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Stabilisation of Infinite-Dimensional Port-Hamiltonian Systems via Dissipative Boundary Feedback

vorgelegt von

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Chapter 1

Introduction

From the very start of the science of mechanics strings and beams played a crucial role in the research, starting from the consideration of e.g. the dynamics of a mass attached to a string. At the beginning models had been considered for which the flexibility of the string or beam had been neglected and quite often they additionally had been idealised to be mass less. Later on, the mathematical instruments had not yet been developed far enough to take also the deflection dynamics of strings or beams in consideration, or, more crucial, the mathematical models for the dynamics had not yet been developed. In fact, lot of these should change in the two hundred years following the publication of Newton's *Principa* (1687). As Truesdell [Tr68] points out, the *Principa* did not constitute the completion of the formalisation of mechanics in mathematical language, but were rather the beginning of a more formal, mathematical description of dynamics, [Tr68] p. 93. In Chapter II of [Tr68] the author draws the history from the work of Newton to the *Méchanique Analitique* (1788) by Lagrange, for the more specific topic of the history of vibration theory following the publication of the *Principa* we also refer to the monograph [CaDo81]. This history includes break-through achievements, especially new ideas and techniques, as well as failures, inaccuracies and severe mistakes. Within this fruitful research period several mathematical physicians contributed to the development of models for beam mechanics, hydrodynamics and rigid body dynamics. Within this thesis we consider linear versions of some of these models for a beam of a string. In fact, it was d'Alembert who in 1746 first derived the wave equation as a partial differential equation model of a string ([Tr68], p. 114), although already both Newton and Taylor had been close to it about thirty years before, see e.g. Chapter 2, p. 10 in [CaDo81]. However, this model does not take any bending forces into account as Daniel Bernoulli and Leonhard Euler did, where the latter introduced the so-called Young's modulus (1727) ([Tr68], p. 124), leading later on to another beam model, the so-called Euler-Bernoulli beam model, again a partial differential equation of second order in time, but not in second (as the wave equation), but in forth order in space. Only some time later more sophisticated beam models had been introduced. E.g. Stephen Timoshenko proposed the Timoshenko beam model. It gives a more precise description of vibrating beams as the models considered before. As this is a Ph.D. thesis on mathematics we will not go into details how these models can be derived from considerations in physics and mechanics. In fact, we will always start with a particular model for the dynamics of a vibrating string or beam and only work based on this particular model by means of mathematical techniques to show well-posedness and stability properties of such systems. By a model we will always mean a description of the dynamics of a system via (a system of) partial differential equations (PDE) plus boundary (feedback) conditions, hence we start right away with the wave equation, the Euler-Bernoulli beam equation or the Timoshenko beam equation etc. ignoring any physical-mechanical justification for such description of a string or beam. However, let us mention that today by the techniques developed by Lagrange (Lagrange formalism based on his *Principle of Least Action* (1761), [Tr68] p. 132) and Hamilton (Hamilton formalism), already in the derivation of such a PDE one usually starts by modelling the total energy $H = T + S$ of a system, consisting of a kinetic part T and a potential part S, depending on state variables $x(t)$ and their time derivatives $\dot{x}(t)$. Then the Lagrange equations or the Hamilton equation, respectively, lead to the PDE model of the system, e.g. for the one-dimensional wave equation

$$\begin{aligned} T &= \int_0^1 \rho(\zeta)\left|\omega_t(t,\zeta)\right|^2 d\zeta \\ S &= \int_0^1 T(\zeta)\left|\omega_\zeta(t,\zeta)\right|^2 d\zeta \end{aligned}$$

and therefore (following the Lagrange formalism) for the Lagrange functional $L = T - S$

$$\frac{\partial}{\partial t}\frac{\partial L}{\partial \omega_t} = \frac{\partial L}{\partial \omega}$$

i.e.

$$\rho(\zeta)\omega_{tt}(t,\zeta) = \left(T(\zeta)\omega_\zeta\right)_\zeta(t,\zeta), \quad t \geq 0,\ \zeta \in (0,1)$$

or following the Hamilton formalism with $(r,p) = (\omega, \rho\omega_t)$

$$\begin{aligned} \frac{\partial}{\partial t}\rho(\zeta)\omega_t(t,\zeta) &= -\frac{\partial H}{\partial \omega}(t,\zeta) \\ \frac{\partial}{\partial t}\omega(t,\zeta) &= \frac{\partial H}{\partial(\rho\omega_t)}(t,\zeta) \end{aligned}$$

leading to

$$\begin{aligned} \rho(\zeta)\omega_{tt}(t,\zeta) &= \left(T(\zeta)\omega_\zeta\right)_\zeta(t,\zeta) \\ \omega_t(t,\zeta) &= \omega_t(t,\zeta), \quad t \geq 0,\ \zeta \in (0,1) \end{aligned}$$

where the second equation is superfluous, obviously. Note that similarly also the Euler-Bernoulli beam equation

$$\rho(\zeta)\omega_{tt}(t,\zeta) + (EI(\zeta)\omega_{\zeta\zeta})_{\zeta\zeta}(t,\zeta) = 0$$

and the linear Timoshenko beam equations

$$\begin{aligned} \rho(\zeta)\omega_{tt}(t,\zeta) &= \left(K(\zeta)(\omega_\zeta - \phi)\right)_\zeta(t,\zeta), \\ I_\rho\phi_{tt}(t,\zeta) &= \left(EI(\zeta)\phi_\zeta\right)_\zeta(t,\zeta) + K(\zeta)(\omega_\zeta - \phi)(t,\zeta), \quad t \geq 0,\ \zeta \in (0,1) \end{aligned}$$

can be derived. All these three equations have in common that the energy change of the (classical) solutions depends only on the energy exchange with the environment at the boundary ($\zeta = 0, 1$), so it is quite easy to characterise dissipative boundary conditions, i.e. boundary conditions for which the energy does not increase for every classical solution. For such *dissipative* equations the solution theory is relatively simple in the sense that the Lumer-Phillips Theorem provides easy necessary and sufficient conditions for the existence of unique solutions given an appropriate initial condition. In the language of semigroup theory which has been developed in the middle of the twenties century this means that there are simple conditions for the operator A in the formulation as abstract Cauchy problem (ACP)

$$\frac{d}{dt}x(t) = Ax(t), \quad t \geq 0$$
$$x(0) = x_0$$

on a suitable Hilbert space X to generate a strongly continuous contraction semigroup. Regarding the above mentioned string and beam equations, starting from the late 1980's the PDE mentioned above were investigated with techniques developed in semigroup theory, also to prepare results in the new arising field of control theory and in fact, still nowadays the investigations of these beam equations is highly popular and relevant, e.g. in modelling nano-tweezers which on nano scale manipulate DNA molecules and for which oscillations were highly undesirable since they would make any precise manipulation of the molecules more or less impossible. Therefore, several research articles (e.g. [RaTa74], [Ch+87], [Ch+87a], [LiMa88], [LiHuCh89], [CoLaMa90], [CoZu95], [FeShZh98], [GuHu04], [GuWaYu05], [Zh07] etc.) of the last decades considered the well-posedness and stability properties of *particular* string or beam equations with *particular* boundary (feedback) conditions. On the other hand, quite recently using the notion of *Dirac structures* efforts have been made to understand so-called *port-Hamiltonian systems* (e.g. [Go02], [VaMa02], [Va06]) better from a more abstract and structural level, starting mainly with ODE systems in port-Hamiltonian form, but also including more and more approaches to infinite-dimensional port-Hamiltonian systems in PDE form (e.g. [LeZwMa05], [Vi+09], [Zw+10], [JaZw12]). In this thesis we take – similar to the research articles [LeZwMa05] and [Vi+09] and the monograph [JaZw12] – a view which more or less lies in between the case-to-case PDE level investigation and the Dirac structural level, therefore combining both approaches and consider *port-Hamiltonian systems* in the abstract differential form

$$\frac{\partial}{\partial t}x(t,\zeta) = \sum_{k=0}^{N} P_k \frac{\partial^k}{\partial \zeta^k}(\mathcal{H}x)(t,\zeta), \quad t \geq 0,\ \zeta \in (0,1) \tag{1.1}$$

plus suitable boundary conditions. The latter we will interpret as feedback laws for boundary control and observation, which will enable us to classify the PDE examples above by the type of feedback law which constitutes the boundary conditions. The idea behind this approach is to understand and unify the case-to-case results on specific equations on the more abstract level, so that the structure behind stability properties becomes more clear. For example, one might then ask whether for a given, say linear and static boundary conditions, the same well-posedness results hold, if this linear static boundary feedback is replaced by a suitable nonlinear and/or dynamic boundary feedback. As we will see, for the well-posedness in contraction semigroup sense, there is an easy condition on generation of contraction semigroups for this kind of systems: Dissipativity of the corresponding operator A is enough.

See Chapter 3 for the case of static and linear boundary feedback, which is mainly a revision of previous results in [LeZwMa05] and its (slight) generalisation in [JaZw12] and [AuJa14]. In fact, with the well-posedness results in this dissipative case, where the both necessary and sufficient conditions are quite easy to check, at hand, we may then come to the core topic of this thesis: Stability of these systems. Actually we employ several techniques to establish either asymptotic (strong) stability of the trajectories (for every given initial datum) or even uniform exponential stability of the system in different scenarios. We classify these systems and structure the thesis based on the classification of the different types of boundary feedback laws mentioned above. In fact, this distinction is made upon the following the questions.

1. Are the boundary (feedback) conditions linear (Chapters 4 and 5) or nonlinear (Chapters 6 and 7)?
2. Is the boundary feedback determining the boundary conditions static (Chapters 4 and 6) or dynamic (Chapters 5 and 7)?
3. How large is the natural number N, the *order* of the port-Hamiltonian system, in equation (1.1)?

As we will see not every method to approach stability of these systems is applicable in every case, but the applicable methods depend heavily on the case distinction above. The following gives a rough overview on the thesis and the contents of the different chapters. Before actually starting with the investigation of the PDE (1.1), we first recall some concepts from operator theory which we need throughout the thesis. The corresponding Chapter 2 consists of three sections on functional analysis and partial differential equations (Section 2.1), evolution equations and semigroup theory (Section 2.2) and systems theory and boundary control and observation systems (Section 2.3). These sections are mainly based on the monographs [We06], [EnNa00] and [TuWe09], respectively, with some further information extracted in particular from [Ad75] and [Sh97]. After that in Chapter 3 we actually start with the investigation of the PDE (1.1). First we introduce the structural assumptions on the PDE (1.1) and motivate them in Section 3.1 with the beam equations of the articles mentioned above as examples and later on possible applications of the abstract theory. Then in Section 3.2 we prepare the generation theorem and also find that the transfer function exists on the right-half complex plane for so called *impedance passive port-Hamiltonian systems*. The latter is not needed in the linear feedback case, but will prove quite useful in the nonlinear setting.
Thereafter in Section 3.3 we give the generation theorem for the case of static and linear boundary feedback and slightly generalise the result of [LeZwMa05] (and [JaZw12], [AuJa14]) to the case of P_0 with spatial dependence $P_0 \in L_\infty(0, 1; \mathbb{F}^{d\times d})$. Section 3.4 sketches how one may get from the boundary control and observation setting considered here to the more standard form in systems theory.
Chapter 4 is dedicated to the stability properties of the semigroups generated by the dissipative operators with static linear feedback in the previous chapter. We begin in 4.1 by recalling the results of [Vi+09] for systems of the form (1.1) for $N = 1$ with its original proof, based on the ideas of [RaTa74], a *sideways-energy estimate* which may also be seen as a *final observability* result. The presentation also takes into account its possible generalisation to systems with nonlinear or dynamic feedback. Then we will see that an immediate generalisation to the case where $N \geq 2$ seems to be not possible, so that other techniques have to be used for them. We shall see in

Section 4.2 that the most intuitive generalisation of the exponential stability result in [Vi+09] does not work, in the sense that one will not obtain uniform exponential stability. Therefore, employing a (simple) version of the Arend-Batty-Lyubich-Vũ Theorem, we first investigate sufficient conditions for asymptotic (strong) stability. Only after that we return to the problem of uniform exponential stability in Section 4.3. First we present two alternative ways to prove the stabilisation result of 4.1 for the case $N = 1$. One of them may be seen as a *Lyapunov method*, whereas the other employs the *Gearhart-Greiner-Prüss-Huang* Theorem on exponential stability of C_0-semigroups on Hilbert spaces. Then we apply the same techniques to systems of order $N \geq 2$ and see that the proof via the Gearhart-Greiner-Prüss Theorem may be generalised to systems with order $N = 2$, or under less restrictive boundary conditions to the Euler-Bernoulli beam equations, and even to port-Hamiltonian systems of arbitrary order $N \geq 1$. We conclude the Chapter with some comments on the $\mathcal{H}$-dependence of uniform exponential stability in Section 4.4 and some applications to the examples of Section 3.1 in Section 4.5.
Whereas in Chapter 3 and Chapter 4 we considered the case of static boundary feedback, in Chapter 5 we investigate systems with dynamic feedback, i.e. port-Hamiltonian systems which are interconnected with a (finite-dimensional) linear control system. We see in Section 5.1 that the generation theorem for contraction semigroups (now on the product Hilbert space) is actually a intuitive generalisation of the static feedback generation theorem of Section 3.3. Thereafter we turn our attention to asymptotic stability properties again. In Section 5.2 we start with the naive approach of considering interconnection systems with a internally stable controller and such that the interconnected system is dissipative in exactly those terms which are enough for asymptotic or uniform exponentially stability in the static feedback scenario. As one could have hoped the results from the static case naturally generalise to this hybrid system setup. However, in the dynamic case the assumptions of Section 5.2 are far too restrictive when it comes to applications. Therefore, in Section 5.3 and Section 5.4 we only consider control systems which are *strictly impedance passive* or *strictly output passive*, respectively. These kind of systems had already been considered in [RaZwLe13] and we mainly follow the lines of [AuJa14], with only small deviation from the path in the latter article. Section 5.5 may be seen as an attempt to combine the strictly input passive (SIP) and the strictly output passive (SOP) controller scenario, so it is a generalisation of the two preceding sections. We comment on the relation between the static feedback and the dynamic feedback case in Section 5.6, at least for the scenario of control systems with collocated input and output. We illustrate the abstract results of the chapter within Section 5.7 using again the examples from Section 3.1.
The results of Chapter 4 and Chapter 5 are for most parts the same as those in the research article [AuJa14] plus some additional comments and slight generalisations. On the other hand the topics of Chapter 6 and Chapter 7 have not been covered by the article [AuJa14], since in contrast to Chapter 4 and Chapter 5 the control feedback is not necessarily linear any more. Of course, this brings additional problems and in fact, the generation theorem has to be generalised to this nonlinear scenario. On the other hand, there is no nonlinear version of the Gearhart-Greiner-Prüss-Huang Theorem known, so that only the Lyapunov technique proof may be generalised to the nonlinear setting.
We start in Chapter 6 with the case of nonlinear and static boundary feedback, starting from an *impedance passive* port-Hamiltonian system. That said, we will

employ the results of Section 2.2 on nonlinear m-dissipative operators and prove the nonlinear generation theorem in Section 6.1. Similar to the static scenario we begin with the port-Hamiltonian systems (1.1) of order $N = 1$ and find that in that case two methods – the Lyapunov technique and the final observability estimate – are applicable for a uniform exponential stability result, see Section 6.2. Then in Section 6.3 and Section 6.4 we apply this Lyapunov method to systems of order $N = 2$ and with the special Euler-Bernoulli beam structure. For a constant Hamiltonian density matrix $\mathcal{H}$ we can more or less generalise the results from the linear case to the nonlinear situation, however, for $\mathcal{H}$ with spatial dependence we need smallness conditions on the (weak) derivative $\mathcal{H}'$ (compared to $\mathcal{H}$). Again we devote a section, here Section 6.5 to some examples for the general theory, based on the beam models of Section 3.1.

In Chapter 7 we combine the two generalisation approaches of the previous chapters and consider the feedback interconnection of a port-Hamiltonian PDE (1.1) with a dynamic controller which now may be nonlinear. We first give the generation theorem in this scenario, see Section 7.1, and afterwards explore some asymptotic and uniform exponential stability results in Section 7.3. Similar to the linear scenario, the results for static nonlinear feedback may be used also to cover the dynamic nonlinear case. We also include some examples to illustrate the abstract theory. Afterwards, in Section 7.2 we follow a different approach then in the previous sections and instead of looking for m-dissipative maps (in the nonlinear sense) look for systems which also have decaying energy, but which are not necessarily related to a strongly continuous contraction semigroup because there may be solutions which for themselves decay in energy, but for which the distance between the two may be non-decreasing, e.g. if one of them decays much faster to zero. On the other hand we will need more restrictions on the port-Hamiltonian system and we therefore shall assume that it is well-posed (in the systems theoretic sense) to obtain a well-posed (in the PDE sense) interconnection system.

In the concluding Chapter 8 we collect some further results, which may be closely related to the topics treated before, but not quite the same. In particular, we have a look on the interconnection of several port-Hamiltonian systems, see Section 8.1, remark that the uniform exponential stability proof of [Vi+09] also works for time-variant $\mathcal{H}(t)$, provided existence of a solution, see Section 8.2. In Section 8.3 we collect some results taken from the article [AuJaLa15] for port-Hamiltonian system which have an additional structural damping, e.g. the wave equation, leading to a holomorphic C_0-semigroup, or more general in the non-autonomous case to maximal L_p-regularity.

Chapter 2

Some Background on Functional Analysis, Evolution Equations and Systems Theory

Preparing for the main part of this thesis, we first recall some well-known notions and results on the following topics. Meanwhile we also fix some notation we use throughout the thesis. The prerequisites include the following. We start with Section 2.1 on Functional Analysis and Partial Differential Equations (PDE), as we later on investigate a special class of PDE with operator theoretical methods which heavily rely on both the theory of Functional Analysis and of Partial Differential Equations. Then we recall some results on some more specific topics, namely the theory of evolutionary equations and systems theory, each in a separate subsection. Within the context of evolution equations we introduce the quite natural concept of a strongly continuous semigroup and its generator and repeat some of the most important results which characterise generators of such semigroups (Section 2.2). Then in Section 2.3 we recall some notions from systems theory and in particular introduce the concept of a Boundary Control and Observation System (BCOS), since the class of PDE under consideration in this thesis will be interpreted as BCOS and boundary conditions as feedback laws for the corresponding input and output maps.

2.1 Background on Functional Analysis and Partial Differential Equations

We will need some background knowledge on Functional Analysis as well as on Partial Differential Equations (PDE) later on. In this section we repeat some definitions and basic results and also use this for fixing some notation we use in this thesis.

First of all, all Banach spaces or Hilbert spaces in this thesis are taken over the field

$\mathbb{F} = \mathbb{R}$, the real numbers, or $\mathbb{F} = \mathbb{C}$, the complex numbers. With $\mathbb{N} = \{1, 2, \ldots\}$ we denote the natural numbers, starting from 1, i.e. $0 \notin \mathbb{N}$. On the other hand, if we want to include 0 we write $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ instead. The most common Banach spaces are the finite dimensional spaces $X = \mathbb{F}^d$ and the L_p-spaces $L_p(\Omega)$, see below. Unless stated otherwise we always equip $\mathbb{F}^d := \{x = (x_1, \ldots, x_d) : x_j \in \mathbb{F}\ (j = 1, \ldots, d)\}$ with the Euclidean norm

$$|z| := \sqrt{\sum_{j=1}^{d} |z_j|^2}, \quad z \in \mathbb{F}^d$$

and write the standard inner product as

$$\langle z_1, z_2 \rangle_{\mathbb{F}^d} = z_1^* z_2 = \sum_{j=1}^{d} \overline{z_{1,j}} z_{2,j}, \quad z_1, z_2 \in \mathbb{F}^d \tag{2.1}$$

where z_1^* denotes the transposed complex conjugate of $z_1 \in \mathbb{F}^d$ and throughout, i.e. also for other Hilbert spaces, we take inner products in such a way that they are linear in the *second* component and anti-linear (linear if $\mathbb{F} = \mathbb{R}$) in the first component.

If X and Y are (real or complex) Banach spaces $\mathcal{B}(X, Y)$ denotes the Banach space of all the linear and bounded operators $B : X \to Y$. For $X = Y$ we simply write $\mathcal{B}(X) := \mathcal{B}(X, X)$. A special role play the dual space $X' := \mathcal{B}(X, \mathbb{F})$ of a Banach space and the bidual $X'' = (X')'$.

More generally, usually $A : D(A) \subset X \to Y$ denotes a linear (not necessarily bounded) operator from a linear subspace $D(A)$ of X (the *domain* of A) to Y and for the case $X = Y$ we define the *resolvent set* $\rho(A)$ as

$$\rho(A) := \{\lambda \in \mathbb{C} : \lambda I - A : D(A) \to X \text{ is bijective}\}$$

and as $\sigma(A) := \mathbb{C} \setminus \rho(A)$ the *spectrum* of A. (In the real case these and the following notions are defined via complexification of the spaces and operators and then the definitions for the complexified operators.) For any $\lambda \in \rho(A)$ the operator

$$R(\lambda, A) := (\lambda I - A)^{-1} : X \to D(A)$$

is called the *resolvent operator*. Further we denote by

$$\begin{aligned}
\sigma_p(A) &:= \{\lambda \in \mathbb{C} : \exists x \in D(A), x \neq 0, Ax = \lambda x\} \\
\sigma_r(A) &:= \{\lambda \in \mathbb{C} : (\lambda I - A)D(A) \text{ not dense}\} \\
\sigma_{ap}(A) &:= \{\lambda \in \mathbb{C} : \exists (x_n)_{n \geq 1} \subset D(A),\ \|x_n\| = 1,\ Ax_n - \lambda x_n \to 0\}
\end{aligned}$$

the *point spectrum*, the *residual spectrum* and the *approximate point spectrum*, respectively. If X is a Hilbert space and $A : D(A) \subseteq X \to X$ a densely defined linear operator we define its *Hilbert space adjoint* as

$$\begin{aligned}
D(A') &= \{x \in X :\ \exists y_x \in X, \forall z \in D(A) :\ \langle Az, x \rangle_X = \langle z, y_x \rangle_X\} \\
A'x &:= y_x
\end{aligned}$$

see Definition V.5.1 in [We11].

The *Lebesgue space* $L_p(\Omega) := L_p(\Omega;\mathbb{F})$ ($p \in [1,\infty)$) (where $\Omega \subseteq \mathbb{R}^n$ is an open subset of $\mathbb{R}^n$) consists of all L_p-integrable functions, i.e. all measurable functions $f:\Omega \to \mathbb{F}$ for which the integral

$$\int_\Omega |f(\zeta)|^p \, d\zeta < \infty$$

with respect to the Lebesgue measure is finite. More precisely, $L_p(\Omega)$ is the family of all equivalence classes of L_p-integrable functions where two functions are said to be equivalent if they coincide on $\Omega \setminus N$ where $N \subseteq \Omega$ is a set of Lebesgue measure zero. In the following we will not distinguish between a function f and its equivalence class, following a convenient, sloppy, but very common convention.

Similarly for the case $p = \infty$ we denote by $L_\infty(\Omega) = L_\infty(\Omega;\mathbb{F})$ the (equivalence classes of) essentially bounded functions, i.e. (the equivalence classes of) those measurable $f:\Omega \to \mathbb{F}$ such that

$$\|f\|_{L_\infty} := \operatorname*{ess\,sup}_{\zeta\in\Omega} |f(\zeta)| < \infty.$$

For the case that $\Omega = (a,b) \subseteq \mathbb{R}$ is an open interval we write $L_p(a,b) := L_p(\Omega)$. By $C(\Omega)$ and $C(\overline{\Omega})$ we denote the space of continuous scalar functions on Ω and $\overline{\Omega}$, respectively, where the latter is a Banach space when equipped with the supremum-norm $\|\cdot\|_C = \|\cdot\|_{C(\overline{\Omega})} = \|\cdot\|_{L_\infty}$ and accordingly by $C^k(\Omega)$ and $C^k(\overline{\Omega})$ ($k \in \mathbb{N}_0 \cup \{\infty\}$) the functions f for which all the derivatives

$$D^\alpha f := \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1}\cdots \partial x_d^{\alpha_n}} f, \quad |\alpha| := \sum_{i=1}^n \alpha_i \le k$$

exist and lie in $C(\Omega)$ and $C(\overline{\Omega})$, respectively. For every $k \in \mathbb{N}_0$ the space $C^k(\overline{\Omega})$ is a Banach space for the norm

$$\|f\|_{C^k} := \max\{\|D^\alpha f\|_C : |\alpha| \le k\}, \quad f \in C^k(\overline{\Omega}).$$

Then $C_c(\Omega)$ is the space of continuous functions with compact support

$$\operatorname{supp} f := \overline{\{z \in \Omega : f(\zeta) \neq 0\}} \tag{2.2}$$

and $C_c^k(\Omega) := C_c(\Omega) \cap C^k(\Omega)$. Moreover, we need the *Sobolev spaces* $W_p^k(\Omega)$ ($k \in \mathbb{N}$, $p \in [1,\infty]$) which are defined as follows.

Definition 2.1.1. *Let an open set $\Omega \subseteq \mathbb{R}^d$ and a number $p \in [1,\infty]$ be given. We then define*

$$W_p^0(\Omega) := L_p(\Omega)$$

and iteratively for $k \in \mathbb{N}$

$$\begin{aligned} W_p^k(\Omega) := \{f \in W_p^{k-1}(\Omega) : \forall \alpha \in \mathbb{N}^d,\ |\alpha| \le k : \exists g_\alpha =: \partial^\alpha f, \forall \phi \in C_c^\infty(\Omega) : \\ \int_\Omega f(\zeta)(D^\alpha \phi)(\zeta) d\zeta = (-1)^{|\alpha|} \int_\Omega g_\alpha(\zeta)\phi(\zeta) d\zeta\} \end{aligned} \tag{2.3}$$

equipped with the norm

$$\|f\|_{W_p^k} := \begin{cases} \left(\sum_{|\alpha|\le k} \|\partial^\alpha f\|_{L_p}^p\right)^{1/p}, & p \in [1,\infty), \\ \operatorname{ess\,sup}_{|\alpha|\le k} \|\partial^\alpha f\|_{L_\infty}, & p = \infty. \end{cases}$$

All the Sobolev spaces $W_p^k(\Omega)$ are Banach spaces, see Theorem 3.2 in [Ad75], in particular for $k \in \mathbb{N}_0$ the spaces $H^k(\Omega) := W_2^k(\Omega)$ are Hilbert spaces for the inner product

$$\langle f, g\rangle_{H^k} := \sum_{|\alpha|\leq k} \int_\Omega \langle \partial^\alpha f(\zeta), \partial^\alpha g(\zeta)\rangle_{\mathbb{F}} d\zeta, \quad f, g \in H^k(\Omega).$$

We list a series of well-known results on Lebesgue spaces $L_p(\Omega)$ and Sobolev spaces $W_k^p(\Omega)$. We start with Hölder's Inequality on products of two functions lying in appropriate Lebesgue spaces.

Theorem 2.1.2 (Hölder's Inequality). *Let $p, p' \in [1, \infty]$ be such that $\frac{1}{p} + \frac{1}{p'} = 1$ (where we use the convention that $\frac{1}{\infty} := 0$). Then for all $f \in L_p(\Omega)$ and $g \in L_{p'}(\Omega)$ we have that $fg \in L_1(\Omega)$ and*

$$\|fg\|_{L_1} \leq \|f\|_{L_p} \|g\|_{L_{p'}}.$$

Proof. See Theorem 2.3 in [Ad75] for the case where $p, p' \in (1, \infty)$ and note that the inequality also holds if $p = \infty$ or $p' = \infty$, see 2.2 in [Ad75]. □

With Hölder's Inequality the following embedding theorem follows quite easily.

Corollary 2.1.3. *Assume that Ω has finite measure, i.e. $\mathbf{1} \in L_1(\Omega)$. Then for $1 \leq p \leq q \leq +\infty$ the embeddings*

$$L_\infty(\Omega) \hookrightarrow L_q(\Omega) \hookrightarrow L_p(\Omega) \hookrightarrow L_1(\Omega)$$

are continuous and for $f \in L_\infty(\Omega)$ one has

$$\|f\|_{L_\infty} = \lim_{p\to\infty} \|f\|_{L_p}.$$

Proof. See Theorem 2.8 in [Ad75] where also the embedding constants have been computed explicitly. □

Another important result is the following approximation result which says that any $L_p(\Omega)$-function ($p \in [1, \infty)$) can be approximated by a smooth function with compact support.

Theorem 2.1.4. *The set of smooth functions with compact support*

$$C_c^\infty(\Omega) = \{f \in C^\infty(\Omega) : \text{ supp } f \subset \Omega \textit{ is compact}\}$$

is dense in $L_p(\Omega)$ for every $p \in [1, \infty)$.

Proof. See Theorem 2.19 in [Ad75]. □

Lemma 2.1.5 (Fundamental Lemma of Calculus of Variations). *Let $\Omega \subset \mathbb{R}^n$ be an open set and let*

$$f \in L_{1,loc}(\Omega) := \{g : \Omega \to \mathbb{F} : \; g|_U \in L_1(U) \textit{ for every bounded open } U \subseteq \Omega\}$$

be such that for all $\phi \in C_c^\infty(\Omega)$

$$\int_\Omega \overline{f(s)}\phi(s)ds = 0.$$

Then $f = 0$ a.e. on Ω.

Proof. See Lemma 3.26 in [Ad75]. □

Lemma 2.1.6 (Fundamental Lemma of Calculus of Variations for Positive Test Functions)**.** *Let $\Omega \subset \mathbb{R}^n$ be an open set and let $f \in L_{1,loc}(\Omega;\mathbb{R})$ be such that for all $\phi \in C_c^\infty(\Omega)$ with $\phi \geq 0$ the integral*

$$\int_\Omega \overline{f(s)}\phi(s)ds = 0$$

vanishes. Then $f = 0$ a.e. on Ω.

Proof. By the fundamental lemma of calculus of variations we only have to show that

$$\int_\Omega \overline{f(s)}\phi(s)ds = 0, \quad \text{for all } \phi \in C_c^\infty(\Omega).$$

This can be done as follows. Take any $\phi \in C_c^\infty(\Omega)$ which therefore is bounded, so that we find $\psi \in C_c^\infty(\Omega)$ such that $\psi + \phi \geq 0$ and $\psi - \phi \geq 0$ both are positive functions in $C_c^\infty(\Omega)$. By assumptions the integrals

$$\int_\Omega \overline{f(s)}(\psi + \phi)(s)ds = 0$$
$$\int_\Omega \overline{f(s)}(\psi - \phi)(s)ds = 0$$

vanish and subtracting these equations also

$$\int_\Omega \overline{f(s)}\phi(s)ds. = 0$$

Since $\phi \in C_c^\infty(\Omega)$ has been arbitrary this holds for every $\phi \in C_c^\infty(\Omega)$ and the assertion follows from the Fundamental Lemma of Calculus of Variations 2.1.5. □

Very often we also consider $\mathbb{F}^d$-valued spaces, e.g. $L_p(\Omega;\mathbb{F}^d)$, the $\mathbb{F}^d$-valued L_p-spaces, which may be expressed as

$$L_p(\Omega;\mathbb{F}^d) := \{f = (f_1,\ldots,f_d) : \Omega \to \mathbb{F}^d, f_j \in L_p(\Omega;\mathbb{F})\}$$

and for which the standard norm is denoted by $\|\cdot\|_{L_p}$ as well, defined as

$$\|f\|_{L_p} := \left(\int_\Omega |f(\zeta)|^p \, d\zeta\right)^{1/p}, \quad f \in L_p(\Omega;\mathbb{F}^d).$$

For the Hilbert space case $p = 2$ this norm is inherited from the standard inner product $\langle\cdot,\cdot\rangle_{L_2}$ given by

$$\langle f, g\rangle_{L_2} = \int_\Omega \langle f(\zeta), g(\zeta)\rangle_{\mathbb{F}^d} d\zeta, \quad f, g \in L_2(\Omega;\mathbb{F}^d).$$

More generally, all the spaces considered so far can be easily generalised to the $\mathbb{F}^d$-valued case, i.e. $W_k^p(\Omega;\mathbb{F}^d), C^k(\Omega;\mathbb{F}^d), C_c^k(\Omega;\mathbb{F}^d)$ etc. Also most of the preceding results easily generalise to an $\mathbb{F}^d$-valued version. However, more involved is the case of more general Banach spaces E as range space, e.g. $L_p(\Omega;E)$ or $W_k^p(\Omega;E)$, because in this case the Lebesgue integral has to be replaced by a Bochner integral. For details on these Bochner L_p-spaces we refer to Appendix C in [EnNa00].

Lemma 2.1.7. *Let $\Omega \subseteq \mathbb{R}^m$ be open and $M : \Omega \to \mathbb{F}^{d\times d}$ be a measurable and essentially bounded function such that $M(\zeta) = M(\zeta)^*$ is symmetric for a.e. $\zeta \in (0,1)$. Then*

1. *$M(\omega)$ is positive semi-definite for a.e. $\omega \in \Omega$ if and only if*
$$\text{for all } f \in L_2(\Omega;\mathbb{F}^d): \quad \langle f, Mf\rangle_{L_2} \geq 0.$$
2. *$M(\omega) = 0$ equals the zero matrix for a.e. $\omega \in \Omega$ if and only if*
$$\text{for all } f \in L_2(\Omega;\mathbb{F}^d): \quad \langle f, Mf\rangle_{L_2} = 0.$$

Proof. First assume that there is a set of positive measure where $M(\omega)$ is not positive semidefinite. Hence, $\lambda(\Omega) > 0$ for the by continuity of the inner product and measurability of M measurable set

$$\Omega = \left\{\zeta \in (0,1): \quad \exists z \in \mathbb{F}^{Nd}: \ \langle M(\zeta)z, z\rangle_{\mathbb{F}^{Nd}} < 0\right\}.$$

Note that
$$\Omega = \bigcup_{\varepsilon>0} \Omega_\varepsilon = \lim_{\varepsilon\searrow 0} \Omega_\varepsilon$$
for the measurable sets

$$\begin{aligned}\Omega_\varepsilon &= \left\{\zeta \in (0,1): \quad \exists z \in \mathbb{F}^{Nd}: \ \langle M(\zeta)z, z\rangle_{\mathbb{F}^{Nd}} < -\varepsilon\,|z|^2\right\}\\ &= \left\{\zeta \in (0,1): \quad \exists z \in \mathbb{F}_{\mathbb{Q}}^{Nd}: \ \langle M(\zeta)z, z\rangle_{\mathbb{F}^{Nd}} < -\varepsilon\,|z|^2\right\}\end{aligned}$$

where we write
$$\mathbb{R}_{\mathbb{Q}} = \mathbb{Q} \quad \text{and} \quad \mathbb{C}_{\mathbb{Q}} = \mathbb{Q} + i\mathbb{Q}$$
and for the second line used that $\mathbb{Q}$ is dense in $\mathbb{R}$. Since the sets Ω_ε are ordered, there is $\varepsilon > 0$ such that $\lambda(\Omega_\varepsilon) > 0$. For $z \in \mathbb{F}^{Nd}$ we now write

$$\Omega_\varepsilon^z := \left\{\zeta \in (0,1): \quad \langle M(\zeta)z, z\rangle < -\varepsilon\,|z|^2\right\}$$

(measurable) and observe that

$$\Omega_\varepsilon = \bigcup_{z\in\mathbb{F}_{\mathbb{Q}}^{Nd}} \Omega_\varepsilon^z.$$

Since $\mathbb{F}_{\mathbb{Q}}^{Nd}$ is countable, $\lambda(\Omega_\varepsilon^z) > 0$ for some $z \in \mathbb{F}_{\mathbb{Q}}^{Nd}$ and so for $f := \mathbf{1}_{\Omega_\varepsilon^z} z \in L_2(0,1;\mathbb{F}^d)$ we find that

$$\langle Mf, f\rangle_{L_2} = \int_{Z_\varepsilon^z} \langle M(\zeta)z, z\rangle \leq -\lambda(\Omega_\varepsilon^z)\varepsilon < 0.$$

Hence, for $\langle f, Mf\rangle \geq 0$ ($\forall f \in L_2(0,1;\mathbb{F}^d)$) it is necessary (and clearly also sufficient) that $M(\omega)$ is positive semidefinite for a.e. $\omega \in \Omega$. The second part easily follows from the first one since a matrix equals the zero matrix if and only if it is both positive and negative semidefinite. □

Theorem 2.1.8 (Rellich-Kondrachov). *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain, $j \in \mathbb{N}_0$ and $m \in \mathbb{N}$ integers and let $p \in [1, \infty]$. If Ω satisfies the strong local Lipschitz property, see 4.5 in [Ad75], then the following embeddings are compact.*

$$W_p^{j+m}(\Omega) \hookrightarrow C^j(\overline{\Omega}),$$
$$W_p^{j+m}(\Omega) \hookrightarrow W_p^j(\Omega).$$

Proof. See Theorem 6.2 in [Ad75] □

Remark 2.1.9. *In particular, the embeddings $H^k(0,1) \hookrightarrow L_2(0,1)$ are compact for every $k \in \mathbb{N}$.*

Another lemma we will use is the following one-dimensional version of the famous Gagliardo-Nirenberg inequality.

Lemma 2.1.10. *Let $(a,b) \subset \mathbb{R}$ any interval, $p \in [1,\infty)$, $N \in \mathbb{N}$ and $\varepsilon > 0$ be given. Then there is a constant $K = K(\varepsilon_0, N, p, b-a)$ such that for every $\varepsilon \in (0, \varepsilon_0]$, $j \in \{0, 1, \ldots, N-1\}$ and $f \in W_p^N(a,b)$ one has*

$$\left\| \frac{\partial^j}{\partial \zeta^j} f \right\|_{L_p(a,b)} \leq K\varepsilon \left\| \frac{\partial^N}{\partial \zeta^N} f \right\|_{L_p(a,b)} + K\varepsilon^{-j/(N-j)} \|f\|_{L_p(a,b)}.$$

Proof. This is a special version of the general Gagliardo-Nirenberg Theorem 4.14 in [Ad75] and in the one-dimensional special case follows from Lemma 4.10 and Lemma 4.12 in [Ad75]. □

Remark 2.1.11. *The Sobolev space $W_\infty^k(0,1)$ $(k \in \mathbb{N})$ can be characterised as*

$$W_\infty^k(0,1) = \{f \in C^{k-1}[0,1] : \ f^{(k-1)} \text{ is Lipschitz continuous } \}.$$

In particular

$$W_\infty^1(0,1) = \mathrm{Lip}(0,1) = \{f : [0,1] \to \mathbb{F} : \ f \text{ Lipschitz-continuous}\}.$$

Proof. See Proposition 8.4 in [Br11]. □

Proposition 2.1.12 (Strict Contraction Principle). *Let $F : X \to X$ be a map on a complete metric space (X, d) and assume that F is uniformly strictly contractive, i.e. there is $\rho \in (0,1)$ such that*

$$d(F(x), F(\tilde{x})) \leq \rho d(x, \tilde{x}), \quad x, \tilde{x} \in X.$$

Then F has a unique fixed point $x_0 = F(x_0)$.

Proof. See Theorem III.2.2 in [We06] for a slightly more general version. □

Proposition 2.1.13. *Let $P \in \mathcal{B}(X)$ be a coercive operator on a Hilbert space X and $n \in \mathbb{N}$ be a natural number. Then there is a coercive operator $Q =: P^{1/n} \in \mathcal{B}(X)$ such that*

$$P = Q^n, \quad \|P\| = \|Q\|^n.$$

In particular, for every given $\rho > 0$ there is a number $m \in \mathbb{N}$ such that

$$\left\| I - P^{1/m} \right\| < \rho.$$

Proof. For the existence of operator $P^{1/n} \in \mathcal{B}(X)$, see Korollar VII.1.16 in [We11]. Since $P^{1/n} \in \mathcal{B}(X)$ is coercive, we also have

$$\|P\| = \left\|(P^{1/n})^n\right\| = \left\|P^{1/n}\right\|^n$$

so that $\left\|P^{1/n}\right\| \xrightarrow{n\to\infty} 1$. Then we calculate that

$$\begin{aligned}\left\|I - P^{1/n}\right\|^2 &= \sup_{\|x\|=1} \left\|(I - P^{1/n})x\right\| \\ &= \sup_{\|x\|=1} \left(\|x\|^2 + \left\|P^{1/n}x\right\|^2 - 2\left\|P^{1/2n}x\right\|^2\right) \\ &\xrightarrow{n\to\infty} 0\end{aligned}$$

and hence $I - P^{1/n} \to 0$ in $\mathcal{B}(X)$ as $n \to \infty$. □

Besides the *usual* Sobolev spaces $W_p^k(\Omega)$ we also encounter the *Bessel potential spaces* $H_p^s(\Omega)$ from time to time, where

$$H_p^s(\Omega) = \left[W_p^{\lfloor s\rfloor}(\Omega), W_p^{\lfloor s\rfloor+1}(\Omega)\right]_{s-\lfloor s\rfloor}$$

is a complex interpolation space of Sobolev spaces, cf. [Am95], with $\lfloor s\rfloor = \sup\{n \in \mathbb{N}_0 : \ n \le s\}$ denoting the floor function.

Theorem 2.1.14 (Embedding Theorem for H_p^s). *Let E be a Banach space of class $\mathcal{HT}$ (that is, a UMD space, see Section 4.4 in [Am95]), $k \in \mathbb{N}$ and $\Omega \subseteq \mathbb{R}^d$ be an open subset with the k-th extension property. Further let $\alpha, r \le s \in [0,k]$, $p, q \in [1,\infty)$. Then the following embeddings are continuous*

$$\begin{aligned} &\textit{if} \quad s - \frac{d}{p} \ge r - \frac{d}{q} \quad &\textit{then} \quad H_p^s(\Omega; E) \hookrightarrow H_q^r(\Omega; E), \\ &\textit{if} \quad s - \frac{d}{p} > \alpha \quad &\textit{then} \quad H_p^s(\Omega; E) \hookrightarrow C^\alpha(\bar{\Omega}; E).\end{aligned}$$

Remark 2.1.15. *Recall that an open set $\Omega \subset \mathbb{R}^d$ is said to have the k-*th extension property *if there exists a bounded linear operator $\mathcal{E} \in \mathcal{B}(H^k(\Omega); H^k(\mathbb{R}^d))$ such that $\mathcal{E}f|_\Omega = f$ for any $f \in H^k(\Omega)$. Note that open sets Ω with C^k-boundary have the k-th extension property, see Theorem 4.26 in [**?**]. In particular, the theorem holds for $d = 1$ and $\Omega = (a,b)$ any open interval and $E = \mathbb{F}^n$ finite dimensional.*

Proof. For the case $\Omega = \mathbb{R}^d$ and the first embedding see, e.g., Corollary 1.4 in [MeVe12]. For general Ω the result follows by the extension property and the characterisation of Bessel-potential spaces as complex interpolation spaces. For the second embedding, see, e.g. Proposition 2.10 in [MeSc12]. Also the general case seems to be well-known. □

Proposition 2.1.16 (Poincaré-Friedrichs inequality). *Let $k \in \mathbb{N}_0$ and $\Omega \subset \mathbb{R}^d$ be any bounded open set with the k-th extension property and define*

$$H_0^k(\Omega) := \overline{C_c^\infty(\Omega)}^{\|\cdot\|_{H^k}}.$$

Then there exists a constant $c = c(\Omega) > 0$ such that

$$\|f\|_{H^k} \leq c \sum_{|\alpha|=k} \|D^\alpha f\|_{L_2}, \quad f \in H_0^k(\Omega).$$

Moreover, if $d = 1$ and $\Omega = (a, b)$ is an interval of finite length, then also for

$$H_0^k(a, b] := \{f \in H^k(a, b) : f(a) = \ldots = f^{(k-1)}(a) = 0\},$$

there exists $c = c(a, b) > 0$ with

$$\|f\|_{H^k} \leq c \left\| f^{(k)} \right\|_{L_2}, \quad f \in H_0^k(a, b].$$

Proof. We restrict ourselves to the case $k = 1$ and remark that for $k > 1$ one may proceed by induction. The first statement is the most well-known version of the Poincaré-Friedrichs inequality, thus we omit the proof here and only focus on the second statement. Since $D := C_c^\infty(a, b]$ is dense in $H_0^k(a, b]$ we only need to consider functions $f \in D$. For any such f we have

$$\begin{aligned} \|f\|_{L_2}^2 &= \int_a^b |f(\zeta)|^2 \, d\zeta = \int_a^b \left| \int_a^\zeta f'(\xi) d\xi \right|^2 d\zeta \\ &\leq \int_a^b \left(\int_a^\zeta |f'(\xi)| \, d\xi \right)^2 d\zeta \leq \int_a^b \int_a^\zeta 1^2 d\xi \int_a^\zeta |f'(\xi)|^2 \, d\xi d\zeta \\ &\leq \int_a^b (\zeta - a) d\zeta \, \|f'\|_{L_2}^2 = \frac{(b-a)^2}{2} \|f'\|_{L_2}^2. \end{aligned}$$

Now the result follows by approximation. □

As a consequence we find an equivalent norm on $H_0^k(a, b]$ for any bounded interval (a, b).

Corollary 2.1.17. *Let $k \in \mathbb{N}$ and (a, b) any open and bounded interval. Then*

$$\|f\|_*^2 := \sum_{l=0}^{k-1} \left| f^{(l)}(a) \right|^2 + \left\| f^{(k)} \right\|_{L_2(a,b)}^2, \quad f \in H^k(a, b)$$

defines a norm equivalent to the usual norm on $H^k(a, b)$.

Proof. By the continuous embedding $H^k(a, b) \hookrightarrow C^{k-1}[a, b]$ of Theorem 2.1.14 we have the estimate

$$\|f\|_*^2 \lesssim \|f\|_{C^{k-1}}^2 + \left\| f^{(k-1)} \right\|_{L_2}^2 \lesssim \|f\|_{H^k}^2.$$

Moreover, the function

$$g_f(\zeta) := f(\zeta) - \sum_{l=0}^{k-1} \frac{(\zeta - a)^l}{l!} f^{(l)}(a), \quad \zeta \in [a, b],$$

lies in $H_0^k(a,b]$ with $g_f^{(k)} = f^{(k)}$, so that by the Poincaré-Friedrichs estimate we deduce for $l = 0, 1, \dots, k-1$ that

$$\left\|f^{(l)}\right\|_{L_2}^2 \leq \left\|g_f^{(l)}\right\|_{L_2}^2 + \left\|\sum_{m=l}^{k-1} \frac{l!(\zeta - a)^{m-l}}{m!} f^{(m-l)}(a)\right\|_{L_2}^2$$
$$\lesssim \left\|g_f^{(k)}\right\|_{L_2}^2 + \sum_{m=l}^{k-1} \left|f^{(m-l)}(a)\right|^2 \leq \|f\|_*^2, \quad f \in H^k(a,b),$$

which implies the assertion. □

Remark 2.1.18. *Here we used the notation $f \lesssim g$ if there is a fixed constant $c > 0$ such that $f \leq cg$ and which does not depend on the particular functions f and g. Moreover, we write $f \simeq g$ if both $f \lesssim g$ and $g \lesssim f$.*

Lemma 2.1.19. *Let $0 \leq k < N \in \mathbb{N}_0$ and $\theta \in (0,1)$ such that $\eta := \theta N \in (k + \frac{1}{2}, k+1)$. Then there exist a constant $c_\theta > 0$ such that for all $f \in H^N(0,1)$*

$$\|f\|_{C^k} \leq c_\theta \|f\|_{L_2}^{1-\theta} \|f\|_{H^N}^\theta .$$

Further for $\sigma := \frac{k}{N}$ there exists a constant $c_\sigma > 0$ such that for all $f \in H^N(0,1)$

$$\|f\|_{H^k} \leq c_\sigma \|f\|_{L_2}^{1-\sigma} \|f\|_{H^N}^\sigma .$$

Proof. Let $p \in (1, \infty)$ such that $\eta - \frac{1}{2} > k + 1 - \frac{1}{p} > k$. Then by the Sobolev-Morrey Embedding Theorem 2.1.14

$$W_p^{k+1}(0,1) \hookrightarrow C^k[0,1]$$

is continuously embedded. Further, using the notation of [Tr83], we have by the theorems of Subsections 3.3.1 and 3.3.6 in [Tr83] that

$$\begin{aligned} W_p^{k+1}(0,1) &= F_{p,2}^{k+1}(0,1) \hookrightarrow F_{2,2}^\eta(0,1) \\ &= \left(F_{2,2}^0(0,1), F_{2,2}^N(0,1)\right)_{\theta,2} \\ &= \left(L_2(0,1), H^N(0,1)\right)_{\theta,2} \end{aligned}$$

and the first of the assertions follows from the interpolation inequality. The second assertion is a spacial case of the Gagliardo-Nirenberg inequality. In the language and with the theory of [Tr83] it results from

$$\begin{aligned} H^k(0,1) = F_{2,2}^k(0,1) &= \left(F_{2,2}^0(0,1), F_{2,2}^N(0,1)\right)_{\sigma,2} \\ &= \left(L_2(0,1), H^N(0,1)\right)_{\sigma,2} . \end{aligned}$$

□

2.2 Background on Evolution Equations

Within this section we recall some background on evolution equations and the theory of strongly continuous semigroups. As a starting point we take the *abstract Cauchy problem* (ACP)

$$\begin{cases} \frac{d}{dt}x(t) &= Ax(t), \quad t \geq 0 \\ x(0) &= x_0 \end{cases} \tag{2.4}$$

on a Banach space X where $A : D(A) \subseteq X \to X$ denotes a closed linear operator on X which determines the evolution of the state space variable $x(t) \in X$. Our plan is as follows.

1. We begin by defining what we understand to be a *solution* of the evolution equation (2.4). In particular, we introduce the concept of *classical solutions* and the more general concept of *mild solutions.*

2. Then we recall criterions ensuring, or even characterising, the *existence* of *unique* solutions for the problem (2.4), given a suitable initial value $x_0 \in X$. These considerations lead naturally to the concept of *strongly continuous semigroups* (of linear operators) which determine the time evolution of the state $x(t)$ for any arbitrary initial value $x_0 \in X$. In particular, we recall the famous *Hille-Yosida Theorem* and the easier applicable, but more restrictive *Lumer-Phillips Theorem* where the latter will turn out to be very useful later on.

3. After that we focus on the asymptotic properties of solutions x of the ACP (2.4), provided they exist. We recall different stability concepts and some results which connect properties of the generator A in (2.4) and stability properties of the corresponding semigroups, i.e. the solutions of (2.4). Namely we recall the *Arend-Batty-Lyubich-Vũ Theorem* (on asymtotic stability) and the *Gearhart-Greiner-Prüss-Huang Theorem* (on uniform exponential stability).

4. Finally we also consider some known results for the situation where the linear operator A is replaced by a nonlinear and possibly multi-valued map $A : D(A) \subseteq X \rightrightarrows X$, so that the adjusted evolution equation takes the form

$$\begin{cases} \frac{d}{dt}x(t) & \in A(x(t)), \quad t \geq 0 \\ x(0) & = x_0. \end{cases} \tag{2.5}$$

   Also for this case we recall solution concepts and a theorem ensuring existence of a unique solution, namely the *Komura-Kato Theorem.* Similarly the concept of a strongly continuous semigroup of linear operators is generalised to the notion of a nonlinear strongly continuous semigroup.

Let us proceed by coming back to the abstract Cauchy problem (2.4). As noted before $A : D(A) \subseteq X \to X$ is assumed to be a linear operator on a Banach space X. (In fact, within this thesis X usually denotes a Hilbert space.) We consider the slightly more general inhomogeneous evolution equation

$$\begin{cases} \frac{d}{dt}x(t) & = Ax(t) + f(t), \quad t \geq 0 \\ x(0) & = x_0 \end{cases} \tag{2.6}$$

where $f : \mathbb{R}_+ \to X$ is a measurable function. For a function $x : \mathbb{R}_+ \to X$ to satisfy equation (2.6) pointwise, x should lie in $C^1(\mathbb{R}_+; X)$ and $x(t) \in D(A)$ should lie in the domain of the linear operator A for every $t \geq 0$. In that case, formally integrating equation (2.6) over $s \in [0, t]$ for some $t \geq 0$ we get the integral formulation

$$x(t) = x_0 + A \int_0^t x(s)ds + \int_0^t f(s)ds$$

provided that time-integration and the operator A may be interchanged. This leads to two notion of solutions as in the following definition (cf. Definitions II.6.1 and II.6.3 in [EnNa00]).

Definition 2.2.1. *Let $f \in L_{1,loc}(\mathbb{R}_+; X)$, $x_0 \in X$ and $x \in C(\mathbb{R}_+; X)$.*

1. *The function x is called a* mild solution *of the evolution equation (2.6) if the value of the integral $\int_0^t x(s)ds$ lies in the domain $D(A)$ of the linear operator A and*
$$x(t) = x_0 + A\int_0^t x(s)ds + \int_0^t f(s)ds, \quad t \geq 0.$$
2. *Assume that $f \in C(\mathbb{R}_+; X)$. The function x is called* classical solution *if $x \in C^1(\mathbb{R}_+; X) \cap C(\mathbb{R}_+; D(A))$ such that*
$$\begin{cases} \frac{d}{dt}x(t) & = Ax(t) + f(t), \quad t \geq 0 \\ x(0) & = x_0. \end{cases} \tag{2.7}$$

We will recall and re-interpret these solution concepts after introducing the terms *well-posedness* (cf. Definition II.6.8 in [EnNa00]) and *strongly continuous contraction semigroups.*

Definition 2.2.2. *Consider the abstract Cauchy problem (2.4). It is called* well-posed *if $D(A) \subseteq X$ is dense in X, for every $x_0 \in D(A)$ there is a unique classical solution $x = x(\cdot; x_0) \in C^1(\mathbb{R}_+; X) \cap C(\mathbb{R}_+; D(A))$ and the solution depends continuously on the initial value x_0, i.e. for all $\tau > 0$ and $\varepsilon > 0$ there is $\delta > 0$ such that*

$$\sup_{t\in[0,\tau]} \|x(t; x_0) - x(t; \tilde{x}_0)\|_X < \varepsilon, \quad t \in [0, \tau], \ \tilde{x}_0 \in B_\delta(x_0) \cap D(A).$$

Next we introduce the concept of a strongly continuous semigroup. As a motivation assume that the abstract Cauchy problem (2.4) is well-posed. Then for every $x_0 \in D(A)$ there exists a unique solution $x(\cdot; x_0) \in C^1(\mathbb{R}_+; X) \cap C(\mathbb{R}_+; D(A))$ and therefore for every fixed $t \geq 0$ the map

$$T(t)x_0 := x(t; x_0) \in D(A), \quad x_0 \in D(A)$$

is well-defined and has the following properties.

1. $T(0)x_0 = x(0; x_0) = x_0$ for all $x_0 \in D(A)$, i.e. $T(0)$ is the identity map on $D(A)$.
2. For every given $s \geq 0$ and $x_0 \in D(A)$ the function
$$x_s(t) := \begin{cases} x(t; x_0), & t < s \\ x(t - s; x(s; x_0)), & t \geq s \end{cases}$$
is a classical solution of (2.4) for the initial value x_0 and by uniqueness of classical solutions coincides with $x(\cdot; x_0)$, i.e. $x(t; x_0) = x(t - s; x(s; x_0))$ for all $0 \leq s \leq t$ and $x_0 \in D(A)$, so that
$$T(t) = T(t - s)T(s), \quad 0 \leq s \leq t.$$

3. For all $x_0 \in D(A)$ the function $T(\cdot)x_0 = x(\cdot; x_0)$ lies in the intersection of $C^1(\mathbb{R}_+; X)$ and $C(\mathbb{R}_+; D(A))$, and
4. $\sup_{t\in[0,\tau]} \|T(t)x_0\| = \sup_{t\in[0,\tau]} \|x(t; x_0) - x(t; 0)\| \to 0$ as $x_0 \to 0$ in $\|\cdot\|_X$.

Together these properties imply that $T(\cdot) : D(A) \to D(A)$ has a unique continuation to a *strongly continuous semigroup* (see Definition II.5.1 in [EnNa00]).

Definition 2.2.3 (C_0-semigroup)**.** *A family* $(T(t))_{t\geq 0} \subseteq \mathcal{B}(X)$ *of bounded linear operators on a Banach space* X *is called* strongly continuous semigroup*, or* C_0-semigroup *(of bounded linear operators), if it has the following properties*

1. *semigroup property:* $T(0) = I$ *is the identity map on* X *and*
$$T(t + s) = T(t)T(s), \quad s, t \geq 0.$$
2. *strong continuity: For every* $x_0 \in X$*, the map* $T(\cdot)x \in C(\mathbb{R}_+; X)$ *is continuous.*

In that case

$$Ax := \lim_{t\to 0} \frac{T(t)x - x}{t}$$
$$D(A) := \{x \in X : \lim_{t\to 0} \frac{T(t)x - x}{t} \in X \text{ exists}\}$$

is called the generator *of the* C_0*-semigroup* $(T(t))_{t\geq 0}$*. Moreover, a* C_0*-semigroup* $(T(t))_{t\geq 0}$ *is called* bounded *if*

$$\|T(t)\| \leq M, \quad t \geq 0$$

for some constant $M \geq 1$ *and* contractive *if the choice* $M = 1$ *is admissible.*

The infimum of those ω such that there is such a constant $M_\omega \geq 1$ is called the *growth bound* $\omega_0(T(\cdot))$ of the C_0-semigroup,

$$\omega_0(T(\cdot)) := \inf\{\omega \in \mathbb{R} : \exists M_\omega \geq 1 \text{ such that } \|T(t)\| \leq M_\omega e^{\omega t} \ (t \geq 0)\}.$$

Remark 2.2.4. *Note that every* C_0*-semigroup* $\|T(t)\| \leq Me^{\omega t}$ *is the exponential bounded in the following sense: There are constants* $M \geq 1$ *and* $\omega \in \mathbb{R}$ *such that*

$$\|T(t)\| \leq Me^{\omega t}, \quad t \geq 0.$$

In particular, the infimum of these ω *exists in* $[-\infty, +\infty)$*.*

Proof. This result is based on the Principle of Uniform Boundedness, see Proposition I.5.5 in [EnNa00]. □

Above we have seen that the solutions of a well-posed ACP naturally define a C_0-semigroup on the Banach space X, but in fact this actually characterises the generators of a C_0-semigroup.

Theorem 2.2.5. *For the abstract Cauchy problem (2.4) and the closed linear operator* $A : D(A) \subseteq X \to X$ *the following are equivalent.*

1. *The abstract Cauchy problem (2.4) is well-posed.*

2. *The operator A generates a C_0-semigroup on X.*

3. *The abstract Cauchy problem has a unique classical solution for every initial value $x_0 \in D(A)$ and the resolvent set $\rho(A)$ is not empty.*

4. *The abstract Cauchy problem has a unique classical solution for every initial value $x_0 \in D(A)$ and there is a sequence $(\lambda_n) \subseteq \mathbb{R}$ converging to $+\infty$ such that all operators $\lambda_n I - A$ are surjective.*

Proof. This is Theorem II.6.7 in [EnNa00]. □

Hence, the question of whether the abstract Cauchy problem (2.4) is well-posed can be equivalently rephrased as: Does A generate a C_0-semigroup?

For more background on the following we refer to Section II.3 in [EnNa00], from where we cite the following generation results. In fact, the question of characterising the C_0-semigroup generators has been solved around 1950, first by Hille and Yosida (1948) for the contractive case, and then by Feller, Miyadera, Phillips (1952) for the general case. Still, today also the general generation theorem due to Feller, Miyadera and Phillips is mainly known as Hille-Yosida Theorem. It states the following.

Theorem 2.2.6 (Hille-Yosida Theorem (General Version due to Feller, Miyadera, Phillips))**.** *Let $A : D(A) \subseteq X \to X$ be a linear operator and $M \geq 1$ and $\omega \in \mathbb{R}$ be constants. Then the following are equivalent.*

1. *A generates a C_0-semigroup $(T(t))_{t\geq 0}$ with*
$$\|T(t)\| \leq Me^{\omega t}, \quad t \geq 0.$$

2. *The operator A is closed and densely defined and every $\lambda > \omega$ lies in the resolvent set and*
$$\|[(\lambda - \omega)R(\lambda, A)]^n\| \leq M, \quad n \in \mathbb{N}, \lambda > \omega.$$

3. *The operator A is closed and densely defined and for every $\lambda \in \mathbb{C}$ with $\operatorname{Re} \lambda > \omega$ one has $\lambda \in \rho(A)$ and*
$$\|R(\lambda, A)^n\| \leq \frac{M}{(\operatorname{Re} \lambda - \omega)^n}, \quad n \in \mathbb{N},\ \operatorname{Re} \lambda > \omega.$$

Proof. This is Theorem II.3.8 in [EnNa00]. □

For contractive C_0-semigroups the Lumer-Phillips Theorem provides a much easier criterion to check whether a given operator A generates a contraction semigroup. As we are mainly concerned with Hilbert spaces in this thesis, we do not give the Banach space version here and from the start assume that A is closed, referring to Theorem II.3.15 in [EnNa00] for its general version on Banach spaces.

Theorem 2.2.7 (Lumer, Phillips (1962))**.** *Let A be a densely defined, closed linear operator on a Hilbert space X. Then A generates a contractive C_0-semigroup on X if and only if A is m-dissipative, i.e. A is dissipative,*
$$\operatorname{Re} \langle Ax, x\rangle_X \leq 0, \quad x \in D(A) \tag{2.8}$$

and the following range condition holds: $\lambda I - A : D(A) \subseteq X \to X$ *is surjective for some (then: all)* $\lambda > 0$.

Proof. See Theorem II.3.15 in [EnNa00] for a slightly more general version. □

Remark 2.2.8. *Closely related to the notion of a* C_0*-semigroup, but slightly more restrictive is the concept of a* strongly continuous *(s.c.) group of linear operators (or,* C_0*-group). Consider the abstract Cauchy problem on the whole of* $\mathbb{R}$,

$$\begin{aligned} \frac{d}{dt}x(t) &= Ax(t), \quad t \in \mathbb{R} \\ x(0) &= x_0 \in X \end{aligned}$$

i.e. solving the abstract Cauchy problem not only forward in time, but backward in time as well, hence the solutions not only live on $\mathbb{R}_+$*, but on the real line* $\mathbb{R}$*. Well-posedness is defined analogously to well-posedness on* $\mathbb{R}_+$ *and the following definition is similar as well. Actually, well-posedness of the problem is equivalent to the existence of a* C_0*-group.*

Definition 2.2.9. *A family* $(T(t))_{t\in\mathbb{R}} \subseteq \mathcal{B}(X)$ *of bounded linear operators on a Banach space* X *is called a* C_0-group*, if it has the following properties.*

1. *group property:* $T(0) = I$ *is the identity map on* X *and*

$$T(t+s) = T(t)T(s), \quad s, t \in \mathbb{R}.$$

2. *strong continuity: For every* $x_0 \in X$*, the map* $T(\cdot)x \in C(\mathbb{R}; X)$ *is continuous.*

In that case

$$\begin{aligned} Ax &:= \lim_{t\to 0} \frac{T(t)x - x}{t} \\ D(A) &:= \{x \in X : \lim_{t\to 0} \frac{T(t)x - x}{t} \in X \text{ exists}\} \end{aligned}$$

is called the generator *of the* C_0*-group* $(T(t))_{t\geq 0}$*. Moreover, a* C_0*-group* $(T(t))_{t\geq 0}$ *is called* isometric *if*

$$\|T(t)x\| = \|x\|, \quad t \in \mathbb{R}, \ x \in X$$

and unitary *if each operator* $T(t)$ $(t \in \mathbb{R})$ *is unitary, i.e.*

$$T(-t) = T(t)', \quad t \in \mathbb{R}.$$

We then have

Theorem 2.2.10 (Stone)**.** *A closed linear operator* $A : D(A) \subseteq X \to X$ *generates a unitary* C_0*-group if and only if* $A = -A'$ *is skew-Hermitian, if and only if both* A *and* $-A$ *are dissipative.*
In particular, the isometric C_0*-groups are exactly the unitary* C_0*-groups.*

Proof. For the proof of the first equivalence (Stone's Theorem) see Theorem II.3.24 in [EnNa00]. There it is also shown that $A = -A'$ implies that A and $-A$ are dissipative, and then it is shown that this already implies that A generates a unitary C_0-group. □

After introducing characterisations of semigroup generators and hence well-posed abstract Cauchy problems we proceed by introducing some terminology concerning the long-time behaviour of the solutions of the semigroup. Here we are mainly interested in the question whether the solutions tend to zero as $t \to +\infty$ and how fast this occurs. We give the corresponding definitions in the framework of C_0-semigroups. Considering the one-to-one correspondence between C_0-semigroups and well-posed abstract Cauchy problems the conditions may be easily reformulated in the context of the well-posed abstract Cauchy problem (2.4). There are several stability concepts: (uniform) exponential stability, polynomial stability, asymptotic (strong) stability or weak stability, to name just a few. The difference between the stability concept is the topology on which convergence to zero is investigated, e.g., in operator norm, the strong or only in the weak topology. For more general stability concepts and the theory thereof we refer to [Ei10] and [Va96] whereas here we mainly focus on (uniform) exponential stability and asymptotic (strong) stability which are defined as follows.

Definition 2.2.11 (Stability Concepts)**.** *A C_0-semigroup $(T(t))_{t\geq 0}$ on a Banach space X is called*

- asymptotically (strongly) stable *if for all $x \in X$*

$$T(t)x \xrightarrow{t\to\infty} 0,$$

- (uniformly) exponentially stable *if there exist $\omega < 0$ and $M \geq 1$ such that*

$$\|T(t)\| \leq Me^{\omega t}, \quad t \geq 0.$$

Remark 2.2.12. *Let $(T(t))_{t\geq 0}$ be a C_0-semigroup on some Banach space X and assume that $\|T(\tau)\| < 1$ for some $\tau > 0$. Then the C_0-semigroup is uniformly exponentially stable.*

Proof. See Proposition V.1.7 in [EnNa00]. □

Remark 2.2.13. *If $\|T(\tau)\| = \rho \in (0,1)$ for some $\tau > 0$, the constants $(M,\omega) \in [1,\infty) \times (-\infty,0)$ may be chosen as*

$$\omega = \frac{\ln \rho}{\tau}, \quad M = e^{-\omega}.$$

A sufficient condition for asymptotic stability using spectral properties of the generator A has been given both by Arendt and Batty [ArBa88] and by Lyubich and Vũ [LyPh88] who independently obtained the following result.

Theorem 2.2.14 (Asymptotic Stability)**.** *Let A generate a bounded C_0-semigroup $(T(t))_{t\geq 0}$ on a Banach space X and assume that $\sigma_r(A) \cap i\mathbb{R} = \emptyset$. If $\sigma(A) \cap i\mathbb{R}$ is countable, then $(T(t))_{t\geq 0}$ is asymptotically (strongly) stable.*

Proof. See Stability Theorem 2.4 in [ArBa88] or the theorem in [LyPh88]. □

Remark 2.2.15. *For reflexive Banach spaces X the assumption $\sigma_r(A) \cap i\mathbb{R} = \emptyset$ reduces to $\sigma_p(A) \cap i\mathbb{R} = \emptyset$.*

Proof. This is a special case of Corollary 2.6 in [ArBa88]. □

In the case that the generator A has compact resolvent, i.e. $D(A)$ is relatively compact in X this can even be stated as a characterisation of asymptotic stability of the C_0-semigroup by means of the spectrum of its generator.

Corollary 2.2.16. *Let A have compact resolvent and generate a bounded C_0-semigroup $(T(t))_{t\geq 0}$ on a Banach space X. Then the C_0-semigroup $(T(t))_{t\geq 0}$ is asymptotically (strongly) stable if and only if*

$$\sigma_p(A) \cap i\mathbb{R} = \emptyset.$$

Proof. For operators with compact resolvent the equality $\sigma(A) = \sigma_p(A)$ holds true which follows from the Fredholm alternative, see e.g. Satz VI.2.4 in [We06]. Now the corollary follows from Theorem 2.2.14 and the fact that an operator A with $\sigma_p(A) \cap i\mathbb{R} \neq \emptyset$ clearly cannot generate an asymptotically stable semigroup. □

On the other hand it is not possible to characterise exponential stability merely by the spectrum of the generator A (although there are some semigroups which have the *Spectral Bound Equals Growth Bound Property*, e.g. analytic semigroups or, more general, eventually norm continuous semigroups or semigroups with the Riesz basis property, see, e.g. some results in Chapter 5 of [EnNa00]) and for Banach spaces the characterisation of exponentially stable semigroup-generators is quite complicated. However, in the Hilbert space case the characterisation is much easier, namely by a spectral condition and uniform boundedness of the resolvent operators on the right half plane. The result is initially due to Gearhart who considered only the contraction case. The theorem has then be generalised (in several ways) by Huang and Prüss. Accordingly the theorem is often referred to as Gearhart's Theorem or Gearhart-Greiner-Prüss Theorem or (especially in the systems theory community) as Huang's Theorem. For its formulation we introduce

$$\mathbb{F}_\omega^+ := \{z \in \mathbb{F} : \text{ Re } z > \omega\}, \quad \omega \in \mathbb{R}$$

and later similarly write

$$\mathbb{F}_\omega^- := \{z \in \mathbb{F} : \text{ Re } z < \omega\}, \quad \omega \in \mathbb{R}.$$

Theorem 2.2.17 (Exponential Stability)**.** *Let A generate a C_0-semigroup $(T(t))_{t\geq 0}$ on a Hilbert space X. Then $(T(t))_{t\geq 0}$ is uniformly exponentially stable if and only if*

$$s(A) := \sup\{\text{Re } \lambda : \lambda \in \sigma(A)\} < 0 \quad \textit{and} \quad \sup_{\lambda \in \mathbb{C}_0^+} \|R(\lambda, A)\| < +\infty.$$

Proof. See, e.g. Theorem 4 and Corollary 5 in [Pr84]. □

For bounded semigroups this result reads as follows.

Corollary 2.2.18. *Let A generate a bounded C_0-semigroup $(T(t))_{t\geq 0}$ on a Hilbert space X. Then $(T(t))_{t\geq 0}$ is uniformly exponentially stable if and only if*

$$\sigma(A) \cap i\mathbb{R} = \emptyset \quad \textit{and} \quad \sup_{\beta \in \mathbb{R}} \|R(i\beta, A)\| < +\infty.$$

There also is a sequence criterion which proves quite convenient for differential operators with dissipative boundary conditions.

Corollary 2.2.19 (Sequence Criterion)**.** *Let A generate a bounded C_0-semigroup $(T(t))_{t\geq 0}$ on a Hilbert space X. Then $(T(t))_{t\geq 0}$ is uniformly exponentially stable if and only if $\sigma(A) \subseteq \mathbb{C}_0^-$ and the following sequence criterion holds: If $(x_n, \beta_n)_{n\geq 1} \subseteq D(A) \times \mathbb{R}$ is a sequence such that*

$$\sup_{n\in\mathbb{N}} \|x_n\|_X < +\infty,$$
$$|\beta_n| \xrightarrow{n\to\infty} \infty,$$
$$Ax_n - i\beta_n x_n \xrightarrow{n\to\infty} 0,$$

then already $x_n \xrightarrow{n\to\infty} 0$ tends to zero in X.

Proof. We show that the sequence criterion is equivalent to the uniform boundedness of the resolvent operators on the imaginary axis. First assume that

$$\sup_{i\mathbb{R}} \|R(\cdot, A)\| < +\infty$$

and take any sequence $(x_n, \beta_n) \subseteq D(A) \times \mathbb{R}$ where $\sup_{n\in\mathbb{N}} \|x_n\|_X < +\infty$ and $|\beta_n| \to \infty$ and such that $Ax_n - i\beta_n x_n \to 0$ as $n \to \infty$. Then we obtain

$$\|x_n\|_X \leq \sup_{\beta\in\mathbb{R}} \|R(i\beta, A)\| \, \|(A - i\beta_n)x_n\|_X \xrightarrow{n\to\infty} 0.$$

On the other hand, if $\sup_{i\mathbb{R}} \|R(\cdot, A)\| = +\infty$, then there is a sequence $(z_n, \beta_n)_{n\geq 1} \subseteq D(A) \times \mathbb{R}$ such that $\|z_n\|_X = 1$ ($n \in \mathbb{N}$) and $|\beta_n| \xrightarrow{n\to\infty} \infty$ with

$$\|R(i\beta_n, A)z_n\| \xrightarrow{n\to\infty} +\infty.$$

Then we set $x_n := \frac{R(i\beta_n,A)z_n}{\|R(i\beta_n,A)z_n\|} \in D(A)$ and observe that $\|x_n\|_X = 1$ ($n \in \mathbb{N}$), but still

$$\|Ax_n - i\beta_n x_n\| = \frac{\|z_n\|}{\|R(i\beta_n, A)z_n\|} \xrightarrow{n\to\infty} 0,$$

so that the sequence criterion cannot hold true. The corollary follows from the Gearhart-Greiner-Prüss-Huang Theorem 2.2.17. □

Let us finally leave the linear situation and consider the abstract Cauchy problem (2.5) where the linear operator $A : D(A) \subseteq X \to X$ is replaced by a nonlinear map $A : D(A) \subseteq X \rightrightarrows X$ as follows. We restrict ourselves to the contractive case here since only dissipative nonlinear systems will be considered within this thesis. More general results on nonlinear semigroups may be found in [Mi92]. Also we restrict ourselves to the Hilbert space scenario where we state the nonlinear version of the Lumer-Phillips Theorem 2.2.7, i.e. the Komura-Kato Theorem 2.2.29 below. We start with the generalisation of the concept of dissipative (resp. monotone) operators to the nonlinear scenario. For more details see, e.g. Chapter IV in [Sh97].

Definition 2.2.20. *Let $A : X \to \mathcal{P}(X) := \{B \subseteq X\}$ be a power set valued-map. We denote by $D(A)$ its* domain

$$D(A) := \{x \in X : A(x) \neq \emptyset\}$$

and then also use the notation $A : D(A) \subseteq X \rightrightarrows X$. In case that $A(x) = \{y_x\}$ is single-valued for all $x \in D(A)$ we call A an operator *and write – analogously to the linear situation –*

$$Ax := y_x, \quad \textit{for the unique } y_x \in A(x).$$

Otherwise we say that A is multi-valued.

We use the notation $A+B$ for the *sum* of two maps $A : D(A) \subset X \rightrightarrows X$ and $B : D(B) \subset X \rightrightarrows X$ which we define as follows

$$(A+B)(x) = \{y_1 + y_2 \in X : y_1 \in A(x),\ y_2 \in B(x)\}.$$

In particular, the domain of the sum $D(A+B) = D(A) \cap D(B)$ is the intersection of the domains of the maps and for the particular case where $B : X \to X$ is a bounded linear operator $(A+B)(x) = \{y_1 + Bx : y_1 \in A(x)\}$ for all $x \in D(A+B)$. Moreover, for any scalar $\alpha \in \mathbb{F}$ we define the operator αA by

$$(\alpha A)(x) := \{\alpha y \in X : y \in A(x)\},$$

i.e. $D(\alpha A) = D(A)$. Note that for linear operators these definitions coincide with the usual notation.

Definition 2.2.21. *Let X be a Hilbert space and $A : D(A) \subseteq X \rightrightarrows X$ be a map which may be nonlinear and/or multi-valued. We say that the map A is* dissipative *(and $-A$ monotone (or, accretive)), if for all $x, x' \in D(A)$ and $y \in A(x)$, $y' \in A(x')$ one has*

$$\operatorname{Re} \langle y - y', x - x' \rangle_X \le 0.$$

If additionally for some $\lambda > 0$ (then: all $\lambda > 0$) the map A satisfies the range-condition

$$\{y \in X : \exists x \in D(A) : y \in (\lambda I - A)(x)\} =: \operatorname{ran}(\lambda I - A) = X$$

*then A is called m-*dissipative *(and $-A$ is called m-monotone (or, m-accretive)). We call $A : D(A) \subset X \rightrightarrows X$ maximal dissipative, if it is dissipative and has no proper dissipative extension, i.e. if $B : D(B) \subset X \rightrightarrows X$ with $D(A) \subset D(B)$ and $A(x) \subset B(x)$ for all $x \in D(A)$ is a dissipative extension, then already $A = B$.*

Remark 2.2.22. *Let $A : D(A) \rightrightarrows X$ be an m-dissipative map on some Hilbert space X. Then for all $x \in D(A)$ the set $A(x)$ is closed and convex and therefore there is a unique $z \in A(x)$ with minimal norm. This defines the* minimal section *A^0 of A:*

$$A^0 x := z, \quad \|z\| = \inf_{y \in A(x)} \|y\|, \quad D(A^0) = D(A).$$

Consequently we may define

$$|Ax| := \left\|A^0 x\right\| = \inf_{y \in A(x)} \|y\|.$$

Moreover, for all $x \in X$ and $\lambda \in \mathbb{F}_0^+$ the element $y \in D(A)$ such that

$$x \in (\lambda I - A)(y)$$

is uniquely determined and we may write $y = (\lambda I - A)^{-1} x$ defining a nonlinear and contractive operator $(\lambda I - A)^{-1}$ on X. In particular, every m-dissipative operator is maximal dissipative.

Proof. Let $x \in D(A)$ be arbitrary and $y \in \overline{A(x)}$ lie in the closure of $A(x)$. We show that actually $y \in A(x)$. First, take a sequence $(y_n)_{n \ge 1} \subseteq A(x)$ such that $y_n \xrightarrow{n\to\infty} y$ and observe that for every $x' \in D(A)$ and $y' \in A(x')$ we get

$$\operatorname{Re} \langle x - x', y - y' \rangle_X = \lim_{n\to\infty} \operatorname{Re} \langle x - x', y_n - y \rangle_X \le 0.$$

Moreover, from the m-dissipativity of A we find $x' \in D(A)$ such that $x - y \in \operatorname{ran}(I - A)$, i.e. $x' - x + y \in A(x')$, and hence

$$\|x - x'\| = \operatorname{Re}\langle x - x', y - (x' - x + y)\rangle \leq 0$$

which implies that $x' = x$ and $y \in A(x)$, proving closedness of $A(x)$. Next, we show that $A(x)$ is convex. Let $x \in D(A)$ be arbitrary and $y, z \in A(x)$, $\lambda \in [0, 1]$. Then, since A is m-dissipative there is $\tilde{x} \in D(A)$ such that

$$\lambda y + (1 - \lambda)z - x \in (A - I)(\tilde{x}),$$

i.e. $\lambda y + (1 - \lambda)z + \tilde{x} - x \in A(\tilde{x})$. From the dissipativity of A we thus obtain

$$\begin{aligned} 0 \leq \|x - \tilde{x}\|^2 &= \langle \lambda y + (1 - \lambda)z - (\lambda y + (1 - \lambda)z - x + \tilde{x}), x - \tilde{x}\rangle \\ &= \lambda\langle y - (\lambda y + (1 - \lambda)z - x + \tilde{x}), x - \tilde{x}\rangle \\ &\quad + (1 - \lambda)\langle z - (\lambda y + (1 - \lambda)z - x + \tilde{x}), x - \tilde{x}\rangle \leq 0, \end{aligned}$$

hence $\tilde{x} = x$ and $\lambda y + (1 - \lambda)z \in A(x)$.
For the second statement let $\lambda > 0$ and $x \in (A - \lambda I)(y) \cap (A - \lambda I)(z)$, then $x + \lambda y \in A(y)$ and $x + \lambda z \in A(z)$, so

$$0 \leq \langle y - z, y - z\rangle = \frac{1}{\lambda}\langle (x + \lambda y) - (x + \lambda z), y - z\rangle \leq 0$$

and it follows $y = z$. For the last statement note that any dissipative extension $\tilde{A}$ of an m-dissipative operator is again m-dissipative, thus $\lambda I - \tilde{A}$ is injective as we just saw, but as extension of the surjective map $\lambda I - A$ both maps then have to be equal, i.e. $\tilde{A} = A$ is no proper extension. □

Lemma 2.2.23. *If $A : D(A) \rightrightarrows X$ is m-dissipative and $B : X \to X$ is dissipative and Lipschitz continuous, then also $A + B : D(A) \rightrightarrows X$ is m-dissipative.*

Proof. We follow the line of proof for Lemma IV.2.1 in [Sh97]. First, we note that the sum of two dissipative operators is again dissipative (take the intersection of their domains as the domain of the sum) and also multiplicating an (m-)dissipative operator by $\alpha > 0$ gives another (m-)dissipative operator. Writing $A+B = \frac{1}{\alpha}(\alpha A + \alpha B)$ where $\alpha > 0$ we may therefore assume that B is a strict contraction (i.e. its Lipschitz constant is strictly less than 1). For given $f \in X$ we shall find $x \in D(A + B) = D(A)$ such that $f \in x - A(x) - Bx$, i.e.

$$x = \Phi(x) := (I - A)^{-1}(f - Bx).$$

Here Φ is a strict contraction and thus from the Strict Contraction Principle Proposition 2.1.12 this equation has a unique solution $x \in D(A)$. □

To round out let us also mention Minty's Theorem.

Theorem 2.2.24 (Minty)**.** *On a Hilbert space X the m-dissipative operators are exactly the maximal dissipative operators.*

Proof. Combine Lemma 2.2.12(iii) and Corollary 3.2.27 in [Mi92]. □

As in the linear (C_0-semigroup) case, m-dissipative operators are closely related to the generators of contraction semigroups which are defined as follows.

Definition 2.2.25 (Semigroup)**.** *Let X be a Banach space and let $X_0 \subset X$ be a closed subset. A family $(S(t))_{t\geq 0}$ of mappings $S(t) : X_0 \to X_0$ $(t \geq 0)$ is called* semigroup *(or, dynamical system) if it satisfies the properties*

1. *$S(0) = I_{X_0}$, the identity map on X_0, and*
2. *$S(t+s) = S(t)S(s)$ for all $s, t \geq 0$.*

We speak of a strongly continuous *(abbr.: s.c.) (nonlinear)* semigroup *(or, dynamical system) if $S(t) \in C(X_0; X_0)$ $(t \geq 0)$ and for all $x \in X_0$ the map $S(\cdot)x \in C(\mathbb{R}_+; X)$ is continuous on $\mathbb{R}_+$. A semigroup $(S(t))_{t\geq 0}$ is called* contractive*, if all maps $S(t)$ $(t \geq 0)$ are contractions, i.e.*

$$\|S(t)x - S(t)x'\|_X \leq \|x - x'\|_X, \quad x, x' \in X_0,\ t \geq 0.$$

Remark 2.2.26. *Note that if additionally $X_0 = X$ and all maps $S(t)$ $(t \geq 0)$ are linear, i.e. $S(t) \in \mathcal{B}(X)$, then the definition above coincides with the usual definition of a C_0-semigroup of linear operators and also the definitions of contractive semigroups are compatible.*

Definition 2.2.27. *Let $(S(t))_{t\geq 0}$ be a (nonlinear) strongly continuous contraction semigroup on X. Set*

$$\hat{D} := \{x \in X : S(\cdot)x \in \operatorname{Lip}(\mathbb{R}_+; X)\}.$$

We define the (infinitesimal) generator *of the s.c. contraction semigroup $(S(t))_{t\geq 0}$ as*

$$A_0(x) := \lim_{t\searrow 0} \frac{S(t)x - x}{t}, \quad D(A_0) := \{x \in X : \lim_{t\searrow 0} \frac{S(t)x - x}{t} \in X \text{ exists}\}$$

and the (g)-operator $A : D(A) \subset X \rightrightarrows X$ as the maximal dissipative extension of A_0 with $D(A) \subset \hat{D}$.

Remark 2.2.28. *By Zorn's Lemma every dissipative operator has a maximal dissipative extension (see Lemma 2.2.12(ii) in [Mi92]). Hence, the (g)-operator always exists. Also note that the infinitesimal operator A_0 (or the (g)-operator A) uniquely determines the s.c. contraction semigroup, see Corollary 3.4.17 in [Mi92].*

The following results shows that for m-dissipative maps the nonlinear version of the abstract Cauchy problem is well-posed and – similar to the Lumer-Phillips Theorem – the solution is given by a nonlinear contraction semigroup.

Theorem 2.2.29 (Komura-Kato)**.** *Let $A : D(A) \subseteq X \rightrightarrows X$ be a (possibly multi-valued) map on a Hilbert space X. If A is m-dissipative, then it generates a nonlinear strongly continuous contraction semigroup $(S(t))_{t\geq 0}$ on $\underline{X} := \overline{D(A)}^X$. More precisely, for each $x_0 \in D(A)$ there is a unique absolutely continuous solution $x \in W^1_\infty(\mathbb{R}_+; X)$ of the abstract nonlinear Cauchy problem*

$$\begin{aligned} \frac{d}{dt}x(t) &\in A(x(t)), \quad t \geq 0 \\ x(0) &= x_0. \end{aligned} \tag{2.9}$$

Also $\left\|\frac{d}{dt}x\right\|_{L_\infty(\mathbb{R}_+;X)} \leq \left\|A^0 x_0\right\|_X$, *the function* $\left\|A^0 x\right\|_X$ *is decreasing and for every* $t \geq 0$ *and the right-derivative* $\frac{d^+}{dt}$ *one has*

$$\frac{d^+}{dt}x(t) := \lim_{t \searrow 0} \frac{x(t+s) - x(t)}{s} = A^0 x(t), \quad t \geq 0.$$

Proof. See Proposition IV.3.1 in [Sh97]. □

Remark 2.2.30. *If A is m-dissipative and $0 \in A(0)$, then $S(t)(0) = 0$ for all $t \geq 0$. Consequently in this case*

$$\|S(t)x\|_X \leq \|x\|_X, \quad t \geq 0.$$

Remark 2.2.31. *Let $(S(t))_{t\geq 0}$ be a s.c. contraction semigroup on some closed subset X_0 of a Hilbert space X. Assume that there is $\tau > 0$ and a constant $\rho \in (0,1)$ such that for all $x, \tilde{x} \in X$ the estimate*

$$\|S(\tau)x - S(\tau)\tilde{x}\| \leq \rho \|x - \tilde{x}\|$$

is valid. Then there are constants $M \geq 1$ and $\omega < 0$ such that for all $x, \tilde{x} \in X$

$$\|S(\tau)x - S(\tau)\tilde{x}\| \leq M e^{\omega t} \|x - \tilde{x}\|, \quad t \geq 0.$$

Proof. Take any $t = \tau k + s \in \mathbb{R}_+$ where $k \in \mathbb{N}_0$ and $s \in [0, \tau)$. Then for all $x, \tilde{x} \in X$ we obtain iteratively that

$$\begin{aligned}
\|S(t)x - S(t)\tilde{x}\| &= \|S(k\tau + s)x - S(k\tau + s)\tilde{x}\| \\
&= \left\|S(\tau)^k S(s)x - S(\tau)^k S(s)\tilde{x}\right\| \\
&\leq \rho^k \|S(s)x - S(s)\tilde{x}\| \leq e^{k \ln \rho} \|x - \tilde{x}\| \\
&= e^{-s\frac{\ln \rho}{\tau}} e^{t\frac{\ln \rho}{\tau}} \|x - \tilde{x}\| \\
&\leq e^{-\frac{\ln \rho}{\tau}} e^{t\frac{\ln \rho}{\tau}} \|x - \tilde{x}\| =: M e^{\omega t} \|x - \tilde{x}\|, \quad t \geq 0.
\end{aligned}$$

where $M := e^{-\frac{\ln \rho}{\tau}} \geq 1$ and $\omega = \frac{\ln \rho}{\tau} < 0$, indeed. □

Let us mention a stability result due to Dafermos and Slemrod [DaSl73] which is some sense is the nonlinear version of the Arendt-Batty-Lyubich-Vũ Theorem 2.2.14 and at the same time is an improvement over the topological version of LaSalle's Invariance Principle, see Theorem 9.2.3 in [CaHa98].

Theorem 2.2.32. *Let $A : D(A) \subseteq X \rightrightarrows X$ be an m-dissipative operator on a Hilbert space X generating a strongly continuous contraction semigroup $(S(t))_{t\geq 0}$ on $X_0 := \overline{D(A)}$. Assume that X_0 is convex, $0 \in \operatorname{ran} A$ and that the map $(\lambda I - A)^{-1}$ is compact for some $\lambda > 0$, i.e.*

$$\overline{(\lambda I - A)^{-1}(B)} \subseteq X \tag{2.10}$$

is compact for every bounded set $B \subseteq X$. Then for every $x_0 \in D(A)$ and $f \in L_1(\mathbb{R}_+; X)$ the mild solution $x \in C(\mathbb{R}_+; X)$ of

$$\frac{d}{dt}x(t) + f(t) \in A(x(t)) \quad (t \geq 0), \quad x(0) = x_0$$

approaches a compact subset $C \subseteq \{z \in X : \|z - z_0\| = r\}$ *of a sphere with centre* $z_0 \in A^{-1}(0)$ *and radius* $r \leq \|x_0 - z_0\| + \|f\|_{L_1}$. *Moreover,* $(S(\cdot)|_C)_{t\geq 0}$ *defines an isometric affine group on* C, *i.e.*

$$T_C(t)z := S(t)(z + z_0) - z_0, \quad t \geq 0,\ z \in C - z_0$$

extends to an isometric C_0*-group (of linear operators) on* $\operatorname{lin}\{C - z_0\}$. *Moreover, if* $x_0 \in D(A)$ *and* $f \in W_1^1(\mathbb{R}_+; X)$, *then* $C \subseteq D(A)$, *the image of* C *under the minimal section* $A^0(C) \subseteq X$ *lies on a sphere with centre* $0 \in X$, *and also the closed convex hull of* C *is contained in* $D(A)$.

Proof. See Theorems 4 and 5 in [DaSl73]. □

2.3 Background on Systems Theory

Within the standard framework of PDE and operator theory mostly Cauchy problems of the form

$$\begin{cases} \frac{d}{dt}x(t) & = Ax(t) + f(t), \quad t \geq 0 \\ x(0) & = x_0 \end{cases}$$

are considered where the evolution of the system is mainly determined by the state of its state variable (e.g. $f = 0$) and an interaction with the environment of the system is only possible if we think of f modelling some influence of the environment on the system. On the other hand in systems theory the interaction of a system with its environment is heavily emphasised. Thus, instead of only taking the state variable $x(t)$ on the state space X into consideration, also an input space U and an output space Y appear. Usually X, U and Y may be arbitrary Banach spaces, but here we restrict ourselves to the situation where X, U and Y are actually Hilbert spaces. We first introduce the *standard formulation* of such a system which takes the form

$$\begin{aligned} \frac{d}{dt}x(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t), \quad t \geq 0 \end{aligned}$$

plus some initial condition $x(0) = x_0 \in X$. Here $x(t) \in X$ denotes the *state space variable* in the state space X, $u(t) \in U$ denotes the *input* and $y(t)$ the *output* at time $t \geq 0$. Accordingly U and Y are called *input space* and *output space*, respectively. All the maps A, B, C and D of the *linear system* $\Sigma = (A, B, C, D)$ are assumed to be linear, but A, B and C may be unbounded, whereas $D \in \mathcal{B}(U, Y)$ is bounded. Since the system should be well-posed (in the sense of an abstract Cauchy problem) for the particular choice $u(t) = 0$ ($t \geq 0$) as input, the operator A is assumed to be the generator of a C_0-semigroup $(T(t))_{t\geq 0}$ of linear operators. The operators B and C will be assumed to be continuous in some weaker sense, we make precise in a moment. For this, we first need to introduce the *interpolation space* X_1^A and the *extrapolation space* X_{-1}^A. The following material is extracted from [TuWe09].

Definition 2.3.1 (The Space X_1^A)**.** *Let* $A : D(A) \subseteq X \to X$ *be a densely defined, closed operator on a Hilbert space* X *and assume that* $\rho(A) \neq \emptyset$, *i.e.* A *has a non* *empty resolvent set. Fix any $\lambda \in \rho(A)$ and define the* interpolation space X_1^A *as* $D(A)$ *equipped with the norm*

$$\|x\|_{X_1^A} := \|(\lambda - A)x\|_X, \quad x \in X_1^A.$$

By Proposition 2.10.1 in [TuWe09] the space X_1^A is a Hilbert space and is continuously embedded into X. Moreover, different choices of $\lambda \in \rho(A)$ lead to equivalent spaces and any bounded operator $B \in \mathcal{B}(X)$ for with $BD(A) \subseteq D(A)$, i.e. the domain is B-invariant, restricts to a bounded operator $B_1 := B|_{X_1^A} \in \mathcal{B}(X_1^A)$.

Definition 2.3.2 (The Space X_{-1}^A)**.** *Let $A : D(A) \subseteq X \to X$ be a densely defined, closed operator on a Hilbert space X and assume that $\rho(A) \neq \emptyset$. Fix any $\lambda \in \rho(A)$ and on X define the norm*

$$\|x\|_{X_{-1}^A} := \left\|(\lambda I - A)^{-1}x\right\|_X, \quad x \in X.$$

Then the extrapolation space X_{-1}^A *is defined as the completion of X with respect to the norm $\|\cdot\|_{X_{-1}^A}$.*

Then thanks to Proposition 2.10.2 in [TuWe09] X_{-1}^A is a Hilbert space and every bounded operator $B \in \mathcal{B}(X)$ for which $D(A')$ is invariant under its Hilbert space adjoint B', i.e. $B'D(A') \subseteq D(A')$, admits a unique continuous extension to a bounded linear operator $B_{-1} \in \mathcal{B}(X_{-1}^A)$.

Remark 2.3.3. *By Proposition 2.10.3 in [TuWe09] the operator A as in the preceding definitions is bounded as operator $A \in \mathcal{B}(X_1^A, X)$ and has a unique continuous extension $A_{-1} \in \mathcal{B}(X, X_{-1}^A)$. Moreover, for every $\lambda \in \rho(A)$ we have $R(\lambda, A) \in \mathcal{B}(X, X_1^A)$ and $R(\lambda, A_{-1}) \in \mathcal{B}(X_{-1}^A, X)$ exists.*

Proposition 2.3.4. *Let A generate a C_0-semigroup $(T(t))_{t\geq 0}$ on a Hilbert space X. Denote by $(T_1(t))_{t\geq 0} \subseteq \mathcal{B}(X_1^A)$ its restriction to X_1^A and by $(T_{-1}(t))_{t\geq 0} \subset \mathcal{B}(X_{-1}^A)$ its unique continuous extension to X_{-1}^A. Then $(T_1(t))_{t\geq 0}$ and $(T_{-1}(t))_{t\geq 0}$ are C_0-semigroups on X_1^A and X_{-1}^A, respectively, with generators $A_1 := A|_{D(A^2)}$ and A_{-1}, respectively.*

Proof. See Proposition 2.10.4 in [TuWe09]. Note that in particular $D(A)$ is $T(\cdot)$-invariant, i.e. $T(t)D(A) \subseteq D(A)$ for every $t \geq 0$. □

Assuming that $B \in \mathcal{B}(U, X_{-1}^A)$ and $C \in \mathcal{B}(X_1^A, Y)$ we can make sense of the *control system* $\Sigma = (A, B, C, D)$

$$\begin{aligned} \frac{d}{dt}x(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t), \quad t \geq 0. \end{aligned}$$

Namely we may interpret the first equation $\dot{x} = Ax + Bu$ as an equation on the extrapolation space X_{-1}^A where $B \in \mathcal{B}(U, X_{-1}^A)$ is a bounded input, so that for all $u \in L_{1,loc}(\mathbb{R}_+; U)$ and $x_0 \in X \subseteq X_{-1}^A$ the mild solution in X_{-1}^A is given by

$$x(t) = T(t)x_0 + \int_0^t T_{-1}(t-s)Bu(s)ds, \quad t \geq 0.$$

Even for initial values $x_0 \in X$ this function not necessarily has values $x(t)$ which also lie in X since $Bu(s) \in X_{-1}^A$ only lies in the extrapolation space, in general. Therefore, one introduces the following notion of admissibility.

Definition 2.3.5. *Let $p \in (1,\infty)$ and $B \in \mathcal{B}(U, X_{-1}^A)$ be given. Then the input operator B is called* p-admissible *if for all $t > 0$ the map*

$$\Phi_t : \quad L_p(0,t;U) \to X_{-1}, \quad u \mapsto \int_0^t T_{-1}(t-s)Bu(s)ds$$

has a range ran $\Phi_t \subset X$ *which lies in X. In that case the maps Φ_t are bounded linear operators mapping from $L_p(0,t;U)$ to X.*

Proof. See Proposition 4.2.2 in [TuWe09]. □

Remark 2.3.6. *For an admissible input operator $B \in \mathcal{B}(U; X_{-1}^A)$ and the corresponding maps Φ_t $(t > 0)$ we may define $\Phi : \mathbb{R}_+ \to \mathcal{B}(L_{p,loc}(\mathbb{R}_+;U);X)$ as*

$$\Phi(t)f := \Phi_t f|_{(0,t)} \tag{2.11}$$

where

$$L_{p,loc}(\mathbb{R}_+;U) = \{f : \mathbb{R}_+ \to U : \ f|_{(0,t)} \in L_p(0,t;U) \ (t > 0)\}.$$

We see that for $s, t \geq 0$ we obtain

$$\begin{aligned}
\Phi(t+s)f &= \Phi_{t+s} f|_{(0,t+s)} \\
&= \int_0^{t+s} T_{-1}(t+s-r)f(r)dr \\
&= \int_0^s T_{-1}(t+s-r)f(r)dr + \int_s^{t+s} T_{-1}(t+s-r)f(r)dr \\
&= T_{-1}(t)\int_0^s T_{-1}(s-r)f(r)dr + \int_{0^t} T^{-1}(t-r')f(r'+s)dr' \\
&= T(t)\Phi(s)f + \Phi(t)f(\cdot + s).
\end{aligned}$$

Similarly the second equation $y = Cx + Du$ makes sense if we take $x \in C(\mathbb{R}_+; X_1^A)$ and assume that $C \in \mathcal{B}(X_1^A, Y)$. In this case an admissibility condition concerns the unique continuation property of the state-output map. Namely given $x_0 \in D(A) = X_1^A$ and neglecting the input $u = 0$, we have that the output is then given as

$$y(t) = CT(t)x_0, \quad t \geq 0, \ x_0 \in X_1^A.$$

Definition 2.3.7. *Let $p \in (1,\infty)$ and $C \in \mathcal{B}(X_1^A, Y)$ be given. Then the output operator C is called* p-admissible *if for all $t > 0$ the map*

$$\Psi_t : \quad X_1^A \to L_p(0,t;Y), \quad x_0 \mapsto CT(\cdot)x_0$$

has a continuous extension to an operator $\Psi_t \in \mathcal{B}(X, L_p(0,t;Y))$.

Remark 2.3.8. *The maps Ψ_t are compatible in the sense that*

$$\Psi_s x_0 = [\Psi_t x_0]\,|_{(0,s)}, \quad 0 \leq s \leq t, \quad x_0 \in X.$$

Therefore, the definition of $\Psi : \mathbb{R}_+ \to \mathcal{B}(X; L_{p,loc}(Y))$ via

$$\Psi(t)x_0 := (\Psi_\tau x_0)(t), \quad \tau > 0, \ x_0 \in X \quad a.e. \ t \in (0,\tau) \tag{2.12}$$

makes sense. For this map we have

$$
\begin{aligned}
(\Psi(t+s)x_0)(r) &= (\Psi_\tau x_0)(r) \\
&= CT(r)x_0 \\
&= \begin{cases} CT(r-t)(T(t)x_0), & r \in [t,t+s] \\ CT(r)x_0, & r \in [0,t] \end{cases} \\
&= \begin{cases} (\Psi_s(T(t)x_0))(r-t), & r \in [t,t+s] \\ (\Psi(t)x_0)(r), & r \in [0,t] \end{cases}
\end{aligned} \tag{2.13}
$$

for every $x_0 \in X_1^A$ and by approximation the equality holds for every $x_0 \in X$.

The semigroup $(T(t))_{t\geq 0}$ and – for admissible input operator $B \in \mathcal{B}(U; X_{-1}^A)$ and output operator $C \in \mathcal{B}(X_1^A; Y)$ – the families of maps $(\Phi(t))_{t\geq 0}$ and $(\Psi(t))_{t\geq 0}$ are almost enough to describe the dynamics of the control system Σ. In fact, if $B \in \mathcal{B}(U; X)$ and $C \in \mathcal{B}(X; Y)$ are admissible, one needs a forth family of maps $(F(t))_{t\geq 0}$ such that for every input function $u \in L_{p,loc}(\mathbb{R}_+; U)$ and initial value $x_0 \in X$ the mild solution of the system $\Sigma = (A, B, C, D)$ is given by

$$
\begin{aligned}
x &= T(\cdot)x_0 + \Phi(\cdot)u \\
y &= \Psi(\cdot)x_0 + F(\cdot)u.
\end{aligned}
$$

Here we only give a formula for $F(t)$ in the scenario where the input operator $B \in \mathcal{B}(U; X)$ and the output operator $C \in \mathcal{B}(X; Y)$ are both bounded. Then define

$$
(F_t u)(s) := Du(s) + \int_0^s C(s-r)Bu(r)dr, \quad t > 0,\ u \in L_p(0,t;U) \quad \text{a.e. } s \in (0,t)
$$

with defines a map $F_t : L_p(0,t;U) \to L_p(0,t;Y)$ and similar to the definition of Ψ one may set

$$
(Ff)(s) := (F_t f|_{(0,t)})(s), \quad f \in L_{p,loc}(\mathbb{R}_+; U),\ t > 0, \quad \text{a.e. } s \in (0,t)
$$

defining a map $F : L_{p,loc}(\mathbb{R}_+; U) \to L_{p,loc}(\mathbb{R}_+; Y)$.
In the following we introduce the notion of a well-posed linear system and for any two functions $f, g : \mathbb{R}_+ \to Z$ and $t \geq 0$ we write $f \underset{t}{\diamondsuit} g$ for the function

$$
(f \underset{t}{\diamondsuit} g)(s) = \begin{cases} f(s), & s \in [0,t] \\ g(s-t), & s > t. \end{cases}
$$

Definition 2.3.9. *Let $p \in [1, \infty)$ be fixed. The quadruple*

$$
\Sigma = \begin{pmatrix} T(\cdot) & \Phi(\cdot) \\ \Psi(\cdot) & F(\cdot) \end{pmatrix}
$$

*is called a (L_p-)*well-posed linear system *if the following hold:*

1. *$(T(t))_{t\geq 0} \subset \mathcal{B}(X)$ is a C_0-semigroup on X with generator A,*
2. *$\Phi(t) \in \mathcal{B}(L_{p,loc}(\mathbb{R}_+; U); X)$ $(t \geq 0)$ such that*

$$
\Phi(t+s)(u \underset{s}{\diamondsuit} v) = T(t)\Phi(s)u + \Phi(t)v
$$

3. $\Psi(t) \in \mathcal{B}(X; L_{p,loc}(\mathbb{R}_+; Y))$ $(t \geq 0)$ *such that*

$$\Psi(t+s)x_0 = \Psi(t)x_0 \underset{t}{\diamondsuit} \Psi(s)T(t)x_0, \quad \Psi(0) = 0$$

4. $F(t) \in \mathcal{B}(L_{p,loc}(\mathbb{R}_+; U); L_{p,loc}(\mathbb{R}_+; Y))$ $(t \geq 0)$ *such that*

$$F(t+s)(u \underset{t}{\diamondsuit} v) = F(t)u \underset{t}{\diamondsuit} (\Psi(s)\Phi(t)u + F(s)v), \quad F(0) = 0.$$

For well-posed linear systems we may define the notion of impedance passivity.

Definition 2.3.10. *A well-posed linear system* $\Sigma = (T, \Phi, \Psi, F)$ *is called* impedance passive *if* $U = Y$ *and*

$$\frac{1}{2}\|x(t)\|_X^2 \leq \frac{1}{2}\|x(0)\|_X^2 + \int_0^t \operatorname{Re}\langle y(s), u(s)\rangle_U ds$$

where $x = T(\cdot)x_0 + (\Phi(T)u)(t)$ *and* $y(t) = (\Psi x_0 + F(t)u)(t)$*, for every* $t \geq 0$ *and* $u \in L_{p,loc}(\mathbb{R}_+; U)$*. Moreover, the system is called* impedance energy preserving *if*

$$\frac{1}{2}\|x(t)\|_X^2 = \frac{1}{2}\|x(0)\|_X^2 + \int_0^t \operatorname{Re}\langle y(s), u(s)\rangle_U ds$$

or scattering passive *if*

$$\frac{1}{2}\|x(t)\|_X^2 \leq \frac{1}{2}\|x(0)\|_X^2 + \int_0^t \|u_c(s)\|_U^2 - \|y_c(s)\|_Y^2 \, ds$$

(not necessarily $U = Y$*) hold instead.*

Remark 2.3.11. *The norms* $\|F(t)\|$ *are non-decreasing in* $t \geq 0$*, but in general it may happen that*

$$\lim_{t\to 0}\|F(t)\| = \inf_{t>0}\|F(t)\| > 0.$$

Example 2.3.12 (Boundary Control of the Uniform 1D Wave Equation)**.** *Consider the one-dimensional wave equation with constant coefficients* $\rho, T > 0$

$$\rho\omega_{tt}(t,\zeta) = (T\omega_\zeta)_\zeta(t,\zeta), \quad t \geq 0, \ \zeta \in (0,1).$$

We set $c = \sqrt{\frac{T}{\rho}}$ *and calculate* $F(t)$ *for the following choice of the input and output function*

$$u(t) = T\omega_\zeta(t,0), \quad y(t) = \omega_t(t,0), \quad t \geq 0$$

and some conservative or dissipative boundary condition at the right end, say

$$\omega_t(t,1) = 0, \quad t \geq 0. \tag{2.14}$$

Our starting point is the d'Alembert solution formula for the wave equation on $\mathbb{R}$ *which is given by*

$$\omega(t,\zeta) = \frac{f(\zeta+ct) + f(\zeta-ct)}{2} + \frac{g(\zeta+ct) - g(\zeta-ct)}{2}, \quad t \geq 0, \ \zeta \in \mathbb{R}$$

where $f, g \in W^2_{p,loc}(\mathbb{R}_+)$ are two functions determined by the initial condition $(\omega, \omega_t) = (\omega_0, \omega_1) \in W^2_{p,loc}(\mathbb{R}_+) \times W^1_{p,loc}(\mathbb{R}_+)$. If we assume that the input function $u \in W^1_{p,loc}(\mathbb{R}_+)$ has sufficient regularity properties, then the solution ω has the same form, but the functions $f, g \in W^2_{p,loc}(0,1)$ have to be extended to functions $f, g \in W^2_{p,loc}(\mathbb{R}_+)$ which outside of $(0,1)$ may also depend on the input function $u \in W^1_{p,loc}(\mathbb{R}_+)$. Indeed, we calculate

$$\begin{aligned}
\omega_t(t,\zeta) &= c\frac{f'(\zeta+ct)-f'(\zeta-ct)}{2} + c\frac{g'(\zeta+ct)+g'(\zeta-ct)}{2} \\
T\omega_\zeta(t,\zeta) &= T\frac{f'(\zeta+ct)+f'(\zeta-ct)}{2} + T\frac{g'(\zeta+ct)-g'(\zeta-ct)}{2}, \\
u(t) = T\omega_\zeta(t,0) &= T\frac{f'(ct)+f'(-ct)}{2} + T\frac{g'(ct)-g'(-ct)}{2} \\
y(t) = \omega_t(t,0) &= c\frac{f'(ct)-f'(-ct)}{2} + c\frac{g'(ct)+g'(-ct)}{2}, \quad t \geq 0
\end{aligned}$$

and using the initial condition $(\omega, \omega_t)(t,\cdot) = (\omega_0, \omega_1)$ we find that

$$\begin{aligned}
\omega_0(\zeta) &= \omega(0,\zeta) = f(\zeta) \\
\omega_1(\zeta) &= \omega_t(0,\zeta) = cg'(\zeta), \quad \zeta \in [0,1].
\end{aligned}$$

We then deduce that for a.e. $s \in [0,1]$ we have

$$\begin{aligned}
f(-s) - g(-s) &= (f-g)(0) - \int_0^s (f'(-r) - g'(-r))dr \\
&= \omega_0(0) - g(0) - 2T\int_0^s u(r/c)dr - \int_0^s (f'+g')(r)dr \\
&= \omega_0(0) - g(0) - 2T\int_0^s u(r/c)dr - \int_0^s \omega_0'(r) + \frac{1}{c}\omega_1(r)dr.
\end{aligned}$$

and hence

$$\omega(t,\zeta) = \begin{cases} \frac{\omega_0(\zeta+ct)+\omega_0(\zeta-ct)}{2} + \frac{1}{2c}\int_{\zeta-ct}^{\zeta+ct}\omega_1(s)ds, & \zeta \in (ct, 1-ct) \\ \frac{\omega_0(\zeta+ct)+\omega_0(0)}{2} + \frac{1}{2c}\int_{\zeta-ct}^{\zeta+ct}\omega_1(|s|)ds & \\ \quad +c^{-1}\int_0^{ct-\zeta} u(s/c)ds, & \zeta \in [0, ct] \end{cases}$$

so that in particular

$$\begin{aligned}
y(t) &= \omega_t(t,0) \\
&= \omega_0'(ct) + \omega_1(ct) + c^{-1}u(t), \quad t \in [0, c^{-1}).
\end{aligned}$$

It can be easily seen that the dynamics of this one-dimensional wave equation with boundary input and output give a well-posed linear system and we obtain that

$$(F(t)u)(s) = \mathbf{1}_{s \leq t}u(s), \quad t \in [0, c^{-1}), \ u \in L_{p,loc}(\mathbb{R}_+), \ a.e.\ s \geq 0.$$

In particular, $\|F(t)\|_{\mathcal{B}(L_p(0,t))} = c^{-1}$ for every $t \in (0, c^{-1})$ and hence

$$\inf_{t>0} \|F(t)\|_{\mathcal{B}(L_p(0,t))} = c^{-1} > 0.$$

However, within this thesis we will take a different approach which is closer to the physical interpretation of the systems later on, namely systems which behave according to some PDE on some domain and which can be manipulated via control and observation at the boundary. Therefore, these systems naturally fit into the setting of Boundary Control and Observation Systems.

Definition 2.3.13 (Boundary Control and Observation System)**.** *Let X, U and Y be Hilbert spaces. A triple $\mathfrak{S} = (\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ of linear operators $\mathfrak{A} : D(\mathfrak{A}) \subseteq X \to X$, $\mathfrak{B} : D(\mathfrak{B}) \subseteq X \to U$ and $\mathfrak{C} : D(\mathfrak{C}) \subseteq X \to Y$ is called a* Boundary Control and Observation System *if it has the following properties.*

1. *$D(\mathfrak{A}) \subseteq D(\mathfrak{B}), D(\mathfrak{C})$, i.e. $\mathfrak{B}$ and $\mathfrak{C}$ may only have larger domains than $\mathfrak{A}$,*
2. *the restriction $A = \mathfrak{A}|_{\ker \mathfrak{B} \cap D(\mathfrak{A})}$ of $\mathfrak{A}$ to the kernel of $\mathfrak{B}$ generates a C_0-semigroup $(T(t))_{t\geq 0}$ on X,*
3. *there is a right-inverse $B \in \mathcal{B}(U, X)$ of $\mathfrak{B}$ such that*
$$\text{ran } B \subseteq D(\mathfrak{A}), \ \mathfrak{A}B \in \mathcal{B}(U, X), \ \mathfrak{B}B = I.$$
4. *$\mathfrak{C}$ is bounded from $D(A)$ to Y where $D(A)$ is equipped with the graph norm of A.*

Moreover, a pair $(\mathfrak{A}, \mathfrak{B})$ as above with the first three properties is called Boundary Control System*.*

We interpret a Boundary Control and Observation System $(\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ as the operators in the following evolutionary system.

$$\begin{aligned} \frac{d}{dt}x(t) &= \mathfrak{A}x(t) \\ x(0) &= x_0 \\ \mathfrak{B}x(t) &= u(t) \\ \mathfrak{C}x(t) &= y(t), \quad t \geq 0 \end{aligned} \tag{2.15}$$

where $x_0 \in X$ is the initial state of the system, $u \in L_{1,loc}(\mathbb{R}_+; U)$ is a given input function (control function) and $y : \mathbb{R}_+ \to Y$ is the (unknown) output function (observation function). One may define classical and mild solutions of Boundary and Control and Observation Systems as follows.

Definition 2.3.14. *Let $\mathfrak{S} = (\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ be a Boundary Control and Observation System. If $x_0 \in D(\mathfrak{A})$ and $u \in C^2(\mathbb{R}_+; U)$, then a pair $x \in C^1(\mathbb{R}_+; X) \cap C(\mathbb{R}_+; D(\mathfrak{A}))$ and $y \in C(\mathbb{R}_+; Y)$ solving* (2.15) *is called* classical solution *of the Boundary Control and Observation System $\mathfrak{S}$.*

Theorem 2.3.15. *Let $\mathfrak{S} = (\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ be a Boundary Control and Observation System. For every given $x_0 \in D(\mathfrak{A})$ and $u \in C^2(\mathbb{R}_+; U)$ with $\mathfrak{B}x_0 = u(0)$, the unique classical solution of the Boundary Control System is given by*

$$\begin{aligned} x(t) &= T(t)(x_0 - Bu(0)) + \int_0^t T(t-s)(\mathfrak{A}Bu(s) - B\dot{u}(s))ds + Bu(t), \\ y(t) &= \mathfrak{C}x(t), \qquad t \geq 0. \end{aligned}$$

Proof. See Theorem 11.2.1 in [JaZw12]. □

The following result can be found in Section 13.1 in [JaZw12].

Lemma 2.3.16. *Assume that $(\mathfrak{A}, \mathfrak{B})$ is a Boundary Control System and let $x_0 \in D(\mathfrak{A})$ and $u \in C^1([0,\tau]; U)$. Then the Boundary Control System has a* unique mild solution *$x \in C([0,\tau]; X)$ which can be written as*

$$x(t) = T(t)x_0 + \int_0^t T(t-s)\mathfrak{A}Bu(s)ds - A_{-1}\int_0^t T(t-s)Bu(s)ds, \quad t \geq 0$$

where $(T(t))_{t\geq 0}$ is the semigroup generated by $A = \mathfrak{A}|_{D(\mathfrak{A})\cap \ker \mathfrak{B}}$.

Also for Boundary Control and Observation Systems we may define the terminology well-posedness.

Definition 2.3.17. *A Boundary Control and Observation System $\mathfrak{S} = (\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ is called* well-posed *if there are constants $\tau > 0$ and $m_\tau > 0$ such that for every initial value $x_0 \in D(\mathfrak{A})$ and input function $u \in C^2([0,\tau]; U)$ with $\mathfrak{B}x_0 = u(0)$ one has the estimate*

$$\|x(\tau)\|_X^2 + \|y\|_{L_2(0,\tau)}^2 \leq m_\tau \left(\|x_0\|_X^2 + \|u\|_{L_2(0,\tau)}^2\right). \tag{2.16}$$

Theorem 2.3.18. *If a Boundary Control and Observation System is well-posed, then for every $\tau > 0$ there is a constant $m_\tau > 0$ such that (2.16) holds. Moreover, every well-posed Boundary Control and Observation system may be equivalently described as a well-posed linear system (T, Φ, Ψ, F).*

Proof. See Theorem 13.1.7 in [JaZw12] for the first statement and its proof and Definition 13.1.8 in [JaZw12] for the formulation as a well-posed linear system. □

For scattering passive system one gets well-posedness for free.

Proposition 2.3.19. *If every classical solution of the Boundary Control System satisfies*

$$\frac{d}{dt}\|x(t)\|_X^2 \lesssim \|u(t)\|^2 - \|y(t)\|_Y^2$$

then the system is well-posed.

Proof. Integrating the inequality from 0 to τ we find

$$\|x(t)\|_X^2 - \|x_0\|_X^2 \lesssim \int_0^\tau \|u(t)\|_U^2\, dt - \int_0^\tau \|y(t)\|_Y^2\, dt.$$

From here well-posedness follows easily. □

Let us also introduce the concept of a *transfer function* which is closely related to the Laplace transform of the semigroup $(T(t))_{t\geq 0}$ generated by A. (For more background on transfer functions we refer to Chapter 12 in [JaZw12].)

Definition 2.3.20 (Transfer function)**.** *Consider the abstract boundary control and observation problem*

$$\begin{aligned} \frac{d}{dt}x(t) &= \mathfrak{A}x(t) \\ u(t) &= \mathfrak{B}x(t) \\ y(t) &= \mathfrak{C}x(t), \quad t \geq 0 \end{aligned}$$

where $\mathfrak{S} = (\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ *is a Boundary Control and Observation system and let* $\lambda \in \mathbb{F}$. *We write* $\lambda \in D(G)$ *if there is* $G(\lambda) \in \mathcal{B}(U, Y)$ *such that for all* $u \in U$ *there is a unique solution of*

$$\begin{aligned} \lambda x &= \mathfrak{A}x \\ u &= \mathfrak{B}x \\ y &= \mathfrak{C}x \end{aligned} \tag{2.17}$$

where $x \in D(\mathfrak{A})$ *and* $y \in Y$ *is given by* $y = G(\lambda)u$.

Remark 2.3.21. *It is known (Theorem 3.6 in [We94]) that for well-posed linear systems the Laplace transform of* $F(t)$ *coincides with the transfer function* $G(\lambda)$ *on some complex right-half plane.*

Chapter 3

Hyperbolic Partial Differential Equations on a One-dimensional Spatial Domain

Throughout this thesis we are concerned with evolution equations of the form

$$\frac{\partial}{\partial t}x(t,\zeta) = \sum_{k=0}^{N} P_k \frac{\partial^k}{\partial \zeta^k}(\mathcal{H}x)(t,\zeta), \quad t \geq 0,\ \zeta \in (0,1). \tag{3.1}$$

The evolution of the system always depends on suitable boundary conditions we introduce later on and which may be static or dynamic, i.e. boundary conditions determined by a feedback via a dynamic control system which itself is governed by an evolution equation (usually an ODE). Step by step we are going to introduce the assumptions we impose on the choice of the matrix-valued function $\mathcal{H}$ and the matrices P_k $(k = 0,1,\dots,N)$. However, let us first begin with $x(t,\zeta)$ which has the following interpretation. First of all, let us take $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ to be either the real or the complex field. Unless stated otherwise, all results in this thesis hold for both choices of $\mathbb{F}$, however note that when it comes to technical real life applications the choice $\mathbb{F} = \mathbb{R}$ very often makes more sense, although this is not necessarily true in all the cases. In fact, for transmission lines in electronic circuits usually the choice $\mathbb{F} = \mathbb{C}$ is more practical. Then $x(t,\zeta) \in \mathbb{F}^d$ describes the state of a certain object, e.g. the displacement of a string or beam, in time $t \geq 0$ and at position $\zeta \in (0,1)$. Here $d \in \mathbb{N} = \{1,2,\dots\}$ denotes any natural number. For $\mathcal{H}$ let us for the moment assume that $\mathcal{H} = I \in \mathbb{F}^{d\times d}$ is the identity matrix, which will be a legitimate choice later on, so that the evolution equation simplifies to

$$\frac{\partial}{\partial t}x(t,\zeta) = \sum_{k=0}^{N} P_k \frac{\partial^k}{\partial \zeta^k}x(t,\zeta), \quad t \geq 0,\ \zeta \in (0,1). \tag{3.2}$$

Let us introduce the objects P_k $(k = 0,1,\dots,N)$ next. First of all we need to fix the *order* $N \in \mathbb{N}$ of the system. Then the evolution equation is a first order evolution equation in time where we expect the right hand side to be an differential operator of order $N \in \mathbb{N}$. Thus, we assume that $P_k \in \mathbb{F}^{d\times d} = \mathcal{B}(\mathbb{F}^d; \mathbb{F}^d)$ $(k = 0, 1, \dots, N)$ are matrices, which we identify with the (bounded) linear operators on $\mathbb{F}^d$. Also let us assume that $P_N \in \mathbb{F}^{d\times d}$ is invertible. (For the case of non-invertible $P_N \in \mathbb{F}^{d\times d}$ we refer to Section 6 in the PhD thesis [Vi07].) Then the right hand side plus a suitable domain actually give a differential operator for different choices of the function space X in which the functions $x(t) := x(t, \cdot)$ should lie. One could think of $X = C([0, 1]; \mathbb{F}^d)$ to be the Banach space of continuous $\mathbb{F}^d$-valued functions or $X = L_p(0, 1; \mathbb{F}^d)$ to be the Banach space of (equivalence classes of) $\mathbb{F}^d$-valued L_p-functions (where $p \in [1, \infty]$). Indeed, depending on the particular application the choices $X = C([0, 1]; \mathbb{F}^d)$, $L_1(0, 1; \mathbb{F}^d)$ or $L_\infty(0, 1; \mathbb{F}^d)$ can be very intuitive, e.g. for a string we might assume that it has a smooth slope, in the sense that there are no jumps, so the choice $X = C([0, 1]; \mathbb{F}^d)$ makes a lot of sense, while in the case of a transport equation where $x(t, \zeta)$ models the mass density at $\zeta \in (0, 1)$ in time $t \geq 0$ it might make more sense to consider $X = L_1(0, 1; \mathbb{F}^d)$ where $\|\cdot\|_{L_1}$ is related to the total mass in the system. However, as will be emphasised below we put our focus on the *energy* of a system which in our case will be given by a quadratic functional, namely (up to a constant and, later on, weights) the square of the L_2-norm $\|\cdot\|_{L_2}^2$. So assume that $x(t, \cdot) \in L_2(0, 1; \mathbb{F}^d)$ is a square-integrable function (for all $t \geq 0$) and (formally) consider the change of the energy $H(t) := \frac{1}{2}\|x(t, \cdot)\|_{L_2}^2$ whilst the time evolution $t \in \mathbb{R}_+ := \{s \in \mathbb{R} : \ s \geq 0\}$. For sufficiently smooth solutions we formally get

$$\frac{d}{dt}\frac{1}{2}\|x(t, \cdot)\|_{L_2}^2 = \operatorname{Re}\langle \frac{\partial}{\partial t} x(t, \cdot), x(t, \cdot)\rangle_{L_2} = \sum_{k=0}^{N} \operatorname{Re}\langle P_k \frac{\partial^k}{\partial \zeta^k} x(t, \cdot), x(t, \cdot)\rangle_{L_2}.$$

Forgetting about boundary conditions for the moment we assume that $x(t, \cdot) \in C_c^\infty(0, 1; \mathbb{F}^d) := \{g \in C^\infty(0, 1; \mathbb{F}^d) : \ \operatorname{supp} g \text{ is compact}\}$. Then for the right hand side we obtain via integration by parts that

$$\begin{aligned}
&2\sum_{k=0}^{N} \operatorname{Re}\langle P_k \frac{\partial^k}{\partial \zeta^k} x(t, \cdot), x(t, \cdot)\rangle_{L_2} \\
&= \sum_{k=0}^{N} \langle P_k \frac{\partial^k}{\partial \zeta^k} x(t, \cdot), x(t, \cdot)\rangle_{L_2} + \langle x(t, \cdot), P_k \frac{\partial^k}{\partial \zeta^k} x(t, \cdot)\rangle_{L_2} \\
&= \sum_{k=0}^{N} \langle P_k \frac{\partial^k}{\partial \zeta^k} x(t, \cdot), x(t, \cdot)\rangle_{L_2} + (-1)^k \langle \frac{\partial^k}{\partial \zeta^k} x(t, \cdot), P_k x(t, \cdot)\rangle_{L_2} \\
&= \sum_{k=0}^{N} \langle (P_k + (-1)^k P_k^*) \frac{\partial^k}{\partial \zeta^k} x(t, \cdot), x(t, \cdot)\rangle_{L_2}.
\end{aligned}$$

For the differential operator to be formally skew-symmetric (as the title of this chapter indicates) on $X = L_2(0, 1; \mathbb{F}^d)$ we choose the matrices $P_k \in \mathbb{F}^{d\times d}$ such that $P_k^* = (-1)^{k+1} P_k$ $(k = 0, 1, \dots, N)$ are skew-adjoint (for $k \in 2\mathbb{N}$ even) or self-adjoint (for $K \in 2\mathbb{N} + 1$ odd) (with respect to the standard inner product on $\mathbb{F}^d$), respectively. Finally let us come to the case where $\mathcal{H}$ does not necessarily equal the identity matrix. The idea is illustrated in the examples within the next subsection, however, for the moment one might think of $\mathcal{H}$ as a kind of energy density, i.e. if $x(t,\cdot)$ takes the same value in two different regions, e.g. (disjoint) subintervals $(a,b),(c,d) \subseteq (0,1)$ of same length, it still might be possible that the regions contribute differently to the total energy which now is given as a weighted integral of $|x(t,\zeta)|^2$, i.e.

$$H(t) = \frac{1}{2}\int_0^1 \langle x(t,\zeta), \mathcal{H}(\zeta)x(t,\zeta)\rangle_{\mathbb{F}^d} d\zeta.$$

From here we may already derive the first assumptions on $\mathcal{H}$ which are intuitive. First of all, an energy $H(t)$ should be real for all $x(t) = x(t,\cdot) \in L_2(0,1;\mathbb{F}^d)$. Therefore, on the one hand (for the integral to always exist) $\mathcal{H} \in L_\infty(0,1;\mathbb{F}^{d\times d})$ should be an essentially bounded measurable function and (for the integral to be real) $\mathcal{H}(\zeta) = \mathcal{H}(\zeta)^*$ should be self-adjoint for a.e. $\zeta \in (0,1)$. In fact, as Lemma 2.1.7 – applied to the imaginary part of $H(t)$ – shows, in the complex case $\mathbb{F} = \mathbb{C}$ for $H(t)$ to be real valued for every possible choice of $x(t) \in L_2(0,1;\mathbb{C}^d)$ it is necessary that $\mathcal{H}(\zeta)$ is self-adjoint for a.e. $\zeta \in (0,1)$. Secondly, if we think of $x = 0$ as being the unique equilibrium of the system, $H(t) > 0$ should be strictly positive whenever $x \neq 0$, so $z^*\mathcal{H}(\zeta)z > 0$ for all $z \in \mathbb{F}^d,\ z \neq 0$, and a.e. $\zeta \in (0,1)$. Also we additionally assume that a measurable function $x : [0,1] \to \mathbb{F}^d$ should lie in $L_2(0,1;\mathbb{F}^d)$ if and only if it has finite energy. This leads to the assumption that there is $m > 0$ such that

$$\langle z, \mathcal{H}(\zeta)z\rangle_{\mathbb{F}^d} \geq m\,|z|^2, \quad z \in \mathbb{F}^d,\ \text{a.e. } \zeta \in (0,1).$$

Later on we use the following notation for estimates like this.

Remark 3.0.22. *For any matrices $M_1, M_2 \in \mathbb{F}^{m\times m}$ we write $M_1 \geq M_2$ if both matrices $M_1 = M_1^*$ and $M_2 = M_2^*$ are symmetric and*

$$\langle z, M_1 z\rangle_{\mathbb{F}^m} \geq \langle z, M_2 z\rangle_{\mathbb{F}^m}, \quad z \in \mathbb{F}^d.$$

In particular, for a symmetric matrix $M \in \mathbb{F}^{m\times m}$ we write $M \geq \alpha I$ where $\alpha \in \mathbb{R}$ if

$$\langle z, Mz\rangle_{\mathbb{F}^m} \geq \alpha\,|z|^2, \quad z \in \mathbb{F}^d.$$

Note that $M \geq 0$ if and only if the matrix M is positive semi-definite. Further we write $M_1 > M_2$ for matrices $M_1, M_2 \in \mathbb{F}^{m\times m}$ such that $M_1 \geq M_2 + \varepsilon I$ for some $\varepsilon > 0$. In this case $M > 0$ if and only if the matrix is (strictly) positive definite.

Coming back to the differential operator in the evolution equation (3.1) above we see that if X is equipped with the standard inner product $\langle\cdot,\cdot\rangle_{L_2}$, the operator $\sum_{k=0}^N P_k \frac{\partial^k}{\partial\zeta^k}(\mathcal{H}\cdot)$ is not formally skew-symmetric any more, because via integration by parts of the terms (assuming that $\mathcal{H}x \in C_c^\infty(0,1;\mathbb{F}^d)$)

$$\sum_{k=0}^N \operatorname{Re}\langle P_k \frac{\partial^k}{\partial\zeta^k}(\mathcal{H}x)(t,\cdot), x(t,\cdot)\rangle_{L_2}$$

in general not all the terms cancel. Therefore, we adjust the inner product on $X = L_2(0,1;\mathbb{F}^d)$ properly, namely we choose

$$\langle\cdot,\cdot\rangle_X = \langle\cdot,\cdot\rangle_{\mathcal{H}} = \langle\cdot,\mathcal{H}\cdot\rangle_{L_2},$$

i.e. for all $x, y \in X = L_2(0,1;\mathbb{F}^d)$ we have

$$\langle x,y\rangle_X = \langle x,y\rangle_{\mathcal{H}} = \int_0^1 \langle x(\zeta), \mathcal{H}(\zeta)y(\zeta)\rangle_{\mathbb{F}^d} d\zeta.$$

Remark 3.0.23. *Note that we choose inner products in such a way that they are linear in the second and anti-linear in the first component. This convention is quite standard in the context of physics, especially quantum mechanics, however in mathematics very often the convention is the other way round, i.e. linearity in the first components and so on.*

We gather all the assumptions made so far in

Assumption 3.0.24. *We consider systems which are described by an evolution equation of the form*

$$\frac{\partial}{\partial t}x(t,\zeta) = \sum_{k=0}^{N} P_k \frac{\partial^k}{\partial \zeta^k}(\mathcal{H}x)(t,\zeta), \quad t \geq 0 \tag{3.3}$$

where the energy state space variable $x(t) := x(t,\cdot)$ is assumed to be a $\mathbb{F}^d$-valued L_2-function, i.e. $x(t) \in L_2(0,1;\mathbb{F}^d)$ for all $t \geq 0$. $N \in \mathbb{N}$ and $d \in \mathbb{N}$ are assumed to be natural numbers, $P_k \in \mathbb{F}^{d\times d}$ $(k = 0,1,\ldots,N)$ matrices with P_N invertible and $\mathcal{H} : [0,1] \to \mathbb{F}^{d\times d}$ a measurable function such that the following holds.

- $P_k^* = (-1)^{k+1}P_k$ *for* $k = 0,1,\ldots,N$.
- P_N *is invertible*
- *There are* $0 < m \leq M < +\infty$ *such that for all* $z \in \mathbb{F}^d$ *and a.e.* $\zeta \in (0,1)$

$$m\,|z|^2 \leq z^*\mathcal{H}(\zeta)z \leq M\,|z|^2 .$$

Then the energy state space X *is the Hilbert space* $L_2(0,1;\mathbb{F}^d)$ *equipped with the inner product* $\langle\cdot,\cdot\rangle_X = \langle\cdot,\cdot\rangle_{\mathcal{H}}$.

Remark 3.0.25. *In the context of infinite dimensional port-Hamiltonian systems authors very often use the energy state space $X = L_2(a,b;\mathbb{F}^d)$ with $a < b$ real numbers and thus put more emphasis on the physical interpretation of the interval $I = (a,b)$. However, the restriction to the case $I = (0,1)$ may be done without loss of generality since for given real numbers $a < b$ and a measurable $\tilde{\mathcal{H}} : [a,b] \to \mathbb{F}^{d\times d}$ with $\tilde{m}I \leq \tilde{\mathcal{H}}(\zeta) \leq \tilde{M}I$ for a.e. $\zeta \in (a,b)$, the map*

$$L_2(a,b;\mathbb{F}^d) =: \tilde{X} \ni \tilde{x} \mapsto x = \tilde{x}(\cdot(b-a)+a) \in X := L_2(0,1;\mathbb{F}^d)$$

where $\tilde{X} = L_2(a,b;\mathbb{F}^d)$ is equipped with the inner product

$$\langle f,g\rangle_{\tilde{X}} := \int_a^b \langle f(\zeta), \tilde{\mathcal{H}}(\zeta)g(\zeta)\rangle_{\mathbb{F}^d} d\zeta, \quad f,g \in \tilde{X}$$

is an isometric isomorphism whenever $X = L_2(0,1;\mathbb{F}^d)$ is equipped with $\langle\cdot,\cdot\rangle_{\mathcal{H}}$ for

$$\mathcal{H}(\zeta) = (b-a)\tilde{\mathcal{H}}(a+\zeta(b-a)), \quad \zeta \in (0,1).$$

However, note that if one is interested in explicit decay rates of the energy (for suitable boundary conditions) these may very well depend on the length $b-a$ of the interval.

3.1 Examples

So far, we did not motivate why we are interested in evolution equations of the particularly skew-symmetric form introduced above. Within this subsection we focus on this task and do so by giving some examples which fall under this structure and which appear from time to time again as possible applications to the abstract theory. We start with examples where $N = 1$ and only afterwards consider applications where $N = 2$.

Example 3.1.1 (Transport Equation)**.** *We consider a thin pipe of length $l > 0$, where in view of Remark 3.0.25 w.l.o.g. we may assume that $l = 1$, and water streaming though the pipe from right to left with a constant velocity $c > 0$. Then (cf. Abschnitt 1.2 in [ArUr00]) for $x(t,\zeta)$ describing the density of the water at the point $\zeta \in (0,1)$ in time $t \geq 0$ the PDE modelling the dynamics of the water is given as*

$$\frac{\partial}{\partial t}x(t,\zeta) = c\frac{\partial}{\partial \zeta}x(t,\zeta) + f(t,\zeta), \quad t \geq 0,\ \zeta \in (0,1)$$

where f describes possible sinks and sources of water, i.e. water flowing to and off the pipe. If we set $f = 0$, i.e. assume that no sinks or sources are present, the equation simplifies to the linear transport equation

$$\frac{\partial}{\partial t}x(t,\zeta) = \frac{\partial}{\partial \zeta}(c\ x(t,\zeta)), \quad t \geq 0,\ \zeta \in (0,1)$$

which for the choice $N = d = 1$, $\mathcal{H}(\zeta) = c$ (a.e. $\zeta \in (0,1)$) and $P_1 = 1$, $P_0 = 0$ is the simplest case of an evolution equation governed by a formally skew-symmetric differential operator on the interval $(0,1)$. Already the first generalisation is the case where the velocity of the water is not assumed to be constant throughout the pipe, but – due to the geometric properties of the pipe, e.g. spatial dependant diameter – depends on the spatial variable $\zeta \in (0,1)$ instead, in such a way that the velocity defines a measurable bounded, e.g. continuous, function $c : [0,1] \to \mathbb{R}$ such that $c(\zeta) \geq \varepsilon$ for some $\varepsilon > 0$ and a.e. $\zeta \in (0,1)$. In this slightly more sophisticated form the nonuniform linear transport equation then reads as

$$\frac{\partial}{\partial t}x(t,\zeta) = \frac{\partial}{\partial \zeta}(c(\zeta)x(t,\zeta)), \quad t \geq 0,\ \zeta \in (0,1) \tag{3.4}$$

with $\mathcal{H}(\zeta) = c(\zeta) \in \mathbb{F} = \mathbb{F}^{1\times 1}$ for this case. Moreover, the energy of the system then is given by

$$H(t) = \frac{1}{2}\int_0^1 c(\zeta)\,|x(t,\zeta)|^2\,d\zeta.$$

Example 3.1.2 (Wave Equation)**.** *We consider a nonuniform vibrating string, e.g. a string of a violin, and w.l.o.g. assume that (possibly after rescaling) it has length $l = 1$. By $\omega(t,\zeta)$ we denote the transverse displacement of the string at position $\zeta \in (0,1)$ in time $t \geq 0$. Here, for simplicity, we assume that the string only moves in a 2D-plane and all effects concerning change of the length of the string due to stretching are neglected. Then a simple model to describe the time evolution of the string is the wave equation*

$$\frac{\partial^2}{\partial t^2}\omega(t,\zeta) = \frac{1}{\rho(\zeta)}\frac{\partial}{\partial \zeta}\left(T(\zeta)\frac{\partial}{\partial \zeta}\omega(t,\zeta)\right), \quad t \geq 0,\ \zeta \in (0,1) \tag{3.5}$$

where $\rho(\zeta) > 0$ denotes the mass density of the string, or more precisely the mass density times the cross sectional area, at position $\zeta \in (0,1)$ and $T(\zeta) > 0$ denotes the Young's modulus of the string at position $\zeta \in (0,1)$. In the simplest case both $T(\zeta) = T_0$ and $\rho(\zeta) = \rho_0$ are constant (uniform string), but in general both $\rho, T : [0,1] \to \mathbb{R}$ (measurable) may depend on the space variable $\zeta \in (0,1)$, however (in our context) they should at least be bounded and uniformly positive, i.e.

$$\frac{1}{\varepsilon} \geq T(\zeta), \rho(\zeta) \geq \varepsilon, \quad a.e.\ \zeta \in (0,1) \tag{3.6}$$

for some $\varepsilon > 0$. Then the energy of the system consists of two parts.

$$E_{kin}(t) := \frac{1}{2} \int_0^1 \rho(\zeta) \left|\omega_t(t,\zeta)\right|^2 d\zeta \qquad \text{(kinetic energy)}$$
$$E_{pot}(t) := \frac{1}{2} \int_0^1 T(\zeta) \left|\omega_\zeta(t,\zeta)\right|^2 d\zeta \qquad \text{(potential energy).}$$

Obviously equation (3.5) does not take the form (3.3), therefore we change the observables describing the state $\omega(t,\zeta)$ of the system. Note that the (e.g. classical) solution space of (3.5) is invariant with respect to adding a constant $c \in \mathbb{F}$, i.e. for every solution $\{\omega(t,\cdot)\}_{t\geq 0}$ of (3.5), also $\{\omega(t,\cdot)+c\}_{t\geq 0}$ solves (3.5), so the evolution equation is not affected by adding or subtracting any constant $c \in \mathbb{F}$. (So far we did not speak about boundary conditions.) On the other hand, the slope of a solution $\{\omega(t,\cdot)\}_{t\geq 0}$ modulo a constant $c \in \mathbb{F}$, i.e. identifying solutions which only differ by a constant $c \in \mathbb{F}$, clearly is determined by $\{\omega_\zeta(t,\cdot)\}_{t\geq 0}$. Therefore, we may choose the following variables to reformulate the wave equation (3.5) as evolution equation of the form (3.3).

$$x(t,\zeta) := \begin{pmatrix} x_1(t,\zeta) \\ x_2(t,\zeta) \end{pmatrix} := \begin{pmatrix} \rho(\zeta)\omega_t(t,\zeta) \\ \omega_\zeta(t,\zeta) \end{pmatrix}$$

and then

$$\mathcal{H}(\zeta) := \begin{bmatrix} \rho^{-1}(\zeta) & \\ & T(\zeta) \end{bmatrix},$$
$$P_1 := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad P_0 := \begin{bmatrix} 0 & \\ & 0 \end{bmatrix}.$$

For this choice the dynamics of the wave equation (3.5) is equivalently described by the evolution equation (3.3).

Example 3.1.3 (Timoshenko Beam Equation, cf. Example 7.1.4 in [JaZw12])**.** *The next example is closely related to the wave equation before. Instead of a string we now consider a (more rigid) beam of length $l > 0$, w.l.o.g. $l = 1$ again. Additionally to the model before we also consider torsion forces which result from the beam not only moving up and down but also twisting a little bit. These twisting effects have been neglected for a string since the parameters connected to the torsion appearing below are relatively small compared to all other parameters of the system. However, a beam, e.g. a wooden or plastic beam, is much more difficult to twist, so that the forces resulting from this twisting cannot be neglected any more. Additionally to the transverse displacement $\omega(t,\zeta)$ we therefore also introduce the rotation angle $\phi(t,\zeta)$*

of a segment of the beam. The evolutionary dynamics are then described by the Timoshenko beam equations

$$\rho(\zeta)\frac{\partial^2}{\partial t^2}\omega(t,\zeta) = \frac{\partial}{\partial\zeta}\left(K(\zeta)\left(\frac{\partial}{\partial\zeta}\omega(t,\zeta) - \phi(t,\zeta)\right)\right),$$
$$I_\rho(\zeta)\frac{\partial^2}{\partial t^2}\phi(t,\zeta) = \frac{\partial}{\partial\zeta}\left(EI(\zeta)\frac{\partial}{\partial\zeta}\phi(\zeta)\right) + K(\zeta)\left(\frac{\partial}{\partial\zeta}\omega(t,\zeta) - \phi(t,\zeta)\right), \tag{3.7}$$

for $t \geq 0$ and $\zeta \in (0,1)$ where $\rho, I_\rho, EI, K : [0,1] \to \mathbb{R}$ are measurable functions and again both bounded and uniformly positive, and have the following physical interpretation. As before $\rho(\zeta)$ denotes the mass density times the cross sectional area whereas $I_\rho(\zeta), EI(\zeta)$ and $K(\zeta)$ are the rotatory moment of inertia of cross section, the product of Young's modulus and the moment of inertia, and the shear modulus, respectively. In this case the total energy of the system is given by

$$H(t) = \frac{1}{2}\int_0^1 \Big(K(\zeta)\left|\omega_\zeta(t,\zeta) - \phi(t,\zeta)\right|^2 + \rho(\zeta)\left|\omega_t(t,\zeta)\right|^2$$
$$+EI(\zeta)\left|\phi_\zeta(t,\zeta)\right|^2 + I_\rho(\zeta)\left|\phi_t(t,\zeta)\right|^2\Big)\,d\zeta$$

and from there the following energy state space variables are intuitive

$$\begin{aligned}
x_1(t,\zeta) &= \omega_\zeta(t,\zeta) - \phi(t,\zeta) && \textit{(shear displacement)}\\
x_2(t,\zeta) &= \rho(\zeta)\omega_t(t,\zeta) && \textit{(momentum)}\\
x_3(t,\zeta) &= \phi_\zeta(t,\zeta) && \textit{(angular displacement)}\\
x_4(t,\zeta) &= I_\rho(\zeta)\phi_t(t,\zeta) && \textit{(angular momentum)}
\end{aligned}$$

and the system may be reformulated in style of (3.3) by choosing

$$\mathcal{H}(\zeta) = \begin{bmatrix} K(\zeta) & & & \\ & \rho^{-1}(\zeta) & & \\ & & EI(\zeta) & \\ & & & I_\rho^{-1}(\zeta) \end{bmatrix}$$
$$P_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad P_0 = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

So far these all have been examples for which $N = 1$ and which have been investigated thoroughly in the monograph [JaZw12]. Additionally we give the following example, see Example 7.8 in [Vi07], which on the first look may not directly be written in the form (3.3), but introducing additional input and output operators it will have the form of a bounded perturbation of a system in the form (3.3).

Example 3.1.4 (Suspension System)**.** *We consider a suspension system modelled by the following system of PDE (cf. Example 7.8 in [Vi07]).*

$$\rho_u(\zeta)\frac{\partial^2}{\partial t^2}u(t,\zeta) = \frac{\partial}{\partial\zeta}\left(T_u(\zeta)\frac{\partial}{\partial\zeta}u(t,\zeta)\right) - \alpha(\zeta)(u(t,\zeta) - v(t,\zeta))$$
$$\rho_v(\zeta)\frac{\partial^2}{\partial t^2}v(t,\zeta) = \frac{\partial}{\partial\zeta}\left(T_v(\zeta)\frac{\partial}{\partial\zeta}v(t,\zeta)\right) + \alpha(\zeta)(u(t,\zeta) - v(t,\zeta)) \tag{3.8}$$

where $\rho_u, \rho_v \in L_\infty(0,1)$ (the mass densities of the two strings) and $T_u, T_v \in L_\infty(0,1)$ are both uniformly positive and $\alpha \in L_\infty(0,1)$ is a uniformly positive function modelling the interaction of the two strings as (a continuous version of) springs connecting the two strings (cf. Example 7.8 in [Vi07] where the parameters are taken to be constants). There are at least two ways to interpret this system in a form (3.3), although in both cases not all conditions are satisfied or additional constructions are needed. On the one hand one may introduce the energy state space variables

$$x(t,\zeta) = \begin{pmatrix} x_1(t,\zeta) \\ x_2(t,\zeta) \\ x_3(t,\zeta) \\ x_4(t,\zeta) \\ x_5(t,\zeta) \end{pmatrix} = \begin{pmatrix} \rho_u(\zeta)u_t(t,\zeta) \\ u_\zeta(t,\zeta) \\ \rho_v(\zeta)v_t(t,\zeta) \\ v_\zeta(t,\zeta) \\ u(t,\zeta)-v(t,\zeta) \end{pmatrix}$$

and then

$$\mathcal{H}(\zeta) = \begin{bmatrix} \rho_u^{-1}(\zeta) & 0 & 0 & 0 & 0 \\ 0 & T_u(\zeta) & 0 & 0 & 0 \\ 0 & 0 & \rho_v^{-1}(\zeta) & 0 & 0 \\ 0 & 0 & 0 & T_v(\zeta) & 0 \\ 0 & 0 & 0 & 0 & \alpha(\zeta) \end{bmatrix},$$

$$P_1 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$P_0 = \begin{bmatrix} 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 \end{bmatrix}$$

Then the total energy of the system is given by

$$H(t) = \frac{1}{2}\int_0^1 \rho_u(\zeta)\left|u_t(t,\zeta)\right|^2 + T_u(\zeta)\left|u_\zeta(t,\zeta)\right|^2 \\ + \rho_v(\zeta)\left|v_\zeta(t,\zeta)\right|^2 + T_v(\zeta)\left|v_\zeta(t,\zeta)\right|^2 + \alpha(\zeta)\left|(u-v)(t,\zeta)\right|^2 d\zeta.$$

Here the difference to the usual assumptions on (3.3) is the fact that $P_1 \in \mathbb{F}^{5\times 5}$ is not invertible. A possible way to overcome this is considering the following system with dynamic feedback.

$$x(t,\zeta) = \begin{pmatrix} x_1(t,\zeta) \\ x_2(t,\zeta) \\ x_3(t,\zeta) \\ x_4(t,\zeta) \end{pmatrix} = \begin{pmatrix} \rho_u(\zeta)u_t(t,\zeta) \\ u_\zeta(t,\zeta) \\ \rho_v(\zeta)v_t(t,\zeta) \\ v_\zeta(t,\zeta) \end{pmatrix}$$

with

$$P_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

and $P_0 = 0 \in \mathbb{F}^{4\times 4}$ *which at first only gives the uncoupled system of two nonuniform strings modelled by wave equations*

$$\rho_u(\zeta)\frac{\partial^2}{\partial t^2}u(t,\zeta) = \frac{\partial}{\partial \zeta}\left(T_u(\zeta)\frac{\partial}{\partial \zeta}u(t,\zeta)\right)$$
$$\rho_v(\zeta)\frac{\partial^2}{\partial t^2}v(t,\zeta) = \frac{\partial}{\partial \zeta}\left(T_v(\zeta)\frac{\partial}{\partial \zeta}v(t,\zeta)\right), \quad t \geq 0.$$

We extend the equation (3.3) by an additional input summand

$$\frac{\partial}{\partial t}x(t,\zeta) = \sum_{k=0}^{N} P_k \frac{\partial^k}{\partial \zeta^k}(\mathcal{H}x)(t,\zeta) + (Bu(t))(\zeta), \quad t \geq 0 \tag{3.9}$$

where in our case the input space is $U = L_2(0,1)\times\mathbb{F}$ *and* $B \in \mathcal{B}(U,X)$ *is a bounded linear operator. The input* $u(t) \in U$ *is determined by a (linear) control system* Σ_c *of the form*

$$\frac{\partial}{\partial t}x_c(t) = A_c x_c(t) + B_c u_c(t)$$
$$y_c(t) = C_c x_c(t) + D_c u_c(t) \tag{3.10}$$

where in this case $x_c(t) \in X_c = \mathbb{F}$ *and* $u_c(t) \in U_c,\ y_c(t) \in Y_c$ *and here* $U_c = Y_c = L_2(0,1)\times\mathbb{F}$*. Also in this case* A_c, B_c, C_c *and* D_c *all are bounded linear operators, namely*

$$A_c = 0,$$
$$B_c = \begin{bmatrix} 0 & I_{\mathbb{F}} \end{bmatrix} \in \mathcal{B}(U_c, X_c)$$
$$(C_c x_c)(\zeta) = x_c, \quad C_c \in \mathcal{B}(X_c, Y_c),$$
$$(D_c u_c)(\zeta) = \int_0^\zeta u_c(s)ds, \quad D_c \in \mathcal{B}(U_c, Y_c). \tag{3.11}$$

This system is equivalent to the original equation (3.8) if we take the feedback interconnection $u_c = y = Cx$ *and* $u = -y_c$ *and the operators*

$$B : D(B) = U = L_2(0,1)\times\mathbb{F} \to X : \ B\left(\begin{smallmatrix} u \\ z \end{smallmatrix}\right) = \begin{pmatrix} \alpha(\cdot)u(\cdot) \\ 0 \\ -\alpha(\cdot)u(\cdot) \\ 0 \end{pmatrix},$$
$$C : D(C) = H^1(0,1) \subset X \to Y = L_2(0,1)\times\mathbb{F} : \ Cx = \left(\begin{smallmatrix} (x_1-x_2)(\cdot) \\ (x_1-x_2)(0) \end{smallmatrix}\right)$$

and identifying $x_c(t)$ *with* $u(t,0) - v(t,0)$*. The advantage is that in this case* P_1 *is invertible. However, this comes to the price that we have a system with dynamic feedback and also the dissipativity of the total system (for appropriate boundary conditions) is not as immediate as in the first case, here we have*

$$H(t) = \frac{1}{2}\int_0^1 \rho_u(\zeta)\left|u_t(t,\zeta)\right|^2 + T_u(\zeta)\left|u_\zeta(t,\zeta)\right|^2 + \rho_v(\zeta)\left|v_\zeta(t,\zeta)\right|^2$$
$$+ T_v(\zeta)\left|v_\zeta(t,\zeta)\right|^2 + \alpha(\zeta)\left|x_c(t) + \int_0^\zeta (u_t(t,r) - v_t(t,r))dr\right|^2 d\zeta.$$

It is also possible to consider systems where $N = 1$ is not enough to be described in the form (3.3). We also give examples for them.

Example 3.1.5 (Schrödinger Equation). *Similar to the transport equation for the case $N = 1$, in case that $N = 2$ the simplest example is given by the* Schrödinger equation. *However, in contrast to all examples considered before, the Schrödinger equation demands $\mathbb{F} = \mathbb{C}$ to be the complex numbers. Let $\psi(t,\zeta)$ be the wave function at the position $\zeta \in (0,1)$ at time $t \geq 0$. (We exclusively consider the one-dimensional Schrödinger equation here.) Its modulus squared $|x(t,\zeta)|^2$ may be interpreted as the (time-dependant) probability density of a particle in a one-dimensional box if $\|\psi(t,\cdot)\|_{L_2(0,1)} = 1$. The Schrödinger equation of a free particle of mass $m > 0$ moving in a one-dimensional direction is then given by*

$$i\hbar \frac{\partial}{\partial t}\psi(t,\zeta) = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial \zeta^2}\psi(t,\zeta), \quad t \geq 0.$$

Here $\hbar > 0$ denotes the Planck constant. To get the standard form (3.3) one simply has to divide by $i\hbar$ and gets for $x(t,\zeta) = \psi(t,\zeta)$ and $H(\zeta) = \frac{\hbar}{2m}$ and $P_1 = i$, $P_0 = 0$ equation (3.3) as wished. The energy is given by the (weighted) squared L_2-norm of the wave equation.

$$H(t) = \frac{\hbar}{4m}\int_0^1 |\psi(t,\zeta)|^2 \, d\zeta.$$

Actually the Schrödinger equation will not play an important role within this thesis, however it will prove useful to obtain some counterexample on uniform exponential stability. More frequently we consider the Euler-Bernoulli beam equation as an example for a system with $N = 2$.

Example 3.1.6 (Euler-Bernoulli Beam). *The Euler-Bernoulli beam model may be seen as a refinement of the wave equation for structures which are not as flexible as a string, e.g. beams, and for which bending forces cannot be neglected any more. However, in comparison to the Timoshenko beam equation, it is still less precise to describe a beam since any shear effects are neglected. Again (as for the wave equation or the Timoshenko beam equation) $\omega(t,\zeta)$ denotes the transverse displacement of the beam at position $\zeta \in (0,1)$ in time $t \geq 0$. Then the Euler-Bernoulli beam equation reads*

$$\rho(\zeta)\omega_{tt}(t,\zeta) + \frac{\partial^2}{\partial \zeta^2}(EI(\zeta)\omega_{\zeta\zeta}(t,\zeta)) = 0, \quad t \geq 0.$$

Here the uniformly bounded and strictly positive functions $\rho, E, I : [0,1] \to \mathbb{R}$ have the following interpretation: $\rho(\zeta)$ is the mass density times the cross sectional area, $E(\zeta)$ the modulus of elasticity and $I(\zeta)$ the area moment of the cross section. The energy of the system is (similar to the wave equation)

$$H(t) := \frac{1}{2}\int_0^1 \rho(\zeta)\,|\omega_t(t,\zeta)|^2 + EI(\zeta)\,|\omega_{\zeta\zeta}(t,\zeta)|^2 \, d\zeta, \quad t \geq 0$$

where in contrast to the wave equation the (weighted) integral over the second derivative (squared) instead over the first derivative (squared) determines the potential part of the energy. Now one may easily rewrite this system as an evolution equation governed by a second order formally skew-symmetric operator, namely setting

$$x(t,\zeta) = \begin{pmatrix} x_1(t,\zeta) \\ x_2(t,\zeta) \end{pmatrix} = \begin{pmatrix} \rho(\zeta)\omega_t(t,\zeta) \\ \omega_{\zeta\zeta}(t,\zeta) \end{pmatrix}$$

and choosing the Hamiltonian density matrix function $\mathcal{H}$ and the structural matrices $P_i \in \mathbb{F}^{2\times 2}$ $(i = 0, 1, 2)$ as

$$\mathcal{H}(\zeta) = \begin{bmatrix} \mathcal{H}_1(\zeta) & \\ & \mathcal{H}_2(\zeta) \end{bmatrix} = \begin{bmatrix} \rho^{-1}(\zeta) & \\ & EI(\zeta) \end{bmatrix}$$
$$P_2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad P_0 = P_1 = \begin{bmatrix} 0 & \\ & 0 \end{bmatrix}.$$

Remark 3.1.7. *For the derivation on the underlying modelling assumptions for the Timoshenko beam equation and the Euler-Bernoulli beam equation as well as for the related Rayleigh beam and shear beam models, where the latter have not been considered here, we refer e.g. to the overview article [HaBeWe99].*

Note that an example for an equation where $N = 3$ is the one-dimensional *Airy equation*

$$\omega_t(t,\zeta) + \omega_{\zeta\zeta\zeta}(t,\zeta) = 0, \quad t \geq 0, \zeta \in (0,1).$$

However, within this thesis we mainly focus on the cases $N = 1$ and $N = 2$.

3.2 Port-Hamiltonian Systems

Up to now we introduced the evolution equation (3.3) without any boundary conditions, so that an initial value problem with the evolution rule (3.3) alone will never be well-posed in the sense of existence of a unique solution continuously depending on the initial value $x(0,\cdot) = x_0 \in X$. (For a more precise version of this statement we refer to Section 3.3, in particular Lemma 3.3.1.) As a preparation we introduce the (maximal) *port-Hamiltonian operator* $\mathfrak{A}$ (given by the right hand side of (3.3) on an appropriate domain $D(\mathfrak{A}) \subseteq X$) and the boundary control and observation operators $\mathfrak{B}$ and $\mathfrak{C}$, respectively, which play a crucial role in describing the boundary conditions and prove quite useful especially in the context of nonlinear boundary feedback later on. In contrast to Assumption 3.0.24 we also allow for P_0 to be not necessarily constant and skew-symmetric, but possible only an essentially bounded measurable function $P_0 : [0,1] \to \mathbb{F}^{d\times d}$. We also recall the assumptions on $\mathcal{H}$ and the matrices P_k $(k = 1, \dots, N)$ in the following definition.

Definition 3.2.1 (Port-Hamiltonian Operator)**.** *Let $N \in \mathbb{N}$ be a natural number and $P_k \in \mathbb{F}^{d\times d}$ $(k = 1, \dots, N)$ matrices with $P_k^* = (-1)^{k+1} P_k$ $(k = 1, \dots, N)$. Further let $P_0 \in L_\infty(0,1;\mathbb{F}^{d\times d})$ and the energy state space $X = L_2(0,1;\mathbb{F}^d)$ be equipped with the inner product $\langle\cdot,\cdot\rangle_X = \langle\cdot,\cdot\rangle_{\mathcal{H}}$ where $\mathcal{H} : [0,1] \to \mathbb{F}^{d\times d}$ is measurable, pointwise symmetric, essentially bounded and uniformly positive definite, i.e.*

$$m\,|z|^2 \leq \langle z, \mathcal{H}(\zeta) z\rangle_{\mathbb{F}^d} \leq M\,|z|^2, \quad z \in \mathbb{F}^d, \text{ a.e. } \zeta \in (0,1)$$

for some constants $0 < m \leq M < +\infty$. Then the operator $\mathfrak{A} : D(\mathfrak{A}) \subseteq X \to X$ defined via

$$\mathfrak{A}x = \sum_{k=0}^{N} P_k (\mathcal{H}x)^{(k)}$$
$$D(\mathfrak{A}) = \left\{ x \in X : \ \mathcal{H}x \in H^N(0,1;\mathbb{F}^d) \right\}$$

is called (maximal) port-Hamiltonian operator*.*

Remark 3.2.2. *Observe that the operator $\mathfrak{A}$ (or, more precisely, its domain $D(\mathfrak{A})$) does not inherit any boundary conditions. Therefore, it does not generate a C_0-semigroup, or, in other words, the initial value problem*

$$\frac{\partial}{\partial t}x(t) = \mathfrak{A}x(t) \ (t \geq 0), \quad x(0) = x_0 \in X$$

is not well-posed since it does not have a unique (strong) solution, but (as we will see below) infinitely many classical solutions $x \in C^1(\mathbb{R}_+; X) \cap C(\mathbb{R}_+; D(\mathfrak{A}))$ for suitable initial values $x \in D(\mathfrak{A})$. However, the operator $\mathfrak{A}$ is closed and its graph norm $\|\cdot\|_{\mathfrak{A}}$ is equivalent to the norm $\|\mathcal{H}\cdot\|_{H^N}$, see the following lemma.

Lemma 3.2.3. *The operator $\mathfrak{A}$ is a closed and densely defined operator on X and there are constants $c_1, c_2 > 0$ such that for every $x \in D(\mathfrak{A})$ one has*

$$c_1 \|\mathcal{H}x\|_{H^N(0,1;\mathbb{F}^d)} \leq \|x\|_{\mathfrak{A}} := \sqrt{\|x\|_X^2 + \|\mathfrak{A}x\|_X^2} \leq c_2 \|\mathcal{H}x\|_{H^N(0,1;\mathbb{F}^d)}.$$

Proof. Since the multiplication operator $X = \left(L_2(0,1;\mathbb{F}^d); \langle\cdot,\cdot\rangle_{\mathcal{H}}\right) \ni x \mapsto P_N\mathcal{H}x \in \left(L_2(0,1;\mathbb{F}^d); \langle\cdot,\cdot\rangle_{L_2}\right)$ is an isomorphism we first consider the special case $\mathcal{H} = I$ and $P_N = I$. Here the denseness is clear since $C_c^\infty(0,1;\mathbb{F}^d) \subseteq H^N(0,1;\mathbb{F}^d)$ is dense in $L_2(0,1;\mathbb{F}^d)$. As a first step we assume that $P_k = 0$ for $k < N$, so that

$$\mathfrak{A}x = \frac{\partial^N}{\partial\zeta^N}x, \quad D(\mathfrak{A}) = H^N(0,1;\mathbb{F}^d)$$

is the N^{th} order derivative operator. By Lemma 2.1.10 the L_2-norms of the j^{th} derivative ($1 < j < N$) can be estimated by the L_2-norm of the function and the L_2-norm of its N^{th} derivative. Therefore, in this case the two norms are equivalent. Now the assertion follows from the two subsequent lemmas. □

Lemma 3.2.4. *Let $A : D(A) = H^N(0,1;\mathbb{F}^d) \subset L_2(0,1;\mathbb{F}^d) \to L_2(0,1;\mathbb{F}^d)$ be any closed operator (i.e. a bounded operator $H^N(0,1;\mathbb{F}^d) \to L_2(0,1;\mathbb{F}^d)$) and $B : D(B) \subseteq X \to X$ another closable operator with $D(B) \supseteq H^k(0,1;\mathbb{F}^d)$ for some $k \in \{0,\dots,N-1\}$. Then also the sum $A + B : D(A+B) = D(A) \subseteq X \to X$ is a closed operator. In fact, B is relatively A-bounded with A-bound $a_0 = 0$.*

Proof. Since A is bounded as linear operator from $H^N(0,1;\mathbb{F}^d)$ to $L_2(0,1;\mathbb{F}^d)$ it is enough to show that for every $\varepsilon > 0$ there is $c_\varepsilon > 0$ such that

$$\|Bx\|_{L_2} \leq c_\varepsilon \|x\|_{L_2} + \varepsilon \|x\|_{H^N(0,1;\mathbb{F}^d)}.$$

Since $B \in \mathcal{B}(H^k(0,1;\mathbb{F}^d); L_2(0,1;\mathbb{F}^d))$ is a bounded linear operator, this follows from Lemma 2.1.10. Then closedness of the operator sum $A + B$ follows from Lemma III.3.4 in [EnNa00]. □

Lemma 3.2.5. *Let X_0, X_1, X_2 and X_3 be Banach spaces and $A : D(A) \subset X_1 \to X_2$ be a closed operator. Further let $B \in \mathcal{B}(X_0, X_1)$ and $C : D(C) \subset X_2 \to X_3$ be such that $C^{-1} \in \mathcal{B}(X_3, X_2) : X_3 \to D(C) \subseteq X_2$ exists as bounded linear operator. Then also the following operators are closed.*

$$\begin{aligned} &AB \text{ with domain } D(AB) = \{x \in X_0 : Bx \in D(A)\}, \\ &CA \text{ with domain } D(CA) = \{x \in D(A) : Ax \in D(C)\}. \end{aligned}$$

Proof. The proof is standard, we give it here for sake of completeness. We both times use the sequence criterion for closed operators. To begin with, let $(x_n)_{n\geq 1} \subseteq D(AB)$ be a sequence such that

$$x_n \to x \in X_0, \quad ABx_n \to y \in X_2 \quad (n \to \infty).$$

Then from $B \in \mathcal{B}(X_0; X_1)$ is also follows $D(A) \ni z_n := Bx_n \xrightarrow{n\to\infty} Bx =: z$ and closedness of A implies that $z \in D(A)$ with $Az = y$, so $x \in D(AB)$ with $ABx = y$. Secondly, let $(x_n)_{n\geq 1} \subseteq D(CA)$ be a sequence such that

$$x_n \to x \in X_1, \quad CAx_n \to y \in X_3 \quad (n \to \infty). \tag{3.12}$$

Then from $C^{-1} \in \mathcal{B}(X_3; X_2)$ it also follows that $Ax_n = C^{-1}(CAx_n) \xrightarrow{n\to\infty} C^{-1}y =: z$ and from the closedness of A we have $x \in D(A)$ with $Ax = z = C^{-1}y$, i.e. $x \in D(CA)$ with $CAx = y$. Hence, both operators AB and CA are closed. □

Let us also note that thanks to the Rellich-Kondrachov Theorem 2.1.8 the domain $D(\mathfrak{A})$ is compactly embedded into X.

Lemma 3.2.6. *The embedding $D(\mathfrak{A}) \hookrightarrow X$ is compact.*

Proof. The operators $\mathcal{H} : L_2(0,1;\mathbb{F}^d) \to X$ and $\mathcal{H}^{-1} : D(\mathfrak{A}) \to H^N(0,1;\mathbb{F}^d)$ are continuous and by the Rellich-Kondrachov Theorem 2.1.8 the embedding $i_{H^N} : H^N(0,1;\mathbb{F}^d) \hookrightarrow L_2(0,1;\mathbb{F}^d)$ is compact. Therefore, also the embedding $i_{D(\mathfrak{A})} : D(\mathfrak{A}) \hookrightarrow X$ is compact since $i_{D(\mathfrak{A})} = \mathcal{H} \circ i_{H^N} \circ \mathcal{H}^{-1}$ is the composition of a compact operator with two bounded operators and therefore compact, see Satz V.6.3 in [We06]. □

After this side remark let us introduce the input and output maps $\mathfrak{B}$ and $\mathfrak{C}$ via the boundary flow and the boundary effort.

Definition 3.2.7 (Boundary Flow and Effort)**.** *We define the* trace operator $\tau : H^N(0,1;\mathbb{F}^d) \to \mathbb{F}^{2Nd} = (\mathbb{F}^d)^{2N}$ *as the linear map*

$$\tau(x) = \begin{pmatrix} \tau_1(x) \\ \tau_0(x) \end{pmatrix} = \begin{pmatrix} x(1) \\ x'(1) \\ \vdots \\ x^{(N-1)}(1) \\ x(0) \\ x'(0) \\ \vdots \\ x^{(N-1)}(0) \end{pmatrix}.$$

Given a maximal port-Hamiltonian operator $\mathfrak{A}$ we then define the boundary flow $f_{\partial,\mathcal{H}x}$ *and the* boundary effort $e_{\partial,\mathcal{H}x}$ *as*

$$\begin{pmatrix} f_{\partial,\mathcal{H}x} \\ e_{\partial,\mathcal{H}x} \end{pmatrix} = R_{ext}\tau(\mathcal{H}x) \tag{3.13}$$

where the matrix $R_{ext} \in \mathbb{F}^{2Nd\times 2Nd}$ is defined as

$$R_{ext} = \frac{1}{\sqrt{2}}\begin{bmatrix} Q & -Q \\ I & I \end{bmatrix},$$

$$Q = \begin{bmatrix} P_1 & P_2 & \cdots & \cdots & P_N \\ -P_2 & -P_3 & \cdots & -P_N & 0 \\ \vdots & & \vdots & & \\ (-1)^{N-1}P_N & 0 & \cdots & 0 & 0 \end{bmatrix}.$$

Remark 3.2.8. *The matrix $Q \in \mathbb{F}^{2Nd\times 2Nd}$ and the map $x \mapsto \binom{f_{\partial,\mathcal{H}x}}{e_{\partial,\mathcal{H}x}}$ depend on $\mathcal{H} \in L_\infty(0,1;\mathbb{F}^{d\times d})$ and all matrices $P_k \in \mathbb{F}^{d\times d}$ $(k = 1,\dots,N)$. However, they do not depend on the matrix-valued function $P_0 \in L_\infty(0,1;\mathbb{F}^{d\times d})$.*

Lemma 3.2.9. *Thanks to the condition $P_k^* = (-1)^{k+1}P_k$ $(k \geq 1)$ the matrix $Q = Q^*$ is symmetric. Moreover, for invertible P_N and since $|\det Q| = |\det P_N|^N$, also the matrix Q is invertible and then R_{ext} is invertible with inverse matrix*

$$R_{ext}^{-1} = \frac{1}{\sqrt{2}}\begin{bmatrix} Q^{-1} & I \\ -Q^{-1} & I \end{bmatrix}.$$

Proof. The last statement is part of Lemma 3.4 in [LeZwMa05]. All other assertions are obvious. □

We are now in the position to introduce the boundary control and boundary observation operators $\mathfrak{B}$ and $\mathfrak{C}$, respectively, and the terminology of a port-Hamiltonian system in boundary control and observation form.

Definition 3.2.10 (Port-Hamiltonian System)**.** *Let $\mathfrak{A}$ be a (maximal) port-Hamiltonian operator with associated boundary flow and effort $\binom{f_{\partial,\mathcal{H}x}}{e_{\partial,\mathcal{H}x}}$. Further let $W_B, W_C \in \mathbb{F}^{Nd\times 2Nd}$ be two full rank matrices such that $\left[\begin{smallmatrix} W_B \\ W_C \end{smallmatrix}\right]$ is invertible. Then we define the* input map *$\mathfrak{B} : D(\mathfrak{B}) = D(\mathfrak{A}) \subseteq X \to U := \mathbb{F}^{Nd}$ and the* output map *$\mathfrak{C} : D(\mathfrak{C}) = D(\mathfrak{A}) \subseteq X \to Y := \mathbb{F}^{Nd}$ via*

$$\mathfrak{B}x = W_B \begin{pmatrix} f_{\partial,\mathcal{H}x} \\ e_{\partial,\mathcal{H}x} \end{pmatrix}$$
$$\mathfrak{C}x = W_C \begin{pmatrix} f_{\partial,\mathcal{H}x} \\ e_{\partial,\mathcal{H}x} \end{pmatrix}$$

and call $\mathfrak{S} = (\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ a port-Hamiltonian system *in boundary control and observation form to which we associate the boundary control and observation problem*

$$\begin{aligned} \frac{d}{dt}x(t) &= \mathfrak{A}x(t) \\ x(0) &= x_0 \\ u(t) &= \mathfrak{B}x(t) \\ y(t) &= \mathfrak{C}x(t), \quad t \geq 0. \end{aligned} \tag{3.14}$$

Remark 3.2.11. *Observe that the maps $\mathfrak{B}$ and $\mathfrak{C}$ may also be described in the form*

$$\mathfrak{B}x = \hat{W}_B \tau(\mathcal{H}x), \quad \mathfrak{C}x = \hat{W}_C \tau(\mathcal{H}x)$$

where both $\hat{W}_B = W_B R_{ext}$ and $\hat{W}_C = W_C R_{ext} \in \mathbb{F}^{Nd\times 2Nd}$ have full rank thanks to $R_{ext} \in \mathbb{F}^{2Nd\times 2Nd}$ being invertible.

Definition 3.2.12 (Impedance Passivity)**.** *Quite often we encounter systems which have a special form, namely where the system* $\mathfrak{S} = (\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ *is* impedance passive *(in the boundary control and observation sense), i.e.*

$$\operatorname{Re} \langle \mathfrak{A}x, x\rangle_X \leq \operatorname{Re} \langle \mathfrak{B}x, \mathfrak{C}x\rangle_{\mathbb{F}^{Nd}}, \quad x \in D(\mathfrak{A}).$$

In particular, in this case the operator $A = \mathfrak{A}|_{\ker \mathfrak{B}}$ *is dissipative and as we will see in the next subsection this already implies that* A *generates a contraction* C_0*-semigroup on the Hilbert space* $X = (L_2(0,1;\mathbb{F}^d); \langle\cdot,\cdot\rangle_X)$*. In case that* $P_0(\zeta)^* = -P_0(\zeta)$ *is skew-symmetric for a.e.* $\zeta \in (0,1)$ *it may also happen that the system* $\mathfrak{S} = (\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ *not only is impedance passive, but even* impedance energy preserving *(in the boundary control and observation sense), i.e.*

$$\operatorname{Re} \langle \mathfrak{A}x, x\rangle_X = \operatorname{Re} \langle \mathfrak{B}x, \mathfrak{C}x\rangle_{\mathbb{F}^{Nd}}, \quad x \in D(\mathfrak{A}).$$

One particular choice of $\mathfrak{B}$ and $\mathfrak{C}$ to make $\mathfrak{S} = (\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ (for dissipative $P_0 \in L_\infty(0,1;\mathbb{F}^{d\times d})$) an impedance passive (or impedance energy preserving) system results from the following lemma which describes how for the formally skew-symmetric part of the operator $\mathfrak{A}$ the boundary conditions influence the change of energy.

Lemma 3.2.13. *The operator* $\mathfrak{A}$ *satisfies*

$$\operatorname{Re} \langle \mathfrak{A}x, x\rangle_X = \operatorname{Re} \langle f_{\partial,\mathcal{H}x}, e_{\partial,\mathcal{H}x}\rangle_{\mathbb{F}^{Nd}} + \operatorname{Re} \langle P_0\mathcal{H}x, \mathcal{H}x\rangle_{L_2}, \qquad x \in D(\mathfrak{A}). \tag{3.15}$$

Proof. Since the multiplication operator $P_0\mathcal{H}$ is a bounded operator on $L_2(0,1;\mathbb{F}^d)$ and P_0 has no influence on the boundary flow and the boundary effort we may and will assume that that $P_0 = 0$ in the following. Then the identity is validated by straightforward computation via integration by parts and the property $P_k^* = (-1)^{k-1}P_k$ $(k \geq 1)$. One readily verifies that for every $x \in D(\mathfrak{A})$

$$\begin{aligned}
\operatorname{Re} \langle \mathfrak{A}x, x\rangle_X &= \operatorname{Re} \langle \sum_{k=1}^N P_k(\mathcal{H}x)^{(k)}, \mathcal{H}x\rangle_{L_2} \\
&= \frac{1}{2}\left[\sum_{k=1}^N \langle P_k(\mathcal{H}x)^{(k)}, \mathcal{H}x\rangle_{L_2} + \langle \mathcal{H}x, P_k(\mathcal{H}x)^{(k)}\rangle\right] \\
&= \frac{1}{2}\left[\sum_{k=1}^N \langle (P_k + (-1)^k P_k^*)(\mathcal{H}x)^{(k)}, \mathcal{H}x\rangle_{L_2}\right] \\
&\quad + \frac{1}{2}\left[\sum_{k=1}^N \sum_{l=0}^{k-1} (-1)^l \langle (\mathcal{H}x)^{(l)}(\zeta), P_k(\mathcal{H}x)^{(k-l-1)}(\zeta)\rangle_{\mathbb{F}^d}\right]_0^1 \\
&= \frac{1}{2}\left[\sum_{k=1}^N \sum_{l=0}^{k-1} (-1)^l \langle (\mathcal{H}x)^{(l)}(\zeta), P_k(\mathcal{H}x)^{(k-l-1)}(\zeta)\rangle_{\mathbb{F}^d}\right]_0^1
\end{aligned}$$

using integration by parts and the condition $P_k^* = (-1)^{k+1}P_k$ on the matrices for

$k \geq 1$. On the other hand, also using that $Q = Q^*$, we compute

$$\begin{aligned}
&\operatorname{Re}\langle f_{\partial,\mathcal{H}x}, e_{\partial,\mathcal{H}x}\rangle_{\mathbb{F}^{Nd}} \\
&= \frac{1}{2}\operatorname{Re}\langle Q\tau_1(\mathcal{H}x) - Q\tau_0(\mathcal{H}x), \tau_1(\mathcal{H}x) + \tau_0(\mathcal{H}x)\rangle_{\mathbb{F}^{Nd}} \\
&= \frac{1}{2}\langle \tau_1(\mathcal{H}x), Q\tau_1(\mathcal{H}x)\rangle_{\mathbb{F}^{Nd}} - \frac{1}{2}\langle \tau_0(\mathcal{H}x), Q\tau_0(\mathcal{H}x)\rangle_{\mathbb{F}^{Nd}} \\
&\quad + \frac{1}{2}\operatorname{Re}\langle (Q - Q^*)\tau_1(\mathcal{H}x), \tau_0(\mathcal{H}x)\rangle_{\mathbb{F}^{Nd}} \\
&= \frac{1}{2}\left[\sum_{n=1}^{N}\langle (\mathcal{H}x)^{(n-1)}(\zeta), \sum_{m=1}^{N-n+1}(-1)^{n-1}P_{m+n-1}(\mathcal{H}x)^{(m-1)}(\zeta)\rangle_{\mathbb{F}^d}\right]_0^1 \\
&= \frac{1}{2}\left[\sum_{n=1}^{N}\sum_{k=n}^{N}(-1)^{n-1}\langle (\mathcal{H}x)^{(n-1)}(\zeta), P_k(\mathcal{H}x)^{(k-n)}(\zeta)\rangle_{\mathbb{F}^d}\right] \\
&= \frac{1}{2}\left[\sum_{k=1}^{N}\sum_{n=1}^{k}(-1)^{n-1}\langle (\mathcal{H}x)^{(n-1)}(\zeta), P_k(\mathcal{H}x)^{(k-n)}(\zeta)\rangle_{\mathbb{F}^d}\right]_0^1 \\
&= \frac{1}{2}\left[\sum_{k=1}^{N}\sum_{l=0}^{k-1}(-1)^{l}\langle (\mathcal{H}x)^{(l)}(\zeta), P_k(\mathcal{H}x)^{(k-l-1)}(\zeta)\rangle_{\mathbb{F}^d}\right]_0^1
\end{aligned}$$

where we used the symmetry of the matrix $Q = Q^* \in \mathbb{F}^{2Nd\times 2Nd}$ and the substitutions $k = m + n - 1$ and (in the last step) $l = n - 1$. □

For the choice of boundary flow and boundary effort as input and output this results in the following corollary.

Corollary 3.2.14. *For every port-Hamiltonian operator $\mathfrak{A}$ with dissipative P_0, i.e.* $\operatorname{Sym} P_0(\zeta) := \frac{1}{2}(P_0(\zeta) + P_0(\zeta)^*) \leq 0$ *negative semidefinite for a.e. $\zeta \in (0,1)$, the choice $\mathfrak{B}x = f_{\partial,\mathcal{H}x}$ and $\mathfrak{C}x = e_{\partial,\mathcal{H}x}$, i.e. $W_B = \begin{bmatrix} I & 0 \end{bmatrix}$ and $W_C = \begin{bmatrix} 0 & I \end{bmatrix}$, (or the other way around) gives an impedance passive system $\mathfrak{S} = (\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ which is even impedance energy preserving if (and only if) $P_0(\zeta)^* = -P_0(\zeta)$ is skew-symmetric for a.e. $\zeta \in (0,1)$.*

Remark 3.2.15. *Actually the conditions* $\operatorname{Sym} P_0(\zeta) \leq 0$ *negative semidefinite for a.e $\zeta \in (0,1)$ or $P_0(\zeta) = -P_0(\zeta)$ skew-symmetric for a.e. $\zeta \in (0,1)$ are necessary for $\mathfrak{S}$ to get a system which is impedance passive or impedance energy preserving, respectively.*

Proof. Since $C_c^\infty(0,1;\mathbb{F}^d) \subseteq \mathcal{H}D(A)$ is dense in $X = L_2(0,1;\mathbb{F}^d)$ it follows from

$$\begin{aligned}
\operatorname{Re}\langle P_0\mathcal{H}x, \mathcal{H}x\rangle_{L_2} &= \operatorname{Re}\langle \mathfrak{A}x, x\rangle_X \\
&\leq \operatorname{Re}\langle \mathfrak{B}x, \mathfrak{C}x\rangle_{\mathbb{F}^{Nd}} = 0, \quad x \in \mathcal{H}^{-1}C_c^\infty(0,1;\mathbb{F}^d)
\end{aligned}$$

for impedance passive port-Hamiltonian systems that

$$\operatorname{Re}\langle P_0\tilde{x}, \tilde{x}\rangle_{L_2} \leq 0, \quad \tilde{x} \in L_2(0,1;\mathbb{F}^d).$$

Then by Lemma 2.1.6

$$\operatorname{Sym} P_0(\zeta) \leq 0, \quad \text{a.e. } \zeta \in (0,1). \tag{3.16}$$

The impedance energy preserving case follows from the observation that the system $\mathfrak{S} = (\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ is impedance energy preserving if and only if both the system $\mathfrak{S}$ and the system $(-\mathfrak{A}, -\mathfrak{B}, \mathfrak{C})$ are impedance passive. □

Starting from Corollary 3.2.14 we may use the general definition of $\mathfrak{B}$ and $\mathfrak{C}$ to conclude that

$$\begin{aligned}
\operatorname{Re}\langle f_{\partial,\mathcal{H}x}, e_{\partial,\mathcal{H}x}\rangle_{\mathbb{F}^{Nd}} &= \frac{1}{2}\langle \begin{pmatrix} f_{\partial,\mathcal{H}x} \\ e_{\partial,\mathcal{H}x} \end{pmatrix}, \Sigma \begin{pmatrix} f_{\partial,\mathcal{H}x} \\ e_{\partial,\mathcal{H}x} \end{pmatrix} \rangle_{\mathbb{F}^{2Nd}} \\
&= \frac{1}{2}\langle \begin{bmatrix} W_B \\ W_C \end{bmatrix}^{-1} \begin{pmatrix} \mathfrak{B}x \\ \mathfrak{C}x \end{pmatrix}, \Sigma \begin{bmatrix} W_B \\ W_C \end{bmatrix}^{-1} \begin{pmatrix} \mathfrak{B}x \\ \mathfrak{C}x \end{pmatrix} \rangle_{\mathbb{F}^{2Nd}} \\
&= \frac{1}{2}\langle \begin{pmatrix} \mathfrak{B}x \\ \mathfrak{C}x \end{pmatrix}, P_{W_B,W_C} \begin{pmatrix} \mathfrak{B}x \\ \mathfrak{C}x \end{pmatrix} \rangle_{\mathbb{F}^{2Nd}}
\end{aligned}$$

where

$$P_{W_B,W_C}^{-1} = \begin{bmatrix} W_B \\ W_C \end{bmatrix} \Sigma \begin{bmatrix} W_B \\ W_C \end{bmatrix}^* = \begin{bmatrix} W_B\Sigma W_B^* & W_B\Sigma W_C^* \\ W_C\Sigma W_B^* & W_C\Sigma W_C^* \end{bmatrix}.$$

Hence, the following holds.

Proposition 3.2.16 (Characterisation of Impedance Passive and Impedance Energy Preserving Systems)**.** *Impedance passive and impedance energy preserving systems may be characterised by the matrices $W_B, W_C \in \mathbb{F}^{Nd\times 2Nd}$ and the matrix-valued function $P_0 \in L_\infty(0,1;\mathbb{F}^d)$.*

1. *The system $\mathfrak{S} = (\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ is impedance energy preserving if and only if $P_0(\zeta)^* = -P_0(\zeta)$ is skew-symmetric for a.e. $\zeta \in (0,1)$ and $P_{W_B,W_C} = \Sigma$, i.e.*
$$W_B\Sigma W_B^* = W_C\Sigma W_C^* = 0, \quad W_B\Sigma W_C^* = I.$$
2. *The system $\mathfrak{S} = (\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ is impedance passive if and only if the symmetric part* $\operatorname{Sym} P_0(\zeta) := \frac{1}{2}(P_0(\zeta) + P_0(\zeta)^*) \le 0$ *is negative semidefinite for a.e. $\zeta \in (0,1)$ and the matrix $P_{W_B,W_C} - \Sigma$ is negative semidefinite. In particular, for $\mathfrak{S}$ to be impedance passive $W_B\Sigma W_B^* \ge 0$ and $W_C\Sigma W_C^* \ge 0$ are necessarily positive semidefinite.*

Proof. Since a system $\mathfrak{S} = (\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ is impedance energy preserving if and only if both $\mathfrak{S}$ and $\mathfrak{S}' = (-\mathfrak{A}, -\mathfrak{B}, \mathfrak{C})$ are impedance energy preserving (using the linearity of $\mathfrak{A}, \mathfrak{B}$ and $\mathfrak{C}$) and $P_0(\zeta) = -P_0(\zeta)$ is skew-symmetric for a.e. $\zeta \in (0,1)$ if and only if both $\operatorname{Sym} P_0(\zeta) \le 0$ and $\operatorname{Sym}(-P_0(\zeta)) = -\operatorname{Sym} P_0(\zeta) \le 0$ are negative semidefinite, it is enough to verify part 2.). However, this follows from the considerations just above. The last assertion may be shown as in the proof of Theorem 7.2.4 in [JaZw12] (or, Theorem 3.3.6), noting that for impedance passive port-Hamiltonian systems we have that

$$\operatorname{Re}\langle f_{\partial,\mathcal{H}x}, e_{\partial,\mathcal{H}x}\rangle_{\mathbb{F}^{Nd}} \le 0, \quad x \in \ker\mathfrak{B} \cup \ker\mathfrak{C}.$$

□

Lemma 3.2.17. *Let $k \in \mathbb{N}$, $W = \begin{bmatrix} W_1 & W_2 \end{bmatrix} \in \mathbb{C}^{k\times 2k}$ and $\Sigma = \left[\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right] \in \mathbb{C}^{2k\times 2k}$.*

1. *Then W has a decomposition*
$$W = S\begin{bmatrix} I+V & I-V \end{bmatrix}$$

with $S \in \mathbb{F}^{k\times k}$ invertible and $V \in \mathbb{F}^{k\times k}$ if and only if the matrix $W_1 + W_2$ is invertible. In particular, then the matrix W has full rank rk $W = k$.

2. *The matrix W has full rank* rk $W = k$ *and the matrix $W\Sigma W^*$ is positive semidefinite if and only if*

$$W = S \begin{bmatrix} I + V & I - V \end{bmatrix}$$

with $S \in \mathbb{F}^{k\times k}$ invertible and $V \in \mathbb{F}^{k\times k}$ where $I - VV^* \geq 0$ is positive semidefinite.

Proof. Part 2.) is Lemma 7.3.1 in [JaZw12]. Clearly part 1.) is closely related to the second assertion and similar ideas are used for its proof. First, let $W = S \begin{bmatrix} I + V & I - V \end{bmatrix}$ with S invertible. Then

$$W_1 + W_2 = S(I + V) + S(I - V) = 2S$$

is invertible by hypothesis. On the other hand, if $W_1 + W_2$ is invertible we may set $S = \frac{1}{2}(W_1 + W_2)$ and $V = (W_1 + W_2)^{-1}(W_1 - W_2)$ to get the desired decomposition. Also in that case $k \geq \text{rk}\, W \geq \text{rk}\,(W_1 + W_2) = k$, so that W has full rank. □

We give a characterisation of the matrices W_B and W_C leading to impedance energy preserving systems in the next lemma which is mainly (i.e. for the case $-P_0(\zeta)^* = P_0(\zeta) \equiv P_0$ constant) due to Villegas ([Vi07]).

Lemma 3.2.18 (Characterisation of Impedance Energy Preserving Systems)**.** *Let $\mathfrak{S} = (\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ be a port-Hamiltonian system. Then*

1. *$\mathfrak{S}$ is impedance energy preserving if and only if $P_0(\zeta)^* = -P_0(\zeta)$ for a.e. $\zeta \in (0,1)$ and there are matrices $S_B, S_C \in \mathbb{F}^{2Nd\times 2Nd}$ and unitary matrices $V_B, V_C \in \mathbb{F}^{Nd\times Nd}$ such that*

$$\begin{aligned} W_B &= S_B \begin{bmatrix} I + V_B & I - V_B \end{bmatrix} \\ W_C &= S_C \begin{bmatrix} I + V_C & I - V_C \end{bmatrix} \\ I &= 2S_C(I - V_C V_B^*)S_B^*. \end{aligned}$$

Then one has in particular that

$$W_B^* W_C + W_C^* W_B = \Sigma := \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \in \mathbb{F}^{2Nd\times 2Nd}.$$

2. *$\mathfrak{S}$ is impedance passive if and only if $P_0(\zeta)$ is dissipative for a.e. $\zeta \in (0,1)$ and there are matrices $S_B, S_C \in \mathbb{F}^{2Nd\times 2Nd}$ and matrices $V_B, V_C \in \mathbb{F}^{Nd\times Nd}$ with $V_B V_B^*, V_C V_C^* \leq I$ such that*

$$\begin{aligned} W_B &= S_B \begin{bmatrix} I + V_B & I - V_B \end{bmatrix} \\ W_C &= S_C \begin{bmatrix} I + V_C & I - V_C \end{bmatrix} I = 2S_C(I - V_C V_B^*)S_B^*. \end{aligned}$$

In that case one has in particular that

$$W_B^* W_C + W_C^* W_B = \Sigma := \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \in \mathbb{F}^{2Nd\times 2Nd}.$$

Although this result is not new by any means and in principle can be found as Theorem 2.16 in [Vi07] we give a full prove anyway because in [Vi07] (and also in [LeZwMa05]) only the case $\mathbb{F} = \mathbb{R}$ and $P_0(\zeta) = -P_0(\zeta)^*$ constant is considered (which do not too much harm), but more importantly the statements are only proved after the well-posedness (in the sense of existence of unique solutions) of the system $\mathfrak{S}$.

Proof. As we have seen above the conditions are sufficient to get an impedance energy preserving system. It remains to check necessity. Let $\mathfrak{S} = (\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ be an impedance energy preserving system. Then in particular for every $x \in X$ such that $\mathcal{H}x \in C_c^\infty(0, 1; \mathbb{F}^d)$ we find

$$\begin{aligned} \operatorname{Re} \langle P_0 \mathcal{H}x, \mathcal{H}x\rangle_{L_2} &= \operatorname{Re} \langle P_0 \mathcal{H}x, \mathcal{H}x\rangle_{L_2} + \operatorname{Re} \langle f_{\partial,\mathcal{H}x}, e_{\partial,\mathcal{H}x}\rangle_{\mathbb{F}^{Nd}} \\ &= \operatorname{Re} \langle \mathfrak{A}x, x\rangle_X \\ &= \operatorname{Re} \langle \mathfrak{B}x, \mathfrak{C}x\rangle_{\mathbb{F}^{Nd}} = 0. \end{aligned}$$

Hence, $P_0(\zeta) = -P_0(\zeta)^*$ has to be skew-symmetric for a.e. $\zeta \in (0, 1)$ by Lemma 2.1.7. Finally we have to find whether the conditions given in Lemma 3.2.18 on W_B and W_C are both necessary and sufficient for $\mathfrak{S}$ to be impedance energy preserving. This comes down to checking when the condition $P_{W_B,W_C} = \Sigma$ is satisfied. For W_B, W_C as in the lemma we have

$$\begin{aligned} W_B \Sigma W_B^* &= S_B \begin{bmatrix} I + V_B & I - V_B \end{bmatrix} \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \begin{bmatrix} I + V_B & I - V_B \end{bmatrix}^* S_B^* \\ &= 2S_B(I - V_B V_B^*)S_B^* \\ W_C \Sigma W_C^* &= 2S_C(I - V_C V_C^*)S_C^* \\ W_B \Sigma W_C &= S_B \begin{bmatrix} I + V_B & I - V_B \end{bmatrix} \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \begin{bmatrix} I + V_C & I - V_C \end{bmatrix}^* S_C^* \\ &= 2S_B(I - V_B V_C^*)S_C^* \end{aligned}$$

and therefore $V_B V_B^* = I$ and $V_C V_C^* = I$, i.e. both V_B and V_C should be unitary, and $I = 2S_B(I - V_B V_C^*)S_C^*$ are both necessary and sufficient conditions for $\mathfrak{S}$ to be impedance energy preserving when $P_0(\zeta)^* = -P_0(\zeta)$ for a.e. $\zeta \in (0, 1)$. □

When we introduced the port-Hamiltonian system $\mathfrak{S} = (\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ we said that it were in boundary control and observation form. However, for $\mathfrak{S}$ to be a Boundary Control and Observation system it has to satisfy the following two properties.

1. The restriction $A = \mathfrak{A}|_{\ker \mathfrak{B}}$ of the maximal port-Hamiltonian operator $\mathfrak{A}$ to the kernel of the boundary input map $\mathfrak{B}$ has to generate a C_0-semigroup on the Hilbert space $X = L_2(0, 1; \mathbb{F}^d)$. A quite large class of input operators for which this property holds will be investigated in the next section, namely the case where the C_0-semigroup is contractive, or at least quasi-contractive. Actually we are going to characterise all the boundary conditions leading to (quasi-)contractive C_0-semigroups.

2. For the boundary control operator $\mathfrak{B}$ there exists a continuous right-inverse $B \in \mathcal{B}(\mathbb{F}^{Nd}; D(\mathfrak{A}))$.

We show that the second condition is always satisfied, so that the only condition left to check is the generator property of $A = \mathfrak{A}|_{\ker \mathfrak{B}}$.

Lemma 3.2.19. *Let d and $N \in \mathbb{N}$ be natural numbers and assume that the matrix $W_B \in \mathbb{F}^{Nd\times 2Nd}$ have full rank. Define $\tau : H^N(0,1;\mathbb{F}^d) \to \mathbb{F}^{2Nd} = (\mathbb{F}^d)^{2N}$ by $\tau_j(x) = x^{(j-1)}(1), \tau_{j+N}(x) = x^{(j-1)}(0)$ for $j = 1,\dots,N$. Then there is an operator $B \in \mathcal{B}(\mathbb{F}^{Nd}; H^N(0,1;\mathbb{F}^d))$ such that*

$$(W_B \circ \tau)B = I_{\mathbb{F}^{Nd}}.$$

Proof. (Cf. step 2 in the proof of Theorem 4.2 in [LeZwMa05].) Let $\{e_j\}_{j=1,\dots,2Nd}$ be the standard orthogonal basis in $\mathbb{F}^{2Nd}$ and choose $f_j \in H^N(0,1;\mathbb{F}^d)$ with $\tau(f_j) = e_j$ for $j = 1,\dots,2Nd$. Since W_B has full rank there is a matrix $V \in \mathbb{F}^{2Nd\times Nd}$ such that $W_B V = I_{\mathbb{F}^{Nd}}$. Decompose V as

$$V = \begin{bmatrix} V_1 \\ \vdots \\ V_{2Nd} \end{bmatrix}$$

where $V_j \in \mathbb{F}^{1\times Nd}$ for $j = 1,\dots,2Nd$. Now set

$$Bz := \sum_{j=1}^{2Nd} V_j z f_j \in H^N(0,1;\mathbb{F}^d), \qquad z \in \mathbb{F}^{Nd}.$$

Then clearly $B \in \mathcal{B}(\mathbb{F}^{Nd}; H^N(0,1;\mathbb{F}^d))$ and

$$\begin{aligned}(W_B \circ \tau)(Bz) &= W_B \sum_{j=1}^{2Nd} V_j z \tau(f_j) = W_B \sum_{j=1}^{2Nd} V_j z e_j \\ &= W_B V z = z, \qquad z \in \mathbb{F}^{Nd}.\end{aligned}$$

□

Corollary 3.2.20. *Let $\mathfrak{A}$ be a port-Hamiltonian operator and let $W_B \in \mathbb{F}^{Nd\times 2Nd}$ have full rank,*

$$\mathfrak{B}x := W_B \begin{pmatrix} f_{\partial,\mathcal{H}x} \\ e_{\partial,\mathcal{H}x} \end{pmatrix}, \quad x \in D(\mathfrak{A}).$$

Then there is a right-inverse $B \in \mathcal{B}(\mathbb{F}^{Nd}; D(\mathfrak{A}))$ of $\mathfrak{B}$, i.e. $\mathfrak{B}B = I_{\mathbb{F}^{Nd}}$.

Proof. We write

$$\mathfrak{B} = W_B R_{ext} \circ \tau \circ \mathcal{H} =: \hat{W}_B \circ \tau \circ \mathcal{H},$$

choose $\hat{B}$ from Lemma 3.2.19 for $\hat{W}_B$ and set

$$B := \mathcal{H}^{-1} \circ \hat{B}.$$

Then $\mathfrak{B}B = W_B \circ \tau \circ \mathcal{H}^{-1} \circ B = \hat{W}_B \circ \tau \circ \hat{B} = I_{\mathbb{F}^{Nd}}$, indeed. □

In particular, the preceding Corollary 3.2.20 states that under the assumptions of Theorem 3.3.6 for any vector $\left(\begin{smallmatrix} f \\ e \end{smallmatrix}\right) \in \ker W_B$ there exists $x \in D(A)$ such that

$$\begin{pmatrix} f \\ e \end{pmatrix} = \begin{pmatrix} f_{\partial,\mathcal{H}x} \\ e_{\partial,\mathcal{H}x} \end{pmatrix}.$$

Therefore, we easily conclude the following theorem which states that for a port-Hamiltonian system $\mathfrak{S} = (\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ to be a Boundary Control and Observation System it is enough that $A := \mathfrak{A}|_{\ker \mathfrak{B}}$ generates a C_0-semigroup.

Theorem 3.2.21. *A port-Hamiltonian system $\mathfrak{S} = (\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ in boundary control and observation form is a Boundary Control and Observation System if and only if the operator $A = \mathfrak{A}|_{\ker \mathfrak{B}}$ generates a C_0-semigroup on X.*

Proof. Since $\tau \circ \mathcal{H} : (D(\mathfrak{A}), \|\cdot\|_{\mathfrak{A}}) \to (\mathbb{F}^{2Nd}, |\cdot|)$ is a bounded and linear map, it is enough to find $B \in \mathcal{B}(U, D(\mathfrak{A}))$ such that $\mathfrak{B}B = I_{\mathbb{F}^{Nd}}$. This is provided by Corollary 3.2.20. □

Remark 3.2.22. *Similarly a port-Hamiltonian system $(\mathfrak{A}, \mathfrak{B})$ without output map $\mathfrak{C}$ is a Boundary Control System if and only if $A = \mathfrak{A}|_{\ker \mathfrak{B}}$ generates a C_0-semigroup.*

Theorem 3.2.23. *Consider the port-Hamiltonian system $\mathfrak{S} = (\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ and the corresponding evolution equations*

$$\begin{aligned}
\frac{\partial}{\partial t} x(t,\zeta) &= \sum_{k=0}^{N} P_k \frac{\partial^k}{\partial \zeta^k} (\mathcal{H}x)(t,\zeta), \quad \zeta \in (0,1) \\
u(t) &= \mathfrak{B}x(t) = W_B \begin{pmatrix} f_{\partial,\mathcal{H}x(t)} \\ e_{\partial,\mathcal{H}x(t)} \end{pmatrix} \\
y(t) &= \mathfrak{C}x(t) = W_C \begin{pmatrix} f_{\partial,\mathcal{H}x(t)} \\ e_{\partial,\mathcal{H}x(t)} \end{pmatrix}, \qquad t \geq 0
\end{aligned}$$

and assume that $A := \mathfrak{A}|_{\ker \mathfrak{B}}$ generates a C_0-semigroup on X. Then for all $u \in C^2(\mathbb{R}_+; \mathbb{F}^{Nd})$ and $x_0 \in D(\mathfrak{A})$ with $u(0) = W_B \left(\begin{smallmatrix} f_{\partial,\mathcal{H}x_0} \\ e_{\partial,\mathcal{H}x_0} \end{smallmatrix}\right)$ the system has a unique classical solution

$$x \in C^1(\mathbb{R}_+; X) \cap C(\mathbb{R}_+; D(\mathfrak{A})), \quad y \in C(\mathbb{R}_+; \mathbb{F}^{Nd}).$$

If additionally $P_0(\zeta) = -P_0(\zeta)^*$ for a.e. $\zeta \in (0,1)$, then

$$\frac{d}{dt} \|x(t)\|_X^2 = \langle \begin{pmatrix} u(t) \\ y(t) \end{pmatrix}, P_{W_B,W_C} \begin{pmatrix} u(t) \\ y(t) \end{pmatrix} \rangle, \quad t \geq 0$$

where

$$P_{W_B,W_C} := \begin{bmatrix} W_B \Sigma W_B^* & W_B \Sigma W_C^* \\ W_C \Sigma W_B^* & W_C \Sigma W_C^* \end{bmatrix}^{-1}.$$

Proof. By Theorem 3.2.21 the system $\mathfrak{S} = (\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ is a Boundary Control and Observation System. From Theorem 2.3.15 we find that for any u and x_0 as above there exists a classical solution $x \in C^1(\mathbb{R}_+; X) \cap C(\mathbb{R}_+; D(\mathfrak{A}))$. Moreover, $\mathfrak{C} \in \mathcal{B}(D(\mathfrak{A}), \mathbb{F}^{Nd})$ and we thus have

$$y = \mathfrak{C}x \in C(\mathbb{R}_+; \mathbb{F}^{Nd}).$$

If additionally $P_0(\zeta) = -P_0(\zeta)^*$ for a.e. $\zeta \in (0,1)$ we compute

$$\begin{aligned}
\frac{d}{dt} \|x(t)\|_X^2 &= 2 \operatorname{Re} \langle e_{\partial,\mathcal{H}x}, f_{\partial,\mathcal{H}x} \rangle_{\mathbb{F}^d} \\
&= \langle \begin{pmatrix} u(t) \\ y(t) \end{pmatrix}, P_{W_B,W_C} \begin{pmatrix} u(t) \\ y(t) \end{pmatrix} \rangle_{\mathbb{F}^{Nd}},
\end{aligned}$$

cf. the calculations before Proposition 3.2.16. □

The following lemma shows that for impedance passive port-Hamiltonian systems the transfer function $G(\lambda)$ is defined for every Re $\lambda > 0$ and its symmetric part Sym $G(\lambda) \geq 0$ is positive semidefinite.

Lemma 3.2.24. *Let $\mathfrak{S} = (\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ be an impedance passive port-Hamiltonian system. Then $\mathbb{F}_0^+ \subseteq D(G)$ and* Sym $G(\lambda) > 0$ *for all $\lambda \in \mathbb{F}_0^+$, i.e. for all $\lambda \in \mathbb{F}_0^+$ there is $m_\lambda > 0$ such that*

$$\operatorname{Re} \langle z, G(\lambda) z \rangle_U > m_\lambda |z|^2, \quad z \in U = \mathbb{F}^{Nd}.$$

More precisely, for every $\lambda \in \mathbb{F}_0^+$ there are operators $\Phi(\lambda) \in \mathcal{B}(X), \Psi(\lambda) \in \mathcal{B}(U, X)$ and $F(\lambda) \in \mathcal{B}(X, Y)$ such that for all $f \in X$ and $u \in U$ there is a unique solution of the problem

$$\begin{aligned} (\lambda - \mathfrak{A})x &= f \\ u &= \mathfrak{B}x \\ y &= \mathfrak{C}x \end{aligned}$$

which is given by

$$\begin{aligned} x &= \Phi(\lambda) f + \Psi(\lambda) u \\ y &= F(\lambda) f + G(\lambda) u. \end{aligned}$$

Without loss of generality we may assume that $\mathcal{H} = I$ for the proof. In fact, for any impedance passive Boundary control and Observation System $\mathfrak{S} = (\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ (on Hilbert spaces X and $U = Y$) and $P \in \mathcal{B}(X)$ any coercive operator on X, also $\mathfrak{S}_P = (\mathfrak{A}P, \mathfrak{B}P, \mathfrak{C}P)$ is an impedance passive Boundary Control and Observation System (on $X_P = X$ equipped with $\langle \cdot, \cdot \rangle_{X_P} := \langle \cdot, P \cdot \rangle_X$) and the transfer function exists on $\mathbb{F}_0^+$ for $(\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ if and only if it exists on $\mathbb{F}_0^+$ for $\mathfrak{S}_P = (\mathfrak{A}P, \mathfrak{B}P, \mathfrak{C}P)$ (the situation is similar for Φ, Ψ and F as in Lemma 3.2.24).

Proposition 3.2.25. *Let X and $U = Y$ be Hilbert spaces and assume that $(\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ is a Boundary Control and Observation System. Further let $0 < P = P^* \in \mathcal{B}(X)$ be a strictly coercive operator on X and $X_P := X$ equipped with the inner product $\langle \cdot, \cdot \rangle_{X_P} := \langle \cdot, P \cdot \rangle_X$. If $(\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ is an impedance passive Boundary Control and Observation system on (X, U, Y) and for some $\lambda \in \mathbb{F}_0^+$ there are operators $\Phi(\lambda) \in \mathcal{B}(X), \Psi(\lambda) \in \mathcal{B}(U, X), F(\lambda) \in \mathcal{B}(X, Y)$ and $G(\lambda) \in \mathcal{B}(U, Y)$ such that for all $f \in X$ and $u \in U$ there is a unique solution of the problem*

$$\begin{aligned} (\mathfrak{A} - \lambda)x &= f \\ u &= \mathfrak{B}x \\ y &= \mathfrak{C}x \end{aligned}$$

which is given by

$$\begin{aligned} x &= \Phi(\lambda) f + \Psi(\lambda) u \\ y &= F(\lambda) f + G(\lambda) u, \end{aligned}$$

then also $(\mathfrak{A}P, \mathfrak{B}P, \mathfrak{C}P)$ is an impedance passive Boundary Control and Observation System on (X_P, U, Y) and there are operators $\hat{\Phi}(\lambda) \in \mathcal{B}(X), \hat{\Psi}(\lambda) \in \mathcal{B}(U, X), \hat{F}(\lambda) \in \mathcal{B}(X, Y)$ and $\hat{G}(\lambda) \in \mathcal{B}(U, Y)$ such that for all $\hat{f} \in X$ and $\hat{u} \in U$ there is a unique solution of the problem

$$\begin{aligned} (\mathfrak{A}P - \lambda)\hat{x} &= \hat{f} \\ \hat{u} &= \mathfrak{B}P\hat{x} \\ \hat{y} &= \mathfrak{C}P\hat{x} \end{aligned}$$

which is given by

$$\hat{x} = \hat{\Phi}(\lambda)\hat{f} + \hat{\Psi}(\lambda)\hat{u}$$
$$\hat{y} = \hat{F}(\lambda)\hat{f} + \hat{G}(\lambda)\hat{u}.$$

Proof. First note that $(\mathfrak{A}P, \mathfrak{B}P, \mathfrak{C}P)$ is an impedance passive Boundary Control and Observation System on X_P. Namely for all $\hat{x} \in D(\mathfrak{A}P)$ one has $P\hat{x} \in D(\mathfrak{A})$, thus

$$\mathrm{Re}\,\langle \mathfrak{A}P\hat{x}, \hat{x}\rangle_{X_P} = \mathrm{Re}\,\langle \mathfrak{A}P\hat{x}, P\hat{x}\rangle_X \leq \mathrm{Re}\,\langle \mathfrak{B}P\hat{x}, \mathfrak{C}P\hat{x}\rangle_U$$

Further $A = \mathfrak{A}|_{D(A)}$ where $D(A) = \{x \in D(\mathfrak{A}) : \mathfrak{B}x = 0\}$ generates a contractive C_0-semigroup and so does $AP = \mathfrak{A}P|_{D(AP)}$ by Lemma 7.2.3 in [JaZw12] (also see Lemma 3.3.5 below). Clearly $P^{-1}B$ (where B is the right-inverse of $\mathfrak{B}$) serves as right-inverse of $\mathfrak{B}P$ and $\mathfrak{C}P : D(AP) \to U$ is bounded since

$$\begin{aligned}\|\mathfrak{C}P\hat{x}\|_U &\lesssim \|P\hat{x}\|_A \\ &= \sqrt{\|P\hat{x}\|_X^2 + \|\mathfrak{A}Px\|_X^2} \simeq \|\hat{x}\|_{AP}\end{aligned}$$

for all $x \in D(AP)$ since the norms $\|\cdot\|_X$, $\left\|P^{1/2}\cdot\right\| =: \|\cdot\|_{X_P}$ and $\|P\cdot\|_X$ are equivalent. Further observe that $\Phi(\lambda) = R(\lambda, A)$, in particular $\|\Phi(\lambda)\| \leq \frac{1}{\mathrm{Re}\,\lambda}$ by the Hille-Yosida Theorem 2.2.6. Let $n \in \mathbb{N}$ be such that for the n^{th}-root $Q = P^{1/n}$ of P, $0 < Q = Q^* \in \mathcal{B}(X)$, one has $\left\|Q^{-1} - I\right\|_{\mathcal{B}(X)} =: \rho < \frac{\mathrm{Re}\,\lambda}{|\lambda|} \in (0, 1]$, see Proposition 2.1.13. Note that then for $X_k := X$ equipped with the inner product $\langle\cdot,\cdot\rangle_k := \langle\cdot, Q^k\cdot\rangle_X$ for $k = 0, 1, \ldots, n$ one has

$$\left\|(Q^{-1} - I)x\right\|_k = \left\|Q^{k/2}(Q^{-1} - I)x\right\|_X = \left\|(Q^{-1} - I)Q^{k/2}x\right\|_X,$$

i.e. $\left\|Q^{-1} - I\right\|_{\mathcal{B}(X_k)} = \left\|Q^{-1} - I\right\|_{\mathcal{B}(X)} = \rho < \frac{\mathrm{Re}\,\lambda}{|\lambda|}$ for $k = 0, 1, \ldots, n$. We consider the case $n = 1$, the general case then easily follows by induction. Hence

$$\left\|\lambda\Phi(\lambda)(P^{-1} - I)\right\|_{\mathcal{B}(X)} \leq \frac{\rho\,|\lambda|}{\mathrm{Re}\,\lambda} < 1$$

and by Neumann's series $(I - \lambda\Phi(\lambda)(P^{-1} - I))^{-1} \in \mathcal{B}(X)$ exists. Note that given $\hat{f} \in X_P, \hat{u} \in U$, also writing $x = P\hat{x}$, one has the following equivalence of problems

$$\begin{aligned}
&(\mathfrak{A}P - \lambda)\hat{x} = \hat{f},\ \mathfrak{B}P\hat{x} = \hat{u},\ \mathfrak{C}P\hat{x} = \hat{y} \\
\iff\quad &(\mathfrak{A} - \lambda)x = \hat{f} + \lambda(P^{-1} - I)x,\ \mathfrak{B}x = \hat{u},\ \mathfrak{C}x = \hat{y} \\
\iff\quad &\begin{cases} x = \Phi(\lambda)\hat{f} + \lambda\Phi(\lambda)(P^{-1} - I)x + \Psi(\lambda)\hat{u}, \\ y = F(\lambda)\hat{f} + \lambda F(\lambda)(P^{-1} - I)x + G(\lambda)\hat{u} \end{cases} \\
\iff\quad &\begin{cases} \hat{x} = P^{-1}x = P^{-1}(I - \lambda\Phi(\lambda)(P^{-1} - I))^{-1}(\Phi(\lambda)\hat{f} + \Psi(\lambda)\hat{u}), \\ \hat{y} = F(\lambda)\hat{f} + \lambda F(\lambda)(P^{-1} - I)P\hat{x} + G(\lambda)\hat{u}. \end{cases}
\end{aligned}$$

From here the assertion follows. □

Proof of Lemma 3.2.24. By Proposition 3.2.25 we may and will assume that $\mathcal{H} = I$ is the identity map on $L_2(0, 1; \mathbb{F}^d)$. Also we only consider the case of constant $P_0 \in \mathbb{F}^{d\times d}$. Let $\lambda \in \mathbb{F}_0^+$, $u \in \mathbb{F}^{Nd}$ and $f \in X$ be given. First, observe that the equation

$$(\lambda - \mathfrak{A})x = f$$

has the general solution $x = h_1$ for $h := (x, x', \dots, x^{(N-1)}) : [0,1] \to (\mathbb{F}^d)^N \tilde{=} \mathbb{F}^{Nd}$ and $h(\zeta) = e^{\zeta B_\lambda} h(0) + q_f(\zeta)$ where

$$B_\lambda = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & 1 \\ \lambda P_N^{-1} - P_N^{-1} P_0 & -P_N^{-1} P_1 & \cdots & \cdots & -P_N^{-1} P_{N-1} \end{bmatrix}.$$

and $q_f(\zeta) = \int_0^\zeta e^{(\zeta - s)B_\lambda} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ -f(s) \end{pmatrix} ds$. Writing $E_\lambda = e^{B_\lambda}$ input $u = \mathfrak{B}x$ and output $y = \mathfrak{C}x$ may be written as

$$\begin{aligned} u &= W_B R_{ext} \left[\begin{smallmatrix} E_\lambda \\ I \end{smallmatrix}\right] h(0) + W_B R_{ext} \left[\begin{smallmatrix} q_f(1) \\ 0 \end{smallmatrix}\right], \\ y &= W_C R_{ext} \left[\begin{smallmatrix} E_\lambda \\ I \end{smallmatrix}\right] h(0) + W_C R_{ext} \left[\begin{smallmatrix} q_f(1) \\ 0 \end{smallmatrix}\right]. \end{aligned}$$

Since the system $(\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ is impedance passive both the matrices $W_B R_{ext} \left[\begin{smallmatrix} E_\lambda \\ I \end{smallmatrix}\right]$ and $W_C R_{ext} \left[\begin{smallmatrix} E_\lambda \\ I \end{smallmatrix}\right]$ are invertible since otherwise (choosing $f = 0$ and $h(0)$ in the kernel of one of these matrices) $\lambda \in \mathbb{F}_0^+ \cap \sigma(\mathfrak{A}|_{\ker \mathfrak{B}})$ or $\lambda \in \mathbb{F}_0^+ \cap \sigma(\mathfrak{A}|_{\ker \mathfrak{C}})$, in contradiction to $\mathfrak{A}|_{\ker \mathfrak{B}}$ and $\mathfrak{A}|_{\ker \mathfrak{C}}$ being dissipative. As a result, for any given $u \in \mathbb{F}^{Nd}$ and $f \in X$ there is a unique solution $(x, y) \in D(\mathfrak{A}) \times \mathbb{F}^{Nd}$ and clearly the map $(f, u) \mapsto (x, y) =: \left[\begin{smallmatrix} \Phi(\lambda) & \Psi(\lambda) \\ F(\lambda) & G(\lambda) \end{smallmatrix}\right] \left[\begin{smallmatrix} f \\ u \end{smallmatrix}\right]$ is linear and bounded. By the same reasoning one finds (for $f = 0$ fixed) the inverse map $G(\lambda)^{-1} : y \mapsto u$, so that $G(\lambda)$ is bijective. Further we have for all $u \in \mathbb{F}^{Nd} \setminus \{0\}$ and the corresponding solution $(x, y) \in D(\mathfrak{A}) \times \mathbb{F}^{Nd}$ of (2.17) that

$$\operatorname{Re} \langle u, G(\lambda)u \rangle_{\mathbb{F}^{Nd}} \geq \operatorname{Re} \langle \mathfrak{A}x, x \rangle_{L_2} = \operatorname{Re} \langle \lambda x, x \rangle_{L_2} = \operatorname{Re} \lambda \, \|x\|_{L_2}^2 > 0$$

so that in fact the symmetric part Sym $G(\lambda) > 0$ is strictly positive definite. □

3.3 The Generation Theorem

Up to now we did not impose any boundary conditions on the port-Hamiltonian partial differential equation, so we could not expect $\mathfrak{A}$ to be the generator of a C_0-semigroup. To make this more clear, let us state the following lemma which (together with the Hille-Yosida Theorem 2.2.6) implies that $\mathfrak{A}$ itself cannot be a generator, indeed.

Lemma 3.3.1. *Let $\mathfrak{A}$ be a (maximal) port-Hamiltonian operator where the matrix-valued functions $\mathcal{H}$ and $P_0 \in W_\infty^1(0,1; \mathbb{F}^{d\times d})$ are Lipschitz continuous. Then*

$$\sigma_p(\mathfrak{A}) = \mathbb{F}.$$

Proof. The proof is based on the theory of non-autonomous ODE. In fact for every $\lambda \in \mathbb{F}$ we have that the equation

$$\mathfrak{A}x = \lambda x$$

may be written as the ODE

$$\frac{d}{d\zeta}h(\zeta) = B_\lambda(\zeta)h(\zeta), \quad \zeta \in [0,1]$$

where we identify $h = (\mathcal{H}x, (\mathcal{H}x)', \dots, (\mathcal{H}x)^{(N-1)}) \in L_2(0,1;\mathbb{F}^d)^N$ and the matrix-valued function $B_\lambda \in W^1_\infty(0,1;\mathbb{F}^{Nd\times Nd})$ is given by

$$B_\lambda(\zeta) = \begin{bmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ \vdots & & \ddots & \ddots & \\ 0 & \cdots & 0 & \ddots & 1 \\ P_N^{-1}(\lambda\mathcal{H}^{-1}(\zeta) - P_0(\zeta)) & -P_N^{-1}P_1 & \cdots & -P_N^{-1}P_{N-1} & 0 \end{bmatrix}.$$

Since $\mathcal{H}$ and P_0 are Lipschitz continuous by the Picard-Lindelöf Theorem, see e.g. Satz III.2.5 in [We06], the equation has a unique solution on $[0,1]$, for every given initial value $h(0) = h_0 \in \mathbb{F}^{Nd}$. Choosing $h_0 \neq 0$ then gives an eigenfunction $x = \mathcal{H}^{-1}(h_j)_{j=1,\dots,d} \in D(\mathfrak{A}) \setminus \{0\}$ of $\mathfrak{A}$ for the eigenvalue λ, thus $\sigma_p(\mathfrak{A}) = \mathbb{F}$. □

Corollary 3.3.2. *Let $\mathfrak{A}$ be a (maximal) port-Hamiltonian operator where $\mathcal{H} \in W^1_\infty(0,1;\mathbb{F}^{d\times d})$ is Lipschitz continuous and $P_0 \in C([0,1];\mathbb{F}^{d\times d})$. Then*

$$\sigma(\mathfrak{A}) = \mathbb{F}.$$

Proof. By the Stone-Weierstrass Theorem (see Theorem VIII.4.7 in [We11]) there are polynomials $P_0^n \in C^\infty([0,1];\mathbb{F}^{d\times d})$ converging to P_0 in $\|\cdot\|_\infty$. *Assume* that there was $\lambda \in \rho(\mathfrak{A})$, i.e. $(\lambda - \mathfrak{A})^{-1} \in \mathcal{B}(X)$ exists. Thanks to the Neumann series then $(\lambda - \mathfrak{A} - (P_0^n - P_0)\mathcal{H})^{-1} \in \mathcal{B}(X)$ exists also for $n \geq n_0 \in \mathbb{N}$ sufficiently large. However, by Lemma 3.3.1 $\sigma(\mathfrak{A} + (P_0^n - P_0)\mathcal{H}) = \sigma(\mathfrak{A}_n) = \mathbb{F}$ for $\mathfrak{A}_n = \sum_{k=1}^N P_k(\mathcal{H}x)^{(k)} + P_0^n$ with $D(\mathfrak{A}_n) = D(\mathfrak{A})$. A contradiction. □

We hope that for suitable boundary conditions, defining a subspace $D(A) \subset D(\mathfrak{A})$ the restricted operator $A = \mathfrak{A}|_{D(A)}$ has the generator property. In the following let $\mathfrak{B}$ and $\mathfrak{C}$ be such that $\mathfrak{S} = (\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ is a port-Hamiltonian system.

Remark 3.3.3. *In fact, we do not necessarily need the output operator $\mathfrak{C}$ here and therefore remark that whenever we have $W_B \in \mathbb{F}^{Nd\times 2Nd}$ defining the input operator $\mathfrak{B}x = W_B \begin{pmatrix} f_{\partial,\mathcal{H}x} \\ e_{\partial,\mathcal{H}x} \end{pmatrix}$ we can always find $W_C \in \mathbb{F}^{Nd\times 2Nd}$ such that $\begin{bmatrix} W_B \\ W_C \end{bmatrix} \in \mathbb{F}^{2Nd\times 2Nd}$ is invertible.*

For the port-Hamiltonian system $\mathfrak{S} = (\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ we consider the *port-Hamiltonian operator* with boundary conditions

$$A := \mathfrak{A}|_{\ker(\mathfrak{B}+K\mathfrak{C})}$$

where $K \in \mathbb{F}^{Nd\times Nd}$ is a feedback matrix, i.e. the domain $D(A)$ realises the boundary condition

$$\mathfrak{B}x = -K\mathfrak{C}x, \quad x \in D(A)$$

Remark 3.3.4. *Within this subsection we may and will always assume that $K = 0$, i.e. $D(A) = \ker \mathfrak{B}$. Note that we did not demand anything more of $\mathfrak{S}$ than being a* *port-Hamiltonian system. Therefore, replacing $\mathfrak{B}$ by $\mathfrak{B} + K\mathfrak{C}$, i.e. replacing W_B by $W_B + KW_C$, we may reduce the feedback case to the case where $K = 0$. However, let us also note that in the context of nonlinear boundary feedback an extra assumption of impedance passivity for $\mathfrak{S}$ will appear. In that case $K \in \mathbb{F}^{Nd\times Nd}$ will be replaced by a nonlinear feedback operator.*

A very useful tool for the characterisation of contraction semigroups is the following lemma, which states that the contractive semigroup generator property is preserved under right multiplicative perturbation by a coercive operator provided the Hilbert space $X_P = X$ is equipped with the inner product $\langle\cdot,\cdot\rangle_P = \langle\cdot, P\cdot\rangle$. However, remark that this fact heavily depends on the contraction property and for general semigroups the assertion is false in general, see Section 6 in [Zw+10].

Lemma 3.3.5. *Let X be some Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and some strictly positive operator $P \in \mathcal{B}(X)$, i.e. P is self-adjoint and $P > \varepsilon I$ for some $\varepsilon > 0$. Denote by X_P the Hilbert space X equipped with the inner product $\langle\cdot,\cdot\rangle_P = \langle\cdot, P\cdot\rangle$. Given some linear operator $A : D(A) \subset X \to X$, consider the operator $AP : D(AP) \subset X_P \to X_P$ with $D(AP) = \{x \in X_P : Px \in D(A)\}$. Then the following are equivalent:*

1. *The operator A generates a contractive C_0-semigroup on X.*
2. *The operator AP generates a contractive C_0-semigroup on X_P.*

Proof. For this result, see Lemma 7.2.3 in [JaZw12]. □

For port-Hamiltonian operators with boundary conditions the contraction semigroup generators can be characterised either by a simple matrix condition or by dissipativity of the operator A. Note that usually the hard part of proving that an operator A generates a contraction semigroup (via the Lumer-Phillips Theorem 2.2.7) is the range condition $\operatorname{ran}(\lambda I - A) = X$ for some $\lambda > 0$.

Theorem 3.3.6. *Let $\mathfrak{S} = (\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ be a port-Hamiltonian system and $K \in \mathbb{F}^{Nd\times Nd}$ some matrix. Consider the operator $A = \mathfrak{A}|_{\ker(\mathfrak{B}+K\mathfrak{C})}$. The following are equivalent.*

1. *A generates a contraction C_0-semigroup,*
2. *A is dissipative, i.e. $\operatorname{Re}\langle Ax, x\rangle_X \le 0$ for all $x \in D(A)$,*
3. *$(W_B + KW_C)\Sigma(W_B + KW_C)^* \ge 0$ is positive semi-definite and $P_0(\zeta)$ is dissipative for a.e. $\zeta \in (0, 1)$, where $\Sigma = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \in \mathbb{C}^{2d\times 2d}$.*

Note that this result is a combination of Theorem 7.2.4 in [JaZw12], where the authors focused on the case $N = 1$, and Theorem 4.2 in [LeZwMa05], where the general case of N-th order Port-Hamiltonian systems is treated for the equivalence of 1.) and 3.). However, in both cases the authors only treat the case $P_0 = -P_0^*$ being a constant skew-symmetric matrix. For the general case where $P_0^* \neq -P_0$ is not skew-adjoint we shall use a perturbation argument. In fact, we also generalise Theorem 2.3 in [AuJa14] where only the case P_0 constant had been considered to the ζ-dependant P_0 case. Still we use the same strategy as in the proof of Theorem 7.2.4 in [JaZw12] with obvious modifications also employing some results of [LeZwMa05].

Lemma 3.3.7. *Let $W = S\left[\begin{smallmatrix} I+V & I-V \end{smallmatrix}\right]$ with $S \in \mathbb{F}^{m\times m}$ invertible and $V \in \mathbb{F}^{m\times m}$. Then the equality*

$$\ker W = \operatorname{ran} \begin{bmatrix} I-V \\ -I-V \end{bmatrix} \tag{3.17}$$

holds.

Proof. Cf. the proof of Lemma 7.3.2 in [JaZw12]. Note that

$$\operatorname{rk} W = \operatorname{rk} \begin{bmatrix} I+V & I-V \end{bmatrix} = \operatorname{rk} \begin{bmatrix} I+V & 2I \end{bmatrix} = m,$$

and similarly $\operatorname{rk} \left[\begin{smallmatrix} I-V \\ -I-V \end{smallmatrix}\right] = m$. Then the result follows from $\operatorname{ran} \left[\begin{smallmatrix} I-V \\ -I-V \end{smallmatrix}\right] \subseteq \ker W$ and noticing that these two linear subspaces have the same dimension m. □

Proof of Theorem 3.3.6. First of all, let us note that by Lemma 3.3.5 we may and will assume that $\mathcal{H} = I$, so that $X = L_2(0,1;\mathbb{F}^d)$ with the standard inner product. Moreover, we may and will assume that $K = 0$ to make the presentation clearer. Also, let us further assume that $P_0 \equiv 0$ for the moment. Our strategy is as follows, cf. Theorem 7.2.4 in [JaZw12]. We use the Lumer-Phillips Theorem 2.2.7 to establish the equivalence of 1.) and 2.). Then we show that also 2.) and 3.) are equivalent, indeed. As a last step we remove the restriction $P_0 \equiv 0$ to obtain the claimed result.

For the equivalence of 1.) and 2.) note that by the Lumer-Phillips Theorem 2.2.7 the operator A generates a contractive C_0-semigroup if and only if it is dissipative and satisfies the range condition

$$\operatorname{ran}(\lambda I - A) = X$$

for some $\lambda > 0$ (and then in fact for all $\lambda \in \mathbb{F}_0^+$). Therefore, it remains to check that for the port-Hamiltonian operator A dissipativity already implies the range condition. So let A be dissipative and take an arbitrary $f \in X$ and (for simplicity) set $\lambda = 1$. We consider the problem

$$\text{find } x \in D(A): \qquad (I - A)x = f$$

which is equivalent to the problem

$$\text{find } x \in D(\mathfrak{A}): \qquad (I - \mathfrak{A})x = f, \quad \mathfrak{B}x = -K\mathfrak{C}x$$

where we may and will assume that $K = 0$, see Remark 3.3.4. Observe that the problem $(I - \mathfrak{A}) = y$ may be equivalently expressed as the ODE

$$x(\zeta) - \sum_{k=1}^{N} P_k x^{(k)}(\zeta) = f(\zeta), \quad \zeta \in [0,1].$$

Writing $h := (x, x', \ldots, x^{(N-1)}) \in L_2(0,1;\mathbb{F}^{Nd})$ and using the invertibility of P_N we may rewrite this N^{th}-order ODE as the first order ODE

$$\frac{d}{d\zeta} h(\zeta) = Bh(\zeta) + g(\zeta), \quad \zeta \in [0,1].$$

where

$$B = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & 1 \\ P_N^{-1} & -P_N^{-1}P_1 & \cdots & \cdots & -P_N^{-1}P_{N-1} \end{bmatrix} \in \mathbb{F}^{Nd\times Nd} \cong (\mathbb{F}^{d\times d})^{N\times N}$$

$$g(\zeta) = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ -P_N^{-1}y(\zeta) \end{pmatrix} \in L_2(0,1;\mathbb{F}^{Nd}) \cong (L_2(0,1;\mathbb{F}^d))^N.$$

Therefore, we obtain the general solution formula

$$h(\zeta) = e^{\zeta B}h(0) + q(\zeta)$$

with $q(\zeta) = \int_0^\zeta e^{(\zeta-s)B}g(s)ds$ for $\zeta \in (0,1)$. For the corresponding $x \in D(\mathfrak{A})$ we then have $x \in D(A)$ if and only if

$$0 = W_B R_{ext} \begin{pmatrix} h(1) \\ h(0) \end{pmatrix} = W_B R_{ext} \begin{pmatrix} e^B h(0) + q(1) \\ h(0) \end{pmatrix},$$

or equivalently

$$W_B R_{ext} \begin{bmatrix} e^B \\ I \end{bmatrix} h(0) = W_B R_{ext} \begin{pmatrix} q(1) \\ 0 \end{pmatrix}$$

As a result, invertibility of the matrix $W_B R_{ext} \left[\begin{smallmatrix} e^B \\ I \end{smallmatrix}\right]$ will imply that there is $x \in D(A)$ such that $(I - A)x = y$, so that $\operatorname{ran}(I - A) = X$. *Assume* that $W_B R_{ext} \left[\begin{smallmatrix} e^B \\ I \end{smallmatrix}\right]$ were not invertible. Then there were $h_0 \neq 0$ such that $h_0 \in \ker W_B R_{ext} \left[\begin{smallmatrix} e^B \\ I \end{smallmatrix}\right]$. For the special choice $h(0) = h_0$ and $q \equiv 0$, i.e. $g \equiv 0$, then equation (3.18) would lead to an eigenvector $x \in D(A)$ of A with eigenvalue $\lambda = 1$, a contradiction to A being dissipative. Hence, 1.) and 2.) are equivalent.

Secondly, let us focus on the equivalence of 2.) and 3.), still under the assumption that $P_0 \equiv 0$. We have by Lemma 3.2.13 that

$$\operatorname{Re}\langle f_{\partial,\mathcal{H}x}, e_{\partial,\mathcal{H}x}\rangle_{\mathbb{F}^{Nd}} = \operatorname{Re}\langle \mathfrak{A}x, x\rangle_X, \quad x \in D(A) \subset D(\mathfrak{A}).$$

and by Corollary 3.2.20 and Lemma 3.2.17

$$\begin{aligned} \left\{ \begin{pmatrix} f_{\partial,\mathcal{H}x} \\ e_{\partial,\mathcal{H}x} \end{pmatrix} : x \in D(A) \right\} &= \ker W_B = \ker S_B \begin{bmatrix} I + V_B & I - V_B \end{bmatrix} \\ &= \ker \begin{bmatrix} I + V_B & I - V_B \end{bmatrix} \\ &= \operatorname{ran} \begin{bmatrix} I - V_B \\ -I - V_B \end{bmatrix} \end{aligned}$$

so that

$$\begin{aligned} A \text{ dissipative} &\iff \forall \begin{pmatrix} f \\ e \end{pmatrix} \in \operatorname{ran} \begin{bmatrix} I - V_B \\ -I - V_B \end{bmatrix} : \operatorname{Re}\langle f, e\rangle_{\mathbb{F}^{Nd}} \leq 0 \\ &\iff \forall l \in \mathbb{F}^{Nd} : \operatorname{Re}\langle (I - V_B)l, -(I + V_B)l\rangle_{\mathbb{F}^{Nd}} \leq 0 \\ &\iff W_B \Sigma W_B \geq 0 \text{ is positive semidefinite} \end{aligned}$$

where we used Corollary 3.2.20 again. Hence, we established the equivalence of 2.) and 3.).

Finally let us consider the general case that $P_0 \not\equiv 0$. Whenever $P_0(\zeta)$ is dissipative for a.e. $\zeta \in (0,1)$, the operator-valued function P_0 may be considered as bounded dissipative perturbation of the operator $A - P_0$ and thus whenever $W_B \Sigma W_B \geq 0$ is positive semidefinite, the operator $A - P_0$ generates a contractive C_0-semigroup and so does $A = (A - P_0) + P_0$, see Theorem III.2.7 in [EnNa00]. Therefore, it remains to prove that whenever A is dissipative, then $A - P_0$ is dissipative and $P_0(\zeta)$ is dissipative for a.e. $\zeta \in (0,1)$. On the one hand, if A is dissipative then $C_c^\infty(0,1;\mathbb{F}^d) \subseteq D(A)$ and

$$\operatorname{Re} \langle Ax, x\rangle_{L_2} = \operatorname{Re} \langle P_0 x, x\rangle_{L_2} \leq 0, \quad x \in C_c^\infty(0,1;\mathbb{F}^d)$$

and since $C_c^\infty(0,1;\mathbb{F}^d) \subseteq L_2(0,1;\mathbb{F}^d)$ is dense this implies that $\P_0(\zeta)$ is dissipative for a.e. $\zeta \in (0,1)$ thanks to Lemma 2.1.7. On the other hand if $A - P_0$ were not dissipative there were $(f,e) \in \ker W_B$ such that

$$\operatorname{Re} \langle f, e\rangle_{\mathbb{F}^{Nd}} = 1$$

and taking $x \in C^\infty([0,1];\mathbb{F}^d)$ such that $(f_{\partial,\mathcal{H}x}, e_{\partial,\mathcal{H}x}) = (f,e)$ and $\|P_0\|_{L_\infty} \|x\|_{L_2}^2 \leq \frac{1}{2}$ this leads to the contradiction

$$0 \geq \operatorname{Re} \langle Ax, x\rangle_{L_2} = \operatorname{Re} \langle (A - P_0)x, x\rangle_{L_2} + \operatorname{Re} \langle P_0 x, x\rangle_{L_2} \geq \frac{1}{2} > 0.$$

To conclude the proof we show that A has compact resolvent whenever there is $\lambda \in \rho(A)$. In fact, this easily follows from the fact that for $\lambda \in \rho(A)$, the operator $R(\lambda, A)$ is bounded as linear operator from X to $D(A) \subseteq D(\mathfrak{A})$ and $D(\mathfrak{A})$ is compactly embedded into X, see Lemma 3.2.6, so that the operator $R(\lambda, A) : X \to X$ is compact. □

As a byproduct, with this result we can also characterise the port-Hamiltonian operators A which generate a unitary C_0-semigroup $(T(t))_{t\geq 0}$. For the case of constant and skew-adjoint $P_0 = -P_0^*$ this has already been stated in Theorem 4.4 of [LeZwMa05].

Corollary 3.3.8. *Let $A = \mathfrak{A}|_{\ker \mathfrak{B}}$ be a port-Hamiltonian operator where $\mathfrak{B}$ has the form*

$$\mathfrak{B}x = W_B \begin{pmatrix} f_{\partial,\mathcal{H}x} \\ e_{\partial,\mathcal{H}x} \end{pmatrix}.$$

Then A generates a unitary C_0-semigroup if and only if $P_0(\zeta)^* = -P_0(\zeta)$ for a.e. $\zeta \in (0,1)$ and $W_B \Sigma W_B^* = 0$ for $\Sigma = \left[\begin{smallmatrix} & I \\ I & \end{smallmatrix}\right]$.

Proof. By Stone's Theorem 2.2.10 the operator A generates a unitary C_0-group if and only if both A and $-A$ are dissipative. By Theorem 3.3.6 these conditions are equivalent to the symmetric part of P_0 being both positive semi-definite $\operatorname{Sym} P_0(\zeta) \geq 0$ and negative semi-definite $\operatorname{Sym} P_0(\zeta) \leq 0$ for a.e. $\zeta \in (0,1)$, and the matrix $W_B \Sigma W_B^*$ being positive semi-definite and negative semi-definite at the same time. This can only hold true if and only if $P_0(\zeta) = -P_0(\zeta)^*$ is skew-symmetric for a.e. $\zeta \in (0,1)$ and $W_B \Sigma W_B^* = 0$ is the zero matrix.

Let us also point out the following consequence of the generation theorem for impedance passive port-Hamiltonian systems.

Corollary 3.3.9. *Any impedance passive port-Hamiltonian system* $\mathfrak{S} = (\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ *is an impedance passive Boundary Control and Observation System.*

Proof. For every impedance passive port-Hamiltonian system the operator $A = \mathfrak{A}|_{\ker \mathfrak{B}}$ is dissipative and thus generates a contractive C_0-semigroup, thanks to Theorem 3.3.6. Then by Theorem 3.2.21 the system $\mathfrak{S} = (\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ is a Boundary Control and Observation system. □

Another, even more direct consequence of Theorem 3.3.6 is

Corollary 3.3.10. *Let* $A = \mathfrak{A}|_{\ker(\mathfrak{B}+K\mathfrak{C})}$ *be a port-Hamiltonian operator. Then* A *generates a quasi-contractive* C_0*-semigroup* $(T(t))_{t\geq 0}$*, i.e.*

$$\|T(t)\| \leq e^{\omega t}, \quad t \geq 0$$

for some $\omega \in \mathbb{R}$*, if and only if* $(W_B + KW_C)\Sigma(W_B + KW_C)^* \geq 0$ *is positive semidefinite.*

Proof. Let $\omega > 0$. Then A generates a C_0-semigroup of type $(1, \omega)$ if and only if $A - \omega I$ is dissipative, if and only $(A - P_0) + (P_0 - \omega I)$ is dissipative by Theorem 3.3.6, if and only if $(W_B + KW_C)\Sigma(W_B + KW_C)^* \geq 0$ is positive semidefinite and $P_0 - \omega I \in L_\infty(0, 1; \mathbb{F}^{d\times d})$ is dissipative. From here the assertion follows. □

3.4 Standard Control Operator Formulation

Let us continue with the study of port-Hamiltonian systems which so far we considered in the boundary control and observation form

$$\begin{aligned}
\mathfrak{A}x &= \sum_{k=1}^{N} P_k(\mathcal{H}x)^{(k)} + P_0(\cdot)(\mathcal{H}x) \\
\mathfrak{B}x &= W_B R_{ext}\tau(\mathcal{H}x) \\
\mathfrak{C}x &= W_C R_{ext}\tau(\mathcal{H}x) \\
D(\mathfrak{A}) = D(\mathfrak{B}) = D(\mathfrak{C}) &= \{x \in L_2(0, 1; \mathbb{F}^d) : \ \mathcal{H}x \in H^N(0, 1; \mathbb{F}^d)\}.
\end{aligned}$$

We are interested in rewriting the boundary control part

$$\dot{x}(t) = \mathfrak{A}x(t), \quad \mathfrak{B}x(t) = u(t), \quad t \geq 0 \tag{3.18}$$

in the more standard form

$$\dot{x}(t) = A_{-1}x(t) + Bu(t), \quad t \geq 0 \tag{3.19}$$

where A_{-1} denotes the extension of $A = \mathfrak{A}|_{\ker \mathfrak{B}}$ to the extrapolation space X_{-1}^A and $B \in \mathcal{B}(U; X_{-1})$. Of course, to define X_{-1}^A and A_{-1} the resolvent set $\rho(A)$ of A should be nonempty. This is always satisfied whenever A is a C_0-semigroup generator, i.e. whenever $(\mathfrak{A}, \mathfrak{B})$ is a Boundary Control System. We want to identify the correct input operator B corresponding to the boundary input operator $\mathfrak{B}$ and we utilise the following characterisation of B for given operators $\mathfrak{B}$.

Proposition 3.4.1. *The Hilbert space adjoint $B' \in \mathcal{B}(D(A'), U)$ for the control operator B (in the standard formulation as control system) of a Boundary Control System $(\mathfrak{A}, \mathfrak{B})$ is given by*

$$\langle \mathfrak{B}x, B'y\rangle_U = \langle \mathfrak{A}x, y\rangle_X - \langle x, A'y\rangle_X, \quad x \in D(\mathfrak{A}), y \in D(A')$$

where $A = \mathfrak{A}|_{\ker \mathfrak{B}}$ and A' is its Hilbert space adjoint operator.

Proof. See Remark 10.1.6 in [TuWe09]. □

This approach obviously makes it necessary to know the Hilbert space adjoint A' of A, hence we first determine A' in the port-Hamiltonian case. This can be done quite easily via integration by parts and rewriting the boundary conditions. Let us assume that $W_B = \left[\, W_{B,1} \; W_{B,2} \,\right]$ such that $W_{B,1} + W_{B,2}$ is invertible, so that due to Lemma 3.3.7 $W = W_B \in \mathbb{F}^{Nd\times 2Nd}$ has the form

$$W = S \left[\begin{array}{cc} I+V & I-V \end{array}\right]$$

for some square matrices $S, V \in \mathbb{F}^{d\times d}$ with S invertible, where for dissipative A we have $VV^* \leq I$. Also we recall the definition of the boundary port variables, namely

$$\begin{pmatrix} f_{\partial,\mathcal{H}x} \\ e_{\partial,\mathcal{H}x} \end{pmatrix} = R_{ext}\tau(\mathcal{H}x)$$

where R_{ext} is defined as

$$R_{ext} = \frac{1}{\sqrt{2}} \left[\begin{array}{cc} Q & -Q \\ I & I \end{array}\right], \quad Q = \left[\begin{array}{cccc} P_1 & P_2 & \cdots & P_N \\ -P_2 & \cdots & -P_N & \\ \vdots & & & \\ (-1)^{N-1}P_N & & & \end{array}\right].$$

To describe the adjoint operator we analogously set $\tilde{Q} = -Q = -Q^*$ and

$$\begin{pmatrix} \tilde{f}_{\partial,\mathcal{H}x} \\ \tilde{e}_{\partial,\mathcal{H}x} \end{pmatrix} = \tilde{R}_{ext}\tau(\mathcal{H}x), \quad \tilde{R}_{ext} = \frac{1}{\sqrt{2}} \left[\begin{array}{cc} \tilde{Q} & -\tilde{Q} \\ I & I \end{array}\right].$$

Remark 3.4.2. *In particular, this means that*

$$\begin{pmatrix} \tilde{f}_{\partial,\mathcal{H}x} \\ \tilde{e}_{\partial,\mathcal{H}x} \end{pmatrix} = \begin{pmatrix} -f_{\partial,\mathcal{H}x} \\ e_{\partial,\mathcal{H}x} \end{pmatrix}, \quad x \in D(\mathfrak{A}),$$

but as we will see in just a moment the boundary port variables $\tilde{f}_{\partial,\mathcal{H}x}$ and $\tilde{e}_{\partial,\mathcal{H}x}$ are exactly those belonging to the operator structure of the adjoint operator A'.

We then have

Proposition 3.4.3. *Assume that $A = \mathfrak{A}|_{\ker \mathfrak{B}}$ generates a C_0-semigroup and $W_B = S\left[\, I+V \; I-V \,\right]$ with $S, V \in \mathbb{F}^{d\times d}$ and S invertible. Then the (Hilbert space) adjoint operator A' of A is given by*

$$A'x = -\sum_{k=1}^{N} P_k(\mathcal{H}x)^{(k)} + P_0^*(\cdot)\mathcal{H}x$$

$$D(A') = \left\{ x \in X : (\mathcal{H}x) \in H^N(0,1;\mathbb{F}^d), \; \left[\begin{array}{cc} I+V^* & I-V^* \end{array}\right] \begin{pmatrix} \tilde{f}_{\partial,\mathcal{H}x} \\ \tilde{e}_{\partial,\mathcal{H}x} \end{pmatrix} = 0 \right\}.$$

Remark 3.4.4. *Note that this statement in principle is the same as Theorem 2.24 in [Vi07]. Further observe that whenever $P_0(\zeta)^* = -P_0(\zeta)$ for a.e. $\zeta \in (0,1)$, then $A' = -\mathfrak{A}|_{D(A')}$.*

Lemma 3.4.5. *Denote by $A_{00} := \mathfrak{A}|_{D(A_{00})}$ with $D(A_{00}) := \{x \in X : \mathcal{H}x \in C_c^\infty(0,1)\}$ the minimal port-Hamiltonian operator. Then its Hilbert space adjoint is given by*

$$A_{00}'x = -\sum_{k=1}^{N} P_k(\mathcal{H}x)^{(k)} + P_0^*(\cdot)\mathcal{H}x, \quad x \in D(A_{00}') = D(\mathfrak{A}).$$

In particular, if $P_0(\zeta)^* = -P_0(\zeta)$ for a.e. $\zeta \in (0,1)$, then $A_{00}' = -\mathfrak{A}$.

Proof. First of all we may and will assume that $P_0 = 0$ since $P_0\mathcal{H}$ is bounded as linear operator on X. Also we only have to consider the case that $\mathcal{H} = I$ is the identity matrix, using that

$$\langle (\mathfrak{A}_{00}\mathcal{H}^{-1})\mathcal{H}x, \mathcal{H}y\rangle_{L_2} = \langle \mathfrak{A}_{00}x, y\rangle_X, \quad x, y \in D(\mathfrak{A}_{00}).$$

Then for the scalar-valued case we may refer to Theorem VI.1.9 in [Go66] (for the Banach space adjoint) and using the Riesz Representation Theorem, which solves the case $d = 1$. In fact, the proof given there can be easily extended to the vector-valued case if one takes care of additional transpositions appearing and replaces scalar multiplication by the dot product whenever necessary. □

Proof of Proposition 3.4.3. Let A' denote the adjoint operator of A. Since $A_{00} \subseteq A$ is an extension of the minimal operator A_{00} we have that $A' \subseteq A_{00}'$ is a restriction of the Hilbert space adjoint of the minimal operator. In particular,

$$A'x = -\sum_{k=1}^{N} P_k(\mathcal{H}x)^{(k)} - P_0^*(\cdot)(\mathcal{H}x), \quad x \in D(A') \subseteq D(A_{00}') = D(\mathfrak{A}).$$

Moreover, for all $x \in D(A)$ and $y \in D(A')$ we obtain

$$\begin{aligned}
0 &= \langle Ax, y\rangle_X - \langle x, A'y\rangle_X \\
&= \sum_{k=1}^{N} \langle P_k(\mathcal{H}x)^{(k)}, (\mathcal{H}y)\rangle_{L_2} - \langle (\mathcal{H}x), -P_k(\mathcal{H}y)^{(k)}\rangle_{L_2} \\
&= \sum_{k=1}^{N}\sum_{l=0}^{k-1} (-1)^l \left[\langle P_k(\mathcal{H}x)^{(k-l-1)}(\zeta), (\mathcal{H}y)^{(l)}(\zeta)\rangle_{\mathbb{F}^d}\right]_0^1 \\
&= \sum_{l=0}^{N-1} \left[\langle \sum_{k=l+1}^{N} (-1)^l P_k(\mathcal{H}x)^{(k-l-1)}(\zeta), (\mathcal{H}y)^{(l)}(\zeta)\rangle_{\mathbb{F}^d}\right]_0^1 \\
&= \sum_{l=0}^{N-1} \left[\langle \left(Q\begin{pmatrix} (\mathcal{H}x)(\zeta) \\ \vdots \\ (\mathcal{H}x)^{(N-1)}(\zeta)\end{pmatrix}\right)_{l+1}, (\mathcal{H}y)^{(l)}(\zeta)\rangle_{\mathbb{F}^d}\right]_0^1 \\
&= \langle \tau(\mathcal{H}x), \begin{bmatrix} Q & \\ & -Q\end{bmatrix}\tau(\mathcal{H}y)\rangle_{\mathbb{F}^d}.
\end{aligned}$$

Thus

$$\begin{aligned} D(A') &= \left\{ y \in D(\mathfrak{A}) : \left[\begin{smallmatrix} Q & \\ & -Q \end{smallmatrix}\right] \tau(\mathcal{H}y) \perp \tau(\mathcal{H}x),\ x \in D(A) \right\} \\ &= \left\{ y \in D(\mathfrak{A}) : R_{ext}^{-*} \left[\begin{smallmatrix} Q & \\ & -Q \end{smallmatrix}\right] \tau(\mathcal{H}y) \perp \left(\begin{smallmatrix} f_{\partial,\mathcal{H}x} \\ e_{\partial,\mathcal{H}x} \end{smallmatrix}\right),\ x \in D(A) \right\} \\ &= \left\{ y \in D(\mathfrak{A}) : R_{ext}^{-*} \left[\begin{smallmatrix} Q & \\ & -Q \end{smallmatrix}\right] \tau(\mathcal{H}y) \perp \operatorname{ran} \left[\begin{smallmatrix} I-V \\ -I-V \end{smallmatrix}\right] \right\} \\ &= \left\{ y \in D(\mathfrak{A}) : \left[\begin{smallmatrix} I-V^* & -(I+V^*) \end{smallmatrix}\right] R_{ext}^{-*} \left[\begin{smallmatrix} Q & \\ & -Q \end{smallmatrix}\right] \tau(\mathcal{H}y) = 0 \right\} \end{aligned}$$

where we used that

$$\left\{ \left(\begin{smallmatrix} f_{\partial,\mathcal{H}x} \\ e_{\partial,\mathcal{H}x} \end{smallmatrix}\right) :\ x \in D(A) \right\} = \ker \left[\begin{smallmatrix} I+V & I-V \end{smallmatrix}\right] = \operatorname{ran} \left[\begin{smallmatrix} I-V \\ -(I+V) \end{smallmatrix}\right].$$

Now the statement follows since

$$\left[\begin{smallmatrix} I-V^* & -(I+V^*) \end{smallmatrix}\right] R_{ext}^{-*} \left[\begin{smallmatrix} Q & \\ & -Q \end{smallmatrix}\right] = \left[\begin{smallmatrix} I+V^* & I-V^* \end{smallmatrix}\right] \tilde{R}_{ext}.$$

□

We are almost ready to state the result on B' for port-Hamiltonian systems, but first need the following auxiliary result.

Lemma 3.4.6. *Let $V \in \mathbb{F}^{m\times m}$. Then the matrix $\left[\begin{smallmatrix} I+V^* & I-V \\ I-V^* & -(I+V) \end{smallmatrix}\right]$ is invertible and*

$$\begin{aligned} &\begin{bmatrix} I+V^* & I-V \\ I-V^* & -(I+V) \end{bmatrix}^{-1} \\ &= \frac{1}{2} \begin{bmatrix} (I+VV^*)^{-1}(I+V) & (I+VV^*)^{-1}(I-V) \\ (I+V^*V)^{-1}(I-V^*) & -(I+V^*V)^{-1}(I+V^*) \end{bmatrix} \end{aligned}$$

Proof. Note that $I + VV^*$ and $I + V^*V$ are invertible since $VV^*, V^*V \geq 0$ are positive semidefinite. Also

$$V(I+V^*V)^{-1} = (I+VV^*)^{-1}V.$$

Then one easily calculates that above matrix is the inverse matrix, indeed. □

Finally it is time to state the result on the adjoint operator B' of the input operator $B \in \mathcal{B}(U; X_{-1})$ in the standard formulation.

Proposition 3.4.7. *Let $(\mathfrak{A}, \mathfrak{B})$ be a port-Hamiltonian Boundary Control System with $W_B = S \left[\begin{smallmatrix} I+V & I-V \end{smallmatrix}\right]$ for some invertible $S \in \mathbb{F}^{Nd\times Nd}$ and some $V \in \mathbb{F}^{Nd\times Nd}$. Then the Hilbert space adjoint $B' \in \mathcal{B}(D(A'), U)$ of the control operator $B \in \mathcal{B}(U; X_{-1})$ in the standard formulation is given by*

$$B'x = \frac{1}{4} S^{-*}(I+VV^*)^{-1} \begin{bmatrix} I-V & I+V \end{bmatrix} \begin{bmatrix} Q & -Q \\ I & I \end{bmatrix} \tau(\mathcal{H}x), \quad x \in D(A').$$

Proof. We may and will assume that $S = I$ in the decomposition of $W = W_B$. For every $x \in D(\mathfrak{A})$ and $y \in D(A')$ one has

$$\begin{aligned}\langle \mathfrak{A}x, y\rangle_X - \langle x, A'y\rangle_X &= \sum_{k=1}^{N} \left(\langle P_k \frac{\partial^k}{\partial\zeta^k}(\mathcal{H}x), (\mathcal{H}y)\rangle_{L_2} + \langle x, P_k^* \frac{\partial^k}{\partial\zeta^k}(\mathcal{H}y)\rangle_{L_2} \right) \\ &\quad + \langle P_0(\mathcal{H}x), (\mathcal{H}y)\rangle_{L_2} - \langle (\mathcal{H}x), P_0^*(\mathcal{H}y)\rangle_{L_2} \\ &= \sum_{k=1}^{N}\sum_{l=1}^{k} \left[\langle P_k \frac{\partial\zeta^{k-l}}{\partial\zeta^{k-l}}(\mathcal{H}x)(\zeta), \frac{\partial^{l-1}}{\partial\zeta^{l-1}}(\mathcal{H}y)(\zeta)\rangle_{\mathbb{F}^d} \right]_0^1 \\ &= \langle \begin{bmatrix} Q & \\ & -Q \end{bmatrix} \tau(\mathcal{H}x), \tau(\mathcal{H}y)\rangle_{\mathbb{F}^{2Nd}}.\end{aligned}$$

Thus, from Proposition 3.4.1 we have for all $x \in D(\mathfrak{A})$ and $y \in D(A')$

$$\begin{aligned} 0 &= \langle \mathfrak{B}x, B'y\rangle_{\mathbb{F}^{Nd}} - (\langle \mathfrak{A}x, y\rangle_X - \langle x, A'y\rangle_{\mathcal{H}}) \\ &= \langle \begin{bmatrix} I+V & I-V \end{bmatrix} R_{ext}\tau(\mathcal{H}x), B'y\rangle_{\mathbb{F}^{Nd}} \\ &\quad - \langle \begin{bmatrix} Q & \\ & -Q \end{bmatrix} \tau(\mathcal{H}x), \tau(\mathcal{H}y)\rangle_{\mathbb{F}^{2Nd}} \\ &= \langle \begin{bmatrix} I+V & I-V \\ I-V^* & -(I+V^*) \end{bmatrix} R_{ext}\tau(\mathcal{H}x), \begin{bmatrix} B'y \\ 0 \end{bmatrix}\rangle_{\mathbb{F}^{2Nd}} \\ &\quad - \langle \begin{bmatrix} I+V^* & I-V^* \\ I-V & -(I+V) \end{bmatrix} R_{ext}\tau(\mathcal{H}x), \begin{bmatrix} I+V^* & I-V^* \\ I-V & -(I+V) \end{bmatrix}^{-1} R_{ext}^{-*} \begin{bmatrix} Q & \\ -Q & \end{bmatrix} \tau(\mathcal{H}y)\rangle_{\mathbb{F}^{2Nd}} \\ &= \langle \Psi(\mathcal{H}x), \begin{pmatrix} B'y \\ 0 \end{pmatrix} - \frac{1}{4} \begin{bmatrix} (I+VV^*)^{-1}(I+V) & (I+VV^*)^{-1}(I-V) \\ (I+V^*V)^{-1}(I-V^*) & -(I+V^*V)^{-1}(I+V^*) \end{bmatrix} R_{ext}\tau(\mathcal{H}y)\rangle_{\mathbb{F}^{2Nd}} \end{aligned}$$

where $\Psi(\mathcal{H}x) := \begin{bmatrix} I+V & I-V \\ I-V^* & -(I+V^*) \end{bmatrix} R_{ext}\tau(\mathcal{H}x)$ and since $\operatorname{ran}\Psi = \mathbb{F}^{2Nd}$ it follows

$$B'y = \frac{1}{4}(I+VV^*)^{-1} \begin{bmatrix} I+V & I-V \end{bmatrix} \begin{bmatrix} Q & -Q \\ I & I \end{bmatrix} \tau(\mathcal{H}y), \quad y \in D(A')$$

as claimed. □

Chapter 4

Stabilisation of Port-Hamiltonian Systems via Static Linear Boundary Feedback

In the previous chapter we have seen that any dissipative port-Hamiltonian operator $A = \mathfrak{A}|_{D(A)}$ with a dissipative boundary condition generates a contractive C_0-semigroup on the energy state space $X = L_2(0,1;\mathbb{F}^d)$ equipped with the energy norm $\|\cdot\|_X = \|\cdot\|_{\mathcal{H}}$. Also we have seen that the dissipative boundary condition may often appear in the form of a static boundary feedback $\mathfrak{B}x = -K\mathfrak{C}x$ for $\mathfrak{B}$ and $\mathfrak{C}$ being the (boundary) input and (boundary) output maps of a port-Hamiltonian system $\mathfrak{S} = (\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ in boundary control and observation form. We also noticed that the contraction property physically means that the energy $H(t) = \frac{1}{2}\|x(t)\|_X^2$ of the system decreases, or more precisely does not increase. Within this section we go a step further and ask whether a given system not only has non-increasing energy, but actually is decreasing in the long term. (From a stabilisation design point of view the problem may be reformulated as finding suitable sufficient conditions on the feedback operator K (and the system $\mathfrak{S}$) to obtain the desired stabilisation property.) In other words, we investigate adequate conditions under which a port-Hamiltonian system with dissipative boundary conditions is stable (in some sense). We stress that the fact that this includes only stability in the semigroup sense, i.e. the boundary conditions are fixed and may result from a static linear boundary feedback ($\mathfrak{B}x = -K\mathfrak{C}x$). In fact, a variety of stability concepts are known for C_0-semigroups of which we only consider two, arguably the two most important ones. On the one hand we have asymptotic (strong) stability, i.e. given a C_0-semigroup $(S(t))_{t\geq 0}$ on some Banach space, do all the trajectories $(S(t)x)_{t\geq 0}$ converge to 0 for every x from that Banach space? On the other hand (uniform) exponential stability, i.e. given an (asymptotically stable) C_0-semigroup $(S(t))_{t\geq 0}$ on some Banach space, are there constants $M \geq 1$ and $\omega < 0$ such that the decay is uniform in x, so that for all x from this space

$$\|S(t)x\| \leq Me^{\omega t}\|x\|, \quad t \geq 0?$$

In the following $(T(t))_{t\geq 0}$ always denotes the contractive C_0-semigroup generated by a dissipative port-Hamiltonian operator $A = \mathfrak{A}|_{\mathfrak{B}+K\mathfrak{C}}$ resulting from a port-Hamiltonian system $\mathfrak{S} = (\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ and a suitable feedback matrix $K \in \mathbb{F}^{Nd\times Nd}$ ensuring dissipativity of the operator

$$\begin{aligned} Ax &= \sum_{k=1}^{N} P_k(\mathcal{H}x)^{(k)} + P_0(\cdot)(\mathcal{H}x) \\ D(A) &= \left\{x \in X : \ \mathcal{H}x \in H^N(0,1;\mathbb{F}^d), \ \mathfrak{B}x = -K\mathfrak{C}x\right\}. \end{aligned}$$

(Note that w.l.o.g. we may always assume that $K = 0$.) Since the resolvent set $\rho(A) \neq \emptyset$ is non-empty and A then has a compact resolvent, see Theorem 3.3.6, we have the spectral theorem

$$\sigma(A) = \sigma_p(A)$$

which proves very useful in the context of asymptotic stability.

4.1 Known Results for the Case $N = 1$

We start by giving an overview on previous results on stability of port-Hamiltonian systems for the case $N = 1$, i.e. within this section $\mathfrak{A}$ always has the form

$$\mathfrak{A}x = P_1(\mathcal{H}x)' + P_0(\cdot)(\mathcal{H}x), \quad x \in D(\mathfrak{A}).$$

Systems of this form have been considered especially in the article [Vi+09] and most of the results mentioned here may be found there. However, for analytic Hamiltonian density matrix functions $\mathcal{H}$ this topic had already been addressed in [RaTa74] and, in fact, some ideas of the latter article were used in [CoZu95] and then again in the aforementioned [Vi+09]. (Also note that exponential stability plays a crucial part in the PhD thesis [Vi07] by one of the coauthors of [Vi+09].) Moreover, the results have been presented in the monograph [JaZw12]. All these articles have in common that they prove exponential stability (for suitable dissipative boundary conditions) in the same way, using a *final observability* estimate which in [CoZu95] is called a *sideways energy estimate*. It says the following.

Lemma 4.1.1. *Let $\mathcal{H}$ be Lipschitz-continuous. There are constants $\tau > 0$ and $c > 0$ such that for every solution $x \in W^1_\infty(\mathbb{R}_+; X) \cap L_\infty(\mathbb{R}_+; D(\mathfrak{A}))$ of*

$$\frac{d}{dt}x(t) = \mathfrak{A}x(t), \quad t \geq 0$$

with $\|x(t)\|_X$ non-increasing the inequality

$$\|x(\tau)\|^2_X \leq c \int_0^\tau |(\mathcal{H}x)(t, 0)|^2 \, dt$$

holds.

Before actually proving this result we comment on some variants and the history of this lemma.

Remark 4.1.2. *1. If one is only interested in one particular contraction C_0-semigroup the lemma may be expressed in the following way. Let A be be a dissipative port-Hamiltonian operator of order $N = 1$ (with Lipschitz continuous Hamiltonian density matrix function $\mathcal{H}$) generating the contractive C_0-semigroup $(T(t))_{t\geq 0}$. Then there are constants $\tau > 0$ and $c > 0$ such that for every $x_0 \in D(A)$ and the trajectory $x(t) := T(t)x_0$ $(t \geq 0)$ one has*

$$\|x(\tau)\|_X^2 \leq c \int_0^\tau |(\mathcal{H}x)(t,0)|^2 \, dt.$$

For this version see, e.g. Lemma III.1 in [Vi+09] and Lemma 9.1.2 in [JaZw12]. Also this estimate had already been used in the proof of Theorem 3 in [RaTa74].

2. *The slightly more general, but arguably more complicated formulation above takes into account that later on we want to use the same result in the context of nonlinear boundary feedback.*

3. *Note that the inequality in Lemma 4.1.1 is quite similar to a usual observability inequality of the form*

$$\|x(0)\|_X \leq c \|Cx\|_{L_p(0,\tau;Y)},$$

but in the port-Hamiltonian case above only the evolved state at time $\tau > 0$ may be estimated from the observation of $(\mathcal{H}x)(t,0)$ for times $t \in [0,\tau]$. Therefore, the terminology final observability.

Proof of Lemma 4.1.1. We use the same strategy as in the proof of Lemma 9.1.2 in [JaZw12]. In fact, the proof carries over almost literally. We begin by choosing $\gamma > 0$ such that

$$P_1^{-1} + \gamma\mathcal{H}(\zeta) \geq 0, \quad -P_1^{-1} + \gamma\mathcal{H}(\zeta) \geq 0, \quad \text{a.e } \zeta \in (0,1)$$

are positive semidefinite and $\kappa > 0$ such that

$$2\,\mathrm{Re}\,(P_1^{-1}P_0(\zeta)\mathcal{H}(\zeta)) + \mathcal{H}'(\zeta) \leq \kappa\mathcal{H}(\zeta), \quad \text{a.e. } \zeta \in (0,1) \tag{4.1}$$

and then $\tau > 2\gamma$. Now let $x \in W_\infty^1(\mathbb{R}_+; X) \cap L_\infty(\mathbb{R}_+; D(\mathfrak{A}))$ be a solution of $\dot{x} = \mathfrak{A}x$ and define

$$F(\zeta) = \int_{\gamma(1-\zeta)}^{\tau-\gamma(1-\zeta)} \langle x(t,\zeta), \mathcal{H}(\zeta)x(t,\zeta)\rangle_{\mathbb{F}^d} dt, \quad \zeta \in [0,1]. \tag{4.2}$$

Since $\mathcal{H}$ is Lipschitz continuous, the function $F : [0,1] \to \mathbb{R}$ is a.e. differentiable with

$$\begin{aligned}
F'(\zeta) = &\int_{\gamma(1-\zeta)}^{\tau-\gamma(1-\zeta)} \langle x(t,\zeta), \frac{\partial}{\partial\zeta}\left(\mathcal{H}(\zeta)x(t,\zeta)\right)\rangle_{\mathbb{F}^d} + \langle \frac{\partial}{\partial\zeta} x(t,\zeta), \mathcal{H}(\zeta)x(t,\zeta)\rangle_{\mathbb{F}^d} dt \\
&+ \gamma\langle x(\gamma(1-\zeta),\zeta), \mathcal{H}(\zeta)x(\gamma(1-\zeta),\zeta)\rangle_{\mathbb{F}^d} \\
&+ \gamma\langle x(\tau-\gamma(1-\zeta),\zeta), \mathcal{H}(\zeta)x(\tau-\gamma(1-\zeta),\zeta)\rangle_{\mathbb{F}^d} \\
= &\int_{\gamma(1-\zeta)}^{\tau-\gamma(1-\zeta)} \langle x(t,\zeta), P_1^{-1}\left(\frac{\partial}{\partial t}x(t,\zeta) - P_0(\zeta)(\mathcal{H}x)(t,\zeta)\right)\rangle_{\mathbb{F}^d} dt \\
&+ \int_{\gamma(1-\zeta)}^{\tau-\gamma(1-\zeta)} \langle P_1^{-1}\frac{\partial}{\partial t}x(t,\zeta), x(t,\zeta)\rangle_{\mathbb{F}^d} \\
&\quad - \langle \mathcal{H}'(\zeta)x(t,\zeta) + P_1^{-1}P_0(z)(\mathcal{H}x)(t,\zeta), x(t,\zeta)\rangle_{\mathbb{F}^d} dt \\
&+ \gamma\langle x(\gamma(1-\zeta),\zeta), \mathcal{H}(\zeta)x(\gamma(1-\zeta),\zeta)\rangle_{\mathbb{F}^d} \\
&+ \gamma\langle x(\tau-\gamma(1-\zeta),\zeta), \mathcal{H}(\zeta)x(\tau-\gamma(1-\zeta),\zeta)\rangle_{\mathbb{F}^d} \\
= &\int_{\gamma(1-\zeta)}^{\tau-\gamma(1-\zeta)} \frac{d}{dt}\langle x(t,\zeta), P_1^{-1}x(t,\zeta)\rangle_{\mathbb{F}^d} dt \\
&- \int_{\gamma(1-\zeta)}^{\tau-\gamma(1-\zeta)} \langle x(t,\zeta), \mathcal{H}'(\zeta)x(t,\zeta)\rangle_{\mathbb{F}^d} dt \\
&- \int_{\gamma(1-\zeta)}^{\tau-\gamma(1-\zeta)} 2\,\mathrm{Re}\,\langle x(t,\zeta), P_1^{-1}P_0(\zeta)(\mathcal{H}x)(t,\zeta)\rangle_{\mathbb{F}^d} dt \\
&+ \gamma\langle x(\gamma(1-\zeta),\zeta), \mathcal{H}(\zeta)x(\gamma(1-\zeta),\zeta)\rangle_{\mathbb{F}^d} \\
&+ \gamma\langle x(\tau-\gamma(1-\zeta),\zeta), \mathcal{H}(\zeta)x(\tau-\gamma(1-\zeta),\zeta)\rangle_{\mathbb{F}^d} \\
= &-2\,\mathrm{Re}\int_{\gamma(1-\zeta)}^{\tau-\gamma(1-\zeta)} \langle x(t,\zeta), \left(P_1^{-1}P_0(\zeta)\mathcal{H}(\zeta) + \frac{1}{2}\mathcal{H}'(\zeta)\right)x(t,\zeta)\rangle_{\mathbb{F}^d} dt \\
&+ \langle x(\tau-\gamma(1-\zeta),\zeta), (P_1^{-1} + \gamma\mathcal{H}(\zeta))x(\tau-\gamma(1-\zeta),\zeta)\rangle_{\mathbb{F}^d} \\
&+ \langle x(\gamma(1-\zeta),\zeta), (-P_1^{-1} + \gamma\mathcal{H}(\zeta))x(\gamma(1-\zeta),\zeta)\rangle_{\mathbb{F}^d}.
\end{aligned}$$

By the choice of $\tau, \gamma > 0$ we find that

$$F'(\zeta) \geq -2\,\mathrm{Re}\int_{\gamma(1-\zeta)}^{\tau-\gamma(1-\zeta)} \langle x(t,\zeta), \left(P_1^{-1}P_0(\zeta)\mathcal{H}(\zeta) + \frac{1}{2}\mathcal{H}'(\zeta)\right)x(t,\zeta)\rangle_{\mathbb{F}^d} dt$$

and since $\mathcal{H}', P_0 \in L_\infty(0,1;\mathbb{F}^{d\times d})$ are essentially bounded, this implies

$$F'(\zeta) \geq -\kappa\int_{\gamma(1-\zeta)}^{\tau-\gamma(1-\zeta)} \langle x(t,\zeta), \mathcal{H}(\zeta)x(t,\zeta)\rangle_{\mathbb{F}^d} dt = -\kappa F(\zeta).$$

Then

$$F(1) \geq e^{-\kappa(1-\zeta)}F(\zeta) \geq e^{-\kappa}F(\zeta), \quad \zeta \in [0,1]$$

and since the energy $\frac{1}{2}\|x(t)\|_X^2$ is non-increasing we get

$$\begin{aligned}
\int_\gamma^{\tau-\gamma} \|x(t)\|_X^2\, dt &\geq (\tau - 2\gamma)\|x(\tau-\gamma)\|_X^2 \\
&\geq (\tau - 2\gamma)\|x(\tau)\|_X^2
\end{aligned}$$

and then

$$\begin{aligned}(\tau - 2\gamma)\,\|x(\tau)\|_X^2 &\leq \int_\gamma^{\tau-\gamma} \|x(t)\|_X^2\,dt \\ &= \int_0^1 \int_\gamma^{\tau-\gamma} \langle x(t,\zeta), \mathcal{H}(\zeta)x(t,\zeta)\rangle_{\mathbb{F}^d} dt d\zeta \\ &\leq \int_0^1 \int_{\gamma(1-\zeta)}^{\tau-\gamma(1-\zeta)} \langle x(t,\zeta), \mathcal{H}(\zeta)x(t,\zeta)\rangle_{\mathbb{F}^d} dt d\zeta \\ &= \int_0^1 F(\zeta)d\zeta \leq e^\kappa F(1) \\ &= e^\kappa \int_0^\tau \langle x(t,\zeta), \mathcal{H}(\zeta)x(t,\zeta)\rangle_{\mathbb{F}^d} dt \\ &\leq \frac{1}{m} e^\kappa \int_0^\tau |(\mathcal{H}x)(t,\zeta)|^2\,dt\end{aligned}$$

and the result follows for

$$c = \frac{e^\kappa}{2m(\tau - 2\gamma)}.$$

□

Corollary 4.1.3. *If the contraction condition of Lemma 4.1.1 is dropped, the assertion is the following. There are constants τ, γ and $c > 0$ with $\tau > 2\gamma$ such that for every solution $x \in W^1_{\infty,loc}(\mathbb{R}_+; X) \cap L_{\infty,loc}(\mathbb{R}_+; D(\mathfrak{A}))$ of $\dot{x} = \mathfrak{A}x$ the estimate*

$$\int_\gamma^{\tau-\gamma} \|x(t)\|_X^2\,dt \leq c \int_0^\tau |(\mathcal{H}x)(t,0)|^2\,dt$$

is satisfied.

Proof. Note that we only needed the contraction property for the proof of Lemma 4.1.1 to show that

$$(\tau - 2\gamma)\,\|x(\tau)\|_X^2 \leq \int_\gamma^{\tau-\gamma} \|x(t)\|_X^2\,dt.$$

Without this estimate at hand, the observation inequality takes the form above. □

Remark 4.1.4. *Similarly, if*

$$\|x(t+s)\|_X^2 \leq \|x(t)\|_X^2 + \|f\|_{L_2(t,t+s)}^2, \quad s, t \geq 0$$

for some function $f \in L_{2,loc}(\mathbb{R}_+; X)$, then

$$\|x(\tau)\|_X^2 \leq c \int_0^\tau |(\mathcal{H}x)(t,0)|^2\,dt + \|f\|_{L_2(\gamma,\tau)}^2.$$

Proof. Use that

$$\begin{aligned}\|x(\tau)\|_X^2 &= \frac{1}{\tau - 2\gamma} \int_\gamma^{\tau-\gamma} \|x(\tau)\|_X^2\,dt \\ &\leq \frac{1}{\tau - 2\gamma} \int_\gamma^{\tau-\gamma} \|x(t)\|_X^2 + \|f\|_{L_2(t,\tau)}^2\,dt \\ &\leq \frac{1}{\tau - 2\gamma} \int_\gamma^{\tau-\gamma} \|x(t)\|_X^2\,dt + \|f\|_{L_2(\gamma,\tau)}^2.\end{aligned}$$

□

Actually it is quite simple to deduce uniform exponential stability (under appropriate dissipation conditions) using the inequality of Lemma 4.1.1.

Theorem 4.1.5. *Let $\mathcal{H}$ be Lipschitz-continuous. If the operator A satisfies the assumption*

$$\operatorname{Re} \langle Ax, x\rangle_X \leq -\kappa \left|(\mathcal{H}x)(0)\right|^2, \qquad x \in D(A)$$

for some $\kappa > 0$, then A generates a uniformly exponentially stable and contractive C_0-semigroup on the Hilbert space X.

Proof. We again follow the lines of proof in [JaZw12], there Theorem 9.1.3. From Lemma 4.1.1 we have constants $\tau, c > 0$ such that for every $x_0 \in D(A)$ and the corresponding classical solution $x = T(\cdot)x \in C^1(\mathbb{R}_+; X) \cap C(\mathbb{R}_+; D(A))$ of the Cauchy problem $\dot{x} = Ax,\ x(0) = x_0$ we have

$$\|x(\tau)\|_X^2 \leq c \, \|(\mathcal{H}x)(\cdot, 0)\|_{L_2(0,\tau;\mathbb{F}^d)}^2 .$$

Moreover, since $x \in C^1(\mathbb{R}_+; X) \cap C(\mathbb{R}_+; D(A))$ is a classical solution the derivative

$$\frac{1}{2}\frac{d}{dt} \|x(t)\|_X^2 = \operatorname{Re} \langle Ax(t), x(t)\rangle_X$$

exists for all $t \geq 0$ and hence

$$\begin{aligned}\frac{1}{2} \|x(t)\|_X^2 - \frac{1}{2} \|x(0)\|_X^2 &= \int_0^\tau \operatorname{Re} \langle Ax(t), x(t)\rangle_X dt \\ &\leq -\kappa \int_0^\tau \left|(\mathcal{H}x)(t,0)\right|^2 dt \\ &\leq -\frac{\kappa}{c} \|x(\tau)\|_X^2 .\end{aligned}$$

This implies that

$$\|T(\tau)x_0\|_X = \|x(\tau)\|_X \leq \sqrt{\frac{c}{c+\kappa}} \|x(0)\|_X = \sqrt{\frac{c}{c+\kappa}} \|x_0\|_X$$

and since this inequality holds for every choice of $x_0 \in D(A)$ which is a dense subset of X, we obtain that

$$\|T(\tau)\| \leq \sqrt{\frac{c}{c+\kappa}} < 1,$$

so that the C_0-semigroup $(T(t))_{t\geq 0}$ is uniformly exponentially stable by Remark 2.2.12. □

This is the original proof of Theorem 4.1.5 as it already appeared in the articles [RaTa74] and [Vi+09]. However, within this PhD thesis we also encounter two new proofs of the same theorem, using two different methods which also can be used to tackle the stability problem for systems where $N \geq 2$. Clearly the proof of Theorem 4.1.5 does not make any use of the fact that $N = 1$ except for the validity of the observability inequality which is of the form

$$\|T(\tau)x_0\|_X \leq c \, \|CT(\cdot)x_0\|_{L_2(0,\tau;Y)}$$

where in this case $Cz = z(0)$ is the point evaluation at $\zeta = 0$, but more generally may be an admissible observation operator. The proof of Theorem 4.1.5 actually shows the following. If A is any generator of a contraction C_0-semigroup $(T(t))_{t\geq 0}$ on a Hilbert space X and there are constants $c, \tau > 0$ and a linear map $C : D(C) \subseteq D(A) \to Y$ (where Y is another Hilbert space) such that

$$\|T(\tau)x_0\|_X \leq c\,\|CT(\cdot)x_0\|_{L_2(0,\tau;Y)}\,, \quad x_0 \in D(A)$$

and the operator A satisfies the dissipation inequality

$$\mathrm{Re}\,\langle Ax, x\rangle_X \leq -\kappa\,\|Cx\|_Y^2\,, \quad x \in D(A)$$

for some $\kappa > 0$, then the C_0-semigroup $(T(t))_{t\geq 0}$ is uniformly exponentially stable. Therefore, it is sufficient to establish an observability inequality as above to obtain uniform exponential stability. Unfortunately, for port-Hamiltonian systems with $N \geq 2$ we were not able for prove a similar inequality. Because of this other methods are used for $N \geq 2$ to get asymptotic and uniform exponential stability results.

4.2 Asymptotic Stability

We continue with the investigation of stability properties for port-Hamiltonian systems of a higher order $N \geq 2$. The following example should serve as a motivation why in this section we do not tackle the problem of uniform exponential stability directly, but rather start with the much less restrictive problem of finding dissipation conditions under which the systems is asymptotically stable at least. In fact, this example shows that for port-Hamiltonian systems asymptotic and exponential stability are not equivalent (as they are for finite dimensional systems). Since it is well-known that for C_0-semigroups on infinite dimensional systems asymptotic stability does not necessarily imply uniform exponential stability, this result is no surprise. More important is the other information we receive from the example, namely that for exponential stability of port-Hamiltonian systems with order $N \geq 2$ – in contrast to port-Hamiltonian systems of order $N = 1$ – it is not enough to have strictly dissipative boundary conditions at one end and arbitrary conservative or dissipative boundary conditions at the other end. This leads to the conclusion that for port-Hamiltonian systems of order $N \geq 2$ it is more difficult to design boundary feedback controllers such that the closed loop system becomes exponentially stable. Another reason for first having a look on asymptotic stability, is the technique of proof we utilise. Namely for exponential stability, the Gearhart-Greiner-Prüss-Theorem will help, but as assumption requires that the spectrum of the generator $\sigma(A) \subset \mathbb{C}_0^-$ lies in the open left half plane, and by compactness of the resolvent for the port-Hamiltonian operators this is equivalent to the semigroup $(T(t))_{t\geq 0}$ generated by A being asymptotically stable. From that perspective this subsection may also be seen as a preparation for the uniform exponential stability results that follow in the subsequent sections.

Example 4.2.1 (Schrödinger Equation)**.** *(Cf. Example 2.18 in [AuJa14].) We have seen that for $N = 1$ it was enough to have strict dissipative boundary conditions at one end whereas the boundary conditions at the other end may be conservative or dissipative, as we wish. Therefore, one might ask whether a similar result also* *holds for port-Hamiltonian systems of order $N \geq 2$, i.e. given a port-Hamiltonian operator of order $N \geq 2$ and with boundary conditions such that*

$$\operatorname{Re}\langle Ax, x\rangle_X \leq -\kappa \sum_{k=0}^{N-1} \left|(\mathcal{H}x)^{(k)}(0)\right|^2, \quad x \in D(A)$$

for some $\kappa > 0$, is the C_0-semigroup $(T(t))_{t\geq 0}$ generated by A then always uniformly exponentially stable? Unfortunately this is not the case, as we show now, repeating Example 2.18 in [AuJa14]. We consider the simplest port-Hamiltonian system with order $N \geq 2$ we can think of, namely the one-dimensional Schrödinger equation on the unit interval

$$i\frac{\partial \omega}{\partial t}(t,\zeta) + \frac{\partial^2 \omega}{\partial \zeta^2}(t,\zeta) = 0, \quad t \geq 0,\ \zeta \in (0,1) \tag{4.3}$$

where $\mathbb{F} = \mathbb{C}$ and we choose the following boundary conditions

$$\begin{aligned}\frac{\partial \omega}{\partial \zeta}(t,0) &= -ik\omega(t,0),\\ \frac{\partial \omega}{\partial \zeta}(t,1) &= \alpha\omega(t,1), \qquad t \geq 0\end{aligned} \tag{4.4}$$

for some constants $k > 0$ and $\alpha \in \mathbb{R} \setminus \{0\}$. In the introductory examples we have already seen that the energy functional is given by

$$E[\omega(t,\cdot)] = \frac{1}{2}\int_0^1 |\omega(t,\zeta)|^2\, d\zeta, \qquad t \geq 0 \tag{4.5}$$

and this is a second order port-Hamiltonian operator

$$Ax = ix'', \quad D(A) = \{z \in H^2(0,1;\mathbb{C}) : z'(0) = -ikz(0), z'(1) = \alpha z(1)\} \tag{4.6}$$

i.e. the Hamiltonian density function $\mathcal{H} \equiv 1$ is identically one, $P_1 = P_0 \equiv 0$ are identically zero and $P_2 = i$ is the multiplication operator for the factor i. Integration by parts and using the boundary conditions we deduce

$$\begin{aligned}\operatorname{Re}\langle Ax, x\rangle_{L_2} &= \operatorname{Im}\left(\langle x'(0), x(0)\rangle_{\mathbb{C}} - \langle x'(1), x(1)\rangle_{\mathbb{C}}\right)\\ &= -\frac{1}{2}\left(k\,|x(0)|^2 + \frac{1}{k}\,|x'(0)|^2\right), \quad x \in D(A).\end{aligned} \tag{4.7}$$

We claim that the semigroup $(T(t))_{t\geq 0}$ generated by A is not uniformly exponentially stable, but only asymptotically (strongly) stable. We show this assertion by applying the Gearhart-Greiner-Prüss-Huang Stability Theorem 2.2.17 and prove that $\sigma(A) \subseteq \mathbb{C}_0^-$, but

$$\sup_{i\mathbb{R}} \|R(\cdot, A)\| = \infty.$$

Let us first prove asymptotic stability. Thanks to the following characterisation of asymptotically stable semigroups whenever its generator A has compact resolvent, this is quite standard and easy.

$$(T(t))_{t\geq 0} \text{ asymptotically stable} \quad \Longleftrightarrow \quad \sigma_p(A) \subseteq \mathbb{C}_0^-, \tag{4.8}$$

see Corollary 2.2.16. First we consider the case $\beta = 0$ and solve the problem

$$\text{find } x \in D(A): \quad -Ax = f$$

for $f \in X = L_2(0,1;\mathbb{C})$, finding that then

$$x(\zeta) = \frac{i + k\zeta}{ik + \alpha(1 - ik)} \left[\int_0^1 \int_0^s f(r)drds - \int_0^1 f(s)ds\right] - i\int_0^\zeta \int_0^s f(r)dr.$$

Now take any $\beta \in \mathbb{R} \setminus \{0\}$ and $f \in X$. We solve the problem

$$\text{find } x \in D(A): \quad (i\beta - A)x = f \tag{4.9}$$

and obtain the solution

$$x(\zeta) = (\cosh(\sqrt{\beta}\zeta) - \frac{ik}{\sqrt{\beta}}\sinh(\sqrt{\beta}\zeta))x_{\beta,f}(0) + \int_0^\zeta \frac{i}{\sqrt{\beta}}\sinh(\sqrt{\beta}(\zeta - s))f(s)ds$$

with the value $x(0) = x_{\beta,f}(0)$ given by

$$x_{\beta,f}(0) = \frac{\int_0^1 i(\cosh(\sqrt{\beta}(1-\xi)) - \frac{1}{\sqrt{\beta}}\sinh(\sqrt{\beta}(1-\xi)))f(\xi)d\xi}{(\alpha + ik)\cosh(\sqrt{\beta}) - \left(\frac{i\alpha k}{\sqrt{\beta}} + \sqrt{\beta}\right)\sinh(\sqrt{\beta})}.$$

First of all this shows that the resolvent $R(i\beta, A) \in \mathcal{B}(X)$ exists for all $\beta \in \mathbb{R}$, i.e. $i\mathbb{R} \cap \sigma_p(A) = \emptyset$, and since A has compact resolvent and is dissipative this already implies

$$\sigma(A) = \sigma_p(A) \subseteq \mathbb{C}_0^-$$

and the C_0-semigroup $(T(t))_{t\geq 0}$ is asymptotically stable thanks to Corollary 2.2.16. Moreover, having the explicit formula for the resolvents on $i\mathbb{R}$ at end we can even say more, namely we show that the resolvents are not uniformly bounded on the imaginary axis and thus $(T(t))_{t\geq 0}$ is not uniformly exponentially stable. For this end we choose $f = \mathbf{1} \in L_2(0,1)$ and obtain for $\beta \neq 0$ that

$$\begin{aligned}(R(i\beta, A)\mathbf{1})(\zeta) &= (\cosh(\sqrt{\beta}\zeta) - \frac{ik}{\sqrt{\beta}}\sinh(\sqrt{\beta}\zeta)) \\ &\times \frac{i(\frac{1}{\sqrt{\beta}}\sinh(\sqrt{\beta}) - \frac{1}{\beta}\cosh(\sqrt{\beta}) + \frac{1}{\beta})}{(\alpha + ik)\cosh(\sqrt{\beta}) - \left(\frac{i\alpha k}{\sqrt{\beta}} + \sqrt{\beta}\right)\sinh(\sqrt{\beta})} + \frac{i}{\beta}\cosh(\sqrt{\beta}\zeta) - \frac{i}{\beta}.\end{aligned}$$

Thus, for all $\zeta \in (0,1)$

$$\begin{aligned}&\beta^{3/2}\frac{(R(i\beta, A)\mathbf{1})(\zeta)}{e^{\sqrt{\beta}\zeta}} \\ &= i\frac{\cosh(\sqrt{\beta}\zeta)}{e^{\sqrt{\beta}\zeta}}\left[\frac{\beta\sinh(\sqrt{\beta}) - \sqrt{\beta}\cosh(\sqrt{\beta}) + \sqrt{\beta}}{(\alpha + ik)\cosh(\sqrt{\beta}) - \left(\frac{i\alpha k}{\sqrt{\beta}} + \sqrt{\beta}\right)\sinh(\sqrt{\beta})} + \sqrt{\beta}\right] \\ &\quad + k\frac{\sinh(\sqrt{\beta}\zeta)}{e^{\sqrt{\beta}\zeta}}\left[\frac{\sqrt{\beta}\sinh(\sqrt{\beta}) - \cosh(\sqrt{\beta}) + 1}{(\alpha + ik)\cosh(\sqrt{\beta}) - \left(\frac{i\alpha k}{\sqrt{\beta}} + \sqrt{\beta}\right)\sinh(\sqrt{\beta})}\right] - \frac{i\sqrt{\beta}}{e^{\sqrt{\beta}\zeta}}\end{aligned}$$

$$
\begin{aligned}
&= k + o(1) + i\frac{\cosh(\sqrt{\beta}\zeta)}{e^{\sqrt{\beta}\zeta}} \\
&\quad\times \frac{-\sqrt{\beta}\cosh(\sqrt{\beta}) + \sqrt{\beta} + (\alpha + ik)\sqrt{\beta}\cosh(\sqrt{\beta}) - i\alpha k \sinh(\sqrt{\beta})}{(\alpha + ik)\cosh(\sqrt{\beta}) - \left(\frac{i\alpha k}{\sqrt{\beta}} + \sqrt{\beta}\right)\sinh(\sqrt{\beta})} \\
&\xrightarrow{\beta\to\infty} k + i(1 - (\alpha + ik)) = 2k + i(1 - \alpha) \neq 0, \qquad (4.10)
\end{aligned}
$$

so that in particular

$$\|R(i\beta, A)\mathbf{1}\|_{L_2} \xrightarrow{\beta\to+\infty} \infty.$$

as a result, the resolvents cannot be uniformly bounded on the imaginary axis and then A does not generate a uniformly exponentially stable C_0-semigroup.

The lesson we learn from the example above is that it may be advisable as a first step to consider only asymptotic stability, still demanding dissipation conditions similar to those which for the case $N = 1$ actually were sufficient to show exponential stability, but already for the most simplest example for the case $N = 2$ only lead to asymptotic stability. We therefore show

Theorem 4.2.2. *Let A be a port-Hamiltonian operator of order $N \in \mathbb{N}$ with Lipschitz-continuous Hamiltonian density matrix function $\mathcal{H}$ and P_0 and boundary conditions such that for some $\kappa > 0$*

$$\operatorname{Re}\langle Ax, x\rangle_X \leq -\kappa \sum_{k=0}^{N-1} \left|(\mathcal{H}x)^{(k)}(0)\right|^2, \quad x \in D(A) \qquad (4.11)$$

then A generates a contractive and asymptotically stable C_0-semigroup $(T(t))_{t\geq 0}$.

Proof. From the dissipativity of A it follows that A generates a contractive C_0-semigroup on X, thanks to Theorem 3.3.6. Since by the same theorem A has compact resolvent it suffices to prove that $\sigma_p(A) \subseteq \mathbb{C}_0^-$, i.e. we have to show that $i\mathbb{R} \subseteq \rho(A)$. Let $\beta \in \mathbb{R}$ be arbitrary and consider a solution $x \in D(A)$ of the problem

$$i\beta x = Ax.$$

Then, from the dissipation condition on A we conclude that

$$\sum_{k=0}^{N-1} \left|(\mathcal{H}x)^{(k)}(0)\right|^2 \leq -\frac{1}{\kappa}\operatorname{Re}\langle Ax, x\rangle_X = 0,$$

so that $x \in \ker(i\beta - \mathfrak{A})$ and $\sum_{k=0}^{N-1} \left|(\mathcal{H}x)^{(k)}(0)\right|^2 = 0$. Next we show that this already implies that $x = 0$, so that $i\beta \notin \sigma_p(A) = \sigma(A)$. In fact,

$$
\begin{aligned}
&x \in \ker(i\beta - \mathfrak{A}), \ \sum_{k=0}^{N-1} \left|(\mathcal{H}x)^{(k)}(0)\right|^2 = 0 \\
\Leftrightarrow\ & i\beta x(\zeta) - \sum_{k=0}^{N} P_k(\mathcal{H}x)^{(k)}(\zeta) = 0, \text{ a.e. } \zeta \in (0,1), \ \sum_{k=0}^{N-1} \left|(\mathcal{H}x)^{(k)}(0)\right|^2 = 0
\end{aligned}
$$

and since $\mathcal{H}$ and P_0 are Lipschitz continuous this ordinary differential equation has the unique solution $x = 0$. We deduce that $i\beta \notin \sigma_p(A)$ and since $\beta \in \mathbb{R}$ had been arbitrary this implies $i\mathbb{R} \cap \sigma_p(A) = \emptyset$. Asymptotic stability follows from Corollary 2.2.16. □

Remark 4.2.3. *Although already quite short in total, the proof of 4.2.2 consists of two parts, namely first noting that A generates a contractive C_0-semigroup and that whenever $x \in \ker(i\beta - A)$ we have $\sum_{k=0}^{N-1} \left|(\mathcal{H}x)^{(k)}(0)\right|^2 = 0$. Then in the second part we actually showed that for any $x \in \ker(i\beta - \mathfrak{A})$ with $\sum_{k=0}^{N-1} \left|(\mathcal{H}x)^{(k)}(0)\right|^2 = 0$ we must have $x = 0$. Therefore, the second part is actually a statement on $\mathfrak{A}$ rather than on A which leads us to the definition and corollary below.*

Definition 4.2.4. *Let $B : D(B) \subset X \to X$ be a closed linear operator and $R \in \mathcal{B}(D(B); H)$ for some Hilbert space H. We say that the pair (B, R) has property ASP if for all $\beta \in \mathbb{R}$ we have $\ker(i\beta - B) \cap \ker R = \{0\}$, i.e.*

$$i\beta x = Bx \text{ and } Rx = 0 \quad \Rightarrow \quad x = 0. \tag{ASP}$$

Remark 4.2.5. *In the article [AuJa14] we used a slightly different terminology, namely there we would say that R has property ASP for the operator B. Also in that case we did not necessarily demand that R would be linear and directly started from implication* (ASP) *as definition.*

Remark 4.2.6. *Let (B, R) a pair with property ASP and $B_0 \subseteq B$ a closed restriction of B and $R_0 \in \mathcal{B}(D(B_0); H_0)$ (H_0 another Hilbert space) such that $\ker R_0 \subset \ker R$, then also the pair (B_0, R_0) has property ASP. In particular, for every $\kappa \neq 0$ the pair $(B, \kappa R)$ has property ASP.*

Proof. For every $\beta \in \mathbb{R}$ we have

$$\ker(i\beta - B_0) \cap \ker R_0 \subset \ker(i\beta - B) \cap \ker R.$$

From here the statement is obvious. □

Therefore, the second part of the proof of Theorem 4.2.2 says the following.

Corollary 4.2.7. *Let $\mathfrak{A}$ be a (maximal) port-Hamiltonian operator of order $N \in \mathbb{N}$ and assume that $\mathcal{H}$ and P_0 are Lipschitz continuous. Then for $R \in \mathcal{B}(D(\mathfrak{A}); \mathbb{F}^{Nd})$ given by*

$$Rx = \tau_0(\mathcal{H}x) = \begin{pmatrix} (\mathcal{H}x)(0) \\ (\mathcal{H}x)'(0) \\ \vdots \\ (\mathcal{H}x)^{(N-1)}(0) \end{pmatrix}$$

the pair $(\mathfrak{A}, R)$ has property ASP.

Moreover, we have the abstract result connecting property ASP to asymptotic stability of the C_0-semigroup $(T(t))_{t\geq 0}$ generated by a port-Hamiltonian operator A.

Proposition 4.2.8. *Let A be a port-Hamiltonian operator with boundary conditions such that*

$$\operatorname{Re} \langle Ax, x\rangle_X \leq - \|Rx\|^2$$

for some $R \in \mathcal{B}(D(\mathfrak{A}); H)$ such that the pair $(\mathfrak{A}, R)$ has property ASP. Then A generates an asymptotically stable contraction C_0-semigroup $(T(t))_{t\geq 0}$.

Proof. Due to $\operatorname{Re} \langle Ax, x\rangle \leq 0$ we especially have $\sigma(A) \subseteq \overline{\mathbb{C}_0^-}$. Since R has property ASP, then for every $x \in \ker(i\beta - A)$ where $\beta \in \mathbb{R}$ we also have

$$\|Rx\|^2 \leq - \operatorname{Re} \langle Ax, x\rangle_X = 0$$

i.e. $x \in \ker(i\beta - \mathfrak{A}) \cap \ker R$ and by the property ASP it follows $x = 0$, so that $i\mathbb{R} \cap \sigma_p(A) = \emptyset$ and asymptotic stability follows by Corollary 2.2.16. □

For special structures of the port-Hamiltonian system we may also conclude asymptotic stability under slightly weaker, or different at least, boundary conditions.

For the particular example of an undamped nonuniform Euler-Bernoulli Beam we conclude the following.

Lemma 4.2.9. *On $X = L_2(0,1;\mathbb{F}^2)$ consider the port-Hamiltonian operator of Euler-Bernoulli type*

$$\begin{aligned} \mathfrak{A}x &= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \frac{\partial^2}{\partial \zeta^2} \begin{bmatrix} \mathcal{H}_1(\zeta) & 0 \\ 0 & \mathcal{H}_2(\zeta) \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ D(\mathfrak{A}) &= \{x \in X : \mathcal{H}x \in H^2(0,1;\mathbb{F}^2)\} \\ &= \{x = (x_1, x_2) \in X : \mathcal{H}_1 x_1, \mathcal{H}_2 x_2 \in H^2(0,1)\} \end{aligned}$$

where $\mathcal{H}_1$ and $\mathcal{H}_1$ are uniformly Lipschitz continuous and strictly positive scalar functions on $[0,1]$. Then for every solution $x \in D(\mathfrak{A})$ of $\mathfrak{A}x = i\beta x$ with

$$\begin{aligned} (\mathcal{H}_1 x_1)(0) = (\mathcal{H}_1 x_1)'(0) = 0 \\ i\beta(\mathcal{H}_2 x_2)(0) \geq 0 \\ i\beta(\mathcal{H}_2 x_2)'(0) \geq 0 \end{aligned}$$

the functions

$$(\mathcal{H}_1 x_1),\ (\mathcal{H}_1 x_1)',\ \mathcal{H}_2(\mathcal{H}_1 x_1)'' = i\beta \mathcal{H}_2 x_2 \quad \text{and} \quad (\mathcal{H}_2(\mathcal{H}_1 x_1)'')' = i\beta(\mathcal{H}_2 x_2)' \tag{4.12}$$

are either all strictly positive on $(0,1]$ or all equal zero.

Proof. Let $\beta \in \mathbb{R}$ be arbitrary and $x \in D(\mathfrak{A})$ be a solution of $\mathfrak{A}x = i\beta x$. Then

$$\begin{aligned} -(\mathcal{H}_2 x_2)'' &= i\beta x_1 \in \mathcal{H}_1^{-1} H^2(0,1) \\ (\mathcal{H}_1 x_2)'' &= i\beta x_2 \in \mathcal{H}_2^{-1} H^2(0,1) \end{aligned}$$

so that from the continuous embedding $H^2(0,1) \hookrightarrow C^1[0,1]$ and the Lipschitz continuity of $\mathcal{H}_1$ and $\mathcal{H}_2$ we conclude that

$$\begin{aligned} (\mathcal{H}_1 x_1)(\zeta) &= (\mathcal{H}_1 x_1)(0) + \int_0^\zeta (\mathcal{H}_1 x_1)'(0) + \int_0^{s_1} (\mathcal{H}_1 x_1)''(0) \\ &\quad + \mathcal{H}_2^{-1}(s_1) \int_0^{s_2} (\mathcal{H}_2(\mathcal{H}_1 x_1)'')'(0) + \int_0^{s_3} (\mathcal{H}_2(\mathcal{H}_1 x_1)'')''(s_4) ds_4 ds_3 ds_2 ds_1 \\ &= \int_0^\zeta \int_0^{s_1} i\beta x_2(0) + \mathcal{H}_2^{-1}(s_2) \int_0^{s_2} \int_0^{s_3} i\beta(\mathcal{H}_2 x_2)'(0) \\ &\quad + \beta^2 x_1(s_4) ds_4 ds_3 ds_2 ds_1 \end{aligned}$$

and accordingly

$$\begin{aligned}(\mathcal{H}_1 x_1)'(\zeta) &= \int_0^\zeta i\beta x_2(0) + \mathcal{H}_2^{-1}(s_1) \int_0^{s_1} \int_0^{s_2} i\beta(\mathcal{H}_2 x_2)'(0) \\ &\quad + \beta^2 x_1(s_3) ds_3 ds_2 ds_1 \\ (\mathcal{H}_2(\mathcal{H}_1 x_1)'')(\zeta) &= i\beta x_2(0) + \int_0^\zeta \int_0^{s_1} i\beta(\mathcal{H}_2 x_2)'(0) + \beta^2 x_1(s_2) ds_2 ds_1 \\ (\mathcal{H}_2(\mathcal{H}_1 x_1)'')'(\zeta) &= \int_0^\zeta i\beta(\mathcal{H}_2 x_2)'(0) + \beta^2 x_1(s_1) ds_1.\end{aligned}$$

From here we conclude the assertion. In fact, assume that at least $i\beta(\mathcal{H}_2 x_2)(0) > 0$ or $i\beta(\mathcal{H}_2 x_2)'(0) > 0$. Then $(\mathcal{H}_1 x_1)$, $(\mathcal{H}_1 x_1)'$, $\mathcal{H}_2(\mathcal{H}_1 x_1)''$ and $(\mathcal{H}_2(\mathcal{H}_1 x_1)'')' > 0$ are strictly positive on some interval $(0, \varepsilon]$ for some $\varepsilon \in (0,1]$. Let $\overline{\varepsilon}$ be the supremum of these ε. Then by continuity the functions above are non-negative of $(0, \overline{\varepsilon}]$ and strictly positive on $(0, \overline{\varepsilon})$. From the formulas it then follows that all these functions are strictly positive on $[0, \overline{\varepsilon}]$. Then $\overline{\varepsilon} = 1$ since otherwise these functions are strictly positive on some proper supintervall $(0, \hat{\varepsilon}) \supset (0, \overline{\varepsilon}]$, contradicting the definition of $\overline{\varepsilon}$. □

Corollary 4.2.10. *Let $\mathfrak{A}$ be a port-Hamiltonian operator of Euler-Bernoulli type with Lipschitz-continuous $\mathcal{H}$ and P_0 as in the preceding lemma and let $R : D(\mathfrak{A}) \to \mathbb{F}^4$ be given by*

$$Rx := \begin{pmatrix} (\mathcal{H}_1 x_1)(0) \\ (\mathcal{H}_1 x_1)'(0) \\ (\mathcal{H}_2 x_2)(0) \\ R_4 x \end{pmatrix}$$

where $R_4 x = (\mathcal{H}_2 x_2)'(0)$, $(\mathcal{H}_1 x_1)(1)$, $(\mathcal{H}_1 x_1)'(1)$ or $(\mathcal{H}_2 x_2)'(0)$. Then the pair $(\mathfrak{A}, R)$ has property ASP. Also the pair $(\mathfrak{A}, \tilde{R})$ with

$$\tilde{R}x = \begin{pmatrix} (\mathcal{H}_1 x_1)(0) \\ (\mathcal{H}_1 x_1)'(0) \\ (\mathcal{H}_2 x_2)'(0) \\ (\mathcal{H}_2 x_2)(1) \end{pmatrix}$$

has property ASP.

Proof. Let $\beta \in \mathbb{R}$ and $x \in D(\mathfrak{A})$ such that $\mathfrak{A}x = i\beta x$ and $Rx = 0$. Then from the preceding lemma either $\mathcal{H}_1 x_1 = 0$ or (possibly after multiplication with some $\alpha \in \mathbb{F}$) all the functions $\mathcal{H}_1 x_1, (\mathcal{H}_1 x_1)', i\beta(\mathcal{H}_2 x_2)$ and $i\beta(\mathcal{H}_2 x_2)'$ are strictly positive on $(0,1]$, in particular $i\beta(\mathcal{H}_2 x_2)(1) > 0$ and $i\beta(\mathcal{H}_2 x_2)'(1) > 0$, but the latter is impossible whenever $Rx = 0$. The second statement follows in similar fashion. □

Remark 4.2.11. *Although the stability properties of port-Hamiltonian systems with conservative boundary conditions, but which are damped through the dissipative term $P_0 \in L_\infty(0,1;\mathbb{F}^{d\times d})$ are in general not in the focus of this thesis, we state the following asymptotic stability result, nevertheless. Let $X = L_2(0,1;\mathbb{F}^2)$ and consider the Euler-Bernoulli type port-Hamiltonian operator with viscous damping, i.e. for* *some Lipschitz continuous function* $\gamma \in W^1_\infty(0,1;\mathbb{F})$ *such that* $\operatorname{Re}\gamma > 0$ *on some interval* $(\varepsilon_1, \varepsilon_2) \subseteq (0,1)$ *and* $\operatorname{Re}\gamma \geq 0$ *on* $(0,1)$ *we consider*

$$\begin{aligned}\mathfrak{A}x &= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \frac{\partial^2}{\partial\zeta^2} \begin{bmatrix} \mathcal{H}_1 & 0 \\ 0 & \mathcal{H}_2 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} - \begin{bmatrix} \gamma & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \mathcal{H}_1 & 0 \\ 0 & \mathcal{H}_2 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ D(\mathfrak{A}) &= \{x \in X : \mathcal{H}x \in H^2(0,1;\mathbb{F}^2)\} \\ &= \{x = (x_1, x_2) \in X : \mathcal{H}_1 x_1, \mathcal{H}_2 x_2 \in H^2(0,1)\}\end{aligned}$$

Then for every choice of $(\zeta_1, \zeta_2) \in \{(0,0),(0,1),(1,1)\}$ *and* $\{c_1, c_2\} = \{0,1\}$ *and* $R : D(\mathfrak{A}) \to \mathbb{F}^4$ *defined as*

$$Rx := \begin{pmatrix} (\mathcal{H}_1 x_1)(\zeta_1) \\ c_1(\mathcal{H}_1 x_1)(1-\zeta_1) + (1-c_1)(\mathcal{H}_2 x_2)'(1-\zeta_1) \\ (\mathcal{H}_2 x_2)(\zeta_2) \\ c_2(\mathcal{H}_2 x_2)(1-\zeta_2) + (1-c_2)(\mathcal{H}_1 x_1)'(1-\zeta_2) \end{pmatrix}$$

the pair $(\mathfrak{A}, R)$ *has property ASP.*

Proof. Let $x \in D(\mathfrak{A})$ be a solution of $\mathfrak{A}x = i\beta x$ with $Rx = 0$. Then we observe that

$$\begin{aligned}0 &= \operatorname{Re}\langle i\beta x, x\rangle_X = \operatorname{Re}\langle \mathfrak{A}x, x\rangle_X \\ &= -\operatorname{Re}\langle \gamma(\mathcal{H}_1 x_1), (\mathcal{H}_1 x_1)\rangle_{L_2} + \operatorname{Re}\langle f_{\partial,\mathcal{H}x}, e_{\partial,\mathcal{H}x}\rangle_{\mathbb{F}^2} \\ &= -\operatorname{Re}\langle \gamma(\mathcal{H}_1 x_1), (\mathcal{H}_1 x_1)\rangle_{L_2}\end{aligned}$$

because $\operatorname{Re}\langle f_{\partial,\mathcal{H}x}, e_{\partial,\mathcal{H}x}\rangle_{\mathbb{F}^2} = 0$ follows from $Rx = 0$. Since $\gamma > 0$ on $(\varepsilon_1, \varepsilon_2)$ it follows that $(\mathcal{H}_1 x_1) = (\mathcal{H}_1 x_1)' = (\mathcal{H}_1 x_1)'' = 0$ on $(\varepsilon_1, \varepsilon_2)$. Then from

$$i\beta x_1 = -(\mathcal{H}_2 x_2)'' - 2\gamma(\mathcal{H}_1 x_1), \quad i\beta x_2 = (\mathcal{H}_1 x_1)''$$

we deduce that in the case that $\beta \neq 0$ we have also $(\mathcal{H}_2 x_2) = (\mathcal{H}_2 x_2)' = (\mathcal{H}_2 x_2)'' = 0$ on $(\varepsilon_1, \varepsilon_2)$. Solving the initial value problems

$$\begin{aligned}i\beta x(\zeta) &= -\left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right](\mathcal{H}x)''(\zeta) - \left[\begin{smallmatrix} \gamma(\zeta) & -1 \end{smallmatrix}\right](\mathcal{H}x)(\zeta), \quad \zeta \in (0, \varepsilon_1) \\ (\mathcal{H}x)(\varepsilon_1) &= (\mathcal{H}x)'(\varepsilon_1) = 0 \\ i\beta x(\zeta) &= -\left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right](\mathcal{H}x)''(\zeta) - \left[\begin{smallmatrix} \gamma(\zeta) & -1 \end{smallmatrix}\right](\mathcal{H}x)(\zeta), \quad \zeta \in (\varepsilon_2, 1) \\ (\mathcal{H}x)(\varepsilon_2) &= (\mathcal{H}x)'(\varepsilon_2) = 0\end{aligned}$$

we conclude that $x = 0$ provided $\beta \neq 0$. On the other hand, if $\beta = 0$, we have $(\mathcal{H}_1 x_1)'' = 0$ on $(0,1)$, so that $(\mathcal{H}_1 x_1)$ takes the form

$$(\mathcal{H}_1 x_1)(\zeta) = (\mathcal{H}_1 x_1)(0) + \zeta(\mathcal{H}_1 x_1)'(0), \quad \zeta \in [0,1]$$

but since $(\mathcal{H}_1 x_1) = 0$ on $(\varepsilon_1, \varepsilon_2)$ this implies that already $(\mathcal{H}_1 x_1) = 0$ on the whole interval $(0,1)$. Then also $(\mathcal{H}_2 x_2)'' = 0$ on $(0,1)$, i.e.

$$(\mathcal{H}_2 x_2)(\zeta) = (\mathcal{H}_2 x_2)(0) + \zeta(\mathcal{H}_2 x_2)'(0), \quad \zeta \in [0,1]$$

and from the additional constraint that $Rx = 0$ and $c_1 \neq c_2$ we obtain that $x = 0$ in any case. □

4.3 Uniform Exponential Stability

We continue with the investigation of stability properties and after our short trip to asymptotic stability again focus on uniform exponential stability. Here the main focus lies on systems with order $N \geq 2$, however we also encounter new methods to prove the known results for the case $N = 1$. Therefore, we start by considering the special case $N = 1$ once again and using two new techniques to obtain the very same results as had been established by the sideways-energy method before. Only then we look at higher order systems and afterwards consider systems with strict dissipation at both ends. We will see that strict dissipation at both ends in any case leads to uniform exponential stabilisation, regardless of the order $N \in \mathbb{N}$ of the port-Hamiltonian system. Although this result is not surprising at all, it helps to understand the ideas behind the proof techniques better. We can give uniform exponential stability results under much less restrictive conditions than strict dissipation at both ends.

4.3.1 First Order Port-Hamiltonian Systems

Within this subsection, the port-Hamiltonian operator $\mathfrak{A}$ always is of order $N = 1$, i.e. $\mathfrak{A}$ always has the form

$$\mathfrak{A}x = P_1(\mathcal{H}x)' + P_0(\mathcal{H}x).$$

As for the known uniform exponentially stability results of Section 4.1 – and also for the asymptotic stability results in Section 4.2 – we assume that $\mathcal{H}$ and P_0 are Lipschitz continuous. Our aim is to give alternative proofs for Theorem 4.1.5 for which the technique of proof may also apply to port-Hamiltonian systems of order $N \geq 2$.

Theorem 4.3.1 (= Theorem 4.1.5)**.** *Assume that the Hamiltonian-density matrix function $\mathcal{H}$ and P_0 are Lipschitz-continuous. If the operator A satisfies the assumption*

$$\operatorname{Re} \langle Ax, x \rangle_X \leq -\kappa \left| (\mathcal{H}x)(0) \right|^2, \qquad x \in D(A)$$

for some $\kappa > 0$, then A generates a uniformly exponentially stable and contractive C_0-semigroup on the Hilbert space X.

Alternative Proof via the Gearhart-Greiner-Prüss Theorem. The first alternative proof we present is based on the Gearhart-Greiner-Prüss Theorem 2.2.17. It had already been presented in Proposition 2.12 of [AuJa14] for the case that P_0 is a constant matrix. Here the results of the preceding Section 4.2 come at hand since from Theorem 4.2.2 we can already deduce that the semigroup $(T(t))_{t\geq 0}$ generated by the operator A is asymptotically stable and $\sigma(A) = \sigma_p(A) \subseteq \mathbb{C}_0^-$. Therefore, it remains to check the uniform boundedness of the resolvents on the imaginary axis, i.e.

$$\sup_{\beta \in \mathbb{R}} \|R(i\beta, A)\| < +\infty$$

or, equivalently, the following sequence criterion, see Corollary 2.2.19,

$$\forall (x_n, \beta_n)_{n\geq 1} \subseteq D(A) \times \mathbb{R} \text{ with } \sup_{n\in\mathbb{N}} \|x_n\|_X < +\infty \text{ and } |\beta_n| \xrightarrow{n\to\infty} +\infty :$$
$$i\beta_n x_n - Ax_n \xrightarrow{n\to+\infty} 0 \quad \Longrightarrow \quad \|x_n\|_X \xrightarrow{n\to\infty} 0.$$

So let $(x_n, \beta_n)_{n\geq 1} \subseteq D(A) \times \mathbb{R}$ be any sequence with $\sup_{n\in\mathbb{N}} \|x_n\|_X < +\infty$, $|\beta_n| \to +\infty$ (as $n \to \infty$) and such that

$$i\beta_n x_n - Ax_n \xrightarrow{n\to+\infty} 0.$$

Then

$$\kappa \left|(\mathcal{H}x_n)(0)\right|^2 \leq -\operatorname{Re}\langle Ax_n - i\beta_n x_n, x_n\rangle_X \xrightarrow{n\to+\infty} 0.$$

Thus, we have a sequence $(x_n, \beta_n)_{n\geq 1} \subseteq D(\mathfrak{A}) \times \mathbb{R}$ with $\sup_n \|x_n\|_X < +\infty$ and $|\beta_n| \to +\infty$ (as $n \to \infty$) such that

$$i\beta_n x_n - Ax_n \xrightarrow{n\to\infty} 0,\ (\mathcal{H}x_n)(0) \xrightarrow{n\to\infty} 0.$$

Next we show that this already implies that $\|x_n\|_X \to 0$ $(n \to \infty)$. For this end, we employ the following useful lemmas.

Lemma 4.3.2. *Let $Q \in W^1_\infty(0,1;\mathbb{F}^{d\times d})$ be a Lipschitz-continuous function of symmetric matrices and $x \in H^1(0,1;\mathbb{F}^d)$. Then*

$$\operatorname{Re}\langle x', Qx\rangle_{L_2} = -\frac{1}{2}\langle x, Q'x\rangle_{L_2} + \frac{1}{2}\left[\langle x(\zeta), Q(\zeta)x(\zeta)\rangle_{\mathbb{F}^d}\right]_0^1.$$

Proof. Using the self-adjointness of $Q(\zeta)$ for a.e. $\zeta \in (0,1)$ we compute

$$\begin{aligned} 2\operatorname{Re}\langle x', Qx\rangle_{L_2} &= \langle x', Qx\rangle_{L_2} + \langle x, Qx'\rangle_{L_2} \\ &= -\langle x, Q'x\rangle_{L_2} + \left[\langle x(\zeta), Q(\zeta)x(\zeta)\rangle_{\mathbb{F}^d}\right]_0^1. \end{aligned}$$

□

Lemma 4.3.3. *Let $\alpha > 0$ and $\beta, \gamma \geq 0$ be given. Then there is a scalar function $\eta \in C^\infty([0,1];\mathbb{R})$ with $\eta(0) = 0$ and strictly positive derivative $\eta' > 0$ such that*

$$\alpha\eta'(\zeta) - \beta\eta(\zeta) \geq \gamma, \quad \zeta \in [0,1]. \tag{4.13}$$

Proof. Scaling η by the factor $\frac{1}{\gamma}$ it is enough to consider the case $\gamma = 1$. We make the ansatz $\eta(\zeta) = e^{\lambda\zeta} - 1$ for $\lambda > 0$ which we are going to specify. Then equation (4.13) is equivalent to

$$(\alpha\lambda - \beta)e^{\lambda\zeta} \geq 1 - \beta\ (\zeta \in [0,1]).$$

Choosing $\lambda > \frac{\max\{1,\beta\}}{\alpha}$ this condition holds. □

Let us also introduce some notation. For sequences $(s_n)_{n\in\mathbb{N}}$ and $(r_n)_{n\in\mathbb{N}}$ we write

$$r_n = s_n + o(1)$$

if the sequence $(r_n - s_n)_{n\in\mathbb{N}}$ of differences is a null sequence.

Continuation of the proof. Since $\mathfrak{A}x_n - i\beta_n x_n \to 0$ converges to zero in X and the sequence $(x_n)_{n\geq 1} \subset X$ is bounded, also the sequence $\frac{x_n}{\beta_n}$ is bounded in the graph norm $\|\cdot\|_{\mathfrak{A}}$ and by Lemma 3.2.3 we get

$$\sup_{n\in\mathbb{N}} \left\| \frac{(\mathcal{H}x_n)'}{\beta_n} \right\|_{L_2} < +\infty.$$

Moreover, observe that since $P_0(\zeta)$ is dissipative for a.e. $\zeta \in (0,1)$ we may also conclude that (recall that A is dissipative if and only if P_0 is dissipative and $A - P_0$ is dissipative)

$$\begin{aligned} 0 &\geq \operatorname{Re}\langle P_0 \mathcal{H} x_n, \mathcal{H} x_n\rangle_{L_2} \\ &= -\operatorname{Re}\langle f_{\partial,\mathcal{H}x_n}, e_{\partial,\mathcal{H}x_n}\rangle_{\mathbb{F}^{Nd}} + \operatorname{Re}\langle A x_n, x_n\rangle_X \\ &\geq \operatorname{Re}\langle A x_n, x_n\rangle_X \xrightarrow{n\to\infty} 0 \end{aligned}$$

so that $(\mathrm{Sym}\ P_0)\mathcal{H}x_n \xrightarrow{n\to\infty} 0$ in X. Letting $q \in C^1([0,1];\mathbb{R})$ with $q(1) = 0$ and having Lemma 4.3.2 in mind we find

$$\begin{aligned} 0 &\leftarrow \frac{1}{\beta_n}\operatorname{Re}\langle \mathfrak{A}x_n - i\beta_n x_n, iq(\mathcal{H}x_n)'\rangle_{L_2} \\ &= \frac{1}{\beta_n}\operatorname{Re}\langle P_1(\mathcal{H}x_n)', iq(\mathcal{H}x_n)'\rangle_{L_2} \\ &\quad + \frac{1}{\beta_n}\operatorname{Re}\langle \frac{P_0 - P_0^*}{2}(\mathcal{H}x_n), iq(\mathcal{H}x_n)'\rangle_{L_2} - \operatorname{Re}\langle x_n, q(\mathcal{H}x_n)'\rangle_{L_2} + o(1) \\ &= \frac{1}{2\beta_n}\left(\langle \mathcal{H}x_n, i\left(q\frac{P_0 - P_0^*}{2}\right)'(\mathcal{H}x_n)\rangle_{L_2}\right. \\ &\quad \left. - \left[\langle (\mathcal{H}x_n)(\zeta), i\left(q\frac{P_0 - P_0^*}{2}\right)(\zeta)(\mathcal{H}x_n)(\zeta)\rangle_{\mathbb{F}^d}\right]_0^1\right) \\ &\quad - \operatorname{Re}\langle x_n, q\mathcal{H}x_n'\rangle_{L_2} - \langle x_n, q\mathcal{H}'x_n\rangle_{L_2} + o(1) \\ &= \frac{1}{2}\langle x_n, (q\mathcal{H})'x_n\rangle_{L_2} - \frac{1}{2}\left[\langle x_n(\zeta), q(\zeta)\mathcal{H}(\zeta)x_n(\zeta)\rangle_{\mathbb{F}^d}\right]_0^1 - \langle x_n, q\mathcal{H}'x_n\rangle_{L_2} + o(1) \\ &= -\frac{1}{2}\langle x_n, (q\mathcal{H}' - q'\mathcal{H})x_n\rangle_{L_2} + o(1), \end{aligned}$$

since $(\mathcal{H}x_n)(0) \to 0$, $q(1) = 0$ and $|\beta_n| \to \infty$, using integration by parts and $P_1 = P_1^*$. In particular, we may choose $q \leq 0$ with $q' > 0$ such that

$$\lambda q(\zeta) + mq'(\zeta) > 0, \qquad \zeta \in [0,1].$$

where $\mathcal{H}(\zeta) \geq mI$ and $\pm\mathcal{H}'(\zeta) \leq \lambda I$ for a.e. $\zeta \in [0,1]$, see Lemma 4.3.3, so $q'\mathcal{H} - q\mathcal{H}'$ is coercive as multiplication operator on X. This implies that

$$\|x_n\|_X \simeq \sqrt{\langle x_n, (q'\mathcal{H} - q\mathcal{H}')x_n\rangle_{L_2}} \xrightarrow{n\to\infty} 0.$$

Hence, the sequence criterion for uniform exponential stability is satisfied and therefore the C_0-semigroup $(T(t))_{t\geq 0}$ is uniformly exponentially stable by Corollary 2.2.19. □

As the proof for the asymptotic stability result, Theorem 4.2.2, this proof of Theorem 4.1.5 leads us to a definition closely related to uniform exponential stability of the C_0-semigroup $(T(t))_{t\geq 0}$.

Definition 4.3.4. *Let $B : D(B) \subset X \to X$ be a closed linear operator and $R \in \mathcal{B}(D(B); H)$ where H is another Hilbert space. We then say that the pair (B, R) has property AIEP if the following holds. For all sequences $(x_n, \beta_n)_{n\geq 1} \subseteq D(B) \times \mathbb{R}$ with $\sup_{n\in\mathbb{N}} \|x_n\| < +\infty$ and $|\beta_n| \xrightarrow{n\to+\infty} +\infty$*

$$i\beta_n x_n - Bx_n \to 0 \text{ and } Rx_n \to 0 \quad \Rightarrow \quad x_n \to 0. \qquad \text{(AIEP)}$$

Moreover, we say that the pair (B, R) has property ESP if it has properties

$$ASP \text{ and } AIEP. \tag{ESP}$$

Remark 4.3.5. *Note that as for the notion ASP another notation had been used in [AuJa14]. Instead of saying that the pair $(\mathfrak{A}, R)$ has property AIEP, there we would say that R has property AIEP for the operator $\mathfrak{A}$.*

With the properties AIEP and ESP we may find the following general result.

Proposition 4.3.6. *Let A be a port-Hamiltonian operator with suitable boundary conditions and $R \in \mathcal{B}(D(\mathfrak{A}); H)$ for some Hilbert space H such that*

$$\operatorname{Re}\langle Ax, x\rangle_X \le -\|Rx\|_H^2, \quad x \in D(A).$$

If $\sigma_p(A) \subseteq \mathbb{C}_0^-$ and the pair $(\mathfrak{A}, R)$ has property AIEP, then the contraction C_0-semigroup $(T(t))_{t\ge0}$ generated by A is uniformly exponentially stable. In particular, if the pair $(\mathfrak{A}, R)$ has property ESP, then the C_0-semigroup $(T(t))_{t\ge0}$ is uniformly exponentially stable.

Proof. Let $(x_n, \beta_n)_{n\ge1} \subseteq D(\mathfrak{A}) \times \mathbb{R}$ be any sequence with $\sup_{n\in\mathbb{N}} \|x_n\|_{L_2} < +\infty$ and $|\beta_n| \to +\infty$ such that $\mathfrak{A}x_n - i\beta_n x_n \to 0$. Then the dissipation condition implies that

$$0 \leftarrow \operatorname{Re}\langle i\beta_n x_n - Ax_n, x_n\rangle_X \ge \|Rx_n\|_H^2 \ge 0,$$

i.e. $Rx_n \to 0$ as $n \to +\infty$. Since the pair $(\mathfrak{A}, R)$ has property AIEP this leads to $x_n \to 0$ and therefore the sequence criterion for uniform exponential stability is satisfied, i.e. the C_0-semigroup $(T(t))_{t\ge0}$ is uniformly exponentially stable. □

Remark 4.3.7. *The advantage of the method presented above is the fact that it is possible to generalise it to port-Hamiltonian systems of order $N \ge 2$. However, since the Gearhart-Greiner-Prüss Theorem states how exponential stability of C_0-semigroups* of linear operators *is connected to spectral properties of its* linear *generator A, the method cannot be applied to systems which inherit nonlinearities. Since later on in Chapter 6 and Chapter 7 we also want to consider systems with nonlinear feedback, this is an unfortunate restriction. Therefore, below we present another method which may also apply for systems with nonlinear feedback, however with the drawback that not all situations which can be covered by the method above for the linear case, can also be treated with the latter technique.*

Alternative proof using a Lyapunov function. The proof is based on the following result.

Proposition 4.3.8. *Let $\mathfrak{A}$ be a port-Hamiltonian operator of order $N = 1$ where $\mathcal{H}$ and P_0 are Lipschitz continuous. Then there is $q : X \to \mathbb{R}$ with $|q(x)| \le c\|x\|_X^2$ such that for every solution $x \in W_\infty^1(\mathbb{R}_+; X) \cap L_\infty(\mathbb{R}_+; D(\mathfrak{A}))$ of*

$$\frac{d}{dt}x(t) = \mathfrak{A}x(t)$$

one has $q(x) \in W_\infty^1(\mathbb{R}_+; \mathbb{R})$ with

$$\|x(t)\|_X + \frac{d}{dt}q(x(t)) \le c\,|(\mathcal{H}x)(t, 0)|^2, \quad a.e.\ t \ge 0.$$

Remark 4.3.9. *In fact, also the following variants hold true for*

$$q(x) = \operatorname{Re}\langle x, \eta P_1^{-1}x\rangle_{L_2}, \quad x \in X$$

as in the proof below. If the solution lies in $C^1(\mathbb{R}_+; X) \cap C(\mathbb{R}_+; D(\mathfrak{A}))$, i.e it is classical, then also $q(x) \in C^1(\mathbb{R}_+; R)$. If the solution merely lies in $W^1_{\infty,loc}(\mathbb{R}_+; X) \cap L_{\infty,loc}(\mathbb{R}_+; D(\mathfrak{A}))$, then $q(x) \in W^1_{\infty,loc}(\mathbb{R}_+; X)$. In both cases the claimed estimate holds for (a.e.) $t \geq 0$.

Proof. On $X = L_2(0, 1; \mathbb{F}^d)$ we define the quadratic functional

$$q(x) := \langle x, \eta P_1^{-1}x\rangle_{L_2}, \quad x \in X$$

where $\eta \in C^1([0, 1]; \mathbb{R})$ is a differentiable function with $\eta(1) = 0$ and $\eta' > 0$ uniformly on $[0, 1]$ (so that in particular $\eta \leq 0$) to be chosen at a later point. Let $x \in W^1_\infty(\mathbb{R}_+; X)$ be any solution of $\dot{x} = \mathfrak{A}x$. Then also $q(x) \in W^1_\infty(\mathbb{R}_+; \mathbb{R})$ and using Lemma 4.3.2 (in the last line) we obtain that for a.e. $t \geq 0$

$$\begin{aligned}
\frac{d}{dt}q(x(t)) &= 2\operatorname{Re}\langle P_1^{-1}\dot{x}(t), \eta x(t)\rangle_{L_2} \\
&= 2\operatorname{Re}\langle (\mathcal{H}x(t))' + P_1^{-1}P_0(\mathcal{H}x(t)), \eta x(t)\rangle_{L_2} \\
&= -\langle \mathcal{H}x(t), (\eta'\mathcal{H}^{-1} + \eta(\mathcal{H}^{-1})' - 2\eta \operatorname{Re}(\mathcal{H}^{-1}P_1^{-1}P_0))\mathcal{H}x(t)\rangle_{L_2} \\
&\quad + \left[\langle \mathcal{H}x(t,\zeta), (\eta\mathcal{H}^{-1})(\zeta)\mathcal{H}x(t,\zeta)\rangle_{\mathbb{F}^d}\right]_0^1.
\end{aligned}$$

Since $\eta(1) = 0$ we conclude that

$$\begin{aligned}
&\|x(t)\|_X^2 + \frac{d}{dt}q(x(t)) \\
&\leq c\,|(\mathcal{H}x)(t,0)|^2 \\
&\quad + \langle \mathcal{H}x(t), ((1-\eta')\mathcal{H}^{-1} - \eta(\mathcal{H}^{-1})' + 2\eta \operatorname{Re}(\mathcal{H}^{-1}P_1^{-1}P_0))\mathcal{H}x(t)\rangle_{L_2}.
\end{aligned}$$

There are constants $m_0, M_1, M_2, M_3 > 0$ such that

$$\begin{aligned}
m_0 I \leq \mathcal{H}^{-1}(\zeta) &\leq M_0 I, \\
(\mathcal{H}^{-1})'(\zeta) &\leq M_1 I \\
-M_3 I \leq \operatorname{Re}\left(\mathcal{H}^{-1}(\zeta)P_1^{-1}P_0\right) &\leq M_3 I, \quad \text{a.e. } \zeta \in (0,1).
\end{aligned}$$

Using Lemma 4.3.3 (for $f(\zeta) = -\eta(1-\zeta)$) we find $f \in C^2([0, 1]; \mathbb{R})$ with $f(0) = 0$ and $f' > 0$ such that

$$\begin{aligned}
&f'(\zeta)m_0 - f(\zeta)[M_1 + 2M_3] \geq M_0 \\
\text{i.e.} \qquad &M_0 - \eta'(\zeta)m_0 - \eta(\zeta)[M_1 + 2M_3] \leq 0, \quad \zeta \in (0,1)
\end{aligned}$$

and we find that

$$\begin{aligned}
&(1-\eta'(\zeta))\mathcal{H}^{-1}(\zeta) + \eta(\zeta)\left[-(\mathcal{H}^{-1})'(\zeta) + 2\operatorname{Re}\left(\mathcal{H}^{-1}(\zeta)P_1^{-1}P_0\right)\right] \\
&\quad \leq (M_0 - \eta'(\zeta)m_0 - \eta(\zeta)\,[M_1 + 2M_3])\, I \leq 0, \quad \text{a.e. } \zeta \in (0,1).
\end{aligned}$$

This concludes the proof of the proposition. □

We proceed with the proof of Theorem 4.1.5 by a Lyapunov functional technique. Let $x_0 \in D(A)$ be arbitrary and set

$$x = T(\cdot)x_0 \in C_b^1(\mathbb{R}_+; X) \cap C_b(\mathbb{R}_+; D(A)) \subseteq W_\infty^1(\mathbb{R}_+; X) \cap L_\infty(\mathbb{R}_+; D(\mathfrak{A}))$$

where for any Hilbert space H we denote by $C_b(\mathbb{R}_+; H) = C(\mathbb{R}_+; H) \cap L_\infty(\mathbb{R}_+; H)$ the space of bounded continuous functions with values in H and $C_b^1(\mathbb{R}_+; H) := \{x \in C^1(\mathbb{R}_+; H) : \ x, x' \in C_b(\mathbb{R}_+; H)\}$ is the space of all continuously differentiable H-valued functions such the function and its first derivative are bounded. Let $q : X \to \mathbb{R}$ be given by Proposition 4.3.8, in particular independent of $x_0 \in D(A)$, and set

$$\Phi(t) := t \left\| x(t) \right\|_X^2 + q(x(t)), \quad t \geq 0. \tag{4.14}$$

Then $\Phi \in W_{\infty,loc}^1(\mathbb{R}_+; \mathbb{R})$ with

$$\begin{aligned} \frac{d}{dt}\Phi(t) &= \left\| x(t) \right\|_X^2 + 2t \operatorname{Re} \langle Ax(t), x(t) \rangle_X + \frac{d}{dt} q(x(t)) \\ &\leq (c - 2t\kappa) \left| (\mathcal{H}x)(t, 0) \right|^2, \quad \text{a.e. } t \geq 0. \end{aligned}$$

Then for $t_0 := \frac{c}{2\kappa}$, which is independent of the initial value $x_0 \in D(A)$, we have

$$\frac{d}{dt}\Phi(t) \leq 0, \quad t \geq t_0$$

and thus Φ decreases on (t_0, ∞). Since $|q(f)| \leq c \left\| f \right\|_X^2$ for some $c > 0$ and all $f \in X$ we also have the following estimate for $t \geq t_0$.

$$t \left\| x(t) \right\|_X^2 \leq \Phi(t) + c \left\| x(t) \right\|_X^2 \leq \Phi(t_0) + c \left\| x(t) \right\|_X^2$$

and then for $t > \max\{t_0, c\}$

$$\left\| x(t) \right\|_X^2 \leq \frac{\Phi(t_0)}{t - c} \leq \frac{t_0 + c}{t - c} \left\| x(t_0) \right\|_X^2 \leq \frac{t_0 + c}{t - c} \left\| x_0 \right\|_X^2 .$$

Since t_0 is independent of $x_0 \in D(A)$ we conclude from the density of $D(A)$ in X that for $t > \max\{t_0, c\}$ the estimate

$$\left\| T(t)x \right\|_X \leq \sqrt{\frac{t_0 + c}{t - c}} \left\| x \right\|_X , \quad x \in X$$

is valid, so that $\|T(t)\| < 1$ for sufficiently large t. As a result, the C_0-semigroup $(T(t))_{t\geq 0}$ is uniformly exponentially stable thanks to Remark 2.2.12. □

Let us further address the following question to which we give a partial answer afterwards.

Problem 4.3.10. *Let A be a port-Hamiltonian operator of order $N = 1$ (with boundary conditions) such that $\mathcal{H}$ and P_0 are Lipschitz continuous. Assume there exists a constant $\kappa > 0$ and an orthogonal projection matrix $Q = Q^2 \in \mathbb{F}^{d \times d}$ such that*

$$\operatorname{Re} \langle Ax, x \rangle_X \leq -\kappa \left(\left| Q\mathcal{H}x(0) \right|^2 + \left| (I - Q)\mathcal{H}x(1) \right|^2 \right), \quad x \in D(A).$$

Does A generate an exponentially stable C_0-semigroup then?

First we provide an easy example showing that this cannot be true, in general.

Example 4.3.11. *Let $d = 2$, $\mathcal{H} \equiv I$, $P_0 = -P_0^* \in \mathbb{F}^{d\times d}$ and for some $p_{12} \in \mathbb{F}$ let*

$$P_1 := \left[\begin{smallmatrix} 0 & p_{12} \\ \overline{p_{12}} & 0 \end{smallmatrix}\right].$$

Define $A : D(A) \subseteq X \to X = L_2(0,1;\mathbb{F}^2)$ as

$$\begin{aligned} Ax &:= P_1 x' + P_0 x \\ D(A) &:= \{x \in H^1(0,1;\mathbb{F}^2) : x_1(0) = x_2(1) = 0\} \end{aligned}$$

and let $Q = \left[\begin{smallmatrix} 1 & \\ & 0 \end{smallmatrix}\right]$ be the projection matrix on the first component. Then we have for all $x = (x_1, x_2) \in D(A)$

$$\begin{aligned} \operatorname{Re}\langle Ax, x\rangle_{L_2} &= \frac{1}{2}\int_0^1 \langle x(\zeta), P_1 x'(\zeta)\rangle_{\mathbb{F}^2} + \langle x'(\zeta), P_1 x(z)\rangle_{\mathbb{F}^2} d\zeta \\ &\quad + \int_0^1 \operatorname{Re}\ \langle x(\zeta), P_0 x(\zeta)\rangle_{\mathbb{F}^2} d\zeta \\ &= \frac{1}{2}\left[\langle x(\zeta), P_1 x(\zeta)\rangle_{\mathbb{F}^2}\right]_0^1 \\ &= \frac{1}{2}\left[\langle x_1(\zeta), p_{12} x_2(\zeta)\rangle_{\mathbb{F}} + x_2(\zeta) p_{12}^* x_1(\zeta)_{\mathbb{F}}\right]_0^1 = 0 \\ &= -\kappa\left(|Qx(0)|^2 + |(I-Q)x(1)|^2\right) \end{aligned}$$

and the C_0-semigroup is isometric (so neither strongly nor exponentially stable), although

$$\operatorname{Re}\langle Ax, x\rangle_{L_2} \le -\kappa\left(|Qx(0)|^2 + |(I-Q)x(1)|^2\right), \quad x \in D(A).$$

Remark 4.3.12. *A more detailed analysis of the above example shows that an operator*

$$\begin{aligned} Ax &= \left[\begin{smallmatrix} p_{11} & p_{12} \\ \overline{p_{12}} & p_{22} \end{smallmatrix}\right] x' + P_0 x \\ D(A) &= \{y \in H^1(0,1)^2 : y_1(0) = y_2(0) = 0\} \end{aligned}$$

with $p_{11}, p_{22} \in \mathbb{R}$ generates a contractive and asymptotically (then: uniformly exponentially) stable C_0-semigroup if and only if

$$p_{11} \le 0,\ p_{22} \ge 0 \text{ and } |p_{11}|^2 + |p_{22}|^2 > 0.$$

Proof. First observe that A is dissipative (and then the generator of a contractive C_0-semigroup) if and only if

$$\begin{aligned} \operatorname{Re}\langle Ax, x\rangle_{L_2} &= \left[\operatorname{Re}\langle \left[\begin{smallmatrix} p_{11} & p_{12} \\ p_{12}^* & p_{22} \end{smallmatrix}\right] \left(\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right), \left(\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right)\rangle_{\mathbb{F}^2}\right]_0^1 \\ &= p_{11}|x_1(1)|^2 - p_{22}|x_2(0)|^2 \le 0, \quad x \in D(A) \end{aligned}$$

and this is the case if and only if $p_{11} \le 0$ and $p_{22} \ge 0$. Moreover, if additionally at least one of the parameters $p_{11} \ne 0$ or $p_{22} \ne 0$ does not equal zero, then

$$\operatorname{Re}\langle Ax, x\rangle_{L_2} \le -|p_{11}|\,|x(1)|^2 - |p_{22}|\,|x(0)|^2, \quad x \in D(A)$$

then Theorem 4.1.5 implies uniform exponential stability of the C_0-semigroup. On the other hand, if both $p_{11} = p_{22} = 0$ equal zero, then the C_0-semigroup is isometric (as we have seen before), so cannot be asymptotically or even uniformly exponentially stable. □

On the other hand, for projection matrices that commute with all other structural matrices P_1, $P_0(\zeta)$ and $\mathcal{H}(\zeta)$ (a.e. $\zeta \in (0,1)$) in the definition of the operator A we have a positive result which essentially follows from decomposing the system and switching the setting of left and right for the components.

Proposition 4.3.13. *Let $\mathcal{H}$ be uniformly positive definite and Lipschitz continuous, $P_1 = P_1^* \in \mathbb{F}^{d\times d}$ invertible, P_0 Lipschitz continuous and on $X = \big(L_2(0,1;\mathbb{F}^d); \langle\cdot,\cdot\rangle_X\big)$ let the operator A be given by*

$$Ax = P_1(\mathcal{H}x)' + P_0(\mathcal{H}x),$$
$$D(A) = \{y \in \mathcal{H}^{-1}H^1(0,1)^d : W_B \left(\begin{smallmatrix} f_{\partial,\mathcal{H}x} \\ e_{\partial,\mathcal{H}x}\end{smallmatrix}\right) = 0\}$$

where $W \in \mathbb{F}^{d\times 2d}$ has full rank. Assume that there exists a orthogonal projection matrix $Q = Q^2 \in \mathbb{F}^{d\times d}$ commuting with all other matrices

$$\mathcal{H}Q - Q\mathcal{H} \equiv 0,\ P_1Q - QP_1 = 0,\ P_0Q - QP_0 \equiv 0$$

and $\kappa > 0$ such that for all $x \in D(A)$

$$\operatorname{Re}\langle Ax, x\rangle_X \le -\kappa\left(\left|Q\mathcal{H}x(0)\right|^2 + \left|(I-Q)\mathcal{H}x(1)\right|^2\right).$$

Then the C_0-semigroup generated by A is uniformly exponentially stable.

Proof. Clearly A generates a contraction C_0-semigroup $(T(t))_{t\ge 0}$. Since $\mathbb{F}^d = \operatorname{ran} Q \oplus \ker Q$ there is an isometric isomorphism $\Phi : \mathbb{F}^d \to \mathbb{F}^d$ such that $\Phi(\operatorname{ran} Q) = \mathbb{F}^{d_1} \times \{0\} \subseteq \mathbb{F}^d$ and $\Phi(\ker Q) = \{0\} \times \mathbb{F}^{d_2} \subseteq \mathbb{F}^d$ where $d_1 = \dim \operatorname{ran} Q$ and $d_2 = d - d_1 = \dim\ker Q$. We write

$$\Phi x = (\Phi_1 x, \Phi_2 x) \in \mathbb{F}^{d_1} \times \mathbb{F}^{d_2}$$

and set

$$(\tilde{\Phi}x)(\zeta) := (\Phi_1(\zeta), \Phi_2(1-\zeta)), \quad x \in L_2(0,1;\mathbb{F}^d),\ \zeta \in (0,1)$$

defining a continuous map $\tilde{\Phi} : L_2(0,1;\mathbb{F}^d) \to L_2(0,1;\mathbb{F}^d)$. Further we define $\tilde{\mathcal{H}} \in W^1_\infty(0,1;\mathbb{F}^{d\times d})$ by setting

$$\tilde{\mathcal{H}}(\zeta) := \tilde{\Phi}(\mathcal{H}(\tilde{\Phi}^{-1}\tilde{x})), \quad \tilde{x} \in L_2(0,1;\mathbb{F}^d)$$

and similar $\tilde{P}_1 \in \mathbb{F}^{d\times d}$ and $\tilde{P}_0 \in L_\infty(0,1;\mathbb{F}^{d\times d})$ via

$$\tilde{P}_1\tilde{z} := \Phi(P_1\Phi^{-1}\tilde{z}), \quad z \in \mathbb{F}^d$$
$$\tilde{P}_0\tilde{x} := \tilde{\Phi}(P_0\tilde{\Phi}^{-1}\tilde{x}), \quad \tilde{x} \in L_2(0,1;\mathbb{F}^d).$$

Then $\tilde{\mathcal{H}}$ is uniformly positive definite since for every $\tilde{z} \in \mathbb{F}^d$ we have

$$\begin{aligned}
\langle \tilde{z}, \tilde{\mathcal{H}}(\zeta)\tilde{z}\rangle_{\mathbb{F}^d} &= \langle \tilde{z}, (\tilde{\Phi}\mathcal{H}\tilde{\Phi}^{-1}\mathbf{1}z)(\zeta)\rangle_{\mathbb{F}^d} \\
&= \langle \tilde{z}, (\tilde{\Phi}\mathcal{H}\tilde{\Phi}^{-1}\mathbf{1}Qz)(\zeta)\rangle_{\mathbb{F}^d} + \langle \tilde{z}, (\tilde{\Phi}\mathcal{H}\tilde{\Phi}^{-1}\mathbf{1}(I-Q)z)(\zeta)\rangle_{\mathbb{F}^d} \\
&= \langle \tilde{z}, \Phi\mathcal{H}(\zeta)\Phi^{-1}Qz\rangle_{\mathbb{F}^d} + \langle \tilde{z}, \Phi\mathcal{H}(1-\zeta)\Phi^{-1}(I-Q)z)(\zeta)\rangle_{\mathbb{F}^d} \\
&= \langle \Phi^{-1}\tilde{z}, \mathcal{H}(\zeta)\Phi^{-1}Qz\rangle_{\mathbb{F}^d} + \langle \Phi^{-1}\tilde{z}, \mathcal{H}(1-\zeta)\Phi^{-1}(I-Q)z)(\zeta)\rangle_{\mathbb{F}^d} \\
&\ge m_0\left|Qz\right|^2 + m_0\left|(I-Q)z\right|^2 = m_0\left|z\right|^2, \quad \text{a.e. } \zeta \in (0,1).
\end{aligned}$$

Similar we deduce that P_1 is symmetric and it is also invertible since $\tilde{P}_1 = \Phi P_1 \Phi^{-1}$ is the product of invertible matrices. Now define on $\tilde{X} = L_2(0,1;\mathbb{F}^{d\times d})$ with $\langle\cdot,\cdot\rangle_{\tilde{X}} = \langle\cdot,\tilde{\mathcal{H}}\cdot\rangle_{L_2}$ the operator

$$\begin{aligned}
\tilde{A}\tilde{x} &= \tilde{\Phi}(A\tilde{\Phi}^{-1}\tilde{x}) \\
&= \tilde{\Phi}(P_1(\mathcal{H}\tilde{\Phi}^{-1}\tilde{x})' + P_0(\mathcal{H}\tilde{\Phi}^{-1}\tilde{x})) \\
&= \tilde{P}_1(\tilde{\mathcal{H}}\tilde{x})' + \tilde{P}_0(\tilde{\mathcal{H}}\tilde{x}) \\
D(\tilde{A}) &= \tilde{\Phi}(D(A)) \\
&= \{\tilde{x} \in \tilde{X} :\ \tilde{\mathcal{H}}x \in H^1(0,1;\mathbb{F}^d),\ \tilde{\mathfrak{B}}\tilde{x} = 0\}
\end{aligned}$$

for $\tilde{\mathfrak{B}}\tilde{x} := \mathfrak{B}\tilde{\Phi}^{-1}\tilde{x}$. Observe that

$$\begin{aligned}
\|\tilde{x}\|_{\tilde{X}}^2 &= \langle \tilde{x}, \tilde{\Phi}\mathcal{H}\tilde{\Phi}^{-1}\tilde{x}\rangle_{L_2} \\
&= \langle \tilde{\Phi}^{-1}\tilde{x}, \mathcal{H}\tilde{\Phi}^{-1}\tilde{x}\rangle_{L_2} = \left\|\tilde{\Phi}^{-1}\tilde{x}\right\|_X^2, \quad \tilde{x} \in \tilde{X}.
\end{aligned} \tag{4.15}$$

Then due to

$$\begin{aligned}
\operatorname{Re}\langle \tilde{A}\tilde{x}, \tilde{x}\rangle_{\tilde{X}} &= \operatorname{Re}\langle \tilde{\Phi}A\tilde{\Phi}^{-1}\tilde{x}, \tilde{\Phi}\mathcal{H}\tilde{\Phi}^{-1}\tilde{x}\rangle_{L_2} \\
&= \operatorname{Re}\langle A\tilde{\Phi}^{-1}\tilde{x}, \tilde{\Phi}^{-1}\tilde{x}\rangle_X \\
&\le -\kappa\left(\left|(Q\tilde{\Phi}^{-1}\tilde{x})(0)\right|^2 + \left|((I-Q)\tilde{\Phi}^{-1}\tilde{x})(1)\right|^2\right) \\
&= -\kappa\left|\Phi^{-1}\tilde{x}(0)\right|^2 = -\kappa\left|\tilde{x}(0)\right|^2
\end{aligned}$$

uniform exponential stability of the semigroup $(\tilde{T}(t))_{t\ge 0}$ generated by $\tilde{A}$ follows from Theorem 3.3.6 and hence because of the identity

$$\tilde{T}(t)(\tilde{\Phi}x) = (\tilde{\Phi}T(t)\tilde{\Phi}^{-1})\tilde{\Phi}x = \tilde{\Phi}T(t)x$$

(by similarity $\tilde{A} = \tilde{\Phi}A\tilde{\Phi}^{-1}$ of the generators and hence the semigroups) also the C_0-semigroup $(T(t))_{t\ge 0}$ is uniformly exponentially stable. □

Example 4.3.14. *Consider three vibrating strings described by the wave equation, cf. Example 9.3 in [JaZw12],*

$$\rho_i(\zeta)\frac{\partial^2}{\partial t^2}\omega_i(t,\zeta) = \frac{\partial}{\partial\zeta}\left(T_i(\zeta)\frac{\partial}{\partial\zeta}\omega_i(t,\zeta)\right), \quad t \ge 0,\ \zeta \in (0,1),\ i = 1,2,3.$$

Here the physical parameters ρ_i, T_i are assumed to be uniformly positive and Lipschitz continuous. Now let the right end of the first beam and the left end of the other beams be connected via a (mass-less) bar, i.e. by balance of forces

$$T_1(1)\frac{\partial\omega_1}{\partial\zeta}(1) + T_2(0)\frac{\partial\omega_2}{\partial\zeta}(0) + T_3(0)\frac{\partial\omega_3}{\partial\zeta}(0) = 0.$$

Further assume that at the other ends of the beams point dampers are attached, so

$$\begin{aligned}
T_1(0)\frac{\partial\omega_1}{\partial\zeta}(0) &= -\alpha_1\frac{\partial\omega_1}{\partial t}(0) \\
T_k(1)\frac{\partial\omega_1}{\partial\zeta}(1) &= -\alpha_k\frac{\partial\omega_k}{\partial t}(1), \quad k = 2,3
\end{aligned}$$

for some constants $\alpha_i \geq 0$ $(i = 1, 2, 3)$. The energy of the system is given by

$$H(t) = \sum_{i=1}^{3} \frac{1}{2} \int_0^1 \rho_i(\zeta) \left| \frac{\partial \omega_i}{\partial t}(t, \zeta) \right|^2 + T_i(\zeta) \left| \frac{\partial \omega_i}{\partial \zeta}(t, \zeta) \right|^2 d\zeta$$

and using the boundary conditions one easily obtains the balance equation

$$\frac{dE}{dt}(t) = -\alpha_1 \left| \frac{\partial \omega_1}{\partial t}(t, 0) \right|^2 - \alpha_2 \left| \frac{\partial \omega_2}{\partial t}(t, 2) \right|^2 - \alpha_3 \left| \frac{\partial \omega_3}{\partial t}(t, 3) \right|^2.$$

To reformulate the problem in our port-Hamiltonian standard form we set similar to Example 3.1.2

$$x_{i,1} := \rho_i \frac{\partial}{\partial t} \omega_i, \; x_{i,2} := \frac{\partial}{\partial \zeta} \omega_i, \quad i = 1, 2, 3$$

and $\mathcal{H} = \operatorname{diag}(\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3) \in W^1_\infty(0, 1; \mathbb{F}^{6\times 6})$ where

$$\mathcal{H}_i := \begin{bmatrix} \frac{1}{\rho_i} & \\ & T_i \end{bmatrix}.$$

Moreover, $P_0 = 0 \in \mathbb{F}^{6\times 6}$ and the matrix $P_1 \in \mathbb{F}^{6\times 6}$ is given as

$$P_1 = \begin{bmatrix} & 1 & & & & \\ 1 & & & & & \\ & & & 1 & & \\ & & 1 & & & \\ & & & & & 1 \\ & & & & 1 & \end{bmatrix}.$$

Clearly P_1 and $\mathcal{H}$ commute with the orthogonal projection $Q : \mathbb{F}^6 \to \mathbb{F}^2 \times \{0\}$ on the first two components and we have for the operator $Ax = P_1(\mathcal{H}x)'$ with

$$D(A) = \{x \in L_2(0, 1; \mathbb{F}^6) : \mathcal{H}x \in H^1(0, 1; \mathbb{F}^6) \text{ satisfies the b.c.}\}$$

the dissipation relation

$$\operatorname{Re} \langle Ax, x \rangle_X = -\alpha_1 |(\mathcal{H}_1 x_1)_2(0)|^2 - \alpha_2 |(\mathcal{H}_2 x_2)_2(1)|^2 - \alpha_3 |(\mathcal{H}_3 x_3)_2(1)|^2$$

for all $x \in D(A)$. If we assume that $\alpha_i > 0$ $(i = 1, 2, 3)$ we obtain for $\kappa := \min\{\alpha_1, \alpha_2, \alpha_3\} > 0$ that

$$\begin{aligned} \operatorname{Re} \langle Ax, x \rangle_{\mathcal{H}} &\leq -\kappa \left(|(\mathcal{H}_1 x_1)(0)|^2 + |(\mathcal{H}_2 x_2)(1)|^2 + |(\mathcal{H}_3 x_3)(1)|^2 \right) \\ &= -\kappa \left(|Q(\mathcal{H}x)(0)|^2 + |(I - Q)(\mathcal{H}x)(1)|^2 \right) \end{aligned}$$

for all $x \in D(A)$. As a result, from Proposition 4.3.13 we obtain uniform exponential stability. □

4.3.2 Second Order Port-Hamiltonian Systems

As we have seen in the preceding subsection, for first order ($N = 1$) port-Hamiltonian systems the sufficient criterion for asymptotic stability in Theorem 4.2.2 even guarantees uniform exponential stability, see Theorem 3.3.6. In this subsection – just as in the article [AuJa14] – we consider second order port-Hamiltonian systems, i.e.

$$\mathfrak{A}x = P_2(\mathcal{H}x)'' + P_1(\mathcal{H}x)' + P_0(\mathcal{H}x), \quad x \in D(\mathfrak{A}) = \mathcal{H}^{-1} H^2(0, 1; \mathbb{F}^d)$$

and

$$A = \mathfrak{A}|_{D(\mathfrak{A})} \text{ for } D(A) = \{x \in D(\mathfrak{A}) : W_B \begin{pmatrix} f_{\partial,\mathcal{H}x} \\ e_{\partial,\mathcal{H}x} \end{pmatrix} = 0\}.$$

We observe that for port-Hamiltonian systems of order $N = 2$ we may obtain exponential stability by adding an additional term in the dissipativity relation (4.11), which had been enough for asymptotic stability, see Theorem 4.2.2, but not sufficient for uniform exponential stability, see Example 4.2.1.

Theorem 4.3.15. *Let A be a port-Hamiltonian operator of order $N = 2$ and assume that P_0 and $\mathcal{H} \in W^1_\infty(0, 1; \mathbb{F}^{d\times d})$ are Lipschitz continuous and A satisfies the stronger dissipativity condition*

$$\begin{aligned} \operatorname{Re} \langle Ax, x\rangle_X \leq -\kappa \Big[&|(\mathcal{H}x)(0)|^2 + |(\mathcal{H}x)'(0)|^2 \\ &+ |\Pi(\mathcal{H}x)(1)|^2 + |(I - \Pi)P_2(\mathcal{H}x)'(1)|^2 \Big] \end{aligned} \tag{4.16}$$

for some positive constant $\kappa > 0$ and some orthogonal projection $\Pi : \mathbb{F}^{Nd} \to \mathbb{F}^{Nd}$. Then for $R : D(\mathfrak{A}) \to \mathbb{F}^{4d}$

$$Rx = \begin{pmatrix} (\mathcal{H}x)(0) \\ (\mathcal{H}x)'(0) \\ \Pi(\mathcal{H}x)(1) \\ (I - \Pi)(\mathcal{H}x)'(1) \end{pmatrix} \tag{4.17}$$

the pair $(\mathfrak{A}, R)$ has property ESP and hence the C_0-semigroup $(T(t))_{t\geq 0}$ generated by A is uniformly exponentially stable.

Proof. Let us show that the pair $(\mathfrak{A}, R)$ has property ESP. Since $(\mathfrak{A}, R)$ clearly has property ASP by Theorem 4.2.2 and Remark 4.2.6 it only remains to verify property AIEP.

Let $(x_n, \beta_n)_{n\geq 1} \subseteq D(\mathfrak{A})\times\mathbb{R}$ be a sequence with $\sup_{n\in\mathbb{N}} \|x_n\|_X < \infty$ and $|\beta_n| \to +\infty$ as $n \to +\infty$ such that

$$i\beta_n x_n - \mathfrak{A}x_n \xrightarrow{n\to\infty} 0 \quad \text{in } L_2(0, 1; \mathbb{F}^d) \tag{4.18}$$

and $Rx_n \xrightarrow{n\to\infty} 0$. Then

$$(\mathcal{H}x_n)(0), (\mathcal{H}x_n)'(0), (\mathcal{H}x_n)(1) \xrightarrow{n\to\infty} 0. \tag{4.19}$$

Further by equation (4.18) and Lemma 3.2.3 the sequence $\left(\frac{\mathcal{H}x_n}{\beta_n}\right)_{n\geq 1} \subseteq H^2(0, 1; \mathbb{F}^d)$ is bounded. Choosing $\eta \in (\frac{3}{2}, 2)$ and using Lemma 2.1.19 we obtain

$$\left\| \frac{\mathcal{H}x_n}{|\beta_n|^{\eta/2}} \right\|_{C^1} \lesssim \left\| \frac{\mathcal{H}x_n}{\beta_n} \right\|_{H^2}^{\eta/2} \|\mathcal{H}x_n\|_{L_2}^{1-\eta/2} .$$

In particular, (since $|\beta_n| \to \infty$) it follows that $\frac{\mathcal{H}x_n}{\beta_n}$ converges to zero in $C^1([0, 1]; \mathbb{F}^d)$, and so does $\frac{(\mathcal{H}x_n)'}{\beta_n}$ in $L_2(0, 1; \mathbb{F}^d)$. Now we establish $x_n \xrightarrow{n\to\infty} 0$ in $L_2(0, 1; \mathbb{F}^d)$ using a multiplier technique similar to the first alternative proof of Theorem 4.1.5. For this end, let $q \in C^2([0, 1]; \mathbb{R})$ be any function with $q(1) = 0$. Then integrating by parts and employing the assumptions on the matrices P_1, P_2 and equation (4.19) we conclude – also using Lemma 4.3.2 –

$$\begin{aligned}
&2\operatorname{Re}\langle P_2(\mathcal{H}x_n)'', \frac{1}{\beta_n} iq(\cdot)(\mathcal{H}x_n)'\rangle_{L_2} \\
&= \frac{1}{\beta_n}\langle(\mathcal{H}x_n)', (iP_2)q'(\cdot)(\mathcal{H}x_n)'\rangle_{L_2} - \frac{1}{\beta_n}\left[q(\zeta)\langle(\mathcal{H}x_n)'(\zeta), iP_2(\mathcal{H}x_n)'(\zeta)\rangle_{\mathbb{F}^d}\right]_0^1 \\
&= \frac{1}{\beta_n}\langle(\mathcal{H}x_n)', (iP_2)q'(\cdot)(\mathcal{H}x_n)'\rangle_{L_2} + o(1)
\end{aligned}$$

since $q(1) = 0$ and $(\mathcal{H}x_n)'(0) \xrightarrow{n\to\infty} 0$. Further

$$\operatorname{Re}\langle P_1(\mathcal{H}x_n)', \frac{1}{\beta_n} iq(\cdot)(\mathcal{H}x_n)'\rangle_{L_2} = 0$$

since $P_1^* = P_1$ and

$$\operatorname{Re}\langle P_0(\mathcal{H}x_n), \frac{1}{\beta_n} iq(\cdot)(\mathcal{H}x_n)'\rangle_{L_2} = o(1)$$

due to $\frac{(\mathcal{H}x_n)'}{\beta_n} \xrightarrow{n\to\infty} 0$ in X. Similar we obtain

$$\begin{aligned}
&2\operatorname{Re}\langle x_n, q(\cdot)(\mathcal{H}x_n)'\rangle_{L_2} \\
&= 2\operatorname{Re}\langle x_n, q(\cdot)\mathcal{H}'(\cdot)x_n\rangle_{L_2} + 2\operatorname{Re}\langle x_n, q(\cdot)\mathcal{H}(\cdot)x_n'\rangle_{L_2} \\
&= 2\langle x_n, q(\cdot)\mathcal{H}'(\cdot)x_n\rangle_{L_2} - \langle x_n, (q\mathcal{H})'x_n\rangle_{L_2} + \left[q(\zeta)\langle x_n(\zeta), \mathcal{H}(\zeta)x_n(\zeta)\rangle_{\mathbb{F}^d}\right]_0^1 \\
&= \langle x_n, (q(\cdot)\mathcal{H}'(\cdot) - q'(\cdot)\mathcal{H}(\cdot))x_n\rangle_{L_2} + o(1)
\end{aligned}$$

since $q(1) = 0$ and $x_n(0) \to 0$.

$$\begin{aligned}
&\operatorname{Re}\langle P_2(\mathcal{H}x_n)'', \frac{1}{\beta_n} iq'(\cdot)(\mathcal{H}x_n)\rangle_{L_2} \\
&= -\operatorname{Re}\langle P_2(\mathcal{H}x_n)', \frac{1}{\beta_n}(iq'(\cdot)(\mathcal{H}x_n))'\rangle_{L_2} \\
&\quad + \frac{1}{\beta_n}\operatorname{Re}\left[q'(\zeta)\langle(\mathcal{H}x_n)'(\zeta), (-iP_2)(\mathcal{H}x_n)(\zeta)\rangle_{\mathbb{F}^d}\right]_0^1 \\
&= \operatorname{Re}\langle(\mathcal{H}x_n)', \frac{1}{\beta_n}(iP_2)q'(\cdot)(\mathcal{H}x_n)'\rangle_{L_2} \\
&\quad + \operatorname{Re}\langle(\mathcal{H}x_n)', \frac{1}{\beta_n} iq''(\cdot)(\mathcal{H}x_n)\rangle_{L_2} + o(1) \qquad (4.20) \\
&= \langle(\mathcal{H}x_n)', \frac{1}{\beta_n}(iP_2)q'(\cdot)(\mathcal{H}x_n)'\rangle_{L_2} + o(1),
\end{aligned}$$

due to $(\mathcal{H}x_n)(0), (\mathcal{H}x_n)'(0), \Pi(\mathcal{H}x_n)(1), (I-\Pi)P_2(\mathcal{H}x_n)'(1) \xrightarrow{n\to\infty} 0$, where we also used that Π is orthogonal. Moreover, we have

$$\operatorname{Re}\langle P_1(\mathcal{H}x_n)', \frac{1}{\beta_n} iq'(\cdot)(\mathcal{H}x_n)\rangle_{L_2} = o(1)$$

since $\frac{(\mathcal{H}x_n)'}{\beta_n} \xrightarrow{n\to\infty} 0$ in X and $\sup_{n\in\mathbb{N}} \|x_n\| < +\infty$, and

$$\operatorname{Re}\langle P_0(\mathcal{H}x_n), \frac{1}{\beta_n} iq'(\cdot)(\mathcal{H}x_n)\rangle_{L_2} = o(1),$$
$$\operatorname{Re}\langle x_n, q'(\cdot)(\mathcal{H}x_n)\rangle_{L_2} = \langle x_n, q'(\cdot)\mathcal{H}x_n\rangle_{L_2}.$$

This implies that for $n \to \infty$

$$\begin{aligned} 0 &\leftarrow 2\operatorname{Re}\langle P_2(\mathcal{H}x_n)'' + P_1(\mathcal{H}x_n)' + P_0(\mathcal{H}x_n) - i\beta_n x_n, \frac{1}{\beta_n} iq(\cdot)(\mathcal{H}x_n)'\rangle_{L_2} \\ &= \frac{1}{\beta_n}\langle(\mathcal{H}x_n)', (iP_2)q'(\cdot)(\mathcal{H}x_n)'\rangle_{L_2} \\ &\quad - \langle x_n, (q(\cdot)\mathcal{H}'(\cdot) - q'(\cdot)\mathcal{H}(\cdot))x_n\rangle_{L_2} + o(1) \end{aligned} \tag{4.21}$$

and

$$\begin{aligned} 0 &\leftarrow \operatorname{Re}\langle P_2(\mathcal{H}x_n)'' + P_1(\mathcal{H}x_n)' + P_0(\mathcal{H}x_n) - i\beta_n x_n, \frac{1}{\beta_n} iq'(\cdot)(\mathcal{H}x_n)\rangle_{L_2} \\ &= \frac{1}{\beta_n}\langle(\mathcal{H}x_n)', (iP_2)q'(\cdot)(\mathcal{H}x_n)'\rangle_{L_2} - \langle x_n, q'(\cdot)\mathcal{H}(\cdot)x_n\rangle_{L_2} + o(1). \end{aligned} \tag{4.22}$$

By subtracting equations (4.21) and (4.22) we arrive at

$$\langle x_n, (q(\cdot)\mathcal{H}'(\cdot) - 2q'(\cdot)\mathcal{H}(\cdot))x_n\rangle_{L_2} \xrightarrow{n\to\infty} 0.$$

If there is some q such that $q(\cdot)\mathcal{H}'(\cdot) - 2q'(\cdot)\mathcal{H}(\cdot) > 0$, this enables us to conclude $\|x_n\|_{L_2} \to 0$. Indeed, Lemma 4.3.3 says that we can choose $q \in C^2([0,1];\mathbb{R})$ such that

$$q(1) = 0,\ q' < 0 \text{ and } qM_1 - 2q'(\cdot)m > 0$$

where $\mathcal{H}(\zeta) \geq mI$ and $\mathcal{H}'(\zeta) \leq M_1 I$ for almost all $\zeta \in [0,1]$ and some constants $m, M_1 > 0$. More precisely let f be the function given by the lemma and set $q(\zeta) = -f(1-\zeta)$. From this we infer $x_n \xrightarrow{n\to\infty} 0$ in X. Property ESP of the pair $(\mathfrak{A}, R)$ and by Proposition 4.3.6 uniform exponential stability follow. □

Remark 4.3.16. *From the proof we may also extract stability properties provided $\mathcal{H}$ satisfies some additional properties. In fact we have the following. Let $\mathfrak{A}$ be a maximal port-Hamiltonian operator with Lipschitz continuous $\mathcal{H}$ and P_0. If there is $q \in C^\infty([0,1];\mathbb{R})$ such that $q(0) = q(1) = 0$ and still the matrix-valued function $2q'\mathcal{H} - q\mathcal{H}'$ is coercive as multiplication operator on X, then for*

$$R : D(\mathfrak{A}) \to \mathbb{F}^{4d},\ x \mapsto \begin{pmatrix} \Pi(\mathcal{H}x)(0) \\ (I-\Pi)P_2(\mathcal{H}x)'(0) \\ \Pi'(\mathcal{H}x)(1) \\ (I-\Pi')P_2(\mathcal{H}x)'(1) \end{pmatrix}$$

where $\Pi, \Pi' \in \mathbb{F}^{d\times d}$ are orthogonal projections, the pair $(\mathfrak{A}, R)$ has property AIEP.

We also give an alternative proof for the case $N = 2$ under the additional restriction that $\mathcal{H}$ is constant along $\zeta \in (0,1)$ and that $P_1 = P_0 = 0$. The proof is similar to the second alternative proof (Lyapunov technique) for the case $N = 1$ in Theorem 4.1.5.

Alternative Proof. In the following we also apply the Lyapunov technique to port-Hamiltonian operators of order $N = 2$. However, we have to restrict ourselves to port-Hamiltonian systems for which $P_1 = P_0 = 0$ and the derivative $\mathcal{H}'$ is small compared to $\mathcal{H}$.

Lemma 4.3.17. *Let $\mathfrak{A}$ be a port-Hamiltonian operator of order $N = 2$ and assume that $P_1 = P_0 = 0$ and the Hamiltonian density matrix function $\mathcal{H}$ is Lipschitz continuous and satisfies the condition*

$$\mathcal{H}(\zeta) - (1-\zeta)\mathcal{H}'(\zeta) \geq \varepsilon' I, \quad a.e.\ \zeta \in (0,1)$$

for some $\varepsilon' > 0$. Then there is a function $q : X \to \mathbb{R}$ with $|q(x)| \leq \hat{c}\,\|x\|_X^2$ such that for every solution $x \in W^1_\infty(\mathbb{R}_+; X) \cap L_\infty(\mathbb{R}_+; D(\mathfrak{A}))$ of $\dot{x} = \mathfrak{A}x$ one has

$$\frac{d}{dt}q(x(t)) \leq -\|x(t)\|_X^2 + c\left(|(\mathcal{H}x)(0)|^2 + |(\mathcal{H}x)'(0)|^2 + |(\mathcal{H}x)(1)|^2\right), \quad a.e.\ t \geq 0.$$

Proof. Let $x \in W^1_\infty(\mathbb{R}_+; X) \cap L_\infty(\mathbb{R}_+; D(\mathfrak{A}))$ be an arbitrary mild solution of $\dot{x} = \mathfrak{A}x$ and write $x(t,\zeta) := x(t)(\zeta)$. Further let $\eta \in C^\infty([0,1];\mathbb{R})$ be a real-valued function, which we specify at a later point, and define the continuous quadratic functional

$$\begin{aligned} q(x) &:= 2\operatorname{Re}\langle \eta P_2^{-1}x, \int_0^\cdot x d\xi\rangle_{L_2} \\ &= 2\operatorname{Re}\int_0^1 \langle \eta(\zeta)P_2^{-1}x(\zeta), \int_0^\zeta x(\xi)d\xi\rangle_{\mathbb{F}^d} d\zeta. \end{aligned}$$

Then we obtain

$$\begin{aligned} &\frac{d}{dt}q(x(t)) \\ &= 2\operatorname{Re}\langle \eta P_2^{-1}x_t(t), \int_0^\cdot x(t,\xi)d\xi\rangle_{L_2} + 2\operatorname{Re}\langle \eta P_2^{-1}x(t), \int_0^\cdot x_t(t,\xi)d\xi\rangle_{L_2} \\ &= 2\operatorname{Re}\langle \eta(\mathcal{H}x)''(t), \int_0^\cdot x(t,\xi)d\xi\rangle_{L_2} - 2\operatorname{Re}\langle \eta x(t), \int_0^\cdot (\mathcal{H}x)''(t,\xi)d\xi\rangle_{L_2} \\ &= -2\operatorname{Re}\langle \eta'(\mathcal{H}x)'(t), \int_0^\cdot x(t,\xi)d\xi\rangle_{L_2} - 2\operatorname{Re}\langle \eta(\mathcal{H}x)'(t), x(t)\rangle_{L_2} \\ &\quad + 2\operatorname{Re}\left[\eta(\zeta)\langle(\mathcal{H}x)'(t,\zeta), \int_0^\zeta x(t,\xi)d\xi\rangle_{\mathbb{F}^d}\right]_0^1 \\ &\quad - 2\operatorname{Re}\langle \eta x(t), (\mathcal{H}x)'(t)\rangle_{L_2} + 2\operatorname{Re}\langle\int_0^1 \eta x(t,\xi)d\xi, (\mathcal{H}x)'(t,0)\rangle_{\mathbb{F}^d} \\ &= 2\operatorname{Re}\langle \eta''(\mathcal{H}x)(t), \int_0^\cdot x(t,\xi)d\xi\rangle_{L_2} + 2\langle \eta'\mathcal{H}x(t), x(t)\rangle_{L_2} \\ &\quad - 2\operatorname{Re}\left[\eta'(\zeta)\langle(\mathcal{H}x)(t,\zeta), \int_0^\zeta x(t,\xi)d\xi\rangle_{\mathbb{F}^d}\right]_0^1 \\ &\quad - 4\operatorname{Re}\langle \eta(\mathcal{H}x)', x\rangle_{L_2} + 2\operatorname{Re}\left[\eta(1)\langle(\mathcal{H}x)'(t,1), \int_0^1 x(t,\xi)d\xi\rangle_{\mathbb{F}^d}\right] \end{aligned}$$

$$
\begin{aligned}
&+ 2\operatorname{Re}\langle\int_0^1 \eta(\xi)x(t,\xi)d\xi, (\mathcal{H}x)'(t,0)\rangle_{\mathbb{F}^d} \\
&= -\langle(\eta''\mathcal{H})'\int_0^\cdot x(t,\xi)d\xi, \int_0^\cdot x(t,\xi)d\xi\rangle_{L_2} \\
&+ \left[\eta''(\zeta)\langle\int_0^\zeta x(t,\xi)d\xi, \mathcal{H}\int_0^\zeta x(t,\xi)d\xi\rangle_{\mathbb{F}^d}\right]_0^1 \\
&- 4\langle\eta\mathcal{H}'x,x\rangle_{L_2} + \langle(\eta\mathcal{H})'x,x\rangle_{L_2} - 2\left[\eta(\zeta)\langle x(\zeta),\mathcal{H}(\zeta)x(\zeta)\rangle_{\mathbb{F}^d}\right]_0^1 \\
&+ 2\langle\eta'\mathcal{H}x(t),x(t)\rangle_{L_2} - 2\operatorname{Re}\left[\eta'(1)\langle(\mathcal{H}x)(t,1), \int_0^1 x(t,\xi)d\xi\rangle_{\mathbb{F}^d}\right] \\
&+ 2\operatorname{Re}\left[\eta(1)\langle(\mathcal{H}x)'(t,1), \int_0^1 x(t,\xi)d\xi\rangle_{\mathbb{F}^d}\right] \\
&+ 2\operatorname{Re}\langle\int_0^1 \eta(\xi)x(t,\xi)d\xi, (\mathcal{H}x)'(t,0)\rangle_{\mathbb{F}^d} \\
&\le -\langle(\eta''\mathcal{H})'\int_0^\cdot x(t,\xi)d\xi, \int_0^\cdot x(t,\xi)d\xi\rangle_{L_2} \\
&+ \eta''(1)\langle\int_0^1 x(t,\xi)d\xi, \mathcal{H}\int_0^1 x(t,\xi)d\xi\rangle_{\mathbb{F}^d} + 2\langle(\eta'\mathcal{H}-\eta\mathcal{H}')x(t),x(t)\rangle_{L_2} \\
&+ 2\left|\eta(0)\right|\langle x(0),(\mathcal{H}x)(0)\rangle_{\mathbb{F}^d} + 2\left|\eta(1)\right|\langle x(1),(\mathcal{H}x)(1)\rangle_{\mathbb{F}^d} + \frac{1}{\alpha}\left|\eta'(1)(\mathcal{H}x)(t,1)\right|^2 \\
&+ \alpha\left|\int_0^1 x(t,\xi)d\xi\right|^2 + \frac{1}{\alpha}\left|\eta(1)(\mathcal{H}x)'(t,1)\right|^2 \\
&+ \alpha\left|\int_0^1 x(t,\xi)d\xi\right|^2 + \alpha\int_0^1 \left|x(t,\xi)\right|^2 d\xi \\
&+ \frac{\|\eta\|_{L_\infty}^2}{\varepsilon}\left|(\mathcal{H}x)'(t,0)\right|^2
\end{aligned}
$$

Now we may choose $\eta(\zeta) = 1-\zeta$ and conclude that since

$$- \eta'(\zeta)\mathcal{H}(\zeta) - \eta(\zeta)\mathcal{H}'(\zeta) = \mathcal{H}(\zeta) - (1-\zeta)\mathcal{H}'(\zeta) \ge \varepsilon' I, \quad \text{a.e. } \zeta \in (0,1)$$

that for some $\varepsilon > 0$ one has

$$\frac{d}{dt}q(x(t)) \le -\varepsilon\left\|x(t)\right\|_X^2 + c_\varepsilon\left(\left|(\mathcal{H}x)(0)\right|^2 + \left|(\mathcal{H}x)'(0)\right|^2 + \left|(\mathcal{H}x)(1)\right|^2\right), \quad \text{a.e. } t \ge 0.$$

Then the result follows with q replaced by $\frac{1}{\varepsilon}q$. □

In this case uniform exponential stability can be derived from the following general observation.

Lemma 4.3.18. *Let $\mathfrak{S} = (\mathfrak{A},\mathfrak{B},\mathfrak{C})$ be a port-Hamiltonian system in boundary control and observation form and assume that for $A := \mathfrak{A}|_{\ker\mathfrak{B}+K\mathfrak{C}}$ and some orthogonal projection $\Pi \in \mathbb{F}^{Nd} \to \mathbb{F}^{Nd}$ and a constant $\kappa > 0$ the estimate*

$$\operatorname{Re}\langle Ax,x\rangle_X \le -\kappa\left(\left|\mathfrak{B}x\right|^2 + \left|\Pi\mathfrak{C}x\right|^2\right), \quad x \in D(A)$$

holds good. Further assume that there are $c > 0$ and $q : X \to \mathbb{R}$ with $|q(x)| \le \hat{c}\left\|x\right\|_X^2$ $(x \in X)$ such that for every solution $x \in W_\infty^1(\mathbb{R}_+;X) \times L_\infty(\mathbb{R}_+;D(\mathfrak{A}))$ of

$\dot{x} = \mathfrak{A}x$ *one has* $q(x) \in W^1_\infty(\mathbb{R}_+)$ *with*

$$\frac{d}{dt}q(x(t)) \leq -\|x(t)\|_X^2 + c\left(|\mathfrak{B}x|^2 + |\Pi\mathfrak{C}x|^2\right), \quad a.e.\ t \geq 0.$$

Then A *generates a contractive and uniformly exponentially stable* C_0*-semigroup on* X.

Proof. Let $x_0 \in D(A)$ and $T(\cdot)x \in C^1(\mathbb{R}_+; X) \cap C(\mathbb{R}_+; D(\mathfrak{A}))$ be the classical solution of the Cauchy problem for this initial value. As in the case of $N = 1$ one sees that for some $t_0 > 0$ (independent of $x_0 \in D(A)$) one has for

$$\Phi(x(t)) := \|x(t)\|_X^2 + q(x(t)), \quad t \geq 0$$

that

$$\frac{d}{dt}\Phi(x(t)) \leq (c - 2\kappa t)\left(|\mathfrak{B}x|^2 + |\Pi\mathfrak{C}x|^2\right) \leq 0, \quad t \geq t_0$$

where the time $t_0 := \frac{c}{2\kappa} > 0$ does not depend on the initial value $x \in D(A)$. Then uniform exponential stability follows just as in the case $N = 1$. $\square$

4.3.3 Euler-Bernoulli Beam Equations

For the special class of port-Hamiltonian systems of Euler-Bernoulli type which have some anti-diagonal structure we prove the following result which allows weaker assumptions on the boundary dissipation.

Proposition 4.3.19. *Let* $d \in 2\mathbb{N}$ *be even and assume that* $\mathfrak{S} = (\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ *is a port-Hamiltonian operator of the following form. Assume that* $\mathcal{H} = \operatorname{diag}(\mathcal{H}_1, \mathcal{H}_2)$ *for Lipschitz continuous matrix-valued functions* $\mathcal{H}_1, \mathcal{H}_2 \in W^1_\infty(0, 1; \mathbb{F}^{d/2\times d/2})$ *and* P_2 *has the form*

$$P_2 = \begin{bmatrix} 0 & M_2 \\ -M_2^* & 0 \end{bmatrix}$$

for an invertible matrix $M_2 \in \mathbb{F}^{d/2\times d/2}$ *and* $P_0 \in W^1_\infty(0, 1; \mathbb{F}^{d\times d})$ *is Lipschitz continuous. Let* $A = \mathfrak{A}|_{\ker(\mathfrak{B}+K\mathfrak{C})}$ *be its restriction to some dissipative boundary conditions. Assume that there is some* $\kappa > 0$ *such that, for all* $x = (x_1, x_2) \in D(A)$

$$\operatorname{Re}\langle Ax, x\rangle_X \leq -\kappa\left(|(\mathcal{H}x)(0)|^2 + \left\{\begin{matrix} |(\mathcal{H}_1x_1)'(0)|^2 \\ or \\ |(\mathcal{H}_2x_2)'(0)|^2 \end{matrix}\right\} + \left\{\begin{matrix} |(\mathcal{H}_1x_1)(1)|^2 \\ or \\ |(\mathcal{H}_2x_2)'(1)|^2 \end{matrix}\right\} + \left\{\begin{matrix} |(\mathcal{H}_1x_1)'(1)|^2 \\ or \\ |(\mathcal{H}_2x_2)(1)|^2 \end{matrix}\right\}\right),$$

where we write $\mathbb{F}^d \ni x(\zeta) = (x_1(\zeta), x_2(\zeta)) \in \mathbb{F}^{d/2} \times \mathbb{F}^{d/2}$*, and assume that* A *has no eigenvalue on the imaginary axis. Then the contractive* C_0*-semigroup* $(T(t))_{t\geq 0}$ *generated by* A *is uniformly exponentially stable. More precisely, for*

$$R : D(\mathfrak{A}) \to \mathbb{F}^{2d},\ x \mapsto Rx := \begin{pmatrix} (\mathcal{H}x)(0) \\ \Pi(\mathcal{H}_1x_1)'(0) + (I - \Pi)M_2(\mathcal{H}_2x_2)'(0) \\ \Pi'(\mathcal{H}_1x_1)(1) + (I - \Pi')M_2(\mathcal{H}_2x_2)'(1) \\ \Pi''(\mathcal{H}_1x_1)'(1) + (I - \Pi'')M_2(\mathcal{H}_2x_2)(1) \end{pmatrix}$$

where $\Pi, \Pi', \Pi'' : \mathbb{F}^{d/2\times d/2}$ *are orthogonal projections, the pair* $(\mathfrak{A}, R)$ *has property AIEP.*

Proof. Let $((x_{n,1}, x_{n,2}), \beta_n)_{n\geq 1} \subseteq D(\mathfrak{A}) \times \mathbb{R}$ be a sequence with finite supremum $\sup_{n\in\mathbb{N}} \|x_n\|_{L_2} < +\infty$ and $|\beta_n| \xrightarrow{n\to+\infty} +\infty$ such that

$$\mathfrak{A}x_n - i\beta_n x_n \xrightarrow{n\to\infty} 0. \tag{4.23}$$

It then follows from Lemma 2.1.10 that

$$\frac{\mathcal{H}x_n}{\beta_n} \xrightarrow{n\to\infty} 0 \text{ in } C^1([0,1];\mathbb{F}^d)$$

so that for every function $q \in C^2([0,1];\mathbb{F}^d)$ with $q(1) = 0$ we have

$$\begin{aligned}
&\operatorname{Re}\langle P_1(\mathcal{H}x_n)' + P_0(\mathcal{H}x_n), \frac{iq}{\beta_n}(\mathcal{H}x_n)'\rangle_{L_2} \\
&= \operatorname{Re}\langle P_1(\mathcal{H}x_n)', \frac{iq}{\beta_n}(\mathcal{H}x_n)'\rangle_{L_2} + o(1) \\
&= o(1)
\end{aligned} \tag{4.24}$$

since iqP_2 is skew-adjoint. Moreover, we deduce from Lemma 4.3.2 the equality

$$\begin{aligned}
&\operatorname{Re}\langle i\beta_n x_n, \frac{iq}{\beta_n}(\mathcal{H}x_n)'\rangle_{L_2} \\
&= \operatorname{Re}\langle x_n, q(\mathcal{H}x_n)'\rangle_{L_2} \\
&= -\frac{1}{2}\langle x_n, (q'\mathcal{H} - q\mathcal{H}')x_n\rangle_{L_2} + \frac{1}{2}\left[\langle x_n(\zeta), (q\mathcal{H})(\zeta)x_n(\zeta)\rangle_{\mathbb{F}^d}\right]_0^1.
\end{aligned}$$

Using that $\left(\frac{iq}{\beta_n}(\mathcal{H}x_n)'\right)_{n\geq 1}$ is a bounded sequence in X and employing equation (4.24) we then find that

$$\begin{aligned}
&-\frac{1}{2}\langle x_n, (q'\mathcal{H} - q\mathcal{H}')x_n\rangle_{L_2} \\
&= \operatorname{Re}\langle i\beta_n(\mathcal{H}x_n), \frac{iq}{\beta_n}(\mathcal{H}x_n)'\rangle_{L_2} - \frac{1}{2}\left[\langle x_n(\zeta), q\mathcal{H}(\zeta)x_n(\zeta)\rangle_{\mathbb{F}^d}\right]_0^1 \\
&= \operatorname{Re}\langle \mathfrak{A}x_n, \frac{iq}{\beta_n}(\mathcal{H}x_n)'\rangle_{L_2} - \frac{1}{2}\left[\langle x_n(\zeta), q\mathcal{H}(\zeta)x_n(\zeta)\rangle_{\mathbb{F}^d}\right]_0^1 + o(1) \\
&= \operatorname{Re}\langle M_2(\mathcal{H}_2x_{n,2})'', \frac{iq}{\beta_n}(\mathcal{H}_1x_{n,1})'\rangle_{L_2} + \operatorname{Re}\langle -M_2^*(\mathcal{H}_1x_{n,1})'', \frac{iq}{\beta_n}(\mathcal{H}_2x_{n,2})'\rangle_{L_2} \\
&\quad - \frac{1}{2}\left[\langle x_n(\zeta), q\mathcal{H}(\zeta)x_n(\zeta)\rangle_{\mathbb{F}^d}\right]_0^1 + o(1) \\
&= -\operatorname{Re}\langle M_2(\mathcal{H}_2x_{n,2})', \frac{iq'}{\beta_n}(\mathcal{H}_1x_{n,1})'\rangle_{L_2} \\
&\quad + \operatorname{Re}\left[\langle M_2(\mathcal{H}_2x_{n,2})'(\zeta), \frac{iq(\zeta)}{\beta_n}(\mathcal{H}_1x_{n,1})'(\zeta)\rangle_{\mathbb{F}^{d/2}}\right]_0^1 \\
&\quad - \frac{1}{2}\left[\langle x_n(\zeta), q\mathcal{H}(\zeta)x_n(\zeta)\rangle_{\mathbb{F}^d}\right]_0^1 + o(1)
\end{aligned} \tag{4.25}$$

On the other hand we also find

$$\operatorname{Re}\langle P_1(\mathcal{H}x_n)' + P_0(\mathcal{H}x_n), \frac{iq'}{\beta_n}(\mathcal{H}x_n)\rangle_{L_2} = o(1) \tag{4.26}$$

and

$$\operatorname{Re}\langle i\beta_n x_n, \frac{iq'}{\beta_n}(\mathcal{H}x_n)\rangle_{L_2} = \operatorname{Re}\langle x_n, q'\mathcal{H}x_n\rangle_{L_2}$$

so that we also obtain – this time using that the sequence $(iq'\beta_n^{-1}(\mathcal{H}x_n))_{n\geq 1}$ is bounded, in fact a null sequence, in X – and equation (4.26) that also

$$\begin{aligned}
&\operatorname{Re}\langle x_n, q'\mathcal{H}x_n\rangle_{L_2} \\
&= \operatorname{Re}\langle i\beta_n x_n, \frac{iq'}{\beta_n}\mathcal{H}x_n\rangle_{L_2} \\
&= \operatorname{Re}\langle \mathfrak{A}x_n, \frac{iq'}{\beta_n}\mathcal{H}x_n\rangle_{L_2} + o(1) \\
&= \operatorname{Re}\langle M_2(\mathcal{H}_2 x_{n,2})'', \frac{iq'}{\beta_n}(\mathcal{H}_1 x_{n,1})\rangle_{L_2} \\
&\quad + \operatorname{Re}\langle -M_2^*(\mathcal{H}_1 x_{n,1})'', \frac{iq'}{\beta_n}(\mathcal{H}_2 x_{n,2})\rangle_{L_2} + o(1) \\
&= -2\operatorname{Re}\langle M_2(\mathcal{H}_2 x_{n,2})'(\zeta), \frac{iq'}{\beta_n}(\mathcal{H}_1 x_{n,1})'\rangle_{L_2} \\
&\quad + \left[\operatorname{Re}\langle M_2(\mathcal{H}_2 x_{n,2})'(\zeta), \frac{iq'(\zeta)}{\beta_n}\mathcal{H}_1 x_{n,1}(\zeta)\rangle_{\mathbb{F}}\right]_0^1 \\
&\quad + \left[\operatorname{Re}\langle M_2(\mathcal{H}_2 x_{n,2})(\zeta), \frac{iq'(\zeta)}{\beta_n}(\mathcal{H}_1 x_{n,1})'(\zeta)\rangle_{\mathbb{F}}\right]_0^1 + o(1)
\end{aligned} \tag{4.27}$$

To eliminate the integral terms with M_2 in equations (4.25) and (4.27) we subtract two times equation (4.25) from equation (4.27) and obtain

$$\begin{aligned}
&\operatorname{Re}\langle x_n, 2q'\mathcal{H} - q\mathcal{H}'x_n\rangle_{L_2} \\
&= \langle x_n, (q'\mathcal{H} - q\mathcal{H}')x_n\rangle_{L_2} + \operatorname{Re}\langle x_n, q'\mathcal{H}x_n\rangle_{L_2} \\
&= -2\operatorname{Re}\left[\langle M_2(\mathcal{H}_2 x_{n,2})'(\zeta), \frac{iq(\zeta)}{\beta_n}(\mathcal{H}_1 x_{n,1})'(\zeta)\rangle_{\mathbb{F}}\right]_0^1 \\
&\quad + \left[\langle x_n(\zeta), q\mathcal{H}(\zeta)x_n(\zeta)\rangle_{\mathbb{F}}\right]_0^1 \\
&\quad + \left[\operatorname{Re}\langle M_2(\mathcal{H}_2 x_{n,2})'(\zeta), \frac{iq'(\zeta)}{\beta_n}\mathcal{H}_1 x_{n,1}(\zeta)\rangle_{\mathbb{F}}\right]_0^1 \\
&\quad + \left[\operatorname{Re}\langle M_2(\mathcal{H}_2 x_{n,2})(\zeta), \frac{iq'(\zeta)}{\beta_n}(\mathcal{H}_1 x_{n,1})'(\zeta)\rangle_{\mathbb{F}}\right]_0^1 + o(1)
\end{aligned} \tag{4.28}$$

We are now in the position to determine some functions R such that the pair $(\mathfrak{A}, R)$ has property AIEP. For this recall that $\frac{\mathcal{H}x_n}{\beta_n} \xrightarrow{n\to+\infty} 0$ in $C^1([0,1];\mathbb{F}^d)$, so that using the Cauchy-Schwarz inequality we may estimate one factor in each boundary component by (the square of) a term which thanks to the factor $\frac{1}{\beta_n}$ convergences to zero and another (square of a term) which (without a factor $\frac{1}{\beta_n}$) converges to zero, whenever $Rx_n \xrightarrow{n\to\infty} 0$. On the other hand the matrix-valued function $2q'\mathcal{H} - q\mathcal{H}' \in L_\infty(0,1;\mathbb{F}^{d\times d})$ should be a coercive operator on X, so that we can conclude that $x_n \xrightarrow{n\to\infty} 0$ in X if the right-hand side converges to zero. Note that in general it is not possible to choose $q \in C^\infty([0,1];\mathbb{R})$ such that $2q'\mathcal{H} - q\mathcal{H}'$ is coercive, i.e. $2q'\mathcal{H} - q\mathcal{H}' \geq \varepsilon I$ for a.e. $\zeta \in (0,1)$ for some $\varepsilon > 0$, and both $q(0) = 0$ and $q(1) = 0$ at the same time. (Note, however, that if $\mathcal{H} \in W_\infty([0,1]; \mathbb{F}^{d\times d})$ has a special form this is possible, indeed, and then the conditions on R for $(\mathfrak{A}, R)$ having property AIEP are less restrictive.) Here we use Lemma 4.3.3 and choose $q \in C^\infty([0,1]; \mathbb{R})$ such that $q(1) = 0$ and then conclude that R has property AIEP if it contains each of the following terms.

1. $\Pi(\mathcal{H}_1 x_1)'(0) + (I - \Pi) M_2 (\mathcal{H}_2 x_2)'(0)$
2. $(\mathcal{H}x)(0)$
3. $\Pi'(\mathcal{H}_1 x_1)(1) + (I - \Pi') M_2 (\mathcal{H}_2 x_2)'(1)$
4. $\Pi''(\mathcal{H}_1 x_1)'(1) + (I - \Pi'') M_2 (\mathcal{H}_2 x_2)(1)$.

We therefore conclude that the assertions of the theorem hold. □

Remark 4.3.20. *In Proposition 4.3.19 the following results hold if we assume we demand particular conditions on $\mathcal{H}$.*

1. *If there is $q \in C^\infty([0,1]; \mathbb{R})$ with $q(0) = q(1) = 0$ and such that $2q'\mathcal{H} - q\mathcal{H}'$ is a coercive multiplication operator on X, then for the following choices of R the pair $(\mathfrak{A}, R)$ has property AIEP.*

$$Rx = \begin{pmatrix} \Pi(\mathcal{H}_1 x_1)'(0) + (I - \Pi) M_2 (\mathcal{H}_2 x_2)'(0) \\ \Pi'(\mathcal{H}_1 x_1)'(0) + (I - \Pi') M_2 (\mathcal{H}_2 x_2)(0) \end{pmatrix} \in \mathbb{F}^d$$

where Π and $\Pi' \in \mathbb{F}^{d/2 \times d/2}$ are orthogonal projections.

2. *If there is $q \in C^\infty([0,1]; \mathbb{R})$ with $q'(0) = q(1) = 0$ and such that $2q'\mathcal{H} - q\mathcal{H}'$ (in particular $q\mathcal{H}'(\zeta) \geq \varepsilon I$ for some $\varepsilon > 0$ and a.e. $\zeta \in (0,1)$) is a coercive multiplication operator on X, then for the following choices of R the pair $(\mathfrak{A}, R)$ has property AIEP.*

$$Rx = \begin{pmatrix} (\mathcal{H}x)(0) \\ \Pi(\mathcal{H}_1 x_1)'(0) + (I - \Pi) M_2 (\mathcal{H}_2 x_2)'(0) \end{pmatrix} \in \mathbb{F}^{3d/2}$$

where $\Pi \in \mathbb{F}^{d/2 \times d/2}$ is an orthogonal projection.

We also apply the Lyapunov method proof technique for exponential stability to the Euler-Bernoulli beam model and restrict ourselves in the following exposition to the case of a single Euler-Bernoulli beam equation, i.e. we consider the port-Hamiltonian system of the special form

$$\mathfrak{A} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \frac{\partial^2}{\partial \zeta^2} \begin{bmatrix} \mathcal{H}_1 & 0 \\ 0 & \mathcal{H}_2 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \tag{4.29}$$

where $\mathcal{H}_1, \mathcal{H}_2$ are two positive Lipschitz continuous functions. We then define on the energy state space $X = (L_2(0,1; \mathbb{F}^2); \langle \cdot, \cdot \rangle_{\mathcal{H}})$ the quadratic functional

$$q(x) = \operatorname{Re} \langle x_1, \eta \int_0^{\cdot} x_2 d\xi \rangle_{L_2}, \quad x \in X \tag{4.30}$$

where $\eta \in C^\infty([0,1]; \mathbb{R})$ is a suitable smooth function which we chose at a later point.

Remark 4.3.21. *If q is linear this is exactly the choice of q in the article [Ch+87], where actually a chain of Euler-Bernoulli beams had been considered.*

We then have for every solution $x \in W^1_\infty(\mathbb{R}_+; X) \cap L_\infty(\mathbb{R}_+; D(\mathfrak{A}))$ of $\dot{x} = \mathfrak{A}x$ that

$$
\begin{aligned}
&\frac{d}{dt} q(x) \\
&= \operatorname{Re} \langle x_{1,t}, \eta \int_0^\cdot x_2 d\xi \rangle_{L_2} + \operatorname{Re} \langle x_1, \eta \int_0^\cdot x_{2,t} d\xi \rangle_{L_2} \\
&= \operatorname{Re} \langle (\mathcal{H}_2 x_2)'', \eta \int_0^\cdot x_2 d\xi \rangle_{L_2} - \operatorname{Re} \langle \eta x_1, \int_0^\cdot (\mathcal{H}_1 x_1)'' d\xi \rangle_{L_2} \\
&= - \operatorname{Re} \langle (\mathcal{H}_2 x_2)', \eta' \int_0^\cdot x_2 d\xi \rangle_{L_2} - \operatorname{Re} \langle (\mathcal{H}_2 x_2)', \eta x_2 \rangle_{L_2} \\
&\quad + \operatorname{Re} \langle (\mathcal{H}_2 x_2)'(1), \eta(1) \int_0^1 x_2(\zeta) d\zeta \rangle_{\mathbb{F}} \\
&\quad - \operatorname{Re} \langle \eta x_1, (\mathcal{H}_1 x_1)' \rangle_{L_2} + \operatorname{Re} \langle \eta x_1, (\mathcal{H}_1 x_1)'(0) \rangle_{L_2} \\
&= \operatorname{Re} \langle \mathcal{H}_2 x_2, \eta'' \int_0^\cdot x_2 d\xi \rangle_{L_2} + \operatorname{Re} \langle \mathcal{H}_2 x_2, \eta' x_2 \rangle_{L_2} \\
&\quad - \operatorname{Re} \langle (\mathcal{H}_2 x_2)(1), \eta'(1) \int_0^1 x_2(\zeta) d\zeta \rangle_{\mathbb{F}} + \frac{1}{2} \langle \mathcal{H}_2 x_2, (\eta \mathcal{H}_2^{-1})' \mathcal{H}_2 x_2 \rangle_{L_2} \\
&\quad - \frac{1}{2} \left[\operatorname{Re} \langle (\mathcal{H}_2 x_2)(\zeta), \eta(\zeta) x_2(\zeta) \rangle_{\mathbb{F}} \right]_0^1 + \operatorname{Re} \langle (\mathcal{H}_2 x_2)'(1), \eta(1) \int_0^1 x_2(\zeta) d\zeta \rangle_{\mathbb{F}} \\
&\quad + \frac{1}{2} \langle (\eta \mathcal{H}_1^{-1})' \mathcal{H}_1 x_1, \mathcal{H}_1 x_1 \rangle_{L_2} - \frac{1}{2} \left[\operatorname{Re} \langle \eta(\zeta) x_1(\zeta), (\mathcal{H}_1 x_1)(\zeta) \rangle_{\mathbb{F}} \right]_0^1 \\
&\quad + \operatorname{Re} \langle \eta x_1, (\mathcal{H}_1 x_1)'(0) \rangle_{L_2} \\
&= -\frac{1}{2} \langle \int_0^\cdot x_2 d\xi, (\eta'' \mathcal{H}_2)' \int_0^\cdot x_2 d\xi \rangle_{L_2} + \frac{1}{2} \langle \int_0^1 x_2(\zeta) d\zeta, (\eta'' \mathcal{H}_2)(1) \int_0^1 x_2(\zeta) d\zeta \rangle_{\mathbb{F}} \\
&\quad + \operatorname{Re} \langle \mathcal{H}_2 x_2, \eta' x_2 \rangle_{L_2} - \operatorname{Re} \langle (\mathcal{H}_2 x_2)(1), \eta'(1) \int_0^1 x_2(\zeta) d\zeta \rangle_{\mathbb{F}} \\
&\quad + \frac{1}{2} \langle \mathcal{H}_2 x_2, (\eta \mathcal{H}_2^{-1})' \mathcal{H}_2 x_2 \rangle_{L_2} - \frac{1}{2} \left[\operatorname{Re} \langle (\mathcal{H}_2 x_2)(\zeta), \eta(\zeta) x_2(\zeta) \rangle_{L_2} \right]_0^1 \\
&\quad + \operatorname{Re} \langle (\mathcal{H}_2 x_2)'(1), \eta(1) \int_0^1 x_2(\zeta) d\zeta \rangle_{\mathbb{F}} \\
&\quad + \frac{1}{2} \langle (\eta \mathcal{H}_1^{-1})' \mathcal{H}_1 x_1, \mathcal{H}_1 x_1 \rangle_{L_2} - \frac{1}{2} \left[\operatorname{Re} \langle \eta(\zeta) x_1(\zeta), (\mathcal{H}_1 x_1)(\zeta) \rangle_{\mathbb{F}} \right]_0^1 \\
&\quad + \operatorname{Re} \langle \eta x_1, (\mathcal{H}_1 x_1)'(0) \rangle_{L_2}. \\
&= \frac{1}{2} \langle (\eta \mathcal{H}_1^{-1})' \mathcal{H}_1 x_1, \mathcal{H}_1 x_1 \rangle_{L_2} + \frac{1}{2} \langle ((\eta \mathcal{H}_2^{-1})' + 2\eta \mathcal{H}_2^{-1}) \mathcal{H}_2 x_2, \mathcal{H}_2 x_2 \rangle_{L_2} \\
&\quad - \frac{1}{2} \langle \int_0^\cdot x_2(\xi) d\xi, (\eta'' \mathcal{H}_2)' \int_0^\cdot x_2(\xi) d\xi \rangle_{L_2} \\
&\quad + \frac{1}{2} \langle \int_0^1 x_2(\zeta) d\zeta, (\eta'' \mathcal{H}_2)(1) \int_0^1 x_2(\zeta) d\zeta \rangle_{\mathbb{F}^{d/2}} \\
&\quad + \eta(1) \left[\langle (\mathcal{H}_2 x_2)'(1), \int_0^1 x_2(\zeta) d\zeta \rangle_{\mathbb{F}^{d/2}} - \frac{1}{2} \langle x(1), (\mathcal{H} x)(1) \rangle_{\mathbb{F}^d} \right]
\end{aligned}
$$

$$+\frac{1}{2}\langle\int_0^1 x_2 d\zeta, (\eta''\mathcal{H}_2)(1)\int_0^1 x_2 d\zeta\rangle_{\mathbb{F}^{d/2}} - \eta'(1)\operatorname{Re}\langle(\mathcal{H}_2 x_2)(1), \int_0^1 x_2 d\zeta\rangle_{\mathbb{F}^{d/2}}$$
$$-\eta(0)\frac{1}{2}\langle x(0), (\mathcal{H}x)(0)\rangle_{\mathbb{F}^d} + \operatorname{Re}\langle \eta x_1, (\mathcal{H}_1 x_1)'(0)\rangle_{L_2}$$

Now let us for a moment discuss the case that $\mathcal{H}_1$ and $\mathcal{H}_2$ are constant along the line $[0,1]$. Then the problem of finding q such that $\frac{d}{dt}q(x) + \|x(t)\|^2$ is bounded by appropriate boundary terms reduces to finding $\eta \in C^\infty([0,1];\mathbb{R})$ such that $\eta' < 0$ on $[0,1]$ and by choice of η and the boundary terms we should control the following terms.

1. $-\left[\langle(\mathcal{H}_2 x_2)(\zeta), \eta(\zeta)x_2(\zeta)\rangle_{\mathbb{F}}\right]_0^1$
2. $+\left[\langle(\mathcal{H}_1 x_1)(\zeta), \eta(\zeta)x_1(\zeta)\rangle_{\mathbb{F}}\right]_0^1$
3. $-\operatorname{Re}\langle(\mathcal{H}_2 x_2)(1), \eta'(1)\int_0^1 x_2(\zeta)d\zeta\rangle_{\mathbb{F}}$
4. $\operatorname{Re}\langle(\mathcal{H}_2 x_2)'(1), \eta(1)\int_0^1 x_2(\zeta)d\zeta\rangle_{\mathbb{F}}$
5. $\operatorname{Re}\langle \eta x_1, (\mathcal{H}_1 x_1)'(0)\rangle_{L_2}$

Since the choice $\eta = 0$ is not admissible, from the last term we see that for this approach to work, we necessarily need dissipation in the term $(\mathcal{H}_1 x_1)'(0)$, i.e. we can only hope for an estimate of the form

$$\frac{d}{dt}q(x) + \|x(t)\|^2 \le c\,|(\mathcal{H}_1 x_1)'(t,0)|^2, \quad \text{a.e. } t \ge 0.$$

Moreover, as we demanded that $\eta' < 0$, in general also the third term cannot be handled by demanding $\eta'(1) = 0$, so that only a estimate of the form

$$\frac{d}{dt}q(x) + \|x(t)\|^2 \le c\left(|(\mathcal{H}_1 x_1)'(t,0)|^2 + |(\mathcal{H}_2 x_2)(t,1)|^2\right), \quad \text{a.e. } t \ge 0$$

might possibly hold. We then may choose $\eta(\zeta) = 1 - \zeta$, so that all terms including η'' or η''' are zero. These considerations lead to the following conclusion.

Lemma 4.3.22. *Let $\mathfrak{A}$ be a port-Hamiltonian operator of order $N = 2$ with $P_2 = \left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right], P_0 = P_1 = 0$ and assume that $\mathcal{H} = \operatorname{diag}(\mathcal{H}_1, \mathcal{H}_2)$ is constant. Then there is $q(x) = \operatorname{Re}\langle x_1, \eta\int_0^\cdot x_2 d\xi\rangle_{L_2}$ such that for every solution $x \in W^1_\infty(\mathbb{R}_+; X) \cap L_\infty(\mathbb{R}_+; D(\mathfrak{A}))$ the estimate*

$$\frac{d}{dt}q(x) + \|x(t)\|^2 \le c\left(|(\mathcal{H}x)(t,0)|^2 + |(\mathcal{H}_1 x_1)'(t,0)|^2 + |(\mathcal{H}_2 x_2)(t,1)|^2\right)$$

holds for a.e. $t \ge 0$, where $c > 0$ is independent of x.

We investigate this example a little bit further for the case that $\mathcal{H}$ is not constant. In that case we cannot necessarily choose $\eta \in C^\infty([0,1];\mathbb{R})$ to be linear and therefore have to to handle the term

$$\operatorname{Re}\langle \mathcal{H}_2 x_2, \eta''\int_0^\cdot x_2 d\xi\rangle_{L_2} \le \left\||\eta'|^{1/2} x_2\right\|_{\mathcal{H}_2}\left\|\frac{\eta''}{\sqrt{|\eta'|}}\int_0^\cdot x_2 d\xi\right\|_{\mathcal{H}_2}$$
$$\le \left\||\eta'|^{1/2} x_2\right\|_{\mathcal{H}_2}^2\left\|\frac{\eta''}{\sqrt{|\eta'|}}\left(\int_0^\cdot \frac{\mathcal{H}_2^{-1}(\xi)}{|\eta'\xi|}\right)^{1/2} d\xi\right\|_{\mathcal{H}_2}$$

Here for the choice

$$\eta(\zeta) = e^{-\alpha\zeta} - e^{-\alpha}, \quad \zeta \in [0,1]$$

where $\alpha > 0$ is a fixed constant and under the assumption that

$$m_2 I \le \mathcal{H}_2(\zeta) \le M_2, \quad \text{a.e. } \zeta \in (0,1)$$

we find that

$$\begin{aligned}
\left\| \frac{\eta''}{\sqrt{|\eta'|}} \left(\int_0^{\cdot} \frac{\mathcal{H}_2^{-1}(\xi)}{|\eta'(\xi)|} d\xi \right)^{1/2} d\xi \right\|_{\mathcal{H}_2}^2 &\le \frac{M_2}{m_2} \int_0^1 \alpha^3 e^{-\alpha\zeta} \int_0^{\zeta} \alpha^{-1} e^{\alpha\xi} d\xi d\zeta \\
&= \frac{\alpha M_2}{m_2} \int_0^1 e^{-\alpha\zeta}(e^{\alpha\zeta} - 1) d\zeta \\
&= \frac{M_2}{m_2} \left(\alpha + 1 - e^{-\alpha} \right)
\end{aligned}$$

and then

$$\begin{aligned}
\frac{d}{dt} q(x) \le &\left[1 - \frac{M_2}{m_2} \left(\alpha + 1 - e^{-\alpha} \right) \right] \langle \eta' x_2, x_2 \rangle_{\mathcal{H}_2} + \frac{1}{2} \langle \mathcal{H}_1 x_1, [(\eta \mathcal{H}_1^{-1})' + \varepsilon](\mathcal{H}_1 x_1) \rangle_{L_2} \\
&+ c_{\varepsilon,\alpha} \left(|(\mathcal{H}x)(t,0)|^2 + |(\mathcal{H}_1 x_1)'(0)|^2 + |(\mathcal{H}_2 x_2)'(t,1)|^2 \right)
\end{aligned}$$

for every $\varepsilon > 0$ and a constant $c_{\varepsilon,\alpha} > 0$ which depends on ε and α, but not on x. Moreover, we have that if we have

$$\eta(\mathcal{H}^{-1})'(\zeta) + \eta' \mathcal{H}^{-1}(\zeta) = (e^{-\alpha\zeta} - e^{-\alpha})(\mathcal{H}^{-1})'(\zeta) - \alpha e^{-\alpha\zeta} \mathcal{H}^{-1}(\zeta) \le -\varepsilon$$

for some $\varepsilon > 0$ and a.e. $\zeta \in (0,1)$ and at the same time $\frac{M_2}{m_2}\left(\alpha + 1 - e^{-\alpha}\right) \le 1$, then

$$\frac{d}{dt} q(x) \le c_{\varepsilon,\alpha} \left(|(\mathcal{H}x)(t,0)|^2 + |(\mathcal{H}_1 x_1)'(0)|^2 + |(\mathcal{H}_2 x_2)'(t,1)|^2 \right). \tag{4.31}$$

We therefore conclude

Proposition 4.3.23. *Let $\mathfrak{A}$ be a port-Hamiltonian operator of order $N = 2$ with $P_2 = \left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right]$, $P_0 = P_1 = 0$ and assume that $\mathcal{H} = \operatorname{diag}(\mathcal{H}_1, \mathcal{H}_2) \in W_\infty^1(0,1;\mathbb{F}^{2\times 2})$ is Lipschitz continuous. Assume that there are $\varepsilon, \alpha > 0$ such that*

$$\frac{\max_{\zeta\in[0,1]} \mathcal{H}_2(\zeta)}{\min_{\zeta\in[0,1]} \mathcal{H}_2(\zeta)} \left(\alpha + 1 - e^{-\alpha} \right) \le 1 \tag{4.32}$$

and at the same time

$$\eta(\mathcal{H}^{-1})'(\zeta) + \eta' \mathcal{H}^{-1}(\zeta) = (e^{-\alpha\zeta} - e^{-\alpha})(\mathcal{H}^{-1})'(\zeta) - \alpha e^{-\alpha\zeta} \mathcal{H}^{-1}(\zeta) \le -\varepsilon$$

for a.e. $\zeta \in (0,1)$. Then there is $q(x) = \operatorname{Re} \langle x_1, \eta \int_0^{\cdot} x_2 d\xi \rangle_{L_2}$ (for $\eta(\zeta) = e^{-\alpha\zeta} - e^{-\alpha}$) such that for every solution $x \in W_\infty^1(\mathbb{R}_+; X) \cap L_\infty(\mathbb{R}_+; D(\mathfrak{A}))$ of $\dot{x} = \mathfrak{A}x$ the estimate

$$\frac{d}{dt} q(x) + \|x(t)\|^2 \le c \left(|(\mathcal{H}x)(t,0)|^2 + |(\mathcal{H}_1 x_1)'(t,0)|^2 + |(\mathcal{H}_2 x_2)(t,1)|^2 \right)$$

holds for a.e. $t \ge 0$, where $c > 0$ is independent of x.

4.3.4 Arbitrary $N \in \mathbb{N}$ – The Full Dissipative Case

In the subsections before we saw how two alternative methods may be used to prove exponential stability in Theorem 4.1.5 and to which extend they generalise to the case $N = 2$. Now we continue with these two methods and drop the restriction that $N = 1$ or $N = 2$. On the other hand we try to keep the computations as simple as possible and therefore start with the full dissipative case, i.e. strict dissipation at both ends of the line $(0, 1)$. Clearly this is the most restrictive dissipation assumption we may think of, however, we therefore establish the quite intuitive result that with strict dissipation at both ends we can always ensure uniform exponential stability of the C_0-semigroup $(T(t))_{t\geq 0}$ and luckily so since if this were not true there were even less possible boundary conditions leading to uniform exponential stabilisation. Note that before we had already seen that in general it is enough to have strictly dissipative boundary conditions at one end and conservative boundary conditions at the other end for asymptotic stability, so we only have to ensure boundedness of the resolvents on the imaginary axis which by Gearhart's Theorem implies exponential stability of the semigroup $(T(t))_{t\geq 0}$.

Theorem 4.3.24. *Let $Ax = \sum_{k=0}^{N} P_k(\mathcal{H}x)^{(k)}$ be a port-Hamiltonian operator of arbitrary order $N \in \mathbb{N}$ with suitable boundary conditions such that*

$$\operatorname{Re} \langle Ax, x\rangle_X \leq -\kappa \sum_{k=0}^{N-1} \sum_{\zeta=0,1} \left|(\mathcal{H}x)^{(k)}(\zeta)\right|^2, \quad x \in D(A)$$

for some $\kappa > 0$. Further assume that $\mathcal{H} \in W^1_\infty(0, 1; \mathbb{F}^{d\times d}) \cap W^K_\infty(0, 1; \mathbb{F}^{d\times d})$ where $N = 2K + 1$ or $N = 2K$. Then A generates a uniformly exponentially stable contraction C_0-semigroup $(T(t))_{t\geq 0}$ on X.

In proof the statement is again established via the Gearhart-Greiner-Prüss Theorem 2.2.17 where we show the following two statements based on the notion of the property ESP for a pair (B, R).

Lemma 4.3.25. *Let $\mathfrak{A}$ be a maximal port-Hamiltonian operator of order $N \in \mathbb{N}$ with $\mathcal{H} \in W^1_\infty(0, 1; \mathbb{F}^{d\times d}) \cap W^K_\infty(0, 1; \mathbb{F}^{d\times d})$ where $N = 2K + 1$ or $N = 2K$. Further let $H = \mathbb{F}^{2Nd}$ and $R \in \mathcal{B}(D(\mathfrak{A}); H)$ be given by the trace map $Rx = \tau(\mathcal{H}x)$. Then the pair $(\mathfrak{A}, R)$ has property ESP.*

Clearly the combination of Proposition 4.3.6 and Lemma 4.3.25 implies Theorem 4.3.24.

Proof. We already know that the pair $(\mathfrak{A}, R)$ has property ASP thanks to Theorem 4.2.2 and Remark 4.2.6. We therefore proceed by checking that also property AIEP is satisfied. Let $(x_n, \beta_n)_{n\geq 1} \subseteq D(\mathfrak{A}) \times \mathbb{R}$ be any sequence with

$$\sup_{n\in\mathbb{N}} \|x_n\| < +\infty, \ |\beta_n| \xrightarrow{n\to\infty} +\infty \text{ and } \mathfrak{A}x_n - i\beta_n x_n \xrightarrow{n\to\infty} 0 \tag{4.33}$$

and assume that

$$Rx_n = \tau(\mathcal{H}x_n) \xrightarrow{n\to+\infty} 0$$

so that all the terms $(\mathcal{H}x_n)^{(k)}(\zeta)$ converge to zero as $n \to \infty$ ($k = 0, 1, \ldots, N - 1$ and $\zeta = 0, 1$). From there it follows that whenever we integrate by parts these terms vanish as $n \to \infty$. Moreover, equation (4.33) gives that

$$\frac{\mathfrak{A}x_n}{\beta_n} - ix_n \xrightarrow{n\to\infty} 0 \tag{4.34}$$

and since $\|\cdot\|_{\mathfrak{A}}$ and $\|\mathcal{H}\cdot\|_{H^N}$ are equivalent we conclude from (4.33) and (4.34) that

$$\sup_{n\in\mathbb{N}} \left\| \frac{(\mathcal{H}x_n)^{(N)}}{\beta_n} \right\|_{L_2} < +\infty.$$

From there it follows with equation (4.33), Lemma 2.1.10 and Lemma 2.1.19 that

$$\frac{1}{\beta_n} \left\|(\mathcal{H}x_n)^{(k)}_{L_2}\right\| \left\|(\mathcal{H}x_n)^{(m)}\right\|_{L_2} \xrightarrow{n\to\infty} 0 \quad (k,m \geq 0 \text{ s.t. } k+m < N).$$

We distinguish the two cases N even and N odd, starting with the odd case $N = 2K+1$. Then for every $q \in C^\infty([0,1];\mathbb{R})$ we deduce that

$$\begin{aligned}
0 &= \frac{1}{\beta_n} \operatorname{Re} \langle \mathfrak{A}x_n - i\beta_n x_n, iq'(\mathcal{H}x_n)\rangle_{L_2} + o(1) \\
&= -\operatorname{Re}\langle x_n, q'(\mathcal{H}x_n)\rangle_{L_2} + \frac{1}{\beta_n}\sum_{k=0}^{2K+1} \operatorname{Re}\langle P_k(\mathcal{H}x)^{(k)}, iq'(\mathcal{H}x_n)\rangle_{L_2} + o(1) \\
&= -\operatorname{Re}\langle x_n, q'(\mathcal{H}x_n)\rangle_{L_2} + \frac{1}{\beta_n} \operatorname{Re}\langle P_{2K+1}(\mathcal{H}x)^{(2K+1)}, iq'(\mathcal{H}x_n)\rangle_{L_2} + o(1) \\
&= -\operatorname{Re}\langle x_n, q'(\mathcal{H}x_n)\rangle_{L_2} + \frac{(-1)^K}{\beta_n} \operatorname{Re}\langle (\mathcal{H}x_n)^{(K+1)}, iP_{2K+1}(q'(\mathcal{H}x_n))^{(K)}\rangle_{L_2} \\
&\quad + \sum_{k=0}^{K-1} \frac{(-1)^k}{\beta_n} \operatorname{Re}\left[\langle (\mathcal{H}x_n)^{(2K-k)}(\zeta), iP_{2K+1}(q'(\mathcal{H}x_n))^{(k)}(\zeta)\rangle_{\mathbb{F}^d}\right]_0^1 + o(1) \\
&= -\operatorname{Re}\langle x_n, q'(\mathcal{H}x_n)\rangle_{L_2} + \frac{(-1)^K}{\beta_n} \operatorname{Re}\langle (\mathcal{H}x_n)^{(K+1)}, iq' P_{2K+1}(\mathcal{H}x_n)^{(K)}\rangle_{L_2} + o(1)
\end{aligned}$$

and also

$$\begin{aligned}
0 &= \frac{1}{\beta_n} \operatorname{Re} \langle \mathfrak{A}x_n - i\beta_n x_n, iq(\mathcal{H}x_n)'\rangle_{L_2} + o(1) \\
&= -\operatorname{Re}\langle x_n, q(\mathcal{H}x_n)'\rangle_{L_2} + \frac{1}{\beta_n}\sum_{k=0}^{2K+1} \operatorname{Re}\langle P_k(\mathcal{H}x_n)^{(k)}, iq(\mathcal{H}x_n)'\rangle_{L_2} + o(1) \\
&= -\operatorname{Re}\langle x_n, q(\mathcal{H}x_n)'\rangle_{L_2} \\
&\quad + \frac{1}{\beta_n} \operatorname{Re}\langle P_{2K+1}(\mathcal{H}x_n)^{(2K+1)} + P_{2K}(\mathcal{H}x_n)^{(2K)}, iq(\mathcal{H}x_n)'\rangle_{L_2} + o(1) \\
&= -\operatorname{Re}\langle x_n, q(\mathcal{H}x_n)'\rangle_{L_2} + \frac{(-1)^K}{\beta_n} \operatorname{Re}\langle (\mathcal{H}x_n)^{(K+1)}, P_{2K+1}i(q(\mathcal{H}x_n)')^{(K)}\rangle_{L_2} \\
&\quad + \sum_{k=0}^{K-1} \frac{(-1)^k}{\beta_n} \operatorname{Re}\left[\langle (\mathcal{H}x_n)^{(2K-l)}(\zeta), P_{2K+1}i(q(\mathcal{H}x_n)')^{(l)}(\zeta)\rangle_{\mathbb{F}^d}\right]_0^1 \\
&\quad + \frac{(-1)^K}{\beta_n} \operatorname{Re}\langle (\mathcal{H}x_n)^{(K+1)}, P_{2K}i(q(\mathcal{H}x_n)')^{(K-1)}\rangle_{L_2}
\end{aligned}$$

$$
\begin{aligned}
&\quad + \sum_{k=0}^{K-2} \frac{(-1)^l}{\beta_n} \operatorname{Re} \Big[\langle (\mathcal{H}x_n)^{(2K-l)}(\zeta), P_{2K} i (q(\mathcal{H}x_n)')^{(l)}(\zeta)\rangle_{\mathbb{F}^d}\Big]_0^1 + o(1) \\
&= -\operatorname{Re}\langle x_n, q(\mathcal{H}x_n)'\rangle_{L_2} \\
&\quad + \frac{(-1)^K}{\beta_n} \operatorname{Re}\langle (\mathcal{H}x_n)^{(K+1)}, i P_{2K+1}(q(\mathcal{H}x_n)^{(K+1)} + Kq'(\mathcal{H}x_n)^{(K)})\rangle_{L_2} \\
&\quad + \frac{(-1)^K}{\beta_n} \operatorname{Re}\langle (\mathcal{H}x_n)^{(K+1)}, iqP_{2K}(\mathcal{H}x_n)^{(K)}\rangle_{L_2} + o(1) \\
&= -\operatorname{Re}\langle x_n, q(\mathcal{H}x_n)'\rangle_{L_2} + \frac{K(-1)^K}{\beta_n} \operatorname{Re}\langle (\mathcal{H}x_n)^{(K+1)}, iq' P_{2K+1}(\mathcal{H}x_n)^{(K)}\rangle_{L_2} \\
&\quad + \frac{(-1)^{K-1}}{2\beta_n} \langle (\mathcal{H}x_n)^{(K)}, iq' P_{2K}(\mathcal{H}x_n)^{(K)}\rangle_{L_2} \\
&\quad + \frac{(-1)^K}{2\beta_n} \Big[\langle (\mathcal{H}x_n)^{(K)}(\zeta), iqP_{2K}(\mathcal{H}x_n)^{(K)}(\zeta)\rangle_{\mathbb{F}^d}\Big]_0^1 + o(1) \\
&= \langle x_n, \frac{q'\mathcal{H} - q\mathcal{H}'}{2} x_n\rangle_{L_2} + K\frac{(-1)^K}{\beta_n} \operatorname{Re}\langle (\mathcal{H}x_n)^{(K+1)}, iq' P_{2K+1}(\mathcal{H}x_n)^{(K)}\rangle_{L_2} + o(1)
\end{aligned}
$$

Subtracting $2K$ times the first (asymptotic) equality from two times the second, we then find

$$\langle x_n, (Nq'\mathcal{H} - q\mathcal{H}')x_n\rangle_{L_2} \xrightarrow{n\to\infty} 0.$$

For suitable $q \in C^\infty([0,1];\mathbb{R})$ from Lemma 4.3.3 the expression $Nq'\mathcal{H} - q\mathcal{H}'$ defines a coercive multiplication operator on $L_2(0,1;\mathbb{F}^d)$ and thus $x_n \xrightarrow{n\to\infty} 0$ in X. The pair $(\mathfrak{A}, R)$ therefore has property ESP.

Secondly, we consider the case that $N = 2K$ is even. In fact, the reasoning is very similar to the one for the odd case, apart from that the matrix $P_N = P_{2K}$ for the fundamental part is skew-Hermitian instead of Hermitian, so the reasoning needs to be adapted to this new situation. Let again $q \in C^\infty([0,1];\mathbb{R})$, then

$$
\begin{aligned}
0 &= \frac{1}{\beta_n} \operatorname{Re}\langle \mathfrak{A}x_n - i\beta_n x_n, iq'(\mathcal{H}x_n)\rangle_{L_2} + o(1) \\
&= -\langle x_n, q'(\mathcal{H}x_n)\rangle_{L_2} + \frac{1}{\beta_n} \sum_{k=0}^{2K} \operatorname{Re}\langle P_k(\mathcal{H}x_n)^{(k)}, iq'(\mathcal{H}x_n)\rangle_{L_2} + o(1) \\
&= -\langle x_n, q'(\mathcal{H}x_n)\rangle_{L_2} + \frac{1}{\beta_n} \operatorname{Re}\langle P_{2K}(\mathcal{H}x_n)^{(2K)}, iq'(\mathcal{H}x_n)\rangle_{L_2} + o(1) \\
&= -\langle x_n, q'(\mathcal{H}x_n)\rangle_{L_2} + \frac{(-1)^K}{\beta_n} \operatorname{Re}\langle P_{2K}(\mathcal{H}x_n)^{(K)}, i(q'(\mathcal{H}x_n))^{(K)}\rangle_{L_2} \\
&\quad + \sum_{k=0}^{K-1} \frac{(-1)^k}{\beta_n} \operatorname{Re} \Big[\langle P_{2K}(\mathcal{H}x)^{(2K-1-k)}(\zeta), i(q'(\mathcal{H}x_n))^{(k)}(\zeta)\rangle_{\mathbb{F}^d}\Big]_0^1 + o(1) \\
&= -\langle x_n, q'(\mathcal{H}x_n)\rangle_{L_2} + \frac{(-1)^{K+1}}{\beta_n} \operatorname{Re}\langle (\mathcal{H}x)^{(K)}, iq' P_{2K}(\mathcal{H}x_n)^{(K)}\rangle_{L_2} + o(1)
\end{aligned}
$$

and secondly

$$0 = \frac{1}{\beta_n} \operatorname{Re}\langle \mathfrak{A}x_n - i\beta_n x_n, iq(\mathcal{H}x_n)'\rangle_{L_2} + o(1)$$

$$
\begin{aligned}
&= -\operatorname{Re}\langle x_n, q(\mathcal{H}x_n)'\rangle_{L_2} + \frac{1}{\beta_n}\sum_{k=0}^{2K}\operatorname{Re}\langle P_k(\mathcal{H}x_n)^{(k)}, iq(\mathcal{H}x_n)'\rangle_{L_2} + o(1) \\
&= -\operatorname{Re}\langle x_n, q(\mathcal{H}x_n)'\rangle_{L_2} \\
&\quad + \frac{1}{\beta_n}\operatorname{Re}\langle P_{2K}(\mathcal{H}x_n)^{(2K)} + P_{2K-1}(\mathcal{H}x_n)^{(2K-1)}, iq(\mathcal{H}x_n)'\rangle_{L_2} + o(1) \\
&= -\operatorname{Re}\langle x_n, q(\mathcal{H}x_n)'\rangle_{L_2} + \frac{(-1)^{K+1}}{\beta_n}\operatorname{Re}\langle(\mathcal{H}x_n)^{(K)}, iP_{2K}(q(\mathcal{H}x_n)')^{(K)}\rangle_{L_2} \\
&\quad + \sum_{k=0}^{K-1}\frac{(-1)^k}{\beta_n}\operatorname{Re}\Big[\langle P_{2K}(\mathcal{H}x_n)^{(2K-1-k)}(\zeta), i(q(\mathcal{H}x_n)')^{(k)}(\zeta)\rangle_{\mathbb{F}^d}\Big]_0^1 \\
&\quad + \frac{(-1)^{(K-1)}}{\beta_n}\operatorname{Re}\langle(\mathcal{H}x_n)^{(K)}, iP_{2K-1}(q(\mathcal{H}x_n)')^{(K-1)}\rangle_{L_2} \\
&\quad + \sum_{k=0}^{K-1}\frac{(-1)^k}{\beta_n}\operatorname{Re}\Big[\langle P_{2K-1}(\mathcal{H}x_n)^{(2K-2-k)}(\zeta), i(q(\mathcal{H}x_n)')^{(k)}(\zeta)\rangle_{\mathbb{F}^d}\Big]_0^1 + o(1) \\
&= -\operatorname{Re}\langle x_n, q(\mathcal{H}x_n)'\rangle_{L_2} \\
&\quad + \frac{(-1)^K}{\beta_n}\operatorname{Re}\langle(\mathcal{H}x_n)^{(K)}, iP_{2K}(q(\mathcal{H}x_n)^{(K+1)} + Kq'(\mathcal{H}x_n)^{(K)})\rangle_{L_2} \\
&\quad + \frac{(-1)^{K-1}}{\beta_n}\operatorname{Re}\langle(\mathcal{H}x_n)^{(K)}, iqP_{2K-1}(\mathcal{H}x_n)^{(K)}\rangle_{L_2} + o(1) \\
&= -\operatorname{Re}\langle x_n, q(\mathcal{H}x_n)'\rangle_{L_2} + \frac{(-1)^{K+1}}{2\beta_n}\langle(\mathcal{H}x_n)^{(K)}, iq'P_{2K}(\mathcal{H}x_n)^{(K)}\rangle_{L_2} \\
&\quad + \frac{(-1)^K}{\beta_n}\Big[\langle(\mathcal{H}x_n)^{(K)}(\zeta), iqP_{2K}(\mathcal{H}x_n)^{(K)}(\zeta)\rangle_{\mathbb{F}^d}\Big]_0^1 \\
&\quad + \frac{K(-1)^K}{\beta_n}\langle(\mathcal{H}x_n)^{(K)}, iq'P_{2K}(\mathcal{H}x_n)^{(K)}\rangle_{L_2} + o(1) \\
&= -\operatorname{Re}\langle x_n, \frac{q'\mathcal{H} - q\mathcal{H}'}{2}x_n\rangle_{L_2} \\
&\quad + \frac{2K-1}{2}\frac{(-1)^K}{\beta_n}\langle(\mathcal{H}x_n)^{(K)}, iq'P_{2K}(\mathcal{H}x_n)^{(K)}\rangle_{L_2} + o(1)
\end{aligned}
$$

This time adding $(2K-1)$-times the first equality to two times the second we get

$$\langle x_n, (Nq'\mathcal{H} - q\mathcal{H}')x_n\rangle_{L_2} \xrightarrow{n\to\infty} 0$$

and for suitable choice of $q \in C^\infty([0,1];\mathbb{R})$ from Lemma 4.3.3 we obtain $x_n \xrightarrow{n\to\infty} 0$ and again the pair $(\mathfrak{A}, R)$ has property ESP. □

To summarise, for the full dissipation case with strict dissipation at both ends, uniform exponential stability can be proved via the Gearhart-Greiner-Prüss Theorem.

Remark 4.3.26. *It would be desirable also to get the same result via the Lyapunov technique of proof. For this one would need a function $q : X \to \mathbb{R}$ such that $|q(x)| \leq c\,\|x\|_X^2$ $(x \in X)$ and that for every mild solution $x \in W^1_\infty(\mathbb{R}_+; X) \cap L_\infty(\mathbb{R}_+; D(\mathfrak{A}))$ the estimate*

$$\frac{d}{dt}q(x(t)) \leq -\|x(t)\|_X^2 + c\,|\tau(\mathcal{H}x)|^2, \quad a.e.\ t \geq 0$$

holds true. A natural candidate for such a functional q would be

$$q(x) = \begin{cases} \langle I_{K-1}x, \eta P_N^{-1} I_K x\rangle_{L_2}, & \text{if } N = 2K \text{ even},\\ \langle I_{K-1}x, \eta P_N^{-1} I_{K-1} x\rangle_{L_2}, & \text{if } N = 2K+1 \text{ odd}, \end{cases} \qquad x \in X$$

for a suitable function $\eta \in C^\infty([0,1];\mathbb{R})$, where

$$(I_k f)(\zeta) := \int_0^\zeta \int_0^{s_1} \cdots \int_0^{s_{k-1}} f(s_k) ds_k \dots ds_1, \quad k \in \mathbb{N}, \quad f \in L_1(0,1;\mathbb{F}^d).$$

Unless $\mathcal{H} \in W_\infty^{\lfloor \frac{N}{2} \rfloor}(0,1;\mathbb{F}^{d\times d})$ is sufficiently smooth with derivatives up to order $\lfloor \frac{N}{2} \rfloor$ sufficiently small compared to $\mathcal{H}$ it is, however, not clear whether this approach actually works, even if one adds additional correction terms.

Steps Towards Less Restrictive Boundary Conditions. Whenever we used integration by parts in the first proof of Theorem 4.3.24 on exponential stability for strict dissipation at both ends of a port-Hamiltonian system of arbitrary order $N \in \mathbb{N}$ we used that under the assumption the proposition above the terms $(\mathcal{H}x_n)(\zeta)^{(k)}$ for $k = 0, 1, \dots, N-1$ and $\zeta = 0, 1$ vanish as $n \to \infty$. However, in fact those terms most of the time appear with an additional factor β_n^{-1} in front, so it might be useful to search for exponents $\gamma > 0$ such that terms $\beta_n^{-\gamma} \left|(\mathcal{H}x_n)^{(k)}(\zeta)\right|$ already vanish thanks to the boundedness of $(x_n)_{n\geq 1} \subseteq L_2(0,1;\mathbb{F}^d)$ and $(\beta_n^{-1}(\mathcal{H}x_n))_{n\geq 1} \subseteq H^N(0,1;\mathbb{F}^d)$ and the generalisation of the Sobolev-Morrey Embedding Theorems to fractional Sobolev spaces. Namely, we have thanks to Lemma 2.1.19 that

$$\beta_n^{-1} \left\|(\mathcal{H}x_n)^{(m)}\right\|_{C[0,1]} \left\|(\mathcal{H}x_n)^{(l)}\right\|_{C[0,1]} \xrightarrow{n\to+\infty} 0.$$

Then, repeating the proof of Lemma 4.3.25, but this time without having any further decay of the boundary terms to zero at hand, except for the equality above resulting from interpolation, we find for $N = 2K+1$ odd that

$$\begin{aligned} o(1) &= \frac{2}{\beta_n} \operatorname{Re} \langle \mathfrak{A}x_n - i\beta_n x_n, iq(\mathcal{H}x_n)'\rangle_{L_2} - \frac{2K}{\beta_n} \operatorname{Re} \langle \mathfrak{A}x_n - i\beta_n x_n, iq'(\mathcal{H}x_n)\rangle_{L_2} \\ &= \langle x_n, (q'\mathcal{H} - q\mathcal{H}')x_n\rangle_{L_2} + 2K \langle x_n, q'\mathcal{H}x_n\rangle_{L_2} \\ &\quad + \frac{2}{\beta_n} \sum_{k=0}^{K-1} \operatorname{Re} \left[\langle (\mathcal{H}x_n)^{(2K-k)}, iP_{2K+1}(q(\mathcal{H}x_n)')^{(k)}(\zeta)\rangle_{\mathbb{F}^d}\right]_0^1 \\ &\quad + \frac{2}{\beta_n} \sum_{k=0}^{K-2} (-1)^k \operatorname{Re} \left[\langle (\mathcal{H}x_n)^{(2K-k)}(\zeta), iP_{2K}(q(\mathcal{H}x_n)')^{(k)}(\zeta)\rangle_{\mathbb{F}^d}\right]_0^1 \\ &\quad + \frac{(-1)^K}{\beta_n} \operatorname{Re} \left[q(\zeta)\langle (\mathcal{H}x_n)^{(K)}, iP_{2K}(\mathcal{H}x_n)^{(K)}(\zeta)\rangle_{\mathbb{F}^d}\right]_0^1 \\ &\quad - \frac{2K}{\beta_n} \sum_{k=0}^{K-1} (-1)^k \operatorname{Re} \left[\langle (\mathcal{H}x_n)^{(2K-k)}(\zeta), iP_{2K+1}(q'(\mathcal{H}x_n))^{(k)}(\zeta)\rangle_{\mathbb{F}^d}\right]_0^1 \\ &= \langle x_n, (Nq'\mathcal{H} - q\mathcal{H}')x_n\rangle_{L_2} \\ &\quad + \frac{(-1)^K}{\beta_n} \left[q(\zeta)\langle (\mathcal{H}x_n)^{(K)}(\zeta), iP_{2K}(\mathcal{H}x_n)^{(K)}(\zeta)\rangle_{\mathbb{F}^d}\right]_0^1 \end{aligned}$$

$$
\begin{aligned}
&+ \frac{2}{\beta_n} \sum_{k=0}^{K-1} (-1)^k \operatorname{Re} \left[\langle (\mathcal{H}x_n)^{(2K-k)}(\zeta), iP_{2K+1} q(\zeta) (\mathcal{H}x_n)^{(k+1)}(\zeta) \rangle_{\mathbb{F}^d} \right]_0^1 \\
&+ \frac{2}{\beta_n} \sum_{k=0}^{K-1} (-1)^k \operatorname{Re} \left[\langle (\mathcal{H}x_n)^{(2K-k)}(\zeta), iP_{2K+1} (k-K) q'(\zeta) (\mathcal{H}x_n)^{(k)}(\zeta) \rangle_{\mathbb{F}^d} \right]_0^1 \\
&+ \frac{2}{\beta_n} \sum_{k=0}^{K-2} (-1)^k \operatorname{Re} \left[\langle (\mathcal{H}x_n)^{(2K-k)}(\zeta), iP_{2K} q(\zeta) (\mathcal{H}x_n)^{(k+1)}(\zeta) \rangle_{\mathbb{F}^d} \right]_0^1 . \\
&+ \frac{2}{\beta_n} \sum_{k=0}^{K-2} (-1)^k \operatorname{Re} \left[\langle (\mathcal{H}x_n)^{(2K-k)}(\zeta), iP_{2K} q'(\zeta) (\mathcal{H}x_n)^{(k)}(\zeta) \rangle_{\mathbb{F}^d} \right]_0^1 .
\end{aligned}
$$

Choosing $q(\zeta) = 1 - e^{\alpha\zeta}, \ \zeta \in [0,1]$, for $\alpha > 0$ large enough, as before, we then conclude the following result.

Theorem 4.3.27. *Let $\mathfrak{A}$ be a port-Hamiltonian operator of odd order $N = 2K+1$. If $R \in D(\mathfrak{A}, H)$ such that*

$$
\begin{aligned}
(-1)^K \langle (\mathcal{H}x)^{(K)}(1), iP_{2K} (\mathcal{H}x)^{(K)} \rangle_{\mathbb{F}^d} &\le 0 \\
\forall \zeta \in \{0,1\}, \ k = 0, \dots, K-1 &: \\
\operatorname{Re} \langle (\mathcal{H}x)^{(N-1-k)}(\zeta), i(P_N(k-K) + kP_{N-1})(\mathcal{H}x)^{(k)}(\zeta) \rangle_{\mathbb{F}^d} &= 0 \\
\forall k = 1, \dots, K-1 &: \\
\operatorname{Re} \langle (\mathcal{H}x)^{(N-k)}(1), i(P_N + P_{N-1})(\mathcal{H}x)^{(k)} \rangle_{\mathbb{F}^d} &= 0
\end{aligned}
$$

for all $x \in \ker R$, then the pair $(\mathfrak{A}, R)$ has property AIEP.

We give some examples: First, let $K = 0$, i.e. $N = 1$, then the only condition to be satisfied is that $(\mathcal{H}x_n)(1) \to 0$. This is

Proposition 4.3.28. *Let $N = 1$ and $\mathcal{H}$ Lipschitz continuous. Then, if $\sigma_p(A) \subseteq \mathbb{C}_0^-$ and*

$$\operatorname{Re} \langle Ax, x \rangle_{\mathcal{H}} \le -\kappa \left| (\mathcal{H}x)(1) \right|^2, \quad x \in D(A)$$

for some $\kappa > 0$, then A generates a uniformly exponentially stable contraction C_0-semigroup on X.

Secondly we consider the case $K = 1$, so $N = 3$, then the first condition is that both $(\mathcal{H}x_n)(1)$ and $(\mathcal{H}x_n)'(1) \to 0$ and from the other terms we get the additional condition that $(\mathcal{H}x_n)(0)$ or $(\mathcal{H}x_n)''(0) \to 0$. This is

Corollary 4.3.29. *Let $N = 3$ and $\mathcal{H}$ Lipschitz continuous. Then, if $\sigma_p(A) \subseteq \mathbb{C}_0^-$ and*

$$\operatorname{Re} \langle Ax, x \rangle_{\mathcal{H}} \le -\kappa \left(\left| (\mathcal{H}x)(1) \right|^2 + \left| (\mathcal{H}x)'(1) \right|^2 + \left| (\mathcal{H}x)^{(2l)}(0) \right|^2 \right)$$

where $\kappa > 0$ and $l = 0$ or 1, then A generates an exponentially stable contraction C_0-semigroup on X.

Now we will do the same investigation for $N = 2K$ even. We then obtain

$$
\begin{aligned}
o(1) &= \frac{N-2}{\beta_n} \operatorname{Re} \langle \mathfrak{A}x_n - i\beta_n x_n, iq'(\mathcal{H}x_n)\rangle_{L_2} + \frac{2}{\beta_n} \operatorname{Re} \langle \mathfrak{A}x_n - i\beta_n x_n, iq(\mathcal{H}x_n)'\rangle_{L_2} \\
&= (2K-1)\langle x_n, q'\mathcal{H}x_n\rangle_{L_2} - \langle x_n, (q'\mathcal{H} - q\mathcal{H}')x_n\rangle_{L_2} \\
&\quad + \frac{2K-1}{\beta_n} \sum_{k=0}^{K-1} (-1)^k \operatorname{Re} \left[\langle P_{2K}(\mathcal{H}x_n)^{(2K-1-k)}(\zeta), i(q'(\mathcal{H}x_n))^{(k)}(\zeta)\rangle_{\mathbb{F}^d} \right]_0^1 \\
&\quad + \frac{2}{\beta_n} \sum_{k=0}^{K-1} (-1)^k \operatorname{Re} \left[\langle P_{2K}(\mathcal{H}x_n)^{(2K-k-1)}(\zeta), i(q(\mathcal{H}x_n)')^{(k)}(\zeta)\rangle_{\mathbb{F}^d} \right]_0^1 \\
&\quad + \frac{2}{\beta_n} \sum_{k=0}^{K-1} (-1)^k \operatorname{Re} \left[\langle P_{2K-1}(\mathcal{H}x_n)^{(2K-k-2)}(\zeta), i(q(\mathcal{H}x_n)')^{(k)}(\zeta)\rangle_{\mathbb{F}^d} \right]_0^1 \\
&\quad + \frac{2(-1)^K}{\beta_n} \left[\langle (\mathcal{H}x_n)^{(K)}(\zeta), iq(\zeta)P_{2K}(\mathcal{H}x)^{(K)}(\zeta)\rangle_{\mathbb{F}^d} \right]_0^1 \\
&= -\langle x_n, (Nq'\mathcal{H} - q\mathcal{H}')x_n\rangle_{L_2} \\
&\quad + \frac{2(-1)^K}{\beta_n} \left[q(z)\langle (\mathcal{H}x_n)^{(K)}(\zeta), iP_{2K}(\mathcal{H}x_n)^{(K)}\rangle_{\mathbb{F}^d} \right]_0^1 \\
&\quad + \sum_{k=0}^{K-1} \frac{(-1)^k}{\beta_n} \left[q(\zeta) \operatorname{Re} \langle P_{2K}(\mathcal{H}x_n)^{(2K-k-1)}(\zeta), i(\mathcal{H}x_n)^{(k+1)}(\zeta)\rangle_{\mathbb{F}^d} \right]_0^1 \\
&\quad + \sum_{k=0}^{K-1} \frac{(-1)^k}{\beta_n} \left[q(\zeta) \operatorname{Re} \langle P_{2K-1}(\mathcal{H}x_n)^{(2K-k-2)}(\zeta), i(\mathcal{H}x_n)^{(k+1)}(\zeta)\rangle_{\mathbb{F}^d} \right]_0^1 \\
&\quad + \sum_{k=0}^{K-1} \frac{(-1)^k(2K-1+2k)}{\beta_n} \operatorname{Re} \left[q'(\zeta)\langle P_{2K}(\mathcal{H}x_n)^{(2K-k-1)}(\zeta), i(\mathcal{H}x_n)^{(k)}\rangle_{\mathbb{F}^d} \right]_0^1 .
\end{aligned}
$$

Hence, we have the following.

Theorem 4.3.30. *Let $\mathfrak{A}$ be a port-Hamiltonian operator of even order $N = 2K$. If $R \in \mathcal{B}(D(\mathfrak{A}), H)$ such that*

$$
\begin{aligned}
(-1)^K \langle (\mathcal{H}x_n)^{(K)}(\zeta), iP_{2K}(\mathcal{H}x_n)^{(K)}(\zeta)\rangle_{\mathbb{F}^d} &\geq 0 \\
\operatorname{Re} \langle P_N(\mathcal{H}x)^{(N-k-1)}(\zeta) + P_{N-1}(\mathcal{H}x)^{(N-k-2)}(\zeta), i(\mathcal{H}x)^{(k+1)}(\zeta)\rangle_{\mathbb{F}^d} &= 0 \\
\operatorname{Re} \langle P_{2K}(\mathcal{H}x)^{(N-k-1)}(\zeta), i(\mathcal{H}x)^{(k)}(\zeta)\rangle_{\mathbb{F}^d} &= 0
\end{aligned}
$$

for $\zeta = 0, 1$ and $k = 0, 1, \ldots, K-1$, for all $x \in \ker R$, then the pair $(\mathfrak{A}, R)$ has property AIEP.

From this we deduce for the special case $N = 2$, i.e. $K = 1$,

Corollary 4.3.31. *Let A be a port-Hamiltonian operator of order $N = 2$ and assume that $\mathcal{H}$ and P_0 are Lipschitz continuous and $\sigma_p(A) \subseteq \mathbb{C}_0^-$. If*

$$
\begin{aligned}
\operatorname{Re} \langle Ax, x\rangle_X \leq -\kappa \Big(&|(\mathcal{H}x)(1)|^2 + |(\mathcal{H}x)'(1)|^2 \\
&+ |\Pi(\mathcal{H}x)(0)|^2 + |(I - \Pi)P_2(\mathcal{H}x)'(0)|^2 \Big)
\end{aligned}
$$

for some $\kappa > 0$ and all $x \in D(A)$, then A generates a uniformly exponentially stable C_0-semigroup on X.

For the case $N = 4$, i.e. $K = 2$, we obtain

Corollary 4.3.32. *Let A be a port-Hamiltonian operator of order $N = 4$ and assume that $\mathcal{H}$ and P_0 are Lipschitz-continuous and $\sigma_p(A) \subseteq \mathbb{C}_0^-$. If $\sigma_p(A) \cap i\mathbb{R} = \emptyset$ and*

$$\begin{aligned}\operatorname{Re}\langle Ax, x\rangle_X \leq -\kappa \Big(&\left|(\mathcal{H}x)(1)\right|^2 + \left|(\mathcal{H}x)''(1)\right|^2 \\ &+ \left|\Pi(\mathcal{H}x)(0)\right|^2 + \left|(I-\Pi)P_2(\mathcal{H}x)'''(0)\right|^2 \\ &+ \left|\Pi(\mathcal{H}x)'(0)\right|^2 + \left|(I-\Pi)P_2(\mathcal{H}x)''(0)\right|^2 \Big), \quad x \in D(A)\end{aligned}$$

for some $\kappa > 0$ and an orthogonal projection $\Pi : \mathbb{F}^{\rightarrow}\mathbb{F}^d$, then A generates a uniformly exponentially stable C_0-semigroup on X.

4.4 On the $\mathcal{H}$-dependence of stability properties

In Section 3.3 we saw that for a port-Hamiltonian operator $A = \mathfrak{A}|_{\ker \mathfrak{B}+K\mathfrak{C}}$ the property of generating a contractive C_0-semigroup does not depend on the coercive multiplication operator $\mathcal{H} \in L_\infty(0,1;\mathbb{F}^{d\times d}) \subseteq \mathcal{B}(L_2(0,1;\mathbb{F}^d))$. We also saw in the previous sections sufficient conditions for exponential stability of the corresponding C_0-semigroups which did not depend on the Hamiltonian density matrix because the operator A (with $\mathcal{H} = I$) satisfies an estimate

$$\operatorname{Re}\langle Ax, x\rangle_{L_2} \leq -\kappa \left|x(0)\right|^2, \quad x \in D(A)$$

for some $\kappa > 0$ if and only if the operator $A\mathcal{H}$ satisfies the estimate

$$\operatorname{Re}\langle A\mathcal{H}x, x\rangle_{\mathcal{H}} \leq -\kappa \left|(\mathcal{H}x)(0)\right|^2, \quad x \in D(A).$$

Therefore, one might ask whether this property holds generally.

Problem 4.4.1. *Let A be a port-Hamiltonian operator with Hamiltonian density matrix function I and generating a contraction C_0-semigroup $(T_I(t))_{t\geq 0}$ on $X_I = (L_2(0,1;\mathbb{F}^d), \langle\cdot,\cdot\rangle_{L_2})$ and let $\mathcal{H} \in L_\infty(0,1;\mathbb{F}^{d\times d})$ be coercive as multiplication operator on $L_2(0,1;\mathbb{F}^{d\times d})$, so that $A\mathcal{H}$ generates a contractive C_0-semigroup $(T_{\mathcal{H}}(t))_{t\geq 0}$ on $X_{\mathcal{H}} = (L_2(0,1;\mathbb{F}^d); \langle\cdot,\cdot\rangle_{\mathcal{H}})$. Is $(T_I(t))_{t\geq 0}$ asymptotically or uniformly exponentially stable if and only if $(T_{\mathcal{H}}(t))_{t\geq 0}$ is asymptotically or uniformly exponentially stable, respectively? Does this property at least hold if $\mathcal{H} \in C^k([0,1];\mathbb{F}^{d\times d})$, where $k \in \mathbb{N}_0 \cup \{\infty\}$, is regular enough?*

Unfortunately – or luckily, depending on the point of view – the answer to this questions is negative and a counter-example is provided by Example 5.1 in [En13]. From a practical point of view this is bad news since the concrete Hamiltonian has to be considered to ensure exponential stability. Also the counter-example shows that a generalisation from constant parameter to distributed parameter systems is nontrivial (though very often intuitive). It may also well be that for some structure of dissipation exponential stability is independent from $\mathcal{H}$, indeed. However, with the following counter example in mind we will not pursue this question in detail any more. We present it here since clearly any regularity assumption on $\mathcal{H}$ would not make any sense if the contrary was true and also stability were independent of $\mathcal{H} \in L_\infty(0,1;\mathbb{F}^{d\times d})$. I.e. a positive result with regularity assumptions would have allowed us to restrict ourselves to the much easier case $\mathcal{H} = I$.

Example 4.4.2 (A Counterexample)**.** *In [En13] the author gives a nice counterexample to the hypothesis that exponential stability is independent of $\mathcal{H}$. In fact, the author considers the following operator corresponding to the transport equation*

$$A_0 x = \frac{d}{ds} x, \quad D(A_0) = \{x \in H^1(0,1;\mathbb{C}^2) : \ x(1) = \mathbb{B} f(0)\}$$

where $\mathbb{B} := \frac{1}{2} \left[\begin{smallmatrix} 1 & 1 \\ -1 & -1 \end{smallmatrix} \right]$. In fact, the corresponding semigroup $(T_0(t))_{t \geq 0}$ is nilpotent, so $\omega_0(T_0) = -\infty$. On the other hand the author shows that the semigroups generated by $A_\theta = A_0 \mathcal{H}_\theta$ for $\mathcal{H}_\theta = \left[\begin{smallmatrix} (1+\theta)^{-1} & \\ & 1 \end{smallmatrix} \right]$ $(\theta \in (-1, \infty))$ are not exponentially stable for all θ of the form $\theta_k = (2k+1)^{-1}$ $(k \in \mathbb{N})$. Note that even $\mathcal{H}_{\theta_k} \to \mathcal{H}_0 = I$ $(k \to \infty)$ and all $\mathcal{H}_\theta$ are constant along the line $\zeta \in (0,1)$.

We elaborate even more on this example to put more emphasise on the differences to those systems we usually consider here.
In fact, even for the case $\theta = 0$ the corresponding operator A_0 does not satisfy any dissipation inequality of the form

$$\operatorname{Re} \langle A_0 x, x \rangle_{\mathcal{H}_0} \leq -\kappa \left| (\mathcal{H}_0 x)_i(\zeta_0) \right|^2, \quad x \in D(A_0)$$

for some constant $\kappa > 0$ and some $\zeta_0 = 0$ or 1. In fact, for any $\theta > -1$ and $x \in D(A_\theta)$ we compute (using the boundary condition $\mathcal{H}_\theta x(1) = \mathbb{B} \mathcal{H}_\theta x(0)$)

$$\begin{aligned} \operatorname{Re} \langle A_\theta x, x \rangle_{\mathcal{H}_\theta} &= (1+\theta)^{-2} \left(|x_1(1)|^2 - |x_1(0)|^2 \right) + |x_2(1)|^2 - |x_2(0)|^2 \\ &= \frac{(1+\theta)^{-2} + 1}{4} |x_1(0) + x_2(0)|^2 - (1+\theta)^{-2} |x_1(0)|^2 - |x_2(0)|^2 . \end{aligned}$$

For the special case $\theta = 0$ this inequality reads

$$\operatorname{Re} \langle A_0 x, x \rangle_{\mathcal{H}_0} = \frac{1}{2} |x_1(0) + x_2(0)|^2 - |x(0)|^2$$

and choosing $x \in H^1(0,1)$ with $x(0) = (1,1)$ and $x(1) = (1,-1)$ (and hence $x \in D(A_0)$) shows that for this special choice the right hand equals zero which proves the assertion. In fact, for every choice of $\theta > -1$ the vector $x \in D(A_0)$ defined above also lies in $D(A_\theta)$ and $\operatorname{Re} \langle A_\theta x, x \rangle_{\mathcal{H}_\theta} = 0$ for all $\theta > -1$. Since for every $\theta > -1$ the candidates for eigenfunctions f_λ to a eigenvalue have the form $f_\lambda(\zeta) = e^{\mathcal{H}_\theta^{-1} \zeta} f_\lambda(0)$ $(\zeta \in [0,1])$ and thus to have an eigenvalue $\lambda \in i\mathbb{R}$ we should have

$$e^{\lambda \mathcal{H}_\theta^{-1}} = \begin{bmatrix} e^{\lambda(1+\theta)} & \\ & e^{\lambda} \end{bmatrix} = \begin{bmatrix} 1 & \\ & -1 \end{bmatrix}$$

where from the second component it follows that $\lambda \in (2\mathbb{Z}+1)\pi i$ and from the first component $\lambda(\theta+1) \in (2\mathbb{Z}+1)i\pi$. From here we get all the possible combinations as

$$\lambda_l = (2l+1)i\pi, \ \theta_{k,l} = \frac{2l+1}{2k+1}, \quad k, l \in \mathbb{Z} \quad \text{such that} \quad \theta_{k,l} > -1.$$

4.5 Examples

We return to the examples from Section 3.1 and give some sufficient boundary conditions leading to asymptotic or uniform exponential energy decay. The first of these is the prototype of a port-Hamiltonian operator of order $N = 1$, namely the transport equation.

Example 4.5.1 (Transport Equation)**.** *In Example 3.1.1 we considered the nonuniform transport equation*

$$\frac{\partial}{\partial t}x(t,\zeta) = \frac{\partial}{\partial \zeta}(c(\zeta)x(t,\zeta)) =: (\mathfrak{A}x(t))(\zeta)$$

for which we have for all $x \in D(\mathfrak{A}) = \{x \in L_2(0,1) : cx \in H^1(0,1)\}$ *that*

$$\operatorname{Re}\langle \mathfrak{A}x, x\rangle_c = \operatorname{Re}\langle cx, (cx)'\rangle_{L_2} = \frac{1}{2}\left[|(cx)(1)|^2 - |(cx)(0)|^2\right].$$

From here it is clear that any dissipative boundary condition takes the form

$$(cx)(1) = \lambda(cx)(0)$$

where $\lambda \in \mathbb{F}$ *with* $|\lambda| \leq 1$*. Moreover, we observe that for the corresponding operators* $A_\lambda = \mathfrak{A}|_{D(A_\lambda)}$ *with* $D(A_\lambda) = \{x \in D(\mathfrak{A}) : (cx)(1) = \lambda(cx)(0)\}$ *we have that*

$$\operatorname{Re}\langle A_\lambda x, x\rangle \begin{cases} = 0, & |\lambda| = 1 \\ \leq -\sigma_\lambda \left(|(cx)(1)|^2 + |(cx)(0)|^2\right), & |\lambda| \in (0,1)\,, \\ \leq -|(cx)(1)|^2\,, & \lambda = 0 \end{cases} \quad x \in D(A_\lambda) \tag{4.35}$$

where $\sigma_\lambda > 0$ *for every* $\lambda \in \mathbb{F}$ *with* $|\lambda| < 1$*. As a result, the* C_0*-semigroups* $(T_\lambda(t))_{t\geq 0}$ *generated by* A_λ *are isometric for* $|\lambda| = 1$ *and uniformly exponentially stable for* $|\lambda| < 1$*, thanks to Theorem 4.1.5.*

Next we turn our attention to the wave equation and then to the Timoshenko beam equation.

Example 4.5.2 (Wave Equation)**.** *For the wave equation*

$$\rho(\zeta)\omega_{tt}(t,\zeta) = \frac{\partial}{\partial \zeta}\left(T(\zeta)\frac{\partial}{\partial \zeta}\omega(t,\zeta)\right), \quad \zeta \in (0,1),\ t \geq 0 \tag{4.36}$$

and the corresponding port-Hamiltonian operator

$$\mathfrak{A} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \frac{\partial}{\partial \zeta} \begin{pmatrix} (\mathcal{H}_1 x_1) \\ (\mathcal{H}_2 x_2) \end{pmatrix} \tag{4.37}$$

for $\operatorname{diag}(\mathcal{H}_1, \mathcal{H}_2) = \operatorname{diag}(\rho^{-1}, T)$ *and* $x = (\rho\omega_t, \omega_\zeta)$*, see Example 3.1.2, we find that*

$$\begin{aligned} \operatorname{Re}\langle \mathfrak{A}x, x\rangle_{\mathcal{H}} &= \operatorname{Re}\langle (\mathcal{H}_2 x_2)', (\mathcal{H}_1 x_1)\rangle_{L_2} + \operatorname{Re}\langle (\mathcal{H}_1 x_1)', (\mathcal{H}_2 x_2)\rangle_{L_2} \\ &= \operatorname{Re}\left[\langle (\mathcal{H}_1 x_1)(\zeta), (\mathcal{H}_2 x_2)(\zeta)\rangle_{\mathbb{F}}\right]_0^1 \end{aligned}$$

which translates to the energy decay relation

$$\frac{d}{dt}H(t) = \operatorname{Re}\langle \omega_t(t,1), T\omega_\zeta(t,1)\rangle_{\mathbb{F}} - \operatorname{Re}\langle \omega_t(t,0), T\omega_\zeta(t,0)\rangle_{\mathbb{F}}.$$

A natural way to choose dissipative boundary conditions is therefore the following

$$\begin{aligned} \alpha_0\omega_t(t,0) + \beta_0(-(T\omega_\zeta)(t,0)) &= 0 \\ \alpha_1\omega_t(t,1) + \beta_1(T\omega_\zeta)(t,1) &= 0 \end{aligned}$$

where $\alpha_i, \beta_i \geq 0$ $(i = 0, 1)$ *are non-negative constants such that* $\alpha_i + \beta_i > 0$ $(i = 0, 1)$*. Clearly if* $\alpha_i \beta_i = 0$ $(i = 0, 1)$ *then we have conservative boundary conditions at both ends, so that*

$$\operatorname{Re} \langle Ax, x \rangle_X = 0, \quad x \in D(A)$$

for the resulting operator A *and thus it generates an isometric* C_0*-semigroup on* X*. Secondly, if* $\alpha_i > 0$ *and* $\beta_i > 0$ *are both strictly positive for some* $i \in \{0, 1\}$*, say* $i = 0$*, then*

$$\operatorname{Re} \langle Ax, x \rangle_X \leq -\sigma \left| (\mathcal{H}x)(0) \right|^2, \quad x \in D(A)$$

for the operator A *with these boundary conditions and Theorem 4.1.5 says that the resulting* C_0*-semigroup* $(T(t))_{t \geq 0}$ *on* X *is uniformly exponentially stable.*

Remark 4.5.3. *Let us note that from the consideration above the following choices for the input and output maps* $\mathfrak{B}$ *and* $\mathfrak{C}$ *make the port-Hamiltonian system* $\mathfrak{S} = (\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$*, where* $\mathfrak{A}$ *is the port-Hamiltonian operator associated to the nonuniform wave equation, an impedance passive, in fact even impedance energy preserving, port-Hamiltonian system in boundary control and observation form.*

$$\begin{pmatrix} \mathfrak{B}x \\ \mathfrak{C}x \end{pmatrix} = \begin{pmatrix} (\mathcal{H}_1 x_1)(0) \\ (\mathcal{H}_1 x_1)(1) \\ -(\mathcal{H}_2 x_2)(0) \\ (\mathcal{H}_2 x_2)(1) \end{pmatrix} \hat{=} \begin{pmatrix} \omega_t(t,0) \\ \omega_t(t,1) \\ -(T\omega_\zeta)(t,0) \\ (T\omega_\zeta)(t,1) \end{pmatrix}$$

$$\text{or} \quad \begin{pmatrix} \mathfrak{B}x \\ \mathfrak{C}x \end{pmatrix} = \begin{pmatrix} -(\mathcal{H}_2 x_2)(0) \\ (\mathcal{H}_1 x_1)(1) \\ (\mathcal{H}_1 x_1)(0) \\ (\mathcal{H}_2 x_2)(1) \end{pmatrix} \hat{=} \begin{pmatrix} -(T\omega_\zeta)(t,0) \\ \omega_t(t,1) \\ \omega_t(t,0) \\ (T\omega_\zeta)(t,1) \end{pmatrix}$$

$$\text{or} \quad \begin{pmatrix} \mathfrak{B}x \\ \mathfrak{C}x \end{pmatrix} = \begin{pmatrix} (\mathcal{H}_1 x_1)(0) \\ (\mathcal{H}_2 x_2)(1) \\ -(\mathcal{H}_2 x_2)(0) \\ (\mathcal{H}_1 x_1)(1) \end{pmatrix} \hat{=} \begin{pmatrix} \omega_t(t,0) \\ (T\omega_\zeta)(t,1) \\ -(T\omega_\zeta)(t,0) \\ \omega_t(t,1) \end{pmatrix}$$

$$\text{or} \quad \begin{pmatrix} \mathfrak{B}x \\ \mathfrak{C}x \end{pmatrix} = \begin{pmatrix} -(\mathcal{H}_2 x_2)(0) \\ (\mathcal{H}_1 x_1)(1) \\ (\mathcal{H}_1 x_1)(0) \\ (\mathcal{H}_1 x_1)(1) \end{pmatrix} \hat{=} \begin{pmatrix} -(T\omega_\zeta)(t,0) \\ (T\omega_\zeta)(t,1) \\ \omega_t(t,0) \\ \omega_t(t,1) \end{pmatrix}$$

Note that this list is not conclusive, but clearly these are the most natural choices for $\mathfrak{B}$ *and* $\mathfrak{C}$*.*

Example 4.5.4 (Feedback Stabilisation of the Timoshenko Beam Equation)**.** *We start by determining the energy change of a Timoshenko beam, i.e.*

$$\begin{aligned} \rho(\zeta)\omega_{tt}(t,\zeta) &= (K(\zeta)(\omega_\zeta - \phi)(t,\zeta))_\zeta \\ I_\rho(\zeta)\phi_{tt}(t,\zeta) &= (EI(\zeta)\phi_\zeta(t,\zeta))_\zeta - K(\zeta)(\omega_\zeta - \phi)(t,\zeta), \quad \zeta \in (0,1), \ t \geq 0. \end{aligned}$$

The corresponding port-Hamiltonian operator is

$$\mathfrak{A}x = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} (\mathcal{H}x)' + \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} (\mathcal{H}x)$$

where $x = (x_1, x_2, x_3, x_4) \hat{=} (\omega_\zeta - \phi, \rho\omega_t, \phi_\zeta, I_\rho\phi_t)$ *and* $\mathcal{H} = \operatorname{diag}(K, \rho^{-1}, EI, I_\rho^{-1})$, *see Example 3.1.3. We then compute*

$$\begin{aligned}\operatorname{Re}\langle \mathfrak{A}x, x\rangle_{\mathcal{H}} &= \operatorname{Re}\left[\langle (\mathcal{H}_1x_1)(\zeta), (\mathcal{H}_2x_2)(\zeta)\rangle_{\mathbb{F}} + \langle (\mathcal{H}_3x_3)(\zeta), (\mathcal{H}_4x_4)(\zeta)\rangle_{\mathbb{F}}\right]_0^1 \\ &\hat{=} \operatorname{Re}\left[\langle (K(\omega_\zeta - \phi)(t,\zeta), \omega_t(t,\zeta)\rangle_{\mathbb{F}} + \langle (EI\omega_\zeta)(t,\zeta), \phi_t(t,\zeta)\rangle_{\mathbb{F}}\right]_0^1 \quad (4.38)\end{aligned}$$

so that natural choices for the input and output maps $\mathfrak{B}$ *and* $\mathfrak{C}x$ *such that the port-Hamiltonian system* $\mathfrak{S} = (\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ *is impedance passive, in fact impedance energy preserving, are the following.*

$$\begin{aligned}\{\mathfrak{B}_1x, \mathfrak{C}_1x\} &= \{(\mathcal{H}_1x_1)(1), (\mathcal{H}_2x_2)(1)\} \hat{=} \{(K(\omega_\zeta - \phi))(t,1), \omega_t(t,1)\} \\ \{\mathfrak{B}_2x, \mathfrak{C}_2x\} &= \{(\mathcal{H}_3x_3)(1), (\mathcal{H}_4x_4)(1)\} \hat{=} \{(EI\phi_\zeta)(t,1), \phi_t(t,1)\} \\ \{\mathfrak{B}_3x, \mathfrak{C}_3x\} &= \{(\mathcal{H}_1x_1)(0), -(\mathcal{H}_2x_2)(0)\} \hat{=} \{(K(\omega_\zeta - \phi))(t,0), -\omega_t(t,0)\} \\ \{\mathfrak{B}_4x, \mathfrak{C}_4x\} &= \{(\mathcal{H}_3x_3)(0), -(\mathcal{H}_4x_4)(0)\} \hat{=} \{(EI\phi_\zeta)(t,1), \phi_t(t,1)\}\end{aligned}$$

Then for every $K \in \mathbb{F}^{4\times 4}$ *such that* $K = K^* \geq 0$ *is positive semidefinite the operator*

$$A_K = \mathfrak{A}|_{\ker(\mathfrak{B}+K\mathfrak{C})}$$

is dissipative and therefore generates a contractive C_0*-semigroup* $(T_K(t))_{t\geq 0}$ *on* X*. Moreover, if there is* $\sigma > 0$ *such that*

$$\langle Kz, z\rangle_{\mathbb{F}^4} \geq \sigma\left(|z_1|^2 + |z_2|^2\right), \quad z = (z_1, z_2, z_3, z_4) \in \mathbb{F}^4$$

then

$$\operatorname{Re}\langle A_Kx, x\rangle_X \leq -\sigma\,|(\mathcal{H}x)(0)|^2, \quad x \in D(A_K)$$

and by Theorem 4.1.5 the C_0*-semigroup is uniformly exponentially stable. For the original problem this means that whenever we impose strictly dissipative boundary conditions at one end of the beam and conservative or dissipative boundary conditions at the other, the energy of the system decays uniformly exponentially to zero as* $t \to \infty$*.*

Next we come to the Euler-Bernoulli beam equation as an example for a port-Hamiltonian operator of order $N = 2$.

Example 4.5.5 (Euler-Bernoulli Beam Equation)**.** *We have already investigated the port-Hamiltonian operators* $\mathfrak{A}$ *associated to the Euler-Bernoulli beam equation, see Example 3.1.6,*

$$\rho\omega_{tt} + [EI\omega_{\zeta\zeta}]_{\zeta\zeta} = 0, \quad \zeta \in (0,1),\ t \geq 0.$$

For the corresponding maximal port-Hamiltonian operator $\mathfrak{A}$ *we find*

$$\begin{aligned}\operatorname{Re}\langle \mathfrak{A}x, x\rangle_{\mathcal{H}} &= \operatorname{Re}\langle \left[\begin{smallmatrix}0 & -1\\ 1 & 0\end{smallmatrix}\right](\operatorname{diag}(\mathcal{H}_1, \mathcal{H}_2)x)'', \operatorname{diag}(\mathcal{H}_1, \mathcal{H}_2)x\rangle_{L_2} \\ &= \operatorname{Re}\langle (\mathcal{H}_1x_1)'', \mathcal{H}_2x_2\rangle_{L_2} - \operatorname{Re}\langle \mathcal{H}_1x_1, (\mathcal{H}_2x_2)''\rangle_{L_2} \\ &= \operatorname{Re}\left[\langle (\mathcal{H}_1x_1)'(\zeta), (\mathcal{H}_2x_2)(\zeta)\rangle_{\mathbb{F}} - \langle (\mathcal{H}_1x_1)(\zeta), (\mathcal{H}_2x_2)'(\zeta)\rangle_{\mathbb{F}}\right]_0^1\end{aligned}$$

for every $x \in D(\mathfrak{A})$*. Hence, in the following way we may choose the input and output maps* $\mathfrak{B}$ *and* $\mathfrak{C}$ *in such a way that* $\mathfrak{S} = (\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ *is an impedance energy* *preserving system.*

$$\{\mathfrak{B}_1 x, \mathfrak{C}_1 x\} = \{(\mathcal{H}_1 x_1)(0), (\mathcal{H}_2 x_2)'(0)\} \hat{=} \{(\omega_t(t,0), (EI\omega_{\zeta\zeta})_\zeta(t,0)\}$$
$$\{\mathfrak{B}_1 x, \mathfrak{C}_1 x\} = \{(\mathcal{H}_1 x_1)'(0), -(\mathcal{H}_2 x_2)(0)\} \hat{=} \{\omega_{t\zeta}(t,0), -(EI\omega_{\zeta\zeta})(t,0)\}$$
$$\{\mathfrak{B}_1 x, \mathfrak{C}_1 x\} = \{(\mathcal{H}_1 x_1)(1), -(\mathcal{H}_2 x_2)'(1)\} \hat{=} \{\omega_t(t,1), -(EI\omega_{\zeta\zeta})(t,1)\}$$
$$\{\mathfrak{B}_1 x, \mathfrak{C}_1 x\} = \{(\mathcal{H}_1 x_1)'(1), (\mathcal{H}_2 x_2)(1)\} \hat{=} \{\omega_{t\zeta}(t,1), (EI\omega_{\zeta\zeta})(t,1)\}.$$

Then boundary conditions of the following form are natural to obtain dissipativity of the operator A, which is $\mathfrak{A}$ restricted to the boundary conditions

$$\alpha_1 \omega_{t\zeta}(t,1) + \beta_1 (EI\omega_{\zeta\zeta})(t,1) = 0 \quad \text{(i)}$$
$$\alpha_2 \omega_t(t,1) + \beta_2 (-EI\omega_{\zeta\zeta})_\zeta(t,1) = 0 \quad \text{(ii)}$$
$$\alpha_3 \omega_{t\zeta}(t,0) + \beta_3 (-EI\omega_{\zeta\zeta})(t,0) = 0 \quad \text{(iii)}$$
$$\alpha_4 \omega_t(t,0) + \beta_4 (EI\omega_{\zeta\zeta})_\zeta(t,0) = 0 \quad \text{(iv)}$$

where $\alpha_i, \beta_i \geq 0$ $(i = 1,2,3,4)$ are non negative constants such that $\alpha_i + \beta_i > 0$ for $i = 1,2,3,4$. Note that whenever both $\alpha_i > 0$ and $\beta_i > 0$ for some $i \in \{1,2,3,4\}$ this leads to dissipation in the corresponding boundary condition, i.e. the boundary condition is not conservative. Let us now translate the conditions of Proposition 4.3.19 to these boundary conditions. We conclude for the relevant terms the following.

1. *$(\mathcal{H}x)(t,0)$ represents $(\omega_t(t,0), EI\omega_{\zeta\zeta}(t,0))$, so that we should have*

$$\alpha_4 > 0 \quad \text{and} \quad \beta_3 > 0.$$

2. *$(\mathcal{H}_1 x_1)'(t,0)$ or $(\mathcal{H}_2 x_2)'(t,0)$ should be included, so that the terms $\omega_{t\zeta}(t,0)$ or $(EI\omega_{\zeta\zeta})_\zeta(t,0)$ should obey some boundary condition, i.e.*

$$\alpha_3 > 0 \quad \text{or} \quad \beta_4 > 0.$$

3. *Similarly $(\mathcal{H}_1 x_1)(t,1)$ or $(\mathcal{H}_2 x_2)'(t,1)$ translates to $\omega_t(t,1)$ or $(EI\omega_{\zeta\zeta})_\zeta(t,1)$, i.e.*

$$\alpha_2 > 0 \quad \text{or} \quad \beta_1 > 0.$$

4. *Finally $(\mathcal{H}_1 x_1)'(t,1)$ or $(\mathcal{H}_2 x_2)(t,1)$ stand for $\omega_{t\zeta}(t,1)$ or $(EI\omega_{\zeta\zeta})(t,1)$, i.e.*

$$\alpha_1 > 0 \quad \text{or} \quad \beta_2 > 0.$$

The interpretation is as follows. On the one hand at 0 we should have dissipative boundary conditions in equation (iii) *or* (iv) *and there is a restriction on the conservative boundary condition, e.g. if $\alpha_4, \beta_4 > 0$ are both strictly positive, the boundary condition $\omega_{t\zeta}(t,0) = 0$ is not admissible for the application of Proposition 4.3.19. At the right end several boundary conditions are possible, e.g. if one of the boundary conditions* (i) *and* (ii) *is dissipative, the other may be arbitrarily conservative or dissipative. However, the admissible (for application of Proposition 4.3.19) conservative boundary conditions are the following ones.*

1. *clamped right end, i.e. $\omega_t(t,1) = \omega_{t\zeta}(t,1) = 0$,*

2. *free right end (or shear hinge right end), i.e.* $(EI\omega_{\zeta\zeta})(t,1) = (EI\omega_{\zeta\zeta})_\zeta(t,1) = 0$.

We remark that this does not cover all the possible boundary conditions that have been mentioned in Discussion 4.1 of [Ch+87], but we only cover the boundary conditions (ii)-(iv) mentioned there plus the boundary conditions used in the main part of [Ch+87], but not the boundary conditions (i), (v) and (vi). Also note that we did not actually show exponential stability, but only under the condition that these boundary conditions already imply asymptotic stability. Also in the port-Hamiltonian formulation the boundary conditions (ii) and (iii), (i) and (v) are the same. Moreover, in contrast to [Ch+87] we allow for a non uniform Euler-Bernoulli beam whereas in [Ch+87] the authors considered a chain of uniformly distributed Euler-Bernoulli beams. Also the above boundary conditions cover the case considered in Theorem 4 of [Ch+87a] which corresponds to the situation that $\alpha_1, \alpha_2, \beta_3, \alpha_4, \beta_4 > 0$ *are all strictly positive.*

Example 4.5.6 (Asymptotic Stabilisation of the Nonuniform Euler-Bernoulli Beam by Shear Force Feedback)**.** *In [Ch+87a] the authors considered the following (uniform) Euler-Bernoulli beam model as model for a space shuttle attached to some flexible mast (the latter modelled as Euler-Bernoulli Beam).*

$$\begin{aligned} &\rho\omega_{tt} + [EI\omega_{\zeta\zeta}]_{\zeta\zeta} = 0 \\ &\omega(t,0) = \omega_\zeta(t,0) = 0 \\ &-[EI\omega_{\zeta\zeta}]_\zeta(t,1) = -k_1\omega_t(t,1) \\ &-[EI\omega_{\zeta\zeta}](t,1) = -k_2\omega_{t\zeta}(t,1) \end{aligned}$$

and given initial datum $(\omega(0,\cdot), \omega_t(0,\cdot)) = (\omega_0, \omega_1) \in L_2(0,1;\mathbb{F}^2)$*. Here* $k_1, k_2 \geq 0$ *are non-negative constants, so that the closed loop system becomes dissipative. The focus of the article [Ch+87a] lies on proving exponential stability for the case that* $k_1 \geq 0$ *and* $k_2 > 0$*, whereas uniform exponential stability for the case* $k_1 > 0$ *and* $k_2 \geq 0$ *had been investigated in [Ch+87]. In both [Ch+87a] and [Ch+87] the authors restricted to the uniform case, i.e.* $\rho, EI > 0$ *being constant along the beam. By Corollary 4.2.10 the system is asymptotically stable for either of the cases* $k_1 > 0, k_2 \geq 0$ *or* $k_1 \geq 0,\ k_2 > 0$*, whenever* $\mathcal{H}_i$ $(i = 1,2)$ *are Lipschitz continuous and uniformly uniformly positive. For uniform exponential stability we apply Proposition 4.3.19 and deduce that in the case that* $k_1 > 0$ *and* $k_2 \geq 0$ *we have uniform exponential stability of the corresponding* C_0*-semigroup, i.e. uniform exponential energy decay for the original problem. Unfortunately our theoretical results do not cover the case* $\alpha_1 = 0$ *and* $\alpha_2 > 0$*. Although we may prove asymptotic stability in a similar way as before, for the corresponding operator* A_{α_1,α_2} *we only have the estimate*

$$\operatorname{Re}\langle A_{\alpha_1,\alpha_2}x, x\rangle_{\mathcal{H}} \leq -\kappa(|\mathcal{H}_1x_1(0)|^2 + |(\mathcal{H}_1x_1)'(0)|^2 + |(\mathcal{H}_2x_2)(1)|^2 + |(\mathcal{H}_1x_1)'(1)|^2).$$

In fact, in the constant parameter case $\mathcal{H} = $ *const. for this situation exponential energy decay can be observed as is proved (in a quite tedious way) in [Ch+87a].*

Chapter 5

Passivity Based Dynamic Linear Feedback Stabilisation

In applications, we often encounter situations where the energy of a system splits into two (or more) parts. In that case we may interpret the total system as an interconnection of two (or more) subsystems which interact with each other in a specific way (in our case by boundary control and observation). If all the systems are infinite dimensional and port-Hamiltonian, very often the whole system is an infinite dimensional port-Hamiltonian system again. However, sometimes we encounter situations where the system has both subsystems of infinite dimensional type and of finite dimensional type. That is, some subsystems are modelled by a (port-Hamiltonian) Partial Differential Equation, whereas others are described by an ordinary differential equation. The interconnection of such systems is sometimes called a *hybrid system*. Our aim in this section is to depict how the theory for the pure infinite dimensional case naturally carries over to these hybrid systems.

Let us first build up the setting for this situation. The assumptions on the infinite dimensional part of the interconnected system are essentially the same as before, except for that we use the input map $\mathfrak{B}$ and the output map $\mathfrak{C}$ for interconnection of the two (finite and infinite) subsystems. In fact, all the results in this section are generalisations of the one component system (i.e. pure infinite dimensional) case. Let $\mathfrak{S} = (\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ be a port-Hamiltonian system of order $N \in \mathbb{N}$, just as in Definition 3.2.10,

$$\begin{aligned}
\frac{\partial}{\partial t} x(t,\zeta) &= \sum_{k=0}^{N} P_k \frac{\partial^k}{\partial \zeta^k} (\mathcal{H}(\zeta) x(t,\zeta)) =: (\mathfrak{A} x(t))(\zeta), \qquad && t \geq 0,\ \zeta \in (0,1) \\
u(t) &= W_B \begin{pmatrix} f_{\partial,\mathcal{H}x} \\ e_{\partial,\mathcal{H}x} \end{pmatrix} (t) =: \mathcal{B} x(t), && \\
y(t) &= W_C \begin{pmatrix} f_{\partial,\mathcal{H}x} \\ e_{\partial,\mathcal{H}x} \end{pmatrix} (t) =: \mathfrak{C} x(t), && t \geq 0
\end{aligned}$$

where as before $\mathfrak{A}$ is defined on its maximal domain

$$D(\mathfrak{A}) = \{x \in X : \mathcal{H}x \in H^N(0,1;\mathbb{F}^d)\}$$

and $D(\mathfrak{B}) = D(\mathfrak{C}) = D(\mathfrak{A})$. Additionally we consider a finite dimensional Hilbert space $X_c = \mathbb{F}^n$ (the state space of the dynamic controller) equipped with some inner product $\langle \cdot, \cdot \rangle_{X_c}$. (Since all norms on $\mathbb{F}^n$ are equivalent, see e.g. Satz V.1.8 in [We06], the corresponding norm $\|\cdot\|_{X_c}$ is equivalent to the usual Euclidean norm.) To keep everything as general as possible we assume that the finite dimensional part takes the standard form

$$\begin{aligned} \frac{\partial}{\partial t} x_c(t) &= A_c x_c(t) + B_c u_c(t), \\ y_c(t) &= C_c x_c(t) + D_c u_c(t), \quad t \geq 0 \end{aligned}$$

for some matrices $A_c \in \mathbb{F}^{n \times n}, B_c \in \mathbb{F}^{n \times Nd}, C_c \in \mathbb{F}^{Nd \times n}$ and $D_c \in \mathbb{F}^{Nd \times Nd}$.

Remark 5.0.7. *We may and will also allow $n = 0$ in the sense that we interpret $\mathbb{F}^0 := \{0\}$ to be the null space and $\mathbb{F}^{0 \times r} := \mathcal{B}(\mathbb{F}^r, \{0\}) = \{M : \mathbb{F}^r \ni z \mapsto 0\}$ as well as $\mathbb{F}^{r \times 0} := \mathcal{B}(\{0\}, \mathbb{F}^r) = \{M : 0 \mapsto 0 \in \mathbb{F}^r\}$. In that case the following feedback interconnection can be interpreted as a fancy way of writing down static boundary conditions. Moreover, the choice $U_c = Y_c = \mathbb{F}^{Nd}$ for the controller input and controller output space, respectively, is not restrictive. In fact, if $\tilde{U}_c, \tilde{Y}_c$ are any finite dimensional Hilbert spaces with dimensions less or equal Nd these can be embedded into $\mathbb{F}^{Nd}$ and using that embedding any operator in $\mathcal{B}(U_c, X_c)$ or $\mathcal{B}(X_c, Y_c)$ can be interpreted as matrix in $\mathbb{F}^{n \times Nd}$ or $\mathbb{F}^{Nd \times n}$, respectively.*

We are interested in situations without external input signal and interconnect the two subsystems by *standard feedback interconnection*, i.e.

$$u_c = y, \qquad u = -y_c.$$

Remark 5.0.8. *Note that also for $n > 1$ this feedback interconnection may include static boundary conditions, namely if* $\operatorname{ran} \begin{bmatrix} C_c & D_c \end{bmatrix} \neq \mathbb{F}^{Nd}$*, then always* $y_c \in \operatorname{ran} \begin{bmatrix} C_c & D_c \end{bmatrix}$ *lies in a proper subspace of* $\mathbb{F}^{Nd}$.

We obtain an operator on the product space $X \times X_c$ which we equip with the canonical inner product

$$\langle (x, x_c), (z, z_c) \rangle_{X \times X_c} = \langle x, z \rangle_X + \langle x_c, z_c \rangle_{X_c}, \quad (x, x_c), (z, z_c) \in X \times X_c.$$

Our plan is as follows. First, we state the generation result for the dynamic feedback operator, then we generalise the stability results from the static case to the dynamic situation.

5.1 The Generation Theorem – Dynamic Case

We generalise the Generation Theorem 3.3.6 for port-Hamiltonian systems with dissipative, static linear feedback to the dynamic feedback situation, also see Theorem 5.8 in [Vi07] for a very similar treatment, but with slightly more restrictive conditions on the dynamic controller. Afterwards we look at the example of a energy-preserving interconnection of impedance passive systems and also note that for appropriate external input and output functions the interconnected (hybrid) system becomes a Boundary Control and Observation System.

Theorem 5.1.1. *Let $\mathfrak{S} = (\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ be a port-Hamiltonian system and $\Sigma_c = (A_c, B_c, C_c, D_c)$ be a linear control system on a finite dimensional space X_c and with input and output space $U_c = Y_c = \mathbb{F}^{Nd}$. On $X \times X_c$ define the* hybrid operator $\mathcal{A} : D(\mathcal{A}) \subseteq X \times X_c \to X \times X_c$ *by*

$$\begin{aligned} D(\mathcal{A}) &:= \{(x, x_c) \in D(\mathfrak{A}) \times X_c : \begin{pmatrix} f_{\partial,\mathcal{H}x} \\ e_{\partial,\mathcal{H}x} \\ x_c \end{pmatrix} \in \ker W_{cl}\}, \\ &= \{(x, x_c) \in D(\mathfrak{A}) \times X_c : (\mathfrak{B} + D_c\mathfrak{C})x = -C_c x_c\} \\ \mathcal{A}(x, x_c) &:= \begin{bmatrix} \mathfrak{A} & 0 \\ B_c\mathfrak{C} & A_c \end{bmatrix} \begin{pmatrix} x \\ x_c \end{pmatrix}, \end{aligned}$$

where the matrix W_{cl} is given by

$$W_{cl} := \begin{bmatrix} W_B + D_c W_C & C_c \end{bmatrix}.$$

If the operator $\mathcal{A}$ is dissipative,

$$\langle \mathcal{A}(x, x_c), (x, x_c) \rangle_{X \times X_c} \leq 0, \quad (x, x_c) \in D(\mathcal{A}), \tag{5.1}$$

then it generates a contractive C_0-semigroup on $X \times X_c$. Moreover, in this case the operator $\mathcal{A}$ has compact resolvent.

Remark 5.1.2. *Similar to the static feedback Generation Theorem 3.3.6 one sees that the condition*

$$\text{Sym } P_0(\zeta) = \frac{1}{2}(P_0(\zeta) + P_0(\zeta)^*) \leq 0, \quad a.e.\ \zeta \in (0, 1)$$

is necessary for $\mathcal{A}$ to generate a contraction C_0-semigroup $(\mathcal{T}(t))_{t \geq 0}$. In fact, for every dissipative hybrid operator one necessarily has

$$\text{Re}\,\langle \mathcal{A}(x, x_c), (x, x_c) \rangle_{X \times X_c} \leq \text{Re}\,\langle P_0 \mathcal{H}x, \mathcal{H}x \rangle_{L_2}, \quad x \in D(\mathcal{A}).$$

Proof of Theorem 5.1.1. The Lumer-Phillips Theorem 2.2.7 says that for the dissipative operator $\mathcal{A}$ to generate a contractive C_0-semigroup, we only need to check that $\text{ran}\,(\lambda I - \mathcal{A}) = X \times X_c$ for some $\lambda > 0$. Fix any $\lambda > \max\{0, s(A_c)\}$, so that the resolvent $(\lambda - A_c)^{-1}$ exists. Further let an arbitrary element $(f, f_c) \in X \times X_c$ be given. We need to find $(x, x_c) \in D(\mathcal{A})$ such that

$$\lambda(x, x_c) - \mathcal{A}(x, x_c) = (f, f_c),$$

i.e.

$$f = (\lambda I_X - \mathfrak{A})x, \qquad f_c = (\lambda I - A_c)x_c - B_c\mathfrak{C}x \tag{5.2}$$

and for $(x, x_c) \in D(\mathfrak{A}) \times X_c$ to lie in $D(\mathcal{A})$ we must also have

$$(\mathfrak{B} + D_c\mathfrak{C})x + C_c x_c = 0. \tag{5.3}$$

When solving (5.2) for x_c and and substituting x_c in (5.3) we arrive at the following problem

$$\begin{aligned} f &= (\lambda I_X - \mathfrak{A})x, \\ (\mathfrak{B} + D_c\mathfrak{C})x + C_c(\lambda I - A_c)^{-1}B_c\mathfrak{C}x &= -C_c(\lambda I - A_c)^{-1}f_c =: \tilde{f}_c. \end{aligned}$$

Corollary 3.2.20 shows that there is a right inverse $B_{cl} \in \mathcal{B}(\mathbb{F}^{Nd}, D(\mathfrak{A}))$ of

$$\mathfrak{B}_{cl} = \mathfrak{B} + (D_c + C_c(\lambda - A_c)^{-1}B_c)\mathfrak{C}$$

so that we may set $x_{new} := x - B_{cl}\tilde{f}_c$ and get the equivalent problem

$$\begin{aligned}(\lambda I - \mathfrak{A})x_{new} &= f - (\lambda I - \mathfrak{A})B_{cl}\tilde{f}_c, \\ \mathfrak{B}_{cl}x_{new} &= \mathfrak{B}_{cl}x - \mathfrak{B}_{cl}B_{cl}\tilde{f}_c = 0.\end{aligned} \tag{5.4}$$

Next we show – using Theorem 3.3.6 – that the operator $A_{cl} = \mathfrak{A}|_{D(A_{cl})}$ with domain

$$D(A_{cl}) = \ker \mathfrak{B}_{cl}$$

generates a contractive C_0-semigroup on X by proving that A_{cl} is dissipative. For every $x \in D(A_{cl})$ we take $x_c = (\lambda - A_c)^{-1}B_c\mathfrak{C}x \in X_c$ and then obtain

$$(\mathfrak{B} + D_c\mathfrak{C})x + C_c x_c = \mathfrak{B}_{cl}x = 0$$

and hence $(x, x_c) \in D(\mathcal{A})$ and we conclude

$$\begin{aligned}\operatorname{Re}\langle A_{cl}x, x\rangle_X &= \langle \mathfrak{A}x, x\rangle_X \\ &= \operatorname{Re}\langle \mathcal{A}(x, x_c), (x, x_c)\rangle_{X\times X_c} - \operatorname{Re}\langle A_c x_c + B_c\mathfrak{C}x, x_c\rangle_{X_c} \\ &\le -\operatorname{Re}\langle A_c x_c + B_c\mathfrak{C}x, x_c\rangle_{X_c} \\ &= -\operatorname{Re}\langle A_c(\lambda - A_c)^{-1}B_c\mathfrak{C}x + B_c\mathfrak{C}x, (\lambda - A_c)^{-1}B_c\mathfrak{C}x\rangle_{X_c} \\ &= -\operatorname{Re}\lambda \left\|(\lambda - A_c)^{-1}B_c\mathfrak{C}x\right\|_{X_c}^2 \le 0\end{aligned}$$

for every $x \in D(A_{cl})$ and $x_c := (\lambda - A_c)^{-1}B_c\mathfrak{C}x$. Then the port-Hamiltonian operator A_{cl} is dissipative and consequently generates a contractive C_0-semigroup on X by Theorem 3.3.6. Now the resolvent operator $(\lambda I - A_{cl})^{-1} \in \mathcal{B}(X)$ exists and the unique solution of (5.4) is given by

$$x_{new} = (\lambda - A_{cl})^{-1}(f - (\lambda - \mathfrak{A})\hat{B}_{cl}\tilde{f}_c), .$$

Finally the choice

$$x = x_{new} + \hat{B}_{cl}\tilde{f}_c, \quad x_c = (\lambda - A_c)^{-1}(f_c + B_c\mathfrak{C}x),$$

defines an element $(x, x_c) \in D(\mathcal{A})$ such that $(\lambda I - \mathcal{A})(x, x_c) = (f, f_c)$. Therefore, the range condition $\operatorname{ran}(\lambda I - \mathcal{A}) = X$ is satisfied and from the Lumer-Phillips Theorem 2.2.7 we conclude that $\mathcal{A}$ is the generator of a contractive C_0-semigroup on $X \times X_c$. Compactness of the resolvent follows similar to the static case in Generation Theorem 3.3.6 from the fact that $D(\mathfrak{A}) \times X_c \hookrightarrow X \times X_c$ is compactly embedded. The latter holds since $D(\mathfrak{A}) \hookrightarrow X$ is compactly embedded (Lemma 3.2.6) and $X_c \hookrightarrow X_c$ is compactly embedded into itself as finite dimensional space. □

Remark 5.1.3. *If the hybrid operator is not dissipative, but merely the condition*

$$\operatorname{Re}\langle \mathcal{A}(x, x_c), (x, x_c)\rangle_{X\times X_c} \le \operatorname{Re}\langle P_0\mathcal{H}x, x\rangle_X$$

holds for all $x \in D(\mathfrak{A})$, i.e. $P_0(\zeta)$ may be non dissipative on a set of positive measure, then the same result holds except for that the C_0-semigroup $(\mathcal{T}(t))_{t\ge0}$ generated by $\mathcal{A}$ is not contractive in that case, but only quasi-contractive, i.e.

$$\|\mathcal{T}(t)\| \le e^{\omega t},$$

where $\omega \in \mathbb{R}$ is such that $P_0(\zeta) - \omega I$ is dissipative for a.e. $\zeta \in (0, 1)$.

Remark 5.1.4. *One particular case where the standard feedback interconnection of a port-Hamiltonian system $\mathfrak{S} = (\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ with a finite dimensional linear control system $\Sigma_c = (A_c, B_c, C_c, D_c)$ is dissipative is the interconnection of two impedance passive subsystems, i.e.*

$$\begin{aligned} \operatorname{Re} \langle \mathfrak{A}x, x\rangle_X &\le \operatorname{Re} \langle \mathfrak{B}x, \mathfrak{C}x\rangle_{\mathbb{F}^{Nd}}, && x \in D(\mathfrak{A}) \\ \operatorname{Re} \langle A_c x_c + B_c u_c, x_c\rangle_{X_c} &\le \operatorname{Re} \langle C_c u_c + D_c u_c, u_c\rangle_{\mathbb{F}^{Nd}}, && x_c \in X_c,\ u_c \in \mathbb{F}^{Nd}. \end{aligned}$$

In that case the operator A_c is dissipative (take $u_c = 0$). However, in general for equation (5.1) to hold not necessarily the operator A_c itself has to be dissipative, i.e. there may be $x_c \in X_c$ such that

$$\operatorname{Re} \langle A_c x_c, x_c\rangle_{X_c} > 0.$$

We give an example for such a system below.

Example 5.1.5 (Dissipative Interconnected System)**.** *We start with an impedance passive port-Hamiltonian system $\mathfrak{S} = (\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ such that*

$$\operatorname{Re} \langle \mathfrak{A}x, x\rangle_X \le \langle \mathfrak{B}x, \mathfrak{C}x\rangle_{\mathbb{F}^{Nd}} - \sigma \left|\mathfrak{B}x\right|^2, \quad x \in D(\mathfrak{A}).$$

Note that such a choice of $\mathfrak{B}$ and $\mathfrak{C}$ is possible, e.g. take $P_0(\zeta)$ to be dissipative for a.e. $\zeta \in (0,1)$ and $\mathfrak{B}x = f_{\partial,\mathcal{H}x}$, $\mathfrak{C}x = e_{\partial,\mathcal{H}x} + \sigma f_{\partial,\mathcal{H}x}$, then

$$\begin{aligned} \operatorname{Re} \langle \mathfrak{A}x, x\rangle_X &\le \operatorname{Re} \langle e_{\partial,\mathcal{H}x}, f_{\partial,\mathcal{H}x}\rangle_{\mathbb{F}^{Nd}} \\ &= \operatorname{Re} \langle \mathfrak{B}x, \mathfrak{C}x\rangle_{\mathbb{F}^{Nd}} - \sigma \left|\mathfrak{B}x\right|^2, \quad x \in D(\mathfrak{A}). \end{aligned}$$

Now we choose the system Σ_c in the following special way.

$$\begin{aligned} A_c &= \sqrt{\sigma} C_c' C_c, \\ \sigma \operatorname{Sym} D_c &\ge D_c' D_c \\ B_c' &= (1 + 2\sigma D_c') C_c \end{aligned}$$

where M' denotes the Hilbert space adjoint of an operator M. The simplest case here is the choice $D_c = 0$ and $B_c = C_c'$, i.e. collocated input and output and no feedthrough term. Then

$$\begin{aligned} &\operatorname{Re} \langle \mathcal{A}(x, x_c), (x, x_c)\rangle_{X \times X_c} \\ &\le \operatorname{Re} \langle \mathfrak{A}x, x\rangle_X + \operatorname{Re} \langle A_c x_c + B_c \mathfrak{C}x, x_c\rangle_{X_c} \\ &\le \operatorname{Re} \langle \mathfrak{B}x, \mathfrak{C}x\rangle_{\mathbb{F}^{Nd}} - \sigma \left|\mathfrak{B}x\right|^2 \\ &\quad + \sigma \left\|C_c x_c\right\|^2 + \operatorname{Re} \langle B_c' x_c, \mathfrak{C}x\rangle_{\mathbb{F}^{Nd}} \\ &= -\operatorname{Re} \langle C_c x_c + D_c \mathfrak{C}x, \mathfrak{C}x\rangle_{\mathbb{F}^{Nd}} - \sigma \left|C_c x_c + D_c \mathfrak{C}x\right|^2 \\ &\quad + \sigma \left\|C_c x_c\right\|^2 + \operatorname{Re} \langle B_c' x_c, \mathfrak{C}x\rangle_{\mathbb{F}^{Nd}} \\ &= -\operatorname{Re} \langle (\sigma I + D_c') D_c \mathfrak{C}x, \mathfrak{C}x\rangle_{\mathbb{F}^{Nd}} \\ &\quad - \operatorname{Re} \langle (B_c' - C_c - 2\sigma D_c' C_c) \mathfrak{C}x, x_c\rangle_{X_c} \\ &\le 0, \quad (x, x_c) \in D(\mathcal{A}). \end{aligned}$$

As for the pure infinite-dimensional port-Hamiltonian system $\mathfrak{S} = (\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$, the interconnected system plus a suitable external input map define a Boundary Control system, if the operator $\mathcal{A}$ generates a C_0-semigroup on the product Hilbert space $X \times X_c$.

Proposition 5.1.6. *Let $\mathfrak{S} = (\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ be a port-Hamiltonian system and $\Sigma_c = (A_c, B_c, C_c, D_c)$ be a finite dimensional linear control system with state space $X_c = \mathbb{F}^n$ and input and output space $U_c = Y_c = \mathbb{F}^{Nd}$. Consider the system $\mathfrak{S}_e = (\mathfrak{A}_e, \mathfrak{B}_e, \mathfrak{C}_e)$ given by*

$$\mathfrak{A}_e \begin{pmatrix} x \\ x_c \end{pmatrix} = \begin{bmatrix} \mathfrak{A} & 0 \\ B_c\mathfrak{C} & A_c \end{bmatrix} \begin{pmatrix} x \\ x_c \end{pmatrix}$$
$$\begin{pmatrix} x \\ x_c \end{pmatrix} \in D(\mathfrak{A}_e) = D(\mathfrak{A}) \times X_c$$

and

$$\mathfrak{B}_e \begin{pmatrix} x \\ x_c \end{pmatrix} = \begin{bmatrix} \mathfrak{B} + D_c\mathfrak{C} & C_c \end{bmatrix} \begin{pmatrix} x \\ x_c \end{pmatrix}$$
$$\begin{pmatrix} x \\ x_c \end{pmatrix} \in D(\mathfrak{B}_e) = D(\mathfrak{A}_e)$$

and $\mathfrak{C}_e \in \mathcal{B}(D(\mathfrak{A}_e); Y_e)$ any closed operator on $X \times X_c$ mapping into a Hilbert space Y_e. Then $\mathfrak{S}_e = (\mathfrak{A}_e, \mathfrak{B}_e, \mathfrak{C}_e)$ is a Boundary Control and Observation system if and only if $\mathcal{A} = \mathfrak{A}_e|_{\ker \mathfrak{B}_e}$ generates a C_0-semigroup.

Proof. We only need to show that there is a right-inverse $B_e \in \mathcal{B}(\mathbb{F}^{Nd}; D(\mathfrak{A}_e))$ of $\mathfrak{B}_e$. For this we may simply take $B_e = \left[\begin{smallmatrix} B \\ 0 \end{smallmatrix}\right]$ where B is the right-inverse of $\mathfrak{B} + D_c\mathfrak{C}$ which exists by Theorem 3.2.21. □

In particular, we have

Proposition 5.1.7. *Let $\mathfrak{S} = (\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ be an impedance passive port-Hamiltonian system and $\Sigma_c = (A_c, B_c, C_c, D_c)$ be a finite dimensional impedance passive control system with $X_c = \mathbb{F}^n$ and $U_c = Y_c = \mathbb{F}^{Nd}$. Then $(\mathfrak{A}_e, \mathfrak{B}_e)$ with*

$$\mathfrak{A}_e \begin{pmatrix} x \\ x_c \end{pmatrix} = \begin{bmatrix} \mathfrak{A} & 0 \\ B_c\mathfrak{C} & A_c \end{bmatrix} \begin{pmatrix} x \\ x_c \end{pmatrix},$$
$$\begin{pmatrix} x \\ x_c \end{pmatrix} \in D(\mathfrak{A}_e) = D(\mathfrak{A}) \times X_c$$

and

$$\mathfrak{B}_e \begin{pmatrix} x \\ x_c \end{pmatrix} = \begin{bmatrix} \mathfrak{B} + D_c\mathfrak{C} & C_c \end{bmatrix} \begin{pmatrix} x \\ x_c \end{pmatrix},$$
$$\begin{pmatrix} x \\ x_c \end{pmatrix} \in D(\mathfrak{A}_e) = D(\mathfrak{A}) \times X_c$$

is a Boundary Control system on the extended state space $X \times X_c$ and the associated C_0-semigroup generator $\mathcal{A} = \mathfrak{A}_e|_{\ker \mathfrak{B}_e}$ defined by

$$\mathcal{A} \begin{pmatrix} x \\ x_c \end{pmatrix} = \begin{bmatrix} \mathfrak{A} & 0 \\ B_c\mathfrak{C} & A_c \end{bmatrix} \begin{pmatrix} x \\ x_c \end{pmatrix}$$

on the domain

$$D(\mathcal{A}) = \left\{ \begin{pmatrix} x \\ x_c \end{pmatrix} \in D(\mathfrak{A}) \times X_c : W_{\Sigma_c} \left(\begin{smallmatrix} f_{\partial,\mathcal{H}x} \\ e_{\partial,\mathcal{H}x} \\ x_c \end{smallmatrix} \right) = 0 \right\}$$
$$= \left\{ \begin{pmatrix} x \\ x_c \end{pmatrix} \in D(\mathfrak{A}) \times X_c : \mathfrak{B}x = -C_c x_c - D_c\mathfrak{C}x \right\}$$

with W_{Σ_c} given by

$$W_{\Sigma_c} = \left[\begin{array}{cc} W_B + D_c W_C & C_c \end{array}\right]$$

generates a contraction C_0-semigroup on $X \times X_c$.

Proof. Theorem 5.1.1 says that in this case $A = \mathfrak{A}_e|_{\ker \mathfrak{B}_e}$ generates a contraction C_0-semigroup. Therefore, the result can be derived from Proposition 5.1.6. □

5.2 Asymptotic Behaviour

We are now in a similar situation as after the generation theorem for the static case and may investigate stability properties next. To obtain asymptotic stability we first of all have to exclude the case where the controller itself has undesirable spectral properties, i.e. there should be no eigenmode of the finite dimensional part when connected to an infinite-dimensional port-Hamiltonian system at rest. Namely observe the following

Example 5.2.1. *If the matrix A_c has an eigenvalue $\lambda \in i\mathbb{R}$ with eigenvector $x_{c,\lambda} \neq 0$ such that $x_{c,\lambda} \in \ker C_c$, then the interconnected system cannot be asymptotically stable since*

$$(x, x_c)(t) := e^{\lambda t}(0, x_{c,\lambda})$$

defines a periodic classical solution of the interconnected problem

$$\begin{aligned} \dot{x}(t) &= \mathfrak{A}x(t), \\ \dot{x}_c(t) &= A_c x_c(t) + B_c u_c(t), \\ u_c(t) &= y(t) = \mathfrak{C}x(t), \\ \mathfrak{B}x(t) = u(t) &= -y_c(t) = -C_c x_c(t) - D_c u_c(t), \quad t \geq 0. \end{aligned}$$

Better spectral properties can be ensured taking A_c to be a Hurwitz matrix, i.e. its eigenvalues lie in the open left half plane. However, as it turns out, for asymptotic stability of the total system it is not only necessary that $\ker(i\beta - A_c) \cap \ker C_c = \{0\}$, but even enough for asymptotic stability, provided the system dissipates enough energy. This is the statement of the following result.

Proposition 5.2.2. *Let $\mathcal{A}$ be a linear dissipative hybrid operator resulting from standard feedback interconnection of a port-Hamiltonian system $\mathfrak{S} = (\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ and a finite dimensional linear control system $\Sigma_c = (A_c, B_c, C_c, D_c)$ as in Theorem 5.1.1 and assume that*

$$\ker(i\beta - A_c) \cap \ker C_c = \{0\}, \quad \beta \in \mathbb{R}.$$

If there is $R \in \mathcal{B}(D(\mathfrak{A}); H)$ (for some Hilbert space H) such that the pair $(\mathfrak{A}, R)$ has property ASP and such that for all $(x, x_c) \in D(\mathcal{A})$

$$\operatorname{Re} \langle \mathcal{A}(x, x_c), (x, x_c) \rangle_{X \times X_c} \leq \|Rx\|_H^2$$

then the C_0-semigroup $(\mathcal{T}(t))_{t \geq 0}$ generated by $\mathcal{A}$ is asymptotically stable.

Proof. We already know by Theorem 5.1.1 that $\mathcal{A}$ generates a contractive C_0-semigroup, $\mathcal{A}$ has compact resolvent and therefore $\sigma(\mathcal{A}) = \sigma_p(\mathcal{A})$. We would like to use Corollary 2.2.16 to the Arendt-Batty-Lyubich-Vũ Stability Theorem 2.2.14

and thus have to prove that $i\mathbb{R} \cap \sigma_p(\mathcal{A}) = \emptyset$. Let $\beta \in \mathbb{R}$ and $(x_0, x_{c,0}) \in D(\mathcal{A})$ be such that

$$i\beta(x_0, x_{c,0}) = \mathcal{A}(x_0, x_{c,0}).$$

The finite dimensional component then reads as

$$i\beta x_{c,0} = B_c \mathfrak{C} x_0 + A_c x_{c,0}.$$

Further for the infinite dimensional part we have that the estimate

$$\begin{aligned} 0 &= \operatorname{Re} \langle i\beta x_0, x_0 \rangle_X \\ &= \operatorname{Re} \langle \mathfrak{A} x_0, x_0 \rangle_X \\ &= \operatorname{Re} \langle \mathfrak{A} x_0, x_0 \rangle_X + \operatorname{Re} \langle i\beta x_{c,0}, x_{c,0} \rangle_{X_c} \\ &= \operatorname{Re} \langle \mathfrak{A} x_0, x_0 \rangle_X + \operatorname{Re} \langle A_c x_{c,0} + B_c \mathfrak{C} x_0, x_{c,0} \rangle_{X_c} \\ &= \operatorname{Re} \langle \mathcal{A}(x_0, x_{c,0}), (x_0, x_{c,0}) \rangle_{X \times X_c} \\ &\leq - \|R x_0\|_H^2 \end{aligned}$$

holds, i.e. $Rx_0 = 0$, and by property ASP of the pair $(\mathfrak{A}, R)$ it follows that $x_0 = 0$ is zero and then also

$$y_{c,0} = C_c x_{c,0} + D_c \mathfrak{C} x_0 = C_c x_{c,0} = -\mathfrak{B} x_0 = 0, \tag{5.5}$$

so that

$$x_{c,0} \in \ker(i\beta - A_c) \cap \ker C_c = \{0\}, \tag{5.6}$$

i.e. $i\beta \notin \sigma_p(A)$. As a result, $\sigma(\mathcal{A}) = \sigma_p(\mathcal{A}) \subseteq \mathbb{C}_0^-$ and the semigroup is asymptotically stable due to Corollary 2.2.16. □

Corollary 5.2.3. *Let $\mathfrak{S} = (\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ be a port-Hamiltonian system of order $N \in \mathbb{N}$ with Lipschitz continuous P_0 and $\mathcal{H}$ and let $\Sigma_c = (A_c, B_c, C_c, D_c)$ be a finite dimensional controller such that*

$$\ker(i\beta - A_c) \cap \ker C_c = \{0\}, \quad \beta \in \mathbb{R}.$$

If there is $\kappa > 0$ such that for all $(x, x_c) \in D(\mathcal{A})$ in the domain of the corresponding hybrid operator $\mathcal{A}$ the estimate

$$\operatorname{Re} \langle \mathcal{A}(x, x_c), (x, x_c) \rangle_{X \times X_c} \leq -\kappa \sum_{k=0}^{N-1} \left| (\mathcal{H}x)^{(k)}(0) \right|^2$$

holds, then the contractive C_0-semigroup $(\mathcal{T}(t))_{t \geq 0}$ generated by $\mathcal{A}$ is asymptotically (strongly) stable.

Proof. By Theorem 4.2.2 and Corollary 4.2.7 for $R : D(\mathfrak{A}) \to \mathbb{F}^{Nd}$,

$$Rx = \tau_0(\mathcal{H}x) = \begin{pmatrix} (\mathcal{H}x)(0) \\ \vdots \\ (\mathcal{H}x)^{(N-1)}(0) \end{pmatrix}$$

the pair $(\mathfrak{A}, R)$ has property ASP, see Theorem 4.2.2. Then the result follows from Proposition 5.2.2. □

Similarly we obtain uniform exponential stability if the pair $(\mathfrak{A}, R)$ also has property AIEP.

Theorem 5.2.4. *Assume that $\mathfrak{S} = (\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ is a port-Hamiltonian system which is interconnected with a finite dimensional linear system $\Sigma_c = (A_c, B_c, C_c, D_c)$. Assume that the resulting hybrid operator $\mathcal{A}$ generates an asymptotically stable C_0-semigroup. Further assume that there is $R \in \mathcal{B}(D(\mathfrak{A}); H)$ such that $(\mathfrak{A}, R)$ has property AIEP and such that*

$$\operatorname{Re} \langle \mathcal{A}(x, x_c), (x, x_c) \rangle_{X \times X_c} \leq - \|Rx\|_H^2, \quad (x, x_c) \in D(\mathcal{A}).$$

Then the C_0-semigroup $(\mathcal{T}(t))_{t \geq 0}$ generated by $\mathcal{A}$ is uniformly exponentially stable.

Proof. Since $\mathcal{A}$ generates an asymptotically stable C_0-semigroup $(\mathcal{T}(t))_{t \geq 0}$ and has compact resolvent, the spectrum

$$\sigma(\mathcal{A}) = \sigma_p(\mathcal{A}) \subseteq \mathbb{C}_0^-$$

lies in the open left half-plane. For the application of the Gearhart-Greiner-Prüss Theorem 2.2.17 we need to show that

$$\sup_{i\mathbb{R}} \|R(\cdot, \mathcal{A})\| < +\infty.$$

As in the static feedback case we use the equivalent sequence criterion instead. Let $((x_n, x_{c,n}, \beta_n))_{n \geq 1} \subseteq D(\mathcal{A}) \times \mathbb{R}$ be any sequence with $\sup_{n \in \mathbb{N}} \|(x_n, x_{c,n})\|_{X \times X_c} < +\infty$ and $|\beta_n| \xrightarrow{n \to \infty} +\infty$ such that

$$i\beta_n (x_n, x_{c,n}) - \mathcal{A}(x_n, x_{c,n}) \xrightarrow{n \to \infty} 0. \tag{5.7}$$

Then it follows that

$$\begin{aligned} \|Rx_n\|_H^2 &\leq - \operatorname{Re} \langle \mathcal{A}(x_n, x_{c,n}), (x_n, x_{c,n}) \rangle_{X \times X_c} \\ &= \operatorname{Re} \langle i\beta_n (x_n, x_{c,n}) - \mathcal{A}(x_n, x_{c,n}), (x_n, x_{c,n}) \rangle_{X \times X_c} \xrightarrow{n \to \infty} 0, \end{aligned}$$

so we deduce

$$Rx_n \xrightarrow{n \to \infty} 0.$$

Since

$$\mathfrak{A}x_n - i\beta_n x_n \xrightarrow{n \to \infty} 0,$$

property AIEP of the pair $(\mathfrak{A}, R)$ implies that

$$x_n \xrightarrow{n \to \infty} 0.$$

Let us now investigate the asymptotic behaviour of the sequence $(x_{c,n})_{n \geq 1} \subset X_c$. By choice of the sequence it holds

$$i\beta_n x_{c,n} - B_c \mathfrak{C} x_n - A_c x_{c,n} \xrightarrow{n \to \infty} 0,$$

and dividing by β_n, which is nonzero for sufficiently large n, we get

$$\frac{B_c \mathfrak{C} x_n}{\beta_n} + i x_{c,n} \xrightarrow{n \to \infty} 0.$$

From equation (5.7) we deduce that $\left\| \frac{\mathcal{A}(x_n, x_{c,n})}{\beta_n} \right\|_{X \times X_c}$ is bounded and further

$$\|\cdot\|_{\mathcal{A}} \simeq \|\cdot\|_{\mathcal{H}^{-1} H^N \times X_c}$$

by Lemma 3.2.3. Hence, $(\frac{\mathcal{H}x_n}{\beta_n})_{n\geq 1}$ is a bounded sequence in $H^N(0,1;\mathbb{F}^d)$ and a null sequence in $L_2(0,1;\mathbb{F}^d)$. By Lemma 2.1.19 it is therefore a null sequence in $C^{N-1}([0,1];\mathbb{F}^d)$. Since $B_c\mathfrak{C}x_n$ continuously depends on $\mathcal{H}x_n \in C^{N-1}([0,1];\mathbb{F}^d)$ this implies $x_{c,n} \to 0$, so that

$$(x_n, x_{c,n}) \xrightarrow{n\to\infty} 0, \qquad \text{in } X \times X_c.$$

Thus, the sequence criterion Corollary 2.2.19 says that the resolvents are uniformly bounded on $i\mathbb{R}$ and the C_0-semigroup $(\mathcal{T}(t))_{t\geq 0}$ is uniformly exponentially stable. □

Using the results of Chapter 4 we may now give sufficient conditions for exponential stability for the special cases $N = 1$ or $N = 2$.

Corollary 5.2.5. *Let $\mathcal{A}$ be a dissipative hybrid operator resulting from the standard feedback interconnection of a port-Hamiltonian system $\mathfrak{S} = (\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ of order $N = 1$, where $P_0, \mathcal{H}$ are Lipschitz continuous, with a finite dimensional linear controller $\Sigma_c = (A_c, B_c, C_c, D_c)$ such that*

$$\ker(i\beta - A_c) \cap \ker C_c = \{0\}, \quad \beta \in \mathbb{R}.$$

If there is some $\kappa > 0$ such that for all $(x, x_c) \in D(\mathcal{A})$

$$\operatorname{Re} \langle \mathcal{A}(x, x_c), (x, x_c)\rangle_{X\times X_c} \leq -\kappa \left|(\mathcal{H}x)(0)\right|^2,$$

then the semigroup $(\mathcal{T}(t))_{t\geq 0}$ generated by $\mathcal{A}$ is uniformly exponentially stable.

Proof. By the proof of Theorem 4.1.5 the pair $(\mathfrak{A}, \tau_0 \circ \mathcal{H})$ has property ESP, hence the result follows from Proposition 5.2.2 and Theorem 5.2.4. □

Corollary 5.2.6. *Let $\mathcal{A}$ be a dissipative hybrid operator resulting from the standard feedback interconnection of a port-Hamiltonian system $\mathfrak{S} = (\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ of order $N = 2$, where $P_0, \mathcal{H}$ are Lipschitz continuous, with a finite dimensional linear control system $\Sigma_c = (A_c, B_c, C_c, D_c)$ such that*

$$\ker(i\beta - A_c) \cap \ker C_c = \{0\}, \quad \beta \in \mathbb{R}.$$

If there exists some $\kappa > 0$ such that for all $(x, x_c) \in D(\mathcal{A})$

$$\begin{aligned}&\operatorname{Re} \langle \mathcal{A}(x, x_c), (x, x_c)\rangle_{X\times X_c} \\ &\leq -\kappa \left[\left|(\mathcal{H}x)(0)\right|^2 + \left|(\mathcal{H}x)'(0)\right|^2 + \left|\Pi(\mathcal{H}x)(1)\right|^2 + \left|(I-\Pi)P_2(\mathcal{H}x)'(1)\right|^2\right],\end{aligned}$$

where $\Pi : \mathbb{F}^d \to \mathbb{F}^d$ is an orthogonal projection, then the semigroup generated by $\mathcal{A}$ is uniformly exponentially stable.

Proof. We have already seen in Theorem 4.3.15 that for the choice

$$R : D(\mathfrak{A}) \to \mathbb{F}^{4d}, \quad x \mapsto \begin{pmatrix} (\mathcal{H}x)(0) \\ (\mathcal{H}x)'(0) \\ \Pi(\mathcal{H}x)(1) \\ (I-\Pi)P_2(\mathcal{H}x)'(1) \end{pmatrix}$$

the pair $(\mathfrak{A}, R)$ has property ESP. Then the result follows from Proposition 5.2.2 and Theorem 5.2.4. □

As a conclusion we may summarise the results of this subsection in the following way. For a fixed impedance passive port-Hamiltonian system $\mathfrak{S} = (\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ the stability properties of the system with generator $A = \mathfrak{A}|_{\ker(\mathfrak{B}+K\mathfrak{C})}$ for some matrix $K = K^* \geq 0$ are more or less the same as the stability properties of the system with generator $\mathfrak{A}$ resulting from standard feedback interconnection of $\mathfrak{S}$ with a finite dimensional, impedance passive linear control system $\Sigma_c = (A_c, B_c, C_c, D_c)$ if one takes $D_c = K$.

5.3 Strictly Input Passive Controllers

In the previous section we have seen that dissipativity conditions like

$$\operatorname{Re} \langle \mathcal{A}(x, x_c), (x, x_c) \rangle_{X \times X_c} \leq -\kappa \left| (\mathcal{H}x)(0) \right|^2 , \quad (x, x_c) \in D(\mathcal{A})$$

for the case $N = 1$ plus some reasonably weak conditions on the control system and regularity of the matrix-valued functions P_0 and $\mathcal{H}$ lead to uniform exponential stabilisation of the C_0-semigroup $(\mathcal{T}(t))_{t\geq 0}$ generated by the hybrid operator $\mathcal{A}$. The drawback, however, of these results is that we actually need strict dissipation in every component of $(\mathcal{H}x)(0)$ here, or, for the case that $N = 2$, even in every component of $(\mathcal{H}x)(0)$, $(\mathcal{H}x)'(0)$ and $\Pi(\mathcal{H}x)(1) + (I - \Pi)P_2(\mathcal{H}x)'(1)$. In practise this seems too restrictive and we therefore show in this section how using impedance passivity of the two systems $\mathfrak{S} = (\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ and $\Sigma_c = (A_c, B_c, C_c, D_c)$ (which we demand from now on) may help to weaken the dissipation conditions, still obtaining stability of the interconnected system. The investigation of the systems done in this subsection has been heavily influenced by the conference paper [RaZwLe13] for which we re-obtain and even generalise the main result using the frequency domain method by combining part of the proof of Theorem 14 therein [RaZwLe13] with our proof for first order port-Hamiltonian systems with static boundary feedback. Using the notions of pairs having properties ASP and AIEP the results there extend to second (or higher) order port-Hamiltonian systems with SIP controllers.

Within this subsection we assume that both the port-Hamiltonian system $\mathfrak{S} = (\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ and the finite dimensional linear control system $\Sigma_c = (A_c, B_c, C_c, D_c)$ are impedance passive, i.e.

$$\begin{aligned} \operatorname{Re} \langle \mathfrak{A}x, x \rangle_X &\leq \operatorname{Re} \langle \mathfrak{B}x, \mathfrak{C}x \rangle_{\mathbb{F}^{Nd}}, && x \in D(\mathfrak{A}) \\ \operatorname{Re} \langle A_c x_c + B_c u_c, x_c \rangle_{X_c} &\leq \operatorname{Re} \langle C_c x_c + D_c u_c, u_c \rangle_{\mathbb{F}^{Nd}}, && x_c \in X_c,\ u_c \in \mathbb{F}^{Nd}. \end{aligned}$$

As a result the interconnected system represented by the operator $\mathcal{A}$ associated to the standard feedback interconnection

$$\mathfrak{B}x = -y_c, \quad u_c = \mathfrak{C}x$$

is dissipative and thanks to Theorem 5.1.1 generates a contractive C_0-semigroup on the product Hilbert space $X \times X_c$. We decompose the full rank matrices $W_B, W_C \in \mathbb{F}^{d \times 2d}$ as

$$W_B = \begin{bmatrix} W_{B,1} \\ W_{B,2} \end{bmatrix}, \quad W_C = \begin{bmatrix} \tilde{W}_{C,1} \\ \tilde{W}_{C,2} \end{bmatrix}$$

where $W_{B,1}, W_{C,1} \in \mathbb{F}^{m\times 2d}$ for some $1 \leq m \leq d$, and accordingly also the input and the output maps split into two parts

$$\mathfrak{B}x = \begin{pmatrix} \mathfrak{B}_1 x \\ \mathfrak{B}_2 x \end{pmatrix} := \begin{bmatrix} W_{B,1} \\ W_{B,2} \end{bmatrix} \begin{pmatrix} f_{\partial,\mathcal{H}x} \\ e_{\partial,\mathcal{H}x} \end{pmatrix},$$
$$\mathfrak{C}x = \begin{pmatrix} \mathfrak{C}_1 x \\ \mathfrak{C}_2 x \end{pmatrix} := \begin{bmatrix} W_{C,1} \\ W_{C,2} \end{bmatrix} \begin{pmatrix} f_{\partial,\mathcal{H}x} \\ e_{\partial,\mathcal{H}x} \end{pmatrix}.$$

Identifying $\mathbb{F}^m$ with the m-dimensional subspace $\mathbb{F}^m \times \{0\}$ of $\mathbb{F}^{Nd}$ we may and will assume that $U_c = Y_c = \mathbb{F}^m \cong \mathbb{F}^m \times \{0\} \subseteq \mathbb{F}^{Nd}$ by assuming that $\{0\} \times \mathbb{F}^{Nd-m}$ lies in $\ker B_c \cap \ker D_c$ and $\operatorname{ran} C_c \cup \operatorname{ran} D_c$ lies in $\mathbb{F}^m \times \{0\}$. Then we assume that the finite dimensional linear control system $\Sigma_c = (A_c, B_c, C_c, D_c)$ is strictly impedance passive.

Definition 5.3.1. *A linear control system $\tilde{\Sigma} = (\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ is called* strictly input passive *(SIP) if $\tilde{U} = \tilde{Y}$ and there is $\sigma > 0$ that for all $x \in D(\tilde{A})$ and $u \in \tilde{U}$ the inequality*

$$\operatorname{Re}\langle \tilde{A}x + \tilde{B}u, x\rangle_{\tilde{X}} - \operatorname{Re}\langle \tilde{C}x + \tilde{D}u, u\rangle_{\tilde{U}} \leq -\sigma \|u\|^2_{\tilde{U}}$$

holds.

Remark 5.3.2 (Typical example for a SIP controller)**.** *Let us have a look on which systems $\tilde{\Sigma} = (\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ are SIP. First let us note that the SIP condition may be rewritten as*

$$\operatorname{Sym} \begin{bmatrix} \tilde{A} & \tilde{B} \\ -\tilde{C} & -\tilde{D} + \sigma I \end{bmatrix} \leq 0$$

being dissipative, so that necessarily $\tilde{A}$ and $\tilde{D} - \sigma I$ (for some $\sigma > 0$) are dissipative. In fact, under the additional assumption that $\tilde{C}' = \tilde{B}$, i.e. input and output are collocated, this is in fact equivalent to saying that $\tilde{\Sigma}$ is SIP. Note that in any case the matrix D_c is invertible and its symmetric part is positive definite.

Remark 5.3.3. *To rewrite the interconnected system in the way we have seen in the preceding subsection we should replace $B_c \in \mathbb{F}^{n\times m}$ by $\left[\begin{smallmatrix} B_c & 0 \end{smallmatrix}\right]$ where $0 \in \mathbb{F}^{n\times(Nd-m)}$, $C_c \in \mathbb{F}^{m\times n}$ by $\left[\begin{smallmatrix} C_c \\ 0 \end{smallmatrix}\right]$ where $0 \in \mathbb{F}^{(Nd-m)\times n}$ and $D_c \in \mathbb{F}^{m\times m}$ by $\left[\begin{smallmatrix} D_c & \\ & 0 \end{smallmatrix}\right] \in \mathbb{F}^{Nd\times Nd}$.*

Theorem 5.3.4. *Let $\mathfrak{S} = (\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ be an impedance passive port-Hamiltonian system, interconnected as in Theorem 5.1.7 with an impedance passive linear SIP controller $\Sigma_c = (A_c, B_c, C_c, D_c)$, i.e.*

$$\mathfrak{B}_1 x = -y_c, \ u_c = \mathfrak{C}_1 x \text{ and } \mathfrak{B}_2 x = 0.$$

Assume that A_c is a Hurwitz matrix, i.e. $\sigma(A_c) \subseteq \mathbb{C}_0^-$, and let the inequality

$$|\mathfrak{B}x|^2 + |\mathfrak{C}_1 x|^2 \geq \|Rx\|^2_H, \quad x \in D(\mathfrak{A})$$

hold for some $\kappa > 0$ which is independent of $x \in D(\mathfrak{A})$ and $R \in \mathcal{B}(D(\mathfrak{A}); H)$ (for some Hilbert space H).

1. *If the pair $(\mathfrak{A}, R)$ has property ASP, then the finite dimensional controller asymptotically stabilises the port-Hamiltonian system, i.e. the C_0-semigroup $(\mathcal{T}(t))_{t\geq 0}$ is asymptotically stable.*

2. *If the C_0-semigroup is asymptotically stable and the pair $(\mathfrak{A}, R)$ has property AIEP, then the finite dimensional controller uniformly exponentially stabilises the port-Hamiltonian system, i.e. the C_0-semigroup $(\mathcal{T}(t))_{t\geq 0}$ is uniformly exponentially stable.*

Proof. We already know that $\mathcal{A}$ generates a contractive C_0-semigroup. By the compact embedding $D(\mathcal{A}) \hookrightarrow X \times X_c$ we have $\sigma(\mathcal{A}) = \sigma_p(\mathcal{A})$. Let us first prove that $\mathcal{A}$ has no eigenvalues on the imaginary axis, provided that the pair $(\mathfrak{A}, R)$ has property ASP. Before note that for every $(x, x_c) \in D(\mathcal{A})$ we get

$$\begin{aligned}\operatorname{Re}\langle \mathcal{A}(x, x_c), (x, x_c)\rangle_{X\times X_c} &= \operatorname{Re}\langle \mathfrak{A}x, x\rangle_{\mathcal{H}} + \operatorname{Re}\langle A_c x_c + B_c\mathfrak{C}_1 x, x_c\rangle_{X_c} \\ &\leq \operatorname{Re}\langle \mathfrak{B}_1 x, \mathfrak{C}_1 x\rangle_{\mathbb{F}^m} + \operatorname{Re}\langle \mathfrak{C}_1 x, -\mathfrak{B}_1 x\rangle_{\mathbb{F}^m} - \sigma \left|\mathfrak{C}_1 x\right|^2 \\ &= -\sigma \left|\mathfrak{C}_1 x\right|^2 .\end{aligned}$$

Let $\beta \in \mathbb{R}$ and $(x, x_c) \in D(\mathcal{A})$ with

$$\mathcal{A}(x, x_c) = i\beta(x, x_c)$$

be arbitrary. Then

$$0 = \operatorname{Re}\langle i\beta(x, x_c), (x, x_c)\rangle_{X\times X_c} = \operatorname{Re}\langle \mathcal{A}(x, x_c), (x, x_c)\rangle_{X\times X_c} \leq -\sigma \left|\mathfrak{C}_1 x\right|^2$$

and hence $\mathfrak{C}_1 x = 0$. From the equation

$$B_c\mathfrak{C}_1 x + A_c x_c = i\beta x_c$$

and the Hurwitz property of A_c we then deduce that $x_c = R(i\beta, A_c)B_c\mathfrak{C}_1 x = 0$ and this also implies $\mathfrak{B}_1 x = 0$, so that (since $\mathfrak{B}_2 x = 0$ from the boundary conditions) $\mathfrak{B}x = 0$. Therefore,

$$\left\|Rx\right\|^2 \leq \left|\mathfrak{B}x\right|^2 + \left|\mathfrak{C}_1 x\right|^2 = 0$$

and x solves the eigenvalue value problem

$$\mathfrak{A}x = i\beta x.$$

Since the pair $(\mathfrak{A}, R)$ has property ASP this can only be true if $x = 0$, so that $(x, x_c) = 0$ must be the zero element in $X \times X_c$ and $i\beta \notin \sigma_p(\mathcal{A}) = \sigma(\mathcal{A})$.

Next we assume that the pair $(\mathfrak{A}, R)$ has property AIEP and take an arbitrary sequence $((x_n, x_{c,n}), \beta_n)_{n\geq 1} \subseteq D(\mathcal{A}) \times \mathbb{R}$ with $\sup_{n\in\mathbb{N}} \left\|(x_n, x_{c,n})\right\|_{X\times X_c} < +\infty$ and $|\beta_n| \to \infty$ such that

$$\mathcal{A}(x_n, x_{c,n}) - i\beta_n(x_n, x_{c,n}) \xrightarrow{n\to\infty} 0 \quad \text{in } X \times X_c.$$

We then especially have

$$0 \leftarrow \langle(\mathcal{A} - i\beta_n)(x_n, x_{c,n}), (x_n, x_{c,n})\rangle_{X\times X_c} \leq -\sigma \left|\mathfrak{C}_1 x_n\right|^2$$

thus $\mathfrak{C}_1 x_n \xrightarrow{n\to+\infty} 0$ converges to zero. Also we have

$$B_c\mathfrak{C}_1 x_n + A_c x_{c,n} - i\beta_n x_{c,n} =: z_n \xrightarrow{n\to\infty} 0$$

and since $\sup_{i\mathbb{R}} \|R(\cdot, A_c)\| < +\infty$ (which in this case is mainly due to the fact that A_c acts on a finite dimensional space) we obtain

$$x_{c,n} = R(i\beta_n, A_c)(B_c\mathfrak{C}_1 x_n - z_n) \xrightarrow{n\to\infty} 0$$

and then also

$$-\mathfrak{B}_1 x_n = C_c x_{c,n} + D_c \mathfrak{C}_1 x_n \xrightarrow{n\to\infty} 0.$$

From our assumption and the boundary condition $\mathfrak{B}_2 x_n = 0$ we thus obtain that $\|Rx_n\| \to 0$ and since the pair $(\mathfrak{A}, R)$ has property AIEP and $\mathfrak{A}x_n - i\beta_n x_n \xrightarrow{n\to+\infty} 0$ converges to zero we obtain that also $\|x_n\|_X \xrightarrow{n\to\infty} 0$ converges to zero which means that

$$\|(x_n, x_{c,n})\|_{X\times X_c} \xrightarrow{n\to\infty} 0.$$

From the sequence criterion Corollary 2.2.19 the C_0-semigroup $(\mathcal{T}(t))_{t\geq 0}$ is uniformly exponentially stable, if it is asymptotically stable. □

In particular, we hereby proved Theorem 14 of [RaZwLe13].

Theorem 5.3.5. *Let $\mathcal{A}$ be a hybrid operator operator resulting from the standard feedback interconnection of an impedance passive port-Hamiltonian system $\mathfrak{S} = (\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ of order $N = 1$ with a SIP finite dimensional linear control system $\Sigma_c = (A_c, B_c, C_c, D_c)$. Further assume that A_c is Hurwitz, $\mathcal{H}$ and P_0 are Lipschitz continuous and there is $\kappa > 0$ such that for all $x \in D(\mathfrak{A})$*

$$|\mathfrak{B}x|^2 + |\mathfrak{C}_1 x|^2 \geq \kappa\, |(\mathcal{H}x)(0)|^2, \quad x \in D(\mathfrak{A}).$$

Then the C_0-semigroup $(\mathcal{T}(t))_{t\geq 0}$ generated by $\mathcal{A}$ is uniformly exponentially stable, i.e. the finite dimensional controller uniformly exponentially stabilises the port-Hamiltonian system.

Proof. Note that the pair $(\mathfrak{A}, R)$ has properties ASP and AIEP by (the proof of) Theorem 4.3.1. Then the assertion follows from Theorem 5.3.4. □

Of course similar results hold for second (or higher) order systems. In fact, one only has to take care that the additional assumption guarantees the right boundary values of $\mathcal{H}x$ and its derivatives to converge to zero (resp. be zero for the eigenvalue problem on $i\mathbb{R}$), e.g. in the second order case.

Corollary 5.3.6. *Let $\mathcal{A}$ be a hybrid operator operator resulting from the standard feedback interconnection of an impedance passive port-Hamiltonian system $\mathfrak{S} = (\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ of order $N = 2$ with a SIP finite dimensional linear control system $\Sigma_c = (A_c, B_c, C_c, D_c)$. Further assume that A_c is Hurwitz, $\mathcal{H}$ and P_0 are Lipschitz continuous and there is $\kappa > 0$ such that for all $x \in D(\mathfrak{A})$*

$$\begin{aligned}|\mathfrak{B}x|^2 + |\mathfrak{C}_1 x|^2 \geq \kappa \Big(&|(\mathcal{H}x)(0)|^2 + |(\mathcal{H}x)'(0)|^2 \\ &+ |\Pi(\mathcal{H}x)(1)|^2 + |(I-\Pi)P_2(\mathcal{H}x)'(1)|^2 \Big)\end{aligned}$$

where $\Pi : \mathbb{F}^d \to \mathbb{F}^d$ is an orthogonal projection. Then the C_0-semigroup $(\mathcal{T}(t))_{t\geq 0}$ generated by $\mathcal{A}$ is uniformly exponentially stable, i.e. the finite dimensional controller uniformly exponentially stabilises the port-Hamiltonian system.

So any time the frequency domain method works for the pure infinite-dimensional port-Hamiltonian system one also gets a result for the correspondent interconnected system. One particular case is the following exponential stability result on Euler-Bernoulli beam-like equations.

Corollary 5.3.7. *Let $d \in 2\mathbb{N}$ and $(\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ be an impedance passive port-Hamiltonian system of order $N = 2$ where $P_2 \in \mathbb{F}^{d\times d}$ has anti-block diagonal structure, i.e.*

$$P_2 = \begin{bmatrix} 0 & M \\ -M^* & 0 \end{bmatrix}.$$

Also $\mathcal{H} = \left[\begin{smallmatrix} \mathcal{H}_1 & \\ & \mathcal{H}_2 \end{smallmatrix}\right]$ is assumed to be a block diagonal (uniformly positive definite) matrix-valued function with Lipschitz continuous $\mathcal{H}_1, \mathcal{H}_2$ and P_0. If the system is interconnected with a finite dimensional linear SIP control system $\Sigma_c = (A_c, B_c, C_c, D_c)$ and there is $\kappa > 0$ such that

$$\begin{aligned} |\mathfrak{B}x|^2 + |\mathfrak{C}_1 x|^2 \geq \kappa \Big(&|(\mathcal{H}x)(0)|^2 + |(\mathcal{H}x)'(0)|^2 \\ &+ |\Pi(\mathcal{H}_1 x_1)(1)|^2 + |(I - \Pi)M(\mathcal{H}_1 x_1)'(1)|^2 \Big) \end{aligned}$$

for some orthogonal projection $\Pi : \mathbb{F}^{d/2} \to \mathbb{F}^{d/2}$ and all $x \in D(\mathfrak{A})$, then the interconnected system is uniformly exponentially stable, i.e. the C_0-semigroup $(\mathcal{T}(t))_{t\geq 0}$ generated by the hybrid operator $\mathcal{A}$ is uniformly exponentially stable.

For the case $N = 1$ it is also possible to use the controllability inequality

$$\|x(\tau)\|_X^2 \leq c \int_0^\tau |(\mathcal{H}x)(t, 1)|^2 \, dt$$

to deduce exponential stability of a impedance passive port-Hamiltonian system connected with an impedance passive and internally exponentially stable linear system, namely using the following result.

Proposition 5.3.8. *Let $\mathfrak{S} = (\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ be an impedance passive port-Hamiltonian system and $\Sigma_c = (A_c, B_c, C_c, D_c)$ be a finite dimensional strictly input passive control system which is internally exponentially stable, i.e. $\sigma(A_c) \subseteq \mathbb{C}_0^-$, and let them be interconnected via the interconnection law*

$$\mathfrak{B}_1 x = -y_c, \quad u_c = \mathfrak{C}_1 x, \quad \mathfrak{B}_2 x = 0.$$

If for every $c > 0$ there is $\tau > 0$ such that for every solution $(x, x_c) = \mathcal{T}(\cdot)(x_0, x_{c,0}) \in C_b^1(\mathbb{R}_+; X \times X_c) \cap C_b(\mathbb{R}_+; D(\mathcal{A}))$ of $\frac{d}{dt}(x, x_c) = \mathcal{A}(x, x_c)$ the estimate

$$\|x(\tau)\|^2 \leq c \int_0^\tau |\mathfrak{B}x(t)|^2 + |\mathfrak{C}_1 x(t)|^2 \, dt \tag{5.8}$$

holds, then $\mathcal{A}$ generates a uniformly exponentially stable contraction semigroup $(\mathcal{T}(t))_{t\geq 0}$ on $X \times X_c$.

Proof. Let $(x_0, x_{c,0}) \in D(\mathcal{A})$ be arbitrary and $x = \mathcal{T}(\cdot)(x_0, x_{c,0}) \in C_b^1(\mathbb{R}_+; X \times$

$X_c) \cap C_b(\mathbb{R}_+; D(\mathfrak{A}) \times X_c)$ be the classical solution of

$$\begin{aligned}
\frac{d}{dt}x(t) &= \mathfrak{A}x(t) \\
\frac{d}{dt}x_c(t) &= A_c x(t) + B_c \mathfrak{C}_1 x(t) \\
\mathfrak{B}_1 x(t) &= -(C_c x(t) + D_c \mathfrak{C}_1 x(t)) \\
\mathfrak{B}_2 x(t) &= 0, \quad t \geq 0 \\
(x, x_c)(0) &= (x_0, x_{c,0}).
\end{aligned}$$

Then for every fixed $c > 0$ and admissible $\tau > 0$ from equation (5.8) we have

$$\begin{aligned}
&\frac{1}{2}\left(\|(x, x_c)(t)\|^2_{X \times X_c} - \|(x_0, x_{c,0})\|^2_{X \times X_c}\right) \\
&= \int_0^\tau \operatorname{Re}\langle \mathcal{A}(x, x_c)(s), (x, x_c)(s)\rangle_{X \times X_c} ds \\
&\leq -\sigma \int_0^\tau |\mathfrak{C}_1 x(s)|^2 \, ds \\
&= -\sigma_1 \int_0^\tau |\mathfrak{C}_1 x(s)|^2 + |\mathfrak{B}x(s)|^2 \, ds - \sigma_2 \int_0^\tau |\mathfrak{C}_1 x(s)|^2 \, ds \\
&\quad + \sigma_1 \int_0^\tau |\mathfrak{B}x(s)|^2 - \frac{\sigma_3}{\sigma_1} |\mathfrak{C}_1 x(s)|^2 \, ds - \sigma_4 \int_0^\tau |\mathfrak{C}_1 x(s)|^2 \, ds
\end{aligned} \tag{5.9}$$

where $\sigma > 0$ denotes the SIP-constant in the inequality

$$\operatorname{Re}\langle A_c x_c + B_c u_c, x_c\rangle_{X_c} \leq \operatorname{Re}\langle C_c x_c + D_c u_c, u_c\rangle_{\mathbb{F}^{Nd}} - \sigma |u_c|^2$$

and $\sigma_i > 0$ $(i = 1, \ldots, 4)$ are positive constants which sum up to σ and which we chose suitable at a later point. We then have the following estimates.

$$-\sigma_1 \int_0^\tau |\mathfrak{C}_1 x(s)|^2 + |\mathfrak{B}x(s)|^2 \, ds \leq -\frac{\sigma_1}{c} \|x(\tau)\|^2_X$$

where we used inequality (5.8),

$$\begin{aligned}
-\sigma_2 \int_0^\tau |\mathfrak{C}_1 x(s)|^2 \, ds &=: -\alpha \frac{M^2 \|B_c\|^2}{|\omega|} \int_0^\tau |\mathfrak{C}_1 x(s)|^2 \, ds \\
&\leq -\frac{\alpha M^2 \|B_c\|^2}{|\omega|} \left[e^{2\omega(\tau - s)}\right]^\tau_{s=0} \int_0^\tau |u_c(s)|^2 \, ds \\
&\leq -2\alpha \int_0^\tau (M e^{\omega(\tau - s)})^2 ds \, \|B_c\|^2 \int_0^\tau |u_c(s)|^2 \, ds \\
&\leq -2\alpha \int_0^\tau \left\|e^{(\tau - s)A_c}\right\|^2 ds \, \|B_c\|^2 \int_0^\tau |u_c(s)|^2 \, ds \\
&\leq -2\alpha \left(\left\|\int_0^\tau e^{(\tau - s)A_c} B_c u_c(s) ds\right\|^2\right)^2 \\
&= -2\alpha \left\|x_c(\tau) - e^{\tau A_c} x_{c,0}\right\|^2_{X_c} \\
&\leq -\alpha \left(\|x_c(\tau)\|^2_{X_c} - 2\left\|e^{\tau A_c} x_{c,0}\right\|^2_{X_c}\right) \\
&\leq 2\alpha M^2 e^{2\omega\tau} \|x_{c,0}\|^2_{X_c} - \alpha \|x_c(\tau)\|^2_{X_c}
\end{aligned}$$

where we assumed that $B_c \neq 0$ for the moment, chose $(M,\omega) \in [1,\infty) \times (-\infty,0)$ such that $\left\|e^{tA_c}\right\| \leq Me^{\omega t}$ $(t \geq 0)$ and used Duhamel's formula. Using that

$$(a+b+c)^2 \leq 3(a^2+b^2+c^2), \quad a,b,c \geq 0$$

we also find that

$$\begin{aligned}
\int_0^\tau |\mathfrak{B}x(s)|^2\, ds &= \int_0^\tau \left| C_c e^{sA_c} x_{c,0} + \int_0^s C_c e^{(s-r)A_c} B_c \mathfrak{C}_1 x(r) dr + D_c \mathfrak{C}_1 x(s) \right|^2 ds \\
&\leq 3 \int_0^\tau \Big[\|C_c\|^2 M^2 e^{2\omega s} \|x_{c,0}\|_{X_c}^2 \\
&\quad + \|C_c\|^2 \|B_c\|^2 \int_0^s \left\| e^{(s-r)A_c} \right\|^2 dr \int_0^\tau |\mathfrak{C}_1 x(r)|^2 \, dr \\
&\quad + \|D_c\|^2 |\mathfrak{C}_1 x(s)|^2 \Big] \, ds \\
&\leq \frac{3M^2 \|C_c\|^2}{2|\omega|} \|x_{c,0}\|_{X_c}^2 \\
&\quad + 3 \left(\frac{\tau M^2}{2|\omega|} \|B_c\|^2 \|C_c\|^2 + \|D_c\|^2 \right) \int_0^\tau |\mathfrak{C}_1 x(s)|^2 \, ds.
\end{aligned}$$

We therefore find with equation (5.9) that

$$\begin{aligned}
&\frac{1}{2} \left(\|(x,x_c)(\tau)\|_{X\times X_c}^2 - \|(x_0,x_{c,0})\|_{X\times X_c}^2 \right) \\
&\leq -\frac{\sigma_1}{c} \|x(\tau)\|_X^2 + 2\alpha M^2 e^{2\omega\tau} \|x_{c,0}\|_{X_c}^2 - \alpha \|x_c(\tau)\|_{X_c}^2 \\
&\quad + \sigma_3 \frac{3M^2 \|C_c\|^2}{2|\omega|} \|x_{c,0}\|_{X_c}^2 \\
&\quad + 3\sigma_3 \left(\frac{\tau M^2}{2|\omega|} \|B_c\|^2 \|C_c\|^2 + \|D_c\|^2 \right) \int_0^\tau |\mathfrak{C}_1 x(s)|^2 \, ds \\
&\quad - \sigma_4 \int_0^\tau |\mathfrak{C}_1 x(s)|^2 \, ds.
\end{aligned}$$

Now we chose $c > 0$ (small enough), $\tau > 0$ (large enough) and the constants $\sigma_i > 0$ such that

$$\sigma_4 = 3\sigma_3 \left(\frac{\tau M^2}{2|\omega|} \|B_c\|^2 \|C_c\|^2 + \|D_c\|^2 \right)$$

$$2\alpha M^2 e^{2\omega\tau} + \sigma_3 \frac{3M^2 \|C_c\|^2}{2|\omega|} < \min\left\{ \alpha, \frac{\sigma}{c} \right\}.$$

Note that for any fixed $c > 0$ the constant $\tau > 0$ may be chosen larger if we we wish. Also all the chosen constants do not depend on $(x_0, x_{c,0}) \in D(\mathcal{A})$. Then there is $\varepsilon > 0$ such that

$$\frac{1}{2} \left(\|(x,x_c)(\tau)\|_{X\times X_c}^2 - \|(x_0,x_{c,0})\|_{X\times X_c}^2 \right) \leq -\varepsilon \|(x,x_c)(\tau)\|_{X\times X_c}^2$$

and therefore

$$\|\mathcal{T}(\tau)(x_0,x_{c,0})\|_{X\times X_c} \leq \frac{1}{\sqrt{1+2\varepsilon}} \|(x_0,x_{c,0})\|_{X\times X_c} =: \rho \|(x_0,x_{c,0})\|_{X\times X_c}$$

and since this estimate holds for all $(x_0, x_{c,0})$ in the dense subset $D(\mathcal{A})$ of $X \times X_c$ we find

$$\|\mathcal{T}(t)\| \leq \rho \in (0,1)$$

and uniform exponential stability follows from Remark 2.2.12. The case $B_c = 0$ can be handled quite similar and is actually easier. We leave the details to the interested reader. □

In the original Lemma 4.1.1 there is a contractivity condition on the solution $x \in W^1_\infty(\mathbb{R}_+; X) \times L_\infty(\mathbb{R}_+; D(\mathfrak{A})$, however one may overcome this obstacle using the following lemma.

Lemma 5.3.9. *Let $\mathfrak{S} = (\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ be an impedance passive port-Hamiltonian system with*

$$\mathfrak{B}x = \begin{pmatrix} \mathfrak{B}_1 x \\ \mathfrak{B}_2 x \end{pmatrix}, \quad \mathfrak{C}x = \begin{pmatrix} \mathfrak{C}_1 x \\ \mathfrak{C}_2 x \end{pmatrix}.$$

Assume that there are constants $\tau > 0$ and $c > 0$ such that for every solution $x \in W^1_\infty(\mathbb{R}_+; X) \cap L_\infty(\mathbb{R}_+; D(\mathfrak{A}))$ the estimate

$$\int_0^\tau \|x(t)\|_X^2 \, dt \leq c \left(\|\mathfrak{B}x(t)\|^2_{L_2(0,\tau;U)} + \|\mathfrak{C}_1 x(t)\|^2_{L_2(0,\tau;Y)} \right)$$

then for the constant $c' = \frac{2c+1}{2\tau} > 0$ such that for every solution $x \in W^1_\infty(\mathbb{R}_+; X) \cap L_\infty(\mathbb{R}_+; D(\mathfrak{A}))$ of $\dot{x} = \mathfrak{A}x$ with $\mathfrak{B}_2 x(t) = 0$ for a.e. $t \geq 0$ the following holds

$$\|x(\tau)\|_X^2 \leq c' \left(\|\mathfrak{B}x(t)\|^2_{L_2(0,\tau;U)} + \|\mathfrak{C}_1 x(t)\|^2_{L_2(0,\tau;Y)} \right).$$

Proof. Take any solution $x \in W^1_\infty(\mathbb{R}_+; X) \cap L_\infty(\mathbb{R}_+; D(\mathfrak{A}))$ of $\dot{x} = \mathfrak{A}x$ such that $\mathfrak{B}_2 x = 0$. Then

$$\begin{aligned}
\|x\|^2_{L_2(0,\tau;X)} &= \int_0^\tau \|x(\tau)\|_X^2 - \left(\|x(\tau)\|_X^2 - \|x(t)\|_X^2 \right) dt \\
&\geq \tau \|x(\tau)\|_X^2 - \langle \mathfrak{B}x, \mathfrak{C}x \rangle_{L_2(0,\tau;U)} \\
&\geq \tau \|x(\tau)\|_X^2 - \langle \mathfrak{B}_1 x, \mathfrak{C}_1 x \rangle_{L_2(0,\tau;U)} \\
&\geq \tau \|x(\tau)\|_X^2 - \frac{1}{2} \left(\|\mathfrak{B}x\|^2_{L_2(0,\tau;U)} + \|\mathfrak{C}_1 x\|^2_{L_2(0,\tau;U)} \right)
\end{aligned}$$

and hence the assertion follows. □

It is also possible to obtain uniformly exponential stability through the Lyapunov method for the Lyapunov function $\Phi(t) = t\|x(t)\| + q(x(t)) + \alpha \int_t^{t+t_0} \|x_c(s)\|^2 \, ds$ where q comes from the exponential stability proof of the static stability theorem and $\alpha, t_0 > 0$ are suitable constants.

Proposition 5.3.10. *Let an impedance passive port-Hamiltonian system $\mathfrak{S} = (\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ be interconnected with an impedance passive and internally exponentially stable linear control system $\Sigma_c = (A_c, B_c, C_c, D_c)$ and assume that*

$$\operatorname{Re} \langle \mathcal{A}(x, x_c), (x, x_c) \rangle_{X \times X_c} \leq -\sigma |\mathfrak{C}_1 x|^2, \quad t \geq 0.$$

Further assume that there is $q : X \to \mathbb{R}$ such that $|q(x)| \leq c\|x\|_X^2$ and for every solution $x \in C^1_b(\mathbb{R}_+; X) \cap C_b(\mathbb{R}_+; D(\mathfrak{A}))$ one has $q(x) \in W^1_\infty(\mathbb{R}_+; X)$ with

$$\frac{d}{dt} q(x(t)) \leq -\frac{1}{2} \|x(t)\|_X^2 + c \left(|\mathfrak{B}x(t)|^2 + |\mathfrak{C}_1 x(t)|^2 \right), \quad a.e. \ t \geq 0.$$

Then there are constants $M \geq 1$ and $\omega < 0$ such that for every $(x_0, x_{c,0}) \in X \times X_c$ and the corresponding solution $(x, x_c) = \mathcal{T}(\cdot)(x_0, x_{c,0}) \in C^1(\mathbb{R}_+; X) \cap C(\mathbb{R}_+; D(\mathfrak{A}))$ the following uniform exponential energy decay holds true.

$$\|(x, x_c)(t)\|_{X \times X_c} \leq M e^{\omega t} \|(x_0, x_{c,0})\|_{X \times X_c}, \quad t \geq 0.$$

For the proof we employ the following lemma on internally stable linear systems.

Lemma 5.3.11. *Let $\tilde{\Sigma} = (\tilde{A}, \tilde{B})$ be a linear system with bounded input operator $\tilde{B} \in \mathcal{B}(\tilde{U}, \tilde{X})$ and $\tilde{A}$ be the generator of a uniformly exponentially stable C_0-semigroup $(\tilde{T}(t))_{t \geq 0}$ on $\tilde{X}$. Then there are constants $t_0, \delta, c > 0$ such that for every initial value $\tilde{x}_0 \in \tilde{X}$ and input function $\tilde{u} \in L_{2,loc}(\mathbb{R}_+; \tilde{U})$ and the corresponding mild solution $\tilde{x} \in C(\mathbb{R}_+; \tilde{X})$ given by Duhamel's formula*

$$\tilde{x}(t) = \tilde{T}(t)\tilde{x}_0 + \int_0^t T(t-s)\tilde{B}\tilde{u}(s)ds, \quad t \geq 0$$

the estimate

$$\frac{d}{dt} \int_t^{t+t_0} \|\tilde{x}(s)\|_{\tilde{X}}^2 \, ds \leq -\delta \|\tilde{x}(t)\|_{\tilde{X}}^2 + c \int_t^{t+t_0} \|\tilde{u}(s)\|_{\tilde{U}}^2 \, ds, \quad t \geq 0$$

is valid.

Proof. Since $(\tilde{T}(t))_{t \geq 0}$ is uniformly exponentially stable there are constants $M \geq 1$ and $\omega < 0$ such that

$$\left\| \tilde{T}(t) \right\| \leq M e^{\omega t}, \quad t \geq 0.$$

We then calculate for every such solution as in the lemma that

$$\begin{aligned}
\frac{d}{dt} \int_t^{t+t_0} \|\tilde{x}(s)\|_{\tilde{X}}^2 \, ds &= \|\tilde{x}(t+t_0)\|_{\tilde{X}}^2 - \|\tilde{x}(t)\|_{\tilde{X}}^2 \\
&= \left\| \tilde{T}(t_0)\tilde{x}(t) + \int_t^{t+t_0} \tilde{T}(t+t_0-s)\tilde{B}\tilde{u}(s)ds \right\|_{\tilde{X}}^2 - \|\tilde{x}(t)\|_{\tilde{X}}^2 \\
&\leq \left(2 \left\| \tilde{T}(t_0) \right\|^2 - 1 \right) \|x(t)\|_{\tilde{X}}^2 \\
&\quad + 2 \left\| \int_t^{t+t_0} \tilde{T}(t+t_0-s)\tilde{B}\tilde{u}(s)ds \right\|_{\tilde{X}}^2 \\
&\leq (2M^2 e^{2\omega t_0} - 1) \|x(t)\|_{\tilde{X}}^2 + 2t_0 M^2 \left\| \tilde{B} \right\|^2 \int_t^{t+t_0} \|\tilde{u}(s)\|_{\tilde{U}}^2 \, ds
\end{aligned}$$

and the result follows by choosing $t_0 > 0$ such that

$$\delta := 1 - 2M^2 e^{2\omega t_0} > 0, \quad c := 2t_0 M^2 \left\| \tilde{B} \right\|^2 > 0.$$

□

Proof of the Proposition. Take any arbitrary $(x_0, x_{c,0}) \in D(\mathcal{A})$ and for some $\alpha > 0$ define the continuously differentiable functional $\Phi : \mathbb{R}_+ \to \mathbb{R}$ by

$$\Phi(t) := t \|(x, x_c)(t)\|_{X \times X_c}^2 + q(x(t)) + \alpha \int_t^{t+t_0} \|x_c(s)\|_{X_c}^2 \, ds, \quad t \geq 0$$

where $(x, x_c)(\cdot) := \mathcal{T}(\cdot)(x_0, x_{c,0}) \in C^1(\mathbb{R}_+; X \times X_c) \cap C(\mathbb{R}_+; D(\mathcal{A}))$ is the classical solution of the interconnected system for the initial value $(x_0, x_{c,0})$. We use the following estimates

$$\begin{aligned}
\frac{d}{dt} q(x(t)) &\leq - \|x(t)\|_X^2 + c_1 \left(|\mathfrak{B}x(t)|^2 + |\Pi \mathfrak{C} x(t)|^2 \right), \\
\frac{d}{dt} \int_t^{t+t_0} \|x_c(s)\|^2 \, ds &\leq -\delta \|x_c(t)\|_{X_c}^2 + c_2 \int_t^{t+t_0} |\Pi \mathfrak{C} x(s)|^2 \, ds \\
&\leq -\delta \|x_c(t)\|_{X_c}^2 - \frac{c_2}{\sigma} \operatorname{Re} \langle \mathcal{A}(x, x_c), (x, x_c) \rangle_{L_2(t,t_0;X\times X_c)} ds \\
&= -\delta \|x_c(t)\|_{X_c}^2 + \frac{c_2}{2\sigma} \left(\|x_c(t)\|_{X_c}^2 - \|x_c(t+t_0)\|_{X_c}^2 \right) \\
|\mathfrak{B}x(t)|^2 &\leq 2 \|C_c\|^2 \|x_c(t)\|_{X_c}^2 + 2 \|D_c\|^2 |\mathfrak{C}x(t)|^2 \\
&= c_3 \|x_c(t)\|_{X_c}^2 + c_4 |\mathfrak{C}x(t)|^2
\end{aligned}$$

and find that for every $t \geq 0$ the following estimates are valid.

$$\begin{aligned}
\frac{d}{dt} \Phi(t) &\leq \|(x, x_c)(t)\|_{X\times X_c}^2 + 2t \operatorname{Re} \langle \mathcal{A}(x, x_c)(t), (x, x_c)(t) \rangle_{X\times X_c} \\
&\quad - \|x(t)\|_X^2 + c_1 \left(|\mathfrak{B}x(t)|^2 + |\Pi \mathfrak{C}x(t)|^2 \right) \\
&\quad - \alpha\delta \|x_c(t)\|_{X_c}^2 + \frac{\alpha c_2}{2\sigma} \left(\|(x, x_c)(t)\|_{X_c}^2 - \|(x, x_c)(t+t_0)\|_{X\times X_c}^2 \right) \\
&\leq (1 + c_1 c_3 - \alpha\delta) \|x_c(t)\|_{X_c}^2 + (c_1 c_4 - 2t\sigma) |\Pi \mathfrak{C}x(t)|^2 \\
&\quad + \frac{\alpha c_2}{2\sigma} \left(\|(x, x_c)(t)\|_{X_c}^2 - \|(x, x_c)(t+t_0)\|_{X\times X_c}^2 \right)
\end{aligned}$$

and then choosing

$$\alpha = \frac{1 + c_1 c_3}{\delta} > 0, \quad \tau = \frac{c_1 c_4}{2\sigma} > 0$$

we find for $t \geq \tau$

$$\begin{aligned}
\Phi(t) - \Phi(\tau) &\leq \int_\tau^t (c_1 c_4 - 2s\sigma) |\Pi \mathfrak{C}x(s)|^2 \, ds \\
&\quad + \frac{\alpha c_2}{2\sigma} \left(\|(x, x_c)(\tau)\|_{X\times X_c}^2 - \|(x, x_c)(t+t_0)\|_{X\times X_c}^2 \right) \\
&\leq \frac{\alpha c_2}{2\sigma} \|(x_0, x_{c,0})\|_{X\times X_c}^2 .
\end{aligned}$$

Therefore,

$$\begin{aligned}
t \|(x, x_c)(t)\|_{X\times X_c}^2 - c \|(x, x_c)(t)\|_{X\times X_c}^2 &\leq \Phi(t) \\
&\leq \Phi(\tau) + \frac{\alpha c_2}{c_2} 2\sigma \|(x_0, x_{c,0})\|_{X\times X_c}^2 \\
&\leq \left(\tau + c + \frac{\alpha c_2}{2\sigma} \right) \|(x_0, x_{c,0})\|_{X\times X_c}^2
\end{aligned}$$

so that for $t > \max\{\tau, c\}$ one has from the density of $D(\mathcal{A})$ in $X \times X_c$ that

$$\|\mathcal{T}(t)\|^2 \leq \frac{\tau + c + \frac{\alpha}{2\sigma}}{t - c} \xrightarrow{t\to+\infty} 0$$

and uniform exponential stability follows with Remark 2.2.12. □

5.4 Strictly Output Passive Controllers

In this section we consider the standard interconnection of a infinite-dimensional port-Hamiltonian system with a strictly output-passive controller. Again, we assume that the port-Hamiltonian system $(\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ is impedance passive,

$$\operatorname{Re} \langle \mathfrak{A}x, x \rangle_X \leq \operatorname{Re} \langle \mathfrak{B}x, \mathfrak{C}x \rangle_{\mathbb{F}^{Nd}}, \quad x \in D(\mathfrak{A}).$$

In the following definition we introduce the terminology of an *SOP controller.*

Definition 5.4.1. *A linear control system $\tilde{\Sigma} = (\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$*

$$\begin{aligned} \dot{x} &= \tilde{A}x + \tilde{B}u \\ y &= \tilde{C}x + \tilde{D}u \end{aligned}$$

for Hilbert spaces $\tilde{X}$ and $\tilde{U} = \tilde{Y}$, a semigroup generator $\tilde{A}$ on $\tilde{X}$ and bounded linear operators $\tilde{B} \in \mathcal{B}(\tilde{U}, \tilde{X}), \tilde{C} \in \mathcal{B}(\tilde{U}, \tilde{X})$ and $\tilde{D} \in \mathcal{B}(\tilde{U})$ is called strictly output passive *(SOP) if there exists a constant $\sigma > 0$ such that for all $x \in D(\tilde{A})$ and $u \in \tilde{U}$ one has the estimate*

$$\operatorname{Re} \langle \tilde{A}x + \tilde{B}u, x \rangle_{\tilde{X}} \leq \operatorname{Re} \langle u, y \rangle_{\tilde{U}} - \sigma \|y\|_{\tilde{Y}}^2$$

where $y = \tilde{C}x + \tilde{D}u$.

Assumption 5.4.2. *The finite dimensional controller (A_c, B_c, C_c, D_c) is strictly output-passive with state space $X_c = \mathbb{F}^n$ for some inner product $\langle \cdot, \cdot \rangle_{X_c}$, e.g.*

$$\langle x_c, z_c \rangle_{X_c} = z_c^* Q_c x_c, \quad x_c, z_c \in X_c \tag{5.10}$$

for some symmetric and positive definite matrix $Q_c \in \mathbb{F}^{n \times n}$, and input and output space $U_c = Y_c = \mathbb{F}^m$ with standard inner product for some $m \in \mathbb{N}$ with $1 \leq m \leq Nd$.

Theorem 5.4.3. *Let $\mathcal{A}$ be the operator resulting from the feedback interconnection $\mathfrak{B}_1 x = -y_c$, $u_c = \mathfrak{C}_1 x$, $\mathfrak{B}_2 x = 0$ of an impedance passive port-Hamiltonian system $\mathfrak{S} = (\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ with a finite dimensional linear SOP control system $\Sigma_c = (A_c, B_c, C_c, D_c)$. Let $R \in D(\mathfrak{A}; H)$ for some Hilbert space H.*

1. *If*
$$\ker(i\beta - A_c) \cap \ker C_c = \{0\}, \quad \beta \in \mathbb{R}$$
and
$$\|Rx\|_H^2 \leq |\mathfrak{B}x|^2, \quad x \in D(\mathfrak{A})$$
and the pair $(\mathfrak{A}, R)$ has property ASP, then $(\mathcal{T}(t))_{t\geq 0}$ is asymptotically stable.

2. *If $(\mathcal{T}(t))_{t\geq 0}$ is asymptotically stable, D_c is invertible,*
$$\|Rx\|_H^2 \leq |\mathfrak{B}x|^2 + |\mathfrak{C}_1 x|^2, \quad x \in D(\mathfrak{A})$$
and the pair $(\mathfrak{A}, R)$ has property AIEP, then $(\mathcal{T}(t))_{t\geq 0}$ is uniformly exponentially stable.

Remark 5.4.4. *The first condition (for asymptotic stability) is far from being weak and any weaker condition would be desirable. For example, for the one-dimensional wave equation as first order port-Hamiltonian systems ($N = 1$) one would like to take, say $EI\omega_\zeta(1), EI\omega_\zeta(0)$ as input, however for the property ASP we only know that $\|Rx\|_H^2 \geq |\omega_t(1)|^2 + |EI\omega_\zeta(1)|^2$ would be sufficient. Therefore, one probably has to check by hand that $\sigma_p(\mathcal{A}) \cap i\mathbb{R}$ for suitable boundary conditions.*

Proof of Theorem 5.4.3. In the following we write $u_c = \mathfrak{C}_1 x$ and $y_c = C_c x_c + D_c \mathfrak{C}_1 x$. First note that for every $(x, x_c) \in D(\mathcal{A})$

$$\begin{aligned}\operatorname{Re}\langle \mathcal{A}(x,x_c),(x,x_c)\rangle_{X\times X_c} &= \operatorname{Re}\langle \mathfrak{A}x, x\rangle_X + \operatorname{Re}\langle A_c x_c + B_c u_c, x_c\rangle_{X_c}\\ &\le \operatorname{Re}\langle \mathfrak{B}x, \mathfrak{C}x\rangle_{\mathbb{F}^{Nd}} + \operatorname{Re}\langle u_c, y_c\rangle_{\mathbb{F}^m} - \sigma\left|y_c\right|^2 = -\sigma\left|y_c\right|^2.\end{aligned}$$

1.) If $\mathcal{A}(x, x_c) = i\beta(x, x_c)$ for some $\beta \in \mathbb{R}$ then $\operatorname{Re}\langle \mathcal{A}(x,x_c),(x,x_c)\rangle_{X\times X_c} = 0$, so $\mathfrak{B}x = -y_c = 0$. Since the pair $(\mathfrak{A}, R)$ has property ASP this implies $x = 0$ and then also $u_c = \mathfrak{C}_1 x = 0$. Thus, $A_c x_c = i\beta x_c$ and from $\ker(i\beta - A_c) \cap \ker C_c = \{0\}$ it follows that $x_c = 0$. Hence, $\mathcal{A}$ has no eigenvalue on the imaginary axis, so $(\mathcal{T}(t))_{t\ge 0}$ is asymptotically stable by Corollary 2.2.16.
2.) Let $((x_n, x_{c,n}), \beta_n)_{n\ge1} \subseteq D(\mathcal{A})\times\mathbb{R}$ be a sequence with $\sup_{n\in\mathbb{N}} \|(x_n, x_{c,n})\|_{X\times X_c} < +\infty$, $|\beta_n| \to +\infty$ and such that $\mathcal{A}(x_n, x_{c,n}) - i\beta_n(x_n, x_{c,n}) \xrightarrow{n\to\infty} 0$ converges to zero in $X \times X_c$. Then

$$\sigma\left|y_{c,n}\right|^2 \le -\operatorname{Re}\langle \mathcal{A}(x_n,x_{c,n}),(x_n,x_{c,n})\rangle_{X\times X_c} \to 0,$$

i.e. $y_{c,n} \xrightarrow{n\to\infty} 0$. Since D_c is invertible it follows that also

$$D_c^{-1} C_c x_{c,n} + u_{c,n} \xrightarrow{n\to\infty} 0$$

and hence the sequence $(u_{c,n})_{n\ge1} \subseteq \mathbb{F}^m$ is bounded. Then from

$$A_c x_{c,n} + B_c u_{c,n} - i\beta_n x_{c,n} =: \eta_n \xrightarrow{n\to\infty} 0$$

and the fact that $(i\beta_n - A_c)^{-1} \xrightarrow{n\to\infty} 0$ we obtain that

$$x_{c,n} = (i\beta_n - A_c)^{-1}(B_c u_{c,n} - \eta_n) \to 0$$

and in particular this also implies $u_{c,n} \xrightarrow{n\to\infty} 0$. Now

$$\|Rx_n\|_H^2 \le \left|\mathfrak{B}x_n\right|^2 + \left|\mathfrak{C}_1 x_n\right|^2 = \left|y_{c,n}\right|^2 + \left|u_{c,n}\right|^2 \xrightarrow{n\to\infty} 0$$

and since the pair $(\mathfrak{A}, R)$ has property *AIEP* it follows that $x_n \xrightarrow{n\to\infty} 0$. By the sequence criterion Corollary 2.2.19 this implies that $(\mathcal{T}(t))_{t\ge0}$ is uniformly exponentially stable. □

Remark 5.4.5 (Example for an SOP controller)**.** *We ask ourselves the question: What is a typical example for an SOP controller? Thus, given a system* $\Sigma_c = (A_c, B_c, C_c, D_c)$ *we look for (easy) conditions whether the system is SOP. First of all note that the SOP property may be expressed as the matrix*

$$\begin{bmatrix} \frac{1}{2}(A_c + A_c') + \sigma C_c' C_c & \frac{1}{2}(B_c - C_c') + \sigma C_c' D_c \\ \frac{1}{2}(B_c' - C_c) + \sigma D_c' C_c & -\frac{1}{2}(D_c + D_c') + \sigma D_c' D_c \end{bmatrix} \le 0$$

being negative semi-definite for some $\sigma > 0$. *We then have that*

$$\operatorname{Re}\langle A_c x_c + B_c u_c, u_c\rangle \le \operatorname{Re}\langle C_c x_c + D_c u_c, u_c\rangle - \sigma\langle C_c x_c + D_c u_c, C_c x_c + D_c u_c\rangle$$

in particular for $u_c = 0$ *we must have that* $A_c + \sigma C_c' C_c$ *is dissipative, thus a state/output-matrix* $C_c \ne 0$ *requires some strict dissipation from* A_c. *On the other hand for* $x_c = 0$ *we obtain that*

$$\operatorname{Re}\langle D_c u_c, u_c\rangle \ge \sigma\langle D_c u_c, D_c u_c\rangle$$

so that one needs to have $0 \leq \operatorname{Sym} D_c \leq \frac{1}{\sigma} I$*, which is not a big issue since in principle we may choose* $\sigma > 0$ *as small as we wish. We then write*

$$\begin{aligned} 0 &\leq \operatorname{Re} \langle C_c x_c + D_c u_c, u_c \rangle - \operatorname{Re} \langle A_c x_c + B_c u_c, x_c \rangle - \sigma \left| C_c x_c + D_c u_c \right|^2 \\ &= \operatorname{Re} \langle (A_c - C_c^* C_c) x_c, x_c \rangle - \operatorname{Re} \langle (I - \sigma D_c) u_c, D_c u_c \rangle \\ &\quad + \operatorname{Re} \langle (B_c' + (2\sigma D_c^* - I) C_c) x_c, u_c \rangle \end{aligned}$$

Therefore, a quite natural choice of Σ_c *ensuring SOP is the following where the first two are necessary for SOP with given* $\sigma > 0$*:*

1. (A_c, C_c) *such that* $A_c + \sigma C_c' C_c$ *is dissipative,*
2. D_c *such that* $0 \leq \operatorname{Sym} D_c = \frac{D_c + D_c^*}{2} \leq \frac{1}{\sigma} I$ *and*
3. B_c *such that* $B_c' = (I - 2\sigma D_c^*) C_c$.

For example, if $\sigma = 1$ *and* $A_c + C_c' C_c$ *dissipative are given a possible choice were* $D_c = I$ *and then* $C_c = -B_c'$*. Note that for SIP controllers the choice* $C_c = B_c'$ *(collocated input/output) makes more sense then the choice made here.*

Example 5.4.6. *In contrast to SIP controllers, for SOP controllers the feed-through operator* D_c *does not necessarily be invertible, in fact take any* $A_c \in \mathbb{F}^{n \times n}, C_c \in \mathbb{F}^{m \times n}$ *and* $\sigma > 0$ *such that* $A_c + \sigma C_c' C_c$ *is dissipative. Then for* $B_c := C_c'$ *and* $D_c = 0$ *the system* $\Sigma_c = (A_c, B_c, C_c, D_c)$ *is SOP, namely*

$$\operatorname{Re} \langle A_c x_c + C_c' u_c, x_c \rangle_{\mathbb{F}^n} \leq \operatorname{Re} \langle C_c x_c, u_c \rangle_{\mathbb{F}^k} - \sigma \left| C_c x_c \right|_{\mathbb{F}^m}^2 .$$

5.5 More General Impedance Passive Controllers

In the two preceding sections on SIP and SOP controllers for stabilisation of impedance passive port-Hamiltonian systems we did not cover some cases which might also be interesting for applications. On the one end we excluded large multi-component systems where not only one, but several finite dimensional controllers are used for stabilisation. In total these controllers form a single finite-dimensional system, of course, but if some of the control parts are SIP and others are SOP in general the total control system will be neither SIP nor SOP. Still it is quite reasonable for the controllers to still be stabilising. Secondly, setting some components of $\mathfrak{B}x$ to be zero, for SIP or SOP control systems we could not use the corresponding components of $\mathfrak{C}x$ for our dynamic feedback law which might also be an unnecessary restriction on the control law. Therefore, we present a slight generalisation of the stabilisation theorems above which does not have these drawback, i.e. does overcome these two presented problems. Again we start with a result on asymptotic stabilisation.

Theorem 5.5.1. *Let* $\mathfrak{S} = (\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ *be an impedance passive port-Hamiltonian system and assume that the control system* $\Sigma_c = (A_c, B_c, C_c, D_c)$ *has the following* *block diagonal form*

$$\begin{aligned} A_c &= \begin{pmatrix} A_{c,1} & \\ & A_{c,2} \end{pmatrix}, \\ B_c &= \begin{pmatrix} B_{c,1} & \\ & B_{c,2} \end{pmatrix}, \\ C_c &= \begin{pmatrix} C_{c,1} & \\ & C_{c,2} \end{pmatrix}, \\ D_c &= \begin{pmatrix} D_{c,1} & \\ & D_{c,2} \end{pmatrix}, \end{aligned} \tag{5.11}$$

where $A_{c,i} \in \mathbb{F}^{n_i \times n_i}$, $B_{c,i} \in \mathbb{F}^{m_i \times n_i}, C_{c,i} \in \mathbb{F}^{n_i \times m_i}$ *and* $D_{c,i} \in \mathbb{F}^{m_i \times m_i}$ *for* $i = 1, 2$ *and* $n_1 + n_2 = n$, $m_1 + m_2 = Nd$. *Assume that the control system is impedance passive with*

$$\begin{aligned} &\operatorname{Re} \langle A_c x_c + B_c u_c, x_c \rangle_{X_c} \\ &\leq \operatorname{Re} \langle C_c x_c + D_c u_c, u_c \rangle_{\mathbb{F}^{Nd}} - \sigma \left(|u_{c,1}|^2 + |C_{c,2} x_{c,2} + D_{c,2} u_{c,2}|^2 \right) \end{aligned}$$

for some $\sigma > 0$ *and all* $u_c \in \mathbb{F}^{Nd}$ *and* $x_c \in \mathbb{F}^n$. *If the matrix* A_c *is Hurwitz, i.e.* $\sigma_p(A_c) \subseteq \mathbb{C}_0^-$, *and*

1. $D_c = D_c^* \geq 0$ *is symmetric and positive semi-definite*
2. *for all* $x \in D(\mathfrak{A})$
$$|\mathfrak{B}x|^2 + \|\mathfrak{C}_1 x\|_{H_c}^2 \geq \|Rx\|_H^2$$

where $R \in \mathcal{B}(D(\mathfrak{A}); H)$ *for some Hilbert space* H *such that the pair* $(\mathfrak{A}, R)$ *has property ASP, then the controller* Σ_c *asymptotically stabilises the system* $\mathfrak{S}$ *for the standard feedback interconnection* $u_c = \mathfrak{C}x$ *and* $\mathfrak{B}x = -y_c$, *i.e. the* C_0*-semigroup* $(\mathcal{T}(t))_{t \geq 0}$ *generated by the interconnection operator* $\mathcal{A}$ *is asymptotically stable.*

Proof. Let $(x, x_c) \in D(\mathcal{A})$ be such that $\mathcal{A}(x, x_c) = i\beta(x, x_c)$ for some $\beta \in \mathbb{R}$. Then

$$0 = \operatorname{Re} \langle \mathcal{A}(x, x_c), (x, x_c) \rangle_{X \times X_c} \leq -\sigma \left(|\mathfrak{C}_1 x|^2 + |\mathfrak{B}_2 x|^2 \right)$$

so that $\mathfrak{C}_1 x = 0$ and $\mathfrak{B}_2 x = 0$ are zero. Then also

$$x_{c,1} = (i\beta - A_{c,1})^{-1} B_{c,1} \mathfrak{C}_1 x = 0$$

and on the other hand

$$\mathfrak{B}_1 x = -(C_{c,1} x_{c,1} + D_{c,1} \mathfrak{C}_1 x) = 0.$$

Since the pair $(\mathfrak{A}, R)$ has property ASP it follows that $x = 0$, in particular also $\mathfrak{C}_2 x = 0$ and hence $x_c = (i\beta - A_c)^{-1} B_c \mathfrak{C} x = 0$. As a result, $i\beta \in i\mathbb{R}$ can never be an eigenvalue of $\mathcal{A}$, i.e. $\sigma_p(\mathcal{A}) \cap i\mathbb{R} = \emptyset$. Asymptotic stability follows from Corollary 2.2.16. □

Now assume that it is already known that $(\mathcal{T}(t))_{t \geq 0}$ is asymptotically stable. Then the following theorem provides sufficient conditions for $(\mathcal{T}(t))_{t \geq 0}$ also being exponentially stable.

Theorem 5.5.2. *Let $\mathfrak{S} = (\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ be an impedance passive port-Hamiltonian Boundary Control System interconnected with a finite dimensional control system $\Sigma_c = (A_c, B_c, C_c, D_c)$ of diagonal structure as in Theorem 5.5.1 which is impedance passive with*

$$\operatorname{Re}\langle A_c x_c + B_c u_c, x_c\rangle_{X_c} \leq \operatorname{Re}\langle C_c x_c + D_c u_c, u_c\rangle_{\mathbb{F}^{Nd}} - |u_{c,1}|^2$$

and $D_c = D_c^*$ is symmetric. Assume that $(\mathcal{T}(t))_{t\geq 0}$ is asymptotically stable, i.e. $\sigma_p(\mathcal{A}) \subseteq \mathbb{C}_0^-$. If there is $R \in \mathcal{B}(D(\mathfrak{A}); H)$ for some Hilbert space H such that

$$|\mathfrak{B}x|^2 + |D_c \mathfrak{C}x|^2 + |\mathfrak{C}_1 x|^2 \geq \|Rx\|_H^2, \quad x \in D(\mathfrak{A})$$

and the pair $(\mathfrak{A}, R)$ has property AIEP, then the C_0-semigroup $(\mathcal{T}(t))_{t\geq 0}$ generated by $\mathcal{A}$ is uniformly exponentially stable.

Remark 5.5.3. *Note that we do not explicitly assume that $i\mathbb{R} \cap \sigma(A_c) = \emptyset$ here since the asymptotic stability of $(\mathcal{T}(t))_{t\geq 0}$ already implies that $\sigma(\mathcal{A}) = \sigma_p(\mathcal{A}) \subseteq \mathbb{C}_0^-$. Also since X_c is finite dimensional the spectrum $\sigma(A_c)$ of A_c is bounded, so that $i\beta \in \rho(A_c)$ for sufficiently large $|\beta|$. To check asymptotic stability for systems where $\sigma(A_c) \cap i\mathbb{R} \neq \emptyset$ one needs another proof since Theorem 5.5.1 only works under the assumption that A_c is Hurwitz.*

Proof of Theorem 5.5.2. We employ the sequence criterion for the Gearhart-Greiner-Prüss-Huang Theorem again. Let $(x_n, x_{c,n}, \beta_n)_{n\geq 1} \subseteq D(\mathcal{A}) \times \mathbb{R}$ be a sequence such that

$$\sup_{n\in\mathbb{N}} \|(x_n, x_{c,n})\|_{X\times X_c} < +\infty, \quad |\beta_n| \xrightarrow{n\to+\infty} +\infty$$

and

$$\mathcal{A}(x_n, x_{c,n}) - i\beta_n(x_n, x_{c,n}) \xrightarrow{n\to\infty} 0.$$

It is immediate from the passivity condition on the port-Hamiltonian system and the finite dimensional controller that then $\mathfrak{C}_1 x_n \xrightarrow{n\to\infty} 0$. We want to show that $(x_n, x_{c,n}) \to 0$ and for this purpose proceed in three steps.

1. Show that $x_{c,n} \xrightarrow{n\to\infty} 0$.
2. Show that $\mathfrak{B}x_n, D_c\mathfrak{C}x_n \xrightarrow{n\to\infty} 0$.
3. Use property AIEP to conclude that also $x_n \xrightarrow{n\to\infty} 0$.

1.) Let us first focus on the infinite dimensional part where

$$\mathfrak{A}x_n - i\beta_n x_n \xrightarrow{n\to\infty} 0.$$

First of all this implies that

$$\sup_{n\geq 1} \frac{\|\mathcal{H}x_n\|_{H^N}}{|\beta_n|} \simeq \sup_{n\geq 1} \frac{\|x_n\|_{\mathfrak{A}}}{|\beta_n|} < +\infty.$$

and by Lemma 2.1.19 we have that $\frac{1}{\beta_n}\|\mathcal{H}x_n\|_{C^{N-1}} \xrightarrow{n\to\infty} 0$ which leads to

$$\frac{\mathfrak{C}x_n}{\beta_n} \to 0.$$

For the finite dimensional part we write

$$A_c x_{c,n} + B_c \mathfrak{C} x_n - i\beta_n x_{c,n} =: f_{c,n} \longrightarrow 0$$

and obtain that

$$\begin{aligned} x_{c,n} &= R(i\beta_n, A_c)(B_c \mathfrak{C} x_n - f_{c,n}) \\ &= \left(iI - \frac{A_c}{\beta_n}\right)^{-1} \frac{B_c \mathfrak{C} x_n - f_{c,n}}{\beta_n} \xrightarrow{n\to+\infty} 0 \end{aligned}$$

where the resolvent $R(i\beta_n, A_c)$ exists for all $n \geq n_0$ sufficiently large and we used the fact that from $A_c \in \mathbb{F}^{n\times n}$ and $|\beta_n| \to +\infty$ we have

$$\left(i - \frac{A_c}{\beta_n}\right)^{-1} \xrightarrow{n\to+\infty} -iI$$

and the terms $\frac{B_c \mathfrak{C} x_n}{\beta_n}$, $\frac{f_{c,n}}{\beta_n} \to 0$ tend to zero as $n \to +\infty$.

2.) We proceed by showing that $\ker D_c \subseteq \ker(B_c - C_c')$ where $C_c' \in \mathcal{B}(\mathbb{F}^{Nd}; X_c)$ is the Hilbert space adjoint of $C_c \in \mathcal{B}(X_c; \mathbb{F}^{Nd})$ w.r.t. the inner product $\langle \cdot, \cdot \rangle_{X_c}$ on X_c. This is

Lemma 5.5.4. *Let $\tilde\Sigma = (\tilde A, \tilde B, \tilde C, \tilde D)$ be a linear control system on Hilbert spaces $\tilde X$ and $\tilde U = \tilde Y$. If $\tilde\Sigma$ is impedance passive, then*

$$\ker \tilde D \subseteq \ker(\tilde B - \tilde C')$$

Proof. Let $u \in \ker \tilde D$. Then for all $x \in \tilde X$ and $\lambda \in \mathbb{R}$ the impedance passivity of $\tilde\Sigma = (\tilde A, \tilde B, \tilde C, \tilde D)$ implies that

$$\begin{aligned} 0 &\geq \operatorname{Re}\langle \tilde A x + \tilde B(\lambda u), x\rangle_{\tilde X} - \operatorname{Re}\langle \tilde C x + \tilde D(\lambda u), \lambda u\rangle_{\tilde U} \\ &= \operatorname{Re}\langle \tilde A x, x\rangle_{\tilde X} + \lambda \operatorname{Re}\langle (\tilde B - \tilde C')u, x\rangle_{\tilde X}. \end{aligned}$$

Since this inequality holds for all $\lambda \in \mathbb{R}$ we deduce

$$\operatorname{Re}\langle (\tilde B - \tilde C')u, x\rangle_{\tilde X} = 0, \quad x \in \tilde X$$

and hence $u \in \ker(\tilde B - \tilde C')$, i.e. $\ker \tilde D \subseteq \ker(\tilde B - \tilde C')$. □

Proof of Theorem 5.5.2 (continued). Let Π be the orthogonal projection on $(\ker D_c)^\perp$. Since $D_c = D_c^* \geq 0$ is positive definite and $D_c|_{(\ker D_c)^\perp} : (\ker D_c)^\perp \to (\ker D_c)^\perp$ is injective, the operator $D_c|_{(\ker D_c)^\perp} > 0$ is strictly positive definite on $\operatorname{ran}\Pi = (\ker D_c)^\perp$, so to show that $D_c \mathfrak{C} x \to 0$ it suffices to show that $\Pi \mathfrak{C} x \to 0$. *Assume* the contrary and w.l.o.g. assume that

$$|\Pi \mathfrak{C} x_n| \xrightarrow{n\to\infty} \limsup_{k\to\infty} |\Pi \mathfrak{C} x_k| > 0.$$

Then observe that

$$\begin{aligned} 0 &\longleftarrow \operatorname{Re}\langle \mathcal{A}(x_n, x_{c,n}) - i\beta_n(x_n, x_{c,n}), (x_n, x_{c,n})\rangle_{X\times X_c} \\ &= \operatorname{Re}\langle \mathfrak{A} x_n, x_n\rangle_X + \operatorname{Re}\langle A_c x_{c,n} + B_c \mathfrak{C} x_n, x_{c,n}\rangle_{X_c} \\ &\leq \operatorname{Re}\langle \mathfrak{B} x_n, \mathfrak{C} x_n\rangle_{\mathbb{F}^{Nd}} + \operatorname{Re}\langle B_c \mathfrak{C} x_n, x_{c,n}\rangle \\ &= \operatorname{Re}\langle -C_c x_{c,n} - D_c \mathfrak{C} x_n, \mathfrak{C} x_n\rangle_{\mathbb{F}^{Nd}} + \operatorname{Re}\langle B_c \mathfrak{C} x_n, x_{c,n}\rangle \\ &= \operatorname{Re}\langle (B_c - C_c')\Pi \mathfrak{C} x_n, x_{c,n}\rangle_{X_c} - \langle D_c \Pi \mathfrak{C} x_n, \Pi \mathfrak{C} x_n\rangle_{\mathbb{F}^{Nd}} \end{aligned}$$

and dividing by $|\Pi\mathfrak{C}x_n| \neq 0$ (for large n) and since $x_{c,n} \to 0$ this leads to

$$\Pi\mathfrak{C}x_n \xrightarrow{n\to+\infty} 0$$

in contradiction to the assumption that $\limsup_{k\to\infty} |\mathfrak{C}_1 x_k| > 0$. As a result, $\Pi\mathfrak{C}x_n$ tends to zero and so does $D_c\mathfrak{C}x_n$. Then also

$$\mathfrak{B}x_n = -C_c x_{c,n} - D_c\mathfrak{C}x_n \xrightarrow{n\to+\infty} 0.$$

This finishes the second step.
3.) From the first to steps we have $x_{c,n} \xrightarrow{n\to\infty} 0$ and

$$\|Rx_n\|_H^2 \lesssim |\mathfrak{B}x_n|^2 + |D_c\mathfrak{C}x_n|^2 + |\mathfrak{C}_1 x_n|^2 \xrightarrow{n\to+\infty} 0$$

and property AIEP for the pair $(\mathfrak{A}, R)$ implies that also $x_n \xrightarrow{n\to\infty} 0$. Hence, uniform exponential stability follows from Corollary 2.2.19. □

Remark 5.5.5. *One may drop the condition $D_c = D_c^*$, but then the terms $\mathfrak{B}x$ and $D_c\mathfrak{C}x$ have to be replaced by $\tilde{\mathfrak{B}}x := \mathfrak{B}x - \frac{D_c - D_c^*}{2}\mathfrak{C}x$ and $\tilde{D}_c\tilde{\mathfrak{C}}x := (\mathrm{Sym}\ D_c)\mathfrak{C}x$, respectively. To see this note that*

$$\begin{aligned}\mathrm{Re}\,\langle\tilde{\mathfrak{B}}x, \tilde{\mathfrak{C}}x\rangle_{\mathbb{F}^{Nd}} &= \mathrm{Re}\,\langle\mathfrak{B}x - \frac{D_c - D_c^*}{2}\mathfrak{C}x, \mathfrak{C}x\rangle_{\mathbb{F}^{Nd}} \\ &= \mathrm{Re}\,\langle\mathfrak{B}x, \mathfrak{C}x\rangle_{\mathbb{F}^{Nd}}\end{aligned}$$

and

$$D(\mathcal{A}) = \{(x, x_c) \in D(\mathfrak{A}) \times X_c : \tilde{\mathfrak{B}}x = -C_c x_c - \tilde{D}_c\tilde{\mathfrak{C}}x\}.$$

5.6 The Static and the Dynamic Case

Before we consider the nonlinear case let us mention some implications of Lemma 3.2.24 for the resolvents of the port-Hamiltonian operators with static or dynamic linear feedback. First, we reformulate Lemma 3.2.24 as

Lemma 5.6.1. *Let $(\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ be an impedance passive port-Hamiltonian Boundary Control system. Then for all $\mathrm{Re}\,\lambda > 0$, $u \in \mathbb{F}^{Nd}$ and $f \in L_2(0,1;\mathbb{F}^d)$ the problem*

$$\begin{aligned}(\mathfrak{A} - \lambda I)x &= f \\ \mathfrak{B}x &= u \\ \mathfrak{C}x &= y\end{aligned}$$

has a unique solution $(x, y) \in D(\mathfrak{A}) \times \mathbb{F}^{Nd}$ which is given by

$$\begin{aligned}x &= \Phi(\lambda)f + \Psi(\lambda)u \\ y &= F(\lambda)f + G(\lambda)u\end{aligned}$$

for some holomorphic functions

$$\begin{aligned}&\Phi \in \mathbb{H}(\mathbb{C}_0^+; \mathcal{B}(X, D(\mathfrak{A}))),\ \Psi \in \mathbb{H}(\mathbb{C}_0^+; \mathcal{B}(\mathbb{F}^{Nd}; D(\mathfrak{A}))), \\ &F \in \mathbb{H}^\infty(\mathbb{C}_0^+; \mathcal{B}(X; \mathbb{F}^{Nd})),\ G \in \mathbb{H}(\mathbb{C}_0^+; \mathcal{B}(\mathbb{F}^{Nd}))\end{aligned}$$

where

$$\mathrm{Sym}\ G(\lambda) > 0, \quad \mathrm{Re}\,\lambda > 0.$$

Remark 5.6.2. *Here we used the following notation for open subsets $\Omega \subseteq \mathbb{C}$ of the complex plane and Hilbert spaces H.*

$$\begin{aligned} \mathbb{H}(\Omega; H) &:= \{f : \Omega \to H : \ f \text{ is holomorphic } \} \\ \mathbb{H}^\infty(\Omega; H) &:= \{f \in \mathbb{H}(\Omega; H) : \ f \text{ is bounded } \}. \end{aligned}$$

With this notation we may easily express the resolvent of the generator resulting from the feedback $u = -Ky$.

Corollary 5.6.3. *Let $K = K^* \geq 0$ be an $Nd \times Nd$-matrix and $A_K := \mathfrak{A}|_{\ker(\mathfrak{B}+K\mathfrak{C})}$. Then for all $\operatorname{Re} \lambda > 0$ the resolvent of A_K is given by*

$$\begin{aligned} R(\lambda, A_K) &= -\Psi(\lambda)K(K + G(\lambda)^{-1})^{-1}G(\lambda)^{-1}F(\lambda) + \Phi(\lambda) \\ &= -\Psi(\lambda)K(KG(\lambda) + I)^{-1}F(\lambda) + \Phi(\lambda). \end{aligned}$$

Likewise we may express the resolvent for the operator including dynamic feedback for the case of collocated input/output of the control system.

Corollary 5.6.4. *Let (A_c, B_c, C_c, D_c) be an impedance passive, exponentially stable (finite-dimensional) controller with $\sigma(A_c) \subseteq \mathbb{C}_0^-$, $U_c = Y_c = \mathbb{F}^{Nd}$ and collocated input/output $B_c' = C_c$ (w.r.t. $\langle \cdot, \cdot \rangle_{X_c}$) and $\operatorname{Sym} D_c \geq 0$ positive semidefinite. Then for*

$$\begin{aligned} \mathcal{A} &:= \begin{bmatrix} \mathfrak{A} & \\ B_c\mathfrak{C} & A_c \end{bmatrix} \\ D(\mathcal{A}) &:= \{(x, x_c) \in D(\mathfrak{A}) \times X_c : \ (\mathfrak{B} + D_c\mathfrak{C})x = -C_c x_c\} \end{aligned}$$

one has $\mathbb{C}_0^+ \subseteq \rho(\mathcal{A})$ with

$$\begin{aligned} &R(\lambda, \mathcal{A}) \\ &= \begin{bmatrix} R(\lambda, A_{D_c}) + \Delta(\lambda) & -\Psi(\lambda)(I + D_c(G(\lambda)^{-1} + D_c)^{-1})C_c R(\lambda, A_c^\lambda) \\ R(\lambda, A_c^\lambda)B_c(I + D_c G(\lambda))^{-1}F(\lambda) & R(\lambda, A_c^\lambda) \end{bmatrix} \end{aligned}$$

where

$$\begin{aligned} A_c^\lambda &:= A_c - B_c(G(\lambda)^{-1} + D_c)^{-1}C_c. \\ \Delta(\lambda) &:= -\Psi(\lambda)(I + D_c(G(\lambda)^{-1} + D_c)^{-1})C_c R(\lambda, A_c^\lambda)B_c(I + D_c G(\lambda))^{-1}F(\lambda) \end{aligned}$$

Note: A_c^λ also has spectrum in $\mathbb{C}_0^-$, so generates a uniformly exponentially stable semigroup.

Proof. The passivity of (A_c, B_c, C_c, D_c) implies that A_c is dissipative and since $B_c' = C_c$ also A_c^λ is dissipative. Moreover, for $i\beta x_c = A_c^\lambda x_c$ we obtain

$$\begin{aligned} 0 &= \operatorname{Re} \langle A_c^\lambda x_c, x_c \rangle_{X_c} \\ &= \operatorname{Re} \langle A_c x_c - B_c(G(\lambda)^{-1} + D_c)^{-1}C_c x_c, x_c \rangle_{X_c} \\ &\leq -\operatorname{Re} \langle (G(\lambda)^{-1} + D_c)^{-1}C_c x_c, C_c x_c \rangle_{\mathbb{F}^{Nd}} \end{aligned}$$

and since $(G(\lambda)^{-1} + D_c)^{-1} \geq 0$ this implies $(G(\lambda)^{-1} + D_c)^{-1}C_c x_c = 0$, thus

$$i\beta x_c = A_c^\lambda x_c = A_c x_c$$

and it follows $x_c = 0$, so exponential stability. □

Finally, this gives the following corollary

Corollary 5.6.5.

$$R(\cdot, A_{D_c}) \in \mathbb{H}^\infty(\mathbb{C}_0^+; \mathcal{B}(X)) \quad \Longleftrightarrow \quad R(\cdot, \mathcal{A}) \in \mathbb{H}^\infty(\mathbb{C}_0^+; \mathcal{B}(X \times X_c)),$$

i.e. A_{D_c} *generates a uniformly exponentially stable* C_0*-semigroup* $(T(t))_{t\geq 0}$ *on* X *if and only if* $\mathcal{A}$ *generates a uniformly exponentially stable* C_0*-semigroup* $(\mathcal{T}(t))_{t\geq 0}$ *on* $X \times X_c$.

5.7 Examples

Within this section we return to some of the examples considered in the introductory examples section and show how some results for particular dynamic boundary conditions can be re-obtained using the abstract results we derived in the sections before. Sometimes we are even able to generalise the previously known results or to impose less restrictive regularity conditions, at least.

Example 5.7.1 (Dynamic Feedback Stabilisation of the Timoshenko Beam Equation)**.** *We consider the nonuniform Timoshenko beam, see Examples 3.1.3 and 4.5.4, with the following stabilisation scheme as in [Zh07].*

$$\begin{aligned}
(K(\phi - \omega_\zeta))(t,0) &= 0 \\
-(EI\phi_\zeta)(t,0) &= 0 \\
(K(\phi - \omega_\zeta))(t,1) &= k_1\omega_t(t,1) + k_2\omega(t,1) \\
-(EI\phi_\zeta)(t,1) &= k_3\phi_t(t,1) + k_4\phi(t,1)
\end{aligned}$$

where a attached mass at the tip ($\zeta = 1$) *adds additional components to the the total energy which now is*

$$\begin{aligned}
H_{tot}(t) = \frac{1}{2}\int_0^1 & K(\zeta)\left|(\omega_\zeta - \phi)(t,\zeta)\right|^2 + EI(\zeta)\left|\phi_\zeta(t,\zeta)\right|^2 \\
& + \rho(\zeta)\left|\omega_t(t,\zeta)\right|^2 + I_\rho(\zeta)\left|\phi_t(t,\zeta)\right|^2 d\zeta \\
& + k_1\left|\omega(t,0)\right|^2 + k_3\left|\phi(t,0)\right|^2
\end{aligned}$$

and the latter two terms $\omega(t,0)$ *and* $\phi(t,0)$ *are not represented in the standard port-Hamiltonian formulation* $x = (\omega_\zeta - \phi, \rho\phi_t, \phi_\zeta, I_\rho\phi_t)$ *in the sense that they cannot be computed from knowing only the value of* x*. On the other hand these term contribute in a discrete way to the total energy, so that it makes sense to consider them as additional variable* $x_c \,\hat{=}\, (\omega(t,0), \phi(t,0)) \in \mathbb{F}^2 = X_c$ *and the latter two boundary conditions as evolutionary laws for these control state space variables.*

$$\frac{d}{dt}x_c(t) = \begin{pmatrix} -\frac{k_1}{k_2} & \\ & -\frac{k_3}{k_4} \end{pmatrix} x_c(t) + \begin{pmatrix} -\frac{1}{k_2} & \\ & -\frac{1}{k_4} \end{pmatrix} \begin{pmatrix} K(\omega_\zeta - \phi)(t,1) \\ EI\phi_\zeta(t,1) \end{pmatrix}$$

We therefore rewrite these boundary conditions as a dynamic feedback control for the state space $X_c = \mathbb{F}^2$ *with the weighted inner product*

$$\langle x_c, z_c\rangle_{X_c} = \langle x_c, Q_c z_c\rangle_{\mathbb{F}^2}, \quad x_c, z_c \in X_c, \quad Q_c = \operatorname{diag}(k_1, k_3)$$

the control input and output space $U_c = Y_c = \mathbb{F}^2$ with the Euclidean inner product and for the matrices

$$A_c = -C_c = -\operatorname{diag}\left(\frac{k_1}{k_2}, \frac{k_3}{k_4}\right), \quad B_c = -D_c = -\operatorname{diag}\left(\frac{1}{k_2}, \frac{1}{k_4}\right)$$

which is interconnected with the port-Hamiltonian system $\mathfrak{S} = (\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ for the impedance energy preserving choice

$$\mathfrak{B}x = \begin{pmatrix} (\mathcal{H}_2x_2)(1) \\ (\mathcal{H}_4x_4)(1) \\ (\mathcal{H}_1x_1)(0) \\ (\mathcal{H}_3x_3)(0) \end{pmatrix}, \quad \mathfrak{C}x = \begin{pmatrix} (\mathcal{H}_1x_1)(1) \\ (\mathcal{H}_3x_3)(1) \\ -(\mathcal{H}_2x_2)(0) \\ -(\mathcal{H}_4x_4)(0) \end{pmatrix}$$

cf. Example 4.5.4, and the feedback interconnection

$$\mathfrak{B}x = \begin{pmatrix} -u_c \\ 0 \end{pmatrix}, \quad u_c = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \mathfrak{C}x.$$

Then the linear control system is impedance passive, even strictly output passive, since for every $x_c \in X_c, u_c \in U_c$ and $y_c = C_cx_c + D_cu_c$ we obtain, using that $Q_c^{-1} = \operatorname{diag}(k_2, k_4)C_c$ and $-I = \operatorname{diag}(k_2, k_4)D_c$,

$$\begin{aligned} &\operatorname{Re}\langle A_cx_c + B_cu_c, x_c\rangle_{X_c} - \operatorname{Re}\langle C_cx_c + D_cu_c, u_c\rangle_{\mathbb{F}^2} \\ &= -\operatorname{Re}\langle C_cx_c + D_cu_c, Q_c^{-1}x_c - u_c\rangle_{X_c} \\ &= -\operatorname{Re}\langle y_c, \operatorname{diag}(k_1, k_3)y_c\rangle_{\mathbb{F}^2} = -k_2\left|y_{c,1}\right|^2 - k_4\left|y_{c,2}\right|^2. \end{aligned}$$

Therefore, the hybrid operator $\mathcal{A}$ generates a contractive C_0-semigroup on the product Hilbert space $X \times X_c$ and $\sigma := \min\{k_2, k_4\} > 0$

$$\operatorname{Re}\langle \mathcal{A}(x, x_c), (x, x_c)\rangle_{X\times X_c} \leq -\sigma\left|\mathfrak{B}x\right|^2, \quad (x, x_c) \in D(\mathcal{A}).$$

In the next step we prove that the C_0-semigroup is even uniformly exponentially stable. We begin by showing asymptotic stability. Here we cannot use Theorem 4.2.2 and 5.4.3 since the term $\mathfrak{B}x$ does not include the value of all components of $\mathcal{H}x$ at the side $\zeta = 0$ or $\zeta = 1$, so that we prove that $i\beta \notin \sigma_p(\mathcal{A})$ for every $\beta \in \mathbb{R}$. In fact, for the case that $\beta = 0$ we obtain that if $\mathcal{A}(x, x_c) = 0$, then $(\mathcal{H}x)' = 0$ and from the dissipation inequality that also $\mathfrak{B}x = 0$, so that $\mathcal{H}x = 0$. Then also $\mathfrak{C}x = 0$ and then $A_cx_c = 0$ so that $(x, x_c) = 0$ and $\beta = 0$ cannot be an eigenvalue of $\mathcal{A}$. Moreover, for the case $\beta \neq 0$ we find for every solution $\mathcal{A}(x, x_c) = i\beta(x, x_c)$ that $\|i\beta x_c\| = \|\mathfrak{B}x\| = 0$ and since $B_c \in \mathcal{B}(U_c; X_c)$ is invertible then also the first two components of $\mathfrak{C}x$ equal zero and $x \in D(\mathfrak{A})$ solves the problem

$$i\beta x = \mathfrak{A}x, \quad (\mathcal{H}x)(1) = 0,$$

so that also $x = 0$ and we conclude that $i\mathbb{R} \cap \sigma_p(\mathcal{A}) = \emptyset$. By Corollary 2.2.16 the C_0-semigroup is asymptotically stable and then by Theorem 5.4.3 we also get uniform exponential stability since the pair $(\mathfrak{A}, R)$ for $Rx = (\mathcal{H}x)(1)$ has property AIEP and

$$\left|\mathfrak{B}x\right|^2 + \left|\Pi_{\mathbb{F}^2\times\{0\}}\mathfrak{C}x\right|^2 \geq \left|(\mathcal{H}x)(1)\right|^2, \quad x \in D(\mathfrak{A}).$$

Example 5.7.2 (Euler-Bernoulli Beam with Both Ends Free)**.** *In this example we consider the nonuniform Euler-Bernoulli Beam with both ends free where at the tip (here: the left end) a mass is attached to damp the beam. Originally this problem had been considered in the article [GuHu04] where the authors assumed that $EI \in C^2([0,1];R)$ is continuously differentiable twice and $\rho \in C^1([0,1];R)$ is continuously differentiable. The assertions below have already been stated, but only partly proved, in Example 4.3 of [AuJa14] under the slightly less restrictive regularity assumptions $EI, \rho \in W^1_\infty(0,1;\mathbb{R})$. (Both functions should still be uniformly positive, of course.) Here we will present a treatment including the missing parts of the proof in [AuJa14]. The beam model under consideration is the following. We consider the usual Euler-Bernoulli beam equation subject to the dynamic boundary conditions*

$$\begin{aligned}
EI(\zeta)\frac{\partial^2}{\partial\zeta^2}\omega(t,\zeta)|_{\zeta=1} &= -\frac{\partial}{\partial\zeta}\left(EI(\zeta)\frac{\partial^2}{\partial\zeta^2}\omega(t,\zeta)\right)|_{\zeta=1} = 0\\
EI(\zeta)\frac{\partial^2}{\partial\zeta^2}\omega(t,\zeta)|_{\zeta=0} &= k_1\frac{\partial}{\partial\zeta}\omega(t,\zeta)|_{\zeta=0} + k_2\frac{\partial^2}{\partial t\partial\zeta}\omega(t,\zeta)|_{\zeta=0}\\
-\frac{\partial}{\partial\zeta}\left(EI(\zeta)\frac{\partial^2}{\partial\zeta^2}\omega(t,\zeta)\right)|_{\zeta=0} &= k_3\omega(t,\zeta)|_{\zeta=0} + k_4\frac{\partial}{\partial t}\omega(t,\zeta)|_{\zeta=0}
\end{aligned}$$

where $k_i > 0$ $(i = 1,\dots,4)$ are positive constants. Obviously the terms $\omega(t,0)$ and $\omega_\zeta(t,0)$ appearing in the boundary conditions may not be represented by the variables $\rho(\zeta)\omega_t(t,\zeta)$ and $\omega_{\zeta\zeta}(t,\zeta)$ of the formulation as port-Hamiltonian system of second order $N = 2$. In fact, they also contribute to the total energy of the beam-mass-system which is given by

$$\begin{aligned}
E_{tot}(t) = \frac{1}{2}\int_0^1 & EI(\zeta)\left|\omega_{\zeta\zeta}(t,\zeta)\right|^2 + \rho(\zeta)\left|\omega_t(t,\zeta)\right|^2 d\zeta\\
&+ \frac{1}{2}\left(k_1\left|\omega_\zeta(t,0)\right|^2 + k_3\left|\omega(t,0)\right|^2\right).
\end{aligned}$$

Therefore, the energy decomposes into two parts. On the one hand we have an continuous part corresponding to the energy of the beam itself and on the other hand we also have a discrete part as weighted sum of the squared Euclidean norms of $\omega(t,0)$ and $\frac{\partial}{\partial\zeta}\omega(t,0)$. We already saw in Example 3.1.6 that its port-Hamiltonian formulation is

$$\begin{aligned}
x_1(t,\zeta) &:= \frac{\partial^2}{\partial\zeta^2}\omega(t,\zeta),\\
x_2(t,\zeta) &:= \frac{\partial}{\partial t}\omega(t,\zeta),
\end{aligned}$$

plus the additional controller state space variables

$$\begin{aligned}
x_{c,1}(t) &:= \frac{\partial}{\partial\zeta}\omega(t,0),\\
x_{c,2}(t) &:= \omega(t,0).
\end{aligned}$$

The latter two boundary conditions may then also be interpreted as evolution law for the new variable $x_c(t)$. We next show that the total system may be represented by passive dynamic boundary control of an impedance energy preserving system. Recall that for the Euler-Bernoulli beam its energy change is determined by

$$\begin{aligned}\frac{d}{dt}H(t) &= \mathrm{Re}\,\left[\langle EI\omega_{\zeta\zeta}(\zeta), \omega_{t\zeta}(\zeta)\rangle_{\mathbb{F}} - \langle (EI\omega_{\zeta\zeta})_\zeta(\zeta), \omega_t(\zeta)\rangle_{\mathbb{F}}\right]_0^1 \\ &= \mathrm{Re}\,\langle (EI\omega_{\zeta\zeta})(1), \omega_{t\zeta}(1)\rangle - \mathrm{Re}\,\langle (EI\omega_{\zeta\zeta})_\zeta(1), \omega_t(1)\rangle \\ &\quad - \mathrm{Re}\,\langle (EI\omega_{\zeta\zeta})(0), \omega_{t\zeta}(0)\rangle + \mathrm{Re}\,\langle (EI\omega_{\zeta\zeta})_\zeta(0), \omega_t(0)\rangle\end{aligned}$$

so that the choice

$$\mathfrak{B}x(t) = \begin{pmatrix} (\mathcal{H}_1 x_1)'(t,0) \\ (\mathcal{H}_1 x_1)(t,0) \\ (\mathcal{H}_2 x_2)(t,1) \\ -(\mathcal{H}_2 x_2)'(t,1) \end{pmatrix} \hat{=} \begin{pmatrix} \omega_{t\zeta}(t,0) \\ \omega_t(t,0) \\ (EI\omega_{\zeta\zeta}(t,1) \\ -(EI\omega_{\zeta\zeta})_\zeta(t,1) \end{pmatrix}$$

$$\mathfrak{C}x(t) = \begin{pmatrix} -(\mathcal{H}_2 x_2)(t,0) \\ (\mathcal{H}_2 x_2)'(t,0) \\ (\mathcal{H}_1 x_1)'(t,1) \\ (\mathcal{H}_1 x_1)(t,1) \end{pmatrix} \hat{=} \begin{pmatrix} -(EI\omega_{\zeta\zeta})(t,0) \\ (EI\omega_{\zeta\zeta})_\zeta(t,1) \\ \omega_{t\zeta}(t,1) \\ \omega_t(t,1) \end{pmatrix}$$

leads to an impedance energy preserving port-Hamiltonian system $\mathfrak{S} = (\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$, where $\mathfrak{A}$ had been defined in Example 3.1.6. We choose $m = 2$ and $\mathfrak{B}_1 x$ and $\mathfrak{C}_1 x$ to be the vector in $\mathbb{F}^2$ consisting of the first two components of $\mathfrak{B}x$ and $\mathfrak{C}x$, respectively. The two-dimensional controller state space $X_c = \mathbb{F}^2$ is equipped with the weighted norm

$$\langle \cdot, \cdot \rangle_{X_c} = \langle \cdot, Q_c \cdot \rangle_{X_c}, \quad Q_c = \begin{bmatrix} k_1 & \\ & k_3 \end{bmatrix}$$

and then the finite dimensional control system is given as $\Sigma_c = (A_c, B_c, C_c, D_c)$ where

$$\begin{aligned} A_c = -C_c &= \begin{bmatrix} -\frac{k_1}{k_2} & \\ & -\frac{k_3}{k_4} \end{bmatrix} \\ B_c = -D_c &= \begin{bmatrix} -\frac{1}{k_2} & \\ & -\frac{1}{k_4} \end{bmatrix}. \end{aligned}$$

We check that the control system is strictly output passive. In fact, we have for every $x_c \in X_c$ and $u_c \in U_c = \mathbb{F}^2$ that

$$\begin{aligned} &\mathrm{Re}\,\langle A_c x_c + B_c u_c, x_c\rangle_{X_c} - \mathrm{Re}\,\langle C_c u_c + D_c u_c, u_c\rangle_{U_c} \\ &= \mathrm{Re}\,\langle -y_c, Q_c x_c\rangle_{\mathbb{F}^2} - \mathrm{Re}\,\langle y_c, u_c\rangle_{\mathbb{F}^2} \\ &= -\mathrm{Re}\,\langle y_c, Q_c x_c + u_c\rangle_{\mathbb{F}^2} \\ &= -\mathrm{Re}\,\langle \mathrm{diag}\,(k_2, k_4) y_c, C_c x_c + D_c u_c\rangle_{\mathbb{F}^2} = -k_2 \left|y_{c,1}\right|^2 - k_4 \left|y_{c,2}\right|^2 \end{aligned}$$

so that the system is strictly output passive, indeed. We therefore conclude that the hybrid operator $\mathcal{A}$ resulting from the interconnection $\mathfrak{B}_1 x = -y_c$, $u_c = \mathfrak{C}_1 x$ and the static boundary condition $\mathfrak{B}_2 x = 0$ (corresponding to the first two boundary conditions and the latter two components of $\mathfrak{B}x$) generates a contractive C_0-semigroup on the product Hilbert space $X \times X_c$. In fact, it is dissipative with energy dissipation

$$\mathrm{Re}\,\langle \mathcal{A}(x, x_c), (x, x_c)\rangle_{X \times X_c} = -k_2 \left|y_{c,1}\right|^2 - k_4 \left|y_{c,2}\right|^2, \quad x \in D(\mathfrak{A}),\ y_c = -\mathfrak{B}_1 x.$$

We investigate the asymptotic properties of the corresponding contractive C_0-semigroup $(\mathcal{T}(t))_{t\geq 0}$ next. We start with the asymptotic stability property where we first remark that the pair $(\mathfrak{A},\mathfrak{B})$ in general does not have property ASP, so that Theorem 5.4.3 is not applicable in this step. Since the hybrid operator $\mathcal{A}$ has compact resolvent and is dissipative, we need to prove that it has no eigenvalues on the imaginary axis. First we show that 0 is no eigenvalue of $\mathcal{A}$. Namely, let $(x,x_c)\in D(\mathcal{A})$ such that $\mathcal{A}(x,x_c)=0$. Then, in particular

$$0 = A_c x_c + B_c u_c = -(C_c x_c + D_c u_c) = \mathfrak{B}_1 x$$

and $x\in D(\mathfrak{A})$ solves the eigenvalue problem

$$\mathfrak{A}x = 0, \quad \mathfrak{B}x = 0$$

i.e.

$$\begin{aligned} (\mathcal{H}_1 x_1)'' = 0, &\quad (\mathcal{H}_1 x_1)(0) = (\mathcal{H}_1 x_1)'(0) = 0\\ (\mathcal{H}_2 x_2)'' = 0, &\quad (\mathcal{H}_2 x_2)(1) = (\mathcal{H}_2 x_2)'(1) = 0. \end{aligned}$$

The unique solution of this problem is $\mathcal{H}x = 0$ and then also $\mathfrak{C}x = 0$, so that $x_c = -A_c^{-1}B_c\mathfrak{C}_1 x = 0$ and $0\notin\sigma_p(\mathcal{A})$. Next we take any $\beta\in\mathbb{R}$ and show that $i\beta\notin\sigma_p(\mathcal{A})$ also does not lie in the point spectrum of $\mathcal{A}$. Let $(x,x_c)\in D(\mathcal{A})$ such that $\mathcal{A}(x,x_c) = i\beta(x,x_c)$ and in particular $\operatorname{Re}\langle\mathcal{A}(x,x_c),(x,x_c)\rangle_{X\times X_c} = 0$, so that $\mathfrak{B}_1 x = 0$ and then

$$x_c = \frac{1}{i\beta}\left(A_c x_c + B_c u_c\right) = \frac{1}{i\beta}\mathfrak{B}_1 x = 0.$$

Since B_c is invertible it also follows that $u_c = \mathfrak{C}_1 x = 0$, so that $\mathfrak{A}x = i\beta x$ and $\mathfrak{B}x = 0$, $\mathfrak{C}_1 x = 0$. For $R = (\mathfrak{B},\mathfrak{C}_1)$ we have already seen that the pair $(\mathfrak{A},R)$ has property ASP, so that again $x=0$ and we conclude $i\mathbb{R}\cap\sigma_p(\mathcal{A}) = \emptyset$ and asymptotic stability follows from Corollary 2.2.16. For uniform exponential stability we may then employ Theorem 5.4.3 since $R = (\mathfrak{B},\mathfrak{C}_1)$ as above has property AIEP and therefore $(\mathcal{T}(t))_{t\geq 0}$ is uniformly exponentially stable. □

Remark 5.7.3. *Let us return to the previous example on the Euler-Bernoulli Beam with a mass at the tip. The damping by dynamic boundary feedback took place at the left end of the beam, whereas at the right end we imposed the free end boundary conditions*

$$(EI\omega_{\zeta\zeta})(t,1) = -(EI\omega_{\zeta\zeta})_\zeta(t,1) = 0$$

or, in the port-Hamiltonian language,

$$(\mathcal{H}_2 x_2)(1) = -(\mathcal{H}_2 x_2)'(1) = 0.$$

Clearly any of these conservative boundary conditions may be replaced by a dissipative boundary condition, e.g.

$$(\mathcal{H}_2 x_2)(1) = -\alpha(\mathcal{H}_1 x_1)'(1)$$

for some $\alpha > 0$ and the contraction and uniform exponential stability property of the corresponding semigroup persist. On the other hand, if instead the conservative boundary condition

$$(\mathcal{H}_1 x_1)'(1) = 0$$

is imposed, the system is not asymptotically stable any more since 0 *is an eigenvalue for the following choice of* $(x, x_c) \in D(\mathcal{A}) \setminus \{0\}$.

$$x(\zeta) = (0, \mathcal{H}_2^{-1}(\zeta)1), \quad x_c = -A_c^{-1} B_c \mathfrak{C}_1 x = \begin{pmatrix} \frac{1}{k_1} \\ 0 \end{pmatrix}.$$

Similarly, if the other conservative boundary condition $(\mathcal{H}_2 x_2)'(1) = 0$ *is replaced by the conservative boundary condition*

$$(\mathcal{H}_1 x_1)(1) = 1$$

we obtain an eigenfunction $(x, x_c) \in D(\mathcal{A})$ *for the eigenvalue* $\beta = 0$ *by setting*

$$x(\zeta) = (0, \mathcal{H}_2^{-1}(\zeta)(1-\zeta)), \quad x_c = -A_c^{-1} B_c \mathfrak{C}_1 x = \begin{pmatrix} 0 \\ \frac{1}{k_3} \end{pmatrix}.$$

Example 5.7.4 (Dynamic boundary control of a flexible rotating beam)**.** *Next we consider the example of beam, modelled by an Euler-Bernoulli beam equation, which should be controlled via the following control equation, which for the constant parameter case, i.e. a uniform beam, may be found in Section 5.3 of [LuGuMo99].*

$$\begin{aligned}
\frac{\partial^2 z}{\partial t^2}(t,\zeta) + \frac{1}{\rho(\zeta)} \frac{\partial^2}{\partial \zeta^2} \left(EI \frac{\partial^2 z}{\partial \zeta^2} \right)(t,\zeta) &= -\zeta \Theta_{tt}(t), & \zeta \in (0,1) \\
z(t,0) = \frac{\partial z}{\partial \zeta}(t,0) &= 0 \\
-\frac{\partial}{\partial \zeta} \left(EI \frac{\partial^2 z}{\partial \zeta^2} \right)(t,1) &= f_1(t) \\
\left(EI \frac{\partial^2 z}{\partial \zeta^2} \right)(t,1) &= f_2(t) \\
\Theta_{tt}(t) &= (EI z_{\zeta\zeta})(t,0) + \tau(t), & t \geq 0
\end{aligned}$$

As a first step we get rid of the Θ_{tt}*-term in the first equation by considering the evolution of the new variable* $\omega(t,\zeta) := z(t,\zeta) + \zeta\Theta(t)$ $(\zeta \in [0,1],\ t \geq 0)$. *We obtain the new system*

$$\begin{aligned}
\rho\omega_{tt} + (EI\omega_{\zeta\zeta})_{\zeta\zeta} &= 0, & \zeta \in (0,1) \\
\omega(t,0) &= 0 \\
\omega_\zeta(t,0) &= \Theta(t) \\
-(EI\omega_{\zeta\zeta})_\zeta(t,1) &= f_1(t) \\
(EI\omega_{\zeta\zeta})(t,1) &= f_2(t) \\
\Theta_{tt}(t) &= (EI\omega_{\zeta\zeta})(t,0) + \tau(t), & t \geq 0
\end{aligned}$$

This new system may be seen as a port-Hamiltonian system (of Euler-Bernoulli type) with some boundary control at the right hand side ($\zeta = 1$) *through the input functions* f_1 *and* f_2 *and with some dynamic control at the left hand side through the additional variable* Θ, *which may also be influenced by an additional input function* τ. *The maximal port-Hamiltonian operator* $\mathfrak{A}$ *is given by*

$$\begin{aligned}
x &= (x_1, x_2) := (\rho\omega_t, \omega_{\zeta\zeta}), \\
\mathcal{H} &:= \begin{bmatrix} \frac{1}{\rho} & \\ & EI \end{bmatrix}, \quad P_1 = P_0 = 0, \quad P_2 = \begin{bmatrix} & -1 \\ 1 & \end{bmatrix}
\end{aligned}$$

and for the choice

$$\mathfrak{B}x = \begin{pmatrix} -(\mathcal{H}x)_2'(1) \\ (\mathcal{H}x)_2(1) \\ (\mathcal{H}x)_1'(0) \\ (\mathcal{H}x)_1(0) \end{pmatrix} \hat{=} \begin{pmatrix} -(EI\omega_{\zeta\zeta})(t,1) \\ (EI\omega_{\zeta\zeta})(t,1) \\ \omega_{t\zeta}(t,0) \\ \omega_t(t,0) \end{pmatrix}$$
$$\mathfrak{C}x = \begin{pmatrix} -(\mathcal{H}x)_1(1) \\ (\mathcal{H}x)_1'(1) \\ (\mathcal{H}x)_2(0) \\ (\mathcal{H}x)_2'(0) \end{pmatrix} \hat{=} \begin{pmatrix} -\omega_t(t,1) \\ \omega_{t\zeta}(t,1) \\ (EI\omega_{\zeta\zeta})(t,0) \\ (EI\omega_{\zeta\zeta})_\zeta(t,0) \end{pmatrix}$$

the system $\mathfrak{S} = (\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ becomes an impedance passive port-Hamiltonian Boundary Control and Observation System. We consider two stabilisation problems, which actually require different choices of the functions $f_1(t), f_2(t)$ and $\tau(t)$, which should be determined by the state of the system at time $t \geq 0$, but for which we also choose different controller state spaces.

1) ***The stabilisation problem****. Our first stabilisation aim is to bring the system at rest, i.e. "$y_t, y_{\zeta\zeta}, \Theta_t \xrightarrow{t\to\infty} 0$". Since we do not mind the asymptotic value of Θ, but only $\dot{\Theta}$ is relevant, we choose the state space $X_c = \mathbb{F}$, identifying $x_c \hat{=} \Theta_t$ and controller input and output space $U_c = Y_c = \mathbb{F}^4$. Then $\dot{x}_c(t) = -\mathfrak{B}_3 x(t) + \tau(t)$. We also chose the control functions f_1, f_2 and τ similar to [LuGuMo99] and let*

$$\begin{aligned} f_1(t) &= -\alpha_1 z_t(t,1) \hat{=} -\alpha_1 \mathfrak{C}_1 x(t) - \alpha_1 x_c(t) \\ f_2(t) &= -\alpha_2 z_{t\zeta}(t,1) \hat{=} -\alpha_2 \mathfrak{C}_2 x(t) - \alpha_2 x_c(t) \\ \tau(t) &= -(f_1(t) + f_2(t) + \alpha_3 x_c) =: f_3(t) - (f_1(t) + f_2(t)) \end{aligned}$$

where $\alpha_j \geq 0$ are non-negative constants for $j = 1, \dots, 3$. Then the controller has the form

$$\begin{aligned} \dot{x}_c &= -(\alpha_1 + \alpha_2 + \alpha_3) x_c + \begin{bmatrix} \alpha_1 & \alpha_2 & -1 & 0 \end{bmatrix} u_c \\ &=: A_c x_c + B_c u_c \\ y_c &= \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ -1 \\ 0 \end{pmatrix} x_c + \begin{bmatrix} \alpha_1 & & & \\ & \alpha_2 & & \\ & & 0 & \\ & & & 0 \end{bmatrix} u_c \\ &=: C_c x_c + D_c u_c \end{aligned}$$

which is impedance passive and interconnected via $u_c = \mathfrak{C}x$ and $\mathfrak{B}x = -y_c$ with the infinite dimensional Euler-Bernoulli port-Hamiltonian system. We then have that

$$\begin{aligned} |\mathfrak{B}x|^2 + |D_c \mathfrak{C}x|^2 \gtrsim& \left|(\mathcal{H}x)'(1)\right|^2 + \left|(\mathcal{H}x)_2(1)\right|^2 + \left|(\mathcal{H}x)_1'(0)\right|^2 \\ &+ \left|(\mathcal{H}x)_1(0)\right|^2 + \alpha_1 \left|(\mathcal{H}x)_1(1)\right|^2 + \alpha_2 \left|(\mathcal{H}x)_1'(1)\right|^2 \end{aligned}$$

and wee see that if both $\alpha_1, \alpha_2 > 0$ by Proposition 4.2.2 (ASP) and Proposition 4.3.19 (AIEP) the pair $(\mathfrak{A}, R)$ for the function

$$Rx = ((\mathcal{H}x)(1), (\mathcal{H}x)'(1), (\mathcal{H}x)_1(0, (\mathcal{H}x)'(1))$$

has property ESP and also the controller is internally stable, so that the mixed dynamic and static feedback law uniformly exponentially stabilises the system, i.e. solves the "stabilisation problem." The control functions retranslate into the original *problem as*

$$\begin{aligned}
\frac{\partial^2 z}{\partial t^2}(t,\zeta) + \frac{1}{\rho}\frac{\partial^2}{\partial \zeta^2}\left(EI\frac{\partial^2 z}{\partial \zeta^2}\right)(t,\zeta) &= -\zeta\Theta_{tt}(t), \quad \zeta \in (0,1) \\
z(t,0) = \frac{\partial y}{\partial \zeta}(t,0) &= 0 \\
-\frac{\partial}{\partial \zeta}\left(EI\frac{\partial^2 z}{\partial \zeta^2}\right)(t,1) &= -\alpha_1 z_t(t,1) \\
\left(EI\frac{\partial^2 z}{\partial \zeta^2}\right)(t,1) &= -\alpha_2 z_{t\zeta}(t,1) \\
\Theta_{tt}(t) &= (EIz_{\zeta\zeta})(t,0) + \alpha_1 z_t(t,1) \\
&\quad + \alpha_2 z_{t\zeta}(t,1) - \alpha_3\Theta(t)
\end{aligned}$$

for all $t \geq 0$. However, by this we did not control for which $\Theta_0 \in \mathbb{F}$ we have convergence $\Theta(t) \to \Theta_\infty$ as $t \to +\infty$.

2) ***The orientation problem.*** *Now we try not only to ensure stability "$z_t(t)$ and $z_{\zeta\zeta}$ and $\Theta_t(t) \xrightarrow{t\to\infty} 0$", but also to push Θ to a given target Θ_∞, i.e. $\Theta(t) \xrightarrow{t\to\infty} \Theta_\infty$. We therefore now choose $X_c = \mathbb{F}^2$ as controller state space with controller state space variables $x_c = (x_{c,1}, x_{c,2}) \hat{=} (\Theta - \Theta_\infty, \Theta_t)$ and $U_c = Y_c = \mathbb{F}^4$. Moreover, we choose the control functions f_1, f_2 and τ as*

$$\begin{aligned}
f_1(t) &= -\alpha_1 z_t(t,1) \hat{=} -\alpha_1 \mathfrak{C}_1 x - \alpha_1 x_{c,2} \\
f_2(t) &= -\alpha_2 z_{t\zeta}(t,1) \hat{=} -\alpha_2 \mathfrak{C}_2 x - \alpha_2 x_{c,2} \\
\tau(t) &= -(f_1(t) + f_2(t) + \alpha_3 x_{c,2} + \alpha_4 x_{c,1}) =: f_3(t) - (f_1(t) + f_2(t))
\end{aligned}$$

where $\alpha_j \geq 0$ for $j = 1, \ldots, 3$ and $\alpha_4 > 0$. We equip X_c with the norm

$$\|(\theta, x_c)\|_{X_c}^2 = \alpha_4 |\theta|^2 + |x_c|^2 .$$

In this case the controller has the form

$$\begin{aligned}
\dot{x}_c &= \begin{bmatrix} & 1 \\ -\alpha_4 & -(\alpha_1+\alpha_2+\alpha_3) \end{bmatrix} x_c + \begin{bmatrix} 0 & 0 & 0 & 0 \\ \alpha_1 & \alpha_2 & -1 & 0 \end{bmatrix} u_c \\
&=: A_c x_c + B_c u_c \\
y_c &= \begin{bmatrix} 0 & \alpha_1 \\ 0 & \alpha_2 \\ 0 & -1 \\ 0 & 0 \end{bmatrix} x_c + \begin{bmatrix} \alpha_1 & & & \\ & \alpha_2 & & \\ & & 0 & \\ & & & 0 \end{bmatrix} u_c \\
&=: C_c x_c + D_c u_c.
\end{aligned}$$

Note that by our choice of $\|\cdot\|_{X_c}$ this controller is impedance passive and again its interconnection with the infinite dimensional system is given by $u_c = \mathfrak{C}x$ and $\mathfrak{B}x = -y_c$. Also in this case we obtain for $\alpha_1, \alpha_2 > 0$ that

$$\begin{aligned}
|\mathfrak{B}x|^2 + |D_c\mathfrak{C}x|^2 \gtrsim\ & |(\mathcal{H}x)'(1)|^2 + |(\mathcal{H}x)_2(1)|^2 + |(\mathcal{H}x)_1'(0)|^2 + |(\mathcal{H}x)_1(0)|^2 \\
&+ |(\mathcal{H}x)_1(1)|^2 + |(\mathcal{H}x)_1'(1)|^2
\end{aligned}$$

and for the controller to be uniformly exponentially stable (and thus uniformly exponentially stabilising the infinite dimensional port-Hamiltonian system) we need to ensure that $\sigma(A_c) \subseteq \mathbb{C}_0^-$. *For* $\bar{\alpha} := \alpha_1 + \alpha_2 + \alpha_3 > 0$ *and* $\alpha_4 > 0$ *one easily obtains the eigenvalues*

$$\lambda_{1,2} = -\frac{\bar{\alpha}}{2} \pm \sqrt{\frac{\bar{\alpha}^2}{4} - \alpha_4} \in \mathbb{C}_0^-.$$

Thus, this feedback law exponentially stabilises the system and solves the orientation problem. In the original formulation this means that

$$\begin{aligned}
\frac{\partial^2 z}{\partial t^2}(t,\zeta) + \frac{1}{\rho}\frac{\partial^2}{\partial \zeta^2}\left(EI\frac{\partial^2 z}{\partial \zeta^2}\right)(t,\zeta) &= -\zeta\Theta_{tt}(t), \quad \zeta \in (0,1) \\
z(t,0) = \frac{\partial y}{\partial \zeta}(t,0) &= 0 \\
-\frac{\partial}{\partial \zeta}\left(EI\frac{\partial^2 z}{\partial \zeta^2}\right)(t,1) &= -\alpha_1 z_t(t,1) \\
\left(EI\frac{\partial^2 z}{\partial \zeta^2}\right)(t,1) &= -\alpha_2 z_{t\zeta}(t,1) \\
\Theta_{tt}(t) &= (EIz_{\zeta\zeta})(t,0) + \alpha_1 z_t(t,1) + \alpha_2 z_{t\zeta}(t,1) \\
&\quad - \alpha_3\dot{\Theta}(t) - \alpha_4(\Theta(t) - \Theta_\infty)
\end{aligned}$$

for all $t \geq 0$, *uniformly exponentially converges in energy norm to the desired state* $\omega(\zeta) = \omega_\infty(\zeta) = 0$ *and* $\Theta = \Theta_\infty$. □

Chapter 6

Nonlinear Boundary Feedback: the Static Case

We continue with the investigation of infinite-dimensional linear port-Hamiltonian systems with dissipative boundary conditions and generalise the results of Chapter 3 and Chapter 4 to the case where the static boundary feedback operator $K \in \mathbb{F}^{Nd \times Nd}$ is replaced by an m-monotone map $\phi : \mathbb{F}^{Nd} \rightrightarrows \mathbb{F}^{Nd}$. Since in Chapter 4 and the corresponding original article [AuJa14] the Arendt-Batty-Lyubich-Vũ Theorem and the Gearhart-Greiner-Prüss Theorem, which both only hold for the case of linear evolution equations, have been used as main tools we need to find alternative methods to tackle the nonlinear feedback situation. We stress that the infinite-dimensional system in principle remains linear, i.e. we do not consider nonlinear port-Hamiltonian systems for which the Hamiltonian energy functional is non quadratic. However, since the feedback in the new situation is nonlinear we consider the equations in the framework of nonlinear contraction semigroups (see, e.g. [Mi92] and [Sh97]) instead of the easier framework of linear semigroups, following ideas similar to those in [Tr14] for the generation theorem and then exploiting ideas which had actually been used in [Ch+87] and in the linear situation for stability properties. We also point out that the approach of [Vi07] and [Vi+09], where $N = 1$ and linear feedback had been considered, may be used as well to obtain stability results for both the static and dynamic scenario to be investigated in Chapter 7, also see [Le14]. Its drawback is that this method is most likely restricted to the case $N = 1$ and therefore for higher order port-Hamiltonian systems with $N \geq 2$ another approach is needed.

One possible motivation to consider linear port-Hamiltonian systems with nonlinear boundary feedback is the following. We may think of $\phi : \mathbb{F}^{Nd} \rightrightarrows \mathbb{F}^{Nd}$ as being an almost linear feedback control operator, for which its perturbation from the linear case is quite small. Then one would expect that for stabilising purposes this feedback should stabilise in about the same way as a perfectly linear controller would do. Therefore, the following results show that to some extend nonlinear perturbations from the linear case do not harm the stabilisation properties. A word of caution. In some cases the (usually finite dimensional) control systems considered here actually include both a finite dimensional controller and a finite dimensional control target which are connected mechanically via a beam modelled by a infinite-dimensional port-Hamiltonian system, e.g. a wave equation, a Timoshenko beam model or a Euler-Bernoulli beam equation, see e.g. [Le14]. Therefore, the terminology *control system* should not be taken too literally when it comes to applications.

6.1 The Generation Theorem – Static, Nonlinear Case

First we generalise the generation theorem for port-Hamiltonian systems with linear dissipative boundary conditions to linear port-Hamiltonian systems with nonlinear dissipative boundary conditions. The strategy is very similar to the linear case, the main differences being the following. On the one hand the Lumer-Phillips Theorem 2.2.7 is restricted to the linear case, so we need an adequate replacement. Here the Komura-Kato Theorem 2.2.29 does the job, so that again we only need to show that the operator A, which now inherits nonlinear boundary conditions, is m-dissipative and in fact, as we will see, similar to the linear case, an m-dissipative boundary feedback will lead to an m-dissipative operator. For this, in the linear case we reduced the case of the generation theorem to the special case where $\mathcal{H} = I$ is uniformly the identity matrix, i.e. the identity on X as multiplication operator. To do this also in the nonlinear case, the relevant Lemma 3.3.5 has to be formulated in a nonlinear version. This is established by the following result.

Lemma 6.1.1. *Let X be a Hilbert space and $A : D(A) \subseteq X \rightrightarrows X$ be a dissipative, possibly nonlinear and/or multivalued, map. Further assume that $P \in \mathcal{B}(X)$ is coercive. If $A-I$ is surjective, so is $AP-I$ and therefore the map is $AP : D(AP) \subseteq X_P \rightrightarrows X_P$ is m-dissipative on the space $X_P = X$ equipped with the inner product $\langle \cdot, \cdot \rangle_{X_P} := \langle \cdot, P \cdot \rangle$.*

Remark 6.1.2. *Note that this a very special and simple case of Theorem 2 in [CaGu72]. Since the proof of Lemma 6.1.1 is quite elementary we give it nevertheless.*

Proof of Lemma 6.1.1. First we show that the map AP is dissipative on X_P. Take any x and $x' \in D(AP)$, $y \in AP(x)$ and $y' \in AP(x')$. Then $Px, Px' \in D(A)$, $y \in A(Px)$ and $y' \in A(Px')$, so that

$$\operatorname{Re} \langle x - x', y - y' \rangle_P = \operatorname{Re} \langle Px - Px', y - y' \rangle \leq 0.$$

This establishes the dissipativity of AP. For the moment let us assume that $\|P - I\| < \frac{1}{2}$. Then from Neumann's series we conclude that the inverse of $P \in \mathcal{B}(X)$ exists and that its norm respects the inequality $\left\|P^{-1}\right\| \leq \frac{1}{1-\|P-I\|} < \frac{1}{1-\frac{1}{2}} = 2$ so that

$$\|P - I\| \left\|P^{-1}\right\| =: \rho \in (0, 1).$$

We show that for any given $f \in X$ there is $x \in D(AP)$ such that

$$(AP - I)(x) \ni f$$

which is equivalent to the problem

$$(AP - P)(x) \ni f + (I - P)x,$$

or, since $(A-I)^{-1}$ exists,

$$x = \Phi_f(x) := P^{-1}(A-I)^{-1}\left(f + (I-P)x\right).$$

We show that the map $\Phi_f : X \to X$ is a strict contraction and therefore admits a unique fixed point $x_f =: (AP-I)^{-1}f$. In fact, we have for every x and $x' \in X$ that

$$\begin{aligned}
&\|\Phi_f(x) - \Phi_f(x')\| \\
&\le \left\|P^{-1}\right\| \left\|(A-I)^{-1}(f+(I-P)x) - (A-I)^{-1}(f+(I-P)x')\right\| \\
&\le \left\|P^{-1}\right\| \|(f+(I-P)x) - (f+(I-P)x')\| \\
&\le \left\|P^{-1}\right\| \|I-P\| \, \|x-x'\| = \rho\, \|x-x'\|
\end{aligned}$$

where we used Remark 2.2.22 in the second step. Therefore, Φ_f is a strict contraction and the Strict Contraction Principle Proposition 2.1.12 gives a unique solution $x_f =: (AP-I)^{-1}f$. In the second step we remove the restriction on P. Namely thanks to Proposition 2.1.13 there are a number $n \in \mathbb{N}$ and a coercive operator $Q = P^{1/n} \in \mathcal{B}(X)$ such that $\|I-Q\| < \frac{1}{2}$ and $P = Q^n$. Note that for all the norms induced by the inner products

$$\langle \cdot, \cdot \rangle_k := \langle \cdot, Q^k \cdot \rangle, \quad k = 0, 1, \ldots, n.$$

we have that

$$\begin{aligned}
\|I-Q\|_k^2 &= \sup_{x\neq 0} \frac{\|(I-Q)x\|_k^2}{\|x\|_k^2} = \sup_{x\neq 0} \frac{\langle (I-Q)x, Q^k(I-Q)x\rangle}{\langle x, Q^k x\rangle} \\
&= \sup_{x\neq 0} \frac{\langle (I-Q)Q^{k/2}x, (I-Q)Q^{k/2}x\rangle}{\langle Q^{k/2}x, Q^{k/2}x\rangle} \\
&= \|I-Q\|^2
\end{aligned}$$

where $Q^{k/2}$ may be given by Proposition 2.1.13 (if k is odd). Writing

$$AP - I = (AQ^{n-1})Q - I$$

the general case follows by induction using the spaces $X_k := (X, \|\cdot\|_k)$, $k = 0, 1, \ldots, n$. Above we have seen that that since $\|I-Q\|_k < \frac{1}{2}$ whenever AQ^k is m-dissipative on X_k that AQ^{k+1} is m-dissipative on X_{k+1} ($k = 0, 1, \ldots, n-1$) and therefore $AP = AQ^n$ is m-dissipative on $X_P = X_n$. □

In our particular situation $P = \mathcal{H}$ is the Hamiltonian density multiplication operator, just as in the linear situation.

Theorem 6.1.3. *Let $\mathfrak{S} = (\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ be an impedance passive port-Hamiltonian system. Assume that $\phi : \mathbb{F}^{Nd} \rightrightarrows \mathbb{F}^{Nd}$ is a (possibly multi-valued, nonlinear) m-monotone map. Then the (single-valued) operator*

$$\begin{aligned}
A &= \mathfrak{A}|_{D(A)} \\
D(A) &= \{x \in D(\mathfrak{A}) : \ \mathfrak{B}x \in -\phi(\mathfrak{C}x)\}
\end{aligned}$$

is m-dissipative and therefore it generates a strongly continuous contraction semigroup $(S(t))_{t\ge 0}$ on $X = L_2(0,1;\mathbb{F}^d)$ which is equipped with the inner product $\langle \cdot, \cdot \rangle_X = \langle \cdot, \mathcal{H} \cdot \rangle_{L_2}$.

Remark 6.1.4. *Note that for the case $N = 1$ a characterisation of m-dissipative boundary conditions yielding an m-dissipative operator A has been given in Theorem 5.4 of [Tr14]. Also note the more general result Theorem 3.1 therein.*

Proof. From Lemma 6.1.1 we know that it suffices to consider the case where $\mathcal{H} = I$ equals the identity. Also note that there is $x_0 \in D(A) \neq \emptyset$ which can be constructed by taking any $(u, y) \in \mathbb{F}^{Nd}$ such that $u \in -\phi(y)$ and then taking $x_0 \in H^1(0, 1; \mathbb{F}^d)$ such that $\left(\begin{smallmatrix} \mathfrak{B}x \\ \mathfrak{C}x \end{smallmatrix}\right) = \left(\begin{smallmatrix} u \\ y \end{smallmatrix}\right)$, cf. Lemma 3.2.19. This implies that $x_0 + C_c^\infty(0, 1; \mathbb{F}^d) \subseteq D(A)$ is a dense subset of $X = L_2(0, 1; \mathbb{F}^d)$. Clearly A is dissipative since for every x and $\tilde{x} \in D(A)$ we have

$$\begin{aligned} \operatorname{Re} \langle Ax - A\tilde{x}, x - \tilde{x} \rangle_{L_2} &= \operatorname{Re} \langle \mathfrak{A}(x - \tilde{x}), x - \tilde{x} \rangle_{L_2} \\ &\leq \operatorname{Re} \langle \mathfrak{B}x - \mathfrak{B}\tilde{x}, \mathfrak{C}x - \mathfrak{C}\tilde{x} \rangle_{\mathbb{F}^{Nd}} \leq 0 \end{aligned}$$

using that $\mathfrak{B}x \in -\phi(\mathfrak{C}x)$, $\mathfrak{B}\tilde{x} \in -\phi(\mathfrak{C}\tilde{x})$ and ϕ is monotone. It remains to show that $\operatorname{ran}(I - A) = X$, i.e. for every $f \in X$ we have to find $x \in D(\mathfrak{A})$ such that

$$\begin{aligned} (I - \mathfrak{A})x &= f \\ \mathfrak{B}x &\in -\phi(\mathfrak{C}x). \end{aligned}$$

From Lemma 3.2.24 we know that all solutions of the first of these equations have the form

$$\begin{aligned} x &= \Phi(1)f + \Psi(1)\mathfrak{B}x \\ \mathfrak{C}x &= F(1)f + G(1)\mathfrak{B}x \end{aligned}$$

and the problem thus reduces to finding $u = \mathfrak{B}x$ and $y = \mathfrak{C}x$ such that

$$u = G(1)^{-1}y - G(1)^{-1}F(1)f \in -\phi(y),$$

i.e. $(G(1)^{-1} + \phi)(y) \ni G(1)^{-1}F(1)f$. Since ϕ is m-monotone and $\operatorname{Sym} G(1)^{-1}$ is coercive by Lemma 3.2.24, also $\phi + G(1)^{-1} - \varepsilon I$ is m-monotone by Lemma 2.2.23 for some small $\varepsilon > 0$. We conclude that there is a (unique) $y \in \mathbb{F}^{Nd}$ such that for $u := G(1)^{-1}y - G(1)^{-1}F(1)f$ one has $u \in -\phi(y)$ and hence there is a (unique) $x \in D(A)$ with $f \in (I - A)(x)$. We have shown that A is m-dissipative and the assertion therefore follows from the Komura-Kato Theorem 2.2.29. □

6.2 Exponential Stability: the Case $N = 1$

In the preceding section we established the generation theorem for nonlinear dissipative boundary feedback. Next, we generalise the stabilisation results from the linear case to the situation of nonlinear boundary feedback. We start with the case $N = 1$, i.e. $\mathfrak{A} = P_1(\mathcal{H}\cdot)' + P_0(\mathcal{H}\cdot)$ on $D(\mathfrak{A}) = \{x \in L_2(0, 1; \mathbb{F}^d) : \mathcal{H}x \in H^1(0, 1; \mathbb{F}^d)\}$. Also we always assume that $\mathcal{H}$ and P_0 are Lipschitz continuous, which also have been assumptions for the stabilisation theorems via linear feedback.

We aim to prove the following uniform exponential stability result.

Theorem 6.2.1. *Let $\mathfrak{S} = (\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ be an impedance passive port-Hamiltonian system and $\phi : \mathbb{F}^d \rightrightarrows \mathbb{F}^d$ be an m-monotone map with $0 \in \phi(0)$. For the nonlinear operator*

$$A := \mathfrak{A}|_{D(A)}, \quad D(A) := \{x \in D(\mathfrak{A}) : \mathfrak{B}x \in -\phi(\mathfrak{C}x)\}$$

assume that there is $\kappa > 0$ such that

$$\operatorname{Re} \langle Ax, x\rangle_X \leq -\kappa (x^* \mathcal{H} x)(1), \quad x \in D(A).$$

Then A generates a strongly continuous contraction semigroup $(S(t))_{t\geq 0}$ with globally exponentially stable equilibrium 0, i.e. there are constants $M \geq 1$ and $\omega < 0$ such that

$$\|S(t)x\|_X \leq M e^{\omega t} \|x\|_X, \quad x \in X, \ t \geq 0.$$

Remark 6.2.2. *If $\phi \in \mathcal{B}(U) = \mathbb{F}^{d\times d}$ is linear this is exactly Theorem 4.1.5. We actually give two proofs of this result. The first one is based on the idea of proof for the linear version of this result in [Vi+09] and the Ph.D. thesis [Vi07], here we refer to the same proof in [JaZw12]. That* sideways energy estimate *(or,* final observability estimate*) already originates back to [RaTa74] and had also been used in [CoZu95], both times in the linear scenario. Afterwards we give a proof which is based on the Lyapunov technique proof of Theorem 4.1.5.*

"Sideways energy estimate"-based proof of Theorem 6.2.1. The proof of Lemma 9.1.2 in [JaZw12] extends straight-forward to the situation with nonlinear boundary feedback, see Lemma 4.1.1.

Lemma 6.2.3. *(See Lemma 4.1.1.) Assume that $\mathcal{H} \in W^1_\infty(0,1;\mathbb{F}^{d\times d})$. Then there are constants $c, \tau > 0$ such that for every solution $x \in W^1_{\infty,loc}(\mathbb{R}_+; L_2(0,1;\mathbb{F}^d)) \cap L_{\infty,loc}(\mathbb{R}_+; D(\mathfrak{A}))$ of $\dot{x} = \mathfrak{A}x$ with non-increasing $\|x(t)\|_X$ we have*

$$\|x(\tau)\|^2_{L_2} \leq c \int_0^\tau |(\mathcal{H}x)(t,1)|^2 \, dt.$$

Thus, there are constants $c > 0$ and $\tau > 0$ such that for every $x_0 \in D(A)$ and $x := S(\cdot)x_0 \in W^1_\infty(\mathbb{R}_+; X) \cap L_\infty(\mathbb{R}_+; D(\mathfrak{A}))$ the estimate

$$\|x(\tau)\|^2_X \leq c \int_0^\tau \langle x(t,1), (\mathcal{H}x)(t,1)\rangle_{\mathbb{F}^d} dt$$

holds true. Then

$$\begin{aligned}
\|x(\tau)\|^2_X - \|x_0\|^2_X &= \int_0^\tau \operatorname{Re} \langle Ax(t), x(t)\rangle_X dt \\
&\leq -\kappa \int_0^\tau \langle x(t,1), (\mathcal{H}x)(t,1)\rangle_{\mathbb{F}^d} dt \\
&\leq -\frac{\kappa}{c} \|x(\tau)\|^2_X
\end{aligned}$$

and hence $\|S(\tau)x_0\|_X \leq \sqrt{\frac{c}{c+k}} \|x_0\|_X$, and since $D(A)$ is dense in X this implies that this inequality actually holds for all $x_0 \in X$. From time invariance of the problem and the semigroup property it follows, with $\mathbb{R} \ni s \mapsto \lfloor s \rfloor := \max\{n \in \mathbb{Z} : n \leq s\} \in \mathbb{Z}$ denoting the floor function,

$$\begin{aligned}
\|S(t)x_0\|_X &= \left\| S(\tau)^{\lfloor \frac{t}{\tau} \rfloor} S(t - \tau\lfloor t/\tau \rfloor) x_0 \right\|_X \\
&\leq \left(\sqrt{\frac{c}{c+k}} \right)^{\lfloor \frac{t}{\tau} \rfloor} \|x_0\|_X \\
&\leq \sqrt{\frac{c+k}{c}} e^{-\frac{t}{2\tau} \ln(\frac{c+k}{c})} \|x_0\|_X, \quad t \geq 0, \ x_0 \in X.
\end{aligned}$$

As a result, 0 is a globally uniformly exponentially stable equilibrium of $(S(t))_{t\geq 0}$.
$\square$

Lyapunov technique proof of Theorem 6.2.1. We use Proposition 4.3.8 and proceed just as in the corresponding proof of the linear version of the theorem. This proposition provides us with a function $q : X \to \mathbb{R}$ such that for every bounded Lipschitz-continuous function x the function $q(x)$ is bounded and Lipschitz continuous and satisfies the estimate

$$\|x(t)\|_X^2 + \frac{d}{dt} q(x(t)) \leq c \left|(\mathcal{H}x)(t, 0)\right|^2, \quad \text{a.e. } t \geq 0.$$

Then we use the following general result.

Proposition 6.2.4. *Let $\mathfrak{S} = (\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ be an impedance passive port-Hamiltonian system and A be the m-dissipative operator, resulting from the dissipative feedback $\mathfrak{B}x \in -\phi(\mathfrak{C}x)$ for some m-monotone map $\phi : D(\phi) \subseteq \mathbb{F}^{Nd} \rightrightarrows \mathbb{F}^{Nd}$ with $0 \in \phi(0)$. Further assume that*

$$\operatorname{Re} \langle Ax, x\rangle_X \leq -\kappa \left(|\mathfrak{B}x|^2 + |\Pi\mathfrak{C}x|^2 \right), \quad x \in D(A)$$

for some orthogonal projection $\Pi \in \mathbb{F}^{Nd\times Nd}$. If there is $q : X \to \mathbb{R}$ such that $|q(x)| \leq \hat{c} \|x\|_X^2$ $(x \in X)$ and for all solutions $x \in W^1_\infty(\mathbb{R}_+; X) \cap L_\infty(\mathbb{R}_+; D(\mathfrak{A}))$ of $\dot{x} = \mathfrak{A}x$ one has $q(x) \in W^1_\infty(\mathbb{R}_+)$ and

$$\|x(t)\|_X^2 + \frac{d}{dt} q(x(t)) \leq c \left(|\mathfrak{B}x(t)|^2 + |\Pi\mathfrak{C}x(t)|^2 \right), \quad \text{a.e. } t \geq 0,$$

then 0 is a globally uniformly exponentially stable equilibrium of $(\mathcal{S}(t))_{t\geq 0}$, i.e. there are constants $M \geq 1$ and $\omega < 0$ such that

$$\|S(t)x_0\|_X \leq M e^{\omega t} \|x_0\|_X, \quad x_0 \in X, \ t \geq 0.$$

Proof. Take any $x_0 \in D(A)$ and let $x = S(\cdot)x_0 \in W^1_\infty(\mathbb{R}_+; X) \cap L_\infty(\mathbb{R}_+; D(\mathfrak{A}))$ be the solution of the nonlinear abstract Cauchy problem, so that $q(x) \in W^1_\infty(\mathbb{R}_+; \mathbb{R})$ is Lipschitz continuous. Define the functional

$$\Phi(t) := t \|x(t)\|_X^2 + q(x(t)), \quad t \geq 0.$$

We conclude that $\Phi \in W^1_{\infty,loc}(\mathbb{R}_+; \mathbb{R})$ is locally Lipschitz continuous and for a.e. $t \geq 0$ we have

$$\begin{aligned}\frac{d}{dt}\Phi(t) &= \|x(t)\|_X^2 + 2t \operatorname{Re} \langle Ax(t), x(t)\rangle_X + \frac{d}{dt} q(x(t)) \\ &\leq (c - 2\kappa t) \left(|\mathfrak{B}x(t)|^2 + |\Pi\mathfrak{C}x(t)|^2 \right)\end{aligned}$$

and therefore Φ does not increase on $[t_0, \infty)$ where $t_0 := \frac{c}{2\kappa} > 0$ is independent of the initial value $x_0 \in D(A)$. Using that $q(x) \leq \hat{c} \|x\|_X^2$ and that the semigroup $(S(t))_{t\geq 0}$ is contractive with $S(\cdot)0 \equiv 0$, we then obtain the estimate

$$\begin{aligned} t \|x(t)\|_X^2 &= \Phi(t) - q(x(t)) \leq \Phi(t_0) + \hat{c} \|x(t)\|_X^2 \\ &= t_0 \|x(t_0)\|_X^2 + q(x(t_0)) + \hat{c} \|x(t)\|_X^2 \\ &\leq (t_0 + \hat{c}) \|x_0\|_X^2 + \hat{c} \|x(t)\|_X^2, \quad t \geq 0 \end{aligned}$$

so that for $t > \max\{t_0, \hat{c}\}$ we obtain the estimate

$$\|S(t)x_0\|_X \leq \sqrt{\frac{t_0 + \hat{c}}{t - \hat{c}}}\, \|x_0\|_X .$$

As a consequence there is $\tau > 0$ such that

$$\|S(\tau)x_0\|_X \leq \rho\, \|x_0\|_X$$

for some $\rho \in (0, 1)$ and all $x_0 \in D(A)$. Since $D(A) \subseteq X$ is dense and $S(\tau) \in C(X; X)$ we deduce that the same estimate holds even for all $x_0 \in X$ which by time invariance implies that 0 is a globally uniformly exponentially stable equilibrium, cf. the conclusion of the first proof of Theorem 4.1.5. □

6.3 Stabilisation: the Case $N > 1$

The idea of this section is to obtain stability results similar to those for the static case, this time in the dynamic controller setup. We start with the generalisation of Theorem 4.2.2 to the case of nonlinear static feedback stabilisation.

Theorem 6.3.1. *Assume that $\mathfrak{S} = (\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ is an impedance passive port-Hamiltonian system of order $N \in \mathbb{N}$ and let $\phi : D(\phi) \subseteq \mathbb{F}^{Nd} \rightrightarrows \mathbb{F}^{Nd}$ be m-monotone and such that $0 \in \phi(0)$. For the operator*

$$A = \mathfrak{A}, \quad D(A_\phi) = \{x \in D(\mathfrak{A}) : \ -\mathfrak{B}x \in \phi(\mathfrak{C}x)\}$$

assume that there is $R : D(\mathfrak{A}) \to H$ (for some Hilbert space X) such that the pair $(\mathfrak{A}, R)$ has property ASP and such that

$$\operatorname{Re} \langle Ax, x\rangle_X \leq -p(\|Rx\|_H), \quad x \in D(A)$$

for some $p : \mathbb{R}_+ \to \mathbb{R}_+$ with $p > 0$ on $(0, \infty)$. Then A generates a strongly continuous (nonlinear) contraction semigroup $(S(t))_{t\geq 0}$ on X and 0 is a globally asymptotically stable equilibrium for $(S(t))_{t\geq 0}$.

Proof. In view of Theorem 2.2.32 we remark that $\overline{D(A)} = X$ is convex and since $(I - A)^{-1} : X \to X$ is a contractive mapping from X into the domain $D(A) \subseteq D(\mathfrak{A})$ which is compactly embedded into X, it maps bounded sets to precompact sets. Since $0 \in \phi(0)$, clearly $0 \in D(A)$ and $A(0) = \{0\}$. Let $x_0 \in D(A)$ be arbitrary, so that by Theorem 2.2.32 the solution $S(\cdot)x_0 \in W^1_\infty(\mathbb{R}_+; X) \cap L_\infty(\mathbb{R}_+; D(\mathfrak{A}))$ converges to a compact, $S(\cdot)$-invariant set C which is included in $D(A)$ and for which $(S(t)|_C)_{t\geq 0}$ extends to an isometric group on lin C and such that $C \subseteq \{z_0 \in X : \|z\| = r\}$ for some $r \geq 0$. In particular, for every $z_0 \in C \subseteq D(A)$ we have

$$0 = \operatorname{Re} \langle Az_0, z_0\rangle_X \leq -p(\|Rz_0\|_H)$$

and it follows that $Rz_0 = 0$ for every $z_0 \in C$, so that for every $z_0 \in C$, the function $z = S(\cdot)z_0 = T_C(\cdot)z_0$ is a solution of the problem

$$\frac{d}{dt} z(t) = \mathfrak{A}z(t)$$
$$Rz = 0, \quad t \geq 0$$

and hence also $Rz_0 = 0$ for every $z_0 \in A_C$, the infinitesimal generator of the isometric group $(T_C(t))_{t\geq 0}$ on C. It follows that for every $\beta \in \mathbb{R}$ and $z_0 \in \ker(A_C - i\beta)$ one also has $Rz_0 = 0$, but then $\ker(A_C - i\beta) = \{0\}$ for every $\beta \in \mathbb{R}$, so that $i\mathbb{R} \cap \sigma_p(A_C) = \emptyset$. Then $(T_C(t))_{t\geq 0}$ is both isometric and asymptotically stable thanks to Corollary 2.2.16, so that $C = \{0\}$ must be the null space, and hence the semigroup $(S(t))_{t\geq 0}$ has a globally asymptotically stable equilibrium at $0 \in X$. □

Remark 6.3.2. *The idea to use Theorem 2.2.32 for asymptotic stability has been taken from the proof of Lemma 2.1 in [FeShZh98], although it probably had been already applied for similar problems before.*

Next we investigate uniform exponentially stability. Unfortunately there is no nonlinear generalisation of the Gearhart-Greiner-Prüss-Huang Theorem 2.2.17 available, so that we have to employ other methods. Our results are based on the idea which we used for the (Lyapunov technique) proof of Theorem 6.2.1, where we took $x_0 \in D(A)$ and for $x = S(\cdot)x_0$ and some suitable $\eta \in C^1([0,1];\mathbb{R})$ defined

$$\Phi(t) = t\,\|x(t)\|_X^2 + \langle x(t), \eta P_1^{-1} x(t)\rangle_{L_2}.$$

Stabilisation of Second Order Systems. We aim for a generalisation of Theorem 6.2.1 to the case where

$$\mathfrak{A}x = P_2(\mathcal{H}x)'' + P_1(\mathcal{H}x)' + P_0(\mathcal{H}x)$$

is a port-Hamiltonian operator of second order ($N = 2$). Again we assume that $\mathcal{H}$ and P_0 are Lipschitz continuous. For the case of (static and dynamic) linear feedback stabilisation of Chapter 4 and Chapter 5, also see [AuJa14], we proved uniform exponential stability under the assumption that

$$|(\mathcal{H}x)(0)|^2 + |(\mathcal{H}x)'(0)|^2 + |\Pi(\mathcal{H}x)(1)|^2 + |(I-\Pi)P_2(\mathcal{H}x)'(1)|^2 \lesssim |\mathfrak{B}x|^2 + |\Pi\mathfrak{C}x|^2$$

for all $x \in D(\mathfrak{A})$ and some orthogonal projection $\Pi : \mathbb{F}^d \to \mathbb{F}^d$ and sufficient stability and passivity conditions on the linear control system, e.g. internally stable and SIP. Of course, the proof there used the Gearhart-Greiner-Prüss Theorem, so lacks any possible generalisation to the nonlinear scenario. However, for the Lyapunov technique Proposition 6.2.4 amounts to finding a suitable $q \in C^1(X;\mathbb{R})$ satisfying the assumptions of Proposition 6.2.4. This had already been done in Section 4.3 under the additional assumption that $\mathcal{H}$ is constant and $P_0, P_1 = 0$ equal the zero matrix. Here we extend that result to the case that $P_0, P_1 \neq 0$ may not vanish, but are sufficiently small compared to P_2 at least.

We will use the following notation. For a symmetric matrix $M = M^*$ we denote by $\operatorname{Pos}(M)$ and $\operatorname{Neg}(M)$ its positive and negative semi-definite part, defined via the *LDL-decomposition* (a variant of the Cholesky decomposition) of M as

$$M = LDL^*$$

where L is a unitriangular matrix and $D = D_+ + D_-$ is a diagonal matrix and $D_\pm$ are the diagonal matrices with positive and negative diagonal entries of D, respectively. Then we set $\operatorname{Pos}(M) := LD_+L^*$ and $\operatorname{Neg}(M) := LD_-L^*$. More general, for arbitrary quadratic matrices M we set $\operatorname{Pos}(M) := \operatorname{Pos}(\operatorname{Sym} M)$ and $\operatorname{Neg}(\operatorname{Sym} M)$.

Lemma 6.3.3. *Let $\mathfrak{S} = (\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ be an impedance passive port-Hamiltonian system of order $N = 2$. Further assume that*

1. *$\mathcal{H}$ is constant and P_0, P_1 are small compared to P_2, i.e.*

$$2 > \left\|\mathrm{Neg}\,(P_2^{-1}P_1)\right\| + \frac{1}{\sqrt{2}}\left\|P_0^* P_2^{-1} + P_2^{-1}P_0 - P_1P_2^{-1}P_1P_2^{-1}\right\| + \frac{1}{2}\left\|(P_2^{-1}P_0)^* P_2^{-1}P_0\right\| \tag{6.1}$$

or

2. *$\mathcal{H}', P_0$ and P_1 satisfy the following smallness condition,*

$$\begin{aligned} 2 > &\left\|(\mathcal{H}'\mathcal{H}^{-1} + \mathrm{Neg}\,(P_2^{-1}P_1\mathcal{H})\mathcal{H}^{-1})(\zeta - 1)\right\|_{L_\infty(0,1;\mathbb{F}^{d\times d})} \\ &+ \frac{1}{\sqrt{2}}\left\|(P_0^*P_2^{-1} + P_2^{-1}P_0 - P_1P_2^{-1}P_1P_2^{-1})(\zeta - 1)\right\|_{L_\infty(0,1;\mathbb{F}^{d\times d})} \\ &+ \frac{1}{2}\left\|((P_2^{-1}P_0)^*P_2^{-1}P_0)(\zeta - 1)\right\|_{L_\infty(0,1;\mathbb{F}^{d\times d})} \end{aligned} \tag{6.2}$$

or, more general,

3. *there is a scalar function $\eta \in C^2([0,1];\mathbb{R})$ with $\eta(1) = 0$ and such that*

$$\begin{aligned} 2\eta' \geq\ &\varepsilon + \left\|\mathcal{H}'\mathcal{H}^{-1} + \mathrm{Neg}\,(P_2^{-1}P_1\mathcal{H})\mathcal{H}^{-1}\right\|_{L_\infty(0,1;\mathbb{F}^{d\times d})} \\ &\frac{1}{\sqrt{2}}\left\|-\eta'' + (P_0^*P_2^{-1} + P_2^{-1}P_0 - P_1P_2^{-1}P_1P_2^{-1})\eta\right\|_{L_\infty(0,1;\mathbb{F}^{d\times d})} \\ &+ \frac{1}{2}\left\|(P_2^{-1}P_0)^*P_2^{-1}P_0\eta\right\|_{L_\infty(0,1;\mathbb{F}^{d\times d})} \end{aligned}$$

for some $\varepsilon > 0$.

Then there is $q : X \to \mathbb{R}_+$ such that $|q(x)| \leq \hat{c}\left\|x\right\|_X^2$ $(x \in X)$ and for all solutions $x \in W_\infty^1(\mathbb{R}_+;X) \cap L_\infty(\mathbb{R}_+;D(\mathfrak{A}))$ of $\dot{x} = \mathfrak{A}x$ the function $q(x)$ lies in $W_\infty^1(\mathbb{R}_+)$ with

$$\left\|x(t)\right\|_X^2 + \frac{d}{dt}q(x(t)) \leq c\left(\left|(\mathcal{H}x)(t,0)\right|^2 + \left|(\mathcal{H}x)'(t,0)\right|^2 + \left|(\mathcal{H}x)(t,1)\right|^2\right)$$

for a.e. $t \geq 0$.

Proof. Define

$$\begin{aligned} q(x) := &\ \mathrm{Re}\,\langle x, \eta P_2^{-1}\int_0^\cdot x(\xi)d\xi\rangle_{L_2} \\ &- \frac{1}{2}\langle P_2^{-1}\int_0^\cdot x(\xi)d\xi, \eta P_1P_2^{-1}\int_0^\cdot x(\xi)d\xi\rangle_{L_2}, \quad x \in X \end{aligned} \tag{6.3}$$

where $\eta \in C^\infty([0,1];\mathbb{R})$ is a scalar function to be chosen suitable later on. Then for every $x \in W_\infty^1(\mathbb{R}_+;X) \cap L_\infty(\mathbb{R}_+;D(\mathfrak{A}))$ with $\dot{x} = \mathfrak{A}x$ we obtain (omitting the parameter t for brevity and employing Lemma 4.3.2) that

$$
\begin{aligned}
&\frac{d}{dt}q(x)\\
&= \operatorname{Re}\langle P_2(\mathcal{H}x)'', \eta P_2^{-1}\int_0^{\cdot} x(\xi)d\xi\rangle_{L_2} + \operatorname{Re}\langle x, \eta\int_0^{\cdot} (\mathcal{H}x)''(\xi)d\xi\rangle_{L_2}\\
&\quad + \operatorname{Re}\langle P_1(\mathcal{H}x)' + P_0(\mathcal{H}x), \eta P_2^{-1}\int_0^{\cdot} x(\xi)d\xi\rangle_{L_2}\\
&\quad + \operatorname{Re}\langle x, \eta P_2^{-1}\int_0^{\cdot} P_1(\mathcal{H}x)'(\xi) + P_0(\mathcal{H}x)(\xi)d\xi\rangle_{L_2}\\
&\quad - \operatorname{Re}\langle \int_0^{\cdot} (\mathcal{H}x)''(\xi)d\xi, \eta P_1 P_2^{-1}\int_0^{\cdot} x(\xi)d\xi\rangle_{L_2}\\
&\quad - \operatorname{Re}\langle P_2^{-1}\int_0^{\cdot} P_1(\mathcal{H}x)'(\xi) + P_0(\mathcal{H}x)(\xi)d\xi, \eta P_1 P_2^{-1}\int_0^{\cdot} x(\xi)d\xi\rangle_{L_2}\\
&= 2\operatorname{Re}\langle (\mathcal{H}x)', \eta x\rangle_{L_2} + \operatorname{Re}\langle (\mathcal{H}x)', \eta'\int_0^{\cdot} x(\xi)d\xi\rangle_{L_2}\\
&\quad - \operatorname{Re}\,\eta(1)\langle (\mathcal{H}x)'(1), \int_0^1 x(\xi)d\xi\rangle_{\mathbb{F}^d} - \operatorname{Re}\langle (\mathcal{H}x)'(0), \eta x\rangle_{L_2}\\
&\quad + \operatorname{Re}\langle P_1(\mathcal{H}x)', \eta P_2^{-1}\int_0^{\cdot} x(\xi)d\xi\rangle_{L_2} + \operatorname{Re}\langle x, \eta\int_0^{\cdot} P_2^{-1}P_1(\mathcal{H}x)'(\xi)d\xi\rangle_{L_2}\\
&\quad - \operatorname{Re}\langle \int_0^{\cdot} x(\xi)d\xi, \eta P_2^{-1}P_0(\mathcal{H}x)\rangle_{L_2} + \operatorname{Re}\langle x, \eta P_2^{-1}P_0\int_0^{\cdot} (\mathcal{H}x)(\xi)d\xi\rangle_{L_2}\\
&\quad - \operatorname{Re}\langle P_1(\mathcal{H}x)', \eta P_2^{-1}\int_0^{\cdot} x(\xi)d\xi\rangle_{L_2} + \operatorname{Re}\langle P_1(\mathcal{H}x)'(0), \eta P_2^{-1}\int_0^{\cdot} x(\xi)d\xi\rangle_{L_2}\\
&\quad - \operatorname{Re}\langle P_2^{-1}P_1(\mathcal{H}x), \eta P_1 P_2^{-1}\int_0^{\cdot} x(\xi)d\xi\rangle_{L_2}\\
&\quad + \operatorname{Re}\langle P_2^{-1}P_1(\mathcal{H}x)(0), \eta P_1 P_2^{-1}\int_0^{\cdot} x(\xi)d\xi\rangle_{L_2}\\
&\quad - \operatorname{Re}\langle \int_0^{\cdot} P_2^{-1}P_0(\mathcal{H}x)(\xi), \eta P_1 P_2^{-1}\int_0^{\cdot} x(\xi)d\xi\rangle_{L_2}\\
&= \langle (\eta\mathcal{H}' - 2\eta'\mathcal{H})x, x\rangle_{L_2} + [\langle x(\zeta), (\eta\mathcal{H})(\zeta)x(\zeta)\rangle_{\mathbb{F}^d}]_0^1\\
&\quad - \operatorname{Re}\langle \mathcal{H}x, \eta''\int_0^{\cdot} x(\xi)d\xi\rangle_{L_2} + \eta'(1)\operatorname{Re}\langle (\mathcal{H}x)(1), \int_0^1 x(\xi)d\xi\rangle_{\mathbb{F}^d}\\
&\quad - \eta(1)\operatorname{Re}\langle (\mathcal{H}x)'(1), \int_0^1 x(\xi)d\xi\rangle_{\mathbb{F}^d} - \operatorname{Re}\langle x, \eta(\mathcal{H}x)'(0)\rangle_{L_2}\\
&\quad + \operatorname{Re}\langle x, \eta P_2^{-1}P_1(\mathcal{H}x)\rangle_{L_2} - \operatorname{Re}\langle x, \eta P_2^{-1}P_1(\mathcal{H}x)(0)\rangle_{L_2}\\
&\quad - \operatorname{Re}\langle \int_0^{\cdot} x(\xi)d\xi, \eta P_2^{-1}P_0(\mathcal{H}x)\rangle_{L_2} + \operatorname{Re}\langle x, \eta P_2^{-1}P_0\int_0^{\cdot} (\mathcal{H}x)(\xi)d\xi\rangle_{L_2}\\
&\quad + \operatorname{Re}\langle P_1(\mathcal{H}x)'(0), \eta P_2^{-1}\int_0^{\cdot} x(\xi)d\xi\rangle_{L_2}\\
&\quad - \operatorname{Re}\langle P_2^{-1}P_1(\mathcal{H}x), \eta P_1 P_2^{-1}\int_0^{\cdot} x(\xi)d\xi\rangle_{L_2}
\end{aligned}
$$

$$
\begin{aligned}
&\quad + \operatorname{Re} \langle P_2^{-1} P_1 (\mathcal{H}x)(0), \eta P_1 P_2^{-1} \int_0^{\cdot} x(\xi) d\xi \rangle_{L_2} \\
&\quad - \operatorname{Re} \langle \int_0^{\cdot} P_2^{-1} P_0 (\mathcal{H}x)(\xi), \eta P_1 P_2^{-1} \int_0^{\cdot} x(\xi) d\xi \rangle_{L_2} \\
&\leq \langle (\varepsilon \mathcal{H} + \eta \mathcal{H}' - 2\eta' \mathcal{H} + \eta \operatorname{Re} (P_2^{-1} P_1 \mathcal{H})) x, x \rangle_{L_2} \\
&\quad + \operatorname{Re} \langle \mathcal{H}x, (-\eta'' + P_0^* P_2^{-1} \eta + P_2^{-1} P_0 \eta - P_1 P_2^{-1} P_1 P_2^{-1} \eta) \int_0^{\cdot} x(\xi) d\xi \rangle_{L_2} \\
&\quad - \operatorname{Re} \langle P_2^{-1} P_0 \int_0^{\cdot} \mathcal{H}x(\xi), \eta P_2^{-1} P_0 \int_0^{\cdot} x(\xi) d\xi \rangle_{L_2} \\
&\quad + c_{\varepsilon,\eta} \Big((1 + |\eta(0)|^2) \left|(\mathcal{H}x)(0)\right|^2 + \left|(\mathcal{H}x)'(0)\right|^2 \\
&\qquad + |\eta(1)|^2 \left|(\mathcal{H}x)(1)\right|^2 + |\eta'(1)|^2 \left|(\mathcal{H}x)(1)\right|^2 \Big) \qquad (6.4)
\end{aligned}
$$

for every $\varepsilon > 0$ and a constant $c_{\varepsilon,\eta} > 0$ which may depend on $\varepsilon > 0$ and η, but which is independent of x. We now estimate in the following ways. On the one hand

$$
\begin{aligned}
&\operatorname{Re} \langle \mathcal{H}x, (-\eta'' + P_0^* P_2^{-1} \eta + P_2^{-1} P_0 \eta + P_2^{-1} P_0 \eta - P_1 P_2^{-1} P_1 P O_2^{-1} \eta) \int_0^{\cdot} x(\xi) d\xi \rangle_{L_2} \\
&\leq \|\mathcal{H}x\|_{L_2} \left\| -\eta'' + (P_0^* P_2^{-1} + P_2^{-1} P_0 - P_1 P_2^{-1} P_1 P_2^{-1}) \eta \right\|_{L_\infty(0,1;\mathbb{F}^{d \times d})} \left\| \int_0^{\cdot} x(\xi) d\xi \right\|_{L_2} \\
&\leq \|\mathcal{H}x\|_{L_2} \left\| -\eta'' + (P_0^* P_2^{-1} + P_2^{-1} P_0 - P_1 P_2^{-1} P_1 P_2^{-1}) \eta \right\|_{L_\infty(0,1;\mathbb{F}^{d \times d})} \frac{1}{\sqrt{2}} \|x\|_{L_2}
\end{aligned}
$$

and on the other hand

$$
\begin{aligned}
&- \operatorname{Re} \langle P_2^{-1} P_0 \int_0^{\cdot} (\mathcal{H}x)(\xi) d\xi, \eta P_2^{-1} P_0 \int_0^{\cdot} x(\xi) d\xi \rangle_{L_2} \\
&\leq \left\| (P_2^{-1} P_0)^* P_2^{-1} P_0 \eta \right\|_{L_\infty(0,1;\mathbb{F}^{d \times d})} \left\| \int_0^{\cdot} (\mathcal{H}x)(\xi) d\xi \right\|_{L_2} \left\| \int_0^{\cdot} x(\xi) d\xi \right\|_{L_2} \\
&\leq \left\| (P_2^{-1} P_0)^* P_2^{-1} P_0 \eta \right\|_{L_\infty(0,1;\mathbb{F}^{d \times d})} \frac{1}{2} \|\mathcal{H}x\|_{L_2} \|x\|_{L_2}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\frac{d}{dt} q(x) \\
&\leq \langle \left[\varepsilon - 2\eta' + \eta (\mathcal{H}' \mathcal{H}^{-1} + 2 \operatorname{Re} (P_2^{-1} P_1 \mathcal{H}) \mathcal{H}^{-1}) \right] \mathcal{H}x, x \rangle_{L_2} \\
&\quad - \left[\frac{\left\| -\eta'' + (P_0^* P_2^{-1} + P_2^{-1} - P_1 P_2^{-1} P_1 P_2^{-1}) \eta \right\|_{L_\infty(0,1;\mathbb{F}^{d \times d})}}{\sqrt{2}} \right. \\
&\qquad \left. + \frac{\left\| (P_2^{-1} P_0)^* P_2^{-1} P_0 \eta \right\|_{L_\infty(0,1;\mathbb{F}^{d \times d})}}{2} \right] \|\mathcal{H}x\|_{L_2} \|x\|_{L_2}
\end{aligned}
$$

We write $\operatorname{Neg} M$ for the negative semi-definite part of a matrix M and with this notation need to show that

$$\begin{aligned}
2\eta' \geq \varepsilon &+ \left\| \mathcal{H}'\mathcal{H}^{-1} + \operatorname{Neg}(P_2^{-1}P_1\mathcal{H})\mathcal{H}^{-1} \right\|_{L_\infty(0,1;\mathbb{F}^{d\times d})} \\
&+ \frac{1}{\sqrt{2}} \left\| -\eta'' + (P_0^* P_2^{-1} + P_2^{-1}P_0 - P_1 P_2^{-1} P_1 P_2^{-1})\eta \right\|_{L_\infty(0,1;\mathbb{F}^{d\times d})} \\
&+ \frac{1}{2} \left\| (P_2^{-1}P_0)^* P_2^{-1} P_0 \eta \right\|_{L_\infty(0,1;\mathbb{F}^{d\times d})}
\end{aligned}$$

for a suitable choice of the scalar function η. In particular, for the choice $\eta(\zeta) = 1-\zeta$ we obtain the condition

$$\begin{aligned}
&\left\| (\mathcal{H}'\mathcal{H}^{-1} + \operatorname{Neg}(P_2^{-1}P_1\mathcal{H})\mathcal{H}^{-1})(\zeta - 1) \right\|_{L_\infty(0,1;\mathbb{F}^{d\times d})} \\
&+ \frac{1}{\sqrt{2}} \left\| (P_0^* P_2^{-1} + P_2^{-1}P_0 - P_1 P_2^{-1} P_1 P_2^{-1})(\zeta - 1) \right\|_{L_\infty(0,1;\mathbb{F}^{d\times d})} \\
&+ \frac{1}{2} \left\| (P_2^{-1}P_0)^* P_2^{-1} P_0 (\zeta - 1) \right\|_{L_\infty(0,1;\mathbb{F}^{d\times d})} \\
&< 2
\end{aligned}$$

The assertion follows. □

Remark 6.3.4. *As we have seen in Subsection 4.3.2 at least for the linear case such conditions on $\mathcal{H}$ are not needed and therefore one would expect that the same stability result should also hold for the case of nonlinear static feedback without any restrictions on P_0, P_1 and $\mathcal{H}$. Therefore, the previous result is not fully satisfactory and it might be possible to find a reasoning which does not depend on smallness conditions on P_0, P_1 and $\mathcal{H}'$.*

6.4 Stabilisation of the Euler-Bernoulli Beam

We investigate how the general result Proposition 6.2.4 may be used to design uniformly exponentially stabilising controllers for the Euler-Bernoulli Beam equation, i.e. the dynamical system governed by the PDE

$$\rho(\zeta)\omega_{tt}(t,\zeta) + (EI\omega_{\zeta\zeta})_{\zeta\zeta}(t,\zeta) = 0, \quad \zeta \in (0,1),\ t \geq 0. \tag{6.5}$$

The energy of the system is given by

$$H(t) = \frac{1}{2}\int_0^1 \rho(\zeta)\left|\omega_t(t,\zeta)\right|^2 + EI(\zeta)\left|\omega_{\zeta\zeta}(t,\zeta)\right|^2 d\zeta, \quad t \geq 0$$

and we have seen in Example 3.1.6 that for the choice

$$\begin{aligned}
x &= \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} := \begin{pmatrix} \rho\omega_t \\ \omega_{\zeta\zeta} \end{pmatrix}, \qquad \mathcal{H} = \begin{pmatrix} \mathcal{H}_1 & \\ & \mathcal{H}_2 \end{pmatrix} := \begin{pmatrix} \rho^{-1} & \\ & EI \end{pmatrix}, \\
P_2 &= \begin{pmatrix} & -P^* \\ P & \end{pmatrix} := \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}
\end{aligned}$$

and $P_0 = P_1 = 0$ equation (6.5) takes the port-Hamiltonian form

$$\frac{\partial}{\partial t}x(t,\zeta) = P_2 \frac{\partial^2}{\partial \zeta^2}\mathcal{H}x(t,\zeta) =: (\mathfrak{A}x(t))(\zeta), \quad \zeta \in (0,1),\ t \geq 0.$$

Note that

$$\begin{aligned}\operatorname{Re}\langle \mathfrak{A}x, x\rangle_X &= \operatorname{Re}\left[\langle (\mathcal{H}_1 x_1)'(\zeta), (\mathcal{H}_2 x_2)(\zeta)\rangle_{\mathbb{F}} - \langle (\mathcal{H}_1 x_1)(\zeta), (\mathcal{H}_2 x_2)'(\zeta)\rangle_{\mathbb{F}}\right]_0^1 \\ &\mathrel{\hat=} \operatorname{Re}\left[\langle \omega_{t\zeta}(\zeta), (EI\omega_{\zeta\zeta})(\zeta)\rangle_{\mathbb{F}} - \langle \omega_t(\zeta), (EI\omega_{\zeta\zeta})_\zeta(\zeta)\rangle_{\mathbb{F}}\right]_0^1\end{aligned}$$

for all $x \mathrel{\hat=} \left(\begin{smallmatrix}\rho\omega_t \\ \omega_{\zeta\zeta}\end{smallmatrix}\right) \in D(\mathfrak{A})$. From here, several choices of $\mathfrak{B}$ and $\mathfrak{C}$ are possible to make $\mathfrak{S} = (\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ an impedance passive port-Hamiltonian Boundary Control and Observation System. In that case (and provided that $\rho^{-1}, EI \in L_\infty(0,1)$ are uniformly positive) for any m-monotone $\phi : D(\phi) \subseteq \mathbb{F}^2 \rightrightarrows \mathbb{F}^2$ the operator $A = \mathfrak{A}|_{D(A)}$, $D(A) = \{x \in D(\mathfrak{A}) : \mathfrak{B}x \in -\phi(\mathfrak{C}x)\}$ generates a s.c. contraction semigroup on $X = L_2(0,1;\mathbb{F}^2)$ (which is a C_0-semigroup if $\phi \in \mathbb{F}^{2\times 2}$ is linear). Lemma 6.3.3 gives some conditions under which the system can be uniformly exponentially stabilised, however these conditions are rather strong and the proof of Lemma 6.3.3 does not take into account the additional structure of the Euler-Bernoulli beam, in particular those of the matrices P_i. We therefore give a result analogous to Lemma 6.3.3 making use of the Euler-Bernoulli beam structure.

Lemma 6.4.1. *Assume that $\mathfrak{S} = (\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ is an impedance passive second order port-Hamiltonian system of the form*

$$\mathcal{H} = \begin{pmatrix} \mathcal{H}_1 & \\ & \mathcal{H}_2 \end{pmatrix}, \quad P_2 = \begin{pmatrix} & -P^* \\ P & \end{pmatrix}, \quad P_1 = 0.$$

Further assume that $\mathcal{H}_i \in W^1_\infty(0,1;\mathbb{F}^{d/2\times d/2})$ $(i = 1,2)$ where $d \in 2\mathbb{N}$ is even, and satisfies one of the following additional conditions.

1. *There is $\eta \in C^2([0,1];\mathbb{R})$ with $\eta(1) = 0$ such that*
$$\eta' \le -\left(|\eta| \left\|\mathcal{H}_2'\mathcal{H}_2^{-1}\right\|_{L_\infty} + \left\|\eta''\sqrt{\zeta}\right\|_{L_\infty} + \left\|\eta\sqrt{\zeta}P_0\right\|_{L_\infty} + \varepsilon\right)$$
$$\eta' \le -\left(|\eta| \left\|\mathcal{H}_1'\mathcal{H}_1^{-1}\right\|_{L_\infty} + \left\|\eta\sqrt{\zeta}P_0\right\|_{L_\infty} + \varepsilon\right).$$
for some $\varepsilon > 0$ (then: all $\varepsilon > 0$), or
2. *the estimate*
$$\sup\left\{\left\|\mathcal{H}_2'\mathcal{H}_2^{-1}\right\|_{L_\infty}, \left\|\mathcal{H}_1'\mathcal{H}_1^{-1}\right\|_{L_\infty}\right\} + \left\|\sqrt{\zeta}P_0\right\|_{L_\infty} < 1$$
holds good.

Then there is $q : X \to \mathbb{R}$ with $|q(x)| \le \hat{c}\,\|x\|_X^2$ $(x \in X)$ such that for all solutions $x \in W^1_{\infty,loc}(\mathbb{R}_+;X) \cap L_{\infty,loc}(\mathbb{R}_+;D(\mathfrak{A}))$ of $\dot{x} = \mathfrak{A}x$ one has $q(x) \in W^1_{\infty,loc}(\mathbb{R}_+)$ and

$$\|x(t)\|_X^2 + \frac{d}{dt}q(x(t)) \le c\left(|(\mathcal{H}x)(0)|^2 + |(\mathcal{H}_1x_1)'(0)|^2 + |(\mathcal{H}_2x_2)(1)|^2\right), \quad \text{a.e. } t \ge 0.$$

Remark 6.4.2. *By symmetry, one also gets the following estimates (for properly adjusted $q : X \to \mathbb{R}$) in Lemma 6.4.1.*

$$\begin{aligned}\|x(t)\|_X^2 + \frac{d}{dt}q(x(t)) &\le c\left(|(\mathcal{H}x)(1)|^2 + |(\mathcal{H}_1x_1)'(1)|^2 + |(\mathcal{H}_2x_2)(0)|^2\right), \\ \|x(t)\|_X^2 + \frac{d}{dt}q(x(t)) &\le c\left(|(\mathcal{H}x)(0)|^2 + |(\mathcal{H}_2x_2)'(0)|^2 + |(\mathcal{H}_1x_1)(1)|^2\right), \\ \|x(t)\|_X^2 + \frac{d}{dt}q(x(t)) &\le c\left(|(\mathcal{H}x)(1)|^2 + |(\mathcal{H}_2x_2)'(1)|^2 + |(\mathcal{H}_1x_1)(0)|^2\right).\end{aligned}$$

Proof of Lemma 6.4.1. This proof is based on the technique used in [Ch+87] for a chain of Euler-Bernoulli beams with the particular boundary condition $\omega_t(0) = \omega_\zeta(0) = 0$ at the left end. There both functions ρ and EI are constant on each chain link, so that $\eta(\zeta) = 1 - \zeta$ can be used in what follows.
For the general case let $\eta \in C^2([0,1];\mathbb{R})$ which we choose at a later point and define

$$q(x) := \operatorname{Re}\langle x_1, \eta P^{-1} \int_0^\cdot x_2(\xi)d\xi\rangle, \quad x = (x_1, x_2) \in X.$$

First consider the case $P_0 = 0$. Note that for $P_0 \neq 0$ additional terms have to be taken into consideration, so that the conditions in the lemma have to be adjusted accordingly. Then for every solution $x \in W^1_\infty(\mathbb{R}_+; X) \cap L_\infty(\mathbb{R}_+; D(\mathfrak{A}))$ of $\dot{x} = \mathfrak{A}x$ we have (using Lemma 4.3.2 again)

$$\begin{aligned}
&\frac{d}{dt} q(x) \\
&= \operatorname{Re}\langle P^{-*}x_{1,t}, \eta \int_0^\cdot x_2(\xi)d\xi\rangle_{L_2} + \operatorname{Re}\langle x_1, \eta \int_0^\cdot P^{-1}x_{2,t}(\xi)d\xi\rangle_{L_2} \\
&= -\operatorname{Re}\langle (\mathcal{H}_2x_2)'', \eta \int_0^\cdot x_2(\xi)d\xi\rangle_{L_2} + \operatorname{Re}\langle x_1, \eta \int_0^\cdot (\mathcal{H}_1x_1)''(\xi)d\xi\rangle_{L_2} \\
&= \operatorname{Re}\langle (\mathcal{H}_2x_2)', \eta x_2\rangle_{L_2} + \operatorname{Re}\langle (\mathcal{H}_2x_2)', \eta' \int_0^\cdot x_2(\xi)d\xi\rangle_{L_2} \\
&\quad - \operatorname{Re}\langle \eta(1)(\mathcal{H}_2x_2)'(1), \int_0^1 x_2(\xi)d\xi\rangle_{\mathbb{F}^{d/2}} \\
&\quad + \operatorname{Re}\langle x_1, \eta(\mathcal{H}_1x_1)'\rangle_{L_2} - \operatorname{Re}\langle \eta x_1, (\mathcal{H}_1x_1)'(0)\rangle_{L_2} \\
&= -\frac{1}{2}\langle x_2, ((\eta\mathcal{H}_2)' - 2\eta\mathcal{H}_2')x_2\rangle_{L_2} + \frac{1}{2}\left[\langle x_2(\zeta), (\eta\mathcal{H}_2)(\zeta)x_2(\zeta)\rangle_{\mathbb{F}^{d/2}}\right]_0^1 \\
&\quad - \operatorname{Re}\langle \mathcal{H}_2x_2, \eta'' \int_0^\cdot x_2(\xi)d\xi\rangle_{L_2} - \operatorname{Re}\langle \mathcal{H}_2x_2, \eta' x_2\rangle_{L_2} \\
&\quad + \operatorname{Re}\langle \eta'(1)(\mathcal{H}_2x_2)(1), \int_0^1 x_2(\xi)d\xi\rangle_{\mathbb{F}^{d/2}} - \operatorname{Re}\langle \eta(1)(\mathcal{H}_2x_2)'(1), \int_0^1 x_2(\xi)d\xi\rangle_{\mathbb{F}^{d/2}} \\
&\quad - \frac{1}{2}\langle x_1, ((\eta\mathcal{H}_1)' - 2\eta\mathcal{H}_1')x_1\rangle_{L_2} + \frac{1}{2}\left[\langle x_1(\zeta), (\eta\mathcal{H}_1)(\zeta)x_1(\zeta)\rangle_{\mathbb{F}^{d/2}}\right]_0^1 \\
&\quad - \operatorname{Re}\langle \eta x_1, (\mathcal{H}_1x_1)'(0)\rangle_{L_2} \\
&\leq \frac{1}{2}\langle (-\eta' - \eta\mathcal{H}_2'\mathcal{H}_2^{-1} + \varepsilon)\mathcal{H}_2x_2, x_2\rangle_{L_2} \\
&\quad + \frac{1}{2}\langle (-\eta' + \eta\mathcal{H}_1'\mathcal{H}_1^{-1} + \varepsilon)\mathcal{H}_1x_1, x_1\rangle_{L_2} - \operatorname{Re}\langle \mathcal{H}_2x_2, \eta'' \int_0^\cdot x_2(\xi)d\xi\rangle_{L_2} \\
&\quad + c_\varepsilon \Big(\left|\eta(1)(\mathcal{H}_2x_2)'(1) - \eta'(1)(\mathcal{H}_2x_2)(1)\right|^2 + \|\eta\|_{L_\infty} \left|(\mathcal{H}_1x_1)'(0)\right|^2 \\
&\qquad + |\eta(0)| \left|(\mathcal{H}_1x_1)(0)\right|^2 + |\eta(1)| \left|(\mathcal{H}_1x_1)(1)\right|^2 \Big).
\end{aligned}$$

In case that $P_0 \neq 0$ we need to handle the additional terms

$$\begin{aligned}
&\operatorname{Re}\langle (P_0\mathcal{H}x)_1, \eta P^{-1} \int_0^\cdot x_2(\xi)d\xi\rangle_{L_2} + \operatorname{Re}\langle x_1, \eta P^{-1} \int_0^\cdot (P_0\mathcal{H}x)_2(\xi)d\xi\rangle_{L_2} \\
&\leq \left\|\eta\sqrt{\zeta}P_0\mathcal{H}x\right\|_{L_2} \|x\|_{L_2}
\end{aligned}$$

$$\leq \left\|\eta\sqrt{\zeta}P_0\right\|_{L_\infty} \|\mathcal{H}x\|_{L_2} \|x\|_{L_2}.$$

We therefore need to find η such that $\eta(1) = 0$ and the following conditions hold true.

$$\eta' \leq -\left(|\eta| \left\|\mathcal{H}_2'\mathcal{H}_2^{-1}\right\|_{L_\infty} + \left\|\eta''\sqrt{\zeta}\right\|_{L_\infty} + \left\|\eta\sqrt{\zeta}P_0\right\|_{L_\infty} + \varepsilon\right)$$
$$\eta' \leq -\left(|\eta| \left\|\mathcal{H}_1'\mathcal{H}_1^{-1}\right\|_{L_\infty} + \left\|\eta\sqrt{\zeta}P_0\right\|_{L_\infty} + \varepsilon\right).$$

This gives the assertion of the lemma under the first condition on $\mathcal{H}$. Under the second condition simply choose $\eta(\zeta) = 1 - \zeta$. □

Theorem 6.4.3. *Let $\mathfrak{S} = (\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ be an impedance passive port-Hamiltonian system of order $N = 2$ of Euler-Bernoulli type as in Lemma 6.4.1 and $M_c : D(M_c) \subseteq X_c \times \mathbb{F}^{Nd} \rightrightarrows X_c \times \mathbb{F}^{Nd}$ as in Assumption 7.3.2. Further assume that for some $c' > 0$ and all $(x_c, u_c) \in D(M_c)$, $(z_c, w_c) \in M_c(x_c, u_c)$*

$$\|x_c\|_{X_c}^2 + |\Pi u_c|^2 \leq c' \left|\Pi_{\mathbb{F}^{Nd}} \begin{pmatrix} z_c \\ w_c \end{pmatrix}\right|^2.$$

and

$$|(\mathcal{H}x)(0)|^2 + |(\mathcal{H}_1 x_1)'(0)|^2 + |(\mathcal{H}_2 x_2)(1)|^2 \lesssim |\mathfrak{B}x|^2 + |\Pi\mathfrak{C}x|^2, \quad x \in D(\mathfrak{A}).$$

Then the interconnected map $\mathcal{A}$ from Theorem 7.1.4 generates a s.c. contraction semigroup $(S(t))_{t\geq 0}$ on $X \times \underline{X}_c$ with globally exponential stable equilibrium 0.

Proof. Combine Lemma 6.4.1 with Proposition 7.3.4. □

6.5 Examples

We apply the abstract results for port-Hamiltonian systems to some particular stabilisation examples.

Example 6.5.1 (Wave Equation)**.** *Consider the one-dimensional wave equation*

$$\rho\omega_{tt}(t,\zeta) - (EI\omega_\zeta)_\zeta(t,\zeta) = 0, \quad \zeta \in (0,1),\ t \geq 0$$

where $EI, \rho \in L_\infty(0,1)$ are uniformly positive, in particular also $EI^{-1}, \rho^{-1} \in L_\infty(0,1)$. At the left end we assume conservative or dissipative boundary conditions of the form

$$\omega_t(t,0) = 0 \quad or \quad (EI\omega_\zeta)(t,0) \in f(\omega_t(t,0)), \quad t \geq 0$$

where $f : \mathbb{F} \rightrightarrows \mathbb{F}$ is maximal monotone and $f(0) \ni 0$, e.g. f could be single-valued, continuous and non decreasing with $f(0) = 0$, in particular the case $f = 0$ (Neumann-boundary condition) is allowed. We further assume that on the right end a (monotone) damper is attached to the system, so that the boundary condition is given by

$$(EI\omega_\zeta)(t,1) \in -g(\omega_t(t,1))$$

where again $g : \mathbb{F} \rightrightarrows \mathbb{F}$ is maximal monotone with $g(0) \ni 0$. Of course, the choice $f = g = 0$ would lead to Neumann-boundary conditions on both sides for which the system is known to be energy-preserving, in particular not strongly stable. Here as usual the energy is given by

$$H(t) := \int_0^1 \rho(\zeta) \left|\omega_t(t,\zeta)\right|^2 + EI(\zeta) \left|\omega_\zeta(t,\zeta)\right|^2 d\zeta.$$

As we have seen in Example 3.1.2 this model fits into our port-Hamiltonian setting when we choose $x = (\rho\omega_t, \omega_z)$, $\mathcal{H} = \operatorname{diag}(\rho^{-1}, EI)$ and $P_1 = \left({}_1 {}^1 \right)$ and $P_0 = 0$. If we additionally define the input and output map as

$$\mathfrak{B}x = \begin{pmatrix} -(EI\omega_\zeta)(0) \\ (EI\omega_\zeta)(1) \end{pmatrix} = \begin{pmatrix} -(\mathcal{H}x)_2(0) \\ (\mathcal{H}x)_2(1) \end{pmatrix}$$
$$\mathfrak{C}x = \begin{pmatrix} \omega_t(0) \\ \omega_t(1) \end{pmatrix} = \begin{pmatrix} (\mathcal{H}x)_1(0) \\ (\mathcal{H}x)_1(1) \end{pmatrix}$$

then the system $\mathfrak{S} = (\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ is impedance passive, since for the maximal port-Hamiltonian operator $\mathfrak{A}$ one has

$$\operatorname{Re}\langle \mathfrak{A}x, x\rangle_X = \operatorname{Re}\left(\langle (\mathcal{H}x)_1(1), (\mathcal{H}x)_2(1)\rangle_{\mathbb{F}} - (\mathcal{H}x)_1(0)(\mathcal{H}x)_2(0)_{\mathbb{F}}\right).$$

(Note that for the Dirichlet case $\omega_t(t,0) = 0$ one has to exchange the first components of $\mathfrak{B}$ and $\mathfrak{C}$ and then choose $f = 0$.) The corresponding port-Hamiltonian operator $A = \mathfrak{A}|_{D(A)}$ (with nonlinear boundary conditions) is dissipative then, where

$$D(A) = \Big\{x \in L_2(0,1;\mathbb{F}^2) : \ \mathcal{H}x \in H^1(0,1;\mathbb{F}^2), \ (\mathcal{H}x)_2(1) \in -g((\mathcal{H}x)_1(1)),$$
$$\begin{cases} (\mathcal{H}x)_1(0) = 0, & \text{(Dirichlet b.c.), or} \\ (\mathcal{H}x)_2(0) \in f((\mathcal{H}x)_1(0)) & \text{(nonlinear Robin b.c.)} \end{cases}\Big\}$$

and we have at least

$$\operatorname{Re}\langle Ax, x\rangle_X \le -\operatorname{Re}\langle (\mathcal{H}x)_1(1), g^0((\mathcal{H}x)_1(1))\rangle_{\mathbb{F}}, \quad x \in D(A).$$

Theorem 6.1.3 assures that A generates a nonlinear s.c. contraction semigroup on $X = L_2(0,1;\mathbb{F}^2)$ with inner product $\langle\cdot,\cdot\rangle_X = \langle\cdot,\cdot\rangle_{\mathcal{H}}$. To have stability results we need stronger assumptions on the damper, i.e. on the map g. First assume that $0 \notin g(x)$ for all $x \in \mathbb{F} \setminus \{0\}$. Then there is $p : \mathbb{R}_+ \to \mathbb{R}_+$ with $p > 0$ on $(0,\infty)$ such that

$$\operatorname{Re}\langle x, g(x)\rangle_{\mathbb{F}} \le -p\left(\sqrt{|x|^2 + |g^0(x)|^2}\right), \quad x \in \mathbb{R} = D(g)$$

so that

$$\operatorname{Re}\langle Ax, x\rangle_X \le -p(|(\mathcal{H}x)(1)|^2), \quad x \in D(A)$$

and asymptotic stability follows from Theorem 6.3.1. Secondly (additionally to g being m-monotone) assume that there even is $\kappa > 0$ such that $\kappa^{-1}|x| \le |z| \le \kappa|x|$ for all $x \in \mathbb{F}$ and $z \in g(x)$ (i.e. in particular $g(0) = \{0\}$). Then we obtain the dissipativity condition

$$\operatorname{Re}\langle Ax, x\rangle_X \le -\tilde{\kappa}\,|(\mathcal{H}x)(1)|^2, \quad x \in D(A)$$

where $\tilde{\kappa} := \frac{1}{2}\min\{\kappa, \kappa^{-1}\}$ and so Theorem 6.2.1 ensures uniform exponential stability of the corresponding nonlinear semigroup. We refer to Example 3.3 in [CoLaMa90] for sufficient conditions leading to asymptotic stability of the n-dimensional wave equation on a smooth, bounded domain $\Omega \subseteq \mathbb{R}^n$.

Example 6.5.2 (Boundary Stabilisation of the Timoshenko Beam Equation)**.** *Next we consider the example of boundary feedback stabilisation of the Timoshenko beam equation, see Example 3.1.6 for its port-Hamiltonian formulation. In the article [FeShZh98] the authors considered the following nonlinear boundary stabilisation approach.*

$$\begin{aligned} \omega(t,0) = \phi(t,0) &= 0 \\ K(\phi(t,1) - \omega_\zeta)(t,1) &\in f(\omega_t(t,1)) \\ -(EI\phi_\zeta(t,1)) &\in g(\phi_t(t,1)) \end{aligned}$$

where $f, g : \mathbb{F} \to \mathbb{F}$ *are two m-monotone maps describing the nonlinear boundary feedback. As we have seen in Example 4.5.4 the choice*

$$\mathfrak{B}x = \begin{pmatrix} (\mathcal{H}_2 x_2)(0) \\ (\mathcal{H}_4 x_4)(0) \\ (\mathcal{H}_1 x_1)(1) \\ (\mathcal{H}_3 x_3)(1) \end{pmatrix} \hat{=} \begin{pmatrix} \omega_t(t,0) \\ \phi_t(t,0) \\ (K(\phi_t - \omega_\zeta)(t,1) \\ (EI\phi_\zeta)(t,1) \end{pmatrix}$$

$$\mathfrak{C}x = \begin{pmatrix} -(\mathcal{H}_1 x_1)(0) \\ -(\mathcal{H}_3 x_3)(0) \\ (\mathcal{H}_2 x_2)(1) \\ (\mathcal{H}_4 x_4)(1) \end{pmatrix} \hat{=} \begin{pmatrix} -(K(\omega_\zeta - \phi))(t,0) \\ -(EI\phi_\zeta)(t,0) \\ \omega_t(t,1) \\ \phi(t,1) \end{pmatrix}$$

leads to an impedance passive port-Hamiltonian system $\mathfrak{S} = (\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ *in boundary control and observation form. Hence, for the nonlinear boundary feedback*

$$\mathfrak{B}x \in -\psi(\mathfrak{C}x) := -\left(\{0\}^2 \times f(\mathfrak{C}_3 x)) \times g(\mathfrak{C}_4 x)\right) \hat{=} -(0, 0, f(\omega_t(t,1), g(\phi_t(t,1)))$$

the corresponding port-Hamiltonian operator A_ψ *with nonlinear boundary conditions is m-dissipative and therefore generates a nonlinear strongly continuous contraction semigroup on* X *thanks to Theorem 6.1.3. Also we find from Theorem 6.2.1 that whenever* $0 \in f(0) \cap g(0)$ *and*

$$\operatorname{Re}\langle u, y\rangle \geq c\,|u|^2, \quad u \in \mathbb{F},\ y \in f(u) \text{ or } g(u)$$

for some $c > 0$ *that*

$$\operatorname{Re}\langle A_\psi x, x\rangle \leq -\sigma\,|(\mathcal{H}x)(1)|^2, \quad x \in D(A_\psi)$$

and hence the s.c. contraction semigroup has 0 *as globally uniformly exponentially stable equilibrium, i.e.*

$$H(t) \leq M e^{\omega t} H(0), \quad t \geq 0$$

for constants $M \geq 1$ *and* $\omega < 0$ *which do not depend on the initial value* $x_0 \in X$*.*

Example 6.5.3 (Dynamic Feedback Stabilisation of an Euler-Bernoulli Beam Equation)**.** *We consider the stabilisation procedure for the Euler-Bernoulli beam equation as considered in the article [CoMo98]. There the authors investigated the following two boundary feedback designs to stabilise the Euler-Bernoulli beam with clamped end at one side and a damper at the other side where also a mass* $m \geq 0$ *may be attached.*

$$\begin{aligned} \rho\omega_{tt}(t,\zeta) + (EI\omega_{\zeta\zeta})_{\zeta\zeta}(t,\zeta) &= 0, && t \geq 0,\ \zeta \in (0,1) \\ \omega(t,0) = \omega_\zeta(t,0) &= 0, && t \geq 0 \\ (EI\omega_{\zeta\zeta})(t,1) &= 0 \\ -(EI\omega_{\zeta\zeta})_\zeta(t,1) + m\omega_{tt}(t,1) &= f(t) \end{aligned}$$

with the following choice for the control function f

$$f(t) := -\alpha\omega(t,1) + \beta(EI\omega_{\zeta\zeta})_{t\zeta}(t,1), \qquad t \geq 0 \tag{6.6}$$

where the constants $\alpha, \beta \geq 0$ are chosen in such a way that $\alpha > 0$ and $\beta > 0$ if $m > 0$ and $\beta = 0$ if $m = 0$. Previously, for the case $m > 0$ and $\beta = 0$ it had been already seen in [LiMa88] that this stabilisation design leads to asymptotic stability, but is not enough for uniform exponential stability. Note that in [CoMo98] as well as in [LiMa88] the authors only considered the uniform case, i.e. $\rho = EI = 1$, whereas below will not impose that restriction and will assume that $\rho, EI > 0$ are two uniformly positive and Lipschitz continuous functions on the interval $[0,1]$. To begin with we formulate the system for the cases $m > 0$ and $m = 0$ as port-Hamiltonian systems with linear dynamic or static dissipative boundary feedback, respectively. Using the representation $(x_1, x_2) \hat{=} (\rho\omega_t, \omega_{\zeta\zeta})$ and setting $\mathcal{H} = \operatorname{diag}(\rho^{-1}, EI)$ as in Example 3.1.6 we find that the dynamics are equivalently described by

$$\frac{\partial}{\partial t}x(t,\zeta) = (\mathfrak{A}x(t))(\zeta) := \frac{\partial^2}{\partial \zeta^2}\begin{bmatrix} & -1 \\ 1 & 0 \end{bmatrix}(\mathcal{H}x)(t,\zeta)$$
$$(\mathcal{H}_1x_1)(t,0) = (\mathcal{H}_1x_1)'(t,0) = (\mathcal{H}_2x_2)(t,0) = 0$$
$$-(\mathcal{H}_2x_2)'(t,1) + m(\mathcal{H}_1x_1)_t(t,1) = -\alpha(\mathcal{H}_1x_1)(t,1) + \beta(\mathcal{H}_2x_2)_{t\zeta}(t,1)$$

Using the following boundary control and boundary observation maps

$$\begin{aligned}
\mathfrak{B}_1x &= (\mathcal{H}_1x_1)(t,1) \hat{=} \omega_t(t,1) \\
\mathfrak{C}_1x &= -(\mathcal{H}_2x_2)'(t,1) \hat{=} -(EI\omega_{\zeta\zeta})_\zeta(t,1) \\
\mathfrak{B}_2x &= \begin{pmatrix} (\mathcal{H}_1x_1)(0) \\ (\mathcal{H}_1x_1)'(0) \\ (\mathcal{H}_2x_2)(1) \end{pmatrix} \hat{=} \begin{pmatrix} \omega_t(t,0) \\ \omega_{t\zeta}(t,0) \\ (EI\omega_{\zeta\zeta})(t,1) \end{pmatrix} \\
\mathfrak{C}_2x &= \begin{pmatrix} (\mathcal{H}_2x_2)'(0) \\ -(\mathcal{H}_2x_2)(0) \\ (\mathcal{H}_1x_1)'(1) \end{pmatrix} \hat{=} \begin{pmatrix} (EI\omega_{\zeta\zeta})_\zeta(t,0) \\ -(EI\omega_{\zeta\zeta})(t,0) \\ \omega_{t\zeta}(t,1) \end{pmatrix}
\end{aligned}$$

the system $\mathfrak{S} = (\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ becomes an impedance passive port-Hamiltonian system in boundary control and observation form, see Example 4.5.5. Depending on whether $m = 0$ or $m > 0$ the system (6.6) may be formulated as port-Hamiltonian system with static ($m = 0$) or dynamic boundary feedback ($m > 0$).

*1.) **the case** $m = \beta = 0$. This situation had been considered in Subsection 3.1 [CoMo98]. In our port-Hamiltonian language the feedback boundary condition amounts to*

$$\mathfrak{B}_1x = -\alpha\mathfrak{C}_1x, \quad \mathfrak{B}_2x = 0 \in \mathbb{F}^3, \quad x \in D(A) \tag{6.7}$$

where $\alpha > 0$, so that the corresponding operator

$$A = \mathfrak{A}|_{D(A)}, \quad D(A) = \{x \in D(\mathfrak{A}) : \text{ (6.7) } \textit{holds}\}$$

is dissipative and therefore generates a contractive C_0-semigroup $(T(t))_{t\geq 0}$ on X. Moreover,

$$|\mathfrak{B}x|^2 + |\mathfrak{C}_1x|^2 \geq |(\mathcal{H}x)(1)|^2 + |(\mathcal{H}_2x_2)'(1)|^2 + |(\mathcal{H}_1x_1)(0)|^2 + |(\mathcal{H}_1x_1)'(0)|^2$$

and then Corollary 4.2.10 says that the pair $(\mathfrak{A}, R)$ for $R = (\mathfrak{B}, \mathfrak{C}_1)$ has property ASP. Moreover, Proposition 4.3.19 (with 0 and 1 interchanged) states that the same *pair $(\mathfrak{A}, R)$ also has property AIEP. Hence, we deduce from Proposition 4.3.6 that* for linear feedback *the resulting operator A generates a uniformly exponentially stable C_0-semigroup on X, for arbitrary uniformly strictly positive and Lipschitz continuous ρ and EI. On the other hand, for a constant Hamiltonian density matrix function $\mathcal{H}$ Lemma 6.4.1 implies that there is $q : X \to \mathbb{R}$ with $|q(x)| \leq c\,\|x\|_X^2$ $(x \in X)$ such that*

$$\begin{aligned}
&\|x(t)\|_X^2 + \frac{d}{dt} q(x(t)) \\
&\leq c\left(|(\mathcal{H}x)(0)|^2 + |(\mathcal{H}_1 x_1)'(0)|^2 + |(\mathcal{H}_2 x_2)(1)|^2\right), \quad \text{a.e. } t \geq 0
\end{aligned} \tag{6.8}$$

for every mild solution $x \in W^1_{\infty,loc}(\mathbb{R}_+; X) \cap L_{\infty,loc}(\mathbb{R}_+; D(\mathfrak{A}))$ of $\frac{d}{dt}x = \mathfrak{A}x$. Therefore, replacing the constant $\alpha > 0$ by a a nonlinear m-monotone map $\phi : D(\phi) \subseteq \mathbb{F} \rightrightarrows \mathbb{F}$, by Theorem 6.1.3 the corresponding map

$$\begin{aligned}
Ax &= \mathfrak{A}|_{D(A)} \\
D(A) &= \{x \in D(\mathfrak{A}) : \ \mathfrak{B}_1 x \in -\phi(\mathfrak{C}_1 x), \ \mathfrak{C}_2 x = 0\}
\end{aligned}$$

generates a s.c. contraction semigroup $(S(t))_{t\geq 0}$ on X.
For asymptotic stability it is enough to demand that $0 \in \phi(z)$ exactly for $z = 0$. If we additionally assume that $\kappa^{-1}|z|\left|\phi^0(z)\right| \geq \kappa\,|z|$ for some $\kappa > 0$ and every $z \in D(\phi)$, then we deduce from Proposition 6.2.4 that 0 is even a uniformly exponentially stable equilibrium.
2.) ***the case** $m > 0$ **and** $\beta = 0$**.*** *This situation had been considered (for the constant parameter case) in the article [LiMa88]. In this situation the boundary feedback is of dynamic form*

$$\mathfrak{B}_1 x(t) + m(\mathfrak{B}_1 x)_t(t) = -\alpha \mathfrak{C}_1 x(t), \quad \mathfrak{B}_2 x(t) = 0 \in \mathbb{F}^3, \quad t \geq 0 \tag{6.9}$$

and therefore we extend the state space by a controller state space variable. We introduce $x_c \mathrel{\hat{=}} \mathfrak{B}_1 x$ on the controller state space $X_c = \mathbb{F}$ with inner product $\langle x_c, z_c\rangle := m\overline{x_c}z_c$ $(x_c, z_c \in X_c)$. Additionally we choose $u_c = \mathfrak{C}_2 x$ and $y_c = -\mathfrak{B}_1 x$ and obtain from equation (6.9) that the control system should have the form

$$\begin{aligned}
\frac{d}{dt}x_c(t) &= -\frac{\alpha}{m}x_c(t) - \frac{1}{m}u_c(t) \\
y_c(t) &= -x_c(t), \qquad t \geq 0.
\end{aligned}$$

Note that then $\Sigma_c = \begin{bmatrix} -\frac{\alpha}{m} & -\frac{1}{m} \\ -1 & 0 \end{bmatrix}$ and input and output are collocated (w.r.t. the norm $\|\cdot\|_{X_c}$ on X_c and the usual Euclidean norm $|\cdot|$ on $U_c = Y_c = \mathbb{F}$) and the system also is impedance passive

$$\begin{aligned}
&\operatorname{Re}\langle A_c x_c + B_c u_c, x_c\rangle_{X_c} - \operatorname{Re}\langle C_c x_c + D_c u_c, u_c\rangle_{U_c} \\
&= \operatorname{Re}\langle -\alpha x_c - u_c, x_c\rangle_{\mathbb{F}} - \operatorname{Re}\langle -x_c, u_c\rangle_{\mathbb{F}} \\
&= -\alpha\,|x_c|^2 = -\frac{\alpha}{m}\,\|x_c\|_{X_c}^2
\end{aligned}$$

In particular, for the corresponding hybrid operator $\mathcal{A}$ on $X \times X_c$ we have

$$\operatorname{Re}\langle \mathcal{A}(x, x_c), (x, x_c)\rangle_{X\times X_c} = -\frac{\alpha}{m}\,\|x_c\|_{X_c}^2. \tag{6.10}$$

The hybrid operator therefore generates a contractive C_0-semigroup on the product Hilbert space $X \times X_c$. Additionally we assume that ρ and EI are Lipschitz continuous. Since the control system is not SIP we might only hope to employ our results on SOP controllers. (Remark that $u_c = -x_c$, so that the control system is SOP, indeed.) However, we do not know whether the pair

$$(\mathfrak{A}, R) = (\mathfrak{A}, ((\mathcal{H}x)(1), (\mathcal{H}_1 x_1)(0), (\mathcal{H}_1 x_1)'(0)))$$

has property ASP and consequently have to take the special structure into account to find that $\sigma_p(\mathcal{A}) \cap i\mathbb{R} = \emptyset$. Namely let $\beta \in \mathbb{R}$ and $(x, x_c) \in D(\mathcal{A})$ such that $i\beta(x, x_c) = \mathcal{A}(x, x_c)$. Then by (6.10) we obtain $x_c = 0$ and hence also

$$\begin{aligned}\mathfrak{B}_1 x &= -y_c = x_c = 0\\ \mathfrak{C}_1 &= u_c = -(\alpha x_c + mi\beta x_c) = 0\end{aligned}$$

so that $Rx := (\mathfrak{B}x, \mathfrak{C}_1 x) = 0$. Thanks to Corollary 4.2.10 (with 0 and 1 interchanged) the pair $(\mathfrak{A}, R)$ has property ASP and it follows that also $x = 0$, so $(x, x_c) = 0$ and $i\beta \in i\mathbb{R}$ cannot be an eigenvalue. Since $\mathcal{A}$ has compact resolvent and this holds for every $\beta \in \mathbb{R}$ asymptotic stability follows from Corollary 2.2.16. Since it had already been shown in [LiMa88] that (for the uniform, i.e constant parameter, case) uniform exponential stability cannot hold true, we do not pursue this topic for the non-uniform scenario with $\rho, EI \in W^1_\infty(0,1)$, but only remark that we have extended the asymptotic stability result of [LiMa88] to the situation of a non-uniform beam. For a possible nonlinear dynamic generalisation we refer to the next chapter.
3.) ***the case*** $m > 0$ ***and*** $\beta > 0$**.** *If also $\beta > 0$ the controller state space has to be adjusted by identifying*

$$x_c \mathrel{\hat{=}} \mathfrak{B}_1 x + \frac{m}{\beta}\mathfrak{C}_1 x$$

and equipping $X_c = \mathbb{F}$ with the equivalent inner product

$$\langle \cdot, \cdot \rangle_{X_c} = K \langle \cdot, \cdot \rangle_{\mathbb{F}}, \quad \text{where } K = \frac{\beta^2}{m + \alpha\beta} > 0 \tag{6.11}$$

so that we obtain the controller dynamics as

$$\begin{aligned}\frac{d}{dt}x_c(t) &= -\frac{1}{\beta}x_c(t) - \frac{1}{\beta}\frac{\alpha\beta - m}{\beta^2}u_c(t)\\ y_c(t) &= -x_c(t) + \frac{m}{\beta}u_c(t), \quad t \geq 0\end{aligned}$$

i.e. $\Sigma_c = \begin{bmatrix} -\frac{1}{\beta} & -\frac{\alpha\beta - m}{\beta^2} \\ -1 & \frac{m}{\beta}\end{bmatrix}$ *and the feedback interconnection is again given by $u_c = \mathfrak{C}_1 x$ and $y_c = -\mathfrak{B}_1 x$. We then calculate for Σ_c:*

$$\begin{aligned}&\operatorname{Re}\langle A_c x_c + B_c u_c, x_c\rangle_{X_c} - \operatorname{Re}\langle C_c x_c + D_c u_c, u_c\rangle_{U_c}\\ &= \operatorname{Re}\langle -\frac{1}{\beta}(\frac{m}{\beta}u_c - y_c) - \frac{1}{\beta}(\alpha - \frac{m}{\beta})u_c, K(\frac{m}{\beta}u_c - y_c)\rangle_{\mathbb{F}} - \operatorname{Re}\langle y_c, u_c\rangle_{\mathbb{F}}\\ &= -K\frac{m}{\beta}\left(\frac{m}{\beta^2} + \frac{1}{\beta}(\alpha - \frac{m}{\beta})\right)|u_c|^2 - \frac{K}{\beta}|y_c|^2\end{aligned}$$

$$+\left(\frac{Km}{\beta^2}+\frac{K}{\beta}(\alpha-\frac{m}{\beta})+\frac{Km}{\beta^2}-1\right)\operatorname{Re}\langle u_c,y_c\rangle$$
$$=-\frac{K\alpha m}{\beta^2}|u_c|^2-\frac{K}{\beta}|y_c|^2$$

for every $(x_c,u_c)\in X_c\times U_c$ *and* $y_c=C_cx_c+D_cu_c$*, so that*

$$\operatorname{Re}\langle\mathcal{A}(x,x_c),(x,x_c)\rangle_{X\times X_c}\leq-\frac{K\alpha m}{\beta^2}|\mathfrak{C}_1x|^2-\frac{K}{\beta}|\mathfrak{B}x|^2$$

for every $(x,x_c)\in X\times X_c$*, and since the control system is internally stable (obviously* $\sigma_p(A_c)=\{-\frac{1}{\beta}\}\subseteq\mathbb{C}_0^-$*) and the pair* $(\mathfrak{A},(\mathfrak{B},\mathfrak{C}_1))$ *has properties ASP and AIEP it follows uniform exponential stability from Proposition 5.2.2 and Theorem 5.2.4, which extends the stability results of [CoMo98] to the case of a non-uniform Euler-Bernoulli beam equation.*

Chapter 7

Passivity Based Nonlinear Dynamic Feedback Stabilisation

The following chapter may be seen as the nonlinear version of Chapter 5 and at the same time the dynamic feedback version of the preceding Chapter 6. In particular, we combine the ideas of these two chapters to also cover the combined case: impedance passive port-Hamiltonian systems with dynamic nonlinear boundary feedback. In contrast to the situation in Chapter 5 where the port-Hamiltonian system $\mathfrak{S}$ is interconnected by standard feedback interconnection $\mathfrak{B}x = -y_c$ and $u_c = \mathfrak{C}x$ with the linear control system $\Sigma_c = (A_c, B_c, C_c, D_c)$ with the dynamics

$$\begin{pmatrix} \frac{d}{dt}x_c(t) \\ -y_c(t) \end{pmatrix} = \begin{bmatrix} A_c & B_c \\ -C_c & -D_c \end{bmatrix} \begin{pmatrix} x_c(t) \\ u_c(t) \end{pmatrix}, \quad t \geq 0$$

we replace the linear system Σ_c by a nonlinear controller Σ_c^{nl}. There are at least two possible approaches to ensure well-posedness in the sense of existence of unique solutions then. On the one hand we may try to employ the Contraction Principle and find solutions as fix points of the corresponding system of integral equations, i.e.

$$\begin{aligned} x(t) &= T(t)x_0 + \int_0^t T_{-1}(t-s)Bu(s)ds, \\ \begin{pmatrix} \frac{d}{dt}x_c(t) \\ -y_c(t) \end{pmatrix} &\in \mathcal{N} \begin{pmatrix} x_c(t) \\ u_c(t) \end{pmatrix} \\ u(t) &= -y_c(t) \\ u_c(t) &= y(t) = CT(t)x_0 + C\int_0^t T_{-1}(t-s)Bu(s)ds + Du(t), \quad t \geq 0 \end{aligned}$$

(see Section 7.2) where we considered the standard formulation (A, B, C, D), i.e.

$$\begin{aligned} \frac{d}{dt}x(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t), \quad t \geq 0 \end{aligned}$$

of the port-Hamiltonian system, in contrast to the boundary control and observation form we have used throughout this thesis so far. Instead we first continue in similar fashion as in the preceding Chapter 6 and look for conditions where the operator arising from the feedback interconnection of the port-Hamiltonian system $\mathfrak{S} = (\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ with a nonlinear controller Σ_c^{nl} forms an m-dissipative operator on $X \times X_c$. In this sense we look for solutions which do not only have non-increasing energy, but for which also the difference between two solutions is non-increasing, so that the solutions naturally form a nonlinear strongly continuous contraction semigroup. After that we are in the position to state some results on asymptotic and uniform exponential stabilisation via dynamic nonlinear boundary feedback, see Section 7.3.

7.1 m-Dissipative Dynamic Control Systems

Let $\mathfrak{S} = (\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ be a port-Hamiltonian system of arbitrary order $N \in \mathbb{N}$ which is assumed to be impedance passive throughout this section. In this section we replace the static boundary feedback $\mathfrak{B}x \in -\phi(\mathfrak{C}x)$ from Chapter 6 by the feedback interconnection $\mathfrak{B}x = -y_c$ and $u_c = \mathfrak{C}x$ with a nonlinear control system Σ_c of the form

$$\begin{pmatrix} \frac{\partial}{\partial t} x_c(t) \\ -y_c(t) \end{pmatrix} \in M_c \begin{pmatrix} x_c(t) \\ u_c(t) \end{pmatrix}, \quad t \geq 0 \tag{NLC}$$

where $M_c : D(M_c) \subseteq \mathbb{F}^{Nd} \times X_c \rightrightarrows \mathbb{F}^{Nd} \times X_c$ is a possibly multi-valued nonlinear map on the product Hilbert space of the controller state space X_c and the input and output space $U_c = Y_c = \mathbb{F}^{Nd}$. In order to motivate the assumptions on the map M_c we will impose below, for the moment we consider the case of a finite dimensional linear control system $\Sigma_c = (A_c, B_c, C_c, D_c)$ which is given by

$$\begin{aligned} \frac{d}{dt} x_c(t) &= A_c x_c(t) + B_c u_c(t) \\ y_c(t) &= C_c x_c(t) + D_c u_c(t), \quad t \geq 0. \end{aligned} \tag{LC}$$

This system is impedance passive if and only if the matrix

$$M_c := \begin{pmatrix} A_c & B_c \\ -C_c & -D_c \end{pmatrix}$$

is dissipative (and then m-dissipative since $X_c \times \mathbb{F}^{Nd}$ is finite dimensional and the map is linear). Similar conditions make sense also for the nonlinear controller, represented by the nonlinear map M_c. For the moment, let X_c be an arbitrary Hilbert space which is equipped with some inner product $\langle \cdot, \cdot \rangle_{X_c}$. From the map $M_c : X_c \times \mathbb{F}^{Nd} \rightrightarrows X_c \times \mathbb{F}^{Nd}$ we demand that it is a (possibly multi-valued and nonlinear) m-dissipative map, so that it, or, more precisely, its minimal section, generates a nonlinear s.c. contraction semigroup on the product Hilbert space $X_c \times \mathbb{F}^{Nd}$. We give a first example.

Example 7.1.1. *Assume that the nonlinear controller has the block form* $M_c = \left(\begin{smallmatrix} A_c & B_c \\ -C_c & -D_c \end{smallmatrix} \right)$ *where* $A_c : D(A_c) \subseteq X_c \rightrightarrows X_c$ *and* $-D_c : D(-D_c) = \mathbb{F}^{Nd} \rightrightarrows \mathbb{F}^{Nd}$ *are* m*-dissipative on* X_c *and* $\mathbb{F}^{Nd}$*, respectively, and the operators* $B_c : \mathbb{F}^{Nd} \to X_c$ *and* $C_c : X_c \to \mathbb{F}^{Nd}$ *are assumed to be linear, bounded and adjoint to each other, i.e.*

input and output $B_c' = C_c$ are collocated. Then $M_c : D(M_c) = D(A_c) \times \mathbb{F}^{Nd} \subseteq X_c \times \mathbb{F}^{Nd} \rightrightarrows X_c \times \mathbb{F}^{Nd}$ is m-dissipative. In fact, for all $(z_c, w_c) \in M_c(x_c, u_c)$ and $(\tilde{z}_c, \tilde{w}_c) \in M_c(\tilde{x}_c, \tilde{u}_c)$ we have that $z_c - B_c u_c \in A_c(x_c)$, $\tilde{z}_c - B_c\tilde{u}_c \in A_c(\tilde{x}_c)$ and $w_c + B_c' x_c \in -D_c(u_c)$, $\tilde{w}_c + B_c'\tilde{x}_c \in -D_c(\tilde{u}_c)$ with

$$\begin{aligned}
&\operatorname{Re}\langle (z_c - \tilde{z}_c, w_c - \tilde{w}_c), (x_c - \tilde{x}_c, u_c - \tilde{u}_c)\rangle_{X_c \times \mathbb{F}^{Nd}} \\
&= \operatorname{Re}\langle (z_c - B_c u_c) - (\tilde{z}_c - B_c \tilde{u}_c), x_c - \tilde{x}_c\rangle_{X_c} \\
&\quad + \operatorname{Re}\langle (w_c + B_c' x_c) - (\tilde{w}_c + B_c'\tilde{x}_c), u_c - \tilde{u}_c\rangle_{\mathbb{F}^{Nd}} \\
&\leq 0
\end{aligned}$$

and for every given $(z_c, w_c) \in X_c \times \mathbb{F}^{Nd}$ and $\lambda > 0$ the map $v_c \mapsto B_c'(A_c - \lambda)^{-1}(z_c - B_c v_c)$ is dissipative and Lipschitz continuous since the map $(A_c - \lambda)^{-1}$ is contractive and dissipative. Then the problem

$$\text{find} \begin{pmatrix} x_c \\ u_c \end{pmatrix} : \quad \begin{pmatrix} z_c \\ w_c \end{pmatrix} \in (M_c - \lambda) \begin{pmatrix} x_c \\ u_c \end{pmatrix} = \begin{pmatrix} (A_c - \lambda)(x_c) + B_c u_c \\ (-D_c - \lambda)(u_c) - B_c' x_c \end{pmatrix}$$

has the (unique) solution

$$\begin{aligned}
x_c &= (A_c - \lambda)^{-1}(z_c - B_c u_c) \\
u_c &= (-D_c - B_c'(A_c - \lambda)^{-1}(z_c - B_c\cdot) - \lambda)^{-1} w_c
\end{aligned}$$

where we used that the map $-D_c - B_c'(A_c - \lambda)^{-1}(z_c - B_c\cdot)$ is m-dissipative thanks to 2.2.23.
Note that choosing $B_c = 0$ leads to static feedback as investigated in Chapter 6, because the dynamics of the control state space variable $x_c(t)$ are completely decoupled from the dynamics of the infinite-dimensional port-Hamiltonian system.

To define the hybrid operator on the product Hilbert space $X \times X_c$, we introduce the following notation.

Definition 7.1.2. *Let H_1 and H_2 be two Hilbert spaces. Then we denote by*

$$\Pi_{H_1}(x_1, x_2) := x_1, \quad \Pi_{H_2}(x_1, x_2) := x_2, \quad (x_1, x_2) \in H_1 \times H_2$$

the canonical projections $\Pi_{H_j} : H_1 \times H_2 \to H_j$ $(j = 1, 2)$.

In particular, $\Pi_{X_c} : X_c \times \mathbb{F}^{Nd} \to X_c$ and $\Pi_{\mathbb{F}^{Nd}} : X_c \times \mathbb{F}^{Nd} \to \mathbb{F}^{Nd}$ are the canonical projections on X_c and $\mathbb{F}^{Nd}$, respectively. Then we are able to define the nonlinear hybrid operator $\mathcal{A} : D(\mathcal{A}) \subseteq X \times X_c \rightrightarrows X \times X_c$ as follows.

Definition 7.1.3. *Let $\mathfrak{S} = (\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ be a port-Hamiltonian system and $M_c : D(M_c) \subseteq X_c \times \mathbb{F}^{Nd} \rightrightarrows X_c \times \mathbb{F}^{Nd}$ be a possibly multi-valued and nonlinear map on $X_c \times \mathbb{F}^{Nd}$. We then define the possibly multi-valued (in the component corresponding to X_c) and nonlinear map $\mathcal{A} : D(\mathcal{A}) \subseteq X \times X_c \rightrightarrows X \times X_c$ as*

$$\begin{aligned}
\mathcal{A}\begin{pmatrix} x \\ x_c \end{pmatrix} &= \begin{pmatrix} \mathfrak{A}x \\ \Pi_{X_c} M_c(x_c, \mathfrak{C}x) \end{pmatrix} \\
D(\mathcal{A}) &= \{(x, x_c) \in D(\mathfrak{A}) \times \Pi_{X_c} D(M_c) : \ \mathfrak{B}x \in \Pi_{\mathbb{F}^{Nd}} M_c(x_c, \mathfrak{C}x)\}
\end{aligned}$$

Using the notation $\underline{X}_c := \overline{\Pi_{X_c} D(M_c)}$ we then find the following.

Theorem 7.1.4. *Assume that* $\mathfrak{S} = (\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ *is an impedance passive port-Hamiltonian system and* $M_c : D(M_c) \subseteq X_c \times \mathbb{F}^{Nd} \rightrightarrows X_c \times \mathbb{F}^{Nd}$ *is an* m*-dissipative map. Then the map* $\mathcal{A} : D(\mathcal{A}) \subset X \times X_c \rightrightarrows X \times X_c$ *is* m*-dissipative on the product Hilbert space* $X \times X_c$*, thus its minimal section generates a nonlinear s.c. contraction semigroup* $(\mathcal{S}(t))_{t\geq 0}$ *on* $X \times \underline{X}_c$.

Proof. Thanks to Lemma 6.1.1 we may and will as in the generation theorems before assume that $\mathcal{H} = I$. We start with the statement that $\overline{D(\mathcal{A})} = X \times \underline{X}_c$. For this take any $(x, x_c) \in X \times \underline{X}_c$. As a first step, let us additionally assume that $x_c \in \Pi_{X_c} D(M_c)$. Then there are u_c and $y_c \in \mathbb{F}^{Nd}$ such that $(x_c, u_c) \in D(M_c)$ and $-y_c \in \Pi_{\mathbb{F}^{Nd}} M_c(x_c, u_c)$. We need to find a sequence $(x_n)_{n\geq 1} \subseteq D(\mathfrak{A})$ converging to x (in X) and such that $\mathfrak{B}x_n = u_c$ and $\mathfrak{C}x_n = -y_c$. We take an arbitrary $x_0 \in D(\mathfrak{A})$ such that $\mathfrak{B}x_0 = -y_c$ and $\mathfrak{C}x_0 = u_c$. Since $C_c^\infty(0, 1; \mathbb{F}^d) \subseteq D(\mathfrak{A})$ is dense in X there is a sequence $(z_n)_{n\geq 1} \subseteq C_c^\infty(0, 1; \mathbb{F}^d) \subseteq D(\mathfrak{A})$ which converges to $x - x_0$ (in X). Then $x_n := x_0 + z_n \to x$ converges to $x \in X$ and also $\mathfrak{B}x_n = \mathfrak{B}x_0 \in \Pi_{\mathbb{F}_{Nd}} M_c(x_c, \mathfrak{C}x_0) = \Pi_{\mathbb{F}^{Nd}} M_c(x_c, \mathfrak{C}x_n)$ as wished. Next we allow that (x, x_c) merely lies in $X \times \underline{X}_c$. Then we find a sequence $(x_{c,n})_{n\geq 1} \subseteq \Pi_{X_c} D(M_c)$ such that $\|x_{c,n} - x_c\|_{X_c} \leq \frac{1}{n}$. We know from the first step that there are sequences $(x_{n,k}, x_{c,n,k})_{k\geq 1} \subseteq D(\mathcal{A})$ such that $\|(x_{n,k}, x_{c,n,k}) - (x_n, x_{c,n})\|_{X\times X_c} \leq \frac{1}{k}$ and hence the diagonal sequence $(x_{n,n}, x_{c,n,n})_{n\geq 1} \subseteq D(\mathcal{A})$ converges to (x, x_c) in $X \times X_c$. This shows that $D(\mathcal{A})$ is dense in $X \times \underline{X}_c$.
We show m-dissipativity of the map $\mathcal{A}$. For all $(x, x_c), (\tilde{x}, \tilde{x}_c) \in D(\mathcal{A})$, $(\mathfrak{A}x, z_c) \in \mathcal{A}(x, x_c)$, $(\mathfrak{A}\tilde{x}, \tilde{z}_c) \in \mathcal{A}(\tilde{x}, \tilde{x}_c)$ we have the estimate

$$\begin{aligned}
&\operatorname{Re} \langle (\mathfrak{A}x, z_c) - (\mathfrak{A}\tilde{x}, \tilde{z}_c), (x, x_c) - (\tilde{x}, \tilde{x}_c)\rangle_{X\times X_c} \\
&= \operatorname{Re} \langle \mathfrak{A}(x - \tilde{x}), x - \tilde{x}\rangle_X + \operatorname{Re} \langle z_c - \tilde{z}_c, x_c - \tilde{x}_c\rangle_{X_c} \\
&\leq \operatorname{Re} \langle \mathfrak{B}(x - \tilde{x}), \mathfrak{C}(x - \tilde{x})\rangle_{\mathbb{F}^{Nd}} + \operatorname{Re} \langle z_c - \tilde{z}_c, x_c - \tilde{x}_c\rangle_{X_c} \\
&= \operatorname{Re} \langle \left(\begin{smallmatrix} z_c \\ \mathfrak{B}x \end{smallmatrix}\right) - \left(\begin{smallmatrix} \tilde{z}_c \\ \mathfrak{B}\tilde{x} \end{smallmatrix}\right), \left(\begin{smallmatrix} x_c \\ \mathfrak{C}x \end{smallmatrix}\right) - \left(\begin{smallmatrix} \tilde{x}_c \\ \mathfrak{C}\tilde{x} \end{smallmatrix}\right)\rangle_{X_c\times\mathbb{F}^{Nd}} \leq 0
\end{aligned}$$

since $\left(\begin{smallmatrix} z_c \\ \mathfrak{B}x \end{smallmatrix}\right) \in M_c \left(\begin{smallmatrix} x_c \\ \mathfrak{C}x \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} \tilde{z}_c \\ \mathfrak{B}\tilde{x} \end{smallmatrix}\right) \in M_c \left(\begin{smallmatrix} \tilde{x}_c \\ \mathfrak{C}\tilde{x} \end{smallmatrix}\right)$ for the m-dissipative map M_c. As a result, $\mathcal{A} : D(\mathcal{A}) \subseteq X \times X_c \rightrightarrows X \times X_c$ is dissipative. It remains to show the range condition $\operatorname{ran}(I - \mathcal{A}) = X \times X_c$. We take an arbitrary $(f, f_c) \in X \times X_c$ and look for $(x, x_c) \in D(\mathcal{A})$ such that

$$(x, x_c) - (f, f_c) \in \mathcal{A}(x, x_c).$$

This problem is equivalent to finding $(x, x_c) \in D(\mathfrak{A}) \times \Pi_{X_c} D(M_c)$ such that

$$\begin{aligned}
(I - \mathfrak{A})x &= f \\
x_c - f_c &\in \Pi_{X_c} M_c \begin{pmatrix} x_c \\ \mathfrak{C}x \end{pmatrix} \\
\mathfrak{B}x &\in \Pi_{\mathbb{F}^{Nd}} M_c \begin{pmatrix} x_c \\ \mathfrak{C}x \end{pmatrix}
\end{aligned}$$

where from the first equality and Lemma 3.2.24 we obtain $x = \Phi(1)f + \Psi(1)\mathfrak{B}x$ and $\mathfrak{C}x = F(1)f + G(1)\mathfrak{B}x$. Since the matrix $G(1) \in \mathbb{F}^{Nd\times Nd}$ is invertible it suffices to solve the problem

$$\begin{pmatrix} x_c \\ G(1)^{-1}\mathfrak{C}x \end{pmatrix} - \begin{pmatrix} f_c \\ G(1)^{-1}F(1)f \end{pmatrix} \in M_c \begin{pmatrix} x_c \\ \mathfrak{C}x \end{pmatrix}. \tag{7.1}$$

Since for some $\varepsilon > 0$, which should be small enough, the matrix $\varepsilon I - \text{Sym } G(1)^{-1}$ is still dissipative, clearly also

$$\Delta := \begin{pmatrix} 0 & \\ & \varepsilon I - G(1)^{-1} \end{pmatrix}$$

is dissipative and linear from $X_c \times \mathbb{F}^{Nd}$ to $X_c \times \mathbb{F}^{Nd}$. We now exploit that $M_c : D(M_c) \subseteq X_c \times \mathbb{F}^{Nd} \rightrightarrows X_c \times \mathbb{F}^{Nd}$ is m-dissipative, so that also $\Delta + M_c$ is m-dissipative by Lemma 2.2.23, and then there is a unique solution $(x_c, \mathfrak{C}x)$ of equation (7.1). We found a unique $(x, x_c) \in D(\mathcal{A})$ such that $(f, f_c) + (x, x_c) \in \mathcal{A}(x, x_c)$. □

To summarise this section: We have seen that the boundary feedback interconnection of an impedance passive port-Hamiltonian system with a nonlinear dynamic control system which is governed by an m-dissipative map, leads to an m-dissipative hybrid map on the product Hilbert space.

7.2 An Alternative Approach

Within this section we consider an alternative approach to dynamic nonlinear feedback stabilisation of port-Hamiltonian systems, or, more general, abstract linear control systems of the form

$$\begin{aligned} \frac{d}{dt}x(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t) + Du(t), \quad t \geq 0 \end{aligned}$$

where A is the generator of a C_0-semigroup on a Hilbert space X, $D \in B(U, Y)$ is a bounded linear operator from the input space U to the output space Y, which both are assumed to be Hilbert spaces. The operators $B \in \mathcal{B}(U; X_{-1}^A)$ and $C \in \mathcal{B}(X_1^A; Y)$ are assumed to be admissible control and observation operators, cf. Section 2.3. In fact, we immediately leave this differential level for the dynamics of the linear control system and assume that the dynamics of the linear control system is given by a well-posed linear system (T, Φ, Ψ, F), see Section 2.3. We discuss the local well-posedness of a well-posed linear system interconnected with a dynamic nonlinear controller in Subsection 7.2.2. For impedance passive control systems connected with impedance passive port-Hamiltonian systems we draw conclusion on the existence of global solutions and non-increasing energy. Before we encounter an existence and uniqueness result for the pure nonlinear control system in Subsection 7.2.1 and use techniques which are applied also to the interconnected system.

7.2.1 The Nonlinear Control System

In this subsection we discuss a nonlinear system of the form

$$\begin{aligned} \frac{d}{dt}x_c(t) &= A_c(x_c(t))x_c(t) + B_c(x_c(t))u_c(t) \\ y_c(t) &= C_c(x_c(t))x_c(t) + D_c(x_c(t))u_c(t) \end{aligned} \tag{NLS}$$

where $A_c : X_c \to \mathcal{B}(X_c)$, $B_c : X_c \to \mathcal{B}(U_c, X_c)$, $C_c : X_c \to \mathcal{B}(X_c, Y_c)$ and $D_c : X_c \to \mathcal{B}(U_c, Y_c)$ are locally Lipschitz continuous operator-valued functions, i.e for every bounded subset $X_0 \subset X_c$ there is a constant $L = L(X_0) > 0$ such that

$$\|A_c(x_c) - A_c(\tilde{x}_c)\|_{\mathcal{B}(X_c)} \leq L \|x_c - \tilde{x}_c\|_{X_c}, \quad x_c, \tilde{x}_c \in X_c$$

and accordingly for the other maps.

We start by giving an example for these nonlinear control systems, as it has been proposed by Le Gorrec, Ramirez and Zwart [Le14].

Example 7.2.1. *Le Gorrec, Ramirez and Zwart [Le14] proposed the following nonlinear ODE as a model for a nonlinear controller of a pieco-elastic beam.*

$$\dot{x}_c = (J_c - R_c(x_c))\frac{\partial H_c}{\partial x_c}(x_c) + B_c(x_c)u_c$$
$$y_c = B_c^*(x_c)\frac{\partial H_c}{\partial x_c}(x_c) + S_c(x_c)u_c$$

where $\frac{\partial H_c}{\partial x_c}(x_c) = \begin{pmatrix} k(x_{c,1})x_{c,1} \\ x_{c,2} \end{pmatrix}$ *and*

$$R_c(x_c) = \begin{bmatrix} 0 & \\ & R_{c,0} \end{bmatrix} + \alpha(x_c)R_2 \quad \text{where } R_2 = \begin{bmatrix} 1 & B^* \\ B & BB^* \end{bmatrix}$$
$$B_c(x_c) = \alpha(x_c)B_{c,0} \quad \text{where } B_{c,0} = \begin{bmatrix} 1 \\ B^* \end{bmatrix}$$
$$S_c(x_c) = \alpha(x_c)S$$

with $J_c = -J_c^*$, $R_1 = R_1^* \geq 0$, $S = S^* > 0$ *and* B *are matrices,* $\alpha : \mathbb{R}^n \to [\alpha_1, \alpha_2] \subset \mathbb{R}_+$ *a bounded positive function and* $k : \mathbb{R}^n \to (0, \infty)$ *positive. Here the energy functional* H_c *on* $X_c = \mathbb{R}^2$ *is given by*

$$H_c((x_{c,1}, x_{c,2})) = \int_0^{x_{c,1}} sk(s)ds + \frac{1}{2}|x_{c,2}|^2, (x_{c,1}, x_{c,2}) \in \mathbb{R}^2.$$

We then have the following local existence result for solutions of the nonlinear system (NLS).

Proposition 7.2.2. *Assume that* A_c, B_c, C_c *and* D_c *are locally Lipschitz in* x_c*. Then for every input function* $u_c \in L_{1,loc}(\mathbb{R}_+; U_c)$ *and* $x_{c,0} \in X_c$ *the problem* (NLS) *has a unique mild solution* $x_c = x_c(\cdot; x_{c,0}, u_c) \in C([0, \tau]; X_c)$ *on some interval* $[0, \tau]$, *i.e.*

$$x_c(t) = x_{c,0} + \int_0^t [A_c\&B_c](x_c(s), u_c(s))ds.$$

Proof. Let an initial value $x_{c,0} \in X_c$ and an input function $u_c \in L_{1,loc}(\mathbb{R}_+; U_c)$ be given. For the moment fix any $\rho, \tau > 0$ which we will choose suitable later on. Denote by $B_\rho(x_{c,0})$ the closed ball in X_c with radius ρ and centre at $x_{c,0}$ and define $\Phi : C([0, \tau]; B_\rho(x_{c,0})) \to C([0, \tau]; X_c)$ by

$$(\Phi(x_c))(t) := x_{c,0} + \int_0^t [A_c\&B_c](x_c(s), u_c(s))ds.$$

We aim for the Strict Contraction Principle Proposition 2.1.12 and first compute for $x_c \in C([0, \tau]; B_\rho(x_{c,0}))$ that for every $t \in [0, \tau]$

$$\begin{aligned} \|(\Phi(x_c))(t) - x_{c,0}\|_{X_c} &= \left\| \int_0^t [A_c\&B_c](x_c(s), u_c(s))ds \right\|_{X_c} \\ &\leq \tau \|A_c\|_{C(B_\rho(x_{c,0}))} \|x_c\|_{C[0,\tau]} \\ &\quad + \|B_c\|_{C(B_\rho(x_{c,0}))} \|u_c\|_{L_1(0,\tau;\mathbb{R}^k)} \end{aligned} \tag{7.2}$$

and then for $x_c, \tilde{x}_c \in C([0,\tau]; B_\rho(x_{c,0}))$ that for every $t \in [0,\tau]$

$$\begin{aligned}
&\|(\Phi(x_c))(t) - (\Phi(\tilde{x}_c))(t)\|_{X_c} \\
&\le \int_0^t \|A_c(x_c(s))\,(x_c(s) - \tilde{x}_c(s))\|_{X_c}\,ds \\
&\quad + \int_0^t \|(A_c(x_c(s)) - A_c(\tilde{x}_c(s)))\|\,\|x_c(s)\|_{X_c}\,ds \\
&\quad + \int_0^t \|B_c(x_c(s)) - B_c(\tilde{x}_c(s)))u_c(s)\|_{X_c}\,ds \\
&\le \tau\,\|A_c\|_{C(B_\rho(x_{c,0}))}\,\|x_c - \tilde{x}_c\|_{C([0,\tau];X_c)} \\
&\quad + \tau\,\|A_c\|_{\mathrm{Lip}(B_\rho(x_{c,0}))}\,\|x_c - \tilde{x}_c\|_{C([0,\tau];X_c)} \\
&\quad + \|B_c\|_{\mathrm{Lip}(B_\rho(x_{c,0}))}\,\|u_c\|_{L_1(0,\tau;U_c)}\,\|x_c - \tilde{x}_c\|_{C[0,\tau];X_c)}\,.
\end{aligned}$$

Thus, for suitable small $\rho, \tau > 0$ (which may be chosen in such a way that they only depend on $\|x_{c,0}\|_{X_c}$ and $\|u_c\|_{L_1(0,\tau;U_c)}$) the map Φ is a strictly contractive mapping $C([0,\tau]; B_\rho(x_{c,0})) \to C([0,\tau]; B_\rho(x_{c,0}))$, so it admits a unique fixed point $x_c(\cdot; x_{c,0}, u_c)$ which is a mild solution of (NLC) on some time interval $[0,\tau]$. Uniqueness of the solution follows by standard procedure based on the fact that from continuity any two solutions stay in $B_\rho(x_{c,0})$ for some time $(0,\tilde{\tau})$ where $\tilde{\tau} > 0$ depends on the solution.

Remark 7.2.3. *Note that if $\Xi \subseteq X_c$ and $\Omega \subseteq L_1(0,T;U_c)$ (for some $T > 0$) are bounded sets, in the constants $\rho, \tau > 0$ in the preceding proof may be chosen globally for all $x_{c,0} \in \Xi$ and $u_c \in \Omega$. More precisely, the solution $x_c = x_c(\cdot; x_{c,0}, u_c)$ depends continuously on the initial datum $x_{c_0} \in X_c$ and the input function $u_c \in L_{1,loc}(\mathbb{R}_+; U_c)$. To see this, fix any $(x_{c,0}, u_c) \in X_c \times L_{1,loc}(\mathbb{R}_+; U_c)$ and let $\Xi \times \Omega$ be a closed and bounded neighbourhood of $(x_{c,0}, u_c)$ with $\rho, \tau > 0$ chosen globally for $\Xi \times \Omega$ as sketched before. Without loss of generality assume that*

$$\|B_c\|_{\mathrm{Lip}(\Xi)}\,\|u_c\|_{L_1(0,\tau;U_c)} < 1.$$

Then for every $(\tilde{x}_{c,0}, \tilde{u}_c) \in \Xi \times \Omega$ and $t \in [0,\tau]$ one has

$$\begin{aligned}
&\|x_c(t) - \tilde{x}_c(t)\|_{X_c} \\
&\le \|x_{c,0} - \tilde{x}_{c,0}\|_{X_c} + \int_0^t \|[A_c\&B_c](x_c(s), u_c(s)) - [A_c\&B_c](\tilde{x}_c(s), \tilde{u}_c(s))\|\,ds \\
&\le \|x_{c,0} - \tilde{x}_{c,0}\|_{X_c} + \int_0^t \|A_c(x_c(s))\|\,\|x_c(s) - \tilde{x}_c(s)\|_{X_c} \\
&\quad + \|A_c(x_c(s)) - A_c(\tilde{x}_c(s))\|\,\|\tilde{x}_c(s)\|_{X_c} \\
&\quad + \|B_c(x_c(s)) - B_c(\tilde{x}_c(s))\|\,\|u_c(s)\|_{U_c} + \|B_c(\tilde{x}_c(s))\|\,\|u_c(s) - \tilde{u}_c(s)\|\,ds \\
&\le \|x_{c,0} - \tilde{x}_{c,0}\|_{X_c} + \tau\rho\,\|A_c\|_{\mathrm{Lip}(B_\rho(0))}\,\|x_c - \tilde{x}_c\|_{L_\infty(0,\tau;X_c)} \\
&\quad + \tau\,\|A_c\|_{C(B_\rho(0))}\,\|x - \tilde{x}\|_{L_\infty(0,\tau;X_c)} \\
&\quad + \|B_c\|_{\mathrm{Lip}(B_\rho(0))}\,\|u_c\|_{L_1(0,\tau;U_c)}\,\|x - \tilde{x}\|_{L_\infty(0,\tau;X_c)} + \|B_c\|_{C(B_\rho(0))}\,\|u_c - \tilde{u}_c\|_{L_1(0,\tau)} \\
&\le \|x_{c,0} - \tilde{x}_{c,0}\|_{X_c} + [\tau\rho\,\|A_c\|_{\mathrm{Lip}(B_\rho(0))} + \tau\,\|A_c\|_{C(B_\rho(0))} \\
&\qquad + \|B_c\|_{\mathrm{Lip}(B_\rho(0))}\,\|u_c\|_{L_1(0,\tau)}]\,\|x_c - \tilde{x}_c\|_{L_\infty(0,\tau)} + \|B_c\|_{C(B_\rho(0))}\,\|u_c - \tilde{u}_c\|_{L_1(0,\tau)}
\end{aligned}$$

and consequently for

$$\tau < \frac{1 - \|B_c\|_{\mathrm{Lip}(C(B_\rho(0)))} \|u_c\|_{L_1(0,\tau)}}{\rho \|A_c\|_{\mathrm{Lip}(C(B_\rho(0)))} + \|A_c\|_{C(B_\rho(x_{c,0}))}}$$

one sees that $\|x_c - \tilde{x}_c\|_{L_\infty} \to 0$ *as* $(\tilde{x}_{c,0}, \tilde{u}_c) \to (x_{c,0}, u_c)$*, i.e. the solution depends continuously on the initial datum* $x_c \in X_c$ *and the input function* $u_c \in L_{1,loc}(\mathbb{R}_+; U_c)$*.*

Remark 7.2.4. *It is a standard procedure to show that for any given* $x_{c,0} \in X_c$ *and* $u_c \in L_{1,loc}(\mathbb{R}_+; U_c)$ *the problem (NLS) has a maximal solution* $x_c(\cdot; x_{c,0}, u_c) \in C([0, \tau_{max}); X_c)$ *which cannot be extended to a continuous solution on a larger time interval. Also it can be seen from the proof of Proposition 7.2.2, that if* $\tau_{max} < \infty$*, then*

$$\|x_c(t; x_{c,0}, u_c)\| \xrightarrow{t \to \tau_{max}} +\infty,$$

i.e. the system (NLS) has the blow-up property *(or,* unique continuation property*).*

In particular, if the energy does not increase, all solutions are global.

Corollary 7.2.5. *Let* $u_c = 0$ *and assume that*

$$H_c(x_c(s)) \leq H_c(x_c(t)), \quad s \geq t \geq 0$$

for every mild solution $(x, x_c) \in C(\mathbb{R}_+; X \times X_c)$ *of* (NLS) *and* H_c *is* radially unbounded*, i.e.*

$$\lim_{\|x_c\| \to \infty} H(x_c) = +\infty,$$

so that the pre-image of every bounded set is bounded. Then the system (NLS) has a bounded global solution, for every given initial value $x_c \in X_c$ *and no input.*

Proof. Since we already observed that (NLS) has the blow-up property, the assertion follows from the fact that $H(x_c(t)) \leq H(x_{c,0})$ and $H_c^{-1}([0, H_c(x_{c,0})])$ being bounded. □

7.2.2 Local Existence of Solutions

Next we consider the feedback interconnection of well-posed linear systems with nonlinear control systems as investigated in the previous subsection. I.e. we want to stabilise a L_p-well-posed linear system (T, Φ, Ψ, F) by feedback interconnection with a nonlinear control system of the form

$$\begin{aligned} \dot{x}_c &= A_c(x_c)x_c + B_c(x_c)u_c \\ &=: [A_c \& B_c](x_c, u_c) \\ y_c &= C_c x_c + D_c(x_c)u_c \\ &=: [C_c \& D_c](x_c, u_c), \quad t \geq 0 \end{aligned} \tag{NLC}$$

where all the operator-valued functions $A_c \in \mathrm{Lip}_{loc}(X_c; \mathcal{B}(X_c))$ (determining the inner dynamics of the controller state space variable) $B_c \in \mathrm{Lip}_{loc}(X_c; \mathcal{B}(U_c, X_c))$ (input map), $C_c \in \mathrm{Lip}_{loc}(X_c; \mathcal{B}(C_c; U_c))$ (output map) and $D_c \in \mathrm{Lip}_{loc}(X_c; \mathcal{B}(U_c))$

(feedthrough) are locally Lipschitz continuous on X_c, in the sense that for every bounded subset X_0 of X_c there is a constant $L = L(X_0) > 0$ such that

$$\|A_c(x_c) - A_c(\tilde{x}_c)\|_{X_c} \leq L \|x_c - \tilde{x}_c\|_{X_c}, \quad x_c, \tilde{x}_c \in X_0$$

and accordingly for the other functions, cf. the previous subsection. In particular, this does imply that the functions A_c, B_c, C_c and D_c are bounded on every bounded subset of X_0. In general the finite dimensional case $X_c = \mathbb{F}^n$ (equipped with some inner product $\langle \cdot, \cdot \rangle_{X_c}$) and $U_c = Y_c = \mathbb{F}^k$ for $\mathbb{F}$ being the field of real or of complex numbers is most practicable, since for infinite dimensional spaces local Lipschitz-continuity very often is not easy to ensure. At the moment we will not impose any passivity conditions on the finite dimensional controller or the linear well-posed system, but we will use properties like these later on to establish existence of global solutions, whereas in the first step (local existence) we do not make use of them, anyway.

Assumption 7.2.6. *We assume that one of the following conditions holds for the L_p-well-posed linear system $\Sigma = (T, \Phi, \Psi, F)$.*

$$\lim_{t \to 0} \|F(t)\| = \inf_{t > 0} \|F(t)\| = 0.$$

or

$$\sup_{x_c \in X_c} D_c(x_c) \inf_{t > 0} \|F(t)\| < 1.$$

Now, for any initial value $(x_0, x_{c,0}) \in X \times X_c$ we consider the following interconnected system

$$\begin{aligned} x(t) &= T(t)x_0 + \Phi(t)u \\ x_c(t) &= x_{c,0} + \int_0^t [A_c \& B_c](x_c(s), u_c(s)) ds, \quad t \geq 0 \\ u &= -y_c = -[C_c \& D_c](x_c, u_c) \\ u_c &= y := \Psi x_0 + Fu \end{aligned} \tag{7.3}$$

where we denote by $\Psi \in \mathcal{B}(X; L_{p,loc}(\mathbb{R}_+; Y))$ and $F \in \mathcal{B}(L_{p,loc}(\mathbb{R}_+; U), L_{p,loc}(\mathbb{R}_+; Y))$ those operators such that $(\Psi x)(t) = (\Psi(T)x)(t)$ and $(Fu)(t) = (F(T)u)(t)$ for every fixed $T > 0$ and a.e. $t \in [0, T]$.

Theorem 7.2.7. *Let $p \in (1, \infty)$. Then for every given $(x_0, x_{c,0}) \in X \times X_c$ there is $\tau > 0$ such that the system (7.3) has a unique solution $(x, x_c) = (x, x_c)(\cdot; x_0, x_{c,0}) \in C([0, \tau]; X \times X_c)$ with $u \in L_p(0, \tau; U)$ and $y \in L_p(0, \tau; Y)$.*

Proof. Let an arbitrary initial value $(x_0, x_{c,0}) \in X \times X_c$ be given. Let $\rho > 0$ and $\sigma > 0$ be two positive constants and take $\tau_0 > 0$ such that

$$\rho > 2 \|\Psi x_0\|_{L^p(0,\tau_0;U)}.$$

In the following we denote by $|f|_{\mathrm{Lip}(M)}$ the optimal Lipschitz constant of a function defined on a closed set M. Moreover, by $B_\rho^Z(z)$ we denote the closed ball with radius $\rho > 0$ and centre $z \in Z$ for any Banach space Z. For $\tau \in [0, \tau_0]$ define the map $\phi : C([0,\tau]; B_\rho^{X_c}(x_{c,0})) \times B_\sigma^{L_p(0,\tau;U)}(0) \to C([0,\tau]; X_c)) \times L_p(0,\tau;U)$ by

$$(\phi(x_c, u))(t) = \begin{pmatrix} \phi_1(x,u)(t) \\ \phi_2(x,u)(t) \end{pmatrix}$$
$$:= \begin{pmatrix} x_{c,0} + \int_0^t [A_c\&B_c](x_c(s), (\Psi x_0 + Fu)(s))ds \\ -[C_c\&D_c](x_c(t), (\Psi x_0 + Fu)(t))) \end{pmatrix}$$

We then have for every (x_c, u) and $(\tilde{x}_c, \tilde{u}) \in C([0,\tau]; B_\rho(x_{c,0})) \times B_\sigma^{L_p(0,\tau;U)}(0)$ that

$$\begin{aligned}
&\|\phi_1(x_c,u) - \phi_1(\tilde{x}_c,\tilde{u})\|_{C[0,\tau]} \\
&= \sup_{t\in[0,\tau]} \left\| \int_0^t [A_c\&B_c](x_c(s), (\Psi x_0 + Fu)(s)) - [A_c\&B_c](\tilde{x}_c(s), (\Psi x_0 + F\tilde{u})(s))ds \right\| \\
&\leq \int_0^\tau \|A_c(x_c(s))x_c(s) - A_c(\tilde{x}_c(s))\tilde{x}_c(s)\|_{X_c} \\
&\quad + \|B_c(x_c(s))(\Psi x_0 + Fu)(s) - B_c(\tilde{x}_c(s))(\Psi x_0 + F\tilde{u})(s)\| \, ds \\
&\leq \int_0^t \|A_c(x_c(s))\| \, \|x_c(s) - \tilde{x}_c(s)\|_{X_c} + \|A_c(x_c(s)) - A_c(\tilde{x}_c(s))\| \, \|\tilde{x}_c(s)\| \\
&\quad + \|B_c(x_c(s))((Fu)(s) - (F\tilde{u})(s))\| \\
&\quad + \|B_c(x_c(s)) - B_c(\tilde{x}_c(s))\| \, \|(\Psi x_0 + F\tilde{u})(s)\| \, ds \\
&\leq \int_0^\tau \|A_c\|_{C(B_\rho(x_{c,0}))} \|x_c(s) - \tilde{x}_c(s)\|_{X_c} \\
&\quad + |A_c|_{\mathrm{Lip}(B_\rho(x_{c,0}))} \|x_c(s) - \tilde{x}_c(s)\| \, \|\tilde{x}_c(s)\|_{X_c} \\
&\quad + \|B_c\|_{C(B_\rho(x_{c,0}))} \|(F(u-\tilde{u}))(s)\|_U \\
&\quad + |B_c|_{\mathrm{Lip}(B_\rho(x_{c,0}))} \|(\Psi x_0 + F\tilde{u})(s)\|_U \, \|x_c(s) - \tilde{x}_c(s)\| \, ds \\
&\leq \tau \left(\|A_c\|_{C(B_\rho(x_{c,0}))} + |A_c|_{\mathrm{Lip}(B_\rho(x_{c,0}))} \|\tilde{x}_c\|_{C[0,\tau]} \right) \|x_c - \tilde{x}_c\|_{C[0,\tau]} \\
&\quad + \|B_c\|_{C(B_\rho(x_{c,0}))} \|F(u-\tilde{u})\|_{L_1(0,\tau)} \\
&\qquad + |B_c|_{\mathrm{Lip}(B_\rho(x_{c,0}))} (\|\Psi x_0\|_{L_1(0,\tau)} + \|F\tilde{u}\|_{L_1(0,\tau)}) \|x_c - \tilde{x}_c\|_{C[0,\tau]} \\
&\leq \tau \left(\|A_c\|_{C(B_\rho(x_{c,0}))} + |A_c|_{\mathrm{Lip}(B_\rho(x_{c,0}))} \|\tilde{x}_c\|_{C[0,\tau]} \right) \|x_c - \tilde{x}_c\|_{C[0,\tau]} \\
&\quad + \tau^{1-1/p} \|B_c\|_{C(B_\rho(x_{c,0}))} \|F(\tau)\|_{\mathcal{B}(L_p(0,\tau))} \|u - \tilde{u}\|_{L_p(0,\tau)} \\
&\quad + \tau^{1-1/p} |B_c|_{\mathrm{Lip}(B_\rho(x_{c,0}))} (\|\Psi x_0\|_{L_p(0,\tau)} + \|F(\tau)\|_{\mathcal{B}(L_p(0,\tau))} \|\tilde{u}\|_{L_p(0,\tau)}) \\
&\qquad \cdot \|x_c - \tilde{x}_c\|_{C[0,\tau]} \\
&\leq \tau \left(\|A_c\|_{C(B_\rho(x_{c,0}))} + |A_c|_{\mathrm{Lip}(B_\rho(x_{c,0}))} (\|x_c\|_{X_c} + \rho) \right) \|x_c - \tilde{x}_c\|_{C[0,\tau]} \\
&\quad + \tau^{1-1/p} \|B_c\|_{C(B_\rho(x_{c,0}))} \|F(\tau)\|_{\mathcal{B}(L_p(0,\tau))} \|u - \tilde{u}\|_{L_p(0,\tau)} \\
&\quad + \tau^{1-1/p} |B_c|_{\mathrm{Lip}(B_\rho(x_{c,0}))} (\|\Psi x_0\|_{L_p(0,\tau)} + \|F(\tau)\|_{\mathcal{B}(L_p(0,\tau))} \sigma) \|x_c - \tilde{x}_c\|_{C[0,\tau]} .
\end{aligned}$$

Note that since $p > 1$ the terms τ and $\tau^{1-\frac{1}{p}}$ can be made as small as we wish, if we choose $\tau > 0$ sufficiently small. We also have

$$\begin{aligned}
&\|\phi_2(x_c,u) - \phi_2(\tilde{x}_c,\tilde{u})\|_{L_p(0,\tau)} \\
&= \|[C_c\&D_c](x_c,\Psi x_0 + Fu) - [C_c\&D_c](\tilde{x}_c,\Psi x_0 + F\tilde{u})\|_{L_p(0,\tau)} \\
&= \|C_c(x_c)x_c + D_c(x_c)(\Psi x_0 + Fu) - C_c(\tilde{x}_c)\tilde{x}_c - D_c(\tilde{x}_c)(\Psi x_0 + F\tilde{u})\|_{L_p(0,\tau)} \\
&\leq \|C_c(x_c)(x_c - \tilde{x}_c)\|_{L_p(0,\tau)} + \|(C_c(x_c) - C_c(\tilde{x}_c))\tilde{x}_c\|_{L_p(0,\tau)} \\
&\quad + \|D_c(x_c)F(u-\tilde{u})\|_{L_p(0,\tau)} + \|(D_c(x_c) - D_c(\tilde{x}_c))F\tilde{u}\|_{L_p(0,\tau)} \\
&\quad + \|(D_c(x_c) - D_c(\tilde{x}_c))\Psi x_0\|_{L_p(0,\tau)} \\
&\leq \|C_c\|_{C(B_\rho(x_{c,0}))} \|x_c - \tilde{x}_c\|_{L_p(0,\tau)} + |C_c|_{\mathrm{Lip}(B_\rho(x_{c,0}))} (\|x_{c,0}\| + \rho) \|x_c - \tilde{x}_c\|_{L_p(0,\tau)} \\
&\quad + \|D_c\|_{C(B_\rho(X_{c,0}))} \|F(\tau)\|_{\mathcal{B}(L_p(0,\tau))} \|u - \tilde{u}\|_{L_p(0,\tau)} \\
&\quad + |D_c|_{\mathrm{Lip}(B_\rho(x_{c,0}))} (\|\Psi x_0\|_{L_p(0,\tau)} + \|F\tilde{u}\|_{L_p(0,\tau)}) \|x_c - \tilde{x}_c\|_{C[0,\tau]} \\
&\leq \tau^{1-1/p}(\|C_c\|_{C(B_\rho(x_{c,0}))} + (\|x_{c,0}\| + \rho) |C_c|_{\mathrm{Lip}(B_\rho(x_{c,0}))}) \|x_c - \tilde{x}_c\|_{C[0,\tau]} \\
&\quad + \|D_c\|_{C(B_\rho(x_{c,0}))} \|F(\tau)\|_{\mathcal{B}(L_p(0,\tau))} \|u - \tilde{u}\|_{L_p(0,\tau)} \\
&\quad + |D_c|_{\mathrm{Lip}(B_\rho(x_{c,0}))} \Big(\|\Psi x_0\|_{L_p(0,\tau)} + \|F(\tau)\|_{\mathcal{B}(L_p(0,\tau))} \sigma\Big) \|x_c - \tilde{x}_c\|_{C[0,\tau]}
\end{aligned}$$

thus for $\tau > 0$ and $\sigma > 0$ sufficiently small the map ϕ is strictly contractive. We show that for an appropriate choice of τ, ρ and $\sigma > 0$ the function ϕ maps $C([0,\tau]; B_\rho(x_{c,0})) \times B_\sigma^{L_p(0,\tau;U)}(0)$ into itself. In fact, we find

$$\begin{aligned}
&\|\phi_1(x_c,u) - x_{c,0}\| \\
&= \sup_{t\in[0,\tau]} \left\| \int_0^t [A_c\&B_c](x_c(s), (\Psi x_0 + Fu)(s)) ds \right\| \\
&\leq \int_0^\tau \|A_c(x_c(s))\| \, \|x_c(s)\|_{X_c} \\
&\quad + \|B_c(x_c(s))\| \, \|(\Psi x_0 + Fu)(s)\|_{U_c} \, ds \\
&\leq \tau \|A_c\|_{C(B_\rho(x_{c,0}))} (\|x_{c,0}\|_{X_c} + \rho) + \|B_c\|_{C(B_\rho(x_{c,0}))} \|\Psi x_0 + Fu\|_{L_1(0,\tau)} \\
&\leq \tau^{1-1/p}(\tau^{1/p} \|A_c\|_{C(B_\rho(x_{c,0}))} (\|x_{c,0}\|_{X_c} + \rho) \\
&\quad + \|B_c\|_{C(B_\rho(x_{c,0}))} (\|\Psi x_0\|_{L_p(0,\tau)} + \|F(\tau)\|_{\mathcal{B}(L_p(0,\tau))} \sigma))
\end{aligned}$$

and

$$\begin{aligned}
&\|\phi_2(x_c,u)\|_{L_p(0,\tau)} \\
&= \|[C_c\&D_c](x_c,\Psi x_0 + Fu)\|_{L_p(0,\tau)} \\
&\leq \|C_c\|_{C(B_\rho(x_{c,0}))} \|x_c\|_{C[0,\tau]} \\
&\quad + \|D_c\|_{C(B_\rho(x_{c,0}))} \|\Psi x_0 + Fu\|_{L_p(0,\tau)} \\
&\leq \tau^{1-1/p}(\|C_c\|_{C(B_\rho(x_{c,0}))} \|x_c\|_{C[0,\tau]} \\
&\quad + \|D_c\|_{C(B_\rho(x_{c,0}))} (\|\Psi x_0\|_{L_p(0,\tau)} + \|F(\tau)\|_{\mathcal{B}(L_p(0,\tau))} \sigma))
\end{aligned}$$

so that for $\tau > 0$ sufficiently small, in particular such that

$$\Big(\|\Psi x_0\|_{L_p(0,\tau)} + \|F(\tau)\|_{\mathcal{B}(L_p(0,\tau))} \sigma\Big) \|D_c\|_{C(B_\rho(x_{c,0}))} < \sigma \tag{7.4}$$

the nonlinear map ϕ not only is strictly contractive, but also maps the complete metric space $C([0,\tau]; B_\rho^{X_c}(x_{c,0})) \times B_\rho^{L_p(0,\tau;U)}(0)$ into itself, so by the Strict Contraction Principle 2.1.12 has a unique fixed point

$$(x_c, u) =: (x_c, u)(\cdot; x_0, x_{c,0}) \in C([0,\tau]; B_\rho^{X_c}(x_{c,0})) \times B_\rho^{L_p(0,\tau;U)}(0).$$

Then $(x, x_c)(\cdot; x_0, x_{c,0}) := (x, x_c) := (Tx_0 + \Phi u, x_c) \in C([0,\tau]; X \times X_c)$ solves (7.3) on the interval $[0,\tau]$. □

Corollary 7.2.8. *For every* $(x_0, x_{c,0}) \in X \times X_c$ *the interconnection problem* (7.3) *has a unique* maximal solution $(x, x_c) = (x, x_c)(\cdot; x_0, x_{c,0}) \in C([0, t_{max}); X \times X_c)$ *with* $(u, y) \in L_{p,loc}([0, t_{max}); U \times Y)$ *which cannot be extended beyond the* maximal existence time $t_{max} \in (0, \infty]$. *Whenever* $t_{max} < +\infty$ *it holds*

$$\limsup_{t \to t_{max}} \|(x, x_c)(t)\|_{X \times X_c} = \lim_{t \to t_{max}} \|(x, x_c)(t)\|_{X \times X_c} = +\infty$$

i.e. the problem (7.3) *has the* blow-up-property *(or,* unique continuation property*): For every* $\tau > 0$ *and every bounded solution* $(x, x_c) \in C([0,\tau); X \times X_c)$ *of* (7.3)*, there is* $\varepsilon > 0$ *and the solution uniquely extends to a solution* $(x, x_c) \in C([0, \tau+\varepsilon]; X \times X_c)$*. In particular, all bounded solutions* $(x, x_c) \in C([0, t_{max}); X \times X_c)$ *are* global*, i.e.* $t_{max} = \infty$.

Proof. The maximal solution can be constructed using Zorn's lemma and this procedure is standard. Uniqueness follows from the following observation. Assume that $(x, x_c) \in C([0, t_{max}); X \times X_c)$ and $(\tilde{x}, \tilde{x}) \in C([0, \tilde{t}_{max}); X \times X_c)$ are two maximal solutions for the same initial value $(x_0; x_{c,0})$. Let

$$t_0 := \sup\{\tau \in [0, \min\{t_{max}, \tilde{t}_{max}\}) : \ (x, x_c) = (\tilde{x}, \tilde{x}_c) \text{ on } [0,\tau]\}$$

From the Local Existence Theorem 7.2.7 we deduce that $t_0 > 0$. Assume that $t_0 < \min\{t_{max}, \tilde{t}_{max}\}$. Hence, both solutions are continuous on $[0, t_0]$ and coincide on $[0, t_0]$. By the Local Existence Theorem 7.2.7 there is a unique solution $(\hat{x}, \hat{x}_c) \in C([0, \hat{\varepsilon}]; X \times X_c)$ of (7.3) for the initial value $(x, x_c)(t_0) = (\tilde{x}, \tilde{x}_c)(t_0)$. On the other hand, also the shifted solutions $(x, x_c)(\cdot + t_0)$ and $(\tilde{x}, \tilde{x}_c)(\cdot + t_0)$ are solutions of (7.3) on the intervals $[0, t_{max} - t_0)$ and $[0, \tilde{t}_{max} - t_0)$, respectively. In particular

$$(\hat{x}, \hat{x}_c)(s) = (x, x_c)(s + t_0) = (\tilde{x}, \tilde{x}_c)(s + t_0), \quad s \in [0, \min\{t_{max} - t_0, \tilde{t}_{max} - t_0, \hat{\varepsilon}\})$$

in contradiction to the choice of t_0. As a result, $t_0 \geq \min\{t_{max}, \tilde{t}_{max}\}$, but then $t_0 = \min\{t_{max}, \tilde{t}_{max}\}$. Since both solutions (x, x_c) and $(\tilde{x}, \tilde{x}_c)$ are maximal, neither of them can be a proper extension of the other, so we conclude that $t_{max} = \tilde{t}_{max} = t_0$ and the first statement follows.

Next, take any bounded solution $(x, x_c) \in C([0, t_{max}); X \times X_c)$. From the proof of Theorem 7.2.7 we extract that the guaranteed existence time $\tau > 0$ therein depends on the following parameters:

$$\rho > 0, \ \sigma > 0, \ \|(A_c, B_c, C_c, D_c)\|_{C(B_\rho(x_{c,0}))}, \ |(A_c, B_c, C_c, D_c)|_{\mathrm{Lip}(B_\rho(x_{c,0}))}.$$

For the additional parameter $r := \sup_{t \in [0, t_{max})} \|(x, x_c)(t)\| < +\infty$, the existence time $\tau > 0$ can be chosen depending on the following parameters instead

$$\rho > 0, \ \sigma > 0, \ \|(A_c, B_c, C_c, D_c)\|_{C(B_{\rho+r}(0))}, \ |(A_c, B_c, C_c, D_c)|_{\mathrm{Lip}(B_{\rho+r}(0))}.$$

Then the construction from the proof of Theorem 7.2.7 says that for every $t \in [0, t_{max})$ the solution $(x, x_c) \in C([0, t_{max}]; X \times X_c)$ is defined on $[0, t + \tau)$. Clearly this can only be true, if $t_{max} = +\infty$. □

7.2.3 Interconnection of Impedance Passive Systems

In this subsection we impose additional constraints on the well-posed linear system and the nonlinear control system by demanding that they both are impedance passive, where for the latter we begin by defining this terminology in a way adequate for nonlinear systems.

Assumption 7.2.9. *On the space X_c consider a continuous map $H_c : X_c \to \mathbb{R}_+$ and such that it is* radially unbounded, *i.e.*

$$\liminf_{\|x_c\|_{X_c}\to\infty} H_c(x_c) = +\infty. \tag{7.5}$$

Remark 7.2.10. *The function H_c may be seen as a (in general non-quadratic) energy functional, e.g. it might be the quadratic functional $H_c(x_c) = \frac{1}{2}\|x_c\|^2_{X_c}$ which were the appropriate choice if the system were linear or dissipative is the sense of Definition 2.2.21, indeed. Also note that from the accretivity of H_c it follows that for every $c > 0$ there is $\rho > 0$ such that $H_c(x_c) > c$ for every $x_c \in X_c$ with norm greater or equal ρ. In other words, preimages of bounded sets under H_c are bounded:*

$$H_c^{-1}([0,\rho]) \subseteq X_c \quad \textit{is bounded for every } \rho > 0.$$

We may then define impedance passivity with respect to this functional H_c.

Definition 7.2.11. *The nonlinear control system is called* impedance passive *(with respect to a functional $H_c : X_c \to \mathbb{R}_+$) if for every given initial value $x_{c,0} \in X_c$ and an input function $u_c \in L_2(0,\tau;U_c)$ for the* mild solution *$x_c \in C([0,\tau];X_c)$, i.e.*

$$x_c(t) = x_c(0) + \int_0^t A_c(x_c(s))x_c(s) + B_c(u_c(s))u_c(s)ds, \quad t \geq 0$$

one has that

$$H_c(x_c(t)) \leq H_c(x_c(0)) + \int_0^t \mathrm{Re}\,\langle u_c(s), y_c(s)\rangle_{U_c} ds, \quad t \in [0,\tau]$$

where $y_c = C_c(x_c)x_c + D_c(x_c)u_c \in L_2(0,\tau;U_c)$.

Example 7.2.12. *Consider the nonlinear control system from [Le14] of Example 7.2.1, i.e.*

$$\begin{aligned} \dot{x}_c &= (J_c - R_c(x_c))\frac{\partial H_c}{\partial x_c}(x_c) + B_c(x_c)u_c \\ y_c &= B_c(x_c)^*\frac{\partial H_c}{\partial x_c}(x_c) + S_c(u_c)u_c. \end{aligned} \tag{7.6}$$

Then the energy functional

$$H_c(x_c) = \int_0^{x_{c,1}} sk(s)ds + \frac{1}{2}|x_{c,2}|^2$$

is radially unbounded if and only if

$$\int_0^\infty |sk(s)|\,ds = +\infty, \textit{ and } \int_{-\infty}^0 |sk(s)|\,ds = +\infty.$$

Now, for impedance passivity we compute that for every Lipschitz continuous (in the x_c component) solution of (7.6) *we have that*

$$\begin{aligned}
&H_c(x_c(t)) - H_c(x_c(0)) \\
&= \int_0^t \frac{d}{ds} H_c(x_c(s))ds \\
&= \mathrm{Re} \int_0^t \langle \frac{\partial H_c}{\partial x_c}(x_c(s)), \dot{x}_c(s)\rangle_{\mathbb{R}^2} ds \\
&= \mathrm{Re} \int_0^t \langle \frac{\partial H_c}{\partial x_c}(x_c(s)), (J_c - R_c(x_c(s))\frac{\partial H_c}{\partial x_c}(x_c(s)) + B_c(x_c(s))u_c(s)\rangle_{\mathbb{R}^2} ds \\
&\leq \mathrm{Re} \int_0^t \langle B_c(x_c(s))^* \frac{\partial H_c}{\partial x_c}(x_c(s)), u_c(s)\rangle_{\mathbb{R}} ds \\
&\leq \mathrm{Re} \int_0^t \langle u_c(s), y_c(s)\rangle_{\mathbb{R}} ds.
\end{aligned}$$

By density, the same result holds for every mild solution $(x_c, u_c, y_c) \in C([0,\tau]; X_c) \times L_2([0,\tau]; U_c \times Y_c)$.

Under these assumptions the global existence of solutions follows easily from the local existence result and the blow-up property.

Theorem 7.2.13. *Let* (T, Φ, Ψ, F) *be an impedance passive* L_2*-well-posed linear system and let the nonlinear control system be impedance passive with respect to a radially unbounded functional* $H_c : X_c \to \mathbb{R}_+$ *and with locally Lipschitz-continuous* A_c, B_c, C_c *and* D_c*. Then for every initial value* $(x_0, x_{c,0}) \in X \times X_c$ *the interconnected system* (7.3) *has a unique global and bounded mild solution* $(x, x_c) \in C_b([0,\infty); X \times X_c)$ *with* $(u, y) \in L_{2,loc}(\mathbb{R}_+; U \times Y)$ *for which the functional*

$$H_{tot}((x, x_c)) := H(x) + H_c(x_c) := \frac{1}{2}\|x\|_X^2 + H_c(x_c)$$

does not increase.

Proof. Let any arbitrary initial value $(x_0, x_{c,0}) \in X \times X_c$ be given and let $(x, x_c) \in C([0, t_{max}); X \times X_c)$ be the corresponding maximal solution with $(u, y) \in L_{2,loc}([0, t_{max}); U \times Y)$. Since $u_c = y$ and $u = -y_c$ we obtain

$$\begin{aligned}
H_{tot}((x, x_c)(t)) - H_{tot}((x_0, x_{c,0})) &= \frac{1}{2}\|x(t)\|_X^2 - \frac{1}{2}\|x(0)\|_X^2 + H_c(x_c(t)) - H_c(x_c(0)) \\
&\leq \int_0^t \mathrm{Re}\,\langle u(s), y(s)\rangle_U ds + \int_0^t \mathrm{Re}\,\langle u_c(s), y_c(s)\rangle_{U_c} ds \\
&= 0
\end{aligned}$$

and therefore $H_{tot}(x, x_c)$ is non increasing on $[0, t_{max})$. Since H and H_c are radially unbounded it follows that $\sup_{t\in[0,t_{max})}\|(x, x_c)(t)\|_{X\times X_c} < +\infty$ and then the blow-up property implies that the solution is global, i.e. $t_{max} = \infty$. □

The results so far are all based on the annoying Assumption 7.2.6 which is unpractical if the feedthrough term $D_c(x_c)$ does not vanish. To overcome this obstacle we remark the following perturbation result which will help up to remove this restriction for the interconnection of impedance passive systems.

Remark 7.2.14. *Assume that (T, Φ, Ψ, F) is an impedance passive L_2-well-posed linear system and $K \in \mathcal{B}(Y;U) = \mathcal{B}(U)$ such that for some $\sigma > 0$ the following estimate holds true*

$$\operatorname{Re} \langle Kw, w\rangle_U \leq -\sigma \|Kw\|_U^2, \quad w \in U.$$

By replacing the input u by $u - Ky$ we again get an impedance passive well-posed linear system

$$\begin{aligned}\hat{\Sigma}(t) &= \begin{bmatrix} \hat{T}(t) & \hat{\Phi}(t) \\ \hat{\Psi}(t) & \hat{F}(t) \end{bmatrix} \\ &= \begin{bmatrix} T(t) - \Phi(t)K(\mathbf{1}_{[0,t]} + F(t)K)^{-1}\Psi(t) & \Phi(t)(\mathbf{1}_{[0,t]} + KF(t))^{-1}KF(t) \\ (\mathbf{1}_{[0,t]} + F(t)K)^{-1}\Psi(t) & (\mathbf{1}_{[0,t]} + F(t)K)^{-1}F(t) \end{bmatrix}.\end{aligned}$$

Here we identify operators in $\mathcal{B}(L_p(0,t;U))$ with its zero-extensions to $\mathcal{B}(L_2(\mathbb{R}_+;U))$, i.e.

$$Lf := L(f|_{[0,t]}), \quad L \in \mathcal{B}(L_2(0,t;U)),\ f \in L_2(\mathbb{R}_+;U)$$

and the inverses appearing are considered as inverses in $\mathcal{B}(L_p(0,t;U))$.

Proof. Since the system (T, Φ, Ψ, F) is impedance passive, we obtain that for every $u \in L_{2,loc}(\mathbb{R}_+;U)$ and $x_0 = 0$

$$\begin{aligned}\operatorname{Re} \langle F(\tau)u, u\rangle_{L_2(0,\tau;U)} &= \operatorname{Re} \langle u, y\rangle_{L_2(0,\tau;U)} \\ &\geq \frac{1}{2} \|\Phi(\tau)u\|_X^2 \geq 0, \quad \tau > 0\end{aligned}$$

so that that the operators $F(\tau) \in \mathcal{B}(L_2(0,\tau;U))$ are accretive for all $\tau > 0$. Then the operator $I + F(\tau)K \in \mathcal{B}(L_2(0,\tau;U))$ is boundedly invertible for every $\tau > 0$ since $F(\tau)K$ is accretive on $L_2(0,\tau;U)$ for U equipped with the equivalent inner product $\langle\left(\frac{K+K'}{2} + \Pi_{\ker K}\right)\cdot,\cdot\rangle_U$, where $\Pi_{\ker K}$ denotes the orthogonal projection onto $\ker K = \ker \frac{K+K'}{2}$ (by assumption). Therefore, for every $x_0 \in X$ and $u \in L_{2,loc}(\mathbb{R}_+;U)$ the problem

$$\begin{pmatrix} x(t) \\ y|_{[0,t]} \end{pmatrix} = \begin{bmatrix} T(t) & \Phi(t) \\ \Psi(t) & F(t) \end{bmatrix} \begin{pmatrix} x_0 \\ u - Ky \end{pmatrix}, \quad t \geq 0$$

has a unique solution

$$\begin{aligned}&\begin{pmatrix} x(t) \\ y|_{[0,t]} \end{pmatrix} \\ &= \begin{bmatrix} T(t) - \Phi(t)K(\mathbf{1}_{[0,t]} + F(t)K)^{-1}\Psi(t) & \Phi(t)(\mathbf{1}_{[0,t]} - K(\mathbf{1}_{[0,t]} + F(t)K)^{-1}F(t)) \\ (\mathbf{1}_{[0,t]} + F(t)K)^{-1}\Psi(t) & (\mathbf{1}_{[0,t]} + F(t)K)^{-1}F(t) \end{bmatrix} \begin{pmatrix} x_0 \\ u \end{pmatrix} \\ &= \begin{bmatrix} T(t) - \Phi(t)K(\mathbf{1}_{[0,t]} + F(t)K)^{-1}\Psi(t) & \Phi(t)(\mathbf{1}_{[0,t]} + KF(t))^{-1}KF(t) \\ (\mathbf{1}_{[0,t]} + F(t)K)^{-1}\Psi(t) & (\mathbf{1}_{[0,t]} + F(t)K)^{-1}F(t) \end{bmatrix} \begin{pmatrix} x_0 \\ u \end{pmatrix}, \quad t \geq 0.\end{aligned}$$

Obviously $\hat{T}(0) = I$, $\hat{\Psi}(0) = 0$ and $\hat{F}(0) = 0$. Moreover, $\hat{T}(t)$ is strongly continuous since T and Φ are strongly continuous and $(\mathbf{1}_{[0,t]} + F(t)K)^{-1}$ and $\Psi(t)$ are uniformly bounded on every bounded interval. To show that $\hat{T}$ is a C_0-semigroup it therefore suffices to show that $\hat{T}(2t) = \hat{T}(t)^2$ for every $t \geq 0$. We find – using that $\Sigma = (T, \Phi, \Psi, F)$ is a linear system –

$$\begin{aligned}
&\hat{T}(2t)x_0 \\
&= \big(T(2t) - \Phi(2t)K(\mathbf{1} + F(2t)K)^{-1}\Psi(2t)\big)\, x_0 \\
&= T(t)^2 x_0 - \Phi(2t)K(\mathbf{1} + F(2t)K)^{-1}(\Psi(t)x_0 \underset{t}{\diamondsuit} \Psi(t)T(t)x_0) \\
&= T(t)^2 x_0 \\
&\quad - \Phi(2t)K[(\mathbf{1}_{[0,t]} + F(t)K)^{-1}\Psi(t)x_0 \\
&\quad\ \underset{t}{\diamondsuit}(\mathbf{1}_{[0,t]} + F(t)K)^{-1} \cdot \big(\Psi(t)T(t)x_0' - \Psi(t)\Phi(t)K(\mathbf{1}_{[0,t]} + F(t)K)^{-1}\Psi(t)x_0\big) \\
&= T(t)^2 x_0 - T(t)\Phi(t)K(\mathbf{1}_{[0,t]} + F(t)K)^{-1}\Psi(t)x_0 \\
&\quad - \Phi(t)K(\mathbf{1}_{[0,t]} + F(t)K)^{-1}\big(\Psi(t)T(t)x_0' - \Psi(t)\Phi(t)K(\mathbf{1}_{[0,t]} + F(t)K)^{-1}\Psi(t)x_0\big) \\
&= (T(t) - \Phi(t)K(\mathbf{1}_{[0,t]} + F(t)K)^{-1}\Psi(t))^2 x_0 \\
&= \hat{T}(t)^2 x_0, \quad t \geq 0,\ x_0 \in X.
\end{aligned}$$

Hence, $\hat{T}$ is a C_0-semigroup. Here we used the fact that for every $s, t > 0$ and functions $u, v \in L_{2,loc}(\mathbb{R}_+; U)$ we have

$$\begin{aligned}
&(\mathbf{1}_{[0,t+s]} - F(t+s)K)^{-1}(u \underset{t}{\diamondsuit} v) \\
&= (\mathbf{1}_{[0,t]} - F(t)K)^{-1}u \underset{t}{\diamondsuit} \big(\mathbf{1}_{[0,s]} - F(s)K)^{-1}(v - \Psi(s)\Phi(t)K(\mathbf{1}_{[0,t]} - F(t)K)^{-1}u\big).
\end{aligned}$$

We continue with the properties of $\hat{\Phi}, \hat{\Psi}$ and $\hat{F}$.

$$\begin{aligned}
&\hat{\Phi}(t+s)(u \underset{t}{\diamondsuit} v) \\
&= \Phi(t+s)(\mathbf{1}_{[0,t+s]} + KF(t+s))^{-1}KF(t+s)(u \underset{t}{\diamondsuit} v) \\
&= \Phi(t+s)(\mathbf{1}_{[0,t+s]} + KF(t+s))^{-1}K\left(F(t)u \underset{t}{\diamondsuit}(\Psi(s)\Phi(t)u + F(s)u)\right) \\
&= \Phi(t+s)[(\mathbf{1}_{[0,t]} + KF(t))^{-1}KF(t)u \underset{t}{\diamondsuit}((\mathbf{1}_{[0,s]} + KF(s))^{-1} \\
&\qquad \cdot (\Psi(s)\Phi(t)u + F(s)u - \Psi(s)\Phi(t)K(\mathbf{1}_{[0,t]} + KF(t))^{-1}KF(t)u)] \\
&= T(t)\Phi(s)(\mathbf{1}_{[0,t]} + KF(t))^{-1}KF(t)u \\
&\quad + \Phi(t)\big((\mathbf{1}_{[0,s]} + KF(s))^{-1}K(\Psi(s)\Phi(t)u + F(s)u \\
&\qquad - \Psi(s)\Phi(t)K(\mathbf{1}_{[0,t]} + KF(t))^{-1}KF(t)u\big) \\
&= \hat{T}(t)\hat{\Phi}(s)u + \hat{\Phi}(t)v
\end{aligned}$$

and

$$\begin{aligned}
\hat{\Psi}(t+s)x_0 &= (\mathbf{1}_{[0,s+t]} + F(s+t)K)^{-1}\Phi(t+s)x_0 \\
&= (\mathbf{1}_{[0,s+t]} + F(s+t)K)^{-1}\left(\Psi(t)x_0 \underset{t}{\diamondsuit} \Psi(s)T(t)x_0\right) \\
&= (\mathbf{1}_{[0,t]} + F(t)K)^{-1}\Psi(t)x_0 \\
&\quad \underset{t}{\diamondsuit}(\mathbf{1}_{[0,s]} + F(s)K)^{-1}\left[\Psi(s)T(t)x_0 - \Psi(s)\Phi(t)K(\mathbf{1}_{[0,t]} + F(t)K)^{-1}\Psi(t)x_0\right] \\
&= \hat{\Psi}(t)x_0 \underset{t}{\diamondsuit} \hat{\Psi}(s)\hat{T}(t)x_0
\end{aligned}$$

as well as

$$\begin{aligned}
&\hat{F}(t+s)(u \underset{t}{\diamondsuit} v)\\
&= (\mathbf{1}_{[0,s+t]} + F(s+t)K)^{-1} F(s+t)(u \underset{t}{\diamondsuit} v)\\
&= (\mathbf{1}_{[0,s+t]} + F(s+t)K)^{-1} \left(F(t)u \underset{t}{\diamondsuit} (\Psi(s)\Phi(t)u + F(s)v) \right)\\
&= (\mathbf{1}_{[0,t]} + F(t)K)^{-1} F(t)u\\
&\quad \underset{t}{\diamondsuit} (\mathbf{1}_{[0,s]} + F(s)K)^{-1} \left(\Psi(s)\Phi(t)u + F(s)v - \Psi(s)\Phi(t)K(\mathbf{1}_{[0,t]} + F(t)K)^{-1} F(t)u \right)\\
&= \hat{F}(t)u \underset{t}{\diamondsuit} \left(\hat{F}(s)v + (\mathbf{1}_{[0,s]} + F(s)K)^{-1} \Psi(s) \left(\Phi(t) - \Phi(t)K(\mathbf{1}_{[0,t]} + F(t)K)^{-1} F(t) \right) u \right)\\
&= \hat{F}(t)u \underset{t}{\diamondsuit} \left(\hat{F}(s)v + \hat{\Psi}(s)\hat{\Phi}(s)u \right).
\end{aligned}$$

Therefore, it only remains to check that $\hat{\Sigma}$ is well-posed. For this end, it is enough to show the inequality

$$\|x(t)\|_X^2 + \|y\|_{L_2(0,t;Y)}^2 \le \hat{c}_t \left(\|x_0\|^2 + \|u - Ky\|_{L_2(0,t;U)}^2 \right)$$

for every solution $(x(t), y_{[0,t]}) = \Sigma(t)(x_0, u)$ $(x_0 \in X,\ u \in L_{2,loc}(\mathbb{R}_+; U))$ of the original linear system. In fact, for every fixed $t \ge 0$ we find that

$$\begin{aligned}
\|x(t)\|_X^2 &\le \|x_0\|^2 + 2\operatorname{Re}\langle u, y\rangle_{L_2(0,t;U)}\\
&\le \|x_0\|^2 + 2\operatorname{Re}\langle u + Ky, y\rangle_{L_2(0,t;U)} - 2\sigma \|Ky\|_{L_2(0,t;U)}^2\\
&\le \|x_0\|^2 + \varepsilon \|y\|_{L_2(0,t;U)}^2 + \varepsilon^{-1} \|u - Ky\|_{L_2(0,t;U)}^2 - 2\sigma \|Ky\|_{L_2(0,t;U)}^2 \qquad (7.7)
\end{aligned}$$

for every $\varepsilon > 0$. Moreover,

$$\|u\|_{L_2(0,t;U)}^2 \le 2\|u - Ky\|_{L_2(0,t;U)}^2 + 2\|Ky\|^2 \qquad (7.8)$$

Combined with the well-posedness of the system $\Sigma = (T, \Phi, \Psi, F)$,

$$\|y\|_{L_2(0,t;U)}^2 + \|x(t)\|_X^2 \le c_t \left(\|x_0\|_X^2 + \|u\|_{L_2(0,t;U)}^2 \right) \qquad (7.9)$$

we therefore find by adding c_t times equation (7.7) to $s\sigma$ times equation (7.9) and using inequality (7.8) that

$$\begin{aligned}
&(\sigma - \varepsilon c_t)\|y\|_{L_2(0,t;Y)}^2 + (\sigma + c_t)\|x(t)\|_X^2\\
&\le (\sigma + 1)c_t \|x_0\|_X^2 + (2\sigma + \varepsilon^{-1})c_t \|u - Ky\|_{L_2(0,t;U)}^2
\end{aligned}$$

and well-posedness of the system $\hat{\Sigma} = (\hat{T}, \hat{\Phi}, \hat{\Psi}, \hat{F})$ follows by choosing $\varepsilon \in (0, \frac{\sigma}{c_t})$. □

From here we may easily remove Assumption 7.2.6 for impedance passive systems.

Corollary 7.2.15. *Assume that $D_c : X_c \to B(U_c)$ is such that for every $x_c \in X_c$ there is $\sigma(x_c) > 0$ such that*

$$\operatorname{Re}\langle D_c(x_c)z_c, z_c\rangle \ge \sigma(x_c)\,|z_c|^2, \quad z_c \in U_c.$$

Further assume that the well-posed linear system (T, Φ, Ψ, F) *is impedance passive. Then for every initial value* $(x_0, x_{c,0}) \in X \times X_c$ *problem* (7.3) *has a unique maximal solution* $(x, x_c) \in C([0, t_{max}(x_0, x_{c,0})); X \times X_c)$*. Moreover, if additionally also the nonlinear control system is impedance passive with respect to some radially unbounded and continuous functional* $H_c : X_c \to \mathbb{R}_+$*, then all these solutions are global, i.e.* $t_{max}(x_0, x_{c,0}) = +\infty$ *for all* $(x_0, x_{c,0}) \in X \times X_c$*, and have non-increasing* total energy

$$H_{tot}((x, x_c)(t)) = H(x(t)) + H_c(x_c(t)) = \frac{1}{2} \|x(t)\|_X^2 + H_c(x_c(t)), \quad t \geq 0.$$

Proof. Fix any initial value $(x_0, x_{c,0}) \in X \times X_c$. Then the perturbed linear system $\hat{\Sigma}$ resulting from the adjusted input $\hat{u} = u - D_c(x_{c,0})y$ is well-posed by Remark 7.2.14 and problem (7.3) may be equivalently re-written as

$$\begin{aligned} x(t) &= \hat{T}(t)x_0 + \hat{\Phi}(t)\hat{u} \\ x_c(t) &= x_{c,0} + \int_0^t [A_c\&B_c](x_c(s), u_c(s))ds, \quad t \geq 0 \\ \hat{u} &= -C_c(x_c) - [D_c(x_c) - D_c(x_{c,0})]u_c \\ u_c &= \hat{y} = \hat{\Psi} + \hat{F}\hat{u}. \end{aligned}$$

Since $\sup_{x_c \in B_\rho(x_{c,0})} |D_c(x_c) - D_c(x_{c,0})| \inf_{t>0} \left\| \hat{F}(t) \right\| < 1$ for sufficiently small $\rho > 0$, the proof of Theorem 7.2.7 shows that this problem admits a unique mild solution $(x, x_c) \in C([0, \tau]; X \times X_c)$ on some interval $[0, \tau]$. Using Zorn's Lemma this solution may be extended in a unique way to a maximal solution $(x, x_c) \in C([0, t_{max}); X \times X_c)$. Now assume that both systems are impedance passive, then

$$H_{tot}((x, x_c)(t)) \leq H_{tot}(x_0, x_{c,0}), \quad t \in [0, t_{max})$$

(cf. Corollary 7.2.8) and a procedure similar to that in Corollary 7.2.8 shows that then $t_{max} = +\infty$. □

Also the term $D_c(x_c)$ may even help to remove the well-posedness assumption on the impedance passive port-Hamiltonian system.

Corollary 7.2.16. *Let* $\mathfrak{S} = (\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ *be an impedance passive port-Hamiltonian system and* Σ_c^{nl} *be an impedance passive (w.r.t. some radially unbounded continuous functional* $H_c : X_c \to \mathbb{R}_+$*) control system (with* $U_c = Y_c = \mathbb{F}^k$ *for some* $1 \leq k \leq Nd$*) as above. Further assume that*

$$\operatorname{Re} \langle D_c(x_c)z_c, z_c \rangle \geq \sigma(x_c) |z_c|, \quad z_c \in U_c$$

for constants $\sigma(x_c) > 0$ *depending on* $x_c \in X_c$*. Then for every initial value* $(x_0, x_{c,0}) \in X \times X_c$ *the problem*

$$\begin{aligned} \frac{d}{dt}x(t) &= \mathfrak{A}x(t), \\ x_c(t) &= x_{c,0} + \int_0^t A_c(x_c(s))x_c(s) + B_c(x_c(s))u_c(s)ds, \quad t \geq 0 \\ \mathfrak{B}_1 x &= -y_c = -C_c(x_c)x_c - D_c(x_c)u_c, \quad \mathfrak{B}_2 x = 0 \\ u_c &= \mathfrak{C}_1 x \end{aligned}$$

where the input map $\mathfrak{B}$ is decomposed as $\mathfrak{B}x = (\mathfrak{B}_1 x, \mathfrak{B}_2 x) \in \mathbb{F}^k \times \mathbb{F}^{Nd-k}$, and also the output map $\mathfrak{C}$ accordingly, has a unique mild solution $(x, x_c) \in C(\mathbb{R}_+; X \times X_c)$ for which $(u, y) \in L_{2,loc}(\mathbb{R}_+; U \times Y)$ and the total energy

$$H_{tot}((x, x_c)(t)) = \frac{1}{2}\|x(t)\|_X^2 + H_c(x_c(t)), \quad t \geq 0$$

is non-increasing.

Proof. Let an initial value $(x_0, x_{c,0}) \in X \times X_c$ be given. Analogously to the previous corollary we consider the following equivalent problem

$$\begin{aligned}
\frac{d}{dt}x(t) &= \mathfrak{A}x(t), \\
x_c(t) &= x_{c,0} + \int_0^t A_c(x_c(s))x_c(s) + B_c(x_c(s))u_c(s)ds, \quad t \geq 0 \\
\mathfrak{B}_1 x + D_c(x_{c,0})\mathfrak{C}_1 x &= D_c(x_{c,0})u_c - y_c = -C_c(x_c)x_c - u_c, \quad \mathfrak{B}_2 x = 0 \\
u_c &= \mathfrak{C}_1 x
\end{aligned}$$

Then the Boundary Control System $(\mathfrak{A}_1 := \mathfrak{A}|_{\ker \mathfrak{B}_2}, \mathfrak{B}_1 + D_c(x_{c,0})\mathfrak{C}_1, \mathfrak{C}_1)$ not only is impedance passive, but

$$\begin{aligned}
\operatorname{Re}\langle \mathfrak{A}_1 x, x\rangle_X &\leq \operatorname{Re}\langle \mathfrak{B}_1 x, \mathfrak{C}_1 x\rangle_{\mathbb{F}^k} \\
&= \operatorname{Re}\langle \mathfrak{B}_1 x + D_c(x_{c,0})\mathfrak{C}_1, \mathfrak{C}_1 x\rangle_{\mathbb{F}^k} - \operatorname{Re}\langle K\mathfrak{C}_1 x, \mathfrak{C}_1 x\rangle_{\mathbb{F}^k} \\
&\leq \operatorname{Re}\langle \mathfrak{B}_1 x + D_c(x_{c,0})\mathfrak{C}_1, \mathfrak{C}_1 x\rangle_{\mathbb{F}^k} - \sigma |\mathfrak{C}_1 x|, \quad x \in D(\mathfrak{A})
\end{aligned}$$

so that

$$\begin{aligned}
\sigma \|\mathfrak{C}_1\|_{L_2(0,t;Y)}^2 + \|x(t)\|_X^2 &\leq \sigma \|\mathfrak{C}_1\|_{L_2(0,t;Y)}^2 + \|x_0\|_X^2 + 2\operatorname{Re}\langle \mathfrak{B}_1 x, \mathfrak{C}_1 x\rangle_{L_2(0,t;U)} \\
&\leq \sigma \|\mathfrak{C}_1\|_{L_2(0,t;Y)}^2 + \|x_0\|_X^2 - 2\sigma \|\mathfrak{C}_1 x\|_{L_2(0,t;U)}^2 \\
&\quad + 2\operatorname{Re}\langle \mathfrak{B}_1 x + D_c(x_{c,0})\mathfrak{C}_1 x, \mathfrak{C}_1 x\rangle_{L_2(0,t;U)} \\
&\leq \|x_0\|_X^2 + \|\mathfrak{B}_1 x + D_c(x_{c,0})\mathfrak{C}_1 x\|_{L_2(0,t;U)}^2
\end{aligned}$$

for every classical solution $x \in C^1(\mathbb{R}_+; X) \cap C(\mathbb{R}_+; D(\mathfrak{A}))$ of the Boundary Control System $(\mathfrak{A}_1, \mathfrak{B}_1 + D_c(x_{c,0})\mathfrak{C}_1, \mathfrak{C}_1)$ and therefore the system $(\mathfrak{A}_1, \mathfrak{B}_1 + D_c(x_{c,0})\mathfrak{C}_1, \mathfrak{C}_1)$ is well-posed as Boundary Control and Observation System and so it is equivalent to a well-posed linear system $\Sigma_{D_c(x_{c,0})}$. As before we first obtain a uniquely determined local solution which can be extended to a maximal solution and impedance passivity of the two subsystems leads to global existence and non-increasing energy. □

Definition 7.2.17. *If $\mathfrak{S} = (\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ is an impedance passive port-Hamiltonian system and is interconnected with an impedance passive(w.r.t. some radially unbounded functional H_c on X_c) nonlinear controller* (NLC), *we denote by*

$$\mathcal{A}\begin{pmatrix} x \\ x_c \end{pmatrix} = \begin{pmatrix} \mathfrak{A}x \\ A_c(x_c)x_c + B_c(x_c)\mathfrak{C}x \end{pmatrix}$$

$$D(\mathcal{A}) = \{(x, x_c) \in D(\mathfrak{A}) \times X_c : \ \mathfrak{B}x = -(C_c(x_c)x_c + D_c(x_c)\mathfrak{C}x)\}$$

the corresponding nonlinear hybrid operator on $X \times \underline{X_c} := X \times X_c$. Since the solution depends continuously on the initial value $(x_0, x_{c,0})$, the maps $\mathcal{S}(t) : X \times X_c \to X \times X_c$ defined by

$$\mathcal{S}(t)(x_0, x_{c,0}) := x(t; x_0, x_{c,0}), \quad t \geq 0, \ (x_0, x_{c,0}) \in X \times X_c$$

where $x(t;x_0,x_{c,0}) \in X \times X_c$ *denotes the solution at time* $t \geq 0$ *for the initial value* $(x_0,x_{c,0}) \in X \times X_c$*, defines a s.c. (in general not contractive) semigroup* $(\mathcal{S}(t))_{t\geq 0}$ *on* $X \times X_c$*.*

Example 7.2.18. *In the model proposed in [Le14] and considered in Example 7.2.1 and Example 7.2.12 the condition on* D_c *in Corollary 7.2.15 is satisfied if the function* k *has range in the interval* $[\alpha_1,\alpha_2]$ *where* $\alpha_i > 0$ $(i = 1,2)$*. Hence, if the control system is impedance passive, the standard feedback interconnection with an impedance passive well-posed system is globally well-posed, i.e. there is a unique global mild solution of the interconnected system.*

7.3 Exponential Stability

We proceed by investigating the stability properties of the solutions of hybrid systems consisting of an impedance passive port-Hamiltonian system $\mathfrak{S} = (\mathfrak{A},\mathfrak{B},\mathfrak{C})$ which is coupled with an impedance passive nonlinear controller, may it be given by the strongly continuous contraction semigroup $(\mathcal{S}(t))_{t\geq 0}$ generated by the m-dissipative map $\mathcal{A}$ of Theorem 7.1.4, or be the solution of an impedance passive port-Hamiltonian system interconnected with an impedance passive nonlinear controller as in Subsection 7.2.3.
We start with an asymptotic stability result for the solutions of the nonlinear Cauchy problem governed by an m-dissipative hybrid map $\mathcal{A}$ as in Theorem 7.1.4.

Proposition 7.3.1. *Assume that* $\mathcal{A}$ *is the* m*-dissipative map resulting from the standard feedback interconnection of an impedance passive port-Hamiltonian system with an* m*-dissipative map* M_c*. Further assume the following.*

1. $\Pi_{X_c} \operatorname{ran} M_c \hookrightarrow X_c$ *is compactly embedded and convex*
2. *There is an orthogonal projection* $\Pi : \mathbb{F}^{Nd} \to \mathbb{F}^{Nd}$ *such that*
$$M_c(0,u_c) \subseteq X_c \times \{0\}, \quad u_c \in \ker \Pi$$
and for some function $p : \mathbb{R}_+ \to \mathbb{R}_+$ *with* $p(z) > 0$ *for all* $z > 0$
$$\operatorname{Re}\langle (z_c,w_c),(x_c,u_c)\rangle_{X_c\times U_c} \leq -p\left(|\Pi u_c|^2\right), \quad (z_c,w_c) \in M_c(x_c,u_c)$$
3. $0 \in M_c(0)$ *and if* $(i\beta x_c, w_c) \in M_c(x_c,u_c)$ *for some* $\beta \in \mathbb{R}$ *and* $u_c \in \ker \Pi$*, then already* $x_c = 0$*.*

If the pair $(\mathfrak{A},(\mathfrak{B},\Pi\mathfrak{C}))$ *has property ASP, then*

$$\mathcal{S}(t)(x_0,x_{c,0}) \xrightarrow{t\to\infty} 0, \quad (x_0,x_{c,0}) \in X \times \underline{X_c}$$

i.e. 0 *is a globally asymptotically stable equilibrium of the nonlinear s.c. contraction semigroup* $(\mathcal{S}(t))_{t\geq 0}$ *generated by* $\mathcal{A}$*.*

Proof. First of all note that $D(\mathcal{A}) \subseteq D(\mathfrak{A}) \times \Pi_{X_c} \operatorname{ran} M_c \hookrightarrow X \times X_c$ is compactly embedded and convex. Therefore, by Theorem 2.2.32 for every $(x_0,x_{c,0}) \in D(\mathcal{A})$ the trajectory $\mathcal{S}(t)(x_0,x_{c,0})$ converges to some compact subset $\Omega_{x_0,x_{c,0}}$ of a sphere

$S_r(0)$ around 0 of radius $r \geq 0$, where we understand the sphere of radius 0 to be $\{0\}$, and which also is a subset of $D(\mathcal{A})$. Moreover,

$$\mathcal{S}(t)|_{\Omega_{x_0,x_{c,0}}} = \hat{\mathcal{T}}(t)|_{\Omega_{x_0,x_{c,0}}}, \quad t \geq 0$$

for some isometric C_0-group $(\hat{\mathcal{T}}(t))_{t\geq 0}$ on lin $\Omega_{x_0,x_{c,0}}$ with generator $\hat{\mathcal{A}}$. Let us investigate the stability properties of $(\hat{\mathcal{T}}(t))_{t\geq 0}$ and consider the problem

$$i\beta(\hat{x}, \hat{x}_c) = \hat{\mathcal{A}}(\hat{x}, \hat{x}_c)$$

where w.l.o.g. we may assume that $(\hat{x}, \hat{x}_c) \in \Omega_{x_0,x_{c,0}}$, so that

$$i\beta(\hat{x}, \hat{x}_c) = \mathcal{A}(\hat{x}, \hat{x}_c).$$

Then

$$0 = \operatorname{Re} \langle \mathcal{A}(\hat{x}, \hat{x}_c), (\hat{x}, \hat{x}_c)\rangle_{X\times X_c} \leq -p\left(|\Pi\mathfrak{C}\hat{x}|^2\right)$$

i.e. $\Pi\mathfrak{C}x = 0$. From $(i\beta\hat{x}_c, -\mathfrak{B}\hat{x}) \in M_c(\hat{x}_c, \mathfrak{C}x)$ with $\mathfrak{C}x \in \ker \Pi$, we conclude from the assumptions of the theorem that $\hat{x}_c = 0$. Since $(0, -\mathfrak{B}\hat{x}) \in M_c(0, \mathfrak{C}x)$ and $\mathfrak{C}x \in \ker \Pi$ we may also employ the second assumption and obtain that $\mathfrak{B}\hat{x} = 0$, so that

$$\begin{aligned} \mathfrak{A}\hat{x} &= i\beta\hat{x} \\ \mathfrak{B}\hat{x} &= 0, \quad \Pi\mathfrak{C}\hat{x} = 0 \end{aligned}$$

and since the pair $(\mathfrak{A}, (\mathfrak{B}, \Pi\mathfrak{C}))$ has property ASP we find that $\hat{x} = 0$. Since $\beta \in \mathbb{R}$ had been arbitrary this proves that $i\mathbb{R} \cap \sigma_p(\hat{\mathcal{A}}) = \emptyset$. However, this implies that $(\hat{\mathcal{T}}(t))_{t\geq 0}$ is asymptotically stable and isometric at the same time which can only hold true if $\overline{\Omega_{x_0,x_{c,0}}} = \{0\}$, i.e. $(\mathcal{S}(t))_{t\geq 0}$ is asymptotically stable. □

We continue with the investigation of stability properties of the s.c semigroup generated by the m-dissipative hybrid map $\mathcal{A}$ of Theorem 7.1.4 and aim for uniform exponential stability next. The first approach is to impose the following dissipativity constraints on the controller.

Assumption 7.3.2. *Assume that $\mathcal{A}$ is an m-dissipative operator as in Theorem 7.1.4 and further assume that $0 \in M_c(0)$ and there is $\rho > 0$ and an orthogonal projection $\Pi : \mathbb{F}^{Nd} \to \mathbb{F}^{Nd}$ on some subspace of $\mathbb{F}^{Nd}$ such that*

$$\operatorname{Re}\, [(z_c, y_c), (x_c, u_c)]_{X_c\times\mathbb{F}^{Nd}} \leq -\rho\left(\|x_c\|^2_{X_c} + \|\Pi u_c\|^2_{\mathbb{F}^{Nd}}\right)$$

for all $(x_c, u_c) \in D(M_c)$ and $(z_c, y_c) \in M_c(x_c, u_c)$ and some equivalent inner product $[\cdot,\cdot]_{X_c\times\mathbb{F}^{Nd}}$ on $X_c \times \mathbb{F}^{Nd}$. Further we assume that for some $c' > 0$ and all $(x_c, u_c) \in D(M_c)$, $w_c \in \Pi_{\mathbb{F}^{Nd}} M_c(x_c, u_c)$ one has

$$|w_c| \leq c'\left(\|x_c\|_{X_c} + |\Pi u_c|\right).$$

Example 7.3.3 (Collocated case). *One particular case which is covered by the preceding assumption is the following. Let $\Sigma_c = (A_c, B_c, C_c, D_c)$ be an impedance passive system with $C_c = B_c' \in \mathcal{B}(X_c; \mathbb{F}^{Nd})$ (the Hilbert space adjoint operator of B_c with respect to the inner products $\langle\cdot,\cdot\rangle_{X_c}$ and $\langle\cdot,\cdot\rangle_{\mathbb{F}^{Nd}}$ on X_c and $\mathbb{F}^{Nd}$, respectively), i.e. collocated input and output, and $A_c, -D_c$ be m-dissipative, possibly multi-valued and nonlinear maps. Further we assume that*

1. *$A_c(0) = \{0\}$ and there is an equivalent inner product $[\cdot,\cdot]_{X_c}$ (inducing the norm $|\cdot|_{X_c}$) on X_c such that for some $\rho > 0$ and all $x_c \in D(A_c)$, $z_c \in A_c(x_c)$ one has*
$$\operatorname{Re}\,[z_c, x_c]_{X_c} \le -\rho\,\|x_c\|_{X_c}^2\,, \tag{7.10}$$

2. *$0 \in D_c(0)$, $\Pi : \mathbb{F}^{Nd} \to \mathbb{F}^{Nd}$ is an orthogonal projection such that $|w_c| \lesssim |\Pi u_c|$ $(u_c \in D(D_c),\ w_c \in D_c(u_c))$ and there is $\sigma > 0$ such that for all $x_c \in X_c$, $z_c \in A_c(x_c)$, $u_c \in \mathbb{F}^{Nd}$ and $w_c \in D_c(u_c)$ one has*
$$\operatorname{Re}\,\langle z_c + B_c u_c, x_c\rangle_{X_c} \le \operatorname{Re}\,\langle C_c x_c + w_c, u_c\rangle_{\mathbb{F}^{Nd}} - \sigma\,|w_c|^2\,. \tag{7.11}$$

Then $M_c = \begin{pmatrix} A_c & B_c \\ -C_c & -D_c \end{pmatrix}$ satisfies Assumption 7.3.2.

Proof. To see that Assumption 7.3.2 is actually satisfied make the ansatz

$$[(x_c, u_c), (\tilde{x}_c, \tilde{u}_c)]_{X_c\times\mathbb{F}^{Nd}} := \alpha\,[x_c, \tilde{x}_c]_{X_c} + \langle (x_c, u_c), (\tilde{x}_c, \tilde{u}_c)\rangle_{X_c\times\mathbb{F}^{Nd}}.$$

We then have for every $(x_c, u_c) \in D(M_c)$ and $z_c \in A_c(x_c)$ and $w_c \in D_c(u_c)$ that

$$\begin{aligned}
&[(z_c + B_c u_c, -C_c - w_c), (x_c, u_c)]_{X_c\times\mathbb{F}^{Nd}} \\
&= \alpha\,[z_c + B_c u_c, x_c]_{X_c} + \langle (z_c + B_c u_c, C_c x_c + w_c), (x_c, u_c)\rangle_{X_c\times\mathbb{F}^{Nd}} \\
&\le -\rho\,\|x_c\|_{X_c}^2 + \alpha\,[B_c u_c, x_c] - \sigma\,|w_c|^2 \\
&\le (\alpha c^2 - \rho)\,\|x_c\|_{X_c}^2 + (\alpha c^2\,\|B_c\| - \sigma)\,\|w_c\|^2
\end{aligned}$$

where $|\cdot| \le c\,\|\cdot\|_{X_c}$ and for $\alpha \in \left(0, \min\{\frac{\rho}{c^2}, \frac{\sigma}{\|B_c\|c^2}\}\right)$ sufficiently small this equivalent inner product on $X_c \times \mathbb{F}^{Nd}$ does the job. □

Under this Assumption 7.3.2 we can show the following.

Proposition 7.3.4. *Let $\mathfrak{S} = (\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ be an impedance passive Boundary Control and Observation system and $M_c : D(M_c) \subseteq X_c \times \mathbb{F}^{Nd} \rightrightarrows X_c \times \mathbb{F}^{Nd}$ be as in Assumption 7.3.2. Denote by $(\mathcal{S}(t))_{t\ge 0}$ the nonlinear s.c. contraction semigroup associated to $\mathcal{A}$ as in Theorem 7.1.4. If there is $q : X \to \mathbb{R}$ such that $|q(x)| \le \hat{c}\,\|x\|_X^2$ $(x \in X)$ and for all mild solutions $x \in W_\infty^1(\mathbb{R}_+; X) \cap L_\infty(\mathbb{R}_+; D(\mathfrak{A}))$ of $\dot{x} = \mathfrak{A}x$ one has $q(x) \in W_\infty^1(\mathbb{R}_+)$ and*

$$\|x(t)\|_X + \frac{d}{dt} q(x(t)) \le c\left(|\mathfrak{B}x(t)|^2 + |\Pi\mathfrak{C}x(t)|^2\right), \quad a.e.\ t \ge 0,$$

then 0 is a globally uniformly exponentially stable equilibrium of $(\mathcal{S}(t))_{t\ge 0}$.

Remark 7.3.5. *In the collocated input/output case (see Example 7.3.3) the first condition means that there is $\tilde{c} \ge 0$ such that for all $u_c \in \mathbb{F}^{Nd}, w_c \in D_c(u_c)$*

$$|w_c| \le \tilde{c}\,\|B_c u_c\|_{X_c} \tag{7.12}$$

i.e. the control state is only directly influenced by those components in which the nonlinear controller is strictly input passive, which means that for the input Πu_c the controller is SIP, and the last condition may be replaced by

$$\|x(t)\|_X^2 + \frac{d}{dt} q(x(t)) \le c\left(|\mathfrak{B}x(t)|^2 + \left|D_c^0\mathfrak{C}x(t)\right|^2\right), \quad a.e.\ t \ge 0,$$

where D_c^0 denotes the minimal section of D_c. To see this let $\Pi : \mathbb{F}^{Nd} \to \mathbb{F}^{Nd}$ be the projection on $(\ker B_c)^\perp$, then for $u_c \in D(D_c)$

$$\left|D_c^0 u_c\right| \lesssim \|B_c u_c\|_{X_c} \lesssim |\Pi u_c| .$$

Remark 7.3.6. *We give some interpretation for the preceding conditions in the collated input/output case. The* Lyapunov condition *(7.10) says that* 0 *is a globally uniformly exponentially stable equilibrium for the s.c. contraction semigroup $(S_c(t))_{t\geq 0}$ associated to A_c. If one has a globally exponentially stable minimum at some other point $x_c^* \in X_c$ one may simply introduce $x_c^{new} := x_c - x_c^*$ as new variable to get to the situation as above. (Similar, one may choose a nonzero desired equilibrium x_c^*.) Conditions (7.11) and (7.12) together may be seen as a strict input passivity condition on the controller system (after getting rid of the redundant parts of the input which only constitute static boundary conditions on the system $\mathfrak{S} = (\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$). In particular, if $D_c = D_c^*$ were linear and symmetric the second condition would read as*

$$\operatorname{Re}\langle z_c + B_c u_c, x_c\rangle_{X_c} \leq \operatorname{Re}\langle C_c x_c + D_c u_c, u_c\rangle - \tilde{\sigma}\,|\Pi_{D_c} u_c|^2$$

for some $\tilde{\sigma} > 0$ and Π_{D_c} the projection on $\ker {D_c}^\perp$.

Proof of Proposition 7.3.4. Let $\delta > 0$ which we choose suitable later on. Let $(x_0, x_{c,0}) \in D(\mathcal{A})$ be arbitrary and set $(x, x_c)(t) := \mathcal{S}(t)(x_0, x_{c,0})$ $(t \geq 0)$. Define

$$\Phi(t) := t\left(\|x(t)\|_X^2 + |x_c(t)|_{X_c}^2\right) + q(x(t)), \quad t \geq 0$$

and note that $\frac{d}{dt}(x, x_c)(t) = (\mathfrak{A}x(t), z_c(t)) := \mathcal{A}^0((x, x_c)(t))$ (a.e. $t \geq 0$) and $\Phi \in W^1_{\infty,loc}(\mathbb{R}_+)$ with

$$\begin{aligned}
\frac{d}{dt}\Phi(t) &\leq |x_c(t)|_{X_c}^2 + 2t\operatorname{Re}\langle \mathfrak{A}x(t), x(t)\rangle_X \\
&\quad + 2t\operatorname{Re}\,[z_c(t), x_c(t)]_{X_c} + c\left(|\mathfrak{B}x(t)|^2 + |\Pi\mathfrak{C}x(t)|^2\right) \\
&\leq |x_c(t)|_{X_c}^2 + c\left(|\mathfrak{B}x(t)|^2 + |\Pi\mathfrak{C}x(t)|^2\right) - 2\rho t\left(\|x_c(t)\|_{X_c}^2 + |\Pi\mathfrak{C}x(t)|^2\right) \\
&\leq |x_c(t)|_{X_c}^2 + c\left(|\mathfrak{B}x(t)|^2 + |\Pi\mathfrak{C}x(t)|^2\right) \\
&\quad - \rho t\left(\|x_c(t)\|_{X_c}^2 + |\Pi\mathfrak{C}x(t)|^2 + \frac{1}{c'}|\mathfrak{B}x(t)|^2\right) \\
&\leq (\tilde{c} - \rho t)\,\|x_c(t)\|_{X_c}^2 + (c - \frac{\rho t}{c'})\,|\mathfrak{B}x(t)|^2 + (c - \rho t)\,|\Pi\mathfrak{C}x(t)|^2, \quad \text{a.e. } t \geq 0.
\end{aligned}$$

Choosing $t_0 := \max\{\frac{\tilde{c}}{\rho}, \frac{c}{\rho}, \frac{cc'}{\rho}\} > 0$ (independent of the initial value x_0) we have that Φ is decreasing on (t_0, ∞). Since $\frac{\Phi(t)}{t}$ behaves as $\|(x, x_c)(t)\|^2$ as $t \to \infty$, we easily deduce uniform exponential stability from this. In fact, for $t \geq t_0$ we have

$$\begin{aligned}
\|(x, x_c)(t)\|_{X\times X_c}^2 &= \frac{\Phi(t)}{t} - \frac{q(x(t)) + |x_c(t)|_{X_c}^2}{t} \\
&\leq \frac{\Phi(t)}{t} + \frac{\hat{c}\,\|x(t)\|_X}{t} \\
&\leq \frac{1}{t}\Phi(t_0) + \frac{\hat{c}}{t}\,\|(x, x_c)(t)\|_{X\times X_c}^2 \\
&\leq \frac{t_0}{t}\max\{1 + \hat{c}, 1 + \tilde{c}\}\,\|(x, x_c)(0)\|_{X\times X_c}^2 + \frac{\hat{c}}{t}\,\|(x, x_c)(t)\|_{X\times X_c}^2,
\end{aligned}$$

so that $\|(x,x_c)(t)\|^2_{X\times X_c} \leq \frac{t_0 \max\{1+\hat{c},1+\tilde{c}\}}{t-\hat{c}} \|(x,x_c)(0)\|^2_{X\times X_c}$ $(t > \max\{t_0,\hat{c}\})$ from where exponential stability with constants $M \geq 1$ and $\omega < 0$ independent of $x_0 \in D(\mathcal{A})$ follows. From density of $D(\mathcal{A})$ in $X \times \underline{X_c}$ and continuity of $\mathcal{S}(t)$ $(t \geq 0)$ we conclude

$$\|\mathcal{S}(t)(x_0,x_{c,0})\|_{X\times X_c} \leq Me^{\omega t}\|(x_0,x_{c,0})\|_{X\times X_c}, \quad (x_0,x_{c,0}) \in X \times \underline{X_c},\ t \geq 0.$$

This means that 0 is a globally uniformly exponentially stable equilibrium. □

Thus, Proposition 7.3.4 and Proposition 4.3.8 together say the following.

Theorem 7.3.7. *Let $\mathfrak{S} = (\mathfrak{A},\mathfrak{B},\mathfrak{C})$ be an impedance passive port-Hamiltonian system of order $N = 1$ and $M_c : D(M_c) \subseteq X_c \times \mathbb{F}^{Nd} \rightrightarrows X_c \times \mathbb{F}^{Nd}$ as in Assumption 7.3.2. Further assume that for some $c' > 0$ and all $(x_c,u_c) \in D(M_c)$, $(z_c,w_c) \in M_c(x_c,u_c)$*

$$|w_c|^2 \leq c'\|x_c\|^2_{X_c} + |\Pi u_c|^2.$$

and

$$|(\mathcal{H}x)(1)|^2 \lesssim |\mathfrak{B}x|^2 + |\Pi\mathfrak{C}x|^2, \quad x \in D(\mathfrak{A}).$$

Then the interconnected map $\mathcal{A}$ from Theorem 7.1.4 generates a s.c. contraction semigroup $(S(t))_{t\geq 0}$ on $X \times \underline{X}_c$ with globally uniformly exponentially stable equilibrium 0.

The interplay of Lemma 6.3.3 with Proposition 7.3.4 then implies the following.

Theorem 7.3.8. *Let $\mathfrak{S} = (\mathfrak{A},\mathfrak{B},\mathfrak{C})$ be an impedance passive port-Hamiltonian system of order $N = 2$ and $M_c : D(M_c) \subseteq X_c \times \mathbb{F}^{Nd} \rightrightarrows X_c \times \mathbb{F}^{Nd}$ as in Assumption 7.3.2. Further assume that $\mathfrak{A}$ satisfies the regularity assumptions of Lemma 6.3.3 and that for some $c' > 0$ and all $(x_c,u_c) \in D(M_c)$, $(z_c,w_c) \in M_c(x_c,u_c)$*

$$|w_c|^2 \leq c'\left(\|x_c\|^2_{X_c} + |\Pi u_c|^2\right).$$

and

$$|(\mathcal{H}x)(0)|^2 + |(\mathcal{H}x)(1)|^2 + |(\mathcal{H}x)'(0)|^2 \lesssim |\mathfrak{B}x|^2 + |\Pi\mathfrak{C}x|^2, \quad x \in D(\mathfrak{A}).$$

Then the interconnected map $\mathcal{A}$ from Theorem 7.1.4 generates a s.c. contraction semigroup $(S(t))_{t\geq 0}$ on $X \times \underline{X}_c$ with globally uniformly exponentially stable equilibrium 0.

Unfortunately the conditions of Assumptions 7.3.2 are quite restrictive since even for linear systems the dissipation condition

$$\begin{aligned}&\mathrm{Re}\ [(A_c x_c + B_c u_c, -C_c x_c - D_c u_c),(x_{,c}, u_c)]_{X_c\times\mathbb{F}^{Nd}}\\ &\leq -\rho\left(\|x_c\|^2_{X_c} + \|\Pi u_c\|^2_{\mathbb{F}^{Nd}}\right)\end{aligned}$$

for every $x_c \in X_c$ and $u_c \in U_c$, is usually not satisfied since for this also $A_c + \varepsilon I$ had to be dissipative for some $\varepsilon > 0$, which in general does not hold, even for dissipative and uniformly exponentially stable $k \times k$-matrices for $k \geq 2$. We therefore weaken the assumptions in the following way.

Assumption 7.3.9. *In the following we assume that either $\mathcal{A}$ is the m-dissipative operator as in Theorem 7.1.4 generating the s.c. contraction semigroup $(\mathcal{S}(t))_{t\geq 0}$ on $X \times \underline{X_c}$ and $0 \in M_c(0)$ in that case, or that $\mathcal{A}$ is the operator resulting from the the interconnection of an impedance passive port-Hamiltonian systems $\mathfrak{S} = (\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ with an impedance passive (w.r.t. a radially unbounded functional H_c on X_c) non-linear control system* (NLC) *of Theorem 7.2.13 or Corollary 7.2.16 generating the strongly continuous, but not necessarily contractive semigroup $(\mathcal{S}(t))_{t\geq 0}$.*
Moreover, we assume the following, for some orthogonal projection $\Pi : \mathbb{F}^{Nd} \to \mathbb{F}^{Nd}$.

1. *There are constants $t_0, \delta, c_\delta > 0$ such that for every mild solution $(x_c, u_c, y_c) \in C(\mathbb{R}_+; X) \times L_{2,loc}(\mathbb{R}_+; U_c) \times L_{2,loc}(\mathbb{R}_+; Y_c)$ of the nonlinear control system (M_c or* (NLS)*, respectively)*
$$\frac{d}{dt}\int_t^{t+t_0} H_c(x_c(s))ds \leq -\delta H_c(x_c(t)) + c\left\|\Pi u_c(s)\right\|^2_{L_2(t,t+t_0)}, \quad t \geq 0$$
where $H_c(x_c) = \frac{1}{2}\left\|x_c\right\|^2_{X_c}$ or H_c is the radially unbounded functional on X_c, respectively,

2. *there is $\sigma > 0$ such that for all such mild solutions*
$$H_{tot}(t+s) \leq H_{tot}(t) - \sigma\left\|\Pi u_c\right\|^2_{L_2(t,t+s)}$$
where $H_{tot}(x, x_c) = \frac{1}{2}\left\|x\right\|^2_X + H_c(x_c)$ is the total energy, and

3. *there is a constant $c' > 0$ such that*
$$|y_c|^2 \leq c'\left(H_c(x_c) + |\Pi u_c|^2\right).$$
where $y_c \in \Pi_{\mathbb{F}^{Nd}} M_c(x_c, u_c)$ or $y_c = C_c(x_c)x_c + D_c(x_c)u_c$, respectively.

Remark 7.3.10. *Note that Assumption 7.3.9 (for m-dissipative $\mathcal{A}$) is actually weaker then Assumption 7.3.2.*

Proof. Assume that Assumption 7.3.2 holds good. Then Conditions 2.) and 3.) are satisfied by Assumption 7.3.2. We only need to show that condition 1.) holds true. By Assumption 7.3.2 we have that for every solution $(x_c, u_c, y_c) \in C(\mathbb{R}_+; X_c) \times L_{p,loc}(\mathbb{R}_+; U_c \times Y_c)$
$$\begin{aligned}\rho\left\|x_c\right\|^2_{X_c} &\leq \int_0^{t_0} \operatorname{Re}\left\langle(\dot{x}(s), -y_c(s)), (x_c(s), u_c(s))\right\rangle_{X_c\times\mathbb{F}^{Nd}} ds \\ &\leq \int_0^{t_0} \operatorname{Re}\left\langle\dot{x}_c(s), x_c(s)\right\rangle_{X_c} - \operatorname{Re}\left\langle y_c(s), u_c(s)\right\rangle_{\mathbb{F}^{Nd}} ds \\ &= \frac{1}{2}\left\|x_c(t_0)\right\|^2_{X_c} - \frac{1}{2}\left\|x_c(0)\right\|^2_{X_c} - \operatorname{Re}\left\langle u_c, y_c\right\rangle_{L_2(0,t_0)}.\end{aligned}$$
Moreover, since
$$|y_c(t)|^2 \leq c'\left(\left\|x_c(t)\right\|^2_{X_c} + |\Pi u_c(t)|^2\right), \quad \text{a.e. } t \geq 0$$
we find that
$$2\operatorname{Re}\left\langle u_c, y_c\right\rangle_{L_2(0,t_0)} \leq \left(\frac{1}{\varepsilon} + \varepsilon c'\right)\left\|\Pi u_c\right\|^2_{L_2(0,t_0)} + \varepsilon c'\left\|x_c\right\|^2_{L_2(0,t_0)}, \quad \forall\varepsilon > 0$$

and then in particular that for all $\varepsilon > 0$ and $t \in [0, t_0]$

$$\begin{aligned}\|x_c(t)\|^2_{X_c} &\geq \|x_c(t_0)\|^2_{X_c} - \operatorname{Re}\langle u_c, y_c\rangle_{L_2(t,t_0)} \\ &\geq \|x_c(t_0)\|^2_{X_c} - \left(\frac{1}{\varepsilon} + \varepsilon c'\right) \|\Pi u_c\|^2_{L_2(0,t_0)} \\ &\quad - \varepsilon c' \|x_c\|^2_{L_2(0,t_0)}\end{aligned}$$

which again leads to

$$\begin{aligned}\|x_c\|^2_{L_2(0,t_0)} \geq t_0 \|x_c(t_0)\|^2_{X_c} - t_0 \left(\frac{1}{\varepsilon} + \varepsilon c'\right) \|\Pi u_c\|^2_{L_2(0,t_0)} \\ - \varepsilon c' t_0 \|x_c\|^2_{L_2(0,t_0)}\end{aligned}$$

i.e.

$$\|x_c\|^2_{L_2(0,t_0)} \geq \frac{1}{1+\varepsilon c' t_0} \left[t_0 \|x_c(t_0)\|^2_{X_c} - t_0 \left(\frac{1}{\varepsilon} + \varepsilon c'\right) \|\Pi u_c\|^2_{L_2(0,t_0)}\right]$$

and then putting everything together, choosing $\varepsilon \in (0, \frac{\rho}{c'})$,

$$\begin{aligned}\|x_c\|^2_{L_2(0,t_0)} &\leq \|x_c(0)\|^2_{X_c} + \left(\frac{1}{\varepsilon} + \varepsilon c'\right) \|\Pi u u_c\|^2_{L_2(0,t_0)} \\ &\quad + \varepsilon c' \|x_c\|^2_{L_2(0,t_0)} - \rho \|x_c(t_0)\|^2_{X_c} \\ &\leq \|x_c(0)\|^2_{X_c} + \left(\frac{1}{\varepsilon} + \varepsilon c'\right) \|\Pi u_c\|^2_{L_2(0,t_0)} + \frac{(\varepsilon c' - \rho)t_0}{1+\varepsilon c' t_0} \|x_c(t_0)\|^2_{X_c} \\ &\quad + \frac{(\varepsilon c' - \rho)t_0(\frac{1}{\varepsilon} + \varepsilon c')}{1+\varepsilon c' t_0}) \|\Pi u_c\|^2_{L_2(0,t_0)}\end{aligned}$$

or, equivalently,

$$\begin{aligned}&\left(1 + \frac{(\rho - \varepsilon c')t_0}{1+\varepsilon c' t_0}\right) \|x\|^2_{L_2(0,t_0)} \\ &\leq \|x_c(0)\|^2_{X_c} + \left(\frac{1}{\varepsilon} + \varepsilon c' + \frac{(\varepsilon c' - \rho)t_0(\frac{1}{\varepsilon} + \varepsilon c')}{1+\varepsilon c' t_0}\right) \|\Pi u_c\|^2_{L_2(0,t_0)}.\end{aligned}$$

Time-invariance of the problem implies that condition 1.) holds true. □

Remark 7.3.11. *Assumption 7.3.9 may be interpreted as follows. The first assumption says that $0 \in X_c$ is a globally uniformly exponentially stable equilibrium point of the nonlinear controller without input, i.e. $u_c = 0$, and that for input $u_c \in L_{2,loc}(\mathbb{R}_+; U_c)$ the state variable at time $t \geq 0$ depends continuously on the input $u|_{[0,t]} \in L_2(0,t;U_c)$. In this sense the nonlinear controller is internally uniformly exponentially stable and is state/input-state stable. The last assumption says that the output continuously depends on the state and the input variable, so that in total the system is state/input-state/output stable. Together assumptions 2.) and 3.) roughly say that the control system is strictly impedance passive if one considers Πu_c as input function, which makes sense as the output variable y_c can be bounded by the state space variable x_c and Πu_c.*

Theorem 7.3.12. *Let $\mathcal{A}$ be as in Assumption 7.3.9. Further assume that there is $q : X \to \mathbb{R}$ with $|q(x)| \leq c \|x\|_X^2$ such that for every mild solution $x \in W^1_\infty(\mathbb{R}_+; X) \cap L_\infty(\mathbb{R}_+; D(\mathfrak{A}))$ of $\dot{x} = \mathfrak{A}x$ one has*

$$\frac{d}{dt} q(x) \leq -\frac{1}{2} \|x(t)\|_X^2 + c \left(|\mathfrak{B}x(t)|^2 + |\Pi \mathfrak{C}x(t)|^2 \right), \quad t \geq 0.$$

Then the nonlinear semigroup $(\mathcal{S}(t))_{t \geq 0}$ is uniformly exponentially stable, i.e. there are constants $M \geq 1$ and $\omega < 0$ such that

$$H_{tot}(\mathcal{S}(t)(x_0, x_{c,0})) \leq M e^{\omega t} H_{tot}((x_0, x_{c,0})), \quad (x_0, x_{c,0}) \in X \times \underline{X_c}, \ t \geq 0.$$

Proof. We start with an observation on the choice of t_0 in Assumption 7.3.9. Using the first condition iteratively, we obtain that

$$\begin{aligned} H_c(x_c(t + nt_0) &\leq (1 - \delta)^n H_c(x_c(t)) + c \|\Pi u_c\|^2_{L_2(t; t+nt_0; U_c)} \\ &\leq (1 - \delta) H_c(x_c(t)) + c \|\Pi u_c\|^2_{L_2(t; t+nt_0; U_c)}, \quad t \geq 0, \ n \in \mathbb{N}. \end{aligned}$$

so that w.l.o.g. we may always assume that $t_0 > 0$ is as large as we wish. Moreover, the inequality then w.l.o.g. may also hold for any $\tilde{t}_0 \geq t_0$. We will make use of that observation in the conclusion of this proof. Since all maps $\mathcal{S}(t) : X \times \underline{X_c} \to X \times \underline{X_c}$ are continuous, it suffices to consider $(x_0, x_{c,0}) \in D(\mathcal{A})$. So let $(x_0, x_{c,0}) \in D(\mathcal{A})$ be arbitrary. The condition on the map $q : X \to \mathbb{R}$ implies that

$$q(x(t)) - q(x(\tau)) \leq \int_\tau^t H(x(s)) ds + c \left(\|u\|^2_{L_2(\tau, t)} + \|\Pi y\|^2_{L_2(\tau, t)} \right), \quad t \geq \tau \geq 0.$$

For $t \geq 0$ and the constants $t_0, \delta, c, c' > 0$ from Assumption 7.3.9 we define the function

$$\Phi(t) := q(x(t)) + t H_{tot}((x, x_c)(t)) + \frac{1 + cc'}{\delta} \int_t^{t+t_0} H_c(x_c(s)) ds, \quad t \geq 0.$$

Then for a.e. $t \geq \tau \geq 0$ we conclude that for every $t \geq 2t_0 > 0$

$$\begin{aligned} &\Phi(t) - \Phi(t_0) \\ &= t_0 \left(H_{tot}((x, x_c)(t)) - H_{tot}((x, x_c)(t_0)) \right) + (t - t_0) H_{tot}((x, x_c)(t)) \\ &\quad + q(x(t)) - q(x(t_0)) + \frac{1 + cc'}{\delta} \int_0^{t_0} \left(H_c(x_c(s + t)) - H_c(x_c(s + t_0)) \right) ds \\ &\leq -\sigma t_0 \|\Pi \mathfrak{C} x\|^2_{L_2(t_0, t)} + (t - t_0) H_{tot}((x, x_c)(t)) - \int_{t_0}^t H(x(s)) ds \\ &\quad + c \int_{t_0}^t |\mathfrak{B}x(s)|^2 + |\Pi \mathfrak{C}x(s)|^2 ds \\ &\quad + \frac{1 + cc'}{\delta} \int_0^{t_0} \left(-\delta H_c(x_c(s + t_0)) + c \int_0^{t - t_0} \|\Pi \mathfrak{C} x(t_0 + s + r)\|^2 dr \right) ds \\ &\leq -\sigma t_0 \|\Pi \mathfrak{C} x\|^2_{L_2(t_0, t)} + (t - t_0) H_{tot}((x, x_c)(t)) - \int_{t_0}^t H_{tot}((x, x_c)(s)) ds \\ &\quad + \frac{(1 + cc')(1 + \delta)}{\delta} \|\Pi \mathfrak{C} x\|^2_{L_2(t_0, t)} \leq 0 \end{aligned}$$

if $t_0 > 0$ (independent of the initial value $(x_0, x_{c,0}) \in X \times X_c$) is large enough. Therefore, for that fixed $t_0 > 0$ we have $\Phi(t) \leq \Phi(t_0)$ $(t \geq 2t_0)$, from where we easily deduce uniform exponential stability. Namely, we may estimate for $t \geq 2t_0$

$$\begin{aligned}
&tH_{tot}((x, x_c)(t)) - cH_{tot}((x, x_c)(0)) - \frac{(1 + cc')t_0}{\delta} H_{tot}((x, x_c)(0)) \\
&\leq \Phi(t) \leq \Phi(t_0) \\
&\leq t_0 H_{tot}((x, x_c)(0)) + cH_{tot}((x, x_c)(0)) + \frac{(1 + cc')t_0}{\delta} H_{tot}((x, x_c)(0))
\end{aligned}$$

leading to

$$H_{tot}((x, x_c)(t)) \leq \frac{1}{t} \left(t_0 + 2c + \frac{2(1 + cc')t_0}{\delta} \right) H_{tot}((x, x_c)(0)), \quad t \geq 2t_0.$$

and by time-invariance of the problem we conclude uniform exponential stability, i.e.

$$H_{tot}((x, x_c)(t)) \leq M e^{\omega t} H_{tot}((x_0, x_{c,0})), \quad t \geq 0$$

for some constants $M \geq 1$ and $\omega < 0$, cf. the proof of Remark 2.2.12. □

It is also possible to employ the final observability result Lemma 4.1.1 to obtain uniform exponential stability, if $\mathfrak{S} = (\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ is an impedance passive port-Hamiltonian system of order $N = 1$, interconnected with an input/state-input/state stable nonlinear control system. To show that result we first need the following auxiliary lemma.

Lemma 7.3.13. *Assume that $\mathcal{A}$ is as in Assumption 7.3.9. Then for every $\tau > 0$ there is a constant $c'' > 0$ such that*

$$\|y_c\|^2_{L_2(0,\tau)} \leq c'' \left(H_c(x_{c,0}) + \|\Pi u_c\|^2_{L_2(0,\tau)} \right)$$

for every mild solution of the control system M_c or (NLC)*, respectively.*

Proof. From the impedance passivity and the fact that due to assumption 1.) in Assumption 7.3.9 the part $(I - \Pi)u_c$ of the input does not influence the controller state space variable x_c, we find that

$$\begin{aligned}
\|H_c(x_c)\|_{L_1(0,\tau)} &\leq \int_0^\tau H_c(x_{c,0}) + \operatorname{Re} \langle \Pi u_c, \Pi y_c \rangle_{L_2(0,t)} dt \\
&\leq \tau H_c(x_{c,0}) + \tau \|\Pi u_c\|_{L_2(0,\tau)} \|y_c\|_{L_2(0,\tau)} \\
&\leq \tau H_{c,0} + \frac{\tau}{2\varepsilon} \|\Pi u_c\|^2_{L_2(0,\tau)} + \frac{\tau\varepsilon}{2} \|y_c\|^2_{L_2(0,\tau)}
\end{aligned}$$

for every $\varepsilon > 0$ and then condition 3.) implies that

$$\begin{aligned}
\|y_c\|^2_{L_2(0,\tau)} &\leq c' \left(\|H_c(x_c)\|_{L_1(0,\tau)} + \|\Pi u_c\|^2_{L_2(0,\tau)} \right) \\
&\leq c' \left(\tau H_c(x_{c,0}) + \left(1 + \frac{\tau}{2\varepsilon}\right) \|\Pi u_c\|^2_{L_2(0,\tau)} + \frac{\tau\varepsilon}{2} \|y_c\|^2_{L_2(0,\tau)} \right)
\end{aligned}$$

and the estimate follows for $\varepsilon > 0$ suitable small, e.g. $\varepsilon = \frac{1}{\tau c'}$. □

Proposition 7.3.14. *Let an $\mathcal{A}$ be as in Assumption 7.3.9. Further assume that the following final observability estimate holds good.*

$$\exists c_\tau, \tau > 0 : \quad \frac{1}{2} \|x(\tau_0)\|_X^2 \leq c_\tau \int_0^{\tau_0} |\Pi \mathfrak{C} x(s)|^2 + |\mathfrak{B} x(s)|^2 \, ds. \tag{7.13}$$

Then there are constants $M \geq 1$ and $\omega < 0$ such that for every $(x_0, x_{c,0}) \in X \times X_c$ and the corresponding mild solution (x, x_c) the following exponential energy decay holds true.

$$H_{tot}((x, x_c)(t)) \leq M e^{\omega t} H_{tot}((x_0, x_{c,0})), \quad t \geq 0.$$

Proof. First remark that inequality (7.13) also holds if τ_0 is replaced by an arbitrary larger $\tau \geq \tau_0$. Take any $(x_0, x_{c,0}) \in X \times X_c$ such that for the corresponding solution $(x, x_c)(t) = \mathcal{S}(t)(x_0, x_{c,0})$ $(t \geq 0)$ the functions $(u_c, y_c) \in L_{2,loc}(\mathbb{R}_+; U_c \times Y_c)$ are locally square-integrable. From the the second assumption in Assumption 7.3.9 we find that

$$H_{tot}((x, x_c)(t)) \leq H_{tot}((x_0, x_{c,0})) - \sigma \|\Pi u_c\|_{L_2(0,\tau)}^2 \tag{7.14}$$

We now employ the final observability estimate and Assumption 7.3.9 to obtain that

$$\begin{aligned} c_\tau \|\Pi u_c\|_{L_2(0,\tau)}^2 &\geq H(x(\tau)) \\ c \|\Pi u_c\|_{L_2(0,\tau)}^2 &\geq H_c(x_c(\tau)) - (1-\delta) H_c(x_{c,0}) \\ c'' \|\Pi u_c\|_{L_2(0,\tau)}^2 &\geq \|y_c\|_{L_2(0,\tau)}^2 - c'' H_c(x_{c,0}) \end{aligned}$$

where in the second inequality we assumed that w.l.o.g. $\tau = t_0$ since t_0 can be replaced by $k\tau$ $(k \in \mathbb{N})$ in the first inequality of Assumption 7.3.9 and in the last line used the auxiliary Lemma 7.3.13. Summing up with factors 1, $\alpha > 0$ (arbitrary) and c_τ we arrive at the inequality

$$\begin{aligned} &(c_\tau + c + c_\tau c'') \left(H_{tot}((x, x_c)(\tau)) - H_{tot}((x_0, x_{c,0})) \right) \\ &\leq -\sigma \left(c_\tau + c + c_\tau c'' \right) \|\Pi u_c\|_{L_2(0,\tau)}^2 \\ &\leq -\sigma (H(x(\tau)) + \alpha H_c(x_c(\tau))) + \left((1-\delta) + \frac{c_\tau c''}{\alpha} \right) \sigma \alpha H_c(x_{c,0}) \end{aligned}$$

and then

$$\begin{aligned} &(c_\tau + c + c_\tau c'') \, H_{tot}((x, x_c)(\tau)) + \sigma (H(x(\tau)) + \alpha H_c(x_c(\tau))) \\ &\leq (c_\tau + c + c_\tau c'') \, H_{tot}((x_0, x_{c,0})) + \left((1-\delta) + \frac{c_\tau c''}{\alpha} \right) \sigma \alpha H_c(x_{c,0}) \\ &\leq (1-\gamma) (c_\tau + c + c_\tau c'') \, H_{tot}((x_0, x_{c,0})) + \left((1-\delta) + \frac{c_\tau c''}{\alpha} \right) \sigma \alpha H_c(x_{c,0}) \\ &\quad + \frac{\gamma (c_\tau + c + c_\tau c'')}{\sigma} \sigma H(x_0) \end{aligned}$$

where $\gamma \in \left(0, \min\left\{1, \frac{\sigma}{c_\tau + c + c_\tau c''}\right\}\right)$. Finally choosing $\alpha > 0$ large enough such that $\frac{c_\tau c''}{\alpha} < \delta$, we find $\rho \in (0, 1)$ (independent of the chosen initial value $(x_0, x_{c,0})$) such that

$$H_{tot,\alpha,\sigma}((x, x_c)(\tau)) \leq \rho H_{tot,\alpha,\sigma}((x_0, x_{c,0}))$$

for

$$H_{tot,\alpha,\sigma}(x, x_c) := (c_\tau + c + c_\tau c'')H_{tot}(x, x_c) + \sigma(H(x_c) + \alpha H_c(x_c)).$$

From the time-invariance of the problem we deduce that

$$H_{tot,\alpha,\sigma}((x, x_c)(t)) \leq Me^{\omega t} H_{tot,\alpha,\sigma}((x_0, x_{c,0})), \quad t \geq 0$$

cf. the technique used for the proof of Remark 2.2.31, and the result follows since $H_{tot,\alpha,\sigma}$ and H_{tot} are equivalent, i.e. there are $c_1, c_2 > 0$ such that

$$H_{tot}(x, x_c) \leq c_1 H_{tot,\alpha,\sigma}(x, x_c) \leq c_2 H_{tot}(x, x_c).$$

□

We return to the example of dynamic feedback stabilisation of a linear Euler-Bernoulli beam with dissipative boundary feedback and apply our abstract theory to it.

Example 7.3.15. *We consider the dynamic feedback stabilisation of an Euler-Bernoulli beam equation with damped left end a mass and dynamic feedback at the right hand side, modelling the tip of the beam.*

$$\begin{aligned} \rho(\zeta)\omega_{tt}(t, \zeta) + (EI(\zeta)\omega_{\zeta\zeta}(t, \zeta))_{\zeta\zeta} &= 0, \quad t \geq 0,\ \zeta \in (0, 1) \\ \omega(t, 0) = \omega_{\zeta\zeta}(t, 0) &= 0 \\ (EI\omega_{\zeta\zeta})(t, 1) &= 0 \\ -(EI\omega_{\zeta\zeta})_\zeta(t, 1) + m\omega_{tt}(t, 1) &= -\alpha\omega_t(t, 1) + \beta(EI\omega_{\zeta\zeta})_{t\zeta}(t, 1) \end{aligned}$$

where $\alpha > 0$ and we investigated the cases $m = \beta = 0$ and $m > 0$, $\beta = 0$ and $m, \beta > 0$, leading to asymptotic stabilisation, at least, see Example 6.5.3. There we considered all three cases in the linear scenario, but left the nonlinear version of the second case open. We therefore now consider the case where $\beta = 0$, but the mass at the tip $m > 0$ does not vanish. From its physical interpretation we leave the latter to be a constant, but only replace $\alpha > 0$ by a nonlinear feedback map $\phi : D(\phi) \subset \mathbb{F} \rightrightarrows \mathbb{F}$, i.e.

$$m\omega_{tt}(t, 1) \in -\phi(\omega_t(t, 1)) + (EI\omega_{\zeta\zeta})_\zeta(t, 1)$$

from where the nonlinear control system takes the form

$$\begin{aligned} \frac{d}{dt}x_c(t) &\in -\frac{1}{m}\phi(x_c(t)) + \frac{1}{m}u_c(t) \\ y_c(t) &= -x_c(t), \qquad t \geq 0. \end{aligned}$$

As long as $\phi : \mathbb{F} \rightrightarrows \mathbb{F}$ is m-monotone this gives a nonlinear m-dissipative map $\mathcal{A}_\phi$

$$\begin{aligned} \mathcal{A}_\phi &= \begin{bmatrix} \mathfrak{A} & 0 \\ \frac{1}{m}\mathfrak{C}_1 x & -\frac{1}{m}\phi(\cdot) \end{bmatrix} \\ D(\mathcal{A}_\phi) &= \{(x, x_c) \in D(\mathfrak{A}) \times \mathbb{F} :\ \mathfrak{B}_1 x = x_c,\ \mathfrak{B}_j x = 0\ (j \geq 2)\} \end{aligned}$$

on the product Hilbert space $X \times X_c = X \times \mathbb{F}$, so that it generates a nonlinear s.c. contraction semigroup $(\mathcal{S}(t))_{t\geq 0}$ on $X \times X_c$. Moreover, if $|z_c| > 0$ for all $z_c \in \phi(x_c)$ for $x_c \neq 0$ and $0 \in \phi(0)$, then

$$\operatorname{Re} \langle(\mathfrak{A}, z_c), (x, x_c)\rangle_{X\times X_c} < 0, \quad (\mathfrak{A}x, z_c) \in \mathcal{A}_\phi(x, x_c) \ \textit{with}\ \mathfrak{B}_1 x = 0$$

and from Corollary 4.2.10 and Proposition 7.3.1 we conclude that

$$\mathcal{S}_\phi(t)(x, x_c) \xrightarrow{t\to\infty} 0, \quad (x, x_c) \in X \times \underline{X_c}$$

i.e. 0 *is a globally asymptotic stable equilibrium. On the other hand, if actually* $|z_c| > \varepsilon |x_c|$ *for some* $\varepsilon > 0$ *and all* $z_c \in \phi(x_c)$ *and* ρ *and* EI *are constant along the spatial variable, then we find that*

$$\|\mathcal{S}_\phi(t)(x, x_c)\|_{X\times X_c} \le Me^{\omega t} \|(x, x_c)\|_{X\times X_c}, \quad t \ge 0$$

for some constants $M \ge 1$ *and* $\omega < 0$ *by Lemma 6.4.1 and Proposition 7.3.4.*

Chapter 8

Further Results

Within this last chapter we collect some further results which are closely related to the questions investigated in the previous chapters, but with a slightly different point of view. First we investigate infinite-dimensional systems which in this case are not interconnected with a finite dimensional control system, but with other infinite-dimensional port-Hamiltonian systems, possibly of different order, instead. Then we consider non-autonomous equations, i.e. where the structural properties may depend on the time variable $t \in \mathbb{R}_+$ and on the one hand show how uniform exponential stability for non-autonomous problems may be investigated using similar techniques as for the autonomous, i.e. time-invariant, case. Within the last section this approach is combined with the investigation of L_p-maximal regularity for a class of port-Hamiltonian systems which are not damped at the boundary, but structurally damped including higher order derivatives of the state space variable $x(t)$.

8.1 Interconnection of Infinite Dimensional Port-Hamiltonian Systems

We consider the interconnection of $L \in \mathbb{N}$ port-Hamiltonian systems of (possibly distinct) orders $N_l \in \mathbb{N}$ $(l = 1, \dots, L)$, i.e.

$$\begin{aligned}\frac{\partial x_l}{\partial t}(t,\zeta) &= \sum_{k=0}^{N_l} P_{l,k} \frac{\partial^k (\mathcal{H}_l x_l)}{\partial \zeta^k}(t,\zeta) =: (\mathfrak{A}_l x_l(t))(\zeta), \quad \zeta \in (0,1)\\ \mathfrak{B}_l x_l(t) &:= W_{B,l} \tau_l (\mathcal{H}_l x_l)(t)\\ \mathfrak{C}_l x_l(t) &:= W_{C,l} \tau_l (\mathcal{H}_l x_l)(t), \quad t \geq 0\end{aligned}$$

where the Hamiltonian density matrix functions $\mathcal{H}_l \in L_\infty(0,1;\mathbb{F}^{d_l \times d_l})$ are uniformly positive definite, the matrices $\begin{bmatrix} W_{B,l} \\ W_{C,l} \end{bmatrix} \in \left(\mathbb{F}^{d_l N_l \times 2 d_l N_l}\right)^2$ are invertible and as usual $P_{l,k}^* = (-1)^{k+1} P_{l,k} \in \mathbb{F}^{d_l \times d_l}$, $k \geq 1$, with P_{l,N_l} invertible. Also the trace maps

$\tau_l : H^{N_l}(0,1;\mathbb{F}^d) \to \mathbb{F}^{Nd}$ $(l = 1,\dots,L)$ are again given by

$$\tau_l(x) = \begin{pmatrix} \tau_{l,0}(x) \\ \tau_{l,1}(x) \end{pmatrix} := \begin{pmatrix} x(1) \\ x'(1) \\ \vdots \\ x^{(N_l-1)}(1) \\ x(0) \\ x'(0) \\ \vdots \\ x^{(N_l-1)}(0) \end{pmatrix}.$$

The closed operators $\mathfrak{A}_l$ are defined on their maximal domains

$$D(\mathfrak{A}_l) = \left\{ f \in X_l = L_2(0,1;\mathbb{F}^{d_l}),\ (\mathcal{H}_l f_l) \in H^{N_l}(0,1;\mathbb{F}^{d_l}) \right\}$$

and on the product Hilbert space $X = L_2(0,1;\mathbb{F}^d) = \Pi_{l=1}^L L_2(0,1;\mathbb{F}^{d_l})$ (where $d = \sum_{l=1}^L d_l$) we have the block diagonal operator

$$\mathfrak{A}x = \operatorname{diag}(\mathfrak{A}_1,\dots,\mathfrak{A}_L), \quad D(\mathfrak{A}) = D(\mathfrak{A}_1) \times \dots \times D(\mathfrak{A}_L).$$

Similar to the case of a single infinite-dimensional port-Hamiltonian system, we equip the product space X with the energy norm $\|\cdot\|_X = \|\cdot\|_{\mathcal{H}}$ inherited from the inner product $\langle \cdot,\cdot \rangle_X = \langle \cdot,\cdot \rangle_{\mathcal{H}}$ given by

$$\langle x, y \rangle_{\mathcal{H}} := \sum_{l=1}^L \langle x_l, y_l \rangle_{\mathcal{H}_l} = \sum_{l=1}^L \int_0^1 \langle x_l(\zeta), \mathcal{H}_l(\zeta) y_l(\zeta) \rangle_{\mathbb{F}^{d_l}} d\zeta.$$

Note that for $\mathcal{H} = \operatorname{diag}(\mathcal{H}_1,\dots,\mathcal{H}_L)$ this definition coincides with the usual one for port-Hamiltonian systems.

8.1.1 Directed Acyclic Graphs of Port-Hamiltonian Systems

In this subsection we consider a family $\{(\mathfrak{A}_l, \mathfrak{B}_l, \mathfrak{C}_l)\}_{l=1,\dots,L}$ of port-Hamiltonian systems which are interconnected in a very specific way. Namely we assume that the interconnection structure takes the form of a directed acyclic graph, see the following definition from graph theory.

Definition 8.1.1. *Let V be any nonempty set and $E \subseteq V \times V$. Then $G = (V,E)$ is called a (directed)* graph *with* vertices *$v \in V$ and* edges *$e \in E$. The graph $G = (V,E)$ is called* acyclic, *if for every $(y,x) \in E$ and all $n \in \mathbb{N}$ and $v_0 = x, v_1, \dots, v_{n-1}, v_n = y \in V$ there is at least one $i \in \{1,\dots,n\}$ such that $(v_{i-1}, v_i) \notin E$, i.e. the graph has no directed cycles.*

Remark 8.1.2. *Assume that $G = (V,E)$ is an acyclic graph and $V = \{v_1,\dots,v_n\}$ is finite, i.e. $G = (V,E)$ is a* finite *graph. Then, possibly after renaming the vertices, we may w.l.o.g. assume that $(v_i, v_j) \notin E$ whenever $i \geq j$.*

To identify an interconnection structure of port-Hamiltonian systems with a graph we also introduce the following definition.

Definition 8.1.3. *Let $G = (V,E)$ be any directed graph. Then for every edge $e = (x,y) \in E$ we call x the* tail *and y the* head *of the edge e.*

Let us transfer this concept to an interconnection of port-Hamiltonian systems. For this end, consider the interconnection given by

$$\mathfrak{B}_l x_l = \sum_{i=1}^{L} K_{il} \mathfrak{C}_i x_i, \qquad l = 1, \ldots, L$$

where $K_{il} \in \mathbb{F}^{d_l N_l \times d_i N_i}$ are matrices of proper dimension. With this interconnection structure we may associate the following graph $G = (V, E)$.

$$\begin{aligned} V &:= \{1, \ldots, L\}, \\ E &:= \{(i,j) \in V \times V : K_{ij} \neq 0\} \end{aligned}$$

The interpretation for this choice is that we say that $(i,j) \in E$ if and only if the output from the system i influences the input of the system j through the matrix $K_{ij} \neq 0$. Hence, whenever $K_{ij} = 0$, i.e. $(i,j) \notin E$, the output of the system i does not directly influence the input of the system j. From here it is clear what we should understand to be an acyclic graph of port-Hamiltonian systems.

Definition 8.1.4. *Let $\{(\mathfrak{A}_l, \mathfrak{B}_l, \mathfrak{C}_l)\}_{l=1,\ldots,L}$ be a family of port-Hamiltonian systems and $K_{ij} \in \mathbb{F}^{d_j N_j \times d_i N_i}$ $(i, j = 1, \ldots, L)$ be matrices defining its interconnection structure*

$$\mathfrak{B}_l x_l = \sum_{i=1}^{L} K_{il} \mathfrak{C}_i x_i, \qquad l = 1, \ldots, L.$$

If the corresponding graph $G = (V, E)$ (defined above) is acyclic, then the system

$$\begin{aligned} \frac{\partial x_l(t)}{\partial t} &= \mathfrak{A}_l x_l(t) \\ \mathfrak{B}_l x_l(t) &= \sum_{i=1}^{L} K_{il} \mathfrak{C}_i x_i(t), \quad l = 1, \ldots, L, \ t \geq 0 \end{aligned}$$

is called acyclic interconnection of port-Hamiltonian systems. *We denote by $A : D(A) \subseteq X \to X$ the corresponding operator which is given by*

$$\begin{aligned} A &= \operatorname{diag}(\mathfrak{A}_1, \ldots, \mathfrak{A}_L) \\ D(A) &= \{x \in D(\mathfrak{A}) : \ \mathfrak{B}_l x_l = \sum_{i=1}^{L} K_{il} \mathfrak{C}_i x_i, \ l = 1, \ldots, L\}. \end{aligned}$$

Remark 8.1.5. *Note that whenever (V, E) is acyclic and the vertices are ordered accordingly, i.e. $(i,j) \notin E$ for all $i \geq j$, then for all $x \in D(A)$ with $x_l = 0$ $(\forall l < l_0)$ one also has $\mathfrak{B}_{l_0} x_{l_0} = 0$.*

As in the case of a single port-Hamiltonian system (or interconnection with a finite dimensional controller) we do not have to worry about the range condition in the Lumer-Phillips Theorem, namely we have the following generation result.

Proposition 8.1.6. *If A is dissipative, then it generates a contractive C_0-semigroup $(T(t))_{t \geq 0}$ on X. Moreover, in that case A has compact resolvent.*

Proof. First of all, let us mention that $\mathcal{H} := \operatorname{diag}(\mathcal{H}_1, \ldots, \mathcal{H}_L)$ is a strictly coercive (matrix-valued) multiplication operator on X and thus due to Lemma 3.3.5 it suffices to consider the case $\mathcal{H} = I$. Since $C_c^\infty(0,1;\mathbb{F}^d) = C_c^\infty(0,1;\mathbb{F}^{d_1}) \times \ldots \times C_c^\infty(0,1;\mathbb{F}^{d_L})$ lies dense in X, the operator A is densely defined. Due to the Lumer-Phillips Theorem 2.2.7 it remains to check the range condition $\operatorname{ran}(I - A) = X$. For $x = (x_1, \ldots, x_L) \in D(\mathfrak{A})$ and $f = (f_1, \ldots, f_L) \in X$ we write $h = (h_1, \ldots, h_L)$ and $g = (g_1, \ldots, g_L)$ where

$$\begin{aligned} h_l &= (x_l, x_l', \ldots, x_l^{(N_l-1)}), \\ g_l &= (0, \ldots, 0, P_{l,N_l}^{-1} f_l). \end{aligned}$$

Then

$$\begin{aligned}
&(\mathfrak{A} - I)x = f \\
&\Leftrightarrow \sum_{k=0}^{N_l} P_{l,k} x_l^{(k)}(\zeta) - x_l(\zeta) = f_l(\zeta), \quad \text{a.e. } \zeta \in (0,1),\ l = 1, \ldots, L \\
&\Leftrightarrow x_l^{(N_l)}(\zeta) = P_{l,N_l}^{-1} \left(x_l(\zeta) - \sum_{k=0}^{N_l-1} P_{l,k} x_l^{(k)}(\zeta) + f_l(\zeta) \right), \\
&\quad \text{a.e. } \zeta \in (0,1),\ l = 1, \ldots, L \\
&\Leftrightarrow h_l'(\zeta) = B_l h_l(\zeta) + g_l(\zeta), \quad \text{a.e. } \zeta \in (0,1),\ l = 1, \ldots, L \\
&\Leftrightarrow h_l(\zeta) = e^{\zeta B_l} h_l(0) + \int_0^\zeta e^{(\zeta - s)B_l} g_l(s) ds, \quad \text{a.e. } \zeta \in (0,1),\ l = 1, \ldots, L
\end{aligned} \tag{8.1}$$

where

$$B_l = \begin{bmatrix} 0 & 1 & & \\ & & \ddots & \\ & & & 1 \\ P_{l,N_l}^{-1} - P_{l,N_l}^{-1} P_{l,0} & -P_{l,N_l}^{-1} P_{l,1} & \ldots & -P_{l,N_l}^{-1} P_{l,N_l-1} \end{bmatrix}. \tag{8.2}$$

In that case we have $x \in D(A)$ if and only if

$$0 = W_{B,1} \begin{bmatrix} e^{B_1} \\ I \end{bmatrix} h_1(0) + W_{B,1} \begin{pmatrix} \int_0^1 e^{(1-s)B_1} g_1(s) ds \\ 0 \end{pmatrix}$$

and

$$\begin{aligned} W_{B,l} \begin{pmatrix} e^{B_l} \\ I \end{pmatrix} h_l(0) + W_{B,l} \begin{pmatrix} \int_0^1 e^{(1-s)B_l} g_l(s) ds \\ 0 \end{pmatrix} \\ = \sum_{i=1}^{l-1} K_{il} \mathfrak{C}_i x_i. \end{aligned}$$

We conclude the following: If $W_{B,l} \left[\begin{smallmatrix} e^{B_l} \\ I \end{smallmatrix} \right]$ is invertible for all $l = 1, \ldots, L$, then for all $f \in X$ the equation $Ax - x = f$ has a (unique) solution $x \in D(A)$. *Assume* that there was $l_0 \in \{1, \ldots, L\}$ with

$$\mathcal{N}_{l_0} = \ker W_{l_0} \begin{pmatrix} e^{B_{l_0}} \\ I \end{pmatrix} \neq \{0\}$$

(w.l.o.g. let l_0 be maximal under these l). Choose $f = 0$ (hence $g = 0$) and $h_l(0) = 0$ for $l < l_0$, $h_{l_0}(0) \in \mathcal{N}_{l_0} \setminus \{0\}$ and inductively for $l > l_0$

$$h_l(0) = -\left(W_{B,l}\begin{pmatrix} e^{B_l} \\ I \end{pmatrix}\right)^{-1} \sum_{k=1}^{l-1} K_{il}\mathfrak{C}_i x_i$$

where $h_i = (h_i)_{j=1,\dots,d_i} \in H^1(0,1;\mathbb{F}^{d_i})$. Then (8.1) defines a solution $x \in D(A)$ of $Ax - x = 0$ with $x \neq 0$, so $1 \in \sigma_p(A)$, contradicting the dissipativity of A. Hence, $\operatorname{ran}(I - A) = X$ and A generates a contractive C_0-semigroup. Compactness of the resolvent follows from the compact embedding $D(\mathfrak{A}) = \Pi_{l=1}^{L} D(\mathfrak{A}_l) \hookrightarrow \Pi_{l=1}^{L} X_l = X$. □

Example 8.1.7. *Within the framework of hybrid systems we have seen that for feedback systems with finite dimensional controllers a useful structure for the infinite dimensional port-Hamiltonian system $\mathfrak{S} = (\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ and the control system $\Sigma_c = (A_c, B_c, C_c, D_c)$ is impedance passivity. However, these are systems with loops, in contrast to the acyclic systems considered here. Therefore, to find suitable classes of systems $\{\mathfrak{S}_i = (\mathfrak{A}_i, \mathfrak{B}_i, \mathfrak{C}_i)\}_{i=1,\dots,L}$ which can easily be composed to obtain a dissipative acyclic system, we introduce the notion of scattering passivity.*
We say that a Boundary Control and Observation System (e.g. a port-Hamiltonian system in boundary control and observation form) $\mathfrak{S} = (\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ is scattering passive *if for all $x \in D(\mathfrak{A})$ the inequality*

$$\operatorname{Re}\langle \mathfrak{A}x, x\rangle_X \leq \|\mathfrak{B}x\|_U^2 - \|\mathfrak{C}x\|_Y^2$$

holds. Using this notion we can easily give an abstract example of an acyclic and dissipative port-Hamiltonian system. Let $\{\mathfrak{S}_i = (\mathfrak{A}_i, \mathfrak{B}_i, \mathfrak{C}_i)\}_{i=1,\dots,L}$ be a finite family of scattering passive port-Hamiltonian systems and consider operators $K_{ij} \in \mathcal{B}(U_j, U_i)$ such that $K_{ij} = 0$ for $i \leq j$ and

$$\forall j = 1, \dots, L,\ \forall y_j \in Y_j \quad \sum_{i=j+1}^{L} \|K_{ij} y_j\|^2 \leq \frac{1}{2}\|y_j\|^2 .$$

Then for the interconnection $\mathfrak{B}_i x_i = \sum_{j=1}^{i-1} K_{ij}\mathfrak{C}_j x_j$ $(i = 1, \dots, L)$ we obtain for $x \in D(A)$ (the corresponding domain including the interconnection structure)

$$\begin{aligned}
\operatorname{Re}\langle \mathfrak{A}x, x\rangle_{\mathcal{H}} &\leq \sum_{i=1}^{L} |\mathfrak{B}_i x_i|^2 - |\mathfrak{C}_i x_i|^2 \\
&= \sum_{i=1}^{L} \left|\sum_{j=1}^{i-1} K_{ij}\mathfrak{C}_j x_j\right|^2 - |\mathfrak{C}_i x_i|^2 \\
&\leq \sum_{j=1}^{L}\left(\sum_{i=j+1}^{L} 2\,|K_{ij}\mathfrak{C}_j x_j|^2 - |\mathfrak{C}_j x_j|^2\right) \leq 0.
\end{aligned}$$

8.1.2 Stability Properties

We have seen before that the generation theorem for acyclic graphs of port-Hamiltonian systems takes the same easy form as the one for a single port-Hamiltonian system.

Now we take the next step and investigate how the results concerning stability carry over from a one-component system to a multi-component system. First, let us recall the properties ASP and AIEP. Let $B : D(B) \subseteq X \to X$ and $R : D(B) \to H$, where H is any Hilbert space, be given. We say that the pair $(\mathfrak{A}, R)$ has property ASP for B if for all eigenvectors x_λ to eigenvalues $\lambda \in i\mathbb{R}$, the function $Rx_\lambda \neq 0$ does not vanish. On the other hand, we say that that the pair $(\mathfrak{A}, R)$ has property AIEP if for all sequences $(x_n)_{n\geq 1} \subseteq D(B)$ which are bounded in X and such that both $Rx_n \xrightarrow{n\to\infty} 0$ and $i\beta_n - Bx_n \xrightarrow{n\to\infty} 0$ for some sequence $(\beta_n)_{n\geq 1}$ such that $|\beta_n| \xrightarrow{n\to\infty} \infty$, it follows that $x_n \xrightarrow{n\to\infty} 0$. Then the first – namely asymptotic – stability result reads as follows.

Proposition 8.1.8. *Let $(\mathfrak{A}_l, \mathfrak{B}_l, \mathfrak{C}_l)_{l=1,\dots,L}$ with interconnection matrices K_{ij} $(i, j = 1, \dots, L)$ form an acyclic graph of port-Hamiltonian systems. Let $R_l^j : D(\mathfrak{A}_l) \to H_l^j$ $(j = 1, 2)$ be given and assume that*

$$\operatorname{Re} \langle Ax, x\rangle_X \leq -\sum_{l=1}^{L} \left\| R_l^1 x_l \right\|_{H_l^1}^2,$$

$$|\mathcal{B}_l x_l|^2 \geq \left\| R_l^2 x_l \right\|_{H_l^2}^2, \qquad l = 1, \dots, L,\ x \in D(A).$$

Set $R_l = (R_l^1, R_l^2) : D(\mathfrak{A}_l) \to H_l := H_l^1 \times H_l^2$. If all pairs $(\mathfrak{A}_l, R_l)$ $(l = 1, \dots, L)$ have property ASP, then A generates an asymptotically stable and contractive C_0-semigroup $(T(t))_{t\geq 0}$.

Proof. The dissipativity implies the generator property. Let $x \in D(A)$ and $\beta \in \mathbb{R}$ such that

$$Ax = i\beta x.$$

Then $R_l^1 x_l = 0$ for all $l = 1, \dots, L$. Further since $\mathfrak{B}_1 x_1 = 0$, also $R_1^2 x_1 = 0$, so that $R_1 x_1 = 0$. Since the pair $(\mathfrak{A}_1, R_1)$ has property ASP and $\mathfrak{A}_1 x_1 = i\beta x_1$ it follows that $x_1 = 0$ is the zero function and thus also $\mathfrak{B}_2 x_2 = 0$. Repeating the argument iteratively we deduce $x_l = 0$ for all $l = 1, \dots, L$, i.e. $x = 0$. As a result $i\mathbb{R} \cap \sigma_p(A) = \emptyset$ and since A has compact resolvent and generates a contraction semigroup, it follows from Corollary 2.2.16 that $(T(t))_{t\geq 0}$ is asymptotically stable. □

For uniform exponential stability we have to sharpen the previous condition AIEP in the following way which also takes into account that for an acyclic graph of port-Hamiltonian systems the information $x_{n,l} \to 0$ does not necessarily imply that $\mathfrak{C}_l x_{n,l} \to 0$, i.e. with the property AIEP alone we would possibly lose information on the behaviour of $(\mathfrak{C}_l x_{n,l})_{n\geq 1}$ which could be helpful to deduce asymptotic behaviour of $(x_{n,l+1})_{n\geq 1}$.

Definition 8.1.9. *Let a Hilbert space X and a linear operator $B : D(B) \subseteq X \to X$ be given. Further let $\mathcal{D} : (D(B), \|\cdot\|_B) \to \mathbb{F}^m$ be continuous and linear. For a linear function $R : D(B) \to H$ we say that the triple $(\mathfrak{A}, R, \mathcal{D})$ has* property AIEP *if*

$$\left.\begin{array}{c} (x_n, \beta_n) \subseteq D(B) \times \mathbb{R}, \\ \|x_n\| \leq c,\ |\beta_n| \to \infty \\ Bx_n - i\beta_n x_n \to 0, \\ Rx_n \to 0 \end{array}\right\} \Longrightarrow \left\{ \begin{array}{c} x_n \to 0, \\ \mathcal{D}x_n \to 0. \end{array}\right.$$

Example 8.1.10. *Consider a port-Hamiltonian operator $\mathfrak{A}$ of order $N = 1$ with Lipschitz continuous $\mathcal{H}$ and P_0, i.e. $\mathfrak{A}x = P_1(\mathcal{H}x)' + P_0(\mathcal{H}x)$, and the linear maps $Rx = (\mathcal{H}x)(0) \in \mathbb{F}^d$ and $\mathcal{D}x = ((\mathcal{H}x)(0), (\mathcal{H}x)(1)) \in \mathbb{F}^{2d}$. Then the triple $(\mathfrak{A}, R, \mathcal{D})$ has property AIEP.*

Proof. Take any sequence $(x_n, \beta_n)_{n\geq 1}$ with $\sup_n \|x_n\| < +\infty$ and such that $|\beta_n| \to \infty$, $Ax_n - i\beta_n x_n \to 0$ and $f(x_n) \to 0$ as $n \to \infty$. Since we already know that the pair $(\mathfrak{A}, R)$ has property AIEP we conclude that $x_n \xrightarrow{n\to\infty} 0$. Then we also get

$$\begin{aligned}
|(\mathcal{H}x_n)(1)|^2 &= 2\operatorname{Re}\langle(\mathcal{H}x_n)', \mathcal{H}x_n\rangle_{L_2} + |(\mathcal{H}x_n)(0)|^2 \\
&= 2\operatorname{Re}(\langle P_1^{-1} i\beta_n x_n, \mathcal{H}x_n\rangle_{L_2} - \langle P_1^{-1}(i\beta_n x_n - \mathfrak{A}x_n), \mathcal{H}x_n\rangle_{L_2} \\
&\quad - \langle P_0\mathcal{H}x_n, \mathcal{H}x_n\rangle_{L_2}) + |(\mathcal{H}x_n)(0)|^2 \\
&\leq 2\left\|P_1^{-1}\right\| \|i\beta_n x_n - \mathfrak{A}x_n\|_X \|x_n\|_X + 2\|P_0\| \|\mathcal{H}x_n\|_{L_2}^2 + |(\mathcal{H}x_n)(0)|^2 \\
&\longrightarrow 0, \quad \text{as } n \to \infty
\end{aligned}$$

i.e. also $(\mathcal{H}x_n)(1) \xrightarrow{n\to\infty} 0$ converges to zero. □

Using this refined notion we can formulate our exponential stability result.

Theorem 8.1.11. *Let $(\mathcal{D}_l, R_l^1, R_l^2) : D(\mathfrak{A}_l) \to H_l$ be given with*

$$\begin{aligned}
\operatorname{Re}\langle Ax, x\rangle_X &\leq -\sum_{l=1}^{L} \left\|R_l^1 x_l\right\|^2, \\
|\mathcal{D}_l(x_l)| &\geq |\mathfrak{C}_{l,1}x_l|, \\
|\mathfrak{B}_l x_l| &\geq \left\|R_l^2 x_l\right\|, \quad l = 1, \ldots, L.
\end{aligned}$$

If $(T(t))_{t\geq 0}$ is asymptotically stable and for $R_l = (R_l^1, R_l^2)$, the triples $(\mathfrak{A}_l, R_l, \mathcal{D}_l)$ have property AIEP for $l = 1, \ldots, L$, then $(T(t))_{t\geq 0}$ is uniformly exponentially stable.

Proof. Let $(x_n, \beta_n) \subseteq D(A) \times \mathbb{R}$ be any sequence with $\sup_{n\in\mathbb{N}} \|x_n\|_X < \infty$ and $|\beta_n| \to \infty$ such that

$$Ax_n - i\beta_n x_n \xrightarrow{n\to\infty} 0,$$

in particular $\mathfrak{A}_l x_{n,l} - i\beta_n x_{n,l} \to 0$ for $l = 1, \ldots, L$. From $\operatorname{Re}\langle Ax_n, x_n\rangle_X \xrightarrow{n\to\infty} 0$ it follows $R_l^1 x_l \xrightarrow{n\to\infty} 0$ for $l = 1, \ldots, L$. Further $\mathfrak{B}_1 x_{n,1} = 0$ since $x_n \in D(\mathcal{A})$, thus $R_1^1 x_{n,1} \xrightarrow{n\to\infty} 0$ converges to zero. By property AIEP this implies that $x_{n,1} \xrightarrow{n\to\infty} 0$ and $\mathcal{D}_1 x_{n,1} \xrightarrow{n\to\infty} 0$. Now assume that $x_{n,l} \xrightarrow{n\to\infty} 0$ and $\mathcal{D}_l x_{n,l} \xrightarrow{n\to\infty} 0$ hold for some $1 \leq l_0 < L$ and all $l \leq l_0$. This implies that

$$\mathfrak{B}_{l_0+1,1} x_{n,l+1} = \sum_{j=1}^{l_0} K_{l_0,j} \mathfrak{C}_{j,1} x_{n,j} \xrightarrow{n\to\infty} 0, \quad \mathfrak{B}_{l_0+1,2} x_{n,l_0+1} = 0,$$

so also $\mathfrak{B}_{l_0+1} x_{n,l+1} \xrightarrow{n\to\infty} 0$ and then $R_{l_0+1} x_{n,l_0+1} \xrightarrow{n\to\infty} 0$ and thanks to property AIEP we deduce from $\mathcal{A}_{l_0+1} x_{n,l_0+1} - i\beta_n x_{n,l_0+1} \xrightarrow{n\to\infty} 0$ that also the sequences $x_{n,l_0+1} \xrightarrow{n\to\infty} 0$ converges to zero and $\mathcal{D}_{l_0+1} x_{n,l_0+1} \xrightarrow{n\to\infty} 0$. Inductively we finally arrive at $x_n \xrightarrow{n\to\infty} 0$ and by Corollary 2.2.19 the C_0-semigroup $(\mathcal{T}(t))_{t\geq 0}$ is uniformly exponentially stable. □

Example 8.1.12. *Consider a chain of coupled vibrating strings as in [LiHuCh89]*

$$\rho_l(\zeta)\frac{\partial^2\omega_l}{\partial t^2}(t,\zeta) = \frac{\partial}{\partial\zeta}\left(T_l(\zeta)\frac{\partial\omega}{\partial\zeta}(t,\zeta)\right), \quad l = 1,\dots,L,\ \zeta\in(0,1),\ t\geq 0$$

where in contrast to the original article [LiHuCh89] we do not demand that ρ_l, T_l *are positive constants, but allow* $0<\varepsilon\leq\rho_l, T_l\in W^1_\infty(0,1)$ *to be uniformly positive Lipschitz continuous functions instead. At the left end of the chain we impose the dissipative boundary condition*

$$\left(T_1\frac{\partial\omega_1}{\partial\zeta}\right)(t,0) = -\kappa_0\frac{\partial\omega_1}{\partial t}(t,0)$$

for some $\kappa_0>0$*. Moreover, the strings are linked in a conservative or dissipative way, namely either*

$$\begin{aligned}\left(T_l\frac{\partial\omega_l}{\partial\zeta}\right)(t,1) - \left(T_{l+1}\frac{\partial\omega_{l+1}}{\partial\zeta}\right)(t,0) &= 0\\ \frac{\partial\omega_l}{\partial t}(t,1) - \frac{\partial\omega_{l+1}}{\partial t}(t,0) &= -\kappa_l'\left(T_l\frac{\partial\omega_l}{\partial\zeta}\right)(t,1)\end{aligned}$$

or

$$\begin{aligned}\left(T_l\frac{\partial\omega_l}{\partial\zeta}\right)(t,1) - \left(T_{l+1}\frac{\partial\omega_{l+1}}{\partial\zeta}\right)(t,0) &= -\kappa_l\frac{\partial\omega_l}{\partial t}(t,1),\\ \frac{\partial\omega_l}{\partial t}(t,1) - \frac{\partial\omega_{l+1}}{\partial t}(t,0) &= 0.\end{aligned}$$

Finally, at the right end we impose a conservative boundary condition

$$\frac{\partial\omega_L}{\partial t}(t,1) = 0 \quad \text{or} \quad \left(T_L\frac{\partial\omega_L}{\partial\zeta}\right)(t,1) = 0.$$

To reformulate this as interconnection of infinite dimensional port-Hamiltonian systems we set

$$\begin{aligned}x_{l,1} &:= \rho_l\frac{\partial\omega_l}{\partial t}, \quad x_{l,2} := \frac{\partial\omega_l}{\partial\zeta},\\ \mathcal{H}_l &:= \begin{bmatrix}\frac{1}{\rho_l} & \\ & T_l\end{bmatrix}, \quad P_{l,1} := \begin{bmatrix} & 1\\ 1 & \end{bmatrix}, \quad P_{l,0} := 0\end{aligned}$$

and the input and output maps as

$$\begin{aligned}\mathfrak{B}_{1,1}x_1 &:= \left(T_1\frac{\partial\omega_1}{\partial\zeta}\right)(0) + \kappa_0\frac{\partial\omega_1}{\partial t}(0),\\ \mathfrak{B}_{l,2}x_l &:= \left(T_l\frac{\partial\omega_l}{\partial\zeta}\right)(0),\\ \mathfrak{C}_{l,2}x_l &:= \frac{\partial\omega_l}{\partial t}(1) + \kappa_l'\left(T_l\frac{\partial\omega_l}{\partial\zeta}\right)(1)\\ \mathfrak{C}_{l,3}x_l &:= \frac{\partial\omega}{\partial t}(0).\\ \mathfrak{B}_{l,3}x_l &:= \left(T_l\frac{\partial\omega_l}{\partial\zeta}\right)(1) + \kappa_l\frac{\partial\omega_l}{\partial t}(1)\\ \mathfrak{B}_{L,1}x_L &:= \begin{cases}\frac{\partial\omega_L}{\partial t}(1) & \text{or}\\ \left(T_L\frac{\partial\omega_L}{\partial\zeta}\right)(1)\end{cases}\end{aligned}$$

where $\kappa_0 > 0$ and for each $l \geq 1$ the constants $\kappa_l, \kappa_l' \geq 0$ are non-negative. From a physical point of view it makes sense to choose the constants such that for all l at least one of the constants κ_l and κ_l' equals zero and in that case one obtains the energy balance

$$\begin{aligned}\operatorname{Re}\langle Ax, x\rangle_X &\leq -\kappa_0 \left|(\mathcal{H}_1 x_1)(0)\right|^2 \\ &\hat{=} -\kappa_0 \left|\frac{\partial \omega}{\partial t}(0)\right|^2 = -\frac{1}{2}\left(\kappa_0 \left|\frac{\partial \omega}{\partial t}(0)\right|^2 + \frac{1}{\kappa_0}\left|T_1 \frac{\partial \omega}{\partial \zeta}(0)\right|^2\right)\end{aligned}$$

Moreover, for $l = 2, \dots, L$ we have

$$\left|\mathfrak{B}_{l,2} x_l\right|^2 + \left|\mathfrak{C}_{l,3} x_l\right|^2 = \left|(\mathcal{H}_l x_l)(0)\right|^2.$$

Thus, we obtain asymptotic and then exponential stability from Proposition 8.1.8 and Theorem 8.1.11, using also Example 8.1.10.

8.2 Non-autonomous Systems: The Case $N = 1$

In the following we see that the stability results from Chapter 9 in [JaZw12] also hold for *non-autonomous port-Hamiltonian systems* provided existence of solutions. The latter is a standing hypothesis in this section, and we do not touch the subject of well-posedness for non-autonomous port-Hamiltonian systems here since due to its hyperbolic nature and possibly time-varying domains very little abstract results are known which could provide a good existence theory. We consider systems of the form

$$\frac{\partial x(t,\zeta)}{\partial t} = P_1 \frac{\partial (\mathcal{H}(t,\zeta) x(t,\zeta))}{\partial \zeta} + P_0(\zeta)\mathcal{H}(t,\zeta)x(t,\zeta), \quad \zeta \in (0,1),\ t \geq 0. \tag{8.3}$$

Thus, from now on the Hamiltonian density matrix function $\mathcal{H}$ may also depend on time, but at least we assume that $\mathcal{H} \in C(\mathbb{R}_+; L_\infty(0,1;\mathbb{F}^{d\times d}))$ which may be considered as a subset of $C(\mathbb{R}_+; \mathcal{B}(L_2(0,1;\mathbb{F}^d)))$ is continuous in the time-variable $t \in \mathbb{R}_+$ and that $\mathcal{H}$ is uniformly positive in space and time, i.e. there exist constants $0 < m \leq M$ such that

$$mI \leq \mathcal{H}(t,\zeta) = \mathcal{H}(t,\zeta)^* \leq MI, \quad \text{a.e. } \zeta \in (0,1),\ t \geq 0.$$

Again $P_1 \in \mathbb{F}^{d\times d}$ is a self-adjoint and invertible matrix, wheres $P_0 \in L_\infty(0,1;\mathbb{F}^{d\times d})$. In the book [JaZw12] the main ingredient for the exponential stability result is Lemma 9.1.2 on a *finite observability estimate* for first order port-Hamiltonian systems which admits the following generalisation to non-autonomous problems.

Lemma 8.2.1. *Assume that $\mathcal{H} \in W^1_\infty(\mathbb{R}_+; W^1_\infty(0,1;\mathbb{F}^{d\times d}))$, i.e. $\mathcal{H}$ is Lipschitz continuous on $\mathbb{R}_+ \times [0,1]$. Then for any locally Lipschitz continuous solution $x \in W^1_{\infty,loc}(\mathbb{R}_+; L_2(0,1;\mathbb{F}^d)) \cap L_{\infty,loc}(\mathbb{R}_+; H^1(0,1;\mathbb{F}^d))$ of (8.3) with*

$$\|x(t+s)\|_{L_2} \leq K\,\|x(t)\|_{L_2}, \quad t \geq 0, s \in (0,\varepsilon) \tag{8.4}$$

for some $K > 0$ and some $\varepsilon > 0$, there are constants $c, \tau > 0$ which only depend on $m, M, P_0, P_1, \frac{\partial \mathcal{H}}{\partial \zeta}, \varepsilon$ and K such that

$$\|x(\tau)\|_{L_2}^2 \leq c \int_0^\tau \left|(\mathcal{H}(t,1)x(t,1)\right|^2 dt.$$

Proof. We apply the same strategy as in Lemma 9.1.2 of [JaZw12]. Choose $\gamma > 0$ and $\kappa > 0$ such that

$$\pm P_1^{-1} + \gamma\mathcal{H}(t,\zeta) \geq 0,$$
$$2\,\mathrm{Sym}\,(P_1^{-1}P_0(\zeta)\mathcal{H}(t,\zeta)) + \frac{\partial\mathcal{H}}{\partial\zeta}(t,\zeta) \leq \kappa\mathcal{H}(t,\zeta), \quad \text{a.e. } \zeta \in (0,1),\ t \geq 0$$

and finally $\tau > 2\gamma$. W.l.o.g. we may assume that $\tau < \varepsilon$. Then defining

$$F(\zeta) := \int_{\gamma(1-\zeta)}^{\tau-\gamma(1-\zeta)} \langle x(t,\zeta), \mathcal{H}(t,\zeta)x(t,\zeta)\rangle_{\mathbb{F}^d} dt, \quad \zeta \in (0,1)$$

we see that $F \in W^1_{\infty,loc}(0,1;\mathbb{R})$ is locally Lipschitz continuous with

$$\begin{aligned}
\frac{dF}{d\zeta}(\zeta) &= \int_{\gamma(1-\zeta)}^{\tau-\gamma(1-\zeta)} \langle x_\zeta(t,\zeta), \mathcal{H}(t,\zeta)x(t,\zeta)\rangle_{\mathbb{F}^d} + \langle x(t,\zeta), (\mathcal{H}(t,\zeta)x(t,\zeta))_\zeta\rangle_{\mathbb{F}^d} dt \\
&\quad + \gamma\langle x(\tau-\gamma(1-\zeta),\zeta), \mathcal{H}(\tau-\gamma(1-\zeta),\zeta)x(\tau-\gamma(1-\zeta),\zeta)\rangle_{\mathbb{F}^d} \\
&\quad + \gamma\langle x(\gamma(1-\zeta),\zeta), \mathcal{H}(\gamma(1-\zeta),\zeta)x(\gamma(1-\zeta),\zeta)\rangle_{\mathbb{F}^d} \\
&=: \int_{\gamma(1-\zeta)}^{\tau-\gamma(1-\zeta)} \langle P_1^{-1}x_t(t,\zeta), x(t,\zeta)\rangle_{\mathbb{F}^d} \\
&\qquad - \langle \frac{\partial\mathcal{H}}{\partial\zeta}(t,\zeta)x(t,\zeta) - P_1^{-1}P_0(\zeta)\mathcal{H}x(t,\zeta), x(t,\zeta)\rangle_{\mathbb{F}^d} dt \\
&\quad + \int_{\gamma(1-\zeta)}^{\tau-\gamma(1-\zeta)} \langle x(t,\zeta), P_1^{-1}\left(x_t(t,\zeta) - P_0(\zeta)\mathcal{H}(t,\zeta)x(t,\zeta)\right)\rangle_{\mathbb{F}^d} dt + b(\zeta) \\
&= \int_{\gamma(1-\zeta)}^{\tau-\gamma(1-\zeta)} \frac{d}{dt}\langle x(t,\zeta), P_1^{-1}x(t,\zeta)\rangle_{\mathbb{F}^d} dt + b(\zeta) \\
&\quad - \int_{\gamma(1-\zeta)}^{\tau-\gamma(1-\zeta)} \langle x(t,\zeta), \left(2\,\mathrm{Sym}\,(P_1^{-1}P_0(\zeta)\mathcal{H}(t,\zeta)) + \frac{\partial\mathcal{H}}{\partial\zeta}(t,\zeta)\right) x(t,\zeta)\rangle_{\mathbb{F}^d} dt \\
&= \langle x(\tau-\gamma(1-\zeta),\zeta), (P_1^{-1} + \gamma\mathcal{H}(\tau-\gamma(1-\zeta),\zeta))x(\tau-\gamma(1-\zeta),\zeta)\rangle_{\mathbb{F}^d} \\
&\quad - \int_{\gamma(1-\zeta)}^{\tau-\gamma(1-\zeta)} \langle x(t,\zeta), \left(2\,\mathrm{Sym}\,(P_1^{-1}P_0(\zeta)\mathcal{H}(t,\zeta)) + \frac{\partial\mathcal{H}}{\partial\zeta}(t,\zeta)\right) x(t,\zeta)\rangle_{\mathbb{F}^d} dt \\
&\geq -\kappa \int_{\gamma(1-\zeta)}^{\tau-\gamma(1-\zeta)} \langle x(t,\zeta), \mathcal{H}(t,\zeta)x(t,\zeta)\rangle_{\mathbb{F}^d} dt = -\kappa F(\zeta).
\end{aligned}$$

This implies

$$F(\zeta) \leq e^\kappa F(1), \quad \zeta \in [0,1]$$

and thus

$$\begin{aligned}
(\tau - 2\gamma)\left\|x(\tau)\right\|_{L_2}^2 &\leq K^2 \int_\gamma^{\tau-\gamma} \left\|x(t)\right\|_{L_2}^2 dt \leq \frac{K^2}{m}\int_\gamma^{\tau-\gamma} \left\|x(t)\right\|_{\mathcal{H}(t,\cdot)}^2 dt \\
&= \frac{K^2}{m}\int_\gamma^{\tau-\gamma}\int_0^1 \langle x(t,\zeta), \mathcal{H}(t,\zeta)x(t,\zeta)\rangle_{\mathbb{F}^d} d\zeta dt \\
&= \frac{K^2}{m}\int_0^1\int_\gamma^{\tau-\gamma} \langle x(t,\zeta), \mathcal{H}(t,\zeta)x(t,\zeta)\rangle_{\mathbb{F}^d} dt d\zeta
\end{aligned}$$

$$= \frac{K^2}{m} \int_0^1 F(\zeta) d\zeta \leq \frac{K^2}{m} e^{\kappa} F(1) d\zeta$$
$$\leq \frac{K^2 e^{\kappa}}{m^2} \int_0^{\tau} |\mathcal{H}(t,1) x(t,1)|^2 \, dt$$

so we find that

$$\|x(\tau)\|_{L_2}^2 \leq \frac{K^2 e^{\kappa}}{m^2(\tau - 2\gamma)} \int_0^{\tau} |\mathcal{H}(t,1) x(t,1)|^2 \, dt$$

which is the desired final observability estimate. □

Remark 8.2.2. *Note that choosing τ close to 2γ we can make $c > 0$ arbitrary small.*

Remark 8.2.3. *Condition (8.4) is satisfied if $\frac{\partial \mathcal{H}}{\partial t} \in L_\infty(\mathbb{R}_+ \times (0,1); \mathbb{F}^{d \times d})$ and*

$$\operatorname{Re} \langle x_t(t,\zeta), x(t,\zeta) \rangle_{\mathcal{H}(t,\cdot)} \leq 0.$$

Proof. In that case we have

$$\frac{d}{dt} \|x(t)\|_{\mathcal{H}(t,\cdot)}^2 = 2 \operatorname{Re} \langle x_t(t,\zeta), x(t,\zeta) \rangle_{\mathcal{H}(t,\cdot)} + \int_0^1 \langle x(t,\zeta), \mathcal{H}_t(t,\zeta) x(t,\zeta) \rangle_{\mathbb{F}^d} dt$$
$$\leq c \|x(t)\|_{\mathcal{H}(t,\cdot)}^2$$

and thus

$$\|x(t+s)\|_{\mathcal{H}(t+s)}^2 \leq e^{cs} \|x(t)\|_{\mathcal{H}(t)}^2, \quad s, t \geq 0.$$

The result follows since all norms $\|\cdot\|_{\mathcal{H}(t,\cdot)}$ are uniformly equivalent to the standard L_2-norm. □

Now we are able to prove the non-autonomous counterpart to Theorem 9.1.3 in [JaZw12].

Proposition 8.2.4. *Assume that the port-Hamiltonian density matrix function $\mathcal{H}$ lies in $L_\infty(\mathbb{R}_+; W_\infty^1(0,1;\mathbb{F}^{d\times d})) \cap W_\infty^1(\mathbb{R}_+; L_\infty(0,1;\mathbb{F}^{d\times d}))$. Then for any solution $x \in W_{\infty,loc}^1(\mathbb{R}_+; L_2(0,1;\mathbb{F}^d)) \cap L_{\infty,loc}(\mathbb{R}_+; H^1(0,1;\mathbb{F}^d))$ of (8.3) with*

$$\operatorname{Re} \langle x_t(t,\zeta), x(t,\zeta) \rangle_{\mathcal{H}(t,\cdot)} \leq -\kappa \, |(\mathcal{H}x)(t,1)|^2, \quad a.e. \geq 0$$

for some $\kappa > 0$ we have

$$\|x(t)\|_{L_2} \leq M e^{-\omega t} \|x(0)\|_{L_2}, \quad t \geq 0$$

where $M \geq 1$ and $\omega > 0$ depend on $\mathcal{H}, P_0, P_1$ and κ, but not on x.

Proof. From Lemma 8.2.1 and the preceding remark on dissipative systems there are constants $c, \tau > 0$ with

$$\|x(\tau)\|_{L_2} \leq c \int_0^{\tau} |(\mathcal{H}x)(t,1)|^2 \, dt$$

where $c > 0$ may be chosen arbitrary small for τ close to 2γ. We then have

$$\begin{aligned}
m\left\|x(\tau)\right\|_{L_2}^2 - M\left\|x(0)\right\|_{L_2}^2 &\le \left\|x(\tau)\right\|_{\mathcal{H}(\tau,\cdot)}^2 - \left\|x(0)\right\|_{\mathcal{H}(0,\cdot)}^2 \\
&= \int_0^\tau \frac{d}{dt}\left(\left\|x(t)\right\|_{\mathcal{H}(t,\cdot)}^2\right) dt \\
&= \int_0^\tau 2\operatorname{Re}\langle x_t(t), x(t)\rangle_{\mathcal{H}(t,\cdot)} dt \\
&\quad + \int_0^\tau \int_0^1 \langle x(t,\zeta), \frac{\partial \mathcal{H}}{\partial t}(t,\zeta)x(t,\zeta)\rangle_{\mathbb{F}^d} d\zeta dt \\
&\le -2\kappa \int_0^\tau \left|(\mathcal{H}x)(t,1)\right|^2 dt \\
&\quad + \left\|\frac{\partial \mathcal{H}}{\partial t}\right\|_{L^\infty} \int_0^\tau \left\|x(t)\right\|_{L_2}^2 dt \\
&\le -\frac{2\kappa}{c}\left\|x(\tau)\right\|_{L_2}^2 + \left\|\frac{\partial \mathcal{H}}{\partial t}\right\|_{L^\infty} K^2\tau \left\|x(0)\right\|_{L_2}^2 .
\end{aligned}$$

Rearranging the terms we deduce that

$$\left\|x(\tau)\right\|_{L_2}^2 \le \frac{M + \left\|\mathcal{H}_t\right\|_{L_\infty} K^2\tau}{m + \frac{2\kappa}{c}} \left\|x(0)\right\|_{L_2}^2 =: \rho^2 \left\|x(0)\right\|_{L_2}^2$$

thus choosing $c > 0$ sufficiently small we have $\rho < 1$ and then using the time-invariance of the problem we obtain iteratively for all $t = n\tau + s,\ s \in [0,\tau)$, that

$$\begin{aligned}
\left\|x(t)\right\|_{L_2} = \left\|x(n\tau + s)\right\|_{L_2} &\le K\left\|x(n\tau)\right\|_{L_2} \le K\rho^n \left\|x(0)\right\|_{L_2} \\
&\le \frac{K}{\rho} e^{\frac{\log(\rho)}{\tau}t} \left\|x(0)\right\|_{L_2} =: Me^{-\omega t}\left\|x(0)\right\|_{L_2} .
\end{aligned}$$

This finishes the proof of the asserted statement. □

8.3 Systems with Structural Damping

By allowing $P_0(\zeta)$ to be dissipative for $\zeta \in (0,1)$, the port-Hamiltonian systems considered in this thesis may be damped through this term, e.g. for the one-dimensional wave equation

$$\rho(\zeta)\omega_{tt}(t,\zeta) = (T(\zeta)\omega_\zeta(t,\zeta))_\zeta - \alpha(\zeta)\omega_t(t,\zeta), \quad \zeta \in (0,1),\ t \ge 0$$

where $\alpha : [0,1] \to \mathbb{R}$ is a bounded measurable and non-negative function. With this type of damping the system may be asymptotically, or even uniformly exponentially, stabilised. However, within this section we consider systems with another type of damping. Namely we consider structural damping, e.g. for the wave equation systems of the form

$$\rho(\zeta)\omega_{tt}(t,\zeta) = (T(\zeta)\omega_\zeta(t,\zeta))_\zeta + (k(\zeta)\omega_{t\zeta})_\zeta, \quad t \ge 0,\ \zeta \in (0,1).$$

If we naively reformulate this system as a port-Hamiltonian system by setting $x = (x_1, x_2) \hat{=} (\rho\omega_t, \omega_{t\zeta})$ we arrive at the following partial differential equation which is of first order in time

$$
\begin{aligned}
\frac{\partial}{\partial t}x(t,\zeta) &\hat{=} \begin{pmatrix} \rho(\zeta)\omega_{tt}(t,\zeta) \\ \omega_{t\zeta}(t,\zeta) \end{pmatrix} \\
&= \frac{\partial^2}{\partial\zeta^2}\left[\begin{bmatrix} k(\zeta) & \\ & 0 \end{bmatrix}\begin{bmatrix} \rho^{-1}(\zeta) & \\ & T(\zeta) \end{bmatrix}\begin{pmatrix} \rho\omega(t,\zeta) \\ \omega_{t\zeta}(t,\zeta) \end{pmatrix}\right] \\
&\quad + \frac{\partial}{\partial\zeta}\left[\begin{bmatrix} & 1 \\ -1 & \end{bmatrix}\begin{bmatrix} \rho^{-1}(\zeta) & \\ & T(\zeta) \end{bmatrix}\begin{pmatrix} \rho(\zeta)\omega_t(t,\zeta) \\ \omega_{t\zeta}(t,\zeta) \end{pmatrix}\right] \\
&= \frac{\partial}{\partial\zeta}\left(\begin{bmatrix} & 1 \\ -1 & \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} k(\zeta) \begin{bmatrix} 1 & 0 \end{bmatrix} \frac{\partial}{\partial\zeta}\right) \\
&\quad \begin{bmatrix} \rho(\zeta)^{-1} & \\ & T(\zeta) \end{bmatrix}\begin{pmatrix} \rho(\zeta)\omega_t(t,\zeta) \\ \omega_\zeta(t,\zeta) \end{pmatrix}
\end{aligned}
$$

for $\zeta \in (0,1)$ and $t \geq 0$. In particular, two conditions of Definition 3.2.10 for the port-Hamiltonian operator $\mathfrak{A}$ are not satisfied. On the one hand $P_2(\zeta) = \operatorname{diag}(-k(\zeta), 0) \in \mathbb{F}^{2\times 2}$ is not invertible, on the other hand it does depend on the space variable $\zeta \in (0,1)$. Therefore, these systems only fall into the more general class of PDE of the form

$$
\frac{\partial}{\partial t}x(t,\zeta) = (\mathcal{J} - \mathcal{G}S\mathcal{G}^*)(\mathcal{H}x)(t,\zeta), \quad \zeta \in (0,1),\ t \geq 0.
$$

Here $\mathcal{J}, \mathcal{G}$ and $\mathcal{G}^*$ are differential operators of the form

$$
\mathcal{J}x = \sum_{k=0}^{N} \frac{\partial^k}{\partial\zeta^k} P_k x, \quad \mathcal{G}x = \sum_{k=0}^{N} \frac{\partial^k}{\partial\zeta^k} G_k x, \quad \mathcal{G}^* z = \sum_{k=0}^{N} (-1)^{k+1} \frac{\partial^k}{\partial\zeta^k} G_k^* z
$$

so that $\mathcal{G}^*$ is the formal adjoint of $\mathcal{G}$. Here $P_k = (-1)^{k+1}P_k^* \in \mathbb{F}^{d\times d}$ $(k = 0, 1, \ldots, N)$ are quadratic matrices, while the matrices $G_k \in \mathbb{F}^{d\times m}$ $(k = 0, 1, \ldots, N)$, map from the (in general smaller) space $\mathbb{F}^m$, where $1 \leq m \leq d$, to $\mathbb{F}^d$. Similar to $\mathcal{H} \in L_\infty(0,1;\mathbb{F}^{d\times d})$ also the matrix-valued function $S \in L_\infty(0,1;\mathbb{F}^{m\times m})$ is assumed to be a coercive multiplication operator on $L_2(0,1;\mathbb{F}^m)$. These systems have been considered, e.g. in [Vi07] and [AuJaLa15], but with emphasis on different aspects of the equation. In Chapter 6 of [Vi07] the author considered conditions under which the corresponding operator A^{ext} (with suitable boundary conditions)

$$
\begin{aligned}
A^{ext}x &= \mathcal{J}\mathcal{H}x - \mathcal{G}S\mathcal{G}^*\mathcal{H}x \\
D(A^{ext}) &= \big\{x \in X = L_2(0,1;\mathbb{F}^d) :\ \mathcal{H}x \in H^N(0,1;\mathbb{F}^d), \\
&\qquad S\mathcal{G}^*\mathcal{H}x \in H^N(0,1;\mathbb{F}^m),\ (f^{ext}_{\partial,\mathcal{H}x}, e^{ext}_{\partial,\mathcal{H}x}) \in \ker W^{ext}\big\}
\end{aligned}
$$

generates a contractive C_0-semigroup $(T^{ext}(t))_{t\geq 0}$, where the *extended boundary flow* and *boundary effort* variables $f^{ext}_{\partial,\mathcal{H}x}$ and $e^{ext}_{\partial,\mathcal{H}x}$, respectively, may not only depend on $\tau(\mathcal{H}x)$, but also on $\tau(-S\mathcal{G}^*\mathcal{H}x)$, or more precisely

$$\begin{pmatrix} f^{ext}_{\partial,\mathcal{H}x} \\ e^{ext}_{\partial,\mathcal{H}x} \end{pmatrix} = R^{ext}\tau(\mathcal{H}x, -S\mathcal{G}^*\mathcal{H}x) = R^{ext} \begin{pmatrix} (\mathcal{H}x)(1) \\ -(S\mathcal{G}^*\mathcal{H}x)(1) \\ \vdots \\ (\mathcal{H}x)^{(N-1)}(1) \\ -(S\mathcal{G}^*\mathcal{H}x)^{(N-1)}(1) \\ (\mathcal{H}x)(0) \\ -(S\mathcal{G}^*\mathcal{H}x)(0) \\ \vdots \\ (\mathcal{H}x)^{(N-1)}(0) \\ -(S\mathcal{G}^*\mathcal{H}x)^{(N-1)}(0) \end{pmatrix}$$

for the matrix $R^{ext} = \begin{bmatrix} Q^{ext} & -Q^{ext} \\ I & I \end{bmatrix} \in \mathbb{F}^{2N(d+m)\times 2N(d+m)}$ and

$$Q^{ext} = \begin{bmatrix} P^{ext}_1 & P^{ext}_2 & \cdots & \cdots & P^{ext}_N \\ -P^{ext}_2 & -P^{ext}_3 & \cdots & -P^{ext}_N & 0 \\ \vdots & & \vdots & & \\ (-1)^{N-1}P^{ext}_N & 0 & \cdots & 0 & 0 \end{bmatrix}$$

$$P^{ext}_k = \begin{bmatrix} P_k & G_k \\ (-1)^{k+1}G^*_k & 0 \end{bmatrix}, \quad k = 0, 1, \ldots, N$$

and it is additionally assumed that P^{ext}_N is invertible, if $G_k \neq 0$ for at least one $k \in \{1, \ldots, N\}$, or P_N is invertible, if $G_k = 0$ for all $k \geq 1$. Since the latter case is already covered by our standard port-Hamiltonian setting, we assume without loss of generality that $G_k \neq 0$ for at least one $k \geq 1$ and $P^{ext}_N \in \mathbb{F}^{(d+m)\times(d+m)}$ is invertible. Then one obtains the following generation result.

Theorem 8.3.1. *Let $W^{ext} \in \mathbb{F}^{N(d+m)\times 2N(d+m)}$ be a full-rank matrix and be such that $W_{ext}\Sigma W^*_{ext} \geq 0$ is positive semidefinite. Then the operator A^{ext} generates a contractive C_0-semigroup on $X = \big(L_2(0,1;\mathbb{F}^d), \langle\cdot,\cdot\rangle_{\mathcal{H}}\big)$.*

Proof. See Theorem 6.11 in [Vi07]. □

Fore a more detailed analysis, including Boundary Control and Observation Systems connected to above problem, we refer to Chapter 6 in [Vi07] and consider the situation in [AuJaLa15] next. There the authors had a different aim and were interested in L_p-maximal regularity of the time-invariant as well as of the corresponding non-autonomous problem for the case that $N = 1$. (In fact, this was done within a more abstract framework, considering more general right-multiplicative perturbations of holomorphic semigroup generators.) As we encounter the concepts of a holomorphic (or, analytic) C_0-semigroup and of L_p-maximal regularity, we first recall these concepts (in the autonomous setting).

Definition 8.3.2. *Let for the moment X be any Banach space. A family of linear bounded operators $(T(z))_{z\in\Sigma_\delta\cup\{0\}}$ on the sector $\Sigma_\delta \cup \{0\} := \{z \in \mathbb{C} : |\arg z| < \delta\} \cup \{0\}$ is called* holomorphic C_0-semigroup *(or,* analytic C_0-semigroup*) (of* angle *$\delta \in (0, \pi/2]$ if*

1. *$T(0) = I$ and $T(z_1 + z_2) = T(z_1)T(z_2)$ for all $z_1, z_2 \in \Sigma_\delta$.*

2. *The map $z \mapsto T(z)$ is analytic in Σ_δ.*

3. $\lim_{\Sigma_{\delta'} \ni z \to 0} T(z)x = x$ *for all $x \in X$ and $\delta' \in (0, \delta)$.*

See e.g. Definition II.4.5 in [EnNa00]. For a more abstract definition, including non-autonomous problems, see Subsection III.1.5 in [Am95].

Definition 8.3.3. *Let X be any Banach space. A closed, linear and densely defined operator $A : D(A) \subseteq X \to X$ is said to have* L_p-maximal regularity *if for each $f \in L_p(0, \tau; X)$, $\tau > 0$ the (autonomous) problem*

$$\frac{d}{dt}x(t) + Ax(t) = f(t), \ t \geq 0, \quad x(0) = 0$$

has a unique solution $x \in W_p^1(0, \tau; X) \cap L_p(0, \tau; D(A))$.

See, e.g. Definition 1.1 in [Pr02].

In the following do not give any proofs here, but only give some results closely connected to above generalised port-Hamiltonian operator, leaving details out and only referring to [AuJaLa15].

Assumption 8.3.4. *Let $\tau > 0$ be fixed. Let $S : [0, \tau] \times [0, 1] \to \mathbb{F}^{m \times m}$ and $\mathcal{H} : [0, 1] \to \mathbb{F}^{d \times d}$ be measurable matrix-valued functions with the following properties. We assume that $S(t, \zeta) = S(t, \zeta)^*$ and $\mathcal{H}(t, \zeta) = \mathcal{H}(t, \zeta)^*$ are pointwise symmetric for a.e. $(t, \zeta) \in [0, \tau] \times [0, 1]$, uniformly bounded and considered as matrix-valued multiplication operators on $L_2(0, 1; \mathbb{F}^{m \times m})$ and $L_2(0, 1; \mathbb{F}^{d \times d})$, respectively, the maps $S(t, \cdot)$ and $\mathcal{H}(t, \cdot)$ are uniformly coercive. Furthermore we assume that S and $\mathcal{H}$ are Lipschitz-continuous in the time variable $t \in [0, \tau]$. In order words, we assume that there are constants $0 < m_i < M_i$ $(i = 1, 2)$ such that*

$$\begin{aligned} m_1 I \leq \mathcal{H}(t, \zeta) &\leq M_1 I \\ m_2 I \leq S(t, \zeta) &\leq M_2 I \\ \|\mathcal{H}(t, \zeta) - \mathcal{H}(s, \zeta)\| &\leq L_1 |s - t| \\ \|S(t, \zeta) - S(s, \zeta)\| &\leq L_2 |s - t|, \quad s, t \in [0, \tau], \ \text{a.e. } \zeta \in [0, 1]. \end{aligned}$$

Moreover, we assume that $P_0 \in L_\infty(0, 1; \mathbb{F}^{d \times d})$ is essentially bounded, P_1 bounded and Lipschitz continuous and the matrix $G \in \mathbb{F}^{d \times m}$ has full rank such that $GG^* \in \mathbb{F}^{d \times d}$ is an orthogonal projection with

$$\ker(I - GG^*)P_1(\zeta) \subseteq \ker GG^*, \quad \text{a.e. } \zeta \in (0, 1).$$

Further let $F \in \mathbb{F}^{2m \times r}$, for some $r \in \{0, 1, \ldots, 2m\}$, have full rank and be such that $FF^* \in \mathbb{F}^{d \times d}$ is a projection. Finally let $W_R : [0, \tau] \to \mathbb{F}^{r \times r}$ be Lipschitz continuous with $W_R(t) = W_R(t)^* \geq 0$ positive semidefinite for all $t \in [0, \tau]$.

Then we consider the operator family $\{A(t)\}_{t \in [0, \tau]}$ for the operators

$$\begin{aligned} A(t) &= \frac{\partial}{\partial \zeta} (GS(t)G + P_1) + P_0 \\ D(A(t)) &= \{x \in L_2(0, 1; \mathbb{F}^{d \times d}) : \quad G^* x \in H^1(0, 1; \mathbb{F}^m), \\ &\qquad GS(t)G^* \in H^1(0, 1; \mathbb{F}^d), \\ &\qquad F^* \mathfrak{B}(t)x = -W_R(t)F^* \mathfrak{C}x, \ (I - FF^*)\mathfrak{C}x = 0\} \end{aligned}$$

where the boundary operators

$$\mathfrak{B}(t)x = \begin{pmatrix} G^*(GS(t)x' + P_1x)(1) \\ -G^*(GS(t)x' + P_1x)(0) \end{pmatrix}$$
$$\mathfrak{C}x = \begin{pmatrix} (G^*x)(1) \\ (G^*x)(0) \end{pmatrix}$$

are defined on the domains

$$\begin{aligned} D(\mathfrak{B}(t)) &= \{x \in L_2(0,1;\mathbb{F}^d) : \; G^*x \in H^1(0,1;\mathbb{F}^m), \\ &\qquad G^*(GS(t)(G^*x)' + P_1x) \in H^1(0,1;\mathbb{F}^m)\} \\ D(\mathfrak{C}) &= \{x \in L_2(0,1;\mathbb{F}^d) : \; G^*x \in H^1(0,1;K^m)\}. \end{aligned}$$

For fixed $t \in [0,\tau]$ the operator $A(t)$ generates a holomorphic semigroup and also the right-multiplicative perturbed operator $A(t)\mathcal{H}(t)$.

Proposition 8.3.5. *For every fixed $t \in [0,\tau]$ the operators $A(t)$ and $A(t)\mathcal{H}(t)$ generate holomorphic C_0-semigroups on $X = L_2(0,1;\mathbb{F}^d)$.*

Proof. See Proposition 4.4 and Proposition 4.5 in [AuJaLa15]. □

Corollary 8.3.6. *If additionally the following conditions are satisfied*

$$W_R(t) + F^* \begin{bmatrix} G^*P_1(1)G & 0 \\ 0 & -G^*P_1(0)G \end{bmatrix} F \geq 0$$
$$\operatorname{Sym}\left(P_0(\cdot) + GG^*P_1'(\cdot) + \frac{1}{2}GG^*P_1'(\cdot)GG^*\right) \leq 0$$
$$P_1(\cdot) = P_1(\cdot)^*$$

then the C_0-semigroup $(T_{A(t)\mathcal{H}(t)}(s))_{s\geq 0}$ generated by $A(t)\mathcal{H}(t)$ is contractive with respect to the inner product $\langle\cdot,\cdot\rangle_{\mathcal{H}(t)}$.

Proof. See Proposition 4.6 in [AuJaLa15]. □

For the non-autonomous problem one then obtains the following (non-autonomous) *L_p-maximal regularity* results under Assumption 8.3.4

Theorem 8.3.7. *Let $p \in (1,\infty)$ and additionally to Assumption 8.3.4 assume that $S(t,\cdot) = S(\cdot)$ and $W_R(t,\cdot) = W_R(\cdot)$ do not depends on $t \in [0,\tau]$, so that the domain $D(A(t)) = D$ of the operator $A(t)$ is the same for all $t \in [0,\tau]$. Then for every $x_0 \in (X,D)_{1-1/p,p}$ and $f \in L_p(0,\tau;X)$ there is a unique solution $u \in MR_{\mathcal{H}}(p,X)$, i.e. $u \in W_p^1(0,\tau;X)$ such that $u(t) \in D(A(t)\mathcal{H}(t))$ for a.e. $t \in [0,\tau]$ and $A(\cdot)u \in L_p(0,\tau;X)$, of the problem*

$$\begin{aligned} u_t(t) - A(t)\mathcal{H}(t)u(t) &= f(t) \\ \mathcal{H}(0)u(0) &= x_0 \\ F^*\mathfrak{B}(t)\mathcal{H}(t)u(t) &= -W_RF^*\mathfrak{C}\mathcal{H}(t)u \\ (I - FF^*)\mathfrak{C}\mathcal{H}(t)u(t) &= 0, \quad t \in [0,\tau]. \end{aligned}$$

Proof. See Theorem 4.7 in [AuJaLa15]. □

Similar for the case $p = 2$ one may also allow t-dependence of $W_R(t)$ and $S(t)$.

Theorem 8.3.8. *Let $p = 2$ and assume that Assumption 8.3.4 is satisfied. Then for every $x_0 \in V = \{v \in L_2(0,1;\mathbb{F}^d) : (I - GG^*)P_1 v \in H^1(0,1;\mathbb{F}^d)\}$ and $f \in L_2(0,\tau;X)$ the non-autonomous system*

$$\begin{aligned} u_t(t) - A(t)\mathcal{H}(t)u(t) &= f(t) \\ \mathcal{H}(0)u(0) &= x_0 \\ F^*\mathfrak{B}(t)\mathcal{H}(t)u(t) &= -W_R(t)F^*\mathfrak{C}\mathcal{H}(t)u(t) \\ (I - FF^*)\mathfrak{C}\mathcal{H}(t)u(t) &= 0, \quad t \in [0,\tau] \end{aligned}$$

has a unique solution $u \in MR_{\mathcal{H}}(2,X)$, i.e. $u \in H^1(0,\tau;X)$ such that $u(t) \in D(A(t)\mathcal{H}(t))$ for a.e. $t \in [0,\tau]$ and $A(\cdot)\mathcal{H}(\cdot)u \in L_2(0,\tau;X)$.

Proof. See Theorem 4.8 in [AuJaLa15]. □

These results can be applied to the particular example of a one-dimensional wave equation with structural damping. To reformulate the one-dimensional *wave equation with structural damping* in the framework considered above, we assume that $0 < \varepsilon \le k \in L_\infty(0,1;\mathbb{R})$ for some $\varepsilon > 0$ and we may set

$$\begin{aligned} \mathcal{H}(\zeta) &= \begin{pmatrix} \frac{1}{\rho(\zeta)} & \\ & T(\zeta) \end{pmatrix}, \\ P_1(\zeta) &= \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}, \quad G = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \\ S(\zeta) &= k(\zeta), \quad P_0 = 0 \end{aligned}$$

and have $d = 2$ and $m = 1$. In particular

$$(I - GG^*)P_1 = P_1 GG^*$$

so that Assumption 8.3.4 is satisfied for every $F \in \mathbb{F}^{2\times r}$ such that $FF^* \in \mathbb{F}^{2\times 2}$ is a projection. Writing

$$H^1_F(0,1) = \left\{ v \in H^1(0,1) : (I - FF^*)\begin{pmatrix} v(1) \\ v(0) \end{pmatrix} = 0 \right\}$$

and

$$\begin{aligned} D_{F,W_R,k} = \{u \in H^1_F(0,1) \times L_2(0,1) :\ & ku_1' + u_2 \in H^1(0,1), \\ & \begin{pmatrix} (ku_1' + u_2)(1) \\ -(ku_1' + u_2)(0) \end{pmatrix} = -W_R \begin{pmatrix} u_1(1) \\ u_1(0) \end{pmatrix} \} \end{aligned}$$

we then find

Proposition 8.3.9. *Let $p \in (1,\infty)$ and assume that $\rho, T : [0,\tau] \times [0,1] \to \mathbb{R}$ and $k : [0,1] \to \mathbb{R}$ are bounded, measurable and uniformly positive with ρ and T Lipschitz continuous in $t \in [0,\tau]$. Further let $F \in \mathbb{F}^{2\times r}$ for some $r \in \{0,1,2\}$ such that $FF^* \in \mathbb{F}^{2\times 2}$ is a projection and $W_R = W_R^* \ge 0$ be an $r \times r$-matrix. Then for every $(x_1,x_2) \in \left(L_2(0,1;\mathbb{F}^2), D_{F,W_R,k}\right)_{1-1/p,p}$ and $f \in L_p(0,\tau;L_2(0,1))$*

the problem

$$\begin{aligned}
\rho(t)\omega_{tt}(t) - (T(t)\omega_\zeta(t) - k\omega_{t\zeta}(t))' &= f(t), \\
(I - FF^*) \begin{pmatrix} \omega_t(t,0) \\ \omega_t(t,1) \end{pmatrix} &= 0 \\
F^* \begin{pmatrix} (k\omega_{t\zeta}(t) + T(t)\omega_\zeta)(1) \\ -(k\omega_{t\zeta}(t) + T(t)\omega_\zeta)(0) \end{pmatrix} &= -W_R F^* \begin{pmatrix} \omega_t(t,1) \\ \omega_t(t,0) \end{pmatrix} \\
\omega_t(0,\cdot) &= x_1 \\
(T\omega_\zeta)(0,\cdot) &= x_2
\end{aligned}$$

has a solution ω such that

$$\begin{aligned}
(\omega_t, T\omega_\zeta) &\in W_p^1(0,\tau; L_2(0,1;\mathbb{F}^2)) \\
k\omega_t + T\omega_\zeta &\in L_p(0,\tau; H^1(0,1;\mathbb{F}^2))
\end{aligned}$$

which is unique up to an additive constant $\Delta \in \mathbb{F}$.
If $p = 2$ the same result also holds true if $W_R : [0,\tau] \to \mathbb{F}^{r\times r}$ is merely Lipschitz continuous, and then (x_1, x_2) should lie in $\left(L_2(0,1;\mathbb{F}^2), D_{F,W_R,k}\right)_{1/2,2} = H_F^1(0,1) \times L_2(0,1)$.

Proof. See Proposition 4.9 and Proposition 4.10 in [AuJaLa15]. □

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Bibliography

[Ad75] R.A. Adams: *Sobolev Spaces.* Pure and Applied Mathematics 65, Academic Press, New York 1975.

[Am95] H. Amann: *Linear and Quasilinear Parabolic Problems. Volume I: Abstract Linear Theory.* Monographs in Mathematics 89, Birkhäuser, Basel 1995.

[ArBa88] W. Arendt, C.J.K. Batty: *Tauberian Theorems and Stability of One-Parameter Semigroups.* Trans. Amer. Soc. Math. 306(2): 337–352, 1988.

[Ar+86] W. Arendt, A. Grabosch, G. Greiner, U. Groh, H.P. Lotz, U. Moustakis, R. Nagel, F. Neubrander, U. Schlotterbeck: *One-Parameter Semigroups of Positive Operators.* Lecture Notes in Mathematics 1184, Springer, Berlin 1986.

[ArUr00] W. Arendt, K. Urban: *Partielle Differenzialgleichungen. Eine Einführung in analytische und numerische Methoden.* Spektrum Akademischer Verlag, Heidelberg 2010.

[AuJa14] B. Augner, B. Jacob: *Stability and Stabilization of Infinite-dimensional Linear Port-Hamiltonian Systems.* Evolution Equations and Control Theory 3(2): 207–229, 2014.

[AuJaLa15] B. Augner, B. Jacob, H. Laasri: *On the right multiplicative perturbation of non-autonomous L_p-maximal regularity.* Journal of Operator Theory 74(2): 391–415, 2015.

[Br11] H. Brezis: *Functional Analysis, Sobolev Spaces and Partial Differential Equations.* Springer, New York 2011.

[CaDo81] J.T. Cannon, S. Dostrovsky: *The evolution of dynamics: vibration theory from 1687 to 1742.* Studies in the History of Mathematics and Physical Sciences 6, Springer, New York-Berlin 1981.

[CaGu72] B. Calvert, K. Gustafson: *Multiplicative perturbation of nonlinear m-accretive operators.* J. Functional Analysis 10: 149–158, 1972.

[CaHa98] T. Cazenave, A. Haraux: *An introduction to semilinear evolution equations.* Oxford University Press, New York 1998.

[Ch+87] G. Chen, M.C. Delfour, A.M. Krall, G. Payres: *Modeling, stabilization and control of serially connected beams.* SIAM J. Control Optim. 25(3): 526–546, 1987.

[Ch+87a] G. Chen, S.G. Krantz, D.W. Ma, C.E. Wayne, H.H. West: *The Euler-Bernoulli Beam Equation with Boundary Energy Dissipation.* Operator methods for optimal control problems (New Orleans, La., 1986): 67–96, Lecture Notes in Pure and Appl. Math. 108, Dekker, New York, 1987.

[Ch01] B. Chentouf: *A note on stabilization of a hybrid PDE-ODE system.* Proceedings of the 40th IEEE Conference on Decision and Control, Orlanda, Florida USA, December 2001.

[ChXuSa00] B. Chentouf, C.Z. Xu, G. Sallet: *On the stabilization of a vibrating equation.* Nonlinear Analysis 39: 537–558, 2000.

[ChYe08] M. Cherkaoui, N. Yebari: *Energy decay estimates for a wave equation with nonlinear boundary feedback.* Glasnik Matematički 43: 375–386, 2008.

[CoLaMa90] F. Conrad, J. Leblond, J.-P. Marmorat: *Energy decay estimates for a beam with nonlinear boundary feedback.* Lecture Notes in Control and Inform. Sci. 147, Springer, Berlin 1990.

[CoMo98] F. Conrad, Ö. Morgül: *On the stabilization of a flexible beam with a tip mass.* SIAM J. Control Optimization 36(6): 1962–1968, 1998.

[CoZu95] S. Cox, E. Zuazua: *The rate at which energy decays in a string damped at one end.* Indiana Univ. Math. J. 44(2): 545–573, 1995.

[CuZw95] R.F. Curtain, H. Zwart: *An introduction to infinite-dimensional linear systems theory.* Texts in Applied Mathematics 21, Springer, New York 1995.

[DaSl73] C. M. Dafermos, M. Slemrod: *Asymptotic Behavior of Nonlinear Contraction Semigroups.* Journal of Functional Analysis 13: 97–106, 1973.

[Ei10] T. Eisner: *Stability of Operators and Operator Semigroups.* Operator Theory: Advances and Applications 209, Birkhaäuser, Basel 2010.

[En13] K.-J. Engel: *Generator property and stability for generalized difference operators.* J. Evol. Equ. 13: 311–334, 2013.

[EnNa00] K.-J. Engel, R. Nagel: *One-Parameter Semigroups for Linear Evolution Equations.* Graduate Texts in Mathematics 194, Springer, New York 2000.

[Ge78] L. Gearhart: *Spectral theory for contraction semigroups on Hilbert spaces.* Trans. Amer. Math. Soc. 236: 385–394, 1978.

[Go66] S. Goldberg: *Unbounded Linear Operators: Theory and Applications* McGraw-Hill Series in Higher Mathematics, McGraw-Hill, New York, 1966.

[Go02] G. Golo: *Interconnection Structures in Port-Based Modelling: Tools for Analysis and Simulation.* Ph.D. thesis, Universiteit Twente, 2002.

[GuHu04] F. Guo, F. Huang: *Boundary feedback stabilization of the undamped Euler-Bernoulli beam with both ends free.* SIAM J. Control Optim. 43(1): 341–356, 2004.

[GuWaYu05] B.-Z. Guo, J.-M. Wang, S.-P. Yung: *On the C_0-semigroup generation and exponential stability resulting from a shear force feedback on a rotating beam.* Systems Control Lett. 54: 557–574, 2005.

[GuLu72] K. Gustafson, G. Lumer: *Multiplicative perturbation of semigroup generators.* Pacific J. Math. 41: 731–742, 1972.

[HaBeWe99] S. M. Han, H. Benaroya, T. Wei: *Dynamics of transversely vibrating beams using four engeneering theories.* J. Sound and Vibration 225(5): 935–938, 1999.

[JaZw12] B. Jacob, H. Zwart: *Linear Port-Hamiltonian Systems on Infinite-dimensional Spaces.* Operator Theory: Advances and Applications 223, Birkhäuser, Springer Basel 2012.

[Ka66] T. Kato: *Perturbation theory for linear operators.* Die Grundlehren der mathematischen Wissenschaften 132, Springer, New York 1966.

[Le14] Y. Le Gorrec: *Exponential stabilization of boundary controlled port Hamiltonian systems with non-linear dynamic control: some preliminary results.* (Joint work with: H. Zwart, H. Ramirez.) Talk at the workshop Port-Hamiltonian Systems: Approximations, Theory and Practice at the Lorentz Center, Leiden 2014.

[LeZwMa05] Y. Le Gorrec, H. Zwart, B. Maschke: *Dirac structures and boundary control systems associated with skew-symmetric differential operators.* SIAM J. Control Optim. 44 (5): 1864–1892, 2005.

[FeShZh98] D. Feng, D. Shi, W. Zhang: *Boundary feedback stabilization of Timoshenko beam with boundary dissipation.* Science in China (Ser. A) 41(5): 483–490, 1998.

[LiMa88] W. Littman, L. Markus: *Stabilization of a hybrid system of elasticity by feedback boundary damping.* Ann. Math. Pura Appl. 152: 281–330, 1988.

[LiHuCh89] K.S. Liu, F.L. Huang, G. Chen: *Exponential stability analysis of a long chain of coupled vibrating strings with dissipative linkage.* SIAM J. Appl. Math. 49 (6): 1694–1707, 1989.

[Li00] K. Liu, Z. Liu: *Boundary stabilization of a nonhomogeneous beam with rotatory inertia at the tip.* J. Comp. Appl. Math. 114: 1–10, 2000.

[LuGuMo99] Z.-H. Luo, B.-Z. Guo, O. Morgul: *Stability and Stabilization of Infinite Dimensional Systems with Applications.* Communications and Control Engeneering, Springer, London 1999.

[LyPh88] Y.I. Lyubich, V.Q. Phong: *Asymptotic stability of linear differential equations in Banach spaces.* Studia Math. 88(1): 37–42, 1988.

[MeSc12] M. Meyries, R. Schnaubelt: *Maximal regularity with temporal weights for parabolic problems with inhomogeneous boundary conditions.* Math. Nachr. 285 (8-9): 1032–1051, 2012.

[MeVe12] M. Meyries, M. Veraar: *Sharp embedding results for spaces of smooth functions with power weights.* Studia Math. 208 (3): 257–293, 2012.

[Mi92] I. Miyadera: *Nonlinear Semigroups.* Translations of Mathematical Monographs 109, American Mathematical Society, Providence (RI) 1992.

[MoRaCo94] Ö. Morgül, B.P. Rao, F. Conrad: *On the Stabilization of a Cable with a Tip Mass.* IEEE Trans. Automatic Control 39(10): 2140–2145, 1994.

[Pe89] G.K. Pedersen: *Analysis Now.* Graduate Texts in Mathematics 118, Springer, New York 1989.

[Pr84] J. Prüss: *On the spectrum of C_0-semigroups.* Trans. Amer. Math. Soc. 284(2): 847–857, 1984.

[Pr02] J. Prüss: *Maximal regularity for Evolution Equations in L_p-spaces.* Lecture Notes, Summer School Positivity and Semigroups, Monopoli 2002.

[RaZwLe13] H. Ramirez, H. Zwart, Y. Le Gorrec: *Exponential stability of boundary controlled port Hamiltonian systems with dynamic feedback.* IFAC Workshop on Control of Sys. Modeled by Part. Diff. Equ. (CPDE), 2013.

[RaTa74] J. Rauch, M. Taylor: *Exponential decay of solutions to hyperbolic equations in bounded domains.* Indiana Univ. Math. J. 24: 79–86, 1974.

[Sh97] R.E. Showalter: *Monotone operators in Banach space and nonlinear partial differential equations.* Mathematical Surveys and Monographs 49, American Mathematical Society, Providence, 1997.

[Tr83] H. Triebel: *Theory of Function Spaces* Monographs in Mathematics 78, Birkhäuser, Basel, 1983.

[Tr14] S. Trostorff: *A characterization of boundary conditions yielding maximal monotone operators.* J. Funct. Anal. 267 (8): 2787–2822, 2014.

[Tr68] C. Truesdell: *Essays in the history of mechanics.* Springer-Verlag New York Inc., New York 1968.

[TuWe09] M. Tuscnak, G. Weiss: *Observation and Control for Operator Semigroups.* Birkhäuser Advanced Texts: Basler Lehrbücher. Birkhäuser, Basel 2009.

[TuWe14] M. Tucsnak, G. Weiss: *Well-posed systems – the LTI case and beyond.* Automatica J. IFAC 50(7): 1757–1779, 2014.

[VaMa02] A.J. van der Schaft, B.M. Maschke: *Hamiltonian formulation of distributed parameter systems with boundary energy flow.* J. Geom. Phys. 42: 166-194, 2002.

[Va06] A.J. van der Schaft: *Port-Hamiltonian systems: an introductory survey.* International Congress of Mathematicans. Vol. III: 1339–1365, Eur. Math. Soc., Zürich 2006.

[Va96] J. van Neerven: *The Asymptotic Behaviour of Semigroups of Linear Operators.* Operator Theory: Advances and Applications 88, Birkhäuser, Basel 1996.

[Vi07] J.A. Villegas: *A port-Hamiltonian Approach to Distributed Parameter Systems.* Ph.D. thesis, Universiteit Twente, 2007.

[Vi+09] J.A. Villegas, H. Zwart, Y. Le Gorrec, B. Maschke: *Exponential stability of a class of boundary control systems.* IEEE Trans. Automat. Control 54(1): 142–147, 2009.

[We94] G. Weiss: *Transfer funtions of regular linear systems. Parrt I: Characterizations of regularity.* Trans. Am. Math. Soc. 342(2): 827–854, 1994.

[We06] D. Werner: *Einführung in die höhere Analysis.* Springer, Berlin Heidelberg 2006.

[We11] D. Werner: *Funktionalanalysis.* Springer, Berlin Heidelberg 2011.

[YaZoFe] Q. Yan, H. Zou, D. Feng: *Nonlinear Boundary Stabilization of Nonuniform Timoshenko Beam.* Acta Math. Appl. Sinicia Engl. 19(2): 239–246, 2003.

[Ze85] E. Zeidler: *Nonlinear Functional Analysis and its Applications III: Variational Methods and Optimization.* Springer, New York 1985.

[Zh07] C.-G. Zhang: *Boundary feedback stabilization of the undamped Timoshenko beam with both ends free.* J. Math. Anal. Appl. 326: 488–499, 2007.

[Zw+10] H. Zwart, Y. Le Gorrec, B. Maschke, J. Villegas: *Well-posedness and regularity of hyperbolic boundary control systems on a one-dimensional spatial domain.* ESAIM Contr. Optim. Calc. Var. 16: 1077–1093, 2010.

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