Bergische Universität Wuppertal

Doctoral Thesis

General Extrapolation Spaces and Perturbations of Bi-Continuous Semigroups

Author:
Christian Budde

Supervisor:
Prof. Dr. Bálint Farkas

Second Reader:
Prof. Dr. Abdelaziz Rhandi

A thesis submitted in fulfillment of the requirements for the degree of Doctor of Science (Dr. rer. nat.)

in the

Workgroup Functional Analysis
School of Mathematics and Natural Sciences

The PhD thesis can be quoted as follows:

urn:nbn:de:hbz:468-20191205-110300-4
[http://nbn-resolving.de/urn/resolver.pl?urn=urn%3Anbn%3Ade%3Ahbz%3A468-20191205-110300-4]

DOI: 10.25926/7ss7-bt33
[https://doi.org/10.25926/7ss7-bt33]

Date of oral examination: 19. September 2019
Dean: Prof.Dr. Michael Günther
Members of the examining board: Prof.Dr. Bálint Farkas
Prof.Dr. Abdelaziz Rhandi
Prof.Dr. Birgit Jacob
Prof.Dr. Barbara Rüdiger-Mastandrea

Dedicated to my family
and the love of my life

Acknowledgements

First and foremost I would like to express my sincere gratitude to my advisor Prof.Dr. Bálint Farkas for the continuous support of my Ph.D study and related research, for his patience, motivation, and immense knowledge. It has been an honor to be his Ph.D student. His guidance helped me in all the time of research and writing of this thesis. I could not have imagined having a better advisor and mentor for my Ph.D study. The joy and enthusiasm he has for his research was contagious and motivational for me. I am also thankful for the excellent example he has provided as a successful mathematician and professor.

My sincere thanks also goes to Prof.Dr. Abdelaziz Rhandi, Dr. Cristian Tacelli, Prof.Dr. Marjeta Kramar Fijavž, Dr. Retha Heyman and Dr. Ronalda Benjamin who provided me an opportunity to join their team as intern in Salerno (Italy), Ljubljana (Slovenia) and Stellenbosch (South Africa). Without their precious support it would not be possible to reach my research goals. Especially, I want to thank the *Team Outgoing* of the International Office of the University Wuppertal which supported my research visits with the ERASMUS+ program.

I am indebted to Prof.Dr. Moritz Weber und Viktoria Kühner for the additional proofreading of this thesis.

I would like to thank my family for supporting me spiritually throughout writing this thesis and my life in general. Last but not least I would like to thank Lena, who was always there to help me.

Christian Budde
Bergische Universität Wuppertal
July 2019

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Introduction

A lot of well-known partial differential equations modeling physical systems, such as the heat equation, the Schrödinger equation or the wave equation, use temporal change of states. *Evolution equation* is an umbrella term for such equations that can be interpreted as differential laws describing the development of a system or as a mathematical treatment of motion in time. In 1921 Albert Einstein said about physical models:

> *As far as the laws of mathematics refer to reality, they are not certain; and as far as they are certain, they do not refer to reality.*

Certainly, this quote encourages a discussion on the benefit of evolution equations in connection with the applications to physics, i.e., it exacerbates the discussion on the connection between mathematics and reality. However, one can say that the science provides models and we process and work them out. Hence, we are, as it seems, not responsible for the relation to reality. Of course, this is a short and crisp consideration of this topic, which normally requires much more discussion. This is for example done by G. Nickel [52, pp. 531-554]. Since this topic is beyond the goals of this thesis, we now leave this philosophic area and turn to mathematics. To quote Henri Poincaré:

> *Mathematics has a threefold purpose. It must provide an instrument for the study of nature. But this is not all: it has a philosophical purpose, and, I daresay, an aesthetic purpose.*

With a solution of an evolution equation, one can predict the future of the corresponding physical system which makes it deterministic. One can find various books on the theory of evolution equations. At the same time, there are also monographs consisting only of mathematical applications of evolution equations in physics and life sciences, cf. [84], which in fact emphasizes the strength of the theory. Evolution equations can be treated by an operator theoretical approach. They can be rewritten as so-called abstract Cauchy problems. We illustrate this by an example. Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain with smooth boundary $\partial\Omega$. Let $\Delta := \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$ denote the Laplacian and consider the following problem

$$\begin{cases} \dfrac{\partial}{\partial t} w(t,x) = \Delta w(t,x), & (t,x) \in [0,\infty) \times \Omega, \\ w(0,x) = f(x), & x \in \Omega, \\ w(t,x) = 0, & x \in \partial\Omega,\ t \geq 0, \end{cases} \tag{PDE}$$

where $f \in \mathrm{L}^2(\Omega)$ is given. Now consider the Banach space $X := \mathrm{L}^2(\Omega)$ and define $u(t) := w(t,\cdot)$ to be a function with variable x. Furthermore, define a linear operator by

$$Au := \Delta u, \quad D(A) := \mathrm{H}_0^2(\Omega),$$

where $\mathrm{H}_0^2(\Omega)$ denotes the Sobolev space of functions with zero trace on the boundary, which is in fact the closure of $\mathrm{C}_0^\infty(\Omega)$ with respect to the Sobolev norm. Observe that the boundary conditions in (PDE) are now incorporated into the domain of the operator, so that (PDE) can be rewritten as

$$\begin{cases} \dot{u}(t) = Au(t), & t \geq 0, \\ u(0) = x \in X, \end{cases} \tag{ACP}$$

which is a Banach space valued initial value problem, also called an *abstract Cauchy problem*. The task is to analyse whether such an abstract Cauchy problem has a solution. Generally speaking, given an unbounded operator $(A, D(A))$ on a Banach space X, a (classical) solution of the corresponding abstract Cauchy problem (ACP) is by definition a function $u : \mathbb{R}_{\geq 0} \to X$ such that u is continuously differentiable, $u(t) \in D(A)$ for all $t \geq 0$ and (ACP) is satisfied. In fact, this leads to the generic term *well-posedness*, including the existence of a unique solution of (ACP). By definition the equation is well-posed if $D(A)$ is dense in X, for every $x \in D(A)$ there exists a unique solution $u(\cdot, x)$ of (ACP) and for every sequence $(x_n)_{n\in\mathbb{N}}$ in $D(A)$ with $\lim_{n\to\infty} x_n = 0$ one has $\lim_{n\to\infty} u(t, x_n) = 0$ uniformly for t on compact intervals $[0, t_0]$ for each $t_0 > 0$.

Now operator semigroups, the generalization of the exponential function, come into the picture. By definition, a family $(T(t))_{t\geq 0}$ of bounded linear operators on a Banach space is called a *strongly continuous one-parameter semigroup*, or *C_0-semigroups* for short, if $T(0) = I$, $T(t + s) = T(t)T(s)$ for all $s, t \geq 0$ and $\|T(t)x - x\| \to 0$ as $t \to 0$ for each $x \in X$. To each C_0-semigroup $(T(t))_{t\geq 0}$ we can assign a linear operator $(A, D(A))$ by setting

$$Ax := \lim_{t\to 0} \frac{T(t)x - x}{t}, \quad D(A) := \left\{ x \in X : \lim_{t\to 0} \frac{T(t)x - x}{t} \text{ exists} \right\},$$

which is in some sense the derivative of $(T(t))_{t\geq 0}$ in $t = 0$. This operator, which enjoys nice properties like closedness or having a dense domain, is called the generator of $(T(t))_{t\geq 0}$. The converse question, which linear operators $(A, D(A))$ are generators of a C_0-semigroup, is answered by the *Hille–Yosida theorem*, cf. [52, Chapter II, Sect. 3, Thm. 3.8].

Solutions of (ACP) and semigroups are strongly connected. In particular, for a closed operator $(A, D(A))$ the problem (ACP) is well-posed if and only if A generates a C_0-semigroup. In this case for every $x \in D(A)$ the function $u : \mathbb{R}_{\geq 0} \to X$ defined by $u(t) := T(t)x$ is the unique classical solution.

Stochastic differential equations, Ornstein–Uhlenbeck processes or Feller processes, give rise to transition semigroups which are in general not strongly continuous. An example of such a semigroup is the one coming from the differential operator on $\mathrm{C_b}(\mathbb{R}^n)$, $n \in \mathbb{N}$, given by

$$\mathcal{A}u(x) := \sum_{i,j=1}^{n} q_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} u(x) + \sum_{i=1}^{n} b_i(x) \frac{\partial}{\partial x_i} u(x),$$

on the domain

$$D(\mathcal{A}) := \left\{ u \in \bigcap_{1<p<\infty} \mathrm{W}^{2,p}_{\mathrm{loc}}(\mathbb{R}^n) \cap \mathrm{C_b}(\mathbb{R}^n) : \ \mathcal{A}u \in \mathrm{C_b}(\mathbb{R}^n) \right\},$$

where q_{ij} and b_i are sufficiently regular functions. In which way stochastic differential equations are associated with this differential operator is discussed in [82, Sect. 2.5]. Now consider the corresponding parabolic problem

$$\begin{cases} \dfrac{\partial}{\partial t} w(t,x) = \mathcal{A}w(t,x), & t \geq 0, \\ w(0,x) = f(x) \in \mathrm{C_b}(\mathbb{R}^n). \end{cases}$$

By [82, Thm. 2.2.1], for every $f \in \mathrm{C_b}(\mathbb{R}^n)$ there exists a solution $u \in \mathrm{C}\left(\mathbb{R}_{\geq 0} \times \mathbb{R}^n\right) \cap \mathrm{C}^{1+\alpha/2,2+\alpha}_{\mathrm{loc}}(\mathbb{R}_{\geq 0} \times \mathbb{R}^n)$ of this problem. Furthermore, by [82, Thm. 2.2.5] this solution can be represented by a semigroup $(T(t))_{t\geq 0}$ on $\mathrm{C_b}(\mathbb{R}^n)$, i.e., $u(t,x) = (T(t)f)(x)$, $t \geq 0$, $x \in \mathbb{R}^n$. However, by [82, Thm. 9.2.6] this semigroup of bounded linear operators fails to be strongly continuous on $\mathrm{C_b}(\mathbb{R}^n)$ in general. Here bi-continuous semigroups come into play, which form in fact the key subject of this thesis. Let us consider our topics and results in more mathematical detail and introduce the main themes and explain the structure of this thesis.

The research on bi-continuous semigroups was motivated by the work of F. Kühnemund. In fact, she was the initiator for the development of the theory of bi-continuous semigroups, cf. [78, 79]. The main idea is to equip the Banach space X, on which the semigroup $(T(t))_{t\geq 0}$ fails to be strongly continuous with respect to the norm, with an additional locally convex topology τ, which is compatible with the norm topology, such that the semigroup becomes strongly continuous with respect to τ. We will investigate some explicit examples of Banach spaces and bi-continuous semigroups in Chapter 1. Moreover, we will recall in Section 1.3 that, as in the case of C_0-semigroups, one can assign a generator to each bi-continuous semigroup which has properties similar to the generators of strongly continuous semigroups. As might have been expected, the semigroup $(T(t))_{t\geq 0}$, corresponding to the differential operator from above, yields, under an additional spectral assumption, a generator which coincides with $(\mathcal{A}, D(\mathcal{A}))$, cf. [82, Prop. 2.3.6]. Likewise, we also discuss abstract Cauchy problems for bi-continuous semigroups. Especially, there is also a notion of well-posedness which is related to the generator as well, see Section 1.5. Moreover, F. Kühnemund proved a Hille–Yosida generation type theorem [79, Thm. 16] which we will address from a new point of view in Section 2.4.1, cf. Theorem 2.38.

The work of F. Kühnemund was followed by research by B. Farkas. He investigated perturbation theory for bi-continuous semigroups. The general idea for perturbations of semigroups is the following: think of an explicit partial differential equation and the corresponding abstract Cauchy problem. In order to show that our problem has a unique solution we want to apply the Hille–Yosida generation theorem. Verifying the conditions of this theorem can be really involved in specific situations. The idea is to split the operator of the given abstract Cauchy problem into a sum of simpler operators. Quite often, it is the case that one of theses operators generates a semigroup. In the abstract one can formulate the question for bi-continuous semigroups as follows: consider a generator $(A, D(A))$ of a bi-continuous semigroup and $(B, D(B))$ a second operator. The task is to find conditions on the operator $(B, D(B))$ such that the following abstract Cauchy problem

$$\begin{cases} \dot{u}(t) = Au(t) + Bu(t), & t \geq 0, \\ u(0) = x \in X. \end{cases}$$

is well-posed, i.e., the sum $A+B$ together with an appropriate domain generates a semigroup.

There is no universal theory concerning the operator $(B, D(B))$ to achieve this. Therefore, one regards several classes of operators. In case that B is a bounded operator, we talk about bounded perturbations. If the domain $D(B)$ coincides with $D(A)$ we consider the so-called Miyadera–Voigt perturbations. Both perturbation types for bi-continuous semigroups were treated by Farkas [56], [55] and [54, Chapter 3]. We will recall the corresponding results in Section 5.1.2 and Section 5.1.3 and take positivity of the Miyadera–Voigt perturbations in Chapter 5 into account. The corresponding perturbation result in this thesis is Theorem 5.19. As remarked by B. Farkas in [54, Sect. 3.2], a third type of semigroup perturbation, the Desch–Schappacher perturbation, was left open. In Chapter 4 we discuss this type of perturbation in detail which culminate in Theorem 4.4 and Theorem 4.7. As a matter of fact, the idea of this perturbation type is to enlarge the Banach space by a technique called extrapolation. In the case of a strongly continuous semigroups $(T(t))_{t\geq 0}$ with generator $(A, D(A))$ one introduces a new norm $\|\cdot\|_{-1}$ on X by $\|x\|_{-1} := \|A^{-1}x\|$, $x \in X$ (observe that we may assume without loss of generality that $(A, D(A))$ is invertible). The completion of X with respect to $\|\cdot\|_{-1}$ is then denoted by X_{-1} and is called the first extrapolation space. In general, the resulting spaces are abstract objects which can not easily be identified as well-known spaces. Some special examples where this is possible are discussed by Nagel, Nickel and Romanelli [93, Sect. 2] or Engel and Nagel [52, Chapter II, Sect. 5(a)]. For non-strongly continuous semigroups, and especially for bi-continuous semigroups one has to take a closer look. As a matter of fact, we discuss these spaces in detail in Chapter 2 and identify some of these spaces for the specific examples as well-known function spaces, cf. Section 2.5. The semigroup $(T(t))_{t\geq 0}$ can be continuously extended to a semigroup on the extrapolation space. The corresponding generator is denoted by $(A_{-1}, D(A_{-1}))$. One combines this extrapolation procedure with perturbations by considering operators $B : X \to X_{-1}$ that are admissible in the sense that $(A_{-1} + B)_{|X}$, with an appropriate domain, is the generator of a semigroup.

Having the abstract theory of bi-continuous semigroups at hand, one can consider applications. In particular, we consider dynamical processes on graphs. In the modern time of the world wide web, everyone and everything is connected electronically or socially. Nowadays the transport of information, goods or passengers is of great interest if these are carried between a large number of customers. These transport processes between the consumers are described by the transport equation. The connections can be modelled by graphs. A topological structure turns these graphs into networks. Precursors in the operator theoretical approach of dynamics on networks are M. Kramar Fijavž and E. Sikolya and their coauthors [76, 77, 50, 43] as well as B. Dorn [41, 42] at a later date, who even treated flows on infinite networks. Briefly, they considered the transport equation on each edge together with Kirchhoff law boundary conditions in the vertices. From this one obtains an abstract Cauchy problem. For the simple case where the velocities on the edges are the same and equal to 1, one can even give an explicit expression for the C_0-semigroup solving the abstract Cauchy problem, cf. [41, Prop. 3.3]. In particular, one can even recover well-posedness, spectral and asymptotic properties. To do so, one consideres $\mathrm{L}^1([0,1], \mathbb{C}^m)$ or $\mathrm{L}^1([0,1], \ell^1)$ as the state space for the dynamics on the edges of the network. Together with M. Kramar Fijavž we regarded the state space $\mathrm{L}^\infty([0,1], \ell^1)$. Here adjoint semigroups and hence bi-continuous semigroups play an important role. By Theorem 7.9 and Corollary 7.16 we show that the transport problem is well-posed on our new state space $\mathrm{L}^\infty([0,1], \ell^1)$ in the simple case where all velocities are equal as well as for the general case.

Overview of this thesis

In **Chapter 1** we introduce the concept of bi-continuous semigroups and the underlying structure. The Banach spaces we are looking at are all equipped with a locally convex topology which interacts with the norm topology in a certain way. Typical examples are the compact-open topology on the space of bounded continuous functions and the weak*-topology on the dual of a Banach space. Beside that we give illustrations of bi-continuous semigroups on these spaces, for instance, the left-translation semigroup and the adjoint semigroup. Another interesting example, the implemented semigroup on the space of bounded linear operators, is reviewed in Chapter 6. These semigroups belong to the guideline of this chapter, and of this thesis in general. In fact, we determine their generators and describe the corresponding space of strong continuity. The last two points are only treated for two of the major examples, because the implemented semigroup needs more attention, hence the discussion of their properties will take place later in Chapter 6.

Chapter 2 consists of the construction and study of extrapolation- and intermediate spaces. For C_0-semigroups these spaces are well-known and can be applied for example to maximal regularity problems. However, the case of bi-continuous semigroups, or more generally non-densely defined operators, is new. One only has to assume some density condition which is in most of the examples, and especially for bi-continuous semigroups, fulfilled. At the end of Section 2.1.2 we give a universal construction for extrapolation spaces which allows us to determine these spaces explicitly. Additionally, in Section 2.4.1 we give a direct proof of the Hille–Yosida generation theorem for bi-continuous semigroups by utilizing these new techniques. This chapter is the starting point for various applications as they come up in Chapter 3, Chapter 4 and Chapter 6.

In **Chapter 3** we consider an explicit application of extrapolation for unbounded operator-valued multiplication operators on Bochner L^p-spaces. In particular, we show that this extrapolation procedure gives rise to the study of fiber L^p-spaces as they are introduced by R. Heymann [65]. In the same manner, T. Graser studied multiplication operators and their extrapolation spaces on the space $\mathrm{C}_0(\mathbb{R}, X)$ in [59] and obtained comparable results. This topic is ongoing research with R. Heymann.

Our main objective in **Chapter 4** is a Desch–Schappacher type perturbation result for bi-continuous semigroups. For this we use the theory from Chapter 2. In this context we consider operators $B \in \mathscr{L}(X, X_{-1})$ such that the map $B : X \to X_{-1}$ is continuous with respect to the corresponding locally convex topologies as described in Section 2.4.2. We show that under certain conditions the operator $(A_{-1}+B)_{|X}$ generates again a bi-continuous semigroup on the Banach space X. In addition, we combine these perturbation results with the intermediate spaces from Chapter 2.

We carry on perturbation theory for bi-continuous semigroups in **Chapter 5**. In particular, we consider Miyadera–Voigt perturbations as they were already treated by Farkas [54, Sect. 3.2] and [55], but with the extra assumption of positivity. The corresponding result for strongly continuous semigroups is due to J. Voigt, cf. [119]. To proceed we consider an order on the underlying Banach space and say that a semigroup is positive if and only if the semigroup operators maps positive elements to positive elements. One observes from [54, Thm. 1.4.2] that positivity of a semigroup is equivalent to the fact that the resolvent operator is also positive. At the end of this chapter, we prove that every unbounded positive rank-one perturbation gives rise to a positive Miyadera–Voigt perturbation. As a second example we consider a semigroup on the space of bounded Borel measures on the real line. As a matter of fact, this specific semigroup is the adjoint of the Gauss–Weierstrass semigroup, also known as diffusion semigroup or heat semigroup.

As already mentioned above, **Chapter 6** is devoted to implemented semigroups which are bi-continuous with respect to the strong operator topology on the space of bounded linear operators. In particular, we discuss generators and the space of strong continuity for such semigroups. At the end of this chapter we consider also Desch–Schappacher perturbations for this class of semigroups. We show that there is a one-to-one correspondence between the implemented semigroups and the underlying strongly continuous semigroups. For this connection we study ideals in the space of bounded linear operators and corresponding module homomorphisms.

Last but not least we consider flows on (infinite) networks in **Chapter 7**. For that we first recall some notations from graph theory, and go on with rephrasing the corresponding partial differential equations of the transport process on the network to an abstract Cauchy problem. Assuming the same transport velocities on each edge of the graph we can give an explicit formula of the appropriate semigroup solving this abstract Cauchy problem. For the case where we have different speeds on the edges, we have to restrict ourselves to finite networks since we apply a Trotter–Kato approximation theorem to handle this case. The conditions in this theorem forces the networks to be finite.

Concerning originality

We remark that not all results in this thesis are to be considered original. Moreover, not all new results use "new techniques". This is the reason that we want to separate here the wheat from the chaff. Firstly, we notice that some of the chapters of this thesis are based on papers:

- Chapter 2 is based on the paper *Intermediate and extrapolated spaces for bi-continuous operator semigroups* [28] (with minor modification and without the part about the implemented semigroup), which is joint work with B. Farkas and has been published in *Journal of Evolution Equations* (DOI:10.1007/s00028-018-0477-8).

- Chapter 4 is based on the paper *A Desch–Schappacher perturbation theorem for bi-continuous semigroups* [27] (with minor modifications and without the part about the implemented semigroup), which is joint work with B. Farkas and accepted for publication in *Mathematische Nachrichten*, and is available online as ArXiv preprint (ArXiv:1811.08455).

- Chapter 7 is based on the paper *Bi-continuous semigroups for flows in infinite networks* [29] (with some more preliminaries on networks), which is joint work with M. Kramar Fijavž, which is submitted to *Operators and Matrices*, and is available online as ArXiv preprint (ArXiv:1901.10292).

As already mentioned above, Chapter 1 has an introducing character. This means among others, that the first chapter consists of already known results. However, we choose to include this chapter to stay as self-contained as possible. In this chapter we present two important examples of bi-continuous semigroups in detail. These examples already occured in [78, Sect. 3.5.1] and [54, Rem. 2.5.3], but to be self-contained we added some details concerning the underlying Banach spaces and locally convex topologies. To do so, some of the arguments are taken from [103, Chapter 2]. Particularly, in Section 1.2.1 we use the arguments from [103, Exam. 2.2.4] whereas we use [103, Thm. 2.1.6] at a later time for the discussion of implemented semigroups in Section 6.1.1.

The novelty of Chapter 2 starts with the the construction of extrapolation spaces for non-densely defined operator due to Definition 2.9. A way more general consideration of extrapolation spaces is treated by Theorem 2.16 using a new approach which is connected with universal extrapolation spaces. Until here we do not use the assumption that the operators we consider are generators of (bi-continuous) semigroups. We connect the extrapolation spaces with semigroups in a general context in Section 2.3. Moreover, we consider the Hille–Yosida generation theorem for bi-continuous semigroups, cf. Theorem 2.38. Even though the statement of this theorem is already known, the techniques for the proof using extrapolation are new. The extrapolation of bi-continuous semigroups is a result of Proposition 2.40. The results in Section 2.4.3 and Section 2.4.4, treating Hölder spaces for bi-continuous semigroups, are new. The proofs uses techniques similar to theses in [52, Chapter II, Sect. 5(b)]. The concluding examples in Section 2.5 generalize the examples of extrapolation spaces in the strongly continuous case, cf. [52, Chapter II, Exam. 5.8 & Ex. 5.23(5)].

The definition of fiber L^p-spaces in Section 3.2 is originally due to R. Heymann. However, Theorem 3.14 regarding extrapolation spaces of unbounded operator-valued multiplication operators, by means of these fiber L^p-spaces, is new and the key result in this chapter. The results from Section 3.3 help us to prove our main result. In fact, the main result gives an insight how to describe the domain of evolution semigroups coming from the solutions of non-autonomous abstract Cauchy problem.

The main results of Chapter 4 are Theorem 4.4 and Theorem 4.7. These are new results since they discuss Desch–Schappacher perturbations for bi-continuous semigroups. Here we use the fact from Chapter 2 that we can extrapolate bi-continuous semigroups. The proofs of the main results are at some places verbatime the same as for the strongly continuous case [52, Chapter III, Thm. 3.1 & Cor. 3.3]. However, we always have to take care of the locally convex topology in the bi-continuous case which leads to some technical subtleties. The example in Section 4.3 is based on [52, Chapter III, Exam. 3.5], however, we consider a different underlying function space which fits in the bi-continuous framework. A novel application of the Desch–Schappacher theorem is later on given in Chapter 6.

Chapter 5 is devoted to Theorem 5.19 as main result. As a matter of fact, the result is based on [119, Thm. 0.1]. We introduce the new notion of bi-AL-spaces in Definition 5.9, which is related to the classical AL-spaces. In order to prove the main result, we use Lemma 5.20 as an auxiliary tool. For the proofs we proceed similar to the original proofs [119, Lemma 2.1] and [119, Thm. 0.1]. However, the argumentation differs, since we work with an additional locally convex topology. The first example of Section 5.3.1 is based on [13, Thm. 2.2] whereas Section 5.3.2 covers a new example on the space of bounded Borel measures, which is in fact the adjoint semigroup of the Gauss–Weierstrass semigroup.

The novelty of Chapter 6 is the consideration of extrapolation and perturbations of implemented semigroups. Although, J. Alber deals with extrapolation spaces [6], our approach is new since we do not only consider extrapolation on the space of strong continuity but in the new setting of bi-continuous semigroups as dicussed in Chapter 2. The highlights of this chapter is the extrapolation procedure of the left implemented semigroup, cf. Section 6.2.2, Lemma 6.4 where we discuss the properties of the domain of the generator of bi-continuous semigroups on the space $\mathscr{L}(E)$ by means of ideals as well as Theorem 6.6, Theorem 6.7 and Theorem 6.14 where we relate perturbations of implemented semigroups with these of strongly continuous semigroups.

The study of large networks by means of operator semigroups, as it is discussed in Chapter 7. in not new, cf. [41, 43, 42]. However, an approach via bi-continuous semigroups considered on the phase space $\mathrm{L}^{\infty}\left([0,1],\ell^1\right)$ is novel. In this context, Theorem 7.9 is a main result. Added to that we consider also Corollary 7.13 and Theorem 7.15 as highlight of this chapter. The novelty of the approach is in the first instance the use of bi-continuous semigroups and moreover that we can not make use of the Lumer–Phillips generation theorem as it is used in the strongly continuous case, cf. [21, Cor. 18.15]. For this reason we use the Trotter–Kato approximation theorem for bi-continuous semigroups, cf. [78, Thm. 2.3 & Thm. 2.6], [54, Thm. 1.2.10] and [4].

Chapter 1

Bi-Continuous Semigroups

§ 1.1 Bi-admissible spaces

In this first chapter we concentrate on the basic theory of *bi-continuous semigroups*. This class of semigroups was first introduced by F. Kühnemund in [78]. The following assumptions, as proposed by F. Kühnemund, cf. [78, Assum. 1.1], will be made during the whole thesis.

Assumption 1.1. Consider a triple $(X_0, \|\cdot\|, \tau)$ where X_0 is a Banach space, and

1. τ is a locally convex Hausdorff topology coarser than the norm-topology on X_0, i.e., the identity map $(X_0, \|\cdot\|) \to (X_0, \tau)$ is continuous;
2. τ is sequentially complete on the $\|\cdot\|$-closed unit ball, i.e., every $\|\cdot\|$-bounded τ-Cauchy sequence is τ-convergent;
3. The dual space of (X_0, τ) is norming for X_0, i.e.,

$$\|x\| = \sup_{\substack{\varphi \in (X_0,\tau)' \\ \|\varphi\| \leq 1}} |\varphi(x)|. \tag{1.1.1}$$

Remark 1.2. (i) There is the related notion of so-called Saks spaces, see [34]. By definition a *Saks space* is a triple $(X_0, \|\cdot\|, \tau)$ such that X_0 is a vector space with a norm $\|\cdot\|$ and locally convex topology τ in such a way that τ is coarser than the $\|\cdot\|$-topology, but the closed unit ball is τ-complete. In particular, X_0 is a Banach space.

(ii) There is also a connection to norming dual pairs discussed in [80]. More information about this can also be found in Section 2.3. In particular, (X_0, Y) with $Y = (X_0, \tau)'$ is a norming dual pair.

(iii) R. Kraaij puts this setting in a more general framework of locally convex spaces with mixed topologies, see [75, Sec. 4], and also [54, App. A]

(iv) Recall from [101, Thm. 1.36 & 1.37] that every locally convex topology gives rise to a family of seminorms and vice versa. In this regard Assumption (1.1.1) is equivalent to the following: There is a set $\mathcal{P}$ of τ-continuous seminorms defining the topology τ, such that

$$\|x\| = \sup_{p \in \mathcal{P}} p(x). \tag{1.1.2}$$

This description is also used by R. Kraaij in [75], cf. his Lemma 4.4. Note also that by this remark and by Lemma 3.1 in [34] we see that a Saks space satisfies Assumption 1.1. Indeed, assume (1.1.1) and let $\mathcal{P}$ be the collection of *all* τ-continuous seminorms p such that $p(x) \leq \|x\|$. Then $|\varphi(\cdot)| \in \mathcal{P}$ for each $\varphi \in (X_0, \tau)'$ with $\|\varphi\| \leq 1$, and (1.1.2) is trivially satisfied. If q is any τ-continuous seminorm, then $q(x) \leq M\|x\|$ for some constant M and for all $x \in X_0$. So that $q/M \in \mathcal{P}$, proving that $\mathcal{P}$ defines precisely the topology τ. For the converse implication suppose that (1.1.2) holds. Then by the application of the Hahn–Banach theorem we obtain (1.1.1).

For the sake of completeness and to be self-contained we discuss two typical examples of spaces which satisfy Assumption 1.1 in the following section. For related examples we refer to [54, Sect. 1.2].

1.1.1 The space $\mathrm{C_b}(\Omega)$ of bounded continuous functions

Consider an arbitrary topological space Ω with a Hausdorff topology κ on it. With $\mathrm{C_b}(\Omega)$ we denote the space of all bounded continuous functions $f : \Omega \to \mathbb{R}$. Equipped with the supremum-norm $\|\cdot\|_\infty$, defined by

$$\|f\|_\infty := \sup_{x\in\Omega} |f(x)|, \quad f \in \mathrm{C_b}(\Omega),$$

this becomes a Banach space. Another topology on $\mathrm{C_b}(\Omega)$ is the *compact-open topology* τ_{co}. To describe this topology, we define for a compact subset $K \subseteq \Omega$ and an open subset $U \subseteq \mathbb{R}$ the set

$$\mathcal{V}(K, U) := \{f \in \mathrm{C_b}(\Omega) : \ f(K) \subseteq U\}.$$

The collection of all such sets forms a subbase for τ_{co}, i.e., every proper open set of the compact-open topology can be written as a union of finite intersections of elements of the form $\mathcal{V}(K, U)$. Another description of this topology uses seminorms. In fact, a generating family of seminorms $\mathcal{P}$ is given by

$$\mathcal{P} = \{p_K : \ K \subseteq \Omega \text{ compact}\}, \quad p_K(f) := \sup_{x\in K} |f(x)|.$$

If the topological space (Ω, κ) is a Tychonoff space, i.e., every singleton subset of Ω is closed and for every closed subset $C \subseteq \Omega$ and each $x \in \Omega \setminus C$ there exists a continuous function $f : \Omega \to [0, 1]$ such that $f(x) = 0$ and $f_{|C} \equiv 1$, then τ_{co} is a Hausdorff topology. If moreover (Ω, κ) is a k_f-space, i.e., the continuity of a function $f : \Omega \to \mathbb{R}$ depends only on the continuity of $f_{|K}$ for all compact $K \subseteq \Omega$, then $\mathrm{C_b}(\Omega)$ is τ_{co}-complete on norm-bounded sets. Notice that every locally compact or metrizable space has this property. Observe that the point measures are τ_{co}-continuous and hence we conclude that $(\mathrm{C_b}(\Omega), \tau_{\mathrm{co}})$ is norming for $(\mathrm{C_b}(\Omega), \|\cdot\|_\infty)$. Even for a Banach space X, the space $\mathrm{C_b}(\Omega, X)$ of vector-valued, continuous, bounded functions fits into the framework of Assumption 1.1.

1.1.2 The dual space X'

The second example makes use of duality in Banach spaces. Let X be a Banach space and X' the norm dual of X, consisting of all linear functionals $\varphi : X \to \mathbb{C}$ which are continuous with respect to the Banach space norm. Then X' becomes a Banach space with respect to the norm defined by

$$\|\varphi\|_{X'} := \sup_{\substack{x\in X\\ \|x\|\leq 1}} |\langle x,\varphi\rangle|, \quad \varphi\in X',$$

where $\langle\cdot,\cdot\rangle : X\times X'\to\mathbb{C}$ is the (canonical) dual pairing between X and X' defined by $\langle x,\varphi\rangle := \varphi(x)$ for $x\in X$ and $\varphi\in X'$. In addition X' becomes a locally convex space, with the family of generating seminorms $\mathcal{P}$ defined by

$$\mathcal{P} = \{p_x : \ x\in X\}, \quad p_x(\varphi) := |\varphi(x)| = |\langle x,\varphi\rangle|.$$

The corresponding topology is called the *weak*-topology* τ_{w^*} and is the coarsest topology making the evaluation maps $\varphi\mapsto\varphi(x)$ continuous. We notice that a sequence $(\varphi_n)_{n\in\mathbb{N}}$ in X' converges to φ with respect to the weak*-topology if and only if $\varphi_n(x)\to\varphi(x)$ for each $x\in X$. This topology is sometimes also denoted by $\sigma(X',X)$. We now show that $\sigma(X',X)$ satisfies Assumption 1.1. To do so, notice that $\sigma(X',X)$ is not first countable in general, so we have to argue by nets instead of sequences. To show that $\sigma(X',X)$ is weaker than the norm-topology observe that $|\langle x,\varphi\rangle| \leq \|x\|\,\|\varphi\|$ for each $x\in X$ and $\varphi\in X'$. Hence, if $(\varphi_\iota)_{\iota\in I}$ is a net converging in the norm of X', then $(\varphi_\iota)_{\iota\in I}$ converges also with respect to the weak*-topology on X', meaning that $\sigma(X',X)$ is weaker than the norm topology. That $\sigma(X',X)$ is a Hausdorff topology is trivial, i.e., let $\varphi,\psi\in X'$ such that $\varphi\neq\psi$, then there exists $x\in X$ with $p_x(\varphi) = |\varphi(x)| \neq |\psi(x)| = p_x(\psi)$. The norming property follows from the definition of the norm

$$\|\varphi\|_{X'} = \sup_{\|x\|\leq 1} |\varphi(x)| = \sup_{\|x\|\leq 1} p_x(\varphi).$$

Now let $(\varphi_n)_{n\in\mathbb{N}}$ be a norm-bounded weak*-Cauchy sequence in X'. Then there exists $M\geq 0$ such that $\|\varphi_n\|\leq M$ for each $n\in\mathbb{N}$ and $(\varphi_n(x))_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{R}$ for each $x\in X$ and hence $\lim_{n\to\infty}\varphi_n(x)$ exists for each $x\in X$. Define $\varphi\in X'$ by $\varphi(x) := \lim_{n\to\infty}\varphi_n(x)$ and observe that dy definition $\varphi_n\to\varphi$ with respect to the weak*-topology. In addition, we obtain that also the limit φ is bounded with $\|\varphi\|\leq M$.

§ 1.2 Bi-continuous semigroups

Now we formulate the definition of a bi-continuous semigroup which is originally due to F. Kühnemund, cf. [79, Def. 3].

Definition 1.3. Let X_0 be a Banach space with norm $\|\cdot\|$ together with a locally convex topology τ, such that the conditions in Assumption 1.1 are satisfied. We call a family of bounded linear operators $(T(t))_{t\geq 0}$ a *bi-continuous semigroup* if it has the following properties.

1. $T(t+s) = T(t)T(s)$ and $T(0) = I$ for all $s,t\geq 0$.
2. $(T(t))_{t\geq 0}$ is strongly τ-continuous, i.e., the map $\varphi_x : [0,\infty)\to(X_0,\tau)$ defined by $\varphi_x(t) = T(t)x$ is continuous for every $x\in X_0$.
3. $(T(t))_{t\geq 0}$ is exponentially bounded, i.e., there exist $M\geq 1$ and $\omega\in\mathbb{R}$ such that $\|T(t)\|\leq M\mathrm{e}^{\omega t}$ for each $t\geq 0$.
4. $(T(t))_{t\geq 0}$ is locally-bi-equicontinuous, i.e., if $(x_n)_{n\in\mathbb{N}}$ is a norm-bounded sequence in X_0 which is τ-convergent to 0, then $(T(s)x_n)_{n\in\mathbb{N}}$ is τ-convergent to 0 uniformly for $s\in[0,t_0]$ for each fixed $t_0\geq 0$.

The following definition will occur frequently in the forthcoming chapters and originates from the theory of strongly continuous semigroups, cf. [52, Chapter I, Def. 5.6].

Definition 1.4. For an exponentially bounded semigroup of bounded linear operators $(T(t))_{t\geq 0}$ we define the *growth bound* to be

$$\omega_0(T) := \inf\left\{\omega \in \mathbb{R} :\ \exists M \geq 1\ \forall t \geq 0 :\ \|T(t)\| \leq M\mathrm{e}^{\omega t}\right\}.$$

We now continue with examples of bi-continuous semigroups on the space $\mathrm{C_b}(\mathbb{R})$ and X' which we already discussed before in Section 1.1.1 and Section 1.1.2.

1.2.1 The left-translation semigroup

As we already saw the space $X_0 = \mathrm{C_b}(\mathbb{R})$ equipped with the supremum-norm and the compact-open topology τ_{co} satisfies Assumption 1.1. Now consider the *left-translation semigroup* $(T(t))_{t\geq 0}$ on $\mathrm{C_b}(\mathbb{R})$ defined by

$$(T(t)f)(x) := f(x+t), \quad t \geq 0,\ f \in \mathrm{C_b}(\mathbb{R}),\ x \in \mathbb{R}.$$

This semigroup actually fails to be strongly continuous with respect to the supremum-norm. In particular, since $T(t)f(x) - f(x) = f(x+t) - f(x)$ the continuity of $t \mapsto T(t)f$ entails that the function f has to be uniformly continuous. By taking the function $f(x) := \sin(x^2)$ for $x \in \mathbb{R}$, which is bounded continuous and not uniformly continuous, we conclude that $\|T(t)f - f\|_\infty$ does not vanish for $t \to 0$. Although $(T(t))_{t\geq 0}$ is not a C_0-semigroup, we can show that it is bi-continuous with respect to the compact-open topology. Since $\|T(t)f\|_\infty = \|f\|_\infty$ for each $f \in \mathrm{C_b}(\mathbb{R})$, $(T(t))_{t\geq 0}$ becomes a contraction semigroup and therefore it is exponentially bounded. Observe that for a compact subset $K \subseteq \mathbb{R}$ holds that

$$p_K(T(t)f - f) = \sup_{x\in K} |f(x+t) - f(x)|, \quad t \geq 0,\ f \in \mathrm{C_b}(\mathbb{R}).$$

Since f is continuous and K is compact, $f_{|K}$ is uniformly continuous, hence for each $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\sup_{x\in K} |f(x+t) - f(x)| < \varepsilon$$

whenever $|t| < \delta$ and we obtain the strong continuity of $(T(t))_{t\geq 0}$ with respect to τ_{co}. For the local bi-equicontinuity let $(f_n)_{n\in\mathbb{N}}$ be a $\|\cdot\|_\infty$-bounded τ_{co}-null sequence. For $t_0 > 0$ and $t \in [0, t_0]$ we obtain

$$p_K(T(t)f_n) = \sup_{x\in K} |f_n(x+t)| \leq \sup_{x\in H} |f_n(x)| = p_H(f_n),$$

where $H := \bigcup_{t\in[0,t_0]} K + t$ is compact. This shows that $T(t)f_n \overset{\tau_{\mathrm{co}}}{\rightrightarrows} 0$ uniformly for $t \in [0, t_0]$. We conclude that the left-translation semigroup is indeed a bi-continuous semigroup on $\mathrm{C_b}(\mathbb{R})$.

1.2.2 The adjoint semigroup

Let $(T(t))_{t\geq 0}$ be a C_0-semigroup on a Banach space X. Recall that the adjoint of a bounded linear operator $T \in \mathscr{L}(X)$ is defined to be the unique bounded linear operator $T' \in \mathscr{L}(X')$

such that

$$\langle Tx, \varphi \rangle = \langle x, T'\varphi \rangle\,,$$

for all $x \in X$ and $\varphi \in X'$. The *adjoint semigroup* $(T(t)')_{t\geq 0}$ on the dual Banach space X' consist of all adjoint operators $T(t)'$ on X'. To show that in general $(T(t)')_{t\geq 0}$ is not strongly continuous with respect to the norm on X' consider the strongly continuous left-translation semigroup $(T(t))_{t\geq 0}$ on $\mathrm{L}^1(\mathbb{R})$ and have a look at its adjoint, the right-translation semigroup on $\mathrm{L}^\infty(\mathbb{R})$. By taking the the characteristic function on the interval $[0,1]$, i.e., $f(x) := \mathbf{1}_{[0,1]}(x)$ for $x \in \mathbb{R}$, one observes that $\operatorname{ess\,sup}_{x\in\mathbb{R}} |f(x-t) - f(x)| = 1$ for $t > 0$. Hence this semigroup is not strongly continuous. Since we already saw that X' equipped with the weak*-topology τ_{w^*} satisfies Assumption 1.1 we show that the adjoint semigroup is bi-continuous on X' with respect to τ_{w^*} (see also [78, Prop. 3.18]). The exponential boundedness is clear, since $(T(t))_{t\geq 0}$ was exponentially bounded. By the strong continuity of $(T(t))_{t\geq 0}$ we obtain for $x' \in X'$ and $y \in X$

$$|\langle y, T(t)'x' - x'\rangle| = |\langle T(t)y - y, x'\rangle| \leq \|x'\| \cdot \|T(t)y - y\|$$

and hence the strong continuity of $(T(t)')_{t\geq 0}$ with respect to the weak*-topology. By using the same argument and the fact that $\{T(t)x : \ t \in [0, t_0]\}$ is compact, we also conclude that $(T(t)')_{t\geq 0}$ is locally bi-equicontinuous.

Remark 1.5. Other important examples are evolution semigroups on $\mathrm{C_b}(\mathbb{R}, X)$, semigroups induced by flows, see [78, Sect. 3.2], and the Ornstein–Uhlenbeck semigroup on $\mathrm{C_b}(\mathbb{R})$. For more details we refer to [78, Sect. 3.3] and [54, Sect. 2.3]. In Chapter 6 we consider another interesting example, the implemented semigroups.

§ 1.3 The generator

As in the case of C_0-semigroups, [52, Chapter II, Def. 1.2], we can define a generator for a bi-continuous semigroup in the following way, cf. [54, Def. 1.2.6].

Definition 1.6. Let $(T(t))_{t\geq 0}$ be a bi-continuous semigroup on X_0. The *generator* A is defined by

$$Ax := \tau\lim_{t\to 0} \frac{T(t)x - x}{t}$$

with the domain

$$D(A) := \Big\{ x \in X_0 : \ \tau\lim_{t\to 0} \frac{T(t)x - x}{t} \text{ exists and } \sup_{t\in(0,1]} \frac{\|T(t)x - x\|}{t} < \infty \Big\}.$$

This definition of the generator leads to a couple of important properties. Recall that for an linear operator $(A, D(A))$ on a Banach space X the *resolvent set* $\rho(A)$ consists of $\lambda \in \mathbb{C}$ such that $\lambda - A$ is invertible, i.e., there exists a bounded operator B with $Bx \in D(A)$ for each $x \in X$ such that $(\lambda - A)Bx = x$ for each $x \in X$ and $B(\lambda - A)x = x$ for each $x \in D(A)$. The inverse of $\lambda - A$ is often denoted by $R(\lambda, A) := (\lambda - A)^{-1}$. The complement $\sigma(A) := \mathbb{C} \setminus \rho(A)$ is called the *spectrum*. The following theorem summarizes some properties the generator of a bi-continuous semigroup, see [79, Sect. 1.2] and [54, Thm. 1.2.7]:

Theorem 1.7. *Let $(T(t))_{t\geq 0}$ be a bi-continuous semigroup on a Banach space X_0 with respect to τ and with generator $(A, D(A))$. Then the following hold:*

(a) *The operator A is bi-closed, i.e., whenever $(x_n)_{n\in\mathbb{N}}$ is a sequence in $D(A)$ such that $x_n \xrightarrow{\tau} x$ and $Ax_n \xrightarrow{\tau} y$ and both sequences are norm-bounded, then $x \in D(A)$ and $Ax = y$.*

(b) *The domain $D(A)$ is bi-dense in X_0, i.e., for each $x \in X_0$ there exists a norm-bounded sequence $(x_n)_{n\in\mathbb{N}}$ in $D(A)$ such that $x_n \xrightarrow{\tau} x$.*

(c) *For $x \in D(A)$ we have $T(t)x \in D(A)$ and $T(t)Ax = AT(t)x$ for all $t \geq 0$.*

(d) *For $t > 0$ and $x \in X_0$ one has*

$$\int_0^t T(s)x \,\mathrm{d}s \in D(A) \quad \text{and} \quad A\int_0^t T(s)x \,\mathrm{d}s = T(t)x - x. \tag{1.3.1}$$

(e) *For $\lambda > \omega$ one has $\lambda \in \rho(A)$ (thus A is closed) and for $x \in X_0$ holds:*

$$R(\lambda, A)x = \int_0^\infty \mathrm{e}^{-\lambda s} T(s)x \,\mathrm{d}s \tag{1.3.2}$$

where the integral is a τ-improper integral.

From Definition 1.6 and Theorem 1.7 we conclude that every bi-continuous semigroup gives rise to a (bi)-closed and bi-densely defined operator $(A, D(A))$. The question under what conditions an operator $(A, D(A))$ is the generator of a bi-continuous semigroup is given by the *Hille–Yosida generation theorem.* For C_0-semigroups this theorem was originally proven by K. Yosida [124] and E. Hille [66, Thm. 12.2.1] and can for example also be found in [52, Chapter II, Thm. 3.8]. Kühnemund also proved a similar generation theorem for bi-continuous semigroups in [78, Thm. 1.28]. There integrated semigroups are used for the proof of the theorem. In Chapter 2 we give a new direct proof of this theorem. Since we use extrapolation methods for this, we talk about the Hille–Yosida theorem for bi-continuous semigroups later on in Section 2.4.1. As a matter of fact, this will be Theorem 2.38 in this thesis. The Hille–Yosida Theorem is closely related to the problem of the existence and uniqueness of solutions of a evolution equations. This correspondence will be discussed in Section 1.5.

In what follows we determine the generator of the translation semigroup from Section 1.2.1 and of the adjoint semigroup from Section 1.2.2.

1.3.1 The left-translation semigroup on $\mathrm{C_b}(\mathbb{R})$

Let us start with the generator of the translation semigroup on $\mathrm{C_b}(\mathbb{R})$ from Section 1.2.1. The following result identifies the generator explicitly.

Proposition 1.8. *Let $(T(t))_{t\geq 0}$ be the left-translation semigroup on $\mathrm{C_b}(\mathbb{R})$. The generator is given by $Af := f'$ with domain $D(A) := \mathrm{C_b^1}(\mathbb{R})$, the space of differentiable functions with bounded and continuous derivative.*

Proof. If $f \in \mathrm{C_b^1}(\mathbb{R})$, then we obtain by the fundamental theorem of calculus

$$\frac{T(t)f(x) - f(x)}{t} = \frac{f(x+t) - f(x)}{t} = \frac{1}{t}\int_0^t f'(x+s)\,\mathrm{d}s, \quad x \in \mathbb{R},\ t > 0.$$

Now let $K \subseteq \mathbb{R}$ be an arbitrary compact subset. Since by assumption the derivative of f is continuous, the restriction of f' to K is uniformly continuous, i.e., for arbitrary $\varepsilon > 0$ there exists $\delta > 0$ such that $|f'(x) - f'(y)| < \varepsilon$ for every $x, y \in K$ whenever $|x - y| < \delta$. Hence for $0 < t < \delta$ we obtain

$$\sup_{x\in K} \left| \frac{f(x+t) - f(x)}{t} - f'(x) \right| \leq \frac{1}{t} \int_0^t \sup_{x\in K} |f'(x+s) - f'(x)| \, \mathrm{d}s < \varepsilon.$$

Hence $\frac{1}{t}(T(t)f - f)$ converges uniformly on compact sets to f' as $t \to 0$. Since $f \in \mathrm{C}^1_{\mathrm{b}}(\mathbb{R})$ one also has $f \in \mathrm{Lip}(\mathbb{R})$ which means that there exists $K \geq 0$ such that $|f(x) - f(y)| \leq K |x - y|$ for all $x, y \in \mathbb{R}$. In particular

$$\sup_{t\in(0,1]} \frac{\|T(t)f - f\|_\infty}{t} \leq \sup_{t\in(0,1]} \sup_{x\in\mathbb{R}} \frac{Kt}{t} = K.$$

We conclude that $\mathrm{C}^1_{\mathrm{b}}(\mathbb{R}) \subseteq D(A)$.

Conversely, suppose that $f \in D(A)$. Then $\frac{1}{t}(f(\cdot + t) - f(\cdot))$ converges uniformly on compact intervals to Af in $\mathrm{C}_{\mathrm{b}}(\mathbb{R})$ as t tends to 0. In particular, for each compact subset $K \subseteq \mathbb{R}$ and each $x \in K$ the difference quotient $\frac{1}{t}(f(\cdot + t) - f(\cdot))$ converges to $(Af)(x)$. We conclude that f is differentiable in $\mathbb{R}$ and $f' = Af \in \mathrm{C}_{\mathrm{b}}(\mathbb{R})$. Hence $D(A) \subseteq \mathrm{C}^1_{\mathrm{b}}(\mathbb{R})$. □

1.3.2 The adjoint semigroup

Consider a C_0-semigroup $(T(t))_{t\geq 0}$ with generator $(A, D(A))$ on a Banach space X and the corresponding adjoint semigroup $(T'(t))_{t\geq 0}$ on X'. We already saw in Section 1.2.2 that this leads to a bi-continuous semigroup with respect to the weak*-topology on X'. Recall the following definition [52, Appendix, Def. B.8].

Definition 1.9. For a densely defined operator $(A, D(A))$ on X, the *adjoint operator* $(A', D(A'))$ on X' is defined by

$$D(A') := \{x' \in X' : \ \exists y' \in X' \ \forall x \in D(A) : \ \langle Ax, x' \rangle = \langle x, y' \rangle\}, \quad A'x' := y'.$$

We prove the following assertion which is already mentioned as an exercise in [52, Chapter I, Sect. 2.5] and can also be found in [116, Thm. 1.2.3].

Proposition 1.10. *Let $(T(t))_{t\geq 0}$ be a strongly continuous semigroup on X. The generator $(B, D(B))$ of the adjoint semigroup $(T'(t))_{t\geq 0}$ on X' is given by $D(B) = D(A')$ and $Bx' := A'x'$.*

Proof. Firstly, fix $x' \in D(A')$. By an application of [116, Prop. 1.2.2] we obtain that for arbitrary $x \in X$ the following holds.

$$\lim_{t\to 0} \frac{1}{t} \langle x, T'(t)x' - x' \rangle = \lim_{t\to 0} \frac{1}{t} \left\langle x, A' \int_0^t T'(s)x' \, \mathrm{d}s \right\rangle = \lim_{t\to 0} \frac{1}{t} \int_0^t \langle x, T'(s)A'x' \rangle \, \mathrm{d}s = \langle x, A'x' \rangle.$$

Hence the limit $\lim_{t\to 0} \frac{1}{t}(T'(t)x' - x')$ exists with respect to the weak*-topology and is equal to $A'x'$. We conclude that $x' \in D(B)$ and $Bx' = A'x'$. This shows that $A' \subseteq B$. For the converse fix $x' \in D(B)$, let $x \in D(A)$ be arbitrary and take notice of the following equality

$$\langle x, Bx' \rangle = \lim_{t\to 0} \frac{1}{t} \langle x, T'(t)x' - x' \rangle = \langle Ax, x' \rangle.$$

This shows that $x' \in D(A')$ and $A'x' = Bx'$, showing that $B \subseteq A'$. □

§ 1.4 The space of strong continuity

The following statement relates bi-continuous semigroups and C_0-semigroups. In particular, by [78, Thm. 1.18(d)] we can always restrict a given bi-continuous semigroup to a large subspace of X_0 which is invariant under the semigroup and cause by restriction a strongly continuous semigroup on this subspace.

Proposition 1.11. *Let $(T(t))_{t\geq 0}$ be a bi-continuous semigroup on X_0 with generator $(A, D(A))$. Then $\underline{X}_0 := \overline{D(A)}^{\|\cdot\|}$ is invariant under $(T(t))_{t\geq 0}$. Moreover, the restriction $(\underline{T}(t))_{t\geq 0} := (T(t)_{|\underline{X}_0})_{t\geq 0}$ is a strongly continuous semigroup on $\underline{X}_0$ generated by $(\underline{A}, D(\underline{A}))$, the part of A in $\underline{X}_0$.*

This property will be very useful in Chapter 2 when we discuss extrapolation spaces. There we also show that the space $\underline{X}_0$ from Proposition 1.11 coincides with the space of strong continuity, i.e., $\underline{X}_0$ consists of all $x \in X_0$ such that the map $t \mapsto T(t)x$ is strongly continuous with respect to the norm topology on X_0, cf. Lemma 2.31. We will take this result here for granted, since we prove this result later on in Chapter 2 in a more general setting, and provide the spaces of strong continuity $\underline{X}_0$ for our reccurent examples from this chapter.

1.4.1 The left-translation semigroup on $\mathrm{C_b}(\mathbb{R})$

The following result determines the space of strong continuity for this semigroup.

Lemma 1.12. *The space of strong continuity for the left-translation semigroup is given by $\underline{X}_0 = \mathrm{UC_b}(\mathbb{R})$, the space of bounded uniformly continuous functions.*

Proof. It is not hard to see that the left-translation semigroup is strongly continuous for $f \in \mathrm{UC_b}(\mathbb{R})$. Indeed $T(t)f$ converges to f with respect to the supremum-norm for $t \to 0$ if and only if $T(t)f$ converges uniformly to f in $\mathbb{R}$ for $t \to 0$ and this happens if and only if $\|T(t)f - f\|_\infty = \sup_{x\in\mathbb{R}} |f(x+t) - f(x)|$ vanishes for $t \to 0$. By rewriting the definition of uniform continuity we see that this condition holds. Hence $\mathrm{UC_b}(\mathbb{R}) \subseteq \underline{X}_0$. Conversely, let $f \in \underline{X}_0$, i.e., there exists a sequence $(f_n)_{n\in\mathbb{N}}$ in $\mathrm{C_b^1}(\mathbb{R})$ such that $f_n \to f$ with respect to the supremum-norm. This means that f is the uniform limit of the sequence $(f_n)_{n\in\mathbb{N}}$. Since $f_n \in \mathrm{C_b^1}(\mathbb{R})$ we know that $f_n \in \mathrm{UC_b}(\mathbb{R})$. Since uniform convergence of functions perserves uniform continuity and boundedness we conclude that for the limit $f \in \mathrm{UC_b}(\mathbb{R})$ must hold, showing that $\underline{X}_0 \subseteq \mathrm{UC_b}(\mathbb{R})$. □

Remark 1.13. There is another elementary way, which avoids Lemma 2.31, to show that the norm-closure of $\mathrm{C_b^1}(\mathbb{R})$ coincides with $\mathrm{UC_b}(\mathbb{R})$, namely by using mollifiers. As shown above, it thus remains to approximate every $f \in \mathrm{UC_b}(\mathbb{R})$ by elements of $\mathrm{C_b^1}(\mathbb{R})$, since $\mathrm{UC_b}(\mathbb{R})$ is norm-closed. To proceed define the function $\eta : \mathbb{R} \to \mathbb{R}$ by

$$\eta(x) := \begin{cases} C\mathrm{e}^{\frac{1}{|x|^2-1}}, & |x| < 1, \\ 0, & |x| \geq 1, \end{cases}$$

where $C > 0$ is a constant such that $\int_{\mathbb{R}} \eta(x)\, \mathrm{d}x = 1$. Then $\eta \in \mathrm{C}^\infty(\mathbb{R})$ and is called the *(standard) mollifier*. Moreover, for each $\varepsilon > 0$ the function

$$\eta_\varepsilon(x) := \frac{1}{\varepsilon}\eta\left(\frac{x}{\varepsilon}\right)$$

is again a C^∞-function and satisfies $\operatorname{supp}(\eta_\varepsilon) \subseteq (-\varepsilon, \varepsilon)$ as well as $\int_{\mathbb{R}} \eta_\varepsilon(x)\, \mathrm{d}x = 1$. For $n \in \mathbb{N}$ consider the function $f_n := \eta_{\frac{1}{n}} * f$. By the observation that for each $n \in \mathbb{N}$ one has $\frac{\mathrm{d}}{\mathrm{d}x}\left(\eta_{\frac{1}{n}} * f\right) = \left(\frac{\mathrm{d}}{\mathrm{d}x}\eta_{\frac{1}{n}}\right) * f$ one concludes that $\eta_{\frac{1}{n}} \in \mathrm{C}^1_{\mathrm{b}}(\mathbb{R})$ for each $n \in \mathbb{N}$. Moreover one shows that $\|f_n - f\|_\infty \to 0$ as n tends to infinity.

1.4.2 The adjoint semigroup

We know that the adjoint semigroup $(T'(t))_{t\geq 0}$ on X' of a C_0-semigroup is not strongly continuous with respect to the norm but is bi-continuous with respect to the weak*-topology. The space of strong continuity is handled by the so-called *sun dual* defined by

$$X^\odot := \left\{x' \in X' :\ \lim_{t\to 0} \|T'(t)x' - x'\| = 0\right\}.$$

This concept is e.g. discussed in [116, Sect. 1.3] and [52, Chapter II, Sect. 2.6]. It was shown that the generator $(A^\odot, D(A^\odot)$ of the corresponding strongly continuous semigroup $(T(t)^\odot)_{t\geq 0}$ on $X^\odot$ is given by the part of the generator $(A', D(A'))$ on $(T'(t))_{t\geq 0}$ in $X^\odot$, i.e.,

$$A^\odot x' = A'x', \quad D(A^\odot) = \left\{x' \in D(A') :\ A'x' \in X^\odot\right\}.$$

Since $D(A^\odot)$, as a generator of a C_0-semigroup, is dense in $X^\odot$ one directly concludes that $X^\odot = \overline{D(A')}^{\|\cdot\|}$ which by Lemma 2.31 coincides with the space of strong continuity.

§ 1.5 Well-posedness

There is a common technique to connect strongly continuous semigroups on Banach spaces, their generators and well-posedness of so-called (autonomous) abstract Cauchy problems arsing from partial differential equations. These abstract Cauchy problems are studied for example by Engel and Nagel [52, Chapter II, Sec. 6] or also by Melnikova and Filinkov [87]. For bi-continuous semigroups this concept was also studied by Farkas in [54, Sect. 4.1]. The abstract Cauchy problem for a linear operator A is given by

$$\begin{cases} \dot{u}(t) = Au(t), & t \geq 0, \\ u(0) = x \in D(A). \end{cases} \tag{ACP}$$

As an example of a abstract Cauchy problem, coming from a partial differential equation, we consider the following equation

$$\begin{cases} \dfrac{\partial w(t,x)}{\partial t} - \dfrac{\partial w(t,x)}{\partial x} = 0, & t \geq 0,\ x \in \mathbb{R}, \\ w(x,0) = f(x). \end{cases} \tag{1.5.1}$$

on the space $C_b(\mathbb{R})$ of bounded continuous functions on $\mathbb{R}$. This equation can be rewritten in the form of (ACP) as already described in the introduction, i.e., consider the Banach space $X := C_b(\mathbb{R})$ and define $u(t) := w(t, \cdot)$ to be a function with variable x. Furthermore, let $(A, D(A))$ be the operator defined by

$$A := \frac{\mathrm{d}}{\mathrm{d}x}, \quad D(A) := C_b^1(\mathbb{R}). \tag{1.5.2}$$

Then the partial differential equation (1.5.1) is of the form (ACP). We now recall the definition of well-posedness for bi-continuous semigroups, cf. [54, Def. 4.1.1], as already promised in Section 1.3.

Definition 1.14. The abstract Cauchy problem (ACP) is called *well-posed* if the following three assertions hold

1. For all $x \in D(A)$ there exists a function $u(t) := u(t, x)$ which solves (ACP) and such that $u \in B_{\mathrm{loc}}(\mathbb{R}_{\geq 0}, X_0) \cap C^1(\mathbb{R}_{\geq 0}, (X_0, \tau))$ and $u' \in B_{\mathrm{loc}}(\mathbb{R}_{\geq 0}, X_0)$, where the differentiation is understood in the vector-valued sense with respect to τ.

2. The solution of (ACP) is unique.

3. The solution u of (ACP) depends continuously on $x \in D(A)$, i.e. if $(x_n)_{n\in\mathbb{N}}$ is a norm-bounded τ-null sequence in $D(A)$, then the solutions $u_n(t) := u(t, x_n)$ are τ-convergent to zero uniformly on compact intervals of $\mathbb{R}_{\geq 0}$.

The following result, due to B. Farkas [54, Thm. 4.1.2], now relates generators of bi-continuous semigroups to well-posedness of abstract Cauchy problems. Observe that the previous definition and the following theorem already contains the well-posedness result (at least in one direction) for C_0-semigroups, cf. [52, Chapter II, Thm. 6.7].

Theorem 1.15. *If $(A, D(A))$ generates a bi-continuous semigroup $(T(t))_{t\geq 0}$, then the abstract Cauchy problem* (ACP) *is well-posed. In this case the function $u(t) := T(t)x$ is the solution of* (ACP).

In the case of our example the corresponding abstract Cauchy problem is well-posed since the operator $(A, D(A))$ from (1.5.2) is the generator of the left-translation semigroup on $C_b(\mathbb{R})$, see Section 1.3.1.

§ 1.6 Adjoint bi-continuous semigroups

This section is distinguished from the previous sections even if we consider again adjoint semigroups. The reason is that we now investigate adjoints of bi-continuous semigroups, instead of strongly continuous ones. In fact, the results we present in this Section are due to B. Farkas. For more details and the proofs we refer to the corresponding work [57].

Let us start with a Banach space X and a locally convex topology τ satisfying Assumption 1.1. Moreover let $(T(t))_{t\geq 0}$ be a bi-continuous semigroup on X with respect to τ. As in the case of adjoints of strongly continuous semigroups, where we considered the sun dual $X^{\odot}$, we now examine the subspace X° of the dual space X' consisting of all norm-bounded linear functionals which are τ-sequentially continuous on norm-bounded sets of X. As a matter of fact X° is a closed linear subspace of X' and hence a Banach space if equipped with the inherited norm of X'. Furthermore, X° can be equipped with the topology $\tau^\circ := \sigma(X^\circ, X)$. In order to show that τ° satisfies Assumption 1.1, we have to postulate that $X^\circ \cap \overline{B(0,1)}$ is sequentially complete with respect to $\sigma(X^\circ, X)$. We remark that this assumption in general does not follow from the general Assumptions 1.1.

Proposition 1.16. *[57, Prop. 2.3] Let $B \in \mathscr{L}(X)$ be a norm-bounded linear operator which is τ-sequentially continuous on norm-bounded sets. Then the adjoint $B' \in \mathscr{L}(X')$ leaves X° invariant.*

From the previous result we conclude that we can restrict $(T'(t))_{t\geq 0}$ to the space X°. We denote this restricted semigroup by $(T^\circ(t))_{t\geq 0}$. To conclude that $(T^\circ(t))_{t\geq 0}$ is bi-continuous on X° with respect to τ° we again have to impose a additional hypothesis. Especially, we have to assume that every norm-bounded τ°-null sequence $(\varphi_n)_{n\in\mathbb{N}}$ in X° is τ-equicontinuous on norm bounded sets.

Let us continue with some examples. Actually, we consider semigroups $(T(t))_{t\geq 0}$ on the Banach space $X := \mathrm{C_b}(\Omega)$, where Ω is a Polish space, which are bi-continuous with respect to the compact-open topology τ_{co}. In [57, Sect. 3] B. Farkas illustrated that $\mathrm{C_b}(\Omega)^\circ$ coincides with $\mathrm{M}(\Omega)$, the space of bounded Baire measures. The following results connect bi-continuous semigroups on $\mathrm{C_b}(\Omega)$ with these on $\mathrm{M}(\Omega)$.

Theorem 1.17. *[57, Thm. 3.5] Let Ω be a Polish space and $(T(t))_{t\geq 0}$ bi-continuous on $\mathrm{C_b}(\mathbb{R})$ with respect to τ_{co}. Then the semigroup $(T^\circ(t))_{t\geq 0}$ defined as $T^\circ(t) := T'(t)_{|\mathrm{M}(\Omega)}$, $t \geq 0$, is a bi-continuous semigroup on $\mathrm{M}(\Omega)$ with respect to τ°.*

Counter-intuitively the converse of the previous theorem also holds true.

Theorem 1.18. *[57, Thm. 3.6] Let Ω be a Polish space. Let $(S(t))_{t\geq 0}$ be a bi-continuous semigroup on $\mathrm{M}(\Omega)$ with respect to τ°. Then there exists a semigroup $(T(t))_{t\geq 0}$ on $\mathrm{C_b}(\mathbb{R})$ which is bi-continuous with respect to τ_{co} and such that $T^\circ(t) = S(t)$ for all $t \geq 0$*

The following result yields a characterization for the generator of adjoints of bi-continuous semigroups. The result is related to Proposition 1.10. In particular, the proof of the result is the same.

Lemma 1.19. *Let $(T(t))_{t\geq 0}$ be bi-continuous on X with respect to τ and assume that the additional hypothesis on X° and τ° hold. Let us denote the generator of $(T^\circ(t))_{t\geq 0}$ by $(A^\circ, D(A^\circ))$. Then*

$$D(A^\circ) = \{x' \in X^\circ :\ \exists y' \in X^\circ\ \forall x \in D(A) :\ \langle Ax, x'\rangle = \langle x, y'\rangle\}, \quad A^\circ x' := y'.$$

§ 1.7 Approximation theorems

In this section we recall the convergence of sequences of bi-continuous semigroups $(T_n(t))_{t\geq 0}$, since we need them later in Chapter 7. For C_0-semigroup this topic is for example treated in [52, Chapter III, Sect. 4]. The original results trace back to the papers by H. Trotter [114] and T. Kato [71]. For the bi-continuous case we need the following definition.

Definition 1.20. For each $n \in \mathbb{N}$ let $(T_n(t))_{t\geq 0}$ be a bi-continuous semigroup on a Banach space X_0 with respect to τ. We say that a sequence $(T_n(t))_{t\geq 0}$ is *uniformly bi-continuous* (of type ω) if the following conditions hold:

1. There exists $M \geq 1$ and $\omega \in \mathbb{R}$ such that $\|T_n(t)\| \leq M\mathrm{e}^{t\omega}$ for each $t \geq 0$ and $n \in \mathbb{N}$.
2. For every $t_0 > 0$ and for every $\|\cdot\|$-bounded τ-null sequence $(x_n)_{n\in\mathbb{N}}$ in X_0 we have

$$\tau\lim_{n\to\infty} T_k(t)x_n = 0,$$

   uniformly for $t \in [0, t_0]$ and $k \in \mathbb{N}$.

Next we formulate the Trotter–Kato Approximation Theorems for bi-continuous semigroups, see [78, Thm. 2.3 & Thm. 2.6], [54, Thm. 1.2.10] and [4]. Even if only Theorem 1.22 is needed in what follows we stay self-contained and also recall Theorem 1.21.

Theorem 1.21 (First Trotter–Kato Approximation Theorem)**.** *Let $(T_n(t))_{t\geq 0}$, $n \in \mathbb{N}$, and $(T(t))_{t\geq 0}$ be uniformly bi-continuous semigroups (of type ω) with generator $(A_n, D(A_n))$ and $(A, D(A))$, respectively and let D be a $\|\cdot\|$-dense subset of $\overline{D(A)}^{\|\cdot\|}$. If*

$$R(\lambda, A_n)x \xrightarrow{\|\cdot\|} R(\lambda, A)x, \quad n \to \infty,$$

for each $x \in D$ and some $\lambda > \omega$, then

$$T_n(t)x \xrightarrow{\tau} T(t)x, \quad n \to \infty,$$

for all $x \in X_0$ uniformly for t in compact intervals of $\mathbb{R}_{\geq 0}$.

Theorem 1.22 (Second Trotter–Kato Approximation Theorem)**.** *For $n \in \mathbb{N}$ let $(T_n(t))_{t\geq 0}$ be uniformly bi-continuous semigroups (of type ω) on X_0 with generators $(A_n, D(A_n))$. Consider the following assertions.*

(a) *There exists a bi-densely defined operator $(A, D(A))$ such that $A_n x \to Ax$ for all x in a bi-core of A and such that* $\mathrm{Ran}(\lambda_0 - A)$ *is bi-dense in X_0.*

(b) *There exists an operator $R \in \mathscr{L}(X_0)$ such that $R(\lambda_0, A_n)x \xrightarrow{\|\cdot\|} Rx$ for all x in a subset of* $\mathrm{Ran}(R)$ *which is bi-dense in X_0.*

(c) *There exists a bi-continuous semigroups $(T(t))_{t\geq 0}$ with generator $(B, D(B))$ such that $T_n(t)x \xrightarrow{\tau} T(t)x$ for all $x \in X_0$ uniformly for t in compact intervals*

Then the implications (a) $\Rightarrow$ (b) $\Rightarrow$ (c) *hold. In particular, if* (a) *holds, then $B = \overline{A}^{\tau}$ (the bi-closure of A).*

Remark 1.23. In the proof of [78, Thm. 2.6] one observes that operator R in assertion (b) gives rise to a pseudo-resolvent that is used to define operator $(B, D(B))$ in assertion (c).

§ 1.8 Notes

The (left)-translation semigroup on $\mathrm{C_b}(\mathbb{R})$ is going to accompany us through out this thesis. Actually, one can find more information in combination with different topics of this thesis in

Section 2.5.1 and Section 4.3. The results of Section 1.6 about adjoints of bi-continuous semigroups will play an important role in Section 5.3.2 when we consider the Gauss–Weierstrass semigroup on the space of bounded Borel measures on $\mathbb{R}$. Likewise we use the approximation theory from Section 1.7 in Chapter 7 in connection with flows on networks.

The use of bi-continuous semigroups can also be motivated by the work of H. Lotz [83]. He proved that on certain Banach spaces, every strongly continuous semigroup is uniformly continuous, i.e., the generator of the semigroups are bounded operators. Typical examples are $\mathrm{L}^\infty(\Omega)$ where (Ω, Σ, μ) is a measure space, and $\mathrm{C}(X)$ where X is either a compact σ-Stonian space or a compact F-space, just to mention a few. For this reason, strong continuity with respect to the norm is not a fruitful property on these spaces and hence another variant of strong continuity needs to be considered.

Chapter 2

Intermediate and Extrapolated Spaces

Introduction

Extrapolation spaces for generators of C_0*-semigroups* (used here synonymously to *"strongly continuous, one-parameter semigroups of bounded linear operators"*) on Banach spaces, or for more general operators, have been designed to study e.g., maximal regularity questions by Da Prato and Grisvard [37]; see also Walter [122], Amann [7], van Neerven [116], Nagel, Sinestrari [94], Nagel [92], Sinestrari [104], Magal, Ruan [86, Ch. 3]. These spaces (and the corresponding extrapolated operators) play a central role in recent abstract perturbation results, most prominently in boundary-type or domain perturbations, see e.g., Desch, Schappacher [38], Greiner [60], Staffans, Weiss [109], Adler, Bombieri, Engel [3], Hadd, Manzo, Rhandi [63]. Extrapolation spaces are also important in the theory of coupled operator matrices, see Engel [46].

In this chapter, we concentrate on the construction of extrapolation spaces for linear operators having a non-empty resolvent set on a Banach space, but we do not assume the operator to fulfill the Hille–Yosida conditions or to be densely defined. In case the operator is densely defined such a construction is known from the seminal papers of Da Prato, Grisvard, [36], Amann [7] and Nagel, Sinestrari [94]. In the case of non-densely defined, sectorial operators there is a very general—*almost purely algebraic*—construction due to Haase [62] leading also to universal extrapolation spaces. Here, we present a slightly different construction of extrapolation and extrapolated Favard spaces, allowing the construction of *extrapolated semigroups* in the absence of strong continuity with respect to the norm. For a non-densely defined Hille–Yosida operator A on the Banach space X_0 such a construction is possible by taking the part of A in $\underline{X}_0 := \overline{D(A)}$, so that the restricted operator becomes the generator of a C_0-semigroup on $\underline{X}_0$, thus leading to an extrapolated semigroup on the extrapolation space $\underline{X}_{-1}$, see Nagel, Sinestrari [95]. But this semigroup will usually not leave the original Banach space X_0 invariant. This is why we restrict our attention to the situation where strong continuity of the semigroup is guaranteed with respect to some coarser locally convex topology τ on X_0. Here the framework of bi-continuous semigroups, or that of Saks spaces, (see Kühnemund [79] and Section 2.4 below) appears to be adequate. However, most of the results presented here are valid also for generators of other classes of semigroups: integrable semigroups of Kunze [80], "C-class" semigroups of Kraaij [75], π-semigroups of Priola [100], weakly continuous semigroups of Cerrai [32], to mention a few.

Given a Banach space X_0 and a Hausdorff locally convex topology τ on X_0 (with certain properties described in Section 2.4), and a bi-continuous semigroup $(T(t))_{t\geq 0}$ with generator A, we construct the full scale of abstract Sobolev (or Hölder) and Favard spaces X_α, $\underline{X}_\alpha$, F_α for $\alpha \in \mathbb{R}$, and the corresponding extrapolated semigroups $(T_\alpha(t))_{t\geq 0}$. (If τ is the norm topology, there is nothing new here, and everything can be found in [52, Section II.5].) These constructions, along with some applications, form the main content of this chapter. Here we illustrate the results on the following well-known example (see also Nagel, Nickel, Romanelli [93] and Section 2.5 for details): Consider the Banach space $X_0 := \mathrm{C_b}(\mathbb{R})$ of bounded, continuous functions and the (left)-translation semigroup $(S(t))_{t\geq 0}$ thereon, defined by $(S(t)f)(x) = f(x+t)$, $x \in \mathbb{R}$, $t \geq 0$, $f \in X_0$. For $\alpha \in (0,1)$ we have the continuous embeddings

$$\mathrm{C}^1_\mathrm{b}(\mathbb{R}) \hookrightarrow \mathrm{Lip_b}(\mathbb{R}) \hookrightarrow \mathrm{h}^\alpha_\mathrm{b}(\mathbb{R}) \hookrightarrow \mathrm{h}^\alpha_\mathrm{b,loc}(\mathbb{R}) \hookrightarrow \mathrm{C}^\alpha_\mathrm{b}(\mathbb{R}) \hookrightarrow \mathrm{UC_b}(\mathbb{R}) \hookrightarrow \mathrm{C_b}(\mathbb{R}) \hookrightarrow \mathrm{L}^\infty(\mathbb{R}),$$

where $\mathrm{C}^1_\mathrm{b}(\mathbb{R})$ is the space of differentiable functions with derivative in $\mathrm{C_b}(\mathbb{R})$, $\mathrm{Lip_b}(\mathbb{R})$ is the space of bounded, Lipschitz functions, $\mathrm{h}^\alpha_\mathrm{b}(\mathbb{R})$ is the space of bounded, little-Hölder continuous functions, $\mathrm{h}^\alpha_\mathrm{b,loc}(\mathbb{R})$ is the space of bounded, locally little-Hölder continuous functions, $\mathrm{C}^\alpha_\mathrm{b}(\mathbb{R})$ is the space of bounded, Hölder continuous functions, $\mathrm{UC_b}(\mathbb{R})$ is the space of bounded, uniformly continuous functions. In the abstract perspective and using the notation in this chapter, this corresponds to the inclusions of Banach spaces:

$$X_1 \hookrightarrow F_1 \hookrightarrow \underline{X}_\alpha \hookrightarrow X_\alpha \hookrightarrow F_\alpha \hookrightarrow \underline{X}_0 \hookrightarrow X_0 \hookrightarrow F_0.$$

The extension of the previous diagram for the full scale $\alpha \in \mathbb{R}$ is possible by extrapolation. The (abstract) spaces $\underline{X}_\alpha$ and F_α ($\alpha \in (0,1)$) are well studied and we refer to the books by Lunardi [85] and Engel, Nagel [52, Section II.5] for a systematic treatment. However, the definition of X_α is new, and requires a recollection of results concerning the other spaces, $\underline{X}_\alpha$ and F_α.

Extrapolated Favard spaces are not only important for perturbation theory. They help to reduce problems concerning semigroups being not strongly continuous to the study of an underlying C_0-semigroup. This perspective is propagated by Nagel and Sinestrari in [95]: To any Hille–Yosida operator on X_0 one can construct a Banach space F_0 (the Favard class) containing X_0 as a closed subspace, and a semigroup $(T(t))_{t\geq 0}$ on F_0. (Note, however, that the semigroup $(T(t))_{t\geq 0}$ defined on F_0 may not leave X_0 invariant.) We adapt this point of view also in this chapter. In particular, we provide an alternative (and short) proof of the Hille–Yosida type generation theorem for bi-continuous semigroups (due to Kühnemund [79]) by employing solely the C_0-theory.

Applications of the Sobolev (Hölder) scale, as presented here, to perturbation theory, in the spirit of the results of Desch, Schappacher [38], or of Jacob, Wegner, Wintermayr [70], will be presented in Chapter 4.

This chapter is organized as follows: In Section 2.1 we recall the standard constructions and results for extrapolation spaces for densely defined (invertible) operators. Moreover, we construct extrapolation spaces for not densely defined operators A with $D(A^2)$ dense in $D(A)$ for the norm of X_0. Our argument differs form the one in Haase [62] in that we build the space X_{-1} based on $\underline{X}_{-2}$ (which, in turn, arises from $\underline{X}_0$ and $\underline{X}_{-1}$), i.e., in a bottom-to-top and then back-to-bottom manner, resulting in the continuous inclusions

$$\underline{X}_0 \hookrightarrow X_0 \hookrightarrow \underline{X}_{-1} \hookrightarrow X_{-1} \hookrightarrow \underline{X}_{-2}.$$

(None of these inclusions is surjective in general.) This approach becomes convenient when we compare the arising extrapolation spaces X_{-1} and $\underline{X}_{-1}$ and construct the extrapolated semigroups thereon. In Section 2.2 we turn to intermediate spaces; the results there are classical, but are put in the general perspective of this chapter. We also present a method for a "concrete" representation of extrapolation spaces (see Theorem 2.16). Section 2.3 discusses the Sobolev (Hölder) scale for semigroup generators. In Section 2.4 we recall the concept of bi-continuous semigroups, construct the corresponding extrapolated semigroups and give a direct proof of the Hille–Yosida generation theorem (due to Kühnemund, see [79]) using extrapolation techniques. We conclude this chapter with some examples in Section 2.5, where we determine the extrapolation spaces of concrete semigroup generators. In particular, we discuss the previously mentioned example of the translation semigroup (complementing results of Nagel, Nickel, Romanelli [93, Sec. 3.1, 3.2]) and then *left implemented semigroups* (cf. Alber [6]).

§ 2.1 Sobolev and extrapolation spaces for invertible operators

In this section we construct abstract Sobolev (Hölder) and extrapolation spaces (the so-called Sobolev scale) for a boundedly invertible linear operator defined on a Banach space. Some of the results are well-known and even standard, but we chose to include them here for the sake of completeness, and because they are needed for the construction of spaces when we deal with not densely defined operators. The emphasis will be, however, on this latter case, when the construction is new, see Section 2.1.2 below. We also note that everything what follows is also valid for operators on Fréchet spaces.

The following is a standing assumption in this chapter.

Assumption A. We suppose that $A : D(A) \to X_0$ is a (not necessarily densely defined) linear operator on a Banach space X_0 with 0 in the resolvent set $\rho(A)$.

As a matter of fact, it is only for convenience to suppose $0 \in \rho(A)$ instead of $\rho(A) \neq \emptyset$. Indeed, if $\lambda \in \rho(A)$ we may consider $A - \lambda$ and carry out the constructions for this new operator satisfying $0 \in \rho(A-\lambda)$. The arising spaces will not depend on $\lambda \in \rho(A)$ (up to isomorphism).

2.1.1 Abstract Sobolev spaces

The material presented here is standard, see Nagel [91], Nagel, Nickel, Romanelli [93] or Engel, Nagel [52, Section II.5], and some parts are valid even for operators on locally convex spaces, when one has to argue with a family of generating seminorms instead of one norm. We set $X_1 := D(A)$ which becomes a Banach space if endowed with the graph norm

$$\|x\|_A := \|x\| + \|Ax\|.$$

An equivalent norm is given by $\|x\|_{X_1} := \|Ax\|$ since we have assumed $0 \in \rho(A)$. Then we have the isometric isomorphism

$$A : X_1 \to X_0 \quad \text{with inverse} \quad A^{-1} : X_0 \to X_1.$$

Definition 2.1. Recall the assumption that $0 \in \rho(A)$, and take $n \in \mathbb{N}$, $n \geq 1$.

1. We define
$$X_n := D(A^n) \quad \text{and} \quad \|x\|_{X_n} := \|A^n x\| \quad \text{for } x \in X_n.$$
If we want to stress the dependence on A, then we write $X_n(A)$ and $\|\cdot\|_{X_n(A)}$.

2. Let
$$X_\infty(A) := \bigcap_{n\in\mathbb{N}} X_n,$$
often abbreviated as X_∞.

3. We further set
$$\underline{X}_0 := \overline{D(A)}, \quad \underline{A} := A|_{\underline{X}_0},$$
the part of A in $\underline{X}_0$, i.e.,
$$D(\underline{A}) = \{x \in D(A) : Ax \in \underline{X}_0\}.$$
Moreover, we let
$$\underline{X}_n := D(\underline{A}^n), \quad \|x\|_{\underline{X}_n} := \|\underline{A}^n x\|.$$
To be specific about the underlying operator A we write $\underline{X}_n(A)$ and $\|x\|_{\underline{X}_n(A)}$.

4. For $n \in \mathbb{N}$ we set $A_n := A|_{X_n}$, the part of A in X_n, in particular $A_0 = A$. Similarly, we let $\underline{A}_n := \underline{A}|_{\underline{X}_n}$, for example $\underline{A}_0 = \underline{A}$. By this notation we also understand implicitly that the surrounding space is $X_n(A)$ respectively $\underline{X}_n(A)$ with its norm, see Remark 2.2.

Remark 2.2. (i) By "underlining" we always indicate an object which is in some sense smaller than the one without underlining. The space $\underline{X}_0(A)$ is connected with the domain of $D(A)$, and the whole issue of distinguishing between X_0 and $\underline{X}_0$ becomes relevant only if A is not densely defined but its part $\underline{A}$ *is* (cf. Remark 2.6). We keep to the notation $\underline{A}$ for the part of the operator A instead of $A|_{\underline{X}_0}$.

(ii) If A is densely defined, then $X_n(A) = \underline{X}_n(A)$ for each $n \in \mathbb{N}$. In particular, if $\underline{X}_1(A) = D(\underline{A})$ is dense in $\underline{X}_0(A)$, then $\underline{X}_n(A) = \underline{X}_n(\underline{A})$ for each $n \in \mathbb{N}$.

(iii) For $n \in \mathbb{N}$ we evidently have $X_1(A^n) = X_n(A)$. Also $\underline{X}_1(A^n) = \underline{X}_n(A)$ holds, because $D(\underline{A^n}) = D(\underline{A}^n)$. Indeed, the inclusion "$D(\underline{A}^n) \subseteq D(\underline{A^n})$" is trivial. While for $x \in D(\underline{A^n})$ we have $x \in \underline{X}_0$ and $A^n x \in \underline{X}_0$, implying $A^{n-1}x \in D(\underline{A})$, and then recursively $x \in D(\underline{A}^n)$.

(iv) For $x \in D(A_n) = D(A^{n+1})$ we have $\|x\|_{X_1(A_n)} = \|A_n x\|_{X_n(A)} = \|A^{n+1}x\| = \|x\|_{X_{n+1}(A)}$. Similarly $D(\underline{A}_n) = D(\underline{A}^{n+1})$.

In order to prove our next result we recall that a projective system $(E_n, \theta_n)_{n\in\mathbb{N}}$ consists of a countable family of sets $(E_n)_{n\in\mathbb{N}}$ and maps $\theta_n : E_{n+1} \to E_n$. The corresponding projective limit, denoted by $\varprojlim_{n\in\mathbb{N}}(E_n,\theta_n)$, is the set of all elements $x = (x_n)_{n\in\mathbb{N}}$ of the cartesian product $\prod_{n\in\mathbb{N}} E_n$ satisfying $x_n = \theta_n(x_{n+1})$ for each $n \in \mathbb{N}$. With π_m we denote the m-th coordinate projection $\pi_m : \prod_{n\in\mathbb{N}} E_n \to E_m$. The following Mittag–Leffler type result is due to J. Esterle.

Theorem 2.3. *[53, Cor. 2.2] Let $(E_n,\theta_n)_{n\in\mathbb{N}}$ be a projective system, where E_n is a complete metric space and $\theta_n : E_{n+1} \to E_n$ continuous for each $n \in \mathbb{N}$. If $\theta_n(E_{n+1})$ is dense in E_n for each $n \in \mathbb{N}$, then $\pi_m\left(\varprojlim_{n\in\mathbb{N}}(E_n,\theta_n)\right)$ is dense in E_m for each $m \in \mathbb{N}$*

One can the statement of this theorem reformulate for our purposes as follows: For all $x \in E_1$ and all $\varepsilon > 0$ there exists $x_n \in E_n$ $(n \in \mathbb{N})$ such that $x_n = \theta_n(x_{n+1})$ and $d_1(x, x_1) < \varepsilon$, where d_1 denotes the metric on E_1.

Proposition 2.4. *Suppose $\underline{A}$ is densely defined in $\underline{X}_0$.*

(a) *For $n \in \mathbb{N}$ the mappings $A^n : X_n \to X_0$ and $\underline{A}^n : \underline{X}_n \to \underline{X}_0$ are isometric isomorphisms.*

(b) *For $n \in \mathbb{N}$ the operators $A_n : X_{n+1} \to X_n$ and $\underline{A}_n : \underline{X}_{n+1} \to \underline{X}_n$ are isometric isomorphisms that intertwine A_{n+1} and A_n, respectively, $\underline{A}_{n+1}$ and $\underline{A}_n$.*

(c) *If $D(\underline{A})$ is dense in $\underline{X}_0$, then X_∞ is dense in $\underline{X}_n$ for each $n \in \mathbb{N}$. As a consequence, $\underline{X}_m$ is dense in $\underline{X}_n$ for each $m, n \in \mathbb{N}$ with $m \geq n$.*

Part (a) and (b) of Proposition 2.4 are trivial by construction. In order to prove the third statement We use the statement of Theorem 2.3 to prove part (c) of Proposition 2.4, which was originally proven by Arendt, El-Mennaoui and Kéyantuo [9, Thm. 6.2].

Proof of Proposition 2.4(c)**.** Owing to the fact that $\underline{A}$ is densely defined in $\underline{X}_0$ we are able to use Theorem 2.3. The space $D(\underline{A}^n)$ equipped with the norm $\|\cdot\|_{\underline{X}_n}$ is a Banach space. Let $\theta_n : D(\underline{A}^{n+1}) \hookrightarrow D(\underline{A}^n)$ be the natural (continuous) inclusion. For $\lambda \in \rho(\underline{A})$ is $(\lambda - \underline{A})^n$ an isomorphism from $D(\underline{A}^n)$ to $\underline{X}_0$ which maps $D(\underline{A}^{n+1})$ onto $D(\underline{A})$. Since $D(\underline{A})$ was supposed to be dense in $\underline{X}_0$ also $D(\underline{A}^{n+1})$ is dense in $D(\underline{A}^n)$. Let $x \in \underline{X}_0$ and $\varepsilon > 0$ be arbitrary. By Theorem 2.3 there exists $x_n \in D(\underline{A}^n)$ for each $n \in \mathbb{N}$ such that $x_n = \theta_n(x_{n+1})$ and $\|x - x_1\| < \varepsilon$. Thus $x_n = x_1$ for each $n \in \mathbb{N}$ and hence $x_1 \in X_\infty$. The last part of the statement follows by replacing X by $D(\underline{A}^m)$ and $\underline{A}$ by the part of $\underline{A}$ in $D(\underline{A}^m)$. □

Remark 2.5. We note that the Mittag–Leffler type result Theorem 2.3 is valid in complete metric spaces. Hence the statements (a), (b) and (c) are all remain true for Fréchet spaces with verbatim the same proof as in [9].

Henceforth, another standing assumption will be the following (though not everywhere needed).

Assumption B. The operator $\underline{A} := A|_{\underline{X}_0} : D(\underline{A}) \to \underline{X}_0$ is densely defined, i.e.,

$$\overline{D(\underline{A})} = \underline{X}_0.$$

Remark 2.6. The condition of $D(\underline{A})$ being dense in $\underline{X}_0$ holds for example if there are $M, \omega > 0$ such that $(\omega, \infty) \subseteq \rho(A)$ and

$$\|\lambda R(\lambda, A)\| \leq M \quad \text{for all } \lambda > \omega. \tag{2.1.1}$$

Indeed, in this case we have for $x \in D(A)$

$$\|\lambda R(\lambda, A)x - x\| = \|R(\lambda, A)Ax\| \leq \frac{M\|Ax\|}{\lambda} \to 0 \quad \text{for } \lambda \to \infty.$$

Hence $D(A^2) \subseteq D(\underline{A})$ is dense in $D(A)$ for the norm of X_0, and this implies the density of $D(\underline{A})$ in $\underline{X}_0$. An operator A satisfying (2.1.1) is often said to have a *ray of minimal growth*, see, e.g., [85, Chapter 3], and also Section 2.2 below. Another term used is *"weak Hille–Yosida operator"*.

Proposition 2.7. *If $T \in \mathscr{L}(X_0)$ is a linear operator commuting with A^{-1}, then the spaces X_n and $\underline{X}_n$ are T-invariant, and $T \in \mathscr{L}(X_n)$ for $n \in \mathbb{N}$.*

Proof. The condition means that $Tx \in D(A)$ for each $x \in D(A)$ and for such x we have $ATx = TAx$. This implies the invariance of X_1 and that $\|Tx\|_{X_1(A)} \leq \|T\|\|x\|_{X_1(A)}$. Using the boundedness assumption we see that $\underline{X}_1$ remains invariant under T. For general $n \in \mathbb{N}$ we may argue by recursion, or simply invoke Remark 2.2.

□

2.1.2 Extrapolation spaces

The construction for the extrapolation spaces here is standard if A is densely defined, or if A is a Hille–Yosida operator, see, e.g., [95]. For $x \in X_0$ we define $\|x\|_{\underline{X}_{-1}(A)} := \|A^{-1}x\|$. Then the surjective mapping

$$A : (D(A), \|\cdot\|) \to (X_0, \|\cdot\|_{\underline{X}_{-1}(A)})$$

becomes isometric, and hence has a uniquely continuous extension

$$\underline{A}_{-1} : (\underline{X}_0, \|\cdot\|) \to (\underline{X}_{-1}, \|\cdot\|_{\underline{X}_{-1}(\underline{A})}),$$

which is an isometric isomorphism, where $(\underline{X}_{-1}, \|\cdot\|_{\underline{X}_{-1}(\underline{A})})$ denotes the completion of $(\underline{X}_0, \|\cdot\|_{\underline{X}_{-1}(A)})$. By construction we obtain immediately:

Proposition 2.8. *The space X_0 is continuously and densely embedded in $\underline{X}_{-1}$. If $\underline{A}$ is densely defined in $\underline{X}_0$, then also X_∞ is dense in $\underline{X}_{-1}$. As a consequence $(\underline{X}_{-1}, \|\cdot\|_{\underline{X}_{-1}(\underline{A})})$ is the completion of $(\underline{X}_0, \|\underline{A}^{-1} \cdot\|)$.*

Proof. The space X_0 is dense in $\underline{X}_{-1}$ by construction. For $x \in X_0$ we have

$$\|x\|_{\underline{X}_{-1}(\underline{A})} = \|AA^{-1}x\|_{\underline{X}_{-1}(\underline{A})} = \|\underline{A}_{-1}A^{-1}x\|_{\underline{X}_{-1}(A)} \leq \|\underline{A}_{-1}\| \cdot \|A^{-1}x\| \leq \|\underline{A}_{-1}\| \cdot \|A^{-1}\| \cdot \|x\|,$$

showing the continuity of the embedding. The last assertion follows since X_∞ is dense in $D(A)$ with respect to $\|\cdot\|$.

□

Of course one can iterate the whole procedure and obtain the following chain of dense and continuous embeddings

$$\underline{X}_0 \hookrightarrow \underline{X}_{-1} \hookrightarrow \underline{X}_{-2} \hookrightarrow \cdots \hookrightarrow \underline{X}_{-n} \quad \text{for } n \in \mathbb{N},$$

where for $n \geq 1$ the space $\underline{X}_{-n}$ is a completion of $\underline{X}_{-n+1}$ with respect to the norm $\|\cdot\|_{\underline{X}_{-n}(\underline{A})}$ defined by $\|x\|_{\underline{X}_{-n}(\underline{A})} = \|\underline{A}_{-n+1}^{-1}x\|_{\underline{X}_{-n+1}(\underline{A})}$ and

$$\underline{A}_{-n} : \underline{X}_{-n+1} \to \underline{X}_{-n}$$

is a unique continuous extension of $\underline{A}_{-n+1} : D(\underline{A}_{-n+1}) \to \underline{X}_{-n+1}$ to $\underline{X}_{-n}$.

These spaces, just as well the ones in the next definition, are called *extrapolation spaces* for the operator A, see, e.g., [95] or [52, Section II.5] for the case of semigroup generators. The spaces $\underline{X}_{-1}$, $\underline{X}_{-2}$ and the operator $\underline{A}_{-2}$ will be used to define the extrapolation space $X_{-1}(A)$. To this purpose we identify X_0 with a subspace of $\underline{X}_{-1}$ and of $\underline{X}_{-2}$.

Definition 2.9. Consider X_0 as a subspace of $\underline{X}_{-2}$, and define

$$X_{-1} := \underline{A}_{-2}(X_0) := \{\underline{A}_{-2}x : \ x \in X_0\} \quad \text{and} \quad \|x\|_{X_{-1}} := \|\underline{A}_{-2}^{-1}x\|.$$

Furthermore, we set $D(A_{-1}) := X_0$ and for $x \in X_0$ we define $A_{-1}x := \underline{A}_{-2}x$. To note the dependence on the operator A we write $X_{-1}(A)$ and $\|\cdot\|_{X_{-1}(A)}$.

Remark 2.10. It is easy to see that the operator A_{-1} is the part of $\underline{A}_{-2}$ in X_{-1}.

In what follows, we will define higher order extrapolation spaces and prove that all these spaces line up in a scale, where one can switch between the levels with the help of (a version) of the operator A (or A_{-1}).

Proposition 2.11. *The operator A_{-1} is an extension of $\underline{A}_{-1}$, $(X_{-1}, \|\cdot\|_{X_{-1}})$ is a Banach space, the norms of $\underline{X}_{-1}$ and X_{-1} coincide on $\underline{X}_{-1}$, and $\underline{X}_{-1}$ is a closed subspace of X_{-1}. The mapping $A_{-1} : X_0 \to X_{-1}$ is an isometric isomorphism.*

Proof. The first assertion is true because $\underline{A}_{-2}$ is an extension of $\underline{A}_{-1}$. That X_{-1} is a Banach space is immediate from the definition. Since $\underline{A}_{-2}^{-1}\underline{A}_{-1} = I$ on $\underline{X}_0$, we have $\underline{A}_{-1}^{-1}x \in \underline{X}_0 \subseteq X_0$ for $x \in \underline{X}_{-1}$, so that $\|\underline{A}_{-2}^{-1}x\| = \|\underline{A}_{-2}^{-1}\underline{A}_{-1}\underline{A}_{-1}^{-1}x\| = \|\underline{A}_{-1}^{-1}x\| = \|x\|_{\underline{X}_{-1}}$. This establishes that the norms coincide. Since $\underline{X}_{-1}$ is a Banach space (with its own norm), it is a closed subspace of X_{-1}. That A_{-1} is an isometric isomorphism follows from the definition.

□

Remark 2.12. By construction we have $\underline{X}_{-1}(\underline{A}_{-n}) = \underline{X}_{-(n+1)}(\underline{A})$ as well as $X_{-1}(A_{-n}) = X_{-(n+1)}(A)$ for each $n \in \mathbb{N}$.

Proposition 2.13. *For $n \in \mathbb{Z}$ the operators $A_n : X_{n+1} \to X_n$ and $\underline{A}_n : \underline{X}_{n+1} \to \underline{X}_n$ are isometric isomorphisms that intertwine A_{n+1} and A_n, respectively, $\underline{A}_{n+1}$ and $\underline{A}_n$.*

Proof. For $n \in \mathbb{N}$ this is Proposition 2.13. So we assume $n < 0$. For $n = -1$ the statement about isometric isomorphisms is just the definition, and the intertwining property is also evident. By recursion we obtain the validity of the assertion for general $n \leq -1$ and for the operator $\underline{A}_n$. By Remark 2.12 it suffices to prove that A_{-1} intertwines A_{-1} and $A_0 = A$. For $x \in D(A_0) = D(A)$ we have $A_{-1}x \in X_0 = D(A_{-1})$ and $Ax = A_{-1}^{-1}A_{-1}A_{-1}x$.

□

Thus for $n \in \mathbb{N}$ we have the following chain of embeddings (continuous and dense, denoted by $\hookrightarrow$) and inclusions as closed subspaces (denoted by $\subseteq$):

$$\cdots \hookrightarrow \underline{X}_n \subseteq X_n \hookrightarrow \underline{X}_0 \subseteq X_0 \hookrightarrow \underline{X}_{-1} \subseteq X_{-1} \hookrightarrow \underline{X}_{-2} \subseteq X_{-2} \hookrightarrow \cdots \underline{X}_{-n} \subseteq X_{-n} \hookrightarrow \cdots ,$$

where in general the inclusions are strict (see the examples in Section 2.5). We also have the following chain of isometric isomorphisms

$$\cdots \longrightarrow \underline{X}_{n+1} \xrightarrow{\underline{A}_n^{-1}} \underline{X}_n \longrightarrow \cdots \longrightarrow \underline{X}_1 \xrightarrow{\underline{A}_0^{-1}} \underline{X}_0 \xrightarrow{\underline{A}_{-1}^{-1}} \underline{X}_{-1} \longrightarrow \cdots \longrightarrow \underline{X}_{-n+1} \xrightarrow{\underline{A}_{-n}^{-1}} \underline{X}_{-n} \longrightarrow \cdots$$

and

$$\cdots \longrightarrow X_{n+1} \xrightarrow{A_n^{-1}} X_n \longrightarrow \cdots \longrightarrow X_1 \xrightarrow{A_0^{-1}} X_0 \xrightarrow{A_{-1}^{-1}} X_{-1} \longrightarrow \cdots \longrightarrow X_{-n+1} \xrightarrow{A_{-n}^{-1}} X_{-n} \longrightarrow \cdots .$$

Proposition 2.14. (a) $\underline{X}_1(\underline{A}_{-1}) = \underline{X}_0$ *and* $X_1(A_{-1}) = X_0$ *with the same norms.*

(b) $\underline{X}_{-1}(\underline{A}_1) = \underline{X}_0$ *with the same norms.*

(c) $(\underline{A}_1)_{-1} = \underline{A}$.

(d) $X_{-1}(A_1) = X_0$ *with the same norms, and* $(A_1)_{-1} = A$.

Proof. (a) By definition $X_1(A_{-1}) = D(A_{-1}) = X_0$ with the graph norm of A_{-1}. Since A_{-1} extends A, for $x \in X_0$ we have $\|A_{-1}x\|_{X_{-1}(A)} = \|Ax\|_{X_{-1}} = \|A^{-1}Ax\| = \|x\|$. The first statement then follows, because $\underline{X}_1(\underline{A}_{-1}) = X_1(\underline{A}_{-1}) = \overline{D(\underline{A})} = \underline{X}_0$ with the same norms.

(b) For $x \in \underline{X}_1(\underline{A}) = D(\underline{A}^2)$ we have

$$\|x\|_{\underline{X}_{-1}(\underline{A}_1)} = \|\underline{A}_1^{-1}x\|_{\underline{X}_1(\underline{A})} = \|\underline{A}\,\underline{A}_1^{-1}x\| = \|x\|,$$

which can be extended by density for all $x \in \underline{X}_0$, showing the equality of the spaces $\underline{X}_{-1}(\underline{A}_1) = \underline{X}_0$ (with the same norm).

(c) By construction the operator $(\underline{A}_1)_{-1} : \underline{X}_1(A) \to \underline{X}_{-1}(\underline{A}_1)$ is the unique continuous extension of

$$\underline{A}_1 : D(\underline{A}_1) = D(\underline{A}^2) \to \underline{X}_1(A),$$

and $(\underline{A}_1)_{-1}$ is an isometric isomorphism. For $x \in \underline{X}_1(A)$ we have $\|x\|_{\underline{X}_{-1}(A_1)} = \|\underline{A}_1^{-1}x\|_{\underline{X}_1(A)} = \|x\|$. But then it follows that $(\underline{A}_1)_{-1} = \underline{A} : D(\underline{A}) \to \underline{X}_0$.

(d) The space $X_{-1}(A_1)$ is defined by

$$X_{-1}(A_1) := (\underline{A_1})_{-2}(X_1(A)) = ((\underline{A_1})_{-1})_{\underline{-1}}(X_1(A)) = \underline{A}_{-1}(X_1(A)) = AX_1(A) = X_0,$$

by part (c). For the norm equality let $x \in X_0$. Then

$$\|x\| = \|AA^{-1}x\| = \|A^{-1}x\|_{X_1(A)} = \|\underline{A}_{-1}^{-1}x\|_{X_1(A)} = \|(\underline{A}_1)_{-2}^{-1}x\|_{X_1(A)} = \|x\|_{X_{-1}(A_1)}.$$

For the last assertion we note: $(A_1)_{-1} = (\underline{A}_1)_{-2}|_{X_1(A)} = A$.

□

Recall the standing assumption that $\underline{A} = A|_{\underline{X}_0}$ is densely defined in $\underline{X}_0 = \overline{D(A)}$. The following proposition plays the key role for the extension of operators to the extrapolation spaces, particularly for the construction of extrapolated semigroups in Section 2.3.

Proposition 2.15. (a) *Let* $n \in \mathbb{N}$. *If* $T \in \mathscr{L}(X_0)$ *is a linear operator commuting with* A^{-1}, *then the operator* T *has a unique continuous extension to* $\underline{X}_{-n}$ *denoted by* $\underline{T}_{-n}$. *The* *operator $\underline{T}_{-n}$ is the restriction of $\underline{T}_{-n-1}$. The space X_{-n} is invariant under $\underline{T}_{-n-1}$, whose restriction is denoted by T_{-n}, for which $T_{-n} \in \mathscr{L}(X_{-n})$. For $k, n \in -\mathbb{N}$ the operators $\underline{T}_n$, $\underline{T}_k$ are all similar; the same holds for T_n and T_k.*

(b) *Let $\underline{T} \in \mathscr{L}(\underline{X}_0)$ such that it leaves $D(A)$ invariant and commutes with $\underline{A}^{-1} = A^{-1}|_{X_0}$. Then $\underline{T}_{-1}x = A\underline{T}A^{-1}x$ for each $x \in X_0$, and as a consequence, $\underline{T}_{-1} : \underline{X}_{-1} \to \underline{X}_{-1}$ leaves X_0 invariant (and, of course, extends $\underline{T}$).*

Proof. (a) For $x \in X_0$ we have

$$\|Tx\|_{X_{-1}(A)} = \|A^{-1}Tx\| = \|TA^{-1}x\| \leq \|T\| \cdot \|A^{-1}x\| = \|T\| \cdot \|x\|_{X_{-1}(A)}.$$

Therefore $T : (X_0, \|\cdot\|_{X_{-1}(A)}) \to (X_0, \|\cdot\|_{X_{-1}(A)})$ is continuous, and hence has a unique continuous extension $\underline{T}_{-1}$ to $\underline{X}_{-1}$. This extension commutes with $\underline{A}_{-1}^{-1}$, because T commutes with A^{-1} and $\underline{A}_{-1}^{-1}$ is the unique continuous extension of A^{-1}. By iteration we obtain the continuous extensions $\underline{T}_{-n}$ onto $\underline{X}_{-n}$, which then all commute with the corresponding $\underline{A}_{-n}^{-1}$. By construction $\underline{T}_{-n}$ is a restriction of $\underline{T}_{-n-1}$. We prove that X_{-1} is invariant under $\underline{T}_{-2}$. Let $x \in X_{-1}$, hence $x = \underline{A}_{-2}y$ for some $y \in X_0$. Then $Ty = \underline{T}_{-2}y = \underline{T}_{-2}\underline{A}_{-2}^{-1}x = \underline{A}_{-2}^{-1}\underline{T}_{-2}x$, hence $\underline{T}_{-2}x = \underline{A}_{-2}Ty \in X_{-1}$, i.e., the invariance of X_{-1} is proved. We have for $x \in X_{-1}$ that $\|T_{-1}x\|_{X_{-1}} = \|A_{-2}^{-1}T_{-1}x\| = \|A_{-2}^{-1}\underline{T}_{-2}x\| = \|\underline{T}_{-2}A_{-2}^{-1}x\| \leq \|\underline{T}_2\| \cdot \|\underline{A}_{-2}^{-1}x\| = \|\underline{T}_2\| \cdot \|x\|_{X_{-1}}$, therefore $T_{-1} \in \mathscr{L}(X_{-1})$. The assertion about T_{-n} follows by recursion.

It is enough to prove the similarity of $T_0 = T$ and T_{-1}, and the similarity of $\underline{T}_0$ and $\underline{T}_{-1}$. The latter assertions can be proved as follows: For $x \in D(A)$ we have

$$\underline{A}_{-1}^{-1}\underline{T}_{-1}\underline{A}_{-1}x = \underline{A}_{-1}^{-1}\underline{T}_{-1}Ax = \underline{A}_{-1}^{-1}TAx = \underline{A}_{-1}^{-1}ATx = \underline{A}_{-1}^{-1}A_{-1}Tx = \underline{T}x,$$

then by continuity and denseness the equality follows even for $x \in \underline{X}_0$. For the similarity of T and T_{-1} take $x \in X_0$. Then

$$A_{-1}^{-1}T_{-1}A_{-1}x = \underline{A}_{-2}^{-1}\underline{T}_{-2}\underline{A}_{-2}x = \underline{T}_{-1}x = Tx.$$

(b) Let $x \in X_0 \subseteq \underline{X}_{-1}$. Then there is a sequence (x_n) in $\underline{X}_0$ with $x_n \to x$ in $\underline{X}_{-1}$ (see Proposition 2.8). But then $A^{-1}x_n \to A^{-1}x$ in $\underline{X}_0$ and $\underline{T}x_n \to \underline{T}_{-1}x$ in $\underline{X}_{-1}$ by part (a). These imply $\underline{T}A^{-1}x_n = A^{-1}\underline{T}x_n \to \underline{A}_{-1}^{-1}\underline{T}_{-1}x$. Hence we conclude $\underline{T}A^{-1}x = \underline{A}_{-1}^{-1}\underline{T}_{-1}x$ and $A\underline{T}A^{-1}x = \underline{T}_{-1}x$ for $x \in X_0$.

□

M. Haase in [62] and S.-A. Wegner in [123] have constructed the so-called *universal extrapolation space* $X_{-\infty}$ as follows: Suppose A is densely defined (this assumption is *not* made by M. Haase), then $X_n = \underline{X}_n$ for each $n \in \mathbb{Z}$ and let $X_{-\infty}$ to be the inductive limit of the sequence of Banach spaces $(X_{-n})_{n\in\mathbb{N}}$ (algebraic inductive limit in [62]). In particular, the space $X_{-\infty}$ is algebraically defined by

$$X_{-\infty} := \varinjlim_{n\in\mathbb{N}} X_{-n} = \bigcup_{n\in\mathbb{N}} X_{-n}.$$

A topology on this space is defined by means of nets as follows: A net $(x_\iota)_{\iota\in I}$ in $X_{-\infty}$ converges to $x \in X_{-\infty}$ if and only if there exists $n \in \mathbb{N}$ and $\iota_0 \in I$ such that $x, x_\iota \in X_{-n}$ for each $\iota \geq \iota_0$ and $\|x - x_\iota\|_{X_{-n}} \to 0$. In other words, the topology on $X_{-\infty}$ is the finest topology which makes all inclusions $X_{-n} \hookrightarrow X_{-\infty}$ continuous. One can extend the operator A to an operator $A_{-\infty} : X_{-\infty} \to X_{-\infty}$ such that

$$A_{-\infty}|_{X_n} = A_n, \quad n \in \mathbb{Z}.$$

Observe that this operator is continuous with respect to the notion of convergence defined above. We now look at a converse situation, and our starting point is the following: Let $\mathscr{E}$ be a locally convex space such that we can embed the Banach space X_0 continuously in $\mathscr{E}$, i.e., there is a continuous injective map $i : X_0 \to \mathscr{E}$, and so we can identify X_0 with a subspace of $\mathscr{E}$. We also assume that we have a continuous operator $\mathcal{A} : \mathscr{E} \to \mathscr{E}$ such that $\lambda - \mathcal{A} : i(X_0) \to \mathscr{E}$ is injective and that

$$D(A) = \{x \in X_0 : \ \mathcal{A} \circ i(x) \in i(X_0)\},$$

and

$$i \circ A = \mathcal{A} \circ i|_{D(A)}.$$

In the next theorem we use this setting to describe the extrapolation spaces $\underline{X}_{-n}$, X_{-n}. Notice that we do not assume that A is a Hille–Yosida operator or densely defined.

Theorem 2.16. *Let X_0 be a Banach space with a continuous embedding $i : X_0 \to \mathscr{E}$ into a locally convex space $\mathscr{E}$, let $A : D(A) \to X_0$ be a linear operator with $\lambda \in \rho(A)$ such that $A = \mathcal{A}|_{X_0}$ (after identifying X_0 with a subspace of $\mathscr{E}$ as described above). We suppose furthermore that $\lambda - \mathcal{A}$ is injective on X_0. Then there is a continuous embedding $i_{-1} : X_{-1} \to \mathscr{E}$ which extends i. After identifying X_{-1} with a subspace of $\mathscr{E}$ (under i_{-1}) we have*

$$X_{-1} = \{(\lambda - \mathcal{A})x : \ x \in X_0\}, \quad \underline{X}_{-1} = \{(\lambda - \mathcal{A})x : \ x \in \underline{X}_0\} \quad \textit{and} \quad A_{-1} = \mathcal{A}|_{X_{-1}}.$$

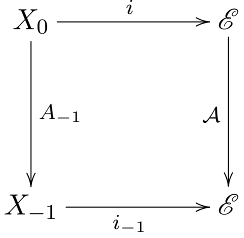
Proof. Without lost of generality we may assume that $\lambda = 0$. Recall that $A_{-1}|_{X_0} = A$ and A_{-1} is an isometric isomorphism $A_{-1} : X_0 \to X_{-1}$. We now define the embedding $i_{-1} : X_{-1} \to \mathscr{E}$ by

$$i_{-1} := \mathcal{A} \circ i \circ A_{-1}^{-1},$$

which is indeed injective and continuous by assumption. Of course, i_{-1} extends i since we have $i = \mathcal{A} \circ i \circ A^{-1}$. We can write

$$i_{-1} \circ A_{-1} = \mathcal{A} \circ i \circ A_{-1}^{-1} \circ A_{-1} = \mathcal{A} \circ i,$$

which yields the following commutative diagram:



Now all assertions follow easily.

□

The last corollary in this section can be proved by induction based on the previous facts.

Corollary 2.17. *Let $\mathcal{A}$, X_0, $\mathscr{E}$ and i be as in Theorem 2.16. Then $X_n \subseteq \mathscr{E}$ and $A_n = \mathcal{A}|_{X_n}$ for each $n \in \mathbb{Z}$ (after identifying X_n with a subspace of $\mathscr{E}$ under an embedding i_n compatible with i).*

§ 2.2 Intermediate spaces for operators with rays of minimal growth

The following definition of intermediate, and as a matter of fact interpolation spaces, just as well many results in this section are standard, and we refer, e.g., to the book by Lunardi [85, Chapter 3], and to Engel, Nagel [52, Section II.5] for the case of semigroup generators. In this section we suppose the following.

Assumption C. The operator A on the Banach space X_0 has a *ray of minimal growth*, i.e., $(0,\infty) \subseteq \rho(A)$ and for some $M \geq 0$

$$\|\lambda R(\lambda, A)\| \leq M \quad \text{for all } \lambda > 0. \tag{2.2.1}$$

Definition 2.18. For $\alpha \in (0,1]$ and $x \in X_0$ we define

$$\|x\|_{F_\alpha(A)} := \sup_{\lambda>0} \|\lambda^\alpha AR(\lambda, A)x\|,$$

and the *abstract Favard space* of order α by

$$F_\alpha(A) := \{x \in X_0 : \|x\|_{F_\alpha(A)} < \infty\}.$$

In the literature the notation $D_A(\alpha, \infty)$ is also used, see, e.g., [85]. We further set

$$F_0(A) := F_1(A_{-1}),$$

see [52, Section II.5(b)] for the case of semigroup generators.

Proposition 2.19. (a) *The Favard space $F_\alpha(A)$ becomes a Banach space if endowed with the norm $\|\cdot\|_{F_\alpha(A)}$.*

(b) *The space X_0 is isomorphic to a closed subspace of $F_0(A)$.*

The statement that X_0 is a closed subspace of $F_0(A)$ when A is a Hille–Yosida operator is due to Nagel and Sinestrari [95, Proof of Prop. 2.7].

Proof. (a) is trivial.

(b) For $x \in X_0$ we have

$$\|\lambda A_{-1}R(\lambda, A_{-1})x\|_{X_{-1}(A)} = \|\lambda AR(\lambda, A)x\|_{X_{-1}(A)} = \|\lambda A^{-1}AR(\lambda, A)x\| \leq M\|x\|,$$

yielding

$$\|x\|_{F_0(A)} = \|x\|_{F_1(A_{-1})} \leq M\|x\|.$$

On the other hand, since A and A_{-1} are similar, we have $\sup_{\lambda>0} \|\lambda R(\lambda, A_{-1})\|_{X_{-1}(A)} \leq M'$ for some $M' \geq 0$ and for all $\lambda > 0$. In particular, by Remark 2.6, $\lambda R(\lambda, A_{-1})x \to x$ for each $x \in \underline{X}_{-1}$. From this we obtain for $x \in X_0$ that

$$\begin{aligned}\|x\| &= \|A_{-1}x\|_{X_{-1}(A)} = \Big\|\lim_{\lambda\to 0} \lambda R(\lambda, A_{-1})A_{-1}x\Big\|_{X_{-1}(A)} \leq \sup_{\lambda>0}\Big\|\lambda A_{-1}R(\lambda, A_{-1})x\Big\|_{X_{-1}(A)} \\ &= \|x\|_{F_1(A_{-1})} = \|x\|_{F_0(A)},\end{aligned}$$

showing the equivalence of the norms $\|\cdot\|$ and $\|x\|_{F_0(A)}$ on X_0.

□

We also need the following well-known result, see, e.g., [85, Chapters 1 and 3], for which we give a short proof.

Proposition 2.20. *For $\alpha \in (0, 1]$ we have $F_\alpha(A) \subseteq \overline{D(A)} = \underline{X}_0$.*

Proof. We have

$$AR(\lambda, A)x = \lambda R(\lambda, A)x - x,$$

so that

$$\|\lambda R(\lambda, A)x - x\| \leq \frac{\|x\|_{F_\alpha(A)}}{\lambda^\alpha} \to 0 \quad \text{as } \lambda \to \infty.$$

□

Definition 2.21. Let A be a linear operator on the Banach space X_0 satisfying (2.2.1). For $\alpha \in (0, 1)$ we set

$$\underline{X}_\alpha(A) := \Big\{x \in F_\alpha(A) : \lim_{\lambda\to\infty} \lambda^\alpha AR(\lambda, A)x = 0\Big\},$$

and we recall from Section 2.1 that

$$\underline{X}_0(A) := \overline{D(A)}, \quad \underline{X}_1(A) = D(A|_{\underline{X}_0(A)}).$$

The proof of the next proposition is straightforward and well-known.

Proposition 2.22. *For $\alpha, \beta \in (0, 1)$ with $\alpha > \beta$ we have*

$$\underline{X}_1(A) \hookrightarrow \underline{X}_\alpha(A) \subseteq F_\alpha(A) \hookrightarrow \underline{X}_\beta(A) \subseteq F_\beta(A) \hookrightarrow \underline{X}_0(A) \subseteq X_0(A)$$

with $\hookrightarrow$ denoting continuous and dense embeddings of Banach spaces, and $\subseteq$ denoting inclusion of closed subspaces.

Proof. For $x \in F_\alpha(A)$ we have

$$\|\lambda^\beta AR(\lambda, A)x\| = \lambda^{\beta-\alpha}\|\lambda^\alpha AR(\lambda, A)x\| \leq \lambda^{\beta-\alpha}\|x\|_\alpha \to 0 \quad \text{as } \lambda \to \infty,$$

which also proves the continuity of $F_\alpha(A) \hookrightarrow \underline{X}_\beta(A)$. The other statements can be proved by similar reasonings.

□

Proposition 2.23. (a) *The spaces* $F_\alpha(A)$ *and* $\underline{X}_\alpha(A)$ *are invariant under each* $T \in \mathscr{L}(X_0)$ *which commutes with* A^{-1}.

(b) *If* $T \in \mathscr{L}(X_0)$ *commutes with* A^{-1}, *then the space* $F_0(A)$ *is invariant under* T_{-1}.

Proof. (a) Suppose that $T \in \mathscr{L}(X_0)$ commutes with $R(\cdot, A)$ and let $x \in \underline{X}_\alpha(A)$. We have to show that $Tx \in \underline{X}_\alpha(A)$. Since T is assumed to be bounded, we obtain:

$$\|\lambda^\alpha AR(\lambda, A)Tx\| = \|\lambda^\alpha ATR(\lambda, A)x\| \leq \|T\| \cdot \|\lambda^\alpha AR(\lambda, A)x\|.$$

This implies both assertions.

(b) Follows from part (a) applied to T_{-1} on the space X_{-1}.

□

Definition 2.24. For $\alpha \in \mathbb{R}$ we write $\alpha = m + \beta$ with $m \in \mathbb{Z}$ and $\beta \in (0, 1]$, and define

$$F_\alpha(A) := F_\beta(A_m),$$

with the corresponding norms. For $\alpha \notin \mathbb{Z}$ we define

$$\underline{X}_\alpha(A) := \underline{X}_\beta(A_m),$$

also with the corresponding norms.

In particular we have for $\alpha \in (0, 1)$ that

$$\underline{X}_{-\alpha}(A) = \underline{X}_{1-\alpha}(A_{-1}) \quad \text{and} \quad F_{-\alpha}(A) = F_{1-\alpha}(A_{-1}).$$

This definition is consistent with Definitions 2.18 and 2.21. The following property of these spaces can be directly deduced from the definitions and the previous assertions (by induction):

Proposition 2.25. *For any* $\alpha, \beta \in \mathbb{R}$ *with* $\alpha > \beta$ *we have*

$$\underline{X}_\alpha(A) \subseteq F_\alpha(A) \hookrightarrow \underline{X}_\beta(A) \subseteq F_\beta(A)$$

with $\hookrightarrow$ *denoting continuous and dense embeddings of Banach spaces, and* $\subseteq$ *denoting inclusion of closed subspaces.*

Now we put these spaces in the context presented at the end of Section 2.1.

Proposition 2.26. (a) *For* $\alpha \in (0, 1]$ *we have* $A_{-1}F_\alpha = F_{\alpha-1}$ *and* $A_{-1}\underline{X}_\alpha = \underline{X}_{\alpha-1}$.

(b) *For* $\alpha \in (0, 1]$ *and* $\mathcal{A}$, λ *and* $\mathscr{E}$ *as in Theorem 2.16 we have*

$$F_{-\alpha} = \Big\{(\lambda - \mathcal{A})y \in X_{-1} : \ y \in F_{1-\alpha}\Big\}.$$

If $\alpha \in (0, 1)$, *then*

$$\underline{X}_{-\alpha} = \Big\{(\lambda - \mathcal{A})y \in X_{-1} : \ y \in \underline{X}_{1-\alpha}\Big\}.$$

§ 2.3 Intermediate and extrapolation spaces for semigroup generators

In this section we consider intermediate and extrapolation spaces when the linear operator $A : D(A) \to X_0$ is the generator of a semigroup $(T(t))_{t\geq 0}$ (meaning that $T : [0,\infty) \to \mathscr{L}(X_0)$ is a monoid homomorphism) in the sense described in the following.

Assumption 2.27. 1. Let X_0 be a Banach space, and let $Y \subseteq X_0'$ be a norming subspace, i.e.,

$$\|x\| = \sup_{y\in Y, \|y\|\leq 1} |\langle x, y\rangle| \quad \text{for each } x \in X_0.$$

2. Let $T : [0,\infty) \to \mathscr{L}(X_0)$ be a semigroup of contractions for which a *generator* $A : D(A) \to X_0$ exists in the sense that

$$R(\lambda, A)x = \int_0^\infty \mathrm{e}^{-\lambda s} T(s)x \; \mathrm{d}s \tag{2.3.1}$$

exists for each $\lambda \geq 0$ as a weak integral with respect to the dual pair (X_0, Y), i.e., for each $y \in Y$ and $x \in X_0$

$$\langle R(\lambda, A)x, y\rangle = \int_0^\infty \mathrm{e}^{-\lambda s} \langle T(s)x, y\rangle \; \mathrm{d}s,$$

and $R(\lambda, A) \in \mathscr{L}(X_0)$ is the resolvent of a linear operator A (see [80] by Kunze).

3. We also suppose that $T(t)$ commutes with A^{-1} for each $t \geq 0$.

If the semigroup $(T(t))_{t>0}$ is only exponentially bounded of type (M, ω), that is

$$\|T(t)\| \leq M\mathrm{e}^{\omega t} \quad \text{for all } t \geq 0,$$

then one can rescale the semigroup (consider$(\mathrm{e}^{-(\omega+1)t}T(t))_{t\geq 0}$), and renorm the Banach space such that the rescaled semigroup becomes a contraction semigroup. Moreover, the new semigroup has negative growth bound, meaning that $T(t) \to 0$ in norm exponentially fast as $t \to \infty$. Then it also has an invertible generator.

Remark 2.28. (i) There are several important classes of semigroups, satisfying Assumption 2.27, hence can be treated in a unified manner: π-semigroups of E. Priola [100], weakly continuous semigroups of S. Cerrai [32], bi-continuous semigroups of F. Kühnemund. We will concentrate on this latter class of semigroups in Section 2.4.

(ii) In this framework M. Kunze [80] introduced the notion of integrable semigroups, which we briefly describe next. Since we have

$$\|y\| = \sup_{x\in X_0, \|x\|\leq 1} |\langle x, y\rangle|$$

and, by the norming assumption,

$$\|x\| = \sup_{y\in Y, \|y\|\leq 1} |\langle x, y\rangle|,$$

the pair (X_0, Y) is called a norming dual pair. M. Kunze has worked out the theory of semigroups on such norming dual pairs in [80]. We recall at least the basic definitions here: assume without loss of generality that Y is a Banach space and consider the weak topology $\sigma = \sigma(X_0, Y)$ on X_0. An *integrable semigroup* of type (M, ω) on the pair (X_0, Y) is a semigroup $(T(t))_{t\geq 0}$ of σ-continuous linear operators satisfying:

1. $(T(t))_{t\geq 0}$ is a semigroup, i.e. $T(t+s) = T(t)T(s)$ and $T(0) = I$ for all $t, s \geq 0$.
2. For all λ with $\mathrm{Re}(\lambda) > \omega$, there exists an σ-continuous linear operator $R(\lambda)$ such that for all $x \in X_0$ and all $y \in Y$

$$\langle R(\lambda)x, y\rangle = \int_0^\infty \mathrm{e}^{-\lambda t}\langle T(t)x, y\rangle \, \mathrm{d}t.$$

Kunze defines the generator A of the semigroup as the (unique) operator $A : D(A) \to X_0$ (if it exists at all) with $R(\lambda) = (\lambda - A)^{-1}$, precisely as in Assumption 2.27. Note that σ-continuity of $T(t)$ can be used to assure that Y is invariant under $T'(t)$, cf. the next remark.

Remark 2.29. The semigroup $(T(t))_{t\geq 0}$ commutes with the inverse of the generator if Y can be chosen such that it is invariant under $T'(t)$ for each $t \geq 0$:

$$\begin{aligned}\langle A^{-1}T(t)x, y\rangle &= \int_0^\infty \langle T(s)T(t)x, y\rangle \, \mathrm{d}s = \int_0^\infty \langle T(s+t)x, y\rangle \, \mathrm{d}s \\ &= \Big\langle \int_0^\infty T(s)x \, \mathrm{d}s, T'(t)y \Big\rangle = \langle T(t)A^{-1}x, y\rangle,\end{aligned}$$

for each $x \in X_0$ and $y \in Y$.

Remark 2.30. (i) From (2.3.1) it follows that for each $x \in X_0$

$$T(t)x - x = A\int_0^t T(s)x \, \mathrm{d}s. \tag{2.3.2}$$

Indeed, we have by (2.3.1) that

$$\begin{aligned}x &= A\int_0^\infty T(s)x \, \mathrm{d}s \\ T(t)x &= A\int_0^\infty T(s)T(t)x \, \mathrm{d}s = \int_t^\infty T(s)x \, \mathrm{d}s.\end{aligned}$$

Subtracting the first of these equation from the second one we obtain the statement.

(ii) If moreover A commutes with $T(t)$ for each $t \geq 0$, then for each $x \in D(A)$ we have

$$T(t)x - x = \int_0^t T(s)Ax \, \mathrm{d}s. \tag{2.3.3}$$

Indeed, as in the above, we have by (2.3.1)

$$\begin{aligned}-x &= -A^{-1}Ax = \int_0^\infty T(s)Ax \, \mathrm{d}s \\ -T(t)x &= -A^{-1}T(t)Ax = \int_0^\infty T(s)T(t)Ax \, \mathrm{d}s = \int_t^\infty T(s)Ax \, \mathrm{d}s.\end{aligned}$$

By a simple subtraction we obtain the statement.

The next lemma and its proof are standard for various classes of semigroups.

Lemma 2.31. *If $(T(t))_{t\geq 0}$ is (locally) norm bounded, then*

$$X_{\mathrm{cont}} := \{x \in X_0 : t \mapsto T(t)x \text{ is } \|\cdot\|\text{-continuous}\}$$

is a closed a subspace of X_0 invariant under the semigroup. Under Assumption 2.27 we have

$$\underline{X}_0 = \overline{D(A)} = X_{\mathrm{cont}}.$$

Proof. The closedness and invariance of X_{cont} are evident. We first show $D(A) \subseteq X_{\mathrm{cont}}$, which implies $\overline{D(A)} \subseteq X_{\mathrm{cont}}$ by closedness of X_{cont}. By (2.3.3) we conclude for $x \in D(A)$ that $T(t)x - x = \int_0^t T(s)Ax \, \mathrm{d}s$. Since

$$\|T(t)x - x\| = \sup_{\|y\|\leq 1} |\langle T(t)x - x, y\rangle| \leq \sup_{\|y\|\leq 1} \int_0^t |\langle T(s)Ax, y\rangle| \, \mathrm{d}s \leq t\|Ax\| \to 0,$$

as $t \to 0$, we obtain $D(A) \subseteq X_{\mathrm{cont}}$ and $\overline{D(A)} \subseteq X_{\mathrm{cont}}$. For the converse inclusion suppose that $x \in X_{\mathrm{cont}}$. Again by (2.3.2) we obtain that the sequence of vectors $x_n := n\int_0^{\frac{1}{n}} T(s)x \, \mathrm{d}s \in D(A)$ $(n \in \mathbb{N})$ converges to x. Indeed:

$$\|x_n - x\| = \sup_{\|y\|\leq 1} |\langle x_n - x, y\rangle| \leq \sup_{\|y\|\leq 1} n\int_0^{\frac{1}{n}} |\langle T(s)x - x, y\rangle| \, \mathrm{d}s \leq n\int_0^{\frac{1}{n}} \|T(s)x - x\| \, \mathrm{d}s.$$

By the continuity of $s \mapsto T(s)x$ we obtain the inclusion $X_{\mathrm{cont}} \subseteq \overline{D(A)}$.

□

Based on this lemma one can prove the following characterization of the Favard and Hölder spaces:

Proposition 2.32. *Let $(T(t))_{t\geq 0}$ be a semigroup satisfying Assumption 2.27 with negative growth bound and generator A. For $\alpha \in (0,1]$ define*

$$F_\alpha(T) := \Big\{x \in X_0 : \sup_{s>0} \frac{\|T(s)x - x\|}{s^\alpha} < \infty\Big\} = \Big\{x \in X_0 : \sup_{s\in(0,1)} \frac{\|T(s)x - x\|}{s^\alpha} < \infty\Big\}, \quad (2.3.4)$$

and for $\alpha \in (0,1)$ define

$$\begin{aligned}\underline{X}_\alpha(T) &:= \Big\{x \in X_0 : \sup_{s>0} \frac{\|T(s)x - x\|}{s^\alpha} < \infty \text{ and } \lim_{s\downarrow 0} \frac{\|T(s)x - x\|}{s^\alpha} = 0\Big\} \\ &= \Big\{x \in X_0 : \lim_{s\downarrow 0} \frac{\|T(s)x - x\|}{s^\alpha} = 0\Big\},\end{aligned} \quad (2.3.5)$$

which become Banach spaces if endowed with the norm

$$\|x\|_{F_\alpha(T)} := \sup_{s>0} \frac{\|T(s)x - x\|}{s^\alpha}.$$

The space $\underline{X}_\alpha(T)$ is a closed subspace of $F_\alpha(T)$. These spaces are invariant under the semigroup $(T(t))_{t\geq 0}$, and $\underline{X}_\alpha(T)$ is the space of $\|\cdot\|_{F_\alpha(T)}$-strong continuity in $F_\alpha(T)$. For $\alpha \in (0,1]$

we have $F_\alpha(A) = F_\alpha(T)$ and for $\alpha \in (0,1)$ we have $\underline{X}_\alpha(A) = \underline{X}_\alpha(T)$ with equivalent norms.

Proof. For $x \in F_\alpha(T)$ we have

$$\|T(t)x\|_{F_\alpha(T)} = \sup_{s>0} \frac{\|T(s)T(t)x - T(t)x\|}{s^\alpha} \leq \|T(t)\| \cdot \sup_{s>0} \frac{\|T(s)x - x\|}{s^\alpha} \leq M\|x\|_{F_\alpha(T)},$$

proving the invariance of $F_\alpha(T)$. Similar reasoning proves the invariance of $\underline{X}_\alpha$. Since $F_\alpha(T) \subseteq X_{\mathrm{cont}} = \underline{X}_0 = \overline{D(A)}$ and $F_\alpha(A) \subseteq \underline{X}_0 = \overline{D(A)}$, the rest of the assertions follow from the corresponding results concerning C_0-semigroups, see, e.g., [52, Sec. II.5].

□

We conclude with the construction of the extrapolated semigroup as a direct consequence of Proposition 2.15.

Proposition 2.33. *Let A generate the semigroup $(T(t))_{t\geq 0}$ of negative growth bound in the sense of Assumption 2.27. Then there is an extension $(T_{-1}(t))_{t\geq 0}$ of the semigroup $(T(t))_{t\geq 0}$ on the extrapolated space X_{-1}, whose generator is A_{-1}.*

2.3.1 Intermezzo on real interpolation spaces

To complete the picture we recall the next result [85, Prop. 5.7], which is formulated there only for C_0-semigroups as a theorem, but A. Lunardi also remarks, without stating the precise assumptions, that this result still holds if one omits the strong continuity assumption. We require here the conditions from Assumption 2.27, under which the proof is verbatim the same as for the C_0-case, and is based on formulas (2.3.2) and (2.3.3). Before we state the result, recall the following definitions from [85].

Definition 2.34. Let X and Y be two real or complex Banach spaces. The couple (X,Y) is an *interpolation couple* if both X and Y are continuously embedded in a Hausdorff topological vector space $\mathcal{V}$.

If (X,Y) is such an interpolation couple, also $X + Y := \{x + y : \ x \in X, \ y \in Y\}$ is a linear subspace of $\mathcal{V}$ and becomes a Banach space equipped with the norm $\|\cdot\|_{X+Y}$ given by

$$\|z\|_{X+Y} := \inf_{\substack{x\in X,\ y\in Y\\ z=x+y}} \|x\|_X + \|y\|_Y$$

Definition 2.35. For each $x \in X + Y$ and $t > 0$ we set

$$K(t,x) := \inf_{\substack{a\in X,\ b\in Y\\ x=a+b}} \|a\|_X + t\,\|b\|_Y.$$

Definition 2.36. Let $0 < \theta < 1$, $1 \leq p \leq \infty$. The *real interpolation spaces* $(X,Y)_{\theta,p}$ are defined by

$$(X,Y)_{\theta,p} := \left\{x \in X + Y : \ t \mapsto t^{-\theta}K(t,x) \in \mathrm{L}^p_*(0,\infty)\right\}$$

and are normed by

$$\|x\|_{\theta,p} := \left\|t^{-\theta}K(t,x)\right\|_{\mathrm{L}^p_*(0,\infty)}$$

where $\mathrm{L}^p_*(0,\infty)$ denotes the L^p-space with respect to the Haar measure $\frac{\mathrm{d}t}{t}$ on the multiplicative group $(0,\infty)$.

Now we are able to formulate and to prove the promised result.

Proposition 2.37. *Let A generate the semigroup $(T(t))_{t>0}$ of negative growth bound in the sense describe in Assumption 2.27. Then for $p \in [1,\infty]$ and $\alpha \in (0,1)$ we have for the interpolation space:*

$$(X, D(A))_{\alpha,p} = \{x \in X : \ t \mapsto \psi(t) := t^{-\alpha}\|T(t)x - x\| \in \mathrm{L}^p_*(0,\infty)\},$$

where $\mathrm{L}^p_*(0,\infty)$ denotes the L^p-space with respect to the Haar measure $\frac{\mathrm{d}t}{t}$ on the multiplicative group $(0,\infty)$. Moreover, the norms $\|x\|_{\alpha,p}$ and

$$\|x\|^{**}_{\alpha,p} = \|x\| + \|\psi\|_{\mathrm{L}^p_*(0,\infty)}$$

are equivalent.

Proof. For each $b \in D(A)$ we have

$$T(t)b - b = \int_0^t AT(s)b \,\mathrm{d}s = \int_0^t T(s)Ab \,\mathrm{d}s, \quad t > 0.$$

Let $x \in (X, D(A))_{\theta,p}$. If $x = a + b$ with $a \in X$ and $b \in D(A)$, then

$$t^{-\theta}\|T(t)x - x\| \le t^{-\theta}(\|T(t)a - a\| + \|T(t)b - b\|) \le (M+1)t^{-\theta}K(t,x)$$

for each $t > 0$ which means in particular that $\psi(t) = t^{-\theta}\|T(t)x - x\| \in \mathrm{L}^p_*(0,\infty)$ and

$$\|x\|^{**}_{\theta,p} \le (M+1)\|x\|_{\theta,p}.$$

For the converse assume that $\psi \in \mathrm{L}^p_*(0,\infty)$. We obtain

$$\lambda^\theta \|AR(\lambda,A)x\| \le \int_0^\infty \lambda^{\theta+1}t^{\theta+1}\mathrm{e}^{-\lambda t}\frac{\|T(t)x - x\|}{t^\theta}\frac{\mathrm{d}t}{t},$$

which is in fact the convolution $f * \psi$ for $f(t) = t^{\theta+1}\mathrm{e}^{-t}$ and $\psi(t) = t^{-\theta}\|T(t)x - x\|$. Since $f \in \mathrm{L}^1_*(0,\infty)$ and $\psi \in \mathrm{L}^p_*(0,\infty)$ one gets $f * \psi \in \mathrm{L}^p_*(0,\infty)$ and

$$\|f * \psi\|_{\mathrm{L}^p_*(0,\infty)} \le \|f\|_{\mathrm{L}^1_*(0,\infty)}\|\psi\|_{\mathrm{L}^p_*(0,\infty)}.$$

From that we obtain

$$\|x\|^*_{\theta,p} \le \Gamma(\theta+1)\|x\|^{**}_{\theta,p}.$$

□

§ 2.4 Intermediate and extrapolation spaces for bi-continuous semigroups

2.4.1 The Hille–Yosida Theorem

Recall the following result of F. Kühnemund from [79, Thm. 16], whose proof is originally based on integrated semigroups. We present here a different proof based on extrapolation spaces.

Theorem 2.38. *Let $(X_0, \|\cdot\|, \tau)$ be a triple satisfying Assumption 1.1, and let A be a linear operator on the Banach space X_0. The following are equivalent:*

(a) *The operator A is the generator of a bi-continuous semigroup $(T(t))_{t\geq 0}$ of type (M,ω).*

(b) *The operator A is a Hille–Yosida operator of type (M,ω), i.e.,*

$$\|R(s,A)^k\| \leq \frac{M}{(s-\omega)^k}$$

for all $k \in \mathbb{N}$ and for all $s > \omega$. Moreover, A is bi-densely defined and the family

$$\{(s-\alpha)^k R(s,A)^k : \ k \in \mathbb{N}, \ s \geq \alpha\} \tag{2.4.1}$$

is bi-equicontinuous for each $\alpha > \omega$, meaning that for each norm bounded τ-null sequence (x_n) one has $(s-\alpha)^k R(s,A)^k x_n \to 0$ in τ uniformly for $k \in \mathbb{N}$ and $s \geq \alpha$ as $n \to \infty$.

In this case, we have the Euler formula:

$$T(t)x := \tau\lim_{m\to\infty} \left(\frac{m}{t} R\left(\frac{m}{t}, A\right)\right)^m x \quad \text{for each } x \in X_0.$$

Moreover, the subspace $\underline{X}_0 := \overline{D(A)} \subseteq X_0$ is the space of norm strong continuity for $(T(t))_{t\geq 0}$, it is invariant under the semigroup, and $(\underline{T}(t))_{t\geq 0} := (T(t)|_{\underline{X}_0})_{t\geq 0}$ is the strongly continuous semigroup on $\underline{X}_0$ generated by the part $\underline{A}$ of A in $\underline{X}_0$.

Proof. It follows from Lemma 2.31 that $\underline{X}_0$ is the space of norm strong continuity for a bi-continuous semigroup $(T(t))_{t\geq 0}$.

We only prove the implication (b) $\Rightarrow$ (a) and the Euler formula; the other implication is standard and the easier one and can be found here [78, Thm. 1.28]. We may suppose that $\omega < 0$. Since A is a Hille–Yosida operator, the part $\underline{A}$ of A in $\underline{X}_0$ generates a C_0-semigroup $(\underline{T}(t))_{t\geq 0}$ of type (M,ω) on the space $\underline{X}_0 := \overline{D(A)}$. Define the function

$$F(s) := \begin{cases} \frac{1}{s} R(\frac{1}{s}, A) & \text{for } s > 0, \\ I & \text{for } s = 0, \end{cases}$$

which is strongly continuous on $\underline{X}_0$ by Remark 2.6. Moreover, we have the Euler formula

$$\underline{T}_0(t)x = \lim_{m\to\infty} F(\tfrac{t}{m})^m x$$

for $x \in \underline{X}_0$ with convergence being uniform for t in compact intervals $[0,t_0]$, see, e.g., [52, Section III.5(a)]. Since $R(\lambda, A)|_{\underline{X}_0} = R(\lambda, \underline{A})$ and since $D(A)$ is bi-dense in X_0, by the local bi-equicontinuity assumption in (2.4.1) we conclude that for $x \in X_0$ and $t > 0$ the limit

$$S(t)x := \tau\lim_{m\to\infty} F(\tfrac{t}{m})^m x \tag{2.4.2}$$

exists, and the convergence is uniform for t in compact intervals $[0,t_0]$. It follows that $t \mapsto S(t)x$ is τ-strongly continuous for each $x \in X_0$. The operator family $(S(t))_{t\geq 0}$ is locally bi-equicontinuous because of the bi-equicontinuity assumption in (2.4.1).

Next, we prove that $\underline{T}(t)$ leaves $D(A)$ invariant. Let $x \in D(A)$, so that $x = A^{-1}y$ for some $y \in X_0$, and insert x in the formula (2.4.2) to obtain

$$\underline{T}(t)x = S(t)A^{-1}y = \tau\lim_{m\to\infty} F\left(\tfrac{t}{m}\right)^m A^{-1}y = A^{-1}\, \tau\lim_{m\to\infty} F\left(\tfrac{t}{m}\right)^m y = A^{-1}S(t)y \in D(A), \quad (2.4.3)$$

where we have used the bi-continuity of A^{-1} and the boundedness of $(\left[\frac{m}{t}R(\frac{m}{t},A)\right]^m y)_{m\in\mathbb{N}}$. By Proposition 2.15 (b) we can extend $\underline{T}(t)$ to X_0 by setting $T(t) := A\underline{T}(t)A^{-1} \in \mathscr{L}(X_0)$. It follows that $(T(t))_{t\geq 0}$ is a semigroup. By formula (2.4.3), we have $T(t)y = A\underline{T}(t)A^{-1}y = AA^{-1}S(t)y = S(t)y$ for each $y \in X_0$. So that $(T(t))_{t\geq 0}$, coinciding with $(S(t))_{t\geq 0}$, is locally bi-equicontinuous, and hence a bi-continuous semigroup.

It remains to show that the generator of $(T(t))_{t\geq 0}$ is A. Let B denote the generator of $(T(t))_{t\geq 0}$. Then, for large $\lambda > 0$ and $x \in \underline{X}_0$, we have

$$R(\lambda, B)x = \int_0^\infty \mathrm{e}^{-\lambda s}T(s)x \,\mathrm{d}s = \int_0^\infty \mathrm{e}^{-\lambda s}\underline{T}(s)x \,\mathrm{d}s = R(\lambda, \underline{A}_0)x = R(\lambda, A)x.$$

Since $R(\lambda, B)$ and $R(\lambda, A)$ are sequentially τ-continuous on norm bounded sets and since $D(A)$ is bi-dense in X_0, we obtain $R(\lambda, B) = R(\lambda, A)$. This finishes the proof.

□

The first statement in the next proposition is proved by R. Nagel and E. Sinestrari, see [92] and [94], while the second one follows directly from the results in Section 2.1.

Proposition 2.39. *Let A be a Hille–Yosida operator on the Banach space X_0 with domain $D(A)$. Denote by $(\underline{T}(t))_{t\geq 0}$ the C_0-semigroup on $\underline{X}_0 = \overline{D(A)}$ generated by the part $\underline{A}$ of A.*

(a) *There is a one-parameter semigroup $(\overline{T}(t))_{t\geq 0}$ on $F_0(A)$ which extends $(\underline{T}(t))_{t\geq 0}$. This semigroup is strongly continuous for the $\|\cdot\|_{X_{-1}(\underline{A})}$ norm.*

(b) *Suppose that for each $t \geq 0$ the operator $\underline{T}(t)$ leaves $D(A)$ invariant. Then the space X_0 is invariant under the semigroup operators $\overline{T}(t)$ for every $t \geq 0$, i.e., for $T(t) := \overline{T}(t)|_{X_0}$ we have $T(t) \in \mathscr{L}(X_0)$.*

2.4.2 Extrapolated semigroups

In this subsection we extend a bi-continuous semigroup on X_0 to the extrapolation space X_{-1} as a bi-continuous semigroup. We have to handle two topologies, and the next proposition leads to an additional locally convex topology on X_{-1} still satisfying Assumption 1.1.

Proposition 2.40. *Let the triple $(X_0, \|\cdot\|, \tau)$ satisfy Assumption 1.1, let $\mathcal{P}$ be as in Remark 1.2(iv), let E be a vector space over $\mathbb{C}$, and let $B : X_0 \to E$ be a bijective linear mapping. We define for $e \in E$ and $p \in \mathcal{P}$*

$$\|e\|_E := \|B^{-1}e\| \quad \textit{and} \quad p_E(e) := p(B^{-1}e).$$

Then the following assertions hold:

(a) *$\|\cdot\|_E$ is a norm, p_E is a seminorm for each $p \in \mathcal{P}$.*

(b) *For the topology τ_E generated by $\mathcal{P}_E := \{p_E : p \in \mathcal{P}\}$ the triple $(E, \|\cdot\|_E, \tau_E)$ satisfies the conditions in Assumption 1.1.*

(c) *If* $(T(t))_{t\geq 0}$ *is a bi-continuous semigroup on* X_0 *with respect to the topology* τ*, then* $T_E(t) := BT(t)B^{-1}$ *defines a bi-continuous semigroup on* E*. If* A *is the generator of* $(T(t))_{t\geq 0}$*, then* BAB^{-1} *is the generator of* $(T_E(t))_{t\geq 0}$*.*

Proof. Assertion (a) is evident. The conditions (1) and (2) from Assumption 1.1 are satisfied by the definition of $\|\cdot\|_E$ and p_E. Since

$$\|e\|_E = \|B^{-1}e\| = \sup_{p\in\mathcal{P}} p(B^{-1}e) = \sup_{p_E\in\mathcal{P}_E} p_E(e),$$

and by Remark 1.2(iii) in Assumption 1.1 is fulfilled. The proof of (b) is complete.

(c) For $e \in E$ we have $\|T_E(t)\|_E = \|B^{-1}BT(t)B^{-1}e\| = \|T(t)B^{-1}e\| \leq \|T(t)\| \cdot \|e\|_E$, which shows that $T_E(t) \in \mathscr{L}(E)$. Clearly, $(T_E(t))_{t\geq 0}$ satisfies the semigroup property. For $e \in E$ and $p_E \in \mathcal{P}_E$ we have

$$p_E(T_E(t)e - e) = p(B^{-1}BT(t)B^{-1}e - B^{-1}e) = p(T(t)B^{-1}e - B^{-1}e) \to 0 \quad \text{for } t \to 0,$$

showing the τ_E-strong continuity of $(T_E(t))_{t\geq 0}$. If (e_n) is a $\|\cdot\|_E$-bounded, τ_E-null sequence, then $(B^{-1}e_n)$ is a $\|\cdot\|$-bounded τ-null sequence, so that by assumption $T_E(t)e_n = T(t)B^{-1}e_n \to 0$ uniformly for t in compact intervals. If A is the generator of $(T(t))_{t\geq 0}$, then by means of (1.3.2) we can conclude that $B^{-1}AB$ is the generator of $(T_E(t))_{t\geq 0}$.

□

Definition 2.41. Let $(T(t))_{t\geq 0}$ be a bi-continuous semigroup in X_0 with generator A.

(a) For $B = A^{-1} : X_0 \to X_1$ and $E = X_1$ in Proposition 2.40 define $\mathcal{P}_1 := \mathcal{P}_E$, $\tau_1 := \tau_E$, $(T_1(t))_{t\geq 0} := (T_E(t))_{t\geq 0}$.

(b) For $B = A_{-1} : X_0 \to X_{-1}$ and $E = X_{-1}$ in Proposition 2.40 define $\mathcal{P}_{-1} := \mathcal{P}_E$, $\tau_{-1} := \tau_E$, $(T_{-1}(t))_{t\geq 0}; := (T_E(t))_{t\geq 0}$.

We obtain immediately the next result.

Proposition 2.42. *The semigroups* $(T_1(t))_{t\geq 0}$ *and* $(T_{-1}(t))_{t\geq 0}$ *are bi-continuous with generators* $A_1 = A|_{D(A)}$ *and* A_{-1}*, respectively.*

Iterating the procedure in Definition 2.41 we obtain the full scale of (extrapolated) semigroups $(T_n(t))_{t\geq 0}$ for $n \in \mathbb{Z}$.

Definition 2.43. Let $(T(t))_{t\geq 0}$ be a bi-continuous semigroup on X_0 with generator A and suppose that $(T_{\pm n}(t))_{t\geq 0}$ and $\mathcal{P}_{\pm n}$ have been defined for some $n \in \mathbb{N}$ already.

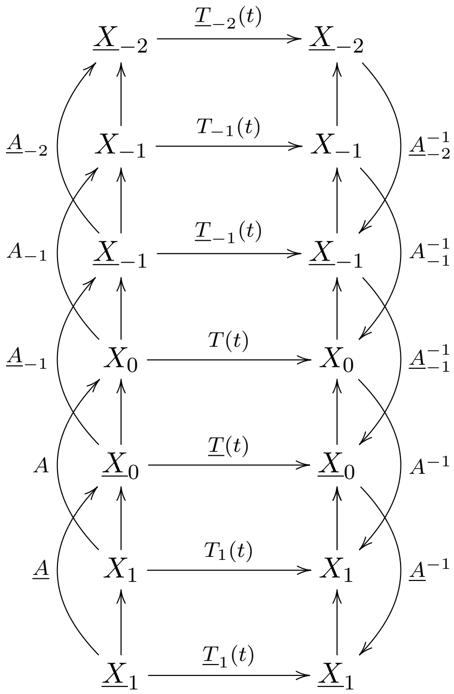
(a) For $B = A_n^{-1} : X_n \to X_{n+1}$, $E = X_{n+1}$ and the semigroup $(T_n(t))_{t\geq 0}$ in Proposition 2.40 define $\mathcal{P}_{n+1} := \mathcal{P}_E$, $\tau_{n+1} := \tau_E$, $(T_{n+1}(t))_{t\geq 0} := (T_E(t))_{t\geq 0}$.

(b) For $B = A_{-n-1} : X_{-n} \to X_{-n-1}$, $E = X_{-n-1}$ and the semigroup $(T_{-n}(t))_{t\geq 0}$ in Proposition 2.40 define $\mathcal{P}_{-n-1} := \mathcal{P}_E$, $\tau_{-n-1} := \tau_E$, $(T_{-n-1}(t))_{t\geq 0} := (T_E(t))_{t\geq 0}$.

Proposition 2.44. *For each* $n \in \mathbb{Z}$ *the semigroup* $(T_n(t))_{t\geq 0}$ *is bi-continuous on* $(X_n, \|\cdot\|_n, \tau_n)$ *with generator* $A_n : X_{n+1} \to X_n$*. Its space of norm strong continuity is* $\underline{X}_n$*.*

Proof. The first statement follows directly from Proposition 2.42 by induction. For $n = 0$ the second assertion is the content of Lemma 2.31, for general $n \in \mathbb{Z}$ one can argue inductively. □

The following diagram summarizes the situation:



The spaces $\underline{X}_{n+1}$ are bi-dense in X_n for the topology τ_n and dense in $\underline{X}_n$ for the norm $\|\cdot\|_{X_n}$. The semigroups $(T_n(t))_{t\geq 0}$ are bi-continuous on X_n, while $(\underline{T}_n(t))_{t\geq 0}$ are C_0-semigroups (strongly continuous for the norm) on $\underline{X}_n$.

2.4.3 Hölder spaces of bi-continuous semigroups

Suppose A generates the bi-continuous semigroup $(T(t))_{t\geq 0}$ of negative growth bound on X_0. Recall from Theorem 2.38 that the restricted operators $\underline{T}(t) := T(t)|_{\underline{X}_0}$ form a C_0-semigroup $(\underline{T}(t))_{t\geq 0}$ on $\underline{X}_0$. Also recall from Proposition 2.32 that for $\alpha \in (0,1]$

$$F_\alpha(A) = F_\alpha(T) = \Big\{x \in \underline{X}_0 :\ \sup_{t>0} \frac{\|\underline{T}(t)x - x\|}{t^\alpha} < \infty\Big\} = \Big\{x \in X_0 :\ \sup_{t>0} \frac{\|T(t)x - x\|}{t^\alpha} < \infty\Big\}$$

with the norm

$$\|x\|_{F_\alpha} = \sup_{t>0} \frac{\|\underline{T}(t)x - x\|}{t^\alpha},$$

and for $\alpha \in (0,1)$:

$$\underline{X}_\alpha(A) := \Big\{x \in \underline{X}_0 :\ \lim_{t\to 0} \frac{\|\underline{T}(t)x - x\|}{t^\alpha} = 0\Big\} = \Big\{x \in X_0 :\ \lim_{t\to 0} \frac{\|T(t)x - x\|}{t^\alpha} = 0\Big\}.$$

We have the (continuous) inclusions

$$\underline{X}_1 \hookrightarrow X_1 \to \underline{X}_\alpha(A) \hookrightarrow F_\alpha(A) \to \underline{X}_0 \hookrightarrow X_0;$$

all these spaces are invariant under $(T(t))_{t\geq 0}$.We now extend this diagram by a space which lies between $\underline{X}_\alpha$ and F_α.

Definition 2.45. Let $(T(t))_{t\geq 0}$ be a bi-continuous semigroup of negative growth bound on a Banach space X_0 with respect to a locally convex topology τ that is generated by a family $\mathcal{P}$ of seminorms satisfying (1.1.2). For $\alpha \in (0,1)$ we define the space

$$X_\alpha := X_\alpha(T) := \Big\{x \in X_0 : \ \tau\lim_{t\to 0} \frac{T(t)x - x}{t^\alpha} = 0 \text{ and } \sup_{t>0} \frac{\|T(t)x - x\|}{t^\alpha} < \infty\Big\}, \tag{2.4.4}$$

and endow it with the norm $\|\cdot\|_{F_\alpha}$. We further equip F_α and X_α with the locally convex topology τ_{F_α} generated by the family of seminorms $\mathcal{P}_{F_\alpha} := \{p_{F_\alpha} : p \in \mathcal{P}\}$, where $p_{F\alpha}$ is defined as

$$p_{F_\alpha}(x) := \sup_{t>0} \frac{p(T(t)x - x)}{t^\alpha}. \tag{2.4.5}$$

It is easy to see that X_α is a Banach space, i.e., as closed subspace of F_α. By construction we have that indeed $\underline{X}_\alpha(A) \subseteq X_\alpha \subseteq F_\alpha(A)$. Next we discuss some properties of this space.

Lemma 2.46. (a) *Let (x_n) be a $\|\cdot\|_{F_\alpha}$-norm bounded sequence in F_α with $x_n \to x \in X_0$ in the topology τ. Then $x \in F_\alpha$.*

(b) *The triple $(F_\alpha, \|\cdot\|_{F_\alpha}, \tau_{F_\alpha})$ satisfies the conditions in Assumption 1.1.*

(c) *X_α is bi-closed in F_α, i.e., every $\|\cdot\|_{F_\alpha}$-bounded an τ_{F_α}-convergent sequence in X_α has its limit in X_α.*

Proof. (a) The statement follows from the fact that the norm $\|\cdot\|_{F_\alpha}$ is lower semicontinuous for the topology τ. If

$$\frac{\|T(t)x_n - x_n\|}{t^\alpha} \leq \|x_n\|_{F_\alpha} \leq M$$

for each $n \in \mathbb{N}$, $t > 0$ and for some $M \geq 0$ we can estimate:

$$\begin{aligned}\sup_{t>0} \frac{\|T(t)x - x\|}{t^\alpha} &= \sup_{t>0}\sup_{p\in\mathcal{P}} p\Big(\frac{T(t)x - x}{t^\alpha}\Big) = \sup_{t>0}\sup_{p\in\mathcal{P}} \lim_{n\to\infty} p\Big(\frac{T(t)x_n - x_n}{t^\alpha}\Big)\\ &\leq \sup_{t>0}\sup_{p\in\mathcal{P}} \limsup_{n\to\infty}\Big\|\frac{T(t)x_n - x_n}{t^\alpha}\Big\| \leq \sup_{t>0}\sup_{n\in\mathbb{N}}\Big\|\frac{T(t)x_n - x_n}{t^\alpha}\Big\| \leq M.\end{aligned}$$

(b) We have for $p \in \mathcal{P}$ and $x \in F_\alpha$ that

$$p_{F_\alpha}(x) = \sup_{t>0} \frac{p(T(t)x - x)}{t^\alpha} \leq \sup_{t>0} \frac{\|T(t)x - x\|}{t^\alpha} = \|x\|_{F_\alpha}.$$

This proves that τ_{F_α} is coarser than the $\|\cdot\|_{F_\alpha}$-topology, but is still Hausdorff by construction. For the second property of Assumption 1.1 let $(x_n)_{n\in\mathbb{N}}$ be a τ_{F_α}-Cauchy sequence in F_α such that there exists $M>0$ with $\|x_n\|_{F_\alpha}\leq M$ for each $n\in\mathbb{N}$. Since τ is coarser than τ_{F_α}, we conclude that (x_n) is τ-Cauchy sequence which is also bounded in $\|\cdot\|_{F_\alpha}$, hence in $\|\cdot\|$. By assumption there is $x\in X_0$ such that $x_n\to x$ in τ. By part (a) we obtain $x\in F_\alpha$. It remains to prove that $x_n\to x$ in τ_{F_α}. Let $\varepsilon>0$, and take $N\in\mathbb{N}$ such that for each $n,m\in\mathbb{N}$ with $n,m\geq N$ we have $p_{F_\alpha}(x_n-x_m)<\varepsilon$. For $t>0$

$$p\Big(\frac{T(t)(x_n-x)-(x_n-x)}{t^\alpha}\Big)=\lim_{m\to\infty}p\Big(\frac{T(t)(x_n-x_m)-(x_n-x_m)}{t^\alpha}\Big)\leq p_{F_\alpha}(x_n-x_m)<\varepsilon$$

for each $n\geq N$. Taking the supremum in $t>0$ we obtain $p_{F_\alpha}(x-x_n)\leq\varepsilon$ for each $n\geq N$.

The norming property in (1.1.1) follows again from Remark 1.2 and the fact that the family $\mathcal{P}$ is norming by assumption.

(c) Let $(x_n)_{n\in\mathbb{N}}$ be a $\|\cdot\|_{F_\alpha}$-bounded and τ_{F_α} convergent sequence in X_α with limit $x\in X_0$. For $p\in\mathcal{P}$ we then have

$$\sup_{t>0}p\Big(\frac{T(t)(x_n-x)-(x_n-x)}{t^\alpha}\Big)\to 0.$$

Since $x_n\in X_\alpha$ for each $n\in\mathbb{N}$, we have

$$\lim_{t\to 0}p\Big(\frac{T(t)x_n-x_n}{t^\alpha}\Big)=0,\quad\text{and}\quad\sup_{t>0}\Big\|\frac{T(t)x_n-x_n}{t^\alpha}\Big\|<\infty.$$

We now conclude for a fixed $p\in\mathcal{P}$

$$\begin{aligned}p\Big(\frac{T(t)x-x}{t^\alpha}\Big)&=p\Big(\frac{T(t)(x-x_n)-(x-x_n)+T(t)x_n-x_n}{t^\alpha}\Big)\\&\leq p\Big(\frac{T(t)(x-x_n)-(x-x_n)}{t^\alpha}\Big)+p\Big(\frac{T(t)x_n-x_n}{t^\alpha}\Big)\\&\leq p_{F_\alpha}(x-x_n)+p\Big(\frac{T(t)x_n-x_n}{t^\alpha}\Big)<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon,\end{aligned}$$

where we first fix $n\in\mathbb{N}$ such that $p_{F_\alpha}(x-x_n)<\frac{\varepsilon}{2}$, and then we take $\delta>0$ such that $0<t<\delta$ implies $p(\frac{T(t)x_n-x_n}{t^\alpha})<\frac{\varepsilon}{2}$.

□

The next goal is to verify that $(T(t))_{t\geq 0}$ can be restricted to X_α to obtain a bi-continuous semigroup with respect to the topology τ_{F_α}.

Lemma 2.47. *If $(T(t))_{t\geq 0}$ is a bi-continuous semigroup, then X_α is invariant under the semigroup.*

Proof. We notice that in order to prove

$$\tau\lim_{s\to 0}\frac{T(s)x-x}{s^\alpha}=0$$

we only have to check that

$$\frac{p(T(s_n)x-x)}{s_n^\alpha}\to 0$$

for $n \to \infty$ for every null-sequence $(s_n)_{n\in\mathbb{N}}$ in $[0,\infty)$ and for each $p \in \mathcal{P}$. Let $x \in X_\alpha$. Then we have that $y_n := \frac{T(s_n)x-x}{s_n^\alpha}$ converges to 0 with respect to τ if $(s_n)_{n\in\mathbb{N}}$ is any null-sequence and $n \to \infty$. Moreover, this sequence $(y_n)_{n\in\mathbb{N}}$ is $\|\cdot\|$-bounded by the assumption that $x \in X_\alpha$. Whence we conclude

$$\tau\lim_{n\to\infty} T(t)y_n = \tau\lim_{n\to\infty} \frac{T(s_n)T(t)x - T(t)x}{s_n^\alpha} = 0,$$

so that $T(t)x \in X_\alpha$.

□

We now prove that $(T(t))_{t\geq 0}$ is bi-continuous on X_α and notice first that the local boundedness and the semigroup property are trivial.

Lemma 2.48. *If $(T(t))_{t\geq 0}$ is a bi-continuous semigroup on X_0 and $\alpha \in (0,1)$, then $(T(t))_{t\geq 0}$ is strongly τ_{F_α}-continuous on X_α.*

Proof. We have to show that $p_{F_\alpha}(T(t_n)x-x) \to 0$ for all $p \in \mathcal{P}$ whenever $t_n \downarrow 0$. Let $s_n, t_n > 0$ be with $s_n, t_n \to 0$. Then

$$\begin{aligned}\frac{p(T(s_n)T(t_n)x - T(s_n)x - T(t_n)x + x)}{s_n^\alpha} &\leq \frac{p(T(t_n)T(s_s)x - T(t_n)x)}{s_n^\alpha} + \frac{p(T(s_n)x - x)}{s_n^\alpha} \\ &= \frac{p(T(t_n)(T(s_n)x - x))}{s_n^\alpha} + \frac{p(T(s_n)x - x)}{s_n^\alpha}. \quad (2.4.6)\end{aligned}$$

The sequence (y_n) given by $y_n := \frac{T(s_n)x-x}{s_n^\alpha}$ is $\|\cdot\|$-bounded and τ-convergent to 0, because $x \in X_\alpha$. So that the last term in the previous equation (2.4.6) converges to 0. But since $\{T(t_n) : n \in \mathbb{N}\}$ is bi-equicontinuous, also the first term in (2.4.6) converges to 0. This proves strong continuity with respect to τ_{F_α}.

□

Lemma 2.49. *Let $(T(t))_{t\geq 0}$ be a bi-continuous semigroup on X_0. Then $(T(t))_{t\geq 0}$ is locally bi-equicontinuous on F_α.*

Proof. Let $(x_n)_{n\in\mathbb{N}}$ be a $\|\cdot\|_{F_\alpha}$-bounded sequence which converges to zero with respect to τ_{F_α} and assume that $(T(t)x_n)_{n\in\mathbb{N}}$ does not converge to zero uniformly for $t \in [0,t_0]$ for some $t_0 > 0$. Hence there exists $p \in \mathcal{P}$, $\delta > 0$ and a sequence $(t_n)_{n\in\mathbb{N}}$ of positive real numbers such that

$$p_{F_\alpha}(T(t_n)x_n) > \delta$$

for all $n \in \mathbb{N}$. As a consequence there exists a null-sequence $(s_n)_{n\in\mathbb{N}}$ in $\mathbb{R}$ such that

$$\frac{p(T(s_n)T(t_n)x_n - T(t_n)x_n)}{s_n^\alpha} > \delta$$

for each $n \in \mathbb{N}$. Now notice that the sequence $(y_n)_{n\in\mathbb{N}}$ defined by $y_n := \frac{T(s_n)x_n-x_n}{s_n^\alpha}$ is a τ-null sequence since:

$$\frac{q(T(s_n)x_n - x_n)}{s_n^\alpha} \leq \sup_{s>0} \frac{q(T(s)x_n - x_n)}{s^\alpha}, \quad q \in \mathcal{P},$$

and the term on the right hand side converges to zero as $n \to \infty$ by assumption. Using the local bi-equicontinuity of the semigroup $(T(t))_{t\geq 0}$ with respect to τ, we conclude that $\frac{T(t)T(s_n)x_n - T(t)x_n}{s_n}$ converges to zero uniformly for $t \in [0, t_0]$, which is a contradiction. Hence $(T(t))_{t\geq 0}$ is locally bi-equicontinuous on X_α.

□

Remark 2.50. Notice that the local bi-equicontinuity with respect to τ_{F_α} holds on the whole space F_α, while strong τ_{F_α}-continuity holds on X_α only. In particular, we will see in Theorem 2.52 that X_α is the space of strong τ_{F_α}-continuity.

We can summarize the previous results in the following theorem.

Theorem 2.51. *Let $(T(t))_{t\geq 0}$ be a bi-continuous semigroup on X_0. Then the restricted operators $T_\alpha(t) := T(t)|_{X_\alpha}$ on X_α form a bi-continuous semigroup. Moreover, the generator A_α of $(T_\alpha(t))_{t\geq 0}$ is the part of A in X_α.*

Proof. Because of the previous series of lemmas it remains to prove that the part of A in X_α generates the restricted semigroup on X_α. We can argue as in the proof of the proposition in [52, Chap. II, Par. 2.3]. Since the embedding $X_\alpha \subseteq X_0$ is continuous for the topologies τ_{F_α} and τ, we conclude that $A_\alpha \subseteq A|_{X_\alpha}$. For the converse let C denote the generator of $(T_\alpha(t))_{t\geq 0}$ and take $\lambda \in \mathbb{R}$ large enough such that

$$R(\lambda, C)x = \int_0^\infty \mathrm{e}^{-\lambda s} T(s)x \, \mathrm{d}s = R(\lambda, A)x, \quad x \in X_\alpha.$$

For $x \in D(A|_{X_\alpha})$ we obtain

$$x = R(\lambda, A)(\lambda - A)x = R(\lambda, C)(\lambda - A)x \in D(C)$$

and hence $A|_{X_\alpha} \subseteq A_\alpha$. This proves that the part of A in X_α generates the restricted semigroup.

□

By similar reasoning as in Lemma 2.31 one can prove the following:

Theorem 2.52. *Let $\alpha \in (0, 1)$ and let $(T(t))_{t\geq 0}$ be a bi-continuous semigroup on X. Then $D(A)$ is τ_{F_α}-bi-dense in X_α and*

$$X_\alpha = \{x \in F_\alpha : \ \tau_{F_\alpha} \lim_{t\to 0} T(t)x = x\}, \tag{2.4.7}$$

i.e., for $x \in F_\alpha$ the mapping $t \mapsto T(t)x$ is τ_{F_α}-continuous if and only if $x \in X_\alpha$.

Proof. Denote by $X_{\alpha,\mathrm{cont}}$ the right-hand side of (2.4.7), i.e., the space of τ_{F_α}-strong continuity. Notice that $D(A) \subseteq \underline{X}_\alpha \subseteq X_\alpha \subseteq X_{\alpha,\mathrm{cont}}$.

Suppose $x \in X_{\alpha,\mathrm{cont}}$. For each $n \in \mathbb{N}$ we have

$$x_n := n \int_0^{\frac{1}{n}} T_\alpha(t)x \, \mathrm{d}t = n \int_0^{\frac{1}{n}} T(t)x \, \mathrm{d}t \in D(A)$$

as a τ- and τ_{F_α}-convergent Riemann integral. Whence it follows that $x_n \overset{\tau_{F_\alpha}}{\to} x$, whereas the $\|\cdot\|_{F_\alpha}$-boundedness of $(x_n)_{n\in\mathbb{N}}$ clear. We conclude that $x \in X_\alpha$ (because X_α is bi-closed in F_α), implying $X_{\alpha,\mathrm{cont}} \subseteq X_\alpha$. As a byproduct we also obtain that $D(A)$ is bi-dense in X_α.

□

Proposition 2.53. *For $0 \le \alpha < \beta \le 1$ we have*

$$X_1 = D(A) \hookrightarrow F_\beta \hookrightarrow \underline{X}_\alpha \subseteq X_\alpha,$$

where the embeddings are continuous for the respective norms and for the respective topologies τ_1, τ_{F_β}, τ_{F_α}. The space $D(A)$ bi-dense in X_α, and as a consequence X_β is bi-dense in X_α.

2.4.4 Representation of Hölder spaces by generators

Analogously to Proposition 2.32 we have a representation of the Hölder space X_α by means of the semigroup generator.

Theorem 2.54. *Let $(T(t))_{t\ge 0}$ be a bi-continuous semigroup with negative growth bound and generator A. For $\alpha \in (0,1)$ we have*

$$X_\alpha = \Big\{x \in X_0 : \ \tau\lim_{\lambda\to\infty} \lambda^\alpha AR(\lambda,A)x = 0 \text{ and } \sup_{\lambda>0} \|\lambda^\alpha AR(\lambda,A)x\| < \infty\Big\}. \tag{2.4.8}$$

Proof. Suppose $x \in X_\alpha$. From Proposition 2.32 we deduce immediately

$$\sup_{\lambda>0} \|\lambda^\alpha AR(\lambda,A)x\| < \infty.$$

Let now $\varepsilon > 0$ be arbitrary. For $x \in X_\alpha$ and $p \in \mathcal{P}$ we can find $\delta > 0$ such that $0 \le t < \delta$ implies $\frac{p(T(t)x-x)}{t^\alpha} < \varepsilon$. Recall the following formula:

$$\lambda^\alpha AR(\lambda,A)x = \lambda^{\alpha+1}\int_0^\infty \mathrm{e}^{-\lambda s}(T(s)x-x)\ \mathrm{d}s.$$

From this we deduce

$$\begin{aligned}
p(\lambda^\alpha AR(\lambda,A)x) &\le \lambda^{\alpha+1}\int_0^\infty \mathrm{e}^{-\lambda s}\cdot\frac{p(T(s)x-x)}{s^\alpha}s^\alpha\ \mathrm{d}s\\
&= \lambda^{\alpha+1}\int_0^\delta \mathrm{e}^{-\lambda s}\cdot\frac{p(T(s)x-x)}{s^\alpha}s^\alpha\ \mathrm{d}s + \lambda^{\alpha+1}\int_\delta^\infty \mathrm{e}^{-\lambda s}\cdot\frac{p(T(s)x-x)}{s^\alpha}s^\alpha\ \mathrm{d}s\\
&< \lambda^{\alpha+1}\varepsilon\int_0^\delta \mathrm{e}^{-\lambda s}s^\alpha\ \mathrm{d}s + \lambda^{\alpha+1}\int_\delta^\infty \mathrm{e}^{-\lambda s}\cdot\frac{\|T(s)x-x\|}{s^\alpha}s^\alpha\ \mathrm{d}s\\
&\le \lambda^{\alpha+1}\varepsilon\int_0^\delta \mathrm{e}^{-\lambda s}s^\alpha\ \mathrm{d}s + \|x\|_{F_\alpha}\lambda^{\alpha+1}\int_\delta^\infty \mathrm{e}^{-\lambda s}\cdot s^\alpha\ \mathrm{d}s\\
&= \varepsilon\int_0^{\lambda\delta} \mathrm{e}^{-t}t^\alpha\ \mathrm{d}t + \|x\|_{F_\alpha}\int_{\lambda\delta}^\infty \mathrm{e}^{-t}t^\alpha\ \mathrm{d}t\\
&\le L\varepsilon + \|x\|_{F_\alpha}\int_{\lambda\delta}^\infty \mathrm{e}^{-t}t^\alpha\ \mathrm{d}t
\end{aligned}$$

where $L := \int_0^\infty \mathrm{e}^{-\lambda s}s^\alpha\ \mathrm{d}s < \infty$. Notice that the last part of the sum tends to zero if $\lambda \to \infty$ since $\delta > 0$ is fixed. So we obtain $\tau\lim_{\lambda\to\infty} \lambda^\alpha AR(\lambda,A)x = 0$.

For the converse inclusion suppose that

$$\tau\lim_{\lambda\to\infty} \lambda^\alpha AR(\lambda, A)x = 0 \text{ and } \sup_{\lambda>0} \|\lambda^\alpha AR(\lambda, A)x\| < \infty,$$

the latter immediately implying $\|x\|_{F_\alpha(T)} < \infty$ (see Proposition 2.32). We have to show that $\tau\lim_{t\to 0} \frac{T(t)x-x}{t^\alpha} = 0$. For $\lambda > 0$ define $x_\lambda = \lambda R(\lambda, A)$ and $y_\lambda = AR(\lambda, A)$, then we have

$$x = \lambda R(\lambda, A)x - AR(\lambda, A)x = x_\lambda - y_\lambda.$$

First notice that for $p \in \mathcal{P}$

$$\frac{p(T(t)x_\lambda - x_\lambda)}{t^\alpha} \leq \frac{1}{t^\alpha} p(T(t)\lambda R(\lambda, A)x - \lambda R(\lambda, A)x) \leq \frac{\lambda^{1-\alpha}}{t^\alpha} \int_0^t p(T(s)\lambda^\alpha AR(\lambda, A)x) \,\mathrm{d}s. \tag{2.4.9}$$

By assumption the term $\lambda^\alpha AR(\lambda, A)x$ is norm-bounded and converges in the topology τ to zero as $\lambda \to \infty$, hence by the local bi-equicontinuity we conclude that $p(T(s)\lambda^\alpha AR(\lambda, A)x) \to 0$ uniformly for $s \in [0, 1]$. Now let $\varepsilon > 0$ and $\lambda_0 > 1$ so large that for $\lambda > \lambda_0$ and $s \in [0, 1]$ we have $p(T(s)\lambda^\alpha AR(\lambda, A)x) < \varepsilon$. If $t < \frac{1}{\lambda_0}$, then $\lambda := \frac{1}{t} > \lambda_0$ and we obtain that the expression in (2.4.9) becomes smaller than ε.
For the estimate of the second part involving y_λ we observe that

$$\frac{p(T(t)y_\lambda - y_\lambda)}{t^\alpha} \leq \frac{1}{(t\lambda)^\alpha} p(T(t)\lambda^\alpha AR(\lambda, A)x) + \frac{1}{(t\lambda)^\alpha} p(\lambda^\alpha AR(\lambda, A)x).$$

By taking $t < \frac{1}{\lambda_0}$ and $\lambda := \frac{1}{t}$ we obtain the estimate

$$\frac{p(T(t)y_\lambda - y_\lambda)}{t^\alpha} \leq p(T(\tfrac{1}{\lambda})\lambda^\alpha AR(\lambda, A)x) + p(\lambda^\alpha AR(\lambda, A)x) < \varepsilon + \varepsilon, \tag{2.4.10}$$

by the choice of λ_0. Altogether we obtain for $t < \frac{1}{\lambda_0}$ that $\frac{p(T(t)x-x)}{t^\alpha} < 3\varepsilon$, showing

$$\tau\lim_{t\to 0} \frac{T(t)x - x}{t^\alpha} = 0,$$

i.e., $x \in X_\alpha$ as required.

□

Remark 2.55. It is possible to define the space $X_\alpha(A)$ as in (2.4.8) also for operators which are not necessarily generators of bi-continuous semigroups. However, we have to suppose that the resolvent fulfills certain continuity assumptions with respect to a topology satisfying, say, Assumption 1.1.

Again, we put our spaces X_α in the general context of Theorem 2.16.

Proposition 2.56. *For $\alpha \in (0, 1)$ and $\mathcal{A}$, λ and $\mathscr{E}$ as in Theorem 2.16 we have*

$$X_{-\alpha} = \Big\{(\lambda - \mathcal{A})y \in X_{-1} : \sup_{t>0} \frac{\|T(t)y - y\|}{t^{1-\alpha}} < \infty,\ \tau\lim_{t\to 0} \frac{T(t)y - y}{t^{1-\alpha}} = 0\Big\}.$$

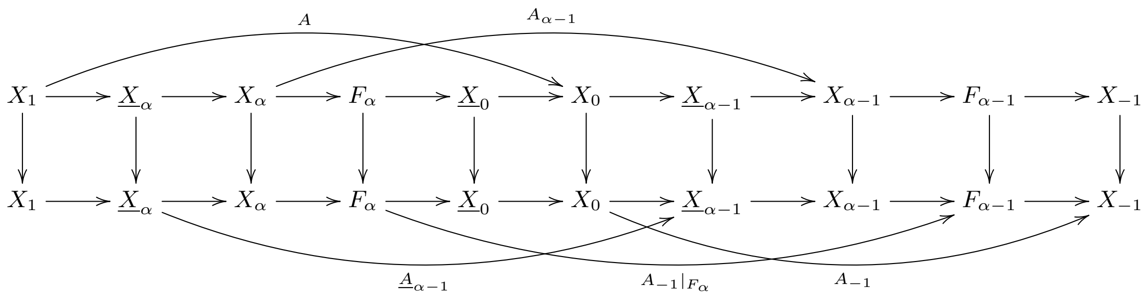
Finally, we extend the scale of spaces X_α to the whole range $\alpha \in \mathbb{R}$.

Definition 2.57. For $\alpha \in \mathbb{R} \setminus \mathbb{Z}$ we write $\alpha = m + \beta$ with $m \in \mathbb{Z}$ and $\beta \in (0, 1]$, and define

$$X_\alpha(A) := X_\beta(A_m),$$

with the corresponding norms. The locally convex topology on X_α comes from X_β via the mapping A_m.

Remark 2.58. We summarize all previous results in the following diagram:



where $\alpha \in (0, 1)$. Here $A_{\alpha-1}$ and $\underline{A}_{\alpha-1}$ are defined to be the part of A_{-1} in $X_{\alpha-1}$ and the part of $\underline{A}_{-1}$ in $\underline{X}_{\alpha-1}$, respectively. They are all continuous with respect to the norms and topologies on these spaces. In addition, we recall that $X_{\alpha-1}$ and $\underline{X}_{\alpha-1}$ are the extrapolation spaces of $X_\alpha(A_{-1})$ and $\underline{X}_\alpha(A_{-1})$, respectively. All horizontal arrows represent continuous inclusions, while the vertical arrows represent the action(s) of the semigroup(s). All the spaces are dense in the underlined ones containing them, while the spaces with underlining are bi-dense in each of the bigger ones.

§ 2.5 Examples

In this section we present examples for extrapolation and intermediate spaces for (generators of) bi-continuous semigroups. We will use Theorem 2.16 and its variants to identify the space X_α for $\alpha < 0$.

2.5.1 The left-translation semigroup

We now consider the left-translation semigroup from Section 1.3.1 and use Theorem 2.16 to determine the corresponding extrapolation spaces. To this purpose let $\mathscr{E} = \mathscr{D}'(\mathbb{R})$ be the space of all distributions on $\mathbb{R}$, let $\mathcal{A} := D : \mathscr{D}'(\mathbb{R}) \to \mathscr{D}'(\mathbb{R})$ be the distributional derivative, and let $i : \mathrm{C_b}(\mathbb{R}) \to \mathscr{D}'(\mathbb{R})$ be the regular embedding. From Theorem 2.16 it then follows

$$\underline{X}_{-1} = \{F \in \mathscr{D}'(\mathbb{R}) : \ F = f - Df \text{ for some } f \in \mathrm{UC_b}(\mathbb{R})\},$$
$$X_{-1} = \{F \in \mathscr{D}'(\mathbb{R}) : \ F = f - Df \text{ for some } f \in \mathrm{C_b}(\mathbb{R})\}.$$

For the Favard and Hölder spaces we have

$$F_\alpha = \Big\{ f \in \mathrm{C_b}(\mathbb{R}) : \ \sup_{\substack{x,y\in\mathbb{R}\\ x\neq y}} \frac{|f(x) - f(y)|}{|x-y|^\alpha} < \infty \Big\} = \mathrm{C}^\alpha_{\mathrm{b}}(\mathbb{R}),$$

$$\underline{X}_\alpha = \Big\{ f \in \mathrm{UC_b}(\mathbb{R}) : \lim_{t\to 0} \sup_{\substack{x,y\in\mathbb{R}\\ 0<|x-y|<t}} \frac{|f(x)-f(y)|}{|x-y|^\alpha} = 0 \Big\} = \mathrm{h}^\alpha_\mathrm{b}(\mathbb{R}).$$

Hence F_α is the space of bounded α-Hölder-continuous functions and $\underline{X}_\alpha$ with the so-called little Hölder space $\mathrm{h}^\alpha_\mathrm{b}(\mathbb{R})$ (see also [85]). The abstract Hölder space X_α corresponding to the bi-continuous semigroup yields the local version $\mathrm{h}^\alpha_{\mathrm{b,loc}}(\mathbb{R})$ of the little Hölder space

$$\mathrm{h}^\alpha_{\mathrm{b,loc}} = \Big\{ f \in \mathrm{C}^\alpha_\mathrm{b}(\mathbb{R}) : \lim_{t\to 0} \sup_{\substack{x,y\in K\\ 0<|x-y|<t}} \frac{|f(x)-f(y)|}{|x-y|^\alpha} = 0 \text{ for each } K \subseteq \mathbb{R} \text{ compact} \Big\}.$$

Then $X_\alpha = h^\alpha_{\mathrm{b,loc}}(\mathbb{R})$.

It is easy to see $\underline{X}_\alpha \subsetneq X_\alpha \subsetneq F_\alpha$. The extrapolated Favard class F_0 can be identified with $\mathrm{L}^\infty(\mathbb{R})$. To prove this we argue as follows: We know from the general theory that $F_0(T) = (1-D)F_1(T)$ where $F_1(T)$ are precisely the bounded Lipschitz functions on $\mathbb{R}$. Now using the fact that $\mathrm{Lip_b}(\mathbb{R}) = \mathrm{W}^{1,\infty}(\mathbb{R})$ with equivalent norms we obtain that indeed $F_0 = \mathrm{L}^\infty(\mathbb{R})$. For an alternative proof of this fact we refer to [52, Chapter II.5(b)].

Moreover, from Corollary 2.26 we obtain for $\alpha \in (0,1)$

$$F_{-\alpha} = \Big\{ f \in \mathscr{D}'(\mathbb{R}) : \quad F = f - Df \text{ for } f \in \mathrm{C}^{1-\alpha}_\mathrm{b}(\mathbb{R}) \Big\},$$

and

$$X_{-\alpha} = \Big\{ f \in \mathscr{D}'(\mathbb{R}) : \quad F = f - Df \text{ for } f \in \mathrm{h}^{1-\alpha}_{\mathrm{b,loc}}(\mathbb{R}) \Big\}.$$

We summarize this example by the diagram:

$$\mathrm{C}^1_\mathrm{b}(\mathbb{R}) \hookrightarrow \mathrm{Lip_b}(\mathbb{R}) \hookrightarrow \mathrm{h}^\alpha_\mathrm{b}(\mathbb{R}) \hookrightarrow \mathrm{h}^\alpha_{\mathrm{b,loc}}(\mathbb{R}) \hookrightarrow \mathrm{C}^\alpha_\mathrm{b}(\mathbb{R}) \hookrightarrow \mathrm{UC_b}(\mathbb{R}) \hookrightarrow \mathrm{C_b}(\mathbb{R}) \hookrightarrow \mathrm{L}^\infty(\mathbb{R})$$

according to the abstract chain of spaces

$$X_1 \hookrightarrow F_1 \hookrightarrow \underline{X}_\alpha \hookrightarrow X_\alpha \hookrightarrow F_\alpha \hookrightarrow \underline{X}_0 \hookrightarrow X_0 \hookrightarrow F_0$$

for $\alpha \in (0,1)$. For the higher order spaces we have

$$\begin{aligned} X_n := D(A^n) &= \Big\{ f \in \mathrm{C_b}(\mathbb{R}) : f \text{ is } n\text{-times differentiable and } f^{(n)} \in \mathrm{C_b}(\mathbb{R}) \Big\} \\ &= \Big\{ f \in \mathrm{C_b}(\mathbb{R}) : f^{(k)} \in \mathrm{C_b}(\mathbb{R}),\ k = 1,\dots,n \Big\} = \mathrm{C}^n_\mathrm{b}(\mathbb{R}) \end{aligned}$$

for $n \in \mathbb{N}$. For $n \in \mathbb{N}$ and $\alpha \in [0,1)$

$$F_{n+\alpha} = \Big\{ f \in \mathrm{C}^n_\mathrm{b}(\mathbb{R}) : \sup_{\substack{x,y\in\mathbb{R}\\ x\neq y}} \frac{|f^{(n)}(x) - f^{(n)}(y)|}{|x-y|^\alpha} < \infty \Big\} = \mathrm{C}^{n,\alpha}_\mathrm{b}(\mathbb{R}).$$

This example complements the corresponding one in Nagel, Nickel, Romanelli [93, Sec. 3.2].

2.5.2 The multiplication semigroup

Let Ω be a locally compact space and $X_0 = \mathrm{C_b}(\Omega)$. Let $q : \Omega \to \mathbb{C}$ be continuous such that $\sup_{x\in\Omega} \mathrm{Re}(q(x)) < 0$. We define the multiplication operator $M_q : D(M_q) \to \mathrm{C_b}(\Omega)$ by $M_q f = qf$ on the maximal domain

$$D(M_q) = \{f \in \mathrm{C_b}(\Omega) : \; qf \in \mathrm{C_b}(\Omega)\}.$$

This operator generates the semigroup $(T_q(t))_{t\geq 0}$ defined by

$$(T_q(t)f)(x) = \mathrm{e}^{tq(x)} f(x), \quad t \geq 0, \; x \in \Omega, \; f \in \mathrm{C_b}(\Omega),$$

which is bi-continuous on $\mathrm{C_b}(\Omega)$ with respect to the compact-open topology. Now let $\mathscr{E} = \mathrm{C}(\Omega)$ the space of all continuous functions on Ω, let $\mathcal{M}_q : \mathrm{C}(\Omega) \to \mathrm{C}(\Omega)$ be the multiplication operator $\mathcal{M}_q f := qf$ and $i : \mathrm{C_b}(\Omega) \to \mathrm{C}(\Omega)$ the identical embedding. Then by Theorem 2.16 we obtain

$$X_{-1} = \{g \in \mathrm{C}(\Omega) : \; q^{-1} g \in \mathrm{C_b}(\Omega)\}.$$

For $\alpha \in (0,1)$, the (abstract) Favard space is

$$F_\alpha = \{f \in \mathrm{C_b}(\Omega) : \; |q|^\alpha f \in \mathrm{C_b}(\Omega)\}. \tag{2.5.1}$$

To see this suppose first that $f \in F_\alpha$, which means

$$\sup_{t>0} \sup_{x\in\Omega} \frac{|\mathrm{e}^{tq(x)} f(x) - f(x)|}{t^\alpha} < \infty.$$

By taking supremum only for $t = \frac{1}{|q(x)|}$ we obtain

$$\sup_{x\in\Omega} \left|\mathrm{e}^{\frac{q(x)}{|q(x)|}} - 1\right| \cdot |f(x)| \cdot |q(x)|^\alpha < \infty,$$

since

$$\frac{|\mathrm{e}^{tq(x)} f(x) - f(x)|}{t^\alpha} = \frac{|\mathrm{e}^{tq(x)} - 1| \cdot |f(x)||q(x)|^\alpha}{|q(x)|^\alpha t^\alpha}. \tag{2.5.2}$$

Hence $|q|^\alpha f \in \mathrm{C_b}(\Omega)$, so that the inclusion "$\subseteq$" in (2.5.1) is established. For the converse assume that $|q|^\alpha f \in \mathrm{C_b}(\Omega)$. Since the function $g(z) = \frac{|\mathrm{e}^z - 1|}{|z|^\alpha}$ is bounded on the left half-plane, we obtain that $f \in F_\alpha$ by (2.5.2). This proves the equality. We also conclude that $F_\alpha = X_\alpha$ since

$$\sup_{x\in K} \left| \frac{\mathrm{e}^{tq(x)} f(x) - f(x)}{t^\alpha} \right| = \sup_{x\in K} \left| \frac{\mathrm{e}^{tq(x)} - 1}{tq(x)} \right| \cdot |f(x)| \cdot |q(x)|^\alpha \, t^{1-\alpha}$$

for each compact set $K \subseteq \Omega$. The extrapolated Favard spaces are then given by

$$F_{-\alpha} = \{f \in \mathrm{C_b}(\Omega) : \; |q|^{1-\alpha} f \in \mathrm{C_b}(\Omega)\} = X_{-\alpha}.$$

The spaces $\underline{X}_\alpha$ are more difficult to describe in general since the space of strong continuity $\underline{X}_0$ depends substantially on the choice of q. For example, if $\frac{1}{q} \in \mathrm{C_0}(\Omega)$, then $\underline{X}_0 = \mathrm{C_0}(\Omega)$.

To see this notice that $\mathrm{C}_0(\Omega) \subseteq \underline{X}_0$ trivially. On the other hand

$$|f| = \left|\frac{1}{q}\right| \cdot |fq|$$

which shows that $D(M_q) \subseteq \mathrm{C}_0(\Omega)$ and hence that $\underline{X}_0 \subseteq \mathrm{C}_0(\Omega)$. For $\alpha \in [0,1]$ this yields

$$\underline{X}_\alpha = \{f \in \mathrm{C}_0(\Omega) : |q|^\alpha f \in \mathrm{C}_0(\Omega)\},$$

and

$$\underline{X}_{-\alpha} = \{qf : f \in \mathrm{C}_0(\Omega),\ |q|^{1-\alpha} f \in \mathrm{C}_0(\Omega)\} = \{f \in \mathrm{C}(\Omega) : |q|^{-\alpha} f \in \mathrm{C}_0(\Omega)\}.$$

This example extends Section 3.2 in [93] by Nagel, Nickel and Romanelli.

2.5.3 The Gauss–Weierstrass semigroup

On $X_0 = \mathrm{C_b}(\mathbb{R}^d)$ $(d \geq 1)$ we consider the Gauß–Weierstraß semigroup defined by $T(0) = I$ and

$$T(t)f(x) = \frac{1}{(4\pi t)^{\frac{d}{2}}} \int_{\mathbb{R}^d} \mathrm{e}^{-\frac{|x-y|^2}{4t}} f(y)\ \mathrm{d}y, \quad t > 0,\ x \in \mathbb{R}^d. \tag{2.5.3}$$

If we equip $\mathrm{C_b}(\mathbb{R}^d)$ with the compact-open topology τ_{co}, then $(T(t))_{t\geq 0}$ becomes a bi-continuous semigroup, and its space of strong continuity is $\mathrm{UC_b}(\mathbb{R}^d)$. From [82, Proposition 2.3.6] we know that the generator A of this semigroup is given $Af = \Delta f$ on the maximal domain

$$D(A) = \{f \in \mathrm{C_b}(\mathbb{R}^d) : \quad \Delta f \in \mathrm{C_b}(\mathbb{R}^d)\},$$

where Δ is the distributional Laplacian. Now the extrapolation space can again be obtained by Theorem 2.16. If we take $\mathscr{E} = \mathscr{D}'(\mathbb{R}^d)$, $\mathcal{A} = \Delta$ and $i : \mathrm{C_b}(\mathbb{R}^d) \to \mathscr{D}'(\mathbb{R}^d)$ the regular embedding we then have

$$X_{-1} = \{F \in \mathscr{D}'(\mathbb{R}^d) :\ F = f - \Delta f \text{ for some } f \in \mathrm{C_b}(\mathbb{R}^d)\}.$$

The domain of the generator can be given explicitly, see, e.g., [82] or [85]. For $d = 1$ it is

$$D(A) = \mathrm{C}^2_\mathrm{b}(\mathbb{R}),$$

while for $d \geq 2$

$$D(A) = \Big\{f \in \mathrm{C_b}(\mathbb{R}^d) \cap \mathrm{W}^{2,p}_{\mathrm{loc}}(\mathbb{R}^d),\ \text{for all } p \in [1,\infty) \text{ and } \Delta f \in \mathrm{C_b}(\mathbb{R}^d)\Big\}.$$

For $\alpha \in (0,1) \setminus \{\frac{1}{2}\}$ the Favard spaces are

$$F_\alpha = \mathrm{C}^{2\alpha}_\mathrm{b}(\mathbb{R}^d),$$

while for $\alpha = \frac{1}{2}$ one obtains

$$F_{\frac{1}{2}} = \Big\{f \in \mathrm{C_b}(\mathbb{R}^d) :\ \sup_{x \neq y} \frac{|f(x) + f(y) - 2f(\frac{x+y}{2})|}{|x-y|} < \infty\Big\}.$$

From Corollary 2.26 it follows that for $\alpha \in (0,1)$, $\alpha \neq \frac{1}{2}$

$$F_{-\alpha} = \Big\{ F \in \mathscr{D}'(\mathbb{R}^d) :\ F = f - \Delta f \text{ for some } f \in \mathrm{C}_{\mathrm{b}}^{2(1-\alpha)}(\mathbb{R}^d) \Big\},$$

and

$$F_{-\frac{1}{2}} = \Big\{ F \in \mathscr{D}'(\mathbb{R}^d) :\ F = f - \Delta f \text{ for some } f \in F_{\frac{1}{2}} \Big\}.$$

§ 2.6 Notes

As already mentioned in the introduction this chapter is based the joint work with B. Farkas [28]. In this thesis we added Section 2.3.1 where we highlight the theory of interpolation spaces. In particular we give a proof of Proposition 2.37.

Another interesting example, which is originally part of [28], is now part of Chapter 6 where we discuss the implemented semigroups. A further application of extrapolation spaces is going to be the issue of the following Chapter 3, where we deal with so called fiberwise extrapolation spaces of unbounded operator-valued multiplication operators. An abstract usage of extrapolation spaces takes place in Chapter 4 in conjunction with perturbation theory.

In actual fact, the article [28], where this chapter is based on, came into being while working on the Desch–Schappacher perturbation which we present in Chapter 4. In particular, the absence of extrapolation spaces for non-densely defined Hille–Yosida operators was decisive for the emergence of this work. Even if we can construct extrapolation spaces by restricting to the space of strong continuity this was not adequate in order to stay in the category of bi-continuous semigroups. During the process of writing also intermediate spaces attract attention since they are commonly discussed in the same breath with extrapolation spaces, cf. [52, Chapter II, Sect. 5], hence we added this part of research to put it into a bigger picture.

Chapter 3

Fiberwise Multiplication

Introduction

Non-autonomous problems arise naturally for example in the context of diffusion processes with time-dependent diffusion coefficients or boundary conditions or in connection with quantum mechanical systems with time-varying potential. They can be described by an abstract Cauchy problem on a Banach space X of the form

$$\begin{cases} \dot{u}(t) = A(t)u(t), & t, s \in \mathbb{R},\ t \geq s, \\ u(s) = x, \end{cases} \tag{nACP}$$

where $(A(t), D(A(t)))_{t\in\mathbb{R}}$ is a family of linear operators. The solution of such a problem, if it exists, is no longer given by a semigroup but a so-called evolution family $(U(t,s))_{t\geq s}$, see for example [52, Chapter VI, Def. 9.2]. The existence of solutions and the well-posedness of (nACP), cf. [52, Chapter VI, Def. 9.1] and [113, Def. 3.4.1], is a challenging topic. In particular, there are just a few different independent and not unified results on existence of solutions for the non-autonomous case due to P. Acquistapace and B. Terreni [2, 1] or T. Kato and H. Tanabe [111, 112, 72, 73]. Nevertheless, if we have a solution by means of an evolution family, we obtain a strongly continuous semigroup $(T(t))_{t\geq 0}$ on the Bochner space $\mathrm{L}^p(\mathbb{R}, X)$ by

$$(T(t)f)(s) := U(s, s-t)f(s-t), \quad t \geq 0,\ f \in \mathrm{L}^p(\mathbb{R}, X),\ s \in \mathbb{R}. \tag{3.0.1}$$

One important challenge is to determine the exact domain of the corresponding generator $(G, D(G))$. By [113, Thm. 3.4.7] operator-valued multiplication operators come up naturally since $Gf = A(\cdot)f - f'$ on an invariant core $D \subseteq \mathrm{W}^{1,p}(\mathbb{R}, X) \cap D(A(\cdot))$. However, one wants to have the generator $(G, D(G))$ explicitly in hand in order to infer properties of the evolution semigroup and hence of the evolution family. For this purpose, one has to consider extrapolation spaces of operator-valued multiplication operators. One of the attempts, in the special case where $A(t) \equiv A$ for some semigroup generator $(A, D(A))$, is due to Nagel, Nickel and Romanelli [93, Sect. 4]. In [59] T. Graser studied bounded and unbounded operator-valued multiplication operators on the space of continuous functions $\mathrm{C}_0(\mathbb{R}, X)$ as well as their extrapolation spaces. We will see that extrapolation spaces of multiplication operators on $\mathrm{L}^p(\mathbb{R}, X)$ behave similarly. Later on, S. Thomaschewski studied properties of such multiplication operators on Bochner L^p-spaces [113, Sect. 2.2 & 2.3] in connection with non-autonomous problems. Especially, she connects multiplication semigroups with unbounded operator-valued multiplication operators. Although we are now interested in extrapolation spaces of such unbounded multiplication operators on Bochner L^p-spaces, we can use some of the results from S. Thomaschewski regarding multiplication operators. In order to construct extrapolation spaces, we recall the notion of fiber integrable functions due to R. Heymann [65].

We start this chapter with some preliminaries on fiber integrable functions and continue with unbounded multiplication operators in the second section. In Section 3.3 we discuss multiplication semigroups whose generators are actually multiplication operators. Furthermore, we determine the extrapolation spaces of such multiplication operators by means of L^p-fiber spaces.

§ 3.1 L^p-fiber spaces

Firstly, we introduce the essential notion of a measurable Banach fiber set which was studied by R. Heymann, cf. [65, Def. VI.1.i].To do so, let (Ω, Σ, μ) be a σ-finite measure space. Furthermore, let V be a complex vector space together with a family of seminorms $\{|||\cdot|||_s : \ s \in \Omega\}$ on V. Assume that there exists a countable set of elements $\mathcal{B} := \{b_k : \ k \in \mathbb{N}\} \subseteq V$ such that $\mathcal{B}$ is a vector space over $\mathbb{Q}+i\mathbb{Q}$ and such that for each $k \in \mathbb{N}$ the map $s \mapsto |||b_k|||_s$ is measurable as a map from Ω to $\mathbb{R}$. For every $s \in \Omega$ we define the set $N_s := \{b_k \in \mathcal{B} : \ |||b_k|||_s = 0\}$ and take the completion of the quotient space $\mathcal{B}/N_s$ with respect to the induced norms $\|\cdot\|_s$ on $\mathcal{B}/N_s$. This Banach space is denoted by X_s.

Definition 3.1. The family of Banach spaces $(X_s, \|\cdot\|_s)_{s\in\Omega}$ is called a *measurable Banach fiber set.*

Next, we follow [65, Def. VI.1.iii] in order to define what it means for a function $f : \Omega \to \bigcup_{s\in\Omega} X_s$ to be measurable

Definition 3.2. Let $(X_s, \|\cdot\|_s)_{s\in\Omega}$ be a measurable Banach fiber set. We define a function $f : \Omega \to \bigcup_{s\in\Omega} X_s$ with $f(s) \in X_s$ for μ-almost every $s \in \Omega$ to be *fiber measurable* if it is almost everywhere a pointwise limit with respect to $\|\cdot\|_s$ of measurable simple functions with values in $\mathcal{B}$. More precisely, this means that there exists a sequence $(f_j)_{j\in\mathbb{N}}$ of functions $f_j : \Omega \to \bigcup_{s\in\Omega} X_s$ such that

1. $f_j = \displaystyle\sum_{i=1}^{n_j} \Big(b_{\pi(i)} + N_s\Big)\, \mathbf{1}_{\Omega_i}$, where $\pi : \mathbb{N} \to \mathbb{N}$, $n_j \in \mathbb{N}$, $\Omega_i \in \Sigma$, $\Omega_i \cap \Omega_j = \varnothing$, $i \neq j$, and $b_i \in \mathcal{B}$ for each $1 \leq i \leq n_j$,
2. $f(s) = \lim\limits_{j\to\infty} f_j(s)$ with respect to $\|\cdot\|_s$ for μ-almost every $s \in \Omega$.

The set of fiber measurable functions on Ω together with the pointwise addition and scalar multiplication is a $\mathbb{C}$-vector space, which we will call a *measurable Banach fiber space.*

Having the concept of measurability we continue with the notion of integrability for functions from Ω to measurable Banach fiber sets, see [65, Def. VI.1.vi]. Especially, we define what it means to be p-integrable for $1 \leq p < \infty$. To do so, we remark that the map $s \mapsto \|f(s)\|_s^p$ is measurable.

Definition 3.3. Let $1 \leq p < \infty$. We call a fiber measurable function $f : \Omega \to \bigcup_{s\in\Omega} X_s$ *fiber p-integrable* if the integral

$$\int_\Omega \|f(s)\|_s^p \, \mathrm{d}\mu(s),$$

exists and is finite. In this case, we call

$$\|f\|_p := \left(\int_\Omega \|f(s)\|_s^p \, \mathrm{d}\mu(s)\right)^{\frac{1}{p}}$$

the L^p*-fiber norm* of f.

Remark 3.4. (i) Observe that the set of fiber p-integrable functions with pointwise addition and scalar multiplication is a vector space.

(ii) The relation defined by $f \sim g \Leftrightarrow f = g$ μ-almost everywhere is an equivalence relation on the set of fiber p-integrable functions.

(iii) The set of equivalence classes of fiber p-integrable functions with the canonical vector space structure is called a L^p*-fiber space* and is denoted by $\mathrm{L}^p(\Omega, (X_s)_{s\in\Omega})$.

(iv) By [65, Prop. VI.1.xi] the space $\mathrm{L}^p(\Omega, (X_s)_{s\in\Omega})$ is actually a Banach space with respect to the L^p-fiber norm.

§ 3.2 Unbounded operator-valued multiplication operators

The main objects of this section are unbounded operator-valued multiplication operators, cf. [113, Def. 2.3.1].

Definition 3.5. Let X be a Banach space and let $(M(s), D(M(s)))_{s\in\Omega}$ be a family of unbounded linear operators on X, i.e., $M(s) : D(M(s)) \subseteq X \to X$ for $s \in \Omega$. The operator $(\mathcal{M}, D(\mathcal{M}))$ on $\mathrm{L}^p(\Omega, X)$ defined by

$$\begin{aligned} D(\mathcal{M}) &:= \{f \in \mathrm{L}^p(\Omega, X) : \ f(s) \in D(M(s)) \ \mu\text{-a.e.}, (s \mapsto M(s)f(s)) \in \mathrm{L}^p(\Omega, X)\}, \\ (\mathcal{M}f)(s) &:= M(s)f(s), \quad f \in D(\mathcal{M}), s \in \Omega, \ \mu\text{-almost everywhere,} \end{aligned}$$

is called the *unbounded operator-valued multiplication operator.* The operators $(M(s), D(M(s)))$, $s \in \Omega$, are called *fiber operators.*

As already mentioned in the beginning, the concept of unbounded multiplication operators was studied by S. Thomaschewski [113] on Bochner L^p-spaces in connection with non-autonomous Cauchy problems. Here we summarize some important results to have an overview of the properties of the operator $(\mathcal{M}, D(\mathcal{M}))$. Firstly, the closedness of the fiber operators implies the closedness of the multiplication operator, see [113, Lemma 2.3.4].

Lemma 3.6. *If $(M(s), D(M(s)))$ is closed for μ-almost every $s \in \Omega$, then $(\mathcal{M}, D(\mathcal{M}))$ is closed.*

In what follows we actually assume that $(\mathcal{M}, D(\mathcal{M}))$ is a closed operator valued multiplication operator with closed fiber operator $(M(s), D(M(s)))_{s\in\Omega}$. The following result [113, Lemma 2.3.5] shows that the resolvent operator of $(\mathcal{M}, D(\mathcal{M}))$ gives also rise to an multiplication operator. For this we remind the reader of the following definition [113, Def. 2.2.3]:

$$\mathrm{L}^\infty(\Omega, \mathscr{L}_\mathrm{s}(X)) := \{M : \Omega \to \mathscr{L}(X) : \; s \mapsto M(s)x \in \mathrm{L}^\infty(\Omega, X) \text{ for all } x \in X\}.$$

Lemma 3.7. *Let $(\mathcal{M}, D(\mathcal{M}))$ be a multiplication operator and assume that $\lambda \in \rho(\mathcal{M})$. Then $R(\lambda, \mathcal{M})$ is a bounded multiplication operator, i.e., there exists $M \in \mathrm{L}^\infty(\Omega, \mathscr{L}_\mathrm{s}(X))$, such that $(R(\lambda, \mathcal{M})f)(s) = M(s)f(s)$ for all $f \in \mathrm{L}^p(\Omega, X)$.*

Unfortunately, the converse of Lemma 3.7 fails due to the characterization of bounded multiplication operators, cf. [113, Thm. 2.2.17]. However, the following result [113, Thm. 2.3.6] holds.

Theorem 3.8. *Let $(\mathcal{M}, D(\mathcal{M}))$ be a densely defined closed operator on $\mathrm{L}^p(\Omega, X)$. Assume that there exists an unbounded sequence $(\lambda_n)_{n\in\mathbb{N}}$ in $\rho(\mathcal{M})$ such that for all $f \in \mathrm{L}^p(\Omega, X)$ one has $\lim_{n\to\infty} \lambda_n R(\lambda_n, \mathcal{M})f = f$. If $R(\lambda_n, \mathcal{M})$ is a bounded multiplication operator for every $n \in \mathbb{N}$, then there exists a family $(M(s), D(M(s)))_{s\in\Omega}$ of densely defined closed operators on X such that $(\mathcal{M}, D(\mathcal{M}))$ is a multiplication operator with fiber operators $(M(s), D(M(s)))_{s\in\Omega}$. Furthere there exists a μ-null-set $\mathscr{N}$ such that for every $s \in \Omega \setminus \mathscr{N}$ and for each $n \in \mathbb{N}$ one has $\lambda_n \in \rho(M(s))$.*

Last but not least, if $(\mathcal{M}, D(\mathcal{M}))$ is already supposed to be a multiplication operator on $\mathrm{L}^p(\Omega, X)$, then the resolvent of $\mathcal{M}$ and the resolvents of the fiber operators are related by the following result [113, Prop. 2.3.7].

Proposition 3.9. *Let $(\mathcal{M}, D(\mathcal{M}))$ be a closed multiplication operator with closed fiber operators $(M(s), D(M(s)))_{s\in\Omega}$.*

(a) *If $\lambda \in \rho(M(s))$ for μ-almost every $s \in \Omega$ and $R(\lambda, M(\cdot)) \in \mathrm{L}^\infty(\Omega, \mathscr{L}_\mathrm{s}(X))$, then $\lambda \in \rho(\mathcal{M})$ and $(R(\lambda, \mathcal{M})f)(s) = R(\lambda, M(s))f(s)$ for all $f \in \mathrm{L}^p(\Omega, X)$ and μ-almost every $s \in \Omega$.*

(b) *If there exists an unbounded sequence $(\lambda_n)_{n\in\mathbb{N}}$ in $\rho(\mathcal{M})$ such that for all $f \in \mathrm{L}^p(\Omega, X)$ one has $\lambda_n R(\lambda_n, \mathcal{M})f \to f$ for $n \to \infty$, then for μ-almost all $s \in \Omega$ and all $n \in \mathbb{N}$ one has $\lambda_n \in \rho(M(s))$ and $(R(\lambda_n, \mathcal{M})f)(s) = R(\lambda_n, M(s))f(s)$ for all $f \in \mathrm{L}^p(\Omega, X)$ and μ-almost every $s \in \Omega$.*

In [113, Sect. 2.2.3] S. Thomaschewski proceeds with the discussion on multiplication operators on $\mathrm{L}^p(\Omega, X)$. In particular, S. Thomaschewski gives a characterization of multiplication semigroups by means of multiplication operators as their generators.

Definition 3.10. A C_0-semigroup $(\mathcal{T}(t))_{t\geq 0}$ on $\mathrm{L}^p(\Omega, X)$ is called a *multiplication semigroup* if for every $t \geq 0$ the operator $\mathcal{T}(t)$ is a bounded multiplication operator, i.e., for every $t \geq 0$ there exists $T_{(\cdot)}(t) \in \mathrm{L}^\infty(\Omega, \mathscr{L}_\mathrm{s}(X))$ such that $(\mathcal{T}(t)f)(s) = T_s(t)f(s)$ for μ-almost every $s \in \Omega$.

By [113, Thm. 2.3.15] these multiplication semigroups are characterized in the following way.

Theorem 3.11. *Let $(\mathcal{M}, D(\mathcal{M}))$ be the generator of a strongly continuous semigroup $(\mathcal{T}(t))_{t\geq 0}$ on $\mathrm{L}^p(\Omega, X)$ such that $\|\mathcal{T}(t)\| \leq M\mathrm{e}^{\omega t}$ for some $M \geq 0$, $\omega \in \mathbb{R}$ and for all $t \geq 0$. The following are equivalent.*

(a) *$(\mathcal{T}(t))_{t\geq 0}$ is a multiplication semigroup.*

(b) $(\mathcal{M}, D(\mathcal{M}))$ *is an unbounded operator valued multiplication operator with fiber operators* $(M(s), D(M(s)))_{s\in\Omega}$. *Moreover, for* μ*-almost every* $s \in \Omega$, $\lambda \in \rho(M(s))$ *whenever* $\mathrm{Re}(\lambda) > \omega$, $(R(\lambda, \mathcal{M})f)(\cdot) = R(\lambda, M(\cdot))f(\cdot)$ *and* $(M(s), D(M(s)))$ *is the generator of a* C_0*-semigroup* $(T_s(t))_{t\geq 0}$ *such that* $(\mathcal{T}(t)f)(s) = T_s(t)f(s)$ *for all* $t \geq 0$.

§ 3.3 Extrapolation of unbounded multiplication operators

In what follows we consider extrapolation spaces of unbounded multiplication fiber operators. In particular, our standing assumption is that the multiplication operator $(\mathcal{M}, D(\mathcal{M}))$ generates a multiplication semigroup $(\mathcal{T}(t))_{t\geq 0}$, cf. Theorem 3.11. By $(M(s), D(M(s)))_{s\in\Omega}$ we denote the fiber-operators corresponding to $(\mathcal{M}, D(\mathcal{M}))$, i.e., $(\mathcal{M}f)(s) = M(s)f(s)$, $f \in \mathrm{L}^p(\Omega, X)$, $s \in \Omega$. By Theorem 3.11 the operator $(M(s), D(M(s)))$, $s \in \Omega$, generates a C_0-semigroup $(T_s(t))_{t\geq 0}$. We assume without loss of generality that $0 \in \rho(M(s))$ for μ-almost every $s \in \Omega$. The extrapolated operators will be denoted by $(M_{-1}(s), D(M_{-1}(s)))$, $s \in \Omega$. The extrapolation space of $(\mathcal{M}, D(\mathcal{M}))$ will be denoted by $\mathcal{X} := (\mathrm{L}^p(\Omega, X))_{-1}(\mathcal{M})$ is formally given by

$$\mathcal{X} = \left\{ f \in \prod_{s\in\Omega} X_{-1,s} : \; \exists g \in \mathrm{L}^p(\Omega, X) : \; f = M_{-1}(\cdot)g \right\}$$

Later, we will prove another characterization of this space by means of L^p-fiber spaces. A first step towards this is the following.

Lemma 3.12. *Let* $(\mathcal{M}, D(\mathcal{M}))$ *be a multiplication operator on* $\mathrm{L}^p(\Omega, X)$ *with fiber operator* $(M(s), D(M(s)))_{s\in\Omega}$. *Moreover, let* $(\mathcal{T}(t))_{t\geq 0}$ *the multiplication semigroup generated by* $(\mathcal{M}, D(\mathcal{M}))$. *Moreover, denote by* $(T_s(t))_{t\geq 0}$ *the* C_0*-semigroups generated by the fiber operators* $(M(s), D(M(s)))_{s\in\Omega}$. *The associated extrapolated semigroups are denoted by* $(T_{-1,s}(t))_{t\geq 0}$, $s \in \Omega$. *Define*

$$(\mathcal{S}(t)f)(s) := T_{-1,s}(t)f(s), \quad t \geq 0, \; f \in \mathcal{X}, \; s \in \Omega.$$

This defines a C_0*-semigroup on* $\mathcal{X}$ *which is generated by the operator* $(\mathcal{M}_{-1}, D(\mathcal{M}_{-1}))$ *defined by*

$$(\mathcal{M}_{-1}f)(s) = M_{-1}(s)f(s), \quad D(\mathcal{M}_{-1}) = \mathrm{L}^p(\Omega, X). \tag{3.3.1}$$

Proof. First of all, to see that $(\mathcal{S}(t))_{t\geq 0}$ is indeed a semigroup is easy, since $(T_{-1,s}(t))_{t\geq 0}$ is a semigroup for each $s \in \Omega$. As a matter of fact, $(T_{-1,s}(t))_{t\geq}$ is an extension of $(T_s(t))_{t\geq 0}$ for each $s \in \Omega$ and hence $(\mathcal{S}(t))_{t\geq 0}$ extends $(\mathcal{T}(t))_{t\geq 0}$. Since by construction $\mathrm{L}^p(\Omega, X)$ is dense in $\mathcal{X}$ the semigroup $(\mathcal{S}(t))_{t\geq 0}$ is strongly continuous. In order to show that the generator of $(\mathcal{S}(t))_{t\geq 0}$ is of the form mentioned in the lemma, let us denote the generator of $(\mathcal{S}(t))_{t\geq 0}$ by $(\mathcal{A}, D(\mathcal{A}))$. Observe that by definition

$$(\mathcal{A}f)(s) = \lim_{t\to 0} \frac{\mathcal{S}(t)f(s) - f(s)}{t} = \lim_{t\to 0} \frac{T_{-1,s}(t)f(s) - f(s)}{t}.$$

This implies that $f(s) \in D(M_{-1}(s))$ and $s \in \Omega$ and that $(\mathcal{A}f)(s) = M_{-1}(s)f(s)$ for μ-almost every $s \in \Omega$. In particular, we obtain $f \in \mathrm{L}^p(\Omega, X)$ and $\mathcal{A}f = M_{-1}(\cdot)f$ since $f(s) = (R(\lambda, M_{-1}(s))(\lambda - \mathcal{A})f)(s)$. Conversely, the condition $f \in \mathrm{L}^p(\Omega, X)$ implies that $f \in D(\mathcal{A})$

and $\mathcal{A}f = M_{-1}(\cdot)f$. $\square$

The previous result shows actually that the extrapolated multiplication operator is again a multiplication operator. In this case the fiber operators are the extrapolated fiber operators $(M_{-1}(s), D(M_{-1}(s)))_{s\in\Omega}$, i.e., (3.3.1) holds. As promised above, we give a characterization of the space $\mathcal{X} := (\mathrm{L}^p(\Omega, X))_{-1}(\mathcal{M})$. To do so, we assume in this sequel that the Banach space X we are working with is separable, i.e., there exists a countable dense set in X. Denote the extrapolation spaces corresponding to the fiber operator $(M(s), D(M(s))$ by $\left(X_{-1,s}, \|\cdot\|_{-1,s}\right)$, $s \in \Omega$. The following result prepares for the extrapolation procedure.

Lemma 3.13. *Suppose X is a separable Banach space. If $0 \in \rho(M(s))$ for almost every $s \in \Omega$ and $s \mapsto M(s)^{-1}x$ is measurable for each $x \in X$, then the family $\left(X_{-1,s}, \|\cdot\|_{-1,s}\right)_{s\in\Omega}$ is a measurable Banach fiber set.*

Proof. We make use of the separability of X and take a dense countable subset of X and make a $\mathbb{Q}+i\mathbb{Q}$ vector space $\mathcal{B}$ out of it. Observe that $\mathcal{B}$ is still countable, i.e, $\mathcal{B} := \{b_k : \ k \in \mathbb{N}\} \subseteq X$. We define a family of seminorms $\{|||\cdot|||_s : \ s \in \Omega\}$ on X by

$$|||x|||_s := \left\|M(s)^{-1}x\right\|, \quad x \in X, \ s \in \Omega.$$

Then $|||\cdot|||_s$ is actually a norm on X. By the assumption $s \mapsto M(s)^{-1}x$ is measurable for each $x \in X$ and hence so is the map $s \mapsto \|b_k\|_s$ for each $k \in \mathbb{N}$. Since $N_s = \{0\}$ for each $s \in \Omega$ we obtain $\mathcal{B}/N_s = \mathcal{B}$. Finally, the completion of $\mathcal{B}$ with respect to $|||\cdot|||_s$ is just the space $X_{-1,s}$, $s \in \Omega$. By Definition 3.1 we therefore obtain that $\left(X_{-1,s}, \|\cdot\|_{-1,s}\right)_{s\in\Omega}$ is a measurable Banach fiber set. $\square$

Since we know that $\left(X_{-1,s}, \|\cdot\|_{-1,s}\right)_{s\in\Omega}$ is a measurable Banach fiber set, we can consider the space of fiber p-integrable functions over this set of Banach spaces. In what follows we relate this space to the extrapolation space of $\mathrm{L}^p(\Omega, (X_s)_{s\in\Omega})$ with respect to the unbounded operator-valued multiplication operator $(\mathcal{M}, D(\mathcal{M}))$.

Theorem 3.14. *Let $1 \le p < \infty$ and consider the unbounded multiplication operator $(\mathcal{M}, D(\mathcal{M}))$ on $\mathrm{L}^p(\Omega, X)$, induced by the family of unbounded operators $(M(s), D(M(s)))_{s\in\Omega}$ on X. Let $(M(s), D(M(s)))$ be a semigroup generator for μ-almost every $s \in \Omega$. Suppose that $0 \in \rho(M(s))$ for μ-almost every $s \in \Omega$ and that $s \mapsto M(s)b$ and $s \mapsto M(s)^{-1}b$ are measurable for each $b \in \mathcal{B}$. Then*

$$[\mathrm{L}^p(\Omega, X)]_{-1}(\mathcal{M}) = \mathrm{L}^p(\Omega, (X_{-1,s})_{s\in\Omega}).$$

Proof. Let $f \in [\mathrm{L}^p(\Omega, X)]_{-1}(\mathcal{M})$ and find $g \in \mathrm{L}^p(\Omega, X)$ such that $f = \mathcal{M}_{-1}g$, where $\mathcal{M}_{-1} : \mathrm{L}^p(\Omega, X) \to \mathrm{L}^p(\Omega, X)_{-1}(\mathcal{M})$. Since g is measurable, we can find a sequence $(g_n)_{n\in\mathbb{N}}$ of simple functions approximating g pointwise, i.e.,

$$g_n := \sum_{i=1}^{m_n} x_i \mathbf{1}_{\Omega_i} \text{ and } g_n \to g \ \mu\text{-almost everywhere.}$$

In order to show that $f \in \mathrm{L}^p(\Omega, (X_{-1,s})_{s\in\Omega})$ define $f_n := \mathcal{M}_{-1}g_n$. Then

$$f_n(s) := (\mathcal{M}_{-1}g_n)(s) = \sum_{i=1}^{m_n} M_{-1}(s)x_i\mathbf{1}_{\Omega_i}(s),$$

where we use Lemma 3.12 as well as the fact that $M_{-1}(s)x_i \in X_{-1,s}$, $s \in \Omega$. However, in general $M_{-1}(s)x_i \in X_{-1,s}$ is not an element of $\mathcal{B}$, cf. Lemma 3.13. To bypass this problem observe that $\mathcal{B}$ is dense in $X_{-1,s}$ for each $s \in \Omega$. For that reason take $\varepsilon > 0$ arbitrarily and find $b_{k_i} \in \mathcal{B}$ such that $\|b_{k_i} - M_{-1}(s)x_i\|_{-1,s} < \varepsilon$ for μ-almost every $s \in \Omega$ and define

$$\widetilde{f}_n := \sum_{i=1}^{m_n} b_{k_i} \mathbf{1}_{\Omega_i}.$$

Observe, that since the measurable sets Ω_i are disjoint, we have

$$\left\| f_n(s) - \widetilde{f}_n(s) \right\|_{-1,s} = \begin{cases} \left\| M_{-1}(s)x_j - b_{k_j} \right\|_{-1,s}, & \text{if } s \in \Omega_j, \\ 0, & \text{if } s \notin \Omega_i \text{ for all } 1 \le i \le m_n, \end{cases}$$

μ-almost everywhere and hence by the choice of $b_{k_i} \in \mathcal{B}$ that $\|f_n(s) - \widetilde{f}_n(s)\|_{-1,s} < \varepsilon$ for μ-almost every $s \in \Omega$. We now show that $f(s) = \lim_{j\to\infty} \widetilde{f}_j(s)$ with respect to $\|\cdot\|_{-1,s}$ for μ-almost every $s \in \Omega$. To do so, we observe that by construction we can find $N \in \mathbb{N}$ such that for all $n \ge N$:

$$\|f(s) - f_n(s)\|_{-1,s} = \|(\mathcal{M}_{-1}g)(s) - (\mathcal{M}_{-1}g_n)(s)\|_{-1,s} = \|g(s) - g_n(s)\| < \varepsilon, \quad s \in \Omega,$$

whence,

$$\left\| f(s) - \widetilde{f}_n(s) \right\|_{-1,s} \le \|f(s) - f_n(s)\|_{-1,s} + \left\| f_n(s) - \widetilde{f}_n(s) \right\|_{-1,s} < 2\varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, we conclude that f is fiber measurable. Furthemore,

$$\|f\|^p_{\mathrm{L}^p(\Omega,(X_{-1,s})_{s\in\Omega})} = \int_\Omega \|f(s)\|^p_{-1,s} \,\mathrm{d}s = \int_\Omega \|g(s)\|^p \,\mathrm{d}s = \|g\|^p_{\mathrm{L}^p(\Omega,X)} < \infty,$$

showing that f is a fiber p-integrable function, i.e., $f \in \mathrm{L}^p(\Omega, (X_{-1,s})_{s\in\Omega})$. For the converse inclusion, suppose that $f \in \mathrm{L}^p(\Omega, (X_{-1,s})_{s\in\Omega})$. We have to show that there exists $g \in \mathrm{L}^p(\Omega, X)$ such that $f = \mathcal{M}_{-1}g$. Since f is fiber measurable, there exists a sequence $(f_j)_{j\in\mathbb{N}}$ of simple functions $f_j : \Omega \to \bigcup_{s\in\Omega} X_{-1,s}$ with $f(s) \in X_{-1,s}$ for μ-almost every $s \in \Omega$ and

$$f_j = \sum_{i=1}^{n_j} b_{k_i} \mathbf{1}_{\Omega_i},$$

where $b_{k_i} \in \mathcal{B}$, $\Omega_i \in \Sigma$, $\Omega_i \cap \Omega_j = \varnothing$, $i \ne j$, for $1 \le i \le n_j$, and

$$\|f(s) - f_j(s)\|_{-1,s} \to 0, \tag{3.3.2}$$

for $j \to \infty$ and μ-almost every $s \in \Omega$. By the assumption that $0 \in \rho(M(s))$ for μ-almost every $s \in \Omega$ we conclude by Proposition 3.9 that $0 \in \rho(\mathcal{M})$. So we define

$$g_j := (\mathcal{M}_{-1}^{-1} f_j)(\cdot) = \sum_{i=1}^{n_j} \left(M_{-1}^{-1}(\cdot) b_{k_i} \right) \mathbf{1}_{\Omega_i}(\cdot).$$

We observe that $M_{-1}^{-1}(s)b_{k_i} \in X$ for μ-almost every $s \in \Omega$ and hence g_j is a simple function. By (3.3.2) we conclude that $(g_j(s))_{j\in\mathbb{N}}$ is a Cauchy sequence with respect to $\|\cdot\|_s$ for μ-almost every $s \in \Omega$ and hence convergent. This yields a measurable function $g : \Omega \to X$ by taking the pointwise limit, i.e.,

$$g(s) := \lim_{j\to\infty} g_j(s), \quad s \in \Omega.$$

By the continuity of $M_{-1}(s)$, $s \in \Omega$, on X and the fact that $M_{-1}(s)M_{-1}^{-1}(s) = I$ for μ-almost every $s \in \Omega$, we directly obtain that $\mathcal{M}_{-1}g = f$. Moreover,

$$\|g\|^p_{\mathrm{L}^p(\Omega,X)} = \int_\Omega \|g(s)\|^p \,\mathrm{d}s = \int_\Omega \|f(s)\|^p_{-1,s} \,\mathrm{d}s = \|f\|^p_{\mathrm{L}^p(\Omega,(X_{-1,s})_{s\in\Omega}} < \infty,$$

and therefore $g \in \mathrm{L}^p(\Omega, X)$. □

§ 3.4 Notes

The content of this chapter was worked out in cooperation with R. Heymann during the authors' research stay at the *University of Stellenbosch* (South Africa) in summer 2018. The research idea of this chapter was developed during a discussion between B. Farkas, R. Heymann and the author after a talk of R. Heymann at the *Functional Analysis Seminar* of our workgroup at the *Bergische Universität Wuppertal* in November 2017. The first contact with R. Heymann was built during the last *AGFA* meeting in Blaubeuren in December 2016.

Observe that in Lemma 3.12 and Theorem 3.14 we considered the case when for almost every $t \in \mathbb{R}$ the operator $(A(t), D(A(t)))$ is the generator of a strongly continuous semigroup on X. However, this assumption is not too restrictive since the generation results by P. Acquistapace and B. Terreni as well as T. Kato and H. Tanabe, mentioned in the introduction, assume that $(A(t), D(A(t)))_{t\in\mathbb{R}}$ are in fact generators of analytic semigroups.

As already mentioned in the introduction, a possible application of our results it the determination of the explicit domain of the evolution semigroup corresponding (nACP). The idea how to find the explicit domain of the evolution family is to enlarge the space by extrapolation and obtain the generator domain by taking the part of the extrapolated generator in the original space. This appears to be possible only in connection with some uniformity conditions on the extrapolation spaces corresponding to the operators $(A(t), D(A(t)))_{t\in\mathbb{R}}$ as they are for example mentioned by H. Amann [7, Sect. 7] and J. Kisyński [74]. To be more exact, $X_{-1,s} \cong X_{-1,s} \cong X_{-1}$ for all $s \in \mathbb{R}$ such that there exists a constant $\kappa > 0$ such that $\frac{1}{\kappa}\|x\|_{X_{-1}} \le \|x\|_{X_{-1,s}} \le \kappa \|x\|_{X_{-1}}$ for all $x \in X_{-1}$ and $t \in \mathbb{R}$.

Chapter 4

Desch–Schappacher Perturbations

As suggested by G. Greiner in [60] abstract perturbation theory of one-parameter semigroups provides good means to change the domain of a semigroup generator. For this an enlargement of the underlying Banach space may be necessary and extrapolation spaces become important. One of the well-known results in this direction goes back to the papers of W. Desch and W. Schappacher, see [38] and [39]. Another prominent example of such general perturbation techniques is due to Staffans and Weiss [108, 107], and an elegant abstract operator theoretic/algebraic approach has been developed by Adler, Bombieri and Engel in [3]. A general theory of unbounded domain perturbations is given by Hadd, Manzo and Rhandi [64]. A more recent paper by Bátkai, Jacob, Voigt and Wintermayr [20] extends the notion of positivity to extrapolation spaces, and studies positive perturbations for positive semigroups on AM-spaces. Hence, the study of abstract Desch–Schappacher type perturbations is a lively research field, to which we contribute with the present article. The reason for such an active interest in this area is that the range of application is vast. We mention here only a selection from the most recent ones: boundary perturbations by Nickel [96], boundary feedback by Casarino, Engel, Nagel and Nickel [31], boundary control by Engel, Kramar Fijavž, Klöss, Nagel and Sikolya [49] and Engel and Kramar Fijavž [47], port-Hamiltonian systems by Baroun and Jacob [19], control theory by Jacob, Nabiullin, Partington and Schwenninger [68, 67] and Jacob, Schwenninger and Zwart [69] and vertex control in networks by Engel and Kramar Fijavž [50, 48].

All the previously mentioned abstract perturbation results were developed for strongly continuous semigroups of linear operators on Banach spaces, C_0-semigroups for short. This is, for certain applications, e.g., for the theory of Markov transition semigroups, far too restrictive. For this situation the Banach space of bounded and continuous functions over a Polish space is the most adequate, but on this space the strong continuity with the respect to norm is, in general, a too stringent requirement.

F. Kühnemund in [78] has developed the abstract theory of bi-continuous semigroups, which has the advantage that not only Markov transition semigroup, but also semigroups induced by jointly continuous flows or implemented semigroups, just to mention a few, can be handled in a unified manner. Some perturbation result for bi-continuous semigroups are known, see [55, 56, 54], however, none of which is suitable for domain perturbations.

As first step this chapter treats a Desch–Schappacher type perturbation theorem for this class of semigroups. Since the theory of bi-continuous semigroups uses a Banach space norm and a additional locally convex topology, it is fundamental to relate our results to the existing, analogous ones on locally convex spaces. We recall the next result, due to Jacob, Wegner, Wintermayr, from [70].

Theorem. Let X be a sequentially complete, locally convex space with fundamental system Γ of continuous seminorms, and let $(A, D(A))$ be the generator of a locally equicontinuous C_0-semigroup $(T(t))_{t\geq 0}$ on X. Moreover, let $\overline{X}$ be a sequentially complete locally convex space such that

(a) $X \subseteq \overline{X}$ is dense and the inclusion map is continuous,

(b) $\overline{A}$ with domain $D(\overline{A}) = X$ generates a locally equicontinuous C_0-semigroup $(\overline{T}(t))_{t\geq 0}$ on $\overline{X}$ such that $\overline{T}(t)_{|X} = T(t)$ holds for all $t \geq 0$.

Let $B : X \to \overline{X}$ be a linear and continuous operator and $t_0 > 0$ be a number such that

(c) $\forall f \in \mathrm{C}\left([0, t_0], X\right) : \displaystyle\int_0^{t_0} \overline{T}(t_0 - t)Bf(t)\ \mathrm{d}t \in X$,

(d) $\forall p \in \Gamma\ \exists K \in (0, 1)\ \ \forall f \in \mathrm{C}\left([0, t_0], X\right) : p\left(\displaystyle\int_0^{t_0} \overline{T}(t_0 - t)Bf(t)\ \mathrm{d}t\right) \leq K \cdot \sup_{t\in[0,t_0]} p(f(t))$.

Then the operator $(C, D(C))$ defined by

$$Cx = (\overline{A} + B)x \text{ for } x \in D(C) = \left\{x \in X : (\overline{A} + B)x \in X\right\}$$

generates a locally equicontinuous C_0-semigroup on X if and only if $D(C) \subseteq X$ is dense.

We will prove a similar result for bi-continuous semigroups with the advantage that we can relax condition (c) of the previous theorem in the sense that we allow different seminorms on the left- and the right-hand side of the inequality. Moreover, one has to change and expand the conditions for the bi-continuous case carefully to obtain a good interplay between the Banach space norm and the locally convex topology. A space $\overline{X}$ with the properties used in the theorem above is called an extrapolation space. For C_0-semigroups on Banach spaces the classical construction is presented in [52, Chapter II, Sect. 5a] in a self-contained manner. Extrapolation spaces for C_0-semigroups on locally convex spaces are constructed by Wegner in [123]. Extrapolated bi-continuous semigroups and extrapolation spaces were treated in Chapter 2.

This chapter is organized as follows. In the first section we recall some definitions and results for bi-continuous semigroups and give some preliminary constructions needed for the Desch–Schappacher perturbation result, which is stated and proved as Theorem 4.4 in Section 4.1. Section 4.2 contains a sufficient condition for operators to satisfy the hypothesis of the abstract perturbation theorem, see Theorem 4.7. In Section 4.3 we prove that for a large class of bounded functions $g : \mathbb{R} \to \mathbb{C}$ which are continuous up to a discrete set of jump discontinuities and for each bounded (complex) Borel measure μ on $\mathbb{R}$ the operator

$$Cf := f' + \int_{\mathbb{R}} f\,\mathrm{d}\mu \cdot g$$

with appropriate domain generates a bi-continuous semigroup on the Banach space $\mathrm{C_b}(\mathbb{R})$ of bounded, continuous functions on $\mathbb{R}$.

§ 4.1 An abstract Desch–Schappacher perturbation result

Let us first fix some notation. Let $(T(t))_{t\geq 0}$ be a bi-continuous semigroup on a Banach space X_0 with respect to τ and with generator $(A, D(A))$, where τ is generated by the family of seminorms $\mathcal{P}$. Furthermore, let $B \in \mathscr{L}(X_0, X_{-1})$ such that $B : (X_0, \tau) \to (X_{-1}, \tau_{-1})$ is a continuous linear operator and define for $t_0 > 0$ the following space:

$$\mathfrak{X}_{t_0} := \left\{ F : [0, t_0] \to \mathscr{L}(X) \;\middle|\; \begin{array}{l} F \text{ is } \tau\text{-strongly continuous, norm bounded} \\ \text{and } \{F(t) : t \in [0, t_0]\} \text{ is bi-equicontinuous} \end{array} \right\}. \tag{4.1.1}$$

Remark 4.1. In [55, Lemma 3.2] it was shown that for $t_0 > 0$ the space $\mathfrak{X}_{t_0}$ is indeed a Banach space (and in particular a Banach algebra) with respect to the norm

$$\|F\| := \sup_{t\in[0,t_0]} \|F(t)\|.$$

For $F \in \mathfrak{X}_{t_0}$ and $t \in [0, t_0]$ we define the so-called *(abstract) Volterra operator* V_B on $\mathfrak{X}_{t_0}$ by

$$(V_B F)(t)x := \int_0^t T_{-1}(t - r)BF(r)x \, \mathrm{d}r. \tag{4.1.2}$$

The integral has to be understood in the sense of a τ_{-1}-Riemann integral. Notice that in general for $x \in X_0$ we have $(V_B F)(t)x \in X_{-1}$. For the formulation of our main result we need the following definition.

Definition 4.2. Let $B \in \mathscr{L}(X_0, X_{-1})$ such that also $B : (X_0, \tau) \to (X_{-1}, \tau_{-1})$ is continuous. The operator B is said to be *admissible*, if there is $t_0 > 0$ such that the following conditions are satisfied:

1. $V_B F(t)x \in X_0$ for all $t \in [0, t_0]$ and $x \in X_0$.

2. $\mathrm{Ran}(V_B) \subseteq \mathfrak{X}_{t_0}$.

3. $\|V_B\| < 1$.

The set of all admissible operators $B : (X_0, \tau) \to (X_{-1}, \tau_{-1})$ will be denoted by $\mathcal{S}_{t_0}^{DS,\tau}$. We write $B \in \mathcal{S}_{t_0}^{DS,\tau}(T)$ whenever it is important to emphasize for which semigroup $(T(t))_{t\geq 0}$ the operator B is admissible.

Remark 4.3. By construction $A_{-1} : (X_0, \tau) \to (X_{-1}, \tau_{-1})$ is continuous, and actually an isomorphism. In particular, we have

$$\forall p \in \mathcal{P} \; \exists L > 0 \; \exists \gamma \in \mathcal{P}_{-1} \forall x \in X_0 : \; p(x) \leq L(\gamma(x) + \gamma(A_{-1}x)).$$

This section contains the formulation of the Desch–Schappacher type perturbation result and its proof. Observe that the proof of the following theorem is at some points verbatim the same as the one of [52, Chapter III, Thm. 3.1].

Theorem 4.4. *Let $(A, D(A))$ be the generator of a bi-continuous semigroup $(T(t))_{t\geq 0}$ on a Banach space X_0 with respect to τ. Let $B : X_0 \to X_{-1}$ such that $B \in \mathcal{S}_{t_0}^{DS,\tau}$ for some $t_0 > 0$. Then the operator $(A_{-1} + B)_{|X_0}$ with domain*

$$D((A_{-1} + B)_{|X_0}) := \{x \in X_0 : \ A_{-1}x + Bx \in X_0\}$$

generates a bi-continuous semigroup $(S(t))_{t\geq 0}$ on X_0 with respect to τ. Moreover, the semigroup $(S(t))_{t\geq 0}$ satisfies the variation of parameters formula

$$S(t)x = T(t)x + \int_0^t T_{-1}(t - r)BS(r)x \ \mathrm{d}r, \tag{4.1.3}$$

for every $t \geq 0$ and $x \in X_0$.

Proof. Since $\|V_B\| < 1$ by hypothesis, we conclude that $1 \in \rho(V_B)$. Now let $t > 0$ be arbitrary and write $t = nt_0 + t_1$ for $n \in \mathbb{N}$ and $t_1 \in [0, t_0)$. Define

$$S(t) := ((R(1, V_B)T_{|[0,t_0]})^n(t_0) \cdot (R(1, V_B)T_{|[0,t_0]})(t_1).$$

We first show that $(S(t))_{t\geq 0}$ is a semigroup. For $0 \leq s, t \leq s + t \leq t_0$ and $n \in \mathbb{N}$ we prove the following identity (cf. [52, p. 184])

$$(V_B^n T)(t + s) = \sum_{k=0}^{n} (V_B^{n-k}T_{|[0,t_0]})(s) \cdot (V_B^k T)(t), \quad \forall n \in \mathbb{N} \tag{4.1.4}$$

by induction. We abbreviate $V := V_B$. Since $V^0 = I$, equation (4.1.4) is trivially satisfied for $n = 0$. Now assume that (4.1.4) is true for some $n \in \mathbb{N}$. Then we obtain by this hypothesis that

$$\begin{aligned}
&\sum_{k=0}^{n+1}(V^{n+1-k}T)(s) \cdot (V^k T)(t) \\
&\quad = \sum_{k=0}^{n} \left(\int_0^s T_{-1}(s - r)BV^{n-k}T(r) \ \mathrm{d}r\right) \cdot V^k T(t) + T(s) \int_0^t T_{-1}(t - r)BV^n T(r) \ \mathrm{d}r \\
&\quad = \int_0^s T_{-1}(s - r)B \sum_{k=0}^{n} V^{n-k}T(r) \cdot V^k T(t) \ \mathrm{d}r + \int_0^t T_{-1}(s + t - r)BV^n T(r) \ \mathrm{d}r \\
&\quad = \int_0^s T_{-1}(s - r)BV^n T(r + t) \ \mathrm{d}r + \int_0^t T_{-1}(s + t - r)BV^n T(r) \ \mathrm{d}r \\
&\quad = \int_t^{s+t} T_{-1}(s + t - r)BV^n T(r) \ \mathrm{d}r + \int_0^t T_{-1}(s + t - r)BV^n T(r) \ \mathrm{d}r \\
&\quad = V^{n+1}T(s + t).
\end{aligned}$$

By this we can conclude that $(S(t))_{t\geq 0}$ satisfies the semigroup law for $0 \leq s, t \leq s + t \leq t_0$. Indeed, for each $t \in [0, t_0]$ the point evaluation $\delta_t : \mathfrak{X}_{t_0} \to \mathscr{L}(X_0)$ is a contraction and since $\|V\| < 1$ by hypothesis the inverse of $I - V$ is given by the Neumann series. Therefore,

$$S(t) = \delta_t \left(\sum_{n=0}^{\infty} V^n T\right) = \sum_{n=0}^{\infty} (V^n T)(t), \quad t \in [0, t_0] \,.$$

Moreover, we have

$$\|(V^nT)(t)\| \leq \|V^n\| \cdot \left\|T_{|[0,t_0]}\right\|,$$

and we conclude that the series above converges absolutely. Hence

$$\begin{aligned} S(s)S(t) &= \sum_{n=0}^{\infty}(V^nT)(s) \cdot \sum_{n=0}^{\infty}(V^nT)(t) \\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{n}(V^{n-k}T)(s)(V^kT)(t) \\ &= \sum_{n=0}^{\infty}(V^nT)(s+t) = S(s+t). \end{aligned}$$

Now we show that $S(t)S(s) = S(t+s)$ for all $t, s > 0$. For that let $t, s > 0$ be arbitrary and $n, m \in \mathbb{N}$ and $t_1, t_2 \in [0, t_0)$ such that $t = nt_0 + t_1$ and $s = mt_0 + t_2$. Then we obtain the following

$$\begin{aligned} S(t)S(s) &= S(t_0)^n S(t_1) S(t_0)^m S(t_2) \\ &= S(t_0)^n S(t_0)^m S(t_1) S(t_2) \\ &= \begin{cases} S(t_0)^{n+m} S(t_1+t_2), & \text{if } t_1+t_2 < t_0, \\ S(t_0)^{n+m+1} S(t_2-(t_0-t_1)), & \text{if } t_1+t_2 \geq t_0. \end{cases} \end{aligned}$$

But in both cases the right-hand side equals $S(t+s)$ by definition. Hence $(S(t))_{t\geq 0}$ satisfies the semigroup law. The next step is to show that it is a bi-continuous semigroup with respect to τ. Notice that

$$S_{|[0,t_0]}(t) = R(1, V_B)T_{|[0,t_0]}(t)$$

and hence $(S(t))_{t\geq 0}$ is locally bounded and the set $\{S(t) : t \in [0, t_0]\}$ is bi-equicontinuous. For $t > 0$ let $m := \lfloor \frac{t}{t_0} \rfloor$ and notice that $\left\{S(t_0)^k : \ 1 \leq k \leq m\right\}$ is bi-equicontinuous, hence we conclude that the set

$$\left\{S(t_0)^k : \ 1 \leq k \leq m\right\} \cdot \{S(s) : \ s \in [0, t_0]\}$$

is also bi-equicontinuous. By definition of $(S(t))_{t\geq 0}$ we obtain τ-strong continuity, and hence $(S(t))_{t\geq 0}$ is a bi-continuous semigroup with respect to τ. We now prove

$$S(t)x = T(t)x + \int_0^t T_{-1}(t-r)BS(r)x \ \mathrm{d}r$$

for each $t > 0$ and $x \in X_0$ by proceeding similarly to [52, Chapter III, Sect. 3]. For $t = nt_0+t_1$, $n \in \mathbb{N}$ and $t_1 \in [0, t_0)$, we obtain:

$$\begin{aligned} &\int_0^t T_{-1}(t-r)BS(r) \ \mathrm{d}r \\ =&\sum_{k=0}^{n-1}\int_{kt_0}^{(k+1)t_0} T_{-1}(t-r)BS(r) \ \mathrm{d}r + \int_{nt_0}^{t} T_{-1}(t-r)BS(r) \ \mathrm{d}r \end{aligned}$$

$$= \sum_{k=0}^{n-1} T_{-1}(t-(k+1)t_0) \int_0^{t_0} T_{-1}(t_0 - r) BS(r) \ \mathrm{d}r \cdot S(kt_0) + \int_0^{t_1} T_{-1}(t_1 - r) BS(r) \ \mathrm{d}r \cdot S(nt_0)$$
$$= \sum_{k=0}^{n-1} T(t-(k+1)t_0)(S(t_0) - T(t_0))S(kt_0) + (S(t_1) - T(t_1))S(nt_0) = S(t) - T(t).$$

The next step is to show that the resolvent set of $(A_{-1} + B)_{|X_0}$ is non-empty. For this we claim that $R(\lambda, A_{-1})B$ is bounded with $\|R(\lambda, A_{-1})B\| < 1$ for λ large enough. Choose $M \geq 0$ and $\omega \in \mathbb{R}$ such that $\|T(t)\| \leq M\mathrm{e}^{\omega t}$ for all $t > 0$. Then for $\lambda > \omega$ we obtain:

$$R(\lambda, A_{-1})B = \int_0^\infty \mathrm{e}^{\lambda r} T_{-1}(r)B \ \mathrm{d}r = \sum_{n=0}^{\infty} \mathrm{e}^{-\lambda n t_0} T(nt_0)(V_B F_\lambda)(t_0)$$

where $F_\lambda(r) := \mathrm{e}^{-\lambda(t_0 - r)} I \in \mathfrak{X}_{t_0}$. From this we obtain the following estimate:

$$\|R(\lambda, A_{-1})B\| \leq \|V_B\| + \frac{M\mathrm{e}^{(\omega-\lambda)t_0}}{1 - \mathrm{e}^{(\omega-\lambda)t_0}} \|V_B\| .$$

Since $\|V_B\| < 1$ we conclude for sufficient large λ:

$$\|R(\lambda, A_{-1})B\| < 1.$$

This yields $1 \in \rho(R(\lambda, A_{-1})B)$ for large λ and then invertibility of $\lambda - (A_{-1} + B)_{|X_0}$, since

$$\lambda - (A_{-1} + B)_{|X_0} = (\lambda - A)(I - R(\lambda, A_{-1})B).$$

Hence the resolvent set of $(A_{-1} + B)_{|X_0}$ contains each sufficiently large λ. In the last step we will show that $(A_{-1} + B)_{|X_0}$ is actually the generator of the bi-continuous semigroup $(S(t))_{t\geq 0}$ with respect to τ. Denote by $(C, D(C))$ the generator of $(S(t))_{t\geq 0}$. Let $\lambda > \max(\omega_0(T), \omega_0(S))$, then by the variation of constant formula (4.1.3), the resolvent representation as Laplace transform [79, Lemma 7] and the fact that we may interchange the improper τ-Riemann integral and the τ_{-1}-Riemann integral by an application of [78, Lemma 1.7] we obtain

$$R(\lambda, C) = R(\lambda, A) + R(\lambda, A_{-1})BR(\lambda, C).$$

Whence we conclude

$$(I - R(\lambda, A_{-1})B)R(\lambda, C) = R(\lambda, A),$$

and therefore

$$I = (\lambda - A)(I - R(\lambda, A_{-1})B)R(\lambda, C) = (\lambda - (A_{-1} + B)_{|X_0})R(\lambda, C).$$

It follows that $C \subseteq (A_{-1} + B)_{|X_0}$ and by the previous observations $C = (A_{-1} + B)_{|X_0}$. □

4.1.1 Abstract Favard Spaces and comparison of semigroups

In this subsection we want to combine the perturbation theory with the theory of abstract Favard spaces as described in Section 2.2. In the next proposition we show that Desch–Schappacher perturbations of bi-continuous semigroups, which satisfy a special range condition concerning the extrapolated Favard class, gives us semigroups which are close to each other in some sense.

Proposition 4.5. *Let $(T(t))_{t\geq 0}$ be a bi-continuous semigroup on X_0 with respect to τ generated by $(A, D(A))$. Suppose that $B \in \mathcal{S}_{t_0}^{DS,\tau}$ with* $\operatorname{Ran}(B) \subseteq F_0(A)$ *and let $(S(t))_{t\geq 0}$ be the perturbed semigroup. Then there exists $C \geq 0$ such that for each $t \in [0,1]$ one has*

$$\|T(t) - S(t)\| \leq Ct.$$

Proof. We may assume $\omega_0(T) < 0$. We find $M \geq 0$ such that $\|T(t)\| \leq M$ and $\|S(t)\| \leq M$ for every $t \in [0,1]$. Since $\operatorname{Ran}(B) \subseteq F_0(A)$ we conclude that $A_{-1}^{-1}B : X_0 \to F_1(A)$. Hence $A_{-1}^{-1}B$ is bounded by the closed graph theorem and we find $K \geq 0$ such that $\left\|A_{-1}^{-1}Bx\right\|_{F_1(A)} \leq K\|x\|$ for each $x \in X_0$. Let $\mathcal{P}_{-1}$ the family of seminorms corresponding to the first extrapolation space (see Section 2.4.2). By using (4.1.3) we obtain

$$\begin{aligned}
\|S(t)x - T(t)x\| &= \left\|A_{-1}\int_0^t T(t-r)A_{-1}^{-1}BS(r)x \,\mathrm{d}r\right\| \\
&= \left\|\tau_{-1}\lim_{h\to 0}\frac{T_{-1}(h)-I}{h}\int_0^t T(t-r)A_{-1}^{-1}BS(r)x \,\mathrm{d}r\right\| \\
&= \left\|\tau_{-1}\lim_{h\to 0}\int_0^t \frac{T(h)-I}{h}T(t-r)A_{-1}^{-1}BS(r)x \,\mathrm{d}r\right\| \\
&= \sup_{p\in\mathcal{P}_{-1}}\lim_{h\to 0} p\left(\int_0^t \frac{T(h)-I}{h}T(t-r)A_{-1}^{-1}BS(r)x \,\mathrm{d}r\right) \\
&\leq \sup_{p\in\mathcal{P}_{-1}}\lim_{h\to 0}\int_0^t p\left(\frac{T(h)-I}{h}T(t-r)A_{-1}^{-1}BS(r)x\right) \,\mathrm{d}r \\
&\leq \limsup_{h\to 0}\int_0^t \left\|\frac{T(h)-I}{h}T(t-r)A_{-1}^{-1}BS(r)x\right\| \,\mathrm{d}r \\
&\leq M\int_0^t \left\|A_{-1}^{-1}BS(r)x\right\|_{F_1(A)} \,\mathrm{d}r \\
&\leq tKM^2 \cdot \|x\|
\end{aligned}$$

for each $x \in X_0$ and $t \in [0,1]$. □

Corollary 4.6. *Let $(T(t))_{t\geq 0}$ be a bi-continuous semigroup on X_0 with respect to τ generated by $(A, D(A))$. If $B \in \mathcal{S}_{t_0}^{DS,\tau}$ and* $\operatorname{Ran}(B) \subseteq F_0(A)$*, then the perturbed semigroup $(S(t))_{t\geq 0}$ leaves the space of strong continuity $\underline{X}_0 := \overline{D(A)}^{\|\cdot\|}$ invariant.*

§ 4.2 Admissible operators

Next we consider a sufficient condition for $B : (X_0, \tau) \to (X_{-1}, \tau_{-1})$ to be admissible. Throughout this section we denote the space of continuous functions $f : [0, t_0] \to (X_0, \tau)$ which are $\|\cdot\|$-bounded by $\mathrm{C_b}([0,t_0], (X_0,\tau))$. If equipped with the sup-norm, $\mathrm{C_b}([0,t_0], (X_0,\tau))$ becomes a Banach space. The proof of the following theorem is at points verbatim the same as the one of [52, Chapter III, Cor. 3.3].

Theorem 4.7. *Let $(T(t))_{t\geq 0}$ be a bi-continuous semigroup with generator $(A, D(A))$ on a Banach space X_0 with respect to τ. Let $\mathscr{P}$ be the set of generating continuous seminorms corresponding to τ. Let $B \in \mathscr{L}(X_0, X_{-1})$ such that $B : (X_0, \tau) \to (X_{-1}, \tau_{-1})$ is a linear and continuous operator, and let $t_0 > 0$ be such that*

(a) $\int\limits_0^{t_0} T_{-1}(t_0 - r)Bf(r)\ \mathrm{d}r \in X_0$ *for each* $f \in \mathrm{C_b}([0, t_0], (X_0, \tau))$.

(b) *For every $\varepsilon > 0$ and every $p \in \mathcal{P}$ there exists $q \in \mathcal{P}$ and $K > 0$ such that for all $f \in \mathrm{C_b}([0, t_0], (X_0, \tau))$*

$$p\left(\int_0^{t_0} T_{-1}(t_0 - r)Bf(r)\ \mathrm{d}r\right) \leq K \cdot \sup_{r\in[0,t_0]} |q(f(r))| + \varepsilon \|f\|_\infty . \tag{4.2.1}$$

(c) *There exists $M \in (0, 1)$ such that for all $f \in \mathrm{C_b}([0, t_0], (X_0, \tau))$*

$$\left\| \int\limits_0^{t_0} T_{-1}(t_0 - r)Bf(r)\ \mathrm{d}r \right\| \leq M \|f\|_\infty . \tag{4.2.2}$$

Then $B \in \mathcal{S}_{t_0}^{DS,\tau}$, and as a consequence the operator $(A_{-1} + B)_{|X_0}$ defined on the domain

$$D((A_{-1} + B)_{|X_0}) := \{x \in X_0 : \ A_{-1}x + Bx \in X_0\}$$

generates a bi-continuous semigroup with respect to τ.

Proof. We first show $\mathrm{Ran}(V_B) \subseteq \mathfrak{X}_{t_0}$. Let $f \in \mathrm{C_b}([0, t_0], (X_0, \tau))$ and define for $t \in [0, t_0]$ the auxiliary function $f_t : [0, t_0] \to X_0$ by

$$f_t(r) := \begin{cases} f(0), & r \in [0, t_0 - t], \\ f(r + t - t_0), & r \in [t_0 - t, t_0]. \end{cases}$$

Then $f_t \in \mathrm{C_b}([0, t_0], (X_0, \tau))$ and

$$\int_0^t T_{-1}(t - r)Bf(r)\ \mathrm{d}r = \int_0^{t_0} T_{-1}(t_0 - r)Bf_t(r)\ \mathrm{d}r - \int_t^{t_0} T_{-1}(r)Bf(0)\ \mathrm{d}r. \tag{4.2.3}$$

By Theorem 1.7

$$\int_t^{t_0} T_{-1}(r)Bf(0)\ \mathrm{d}r = T(t) \int_0^{t_0-t} T_{-1}(r)Bf(0)\ \mathrm{d}r \in D(A_{-1}) = X_0.$$

We conclude that the map $\psi : [0, t_0] \to X_{-1}$ defined by

$$\psi(t) := \int_0^t T_{-1}(t - r)Bf(r)\ \mathrm{d}r \tag{4.2.4}$$

has values in X_0. Moreover, for $\varepsilon > 0$ and $p \in \mathcal{P}$ we have the following estimate:

$$p(\psi(t) - \psi(s))$$

$$
\begin{aligned}
=&p\left(\int_0^t T_{-1}(t-r)Bf(r)\ \mathrm{d}r - \int_0^s T_{-1}(s-r)Bf(r)\ \mathrm{d}r\right)\\
\leq&p\left(\int_0^{t_0} T_{-1}(t_0-r)B(f_t(r)-f_s(r))\ \mathrm{d}r\right) + p\left(\int_s^t T_{-1}(r)Bf(0)\ \mathrm{d}r\right)\\
\leq&K\cdot \sup_{r\in[0,t_0]} q(f_t(r)-f_s(r)) + p\left(\int_s^t T_{-1}(r)Bf(0)\ \mathrm{d}r\right) + \varepsilon \left\|f_t - f_s\right\|_\infty\\
\leq&K\cdot \sup_{r\in[0,t_0]} q(f_t(r)-f_s(r))\\
&+ L\cdot\left(\gamma\left(\int_s^t T_{-1}(r)Bf(0)\ \mathrm{d}r\right) + \gamma\left((T_{-1}(t)-T_{-1}(s))Bf(0)\right)\right) + \varepsilon\left\|f_t - f_s\right\|\\
\leq&K\cdot \sup_{r\in[0,t_0]} |q(f_t(r)-f_s(r))|\\
&+ L\cdot\left(\int_s^t \gamma(T_{-1}(r)Bf(0))\ \mathrm{d}r + \gamma\left((T_{-1}(t)-T_{-1}(s))Bf(0)\right)\right) + 2\varepsilon\left\|f\right\|_\infty
\end{aligned}
$$

where the $\gamma \in \mathcal{P}_{-1}$ of the second to last inequality comes from Remark 4.3. The extrapolated semigroup $(T_{-1}(t))_{t\geq 0}$ is strongly τ_{-1}-continuous and $\gamma \in \mathcal{P}_{-1}$, so that we can find $\delta_1 > 0$ such that

$$\gamma(T_{-1}(t) - T_{-1}(s)Bf(0)) < \varepsilon \text{ whenever } |t-s| < \delta_1.$$

Moreover, f is τ-continuous and therefore uniformly τ-continuous on compact sets, which gives us $\delta_2 > 0$ such that

$$\sup_{r\in[0,t_0]} |q(f_t(r) - f_s(r))| < \varepsilon \text{ if } |t-s| < \delta_2.$$

Last but not least, $\gamma(T_{-1}(r)Bf(0))$ is bounded by some constant $M > 0$, so for $\delta_3 = \frac{\varepsilon}{M}$ we have

$$|s-t| < \delta_3 \implies \int_s^t \gamma(T_{-1}(r)Bf(0))\ \mathrm{d}r < \varepsilon.$$

Now, we take $\delta := \min\{\delta_1, \delta_2, \delta_3\}$ and obtain

$$p(\psi(t) - \psi(s)) < (K + 2\left\|f\right\|_\infty + 2L)\varepsilon,$$

showing that $\psi : [0, t_0] \to X_0$ is τ-continuous.

Next, we prove the norm-boundedness using the same techniques and arguments as in [52, Chapter III, Sect. 3]. Let $f \in \mathrm{C_b}\left([0,t_0], (X_0,\tau)\right)$ and write

$$f = \widetilde{f}_\delta + h_\delta,$$

where

$$h_\delta(r) := \begin{cases} \left(1 - \frac{r}{\delta}\right) f(r), & 0 \leq r < \delta,\\ 0, & \delta \leq r \leq t_0 \end{cases}$$

for some $\delta > 0$. Then $\widetilde{f}_\delta$ and h_δ are norm-bounded and continuous with respect to τ, $\widetilde{f}_\delta(0) = 0$

and $\|\widetilde{f}_\delta\|_\infty \le \|f\|_\infty$. Now we obtain

$$\begin{aligned}
&\left\|\int_0^t T_{-1}(t-r)Bf(r)\,\mathrm{d}r\right\| \le \left\|\int_0^t T_{-1}(t-r)B\widetilde{f}_\delta(r)\,\mathrm{d}r\right\| + \left\|\int_0^t T_{-1}(t-r)Bh_\delta(r)\,\mathrm{d}r\right\| \\
\le & M\left\|\widetilde{f}_\delta\right\|_\infty + K\left(\left\|\int_0^t T_{-1}(t-r)Bh_\delta(r)\,\mathrm{d}r\right\|_{-1} + \left\|A_{-1}\int_0^t T_{-1}(t-r)Bh_\delta(r)\,\mathrm{d}r\right\|_{-1}\right) \\
\le & M\left\|\widetilde{f}_\delta\right\|_\infty + K\left\|\int_0^\delta T_{-1}(t-r)\left(1-\frac{r}{\delta}\right)Bf(0)\,\mathrm{d}r\right\|_{-1} \\
& + K\left\|T_{-1}(t)Bf(0) - \frac{1}{\delta}\int_0^\delta T_{-1}(t-r)Bf(0)\,\mathrm{d}r\right\|_{-1}.
\end{aligned}$$

By taking $\delta \searrow 0$ we obtain

$$\left\|\int_0^t T_{-1}(t-r)Bf(r)\,\mathrm{d}r\right\| \le M\,\|f\|_\infty. \tag{4.2.5}$$

We proceed with showing local bi-equicontinuity. For that let $(x_n)_{n\in\mathbb{N}}$ be a norm-bounded τ-null-sequence. Let $\varepsilon > 0$ and $p \in \mathscr{P}$, then by taking $f^n(r) = f(r)x_n$, we can find $q \in \mathcal{P}$ such that

$$\begin{aligned}
p\left(V_B F(t)x_n\right) =& p\left(\int_0^t T_{-1}(t-r)BF(r)x_n\,\mathrm{d}r\right) \\
\le & p\left(\int_0^{t_0} T_{-1}(t_0-r)Bf^n(r)\,\mathrm{d}r - \int_t^{t_0} T_{-1}(r)Bf^n(0)\,\mathrm{d}r\right) \\
\le & K\cdot \sup_{r\in[0,t_0]} |q(f^n(r))| + p\left(\int_t^{t_0} T_{-1}(r)Bf^n(0)\,\mathrm{d}r\right) + \varepsilon\,\|f_t^n\| \\
\le & K\cdot \sup_{r\in[0,t_0]} |q(f^n(r))| + \varepsilon\,\|f_t^n\| \\
& + L\cdot\left(\gamma\left(\int_t^{t_0} T_{-1}(r)Bf^n(0)\,\mathrm{d}r\right) + \gamma\left(\left(T_{-1}(t_0) - T_{-1}(t)\right)Bf^n(0)\right)\right).
\end{aligned}$$

Now we can argue by the local bi-equicontinuity of $(T(t))_{t\ge0}$ and $(T_{-1}(t))_{t\ge0}$ and with the arbitrarily small $\varepsilon > 0$ to conclude the local bi-equicontinuity of $V_B F$. Hence we see that V_B maps $\mathfrak{X}_{t_0}$ to $\mathfrak{X}_{t_0}$ and by (4.2.5) that $\|V_B\| < 1$ since by assumption $M \in (0,1)$. □

§ 4.3 Perturbations of the translation semigroup

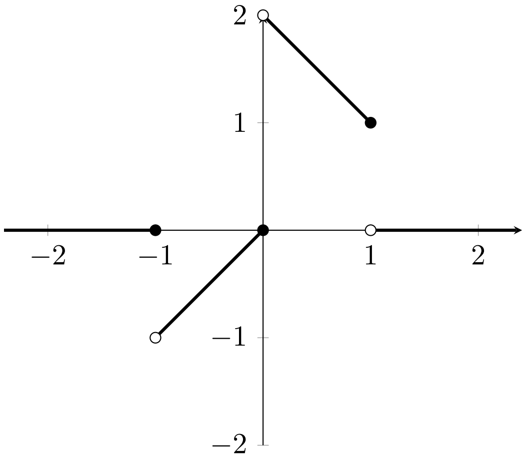
In this section we want to give an application of the Desch–Schappacher perturbation result to an explicit example. Another example is part of Chapter 6. Recall from Section 1.3.1 that on the space $X_0 = \mathrm{C_b}(\mathbb{R})$ the left-translation semigroup $(T(t))_{t\ge0}$ is bi-continuous with respect to the compact-open topology τ_{co}. Moreover, the resulting extrapolation spaces, in the notation we used in Chapter 2, are given by (cf. Section 2.5.1):

$$\begin{aligned}
\underline{X}_{-1} &= \{F \in \mathscr{D}'(\mathbb{R}) : \ F = f - Df \text{ for some } f \in \mathrm{UC_b}(\mathbb{R})\}, \\
X_{-1} &= \{F \in \mathscr{D}'(\mathbb{R}) : \ F = f - Df \text{ for some } f \in \mathrm{C_b}(\mathbb{R})\}.
\end{aligned}$$

where $\mathrm{UC_b}(\mathbb{R})$ denotes the space of bounded uniformly continuous functions and Df the distributional derivative of f. Recall that the generator of $(T(t))_{t\geq 0}$ is $A = \frac{\mathrm{d}}{\mathrm{d}x}$ with domain $D(A) := \mathrm{C}^1_\mathrm{b}(\mathbb{R})$, and $A_{-1} = D$ with domain $D(A_{-1}) = \mathrm{C_b}(\mathbb{R})$, where D denotes the distributional derivative. The extrapolated semigroup $(T_{-1}(t))_{t\geq 0}$ is the restriction to X_{-1} of the left-translation semigroup on the space $\mathscr{D}'(\mathbb{R})$ of distributions.
Consider the function $g : \mathbb{R} \to \mathbb{R}$ defined by

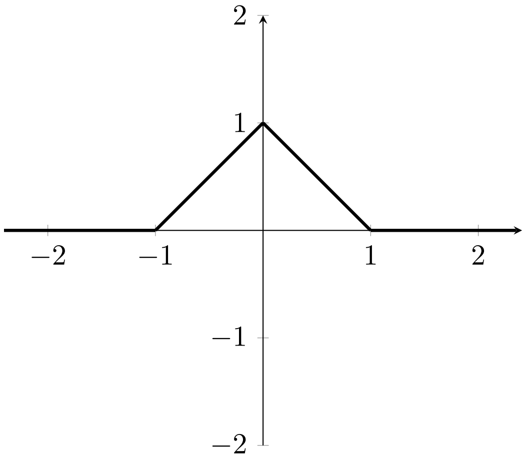
$$g(x) = \begin{cases} 0, & x \leq -1,\ x > 1, \\ x, & -1 < x \leq 0, \\ 2 - x, & 0 < x \leq 1. \end{cases} \tag{4.3.1}$$

The graph of this function is the following.



Notice that $g \in X_{-1}$, since $g = h - Dh$ where h is the tent function on the real line defined by

$$h(x) = \begin{cases} 0, & x \leq -1,\ x > 1, \\ x + 1, & -1 < x \leq 0, \\ -x + 1, & 0 < x \leq 1. \end{cases}$$



We now construct an operator $B \in \mathscr{L}(X, X_{-1})$ satisfying all conditions of Theorem 4.7, i.e., $((A_{-1}+B)_{|X_0}, D((A_{-1}+B)_{|X_0}))$ generates a bi-continuous semigroup. For this purpose let μ be a bounded regular Borel measure on $\mathbb{R}$ and define the continuous functional $\Phi : \mathrm{C_b}(\mathbb{R}) \to \mathbb{R}$ by $\Phi(f) = \int_{\mathbb{R}} f \ \mathrm{d}\mu$ and the operator $B : X_0 \to X_{-1}$ by

$$Bf := \Phi(f)g.$$

This operator B is by construction continuous with respect to the local convex topologies on the spaces X_0 and X_{-1}, and also for the norms. Moreover, B has all properties required in Theorem 4.7. To see this let $f \in \mathrm{C_b}([0, t_0], (X_0, \tau))$ be arbitrary. Define a map $\psi : \mathbb{R} \to \mathbb{R}$ by

$$\psi(\cdot) = \int_0^{t_0} T_{-1}(t_0 - r)Bf(r)(\cdot) \ \mathrm{d}r.$$

Observe that

$$T_{-1}(t_0 - r)Bf(r)(x) = T_{-1}(t_0 - r)\Phi(f(r))g(x) = \Phi(f(r))g(x + t_0 - r).$$

We claim that ψ is continuous. Indeed, let $\varepsilon > 0$ be arbitrary, and notice that by substitution for each $x \in \mathbb{R}$

$$\int_0^{t_0} \Phi(f(r))g(x + t_0 - r) \ \mathrm{d}r = \int_x^{x+t_0} \Phi(f(x + t_0 - s))g(s) \ \mathrm{d}s.$$

After this substitution we can make the following calculation for each $x, y \in \mathbb{R}$

$$\begin{aligned}
\psi(x) - \psi(y) &= \int_x^{x+t_0} \Phi(f(x + t_0 - s))g(s) \ \mathrm{d}s - \int_y^{y+t_0} \Phi(f(x + t_0 - s))g(s) \ \mathrm{d}s \\
&= \int_0^{x+t_0} \Phi(f(x + t_0 - s))g(s) \ \mathrm{d}s - \int_0^{x} \Phi(f(x + t_0 - s))g(s) \ \mathrm{d}s \\
&\quad - \int_0^{y+t_0} \Phi(f(y + t_0 - s))g(s) \ \mathrm{d}s + \int_0^{y} \Phi(f(y + t_0 - s))g(s) \ \mathrm{d}s \\
&= \int_{y+t_0}^{x+t_0} \left(\Phi(f(x + t_0 - s) - f(y + t_0 - s))\right) g(s) \ \mathrm{d}s \\
&\quad + \int_x^{y} \left(\Phi(f(x + t_0 - s) - f(y + t_0 - s))\right) g(s) \ \mathrm{d}s.
\end{aligned}$$

By the assumptions there exists $M > 0$ such that

$$\left\| \left(\Phi(f(x + t_0 - \cdot) - f(y + t_0 - \cdot))\right) g(\cdot) \right\|_\infty \leq M.$$

For $\delta := \frac{\varepsilon}{2M} > 0$ and for $x, y \in \mathbb{R}$ with $|x - y| < \delta$ we have

$$\begin{aligned}
&|\psi(x) - \psi(y)| \\
\leq &\int_{y+t_0}^{x+t_0} \left| \left(\Phi(f(x + t_0 - s) - f(y + t_0 - s))\right) g(s) \right| \ \mathrm{d}s \\
&+ \int_x^{y} \left| \left(\Phi(f(x + t_0 - s) - f(y + t_0 - s))\right) g(s) \right| \ \mathrm{d}s \\
\leq &2 \left| x - y \right| \cdot \left\| \left(\Phi(f(x + t_0 - \cdot) - f(y + t_0 - \cdot))\right) g(\cdot) \right\|_\infty
\end{aligned}$$

$$\leq 2M \cdot |x-y| < \varepsilon.$$

This proves that $\psi \in \mathrm{C_b}(\mathbb{R})$.

Observe that in general we only have

$$Q := \int_0^{t_0} T_{-1}(t_0 - r)Bf(r)\ \mathrm{d}r \in X_{-1},$$

so that point evaluation of this expression at $x \in \mathbb{R}$ does not make sense. We know, however, that $\psi \in \mathrm{C_b}(\mathbb{R})$, and that the pointwise Riemann-sums $R_n(x)$ for the integral

$$\int_0^{t_0} \Phi(f(r))g(x + t_0 - r)\mathrm{d}r$$

converges for all $x \in \mathbb{R}$ to $\psi(x)$. If we can show that the sequence $(R_n)_{n\in\mathbb{N}}$ converges in the sense of distributions we can conclude that $Q := \int_0^{t_0} T_{-1}(t_0 - r)Bf(r)\ \mathrm{d}r = \psi \in X_0$. Let $\widetilde{\psi} \in \mathscr{D}(\mathbb{R})$ be a test function and define $\varphi := \widetilde{\psi} - \mathscr{D}\widetilde{\psi}$. Then

$$\left\langle (1 - A_{-1})^{-1} R_n, \varphi \right\rangle \to \left\langle (1 - A_{-1})^{-1} Q, \varphi \right\rangle.$$

By the meaning of this pairing we conclude that $\langle R_n, \widetilde{\psi}\rangle \to \langle Q, \widetilde{\psi}\rangle$. By the above we conclude that $Q \in X_0$.

The next step is to estimate the norm. Notice that

$$\begin{aligned}
\Big\| \int_0^{t_0} T_{-1}(t_0 - r)Bf(r)\ \mathrm{d}r \Big\|_\infty &= \sup_{x\in\mathbb{R}} \Big| \int_0^{t_0} \Phi(f(r))g(x + t_0 - r)\ \mathrm{d}r \Big| \\
&\leq \sup_{x\in\mathbb{R}} \int_0^{t_0} |\Phi(f(r))| \cdot |g(x + t_0 - r)|\,\mathrm{d}r \leq 2\int_0^{t_0} |\Phi(f(r))|\ \mathrm{d}r \\
&\leq 2\int_0^{t_0} \int_{\mathbb{R}} |f(r)(x)|\ \mathrm{d}\,|\mu|\,(x)\ \mathrm{d}r \leq 2\,|\mu|\,(\mathbb{R}) \int_0^{t_0} \|f(r)\|_\infty\ \mathrm{d}r \\
&= 2\,|\mu|\,(\mathbb{R}) \int_0^{t_0} \|f(r)\|_\infty\ \mathrm{d}r \leq 2\,|\mu|\,(\mathbb{R}) t_0 \,\|f\|_\infty .
\end{aligned}$$

In particular we can choose t_0 so small that $M := 2\,|\mu|\,(\mathbb{R})t_0 < 1$. Hence condition (c) of Theorem 4.7 is fulfilled. Condition (b) from Theorem 4.7 can be proven similarly. Let $K \subseteq \mathbb{R}$ be an arbitrary compact set and $\varepsilon > 0$. Then

$$\begin{aligned}
p_K\left(\int_0^{t_0} T_{-1}(t_0 - r)Bf(r)(x)\ \mathrm{d}r \right) &\leq \sup_{x\in K} \int_0^{t_0} |\Phi(f(r))| \cdot |g(x + t_0 - r)|\,\mathrm{d}r \\
&\leq 2 \sup_{x\in K} \int_0^{t_0} |\Phi(f(r))|\ \mathrm{d}r \\
&\leq 2t_0|\mu|(\mathbb{R}) \sup_{r\in[0,t_0]} \sup_{y\in K'} |f(r)(y)| + \varepsilon \,\|f\|_\infty ,
\end{aligned}$$

since by the regularity of the measure μ we choose $K' \subseteq \mathbb{R}$ such that $|\mu|\,(\mathbb{R} \setminus K') < \varepsilon$. By Theorem 4.7 we conclude that $(A_{-1} + B)_{|X_0}$ generates again a bi-continuous semigroup on $\mathrm{C_b}(\mathbb{R})$ with respect to τ_{co}. We now give an expression for the generator. Observe that

$f \in D((A_{-1}+B)_{|X_0})$ if and only if $f \in \mathrm{C_b}(\mathbb{R})$ and $f' + \Phi(f)g \in \mathrm{C_b}(\mathbb{R})$ and this is precisely then, when the following conditions are satisfied.

$$\begin{cases} \lim\limits_{t \nearrow -1} (f'(t) + \Phi(f)g(t)) = \lim\limits_{t \searrow -1} (f'(t) + \Phi(f)g(t)), \\ \lim\limits_{t \nearrow 0} (f'(t) + \Phi(f)g(t)) = \lim\limits_{t \searrow 0} (f'(t) + \Phi(f)g(t)), \\ \lim\limits_{t \nearrow 1} (f'(t) + \Phi(f)g(t)) = \lim\limits_{t \searrow 1} (f'(t) + \Phi(f)g(t)). \end{cases} \tag{4.3.2}$$

By the explicit expression for $g : \mathbb{R} \to \mathbb{R}$ we can rewrite Equation (4.3.2) as follows:

$$\begin{cases} \lim\limits_{t \nearrow -1} f'(t) = \lim\limits_{t \searrow -1} f'(t) - \Phi(f), \\ \lim\limits_{t \nearrow 0} f'(t) = \lim\limits_{t \searrow 0} f'(t) + 2\Phi(f), \\ \lim\limits_{t \nearrow 1} f'(t) + \Phi(f) = \lim\limits_{t \searrow 1} f'(t). \end{cases} \tag{4.3.3}$$

Or equivalently

$$\lim_{t \nearrow -1} f'(t) - \lim_{t \searrow -1} f'(t) = -\frac{1}{2}\left(\lim_{t \nearrow 0} f'(t) - \lim_{t \searrow 0} f'(t)\right) = \lim_{t \nearrow 1} f'(t) - \lim_{t \searrow 1} f'(t) = -\Phi(f). \tag{4.3.4}$$

We see that the generator $(C, D(C))$ of the perturbed semigroup is given by

$$\begin{aligned} Cf &= f' + \int_{\mathbb{R}} f \, \mathrm{d}\mu \cdot g, \quad f \in D(C), \\ D(C) &= \left\{ f \in \mathrm{C_b}(\mathbb{R}) : \ f \in \mathrm{C^1_b}(\mathbb{R} \setminus \{-1, 0, 1\}) \text{ and (4.3.4) holds} \right\}. \end{aligned}$$

The previous example uses a function $g \in X_{-1}$ which has three points of discontinuity, with one sided limits at each of these points. We generalize this to a countable (discrete) set of jump discontinuities. For that assume that $g \in X_{-1}$ is a function such that $\|g\|_\infty < \infty$ and that the set of discontinuities of g is discrete. One defines again an operator $B : X_0 \to X_{-1}$ by

$$Bf := \Phi(f)g := \int_{\mathbb{R}} f \, \mathrm{d}\mu \cdot g, \quad f \in \mathrm{C_b}(\mathbb{R}).$$

Notice that none of previous calculations and arguments depend on the number of discontinuities (in fact, we only used that g is bounded). So we can conclude that $(A_{-1}+B)_{|X_0}$ generates a bi-continuous semigroup on X_0 with respect to τ_{co}. The only issue we have to care about are the conditions mentioned in (4.3.4), that is an "explicit" description of the domain. Let $Z := \{x_1, x_2, x_3, \ldots\}$ be the set of discontinuities of g that is assumed to be discrete, and we suppose that all of these points are jump discontinuities. Let us define $a_n := \lim_{t \nearrow x_n} g(t)$ and $b_n := \lim_{t \searrow x_n} g(t)$. We observe that $f \in D((A_{-1}+B)_{|X_0})$ if and only if

$$\lim_{t \nearrow x_n} f'(t) + \Phi(f) a_n = \lim_{t \searrow x_n} f'(t) + \Phi(f) b_n, \quad \text{for each } n \in \mathbb{N},$$

or equivalently

$$\lim_{t\nearrow x_n} f'(t) - \lim_{t\searrow x_n} f'(t) = \Phi(f)(b_n - a_n), \quad \text{for each } n \in \mathbb{N}.$$

We conclude that the operator $(C, D(C))$ given by

$$Cf = f' + \int_{\mathbb{R}} f \,\mathrm{d}\mu \cdot g,$$
$$D(C) = \left\{ f \in \mathrm{C_b}(\mathbb{R}) :\ f \in \mathrm{C_b^1}(\mathbb{R} \setminus Z), \lim_{t\nearrow x_n} f'(t) - \lim_{t\searrow x_n} f'(t) = \Phi(f)(b_n - a_n), \quad n \in \mathbb{N} \right\}$$

generates a bi-continuous semigroup on $\mathrm{C_b}(\mathbb{R})$ with respect to τ_{co}.

§ 4.4 Notes

This chapter is based on joint work with B. Farkas [27]. As mentioned in the notes of Chapter 2, i.e., Section 2.6, this joint work was the starting point for extrapolation spaces for bi-continuous semigroups. For the Desch–Schappacher perturbation theorem presented here in this chapter, cf. Theorem 4.7, we firstly started with perturbation operators B which are continuous from (X, τ) to $(\underline{X}_{-1}, \|\cdot\|_{-1})$. During the last *AGFA* meeting in Blaubeuren in December 2016, the author gave a talk on this topic and during an evening of discussion he got an inspiring idea from R. Nagel how to stay with the extrapolation spaces in the category of bi-continuous semigroups. This idea actually culminate in Definition 2.9 and the results in [28].

As mentioned above in the introduction of this chapter, A. Bátkai, B. Jacob, J. Voigt and J. Wintermayr took positivity into account and proved a Desch–Schappacher perturbation theorem on AM-spaces in [20]. To do this for bi-continuous semigroups is one of the prospective research projects. A first step in this direction is a Miyadera–Voigt perturbation result for positive bi-continuous semigroups, which is in fact the subject of the following Chapter 5.

Compared with the original paper we omit the part on implemented semigroups and postpone them to Chapter 6. There we combine the results from the current chapter as well as from Chapter 2 to apply them to this special kind of semigroup.

Chapter 5

Positive Miyadera–Voigt perturbations

Introduction

Various models of physical processes ask for positive solutions in order to have a reasonable interpretation, i.e., consider solutions containing the absolute temperature or a density. The maximum principle for partial differential equations guarantees positive solutions under positive initial data. This demonstrates the importance of positivity in the theory of operator semigroups. The fundamental concepts in this context such as vector lattices, Banach lattices and positive operators are studied in detail for example in [102] and [88]. Positive operator semigroups are treated for example by Arendt et al. [10] and more recently by A. Bátkai, M. Kramar Fijavž and A. Rhandi [21].

This chapter is based on an article by J. Voigt, cf. [119]. Here positive operator semigroups and perturbations are combined and the following Miyadera–Voigt perturbation result for positive C_0-semigroups was proven.

Theorem. [119, Thm. 0.1] Let E be a AL-space, and let A be the generator of a positive C_0-semigroup on E. Let $B : D(A) \to E$ be a positive operator, and assume that $A + B$ is resolvent positive. Then $A + B$ is the generator of a positive C_0-semigroup.

Quite a number of other positive perturbation results for strongly continuous semigroups and their applications are for example handeled by Arlotti and Banasiak [14]. In this chapter we consider positive perturbations of bi-continuous semigroups in the style of Voigt's work. Especially, we use the Miyadera–Voigt perturbation theorem for bi-continuous semigroups developed by Farkas [54].

This chapter is organized as follows: in Section 1 we recall the basic terminology of positivity as well as the bounded and Miyadera–Voigt perturbations for bi-continuous semigroups and introduce Theorem 5.19 as our main result. Section 2 consists of the proof of this result. In the last section we discuss rank-one perturbations and bi-continuous semigroup on the space $\mathrm{M}(\mathbb{R})$ of bounded Borel measures in connection with differentiable measures.

§ 5.1 Preliminaries

In this section we recall the notion of positivity in Banach spaces (see Section 5.1.1) and the already known perturbation results for bi-continuous semigroups, due to B. Farkas in [55, Thm. 3.5] and [54, Thm. 3.2.2]. In particular we recall bounded perturbations in Section 5.1.2 and perturbations of Miyadera–Voigt type in Section 5.1.3. The formulation for C_0-semigroups on Banach spaces can be found in [52, Chapter III, Sect. 1 & Sect. 3c].

5.1.1 Positivity and bi-AL-spaces

As a matter of fact, we need a order for our underlying Banach space in order to have a concept of positivity. The following definitions are taken from cf. [102, Chapter II] and [88, Chapter 1]. Firstly, we recall the notion of a partial order.

Definition 5.1. Let X be a nonempty set. A *partial order* on X is a binary relation $\leq$ satisfying the following properties for all $x, y, z \in X$:

1. $x \leq x$,
2. $x \leq y$ and $y \leq x$ implies $x = y$,
3. $x \leq y$ and $y \leq z$ implies $x \leq z$.

Since we also have a vector space structure we wish to have a order which is consistent with the vector space. This leads to the notion of vector lattices.

Definition 5.2. A *vector lattice* or *Riesz space* is a vector space V equipped with a partial order $\leq$ such that for each $x, y, z \in V$:

1. $x \leq y \Rightarrow x + z \leq y + z$,
2. $x \leq y \Rightarrow \alpha x \leq \alpha y$ for all scalars $\alpha \geq 0$,
3. For any pair $x, y \in V$ there exists a supremum, denoted by $x \vee y$, and a infimum, denoted by $x \wedge y$, in V with respect to the partial order $\leq$.

An element $x \in V$ is called *positive* if $x \geq 0$. The set of all positive elements is denoted by V_+.

Remark 5.3. We observe that for a Riesz space V the set V_+ is a convex proper cone, i.e., the set of positive elements satisfies the following properties:

(i) $V_+ + V_+ \subseteq V_+$,

(ii) $\lambda V_+ \subseteq V_+$ for all $\lambda > 0$,

(iii) $V_+ \cap (-V_+) = \{0\}$

Pay attention to the fact that each convex proper cone $C \subseteq V$, i.e., C satisfies the three previous conditions, gives rise to an order $\leq$ defined by $x \leq y$ if and only if $y - x \in C$. This order turns V into a vector lattice and the set of positive elements coincides with C, i.e., the equality $V_+ = C$ holds.

In the case of normed spaces and especially Banach spaces, we recall the notion of Banach lattices, for the purpose of compatibility of the norm and the order.

Definition 5.4. A *Banach lattice* is a Banach space $(X, \|\cdot\|)$ which is a Riesz space with an partial order such that for all $x, y \in X$: $|x| \leq |y| \Rightarrow \|x\| \leq \|y\|$, where $|x| := x \wedge (-x)$ is called the absolute value.

Remark 5.5. As a matter of fact, by [102, Chapter II, Prop. 5.2] the positive cone $X_+ = \{x \in X : \ x \geq 0\}$ of a Banach lattice X is always a closed set. Aside from that, X is even archimedean, i.e., if $x, y \in X$ satisfy $nx \leq y$ for all $n \in \mathbb{N}$, then $x \leq 0$. We remark that in contrast, vector lattices in general do not necessarily have this property as the example of $\mathbb{R}^n$ equipped with the lexicographical order shows, cf. [125, Example 9.2(ii)]

Recall from [88, pp. 8-9], that if K is a compact Hausdorff space, then the space $(\mathrm{C}(K), \|\cdot\|_\infty)$ with the pointwise order, i.e., $f \leq g$ if and only if $f(x) \leq g(x)$ for all $x \in K$, is an example of a Banach lattice. Likewise, for a measure space (Ω, Σ, μ) the Banach space of integrable functions $(\mathrm{L}^1(\Omega, \mu), \|\cdot\|_1)$ is a Banach lattice with the pointwise order. Moreover, for a Polish space Ω, the space of bounded Borel measures $\mathrm{M}(\Omega)$ is also a Banach lattice. We say that $\mu \in \mathrm{M}(\Omega)$ is positive if and only if $\mu(A) \geq 0$ for all $A \in \Sigma$, $A \neq \varnothing$. Equipped with the total variance norm $\|\cdot\|_{\mathrm{M}(\Omega)}$, defined by

$$\|\mu\|_{\mathrm{M}(\Omega)} := |\mu|(\Omega) = \sup_{\pi} \sum_{A \in \pi} |\mu(A)|,$$

where the supremum runs over all possible partitions π of Ω into a countable number of disjoint measurable sets, $\mathrm{M}(\mathbb{R})$ becomes a Banach lattice. On Banach lattices one can also consider linear operators. Positivity for operators is then defined as follow.

Definition 5.6. A bounded linear operator $T \in \mathscr{L}(X)$ on a Banach lattice X is called *positive*, denoted by $T \geq 0$, if $Tx \geq 0$ for each $x \in X_+$. A semigroup of bounded linear operators $(T(t))_{t \geq 0}$ on such a Banach lattice is called positive if $T(t) \geq 0$ for each $t \geq 0$.

The following term was suggested by Arendt [8].

Definition 5.7. A linear operator $(A, D(A))$ on a Banach lattice X is called *resolvent positive* if there exists $\omega \in \mathbb{R}$ such that $(\omega, \infty) \subseteq \rho(A)$ and such that $R(\lambda, A) \geq 0$ for each $\lambda > \omega$.

In [119] the concept of so-called AL-spaces, as it is mentioned in the introduction of this chapter, is significant. These spaces satisfy a special kind of norm property, cf. [102, Chapter II, Sect. 8].

Definition 5.8. A Banach lattice $(X, \|\cdot\|)$ is called an AL-*space*, or *abstract* L-*space*, if the norm satisfies

$$\|x + y\| = \|x\| + \|y\|,$$

for each $x, y \in X_+$.

Typical examples for AL-spaces are the already mentioned spaces $\mathrm{L}^1(\Omega, \mu)$ and $\mathrm{M}(\Omega)$. To see that they are actually AL-spaces let $f, g \in \mathrm{L}^1(\Omega, \mu)$ be positive, i.e., $f(x) \geq 0$ and $g(x) \geq 0$ for each $x \in \Omega$.

$$\|f + g\|_1 = \int_\Omega |f + g| \ \mathrm{d}\mu$$

$$
\begin{aligned}
&= \int_\Omega f + g \ \mathrm{d}\mu \\
&= \int_\Omega f \ \mathrm{d}\mu + \int_\Omega g \ \mathrm{d}\mu \\
&= \int_\Omega |f| \ \mathrm{d}\mu + \int_\Omega |g| \ \mathrm{d}\mu \\
&= \|f\|_1 + \|g\|_1
\end{aligned}
$$

In order to show that $\mathrm{M}(\Omega)$ is a AL-space let $\mu, \nu \in \mathrm{M}(\Omega)$ be two positive bounded Borel measures and observe:

$$
\begin{aligned}
\|\mu + \nu\|_{\mathrm{M}(\Omega)} &= |\mu + \nu| (\Omega) \\
&= \sup_\pi \sum_{A \in \pi} |(\mu + \nu)(A)| \\
&= \sup_\pi \sum_{A \in \pi} |\mu(A) + \nu(A)| \\
&= \sup_\pi \sum_{A \in \pi} \mu(A) + \nu(A) \\
&= \sup_\pi \left(\sum_{A \in \pi} \mu(A) + \sum_{A \in \pi} \nu(A) \right) \\
&= \sup_\pi \sum_{A \in \pi} \mu(A) + \sup_\pi \sum_{A \in \pi} \nu(A) \\
&= \sup_\pi \sum_{A \in \pi} |\mu(A)| + \sup_\pi \sum_{A \in \pi} |\nu(A)| \\
&= |\mu| (\Omega) + |\nu| (\Omega) \\
&= \|\mu\|_{\mathrm{M}(\Omega)} + \|\nu\|_{\mathrm{M}(\Omega)},
\end{aligned}
$$

where the supremum runs over all possible partitions π of Ω into a countable number of disjoint measurable subsets.

As we have to take care of an additional locally convex topology in the setting of bi-continuous semigroups, we introduce the following notion which is related to Definition 5.8.

Definition 5.9. Let $(X, \|\cdot\|, \leq)$ be a Banach lattice with ordering and locally convex topology τ generated by a family $\mathcal{P}$ of seminorms which satisfies the Assumptions 1.1. We say that X is a *bi*-AL *space* if for all $x, y \in X_+$ the equality

$$\|x + y\| = \|x\| + \|y\|$$

holds and there exists $\mathcal{P}_+ \subseteq \mathcal{P}$ such that $\mathcal{P}_+$ still generates the locally convex topology τ and for all $x, y \in X_+$

$$p(x + y) = p(x) + p(y)$$

for each $p \in \mathcal{P}_+$.

Locally convex spaces satisfying such additivity conditions as in Definition 5.9 are also mentioned in [35] to discuss regular operators on vector lattices. An example of a space satisfying both the norm and seminorm property from Definition 5.9 is $\mathrm{M}(\Omega)$. Recall from

Section 1.6 that $\mathrm{M}(\Omega)$ equipped with the total variation norm and the weak*-topology $\tau^\circ = \sigma(\mathrm{C_b}(\Omega), \mathrm{M}(\Omega))$ satisfies Assumption 1.1. In fact, τ° is the locally convex topology generated by the following set of seminorms

$$\mathcal{P} := \{p_f : \ f \in \mathrm{C_b}(\Omega)\},$$

where

$$p_f(\mu) := \left|\int_\Omega f \ \mathrm{d}\mu\right|, \quad \mu \in \mathrm{M}(\Omega),$$

Now let

$$\mathcal{P}_+ := \{p_f : \ f \in \mathrm{C_b}(\Omega), \ f \geq 0\}.$$

Let τ_+ denote the locally convex topology generated by $\mathcal{P}_+$. We first observe that for each $p_f \in \mathcal{P}_+$ one has for $\mu, \nu \geq 0$

$$\begin{aligned} p_f(\mu) + p_f(\nu) &= \left|\int_\Omega f \ \mathrm{d}\mu\right| + \left|\int_\Omega f \ \mathrm{d}\nu\right| \\ &= \int_\Omega f \ \mathrm{d}\mu + \int_\Omega f \ \mathrm{d}\nu \\ &= \int_\Omega f \ \mathrm{d}(\mu + \nu) \\ &= \left|\int_\Omega f \ \mathrm{d}(\mu + \nu)\right| \\ &= p_f(\mu + \nu) \end{aligned}$$

It remains to show that $\tau^\circ = \tau_+$.

Lemma 5.10. *The family $\mathcal{P}_+$ generates the topology τ°, i.e., for a net $(\mu_\alpha)_{\alpha \in \Lambda}$ in $\mathrm{M}(\Omega)$:*

$$\mu_\alpha \overset{\tau^\circ}{\to} \mu \iff \mu_\alpha \overset{\tau_+}{\to} \mu$$

Proof. Suppose that

$$\int_\Omega f \ \mathrm{d}\mu_\alpha \to \int_\Omega f \ \mathrm{d}\mu, \tag{5.1.1}$$

for each $f \in \mathrm{C_b}(\Omega)$, then obviously (5.1.1) also holds for every positive $f \in \mathrm{C_b}(\Omega)$. For the other implication notice that for arbitrary $f \in \mathrm{C_b}(\Omega)$ one can decompose f into a positive and negative part, i.e., there exist positive $f_+, f_- \in \mathrm{C_b}(\Omega)$ such that

$$f = f_+ - f_-.$$

Then

$$\int_\Omega f_+ \ \mathrm{d}\mu_\alpha \to \int_\Omega f_+ \ \mathrm{d}\mu,$$

and

$$\int_\Omega f_- \ \mathrm{d}\mu_\alpha \to \int_\Omega f_- \ \mathrm{d}\mu.$$

Since

$$\int_\Omega f \ \mathrm{d}\mu_\alpha = \int_\Omega f_+ \ \mathrm{d}\mu_\alpha - \int_\Omega f_- \ \mathrm{d}\mu_\alpha,$$

we observe that

$$\int_\Omega f \ \mathrm{d}\mu_\alpha \to \int_\Omega f_+ \ \mathrm{d}\mu - \int_\Omega f_- \ \mathrm{d}\mu,$$

and hence we are done. □

5.1.2 Bounded perturbations

Let us continue with perturbation results for bi-continuous semigroups. The bounded perturbations are in some sense simple, since one has not to care about the domain of the perturbing operator. Moreover, the following theorem shows that in particular bounded operators with a additional continuity condition give rise to a perturbation.

Theorem 5.11. *[54, Thm. 3.1.5] Let $(T(t))_{t\geq 0}$ be a bi-continuous semigroup on a Banach space X_0 with respect to τ and with generator $(A, D(A))$. Suppose that $B \in \mathscr{L}(E)$ is τ-sequentially continuous on $\|\cdot\|$-bounded sets. Then $(A + B, D(A))$ is also a generator of a bi-continuous semigroup $(S(t))_{t\geq 0}$. Moreover $(S(t))_{t\geq 0}$ is given by the Dyson–Phillips series*

$$S(t) := \sum_{n=0}^{\infty} T_n(t), \quad t \geq 0,$$

which is uniformly norm-convergent on compact intervals. Here the Dyson–Phillips terms $(T_n(t))_{t\geq 0}$ are defined as

$$T_0(t) := T(t), \quad T_n(t) := \int_0^t T(t-s) B T_{n-1}(s) \ \mathrm{d}s, \quad n > 0,$$

where the integral is under stood in the τ-strong topology.

5.1.3 Miyadera–Voigt perturbations

In contrast to Section 5.1.2 we now come to perturbations of bi-continuous semigroups with an unbounded operator $(B, D(B))$. Let $(T(t))_{t\geq 0}$ be a bi-continuous semigroup on a Banach space X_0 with respect to τ and generator $(A, D(A))$. On the domain $D(A)$ of A one can define a locally convex topology τ_A which is determined by the family of seminorms

$$\mathcal{P}_A := \{p(\cdot) + q(A\cdot) : \ p, q \in \mathcal{P}\}.$$

The graph norm $\|\cdot\|_A$ on $D(A)$ is defined by

$$\|x\|_A := \|x\| + \|Ax\|, \quad x \in D(A).$$

Due to [54, Sect. 1.2 a)] we always may assume that $p(x) \leq \|x\|$ for each $p \in \mathcal{P}$ and $x \in X_0$ and hence one observes that τ_A is coarser than the graph norm topology (the topology coming from the norm $\|\cdot\|_A$). For the Miyadera–Voigt perturbations we consider operators $B : (D(A), \tau_A) \to (X_0, \tau)$ which are continuous on $\|\cdot\|_A$-bounded sets. The following definitions are needed to prepare the perturbation theorem.

Definition 5.12. Let $D \subseteq X_0$ be an arbitrary subset and $\eta > 1$. We say that D is *η-bi-dense* in X_0 for the locally convex topology τ, if for each $x \in X_0$ there exists a sequence $(x_n)_{n\in\mathbb{N}}$ in

D which converges to x with respect to τ and such that

$$\|x_n\| \leq \eta \|x\|,$$

for each $n \in \mathbb{N}$.

Definition 5.13. A bounded function $F : [0, t_0] \to \mathscr{L}(X_0)$ is said to be *local*, if for each $p \in \mathcal{P}$ and $\varepsilon > 0$ there exists a constant $K > 0$ and $q \in \mathcal{P}$ such that

$$p(F(t)x) \leq Kq(x) + \varepsilon \|x\|,$$

for all $t \in [0, t_0]$ and $x \in X_0$.

Remark 5.14. (i) As examples of local functions one can consider the family $\{R(\lambda, A) : \lambda \in [\alpha, \beta]\}$ where $(A, D(A))$ is the generator of a bi-continuous semigroup and $\omega_0(T) < \alpha < \beta$.

(ii) For every bi-continuous semigroup $(T(t))_{t\geq 0}$ on $\mathrm{C_b}(\Omega)$, for Ω a Polish space, the set $\{T(t) : t \in [0, t_0]\}$ is local for each $t_0 > 0$.

(iii) If $(T(t))_{t\geq 0}$ is a bi-continuous semigroup on $\mathrm{C_b}(\Omega)$ and $\mathscr{K} \subseteq \mathrm{M}(\Omega)$ and norm-bounded and weak*-compact subset the set $\{T'(t)\nu : \nu \in \mathscr{K}\}$ is tight/local. As example one can take $\mathscr{K} = \mathrm{M}_1(\Omega)$ if Ω is compact.

One may not expect that every operator $(B, D(B))$ has the property that $(A+B, D(A+B))$ generates again a bi-continuous semigroups. For that reason one introduces the notion of admissibility. Recall from Chapter 4 the definition of the space $\mathfrak{X}_{t_0}$, see (4.1.1), and define the Miyadera–Voigt admissibility as follows.

Definition 5.15. Let $(A, D(A))$ be the η-bi-densely defined generator of a bi-continuous semigroup $(T(t))_{t\geq 0}$. An operator $B : (D(A), \tau_A) \to (X_0, \tau)$ which is continuous on $\|\cdot\|_A$-bounded sets is called *Miyadera–Voigt admissible* on $\mathfrak{X}_{t_0}$ if there exists $t_0 > 0$ for which $T(t) \in \mathfrak{X}_{t_0}$ for $t \in [0, t_0]$ and such that the following conditions hold

1. For all $x \in D(A)$ the maps

$$s \mapsto \|BT(s)x\|, \quad s \in [0, t_0]$$

are bounded.

2. The operator

$$B(F, t)x := \int_0^t F(t-s)BT(s)x \, \mathrm{d}s$$

defined on $D(A)$ extends to linear operator $\overline{B}(F, t)$ on X_0 which is τ-continuous on norm-bounded sets for all $t \in [0, t_0]$ and $F \in \mathfrak{X}_{t_0}$. We require moreover that the operator $\overline{B}(F, t)$ is also norm-bounded.

3. The abstract Volterra operator V_{t_0} on $\mathfrak{X}_{t_0}$, cf. (4.1.1), defined by

$$(V_{t_0}F)(t)x := \overline{B}(F, t)x,$$

for all $x \in X_0$ and $t \in [0, t_0]$ is a bounded operator on $\mathfrak{X}_{t_0}$ and we have $\|V_{t_0}\| \leq \frac{1}{\eta}$

The following theorem tells us that under the condition of Definition 5.15 the operator $(A + B, D(A))$ generates a bi-continuous semigroup on X_0. This generalize the perturbation theorem originally due to I. Miyadera [89] and improved by J. Voigt [118].

Theorem 5.16. *[54, Thm. 3.2.2] Let $(T(t))_{t\geq 0}$ be a bi-continuous semigroup with generator $(A, D(A))$ and suppose that $B : (D(A), \tau_A) \to (X_0, \tau)$ is Miyadera–Voigt admissible on $\mathfrak{X}_{t_0}$. In this case $(A + B, D(A))$ generates a bi-continuous semigroup $(S(t))_{t\geq 0}$. Moreover the semigroup $(S(t))_{t\geq 0}$ satisfies*

$$S(t)x = T(t)x + \int_0^t S(t - s)BT(s)x \, \mathrm{d}s,$$

for all $x \in D(A)$ and $t \geq 0$.

The following theorem gives at least for local bi-continuous semigroups a sufficient condition for $B : (D(A), \tau_A) \to (X_0, \tau)$ being Miyadera–Voigt admissible.

Theorem 5.17. *[54, Thm. 3.2.3] Let $(T(t))_{t\geq 0}$ be a bi-continuous semigroup on X_0 with respect to τ and generator $(A, D(A))$. Suppose that $D(A)$ is η-bi-dense in X_0 and that $B : (D(A), \tau_A) \to (X_0, \tau)$ is continuous on $\|\cdot\|_A$-bounded sets. Suppose that there exists $t_0 > 0$ and $0 < K < \frac{1}{\eta}$ such that*

(a) *The map $s \mapsto \|BT(s)x\|$ is bounded on $[0, t_0]$ and for each $x \in D(A)$.*

(b) $\displaystyle\int_0^t \|BT(s)x\| \, \mathrm{d}s < K \|x\|$ *for each $t \in [0, t_0]$ and $x \in D(A)$.*

(c) *For all $\varepsilon > 0$ and $p \in \mathcal{P}$ there exists $q \in \mathcal{P}$ and $M > 0$ such that*

$$\int_0^{t_0} p(BT(s)x) \, \mathrm{d}s < Mq(x) + \varepsilon \|x\|,$$

for each $x \in D(A)$.

Then $(A + B, D(A + B))$ generates a bi-continuous semigroup $(S(t))_{t\geq 0}$. Furthermore, the semigroup $(S(t))_{t\geq 0}$ satisfies the variation of parameter formula

$$T(t)x = S(t)x + \int_0^t T(t - s)BS(s)x \, \mathrm{d}s, \quad x \in D(A).$$

In [55] B. Farkas considered bounded and Miyadera–Voigt of bi-continuous semigroups. In this chapter we take positivity of Miyadera–Voigt perturbations into account as J. Voigt in [119] did for strongly continuous operator semigroups on Banach spaces. In [13] W. Arendt and A. Rhandi characterize positive perturbations by multiplication operator and apply this to elliptic Schrödinger operators.

Remark 5.18. (i) If $(T(t))_{t\geq 0}$ is a bi-continuous semigroup generated by $(A, D(A))$, then one can deduce from [14, Lemma 4.15] the following equivalence for $\lambda \in \mathbb{R}$:

$$\begin{aligned} &\exists M \in (0, 1) \ \forall t \in [0, t_0] \ \forall x \in D(A) : \int_0^t \|BT(s)x\| \, \mathrm{d}s \leq M \|x\| \\ \iff &\exists M' \in (0, 1) \ \forall t \in [0, t_0] \ \forall x \in D(A) : \int_0^t \left\|\mathrm{e}^{-\lambda s} BT(s)x\right\| \, \mathrm{d}s \leq M' \|x\| \end{aligned}$$

(ii) One also easily proves the following equivalence:

$$\begin{aligned}
&\forall\varepsilon>0\ \forall p\in\mathcal{P}\ \exists K\geq 0\ \exists q\in\mathcal{P}\ \forall x\in D(A):\ \int_0^{t_0} p(BT(s)x)\ \mathrm{d}s\leq Kq(x)+\varepsilon\|x\|\\
\iff\ &\forall\varepsilon>0\ \forall p\in\mathcal{P}\ \exists K'\geq 0\ \exists q'\in\mathcal{P}\ \forall x\in D(A):\ \int_0^{t_0} p(\mathrm{e}^{-\lambda s}BT(s)x)\ \mathrm{d}s\leq K'q'(x)+\varepsilon\|x\|
\end{aligned}$$

The main result of this chapter is the following.

Theorem 5.19. *Let $\eta>1$, and let $(A,D(A))$ be the generator of a positive local bi-continuous semigroup $(T(t))_{t\geq 0}$ on a bi-AL space X with η-bi-dense domain $D(A)$. Let $B:D(A)\to X$ be a positive operator, i.e., $Bx\geq 0$ for all $x\in D(A)\cap X_+$, and assume that $BR(\lambda,A)$ is local and $(A+B,D(A))$ is resolvent positive. Then $(A+B,D(A))$ is the generator of a positive bi-continuous semigroup.*

The following section is devoted to the proof of Theorem 5.19.

§ 5.2 The Proof

Recall from [52, Chapter II, Def. 1.12] that the spectral bound $\mathrm{s}(A)$ of an linear operator is defined by

$$\mathrm{s}(A):=\sup\{\mathrm{Re}(\lambda):\ \lambda\in\sigma(A)\}.$$

In order to prove Theorem 5.19 we need the following lemma.

Lemma 5.20. *Let $\eta>1$ and let $(A,D(A))$ be the η-bi-densely defined generator of a positive local bi-continuous semigroup $(T(t))_{t\geq 0}$ on a bi-AL space X. Let $M:=\sup_{t\in[0,1]}\|T(t)\|<\infty$ and let $B:D(A)\to X$ be a positive operator such that there exists $\lambda>\mathrm{s}(A)$ such that the operator $BR(\lambda,A)$ is local and $\|BR(\lambda,A)\|<\frac{1}{2M}<1$. Then $A+B$ is the generator of a positive bi-continuous semigroup.*

Proof. We start by establishing property (ii) in Theorem 5.17. By Remark 5.18(i) it suffices that we have the following estimate for each $x\in D(A)_+$

$$\begin{aligned}
\int_0^t \left\|B\mathrm{e}^{-\lambda s}T(s)x\right\|\ \mathrm{d}s &= \left\|\int_0^t B\mathrm{e}^{-\lambda s}T(s)x\ \mathrm{d}s\right\|\\
&= \left\|B\int_0^t \mathrm{e}^{-\lambda s}T(s)x\ \mathrm{d}s\right\|\\
&\leq \|BR(\lambda,A)x\|\\
&< \frac{1}{2M}\|x\|,
\end{aligned}$$

where the first equality is justified by the AL-property of the space and the fact that if $x\in D(A)_+$ then it is element of the space of strong continuity (which in fact coincides with $\underline{X}_0:=\overline{D(A)}^{\|\cdot\|}$). The second equality follows by the fact that the operator B is a continuous map from $(D(A),\tau_A)$ to (X,τ) which is A-bounded by [14, Lemma 4.1], i.e., there exists $a,b\geq 0$ such that $\|Bx\|\leq a\|Ax\|+b\|x\|$ for each $x\in D(A)$. The first inequality follows by the Laplace transform representation of the resolvent $R(\lambda,A)$. As a consequence we conclude that property (ii) of Theorem 5.17 holds. Now let $\varepsilon > 0$ and $p \in \mathcal{P}_+$ be arbitrary. Then

$$\int_0^{t_0} p\left(\mathrm{e}^{-\lambda s} BT(s)x\right)\, \mathrm{d}s = p\left(\int_0^{t_0} \mathrm{e}^{-\lambda s} BT(s)x\, \mathrm{d}s\right) = p\left(B\int_0^{t_0} \mathrm{e}^{-\lambda s} T(s)x\, \mathrm{d}s\right)$$
$$\leq p\left(BR(\lambda, A)x\right) \leq Kq(x) + \varepsilon \|x\|.$$

Here the first step follows by the properties of the bi-AL space. The second one follows by the same argument as before and the last inequality follows by the assumption of localness of the operator $BR(\lambda, A)$. Hence property (iii) is fullfilled. Moreover also (i) holds since $(B, D(B))$ is A-bounded by [14, Lemma 4.1].

Keep in mind that we showed that the properties (a)−(c) of Theorem 5.17 hold for $x \in D(A)_+$. We have to show that they hold true for each $x \in D(A)$. In the first place we start showing that the norm condition (b) of Theorem 5.17 holds from each $x \in D(A)$. For this purpose, let $x \in D(A)$ and find $x_+, x_- \in X_+$ such that $x = x_+ - x_-$. Especially, take $x_+ := x \vee 0$ and $x_- := -(x \wedge 0)$ and observe that $x_+ + x_- = |x|$. For $n \in \mathbb{N}$ define

$$x_{n,\pm} := n\int_0^{\frac{1}{n}} T(t)x_\pm\, \mathrm{d}t \in D(A),$$

and notice that $x_{n,\pm} \overset{\tau}{\to} x_\pm$ in X and $x_{n,+} - x_{n,-} \overset{\tau_A}{\to} x$. The sequences $(x_{n,+})_{n\in\mathbb{N}}$ and $(x_{n,-})_{n\in\mathbb{N}}$, and hence also $(x_{n,+} - x_{n,-})_{n\in\mathbb{N}}$, are norm-bounded, i.e.,

$$\|x_{n,\pm}\| = \left\| n\int_0^{\frac{1}{n}} T(t)x_\pm\, \mathrm{d}t \right\| \leq n\int_0^{\frac{1}{n}} \|T(t)x_\pm\|\, \mathrm{d}t \leq L\|x_\pm\|,$$

where $L := \sup_{t\in[0,\frac{1}{n}]} \|T(t)\| < \infty$. Hence we obtain

$$\int_0^t \left\|\mathrm{e}^{-\lambda s} BT(s)(x_{n,+} - x_{n,-})\right\|\, \mathrm{d}s < \frac{1}{2M}\left(\|x_{n,+}\| + \|x_{n,-}\|\right) = \frac{1}{2M}\|x\|. \tag{5.2.1}$$

We have to show that the integrand converges to $\left\|\mathrm{e}^{-\lambda s} BT(s)x\right\|$ in order to conclude the desired estimate. Observe that for each $\varphi \in (X, \tau)'$

$$\varphi\left(\int_0^t \mathrm{e}^{-\lambda s} BT(s)x\, \mathrm{d}s\right) = \int_0^t \varphi\left(\mathrm{e}^{-\lambda s} BT(s)x\right)\, \mathrm{d}s$$
$$= \int_0^t \tau\lim_{n\to\infty} \varphi\left(\mathrm{e}^{-\lambda s} BT(s)(x_{n,+} - x_{n,-})\right)\, \mathrm{d}s$$
$$\leq \tau\liminf_{n\to\infty} \int_0^t \left\|\mathrm{e}^{-\lambda s} BT(s)(x_{n,+} - x_{n,-})\right\|\, \mathrm{d}s = \frac{1}{2M}\|x\|$$

Due to the norming property of the local convex topology (cf. Assumption 1.1) and the AL-property we obtain by taking the supremum over all $\varphi \in (X, \tau)'$ with $\|\varphi\| \leq 1$ the following inequality

$$\int_0^t \left\|\mathrm{e}^{-\lambda s} BT(s)x\right\|\, \mathrm{d}s = \frac{1}{2M}\|x\|.$$

To show the third requirement of Theorem 5.17 is valid, let $\varepsilon > 0$ and $p \in \mathcal{P}_+$ be arbitrary and observe that there exists $K > 0$ and $q \in \mathcal{P}$ such that for each $n \in \mathbb{N}$

$$
\begin{aligned}
\int_0^{t_0} p(\mathrm{e}^{-\lambda s} BT(s)(x_{n,+} - x_{n,-}))\ \mathrm{d}s &\leq \int_0^{t_0} p(\mathrm{e}^{-\lambda s} BT(s)x_{n,+})\ \mathrm{d}s + \int_0^{t_0} p(\mathrm{e}^{-\lambda s} BT(s)x_{n,-})\ \mathrm{d}s \\
&\leq K\left(q(x_{n,+}) + q(x_{n,-})\right) + \varepsilon\left(\|x_{n,+}\| + \|x_{n,i}\|\right) \\
&\leq K\left(q(x_{n,+}) + q(x_{n,-})\right) + 2L\varepsilon\,\|x\|
\end{aligned}
$$

Since by construction $x_{n,+} \to x_+$ and $x_{n,-} \to x_-$ we conclude that $q(x_{n,+}) + q(x_{n,-}) \to q(|x|) = q(x)$. The left-hand side also converges. To see this define for a fixed $\lambda > s(A)$ a sequence $(y_n)_{n\in\mathbb{N}}$ in X for $n \in \mathbb{N}$ by

$$
y_n := (\lambda - A)(x_{n,+} - x_{n,-}),
$$

and set

$$
y := \tau\lim_{n\to\infty} y_n = (\lambda - A)x.
$$

Then by localness on the operator $BR(\lambda, A)$ and the semigroup $(T(t))_{t\geq 0}$ we find $K > 0$ and $q, q' \in \mathcal{P}$ such that

$$
\begin{aligned}
&\left|p(\mathrm{e}^{-\lambda s} BR(\lambda, A)T(s)y_n) - p(\mathrm{e}^{-\lambda s} BR(\lambda, A)T(s)y)\right| \\
\leq& p(\mathrm{e}^{-\lambda s} BR(\lambda, A)T(s)y_n - \mathrm{e}^{-\lambda s} BR(\lambda, A)T(s)y) \\
\leq& \mathrm{e}^{-\lambda s} p(BR(\lambda, A)T(s)(y_n - y)) \\
\leq& \mathrm{e}^{-\lambda s}\left(Kq(T(s)(y_n - y)) + \varepsilon\,\|T(s)(y_n - y)\|\right) \\
\leq& K' q(T(s)(y_n - y)) + \mathrm{e}^{(\omega-\lambda)s}\varepsilon\,\|y_n - y\| \\
\leq& K'' q'(y_n - y) + \varepsilon(K' + \mathrm{e}^{(\omega-\lambda)s})\,\|y_n - y\| \\
\leq& K'' q'(y_n - y) + \varepsilon M(K' + \mathrm{e}^{(\omega-\lambda)s}),
\end{aligned}
$$

where $K' := K\mathrm{e}^{-\lambda s}$ and $K'' > 0$ is a product of K' and a constant coming from the localness of the semigroup $(T(t))_{t\geq 0}$. Moreover, $M \geq 0$ is a constant arising from the exponential boundedness of the semigroup. Since $y_n \stackrel{\tau}{\to} y$ and $\varepsilon > 0$ was arbitrary we see that the convergence of the integrand is uniform in s. Thus the integral converges and we are done. □

The main work for proving Theorem 5.19 is included in the previous Lemma 5.20. Now we are able to accomplish the proof of our main theorem which runs in fact parallel to the original proof of [119, Thm. 0.1].

Proof of Theorem 5.19. By [119, Thm. 1.1] there exists a $\lambda \in \rho(A+B)$ and we know that $r(BR(\lambda, A)) < 1$. Moreover

$$
R(\lambda, A) \leq R(\lambda, A)\sum_{n=0}^{\infty}(BR(\lambda, A))^n = R(\lambda, A + B).
$$

Now, by taking sB instead of B for $s \in [0, 1]$ we obtain by the above

$$
R(\lambda, A) \leq R(\lambda, A + sB) \leq R(\lambda, A + B).
$$

Since B is positive and $\mathrm{Ran}(R(\lambda, A + B)) = D(A)$, we conclude that $BR(\lambda, A + B) \in \mathscr{L}(X)$.

Therefore, also $2\eta BR(\lambda, A+B) \in \mathscr{L}(X)$ and there exists $n \in \mathbb{N}$ such that

$$\|2\eta BR(\lambda, A+B)\| < n.$$

Hence

$$\left\| \frac{1}{n} BR(\lambda, A+sB) \right\| < \frac{1}{2\eta},$$

for each $s \in [0,1]$. In particular, one has

$$\left\| \frac{1}{n} BR\left(\lambda, A + \frac{j}{n} B\right) \right\| < \frac{1}{2\eta},$$

for $0 \leq j \leq n-1$. Now we apply Lemma 5.20 for the perturbation $\frac{1}{n}B$ repeatedly for $A, A+\frac{1}{n}B, \dots, A+\frac{n-1}{n}B$ and obtain the generation by $A+B$ in the last step. □

§ 5.3 Examples

5.3.1 Rank-one perturbations

As a first example we consider rank-one perturbations as they are treated for C_0-semigroups by W. Arendt in A. Rhandi [13, Thm. 2.2]. Let $(T(t))_{t \geq 0}$ be a positive bi-continuous semigroup on X with respect to τ with generator $(A, D(A))$. As a matter of fact every locally convex topology is generated by a family of seminorms $\mathcal{P}$. Since the generated topology does not change if we add another continuous seminorm, we can assume that $\mathcal{P}$ also contains all finite linear combinations of their seminorms, see also [33, Prop. 7.1.4]. Keep in mind, that if we actually suppose this, the inequality $p(x) \leq \|x\|$ for $p \in \mathcal{P}$ and $x \in X$ does not hold in general. For $\varphi : D(A) \to \mathbb{R}$ a positive τ_A-continuous linear functional and $y \geq 0$ in X we define the rank-one perturbation $B : D(A) \to X$ by

$$Bx := \varphi(x)y, \quad x \in D(A).$$

The operator $(B, D(A))$ satisfies the assumptions of Theorem 5.19, and hence it is a Miyadera–Voigt perturbation. To see this let $\varepsilon' > 0$ and $p \in \mathscr{P}$ be arbitrary and observe that

$$p\left(BR(\lambda, A)x\right) = p\left(\varphi\left(R(\lambda, A)x\right) y\right) = \left|\varphi\left(R(\lambda, A)x\right)\right| p(y), \quad x \in X.$$

Since we always assume that $\mathcal{P}$ is a directed family of seminorms and φ is τ_A continuous we conclude that there exists $M > 0$ and $p', q' \in \mathcal{P}$ such that

$$|\varphi(R(\lambda, A)x)| \leq M\left(p'(R(\lambda, A)x) + q'(x)\right), \quad x \in X.$$

By [54, Lemma 1.2.23] and Remark 5.14 the operator $R(\lambda, A)$, $\lambda \in \rho(A)$, is local, i.e., for each $\varepsilon' > 0$ there exists $K' > 0$ and $q'' \in \mathcal{P}$ such that

$$p'(R(\lambda, A)x) \leq K' q''(x) + \varepsilon' \|x\|, \quad x \in X.$$

This leads with $K'' := p(y)MK'$ to the following inequality

$$p(BR(\lambda, A)x) \leq K''\left(q''(x) + \frac{1}{M} q'(x)\right) + \varepsilon \|x\|, \quad x \in X.$$

where $\varepsilon = Mp(y)\varepsilon'$. We conclude from the above that $q'' + \frac{1}{M}q' \in \mathcal{P}$ and we are done. Since $(A, D(A))$ is a Hille–Yosida operator one gets

$$\|BR(\lambda, A)x\| \leq |\varphi(R(\lambda, A)x| \cdot \|y\| \leq \|\varphi\| \, \|R(\lambda, A)x\| \, \|y\| \leq \|\varphi\| \, \frac{M}{|\lambda - \omega|} \, \|x\| \, \|y\| \, .$$

Since all constants are fixed from the beginning our expression becomes smaller than 1 by chosing $\lambda > \omega$ big enough. This show that rank-one perturbations fit in our setting. Observe that the argumentation can be extended to finite rank operators.

5.3.2 Gauss–Weierstrass semigroup on $\mathrm{M}(\mathbb{R})$

Now we discuss an explicit example of a bi-continuous semigroup on the space $\mathrm{M}(\mathbb{R})$ of bounded Borel measures on $\mathbb{R}$ with respect to the Borel σ-algebra $\mathscr{B}(\mathbb{R})$. We noticed in Section 1.6 that every bi-continuous semigroup on $\mathrm{C_b}(\mathbb{R})$ with respect to the compact-open topology τ_{co} gives rise to a bi-continuous semigroup on $\mathrm{M}(\mathbb{R})$ with respect to $\tau^\circ = \sigma(\mathrm{C_b}(\mathbb{R}), \mathrm{M}(\mathbb{R}))$ and vice versa.

For what follows recall from [25, Sect. 1.1] or [23, Sect. 3.8] the definition of the (centered) Gaussian measure γ_t, $t > 0$, on $\mathbb{R}$ defined by

$$\gamma_t(\Omega) = \frac{1}{\sqrt{2\pi t}} \int_\Omega \mathrm{e}^{-\frac{|x|^2}{2t}} \, \mathrm{d}\lambda^1, \quad \Omega \in \mathscr{B}(\mathbb{R}),$$

where λ^1 denotes the Lebesgue measure on $\mathbb{R}$. This measure γ_t, $t > 0$, is a strictly positive bounded Borel measure and in particular a Radon measure. As a matter of fact γ_t, $t > 0$, is the Lebesgue measure with density φ_t given by

$$\varphi_t(x) = \frac{1}{\sqrt{2\pi t}} \mathrm{e}^{-\frac{|x|^2}{2t}}, \quad t > 0, \ x \in \mathbb{R}.$$

Recall from [23, Def. 3.9.8] and [24, Sect. 1.2] that for $\mu, \nu \in \mathrm{M}(\mathbb{R})$ one defines the convolution of μ and ν by

$$(\mu * \nu)(\Omega) = \int_\mathbb{R} \int_\mathbb{R} \mathbf{1}_\Omega(x + y) \, \mathrm{d}\mu(x) \, \mathrm{d}\nu(y) = \int_\mathbb{R} \nu(\Omega - x) \, \mathrm{d}\mu(x).$$

If one of the measures $\mu, \nu \in \mathrm{M}(\mathbb{R})$ has some density with respect to the one-dimensional Lebesgue measure λ^1, say $\nu = g \cdot \lambda^1$, as it is the case for γ_t, then the convolution has density $g * \mu$ defined by

$$(g * \mu)(x) = \int_\mathbb{R} g(x - y) \, \mathrm{d}\mu(y).$$

Now we define a family of operators $(T(t))_{t \geq 0}$ on $\mathrm{M}(\mathbb{R})$ by $T(0)\mu = \mu$ and

$$T(t)\mu = \gamma_t * \mu, \quad t > 0, \ \mu \in \mathrm{M}(\mathbb{R}). \tag{5.3.1}$$

Since $\gamma_t \geq 0$ for each $t > 0$ we conclude that the family $(T(t))_{t \geq 0}$ consists of positive operators on $\mathrm{M}(\mathbb{R})$. The next result shows that $(T(t))_{t \geq 0}$ is in fact a bi-continuous semigroup on $\mathrm{M}(\mathbb{R})$ equipped with the total variation norm and with respect to $\sigma(\mathrm{C_b}(\mathbb{R}), \mathrm{M}(\mathbb{R}))$. Recapitulate that the Gauss–Weierstrass semigroup $(T_*(t))_{t\geq 0}$ on $\mathrm{C_b}(\mathbb{R})$ is defined by $T(0) = I$ and

$$(T_*(t)f)(x) := (\varphi_t * f)(x) = \frac{1}{\sqrt{2\pi t}} \int_{\mathbb{R}} \mathrm{e}^{-\frac{|x-y|^2}{2t}} f(y) \, \mathrm{d}y, \quad t > 0, \ f \in \mathrm{C_b}(\mathbb{R}), \ x \in \mathbb{R}.$$

Having this in mind, we can prove the following theorem.

Theorem 5.21. *The semigroup $(T(t))_{t\geq 0}$ is the adjoint of the Gauss–Weierstrass semigroup $(T_*(t))_{t\geq 0}$ on $\mathrm{C_b}(\mathbb{R})$.*

Proof. We observe that for $f \in \mathrm{C_b}(\mathbb{R})$ with $f \geq 0$ and $\mu \in \mathrm{M}(\mathbb{R})$ the following holds.

$$\begin{aligned}
\langle f, T(t)\mu \rangle &= \left| \int_{\mathbb{R}} f(x) \, \mathrm{d}(T(t)\mu)(x) \right| \\
&= \left| \int_{\mathbb{R}} f(x) \, \mathrm{d}(\gamma_t * \mu)(x) \right| \\
&= \left| \int_{\mathbb{R}} f(x) \, \mathrm{d}((\varphi_t \cdot \lambda^1) * \mu)(x) \right| \\
&= \left| \int_{\mathbb{R}} f(x) \, \mathrm{d}((\varphi_t * \mu) \cdot \lambda^1)(x) \right| \\
&= \left| \int_{\mathbb{R}} f(x)(\varphi_t * \mu)(x) \, \mathrm{d}\lambda^1(x) \right| \\
&= \left| \int_{\mathbb{R}} f(x) \int_{\mathbb{R}} \varphi_t(x-y) \, \mathrm{d}\mu(y) \, \mathrm{d}\lambda^1(x) \right| \\
&= \left| \int_{\mathbb{R}} (f * \varphi_t)(y) \, \mathrm{d}\mu(y) \right| \\
&= \left| \int_{\mathbb{R}} T_*(t)f(y) \, \mathrm{d}\mu(y) \right| \\
&= \langle T_*(t)f, \mu \rangle
\end{aligned}$$

Here $\langle \cdot, \cdot \rangle$ denotes the pairing between $\mathrm{C_b}(\mathbb{R})$ and $\mathrm{M}(\mathbb{R})$. This proves the assertion. ☐

Remark 5.22. (i) The notation $(T_*(t))_{t\geq 0}$ is intuitive since $(T_*(t))_{t\geq 0}$ is the preadjoint of $(T(t))_{t\geq 0}$, i.e., $T_*(t)^* = T(t)$ for all $t \geq 0$.

(ii) Since $(T_*(t))_{t\geq 0}$ is known to be bi-continuous on $\mathrm{C_b}(\mathbb{R})$ with respect to the compact-open topology we conclude by [57, Thm. 3.5] that $(T(t))_{t\geq 0}$ on $\mathrm{M}(\mathbb{R})$ is bi-continuous with respect to $\sigma(\mathrm{C_b}(\mathbb{R}), \mathrm{M}(\mathbb{R}))$.

(iii) Recall that the generator $(A_*, D(A_*))$ of the Gauss–Weierstrass semigroup on $\mathrm{C_b}(\mathbb{R})$ is given by $(\Delta, \mathrm{C}^2_{\mathrm{b}}(\mathbb{R}))$ where Δ denotes the Laplacian and $\mathrm{C}^2_{\mathrm{b}}(\mathbb{R})$ the space of twice continuous differentiable functions with bounded derivatives.

(iv) We also notice that $\mathbb{C}_+ \subseteq \rho(A_*)$, where $\mathbb{C}_+$ denotes the right-half plane, i.e., all complex numbers with positive real part.

(v) To see that $(T(t))_{t\geq 0}$ is indeed a semigroup one can also use the uniqueness of the Fourier-transform.

The next step is to determine the generator $(A, D(A))$ of $(T(t))_{t\geq 0}$. To do so we have to consider differentiable measures. The original study of such measures is due to S.V. Fomin [58] and A.V. Skorohod [105], [106, Chapter 4, Sect. 21]. In [24] V.I. Bogachev discusses differentiable measures in more details.

Definition 5.23. A Borel measure μ on $\mathbb{R}$ is called *Skorohod differentiable* or *S-differentiable* if, for every function $f \in \mathrm{C_b}(\mathbb{R})$, the function

$$t \mapsto \int_{\mathbb{R}} f(x-t)\ \mathrm{d}\mu(x),$$

is differentiable.

The following theorem [24, Thm. 3.6.1] shows that the previous definition is equivalent to the existence of a Borel measure ν, called the Skorohod derivative of μ, such that for each bounded continuous function on $\mathbb{R}$ one has

$$\lim_{t\to 0} \int_{\mathbb{R}} \frac{f(x-t) - f(x)}{t}\ \mathrm{d}\mu(x) = \int_{\mathbb{R}} f(x)\ \mathrm{d}\nu(x) \tag{5.3.2}$$

Theorem 5.24. *Let μ be a Skorohod differentiable Borel measure on $\mathbb{R}$. Then there exists a Borel measure ν which is its Skorohod derivative, i.e., ν satisfies* (5.3.2) *for all bounded continuous functions f on $\mathbb{R}$.*

Remark 5.25. (i) The Skorohod derivative ν of μ as it appears in (5.3.2) and Theorem 5.24 is denoted by $\mathscr{D}\mu$, i.e., $\nu = \mathscr{D}\mu$.

(ii) By [24, Prop. 3.4.1] one has that for a bounded Borel measure μ on $\mathbb{R}$ the measure μ is Skorohod differentiable if and only if it has density of bounded variation, in particular every Skorohod differentiable measure on $\mathbb{R}$ admits a bounded density. In that case the density of the Skorohod derivative is just the (generalized) derivative of the original density.

(iii) For higher order derivatives we recall from [24, Prop. 3.7.1] that if the map $t \mapsto \int_{\mathbb{R}} f(x-t)\ \mathrm{d}\mu(x)$ is n-times differentiable, then μ is n-times Skorohod differentiable, i.e., for all $f \in \mathrm{C_b}(\mathbb{R})$ the function

$$(t_1, \ldots, t_n) \mapsto \int_{\mathbb{R}} f(x + t_1 + \cdots + t_n)\ \mathrm{d}\mu(x),$$

has partial derivatives $\partial_{t_1} \cdots \partial_{t_n}$.

(iv) μ is Skorohod differentiable with Skorohod derivative $\mathscr{D}\mu$ if and only if for each $f \in \mathrm{C_b^\infty}(\mathbb{R})$

$$\int_{\mathbb{R}} \frac{\mathrm{d}}{\mathrm{d}x} f(x)\ \mathrm{d}\mu(x) = -\int_{\mathbb{R}} f(x)\ \mathrm{d}(\mathscr{D}\mu)(x).$$

In order to determine the generator of $(T(t))_{t\geq 0}$ on $\mathrm{M}(\mathbb{R})$ we use Lemma 1.19.

Theorem 5.26. *The generator $(A, D(A))$ of $(T(t))_{t\geq 0}$ is given by*

$$A\mu := \Delta\mu, \quad D(A) := \{\mu \in \mathrm{M}(\mathbb{R}) : \mu \textit{ is twice Skorohod differentiable }\},$$

where $\Delta\mu$ denotes the second order Skorohod derivative of μ.

Proof. Let us denote the generator of our semigroup $(T(t))_{t\geq 0}$ by $(C, D(C))$. By Lemma 1.19 and Remark 5.22 we conclude that the domain of the generator is of the following form

$$D(C) = \left\{\mu \in \mathrm{M}(\mathbb{R}) :\ \exists \nu \in \mathrm{M}(\mathbb{R})\ \forall f \in \mathrm{C}_\mathrm{b}^2(\mathbb{R}) :\ \langle f, \nu\rangle = \langle \Delta f, \mu\rangle\right\}.$$

Now let $\mu \in \mathrm{M}(\mathbb{R})$ be twice Skorohod differentiable, i.e., $\mu \in D(A)$, and denote its second derivative with $\nu := \Delta\mu$. For $f \in \mathrm{C}_\mathrm{b}^2(\mathbb{R})$ one has

$$\langle \Delta f, \mu\rangle = \int_\mathbb{R} \Delta f \ \mathrm{d}\mu = \int_\mathbb{R} f \ \mathrm{d}\nu = \langle f, \nu\rangle,$$

showing that $D(A) \subseteq D(C)$. For the converse let $\mu \in D(C)$, i.e., there exists $\nu \in \mathrm{M}(\mathbb{R})$ such that for each $f \in \mathrm{C}_\mathrm{b}^2(\mathbb{R})$ holds that

$$\langle f, \nu\rangle = \langle \Delta f, \mu\rangle = \left\langle \lim_{t\to 0} \frac{T_*(t)f - f}{t}, \mu\right\rangle = \left\langle f, \lim_{t\to 0} \frac{T(t)\mu - \mu}{t}\right\rangle.$$

From this we may conclude that $\sigma(\mathrm{C}_\mathrm{b}(\mathbb{R}), \mathrm{M}(\mathbb{R})) - \lim_{t\to 0} \frac{T(t)\mu-\mu}{t}$ exists. Combining this with [24, Thm. 3.6.4] we obtain that $\mu \in D(A)$. This finishes the proof. $\square$

Remark 5.27. In Remark 5.22 we observed that $\mathbb{C}_+ \subseteq \rho(A)$ and since $\sigma(A_*) = \sigma(A)$ by [52, Chapter II, Sect. 2.5], we also obtain that $\mathbb{C}_+ \subseteq \rho(A)$, especially, $\lambda \in \rho(A)$ for $\lambda > 0$. Moreover we observe that $(T(t))_{t\geq 0}$ is a bounded semigroup and hence we also conclude that $\mathbb{C}_+ \subseteq \rho(A)$.

Now fix a positive, Lebesgue integrable and unbounded function $\psi : \mathbb{R} \to \mathbb{R}$ and define $B : D(A) \to \mathrm{M}(\mathbb{R})$ by

$$B\mu := \psi \cdot \mu, \quad \mu \in D(A).$$

To show that the operator B satisfies the conditions of Theorem 5.19 let $\lambda \in \rho(A)$, especially, we can choose $\lambda \in \mathbb{R}$ with $\lambda > 0$ and observe that by the duality between $\mathrm{C}_\mathrm{b}(\mathbb{R})$ and $\mathrm{M}(\mathbb{R})$

$$p_f(BR(\lambda, A)\mu) = |\langle f, BR(\lambda, A)\mu\rangle| = |\langle f\psi, R(\lambda, A)\mu\rangle| = |\langle R(\lambda, A_*)(f\psi), \mu\rangle|$$

for all $f \in \mathrm{C}_\mathrm{b}(\mathbb{R})$ and $\mu \in D(A)$. Moreover, by an application of Fubini's theorem and an explicit integral one obtains

$$\begin{aligned} |R(\lambda, A_*)(f\psi)(x)| &= \left|\int_0^\infty \int_\mathbb{R} \mathrm{e}^{-\lambda t} \varphi(x-y) f(y)\psi(y) \ \mathrm{d}y \ \mathrm{d}t\right| \\ &= \left|\int_\mathbb{R} \int_0^\infty \mathrm{e}^{-\lambda t} \varphi(x-y) f(y)\psi(y) \ \mathrm{d}t \ \mathrm{d}y\right| \\ &= \left|\int_\mathbb{R} \frac{1}{\sqrt{2\lambda}} \mathrm{e}^{-\sqrt{2\lambda}|x-y|} f(y)\psi(y) \ \mathrm{d}y\right| \\ &= |(\xi_\lambda * f\psi)(x)|, \end{aligned}$$

where $\xi_\lambda(x) := \frac{1}{\sqrt{2\lambda}} \mathrm{e}^{-\sqrt{2\lambda}|x|}$ is continuous and hence the convolution $\xi_\lambda * f\psi$ is continuous. Furthermore, by Young's convolution inequality and the assumption that ψ is integrable we obtain

$$\|\xi_\lambda * f\psi\|_\infty \leq \|\xi_\lambda\|_\infty \|f\psi\|_1 \leq \|\xi_\lambda\|_\infty \|f\|_\infty \|\psi\|_1 < \infty$$

hence $h := \xi_\lambda * f\psi \in \mathrm{C_b}(\mathbb{R})$ and $p_f(BR(\lambda, A)\mu) = p_h(\mu)$. By using that $\mathrm{M}(\mathbb{R})$ is the dual of $\mathrm{C_b}(\mathbb{R})$ we obtain the following norm estimate

$$\|BR(\lambda, A)\mu\| = \sup_{\substack{f\in \mathrm{C_b}(\mathbb{R})\\ \|f\|_\infty\leq 1}} |\langle f, BR(\lambda, A)\mu\rangle| = \sup_{\substack{f\in \mathrm{C_b}(\mathbb{R})\\ \|f\|_\infty\leq 1}} |\langle R(\lambda, A_*)f\psi, \mu\rangle| \leq \sup_{\substack{f\in \mathrm{C_b}(\mathbb{R})\\ \|f\|_\infty\leq 1}} \|R(\lambda, A_*)f\psi\|_\infty \|\mu\|.$$

Moreover,

$$\begin{aligned} |R(\lambda, A_*)(f\psi)(x)| &= \left|\int_0^\infty \mathrm{e}^{-\lambda t} T_*(t)f(x)\psi(x)\ \mathrm{d}t\right| \\ &= \left|\int_0^\infty \mathrm{e}^{-\lambda t} \int_{\mathbb{R}} \varphi_t(x-y)f(y)\psi(y)\ \mathrm{d}y\mathrm{d}t\right| \\ &\leq \|f\|_\infty \int_0^\infty \mathrm{e}^{-\lambda t} (\varphi * \psi)(x)\ \mathrm{d}t \\ &\leq \|f\|_\infty \|\psi\|_1 \int_0^\infty \mathrm{e}^{-\lambda t} \|\varphi_t\|_\infty\ \mathrm{d}t \\ &= \|f\|_\infty \|\psi\|_1 \int_0^\infty \frac{\mathrm{e}^{-\lambda t}}{\sqrt{2\pi t}}\ \mathrm{d}t \\ &= \frac{\|f\|_\infty \|\psi\|_1}{\sqrt{2\lambda}} \end{aligned}$$

Hence, for λ big enough we obtain $\|R(\lambda, A_*)f\psi\| < 1$ and hence $\|BR(\lambda, A)\| < 1$. Now we can apply Theorem 5.19 and conclude that $A + B$ generates a positive bi-continuous semigroup on $\mathrm{M}(\mathbb{R})$.

Remark 5.28. Suppose that $f \in \mathrm{L}^p(\mathbb{R})$ is unbounded. Observe that there exists $g \in \mathrm{L}^1(\mathbb{R})$ and $h \in \mathrm{L}^\infty(\mathbb{R})$ such that $f = g+h$. In particular, we consider the operator $B : D(A) \to \mathrm{M}(\mathbb{R})$ defined by $B\mu := f \cdot \mu$ as above. Since the case for $f \in \mathrm{L}^1(\mathbb{R})$ is treated before and $f \in \mathrm{L}^\infty(\mathbb{R})$ gives rise to a bounded perturbation we conclude that we can extend our previous result to the whole scale of L^p-spaces.

§ 5.4 Notes

As already mentioned in the beginning, this work is inspired by the work of J. Voigt [119] and B. Farkas [54]. As mentioned in Section 4.4, we are also looking for a Desch–Schappacher perturbation theorem for positive bi-continuous semigroups similar to the results for strongly continuous semigroups presented in [20]. To do so we have to extend the notion of AM-spaces for our purpose as we have done it for AL-spaces in this chapter by means of Definition 5.9.

Even if the example of the Gauss–Weierstrass semigroup on $\mathrm{M}(\mathbb{R})$ is the adjoint of the well-known Gauss–Weierstrass semigroup on $\mathrm{C_b}(\mathbb{R})$ it is interesting how differentiable measures come into play. Differentiable measures as well as Gaussian measures as they are used in this chapter are discussed in more detail in [24], [23] and [25]. There not only the space $\mathbb{R}$ is treated but locally convex spaces in general including infinite dimensional spaces. This raises the first difficulties in this theory, especially, as a matter of fact, there is no canonical infinite dimensional substitute for Lebesgue measure. For example, on an infinite dimensional, separable Banach space X the only locally finite and translation invariant Borel measure μ on X is the trivial measure, i.e., $\mu(A) = 0$ for each measurable set $A \subseteq X$. For more general statements on invariant measures on infinite dimensional vector spaces we refer for example to the work of V.N. Sudakov [110] or Y. Umemura [115].

Chapter 6

Implemented Semigroups

Introduction

The question whether automorhisms and operators are implemented arises naturally if one studies the quantum mechanical aspects in mathematical physics. Typically issues of symmetries apply here. For example Ludwig's Theorem states that on every Hilbert space $\mathcal{H}$ with $\dim(\mathcal{H}) \geq 3$ each order-lattice automorphism α of the effect algebra, defined by $\mathcal{E} := \{T \in \mathscr{L}(\mathcal{H}) : \ 0 \leq T \leq 1\}$, is unitarily implemented, i.e., there exists a unitary operator $U \in \mathscr{L}(\mathcal{H})$ such that $\alpha(T) = UTU^*$ for each $T \in \mathcal{E}$, see for example [30]. Another important symmetry phenomenon in the operator algebraic setting of quantum mechanics is the so-called Wigner symmetry [81, Chapter 5]. In this context one also see implemented semigroups appearing. As it is stated in [26, Chapter 3, Section 2], all continuous one-parameter groups of Wigner symmetries are implemented by strongly continuous one-parameter groups of unitary operators. The most common setting in quantum mechanics is the case of Hilbert spaces. Implemented semigroups on Hilbert spaces also appear in other resources. In particular, in [45, Sect. 4.4, 6.1, 7.1, 7.2] T. Eisner discusses stability of implemented operators as well as of implemented semigroups with applications to Lyapunov's equations. In [11, Chapter D-IV] the point of interest is the asymptotic behaviour of positive implemented semigroups on C^*-algebras and von Neumann algebras. The implemented semigroups also come to light if one considers the following operator equation

$$AX + XB = Y, \ X \in \mathscr{L}(F, E),$$

where A and B are generators of strongly continuous semigroups on Banach spaces E and F. This operator-valued equation has been considered by several authors [12, 97, 98, 99, 61]. We now consider extrapolation spaces and Desch–Schappacher perturbations of implemented semigroup on Banach spaces. In particular, we round out the work of J. Alber [6], where extrapolation spaces for this semigroups are studied but only with respect to the space of strong continuity. With the construction from Chapter 2 we are able to extrapolate implemented semigroups in a more general context. We will use this to consider Desch–Schappacher perturbations of implemented semigroups and show that the domains of these semigroups have a module structure.

This chapter is organize as follows: the first section is devoted to the fact that the space $\mathscr{L}(E)$ of bounded linear operators on a Banach space E satisfies Assumption 1.1 with respect to the strong operator topology. Moreover we show that the implemented semigroup is indeed bi-continuous. In the second section we continue with intermediate and extrapolation spaces of this semigroup and apply this to Desch–Schappacher perturbations of implemented semigroups in the last section.

§ 6.1 Preliminaries

6.1.1 The space of bounded linear operators $\mathscr{L}(E)$

We consider the space $X := \mathscr{L}(E)$ of bounded linear operators on a Banach space E. The operator norm defined by

$$\|T\| := \sup_{\|x\|\leq 1} \|Tx\|, \quad T \in \mathscr{L}(E),$$

makes $\mathscr{L}(E)$ a Banach space. The so-called *strong operator topology* τ_{sot} is defined to be the coarsest topology that makes the evaluation maps $T \mapsto Tx$ continuous for each $x \in E$ (see also [90]). For $S \in \mathscr{L}(E)$, $x \in E$ and $\varepsilon > 0$ we defined

$$\mathcal{U}(S, x, \varepsilon) := \{T \in \mathscr{L}(E) : \ \|Tx - Sx\| < \varepsilon\}.$$

This gives us a subbase for the strong operator topology. In particular we observe that a net $(T_\iota)_{\iota\in I}$ in $\mathscr{L}(E)$ converges to $T \in \mathscr{L}(E)$ with respect to the strong operator topology if and only if $\|T_i x - Tx\| \to 0$ for each $x \in E$. This gives rise to a family of seminorms $\mathcal{P}$ which generates the (locally convex) strong operator topology and is given by

$$\mathcal{P} = \{p_x : \ x \in E\}, \quad p_x(T) := \|Tx\|.$$

Of course τ_{sot} is Hausdorff and coarser than the operator norm topology on $\mathscr{L}(E)$. In order to show that τ_{sot} is sequentially complete on norm-bounded set consider $(T_n)_{n\in\mathbb{N}}$ to be a norm-bounded τ_{sot}-Cauchy sequence. We conclude that $(T_n(x))_{n\in\mathbb{N}}$ is a Cauchy sequence in E and hence convergent. Hence we define $T : E \to E$ by

$$Tx := \lim_{n\to\infty} T_n(x), \quad x \in E.$$

By linearity of the operators $(T_n)_{n\in\mathbb{N}}$ we achieve that T is linear as well. In view of the fact that $(T_n)_{n\in\mathbb{N}}$ was supposed to be bounded with respect to the operator norm, we obtain

$$\|T(x)\| \leq \sup_{n\in\mathbb{N}} \|T_n(x)\| < \infty,$$

and hence $T \in \mathscr{L}(E)$. By construction we attain that $T_n \to T$ with respect to τ_{sot}. For the norming property we follow [103, Thm. 2.1.6], in fact fix sequences $(x_n)_{n\in\mathbb{N}}$ in E and $(\varphi_n)_{n\in\mathbb{N}}$ in E' with $\|x_n\| \leq 1$ and $\|\varphi_n\| \leq 1$ for each $n \in \mathbb{N}$ and such that

$$\|T\| = \sup_{n\in\mathbb{N}} \|Tx_n\| \text{ and } \|Tx_n\| = |\varphi_n(Tx_n)|.$$

Now define $\Lambda_n \in (\mathscr{L}(E), \tau_{\mathrm{sot}})'$

$$\Lambda_n(T) := \varphi_n(Tx_n)$$

and observe that $\|\Lambda_n\| \leq 1$ for each $n \in \mathbb{N}$. Moreover

$$\|T\| = \sup_{n\in\mathbb{N}} \|Tx_n\| = \sup_{n\in\mathbb{N}} |\varphi_n(T(x_n))| = \sup_{n\in\mathbb{N}} |\Lambda_n(T)| \leq \sup_{\substack{\varphi\in(\mathscr{L}(E),\tau_{\mathrm{sot}})' \\ \|\varphi\|\leq 1}} |\varphi(T)| \leq \|T\|$$

which gives us the norming property. We remark that the dual of $(\mathscr{L}(E), \tau_{\mathrm{sot}})$ can be identified with the set $\mathcal{F}(E)$ of finite rank operators on E (see [44]).

6.1.2 The implemented semigroup

Let E be some Banach space and $(T(t))_{t\geq 0}$, $(S(t))_{t\geq 0}$ two C_0-semigroups on E. We already showed that $\mathscr{L}(E)$ satisfies Assumption 1.1 if we equip $\mathscr{L}(E)$ with the strong operator topology τ_{sot}. The semigroup $(\mathcal{U}(t))_{t\geq 0}$ on $\mathscr{L}(E)$ defined by

$$\mathcal{U}(t)L := T(t)LS(t), \quad L \in \mathscr{L}(E),\ t \geq 0,$$

is called the *implemented semigroup*. If $T(t) \equiv I$ or $S(t) \equiv I$, then the corresponding semigroups $(\mathcal{U}_R)(t)_{t\geq 0}$ and $(\mathcal{U}_L)(t)_{t\geq 0}$ are called *right implemented* and *left implemented semigroup*, respectively. As F. Kühnemund already mentioned in [78], these semigroups play an important role in quantum mechanics (see also [26]). To show that $(\mathcal{U}(t))_{t\geq 0}$ is not strongly continuous in general, let $(T(t))_{t\geq 0}$ and $(S(t))_{t\geq 0}$ be two strongly continuous semigroups on the Banach space $E = \mathrm{L}^1(\mathbb{R}, \mathbb{C})$ defined by

$$(T(t)f)(x) := f(x+t), \quad (S(t)f)(x) = \overline{(T(t)f)(s)}, \quad t \geq 0,\ f \in \mathrm{L}^1(\mathbb{R},\mathbb{C}),\ x \in \mathbb{R}.$$

By considering the operator $L \in \mathscr{L}(E)$, defined by

$$Lf := \mathbf{1}_{[0,1]}f, \quad f \in \mathrm{L}^1(\mathbb{R},\mathbb{C}),$$

one recognizes that the implemented semigroup on the space $\mathscr{L}(E)$ defined by $\mathcal{U}(t)L := T(t)LS(t)$, $t \geq 0$, $L \in \mathscr{L}(E)$, is not strongly continuous with respect to the operator norm, i.e., let $(f_n)_{n\in\mathbb{N}}$ be a sequence in $\mathrm{L}^1(\mathbb{R},\mathbb{C})$ such that $\|f_n\| = 1$ and $\mathrm{supp}(f_n) \subseteq \left[0, \frac{1}{n}\right]$ for all $n \in \mathbb{N}$ and observe that

$$\left\|\mathcal{U}\left(\frac{1}{n}\right) Lf_n - Lf_n\right\|_1 = \left\|\mathbf{1}_{\left[\frac{1}{n},\frac{1}{n}+1\right]}f_n - \mathbf{1}_{[0,1]}f_n\right\|_1 = \|f_n\|_1 = 1.$$

Consequently, $(\mathcal{U}(t))_{t\geq 0}$ is not strongly continuous with respect to the norm on $\mathrm{L}^1(\mathbb{R},\mathbb{C})$, cf. [5]. We remark that $(\mathcal{U}(t))_{t\geq 0}$ is a C_0-semigroup if and only if both semigroups $(T(t))_{t\geq 0}$ and $(S(t))_{t\geq 0}$ are uniformly continuous meaning that

$$\lim_{t\to 0} \|T(t) - I\| = \lim_{t\to 0} \|S(t) - I\| = 0.$$

Now we show that the implemented semigroup is bi-continuous with respect to τ_{sot} (see also [78, Prop. 3.16]). Since $(T(t))_{t\geq 0}$ and $(S(t))_{t\geq 0}$ are (automatically) exponentially bounded, also $(\mathcal{U}(t))_{t\geq 0}$ is. For $x \in E$ we obtain:

$$p_x(\mathcal{U}(t)L - L) = \|T(t)LS(t)x - Lx\| \leq \|T(t)L(S(t)x - x)\| + \|T(t)Lx - Lx\|$$

which proves the strong continuity. For the local bi-equicontinuity let $(L_n)_{n\in\mathbb{N}}$ be a norm-bounded strongly convergent null sequence. For $t_0 > 0$ and $x \in E$ holds

$$\sup_{t\in[0,t_0]} p_x(\mathcal{U}(t)L_n) = \sup_{t\in[0,t_0]} \|T(t)L_nS(t)x\| \leq M \cdot \sup_{t\in[0,t_0]} \|L_nS(t)x\|,$$

where $M := \sup_{t\in[0,t_0]} \|T(t)\|$, which is finite by Proposition [52, Chapter I, Sect. 5, Prop. 5.5]. Since $(L_n)_{n\in\mathbb{N}}$ converges uniformly on compact sets and both semigroups are strongly continuous on E we conclude that the right-hand side tends to zero, which proves that $(\mathcal{U}(t))_{t\geq 0}$ is locally bi-equicontinuous.

§ 6.2 Intermediate and extrapolation spaces

6.2.1 Favard- and Hölder spaces

In what follows we restrict our self to left implemented semigroups $(\mathcal{U}_L(t))_{t\geq 0}$. Moreover assume that the implementing C_0-semigroup $(T(t))_{t\geq 0}$ has negative growth bound, see Definition 1.4. As we just noticed, $(\mathcal{U}_L(t))_{t\geq 0}$ is a bi-continuous semigroup on $\mathscr{L}(E)$ with respect to the strong operator topology and still has negative growth bound. We determine the intermediate and extrapolation spaces for this semigroup. We can write:

$$\begin{aligned}\|B\|_{F_\alpha(\mathcal{U}_L)} &= \sup_{t>0} \frac{\|\mathcal{U}_L(t)B - B\|}{t^\alpha} = \sup_{t>0} \frac{\|T(t)B - B\|}{t^\alpha} \\ &= \sup_{t>0} \sup_{\|x\|\leq 1} \frac{\|T(t)Bx - Bx\|}{t^\alpha} = \sup_{\|x\|\leq 1} \sup_{t>0} \frac{\|T(t)Bx - Bx\|}{t^\alpha} = \sup_{\|x\|\leq 1} \|Bx\|_{F_\alpha(T)}.\end{aligned}$$

From this we conclude the following.

Proposition 6.1. *Let $(\mathcal{U}_L(t))_{t\geq 0}$ be the semigroup which is left implemented by $(T(t))_{t\geq 0}$. Then for $\alpha \in (0,1]$*

$$F_\alpha(\mathcal{U}_L) = \mathscr{L}(E, F_\alpha(T))$$

with the same norms.

From the definition we obtain that

$$\begin{aligned}X_\alpha(\mathcal{U}_L) &= \Big\{B \in \mathscr{L}(E) : \ \tau\lim_{t\to 0} \frac{\mathcal{U}_L(t)B - B}{t^\alpha} = 0, \ \|B\|_{F_\alpha(\mathcal{U}_L)} < \infty\Big\} \\ &= \Big\{B \in \mathscr{L}(E) : \ \lim_{t\to 0} \frac{\|T(t)Bx - Bx\|}{t^\alpha} = 0 \quad \text{for all } x \in E\Big\}, \\ \underline{X}_\alpha(\mathcal{U}_L) &= \Big\{B \in \mathscr{L}(E) : \ \lim_{t\to 0} \frac{T(t)B - B}{t^\alpha} = 0\Big\}.\end{aligned}$$

Proposition 6.2. *Let $(\mathcal{U}_L(t))_{t\geq 0}$ be the semigroup which is left implemented by $(T(t))_{t\geq 0}$. Then*

$$X_\alpha(\mathcal{U}_L) = \mathscr{L}(E, X_\alpha(T))$$

with the same norms.

6.2.2 Extrapolation

We now turn to the extrapolation spaces. For the C_0-semigroup $(\underline{\mathcal{U}}_L(t))_{t\geq 0}$ on the space $\underline{X}_0$ these have been studied by J. Alber in [6]. He has shown that the generator $\mathcal{G}$ of $(\mathcal{U}_L(t))_{t\geq 0}$ is given by

$$\mathcal{G}V = A_{-1}V$$

on

$$D(\mathcal{G}) = \{V \in \mathscr{L}(E) : \quad A_{-1}V \in \mathscr{L}(E)\},$$

where A_{-1} denotes the generator of the extrapolated C_0-semigroup $(T_{-1}(t))_{t\geq 0}$ on E_{-1}. The extrapolation spaces X_{-1} and $\underline{X}_{-1}$ can now be obtained by Theorem 2.16. For that let

$$\mathscr{E} = \{S : E \to E_{-\infty} : \text{ linear and continuous}\},$$

where $E_{-\infty}$ is the universal extrapolation space of $(T(t))_{t\geq 0}$ (see the paragraph preceding Theorem 2.16), and let $i : \mathscr{L}(E) \to \mathscr{E}$ be the identity. Consider the operator-valued multiplication operator

$$\mathcal{A}V = A_{-\infty}V, \quad V \in \mathscr{E},$$

where $A_{-\infty}x = A_{-(n-1)}x$ for $x \in E_{-n}$. Notice that $\lambda - \mathcal{A} : X_0 \to \mathscr{E}$ is injective for $\lambda > 0$ since $A_{-\infty}$ and A_{-1} coincide on E. Hence by applying Theorem 2.16 we obtain

$$X_{-1} = \{A_{-1}V : \quad V \in \mathscr{L}(E)\}$$

and

$$\underline{X}_{-1} = \{A_{-1}V : \quad V \in \underline{X}_0\}.$$

From this we conclude that

$$X_{-1} = \Big\{V \in \mathscr{L}(E, E_{-1}) : \ \exists (V_n)_{n\in\mathbb{N}} \subseteq \mathscr{L}(E) \text{ with } V_n \to V \text{ strongly}\Big\} = \overline{\mathscr{L}(E)}^{\mathscr{L}_{\text{stop}}(E,E_{-1})}.$$

Since for any $C \in \mathscr{L}(E, E_{-1})$ we have $nR(n, A_{-1})C \in \mathscr{L}(E)$ and $nR(n, A_{-1})C \to C$ strongly as $n \to \infty$, we obtain

$$X_{-1} = \mathscr{L}(E, E_{-1}).$$

For $\underline{X}_{-1}$ we have:

$$\underline{X}_{-1} = \Big\{V \in \mathscr{L}(E, E_{-1}) : \ \exists (V_n)_{n\in\mathbb{N}} \subseteq \mathscr{L}(E) \text{ with } V_n \to V \text{ in } \mathscr{L}(E, E_{-1})\Big\} = \overline{\mathscr{L}(E)}^{\mathscr{L}(E,E_{-1})}.$$

This last statement is a result of J. Alber, see [6, Thm. 2.4], which we could recover as a simple consequence of the abstract techniques described in Section 2.1.2. Finally, we obtain by Corollary 2.26 and Remark 2.58 that for $\alpha \in [0, 1)$

$$F_{-\alpha}(\mathcal{U}_L) = A_{-1}\mathscr{L}(E, F_{1-\alpha}(S)) = \mathscr{L}(E, F_{-\alpha}(T))$$

and

$$X_{-\alpha}(\mathcal{U}_L) = A_{-1}\mathscr{L}(E, X_{1-\alpha}(S)) = \mathscr{L}(E, X_{-\alpha}(T)).$$

§ 6.3 Perturbations

6.3.1 Ideals in $\mathscr{L}(E)$ and module homomorphisms

Our purpose is to relate Desch–Schappacher perturbations of the C_0-semigroup $(T(t))_{t\geq 0}$ and Desch–Schappacher perturbations of the left implemented semigroup $(\mathcal{U}_L(t))_{t\geq 0}$. Before doing so we need some auxiliary results exploiting the algebraic structure of the domain of Hille–Yosida operators on $\mathscr{L}(E)$.

Lemma 6.3. *Let E be a Banach space and $(\mathcal{A}, D(\mathcal{A}))$ a Hille–Yosida operator on $\mathscr{L}(E)$, i.e., suppose there exists $\omega \in \mathbb{R}$ and $M \geq 1$ such that $(\omega, \infty) \subseteq \rho(\mathcal{A})$ and*

$$\|R(\lambda, \mathcal{A})^n\| \leq \frac{M}{(\lambda - \omega)^n},$$

for each $\lambda > \omega$ and $n \in \mathbb{N}$. The following are equivalent:

(a) *$D(\mathcal{A})$ is a right ideal of the Banach algebra $\mathscr{L}(E)$, i.e., $CB \in D(\mathcal{A})$ whenever $C \in D(\mathcal{A})$, $B \in \mathscr{L}(E)$, and $\mathcal{A}$ is a right $\mathscr{L}(E)$-module homomorphism, i.e., $\mathcal{A}(CB) = \mathcal{A}(C)B$ for $C \in D(\mathcal{A})$ and $B \in \mathscr{L}(E)$.*

(b) *There exists a Hille–Yosida operator $(A, D(A))$ such that $\mathcal{A}(C) = AC$, where $D(\mathcal{A}) = \mathscr{L}(E, D(A))$.*

(c) *There exists a Hille–Yosida operator $(A, D(A))$ such that $\mathcal{A}(C) = A_{-1}C$, where $D(\mathcal{A}) = \{C \in \mathscr{L}(E) :\ A_{-1}C \in \mathscr{L}(E)\}$.*

Proof. The implication (c) $\Rightarrow$ (a) is just a checking of properties of an explicitly given operator. The implication (b) $\Leftrightarrow$ (c) follows from the fact that the operator A and A_{-1} coincide on the domain $D(A)$ of A.

(a) $\Rightarrow$ (b) By definition one has $R(\lambda, \mathcal{A}) \in \mathscr{L}(\mathscr{L}(E))$ whenever $\lambda \in \rho(\mathcal{A})$. Define for $\lambda \in \rho(\mathcal{A})$

$$R(\lambda) := R(\lambda, \mathcal{A})(I).$$

Since $R(\lambda, \mathcal{A})$, $\lambda \in \rho(\mathcal{A})$ satisfy the resolvent identity also $R(\lambda)$, $\lambda \in \rho(\mathcal{A})$ do:

$$\begin{aligned} R(\lambda) - R(\mu) &= R(\lambda, \mathcal{A})(I) - R(\mu, \mathcal{A})(I) = (R(\lambda, \mathcal{A}) - R(\mu, \mathcal{A}))(I) \\ &= ((\lambda - \mu)R(\lambda, \mathcal{A})R(\mu, \mathcal{A}))(I) = (\lambda - \mu)R(\lambda)R(\mu) \end{aligned}$$

for each $\lambda, \mu \in \rho(\mathcal{A})$. Hence the family $(R(\lambda))_{\lambda \in \rho(\mathcal{A})}$ is a pseudoresolvent. If $R(\lambda)x = 0$ for some $x \in E$, then

$$0 = \lambda R(\lambda)x = \lambda R(\lambda, \mathcal{A})(I)x.$$

But since $\lambda R(\lambda, \mathcal{A})(I) \to I$ as $\lambda \to \infty$, it follow that $x = 0$. Therefore $R(\lambda)$ is injective, and hence there exists a closed operator $(A, D(A))$ such that $R(\lambda) = R(\lambda, A)$, i.e.,

$$R(\lambda, A) = R(\lambda, \mathcal{A})(I),$$

Let $C \in D(\mathcal{A})$, i.e., $C = R(\lambda, \mathcal{A})D$ for some $D \in \mathscr{L}(E)$. Then

$$\mathcal{A}(C) = \mathcal{A}(R(\lambda, \mathcal{A})D) = \lambda R(\lambda, \mathcal{A})D - D = (\lambda R(\lambda, A) - I)D$$

$$= (\lambda R(\lambda, A) - (\lambda - A)R(\lambda, A))D = AR(\lambda, A)D = AC.$$

□

Lemma 6.4. *Let E be a Banach space and $(\mathcal{A}, D(\mathcal{A}))$ a generator of a bi-continuous semigroup $(\mathcal{T}(t))_{t\geq 0}$ on $\mathscr{L}(E)$ with respect to τ_{sot}. The following are equivalent:*

(a) *$D(\mathcal{A})$ is a right-ideal of $\mathscr{L}(E)$ and $\mathcal{A}$ is a right $\mathscr{L}(E)$-module homomorphism.*

(b) *The semigroup $(\mathcal{T}(t))_{t\geq 0}$ is left implemented, i.e., there exists a C_0-semigroup $(S(t))_{t\geq 0}$ such that $\mathcal{T}(t)C = S(t)C$ for each $t \geq 0$.*

Under these equivalent conditions, if $(B, D(B))$ is the generator of the C_0-semigroup $(S(t))_{t\geq 0}$, then $\mathcal{A}(C) = B_{-1}C$ for each $C \in \mathscr{L}(E, E_{-1}) = X_{-1}(\mathcal{A})$.

Proof. (b) ⇒ (a) : If $C \in D(\mathcal{A})$, then the limit

$$(\mathcal{A}C)(x) := \lim_{t\searrow 0} \frac{\mathcal{T}(t)Cx - Cx}{t},$$

exists for each $x \in X_0$. Since $(\mathcal{T}(t))_{t\geq 0}$ is left implemented we obtain for $B \in \mathscr{L}(E)$

$$(\mathcal{A}(CB))(x) = \lim_{t\to 0} \frac{\mathcal{T}(t)(CB)x - (CB)x}{t} = \lim_{t\to 0} \frac{(\mathcal{T}(t)C)(Bx) - C(Bx)}{t},$$

and we conclude that $CB \in D(\mathcal{A})$ and $\mathcal{A}(CB) = \mathcal{A}(C)B$.

(a) ⇒ (b) : For $\lambda \in \rho(\mathcal{A})$, $C \in D(\mathcal{A})$, $B \in \mathscr{L}(E)$ one has

$$(\lambda - \mathcal{A})(CB) = \lambda CB - \mathcal{A}(C)B = (\lambda C - \mathcal{A}(C))B.$$

Since $\lambda - \mathcal{A}$ is a bijective map we conclude that

$$R(\lambda, \mathcal{A})(DB) = (R(\lambda, \mathcal{A})D)B$$

for each $D \in \mathscr{L}(E)$. By the Euler-Formula (see [28, Thm. 4.6] and [52, Chapter II, Sect. 3]) we obtain

$$\mathcal{T}(t)C = \tau_{\text{sot}}\lim_{n\to\infty} \left(\frac{n}{t} R\left(\frac{n}{t}, \mathcal{A}\right)\right)^n C.$$

From this we deduce the equality

$$\mathcal{T}(t)(CB)(x) = \left(\tau\lim_{n\to\infty} \left(\frac{n}{t} R\left(\frac{n}{t}, \mathcal{A}\right)\right)^n C\right) B = (\mathcal{T}(t)C)B.$$

Set $S(t) := \mathcal{T}(t)I$, and we are done

$$\mathcal{T}(t)C = \mathcal{T}(t)(I \cdot C) = (\mathcal{T}(t)I)C = S(t)C.$$

Finally, $\mathcal{A}$ is multiplication operator by the generator $(B, D(B))$ of the semigroup $(S(t))_{t\geq 0}$ by Lemma 6.3. □

Proposition 6.5. *Let $(T(t))_{t\geq 0}$ and $(S(t))_{t\geq 0}$ be C_0-semigroups on the Banach space E, and let $(A, D(A))$ denote the generator of $(T(t))_{t\geq 0}$. Let $(\mathcal{U}(t))_{t\geq 0}$ and $(\mathcal{V}(t))_{t\geq 0}$ be the semigroups* *left implemented by* $(T(t))_{t\geq 0}$ *and* $(S(t))_{t\geq 0}$*, respectively. Let* $(\mathcal{G}, D(\mathcal{G}))$ *be the generator of* $(\mathcal{U}(t))_{t\geq 0}$ *and let* $\mathcal{K} : \mathscr{L}(E) \to \mathscr{L}(E, E_{-1}(A))$ *be such that* $\mathcal{K} \in \mathcal{S}_{t_0}^{DS,\tau_{\text{sot}}}(\mathcal{U})$ *and such that* $\mathcal{C} := (\mathcal{G}_{-1} + \mathcal{K})_{|\mathscr{L}(E)}$ *(with maximal domain) is the generator of* $(\mathcal{V}(t))_{t\geq 0}$*. Then* $\mathcal{K}$ *has the property that*

$$\mathcal{K}(CD) = \mathcal{K}(C)D,$$

for each $C, D \in \mathscr{L}(E)$*.*

Proof. Since by assumption $\mathcal{G}$ and $\mathcal{C} = (\mathcal{G}_{-1}+\mathcal{K})_{|\mathscr{L}(E)}$ both generate implemented semigroups we conclude by Lemma 6.4 that $\mathcal{G}$, and hence $\mathcal{G}_{-1}$, and $\mathcal{C}$ are all multiplication operators. One has $\mathcal{G}_{-1}(C) = A_{-1}C$ for each $C \in \mathscr{L}(E)$ and there exists an operator $M : E \to E_{-1}(L)$ such that $\mathcal{C}(C) = MC$ for each $C \in D(\mathcal{C})$. We conclude that

$$\mathcal{K}(C) = MC - A_{-1}C$$

for each $C \in D(\mathcal{C})$. Since $(\mathcal{C}, D(\mathcal{C}))$ is bi-dense in $\mathscr{L}(E)$, for each $C \in \mathscr{L}(E)$, there exists a sequence of operators $(C_n)_{n\in\mathbb{N}}$ in $D(\mathcal{C})$ such that $\sup_{n\in\mathbb{N}} \|C_n\| < \infty$ and

$$C_n x \to Cx,$$

for each $x \in E$. The continuity of $\mathcal{K}$ and $\mathcal{G}_{-1}$ yields

$$\mathcal{K}(C_n) \overset{\tau_{\text{sot}}}{\to} \mathcal{K}(C),$$
$$\mathcal{G}(C_n) \overset{\tau_{\text{sot}}}{\to} \mathcal{G}(C)$$

with convergence in $\mathscr{L}_{\text{sot}}(E, E_{-1}(A))$. Therefore, for each $x \in E$ the sequence $(MC_n x)_{n\in\mathbb{N}}$ is Cauchy in $E_{-1}(A)$ and we can define

$$Lx := \lim_{n\to\infty} MC_n x, \quad x \in E.$$

By construction we obtain $L \in \mathscr{L}(E, E_{-1}(A))$ and $\mathcal{C}(C) = LC$ for each $C \in \mathscr{L}(E)$ and therefore

$$\mathcal{K}(C) = A_{-1}C + LC,$$

for $C \in D(\mathcal{C})$. Now we define $B := L + A_{-1}$ as an operator in $\mathscr{L}(E, E_{-1}(A))$ and conclude that

$$\mathcal{K}(C) = BC$$

for each $C \in \mathscr{L}(E)$ and that was to be proven. □

6.3.2 A one-to-one correspondence

We are now prepared to relate Desch–Schappacher perturbations of the implemented semigroup with the perturbations of the underlying C_0-semigroup. To do so we have to use the class of Desch–Schappacher admissible operators $\mathcal{S}_{t_0}^{DS}$ for C_0-semigroups. Recall from [52, Chapter III, Section 3a] the following definitions for a strongly continuous semigroup $(T(t))_{t\geq 0}$ on a Banach space E. We define

$$\mathcal{S}_{t_0}^{DS}(T) := \{B \in \mathscr{L}(E, E_{-1}) : \ V_B \in \mathscr{L}\left(\mathrm{C}\left([0, t_0], \mathscr{L}_{\text{sot}}(E)\right)\right), \ \|V_B\| < 1\},$$

where V_B denotes the corresponding Volterra operator on E defined by

$$(V_B F)(t) := \int_0^t T_{-1}(t-r)BF(r)\,\mathrm{d}r, \quad F \in \mathrm{C}\left([0,t_0], E\right),\ t \in [0,t_0].$$

The following result shows that Desch–Schappacher perturbations of a C_0-semigroup always give us Desch–Schappacher perturbations of the corresponding implemented semigroup.

Theorem 6.6. *Let $(\mathcal{U}(t))_{t\geq 0}$ be the semigroup on $\mathscr{L}(E)$ left implemented by the C_0-semigroup $(T(t))_{t\geq 0}$. Suppose that $B \in \mathcal{S}_{t_0}^{DS}$ and let $(S(t))_{t\geq 0}$ be the perturbed C_0-semigroup. Define the operator $\mathcal{K} : \mathscr{L}(E) \to \mathscr{L}(E, E_{-1})$ by*

$$\mathcal{K}S := BS, \quad S \in \mathscr{L}(E).$$

Then $\mathcal{K} \in \mathcal{S}_{t_0}^{DS,\tau_{\mathrm{sot}}}$ and the perturbed semigroup $(\mathcal{V}(t))_{t\geq 0}$ is left implemented by $(S(t))_{t\geq 0}$.

Proof. First of all we show that $V_{\mathcal{K}}F(t)C \in \mathscr{L}(E)$ for $F \in \mathfrak{X}_{t_0}$, $t \in [0,t_0]$ and $C \in \mathscr{L}(E)$. Define $f \in \mathrm{C}\left([0,t_0], \mathscr{L}_{\mathrm{sot}}(E)\right)$ by $f(r) := F(r)C$ and observe

$$(V_{\mathcal{K}}F)(t)Cx = \int_0^t \mathcal{U}_{-1}(t-r)\mathcal{K}F(r)Cx\,\mathrm{d}r = \int_0^t T_{-1}(t-r)Bf(r)x\,\mathrm{d}r.$$

Since by assumption $B \in \mathcal{S}_{t_0}^{DS}$, we obtain $(V_{\mathcal{K}}F)(t)Cx \in E$. The following estimate will be crucial for what follows

$$\begin{aligned}\|(V_{\mathcal{K}}F)(t)Cx\| &= \left\|\int_0^t \mathcal{U}_{-1}(t-r)\mathcal{K}F(r)Cx\,\mathrm{d}r\right\| = \left\|\int_0^t T_{-1}(t-r)BF(r)Cx\right\| \\ &= \|(V_B f)(t)x\| \leq \|V_B\| \cdot \|f\| \cdot \|Cx\| \leq \|V_B\| \cdot \|f\| \cdot \|C\| \cdot \|x\|.\end{aligned}$$

This estimate shows that $(V_{\mathcal{K}}F)(t)C \in \mathscr{L}(E)$. Moreover, we directly see that $\mathrm{Ran}(V_{\mathcal{K}}) \subseteq \mathfrak{X}_{t_0}$, since τ_{sot}-strong continuity, norm boundedness and bi-equicontinuity of $V_{\mathcal{K}}F$ follow also from the previous estimate. Also the fact that $\|V_{\mathcal{K}}\| < 1$ is immediate, due to the assumption that $B \in \mathcal{S}_{t_0}^{DS}$. Finally, we show that $(\mathcal{G}_{-1} + \mathcal{K})_{|\mathscr{L}(E)}$ generates the semigroup left implemented by $(S(t))_{t\geq 0}$. For this notice that for sufficiently large $\lambda > 0$ we have

$$\begin{aligned}R(\lambda, (A_{-1}+B)_E)Cx &= \int_0^\infty \mathrm{e}^{-\lambda t}S(t)Cx\,\mathrm{d}t = \int_0^\infty \mathrm{e}^{-\lambda t}\mathcal{V}(t)Cx\,\mathrm{d}t \\ &= R(\lambda, (\mathcal{G}_{-1}+\mathcal{K})_{|\mathscr{L}(E)})Cx,\end{aligned}$$

for all $x \in E$ and $C \in \mathscr{L}(E)$. Whence we conclude that $(\mathcal{G}_{-1} + \mathcal{K})_{|\mathscr{L}(E)}$ generates the semigroup left implemented by $(S(t))_{t\geq 0}$. □

The converse of Theorem 6.6 is also true.

Theorem 6.7. *Let $(\mathcal{U}(t))_{t\geq 0}$ and $(\mathcal{V}(t))_{t\geq 0}$ be two semigroups on $\mathscr{L}(E)$, left implemented by the C_0-semigroups $(T(t))_{t\geq 0}$ and $(S(t))_{t\geq 0}$, respectively. Let $(A, D(A))$ be the generator of $(T(t))_{t\geq 0}$ and let $\mathcal{K} \in \mathcal{S}_{t_0}^{DS,\tau_{\mathrm{sot}}}(\mathcal{U})$ be such that $(\mathcal{V}(t))_{t\geq 0}$ is the corresponding perturbed semigroup. Define $B \in \mathscr{L}(E, E_{-1})$ by*

$$Bx := (\mathcal{K}I)x, \quad x \in E.$$

Then $B \in \mathcal{S}_{t_0}^{DS}(T)$ and $(A_{-1}+B)_{|E}$ generates $(S(t))_{t\geq 0}$.

Proof. Let $f \in \mathrm{C}([0,t_0], \mathscr{L}_{\mathrm{sot}}(E))$ and $x \in E$. We observe that by Lemma 6.3 one has that $(\mathcal{K}I)f(r) = \mathcal{K}(If(r)) = \mathcal{K}f(r)$ for each $r \in [0,t_0]$. For $f \in \mathrm{C}([0,t_0], \mathscr{L}_{\mathrm{sot}}(E))$ we define $F \in \mathfrak{X}_{t_0}$ by $F(r) := M_{f(r)}$, the multiplication with $f(r)$, i.e., $F(r)C = f(r)C$ for each $C \in \mathscr{L}(E)$. The following computation is crucial for the proof:

$$\begin{aligned} V_B f(t)x &= \int_0^t T_{-1}(t-r)Bf(r)x \,\mathrm{d}r = \int_0^t T_{-1}(t-r)(\mathcal{K}I)f(r)x \,\mathrm{d}r \\ &= \int_0^t \mathcal{U}_{-1}(t-r)\mathcal{K}f(r)x \,\mathrm{d}r = \int_0^t \mathcal{U}_{-1}(t-r)\mathcal{K}F(r)Ix \,\mathrm{d}r \\ &= (V_{\mathcal{K}}F)(t)Ix. \end{aligned}$$

From this and from the assumption that $\mathcal{K} \in \mathcal{S}_{t_0}^{DS,\tau_{\mathrm{sot}}}$, we conclude that $B \in \mathcal{S}_{t_0}^{DS}$. Moreover we have

$$S(t)x = \mathcal{V}(t)Ix = \mathcal{U}(t)Ix + \int_0^t \mathcal{U}_{-1}(t-r)\mathcal{K}\mathcal{V}(r)Ix \,\mathrm{d}r = T(t)x + \int_0^t T_{-1}(t-r)BS(r)x \,\mathrm{d}r,$$

for each $x \in E$. This yields that $(A_{-1}+B)_{|E}$ generates $(S(t))_{t\geq 0}$. □

Summarizing Theorems 6.7 and 6.6 we can state the following.

Corollary 6.8. *Let $(\mathcal{U}(t))_{t\geq 0}$ and $(\mathcal{V}(t))_{t\geq 0}$ be two semigroups on $\mathscr{L}(E)$ left implemented by the C_0-semigroups $(T(t))_{t\geq 0}$ and $(S(t))_{t\geq 0}$ on E, respectively. Let us denote the generators of $(\mathcal{U}(t))_{t\geq 0}$ and $(T(t))_{t\geq 0}$ by $(\mathcal{G}, D(\mathcal{G}))$ and $(A, D(A))$, respectively. The following are equivalent:*

(i) *There exists $\mathcal{K} \in \mathcal{S}_{t_0}^{DS,\tau}(\mathcal{U})$ such that $(\mathcal{V}(t))_{t\geq 0}$ is generated by $(\mathcal{G}_{-1}+\mathcal{K})_{|\mathscr{L}(E)}$.*

(ii) *There exists $B \in \mathcal{S}_{t_0}^{DS}(T)$ such that $(S(t))_{t\geq 0}$ is generated by $(A_{-1}+B)_{|E}$.*

Remark 6.9. Notice that not every Desch–Schappacher perturbation of a implemented semigroup gives again a implemented semigroup. To see this let $(\mathcal{G}, D(\mathcal{G}))$ be the generator of the left implemented semigroup $(\mathcal{U}(t))_{t\geq 0}$ and $\Phi \in (\mathscr{L}(E), \tau_{\mathrm{sot}})'$. Define, as above, an operator $\mathcal{K} : \mathscr{L}(E) \to \mathscr{L}(E, E_{-1})$ by

$$\mathcal{K}(C) := \Phi(C)\mathcal{G}_{-1}(I), \quad C \in \mathscr{L}(E).$$

Such an operator $\mathcal{K}$ is not multiplicative if $\Phi \neq 0$.

6.3.3 Comparisons

Now we relate comparison properties of the implemented semigroup and properties of the underlying C_0-semigroup. First of all, for $B \in \mathscr{L}(E)$ we define the multiplication operator $M_B \in \mathscr{L}(\mathscr{L}(E), \mathscr{L}(E))$ by $M_B S := BS$. Then one has $\|M_B\| = \|B\|$. By taking $B := T(t) - S(t)$ for $t > 0$ we directly obtain the following result.

Lemma 6.10. *Let $(\mathcal{U}(t))_{t\geq 0}$ and $(\mathcal{V}(t))_{t\geq 0}$ be two semigroups on $\mathscr{L}(E)$ left implemented by the C_0-semigroups $(T(t))_{t\geq 0}$ and $(S(t))_{t\geq 0}$, respectively. Then the following are equivalent:*

(a) *There exists $M \geq 0$ such that $\|\mathcal{U}(t) - \mathcal{V}(t)\| \leq Mt$ for each $t \in [0,1]$.*

(b) *There exists* $M \geq 0$ *such that* $\|T(t) - S(t)\| \leq Mt$ *for each* $t \in [0,1]$.

Recall from [28, Prop. 6.1] that the Favard spaces of the implemented semigroup and of the underlying semigroup for $\alpha \in [0,1]$ are connected by

$$F_\alpha(\mathcal{U}) = \mathscr{L}(E, F_\alpha(T)). \tag{6.3.1}$$

This yields to the following result.

Lemma 6.11. *For* $B \in \mathscr{L}(E, E_{-1})$ *we define* $\mathcal{K} : \mathscr{L}(E) \to \mathscr{L}(E, E_{-1})$ *by* $\mathcal{K}S := BS$. *Then* $\mathrm{Ran}(\mathcal{K}) \subseteq F_\alpha(\mathcal{U})$ *if and only if* $\mathrm{Ran}(B) \subseteq F_\alpha(T)$.

By [6] and [28] the extrapolated implemented semigroup is defined by

$$\mathcal{U}_{-1}(t)S = T_{-1}(t)S, \quad S \in \overline{\mathscr{L}(E)}^{\mathscr{L}_{\mathrm{sot}}(E,E_{-1})} = \mathscr{L}(E, E_{-1}).$$

This gives

$$F_0(\mathcal{U}) = F_1(\mathcal{U}_{-1}) = \mathscr{L}(E, F_1(T_{-1})) = \mathscr{L}(E, F_0(T)).$$

Proposition 6.12. *Let* $(\mathcal{U}(t))_{t\geq 0}$ *and* $(\mathcal{V}(t))_{t\geq 0}$ *be two semigroups on* $\mathscr{L}(E)$ *left implemented by* $(T(t))_{t\geq 0}$ *and* $(S(t))_{t\geq 0}$, *respectively. Furthermore let* $(\mathcal{G}, D(\mathcal{G}))$ *denote the generator of* $(\mathcal{U}(t))_{t\geq 0}$. *Suppose that there exists* $M \geq 0$ *such that*

$$\|\mathcal{U}(t) - \mathcal{V}(t)\| \leq Mt$$

for each $t \in [0,1]$. *Then there exists* $\mathcal{K} \in \mathcal{S}_{t_0}^{DS,\tau}$ *with* $\mathrm{Ran}(\mathcal{K}) \subseteq F_0(\mathcal{G})$.

Proof. Since $\|\mathcal{U}(t) - \mathcal{V}(t)\| \leq Mt$ for each $t \in [0,1]$ we can use Lemma 6.10 to conclude that $\|T(t) - S(t)\| \leq Mt$ for each $t \in [0,1]$. If $(A, D(A))$ denotes the generator of $(T(t))_{t\geq 0}$, then by [52, Chapter III, Thm. 3.9] we find $B \in \mathscr{L}(E, E_{-1})$ such that $B \in \mathcal{S}_{t_0}^{DS}$ and $\mathrm{Ran}(B) \subseteq F_0(A)$. As in Theorem 6.6 this gives rise to an multiplication operator $\mathcal{K} : \mathscr{L}(E) \to \mathscr{L}(E, E_{-1})$ defined by

$$\mathcal{K}S := BS, \quad S \in \mathscr{L}(E).$$

By Lemma 6.11 we conclude that $\mathrm{Ran}(\mathcal{K}) \subseteq F_0(\mathcal{G})$. It remains to show that $(\mathcal{G}_{-1} + \mathcal{K})_{|\mathscr{L}(E)}$ generates $(\mathcal{V}(t))_{t\geq 0}$. But, by [52, Chapter III, Thm. 3.9], $(A_{-1}+B)_{|E}$ generates $(S(t))_{t\geq 0}$. □

Combining Propositions 6.12, 4.5 and [52, Chapter III, Thm. 3.9] we obtain the following theorem.

Theorem 6.13. *Let* $(\mathcal{U}(t))_{t\geq 0}$ *and* $(\mathcal{V}(t))_{t\geq 0}$ *be two semigroups on* $\mathscr{L}(E)$ *left implemented by* $(T(t))_{t\geq 0}$ *and* $(S(t))_{t\geq 0}$, *respectively. Denote by* $(\mathcal{G}, D(\mathcal{G}))$ *the generator of* $(\mathcal{U}(t))_{t\geq 0}$ *and by* $(A, D(A))$ *the generator of* $(T(t))_{t\geq 0}$. *If* $\mathcal{K} \in \mathcal{S}_{t_0}^{DS,\tau}(\mathcal{U})$ *such that* $\mathrm{Ran}(\mathcal{K}) \subseteq F_0(\mathcal{G})$, *then there exists* $B \in \mathcal{S}_{t_0}^{DS}(T)$ *with* $\mathrm{Ran}(B) \subseteq F_0(A)$ *such that* $\mathcal{K}S = BS$ *for each* $S \in \mathscr{L}(E)$.

Proof. By Proposition 4.5 we find $M \geq 0$ such that $\|\mathcal{U}(t) - \mathcal{V}(t)\| \leq Mt$ for each $t \in [0,1]$. Following the proof of Proposition 6.12 there exists $B \in \mathcal{S}_{t_0}^{DS}$ such that $\mathrm{Ran}(B) \subseteq F_0(A)$. □

From this we can deduce the following equivalence.

Theorem 6.14. *Let $(\mathcal{U}(t))_{t\geq 0}$ and $(\mathcal{V}(t))_{t\geq 0}$ be two semigroups on $\mathscr{L}(E)$ left implemented by the C_0-semigroups $(T(t))_{t\geq 0}$ and $(S(t))_{t\geq 0}$ on E, respectively. Let us denote the generators of $(\mathcal{U}(t))_{t\geq 0}$ and $(T(t))_{t\geq 0}$ by $(\mathcal{G}, D(\mathcal{G}))$ and $(A, D(A))$, respectively. The following are equivalent:*

(a) *There exists $\mathcal{K} \in \mathcal{S}_{t_0}^{DS,\tau}(\mathcal{U})$ such that $\operatorname{Ran}(\mathcal{K}) \subseteq F_0(\mathcal{G})$ and such that $(\mathcal{V}(t))_{t\geq 0}$ is generated by $(\mathcal{G}_{-1} + \mathcal{K})_{|\mathscr{L}(E)}$.*

(b) *There exists $B \in \mathcal{S}_{t_0}^{DS}(T)$ such that $\operatorname{Ran}(B) \subseteq F_0(A)$ and such that $(S(t))_{t\geq 0}$ is generated by $(A_{-1} + B)_{|E}$.*

§ 6.4 Notes

The results mentioned in this chapter are taken from [28] and [27]. We remark, that we only considered the left implemented case. The right implemented semigroup as well as the two-sided implemented semigroup are not studied in such detail. Here we refer for extrapolation results to the work of J. Alber [5, 6]. The question of implementation, as it was already mentioned above by Ludwig's Theorem and the Wigner symmetry, is still interesting. Especially, the results from Section 6.3.2 and Section 6.3.3 assume that the perturbed semigroups are implemented as well. The question is then, under which assumptions on the perturbation operator the perturbed implemented semigroup is again implemented. One indication could in point of fact be the aspects of quantum mechanics.

Chapter 7

Flows on Networks

Introduction

Consider a very large network, whose actual size may not be known but some of its structural properties are understood well. One way to model this situation is to consider an infinite graph with combinatorially reasonable assumptions based on the a priori knowledge about the structure of the network. Along the edges of the network some transport processes take place that are coupled in the vertices in which the edges meet. This means that we consider each edge as an interval and describe functions on it, that is, we consider a *metric graph*. Systems of partial differential equations on a metric graph are also known as *quantum graphs*. The transport processes (or *flows*) on the edges are given by partial differential equations of the form $\frac{\partial}{\partial t}u_j(t,x) = c_j\frac{\partial}{\partial x}u_j(t,x)$ and are interlinked in the common nodes via some prescribed transmission conditions.

Such a problem was considered by Dorn et al. [41, 43, 42] on the state space $\mathrm{L}^1\left([0,1],\ell^1\right)$ applying the theory of strongly continuous operator semigroups. A semigroup approach to flows in finite metric graphs was first presented by Kramar and Sikolya [76] and further used in [51, 42, 22, 15, 21] while transport processes in infinite networks were also studied in [16, 18]. However, all these results were obtained in the L^1-setting. By considering problems in infinite graphs, the flow problem in the L^∞-setting is interesting for applications as well. J. von Below and J.A. Lubary [120, 121], for example, study eigenvalues of the Laplacian on infinite networks in an L^∞-setting. To the best of our knowledge, transport equations on infinite metric graphs with an L^∞-state space have not yet been studied. We consider this problem on the state space $\mathrm{L}^\infty\left([0,1],\ell^1\right)$ where the obtained operator semigroup is not strongly continuous. To tackle this we make use of the theory of bi-continuous semigroups.

This chapter is organised as follows. Section 7.1 is a preliminary section where we introduce some notions for networks and metric graphs. In Section 7.2 we present the flow problem for an infinite metric graph. We first prove the well-posedness in the case when all flow velocities c_j equal 1. Next, we generalise this result to the case with rationally dependent velocities satisfying a finiteness condition. Finally, we show that the general problem is well-posed. Actually this will be done only on a finite metric graph due to technical assumptions.

§ 7.1 Preliminaries

In order to talk about finite and infinite networks we use the notation used in [76] and [41]. A network is modelled with an *infinite directed graph* $G = (V, E)$ with a set of *vertices* $V = \{\mathrm{v}_i \mid i \in I\}$ and a set of *directed edges* $E = \{\mathrm{e}_j \mid j \in J\} \subseteq V \times V$ for some countable sets $I, J \subseteq \mathbb{N}$. For a directed edge $\mathrm{e} = (\mathrm{v}_i, \mathrm{v}_k)$ we call v_i the *tail* and v_k the *head* of e. Further, the edge e is an *outgoing edge* of the vertex v_i and an *incoming edge* for the vertex v_k. We assume that the graph G is *simple*, i.e., there are no loops or multiple edges. This means in particular, that there are no edges of the form $\mathrm{e} = (\mathrm{v}_i, \mathrm{v}_i)$, $i \in I$ (i.e., the tail and the head of the edge coincide and so an edge connects a vertex with itself) and no several edges connecting two vertices in the same direction

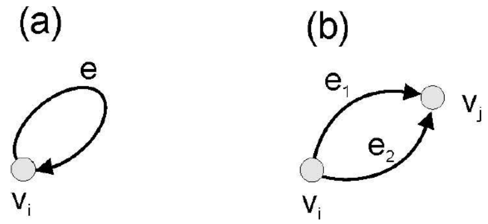


Figure 7.1: A loop (a) and a graph with multiple edges (b), cf. [40, Fig. 2.2]

Moreover, G is assumed to be *locally finite*, i.e., each vertex has only finitely many outgoing edges. Furthermore, the graph G is *weighted*, that is equipped with some weights $0 \leq w_{ij} \leq 1$ such that

$$\sum_{j \in I} w_{ij} = 1 \text{ for all } i \in I. \tag{7.1.1}$$

The structure of a graph can be described by its incidence or its adjacency matrix. The *outgoing incidence matrix* $\Phi^- = (\Phi^-_{ij})$ is defined by

$$\Phi^-_{ij} := \begin{cases} 1 & \text{if } \mathrm{v}_i \xrightarrow{\mathrm{e}_j}, \\ 0 & \text{otherwise,} \end{cases} \tag{7.1.2}$$

whereas the *weighted outgoing incidence matrix* $\Phi^-_\omega = (\Phi^-_{\omega,ij})$ is defined by

$$\Phi^-_{\omega,ij} := \begin{cases} w_{ij} & \text{if } \mathrm{v}_i \xrightarrow{\mathrm{e}_j}, \\ 0 & \text{otherwise.} \end{cases} \tag{7.1.3}$$

By $\mathrm{v}_i \xrightarrow{\mathrm{e}_j}$ we mean that the vertex v_i is the tail of the edge e_j. The *incoming incidence matrix* $\Phi^+ = (\Phi^+_{ij})$ is defined by

$$\Phi^+_{ij} := \begin{cases} 1 & \text{if } \xrightarrow{\mathrm{e}_j} \mathrm{v}_i, \\ 0 & \text{otherwise.} \end{cases} \tag{7.1.4}$$

Here $\xrightarrow{\mathrm{e}_j} \mathrm{v}_i$ means that the vertex v_i is the head of the edge e_j. The *incidence matrix* Φ of the directed graph G, describing the structure of the network completely, is then defined by $\Phi := \Phi^+ - \Phi^-$. There are two other important matrices associated to a general graph and who are needed in what follows. The *transposed adjacency matrix* of the weighted graph G is defined by

$$\mathbb{A} := \Phi^+ \left(\Phi_\omega^-\right)^\top .$$

The nonzero entries of $\mathbb{A}$ correspond exactly to the edges of the graph, keeping track also of the appropriate weights, cf. [21, p. 280]. In fact, $\mathbb{A}$ can be described explicit as

$$\mathbb{A}_{ij} := \begin{cases} w_{jk} & \text{if } \mathrm{v}_j \xrightarrow{\mathrm{e}_k} \mathrm{v}_i, \\ 0 & \text{otherwise.} \end{cases} \tag{7.1.5}$$

Last but not least we use the so-called *weighted (transposed) adjacency matrix of the line graph* $\mathbb{B} = (\mathbb{B}_{ij})$ defined by $\mathbb{B} := \left(\Phi_\omega^-\right)^\top \Phi^+$. One can also give an explicit entrywise description as

$$\mathbb{B}_{ij} := \begin{cases} w_{ki} & \text{if } \xrightarrow{\mathrm{e}_j} \mathrm{v}_k \xrightarrow{\mathrm{e}_i}, \\ 0 & \text{otherwise.} \end{cases} \tag{7.1.6}$$

By (7.1.1), the matrix $\mathbb{B}$ is column stochastic, i.e., the sum of entries of each column is 1, and defines a bounded positive operator on $\ell^1 := \ell^1(J)$ with $r(\mathbb{B}) = \|\mathbb{B}\| = 1$.

We identify every edge of our graph with the unit interval, $\mathrm{e}_j \equiv [0,1]$ for each $j \in J$, and parametrise it contrary to the direction of the flow, if $c_j > 0$, $j \in J$, so that it is assumed to have its tail at the endpoint 1 and its head at the endpoint 0. For simplicity we use the notation $\mathrm{e}_j(1)$ and $\mathrm{e}_j(0)$ for the tail and the head, respectively. In this way we obtain a *metric graph.*

§ 7.2 Transport problems on (in)finite metric graphs

We now consider a transport process (or a flow) along the edges of an infinite network, modelled by a metric graph G. The distribution of material along edge e_j at time $t \geq 0$ is described by function a $u_j(x,t)$ for $x \in [0,1]$. The material is transported along the edge e_j with constant velocity $c_j > 0$, $j \in J$. We assume that

$$0 < c_{\min} \leq c_j \leq c_{\max} < \infty \tag{7.2.1}$$

for all $j \in J$. Let $C := \operatorname{diag}(c_j)_{j\in J}$ be a diagonal velocity matrix and define another weighted adjacency matrix of the line graph by

$$\mathbb{B}^C := C^{-1}\mathbb{B}C.$$

In the vertices the material gets redistributed according to some prescribed rules. This is modelled in the boundary conditions by using the adjacency matrix $\mathbb{B}_C$. The flow process on

G is thus described by the following infinite system of equations

$$\begin{cases} \dfrac{\partial}{\partial t}u_j(x,t) = c_j \dfrac{\partial}{\partial x}u_j(x,t), & x \in (0,1)\,,\ t \geq 0, \\ u_j(1,t) = \displaystyle\sum_{k\in J} \mathbb{B}^C_{jk} u_k(0,t), & t \geq 0,\ \text{(boundary conditions)} \\ u_j(x,0) = f_j(x), & x \in (0,1)\,, \end{cases} \tag{7.2.2}$$

for every $j \in J$, where $f_j(x)$ are the initial distributions along the edges.
One can give different interpretations to the weights w_{ij}, i.e., entries of the matrix $\mathbb{B}$, resulting in different transport problems. The two most obvious are the following.

1. w_{ij} is the proportion of the material arriving from edge e_j leaving on edge e_i.

2. w_{ij} is the proportion of the material arriving in vertex $\mathrm{e}_j(0) = \mathrm{e}_i(1)$ leaving on edge e_i.

Note, that in both situations (7.1.1) represents a conservation of mass and the assumption on local finiteness of the graph guarantees that all the sums are finite. While the latter situation is the most common one (see e.g. [41, 76, 21]) the first one was considered for finite networks in [22, Sect. 5]. Here, we will not give any particular interpretation and will treat all the cases simultaneously.

Remark 7.1. By replacing in (7.2.2) the graph matrix $\mathbb{B}^C$ with some other matrix, one obtains a more general initial-value problem that does not necessarily consider a process in a physical network. Such a problem from population dynamics was for example studied in [16]. Furthermore, the question when such a general problem can be identified with a corresponding problem on a metric graph was raised in [15].

7.2.1 The simple case: constant velocities

First we assume that all the velocities are the same: $c_j = 1$ for each $j \in J$. In what follows we abbreviate the space $\ell^1(J)$ by ℓ^1. As the state space we set $X_0 := \mathrm{L}^\infty\left([0,1]\,, \ell^1\right)$ equipped with the norm

$$\|f\|_{X_0} := \operatorname*{ess\,sup}_{s\in[0,1]} \|f(s)\|_{\ell^1}.$$

On the Banach space X_0 we define the operator $(A, D(A))$ by

$$\begin{aligned} A &:= \operatorname{diag}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right), \\ D(A) &:= \left\{v \in \mathrm{W}^{1,\infty}\left([0,1]\,, \ell^1\right) \mid v(1) = \mathbb{B}v(0)\right\}. \end{aligned} \tag{7.2.3}$$

Now we consider the corresponding abstract Cauchy problem on X_0 corresponding to the operator $(A, D(A))$ defined in (7.2.3)

$$\begin{cases} \dot{u}(t) = Au(t), & t \geq 0, \\ u(0) = (f_j)_{j\in J}. \end{cases} \tag{7.2.4}$$

For the sake of completeness we prove that (7.2.4) is equivalent to the flow problem (7.2.2) in case when all the velocities equal 1.

To see this we show that the domain condition in (7.2.3) is equivalent to the boundary condition in (7.2.2). For $g \in D(A)$ we obtain for its j-th component the following equality

$$g_j(1) = \mathbb{B}_j g(0) = \left(\Phi_\omega^-\right)_j^\top \Phi^+ g(0)$$

By the structure of $\left(\Phi_\omega^-\right)^\top$ one obtains

$$\Phi_{ij}^- g_j(1) = w_{ij} \sum_{k \in J}^{m} \Phi_{ik}^+ g_k(0).$$

For the converse we start by using the boundary condition in (7.2.2). Moreover, we recall from [21, Prop. 18.2] that the j-th row of $\left(\Phi_\omega^-\right)^\top$ corresponds to the edge e_j and has exactly one nonzero element, say w_{ij} , corresponding to the starting vertex v_i for this edge. That is to say that $\Phi_{ij}^- = 1$. Hence we obtain

$$\begin{aligned}(\mathbb{B}g(0))_j &= \left(\Phi_\omega^-\right)_j^\top \Phi^+ g(0)\\ &= w_{ij} \sum_{k \in J}^{m} \Phi_{ik}^+ g_k(0)\\ &= \Phi_{ij}^- g_j(1) = g_j(1)\end{aligned}$$

This shows indeed that the two problems (7.2.2) and (7.2.4) are equivalent.

On the state space $\mathrm{L}^1\left([0,1], \ell^1\right)$ this was considered by B. Dorn [41] where an explicit formula for the solution semigroup in terms of a shift and matrix $\mathbb{B}$ was derived. Dorn made use of the left-translation semigroup $(T_l(t))_{t \geq 0}$, which is not strongly continuous on X_0. However, by using duality arguments we will show that it is a bi-continuous semigroup on X_0. First note that the left-translation semigroup on $X_0 = \mathrm{L}^1\left([0,1], \mathrm{c}_0\right)' = \mathrm{L}^\infty\left([0,1], \ell^1\right)$ is the adjoint semigroup of the right-translation semigroup on $\mathrm{L}^1\left([0,1], \mathrm{c}_0\right)$, see [52, Chapter I, Example 5.14]. Here c_0 is the space of all sequences converging to 0, which is in fact a Banach space equipped with the supremum-norm.

Lemma 7.2. *The right-translation semigroup* $(T_\mathrm{r}(t))_{t \geq 0}$*, defined by*

$$T_\mathrm{r}(t) f(s) := \begin{cases} f(s-t), & s-t \geq 0, \\ 0, & s-t < 0, \end{cases}$$

for $f \in \mathrm{L}^1\left([0,1], \mathrm{c}_0\right)$*,* $t \geq 0$ *and* $s \in [0,1]$*, is strongly continuous on* $\mathrm{L}^1\left([0,1], \mathrm{c}_0\right)$*.*

Proof. Let $\mathbf{x} = (x_n)_{n \in \mathbb{N}} \in \mathrm{c}_0$ and let $f := \mathbf{x} \cdot \mathbf{1}_\Omega$ for a measurable subset $\Omega \subseteq [0,1]$. We first show that $T_\mathrm{r}(t) f \to f$ with respect to the norm on $\mathrm{L}^1\left([0,1], \mathrm{c}_0\right)$ as $t \to 0$:

$$\begin{aligned}&\int_0^1 \left\|T_\mathrm{r}(t) f(s) - f(s)\right\|_{\mathrm{c}_0} \mathrm{d}s = \int_0^1 \left\|f(s-t)\mathbf{1}_{[s-t \geq 0]} - f(s)\right\|_{\mathrm{c}_0} \mathrm{d}s\\ &= \|\mathbf{x}\|_{\mathrm{c}_0} \cdot \int_0^1 \left|\mathbf{1}_\Omega(s-t) - \mathbf{1}_\Omega(s)\right| \mathrm{d}s = \|\mathbf{x}\|_{\mathrm{c}_0} \cdot \int_0^1 \left|\mathbf{1}_{(\Omega+t)\triangle\Omega}(s)\right| \mathrm{d}s\\ &= \|\mathbf{x}\|_{\mathrm{c}_0} \cdot \lambda^1\left((\Omega+t)\triangle\Omega\right) \to 0 \text{ as } t \to 0,\end{aligned}$$

where λ^1 is the one-dimensional Lebesgue measure on $[0,1]$ and Δ the symmetric difference of sets defined by $A\Delta B := (A\cup B)\setminus(A\cap B)$, $A, B \subseteq [0,1]$. Since every function $f \in \mathrm{L}^1([0,1], \mathrm{c}_0)$ is an increasing limit of linear combinations of functions of the form $\mathbf{x}\cdot\mathbf{1}_\Omega$ for some $\mathbf{x} \in \mathrm{c}_0$ and measurable set $\Omega \subseteq [0,1]$, the vector-valued version of Beppo–Levi's monotone convergence theorem yields the result. □

Remark 7.3. (i) The Beppo–Levi theorem we used in the preceding proof is by [117, Prop. 2.6] only applicable in our case owing to the fact that c_0 is a Banach lattice with σ-order continuous norm meaning that if $(x_n)_{n\in\mathbb{N}}$ a decreasing sequence with respect to the ordering on c_0 such that the infimum is 0, then $(x_n)_{n\in\mathbb{N}}$ norm-converges to 0. For general Banach spaces X there is no monotone convergence theorem on $\mathrm{L}^1([0,1], X)$.

(ii) Observe that the classical proof of strong continuity, as it can for example be found in [21, Ex. 9.11], uses a density argument instead of the monotone convergence theorem. Especially, the strong continuity does not depend on the space c_0, hence one could generalise this result to strongly continuous right-translation semigroups on $\mathrm{L}^1([0,1], Y)$ and even to $\mathrm{L}^p([0,1], Y)$ for an arbitrary Banach space Y and $1 \leq p < \infty$.

Lemma 7.4. *The left-translation semigroup $(T_{\mathrm{l}}(t))_{t\geq 0}$, defined by*

$$T_{\mathrm{l}}(t)f(s) := \begin{cases} f(s+t), & s+t \leq 1, \\ 0, & s+t > 1, \end{cases}$$

*for $t \geq 0$ and $s \in [0,1]$, is bi-continuous on $\mathrm{L}^\infty([0,1], \ell^1)$ with respect to the weak**-topology.*

Proof. This result follows directly from the results in Chapter 1 since $(T_l(t))_{t\geq 0}$ is the adjoint semigroup of the strongly continuous semigroup $(T_{\mathrm{r}}(t))_{t\geq 0}$, cf. [52, Chapter II, Sect. 3.6, Exam. (i)], hence we can apply the result from Section 1.2.2, i.e., the adjoint semigroup is bi-continuous with respect to the weak*-topology. □

We now use the formula for the semigroup which was derived by B. Dorn for infinite networks [41] and show that it yields a bi-continuous semigroup on $\mathrm{L}^\infty([0,1], \ell^1)$. For that we have to check all the assertions from Definition 1.3 which we do in several steps.

Lemma 7.5. *The semigroup $(T(t))_{t\geq 0}$ on $X_0 = \mathrm{L}^\infty([0,1], \ell^1)$, defined by*

$$(T(t)f)(s) = \mathbb{B}^n f(t+s-n), \quad n \leq t+s < n+1, \ f \in X_0, \ n \in \mathbb{N}_0, \tag{7.2.5}$$

*is strongly continuous with respect to the weak**-topology.*

Proof. The semigroup property is easy to verify, cf. [40, Prop. 1.2.1]. Observe that for any $f \in X_0$, $g \in \mathrm{L}^1([0,1], \mathrm{c}_0)$, and $t \in (0,1]$ we have

$$\begin{aligned} |\langle T(t)f - f, g\rangle| &= \left|\int_0^1 \langle T(t)f(s) - f(s), g(s)\rangle \,\mathrm{d}s\right| \\ &\leq \left|\int_0^{1-t} \langle f(s+t) - f(s), g(s)\rangle \,\mathrm{d}s\right| + \int_{1-t}^1 |\langle \mathbb{B}f(s+t-1) - f(s), g(s)\rangle| \,\mathrm{d}s \\ &= \left|\int_0^1 \langle T_l(t)f(s) - f(s), g(s)\rangle \,\mathrm{d}s\right| + \int_{1-t}^1 |\langle \mathbb{B}f(s+t-1) - f(s), g(s)\rangle| \,\mathrm{d}s. \end{aligned}$$

Now, notice that the second summand vanishes since $\lambda^1([1-t,1]) \to 0$ as $t \to 0$. Here, λ^1 is again the one-dimensional Lebesgue measure on the unit interval $[0,1]$. By Lemma 7.4, the left-translation semigroup is bi-continuous on X_0, which means, in particular, that it is strongly continuous with respect to the weak*-topology and hence the first summand also vanishes as $t \to 0$. □

Lemma 7.6. *The semigroup* $(T(t))_{t\geq 0}$*, defined by* (7.2.5)*, is a contraction semigroup on* X_0*.*

Proof. Let $f \in X_0$ and $t \geq 0$. Then there exists $n \in \mathbb{N}_0$ such that $n \leq t < n+1$. This means that for $s \in [0,1]$ one has $n \leq s+t < n+2$. By (7.2.5), we can make the following estimate.

$$\begin{aligned}\|T(t)f\|_{X_0} &= \operatorname*{ess\,sup}_{s\in[0,1]} \|T(t)f(s)\|_{\ell^1} \\ &\leq \max\left\{ \operatorname*{ess\,sup}_{s\in[0,n+1-t)} \|\mathbb{B}^n f(s+t-n)\|_{\ell^1},\ \operatorname*{ess\,sup}_{s\in[n+1-t,1)} \left\|\mathbb{B}^{n+1} f(s+t-n-1)\right\|_{\ell^1} \right\}.\end{aligned}$$

Since $\|\mathbb{B}^n\| = \|\mathbb{B}\|^n = 1$, we have

$$\|\mathbb{B}^n f(s+t-n)\|_{\ell^1} \leq \|\mathbb{B}^n\| \cdot \|f\|_{X_0} = \|f\|_{X_0}$$

and hence, $\|T(t)f\|_{X_0} \leq \|f\|_{X_0}$. □

Lemma 7.7. *The semigroup* $(T(t))_{t\geq 0}$*, defined by* (7.2.5)*, is locally bi-equicontinuous with respect to the weak*-topology on* $X_0 = \mathrm{L}^\infty([0,1], \ell^1)$*.*

Proof. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of functions in X_0 that is $\|\cdot\|_{X_0}$-bounded and converges to 0 with respect to the weak*-topology. By Definition 1.3 we need to show that $(T(t)f_n)_{n\in\mathbb{N}}$ also converges to 0 with respect to the weak*-topology uniformly for $t \in [0, t_0]$. To this end, fix $t_0 > 0$ and let $m := \lfloor t_0 \rfloor$. Moreover, for fixed $t \in [0,t_0]$ let $k := \lfloor t \rfloor$. Then, for this $t \in [0, t_0]$ we obtain

$$\begin{aligned}|\langle T(t)f_n, g\rangle| &\leq \int_0^1 \left|\langle T(t)f_n(s), g(s)\rangle_{X_0}\right| \mathrm{d}s \\ &\leq \int_0^1 \left|\left\langle \mathbb{B}^k f_n(s+t-k), g(s)\right\rangle_{X_0}\right| \mathrm{d}s \\ &\leq \int_0^1 \left|\left\langle T_l(t-k)f_n(s), \left(\mathbb{B}^k\right)' g(s)\right\rangle_{X_0}\right| \mathrm{d}s.\end{aligned}$$

By taking the maximum for $0 \leq t \leq t_0$ and the corresponding maximum for $0 \leq k \leq m$ we obtain for $0 \leq s+t \leq m+1$, $s \in [0,1]$ by (7.2.5) the following:

$$\begin{aligned}|\langle T(t)f_n, g\rangle| &\leq \int_0^1 \left|\langle T(t)f_n(s), g(s)\rangle_{X_0}\right| \mathrm{d}s \\ &\leq \int_0^1 \max_{0\leq k\leq m} \left|\left\langle \mathbb{B}^k f_n(s+t-k), g(s)\right\rangle_{X_0}\right| \mathrm{d}s \\ &\leq \int_0^1 \max_{0\leq k\leq m} \left|\left\langle T_l(t-k)f_n(s), \left(\mathbb{B}^k\right)' g(s)\right\rangle_{X_0}\right| \mathrm{d}s,\end{aligned}$$

for each $g \in \mathrm{L}^1([0,1], \mathrm{c}_0)$. Since, by Lemma 7.4, the left-translation semigroup $(T_l(t))_{t\geq 0}$ is bi-continuous, hence locally bi-equicontinuous, $|\langle T(t)f_n, g\rangle|$ tends to 0 uniformly on $[0, t_0]$. This finishes the proof. □

Let us recall here the explicit expression of the resolvent of operator $(A, D(A))$ defined by (7.2.3) which was obtained in [41, Theorem 18]. This result does not rely on the Banach space and remains the same if we take $X_0 = \mathrm{L}^\infty([0,1], \ell^1)$ instead of $\mathrm{L}^1([0,1], \ell^1)$. To be self-contained we reproduce the proof.

Proposition 7.8. *For* $\mathrm{Re}(\lambda) > 0$ *the resolvent* $R(\lambda, A)$ *of the operator* $(A, D(A))$ *defined by* (7.2.3) *is given by*

$$(R(\lambda, A)f)(s) := \sum_{k=0}^{\infty} \mathrm{e}^{-\lambda k} \int_0^1 \mathrm{e}^{-\lambda(t+1-s)} \mathbb{B}^{k+1} f(t)\ \mathrm{d}t + \int_s^1 \mathrm{e}^{\lambda(s-t)} f(t)\ \mathrm{d}t, \quad f \in X_0,\ s \in [0,1].$$

Proof. Let $f \in X_0$ and $g \in D(A)$ such that $(\lambda - A)g = f$, or in other words, $\lambda g - g' = f$. To solve this differential equation we use the variation of constants formula to obtain

$$g(s) = \mathrm{e}^{\lambda s} \cdot d + \int_s^1 \mathrm{e}^{\lambda(s-t)} f(t)\ \mathrm{d}t, \quad s \in [0,1], \tag{7.2.6}$$

for some $d \in \ell^1$. By the assumption that $g \in D(A)$ we have $g(1) = \mathbb{B}g(0)$ which yields,

$$d \cdot \mathrm{e}^{\lambda} = \mathbb{B}d + \mathbb{B} \int_0^1 \mathrm{e}^{-\lambda t} f(t)\ \mathrm{d}t,$$

whence

$$(I - \mathrm{e}^{-\lambda}\mathbb{B})d = \mathbb{B} \int_0^1 \mathrm{e}^{-\lambda(t+1)} f(t)\ \mathrm{d}t.$$

For $\lambda \in \mathbb{C}$ with $\mathrm{Re}(\lambda) > 0$ we obtain from the fact that $r(\mathbb{B}) = 1$ that $r\left(\mathrm{e}^{-\lambda}\mathbb{B}\right) < 1$. By applying the Neumann series one gets

$$d = \sum_{k=1}^{\infty} \left(\mathrm{e}^{-\lambda}\mathbb{B}\right)^k \mathbb{B} \int_0^1 \mathrm{e}^{-\lambda(t+1)} f(t)\ \mathrm{d}t.$$

Combining this with (7.2.6) yields the desired formula. □

We are now in the state prove the first generation theorem.

Theorem 7.9. *The operator* $(A, D(A))$*, defined in* (7.2.3)*, generates a bi-continuous contraction semigroup* $(T(t))_{t\geq 0}$ *on* X_0 *with respect to the weak*-topology. This semigroup is given by* (7.2.5)*, i.e.,*

$$(T(t)f)(s) = \mathbb{B}^n f(t+s-n), \quad n \leq t+s < n+1,\ f \in X_0,\ n \in \mathbb{N}_0.$$

Proof. By Lemmas 7.5, 7.6, and 7.7, semigroup $(T(t))_{t\geq 0}$ defined by (7.2.5) is a bi-continuous semigroup with respect to the weak*-topology. It remains to show that $(A, D(A))$, given in (7.2.3), is the generator of this semigroup. Let $(B, D(B))$ be the generator of $(T(t))_{t\geq 0}$. By (1.3.2), the resolvent of B is the Laplace transform of the semigroup $(T(t))_{t\geq 0}$, that is, for

$\lambda > \omega_0(T)$ we have

$$\begin{aligned}(R(\lambda,B)f)(s) &= \int_0^\infty \left(\mathrm{e}^{-\lambda t}T(t)f\right)(s)\ \mathrm{d}t \\ &= \int_0^{1-s} \mathrm{e}^{-\lambda t} f(t+s)\ \mathrm{d}t + \sum_{n=1}^\infty \int_{n-s}^{n-s+1} \mathrm{e}^{-\lambda t}\mathbb{B}^n f(t+s-n)\ \mathrm{d}t \\ &= \int_s^1 \mathrm{e}^{-\lambda(\xi-s)} f(\xi)\ \mathrm{d}\xi + \sum_{n=1}^\infty \int_0^1 \mathrm{e}^{-\lambda(\xi-s+n)}\mathbb{B}^n f(\xi)\ \mathrm{d}\xi.\end{aligned}$$

To conclude the proof we observe that $D(A)$ is bi-dense in X_0. To see this, we refer to Theorem 7.15 below, where we show this for the more general case (even if we there consider finite networks, the idea of the proof is the same). Now, by Proposition 7.8, the resolvent operators $R(\lambda, A)$ and $R(\lambda, C)$, coincide on the bi-dense set $D(A)$, so we conclude that $C = A$. □

Corollary 7.10. *If all* $c_j = 1$, $j \in J$, *the flow problem* (7.2.2) *is well-posed on* $X_0 = \mathrm{L}^\infty([0,1], \ell^1)$.

Remark 7.11. All the obtained results also hold for finite networks. If $G = (V, E)$ is a finite network with $|E| = m < \infty$, we have $\ell^1(\{1,\dots,m\}) \cong \mathbb{C}^m$, hence we consider the semigroups on the space $X_0 = \mathrm{L}^\infty([0,1], \mathbb{C}^m)$.

7.2.2 The rationally dependent case

We now consider the case when the velocities c_j appearing in (7.2.2) are not all equal to 1 and define on $X_0 := \mathrm{L}^\infty([0,1], \ell^1)$ the operator

$$\begin{aligned} A_C &:= \operatorname{diag}\left(c_j \cdot \frac{\mathrm{d}}{\mathrm{d}x}\right), \\ D(A_C) &:= \left\{ f \in \mathrm{W}^{1,\infty}\left([0,1], \ell^1\right) \mid f(1) = \mathbb{B}^C f(0) \right\}. \end{aligned} \tag{7.2.7}$$

We assume, however, that the velocities are linearly dependent over $\mathbb{Q}$: $\frac{c_i}{c_j} \in \mathbb{Q}$ for all $i, j \in J$, with a finite common multiplier, that is,

$$\text{there exists } 0 < c \in \mathbb{R} \text{ such that } \ell_j := \frac{c}{c_j} \in \mathbb{N} \text{ for all } j \in J. \tag{7.2.8}$$

This enables us to use the procedure that was introduced in the proof of [76, Thm. 4.5] and carried out in detail in [17, Sect. 3]. We construct a new directed graph $\widetilde{G}$ by adding $\ell_j - 1$ vertices on edge e_j for all $j \in J$. The newly obtained edges inherit the direction of the original edge and are parametrised by unit intervals $[0, 1]$. We can thus consider a new problem on $\widetilde{G}$ with corresponding functions $\widetilde{u}_j$ and velocities $\widetilde{c}_j := c$ for each $j \in \widetilde{J}$. After appropriately correcting the initial and boundary conditions the new problem is equivalent to the original one. Since all the velocities on the edges of the new graph are equal, we can treat this case by rescaling the velocities to 1 and use the results from Subsection 7.2.1. Moreover, since (7.2.1) and (7.2.8) hold, the procedure described by J. Banasiak and P. Namayanja in [17, Sect. 3] for the finite case can be as applied in the infinite case as well. Hence, we even obtain an isomorphism between the corresponding semigroups.

Proposition 7.12. *Let the assumptions* (7.2.1) *and* (7.2.8) *on the velocities* c_j *hold. Then operator* $(A_C, D(A_C))$*, defined in* (7.2.7)*, generates a contraction bi-continuous semigroup* $(T_C(t))_{t\geq 0}$ *on* X_0 *with respect to the weak*-topology. Moreover, there exists an isomorphism* $S\colon X_0 \to X_0$ *such that*

$$T_C(ct)f = ST(t)S^{-1}f, \tag{7.2.9}$$

where the semigroup $(T(t))_{t\geq 0}$ *is given by* (7.2.5)

Corollary 7.13. *If the assumptions* (7.2.1) *and* (7.2.8) *on the velocities* c_j *hold, the flow problem* (7.2.2) *is well-posed on* $X_0 = \mathrm{L}^\infty([0,1], \ell^1)$.

7.2.3 The general case for finite networks

We finally consider the case of general $c_j \in \mathbb{R}$ but restrict ourselves to *finite* graphs, i.e., we work on the Banach space $X_0 = \mathrm{L}^\infty([0,1], \mathbb{C}^m)$, where m denotes the number of edges in the graph. In [21, Cor. 18.15] the Lumer–Phillips generation theorem for positive strongly continuous semigroups is applied to show that the transport problem with general $c_j \in \mathbb{R}$ is well-posed on $X_0 = \mathrm{L}^1([0,1], \mathbb{C}^m)$. Since an appropriate variant of a Lumer–Phillips generation theorem for the bi-continuous situation is not known, we proceed differently and use the bi-continuous version of the Trotter–Kato approximation theorem, cf. Theorem 1.22. Let

$$E_\lambda(s) := \operatorname{diag}\left(\mathrm{e}^{(\lambda/c_j)s}\right), \quad s \in [0,1], \quad \text{and} \quad \mathbb{B}_\lambda^C := E_\lambda(-1)\mathbb{B}^C, \operatorname{Re}(\lambda) > 0.$$

By using this notation one can write an explicit expression for the resolvent of operator A_C defined in (7.2.7), cf. [21, Prop. 18.12].

Lemma 7.14. *For* $\operatorname{Re}(\lambda) > 0$ *the resolvent* $R(\lambda, A_C)$ *of operator* A_C *given in* (7.2.7) *equals*

$$R(\lambda, A_C) = \left(I_{X_0} + E_\lambda(\cdot)\left(1 - \mathbb{B}_\lambda^C\right)^{-1}\mathbb{B}_\lambda^C \otimes \delta_0\right) R_\lambda,$$

where $\delta_0 : \mathrm{L}^\infty([0,1], \mathbb{C}^m) \to \mathbb{C}^m$ *denotes the point evaluation at* 0 *and*

$$(R_\lambda f)(s) = \int_s^1 E_\lambda(s-t)C^{-1}f(t)\,\mathrm{d}t, \quad s \in [0,1], \; f \in \mathrm{L}^\infty([0,1], \mathbb{C}^m).$$

Proof. Let $f \in X_0$ and $g \in D(A_C)$ such that $(\lambda - A_C)g = f$. As a result of the variation of constants formula we obtain

$$g(s) = E_\lambda(s)d + \int_s^1 E_\lambda(s-t)C^{-1}f(t)\,\mathrm{d}t, \quad s \in [0,1], \tag{7.2.10}$$

for some $d \in \mathbb{C}^m$. The boundary condition of $D(A_C)$ yields that $g(0) = \mathbb{B}^C g(1)$ meaning that

$$E_\lambda(1)d = \mathbb{B}^C d + \mathbb{B}^C \int_0^1 E_\lambda(-t)C^{-1}f(t)\,\mathrm{d}t.$$

By multiplying the previous equation by $E_\lambda(-1)$ we get

$$\left(I - \mathbb{B}_\lambda^C\right)d = \mathbb{B}_\lambda^C(R_\lambda f)(0).$$

We observe that $r\left(\mathbb{B}_\lambda^C\right) \leq \left\|\mathbb{B}_\lambda^C\right\| < 1$ (see [21, Sect. 18.3]), hence $(I - \mathbb{B}_\lambda^C)^{-1}$ exists for $\operatorname{Re}(\lambda) > 0$ and

$$d = \left(I - \mathbb{B}_\lambda^C\right)^{-1} \mathbb{B}_\lambda^C (R_\lambda f)(0)$$

Together with (7.2.10) we get the desired expression for the resolvent. □

Theorem 7.15. *The operator $(A_C, D(A_C))$, defined in (7.2.7), generates a bi-continuous semigroup $(T_C(t))_{t\geq 0}$ on $X_0 = \mathrm{L}^\infty([0,1], \mathbb{C}^m)$.*

Proof. We first show that operator A_C is bi-densely defined. Take any $f \in \mathrm{L}^\infty([0,1], \mathbb{C}^m)$. For $n \in \mathbb{N}$ let $\Omega_n := \left[\frac{1}{n}, 1 - \frac{1}{n}\right] \subseteq [0,1]$ and define $f_n : [0,1] \to \mathbb{C}^m$ by a linear truncation of f outside Ω_n, i.e.,

$$f_n(x) := \begin{cases} nf\left(\frac{1}{n}\right)x, & x \in \left[0, \frac{1}{n}\right], \\ f(x), & x \in \left[\frac{1}{n}, 1 - \frac{1}{n}\right], \\ nf\left(1 - \frac{1}{n}\right)(1-x), & x \in \left[1 - \frac{1}{n}, 1\right]. \end{cases}$$

Observe that f_n is Lipschitz for each $n \in \mathbb{N}$ and hence $f_n \in \mathrm{W}^{1,\infty}([0,1], \mathbb{C}^m)$. Moreover $f_n(1) = f_n(0) = 0$ for each $n \in \mathbb{N}$ implying that $f_n(0) = \mathbb{B}^C f_n(0)$, hence $f_n \in D(A_C)$. Furthermore one has that $\sup_{n\in\mathbb{N}} \|f_n\|_\infty \leq \|f\|_\infty < \infty$ and $f_n \to f$ as $n \to \infty$ with respect to the weak*-topology, since

$$\left|\int_0^1 \langle (f_n(x) - f(x)), g(x)\rangle \, \mathrm{d}x\right| \leq 2\|f\|_\infty \lambda^1\left(\left[0, \frac{1}{n}\right] \cup \left[1 - \frac{1}{n}, 1\right]\right)\|g\|_1 = \frac{4}{n}\|f\|_\infty \|g\|_1$$

for each $g \in \mathrm{L}^1([0,1], \mathbb{C}^m)$. We now define a sequence of operators A_n approximating A_C in the following way. For each $c_j \in \mathbb{R}$ there exists a sequence $\left(c_j^{(n)}\right)_{n\in\mathbb{N}}$ in $\mathbb{Q}$ such that $\lim_{n\to\infty} c_j^{(n)} = c_j$. Since the network is finite, for each $n \in \mathbb{N}$ the velocities $c_j^{(n)}$, $j \in J$, satisfy condition (7.2.8) and, by Proposition 7.12 we obtain a bi-continuous contraction semigroup $(T_n(t))_{t\geq 0}$ generated by

$$\begin{aligned} A_n &:= \operatorname{diag}\left(c_j^{(n)} \cdot \frac{\mathrm{d}}{\mathrm{d}x}\right), \\ D(A_n) &:= \left\{f \in \mathrm{W}^{1,\infty}([0,1], \mathbb{C}^m) \mid f(1) = \mathbb{B}^{C_n} f(0)\right\}, \end{aligned} \tag{7.2.11}$$

where $C_n := \operatorname{diag}\left(c_j^{(n)}\right)$. Moreover, all semigroups $(T_n(t))_{t\geq 0}$, $n \in \mathbb{N}$, are similar and thus uniformly bi-continuous of type 0, cf. Definition 1.20. Observe, that the general assumptions of Theorem 1.22 are satisfied. Let us now check the assumptions of assertion (b) of Theorem 1.22. Let $R := R(\lambda, A_C)$ and observe that $R : \mathrm{L}^\infty([0,1], \mathbb{C}^m) \to D(A_C)$ is a bijection. By the above, $\operatorname{Ran}(R)$ is bi-dense in $\mathrm{L}^\infty([0,1], \mathbb{C}^m)$. For every $n \in \mathbb{N}$, replacing c_j by $c_j^{(n)}$ for all $j \in J$, Lemma 7.14 yields an explicit expression for $R(\lambda, A_n)$. It is easy to see that $R(\lambda, A_n)f \overset{\|\cdot\|}{\to} Rf$ for $f \in D(A_C)$ as $n \to \infty$. Applying Theorem 1.22 gives us a bi-continuous semigroup $(T_C(t))_{t\geq 0}$ and an operator $(B, D(B))$ generating this semigroup. Note that, since in our case $R = R(\lambda, A_C)$ is a resolvent, by Remark 1.23 we have $R = R(\lambda, A_C) = R(\lambda, B)$ for $\lambda \in \rho(A_C)$ and by the uniqueness of the Laplace transform we conclude that $(B, D(B)) = (A_C, D(A_C))$. □

Corollary 7.16. *The flow problem* (7.2.2) *for finite networks is well-posed on* $\mathrm{L}^\infty([0,1],\mathbb{C}^m)$.

Remark 7.17. Observe that in the same manner, by using the original strongly continuous version of the Trotter–Kato Theorem (see [52, Chapter III, Sect.4b]), one can deduce the well-posedness of the problem on $X_0 = \mathrm{L}^1([0,1],\mathbb{C}^m)$.

§ 7.3 Notes

The content of this chapter is joint work with M. Kramar Fijavž [29]. In fact, we treat the case of the state space $\mathrm{L}^\infty([0,1],\ell^1)$ which gives rise to bi-continuous semigroups. We chose this state space since on the one hand one has the Radon–Nikodym property of ℓ^1 and on the other hand a preadjoint semigroup which can be controlled to obtain our results. To mention is that some of the results become achievable after a fruitful discussion with B. Farkas.

The research question actually appeared as a part of a suggested research proposal for a *PPP project* of the *DAAD* driven forth by B. Farkas and M. Kramar Fijavž. Although this project was not supported we started a collaboration. To be more precise, in April 2018 the author visited the *University of Ljubljana* and was financially supported by the *ERASMUS+* program in the context of a staff training (STT).

M. Kramar Fijavž and the author first met during the final workshop of the *20*$^{\text{th}}$ *Internet-seminar* on *Linear Parabolic Equations* at the *University of Salerno*. As a matter of fact the author was part of *Project F: Non-autonomous diffusion in networks* conducted by Kramar Fijavž. This project awaken interest of the author in networks. Afterwards we stayed in contact and this get our collaboration off the ground.

We remark that there is an open question: can Theorem 7.15 be handled without the assumption of finiteness of the network as mentioned in Section 7.2.3? In fact, this assumption is caused by the use of the Trotter–Kato approximation theorem. In [21, Sect. 18.3] the strongly continuous case for finite networks on the phase space $\mathrm{L}^1([0,1],\mathbb{C}^m)$ is discussed. The author thinks that it should be no problem to generalize this to infinite networks. As a matter of fact, one uses the Lumer–Phillips generation theorem, cf. [21, Thm. 11.10] and [52, Chapter II, Thm. 3.15]. Such a generation theorem is not known so far for the bi-continuous case. Even if we also have the work of B. Dorn [41], the state spaces $\mathrm{L}^\infty([0,1],\ell^\infty)$ and $\mathrm{L}^1([0,1],\ell^\infty)$ are not yet considered and are left open.

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