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Euler complexes and Cohomology of arithmetic groups for certain imaginary quadratics

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Introduction

Since the introduction of schemes, many facets of geometry and number theory were brought under a common framework in the language of schemes. For example, the class groups on the number-theoretic side and the divisor class group on the geometric side are essentially the same construction applied to the spectrum of the ring of integers in the number-theoretic case and the spectrum of regular functions in the geometric case. Through this connection, many ideas from both fields were cross-pollinated. An important example is the theory of L-functions, which has its origin in number theory, and on the other hand, we have K-theory, which has its roots in algebraic topology and geometry.

The main topic of our thesis is also an interplay between number theory and geometry. In this case, the number-theoretic side is the cohomology groups of the arithmetic groups. The geometric side is given in terms of motives, which can be thought of as the universal cohomology theory. Let us elaborate a bit on what both of these sides are.

Arithmetic groups are certain groups of number-theoretic importance. The simplest example of an arithmetic group is the modular group $SL_2(\mathbb{Z})$, which is ubiquitous in mathematics and is related to modular forms. By Eichler-Shimura (Chapter 8 of [58]), the modular forms are related to the group cohomology of $SL_2(\mathbb{Z})$ with $\mathbb{Z}[x, y]$ coefficients. The simplest example of this fact would be that the space of weight 2 modular forms is isomorphic to $H^1(SL_2(\mathbb{Z}), \mathbb{C})$. To study the cohomology of an arithmetic group Γ, one typically studies the cohomology of the associated locally symmetric space $Y(\Gamma)$ and its compactification $X(\Gamma)$, which usually has an orbifold structure. These spaces themselves contain number-theoretic information, e.g. they are modular curves in the case of $SL_2(\mathbb{Z})$. If Γ is torsion-free, then $X(\Gamma)$ is a compact manifold. The group cohomology of Γ in this case can be calculated by the singular homology of the pair $(X(\Gamma), \partial X(\Gamma))$ because,

$$H^i_{gp}(\Gamma, V) = H^i_{sing}(Y(\Gamma), \mathbf{V}) = H^{sing}_{d-i}(X(\Gamma), \partial X(\Gamma)) \tag{1}$$

where $\partial X(\Gamma) := X(\Gamma) - Y(\Gamma)$, d is the real dimension of $X(\Gamma)$, V is a Γ-module and $\mathbf{V}$ is its associated local system. We will denote the singular/cellular complex of $X(\Gamma)$ as $S_\bullet(\Gamma)$ and this is the complex that is crucial for us.

Motives are algebro-geometric objects, envisioned by Grothendieck, as the universal cohomology theory. They were primarily motivated by the fact that different cohomology theories for algebraic varieties had more complicated structure e.g. Hodge structure, Galois action etc which were compatible in a certain sense. Let k be a field. Let $\mathbf{Var}_k$ be the category of algebraic varieties over F i.e. finite type, separated schemes over k. Denote by DC the derived category of C. A cohomology theory with coefficients in category C is a functor $H : \mathbf{Var}_k \to DC$ satisfying the Bloch-Ogus axioms mentioned in Levine's chapter on mixed motives in [18]. These properties include Mayer-Vietoris sequence, Poincaré duality and Künneth formula. Let F be another field. If C is a F-linear category, then the cohomology groups are F-vector spaces. The category of mixed motives $\mathcal{MM}(k, F)$ would then be the category such that for any cohomology functor H there exists a $\tilde{H} : D\mathcal{MM}(k) \to DC$ making the following diagram commute

$$\begin{tikzcd} \mathbf{Var}_k \arrow[r, "H_{mot}"] \arrow[dr, "H"'] & D\mathcal{MM}(k, F) \arrow[d, "\tilde{H}"] \\ & DC \end{tikzcd} \tag{2}$$

The category $\mathcal{MM}(k, F)$ should be a Tannakian category which means that it is equivalent to the category of finite dimensional representations of some pro-affine algebraic group $G = G(k, F)$. The ring of functions of G, usually denoted by $O(G)$, is a non-negatively graded connected Hopf algebra, and it has an associated graded Lie coalgebra $L_\bullet(\mathcal{MM}(k, F))$ of indecomposable elements.

$$L_\bullet(\mathcal{MM}(k, F)) := \frac{O(G)_{>0}}{O(G)_{>0}^2} \tag{3}$$

where $O(G)_{>0}$ is the sum of all graded components of $O(G)$ except the zeroth component. The Lie coalgebra $L_\bullet(\mathcal{MM}(k, F))$ gives us a sequence,

$$L_\bullet(\mathcal{MM}(k, F)) \xrightarrow{\delta} \bigwedge^2 L_\bullet(\mathcal{MM}(k, F)) \to \bigwedge^3 L_\bullet(\mathcal{MM}(k, F)) \to \cdots \tag{4}$$

where $\delta(x) = \Delta(x) - 1 \otimes x - x \otimes 1$. Even though the category $\mathcal{MM}(k, F)$ is conjectural, we can work with a category of some 'simpler' motives that can be constructed in many cases. These are mixed Tate motives. Let us denote this subcategory of $\mathcal{MM}(k, F)$ as $\mathcal{MTM}(k, F)$. Levine proved [44] that $\mathcal{MTM}(k, F)$ exists when k is a number field. It is a subcategory of $\mathcal{MM}(k, F)$ if the latter exists. Deligne and Goncharov [12] gave a construction of $\mathcal{MM}(O_k, F)$, where O_k is the ring of integers of a number field k. From now, we will assume k is a number field. Let n be a natural number and let μ_n is the group of n-th roots of unity in k. The category $\mathcal{MTM}(k, F)$ has a pro-object $\pi_1(\mathbb{P}^1 - (\mu_n \cup \{0, \infty\}))$ mentioned in [12]. This is the motivic version of the pro-completion of the fundamental group $\pi_1(\mathbb{P}^1 - (\mu_n \cup \{0, \infty\}))$. The subquotients of $\pi_1(\mathbb{P}^1 - (\mu_n \cup \{0, \infty\}))$ generate a Tannakian subcategory of $\mathcal{MTM}(k, F)$. From this Tannakian subcategory, we can again construct a graded Lie coalgebra which we will denote by $\mathcal{L}_\bullet(k, F)$. Let $\wedge^* \mathcal{L}_\bullet(k, F)$ be the complex

$$\mathcal{L}_\bullet(k, F) \to \bigwedge^2 \mathcal{L}_\bullet(k, F) \to \bigwedge^3 \mathcal{L}_\bullet(k, F) \to \cdots \tag{5}$$

which is dual to Chevalley-Eilenberg complex for Lie algebras. This is the complex we are interested in.

Iterated integrals and $\wedge^* \mathcal{L}_\bullet$

In [25], Goncharov describes what $\wedge^* \mathcal{L}_\bullet(k, \mathbb{Q})$ should look like. The Lie coalgebra $\mathcal{L}_\bullet(k, \mathbb{Q})$ is generated by symbols $I^{\mathcal{M}}(a_0; a_1, \cdots, a_l; a_{l+1})$ where $a_i \in \mu_n \cup \{0, \infty\}$, where $\mu_n \subset k$, subject to the relations (See section 2.1 of[25]);

1. *Unit*: $I^{\mathcal{M}}(a; b) = 1$ for any $a, b \in \mu_n \cup \{0, \infty\}$.

2. *Relation from shuffle product*: For $m, l \geq 0$ and $a, b \in \mu_n \cup \{0, \infty\}$, one has

$$\sum_{\sigma \in \Sigma_{l,m}} I^{\mathcal{M}}(a; \sigma(a_1), \cdots, \sigma(a_{l+m}); b) = 0$$

where $\Sigma_{l,m}$ is the set of permutations σ on $1, 2, \cdots, l+m$ such that $\sigma(n) > \sigma(m)$ if $1 \leq n, m \leq l$ or if $l < n, m \leq l+m$.

3. *Triviality*: For $l > 0$, $I^{\mathcal{M}}(a; a_1, \cdots, a_l; a) = 0$.

4. *Cobracket*: $\delta(I^{\mathcal{M}}(a_0; a_1, \cdots, a_l; a_{l+1}))$ is defined as

$$\delta(I^{\mathcal{M}}(a_0; a_1, \cdots; a_{l+1})) \sum_{0 \leq i < k \leq l+1} I^{\mathcal{M}}(a_0; a_1, \cdots, a_i, a_k, \cdots; a_{l+1}) \wedge I^{\mathcal{M}}(a_i; a_{i+1}, \cdots, a_{k-1}; a_k)$$

The Lie coalgebra has a grading called **weight grading** with $I^{\mathcal{M}}(a_0; a_1, \cdots, a_l; a_{l+1})$ being a homogeneous element of weight l. These elements are motivic versions of multiple iterated integrals

$$I(a_0; a_1, \cdots, a_k; a_{k+1}) = \int_{\Delta_\gamma} \frac{dt_1}{t_1 - a_1} \wedge \frac{dt_2}{t_2 - a_2} \wedge \cdots \wedge \frac{dt_k}{t_k - a_k} \tag{6}$$

where γ is a path in $\mathbb{P}^1(\mathbb{C}) - \{a_1, \cdots, a_k\}$ starting from a_0 and ending at a_{k+1} and Δ_γ is the region containing points of the form $\{ (\gamma(t_1), \gamma(t_2), \cdots, \gamma(t_k)) \mid 0 \leq t_1 \leq t_2 \leq \cdots \leq t_k \leq 1\}$.

In weight one, the iterated integrals are generated by $I^{\mathcal{M}}(0; z; 1)$, where $z \in \mu_n$. There is a natural map $\mathcal{L}_1(F, \mathbb{Q}) \to K_1(F) \otimes \mathbb{Q} \cong F^\times \otimes \mathbb{Q}$. The image of $I^{\mathcal{M}}(0, z, 1)$ is the cyclotomic unit $1 - z$ if z is a primitive n-th root of unity. The group of cyclotomic units is a finite index subgroup in $O_F^\times$.

In weight two, we have two important types of elements. Let $z, z_1, z_2 \in \mu_n$ then we have the motivic dilogarithm defined as $Li_2^{\mathcal{M}}(z) := I^{\mathcal{M}}(0; 0, z; 1)$ and the double logarithm defined as $Li_{1,1}^{\mathcal{M}}(z_1, z_2) := I^{\mathcal{M}}(0; (z_1 z_2)^{-1}, (z_2)^{-1}; 1)$ called the motivic double logarithm. As integrals, dilogarithm and double logarithm are

$$Li_2(z) = \int_0^z \frac{1}{t} \int_0^t \frac{1}{1-u} du dt,$$
$$Li_{1,1}(z_1, z_2) = \int_0^1 \int_0^t \frac{du}{u - z_2^{-1}} \frac{dt}{t - (z_1 z_2)^{-1}}$$

Cyclotomic Fields and $\Gamma_1(N)$

Let N be a natural number. Let Γ be the group $\Gamma_1(N) \subset GL_2(\mathbb{Z})$, defined as the stabilizer of $(0, 1) \in (\mathbb{Z}/N)^2$. The group $\Gamma_1(N)$ acts on the symmetric space of $GL_2(\mathbb{Z})$, which is $\mathbb{C} - \mathbb{R}$. If we include the cusps $\mathbb{P}^1(\mathbb{Q})$, then the space $\overline{\mathbb{C} - \mathbb{R}} := (\mathbb{C} - \mathbb{R}) \cup \mathbb{P}^1(\mathbb{Q})$ has a $GL_2(\mathbb{Z})$-equivariant triangulation. The action of $GL_2(\mathbb{Z})$ on these triangles is transitive by reduction theory and they are in the orbit of the triangle with vertices $0, 1$ and ∞. The cellular complex for $X(\Gamma)$ would then be a complex where $S_i(\Gamma)$ would be the formal vector space generated by the i-dimensional cells of $X(\Gamma)$.

Let $F = \mathbb{Q}(\zeta_N)$ where ζ_N is a primitive N-th root of unity. For the sake of brevity, we will write $\mathcal{L}_\bullet(\mathbb{Q}(\zeta_N), \mathbb{Q})$ as $\mathcal{L}_\bullet^N$ and the weight two part of $\wedge^* \mathcal{L}_\bullet^N$ is

$$\mathcal{L}_2^N \xrightarrow{\delta} \wedge^2 \mathcal{L}_1^N \tag{7}$$

In [22], Goncharov constructed (depending on the choice of the primitive N-root ζ_N) a map $\theta^\bullet$ of complexes

$$\begin{tikzcd} S_2(\Gamma) \arrow[r] \arrow[d, "\theta^2"] & S_1(\Gamma) \arrow[d, "\theta^1"] \\ \mathcal{L}_2^N \arrow[r] & \wedge^2 \mathcal{L}_1^N \end{tikzcd} \tag{8}$$

By definition $S_2(\Gamma) = \mathbb{Q} \otimes_\Gamma \mathbb{Q}[GL_2(\mathbb{Z})] \otimes_{S_3} \chi_{sgn}$, where S_3 is the symmetric group on 3 letters stabilizing the triangle with vertices $0, 1$ and ∞ and χ_{sgn} is its sign representation. The map θ^2 is defined as

$$\theta^2\left(\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)\right) = \sum_{\sigma \in S_3} sgn(\sigma) Li^{\mathcal{M}}_{1,1}(\zeta_N^{a_{\sigma(1)}}, \zeta_N^{a_{\sigma(2)}})$$

where $a_1 = c$, $a_2 = d$,$a_3 = -c-d$ and

$$\theta^2\left(\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)\right) = \theta^2\left(1 \otimes \begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes 1\right)$$

The map is well-defined. The map θ^1 is defined as

$$\theta^1 \begin{pmatrix} a & b \\ c & d \end{pmatrix} = (1 - \zeta_N^c) \wedge (1 - \zeta_N^d)$$

By Suslin's theorem [63], the kernel of the bottom map is isomorphic to $K_3(F) \otimes \mathbb{Q} = K_3(O_F) \otimes \mathbb{Q}$. The image of $\theta^2_{E,z}$ is therefore a version of the cyclotomic elements in $K_3(O_F) \otimes \mathbb{Q}$ similar to the way the cyclotomic units were for $K_1(O_F) \otimes \mathbb{Q}$. In this sense, the higher cyclotomic theory is related to the cohomology of Γ. This crucial bit motivates the following definition.

Definition. *Denote by $C_2(N)$ the image of the map θ^2 and by $C_1(N)$ the subgroup of $O_F^\times$ generated by the cyclotomic units. The Euler complex is defined to be the subcomplex of the complex 7*

$$C_2(N) \to \wedge^2 C_1(N) \tag{9}$$

Proposition 1. *[22] Let p be prime. The quotient of $\mathcal{L}_2^p$ by $C_2(p)$ is isomorphic to $H^1_{cusp}(\Gamma_1(p), \mathbb{Q})$, where $H^\bullet_{cusp}$ denotes the cuspidal cohomology.*

Similar to how the group of cyclotomic units $C_1(p)$ is a subspace of $\mathcal{L}_1^p = O_F^\times$, $C_2(p)$ sits inside $\mathcal{L}_2^p$. This indicates that $C_2(p)$ plays the analogous role of cyclotomic units in $\mathcal{L}_2^p$. In light of this fact, Proposition 1 can be seen as a higher version of the statement that $C_1(p)$ is finite index subgroup of $\mathcal{L}_1^p$.

The maps of cohomology on the right of the commutative diagram 8 are also important if, instead of rational coefficients, we take $\mathbb{Z}[\frac{1}{N}]$ coefficients. These are the Eisenstein cocycles mentioned in [57]. In [27], [57] and [9] a similar map of complexes is constructed where $\mathbb{Q}(\zeta_N)$ is replaced by the field of fractions of $X_1(N)$, $\mathbb{Q}(X_1(N))$. In that case, the cohomology on the right is the Milnor K_2 of the field $\mathbb{Q}(X_1(N))$ and the image of $C_2(N)$ in $\wedge^2 C_1(N)$ is the set of the Beilinson-Kato elements, as mentioned in loc. cit. The compatibility of $\theta^\bullet$ for different N translates to the fact that Beilinson-Kato elements make an Euler system. If one thinks of cyclotomic fields as Abelian extensions of $\mathbb{Q}$, then this theory is essentially a theory over $\mathbb{Q}$ i.e. the connection is between motivic aspects of Abelian extensions of $\mathbb{Q}$ and arithmetic groups contained in $GL_2(\mathbb{Z})$, where we view $\mathbb{Z}$ as the ring of integers of $\mathbb{Q}$.

Generalization

There are two ways to generalize the $\theta^\bullet$ map. One can either take weights higher than 2. This is the subject of study in [24]. The other option would be to replace $\mathbb{Q}$ by some other number field. As seen above, we need a theory of Abelian extension of the number field in order to generalize. This is true for imaginary quadratic fields, for which we have the theory of complex multiplication. Let K be an imaginary quadratic field, $\mathcal{N}$ be an ideal of $\mathcal{O}_K$ and E be an elliptic curve with $End(E) \cong \mathcal{O}_K$. We denote by $E[\mathcal{N}]$ the group of $\mathcal{N}$-torsion point of E i.e. those elements of E that are annihilated by all elements of $\mathcal{N}$. Let $K_{\mathcal{N}}$ be the extension of K by adjoining coordinates of all $\mathcal{N}$-torsion points to K. The table below shows what the analogous replacement should be to get a theory over an imaginary quadratic field.

$\mathbb{Q}$	K
$\mathbb{G}_m = \mathbb{P}^1 - \{0,\infty\}$	E
μ_N	$E[\mathcal{N}]$
$\mathbb{Q}(\mu_N)$	$K_{\mathcal{N}}$
Cyclotomic units	Elliptic units
$GL_2(\mathbb{Z})$	$GL_2(\mathcal{O}_K)$
$\Gamma_1(N)$	$\Gamma_1(\mathcal{N})$
Triangulation	Bianchi Tessellation

where $\Gamma_1(\mathcal{N})$ is the stabilzer of $(0,1) \in (\mathcal{O}_K/\mathcal{N})$.

For the case $K = \mathbb{Q}(i), \mathbb{Q}(\zeta_3)$, these maps $\theta^\bullet_{E,z}$ were defined in [27] and the higher weight versions for these two fields were defined in [46]. The proof mentioned in [27] cannot be used for any other imaginary quadratic field K because it relies on the fact that every 2 cell is in the $GL_2(\mathcal{O}_K)$-orbit of the ideal triangle with vertices $0, 1, \infty$. Our main goal is to generalize this result to imaginary quadratic fields for which the rings of integers are Euclidean.

Main result

Let $C_i(\mathcal{P})$ be the analog of $C_i(p)$ above and let $\widehat{C_1(\mathcal{P})} := C_1(\mathcal{P}) \oplus \mathbb{Q}$. Similarly, let $\mathcal{L}_2(\mathcal{P})$ be the analog of $\mathcal{L}_2^p$. We will assume the very important assumption, which is 3.2.38.

Assumption 3.2.38. *Let F be a function field of a projective variety. We will assume that $\mathcal{B}_2(F)$ is isomorphic to $\mathcal{L}_2(F) \otimes \mathbb{Q}$, where $\mathcal{B}_2(F)$ is the scissor congruence group defined in Definition 3.2.25.*

The main result of this thesis is Theorem 4.8.10.

Theorem 4.8.10. *Let $K = \mathbb{Q}(\sqrt{-d})$ for $d = 1, 2, 3, 7, 11$ and let $\mathcal{O}_K$ be the rings of integers of K. Let $\mathcal{P}$ be an prime ideal of $\mathcal{O}_K$. Let E be an elliptic curve with $End(E) \cong \mathcal{O}_K$ and z be a $\mathcal{P}$-torsion element in E. Then, assuming the statement 3.2.38, we have the following commutative diagram,*

$$\begin{tikzcd}
S_3(X(\Gamma_1(\mathcal{P}))) \arrow[r, "\partial"] \arrow[d] & S_2(X(\Gamma_1(\mathcal{P}))) \arrow[r, "\partial"] \arrow[d, "\theta^2_{E,z}"'] & S_1(X(\Gamma_1(\mathcal{P}))) \arrow[r, "\partial"] \arrow[d, "\theta^1_{E,z}"] & S_0(X(\Gamma_1(\mathcal{P}))) \arrow[d, "\theta^0_{E,z}"] \\
0 \arrow[r] & C_2(\mathcal{P}) \arrow[r, "\delta"] & \bigwedge^2 \hat{C}_1(\mathcal{P}) \arrow[r, "\delta"] & C_1(\mathcal{P}) \oplus C_1(\mathcal{P})
\end{tikzcd}$$

such that $\theta^1_{E,z}$ is induced from the inclusion of the elliptic units into $K^\times_{\mathcal{P}}$ and $\theta^1_{E,z}$ and $\theta^0_{E,z}$ are isomorphisms while $\theta^2_{E,z}$ is surjective.

The map of complexes $\theta^{\bullet}_{E,z}$ does depend on the choice of $\mathcal{P}$-torsion point z. Lemma 4.8.11 details how the map $\theta^{\bullet}_{E,z}$ changes when a different $\mathcal{P}$-torsion point z' is selected and how it relates to Galois action.

Lemma 4.8.11. *The map $\theta^{\bullet}_{E,z}$ intertwines the action of $\mathcal{D}_{\mathcal{N}}$ on $S_{\bullet}(X_1(\mathcal{N}))$ with the action of $Gal(K_{\mathcal{N}}/K)$ on $C_{\bullet}(\mathcal{N})$.*

where $\mathcal{D}_{\mathcal{N}}$ is the normalizer of $\Gamma_1(\mathcal{N}))$ in $GL_2(O_K)$ modulo $\Gamma_1(\mathcal{N})$. If we take $d = 1, 3$ and we truncate the sequence in degree $2, 1, 0$, we get Goncharov's result Theorem 5.7 in [27] as proved in Section 4.9. The fact that the left box commutes is Conjecture 5.13 in loc.cit mentioned below.

Conjecture. *For a* 3*-cell e_3, $\theta^{2}_{E,z}(\partial(e_3)) = 0$.*

The obstacle to extending the result in [27] to other imaginary quadratic fields lies in the fact that 2-cells in both of these cases are triangles of a certain type and the map $\theta^{2}_{E,z}$ only describes the contribution of these triangles. This does not occur for other imaginary quadratics.

The general strategy is to first give the symmetric space of $\Gamma_1(\mathcal{N})$ an algebraic interpretation using results from [56] in terms of Neron-Severi group of $E \times E$. Under this interpretation, the cusps correspond to subelliptic curves of $E{\times}E$. The 2-cells in the Bianchi tessellation are further broken down into something which we call Bianchi triples. These are ordered triples of distinct cusps, and therefore distinct subelliptic curves (E_1, E_2, E_3), subject to certain conditions. Using the triple (E_1, E_2, E_3) and an $\mathcal{N}$-torsion point u of $E \times E$, we construct a motive $N_2^{x,u}$ where x is in an open set of $E \times E$ in Section 4.5. Restricting to the generic point of $E \times E$, $N_2^{x,u}$ gives an element of $\mathcal{B}_2(K_{\mathcal{N}}(E \times E))$ where $K_{\mathcal{N}}(E \times E)$ is the field of rational functions on $(E \times E) \times K_{\mathcal{N}}$. Using specializations of $N_2^{x,u}$, we construct the map $\theta_{E,z}$ in Section 4.7. Here, we require our assumption 3.2.38, because the motive may not exist for any x on the nose. For a 2-cell, adding up all the contribution of the Bianchi triple contained in the 2-cell gives us the required element. In the case $d = 1, 2, 3, 7, 11$, we can explicitly calculate the Bianchi triples and prove the aforementioned theorem.

The theorem also implies the following proposition, analogous to Proposition 1, by using the same argument as the one in [27].

Proposition. *Let $\mathcal{P}$ be a prime ideal of O_K where K is one of the imaginary quadratics $\mathbb{Q}(\sqrt{-d})$ for $d = 1, 2, 3, 7, 11$. Then*

$$\frac{\mathcal{L}_2(\mathcal{P})}{K_3(K_{\mathcal{P}}) \otimes \mathbb{Q} + C_2(\mathcal{P})} \cong H^2_{cusp}(\Gamma_1(\mathcal{P}), \mathbb{Q}) \tag{10}$$

where $K_3(K_{\mathcal{P}})$ is the third algebraic K-group of the abelian extension $K_{\mathcal{P}}$ of K.

Organization

In the first chapter, we recall basic theory about Abelian varieties, with a view towards proving the equivalence of Neron-Severi group $NS(X)$ of an Abelian variety X with endomorphisms $End(X)$. This is the content of Theorem 1.2.43. In the case of $E \times E$, Theorem 1.2.47 and the discussion after Definition 1.2.25 sets up an equivalence between the ample cone in $NS(E \times E)$ and the space of positive-definite Hermitian matrices. This chapter also reviews elliptic units in section 1.2.2.

The second chapter reviews the construction of the symmetric space of $GL_2(\mathcal{O}_K)$ using symmetric/Hermitian forms, which coincides with the ample cone of $NS(E \times E)$. The chapter also gives an overview of Bianchi tessellation, which helps us to construct the modular complex used to compute the cohomology of the locally symmetric space and hence $\Gamma_1(\mathcal{N})$.

The third chapter reviews the construction of the category of mixed motives following Voevodsky in the first section along with some properties of mixed motives, especially the Gysin spectral sequence. In the second section, we review what a mixed Tate category is and show that its Hopf algebra can be contructed via framed motives (Theorem 3.2.21), which are a formal version of periods. We also give an overview of Bloch-Suslin complex and how dilogarithm relates to it as it is the motivic part of the story. The end of the chapter reviews Goncharov's construction of the maps $\theta^{\bullet}_{E,z}$ in [27] and we discuss some limitations that prevents the same proof to work over imaginary quadratics other than $\mathbb{Q}(i)$ and $\mathbb{Q}(\sqrt{-3})$.

The last chapter is the main result of the thesis. The beginning of the chapter relates to how cusps correspond to subelliptic curves of $E \times E$. Sections 2-6 of this chapter is dedicated to the construction of the generalization of $\theta_{E,z}$ mentioned above. The construction is a two-step process. First, we break our 2-cells into Bianchi triples and then, for a choice of $\mathcal{N}$-torsion point z of E, we map these Bianchi triples to framed motives. The advantage of constructing Bianchi triples is that it is a free $PGL_2(\mathcal{O}_K)$ module and therefore, we do not have to worry about satisfying any relation except scaling by a unit $\varepsilon \in \mathcal{O}_K^{\times}$.

Chapter 1

Abelian varieties

An **Abelian variety** A over a field k is a projective algebraic variety over k i.e. an integral scheme of finite type over k which can be embedded into $\mathbb{P}_k^N$ for some natural number N, such that we have maps;

$$m : A \times A \to A, \qquad inv : A \to A \qquad e : \mathrm{Spec}(F) \to A \tag{1.1}$$

satisfying the usual commutative diagrams for groups i.e. it satisfies the following commutative diagrams,

- Associativity

$$\begin{tikzcd} A \times A \times A \arrow[r, "m\times id"] \arrow[d, "id\times m"'] & A \times A \arrow[d, "m"] \\ A \times A \arrow[r, "m"'] & A \end{tikzcd}$$

- Identity

$$\begin{tikzcd} A \arrow[r, "(id{,}e)"] \arrow[d, "(e{,}id)"'] \arrow[dr, "id"] & A \times A \arrow[d, "m"] \\ A \times A \arrow[r, "m"'] & A \end{tikzcd}$$

- Inverse

$$\begin{tikzcd} A \arrow[r, "(id{,}inv)"] \arrow[d, "(inv{,}id)"'] \arrow[dr] & A \times A \arrow[d, "m"] \\ A \times A \arrow[r, "m"'] & A \end{tikzcd}$$

  where the diagonal maps is the composition $A \to \mathrm{Spec}(F) \to A$, where the first map is the structure map and the second map is e.

It is well known by Corollary 1.4 of [49] that Abelian varieties are commutative. Abelian Varieties of dimension 1 are called **elliptic curves**.

Next, we will describe what a morphism between Abelian varieties looks like.

Definition 1.0.1. *A regular map $\phi : A \to B$ from an Abelian variety A to another Abelian variety B is called a **morphism** if it is also a group homomorphism i.e. the following diagram* *commutes.*

$$\begin{tikzcd} A \times A \arrow[r, "m_A"] \arrow[d, "(\phi,\phi)"'] \arrow[dr] & A \arrow[d, "\phi"] \\ B \times B \arrow[r, "m_B"'] & B \end{tikzcd}$$

There is also a very special type of morphism between Abelian varieties.

Definition 1.0.2. *A morphism ϕ is an* ***isogeny*** *if ϕ is surjective and it has finite kernel. The cardinality of the kernel is called the* ***degree*** *of the isogeny.*

Abelian varieties have many interesting properties and many of them result from the rigidity lemma.

Lemma 1.0.3. *(Rigidity Lemma) Let X be a proper variety and let Y and Z be varieties. If $f : X \times Y \to Z$ is a regular map such that,*

$$f(\{x_0\} \times Y) = \{z_0\} = f(X \times \{y_0\})$$

for some k-rational points x_0, y_0, z_0 of X, Y, Z respectively, then image of f is z_0.

Proof. Let Z_0 be an affine open neighbourhood of z_0. Then, since the projection $pr_2 : X \times Y \to Y$ is closed due to properness of X, the set $U := pr_2(f^{-1}(Z \backslash Z_0))$ is closed and a closed point y is not in U if and only if $f(X \times \{y\}) \subset Z_0$. In particular, $y_0 \notin U$ so that $y_0 \in Y \backslash U$, which is a dense open subset because Y is irreducible. The restriction of f to $X \times \{y\}$, where $y \notin U$, is map from a proper variety X to an affine variety Z_0. This map is thus a constant map with image being $\{z_0\}$. Thus, $f^{-1}(z_0)$ contains $X \times (Y \backslash U)$ and therefore also its closure $X \times Y$. □

The lemma has the following corollaries.

Corollary 1.0.4. *Any Abelian variety A is commutative.*

Proof. Let $X = Y = Z = A$ and $x_0 = z_0 = y_0 = 0$, where 0 is the identity element of A and let f be the commutator map. The rigidity lemma implies that f is the constant map sending all commutators to 0. This is exactly the definition of commutativity. □

The second corollary helps us to study homomorphisms between any two Abelian varieties.

Corollary 1.0.5. *Let $f : A \to B$ be a regular map between Abelian varieties A and B. If f sends identity to identity then f is also a morphism of Abelian varieties.*

Proof. Let $h : A \to B$. Let $X = Y = A$ and $Z = B$, $x_0 = y_0 = 0_A$, $z_0 = 0_B$ and $f = m_B \circ (h \times h) - h \circ m_A$. f satisfies the equations $f(0_A \times A) = f(A \times 0_A) = 0_B$, so we can apply the rigidity lemma to get $f = 0$. Hence, we get the corollary. □

Abelian varieties over complex numbers can be expressed analytically as complex tori.

1.1 Complex Tori

A **lattice** $\Lambda \subset \mathbb{C}^g$ is a rank $2g$ free Abelian subgroup of $\mathbb{C}^g$ such that the $\mathbb{R}-$linear span of Λ in $\mathbb{C}^g$ is $\mathbb{C}^g$ itself. A **complex torus** is the quotient $\mathbb{C}^g/\Lambda$.

A complex torus inherits the group operation from $\mathbb{C}^g$, which is also complex analytic, thus making the complex torus a complex compact Lie group. Theorem 2.8 of [49] gives the equivalence of Abelian varieties over complex numbers and complex tori with polarization (Theorem 1.2.28). In particular, for $g = 1$, we have a correspondence between elliptic curves and complex tori in $\mathbb{C}$.

Using complex tori, it is much easier to see what the maps between the Abelian varieties are.

Theorem 1.1.1. *Let A, B be two Abelian varieties over $\mathbb{C}$ such that $A = \mathbb{C}^n/\Lambda$ and $B = \mathbb{C}^m/\Lambda'$ and ϕ is a regular map from A to B. Then $\phi(x + \Lambda) = Mx + b + \Lambda'$ where $b \in \mathbb{C}^m$ and M is a $\mathbb{C}$-linear map from $\mathbb{C}^n$ to $\mathbb{C}^m$ such that $M\Lambda \subset \Lambda'$. If ϕ is a morphism, then $b = 0$.*

This theorem makes it easier to find the endomorphism ring of an Abelian variety. Let us restrict it to the case of elliptic curves. In this case, M is just a complex number α such that $\alpha\Lambda \subset \Lambda$. If $\alpha \neq 0$ satisfies $\alpha\Lambda \subset \Lambda$, then α can be viewed as a 2×2 matrix with integral entries and so it also satisfies the characteristic polynomial of the matrix. The argument implies that α is an element of the ring of integers of some quadratic extension K of $\mathbb{Q}$. Moreover, K is an imaginary extension because Λ is a lattice. This shows that the endomorphism ring $End(E)$ of an elliptic curve E is a subring of O_K, the ring of integers of K. These elliptic curves have a special name.

Definition 1.1.2. *An elliptic curve E over a field L is said to have **complex multiplication** or CM if $\mathbb{Z}$ is a proper subset of $End_L(E)$, where $n \in \mathbb{Z} \subset End(E)$ corresponds to multiplication-by-n map.*

If E is an elliptic curve defined over $\mathbb{Q}$, it could happen that the endomorphism ring is $\mathbb{Z}$ but the geometric endomorphism ring i.e. the ring consisting of endomorphisms of $E_{\overline{\mathbb{Q}}}$ is larger than $\mathbb{Z}$. In this case, we say that E has **potential CM**. If E has potential CM then all endomorphisms are defined over K where K is the fraction field of $End_{\overline{\mathbb{Q}}}(E)$ by Theorem 2.2(b) of [59]. Since we will always be considering elliptic curve over such fields, potential CM is just CM.

1.2 Line Bundles on Complex Tori

In this section, we will see what holomorphic line bundles over complex tori are and how do we express them in terms of quadratic forms and semi-characters which are easier to work with. Let X be a complex manifold of dimension n. Let O_X be the sheaf of holomorphic functions on X.

Definition 1.2.1. *A **holomorphic line bundle** $\mathcal{L}$ is an O_X-module which is locally free of rank 1 i.e. for every point $x \in X$ there exists an open neighborhood U of x such that $\mathcal{L}|_U \cong O_U$.*

The tensor product of two line bundles is also a line bundle and the dual of a line bundle is also a line bundle. We note that O_X is a line bundle. This leads us to the next definition,

Definition 1.2.2. *We define the **Picard group**, denoted by $Pic(X)$, is the set of isomorphism classes of line bundles on X.*

Lemma 1.2.3. *$Pic(X)$ is a Abelian group with the tensor product as the group operation.*

Proof. The line bundle $\mathcal{O}_X$ acts as the identity element of $Pic(X)$. Let $\mathcal{L}$ be a line bundle. By definition of dual line bundle $\mathcal{L}^\vee$ of $\mathcal{L}$, there exists an evaluation map,

$$\mathcal{L} \otimes \mathcal{L}^\vee \to \mathcal{O}_X$$

To show this map is an isomorphism of sheaves, we can see it locally. But $\mathcal{L}$ and its dual are locally free, so this is reduces to the case $\mathcal{L} = \mathcal{O}_X$ locally, for which the evaluation map is an isomorphism. This shows that $\mathcal{L}^\vee$ is the inverse of $\mathcal{L}$ under $\otimes$. Finally, the associativity follows from the associativity of $\otimes$ product of line bundles upto isomorphism. $\square$

We have another characterization of $Pic(X)$.

Theorem 1.2.4. *$Pic(X) \cong H^1(X, \mathcal{O}_X^\times)$ where $\mathcal{O}_X^\times$ is the sheaf of invertible functions over X with multiplication as the operation.*

Proof. (Sketch) Let $\mathcal{L}$ be a holomorphic line bundle. Then it has local trivialization $\{U_i\}_{i\in I}$ i.e. a cover such that $\alpha_i : \mathcal{L}|_{U_i} \xrightarrow{\sim} \mathcal{O}_{U_i}$. We can compare the two local trivialization α_i and α_j on $U_i \cap U_j$. This will give us an isomorphism

$$\alpha_{ij} : \mathcal{O}_{U_i \cap U_j} \xrightarrow{\sim} \mathcal{O}_{U_i \cap U_j}$$

where $\alpha_{ij} := \alpha_i|_{U_i \cap U_j} \circ \alpha_j|^{-1}_{U_i \cap U_j}$. So, $\alpha_{ij} \in \mathcal{O}^\times_{U_i \cap U_j}$ and restricting to $U_i \cap U_j$ we get $\alpha_{ij} \in \mathcal{O}_X^\times\left(U_i \cap U_j\right)$. Recall the Čech complex associated to a cover [35],

$$0 \to \prod_{i\in I} \mathcal{O}_X^\times(U_i) \to \prod_{i,j\in I} \mathcal{O}_X^\times(U_i \cap U_j) \to \prod_{i,j,k\in I} \mathcal{O}_X^\times(U_i \cap U_j \cap U_k) \to \cdots$$

The collection (α_{ij}) is an element of $\prod_{i,j\in I} \mathcal{O}_X^\times(U_i \cap U_j)$. Moreover, since $\mathcal{L}$ is a sheaf we have that (α_{ij}) is in the kernel of the last map and therefore it is a cocycle. The cocycles corresponding to isomorphic line bundles will differ by an element of the image of $\prod_{i\in I} \mathcal{O}_X^\times(U_i)$ and therefore, the line bundle $\mathcal{L}$ gives us an element of $H^1(X, \mathcal{O}_X^\times)$. The converse is possible due to (Zariski) descent of line bundles e.g. see exercise II.1.22 in [35]. $\square$

This characterization of $Pic(X)$ will give rise to a short exact sequence for $Pic(X)$. Consider the exponential exact sequence of sheaves,

$$0 \to \mathbb{Z} \to \mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_X^\times \to 0. \tag{1.2}$$

Applying the machinery of cohomology gives us the following long exact sequence

$$\cdots \to H^1(X, \mathbb{Z}) \to H^1(X, \mathcal{O}_X) \to H^1(X, \mathcal{O}_X^\times) \xrightarrow{c_1} H^2(X, \mathbb{Z}) \to H^2(X, \mathcal{O}_X) \to \cdots$$

The map c_1 is the **first Chern class** map. We have to emphasize at this point that this is analytic Picard group because we are considering holomorphic line bundles. We can also consider algebraic Picard group, where the line bundles are locally trivial Zariski sheaves. By Théorème 4.4 of Exposé XII of [31], the analytic Picard group and algebraic Picard group are isomorphic if X is a proper $\mathbb{C}$-variety.

Definition 1.2.5. *We call the kernel of c_1, the **Jacobian** of X. It is denoted by $Pic^0(X)$. The image of c_1 is called the **Neron-Severi** group of X and is denoted by $NS(X)$.*

So we have an exact sequence

$$0 \to Pic^0(X) \to Pic(X) \to NS(X) \to 0 \tag{1.3}$$

Remark 1.2.6. *All the constructions in this section until now, except for the exponential sequence, also have an algebraic counterpart and give us the same thing as holomorphic ones in the case of complex varieties. If X is a smooth curve then, algebraically, $Pic(X)$ can be expressed in terms of divisors of X and the algebraic Chern class map is the degree map $Pic(X) \to \mathbb{Z}$ sending the class of point $[x]$ to $1 \in \mathbb{Z}$. This is compatible with the analytic Chern class map mentioned above by Exercise 18.4.G of [65].*

Let us return to the case when X is a complex torus of dimension n with lattice $\Lambda \subset \mathbb{C}^n$. We will also need some more definition before proceeding.

Definition 1.2.7. *Let V be a complex vector space. A **Hermitian form** H on V is a map $H : V \times V \to \mathbb{C}$ such that it is $\mathbb{C}$- linear in the first argument and $H(v,w) = \overline{H(w,v)}$ for all $v, w \in V$.*

Definition 1.2.8. *A Hermitian form H is said to be a **Riemann form** for a lattice Λ if the imaginary part of $H(\Lambda \times \Lambda)$ is contained in $\mathbb{Z}$.*

We can further simplify $H^2(X,\mathbb{Z})$. Since X is a complex torus of dimension n with lattice Λ, we can easily find that $H_1(X,\mathbb{Z}) \cong \Lambda$ which is a free Abelian group. By the universal coefficient theorem, we get $H^1(X,\mathbb{Z}) \cong Hom(\Lambda,\mathbb{Z})$. Using the cup product, we have the map

$$\bigwedge^2 H^1(X,\mathbb{Z}) \to H^2(X,\mathbb{Z}) \tag{1.4}$$

Künneth formula, the fact that $H^q(S_1^m,\mathbb{Z})$ are torsion-free and the cup product is compatible with the Künneth formula, together imply that the map above is an isomorphism. See Lemma 1.3.1 of [42] for more details. Thus, we have an isomorphism $H^2(X,\mathbb{Z}) \cong \bigwedge^2 Hom(\Lambda,\mathbb{Z})$, but the latter expression is isomorphic to the group of alternating forms on $\Lambda \times \Lambda$ with integral values, which we will denote by $Alt^2(\Lambda,\mathbb{Z})$.

Lemma 1.2.9. *For a complex torus $X = \mathbb{C}^n/\Lambda$, $H^2(X,\mathbb{Z})$ is canonically isomorphic to $Alt^2(\Lambda,\mathbb{Z})$.*

Proof. See Corollary 1.3.2 of [42]. □

Lemma 1.2.10. *Let V be a vector space. Then there is a natural $1-1$ correspondence.*

$$\{\text{ Hermitian forms } H\} \leftrightarrow \{\text{ Real Alternating forms } E \text{ such that } E(v,w) = E(iv,iw)\}$$

where the right map is $H \to \mathrm{Im}(H)$ and the left map is $E(v,w) \to E(iv,w) + iE(v,w)$

Proof. If we have a Hermitian form H then let E be the imaginary part of H, then

$$E(v,w) = Im(H(v,w)) = Im(\overline{H(w,v)}) = -Im(H(w,v)) = -E(w,v)$$

So E is alternating. For the converse,

$$\overline{H(v,w)} = \overline{E(iv,w) + iE(v,w)} = E(iv,w) - iE(v,w) = E(iw,v) + iE(w,v) = H(w,v)$$ □

The type of alternating forms mentioned in Lemma 1.2.10 would be precisely the image of the Chern class map after tensoring by $\mathbb{R}$. This is what we will see below.

Definition 1.2.11. *Let G be a group and M be a G-module.*

1. *A 1-**cocycle** f on G valued in M is a function $f : G \to M$ which satisfies the following equation:*
$$f(gh) = g \cdot f(h) + f(g) \qquad \forall g, h \in G \tag{1.5}$$
The group of all such 1-cocycles is denoted by $Z^1(G, M)$.

2. *A 1-**coboundary** $f : G \to M$ is a function of the form $f(g) = g(m) - m$ for some $m \in M$. The group of all 1-coboundaries is denoted by $B^1(G, M)$.*

3. *The **first group cohomology** $H^1(G, M)$ is the quotient of $Z^1(G, M)$ by $B^1(G, M)$.*

4. *A 2-**cocycle** f on G valued in M is a function $f : G \times G \to M$ which satisfy the following equation,*
$$g(f(h, j)) - f(gh, j) + f(g, hj) - f(g, h) = 0 \qquad \forall j, g, h \in G$$
The group of all such 2-cocycles is denoted by $Z^2(G, M)$.

5. *A 2-**coboundary** $f : G \to M$ is a function of the form $f(g, h) = g(\beta(h)) - \beta(gh) + \beta(g)$ for some function $\beta : G \to M$. The group of all 2-coboundaries is denoted by $B^2(G, M)$.*

6. *The second group cohomology $H^2(G, M)$ is the quotient of $Z^2(G, M)$ by $B^2(G, M)$.*

The complex torus X has V as its universal cover i.e. the quotient map $\pi : V \to X$. Take a line bundle $\mathcal{L}$ on X. Pulling $\mathcal{L}$ back along π, we get the line bundle $\pi^*(\mathcal{L})$ on V. Now, $\pi^*(\mathcal{L})$ is isomorphic to the trivial line bundle because any line bundle on a contractible space is trivial(Corollary 4.8, [37]). So we can identify $\pi^*(\mathcal{L})$ with the trivial projection $V \times \mathbb{C} \to V$. The lattice acts linearly fiber-wise on $\pi^*(\mathcal{L})$ because π is a Λ-invariant map. Taking quotient by Λ group action would give us $\mathcal{L}$ back again. Let us make this idea more explicit. The elements of $V \times \mathbb{C}$ are of the form (x, z). Let $\lambda \in \Lambda$. The action of λ explicitly becomes
$$\lambda \cdot (x, z) = (x + \lambda, e_\lambda(x)z) \tag{1.6}$$
where $e_\lambda(x) \in \mathbb{C}^\times$ for all $x \in V$. The compatibility with group action gives us the following condition on $e_\lambda(x)$,
$$e_{\lambda_1+\lambda_2}(x) = e_{\lambda_1}(x + \lambda_2)e_{\lambda_2}(x) \tag{1.7}$$
for $\lambda_1, \lambda_2 \in \Lambda$. This is exactly a 1-cocycle on Λ with values in $\mathcal{O}_V^\times(V)$. The same process also works in reverse and therefore, we get the following theorem.

Theorem 1.2.12. *For a complex torus X with the lattice Λ in a complex vector space V, $Pic(X) \cong H^1(\Lambda, \mathcal{O}_V^\times(V))$ and this isomorphism is natural.*

Remark 1.2.13. *The terms $e_\lambda(x)$ are called **factors of automorphy**.*

We now have two different ways of representing line bundles. One question that arises is how to express the Chern class in the cocycle formulation. We need two lemmas before we can do that.

Lemma 1.2.14. *The groups $H^2(\Lambda, \mathbb{Z})$ is canonically isomorphic to $Alt^2(\Lambda, \mathbb{Z})$*

Proof. See [42], lemma 2.1.3. □

Lemma 1.2.15. *Let X be a complex torus with the lattice Λ in a complex vector space V. Let ϕ_1 be the inverse of the isomorphism in Theorem 1.2.12 and $\phi_2 : Alt^2(\Lambda, \mathbb{Z}) \cong H^2(X, \mathbb{Z})$ be the isomorphism mentioned in Lemma 1.2.9. Then we have the commutative diagram*

$$\begin{tikzcd} H^1(\Lambda, \mathcal{O}_V^\times(V)) \arrow[r, "\delta"] \arrow[d, "\phi_1"] & H^2(\Lambda, \mathbb{Z}) \cong Alt^2(\Lambda, \mathbb{Z}) \arrow[d, "\phi_2"] \\ Pic(X) \arrow[r, "c_1"] & H^2(X, \mathbb{Z}) \end{tikzcd}$$

where δ is the boundary map of group cohomology of Λ.

Proof. See [42], lemma 2.1.4. □

Unwinding the definitions in the theorem, we get an explicit formula for Chern class for any element in $H^1(\Lambda, \mathcal{O}_V^\times(V))$.

Theorem 1.2.16. *Let X be a complex torus with the lattice Λ in a complex vector space V and $\mathcal{L}$ be a line bundle on X. Let $f \in H^1(\Lambda, \mathcal{O}_V^\times(V))$ corresponding to the line bundle $\mathcal{L}$ by Theorem 1.2.12 and g be a holomorphic function on V such that $f = \exp(2\pi i g)$. Then the Chern class $c_1(\mathcal{L})$ in $Alt^2(\Lambda, \mathbb{Z}) \cong H^2(X, \mathbb{Z})$ is*

$$E_L(\lambda, \mu) := c_1(\mathcal{L})(\lambda, \mu) = g(\mu, v + \lambda) + g(\lambda, v) - g(\lambda, v + \mu) - g(\mu, v) \tag{1.8}$$

for all $\lambda, \mu \in \Lambda$ and $v \in V$.

Proof. See Section 2.1 of [42]. □

Using some Hodge theory (page 26-29,[42]), we can also prove the following theorem,

Theorem 1.2.17. *Let X be a complex torus with the lattice Λ in a complex vector space V. Let $E : V \times V \to \mathbb{R}$ be an alternating form. The following statements are equivalent:*

1. *E is the $\mathbb{R}$-linear extension of the Chern class of a complex line bundle $\mathcal{L}$ on X.*
2. *$E(\Lambda \times \Lambda) \subset \mathbb{Z}$ and $E(v, w) = E(iv, iw)$.*

Combining Theorem 1.2.17 with Lemma 1.2.10, we get the following theorem.

Theorem 1.2.18. *Let X be a complex torus with lattice $\Lambda \subset \mathbb{C}^n$. Then $NS(X) \otimes \mathbb{R}$ is isomorphic to the space of Riemann forms over $\mathbb{C}^n$ viewed as an $\mathbb{R}$-vector spaces.*

Now, we would like to move in the converse direction; given a Riemann form H, we would like to construct the line bundle up to isomorphism. Because $Pic^0(X)$ is not zero, we cannot do this construction uniquely. So, is there some additional data that we can use to construct the inverse? The answer is yes.

Definition 1.2.19. *Let X be a complex torus with the lattice Λ in a complex vector space V and H be a Riemann form for Λ. Let S^1 be the circle group $S^1 := \{z \in \mathbb{C} | \, |z| = 1\}$. A **semi-character for** H is a map $\chi : \Lambda \to S^1$ such that*

$$\chi(x + y) = \chi(x)\chi(y) \exp(\pi i Im(H(x, y)))$$

for all $x, y \in \Lambda$.

Using the pair (H,χ) where χ is a semi-character for H, we can construct a line bundle $\mathcal{L}(H,\chi)$. Define $a : \Lambda \times V \to \mathbb{C}^\times$ by,

$$a(\lambda, v) = \chi(\lambda)\exp\left(\pi H(v,\lambda) + \frac{\pi}{2}H(\lambda,\lambda)\right)$$

where $\lambda \in \Lambda$ and $v \in V$. Note that $a(\lambda, -)$ is a holomorphic invertible function on V so it is equivalent to a map $a : \Lambda \to \mathcal{O}_V(V)^\times$. We see that a is then a 1-cocycle because,

$$\begin{aligned}
a(\lambda+\mu, v) &= \chi(\lambda+\mu)\exp\left(\pi H(v,\lambda+\mu) + \frac{\pi}{2}H(\lambda+\mu,\lambda+\mu)\right)\\
&= \chi(\lambda)\chi(\mu)\exp\Big(\pi i\mathrm{Im}(H(\lambda,\mu))\Big)\exp\left(\pi H(v,\lambda+\mu) + \frac{\pi}{2}H(\lambda+\mu,\lambda+\mu)\right)\\
&= \chi(\lambda)\chi(\mu)\exp\left(\pi i\mathrm{Im}(H(\lambda,\mu)) + \pi H(v,\lambda+\mu) + \frac{\pi}{2}H(\lambda+\mu,\lambda+\mu)\right)
\end{aligned}$$

now we can simplify the exponent of exp,

$$\begin{aligned}
&i\mathrm{Im}(H(\lambda,\mu)) + H(v,\lambda+\mu) + \frac{1}{2}H(\lambda+\mu,\lambda+\mu)\\
&= i\mathrm{Im}(H(\lambda,\mu)) + H(v,\lambda) + H(v,\mu) + \frac{1}{2}H(\lambda,\lambda) + \frac{1}{2}H(\mu,\mu) + \mathrm{Re}(H(\lambda,\mu))\\
&= H(\lambda,\mu) + H(v,\lambda) + H(v,\mu) + \frac{1}{2}H(\lambda,\lambda) + \frac{1}{2}H(\mu,\mu)\\
&= \left(H(\lambda,\mu) + H(v,\lambda) + \frac{1}{2}H(\lambda,\lambda)\right) + \left(H(v,\mu) + \frac{1}{2}H(\mu,\mu)\right)\\
&= \left(2i\mathrm{Im}(H(\lambda,\mu)) + H(v+\mu,\lambda) + \frac{1}{2}H(\lambda,\lambda)\right) + \left(H(v,\mu) + \frac{1}{2}H(\mu,\mu)\right)
\end{aligned}$$

Putting back into the expression of $a(\lambda+\mu, v)$,we get,

$$a(\lambda+\mu, v) = a(\lambda, v+\mu)a(\mu, v)$$

which means a is 1-cocycle in $Z^1(\Lambda, \mathcal{O}_V(V)^\times)$, which correspond to a line bundle by theorem 1.2.12. The following theorem then provides us with the fact that all line bundles are of such form

Theorem 1.2.20. *(Appell-Humbert) Any line bundle $\mathcal{L}$ on a complex torus X is isomorphic to $\mathcal{L}(H,\chi)$ for some Riemann form H for Λ and χ a semi-character for H.*

Proof. See pages 21, 22 of [53] □

The Appell-Humbert theorem also gives a characterization of $Pic^0(X)$. Any line bundle in $Pic^0(X)$ would be of the form $\mathcal{L}(0,\chi)$. In this case, χ is a homomorphism from Λ to S^1. Therefore, we get

Proposition 1.2.21. *For a complex torus X, $Pic^0(X) \cong Hom(\Lambda, S^1)$.*

Proof. Take the Riemann form to be zero. The semi-characters are just homomorphisms from Λ to S^1 in this case. □

1.2.1 Dual Torus and $Pic^0(X)$

For a complex torus X, we can construct a dual torus $\widehat{X}$. Since X is a complex torus, it is the quotient of $\mathbb{C}^n$ by a lattice Λ. Let $V^\vee$ be the set of $\mathbb{C}$-antilinear maps from V to $\mathbb{C}$. The evaluation map gives a $\mathbb{R}$-linear pairing,

$$\langle , \rangle : V^\vee \times V \to \mathbb{R}, \langle l, v\rangle := \mathrm{Im}(l(v)) \tag{1.9}$$

This pairing is non-degenerate because if $v \in V$ such that for all $l \in V^\vee$, $\mathrm{Im}(l(v)) = 0$ then the real part of $l(v)$ is also zero by replacing l by il. This means $l(v) = 0$ for all $l \in V^\vee$ which further implies that v is zero. A similar argument implies that if $l \in V^\vee$ such that $\mathrm{Im}(l(v)) = 0$ for all $v \in V$, then l is zero. We also have a **dual lattice** $\widehat{\Lambda}$ in $V^\vee$ which is defined as,

$$\widehat{\Lambda} := \{\, l \in V^\vee \,|\, \langle l, v\rangle \in \mathbb{Z}\, \forall v \in \Lambda\}$$

The **dual torus** $\widehat{X}$ is defined as,

$$\widehat{X} := V^\vee/\widehat{\Lambda}$$

$\widehat{X}$ has the same complex dimension as X but need not be isomorphic to it.

We can also construct dual of maps; if f is a map of complex tori from X to Y then it can be easily checked that the naturality of duality of vector space and the definition of dual lattice give us a map $\widehat{f} : \widehat{Y} \to \widehat{X}$. Thus, duality is a contravariant functor on the category of Abelian varieties. We also see from definition of duality of vector spaces that

$$d : X \xrightarrow{\sim} \widehat{\widehat{X}}, \qquad d^{-1} \circ \widehat{\widehat{f}} \circ d = f$$

and both the isomorphisms d and d^{-1} are also natural. Using the dual torus, we can construct a map

$$V^\vee \to Hom(\Lambda, S^1)$$

by mapping $l \in V^\vee$ to $\exp\big(2\pi i\langle l, -\rangle\big)$ in $Hom(\Lambda, S^1)$. The kernel consists precisely of those elements for which $\langle l, \lambda\rangle \in \mathbb{Z}$ for all $\lambda \in \Lambda$. But this is precisely $\widehat{\Lambda}$ and therefore it factors through $\widehat{X}$ to give

$$\widehat{X} \to Hom(\Lambda, S^1) \tag{1.10}$$

Theorem 1.2.22. *The map* (1.10) *is an isomorphism.*

By the Appell-Humbert theorem, this means that $Pic^0(X)$ itself is a complex torus. We will now identify $Pic^0(X)$ with $\widehat{X}$.

Let $\mathcal{L}$ be a line bundle on X. Then, we can construct a map,

$$\phi_{\mathcal{L}} : X \to \widehat{X}, x \to t_x^*\mathcal{L} \otimes \mathcal{L}^{-1} \tag{1.11}$$

where t_x is the translation by x map. This map is a morphism of Abelian varieties by the theorem of the square (Theorem 5.5, [49]). The line bundle on the right has Chern class 0. To see this, use Appell-Humbert theorem and note that if $\mathcal{L} = \mathcal{L}(H, \chi)$ then $t_x^*\mathcal{L} = \mathcal{L}(H, \exp(2\pi i \mathrm{Im}(H(v, \cdot)))\chi)$, where v be a preimage of x under the covering map $V \to X$, so that

$$t_x^*\mathcal{L} \otimes \mathcal{L}^{-1} = \mathcal{L}(0, \exp(2\pi i \mathrm{Im}(H(v, \cdot)))) \tag{1.12}$$

In particular $\phi_{\mathcal{L}}$ depends on Chern class H of $\mathcal{L}$. Varying $\mathcal{L}$, we then have a map,

$$\phi : NS(X) \to Hom(X, \widehat{X}) \tag{1.13}$$

This map is a group homomorphism. A question one might ask is that for which $\mathcal{L}$, $\phi_{\mathcal{L}}$ is an isogeny? This happens when H is positive definite because in that case H induces an isomorphism between $V \to V^\vee$ by sending x to $H(x, \cdot)$. This map also sends Λ to $\widehat{\Lambda}$, thus $X \cong \widehat{X}$. This map coincides with $\phi_{\mathcal{L}}$ with $c_1(\mathcal{L}) = H$.

Definition 1.2.23. *Let $\mathcal{L}$ be a line bundle on X such that $c_1(\mathcal{L})$ is a positive definite Riemann form. Then, the induced map $\phi_{\mathcal{L}}$ is an isogeny and is called a **polarization** of X. If, moreover, $\phi_{\mathcal{L}}$ is an isomorphism, then we call $\phi_{\mathcal{L}}$ a **principal polarization**. A complex torus is polarized/principally polarized if it has a polarization/principal polarization.*

Sometimes, we will call the form H itself a polarization on X.

Example 1.2.24. *For an elliptic curve E with lattice $\Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$, the dual elliptic curve is also an elliptic curve and moreover it is isomorphic to E by $\phi_{\mathcal{L}}$ where $c_1(\mathcal{L}) = H$ where H is the positive definite Riemann form*

$$H(z, w) = \frac{z\overline{w}}{Im(\omega_1\overline{\omega_2})}.$$

This polarization is a principal polarization.

1.2.2 Theta functions

Theta functions are sections of line bundles i.e. elements of $H^0(X, \mathcal{L})$ where $\mathcal{L}$ is a line bundle. One of their main use is to construct a map $X \to \mathbb{P}^N$ for some $N \in \mathbb{N}$ for an Abelian variety X. The algebraic version of this argument is presented in detail in Section 7.1 of [35]. To do the construction, one needs a line bundle $\mathcal{L}$ which has many sections.

Definition 1.2.25. *A line bundle $\mathcal{L}$ on a proper variety X is said to be **very ample** if there exists a closed immersion $\iota : X \to \mathbb{P}^d$ such that $\mathcal{L} \cong \iota^*(\mathcal{O}(1))$. $\mathcal{L}$ is called **ample** if there exists some positive integer n such that $\mathcal{L}^{\otimes n}$ is very ample.*

We will first try to construct a section for any line bundle $\mathcal{L}(H, \chi)$ on an Abelian variety X. Using the 1-cocycle a that we used in the construction of $\mathcal{L}(H, \chi)$, we can recast the definition of section of $\mathcal{L}(H, \chi)$ as a holomorphic function $\theta : V \to \mathbb{C}$ such that for any $\lambda \in \Lambda$ and $v \in V$

$$\theta(v + \lambda) = \chi(\lambda) \exp\left(\pi H(v, \lambda) + \frac{\pi}{2} H(\lambda, \lambda)\right) \theta(v). \tag{1.14}$$

One way to construct such a section is to take the inverse of the 1-cocycle a,

$$a(\lambda, v)(a(\lambda + \mu, v))^{-1} = (a(\mu, v + \lambda))^{-1} \tag{1.15}$$

Then we sum over μ, to get,

$$a(\lambda, v) \sum_{\mu \in \Lambda} (a(\lambda + \mu, v))^{-1} = \sum_{\mu \in \Lambda} (a(\mu, v + \lambda))^{-1} \tag{1.16}$$

If the sum is absolutely convergent, then we can permute elements in the right sum, define $\theta(v) = \sum_{\mu \in \Lambda} (a(\mu, v))^{-1}$, we get,

$$a(\lambda, v)\theta(v) = \theta(v + \lambda) \tag{1.17}$$

So if the sums in (1.16) are absolutely convergent, then we have a section. This happens if H is a positive definite form i.e. a polarization.

Definition 1.2.26. *A function $\theta : V \to \mathbb{C}$ is called a* ***theta function*** *for the line bundle $\mathcal{L}$ on the complex torus V/Λ if it satisfies,*

$$a(\lambda, v)\theta(v) = \theta(v + \lambda)$$

where $a(\lambda, v)$ is the factor of automorphy for $\mathcal{L}$.

The discussion above implies the following corollary.

Corollary 1.2.27. *A line bundle $\mathcal{L}(H, \chi)$ has non-zero global sections if H is positive definite.*

Using these sections, one can construct a map $\Theta : X \to \mathbb{P}^{d-1}$ where d is the dimension of $H^0(\mathcal{L})$ by mapping a point $x \in X$ to $[\theta_1(x), \cdots, \theta_d(x)]$, where $\{\theta_i(x)\}$ forms a basis of $H^0(\mathcal{L})$. The following theorem gives a necessary and sufficient condition for the map Θ to be a closed embedding.

Theorem 1.2.28. *Let X be a complex torus with $\mathcal{L} = \mathcal{L}(H, \chi)$ and n be a natural number greater than 2. Then the Riemann form H is positive definite if and only if the space of holomorphic sections of $\mathcal{L}^{\otimes n}$ gives an embedding of X as a closed complex submanifold in a projective space.*

Proof. See Theorem of Lefschetz on page $29 - 33$ of [53]. □

By Chow's theorem (Theorem 1.1.3, [54]), X is a complex algebraic variety. On the other hand, if X is already a complex projective variety, then the pullback of $O(1)$ along the embedding $\iota : X \to \mathbb{P}^N$ would give a line bundle with its Chern class being positive definite. This means that the converse of Lefschetz theorem also holds.

Theorem 1.2.29. *A complex torus X is an Abelian variety if and only if X has positive definite Riemann form i.e. a polarization.*

By Theorem 1.2.28, a line bundle is ample if and only if its Chern class, when viewed as a Riemann form, is positive definite. We would like extend this Theorem to positive semi-definite forms. A natural question that arises is what property of line bundles corresponds to positive semi-definiteness. Recall that the first Chern class maps $Pic(Y)$ to $H^2(Y, \mathbb{Z})$, where Y is a variety. If Y is a curve, then $H^2(Y, \mathbb{Z}) \cong \mathbb{Z}$. In this case, we call the first Chern class the **degree** of the line bundle.

Definition 1.2.30. *A line bundle $\mathcal{L}$ on a proper variety X is called* ***nef line bundle*** *if for any projective curve C in Y, the pullback of $\mathcal{L}$ along the closed immersion of C into X has non-negative degree.*

Theorem 1.2.31. *The ample line bundles correspond to positive definite forms and the nef line bundles correspond to positive semidefinite forms.*

Proof. Follows from Theorem 1.2.28 and Kleiman criterion mentioned below. □

Kleiman Criterion:-. *The set of convex combinations of ample(nef) line bundles inside $NS(X) \otimes \mathbb{R}$ is an open(nef) cone such that its closure(interior) is the convex combinations of nef(ample) line bundles. See Proposition 6.1.18 of [43] for more details.*

Jacobi Theta Function

We now restrict to the case of an elliptic curve i.e. an 1-dimensional abelian variety. Consider a complex torus with complex dimension one. It can be thought of as $\mathbb{C}/\Lambda$, where Λ is a lattice in $\mathbb{C}$. More explicitly, $\Lambda = \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2$ where ω_1, ω_2 are complex numbers which are $\mathbb{R}$-linearly independent. If we have a Hermitian form H, then it is a Riemann form iff $A(\omega_1 \wedge \omega_2) \in \mathbb{Z}$, where A is the alternating form associated to H. Scaling by a suitable positive real number, we can assume that H is a Riemann form with $E(\omega_1 \wedge \omega_2) = 1$ and therefore H is a polarization. By Theorem 1.2.29, we $\mathbb{C}/\Lambda$ is an elliptic curve, which we will denote by E.

If we restrict to $\omega_1 = 1$ and $\omega_2 = \tau$ for a complex number τ with positive imaginary part, then we have a 1-cocycle a defined as

$$a(n + m\tau, v) = \exp\left(-\pi i m^2\tau - 2\pi i m v\right) \tag{1.18}$$

where $n, m \in \mathbb{Z}$ and $v \in \mathbb{C}$. The theta function that we get is called the **Jacobi theta function** $\theta_{E,z}$ where the subscript E is written to emphasize the elliptic curve. Using (1.8), we find after some calculation that the corresponding Hermitian form is positive definitive. So the line bundle corresponding to the cocycle a above is ample. The function $\theta_{E,z}$ has period 1 i.e. $\theta_{E,z}(z+1) = \theta_{E,z}(z)$, so it has a Fourier expansion, which is just a power series in $w := \exp(2\pi i z)$. Let $q = \exp(2\pi i\tau)$ and $z = x + iy$. After tweaking $\theta_{E,z}$ slightly and calling it $\theta(w; q)$, we have a product expansion of $\theta(w; q)$,

$$\theta(w; q) = q^{B_2(x)} exp\,(\pi i x(y-1))\,(1-w)\prod_{n=1}^{\infty}(1-q^n w)(1-q^n/w) \tag{1.19}$$

where $B_2(x) = x^2 - x + \frac{1}{6}$. The function $\theta(w; q)$ is analytic in both w and q. The q coordinate corresponds to the elliptic curve parametrized by τ and w coordinate corresponds to a point on the given elliptic curve.

Now, if we fix z, such that $Nx, Ny \in \mathbb{Z}$ for some fixed $N \in \mathbb{N}$ and view $\theta(w; q)$ as a function in q, we see that $\theta(w; q)^N$ is well-defined function on the space of complex tori because $\theta(w; q)^N$ is invariant under the transformation $(x, y) \mapsto (x + n, y + m)$ for $n, m \in \mathbb{Z}$. Conceptually, this means that we are varying the lattices and fixing a family of N-torsion points as the lattices vary. The functions in q, $\theta(w; q)^N$ are called **modular units**. These are non-invertible holomorphic functions over $Y_1(N)$ introduced in the next chapter.

If we fix q and let w vary, then we are fixing an elliptic curve E and varying the point on E. This means that $\theta(w; q)$ is a holomorphic function on the elliptic curve E.

Definition 1.2.32. *Let K be an imaginary quadratic field. Let E be an CM elliptic curve with endomorphism ring $R \subset K$ and let z be an N-torsion point of E. Then $\theta(w; q)$ is called an* ***elliptic unit*** *where q corresponds to E and w corresponds z. We will denote the elliptic unit by $\theta_{E,z}(z)$ or just $\theta(z)$ if E is clear from the context.*

Remark 1.2.33. *Every elliptic unit is contained in some Abelian extension of K. This follows from the theory of complex multiplication and the fact that j-invariant of CM elliptic curve is algebraic. Details can be found in Chapter* 11 *of [41] and [51] for theory of complex multiplication.*

Remark 1.2.34. *If $\eta \in Aut(E)$ then $\theta(z) = \theta(\eta z)$ for all torsion point z of E by Corollary 2.6 of [27]. In particular, $\theta(z) = \theta(-z)$ all torsion point z of E.*

The Jacobi theta function and its slight modification $\theta(w;q)$ have a very small set of zeros. For a given q, it is known that,

$$\theta(w;q) = 0 \Leftrightarrow z \in \Lambda \tag{1.20}$$

Moreover, all zeroes of $\theta(w;q)$ are simple roots. This means that if f is a meromorphic function on E corresponding to q with divisor $\sum_i n_i[a_i]$ then f is a constant multiplied by $\prod_i \theta_{E,z}(z-a_i)^{n_i}$ because $\theta_{E,z}(z) = \theta(w,q)$ has a simple zero at elements of Λ by (1.20) and no poles.

Remark 1.2.35. *What we call $\theta_{E,z}(z)$ here is sometimes also called $\vartheta_{11}(z;\tau)$ in the literature e.g. [52].*

Remark 1.2.36. *Section* 4.1 *of [28] also provides an algebraic description for the elliptic unit $\theta(a)$. Let a be an N-torsion element of $E(k)$ where E is defined over a field k, then the elliptic unit $\theta(a)$ is an element of $\overline{k}^{\times} \otimes_{\mathbb{Z}} \mathbb{Z}\left[\frac{1}{N}\right]$.*

1.2.3 Rosati Involution

Let $End(X)$ be the endomorphisms of an Abelian variety X mapping 0 to itself and $End_{\mathbb{Q}}(X) := End(X) \otimes_{\mathbb{Z}} \mathbb{Q}$. Let H be a polarization of X. The polarization induces an involution on $End_{\mathbb{Q}}(X)$. The goal of this section is to show how this involution is constructed. The polarization H induces an isogeny $\phi_H : X \to \widehat{X}$ by (1.11). Here, we conflate $\phi_{\mathcal{L}}$ with ϕ_H by Theorem 1.2.20 and Definition 1.2.23. We want to construct an inverse of ϕ_H at least in $Hom(\widehat{X}, X) \otimes \mathbb{Q}$.

Definition 1.2.37. *Let $f : X \to Y$ be an isogeny. The* ***exponent*** *of f denoted by $e(f)$ is the exponent of the kernel of f i.e. the l.c.m. of orders of all elements of $ker(f)$.*

Lemma 1.2.38. *For any isogeny $f : X \to Y$ with exponent e, there exists an isogeny $g : Y \to X$ such that,*

$$f \circ g = [e]_Y, \qquad g \circ f = [e]_X$$

where $[e]$ is the multiplication by e map. Moreover, this g is unique.

Proof. By a version of the first isomorphism theorem of groups, f induces an isomorphism $f' : X/ker(f) \xrightarrow{\cong} Y$. We note that $ker(f) \subset ker([e]_X)$, which gives a natural quotient map

$$q : X/ker(f) \to X/ker([e])$$

But, again by the first isomorphism theorem,$[e]$ induces a map $[e]' : X/ker([e]) \xrightarrow{\cong} X$. Define g to be the composition $[e]' \circ q \circ (f')^{-1}$. Then, $g \circ f = [e]_X$ by definition of g. The map g is surjective because $[e]_X$ is surjective and therefore g is an isogeny. By definition of g, we also note that the kernel of g is contained in $ker([e]_Y)$ because $[e]'$ and f' are isomorphisms. Doing the same procedure for g as we did for f above, we get a map h such that $h \circ g = [e]_Y$. Pre-composing this equation by f, we get

$$f \circ [e]_X = [e]_Y \circ f = (h \circ g) \circ f = h \circ (g \circ f) = h \circ [e]_X$$

where the first equation holds because f respects the group structure. Since $[e]_X$ is surjective,

$$f \circ [e]_X = h \circ [e]_X \Rightarrow f = h$$

The claim follows. $\square$

Corollary 1.2.39. *An element in $End(X)$ is an isogeny if and only if it is invertible in $End_{\mathbb{Q}}(X)$.*

Definition 1.2.40. *Let $f : X \to Y$ be an isogeny . Then the unique isogeny mentioned in Lemma 1.2.38 is called the* ***dual isogeny*** *of f.*

This lemma implies that if ϕ_H is an isogeny with exponent $e(H)$ then there exists a dual isogeny ψ_H of ϕ_H such that $\psi_H \circ \phi_H = [e(H)]_X$ and $\phi_H \circ \psi_H = [e(H)]_{\widehat{X}}$. Then, the inverse of ϕ_H would be $\frac{1}{e(H)}\psi_H$.
Now we have enough tools to define the Rosati involution with respect to H. Fix a polarization H on X. Let $f \in End_{\mathbb{Q}}(X)$, then $\widehat{f} \in End_{\mathbb{Q}}(\widehat{X})$. Consider the composition,

$$X \xrightarrow{\phi_H} \widehat{X} \xrightarrow{\widehat{f}} \widehat{X} \xrightarrow{\phi_H^{-1}} X \tag{1.21}$$

This is an endomorphism of X which we denote by $f^\dagger$.

Definition 1.2.41. *Let X be a complex torus and H be a polarization on X. Then* ***the Rosati involution*** *$\dagger$ with respect to H is defined as the map*

$$\dagger : End_{\mathbb{Q}}(X) \to End_{\mathbb{Q}}(X), \qquad f \to f^\dagger := \phi_H^{-1} \circ \widehat{f} \circ \phi_H$$

The Rosati involution is additive because

$$(f+g)^\dagger := \phi_H^{-1} \circ \widehat{f+g} \circ \phi_H = \phi_H^{-1} \circ (\widehat{f} + \widehat{g}) \circ \phi_H = \phi_H^{-1} \circ \widehat{f} \circ \phi_H + \phi_H^{-1} \circ \widehat{g} \circ \phi_H = f^\dagger + g^\dagger$$

where we used additivity of dual maps in the second step and use the additivity of ϕ_H and its inverse at the second last step. By composing f and g and simplifying, one sees that $(fg)^\dagger = g^\dagger f^\dagger$. This is the reason why it is also called anti-involution as in Section 5.1 of [42]. It is an involution because if $f \in End_{\mathbb{Q}}(X)$ then $(f^\dagger)^\dagger$ is equal to

$$\phi_H^{-1} \circ \widehat{\phi_H} \circ \widehat{\widehat{f}} \circ \widehat{\phi_H^{-1}} \circ \phi_H$$

But $\widehat{\widehat{f}} = f$ and $\widehat{\phi_H} = \phi_H$ by corollary 2.4.6(c) of [42]. Therefore,

$$\phi_H^{-1} \circ \widehat{\phi_H} \circ \widehat{\widehat{f}} \circ \widehat{\phi_H^{-1}} \circ \phi_H = f$$

Let f be an element of $End(X)$. Then, viewing X as a complex torus $\mathbb{C}^g/\Lambda$, f can be represented by a matrix $A_f \in M_g(\mathbb{C})$ such that $A_f(\Lambda) \subset \Lambda$. The Rosati involution of f is also the adjoint of f with respect to the inner product H. That is, given $v, w \in V$, we have

$$H(A_f v, w) = H(v, A_{f^\dagger} w)$$

To get a better understanding of $NS(X)$, we will express it in terms of endomorphisms of X, which can be expressed as matrices. From now on we fix a polarization H_0 on X. The map (1.13) and ϕ_{H_0}, give us the map,

$$NS(X) \otimes \mathbb{Q} \to Hom(X, \widehat{X})_{\mathbb{Q}} \xrightarrow{\phi_{H_0,*}} Hom(X, X)_{\mathbb{Q}} \tag{1.22}$$

by sending a Hermitian form H to $\phi_{H_0}^{-1} \circ \phi_H$. The image is contained in the subset of those endomorphisms which are fixed under the Rosati involution with respect to H_0 because

$$\left(\phi_{H_0}^{-1} \circ \phi_H\right)^\dagger = \phi_{H_0}^{-1} \circ \widehat{\phi_H} \circ \widehat{\phi_{H_0}^{-1}} \circ \phi_{H_0} = \phi_{H_0}^{-1} \circ \phi_H \circ \phi_{H_0}^{-1} \circ \phi_{H_0} = \phi_{H_0}^{-1} \circ \phi_H$$

Definition 1.2.42. *An endomorphism $f : X \to X$ is called* ***symmetric*** *if $f^\dagger = f$. The set of all symmetric endomorphisms is denoted by $End^s(X)$. Similarly, $End^s_{\mathbb{Q}}(X) := End^s(X) \otimes \mathbb{Q}$.*

The image of the map (1.22) is contained in $End^s_{\mathbb{Q}}(X)$. The following theorem shows that, in fact, the image is equal to $End^s_{\mathbb{Q}}(X)$.

Theorem 1.2.43. *The map* (1.22) *is an isomorphism. Moreover, if H_0 is a principal polarization, then the map gives an isomorphism between $NS(X)$ and $End^s(X)$.*

Proof. Denote the map (1.22) as Φ. First we will show that Φ is injective. If $\mathcal{L}$ is a line bundle such that $\Phi(\mathcal{L}) = 0$, then it means that $\phi_{H_0}^{-1} \circ \phi_{\mathcal{L}} = 0$. Multiplying this map by the exponent of ϕ_{H_0}, we get $\psi_{H_0} \circ \phi_{\mathcal{L}} = 0$, where ψ_{H_0} is the dual isogeny of ϕ_{H_0}. This means that the image of $\phi_{\mathcal{L}}$ is contained in the kernel of ψ_{H_0}, which is a finite discrete set. The image of $\phi_{\mathcal{L}}$ is connected, therefore it is zero. By the Appell-Humbert theorem 1.2.20, if $\mathcal{L} = \mathcal{L}(H, \chi)$ then $\phi_{\mathcal{L}}(v) = \mathcal{L}(0, \exp(2\pi i \mathrm{Im}(H(v, \cdot))))$. This is isomorphic to the trivial line bundle because the image of $\phi_{\mathcal{L}}$ is zero. This means that $\mathrm{Im}(H(v, \lambda)) \in \mathbb{Z}$ for all $v \in V$ and $\lambda \in \Lambda$. Since $\mathrm{Im}(H)$ is continuous in the first entry and V is connected, $\mathrm{Im}(H(\cdot, \lambda))$ is a constant function for all $\lambda \in \Lambda$. Choosing $v = \lambda$ implies that $\mathrm{Im}(H)$ is zero. So, $E_H = \mathrm{Im}(H) = 0$, which implies that H is zero and therefore $\mathcal{L}$ is in $Pic^0(X)$.
For the surjectivity, take an $f \in End^s(X)$. We want to show that $\phi_{H_0} \circ f = \phi_H$ for some positive definite Hermitian form H. A form on V is Hermitian if and only if it is of the form ϕ_H by theorem 2.5.5 of [42]. Therefore, it is enough to show $\phi_{H_0} \circ f$ induces a Hermitian form on V. The induced form is $H_0(f(\cdot), \cdot)$ on V. But $H_0(f(\cdot), \cdot)$ is Hermitian if and only if

$$H_0(f(v), w) = \overline{H_0(f(w), v)} = \overline{H_0(w, f^\dagger(v))} = H_0(f^\dagger(v), w) \tag{1.23}$$

where the second equality follows from the fact that the Rosati involution is adjoint with respect to H_0. Since H_0 is non-degenerate, $H_0(f(\cdot), \cdot)$ is Hermitian if and only if $f = f^\dagger$ and thus in $End^s(X)$. □

Remark 1.2.44. *If H_0 is a principal polarization in the theorem above, then ϕ_{H_0} already has an inverse in the category of Abelian varieties, so the same argument works without the need of tensoring by $\mathbb{Q}$.*

Example of $E \times E$

Suppose E is a CM elliptic curve. Then, $End(E) = O$ is a free Abelian group of rank two where O is an order in the ring of integers of some imaginary quadratic field K. The field K can be embedded into $\mathbb{C}$, so we will consider O as a subset of $\mathbb{C}$. E has a unique principal polarization by Example 1.2.24, which can be used to give a principal polarization on $E \times E$ term wise. More explicitly, if h is the principal polarization of E then we define $(h \oplus h)(z_1, z_2)$ to be $h(z_1) + h(z_2)$. We will take this principal polarization as H_0.
The endomorphism ring of $E \times E$ is isomorphic to the ring of 2-by-2 matrices $M_2(O)$ with coefficients in O and the group of automorphisms of $E \times E$ is $GL_2(O)$. The principal polarization is $H_0(z_1, z_2) = \frac{|z_1|^2 + |z_2|^2}{\mathrm{Im}(\tau)}$. If $f : \mathbb{C}^2 \to \mathbb{C}^2$ is represented by a matrix A_f, then $f^\dagger$ is represented by the Hermitian conjugate of A_f. To find $NS(E \times E)$, we need to find the symmetric endomorphisms. The symmetric endomorphisms can be identified with 2×2 Hermitian matrices. We conclude the discussion with the following lemma.

Lemma 1.2.45. *$End^s(E \times E)$ is isomorphic to the space of Hermitian matrices in $M_2(O)$.*

Corollary 1.2.46. *$End^s(E \times E)$ and $NS(E \times E)$ have rank* 4.

The isomorphism in the corollary can be upgraded to a $GL_2(O)$-equivariant map. If we denote the space of Hermitian matrices in $M_2(O)$ as $M_2^s(O)$, then $GL_2(O)$ acts on $M_2^s(O)$ as

$$(g, f) \to g^\dagger f g$$

where $g \in GL_2(O)$ and $g^\dagger$ is the conjugate transpose of g. After tensoring by $\mathbb{R}$, we get the following theorem

Theorem 1.2.47. *$NS(E \times E) \otimes \mathbb{R}$ is isomorphic to $M_2^s(\mathbb{C})$ as a $\mathbb{R}[GL_2(O)]$-module.*

Proof. See theorem 5.3 of [56]. □

By Theorem 1.2.47 and the Kleiman criterion, the ample cone of $NS(E \times E) \otimes \mathbb{R}$ can be identified with positive definite Hermitian matrices. This identification is also $GL_2(O)$-equivariant. The space of positive definite Hermitian matrices along with the action of $GL_2(O)$ comes up in the construction of the symmetric space of $GL_2(O)$ as introduced in Section 2.4 of the next chapter if O is a ring of integers of an imaginary quadratic field. This realization helps us to give some algebro-geometric meaning to the symmetric space.

Chapter 2

Symmetric spaces and Bianchi Tessellation

Let K be an imaginary quadratic field and O_K be its rings of integers. Let Γ be an arithmetic group such that $\Gamma \subset GL_2(O_K)$. The Bianchi tessellation is a $GL_2(O_K)$-equivariant tessellation of the symmetric space of $GL_2(O_K)$ similar to the usual tessellation or triangulation of the upper half plane with respect to some arithmetic subgroup of $GL_2(\mathbb{Z})$. It was introduced by Bianchi in [3]. In this chapter, we will introduce the symmetric and locally symmetric spaces and their compactification. Then we will proceed to recall the triangulation of the upper half plane and the Bianchi tessellation following [47]. The tessellation helps us to construct the relative cellular complex $S_\bullet(X(\Gamma))$ mentioned in the introduction, which computes the group cohomology of Γ. In the case of $GL_2(\mathbb{Z})$, the locally symmetric spaces are the modular curves. Modular curves have a moduli interpretation, which we will discuss below.

2.1 Symmetric space and locally symmetric space for $GL_2(\mathbb{Z})$

In this section, we will describe the construction of locally symmetric space for $GL_2(\mathbb{Z})$.

Definition 2.1.1. *Let F be a field and U be an F-vector space. A **quadratic form** Q is a function $Q : U \to F$ satisfying,*

1. $Q(\lambda \mathbf{x}) = \lambda^2 Q(x)$ *for all* $\lambda \in F$ *and* $\mathbf{x} \in U$
2. $B(\mathbf{x}, \mathbf{y}) := Q(\mathbf{x} + \mathbf{y}) - Q(\mathbf{x}) - Q(\mathbf{y})$ *is a symmetric F-bilinear form.*

*Moreover, if $F = \mathbb{R}$ then, we say Q is **positive semi-definite** iff $Q(\mathbf{x}) \geq 0$ and **positive definite** iff Q is positive semi-definite and $Q(\mathbf{x}) = 0 \Leftrightarrow \mathbf{x} = 0$. Analogously, we can define negative definite and negative semi-definite quadratic forms.*

Let V be the vector space of quadratic forms on $\mathbb{R}^2$ and $V_{>0}$ be the subset of positive definite quadratic forms in V. The group $GL_2(\mathbb{R})$ acts on V and $V_{>0}$ on the right by pre-composition. Let us explicitly see how this action looks like.
Fix a basis $\mathbf{e_1}, \mathbf{e_2}$ of $\mathbb{R}^2$. Then, by the definition of quadratic form, $B(\mathbf{x}, \mathbf{y})$ is a symmetric bilinear form. Because $\mathbf{e_1}, \mathbf{e_2}$ is a basis we can write $\mathbf{x} = x_1\mathbf{e_1} + x_2\mathbf{e_2}$ and $\mathbf{y} = y_1\mathbf{e_1} + y_2\mathbf{e_2}$. We find $B(\mathbf{x}, \mathbf{y})$,

$$B(x_1\mathbf{e_1} + x_2\mathbf{e_2}, y_1\mathbf{e_1} + y_2\mathbf{e_2}) = x_1y_1B(\mathbf{e_1}, \mathbf{e_1}) + (x_1y_2 + y_1x_2)B(\mathbf{e_1}, \mathbf{e_2}) + x_2y_2B(\mathbf{e_2}, \mathbf{e_2})$$

Denote $B(\mathbf{e_1}, \mathbf{e_1}) = 2a$,$B(\mathbf{e_1}, \mathbf{e_2}) = 2b$ and $B(\mathbf{e_2}, \mathbf{e_2}) = 2c$, we have

$$\begin{aligned} 2Q(x) &= 4Q(x) - Q(x) - Q(x) = Q(2x) - Q(x) - Q(x) \\ &= Q(x+x) - Q(x) - Q(x) = B(x,x) \\ &= 2a(x_1)^2 + 4bx_1x_2 + 2c(x_2)^2 \\ \Rightarrow Q(x_1\mathbf{e_1} + x_2\mathbf{e_2}) &= a(x_1)^2 + 2bx_1x_2 + c(x_2)^2 \end{aligned}$$

So we can associate to any binary quadratic form a triple of real numbers (a, b, c) and vice versa. The calculation also shows that $2Q(x) = B(x, x)$. Notice that we can also write $B(x, y)$ in terms of matrices as;

$$B(x_1\mathbf{e_1} + x_2\mathbf{e_2}, y_1\mathbf{e_1} + y_2\mathbf{e_2}) = B((x_1, x_2), (y_1, y_2)) = (x_1, x_2)\begin{pmatrix} 2a & 2b \\ 2b & 2c \end{pmatrix}(y_1, y_2)^t$$

Calling the matrix $\mathbf{Q}$, we get,

$$Q(\mathbf{x}) = \mathbf{x}\mathbf{Q}\mathbf{x}^t \tag{2.1}$$

where the superscript t stands for transpose. This proves the following lemma,

Lemma 2.1.2. *There is a 1-1 correspondence between binary quadratic forms and symmetric* 2×2 *matrices with real entries.*

The action of $GL_2(\mathbb{R})$ on V is also easy to describe using this correspondence. The pre-composition means that if $\mathbf{g} \in GL_2(\mathbb{R})$ then we can plug $\mathbf{gx}$ instead of $\mathbf{x}$ in equation (2.1). This upgrades the correspondence above,

Lemma 2.1.3. *There is a 1-1* $GL_2(\mathbb{R})$*-equivariant correspondence between binary quadratic form and symmetric* 2×2 *matrices* $M_2^s(\mathbb{R})$ *with real entries. The action of* $\mathbf{g} \in GL_2(\mathbb{R})$ *on symmetric* 2×2 *matrices is* $\mathbf{Q} \to \mathbf{g}\mathbf{Q}\mathbf{g}^t$.

So we can identify the space of quadratic forms and the space of symmetric 2×2- matrices. We can also describe $M_2^s(\mathbb{R})$ as a homogeneous space. To show this, we will first see that the action of $GL_2(\mathbb{R})$ on $V_{>0}$ is transitive, and then we will take the quotient by stabilizer of any quadratic form.

Lemma 2.1.4. *The action of* $GL_2(\mathbb{R})$ *on* $V_{>0}$ *is transitive.*

Proof. We only need to show that any positive definite quadratic form is in the orbit of the quadratic form $Q_0(\mathbf{x}) = x_1^2 + x_2^2$. Q_0 corresponds to the identity matrix in $M_2^s(\mathbb{R})$ using Lemma 2.1.3. So given any positive definite quadratic form Q, we have a corresponding matrix,

$$X = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$$

By positive definiteness $a > 0$, $c > 0$ and $ac - b^2 > 0$ because $Q(\mathbf{e}_1) > 0$, $Q(\mathbf{e}_2) > 0$ and the fact that for positive definite matrices both eigenvalues should be positive. Let

$$g = \begin{pmatrix} 1 & 0 \\ -b & a \end{pmatrix}.$$

Then $g \cdot X$ can be computed.

$$g \cdot X = \begin{pmatrix} 1 & 0 \\ -b & a \end{pmatrix} \cdot \begin{pmatrix} a & b \\ b & c \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -b & a \end{pmatrix}\begin{pmatrix} a & b \\ b & c \end{pmatrix}\begin{pmatrix} 1 & -b \\ 0 & a \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & a(ac - b^2) \end{pmatrix}$$

Now, denote $\alpha = \sqrt{a^{-1}}$ and $\beta = \sqrt{a^{-1}(ac-b^2)^{-1}}$. This is possible in $\mathbb{R}$ because both a and $ac-b^2$ are positive. Now acting by $diag(\alpha,\beta)$, we get,

$$\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \cdot \begin{pmatrix} a & 0 \\ 0 & a(ac-b^2) \end{pmatrix} = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & a(ac-b^2) \end{pmatrix} \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

So every quadratic form is in the orbit of Q_0. □

Corollary 2.1.5. *$V_{>0}$ is isomorphic to the homogeneous space $GL_2(\mathbb{R})/O(2)$ as a $GL_2(\mathbb{R})$-equivariant topological space.*

Proof. By the lemma above, we just need to find the stabilizer of Q_0 which by definition are matrices γ such that $\gamma\gamma^t = 1$, which are, by definition, the orthogonal matrices. This establishes the isomorphism $GL_2(\mathbb{R})/O(2) \to V_{>0}$ as $GL_2(\mathbb{R})$-sets by mapping $gO(2)$ to gg^t in $V_{>0}$. The map $GL_2(\mathbb{R}) \to V_{>0}$ is a polynomial map in the entries of $GL_2(\mathbb{R})$ and therefore it is continuous. The map is also open by Proposition 1.2 of [50]. This means that the map is homeomorphism. □

We are one step away from creating our symmetric space. The last step is to impose an equivalence relation on the quadratic forms,

Definition 2.1.6. *Two quadratic forms Q_1 and Q_2 are called **homothetic** if there exists $\alpha \in \mathbb{R}^\times$ such that $Q_1 = \alpha Q_2$.*

Definition 2.1.7. *The **symmetric space for** $GL_2(\mathbb{Z})$ is the space $V_{>0}$ up to homothety i.e. $V_{>0}/\mathbb{R}^\times = V_{>0}/\mathbb{R}_{>0}$. Equivalently, it is the quotient space $GL_2(\mathbb{R})/\mathbb{R}^\times O(2)$, where, by $\mathbb{R}^\times \subset GL_2(\mathbb{R})$, we mean the central subgroup of $GL_2(\mathbb{R})$ with entries in $\mathbb{R}^\times$.*

Yet another and more well known way to see symmetric space for $GL_2(\mathbb{Z})$ is to consider $\mathbb{C}-\mathbb{R}$. The group $GL_2(\mathbb{Z})$ acts on $\mathbb{C}-\mathbb{R}$ via the Möbius transformations,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} z = \frac{az+b}{cz+d}$$

The subgroup $SL_2(\mathbb{R})$ of $GL_2(\mathbb{R})$ acts on the upper half plane $\mathbb{H}^2$ via this action. The next lemma shows that $\mathbb{C}-\mathbb{R}$ is actually the symmetric space for $GL_2(\mathbb{Z})$.

Lemma 2.1.8. *$\mathbb{C}-\mathbb{R}$ is the symmetric space for $GL_2(\mathbb{Z})$.*

Proof. First thing to note is that the action of $\mathbb{R}^\times$ on $\mathbb{C}-\mathbb{R}$ via Möbius transformation is trivial. We will see the orbit of i under the action of $GL_2(\mathbb{R})$. The action is transitive because for $z = x+iy$ where $y \neq 0$, we have,

$$\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} i = yi + x = z$$

Morover, the stabilizer of i consists of matrices of the form,

$$\begin{pmatrix} a & -c \\ c & a \end{pmatrix} = \sqrt{a^2+c^2} \begin{pmatrix} \frac{a}{\sqrt{a^2+c^2}} & -\frac{c}{\sqrt{a^2+c^2}} \\ \frac{c}{\sqrt{a^2+c^2}} & \frac{a}{\sqrt{a^2+c^2}} \end{pmatrix} \in \mathbb{R}^\times O(2)$$

so $\mathrm{Stab}(i) \subset \mathbb{R}^\times O(2)$. The other containment is obvious. This realizes $\mathbb{C}-\mathbb{R}$ as $GL_2(\mathbb{R})/(\mathbb{R}^\times O(2))$. □

The action of $GL_2(\mathbb{Z})$ via its inclusion into $GL_2(\mathbb{R})$ on $\mathbb{C}-\mathbb{R}$ is almost free. This is the content of the next lemma.

Lemma 2.1.9. *Consider the action of $GL_2(\mathbb{Z})$ on $\mathbb{C}-\mathbb{R}$ and let $z \in \mathbb{C}-\mathbb{R}$. Then we have the following,*

1. *Stab(z) is cyclic of order 4 if z is in the orbit of i.*
2. *Stab(z) is cyclic of order 6 if z is in the orbit of ω, where ω is the sixth root of unit in the first quadrant.*
3. *Stab(z) is cyclic of order 2 otherwise.*

Proof. First, we will show that any torsion element of $SL_2(\mathbb{Z})$ is of the required orders.
Let g be such an element. Its characteristic polynomial is of degree two and since it is torsion, its complex eigenvalues are roots of unity. Its characteristic polynomial divides the equation x^n-1, so the roots of the characteristic polynomial are ζ and $\overline{\zeta}$ where ζ is a root of unity. So the characteristic equation is of the form,

$$x^2 - 2Re(\zeta)x + 1 = 0$$

Since the trace of g is integral, $2Re(\zeta) \in \mathbb{Z}$ and $|Re(z)| \leq 1$. Both of these facts imply $Re(\zeta) = cos(\theta) \in \{0, \pm\frac{1}{2}, \pm 1\}$. These possibilities together imply the orders that appear.
If $g \in$ Stab(z), then, since the sign of Im(z) is not changed, we have $g \in SL_2(\mathbb{Z})$. Since the action of $SL_2(\mathbb{Z})$ is properly discontinuous, Stab(z) is finite. Moreover, by Proposition 2.3.3 of [14], we have few candidates for such a group, so the lemma follows; see the corollary of the proposition in loc. cit. □

2.2 Cusps

When we construct modular curves later on, one problem that would arise is that they are not compact, so studying them would be difficult. A common trick to resolve this issue is to compactify our curve by adding a few points. In this section, cusps will play the role of these extra points. We add these points to $\mathbb{H}$ in such a way that the action of $GL_2(\mathbb{Z})$ on $\mathbb{H}$ also extends continuously to the cusps. This is not a compactification of $\mathbb{H}$ but, after taking the quotient, this will give us a compact space. The idea is to study objects over $\mathbb{H}$ with cusps because it is easier to work with and to deduce things for $GL_2(\mathbb{Z})$ or congruence groups.
A hint of this idea can be inferred from the fact that $GL_2(\mathbb{Z})$ also acts on $\mathbb{Q}\cup\{\infty\}$ by Möbius transformation (or projective linear transformation to be precise). This is the'smallest' set that you can add with an action of $GL_2(\mathbb{Z})$ such that the action of $GL_2(\mathbb{Z})$ extends.

Definition 2.2.1. *Let $\mathbb{C}^* = (\mathbb{C}-\mathbb{R})\cup\mathbb{Q}\cup\{\infty\}$. The points of $\mathbb{Q}\cup\{\infty\}$ are called the* ***cusps****.*

$\mathbb{C}^*$ also has a topology, which is described in more detail in ([14]), such that the action of $GL_2(\mathbb{Z})$ is properly discontinuous. The topological basis at ∞ consists of set of the form $\{z \in \mathbb{C}|\text{Im}(z) > R\}\cup\{\infty\}$ for $R > 0$. For the other cusps, we use the action of $GL_2(\mathbb{Z})$ to get neighbourhood systems of other cusps. $\mathbb{C}^*$ has a triangulation; Take the triangle with vertices being the cusps $0, 1$ and ∞. Then the $GL_2(\mathbb{Z})$-orbit of this triangle gives a set of triangles which form a triangulation of $\mathbb{C}^*$. This triangulation with $GL_2(\mathbb{Z})$ action on it helps us to calculate singular (co)homology of modular curves with relative ease.

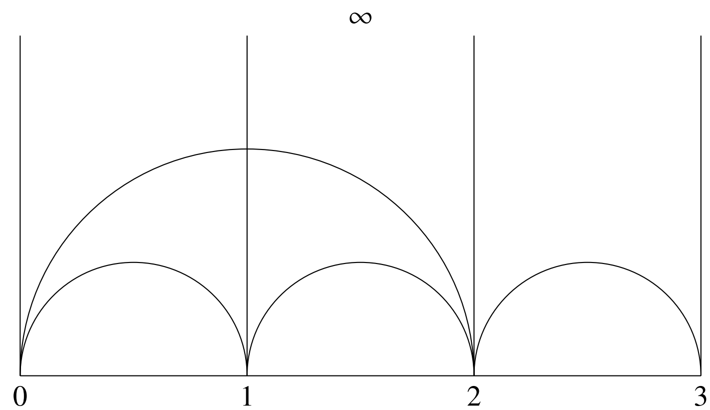

Figure 2.1: The Triangulation as seen on $\mathbb{H}$. This shows the ideal triangles $(0, 1, \infty)$, $(1, 2, \infty)$, $(2, 3, \infty)$ and $(0, 1, 2)$

The stabilizer of the triangle $(0, 1, \infty)$ in $PGL_2(\mathbb{Z})$ is isomorphic to S_3 and it is generated by matrices,

$$S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad U = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$$

2.3 Modular curves

Suppose we want to classify elliptic curves over $\mathbb{C}$ up to isomorphism. We have seen that any such elliptic curve corresponds to a lattice and any isomorphism of elliptic curve is an isomorphism of lattices in $\mathbb{C}$. So, if we classify all lattices up to isomorphism, then we are done. One way to do this is to construct a family of lattice which, hopefully, has a lattice from every isomorphism class.

Definition 2.3.1. *Let $\tau \in \mathbb{C} - \mathbb{R}$. Then we denote $\Lambda_\tau = \langle 1, \tau \rangle \subset \mathbb{C}$*

Definition 2.3.2. *Let Λ, Λ' be two lattices in $\mathbb{C}$. A morphism $\alpha : \Lambda \to \Lambda'$ is an element of $\mathbb{C}$ such that $\alpha\Lambda \subset \Lambda'$.*

Remark 2.3.3. *Recall that the definition of morphisms between complex tori and morphisms between lattices is the same. This sets up an equivalence of category of complex tori and category of lattices.*

The next lemma shows that our hope is not in vain.

Lemma 2.3.4. *Every lattice $\Lambda \subset \mathbb{C}$ is isomorphic to Λ_τ for some $\tau \in \mathbb{C} - \mathbb{R}$.*

Proof. Let $\Lambda = \langle a, b \rangle$. Then choose $\tau = b/a$ and the isomorphism $\Lambda \to \Lambda_\tau$ is just dividing by a. Since Λ is a lattice,$\tau \notin \mathbb{R}$. □

The next lemma shows when two lattices Λ_τ and $\Lambda_{\tau'}$ are isomorphic.

Lemma 2.3.5. *Let τ, τ' be two elements of $\mathbb{C} - \mathbb{R}$. Then, Λ_τ and $\Lambda_{\tau'}$ are isomorphic if there exists a $g \in GL_2(\mathbb{Z})$, such that $g \cdot \tau = \tau'$*

Proof. Let $\Lambda_\tau = \langle 1, \tau \rangle$ and $\Lambda_{\tau'} = \langle 1, \tau' \rangle$. The fact that they are isomorphic means that there is $\alpha \in \mathbb{C}$ such that $\alpha\Lambda_{\tau'} = \Lambda_\tau$. This is equivalent to the equations,

$$\alpha \cdot 1 = c + d\tau$$
$$\alpha \cdot \tau' = a + b\tau$$

or in matrix form,

$$\alpha \begin{pmatrix} 1 \\ \tau' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 \\ \tau \end{pmatrix}$$

Denote the matrix on the right by g. It is invertible because we assumed it to be an isomorphism. Solving for τ' gives,

$$\tau' = \frac{a + b\tau}{c + d\tau} = g \cdot \tau$$

□

Combining both lemmas, we deduce that $(\mathbb{C} - \mathbb{R})/GL_2(\mathbb{Z}) = \mathbb{H}^2/SL_2(\mathbb{Z})$ parameterizes isomorphism classes of elliptic curves over $\mathbb{C}$ where $\mathbb{H}^2 = \{z \in \mathbb{C} | \mathrm{Im}(z) > 0\}$. This set can be given a complex manifold structure (see Chapter 2 of [14]).

Definition 2.3.6. *The **modular curve for** $SL_2(\mathbb{Z})$ denoted by $Y(1)$ is defined to be the complex manifold $\mathbb{H}/SL_2(\mathbb{Z})$.*

As a complex manifold, $Y(1)$ is diffeomorphic to $\mathbb{C}$ by the j-invariant by Section 3.2 of [14]. $Y(1)$ is also a coarse moduli space for the moduli problem of isomorphism classes of elliptic curves over $\mathbb{C}$ e.g. see Section 2.1 of [33] for more details. The notation $Y(1)$ is quite suggestive; it seems that there are $Y(N)$ for $N \in \mathbb{N}$. To explain it, we need to define congruence subgroups.

Definition 2.3.7. *Let $\phi : GL_2(\mathbb{Z}) \to GL_2(\mathbb{Z}/N\mathbb{Z})$ be the entry-wise reduction modulo N map, then we define the congruence groups as;*

1. $\Gamma(N) := ker(\phi)$
2. $\Gamma_1(N) := \phi^{-1}\left\{\begin{pmatrix} 1 & x \\ 0 & y \end{pmatrix} \middle| x, y \in \mathbb{Z}/N\mathbb{Z}\right\}$
3. $\Gamma_0(N) := \phi^{-1}\left\{\begin{pmatrix} z & x \\ 0 & y \end{pmatrix} \middle| x, y, z \in \mathbb{Z}/N\mathbb{Z}\right\}$

In particular, congruence subgroups are finite index subgroups of $GL_2(\mathbb{Z})$.

Let Γ be a congruence subgroup of $GL_2(\mathbb{Z})$. Then, we can define the modular curve for Γ.

Definition 2.3.8. *The **modular curve for** Γ denoted by $Y(\Gamma)$ is defined to be the complex manifold $(\mathbb{C} - \mathbb{R})/\Gamma$. We also denote $Y(N) := Y(\Gamma(N))$, $Y_1(N) := Y(\Gamma_1(N))$ and $Y_0(N) := Y(\Gamma_0(N))$.*

As usual, we can give them complex manifold structure.

Remark 2.3.9. *$Y_0(N)$ and $Y_1(N)$ are actually geometrically integral algebraic curves defined over $\mathbb{Z}[\frac{1}{N}]$. $Y(N)$ is also a geometrically integral algebraic curve but over $\mathbb{Q}(\zeta_N)$, where ζ_N is the primitive N-th root of unity. See Chapter 7 of [14] for the proof of these statements. To see some explicit equations for $Y_1(N)$ for $N > 3$, the reader is referred to [62].*

The curves $Y_0(N)$, $Y_1(N)$ and $Y(N)$ also have moduli interpretation. All of them are moduli spaces for elliptic curves but with an extra structure called **level structure**. In each case, the level structure is as follows.

- If the moduli problem is a functor that associates to a scheme Z the set of isomorphism classes of (E, C) where E is an elliptic curve over the scheme Z and C is an order N cyclic subgroup of the group $E(Z)$. The coarse moduli space in this case is $Y_0(N)$.
- If the moduli problem is a functor that associates to a scheme Z the set of isomorphism classes of (E, z) where E is an elliptic curve over the scheme Z and z is an order N element of $E(Z)$. The coarse moduli space in this case is $Y_1(N)$.
- If the moduli problem is a functor that associates to a scheme Z the set of isomorphism classes of (E, z_1, z_2) where E is an elliptic curve over the scheme Z and z_1, z_2 form a basis of $E(Z)[N]$ such that $w(z_1, z_2) = exp(\frac{2\pi i}{N})$. The coarse moduli space in this case is $Y(N)$. Here $w(-, -)$ is the Weil pairing.

A good source for moduli properties of modular curves can be found in [39].

Remark 2.3.10. *The curve $Y_1(N)$ and $Y(N)$ are also classifying spaces for $\Gamma_1(N)$ and $\Gamma(N)$ respectively for $N > 2$. This stems from the fact that $\Gamma_1(N) \cap SL_2(\mathbb{Z})$ and $\Gamma(N) \cap SL_2(\mathbb{Z})$ are torsion free by Lemma 9 of [60].*

Remark 2.3.11. *The reason $Y_1(N)$ is a fine moduli space for its moduli problem for $N > 3$ by Section 3 of [38]. It stems from the fact that the level structures reduce the automorphisms that are obstructions to the existence of a fine moduli space as an algebraic variety.*

These curves are not compact, as we can see from the example of $Y(1)$; topologically, it is a 2-sphere with a missing point. Here, the cusps play the role of compactifying $Y(1)$. Adding the cusps, we can define $X(1) := ((\mathbb{C} - \mathbb{R}) \cup \mathbb{P}^1(\mathbb{Q}))/GL_2(\mathbb{Z})$ as the compactification of $Y(1)$. Using cusps, we obtain the compactification $X(\Gamma) := ((\mathbb{C} - \mathbb{R}) \cup \mathbb{P}^1(\mathbb{Q}))/\Gamma$ of $Y(\Gamma)$. Again, we denote $X_0(N) = X(\Gamma_0(N))$, $X_1(N) = X(\Gamma_1(N))$ and $X(N) = X(\Gamma(N))$. $X(\Gamma)$ is also called the modular curve. They have the structure of compact Riemann surfaces and so they are also projective varieties over $\mathbb{C}$ by Theorem 7.5 of [50]. Moreover, similar to their non-compact counterparts, $X_0(N)$ and $X_1(N)$ are also defined over $\mathbb{Q}$ and $X(N)$ is defined over $\mathbb{Q}(\zeta_N)$. The moduli interpretation for $Y(\Gamma)$ for the three congruence groups extend to the cusps where they correspond to level structures on the smooth locus of a nodal cubic curve, which is isomorphic to $\mathbb{G}_m$.

2.4 Symmetric space for $GL_2(O_K)$

Let K be an imaginary quadratic. In this section, we will define the symmetric space for $GL_2(O_K)$. We will use quadratic forms again to construct the symmetric space for $GL_2(O_K)$. Since $\mathbb{C} \cong K \otimes_{\mathbb{Z}} \mathbb{R}$, the action of $GL_2(O_K)$ extends to the action on $\mathbb{C}^2$. Hence, $GL_2(O_K)$ can be viewed as a subgroup of $GL_2(\mathbb{C})$. The isomorphism above also lets us view K as a subfield of $\mathbb{C}$ and O_K as a lattice in $\mathbb{C}$.

Definition 2.4.1. *Let Q be the space of hermitian forms in two variables and $Q_{>0}$ be the space of positive definite hermitian forms. The **symmetric space for** $GL_2(O_K)$ is $Q_{>0}/\mathbb{R}_{>0}$. We will denote this by $\tilde{Y}$.*

The elements of $\mathcal{Q}$ can be represented as hermitian 2-by-2 matrices. Then $GL_2(\mathbb{C})$ acts on $\mathcal{Q}$ by

$$g \cdot Q := gQg^{\dagger} \tag{2.2}$$

where $g \in GL_2(\mathbb{C})$ and $Q \in \mathcal{Q}$ and $g^{\dagger}$ is the conjugate transpose of g. We can also define congruence groups in $GL_2(\mathcal{O}_K)$ which we denote by $\Gamma(\mathcal{N})$, $\Gamma_0(\mathcal{N})$ and $\Gamma_1(\mathcal{N})$ similar to the one before. Here $\mathcal{N}$ is an ideal of $\mathcal{O}_K$. Considering $GL_2(\mathcal{O}_K)$ as a subset of $GL_2(\mathbb{C})$, we can consider the action of Γ on $\mathcal{Q}$. Then it is a well known fact that

Theorem 2.4.2. ***(Theorem 1.2.3,[34])*** *The action of $GL_2(\mathcal{O}_K)$ on $\mathcal{Q}$ is properly discontinuous.*

Proof. This is due to Proposition 2.4 of [50] and the fact that $GL_2(\mathcal{O}_K)$ is discrete. □

The action of $GL_2(\mathcal{O}_K)$ on $\mathcal{Q}$ also restricts to an action on $\mathcal{Q}_{>0}$. The action however is not free but close to being one in the sense that stabilizers of positive definite Hermitian forms are finite.

Theorem 2.4.3. ***(Theorem 1.2.3,[34])*** *If Γ is torsion-free then the action of Γ on $\mathcal{Q}_{>0}$ is free. In particular, the action of $\Gamma_1(\mathcal{P})$ for all prime ideals $\mathcal{P}$ of $\mathcal{O}_K$ with norm greater than 3, is free.*

Proof. Proof is similar to Theorem 8 of [60]. □

The theorems above also apply to $\tilde{Y}$ instead of $\mathcal{Q}_{>0}$ because center of $GL_2(\mathcal{O}_K)$ acts trivially on $\tilde{Y}$. On the other hand, $\tilde{Y}$ is contractible. To see this, note that $\mathcal{Q}_{>0}$ is in half space of $\mathcal{Q}$ i.e. there exists a hyperplane h passing through 0 such that $\mathcal{Q}_{>0}$ is on one side of it. This holds because if Q is positive definite then $-Q$ is not. Secondly, $\mathcal{Q}_{>0}$ is also a convex. Both of these facts imply that $\tilde{Y}$ is contractible. If the action of Γ on $\tilde{Y}$ is free, then $Y(\Gamma) := \tilde{Y}/\Gamma$ is the classifying space for Γ.

Remark 2.4.4. *Taking quotients of manifolds usually is a technical matter, in which proper discontinuity and freeness of Γ play an essential role. For details, see Section 1.2 of [50].*

We can also compactify $Y(\Gamma)$ by first compactifying $\tilde{Y}$ such that the compactification is $GL_2(\mathcal{O}_K)$-equivariant. Again, we will add 'smallest' $GL_2(\mathcal{O}_K)$-equivariant set to $\tilde{Y}$ such that we can compactify the $Y(\Gamma)$. To construct this compactification, we first observe that we have a map from the affine space $\mathbb{A}^2(\mathcal{O}_K) := \{(a,b) \in \mathcal{O}_K^2 | (a,b) \neq (0,0)\}$ to the space of hermitian forms $\mathcal{Q}$,

$$\mathbb{A}^2(\mathcal{O}_K) \to \mathcal{Q} \tag{2.3}$$

$$\mathbf{v} \to \mathbf{v}^{\dagger}\mathbf{v} \tag{2.4}$$

The image of $\mathbb{A}^2(\mathcal{O}_K) - \{0\}$ is contained in the set of positive semi-definite hermitian matrices and the map is also $GL_2(\mathcal{O}_K)$-equivariant. Moreover, note that if we scale $\mathbf{v}$ by some $\lambda \in K$, then we will scale the corresponding quadratic form under the map (2.3) by $|\lambda|^2$. Therefore, taking quotient by $\mathbb{R}_{>0}$, the map (2.3) factors through the map

$$\mathbb{P}^1(K) = \mathbb{P}^1(\mathcal{O}_K) \to \mathcal{Q}/\mathbb{R}_{>0} \tag{2.5}$$

Definition 2.4.5. *We define $\tilde{X}$ to be the space $(\mathcal{Q}_{>0}/\mathbb{R}_{>0}) \cup \mathbb{P}^1(K)$ where the topology is the same as the one described in Section 3 of [6]. A concrete way to see $\tilde{X}$ and its topology is mentioned in the next section.*

Remark 2.4.6. *The particular example of $GL_2(\mathbb{C})$ is given in Section I.4.2 in loc. cit.*

Remark 2.4.7. *The topological space $X(\Gamma) := \tilde{X}/\Gamma$ is compact.*

$X(\Gamma)$ constructed this way is compact. The extra points that we introduced are called the **cusps**.

Theorem 2.4.8. *[34] The number of cusps in X_Γ is finite. More precisely, it is the class number of K.*

There are some key differences between the symmetric spaces of $GL_2(\mathbb{Z})$ and $GL_2(\mathcal{O}_K)$. One of the most prominent one is the fact that for the classical case of $GL_2(\mathbb{Z})$, we have an algebraic structure on $Y(\Gamma)$ and $X(\Gamma)$ which is not the case for $GL_2(\mathcal{O}_K)$, because $Y(N)$ and $X(N)$ have real dimension 3 which means it is not even a complex manifold, let alone an algebraic variety.

2.5 Bianchi tessellation

In this section, we will construct the Bianchi tessellation, following [47]. The main idea is to use the idea of a perfect form to give a tessellation, which was originally done by Voronoi in [68].

Definition 2.5.1. *A vector $v \in \mathcal{O}_K^{\oplus 2} - 0$ is called a **minimal vector** of a positive definite Hermitian form Q if $Q(v, v) = min\{Q(u, u) | u \in \mathcal{O}_K^{\oplus 2} - 0\}$. The set of minimal vectors of Q will be denoted by Min_Q*

Definition 2.5.2. *A quadratic form P is called **perfect** if for any other form Q,*

$$Min_P \subset Min_Q \Rightarrow Q = \lambda P$$

for some positive real number λ. That is P is determined by its minimal vectors up to scalar.

Denote by Q_v the image of v under (2.3).

Definition 2.5.3. *For a perfect form P, we can define a 4-cell in $\mathcal{Q}_{>0} \cup \mathbb{A}^2(\mathcal{O}_K)$ by taking the convex hull of the rays passing from the origin through Q_v for all $v \in Min_P$. After taking quotient by $\mathbb{R}^{>0}$, it becomes a 3-cell.*

Remark 2.5.4. *A priori, a perfect form need not have coefficients in K but after scaling by a suitable real number, it would have coefficients in K. Since the perfect form only depends on the minimal vectors, which has entries in K and minimum value, which upto scaling, can be assumed to be in K, linear algebra calculation will force the coefficients to al;so be in K. This would not change the ray nor the convex hull.*

By Proposition 7.1.6 of [47], it is a 3-cell because P is perfect. Note that if we replace P by some multiple of it, then we get the same 3-cell. The 2-cells are then the intersection of two adjacent 3-cells and the 1-cells are defined similarly. This results in a tessellation, as proved in Chapter 7 of loc.cit. This tessellation coincides with the Bianchi tessellation introduced in [3] by explicit computation in [71].

Bianchi Tessellation and Hyperbolic 3-space $\mathbb{H}^3$

The symmetric space $\mathcal{Q}_{>0}/\mathbb{R}_{>0}$ and its compactification $\tilde{X}$ also has a more humble description, similar to how $\mathbb{C} - \mathbb{R}$ and $\mathbb{C}^*$ is a concrete description of symmetric space for $GL_2(\mathbb{Z})$. The role of this more concrete form of $\mathcal{Q}_{>0}/\mathbb{R}_{>0}$ is played by hyperbolic 3-space.

Definition 2.5.5. *The hyperbolic* 3*-space* $\mathbb{H}^3$ *is defined as* $\mathbb{C}\times\mathbb{R}_{>0}$*. The points* $K\times 0\subset\mathbb{C}\times\mathbb{R}_{\geq 0}$ *play the role of cusps along with the point at infinity* ∞*, which is thought of as the point corresponding to* $\mathbb{C}\times\infty$*.*

The group $GL_2(\mathbb{C})$ acts on $\mathbb{H}^3$ by the following rule;

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}(z,r) = \frac{1}{N}((az+b)\overline{(cz+d)} + a\bar{c}r^2, r)$$

where $N = |cz+d|^2 + |c|^2r^2$. The arithmetic groups of $GL_2(O_K)$ act via their inclusion into $GL_2(\mathbb{C})$. In the next section, we will give some examples of polytopes in $\mathbb{H}^3$. The topology of $\tilde{X}$ is generated by open sets of $\mathbb{H}^3$ and open half sphere centered at cusps. For more examples, the reader is referred to [32].

2.5.1 Examples of Bianchi tessellation

In this section, we will see Bianchi tessellations for discriminant $-1, -3, -7$ and also note information about the orbits and stabilizers of the cells in the Bianchi tessellation. Most of this information can be found in [32]. Yasaki in [71] also computed these data for imaginary quadratic fields with higher discriminant.

Remark 2.5.6. *In this section, we will be using the action of* $PGL_2(O_K)$ *instead of* $GL_2(O_K)$ *because the center of* $GL_2(O_K)$ *acts trivially, so all the statements should be understood modulo the center of* $GL_2(O_K)$*.*

2.5.2 $d = -1$

In this case, all the 3-cells are in one orbit. One element of this orbit is an ideal octahedron $\mathbb{V}_1$.

$$\mathbb{V}_1 = (0, 1, i, 1+i, \frac{1+i}{2}, \infty)$$

With the first four forming the "square". The corresponding perfect form is $H(w,z) = |w|^2 + \mathrm{Re}(w(i-1)\bar{z}) + |z|^2$ with the minimum value 1. The stabilizer of this octahedron is isomorphic to the symmetric group S_4 on four letters and its description is,

$$Stab_{PGL_2(O_K)}(\mathbb{V}_1) =< U, Y | U^3 = Y^4 = (UY^2)^3 = 1 >$$

where,

$$Y = \begin{pmatrix} i & 1 \\ 0 & 1 \end{pmatrix}, \qquad U = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$$

The element Y reverses the orientation on $Q_{>0}$ and therefore, it also reverses the orientation of the space$\tilde{Y}$.
The 2-cells are also in one orbit i.e. in the orbit of the triangle $(0, 1, \infty)$. The stabilizer of the given triangle is isomorphic to S_3, the symmetric group on three letters. Its presentation is,

$$\mathrm{Stab}_{PGL_2(O_K)}((0, 1, \infty)) =< S, U >$$

where U is the same U as above and S is the matrix with diagonal entries 0 and off-diagonal entries 1.

The 1-cells also form a single orbit i.e. in orbit of $(0, \infty)$. The stabilizer of this edge is isomorphic to $S_2 \times O_K^\times$. It is generated by matrices,

$$S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad D = \begin{pmatrix} i & 0 \\ 0 & 1 \end{pmatrix}$$

S is orientation reversing while D preserves orientation of $\tilde{Y}$.
The 0-cells are also in one orbit namely in the orbit of ∞. The stabilizer of ∞ is isomorphic to $\mathbb{Z} \times O_K^\times$ and is generated by T and D, where

$$T = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

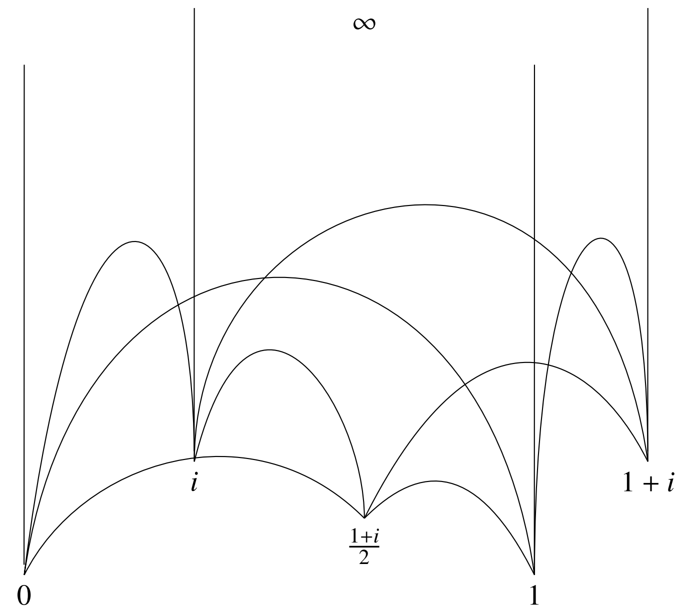

Figure 2.2: Visualization of Bianchi tessellation for $d = -1$ in upper half space. The cusps are on the plane $\mathbb{C} \times 0$ and ∞ is considered far above the plane. The lines going upwards have ∞ as the other vertex. The curves are semi-circle intersecting the plane perpendicularly.

2.5.3 $d = -3$

In this case, all the 3-cells are in one orbit. One element of this orbit is an ideal tetrahedron $\mathbb{V}_1$. Let $\omega = \frac{1}{2}(1 + \sqrt{-3})$. We have

$$\mathbb{V}_1 = (0, 1, -\omega^2, \infty)$$

The stabilizer of this tetrahedron is isomorphic to A_4 and its description is,

$$\mathrm{Stab}_{PGL_2(O_K)}(\mathbb{V}_1) =< U, Y | U^3 = Y^2 = (UY)^3 = 1 >$$

where,

$$Y = \begin{pmatrix} 0 & \omega \\ \omega^2 & 0 \end{pmatrix}, \qquad U = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$$

The whole group preserves the orientation of the space $\tilde{Y}$.
The 2-cells are also in one orbit i.e. in the orbit of the triangle $(0, 1, \infty)$, which has stabilizer S_3. Its presentation is

$$Stab_{PGL_2(O_K)}((0, 1, \infty)) =< S, U >$$

where U is the same U as above and S is the matrix with diagonal entries 0 and off-diagonal entries 1.
The 1-cells also form a single orbit i.e. in orbit of $(0, \infty)$. The stabilizer of this edge is isomorphic to $S_2 \times O_K^\times$. It is generated by matrices,

$$S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad D = \begin{pmatrix} \omega & 0 \\ 0 & 1 \end{pmatrix}$$

where $a \in O_K^\times$. U is orientation reversing while D preserves orientation of $\tilde{Y}$.
The 0-cells are also in one orbit namely in the orbit of ∞. The stabilizer of ∞ is isomorphic to $\mathbb{Z} \times O_K^\times$ and is generated by T and D, where

$$T = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

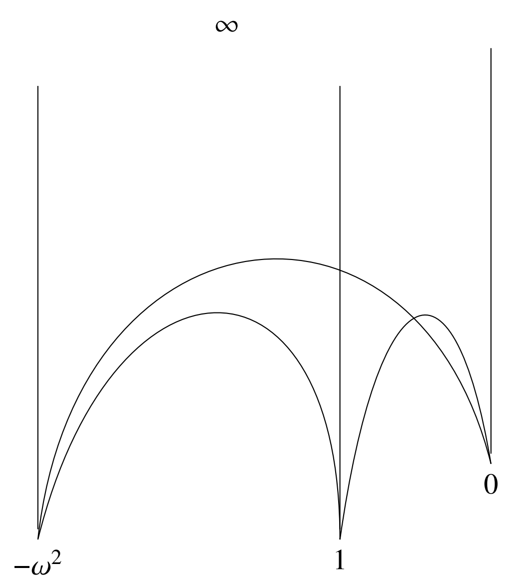

Figure 2.3: Visualization of Bianchi tessellation for $d = -3$ in upper half space. The cusps are on the plane $\mathbb{C} \times 0$ and ∞ is considered far above the plane. The lines going upwards have ∞ as the other vertex. The curves are semi-circle intersecting the plane perpendicularly.

2.5.4 $d = -7$

In this case, all the 3-cells are in one orbit. One element of this orbit is an ideal triangular prism $\mathbb{V}_1$. Let $\omega = \frac{1}{2}(1 + \sqrt{-7})$. We have

$$\mathbb{V}_1 = (0, 1, \omega, \frac{\omega}{2}, \frac{1+\omega}{2}, \infty)$$

with one triangle having vertices $0, 1, \infty$ and the rest forming the other triangle. The stabilizer of this triangular prism is isomorphic to S_3 and its description is,

$$\mathrm{Stab}_{PGL_2(O_K)}(\mathbb{V}_1) =< U, Y | U^3 = Y^2 = (UY)^2 = 1 >$$

where,

$$Y = \begin{pmatrix} \omega & 1-\omega \\ 1 & -\omega \end{pmatrix}, \qquad U = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$$

The element Y reverses the orientation on $\mathbb{Q}^{\geq 0}$ and therefore, it also reverses the orientation of the space $\tilde{Y}$. Geometrically, U rotates the prism by $\frac{2\pi}{3}$ around the axis through the two triangular faces and Y is a reflection which flips the two triangular faces.

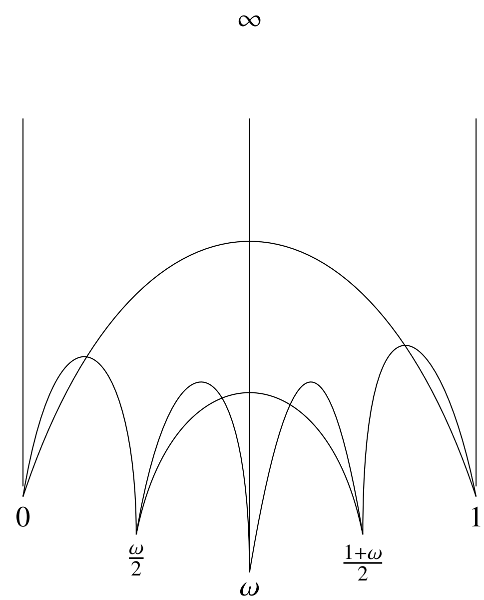

Figure 2.4: Visualization of Bianchi tessellation for $d = -7$ in upper half space. The cusps are on the plane $\mathbb{C} \times 0$ and ∞ is considered far above the plane. The lines going upwards have ∞ as the other vertex. The curves are semi-circle intersecting the plane perpendicularly.

The 2-cells are also in two orbits; the triangles and the quadrilaterals. The triangles are in the orbit of the triangle $(0, 1, \infty)$, which has the stabilizer S_3. Its presentation is,

$$\mathrm{Stab}_{PGL_2(O_K)}((0, 1, \infty)) =< S, U >$$

where U is the same U as above and S is the matrix with diagonal entries 0 and off-diagonal entries 1.

The quadrilaterals are in the orbit of $(0, \infty, \frac{\omega}{2}, \omega)$. The stabilizer of this quadrilateral is isomorphic to Klein 4-group and it is generated by two commuting, order two matrices in $PGL_2(O_K)$, A and B, which are

$$A = \begin{pmatrix} -1 & \omega \\ 0 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} -1 & 0 \\ -\omega^2 & 1 \end{pmatrix}$$

Both A and B are orientation reversing. The 1-cells also form a single orbit i.e. in orbit of $(0, \infty)$. The stabilizer of this edge is isomorphic to $S_2 \times O_K^{\times}$. It is generated by matrices,

$$S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad D = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

S is orientation reversing while D preserves orientation of $\tilde{Y}$.
The 0-cells are also in one orbit namely in the orbit of ∞. The stabilizer of ∞ is isomorphic to $\mathbb{Z} \times O_K^{\times}$ and is generated by T and D, where

$$T = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

Remark 2.5.7. *In general if class number of K is 1, all the cusps are in one orbit. In the case when O_K is also Euclidean, all edges are also in one orbit, namely in the orbit of the edge $(0, \infty)$.*

2.6 Modular Complex

In this section, we will introduce the modular complex and also show how it relates to the singular cohomology of $Y(\Gamma)$ and the group cohomology of Γ.
Since we will be working over $\mathbb{Q}$ coefficients and the stabilizers in Γ are finite, the singular cohomology groups of $Y(\Gamma)$ are isomorphic to group cohomology groups of Γ by Proposition II.4.1 of [8].

Theorem 2.6.1.
$$H^i_{sing}(Y(\Gamma), M) \cong H^i_{gp}(\Gamma, M)$$
where M on the right is a $\mathbb{Q}[\Gamma]$-module and M on the left is the corresponding local system on $Y(\Gamma)$.

A consequence of this theorem is that we will only need to study the singular cohomology of $Y(\Gamma)$ and that will suffice for the group cohomology of Γ. The previous sections give some tessellations of $\tilde{X}$. We can use the cellular complex to calculate its homology groups.

Definition 2.6.2. *The cellular groups are defined as follows;*

1. *$S_0(\tilde{X})$ is the $\mathbb{Q}$-vector space generated by the cusp or 0-cells.*
2. *$S_1(\tilde{X})$ is the $\mathbb{Q}$-vector space generated by the 1-cells i.e. edges in the tessellation.*
3. *$S_2(\tilde{X})$ is the $\mathbb{Q}$-vector space generated by the 2-cells in the tessellation.*
4. *$S_3(\tilde{X})$ is the $\mathbb{Q}$-vector space generated by the 3-cells in the tessellation.*

The cellular chain complex will be denoted by $S_\bullet(\tilde{X})$.

Since the action of $GL_2(O_K)$ preserves the tessellation, it permutes the basis of $S_i(\tilde{X})$ for $i = 0, 1, 2, 3$. The Bianchi tessellation also has finitely many $GL_2(O_K)$-orbits for each i-dimensional cells.

Definition 2.6.3. *The **modular complex** $S_\bullet(X(\Gamma))$ is defined to be $\mathbb{Q} \otimes_\Gamma S_\bullet(\tilde{X})$, where the tensor is over $\mathbb{Q}[\Gamma]$-modules and $\mathbb{Q}$ is the trivial $\mathbb{Q}[\Gamma]-$module.*

The modular complex lets us compute homology of $X(\Gamma)$. In each case of Γ that we consider, it contains $\Gamma(\mathcal{N})$. This means that the index $[GL_2(\mathcal{O}_K) : \Gamma]$ is finite because $[GL_2(\mathcal{O}_K) : \Gamma(\mathcal{N})]$ is less than cardinality of $GL_2(\mathcal{O}_K/\mathcal{N})$. Finite number of orbits in Bianchi tessellation coupled with the fact that $[GL_2(\mathcal{O}_K) : \Gamma]$ is finite implies that $S_i(X(\Gamma))$ is finite dimensional for each $i = 0, 1, 2, 3$.
Now consider the complex

$$S_3(X(\Gamma)) \to S_2(X(\Gamma)) \to S_1(X(\Gamma)) \tag{2.6}$$

concentrated in degrees $3, 2, 1$. It computes the homology groups $H_\bullet(X(\Gamma), \partial X(\Gamma), \mathbb{Q})$, where $\partial X(\Gamma)$ is the (finite) set of cusps. The homology groups of this complex are the Borel-Moore homology groups $H_\bullet^{BM}(Y(\Gamma))$ of $Y(\Gamma)$. By Poincaré duality, $H_\bullet(X(\Gamma), \partial X(\Gamma), \mathbb{Q}) \cong H_\bullet^{BM}(Y(\Gamma)) \cong H^{3-\bullet}(Y(\Gamma))$. Therefore, Equation (2.6) combined with Theorem (2.6.1) implies the following theorem.

Theorem 2.6.4. *The n-th homology of the complex (2.6) is isomorphic to* $H_{gp}^{3-n}(\Gamma, \mathbb{Q})$.

This theorem is quite powerful. It has some profound consequences.

Corollary 2.6.5. *The group cohomology* $H_{gp}^n(\Gamma, \mathbb{Q}) = 0$ *unless* $n = 0, 1, 2$.

Corollary 2.6.6. *The groups* $H_{gp}^n(\Gamma, \mathbb{Q})$ *are finite dimensional for all* $n \in \mathbb{Z}$.

Proof. As we remarked earlier, that $S_\bullet(X(\Gamma))$ is finite dimensional. The corollary follows from this fact. □

It might be surprising to see that the cohomology of the group Γ is 'simple' even though these groups are very complicated. This also shows the utility of locally symmetric spaces in the study of arithmetic groups in general. In more general setting, using Borel-Serre compatification which is a comapct manifold with boundary homotopic to $Y(\Gamma)$, one can also shown these types of finiteness result. See Chapter 9 of [6] for Borel-Serre compatification.

Chapter 3

Framed motives and periods

In this chapter, we will introduce what motives are and discuss some of their aspects. The slogan is,

"Motives is the universal cohomology theory."

But what do the terms in the slogan mean? Let us try to explain them. Let $\mathbf{Var}_k$ be the category of varieties over k i.e. finite type, seperated integral schemes over k for a field k.

A *cohomology theory H* valued in an Abelian category C is a functor,

$$H : \mathbf{Var}_k \to DC \tag{3.1}$$

satisfying some properties, see [18], page 454. Some of these properties are Poincaré duality, Mayer-Vietoris sequence, Künneth formula and are mentioned in Section 3.1.6 below. Here, DC is the derived category of C. We have many examples of such cohomology theories. On the top of the list, we have Betti cohomology, which is the singular cohomology of the complex points of the variety. We also have de Rham cohomology if $char(k) = 0$ which has the structure of mixed Hodge structure (which we will see below)and l-adic cohomology which is valued in $Gal(\bar{k}/k)$-representations. All of these cohomology theories not only satisfy same properties but also have an extra structure like mixed Hodge structure, or $Gal(\bar{k}/k)$ action. Moreover, there are comparison maps e.g. de Rham's theroem

$$H_B^\bullet(X, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C} \to H_{dR}^\bullet(X, k) \otimes_k \mathbb{C} \tag{3.2}$$

Grothendieck had an idea that since all these cohomology theories have a similar pattern (or motif in French), they must all factor through a single category $\mathcal{MM}(k)$ i.e. there is a functor H_{mot} such that for every cohomology theory H, there exists $\tilde{H}$ such that the following diagram commutes,

$$\begin{tikzcd} \mathbf{Var}_k \arrow[r, "H_{mot}"] \arrow[dr, "H"'] & D\mathcal{MM}(k) \arrow[d, "\tilde{H}"] \\ & DC \end{tikzcd} \tag{3.3}$$

The functor $\tilde{H}$ is unique upto unique isomorphism. The (still hypothetical) category $\mathcal{MM}$ is called the **category of mixed motives**.

The target categories for the cohomology functors mentioned above are not only Abelian but they are Tannakian categories.

Definition 3.0.1. *A **neutral Tannakian category** is a rigid Abelian tensor category $\mathcal{T}$ such that $F = End(\mathbb{1})$ is a field along with an exact faithful F-linear tensor functor ω from $\mathcal{T}$ to the category of F-vector spaces $Vect_F$.*

We will omit the word neutral from now on. Even though, Tannakian categories is defined in terms of category theoretic lingo, they are surprisingly "concrete" as shown by the theorem below.

Theorem 3.0.2. *Let $(\mathcal{T}, \omega)$ be a Tannakian category with $F = End(\mathbb{1})$. Then there exists an affine pro-algebraic F-group scheme $\mathcal{G}$ such that $(\mathcal{T}, \omega)$ is $\otimes$-equivalent to the category $(\mathbf{Rep}_{\mathcal{G}}, \text{forgetful})$ of finite dimensional F-representations of $\mathcal{G}$.*

Proof. See Theorem 2.11 of [13]. □

3.1 The category $\mathbf{DM}_k$

In this section, we will outline a construction of a contender for $D\mathcal{MM}_k$, the derived categroy of $\mathcal{MM}(k)$ denoted by $\mathbf{DM}_k$. This is due to Voevodsky [66]. The idea is to first construct a triangulated category, which would play the role of derived category of $\mathcal{MM}_k$ and then show that this category has a 'motivic' t-structure, which would then extract $\mathcal{MM}_k$ out of $D\mathcal{MM}_k$. The search of motivic t-structure continues to this day but the category $\mathbf{DM}_k$ satisfies all the properties that $D\mathcal{MM}_k$ should. See Section II.5 [18] for more details.

Following [66], we will outline the construction of $\mathbf{DM}_k$, in a series of steps;

1. Defining category of finite correspondences $\mathbf{SmCor}(k)$.
2. Constructing the homotopy category $K^b(\mathbf{SmCor}(k))$.
3. Taking Verdier quotient of $K^b(\mathbf{SmCor}(k))$ by the thick subcategory generated by $\mathbb{A}^1$-homotopy and Mayer-Vietoris sequence .
4. Taking pseudo-Abelian envelope of the category we get from the previous step to get $\mathbf{DM}_k^{eff}$, the category of effective motives.
5. Inverting the Tate motive to get $\mathbf{DM}_k$

We will briefly explain these steps one by one. General reference for this construction is the original paper [66] or alternatively, Lecture 14 [48].

Remark 3.1.1. *The category which we denote by $\mathbf{DM}_k$ is denoted by $\mathbf{DM}_k^{gm}$ in literature e.g. in loc. cit..*

3.1.1 The category $\mathbf{SmCor}(k)$

To define $\mathbf{SmCor}(k)$, we first need to define finite correspondences.

Definition 3.1.2. *Let X and Y be smooth varieties over k. A **finite correspondence** from X to Y is a $\mathbb{Z}$-linear combination of integral closed subsets Z_i of $X \times Y$ such that the projection map $Z_i \to X$ is finite and surjective over some connected component of X. We will denote correspondences from X to Y as $Cor(X, Y)$*

Remark 3.1.3. *Replacing $\mathbb{Z}$ with any other ring R will give us category of finite correspondences with coefficients in R. Later, we will consider $R = \mathbb{Q}$.*

Example 3.1.4. *If $f : X \to Y$ is a morphism in $\mathbf{Var}_k$, then the graph $\Gamma_f \subset X \times Y$ is a finite correspondence from X to Y.*

We can also compose finite correspondences. Let $W_1 \in Cor(X, Y)$ and $W_2 \in Cor(Y, Z)$. Then, there is an associative, additive composition,

$$comp : Cor(X, Y) \otimes Cor(Y, Z) \to Cor(X, Z)$$

We can see how this composition works by, first, assuming W_1 and W_2 are integral subvarieties of $X \times Y$ and $Y \times Z$ respectively and then extend by linearity. In the former case, we note that $W_1 \times Z$ and $X \times W_2$ are both subvarieties of $X \times Y \times Z$. We intersect the divisors $W_1 \times Z$ and $X \times W_2$ in $X \times Y \times Z$ and then pushforward to $X \times Z$. By Lemma 1.7 of [48], the divisors intersect properly and the result of the pushforward of the intersection is also a finite correspondence. The fact that it is well-defined and associative is the content of Lemma 4.33 in [19]. Equipped with a notion of composition, we can define the category $\mathbf{SmCor}(k)$.

Remark 3.1.5. *If $f : X \to Y$ is a finite map which is surjective onto some connected component of Y then it has a "inverse" in correspondence given by the image of Γ_f under the isomorphism $X \times Y \to Y \times X$, which switches the coordinates.*

Definition 3.1.6. *The **category of finite correspondences** with coefficient in ring R $\mathbf{SmCor}(k, R)$ is defined as a category where;*

- *The objects of $\mathbf{SmCor}(k)$ are just objects of $\mathbf{Var}_k$.*
- *The set of morphisms from X to Y is just $Cor(X, Y) \otimes R$ for $X, Y \in ob(\mathbf{SmCor}(k))$*
- *The composition of morphisms is defined above.*

There is a functor from $\mathbf{Var}_k \to \mathbf{SmCor}(k, R)$ mapping the object X to X and the morphism f to Γ_f. From now on, we will take correspondences with $R = \mathbb{Q}$ and just write $\mathbf{SmCor}(k)$. The category also has direct sums and a symmetric monoidal structure. The direct sum of X and Y is $X \sqcup Y$ and $X \otimes Y := X \times Y$. The category is also $\mathbb{Q}$-linear.

3.1.2 The homotopy category $K^b(\mathbf{SmCor}(k))$

Since the category $\mathbf{SmCor}(k)$ is pre-additive i.e. the Hom sets are abelian groups and the composition maps are group homomorphisms, we can form chain complexes in $\mathbf{SmCor}(k)$. We denote the category of bounded chain complexes in $\mathbf{SmCor}(k)$ as $C^b(\mathbf{SmCor}(k))$, the homotopy category $K^b(\mathbf{SmCor}(k))$ is the category with same objects as $C^b(\mathbf{SmCor}(k))$ and the morphisms in $K^b(\mathbf{SmCor}(k))$ are equivalence classes of morphisms in $C^b(\mathbf{SmCor}(k))$ with respect to chain homotopy [55]. $K^b(\mathbf{SmCor}(k))$ has a triangulated structure and also has tensor product and direct sum inherited from $C^b(\mathbf{SmCor}(k))$. We also have a functor $\mathbf{SmCor}(k) \to K^b(\mathbf{SmCor}(k))$ where X is mapped to $[X]$ which is a complex with X in degree zero and zero elsewhere.

3.1.3 $\mathbb{A}^1$-homotopy and Mayer-Vietoris sequence

The category homotopy $K^b(\mathbf{SmCor}(k))$ has some chain complexes which 'represent' $\mathbb{A}^1$-homotopy and Mayer-Vietoris sequence.

- $\mathbb{A}^1$-homotopy: We have a complex

$$[X \times \mathbb{A}^1] \xrightarrow{pr} [X]$$

in degree 1 and 0 respectively.

- Mayer-Vietoris sequence: If $X = U \cup V$ for two non-empty open sets U and V contained in X, then we have the sequence,

$$[U \cap V] \xrightarrow{i_{U\cap V,U}+i_{U\cap V,V}} [U] \oplus [V] \xrightarrow{i_{U,X}-i_{V,X}} [X]$$

in degree 2,1 and 0 respectively and all the is are open embeddings.

We require these complexes to be acyclic for $\mathbb{A}^1$-homotopy and Mayer-Vietoris sequence to hold. So we are going to quotient by these complexes. Let $\mathcal{J}$ be the thick full triangulated subcategory of $K^b(\mathbf{SmCor}(k))$ generated by the complexes above for all X and any open covering U, V of X. Then we define S to be the set of morphisms in $K^b(\mathbf{SmCor}(k))$ such that $f : X \to Y$ is in S iff there exists a $Z \in \mathcal{J}$ such that we have the distinguished triangle,

$$\begin{tikzcd} X \arrow[rr] & & Y \arrow[dl] \\ & Z \arrow[ul, "{[1]}"] & \end{tikzcd}$$

Definition 3.1.7. *We define the quotient $K^b(\mathbf{SmCor}(k))/\mathcal{J}$ to be the category $K^b(\mathbf{SmCor}(k))[S^{-1}]$ where S is the set of morphims corresponding to the thick, full triangulated subcategory$\mathcal{J}$. We also have a functor*

$$K^b(\mathbf{SmCor}(k)) \to K^b(\mathbf{SmCor}(k))/\mathcal{J}$$

More details for the construction of Verdier quotients can be found in Section 13.6(05RA) of [61]. It is relevant ot mention that $K^b(\mathbf{SmCor}(k))/\mathcal{J}$ is still triangulated.

3.1.4 The Category $\mathbf{DM}_k^{eff}$

The pseudo-Abelian envelope of a category C enlarges C by splitting idempotents. An idempotent $e : A \to A$ is said to **split** in the category C, if there exists an object B and maps $f : B \to A$ and $g : A \to B$ such that $gf = id$ and $fg = e$. One can think of B as the image of A under e and f as the inclusion of B into A. We need to introduce such elements to ensure that many properties of the cohomology hold e.g. projective bundle formula. It also helps to see $H^0(X, \mathbb{Q})$ as a direct summand coming from the idempotent $X \to \mathrm{Spec}(k) \to X$ if X is a connected variety with a k-rational point.

Definition 3.1.8. *The **pseudo Abelian envelope/Karoubi envelope** $Kar(C)$ is a category whose objects are pairs of the form (X, p) where p is an idempotent on X and morphism sets are defined as,*

$$Mor_{Kar(C)}((X, p), (Y, q)) = \{f \in Mor_C(X, Y) | f = q \circ f \circ p\}$$

There is also a functor $C \to Kar(C)$ by sending X to (X, id) and a morphism f to f itself. By [1], if C is triangulated then so is $Kar(C)$.

Definition 3.1.9. *The **derived category of effective motives** denoted by $\mathbf{DM}_k^{eff}$ is defined to be pseudo-Abelian envelope of $K^b(\mathbf{SmCor}(k))/\mathcal{J}$.*

At every instance of the construction till now, we had a functor. Composing all these functors give us a functor $\mathbf{Var}_k \to \mathbf{DM}_k^{eff}$. We denote the image of X under this functor as $M(X)$, the effective motive of X.

3.1.5 The Tate motive $\mathbb{Q}(1)$ and $\mathbf{DM}_k$

The last step is to introduce Poincaré duality for smooth projective varieties. To do this, we have to invert the Tate motive. It turns out that inverting the Tate motive also makes the category $\mathbf{DM}_k$ rigid i.e. we have dual objects [48].
Let X be a variety over k having a k-rational point, then the composition of structure morphism and the inclusion of the k-rational point gives an idempotent on X. Since idempotents split in $\mathbf{DM}_k^{eff}$, we have

$$M(X) = M(k) \oplus \tilde{M}(X)$$

$M(k)$ is just the unit for the tensor product of $\mathbf{DM}_k^{eff}$, so we denote it by $\mathbb{Q}(0)$. Now $\mathbb{P}^1$ has a k-rational point so we can split off $M(k)$ from $M(\mathbb{P}^1)$ as a direct summand.

Definition 3.1.10. *The **Tate motive** $\mathbb{Q}(-1)$ is defined to be $\tilde{M}(\mathbb{P}^1)[-2]$. We will define $\mathbb{Q}(-n)$ to be $\mathbb{Q}(1)^{\otimes n}$ for non-negative n.*

Now we have all the tools to finally define the rigid $\mathbb{Q}$-linear triangulated category $\mathbf{DM}_k$.

Definition 3.1.11. *The **derived category of motives** $\mathbf{DM}_k$ is the category with objects and morphisms as described below;*

- *Its objects are pairs of the form (M, m) where $M \in \mathbf{DM}_k^{eff}$ and $m \in \mathbb{Z}$.*
- $Hom_{\mathbf{DM}_k}((M, m), (N, n)) = \lim_{r \to -\infty} Hom\mathbf{DM}_k^{eff}(M \otimes \mathbb{Q}(m + r), N \otimes \mathbb{Q}(n + r))$

The definition above implies that tensoring by $\mathbb{Q}(-1)$ is an equivalence of categories between $\mathbf{DM}_k$ and itself. This implies, in particular, that there is an object $\mathbb{Q}(1)$ such that $\mathbb{Q}(1) \otimes \mathbb{Q}(1)$ is isomorphic to the tensor unit. This is the inversion of $\mathbb{Q}(-1)$. Similar to the definition of $\mathbb{Q}(-n)$, we define $\mathbb{Q}(n)$ for positive n to be $\mathbb{Q}(1)^{\otimes n}$.

The last piece of information that we need to extract the category of mixed motives from $\mathbf{DM}_k$ is the existence of a motivic t-structure. To get the idea of t-structure, we note that an Abelian category C embeds into the derived category DC as complexes concentrated in degree zero. But degree is not well-defined a priori for a triangulated category so we need some alternative description to define this embedding. One way is to view complexes in degree zero as intersection of two subcategories; one concentrated in non-negative degrees and one concentrated in non-positive degrees. These two categories have some compatibility condition and it turns out that if we have two subcategories which satisfy these compatibility relation then we can find the underlying Abelian category.

Definition 3.1.12. *A **t-structure** on a triangulated category $\mathcal{D}$ consists of two full subcategories $\mathcal{D}_{\geq 0}$ and $\mathcal{D}_{\leq 0}$,stable under isomorphisms,such that;*

- *If $X \in \mathcal{D}_{\leq 0}$ and $Y \in \mathcal{D}_{\geq 0}$ then $Hom_{\mathcal{D}}(X, Y[-1]) = 0$.*
- *If $X \in \mathcal{D}_{\leq 0}$ then $X[1] \in \mathcal{D}_{\leq 0}$. Similarly, if $Y \in \mathcal{D}_{\geq 0}$ then $Y[-1] \in \mathcal{D}_{\geq 0}$.*
- *If A is an object of $\mathcal{D}$, then there exists $X \in \mathcal{D}_{\leq 0}$ and $Y \in \mathcal{D}_{\geq 0}$ such that all three of them fit into the distinguished triangle,*

$$\begin{tikzcd}
X \arrow[rr] & & A \arrow[dl] \\
& Y[-1] \arrow[ul, "{[1]}"] &
\end{tikzcd}$$

Definition 3.1.13. *The **heart** $\mathcal{D}^{\heartsuit}$ of the t-structure is the category $\mathcal{D}_{\leq 0} \cap \mathcal{D}_{\geq 0}$.*

Existence of motivic t-structure for $\mathbf{DM}_k$ is still not known in general.

3.1.6 Some Properties of DM(k)

In this section, we will provide some properties of the category **DM**(k).

1. **Künneth formula:** $M(X \times Y) = M(X) \otimes M(Y)$.
2. $\mathbb{A}^1$**- homotopy:** $M(X \times \mathbb{A}^1) = M(X)$
3. **Mayer-Vietoris sequence:** For a Zariski cover $X = U \cup V$, we have a distinguished triangle,
$$\begin{tikzcd} M(U \cap V) \arrow[rr] & & M(U) \oplus M(V) \arrow[dl] \\ & M(X) \arrow[ul, "{[1]}"] & \end{tikzcd}$$
4. **Gysin sequence:** Let $Z \subset X$ be a smooth closed subscheme of codimension d and X is also smooth then we have a distinguished triangle,
$$\begin{tikzcd} M(X - Z) \arrow[rr] & & M(X) \arrow[dl] \\ & M(Z)(-d)[2d] \arrow[ul, "{[1]}"] & \end{tikzcd}$$
5. **Blow up:** Let Z be a smooth closed subscheme of X and let $Bl_Z(X)$ be the blow up of X along Z. Let E be the exceptional divisor. Then there is a distinguished triangle,
$$\begin{tikzcd} M(E) \arrow[rr] & & M(Bl_Z(X)) \oplus M(Z) \arrow[dl] \\ & M(X) \arrow[ul, "{[1]}"] & \end{tikzcd}$$
If Z has codimension d in X, then the distinguished triangle implies the following canonical isomorphism,
$$M(Bl_Z(X)) = M(X) \oplus \bigoplus_{i=1}^{d-1} M(Z)(-i)[2i].$$
6. **Duality:** For any A in **DM**(k), we have its dual $A^\vee$ such that for any smooth projective variety X of dimension d, satisfies
$$M(X)^\vee = M(X)(d)[-2d].$$
7. **Adjunction:** We have the following two formula relating tensor product and the dual,
$$\begin{aligned} Hom(A \otimes B^\vee, C) &= Hom(A, B \otimes C) \\ Hom(A \otimes B, C) &= Hom(A, B^\vee \otimes C) \end{aligned}$$

To show some utility of these properties, we will show using Gysin sequence and $\mathbb{A}^1$-homotopy that
$$M(\mathbb{P}^n) = \bigoplus_{i=0}^{n} \mathbb{Q}(-i)[2i]$$

which seems quite similar to the cohomology groups of $\mathbb{P}^n$. The shifts tells us which cohomology group we are looking at and the Tate twists i.e. the $\mathbb{Q}(-i)$ gives us additional information e.g. Hodge structure etc. Take $X = \mathbb{P}^n$ and Z to be a hyperplane in X in the Gysin sequence

$$M(\mathbb{A}^n) \to M(\mathbb{P}^n) \to M(\mathbb{P}^{n-1})(-1)[2] \to M(\mathbb{A}^n)[1]$$

By $\mathbb{A}^1$-homotopy $M(\mathbb{A}^n) = M(k) = \mathbb{Q}(0)$. Since $\mathbb{P}^n$ has a k-rational point, the structure morphism of $\mathbb{P}^n$ splits the left most map in the sequence, so we have,

$$M(\mathbb{P}^n) = \mathbb{Q}(0) \oplus M(\mathbb{P}^{n-1})(-1)[2]$$

By induction on n, the claim follows.

The Gysin Spectral Sequence

Let X be an algebraic variety. The Gysin sequence mentioned above only works if Z is smooth, closed subscheme. A natural question that arises is how to extend the sequence to any SNC divisor $L = \bigcup_{i \in \tilde{I}} L_i$ where L_i are smooth, closed subschemes and $\tilde{I}$ is a finite indexing set. We can also ask the same question for a relative cohomology group $H^k(X - L, M - L)$ where $M = \bigcup_{j \in \tilde{J}} M_j$ is another divisor such that $L \cup M$ is SNC and $\tilde{J}$ is a finite indexing set. The first question is a special case of the second question with $M = \emptyset$. The answer to the second question is given by a spectral sequence.

Proposition 3.1.14. ***(Proposition A.1, [15])*** *There is a convergent spectral sequence*

$$E_1^{-p,q} = \bigoplus_{\substack{r-s=p \\ |I|=r \\ |J|=s}} H^{q-2r}(L_I \cap M_J)(-r) \Rightarrow H^{-p+q}(X - L, M - L) \tag{3.4}$$

where $L_I = \bigcap_{i \in I} L_i$ *where* $I \subset \tilde{I}$ *and* $M_J = \bigcap_{j \in J} M_j$ *where* $J \subset \tilde{J}$.

The differential $d_1 : E_1^{-p,q} \to E_1^{-p+1,q}$ is the differential of the total complex

$$\begin{tikzcd}
 & \vdots \arrow[d, "d_v"] & \vdots \arrow[d, "d_v"] & \\
\cdots \arrow[r, "d_h"] & T^q_{1,1} \arrow[r, "d_h"] \arrow[d, "d_v"] & T^q_{0,1} \arrow[r, "d_h"] \arrow[d, "d_v"] & \cdots \\
\cdots \arrow[r, "d_h"] & T^q_{1,2} \arrow[r, "d_h"] \arrow[d, "d_v"] & T^q_{0,2} \arrow[r, "d_h"] \arrow[d, "d_v"] & \cdots \\
 & \vdots & \vdots &
\end{tikzcd}$$

where

- the terms are $T^q_{r,s} := \bigoplus_{\substack{|I|=r \\ |J|=s}} H^{q-2r}(L_I \cap M_J)(-r)$ in degree $(-r, s)$.

- The horizontal differential $d_h : T^q_{r,s} \to T^q_{r-1,s}$ is the sum of the maps

$$g_u : H^{q-2r}(L_I \cap M_J)(-r) \to H^{q-2r+2}(L_{I\setminus\{u\}} \cap M_J)(-r+1) \tag{3.5}$$

  for every $u \in \tilde{I}$ where g_u is the Gysin morphism induced from the inclusion morphism $L_I \cap M_J \to L_{I\setminus\{u\}} \cap M_J$ multiplied by some appropriate signs.

- The vertical differential $d_v : T^q_{r,s} \to T^q_{r,s+1}$is sum of maps

$$res_v : H^{q-2r}(L_I \cap M_J)(-r) \to H^{q-2r}(L_I \cap M_{J\cup\{v\}})(-r) \tag{3.6}$$

  for every $v \in \tilde{J}$ where res_v is the restriction map induced from the inclusion morphism $L_I \cap M_J \to L_I \cap M_{J\cup\{v\}}$.

Remark 3.1.15. *The spectral sequence degenerates at the second page if X is smooth and projective because all entries on E_1 page are pure motives but the differentials map to pure motive with a different weight.*

Remark 3.1.16. *The dual of the spectral sequence of $H^{-p+q}(X-L, M-L)$ is isomorphic to the spectral sequence of $H^{-p+q}(X-M, L-M)$.*

3.2 Mixed Tate Motives and Framed Motives

In this section, we will deal with a subcategory of $\mathbf{DM}(k)$, which is generated by the Tate motives $\mathbb{Q}(n)$ for $n \in \mathbb{Z}$ and closed under extensions.

Definition 3.2.1. *The* ***derived category of mixed Tate motives*** *denoted by* $\mathbf{DMT}(k)$ *is the smallest full triangulated subcategory of* $\mathbf{DM}(k)$ *containing $\mathbb{Q}(n)$ for all integers n and stable under extensions i.e. if in a distinguished triangle $X \to Y \to Z \to X[1]$ any two of X, Y or Z are in* $\mathbf{DMT}(k)$ *then so is the third one.*

The first step in order to understand $\mathbf{DMT}(k)$ is to see how can we extend $\mathbb{Q}(n)$ by $\mathbb{Q}(m)$, since these are the simple objects of $\mathbf{DMT}(k)$. For any such extension A, $A \otimes \mathbb{Q}(-n)$ is an extension of $\mathbb{Q}(0)$ by $\mathbb{Q}(m-n)$ so without loss of generalization, we can assume $n = 0$. By [67],[45],[4], these extensions (which form a group) can be compared to motivic cohomology and therefore with K-theory. So we get,

$$Ext^i_{\mathbf{DMT}(k)}(\mathbb{Q}(0), \mathbb{Q}(m)) = H^i(k, \mathbb{Q}(m)) = (K_{2m-i}(k) \otimes \mathbb{Q})^{(m)} \tag{3.7}$$

where the superscript (m) denotes an appropriate eigenspace with respect to the Adams operation (see Section IV.5 of [70]) and $H^i(k, \mathbb{Q}(m))$ is the motivic cohomology. For sake of brevity, we will omit the subscript of Ext if the category is obvious. Usually K-groups are not known but for the case when k is a number field, we have the following remarkable result by Borel.

Theorem 3.2.2. *[5] Let k be a number field with r embedding in $\mathbb{R}$ and c conjugate pairs of strictly complex embeddings in $\mathbb{C}$. Then we have the following,*

$$H^i(k, \mathbb{Q}(m)) = (K_{2m-i}(k) \otimes \mathbb{Q})^{(m)} = \begin{cases} \mathbb{Q} & \textit{if } i = 0, m = 0 \\ k^\times \otimes \mathbb{Q} & \textit{if } i = 1, m = 1 \\ \mathbb{Q}^{r+c} & \textit{if } i = 1, m \geq 3 \textit{ odd} \\ \mathbb{Q}^c & \textit{if } i = 1, m \geq 2 \textit{ even} \\ 0 & \textit{else} \end{cases} \tag{3.8}$$

There are few points of note that follow from this theorem,

- $Ext^i(\mathbb{Q}(0), \mathbb{Q}(n)) = 0$ if $i > 1$.
- $Ext^0(\mathbb{Q}(0), \mathbb{Q}(n)) = Hom(\mathbb{Q}(0), \mathbb{Q}(n))$ which is zero if $n \neq 0$, otherwise it is equal to $\mathbb{Q}$.
- $Ext^1(\mathbb{Q}(0), \mathbb{Q}(n)) = 0$ if $n \leq 0$.
- All *Ext* groups are finite-dimensional except for $Ext^1(\mathbb{Q}(0), \mathbb{Q}(1))$, which is $k^\times \otimes \mathbb{Q}$.

Borel's theorem also implies Beilinson-Soulé conjecture for number fields,

Conjecture 3.2.3. *Let k be a field. Then, $K_n(k)^{(r)}_{\mathbb{Q}}$ is trivial for $n > 2r$.*

In terms of motivic cohmology, this just means that $H^i(k, \mathbb{Q}(0)) = 0$ for $i < 0$. In [44], Levine showed that if the Beilinson-Soulé conjecture is satisfied for a field k then the category $\mathbf{DMT}(k)$ has a motivic t-structure. Since in the case of number fields, it is true, we get the following definition.

Definition 3.2.4. *Let k be a number field. The* ***category of mixed Tate motives*** *denoted by $\mathcal{MTM}(k)$ is the heart of the motivic t-structure on* $\mathbf{DMT}(k)$ *defined in [44].*

The rigidity of the category $\mathbf{DM}(k)$ also applies to $\mathcal{MTM}(k)$. So, $\mathcal{MTM}(k)$ is a rigid symmetric monoidal $\mathbb{Q}$-linear category with $End(\mathbb{Q}(0)) = \mathbb{Q}$. Moreover the fact that $Ext^1(\mathbb{Q}(m), \mathbb{Q}(n)) = 0$ for $n \leq m$ implies that for $M \in \mathcal{MTM}(k)$, we have an increasing **weight filtration** $W_\bullet M$ such that,

$$gr^W_{2n} M = \mathbb{Q}(-n)^{\oplus k_n}, \qquad gr^W_{2n+1} M = 0$$

for some natural numbers k_n. Using this fact,we can define a fiber functor $\omega : \mathcal{MTM}(k) \to Vect_{\mathbb{Q}}$ defined as

$$\omega(M) := \bigoplus_n Hom(\mathbb{Q}(-n), gr^W_{2n}(M))$$

This functor is faithful because if all *Hom*-sets are zero then M itself is zero. It is also $\mathbb{Q}$-linear, exact and respects the tensor product. This means that $\mathcal{MTM}(k)$ along with ω is a neutral Tannakian category and so by Tannakian formalism it is equivalent to the category of finite-dimensional representation of some pro-algebraic affine group k-scheme $\mathcal{G}$ (see [13] for an introduction to Tannakian categories). Inside, $\mathcal{MTM}(k)$ we have a simpler subcategory $\mathcal{TM}(k)$, the category of pure Tate motives where all the objects are direct sums of $\mathbb{Q}(n)$ for $n \in \mathbb{Z}$. ω is also a fiber functor on $\mathcal{TM}(k)$ and the associated pro-algebraic group scheme is just $\mathbb{G}_m$ because of the grading. The inclusion $\mathcal{TM}(k) \to \mathcal{MTM}(k)$ implies the following surjection.

$$\mathcal{G} \to \mathbb{G}_m \tag{3.9}$$

Moreover, $Hom(\mathbb{Q}(-n), gr^W_{2n}(M))$ is a vector space, but we can consider it as the Tate motive $Hom(\mathbb{Q}(-n), gr^W_{2n}(M)) \otimes \mathbb{Q}(-n)$ in weight $2n$. This helps us to upgrade ω to the functor,

$$\omega : \mathcal{MTM}(k) \to \mathcal{TM}(k)$$

which gives us the following map of group scheme,

$$\mathbb{G}_m \to \mathcal{G}$$

This map provides an action of $\mathbb{G}_m$ on $\mathcal{G}$. Let A be the ring of functions on $\mathcal{G}$. The action of $\mathbb{G}_m$ on $\mathcal{G}$ induces an action on A and, therefore, induces a grading on it. Using Proposition 2.2, 2.3 and section 2.4 of [12], we conclude the following theorem,

Theorem 3.2.5. *Let A be the Hopf algebra such that $\mathcal{G} := Spec(A)$, then A is a connected non-negatively graded Hopf algebra i.e. $A = \oplus A_n$. Moreover, the graded dual $A^\vee := \oplus A_n^\vee$ of A is the graded tensor algebra generated by the graded vector space $\oplus_n Ext^1(\mathbb{Q}(0), \mathbb{Q}(n))^\vee$, where $Ext^1(\mathbb{Q}(0), \mathbb{Q}(n))^\vee$ is in degree $2n$.*

Definition 1.4 and section of 1.6 of [12], also gives a description of mixed Tate motives unramified over $O_{k,S}$, where S is a finite set of non-archimedean places.

Definition 3.2.6. *The* ***category of mixed Tate motives unramified over*** $O_{k,S}$ *denoted by* $\mathcal{MTM}(O_{k,S})$ *is the full subcategory of $\mathcal{MTM}(k)$ consisting of objects M such that for all $n \in \mathbb{Z}$, the element*

$$\begin{aligned} gr^W_{2n}(M) \otimes \mathbb{Q}(n) \in Ext^1(\mathbb{Q}(0)^{\oplus k_n}, \mathbb{Q}(1)^{\oplus k_{n-1}}) &= Ext^1(\mathbb{Q}(0), \mathbb{Q}(1))^{\oplus k_n k_{n-1}} \\ &= k^\times \otimes \mathbb{Q}^{\oplus k_n k_{n-1}} \end{aligned} \tag{3.10}$$

is contained in $O^\times_{k,S} \otimes \mathbb{Q}^{\oplus k_n k_{n-1}}$

For $\mathcal{MTM}(O_{k,S})$, even $O^\times_{k,S} \otimes \mathbb{Q}$ is finite-dimensional by [5] or by Dirichlet's theorem, so in this case the graded dual of the corresponding A would be finite dimensional in each grade. In the next section, we will see another way to describe A.

3.2.1 Mixed Tate Category

In this section, we will recall what framed objects are and how they can also be used to construct the Hopf algebra A mentioned in the previous section. We will construct this generally for a mixed Tate category and later on show that in our case they are also motivic representation of periods. In the end, we will give an example of such a framed object. More details for this section can be found in appendix of [25].

Definition 3.2.7. *Let k be a field. Let $\mathcal{T}$ be a Tannakian k-linear category and $k(1)$ be an invertible object in $\mathcal{T}$ with inverse $k(-1)$. Set $k(\pm n) := k(\pm 1)^{\otimes n}$ for $n \in \mathbb{N}$ and $k(0)$ is the unit for tensor product. Then the pair $(\mathcal{T}, k(1))$ is called* ***mixed Tate category*** *if it satisfies the following conditions,*

1. *$k(n)$ are all the simple objects for $n \in \mathbb{Z}$.*
2. *$k(n)$ are mutually non-isomorphic objects for $n \in \mathbb{Z}$.*
3. *$Ext^1(k(0), k(n)) = 0$ if $n \leq 0$.*

Example 3.2.8. *$\mathcal{MTM}(k)$ is a mixed Tate category.*

A **functor** of mixed Tate categories $\phi : (\mathcal{T}_1, k(1)_1) \to (\mathcal{T}_2, k(1)_2)$ is a functor such that $\phi(k(1)_1) = k(1)_2$. Any object M in a mixed Tate category $(\mathcal{T}, k(1))$ is an extension of simple objects i.e. $k(n)$. The extensions of the simple objects are given by the third condition in Definition 3.2.7. As we discussed in the previous section, this condition gives us a weight filtration for every object in $\mathcal{MTM}(k)$. We can apply the same argument to any mixed Tate category to get an increasing weight filtration $W_\bullet M$ such that,

$$gr^W_{2n} M = k(-n)^{\oplus i_n}, \qquad gr^W_{2n+1} M = 0$$

for some non-negative integers i_n. We also have a fiber functor $\omega : \mathcal{T} \to Vect_k$, where $Vect_k$ is the category of finite dimensional k-linear vector spaces analogous to the ω functor for $\mathcal{MTM}(k)$,

$$\omega(M) := \bigoplus_n Hom_{\mathcal{T}}(k(-n), gr^W_{2n} M)$$

By Tannakian formalism $\mathcal{T}$ is equivalent to the category of finite dimensional representations of the pro-algebraic group $\mathcal{G} = \mathrm{Spec}(A)$, where $A = A(\mathcal{T})$ is a Hopf algebra over k. As we have seen in the previous section, there is an action of $\mathbb{G}_m$ on $\mathcal{G}$ and therefore also on A. This gives a grading on A. Another way to see the grading on A is to consider the codomain of ω to be the category of finite dimensional graded k-vector spaces. In [11], Deligne gives a recipe to construct A as an ind-object in the target category of ω which in our case is the category of finite dimensional graded vector spaces, giving A a grading.

Remark 3.2.9. *For the rest of this chapter, vector spaces, modules and comodules will be assumed to be finite dimensional over k.*

A Brief Description of the Hopf Algebra A and its Lie algebra

We will use Lemma 2.12 to Remark 2.17 of [13] to give a description of the Hopf algebra A. It is an ind-Hopf algebra to be precise therefore $\mathcal{G} = \mathrm{Spec}(A)$ is a pro-algebraic group. In this section, we assume the target category of ω is the category of graded vector spaces.

Let $\mathcal{T}$ be a mixed Tate category. Let X be an object of $\mathcal{T}$ and let $\mathcal{T}_X$ be the smallest full subcategory of $\mathcal{T}$ generated by subquotients of $X^n \oplus (X^\vee)^m$ where $n, m \in \mathbb{N}$. It is important to note that $\mathcal{T}_X$ is closed under $\otimes$. The restriction of ω to $\mathcal{T}_X$, will be denoted by ω_X.

Let $B_X := End(\omega_X)$ be the k-algebra of endomorphisms of the functor ω. The composition of endomorphisms gives B_X an algebra structure. It is a sub-k-algebra of $End(\omega(X))$, which is finite dimensional and therefore B_X are finite dimensional k-algebras. This also implies that its dual (in the sense of vector spaces) is a coalgebra which we will denote by A_X. By Lemma 2.13 of loc. cit., $\mathcal{T}_X$ is equivalent to the category of comodules of A_X. The inclusions of subcategories $\mathcal{T}_X \to \mathcal{T}_Y$ for objects Y and $X \in \mathcal{T}_Y$ implies that $(A_X)_{X\in ob(\mathcal{T})}$ is a directed system.

Theorem 3.2.10. ***(Proposition 2.14, [13])*** *Let A be the ind-coalgebra* $\varinjlim A_X$. *Then the category* $(\mathcal{T}, \omega)$ *is tensor equivalent to the category of graded comodules* $(\mathrm{CoMod}_A, \text{forgetful})$ *of A.*

The coalgebra A also inherits an algebra structure due to the symmetric monoidal structure $\otimes$, which also makes A a commutative algebra. More precisely, $\otimes$ is a functor

$$\otimes : \mathrm{CoMod}_A \times \mathrm{CoMod}_A \to \mathrm{CoMod}_A$$

By Proposition 2.16 of loc. cit., this induces a k-linear map $\mu : A \otimes A \to A$ satisfying associativity and commutativity. The symmetric monoidal structure is also compatible with the inclusions of Tannakian subcategories mentioned above, therefore each A_X is also a bialgebra. The antipode exists due to rigidity of $\mathcal{T}$ as mentioned in discussion following Proposition 2.16 in loc.cit.

As mentioned in Appendix A of [25], the graded pro-Lie algebra is defined as,

$$L_\bullet(\mathcal{T}) := Der(\omega) = \{F \in End(\omega) | F_{X\otimes Y} = F_X \otimes id_Y - id_X \otimes F_Y \quad \forall X, Y \in ob(\mathcal{T})\} \tag{3.11}$$

We can again use similar argument as above to get a Lie coalgebra $\mathcal{L}_\bullet(\mathcal{T})$. It is also the graded dual of $L_\bullet(\mathcal{T})$ Alternatively, we can define $\mathcal{L}_\bullet(\mathcal{T})$ as,

$$\mathcal{L}_\bullet(\mathcal{T}) = \frac{A^\vee_{>0}}{A^\vee_{>0} \cdot A^\vee_{>0}} \tag{3.12}$$

where $A^\vee_{>0}$ is the kernel of the surjective map $A^\vee \to k$ where k is in degree zero. If Δ is the coproduct of $A^\vee$, then we can define the cobracket δ as,

$$\delta(X) = \Delta(X) - (1 \otimes X + X \otimes 1)$$

All these descriptions of $\mathcal{L}_\bullet(\mathcal{T})$ are the same. The $\mathbb{G}_m$ action on A gives a grading on A and also its dual which is $End(\omega)$. The n-th graded component denoted by $End(\omega)_n$ consists of endomorphism F such that $F(W_m M) \subset W_{n+m}M$ for all $M \in \mathcal{T}$. These facts and constructions are mentioned in detail in Appendix A of [25].

3.2.2 Framed Motives

Let $\mathcal{T}$ be a mixed Tate category. We will create a connected graded Hopf algebra $\mathcal{A}_\bullet$ and show in theorem (3.2.21) that $\mathcal{A}_\bullet \cong A(\mathcal{T})$ as a graded Hopf algebra.

Definition 3.2.11. *Let $n \geq 0$. An n-**framed motive** $[M, v_0, f_n]$ consists of three things,*

- *An object M of $\mathcal{T}$.*
- *A non-zero map $v_0 : k(0) \to gr_0^W(M)$*
- *A non-zero map $f_n : gr_{-2n}^W(M) \to k(n)$*

There is an equivalence relation on framed motives which is inspired from the equivalence of periods. For more details, one should refer to [40, 36].

Definition 3.2.12. *Let $k \subset \mathbb{C}$. A **naive period** P over k is a complex number whose real and imaginary parts are integrals of the form*

$$P = \int_G \eta$$

where G is a semi-algebraic subset of $\mathbb{R}^m$ and is equidimensional of dimension d and η is a rational differential d-form with coefficients in k and has no poles in G.

Remark 3.2.13. *We usually take k to be a number field.*

Let X be a compact algebraic variety over $\mathbb{R}$. A period needs a domain of integration G such that ∂G is an algebraic set and an algebraic differential form η. The domain G gives us an element of $H_d(X, \partial G)$ and the form η is an element of $H^d(X - \partial G)$ because it has be defined on G. Then, the integral is an k-linear map

$$H^d(X - \partial G) \otimes H_d(X, \partial G) \to \mathbb{C}.$$

maps $\eta \otimes G$ to the integral P. This map also induces an isomorphism

$$H^d(X - \partial G) \otimes \mathbb{C}(H_d(X, \partial G) \otimes \mathbb{C})^\vee. \tag{3.13}$$

We choose a basis $\{e^1, e^2, \cdots, e^m\}$ of $H^d(X - \partial G)$ with $\eta = e^1$. Similarly, we choose a basis $\{f_1, f_2, \cdots, f_m\}$ of $H_d(X, \partial G)$ with $G = f_1$, which in turn induces a dual basis $\{f^1, f^2, \cdots, f^m\}$ of $(H_d(X, \partial G) \otimes \mathbb{C})^\vee$. The choice of basis of $H^d(X - \partial G)$ induces a splitting of $H^d(X - \partial G)$ and so η induces a map $\mathbb{Q}(d) \to H^d(X - \partial G)$ which factors through $gr_{-2d}^W(H^d(X - \partial G))$ if η is of the form $\wedge^i dlog(g_i)$ for rational functions g_i on $X - \partial G$. Similarly, for G, by duality, a similar argument shows that G induces a map $gr_0^W(H^d(X - \partial G)) \to \mathbb{Q}$. The choice of basis will a matrix out of the isomorphism 3.13 called the period matrix. The $(1, 1)$ entry of this matrix is the integral P. This digression links the framing of the motive $H^d(X - \partial G)$ mentioned in Definition 3.2.11 and the period P and therefore a framed motive can be thought of as a formal version of a period. For a more explicit description, see chapter 12,13 of [36].

The periods have an equivalence relation given by the usual change of variable formula. Let $f : (X, D) \to (Y, E)$ be a map. Suppose we have a form $\eta \in H^d(Y, E)$ and an appropriate subset $G \subset X$ such that $\partial G \cup D$ is an SNC divisor. Then we have,

$$\int_G f^*\eta = \int_{f_*G} \eta$$

We want the framed motives to reflect this equality of periods, so we will impose an equivalence relation on framed motives which mirrors the equality of periods.

Definition 3.2.14. *The equivalence relation on n-framed motives is the coarsest equivalence relation such that for any $M, M' \in ob(\mathcal{T})$ and any morphism $q : M \to M'$ and any maps $f'_n : gr^W_{-2n}(M') \to k(n)$ and $v_0 : k(0) \to gr^W_0(M)$, $[M, v_0, q^*(f'_n)] \sim [M', q_*(v_0), f'_n]$, where q_* and q^* are induced from the map q*

$$q_* : Hom(k(0), gr^W_0(M)) \to Hom(k(0), gr^W_0(M'))$$
$$q^* : Hom(gr^W_{-2n}(M), k(n)) \to Hom(gr^W_{-2n}(M'), k(n))$$

We will denote, by $\mathcal{A}_n$, the set of equivalence classes of n-framed motives.

Lemma 3.2.15. *Any framed motive $[M, v_0, f_n]$ is equivalent to $[M', v'_0, f'_n]$ where the weight filtration on M' is concentrated in degrees $[-2n, 0]$ and the 0-th graded components and $-2n$-th graded components are one dimensional.*

Proof. We have the canonical quotient and inclusion maps respectively

$$M \to M/W_{-2(n+1)}M \text{ and } W_0M/W_{-2(n+1)}M \to M/W_{-2(n+1)}M.$$

Both of these maps respect the framing so, by equivalence relation, we can replace M by $W_0M/W_{-2(n+1)}M$. Therefore, we can assume the weight filtration is concentrated in degrees $[-2n, 0]$. To show the 0-th graded component can be made to be one-dimensional, we have the following pullback square,

$$\begin{tikzcd} M_1 \arrow[r] \arrow[d] & k(0) \arrow[d, "v_0"] \\ M \arrow[r] & gr^W_0(M) \end{tikzcd}$$

So replacing M by M_1, we can reduce to the case where 0-th graded component is one-dimensional. For the $-2n$-th graded component, we have the following pushout diagram,

$$\begin{tikzcd} gr^W_{-2n}(M) \arrow[r] \arrow[d, "f_n"] & M \arrow[d] \\ k(n) \arrow[r] & M_2 \end{tikzcd}$$

Again, by replacing M by M_2, we get the case where $-2n$-th graded component is one dimensional. □

Because of the Lemma 3.2.15, we can assume that M is concentrated in weights $[0, -2n]$ with the top and bottom graded components being one-dimensional.
The set $\mathcal{A}_n$ is not only a set but an Abelian group with addition defined as,

$$[M, v_0, f_n] + [M', v'_0, f'_n] = [M \oplus M', (v_0, v'_0), f_n + f'_n]$$

The addition is well defined and the identity is the split extension $[k(0) \oplus k(n), i_1, \pi_2]$ where i_1 is the inclusion map and π_2 is the projection map. The inversion is,

$$-[M, v_0, f_n] = [M, -v_0, f_n] = [M, v_0, -f_n]$$

See Section 4 of chapter 3 of [23] for the proofs of these facts.

Definition 3.2.16. *We define $\mathcal{A}_\bullet$ to be $\bigoplus_{n\geq 0} \mathcal{A}_n$.*

We also note that $\mathcal{A}_0 = k$. Indeed, for any 0-framed motive $[M, v_0, f_0]$, we have the composition $f_0 \circ v_0 \in End_{\mathcal{T}}(k(0)) = k$. This gives us a map $\mathcal{A}_0 \to k$. It is well defined because if $[M, v_0, q^*(f_0')] \sim [M', q_*(v_0), f_0']$ for some morphism $q : M \to M'$, then

$$k(0) \xrightarrow{v_0} gr_0^W(M) \xrightarrow{q} gr_0^W(M') \xrightarrow{f_0'} k(0) \tag{3.14}$$

can be interpreted both as $q^*(f_0') \circ v_0$ or as $f_0' \circ q_*(v_0)$. Moreover, by the equivalence relation and simplicity of $k(0)$, we can assume $M = k(0) \oplus k(0)$ in the case $f_0 \circ v_0 = 0$ and $M = k(0)$ in the case $f_0 \circ v_0 \neq 0$. In the first case it is the zero element of $\mathcal{A}_0$ and for the second case, the framed motive is $[k(0), v_0, f_0]$ which by scaling v_0 we can assume to be equivalent to $[k(0), 1, f_0]$. Writing f_0 as lu where $l \in k$ and $u \in k^\vee$ such that $u(1) = 1$, we have $[k(0), 1, f_0] = l[k(0), 1, u]$. Therefore, every element in $\mathcal{A}_0$ is a multiple of $[k(0), 1, u]$. This gives us the bijection.

We also have a graded product on $\mathcal{A}_\bullet$. Let $[M, v_0, f_n]$ and $[M', v_0', f_m']$ be an n-framed motive and an m-framed motive respectively. We define their product as

$$[M, v_0, f_n] \otimes [M', v_0', f_m'] = [M \otimes M', v_0 \otimes v_0', f_n \otimes f_m']$$

The associativity and commutativity follows from the symmetric monoidal structure on $\mathcal{MTM}(k)$ and its properties.

Remark 3.2.17. *This tensor product defines an action of $\mathcal{A}_0$ on $\mathcal{A}_n$. It can be checked that it defines a k-vector space structure on $\mathcal{A}_n$.This makes $\mathcal{A}_\bullet$ a graded connected k-algebra.*

It remains to see the definition of the coproduct. It is defined as the sum of maps

$$\Delta_{p,n-p} : \mathcal{A}_n \to \mathcal{A}_p \otimes \mathcal{A}_{n-p} \tag{3.15}$$

for $0 \leq p \leq n$. Let $[M, v_0, f_n]$ be an element of $\mathcal{A}_n$ and let $\{b_i\}_{1\leq i\leq m}$ be a basis of $gr_{-2p}^W(M)$ and $\{b_i^\vee\}_{1\leq i\leq m}$ be its dual basis. Then,

$$\Delta_{p,n-p}([M, v_0, f_n]) := \sum_{i=1}^{m} [M, b_i, f_n](-p) \otimes [M, v_0, b_i^\vee] \tag{3.16}$$

Lemma 3.2.18. *$\Delta_{p,n-p}$ is well defined.*

Proof. Let $\{c_i\}$ be another basis of $gr_{-2p}^W(M)$. Then there exists an invertible matrix $D = \{d_{kl}\}$ such that $Db_i = c_i$. Then, by definition of dual space, we find that $c_i^\vee$ is equal to $D^{-1}b_i^\vee$.

Denoting $D^{-1} = G$, we get,

$$\begin{aligned}
\Delta_{p,n-p}([M, v_0, f_n]) &= \sum_{i=1}^{m} [M, c_i, f_n](-p) \otimes [M, v_0, c_i^\vee] \\
&= \sum_{i=1}^{m} \sum_{j=1}^{m} \sum_{k=1}^{m} d_{ik}[M, b_k, f_n](-p) \otimes g_{ij}[M, v_0, b_j^\vee] \\
&= \sum_{j=1}^{m} \sum_{k=1}^{m} \left(\sum_{i=1}^{m} g_{ij} d_{ik} \right) [M, b_k, f_n](-p) \otimes [M, v_0, b_j^\vee] \\
&= \sum_{j=1}^{m} \sum_{k=1}^{m} \delta_{kj}[M, b_k, f_n](-p) \otimes [M, v_0, b_j^\vee] \\
&= \sum_{j=1}^{m} [M, b_j, f_n](-p) \otimes [M, v_0, b_j^\vee]
\end{aligned}$$

where we used $\sum_{i=1}^{m} g_{ij} d_{ik} = \delta_{kl}$ by definition of inverse. □

Summing over all possibilities of p in $\Delta_{p,n-p}$, we get

$$\Delta : \mathcal{A}_\bullet \to \mathcal{A}_\bullet \otimes \mathcal{A}_\bullet$$

which is defined as,

$$\Delta([M, v_0, f_n]) := \sum_{0 \le p \le n} \Delta_{p,n-p}([M, v_0, f_n]) \tag{3.17}$$

The coproduct is compatible with the product. Let $\{b_i^n\}$ be a basis of $gr_n^W(M)$ and let $\{c_j^m\}$ be a basis of $gr_m^W(M')$. Then,

$$\begin{aligned}
&\Delta\left([M, v, f][M', v', f']\right) = \Delta\left([M \otimes M', v \otimes v', f \otimes f']\right) \\
&= \sum_{n,m} \sum_{i,j} [M \otimes M', b_i^n \otimes c_j^m, f \otimes f'](-n-m) \otimes [M \otimes M', v \otimes v', (b_i^n \otimes c_j^m)^\vee] \\
&= \sum_{n,m} \sum_{i,j} [M \otimes M', b_i^n \otimes c_j^m, f \otimes f'](-n-m)) \otimes [M \otimes M', v \otimes v', (b_i^n)^\vee \otimes (c_j^m)^\vee] \\
&= \sum_{n,m} \sum_{i,j} [M, b_i^n, f](-n)[M', c_j^m, f'](-m) \otimes [M, v, (b_i^n)^\vee][M', v', (c_j^m)^\vee] \\
&= \sum_{n,m} \sum_{i,j} \left(\otimes[M, b_i^n, f](-n) \otimes [M, v, (b_i^n)^\vee]\right)\left([M', c_j^m, f'](-m) \otimes [M', v', (c_j^m)^\vee]\right) \\
&= \left(\sum_{i,n} [M, b_i^n, f](-n) \otimes [M, v, (b_i^n)^\vee] \right) \left(\sum_{j,m} [M', c_j^m, f'](-m) \otimes [M', v', (c_j^m)^\vee] \right) \\
&= \Delta\left([M, v, f]\right) \Delta\left([M', v', f']\right)
\end{aligned}$$

where we used the fact

$$gr_n^W(M \otimes M') = \oplus_{i+j=n} gr_i^W(M) \otimes gr_j^W(M').$$

This shows that $\mathcal{A}_\bullet$ is a connected non-negatively graded bi-algebra. By Proposition 1.4.14 of [30], it is automatically Hopf algebra.

Remark 3.2.19. *The antipode of* $[M, v, f]$ *is* $[M^\vee, f, v]$ *as mentioned on page 243 of [19].*

Definition 3.2.20. *The Hopf algebra* $\mathcal{A}_\bullet$ *is called the fundamental Hopf algebra of the mixed Tate category* $\mathcal{T}$.

The next task in order is to show that $\mathcal{A}_\bullet$ is isomorphic to A above.

Theorem 3.2.21. *The graded Hopf algebra* $\mathcal{A}_\bullet$ *is isomorphic to* $A(\mathcal{T})$*, where* $A(\mathcal{T})$ *is the ring of functions of the Tannakian group* $\mathcal{G}$.

Proof. Let $X \in \mathcal{T}$ and let $A_X := End(\omega_X)^\vee$ be the coalgebra corresponding to the category $\mathcal{T}_X$ as mentioned in Theorem 3.2.10 and the discussion preceding it. Let $\mathcal{A}_{\bullet,X}$ be the sub-coalgebra of $\mathcal{A}_\bullet$ generated by framed objects $[M, v_0, f_n]$ where $M \in \mathcal{T}_X$. We will first construct an isomorphism ϕ_X of graded coalgebra from $\mathcal{A}_{\bullet,X}$ to A_X. Taking limits over X, we will get an isomorphism between $\mathcal{A}_\bullet$ and A. The last step is to show that ϕ also respects the algebra structure. Let $[M, v_0, f_n] \in \mathcal{A}_{n,X}$. Since F is a natural transformation, it is collection of maps $F_M : \omega(M) \to \omega(M)$ for all M in $\mathcal{T}_X$. Furthermore, if $F \in End(\omega_X)_n$ then $F_M(\omega(W_m(M))) \subset \omega(W_{n+m}(M))$ for all M in $\mathcal{T}_X$. We construct the map $\phi_X : \mathcal{A}_{\bullet,X} \to A_X$ by defining $\phi_X([M, v_0, f_n])$ to be the element such that

$$< \phi_X([M, v_0, f_n]), F >:=< F_M(v_0), f_n >$$

where $< \cdot, \cdot >$ on the right hand side is the evaluation on $gr^W_{2n}(M) \otimes gr^W_{2n}(M)^\vee$. To unpack the definition a bit, by the definition of $End(\omega_X)_n$ and the fact that we can think of v_0 as an element of $gr^W_0(M)$ as the image of $1 \in \mathbb{Q}(0)$ then $F_M(v_0) \in W_{2m}(M)$ and therefore it gives an element of $gr^W_{2m}(M)$.

First point of note is that ϕ_X is well defined. If we have a map $q : M \to M'$ which respects the framing then we have the commutative diagram

$$\begin{tikzcd}
\omega(M) \arrow[r, "q_*"] \arrow[d, "F_M"] & \omega(M') \arrow[d, "F_{M'}"'] \\
\omega(M) \arrow[r, "q_*"] & \omega(M')
\end{tikzcd}$$

So we have,

$$\begin{aligned}
< \phi_X([M, v_0, q^*(f'_n)]), F > &=< F_M(v_0), q^*(f'_n) > \\
&=< q_*F_M(v_0), f'_n > \\
&=< F_{M'}(q_*(v_0)), f'_n > \\
&=< \phi_X([M', q_*(v_0), f'_n]), F >
\end{aligned}$$

Hence the claim holds.

Secondly, ϕ_X is a morphism of graded coalgebras. It is graded by definition so it suffices to show that ϕ_X is a morphism of coalgebras. The coproduct structure on $End(\omega_X)^\vee$ is the dual of the product structure (which is composition) on $End(\omega_X)$ i.e. if $\eta \in End(\omega_X)^\vee$ and $F, G \in End(\omega_X)$ then $\Delta\eta(G \otimes F) = \eta(G \circ F)$ which in terms of evaluation is $< \Delta\eta, G \otimes F >=< \eta, G \circ F >$. To show that ϕ_X preserves coproduct we need to show that

$$< (\phi_X \otimes \phi_X)(\Delta[M, v_0, f_n]), (F \otimes G) >=< \Delta(\phi_X([M, v_0, f_n])), (F \otimes G) >=< \phi_X([M, v_0, f_n]), F \circ G >$$

Assume without loss of generality that $F \in End(\omega_X)_c$ and $G \in End(\omega_X)_d$ such that $c + d = n$. Using definition (3.17), we get

$$< (\phi_X \otimes \phi_X)(\Delta[M, v_0, f_n]), (F \otimes G) >= \sum_{0 \le p \le n} < (\phi_X \otimes \phi_X)\Delta_{p,n-p}([M, v_0, f_n]), F \otimes G > \quad (3.18)$$

A typical term in $\Delta_{p,n-p}([M, v_0, f_n])$ looks like $[M, b_i, f_n](-p) \otimes [M, v_0, b_i^\vee]$. This implies that

$$\begin{aligned} &< (\phi_X \otimes \phi_X)([M(-p), b_i, f_n] \otimes [M, v_0, b_i^\vee]),(F \otimes G) > \\ &\qquad =< \phi_X([M, b_i, f_n](-p)) \otimes \phi_X([M, v_0, b_i^\vee]), (F \otimes G) > \\ &\qquad =< \phi_X([M, b_i, f_n](-p)), G >< \phi_X([M, v_0, b_i^\vee], F > \\ &\qquad =< G_{M(-p)}(b_i(-p)), f_n(-p) >< F_M(v_0), b_i^\vee > \end{aligned}$$

If $c > p$, then $b_i^\vee = 0$ in $gr^W_{2c}(M)$, therefore, $< F_M(v_0), b_i^\vee >= 0$. If $c < p$, then by a similar argument $< G_{M(-p)}(b_i), f_n >= 0$. Therefore, the only non-zero term that we have is from the contribution of $\Delta_{p,n}$. Let $\{b_s\}_{s=1}^k$ be a basis of $gr^W_{2c}(M)$. To emphasize that an element x is in $gr^W_{2c}(M)$, we will denote it as x^c. Then $F_M(v_0) = \sum_{s=1}^k a_s b_s^c$ which also means that $< F_M(v_0), b_s^\vee >= a_s$. We do the same with f_n and write it as f_n^n. Simplifying $< (GF)_M(v_0), f_n^n >$ we get

$$< (GF)_M(v_0), f_n >=< G_M \circ F_M(v_0), f_n^n >=< G_M\left(\sum_{i=1}^k a_i b_i^c\right), f_n(n) >= \sum_{i=1}^k a_i < G_M(b_i^c, f_n^n > \quad (3.19)$$

where we used the fact that $< F_M(v_0), (b_s^c)^\vee >= a_s$. The term $< G_M(b_s^c), f_n^n >$ is equal to $< G_{M(-s)} b_s^0, f_n^{n-c} >=< \phi([M, b_s, f_n](-c)), G >$ putting it back into (3.19), we get,

$$\begin{aligned} < (GF)_M(v_0), f_n > &= \sum_{s=1}^k a_i < \phi_X([M, b_s^c, f_n](-c)) > \\ &= \sum_{s=1}^k < F_M(v_0), b_s^\vee >< \phi_X([M, b_s, f_n](-c)), G > \\ &= \sum_{s=1}^k < \phi_X([M, b_s, f_n](-c)), G >< \phi_X([M, v_0, b_s^\vee]), F > \\ &= \sum_{s=1}^k < \phi([M, b_s, f_n](-c) \otimes [M, v_0, b_s^\vee]), G \otimes F > \end{aligned}$$

so we get,

$$< (GF)_M(v_0), f_n >=< \phi_X\left(\sum_{s=1}^k [M, b_s, f_n](-c) \otimes [M, v_0, b_s^\vee]\right), G \otimes F >=< \phi_X(\Delta[M, v_0, f_n]), G \otimes F > \quad (3.20)$$

On the other hand, we also have

$$< (GF)_M(v_0), f_n >=< \phi_X([M, v_0, f_n]), G \circ F >=< \Delta(\phi_X([M, v_0, f_n])), G \otimes F > \quad (3.21)$$

Combining (3.20) and (3.21), we can conclude that ϕ_X preserves coproduct. Therefore ϕ_X is a coalgebra morphism.

Now, we will prove that ϕ_X is surjective. First we construct a functor $\Psi : \mathcal{T} \to \mathrm{CoMod}(\mathcal{A}_\bullet)$ where $\mathrm{CoMod}(\mathcal{A}_\bullet)$ is the category of graded comodules of $\mathcal{A}_\bullet$. Let M be an object in $\mathcal{T}$. Then we will construct the $\mathcal{A}_\bullet$-comodule structure Δ_ψ on $\omega(M) = \bigoplus_n Hom_{\mathcal{T}}(k(-n), gr^W_{2n}M)$. Let $\{f^n_i\}_{i,n}$ be the basis of $\omega(M)$ such that $f^n_i \in Hom_{\mathcal{T}}(k(-n), gr^W_{2n}M)$ for all i. Then the comodule structure is defined as,

$$\begin{aligned}\omega(M) &\xrightarrow{\Delta_{\psi_X}} \omega(M) \otimes \mathcal{A}_\bullet \\ f^n_i &\mapsto \sum_{m>n} \sum_j f^m_j \otimes [M, f^n_i, (f^m_j)^\vee](-n)\end{aligned}$$

It can be checked using this definition that this map provides a $\mathcal{A}_\bullet$-comodule structure on $\omega(M)$. The functor Ψ sends M to $\omega(M)$ with this comodule structure. The fact that morphism is sent to a morphism follows from the equivalence relation imposed in the definition of $\mathcal{A}_\bullet$ by an argument similar to the proof of well definedness of ϕ_X. Restricting Ψ to $\mathcal{T}_X$, the functor Ψ then factors through $\mathrm{CoMod}(\mathcal{A}_{\bullet,X})$. Let $\Psi_X : \mathcal{T}_X \to \mathrm{CoMod}(\mathcal{A}_{\bullet,X})$ be the functor induced from factorization of Ψ. The functor also has an additional property; composing Ψ_X with the forgetful functor $q_X : \mathrm{CoMod}(\mathcal{A}_{\bullet,X}) \to Vect_k$ yields ω_X i.e. $q_X \circ \Psi_X = \omega_X$. This also means Ψ_X is faithful because ω_X is faithful. Combining $q_X \circ \Psi_X = \omega_X$ with the map $End(q_X) \to End(q_X \circ \Psi_X)$ and dualizing we get a graded coalgebra map $\psi : End(\omega_X)^\vee \to \mathcal{A}_{\bullet,X}$. On the other hand, the map ϕ_X induces a functor $\Phi_X : \mathrm{CoMod}(\mathcal{A}_{\bullet,X}) \to \mathrm{CoMod}(A_X) \cong \mathcal{T}_X$. We will now calculate $\phi_X \circ \psi_X$. As $M \in \mathcal{T}_X$, by Tannakian formalism, we can view it as $\omega_X(M)$ with an $End(\omega_X)$-module structure (or equivalently, $End(\omega_X)^\vee$-comodule structure because $End(\omega_X)$ is finite dimensional). If $F \in End(\omega_X)_n$, then

$$\begin{aligned}F(f^n_i) &= \sum_{m>n} \sum_j f^m_j \otimes < F(f^n_i), (f^m_j)^\vee > \\ &= \sum_{m>n} \sum_j f^m_j \otimes < \phi_X([M, f^n_i, (f^m_j)^\vee](-n)), F > \\ &= \sum_{m>n} \sum_j f^m_j \otimes ev(\phi_X([M, f^n_i, (f^m_j)^\vee](-n)) \otimes F) \\ &= \sum_{m>n} \sum_j (id \otimes ev)(f^m_j \otimes \phi_X([M, f^n_i, (f^m_j)^\vee](-n)) \otimes F) \\ &= (id \otimes ev)(\sum_{m>n} \sum_j f^m_j \otimes \phi_X([M, f^n_i, (f^m_j)^\vee](-n)) \otimes F) \\ &= (id \otimes ev)((id \otimes \phi_X \otimes id)(\sum_{m>n} \sum_j f^m_j \otimes [M, f^n_i, (f^m_j)^\vee](-n)) \otimes F) \\ &= (id \otimes ev)(id \otimes \phi_X \otimes id)(\Delta_{\psi_X}[M, f^n_i, (f^m_j)^\vee](-n) \otimes F)\end{aligned}$$

The last expression is precisely the expression for the $End(\omega_X)$-module structure on $\omega_X(M)$ induced from the $End(\omega_X)^\vee$-comodule structure Δ_{ψ_X}. This means that $(\Phi_X \circ \Psi_X)(M)$ and M are the same as $End(\omega_X)$-modules and therefore also as $End(\omega_X)^\vee$-comodules. In short, $\Phi_X \circ \Psi_X$ acts as identity on the objects. The morphisms in $\mathcal{T}_X$ are linear maps respecting the $End(\omega_X)^\vee$-comodule structure and as linear maps they are just sent to themselves under $\Phi_X \circ \Psi_X$. Therefore, we conclude that $\Phi_X \circ \Psi_X = id$ and the corresponding maps of coalgebras is $\phi_X \circ \psi_X = id$, which implies that ϕ_X is surjective.

Taking inductive limit over $X \in \mathcal{T}$, we get a graded surjective map $\phi : \mathcal{A}_\bullet \to A$ respecting the coalgebra structure.

We will now show that ϕ is injective. Let $[M, v_0, f_n]$ be a framed motive such that $\phi([M, v_0, f_n])$ is zero. Without loss of generality, we can assume M is concentrated in weights $[-2n, 0]$, $gr^W_0(M) = k(0)$ and $gr^W_{-2n}(M) = k(n)$. Then $M \in \mathcal{T}_M$ and so $\phi([M, v_0, f_n]) = \phi_M([M, v_0, f_n])$. Viewing M as a $End(\omega_M)$-module, let N be the cyclic $End(\omega_M)$-submodule generated by $k(0)$. If we show that $gr^W_{-2n}(N) = 0$ then we would have maps,

$$k(0) \oplus k(n) \leftarrow N \oplus k(n) \rightarrow M$$

where the left map is induced by the quotient map of N by $W_1 N$ and the right map is inclusion. So $[M, v_0, f_n]$ would be equivalent to the trivial framed motive. To show that $gr^W_{-2n}(N)$ is zero, recall that $\phi([M, v_0, f_n]) = 0$ implies that $< \phi([M, v_0, f_n]), F >= 0$ for any endomorphism $F \in End(\omega)_n$. This means that $< F_M v_0, f_n >= 0$ for any $F \in End(\omega)_n$. On the other hand, the n-th graded part of N precisely consists of the elements $F_M v_0$ for $F \in End(\omega)_n$ due to grading consideration. Combining this fact with the equation $< F_M v_0, f_n >= 0$ implies that $gr^W_{-2n}(N) = 0$. Therefore, ϕ is injective.

Combining all arguments above implies that ϕ is an isomorphism of graded coalgebra with ψ as its inverse.

To show ϕ also respects the algebra structure, we will instead show that ψ respects the algebra structure. This will follow from the fact that Ψ is a $\otimes$-functor by Proposition 2.16 of [13]. First note that,

$$gr^W_n(M \otimes N) = \oplus_{i+j=n} gr^W_i(M) \otimes gr^W_j(N) \tag{3.22}$$

where $M, N \in \mathcal{T}$. Secondly, the A-comodule structure on $\omega(M) \otimes \omega(N)$ is defined via

$$\omega(M) \otimes \omega(N) \rightarrow (\omega(M) \otimes A) \otimes (\omega(N) \otimes A) \rightarrow \omega(M) \otimes \omega(N) \otimes A \otimes A \rightarrow \omega(M) \otimes \omega(N) \otimes A \tag{3.23}$$

where the first map is due to A-comodule structure of $\omega(M)$ and $\omega(N)$ and the last map is the induced from multiplication map $A \otimes A \rightarrow A$. Let $\{f^n_i\}_{i \in I}$ be a basis of $gr^W_n(M)$ and $\{h^k_j\}_{i \in I}$ be a basis of $gr^W_k(N)$. Then, the image of $\omega(M \otimes N)$ under Ψ has the $\mathcal{A}_\bullet$-comodule structure defined by

$$\begin{aligned} f^n_i \otimes h^k_j \rightarrow & \sum_{m>n} \sum_{l>k} \sum_a \sum_b f^m_a \otimes h^l_b \otimes [M \otimes N, f^n_i \otimes h^l_j, (f^m_a)^\vee \otimes (h^k_b)^\vee](-n-k) \\ & \sum_{m>n} \sum_{l>k} \sum_a \sum_b f^m_a \otimes h^l_b \otimes [M, f^n_i, (f^m_a)^\vee](-n)[N, h^l_j, (h^k_b)^\vee](-k) \end{aligned}$$

Here we used the fact that Equation 3.22 gives us a basis $\omega(M \otimes N)$ in terms of $f^m_a \otimes h^l_b$. Comparing with the map (3.23), we see that the $\mathcal{A}_\bullet$-comodule structure on $\omega(M \otimes N)$ is induced from the product of $\mathcal{A}_\bullet$. Therefore, Ψ is a $\otimes$-functor. □

Remark 3.2.22. *For the sake of conformity with [25], we will write the coproduct as*

$$\Delta_{p,n-p}([M, v_0, f_n]) := \sum_{i=1}^{m} [M, v_0, b^\vee_i] \otimes [M, b_i, f_n](-p)$$

with the factors switched instead of the usual $\Delta_{p,n-p}([M, v_0, f_n]) := \sum_{i=1}^{m} [M, b_i, f_n](-p) \otimes [M, v_0, b^\vee_i]$.

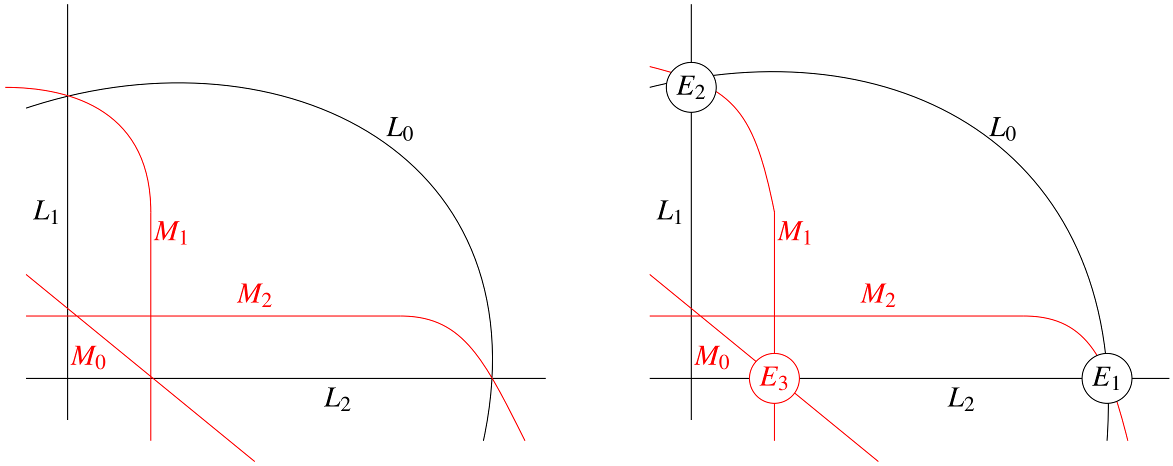

Figure 3.1: The image on the left shows the lines L_i in black and the lines M_i in red for $i = 0, 1, 2$. The image on the right shows the same image after blowing up at the three aforementioned points. The union of black lines is the divisor L and the union of red lines is the divisor M.

3.2.3 An Example of Framed Motive

We will now provide an example of a framed motive, which is also mentioned in [29] and is also a basic example of Aomoto polylogarithm mentioned in [2]. Let k be a field and let $x \in k - \{0, 1\}$ We will consider two divisors in $\mathbb{P}^2$ and call them L and M. Consider the lines $L_i := \{z_i = 0\}$ where z_i are the homogeneous coordinates of $\mathbb{P}^2$ for $i = 0, 1, 2$ and the lines $M_0 = \{z_1 + z_2 = z_0\}$, $M_1 = \{z_1 = z_0\}$ and $M_2 = \{z_2 = xz_0\}$. We will then blowup at the points where three of the lines L_i and M_i intersect which are

$$L_0 \cap L_2 \cap M_2 = [0 : 1 : 0], \qquad L_0 \cap L_1 \cap M_1 = [0 : 0 : 1], \qquad M_0 \cap M_1 \cap L_2 = [1 : 1 : 0]$$

Denote the blowup at these points by $\widehat{\mathbb{P}^2}$ and the exceptional divisors at these points by E_1, E_2 and E_3 respectively. These are nine lines in total as shown in figure 3.1. Now, we define L to be the union of strict preimages of L_i for $i = 0, 1, 2$ and the exceptional divisors E_1 and E_2 and we define M to be that union of rest of the lines. In $\widehat{\mathbb{P}^2}$, the divisor $L \cup M$ is a simple normal crossing divisor, so we can define a motive $Li_2^{mot}(x)$,

$$Li_2^{mot}(x) := H^2\left(\widehat{\mathbb{P}^2} - L, M - (M \cap L)\right)$$

By the Gysin spectral sequence, this is a mixed Tate motive concentrated in degrees $[-4, 0]$ where 0-th and -4-th graded components are one dimensional. From the discussion following Definition (4.3.9), the differential form η which is the pullback of $\frac{dz_1}{z_1} \wedge \frac{dz_2}{z_2}$ along $\widehat{\mathbb{P}^2} \to \mathbb{P}^2$ gives us a map $\eta : \mathbb{Q}(0) \to gr_0^W(Li_2^{mot}(x))$ and the polygon formed by the divisors of M, by duality, gives us a map $P_M : gr_{-4}^W(Li_2^{mot}(x)) \to \mathbb{Q}(2)$ and the framed motive would then correspond to the integral

$$\int_0^1 \int_0^x \frac{dz_1}{1 - z_1} \frac{dz_2}{z_2}$$

which works out to be the sum,

$$\sum_{n=1}^{\infty} \frac{x^n}{n^2}$$

This is precisely the definition of the dilogarithm $Li_2(x)$ up to monodromy.

3.2.4 Dilogarithm and Bloch-Suslin complex

In the end of the previous section, we saw how dilogarithm can be expressed as a period. We will now formally introduce dilogarithm following [17].

Definition 3.2.23. *Let z be a complex number such that $|z| < 1$, then we define* ***dilogarithm*** *$Li_2(z)$ as,*

$$Li_2(z) := \sum_{n=1}^{\infty} \frac{z^n}{n^2}$$

The function has an analytic continuation to $\mathbb{C} - [1, \infty)$ as mentioned in [72]. Ideally, we would like to analytically continue Li_2 over $\mathbb{C}$ but this results in a multivalued function due to monodromy. But we have a real valued variant of dilogarithm that is well defined and continuous over $\mathbb{C}$ at the cost of being just a real analytic function. This variant is called **Bloch-Wigner** function which is defined as,

$$D(z) := \mathtt{Im}(Li_2(z) + log(1 - z)log|z|)$$

where $\mathtt{Im}$ denotes the imaginary part of the expression. Bloch-Wigner function satisfies some functional equations,

$$D(x) + D(1 - x) = 0$$

$$D(x) + D(x^{-1}) = 0$$

$$D(x) - D(y) + D\left(\frac{y}{x}\right) - D\left(\frac{1 - y}{1 - x}\right) + D\left(\frac{1 - y^{-1}}{1 - x^{-1}}\right) = 0$$

where, in the last equation, $x \neq y$ and $x, y \neq 0, 1$. The second equation is the two term relation and the third one is called the five term relation. We also know that $D(0) = D(1) = 0$ and so by the two term relation, we extend D to ∞ by defining $D(\infty) = 0$. Thus D is a continuous function on $\mathbb{P}^1(\mathbb{C})$ and real analytic everywhere except $0, 1, \infty$. Plugging the degenerate value $y = 1$ and $x \in \mathbb{C} - \{0, 1\}$, gives the two term relation and plugging $x = 0$ and $y \in \mathbb{C} - \{0, 1\}$ gives the first relation. In short, every one of the above relation follows from the five term relation and its degenerations. We will define an algebraic version called the Bloch group for any field F of characteristic 0, which encapsulates the functional equation of the Bloch-Wigner function.

Remark 3.2.24. *For general statements about functional equations of Bloch-Wigner group (and more!), see [10].*

Definition 3.2.25. *Let k be a field of characteristic 0. The* ***scissors congruence group*** *denoted by $\mathcal{B}_2(k)$ is defined to be the quotient,*

$$\mathcal{B}_2(k) = \frac{\mathbb{Q}[\mathbb{P}^1(k)]}{R(k)}$$

where,

$$R(k) = \left\langle [x] - [y] + \left[\frac{y}{x}\right] - \left[\frac{1 - y}{1 - x}\right] + \left[\frac{1 - y^{-1}}{1 - x^{-1}}\right], [0], [1], [\infty] \right\rangle$$

where x, y are neither 0 nor 1 and $x \neq y$.

Remark 3.2.26. *It is called scissor congruence group because it is related to scissors congruence problem in hyperbolic 3-space and 3-sphere. See page 68 of [16].*

Remark 3.2.27. *Another way to define the five term relation is to consider five points a_0, a_1, a_2, a_3, a_4 in $\mathbb{P}^1(k)$. The five term relation is*

$$\sum_{i=0}^{4}(-1)^i[r(a_0, \ldots, \widehat{a_i}, \ldots, a_4)]$$

where $r(a, b, c, d)$ is the cross ratio of the four points a, b, c, d and $\widehat{a_i}$ mean swe are omitting a_i. It is explicitly defined as

$$r(a, b, c, d) = \frac{(a-c)(b-d)}{(a-d)(b-c)}$$

From now on, we will denote $A \otimes_{\mathbb{Z}} \mathbb{Q}$ by $A_{\mathbb{Q}}$ where A is an abelian group.
Consider the following map

$$\begin{aligned} \delta : \mathbb{Q}[\mathbb{P}^1(k)] &\to \bigwedge^2 k^{\times}_{\mathbb{Q}} \\ [x] &\to (1-x) \wedge x \text{ if } x \neq 0, 1, \infty \\ [x] &\to 0 \text{ if } x = 0, 1, \infty \end{aligned}$$

Theorem 3.2.28. $\delta(R(k)) = 0$

Proof. Follows from the discussion following Definition 3.3.23 and Lemma 3.2.32. □

The above theorem then implies that δ factors through $\mathcal{B}_2(k)$,

$$\delta : \mathcal{B}_2(k) \to \bigwedge^2 k^{\times}_{\mathbb{Q}}$$

The kernel of this map is an important group itself.

Definition 3.2.29. *The **Bloch group** denoted by $B_2(k)$ is the kernel of the map $\delta : \mathcal{B}_2(k) \to \bigwedge^2 k^{\times}_{\mathbb{Q}}$.*

The complex,

$$\delta : \mathcal{B}_2(k) \to \bigwedge^2 k^{\times}_{\mathbb{Q}} \tag{3.24}$$

has cokernel $K_2^M(k)_{\mathbb{Q}}$ and kernel $B_2(k) \cong K_3^{ind}(k)_{\mathbb{Q}} \cong K_3(k)_{\mathbb{Q}}$ by a result of Suslin [63].

Rigidity of $\mathcal{B}_2(k)$

In this section, we will show a rigidity result for $\mathcal{B}_2(k)$. The way to do this is by introducing a different version of $\mathcal{B}_2(k)$, $\mathcal{B}'_2(k)$ which is rigid by definition and then show that both are isomorphic. This alternate definition is mentioned in section 1.9 of [21].

To define subgroup $\mathcal{R}(k)$ of $\mathbb{Q}[\mathbb{P}^1(k)]$, we first see what rigidity means conceptually. Let X be a smooth connected curve over k and $\sum_i n_i[f_i]$ be an element of $\mathbb{Q}[\mathbb{P}^1(k(X))]$ such that $\delta(\sum_i n_i[f_i]) = 0$. Keeping Suslin's result in mind, this element belongs to $K_3^{ind}(k(X))_{\mathbb{Q}}$. Rigidity means that restricting this element to two different k-rational points give us the same element of $K_3^{ind}(k)_{\mathbb{Q}}$. Let us denote these two points as a and b. Then $\sum_i n_i[f_i(a)]$ show be equal to $\sum_i n_i[f_i(b)]$ in $\mathcal{B}_2(k)$. This motivates us to the following definition.

Definition 3.2.30. *$\mathcal{R}(k)$ is the subgroup of $\mathbb{Q}[\mathbb{P}^1(k)]$ generated by elements of the form $\sum_i n_i([f_i(a)]-[f_i(b)])$, where $\sum_i n_i[f_i]$ is an element of $\mathbb{Q}[\mathbb{P}^1(k(X))]$ from some connected curve X over k such that $\delta(\sum_i n_i[f_i]) = 0$ and a, b are any two k-rational points of X.*

Definition 3.2.31. *We define $\mathcal{B}_2'(k)$ to be,*

$$\mathcal{B}_2'(k) = \frac{\mathbb{Q}[\mathbb{P}^1(k)]}{\mathcal{R}(k)}$$

By definition $\mathcal{B}_2'(k)$ is rigid. By Lemma 1.16 of [21] states that $\delta(\mathcal{R}(k)) = 0$, so we have a complex,

$$\delta' : \mathcal{B}_2'(k) \to \bigwedge^2 k^\times$$

Now we will construct a map from $\mathcal{B}_2(k) \to \mathcal{B}_2'(k)$.

Lemma 3.2.32. *$R(k) \subset \mathcal{R}(k)$*

Proof. Take the five term relation and replace x by $tx+(1-t)$, where t is the parameter of $\mathbb{P}^1(k)$. Here, y and x are considered constant functions over $\mathbb{P}^1(k)$. Then,

$$[tx+1-t]-[y]+\left[\frac{y}{tx+1-t}\right]-\left[\frac{1-y}{t(1-x)}\right]+\left[\frac{1-y^{-1}}{1-(tx+(1-t))^{-1}}\right] \in \mathbb{Q}[\mathbb{P}^1(k(t))]$$

Restricting at $t = 1 \in \mathbb{P}^1(k)$, gives the usual five term relation in $\mathbb{Q}[\mathbb{P}^1(k)]$ and restricting at $t = 0$ gives

$$[1]-[y]+[y]-[\infty]+[\infty] = [1] = 0$$

Therefore, we get the lemma. □

Remark 3.2.33. *This statement is also true with integral coefficients rather than rational coefficients as evident from the proof.*

Because of the containment $R(k) \subset \mathcal{R}(k)$, we have a natural surjective map $\mathcal{B}_2(k) \to \mathcal{B}_2'(k)$ which sends the class of $[x]$ in $\mathcal{B}_2(k)$ to the class of $[x]$ in $\mathcal{B}_2'(k)$. This map is also an injection and therefore an isomorphism by proposition 6.1 of [26] if k is a number field.

Lemma 3.2.34. *If k is a number field then the map $\mathcal{B}_2(k) \to \mathcal{B}_2'(k)$ is an isomorphism.*

Back to Framed motives

We defined a Lie coalgebra $\mathcal{L}_\bullet(k) := \mathcal{L}_\bullet(\mathcal{MTM}(k))$ previously (3.12). It is graded in even positive degrees. Taking the weight four part of the Lie coalgebra version of Chevalley-Eilenberg complex(see Section 7.7 of [69]), we get,

$$\mathcal{L}_2(k) \xrightarrow{\delta} \bigwedge^2 \mathcal{L}_1(k)$$

where $\delta(x) = \Delta(x) - 1 \otimes x - x \otimes 1$. By Theorem 3.2.21, we can identify $\mathcal{L}_\bullet(k)$ with framed motives. For example, $\mathcal{L}_1(k)$ would be identified with 1-framed motives $[v_0, M, f_1]$, which up to equivalence relation, are of the form

$$0 \to \mathbb{Q}(0) \xrightarrow{v_0} M \xrightarrow{f_1} \mathbb{Q}(-1) \to 0$$

which means M is just an element of $Ext^1_{\mathcal{MTM}(k)}(\mathbb{Q}(-1), \mathbb{Q}(0)) = K_1(k) = k^\times$. More explicitly, they are framed Kummer motives (Example 4.59, [19]). So the complex becomes,

$$\mathcal{L}_2(k) \to \bigwedge^2 k^\times$$

We will see how $\mathcal{L}_2(k)$ compares to $\mathcal{B}_2(k)$.

Lemma 3.2.35. *For a number field k, $\mathcal{B}_2(k)$ is isomorphic to $\mathcal{L}_2(k)$.*

Proof. The motivic dilogarithm, mentioned in Section 3.2.3 above gives us a map of complexes,

$$\begin{tikzcd} \mathbb{Q}[\mathbb{P}^1(k)] \arrow[r] \arrow[d] & \bigwedge^2 k^\times_{\mathbb{Q}} \arrow[d] \\ \mathcal{L}_2(k) \arrow[r] & \bigwedge^2 k^\times_{\mathbb{Q}} \end{tikzcd}$$

Let $\sigma : k \to \mathbb{C}$. Consider the real period map $\mathcal{L}_2(k) \to \mathbb{C}$, which maps a framed motive to the real part of the period. This depends on σ because de Rham's theorem requires base change from k to $\mathbb{C}$, which is provided by σ. Under the composition of real period map with the left vertical map, $[x]$ is sent to $D(\sigma(x))$. Since D satisfies five-term relation, an element of $R_2(k)$ is sent to zero for all $\sigma \in Hom(F, \mathbb{C})$. The image of an element of $R_2(k)$ in $\mathcal{L}_2(k)$, is in the kernel of the bottom horizontal row, which is isomorphic to $K_3(k)_{\mathbb{Q}}$ by Theorem 4.108 of [19]. The sum of real period maps coincides with the regulator map on $K_3(k)_{\mathbb{Q}}$ (see page 18 of [72]). Since the regulator map is injective, the left map factors through $\mathcal{B}_2(k)$. So the diagram above can be further simplified to

$$\begin{tikzcd} \mathcal{B}_2(k) \arrow[r] \arrow[d] & \bigwedge^2 k^\times \arrow[d] \\ \mathcal{L}_2(k) \arrow[r] & \bigwedge^2 k^\times \end{tikzcd}$$

The top and bottom maps are epimorphism because $K_2(k)_{\mathbb{Q}} = 0$ and the kernel of both maps are isomorphic to $K_3(k)$; for the top row, it follows from Suslin's theorem (Theorem 5.2 of [63]) and for the bottom row, it follows by an argument similar to the proof of Theorem 4.108 of [19]. If the map induced on the kernels i.e. $K_3(k)_{\mathbb{Q}}$ is an isomorphism then by diagram chasing, we will prove the theorem. The map on the kernels is the identity map $K_3(k)_{\mathbb{Q}} \to K_3(k)_{\mathbb{Q}}$. Zagier's conjecture for $n = 2$ (Chapter 2 of [21]) implies this map is surjective and because $K_3(k)$ is finite dimensional, it is an isomorphism. □

Remark 3.2.36. *This lemma also holds for mixed Tate motives over $O_{k,S}$.*

Conjecture 3.2.37. *We will also assume a general version of Lemma 3.2.35 i.e. if F is a function field of projective variety X, then $\mathcal{B}_2(F)$ is isomorphic to $\mathcal{L}_2(F)$.*

Assumption 3.2.38. *Let F be a function field of a projective variety. We will assume that $\mathcal{B}_2(F)$ is isomorphic to $\mathcal{L}_2(F)$. This statement is equivalent to the rigidity of $K_3^{ind}(k(X))_{\mathbb{Q}}$ for any projective curve X.*

3.2.5 A Specialization Map

The isomorphism between $\mathcal{B}_2(k)$ and $\mathcal{L}_2(k)$ helps us to define a specialization map which is easier to define for $\mathcal{B}_2(k)$. Let $\mathcal{O}$ be a discrete valuation ring with uniformizer t and F be its field of fractions. Let k be the residue field and if $f \in \mathcal{O}$ then we denote its image in k as $\overline{f}$. Any discrete valuation ring comes equipped with a valuation homomorphism which we will denote by v.

Definition 3.2.39. *The* ***specialization maps*** $sp_1 : F^\times \to k^\times$ *and* $sp_2 : \mathcal{B}_2(F) \to \mathcal{B}_2(k)$ *are as follows,*

- $sp_1 : F^\times \to k^\times$ *is given by the map,*

$$sp_1 : F^\times \to k^\times, \qquad sp_1(f) = \overline{f/t^{v(f)}} \tag{3.25}$$

- $sp_2 : \mathcal{B}_2(F) \to \mathcal{B}_2(k)$ *is given by the map,*

$$sp_2 : \mathcal{B}_2(F) \to \mathcal{B}_2(k), \qquad sp_2([f]) = \begin{cases} [\overline{f}] & \text{if} \quad f \in \mathcal{O}^\times \\ 0 & \text{otherwise} \end{cases} \tag{3.26}$$

Lemma 3.2.40. *(Lemma 3.3, [27]) The following diagram is commutative*

$$\begin{tikzcd} \mathcal{B}_2(F) \arrow[r] \arrow[d, "sp_2"] & \bigwedge^2 F^\times_{\mathbb{Q}} \arrow[d, "\wedge^2 sp_1"] \\ \mathcal{B}_2(k) \arrow[r] & \bigwedge^2 k^\times_{\mathbb{Q}} \end{tikzcd}$$

Proof. If $f \in \mathcal{O}^\times$ then the diagram can easily be seen to commute. If $v(f) = n > 0$, then $f = t^n g$ where $g \in \mathcal{O}^\times$. If $[f] \in \mathcal{B}_2(F)$, then the upper part of the diagram gives $\wedge^2 sp_1\,(f \wedge (1-f)) = \overline{g} \wedge 1 = 0$. The lower part of the diagram gives 0 on the nose. If $v(f) = n < 0$, we use the inversion relation $[f] = -[1/f]$, where $v(1/f) = n > 0$. This reduces to the second case where $v(f) = n > 0$. Thus the lemma holds. □

This lemma is especially useful if X is a irreducible curve and x is a smooth closed point of X as we will see in Chapter 4. In that case, $\mathcal{O}$ would be the local ring at x and $F = k(X)$. Intuitively, this means that we are just "plugging" x in a functional version of framed motive i.e. a framed motives over X. The advantage we get, is that we can restrict to points where the "plugging" part does not make sense and at these points, we have a regularized version of the framed motive.

3.3 Euler complex and its relation to modular complex

In this section, we will define what the Euler complex is and how it is related to the geometry of the locally symmetric space of $GL_2(\mathcal{O}_K)$ for $K = \mathbb{Q}(i), \mathbb{Q}(\sqrt{-3})$. We will assume E is an elliptic curve over a field k. Usually k will be some algebraic extension of K. To this end, we will first construct elements $\theta_{E,z}(a,b,c) \in \mathcal{B}_2(\overline{k})$, where a, b, c are torsion points of E such that $a + b + c = 0$. The construction utilizes the map h in Theorem 3.3.1.

3.3.1 Definition and Properties of $\theta_{E,z}(a,b,c)$

We will follow the construction laid out in [27]. To do this construction, we need the following theorem.

Theorem 3.3.1. ***(Theorem 6.14, [26])****Let k be an algebraically closed field. Let X be either an elliptic curve or a rational curve over k and $F = k(X)$. Then there exists a group homomorphism $h : \bigwedge^3 F^\times_{\mathbb{Q}} \to \mathcal{B}_2(k)$ such that,*

a) $h(k^\times_{\mathbb{Q}} \wedge \bigwedge^2 F^\times_{\mathbb{Q}}) = 0$ and the following diagram is commutative:

$$\begin{tikzcd}
\mathcal{B}_2(F) \otimes F^\times_{\mathbb{Q}} \arrow[r, "\delta\wedge 1"] \arrow[d, "Res"] & \bigwedge^3 F^\times_{\mathbb{Q}} \arrow[d] \arrow[dl, "h"'] \\
\mathcal{B}_2(k) \arrow[r, "\delta"] & \bigwedge^2 k^\times_{\mathbb{Q}}
\end{tikzcd}$$

b) If $X \subset \mathbb{P}^2$ is an elliptic curve then for any linear homogeneous function $l_0, \ldots, l_3$ one has

$$h(l_1/l_0 \wedge l_2/l_0 \wedge l_3/l_0) = -\sum_{i=0}^{3} [r(l_{i0}, \ldots, \widehat{l_{ii}}, \ldots, l_{i3}, D_i)]$$

where D_i is the divisor $L_i \cap X$ inside $\mathbb{P}^2$, L_i being the line $l_i = 0$ and $l_{ij} = L_i \cap L_j$. The cross-ratio is extended linearly for divisors.

c) If $X = \mathbb{P}^1$ and t is the natural parameter on it and $a_i \in \mathbb{P}^1(k)$, then

$$h\left(\frac{t-a_1}{t-a_0} \wedge \frac{t-a_2}{t-a_0} \wedge \frac{t-a_3}{t-a_0}\right) = -[r(a_0, a_1, a_2, a_3)]$$

d) If X is defined over $\mathbb{C}$ then,

$$\int_{X(\mathbb{C})} r_2(f_1 \wedge f_2 \wedge f_3) = D\left(h(f_1 \wedge f_2 \wedge f_3)\right)$$

where $r_2(f_1 \wedge f_2 \wedge f_3)$ is the Chow dilogarithm mentioned in [20].

e) The map h is $Gal(F/k)$-invariant.

Remark 3.3.2. *The map h is also related to Chow reciprocity and other reciprocities e.g. Weil as mentioned in [20].*

Now we have enough information to describe the definition of $\theta_{E,z}(a,b,c)$.

Definition 3.3.3. *Let a, b, c be N-torsion elements of $E(k)$ such that $a + b + c = 0$. Then*

$$\theta_{E,z}(a,b,c) := \frac{1}{N^5} \sum_{x \in E[N]} h(f_{0,x} \wedge f_{a,x} \wedge f_{a+b,x}) \in \mathcal{B}_2(\bar{k}) \tag{3.27}$$

where $f_{x,y} \in k(E)^\times$ such that $div(f_{x,y}) = N([x] - [y])$.

Theorem 3.3.4. *The elements $\theta_{E,z}(a,b,c)$ have the following basic properties,*

1. $\delta(\theta_{E,z}(a,b,c)) = \theta_{E,z}(a) \wedge \theta_{E,z}(b) + \theta_{E,z}(b) \wedge \theta_{E,z}(c) + \theta_{E,z}(c) \wedge \theta_{E,z}(a)$.

2. $\theta_{E,z}(a,b,c)$ *is alternating in its three entries.*

3. $\theta_{E,z}(a,b,c) = \theta_{E,z}(\epsilon(a), \epsilon(b), \epsilon(c))$ *for* $\epsilon \in Aut(E)$.

4. $\theta_{E,z}(a,b,c)$ *satisfies distribution relation; for a morphism* $\psi : E \to E'$, *we have*

$$\sum_{\psi(a_i)=a'_i} \theta_{E,z}(a_1,a_2,a_3) = \theta_{E'}(a'_1,a'_2,a'_3)$$

5. $\theta_{E,z}(a,b,c)$ *does not depend on* N *i.e. if* $N|M$, *then replacing the sum in definition* (3.27) *with sum over* $x \in E[M]$ *gives us the same element.*

6. $\theta_{E,z}(a,b,c) \in \mathcal{B}_2(k')$ *where* k' *is the smallest extension of* k *such that any* N*-torsion point of* E *is a* k'*-rational point.*

Proof. Proofs of all these properties are in Chapter 2 of [27]. □

3.3.2 Euler Complex and Modular Complex

In this part, we will use $\theta_{E,z}(a,b,c)$ defined above to define the Euler complex and its relation to the modular complex defined in chapter 1. Let E be an elliptic curve over K such that $End(E) = \mathcal{O}_K$ for $K = \mathbb{Q}(i), \mathbb{Q}(\sqrt{-3})$. Let $\mathcal{N}$ be an ideal of $\mathcal{O}_K$ and let N be the norm of $\mathcal{N}$. We let $a,b,c \in E[\mathcal{N}]$. By the last property of $\theta_{E,z}(a,b,c)$, we get $\theta_{E,z}(a,b,c) \in \mathcal{B}_2(K_\mathcal{N})$ where $K_\mathcal{N}$ is the field extension of K by adjoining N-roots of unity and $\theta_{E,z}(a)$, for all $a \in E[\mathcal{N}]$.

Definition 3.3.5. *We have the following subspaces of* $\mathcal{B}_2(K_\mathcal{N})$*;*

- $C_1(\mathcal{N})$ *defined to be the subspace of* $K_\mathcal{N}^\times$ *generated by the elliptic units* $\theta(a)$ *for all* $a \in E[\mathcal{N}] - 0 \cong \mathcal{O}_{K_\mathcal{N}}\left[\frac{1}{N}\right]^\times \otimes \mathbb{Q}$.
- $C_1^{un}(\mathcal{N})$ *defined to be* $\mathcal{O}_{K_\mathcal{N}}^\times \otimes \mathbb{Q} \subset K_\mathcal{N}^\times$,
- $\hat{C}_1(\mathcal{N}) := C_1(N) \oplus \mathbb{Q}$ *where this extra summand represents* $\theta(0)$.
- $C_2(\mathcal{N})$ *defined to be the subspace of* $\mathcal{B}_2(K_\mathcal{N})$ *generated by* $\theta_{E,z}(a,b,c)$ *where* a,b,c *are* $\mathcal{N}$*-torsion elements of* E *such that* $a+b+c=0$.

By the first property of $\theta_{E,z}(a,b,c)$, we note that $\delta(C_2(\mathcal{N})) \subset \bigwedge^2 C_1(\mathcal{N})$. This will give rise to two complexes,

Definition 3.3.6. *We have two subcomplexes of Bloch Suslin complex* $\mathcal{B}_2(K_\mathcal{N}) \to \bigwedge^2 K_\mathcal{N}^\times$*, all concentrated in degree* $[1,2]$*,*

1. $\boldsymbol{E}^\bullet(\mathcal{N}) := C_2(\mathcal{N}) \to \bigwedge^2 C_1(\mathcal{N})$. *This is called the* ***Euler complex*** *for* $\mathcal{N}$.

2. $\widehat{\boldsymbol{E}}^\bullet(\mathcal{N}) := C_2(\mathcal{N}) \to \bigwedge^2 \hat{C}_1(\mathcal{N})$.

3. *if* $\mathcal{N}$ *is a prime ideal then* $\delta(C_2(\mathcal{N})) \subset \bigwedge^2 C_1^{un}(\mathcal{N})$ *so we have,*

$$\boldsymbol{E}^\bullet_{un}(\mathcal{N}) := C_2(\mathcal{N}) \to \bigwedge^2 C_1^{un}(\mathcal{N})$$

The main theorem which connects the (motivic) Euler complex to the (geometric) modular complex, the geometric side is the following theorem,

Theorem 3.3.7. *(Theorem 5.7,[27]) Let $K = \mathbb{Q}(i), \mathbb{Q}(\sqrt{-3})$ and O_K be its ring of integers and let $\mathcal{N}$ be an ideal of O_K. For each $z \in E[\mathcal{N}]$. Then there exists a surjective map of complexes*

$$\begin{tikzcd} S_2(X(\Gamma_1(\mathcal{N}))) \arrow[r, "\partial"] \arrow[d, "\theta^2_{E,z}"'] & S_1(X(\Gamma_1(\mathcal{N}))) \arrow[r, "\partial"] \arrow[d, "\theta^1_{E,z}"] & S_0(X(\Gamma_1(\mathcal{N}))) \arrow[d, "\theta^0_{E,z}"] \\ C_2(\mathcal{N}) \arrow[r, "\delta"] & \bigwedge^2 \hat{C}_1(\mathcal{N}) \arrow[r, "\delta"] & C_1(\mathcal{N}) \oplus C_1(\mathcal{N}) \end{tikzcd}$$

The map also intertwines the action of normalizer of $\Gamma_1(\mathcal{N})$ with the Galois action of $Gal(K_{\mathcal{N}}/K)$.

Proof. (Sketch) As seen in the second chapter, the Bianchi tessellation in either case has 2-cells in one $GL_2(O_K)$-orbit i.e. in orbit of $(0, 1, \infty)$. Therefore,

$$S_2(X(\Gamma_1(\mathcal{N}))) := \mathbb{Q} \otimes_{\Gamma_1(\mathcal{N})} \mathbb{Q}[GL_2(O_K)] \otimes \chi_{sgn} := \frac{\mathbb{Q}[(\alpha, \beta, \gamma) \in (O_{K_{\mathcal{N}}}/\mathcal{N})^3 - 0 | \alpha + \beta + \gamma = 0]}{\text{the dihedral relations}} \tag{3.28}$$

where the dihedral relations are;

$$\begin{aligned} (\alpha, \beta, \gamma) &= -(\beta, \alpha, \gamma) \\ (\alpha, \beta, \gamma) &= (\beta, \gamma, \alpha) \\ (\alpha, \beta, \gamma) &= (\epsilon\alpha, \epsilon\beta, \epsilon\gamma) \end{aligned}$$

for $\epsilon \in O^{\times}_{K_{\mathcal{N}}}$. Similarly, using the relations mentioned in Section 2.5.2 and 2.5.3, we get

$$S_2(X(\Gamma_1(\mathcal{N}))) := \mathbb{Q} \otimes_{\Gamma_1(\mathcal{N})} \mathbb{Q}[GL_2(O_K)] \otimes \chi_{sgn} := \frac{\mathbb{Q}[(\alpha, \beta) \in (O_{K_{\mathcal{N}}}/\mathcal{N})^2 - 0]}{(\alpha, \beta) = -(\beta, \alpha) = (\epsilon_1\alpha, \epsilon_2\beta)} \tag{3.29}$$

for $\epsilon_1, \epsilon_2 \in O^{\times}_{K_{\mathcal{N}}}$ Then the map $\theta^1_{E,z}$ and $\theta^2_{E,z}$ are defined as

$$\begin{aligned} \theta^2_{E,z}(\alpha, \beta, \gamma) &:= \theta_{E,z}(\alpha z, \beta z, \gamma z) \\ \theta^1_{E,z}(\alpha, \beta) &:= \theta(\alpha z) \wedge \theta(\beta z) \end{aligned}$$

The term αz makes sense because $O_{K_{\mathcal{N}}}$ acts on E and z is a $\mathcal{N}$-torsion point. $\theta^2_{E,z}$ is well defined because of second and third property of $\theta(a, b, c)$. The map $\theta^1_{E,z}$ is also well defined because of the properties of wedge product and the elliptic units $\theta(a)$. The left square commutes because of the first property of $\theta(a, b, c)$.
The commutativity of the right box is omitted (see proof of Theorem 4.8.10). The surjectivity of the map of complex follows from the definition of Euler complex and the distribution relations. □

Remark 3.3.8. *In the case that $\mathcal{N}$ is a prime ideal, the right and middle maps are even isomorphisms because there are no distribution relations for $\theta_{E,z}(a)$ and the complex above is quasi isomorphic to $\boldsymbol{E}^{\bullet}_{un}(\mathcal{N})$. See [27] for more details.*

The right box in theorem 3.3.7, is always commutative so the main hurdle in generalizing the theorem is in the left square. Here, we face two issues. Firstly, we have to construct an appropriate analog of $\theta_{E,z}(a, b, c)$ such that the diagram commutes and moreover it should also be compatible with the stabilizer of the 2-cell at hand which is usually not a triangle as seen in the Bianchi tessellation of $K = \mathbb{Q}(\sqrt{-7})$. But the alternating relation is baked into the definition of $\theta_{E,z}(a, b, c)$ at its very conception so we have to use some other way to define the analog of $\theta_{E,z}(a, b, c)$.

Chapter 4

Euler Complex for $d = -1, -3, -2, -7, -11$

4.1 Bianchi Tessellations and its algebraic interpretation

Let $E \times E$ be the product of elliptic curve E over K such that $End(E) = \mathcal{O}_K$ for an imaginary quadratic field $K = \mathbb{Q}(\sqrt{-d})$ for positive squarefree d. We also fix an $\mathcal{N}$-torsion point of E where $\mathcal{N}$ is an ideal of $\mathcal{O}_K$ with norm N. The general procedure follows like this,

- Set up a $1-1$ correspondence between cusps in Bianchi tessellation and subelliptic curves of $E \times E$.
- Break a 2-cell into Bianchi triples, which have trivial stabilziers under $GL_2(\mathcal{O}_K)$-action.
- Construct a framed motive for each Bianchi triple.

4.1.1 Cusps

By theorem 1.2.47, we know that positive definite Hermitian forms correspond to ample line bundles and by Kleiman criterion, nef line bundles correspond to semi-positive definite Hermitan forms. The cone of nef line bundles is closed so our cusps correspond to some semi-definite line bundles. We will set up a $1 - 1$ correspondence between the cusps in Bianchi tessellation and subelliptic curves of $E \times E$. To do this, we will first construct endomorphisms from subelliptic curves and show that they exhaust a certain subspace of $End^s(E \times E)$ which we will call rank 1 endomorphisms. Then we will show that these endomorphisms correspond to the cusps in (2.3).

Definition 4.1.1. *A **subelliptic curve** E' of $E \times E$ is a closed subvariety of $E \times E$ which is also an elliptic curve with the same identity element as $E \times E$.*

Definition 4.1.2. *For any sub elliptic curve of E' of $E \times E$, We have an endomorphism $\beta_{E'} := \iota_{E'} \circ \widehat{\iota_{E'}}$ where $\iota_{E'} : E' \hookrightarrow E \times E$ is an inclusion of E' into $E \times E$. It is unique up to an action of $Aut(E')$ on the E'. We call $\beta_{E'}$ a **cusp** of $E \times E$ and the **value of** α **at** E' for $\alpha \in End^s(E \times E)$ is defined as the composition $\widehat{\iota_{E'}} \circ \alpha \circ \iota_{E'}$ in $End^s(E')$. We will denote the value by $(\alpha, \beta_{E'})$ as an element of $End(E')$.*

Remark 4.1.3. *$\beta_{E'}$ does not depend on the inclusion. Let $j : E' \to E \times E$ be another inclusion of E' into $E \times E$. Then j factors through $\iota_{E'}$,*

$$\begin{tikzcd} E' \arrow[rr, "j"] \arrow[dr, "\eta"'] & & E \times E \\ & E' \arrow[ur, "\iota_{E'}"'] & \end{tikzcd}$$

η is an injective non-zero map, therefore, η is an isomorphism because it is an isogeny with degree 1 and its inverse is the dual isogeny. Then,

$$j \circ \widehat{j} = (\iota_{E'} \circ \eta) \circ (\widehat{\eta} \circ \widehat{\iota_{E'}}) = \iota_{E'} \circ (\eta \circ \widehat{\eta}) \circ \widehat{\iota_{E'}} = \beta_{E'}$$

So $\beta_{E'}$ only depends on the subelliptic curve and not on the inclusion.

Remark 4.1.4. *Generally, $\widehat{\iota_{E'}}$ need not be equal to the projection $\pi_{E'}$. More generally, $\widehat{\iota_{E'}} \circ \iota_{E'} = [n]$ for some natural number n and if n is not equal to 1 then $\pi_{E'}$ and $\widehat{\iota_{E'}}$ differ.*

Until now, we have seen that for a subelliptic curve E', we have an element $\beta_{E'}$ in $End(E \times E) \cong M_2(O_K)$. We will see below that $\beta_{E'}$ has rank 1 as a matrix and moreover, any such matrix is $\beta_{E'}$ for some subelliptic curve E' up to a constant.

Definition 4.1.5. *An endomorphism $\alpha \in End(E \times E)$ is a **rank** 1 **endomorphism** if the image of α is of dimension 1 or equivalently its kernel has dimension 1.*

Lemma 4.1.6. *$\beta_{E'}$ is an endomorphism of rank 1.*

Proof. $\widehat{\iota_{E'}} : E \times E \to E'$ must have kernel of dimension 1 because of dimension reasons and non-triviality. Since $\beta_{E'} = \iota_{E'} \circ \widehat{\iota_{E'}}$, the lemma follows. □

Lemma 4.1.7. *For every symmetric rank 1 endomorphism $\alpha \in End(E \times E)$, there exists $n, m \in \mathbb{Z}$ and a subelliptic curve E' of $E \times E$ such that $n\alpha = m\beta_{E'}$.*

Proof. Let $E' = im(\alpha)$, then α can be written as $E \times E \xrightarrow{s} E' \xrightarrow{\iota_{E'}} E \times E$. Since α is symmetric, we can write $\alpha = \widehat{s} \circ \widehat{\iota_{E'}}$. Now,

$$\alpha \circ \alpha = (\iota_{E'} \circ s) \circ (\widehat{s} \circ \widehat{\iota_{E'}}) = \iota_{E'} \circ (s \circ \widehat{s}) \circ \widehat{\iota_{E'}}. \tag{4.1}$$

Since $s \circ \widehat{s}$ is a map from E' to itself and it is symmetric, it is therefore multiplication-by-n map for some natural number n. So $\alpha^2 = \iota_{E'} \circ [n] \circ \widehat{\iota_{E'}} = [n] \circ \iota_{E'} \circ \widehat{\iota_{E'}}$ because $\iota_{E'}$ is a group homomorphism. This implies

$$\alpha^2 = n\beta_{E'} \tag{4.2}$$

Now as an element of $End(E \times E) \cong M_2(O_K)$, α has a characteristic polynomial $\alpha^2 - a\alpha + b = 0$ where $a, b \in O_K = End(E)$. Since α has kernel of dimension 1, then if $x \in ker(\alpha)$, we get $0 = (\alpha^2 - a\alpha + b)x = bx$ because $\alpha x = 0$. Since kernel is of dimension 1, a choice of non-zero $x \in ker(\alpha)$ implies that $b = 0$ and so, $\alpha^2 - a\alpha = 0$, where $a \in \mathbb{Z} \subset O_K$ because α is symmetric; by symmetry we have,

$$a\alpha = -\alpha^2 = \widehat{a}\alpha \Rightarrow (a - \widehat{a})\alpha = 0$$

which implies that on $E' = im(\alpha)$, $a - \widehat{a} : E' \to E \times E$ is the zero map. Since 0 is an element of E', $a - \widehat{a} : E' \to E \times E$ factors thorough $\iota_{E'} : E' \to E \times E$, that is

$$a - \widehat{a} : E' \to E' \xrightarrow{\iota_{E'}} E \times E$$

The composed map is zero and $\iota_{E'}$ is injective implies that the first map in the decomposition is the zero map. But $End(E')$ is an integral domain so $a = \widehat{a}$ in $End(E') \subset O_K$. This implies $a \in \mathbb{Z}$. To emphasize this fact, we replace a by m.

$$\alpha^2 = m\alpha \tag{4.3}$$

Combining, (4.2) and (4.3), we get,

$$m\alpha = n\beta_{E'} \tag{4.4}$$

□

The lemma above implies that all rank 1 endomorphisms are rationally equal to a cusp i.e. they generate the same subspace in $End^s(E \times E) \otimes \mathbb{Q}$.

If there exists $m, n \in \mathbb{Z}$ and G, F two subelliptic curves of $E \times E$ such that $m\beta_G = n\beta_F$, then the image of $m\beta_F$ is the same as the image of β_F because multiplication-by-m map is surjective. Thus G and F have the same image in $E \times E$, so they are the same subelliptic curve but maybe it is possible that they are embedded differently e.g. E can be embedded into $E \times E$ after applying an automorphism of E.

Corollary 4.1.8. *There is a $1-1$ correspondence between subelliptic curves of $E \times E$ and rays passing through rank 1 endomorphisms/cusps in $End^s(E \times E) \otimes \mathbb{R}$. Moreover, the correspondence is $GL_2(O_K)$-equivariant.*

The equivariance of the correspondence follows by applying the action of $GL_2(O_K)$ on $\alpha^2 = n\beta_{E'}$ and noting that $g\beta_{E'}g^\dagger = g\beta_{E'}\widehat{g} = g \circ \iota_{E'} \circ \widehat{\iota_{E'}} \circ \widehat{g} = \beta_{g(E')}$, where the first equality holds because of Theorem 1.2.47 and the discussion preceding it.

Remark 4.1.9. *One can see that the definition of (α, β_E) mirrors the definition of evaluating a quadratic form at a point as seen above.*

Remark 4.1.10. *If α is symmetric, then by construction $(\alpha, \beta_E) : E \to E$ is also symmetric, which translates to the fact that the value is an integer. If these are positive(non-negative) integers for every subelliptic curve of $E \times E$ then we say that α is positive (semi)definite.*

We have an equivalence between points in $\mathbb{P}^1(K)$, maps $E \to E \times E$ upto left multiplication by $O_K = End(E)$ and subelliptic curves of $E \times E$.

Remark 4.1.11. *Let E' be a subelliptic curve, then it is an image of some rank 1 endomorphism α. If class number of K is 1, then O_K is a PID so we can take Smith normal form of α, which is a matrix of the form,*

$$\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$$

where $a \in O_K$. The image of α is also a subelliptic curve E'', which is isomorphic to E'. The elliptic curve is isomorphic to the image of the map $a : E \to E$, which is just E. This means that any subelliptic curve of $E \times E$ is isomorphic to E if the class number of K is 1.

4.2 From 2-cells to Bianchi triples

Definition 4.2.1. *A **Bianchi triple** β is a triple (E_1, E_2, E_3) of 3 subelliptic curves of $E \times E$ such that the cusps (c_1, c_2, c_3) corresponding to it satisfy the following;*

1. *All* 3 *cusps lie in a common face in the tessellation.*

2. c_1 *and* c_2 *share an edge in the tessellation.*

3. c_3 *and* c_2 *share an edge in the tessellation.*

4. c_1, c_2 *and* c_3 *are distinct.*

We denote the set of Bianchi triples as BT and $\mathbb{Q}[BT]$ as BTG. We will conflate Bianchi triple with the triple of the corresponding cusps.

$GL_2(O_K)$ acts on the set BT and BTG via its usual action on cusps in Bianchi tessellation. Since it is just fractional linear transformation, we have the following result,

Lemma 4.2.2. *The stabilizer of a Bianchi triple is trivial and consequently, BTG is a free $\mathbb{Q}[PGL_2(O_K)]$ module.*

Proof. By correspondence of cusps and points of $\mathbb{P}^1(K)$, BT is a subset of ordered triple of points in $\mathbb{P}^1(K)$. The action of $PGL_2(K)$ on $\mathbb{P}^1(K)$ is simply 3-transitive. Since $PGL_2(O_K)$ is a subgroup of $PGL_2(K)$, the action of $PGL_2(O_K)$ on ordered triple in $\mathbb{P}^1(K)$ is still free. □

We will now construct a map $\rho : S_2(\tilde{X}) \to BTG$. Fix an orientation of the symmetric space. It induces an orientation on every 2-cell σ. We say $\beta \subset \sigma$ if all the cusps and corresponding edges are in σ.

Definition 4.2.3. *If $\beta \subset \sigma$ then we define $sign_\sigma(\beta)$ to be* 1 *if the edges (c_1, c_2) and (c_2, c_3) occur in the same order as in the orientation and* -1 *otherwise.*

Definition 4.2.4. *We define $\rho : S_2(\tilde{X}) \to BTG$ as the map,*

$$\rho(\sigma) = \frac{1}{2}\Sigma_{\beta\subset\sigma} sign_\sigma(\beta)[\beta]$$

extended to $S_2(\tilde{X})$ by linearity.

By construction, it already respects the (left) action of $GL_2(O_K)$ on 2-cells.

Remark 4.2.5. *Another way to think about ρ is the following; We have a canonical map $BTG \to S_2(\tilde{X})$ by sending a Bianchi triple β to $sign_\sigma(\beta)\sigma$ where σ is the unique* 2*-cell containing β. The map is surjective and splits because $S_2(\tilde{X})$ is a projective $GL_2(O_K)$-module. ρ is the splitting map up to a multiple of $\mathbb{Q}$.*

Definition 4.2.6. *The vector space of* ***framed Bianchi triple*** *FBT is the space $\mathbb{Q}[\mathbb{F}^2_{\mathcal{N}}] \otimes_{GL_2(O_K)} BTG$ where $\mathbb{F}^2_{\mathcal{N}}$ is the collection of elements $(\alpha, \beta) \in (O_K/\mathcal{N})^2$ such that $<\alpha, \beta> = O_K/\mathcal{N}$. The right action of $g \in GL_2(O_K)$ on $\mathbb{F}^2_{\mathcal{N}}$ is just the usual left action by g^{-1}.*

Let $(0, 1) \in \mathbb{F}^2_{\mathcal{N}}$ and $\Gamma_1(\mathcal{N}) \subset GL_2(O_K)$ be its stabilizer, then $(0, 1)$ gives a right $GL_2(O_K)$-map $\mathbb{Q}[\Gamma\backslash GL_2(O_K)] \to \mathbb{Q}[\mathbb{F}^2_{\mathcal{N}}]$ by sending $\Gamma_1(\mathcal{N})g$ to $g^{-1}(0, 1)$. Combining with the map ρ, we get a new map $\mathbb{Q}[\Gamma_1(\mathcal{N})\backslash GL_2(O_K)] \otimes_{GL_2(O_K)} S_2(\tilde{X}) \to FBT$. The domain of this map is precisely the 2-cells of the locally symmetric space of $\Gamma_1(\mathcal{N})$.

4.3 From Bianchi triples to framed motives

In this section, we will usually work on $(\alpha,\beta)\otimes\tau$, where $(\alpha,\beta)\in\mathbb{F}_{\mathcal{N}}^2$ and $\tau\in BT$. To construct a map from $FBT \to \mathcal{B}_2(\overline{K})$, it is sufficient to define the image of $(\alpha,\beta)\otimes\tau$. If we choose a Bianchi triple γ in each $GL_2(\mathcal{O}_K)$-orbit and call such elements **standard**, then using the relation $g^{-1}(\alpha,\beta)\otimes\gamma = (\alpha,\beta)\otimes g\gamma$, we can assume, without loss of generality, that γ is standard. The only problem would be to check whether such a map is well defined, but it is not a problem because BTG is a free $\mathbb{Q}[GL_2(\mathcal{O}_K)]$-module and the standard Bianchi triple form a basis of this free module. We will now only consider elements of the form consider $(\alpha,\beta)\otimes\tau$ as above but with τ standard.

Let $\tau = (E_1, E_2, E_3)$ then we have map $f_i : E \to E\times E$ which factors through an inclusion $\iota_i : E_i \hookrightarrow E\times E$. The inclusion is unique up to $Aut(E_i)$. The map f_i can be expressed as (a_i, b_i) where $a_i, b_i \in \mathcal{O}_K$. We take f_i such that $(a_i, b_i) \not\subset (x)$ for all non-units $x\in\mathcal{O}_K$. The map f_i is also unique up to the action of $Aut(E)$. We can decompose f_i as $\iota_i\circ\pi_i$ where $\pi_i : E\to E_i$. Consider the following pullback diagram,

$$\begin{tikzcd} \tilde{E}_2 \arrow[r] \arrow[d] & E_1\times E_3 \arrow[d, "\iota_1+\iota_3"] \\ E_2 \arrow[r] & E\times E \end{tikzcd}$$

where $\iota_1+\iota_3 = m\circ(\iota_1\times\iota_3)$. Here we make an assumption that $E_2\cap E_i = 0$ for $i = 1,3$. This assumption holds if K has class number 1 and $\mathcal{O}_K$ is Euclidean because all 1-cells are in the orbit of $(0,\infty)$ as seen in the examples of Bianchi tessellation mentioned in chapter 2. Then $\tilde{E}_2\cap E_i = 0$ for $i = 1,3$, because of the pullback diagram,

$$\begin{tikzcd} E_i\cap E_2 \arrow[r] \arrow[d] & E_i \arrow[d, "id\times 0"] \\ \tilde{E}_2 \arrow[r] \arrow[d] & E_1\times E_3 \arrow[d, "\iota_1+\iota_3"] \\ E_2 \arrow[r] & E\times E \end{tikzcd}$$

For the torsion point $z\in E[\mathcal{N}]$, let u denote $(\alpha z, \beta z)\in E_1\times E_3$. Thus, we have certain elliptic curves $E_1\times 0$, $0\times E_3$ and $\tilde{E}_2$, which we will denote by h, v and $\tilde{E}$ respectively and a torsion point u in $E_1\times E_3$. Consider the family of motives,

$$N^{x,u} = H^2(E_1\times E_3\backslash(t_xh\cup t_xv\cup h\cup v), t_{u+s}h\cup t_{u+s}v\cup t_s\tilde{E})$$

where s is an element of $E_1\times E_3$ and t_xD denotes translation of the divisor D by x . For convenience, we define

$$L = h\cup v\cup t_xh\cup t_xv \tag{4.5}$$

$$M = t_{u+s}h\cup t_{u+s}v\cup t_s\tilde{E} \tag{4.6}$$

$L\cup M$ is an SNC divisor for generic s because L and M both are SNC divisors. The condition translate to the fact that $u\notin\tilde{E}$ and $x\notin h\cup v$. This means that the motive $N^{x,u}$ is a variation of motives over some open set U of $E\times E$.

Remark 4.3.1. *The motive $N^{x,u}$ can be compared to the motive of double logarithm in [22], which is similar to contruction of motivic dilogarithm in Section 3.2.3. This is in line with the idea that we want an elliptic version of double logarithm.*

4.4 Example in Weight 2

The motive N is a mixed elliptic motive. In this next section, we will show that there is a submotive which is of mixed Tate type. To give an idea of what happens in the next section, we will see the toy example for $H^1(E\backslash\{a,b\},\{c,d\})$, where E is an elliptic curve and a,b,c,d are distinct torsion points of E. By Proposition 3.1.14, we have the spectral sequence which degenerates on the second page because E is smooth and projective,

$$\bigoplus_{\substack{i-j=p\\|I|=i\\|J|=j}} H^{q-2i}(L_I \cap M_J)(-i) \Rightarrow H^{q-p}(L,M)$$

Writing relevant terms for $H^1(E\backslash\{a,b\},\{c,d\})$, we get,

$$\begin{tikzcd}
 & p=-1 & p=0 & p=1 \\
q=0 & \underline{H^0(c)\oplus H^0(d)} & H^0(E) \arrow[l] & \\
q=1 & & \underline{H^1(E)} & \\
q=2 & & H^2(E) & \underline{H^0(a)(-1)\oplus H^0(b)(-1)} \arrow[l]
\end{tikzcd}$$

Where the underlined terms are related to $H^1(E\backslash\{a,b\},\{c,d\})$. Initially it may seem like that this motive has no submotive of mixed Tate type due to presence of $H^1(E)$ in the graded piece corresponding to weight 1. To see there is a mixed Tate motive, we first use the relative long exact sequence,

$$\cdots \to H^0(E\backslash\{a,b\}) \to H^0(c)\oplus H^0(d) \to H^1(E\backslash\{a,b\},\{c,d\}) \to H^1(E\backslash\{a,b\}) \to 0 \tag{4.7}$$

From this exact sequence, we see that $W_1(H^1(E\backslash\{a,b\},\{c,d\}))$ is coming from $(p,q) = (1,2)$ part of spectral sequence which is Tate (better yet it is $\mathbb{Q}(-1)$). Using the spectral sequence for $H^1(E\backslash\{a,b\})$, we have the following exact sequence,

$$0 \to H^1(E) \to H^1(E\backslash\{a,b\}) \to H^0(a)(-1)\oplus H^0(b)(-1) \to H^2(E) \to 0 \tag{4.8}$$

From Theorem 4.4.3 below, we can see that $H^1(E\backslash\{a,b\})$ is isomorphic to $\mathbb{Q}(0)\oplus H^1(E)$ becasue because a,b are torsion points. The proof of the motivic version carries over from the proof in loc.cit. Putting it back in 4.7, we get,

$$0 \to \mathbb{Q}(0) \to H^1(E\backslash\{a,b\},\{c,d\}) \to \mathbb{Q}(-1)\oplus H^1(E) \to 0 \tag{4.9}$$

From here, we can see that there is a submotive P such that,

$$0 \to \mathbb{Q}(0) \to P \to \mathbb{Q}(-1) \to 0 \tag{4.10}$$

so P is a mixed Tate motive. We will now provide a framing. Let γ be a path from c to d in E and $\omega = dlog(\frac{\theta(z-a)}{\theta(z-b)})$ provide a framing for P. Then, the period is the integral

$$\int_\gamma dlog(\frac{\theta(z-a)}{\theta(z-b)}) = log(\frac{\theta(d-a)\theta(c-b)}{\theta(d-b)\theta(c-a)}) + monodromyfactor \tag{4.11}$$

where the monodromy factor is imaginary. Thus, the framed motive corresponds to $\frac{\theta(d-a)\theta(c-b)}{\theta(d-b)\theta(c-a)}$ in $L^\times = \mathcal{B}_1(L)$. Choosing a different path γ' will just change the monodromy factor and so it will not affect the corresponding real period.

Lemma 4.4.1. *From now on , we will identify $\mathcal{A}_1$ with $L^\times$. The motive $H^1(E\backslash\{a,b\},\{c,d\})$ has a submotive P of mixed Tate type. The framed motive $[H^1(E\backslash\{a,b\},\{c,d\}),\gamma,\omega] = [P,\gamma,\omega]$ in $\mathcal{A}_1$ corresponds to the element $\frac{\theta(d-a)\theta(c-b)}{\theta(d-b)\theta(c-a)}$ in $L^\times$.*

Another way to view this fact is to note that $P \in Ext^1_{\mathcal{MM}_L}(\mathbb{Q}(-1),\mathbb{Q}(0)) \cong K_1(L)_\mathbb{Q} \cong L^\times_\mathbb{Q}$ and for this case, the real part of the period is the Dirichlet regulator which again shows that the period of our framed motive is an element in $L^\times_\mathbb{Q}$.

Remark 4.4.2. *This formula looks like cross-ratio of four points. This is due to the fact that if a,b,c,d are 4 points in $\mathbb{P}^1$, then there is a framing on $H^1(\mathbb{P}^1 - \{a,b\},\{c,d\})$, which gives the cross-ratio $\{a,b,c,d\}$.*

4.4.1 An Essential Theorem

In this section, we will prove the following theorem,

Theorem 4.4.3. *$H^1(E\backslash\{a,b\}) \cong \mathbb{Q}(-1) \oplus H^1(E)$ if the divisor $[a]-[b]$ is a torsion element in $Pic(E)$.*

Proof. Suppose $[a]-[b]$ is a torsion element in $Pic(E)$. Then there exists a rational function f such that $div(f) = N([a]-[b])$. But a rational function is a map $f : E \to \mathbb{P}^1$, such that $f^{-1}(0) = a$ and $f^{-1}(\infty) = b$. Therefore, the map induces the diagram in the category of mixed Tate motives.

$$\begin{tikzcd}
H^1(E) \arrow[r, hook] & H^1(E\backslash\{a,b\}) \arrow[r] & H^0(a)(-1)\oplus H^0(b)(-1) \arrow[r, two heads] & H^2(E) \\
H^1(\mathbb{P}^1) \arrow[r, hook] \arrow[u] & H^1(\mathbb{P}^1\backslash\{0,\infty\}) \arrow[r] \arrow[u] & H^0(0)(-1)\oplus H^0(\infty)(-1) \arrow[r, two heads] \arrow[u, two heads] & H^2(\mathbb{P}^1) \arrow[u, two heads]
\end{tikzcd}$$

where the vertical maps are induced by f. Recall that $M(\mathbb{P}^1) = \mathbb{Q}(0)\oplus\mathbb{Q}(-1)[2]$. By duality, we know that $H^2(\mathbb{P}^1) \cong \mathbb{Q}(-1) \cong H^2(E)$, so the last vertical map $H^2(\mathbb{P}^1) \to H^2(E)$ is an isomorphism. Moreover, $H^1(\mathbb{P}^1) = 0$, which implies that $H^1(\mathbb{P}^1\backslash\{0,\infty\})$ is a a submotive of $H^0(0)(-1)\oplus H^0(\infty)(-1) = \mathbb{Q}(-1)\oplus\mathbb{Q}(-1)$. The bottom row of the diagram is just a short exact sequence in $\mathbb{Q}$-vector spaces tensored by $\mathbb{Q}(-1)$, therefore, the bottom row splits. By diagram chasing, we can conclude that that $H^1(E\backslash\{a,b\}) \cong \mathbb{Q}(-1)\oplus H^1(E)$. □

4.5 Construction for Weight 4

In this section, we return to the case of $N^{x,u}$ mentioned in section 4.3. Here, we denote $t_xh \cup t_xv \cup h \cup v$ as L and $t_{u+s}h \cup t_{u+s}v \cup t_s\tilde{E}$ as M. Denote by L_1, L_2, L_3, L_4 (resp. M_1, M_2, M_3) the divisors of L (resp. M) in the given order and L_{ij} (resp. M_{ij}) the point of intersections $L_i \cap L_j$ (resp. $M_i \cap M_j$). Let X be $E_1 \times E_3$. Then, the E_2-page of the spectral sequence for N is shown in figure 4.2. The homology at the underlined place will give us the graded components of $N^{x,u}$.

As in the weight 2 case, the obstruction to the existence of submotive of mixed Tate type are $H^1(M_i)$ and $H^1(L_i)(-1)$ and possibly something in $gr^W_2(N)$. We want to again use the splitting

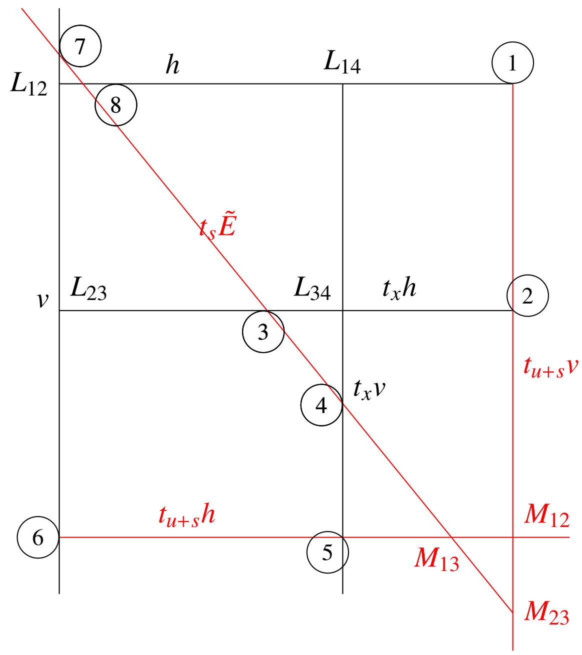

Figure 4.1: Ensemble of divisors L (black) and M (red) and their intersection. Numbered points are points of the form $L_i \cap M_j$. The points L_{ij} are of points of the form $L_i \cap L_j$ and The points M_{ij} are points of the form $M_i \cap M_j$

to show that there is a submotive of mixed Tate type. By Proposition 10.7 of [7] with $k = m = 1$ and $n = 2$, we get a surjective map,

$$\bigoplus W_1 H^1(M_i \backslash L \cap M_i, (M_i \cap \bigcup_{i \neq j} M_j)) \to W_1 N^{x,u} \tag{4.12}$$

Since M_i intersects other divisors of M at torsion points shifted by s, we can apply Lemma 4.4.1, taking Theorem 4.4.3 into consideration, to get that

$$W_1 H^1(M_i \backslash L \cap M_i, (M_i \cap \bigcup_{i \neq j} M_j)) \cong \mathbb{Q}(0)^u \oplus H^1(E)^v \tag{4.13}$$

for some natural number u, v. Because 4.12 is surjective we have a surjective map,

$$\mathbb{Q}(0)^u \oplus H^1(E)^v \cong \bigoplus W_1 H^1(M_i \backslash L \cap M_i, (M_i \cap \bigcup_{i \neq j} M_j)) \to W_1 N^{x,u} \to gr_1^W(N^{x,u}) \tag{4.14}$$

This is a surjective map and it sends $\mathbb{Q}(0)^u$ to zero because of incompatible weights. This means that the map,

$$H^1(E)^v \to gr_1^W N^{x,u} \tag{4.15}$$

is surjective. Denote the kernel of the map by K. Under the map 4.14, K is sent to zero so we can quotient K from the left hand side. By 4.15, the quotient is isomorphic to $\mathbb{Q}(0)^u \oplus gr_1^W(N^{x,u})$ and

$$gr_1^W(N^{x,u}) \to W_1 N^{x,u} \to gr_1^W(N^{x,u}) \tag{4.16}$$

is an isomorphism. This gives us an splitting of $gr_1^W(N^{x,u})$ in $W_1 N^{x,u}$ so that,

$$W_1(N^{x,u}) = \mathbb{Q}(0)^{u'} \oplus gr_1^W(N^{x,u})$$

for some integer u'.

Definition 4.5.1. *We define $N_1^{x,u}$ to be the quotient of $N^{x,u}$ by $gr_1^W(N^{x,u})$.*

Then $W_1(N_1^{x,u}) = \mathbb{Q}(0)^{u'}$ and we have cut off the non-Tate part in weight 1. There may still be some non-Tate part sitting in graded component of weight 2 and 3. To do this, we first consider the map from the relative sequence

$$H^2((E_1 \times E_3) - L, M - L) \to H^2((E_1 \times E_3) - L) \tag{4.17}$$

The motive $H^2((E_1 \times E_3) - L)$ is equal to $H^2((E_1 - \{0, x_1\}) \times (E_3 - \{0, x_2\}))$, where $x = (x_1, x_2)$ is an $\mathcal{N}$-torsion point of $E_1 \times E_3$. Since this is a product of varieties, we apply Künneth formula to get

$$H^2((E_1 - \{0, x_1\}) \times (E_3 - \{0, x_2\})) = H^1(E_1 - \{0, x_1\}) \otimes H^1(E_3 - \{0, x_2\}) \tag{4.18}$$

Again by Theorem 4.4.3, $H^1(E_1 - \{0, x_1\})$ and $H^1(E_3 - \{0, x_2\})$ split and both of which have a direct summand isomorphic to $\mathbb{Q}(-1)$. This means that equation 4.18 implies that $H^2((E_1 - \{0, x_1\}) \times (E_3 - \{0, x_2\}))$ has a direct summand isomorphic to $\mathbb{Q}(-2)$.

Definition 4.5.2. *We define $N_2^{x,u}$ to the preimage of $\mathbb{Q}(-2)$ under the map $N_1^{x,u} \to N^{x,u} \to H^2((E_1 \times E_3) - L)$.*

Theorem 4.5.3. *The motive $N_2^{x,u}$ is mixed Tate.*

Proof. It suffices to show that $W_3 N_2^{x,u}$ is a mixed Tate motive. The image of $W_3 N_2^{x,u}$ under the map

$$W_3 N_2^{x,u} \to N_2^{x,u} \to N_1^{x,u} \to N^{x,u} \to H^2((E_1 \times E_3) - L)$$

is zero by definition of $N_2^{x,u}$. The spectral sequences for $H^2((E_1 \times E_3) - L, M - L)$ and $H^2((E_1 \times E_3) - L)$ are compatible with the map $H^2((E_1 \times E_3) - L, M - L) \to H^2((E_1 \times E_3) - L)$. The spectral sequence for $H^2((E_1 \times E_3) - L, M - L)$ is given in figure 4.2 and the spectral sequence for $H^2((E_1 \times E_3) - L)$ is the same figure with the terms with M_i removed. These two spectral sequence have the same entries in $q = 3$ row. Therefore, the kernel of the natural map is trivial i.e.$gr_3^W(N_2^{x,u}) = 0$ and therefore $W_3 N_2^{x,u} = W_2 N_2^{x,u}$.
For $gr_2^W(N_2^{x,u})$, we again use the spectral sequences for which the relevant piece for $gr_2^W(N_2^{x,u})$ is the kernel of the following map of complexes,

$$\begin{tikzcd}
\bigoplus_i H^2(M_i) \arrow[d] & H^2(X) \oplus \bigoplus_{i,j} H^0(L_i \cap M_j)(-1) \arrow[l] \arrow[d] & \bigoplus_j H^0(L_j)(-1) \arrow[l] \arrow[d] \\
0 & H^2(X) \arrow[l] & \bigoplus_j H^0(L_j)(-1) \arrow[l]
\end{tikzcd}$$

The kernel works out to,

$$\bigoplus_i H^2(M_i) \leftarrow \bigoplus_{i,j} H^0(L_i \cap M_j)(-1) \tag{4.19}$$

The required piece $gr_2^W(N_2^{x,u})$ is the kernel of the induced map on middle cohomology group, which is Tate because both $\bigoplus_i H^2(M_i)$ and $\bigoplus_{i,j} H^0(L_i \cap M_j)(-1)$ are Tate motives. As for $W_1 N_2^{x,u}$, it is mixed Tate because it is equal to $W_1 N_1^{x,u}$, which we have already seen to be mixed Tate above. ◻

The fact that $gr_4^W(N_2^{x,u}) \cong \mathbb{Q}(-2)$ is a consequence of the definition of $N_2^{x,u}$. In Figure 4.1, this component corresponds to the black square formed by the divisor L. Similarly, $gr_0^W(N_2^{x,u}) \cong \mathbb{Q}(0)$ corresponding to the red triangle in the same figure.
As for the $gr_2^W(N_2^{x,u})$, the proof of the Theorem 4.5.3 also gives us a recipe to find graded components of $N_2^{x,u}$. It is the middle cohomology of the map

$$\bigoplus_i H^2(M_i) \leftarrow \bigoplus_{i,j} H^0(L_i \cap M_j)(-1) \leftarrow \bigoplus_j H^0(L_j)(-1) \tag{4.20}$$

The right map is induced from the closed immersions $L_i \cap M_j \to L_i$. The left map is induced by Gysin maps also induced by closed immersions $L_i \cap M_j \to M_j$. Even more explicitly, $1 \in H^0(L_i \cap M_j)$ is sent to an element which is $|L_i \cap M_j|$ in $H^2(M_j)$ coordinate and zero in the rest. For our configuration, this becomes

$$\begin{array}{llll} ① & \to [t_{u+s}v] & ⑤ & \to [t_{u+s}h] \\ ② & \to [t_{u+s}v] & ⑥ & \to [t_{u+s}h] \\ ③ & \to [t_s\tilde{E}] & ⑦ & \to [t_s\tilde{E}] \\ ④ & \to [t_s\tilde{E}] & ⑧ & \to [t_s\tilde{E}] \end{array}$$

where $[D]$ is used to denote 1 in $H^2(D)$. After some linear algebra calculations, we get the following lemma.

Lemma 4.5.4. *The graded component $gr_2^W(N_2^{x,u})$ is isomorphic to $\mathbb{Q}(-1)^3$ and it is generated by ① − ②, ③ − ④, ⑤ − ⑥.*

$$
\begin{array}{cccccc}
 & p=-2 & p=-1 & p=0 & p=1 & p=2 \\[1em]
q=0 & \underline{\bigoplus_{\substack{i,j\\ i\neq j}} H^0(M_{i,j})} \longleftarrow & \bigoplus_i H^0(M_i) \longleftarrow & H^0(X) & & \\[1em]
q=1 & & \underline{\bigoplus_i H^1(M_i)} \longleftarrow & H^1(X) & & \\[1em]
q=2 & & \bigoplus_i H^2(M_i) \longleftarrow & \underline{H^2(X) \oplus \bigoplus_{i,j} H^0(L_i \cap M_j)(-1)} \longleftarrow & \bigoplus_j H^0(L_j)(-1) & \\[1em]
q=3 & & & H^3(X) \longleftarrow & \underline{\bigoplus_j H^1(L_j)(-1)} & \\[1em]
q=4 & & & H^4(X) \longleftarrow & \bigoplus_j H^2(L_j)(-1) \longleftarrow & \underline{\bigoplus_{\substack{i,j\\ i\neq j}} H^0(L_{i,j})(-2)}
\end{array}
$$

Figure 4.2: E_2 page of the spectral sequence for N

4.5.1 Dual of the Motive $N_2^{x,u}$

In this section, we will find another interpretation of the dual of the motive $N_2^{x,u}$. By Proposition 3.1.14, $N^\vee = H^2(X - M, L - M)$ and by Remark 3.1.16, the duality is compatible with both the spectral sequence i.e. the dual of the spectral sequence of N is isomorphic to the spectral sequence of $N^\vee$. we can construct $(N_2^{x,u})^\vee$ as a subquotient of N by dualizing the definitions 4.5.1 and 4.5.2. Thus, $(N_2^{x,u})^\vee$ is a mixed Tate motive. We can also explicitly find the graded component of weight 1 of $(N_2^{x,u})^\vee$ by again dualizing equation 4.19 which becomes

$$\bigoplus_j H^0(M_j) \to \bigoplus_{i,j} H^0(L_i \cap M_j)(-1) \to \bigoplus_i H^2(L_i) \tag{4.21}$$

This equation comes from the spectral sequence of $H^2(X - M, L - M)$. The first map is induced by the closed immersion $L_i \cap M_j \to M_j$. More explicitly, we have

$$\begin{aligned} [t_{u+s}h] &\to (5) + (6) \\ [t_{u+s}v] &\to (1) + (2) \\ \left[t_s\tilde{E}\right] &\to (3) + (4) + (7) + (8) \end{aligned}$$

We also get an analogous lemma to Lemma 4.5.4.

Lemma 4.5.5. *The graded component $gr_2^W((N_2^{x,u})^\vee)$ is isomorphic to $\mathbb{Q}(-1)^3$ and it is generated by the image of $(1) - (8)$, $((3) - (2)) + ((8) - (1))$ and $((7) - (6))$.*

The duality in $\mathbf{DM}(k)$ is induced from Poincaré duality. Since the Gysin maps are dual to the restriction maps, the equations 4.21 and 4.19 are dual in $\mathbf{DM}(k)$. Poincaré duality also induces an isomorphism between $E_{0,2}$ term in the spectral sequence $N_2^{x,u}$ and its dual $(N_2^{x,u})^\vee$, which is again $E_{0,2}$ because it is Tate. The inner product induced from Poincaré duality on $E_{0,2} = \bigoplus_{i,j} H^0(L_i \cap M_j)(-1)$ is the sum of inner product of the individual summands. On each summand, the inner product is identity. This means that this inner product is the standard inner product on $E_{0,2}$ considered as a vector space.

Lemma 4.5.6. *The dual basis, according to the inner product induced from Poincaré duality, of the basis of $gr_2^W(N_2^{x,u})$ mentioned in 4.5.4 is exactly the basis of $gr_2^W((N_2^{x,u})^\vee)$ mentioned in 4.5.5.*

This lemma and the fact that spectral sequence and duality are compatible, play an essential role in the computation of coproduct.

4.6 Coproduct of $N_2^{x,u}$

The goal of this section is to find the coproduct of $N_2^{x,u}$ explicitly in terms of elliptic units using the example in weight 2 in Section 4.4. Let $(\alpha,\beta)\otimes(E_1, E_2, E_3)$ be a framed Bianchi triple and let z be an $\mathcal{N}$-torsion point of E. The motive $N_2^{x,u}$ inherits its 2-framing from $N^{x,u}$. It is framed by the form $v_0 = dlog(f_{x_1}) \wedge dlog(f_{x_2})$, where f_x is a function such that $div(f_x) = |E[\mathcal{N}]|([x] - [0])$, and the 2nd part of the frame f_2 is given by a triangle in $\{(\gamma(t_1), \gamma(t_2)) | 0 < t_1 < t_2 < 1\} \subset E \times E$ where γ is a path from αz to βz. We will try to calculate $[N_2^{x,u}, v_0, ((1) - (2))^\vee]$. This 1-framing is equivalent to $[W_2N_2^{x,u}, v_0, ((1) - (2))^\vee]$ by the inclusion $W_2N^{x,u} \to N^{x,u}$.

Let q be the sum of face maps mentioned in section 10.3 of [7] whose weight 1 we consider in equation 4.12. We consider the composition

$$H^1(M_2\backslash L\cap M_2,(M_2\cap\bigcup_{i\neq j}M_j))\xrightarrow{\iota}\bigoplus_i H^1(M_i\backslash L\cap M_i,(M_i\cap\bigcup_{i\neq j}M_j))\xrightarrow{q}N^{x,u} \tag{4.22}$$

Denote by t the map $q\circ\iota$. All these maps are compatible with the spectral sequences. To find gr_2, we focus on the row with $q=2$ on the first page of the spectral sequence. The maps in (4.22) induce maps between these rows of the first pages of spectral sequences. This results in the following maps of complex

$$\begin{tikzcd}
0 \arrow[d] & \bigoplus_i H^0(L_i\cap M_2)(-1) \arrow[l] \arrow[d] & 0 \arrow[l] \arrow[d] \\
0 \arrow[d] & \bigoplus_{i,j} H^0(L_i\cap M_j)(-1) \arrow[l] \arrow[d] & 0 \arrow[l] \arrow[d] \\
\bigoplus_i H^2(M_i) & H^2(X)\oplus\bigoplus_{i,j} H^0(L_i\cap M_j)(-1) \arrow[l] & \bigoplus_j H^0(L_j)(-1) \arrow[l]
\end{tikzcd}$$

where the rows are the rows $E_1^{\bullet,2}$ and the columns are the induced maps from (4.22). From the middle column, it is easy to observe that 1–2 is in the image of $\bigoplus_i H^0(L_i\cap M_2)(-1)$ because both points 1 and 2 lie on M_2 by Figure 4.1. This means that the 1-framing $(1-2)^\vee$ can be pulled back along t to a 1-framing of $H^1(M_2\backslash L\cap M_2,(M_2\cap\bigcup_{i\neq j}M_j))$.

On the other hand, for gr_0, we have to compare the rows $E_1^{\bullet,0}$. The map induced by the composition t is

$$\begin{tikzcd}
H^0(M_2\cap M_1)\oplus H^0(M_2\cap M_3) \arrow[d] & H^0(M_2) \arrow[l] \arrow[d] & H^0(X) \arrow[l] \arrow[d] \\
\bigoplus_{\substack{i,j\\ i\neq j}} H^0(M_{i,j}) & \bigoplus_i H^0(M_i) \arrow[l] & 0 \arrow[l]
\end{tikzcd}$$

where the first row is the $E_1^{\bullet,0}$ part of the spectral sequence of $H^1(M_2\backslash L\cap M_2,(M_2\cap\bigcup_{i\neq j}M_j))$ and the second row is the $E_1^{\bullet,0}$ part of the spectral sequence of $N^{x,u}$. The cokernels of the left most horizontal maps provide us the weight 0 graded components of the two motives we are considering. This map of complexes induces an isomorphism t_* on the cokernels of the left horizontal maps such that v_0 is the image of $v_0':=M_{23}-M_{12}$ under t_*, where $M_{ij}=M_i\cap M_j$. By equivalence of framed motive when applied to t, we get

$$[N_2^{x,u},v_0,(1-2)^\vee]=[N_2^{x,u},t_*(v_0'),(1-2)^\vee]=[H^1(M_2\backslash L\cap M_2,(M_2\cap\bigcup_{i\neq j}M_j)),v_0',t^*\left((1-2)^\vee\right)] \tag{4.23}$$

We can apply the same technique to 3–4 and 5–6. Let R_i be $H^1(M_i\backslash L\cap M_i,(M_i\cap\bigcup_{i\neq j}M_j))$ for $i=1,2,3$.

Theorem 4.6.1. *We have the following equations,*

1. $[N_2^{x,u},v_0,(1-2)^\vee]=[R_2,v_0',(1-2)^\vee]$

2. $[N_2^{x,u}, v_0, (③ - ④)^\vee] = [R_3, v_0', (③ - ④)^\vee]$

3. $[N_2^{x,u}, v_0, (⑤ - ⑥)^\vee] = [R_1, v_0', (⑤ - ⑥)^\vee]$

Next, we will try to simplify terms of type $[N_2^{x,u}, \_, f_2]$. Again, we will take $[N_2^{x,u}, ① - ②, f_2]$ as an example. This is an 1-framed motive, which are primitive elements of the Hopf algebra $\mathcal{A}_\bullet$. Therefore, antipode of $[N_2^{x,u}, ① - ②, f_2]$ is $-[N_2^{x,u}, ① - ②, f_2]$. On the other hand, the antipode is also described as $[(N_2^{x,u})^\vee, f_2, ① - ②]$. By Lemma 4.5.6, we know that $① - ② = (① - ⑧)^\vee$. The duality of Gysin spectral sequence implies that we can use the same argument as above with L and M interchanged because of Remark 3.1.16. Similar to above, we define P_j as $H^1(L_j \backslash M \cap L_j, (L_j \cap \bigcup_{i \neq j} L_i))$ for $j = 1, 2, 3, 4$.

Theorem 4.6.2. *We have the following equations,*

1. $[N_2^{x,u}, (① - ②), f_2] = -[P_1, f_2, (① - ⑧)^\vee]$

2. $[N_2^{x,u}, (③ - ④), f_2] = [P_1, f_2, (① - ⑧)^\vee] + [P_3, f_2, (② - ③)^\vee]$

3. $[N_2^{x,u}, (⑤ - ⑥), f_2] = -[P_2, f_2, (⑦ - ⑥)^\vee]$

In both Theorem 4.6.1 and 4.6.2, the righthand side can be computed in terms of elliptic units using Lemma 4.4.1. Theorem 4.6.1 and 4.6.2 finally give us the coproduct of $[N_2^{x,u}, v_0, f_2]$,

Corollary 4.6.3.

$$\begin{aligned}\delta([N_2^{x,u}, v_0, f_2]) = &- [R_2, v_0, (① - ②)^\vee] \wedge [P_1, f_2, (① - ⑧)^\vee] \\ &+ [R_3, v_0, (③ - ④)^\vee] \wedge [P_1, f_2, (① - ⑧)^\vee] \\ &+ [R_3, v_0, (③ - ④)^\vee] \wedge [P_3, f_2, (② - ③)^\vee] \\ &- [R_1, v_0, (⑤ - ⑥)^\vee] \wedge [P_2, f_2, (⑦ - ⑥)^\vee]\end{aligned}$$

Proof. Follows from the definition of coproduct (3.16) and Theorem 4.6.1 and 4.6.2. □

The motive $N_2^{x,u}$ is a variation of mixed Tate motive over some open set of X. The mixed Tate motive over the generic fiber Spec$(K(X))$ is mixed Tate over $K(X)$. By Conjecture 3.2.37, this corresponds to an element of $\mathcal{B}_2(K(X))$.An upshot of considering the variation of motives instead of just a motive (with $s = 0$) is that, using specialization map in Section 3.2.5, we can obtain an element of $\mathcal{B}_2(K_N)$ even if $L \cup M$ is not an SNC divisor. This is a form of regularization mentioned in Section 1.7 of [19].

4.7 Definition of $\theta_{E,z}$

In this section, we will define $\theta_E : FBT \to \mathcal{B}_2(\overline{K})$. By composing with the map from $\rho : S_2(X_1(N)) \to FBT$, we will create the desired map $\theta^2_{E,z}$ in Theorem 4.8.10. In Section 4.9, we will compare this definition of $\theta^2_{E,z}$ with the one introduced by Goncharov in [27]. This makes $\theta^2_{E,z}$ a generalization of the θ_E introduced in loc. cit.

Definition 4.7.1. *Let $u \otimes (E_1, E_2, E_3)$ be a framed Bianchi triple where $u = (\alpha z, \beta z)$ as before. Recall that we denote $E_1 \times E_3$ as X. Define $\theta_E(u \otimes (E_1, E_2, E_3))$ as,*

$$\theta_E(u \otimes (E_1, E_2, E_3)) := \begin{cases} -(2D^2)^{-1} \sum_{x \in X[\mathcal{N}]-(h \cup v)} sp_2^x(N_2^{x,u}) & \text{if } u \notin \tilde{E}_2 \\ 0 & \text{otherwise.} \end{cases} \tag{4.24}$$

where D is the order of the group $E[\mathcal{N}]$ and sp_2^x is the specialization in Section 3.2.5 with $\mathcal{O}$ being the local ring at x.

First we will show that θ_E is indeed well-defined.

Lemma 4.7.2. *Under our assumption 3.2.38,θ_E is well defined.*

Proof. By lemma (4.2.2), the only thing to check is $\theta_{E,z}(u \otimes (E_1, E_2, E_3)) = \theta_{E,z}(\eta u \otimes (E_1, E_2, E_3))$, where $\eta \in \mathcal{O}_K^\times$.

We know that multiplication by η induces an isomorphism,

$$H^2(X - L^x, M_z - (L^x \cap M_z)) \to H^2(X - L^{\eta x}, M_{\eta z} - (L^{\eta x} \cap M_{\eta z})) \tag{4.25}$$

where $L^x = t_x h \cup t_x v \cup t_h \cup t_v \cup E_0$ and $M_z = t_{z+s} h \cup t_{z+s} v \cup t_s \tilde{E}$. This isomorphism is compatible with the framing and moreover, it induces an isomorphism between $N_2^{x,u}$ and $N_2^{\eta x, \eta u}$. Therefore, these two elements are equal in $\mathcal{L}_2(K(X))$ by the equivalence of framed motives. Since $\mathcal{L}_2(K(X))$ is isomorphic to $\mathcal{B}_2(K(X))$ by 3.2.38, we can assume this framed motive lies in $\mathcal{B}_2(K(X))$. By our assumption, Since η just changes the order of summation in θ_E because it permutes the elements of $X[\mathcal{N}] - (h \cup v)$, we have,

$$\theta_E(\eta u \otimes (E_1, E_2, E_3)) = \theta_E(u \otimes (E_1, E_2, E_3)) \tag{4.26}$$

which is precisely what we had to prove. □

A priori $\theta_E(u \otimes (E_1, E_2, E_3))$ is an element in $\mathcal{B}_2(\overline{K})$. The next lemma shows that it is in $\mathcal{B}_2(K_\mathcal{N})$.

Lemma 4.7.3. $\theta_{E,z}(u \otimes (E_1, E_2, E_3)) \in \mathcal{B}_2(K_\mathcal{N})$

Proof. The inclusion of L and M into X is defined over $K_\mathcal{N}(X)$ because the translation by an $\mathcal{N}$-torsion point is defined over $K_\mathcal{N}$ and the other maps are defined over $K(X)$. So $N_2^{x,u}$ is a mixed Tate motive over $K_\mathcal{N}(X)$. So we can assume the base field is $K_\mathcal{N}$ instead of K and all $\mathcal{N}$-torsion points are rational points of E. So what remains to be proven is the fact that specializing to a rational point gives an element in $\mathcal{B}_2(K_\mathcal{N})$.
If $[f]$ is an element in $\mathcal{B}_2(K_\mathcal{N}(X))$ and x is a rational point then it is clear that $sp_2^x([f])$ (the specialization map mentioned in Section 3.2.5) should be of the form $[\tilde{f}]$ where $\tilde{f}$ is in the residue field of x, which is $K_\mathcal{N}$. Therefore, $sp_2^x([f]) \in \mathcal{B}_2(K_\mathcal{N})$. The claim follows. □

Before calculating θ_E or more precisely its coproduct explicitly, we prove a lemma which will be very useful in simplification. Here, we would like to mention that by $\theta(0)$ we mean $Sq_0(\theta(s))$ where s is a generic point of E.

Lemma 4.7.4. *Let x, y be two elements of $E[\mathcal{N}]$ and s be a generic element of E, then*

$$\bigwedge^2 k(X)^\times \ni \sum_{x \in E[\mathcal{N}]} \theta(x+s) \wedge \theta(x+y-s) = 0 \tag{4.27}$$

Proof. Replace x with $-y-z$ in the equation to get,

$$\begin{aligned} L = \sum_{x\in E[\mathcal{N}]} \theta(x+s)\wedge\theta(x+y-s) &= \sum_{z\in E[\mathcal{N}]} \theta(-y-z+s)\wedge\theta(-y-z+y-s) \\ &= \sum_{z\in E[\mathcal{N}]} \theta(+y+z-s)\wedge\theta(z+s) \\ &= -\sum_{z\in E[\mathcal{N}]} \theta(z+s)\wedge\theta(+y+z-s) = -L \end{aligned}$$

where we used $\theta(x) = \theta(-x)$ in the second step by Remark 1.2.34. This implies that $2L = 0$, which means that $L = 0$. □

Corollary 4.7.5. *For $z, y \in E[\mathcal{N}]$ we have $\sum_{x\in E[\mathcal{N}]} \theta(x+z)\wedge\theta(x+y) = 0$*

Proof. Apply Theorem 4.7.4 with $s = z$ and $z + y$ instead of y. □

4.8 General construction for $K = \mathbb{Q}(\sqrt{-d})$ for $d = 1, 2, 3, 7, 11$

For class number one, all the cusps are in one $GL_2(\mathcal{O}_K)$-orbit and any subelliptic curve is isomorphic to E by Remark 4.1.11. To define the map $\theta^2_{E,z}$, we have to select some standard Bianchi triple. To do so, we select specific Bianchi triple, which generate all the Bianchi triple by the action of $GL_2(\mathcal{O}_K)$. We also assume $\mathcal{N}$ is a prime ideal of $\mathcal{O}_K$.

Lemma 4.8.1. *If $d = 1, 2, 3, 7, 11$ than any Bianchi triple is in the orbit of a Bianchi triple of the form $(0, \infty, a)$, where $a \in \mathcal{O}_K \subset \mathbb{P}^1(K)$. Moreover, any two Bianchi triples of the form $(0, \infty, a), (0, \infty, b)$ are in the same orbit if and only if a and b differ by a unit i.e. for some $u \in \mathcal{O}_K^\times$, $a = ub$.*

Proof. Since class number of K is one, all cusps are in one orbit so we can use this action to map c_2 to ∞. For $d = 1, 2, 3, 7, 11$, the cusps which are joined by an edge to ∞ are elements of $\mathcal{O}_K \subset \mathbb{P}^1(K)$. So, the Bianchi triple is in the orbit of (x, ∞, y) for some $x, y \in \mathcal{O}_K$. Now, act on the right by the matrix

$$\gamma = \begin{bmatrix} 1 & 0 \\ -x & 1 \end{bmatrix}$$

to get that (x, ∞, y) and $(0, \infty, y-x)$ are in the same orbit. The first claim follows. For the second claim, note that if a matrix A maps a Bianchi triple $(0, \infty, a)$ to another Bianchi triple $(0, \infty, b)$, then it must stabilize 0 and ∞. Therefore

$$A = \begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix}$$

Then $u = xy^{-1}$ does the trick. □

This lemma helps us to fix standard Bianchi triples. First, we choose $(0, \infty, 1)$ to be a standard Bianchi triple. Then, the 3-cells mentioned in Section 2.5.1 of Chapter 2 have 0, 1 and ∞ as its vertices. Let e_3 be this 3-cell. The edge $(0, \infty)$ lies on the $(0, \infty, 1)$ triangle and the other 2-cell e_2. Then e_2 must have a Bianchi triple of the form $(0, \infty, a)$. If this is not in the orbit of $(0, \infty, 1)$, then we will choose it to be the second standard Bianchi triple. Now, we repeat this process for the second Bianchi triple in an another 3-cell e_3'. This process terminates, because of the number of orbits of Bianchi triples is finite.

After choosing the standard Bianchi triples, we can move towards calculating the framed motive $N_2^{x,u}$.

Lemma 4.8.2. *The base change $\tilde{E_2}$ of E_2 along $E_1 \times E_3 \to E \times E$ is the antidiagonal $\Psi := \{(x, -x) \in E \times E\}$.*

Proof. Using the correspondence of cusps and subelliptic curves (Corollary 4.1.8), we deduce that E_c for a cusp c is image of f_c as we mentioned in Section 4.3, where f_c is the map

$$f_c : E \to E \times E, \qquad x \to (cx, x)$$
$$f_\infty : E \to E \times E, \qquad x \to (x, 0)$$

Then,

$$\begin{aligned}\tilde{E_2} =& \{(x, y, z) \in E \times E \times E | f_\infty(x) = f_0(y) + f_a(z)\}\\ =& \{(x, y, z) \in E \times E \times E | (x, 0) = (0, y) + (az, z)\}\\ =& \{(x, y, z) \in E \times E \times E | (x, 0) = (az, y + z)\}\\ =& \{(az, -z, z) \in E \times E \times E\}\end{aligned}$$

Projecting to the last two coordinates, we see that $\tilde{E_2} = \Psi$. □

Remark 4.8.3. *It can be seen that there is no dependence on a in $N_2^{x,u}$, which makes calculating $\theta_{E,z}$ easier in our case.*

Let $x = (x_1, x_2)$ be an element of $E \times E$ such that $x_1, x_2 \in E[\mathcal{N}] - 0$ and z be an element of $E[\mathcal{N}]$. Let $(\alpha, \beta) \otimes (0, \infty, a)$ be a standard Bianchi triple and $u_1 = \alpha z$ and $u_2 = \beta z$. We will denote the pair (u_1, u_2) as u. To calculate the coproduct of $N_2^{x,u}$, we first find the relevant points of intersection;

$$\begin{array}{llll} (1) = (u_1 + s_1, 0), & (2) = (u_1 + s_1, x_2), & (3) = (s_1 + s_2 - x_2, x_2), & (4) = (x_1, s_1 + s_2 - x_1) \\ (5) = (x_1, u_2 + s_2), & (6) = (0, u_2 + s_2), & (7) = (0, s_1 + s_2), & (8) = (s_1 + s_2, 0) \\ L_{34} = (x_1, x_2), & L_{23} = (0, x_2), & L_{14} = (x_1, 0), & L_{12} = (0, 0) \\ M_{12} = (u_1 + s_1, u_2 + s_2), & M_{13} = (s_1 - u_2, s_2 + u_2), & M_{23} = (s_1 + u_1, s_2 - u_1) & \end{array}$$

where $s = (s_1, s_2)$ is a generic point of $E \times E$. All of these points are shown in Figure 4.1. The framing v_0 and f_2 give an anticlockwise orientation on both the triangle with vertices M_{ij} and the square with vertices L_{ij}. For example, the order M_{12} and M_{13} in R_1 is explained by this orientation i.e. the framing v_0 induces the 0-th framing $M_{13} - M_{12}$ in R_1 and not $M_{12} - M_{13}$. The framed motive $[R_2, v_0, ((1) - (2))^\vee]$ is $[H^1(t_{u+s}v - \{(1), (2)\}, \{M_{12}, M_{23}\}), M_{12} - M_{23}, ((1) - (2))^\vee]$. Using Lemma 4.4.1, this simplifies to,

$$[R_2, v_0, ((1) - (2))^\vee] = \frac{\theta(u_2 + s_2 - x_2)\theta(-u_1 + s_2 - 0)}{\theta(u_2 + s_2 - 0)\theta(-u_1 + s_2 - x_2)}$$

All the terms required in order to compute the coproduct via Corollary 4.6.3 are simplified to,

$$\begin{aligned}[R_1, v_0, ((5) - (6))^\vee] =& \frac{\theta(u_1 + s_1 - 0)\theta(-u_2 + s_1 - x_1)}{\theta(u_1 + s_1 - x_1)\theta(-u_2 + s_1 - 0)}\\ [R_2, v_0, ((1) - (2))^\vee] =& \left(\frac{\theta(-u_1 + s_2 - x_2)\theta(u_2 + s_2 - 0)}{\theta(-u_1 + s_2 - 0)\theta(u_2 + s_2 - x_2)}\right)^{-1}\\ [R_3, v_0, ((3) - (4))^\vee] =& \frac{\theta(-u_2 - x_1 + s_1)\theta(u_1 - s_2 + x_2)}{\theta(-u_2 - s_2 + x_2)\theta(u_1 - x_1 + s_1)}\end{aligned}$$

Similarly,

$$
\begin{aligned}
[P_1, f_2, (1 - 8)^\vee] &= \frac{\theta(0 - s_1 - s_2)\theta(x_1 - u_1 - s_1)}{\theta(0 - u_1 - s_1)\theta(x_1 - s_1 - s_2)} \\
[P_2, f_2, (7 - 6)^\vee] &= \frac{\theta(0 - u_2 - s_2)\theta(x_2 - s_1 - s_2)}{\theta(0 - s_1 - s_2)\theta(x_2 - u_2 - s_2)} \\
[P_3, f_2, (2 - 3)^\vee] &= \frac{\theta(x_1 - s_1 - s_2 + x_2)\theta(0 - u_1 - s_1)}{\theta(x_1 - u_1 - s_1)\theta(0 - s_1 - s_2 + x_2)}
\end{aligned}
$$

Now, we will specialize to the antidiagonal. This amounts to plugging $s_1 = s$ and $s_2 = -s$

$$
\begin{aligned}
[R_1, v_0, (5 - 6)^\vee] &= \frac{\theta(u_1 + s)\theta(u_2 + x_1 - s)}{\theta(u_2 - s)\theta(u_1 - x_1 + s)} \\
[R_2, v_0, (1 - 2)^\vee] &= \left(\frac{\theta(u_1 + s)\theta(u_2 - x_2 - s)}{\theta(u_2 - s)\theta(u_1 + x_2 + s)}\right)^{-1} \\
[R_3, v_0, (3 - 4)^\vee] &= \frac{\theta(u_2 + x_1 - s)\theta(u_1 + x_2 + s)}{\theta(u_1 - x_1 + s)\theta(x_2 - u_2 + s)}
\end{aligned}
\tag{4.28}
$$

$$
\begin{aligned}
[P_1, f_2, (1 - 8)^\vee] &= \frac{\theta(0)\theta(x_1 - u_1 - s)}{\theta(x_1)\theta(u_1 + s)} \\
[P_2, f_2, (7 - 6)^\vee] &= \frac{\theta(x_2)\theta(u_2 - s)}{\theta(0)\theta(x_2 - u_2 + s)} \\
[P_3, f_2, (2 - 3)^\vee] &= \frac{\theta(x_1 + x_2)\theta(u_1 + s_1)}{\theta(x_1 - u_1 - s)\theta(x_2)}
\end{aligned}
\tag{4.29}
$$

From now on, we will omit the framing and write, say $[R_1, v_0, (5 - 6)^\vee]$, simply as R_1.Equation 4.29 also imply the following equation;

$$
P_1 P_3 = \frac{\theta(x_1 + x_2)\theta(0)}{\theta(x_1)\theta(x_2)} \tag{4.30}
$$

Putting Equations 4.28,4.29 and 4.30 together we get,

$$
\begin{aligned}
\sum_{x_1, x_2 \in E[\mathcal{N}] - 0} \delta(N_2^{x,u}) = \sum_{x_1, x_2 \in E[\mathcal{N}] - 0} \Bigg(& -\frac{\theta(u_1 + s)\theta(u_2 + x_1 - s)}{\theta(u_2 - s)\theta(u_1 - x_1 + s)} \wedge \frac{\theta(x_2)\theta(u_2 - s)}{\theta(0)\theta(x_2 - u_2 + s)} \\
& + \frac{\theta(u_1 + s)\theta(u_2 - x_2 - s)}{\theta(u_2 - s)\theta(u_1 + x_2 + s)} \wedge \frac{\theta(0)\theta(x_1 - u_1 - s)}{\theta(x_1)\theta(u_1 + s)} \\
& + \frac{\theta(u_2 + x_1 - s)\theta(u_1 + x_2 + s)}{\theta(u_1 - x_1 + s)\theta(x_2 - u_2 + s)} \wedge \frac{\theta(x_1 + x_2)\theta(0)}{\theta(x_1)\theta(x_2)} \Bigg)
\end{aligned}
\tag{4.31}
$$

Note that if we plug $x_1 = 0$ or $x_2 = 0$ in the expression on the right hand side, then it still makes sense, even though, they do not come up in the formula for coproduct. If we put $x_1 = 0$, then $P_3 = P_1 = R_1 = 1$ so the expression on the right hand side of 4.31 is zero. Similarly, if $x_2 = 0$ then $P_1 P_3 = 1$ and $P_2 = R_2 = 1$ which again implies that the expression on the right hand side of 4.31 is zero. This means that in the calculation of the coproduct of $\theta_{E,z}$ we can assume that the sum is over $X[\mathcal{N}]$ instead of $X[\mathcal{N}] - (h \cup v)$. Equation 4.31 becomes

$$\sum_{x_1,x_2\in E[\mathcal{N}]-0} \delta(N_2^{x,u}) = \sum_{x_1,x_2\in E[\mathcal{N}]} \Bigg(-\frac{\theta(u_1+s)\theta(u_2+x_1-s)}{\theta(u_2-s)\theta(u_1-x_1+s)} \wedge \frac{\theta(x_2)\theta(u_2-s)}{\theta(0)\theta(x_2-u_2+s)} + \frac{\theta(u_1+s)\theta(u_2-x_2-s)}{\theta(u_2-s)\theta(u_1+x_2+s)} \wedge \frac{\theta(0)\theta(x_1-u_1-s)}{\theta(x_1)\theta(u_1+s)} + \frac{\theta(u_2+x_1-s)\theta(u_1+x_2+s)}{\theta(u_1-x_1+s)\theta(x_2-u_2+s)} \wedge \frac{\theta(x_1+x_2)\theta(0)}{\theta(x_1)\theta(x_2)} \Bigg) \tag{4.32}$$

We will compute each term separately. By linearity and collecting terms with only x_1 or x_2 as much as possible, we can have

$$\begin{aligned} \sum_{x_1,x_2\in E[\mathcal{N}]} \frac{\theta(u_1+s)\theta(u_2+x_1-s)}{\theta(u_2-s)\theta(u_1-x_1+s)} \wedge \frac{\theta(x_2)\theta(u_2-s)}{\theta(0)\theta(x_2-u_2+s)} = \\ D^2 \frac{\theta(u_1+s)}{\theta(u_2-s)} \wedge \frac{\theta(u_2-s)}{\theta(0)} + D \sum_{x_2\in E[\mathcal{N}]} \frac{\theta(u_1+s)}{\theta(u_2-s)} \wedge \frac{\theta(x_2)}{\theta(x_2-u_2+s)} \\ +D \sum_{x_1\in E[\mathcal{N}]} \frac{\theta(u_2+x_1-s)}{\theta(u_1-x_1+s)} \wedge \frac{\theta(u_2-s)}{\theta(0)} + \sum_{x_1,x_2\in E[\mathcal{N}]} \frac{\theta(u_2+x_1-s)}{\theta(u_1-x_1+s)} \wedge \frac{\theta(x_2)}{\theta(x_2-u_2+s)} \end{aligned} \tag{4.33}$$

where $D = |E[\mathcal{N}]|$. These terms themselves simplify.

$$\begin{aligned} \sum_{x_2\in E[\mathcal{N}]} \frac{\theta(u_1+s)}{\theta(u_2-s)} \wedge \frac{\theta(x_2)}{\theta(x_2-u_2+s)} &= \frac{\theta(u_1+s)}{\theta(u_2-s)} \wedge \prod_{x_2\in E[\mathcal{N}]} \frac{\theta(x_2)}{\theta(x_2-u_2+s)} \\ &= \frac{\theta(u_1+s)}{\theta(u_2-s)} \wedge \prod_{x_2\in E[\mathcal{N}]} \frac{\theta(x_2)}{\theta(x_2+s)} \end{aligned}$$

where $\prod_{x_2\in E[\mathcal{N}]} \theta(x_2-u_2+s) = \prod_{x_2\in E[\mathcal{N}]} \theta(x_2+s)$. By the rule $x \mapsto -x$, we also note that $\prod_{x\in E[\mathcal{N}]} \theta(x+s) = \prod_{x\in E[\mathcal{N}]} \theta(-x+s) = \prod_{x\in E[\mathcal{N}]} \theta(x-s)$. This identity helps to simplify the next term.

$$\begin{aligned} \sum_{x_1\in E[\mathcal{N}]} \frac{\theta(u_2+x_1-s)}{\theta(u_1-x_1+s)} \wedge \frac{\theta(u_2-s)}{\theta(0)} &= \left(\prod_{x_1\in E[\mathcal{N}]} \frac{\theta(u_2+x_1-s)}{\theta(u_1-x_1+s)} \right) \wedge \frac{\theta(u_2-s)}{\theta(0)} \\ &= \left(\prod_{x_1\in E[\mathcal{N}]} \frac{\theta(x_1-s)}{\theta(x_1+s)} \right) \wedge \frac{\theta(u_2-s)}{\theta(0)} \\ &= 1 \wedge \frac{\theta(u_2-s)}{\theta(0)} = 0 \end{aligned}$$

Similarly, the last term is also zero because

$$\sum_{x_1,x_2\in E[\mathcal{N}]} \frac{\theta(u_2+x_1-s)}{\theta(u_1-x_1+s)} \wedge \frac{\theta(x_2)}{\theta(x_2-u_2+s)} = \prod_{x_1\in E[\mathcal{N}]} \frac{\theta(u_2+x_1-s)}{\theta(u_1-x_1+s)} \wedge \prod_{x_2\in E[\mathcal{N}]} \frac{\theta(x_2)}{\theta(x_2-u_2+s)}$$

After these simplifications, Equation 4.33 becomes

$$\sum_{x_1,x_2\in E[\mathcal{N}]} \frac{\theta(u_1+s)\theta(u_2+x_1-s)}{\theta(u_2-s)\theta(u_1-x_1+s)} \wedge \frac{\theta(x_2)\theta(u_2-s)}{\theta(0)\theta(x_2-u_2+s)} = D^2 \frac{\theta(u_1+s)}{\theta(u_2-s)} \wedge \frac{\theta(u_2-s)}{\theta(0)} + D \frac{\theta(u_1+s)}{\theta(u_2-s)} \wedge Q \tag{4.34}$$

where $Q = \prod_{x_2 \in E[\mathcal{N}]} \frac{\theta(x_2)}{\theta(x_2+s)}$. A similar computation for the second term of Equation 4.32 yields

$$\sum_{x_1, x_2 \in E[\mathcal{N}]} \frac{\theta(u_1+s)\theta(u_2-x_2-s)}{\theta(u_2-s)\theta(u_1+x_2+s)} \wedge \frac{\theta(0)\theta(x_1-u_1-s)}{\theta(x_1)\theta(u_1+s)} = D^2 \frac{\theta(u_1+s)}{\theta(u_2-s)} \wedge \frac{\theta(0)}{\theta(u_1+s)} - D\frac{\theta(u_1+s)}{\theta(u_2-s)} \wedge Q \tag{4.35}$$

The third term of Equation 4.32 is

$$\sum_{x_1, x_2 \in E[\mathcal{N}]} \frac{\theta(u_2+x_1-s)\theta(u_1+x_2+s)}{\theta(u_1-x_1+s)\theta(x_2-u_2+s)} \wedge \frac{\theta(x_1+x_2)\theta(0)}{\theta(x_1)\theta(x_2)} \tag{4.36}$$

Denote this expression as E. By changing $x_1 \to -x_2$ and $x_2 \to -x_1$, we get

$$\begin{aligned}
E &= \sum_{x_1, x_2 \in E[\mathcal{N}]} \frac{\theta(u_2+x_1-s)\theta(u_1+x_2+s)}{\theta(u_1-x_1+s)\theta(x_2-u_2+s)} \wedge \frac{\theta(x_1+x_2)\theta(0)}{\theta(x_1)\theta(x_2)} \\
&= \sum_{x_1, x_2 \in E[\mathcal{N}]} \frac{\theta(u_2-x_2-s)\theta(u_1-x_1+s)}{\theta(u_1+x_2+s)\theta(-x_1-u_2+s)} \wedge \frac{\theta(-x_2-x_1)\theta(0)}{\theta(-x_2)\theta(-x_1)} \\
&= \sum_{x_1, x_2 \in E[\mathcal{N}]} \frac{\theta(-u_2+x_2+s)\theta(u_1-x_1+s)}{\theta(u_1+x_2+s)\theta(x_1+u_2-s)} \wedge \frac{\theta(x_2+x_1)\theta(0)}{\theta(x_2)\theta(x_1)} \\
&= -\left(\sum_{x_1, x_2 \in E[\mathcal{N}]} \frac{\theta(u_2+x_1-s)\theta(u_1+x_2+s)}{\theta(u_1-x_1+s)\theta(x_2-u_2+s)} \wedge \frac{\theta(x_1+x_2)\theta(0)}{\theta(x_1)\theta(x_2)} \right) \\
&= -E
\end{aligned}$$

where we used $\theta(x) = \theta(-x)$ for the second equality and inverted the first entry for the third equality. This implies that the Expression 4.36 is zero i.e.

$$\sum_{x_1, x_2 \in E[\mathcal{N}]} \frac{\theta(u_2+x_1-s)\theta(u_1+x_2+s)}{\theta(u_1-x_1+s)\theta(x_2-u_2+s)} \wedge \frac{\theta(x_1+x_2)\theta(0)}{\theta(x_1)\theta(x_2)} = 0 \tag{4.37}$$

Putting everything together we have the following theorem.

Theorem 4.8.4. *The coproduct of* $\sum_{x_1, x_2 \in E[\mathcal{N}]-0} \delta(N_2^{x,u})$ *is*

$$\begin{aligned}
\sum_{x_1, x_2 \in E[\mathcal{N}]-0} \delta(N_2^{x,u}) = -D^2 \frac{\theta(u_1+s)}{\theta(u_2-s)} \wedge \frac{\theta(u_2-s)}{\theta(0)} - D\frac{\theta(u_1+s)}{\theta(u_2-s)} \wedge Q \\
+ D^2 \frac{\theta(u_1+s)}{\theta(u_2-s)} \wedge \frac{\theta(0)}{\theta(u_1+s)} - D\frac{\theta(u_1+s)}{\theta(u_2-s)} \wedge Q
\end{aligned}$$

Proof. Follows from Equation 4.34,4.35, 4.32 and 4.37. □

Corollary 4.8.5. *Let K be an imaginary quadratic field $\mathbb{Q}(\sqrt{d})$ where $d = -1, -2, -3, -7, -11$ and let $\mathcal{N}$ be an ideal of O_K. Let E be an elliptic curve such that $End(E) \cong O_K$ and z be an $\mathcal{N}$-torsion point of E. Let $a \in O_K$ and $\alpha, \beta \in O_K/\mathcal{N}$ such that $< \alpha, \beta >= O_K/\mathcal{N}$. Then*

$$\delta\left(\theta_E((\alpha z, \beta z) \otimes (0, \infty, a))\right) = \theta(\alpha z) \wedge \theta(\beta z) - \frac{1}{2D} \frac{\theta(u_1)}{\theta(u_2)} \wedge \theta(0) \tag{4.38}$$

Proof. Specialize Theorem 4.8.4 to $s = 0$. The left hand side is $-2D^2\delta\left(\theta_E((\alpha z, \beta z)\right)$ after specialization and the right hand side becomes $-2D^2\theta(\alpha z) \wedge \theta(\beta z) + D\theta(u_1) \wedge \theta(0) - D\theta(u_2) \wedge \theta(0)$. The result follows. □

Note that the coproduct of θ_E depends only on the first two cusp (c_1, c_2) of the Bianchi triple (c_1, c_2, c_3); by Theorem 4.8.1, if two Bianchi triples $(0, \infty, a)$ and $(0, \infty, b)$ are in the same $GL_2(\mathcal{O}_K)$-orbit then, there is a diagonal matrix d such that $d(0, \infty, a) = (0, \infty, b)$. This implies the following equality,

$$(\alpha, \beta) \otimes (0, \infty, b) = d^{-1}(\alpha, \beta) \otimes (0, \infty, a)$$

by the definition of framed Bianchi triple. The diagonal matrix d^{-1} scales αz and βz by units of $\mathcal{O}_K$. But $\theta(z) = \theta(vz)$ for all $v \in \mathcal{O}_K^\times$ by Remark 1.2.34, so the coproduct of their images under $\theta_{E,z}$ remains unchanged. Since every edge is in the $GL_2(\mathcal{O}_K)$-orbit of $(0, \infty)$, this result also holds if any two Bianchi triples of the form (c_1, c_2, c_3) and (c_1, c_2, c_3'). This discussion implies the fact that first two cusps of the Bianchi triple determine the coproduct of the images under $\theta_{E,z}$. Lets enshrine this fact in a lemma.

Lemma 4.8.6. *The coproduct of θ_E only depends on (c_0, c_1).*

Equipped with this knowledge, we will now turn our attention to the following definition,

Definition 4.8.7. *We define the following subspaces;*

- *Define $C_1(\mathcal{N})$ to be the subspace of $K_{\mathcal{N}}^\times$ generated by $\theta(a)$ for all $a \in E[\mathcal{N}] - 0$.*
- *Define $\hat{C}_1(\mathcal{N}) := C_1(\mathcal{N}) \oplus \mathbb{Q}$ where this extra factor represents $\widehat{\theta(0)}$.*
- *Define $C_2(\mathcal{N})$ to be the subspace of $\mathcal{B}_2(K_{\mathcal{N}})$ generated by $\theta_{E,z} \circ \rho(z \otimes e)$, where z is an $\mathcal{N}$-torsion element in $E \times E$ and e is a 2-cell in the Bianchi tessellation.*

Proposition 4.8.8. $\delta(C_2(\mathcal{N})) \subset \Lambda^2 \hat{C}_1(\mathcal{N})$

Proof. Follows immediately from (4.38). □

Definition 4.8.9. *The **Euler complex**, denoted by $C_\bullet(\mathcal{N})$ is the complex*

$$0 \to C_2(\mathcal{N}) \xrightarrow{\delta} \Lambda^2 \hat{C}_1(\mathcal{N}) \xrightarrow{\delta} C_1(\mathcal{N}) \oplus C_1(\mathcal{N})$$

where $\delta : C_2(\mathcal{N}) \to \Lambda^2 \hat{C}_1(\mathcal{N})$ is the restriction of the Bloch-Suslin map to $C_2(\mathcal{N})$ (see Section 3.2.4). The other δ is defined as,

$$\begin{aligned} \delta\left(\theta(a) \wedge \theta(b)\right) &= (0, \theta(a) - \theta(b)), \ \text{if } a, b \neq 0 \\ \delta\left(\widehat{\theta(0)} \wedge \theta(b)\right) &= (-\theta(b), \theta(b)), \quad \text{if } b \neq 0 \end{aligned}$$

The Spaces $S_0(X(\Gamma_1(\mathcal{N}))$ and $S_1(X(\Gamma_1(\mathcal{N}))$

Let $\mathcal{P}$ be a prime ideal of $\mathcal{O}_K$. Let $\mathbb{F}_{\mathcal{P}}$ be the residue field $\mathcal{O}_K/\mathcal{P}$. To prove Theorem 4.8.10, we need a presentation of $S_0(X(\Gamma_1(\mathcal{P}))$ and $S_1(X(\Gamma_1(\mathcal{P}))$. Furthermore, we will also describe $\theta_{E,z}^0$ and $\theta_{E,z}^1$ as mentioned in [27].

Since all edges are in the orbit of $e_1 := (0, \infty)$, we have

$$S_1(X(\Gamma_1(\mathcal{N}))) = \mathbb{Q} \otimes_{\Gamma_1(\mathcal{N})} \mathbb{Q}[GL_2(\mathcal{O}_K)] \otimes_{\mathrm{Stab}(e_1)} \chi_{sgn}$$

where χ_{sgn} is the character of $\mathrm{Stab}(e_1)$ encoding the information of the orientation of the edge e_1. As seen in Section 2.5.1, the stabilizer of e_1 is isomorphic to $S_2 \times (\mathcal{O}_K^\times)^2$. The extra copy of

$O_K^\times$ is due to the fact that we are considering $GL_2(O_K)$ and not $PGL_2(O_K)$. The character χ_{sgn} is trivial on $O_K^\times$ and is the sign representation on S_2. This is because matrix U in Section 2.5.1 flips the orientation of e_1. Recall the set $\mathbb{F}_{\mathcal{N}^2}$ from Definition 4.2.6. Because the class number of K is 1, we have,

$$\mathbb{Q} \otimes_{\Gamma_1(\mathcal{N})} \cong \mathbb{Q}[\mathbb{F}_{\mathcal{N}^2}]$$

The character χ_{sgn} gives some relations on this set, which results in the following form of $S_1(X(\Gamma_1(\mathcal{N})))$.

$$S_1(X(\Gamma_1(\mathcal{N}))) = \frac{\mathbb{Q}[\mathbb{F}_{\mathcal{N}^2}]}{(\alpha,\beta) = -(\beta,\alpha) = (\eta_1\alpha, \eta_2\beta)} \tag{4.39}$$

where η_1, η_2 are units in O_K. Similarly,

$$S_0(X_1(\mathcal{N})) = \mathbb{Q}[\Gamma_1(\mathcal{N}) \backslash GL_2(O_K)/B(O_K)]$$

where $B(O_K)$ is the Borel subgroup. It consists of upper triangular matrices with 1's on the diagonal. In the case that $\mathcal{N}$ is a prime ideal $\mathcal{P}$, we further simplify $S_0(X(\Gamma_1(\mathcal{P})))$.

$$S_0(X(\Gamma_1(\mathcal{P}))) = \mathbb{Q}[\Gamma_1(\mathcal{P}) \backslash GL_2(O_K)/B(O_K)] = \mathbb{Q}[\{[0,\alpha],[\alpha,0]\}] \tag{4.40}$$

where $\alpha \in \mathbb{F}_{\mathcal{P}}^\times/\mu$ where μ is the image of $O_K^\times$ in $\mathbb{F}_{\mathcal{P}}^\times$. The boundary map $\partial : S_1(X(\Gamma_1(\mathcal{P}))) \to S_0(X(\Gamma_1(\mathcal{P})))$ more explicitly becomes

$$\partial((\alpha,\beta)) = [0,\alpha] - [0,\beta], \text{ if } \alpha,\beta \neq 0$$
$$\partial((0,\beta)) = -[\beta,0] + [0,\beta], \text{ if } \alpha,\beta \neq 0$$

The map θ_1^E can now be defined in terms of the presentation of $S_1(X(\Gamma_1(\mathcal{P})))$.

$$\theta^1_{E,z}(\alpha,\beta) = \theta(\alpha z) \wedge \theta(\beta z), \text{ if } \alpha,\beta \neq 0$$
$$(\alpha, 0) = \theta(\alpha z) \wedge \widehat{\theta(0)}, \text{ if } \alpha \neq 0$$

For $\mathcal{P}$, a prime ideal of O_K, the map θ_0^E can now be defined in terms of the presentation of $S_0(X(\Gamma_1(\mathcal{P})))$

$$\theta^0_{E,z}([\alpha,0]) = (0, \theta(\alpha z))$$
$$\theta^0_{E,z}([0,\alpha]) = (\theta(\alpha z), 0)$$

Both of these maps are well defined due to Remark 1.2.34.

The Main Theorem

Using all the information we have gathered till now, we can show the following theorem.

Theorem 4.8.10. *Let $K = \mathbb{Q}(\sqrt{-d})$ for $d = 1, 2, 3, 7, 11$. Let E be an elliptic curve with $End(E) \cong O_K$. Let $\mathcal{P}$ be a prime ideal of O_K and z be a $\mathcal{P}$-torsion element z in E. Then, assuming the statement 3.2.38, we have the commutative diagram,*

$$\begin{tikzcd}
S_3(X(\Gamma_1(\mathcal{P}))) \arrow[r, "\partial"] \arrow[d] & S_2(X(\Gamma_1(\mathcal{P}))) \arrow[r, "\partial"] \arrow[d, "\theta^2_{E,z}"'] & S_1(X(\Gamma_1(\mathcal{P}))) \arrow[r, "\partial"] \arrow[d, "\theta^1_{E,z}"] & S_0(X(\Gamma_1(\mathcal{P}))) \arrow[d, "\theta^0_{E,z}"] \\
0 \arrow[r] & C_2(\mathcal{P}) \arrow[r, "\delta"] & \bigwedge^2 \hat{C}_1(\mathcal{P}) \arrow[r, "\delta"] & C_1(\mathcal{P}) \oplus C_1(\mathcal{P})
\end{tikzcd}$$

where $\theta^2_{E,z}$ is the composition of ρ in Definition 4.2.4 and $\theta_E((0,z) \otimes -)$ and $\theta^1_{E,z}$ is induced from the inclusion of the elliptic units into $K_{\mathcal{N}}^\times$. Moreover, $\theta^1_{E,z}$ and $\theta^0_{E,z}$ are isomorphisms and $\theta^2_{E,z}$ is surjective.

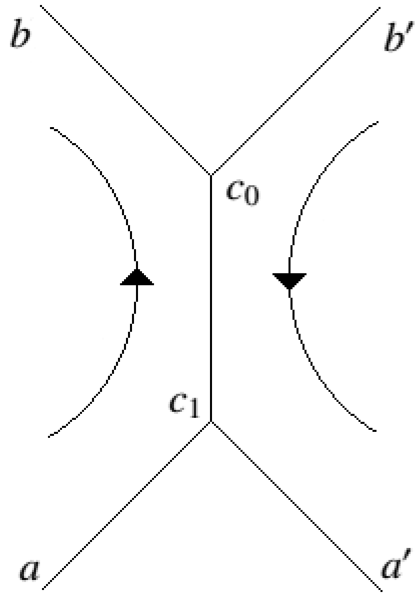

Figure 4.3: The picture of the 3-cell at the edge (c_0, c_1)
with the vertices c_0, c_1, a, a', b and b'. The orientation is shown by the arrows.

Proof. There are three commutative squares to prove. We will start with the rightmost one.

Using the description of $\theta^0_{E,z}$ and $\theta^1_{E,z}$ mentioned above, the rightmost box can be seen to be commutative,

$$\delta \circ \theta^1_{E,z}(\alpha,\beta) = \delta(\theta(\alpha z) \wedge \theta(\beta z)) = (0, \theta(\alpha z) - \theta(\beta z)) = \theta^0_{E,z}(\partial(\alpha,\beta)) = \theta^0_{E,z} \circ \partial(\alpha,\beta)$$

if $\alpha,\beta \neq 0$ and

$$\delta \circ \theta^1_{E,z}(\beta,0) = \delta(\theta(\beta z) \wedge \widehat{\theta(0)}) = (\theta(\beta z), -\theta(\beta z)) = \theta^0_{E,z}(\partial(\beta),0) = \theta^0_{E,z} \circ \partial(\beta,0)$$

if $\alpha = 0, \beta \neq 0$. This takes care of the rightmost square.

For the leftmost box note that for any 3-cell S an edge (c_0, c_1) appears in two faces resulting in 4 Bianchi triples; (c_0, c_1, a) and (c_1, c_0, b) in one face and (c_0, c_1, a') and (c_1, c_0, b') in the other. Let $g \in GL_2(O_K)$ map (c_0, c_1) to $(0, \infty)$. Using g we see that (c_0, c_1, a) and (c_0, c_1, a') can be mapped to $(0, \infty, x)$ and $(0, \infty, x')$ respectively. Both are standard so they will give the same value but since they are in different orientation in both faces so they cancel. This also happens for (c_1, c_0, b) and (c_1, c_0, b') and every other edge. So the total sum is zero and thus commutativity holds. The figure 4.3 shows this phenomenon.

For the middle square, the argument works for any ideal $\mathcal{N}$ of O_K so we will work in this general setting. First note that the Bianchi triples also have an explicit form.

$$\mathbb{Q} \otimes_{\Gamma_1(\mathcal{N})} BTG = \bigoplus \mathbb{Q}[\mathbb{F}^2_{\mathcal{N}}] \tag{4.41}$$

This follows from the fact that BTG is a free $GL_2(O_K)$-module. Consider the map $\epsilon : \mathbb{Q} \otimes_{\Gamma_1(\mathcal{N})} BTG \to S_1(X(\Gamma_1(\mathcal{N})))$ defined by sending (c_0, c_1, c_2) to the class of (c_0, c_1). In a way, this map forgets the second edge in a Bianchi triple. Explicitly, it would just be the sum followed by projection of $\mathbb{Q}[\mathbb{F}^2_{\mathcal{N}}]$ onto $\mathbb{Q}[\mathbb{F}^2_{\mathcal{N}}] \otimes_{Stab(e_1)} \chi_{sgn}$, where e_1 and chi_1 are mentioned on page 85. If

e_2 is a 2-cell with n sides then, by the definition of ϵ, there are $2n$ Bianchi triples 'in' e_2. By Lemma 4.8.6 and Corollary 4.8.5, this implies that the contribution of the two Bianchi triple for an edge (c_0, c_1) ,namely (c_0, c_1, a) and (c_1, c_0, b), in the coproduct is the same. This results in a factor of 2, which is compensated by $\frac{1}{2}$ in the definition of ρ in Definition 4.2.4. Thus, we see that $\epsilon \circ \rho = \partial$. we have the following commutative diagram.

$$\begin{tikzcd}
S_2(X(\Gamma_1(\mathcal{N}))) \arrow[r, "\rho"] & \mathbb{Q} \otimes_{\Gamma_1(\mathcal{N})} BTG \arrow[r, "\epsilon"] \arrow[d, "\theta_E((0,z)\otimes -)"] & S_1(X(\Gamma_1(\mathcal{N}))) \arrow[d, "\theta_1 + \psi"] \\
& C_2(\mathcal{N}) \arrow[r, "\delta"'] & \Lambda^2 \hat{C}_1(\mathcal{N})
\end{tikzcd}$$

The composition $\delta \circ \theta_E((0, z) \otimes -) \circ \rho$ is $\delta \circ \theta^2_{E,z}$. The map ψ is defined as,

$$\psi(\alpha, \beta) = -(2D)^{-1}\theta(\alpha z) \wedge \theta(0) - (2D)^{-1}\theta(\beta z) \wedge \theta(0)$$

The commutativity with ψ as given above follows from (4.38). Now ψ fits in the commutative diagram,

$$\begin{tikzcd}
S_1(X_1(\mathcal{N})) \arrow[r, "\partial"] \arrow[d, "\psi"'] & S_0(X_1(\mathcal{N})) \arrow[dl, "q"] \\
\Lambda^2 \hat{C}_1(\mathcal{N}) &
\end{tikzcd}$$

where $q([0, b]) = -(2D)^{-1}\theta(bz) \wedge \theta(0)$ and $q([b, 0]) = 0$. Using these commutative diagrams, we get,

$$\begin{aligned}
\delta \circ \theta^2_{E,z} =& \delta \circ \theta_{E,z} \circ \rho = (\theta^1_{E,z} + \psi) \circ \epsilon \circ \rho \\
=& \theta^1_{E,z} \circ \epsilon \circ \rho + \psi \circ \epsilon \circ \rho \\
=& \theta^1_{E,z} \circ \partial + q \circ \partial \circ \partial \\
=& \theta^1_{E,z} \circ \partial
\end{aligned}$$

Finally, $\theta^2_{E,z}$ is surjective by definition and, $\theta^0_{E,z}$ and $\theta^1_{E,z}$ are isomorphisms by Theorem 5.7 in [27]. □

The map of complexes $\theta^\bullet_{E,z}$ also intertwines the action of the diamond operators and Galois action. First, we will briefly describe, the diamond operators. Let

$$\mathcal{D}_\mathcal{N} := N(\Gamma_1(\mathcal{N}))/\Gamma_1(\mathcal{N}).$$

The group $\mathcal{D}_\mathcal{N}$ acts on the locally symmetric space $X(\Gamma_1(\mathcal{N}))$. The action of $\mathcal{D}_\mathcal{N}$ on our Bianchi triples is via its action on $\mathbb{F}^2_\mathcal{N}$. The group can be identified with the matrices of the form

$$\begin{pmatrix} 1 & 0 \\ 0 & \alpha \end{pmatrix}$$

where α is an element of the group $\mathbb{F}^\times_\mathcal{N}/\mu$ where $\mathbb{F}_\mathcal{N}$ is the ring $O_K/\mathcal{N}$ and μ is the image of $O^\times_K$ in $\mathbb{F}^\times_\mathcal{N}$. We will identify $\mathcal{D}_\mathcal{N}$ with $\mathbb{F}^\times_\mathcal{N}/\mu$. On the other hand, the Galois group $Gal(K_\mathcal{N}/K)$ acts on the set of strict $\mathcal{N}$-torsion elements, where by strict $\mathcal{N}$-torsion element we mean an $\mathcal{N}$-torsion element which is in some basis of $E[\mathcal{N}]$. The action of O_K on $E[\mathcal{N}]$ induces an action of $\mathcal{D}_\mathcal{N}$ on $E[\mathcal{N}]$. By theory of complex multiplication of elliptic curves, the automorphism group of $E[\mathcal{N}]$ is precisely the Galois group $Gal(K_\mathcal{N}/K)$. Therefore, we have a map $\mathcal{D}_\mathcal{N} \to Gal(K_\mathcal{N}/K)$. This map is an isomorphism as mentioned on page 28 of [27]. Under this isomorphism, the Galois action of the image of $\alpha \in \mathbb{F}_\mathcal{N}^\times/\mu$ sends z to $z' = \alpha z$. The Galois action on $K_\mathcal{N}$ induces a Galois action on the complex $C_\bullet$. So

$$\theta_E^2((\alpha \cdot_{\mathcal{D}_\mathcal{N}} (0, z)) \otimes e_2) = \theta_E^2((0, \alpha z) \otimes e_2) = \theta_E^2((0, z') \otimes e_2) = \alpha \cdot_{Gal(K_\mathcal{N}/K)} \theta_E^2((0, z) \otimes e_2)$$

where e_2 is a 2-cell. The last equality holds because all the subelliptic curves are defined over K and the only thing Galois action changes are the points z and x in the divisors L and M in $N_2^{x,z}$ i.e. $\alpha \cdot_{Gal(K_\mathcal{N}/K)} N_2^{x,z} = N_2^{\alpha x, z'}$. But since we are summing over x, multiplying by α just changes the order of summation of x so only z changes to z'. We can encapsulate this into a lemma.

Lemma 4.8.11. *The map $\theta_{E,z}^\bullet$ intertwines the action of $\mathcal{D}_\mathcal{N}$ on $S_\bullet(X_1(\mathcal{N}))$ with the action of $Gal(K_\mathcal{N}/K)$ on $C_\bullet(\mathcal{N})$.*

One point to note is that $\delta(\theta_{E,z}^2((0, z) \otimes e_2))) \in \Lambda^2 C_1(\mathcal{N})$ by definition but we could have contribution of $\widehat{\theta(0)}$ in $\theta_{E,z}^1(\partial e_2)$. Even though, it may seem like a contradiction but we will show that the contribution of $\widehat{\theta(0)}$ cancels out so that, even in this particular case, the commutativity of middle square in Theorem 4.8.10 holds. First, we will explain the cases where terms with $\widehat{\theta(0)}$ appear. Generally, $\theta_{E,z}^1((0, z) \otimes (c_0, c_1)) = \theta_{E,z}^1((0, z)\gamma \otimes (0, \infty))$ where γ is the matrix which sends the pair $(0, \infty)$ to (c_0, c_1). We will shorten $\theta_{E,z}^1((0, z)\gamma \otimes (0, \infty))))$ to $\theta_{E,z}^1((0, z)\gamma)$ for now. To get $\widehat{\theta(0)}$ from Bianchi triple, we must have $(0, z)\gamma$ to have a zero in one of the coordinate. But

$$(0, z)\gamma = (0, z)\begin{pmatrix} \gamma_1 & \gamma_2 \\ \gamma_3 & \gamma_4 \end{pmatrix} = (\gamma_3 z, \gamma_4 z)$$

So either γ_3 or γ_4 has to be zero modulo $\mathcal{N}$. If $\gamma_3 = 0$ then,

$$\gamma \cdot (0, \infty) = (\gamma_2\gamma_4^{-1}, \infty) = \begin{pmatrix} 1 & \gamma_2\gamma_4^{-1} \\ 0 & 1 \end{pmatrix}(0, \infty)$$

Since the matrix on right hand side is in $\Gamma_1(\mathcal{N})$, we can take to γ to be in $\Gamma_1(\mathcal{N})$, which implies that $\theta_{E,z}^1((0, z)\gamma) = \theta_{E,z}^1((0, z)) = \widehat{\theta(0)} \wedge \theta(z)$. We also note that the Bianchi triple has $c_2 = \infty$. If $\gamma_4 = 0$, then,

$$\gamma \cdot (0, \infty) = (\infty, \gamma_1\gamma_3^{-1}) = \begin{pmatrix} 1 & \gamma_1\gamma_3^{-1} \\ 0 & 1 \end{pmatrix}(\infty, 0) = \begin{pmatrix} 1 & \gamma_1\gamma_3^{-1} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}(0, \infty)$$

Since the matrix on right hand side is in $\Gamma_1(\mathcal{N})S$, we can take to γ to be in $\Gamma_1(\mathcal{N})S$, where

$$S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

This means that $\theta_{E,z}^1((0, z)\gamma) = \theta_{E,z}^1((0, z)U) = \theta_{E,z}^1((0, z)) = -\widehat{\theta(0)} \wedge \theta(z)$. We also note that the Bianchi triple has $c_1 = \infty$.

Summarizing, there are three points of note;

- If $\widehat{\theta(0)}$ has to appear then one cusp has to be in $\Gamma_1(\mathcal{N})$-orbit of ∞.
- If $c_1 = \infty$ then $\theta_{E,z}^1((0, z) \otimes (c_1, c_2)) = -\widehat{\theta(0)} \wedge \theta(z)$

- If $c_2 = \infty$ then $\theta^1_{E,z}((0,z)\otimes(c_1,c_2)) = \widehat{\theta(0)} \wedge \theta(z)$

By the first fact, It is sufficient to consider the cusp involved to be ∞. Then there is a consecutive series of cusps (a, ∞, b). Then, $\theta^1((0,z)\otimes(a,\infty)) = \theta(0)\wedge\theta(z)$ by the third fact and $\theta^1((0,z)\otimes(\infty,b)) = -\theta(0)\wedge\theta(z)$ by the second fact. Since the definition of ∂ includes the sum $(a,\infty)+(\infty,b)$, the contribution in $\theta^1_{E,z}$ of these two terms adds up to 0, so $\theta(0)$ does not appear in the image of ∂e_2 under $\theta^1_{E,z}$.

4.9 Comparison with $\theta^{(1)}_E$

This section is about comparing Goncharov's definition $\theta^{(1)}_E$ (Definition 3.3.3) and our definition of $\theta^2_{E,z}$ and show that both of them are the same.

Theorem 4.9.1. *Let $\mathcal{N}$ be an ideal of O_K where $K = \mathbb{Q}(i), \mathbb{Q}(\sqrt{-3})$. Let N be the norm of $\mathcal{N}$. Let E be an elliptic curve such that $End(E) \cong O_K$ and z be an $\mathcal{N}$-torsion point of E. Then $\theta^{(1)}_E - \theta^2_{E,z}$ is a linear combination of terms $Li^{mot}_2(\zeta^a_{3N})$, in the case when $K = \mathbb{Q}(\sqrt{-3})$, and $Li^{mot}_2(\zeta^a_{4N})$, in the case when $K = \mathbb{Q}(i)$, where $a \in \mathbb{Z}$.*

Proof. Let $z_1, z_2 \in E[\mathcal{N}]$, we will show that $\theta^{(1)}_E(z_1, z_2, -z_1 - z_2) - \theta^2_{E,z}(z_1, z_2)$ can be expressed in terms of dilogarithm terms mentioned in the theorem. We will do this in two cases;

Case 1: z_1 and z_2 are contained in a cyclic group.

Let N be the norm of $\mathcal{N}$. In this case, there exists an order N element z in E such that $z_i = \eta_i z$ for $i = 1, 2$ and $\eta_i \in \mathbb{Z}$. Consider the curve $Y_1(N)$. It is the moduli space of elliptic curves with a fixed N-torsion point. We can upgrade $\theta^{(1)}_E$ and $\theta^2_{E,z}$ to a mixed Tate motive over $Y_1(N)$ by replacing E with the universal elliptic curve $\mathcal{E}$ and the torsion points z_1, z_2 with $\eta_1\mathbf{z}, \eta_2\mathbf{z}$, where $\mathbf{z}$ is the universal N-torsion section of $\mathbf{z} : Y_1(N) \to \mathcal{E}$. This is possible because the antidiagonal can be defined for every family of elliptic curves. We can then pull this motive back along the inclusion of generic point of $Y_1(N)$ to get a mixed Tate motive $\mathbf{M}$ over $\mathbb{Q}(Y_1(N))$. Corollary 4.8.4, and therefore Theorem 4.8.10, also generalize to similar results for $\mathbf{M}$. More explicitly, it implies that the coproduct of $\theta^2_{\mathcal{E}}(\eta_1\mathbf{z}, \eta_2\mathbf{z}) \in \mathcal{B}_2(\mathbb{Q}(Y_1(N)))$ is

$$\delta(\theta^2_{\mathcal{E}}(\eta_1\mathbf{z}, \eta_2\mathbf{z})) = \theta(\mathbf{z}_1)\wedge\theta(\mathbf{z}_2) + \theta(\mathbf{z}_2)\wedge\theta(\mathbf{z}_3) + \theta(\mathbf{z}_3)\wedge\theta(\mathbf{z}_1) \in \bigwedge^2 \mathbb{Q}(Y_1(N))$$

where $\mathbf{z}_1 = \eta_1\mathbf{z}$, $\mathbf{z}_2 = \eta_2\mathbf{z}$ and $\mathbf{z}_1 + \mathbf{z}_2 + \mathbf{z}_3 = 0$, where $\theta(\mathbf{z}_i)$ is the modular unit.

As seen in see Section 3.3 in [27], the coproduct of $\theta^{(1)}_{\mathcal{E}} \in \mathcal{B}_2(\mathbb{Q}(Y_1(N)))$ is

$$\delta(\theta^{(1)}_{\mathcal{E}}(\mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3)) = \theta(\mathbf{z}_1)\wedge\theta(\mathbf{z}_2) + \theta(\mathbf{z}_2)\wedge\theta(\mathbf{z}_3) + \theta(\mathbf{z}_3)\wedge\theta(\mathbf{z}_1)$$

Combining both, we get,

$$\delta(\theta^{(1)}_{\mathcal{E}}(\mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3) - \theta^2_{\mathcal{E}}(\mathbf{z})) = 0 \tag{4.42}$$

Therefore, the difference is an element of $ker(\delta) \subset \mathcal{B}_2(\mathbb{Q}(Y_1(N)))$. By Definition 3.2.30, restricting to the points (z, E) and (z', E') in $Y_1(N)$, implies the following equation,

$$\theta^{(1)}_E(\eta_1 z, \eta_2 z, -\eta_1 z - \eta_2 z) - \theta^2_E(\eta_1 z, \eta_2 z) = \theta^{(1)}_{E'}(\eta_1 z', \eta_2 z', -\eta_1 z' - \eta_2 z') - \theta^2_{E'}(\eta_1 z', \eta_2 z') \tag{4.43}$$

By defining $Z_E(z_1, z_2) := \theta_E^{(1)}(\eta_1 z, \eta_2 z, -\eta_1 z - \eta_2 z) - \theta_E^2(\eta_1 z, \eta_2 z)$, we get;

$$Z_E(\eta_1 z, \eta_2 z) = Z_{E'}(\eta_1 z', \eta_2 z') \tag{4.44}$$

Now the Galois group $G := Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ acts on $Y_1(N)(\overline{\mathbb{Q}})$ (because $Y_1(N)$ is defined over $\mathbb{Q}$) by sending a point (z, E) to (z^σ, E^σ) for $\sigma \in G$. Restricting at (z, E) and (z^σ, E^σ) implies,

$$Z_E(\eta_1 z, \eta_2 z) = Z_{E^\sigma}(\eta_1 z^\sigma, \eta_2 z^\sigma) = \sigma \cdot Z_E(\eta_1 z, \eta_2 z) \tag{4.45}$$

where the action of right hand side is the action of G on $\mathcal{B}_2(\overline{\mathbb{Q}})$. By commutativity of the square in Lemma 3.2.40, the element $Z_E(\eta_1 z, \eta_2 z) \in K_3(\overline{\mathbb{Q}})$ because it is the specialization of an element $\mathcal{B}_2(\mathbb{Q}(Y_1(N)))$ with coproduct zero. The equation (4.45) implies that $Z_E(\eta_1, z, \eta_2 z)$ is invariant under the Galois action. By Galois descent for rational K-theory (Theorem 2.15, [64]),$Z_E(\eta_1, z, \eta_2 z) \in K_3(\overline{\mathbb{Q}})^G = K_3(\mathbb{Q}) = 0$. So, $Z_E(\eta_1, z, \eta_2 z) = 0$.

Case 2: z_1 and z_2 are not contained in a cyclic subgroup.

In this case, we have $z_i = \eta_i x_i$ for $i = 1, 2$ where $\eta_i \in \mathbb{Z}$ and x_1, x_2 for a basis of $E[N]$ with Weil pairing ζ_N where $\zeta_N = exp(\frac{2\pi i}{N})$. We can use the a similar argument as above to get a variation of motives over $Y(N)$ which parametrizes basis of N-torsion points of elliptic curves with Weil pairing ζ_N. Using a similar argument as above with $\eta_i \mathbf{z}$ being replaced by $\eta_i \mathbf{x}_i$ for $i = 1, 2$, the analogue for Equation (4.44) is

$$Z_E(\eta_1 x_1, \eta_2 x_2) = Z_{E'}(\eta_1 x_1', \eta_2 x_2') \tag{4.46}$$

and a similar Galois descent argument works for $Gal(\overline{\mathbb{Q}}/K(\zeta_N))$ because $Y(N)$ is defined over $\mathbb{Q}(\zeta_N)$, which implies that $Z_E(\eta_1 x_1, \eta_2 x_2)$ is an element of $K_3(K(\zeta_N))$. Since K is also cyclotomic, $K(\zeta_N)$ is also cyclotomic. By the discussion preceding Section 1.3 of [27], $K_3(K(\zeta_M))$ is generated by the elements $Li_2^{mot}(\zeta_M^a)$ for $a \in \mathbb{Z}$. The theorem follows. □

Corollary 4.9.2. *Let $\mathcal{P}$ be a split or ramified prime ideal of O_K and $z_1, z_2 \in E[\mathcal{P}]$, then $\theta_E^{(1)} = \theta_{E,z}^2$*

Proof. For such prime ideals $\mathcal{P}$, $E[\mathcal{P}]$ is cyclic because it is a proper, non-trivial subgroup of $E[p] \cong (\mathbb{Z}/p)^2$. So case 1 in the previous lemma implies the corollary. □

Remark 4.9.3. *The commutativity of the left square in Theorem 4.8.10 is equivalent to conjecture* (5.13) *in ([27]) if $\theta_{E,z}^2$ coincides with $\theta_E^{(1)}$ in loc. cit.*

Remark 4.9.4. *Theorem 4.9.1 is as good as it gets. In [24] and [46], the map $\theta_{E,z}$ is considered modulo terms in depth* 1 *and the terms $Li_2^{mot}(\zeta_M^a)$ are in depth* 1*. For more details about depth, see loc. cit.*

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