way but is more unified for all models and more simple technically. This approach can be gengeralised for the graded algebra case. The details of calculation are to cumbersome to give here, but in worth to note that Bethe vectors for algebra symmetry $\mathfrak{gl}(2|1)$ case, given in the next chapter, were obtained using exactly this approach. Work [94], where an alternative method of Bethe vectors calculation was derived, should be also mentioned.

Remark. The definition of the off-shell Bethe vectors are not unique. The only requirement is that under condition on the spectral parameters (the Bethe equations) any off-shell Bethe vector should become on-shell Bethe vector that is an eigenvector of the transfer-matrix. It is clear, that such definition allows some degrees of freedom. But all the methods, mentioned above, result in the same off-shell Bethe vector.

As its already clear from (1.1.23)-(1.1.25), polynomials that define Bethe vectors for algebra symmetry $\mathfrak{gl}(N)$ are quite more complicated than in $\mathfrak{gl}(2)$ algebra symmetry case. Thus, it consists of all possible matrix elements T_{ij} with $i < j$. Such matrix elements also can be considered as creation operators of few types of (quasi)particles, thus in the case of algebra $\mathfrak{gl}(3)$ (or $\mathfrak{gl}(2|1)$) elements T_{12}, T_{23} create particles of the first and the second types respectively, while operator T_{13} creates (quasi)particles of both types (so in NABA approach for algebra symmetry $\mathfrak{gl}(n)$ there exist $n-1$ sort of (quasi)particles). Operators T_{ij} with $i > j$ are the operators of annihilation of (quasi)particles, and the diagonal matrix elements define the transfer-matrix, that is the generation function of the integrals of motion (1.1.8)-(1.1.9). In the vacuum state (quasi)particles are absent.

Moreover, Bethe vectors in the $\mathfrak{gl}(N)$ algebra symmetry case depends already on a set of $N-1$ variables $\{\bar{u}, \bar{v}, \bar{w}, \dots\}$. Thus, in case of algebra symmetry $\mathfrak{gl}(3)$ (or $\mathfrak{gl}(2|1)$) vectors $\mathbb{B}_{a,b}(\bar{u};\bar{v})$ depends on two types of variables with cardinalities a and b correspondingly: $\bar{u} = \{u_1, \dots, u_a\}$ and $\bar{v} = \{v_1, \dots, v_b\}$, with $a, b = 0, 1, \dots$. Explicit form of vectors and Bethe equations for algebra symmetry $\mathfrak{gl}(2|1)$ are given in next chapter.

Parameters $\bar{u}$, $\bar{v}$ are rapidities of (quasi)particles of corresponding types (we have exactly $N-1$ types of (quasi)particles and $N-1$ types of spectral parameters). Bethe equations are nothing but spectral equations. Naturally, Bethe equations also depends on the particular algebra symmetry and the particular physical model (e.g. on set of λ_j). Bethe equations for the algebra symmetry $\mathfrak{gl}(2|1)$ are given in the next chapter.

Terminology of definition (1.1.1) can be used in the highest rank algebra too up to replacement of one set of spectral parameters by $N-1$ sets, that should satisfy a coupled system of Bethe equations.

Since on-shell (dual)Bethe vectors are eigenvectors for (1.1.7) (or (1.1.16)), by definition

$$\mathcal{T}(w)\mathbb{B}_{a,b}(\bar{u};\bar{v}) = \tau(w|\bar{u},\bar{v})\,\mathbb{B}_{a,b}(\bar{u};\bar{v}), \qquad \mathbb{C}_{a,b}(\bar{u};\bar{v})\mathcal{T}(w) = \tau(w|\bar{u},\bar{v})\,\mathbb{C}_{a,b}(\bar{u};\bar{v}), \tag{1.1.29}$$

where $\tau(w|\bar{u},\bar{v})$ are eigenvalue of (1.1.7) (or (1.1.16)). An explicit form of $\tau(w|\bar{u},\bar{v})$ for algebra symmetry $\mathfrak{gl}(2|1)$ together with the explicit form of Bethe equations are established in the next chapter.

1.1.4 L-operators. Gases and t-J model

One of the simplest models is the Heisenberg spin chain (0.1.1) with $\Delta = 1$. The Hamiltonian of this model can be presented as

$$H = J \sum_{j=1}^{L} P_{j,j+1}, \tag{1.1.30}$$

where P_{ij} is the (graded) operator of particle permutation, the Hilbert space of one site is $\mathbb{C}^2$. In the presence of multiple components model is generalised by replacement of space $\mathbb{C}^2 \to \mathbb{C}^{m|n}$ and replacement of permutation operator by operator acting in this space. In case of Fermi particles permutation leads to appearing of sign -1.

Consider the representation of the one-dimensional model of strongly correlated electrons called t-J model it terms of L-operators. THe Hamiltonian of such system is

$$H = \sum_{j=1}^{L} \left\{ -t\mathcal{P} \sum_{\sigma=\pm1} \left(c^\dagger_{j,\sigma} c_{j+1,\sigma} + h.c. \right) \mathcal{P} + J \left(\mathbb{S}_j \mathbb{S}_{j+1} - \frac{1}{4} n_j n_{j+1} \right) \right\}. \tag{1.1.31}$$

Here t is the hopping constant (probability of tunneling of electron on neighboring site), J is spin-spin interaction of neighboring sites, operator $n_{j,\sigma} \equiv c^\dagger_{j,\sigma} c_{j,\sigma}$ defines the electrons number in the j-th site, $n_j \equiv n_{j,1} + n_{j,-1}$,

$$\mathbb{S}^a_j = \frac{1}{2} \sigma^a_{\alpha\beta} c^\dagger_{j,\alpha} c_{j,\beta}, \qquad \alpha, \beta = \uparrow, \downarrow, \qquad a = x, y, z, \tag{1.1.32}$$

is the spin operator (where σ^a are Pauli matrices), $\mathcal{P} = \prod_{j=1}^{L} (1 - n_{j,\uparrow} n_{j,\downarrow})$ is the projector on the states that do not contain double occupancy, i.e. allowing only states Ω, $|\uparrow\rangle_j = c^\dagger_{j,1}\Omega$, $|\downarrow\rangle_j = c^\dagger_{j,-1}\Omega$ i.e. states with no electrons in sites j or single electron with spin up or down, $\{c_{j,\sigma}, c^\dagger_{k,\tau}\} = \delta_{jk}\delta\sigma\tau$. As was shown in [75, 76, 117] in case $t = -J/2$ this system can be described by L-operator

$$L_j(u) = u\mathbb{I} + \begin{pmatrix} (1-n_{j,\downarrow})(1-n_{j,\uparrow}) & (1-n_{j,\downarrow})c_{j,\uparrow} & c_{j,\downarrow}(1-n_{j,\uparrow}) \\ (1-n_{j,\downarrow})c^\dagger_{j,\uparrow} & (1-n_{j,\downarrow})n_{j,\uparrow} & -c_{j,\downarrow}c^\dagger_{j,\uparrow} \\ c^\dagger_{j,\downarrow}(1-n_{j,\uparrow}) & c^\dagger_{j,\downarrow}c_{j,\uparrow} & n_{j,\downarrow}(1-n_{j,\uparrow}) \end{pmatrix}. \tag{1.1.33}$$

This L-operator satisfies the RTT-relation, with R-matrix that posses $\mathfrak{gl}(1|2)$ symmetry. Space gradation is given by $[1] = 0$, $[2] = [3] = 1$.

The permutation operator P_{jk} can be written terms of fermions c_σ, $c^\dagger_\sigma$ in the following way

$$P^\sigma_{jk} = 1 - (c^\dagger_{j,\sigma} - c^\dagger_{k,\sigma})(c_{j,\sigma} - c_{k,\sigma}), \qquad j,k = 1,\ldots,L, \qquad \sigma = \uparrow,\downarrow, \qquad P_{jk} = P^\uparrow_{jk} P^\downarrow_{jk} = P^\downarrow_{jk} P^\uparrow_{jk}. \tag{1.1.34}$$

Define now permutation operator that acts on a restricted Hilbert space where double occupancy is prohibited.

$$\tilde{P}_{jk} = P_{jk}\mathcal{P}. \tag{1.1.35}$$

Using this representation it is possible to rewrite (1.1.31) in form (1.1.30) up to constant terms, and L-operator become the graded (1.1.3) up to common multiplier. In this form the RLL-relation coincides with the Yang-Baxter relation. The Hilbert space is $\mathcal{H} = \mathcal{H}_1 \otimes \cdots \otimes H_n$, where $\mathcal{H}_j$ become space $\mathbb{C}^{1|2}_j$ with components Ω, $|\uparrow\rangle_j = c^\dagger_{j,\uparrow}\Omega$, $|\downarrow\rangle_j = c^\dagger_{j,\downarrow}\Omega$.

1.1.5 Notation

In order to make our formulas more compact we use several auxiliary functions and conventions on the notation.

In addition to the functions $g(x,y)$ we also introduce the functions

$$f(x,y) = 1 + g(x,y) = \frac{x-y+c}{x-y}, \qquad h(x,y) = \frac{f(x,y)}{g(x,y)} = \frac{x-y+c}{c} \tag{1.1.36}$$

$$\text{and} \quad t(x,y) = \frac{g(x,y)}{h(x,y)} = \frac{c^2}{(x-y+c)(x-y)}. \tag{1.1.37}$$

These functions have obvious properties

$$g(x,y) = -g(y,x), \qquad h(x,y) = \frac{1}{g(x,y-c)}, \qquad f(x-c,y) = \frac{1}{f(y,x)}. \tag{1.1.38}$$

We always denote sets of variables by bar: $\bar{w}$, $\bar{u}$, $\bar{v}$ etc. Individual elements of the sets are denoted by subscripts and without bar: w_j, u_k, v_ℓ etc. As a rule, the number of elements in the sets is not shown explicitly; however we give these cardinalities in special comments after the formulae. Subsets of variables are denoted by Roman subscripts: $\bar{u}_{\rm I}$, $\bar{v}_{\rm II}$, $\bar{w}_{\rm ii}$ etc. For example, the notation $\bar{u} \Rightarrow \{\bar{u}_{\rm I},\ \bar{u}_{\rm II}\}$ means that the set $\bar{u}$ is divided into two disjoint subsets $\bar{u}_{\rm I}$ and $\bar{u}_{\rm II}$. We assume that the elements in every subset are ordered in such a way that the sequence of their subscripts is strictly increasing. For the union of two sets into another one we use the notation $\{\bar{u},\bar{w}\} = \bar{\xi}$. Finally we use a special notation $\bar{u}_j$, $\bar{v}_k$ and so on for the sets $\bar{u} \setminus u_j$, $\bar{v} \setminus v_k$ etc.

In order to avoid excessively cumbersome formulae we use shorthand notation for products of functions depending on one or two variables. Namely, whenever such a function depends on a set of variables, this means that we deal with the product of this function with respect to the corresponding set, as follows:

$$\lambda_i(\bar{u}) = \prod_{u_j \in \bar{u}} \lambda_i(u_j); \quad g(x_k, \bar{w}_\ell) = \prod_{\substack{w_j \in \bar{w} \\ w_j \neq w_\ell}} g(x_k, w_j); \quad f(\bar{u}_{\rm II}, \bar{u}_{\rm I}) = \prod_{u_j \in \bar{u}_{\rm II}} \prod_{u_k \in \bar{u}_{\rm I}} f(u_j, u_k). \tag{1.1.39}$$

This notation is also used for the product of commuting operators,

$$T_{ij}(\bar{u}) = \prod_{u_k \in \bar{u}} T_{ij}(u_k). \tag{1.1.40}$$

One can easily see from the commutation relations (1.1.14) that in case $[i]+[j]=0$ $[T_{ij}(u), T_{ij}(v)] = 0$, and hence, the operator product (1.1.40) is well defined. However, if $[i]+[j]=1$, then $[T_{ij}(u), T_{ij}(v)] \neq 0$, therefore we introduce symmetric operator products

$$\mathbb{T}_{j3}(\bar{v}) = \frac{T_{j3}(v_1)\dots T_{j3}(v_n)}{\prod_{n\ge\ell>m\ge1} h(v_\ell, v_m)}, \qquad \mathbb{T}_{3j}(\bar{v}) = \frac{T_{3j}(v_1)\dots T_{3j}(v_n)}{\prod_{n\ge\ell>m\ge1} h(v_m, v_\ell)} \qquad j=1,2. \tag{1.1.41}$$

Due to the commutation relation

$$\begin{aligned}
&[T_{ik}(v_1), T_{ik}(v_2)] = 0, \qquad \text{if} \quad [T_{ik}] = 0, \\
&h(v_1,v_2) T_{j3}(v_1) T_{j3}(v_2) = h(v_2,v_1) T_{j3}(v_2) T_{j3}(v_1), \qquad j=1,2, \\
&h(v_2,v_1) T_{3j}(v_1) T_{3j}(v_2) = h(v_1,v_2) T_{3j}(v_2) T_{3j}(v_1), \qquad j=1,2,
\end{aligned} \tag{1.1.42}$$

in the symmetrised product all operators can be written in arbitrary order.

The shorthand notation can be used also for $\lambda_i(\bar z)$, $r_i(\bar z)$ For instance,

$$\lambda_2(\bar z) = \prod_{z_j\in\bar z}\lambda_2(z_j); \qquad r_1(\bar\eta_{\rm II}) = \prod_{\eta_j\in\bar\eta_{\rm II}} r_1(\eta_j). \tag{1.1.43}$$

In various formulae the Izergin determinant $K_n(\bar x|\bar y)$ appears[1]. [44], [46]. It is defined for two sets $\bar x$ and $\bar y$ with common cardinality $\#\bar x = \#\bar y = n$,

$$K_n(\bar x|\bar y) = h(\bar x,\bar y)\prod_{\ell<m}^{n} g(x_\ell,x_m)g(y_m,y_\ell)\ \det_n\left[t(x_i,y_j)\right]. \tag{1.1.44}$$

According to shorthand convention notation $h(\bar x,\bar y)$ in definition of (1.1.44) implies double product h over sets $\bar x$ and $\bar y$. Subscript n in the determinant denotes the size of matrix under determinant. It is easy to see from definition (1.1.44) that $K_1(x|y) = g(x,y)$ and

$$K_n(\bar x|\bar y + c) = (-1)^n\frac{K_n(\bar y|\bar x)}{f(\bar y,\bar x)}. \tag{1.1.45}$$

Other properties of K_n are given in Appendix 8.1.

For the set $\bar w = \{w_1,\dots,w_n\}$ we define

$$\Delta'(\bar w) = \prod_{j<k}^{n} g(w_j,w_k), \qquad \Delta(\bar w) = \prod_{j>k}^{n} g(w_j,w_k). \tag{1.1.46}$$

1.1.6 Antimorphism ψ

The algebra (1.1.1) possesses an antimorphism [84]

$$\psi\big(T_{ij}(u)\big) = (-1)^{[i][j]+[i]}T_{ji}(u), \qquad \psi\big(AB\big) = (-1)^{[A][B]}\psi\big(B\big)\psi\big(A\big), \tag{1.1.47}$$

where A and B are arbitrary operators of fixed gradings. It follows from (1.1.47) that

$$\psi\big(A_1\dots A_n\big) = (-1)^{\vartheta_n}\psi\big(A_n\big)\dots\psi\big(A_1\big), \qquad \vartheta_n = \sum_{1\le i<j\le n}[A_i]\cdot[A_j]. \tag{1.1.48}$$

For the graded algebra $\mathfrak{gl}(2|1)$ it is easy to check that if $[i]=[j]=0$, then

$$\psi\big(T_{ij}(\bar u)\big) = T_{ji}(\bar u), \quad \psi\big(\mathbb{T}_{i3}(\bar u)\big) = (-1)^{n(n-1)/2}\mathbb{T}_{3i}(\bar u), \quad \psi\big(\mathbb{T}_{3i}(\bar u)\big) = (-1)^{n(n+1)/2}\mathbb{T}_{i3}(\bar u), \tag{1.1.49}$$

where $n = \#\bar u$.

[1] Note that by definition this function depends on two sets of variables. Therefore, the convention on shorthand notations for the products is not apply in this case.

Chapter 2

Multiple action rules in algebra symmetry $\mathfrak{gl}(2|1)$

Since in ABA monodromy matrix is basic object whose entries are building blocks for both physical operators and Bethe vectors (that are in particular on-shell case eigenvectors of Hamiltonian), it is natural to consider commutation relations between T_{ij} carefully. These commutation relations follow directly from RTT-relation, but work with them in case of large operators number is not easy. For instance, in order to commute operators $T_{11}(u)$ and $T_{22}(v)$, it is enough to put $i = j = 1$, $k = l = 2$ in (1.1.14). Obviously, this commutation rules leads to appearance only terms $T_{12}(u)T_{21}(v)$ and $T_{12}(v)T_{21}(u)$ with some coefficients in r.h.s. Commutator of operators' products $[T_{11}(\bar{u}), T_{22}(\bar{v})]$ in case $\#\bar{u}, \#\bar{v} > 1$ is, however, much more involved. Successive application of formula (1.1.14) spawn more and more terms of different form. In general the commutator rules for product of operators can be established [109], however, on practice only the action rules of monodromy matrix entries onto Bethe vectors rather than commutation relation are required.

In this chapter the action rule of T_{ij} onto generic Bethe vectors, e.g. polynomial on T_{ij}, $i < j$ acting on pseudovacuum vector are found. These results are necessary for further calculation on scalar products and form factors. This chapter is based on the papers [87] and [90] published by the thesis author in collaboration.

2.1 Bethe vector in $\mathfrak{gl}(2|1)$ algebra symmetry case

Explicit representations for $\mathfrak{gl}(2|1)$ Bethe vectors were obtained in[1] [84].

Definition 2.1.1. *For $\#\bar{u} = a$ and $\#\bar{v} = b$ define a Bethe vector*

$$\mathbb{B}_{a,b}(\bar{u}; \bar{v}) = \sum g(\bar{v}_{\rm I}, \bar{u}_{\rm I}) \frac{f(\bar{u}_{\rm I}, \bar{u}_{\rm II}) g(\bar{v}_{\rm II}, \bar{v}_{\rm I}) h(\bar{u}_{\rm I}, \bar{u}_{\rm I})}{\lambda_2(\bar{u}) \lambda_2(\bar{v}_{\rm II}) f(\bar{v}, \bar{u})} \, \mathbb{T}_{13}(\bar{u}_{\rm I}) \, T_{12}(\bar{u}_{\rm II}) \, \mathbb{T}_{23}(\bar{v}_{\rm II}) \Omega. \qquad (2.1.1)$$

Here the sum is taken over partitions $\bar{v} \Rightarrow \{\bar{v}_{\rm I}, \bar{v}_{\rm II}\}$ and $\bar{u} \Rightarrow \{\bar{u}_{\rm I}, \bar{u}_{\rm II}\}$ with the restriction $\#\bar{u}_{\rm I} = \#\bar{v}_{\rm I} = n$, where $n = 0, 1, \ldots, \min(a, b)$. Recall also that we use the shorthand notation for the products of all the functions and the operators in (2.1.1).

[1] The formulae for the Bethe vectors obtained in [84] differ from (2.1.1), (2.1.2) by a normalisation factor $\lambda_2(\bar{v})\lambda_2(\bar{u}) f(\bar{v}, \bar{u})$.

An alternative formula for the Bethe vector is

$$\mathbb{B}_{a,b}(\bar{u};\bar{v}) = \sum K_n(\bar{v}_{\rm I}|\bar{u}_{\rm I}) \frac{f(\bar{u}_{\rm I},\bar{u}_{\rm II})g(\bar{v}_{\rm II},\bar{v}_{\rm I})}{\lambda_2(\bar{u}_{\rm II})\lambda_2(\bar{v})f(\bar{v},\bar{u})}\, \mathbb{T}_{13}(\bar{v}_{\rm I})\, \mathbb{T}_{23}(\bar{v}_{\rm II})\, T_{12}(\bar{u}_{\rm II})\Omega, \tag{2.1.2}$$

where K_n is the Izergin determinant (1.1.44). *Other notation is the same as in* (2.1.1).

As mentioned, the distinctive feature of the Bethe vectors is that under certain conditions on $\bar{u}$ and $\bar{v}$ (Bethe equations), they are eigenvectors of the transfer matrix. We will show in section 2.1.5 that the vectors (2.1.1), (2.1.2) do possess this property.

These vectors are identical as it was shown explicitly in [109] by reordering of operators using commutation relations. Initially to distinct form appear due to two different embedding of matrix block 2×2 in matrix 3×3 during the nesting procedure (see (1.1.22) and (1.1.28)).

Further, in addition to the right Bethe vector (ordinary) we will need left (dual) Bethe vector. Here to representation for them are given [84, 109]:

$$\mathbb{C}_{a,b}(\bar{u};\bar{v}) = (-1)^{\frac{b^2-b}{2}} \sum K_n(\bar{v}_{\rm I}|\bar{u}_{\rm I}) \frac{f(\bar{u}_{\rm I},\bar{u}_{\rm II})g(\bar{v}_{\rm II},\bar{v}_{\rm I})}{\lambda_2(\bar{v})\lambda_2(\bar{u}_{\rm II})f(\bar{v},\bar{u})} \Omega^\dagger T_{21}(\bar{u}_{\rm II})\, \mathbb{T}_{32}(\bar{v}_{\rm II})\mathbb{T}_{31}(\bar{v}_{\rm I}), \tag{2.1.3}$$

and

$$\mathbb{C}_{a,b}(\bar{u};\bar{v}) = (-1)^{\frac{b^2-b}{2}} \sum g(\bar{v}_{\rm I},\bar{u}_{\rm I}) \frac{f(\bar{u}_{\rm II},\bar{u}_{\rm I})f(\bar{v}_{\rm I},\bar{u}_{\rm II})g(\bar{v}_{\rm II},\bar{v}_{\rm I})h(\bar{u}_{\rm I},\bar{u}_{\rm I})}{\lambda_2(\bar{u})\lambda_2(\bar{v}_{\rm II})f(\bar{v},\bar{u})}\, \Omega^\dagger \mathbb{T}_{32}(\bar{v}_{\rm II})\, \mathbb{T}_{31}(\bar{u}_{\rm I})\, T_{21}(\bar{u}_{\rm II}). \tag{2.1.4}$$

Here the sum is taken over the same partitions of the sets $\bar{u}$ and $\bar{v}$ as in (2.1.1).

It can be shown [91], that the action of any monodromy matrix entry T_{ij} on a Bethe vector reduces to a *finite* linear combination of Bethe vectors. However, it is not so obvious if we deal with explicit representations (2.1.1), (2.1.2). Furthermore, in spite of the action of $T_{ij}(z)$ onto $\mathbb{B}_{a,b}(\bar{u};\bar{v})$ formally can be derived via (1.1.14) and (1.1.17), actually it is pretty nontrivial problem.

Fortunately, similarly to the $\mathfrak{gl}(3)$ case [95] the $\mathfrak{gl}(2|1)$ Bethe vectors possess recursions over the number of the Bethe parameters [84]. The first recursion has the form

$$T_{12}(z)\, \mathbb{B}_{a,b}(\bar{u};\bar{v}) = \lambda_2(z)f(\bar{v},z)\mathbb{B}_{a+1,b}(\{\bar{u};z\};\bar{v}) + \sum_{j=1}^{b} g(z,v_j)g(\bar{v}_j,v_j)\, T_{13}(z)\mathbb{B}_{a,b-1}(\bar{u};\bar{v}_j). \tag{2.1.5}$$

The second recursion reads

$$T_{23}(z)\mathbb{B}_{a,b}(\bar{u};\bar{v}) = \lambda_2(z)h(\bar{v},z)f(z,\bar{u})\mathbb{B}_{a,b+1}(\bar{u};\{\bar{v},z\}) + \sum_{j=1}^{a} g(u_j,z)f(u_j,\bar{u}_j)T_{13}(z)\mathbb{B}_{a-1,b}(\bar{u}_j;\bar{v}). \tag{2.1.6}$$

We recall that in these formulas $\bar{v}_j$ and $\bar{u}_j$ respectively mean $\bar{v}\setminus v_j$ and $\bar{u}\setminus u_j$. The shorthand notation for the products of the functions g and f is also used.

Equations (2.1.5), (2.1.6) allow us to built recursively Bethe vectors starting with the simplest cases

$$\mathbb{B}_{a,0}(\bar{u};\emptyset) = \frac{T_{12}(\bar{u})}{\lambda_2(\bar{u})}\Omega, \qquad \mathbb{B}_{0,b}(\emptyset;\bar{v}) = \frac{\mathbb{T}_{23}(\bar{v})}{\lambda_2(\bar{v})}\Omega. \tag{2.1.7}$$

One can also easily derive the actions of T_{ij} onto either $\mathbb{B}_{a,0}(\bar{u};\emptyset)$ or $\mathbb{B}_{0,b}(\emptyset;\bar{v})$, and then, using induction over a or b obtain the action rule in the general case. This is main task of this chapter.

2.1.1 Bethe vectors and antimorphism ψ

Define the action of antimorphism (1.1.47) on the Bethe vectors [84]. We can always choose the grading of Ω and $\Omega^\dagger$ such that $[\Omega]=[\Omega^\dagger]=0$. Then we set $[A\Omega]=[\Omega^\dagger A]=[A]$ and define

$$\begin{aligned} &\psi(\Omega)=\Omega^\dagger, &&\psi(A\Omega)=\Omega^\dagger\psi(A),\\ &\psi(\Omega^\dagger)=\Omega, &&\psi(\Omega^\dagger A)=\psi(A)\Omega, \end{aligned} \tag{2.1.8}$$

where A is an arbitrary product of the monodromy matrix entries. It is easy to see that

$$[\mathbb{B}_{a,b}(\bar u;\bar v)]=[\mathbb{C}_{a,b}(\bar u;\bar v)]=b. \tag{2.1.9}$$

It is also easy to check that

$$\psi\big(\mathbb{B}_{a,b}(\bar u;\bar v)\big)=\mathbb{C}_{a,b}(\bar u;\bar v), \qquad \psi\big(\mathbb{C}_{a,b}(\bar u;\bar v)\big)=(-1)^b\,\mathbb{B}_{a,b}(\bar u;\bar v). \tag{2.1.10}$$

Indeed, fixing partitions in (2.1.1), (2.1.3) such that $\#\bar u_{\rm I}=\#\bar v_{\rm I}=n$ and using (1.1.49) it is easy to obtain

$$\begin{aligned} \psi\Big(\mathbb{T}_{13}(\bar u_{\rm I})\,T_{12}(\bar u_{\rm II})\,\mathbb{T}_{23}(\bar v_{\rm II})\Big)&=(-1)^{n(n-1)/2+(b-n)(b-n-1)/2+n(b-n)}\mathbb{T}_{32}(\bar v_{\rm II})\,T_{21}(\bar u_{\rm II})\,\mathbb{T}_{31}(\bar u_{\rm I})\\ &=(-1)^{b(b-1)/2}\mathbb{T}_{32}(\bar v_{\rm II})\,T_{21}(\bar u_{\rm II})\,\mathbb{T}_{31}(\bar u_{\rm I}), \end{aligned} \tag{2.1.11}$$

and, similarly,

$$\begin{aligned} \psi\Big(\mathbb{T}_{32}(\bar v_{\rm II})\,T_{21}(\bar u_{\rm II})\,\mathbb{T}_{31}(\bar u_{\rm I})\Big)&=(-1)^{n(n+1)/2+(b-n)(b-n+1)/2+n(b-n)}\mathbb{T}_{13}(\bar u_{\rm I})\,T_{12}(\bar u_{\rm II})\,\mathbb{T}_{23}(\bar v_{\rm II})\\ &=(-1)^{b(b+1)/2}\mathbb{T}_{13}(\bar u_{\rm I})\,T_{12}(\bar u_{\rm II})\,\mathbb{T}_{23}(\bar v_{\rm II}). \end{aligned} \tag{2.1.12}$$

These equations immediately imply (2.1.10).

2.1.2 Multiple action rules

The main result of this paper consists of explicit formulae of the multiple actions of the monodromy matrix entries onto Bethe vectors. We show that this action always reduces to a finite linear combination of Bethe vectors.

Everywhere in this section we assume that $\bar u$, $\bar v$, and $\bar z$ are three sets of generic complex numbers with cardinalities $\#\bar u=a$, $\#\bar v=b$, and $\#\bar z=n$, $a,b,n=0,1,\dots$. We also set $\bar\eta=\{\bar u,\bar z\}$ and $\bar\xi=\{\bar v,\bar z\}$.

Actions $T_{ij}(\bar z)$ with $i<j$

- Multiple action of $T_{13}(z)$:

$$\mathbb{T}_{13}(\bar z)\mathbb{B}_{a,b}(\bar u;\bar v)=\lambda_2(\bar z)h(\bar v,\bar z)\mathbb{B}_{a+n,b+n}(\bar\eta;\bar\xi). \tag{2.1.13}$$

- Multiple action of $T_{12}(z)$:

$$T_{12}(\bar z)\mathbb{B}_{a,b}(\bar u;\bar v)=\lambda_2(\bar z)h(\bar\xi,\bar z)\sum\frac{g(\bar\xi_{\rm II},\bar\xi_{\rm I})}{h(\bar\xi_{\rm I},\bar z)}\mathbb{B}_{a+n,b}(\bar\eta;\bar\xi_{\rm II}). \tag{2.1.14}$$

  Here the sum is taken over partitions $\bar\xi\Rightarrow\{\bar\xi_{\rm I},\bar\xi_{\rm II}\}$ with $\#\bar\xi_{\rm I}=n$.

- Multiple action of $T_{23}(z)$:

$$\mathbb{T}_{23}(\bar z)\mathbb{B}_{a,b}(\bar u;\bar v)=(-1)^n\lambda_2(\bar z)h(\bar v,\bar z)\sum K_n(\bar z|\bar\eta_{\rm I}+c)f(\bar\eta_{\rm I},\bar\eta_{\rm II})\mathbb{B}_{a,b+n}(\bar\eta_{\rm II};\bar\xi). \tag{2.1.15}$$

  Here the sum is taken over partitions $\bar\eta\Rightarrow\{\bar\eta_{\rm I},\bar\eta_{\rm II}\}$ with $\#\bar\eta_{\rm I}=n$.

2.1.3 Actions of $T_{ii}(\bar{z})$

In formulae (2.1.16)–(2.1.18) the sums are taken over partitions $\bar{\xi} \Rightarrow \{\bar{\xi}_{\rm I}, \bar{\xi}_{\rm II}\}$ and $\bar{\eta} \Rightarrow \{\bar{\eta}_{\rm I}, \bar{\eta}_{\rm II}\}$ with $\#\bar{\xi}_{\rm I} = \#\bar{\eta}_{\rm I} = n$.

- Multiple action of $T_{11}(z)$:

$$T_{11}(\bar{z})\mathbb{B}_{a,b}(\bar{u};\bar{v}) = (-1)^n \lambda_2(\bar{z}) h(\bar{\xi},\bar{z}) \sum r_1(\bar{\eta}_{\rm I}) \frac{f(\bar{\eta}_{\rm II},\bar{\eta}_{\rm I}) g(\bar{\xi}_{\rm II},\bar{\xi}_{\rm I})}{h(\bar{\xi}_{\rm I},\bar{z}) f(\bar{\xi}_{\rm II},\bar{\eta}_{\rm I})} K_n(\bar{\eta}_{\rm I}|\bar{\xi}_{\rm I}+c) \mathbb{B}_{a,b}(\bar{\eta}_{\rm II};\bar{\xi}_{\rm II}). \tag{2.1.16}$$

- Multiple action of $T_{22}(z)$:

$$T_{22}(\bar{z})\mathbb{B}_{a,b}(\bar{u};\bar{v}) = (-1)^n \lambda_2(\bar{z}) h(\bar{\xi},\bar{z}) \sum \frac{f(\bar{\eta}_{\rm I},\bar{\eta}_{\rm II}) g(\bar{\xi}_{\rm II},\bar{\xi}_{\rm I})}{h(\bar{\xi}_{\rm I},\bar{z})} K_n(\bar{z}|\bar{\eta}_{\rm I}+c) \mathbb{B}_{a,b}(\bar{\eta}_{\rm II};\bar{\xi}_{\rm II}). \tag{2.1.17}$$

- Multiple action of $T_{33}(z)$:

$$T_{33}(\bar{z})\mathbb{B}_{a,b}(\bar{u};\bar{v}) = \lambda_2(\bar{z}) h(\bar{\xi},\bar{z}) \sum r_3(\bar{\xi}_{\rm I}) \frac{f(\bar{\eta}_{\rm I},\bar{\eta}_{\rm II}) g(\bar{\xi}_{\rm II},\bar{\xi}_{\rm I}) h(\bar{\eta}_{\rm I},\bar{\eta}_{\rm I})}{h(\bar{\xi}_{\rm I},\bar{\eta}_{\rm I}) h(\bar{\eta}_{\rm I},\bar{z}) f(\bar{\xi}_{\rm I},\bar{\eta}_{\rm II})} \mathbb{B}_{a,b}(\bar{\eta}_{\rm II};\bar{\xi}_{\rm II}). \tag{2.1.18}$$

2.1.4 Actions of $T_{ij}(\bar{z})$ with $i > j$

- Multiple action of $T_{21}(z)$:

$$T_{21}(\bar{z})\mathbb{B}_{a,b}(\bar{u};\bar{v}) = \lambda_2(\bar{z}) h(\bar{\xi},\bar{z}) \sum r_1(\bar{\eta}_{\rm I}) \frac{f(\bar{\eta}_{\rm II},\bar{\eta}_{\rm I}) f(\bar{\eta}_{\rm II},\bar{\eta}_{\rm III}) f(\bar{\eta}_{\rm III},\bar{\eta}_{\rm I}) g(\bar{\xi}_{\rm II},\bar{\xi}_{\rm I})}{h(\bar{\xi}_{\rm I},\bar{z}) f(\bar{\xi}_{\rm II},\bar{\eta}_{\rm I})} \times K_n(\bar{z}|\bar{\eta}_{\rm II}+c) K_n(\bar{\eta}_{\rm I}|\bar{\xi}_{\rm I}+c) \mathbb{B}_{a-n,b}(\bar{\eta}_{\rm III};\bar{\xi}_{\rm II}). \tag{2.1.19}$$

Here the sum is taken over partitions $\bar{\xi} \Rightarrow \{\bar{\xi}_{\rm I}, \bar{\xi}_{\rm II}\}$ and $\bar{\eta} \Rightarrow \{\bar{\eta}_{\rm I}, \bar{\eta}_{\rm II}, \bar{\eta}_{\rm III}\}$ with $\#\bar{\xi}_{\rm I} = \#\bar{\eta}_{\rm I} = \#\bar{\eta}_{\rm II} = n$.

- Multiple action of $T_{32}(z)$:

$$\mathbb{T}_{32}(\bar{z})\mathbb{B}_{a,b}(\bar{u};\bar{v}) = (-1)^{\frac{n(n-1)}{2}} \lambda_2(\bar{z}) h(\bar{\xi},\bar{z}) \sum r_3(\bar{\xi}_{\rm I}) \frac{f(\bar{\eta}_{\rm I},\bar{\eta}_{\rm II}) g(\bar{\xi}_{\rm II},\bar{\xi}_{\rm I}) g(\bar{\xi}_{\rm III},\bar{\xi}_{\rm II}) g(\bar{\xi}_{\rm III},\bar{\xi}_{\rm I})}{h(\bar{\eta}_{\rm I},\bar{z}) h(\bar{\xi}_{\rm I},\bar{\eta}_{\rm I}) h(\bar{\xi}_{\rm II},\bar{z}) f(\bar{\xi}_{\rm I},\bar{\eta}_{\rm II})} \times h(\bar{\eta}_{\rm I},\bar{\eta}_{\rm I})\, \mathbb{B}_{a,b-n}(\bar{\eta}_{\rm II};\bar{\xi}_{\rm III}). \tag{2.1.20}$$

Here the sum is taken over partitions $\bar{\xi} \Rightarrow \{\bar{\xi}_{\rm I}, \bar{\xi}_{\rm II}, \bar{\xi}_{\rm III}\}$ and $\bar{\eta} \Rightarrow \{\bar{\eta}_{\rm I}, \bar{\eta}_{\rm II}\}$ with $\#\bar{\xi}_{\rm I} = \#\bar{\xi}_{\rm II} = \#\bar{\eta}_{\rm I} = n$.

- Multiple action of $T_{31}(z)$:

$$\mathbb{T}_{31}(\bar{z})\mathbb{B}_{a,b}(\bar{u};\bar{v}) = (-1)^{\frac{n(n+1)}{2}} \lambda_2(\bar{z}) h(\bar{\xi},\bar{z}) \sum r_3(\bar{\xi}_{\rm I}) r_1(\bar{\eta}_{\rm II}) \frac{g(\bar{\xi}_{\rm II},\bar{\xi}_{\rm I}) g(\bar{\xi}_{\rm III},\bar{\xi}_{\rm II}) g(\bar{\xi}_{\rm III},\bar{\xi}_{\rm I})}{h(\bar{\eta}_{\rm I},\bar{z}) h(\bar{\xi}_{\rm I},\bar{\eta}_{\rm I}) h(\bar{\xi}_{\rm II},\bar{z})} \times \frac{f(\bar{\eta}_{\rm I},\bar{\eta}_{\rm II}) f(\bar{\eta}_{\rm I},\bar{\eta}_{\rm III}) f(\bar{\eta}_{\rm III},\bar{\eta}_{\rm II}) h(\bar{\eta}_{\rm I},\bar{\eta}_{\rm I})}{f(\bar{\xi}_{\rm I},\bar{\eta}_{\rm II}) f(\bar{\xi}_{\rm I},\bar{\eta}_{\rm III}) f(\bar{\xi}_{\rm III},\bar{\eta}_{\rm II})} K_n(\bar{\eta}_{\rm II}|\bar{\xi}_{\rm II}+c)\, \mathbb{B}_{a-n,b-n}(\bar{\eta}_{\rm III};\bar{\xi}_{\rm III}). \tag{2.1.21}$$

Here the sum is taken over partitions $\bar{\xi} \Rightarrow \{\bar{\xi}_{\rm I}, \bar{\xi}_{\rm II}, \bar{\xi}_{\rm III}\}$ and $\bar{\eta} \Rightarrow \{\bar{\eta}_{\rm I}, \bar{\eta}_{\rm II}, \bar{\eta}_{\rm III}\}$ with $\#\bar{\xi}_{\rm I} = \#\bar{\xi}_{\rm II} = \#\bar{\eta}_{\rm I} = \#\bar{\eta}_{\rm II} = n$.

The proofs of multiple action rules are given in sections 2.2–2.4.

2.1.5 on-shell Bethe vectors

The action formulas (2.1.13)–(2.1.21) are valid for generic complex numbers $\bar{z}$, $\bar{u}$, and $\bar{v}$. In this section we consider them for on-shell Bethe vector $\mathbb{B}_{a,b}(\bar{u};\bar{v})$, that is when the parameters $\bar{u}$ and $\bar{v}$ satisfy a system of Bethe equations (see (2.1.30)).

In order to find explicitly the result of the transfer matrix action onto $\mathbb{B}_{a,b}(\bar{u};\bar{v})$ one should set $n = 1$ in (2.1.16)–(2.1.18). Then the subsets $\bar{\eta}_{\rm I}$ and $\bar{\xi}_{\rm I}$ consist of one element only. Obviously, there are two essentially different types of partitions of the set $\bar{\eta} = \{z, \bar{u}\}$:

$$\bar{\eta}_{\rm I} = z, \qquad \bar{\eta}_{\rm II} = \bar{u}, \tag{2.1.22}$$

$$\bar{\eta}_{\rm I} = u_j, \qquad \bar{\eta}_{\rm II} = \{z, \bar{u}_j\}, \qquad j = 1, \dots, a. \tag{2.1.23}$$

Similarly, there are two different types of partitions of the set $\bar{\xi} = \{z, \bar{v}\}$:

$$\bar{\xi}_{\rm I} = z, \qquad \bar{\xi}_{\rm II} = \bar{v}, \tag{2.1.24}$$

$$\bar{\xi}_{\rm I} = v_k, \qquad \bar{\xi}_{\rm II} = \{z, \bar{v}_k\}, \qquad k = 1, \dots, b. \tag{2.1.25}$$

Thus, the action of $\mathcal{T}(z)$ onto $\mathbb{B}_{a,b}(\bar{u};\bar{v})$ can be written in the form

$$\mathcal{T}(z)\mathbb{B}_{a,b}(\bar{u};\bar{v}) = \tau(z|\bar{u};\bar{v})\mathbb{B}_{a,b}(\bar{u};\bar{v}) + \sum_{j=1}^{a} \Lambda_j \mathbb{B}_{a,b}(\{z,\bar{u}_j\};\bar{v}) + \sum_{k=1}^{b} \tilde{\Lambda}_k \mathbb{B}_{a,b}(\bar{u};\{z,\bar{v}_k\}) + \sum_{j=1}^{a}\sum_{k=1}^{b} M_{jk}\mathbb{B}_{a,b}(\{z,\bar{u}_j\};\{z,\bar{v}_k\}), \tag{2.1.26}$$

where τ, Λ_j, $\tilde{\Lambda}_k$, and M_{jk} are numerical coefficients. In order to find $\tau(z|\bar{u};\bar{v})$ we substitute the partitions (2.1.22) and (2.1.24) into (2.1.16)–(2.1.18). We obtain

$$\tau(z|\bar{u},\bar{v}) = \lambda_1(z)f(\bar{u},z) + \lambda_2(z)f(z,\bar{u})f(\bar{v},z) - \lambda_3(z)f(\bar{v},z), \tag{2.1.27}$$

where we have used $h(z,z) = 1$ and $K_1(z|z+c) = g(z,z+c) = -1$.

In order to find Λ_j we substitute the partitions (2.1.23) and (2.1.24) into (2.1.16)–(2.1.18). We find

$$\Lambda_j = \lambda_2(z)h(\bar{v},z)g(\bar{v},z)g(z,u_j)\left(r_1(u_j)\frac{f(\bar{u}_j,u_j)}{f(\bar{v},u_j)} - f(u_j,\bar{u}_j)\right). \tag{2.1.28}$$

Similarly, in order to find $\tilde{\Lambda}_k$ we substitute the partitions (2.1.22) and (2.1.25) into (2.1.16)–(2.1.18). This gives us

$$\tilde{\Lambda}_k = \lambda_2(z)f(z,\bar{u})g(\bar{v}_k,v_k)h(\bar{v}_k,z)g(z,v_k)\left(1 - \frac{r_3(v_k)}{f(v_k,\bar{u})}\right). \tag{2.1.29}$$

If $\mathbb{B}_{a,b}(\bar{u};\bar{v})$ is an eigenvector of $\mathcal{T}(z)$, then the coefficients Λ_j and $\tilde{\Lambda}_k$ must vanish for arbitrary z. Setting $\Lambda_j = 0$ for $j = 1, \dots, a$ and $\tilde{\Lambda}_k = 0$ for $k = 1, \dots, b$ we arrive at a system of equations

$$\begin{aligned} r_1(u_j) &= \frac{f(u_j,\bar{u}_j)}{f(\bar{u}_j,u_j)}f(\bar{v},u_j), \qquad j = 1, \dots, a, \\ r_3(v_k) &= f(v_k,\bar{u}), \qquad k = 1, \dots, b. \end{aligned} \tag{2.1.30}$$

Let us check that $M_{jk} = 0$ provided the system (2.1.30) is fulfilled. Substituting the partitions (2.1.23) and (2.1.25) into (2.1.16)–(2.1.18) we obtain

$$M_{jk} = \lambda_2(z)h(\bar{v}_k, z)g(\bar{v}_k, v_k)\left(\frac{r_1(u_j)f(\bar{u}_j, u_j)}{f(\bar{v}, u_j)}g(v_k, u_j)g(z, v_k)\right.$$
$$\left. + f(u_j, \bar{u}_j)g(u_j, z)\Big[g(z, v_k) + \frac{r_3(v_k)}{f(v_k, \bar{u})}g(v_k, u_j)\Big]\right). \quad (2.1.31)$$

Substituting here $r_1(u_j)$ and $r_3(v_k)$ from equations (2.1.30), we immediately find that $M_{jk} = 0$ due to the identity

$$g(v_k, u_j)g(z, v_k) + g(u_j, z)g(z, v_k) + g(u_j, z)g(v_k, u_j) = 0. \quad (2.1.32)$$

Thus, the system (2.1.30) can be treated as the system of Bethe equations for the parameters $\bar{u}$ and $\bar{v}$. If (2.1.30) holds, then the corresponding Bethe vector $\mathbb{B}_{a,b}(\bar{u}; \bar{v})$ is on-shell, i.e. it is an eigenvector of the transfer matrix $\mathcal{T}(z)$. The eigenvalue of this on-shell vector is given by (2.1.27). At the same time, it is easy to see that the function $\tau(z|\bar{u}, \bar{v})$ has no poles in the points $z = u_j$, $j = 1, \dots, a$, and $z = v_k$, $k = 1, \dots, b$ due to the system (2.1.30).

2.1.6 Twist

Apart from the usual monodromy matrix it is convenient to consider a twisted monodromy matrix $T_\kappa(u)$ [48, 115]. For the models with $\mathfrak{gl}(2|1)$ symmetry it is defined as follows. Let κ be a 3×3 diagonal matrix $\kappa = \mathrm{diag}(\kappa_1, \kappa_2, \kappa_3)$, where κ_i are complex numbers. Then $T_\kappa(u) = \kappa T(u)$, where $T(u)$ is the standard monodromy matrix.

One can easily check that the twisted monodromy matrix satisfies the RTT-relation (1.1.1) with the R-matrix (1.1.3). The supertrace of the twisted monodromy matrix $\mathcal{T}_\kappa(u) = \mathrm{str}\, T_\kappa(u)$ is called the twisted transfer matrix. The eigenstates (resp. dual eigenstates) of the twisted transfer matrix are called twisted on-shell Bethe vectors (resp. twisted dual on-shell Bethe vectors). A generic (dual) Bethe vector becomes a twisted (dual) on-shell Bethe vector, if the Bethe parameters satisfy a system of twisted Bethe equations

$$\begin{aligned} r_1(u_j) &= \frac{\kappa_2}{\kappa_1}\prod_{\substack{k=1 \\ k\neq j}}^{a}\frac{f(u_j, u_k)}{f(u_k, u_j)}\prod_{l=1}^{b} f(v_l, u_j), \qquad j = 1, \dots, a, \\ r_3(v_j) &= \frac{\kappa_2}{\kappa_3}\prod_{l=1}^{a} f(v_j, u_l), \qquad j = 1, \dots, b. \end{aligned} \quad (2.1.33)$$

Then

$$\mathcal{T}_\kappa(w)\mathbb{B}_{a,b}(\bar{u}; \bar{v}) = \tau_\kappa(w|\bar{u}, \bar{v})\, \mathbb{B}_{a,b}(\bar{u}; \bar{v}), \qquad \mathbb{C}_{a,b}(\bar{u}; \bar{v})\mathcal{T}_\kappa(w) = \tau_\kappa(w|\bar{u}, \bar{v})\, \mathbb{C}_{a,b}(\bar{u}; \bar{v}), \quad (2.1.34)$$

where the eigenvalue $\tau_\kappa(w|\bar{u}, \bar{v})$ is given by (2.1.27), in which one should replace $\lambda_i(w)$ by $\kappa_i\lambda_i(w)$

$$\tau(z|\bar{u}, \bar{v}) = \kappa_1\lambda_1(z)f(\bar{u}, z) + \kappa_2\lambda_2(z)f(z, \bar{u})f(\bar{v}, z) - \kappa_3\lambda_3(z)f(\bar{v}, z), \quad (2.1.35)$$

2.2 Proofs of multiple actions for T_{ij} with $i < j$

Bethe vectors consist of the elements from the upper triangular part of the monodromy matrix applied to pseudovacuum Ω (2.1.1), (2.1.2). Then, it is intuitively clear that actions of the elements T_{ij} with $i < j$ are the simplest. We begin our consideration from the right-upper corner of monodromy matrix and will move along anti-diagonal direction successively proving the action relations.

2.2.1 Proof for T_{13}

For $n = 1$ equation (2.1.13) follows directly from the definitions of the Bethe vectors. Let us take, for instance, (2.1.1) and set there $\bar u = \{z, \bar u'\}$ and $\bar v = \{z, \bar v'\}$. Then the product $1/f(\bar v, \bar u)$ vanishes, as it contains $1/f(z, z)$. This zero, however, can be compensated if and only if $z \in \bar u_{\rm I}$ and $z \in \bar v_{\rm I}$. Indeed, in this case the product $g(\bar v_{\rm I}, \bar u_{\rm I})$ contains a singular factor $g(z, z)$. Thus, we should consider only such partitions, for which $z \in \bar u_{\rm I}$ and $z \in \bar v_{\rm I}$. Therefore we should set: $\bar u_{\rm I} = \{z, \bar u'_{\rm I}\}$ and $\bar v_{\rm I} = \{z, \bar v'_{\rm I}\}$; $\bar u_{\rm II} = \bar u'_{\rm II}$ and $\bar v_{\rm II} = \bar v'_{\rm II}$. Then we obtain

$$\mathbb{B}_{a,b}(\{z, \bar u'\}; \{z, \bar v'\}) = \sum g(\bar v'_{\rm I}, \bar u'_{\rm I}) \frac{f(\bar u'_{\rm I}, \bar u'_{\rm II}) g(\bar v'_{\rm II}, \bar v'_{\rm I}) h(\bar u'_{\rm I}, \bar u'_{\rm I})}{\lambda_2(\bar u')\lambda_2(\bar v'_{\rm II}) f(\bar v', \bar u')}\, g(\bar v'_{\rm I}, z) g(z, \bar u'_{\rm I})$$
$$\times \frac{f(z, \bar u'_{\rm II}) g(\bar v'_{\rm II}, z) h(z, \bar u'_{\rm I}) h(\bar u'_{\rm I}, z)}{\lambda_2(z) f(\bar v', z) f(z, \bar u')}\, \frac{T_{13}(z)}{h(\bar u'_{\rm I}, z)}\, \mathbb{T}_{13}(\bar u'_{\rm I})\, T_{12}(\bar u'_{\rm II})\, \mathbb{T}_{23}(\bar v'_{\rm II})\Omega. \quad (2.2.1)$$

After evident cancellations we arrive at

$$\mathbb{B}_{a,b}(\{z, \bar u'\}; \{z, \bar v'\}) = \frac{T_{13}(z)}{\lambda_2(z) h(\bar v', z)}\, \mathbb{B}_{a-1,b-1}(\bar u'; \bar v'), \quad (2.2.2)$$

which coincides with (2.1.13) at $n = 1$. The same result arises from the analysis of equation (2.1.2).

Now we use induction over n. Assume that (2.1.13) holds for some $n - 1$. Then

$$\mathbb{T}_{13}(\bar z)\mathbb{B}_{a,b}(\bar u; \bar v) = \frac{T_{13}(z_n)\mathbb{T}_{13}(\bar z_n)}{h(\bar z_n, z_n)}\, \mathbb{B}_{a,b}(\bar u; \bar v)$$
$$= \lambda_2(\bar z_n) \frac{h(\bar v, \bar z_n)}{h(\bar z_n, z_n)}\, T_{13}(z_n)\mathbb{B}_{a+n-1,b+n-1}(\{\bar u, \bar z_n\}; \{\bar v, \bar z_n\})$$
$$= \lambda_2(\bar z) h(\bar v, \bar z)\mathbb{B}_{a+n,b+n}(\{\bar u, \bar z\}; \{\bar v, \bar z\}), \quad (2.2.3)$$

and thus, (2.1.13) is proved. □

Using (2.1.13) one can recast recursions (2.1.5) and (2.1.6) as follows:

$$T_{12}(z)\, \mathbb{B}_{a,b}(\bar u; \bar v) = \lambda_2(z) f(\bar v, z)\mathbb{B}_{a+1,b}(\{\bar u, z\}; \bar v)$$
$$+ \lambda_2(z) \sum_{j=1}^{b} g(z, v_j) g(\bar v_j, v_j) h(\bar v_j, z)\, \mathbb{B}_{a+1,b}(\{\bar u, z\}; \{\bar v_j, z\}), \quad (2.2.4)$$

and

$$T_{23}(z)\mathbb{B}_{a,b}(\bar u; \bar v) = \lambda_2(z) h(\bar v, z)\Big(f(z, \bar u)\mathbb{B}_{a,b+1}(\bar u; \{\bar v, z\})$$
$$+ \sum_{j=1}^{a} g(u_j, z) f(u_j, \bar u_j)\mathbb{B}_{a,b+1}(\{\bar u_j, z\}; \{\bar v, z\})\Big). \quad (2.2.5)$$

One can easily recognize in these equations the actions (2.1.14) and (2.1.15) for $n = 1$. Then one should use induction over n.

2.2.2 Proof for T_{12}

Assume that (2.1.14) holds for some $n-1$. Then

$$T_{12}(\bar{z})\mathbb{B}_{a,b}(\bar{u};\bar{v}) = T_{12}(z_n)\lambda_2(\bar{z}_n)h(\bar{\xi},\bar{z}_n)\sum \frac{g(\bar{\xi}_{\rm II},\bar{\xi}_{\rm I})}{h(\bar{\xi}_{\rm I},\bar{z}_n)}\mathbb{B}_{a+n-1,b}(\bar{\eta};\bar{\xi}_{\rm II}). \tag{2.2.6}$$

Here $\bar{\eta} = \{\bar{z}_n,\bar{u}\}$, $\bar{\xi} = \{\bar{z}_n,\bar{v}\}$, and the sum runs through the partitions $\bar{\xi} \Rightarrow \{\bar{\xi}_{\rm I},\bar{\xi}_{\rm II}\}$ with $\#\bar{\xi}_{\rm I} = n-1$. Acting with $T_{12}(z_n)$ we obtain

$$T_{12}(\bar{z})\mathbb{B}_{a,b}(\bar{u};\bar{v}) = \lambda_2(\bar{z})\frac{h(\bar{\xi},\bar{z}_n)}{h(z_n,\bar{z}_n)}\sum \frac{g(\bar{\xi}_{\rm II},\bar{\xi}_{\rm I})}{h(\bar{\xi}_{\rm I},\bar{z}_n)}h(\bar{\xi}_{\rm II},z_n)\frac{g(\bar{\xi}_{\rm ii},\bar{\xi}_{\rm i})}{h(\bar{\xi}_{\rm i},z_n)}\mathbb{B}_{a+n,b}(\bar{\eta};\bar{\xi}_{\rm ii}). \tag{2.2.7}$$

Here already $\bar{\eta} = \{\bar{z},\bar{u}\}$ and $\bar{\xi} = \{\bar{z},\bar{v}\}$. The sum first is taken over partitions $\{\bar{z}_n,\bar{v}\} \Rightarrow \{\bar{\xi}_{\rm I},\bar{\xi}_{\rm II}\}$ with $\#\bar{\xi}_{\rm I} = n-1$, and then over partitions $\{z_n,\bar{\xi}_{\rm II}\} \Rightarrow \{\bar{\xi}_{\rm i},\bar{\xi}_{\rm ii}\}$ with $\#\bar{\xi}_{\rm i} = 1$. One can say that the sum is taken over partitions $\{\bar{z},\bar{v}\} = \bar{\xi} \Rightarrow \{\bar{\xi}_{\rm I},\bar{\xi}_{\rm i},\bar{\xi}_{\rm ii}\}$ with restrictions $z_n \notin \bar{\xi}_{\rm I}$, $\#\bar{\xi}_{\rm I} = n-1$, and $\#\bar{\xi}_{\rm i} = 1$. Presenting $\bar{\xi}_{\rm II}$ as $\bar{\xi}_{\rm II} = \{\bar{\xi}_{\rm i},\bar{\xi}_{\rm ii}\} \setminus \{z_n\}$ we obtain

$$g(\bar{\xi}_{\rm II},\bar{\xi}_{\rm I}) = \frac{g(\bar{\xi}_{\rm i},\bar{\xi}_{\rm I})g(\bar{\xi}_{\rm ii},\bar{\xi}_{\rm I})}{g(z_n,\bar{\xi}_{\rm I})}, \qquad h(\bar{\xi}_{\rm II},z_n) = h(\bar{\xi}_{\rm i},z_n)h(\bar{\xi}_{\rm ii},z_n), \tag{2.2.8}$$

and hence,

$$T_{12}(\bar{z})\mathbb{B}_{a,b}(\bar{u};\bar{v}) = \lambda_2(\bar{z})\frac{h(\bar{\xi},\bar{z}_n)}{h(z_n,\bar{z}_n)}\sum \frac{g(\bar{\xi}_{\rm i},\bar{\xi}_{\rm I})g(\bar{\xi}_{\rm ii},\bar{\xi}_{\rm I})g(\bar{\xi}_{\rm ii},\bar{\xi}_{\rm i})}{g(z_n,\bar{\xi}_{\rm I})h(\bar{\xi}_{\rm I},\bar{z}_n)}h(\bar{\xi}_{\rm ii},z_n)\ \mathbb{B}_{a+n,b}(\bar{\eta};\bar{\xi}_{\rm ii}). \tag{2.2.9}$$

Observe that the condition $z_n \notin \bar{\xi}_{\rm I}$ is ensured by the product $g(z_n,\bar{\xi}_{\rm I})$ in the denominator. Hence, we can say that the sum is taken over partitions $\bar{\xi} \Rightarrow \{\bar{\xi}_{\rm I},\bar{\xi}_{\rm i},\bar{\xi}_{\rm ii}\}$ with the restrictions on the cardinalities of the subsets only. Setting $\{\bar{\xi}_{\rm I},\bar{\xi}_{\rm i}\} = \bar{\xi}_0$ we recast (2.2.9) as follows:

$$T_{12}(\bar{z})\mathbb{B}_{a,b}(\bar{u};\bar{v}) = \lambda_2(\bar{z})\frac{h(\bar{\xi},\bar{z})}{h(z_n,\bar{z}_n)}\sum \frac{g(\bar{\xi}_{\rm i},\bar{\xi}_{\rm I})g(z_n,\bar{\xi}_{\rm i})h(\bar{\xi}_{\rm i},\bar{z}_n)}{g(z_n,\bar{\xi}_0)h(\bar{\xi}_0,\bar{z})}g(\bar{\xi}_{\rm ii},\bar{\xi}_0)\ \mathbb{B}_{a+n,b}(\bar{\eta};\bar{\xi}_{\rm ii}). \tag{2.2.10}$$

The sum over partitions $\bar{\xi}_0 \Rightarrow \{\bar{\xi}_{\rm I},\bar{\xi}_{\rm i}\}$ can be computed via lemma 9.3.1

$$\sum_{\bar{\xi}_0 \Rightarrow \{\bar{\xi}_{\rm I},\bar{\xi}_{\rm i}\}} g(\bar{\xi}_{\rm i},\bar{\xi}_{\rm I})g(z_n,\bar{\xi}_{\rm i})h(\bar{\xi}_{\rm i},\bar{z}_n) = g(z_n,\bar{\xi}_0)h(z_n,\bar{z}_n), \tag{2.2.11}$$

where we took into account that $\#\bar{\xi}_0 = n$. Thus, we arrive at

$$T_{12}(\bar{z})\mathbb{B}_{a,b}(\bar{u};\bar{v}) = \lambda_2(\bar{z})h(\bar{\xi},\bar{z})\sum \frac{g(\bar{\xi}_{\rm ii},\bar{\xi}_0)}{h(\bar{\xi}_0,\bar{z})}\ \mathbb{B}_{a+n,b}(\bar{\eta};\bar{\xi}_{\rm ii}), \tag{2.2.12}$$

which coincides with (2.1.14) up to a relabeling of the subsets. □

2.2.3 Proof for T_{23}

Assume that (2.1.15) holds for some $n-1$. Let $\#\bar{z}=n$. Then we have

$$\begin{aligned}\mathbb{T}_{23}(\bar{z})\mathbb{B}_{a,b}(\bar{u};\bar{v}) &= \frac{T_{23}(z_n)\mathbb{T}_{23}(\bar{z}_n)}{h(\bar{z}_n,z_n)}\,\mathbb{B}_{a,b}(\bar{u};\bar{v})\\ &= (-1)^{n-1}\lambda_2(\bar{z}_n)\frac{h(\bar{v},\bar{z}_n)}{h(\bar{z}_n,z_n)}\sum K_{n-1}(\bar{z}_n|\bar{\eta}_{\rm I}+c)f(\bar{\eta}_{\rm I},\bar{\eta}_{\rm II})T_{23}(z_n)\mathbb{B}_{a,b+n-1}(\bar{\eta}_{\rm II};\bar{\xi}). \end{aligned} \tag{2.2.13}$$

Here $\bar{\eta}=\{\bar{z}_n,\bar{u}\}$, $\bar{\xi}=\{\bar{z}_n,\bar{v}\}$, and the sum runs through the partitions $\bar{\eta}\Rightarrow\{\bar{\eta}_{\rm I},\bar{\eta}_{\rm II}\}$ with $\#\bar{\eta}_{\rm I}=n-1$. Acting with $T_{23}(z_n)$ we obtain

$$\begin{aligned}\mathbb{T}_{23}(\bar{z})\mathbb{B}_{a,b}(\bar{u};\bar{v}) &= (-1)^{n}\lambda_2(\bar{z})\frac{h(\bar{v},\bar{z}_n)}{h(\bar{z}_n,z_n)}\\ &\times\sum K_{n-1}(\bar{z}_n|\bar{\eta}_{\rm I}+c)f(\bar{\eta}_{\rm I},\bar{\eta}_{\rm II})h(\bar{\xi},z_n)K_1(z_n|\bar{\eta}_{\rm i}+c)f(\bar{\eta}_{\rm i},\bar{\eta}_{\rm ii})\mathbb{B}_{a,b+n}(\bar{\eta}_{\rm ii};\bar{\xi}).\end{aligned} \tag{2.2.14}$$

Here already $\bar{\eta}=\{\bar{z},\bar{u}\}$, $\bar{\xi}=\{\bar{z},\bar{v}\}$, and the sum is taken first over partitions $\{\bar{z}_n,\bar{u}\}\Rightarrow\{\bar{\eta}_{\rm I},\bar{\eta}_{\rm II}\}$ with $\#\bar{\eta}_{\rm I}=n-1$, and then over partitions $\{z_n,\bar{\eta}_{\rm II}\}\Rightarrow\{\bar{\eta}_{\rm i},\bar{\eta}_{\rm ii}\}$ with $\#\bar{\eta}_{\rm i}=1$. Substituting here $\bar{\eta}_{\rm II}=\{\bar{\eta}_{\rm i},\bar{\eta}_{\rm ii}\}\setminus\{z_n\}$ we find

$$\begin{aligned}\mathbb{T}_{23}(\bar{z})\mathbb{B}_{a,b}(\bar{u};\bar{v}) &= (-1)^{n}\lambda_2(\bar{z})h(\bar{v},\bar{z})\\ &\times\sum K_{n-1}(\bar{z}_n|\bar{\eta}_{\rm I}+c)K_1(z_n|\bar{\eta}_{\rm i}+c)\frac{f(\bar{\eta}_{\rm I},\bar{\eta}_{\rm i})f(\bar{\eta}_{\rm I},\bar{\eta}_{\rm ii})f(\bar{\eta}_{\rm i},\bar{\eta}_{\rm ii})}{f(\bar{\eta}_{\rm I},z_n)}\,\mathbb{B}_{a,b+n}(\bar{\eta}_{\rm ii};\bar{\xi}).\end{aligned} \tag{2.2.15}$$

Setting $\bar{\eta}_0=\{\bar{\eta}_{\rm I},\bar{\eta}_{\rm i}\}$ and using $K_1(z_n|\bar{\eta}_{\rm i}+c)=-K_1(\bar{\eta}_{\rm i}|z_n)/f(\bar{\eta}_{\rm i},z_n)$ we obtain

$$\begin{aligned}\mathbb{T}_{23}(\bar{z})\mathbb{B}_{a,b}(\bar{u};\bar{v}) &= (-1)^{n-1}\lambda_2(\bar{z})h(\bar{v},\bar{z})\\ &\times\sum K_{n-1}(\bar{z}_n|\bar{\eta}_{\rm I}+c)K_1(\bar{\eta}_{\rm i}|z_n)f(\bar{\eta}_{\rm I},\bar{\eta}_{\rm i})\frac{f(\bar{\eta}_0,\bar{\eta}_{\rm ii})}{f(\bar{\eta}_0,z_n)}\,\mathbb{B}_{a,b+n}(\bar{\eta}_{\rm ii};\bar{\xi}).\end{aligned} \tag{2.2.16}$$

Now we can compute the sum over partitions $\bar{\eta}_0\Rightarrow\{\bar{\eta}_{\rm I},\bar{\eta}_{\rm i}\}$ via (9.3.17)

$$\sum_{\bar{\eta}_0\Rightarrow\{\bar{\eta}_{\rm I},\bar{\eta}_{\rm i}\}} K_{n-1}(\bar{z}_n|\bar{\eta}_{\rm I}+c)K_1(\bar{\eta}_{\rm i}|z_n)f(\bar{\eta}_{\rm I},\bar{\eta}_{\rm i}) = -f(\bar{\eta}_0,z_n)K_n(\bar{z}|\bar{\eta}_0+c), \tag{2.2.17}$$

which gives us

$$\mathbb{T}_{23}(\bar{z})\mathbb{B}_{a,b}(\bar{u};\bar{v}) = (-1)^{n}\lambda_2(\bar{z})h(\bar{v},\bar{z})\sum K_n(\bar{z}|\bar{\eta}_0+c)f(\bar{\eta}_0,\bar{\eta}_{\rm ii})\,\mathbb{B}_{a,b+n}(\bar{\eta}_{\rm ii};\bar{\xi}). \tag{2.2.18}$$

This is exactly (2.1.15) up to the labeling of the subsets. □

2.3 Proof of the multiple action of the operator T_{22}

The proofs for the actions (2.1.16)–(2.1.21) are much more involved than the ones considered in the previous section. Fortunately, they all are quite similar. Therefore, we only detail one as a typical example, the other actions being proven in the same manner. We focus on the operator $T_{22}(u)$.

The strategy of the proof is the following. First, we prove equation (2.1.17) for $a = \#\bar{u} = 0$ and $n = \#\bar{z} = 1$. This can be done either via the standard consideration of the algebraic Bethe ansatz or using induction over $b = \#\bar{v}$. In both cases we use (2.1.7) and the relation

$$T_{22}(u)T_{23}(v) = f(v,u)T_{23}(v)T_{22}(u) + g(u,v)T_{23}(u)T_{22}(v), \tag{2.3.1}$$

that follows from (1.1.14).

The next step of the proof is an induction over a. We assume that (2.1.17) is valid for $n = 1$ and some a and use recursion (2.1.5). Hereby, we use some of commutation relations (1.1.14)

$$T_{22}(u)T_{12}(v) = f(u,v)T_{12}(v)T_{22}(u) + g(v,u)T_{12}(u)T_{22}(v), \tag{2.3.2}$$

$$[T_{22}(u), T_{13}(v)] = g(u,v)\big(T_{12}(v)T_{23}(u) - T_{12}(u)T_{23}(v)\big). \tag{2.3.3}$$

Finally, when equation (2.1.17) is proved for $n = 1$ and arbitrary a and b we use induction over n.

Remark. We begin the proof with the case $n = 1$, $a = 0$, and arbitrary b. However, one could also begin with the case $n = 1$, $b = 0$, and arbitrary a. For the action of the operator $T_{22}(z)$ this is a matter of choice. For other operators these two starting cases could be essentially different. For instance, one can easily see that $T_{21}(z)\mathbb{B}_{0,b}(\emptyset,\bar{v}) = 0$ for arbitrary b. On the other hand, the action $T_{21}(z)\mathbb{B}_{a,0}(\bar{u},\emptyset)$ is highly nontrivial, although it is clear that it should coincide with the similar action in the models with $\mathfrak{gl}(3)$-invariant R-matrix. Obviously, in this case it is better to begin the proof with the vector $\mathbb{B}_{0,b}(\emptyset,\bar{v})$.

2.3.1 Action of $T_{22}(z)$ at $a = 0$ and $z = 1$

In the particular case $a = 0$ and $n = 1$ equation (2.1.17) turns into

$$T_{22}(z)\mathbb{B}_{0,b}(\emptyset;\bar{v}) = \lambda_2(z)h(\bar{v},z)\sum \frac{g(\bar{\xi}_{\rm II},\bar{\xi}_{\rm I})}{h(\bar{\xi}_{\rm I},z)}\mathbb{B}_{0,b}(\emptyset;\bar{\xi}_{\rm II}). \tag{2.3.4}$$

The sum is taken over partitions $\{z,\bar{v}\} = \bar{\xi} \Rightarrow \{\bar{\xi}_{\rm I},\bar{\xi}_{\rm II}\}$ with $\#\bar{\xi}_{\rm I} = 1$. We prove this action using the standard scheme of the algebraic Bethe ansatz. The vector $\mathbb{B}_{0,b}(\emptyset;\bar{v})$ is given by the second equation (2.1.7). Thus, we should move the operator $T_{22}(z)$ to the right through the product of the operators $T_{23}(v_j)$. Using (2.3.1) we easily find

$$T_{22}(z)\mathbb{B}_{0,b}(\emptyset;\bar{v}) = \Lambda\mathbb{B}_{0,b}(\emptyset;\bar{v}) + \sum_{j=1}^{b}\Lambda_j\mathbb{B}_{0,b}(\emptyset;\{\bar{v}_j,z\}), \tag{2.3.5}$$

where Λ and Λ_j are some coefficients to be determined. Obviously, in order to obtain the coefficient of $\mathbb{B}_{0,b}(\emptyset;\bar{v})$ one should use only the first term in the r.h.s. of (2.3.1). From this we immediately find

$$\Lambda = \lambda_2(z)f(\bar{v},z). \tag{2.3.6}$$

Then, due to the symmetry of $\mathbb{T}_{23}(\bar{v})$ over $\bar{v}$ it is enough to find Λ_1 only. Permuting $T_{22}(z)$ with $T_{23}(v_1)$ we should use the second term in the r.h.s. of (2.3.1). We have

$$T_{22}(z)\frac{\mathbb{T}_{23}(\bar{v})}{\lambda_2(\bar{v})}\Omega = T_{22}(z)\frac{T_{23}(v_1)\mathbb{T}_{23}(\bar{v}_1)}{\lambda_2(\bar{v})h(\bar{v}_1,v_1)}\Omega = g(z,v_1)T_{23}(z)\frac{T_{22}(v_1)\mathbb{T}_{23}(\bar{v}_1)}{\lambda_2(\bar{v})h(\bar{v}_1,v_1)}\Omega + UWT, \tag{2.3.7}$$

where UWT means *unwanted terms*, i.e. the terms that cannot give a contribution to the coefficient Λ_1. Now, moving $T_{22}(v_1)$ through the product $\mathbb{T}_{23}(\bar v)$ we should use only the first term in the r.h.s. of (2.3.1), which gives us

$$T_{22}(z)\frac{\mathbb{T}_{23}(\bar v)}{\lambda_2(\bar v)}\Omega = g(z,v_1)g(\bar v_1,v_1)T_{23}(z)\frac{\mathbb{T}_{23}(\bar v_1)}{\lambda_2(\bar v)}\lambda_2(v_1)\Omega + UWT, \tag{2.3.8}$$

where we used $g(\bar v_1,v_1) = f(\bar v_1,v_1)/h(\bar v_1,v_1)$. It remains to combine $T_{23}(z)$ and $\mathbb{T}_{23}(\bar v_1)$ into $\mathbb{T}_{23}(\{z,\bar v_1\})$ and we arrive at

$$T_{22}(z)\frac{\mathbb{T}_{23}(\bar v)}{\lambda_2(\bar v)}\Omega = \lambda_2(z)g(z,v_1)g(\bar v_1,v_1)h(\bar v_1,z)\frac{\mathbb{T}_{23}(\{z,\bar v_1\})}{\lambda_2(\bar v_1)\lambda_2(z)}\Omega + UWT, \tag{2.3.9}$$

leading to

$$\Lambda_1 = \lambda_2(z)g(z,v_1)g(\bar v_1,v_1)h(\bar v_1,z). \tag{2.3.10}$$

Thus, we eventually obtain

$$T_{22}(z)\mathbb{B}_{0,b}(\emptyset;\bar v) = \lambda_2(z)f(\bar v,z)\mathbb{B}_{0,b}(\emptyset;\bar v) + \lambda_2(z)\sum_{j=1}^{b} g(z,v_j)g(\bar v_j,v_j)h(\bar v_j,z)\mathbb{B}_{0,b}(\emptyset;\{\bar v_j,z\}). \tag{2.3.11}$$

It is easy to see that this formula coincides with (2.3.4). Indeed the first term in (2.3.11) corresponds to the partition $\bar\xi_{\rm I} = z$ and $\bar\xi_{\rm II} = \bar v$ in (2.3.4). The other terms arise in the case of the partitions $\bar\xi_{\rm I} = v_j$, $j = 1,\dots,b$, and $\bar\xi_{\rm II} = \{z,\bar v_j\}$. Thus, action (2.3.4) is proved. □

2.3.2 Induction over a

For $n = 1$ equation (2.1.17) takes the form

$$T_{22}(z)\mathbb{B}_{a,b}(\bar u;\bar v) = \lambda_2(z)h(\bar v,z)\sum \frac{f(\bar\eta_{\rm I},\bar\eta_{\rm II})g(\bar\xi_{\rm II},\bar\xi_{\rm I})}{h(\bar\eta_{\rm I},z)h(\bar\xi_{\rm I},z)}\mathbb{B}_{a,b}(\bar\eta_{\rm II};\bar\xi_{\rm II}). \tag{2.3.12}$$

The sum is taken over partitions $\bar\xi \Rightarrow \{\bar\xi_{\rm I},\bar\xi_{\rm II}\}$ and $\bar\eta \Rightarrow \{\bar\eta_{\rm I},\bar\eta_{\rm II}\}$ with $\#\bar\xi_{\rm I} = \#\bar\eta_{\rm I} = 1$. We assume that (2.3.12) is valid for some $a \ge 0$ and b arbitrary. Then, due to recursion (2.1.5) we have

$$\begin{aligned} T_{22}(z_1)\mathbb{B}_{a+1,b}(\{\bar u;z_2\};\bar v) &= T_{22}(z_1)\frac{T_{12}(z_2)\,\mathbb{B}_{a,b}(\bar u;\bar v)}{\lambda_2(z_2)f(\bar v,z_2)} \\ &\quad - T_{22}(z_1)\sum_{j=1}^{b}\frac{g(z_2,v_j)g(\bar v_j,v_j)\,T_{13}(z_2)\mathbb{B}_{a,b-1}(\bar u;\bar v_j)}{\lambda_2(z_2)f(\bar v,z_2)}. \end{aligned} \tag{2.3.13}$$

We see that in order to compute the action of $T_{22}(z_1)$ onto $\mathbb{B}_{a+1,b}(\{z_2,\bar u\};\bar v)$ we should calculate the successive actions of the operators $T_{22}(z_1)T_{12}(z_2)$ and $T_{22}(z_1)T_{13}(z_2)$. This can be done via (2.3.2) and (2.3.3)

$$T_{22}(z_1)T_{12}(z_2)\mathbb{B}_{a,b}(\bar u;\bar v) = \Big(f(z_1,z_2)T_{12}(z_2)T_{22}(z_1) + g(z_2,z_1)T_{12}(z_1)T_{22}(z_2)\Big)\mathbb{B}_{a,b}(\bar u;\bar v), \tag{2.3.14}$$

$$\begin{aligned} T_{22}(z_1)T_{13}(z_2)\mathbb{B}_{a,b-1}(\bar u;\bar v_j) &= T_{13}(z_2)T_{22}(z_1)\mathbb{B}_{a,b-1}(\bar u;\bar v_j) \\ &\quad + g(z_1,z_2)\big(T_{12}(z_2)T_{23}(z_1) - T_{12}(z_1)T_{23}(z_2)\big)\mathbb{B}_{a,b-1}(\bar u;\bar v_j). \end{aligned} \tag{2.3.15}$$

Thus, we have reduced the action $T_{22}(z_1)\mathbb{B}_{a+1,b}(\{\bar u; z_2\}; \bar v)$ to the calculation of several successive actions. In all of them the operator T_{22} acts either on $\mathbb{B}_{a,b}(\bar u; \bar v)$ or on $\mathbb{B}_{a,b-1}(\bar u; \bar v_j)$, which are known due to the induction assumption. The actions of other operators T_{ij} with $i < j$ are already known for a and b arbitrary.

Successive action of T_{12} and T_{23}

We begin our calculation with the successive action of the operators T_{12} and T_{23}. Using (2.1.15) we have

$$T_{12}(z_2)T_{23}(z_1)\mathbb{B}_{a,b}(\bar u; \bar v) = \lambda_2(z_1)h(\bar v, z_1)\sum \frac{f(\bar\eta_{\rm I}, \bar\eta_{\rm II})}{h(\bar\eta_{\rm I}, z_1)} T_{12}(z_2)\mathbb{B}_{a,b+1}(\bar\eta_{\rm II}; \bar\xi). \tag{2.3.16}$$

Here $\bar\eta = \{z_1, \bar u\}$ and $\bar\xi = \{z_1, \bar v\}$. The sum is taken over partitions $\bar\eta \Rightarrow \{\bar\eta_{\rm I}, \bar\eta_{\rm II}\}$ with $\#\bar\eta_{\rm I} = 1$. Then we use (2.1.14) and find

$$T_{12}(z_2)T_{23}(z_1)\mathbb{B}_{a,b}(\bar u; \bar v) = \lambda_2(\bar z)h(\bar v, z_1)h(\bar\xi, z_2)\sum \frac{f(\bar\eta_{\rm I}, \bar\eta_{\rm II})}{h(\bar\eta_{\rm I}, z_1)}\frac{g(\bar\xi_{\rm II}, \bar\xi_{\rm I})}{h(\bar\xi_{\rm I}, z_2)}\mathbb{B}_{a+1,b+1}(\{\bar\eta_{\rm II}, z_2\}; \bar\xi_{\rm II}). \tag{2.3.17}$$

Here already $\bar\xi = \{\bar z, \bar v\}$, however we still have $\bar\eta = \{z_1, \bar u\}$. Replacing $\{\bar\eta_{\rm II}, z_2\}$ by $\bar\eta_{\rm II}$ we recast (2.3.17) in the form

$$T_{12}(z_2)T_{23}(z_1)\mathbb{B}_{a,b}(\bar u; \bar v) = \lambda_2(\bar z)h(\bar v, \bar z)h(z_1, z_2)\sum \frac{f(\bar\eta_{\rm I}, \bar\eta_{\rm II})g(\bar\xi_{\rm II}, \bar\xi_{\rm I})}{f(\bar\eta_{\rm I}, z_2)h(\bar\eta_{\rm I}, z_1)h(\bar\xi_{\rm I}, z_2)}\mathbb{B}_{a+1,b+1}(\bar\eta_{\rm II}; \bar\xi_{\rm II}). \tag{2.3.18}$$

Here $\bar\eta = \{\bar z, \bar u\}$, and the sum is taken over partitions $\bar\eta \Rightarrow \{\bar\eta_{\rm I}, \bar\eta_{\rm II}\}$ and $\bar\xi \Rightarrow \{\bar\xi_{\rm I}, \bar\xi_{\rm II}\}$ with $\#\bar\eta_{\rm I} = \#\bar\xi_{\rm I} = 1$. Note that the condition $z_2 \notin \bar\eta_{\rm I}$ is ensured automatically. Indeed, if $z_2 \in \bar\eta_{\rm I}$, then $1/f(\bar\eta_{\rm I}, z_2) = 0$.

Replacing here $z_1 \leftrightarrow z_2$ we obtain

$$T_{12}(z_1)T_{23}(z_2)\mathbb{B}_{a,b}(\bar u; \bar v) = \lambda_2(\bar z)h(\bar v, \bar z)h(z_2, z_1)\sum \frac{f(\bar\eta_{\rm I}, \bar\eta_{\rm II})g(\bar\xi_{\rm II}, \bar\xi_{\rm I})}{f(\bar\eta_{\rm I}, z_1)h(\bar\eta_{\rm I}, z_2)h(\bar\xi_{\rm I}, z_1)}\mathbb{B}_{a+1,b+1}(\bar\eta_{\rm II}; \bar\xi_{\rm II}). \tag{2.3.19}$$

Thus, we find

$$\begin{aligned}
g(z_1, z_2)\big(T_{12}(z_2)T_{23}(z_1) - T_{12}(z_1)T_{23}(z_2)\big)\mathbb{B}_{a,b}(\bar u; \bar v) \\
= \lambda_2(\bar z)h(\bar v, \bar z)\sum \frac{f(\bar\eta_{\rm I}, \bar\eta_{\rm II})g(\bar\xi_{\rm II}, \bar\xi_{\rm I})}{h(\bar\eta_{\rm I}, \bar z)}\mathbb{B}_{a+1,b+1}(\bar\eta_{\rm II}; \bar\xi_{\rm II}) \\
\times\left\{\frac{f(z_1, z_2)}{g(\bar\eta_{\rm I}, z_2)h(\bar\xi_{\rm I}, z_2)} + \frac{f(z_2, z_1)}{g(\bar\eta_{\rm I}, z_1)h(\bar\xi_{\rm I}, z_1)}\right\}.
\end{aligned} \tag{2.3.20}$$

Successive action of T_{13} and T_{22}

Combining the actions (2.3.12) and (2.1.13) we obtain

$$\begin{aligned}
T_{13}(z_2)T_{22}(z_1)\mathbb{B}_{a,b}(\bar u; \bar v) = \lambda_2(\bar z)h(\bar v, z_1)\sum \frac{f(\bar\eta_{\rm I}, \bar\eta_{\rm II})g(\bar\xi_{\rm II}, \bar\xi_{\rm I})h(\bar\xi_{\rm II}, z_2)}{h(\bar\eta_{\rm I}, z_1)h(\bar\xi_{\rm I}, z_1)} \\
\times \mathbb{B}_{a+1,b+1}(\{\bar\eta_{\rm II}, z_2\}; \{\bar\xi_{\rm II}, z_2\}).
\end{aligned} \tag{2.3.21}$$

Here $\bar\eta = \{\bar u, z_1\}$ and $\bar\xi = \{\bar v, z_1\}$. Replacing $\{\bar\eta_{\rm II}, z_2\}$ with $\bar\eta_{\rm II}$ and $\{\bar\xi_{\rm II}, z_2\}$ with $\bar\xi_{\rm II}$ we arrive at

$$T_{13}(z_2)T_{22}(z_1)\mathbb{B}_{a,b}(\bar u;\bar v) = \lambda_2(\bar z)h(\bar v,\bar z)h(z_1,z_2)\sum \frac{f(\bar\eta_{\rm I},\bar\eta_{\rm II})g(\bar\xi_{\rm II},\bar\xi_{\rm I})\mathbb{B}_{a+1,b+1}(\bar\eta_{\rm II};\bar\xi_{\rm II})}{f(\bar\eta_{\rm I},z_2)g(z_2,\bar\xi_{\rm I})h(\bar\eta_{\rm I},z_1)h(\bar\xi_{\rm I},\bar z)} \qquad (2.3.22)$$

Here already $\bar\eta = \{\bar u, \bar z\}$ and $\bar\xi = \{\bar v, \bar z\}$. The sum is taken over partitions $\bar\eta \Rightarrow \{\bar\eta_{\rm I}, \bar\eta_{\rm II}\}$ and $\bar\xi \Rightarrow \{\bar\xi_{\rm I}, \bar\xi_{\rm II}\}$ with $\#\bar\eta_{\rm I} = \#\bar\xi_{\rm I} = 1$.

Now we are able to calculate the successive action $T_{22}(z_1)T_{13}(z_2)$ on $\mathbb{B}_{a,b}(\bar u;\bar v)$. Indeed, due to (2.3.15) this successive action is given by a combination of (2.3.20) and (2.3.22). A straightforward calculation leads us to the following representation:

$$T_{22}(z_1)T_{13}(z_2)\mathbb{B}_{a,b}(\bar u;\bar v) = \lambda_2(\bar z)h(\bar v,\bar z)h(z_2,z_1)\sum \frac{f(\bar\eta_{\rm I},\bar\eta_{\rm II})g(\bar\xi_{\rm II},\bar\xi_{\rm I})}{h(\bar\eta_{\rm I},z_1)h(\bar\xi_{\rm I},z_1)}\mathbb{B}_{a+1,b+1}(\bar\eta_{\rm II};\bar\xi_{\rm II}). \qquad (2.3.23)$$

Remark 2.3.1. *Taking into account* (2.1.13) *we conclude that if the action* (2.3.12) *is valid on the vector* $\mathbb{B}_{a,b}(\bar u;\bar v)$*, then it is also valid on vectors of the special type* $\mathbb{B}_{a+1,b+1}(\bar u';\bar v')$*, if* $\bar u' \cap \bar v' \neq \emptyset$*.*

Successive action of T_{12} and T_{22}

Using (2.3.12) we obtain

$$T_{12}(z_2)T_{22}(z_1)\mathbb{B}_{a,b}(\bar u;\bar v) = \lambda_2(z_1)h(\bar v,z_1)\sum \frac{f(\bar\eta_{\rm I},\bar\eta_{\rm II})g(\bar\xi_{\rm II},\bar\xi_{\rm I})}{h(\bar\eta_{\rm I},z_1)h(\bar\xi_{\rm I},z_1)}T_{12}(z_2)\mathbb{B}_{a,b}(\bar\eta_{\rm II};\bar\xi_{\rm II}). \qquad (2.3.24)$$

Here $\bar\eta = \{\bar u, z_1\}$ and $\bar\xi = \{\bar v, z_1\}$. Applying (2.1.14) to this formula we find

$$T_{12}(z_2)T_{22}(z_1)\mathbb{B}_{a,b}(\bar u;\bar v) = \lambda_2(\bar z)h(\bar v,z_1)\sum \frac{f(\bar\eta_{\rm I},\bar\eta_{\rm II})g(\bar\xi_{\rm II},\bar\xi_{\rm I})}{h(\bar\eta_{\rm I},z_1)h(\bar\xi_{\rm I},z_1)} \times h(\bar\xi_{\rm II},z_2)\frac{g(\bar\xi_{\rm ii},\bar\xi_{\rm i})}{h(\bar\xi_{\rm i},z_2)}\mathbb{B}_{a+1,b}(\{\bar\eta_{\rm II},z_2\};\bar\xi_{\rm ii}). \qquad (2.3.25)$$

Here we first have partitions $\bar\eta = \{\bar u, z_1\} \Rightarrow \{\bar\eta_{\rm I}, \bar\eta_{\rm II}\}$ and $\bar\xi = \{\bar v, z_1\} \Rightarrow \{\bar\xi_{\rm I}, \bar\xi_{\rm II}\}$. Then we combine $\bar\xi_{\rm II}$ with z_2 and divide this set into new subsets $\{\bar\xi_{\rm II}, z_2\} \Rightarrow \{\bar\xi_{\rm i}, \bar\xi_{\rm ii}\}$. The restrictions are: $\#\bar\xi_{\rm i} = \#\bar\xi_{\rm I} = \#\bar\eta_{\rm I} = 1$, $z_2 \notin \bar\eta_{\rm I}$, and $z_2 \notin \bar\xi_{\rm I}$. As we already did before, we replace $\{\bar\eta_{\rm II}, z_2\}$ with $\bar\eta_{\rm II}$ and use $\bar\xi_{\rm II} = \{\bar\xi_{\rm i}, \bar\xi_{\rm ii}\} \setminus \{z_2\}$. Then

$$T_{12}(z_2)T_{22}(z_1)\mathbb{B}_{a,b}(\bar u;\bar v) = \lambda_2(\bar z)h(\bar v,\bar z)h(z_1,z_2) \times \sum \frac{f(\bar\eta_{\rm I},\bar\eta_{\rm II})g(\bar\xi_{\rm i},\bar\xi_{\rm I})g(\bar\xi_{\rm ii},\bar\xi_{\rm I})g(\bar\xi_{\rm ii},\bar\xi_{\rm i})}{f(\bar\eta_{\rm I},z_2)h(\bar\eta_{\rm I},z_1)h(\bar\xi_{\rm I},z_1)g(z_2,\bar\xi_{\rm I})h(\bar\xi_{\rm I},z_2)h(\bar\xi_{\rm i},z_2)}\,\mathbb{B}_{a+1,b}(\bar\eta_{\rm II};\bar\xi_{\rm ii}). \qquad (2.3.26)$$

Setting $\bar\xi_0 = \{\bar\xi_{\rm i}, \bar\xi_{\rm I}\}$ we recast (2.3.26) as follows:

$$T_{12}(z_2)T_{22}(z_1)\mathbb{B}_{a,b}(\bar u;\bar v) = \lambda_2(\bar z)h(\bar v,\bar z)h(z_1,z_2) \times \sum \frac{f(\bar\eta_{\rm I},\bar\eta_{\rm II})g(\bar\xi_{\rm i},\bar\xi_{\rm I})g(\bar\xi_{\rm ii},\bar\xi_0)}{g(\bar\eta_{\rm I},z_2)h(\bar\eta_{\rm I},\bar z)h(\bar\xi_{\rm I},z_1)g(z_2,\bar\xi_{\rm I})h(\bar\xi_0,z_2)}\,\mathbb{B}_{a+1,b}(\bar\eta_{\rm II};\bar\xi_{\rm ii}). \qquad (2.3.27)$$

The sum over partitions of the set $\bar\xi = \{z_1, z_2, \bar v\}$ is organized as follows: first, we have partitions $\bar\xi \Rightarrow \{\bar\xi_{\rm ii}, \bar\xi_0\}$; second we divide $\bar\xi_0 \Rightarrow \{\bar\xi_{\rm i}, \bar\xi_{\rm I}\}$. The latter sum consists of two terms and can be computed straightforwardly. This leads us to

$$T_{12}(z_2)T_{22}(z_1)\mathbb{B}_{a,b}(\bar u;\bar v) = \lambda_2(\bar z)h(\bar v,\bar z)h(\bar z,\bar z)\sum \frac{f(\bar\eta_{\rm I},\bar\eta_{\rm II})g(\bar\xi_{\rm ii},\bar\xi_0)}{g(\bar\eta_{\rm I},z_2)h(\bar\eta_{\rm I},\bar z)h(\bar\xi_0,\bar z)}\,\mathbb{B}_{a+1,b}(\bar\eta_{\rm II};\bar\xi_{\rm ii}), \tag{2.3.28}$$

and relabeling $\bar\xi_0 \to \bar\xi_{\rm I}$, $\bar\xi_{\rm ii} \to \bar\xi_{\rm II}$ we finally obtain

$$T_{12}(z_2)T_{22}(z_1)\mathbb{B}_{a,b}(\bar u;\bar v) = \lambda_2(\bar z)h(\bar v,\bar z)h(\bar z,\bar z)\sum \frac{f(\bar\eta_{\rm I},\bar\eta_{\rm II})g(\bar\xi_{\rm II},\bar\xi_{\rm I})}{g(\bar\eta_{\rm I},z_2)h(\bar\eta_{\rm I},\bar z)h(\bar\xi_{\rm I},\bar z)}\,\mathbb{B}_{a+1,b}(\bar\eta_{\rm II};\bar\xi_{\rm II}). \tag{2.3.29}$$

Here the sum is taken over partitions $\bar\eta \Rightarrow \{\bar\eta_{\rm I},\bar\eta_{\rm II}\}$, $\bar\xi \Rightarrow \{\bar\xi_{\rm I},\bar\xi_{\rm II}\}$. The cardinalities of the subsets are $\#\bar\eta_{\rm I} = 1$, $\#\bar\xi_{\rm I} = 2$.

Successive action of T_{22} and T_{12}

Using (2.3.14) and (2.3.29) we are able to calculate the action of $T_{22}(z_1)T_{12}(z_2)$ onto $\mathbb{B}_{a,b}(\bar u;\bar v)$. It is clear that for this we should take the following combination: equation (2.3.29) multiplied with $f(z_1,z_2)$ and the same equation with $z_1 \leftrightarrow z_2$ multiplied with $g(z_2,z_1)$. This straightforward calculation gives

$$T_{22}(z_1)T_{12}(z_2)\mathbb{B}_{a,b}(\bar u;\bar v) = \lambda_2(\bar z)h(\bar v,\bar z)h(\bar z,\bar z)\sum \frac{f(\bar\eta_{\rm I},\bar\eta_{\rm II})g(\bar\xi_{\rm II},\bar\xi_{\rm I})}{h(\bar\eta_{\rm I},z_1)h(\bar\xi_{\rm I},\bar z)}\,\mathbb{B}_{a+1,b}(\bar\eta_{\rm II};\bar\xi_{\rm II}). \tag{2.3.30}$$

Here the sum is taken over partitions $\bar\eta \Rightarrow \{\bar\eta_{\rm I},\bar\eta_{\rm II}\}$, $\bar\xi \Rightarrow \{\bar\xi_{\rm I},\bar\xi_{\rm II}\}$. The cardinalities of the subsets are $\#\bar\eta_{\rm I} = 1$, $\#\bar\xi_{\rm I} = 2$.

2.3.3 Recursion formula

Now everything is ready for the use of recursion (2.3.13). Due to (2.2.4) we can write it as follows:

$$\mathbb{B}_{a+1,b}(\{\bar u,z_2\};\bar v) = \frac{1}{\lambda_2(z_2)f(\bar v,z_2)}\Big(T_{12}(z_2)\,\mathbb{B}_{a,b}(\bar u;\bar v) - \Psi\Big), \tag{2.3.31}$$

where

$$\Psi = \lambda_2(z_2)h(\bar v,z_2)\sum_{z_2\notin\bar\xi_{\rm I}} \frac{g(\bar\xi_{\rm II},\bar\xi_{\rm I})}{h(\bar\xi_{\rm I},z_2)}\,\mathbb{B}_{a+1,b}(\bar\eta;\bar\xi_{\rm II}). \tag{2.3.32}$$

Here $\bar\eta = \{z_2,\bar u\}$ and $\bar\xi = \{z_2,\bar v\}$, and we used $h(z_2,z_2) = 1$. The sum is taken over partitions $\bar\xi \Rightarrow \{\bar\xi_{\rm I},\bar\xi_{\rm II}\}$ with $\#\bar\xi_{\rm I} = 1$. One more restriction $z_2 \notin \bar\xi_{\rm I}$ is shown explicitly by the subscript of the sum.

Recall that we assume that the action of $T_{22}(z_1)$ on the vectors $\mathbb{B}_{a,b}(\bar u;\bar v)$ is given by (2.3.12) at some value of $a \ge 0$ and arbitrary b. All the vectors in the linear combination (2.3.32) have the form $\mathbb{B}_{a+1,b}(\{\bar u,z_2\};\{\bar v_j,z_2\})$, that is $\{\bar u,z_2\}\cap\{\bar v_j,z_2\} \ne \emptyset$. Hence, taking into account remark 2.3.1, the action of $T_{22}(z_1)$ on these vectors is known and it is given by (2.3.12):

$$T_{22}(z_1)\Psi = \lambda_2(\bar z)h(\bar v,z_2)\sum_{z_2\notin\bar\xi_{\rm I}} \frac{g(\bar\xi_{\rm II},\bar\xi_{\rm I})}{h(\bar\xi_{\rm I},z_2)}h(\bar\xi_{\rm II},z_1)\frac{f(\bar\eta_{\rm I},\bar\eta_{\rm II})g(\bar\xi_{\rm ii},\bar\xi_{\rm i})}{h(\bar\eta_{\rm I},z_1)h(\bar\xi_{\rm i},z_1)}\mathbb{B}_{a+1,b}(\bar\eta_{\rm II};\bar\xi_{\rm ii}). \tag{2.3.33}$$

In this formula $\bar\eta = \{\bar z, \bar u\}$ and $\bar\xi = \{\bar z, \bar v\}$. At the first step we have partitions $\{z_2, \bar v\} \Rightarrow \{\bar\xi_{\rm I}, \bar\xi_{\rm II}\}$. Then we obtain additional partitions $\{\bar z, \bar v\} \Rightarrow \{\bar\xi_{\rm i}, \bar\xi_{\rm ii}\}$ and $\bar\eta \Rightarrow \{\bar\eta_{\rm I}, \bar\eta_{\rm II}\}$. Hereby $\#\bar\eta_{\rm I} = \#\bar\xi_{\rm I} = \#\bar\xi_{\rm i} = 1$. Thus, one can say that the set $\bar\xi = \{\bar z, \bar v\}$ is divided into subsets $\{\bar\xi_{\rm I}, \bar\xi_{\rm i}, \bar\xi_{\rm ii}\}$ with the restrictions $z_1 \notin \bar\xi_{\rm I}$ and $z_2 \notin \bar\xi_{\rm I}$. Substituting $\bar\xi_{\rm II} = \{\bar\xi_{\rm i}, \bar\xi_{\rm ii}\} \setminus \{z_1\}$ into (2.3.33) we obtain

$$T_{22}(z_1)\Psi = \lambda_2(\bar z)h(\bar v, z_2) \sum_{z_2\notin\bar\xi_{\rm I}} \frac{g(\bar\xi_{\rm ii}, \bar\xi_{\rm I})g(\bar\xi_{\rm i}, \bar\xi_{\rm I})g(\bar\xi_{\rm ii}, \bar\xi_{\rm i})f(\bar\eta_{\rm I}, \bar\eta_{\rm II})h(\bar\xi_{\rm ii}, z_1)}{g(z_1, \bar\xi_{\rm I})h(\bar\xi_{\rm I}, z_2)h(\bar\eta_{\rm I}, z_1)} \mathbb{B}_{a+1,b}(\bar\eta_{\rm II}; \bar\xi_{\rm ii}). \quad (2.3.34)$$

Observe that the restriction $z_1 \notin \bar\xi_{\rm I}$ holds automatically due to the factor $g(z_1, \bar\xi_{\rm I})^{-1}$. In order to get rid of the restriction $z_2 \notin \bar\xi_{\rm I}$ we present $T_{22}(z_1)\Psi$ as a difference of two terms. The first term is just the sum over partitions in (2.3.34), where no restrictions on the partitions of the set $\bar\xi$ are imposed. In the second term we simply set $\bar\xi_{\rm I} = z_2$. Thus,

$$T_{22}(z_1)\Psi = \Psi' - \Psi'', \quad (2.3.35)$$

where

$$\Psi' = \lambda_2(\bar z)h(\bar v, z_2) \sum \frac{g(\bar\xi_{\rm ii}, \bar\xi_{\rm I})g(\bar\xi_{\rm i}, \bar\xi_{\rm I})g(\bar\xi_{\rm ii}, \bar\xi_{\rm i})f(\bar\eta_{\rm I}, \bar\eta_{\rm II})h(\bar\xi_{\rm ii}, z_1)}{g(z_1, \bar\xi_{\rm I})h(\bar\xi_{\rm I}, z_2)h(\bar\eta_{\rm I}, z_1)} \mathbb{B}_{a+1,b}(\bar\eta_{\rm II}; \bar\xi_{\rm ii}), \quad (2.3.36)$$

and

$$\Psi'' = \lambda_2(\bar z)h(\bar v, z_2) \sum \frac{g(\bar\xi_{\rm ii}, z_2)g(\bar\xi_{\rm i}, z_2)g(\bar\xi_{\rm ii}, \bar\xi_{\rm i})f(\bar\eta_{\rm I}, \bar\eta_{\rm II})h(\bar\xi_{\rm ii}, z_1)}{g(z_1, z_2)h(\bar\eta_{\rm I}, z_1)} \mathbb{B}_{a+1,b}(\bar\eta_{\rm II}; \bar\xi_{\rm ii}). \quad (2.3.37)$$

In (2.3.37) we have $\{\bar\xi_{\rm i}, \bar\xi_{\rm ii}\} = \{\bar v, z_1\}$, therefore

$$\Psi'' = \lambda_2(\bar z)h(\bar v, z_1)f(\bar v, z_2) \sum \frac{g(\bar\xi_{\rm ii}, \bar\xi_{\rm i})f(\bar\eta_{\rm I}, \bar\eta_{\rm II})}{h(\bar\xi_{\rm i}, z_1)h(\bar\eta_{\rm I}, z_1)} \mathbb{B}_{a+1,b}(\bar\eta_{\rm II}; \bar\xi_{\rm ii}). \quad (2.3.38)$$

In (2.3.36) we can take the sum over partitions into subsets $\bar\xi_{\rm i}$ and $\bar\xi_{\rm I}$, because it consists of two terms only:

$$\frac{g(\bar\xi_{\rm i}, \bar\xi_{\rm I})}{g(z_1, \bar\xi_{\rm I})h(\bar\xi_{\rm I}, z_2)} + \frac{g(\bar\xi_{\rm I}, \bar\xi_{\rm i})}{g(z_1, \bar\xi_{\rm i})h(\bar\xi_{\rm i}, z_2)} = \frac{h(z_1, z_2)}{h(\bar\xi_0, z_2)}, \quad (2.3.39)$$

where $\bar\xi_0 = \{\bar\xi_{\rm i}, \bar\xi_{\rm I}\}$. Thus,

$$\Psi' = \lambda_2(\bar z)h(\bar v, z_2)h(z_1, z_2) \sum \frac{g(\bar\xi_{\rm ii}, \bar\xi_0)f(\bar\eta_{\rm I}, \bar\eta_{\rm II})h(\bar\xi_{\rm ii}, z_1)}{h(\bar\xi_0, z_2)h(\bar\eta_{\rm I}, z_1)} \mathbb{B}_{a+1,b}(\bar\eta_{\rm II}; \bar\xi_{\rm ii}), \quad (2.3.40)$$

and extracting the product $h(\bar\xi, z_1)$ we recast (2.3.40) as follows:

$$\Psi' = \lambda_2(\bar z)h(\bar v, \bar z)h(\bar z, \bar z) \sum \frac{g(\bar\xi_{\rm ii}, \bar\xi_0)f(\bar\eta_{\rm I}, \bar\eta_{\rm II})}{h(\bar\xi_0, \bar z)h(\bar\eta_{\rm I}, z_1)} \mathbb{B}_{a+1,b}(\bar\eta_{\rm II}; \bar\xi_{\rm ii}). \quad (2.3.41)$$

Here the sum is taken over partitions $\bar\eta \Rightarrow \{\bar\eta_{\rm I}, \bar\eta_{\rm II}\}$, $\bar\xi \Rightarrow \{\bar\xi_0, \bar\xi_{\rm ii}\}$ with $\#\bar\eta_{\rm I} = 1$, $\#\bar\xi_0 = 2$. Comparing this expression with (2.3.30) we see that

$$\Psi' = T_{22}(z_1)T_{12}(z_2)\mathbb{B}_{a,b}(\bar u; \bar v). \quad (2.3.42)$$

Thus, we find from the recursion (2.3.31)

$$T_{22}(z_1)\mathbb{B}_{a+1,b}(\{\bar u, z_2\}; \bar v) = \frac{T_{22}(z_1)T_{12}(z_2)\,\mathbb{B}_{a,b}(\bar u; \bar v) - \Psi' + \Psi''}{\lambda_2(z_2)f(\bar v, z_2)} = \frac{\Psi''}{\lambda_2(z_2)f(\bar v, z_2)}. \quad (2.3.43)$$

Substituting (2.3.38) in this expression, we arrive at

$$T_{22}(z_1)\mathbb{B}_{a+1,b}(\{\bar{u},z_2\};\bar{v}) = \lambda_2(z_1)h(\bar{v},z_1)\sum \frac{g(\bar{\xi}_{\rm II},\bar{\xi}_{\rm I})f(\bar{\eta}_{\rm I},\bar{\eta}_{\rm II})}{h(\bar{\xi}_{\rm I},z_1)h(\bar{\eta}_{\rm I},z_1)}\,\mathbb{B}_{a+1,b}(\bar{\eta}_{\rm II};\bar{\xi}_{\rm II}), \tag{2.3.44}$$

where we have relabeled $\bar{\xi}_{\rm i}\to\bar{\xi}_{\rm I}$ and $\bar{\xi}_{\rm ii}\to\bar{\xi}_{\rm II}$. Thus, the induction step is completed. □

2.3.4 Induction over n

Actually, the induction over n for the action of $T_{22}(\bar{z})$ is a combination of the corresponding proofs for the actions of $T_{12}(\bar{z})$ and $T_{23}(\bar{z})$. Assume that (2.1.17) is valid for some $n-1$. Then

$$T_{22}(\bar{z})\mathbb{B}_{a,b}(\bar{u};\bar{v}) = (-1)^{n-1}\lambda_2(\bar{z}_n)h(\bar{\xi},\bar{z}_n)\sum \frac{f(\bar{\eta}_{\rm I},\bar{\eta}_{\rm II})g(\bar{\xi}_{\rm II},\bar{\xi}_{\rm I})}{h(\bar{\xi}_{\rm I},\bar{z}_n)}K_{n-1}(\bar{z}_n|\bar{\eta}_{\rm I}+c)T_{22}(z_n)\mathbb{B}_{a,b}(\bar{\eta}_{\rm II};\bar{\xi}_{\rm II}). \tag{2.3.45}$$

Here the sum is taken over partitions $\{\bar{z}_n,\bar{v}\}=\bar{\xi}\Rightarrow\{\bar{\xi}_{\rm I},\bar{\xi}_{\rm II}\}$ and $\{\bar{z}_n,\bar{u}\}=\bar{\eta}\Rightarrow\{\bar{\eta}_{\rm I},\bar{\eta}_{\rm II}\}$ with $\#\bar{\xi}_{\rm I}=\#\bar{\eta}_{\rm I}=n-1$. Acting with $T_{22}(z_n)$ onto $\mathbb{B}_{a,b}(\bar{\eta}_{\rm II};\bar{\xi}_{\rm II})$ we obtain

$$\begin{aligned}T_{22}(\bar{z})\mathbb{B}_{a,b}(\bar{u};\bar{v}) = (-1)^{n}\lambda_2(\bar{z})\frac{h(\bar{\xi},\bar{z}_n)}{h(z_n,\bar{z}_n)}\sum \frac{f(\bar{\eta}_{\rm I},\bar{\eta}_{\rm II})g(\bar{\xi}_{\rm II},\bar{\xi}_{\rm I})}{h(\bar{\xi}_{\rm I},\bar{z}_n)}K_{n-1}(\bar{z}_n|\bar{\eta}_{\rm I}+c)\\ \times h(\bar{\xi}_{\rm II},z_n)\frac{f(\bar{\eta}_{\rm i},\bar{\eta}_{\rm ii})g(\bar{\xi}_{\rm ii},\bar{\xi}_{\rm i})}{h(\bar{\xi}_{\rm i},z_n)}K_1(z_n|\bar{\eta}_{\rm i}+c)\mathbb{B}_{a,b}(\bar{\eta}_{\rm ii};\bar{\xi}_{\rm ii}).\end{aligned} \tag{2.3.46}$$

Here already $\bar{\xi}=\{\bar{z},\bar{v}\}$ and $\bar{\eta}=\{\bar{z},\bar{u}\}$, and we have additional partitions $\{\bar{\xi}_{\rm II},z_n\}\Rightarrow\{\bar{\xi}_{\rm i},\bar{\xi}_{\rm ii}\}$ and $\{\bar{\eta}_{\rm II},z_n\}\Rightarrow\{\bar{\eta}_{\rm i},\bar{\eta}_{\rm ii}\}$ with $\#\bar{\xi}_{\rm i}=\#\bar{\eta}_{\rm i}=1$. Thus, we can say that we have the sum over partitions $\bar{\xi}\Rightarrow\{\bar{\xi}_{\rm I},\bar{\xi}_{\rm i},\bar{\xi}_{\rm ii}\}$ and $\bar{\eta}\Rightarrow\{\bar{\eta}_{\rm I},\bar{\eta}_{\rm i},\bar{\eta}_{\rm ii}\}$ with restrictions $z_n\notin\bar{\eta}_{\rm I}$ and $z_n\notin\bar{\xi}_{\rm I}$.

Substituting $\bar{\xi}_{\rm II}=\{\bar{\xi}_{\rm i},\bar{\xi}_{\rm ii}\}\setminus\{z_n\}$, $\bar{\eta}_{\rm II}=\{\bar{\eta}_{\rm i},\bar{\eta}_{\rm ii}\}\setminus\{z_n\}$ and denoting $\bar{\xi}_0=\{\bar{\xi}_{\rm i},\bar{\xi}_{\rm I}\}$, $\bar{\eta}_0=\{\bar{\eta}_{\rm i},\bar{\eta}_{\rm I}\}$ we obtain

$$\begin{aligned}T_{22}(\bar{z})\mathbb{B}_{a,b}(\bar{u};\bar{v}) = (-1)^{n}\lambda_2(\bar{z})\frac{h(\bar{\xi},\bar{z})}{h(z_n,\bar{z}_n)}\sum \frac{f(\bar{\eta}_{\rm I},\bar{\eta}_{\rm i})f(\bar{\eta}_0,\bar{\eta}_{\rm ii})g(\bar{\xi}_{\rm i},\bar{\xi}_{\rm I})g(\bar{\xi}_{\rm ii},\bar{\xi}_0)}{f(\bar{\eta}_{\rm I},z_n)g(z_n,\bar{\xi}_{\rm I})h(\bar{\xi}_{\rm I},\bar{z}_n)h(\bar{\xi}_0,z_n)}\\ \times K_{n-1}(\bar{z}_n|\bar{\eta}_{\rm I}+c)K_1(z_n|\bar{\eta}_{\rm i}+c)\mathbb{B}_{a,b}(\bar{\eta}_{\rm ii};\bar{\xi}_{\rm ii}).\end{aligned} \tag{2.3.47}$$

Observe that the restrictions $z_n\notin\bar{\eta}_{\rm I}$ and $z_n\notin\bar{\xi}_{\rm I}$ hold automatically due to the factors $f(\bar{\eta}_{\rm I},z_n)$ and $g(z_n,\bar{\xi}_{\rm I})$ in the denominator of (2.3.47). Using $K_1(z_n|\bar{\eta}_{\rm i}+c)=-K_1(\bar{\eta}_{\rm i}|z_n)/f(\bar{\eta}_{\rm i},z_n)$ we recast (2.3.47) in the form

$$\begin{aligned}T_{22}(\bar{z})\mathbb{B}_{a,b}(\bar{u};\bar{v}) = (-1)^{n-1}\lambda_2(\bar{z})\frac{h(\bar{\xi},\bar{z})}{h(z_n,\bar{z}_n)}\sum \frac{f(\bar{\eta}_0,\bar{\eta}_{\rm ii})g(\bar{\xi}_{\rm ii},\bar{\xi}_0)}{f(\bar{\eta}_0,z_n)g(z_n,\bar{\xi}_0)h(\bar{\xi}_0,\bar{z})}\\ \times\Big\{g(\bar{\xi}_{\rm i},\bar{\xi}_{\rm I})g(z_n,\bar{\xi}_{\rm i})h(\bar{\xi}_{\rm i},\bar{z}_n)\Big\}\Big\{K_{n-1}(\bar{z}_n|\bar{\eta}_{\rm I}+c)K_1(\bar{\eta}_{\rm i}|z_n)f(\bar{\eta}_{\rm I},\bar{\eta}_{\rm i})\Big\}\,\mathbb{B}_{a,b}(\bar{\eta}_{\rm ii};\bar{\xi}_{\rm ii}).\end{aligned} \tag{2.3.48}$$

The sums over partitions $\bar{\xi}_0\Rightarrow\{\bar{\xi}_{\rm i},\bar{\xi}_{\rm I}\}$ and $\bar{\eta}_0\Rightarrow\{\bar{\eta}_{\rm i},\bar{\eta}_{\rm I}\}$ (see the terms in braces) were already computed (see (2.2.11) and (2.2.17)). Thus, we arrive at

$$T_{22}(\bar{z})\mathbb{B}_{a,b}(\bar{u};\bar{v}) = (-1)^{n}\lambda_2(\bar{z})h(\bar{\xi},\bar{z})\sum \frac{f(\bar{\eta}_0,\bar{\eta}_{\rm ii})g(\bar{\xi}_{\rm ii},\bar{\xi}_0)}{h(\bar{\xi}_0,\bar{z})}K_n(\bar{z}|\bar{\eta}_0+c)\,\mathbb{B}_{a,b}(\bar{\eta}_{\rm ii};\bar{\xi}_{\rm ii}), \tag{2.3.49}$$

which ends the proof. □

2.4 Induction over n for the actions of $T_{ij}(\bar{z})$ with $i > j$.

The action formulas for all other elements of the monodromy matrix can be proved exactly in the same manner. However, it is clear that the technical difficulty of the proofs increases when moving from the right top corner of the monodromy matrix to the left bottom corner. It is due to the form of the recursion formulas and the commutation relations (1.1.14). For example, we have seen that for the derivation of the action of $T_{22}(\bar{z})$ one should know the actions of $T_{12}(\bar{z})$ and $T_{23}(\bar{z})$ onto Bethe vectors. The latest are relatively simple. However, one can easily convince oneself that to get the action of $T_{ij}(\bar{z})$ with $i > j$, it is necessary to know the actions of the diagonal elements $T_{ii}(\bar{z})$, which are more involved. Therefore, we omit the detailed proofs of the multiple actions of the operators $T_{11}(\bar{z})$ (2.1.16), $T_{33}(\bar{z})$ (2.1.18), and the operators $T_{ij}(\bar{z})$ from the lower-triangular part of the monodromy matrix (2.1.19)–(2.1.21). However, as an illustration of the method, we prove the multiple action of the operator $T_{21}(\bar{z})$ assuming that the action of a single operator $T_{21}(z)$ is known.

As previously, the proof goes by induction over $n = \#\bar{z}$. We assume that the action (2.1.19) holds for some $n-1$. Then acting successively with $T_{21}(\bar{z}_n)$ and $T_{21}(z_n)$ on $\mathbb{B}_{a,b}(\bar{u};\bar{v})$ we obtain

$$T_{21}(\bar{z})\mathbb{B}_{a,b}(\bar{u};\bar{v}) = \lambda_2(\bar{z})\frac{h(\bar{\xi},\bar{z}_n)}{h(z_n,\bar{z}_n)}\sum r_1(\bar{\eta}_{\rm I})\frac{f(\bar{\eta}_{\rm II},\bar{\eta}_{\rm I})f(\bar{\eta}_{\rm II},\bar{\eta}_{\rm III})f(\bar{\eta}_{\rm III},\bar{\eta}_{\rm I})g(\bar{\xi}_{\rm II},\bar{\xi}_{\rm I})}{h(\bar{\xi}_{\rm I},\bar{z}_n)f(\bar{\xi}_{\rm II},\bar{\eta}_{\rm I})}$$
$$\times K_{n-1}(\bar{z}_n|\bar{\eta}_{\rm II}+c)K_{n-1}(\bar{\eta}_{\rm I}|\bar{\xi}_{\rm I}+c)\; h(\bar{\xi}_{\rm II},z_n)r_1(\bar{\eta}_{\rm i})\frac{f(\bar{\eta}_{\rm ii},\bar{\eta}_{\rm i})f(\bar{\eta}_{\rm ii},\bar{\eta}_{\rm iii})f(\bar{\eta}_{\rm iii},\bar{\eta}_{\rm i})g(\bar{\xi}_{\rm ii},\bar{\xi}_{\rm i})}{h(\bar{\xi}_{\rm i},z_n)f(\bar{\xi}_{\rm ii},\bar{\eta}_{\rm i})}$$
$$\times K_1(z_n|\bar{\eta}_{\rm ii}+c)K_1(\bar{\eta}_{\rm i}|\bar{\xi}_{\rm i}+c)\mathbb{B}_{a-n,b}(\bar{\eta}_{\rm iii};\bar{\xi}_{\rm ii}). \quad (2.4.1)$$

Here $\#\bar{\eta}_{\rm i} = \#\bar{\eta}_{\rm ii} = \#\bar{\xi}_{\rm i} = 1$ and $\#\bar{\eta}_{\rm I} = \#\bar{\eta}_{\rm II} = \#\bar{\xi}_{\rm I} = n-1$. Originally we have partitions $\{\bar{z}_n,\bar{u}\} = \bar{\eta} \Rightarrow \{\bar{\eta}_{\rm I},\bar{\eta}_{\rm II},\bar{\eta}_{\rm III}\}$ and $\{\bar{z}_n,\bar{v}\} = \bar{\xi} \Rightarrow \{\bar{\xi}_{\rm I},\bar{\xi}_{\rm II}\}$. Then we have additional partitions $\{z_n,\bar{\eta}_{\rm III}\} \Rightarrow \{\bar{\eta}_{\rm i},\bar{\eta}_{\rm ii},\bar{\eta}_{\rm iii}\}$ and $\{z_n,\bar{\xi}_{\rm II}\} \Rightarrow \{\bar{\xi}_{\rm i},\bar{\xi}_{\rm ii}\}$. Thus, in equation (2.4.1) we have $\{\bar{z},\bar{u}\} = \bar{\eta}$ and $\{\bar{z},\bar{v}\} = \bar{\xi}$. Setting there $\bar{\eta}_{\rm III} = \{\bar{\eta}_{\rm i},\bar{\eta}_{\rm ii},\bar{\eta}_{\rm iii}\}\setminus\{z_n\}$ and $\bar{\xi}_{\rm II} = \{\bar{\xi}_{\rm i},\bar{\xi}_{\rm ii}\}\setminus\{z_n\}$ we arrive at

$$T_{21}(\bar{z})\mathbb{B}_{a,b}(\bar{u};\bar{v}) = \lambda_2(\bar{z})\frac{h(\bar{\xi},\bar{z})}{h(z_n,\bar{z}_n)}\sum r_1(\bar{\eta}_{\rm I})r_1(\bar{\eta}_{\rm i})\frac{f(\bar{\eta}_{\rm ii},\bar{\eta}_{\rm i})f(\bar{\eta}_{\rm ii},\bar{\eta}_{\rm iii})f(\bar{\eta}_{\rm iii},\bar{\eta}_{\rm i})g(\bar{\xi}_{\rm ii},\bar{\xi}_{\rm i})}{h(\bar{\xi}_{\rm i},z_n)f(\bar{\xi}_{\rm ii},\bar{\eta}_{\rm i})}$$
$$\times\frac{f(\bar{\eta}_{\rm II},\bar{\eta}_{\rm I})f(\bar{\eta}_{\rm II},\bar{\eta}_{\rm i})f(\bar{\eta}_{\rm II},\bar{\eta}_{\rm ii})f(\bar{\eta}_{\rm II},\bar{\eta}_{\rm iii})f(\bar{\eta}_{\rm i},\bar{\eta}_{\rm I})f(\bar{\eta}_{\rm ii},\bar{\eta}_{\rm I})f(\bar{\eta}_{\rm iii},\bar{\eta}_{\rm I})g(\bar{\xi}_{\rm i},\bar{\xi}_{\rm I})g(\bar{\xi}_{\rm ii},\bar{\xi}_{\rm I})}{h(\bar{\xi}_{\rm I},z_n)h(\bar{\xi}_{\rm I},\bar{z}_n)f(\bar{\xi}_{\rm i},\bar{\eta}_{\rm I})f(\bar{\xi}_{\rm ii},\bar{\eta}_{\rm I})f(\bar{\eta}_{\rm II},z_n)g(z_n,\bar{\xi}_{\rm I})}$$
$$\times K_{n-1}(\bar{z}_n|\bar{\eta}_{\rm II}+c)K_1(z_n|\bar{\eta}_{\rm ii}+c)K_{n-1}(\bar{\eta}_{\rm I}|\bar{\xi}_{\rm I}+c)K_1(\bar{\eta}_{\rm i}|\bar{\xi}_{\rm i}+c)\mathbb{B}_{a-n,b}(\bar{\eta}_{\rm iii};\bar{\xi}_{\rm ii}). \quad (2.4.2)$$

Now we set $\{\bar{\eta}_{\rm I},\bar{\eta}_{\rm i}\} = \bar{\eta}_0$, $\{\bar{\eta}_{\rm II},\bar{\eta}_{\rm ii}\} = \bar{\eta}_{0'}$, and $\{\bar{\xi}_{\rm I},\bar{\xi}_{\rm i}\} = \bar{\xi}_0$. We also transform $K_1(z_n|\bar{\eta}_{\rm ii}+c) = -K_1(\bar{\eta}_{\rm ii}|z_n)/f(\bar{\eta}_{\rm ii},z_n)$ and $K_1(\bar{\eta}_{\rm i}|\bar{\xi}_{\rm i}+c) = -K_1(\bar{\xi}_{\rm i}|\bar{\eta}_{\rm i})/f(\bar{\xi}_{\rm i},\bar{\eta}_{\rm i})$. Then (2.4.2) takes the form

$$T_{21}(\bar{z})\mathbb{B}_{a,b}(\bar{u};\bar{v}) = \lambda_2(\bar{z})\frac{h(\bar{\xi},\bar{z})}{h(z_n,\bar{z}_n)}\sum\frac{r_1(\bar{\eta}_0)f(\bar{\eta}_{0'},\bar{\eta}_0)f(\bar{\eta}_{0'},\bar{\eta}_{\rm iii})f(\bar{\eta}_{\rm iii},\bar{\eta}_0)g(\bar{\xi}_{\rm ii},\bar{\xi}_0)g(\bar{\xi}_{\rm i},\bar{\xi}_{\rm I})}{h(\bar{\xi}_{\rm I},\bar{z})f(\bar{\xi}_{\rm i},\bar{\eta}_0)f(\bar{\eta}_{0'},z_n)f(\bar{\xi}_{\rm ii},\bar{\eta}_0)g(z_n,\bar{\xi}_{\rm I})h(\bar{\xi}_{\rm i},z_n)}$$
$$\times\left\{K_{n-1}(\bar{z}_n|\bar{\eta}_{\rm II}+c)K_1(\bar{\eta}_{\rm ii}|z_n)f(\bar{\eta}_{\rm II},\bar{\eta}_{\rm ii})\right\}\left\{K_{n-1}(\bar{\eta}_{\rm I}|\bar{\xi}_{\rm I}+c)K_1(\bar{\xi}_{\rm i}|\bar{\eta}_{\rm i})f(\bar{\eta}_{\rm i},\bar{\eta}_{\rm I})\right\}\mathbb{B}_{a-n,b}(\bar{\eta}_{\rm iii};\bar{\xi}_{\rm ii}). \quad (2.4.3)$$

The sums over partitions in braces can be computed via (9.3.17):

$$\sum_{\bar{\eta}_{0'}\Rightarrow\{\bar{\eta}_{\rm II},\bar{\eta}_{\rm ii}\}} K_{n-1}(\bar{z}_n|\bar{\eta}_{\rm II}+c)K_1(\bar{\eta}_{\rm ii}|z_n)f(\bar{\eta}_{\rm II},\bar{\eta}_{\rm ii}) = -f(\bar{\eta}_{0'},z_n)K_n(\bar{z}|\bar{\eta}_{0'}+c), \quad (2.4.4)$$

and

$$\sum_{\bar\eta_0\Rightarrow\{\bar\eta_{\rm I},\bar\eta_{\rm i}\}} K_{n-1}(\bar\eta_{\rm I}|\bar\xi_{\rm I}+c)K_1(\bar\xi_{\rm i}|\bar\eta_{\rm i})f(\bar\eta_{\rm i},\bar\eta_{\rm I}) = (-1)^{n-1}\frac{K_n(\bar\xi_0|\bar\eta_0)}{f(\bar\xi_{\rm I},\bar\eta_0)}. \tag{2.4.5}$$

Substituting this into (2.4.3) we arrive at

$$T_{21}(\bar z)\mathbb{B}_{a,b}(\bar u;\bar v) = \lambda_2(\bar z)\frac{h(\bar\xi,\bar z)}{h(z_n,\bar z_n)}\sum\frac{r_1(\bar\eta_0)f(\bar\eta_{0'},\bar\eta_0)f(\bar\eta_{0'},\bar\eta_{\rm iii})f(\bar\eta_{\rm iii},\bar\eta_0)g(\bar\xi_{\rm ii},\bar\xi_0)}{h(\bar\xi_0,\bar z)f(\bar\xi_{\rm ii},\bar\eta_0)g(z_n,\bar\xi_0)}$$

$$\times K_n(\bar z|\bar\eta_{0'}+c)K_n(\bar\eta_0|\bar\xi_0+c)\Big\{g(\bar\xi_{\rm i},\bar\xi_{\rm I})h(\bar\xi_{\rm i},\bar z_n)g(z_n,\bar\xi_{\rm i})\Big\}\mathbb{B}_{a-n,b}(\bar\eta_{\rm iii};\bar\xi_{\rm ii}), \tag{2.4.6}$$

where we again replaced $K_n(\bar\xi_0|\bar\eta_0)$ by $K_n(\bar\eta_0|\bar\xi_0+c)$ via (1.1.45). The sum over partitions in braces can be calculated via lemma 9.3.1

$$\sum_{\bar\xi_0\Rightarrow\{\bar\xi_{\rm I},\bar\xi_{\rm i}\}} g(\bar\xi_{\rm i},\bar\xi_{\rm I})h(\bar\xi_{\rm i},\bar z_n)g(z_n,\bar\xi_{\rm i}) = h(z_n,\bar z_n)g(z_n,\bar\xi_0). \tag{2.4.7}$$

Thus, we finally obtain

$$T_{21}(\bar z)\mathbb{B}_{a,b}(\bar u;\bar v) = \lambda_2(\bar z)h(\bar\xi,\bar z)\sum\frac{r_1(\bar\eta_0)f(\bar\eta_{0'},\bar\eta_0)f(\bar\eta_{0'},\bar\eta_{\rm iii})f(\bar\eta_{\rm iii},\bar\eta_0)g(\bar\xi_{\rm ii},\bar\xi_0)}{h(\bar\xi_0,\bar z)f(\bar\xi_{\rm ii},\bar\eta_0)}$$

$$\times K_n(\bar z|\bar\eta_{0'}+c)K_n(\bar\eta_0|\bar\xi_0+c)\mathbb{B}_{a-n,b}(\bar\eta_{\rm iii};\bar\xi_{\rm ii}), \tag{2.4.8}$$

which coincides with the original formula up to the labeling of the subsets. □

2.4.1 Conclusion

The result of this chapter is explicit multiple action formulae for the monodromy matrix entries T_{ij} on the generic (off-shell) Bethe vectors. Comparison of these multiple action formulae with algebra rank symmetry $\mathfrak{gl}(3)$ case [84] demonstrates, that formulae are similar. The main difference is that replacement of Izergin determinants $K_n(\bar u|\bar v)$ by functions $g(\bar u,\bar v)$ (that is itself determinant of Cauchy matrix). This property provides some simplification of formulae and give a hope that scalar product of tho Bethe vectors in case of graded algebra symmetry is simpler than in algebra symmetry $\mathfrak{gl}(3)$ case and determinant representation can also be found. Computation of this scalar product is further step, considered in next chapter. Formulae, established here, is a necessary step for this calculation.

Chapter 3

Scalar product of the Bethe vectors

As it is described in the previous chapter (dual) Bethe vectors are special form polynomials on the entries from (lower)upper triangular part of the monodromy matrix applied to the (left) right pseudovacuum. Here the entries from lower triangle part (T_{ij}, $i > j$) acts trivially on the right vacuum while the entries from upper triangle part (T_{ij}, $i < j$) act trivially on the left vacuum. The diagonal operators actions on both vacuuma are known. Hereby, for calculation of the scalar product of the usual and the dual Bethe vectors it is required to perform the normal ordering of operators. The multiple action rules is very convenient for this purpose, because as was already mentioned direct application of the commutation relations (1.1.14) can be bulky problem. Instead, action rules allow to act by monodromy matrix entries T_{ij} with $i > j$ from the left (dual) Bethe vector on the right Bethe vector and application of these formulae provides much more efficient way of normal ordering.

The result should be a function that does not contain any operators. Here this function is calculated and a formula for scalar product derived. This chapter is based on the paper [88] published by the thesis author in collaboration.

3.1 Generic form of scalar product of Bethe vectors

The scalar product of Bethe vectors is defined as

$$S_{a,b} \equiv S_{a,b}(\bar{u}^C;\bar{v}^C|\bar{u}^B;\bar{v}^B) = \mathbb{C}_{a,b}(\bar{u}^C;\bar{v}^C)\mathbb{B}_{a,b}(\bar{u}^B;\bar{v}^B)\,, \tag{3.1.1}$$

where all the Bethe parameters are generic complex numbers. We have added the superscripts C and B to the sets $\bar{u}$, $\bar{v}$ in order to stress that the vectors $\mathbb{C}_{a,b}$ and $\mathbb{B}_{a,b}$ may depend on different sets of parameters.

Being a scalar function, the scalar product is invariant under the action of the antimorphism ψ (1.1.47)

$$\psi\Big(S_{a,b}(\bar{u}^C;\bar{v}^C|\bar{u}^B;\bar{v}^B)\Big) = S_{a,b}(\bar{u}^C;\bar{v}^C|\bar{u}^B;\bar{v}^B). \tag{3.1.2}$$

On the other hand, acting with ψ on the r.h.s. of (3.1.1) and using the explicit representations (2.1.1) and (2.1.3) for the Bethe vectors we find

$$\psi\Big(\mathbb{C}_{a,b}(\bar{u}^C;\bar{v}^C)\mathbb{B}_{a,b}(\bar{u}^B;\bar{v}^B)\Big) = \mathbb{C}_{a,b}(\bar{u}^B;\bar{v}^B)\mathbb{B}_{a,b}(\bar{u}^C;\bar{v}^C) = S_{a,b}(\bar{u}^B;\bar{v}^B|\bar{u}^C;\bar{v}^C). \tag{3.1.3}$$

Here we have used $\psi(T_{j3}) = T_{3j}$ and $\psi(T_{3j}) = -T_{j3}$ for $j = 1,2$ (see (1.1.47)). Then using (1.1.48), and the fact that the total number of odd operators T_{3j} and T_{j3} with $j = 1,2$ in the scalar product is equal $2b$, we arrive at (3.1.3). Thus, we conclude that the scalar product is invariant under the permutation of the sets $\{\bar u^C,\bar v^C\}\leftrightarrow\{\bar u^B,\bar v^B\}$:

$$S_{a,b}(\bar u^C;\bar v^C|\bar u^B;\bar v^B)=S_{a,b}(\bar u^B;\bar v^B|\bar u^C;\bar v^C). \tag{3.1.4}$$

In order to calculate the scalar product one can take an explicit formula for the dual Bethe vector ((2.1.3) or (2.1.4)) and then use the formulas of the multiple actions (2.1.19)–(2.1.20). Basing on these formulas we can present the scalar product of Bethe vectors in the following schematic form:

$$S_{a,b}(\bar u^C;\bar v^C|\bar u^B;\bar v^B)=\sum r_1(\bar w_{\rm i})r_3(\bar w_{\rm ii})W_{\rm part}(\bar w_{\rm i};\bar w_{\rm ii};\bar w_{\rm iii}). \tag{3.1.5}$$

Here a set $\bar w$ is the union of all the Bethe parameters: $\bar w=\{\bar u^C,\bar u^B,\bar v^C,\bar v^B\}$. The sum is taken over partitions of this set into three subsets $\bar w\Rightarrow\{\bar w_{\rm i},\bar w_{\rm ii},\bar w_{\rm iii}\}$. The functions $W_{\rm part}$ are some rational coefficient. Their explicit forms are not important for now. We stress in (3.1.5) that a part of the Bethe parameters $\bar w_{\rm i}$ becomes the arguments of the functions r_1, while the parameters $\bar w_{\rm ii}$ become the arguments of the functions r_3. The remaining parameters $\bar w_{\rm iii}$ enter the rational functions $W_{\rm part}$ only.

Let us call the set $\{\bar u^C,\bar u^B\}$ the parameters of u-type. Correspondingly, we call the set $\{\bar v^C,\bar v^B\}$ the parameters of v-type.

Conjecture 3.1.1. *The set $\bar w_{\rm i}$ in* (3.1.5) *consists of parameters of the u-type only, while the set $\bar w_{\rm ii}$ consists of the parameters of v-type, that is, $\bar w_{\rm i}\subset\{\bar u^C,\bar u^B\}$ and $\bar w_{\rm ii}\subset\{\bar v^C,\bar v^B\}$. Moreover, $\#\bar w_{\rm i}=a$ and $\#\bar w_{\rm ii}=b$.*

Proof. Let us prove that $\bar w_{\rm i}\subset\{\bar u^C,\bar u^B\}$. For this we take the dual Bethe vector in the form (2.1.4). Let us fix a partition $\bar u^C\Rightarrow\{\bar u^C_{\rm I},\bar u^C_{\rm II}\}$ in (2.1.4), such that $\#\bar u^C_{\rm I}=n$, $n=0,1,\dots,\min(a,b)$. Calculating the scalar product we first act with the operators $T_{21}(\bar u^C_{\rm II})$ onto the Bethe vector. Then due to (2.1.19) we obtain a sum over partitions of the set $\{\bar u^C_{\rm II},\bar u^B\}$. The terms of this sum are proportional to the products of the functions $r_1(\bar\eta_{\rm I})$, where $\bar\eta_{\rm I}\subset\{\bar u^C_{\rm II},\bar u^B\}$ and $\#\bar\eta_{\rm I}=a-n$. Hence, the parameters $\bar\eta_{\rm I}$ are of the u-type.

Next, we act with the operators $\mathbb{T}_{31}(\bar u^C_{\rm I})$ onto obtained Bethe vectors via (2.1.21). We get new partitions of the set $\big\{\{\bar u^C,\bar u^B\}\setminus\bar\eta_{\rm I}\big\}$ and new products of functions r_1, say, $r_1(\bar\eta_{\rm I'})$. Obviously, $\bar\eta_{\rm I'}\subset\big\{\{\bar u^C,\bar u^B\}\setminus\bar\eta_{\rm I}\big\}$ and $\#\bar\eta_{\rm I'}=n$. Thus, the total number of the functions r_1 is equal to a, and all their arguments are of the u-type.

Finally, we should act with the product of the operators $\mathbb{T}_{32}(\bar v^C_{\rm II})$. But due to (2.1.20) this action does not produce new functions r_1. Thus, we have proved that $\bar w_{\rm i}\subset\{\bar u^C,\bar u^B\}$ and $\#\bar w_{\rm i}=a$.

Similarly, one can prove that $\bar w_{\rm ii}\subset\{\bar v^C,\bar v^B\}$ and $\#\bar w_{\rm ii}=b$. However, for this one should take representation (2.1.3) for the dual Bethe vector. Then all the functions r_3 will be produced under the successive actions of the operators $\mathbb{T}_{32}(\bar v^C_{\rm II})$ and $\mathbb{T}_{31}(\bar v^C_{\rm I})$. Repeating the considerations above we prove that all the parameters $\bar w_{\rm ii}$ are of the v-type and their total number is equal to b. □

Due to proposition 3.1.1 one can recast (3.1.5) in the form

$$S_{a,b}(\bar u^C;\bar v^C|\bar u^B;\bar v^B)=\sum r_1(\bar\eta_{\rm I})r_3(\bar\xi_{\rm I})W_{\rm part}(\bar\eta_{\rm I};\bar\eta_{\rm II}|\bar\xi_{\rm I};\bar\xi_{\rm II}). \tag{3.1.6}$$

Here $\bar\eta=\{\bar u^C,\bar u^B\}$ and $\bar\xi=\{\bar v^C,\bar v^B\}$. The sum is taken over partitions $\bar\eta\Rightarrow\{\bar\eta_{\rm I},\bar\eta_{\rm II}\}$ and $\bar\xi\Rightarrow\{\bar\xi_{\rm I},\bar\xi_{\rm II}\}$, such that $\#\bar\eta_{\rm I}=a$ and $\#\bar\xi_{\rm I}=b$. Setting in (3.1.6)

$$\begin{aligned}
&\bar\eta_{\rm I}=\{\bar u^B_{\rm I},\ \bar u^C_{\rm II}\}, &&\bar\eta_{\rm II}=\{\bar u^B_{\rm II},\ \bar u^C_{\rm I}\}, &&\#\bar u^B_{\rm I}=\#\bar u^C_{\rm I}=k, \quad k=1,\dots,a;\\
&\bar\xi_{\rm I}=\{\bar v^B_{\rm I},\ \bar v^C_{\rm II}\}, &&\bar\xi_{\rm II}=\{\bar v^B_{\rm II},\ \bar v^C_{\rm I}\}, &&\#\bar v^B_{\rm I}=\#\bar v^C_{\rm I}=n, \quad n=1,\dots,b,
\end{aligned} \tag{3.1.7}$$

we arrive at a representation

$$S_{a,b}(\bar{u}^C;\bar{v}^C|\bar{u}^B;\bar{v}^B) = \sum \frac{r_1(\bar{u}^C_{\rm II})r_1(\bar{u}^B_{\rm I})r_3(\bar{v}^C_{\rm II})r_3(\bar{v}^B_{\rm I})}{f(\bar{v}^C,\bar{u}^C)f(\bar{v}^B,\bar{u}^B)} W_{\rm part}\begin{pmatrix} \bar{u}^C_{\rm II},\bar{u}^B_{\rm II}, & \bar{u}^C_{\rm I},\bar{u}^B_{\rm I} \\ \bar{v}^C_{\rm I},\bar{v}^B_{\rm I}, & \bar{v}^C_{\rm II},\bar{v}^B_{\rm II} \end{pmatrix}. \tag{3.1.8}$$

Here the sum runs over all the partitions $\bar{u}^C \Rightarrow \{\bar{u}^C_{\rm I},\bar{u}^C_{\rm II}\}$, $\bar{u}^B \Rightarrow \{\bar{u}^B_{\rm I},\bar{u}^B_{\rm II}\}$, $\bar{v}^C \Rightarrow \{\bar{v}^C_{\rm I},\bar{v}^C_{\rm II}\}$ and $\bar{v}^B \Rightarrow \{\bar{v}^B_{\rm I},\bar{v}^B_{\rm II}\}$ with $\#\bar{u}^C_{\rm I} = \#\bar{u}^B_{\rm I}$ and $\#\bar{v}^C_{\rm I} = \#\bar{v}^B_{\rm I}$. The functions $W_{\rm part}$ are rational coefficients, which depend on the partitions but do not depend on the functions r_1 and r_3. We also have extracted explicitly the product $f(\bar{v}^C,\bar{u}^C)^{-1}f(\bar{v}^B,\bar{u}^B)^{-1}$ that plays the role of a normalisation factor.

Definition 3.1.1. *We call the highest coefficient $Z_{a,b}(\bar{u}^C;\bar{u}^B|\bar{v}^C;\bar{v}^B)$ the function $W_{\rm part}$ that corresponds to the extreme partitions $\bar{u}^C_{\rm I} = \bar{u}^B_{\rm I} = \emptyset$ and $\bar{v}^C_{\rm II} = \bar{v}^B_{\rm II} = \emptyset$:*

$$W_{\rm part}\begin{pmatrix} \bar{u}^C,\bar{u}^B, & \emptyset,\emptyset \\ \bar{v}^C,\bar{v}^B, & \emptyset,\emptyset \end{pmatrix} = Z_{a,b}(\bar{u}^C;\bar{u}^B|\bar{v}^C;\bar{v}^B). \tag{3.1.9}$$

In other words, $Z_{a,b}(\bar{u}^C;\bar{u}^B|\bar{v}^C;\bar{v}^B)$ is the coefficient of the product $r_1(\bar{u}^C)r_3(\bar{v}^B)$.

One also can define a conjugated highest coefficient corresponding to the extreme partition $\bar{u}^C_{\rm II} = \bar{u}^B_{\rm II} = \emptyset$ and $\bar{v}^C_{\rm I} = \bar{v}^B_{\rm I} = \emptyset$, that is the coefficient of the product $r_1(\bar{u}^B)r_3(\bar{v}^C)$. However, due to (3.1.4) it is clear that

$$W_{\rm part}\begin{pmatrix} \emptyset,\emptyset, & \bar{u}^C,\bar{u}^B, \\ \emptyset,\emptyset, & \bar{v}^C,\bar{v}^B \end{pmatrix} = Z_{a,b}(\bar{u}^B;\bar{u}^C|\bar{v}^B;\bar{v}^C). \tag{3.1.10}$$

We will show that all other coefficients $W_{\rm part}$ are equal to bilinear combinations of the highest coefficient and its conjugated.

Conjecture 3.1.2. *For a fixed partition with $\#\bar{u}^C_{\rm I} = \#\bar{u}^B_{\rm I} = k$ and $\#\bar{v}^C_{\rm I} = \#\bar{v}^B_{\rm I} = n$, (where $k = 0,\dots,a$ and $n = 0,\dots,b$), the coefficient $W_{\rm part}$ has the form*

$$W_{\rm part}\begin{pmatrix} \bar{u}^C_{\rm II},\bar{u}^B_{\rm II}, & \bar{u}^C_{\rm I},\bar{u}^B_{\rm I} \\ \bar{v}^C_{\rm I},\bar{v}^B_{\rm I}, & \bar{v}^C_{\rm II},\bar{v}^B_{\rm II} \end{pmatrix} = f(\bar{u}^B_{\rm II},\bar{u}^B_{\rm I})f(\bar{u}^C_{\rm I},\bar{u}^C_{\rm II})g(\bar{v}^B_{\rm II},\bar{v}^B_{\rm I})g(\bar{v}^C_{\rm I},\bar{v}^C_{\rm II})f(\bar{v}^C_{\rm I},\bar{u}^C_{\rm I})f(\bar{v}^B_{\rm II},\bar{u}^B_{\rm II})$$

$$\times\, Z_{a-k,n}(\bar{u}^C_{\rm II};\bar{u}^B_{\rm II}|\bar{v}^C_{\rm I};\bar{v}^B_{\rm I})\; Z_{k,b-n}(\bar{u}^B_{\rm I};\bar{u}^C_{\rm I}|\bar{v}^B_{\rm II};\bar{v}^C_{\rm II})\,. \tag{3.1.11}$$

The main goal of this paper is to find an explicit formula for the highest coefficient $Z_{a,b}$ and to prove the representation (3.1.11) for the coefficient $W_{\rm part}$.

Comparing (3.1.11) with the similar formula for the $\mathfrak{gl}(3)$-based models [85] one can see that they are very similar. It is enough to replace the product $g(\bar{v}^B_{\rm II},\bar{v}^B_{\rm I})g(\bar{v}^C_{\rm I},\bar{v}^C_{\rm II})$ in (3.1.11) with the product $f(\bar{v}^B_{\rm I},\bar{v}^B_{\rm II})f(\bar{v}^C_{\rm II},\bar{v}^C_{\rm I})$ in order to reproduce the formula of the paper [85]. One should remember, however, that the highest coefficients also have different representations in the models described by $\mathfrak{gl}(2|1)$ and $\mathfrak{gl}(3)$ algebras. In particular, it will shown that in the case under consideration the highest coefficient $Z_{a,b}$ admits a single determinant representation, while in the $\mathfrak{gl}(3)$ case such a determinant formula is not known.

3.2 Successive actions

In the previous section we have described how the scalar product depends on the functions r_k. Our goal now is to find explicitly the rational coefficients $W_{\rm part}$. For this we calculate successive action of the operators T_{ij} with $i>j$ onto a generic Bethe vector. This calculation is based on the results of previous chapter where the multiple actions of the monodromy matrix entries were computed.

3.2.1 Successive action of $\mathbb{T}_{31}(\bar x)T_{21}(\bar y)$

We start with the successive action of the products $\mathbb{T}_{31}(\bar x)T_{21}(\bar y)$. Let $\#\bar x=n$ and $\#\bar y=a-n$ where $n=0,1,\dots,\min(a,b)$. Define

$$G_{n,a}(\bar x,\bar y)=\frac{\mathbb{T}_{31}(\bar x)T_{21}(\bar y)}{\lambda_2(\bar x)\lambda_2(\bar y)}\mathbb{B}_{a,b}(\bar u;\bar v). \tag{3.2.1}$$

Using successively (2.1.19) and (2.1.21) we obtain

$$\begin{aligned}G_{n,a}(\bar x,\bar y)=(-1)^{\frac{n(n+1)}{2}}h(\bar v,\bar y)h(\bar y,\bar y)\sum r_1(\bar\eta_{\rm I})\frac{f(\bar\eta_{\rm II},\bar\eta_{\rm I})f(\bar\eta_{\rm II},\bar\eta_{\rm III})f(\bar\eta_{\rm III},\bar\eta_{\rm I})g(\bar\xi_{\rm II},\bar\xi_{\rm I})}{h(\bar\xi_{\rm I},\bar y)f(\bar\xi_{\rm II},\bar\eta_{\rm I})}\\ \times K_{a-n}(\bar y|\bar\eta_{\rm II}+c)K_{a-n}(\bar\eta_{\rm I}|\bar\xi_{\rm I}+c)h(\bar\xi_{\rm II},\bar x)h(\bar x,\bar x)r_3(\bar\xi_{\rm i})r_1(\bar\eta_{\rm ii})\frac{g(\bar\xi_{\rm ii},\bar\xi_{\rm i})g(\bar\xi_{\rm iii},\bar\xi_{\rm ii})g(\bar\xi_{\rm iii},\bar\xi_{\rm i})}{h(\bar\eta_{\rm i},\bar x)h(\bar\xi_{\rm i},\bar\eta_{\rm i})h(\bar\xi_{\rm ii},\bar x)}\\ \times\frac{f(\bar\eta_{\rm i},\bar\eta_{\rm ii})h(\bar\eta_{\rm i},\bar\eta_{\rm i})}{f(\bar\xi_{\rm i},\bar\eta_{\rm ii})f(\bar\xi_{\rm iii},\bar\eta_{\rm ii})}\,K_n(\bar\eta_{\rm ii}|\bar\xi_{\rm ii}+c)\,\mathbb{B}_{0,b-n}(\emptyset;\bar\xi_{\rm iii}).\end{aligned} \tag{3.2.2}$$

The sum is organized as follows. First the sets $\{\bar y,\bar u\}$ and $\{\bar y,\bar v\}$ are divided respectively into subsets $\{\bar\eta_{\rm I},\bar\eta_{\rm II},\bar\eta_{\rm III}\}$ and $\{\bar\xi_{\rm I},\bar\xi_{\rm II}\}$ with the restriction $\#\bar\xi_{\rm I}=\#\bar\eta_{\rm I}=\#\bar\eta_{\rm II}=a-n$. Then the sets $\{\bar x,\bar\eta_{\rm III}\}$ and $\{\bar x,\bar\xi_{\rm II}\}$ are divided respectively into subsets $\{\bar\eta_{\rm i},\bar\eta_{\rm ii}\}$ and $\{\bar\xi_{\rm i},\bar\xi_{\rm ii},\bar\xi_{\rm iii}\}$ with the restriction $\#\bar\xi_{\rm i}=\#\bar\xi_{\rm ii}=\#\bar\eta_{\rm i}=\#\bar\eta_{\rm ii}=n$.

It is convenient to introduce the sets $\bar\eta=\{\bar y,\bar x,\bar u\}$ and $\bar\xi=\{\bar y,\bar x,\bar v\}$. Then we can understand the sum in (3.2.2) as the sum over partitions $\bar\eta\Rightarrow\{\bar\eta_{\rm I},\bar\eta_{\rm II},\bar\eta_{\rm i},\bar\eta_{\rm ii}\}$ and $\bar\xi\Rightarrow\{\bar\xi_{\rm I},\bar\xi_{\rm i},\bar\xi_{\rm ii},\bar\xi_{\rm iii}\}$ with the restrictions mentioned above and an additional constrain $\bar x\cap\{\bar\eta_{\rm I},\bar\eta_{\rm II},\bar\xi_{\rm I}\}=\emptyset$. Hereby $\bar\eta_{\rm III}=\{\bar\eta_{\rm i},\bar\eta_{\rm ii}\}\setminus\bar x$ and $\bar\xi_{\rm II}=\{\bar\xi_{\rm i},\bar\xi_{\rm ii},\bar\xi_{\rm iii}\}\setminus\bar x$. Then we have

$$f(\bar\eta_{\rm II},\bar\eta_{\rm III})f(\bar\eta_{\rm III},\bar\eta_{\rm I})=\frac{f(\bar\eta_{\rm II},\bar\eta_{\rm i})f(\bar\eta_{\rm II},\bar\eta_{\rm ii})f(\bar\eta_{\rm i},\bar\eta_{\rm I})f(\bar\eta_{\rm ii},\bar\eta_{\rm I})}{f(\bar\eta_{\rm II},\bar x)f(\bar x,\bar\eta_{\rm I})}, \tag{3.2.3}$$

and

$$\frac{g(\bar\xi_{\rm II},\bar\xi_{\rm I})}{f(\bar\xi_{\rm II},\bar\eta_{\rm I})}=\frac{g(\bar\xi_{\rm i},\bar\xi_{\rm I})g(\bar\xi_{\rm ii},\bar\xi_{\rm I})g(\bar\xi_{\rm iii},\bar\xi_{\rm I})f(\bar x,\bar\eta_{\rm I})}{g(\bar x,\bar\xi_{\rm I})f(\bar\xi_{\rm i},\bar\eta_{\rm I})f(\bar\xi_{\rm ii},\bar\eta_{\rm I})f(\bar\xi_{\rm iii},\bar\eta_{\rm I})}. \tag{3.2.4}$$

Observe that the restrictions $\bar x\cap\bar\eta_{\rm II}=\emptyset$ and $\bar x\cap\bar\xi_{\rm I}=\emptyset$ hold automatically due to the presence of the product $f(\bar\eta_{\rm II},\bar x)$ in the denominator of (3.2.3) and the product $g(\bar x,\bar\xi_{\rm I})$ in the denominator of (3.2.4). Indeed, $1/f(\bar\eta_{\rm II},\bar x)=0$ as soon as $\bar x\cap\bar\eta_{\rm II}\neq\emptyset$ and $1/g(\bar x,\bar\xi_{\rm I})=0$ as soon as $\bar x\cap\bar\xi_{\rm I}\neq\emptyset$. Actually, one can easily see that the condition $\bar x\cap\bar\eta_{\rm I}=\emptyset$ also holds, although the product $f(\bar x,\bar\eta_{\rm I})$ in the denominator of (3.2.3) is compensated by the same product in the numerator of (3.2.4). Indeed, we have seen that $\bar x\cap\bar\xi_{\rm I}=\emptyset$, that is to say, $\bar x\subset\{\bar\xi_{\rm i},\bar\xi_{\rm ii},\bar\xi_{\rm iii}\}$. But in this case $\bar x\cap\bar\eta_{\rm I}=\emptyset$ due to the products of the f-functions in the denominator of (3.2.4).

Thus, we can recast (3.2.2) as follows:

$$G_{n,a}(\bar{x},\bar{y}) = (-1)^{\frac{n(n+1)}{2}} \frac{h(\bar{\xi},\bar{y})h(\bar{\xi},\bar{x})}{h(\bar{x},\bar{y})} \sum r_1(\bar{\eta}_{\rm I})r_1(\bar{\eta}_{\rm ii})r_3(\bar{\xi}_{\rm i})$$
$$\times \frac{f(\bar{\eta}_{\rm II},\bar{\eta}_{\rm I})f(\bar{\eta}_{\rm II},\bar{\eta}_{\rm i})f(\bar{\eta}_{\rm II},\bar{\eta}_{\rm ii})f(\bar{\eta}_{\rm i},\bar{\eta}_{\rm I})f(\bar{\eta}_{\rm ii},\bar{\eta}_{\rm I})f(\bar{\eta}_{\rm i},\bar{\eta}_{\rm ii})h(\bar{\eta}_{\rm i},\bar{\eta}_{\rm i})}{f(\bar{\eta}_{\rm II},\bar{x})f(\bar{\xi}_{\rm i},\bar{\eta}_{\rm I})f(\bar{\xi}_{\rm ii},\bar{\eta}_{\rm I})f(\bar{\xi}_{\rm iii},\bar{\eta}_{\rm I})f(\bar{\xi}_{\rm i},\bar{\eta}_{\rm ii})f(\bar{\xi}_{\rm iii},\bar{\eta}_{\rm ii})}$$
$$\times \frac{g(\bar{\xi}_{\rm i},\bar{\xi}_{\rm I})g(\bar{\xi}_{\rm ii},\bar{\xi}_{\rm I})g(\bar{\xi}_{\rm iii},\bar{\xi}_{\rm I})g(\bar{\xi}_{\rm ii},\bar{\xi}_{\rm i})g(\bar{\xi}_{\rm iii},\bar{\xi}_{\rm ii})g(\bar{\xi}_{\rm iii},\bar{\xi}_{\rm i})}{h(\bar{\xi}_{\rm I},\bar{y})h(\bar{\xi}_{\rm I},\bar{x})h(\bar{\eta}_{\rm i},\bar{x})h(\bar{\xi}_{\rm i},\bar{\eta}_{\rm i})h(\bar{\xi}_{\rm ii},\bar{x})g(\bar{x},\bar{\xi}_{\rm I})}$$
$$\times K_{a-n}(\bar{y}|\bar{\eta}_{\rm II}+c)K_{a-n}(\bar{\eta}_{\rm I}|\bar{\xi}_{\rm I}+c)\; K_n(\bar{\eta}_{\rm ii}|\bar{\xi}_{\rm ii}+c)\; \mathbb{B}_{0,b-n}(\emptyset;\bar{\xi}_{\rm iii}). \quad (3.2.5)$$

Here we have also used

$$h(\bar{x},\bar{x})h(\bar{\xi}_{\rm II},\bar{x}) = \frac{h(\bar{\xi},\bar{x})}{h(\bar{\xi}_{\rm I},\bar{x})}. \quad (3.2.6)$$

In (3.2.5) the sum is taken over partitions $\bar{\eta} \Rightarrow \{\bar{\eta}_{\rm I},\bar{\eta}_{\rm II},\bar{\eta}_{\rm i},\bar{\eta}_{\rm ii}\}$ and $\bar{\xi} \Rightarrow \{\bar{\xi}_{\rm I},\bar{\xi}_{\rm i},\bar{\xi}_{\rm ii},\bar{\xi}_{\rm iii}\}$. The restriction are imposed on the cardinalities of the subsets only.

One can reduce the number of subsets in (3.2.5). Let $\bar{\eta}_0 = \{\bar{\eta}_{\rm I},\bar{\eta}_{\rm ii}\}$. Then (3.2.5) takes the form

$$G_{n,a}(\bar{x},\bar{y}) = (-1)^{\frac{n(n+1)}{2}} \frac{h(\bar{\xi},\bar{y})h(\bar{\xi},\bar{x})}{h(\bar{x},\bar{y})} \sum r_1(\bar{\eta}_0)r_3(\bar{\xi}_{\rm i}) \frac{f(\bar{\eta}_{\rm II},\bar{\eta}_0)f(\bar{\eta}_{\rm i},\bar{\eta}_0)}{f(\bar{\xi}_{\rm iii},\bar{\eta}_0)f(\bar{\xi}_{\rm i},\bar{\eta}_0)}\; K_{a-n}(\bar{y}|\bar{\eta}_{\rm II}+c)$$
$$\times \frac{f(\bar{\eta}_{\rm II},\bar{\eta}_{\rm i})h(\bar{\eta}_{\rm i},\bar{\eta}_{\rm i})g(\bar{\xi}_{\rm i},\bar{\xi}_{\rm I})g(\bar{\xi}_{\rm ii},\bar{\xi}_{\rm I})g(\bar{\xi}_{\rm iii},\bar{\xi}_{\rm I})g(\bar{\xi}_{\rm ii},\bar{\xi}_{\rm i})g(\bar{\xi}_{\rm iii},\bar{\xi}_{\rm ii})g(\bar{\xi}_{\rm iii},\bar{\xi}_{\rm i})}{f(\bar{\eta}_{\rm II},\bar{x})h(\bar{\xi}_{\rm I},\bar{y})h(\bar{\xi}_{\rm I},\bar{x})h(\bar{\eta}_{\rm i},\bar{x})h(\bar{\xi}_{\rm i},\bar{\eta}_{\rm i})h(\bar{\xi}_{\rm ii},\bar{x})g(\bar{x},\bar{\xi}_{\rm I})}\; \mathbb{B}_{0,b-n}(\emptyset;\bar{\xi}_{\rm iii})$$
$$\times \frac{f(\bar{\eta}_{\rm ii},\bar{\eta}_{\rm I})}{f(\bar{\xi}_{\rm ii},\bar{\eta}_{\rm I})} K_{a-n}(\bar{\eta}_{\rm I}|\bar{\xi}_{\rm I}+c)\; K_n(\bar{\eta}_{\rm ii}|\bar{\xi}_{\rm ii}+c). \quad (3.2.7)$$

We see that the sum over partitions $\bar{\eta}_0 \Rightarrow \{\bar{\eta}_{\rm I},\bar{\eta}_{\rm ii}\}$ involves the terms in the last line only. This sum can be computed via lemma 9.3.2. Using (9.3.17) we find

$$\sum_{\bar{\eta}_0\Rightarrow\{\bar{\eta}_{\rm I},\bar{\eta}_{\rm ii}\}} K_{a-n}(\bar{\eta}_{\rm I}|\bar{\xi}_{\rm I}+c)\; K_n(\bar{\eta}_{\rm ii}|\bar{\xi}_{\rm ii}+c)\frac{f(\bar{\eta}_{\rm ii},\bar{\eta}_{\rm I})}{f(\bar{\xi}_{\rm ii},\bar{\eta}_{\rm I})}$$
$$= \frac{(-1)^n}{f(\bar{\xi}_{\rm ii},\bar{\eta}_0)} \sum_{\bar{\eta}_0\Rightarrow\{\bar{\eta}_{\rm I},\bar{\eta}_{\rm ii}\}} K_{a-n}(\bar{\eta}_{\rm I}|\bar{\xi}_{\rm I}+c)\; K_n(\bar{\xi}_{\rm ii}|\bar{\eta}_{\rm ii})f(\bar{\eta}_{\rm ii},\bar{\eta}_{\rm I}) = \frac{(-1)^a K_a(\{\bar{\xi}_{\rm I},\bar{\xi}_{\rm ii}\}|\bar{\eta}_0)}{f(\bar{\xi}_{\rm ii},\bar{\eta}_0)f(\bar{\xi}_{\rm I},\bar{\eta}_0)}. \quad (3.2.8)$$

Thus, (3.2.7) takes the form

$$G_{n,a}(\bar{x},\bar{y}) = (-1)^{a+\frac{n(n+1)}{2}} \frac{h(\bar{\xi},\bar{y})h(\bar{\xi},\bar{x})}{h(\bar{x},\bar{y})} \sum r_1(\bar{\eta}_0)r_3(\bar{\xi}_{\rm i})K_{a-n}(\bar{y}|\bar{\eta}_{\rm II}+c)$$
$$\times \frac{f(\bar{\eta}_{\rm II},\bar{\eta}_0)f(\bar{\eta}_{\rm II},\bar{\eta}_{\rm i})f(\bar{\eta}_{\rm i},\bar{\eta}_0)h(\bar{\eta}_{\rm i},\bar{\eta}_{\rm i})g(\bar{\xi}_{\rm i},\bar{\xi}_{\rm I})g(\bar{\xi}_{\rm ii},\bar{\xi}_{\rm I})g(\bar{\xi}_{\rm iii},\bar{\xi}_{\rm I})g(\bar{\xi}_{\rm ii},\bar{\xi}_{\rm i})g(\bar{\xi}_{\rm iii},\bar{\xi}_{\rm ii})g(\bar{\xi}_{\rm iii},\bar{\xi}_{\rm i})}{f(\bar{\eta}_{\rm II},\bar{x})f(\bar{\xi}_{\rm ii},\bar{\eta}_0)f(\bar{\xi}_{\rm I},\bar{\eta}_0)f(\bar{\xi}_{\rm iii},\bar{\eta}_0)f(\bar{\xi}_{\rm i},\bar{\eta}_0)h(\bar{\xi}_{\rm I},\bar{y})h(\bar{\xi}_{\rm I},\bar{x})h(\bar{\eta}_{\rm i},\bar{x})h(\bar{\xi}_{\rm i},\bar{\eta}_{\rm i})h(\bar{\xi}_{\rm ii},\bar{x})g(\bar{x},\bar{\xi}_{\rm I})}$$
$$\times K_a(\{\bar{\xi}_{\rm I},\bar{\xi}_{\rm ii}\}|\bar{\eta}_0)\; \mathbb{B}_{0,b-n}(\emptyset;\bar{\xi}_{\rm iii}). \quad (3.2.9)$$

Now we define $\bar\xi_0=\{\bar\xi_{\rm I},\bar\xi_{\rm ii}\}$. Then (3.2.9) can be written as

$$G_{n,a}(\bar x,\bar y)=(-1)^{a+\frac{n(n-1)}{2}}\frac{h(\bar\xi,\bar y)h(\bar\xi,\bar x)}{h(\bar x,\bar y)}\sum r_1(\bar\eta_0)r_3(\bar\xi_{\rm i})K_{a-n}(\bar y|\bar\eta_{\rm II}+c)K_a(\bar\xi_0|\bar\eta_0)$$
$$\times\frac{f(\bar\eta_{\rm II},\bar\eta_0)f(\bar\eta_{\rm II},\bar\eta_{\rm i})f(\bar\eta_{\rm i},\bar\eta_0)h(\bar\eta_{\rm i},\bar\eta_{\rm i})g(\bar\xi_{\rm i},\bar\xi_0)g(\bar\xi_{\rm iii},\bar\xi_0)g(\bar\xi_{\rm iii},\bar\xi_{\rm i})}{f(\bar\eta_{\rm II},\bar x)f(\bar\xi_0,\bar\eta_0)f(\bar\xi_{\rm iii},\bar\eta_0)f(\bar\xi_{\rm i},\bar\eta_0)h(\bar\eta_{\rm i},\bar x)h(\bar\xi_{\rm i},\bar\eta_{\rm i})h(\bar\xi_0,\bar x)}\,\mathbb{B}_{0,b-n}(\emptyset;\bar\xi_{\rm iii})$$
$$\times\frac{g(\bar\xi_{\rm ii},\bar\xi_{\rm I})}{h(\bar\xi_{\rm I},\bar y)g(\bar x,\bar\xi_{\rm I})}. \qquad (3.2.10)$$

The sum over partitions $\bar\xi_0\Rightarrow\{\bar\xi_{\rm I},\bar\xi_{\rm ii}\}$ involves the terms in the last line only. It can be computed via (9.3.16):

$$\sum\frac{g(\bar\xi_{\rm ii},\bar\xi_{\rm I})}{g(\bar x,\bar\xi_{\rm I})h(\bar\xi_{\rm I},\bar y)}=\frac{(-1)^{n(a-n)}}{g(\bar\xi_0,\bar x)}\sum\frac{g(\bar\xi_{\rm ii},\bar\xi_{\rm I})g(\bar\xi_{\rm ii},\bar x)}{h(\bar\xi_{\rm I},\bar y)}$$
$$=\frac{(-1)^{n(a-n)}}{g(\bar\xi_0,\bar x)}\sum g(\bar\xi_{\rm ii},\bar\xi_{\rm I})g(\bar\xi_{\rm ii},\bar x)g(\bar\xi_{\rm I},\bar y-c)=\frac{h(\bar x,\bar y)}{h(\bar\xi_0,\bar y)}. \qquad (3.2.11)$$

Thus, we arrive at

$$G_{n,a}(\bar x,\bar y)=(-1)^{a+\frac{n(n-1)}{2}}h(\bar\xi,\bar y)h(\bar\xi,\bar x)\sum\frac{r_1(\bar\eta_0)r_3(\bar\xi_{\rm i})}{h(\bar\xi_0,\bar x)h(\bar\xi_0,\bar y)}K_{a-n}(\bar y|\bar\eta_{\rm II}+c)K_a(\bar\xi_0|\bar\eta_0)$$
$$\times\frac{f(\bar\eta_{\rm II},\bar\eta_0)f(\bar\eta_{\rm II},\bar\eta_{\rm i})f(\bar\eta_{\rm i},\bar\eta_0)h(\bar\eta_{\rm i},\bar\eta_{\rm i})g(\bar\xi_{\rm i},\bar\xi_0)g(\bar\xi_{\rm iii},\bar\xi_0)g(\bar\xi_{\rm iii},\bar\xi_{\rm i})}{f(\bar\eta_{\rm II},\bar x)f(\bar\xi_0,\bar\eta_0)f(\bar\xi_{\rm iii},\bar\eta_0)f(\bar\xi_{\rm i},\bar\eta_0)h(\bar\eta_{\rm i},\bar x)h(\bar\xi_{\rm i},\bar\eta_{\rm i})}\,\mathbb{B}_{0,b-n}(\emptyset;\bar\xi_{\rm iii}). \qquad (3.2.12)$$

Finally, after a relabeling of the subsets $\bar\eta_0\to\bar\eta_{\rm I}$, $\bar\eta_{\rm i}\to\bar\eta_{\rm II}$, $\bar\eta_{\rm II}\to\bar\eta_{\rm III}$, $\bar\xi_{\rm i}\to\bar\xi_{\rm I}$, $\bar\xi_0\to\bar\xi_{\rm II}$, $\bar\xi_{\rm iii}\to\bar\xi_{\rm III}$, we recast (3.2.12) as follows:

$$G_{n,a}(\bar x,\bar y)=(-1)^{a+\frac{n(n-1)}{2}}h(\bar\xi,\bar y)h(\bar\xi,\bar x)\sum\frac{r_1(\bar\eta_{\rm I})r_3(\bar\xi_{\rm I})}{h(\bar\xi_{\rm II},\bar x)h(\bar\xi_{\rm II},\bar y)}K_{a-n}(\bar y|\bar\eta_{\rm III}+c)K_a(\bar\xi_{\rm II}|\bar\eta_{\rm I})$$
$$\times\frac{f(\bar\eta_{\rm III},\bar\eta_{\rm I})f(\bar\eta_{\rm III},\bar\eta_{\rm II})f(\bar\eta_{\rm II},\bar\eta_{\rm I})h(\bar\eta_{\rm II},\bar\eta_{\rm II})g(\bar\xi_{\rm I},\bar\xi_{\rm II})g(\bar\xi_{\rm III},\bar\xi_{\rm II})g(\bar\xi_{\rm III},\bar\xi_{\rm I})}{f(\bar\eta_{\rm III},\bar x)f(\bar\xi,\bar\eta_{\rm I})h(\bar\eta_{\rm II},\bar x)h(\bar\xi_{\rm I},\bar\eta_{\rm II})}\,\mathbb{B}_{0,b-n}(\emptyset;\bar\xi_{\rm III}). \qquad (3.2.13)$$

We recall that the cardinalities of the subsets are

$$\begin{aligned}&\#\bar\eta_{\rm I}=a, &&\#\bar\eta_{\rm II}=n, &&\#\bar\eta_{\rm III}=a-n,\\ &\#\bar\xi_{\rm I}=n, &&\#\bar\xi_{\rm II}=a, &&\#\bar\xi_{\rm III}=b-n.\end{aligned} \qquad (3.2.14)$$

Remark 3.2.1. *Strictly speaking, the sets $\bar\eta$ and $\bar\xi$ in equation (3.2.13) should be understood as*

$$\begin{aligned}\bar\eta&=\{\bar x+\epsilon_1,\bar y+\epsilon_1,\bar u+\epsilon_1\},\\ \bar\xi&=\{\bar x+\epsilon_2,\bar y+\epsilon_2,\bar v+\epsilon_2\},\end{aligned}\qquad \text{at}\quad \epsilon_k\to0,\qquad k=1,2. \qquad (3.2.15)$$

The point is that individual factors in (3.2.13) may have singularities, if we set $\epsilon_k=0$. For instance, if $\bar\xi_{\rm II}\cap\bar\eta_{\rm I}\neq\emptyset$, then the Izergin determinant $K_a(\bar\xi_{\rm II}|\bar\eta_{\rm I})$ is singular. However, these poles are compensated by the product $f(\bar\xi,\bar\eta_{\rm I})^{-1}$. Therefore, for appropriate evaluating the limit we should have $\epsilon_k\neq0$. In order to lighten the formulas we do not write these auxiliary parameters ϵ_k explicitly, but one has to keep them in mind when doing the calculations.

3.2.2 Successive action of $\mathbb{T}_{32}(\bar{z})\mathbb{T}_{31}(\bar{x})T_{21}(\bar{y})$

Let now $\#\bar{z} = b - n$. Then we define

$$\frac{\mathbb{T}_{32}(\bar{z})\,\mathbb{T}_{31}(\bar{x})\,T_{21}(\bar{y})}{\lambda_2(\bar{z})\,\lambda_2(\bar{x})\,\lambda_2(\bar{y})}\mathbb{B}_{a,b}(\bar{u},\bar{v}) = \frac{\mathbb{T}_{32}(\bar{z})}{\lambda_2(\bar{z})}G_{n,a}(\bar{x},\bar{y}) = H_{n,a,b}(\bar{x},\bar{y},\bar{z})\Omega. \tag{3.2.16}$$

In order to act with $\mathbb{T}_{32}(\bar{z})$ onto $G_{n,a}(\bar{x},\bar{y})$ we should use (2.1.20). Let us denote the union $\{\bar{x},\bar{y}\}$ as $\bar{u}^C$ (as it will be in the case of the scalar product). Then we obtain

$$\begin{aligned} H_{n,a,b}(\bar{x},\bar{y},\bar{z}) = (-1)^{a+\frac{n(n-1)}{2}+\frac{(b-n)(b-n-1)}{2}} h(\bar{v},\bar{u}^C)h(\bar{u}^C,\bar{u}^C)\sum \frac{r_1(\bar{\eta}_{\rm I})r_3(\bar{\xi}_{\rm I})r_3(\bar{\xi}_{\rm i})}{h(\bar{\xi}_{\rm II},\bar{u}^C)} \\ \times K_{a-n}(\bar{y}|\bar{\eta}_{\rm III}+c)K_a(\bar{\xi}_{\rm II}|\bar{\eta}_{\rm I})\frac{f(\bar{\eta}_{\rm III},\bar{\eta}_{\rm I})f(\bar{\eta}_{\rm III},\bar{\eta}_{\rm II})f(\bar{\eta}_{\rm II},\bar{\eta}_{\rm I})h(\bar{\eta}_{\rm II},\bar{\eta}_{\rm II})}{f(\bar{\eta}_{\rm III},\bar{x})f(\bar{\xi},\bar{\eta}_{\rm I})} \\ \times \frac{g(\bar{\xi}_{\rm I},\bar{\xi}_{\rm II})g(\bar{\xi}_{\rm III},\bar{\xi}_{\rm II})g(\bar{\xi}_{\rm III},\bar{\xi}_{\rm I})}{h(\bar{\eta}_{\rm II},\bar{x})h(\bar{\xi}_{\rm I},\bar{\eta}_{\rm II})}g(\bar{\xi}_{\rm ii},\bar{\xi}_{\rm i}). \end{aligned} \tag{3.2.17}$$

Here the partitions of the set $\bar{\eta}$ remain the same as in (3.2.13). The partitions of the remaining variables are organized as follows. We first have the partitions of the set $\{\bar{u}^C,\bar{v}\} = \bar{\xi} \Rightarrow \{\bar{\xi}_{\rm I},\bar{\xi}_{\rm II},\bar{\xi}_{\rm III}\}$. Then we combine $\{\bar{z},\bar{\xi}_{\rm III}\}$ and obtain additional partitions $\{\bar{z},\bar{\xi}_{\rm III}\} \Rightarrow \{\bar{\xi}_{\rm i},\bar{\xi}_{\rm ii}\}$ with the restriction $\#\bar{\xi}_{\rm i} = \#\bar{\xi}_{\rm ii} = b-n$.

We should substitute $\bar{\xi}_{\rm III} = \{\bar{\xi}_{\rm i},\bar{\xi}_{\rm ii}\}\setminus\bar{z}$ into (3.2.17). Then, using

$$\frac{g(\bar{\xi}_{\rm III},\bar{\xi}_{\rm II})g(\bar{\xi}_{\rm III},\bar{\xi}_{\rm I})}{f(\bar{\xi},\bar{\eta}_{\rm I})} = \frac{g(\bar{\xi}_{\rm i},\bar{\xi}_{\rm II})g(\bar{\xi}_{\rm ii},\bar{\xi}_{\rm II})g(\bar{\xi}_{\rm i},\bar{\xi}_{\rm I})g(\bar{\xi}_{\rm ii},\bar{\xi}_{\rm I})}{f(\bar{v},\bar{\eta}_{\rm I})f(\bar{u}^C,\bar{\eta}_{\rm I})g(\bar{z},\bar{\xi}_{\rm I})g(\bar{z},\bar{\xi}_{\rm II})}, \tag{3.2.18}$$

we arrive at

$$\begin{aligned} H_{n,a,b}(\bar{x},\bar{y},\bar{z}) = (-1)^{a+\frac{n(n-1)}{2}+\frac{(b-n)(b-n-1)}{2}} h(\bar{v},\bar{u}^C)h(\bar{u}^C,\bar{u}^C)\sum \frac{r_1(\bar{\eta}_{\rm I})r_3(\bar{\xi}_{\rm I})r_3(\bar{\xi}_{\rm i})}{h(\bar{\xi}_{\rm II},\bar{u}^C)} \\ \times K_{a-n}(\bar{y}|\bar{\eta}_{\rm III}+c)K_a(\bar{\xi}_{\rm II}|\bar{\eta}_{\rm I})\frac{f(\bar{\eta}_{\rm III},\bar{\eta}_{\rm I})f(\bar{\eta}_{\rm III},\bar{\eta}_{\rm II})f(\bar{\eta}_{\rm II},\bar{\eta}_{\rm I})h(\bar{\eta}_{\rm II},\bar{\eta}_{\rm II})f(\bar{z},\bar{\eta}_{\rm I})}{f(\bar{\eta}_{\rm III},\bar{x})f(\bar{\xi},\bar{\eta}_{\rm I})} \\ \times \frac{g(\bar{\xi}_{\rm I},\bar{\xi}_{\rm II})g(\bar{\xi}_{\rm i},\bar{\xi}_{\rm II})g(\bar{\xi}_{\rm ii},\bar{\xi}_{\rm II})g(\bar{\xi}_{\rm i},\bar{\xi}_{\rm I})g(\bar{\xi}_{\rm ii},\bar{\xi}_{\rm I})g(\bar{\xi}_{\rm ii},\bar{\xi}_{\rm i})}{h(\bar{\eta}_{\rm II},\bar{x})h(\bar{\xi}_{\rm I},\bar{\eta}_{\rm II})g(\bar{z},\bar{\xi}_{\rm I})g(\bar{z},\bar{\xi}_{\rm II})}. \end{aligned} \tag{3.2.19}$$

Here we have denoted by $\bar{\xi}$ the union $\{\bar{z},\bar{u}^C,\bar{v}\}$. This set is divided into four subsets $\bar{\xi} \Rightarrow \{\bar{\xi}_{\rm i},\bar{\xi}_{\rm ii},\bar{\xi}_{\rm I},\bar{\xi}_{\rm II}\}$ with the cardinalities $\#\bar{\xi}_{\rm i} = \#\bar{\xi}_{\rm ii} = b-n$, $\#\bar{\xi}_{\rm I} = n$, and $\#\bar{\xi}_{\rm II} = a$.

Let $\bar{\xi}_0 = \{\bar{\xi}_{\rm i},\bar{\xi}_{\rm I}\}$. Then

$$\begin{aligned} H_{n,a,b}(\bar{x},\bar{y},\bar{z}) = (-1)^{a+n(b+1)+\frac{b(b-1)}{2}} h(\bar{v},\bar{u}^C)h(\bar{u}^C,\bar{u}^C)\sum \frac{r_1(\bar{\eta}_{\rm I})r_3(\bar{\xi}_0)}{h(\bar{\xi}_{\rm II},\bar{u}^C)}K_{a-n}(\bar{y}|\bar{\eta}_{\rm III}+c) \\ \times K_a(\bar{\xi}_{\rm II}|\bar{\eta}_{\rm I})\frac{f(\bar{\eta}_{\rm III},\bar{\eta}_{\rm I})f(\bar{\eta}_{\rm III},\bar{\eta}_{\rm II})f(\bar{\eta}_{\rm II},\bar{\eta}_{\rm I})h(\bar{\eta}_{\rm II},\bar{\eta}_{\rm II})f(\bar{z},\bar{\eta}_{\rm I})g(\bar{\xi}_0,\bar{\xi}_{\rm II})g(\bar{\xi}_{\rm ii},\bar{\xi}_{\rm II})g(\bar{\xi}_{\rm ii},\bar{\xi}_0)}{f(\bar{\eta}_{\rm III},\bar{x})f(\bar{\xi},\bar{\eta}_{\rm I})h(\bar{\eta}_{\rm II},\bar{x})g(\bar{z},\bar{\xi}_{\rm II})} \\ \times \frac{g(\bar{\xi}_{\rm i},\bar{\xi}_{\rm I})}{h(\bar{\xi}_{\rm I},\bar{\eta}_{\rm II})g(\bar{z},\bar{\xi}_{\rm I})}. \end{aligned} \tag{3.2.20}$$

The sum over partitions $\bar{\xi}_0 \Rightarrow \{\bar{\xi}_{\rm I},\bar{\xi}_{\rm i}\}$ involves the terms in the last line only. It can be computed via (9.3.16):

$$\sum \frac{g(\bar{\xi}_{\rm i},\bar{\xi}_{\rm I})}{h(\bar{\xi}_{\rm I},\bar{\eta}_{\rm II})g(\bar{z},\bar{\xi}_{\rm I})} = \frac{(-1)^{b-n}}{g(\bar{z},\bar{\xi}_0)}\sum \frac{g(\bar{\xi}_{\rm i},\bar{\xi}_{\rm I})g(\bar{\xi}_{\rm i},\bar{z})}{h(\bar{\xi}_{\rm I},\bar{\eta}_{\rm II})} = \frac{h(\bar{z},\bar{\eta}_{\rm II})}{h(\bar{\xi}_0,\bar{\eta}_{\rm II})}. \tag{3.2.21}$$

Substituting this into (3.2.20) we find

$$H_{n,a,b}(\bar{x},\bar{y},\bar{z}) = (-1)^{a+n(b+1)+\frac{b(b-1)}{2}} h(\bar{v},\bar{u}^C)h(\bar{u}^C,\bar{u}^C) \sum \frac{r_1(\bar{\eta}_{\rm I})r_3(\bar{\xi}_0)}{h(\bar{\xi}_{\rm II},\bar{u}^C)} \\ \times \frac{f(\bar{\eta}_{\rm III},\bar{\eta}_{\rm I})f(\bar{\eta}_{\rm III},\bar{\eta}_{\rm II})f(\bar{\eta}_{\rm II},\bar{\eta}_{\rm I})h(\bar{\eta}_{\rm II},\bar{\eta}_{\rm II})f(\bar{z},\bar{\eta}_{\rm I})g(\bar{\xi}_0,\bar{\xi}_{\rm II})g(\bar{\xi}_{\rm ii},\bar{\xi}_{\rm II})g(\bar{\xi}_{\rm ii},\bar{\xi}_0)h(\bar{z},\bar{\eta}_{\rm II})}{f(\bar{\eta}_{\rm III},\bar{x})f(\bar{\xi},\bar{\eta}_{\rm I})h(\bar{\eta}_{\rm II},\bar{x})h(\bar{\xi}_0,\bar{\eta}_{\rm II})g(\bar{z},\bar{\xi}_{\rm II})} \\ \times K_{a-n}(\bar{y}|\bar{\eta}_{\rm III}+c)K_a(\bar{\xi}_{\rm II}|\bar{\eta}_{\rm I}). \quad (3.2.22)$$

Finally, relabeling $\bar{\xi}_0 \to \bar{\xi}_{\rm I}$ and $\bar{\xi}_{\rm ii} \to \bar{\xi}_{\rm III}$ we arrive at

$$H_{n,a,b}(\bar{x},\bar{y},\bar{z}) = (-1)^{a+n(b+1)+\frac{b^2-b}{2}} \frac{h(\bar{\xi},\bar{u}^C)}{h(\bar{z},\bar{u}^C)} \sum r_1(\bar{\eta}_{\rm I})r_3(\bar{\xi}_{\rm I})\, K_a(\bar{\xi}_{\rm II}|\bar{\eta}_{\rm I})K_{a-n}(\bar{y}|\bar{\eta}_{\rm III}+c) \\ \times \frac{f(\bar{\eta}_0,\bar{\eta}_{\rm I})h(\bar{\eta}_0,\bar{\eta}_{\rm II})g(\bar{\eta}_{\rm III},\bar{\eta}_{\rm II})f(\bar{z},\bar{\eta}_{\rm I})h(\bar{z},\bar{\eta}_{\rm II})g(\bar{\xi}_{\rm I},\bar{\xi}_{\rm II})g(\bar{\xi}_{\rm III},\bar{\xi}_{\rm I})g(\bar{\xi}_{\rm III},\bar{\xi}_{\rm II})}{h(\bar{\xi}_{\rm I},\bar{\eta}_{\rm II})f(\bar{\xi},\bar{\eta}_{\rm I})h(\bar{\eta}_{\rm II},\bar{x})f(\bar{\eta}_{\rm III},\bar{x})g(\bar{z},\bar{\xi}_{\rm II})h(\bar{\xi}_{\rm II},\bar{u}^C)}. \quad (3.2.23)$$

Recall that in this formula $\bar{\eta} = \{\bar{x},\bar{y},\bar{u}\}$, $\bar{\xi} = \{\bar{z},\bar{x},\bar{y},\bar{v}\}$, and we denote $\bar{u}^C = \{\bar{x},\bar{y}\}$. The sum is taken over partitions $\bar{\eta} \Rightarrow \{\bar{\eta}_{\rm I},\bar{\eta}_{\rm II},\bar{\eta}_{\rm III}\}$ and $\bar{\xi} \Rightarrow \{\bar{\xi}_{\rm I},\bar{\xi}_{\rm II},\bar{\xi}_{\rm III}\}$. Hereby $\bar{\eta}_0 = \{\bar{\eta}_{\rm II},\bar{\eta}_{\rm III}\}$. The cardinalities of the subsets are

$$\begin{aligned} &\#\bar{\eta}_{\rm I} = a, && \#\bar{\eta}_{\rm II} = n, && \#\bar{\eta}_{\rm III} = a-n, \\ &\#\bar{\xi}_{\rm I} = b, && \#\bar{\xi}_{\rm II} = a, && \#\bar{\xi}_{\rm III} = b-n. \end{aligned} \quad (3.2.24)$$

3.3 Highest coefficient

Equation (3.2.23) allows us to obtain an explicit representation for the scalar product of Bethe vectors. Using (2.1.4) we find

$$S_{a,b} = \frac{(-1)^{\frac{b^2-b}{2}}}{f(\bar{v}^C,\bar{u}^C)} \sum g(\bar{v}^C_{\rm I},\bar{u}^C_{\rm I})f(\bar{u}^C_{\rm II},\bar{u}^C_{\rm I})f(\bar{v}^C_{\rm I},\bar{u}^C_{\rm II})g(\bar{v}^C_{\rm II},\bar{v}^C_{\rm I})h(\bar{u}^C_{\rm I},\bar{u}^C_{\rm I})\; H_{n,a,b}(\bar{u}^C_{\rm I},\bar{u}^C_{\rm II},\bar{v}^C_{\rm II}). \quad (3.3.1)$$

The sum is taken over partitions $\bar{u}^C \Rightarrow \{\bar{u}^C_{\rm I},\bar{u}^C_{\rm II}\}$ and $\bar{v}^C \Rightarrow \{\bar{v}^C_{\rm I},\bar{v}^C_{\rm II}\}$, where $\#\bar{v}^C_{\rm I} = \#\bar{u}^C_{\rm I} = n$, and $n = 0,1,\dots,\min(a,b)$. The function $H_{n,a,b}(\bar{u}^C_{\rm I},\bar{u}^C_{\rm II},\bar{v}^C_{\rm II})$ itself is given as a sum over partitions described in (3.2.23). Namely, the union $\{\bar{u}^C,\bar{u}^B\}$ is divided into three subsets and the union $\{\bar{v}^C_{\rm II},\bar{u}^C,\bar{v}^B\}$ also is divided into three subsets. Although the resulting formula is explicit, it is inconvenient for later use. Therefore, we will try to simplify it. To do this, we introduce a new function.

Definition 3.3.1. *Let $\bar{x}$, $\bar{y}$, $\bar{t}$, $\bar{s}$, and $\bar{\beta}$ be five sets of generic complex numbers with cardinalities $\#\bar{x} = n$, $\#\bar{y} = m$, and $\#\bar{\beta} = n+m$. The cardinalities of the sets $\bar{t}$ and $\bar{s}$ are not fixed. Define a function*

$$J_{n,m}(\bar{x};\bar{y}|\bar{t};\bar{s}|\bar{\beta}) = \Delta_{n+m}(\bar{\beta})\Delta'_n(\bar{x})\Delta'_m(\bar{y}) \det_{n+m} \mathcal{J}_{jk}, \quad (3.3.2)$$

where

$$\left.\begin{aligned} \mathcal{J}_{jk} &= \frac{g(\beta_j,x_k)}{h(\beta_j,x_k)}, && k=1,\dots,n; \\ \mathcal{J}_{j,k+n} &= g(\beta_j,y_k)\frac{h(\beta_j,\bar{t})}{h(\beta_j,\bar{s})}, && k=1,\dots,m; \end{aligned}\right\} \quad j=1,\dots,n+m. \quad (3.3.3)$$

Developing the determinant in (3.3.2) with respect to the first n columns (see Appendix 9.2 for more details) we obtain a presentation of $J_{n,m}$ as a sum over partitions of the set $\bar{\beta}$:

$$J_{n,m}(\bar{x};\bar{y}|\bar{t};\bar{s}|\bar{\beta}) = \sum K_n(\bar{\beta}_{\rm I}|\bar{x}) \frac{g(\bar{\beta}_{\rm II},\bar{\beta}_{\rm I})g(\bar{\beta}_{\rm II},\bar{y})h(\bar{\beta}_{\rm II},\bar{t})}{h(\bar{\beta}_{\rm I},\bar{x})h(\bar{\beta}_{\rm II},\bar{s})}. \tag{3.3.4}$$

Here the sum is taken over partitions $\bar{\beta} \Rightarrow \{\bar{\beta}_{\rm I},\bar{\beta}_{\rm II}\}$, such that $\#\bar{\beta}_{\rm I} = n$ and $\#\bar{\beta}_{\rm II} = m$.

3.3.1 First representation for the highest coefficient

Let us find the highest coefficient $Z_{a,b}(\bar{u}^B;\bar{u}^C|\bar{v}^B;\bar{v}^C)$. We recall that up to the normalisation factor $\big(f(\bar{v}^C,\bar{u}^C)f(\bar{v}^B,\bar{u}^B)\big)^{-1}$ it is the rational coefficient of the product $r_1(\bar{u}^B)r_3(\bar{v}^C)$ (see (3.1.10), (3.1.11)).

Obviously, for this we should set $\bar{\eta}_{\rm I} = \bar{u}^B$ and $\bar{\xi}_{\rm I} = \bar{v}^C$ in (3.2.23). However, $\bar{\xi}_{\rm I} \subset \{\bar{v}^C_{\rm II},\bar{u}^C,\bar{v}^B\}$. Hence, one can have $\bar{\xi}_{\rm I} = \bar{v}^C$ if and only if $\bar{v}^C_{\rm II} = \bar{v}^C$, and thus, $\bar{v}^C_{\rm I} = \emptyset$. But $\#\bar{v}^C_{\rm I} = \#\bar{u}^C_{\rm I} = n$ in (3.3.1), therefore $\bar{u}^C_{\rm I} = \emptyset$ and $n = 0$. Thus, (3.3.1) takes the form

$$\frac{r_1(\bar{u}^B)r_3(\bar{v}^C)\; Z_{a,b}(\bar{u}^B;\bar{u}^C|\bar{v}^B;\bar{v}^C)}{f(\bar{v}^C,\bar{u}^C)f(\bar{v}^B,\bar{u}^B)} = \frac{(-1)^{\frac{b^2-b}{2}}}{f(\bar{v}^C,\bar{u}^C)}\; H_{0,a,b}(\emptyset,\bar{u}^C,\bar{v}^C)\Big|_{\bar{\eta}_{\rm I}=\bar{u}^B;\; \bar{\xi}_{\rm I}=\bar{v}^C}. \tag{3.3.5}$$

Substituting the conditions $\bar{\eta}_{\rm I} = \bar{u}^B$ and $\bar{\xi}_{\rm I} = \bar{v}^C$ into (3.2.23) we should take into account that $\#\bar{\eta}_{\rm II} = n = 0$ (see (3.2.24)). Hence, $\bar{\eta}_{\rm II} = \emptyset$, which implies $\bar{\eta}_{\rm III} = \bar{\eta}_0 = \bar{u}^C$. Thus, substituting these subsets into (3.2.23) we find

$$\begin{aligned} r_1(\bar{u}^B)r_3(\bar{v}^C)\; Z_{a,b}(\bar{u}^B;\bar{u}^C|\bar{v}^B;\bar{v}^C) &= (-1)^a h(\bar{v}^B,\bar{u}^C)h(\bar{u}^C,\bar{u}^C)r_1(\bar{u}^B)r_3(\bar{v}^C) \\ &\times K_a(\bar{u}^C|\bar{u}^C + c) \sum K_a(\bar{\xi}_{\rm II}|\bar{u}^B)\frac{g(\bar{\xi}_{\rm III},\bar{v}^C)g(\bar{\xi}_{\rm III},\bar{\xi}_{\rm II})}{h(\bar{\xi}_{\rm II},\bar{u}^C)}, \end{aligned} \tag{3.3.6}$$

where the sum is taken over partitions $\{\bar{u}^C,\bar{v}^B\} = \bar{\xi} \Rightarrow \{\bar{\xi}_{\rm II},\bar{\xi}_{\rm III}\}$ with $\#\bar{\xi}_{\rm II} = a$ and $\#\bar{\xi}_{\rm III} = b$. Due to (1.1.45) we conclude that $K_a(\bar{u}^C|\bar{u}^C + c) = (-1)^a$, and we arrive at

$$Z_{a,b}(\bar{u}^B;\bar{u}^C|\bar{v}^B;\bar{v}^C) = h(\bar{v}^B,\bar{u}^C)h(\bar{u}^C,\bar{u}^C)\sum K_a(\bar{\xi}_{\rm II}|\bar{u}^B)\frac{g(\bar{\xi}_{\rm III},\bar{v}^C)g(\bar{\xi}_{\rm III},\bar{\xi}_{\rm II})}{h(\bar{\xi}_{\rm II},\bar{u}^C)}. \tag{3.3.7}$$

Finally, using $\{\bar{u}^C,\bar{v}^B\} = \bar{\xi}$ we recast (3.3.7) as follows:

$$\begin{aligned} Z_{a,b}(\bar{u}^B;\bar{u}^C|\bar{v}^B;\bar{v}^C) &= \sum K_a(\bar{\xi}_{\rm II}|\bar{u}^B)g(\bar{\xi}_{\rm III},\bar{v}^C)g(\bar{\xi}_{\rm III},\bar{\xi}_{\rm II})h(\bar{\xi}_{\rm III},\bar{u}^C) \\ &= h(\bar{v}^B,\bar{u}^B)h(\bar{u}^C,\bar{u}^B)\sum K_a(\bar{\xi}_{\rm II}|\bar{u}^B)\frac{g(\bar{\xi}_{\rm III},\bar{v}^C)g(\bar{\xi}_{\rm III},\bar{\xi}_{\rm II})h(\bar{\xi}_{\rm III},\bar{u}^C)}{h(\bar{\xi}_{\rm II},\bar{u}^B)h(\bar{\xi}_{\rm III},\bar{u}^B)}. \end{aligned} \tag{3.3.8}$$

Comparing (3.3.8) and (3.3.4) we conclude that

$$Z_{a,b}(\bar{u}^B;\bar{u}^C|\bar{v}^B;\bar{v}^C) = h(\bar{v}^B,\bar{u}^B)h(\bar{u}^C,\bar{u}^B)\; J_{a,b}(\bar{u}^B;\bar{v}^C|\bar{u}^C;\bar{u}^B|\{\bar{v}^B,\bar{u}^C\}). \tag{3.3.9}$$

Thus, we have obtained an explicit representation for the highest coefficient $Z_{a,b}(\bar{u}^B;\bar{u}^C|\bar{v}^B;\bar{v}^C)$ in terms of the determinant of the $(a+b)\times(a+b)$ matrix $\mathcal{J}_{jk}$ (3.3.3).

3.3.2 Second highest coefficient

In order to obtain the second highest coefficient $Z_{a,b}(\bar{u}^C;\bar{u}^B|\bar{v}^C;\bar{v}^B)$ it is enough to make the replacements $\bar{u}^C \leftrightarrow \bar{u}^B$ and $\bar{v}^C \leftrightarrow \bar{v}^B$ in (3.3.9). On the other hand, this coefficient should arise if we set $\bar{\eta}_{\rm I} = \bar{u}^C$ and $\bar{\xi}_{\rm I} = \bar{v}^B$ in (3.2.23). However, if we do so, then we do not obtain (3.3.9) with the replacements mentioned above. Instead, we obtain much more sophisticated formula involving many sums over partitions. This 'break of symmetry' occurs because we use a specific representation (2.1.4) for the dual Bethe vector. If we would use equation (2.1.3) for $\mathbb{C}_{a,b}(\bar{u};\bar{v})$, then we would have an analog of (3.3.9) for $Z_{a,b}(\bar{u}^C;\bar{u}^B|\bar{v}^C;\bar{v}^B)$, however, we would have a more complex formula for $Z_{a,b}(\bar{u}^B;\bar{u}^C|\bar{v}^B;\bar{v}^C)$.

A 'complex' formula for the highest coefficient provides us with a very non-trivial identity for $Z_{a,b}(\bar{u}^C;\bar{u}^B|\bar{v}^C;\bar{v}^B)$, that will be used later. In order to obtain this identity we first make several additional summations in (3.3.1). Let us rewrite this equation explicitly

$$S_{a,b} = \frac{h(\bar{v}^B,\bar{u}^C)h(\bar{u}^C,\bar{u}^C)}{f(\bar{v}^C,\bar{u}^C)} \sum (-1)^{a+n(b+1)} g(\bar{v}^C_{\rm I},\bar{u}^C) g(\bar{u}^C_{\rm II},\bar{u}^C_{\rm I}) h(\bar{v}^C_{\rm I},\bar{u}^C_{\rm II}) g(\bar{v}^C_{\rm II},\bar{v}^C_{\rm I}) h(\bar{u}^C,\bar{u}^C_{\rm I})$$
$$\times\, r_1(\bar{\eta}_{\rm I}) r_3(\bar{\xi}_{\rm I})\, K_a(\bar{\xi}_{\rm II}|\bar{\eta}_{\rm I}) K_{a-n}(\bar{u}^C_{\rm II}|\bar{\eta}_{\rm III}+c)$$
$$\times\, \frac{f(\bar{\eta}_0,\bar{\eta}_{\rm I})h(\bar{\eta}_0,\bar{\eta}_{\rm II})g(\bar{\eta}_{\rm III},\bar{\eta}_{\rm II})f(\bar{v}^C_{\rm II},\bar{\eta}_{\rm I})h(\bar{v}^C_{\rm II},\bar{\eta}_{\rm II})g(\bar{\xi}_{\rm I},\bar{\xi}_{\rm II})g(\bar{\xi}_{\rm III},\bar{\xi}_{\rm I})g(\bar{\xi}_{\rm III},\bar{\xi}_{\rm II})}{h(\bar{\xi}_{\rm I},\bar{\eta}_{\rm II})f(\bar{\xi},\bar{\eta}_{\rm I})h(\bar{\eta}_{\rm II},\bar{u}^C_{\rm I})f(\bar{\eta}_{\rm III},\bar{u}^C_{\rm I})g(\bar{v}^C_{\rm II},\bar{\xi}_{\rm II})h(\bar{\xi}_{\rm II},\bar{u}^C)}. \qquad (3.3.10)$$

The sum over partitions into subsets $\bar{\xi}_{\rm II}$ and $\bar{\xi}_{\rm III}$, as well as the sum over partitions $\bar{u}^C \Rightarrow \{\bar{u}^C_{\rm I},\bar{u}^C_{\rm II}\}$ can be computed in terms of the function J (3.3.2). Let $\bar{\xi}_0 = \{\bar{\xi}_{\rm II},\bar{\xi}_{\rm III}\}$. Then

$$\sum_{\bar{\xi}_0\Rightarrow\{\bar{\xi}_{\rm II},\bar{\xi}_{\rm III}\}} \frac{K_a(\bar{\xi}_{\rm II}|\bar{\eta}_{\rm I})g(\bar{\xi}_{\rm I},\bar{\xi}_{\rm II})g(\bar{\xi}_{\rm III},\bar{\xi}_{\rm I})g(\bar{\xi}_{\rm III},\bar{\xi}_{\rm II})}{g(\bar{v}^C_{\rm II},\bar{\xi}_{\rm II})h(\bar{\xi}_{\rm II},\bar{u}^C)}$$
$$= \frac{(-1)^{ab+n+b}g(\bar{\xi}_0,\bar{\xi}_{\rm I})h(\bar{\xi}_0,\bar{\eta}_{\rm I})}{g(\bar{v}^C_{\rm II},\bar{\xi}_0)h(\bar{\xi}_0,\bar{u}^C)} J_{a,b-n}(\bar{\eta}_{\rm I};\bar{v}^C_{\rm II}|\bar{u}^C;\bar{\eta}_{\rm I}|\bar{\xi}_0). \qquad (3.3.11)$$

Similarly, one can verify that

$$\sum_{\bar{u}^C\Rightarrow\{\bar{u}^C_{\rm I},\bar{u}^C_{\rm II}\}} \frac{K_{a-n}(\bar{u}^C_{\rm II}|\bar{\eta}_{\rm III}+c)g(\bar{u}^C_{\rm II},\bar{u}^C_{\rm I})h(\bar{v}^C_{\rm I},\bar{u}^C_{\rm II})h(\bar{u}^C,\bar{u}^C_{\rm I})}{h(\bar{\eta}_{\rm II},\bar{u}^C_{\rm I})f(\bar{\eta}_{\rm III},\bar{u}^C_{\rm I})}$$
$$= (-1)^{a+n+an}\frac{h(\bar{v}^C_{\rm I},\bar{u}^C)}{g(\bar{\eta}_{\rm III},\bar{u}^C)} J_{a-n,n}(\bar{\eta}_{\rm III};\bar{v}^C_{\rm I}|\bar{u}^C+c;\bar{\eta}_0+c|\bar{u}^C-c). \qquad (3.3.12)$$

Substituting these results into (3.3.10) we find

$$S_{a,b} = \frac{h(\bar{v}^B,\bar{u}^C)h(\bar{u}^C,\bar{u}^C)}{f(\bar{v}^C,\bar{u}^C)} \sum (-1)^{b+ab+n(a+b+1)} r_1(\bar{\eta}_{\rm I}) r_3(\bar{\xi}_{\rm I}) J_{a,b-n}(\bar{\eta}_{\rm I};\bar{v}^C_{\rm II}|\bar{u}^C;\bar{\eta}_{\rm I}|\bar{\xi}_0)$$
$$\times\, J_{a-n,n}(\bar{\eta}_{\rm III};\bar{v}^C_{\rm I}|\bar{u}^C+c;\bar{\eta}_0+c|\bar{u}^C-c) f(\bar{\eta}_0,\bar{\eta}_{\rm I})h(\bar{\eta}_0,\bar{\eta}_{\rm II})g(\bar{\eta}_{\rm III},\bar{\eta}_{\rm II})$$
$$\times\, \frac{h(\bar{v}^C_{\rm II},\bar{\eta}_{\rm II})f(\bar{v}^C_{\rm I},\bar{u}^C)g(\bar{v}^C_{\rm II},\bar{v}^C_{\rm I})g(\bar{\xi}_0,\bar{\xi}_{\rm I})h(\bar{\xi}_0,\bar{\eta}_{\rm I})}{g(\bar{\eta}_{\rm III},\bar{u}^C)f(\bar{u}^C,\bar{\eta}_{\rm I})f(\bar{v}^B,\bar{\eta}_{\rm I})h(\bar{\xi}_{\rm I},\bar{\eta}_{\rm II})g(\bar{v}^C_{\rm II},\bar{\xi}_0)h(\bar{\xi}_0,\bar{u}^C)}. \qquad (3.3.13)$$

The sum is taken over partitions:
(1) $\bar{v}^C \Rightarrow \{\bar{v}^C_{\rm I},\bar{v}^C_{\rm II}\}$ with $\#\bar{v}^C_{\rm I} = n$ and $\#\bar{v}^C_{\rm II} = b-n$;
(2) $\{\bar{u}^C,\bar{v}^B,\bar{v}^C_{\rm II}\} = \bar{\xi} \Rightarrow \{\bar{\xi}_{\rm I},\bar{\xi}_0\}$ with $\#\bar{\xi}_{\rm I} = b$ and $\#\bar{\xi}_0 = b+a-n$;
(3) $\{\bar{u}^C,\bar{u}^B\} = \bar{\eta} \Rightarrow \{\bar{\eta}_{\rm I},\bar{\eta}_0\}$ and $\bar{\eta}_0 \Rightarrow \{\bar{\eta}_{\rm II},\bar{\eta}_{\rm III}\}$ with $\#\bar{\eta}_{\rm I} = a$, $\#\bar{\eta}_{\rm II} = n$, and $\#\bar{\eta}_{\rm III} = a-n$.
In all these partitions $n = 0,1,\dots,\min(a,b)$.

Due to proposition 3.1.1 the function r_3 depends on the variables of the v-type only. Hence, $\bar\xi_{\rm I}\cap\bar u^C=\emptyset$, that is $\bar u^C\subset\bar\xi_0$. Therefore, we can set $\bar\xi_0=\{\bar u^C,\bar\xi_{\rm II}\}$. Substituting this into (3.3.13) we obtain

$$S_{a,b}=\frac{h(\bar v^B,\bar u^C)}{f(\bar v^C,\bar u^C)}\sum(-1)^{b+n(a+b+1)}r_1(\bar\eta_{\rm I})r_3(\bar\xi_{\rm I})J_{a,b-n}(\bar\eta_{\rm I};\bar v^C_{\rm II}|\bar u^C;\bar\eta_{\rm I}|\{\bar\xi_{\rm II},\bar u^C\})$$
$$\times J_{a-n,n}(\bar\eta_{\rm III};\bar v^C_{\rm I}|\bar u^C+c;\bar\eta_0+c|\bar u^C-c)f(\bar\eta_0,\bar\eta_{\rm I})h(\bar\eta_0,\bar\eta_{\rm II})g(\bar\eta_{\rm III},\bar\eta_{\rm II})$$
$$\times\frac{h(\bar v^C_{\rm II},\bar\eta_{\rm II})f(\bar v^C_{\rm I},\bar u^C)g(\bar v^C_{\rm II},\bar v^C_{\rm I})g(\bar\xi_{\rm II},\bar\xi_{\rm I})h(\bar\xi_{\rm II},\bar\eta_{\rm I})g(\bar\xi_{\rm I},\bar u^C)}{g(\bar\eta_{\rm III},\bar u^C)g(\bar u^C,\bar\eta_{\rm I})f(\bar v^B,\bar\eta_{\rm I})h(\bar\xi_{\rm I},\bar\eta_{\rm II})g(\bar v^C_{\rm II},\bar\xi_{\rm II})h(\bar\xi_{\rm II},\bar u^C)g(\bar v^C_{\rm II},\bar u^C)}.\qquad(3.3.14)$$

In this formula $\bar\xi=\{\bar v^C_{\rm II},\bar v^B\}$, $\#\bar\xi_{\rm I}=b$ and $\#\bar\xi_{\rm II}=b-n$. All other subsets are the same as in (3.3.13).

Now everything is ready to formulate the second representation for the highest coefficient $Z_{a,b}(\bar u^C;\bar u^B|\bar v^C;\bar v^B)$. For this we set $\bar\eta_{\rm I}=\bar u^C$ and $\bar\xi_{\rm I}=\bar v^B$. Then automatically $\bar\eta_0=\bar u^B$, $\bar\xi_{\rm II}=\bar v^C_{\rm II}$ and we also can set $\bar\eta_{\rm II}=\bar u^B_{\rm I}$, $\bar\eta_{\rm III}=\bar u^B_{\rm II}$. Substituting this into (3.3.14) and keeping in mind remark 3.2.1 we obtain

$$Z_{a,b}(\bar u^C;\bar u^B|\bar v^C;\bar v^B)=f(\bar v^B,\bar u^B)f(\bar u^B,\bar u^C)\sum(-1)^{n(a+b+1)+b}g(\bar v^C_{\rm II},\bar v^C_{\rm I})f(\bar v^C_{\rm I},\bar u^C)$$
$$\times J_{a-n,n}(\bar u^B_{\rm II};\bar v^C_{\rm I}|\bar u^C+c;\bar u^B+c|\bar u^C-c)\frac{h(\bar u^B,\bar u^B_{\rm I})g(\bar u^B_{\rm II},\bar u^B_{\rm I})h(\bar v^C_{\rm II},\bar u^B_{\rm I})g(\bar v^C_{\rm II},\bar v^B)h(\bar v^C_{\rm II},\bar u^C)}{g(\bar u^B_{\rm II},\bar u^C)h(\bar v^B,\bar u^B_{\rm I})g(\bar v^C_{\rm II},\bar u^C)h(\bar v^C_{\rm II},\bar u^C)}$$
$$\times\lim_{\substack{\bar\eta_{\rm I}\to\bar u^C\\ \bar\xi_{\rm II}\to\bar v^C_{\rm II}}}\frac{J_{a,b-n}(\bar\eta_{\rm I};\bar v^C_{\rm II}|\bar u^C;\bar\eta_{\rm I}|\{\bar u^C,\bar\xi_{\rm II}\})}{g(\bar u^C,\bar\eta_{\rm I})g(\bar v^C_{\rm II},\bar\xi_{\rm II})}.\qquad(3.3.15)$$

Using (9.3.3) and (9.3.4) we find

$$\lim_{\substack{\bar\eta_{\rm I}\to\bar u^C\\ \bar\xi_{\rm II}\to\bar v^C_{\rm II}}}\frac{J_{a,b-n}(\bar\eta_{\rm I};\bar v^C_{\rm II}|\bar u^C;\bar\eta_{\rm I}|\{\bar u^C,\bar\xi_{\rm II}\})}{g(\bar u^C,\bar\eta_{\rm I})g(\bar v^C_{\rm II},\bar\xi_{\rm II})}=(-1)^{b+n}g(\bar v^C_{\rm II},\bar u^C),\qquad(3.3.16)$$

and we thus arrive at

$$Z_{a,b}(\bar u^C;\bar u^B|\bar v^C;\bar v^B)=f(\bar u^B,\bar u^C)f(\bar v^B,\bar u^B)\sum(-1)^{n(a+b)}J_{a-n,n}(\bar u^B_{\rm II};\bar v^C_{\rm I}|\bar u^C+c;\bar u^B+c|\bar u^C-c)$$
$$\times h(\bar u^B_{\rm I},\bar u^B_{\rm I})f(\bar u^B_{\rm II},\bar u^B_{\rm I})\;\frac{f(\bar v^C_{\rm I},\bar u^C)h(\bar v^C_{\rm II},\bar u^B_{\rm I})g(\bar v^C_{\rm II},\bar v^C_{\rm I})g(\bar v^C_{\rm II},\bar v^B)}{h(\bar v^B,\bar u^B_{\rm I})g(\bar u^B_{\rm II},\bar u^C)}.\qquad(3.3.17)$$

Here the sum is taken over partitions $\bar u^B\Rightarrow\{\bar u^B_{\rm II},\bar u^B_{\rm I}\}$ and $\bar v^C\Rightarrow\{\bar v^C_{\rm I},\bar v^C_{\rm II}\}$ with $\#\bar v^C_{\rm I}=n$ and $\#\bar u^B_{\rm II}=a-n$.

This is the second representation for the highest coefficient discussed above. It will play the key role below, therefore we formulate it as a proposition.

Conjecture 3.3.1. *For arbitrary sets of complex numbers $\bar t$, $\bar x$, $\bar s$, and $\bar y$ with cardinalities $\#\bar t=\#\bar x=a$ and $\#\bar s=\#\bar y=b$ the following identity holds:*

$$Z_{a,b}(\bar t;\bar x|\bar s;\bar y)=f(\bar x,\bar t)f(\bar y,\bar x)\sum(-1)^{n_1(a+b)}J_{\ell_2,n_1}(\bar x_{\rm II};\bar s_{\rm I}|\bar t+c;\bar x+c|\bar t-c)$$
$$\times h(\bar x_{\rm I},\bar x_{\rm I})f(\bar x_{\rm II},\bar x_{\rm I})\;\frac{f(\bar s_{\rm I},\bar t)h(\bar s_{\rm II},\bar x_{\rm I})g(\bar s_{\rm II},\bar s_{\rm I})g(\bar s_{\rm II},\bar y)}{h(\bar y,\bar x_{\rm I})g(\bar x_{\rm II},\bar t)}.\qquad(3.3.18)$$

Here $\ell_2 = \#\bar{x}_{\rm II}$, $n_1 = \#\bar{s}_{\rm I}$. *The sum is taken over partitions*

$$\bar{x} \Rightarrow \{\bar{x}_{\rm I}, \bar{x}_{\rm II}\}, \qquad \bar{s} \Rightarrow \{\bar{s}_{\rm I}, \bar{s}_{\rm II}\}, \tag{3.3.19}$$

with a restriction $\#\bar{s}_{\rm I} = \#\bar{x}_{\rm I}$ *(which is equivalent to* $\ell_2 + n_1 = a$*).*

Proof. Setting in (3.3.17) $\bar{u}^C = \bar{t}$, $\bar{u}^B = \bar{x}$, $\bar{v}^C = \bar{s}$, and $\bar{v}^B = \bar{y}$ we obtain (3.3.18).

3.3.3 General formula for the scalar product

Now we turn back to equation (3.3.14). To proceed further we should specify all the subsets. Let $\bar{v}^C = \{\bar{v}^C_{\rm i}, \bar{v}^C_{\rm ii}, \bar{v}^C_{\rm iii}\}$ and $\bar{v}^B = \{\bar{v}^B_{\rm i}, \bar{v}^C_{\rm ii}\}$. We set

$$\begin{aligned} &\bar{v}^C_{\rm I} = \bar{v}^C_{\rm iii}, && \bar{v}^C_{\rm II} = \{\bar{v}^C_{\rm i}, \bar{v}^C_{\rm ii}\},\\ &\bar{v}^B = \{\bar{v}^B_{\rm i}, \bar{v}^B_{\rm ii}\}, && \\ &\bar{\xi}_{\rm I} = \{\bar{v}^C_{\rm i}, \bar{v}^B_{\rm ii}\}, && \bar{\xi}_{\rm II} = \{\bar{v}^C_{\rm ii}, \bar{v}^B_{\rm i}\}, \end{aligned} \quad \text{with} \quad \begin{cases} \#\bar{v}^C_s = n_s,\\ \#\bar{v}^B_s = m_s, \end{cases} \quad s = {\rm i}, {\rm ii}, {\rm iii}. \tag{3.3.20}$$

It is easy to see that the following conditions for the cardinalities hold:

$$n_{\rm iii} = n, \quad n_{\rm i} + n_{\rm ii} = b - n, \quad m_{\rm i} + m_{\rm ii} = b, \quad m_{\rm i} = n_{\rm i}. \tag{3.3.21}$$

Let also

$$\begin{aligned} &\bar{u}^C = \{\bar{u}^C_{\rm i}, \bar{u}^C_0\}, && \bar{u}^C_0 = \{\bar{u}^C_{\rm ii}, \bar{u}^C_{\rm iii}\},\\ &\bar{u}^B = \{\bar{u}^B_{\rm i}, \bar{u}^B_0\}, && \bar{u}^B_{\rm i} = \{\bar{u}^B_{\rm ii}, \bar{u}^B_{\rm iii}\},\\ &\bar{\eta}_{\rm I} = \{\bar{u}^C_{\rm i}, \bar{u}^B_0\}, && \bar{\eta}_0 = \{\bar{u}^C_0, \bar{u}^B_{\rm i}\}, \end{aligned} \quad \text{with} \quad \begin{cases} \#\bar{u}^C_p = k_p,\\ \#\bar{u}^B_p = \ell_p, \end{cases} \quad p = 0, {\rm i}, {\rm ii}, {\rm iii}. \tag{3.3.22}$$

We have the following conditions for the cardinalities:

$$k_{\rm i} + k_0 = \ell_{\rm i} + \ell_0 = a, \quad k_{\rm i} = \ell_{\rm i}, \quad k_0 = \ell_0. \tag{3.3.23}$$

Observe that we do not fix a distribution of the parameters $\bar{u}^C$ and $\bar{u}^B$ among the subsets $\bar{\eta}_{\rm II}$ and $\bar{\eta}_{\rm III}$. It is important, however, that $\#\bar{\eta}_{\rm II} = n = n_{\rm iii}$ and $\#\bar{\eta}_{\rm III} = a - n$.

Using (9.3.3) and (9.3.4) we obtain

$$\begin{aligned} \frac{J_{a,b-n}(\bar{\eta}_{\rm I}; \bar{v}^C_{\rm II} | \bar{u}^C; \bar{\eta}_{\rm I} | \{\bar{u}^C, \bar{\xi}_{\rm II}\})}{g(\bar{u}^C, \bar{\eta}_{\rm I}) g(\bar{v}^C_{\rm II}, \bar{\xi}_{\rm II})} &= (-1)^{b-n} \frac{g(\bar{v}^C_{\rm ii}, \bar{u}^C) g(\bar{v}^B_{\rm i}, \bar{u}^C_{\rm i}) h(\bar{v}^C_{\rm ii}, \bar{u}^C_0)}{g(\bar{u}^C_0, \bar{u}^B_0) g(\bar{v}^B_{\rm i}, \bar{v}^C_{\rm i}) h(\bar{v}^C_{\rm ii}, \bar{u}^B_0)}\\ &\quad \times J_{\ell_0, n_{\rm i}}(\bar{u}^B_0; \bar{v}^C_{\rm i} | \bar{u}^C_0; \bar{u}^B_0 | \{\bar{u}^C_0, \bar{v}^B_{\rm i}\}). \end{aligned} \tag{3.3.24}$$

Due to (3.3.9) this function reduces to the highest coefficient

$$J_{\ell_0, n_{\rm i}}(\bar{u}^B_0; \bar{v}^C_{\rm i} | \bar{u}^C_0; \bar{u}^B_0 | \{\bar{u}^C_0, \bar{v}^B_{\rm i}\}) = \frac{Z_{\ell_0, n_{\rm i}}(\bar{u}^B_0, \bar{u}^C_0 | \bar{v}^B_{\rm i}, \bar{v}^C_{\rm i})}{h(\bar{u}^C_0, \bar{u}^B_0) h(\bar{v}^B_{\rm i}, \bar{u}^B_0)}. \tag{3.3.25}$$

Now we substitute (3.3.24), (3.3.25) into (3.3.14). We also write explicitly the products $g(\bar{v}^C_{\rm II}, \bar{v}^C_{\rm I})$, $g(\bar{\xi}_{\rm II}, \bar{\xi}_{\rm I})$, $f(\bar{\eta}_0, \bar{\eta}_{\rm I})$, and combine $\{\bar{v}^C_{\rm ii}, \bar{v}^C_{\rm iii}\} = \bar{v}^C_0$. Then we have

$$\begin{aligned} S_{a,b} = \frac{1}{f(\bar{v}^C, \bar{u}^C)} \sum r_1(\bar{u}^C_{\rm i}) r_1(\bar{u}^B_0) r_3(\bar{v}^C_{\rm i}) r_3(\bar{v}^B_{\rm ii})\ Z_{\ell_0, n_{\rm i}}(\bar{u}^B_0, \bar{u}^C_0 | \bar{v}^B_{\rm i}, \bar{v}^C_{\rm i}) \frac{f(\bar{u}^B_{\rm i}, \bar{u}^C_{\rm i}) f(\bar{v}^B_{\rm ii}, \bar{u}^C)}{f(\bar{v}^B, \bar{u}^B_0) f(\bar{v}^B_{\rm ii}, \bar{u}^C_{\rm i})}\\ \times f(\bar{u}^C_0, \bar{u}^C_{\rm i}) f(\bar{u}^B_{\rm i}, \bar{u}^B_0) g(\bar{v}^C_0, \bar{v}^C_{\rm i}) g(\bar{v}^B_{\rm i}, \bar{v}^B_{\rm ii}) \quad \Big\{ J_{a-n,n}(\bar{\eta}_{\rm III}; \bar{v}^C_{\rm iii} | \bar{u}^C + c; \bar{\eta}_0 + c | \bar{u}^C - c)\\ \times (-1)^{n(a+b-n_{\rm i})} h(\bar{\eta}_{\rm II}, \bar{\eta}_{\rm II}) f(\bar{\eta}_{\rm III}, \bar{\eta}_{\rm II}) \frac{f(\bar{v}^C_{\rm iii}, \bar{u}^C) g(\bar{v}^C_{\rm ii}, \bar{v}^C_{\rm iii}) g(\bar{v}^C_{\rm ii}, \bar{v}^B_{\rm ii}) h(\bar{v}^C_{\rm ii}, \bar{\eta}_{\rm II})}{g(\bar{\eta}_{\rm III}, \bar{u}^C) h(\bar{v}^B_{\rm ii}, \bar{\eta}_{\rm II})} \Big\}. \end{aligned} \tag{3.3.26}$$

Here the sum is organized as follows. First, we have partitions

$$\begin{aligned}
&\bar u^C \Rightarrow \{\bar u^C_{\rm i}, \bar u^C_0\}, &&\bar u^B \Rightarrow \{\bar u^B_{\rm i}, \bar u^B_0\}, &&\#\bar u^C_0 = \#\bar u^B_0 = \ell_0,\\
&\bar v^C \Rightarrow \{\bar v^C_{\rm i}, \bar v^C_0\}, &&\bar v^B \Rightarrow \{\bar v^B_{\rm i}, \bar v^B_{\rm ii}\}, &&\#\bar v^C_{\rm i} = \#\bar v^B_{\rm i} = n_{\rm i}.
\end{aligned} \tag{3.3.27}$$

After this we have two additional partitions: the set $\bar v^C_0$ is divided into subsets $\bar v^C_{\rm ii}$ and $\bar v^C_{\rm iii}$; the union of the subsets $\bar\eta_0 = \{\bar u^C_0, \bar u^B_{\rm i}\}$ is divided into subsets $\bar\eta_{\rm II}$ and $\bar\eta_{\rm III}$ (see the terms in braces in (3.3.26)). Hereby we have one restriction for the cardinalities $\#\bar v^C_{\rm iii} = \#\bar\eta_{\rm II} = n = n_{\rm iii}$. Let us write separately this additional sum over partitions in braces of (3.3.26)

$$\begin{aligned}
\mathcal F(\bar u^C; \bar\eta_0; \bar v^C_0, \bar v^B_{\rm ii}) = \sum_{\substack{\bar\eta_0 \Rightarrow \{\bar\eta_{\rm II}, \bar\eta_{\rm III}\}\\ \bar v^C_0 \Rightarrow \{\bar v^C_{\rm ii}, \bar v^C_{\rm iii}\}}} &J_{a-n,n}(\bar\eta_{\rm III}; \bar v^C_{\rm iii} | \bar u^C + c; \bar\eta_0 + c | \bar u^C - c)\\
&\times (-1)^{n(a+b-n_{\rm i})} h(\bar\eta_{\rm II}, \bar\eta_{\rm II}) f(\bar\eta_{\rm III}, \bar\eta_{\rm II}) \frac{f(\bar v^C_{\rm iii}, \bar u^C) g(\bar v^C_{\rm ii}, \bar v^C_{\rm iii}) g(\bar v^C_{\rm ii}, \bar v^B_{\rm ii}) h(\bar v^C_{\rm ii}, \bar\eta_{\rm II})}{g(\bar\eta_{\rm III}, \bar u^C) h(\bar v^B_{\rm ii}, \bar\eta_{\rm II})}.
\end{aligned} \tag{3.3.28}$$

Comparing (3.3.28) with (3.3.18) one can see that they coincide after appropriate identification of the subsets and their cardinalities. Namely, (3.3.28) turns into (3.3.18) under the replacements $b - n_{\rm i} \to b$, $\bar u^C \to \bar t$, $\bar\eta_0 \to \bar x$, $\bar v^C_0 \to \bar s$, $\bar v^B_{\rm ii} \to \bar y$. Thus, due to Proposition 3.3.1 we obtain

$$\mathcal F(\bar u^C; \bar\eta_0; \bar v^C_0, \bar v^B_{\rm ii}) = \frac{Z_{a,b-n_{\rm i}}(\bar u^C; \bar\eta_0 | \bar v^C_0, \bar v^B_{\rm ii})}{f(\bar\eta_0, \bar u^C) f(\bar v^B_{\rm ii}, \bar\eta_0)}. \tag{3.3.29}$$

Thus, substituting this into (3.3.26) we arrive at

$$\begin{aligned}
S_{a,b} = \frac{1}{f(\bar v^C, \bar u^C)} \sum r_1(\bar u^C_{\rm i}) r_1(\bar u^B_0) r_3(\bar v^C_{\rm i}) r_3(\bar v^B_{\rm ii})\, Z_{\ell_0,n_{\rm i}}(\bar u^B_0, \bar u^C_0 | \bar v^B_{\rm i}, \bar v^C_{\rm i}) \frac{f(\bar u^B_{\rm i}, \bar u^C_{\rm i}) f(\bar v^B_{\rm ii}, \bar u^C)}{f(\bar v^B, \bar u^B_0) f(\bar v^B_{\rm ii}, \bar u^C_{\rm i})}\\
\times f(\bar u^C_0, \bar u^C_{\rm i}) f(\bar u^B_{\rm i}, \bar u^B_0) g(\bar v^C_0, \bar v^C_{\rm i}) g(\bar v^B_{\rm i}, \bar v^B_{\rm ii}) \frac{Z_{a,b-n_{\rm i}}(\bar u^C; \bar\eta_0 | \bar v^C_0, \bar v^B_{\rm ii})}{f(\bar\eta_0, \bar u^C) f(\bar v^B_{\rm ii}, \bar\eta_0)}.
\end{aligned} \tag{3.3.30}$$

It remains to simplify the ratio $Z_{a,b-n_{\rm i}}(\bar u^C; \bar\eta_0 | \bar v^C_0, \bar v^B_{\rm ii}) / f(\bar\eta_0, \bar u^C)$. It can be done via (3.3.9), (9.3.3):

$$\frac{Z_{a,b-n_{\rm i}}(\bar u^C; \bar\eta_0 | \bar v^C_0, \bar v^B_{\rm ii})}{f(\bar\eta_0, \bar u^C) f(\bar v^B_{\rm ii}, \bar\eta_0)} = \frac{f(\bar v^C_0, \bar u^C_0) Z_{a-\ell_0,b-n_{\rm i}}(\bar u^C_{\rm i}; \bar u^B_{\rm i} | \bar v^C_0, \bar v^B_{\rm ii})}{f(\bar u^B_{\rm i}, \bar u^C_{\rm i}) f(\bar v^B_{\rm ii}, \bar u^B_{\rm i}) f(\bar v^B_{\rm ii}, \bar u^C_0)}. \tag{3.3.31}$$

Substituting this into (3.3.30) we obtain

$$\begin{aligned}
S_{a,b} = \sum r_1(\bar u^C_{\rm i}) r_1(\bar u^B_0) r_3(\bar v^C_{\rm i}) r_3(\bar v^B_{\rm ii}) f(\bar u^C_0, \bar u^C_{\rm i}) f(\bar u^B_{\rm i}, \bar u^B_0) g(\bar v^C_0, \bar v^C_{\rm i}) g(\bar v^B_{\rm i}, \bar v^B_{\rm ii})\\
\times \frac{f(\bar v^C_0, \bar u^C_0) f(\bar v^B_{\rm i}, \bar u^B_{\rm i})}{f(\bar v^C, \bar u^C) f(\bar v^B, \bar u^B)}\, Z_{\ell_0,n_{\rm i}}(\bar u^B_0, \bar u^C_0 | \bar v^B_{\rm i}, \bar v^C_{\rm i})\, Z_{a-\ell_0,b-n_{\rm i}}(\bar u^C_{\rm i}, \bar u^B_{\rm i} | \bar v^C_0, \bar v^B_{\rm ii}).
\end{aligned} \tag{3.3.32}$$

It is easy to see that after appropriate relabeling the subsets we arrive at

$$\begin{aligned}
S_{a,b} = \sum r_1(\bar u^C_{\rm II}) r_1(\bar u^B_{\rm I}) r_3(\bar v^C_{\rm II}) r_3(\bar v^B_{\rm I}) f(\bar u^C_{\rm I}, \bar u^C_{\rm II}) f(\bar u^B_{\rm II}, \bar u^B_{\rm I}) g(\bar v^C_{\rm I}, \bar v^C_{\rm II}) g(\bar v^B_{\rm II}, \bar v^B_{\rm I})\\
\times \frac{f(\bar v^C_{\rm I}, \bar u^C_{\rm I}) f(\bar v^B_{\rm II}, \bar u^B_{\rm II})}{f(\bar v^C, \bar u^C) f(\bar v^B, \bar u^B)}\, Z_{a-k,n}(\bar u^C_{\rm II}, \bar u^B_{\rm II} | \bar v^C_{\rm I}, \bar v^B_{\rm I})\, Z_{k,b-n}(\bar u^B_{\rm I}, \bar u^C_{\rm I} | \bar v^B_{\rm II}, \bar v^C_{\rm II}),
\end{aligned} \tag{3.3.33}$$

where $k = \#\bar u^C_{\rm I} = \#\bar u^B_{\rm I}$ and $n = \#\bar v^C_{\rm I} = \#\bar v^B_{\rm I}$. Comparing this result with (3.1.8) and (3.1.11) we see that proposition 3.1.2 is proved.

3.4 Scalar product in the $\mathfrak{gl}(1|1)$ sector

Consider a particular case of the subalgebra $\mathfrak{gl}(1|1)$, generated by the operators $T_{23}(u)$, $T_{22}(u)$, $T_{33}(u)$ and $T_{32}(u)$. In this case one should set $\bar u^C = \bar u^B = \emptyset$ in the formulas for the scalar product. Then the highest coefficient simplifies as

$$Z_{0,b}(\emptyset;\emptyset|\bar s;\bar y) = \Delta_b(\bar s)\Delta'_b(\bar y)\ \det_b g(s_j,y_k) = g(\bar s,\bar y), \tag{3.4.1}$$

where we used an explicit representation for Cauchy determinant

$$\det_m g(u_j,v_k) = \frac{g(\bar u,\bar v)}{\Delta'_m(\bar u)\Delta_m(\bar v)}. \tag{3.4.2}$$

The scalar product (3.3.33) takes the form

$$S_{0,b} = \sum r_3(\bar v^C_{\rm II})r_3(\bar v^B_{\rm I})g(\bar v^C_{\rm I},\bar v^C_{\rm II})g(\bar v^B_{\rm II},\bar v^B_{\rm I})g(\bar v^C_{\rm I},\bar v^B_{\rm I})g(\bar v^B_{\rm II},\bar v^C_{\rm II}), \tag{3.4.3}$$

where the sum is taken over partitions $\bar v^C \Rightarrow \{\bar v^C_{\rm I},\bar v^C_{\rm II}\}$ and $\bar v^B \Rightarrow \{\bar v^B_{\rm I},\bar v^B_{\rm II}\}$ such that $\#\bar v^C_{\rm I} = \bar v^B_{\rm I}$. It is easy to see that this sum reduces to a single determinant

$$S_{0,b} = \Delta'_b(\bar v^C)\Delta_b(\bar v^B)\det_b\Big[g(v^C_j,v^B_k)\big(r_3(v^B_k) - r_3(v^C_j)\big)\Big]. \tag{3.4.4}$$

Indeed, developing the determinant in (3.4.4) via Laplace formula and using (3.4.2), (9.2.5) we obtain the sum (3.4.3).

Thus, the scalar product of Bethe vectors in $\mathfrak{gl}(1|1)$ integrable models admits a determinant representation without any restriction on the Bethe parameters. This is not surprising, as these models are equivalent to free fermions [76, 80].

3.5 Different representations for the highest coefficient

If $\bar v^C = \bar v^B = \emptyset$, then formula (3.3.33) describes the scalar product in the $\mathfrak{gl}(2)$-based models. In this case the scalar product admits a determinant representation, if one of the Bethe vectors is an eigenvector of the transfer matrix. One expects that in the general $\mathfrak{gl}(2|1)$ case the sum over partitions in (3.3.33) also can be reduced to a single determinant for some particular cases of Bethe vectors. To make this reduction one should have different representations for the highest coefficient $Z_{a,b}(\bar t;\bar x|\bar s;\bar y)$. In this section we give several formulas for $Z_{a,b}$ in terms of sums over partitions and multiple contour integrals.

We have already obtained an expression for $Z_{a,b}$ as the determinant of an $(a+b)\times(a+b)$ matrix

$$Z_{a,b}(\bar t;\bar x|\bar s;\bar y) = h(\bar w,\bar t)\Delta_{a+b}(\bar w)\Delta'_a(\bar t)\Delta'_b(\bar y)\ \det_{a+b}\mathcal{J}_{jk}, \tag{3.5.1}$$

where $\bar w = \{\bar x,\bar s\}$ and the matrix $\mathcal{J}_{jk}$ is defined in (3.3.3):

$$\begin{aligned} \mathcal{J}_{jk} &= \frac{g(w_j,t_k)}{h(w_j,t_k)}, && k=1,\dots,a; \\ \mathcal{J}_{j,k+a} &= g(w_j,y_k)\frac{h(w_j,\bar x)}{h(w_j,\bar t)}, && k=1,\dots,b; \end{aligned} \qquad j=1,\dots,a+b. \tag{3.5.2}$$

Developing the determinant with respect to the a first columns we obtain

$$Z_{a,b}(\bar{t};\bar{x}|\bar{s};\bar{y}) = \sum K_a(\bar{w}_{\rm I}|\bar{t})h(\bar{w}_{\rm II},\bar{x})g(\bar{w}_{\rm II},\bar{y})g(\bar{w}_{\rm II},\bar{w}_{\rm I}). \tag{3.5.3}$$

The sum is taken over partitions $\{\bar{x},\bar{s}\} = \bar{w} \Rightarrow \{\bar{w}_{\rm I},\bar{w}_{\rm II}\}$ with $\#\bar{w}_{\rm I} = a$ and $\#\bar{w}_{\rm II} = b$.

Let us give several alternative representations for the highest coefficient.

- As a sum over partitions of $\bar{t}$ and $\bar{y}$:

$$Z_{a,b}(\bar{t};\bar{x}|\bar{s};\bar{y}) = f(\bar{s},\bar{t})f(\bar{y},\bar{x})\sum g(\bar{\eta}_{\rm I},\bar{\eta}_{\rm II})\frac{h(\bar{t},\bar{\eta}_{\rm II})}{h(\bar{s},\bar{\eta}_{\rm II})}K_a(\bar{x}|\bar{\eta}_{\rm I}). \tag{3.5.4}$$

  Here the sum is taken over partitions $\{\bar{t},\bar{y}+c\} = \bar{\eta} \Rightarrow \{\bar{\eta}_{\rm I},\bar{\eta}_{\rm II}\}$ such that $\#\bar{\eta}_{\rm I} = a$ and $\#\bar{\eta}_{\rm II} = b$.

- As a sum over partitions of $\bar{t}$ and $\bar{x}$:

$$\begin{aligned}Z_{a,b}(\bar{t};\bar{x}|\bar{s};\bar{y}) = (-1)^a h(\bar{x},\bar{x})h(\bar{s},\bar{x})g(\bar{x},\bar{y})g(\bar{s},\bar{y})\\ \sum K_a(\bar{t}-c|\bar{\xi}_{\rm I})\frac{h(\bar{\xi}_{\rm I},\bar{t})g(\bar{x},\bar{\xi}_{\rm I})g(\bar{s},\bar{\xi}_{\rm I})}{g(\bar{\xi}_{\rm I},\bar{y})}g(\bar{\xi}_{\rm I},\bar{\xi}_{\rm II}).\end{aligned} \tag{3.5.5}$$

  Here the sum is taken over partitions $\{\bar{t},\bar{x}-c\} = \bar{\xi} \Rightarrow \{\bar{\xi}_{\rm I},\bar{\xi}_{\rm II}\}$ such that $\#\bar{\xi}_{\rm I} = \#\bar{\xi}_{\rm II} = a$.

- As a sum over partitions of $\bar{s}$ and $\bar{y}$:

$$Z_{a,b}(\bar{t};\bar{x}|\bar{s};\bar{y}) = (-1)^{a+b}f(\bar{x},\bar{t})f(\bar{s},\bar{t})\sum g(\bar{\nu}_{\rm I},\bar{\nu}_{\rm II})K_{a+b}(\{\bar{\nu}_{\rm I},\bar{t}-c\}|\{\bar{x},\bar{s}\}) \tag{3.5.6}$$

  Here the sum is taken over partitions $\{\bar{s}-c,\bar{y}\} = \bar{\nu} \Rightarrow \{\bar{\nu}_{\rm I},\bar{\nu}_{\rm II}\}$ such that $\#\bar{\nu}_{\rm I} = \#\bar{\nu}_{\rm II} = b$.

All the sum formulas listed above follow from (3.5.3) and can be proved via reduction of the sums over partitions to multiple contour integrals of Cauchy type. Let us show how this method works.

Consider a b-fold integral

$$\mathcal{I} = \frac{(-1)^b}{(2\pi ic)^b b!}\oint\limits_{\bar{w}} K_{a+b}(\bar{w}|\{\bar{t},\bar{z}+c\})h(\bar{z},\bar{x})\frac{g(\bar{z},\bar{y})g(\bar{z},\bar{w})}{\Delta_b(\bar{z})\Delta'_b(\bar{z})}\,d\bar{z}, \tag{3.5.7}$$

where $\bar{w} = \{\bar{s},\bar{x}\}$ and $d\bar{z} = dz_1,\dots,dz_b$. We have used a subscript $\bar{w}$ on the integral symbol in order to stress that the integration contour for every z_j surrounds the set $\bar{w}$ in the anticlockwise direction. We also assume that the integration contours do not contain any other singularities of the integrand. Similar prescription will be kept for all other integral representations considered below.

The only poles of the integrand within the integration contours are the points $z_j = w_k$. Evaluating the integral by the residues in these poles we obtain (see appendix 9.1 for details)

$$\mathcal{I} = (-1)^b\sum K_{a+b}(\bar{w}|\{\bar{t},\bar{w}_{\rm II}+c\})h(\bar{w}_{\rm II},\bar{x})g(\bar{w}_{\rm II},\bar{y})g(\bar{w}_{\rm II},\bar{w}_{\rm I}), \tag{3.5.8}$$

where the sum is taken over partitions of $\bar{w}$ into subsets $\bar{w}_{\rm I}$ and $\bar{w}_{\rm II}$ with $\#\bar{w}_{\rm I} = a$ and $\#\bar{w}_{\rm II} = b$. Due to (1.1.45) we have

$$K_{a+b}(\bar{w}|\{\bar{t},\bar{w}_{\rm II}+c\}) = (-1)^b K_a(\bar{w}_{\rm I}|\bar{t}), \tag{3.5.9}$$

and comparing the obtained sum with (3.5.3) we immediately obtain $\mathcal{I} = Z_{a,b}(\bar t;\bar x|\bar s;\bar y)$.

Similarly, one can check that the sum over partitions in (3.5.3) can be presented as an a-fold contour integral

$$Z_{a,b}(\bar t;\bar x|\bar s;\bar y) = (-1)^b \frac{h(\bar w,\bar x)g(\bar y,\bar w)}{(2\pi ic)^a a!} \oint\limits_{\bar w} \frac{K_a(\bar z|\bar t)g(\bar z,\bar w)}{h(\bar z,\bar x)g(\bar z,\bar y)\Delta_a(\bar z)\Delta'_a(\bar z)}\, d\bar z, \qquad (3.5.10)$$

where now $d\bar z = dz_1,\dots,dz_a$. Indeed, taking the residues in the points $\bar z = \bar w_{\rm I}$ we obtain

$$Z_{a,b}(\bar t;\bar x|\bar s;\bar y) = (-1)^b h(\bar w,\bar x)g(\bar y,\bar w)\sum \frac{K_a(\bar w_{\rm I}|\bar t)g(\bar w_{\rm I},\bar w_{\rm II})}{h(\bar w_{\rm I},\bar x)g(\bar w_{\rm I},\bar y)}. \qquad (3.5.11)$$

Multiplying the terms of the sum with the prefactor $h(\bar w,\bar x)g(\bar y,\bar w)$ we arrive at (3.5.3).

Let us turn back to the integral (3.5.7). Obviously, it can be calculated taking the residues in the poles outside the original integration contour. It is easy to see that for arbitrary z_j the integrand behaves as $1/z_j^3$ at $z_j\to\infty$. Hence, the residue at infinity vanishes. The poles outside the original integration contours are in $z_j = y_k$ and $z_j = s_k - c$ (the poles at $z_j = x_k - c$ are compensated by the zeros of the product $h(\bar z,\bar x)$). Thus, we can move the original contour surrounding $\bar w$ to the points $\bar\nu = \{\bar y, \bar s - c\}$

$$Z_{a,b}(\bar t;\bar x|\bar s;\bar y) = \frac{1}{(2\pi ic)^b b!}\oint\limits_{\bar\nu} K_{a+b}(\bar w|\{\bar t,\bar z + c\})h(\bar z,\bar x)\frac{g(\bar z,\bar y)g(\bar z,\bar w)}{\Delta_b(\bar z)\Delta'_b(\bar z)}\, d\bar z. \qquad (3.5.12)$$

It is convenient to transform the integrand, applying (8.1.6) to $K_{a+b}(\bar w|\{\bar t,\bar z + c\})$. Then substituting $\bar w = \{\bar x,\bar s\}$ and using elementary properties of $f(z,w)$ we obtain

$$Z_{a,b}(\bar t;\bar x|\bar s;\bar y) = \frac{(-1)^{a+b}}{(2\pi ic)^b b!}\oint\limits_{\bar\nu} \frac{K_{a+b}(\{\bar t - c,\bar z\}|\bar w)f(\bar w,\bar t)h(\bar z,\bar x)g(\bar z,\bar y)g(\bar z,\bar x)g(\bar z,\bar s)}{f(\bar z,\bar x)f(\bar z,\bar s)\Delta_b(\bar z)\Delta'_b(\bar z)}\, d\bar z, \qquad (3.5.13)$$

and after simplification we arrive at

$$Z_{a,b}(\bar t;\bar x|\bar s;\bar y) = \frac{(-1)^{a+b}f(\bar w,\bar t)}{(2\pi ic)^b b!}\oint\limits_{\bar\nu} K_{a+b}(\{\bar t - c,\bar z\}|\bar w)\frac{g(\bar z,\bar\nu)}{\Delta_b(\bar z)\Delta'_b(\bar z)}\, d\bar z. \qquad (3.5.14)$$

Now all the poles are explicitly combined in the product $g(\bar z,\bar\nu)$. Hence, the result of the integration gives the sum over partitions of $\bar\nu \Rightarrow \{\bar\nu_{\rm I},\bar\nu_{\rm II}\}$ with $\#\bar\nu_{\rm I} = \#\bar\nu_{\rm II} = b$, which coincides with (3.5.6).

Applying (1.1.45) to Izergin determinant $K_{a+b}(\{\bar\nu_{\rm I},\bar t - c\}|\{\bar x,\bar s\})$ in (3.5.6), we have

$$\begin{aligned} K_{a+b}(\{\bar\nu_{\rm I},\bar t - c\}|\{\bar x,\bar s\}) &= (-1)^b K_{a+2b}(\{\bar\nu,\bar t - c\}|\{\bar x,\bar s,\bar\nu_{\rm II} + c\}) \\ &= (-1)^b K_{a+2b}(\{\bar y,\bar s - c,\bar t - c\}|\{\bar x,\bar s,\bar\nu_{\rm II} + c\}) = K_{a+b}(\{\bar y,\bar t - c\}|\{\bar x,\bar\nu_{\rm II} + c\}). \end{aligned} \qquad (3.5.15)$$

Then the sum over partitions in (3.5.6) is equivalent to a multiple contour integral

$$Z_{a,b}(\bar t;\bar x|\bar s;\bar y) = \frac{(-1)^a f(\bar x,\bar t)f(\bar s,\bar t)}{(2\pi ic)^b b!}\oint\limits_{\bar\nu} K_{a+b}(\{\bar y,\bar t - c\}|\{\bar x,\bar z + c\})\frac{g(\bar z,\bar\nu)}{\Delta_b(\bar z)\Delta'_b(\bar z)}\, d\bar z. \qquad (3.5.16)$$

Using (8.1.6) we recast (3.5.16) as

$$Z_{a,b}(\bar t;\bar x|\bar s;\bar y) = \frac{(-1)^b f(\bar y,\bar x)f(\bar s,\bar t)}{(2\pi ic)^b b!}\oint\limits_{\bar\nu} \frac{K_{a+b}(\{\bar x - c,\bar z\}|\{\bar y,\bar t - c\})}{f(\bar z,\bar y)f(\bar z,\bar t - c)}\frac{g(\bar z,\bar\nu)}{\Delta_b(\bar z)\Delta'_b(\bar z)}\, d\bar z. \qquad (3.5.17)$$

Setting now $\bar\eta = \{\bar t, \bar y + c\}$ we obtain

$$Z_{a,b}(\bar t; \bar x|\bar s; \bar y) = \frac{(-1)^b f(\bar y, \bar x) f(\bar s, \bar t)}{(2\pi i c)^b b!} \oint\limits_{\bar\nu} K_{a+b}(\{\bar x - c, \bar z\}|\bar\eta - c) \frac{h(\bar z, \bar t) g(\bar z, \bar\eta - 2c)}{h(\bar z, \bar s)\Delta_b(\bar z)\Delta'_b(\bar z)}\, d\bar z. \quad (3.5.18)$$

Now we can evaluate the integral by the residues outside the integration contours. All of them are collected in the product $g(\bar z, \bar\eta - 2c)$. Taking the residues in the points $\bar\eta - 2c$ and using $h(x - 2c, y) = -h(y, x)$ we immediately arrive at (3.5.4).

Similarly, starting with the integral representation (3.5.10) one can obtain the sum formula (3.5.5).

Conclusion

In this chapter the scalar product of two generic (off-shell) Bethe vectors was calculated in model with algebra symmetry $\mathfrak{gl}(2|1)$. The final formula is expressed via the highest coefficient that were calculated using multiple action fomulae. In general expression resemble the similar representation for scalar product of Bethe vectors in case of algebra symmetry $\mathfrak{gl}(3)$ but details and explicit expressions for the highest coefficient differs. An interested and useful property is existence of a determinant representation for highest coefficient in algebra symmetry $\mathfrak{gl}(2|1)$ case in contrary to $\mathfrak{gl}(3)$.

The final formula contain multiple summation and barely can be used in application. The exception is $\mathfrak{gl}(1|1)$ subalgebra case, where the compact formula (3.4.4) was obtained. However, it is expected that similar to $\mathfrak{gl}(2)$ and $\mathfrak{gl}(3)$ under some restriction on the spectral parameters the final formula can be reduced compact form, and similar to $\mathfrak{gl}(2)$ and $\mathfrak{gl}(3)$ case it is naturally to expect that this form should be determinant of some matrix. Such compact representations, of course, is more acceptable for both numerical and analytical calculation of the form factors. The calculation of scalar product performed in next chapter.

Chapter 4

Determinant representation of scalar product

Scalar product representation (3.3.33) is extremely bulky due to presence of multiple sum over partition. It is possible to obtain similar expressions for one-point form factors, but, as already mentioned, application of such formulae to calculation of correlation functions via form factor sum is not easy. Even application of such expression numerical computation does not look promising.

Up to now both set of spectral parameters were free. However, in case of $\mathfrak{gl}(2)$ algebra symmetry it is possible to rewrite (3.3.33) in compact form in case when one of the Bethe vectors is on-shell (i.e. spectral parameters of this vector satisfy Bethe equation) [47]. This compact expression is *determinant representation* of scalar product. In case of algebra symmetry $\mathfrak{gl}(3)$ determinant representation was also derived (see [67, 68, 119]) but with more heavy restriction: the second vector was twisted on-shell (i.e. the second set of spectral parameters satisfy the system of twisted on-shell Bethe equation). This was, at least, enough in order to obtain determinant representation for the ultralocal form factors.

For further study of integrable systems with graded algebra symmetry it is important to obtain as general as possible formulae for the scalar product. Thus, an ideal case could be the situation when the determinant representation exists for the scalar-product of on-shell and off-shell Bethe vectors. It allows to calculate form factors and correlation of any multi-point operators since for any set of local operators $O_1, \ldots, O_\ell$ their action on on-shell Bethe vector create a finite linear combination of off-shell Bethe vectors $|\psi_k\rangle$ with known coefficients C_k.

$$O_1 O_2 \ldots O_\ell |\psi\rangle = \sum_k C_k |\psi_k\rangle, \tag{4.0.1}$$

The knowledge of scalar product between on-shell and off-shell Bethe vectors allows to derive [48, 49, 57, 100, 101, 103, 110] representation for the ultralocal form factors $\langle n|O_1 \ldots O_\ell|m\rangle$ and, at least zero-temperature, correlation functions without form factor series calculation[1].

It turns out, that in the algebra symmetry case $\mathfrak{gl}(2|1)$ it is possible to find determinant representation for the scalar product in more general case than in $\mathfrak{gl}(3)$. The determinant representation can be found if both sets of spectral parameters satisfy only part of the Bethe equation. We call such case product of semi-on-shell Bethe vectors.

[1] It should be noted that correlation functions and form factors in this approach are given by multiple sums or, thermodynamic limit, integrals (so-called integral representations). Computation of these integrals is itself quite complicated problem (see [156–158]).

The result and the details of calculation are given below. This chapter is based on the paper [89] published by the thesis author in collaboration.

4.1 Generalised model

In concrete quantum models the functions $r_1(z)$ and $r_3(z)$ are fixed. Then the system of Bethe equations (2.1.30) determines the admissible values of the parameters $\bar{u}$ and $\bar{v}$. Eventually these values characterize the spectrum of the Hamiltonian of the quantum model under consideration.

In the framework of the generalized model one can consider the Bethe parameters $\{\bar{u}, \bar{v}\}$ and the functions $\{r_1(u_j), r_3(v_k)\}$ as two types of variables [46]. The first type comes from the R-matrix, the second type comes from the monodromy matrix. In the case of generic Bethe vectors these two types of variables are independent. Of course, in the case of (twisted) on-shell Bethe vectors they become related by the (twisted) Bethe equations.

One can also consider an intermediate case, when only a subset of $\{r_1(u_j), r_3(v_k)\}$ is related to the Bethe parameters $\{\bar{u}, \bar{v}\}$ by a part of the Bethe equations. For instance, we can impose the first set of equations (2.1.30) involving the functions $r_1(u_j)$, without imposing the second set of equations for $r_3(v_j)$ (or vice versa). In the case of a concrete model this means that a part of the Bethe parameters remains free, while the other parameters become functions of them. We call a Bethe vector possessing this property a *semi-on-shell Bethe vector*. Thus, the semi-on-shell Bethe vectors occupy an intermediate position between generic and on-shell Bethe vectors.

4.2 Scalar products

4.2.1 Scalar product of semi-on-shell Bethe vectors

The main result of this chapter is a determinant representation for the scalar products of semi-on-shell Bethe vectors. Consider the scalar product (3.1.1), where the following constraints are imposed:

$$\begin{aligned} r_1(u_j^C) &= \varkappa \prod_{\substack{k=1 \\ k\neq j}}^{a} \frac{f(u_j^C, u_k^C)}{f(u_k^C, u_j^C)} \prod_{l=1}^{b} f(v_l^C, u_j^C), \qquad && j = 1, \dots, a, \\ r_3(v_k^B) &= \prod_{l=1}^{a} f(v_k^B, u_l^B), && k = 1, \dots, b. \end{aligned} \tag{4.2.1}$$

Here $\varkappa$ is a complex parameter. If we set $\varkappa = \kappa_2/\kappa_1$, then we easily recognize the first set of equations (2.1.33) for the parameters $\{\bar{u}^C, \bar{v}^C\}$ and the second set of equations (2.1.30) for the parameters $\{\bar{u}^B, \bar{v}^B\}$. Thus, $\mathbb{B}_{a,b}(\bar{u}^B; \bar{v}^B)$ is a semi-on-shell Bethe vector, while $\mathbb{C}_{a,b}(\bar{u}^C; \bar{v}^C)$ is a dual twisted semi-on-shell Bethe vector.

Let $\bar{x} = \{\bar{u}^B, \bar{v}^C\}$. Define an $(a+b) \times (a+b)$ matrix $\mathcal{N}$ with the following entries:

$$\mathcal{N}_{jk} = \frac{(-1)^{a-1} r_1(x_k)}{f(\bar{v}^C, x_k)} t(u_j^C, x_k) h(\bar{u}^C, x_k) + \varkappa t(x_k, u_j^C) h(x_k, \bar{u}^C), \qquad \begin{array}{l} j = 1, \dots, a \\ k = 1, \dots, a+b, \end{array} \tag{4.2.2}$$

and

$$\mathcal{N}_{a+j,k} = \frac{g(x_k,\bar v^B)}{g(x_k,\bar v^C)}\left(1-\frac{r_3(x_k)}{f(x_k,\bar u^B)}\right) \times\left(g(x_k,v_j^C)h(x_k,\bar u^B)+\frac{(-1)^{a-1}r_1(x_k)r_3(v_j^C)h(\bar u^B,x_k)}{\varkappa f(v_j^C,\bar u^C)f(\bar v^B,x_k)h(v_j^C,x_k)}\right), \qquad \begin{array}{l} j=1,\dots,b,\\ k=1,\dots,a+b.\end{array} \tag{4.2.3}$$

Conjecture 4.2.1. *The scalar product* $S_{a,b}$ (3.1.1) *with constraint* (4.2.1) *has the following determinant representation*

$$S_{a,b} = \Delta_{a+b}(\bar x)\Delta'_a(\bar u^C)\Delta'_b(\bar v^C)\det_{a+b}\mathcal{N}, \tag{4.2.4}$$

where Δ *and* Δ' *are defined in* (1.1.46).

The proof of representation (4.2.4) will be given in section 4.5.

It follows from (3.3.33) that the scalar products are symmetric under the simultaneous replacement $\bar u^C \leftrightarrow \bar u^B$ and $\bar v^C \leftrightarrow \bar v^B$. Therefore making this replacement in (4.2.1)–(4.2.4) we obtain a determinant presentation for the scalar product of another set of semi-on-shell Bethe vectors.

It is interesting to see how the matrix elements $\mathcal{N}_{jk}$ depend on the functions r_1 and r_3. Namely, one can easily show that $\mathcal{N}_{jk}$ might depend on $r_3(v_k^C)$, however, they do not depend on $r_3(u_k^B)$. Indeed, if $x_k = u_k^B$ in (4.2.3), then $r_3(x_k)$ is multiplied with the product $1/f(x_k,\bar u^B)$, which vanishes for $x_k \in \bar u^B$. Similarly, the function $r_1(x_k)$ always enters either with the product $1/f(\bar v^C,x_k)$ or $1/g(\bar v^C,x_k)$. Both these products vanish for $x_k \in \bar v^C$, therefore the matrix elements $\mathcal{N}_{jk}$ might depend on $r_1(u_k^B)$ only.

Finally, looking at (4.2.3) for $k>a$ (that is, $x_k \in \bar v^C$), we see that $\mathcal{N}_{jk} \sim \delta_{jk}$ due to the product $1/g(\bar v^C,x_k)$. Thus, the right-lower block of the matrix $\mathcal{N}$ is diagonal.

4.2.2 Scalar product of twisted and usual on-shell Bethe vectors

Equation (4.2.4) has important particular cases. First of all, it describes a scalar product of twisted and usual on-shell Bethe vectors. Let the twist matrix be $\kappa = \mathrm{diag}(\kappa_1,\kappa_2,\kappa_3)$. Then one should set $\varkappa = \kappa_2/\kappa_1$ in (4.2.1) and impose two additional constraints

$$\begin{aligned} r_1(u_j^B) &= \prod_{\substack{k=1\\k\ne j}}^{a}\frac{f(u_j^B,u_k^B)}{f(u_k^B,u_j^B)}\prod_{l=1}^{b} f(v_l^B,u_j^B), \qquad && j=1,\dots,a,\\ r_3(v_k^C) &= \frac{\kappa_2}{\kappa_3}\prod_{l=1}^{a} f(v_k^C,u_l^C), && k=1,\dots,b. \end{aligned} \tag{4.2.5}$$

In this case the vector $\mathbb{B}_{a,b}(\bar u^B;\bar v^B)$ becomes an on-shell Bethe vector, while $\mathbb{C}_{a,b}(\bar u^C;\bar v^C)$ becomes a twisted on-shell Bethe vector. Then their scalar product has the determinant representation (4.2.4), where

$$\mathcal{N}_{jk} = h(x_k,\bar u^B)\left(t(u_j^C,x_k)\frac{f(\bar v^B,x_k)h(\bar u^C,x_k)}{f(\bar v^C,x_k)h(\bar u^B,x_k)}+\frac{\kappa_2}{\kappa_1}t(x_k,u_j^C)\frac{h(x_k,\bar u^C)}{h(x_k,\bar u^B)}\right), \quad j=1,\dots,a, \tag{4.2.6}$$

$$\mathcal{N}_{a+j,k} = h(x_k,\bar u^B)\frac{g(x_k,\bar v^B)}{g(x_k,\bar v^C)}\left(1-\frac{\kappa_2}{\kappa_3}\frac{f(x_k,\bar u^C)}{f(x_k,\bar u^B)}\right)\left(g(x_k,v^C_j)+\frac{\kappa_1}{\kappa_3 h(v^C_j,x_k)}\right), \quad j=1,\dots,b, \tag{4.2.7}$$

and $k=1,\dots,a+b$ in both formulas.

4.2.3 Norm of on-shell Bethe vector

The second particular case of (4.2.4) is the norm of on-shell Bethe vector. For this one should set[2] $\bar\kappa = 1$ and consider the limit $u^B_j \to u^C_j = u_j$, $v^C_j \to v^B_j = v_j$. Then the result has the form

$$\|\mathbb{B}_{a,b}(\bar u;\bar v)\|^2 = (-1)^{a+b}\prod_{j=1}^{b}\prod_{k=1}^{a} f(v_j,u_k)\prod_{\substack{j,k=1\\ j\ne k}}^{a} f(u_j,u_k)\prod_{\substack{j,k=1\\ j\ne k}}^{b} g(v_j,v_k)\ \det_{a+b}\widehat{\mathcal{N}}. \tag{4.2.8}$$

Here $\widehat{\mathcal{N}}$ is an $(a+b)\times(a+b)$ block-matrix. The left-upper block is

$$\widehat{\mathcal{N}}_{jk} = \delta_{jk}\left[c\,\frac{r_1'(u_k)}{r_1(u_k)} + \sum_{\ell=1}^{a}\frac{2c^2}{u_{k\ell}^2-c^2} - \sum_{m=1}^{b} t(v_m,u_k)\right] - \frac{2c^2}{u_{kj}^2-c^2}, \quad j,k=1,\dots,a, \tag{4.2.9}$$

where $u_{kj}=u_k-u_j$ and $r_1'(u_k)$ means the derivative of the function $r_1(u)$ at the point $u=u_k$. The right-lower block is diagonal

$$\widehat{\mathcal{N}}_{j+a,k+a} = \delta_{jk}\left[c\,\frac{r_3'(v_k)}{r_3(v_k)} + \sum_{\ell=1}^{a} t(v_k,u_\ell)\right], \quad j,k=1,\dots,b, \tag{4.2.10}$$

where $r_3'(v_k)$ means the derivative of the function $r_3(v)$ at the point $v=v_k$. The antidiagonal blocks are

$$\widehat{\mathcal{N}}_{j,k+a} = t(v_k,u_j), \quad j=1,\dots,a, \quad k=1,\dots,b, \tag{4.2.11}$$

and

$$\widehat{\mathcal{N}}_{j+a,k} = -t(v_j,u_k), \quad j=1,\dots,b, \quad k=1,\dots,a. \tag{4.2.12}$$

It is easy to relate the determinant of the matrix $\widehat{\mathcal{N}}$ with the Jacobian of the Bethe equations. Namely, let

$$\begin{aligned} \Phi_j &= \log\Big(\frac{r_1(u_j)}{f(\bar v,u_j)}\prod_{\substack{k=1\\ k\ne j}}^{a}\frac{f(u_k,u_j)}{f(u_j,u_k)}\Big), \qquad j=1,\dots,a,\\ \Phi_{a+j} &= \log\left(\frac{r_3(v_j)}{f(v_j,\bar u)}\right), \qquad j=1,\dots,b. \end{aligned} \tag{4.2.13}$$

Then the Bethe equations for the sets $\bar u$ and $\bar v$ take the form

$$\Phi_j = 2\pi i n_j, \qquad j=1,\dots,a+b, \tag{4.2.14}$$

where n_j are integer numbers. A straightforward calculation shows that

$$\begin{aligned} \widehat{\mathcal{N}}_{j,k} &= c\,\frac{\partial\Phi_j}{\partial u_k} \qquad k=1,\dots,a,\\ \widehat{\mathcal{N}}_{j,a+k} &= c\,\frac{\partial\Phi_j}{\partial v_k}, \qquad k=1,\dots,b, \end{aligned} \qquad j=1,\dots,a+b. \tag{4.2.15}$$

[2] Here and below the notation $\bar\kappa = 1$ means $\kappa_1=\kappa_2=\kappa_3=1$.

4.3 Orthogonality of the eigenvectors

Consider the scalar product of twisted on-shell and usual on-shell vectors. In this case the entries of the matrix $\mathcal{N}$ are given by (4.2.6), (4.2.7). Assume that $\{\bar u^C,\bar v^C\}\ne\{\bar u^B,\bar v^B\}$ at $\kappa=1$. Then in the limit $\kappa=1$ we obtain the scalar product of two different on-shell Bethe vectors, which should be orthogonal. Let us show this.

To prove the orthogonality of on-shell Bethe vectors we introduce an $(a+b)$-component vector Ω

$$\begin{aligned}\Omega_j&=\frac{1}{g(u_j^C,\bar u^B)}\prod_{\substack{k=1\\k\ne j}}^{a}g(u_j^C,u_k^C),\qquad j=1,\dots,a,\\ \Omega_{a+j}&=\frac{1}{g(v_j^C,\bar v^B)}\prod_{\substack{k=1\\k\ne j}}^{a}g(v_j^C,v_k^C),\qquad j=1,\dots,b.\end{aligned}\tag{4.3.1}$$

Due to the condition $\{\bar u^C,\bar v^C\}\ne\{\bar u^B,\bar v^B\}$, the vector Ω has at least one non-zero component.

Using the contour integral method (see Appendix 9.3.1) one can easily calculate the following sums:

$$\begin{aligned}&\sum_{j=1}^{a}t(u_j^C,x_k)\Omega_j=\frac{h(\bar u^B,x_k)}{h(\bar u^C,x_k)}-\frac{g(x_k,\bar u^C)}{g(x_k,\bar u^B)}, &&\sum_{j=1}^{b}g(x_k,v_j^C)\Omega_{a+j}=\frac{g(x_k,\bar v^C)}{g(x_k,\bar v^B)}-1,\\ &\sum_{j=1}^{a}t(x_k,u_j^C)\Omega_j=\frac{g(x_k,\bar u^C)}{g(x_k,\bar u^B)}-\frac{h(x_k,\bar u^B)}{h(x_k,\bar u^C)}, &&\sum_{j=1}^{b}\frac{\Omega_{a+j}}{h(v_j^C,x_k)}=1-\frac{h(\bar v^B,x_k)}{h(\bar v^C,x_k)}.\end{aligned}\tag{4.3.2}$$

Using these results we obtain

$$\frac{\sum_{j=1}^{a}\mathcal{N}_{jk}\Omega_j}{h(x_k,\bar u^B)}=\frac{f(\bar v^B,x_k)}{f(\bar v^C,x_k)}\left(1-\frac{f(\bar u^C,x_k)}{f(\bar u^B,x_k)}\right)+\frac{\kappa_2}{\kappa_1}\left(\frac{f(x_k,\bar u^C)}{f(x_k,\bar u^B)}-1\right),\tag{4.3.3}$$

and

$$\frac{\sum_{j=1}^{b}\mathcal{N}_{a+j,k}\Omega_{a+j}}{h(x_k,\bar u^B)}=\left(1-\frac{\kappa_2}{\kappa_3}\frac{f(x_k,\bar u^C)}{f(x_k,\bar u^B)}\right)\left[\left(\frac{\kappa_1}{\kappa_3}-1\right)\frac{g(\bar v^B,x_k)}{g(\bar v^C,x_k)}+1-\frac{\kappa_1}{\kappa_3}\frac{f(\bar v^B,x_k)}{f(\bar v^C,x_k)}\right].\tag{4.3.4}$$

Note that if $x_k\in\{\bar u^B,\bar v^C\}$, then $\big(f(x_k,\bar u^B)f(\bar v^C,x_k)\big)^{-1}=0$, $\big(f(x_k,\bar u^B)g(\bar v^C,x_k)\big)^{-1}=0$, and $\big(f(\bar u^B,x_k)f(\bar v^C,x_k)\big)^{-1}=0$. Therefore the terms proportional to these products vanish. Thus, neglecting such the terms we find

$$\frac{\sum_{j=1}^{a+b}\mathcal{N}_{jk}\Omega_j}{h(x_k,\bar u^B)}=1-\frac{\kappa_2}{\kappa_1}+\left(\frac{\kappa_1}{\kappa_3}-1\right)\left(\frac{g(\bar v^B,x_k)}{g(\bar v^C,x_k)}-\frac{f(\bar v^B,x_k)}{f(\bar v^C,x_k)}\right)+\frac{f(x_k,\bar u^C)}{f(x_k,\bar u^B)}\left(\frac{\kappa_2}{\kappa_1}-\frac{\kappa_2}{\kappa_3}\right).\tag{4.3.5}$$

We see that this linear combination of rows of the matrix $\mathcal{N}$ vanishes at $\bar\kappa=1$. Hence, the determinant vanishes at $\bar\kappa=1$, which means that two different on-shell vectors are orthogonal.

4.4 Form factors of diagonal elements

We define form factors of the diagonal monodromy matrix entries as matrix elements of the operators $T_{ii}(z)$ between two on-shell Bethe vectors. We use a standard method for their calculation [95, 115]. Let

$$Q = \mathbb{C}^{(\kappa)}_{a,b}(\bar u^C;\bar v^C)\bigl(\operatorname{str} T_\kappa(z) - \operatorname{str} T(z)\bigr)\mathbb{B}_{a,b}(\bar u^B;\bar v^B). \tag{4.4.1}$$

Here $T_\kappa(z)$ is the twisted monodromy matrix and $T(z)$ is the usual monodromy matrix. We assume that $\mathbb{B}_{a,b}(\bar u^B;\bar v^B)$ is an on-shell Bethe vector and $\mathbb{C}^{(\kappa)}_{a,b}(\bar u^C;\bar v^C)$ is a dual twisted on-shell Bethe vector. In order to stress this difference we have added a superscript κ on the dual twisted Bethe vector. Obviously, $\mathbb{C}^{(\kappa)}_{a,b}(\bar u^C;\bar v^C)$ turns into the usual dual on-shell Bethe vector at $\bar\kappa = 1$.

On the one hand

$$Q = \sum_{i=1}^{3}(-1)^{[i]}(\kappa_i - 1)\mathbb{C}^{(\kappa)}_{a,b}(\bar u^C;\bar v^C)\; T_{ii}(z)\; \mathbb{B}_{a,b}(\bar u^B;\bar v^B). \tag{4.4.2}$$

On the other hand

$$Q = \Bigl(\tau_\kappa(z|\bar u^C,\bar v^C) - \tau(z|\bar u^B,\bar v^B)\Bigr)\mathbb{C}^{(\kappa)}_{a,b}(\bar u^C;\bar v^C)\mathbb{B}_{a,b}(\bar u^B;\bar v^B), \tag{4.4.3}$$

where $\tau_\kappa(z|\bar u^C,\bar v^C)$ and $\tau(z|\bar u^B,\bar v^B)$ respectively are the eigenvalues of the twisted and usual transfer matrices. Comparing (4.4.2) and (4.4.3) we find

$$\begin{aligned}&\mathbb{C}_{a,b}(\bar u^C;\bar v^C)\; T_{ii}(z)\; \mathbb{B}_{a,b}(\bar u^B;\bar v^B)\\ &\qquad = (-1)^{[i]}\frac{d}{d\kappa_i}\Bigl[\bigl(\tau_\kappa(z|\bar u^C,\bar v^C) - \tau(z|\bar u^B,\bar v^B)\bigr)\mathbb{C}^{(\kappa)}_{a,b}(\bar u^C;\bar v^C)\mathbb{B}_{a,b}(\bar u^B;\bar v^B)\Bigr]_{\bar\kappa=1}.\end{aligned} \tag{4.4.4}$$

Here $\mathbb{C}_{a,b}(\bar u^C;\bar v^C)$ is the value of the vector $\mathbb{C}^{(\kappa)}_{a,b}(\bar u^C;\bar v^C)$ at $\bar\kappa = 1$. Since this vector is a dual on-shell vector, we obtain a form factor of $T_{ii}(z)$ in the l.h.s. of (4.4.4). In the r.h.s. of (4.4.4) we should distinguish between two cases. If $\{\bar u^C,\bar v^C\} = \{\bar u^B,\bar v^B\} = \{\bar u,\bar v\}$ at $\bar\kappa = 1$, then the derivative in (4.4.4) acts on $\tau_\kappa(z|\bar u^C,\bar v^C)$ only, and we find

$$\mathbb{C}_{a,b}(\bar u;\bar v)\; T_{ii}(z)\; \mathbb{B}_{a,b}(\bar u;\bar v) = (-1)^{[i]}\|\mathbb{B}_{a,b}(\bar u;\bar v)\|^2 \frac{d}{d\kappa_i}\tau_\kappa(z|\bar u^C,\bar v^C)\Big|_{\substack{\bar u^C=\bar u,\ \bar v^C=\bar v\\ \bar\kappa=1}}. \tag{4.4.5}$$

Note, that here we have the full derivative over κ_i. Therefore, it acts also on the Bethe parameters $\{\bar u^C,\bar v^C\}$, because due to the twisted Bethe equations they implicitly depend on the twist parameters: $\bar u^C = \bar u^C(\kappa)$ and $\bar v^C = \bar v^C(\kappa)$.

If $\{\bar u^C,\bar v^C\} \ne \{\bar u^B,\bar v^B\}$ at $\bar\kappa = 1$, then *universal form factor* [86] of operators $T_{ij}(z)$ can be defined

$$\mathfrak{F}^{(i,j)}\begin{pmatrix}\bar u^C & \bar u^B\\ \bar v^C & \bar v^B\end{pmatrix}^{a',a}_{b',b} = \frac{\mathcal{F}^{(i,j)}\left(z\middle|\begin{smallmatrix}\bar u^C & \bar u^B\\ \bar v^C & \bar v^B\end{smallmatrix}\right)^{a',a}_{b',b}}{\tau(z|\bar u^C,\bar v^C) - \tau(z|\bar u^B,\bar v^B)}. \tag{4.4.6}$$

where

$$\mathcal{F}^{(i,j)}\left(z\middle|\begin{smallmatrix}\bar u^C & \bar u^B\\ \bar v^C & \bar v^B\end{smallmatrix}\right)^{a',a}_{b',b} = \mathbb{C}_{a',b'}(\bar u^C;\bar v^C)T_{ij}(z)\mathbb{B}_{a,b}(\bar u^B;\bar v^B). \tag{4.4.7}$$

It is easy to show that the functions $\mathfrak{F}^{(ij)}$ do not depend on z. Indeed, it follows from the commutation relations (1.1.14) that

$$[\mathcal{T}(z), T_{ij}(w)] = [\mathcal{T}(w), T_{ij}(z)], \tag{4.4.8}$$

where $\mathcal{T}$ is the transfer matrix (1.1.16). Hence, for arbitrary on-shell Bethe vectors $\mathbb{C}_{a',b'}(\bar{u}^C;\bar{v}^C)$ and $\mathbb{B}_{a,b}(\bar{u}^B;\bar{v}^B)$ we obtain

$$\mathbb{C}_{a',b'}(\bar{u}^C;\bar{v}^C)[\mathcal{T}(z),T_{ij}(w)]\mathbb{B}_{a,b}(\bar{u}^B;\bar{v}^B)=\mathbb{C}_{a',b'}(\bar{u}^C;\bar{v}^C)[\mathcal{T}(w),T_{ij}(z)]\mathbb{B}_{a,b}(\bar{u}^B;\bar{v}^B). \tag{4.4.9}$$

Using (1.1.29) we find

$$\begin{aligned}\big(\tau(z|\bar{u}^C,\bar{v}^C)-\tau(z|\bar{u}^B,\bar{v}^B)\big)\mathbb{C}_{a',b'}(\bar{u}^C;\bar{v}^C)T_{ij}(w)\mathbb{B}_{a,b}(\bar{u}^B;\bar{v}^B)\\ =\big(\tau(w|\bar{u}^C,\bar{v}^C)-\tau(w|\bar{u}^B,\bar{v}^B)\big)\mathbb{C}_{a',b'}(\bar{u}^C;\bar{v}^C)T_{ij}(z)\mathbb{B}_{a,b}(\bar{u}^B;\bar{v}^B),\end{aligned} \tag{4.4.10}$$

where τ are eigenvalues of the transfer matrix. Equation (4.4.10) immediately yields

$$\frac{\mathbb{C}_{a',b'}(\bar{u}^C;\bar{v}^C)T_{ij}(w)\mathbb{B}_{a,b}(\bar{u}^B;\bar{v}^B)}{\tau(w|\bar{u}^C,\bar{v}^C)-\tau(w|\bar{u}^B,\bar{v}^B)}=\frac{\mathbb{C}_{a',b'}(\bar{u}^C;\bar{v}^C)T_{ij}(z)\mathbb{B}_{a,b}(\bar{u}^B;\bar{v}^B)}{\tau(z|\bar{u}^C,\bar{v}^C)-\tau(z|\bar{u}^B,\bar{v}^B)}. \tag{4.4.11}$$

It is clear that the l.h.s. of (4.4.11) depends on w, while the r.h.s. depends on z. Thus, the ratio (4.4.6) does not depend on the argument of the operator T_{ij}.

Form factors (4.4.6) are called universal, because they are determined by the R-matrix only. In other words, for a given R-matrix they do not depend on the monodromy matrix, and hence, they are model independent. Indeed, all the dependence of the form factors on a specific model is hidden in the functions r_1 and r_3. However, since the Bethe parameters satisfy Bethe equations, the dependence on $r_1(u_i)$ and $r_3(v_i)$ actually disappears due to (2.1.30). Hence, the only functions which ‘remember’ about the original model are $r_1(z)$ and $r_3(z)$. But we have seen that the universal form factors do not depend on z, therefore they do not depend on $r_1(z)$ and $r_3(z)$. Thus, as we have claimed above, they do not depend on the monodromy matrix of the model.

Remark. Strictly speaking the universal form factors do not depend on the functions r_k, if $\bar{u}^C\cap\bar{u}^B=\emptyset$ and $\bar{v}^C\cap\bar{v}^B=\emptyset$, that is when the Bethe parameters of both vectors are all different. Otherwise, if, for instance, $u^C_j=u^B_k$, then the universal form factors depend on the logarithmic derivative $\log' r_1(u^B_k)$ of the function $r_1(u)$ [86]. Similarly, if $v^C_j=v^B_k$, then the universal form factors depend on the logarithmic derivative $\log' r_3(v^B_k)$ of the function $r_3(v)$.

Particular case of diagonal form factors, considered in this chapter, is given by following formulae

$$\mathfrak{F}^{(ii)}_{a,b}(\bar{u}^C;\bar{v}^C|\bar{u}^B;\bar{v}^B)=\frac{\mathbb{C}_{a,b}(\bar{u}^C;\bar{v}^C)\;T_{ii}(z)\;\mathbb{B}_{a,b}(\bar{u}^B;\bar{v}^B)}{\tau(z|\bar{u}^C,\bar{v}^C)-\tau(z|\bar{u}^B,\bar{v}^B)}, \tag{4.4.12}$$

here $a'=a$, $b'=b$. In this case, due to the orthogonality of Bethe vectors at $\{\bar{u}^C,\bar{v}^C\}\neq\{\bar{u}^B,\bar{v}^B\}$, the derivative in the r.h.s. of (4.4.4) acts on the scalar product only. Therefore we obtain for the universal form factor of $T_{ii}(z)$

$$\mathfrak{F}^{(ii)}_{a,b}(\bar{u}^C;\bar{v}^C|\bar{u}^B;\bar{v}^B)=(-1)^{[i]}\frac{d}{d\kappa_i}\mathbb{C}^{(\kappa)}_{a,b}(\bar{u}^C;\bar{v}^C)\mathbb{B}_{a,b}(\bar{u}^B;\bar{v}^B)\Big|_{\bar{\kappa}=1}. \tag{4.4.13}$$

Thus, for the calculation of the $T_{ii}(z)$ universal form factors, it is enough to differentiate w.r.t. κ_i the scalar product of the twisted and usual on-shell vectors and then set $\bar{\kappa}=1$.

Suppose that $\Omega_p\neq 0$ for some $p\in\{1,\dots,a+b\}$ in the vector (4.3.1). Then we can add to the p-th row of the matrix $\mathcal{N}$ all other rows multiplied with Ω_j/Ω_p. It is clear that after this the elements of the p-th row are given by (4.3.5) with the common prefactor Ω_p^{-1}. Then to calculate the form factor of $T_{ii}(z)$ it is enough to differentiate the p-th row with respect to κ_i at $\bar{\kappa}=1$. Hereby we should simply set $\bar{\kappa}=1$ everywhere else. We obtain

$$\mathfrak{F}^{(i,i)}_{a,b}(\bar{u}^C;\bar{v}^C|\bar{u}^B;\bar{v}^B)=\Omega_p^{-1}\,f(\bar{v}^C,\bar{u}^B)h(\bar{u}^B,\bar{u}^B)\Delta'(\bar{u}^C)\Delta(\bar{u}^B)\Delta(\bar{v}^C)\Delta'(\bar{v}^C)\;\det_{a+b}\mathcal{N}^{(i)}. \tag{4.4.14}$$

Here the entries $\mathcal{N}^{(i)}_{jk}$ of the matrix $\mathcal{N}^{(i)}$ do not depend on $i = 1, 2, 3$, for all $j \ne p$. In other words, they are the same for all $\mathfrak{F}^{(ii)}_{a,b}$. More precisely, if $1 \le j \le a$, then

$$\mathcal{N}^{(i)}_{jk} = t(u^C_j, x_k)\frac{f(\bar v^B, x_k)h(\bar u^C, x_k)}{f(\bar v^C, x_k)h(\bar u^B, x_k)} + t(x_k, u^C_j)\frac{h(x_k, \bar u^C)}{h(x_k, \bar u^B)}, \qquad k = 1, \dots, a+b, \tag{4.4.15}$$

and if $1 \le j \le b$, then

$$\mathcal{N}^{(i)}_{a+j,k} = -\frac{g(x_k, \bar v^B)}{g(x_k, \bar v^C)}\left(1 - \frac{f(x_k, \bar u^C)}{f(x_k, \bar u^B)}\right) t(v^C_j, x_k), \qquad k = 1, \dots, a+b. \tag{4.4.16}$$

The p-th row depends on the specific universal form factor $\mathfrak{F}^{(ii)}_{a,b}$, $i = 1, 2, 3$:

$$\begin{aligned} &\mathcal{N}^{(1)}_{pk} = 1 + \frac{g(\bar v^B, x_k)}{g(\bar v^C, x_k)} - \frac{f(\bar v^B, x_k)}{f(\bar v^C, x_k)} - \frac{f(x_k, \bar u^C)}{f(x_k, \bar u^B)},\\ &\mathcal{N}^{(2)}_{pk} = -1, \qquad\qquad\qquad\qquad k = 1, \dots, a+b.\\ &\mathcal{N}^{(3)}_{pk} = \mathcal{N}^{(1)}_{pk} + \mathcal{N}^{(2)}_{pk}, \end{aligned} \tag{4.4.17}$$

Recall that here p is an arbitrary integer from the set $\{1, \dots, a+b\}$ such that $\Omega_p \ne 0$. Observe, that equations (4.4.17) imply the vanishing of the form factors of $\operatorname{str} T(z)$ between different states, in accordance with the orthogonality of on-shell Bethe vectors.

Form factors of the non-diagonal matrix entries T_{ij} are considered in the next chapter.

4.5 Calculating the scalar product

In this section we prove Proposition 4.2.1. Our starting point is the sum formula (3.3.33) for the scalar product of generic Bethe vectors that contains multiple summations. With the semi-on-shell condition on the spectral parameters of the Bethe vectors this formula can be rewritten in determinant form.

4.5.1 Summation over the partitions of $\bar u^C$ and $\bar v^B$

It is convenient to use two different representation for highest coefficients in (3.3.33). For the first one we us representation

$$Z_{a,b}(\bar t; \bar x|\bar s; \bar y) = \sum g(\bar\omega_{\rm II}, \bar\omega_{\rm I})h(\bar\omega_{\rm II}, \bar x)g(\bar\omega_{\rm II}, \bar y)\; K_a(\bar\omega_{\rm I}|\bar t). \tag{4.5.1}$$

Summation is taken over partitions $\{\bar x, \bar s\} = \bar\omega \Rightarrow \{\bar\omega_{\rm I}, \bar\omega_{\rm II}\}$ with restrictions $\#\bar\omega_{\rm I} = a$, $\#\bar\omega_{\rm II} = b$. For the second highest coefficient following representation is used:

$$Z_{a,b}(\bar t; \bar x|\bar s; \bar y) = f(\bar s, \bar t)f(\bar y, \bar x)\sum g(\bar\eta_{\rm I}, \bar\eta_{\rm II})\frac{h(\bar t, \bar\eta_{\rm II})}{h(\bar s, \bar\eta_{\rm II})}K_a(\bar x|\bar\eta_{\rm I}). \tag{4.5.2}$$

The sum is taken over partitions $\{\bar t, \bar y + c\} = \bar\eta \Rightarrow \{\bar\eta_{\rm I}, \bar\eta_{\rm II}\}$ with restrictions $\#\bar\eta_{\rm I} = a$, $\#\bar\eta_{\rm II} = b$.

Substituting (4.5.1) for $Z_{a-k,n}(\bar u^C_{\rm II}, \bar u^B_{\rm II}|\bar v^C_{\rm I}, \bar v^B_{\rm I})$ and (4.5.2) for $Z_{k,b-n}(\bar u^B_{\rm I}, \bar u^C_{\rm I}|\bar v^B_{\rm II}, \bar v^C_{\rm II})$ into the representation (3.3.33) we obtain

$$\begin{aligned} S_{a,b} = \sum r_1(\bar u^C_{\rm II})r_1(\bar u^B_{\rm I})r_3(\bar v^C_{\rm II})r_3(\bar v^B_{\rm I})f(\bar u^C_{\rm I}, \bar u^C_{\rm II})f(\bar u^B_{\rm II}, \bar u^B_{\rm I})g(\bar v^C_{\rm I}, \bar v^C_{\rm II})g(\bar v^B_{\rm II}, \bar v^B_{\rm I})\\ \times \frac{K_{a-k}(\bar\omega_{\rm I}|\bar u^C_{\rm II})K_k(\bar u^C_{\rm I}|\bar\eta_{\rm I})}{f(\bar v^C, \bar u^C_{\rm II})f(\bar v^B_{\rm I}, \bar u^B)}\; g(\bar\omega_{\rm II}, \bar\omega_{\rm I})h(\bar\omega_{\rm II}, \bar u^B_{\rm II})g(\bar\omega_{\rm II}, \bar v^B_{\rm I})g(\bar\eta_{\rm I}, \bar\eta_{\rm II})\frac{h(\bar u^B_{\rm I}, \bar\eta_{\rm II})}{h(\bar v^B_{\rm II}, \bar\eta_{\rm II})}. \end{aligned} \tag{4.5.3}$$

Here we have additional summations over the partitions $\{\bar u^B_{\rm II},\bar v^C_{\rm I}\}=\bar\omega\Rightarrow\{\bar\omega_{\rm I},\bar\omega_{\rm II}\}$ and $\{\bar u^B_{\rm I},\bar v^C_{\rm II}+c\}=\bar\eta\Rightarrow\{\bar\eta_{\rm I},\bar\eta_{\rm II}\}$, such that $\#\bar\omega_{\rm I}=a-k$, $\#\bar\omega_{\rm II}=n$, $\#\bar\eta_{\rm I}=k$, and $\#\bar\eta_{\rm II}=b-n$.

Let suppose that the constraints (4.2.1) are fulfilled. Then taking the products of (4.2.1) with respect to the corresponding subsets we obtain

$$\begin{aligned}r_1(\bar u^C_{\rm II})&=\varkappa^{a-k}\frac{f(\bar u^C_{\rm II},\bar u^C_{\rm I})}{f(\bar u^C_{\rm I},\bar u^C_{\rm II})}f(\bar v^C,\bar u^C_{\rm II}),\\ r_3(\bar v^B_{\rm I})&=f(\bar v^B_{\rm I},\bar u^B).\end{aligned}\tag{4.5.4}$$

Substituting these expressions into (4.5.3) we arrive at

$$\begin{aligned}S_{a,b}=\sum\varkappa^{a-k}&r_1(\bar u^B_{\rm I})r_3(\bar v^C_{\rm II})f(\bar u^B_{\rm II},\bar u^B_{\rm I})g(\bar v^C_{\rm I},\bar v^C_{\rm II})g(\bar\omega_{\rm II},\bar\omega_{\rm I})g(\bar\eta_{\rm I},\bar\eta_{\rm II})\\ &\times h(\bar u^B_{\rm I},\bar\eta_{\rm II})h(\bar\omega_{\rm II},\bar u^B_{\rm II})\Big[f(\bar u^C_{\rm II},\bar u^C_{\rm I})K_{a-k}(\bar\omega_{\rm I}|\bar u^C_{\rm II})K_k(\bar u^C_{\rm I}|\bar\eta_{\rm I})\Big]\cdot\Big[g(\bar v^B_{\rm II},\bar v^B_{\rm I})\frac{g(\bar\omega_{\rm II},\bar v^B_{\rm I})}{h(\bar v^B_{\rm II},\bar\eta_{\rm II})}\Big].\end{aligned}\tag{4.5.5}$$

Here in the second line we have allocated with square brackets the terms depending on the subsets of $\bar u^C$ and the subsets of $\bar v^B$ respectively. The sum over partitions $\bar u^C\Rightarrow\{\bar u^C_{\rm I},\ \bar u^C_{\rm II}\}$ can be computed via (9.3.17):

$$\sum K_{a-k}(\bar\omega_{\rm I}|\bar u^C_{\rm II})K_k(\bar u^C_{\rm I}|\bar\eta_{\rm I})f(\bar u^C_{\rm II},\bar u^C_{\rm I})=(-1)^kf(\bar u^C,\bar\eta_{\rm I})K_a(\{\bar\eta_{\rm I}-c,\bar\omega_{\rm I}\}|\bar u^C).\tag{4.5.6}$$

The sum over partitions $\bar v^B\Rightarrow\{\bar v^B_{\rm I},\ \bar v^B_{\rm II}\}$ can be computed via (9.3.16):

$$\sum g(\bar v^B_{\rm II},\bar v^B_{\rm I})\frac{g(\bar\omega_{\rm II},\bar v^B_{\rm I})}{h(\bar v^B_{\rm II},\bar\eta_{\rm II})}=(-1)^n\sum g(\bar v^B_{\rm II},\bar v^B_{\rm I})g(\bar v^B_{\rm I},\bar\omega_{\rm II})g(\bar v^B_{\rm II},\bar\eta_{\rm II}-c)=(-1)^n\ \frac{g(\bar v^B,\bar\omega_{\rm II})h(\bar\omega_{\rm II},\bar\eta_{\rm II})}{h(\bar v^B,\bar\eta_{\rm II})}.\tag{4.5.7}$$

Substituting these results into (4.5.5) we arrive at

$$\begin{aligned}S_{a,b}=\sum\varkappa^{a-k}&(-1)^{k+n}r_1(\bar u^B_{\rm I})r_3(\bar v^C_{\rm II})f(\bar u^B_{\rm II},\bar u^B_{\rm I})g(\bar v^C_{\rm I},\bar v^C_{\rm II})f(\bar u^C,\bar\eta_{\rm I})\\ &\times g(\bar\omega_{\rm II},\bar\omega_{\rm I})g(\bar\eta_{\rm I},\bar\eta_{\rm II})g(\bar v^B,\bar\omega_{\rm II})h(\bar\omega_{\rm II},\bar u^B_{\rm II})\frac{h(\bar u^B_{\rm I},\bar\eta_{\rm II})h(\bar\omega_{\rm II},\bar\eta_{\rm II})}{h(\bar v^B,\bar\eta_{\rm II})}\ K_a(\{\bar\eta_{\rm I}-c,\bar\omega_{\rm I}\}|\bar u^C).\end{aligned}\tag{4.5.8}$$

We recall that in this formula

$$\begin{aligned}&\#\bar u^B_{\rm I}=k, &&\#\bar u^B_{\rm II}=a-k, &&\#\bar v^C_{\rm I}=n, &&\#\bar v^C_{\rm II}=b-n,\\ &\#\bar\omega_{\rm I}=a-k, &&\#\bar\omega_{\rm II}=n, &&\#\bar\eta_{\rm I}=k, &&\#\bar\eta_{\rm II}=b-n.\end{aligned}\tag{4.5.9}$$

4.5.2 Partial summation over the partitions of $\bar u^B$ and $\bar v^C$

In order to go further we specify the subsets in (4.5.8) as follows

$$\begin{aligned}&\bar u^B_{\rm I}=\{\bar u^B_{\rm i},\ \bar u^B_{\rm ii}\}, &&\bar u^B_{\rm II}=\{\bar u^B_{\rm iii},\ \bar u^B_{\rm iv}\}, &&\bar\omega_{\rm I}=\{\bar u^B_{\rm iii},\ \bar v^C_{\rm iv}\}, &&\bar\omega_{\rm II}=\{\bar u^B_{\rm iv},\ \bar v^C_{\rm iii}\},\\ &\bar v^C_{\rm I}=\{\bar v^C_{\rm iii},\ \bar v^C_{\rm iv}\}, &&\bar v^C_{\rm II}=\{\bar v^C_{\rm i},\ \bar v^C_{\rm ii}\}, &&\bar\eta_{\rm I}=\{\bar u^B_{\rm ii},\ \bar v^C_{\rm i}+c\}, &&\bar\eta_{\rm II}=\{\bar u^B_{\rm i},\ \bar v^C_{\rm ii}+c\}.\end{aligned}\tag{4.5.10}$$

The cardinalities of the introduced sub-subsets are $\#\bar u^B_j=k_j$ and $\#\bar v^C_j=n_j$ for $j={\rm i},{\rm ii},{\rm iii},{\rm iv}$. It is easy to see that $k_{\rm ii}+k_{\rm i}=k$, $n_{\rm iii}+n_{\rm iv}=n$, $k_{\rm iv}=n_{\rm iv}$, and $k_{\rm i}=n_{\rm i}$.

Equation (4.5.8) takes the form

$$
\begin{gathered}
S_{a,b}=\sum \varkappa^{a-k}(-1)^{k+n} r_1(\bar u^B_{\rm i}) r_1(\bar u^B_{\rm ii}) r_3(\bar v^C_{\rm i}) r_3(\bar v^C_{\rm ii}) \frac{f(\bar u^C,\bar u^B_{\rm ii})}{f(\bar v^C_{\rm i},\bar u^C)}\\
\times f(\bar u^B_{\rm iii},\bar u^B_{\rm i}) f(\bar u^B_{\rm iv},\bar u^B_{\rm i}) f(\bar u^B_{\rm iii},\bar u^B_{\rm ii}) f(\bar u^B_{\rm iv},\bar u^B_{\rm ii}) g(\bar v^C_{\rm iii},\bar v^C_{\rm i}) g(\bar v^C_{\rm iv},\bar v^C_{\rm i}) g(\bar v^C_{\rm iii},\bar v^C_{\rm ii}) g(\bar v^C_{\rm iv},\bar v^C_{\rm ii})\\
\times g(\bar u^B_{\rm iv},\bar u^B_{\rm iii}) g(\bar u^B_{\rm iv},\bar v^C_{\rm iv}) g(\bar v^C_{\rm iii},\bar u^B_{\rm iii}) g(\bar v^C_{\rm iii},\bar v^C_{\rm iv}) g(\bar u^B_{\rm ii},\bar u^B_{\rm i}) g(\bar u^B_{\rm ii},\bar v^C_{\rm ii}+c) g(\bar v^C_{\rm i}+c,\bar u^B_{\rm i}) g(\bar v^C_{\rm i},\bar v^C_{\rm ii})\\
\times \frac{g(\bar v^B,\bar u^B_{\rm iv}) g(\bar v^B,\bar v^C_{\rm iii})}{h(\bar v^B,\bar u^B_{\rm i}) h(\bar v^B,\bar v^C_{\rm ii}+c)} h(\bar u^B_{\rm iv},\bar u^B_{\rm iii}) h(\bar u^B_{\rm iv},\bar u^B_{\rm iv}) h(\bar v^C_{\rm iii},\bar u^B_{\rm iii}) h(\bar v^C_{\rm iii},\bar u^B_{\rm iv})\\
\times h(\bar u^B_{\rm i},\bar u^B_{\rm i}) h(\bar u^B_{\rm ii},\bar u^B_{\rm i}) h(\bar u^B_{\rm i},\bar v^C_{\rm ii}+c) h(\bar u^B_{\rm ii},\bar v^C_{\rm ii}+c)\\
\times h(\bar u^B_{\rm iv},\bar u^B_{\rm i}) h(\bar u^B_{\rm iv},\bar v^C_{\rm ii}+c) h(\bar v^C_{\rm iii},\bar u^B_{\rm i}) h(\bar v^C_{\rm iii},\bar v^C_{\rm ii}+c)\; K_a(\{\bar u^B_{\rm ii}-c,\bar v^C_{\rm i},\bar v^C_{\rm iv},\bar u^B_{\rm iii}\}|\bar u^C). \qquad (4.5.11)
\end{gathered}
$$

Here we have used the relation $f(x,y+c)=1/f(y,x)$.

Now we combine the sub-subsets of $\bar u^B$ and $\bar v^C$ into new subsets

$$
\begin{aligned}
\bar u^B_{\rm I} &= \{\bar u^B_{\rm i},\ \bar u^B_{\rm iv}\}, &\qquad \bar v^C_{\rm I} &= \{\bar v^C_{\rm i},\ \bar v^C_{\rm iv}\},\\
\bar u^B_{\rm II} &= \{\bar u^B_{\rm ii},\ \bar u^B_{\rm iii}\}, &\qquad \bar v^C_{\rm II} &= \{\bar v^C_{\rm ii},\ \bar v^C_{\rm iii}\}.
\end{aligned} \qquad (4.5.12)
$$

Due to (4.5.10) we have $\#\bar u^B_{\rm I}=\#\bar v^C_{\rm I}=n_{\rm i}+n_{\rm iv}\equiv n_{\rm I}$. Observe that these new subsets are different from the subsets used, for example, in (3.3.33). We use however the same notation, as we deal with sums over partitions, and therefore it does not matter how the separate terms in these sums are denoted.

Then using (1.1.38) we recast (4.5.11) in a partly factorised form

$$
S_{a,b}=(-1)^b \sum_{\substack{\bar u^B\Rightarrow\{\bar u^B_{\rm I},\bar u^B_{\rm II}\}\\ \bar v^C\Rightarrow\{\bar v^C_{\rm I},\,\bar v^C_{\rm II}\}}} f(\bar u^B_{\rm I},\bar u^B_{\rm II}) h(\bar u^B_{\rm I},\bar u^B_{\rm I}) g(\bar v^C_{\rm II},\bar v^C_{\rm I}) g(\bar v^B,\bar u^B_{\rm I}) g(\bar v^B,\bar v^C_{\rm II}) g(\bar v^C_{\rm II},\bar u^B_{\rm II}) h(\bar v^C_{\rm II},\bar u^B)
$$
$$
\times G_{n_{\rm I}}(\bar u^B_{\rm I}|\bar v^C_{\rm I})\mathcal{L}^{(u)}_a(\{\bar u^B_{\rm II},\bar v^C_{\rm I}\}|\bar u^C)\mathcal{L}^{(v)}_b(\bar v^C_{\rm II}|\bar u^B). \qquad (4.5.13)
$$

Here the partitions of the sets $\bar u^B$ and $\bar v^C$ are explicitly shown by the superscripts of the sum. The functions $G_{n_{\rm I}}$, $\mathcal{L}^{(u)}_a$, and $\mathcal{L}^{(v)}_b$ in their turn are given as sums over partitions into sub-subsets. We have for $G_{n_{\rm I}}$

$$
G_{n_{\rm I}}(\bar u^B_{\rm I}|\bar v^C_{\rm I})=\sum_{\substack{\bar u^B_{\rm I}\Rightarrow\{\bar u^B_{\rm iv},\,\bar u^B_{\rm i}\}\\ \bar v^C_{\rm I}\Rightarrow\{\bar v^C_{\rm i},\,\bar v^C_{\rm iv}\}}} \varkappa^{-n_{\rm i}}\hat r_3(\bar v^C_{\rm i})\hat r_1(\bar u^B_{\rm i}) g(\bar u^B_{\rm i},\bar u^B_{\rm iv}) g(\bar v^C_{\rm iv},\bar v^C_{\rm i})\frac{g(\bar u^B_{\rm iv},\bar v^C_{\rm iv})}{h(\bar v^C_{\rm i},\bar u^B_{\rm i})}, \qquad (4.5.14)
$$

where we introduced new functions $\hat r_1(u^B_j)$ and $\hat r_3(v^C_j)$ through the following equations:

$$
r_1(u^B_j)=\hat r_1(u^B_j)\,\frac{f(u^B_j,\bar u^B_j)}{f(\bar u^B_j,u^B_j)} f(\bar v^B,u^B_j), \qquad r_3(v^C_j)=\hat r_3(v^C_j) f(v^C_j,\bar u^C). \qquad (4.5.15)
$$

For the two other functions we have

$$
\mathcal{L}^{(u)}_a(\{\bar u^B_{\rm II},\bar v^C_{\rm I}\}|\bar u^C)=\sum_{\bar u^B_{\rm II}\Rightarrow\{\bar u^B_{\rm ii},\bar u^B_{\rm iii}\}} K_a(\{\bar u^B_{\rm ii}-c,\bar v^C_{\rm I},\bar u^B_{\rm iii}\}|\bar u^C)
$$
$$
\times \varkappa^{a-k_{\rm ii}}(-1)^{k_{\rm ii}}\frac{r_1(\bar u^B_{\rm ii})}{f(\bar v^C,\bar u^B_{\rm ii})} f(\bar u^C,\bar u^B_{\rm ii}) f(\bar u^B_{\rm iii},\bar u^B_{\rm ii}) f(\bar v^C_{\rm I},\bar u^B_{\rm ii}), \qquad (4.5.16)
$$

and

$$\mathcal{L}_b^{(v)}(\bar v_{\rm II}^C|\bar u^B)=\sum_{\bar v_{\rm II}^C\Rightarrow\{\bar v_{\rm ii}^C,\bar v_{\rm iii}^C\}}(-1)^{n_{\rm ii}}\frac{r_3(\bar v_{\rm ii}^C)}{f(\bar v_{\rm ii}^C,\bar u^B)}. \tag{4.5.17}$$

We would like to stress that passing from (4.5.11) to (4.5.13) we did not make any transforms. One can check that substituting (4.5.14), (4.5.16), and (4.5.17) into (4.5.13) we turn back to (4.5.11).

The sums over partitions in (4.5.14), (4.5.16), and (4.5.17) can be easily calculated. The most simple is the sum (4.5.17)

$$\mathcal{L}_b^{(v)}(\bar v_{\rm II}^C|\bar u^B)=\prod_{v_j^C\in\bar v_{\rm II}^C}\left(1-\frac{r_3(v_j^C)}{f(v_j^C,\bar u^B)}\right). \tag{4.5.18}$$

If we introduce

$$\varphi(z)=1-\frac{r_3(z)}{f(z,\bar u^B)}, \tag{4.5.19}$$

and extend our convention on the shorthand notation to this function, then

$$\mathcal{L}_b^{(v)}(\bar v_{\rm II}^C|\bar u^B)=\varphi(\bar v_{\rm II}^C). \tag{4.5.20}$$

The sum (4.5.14) also is quite simple, because actually this is the Laplace expansion of the determinant of the sum of two matrices (see appendix 9.3.2 for more details). Indeed, it is enough to present (see (9.3.15))

$$\begin{aligned} g(\bar u_{\rm iv}^B,\bar v_{\rm iv}^C)&=\Delta_{n_{\rm iv}}(\bar v_{\rm iv}^C)\Delta'_{n_{\rm iv}}(\bar u_{\rm iv}^B)\det_{n_{\rm iv}}\Big(g(\bar u_{{\rm iv}_k}^B,\bar v_{{\rm iv}_j}^C)\Big),\\ \frac{1}{h(\bar v_{\rm i}^C,\bar u_{\rm i}^B)}&=\Delta_{n_{\rm i}}(\bar v_{\rm i}^C)\Delta'_{n_{\rm i}}(\bar u_{\rm i}^B)\det_{n_{\rm i}}\Big(\frac{1}{h(\bar v_{{\rm i}_j}^C,\bar u_{{\rm i}_k}^B)}\Big), \end{aligned} \tag{4.5.21}$$

and we immediately recognize the Laplace formula in (4.5.14). Thus,

$$G_{n_{\rm I}}(\bar u_{\rm I}^B|\bar v_{\rm I}^C)=\Delta_{n_{\rm I}}(\bar v_{\rm I}^C)\Delta'_{n_{\rm I}}(\bar u_{\rm I}^B)\det_{n_{\rm I}}\Big(g(u_{{\rm I}_k}^B,v_{{\rm I}_j}^C)+H(u_{{\rm I}_k}^B,v_{{\rm I}_j}^C)\Big), \tag{4.5.22}$$

where

$$H(u_k^B,v_j^C)=\frac{\hat r_3(v_j^C)\hat r_1(u_k^B)}{\varkappa h(v_j^C,u_k^B)}. \tag{4.5.23}$$

Finally, the sum (4.5.16) can be computed via Lemma 9.3.3. Namely, if we set in (9.3.18): $m=a$, $\bar\xi=\bar u^C$, $\bar w=\{\bar u_{\rm II}^B,\ \bar v_{\rm I}^C\}$ and

$$C_1(w)=\frac{-r_1(w)}{f(\bar v^C,w)},\qquad C_2(w)=\varkappa, \tag{4.5.24}$$

then we obtain equation (4.5.16). Indeed, in this case one has $C_1(v_j^C)=0$ due to the product $1/f(\bar v^C,w)$ in (4.5.24). Hence, we automatically have $\bar v_{\rm I}^C\subset\bar w_{\rm II}$, otherwise the corresponding contribution to the sum vanishes. This means that when splitting the set $\bar w=\{\bar u_{\rm II}^B,\ \bar v_{\rm I}^C\}$ into two subsets we actually should consider only the partitions of the set $\bar u_{\rm II}^B$ into $\bar u_{\rm ii}^B$ and $\bar u_{\rm iii}^B$, as we have in (4.5.16). We obtain

$$\mathcal{L}_a^{(u)}(\bar w|\bar u^C)=\Delta'_a(\bar u^C)\Delta_a(\bar w)\det_a\Big(\mathcal{M}(u_j^C,w_k)\Big), \tag{4.5.25}$$

with

$$\mathcal{M}(u_j^C, w_k) = (-1)^{a-1} \frac{r_1(w_k)}{f(\bar{v}^C, w_k)} t(u_j^C, w_k) h(\bar{u}^C, w_k) + \varkappa t(w_k, u_j^C) h(w_k, \bar{u}^C), \tag{4.5.26}$$

and $\bar{w} = \{\bar{u}^B_{\rm II},\ \bar{v}^C_{\rm I}\}$.

4.5.3 Final summation over the partitions of $\bar{u}^B$ and $\bar{v}^C$

Let $\bar{x} = \{\bar{u}^B, \bar{v}^C\}$. Consider a partition of $\bar{x}$ into subsets $\bar{x}_{\rm I}$ and $\bar{x}_{\rm II}$. Let $\bar{x}_{\rm I} = \{\bar{u}^B_{\rm II}, \bar{v}^C_{\rm I}\}$ and $\bar{x}_{\rm II} = \{\bar{u}^B_{\rm I}, \bar{v}^C_{\rm II}\}$. Then (4.5.13) can be written in a relatively compact form

$$\begin{aligned} S_{a,b} = (-1)^b \sum_{\bar{x} \Rightarrow \{\bar{x}_{\rm I}, \bar{x}_{\rm II}\}} & h(\bar{x}_{\rm II}, \bar{u}^B) g(\bar{v}^B, \bar{x}_{\rm II}) \varphi(\bar{x}_{\rm II}) g(\bar{x}_{\rm II}, \bar{x}_{\rm I}) \\ \times \frac{\Delta_{n_{\rm I}}(\bar{v}^C_{\rm I}) \Delta'_{n_{\rm I}}(\bar{u}^B_{\rm I})}{g(\bar{u}^B_{\rm I}, \bar{v}^C_{\rm I})} & \det_{n_{\rm I}} \Big(g(u^B_{{\rm I}_k}, v^C_{{\rm I}_j}) + H(u^B_{{\rm I}_k}, v^C_{{\rm I}_j}) \Big) \ \Delta'_a(\bar{u}^C) \Delta_a(\bar{x}_{\rm I}) \det_a \Big(\mathcal{M}(u^C_j, x_{{\rm I}_k}) \Big). \end{aligned} \tag{4.5.27}$$

Here we have used $\varphi(u^B_j) = 1$.

Our goal is to reduce (4.5.27) to an equation of the following type:

$$\sum_{\bar{x} \Rightarrow \{\bar{x}_{\rm I}, \bar{x}_{\rm II}\}} g(\bar{x}_{\rm II}, \bar{x}_{\rm I}) \Delta_a(\bar{x}_{\rm I}) \det_a \big(A_j(x_{{\rm I}_k}) \big) \ \Delta_b(\bar{x}_{\rm II}) \det_b \big(B_j(x_{{\rm II}_k}) \big) = \Delta_{a+b}(\bar{x}) \det_{a+b} \begin{pmatrix} A_j(x_k) \\ - \ - \ - \\ B_j(x_k) \end{pmatrix}. \tag{4.5.28}$$

Here in the r.h.s. we have a matrix consisting of two parts: the entries in the first a rows are $A_j(x_k)$, while in the remaining rows one has $B_j(x_k)$.

Looking at (4.5.27) we see that we can set $A_j(x_k) = \mathcal{M}(u^C_j, x_k)$. We also have a product $g(\bar{x}_{\rm II}, \bar{x}_{\rm I})$. The products $h(\bar{x}_{\rm II}, \bar{u}^B)$, $g(\bar{v}^B, \bar{x}_{\rm II})$, and $\varphi(\bar{x}_{\rm II})$ can be easily absorbed into the determinant of the matrix $B_j(x_k)$. It remains to construct this matrix $B_j(x_k)$.

Consider a function $F_b(\bar{z}|\bar{v}^C)$ depending on b variables $\bar{z}$ and b variables $\bar{v}^C$

$$F_b(\bar{z}|\bar{v}^C) = \Delta'_b(\bar{z}) \Delta_b(\bar{v}^C) \det_b \big(B_j(z_k) \big), \tag{4.5.29}$$

where

$$B_j(z_k) = \frac{g(z_k, v^C_j) + H(z_k, v^C_j)}{g(z_k, \bar{v}^C)}. \tag{4.5.30}$$

Obviously, $F_b(\bar{z}|\bar{v}^C)$ is a symmetric function of $\bar{z}$ and a symmetric function of $\bar{v}^C$. Let $\bar{z} = \bar{x}_{\rm II} = \{\bar{u}^B_{\rm I}, \bar{v}^C_{\rm II}\}$. Due to the symmetry of $F_b(\bar{z}|\bar{v}^C)$ we can say that the parameters $\bar{u}^B_{\rm I}$ correspond to the first $n_{\rm I}$ columns of the matrix $B_j(z_k)$, i.e. $z_k = u^B_{{\rm I}_k}$ for $k = 1, \dots, n_{\rm I}$. Then in the remaining columns we should set $z_{n_{\rm I}+k} = v^C_{{\rm II}_k}$, $k = 1, \dots, b - n_{\rm I}$. It is easy to see that in these last columns

$$B_j(v^C_{{\rm II}_k}) = \delta_{jk} \prod_{\substack{\ell=1 \\ v^C_\ell \ne v^C_{{\rm II}_k}}}^{b} \frac{1}{g(v^C_{{\rm II}_k}, v^C_\ell)}, \qquad k = 1, \dots, b - n_{\rm I}, \tag{4.5.31}$$

where $\delta_{jk} = 1$ if $v^C_j = v^C_{{\rm II}_k}$, and $\delta_{jk} = 0$ otherwise. Thus, the determinant reduces to the determinant of the matrix of the size $n_{\rm I} \times n_{\rm I}$. Simple calculation shows that

$$F_b(\{\bar{u}^B_{\rm I}, \bar{v}^C_{\rm II}\}|\bar{v}^C) = \frac{\Delta_{n_{\rm I}}(\bar{v}^C_{\rm I}) \Delta'_{n_{\rm I}}(\bar{u}^B_{\rm I})}{g(\bar{u}^B_{\rm I}, \bar{v}^C_{\rm I})} \det_{n_{\rm I}} \Big(g(u^B_{{\rm I}_k}, v^C_{{\rm I}_j}) + H(u^B_{{\rm I}_k}, v^C_{{\rm I}_j}) \Big), \tag{4.5.32}$$

which is exactly the expression in (4.5.27). Thus, we recast (4.5.27) as follows:

$$S_{a,b} = (-1)^b \Delta'_b(\bar{v}^C)\Delta'_a(\bar{u}^C) \sum_{\bar{x}\Rightarrow\{\bar{x}_{\rm I},\bar{x}_{\rm II}\}} g(\bar{x}_{\rm II},\bar{x}_{\rm I})\Delta_b(\bar{x}_{\rm II})\Delta_a(\bar{x}_{\rm I}) \times \det_b\Big(h(x_{{\rm II}_k},\bar{u}^B)g(\bar{v}^B,x_{{\rm II}_k})\varphi(x_{{\rm II}_k})B_j(x_{{\rm II}_k})\Big) \det_a\Big(\mathcal{M}(u_j^C,x_{{\rm I}_k})\Big). \quad (4.5.33)$$

It remains to use (4.5.28) and we end up with

$$S_{a,b} = \Delta'_b(\bar{v}^C)\Delta'_a(\bar{u}^C)\Delta_{a+b}(\bar{x}) \det_{a+b}\begin{pmatrix} \mathcal{M}(u_j^C,x_k) \\ - - - - - - - - - - - - - - \\ -h(x_k,\bar{u}^B)g(\bar{v}^B,x_k)\varphi(x_k)B_j(x_k)\end{pmatrix}, \quad (4.5.34)$$

where $\bar{x} = \{\bar{u}^B,\bar{v}^C\}$. Substituting here $\mathcal{M}(u_j^C,x_k)$ and $B_j(x_k)$ we arrive at the statement of Proposition 4.2.1.

Conclusions

In this chapter determinant representation for the scalar product of two semi-on-shell Bethe vectors was derived for integrable model with $\mathfrak{gl}(2|1)$ algebra symmetry. The particular case of such scalar product is product of on-shell and twisted on-shell Bethe vectors.

It should be noted, that this result is more general in comparison to the $\mathfrak{gl}(3)$ algebra symmetry case, where only determinant product of the on-shell and twisted on-shell Bethe vectors was derived. Such generalisation was expected from a comparison of the scalar products in two particular cases: $\mathfrak{gl}(2)$ (subalgebra of $\mathfrak{gl}(3)$) and $\mathfrak{gl}(1|1)$ (subalgebra of $\mathfrak{gl}(2|1)$). While in the first case only determinant representation for the scalar product of on-shell and off-shell Bethe vectors was known, in the second case the compact formula (3.4.4) was obtained for two generic Bethe vectors. Since, it was natural to expect that grading provide some simplifications in the scalar product computation.

The form factors of the diagonal monodromy matrix entries also were derived here. Using determinant representation for the form factors and zero modes approach (see [86]) determinant representation for all monodromy matrix entries $T_{ij}(u)$ can be derived.

Chapter 5

Form factors

In the previous chapter form factors of the diagonal entries of monodromy matrix and scalar product of two semi-on-shell Bethe vectors were derived. Now, using zero modes method, all remaining form factors can be derived. This chapter is based on paper [90] published by thesis author in collaboration.

5.1 Form factor of monodromy matrix entries

In previous chapter form factors of the monodromy matrix entries were defined as

$$\mathcal{F}^{(i,j)}\left(z\middle|\begin{smallmatrix}\bar{u}^C & \bar{u}^B\\ \bar{v}^C & \bar{v}^B\end{smallmatrix}\right)^{a',a}_{b',b} = \mathbb{C}_{a',b'}(\bar{u}^C;\bar{v}^C)T_{ij}(z)\mathbb{B}_{a,b}(\bar{u}^B;\bar{v}^B). \tag{5.1.1}$$

Here both $\mathbb{C}_{a',b'}(\bar{u}^C;\bar{v}^C)$ and $\mathbb{B}_{a,b}(\bar{u}^B;\bar{v}^B)$ are on-shell Bethe vectors, the parameter z is an arbitrary complex number, and

$$\begin{aligned} a' &= a + \delta_{i1} - \delta_{j1},\\ b' &= b + \delta_{j3} - \delta_{i3}. \end{aligned} \tag{5.1.2}$$

Universal form factors $\mathfrak{F}^{(i,j)}$ were also defined, for $\{\bar{u}^C,\bar{v}^C\} \neq \{\bar{u}^B,\bar{v}^B\}$ case. The set of connections between form factors can be found even without knowledge about their explicit form.

Conjecture 5.1.1. *Form factors $\mathcal{F}^{(i,j)}$ and $\mathcal{F}^{(j,i)}$ are related by*

$$\mathcal{F}^{(i,j)}\left(z\middle|\begin{smallmatrix}\bar{u}^C & \bar{u}^B\\ \bar{v}^C & \bar{v}^B\end{smallmatrix}\right)^{a',a}_{b',b} = (-1)^{\theta_{ij}}\mathcal{F}^{(j,i)}\left(z\middle|\begin{smallmatrix}\bar{u}^B & \bar{u}^C\\ \bar{v}^B & \bar{v}^C\end{smallmatrix}\right)^{a,a'}_{b,b'}, \tag{5.1.3}$$

where

$$\begin{aligned} &\theta_{ij} = 0, && [i]+[j] = 0, && \mod (2),\\ &\theta_{ij} = b, && [i] = 0, \quad [j] = 1, &&\\ &\theta_{ij} = b+1, && [i] = 1, \quad [j] = 0. && \end{aligned} \tag{5.1.4}$$

Proof. Since a form factor $\mathcal{F}^{(i,j)}$ is a c-number function, it is invariant under the action of the antimorphism ψ:

$$\psi\left(\mathcal{F}^{(i,j)}\left(z\middle|\begin{smallmatrix}\bar{u}^C & \bar{u}^B\\ \bar{v}^C & \bar{v}^B\end{smallmatrix}\right)^{a',a}_{b',b}\right) = \mathcal{F}^{(i,j)}\left(z\middle|\begin{smallmatrix}\bar{u}^C & \bar{u}^B\\ \bar{v}^C & \bar{v}^B\end{smallmatrix}\right)^{a',a}_{b',b}. \tag{5.1.5}$$

On the other hand, action of ψ on the r.h.s. of (5.1.1) gives

$$\psi\Big(\mathbb{C}_{a',b'}(\bar u^C;\bar v^C)T_{ij}(z)\mathbb{B}_{a,b}(\bar u^B;\bar v^B)\Big) = (-1)^{([i]+[j])(b+b')+b'b}\psi\Big(\mathbb{B}_{a,b}(\bar u^B;\bar v^B)\Big)\psi\Big(T_{ij}(z)\Big)\psi\Big(\mathbb{C}_{a',b'}(\bar u^C;\bar v^C)\Big) = (-1)^{\theta_{ij}}\mathbb{C}_{a,b}(\bar u^B;\bar v^B)T_{ji}(z)\mathbb{B}_{a',b'}(\bar u^C;\bar v^C), \quad (5.1.6)$$

where (1.1.47), (2.1.9), (2.1.10) is used, and

$$\theta_{ij} = ([i]+[j])(b+b') + b'b + b' + [i][j] + [i]. \quad (5.1.7)$$

Thus, we have reduced the form factor $\mathcal{F}^{(i,j)}$ to the form factor $\mathcal{F}^{(j,i)}$. In order to simplify the phase factor we can use (5.1.2)

$$b' - b = \delta_{j3} - \delta_{i3} = [j] - [i]. \quad (5.1.8)$$

After elementary algebra we obtain

$$\theta_{ij} = ([j]+[i])b + [i][j] + [i], \qquad \mod (2), \quad (5.1.9)$$

and it is straightforward to check that this expression is equivalent to (5.1.4). □

It follows from (5.1.3) that form factors of diagonal matrix elements $\mathcal{F}^{(i,i)}$ are invariant under the replacement $\bar u^C \leftrightarrow \bar u^B$ and $\bar v^C \leftrightarrow \bar v^B$. This invariance yields the following transformation of the corresponding universal form factors

$$\mathfrak{F}^{(i,i)}\begin{pmatrix}\bar u^C & \bar u^B\\ \bar v^C & \bar v^B\end{pmatrix}_{b,b}^{a,a} = -\mathfrak{F}^{(i,i)}\begin{pmatrix}\bar u^B & \bar u^C\\ \bar v^B & \bar v^C\end{pmatrix}_{b,b}^{a,a}. \quad (5.1.10)$$

The minus sign appears due to the denominator in (4.4.6). For the universal form factors of the off-diagonal matrix elements it is easy to obtain obtain

$$\mathfrak{F}^{(3,1)}\begin{pmatrix}\bar u^C & \bar u^B\\ \bar v^C & \bar v^B\end{pmatrix}_{b,b+1}^{a,a+1} = (-1)^{b+1}\mathfrak{F}^{(1,3)}\begin{pmatrix}\bar u^B & \bar u^C\\ \bar v^B & \bar v^C\end{pmatrix}_{b+1,b}^{a+1,a}, \quad (5.1.11)$$

$$\mathfrak{F}^{(3,2)}\begin{pmatrix}\bar u^C & \bar u^B\\ \bar v^C & \bar v^B\end{pmatrix}_{b,b+1}^{a,a} = (-1)^{b+1}\mathfrak{F}^{(2,3)}\begin{pmatrix}\bar u^B & \bar u^C\\ \bar v^B & \bar v^C\end{pmatrix}_{b+1,b}^{a,a}, \quad (5.1.12)$$

and

$$\mathfrak{F}^{(2,1)}\begin{pmatrix}\bar u^C & \bar u^B\\ \bar v^C & \bar v^B\end{pmatrix}_{b,b}^{a,a+1} = -\mathfrak{F}^{(1,2)}\begin{pmatrix}\bar u^B & \bar u^C\\ \bar v^B & \bar v^C\end{pmatrix}_{b,b}^{a+1,a}. \quad (5.1.13)$$

5.2 Determinant formulas for form factors

Considering form factors of the monodromy matrix entries one should distinguish between two cases[1]: (1) $\{\bar u^C,\bar v^C\} = \{\bar u^B,\bar v^B\}$; (2) $\{\bar u^C,\bar v^C\} \neq \{\bar u^B,\bar v^B\}$. The first case occurs only for form factors $\mathcal{F}^{(i,i)}$ of diagonal matrix elements $T_{ii}(z)$. Indeed, the condition $\{\bar u^C,\bar v^C\} = \{\bar u^B,\bar v^B\}$ implies $a'=a$ and $b'=b$ (see (5.1.2)), which is possible for diagonal entries $T_{ii}(z)$ only. We first present the results for this case.

[1]Here and below for brevity we write $\{\bar u^C,\bar v^C\} = \{\bar u^B,\bar v^B\}$, although one should understand this condition as $\bar u^C = \bar u^B$ and $\bar v^C = \bar v^B$.

5.2.1 Form factors between identical states

Let $\bar u^C=\bar u^B=\bar u$ and $\bar v^C=\bar v^B=\bar v$. The form factors $\mathcal{F}^{(i,i)}$ have the following determinant representations:

$$\mathcal{F}^{(i,i)}\left(z\Big|\begin{smallmatrix}\bar u & \bar u\\ \bar v & \bar v\end{smallmatrix}\right)^{a,a}_{b,b}=(-c)^{a+b}\prod_{j=1}^{b}\prod_{k=1}^{a}f(v_j,u_k)\prod_{\substack{j,k=1\\ j\neq k}}^{a}f(u_j,u_k)\prod_{\substack{j,k=1\\ j\neq k}}^{b}g(v_j,v_k)\ \det_{a+b+1}\widehat{\mathcal{N}}^{(i,i)}. \tag{5.2.1}$$

In order to describe the $(a+b+1)\times(a+b+1)$ matrices $\widehat{\mathcal{N}}^{(i,i)}$ we combine the sets $\bar u$ and $\bar v$ into a set $\bar x=\{u_1,\dots,u_a,v_1,\dots,v_b\}$. Then

$$\begin{aligned}
&\widehat{\mathcal{N}}^{(i,i)}_{j,k}=\frac{\partial\Phi_j}{\partial x_k} && j,k=1,\dots,a+b,\\
&\widehat{\mathcal{N}}^{(i,i)}_{a+b+1,k}=(-1)^{[i]}\frac{\partial\tau(z|\bar u,\bar v)}{\partial x_k}, && k=1,\dots,a+b,\\
&\widehat{\mathcal{N}}^{(i,i)}_{j,a+b+1}=\delta_{i1}-\delta_{i2}, && j=1,\dots,a,\\
&\widehat{\mathcal{N}}^{(i,i)}_{j,a+b+1}=\delta_{i3}-\delta_{i2}, && j=a+1,\dots,a+b,\\
&\widehat{\mathcal{N}}^{(i,i)}_{a+b+1,a+b+1}=(-1)^{[i]}\frac{\partial\tau_\kappa(z|\bar u,\bar v)}{\partial\kappa_i}.
\end{aligned} \tag{5.2.2}$$

Here Φ_j are given by (4.2.13), and the eigenvalues of the usual and twisted transfer matrices $\tau(z|\bar u,\bar v)$ and $\tau_\kappa(z|\bar u,\bar v)$ are defined respectively in (2.1.27) and (2.1.35). The proof of the determinant formula (5.2.1) is given in (5.3).

5.2.2 Form factors between different states

Notation

If $\{\bar u^C,\bar v^C\}\neq\{\bar u^B,\bar v^B\}$, then the universal form factors are well defined. We assume that the sets of Bethe parameters $\bar u^C$, $\bar v^C$, $\bar u^B$, $\bar v^B$ are fixed and their cardinalities are

$$\#\bar u^C=a',\qquad \#\bar u^B=a,\qquad \#\bar v^C=b',\qquad \#\bar v^B=b, \tag{5.2.3}$$

where a' and b' are related to a and b by (5.1.2). Before giving explicit determinant presentations for the universal form factors we introduce several new functions.

We introduce a function

$$H(\bar u^C;\bar u^B;\bar v^C)=f(\bar v^C,\bar u^B)h(\bar u^B,\bar u^B)\Delta'(\bar u^C)\Delta(\bar u^B)\Delta(\bar v^C)\Delta'(\bar v^C). \tag{5.2.4}$$

The function H plays the role of a universal prefactor that appears in all determinant formulas for form factors. One should remember, however, that in spite of this function has the universal representation (5.2.4), the cardinalities of the sets $\bar u^C$, $\bar u^B$, and $\bar v^C$ are different for the different form factors.

Since we consider the case $\{\bar u^C,\bar v^C\}\neq\{\bar u^B,\bar v^B\}$, there exists at least one component of vector (4.3.1) Ω_p such that $\Omega_p\neq 0$.

Finally, for fixed sets of variables $\bar u^C$, $\bar u^B$, $\bar v^C$, and $\bar v^B$ we introduce two rectangular matrices $\mathcal{L}$ and $\mathcal{M}$. The matrix $\mathcal{L}$ has the size $a'\times(a+b')$ and its entries are

$$\mathcal{L}_{j,k}=t(u^C_j,x_k)\frac{(-1)^{a'-1}r_1(x_k)h(\bar u^C,x_k)}{f(\bar v^C,x_k)h(x_k,\bar u^B)}+t(x_k,u^C_j)\frac{h(x_k,\bar u^C)}{h(x_k,\bar u^B)},\qquad \begin{array}{l}j=1,\dots,a',\\ k=1,\dots,a+b'.\end{array} \tag{5.2.5}$$

The matrix $\mathcal{M}$ has the size $b' \times (a+b')$ and its entries are

$$\mathcal{M}_{j,k} = -t(v^C_j, x_k)\frac{g(\bar v^B, x_k)}{g(\bar v^C, x_k)}\left(1 - \frac{r_3(x_k)}{f(x_k, \bar u^B)}\right), \qquad \begin{array}{l} j = 1,\dots,b', \\ k = 1,\dots,a+b'. \end{array} \tag{5.2.6}$$

Here the set $\bar x$ is the union of two sets: $\bar x = \{\bar u^B, \bar v^C\}$. Actually, both matrices $\mathcal{L}$ and $\mathcal{M}$ consist of two blocks depending on whether $x_k \in \bar u^B$ or $x_k \in \bar v^C$. The structures of these blocks are very different, and we give now a more detailed description of them.

First of all, we note that $1/f(\bar v^C, x_k) = 0$ if $x_k \in \bar v^C$, and $1/f(x_k, \bar u^B) = 0$ if $x_k \in \bar u^B$. Therefore we obtain

$$\mathcal{L}_{j,k+a} = t(v^C_k, u^C_j)\frac{h(v^C_k, \bar u^C)}{h(v^C_k, \bar u^B)}, \qquad k = 1,\dots,b', \tag{5.2.7}$$

and

$$\mathcal{M}_{j,k} = -t(v^C_j, u^B_k)\frac{g(\bar v^B, u^B_k)}{g(\bar v^C, u^B_k)}, \qquad k = 1,\dots,a. \tag{5.2.8}$$

The product $1/g(\bar v^C, x_k)$ also vanishes, if $x_k \in \bar v^C$. However, this zero can be compensated by the pole of the function $t(v^C_j, x_k)$, if $x_k = v^C_j$. Therefore, the block of the matrix $\mathcal{M}$ with $k > a$ has diagonal structure:

$$\mathcal{M}_{j,a+k} = -\delta_{jk}\frac{g(\bar v^B, v^C_k)}{g(\bar v^C_k, v^C_k)}\left(1 - \frac{f(v^C_k, \bar u^C)}{f(v^C_k, \bar u^B)}\right), \qquad k = 1,\dots,b'. \tag{5.2.9}$$

Here we replaced the function $r_3(v^C_k)$ with the product $f(v^C_k, \bar u^C)$ due to the Bethe equations. One should remember, however, that this replacement is possible only if $\bar v^C \cap \bar v^B = \emptyset$. Otherwise, if some parameters $v^C_{j_1}, \dots, v^C_{j_\ell}$ from the set $\bar v^C$ coincide with the parameters $v^B_{j_1}, \dots, v^B_{j_\ell}$ from the set $\bar v^B$, then one should first take the limits $v^C_{j_s} \to v^B_{j_s}$ in (5.2.6) and only after this we can impose Bethe equations for the functions $r_3(v^C_k)$.

Similarly, if $\bar u^C \cap \bar u^B = \emptyset$, then the matrix elements $\mathcal{L}_{j,k}$ with $j = 1,\dots,a'$ and $k = 1,\dots,a$ take the form

$$\mathcal{L}_{j,k} = (-1)^{a'+a} t(u^C_j, u^B_k)\frac{f(\bar v^B, u^B_k)h(\bar u^C, u^B_k)}{f(\bar v^C, u^B_k)h(\bar u^B, u^B_k)} + t(u^B_k, u^C_j)\frac{h(u^B_k, \bar u^C)}{h(u^B_k, \bar u^B)}. \tag{5.2.10}$$

Determinant formulas

Here the list of determinant representation for universal form factors of monodromy matrix entries $T_{ij}(z)$ is given. Certainly, it should be enough to give explicit formulas for $\mathfrak{F}^{(i,j)}$ with $i \le j$ only, because making replacements $\bar u^C \leftrightarrow \bar u^B$ and $\bar v^C \leftrightarrow \bar v^B$ one can recast the remaining form factors (see (5.1.11)–(5.1.13)). However, the matrices $\mathcal{L}_{j,k}$ and $\mathcal{M}_{j,k}$, as well as the prefactor H are not symmetric over these replacements. Therefore, changing $\bar u^C \leftrightarrow \bar u^B$ and $\bar v^C \leftrightarrow \bar v^B$ in the determinant formulas given below we obtain more representations for the universal form factors.

- Diagonal form factors. $\mathfrak{F}^{(i,i)}$. In this case $a' = a$ and $b' = b$.

These determinant representations were derived [88] and already were given in previous chapter (see (4.4.14)-(4.4.17)). It should be noted that the form factors are symmetric with respect to any of the four sets of Bethe parameters. This symmetry follows from the symmetry of the Bethe vectors. Therefore, without any loss of generality one can assume in (4.4.14) that $p = a$ or $p = a + b$.

- For the universal form factor $\mathfrak{F}^{(1,2)}$, we have $a' = a+1$ and $b' = b$. Let $\Omega_{a+1} \neq 0$. Then $\mathfrak{F}^{(1,2)}$ has the form

$$\mathfrak{F}^{(1,2)}\begin{pmatrix} \bar{u}^C & \bar{u}^B \\ \bar{v}^C & \bar{v}^B \end{pmatrix}^{a+1,a}_{b,b} = \frac{H}{\Omega_{a+1}} \det_{a+b} \mathcal{N}^{(1,2)}, \tag{5.2.11}$$

where

$$\begin{aligned} \mathcal{N}^{(1,2)}_{j,k} &= \mathcal{L}_{j,k}, \qquad j = 1,\dots,a, \\ \mathcal{N}^{(1,2)}_{j+a,k} &= \mathcal{M}_{j,k}, \qquad j = 1,\dots,b, \end{aligned} \tag{5.2.12}$$

and $k = 1,\dots,a+b$. The set $\bar{x} = \{u^B_1,\dots,u^B_a, v^C_1,\dots,v^C_b\}$.

- For the universal form factor $\mathfrak{F}^{(2,3)}$, we notice that $a' = a$ and $b' = b+1$. Let $\Omega_{a+b+1} \neq 0$. Then $\mathfrak{F}^{(2,3)}$ has the form

$$\mathfrak{F}^{(2,3)}\begin{pmatrix} \bar{u}^C & \bar{u}^B \\ \bar{v}^C & \bar{v}^B \end{pmatrix}^{a,a}_{b+1,b} = (-1)^{b+1}\frac{H}{\Omega_{a+b+1}} \det_{a+b+1} \mathcal{N}^{(2,3)}, \tag{5.2.13}$$

where

$$\begin{aligned} \mathcal{N}^{(2,3)}_{j,k} &= \mathcal{L}_{j,k}, \qquad j = 1,\dots,a, \\ \mathcal{N}^{(2,3)}_{j+a,k} &= \mathcal{M}_{j,k}, \qquad j = 1,\dots,b, \\ \mathcal{N}^{(2,3)}_{a+b+1,k} &= 1, \end{aligned} \tag{5.2.14}$$

and $k = 1,\dots,a+b+1$. The set $\bar{x} = \{u^B_1,\dots,u^B_a, v^C_1,\dots,v^C_{b+1}\}$.

- For the universal form factor $\mathfrak{F}^{(1,3)}$, one sees that $a' = a+1$ and $b' = b+1$. Let $\Omega_{a+1} \neq 0$. Then $\mathfrak{F}^{(1,3)}$ has the form

$$\mathfrak{F}^{(1,3)}\begin{pmatrix} \bar{u}^C & \bar{u}^B \\ \bar{v}^C & \bar{v}^B \end{pmatrix}^{a+1,a}_{b+1,b} = (-1)^{b+1}\frac{H}{\Omega_{a+1}} \det_{a+b+1} \mathcal{N}^{(1,3)}, \tag{5.2.15}$$

where

$$\begin{aligned} \mathcal{N}^{(1,3)}_{j,k} &= \mathcal{L}_{j,k}, \qquad j = 1,\dots,a, \\ \mathcal{N}^{(1,3)}_{j+a,k} &= \mathcal{M}_{j,k}, \qquad j = 1,\dots,b+1, \end{aligned} \tag{5.2.16}$$

and $k = 1,\dots,a+b+1$. The set $\bar{x} = \{u^B_1,\dots,u^B_a, v^C_1,\dots,v^C_{b+1}\}$.

- For the universal form factor $\mathfrak{F}^{(2,1)}$, one has $a' = a-1$ and $b' = b$. It has the form

$$\mathfrak{F}^{(2,1)}\begin{pmatrix} \bar{u}^C & \bar{u}^B \\ \bar{v}^C & \bar{v}^B \end{pmatrix}^{a-1,a}_{b,b} = H \det_{a+b} \mathcal{N}^{(2,1)}, \tag{5.2.17}$$

where

$$\begin{aligned} \mathcal{N}^{(2,1)}_{j,k} &= \mathcal{L}_{j,k}, \qquad j = 1,\dots,a-1, \\ \mathcal{N}^{(2,1)}_{a,k} &= -1, \\ \mathcal{N}^{(2,1)}_{j+a,k} &= \mathcal{M}_{j,k}, \qquad j = 1,\dots,b, \end{aligned} \tag{5.2.18}$$

and $k = 1,\dots,a+b$. The set $\bar{x} = \{u^B_1,\dots,u^B_a, v^C_1,\dots,v^C_b\}$.

- For the universal form factor $\mathfrak{F}^{(3,2)}$ with $a'=a$ and $b'=b-1$. It has the form

$$\mathfrak{F}^{(3,2)}\begin{pmatrix} \bar u^C & \bar u^B \\ \bar v^C & \bar v^B \end{pmatrix}^{a,a}_{b-1,b} = (-1)^{b-1} H \det_{a+b-1} \mathcal{N}^{(3,2)}, \tag{5.2.19}$$

where

$$\begin{aligned} \mathcal{N}^{(3,2)}_{j,k} &= \mathcal{L}_{j,k}, \qquad j=1,\dots,a, \\ \mathcal{N}^{(3,2)}_{j+a,k} &= \mathcal{M}_{j,k}, \qquad j=1,\dots,b-1, \end{aligned} \tag{5.2.20}$$

and $k=1,\dots,a+b-1$. The set $\bar x=\{u^B_1,\dots,u^B_a,v^C_1,\dots,v^C_{b-1}\}$.

- For the universal form factor $\mathfrak{F}^{(3,1)}$ and $a'=a-1$ and $b'=b-1$. It has the form

$$\mathfrak{F}^{(3,1)}\begin{pmatrix} \bar u^C & \bar u^B \\ \bar v^C & \bar v^B \end{pmatrix}^{a-1,a}_{b-1,b} = (-1)^{b-1} H \det_{a+b-1} \mathcal{N}^{(3,1)}, \tag{5.2.21}$$

where

$$\begin{aligned} \mathcal{N}^{(3,1)}_{j,k} &= \mathcal{L}_{j,k}, \qquad j=1,\dots,a-1, \\ \mathcal{N}^{(3,1)}_{j+a,k} &= \mathcal{M}_{j,k}, \qquad j=1,\dots,b-1, \\ \mathcal{N}^{(3,1)}_{a,k} &= \frac{(-1)^{a-1} r_1(x_k) h(\bar u^C,x_k)}{f(\bar v^C,x_k) h(x_k,\bar u^B)} - \frac{h(x_k,\bar u^C)}{h(x_k,\bar u^B)}, \end{aligned} \tag{5.2.22}$$

and $k=1,\dots,a+b-1$. The set $\bar x=\{u^B_1,\dots,u^B_a,v^C_1,\dots,v^C_{b-1}\}$.

The proofs of the determinant representations for the universal form factors of off-diagonal matrix elements will be given in section 5.5.

5.3 Proof of determinant formula for diagonal form factor

Form factors of the operators $T_{ii}(z)$ were calculated in chapter 4.4 (see also paper [89]). It was shown that $\mathcal{F}^{(i,i)}$ are proportional to derivatives on κ_i of twisted transfer matrix eigenvalue:

$$\mathcal{F}^{(i,i)}\left(z\middle|\begin{smallmatrix} \bar u & \bar u \\ \bar v & \bar v \end{smallmatrix}\right)^{a,a}_{b,b} = (-1)^{[i]} \frac{d\tau_\kappa(z|\bar u^C,\bar v^C)}{d\kappa_i}\Big|_{\bar\kappa=1} \mathbb{C}_{a,b}(\bar u;\bar v)\mathbb{B}_{a,b}(\bar u;\bar v). \tag{5.3.1}$$

A peculiarity of this representation is that we have a *full* derivative of $\tau_\kappa(z|\bar u^C,\bar v^C)$ over κ_i. In other words, one should consider the Bethe parameters $\bar u^C$ and $\bar v^C$ as implicit functions of κ_i, whose dependence on the twist parameters is determined by the twisted Bethe equations (2.1.33). In this section we show that representation (5.3.1) and (5.2.1) are equivalent.

Consider a solution $\{\bar u^C(\kappa),\bar v^C(\kappa)\}$ of the twisted Bethe equations such that $\{\bar u^C(\kappa),\bar v^C(\kappa)\}\to\{\bar u,\bar v\}$ as $\bar\kappa\to 1$. Then, similarly to (4.2.13), we introduce an $(a+b)$-component vector Φ^C as

$$\begin{aligned} \Phi^C_j &= \log\left(\frac{r_1(u^C_j)}{f(\bar v^C,u^C_j)}\frac{f(\bar u^C_j,u^C_j)}{f(u^C_j,\bar u^C_j)}\right), \qquad j=1,\dots,a, \\ \Phi^C_{a+j} &= \log\left(\frac{r_3(v^C_j)}{f(v^C_j,\bar u^C)}\right), \qquad j=1,\dots,b. \end{aligned} \tag{5.3.2}$$

Comparing this vector with the vector Φ (4.2.13) we see that $\Phi^C\to\Phi$ as[2] $\bar\kappa\to 1$.

[2]Here and below $\bar\kappa=1$ stands for $\kappa_1=\kappa_2=\kappa_3=1$. We also assume that the condition $\bar\kappa=1$ automatically yields $\bar u^C=\bar u$ and $\bar v^C=\bar v$.

Taking the logarithm of the twisted Bethe equations (2.1.33) we obtain

$$\begin{aligned} \Phi_j^C &= \log\left(\frac{\kappa_2}{\kappa_1}\right), \qquad j=1,\dots,a,\\ \Phi_{a+j}^C &= \log\left(\frac{\kappa_2}{\kappa_3}\right), \qquad j=1,\dots,b. \end{aligned} \tag{5.3.3}$$

Differentiating these equations over κ_i at $\bar\kappa=1$ we find

$$\begin{aligned} &\sum_{k=1}^{a}\frac{\partial\Phi_j}{\partial u_k}\frac{du_k^C}{d\kappa_i}\Big|_{\bar\kappa=1}+\sum_{k=1}^{b}\frac{\partial\Phi_j}{\partial v_k}\frac{dv_k^C}{d\kappa_i}\Big|_{\bar\kappa=1}=\delta_{2i}-\delta_{1i}, \qquad j=1,\dots,a,\\ &\sum_{k=1}^{a}\frac{\partial\Phi_{a+j}}{\partial u_k}\frac{du_k^C}{d\kappa_i}\Big|_{\bar\kappa=1}+\sum_{k=1}^{b}\frac{\partial\Phi_{a+j}}{\partial v_k}\frac{dv_k^C}{d\kappa_i}\Big|_{\bar\kappa=1}=\delta_{2i}-\delta_{3i}, \qquad j=1,\dots,b, \end{aligned} \tag{5.3.4}$$

where we have taken into account that $\Phi_j^C=\Phi_j$, $u_j^C=u_j$, and $v_j^C=v_j$ at $\bar\kappa=1$.

Let $\bar x=\{u_1^C,\dots,u_a^C,v_1^C,\dots,v_b^C\}$. Then using (5.2.2) we recast (5.3.4) as follows:

$$\begin{aligned} &\sum_{k=1}^{a+b}\widehat{\mathcal N}^{(i,i)}_{j,k}\frac{dx_k}{d\kappa_i}\Big|_{\bar\kappa=1}=\delta_{2i}-\delta_{1i}, \qquad j=1,\dots,a,\\ &\sum_{k=1}^{a+b}\widehat{\mathcal N}^{(i,i)}_{a+j,k}\frac{dx_k}{d\kappa_i}\Big|_{\bar\kappa=1}=\delta_{2i}-\delta_{3i}, \qquad j=1,\dots,b. \end{aligned} \tag{5.3.5}$$

Hence, if we multiply the columns $\widehat{\mathcal N}^{(i,i)}_{j,k}$ with $k=1,\dots,a+b$ by the coefficients $dx_k/d\kappa_i$ and add this linear combination to the last column of the matrix $\widehat{\mathcal N}^{(i,i)}$, then we obtain zeros everywhere except the right-lower element. For this non-zero entry we obtain

$$\begin{aligned} &\widehat{\mathcal N}^{(i,i)}_{a+b+1,a+b+1}+\sum_{k=1}^{a+b}\widehat{\mathcal N}^{(i,i)}_{a+b+1,k}\frac{dx_k}{d\kappa_i}\Big|_{\bar\kappa=1}\\ &\quad=(-1)^{[i]}\frac{\partial\tau_\kappa(z|\bar u,\bar v)}{\partial\kappa_i}+(-1)^{[i]}\sum_{k=1}^{a+b}\frac{\partial\tau(z|\bar u^C,\bar v^C)}{\partial x_k}\frac{dx_k}{d\kappa_i}\Big|_{\bar\kappa=1}=(-1)^{[i]}\frac{d\tau_\kappa(z|\bar u^C,\bar v^C)}{d\kappa_i}\Big|_{\bar\kappa=1}. \end{aligned} \tag{5.3.6}$$

Thus, we arrive at

$$\begin{aligned} \mathcal F^{(i,i)}\left(z\Big|\begin{smallmatrix}\bar u & \bar u\\ \bar v & \bar v\end{smallmatrix}\right)^{a,a}_{b,b}&=(-c)^{a+b}\prod_{j=1}^{b}\prod_{k=1}^{a}f(v_j,u_k)\prod_{\substack{j,k=1\\ j\ne k}}^{a}f(u_j,u_k)\prod_{\substack{j,k=1\\ j\ne k}}^{b}g(v_j,v_k)\\ &\quad\times(-1)^{[i]}\frac{d\tau_\kappa(z|\bar u^C,\bar v^C)}{d\kappa_i}\Big|_{\bar\kappa=1}\det_{a+b}\widehat{\mathcal N}^{(i,i)}, \end{aligned} \tag{5.3.7}$$

where now the size of the matrix $\widehat{\mathcal N}^{(i,i)}$ is $(a+b)\times(a+b)$. Comparing this expression with (4.2.8) we reproduce representation (5.3.1). □

5.4 Zero modes

It was shown in the paper [86] that in the models with $\mathfrak{gl}(N)$-invariant R-matrix all the form factors can be obtained from one initial form factor and taking special limits of the Bethe parameters. Our method was based on the use of zero modes of the monodromy matrix. This approach can be applied to the models with $\mathfrak{gl}(m|n)$ symmetry without significant changes. In this section we give a brief description of this method and find simple relations between different form factors.

The basis of the zero modes method is an expansion of the monodromy matrix $T(u)$ into a series over inverse spectral parameter

$$T_{ij}(u) = \delta_{ij}\mathbb{I} + \sum_{n=0}^{\infty} T_{ij}[n]\ \left(\tfrac{c}{u}\right)^{n+1}. \tag{5.4.1}$$

Note that the expansion (5.4.1) yields similar expansions for the functions $\lambda_i(u)$ and $r_k(u)$

$$\begin{aligned} \lambda_i(u) &= 1 + \sum_{n=0}^{\infty} \lambda_i[n]\ \left(\tfrac{c}{u}\right)^{n+1}, \qquad i = 1,2,3 \\ r_k(u) &= 1 + \sum_{n=0}^{\infty} r_k[n]\ \left(\tfrac{c}{u}\right)^{n+1}, \qquad k = 1,3. \end{aligned} \tag{5.4.2}$$

Assumption (5.4.1) implies that the Bethe vectors remain on-shell if one of their parameters tends to infinity. This is because the structure of the Bethe equations (2.1.30) is preserved when $r_k(u) \to 1$ at $u \to \infty$.

The operators $T_{ij}[0]$ are called the zero modes. They span a $\mathfrak{gl}(2|1)$ superalgebra. Sending in (1.1.14) one of the arguments to infinity we obtain commutation relations of the zero modes and the operators $T_{kl}(z)$

$$[T_{ij}[0], T_{kl}(z)\} = (-1)^{[l]([i]+[j])+[i][j]}\big(\delta_{il}T_{kj}(z) - \delta_{kj}T_{il}(z)\big), \tag{5.4.3}$$

showing that the monodromy entries form an adjoint representation of the $\mathfrak{gl}(2|1)$ superalgebra generated by the zero modes.

5.4.1 Action of the zero modes onto Bethe vectors

The explicit formulas for the action the operators $T_{ij}(z)$ onto Bethe vectors were derived in [87]. Taking the limit $z \to \infty$ in those expressions we obtain the action of the zero modes $T_{ij}[0]$. The action of $T_{ij}[0]$ with $i < j$ is given by

$$T_{13}[0]\mathbb{B}_{a,b}(\bar u;\bar v) = -\lim_{w\to\infty} \left(-\tfrac{w}{c}\right)^{b+1}\ \mathbb{B}_{a+1,b+1}(\{\bar u, w\};\{\bar v, w\}), \tag{5.4.4}$$

$$T_{23}[0]\mathbb{B}_{a,b}(\bar u;\bar v) = -\lim_{w\to\infty} \left(-\tfrac{w}{c}\right)^{b+1}\ \mathbb{B}_{a,b+1}(\bar u;\{\bar v, w\}), \tag{5.4.5}$$

$$T_{12}[0]\mathbb{B}_{a,b}(\bar u;\bar v) = \lim_{w\to\infty} \tfrac{w}{c}\ \mathbb{B}_{a+1,b}(\{\bar u, w\};\bar v). \tag{5.4.6}$$

Let us show how one can obtain these equations. For this we consider the simplest case (5.4.4). The action of the operator $T_{13}(w)$ onto a Bethe vector $\mathbb{B}_{a,b}(\bar u;\bar v)$ is (see [87])

$$T_{13}(w)\mathbb{B}_{a,b}(\bar u;\bar v) = \lambda_2(w)h(\bar v, w)\ \mathbb{B}_{a+1,b+1}(\{\bar u, w\};\{\bar v, w\}). \tag{5.4.7}$$

Multiplying both sides by w/c, taking the limit $w \to \infty$, and using the asymptotic properties of the functions $h(v, w)$ (1.1.38) and $\lambda_2(w)$ (5.4.2) we immediately arrive at (5.4.4).

The parameters $\bar u$ and $\bar v$ in (5.4.4)–(5.4.6) are a priori generic complex numbers, but they may satisfy the Bethe equations in specific cases. Then in the r.h.s. of (5.4.5) and (5.4.6) we obtain on-shell Bethe vectors, because the infinite root w together with the sets $\bar u$ and $\bar v$ satisfy Bethe equations due to the condition (5.4.2).

Applying the antimorphism ψ to the actions (5.4.4)–(5.4.6) we obtain

$$\mathbb{C}_{a,b}(\bar u;\bar v)T_{31}[0] = \lim_{w\to\infty} \left(\tfrac{w}{c}\right)^{b+1} \mathbb{C}_{a+1,b+1}(\{\bar u, w\};\{\bar v, w\}), \tag{5.4.8}$$

$$\mathbb{C}_{a,b}(\bar u;\bar v)T_{32}[0] = \lim_{w\to\infty} \left(\tfrac{w}{c}\right)^{b+1} \mathbb{C}_{a+1,b}(\bar u;\{\bar v, w\}), \tag{5.4.9}$$

$$\mathbb{C}_{a,b}(\bar u;\bar v)T_{21}[0] = \lim_{w\to\infty} \tfrac{w}{c}\, \mathbb{C}_{a+1,b}(\{\bar u, w\};\bar v). \tag{5.4.10}$$

As in the above case, if the parameters $\{\bar u, \bar v\}$ satisfy Bethe equations, then $\{\bar u, \bar v, w\}$ also satisfy Bethe equations as $w \to \infty$.

Similarly to the $\mathfrak{gl}(N)$ case (see [96]) the on-shell vectors (resp. dual on-shell vectors) depending on finite Bethe roots are *singular weight* vectors of the zero modes $T_{ij}[0]$ with $i > j$ (resp. $T_{ij}[0]$ with $i < j$):

$$\begin{aligned} T_{ij}[0]\mathbb{B}_{a,b}(\bar u;\bar v) &= 0, \qquad i > j,\\ \mathbb{C}_{a,b}(\bar u;\bar v)T_{ij}[0] &= 0, \qquad i < j. \end{aligned} \tag{5.4.11}$$

These equations can be obtained from the explicit formulas of the actions of T_{ij} onto Bethe vectors 2.1.2.

5.4.2 Relations between different form factors

The zero modes allow us to find simple relations between different form factors. As a starter, we consider an example. Setting in (5.4.3) $j = k = l = 2$ and $i = 1$ we obtain

$$[T_{12}[0], T_{22}(z)] = -T_{12}(z). \tag{5.4.12}$$

Let $\mathbb{C}_{a+1,b}(\bar u^C;\bar v^C)$ and $\mathbb{B}_{a,b}(\bar u^B;\bar v^B)$ be two on-shell vectors with all their Bethe parameters finite. Then (5.4.12) yields

$$\begin{aligned} \mathbb{C}_{a+1,b}(\bar u^C;\bar v^C)T_{12}(z)\mathbb{B}_{a,b}(\bar u^B;\bar v^B) = &-\mathbb{C}_{a+1,b}(\bar u^C;\bar v^C)T_{12}[0]T_{22}(z)\mathbb{B}_{a,b}(\bar u^B;\bar v^B)\\ &+ \mathbb{C}_{a+1,b}(\bar u^C;\bar v^C)T_{22}(z)T_{12}[0]\mathbb{B}_{a,b}(\bar u^B;\bar v^B). \end{aligned} \tag{5.4.13}$$

The first term in the r.h.s. vanishes as $T_{12}[0]$ acts on the dual on-shell Bethe vector. The action of $T_{12}[0]$ on the on-shell vector $\mathbb{B}_{a,b}(\bar u^B;\bar v^B)$ is given by (5.4.6), hence,

$$\mathbb{C}_{a+1,b}(\bar u^C;\bar v^C)T_{12}(z)\mathbb{B}_{a,b}(\bar u^B;\bar v^B) = \mathbb{C}_{a+1,b}(\bar u^C;\bar v^C)T_{22}(z) \lim_{w\to\infty} \tfrac{w}{c}\, \mathbb{B}_{a+1,b}(\{\bar u^B, w\};\bar v^B). \tag{5.4.14}$$

Since the original vector $\mathbb{B}_{a,b}(\bar u^B;\bar v^B)$ was on-shell, the new vector $\mathbb{B}_{a+1,b}(\{\bar u^B, w\};\bar v^B)$ with $w \to \infty$ also is on-shell. Therefore, in the r.h.s. of (5.4.14) we have the form factor of $T_{22}(z)$, and we arrive at

$$\mathcal{F}^{(1,2)}\left(z\middle|\begin{smallmatrix}\bar u^C & \bar u^B\\ \bar v^C & \bar v^B\end{smallmatrix}\right)_{b,b}^{a+1,a} = \lim_{w\to\infty} \frac{w}{c}\mathcal{F}^{(2,2)}\left(z\middle|\begin{smallmatrix}\bar u^C & \{\bar u^B, w\}\\ \bar v^C & \bar v^B\end{smallmatrix}\right)_{b,b}^{a+1,a+1}. \tag{5.4.15}$$

Thus, the form factor $\mathcal{F}^{(1,2)}$ can be obtained from $\mathcal{F}^{(2,2)}$ by sending one of the Bethe parameters to infinity.

The relation (5.4.15) can be easily reformulated for the universal form factors. Indeed, looking at the explicit expression (2.1.27) for the eigenvalue $\tau(z|\bar u,\bar v)$ we see that

$$\lim_{u_j\to\infty}\tau(z|\bar u,\bar v)=\tau(z|\bar u_j,\bar v),\qquad \lim_{v_k\to\infty}\tau(z|\bar u,\bar v)=\tau(z|\bar u,\bar v_k). \tag{5.4.16}$$

Thus, if one of the Bethe parameters goes to infinity, then the transfer matrix eigenvalue $\tau(z|\bar u,\bar v)$ turns into the eigenvalue depending on the remaining Bethe parameters. Hence, we arrive at

$$\mathfrak{F}^{(1,2)}\begin{pmatrix}\bar u^C & \bar u^B\\ \bar v^C & \bar v^B\end{pmatrix}^{a+1,a}_{b,b}=\lim_{w\to\infty}\frac{w}{c}\,\mathfrak{F}^{(2,2)}\begin{pmatrix}\bar u^C & \{\bar u^B,w\}\\ \bar v^C & \bar v^B\end{pmatrix}^{a+1,a+1}_{b,b}. \tag{5.4.17}$$

Similarly, starting with the universal form factor $\mathfrak{F}^{(2,2)}$ and using commutation relations (5.4.3) we can obtain all the universal form factors $\mathfrak{F}^{(i,j)}$ with $|i-j|=1$:

$$\mathfrak{F}^{(2,3)}\begin{pmatrix}\bar u^C & \bar u^B\\ \bar v^C & \bar v^B\end{pmatrix}^{a,a}_{b+1,b}=\lim_{w\to\infty}\left(\frac{-w}{c}\right)^{b+1}\mathfrak{F}^{(2,2)}\begin{pmatrix}\bar u^C & \bar u^B\\ \bar v^C & \{\bar v^B,w\}\end{pmatrix}^{a,a}_{b+1,b+1}, \tag{5.4.18}$$

$$\mathfrak{F}^{(2,1)}\begin{pmatrix}\bar u^C & \bar u^B\\ \bar v^C & \bar v^B\end{pmatrix}^{a-1,a}_{b,b}=\lim_{w\to\infty}\frac{w}{c}\,\mathfrak{F}^{(2,2)}\begin{pmatrix}\{\bar u^C,w\} & \bar u^B\\ \bar v^C & \bar v^B\end{pmatrix}^{a,a}_{b,b}, \tag{5.4.19}$$

$$\mathfrak{F}^{(3,2)}\begin{pmatrix}\bar u^C & \bar u^B\\ \bar v^C & \bar v^B\end{pmatrix}^{a,a}_{b-1,b}=-\lim_{w\to\infty}\left(\frac{w}{c}\right)^{b}\mathfrak{F}^{(2,2)}\begin{pmatrix}\bar u^C & \bar u^B\\ \{\bar v^C,w\} & \bar v^B\end{pmatrix}^{a,a}_{b,b}. \tag{5.4.20}$$

The universal form factors $\mathfrak{F}^{(i,j)}$ with $|i-j|=2$ can be obtained as the limits of $\mathfrak{F}^{(i,j)}$ with $|i-j|=1$, for example,

$$\mathfrak{F}^{(1,3)}\begin{pmatrix}\bar u^C & \bar u^B\\ \bar v^C & \bar v^B\end{pmatrix}^{a+1,a}_{b+1,b}=\lim_{w\to\infty}\left(\frac{-w}{c}\right)^{b+1}\mathfrak{F}^{(1,2)}\begin{pmatrix}\bar u^C & \bar u^B\\ \bar v^C & \{\bar v^B,w\}\end{pmatrix}^{a+1,a}_{b+1,b+1}, \tag{5.4.21}$$

$$\mathfrak{F}^{(3,1)}\begin{pmatrix}\bar u^C & \bar u^B\\ \bar v^C & \bar v^B\end{pmatrix}^{a-1,a}_{b-1,b}=\lim_{w\to\infty}\frac{w}{c}\mathfrak{F}^{(3,2)}\begin{pmatrix}\{\bar u^C,w\} & \bar u^B\\ \bar v^C & \bar v^B\end{pmatrix}^{a,a}_{b-1,b}. \tag{5.4.22}$$

Thus, starting with $\mathfrak{F}^{(2,2)}$ and taking different limits of the Bethe parameters we obtain all the universal form factors of the off-diagonal matrix elements of the monodromy matrix. Formally, $\mathfrak{F}^{(1,1)}$ and $\mathfrak{F}^{(3,3)}$ can be also included in this scheme, for example,

$$\mathfrak{F}^{(1,1)}\begin{pmatrix}\bar u^C & \bar u^B\\ \bar v^C & \bar v^B\end{pmatrix}^{a,a}_{b,b}-\mathfrak{F}^{(2,2)}\begin{pmatrix}\bar u^C & \bar u^B\\ \bar v^C & \bar v^B\end{pmatrix}^{a,a}_{b,b}=\lim_{w\to\infty}\frac{w}{c}\,\mathfrak{F}^{(1,2)}\begin{pmatrix}\{\bar u^C,w\} & \bar u^B\\ \bar v^C & \bar v^B\end{pmatrix}^{a+1,a}_{b,b}. \tag{5.4.23}$$

However, in our case this relation is not needed, because determinant representations for all diagonal universal form factors were already derived in previous chapter.

It should be noted that the possibility of considering the limit of an infinite Bethe parameter is based on the use of the generalised model. On the one hand, in this model, the Bethe parameters are arbitrary complex numbers. Hence, one of them can be sent to infinity. On the other hand, the existence of an infinite root in the Bethe equations agrees with the expansion (5.4.2). At the same time, since the final expression for form factors depends on r_1 and r_3 only through the eigenvalue $\tau(z|\bar u,\bar v)$, the condition (5.4.2) is not a restriction on the form factors. It can be checked for instance in Bose gas models [69], where the relations between form factors and the zero modes method both apply, although the condition (5.4.2) is not fulfilled.

5.5 Form factors of off-diagonal elements

In this section we deduce from the zero modes method determinant representations for the universal form factors of the operators $T_{ij}(z)$ with $i \neq j$. Consideration is restricted with two typical examples of $\mathfrak{F}^{(1,2)}$ and $\mathfrak{F}^{(3,2)}$. All other determinant representations for the universal form factors can be obtained in a similar manner.

5.5.1 Form factor $\mathfrak{F}^{(1,2)}$

Due to (5.4.17) the form factor $\mathfrak{F}^{(1,2)}$ is a limiting case of the form factor $\mathfrak{F}^{(2,2)}$. The determinant representation for the latter is given by (4.4.14)–(4.4.17), where without any loss of generality we can set $p = a + 1$. In these expressions we also should replace a with $a + 1$ and $\bar{u}^B$ with $\{\bar{u}^B, w\}$ Then we have

$$\mathfrak{F}^{(1,2)}\begin{pmatrix} \bar{u}^C & \bar{u}^B \\ \bar{v}^C & \bar{v}^B \end{pmatrix}_{b,b}^{a+1,a} = \lim_{w\to\infty} \frac{wH}{c\,\Omega_{a+1}} \det_{a+b+1} \mathcal{N}^{(2,2)}. \tag{5.5.1}$$

For taking the limit it is convenient to multiply the first a rows of the matrix $\mathcal{N}^{(2,2)}$ by the factors $-w/c$. Then we obtain

$$\mathfrak{F}^{(1,2)}\begin{pmatrix} \bar{u}^C & \bar{u}^B \\ \bar{v}^C & \bar{v}^B \end{pmatrix}_{b,b}^{a+1,a} = \lim_{w\to\infty} \left(\frac{-c}{w}\right)^a \frac{wH}{c\,\Omega_{a+1}} \det_{a+b+1} \overset{\circ}{\mathcal{N}}{}^{(2,2)}_{j,k}. \tag{5.5.2}$$

where

$$\begin{aligned} \overset{\circ}{\mathcal{N}}{}^{(2,2)}_{j,k} &= -\frac{w}{c}\mathcal{N}^{(2,2)}_{j,k}, \qquad j = 1,\dots,a, \\ \overset{\circ}{\mathcal{N}}{}^{(2,2)}_{j,k} &= \mathcal{N}^{(2,2)}_{j,k}, \qquad j = a+1,\dots,a+b+1. \end{aligned} \tag{5.5.3}$$

Now let us give explicit expressions for the prefactor and the matrix elements in (5.5.2). The factor H is

$$\begin{aligned} H(\bar{u}^C;\{\bar{u}^B,w\};\bar{v}^C) &= f(\bar{v}^C,\bar{u}^B)h(\bar{u}^B,\bar{u}^B)\Delta'(\bar{u}^C)\Delta(\bar{u}^B)\Delta(\bar{v}^C)\Delta'(\bar{v}^C) \\ &\times f(\bar{v}^C,w)h(w,\bar{u}^B)h(\bar{u}^B,w)g(w,\bar{u}^B) = f(\bar{v}^C,w)f(w,\bar{u}^B)h(\bar{u}^B,w)H(\bar{u}^C;\bar{u}^B;\bar{v}^C), \end{aligned} \tag{5.5.4}$$

where $H(\bar{u}^C;\bar{u}^B;\bar{v}^C)$ is given by (5.2.4). Hence, due to (1.1.38) we find

$$\lim_{w\to\infty}\left(\frac{-c}{w}\right)^a H(\bar{u}^C;\{\bar{u}^B,w\};\bar{v}^C) = H(\bar{u}^C;\bar{u}^B;\bar{v}^C). \tag{5.5.5}$$

The coefficient Ω_{a+1} is equal to

$$\Omega_{a+1}(\bar{u}^C;\{\bar{u}^B,w\}) = \frac{1}{g(u^C_{a+1},w)}\,\frac{g(u^C_{a+1},\bar{u}^C_{a+1})}{g(u^C_{a+1},\bar{u}^B)} = \frac{\Omega_{a+1}(\bar{u}^C;\bar{u}^B)}{g(u^C_{a+1},w)}, \tag{5.5.6}$$

and therefore

$$\lim_{w\to\infty}\frac{c}{w}\,\Omega_{a+1}(\bar{u}^C;\{\bar{u}^B,w\}) = -\Omega_{a+1}(\bar{u}^C;\bar{u}^B), \tag{5.5.7}$$

where $\Omega_{a+1}(\bar{u}^C;\bar{u}^B)$ is given by (4.3.1). Thus, the prefactor coincides with the one in (5.2.11) up to the sign.

Consider now the matrix elements $\mathring{\mathcal{N}}^{(2,2)}_{j,k}$. First of all $\mathring{\mathcal{N}}^{(2,2)}_{a+1,k} = -1$ for all $k = 1, \dots, a+b+1$. If $j, k \neq a+1$, then

$$\begin{aligned}
&\mathring{\mathcal{N}}^{(2,2)}_{j,k}(\bar u^C; \{\bar u^B, w\}; \bar v^C) \\
&= -\frac{w}{c}\left(\frac{(-1)^a r_1(x_k) t(u^C_j, x_k) h(\bar u^C, x_k)}{f(\bar v^C, x_k) h(x_k, \bar u^B) h(x_k, w)} + \frac{t(x_k, u^C_j) h(x_k, \bar u^C)}{h(x_k, \bar u^B) h(x_k, w)}\right), \qquad \begin{array}{l} j = 1, \dots, a, \\ k = 1, \dots, a+b+1, \\ k \neq a+1, \end{array}
\end{aligned} \tag{5.5.8}$$

$$\begin{aligned}
&\mathring{\mathcal{N}}^{(2,2)}_{a+1+j,k}(\{\bar u^B, w\}; \bar v^C; \bar v^B) \\
&= -t(v^C_j, x_k) \frac{g(\bar v^B, x_k)}{g(\bar v^C, x_k)}\left(1 - \frac{r_3(x_k)}{f(x_k, \bar u^B) f(x_k, w)}\right), \qquad \begin{array}{l} j = 1, \dots, b, \\ k = 1, \dots, a+b+1, \\ k \neq a+1. \end{array}
\end{aligned} \tag{5.5.9}$$

Here $\{x_1, \dots, x_a\} = \{u^B_1, \dots, u^B_a\}$ and $\{x_{a+2}, \dots, x_{a+b+1}\} = \{v^C_1, \dots, v^C_b\}$. Taking the limit $w \to \infty$ we obtain

$$\begin{aligned}
&\lim_{w\to\infty} \mathring{\mathcal{N}}^{(2,2)}_{j,k}(\bar u^C; \{\bar u^B, w\}; \bar v^C) \\
&= \frac{(-1)^a r_1(x_k) t(u^C_j, x_k) h(\bar u^C, x_k)}{f(\bar v^C, x_k) h(x_k, \bar u^B)} + \frac{t(x_k, u^C_j) h(x_k, \bar u^C)}{h(x_k, \bar u^B)}, \qquad \begin{array}{l} j = 1, \dots, a, \\ k = 1, \dots, a+b+1, \\ k \neq a+1, \end{array}
\end{aligned} \tag{5.5.10}$$

$$\begin{aligned}
&\lim_{w\to\infty} \mathring{\mathcal{N}}^{(2,2)}_{a+1+j,k}(\{\bar u^B, w\}; \bar v^C; \bar v^B) \\
&= -t(v^C_j, x_k) \frac{g(\bar v^B, x_k)}{g(\bar v^C, x_k)}\left(1 - \frac{r_3(x_k)}{f(x_k, \bar u^B)}\right), \qquad \begin{array}{l} j = 1, \dots, b, \\ k = 1, \dots, a+b+1, \\ k \neq a+1. \end{array}
\end{aligned} \tag{5.5.11}$$

Finally, for the elements $\mathring{\mathcal{N}}^{(2,2)}_{j,a+1}$ with $j \neq a+1$ we have

$$\mathring{\mathcal{N}}^{(2,2)}_{j,a+1}(\bar u^C; \{\bar u^B, w\}; \bar v^C) = \frac{-w}{c}\left(t(u^C_j, w) \frac{(-1)^a r_1(w) h(\bar u^C, w)}{f(\bar v^C, w) h(w, \bar u^B)} + \frac{t(w, u^C_j) h(w, \bar u^C)}{h(w, \bar u^B)}\right), \quad j < a+1, \tag{5.5.12}$$

$$\mathring{\mathcal{N}}^{(2,2)}_{j,a+1}(\{\bar u^B, w\}; \bar v^C; \bar v^B) = -t(v^C_j, w) \frac{g(\bar v^B, w)}{g(\bar v^C, w)}, \quad j > a+1, \tag{5.5.13}$$

and sending there w to infinity we obtain

$$\lim_{w\to\infty} \mathring{\mathcal{N}}^{(2,2)}_{j,a+1}(\bar u^C; \{\bar u^B, w\}; \bar v^C) = \lim_{w\to\infty} \mathring{\mathcal{N}}^{(2,2)}_{j,a+1}(\{\bar u^B, w\}; \bar v^C; \bar v^B) = 0. \tag{5.5.14}$$

We see that the $(a+1)$-th column of the matrix $\mathring{\mathcal{N}}^{(2,2)}_{j,k}$ contains only one non-zero element $\mathring{\mathcal{N}}^{(2,2)}_{a+1,a+1} = -1$. Thus, the determinant in (5.5.2) reduces to the determinant of the $(a+b) \times (a+b)$ matrix with the matrix elements (5.5.10) and (5.5.11). Obviously, this representation coincides with the expressions (5.2.11) and (5.2.12).

5.5.2 Form factor $\mathfrak{F}^{(3,2)}$

The form factor $\mathfrak{F}^{(3,2)}$ also can be obtained as a limit of the form factor $\mathfrak{F}^{(2,2)}$ via (5.4.20). We use again representation (4.4.14)–(4.4.17), but now it is convenient to set $p = a + b$. We also should replace $\bar v^C$ with $\{\bar v^C, w\}$. Then

$$\mathfrak{F}^{(3,2)}\begin{pmatrix}\bar u^C & \bar u^B\\ \bar v^C & \bar v^B\end{pmatrix}^{a,a}_{b-1,b} = -\lim_{w\to\infty}\left(\frac{w}{c}\right)^b \frac{H}{\Omega_{a+b}} \det_{a+b}\mathcal{N}^{(2,2)}. \tag{5.5.15}$$

For taking the limit we multiply the rows with $j = a+1,\dots,a+b-1$ of the matrix $\mathcal{N}^{(2,2)}$ by the factors c/w. Then we obtain

$$\mathfrak{F}^{(3,2)}\begin{pmatrix}\bar u^C & \bar u^B\\ \bar v^C & \bar v^B\end{pmatrix}^{a,a}_{b-1,b} = -\lim_{w\to\infty}\frac{w}{c}\left(\frac{w}{c}\right)^{2b-2} \frac{H}{\Omega_{a+b}} \det_{a+b}\mathring{\mathcal{N}}^{(2,2)}_{j,k}. \tag{5.5.16}$$

where

$$\begin{aligned}
&\mathring{\mathcal{N}}^{(2,2)}_{j,k} = \mathcal{N}^{(2,2)}_{j,k}, \qquad j = 1,\dots,a,\\
&\mathring{\mathcal{N}}^{(2,2)}_{j,k} = \frac{c}{w}\mathcal{N}^{(2,2)}_{j,k}, \qquad j = a+1,\dots,a+b-1,\\
&\mathring{\mathcal{N}}^{(2,2)}_{a+b,k} = \mathcal{N}^{(2,2)}_{a+b,k} = -1.
\end{aligned} \tag{5.5.17}$$

Now let us give explicit expressions for the prefactor and the matrix elements in (5.5.16). The factor H is

$$\begin{aligned}
H(\bar u^C;\bar u^B;\{\bar v^C,w\}) &= f(\bar v^C,\bar u^B)h(\bar u^B,\bar u^B)\Delta'(\bar u^C)\Delta(\bar u^B)\Delta(\bar v^C)\Delta'(\bar v^C)\\
&\times f(w,\bar u^B)g(w,\bar v^C)g(\bar v^C,w) = f(w,\bar u^B)g(w,\bar v^C)g(\bar v^C,w)H(\bar u^C;\bar u^B;\bar v^C),
\end{aligned} \tag{5.5.18}$$

where $H(\bar u^C;\bar u^B;\bar v^C)$ is given by (5.2.4). Hence, due to (1.1.38) we find

$$\lim_{w\to\infty}\left(\frac{w}{c}\right)^{2b-2} H(\bar u^C;\bar u^B;\{\bar v^C,w\}) = (-1)^{b-1}H(\bar u^C;\bar u^B;\bar v^C). \tag{5.5.19}$$

The coefficient Ω_{a+b} is equal to

$$\Omega_{a+b}(\{\bar v^C,w\};\bar u^B) = \frac{g(w,\bar v^C)}{g(w,\bar v^B)}, \tag{5.5.20}$$

and therefore

$$\lim_{w\to\infty}\frac{c}{w}\,\Omega_{a+b}(\{\bar v^C,w\};\bar u^B) = 1. \tag{5.5.21}$$

Consider now the matrix elements $\mathring{\mathcal{N}}^{(2,2)}_{j,k}$. If $k \ne a+b$, then

$$\begin{aligned}
&\mathring{\mathcal{N}}^{(2,2)}_{j,k}(\bar u^C;\bar u^B;\{\bar v^C,w\})\\
&= \frac{(-1)^{a-1}r_1(x_k)t(u^C_j,x_k)h(\bar u^C,x_k)}{f(\bar v^C,x_k)f(w,x_k)h(x_k,\bar u^B)} + \frac{t(x_k,u^C_j)h(x_k,\bar u^C)}{h(x_k,\bar u^B)}, \qquad \begin{array}{l} j = 1,\dots,a,\\ k = 1,\dots,a+b-1,\end{array}
\end{aligned} \tag{5.5.22}$$

$$\begin{aligned}
&\mathring{\mathcal{N}}^{(2,2)}_{a+j,k}(\bar u^B;\{\bar v^C,w\};\bar v^B)\\
&\quad = -\frac{c}{w}t(v^C_j,x_k)\frac{g(\bar v^B,x_k)}{g(\bar v^C,x_k)g(w,x_k)}\left(1 - \frac{r_3(x_k)}{f(x_k,\bar u^B)}\right), \qquad \begin{array}{l} j = 1,\dots,b-1,\\ k = 1,\dots,a+b-1.\end{array}
\end{aligned} \tag{5.5.23}$$

Here $\{x_1,\dots,x_a\} = \{u^B_1,\dots,u^B_a\}$ and $\{x_{a+1},\dots,x_{a+b-1}\} = \{v^C_1,\dots,v^C_{b-1}\}$. Taking the limit $w\to\infty$ we obtain

$$\begin{aligned}&\lim_{w\to\infty}\overset{\circ}{\mathcal{N}}{}^{(2,2)}_{j,k}(\bar u^C;\bar u^B;\{\bar v^C,w\})\\ &=\frac{(-1)^{a-1}r_1(x_k)t(u^C_j,x_k)h(\bar u^C,x_k)}{f(\bar v^C,x_k)h(x_k,\bar u^B)}+\frac{t(x_k,u^C_j)h(x_k,\bar u^C)}{h(x_k,\bar u^B)},\qquad \begin{aligned}&j=1,\dots,a,\\ &k=1,\dots,a+b-1,\end{aligned}\end{aligned}\tag{5.5.24}$$

$$\begin{aligned}&\lim_{w\to\infty}\overset{\circ}{\mathcal{N}}{}^{(2,2)}_{a+j,k}(\bar u^B;\{\bar v^C,w\};\bar v^B)\\ &\qquad=-t(v^C_j,x_k)\frac{g(\bar v^B,x_k)}{g(\bar v^C,x_k)}\left(1-\frac{r_3(x_k)}{f(x_k,\bar u^B)}\right),\qquad \begin{aligned}&j=1,\dots,b,\\ &k=1,\dots,a+b-1.\end{aligned}\end{aligned}\tag{5.5.25}$$

Finally, for the elements $\overset{\circ}{\mathcal{N}}{}^{(2,2)}_{j,a+b}$ with $j\ne a+b$ we have

$$\overset{\circ}{\mathcal{N}}{}^{(2,2)}_{j,a+b}(\bar u^C;\bar u^B;\{\bar v^C,w\})=\frac{t(w,u^C_j)h(w,\bar u^C)}{h(w,\bar u^B)},\qquad j=1,\dots,a,\tag{5.5.26}$$

$$\overset{\circ}{\mathcal{N}}{}^{(2,2)}_{a+j,a+b}(\bar u^B;\{\bar v^C,w\};\bar v^B)=0,\qquad j=1,\dots,b-1,\tag{5.5.27}$$

and sending there w to infinity we obtain that $\overset{\circ}{\mathcal{N}}{}^{(2,2)}_{j,a+b}\to 0$ as $w\to\infty$ for $j<a+b$.

We see that the last column of the matrix $\overset{\circ}{\mathcal{N}}{}^{(2,2)}_{j,k}$ contains only one non-zero element $\overset{\circ}{\mathcal{N}}{}^{(2,2)}_{a+b,a+b}=-1$. Thus, the determinant in (5.5.16) reduces to the determinant of the $(a+b-1)\times(a+b-1)$ matrix with the matrix elements (5.5.24) and (5.5.25). Obviously, this representation coincides with (5.2.19), (5.2.20).

Remark. In all considerations above we assumed that Bethe parameters of on-shell Bethe vectors $\mathbb{C}_{a',b'}(\bar u^C;\bar v^C)$ and $\mathbb{B}_{a,b}(\bar u^B;\bar v^B)$ were finite. However, if $r_k(z)\to 1$ at $z\to\infty$, then Bethe equations (2.1.30) admit infinite solutions as well. The peculiarity of such infinite roots is that the corresponding Bethe vectors are no longer singular vectors of the zero modes $T_{ij}[0]$ with $i>j$ (respectively, the operators $T_{ij}[0]$ with $i<j$ do not annihilate dual on-shell vectors with infinite parameters). This property played an essential role in our derivations, therefore one might have impression that the case of infinite Bethe roots requires a special study. However, as it was shown in [86] for the models with $\mathfrak{gl}(3)$-invariant R-matrix, all relations between the form factors remain valid even in the presence of infinite Bethe parameters. The method of the work [86] can be used for the models described by the $\mathfrak{gl}(2|1)$ superalgebra without any changes. Therefore we do not give here a special consideration to this problem.

5.6 Form factors in the models described by $\mathfrak{gl}(1|2)$ superalgebra

As already mention, models possessed $\mathfrak{gl}(1|2)$ and $\mathfrak{gl}(2|1)$ symmetry are isomorphic. To distinguish object belonging to these two algebras below everything pertain to $\mathfrak{gl}(1|2)$ algebra symmetry case will be marked with tilde symbol. In particular, the new gradation appears $\widetilde{[1]}=0$ and $\widetilde{[2]}=\widetilde{[3]}=1$. Monodromy matrix entries are denoted as $\widetilde{T}_{ij}$, and their vacuum eigenvalues as $\widetilde{\lambda}_j$, Bethe vectors are denoted as $\widetilde{\mathbb{B}}_{a,b}(\bar u,\bar v)$, etc.

Isomorphism φ defined in following way.

Definition 5.6.1. *Let $\bar{\jmath} = 4 - j$. Then*

$$\varphi \ : \ \begin{cases} [j] & \to \quad \widetilde{[j]} = [\bar{\jmath}] + 1, \\ T_{ij}(u) & \to \quad (-1)^{[j][i]+[j]+1}\, \widetilde{T}_{\bar{\jmath},\bar{\imath}}(u) \\ \lambda_j(u) & \to \quad \widetilde{\lambda}_j(u) = -\lambda_{\bar{\jmath}}(u). \end{cases} \tag{5.6.1}$$

Hence,

$$\varphi(AB) = \varphi(A)\varphi(B). \tag{5.6.2}$$

Remark. There is a big freedom in the definition of φ. Namely, we can use the following action $T_{ij}(u) \to (-1)^{[j][i]+\alpha[i]+\beta[j]+\gamma}\, \widetilde{T}_{\bar{\jmath},\bar{\imath}}(u)$, where α, β, and γ are arbitrary constants. Indeed, if the operators $\widetilde{T}_{ij}(u)$ satisfy the commutation relations of $\mathfrak{gl}(1|2)$, then multiplication by $(-1)^{\alpha[i]+\beta[j]+\gamma}$ is equivalent to the multiplication of the monodromy matrix $\widetilde{T}$ by diagonal twists (from the left by $\mathrm{diag}((-1)^{\alpha[i]})$ and from the right by $\mathrm{diag}((-1)^{\beta[j]+\gamma})$). It is clear that after this multiplication the commutation relations are preserved. We have used this possibility in (5.6.1) in order to have

$$\varphi\big(\mathrm{str}(T(u)\big) = \mathrm{str}\, \widetilde{T}(u). \tag{5.6.3}$$

However, even this additional restriction does not fix completely the action of φ. We could choose, for instance, $T_{ij}(u) \to (-1)^{[j][i]+[i]+1}\, \widetilde{T}_{\bar{\jmath},\bar{\imath}}(u)$.

5.6.1 Bethe vectors

Bethe vectors in $\mathfrak{gl}(1|2)$ were constructed in [84]:

$$\widetilde{\mathbb{B}}_{a,b}(\bar u;\bar v) = (-1)^a \sum \frac{g(\bar u_{\rm I},\bar v_{\rm I}) f(\bar v_{\rm I},\bar v_{\rm II}) g(\bar u_{\rm II},\bar u_{\rm I}) h(\bar v_{\rm I},\bar v_{\rm I})}{\widetilde{\lambda}_2(\bar u_{\rm II})\widetilde{\lambda}_2(\bar v) f(\bar u,\bar v)} \widetilde{\mathbb{T}}_{13}(\bar v_{\rm I})\widetilde{T}_{23}(\bar v_{\rm II})\widetilde{\mathbb{T}}_{12}(\bar u_{\rm II})\widetilde{\Omega}. \tag{5.6.4}$$

The dual vectors have the following explicit form

$$\widetilde{\mathbb{C}}_{a,b}(\bar u;\bar v) = (-1)^{\frac{a(a-1)}{2}} \sum \frac{g(\bar u_{\rm I},\bar v_{\rm I}) f(\bar v_{\rm I},\bar v_{\rm II}) g(\bar u_{\rm II},\bar u_{\rm I}) h(\bar v_{\rm I},\bar v_{\rm I})}{\widetilde{\lambda}_2(\bar u_{\rm II})\widetilde{\lambda}_2(\bar v) f(\bar u,\bar v)} \widetilde{\Omega}^\dagger \widetilde{\mathbb{T}}_{21}(\bar u_{\rm II})\widetilde{T}_{32}(\bar v_{\rm II})\widetilde{\mathbb{T}}_{31}(\bar v_{\rm I}). \tag{5.6.5}$$

Then, assuming that $\varphi(\Omega) = \widetilde{\Omega}$ and $\varphi(\Omega^\dagger) = \widetilde{\Omega}^\dagger$ we find

$$\varphi\big(\mathbb{B}_{a,b}(\bar u;\bar v)\big) = \widetilde{\mathbb{B}}_{b,a}(\bar v;\bar u), \qquad \varphi\big(\mathbb{C}_{a,b}(\bar u;\bar v)\big) = \widetilde{\mathbb{C}}_{b,a}(\bar v;\bar u). \tag{5.6.6}$$

Here we have (dual) Bethe vectors of $\mathfrak{gl}(2|1)$ in the l.h.s., and (dual) Bethe vectors of $\mathfrak{gl}(1|2)$ in the r.h.s. One can also easily check that

$$\psi\big(\widetilde{\mathbb{B}}_{a,b}(\bar u;\bar v)\big) = (-1)^a \widetilde{\mathbb{C}}_{a,b}(\bar u;\bar v), \qquad \psi\big(\widetilde{\mathbb{C}}_{a,b}(\bar u;\bar v)\big) = \widetilde{\mathbb{B}}_{a,b}(\bar u;\bar v). \tag{5.6.7}$$

5.6.2 Form factors

Form factors of the operators $T_{ij}(z)$ depend on the functions $\lambda_k(z)$. Therefore they are not invariant under the action of φ:

$$\varphi\Big(\mathcal{F}^{(i,j)}\left(z\Big|\begin{smallmatrix}\bar u^C & \bar u^B\\ \bar v^C & \bar v^B\end{smallmatrix}\right)^{a',a}_{b',b}\Big) = \mathcal{F}^{(i,j)}\left(z\Big|\begin{smallmatrix}\bar u^C & \bar u^B\\ \bar v^C & \bar v^B\end{smallmatrix}\right)^{a',a}_{b',b}\Big|_{\lambda_k(z)\to-\lambda_{\bar k}(z)}. \tag{5.6.8}$$

On the other hand we have

$$
\begin{aligned}
&\varphi\left(\mathcal{F}^{(i,j)}\left(z\middle|\begin{smallmatrix}\bar{u}^C & \bar{u}^B\\ \bar{v}^C & \bar{v}^B\end{smallmatrix}\right)^{a',a}_{b',b}\right)=\varphi\Big(\mathbb{C}_{a',b'}(\bar{u}^C;\bar{v}^C)T_{ij}(z)\mathbb{B}_{a,b}(\bar{u}^B;\bar{v}^B)\Big)\\
&=(-1)^{[j][i]+[j]+1}\widetilde{\mathbb{C}}_{b',a'}(\bar{v}^C;\bar{u}^C)\widetilde{T}_{\bar{\jmath},\bar{\imath}}(z)\widetilde{\mathbb{B}}_{b,a}(\bar{v}^B;\bar{u}^B)=(-1)^{[j][i]+[j]+1}\widetilde{\mathcal{F}}^{(\bar{\jmath},\bar{\imath})}\left(z\middle|\begin{smallmatrix}\bar{v}^C & \bar{v}^B\\ \bar{u}^C & \bar{u}^B\end{smallmatrix}\right)^{b',b}_{a',a}.
\end{aligned}
\tag{5.6.9}
$$

Thus, we obtain

$$
(-1)^{[j][i]+[j]+1}\widetilde{\mathcal{F}}^{(\bar{\jmath},\bar{\imath})}\left(z\middle|\begin{smallmatrix}\bar{v}^C & \bar{v}^B\\ \bar{u}^C & \bar{u}^B\end{smallmatrix}\right)^{b',b}_{a',a}=\mathcal{F}^{(i,j)}\left(z\middle|\begin{smallmatrix}\bar{u}^C & \bar{u}^B\\ \bar{v}^C & \bar{v}^B\end{smallmatrix}\right)^{a',a}_{b',b}\Bigg|_{\lambda_k(z)=-\widetilde{\lambda}_{\bar{k}}(z)}.
\tag{5.6.10}
$$

Changing here

$$
\bar{u}^{C,B}\leftrightarrow\bar{v}^{C,B},\qquad a\leftrightarrow b,\qquad a'\leftrightarrow b',\qquad \bar{\jmath}\leftrightarrow i,\qquad \bar{\imath}\leftrightarrow j,
\tag{5.6.11}
$$

we find

$$
\widetilde{\mathcal{F}}^{(i,j)}\left(z\middle|\begin{smallmatrix}\bar{u}^C & \bar{u}^B\\ \bar{v}^C & \bar{v}^B\end{smallmatrix}\right)^{a',a}_{b',b}=(-1)^{[\bar{\jmath}][\bar{\imath}]+[\bar{\imath}]+1}\mathcal{F}^{(\bar{\jmath},\bar{\imath})}\left(z\middle|\begin{smallmatrix}\bar{v}^C & \bar{v}^B\\ \bar{u}^C & \bar{u}^B\end{smallmatrix}\right)^{b',b}_{a',a}\Bigg|_{\lambda_k(z)=-\widetilde{\lambda}_{\bar{k}}(z)}.
\tag{5.6.12}
$$

It remains to use $\widetilde{[j]}=[\bar{\jmath}]+1$, and we finally arrive at

$$
\widetilde{\mathcal{F}}^{(i,j)}\left(z\middle|\begin{smallmatrix}\bar{u}^C & \bar{u}^B\\ \bar{v}^C & \bar{v}^B\end{smallmatrix}\right)^{a',a}_{b',b}=(-1)^{\widetilde{[j]}\widetilde{[i]}+\widetilde{[j]}+1}\mathcal{F}^{(\bar{\jmath},\bar{\imath})}\left(z\middle|\begin{smallmatrix}\bar{v}^C & \bar{v}^B\\ \bar{u}^C & \bar{u}^B\end{smallmatrix}\right)^{b',b}_{a',a}\Bigg|_{\lambda_k(z)=-\widetilde{\lambda}_{\bar{k}}(z)}.
\tag{5.6.13}
$$

Thus, the form factors of the monodromy matrix entries in the models with $\mathfrak{gl}(1|2)$ and $\mathfrak{gl}(2|1)$ symmetries are related to each other by the replacement of variables (5.6.11).

Conclusion

The main results of this chapter are formulae for determinant representations of the monodromy matrix entries form factors in the integrable models possessed $\mathfrak{gl}(2|1)$ and $\mathfrak{gl}(1|2)$ algebra symmetries. These results can be directly applied to the computation of form factors of ultralocal operators and, correspondingly, correlation functions of integrable models using form factor series (0.1.7). This problem is considered in next chapter.

Calculation of form factors and scalar product in the models with algebra symmetry $\mathfrak{gl}(m|n)$, $m,n\geq 2$ is an open question. Any results for such algebras will be extremely important for description of integrable higher-spin and multicomponent Fermi gases (see review [122]) and supersymmetric Yang-Mills theories [118]. Zero modes method can be applied also there, so in order to obtain all form factor it is required to compute just one and other can be obtained by taking special limits. However, it is clear that even computation of single form factor is an extremely complicated problem in the higher rank algebra symmetry case. For instance, it is not clear what can be possible generalisation of formulae for the matrix block (5.2.5)-(5.2.6) in the determinant.

One more interesting problem is a computation of the form factors in case of the trigonometric R-matrix.

Chapter 6

Correlation functions in Gaudin-Yang model

The problem of representations of ultralocal operators form factors via monodromy matrix entries form factors was solved in [68, 86] for arbitrary algebra $\mathfrak{gl}(N)$ and generalised for graded algebra symmetry case in [98]. For fundamental models the solution of the quantum inverse problem can be also applied (see [80, 97]). It provides us iwth the possibility to calculate correlation functions using form factor series summation (0.1.7). This summation can be performed both analytically and numerically. Below the numerical approach is developed for integrable 1D Fermi gas described by Gaudin-Yang model.

6.1 1D Fermi gases

One-dimensional gases have been intensively studied during many years. Such models attract a lot of attention since there was found that they are often exactly solvable by the Bethe ansatz technique and other methods [18, 20–25, 27]. The non-perturbative description of Bose and Fermi gases was studied in a giant number of works (see reviews [122, 134]). The experimental progress in the realisation of the 1D optical traps [138–141] and the search for the perspective methods for realisation of quantum bits renew interest on these models.

The object of our interest is an integrable model of one dimensional Fermi gas described by the *Gaudin-Yang model* [25, 32]. This model among with the *Lieb-Liniger model* (Bose analog) is the simplest example of the low-dimensional models with the ultralocal interaction. We restrict ourselves here to the model with spin-1/2. Integrable models with the higher spins, multiple components and mixtures of Bose and Fermi gases also exist but their description requires calculation of the form factors by ABA approach in algebra symmetry case $\mathfrak{gl}(m|n)$ with $m + n > 3$ that is an open problem.

The experimental realisations of 1D Fermi systems were done in [136, 137]. Properties of the Gaudin-Yang model were studied in [123–130]. Spectrum, phase diagrams, thermal behavior, relaxation properties were considered. Correlation functions were examined in [131–133] using CFT prediction or Luttinger liquid approach. However, the description of correlation functions in such models is far from being complete. The goal of the current chapter is to present the numerical algorithm that allows to describe the correlation functions of the 1D spin-1/2 Fermi gas. This method is loosely based on the ABACUS algorithm that was developed in the works [60, 61, 104, 135] (see also [159–162] and review [163]) and gave perfect description of the one-component spinless Bose gas, Heisenberg spin chains, etc. This technique is, in fact, the procedure of the numerical summation of the form factors of ultralocal (one-point) operators, that can be calculated using the algebraic Bethe ansatz approach (ABA). Our current task is the generalisation of the ABACUS technique to the spin-1/2 Fermi gases and t-J supersymmetric model. The final result is a Fourier map of correlators called *dynamical structure factor* (DSF). We present our result as a density diagram in energy-momentum coordinates. Experimentally such image can be obtained via the light scattering measurement.

6.1.1 Ultralocal operator via ABA

Form factors of the monodromy matrix entries T_{ij} with algebra symmetry $\mathfrak{gl}(2|1)$ were calculated in [90] and are given in chapters 4-5 ((4.4.14)-(4.4.17) and (5.2.11)-(5.2.21)). Any ultralocal physical operator can be expressed via these operators. Particular cases are given in the next section. Form factors of the ultralocal operators $\mathcal{O}(m)$ are connected to the universal form factors of the monodromy matrix entries T_{ij} as [98]

$$\mathcal{O}_{ij}\left(m\,\Big|\,\begin{smallmatrix}\bar{u}^C & \bar{u}^B\\ \bar{v}^C & \bar{v}^B\end{smallmatrix}\right)^{a',a}_{b',b}=\left(\frac{\ell_1(\bar{u}^C)\ell_3(\bar{v}^B)}{\ell_1(\bar{u}^B)\ell_3(\bar{v}^C)}-1\right)\mathfrak{F}^{(i,j)}\left(\begin{smallmatrix}\bar{u}^C & \bar{u}^B\\ \bar{v}^C & \bar{v}^B\end{smallmatrix}\right)^{a',a}_{b',b}, \tag{6.1.1}$$

where ℓ_1, ℓ_3 are r_1 and r_3 for the one-site model, m is the discrete coordinate on the lattice. Finally, the connection between physical operators O and the ultralocal operators $\mathcal{O}_{ij}$ depends on the specific model and particular operators.

6.1.2 Gaudin-Yang model

The Hamiltonian of the one-dimensional Fermi gas with the ultralocal interaction is given by

$$H=\int_0^L dx\left\{\partial\psi^\dagger_\alpha\partial\psi_\alpha+c\psi^\dagger_\alpha\psi^\dagger_\beta\psi_\beta\psi_\alpha\right\},\qquad \alpha,\beta=\uparrow,\downarrow, \tag{6.1.2}$$

$$\left\{\psi^\dagger_\alpha(x),\psi_\beta(y)\right\}=\delta_{\alpha\beta}\delta(x-y), \tag{6.1.3}$$

the sum is taken over repeated indices. The Bethe ansatz description of this model was given by [32] (see also [25]). Gaudin-Yang model corresponds to the choice $r_1(u_j)=-1$, $r_3(v_j)=-\exp(iv_jL)$ in (6.2.8). The case of zero magnetic field is considered for simplicity and chemical potential is chosen to be zero since we consider case of fixed particles number and chemical potential gives just trivial shift of the ground state. Variables $\bar{v}$ are momenta of the fermions. Variables $\bar{u}$ are rapidities of the excitations corresponding to spin flipping. Experimentally coupling constant c can be tuned by external magnetic field using Feshbach resonance [142–145]. Here and below $\hbar{=}m{=}1$ convention is used.

A similar system of one-component Bose gas with the delta-interaction was studied numerically in [104, 135]. Below we study only the case with repulsive interaction, so $c>0$.

The connection of form factors of the monodromy matrix entries to form factors of local operators (6.1.1) in the Gaudin-Yang model is given by

$$\mathbb{C}_{a+2-k,b+1}(\bar{u}^C,\bar{v}^C)\psi^\dagger_k(x)\mathbb{B}_{a,b}(\bar{u}^B,\bar{v}^B)=i\sqrt{c}\,\mathfrak{F}^{(k,3)}\left(\begin{smallmatrix}\bar{u}^C & \bar{u}^B\\ \bar{v}^C & \bar{v}^B\end{smallmatrix}\right)^{a+2-k,a}_{b+1,b},\qquad k=1,2, \tag{6.1.4}$$

$$\mathbb{C}_{a-i+j,b}(\bar{u}^C,\bar{v}^C)\psi^\dagger_i(x)\psi_j(x)\mathbb{B}_{a,b}(\bar{u}^B,\bar{v}^B)=-\mathcal{P}(\bar{v}^B,\bar{v}^C)e^{ix\mathcal{P}(\bar{v}^B,\bar{v}^C)}\mathfrak{F}^{(i,j)}\left(\begin{smallmatrix}\bar{u}^C & \bar{u}^B\\ \bar{v}^C & \bar{v}^B\end{smallmatrix}\right)^{a-i+j,a}_{b,b}, \tag{6.1.5}$$

$$i,j=1,2,$$

with the energy and the momentum of the excited states are given by

$$\omega = \omega(\bar{v}^B) - \omega(\bar{v}^C), \qquad \omega(\bar{v}) = \sum_{i=1}^{b} v_i^2, \qquad q(\bar{v}) = \sum_{i=1}^{b} v_i, \qquad \mathcal{P} = q(\bar{v}^B) - q(\bar{v}^C). \tag{6.1.6}$$

These formulae are direct corollary of the general formula (6.1.1). The derivation for the Fermi gas coincides with the one for the Bose gas given in [68].

6.2 Calculation of the DSF in 1D integrable model via ABA

6.2.1 Form factor series

Two point dynamical correlation function of arbitrary operators O (0.1.7) at zero temperature can be presented as the form factor series

$$\langle O(x,t)O^\dagger(0,0)\rangle = \sum_m \frac{\langle 0|O(0,0)|m\rangle\langle m|O^\dagger(0,0)|0\rangle}{\langle m|m\rangle\langle 0|0\rangle} e^{i\omega_m t - ixp_m} \equiv \sum_m \left|\mathcal{F}_m^O\right|^2 e^{i\omega_m t - ixp_m}, \tag{6.2.1}$$

where $\left|\mathcal{F}_m^O\right|$ is the form factor of the ultralocal operator (0.1.8) between the state $|m\rangle$ and the ground state.

The problem is the summation of series (6.2.1). Analytically such form factor series was studied in [50–55, 58, 59]. However, quite often summation of (6.2.1) can be performed only numerically. The method of numerical calculation of the DSF based on the form factor summation formula was developed in the works of M. Karbach, G. Müller [159–162] and J.-S. Caux *et. al.* [60, 61, 104, 135, 163]. Our current task is the generalisation of this method to the higher rank algebra symmetry case.

6.2.2 Eigenstates

Generic (off-shell) Bethe vectors become the eigenvectors if their spectral parameters satisfy the system of BAE. Naturally, the different solutions of BAE correspond to different eigenstates. Hence, in the ABA approach the states are numerated by the solutions of BAE. The number of BAE solutions should coincide with the dimension of the Hilbert space of the model under consideration[1]. This property provides completeness of the Bethe ansatz and was conjectured since the very first work of H. Bethe [13] but was never rigorously proven for a wide range of models, except some cases, such as XXX or XXZ Heisenberg spin chains [152–154, 164]. Quite a general proof was done in [146], where *the string hypothesis* (see 6.2.3) was developed[2]. The string hypothesis provides a classification of all solutions of Bethe equations and allows to proof the completeness, however, it is not proven itself and often is violated [147–149, 151]. An important remark here is that the number of violations growths as square root of a (quasi)particles number that is infinitely slow in comparison with the total number of solutions. Thus, it can be expected that in absolute majority of situations violation of the string hypothesis does not play significant role, and even if some solutions are lost (hence, some terms in form factor series

[1]It should be noticed, that there also exist some solutions of the BAE that do not corresponds to any Bethe state, these solutions are called *nonphysical*. Of course, only the total number of physical solutions coincides with the dimension of the Hilbert space.

[2]Firstly the string hypothesis was conjectured by H. Bethe.

are omitted), the deviation of form factor series (6.2.1) from the correct value is expected to be negligible. The experience shows that this expectation is correct and dropping part of basis in series (6.2.1) is acceptable.

6.2.3 Classification of solutions

We refer to the first and the second equation of (2.1.30) as *the first and the second level (of BAE nesting)* correspondingly. We call in the same way variables $\bar{u}$ and $\bar{v}$ as *the first and the second level Bethe parameters* (or *the spectral parameters*).

The string hypothesis allows to classify the solutions of the BAE and provides significant simplification of the BAE solution method.

Definition 6.2.1. *Among the solutions of BAE there exist solutions, with few Bethe parameters connected by the following relations*

$$u^\alpha_{s,j} = u^\alpha_s + ic(s-2j)/2 + \delta^\alpha_{s,j}, \qquad j = 1,\dots,s, \qquad \delta^\alpha_{s,j} \sim e^{-|c|L}, \qquad \Im m\,(u^\alpha_s) = 0. \quad (6.2.2)$$

Such solutions are called strings. Here s is the length of the string, i.e. the number of Bethe parameters in the string, L is a system size, α is a string counter (indicates the particular string α among all the string of length s) and j counts Bethe parameters inside one string. $\delta^\alpha_{s,j}$ gives deviations of the solution from the ideal string. u^α_s is a string center (defines the real part of the string).

From the definition it is clear that strings are symmetric w.r.t. the real axis. The strings of length one are just real solutions (i.e. all Bethe parameters have zero imaginary part)[3].

The assumption of the string hypothesis is that *all possible solutions of the Bethe equation are given by the strings.* Particular solution of BAE consists from the set of strings of arbitrary lengths. Of course, the total number of Bethe parameters should be conserved

$$\sum_{s=1}^{\infty} sM_s = a, \quad (6.2.3)$$

where M_s is the number of strings of length s. In case of the nested Bethe ansatz Bethe parameters of each level of nesting can form strings.

Remark. The restrictions on the length of strings can appear in particular models in dependence on model parameters tuning. For example, in 1D Bose gas (Lieb-Liniger model [20–23]) with repulsive interaction there are no solutions of BAE with strings longer than 1, so all Bethe parameters are real (or have a common shift in the complex plane). While in case of attractive interaction strings of any length are allowed.

The string hypothesis does not give all possible solutions, for instance, another types of complex solutions with shift in complex direction by $2ic$ can exist, moreover for long strings the dependence of $\delta^\alpha_{s,j}$ on L can be not exponential but is a power law. The type and number of string hypothesis violations depend on the model and specific tuning of model parameters such as magnetic field, coupling constant, etc., but as is already mentioned it often does not play significant role for numerical computation. In case of Gaudin-Yang model with $c > 0$ it can be expected that for a wide range of parameters the complex solutions do not play a significant role at all, similarly to case of the repulsive Bose gas.

[3]For the BAE in form (2.1.30) these Bethe parameters will be sifted in the complex plane by $ic/2$, but we always can make a shift of u by $ic/2$ directly in the BAE and obtain real solutions.

6.2.4 Basis scanning algorithm

In order to calculate the form factor sum over all possible physical solutions they should be found explicitly from the BAE. Of course, numerically the summation in (6.2.1) can be performed only for a finite number of Bethe parameters, i.e. for the system with finite number of (quasi)particles, but hopefully we can hold this number big enough to obtain results that describe the behavior of the system in the thermodynamic limit well enough.

Even numerical summation of the form factor series (6.2.1) for the finite system size will be an impossibly complicated task if all contributions to the sum will be taken into account. Indeed, even in the system with the final dimension of Hilbert space, the total number of states growths exponentially with the grows of the system size. Consider, for instance, an ordinary Heisenberg spin-1/2 chain (0.1.1). The Hilbert space has dimension 2^N for the chain of length N. In this model excitations above the vacuum state are magnons and Bethe parameters are rapidities of the magnons. In the anti-ferromagnetic regime (energy constant in front of Hamiltonian is negative) the number of magnons in the ground state is $M = N/2$ (correspondingly, the cardinality of $\bar{u}$ is $a = N/2$), the number of all possible solutions in this spin sector is C_N^a. Hence, in the ABA approach the number of the Bethe parameters is $a \sim N$, they are defined by the system of nonlinear equation (BAE) and the total number of solutions growth as C_N^a with the growth of system size. Moreover, in the case of Bose or Fermi gases the dimension of the Hilbert space is infinite even for the finite size systems. Hence, calculations of the form factor sum seems to be an extremely complicated task unless some restriction that allows to cut the basis will be found.

However, not all form factors in the series (6.2.1) are equal. Some form factors give significant contributions while the vast majority are neglectable small. Fortunately, in most situations it is easy to choose the form factor types that give the most significant contributions to the series. Moreover, some general procedure that scan the basis in order to choose only some particular contributions to the form factor sum can be found.

For this purpose we consider the classification of the states in terms of solutions of BAE (2.1.30). Details depend on specific model and specific regime of the model. We consider here Gaudin-Yang model (6.1.2) with repulsive interaction ($c > 0$). Then the second level Bethe parameters are always real as in Bose gas (only the strings of length 1 are allowed), while the first level of Bethe parameters form strings of any length [150]. Further, we call solutions that contain only strings of length 1 just *the real solutions*, since all parameters can be made real by proper shift, while solutions, that contain strings of length 2 or more, traditionally are called *the string solutions*.

Since we assume the connection (6.2.2) between part of Bethe parameters, the number of Bethe equations should be reduced to the number of all string centers that are only unknown variables [146]. Gaudin-Yang Bethe equations (2.1.30) can be rewritten [150] as

$$\prod_{j=1}^{b} e\left(\frac{u_\alpha^n - v_j}{nc}\right) = -\prod_{m=1}^{\infty}\prod_{\beta=1}^{n_m} E_{nm}\left(\frac{u_\alpha^n - u_\beta^m}{c}\right), \qquad n = 1, 2, \dots, \qquad \alpha = 1, \dots, n_m, \quad (6.2.4)$$

$$e^{iLv_j} = \prod_{s=1}^{\infty}\prod_{\alpha=1}^{n_s} e\left(\frac{v_j - u_\alpha^n}{nc}\right), \qquad j = 1, 2, \dots, b, \quad (6.2.5)$$

with

$$E_{nm}(x) = \left[\begin{array}{ll} e\left(\frac{x}{|n-m|}\right) e^2\left(\frac{x}{|n-m|+2}\right) e^2\left(\frac{x}{|n-m|+4}\right) \cdots e^2\left(\frac{x}{n+m-2}\right) e\left(\frac{x}{n+m}\right), & n \neq m, \\ e^2\left(\frac{x}{2}\right) e^2\left(\frac{x}{4}\right) \cdots e^2\left(\frac{x}{2n-2}\right) e\left(\frac{x}{2n}\right), \quad n = m, & \end{array} \right. \tag{6.2.6}$$

where $e(x) = \arctan(x)^4$.

There are known from the experience [60, 61, 104, 135, 163] two conjectures that provide good approximation for zero temperature case. They never were proven rigorously but rather established and checked numerically.

Conjecture 6.2.1. *Form factors between ground states and excited state that contain the string longer than one decreases with the growth of the string's length. The form factors decrease with the growth of number of strings with a length more that 1.*

The conjecture (6.2.1) is not itself a strict quantitative criteria, but gives good approximation for the repulsive case (see however discussion at the end of the subsection). The details are discussed below. The method of precision control is discussed in the end of the section.

It is convenient in addition to spectral parameters $\{\bar{u}\}$, $\{\bar{v}\}$ to introduce two sets of the *quantum numbers* $\{I\}$ and $\{J\}$ whose cardinalities are given by $\#\bar{I} = \#\bar{u} = a$, $\#\bar{J} = \#\bar{v} = b$. They are defined as the phases of the l.h.s. of the BAE system. In [150] there were restrictions on the quantum numbers in the Gaudin-Yang model derived

$$\begin{aligned} &J_j = 1, 2, \ldots, \\ &|I_\alpha^n| = \frac{1}{2}\left|b - 1 - \sum t_{nm} M_m\right|, \qquad t_{nm} = 2\mathrm{Min(m,n)} - \delta_{\mathrm{mn}}. \end{aligned} \tag{6.2.7}$$

The logarithm of (6.2.4)-(6.2.5) in case $M_1 = a$, $M_n = 0$ for $n > 1$ (only solutions with no strings) is

$$\begin{aligned} 0 &= \sum_{k\neq j}^{a} e_1(u_j - u_k) + \sum_{k=1}^{b} e_{1/2}(v_k - u_j) + 2\pi i I_j, \\ i v_j L &= \sum_{k=1}^{b} e_{1/2}(v_j - u_k) + 2\pi i J_j. \end{aligned} \tag{6.2.8}$$

The phases $\{I, J\}$ are the (half)integer[5] numbers. Every given set $\{I, J\}$ defines in unique way the set $\{\bar{u}, \bar{v}\}$, every I_j corresponds to some u_j, and J_k to v_k.

The second restriction on all possible solutions is a number of the particle/hole excitations that should be taken into account.

Conjecture 6.2.2. *Form factors decrease with the growth of a particles/holes excitations number, so the most significant contributions to the sum* (6.2.1) *are given by the form factors between the ground state and the excited states with only very few particles/holes. The same is true for the NABA, with a correction, that excitations should be considered on both first and second levels of nesting. Actually, below will consider excitations with no more than two particles/holes on the each level.*

[4] We shift here u_ℓ in BAE by $-ic/2$ for further convenience.

[5] It depends on specific form factor

Remark. Of course, the reduction of series (6.2.1) to the one- or two-particle/hole excitations is enough if the system size is finite, that is obviously used in the case of numerical calculation. In case of system in the thermodynamic limit i.e. $L \to \infty$, $a = \#\bar{u} \to \infty$, $b = \#\bar{v} \to \infty$ but $a/L \to n_a$, $b/L \to n_b$, the families of many particle/hole excitations should be taken into account. See details in [50–54].

The basis scanning. Even between the states with one- or two-particles/holes there are more and less important. In order to choose the most significant, the "basis scanning" algorithm can be used. Consider one particle/hole excitations on the first level of nesting, i.e. replace one of numbers $\{I\}$, say I_k, by I'_k that lies outside the Fermi interval. We memorize among all such excited states these that give the largest contributions and denote them as $\{I'_k, J\}$'s ($k = 1, \ldots, a$ denotes the quantum number that was moved and I'_k denotes the value of quantum number after it was moved outside the Fermi zone). The number of the important (in sense that they give significant contribution to the form factor series) excited states usually is much smaller than the number of all possible one particle/hole excitations at the first level of nesting. The second level of nesting is still unperturbed yet. Now it is enough to add one particle/hole excitation on the second level only to the chosen states $\{I'_k, J\}$. These states give most significant contributions among all possible states with one particle/hole excitations on the first and the second levels simultaneously. Again, we choose among these excitations the most significant contributions. Denote them as $\{I'_j, J'_k\}$ ($k = 1, \ldots, a$, $j = 1, \ldots, b$). The next set of excitations can be build adding one particle/hole excitation on the first or the second level to the states from $\{I'_j, J'_k\}$. Such scanning procedure allows us significantly reduce the number of states that are taken into account in all the sum in (0.1.7) without significant loose of precision.

String contributions. Above we consider the system with an arbitrary cardinality a. This means, that a spins is flipped. It is especially interesting consider the case $a = b/2$. In case of zero magnetic field and zero temperature this case corresponds to zero magnetization of system and is global ground states (for every fixed a and b there are their own ground states, but case $a = b/2$ is global, i.e. the lowest energy state among all choice a, b). As above, we can try restrict ourselves by the BAE solutions with no strings. From the system (6.2.7) it is easy to show, that in this case I_{max} (I_{min}) coincides with right (left) Fermi boundary on this level of nesting. This means that the "real" one particle/hole excitations are not possible on this level. However, the string solutions still exist and should be included into consideration. In this case these excitations already can not be neglected. Among the strings there exists their own gradation of "importance". Thus, the states with a small number of strings of short length always give more contribution, than the states with the long and/or multiple strings.

Another situation is a case of the negative coupling constant c. In this case the ground state contains strings on the second level of nesting and the solutions with strings on the both levels are expected to give most significant contributions to the form factor sum.

Precision control. Since instead of summation over the full Hilbert space only the very limited part of the basis is used without any rigorous proof of such restriction in the general case, the method of precision control of the scanning procedure is required. It can be provided by the *sum rules*, i.e. certain relations on the form factor sum that can be established from the general quantum mechanical conservation laws. The idea of using such formulae as a control method follows back to [159–162]. The particular sum rules depend on the model and the particular form factor, however the main idea is very simple. At least in some cases the sum of all (or part of) the form factors are known from some conservation laws. If all form factor in the series is non-negative and sum is (near) satisfied by some finite number of contributions, it is clear that the rest do not play significant role.

6.2.5 Correlator $\langle\psi_\uparrow^\dagger(x,t)\psi_\uparrow(0,0)\rangle$

Consider the correlation function of two field operators $\langle\psi_\uparrow^\dagger(x,t)\psi_\uparrow(0,0)\rangle$. We call corresponding DSF S_{23} because field operator $\psi_\uparrow$ coincides (up to the factor c) with the local operator obtained from the monodromy matrix element T_{23}

$$S_{23}(q,\omega) = \int_{-\infty}^{\infty} dte^{i\omega t}\int \frac{dx}{(2\pi)}e^{-ixq}\left\langle\psi_\uparrow^\dagger(x,t)\psi_\uparrow(0,0)\right\rangle$$
$$= 2\pi Lc\sum_{\{\bar v^B,\bar u^B\}}\left|\mathfrak{F}^{(2,3)}\begin{pmatrix}\bar u^C & \bar u^B\\ \bar v^C & \bar v^B\end{pmatrix}^{a,a}_{b+1,b}\right|^2\delta_{q-q(\bar v^B,\bar u^B)}\delta\left(\omega(\bar u^B,\bar v^B)-\omega\right). \quad (6.2.9)$$

Here $\{\bar u^B,\bar v^B\}$ is Bethe parameters that corresponds to excited states and we perform the summation over all (important) states. The set $\{\bar u^C,\bar v^C\}$ corresponds to the vacuum state. On Fig.6.1-6.2 S_{23} is shown as function of q,ω for the different values of the coupling constant c and for the different numbers of flipped spins.

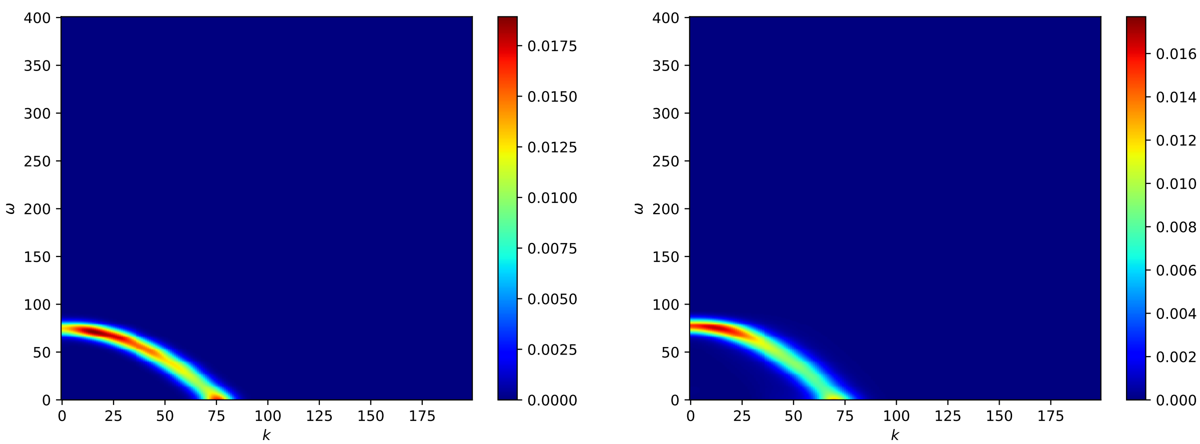

Figure 6.1: The diagram of $S_{23}(q,\omega)$ in plane $\omega-q$ at $c=0.32$ (left) and $c=7.32$ (right) with 2 spins down. Density of particles is 0.4. In contrary to the spinless Bose system all density is concentrated near the lower threshold even in the case of weak interaction. In fact the density with higher energy is not equal zero, but is so low that is is not distinguishable on the diagram without logarithmic scaling of colour.

Here the system of 40 fermions is considered. The system size L is taken 100[6]. The energy and momentum are measured correspondingly in $0.01L^{-2}$ and $0.01L^{-1}$ units.

[6]The statement about system size itself has no sense, since all system parameters can be renormilised in such a way that L will appears only in combination Lc (all Bethe parameters will be also rescaled). Hereby, the length can be measured in any convenient in particular case units (say nanometers), while in order to restore the proper physical units it is enough to rescale c, that itself leads to rescaling of all Bethe parameters and, correspondingly, to rescaling of ω and q.

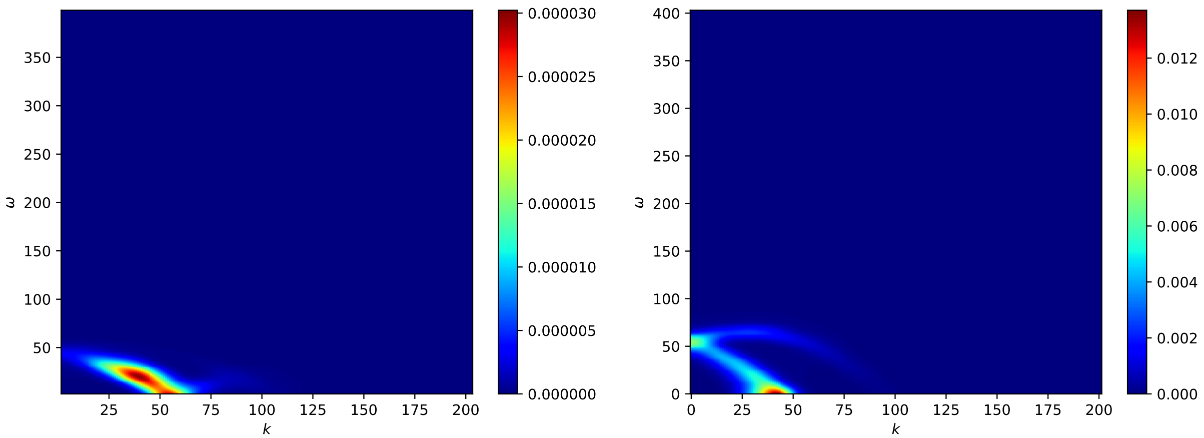

Figure 6.2: The diagram of $S_{23}(q,\omega)$ in plane $\omega - q$ at $c = 0.32$ (left) and $c = 7.32$ (right) with 10 spins down. Density of particles is 0.4. In fact the density with higher energy is not equal zero, but is so low that is is not distinguishable on the diagram without the logarithmic scaling of colour.

Sum rule

For the correlator of $\langle \psi^\dagger(x,t)\psi(0,0)\rangle$ the rule is given by

$$\frac{1}{L}\sum_q \int \frac{d\omega}{(2\pi)} S_{23}(q,\omega) = \frac{n - n_\downarrow}{L} = \frac{b-a}{L}, \tag{6.2.10}$$

i. e. we just reproduce the density of particles with the spin up.

6.2.6 Correlator $\langle \psi^\dagger_\uparrow(x,t)\psi_\downarrow(x,t)\psi^\dagger_\downarrow(0,0)\psi_\uparrow(0,0)\rangle$

Correlator of fields with the different spin projections can be expressed via the correlator of the monodromy matrix entries T_{12} and T_{21}. We call corresponding DSF $S_{12}(q,\omega)$

$$\begin{aligned} S_{12}(q,\omega) &= \int_{-\infty}^{\infty} dt e^{i\omega t} \int \frac{dx}{(2\pi)} e^{-ixq} \left\langle \psi^\dagger_\uparrow(x,t)\psi_\downarrow(x,t)\psi^\dagger_\downarrow(0,0)\psi_\uparrow(0,0)\right\rangle \\ &= 2\pi L \sum_{\{\bar v^B, \bar u^B\}} |\mathcal{P}|^2 \left| \mathfrak{F}^{(1,2)} \begin{pmatrix} \bar u^C & \bar u^B \\ \bar v^C & \bar v^B \end{pmatrix}^{a+1,a}_{b,b} \right|^2 \delta_{q - q(\bar v^B, \bar u^B)} \delta\left(\omega(\bar u^B, \bar v^B) - \omega\right). \end{aligned} \tag{6.2.11}$$

On Fig.6.3-6.4 S_{12} is shown as a function of q, ω for the different values of the coupling constant c and for the different numbers of the flipped spins. The system of length 160 with 74 (Fig.6.3) and 55 (Fig.6.4) fermions is considered.

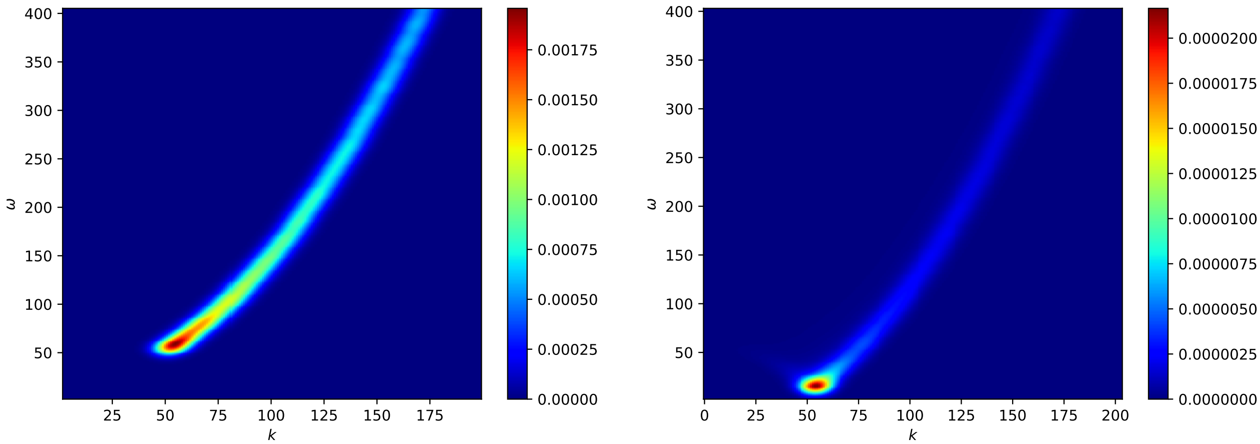

Figure 6.3: The diagram of $S_{12}(q,\omega)$ in plane $\omega - q$ at $c = 0.52$ (left) and $c = 7.52$ (right) with 2 spins down. Density of particles is 0.46.

Sum rule

The simplest situation of $x = 0$ static correlation function can be considered. In this way the inverse Fourier transform is a sum

$$\frac{1}{L}\sum_q \int \frac{d\omega}{(2\pi)} S_{12}(q,\omega) = \langle \psi^\dagger_\uparrow(0,0)\psi_\downarrow(0,0)\psi^\dagger_\downarrow(0,0)\psi_\uparrow(0,0)\rangle. \tag{6.2.12}$$

The later can be rewritten, using the commutation relations (6.1.3), via correlators $\langle n_\uparrow(0,0)n_\downarrow(0,0)\rangle$ and $n_\uparrow$. The DSF for this correlator we call $S_{\uparrow\downarrow}(k,\omega)$ and the correlation function can be calculated using the inverse Fourier transform that coincides with the following sum rule in case $x = 0, t = 0$

$$\frac{1}{L}\int \frac{d\omega}{(2\pi)} S_{\uparrow\downarrow}(k,\omega)\omega = 0. \tag{6.2.13}$$

This sum rule, however, can not be considered as the good instrument, since at some energy region the correlator becomes negative, and it can be used as a good verification method only if all the contributions with a given momenta is taken into account, that is not practically possible.

Instead the following method can be used. According to the Hellmann-Feynman theorem for arbitrary continuous parameter λ and the eigenstates $\psi = \psi_\lambda$

$$\langle \psi_\lambda | \frac{\partial H}{\partial c} | \psi_\lambda \rangle = \frac{\partial E}{\partial c} = 2 g_{\uparrow\downarrow}(0), \tag{6.2.14}$$

where $g_{\uparrow\downarrow}(x) = \langle n_\uparrow(x) n_\downarrow(x)\rangle$. The ground state energy was derived in [127]

$$\begin{aligned} E &= \frac{n^3\pi^2}{3}\left[1 - \frac{4\log 2}{\gamma} + \frac{12\log^2 2}{\gamma^2} - \frac{32\log^3 2}{\gamma^3} + \frac{8\pi^2\zeta(3)}{5\gamma^3}\right] + O(c^{-4}), \qquad P = 0, \\ E &= \frac{n^3\pi^2}{3}\left[1 - \frac{8n_\downarrow}{c} + \frac{48 n_\downarrow^2}{c^2} - \frac{1}{c^3}\left(256 n_\downarrow^3 - \frac{32}{5}\pi^2 n^2 n_\downarrow\right)\right] + O(c^{-4}), \qquad P \gtrsim 0.5, \end{aligned} \tag{6.2.15}$$

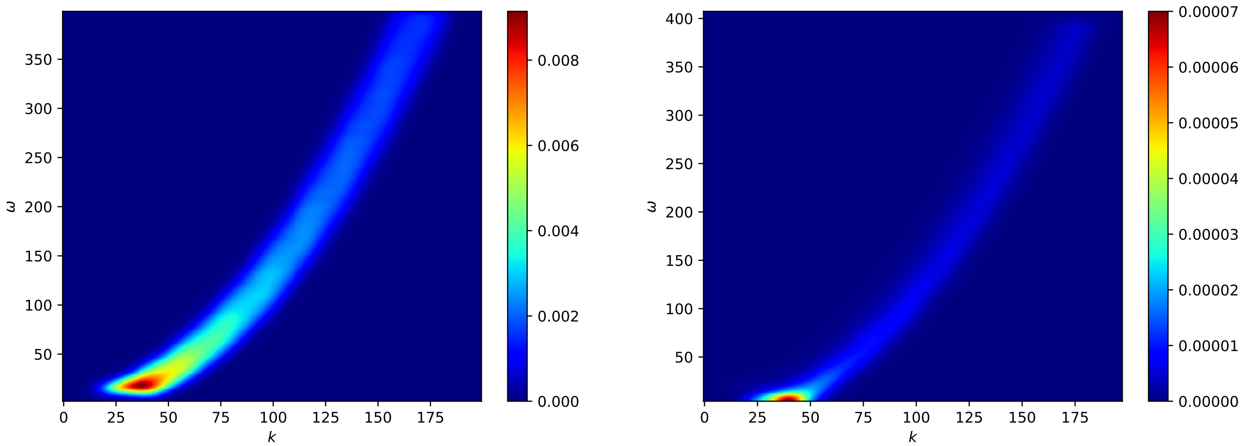

Figure 6.4: The diagram of $S_{12}(q,\omega)$ in plane $\omega - q$ at $c = 0.52$ (left) and $c = 7.52$ (right) with 12 spins down. Density of particles is 0.34.

where

$$P = \frac{n_\uparrow - n_\downarrow}{n}, \tag{6.2.16}$$

is a polarization and $\gamma = c/n$.

6.2.7 Correlator $\langle n_\uparrow(x,t) n_\uparrow(0,0)\rangle$

Correlator of the fields with the different spin projection can be expressed via the correlator of the monodromy matrix entries T_{11}. We call corresponding DSF $S_{11}(q,\omega)$

$$\begin{aligned} S_{11}(q,\omega) &= \int_{-\infty}^{\infty} dt e^{i\omega t} \int \frac{dx}{(2\pi)} e^{-ixq} \langle n_\uparrow(x,t) n_\uparrow(0,0)\rangle \\ &= 2\pi L \sum_{\{\bar v^B, \bar u^B\}} |\mathcal{P}|^2 \left| \mathfrak{F}^{(1,1)} \begin{pmatrix} \bar u^C & \bar u^B \\ \bar v^C & \bar v^B \end{pmatrix}_{b,b}^{a,a} \right|^2 \delta_{q - q(\bar v^B, \bar u^B)} \delta\left(\omega(\bar u^B, \bar v^B) - \omega\right). \end{aligned} \tag{6.2.17}$$

S_{11} is shown as function of q, ω for the different values of the coupling constant c with the 18 flipped spins. The system size is taken 80 and 74 fermions in the system.

Correlator $\langle n_\uparrow(x,t) n_\uparrow(0,0)\rangle$ was measured in the experiment with a gas of atoms of ^{6}Li [165] where gas was confined in a crossed dipole trap. The coupling constant is proportional to the scattering length a that was tuned using the Feshbach resonance. Strong and intermediate coupling regimes were considered $1/(k_F a) = 1$ and $1/(k_F a) = 0$. The DSF was measured at $k = 4.5 k_F$ and for a wide range of energies. The DSF of S_{11} and DSF $S_{\uparrow\downarrow}$ (Fourier image of correlator $\langle n_\uparrow(x,t) n_\downarrow(0,0)\rangle$), measured in [165], are shown on Fig.6.6. In fact, this is the vertical cut of DSF image at fixed momenta. The nontrivial result is a presence of few maximums of the DSF at different energies.

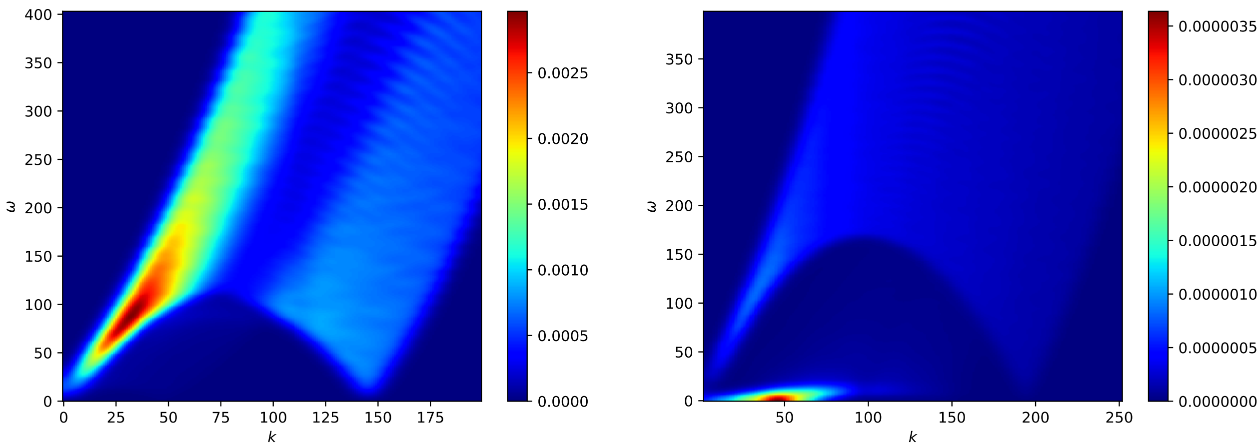

Figure 6.5: The diagram of $S_{11}(q,\omega)$ in plane $\omega - q$ at $c = 0.42$ (left) and $c = 20.52$ (right) with 18 spins down. Density of particles is 0.46. Two independent "profile" of excitations can be seen at strong interaction regime. In fact, only the lower threshold of the lower "profile" can be seen without application of the logarithmic scale. For weak interaction only one "profile" exist at all.

The direct comparison with the experiment is not possible, since measurement was done at the relatively high momentum and at the relatively high energy region, not shown on (6.2.17). At such momentum clearly the two- and tree- particle/hole contributions should already contribute. However, it can be expected that the general pattern should be similar.

From the Fig.6.5 and BAE (6.2.4)-(6.2.5) it is clear, that the shape of the DSF should be "a superposition" of the two type of excitations: the excitations on the second level of nesting, whose spectrum and DSF resemble those for the Bose gas [62, 104, 135] and the excitations on the first level of nesting, whose spectrum of the XXX spin chain [60, 61] spectrum. In particular, with the fixed second level of variables, the first level BAE are just BAE of the inhomogeneous XXX spin chain (see (1.1.27)) and the homogeneous limit of the DSF for spin-spin correlator is shown on Fig.1 in the introduction. Vice versa, if we consider the case without the first level of nested variables (all spin up, so $\#\bar{u} = \emptyset$) we deal with 1D free fermions, whose behaviour is similar to 1D bosons, since even for Bose gas Pauli exclusion principle is present in 1D and there is not too much difference between Bose and Fermi gases expected for the density-density correlator.

Creating the one particle excitation on the second level we "move right" on $\omega - q$ plane since the momentum of such form factor increases. When we increase the energy of this particle we obviously move along the parabolae on $\omega - q$ plane. The DSF profile here just resemble the case of 1D Bose gas. Creation of excitations on the first level leads to duplication of such profiles in the $\omega - q$ plane, since for fixed Bethe parameters on the second level of nesting we have many possibilities of excitations distributions on the first level. This is obviously clear from 6.5 on the low energy/momenta region, where we can see two shifted Bose gas-like profiles. The cuts of such profile at few fixed momenta are shown on Fig.6.7 and resemble two peaks structure measured in [165] and shown on Fig.6.6.

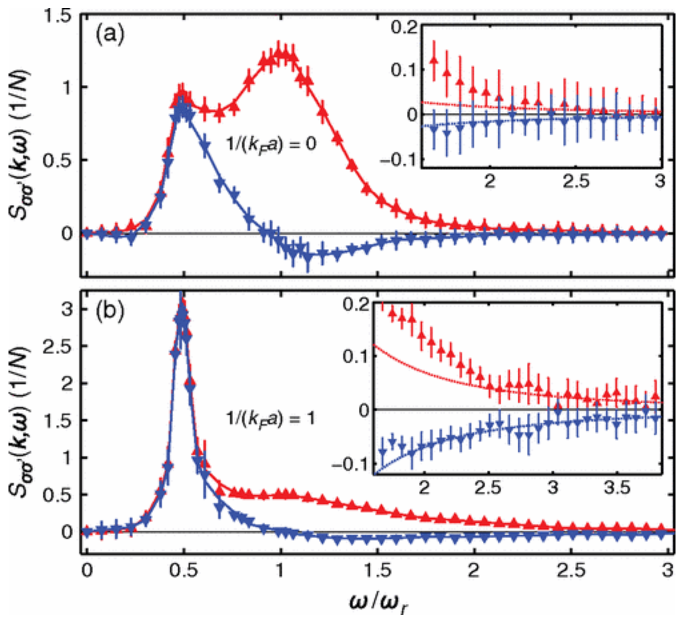

Figure 6.6: Spin-parallel and spin-antiparallel components of the dynamic structure factor of a strongly interacting Fermi gas measured at (a) $1/(k_F a) = 0$, and, (b) at $1/(k_F a) = 1.0$.Red upright triangles are the spin-parallel structure factor $S_{11}(k,\omega)$ and blue inverted triangles are the spin anti-parallel response $S_{\uparrow\downarrow}(k,\omega)$. Solid lines are a guide to the eye. Insets show zoomed in plots of the high frequency region where $S_{\uparrow\downarrow}(k,\omega)$. $\omega_r = \hbar k_F^2/2m$. [165].

Similar picture will appears also with the two-, three- particles/holes excitations and at higher energies/momenta and was measured in [165] for the fixed momenta and clearly seen on Fig.6.6 at $1/(k_F a) = 0$.

For the weak coupling case the second peak is not presented, however, and this coincides with theoretical computation (see left part of 6.5). This shows that the form factors at weak coupling is significantly differs from zero only at one fixed configuration of the first level (spin flips) excitations.

6.3 Conclusion

The first main result of this chapter is a description of the generalisation of the algorithm, developed in [60, 61, 104, 135, 159–162] for the algebra symmetry case $\mathfrak{gl}(2)$. The details of the algorithm slightly varies for different models and different correlators, but the general schema is very similar and loosely resemble the case of the algorithm for the algebra symmetry $\mathfrak{gl}(2)$ based models. The only complexification is the appearance of the second level of nesting. It provides us with several problems. The first one is the computation of the BAE solutions and form factors. Thus, the complexity of the numerical algorithm growths as $\approx a^{6.62\dots}$ for a (quasi)particles number a. Appearance of the second level of nesting leads to growth $(a+b)^{6.62\dots}$ where b is a number of (quasi)particles on the second level of nesting. The second problem is a growth of a number of contributions that should be taken into account. The number of the one-particle/hole contributions is $\sim C_L^a$. Appearance of the second level gives the additional factor $\sim C_a^b$ to the number. Even taken into account, that the amount of the *numerically significant* form factors

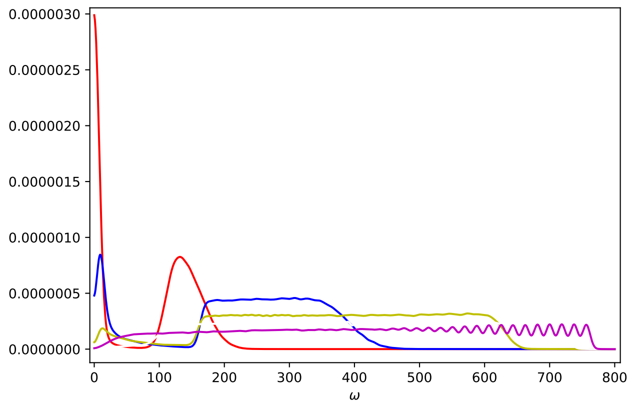

Figure 6.7: S_{11} at $c = 20.52$ at momenta $k = 40, 80, 120, 160, 200$.

is, let say, approximately 1-2% of the overall number of the one-particle/hole form factors (it strongly depends on a particular model and particular form factors), the total number of contributions that should be calculated become giant in comparison with the algebra symmetry $\mathfrak{gl}(2)$ case even for the relatively small systems. This problem strongly restricts the precision of the numerical algorithm, especially in the situations when two-, three-, etc. particle/hole form factors should be taken into account. The solution of such problem is seen in an analytical study of the form factor behaviour. Thus, if it will be possible to understand what types of excitations give the most significant contributions to the form factor series for a particular set of model's parameters and particular form factors, it can be possible significantly increase the productivity of the basis scanning algorithm and choose the very limited number of form factor from the very beginning. This problem is especially important in context of generalisation of the algorithm to the higher algebra rank symmetry case. Thus, if the determinant representation for the form factors of ultralocal operators will be found for algebra symmetry $\mathfrak{gl}(N)$ $N > 3$ (or graded algebras) the method in principle can be applied to these cases too. The same is true for B-, C- and D- algebra series. Moreover, even despite the fact that in general the determinant representations for the form factors of ultralocal operators for the algebra symmetry $\mathfrak{gl}(2|2)$, $\mathfrak{gl}(4)$ related model are unknown, the particular cases with one (quasi)particles on the third level or the second level of nesting can be studied now. Such form factors, for instance, allow to describe the Green function of one impurity particle moving through the 1D volume.

The developed algorithm was applied to the model of spin-1/2 integrable Fermi gas. Few correlation functions were probed. The most interesting case is seems to be the (6.2.17) since it can be compared with the measured in the recent experiment on (quasi) 1D Fermi gas. Another correlation functions are also expected to be measured and it will be interesting to compare such experimental results with our theoretical prediction.

Finally, it is also possible to apply this numerical method to the supersymmetric t-J model and other models related to algebra symmetry $\mathfrak{gl}(2|1)$ or $\mathfrak{gl}(1|2)$.

Chapter 7

Conclusion

Despite an extensive study and significant successes of the ABA and other methods, the description of quantum integrable models are far from being complete. Even in the theory of the simplest one-component models a lot of white spots remains. Moreover, especially interesting multicomponent models are much less studied and a lot of their properties are unknown or are conjectured. The study of these models requires improvement of already existing tools and methods. NABA technique is one of very perspective approaches in this direction, but is quite complicated to use and often requires some improvement and brushing in order to be applicable. The goal of this work was the development of approaches that will allow to study at least two-component models within NABA. Below we remind the result of this thesis and further perspectives.

The main results of this thesis are compact representations for scalar products and form factors of ultralocal operators in the 1D integrable models based on the algebra symmetry $\mathfrak{gl}(2|1)$ or $\mathfrak{gl}(1|2)$. Among the models associated with these symmetries are 1D spin-1/2 Fermi gas with ultralocal interaction, supersymmetric t-J model, lattice gases. The spectra, phase diagrams and some thermodynamic properties of such models have been already studied earlier. Nonperturbative computation of the transport coefficients, non-equilibrium behaviour, dynamical properties of these models were not done yet, however. Calculation of the compact representations of the form factors is the first and necessary step to progress in the study of these models.

Important continuation in this direction could be a computation of the ultralocal form factors in the case of models associated with the algebra symmetry $\mathfrak{gl}(m|n)$ with $n, m \geq 2$ or $\mathfrak{gl}(N)$, $N \geq 4$. Among models with these algebra symmetries are Fermi/Bose gases with higher spins and/or multiple component mixtures and different lattice gases. Nonperturbative study of these gases is a highly nontrivial problem in any approach, and the Bethe ansatz is not an exception. While it is not a big deal to write formal expressions for the form factors in these cases, computation of the compact formulae, suitable for further application, is an extremely complicated problem. Moreover, it is not even clear what generalisation is expected in this direction. Thus, lines and columns of matrix under determinant (4.4.15)-(4.4.16) (or (4.2.2)-(4.2.3)) are numerated by sets $\{\bar{u}^C, \bar{v}^B\}$ and $\{\bar{u}^B, \bar{v}^C\}$ correspondingly. It is hard to guess how should be numerated the lines and the columns in the $\mathfrak{gl}(4)$ (or $\mathfrak{gl}(2|2)$) algebra symmetry case. The first way in this direction can be some representation for the scalar product and form factors in the case of algebra symmetry $\mathfrak{gl}(3)$ (or $\mathfrak{gl}(2|1)$) where lines and columns are numerated in a more natural way by $\{\bar{u}^B, \bar{v}^B\}$ and $\{\bar{u}^C, \bar{v}^C\}$. Also, an interesting development in this direction is computation of such scalar products and form factors in the model associated with trigonometric R-matrix.

One more important direction is the calculation of the scalar product of the on-shell and off-shell Bethe vectors that is important for description of correlation functions in super Yang-Mills theories. Calculation of these scalar products is much more complicated than semi-on-shell case derived in these thesis and is known only for an algebra symmetry $\mathfrak{gl}(2)$.

According to the Kubo formula transport coefficients can be expressed via the correlators of corresponding currents. The last can be calculated now using compact representations of form factors and scalar product derived in the thesis. Correlation functions computation itself is a very nontrivial problem that is not completely solved even in the simpler case of algebra symmetry $\mathfrak{gl}(2)$. Few different approaches exist here. The first one is the multiple integral representation formulae derived in $\mathfrak{gl}(2)$ algebra symmetry case in [48, 49, 57, 100, 101, 103, 110]. Derivation of these formulae requires on-shell–off-shell scalar product (see discussion in introduction to Chapter 4), that is not known, or some manipulations with multiple contour integrals as an intermediate step (see, for instance, [57]) that is a complicated task in higher rank algebra symmetry case[1]. Moreover, integral representations itself are not a complete solution of the problem, since these multiple integrals are not easy to factorise even in the case of algebra symmetry $\mathfrak{gl}(2)$. The important results in this direction were obtain in [156–158] but the generalisation to the higher rank algebra case is far from being complete.

The second approach is a form factor series summation. This approach proved to be fruitful in correlation function computations in case of the algebra symmetry $\mathfrak{gl}(2)$ [50–55, 58, 59]. It can be applied for many model in presence of dynamics, magnetic fields, nonperiodic boundaries, etc. The application of the quantum transfer matrix (QTM) was proven to be quite useful in the case of non-zero temperature. The analysis of thermodynamic behaviour of form factor series is also complicated task and the details strongly depends on the model, set of model parameters, temperature, etc. The generalisation of form factor series analysis to the higher rank algebra symmetry case could give significant progress in the understanding of the multicomponent system. Some advance in this direction was done in [155] for zero temperature situation. Among the obstacles in finite temperature case is a problem of QTM equations analysis in the higher rank algebra symmetry. The further progress in correlation function study requires the derivation of the new form of equations for Bethe parameters in case of the non-zero temperature.

On the other hand, numerical approach to the form factor series summation was developed in the series of works [60–62, 104, 135, 159–163]. It allows to derive results for the dynamical correlation function using the easy and not resource consuming algorithm. The generalisation of this algorithm is done in the last chapter of thesis. This is the second result of these thesis. The resources consumption of the method is much higher, since fast growth of the basis size and correspondingly contributions to the form factor sum. However, the method is still useful for the zero or ultra low temperatures. The main disadvantage of this algorithm remains the requirement of explicit solution of BAE. While it is easy to do for the ordinary Bethe equations using the string hypothesis it is not the case for finite temperature, when QTM Bethe equations are used. The string hypothesis is no more applicable in this case and the solution of the Bethe equation is a very complicated task.

Finally, the huge progress would be a development of similar approaches to the model associated with algebra symmetries $\mathfrak{so}(2n)$, $\mathfrak{so}(2n+1)$, since a lot of integrable systems like the

[1] In the particular case of fundamental systems, multiple integral formulae in the thermodynamic limit at zero temperature can be derived using also the vertex operator approach avoiding ABA. We are not going to discuss it here since it does not allow to describe Fermi gases that is our object of interest.

Hubbard model or model of higher spin gases are related to such algebras.

Chapter 8

Appendix 1

In this appendix some formulae, used in chapters 2-4, are proven. Appendix is based on paper [87] published by thesis author in collaboration.

8.1 Identities for rational functions

Lemma 8.1.1. *Let $\bar{w}$, $\bar{u}$ and $\bar{v}$ be sets of complex variables with $\#\bar{u} = m_1$, $\#\bar{v} = m_2$, and $\#\bar{w} = m_1 + m_2$, where m_1 and m_2 are fixed arbitrary integers. Then*

$$\sum g(\bar{w}_{\rm I}, \bar{u})g(\bar{w}_{\rm II}, \bar{v})g(\bar{w}_{\rm II}, \bar{w}_{\rm I}) = \frac{g(\bar{w}, \bar{u})g(\bar{w}, \bar{v})}{g(\bar{u}, \bar{v})}. \tag{8.1.1}$$

The sum is taken with respect to all partitions of the set $\bar{w}$ into subsets $\bar{w}_{\rm I}$ and $\bar{w}_{\rm II}$ with $\#\bar{w}_{\rm I} = m_1$ and $\#\bar{w}_{\rm II} = m_2$.

The proof of this lemma is given in [109]. Let us show how lemma 9.3.1 works. In equation (2.2.11) we have a sum

$$I = \sum_{\bar{\xi}_0 \Rightarrow \{\bar{\xi}_{\rm I}, \bar{\xi}_{\rm i}\}} g(\bar{\xi}_{\rm i}, \bar{\xi}_{\rm I})g(z_n, \bar{\xi}_{\rm i})h(\bar{\xi}_{\rm i}, \bar{z}_n), \tag{8.1.2}$$

where $\#\bar{\xi}_{\rm i} = 1$ and $\#\bar{\xi}_{\rm I} = n - 1$. First, we reduce this sum to the form (9.3.16) using $h(u, v) = 1/g(u, v - c)$. We have

$$I = -h(\bar{\xi}_0, \bar{z}_n) \sum_{\bar{\xi}_0 \Rightarrow \{\bar{\xi}_{\rm I}, \bar{\xi}_{\rm i}\}} g(\bar{\xi}_{\rm i}, \bar{\xi}_{\rm I}) \frac{g(\bar{\xi}_{\rm i}, z_n)}{h(\bar{\xi}_{\rm I}, \bar{z}_n)} = -h(\bar{\xi}_0, \bar{z}_n) \sum_{\bar{\xi}_0 \Rightarrow \{\bar{\xi}_{\rm I}, \bar{\xi}_{\rm i}\}} g(\bar{\xi}_{\rm i}, \bar{\xi}_{\rm I})g(\bar{\xi}_{\rm i}, z_n)g(\bar{\xi}_{\rm I}, \bar{z}_n - c). \tag{8.1.3}$$

Now we can directly apply (9.3.16), and we arrive at

$$I = -h(\bar{\xi}_0, \bar{z}_n) \frac{g(\bar{\xi}_0, z_n)g(\bar{\xi}_0, \bar{z}_n - c)}{g(\bar{z}_n - c, z_n)} = g(z_n, \bar{\xi}_0)h(z_n, \bar{z}_n). \tag{8.1.4}$$

8.1.1 Izergin determinant properties

Isergin determinant $K_n(\bar{x}|\bar{y})$ is symmetric function on $x_1, \dots, x_n$ and symmetric function on $y_1, \dots, y_n$. It has asymptotic $1/x_n$ (correspondingly $1/y_n$) at $x_n \to \infty$ (correspondingly $y_n \to$

∞) and other variables are fixed. Izergin determinant has simple pole at $x_j = y_k$. It follows directly from the definition (1.1.44) that $K_n(\bar{x}|\bar{y})$ possesses following properties:

$$K_{n+m}(\{\bar{x}, \bar{z} - c\}|\{\bar{y}, \bar{z}\}) = K_{n+m}(\{\bar{x}, \bar{z}\}|\{\bar{y}, \bar{z} + c\}) = (-1)^m K_n(\bar{x}|\bar{y}), \qquad \#\bar{z} = m, \tag{8.1.5}$$

and

$$K_n(\bar{x} - c|\bar{y}) = K_n(\bar{x}|\bar{y} + c) = (-1)^n \frac{K_n(\bar{y}|\bar{x})}{f(\bar{y}, \bar{x})}. \tag{8.1.6}$$

Lemma 8.1.2. *Let $\bar{w}$, $\bar{u}$ and $\bar{v}$ be sets of complex variables with $\#\bar{u} = m_1$, $\#\bar{v} = m_2$, and $\#\bar{w} = m_1 + m_2$. Then*

$$\sum K_{m_1}(\bar{w}_{\rm I}|\bar{u}) K_{m_2}(\bar{v}|\bar{w}_{\rm II}) f(\bar{w}_{\rm II}, \bar{w}_{\rm I}) = (-1)^{m_1} f(\bar{w}, \bar{u}) K_{m_1+m_2}(\{\bar{u} - c, \bar{v}\}|\bar{w}). \tag{8.1.7}$$

The sum is taken with respect to all partitions of the set $\bar{w}$ into subsets $\bar{w}_{\rm I}$ and $\bar{w}_{\rm II}$ with $\#\bar{w}_{\rm I} = m_1$ and $\#\bar{w}_{\rm II} = m_2$.

The proof of this Lemma is given in [66].

Chapter 9

Appendix 2

In this appendix some formulae, used in chapters 2-4, are proven. Appendix is based on paper [88] published by thesis author in collaboration.

9.1 Computation of integrals

Conjecture 9.1.1. *Let $\bar{w} = \{w_1, \dots, w_N\}$ be a set of complex numbers. Let $\mathcal{F}(\bar{z})$ be a function of n variables $z_1, \dots, z_n$ ($n \le N$). Assume that $\mathcal{F}(\bar{z})$ is a symmetric function of $\bar{z}$ and that it is holomorphic with respect to each z_j within a domain containing the points $\bar{w}$. Define*

$$\langle \mathcal{F} \rangle = \frac{1}{(2\pi i c)^n n!} \oint_{\bar{w}} \frac{g(\bar{z}, \bar{w})\, d\bar{z}}{\Delta_n(\bar{z}) \Delta'_n(\bar{z})} \mathcal{F}(\bar{z}). \tag{9.1.1}$$

Here $d\bar{z} = dz_1, \dots, dz_n$ and the integration contour for every z_j surrounds the points $\bar{w}$ in the anticlockwise direction. We assume that there is no other singularities of the integrand within the integration contours. Then

$$\langle \mathcal{F} \rangle = \sum g(\bar{w}_{\rm I}, \bar{w}_{\rm II}) \mathcal{F}(\bar{w}_{\rm I}), \tag{9.1.2}$$

where the sum is taken over partitions $\bar{w} \Rightarrow \{\bar{w}_{\rm I}, \bar{w}_{\rm II}\}$ such that $\#\bar{w}_{\rm I} = n$.

Proof. We use induction over n. For $n = 1$ the statement of the proposition is obvious. Suppose that it is valid for some $n-1$. Then splitting $\bar{z} = \{z_n, \bar{z}_n\}$ we obtain

$$\langle \mathcal{F} \rangle = \frac{1}{(2\pi i c)^n n!} \oint_{\bar{w}} \frac{g(z_n, \bar{w}) g(\bar{z}_n, \bar{w}) \mathcal{F}(\{\bar{z}_n, z_n\})\, d\bar{z}_n\, dz_n}{\Delta_{n-1}(\bar{z}_n) \Delta'_{n-1}(\bar{z}_n) g(z_n, \bar{z}_n) g(\bar{z}_n, z_n)}$$
$$= \sum \frac{g(\bar{w}_{\rm I}, \bar{w}_{\rm II})}{2\pi i c n} \oint_{\bar{w}} \frac{g(z_n, \bar{w}_{\rm II}) \mathcal{F}(\{\bar{w}_{\rm I}, z_n\})\, dz_n}{g(\bar{w}_{\rm I}, z_n)}, \tag{9.1.3}$$

where the sum is taken over partitions $\bar{w} \Rightarrow \{\bar{w}_{\rm I}, \bar{w}_{\rm II}\}$ such that $\#\bar{w}_{\rm I} = n-1$. Performing the integration over z_n we find

$$\langle \mathcal{F} \rangle = \frac{1}{n} \sum \frac{g(\bar{w}_{\rm I}, \bar{w}_{\rm II}) g(\bar{w}_{\rm i}, \bar{w}_{\rm ii})}{g(\bar{w}_{\rm I}, \bar{w}_{\rm i})} \mathcal{F}(\{\bar{w}_{\rm I}, \bar{w}_{\rm i}\}), \tag{9.1.4}$$

where we obtain an additional partition $\bar{w}_{\rm II} \Rightarrow \{\bar{w}_{\rm i}, \bar{w}_{\rm ii}\}$ with $\#\bar{w}_{\rm i} = 1$. Substituting in (9.1.4) $\bar{w}_{\rm II} = \{\bar{w}_{\rm i}, \bar{w}_{\rm ii}\}$ and setting there $\{\bar{w}_{\rm i}, \bar{w}_{\rm I}\} = \bar{w}_0$ we arrive at

$$\langle \mathcal{F} \rangle = \frac{1}{n} \sum g(\bar{w}_0, \bar{w}_{\rm ii}) \mathcal{F}(\bar{w}_0). \tag{9.1.5}$$

Now the sum over partitions is organized as follows. First we have the partitions $\bar{w} \Rightarrow \{\bar{w}_0, \bar{w}_{\rm ii}\}$ with $\#\bar{w}_0 = n$, and then we have the additional partition $\bar{w}_0 \Rightarrow \{\bar{w}_{\rm i}, \bar{w}_{\rm I}\}$ with $\#\bar{w}_{\rm i} = 1$. Obviously, the sum over the later partition gives n, and we obtain the statement of the proposition. □

Note that if $n > N$ in (9.1.1), then $\langle \mathcal{F} \rangle = 0$.

9.2 Summation formulas

Lemma 9.2.1. *Let $\bar{\xi}$, $\bar{\alpha}$ and $\bar{\beta}$ be sets of complex variables with $\#\alpha = n$, $\#\beta = m$, and $\#\xi = n + m$. Then*

$$\sum K_n(\bar{\xi}_{\rm I}|\bar{\alpha}) K_m(\bar{\beta}|\bar{\xi}_{\rm II}) f(\bar{\xi}_{\rm II}, \bar{\xi}_{\rm I}) = (-1)^n f(\bar{\xi}, \bar{\alpha}) K_{n+m}(\bar{\alpha} - c, \bar{\beta}|\bar{\xi}). \tag{9.2.1}$$

The sum is taken with respect to all partitions of the set $\bar{\xi}$ into subsets $\bar{\xi}_{\rm I}$ and $\bar{\xi}_{\rm II}$ with $\#\bar{\xi}_{\rm I} = n$ and $\#\bar{\xi}_{\rm II} = m$.

The proof of this lemma can be found in [65].

Lemma 9.2.2. *For any set of functions $\phi_k(\beta)$, $k = 1, \dots, n + m$, let*

$$\Phi_{n+m}(\bar{\beta}) = \Delta_{n+m}(\bar{\beta}) \det_{n+m} \phi_k(\beta_j), \tag{9.2.2}$$

where $\bar{\beta} = \{\beta_1, \dots, \beta_{n+m}\}$. Then

$$\begin{aligned} \Phi_{n+m}(\bar{\beta}) = \sum \Delta_n(\bar{\beta}_{\rm I}) \det_{k=1,\dots,n} \phi_k(\beta_{{\rm I}_j}) \cdot \Delta_m(\bar{\beta}_{\rm II}) \det_{k=1,\dots,m} \phi_{n+k}(\beta_{{\rm II}_j}) \cdot g(\bar{\beta}_{\rm II}, \bar{\beta}_{\rm I}) \\ = \sum \Phi_n(\bar{\beta}_{\rm I}) \widehat{\Phi}_m(\bar{\beta}_{\rm II}) g(\bar{\beta}_{\rm II}, \bar{\beta}_{\rm I}), \end{aligned} \tag{9.2.3}$$

where $\Phi_n(\bar{\xi}_{\rm I})$ is built on the functions ϕ_k, $k = 1, \dots, n$, while $\widehat{\Phi}_m(\bar{\xi}_{\rm II})$ is built on the functions ϕ_{n+k}, $k = 1, \dots, m$.

Proof. Developing the determinant in (9.2.2) over the first n columns via Laplace formula we obtain

$$\Phi_{n+m}(\bar{\beta}) = \Delta_{n+m}(\bar{\beta}) \sum (-1)^{P_{\rm I,II}} \det_{k=1,\dots,n} \phi_k(\beta_{{\rm I}_j}) \det_{k=1,\dots,m} \phi_{n+k}(\beta_{{\rm II}_j}), \tag{9.2.4}$$

where the sum is taken over partitions $\bar{\beta} \Rightarrow \{\bar{\beta}_{\rm I}, \bar{\beta}_{\rm II}\}$ such that $\#\bar{\beta}_{\rm I} = n$. The sign $P_{\rm I,II}$ is the parity of a permutation mapping the union $\{\bar{\beta}_{\rm I}, \bar{\beta}_{\rm II}\}$ into the naturally ordered set $\bar{\beta}$. One can get rid of this sign presenting $\Delta_{n+m}(\bar{\beta})$ as follows

$$\Delta_{n+m}(\bar{\beta}) = (-1)^{P_{\rm I,II}} \Delta_n(\bar{\beta}_{\rm I}) \Delta_m(\bar{\beta}_{\rm II}) g(\bar{\beta}_{\rm II}, \bar{\beta}_{\rm I}). \tag{9.2.5}$$

Substituting (9.2.5) into (9.2.4) we immediately arrive at (9.2.3). □

We use several particular cases of (9.2.3) in the core of the paper. Let

$$\begin{aligned} \phi_k(\beta) &= \frac{g(\beta, x_k)}{h(\beta, x_k)}, \qquad && k = 1, \dots, n; \\ \phi_{k+n}(\beta) &= g(\beta, y_k)\frac{h(\beta, \bar{t})}{h(\beta, \bar{s})}, \qquad && k = 1, \dots, m, \end{aligned} \tag{9.2.6}$$

where $\bar{x}$, $\bar{y}$, $\bar{t}$, and $\bar{s}$ are some sets of parameters. Then the matrix elements $\phi_k(\beta_j)$ coincide with the entries $\mathcal{J}_{jk}$ (3.3.3). Hence, we obtain for $J_{n,m}(\bar{x};\bar{y}|\bar{t};\bar{s}|\bar{\beta})$ (3.3.2)

$$\begin{aligned} J_{n,m}(\bar{x};\bar{y}|\bar{t};\bar{s}|\bar{\beta}) = \Delta'_n(\bar{x})\Delta'_m(\bar{y}) \sum \Delta_n(\bar{\beta}_{\rm I}) \det_n \left(\frac{g(\beta_{{\rm I}_j}, x_k)}{h(\beta_{{\rm I}_j}, x_k)}\right) \\ \times \Delta_m(\bar{\beta}_{\rm II}) \det_m \left(g(\beta_{{\rm II}_j}, y_k)\frac{h(\beta_{{\rm II}_j}, \bar{t})}{h(\beta_{{\rm II}_j}, \bar{s})}\right) g(\bar{\beta}_{\rm II}, \bar{\beta}_{\rm I}). \end{aligned} \tag{9.2.7}$$

Now we use the definition of Izergin determinant

$$\Delta'_n(\bar{x})\Delta_n(\bar{\beta}_{\rm I}) \det_n \left(\frac{g(\beta_{{\rm I}_j}, x_k)}{h(\beta_{{\rm I}_j}, x_k)}\right) = \frac{K_n(\bar{\beta}_{\rm I}|x)}{h(\bar{\beta}_{\rm I}, \bar{x})}, \tag{9.2.8}$$

and an explicit expression (3.4.2) for Cauchy determinant $\det_m g(\beta_{{\rm II}_j}, y_k)$. Substituting these expressions into (9.2.7) we find

$$J_{n,m}(\bar{x};\bar{y}|\bar{t};\bar{s}|\bar{\beta}) = \sum \frac{K_n(\bar{\beta}_{\rm I}|x)}{h(\bar{\beta}_{\rm I}, \bar{x})} \cdot g(\bar{\beta}_{\rm II}, \bar{y})\frac{h(\beta_{\rm II}, \bar{t})}{h(\beta_{\rm II}, \bar{s})} g(\bar{\beta}_{\rm II}, \bar{\beta}_{\rm I}), \tag{9.2.9}$$

which coincides with (3.3.4).

Another example used in the text is

$$\begin{aligned} \phi_k(\beta) &= g(\beta, x_k), \qquad && k = 1, \dots, n; \\ \phi_{k+n}(\beta) &= g(\beta, y_k), \qquad && k = 1, \dots, m. \end{aligned} \tag{9.2.10}$$

Then using explicit representation of the Cauchy determinant (3.4.2) we have

$$\Phi_{n+m}(\bar{\beta}) = \frac{g(\bar{\beta}, \bar{x})g(\bar{\beta}, \bar{y})}{\Delta'_{n+m}(\{\bar{x}, \bar{y}\})}. \tag{9.2.11}$$

On the other hand, it follows from (9.2.3) that

$$\frac{g(\bar{\beta}, \bar{x})g(\bar{\beta}, \bar{y})}{\Delta'_{n+m}(\{\bar{x}, \bar{y}\})} = \sum \Delta_n(\bar{\beta}_{\rm I}) \det_{k=1,\dots,n} g(\beta_{{\rm I}_j}, x_k) \cdot \Delta_m(\bar{\beta}_{\rm II}) \det_{k=1,\dots,m} g(\beta_{{\rm II}_j}, y_k) \cdot g(\bar{\beta}_{\rm II}, \bar{\beta}_{\rm I}). \tag{9.2.12}$$

Multiplying (9.2.12) with $\Delta'_n(\bar{x})$ and $\Delta'_m(\bar{y})$ and using (3.4.2) we arrive at

$$\sum g(\beta_{\rm I}, \bar{x})g(\beta_{\rm II}, \bar{y})g(\bar{\beta}_{\rm II}, \bar{\beta}_{\rm I}) = \frac{g(\bar{\beta}, \bar{x})g(\bar{\beta}, \bar{y})}{g(\bar{x}, \bar{y})}, \tag{9.2.13}$$

where the sum is taken with respect to the partitions of the set $\bar{\beta}$ into subsets $\bar{\beta}_{\rm I}$ and $\bar{\beta}_{\rm II}$ with $\#\bar{\beta}_{\rm I} = n$ and $\#\bar{\beta}_{\rm II} = m$.

9.3 Reduction properties of $J_{n,m}$

Consider a function $J_{n+1,m}(\{\bar{x},z'\};\bar{y}|\bar{t};\bar{s}|\{\bar{\beta},z\})$ defined by (3.3.2). Let $z'\to z$. Then the matrix element $g(z,z')/h(z,z')$ becomes singular and the determinant reduces to the product of this singular element and the corresponding minor. After elementary algebra we obtain

$$\lim_{z'\to z}\frac{1}{g(z,z')}J_{n+1,m}(\{\bar{x},z'\};\bar{y}|\bar{t};\bar{s}|\{\bar{\beta},z\}) = g(\bar{\beta},z)g(z,\bar{x})J_{n,m}(\bar{x};\bar{y}|\bar{t};\bar{s}|\bar{\beta}). \tag{9.3.1}$$

Similarly, if we consider the function $J_{n,m+1}(\bar{x};\{\bar{y},z'\}|\bar{t};\bar{s}|\{\bar{\beta},z\})$ in the limit $z'\to z$, then the matrix element $g(z,z')h(z,\bar{t})/h(z,\bar{s})$ becomes singular. The determinant again reduces to the product of this singular element and the corresponding minor, and we find

$$\lim_{z'\to z}\frac{1}{g(z,z')}J_{n,m+1}(\bar{x};\{\bar{y},z'\}|\bar{t};\bar{s}|\{\bar{\beta},z\}) = g(z,\bar{\beta})g(\bar{y},z)\frac{h(z,\bar{t})}{h(z,\bar{s})}\;J_{n,m}(\bar{x};\bar{y}|\bar{t};\bar{s}|\bar{\beta}). \tag{9.3.2}$$

Equations (9.3.1), (9.3.2) obviously could be generalized to the case when z and z' are respectively replaced with the sets $\bar{z}$ and $\bar{z}'$ such that $\#\bar{z}=\#\bar{z}'=\rho\ge 1$. Then

$$\lim_{\bar{z}'\to\bar{z}}\frac{1}{g(\bar{z},\bar{z}')}J_{n+\rho,m}(\{\bar{x},\bar{z}'\};\bar{y}|\bar{t};\bar{s}|\{\bar{\beta},\bar{z}\}) = g(\bar{\beta},\bar{z})g(\bar{z},\bar{x})J_{n,m}(\bar{x};\bar{y}|\bar{t};\bar{s}|\bar{\beta}), \tag{9.3.3}$$

and

$$\lim_{\bar{z}'\to\bar{z}}\frac{1}{g(\bar{z},\bar{z}')}J_{n,m+\rho}(\bar{x};\{\bar{y},\bar{z}'\}|\bar{t};\bar{s}|\{\bar{\beta},\bar{z}\}) = g(\bar{z},\bar{\beta})g(\bar{y},\bar{z})\frac{h(\bar{z},\bar{t})}{h(\bar{z},\bar{s})}\;J_{n,m}(\bar{x};\bar{y}|\bar{t};\bar{s}|\bar{\beta}). \tag{9.3.4}$$

One more obvious reduction is

$$J_{n,m}(\bar{x};\bar{y}|\{\bar{t},\bar{z}\};\{\bar{s},\bar{z}\}|\bar{\beta}) = J_{n,m}(\bar{x};\bar{y}|\bar{t};\bar{s}|\bar{\beta}). \tag{9.3.5}$$

9.3.1 Summation rules

Single sums

Here an example of the derivation of identities (4.3.2) are given.

Consider a contour integral

$$I=\frac{1}{2\pi i}\oint_{|z|=R\to\infty}\frac{dz}{x_k-z}\prod_{l=1}^{b}\frac{z-v_l^B}{z-v_l^C}\,. \tag{9.3.6}$$

Taking the residue at infinity we find that $I=-1$. On the other hand, this integral is equal to the sum of the residues within the integration contour. Hence,

$$-1=-\prod_{l=1}^{b}\frac{x_k-v_l^B}{x_k-v_l^C}+\sum_{j=1}^{b}\frac{1}{x_k-v_j^C}\,\frac{\prod_{l=1}^{b}(v_j^C-v_l^B)}{\prod_{l=1,\,l\neq j}^{b}(v_j^C-v_l^C)}\,. \tag{9.3.7}$$

Rewriting everything in terms of the function g we obtain

$$\sum_{j=1}^{b}g(x_k,v_j^C)\Omega_{a+j}=\frac{g(x_k,\bar{v}^C)}{g(x_k,\bar{v}^B)}-1. \tag{9.3.8}$$

This is one of the identities in (4.3.2). All the other identities can be proved exactly in the same way.

9.3.2 Multiple sums

Laplace formula

Let $\#\bar u = \#\bar v = n$. Let A and B be $n\times n$ matrices whose matrix elements are indexed by the parameters u_j and v_k: $A_{jk} = A(u_j, v_k)$ and $B_{jk} = B(u_j, v_k)$. The Laplace formula gives an expression of $\det(A+B)$ in terms of $\det A$ and $\det B$:

$$\det\bigl(A(u_j,v_k)+B(u_j,v_k)\bigr) = \sum(-1)^{[P_u]+[P_v]}\det\bigl(A(u_{\mathrm{I}_j},v_{\mathrm{I}_k})\bigr)\det\bigl(B(u_{\mathrm{II}_j},v_{\mathrm{II}_k})\bigr). \tag{9.3.9}$$

The sum is taken over partitions $\bar u \Rightarrow \{\bar u_{\mathrm{I}}, \bar u_{\mathrm{II}}\}$ and $\bar v \Rightarrow \{\bar v_{\mathrm{I}}, \bar v_{\mathrm{II}}\}$ with the restriction $\#\bar u_{\mathrm{I}} = \#\bar v_{\mathrm{I}}$. Recall that according to our convention the elements of every subset are ordered in the natural order. $[P_u]$ denotes the parity of the permutation mapping the union $\{\bar u_{\mathrm{I}}, \bar u_{\mathrm{II}}\}$ into the naturally ordered set $\bar u$. The notation $[P_v]$ has an analogous meaning. Equivalently, one can say that $[P_u]+[P_v]$ is the parity of the permutation mapping the sequence of the subscripts of the union $\{\bar u_{\mathrm{I}}, \bar u_{\mathrm{II}}\}$ into the sequence of the subscripts of the union $\{\bar v_{\mathrm{I}}, \bar v_{\mathrm{II}}\}$.

Let us introduce two functions

$$\mathcal{A}(\bar u|\bar v) = \Delta_n(\bar u)\Delta'_n(\bar v)\det A(u_j,v_k) \quad\text{and}\quad \mathcal{B}(\bar u|\bar v) = \Delta_n(\bar u)\Delta'_n(\bar v)\det B(u_j,v_k). \tag{9.3.10}$$

These functions depend on two sets of variables $\bar u$ and $\bar v$. They are symmetric over $\bar u$ and symmetric over $\bar v$. Then equation (9.3.9) can be written in the following form:

$$\Delta_n(\bar u)\Delta'_n(\bar v)\det\bigl(A(u_j,v_k)+B(u_j,v_k)\bigr) = \sum \mathcal{A}(\bar u_{\mathrm{I}}|\bar v_{\mathrm{I}})\mathcal{B}(\bar u_{\mathrm{II}}|\bar v_{\mathrm{II}})g(\bar u_{\mathrm{II}},\bar u_{\mathrm{I}})g(\bar v_{\mathrm{I}},\bar v_{\mathrm{II}}). \tag{9.3.11}$$

Indeed, multiplying (9.3.9) with $\Delta_n(\bar u)\Delta'_n(\bar v)$ and using obvious relations

$$\begin{aligned}\Delta_n(\bar u) &= (-1)^{[P_u]}\Delta_{n_{\mathrm{I}}}(\bar u_{\mathrm{I}})\Delta_{n-n_{\mathrm{I}}}(\bar u_{\mathrm{II}})g(\bar u_{\mathrm{II}},\bar u_{\mathrm{I}}),\\ \Delta'_n(\bar v) &= (-1)^{[P_v]}\Delta'_{n_{\mathrm{I}}}(\bar v_{\mathrm{I}})\Delta'_{n-n_{\mathrm{I}}}(\bar v_{\mathrm{II}})g(\bar v_{\mathrm{I}},\bar v_{\mathrm{II}}),\end{aligned} \tag{9.3.12}$$

we arrive at (9.3.11).

In section 4.5 we use (9.3.11) in the particular case of Cauchy determinants. For completeness, we recall that for arbitrary complex $\bar u$ and $\bar v$ with $\#\bar u = \#\bar v = n$ the Cauchy determinant is defined as

$$C_n = \det\left(\frac{1}{u_j - v_k}\right). \tag{9.3.13}$$

It has an explicit presentation in terms of double products

$$C_n = \frac{\prod_{1\le k<j\le n}(u_j-u_k)(v_k-v_j)}{\prod_{j=1}^n\prod_{k=1}^n(u_j-v_k)}. \tag{9.3.14}$$

From this we immediately obtain

$$g(\bar u,\bar v) = \Delta_n(\bar u)\Delta'_n(\bar v)\det\bigl(g(u_j,v_k)\bigr), \qquad \frac{1}{h(\bar u,\bar v)} = \Delta_n(\bar u)\Delta'_n(\bar v)\det\left(\frac{1}{h(u_j,v_k)}\right). \tag{9.3.15}$$

Other sums over partitions

In the core of the proof, we use different equalities, that were proven elsewhere. We recall them in the present appendix.

Lemma 9.3.1. *Let $\bar{w}$, $\bar{u}$ and $\bar{v}$ be sets of complex variables with $\#\bar{u} = m_1$, $\#\bar{v} = m_2$, and $\#\bar{w} = m_1 + m_2$. Then*

$$\sum g(\bar{w}_{\rm I}, \bar{u}) g(\bar{w}_{\rm II}, \bar{v}) g(\bar{w}_{\rm II}, \bar{w}_{\rm I}) = \frac{g(\bar{w}, \bar{u}) g(\bar{w}, \bar{v})}{g(\bar{u}, \bar{v})}\,, \tag{9.3.16}$$

where the sum is taken with respect to all partitions of the set $\bar{w}$ into subsets $\bar{w}_{\rm I}$ and $\bar{w}_{\rm II}$ with $\#\bar{w}_{\rm I} = m_1$ and $\#\bar{w}_{\rm II} = m_2$.

The proof of this Lemma is given in [109].

Lemma 9.3.2. *Let $\bar{w}$, $\bar{u}$ and $\bar{v}$ be sets of complex variables with $\#\bar{u} = m_1$, $\#\bar{v} = m_2$, and $\#\bar{w} = m_1 + m_2$. Then*

$$\sum K_{m_1}(\bar{w}_{\rm I}|\bar{u}) K_{m_2}(\bar{v}|\bar{w}_{\rm II}) f(\bar{w}_{\rm II}, \bar{w}_{\rm I}) = (-1)^{m_1} f(\bar{w}, \bar{u}) K_{m_1+m_2}(\{\bar{u} - c, \bar{v}\}|\bar{w}). \tag{9.3.17}$$

The sum is taken with respect to all partitions of the set $\bar{w}$ into subsets $\bar{w}_{\rm I}$ and $\bar{w}_{\rm II}$ with $\#\bar{w}_{\rm I} = m_1$ and $\#\bar{w}_{\rm II} = m_2$.

The proof of this Lemma is given in [66].

Lemma 9.3.3. *Let $\bar{w}$ and $\bar{\xi}$ be two sets of generic complex numbers with $\#\bar{w} = \#\bar{\xi} = m$. Let also $C_1(w)$ and $C_2(w)$ be two arbitrary functions of a complex variable w. Let us extend our convention on the shorthand notation to the products of these functions. Then*

$$\sum K_m(\bar{w}_{\rm I} - c, \bar{w}_{\rm II}|\bar{\xi}) f(\bar{\xi}, \bar{w}_{\rm I}) f(\bar{w}_{\rm II}, \bar{w}_{\rm I}) C_1(\bar{w}_{\rm I}) C_2(\bar{w}_{\rm II})$$
$$= \Delta'_m(\bar{\xi}) \Delta_m(\bar{w}) \det_m \Big(C_2(w_k) t(w_k, \xi_j) h(w_k, \bar{\xi}) + (-1)^m C_1(w_k) t(\xi_j, w_k) h(\bar{\xi}, w_k) \Big). \tag{9.3.18}$$

Here the sum is taken over all possible partitions of the set $\bar{w}$ into subsets $\bar{w}_{\rm I}$ and $\bar{w}_{\rm II}$.

The proof of this Lemma is given in [66].

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