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Towards the inductive Galois–McKay Condition for groups of type A

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Introduction

This dissertation investigates the inductive Galois–McKay conditions for finite groups of Lie type A. It gives the verification of the first inductive condition and explores techniques that may be useful in order to verify similar results for other types of groups of Lie type, hence contributing to the verification of the conjecture for all finite groups.

The Galois–McKay conjecture

The theory of the local-global conjectures is a specific field in the character theory of finite groups that wishes to study the connection between the character theory of the global group G and the character theory of its local subgroups, that is normalizers of ℓ -subgroups. This fundamental connection was first proposed by McKay in 1972 and states that given a prime ℓ and a Sylow ℓ -subgroup P of G , there exists a bijection

$$\Omega : \{\chi \in \text{Irr}(G) \mid \ell \nmid \chi(1)\} \rightarrow \{\varphi \in \text{Irr}(N_G(P)) \mid \ell \nmid \varphi(1)\}.$$

We denote the sets above as $\text{Irr}_{\ell'}(G)$ and $\text{Irr}_{\ell'}(N_G(P))$ respectively. Indeed, this statement seems simple, as it is a pure counting requirement, but was only proven to be true in 2025 by Cabanes–Späth [CS25]. Building from this conjecture many other strengthenings of the McKay conjecture, like the Alperin–McKay conjecture, have been proposed. In particular, these are all counting conjectures, giving information mostly about the number of characters of certain degree. The version of the McKay conjecture proposed by Gabriel Navarro in 2004, however, considers a much larger set of information, as it proposes that the bijection of the McKay conjecture can be chosen to be equivariant with regard to certain Galois automorphisms. In particular, this means that much more information about the characters needs to be known and studied. The statement of the conjecture is the following

Conjecture ((Navarro’s) Galois–McKay, 2004). Let G be a finite group, let ℓ be a prime and let $P \in \text{Syl}_{\ell}(G)$. Let $\mathcal{H}_{\ell} \leq \text{Gal}(\mathbb{Q}^{ab} : \mathbb{Q})$ be a certain Galois subgroup, see Definition 1.25. Then there exists an \mathcal{H}_{ℓ} -equivariant bijection

$$\Omega : \text{Irr}_{\ell'}(G) \rightarrow \text{Irr}_{\ell'}(N_G(P)).$$

The inductive conditions

As the conjecture of McKay was first proposed, direct methods to prove the statement were searched for, but there seemed to be no proof that uses purely group and character theory methods. However the Classification of Finite Simple Groups (CFSG) opened up a new possibility and in 2007 Isaacs–Malle–Navarro [IMN07]

reduced the verification of the McKay conjecture to the verification of the so-called inductive McKay conditions, which in turn should hold for all quasi-simple groups. Unfortunately the inductive conditions are stronger than the conjecture itself and information on the action of certain automorphisms on G need to be understood. However, about 20 years after the inductive conditions were proposed the McKay conjecture was proven by Cabanes and Späth in [CS25], thereby finalizing the proof.

In view of this, the reduction of the Galois–McKay conjecture by Navarro, Späth and Vallejo to the inductive Galois–McKay conditions seems to be promising.

Theorem. [NSV20] Let ℓ be a prime and assume that the equivariance condition $(\text{Eq})_{\mathcal{H}_\ell}$ and extension condition $(\text{Ext})_{\mathcal{H}_\ell}$, see Definition 1.42, hold for all finite simple non-abelian groups. Then the Galois–McKay conjecture is true for ℓ and any finite group.

In this thesis we give the verification of condition $(\text{Eq})_{\mathcal{H}_\ell}$ for finite groups of Lie type **A**.

Outline of the Thesis

Similar to the process for the verification of the inductive McKay conditions, the inductive Galois–McKay conditions were verified for groups of Lie type in defining characteristic [Ruh21], [Joh22] and for the prime $\ell = 2$, [RSF25]. The first chapter of this thesis will introduce the representation and character theoretic background in order to state the McKay conjecture, inductive conditions and its Galois refinements.

Following this, we develop the theory of extension maps and maximal extendibility, which have been useful in order to systematically construct irreducible characters using Clifford theory. In order to adapt the theory, we need to take special care of these constructions with respect to the action of Galois automorphisms.

The finite groups of Lie type are the largest family in the classification of finite simple groups and therefore represent a central case in any verification of local–global conjectures. In the third chapter we recall basic notions concerning algebraic groups, Steinberg endomorphisms, maximal tori, and focus on the structure of Sylow d -tori and their centralizers and normalizers, specifically for groups of type **A**, as these will be of importance in the verification of the local statement.

Chapter 4 establishes the framework for verifying the inductive Galois–McKay conditions by first introducing the required number-theoretic results and investigating the action of \mathcal{H}_ℓ on specific character fields. Afterwards we analyze the outer automorphism group of $\text{SL}_n(\epsilon q)$ and introduce a certain type of automorphism denoted by type \mathfrak{t}_ℓ , see Definition 4.13. We adapt Späth’s criterion from [Spä12] in order to construct an \mathcal{H}_ℓ -equivariant bijection. While this result is closely related to Theorem 3.4 of [RSST25], we have a distinct construction and a more thorough proof than is currently available in the literature. Importantly, we obtain some freedom by choosing an element t in the universal cover, which "stabilizes" the action of \mathcal{H} in the global and local parametrization of characters. These results are then refined into Galois-compatible conditions $A(\infty)_{\mathcal{H}_t}$, $A(d)_{\mathcal{H}_t}$ and $B(d)_{\mathcal{H}_\ell}$ for finite groups of Lie type in non-defining characteristic ℓ . In the case of type **A**, the integer d is defined as the order of $\epsilon q \pmod{\ell}$.

Theorem A (Theorem 4.18). Let G be a finite group, let ℓ be a prime dividing $|G|$ and let $P \in \text{Syl}_\ell(G)$. Let $M < G$ be a proper subgroup of G and let $\mathbb{G} \subset \text{Irr}(G)$ and $\mathbb{M} \subset \text{Irr}(M)$. Assume that M , \mathbb{G} and \mathbb{M} have sufficiently nice extension properties, see Theorems 4.18 and 4.17, and the characters are well behaved under the action of \mathcal{H}_ℓ . Then the inductive condition $(\text{Eq})_{\mathcal{H}_\ell}$ holds.

The remaining chapters of the thesis concentrate on groups of type A, i.e., we assume that $G \in \{\text{SL}_n(q), \text{SU}_n(q)\}$. The result from Chapter 5 can also be found in [RSST25]. While some of the considerations there also apply to symplectic groups we here concentrate on groups of type A. Another difference is that we here explicitly construct the transversal using unipotent support, while in [RSST25], the properties of certain stabilizers lead them just to the existence of a transversal as required. Through this, we also obtain some information about the choice for the element t .

Theorem B (Theorem 5.13). There exists an element t inducing an outer diagonal automorphism of G of type \mathfrak{t}_ℓ such that condition $\text{A}(\infty)_{\mathcal{H}_t}$ -holds for groups of type A and any prime ℓ .

Focusing on the doubly regular case where the centralizer of the Sylow d -torus is abelian, see Definition 6.1, Chapter 6 presents new results concerning the local conditions $\text{A}(d)_{\mathcal{H}_t}$ and $\text{B}(d)_{\mathcal{H}_\ell}$. Similar to the McKay case in [CS17], we construct an extension map. We then construct a specific element t to ensure that the extension map is \mathcal{H}_t -equivariant, a property previously unverified in this context. Furthermore, we check that the remaining construction of a certain transversal is compatible with \mathcal{H}_ℓ and \mathcal{H}_t actions.

Theorem C (Theorems 6.2 and 6.24). Let ℓ be an odd prime and assume that the integer d is doubly regular for (\mathbf{G}, F) of type \mathbf{A}_{n-1} . Then there exists an element t inducing an outer diagonal automorphism of type \mathfrak{t}_ℓ on \mathbf{G}^F such that conditions $\text{A}(d)_{\mathcal{H}_t}$ and $\text{B}(d)_{\mathcal{H}_\ell}$ hold.

The final chapter combines the global and doubly regular results to provide the verification of the local conditions $\text{A}(d)_{\mathcal{H}_t}$ and $\text{B}(d)_{\mathcal{H}_\ell}$ in full generality for type A. By analyzing the block-diagonal structure of the normalizer of a Sylow d -torus, the automorphisms t_1 and t_2 introduced in the previous chapters are fused in order to define a final \mathcal{H}_t -stable transversal.

Theorem D (Corollaries 7.6 and 7.8). Let ℓ be an odd prime and assume (\mathbf{G}, F) is of type \mathbf{A}_{n-1} . Then there exists an element t inducing an outer diagonal automorphism of type \mathfrak{t}_ℓ on \mathbf{G}^F such that conditions $\text{A}(d)_{\mathcal{H}_t}$ and $\text{B}(d)_{\mathcal{H}_\ell}$ hold.

This completes the verification of the Equivariance statement of the inductive conditions, marking a significant step towards the proof of the Galois–McKay conjecture for these groups. In [RSST25, Corollary C], this result is key to verify the following statement.

Theorem E ([RSST25] and Theorem 7.9). The Equivariance condition $(\text{Eq})_{\mathcal{H}_\ell}$ of the inductive Galois–McKay conditions holds for groups of type A and all primes ℓ .

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1. Character theory and the Galois–McKay conjecture

The concept of character theory dates back to the 1870’s when Georg Frobenius studied the representation theory of finite groups entirely through their characters. Indeed, given a rather complicated representation matrix, its character usually is much easier to understand and study. The usefulness of character theory has been clear from the start, as many group theoretic statements, which are hard to prove using only group theoretic methods, can be shown using character theory. These include Burnside’s $p^a q^b$ theorem and the Feit–Thompson theorem, see for example [Asc00, p. 260]

Although these functions may seem easy to understand, they have been studied since their invention and there are still many open questions to solve. Just recently, over 50 years after the conjecture was originally stated by John McKay in 1972, the McKay conjecture has been proven by Marc Cabanes and Britta Späth in [CS25]. In fact, this proposes to be just the beginning of verifications of these, so called, local-global conjectures. This term stems from the potential to describe character properties of the global group G by the properties of characters of a local subgroup $N_G(P)$, the normalizer of a Sylow p -subgroup. In this thesis we will study the strengthening of the McKay conjecture proposed by Gabriel Navarro in 2004 [Nav04] called the Galois–McKay conjecture or McKay–Navarro conjecture.

In this first chapter we will give an overview of the basic definitions, structures and the Galois–McKay conjecture itself. Although we assume that the reader is familiar with these concepts, we state them nonetheless for completeness sake. We start by introducing the basics of character theory and continue to state the McKay conjecture as well as the Galois refinement stated by Navarro in 2004. Then we continue to describe the inductive statements, which imply the conjecture and were introduced by Navarro, Späth and Vallejo in 2020 [NSV20]. Finally we explore some constructions to extend characters on various group structures, as we will need these extendibility properties throughout the thesis.

1.A Character theory

We will provide a brief overview of the basics of character theory. The definitions and theorems mentioned here can be found in standard texts on the subject, such as [Isa76] or [Nav18].

1.A.1 Complex representations and characters

Throughout this thesis, G will stand for a finite group.

Definition 1.1. For a finite group G and a complex vector space V , a *complex representation of G on V* is a group homomorphism $\mathcal{X} : G \rightarrow \mathrm{GL}(V)$. The *degree* of a representation is $n = \dim(V)$. We say that two finite-dimensional complex representations \mathcal{X} and \mathcal{Y} are *similar*, if there exists a matrix $M \in \mathrm{GL}_n(\mathbb{C})$ such that $\mathcal{Y}(g) = M^{-1}\mathcal{X}(g)M$ for all $g \in G$. Indeed, two similar representations must be of the same degree.

Definition 1.2. Let \mathcal{X} be a complex representation of a finite group G . We define the *complex character* of the representation as the trace of the representation

$$\chi(g) := \mathrm{trace}(\mathcal{X}(g)).$$

Two representations are similar if and only if they afford the same character. We write $\mathrm{Char}(G)$ for the set of all characters of a group. If the representation has no proper subrepresentations, then it and its afforded character are called *irreducible*. Subsequently we write $\mathrm{Irr}(G)$ for the set of all irreducible characters of G . Let $\mathrm{cf}(G)$ be the space of all class functions of G , that is complex functions $\rho : G \rightarrow \mathbb{C}$, such that $\rho(xgx^{-1}) = \rho(g)$ for all $g, x \in G$, that is ρ is constant on conjugacy classes of G . Then each character is a class function of G . The complex space of class functions $\mathrm{cf}(G)$ becomes a hermitian space with the hermitian inner product given by

$$\langle \sigma, \rho \rangle := \frac{1}{|G|} \sum_{x \in G} \sigma(x) \overline{\rho(x)} \quad \text{for } \sigma, \rho \in \mathrm{cf}(G).$$

The beautiful structure and relevance of irreducible characters is now seen by the following theorem.

Theorem 1.3. [Nav18, Theorem 1.1] *Let G be a finite group. Then the set of irreducible characters of G forms an orthonormal basis of $\mathrm{cf}(G)$ with respect to the hermitian inner product above.*

Indeed each element $\rho \in \mathrm{Char}(G)$ can be written in the following way

$$\rho = \sum_{\chi \in \mathrm{Irr}(G)} \langle \rho, \chi \rangle \chi.$$

Furthermore, in the case that $\langle \rho, \chi \rangle \neq 0$ we say that χ is an *irreducible constituent* of ρ . Another elementary but very important theorem is the following

Theorem 1.4 (Schur’s Lemma). [Nav18, Theorem 1.7] *Let G be a finite group with an irreducible complex representation $\mathcal{X} : G \rightarrow \mathrm{GL}(V)$. Let $f : V \rightarrow V$ be a \mathbb{C} -linear endomorphism such that $f \circ \mathcal{X}(g) = \mathcal{X}(g) \circ f$ for all $g \in G$. Then f is a scalar map.*

We can also define characters on arbitrary fields \mathbf{k} . In the case where $\text{char}(\mathbf{k}) \nmid |G|$ the \mathbf{k} -representation theory will not differ much from \mathbb{C} -representation theory. This is a consequence of Maschke's theorem

Theorem 1.5 (Maschke's Theorem). *[Isa76, Theorem 1.9] Let G be a finite group and \mathbf{k} a field such that the characteristic of \mathbf{k} does not divide the order of G . Then all representations of G over \mathbf{k} are a direct sum of irreducible representations.*

This furthermore implies the theorem due to Wedderburn.

Theorem 1.6. *[Nav18, Theorem 1.3] Let G be a finite group, let \mathbf{k} be an algebraically closed field of characteristic zero. Let m be the number of conjugacy classes of G .*

- (a) *There are exactly m irreducible non-similar \mathbf{k} -representations $\{\mathcal{X}_1, \dots, \mathcal{X}_m\}$ of G .*
- (b) *If χ_i is the \mathbf{k} -character afforded by \mathcal{X}_i , then $\{\chi_1, \dots, \chi_m\}$ are \mathbf{k} -linearly independent.*
- (c) *We have $|G| = \sum_{i=1}^m \chi_i(1)^2$.*

1.A.2 Construction of new characters

There are several ways to construct new characters from characters of subgroups and vice versa.

Definition 1.7. Let $H \leq G$ and $\chi \in \text{Char}(G)$. Then the *restricted character* defined by

$$\text{Res}_H^G(\chi)(h) = \chi(h) \quad \text{for all } h \in H,$$

is a character of H . Given an irreducible constituent $\theta \in \text{Irr}(H)$ of $\text{Res}_H^G(\chi)$, we say that χ *lies over* θ . Given any character $\chi \in \text{Irr}(H)$, we define the *induced character* as

$$\text{Ind}_H^G(\chi)(g) = \frac{1}{|H|} \sum_{x \in G, xgx^{-1} \in H} \chi(xgx^{-1}) \quad \text{for all } g \in G.$$

Often when it is clear on which groups our characters are defined we will also write

$$\chi|_H = \text{Res}_H^G(\chi)$$

for the restricted character and

$$\chi^G = \text{Ind}_H^G(\chi)$$

for the induced character. The adjunction between the induction and restriction of characters is exposed by the so called Frobenius Reciprocity.

Theorem 1.8 (Frobenius Reciprocity). *Let H be a subgroup of G and let χ be a character of G , θ a character of H . Then we have*

$$\langle \chi, \text{Ind}_H^G(\theta) \rangle = \langle \text{Res}_H^G(\chi), \theta \rangle.$$

Theorem 1.9. *Let G and H be finite groups. For each $\chi \in \text{Char}(G)$ and $\theta \in \text{Char}(H)$, we get a character of $G \times H$ via*

$$(\chi \times \theta)(g, h) = \chi(g)\theta(h).$$

In particular, the map

$$\Omega : \text{Irr}(G) \times \text{Irr}(H) \rightarrow \text{Irr}(G \times H)$$

$$(\chi, \theta) \mapsto \chi \times \theta,$$

is a bijection.

A fundamental problem in character theory is constructing all irreducible characters of a group G using the characters of its subgroups. While it is possible to induce a character from any subgroup to G , the resulting character is typically challenging to express explicitly as a sum of irreducible characters of G . However, when N is a normal subgroup of G , Clifford theory provides deeper insights and a systematic approach to constructing irreducible characters. In fact, the most comprehensive understanding of induced and restricted characters is obtained when the subgroup is normal in G .

Definition 1.10. Suppose $N \triangleleft G$, $\chi \in \text{Irr}(G)$, and $\theta \in \text{Irr}(N)$. We denote the set of irreducible constituents of the restricted character by

$$\text{Irr}(N \mid \chi) = \text{Irr}(\chi|_N) := \{\theta \in \text{Irr}(N) \mid \langle \theta, \chi|_N \rangle \neq 0\}$$

and the set of irreducible constituents of the induced character by

$$\text{Irr}(G \mid \theta) = \text{Irr}(\theta^G) := \{\chi \in \text{Irr}(G) \mid \langle \chi, \theta^G \rangle \neq 0\}.$$

Definition 1.11. For a finite group G and $N \triangleleft G$, G acts on the characters of N by conjugation. Let $\chi \in \text{Char}(N)$ and $g \in G$. Then

$$\chi^g(n) := \chi(n^g).$$

We write $G_\chi := \{g \in G \mid \chi^g = \chi\}$ for the stabilizer, also called the *inertia subgroup* of χ in G .

Theorem 1.12 (Clifford). *Let G be a finite group with normal subgroup N . Let χ be an irreducible character of G and $\theta \in \text{Irr}(N \mid \chi)$. Let $\{\theta_1, \dots, \theta_k\}$ be the set of distinct G -conjugates of θ and $e = [\chi_N, \theta]$. Then we can write*

$$\chi|_N = \sum_{i=1}^k e \theta_i.$$

Proof. See [Nav18, Theorem 1.19]. □

Hence, the restriction of an irreducible character $\chi \in \text{Irr}(G)$ can be described by only one irreducible character $\theta \in \text{Irr}(N \mid \chi)$. On the other hand, given an irreducible character of N , we can describe the characters lying above it by Clifford correspondence.

Theorem 1.13 (Clifford correspondence). *Let G be a finite group with normal subgroup N and $\theta \in \text{Irr}(N)$. Let $T = G_\theta$ be the inertia subgroup of θ . Then the following hold:*

- (a) *The induction map Ind_T^G defines a bijection between the sets $\text{Irr}(T \mid \theta)$ and $\text{Irr}(G \mid \theta)$.*
- (b) *If $\psi \in \text{Irr}(T \mid \theta)$, then we have $\langle (\psi^G)|_N, \theta \rangle = \langle \psi|_N, \theta \rangle$.*
- (c) *If $\psi \in \text{Irr}(T \mid \theta)$ and $\chi = \psi^G$, then $\chi|_T = \psi + \Delta$, where Δ is either zero or a character of T such that none of its irreducible components lie over θ .*

Proof. See [Nav18, Theorem 1.20]. □

Another important theorem is the so called *Gallagher correspondence*, which gives us a parametrization of the set $\text{Irr}(G \mid \theta)$ in case that $\text{Res}_N^G(\theta)$ is an irreducible character of N . We say that a character $\chi \in \text{Irr}(N)$ *extends* to G in case there exists some $\theta \in \text{Irr}(G)$ with $\text{Res}_N^G(\theta) = \chi$.

Theorem 1.14 (Gallagher correspondence). *Let $N \triangleleft G$ and $\theta \in \text{Irr}(N)$ an irreducible character that extends to some $\chi \in \text{Irr}(G_\theta)$. Then the map $\text{Irr}(G_\theta/N) \rightarrow \text{Irr}(G_\theta \mid \theta)$ given by $\eta \rightarrow \tilde{\eta}\chi$ is a bijection, where $\tilde{\eta} \in \text{Irr}(G_\theta)$ is defined by $\tilde{\eta}(g) = \eta(gN)$.*

Proof. See [Nav18, Corollary 1.23]. □

We see that given $N \triangleleft G$, we are interested in *G -invariant characters*, that is characters $\theta \in \text{Irr}(N)$ such that $\theta^g = \theta$ for all $g \in G$. In the best cases we can extend a given G -invariant character to an irreducible character in G as we will see later on. In many other cases we do not have such well behaving extensions. But in any case we can define a projective representation of G that restricts to the character θ .

Definition 1.15. Let G be a finite group. A finite dimensional *complex projective representation* of G is a map

$$\mathcal{P} : G \rightarrow \text{GL}_n(\mathbb{C})$$

such that for each $x, y \in G$ the matrices $\mathcal{P}(x)$ and $\mathcal{P}(y)$ satisfy

$$\mathcal{P}(x)\mathcal{P}(y) = \alpha(x, y)\mathcal{P}(xy),$$

where α is the *factor set* of \mathcal{P} , that is a function $\alpha : G \times G \rightarrow \mathbb{C}^\times$ with

$$\alpha(xy, z)\alpha(x, y) = \alpha(x, yz)\alpha(y, z)$$

for all $x, y, z \in G$. The factor set is uniquely determined by the projective representation.

Definition 1.16. Let G be a group and A an abelian group. An A -cocycle of G , also sometimes known as an A -factor set, is a function $\alpha : G \times G \rightarrow A$ such that $\alpha(xy, z)\alpha(x, y) = \alpha(x, yz)\alpha(y, z)$ for all $x, y, z \in G$. We define $Z^2(G, A)$ as the set of all A -factor sets of G . This forms a group under pointwise multiplication. Given an arbitrary function $\mu : G \rightarrow A$, we can define $\delta(\mu) : G \times G \rightarrow A$ given by

$$\delta(\mu)(g, h) = \mu(g)\mu(h)\mu(gh)^{-1}.$$

Since A is abelian

$$\begin{aligned} \delta(\mu)(xy, z)\delta(\mu)(x, y) &= \mu(xy)\mu(z)\mu(xyz)^{-1}\mu(x)\mu(y)\mu(xy)^{-1} \\ &= \mu(x)\mu(yz)\mu(xyz)^{-1}\mu(y)\mu(z)\mu(yz)^{-1} \\ &= \delta(\mu)(x, yz)\delta(\mu)(y, z) \end{aligned}$$

and $\delta(\mu)$ is a cocycle, hence δ is a homomorphism of the group of A -valued functions to the group of A -cocycles. Let $B^2(G, A)$ be the image of δ and $H^2(G, A) := Z^2(G, A)/B^2(G, A)$ the second cohomology group under the trivial action of G on A . Two cocycles α, β are *equivalent* if

$$\alpha B^2(G, A) = \beta B^2(G, A)$$

and $H^2(G, A)$ is the set of equivalence classes of cocycles.

Definition 1.17. For a positive integer m , we denote by \mathbb{Q}_m the m -th cyclotomic field, that is, the field obtained by adjoining all m -th roots of unity to \mathbb{Q} . We write \mathbb{Q}^{ab} for the smallest field extension of \mathbb{Q} containing all roots of unity.

Lemma 1.18. [Isa76, Theorem 11.2]. Let $N \triangleleft G$ and $\theta \in \text{Irr}(N)$ a G -invariant character corresponding to a representation \mathcal{X} . Then there exists a projective representation \mathcal{P} of G such that

- $\mathcal{P}|_N = \mathcal{X}$.
- $\mathcal{P}(ng) = \mathcal{P}(n)\mathcal{P}(g)$,
- $\mathcal{P}(gn) = \mathcal{P}(g)\mathcal{P}(n)$,

for all $g \in G$ and $n \in N$. Furthermore, by [NSV20, Corollary 1.2] the projective representation may be chosen such that its entries are in \mathbb{Q}^{ab} and its corresponding factor set only takes root of unity values.

Theorem 1.19. Let $N \triangleleft G$ and $\theta \in \text{Irr}(N)$ a G -invariant character corresponding to a representation \mathcal{X} . Let \mathcal{P} be a projective representation of G that satisfies the conditions from Lemma 1.18. Let α be the factor set of \mathcal{P} . Define the cocycle β on G/N by setting

$$\beta(gN, hN) = \alpha(g, h).$$

Then β is well-defined and its equivalence class $[\beta]$ in $H^2(G/N, \mathbb{C}^\times)$ only depends on the choice of θ . Furthermore, $[\beta] = [1]$ if and only if θ extends to G .

Proof. See [Isa76, Theorem 11.7]. □

1.A.3 Character values

Our focus will be on the action of Galois automorphisms on characters, hence the potential values that a character χ associated with a complex representation \mathcal{X} of G might afford are of particular interest.

For an element $g \in G$, the matrix $\mathcal{X}(g)$ is similar to a diagonal matrix $\text{diag}(\zeta_1, \dots, \zeta_n)$, with $\zeta_i \in \mathbb{C}$ for all i . Since the order of $\mathcal{X}(g)$ must divide the order of g , denoted by $\text{ord}(g)$, each ζ_i is a $\text{ord}(g)$ -th root of unity. Consequently, the character $\chi(g)$ afforded by \mathcal{X} is given by

$$\chi(g) = \zeta_1 + \dots + \zeta_n,$$

which is an algebraic integer in the cyclotomic field $\mathbb{Q}_{\text{ord}(g)}$. Therefore, for any character χ of G , the values of χ lie within the cyclotomic field $\mathbb{Q}_{|G|}$.

Given that $|\zeta_i| = 1$ for all i , and if $\overline{\zeta_i}$ represents the complex conjugate of ζ_i , the matrix $\mathcal{X}(g^{-1}) = \mathcal{X}(g)^{-1}$ is similar to the diagonal matrix $\text{diag}(\overline{\zeta_1}, \dots, \overline{\zeta_n})$. Consequently, we have

$$\chi(g^{-1}) = \overline{\chi(g)}.$$

Definition 1.20. Let G be a finite group and χ a character of G , then we define

$$\mathbb{Q}(\chi) = \mathbb{Q}(\chi(g) \mid g \in G)$$

as the *field of values* or *character field* of χ . It is the smallest subfield of \mathbb{C} containing all character values. By the above observations we have $\mathbb{Q}(\chi) \subset \mathbb{Q}_{\text{exp}(G)}$, where $\text{exp}(G)$ is the exponent of G .

Example 1.21. (1) Let $G = C_n$ be the cyclic group of order n . As G is abelian, all characters are of degree 1 and form an abelian group of order n . Indeed, the set of irreducible characters $\text{Irr}(G)$ is itself a cyclic group of order n , generated by a character χ which has n -th roots of unity as values. Thus, for each $\chi \in \text{Irr}(G)$ we have

$$\mathbb{Q}(\chi) \subset \mathbb{Q}_n.$$

(2) Let $G = \mathfrak{S}_n$ be the symmetric group of n elements. Then each irreducible character of G only has values in \mathbb{Z} . See [Isa76, Corollary 3.6]. So in this case for any $\chi \in \text{Irr}(G)$ we have

$$\mathbb{Q}(\chi) = \mathbb{Q}.$$

Definition 1.22. Let χ be a character of a finite group G and let $\sigma \in \text{Gal}(\mathbb{Q}_{\text{exp}(G)}/\mathbb{Q})$. Then the σ -conjugate of χ is the function $\chi^\sigma : G \rightarrow \mathbb{C}$ given by

$$\chi^\sigma(g) := \sigma(\chi(g)) \text{ for } g \in G.$$

In fact χ^σ is a character of G , see [Nav18, Theorem 3.1].

Proposition 1.23. *Induction and restriction of characters is equivariant with respect to Galois automorphisms. In particular, for $H \leq G$, $\theta \in \text{Irr}(H)$, $\chi \in \text{Irr}(G)$ and $\sigma \in \text{Gal}(\mathbb{Q}^{ab}/\mathbb{Q})$ we have*

$$\text{Ind}_H^G(\theta)^\sigma = \text{Ind}_H^G(\theta^\sigma) \quad \text{and} \quad \text{Res}_H^G(\theta)^\sigma = \text{Res}_H^G(\theta^\sigma)$$

and therefore $\mathbb{Q}(\theta^G) \subseteq \mathbb{Q}(\theta)$ and $\mathbb{Q}(\chi|_H) \subseteq \mathbb{Q}(\chi)$.

Proof. The definition implies that for any $\theta \in \text{Irr}(H)$ and $\sigma \in \text{Gal}(\mathbb{Q}^{ab}/\mathbb{Q})$ a Galois automorphism, we have

$$\begin{aligned} \text{Ind}_H^G(\theta)^\sigma(g) &= \sigma \left(\frac{1}{|H|} \sum_{x \in G, xgx^{-1} \in H} \theta(xgx^{-1}) \right) \\ &= \frac{1}{|H|} \sum_{x \in G, xgx^{-1} \in H} \sigma(\theta(xgx^{-1})) \\ &= \text{Ind}_H^G(\theta^\sigma)(g). \end{aligned}$$

For all $g \in G$ and for any $\chi \in \text{Irr}(G)$ and $h \in H$ we have

$$\text{Res}_H^G(\chi)^\sigma(h) = \sigma(\chi(h)) = \text{Res}_H^G(\chi^\sigma)(h). \quad \square$$

We are now able to state the conjecture that is the focus of this thesis.

1.B The McKay and Galois–McKay conjectures and inductive conditions

Questions about the number of irreducible characters in a group have been investigated since the introduction of the field. Nonetheless there are still many open questions. In 1971 John McKay published a short paper [McK71] about the connection of the number of characters of degree non-divisible by 2 for any finite simple group G and the same number in the normalizer of a Sylow 2-subgroup. This is today known as a special case of the McKay conjecture.

Given a finite group G and a prime ℓ dividing the order of G we write

$$\text{Irr}_{\ell'}(G) := \{\chi \in \text{Irr}(G) \mid \ell \nmid \chi(1)\}$$

for the so called *irreducible ℓ' -characters of G* . Then the McKay conjecture, now a theorem by [CS25], can be stated as follows.

Theorem 1.24 (The McKay conjecture). *Let G be a finite group, ℓ a prime dividing the order of G and P a Sylow ℓ -subgroup of G . Then*

$$|\text{Irr}_{\ell'}(G)| = |\text{Irr}_{\ell'}(N_G(P))|.$$

The McKay conjecture is also known as a *local-global conjecture*, as it relates the character theory of the local subgroup $N_G(P)$ with that of the entire group G . Over time, many refinements of the conjecture have been proposed, most notably the *Alperin–McKay conjecture*. In 2004 [Nav04], Gabriel Navarro introduced a further refinement, extending the conjecture to relate not only the character degrees but also the character fields of the local and global groups. In this context, the action of a certain Galois subgroup, denoted as $\mathcal{H}_\ell \leq \mathcal{G} = \text{Gal}(\mathbb{Q}^{ab}/\mathbb{Q})$, is expected to be particularly well-behaved.

Definition 1.25. Let $\mathcal{H}_\ell \leq \text{Gal}(\mathbb{Q}^{ab}/\mathbb{Q})$ be the subgroup of automorphisms such that for each $\sigma \in \mathcal{H}_\ell$ there exists some $f \in \mathbb{Z}_{\geq 0}$ with $\sigma(\zeta) = \zeta^{\ell^f}$ for all roots of unity ζ of ℓ' order.

Given some $\sigma \in \text{Gal}(\mathbb{Q}^{ab}/\mathbb{Q})$ some finite group G and $K \subset \text{Irr}(G)$, we write $K^\sigma = \{\chi \in K \mid \chi^\sigma = \chi\}$ for the set of σ -fixed characters.

Conjecture 1.26 (The Galois–McKay conjecture). [Nav04, Conjecture A] Let G be a finite group of order n and ℓ a prime. Let P be a Sylow ℓ -subgroup of G . Let $\sigma \in \mathcal{H}_\ell$. Then

$$|\operatorname{Irr}_{\ell'}(G)^\sigma| = |\operatorname{Irr}_{\ell'}(N_G(P))^\sigma|,$$

i.e., σ fixes the same number of characters in $\operatorname{Irr}_{\ell'}(G)$ and $\operatorname{Irr}_{\ell'}(N_G(P))$.

A slightly stronger version of this conjecture is proposed the following way.

Conjecture 1.27. Let G , ℓ , P and \mathcal{H}_ℓ be as above. Then there exists an \mathcal{H}_ℓ -equivariant bijection

$$\Omega : \operatorname{Irr}_{\ell'}(G) \xrightarrow{\sim} \operatorname{Irr}_{\ell'}(N_G(P)).$$

Progress on the McKay conjecture was made in the years following its introduction, but then stagnated. Even though the conjecture is easy to state, a general proof seemed out of reach. After the classification of finite simple groups was verified, it proved to be helpful regarding many group and representation theoretic problems.

In 2007, I. Martin Isaacs, Gunter Malle, and Gabriel Navarro proposed an inductive approach that reduced the conjecture to the verification of certain stronger conditions for all finite simple groups. This strategy proved to be successful, as Marc Cabanes and Britta Späth completed the proof of the McKay conjecture in [CS25].

The required conditions are formulated for an extension of the group, known as the universal covering group.

Definition 1.28. Let G be a group and $\pi : H \rightarrow G$ a surjective group homomorphism such that $\ker(\pi) \subseteq Z(H)$. In that case the pair (H, π) is called a *central extension* of G . A morphism $f : (H', \pi') \rightarrow (H, \pi)$ of two central extensions of G is a map such that $f : H' \rightarrow H$ is a group homomorphism with $\pi \circ f = \pi'$.

A central extension (\hat{G}, π) of G is *universal* if for each central extension (H, δ) , there exists a unique morphism $f : (\hat{G}, \pi) \rightarrow (H, \delta)$. In this case, we say that \hat{G} is the *universal covering group* of G .

Indeed, the universal covering group \hat{G} is unique up to isomorphism [Asc00, 33.1] and G has a universal central extension if and only if it is a perfect group [Asc00, 33.4]. In this case \hat{G} itself is a perfect group.

Example 1.29. Let $G = \operatorname{PSL}_n(q)$ for some prime power q . It is simple with the exceptions for $\operatorname{PSL}_2(2)$ and $\operatorname{PSL}_2(3)$, which are isomorphic to \mathfrak{S}_3 and \mathcal{A}_4 , respectively. In all other cases the special linear group $\operatorname{SL}_n(q)$ is the universal covering group of G [Wil09, 3.3.8].

Remark 1.30. *The universal central extension of a finite simple group G is also a typical example of a quasi-simple group. Thus, $\operatorname{SL}_n(q)$ is quasi-simple for all pairs (n, q) apart from $(2, 2)$ and $(2, 3)$. Note however, that not every quasi-simple group is a universal central extension.*

Now we are able to state the inductive conditions.

Definition 1.31 (The inductive condition for X and ℓ). Let X be a non-abelian simple group and ℓ a prime dividing the order of X . Let G be the universal covering group of X . Let P be a Sylow ℓ -subgroup of G and $\Gamma := \text{Aut}(G)_P$. Then the *inductive McKay conditions* for X and the prime ℓ are the following

(Eq) There exist a Γ -stable subgroup $N_G(P) \leq N \not\leq G$ and a Γ -equivariant bijection

$$\Omega : \text{Irr}_{\ell'}(G) \xrightarrow{\cong} \text{Irr}_{\ell'}(N).$$

(Ext) For all $\chi \in \text{Irr}_{\ell'}(G)$ there exist projective representations \mathcal{P} and \mathcal{P}' of $G \rtimes \Gamma_{\chi}$ and $N \rtimes \Gamma_{\Omega(\chi)}$ associated to χ and $\Omega(\chi)$ respectively such that the factor sets α and α' coincide on $(N \rtimes \Gamma_{\chi}) \times (N \rtimes \Gamma_{\chi})$ and $\mathcal{P}(c)$, $\mathcal{P}'(c)$ are scalar matrices associated to the same scalar for all $c \in C_{G \rtimes \Gamma_{\chi}}(G)$.

Since the conditions pertain more to the quasi-simple group G rather than the group X , we often interchangeably refer to them as the inductive conditions for G .

Theorem 1.32. [IMN07, Theorem B] *Let ℓ be a prime. If the inductive McKay conditions are satisfied by all finite non-abelian simple groups G with $\ell \mid |G|$, then the McKay conjecture holds for all groups and the prime ℓ .*

We note that the inductive conditions, although equivalent, are stated slightly differently in the original paper of Isaacs, Malle, and Navarro in 2007. Here, we have used a different definition due to [Spä12, 2.3, 2.8].

Similarly to the McKay conjecture, Navarro’s Galois–McKay conjecture has also been reduced to inductive conditions on simple groups by Navarro, Späth and Vallejo in 2020 [NSV20]. Unsurprisingly the statements above have to comply with extra \mathcal{H}_{ℓ} -equivariance properties. Indeed, condition (Eq) is just extended by \mathcal{H}_{ℓ} -equivariance. The statement on the factor sets is more complicated and thus we introduce a slightly different language which includes the necessary information in the notion of so called character triples.

Definition 1.33. Let G be a finite group, N a normal subgroup of G and $\theta \in \text{Irr}(N)$ a G -invariant character. Then (G, N, θ) is a *character triple*.

Let $\mathcal{P} : G \rightarrow \text{GL}_{\theta(1)}(\mathbb{C})$ be a projective representation. We say \mathcal{P} is *associated* with θ if \mathcal{P} restricted to N is a representation of N affording θ , and for all $g \in G$ and $n \in N$ we have

$$\mathcal{P}(ng) = \mathcal{P}(n)\mathcal{P}(g) \text{ and } \mathcal{P}(gn) = \mathcal{P}(g)\mathcal{P}(n).$$

We see that by Lemma 1.18, there always exists a projective representation \mathcal{P} of G associated to θ with entries in \mathbb{Q}^{ab} and such that the factor set α only has roots of unity values. If \mathcal{P} is any such representation, then $\mathcal{P}(g)$ has finite order.

We want to rewrite condition (Ext) for character triples with an associated projective representation. For this we need an order relation between two character triples.

Definition 1.34. Let (G, N, θ) and (H, M, φ) be two character triples. We write

$$(G, N, \theta) \geq (H, M, \varphi)$$

if the following are satisfied:

- (a) $G = NH$.
- (b) $N \cap H = M$.
- (c) There exist projective representations \mathcal{P} of G and \mathcal{P}' of H associated to θ and φ respectively, with factor set α and α' such that

$$\alpha'(h_1, h_2) = \alpha(h_1, h_2) \text{ for all } h_1, h_2 \in H.$$

We say that $(\mathcal{P}, \mathcal{P}')$ is *associated* with $(G, N, \theta) \geq (H, M, \varphi)$.

Since the condition also involves the values on central elements $c \in C_G(N)$, we must include this information in the ordering of character triples. Specifically, for a character triple (G, N, θ) with the associated projective representation \mathcal{P} and for $n \in N$, $c \in C_G(N)$, we have

$$\mathcal{P}(n)\mathcal{P}(c) = \mathcal{P}(nc) = \mathcal{P}(cn) = \mathcal{P}(c)\mathcal{P}(n).$$

Thus, by Schur's lemma 1.4, $\mathcal{P}(c)$ must be a scalar matrix, which defines a function $\mu : C_G(N) \rightarrow \mathbb{C}^\times$.

Definition 1.35. For two character triples (G, N, θ) (H, M, φ) we write

$$(G, N, \theta) \geq_c (H, M, \varphi)$$

whenever the following are satisfied:

- (a) $(G, N, \theta) \geq (H, M, \varphi)$.
- (b) $C_G(N) \subset H$.
- (c) There exist $(\mathcal{P}, \mathcal{P}')$ associated to $(G, N, \theta) \geq (H, M, \varphi)$ and $\mu : C_G(N) \rightarrow \mathbb{C}^\times$ such that $\mathcal{P}(x) = \mu(x)\text{Id}_{\theta(1)}$ and $\mathcal{P}'(x) = \mu(x)\text{id}_{\varphi(1)}$ for all $x \in C_G(N)$.

Now we can restate the inductive conditions from Definition 1.31 in terms of character triples.

Definition 1.36. Let X be a non-abelian simple group. Let ℓ be a prime dividing the order of X and let G be a universal covering group of X . Let P be a Sylow ℓ -subgroup of G and $\Gamma := \text{Aut}(G)_P$. Then X satisfies the *inductive McKay conditions* for the prime ℓ if and only if the following hold:

(Eq) There exists a Γ -stable subgroup $N_G(P) \leq N \not\leq G$ and a Γ -equivariant bijection

$$\Omega : \text{Irr}_{\ell'}(G) \xrightarrow{\cong} \text{Irr}_{\ell'}(N).$$

(Ext) For each $\theta \in \text{Irr}_{\ell'}(G)$ we have

$$(G \rtimes \Gamma_\theta, G, \theta) \geq_c (N \rtimes \Gamma_\theta, N, \Omega(\theta)).$$

Definition 1.37. Let (G, N, θ) be a character triple, $\chi \in \text{Irr}(G \mid \theta)$, $\sigma \in (\mathcal{H}_\ell)_\theta$. Then we define the commutator $[\chi, \sigma] \in \text{Lin}(G)$ as the linear character such that $\chi^\sigma = [\chi, \sigma] \cdot \chi$, whose existence is ensured by Clifford's theorem, since both χ and $\chi^\sigma \in \text{Irr}(G \mid \theta)$.

The inductive conditions that imply the Galois–McKay conjecture require that they are compatible with Galois automorphisms. Without the use of character triples, however, the statement can become quite convoluted. Therefore, it is necessary to define the concept of an \mathcal{H} -triple and establish an \mathcal{H} -compatible partial order among \mathcal{H} -character triples.

Definition 1.38. Let (G, N, θ) be a character triple, $\mathcal{H} \leq \text{Gal}(\mathbb{Q}^{ab}/\mathbb{Q})$ a Galois subgroup. Then we denote $\theta^{\mathcal{H}}$ for the \mathcal{H} -orbit and write

$$\text{Irr}(G \mid \theta^{\mathcal{H}}) = \bigcup_{\sigma \in \mathcal{H}} \text{Irr}(G \mid \theta^{\sigma}).$$

Furthermore, let $G_{\theta^{\mathcal{H}}}$ be the stabilizer of the set $\theta^{\mathcal{H}}$ with respect to the action of G on $\text{Irr}(N)$ by conjugation. If $G = G_{\theta^{\mathcal{H}}}$, then we write $(G, N, \theta)_{\mathcal{H}}$ and call this an \mathcal{H} -triple.

Proposition 1.39. *Let (G, N, θ) be a character triple and \mathcal{P} and \mathcal{P}' be two projective representations of G associated to θ . Then there exists a matrix $M \in \text{GL}_{\theta(1)}(\mathbb{C})$ and a function $\mu : G \rightarrow \mathbb{C}$ with $\mu(1) = 1$ and constant on each left coset $gN \in G/N$, such that*

$$\mathcal{P}'(g) = \mu(g)M^{-1}\mathcal{P}(g)M, \quad \forall g \in G.$$

On the other hand if $\mathcal{P}' : G \rightarrow \text{GL}_{\theta(1)}(\mathbb{C})$ is any function with the above properties, then it is a projective representation of G associated to θ .

Proof. Given \mathcal{P}' with the above properties, it must be a projective representation of G . As by definition $\mathcal{P}'|_N$ and $\mathcal{P}|_N$ are representations of N affording the same character, there exists a matrix $M_1 \in \text{GL}_{\theta(1)}(\mathbb{C})$ such that

$$\mathcal{P}'|_N = M_1^{-1}\mathcal{P}|_N M_1.$$

In particular, we may assume $\mathcal{P}'|_N = \mathcal{P}|_N$. For $g \in G$ and $n \in N$, we then have

$$\mathcal{P}(g)\mathcal{P}(n)\mathcal{P}(g)^{-1} = \mathcal{P}'(g)\mathcal{P}(n)\mathcal{P}'(g)^{-1}$$

and thus

$$\mathcal{P}(n)\mathcal{P}(g)^{-1}\mathcal{P}'(g) = \mathcal{P}(g)^{-1}\mathcal{P}'(g)\mathcal{P}(n),$$

and $\mathcal{P}(g)^{-1}\mathcal{P}'(g)$ is a scalar matrix by Schur's Lemma 1.4. Hence, there exists a function $\mu : G \rightarrow \mathbb{C}^{\times}$, such that $\mathcal{P}'(g) = \mu(g)\mathcal{P}(g)$ and $\mu(N) = 1$. Furthermore, from the definition of \mathcal{P} on gn , we see that $\mu(gn) = \mu(g)$ and μ is a linear character of G , with N in its kernel. \square

Lemma 1.40. *[NSV20, Lemma 1.4] Suppose $N \triangleleft G$, $\theta \in \text{Irr}(N)$ and $\theta^{g\sigma} = \theta$ for some $g \in G$ and $\sigma \in \text{Gal}(\mathbb{Q}^{ab}/\mathbb{Q})$. Let \mathcal{P} be the projective representation of G_{θ} associated to θ with factor set α . Then*

$$\mathcal{P}^{g\sigma}(x) = \mathcal{P}(gxg^{-1})^{\sigma},$$

where σ acts entry-wise on the representation \mathcal{P} , once again defines a projective representation of G_{θ} associated to θ and has factor set $\alpha^{g\sigma}(x, y) = \alpha^g(x, y)^{\sigma}$. There also is a unique function

$$\mu_{\sigma g} : G \rightarrow \mathbb{C}^{\times}$$

such that $\mu_{\sigma g}(1) = 1$ and $\mathcal{P}^{g\sigma} \sim \mu_{\sigma g}\mathcal{P}$.

Proof. As \mathcal{P} and α have entries in \mathbb{Q}^{ab} and $G_\theta^g = G_\theta$, this is a well-defined projective representation. By Proposition 1.39, there exists $\mu_{\sigma g}$. \square

From now on we use the notation $[\mathcal{P}, \sigma g] = \mu_{\sigma g}$, with regard to the action of $\mathcal{H} \times G$ on the projective representation, where $\mu_{\sigma g}$ is the function from the Lemma above.

Definition 1.41. Let $(G, N, \theta)_\mathcal{H}$ and $(H, M, \varphi)_\mathcal{H}$ be two \mathcal{H} -triples. Then we write

$$(G, N, \theta)_\mathcal{H} \geq_c (H, M, \varphi)_\mathcal{H}$$

when the following are satisfied

- (a) $(G, N, \theta) \geq_c (H, M, \varphi)$ with associated projective representations $(\mathcal{P}, \mathcal{P}')$.
- (b) $(H \times \mathcal{H})_\theta = (H \times \mathcal{H})_\varphi$.
- (c) For each $\sigma g \in (H \times \mathcal{H}_\theta)$ we have

$$[\mathcal{P}, \sigma g] = [\mathcal{P}', \sigma g].$$

Definition 1.42. [The inductive Galois–McKay conditions] Let X be a non-abelian simple group with universal covering group G . Let ℓ be a prime dividing the order of X , P a Sylow ℓ -subgroup of G and $\Gamma := \text{Aut}(G)_P$. Then X satisfies the *inductive Galois–McKay conditions* for the prime ℓ if the following hold:

(Eq) $_{\mathcal{H}_\ell}$ There exists a Γ -stable subgroup $N_G(P) \leq N \not\cong G$ and a $\Gamma \times \mathcal{H}_\ell$ -equivariant bijection

$$\Omega : \text{Irr}_{\ell'}(G) \xrightarrow{\sim} \text{Irr}_{\ell'}(N).$$

(Ext) $_{\mathcal{H}_\ell}$ For each $\theta \in \text{Irr}_{\ell'}(G)$ we have

$$(G \rtimes \Gamma_{\theta^{\mathcal{H}_\ell}}, G, \theta)_{\mathcal{H}_\ell} \geq_c (N \rtimes \Gamma_{\theta^{\mathcal{H}_\ell}}, N, \Omega(\theta))_{\mathcal{H}_\ell}.$$

Theorem 1.43. [NSV20, Theorem A] *Let ℓ be a prime. If the inductive Galois–McKay conditions are satisfied by all finite non-abelian simple groups G with $\ell \mid |G|$, then the Galois–McKay conjecture holds for all groups and the prime ℓ .*

We will briefly summarize what is currently known regarding to the Galois–McKay conjecture and its inductive conditions.

The conjecture itself has been verified for several classes of groups:

- groups with cyclic Sylow ℓ -subgroups and the sporadic groups by Navarro, see [Nav04],
- solvable groups by Dade and symmetric groups by Fong, see [Nav04],
- alternating groups by Nath and Brunat–Nath, see [Nat09], [BN21].

Furthermore, the inductive conditions have been verified for:

- the sporadic and alternating groups for $\ell = 2$ by Vallejo, building on [Nat09],
- groups of Lie type in defining characteristic by Ruhstorfer and Johansson, see [Ruh21], [Joh22],
- the prime $\ell = 2$ by Ruhstorfer and Schaeffer Fry, see [RSF22], [RSF25].

2. Extension maps and Galois automorphisms

One of the central constructions used throughout this thesis is the so-called *extension map*. We begin by introducing the notion of extension map and maximal extendibility. Alongside we show examples for which these properties hold. Moreover, by applying Clifford theory, we can obtain a parametrization of irreducible characters in certain cases which will be of high importance later on.

On top of that, we will examine the action of (Galois)-automorphisms on the characters constructed this way and establish some useful although technical lemmas. These results can be viewed as Galois analogues of standard constructions appearing in the proof of the general McKay conjecture.

2.A Maximal Extendibility

Since characters induced from normal subgroups are not necessarily irreducible, maximal extendibility offers a useful way to control their behaviour.

Definition 2.1. Let $X \triangleleft Y$ be finite groups and let $\mathcal{X} \subset \text{Irr}(X)$. We say that *maximal extendibility holds with respect to $X \triangleleft Y$* for \mathcal{X} , if \mathcal{X} is Y -stable and every $\chi \in \mathcal{X}$ extends to Y_χ . An *extension map* with respect to $X \triangleleft Y$ for \mathcal{X} is a Y -equivariant function

$$\Lambda : \mathcal{X} \rightarrow \bigcup_{X \leq I \leq Y} \text{Irr}(I)$$

such that for every $\chi \in \mathcal{X}$, the character $\Lambda(\chi)$ belongs to $\text{Irr}(Y_\chi)$ and extends χ . It is easily checked, that if maximal extendibility holds with respect to $X \triangleleft Y$ for \mathcal{X} , then such an extension map exists. If $\mathcal{X} = \text{Irr}(X)$, we simply say *maximal extendibility holds with respect to $X \triangleleft Y$* .

We are interested in the existence of extension maps, as well as their compatibility with Galois automorphisms. One of the most important cases in which there exists an extension map is the following Lemma for normal subgroups with cyclic quotient, which can be proven using basic, although non-character theoretic methods.

Lemma 2.2. *Let $X \triangleleft Y$ be finite groups such that Y/X is cyclic. Then maximal extendibility holds with respect to $X \triangleleft Y$.*

A proof of the general result is given in [Nav18, Theorem 5.1]. Here, we only consider the special case where χ is a linear character. This allows the proof to be completed using purely character-theoretic arguments, which is not possible in the general setting.

Proof. Let $X \triangleleft Y$ be abelian and $\chi \in \text{Irr}(X)$, Y -invariant. As Y/X is cyclic, we have $Y/X = \langle gX \rangle$ for some $g \in Y$. Let $m = \text{ord}(gX)$ and thus $g^m \in X$. Let $\chi(g^m) = z \in \mathbb{C}^\times$ and $\zeta \in \mathbb{C}^\times$ such that $\zeta^m = z$. Then we define the extension of X to Y as

$$\tilde{\chi}(xg^k) = \chi(x)\zeta^k, \quad \text{for } x \in X, k \in \mathbb{Z}.$$

Clearly, as χ is linear, the character is well-defined and as χ is Y -invariant for x_1g^k, x_2g^j we have

$$\tilde{\chi}(x_1g^k x_2g^j) = \chi(x_1)\chi(g^k x_2g^j) = \chi(x_1)\chi(x_2)\zeta^{k+j} = \tilde{\chi}(x_1g^k)\tilde{\chi}(x_2g^j),$$

and $\tilde{\chi}$ is a well-defined character. As there are m choices for an m -th root of z , there are m distinct extensions of χ . \square

The following is now immediate from the proof above.

Remark 2.3. *Let $X \triangleleft Y$ be two abelian groups such that there exists some $y \in Y$ with $Y = \langle y, X \rangle$. Let $\text{ord}(y) = n$, $\lambda \in \text{Lin}(X)$ and $\tilde{\lambda} \in \text{Irr}(Y | \lambda)$, then $\mathbb{Q}(\tilde{\lambda}) \subseteq (\mathbb{Q}(\lambda))(\zeta)$, where ζ is a n -th root of $\lambda(y)$. More precisely $\mathbb{Q}(\tilde{\lambda}) = (\mathbb{Q}(\lambda))(\tilde{\lambda}(y))$.*

Another subset of groups with nice extendibility properties are semidirect products, which we shortly introduce to fix a notation.

Definition 2.4. Let N and H be two finite groups, such that there exists a homomorphism $\phi : H \rightarrow \text{Aut}(N)$. Then the (external) semidirect product is the group $G := N \rtimes H$ where elements of G are elements of the cartesian product $N \times H$ with multiplication

$$(n, h)(n', h') = (n\phi(h)(n'), hh').$$

In case N and H are subgroups of a given group G such that $N \cap H = \{1_G\}$ and $H \leq \text{N}_{\mathbf{G}}(N)$, then H acts on N by conjugation and this defines the "inner" semidirect product $NH = \langle N, H \rangle$

The character theory of semidirect products $G = A \rtimes H$ with abelian normal subgroup A is particularly well behaved, as maximal extendibility holds with respect to $A \triangleleft G$.

Proposition 2.5. *[Ser77, Section 8.2] Let $G = A \rtimes H$ be the semidirect product with an abelian normal subgroup A . Then maximal extendibility holds with respect to $A \triangleleft G$.*

Indeed these two examples of extension maps are the main building blocks for more complex extension maps, as there is a way to "lift" character extensions assuming certain group properties.

Lemma 2.6. [Spä10, Lemma 4.1] Let $X \triangleleft Y$ be finite groups and take $U \leq Y$ such that $Y = XU$. Let $\psi \in \text{Irr}(X)$ an irreducible character such that $\psi_0 := \psi|_{U \cap X}$ is irreducible and has an extension $\tilde{\psi}_0$ to U_{ψ_0} . Then ψ has a unique extension $\tilde{\psi}$ to Y_ψ with

$$\text{Res}_{U_\psi}^{Y_\psi}(\tilde{\psi}) = \text{Res}_{U_\psi}^{U_{\psi_0}}(\tilde{\psi}_0).$$

Proof. Let \mathcal{X} be a representation of X affording ψ and \mathcal{P} the projective representation of Y_ψ from Lemma 1.18. In particular, for \mathcal{P} we have

$$\mathcal{P}(ux) = \mathcal{P}(u)\mathcal{P}(x), \quad \text{and} \quad \mathcal{P}(xu) = \mathcal{P}(x)\mathcal{P}(u), \quad \forall x \in X, u \in U.$$

Let $[\beta] \in H^2(Y_\psi/X, \mathbb{C}^\times)$ be the equivalence class of cocycles defined by Theorem 1.19. The restriction $\mathcal{P}|_{U_\psi}$ is a projective representation with trivial cocycle class, as ψ_0 is extendible, once again by Theorem 1.19. Since furthermore $Y_\psi = (XU)_\psi = XU_\psi$, $[\beta]$ must also be trivial and ψ extends to Y_ψ . Finally the uniqueness follows from Gallagher correspondence 1.14 and $Y_\psi/X \cong U_\psi$. \square

Note that this statement is an extension of Proposition 2.5, as for the semidirect product $X \cap U = \{1\}$, hence each character of degree 1 has a unique extension. Furthermore, the extension is well behaved with respect to Galois automorphisms.

Lemma 2.7. Let X, U and Y be as in Lemma 2.6, $\psi \in \text{Irr}(X)$ and $\tilde{\psi}$ its unique extension to Y_ψ . Let $\sigma \in \text{Gal}(\mathbb{Q}^{ab}/\mathbb{Q}) \times \text{Aut}(Y)_{X,U}$. Assume that σ acts trivial on Y/X and that the extension with respect to $U \cap X \triangleleft U$ is σ -equivariant. Then $\tilde{\psi}^\sigma$ is the extension of ψ such that $\tilde{\psi}^\sigma|_{U_{\psi^\sigma}} = (\tilde{\psi}_0)^\sigma$ and $\tilde{\psi}^\sigma = (\tilde{\psi})^\sigma$.

Proof. We see that $(\tilde{\psi})^\sigma$ is an extension of ψ^σ . Furthermore, we have $U_{\psi^\sigma} = U_\psi$ and $Y_{\psi^\sigma} = Y_\psi$ since σ acts trivially on Y/X . Furthermore, $\tilde{\psi}_0^\sigma = (\tilde{\psi}_0)^\sigma$ and thus

$$\text{Res}_{U_{\psi^\sigma}}^{Y_{\psi^\sigma}}((\tilde{\psi})^\sigma) = \text{Res}_{U_\psi}^{Y_\psi}((\tilde{\psi})^\sigma) = \text{Res}_{U_\psi}^{U_{\psi_0}}(\tilde{\psi}_0)^\sigma = \text{Res}_{U_{\psi^\sigma}}^{U_{\psi_0^\sigma}}(\tilde{\psi}_0^\sigma).$$

Hence, both $\tilde{\psi}^\sigma$ and $(\tilde{\psi})^\sigma$ fulfil the properties of the extension of the above lemma and uniqueness implies the statement. \square

Lemma 2.8. [CS17, Lemma 5.8 b)] Let $Y \triangleleft \tilde{Y}$ such that $\tilde{Y} = \langle Y, t \rangle$ for some $t \in \tilde{Y}$. Let $X \triangleleft Y$ and $X \triangleleft \tilde{Y}$ and assume that t acts trivially on Y/X and let $\psi \in \text{Irr}(X)$ be a t -invariant character with extension $\tilde{\psi}$ to $\text{Irr}(Y_\psi)$. Then there exists $\nu \in \text{Irr}(Y_\psi)$ such that $(\tilde{\psi})^t = \tilde{\psi}\nu$ and $\ker(\nu) = Y_{\hat{\psi}}$, where $\hat{\psi}$ is an extension of ψ to $\langle X, t \rangle$.

Proof. By Gallagher's correspondence 1.14 we have $(\tilde{\psi})^t \in \text{Irr}(Y_{\psi^t} | \psi^t) = \text{Irr}(Y_\psi | \psi)$ and there exists some $\nu \in \text{Irr}(Y_\psi/X)$ such that $(\tilde{\psi})^t = \tilde{\psi}\nu$ and in particular we can regard $\nu \in \text{Irr}(Y_\psi)$.

Let $y \in \ker(\nu)$ then the group $\langle X, y \rangle$ is normalized by t , since t acts trivially on Y/X . The character $\theta = \tilde{\psi}|_{\langle X, y \rangle}$ is also t -invariant and hence extends to a character $\hat{\theta} \in \text{Irr}(\langle X, y, t \rangle)$. Now since $\hat{\theta}^y = \hat{\theta}$ and $\hat{\theta}|_{\langle X, t \rangle} = \hat{\psi}$ for some cyclic extension of ψ to $\langle X, t \rangle$, we have that $y \in Y_{\hat{\psi}}$.

If on the other hand $y \in Y_{\hat{\psi}}$, we have $\langle X, t \rangle \triangleleft \langle X, t, x \rangle$, as the group $\langle X, y, t \rangle / X$ is abelian. We use Lemma 2.7 with

$$\langle X, t, y \rangle = \langle X, t \rangle \langle X, y \rangle,$$

so for any character ψ with extension $\hat{\psi} \in \text{Irr}(\langle X, t \rangle)$ we have $\psi = \hat{\psi}|_X$ and there exists a unique extension θ of ψ to $\langle X, t, y \rangle$, with

$$\theta|_{\langle X, y \rangle} = \tilde{\psi}|_{\langle X, y \rangle}.$$

Hence, $\tilde{\psi}^y|_{\langle X, y \rangle} = \tilde{\psi}|_{\langle X, y \rangle}$ and by definition of ν we have $y \in \ker \nu$. \square

Lemma 2.9. *Let $X \triangleleft XY$ and let U be a finite group, such that X and Y act on $\text{Irr}(U)$.*

- a) *Suppose there exists an X -transversal \mathbb{T}' in $\text{Irr}(U)$ such that for each $\chi \in \mathbb{T}'$ we have $(XY)_\chi = X_\chi Y_\chi$. Then there exists a Y -stable X -transversal \mathbb{T} in $\text{Irr}(U)$.*
- b) *Given a Y -stable X -transversal \mathbb{T} in $\text{Irr}(U)$, then $(XY)_\chi = X_\chi Y_\chi$ for all $\chi \in \mathbb{T}$.*

Proof. Since the XY -orbits in $\text{Irr}(U)$ contain the X -orbits, we may choose a set of $\chi_i \in \mathbb{T}'$ for each such orbit. Let $\mathbb{T} = \{\chi_i | i\}^Y$ the set of all Y -conjugates of the chosen representatives. To verify that \mathbb{T} is an X -transversal assume there are $\psi_1, \psi_2 \in \mathbb{T}$ and $x \in X$ such that $\psi_1^x = \psi_2$. We have $\psi_1 = \chi_i^{y_1}$ and $\psi_2 = \chi_j^{y_2}$ for some $\chi_i, \chi_j \in \mathbb{T}'$ and $y_1, y_2 \in Y$. Then

$$(\chi_i^{y_1})^x = (\psi_1)^x = \psi_2 = \chi_j^{y_2} \Rightarrow \chi_i^{x'y} = \chi_j \text{ for } y = y_1 y_2^{-1} \text{ and } x' = x^{y_1^{-1}} \in X.$$

By our choice of χ_i , it follows that $i = j$, and thus $x'y \in (XY)_{\chi_i} = X_{\chi_i} Y_{\chi_i}$. Thus $\chi_i^{x'} = \chi_i$ and in turn $\psi_1 = \psi_2$, hence \mathbb{T} is a Y -stable X -transversal.

For the inverse statement, let $\chi \in \mathbb{T}$ and $xy \in (XY)_\chi$ for $x \in X$ and $y \in Y$. Then

$$\chi^{xy} = \chi \Leftrightarrow \chi^x = \chi^{y^{-1}}.$$

As \mathbb{T} is Y -stable χ^y must lie in \mathbb{T} . But clearly $\chi^{y^{-1}} = \chi^x$ and χ are X -conjugate and thus $\chi^x = \chi^{y^{-1}} = \chi$ as \mathbb{T} is an X -transversal. Thus $x \in X_\chi$ and $y \in Y_\chi$. \square

2.B Parametrization of Characters

For us the existence of extension maps is of importance, since it allows us to parametrize the characters of Y .

Lemma 2.10. *Let $X \triangleleft Y$ be finite groups, and suppose there exists an extension map Λ with respect to $X \triangleleft Y$. Define the set*

$$\Sigma := \{(\theta, \eta) \mid \theta \in \text{Irr}(X), \eta \in \text{Irr}(Y_\theta \mid 1_X)\}.$$

Then the map

$$\Pi : \Sigma \rightarrow \text{Irr}(Y), \quad \Pi(\theta, \eta) = (\Lambda(\theta)\eta)^Y$$

is surjective and induces a bijection between Y -orbits in Σ and $\text{Irr}(Y)$, in particular $\Pi(\theta, \eta) = \Pi(\theta', \eta')$ if and only if there exists $y \in Y$ such that $\eta^y = \eta'$ and $\theta^y = \theta'$. Furthermore, given $\mathcal{H} \leq \text{Gal}(\mathbb{Q}^{ab}/\mathbb{Q}) \times \text{Aut}(Y)_X$ and assume Λ to be \mathcal{H} -equivariant. Then $\Pi(\theta, \eta)^\sigma = \Pi(\theta^\sigma, \eta^\sigma)$ for all $\sigma \in \mathcal{H}$.

Proof. See for example [CS25, 2.1] for the first part. The second part of the statement follows directly from the construction of Π and Proposition 1.23. \square

We obtain a slightly nicer property in the case that Y/X is abelian.

Proposition 2.11. *Let $X \triangleleft Y$ with abelian quotient and assume that maximal extendibility holds with respect to $X \triangleleft Y$. Then for any $\chi \in \text{Irr}(Y)$ and $\chi_0 \in \text{Irr}(X \mid \chi)$, there exists an irreducible extension $\tilde{\chi}_0$ of χ_0 to Y_{χ_0} , such that $\chi = \tilde{\chi}_0^Y$ and $\chi|_{Y_{\chi_0}} = \tilde{\chi}_0$.*

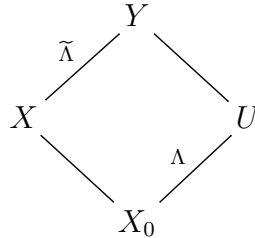
Proof. As the characters $\eta \in \text{Irr}(Y_{\chi}/X)$ are all linear and can be regarded as characters of Y_{χ} , any character $\Lambda(\chi_0)\eta$ is irreducible. Thus, the first statement holds. The second statement is a direct consequence of Clifford correspondence 1.13c). \square

We recall a result, similar to [CS25, Proposition 2.3], and refine it with regard to the actions of group and Galois automorphisms.

Lemma 2.12. *Let $X \triangleleft Y$ and $U \triangleleft Y$ with $XU = Y$. Assume that for $X_0 = U \cap X \triangleleft Y$, the quotient $X/X_0 \cong Y/U$ is abelian and that maximal extendibility holds with respect to $X_0 \triangleleft X$.*

Let \mathbb{X} be an U -stable X -transversal of $\text{Irr}(X_0)$ and Λ an extension map with respect to $X_0 \triangleleft U$ for \mathbb{X} . Then there exists a Y -equivariant extension map $\tilde{\Lambda}$ with respect to $X \triangleleft Y$, such that the following holds:

- a) For each $\chi_0 \in \mathbb{X}$ and $\chi \in \text{Irr}(X \mid \chi_0)$, $\Lambda(\chi_0)|_{U_{\chi}}$ is a constituent of $\tilde{\Lambda}(\chi)|_{U_{\chi}}$.



- b) $\tilde{\Lambda}$ is $\text{Lin}(Y/U)$ -equivariant, so for every $\lambda \in \text{Lin}(Y/U)$ and $\chi \in \text{Irr}(X)$

$$\tilde{\Lambda}(\lambda|_X \chi) = \lambda|_{Y_{\chi}} \tilde{\Lambda}(\chi)$$

- c) Let A be a group acting on Y , stabilizing U and X . Assume that \mathbb{X} is A -stable and Λ is A -equivariant. Then $\tilde{\Lambda}$ is A -equivariant.

- d) Let $\mathcal{G} \leq \text{Gal}(\mathbb{Q}^{ab}/\mathbb{Q}) \times \text{Aut}(Y)_{X,U}$. Assume that \mathbb{X} is \mathcal{G} -stable and let $\sigma \in \mathcal{G}$, $\chi_0 \in \mathbb{X}$ and $\chi \in \text{Irr}(X \mid \chi_0)$. Then

$$[\tilde{\Lambda}(\chi), \sigma]|_{U_{\chi\sigma}} = [\Lambda(\chi_0), \sigma]|_{U_{\chi\sigma}},$$

where $[-, \sigma]$ is defined as in Definition 1.37. So if Λ is \mathcal{G} -equivariant, then $\tilde{\Lambda}$ is \mathcal{G} -equivariant.

Proof. Let $\chi \in \text{Irr}(X)$. As \mathbb{X} is a X -transversal in $\text{Irr}(X_0)$, there exists a unique $\chi_0 \in \mathbb{X}$ with $\chi \in \text{Irr}(X | \chi_0)$. As maximal extendibility holds with respect to $X_0 \triangleleft X$ and X/X_0 is abelian, by Proposition 2.11 there exists some extension $\tilde{\chi}_0$ of χ_0 in X_{χ_0} such that $\tilde{\chi}_0^X = \chi$. Since there also exists an extension of χ_0 to $\Lambda(\chi_0) \in \text{Irr}(U_{\chi_0})$, we apply Lemma 2.6 with

$$X := X_{\chi_0}, \quad U := U_{\chi_0}, \quad Y := U_{\chi_0} X_{\chi_0}, \quad \psi := \tilde{\chi}_0$$

and obtain a common extension χ'_0 of $\tilde{\chi}_0$ and $\Lambda(\chi_0)|_{U_{\tilde{\chi}_0}}$ to $U_{\tilde{\chi}_0} X_{\chi_0}$. We define:

$$\tilde{\Lambda}(\chi) := (\chi'_0)^{Y_\chi}.$$

If $Y_\chi = U_{\tilde{\chi}_0} X$, then $\tilde{\Lambda}(\chi)$ is a well-defined irreducible character extension of χ . As χ_0 lies in the U -stable X -transversal \mathbb{X} , the equality $Y_{\chi_0} = (UX)_{\chi_0} = U_{\chi_0} X_{\chi_0}$ holds. As χ_0 is a constituent of χ , we have $Y_\chi \leq Y_{\chi_0} X = U_{\chi_0} X$. We obtain the equation

$$Y_\chi = (U_{\chi_0} X)_\chi = (U_\chi \cap U_{\chi_0}) X.$$

Any $u \in U$ stabilizing χ_0 and χ must also stabilize $\tilde{\chi}_0$, this implies $Y_\chi = U_{\tilde{\chi}_0} X$ and Λ is a well-defined extension map.

As $\tilde{\Lambda}(\chi)$ is the induced character of χ'_0 and $U \triangleleft Y$, the character $\tilde{\Lambda}(\chi)|_{U_\chi}$ is the sum of X -conjugates of $\chi'_0|_{U_\chi}$ and the set $\text{Irr}(U_\chi | \tilde{\Lambda}(\chi))$ contains $\Lambda(\chi_0)|_{U_\chi}$. Furthermore, $\text{Irr}(U_\chi | \chi_0) \cap \text{Irr}(U_\chi | \tilde{\Lambda}(\chi)) = \{\Lambda(\chi_0)|_{U_\chi}\}$ and no other extension of χ to Y_χ has this property and hence $\tilde{\Lambda}$ is uniquely defined, this concludes part a).

Let λ be linear character of Y/U and identify it with a character of Y . Let $\chi \in \text{Irr}(X)$ and χ_0 as defined above, then $\chi_0 \in \text{Irr}(X_0 | \chi \cdot \lambda|_X)$ and $(\tilde{\chi}_0 \cdot \lambda|_{X_\chi})^X = \chi \cdot \lambda|_X$. As λ is U -invariant, the extension to $U_{\tilde{\chi}_0} X_{\chi_0}$ is uniquely defined by the previous considerations. Hence b) follows.

For c) let $a \in A$. Clearly, $\chi^a|_{X_0}$ has the constituent $\chi_0^a \in \mathbb{X}$ and the extension $\tilde{\chi}_0^a$ of χ_0^a satisfies $(\tilde{\chi}_0^a)^X = \chi^a$. Since Λ is A -equivariant, the extension $(\chi_0^a)'$ is given by $(\chi'_0)^a$ and

$$\tilde{\Lambda}(\chi^a) = ((\chi'_0)^a)^{Y_{\chi^a}} = ((\chi'_0)^{Y_\chi})^a = \tilde{\Lambda}(\chi)^a.$$

Let $\sigma \in \mathcal{G}$ and χ, χ_0 as above. By the considerations in c) and Lemma 2.7, if Λ is σ -equivariant, so is $\tilde{\Lambda}$. As the extensions $\tilde{\Lambda}$ and χ'_0 are the unique extensions, such that $\Lambda(\chi_0)|_{U_\chi}$ is a constituent of their restrictions to U_χ , the commutator statement holds. \square

Let $\mathcal{H} \leq \text{Gal}(\mathbb{Q}^{ab}/\mathbb{Q})$. Assume that we have an \mathcal{H} -equivariant extension map with respect to $X \triangleleft Y$, where X and Y are groups such that the following holds:

- All characters of X are \mathcal{H} -invariant.
- For each $\theta \in \text{Irr}(X)$, all characters of $\text{Irr}(Y_\theta/X)$ are \mathcal{H} -invariant.

Then all characters of Y are \mathcal{H} -invariant.

A particularly nice application for the parametrization of characters, once again happens when Y is a semi-direct product, with abelian normal subgroup X .

Lemma 2.13. *Let $Y = X \rtimes H$ be the semidirect product of an abelian normal subgroup X and a subgroup H . Then all irreducible characters of Y can be induced from the irreducible characters of X and certain subgroups of H .*

Proof. Let Λ be the extension map with respect to $X \triangleleft Y$ from Proposition 2.5. For $\chi \in \text{Irr}(X)$ we have

$$Y_\chi = \{(x, h) \in Y \mid \chi^{(x_1, h)}(x_2) = \chi(x_2) \forall x_2 \in X\} = \{h \in H \mid \chi(x^h) = \chi(x) \forall x \in X\} = X \rtimes H_\chi.$$

Since the extension of characters of X to $\text{Irr}(Y_\chi)$ is ensured by Proposition 2.5, the statement follows from Lemma 2.10. \square

Apart from the regular semidirect product that appears in our constructions, a specific version known as the wreath product is of particular interest in the construction of characters of Weyl groups. Thus, we finish the chapter with a parametrization of irreducible characters of wreath products.

Definition 2.14. Let G be a finite group and $H \leq \mathfrak{S}_n$, where \mathfrak{S}_n acts on the group G^n by permuting the factors. Then we write $G \wr H := G^n \rtimes H$ and call it the *wreath product* of G and H .

Definition 2.15. Let $\underline{\alpha} = (\alpha_1, \dots, \alpha_k)$ with $\cup \alpha_i = \{1, 2, \dots, n\}$ be a partition into k disjoint non-empty subsets. A *Young subgroup* of \mathfrak{S}_n associated to $\underline{\alpha}$ is the subgroup of the form $\mathfrak{S}_{\alpha_1} \times \dots \times \mathfrak{S}_{\alpha_k}$, where each group \mathfrak{S}_{α_j} acts on the elements of α_j as the symmetric group and trivial on all other elements of $\{1, \dots, n\}$. Given a partition $\underline{\alpha}$ we write $S_{\underline{\alpha}}$ for the corresponding Young subgroup.

Definition 2.16. Let G be a finite group and $\text{Irr}(G) = \{\theta_1, \dots, \theta_k\}$. Let χ be an irreducible character of G^n with decomposition

$$\chi = \chi_1 \times \dots \times \chi_n, \quad \chi_i \in \text{Irr}(G).$$

Then we set for each $1 \leq i \leq k$, $n_i = |\{j \mid \chi_j \sim \theta_i\}|$ as the number of factors of χ that are isomorphic to θ_i . Then we write $(n_1, \dots, n_k) = \underline{n}_\chi$ as the *partition type* of χ . Note that $n_i = 0$ is possible.

Lemma 2.17. Let G be a finite group, $H \leq \mathfrak{S}_n$ and χ be an irreducible character of the base group G^n with \underline{n}_χ its partition type. Then the inertia group is given by

$$(G \wr H)_\chi \cong G \wr (H \cap S_{\underline{n}_\chi}).$$

Proof. See [JK81, Lemma 4.3.27]. \square

In the following proposition, we show that the normalizer of a Young Subgroup in \mathfrak{S}_n is given by the direct product of smaller wreath products.

Proposition 2.18. Let $\underline{\alpha} = \cup_{j=1}^k \alpha_j$ be a partition of n and let $S_{\underline{\alpha}}$ be the corresponding Young subgroup in \mathfrak{S}_n . Let $A = \{|\alpha_j|, 1 \leq j \leq k\} = \{x_1, \dots, x_m\}$, the set of all possible cardinalities, where each x_i describes a possible cardinality of one of the α_j . We set $c_i := |\{j \mid |\alpha_j| = x_i\}|$, the number that corresponds to how often each partition size exists in $\underline{\alpha}$. Then

$$N_{\mathfrak{S}_n}(S_{\underline{\alpha}}) \cong S_{\underline{\alpha}} \rtimes (\mathfrak{S}_{c_1} \times \dots \times \mathfrak{S}_{c_m}) \cong (\mathfrak{S}_{x_1} \wr \mathfrak{S}_{c_1}) \times \dots \times (\mathfrak{S}_{x_m} \wr \mathfrak{S}_{c_m}).$$

Proof. We can write our Young subgroup as

$$S_{\underline{\alpha}} \cong \mathfrak{S}_{x_1}^{c_1} \times \dots \times \mathfrak{S}_{x_m}^{c_m}.$$

Let $\sigma \in \mathfrak{S}_n$ and $\rho \in S_{\underline{\alpha}}$ with

$$\rho = \rho_{(1,1)} \times \rho_{(1,2)} \times \dots \times \rho_{(1,c_1)} \times \rho_{(2,1)} \times \dots \times \rho_{(m,c_m)},$$

where each $\rho_{(i,j)}$ acts on \mathfrak{S}_{x_i} . Then we have

$$\rho^\sigma = \rho_{(1,1)}^\sigma \times \dots \times \rho_{(m,c_m)}^\sigma \in N_{\mathfrak{S}_n}(Y)$$

if and only if $\rho_{(i,j)}^\sigma \in \mathfrak{S}_{\alpha_i}$ with $|\alpha_i| = x_i$. Thus σ may permute two factors if they lie in the same size of permutation group. Thus $\sigma = \sigma_1 \times \dots \times \sigma_m \in \mathfrak{S}_{x_1} \times \dots \times \mathfrak{S}_{x_m}$, where each σ_i acts on the set $\{\alpha_j \mid |\alpha_j| = x_i\}$ of order c_i . \square

Proposition 2.19. *Let G be a group and $H \leq \mathfrak{S}_n$. Let $G \wr H$ be the wreath product of G and H . Then maximal extendibility holds with respect to $G^n \triangleleft G \wr H$.*

Proof. Let $\chi \in \text{Irr}(G)$ be any irreducible character of G . First we show that $\chi^n \in \text{Irr}(G^n)$ is extendible. First, we note that since permuting the values does not change the value of the character, we have $(G \wr H)_\chi = G \wr H_\chi = G \wr H$. Then by [JK81, Lemma 4.3.9] for each $(g, \pi) \in G^n \rtimes H$, we define an extension of χ^n by

$$\widetilde{\chi}^n(g_1 \times g_2 \times \dots \times g_n, \pi) = \prod_{i=1}^{|\pi|} \chi(f(i, g, \pi)),$$

where $|\pi|$ corresponds to the number of disjoint cycles of π and $\{f(i, g, \pi) \mid 1 \leq i \leq |\pi|\} \subset G$. Given $(a_i, \pi(a_i), \dots, \pi^{r-1}(a_i))$ the i -th-cycle of π of length r , $f(i, g, \pi) \in G$ is given by

$$f(i, g, \pi) = g_{a_i} g_{\pi(a_i)} \dots g_{\pi^r(a_i)}$$

such that $f(i, g, 1) = g_i$. Hence, $\widetilde{\chi}^n(g, 1) = \chi^n$ and it is an extension of χ^n . Now given any irreducible character $\chi \in \text{Irr}(G^n)$ with partition type \underline{n}_χ , we can extend χ to an irreducible character of $G \wr (H_\chi) = G \wr (H \cap S_{\underline{n}_\chi})$ by setting

$$\widetilde{\chi}(g, \pi) = \prod_{i=1}^{|\pi|} \chi(f(i, g, \pi)),$$

which is well-defined by [JK81, 4.3.29]. \square

Following this construction we can compute the character fields for extensions of characters on the wreath product. This is a slightly stronger version of [JK81, Corollary 4.4.11]

Lemma 2.20. *Let G be a group such that for each irreducible character $\lambda \in \text{Irr}(G)$, its character field $\mathbb{Q}(\lambda)$ is contained in some field K . Let $\underline{\alpha}$ be some partition of n and $S_{\underline{\alpha}}$ the corresponding Young subgroup. Let $\chi \in \text{Irr}(G^n)$ be any irreducible character. Then for its extension $\widetilde{\chi} \in \text{Irr}((G \wr S_{\underline{\alpha}})_\chi)$ we have $\mathbb{Q}(\widetilde{\chi}) \leq K$. Furthermore, for any irreducible character $\widetilde{\phi} \in \text{Irr}(G \wr S_{\underline{\alpha}})$ we have $\mathbb{Q}(\widetilde{\phi}) \leq K$.*

Proof. Let $\chi \in \text{Irr}(G^n)$. Clearly, $\mathbb{Q}(\chi) \leq K$. Let \underline{n}_χ be its partition type. Then the inertia subgroup of χ is given by $G \wr (S_{\underline{\alpha}} \cap S_{\underline{n}_\chi})$. It is obvious that the group $S := (S_{\underline{\alpha}} \cap S_{\underline{n}_\chi})$ must be the product of disjoint symmetric subgroups. In particular, for any character ψ of S we have $\mathbb{Q}(\psi) = \mathbb{Q}$, as we have seen in Example 1.21. Let $\tilde{\psi}$ be the extension of ψ on $G \wr S$. By definition $\tilde{\psi}(g, \pi) = \psi(\pi)$ so again $\mathbb{Q}(\tilde{\psi}) = \mathbb{Q}$. Now let $\tilde{\chi}$ be the extension of χ in $G \wr S$. By Proposition 2.19 the values of $\tilde{\chi}$ are a product of values of χ . Thus $\mathbb{Q}(\tilde{\chi}) = \mathbb{Q}(\chi)$. By Lemma 2.10

$$\left\{ \text{Ind}_{G \wr S}^{G \wr S_{\underline{\alpha}}} \tilde{\chi} \tilde{\psi} \mid \chi \in \text{Irr}(G^n), \tilde{\psi} \in \text{Irr}(G^n \rtimes S) \right\} = \text{Irr}(G \wr S_{\underline{\alpha}}),$$

and the result follows for all characters of $G \wr S_{\underline{\alpha}}$ by Proposition 1.23. \square

Lemma 2.21. *Let $\underline{\alpha}$ be a partition of n and $Y := S_{\underline{\alpha}}$ the corresponding Young subgroup. Then each $\chi \in \text{Irr}(Y)$ extends to a character $\tilde{\chi} \in \text{Irr}(N_{\mathfrak{S}_n}(Y)_\chi)$ with $\mathbb{Q}(\chi) = \mathbb{Q}(\tilde{\chi})$. Furthermore, there exists an irreducible rational valued character in $\text{Irr}(N_{\mathfrak{S}_n}(Y) \mid \chi)$.*

Proof. By Proposition 2.18 there exist integers x_1, \dots, x_k and c_1, \dots, c_k such that Y is isomorphic to

$$Y \cong (\mathfrak{S}_{x_1})^{c_1} \times \cdots \times (\mathfrak{S}_{x_k})^{c_k}.$$

Thus, $\chi \in \text{Irr}(Y)$ can be written as $\chi_1 \times \cdots \times \chi_k$ where each factor χ_i is itself an irreducible character of $\text{Irr}((\mathfrak{S}_{x_i})^{c_i})$ and $\chi_i = \chi_i^1 \times \cdots \times \chi_i^{c_i}$, where $\chi_i^j \in \text{Irr}(\mathfrak{S}_{x_i})$. Again by Proposition 2.18 the normalizer of Y is given by

$$N_{\mathfrak{S}_n}(Y) = (\mathfrak{S}_{x_1} \wr \mathfrak{S}_{c_1}) \times \cdots \times (\mathfrak{S}_{x_k} \wr \mathfrak{S}_{c_k}).$$

Since each factor χ_i corresponds to the factor $(\mathfrak{S}_{x_i} \wr \mathfrak{S}_{c_i})$ we have

$$N_{\mathfrak{S}_n}(Y)_\chi = (\mathfrak{S}_{x_1} \wr \mathfrak{S}_{c_1})_{\chi_1} \times \cdots \times (\mathfrak{S}_{x_k} \wr \mathfrak{S}_{c_k})_{\chi_k}$$

By Lemma 2.17 the inertia subgroup of χ_i in $\mathfrak{S}_{x_i} \wr \mathfrak{S}_{c_i}$ is again a wreath product described by the type of χ_i , so for $(\underline{c}_i)_{\chi_i}$ a partition of c_i we have

$$(\mathfrak{S}_{x_i} \wr \mathfrak{S}_{c_i})_{\chi_i} = \mathfrak{S}_{x_i} \wr (\mathfrak{S}_{c_i} \cap \mathfrak{S}_{(\underline{c}_i)_{\chi_i}}).$$

Set $S_i := (\mathfrak{S}_{c_i} \cap \mathfrak{S}_{(\underline{c}_i)_{\chi_i}})$ which is itself a Young group. By Proposition 2.19 we can extend χ_i to an irreducible character $\tilde{\chi}_i$ of $\mathfrak{S}_{x_i} \wr S_i$. Since each character χ_i is an irreducible character of a product of symmetric groups, they are all rational valued and by Proposition 2.19 $\tilde{\chi}_i$ is also rational valued. Thus, $\tilde{\chi} := \tilde{\chi}_1 \times \cdots \times \tilde{\chi}_k$ is an irreducible rational extension of χ to $N_{\mathfrak{S}_n}(Y)_\chi$. Finally let κ_i be an irreducible character of S_i and $\tilde{\kappa}_i$ the irreducible extension to $\mathfrak{S}_{x_i} \wr S_i$. $\tilde{\kappa}_i$ once again is rational valued and

$$\psi_i := \text{Ind}_{\mathfrak{S}_{x_i} \wr S_i}^{\mathfrak{S}_{x_i} \wr \mathfrak{S}_{c_i}} (\tilde{\chi}_i \tilde{\kappa}_i)$$

is a rational valued irreducible extension of χ_i . Thus, $\psi = \psi_1 \times \cdots \times \psi_k$ is a rational valued irreducible character in $\text{Irr}(N_{\mathfrak{S}_n}(Y))$. \square

Lastly we present a result concerning irreducible characters containing an irreducible character of multiplicity one in its induced character.

Proposition 2.22. *[CS17, Proposition 4.5] Let $X \triangleleft Y$ and let A be a group acting on Y stabilizing X . Let $\rho \in \mathbb{Z}_{\geq 0}\text{Irr}(X)$ an A -stable character and $\theta \in \text{Irr}(Y)$ a constituent of ρ^Y with multiplicity one. Then:*

- (i) *There exists a unique irreducible component χ_0 of $\theta|_X$ and ρ with multiplicity one in both. This character satisfies*

$$(Y \rtimes A)_{\chi_0} = Y_{\chi_0} \rtimes A_{\chi_0}.$$

- (ii) *If ρ extends to $X \rtimes A$, then χ_0 extends to $X \rtimes A_{\chi_0}$.*

3. Finite groups of Lie type

The aim of this chapter is to give an overview of some general theory and facts on finite groups of Lie type that will be required in the subsequent chapters, with particular emphasis on groups of type A. Finite groups of Lie type form the most diverse family in the classification of finite simple groups, and thus are of particular interest in the verification of the inductive conditions.

We begin by briefly recalling said classification. Continuing, we state the necessary background on linear algebraic groups, root systems, maximal tori, and Weyl groups. All of these results can be found in the standard literature e.g., [MT11], [Hum75]. Hence, we will refrain from giving proofs and instead collect the statements needed later on in order to set up our notation.

We continue to define F -stable tori given a Steinberg endomorphism F and highlight Sylow d -tori as they are of high importance later on, since these objects provide natural connections to normalizers of Sylow ℓ -subgroups. The chapter concludes with some technical lemmata needed in Chapter 6.

Throughout, we restrict our attention to the case of type A, where explicit descriptions of our objects and automorphisms are introduced.

3.A Classification of finite simple groups

The classification of finite simple groups due to Aschbacher et al. can be summarized in the following theorem, see for example [Wil09, Section 1.2].

Theorem 3.1. *Let S be a finite simple group. Then S is isomorphic to one of the following.*

- *A cyclic group C_p for p prime.*
- *An alternating group A_n for $n \geq 5$.*
- *A simple group of Lie type over a finite field.*
- *One of 26 sporadic groups.*

The information of irreducible characters of the sporadic groups can often be obtained through various databases and computer programs like GAP. The alternating groups form one infinite family of groups, while the simple groups of Lie type consist of several infinite families of groups themselves.

Using induction methods, one can reduce many problems of finite groups and representation theory of finite groups to questions about simple groups or groups related to simple groups. Indeed, the family of finite groups of Lie type is the most complex as it consists of several infinite subfamilies.

3.B Linear algebraic groups and groups of Lie type

We assume the basics of linear algebraic groups. Here a linear algebraic group is an affine variety together with a compatible group structure. We use the standard topological notions as *connected*, *closed*, *irreducible* etc. Furthermore, morphisms of linear algebraic groups are morphisms of varieties compatible with the group structure. For a linear algebraic group \mathbf{G} , we denote by \mathbf{G}° the connected component of \mathbf{G} containing the identity element. We fix some prime p and let \mathbf{k} be an algebraically closed field of characteristic p .

Each linear algebraic group \mathbf{G} over \mathbf{k} can be embedded as a closed subgroup of some general linear group $\mathrm{GL}_n(\mathbf{k})$, [Gec03, Cor. 2.4.4]. Hence, we can always regard \mathbf{G} as some closed subgroup of some $\mathrm{GL}_n(\mathbf{k})$. In particular, we say that $g \in \mathbf{G}$ is *unipotent* if $\mathrm{ord}(g)$ is some p -power, and $g \in \mathbf{G}$ is *semisimple* if it is diagonalizable as a matrix in $\mathrm{GL}_n(\mathbf{k})$.

By *Jordan decomposition* [MT11, Theorem 2.5], every element $g \in G$ has a unique decomposition into semisimple and unipotent elements $g_s, g_u \in \mathbf{G}$ with $g = g_s g_u = g_u g_s$. One writes \mathbf{G}_u and \mathbf{G}_s for the subsets of unipotent and semisimple elements of \mathbf{G} respectively. In particular, if $\mathbf{G} = \mathbf{G}_u$, we say that \mathbf{G} is *unipotent*.

A maximal closed connected solvable normal subgroup of \mathbf{G} is called the *radical* $R(\mathbf{G})$ of \mathbf{G} . The set $R(\mathbf{G})_u$ of unipotent elements is the maximal closed connected normal unipotent subgroup of \mathbf{G} and is called the *unipotent radical* of \mathbf{G} , denoted by $R_u(\mathbf{G})$. We have $R_u(\mathbf{G}) \leq R(\mathbf{G}) \leq \mathbf{G}^\circ$.

We call a linear algebraic group \mathbf{G} *reductive* if $R_u(\mathbf{G}) = 1$. It is called *semisimple* if \mathbf{G} is connected and $R(\mathbf{G}) = 1$. By [Hum95, Section 4.2], for any connected reductive group \mathbf{G} , the set \mathbf{G}_u is an irreducible closed subset and invariant under the conjugation of \mathbf{G} .

We call a linear algebraic group \mathbf{T} a *torus*, if \mathbf{T} is isomorphic to $\mathbf{k}^\times \times \cdots \times \mathbf{k}^\times$, that is if \mathbf{T} is a direct product of a finite number of copies of the multiplicative group \mathbf{k}^\times . We say a torus $\mathbf{T} \leq \mathbf{G}$ is *maximal* in \mathbf{G} if it is maximal with respect to inclusion. Some important properties of an algebraic group \mathbf{G} are encoded in its maximal tori. Given a linear algebraic group \mathbf{G} , a *Borel* subgroup of \mathbf{G} is a closed, connected and solvable subgroup that is maximal with respect to all these properties. By [MT11, Theo. 6.4, Cor. 6.5], all Borel subgroups and maximal tori are conjugate in \mathbf{G} .

The *Weyl group* of a linear algebraic group with respect to a maximal torus \mathbf{T} is the group $W_{\mathbf{G}}(\mathbf{T}) = N_{\mathbf{G}}(\mathbf{T})/C_{\mathbf{G}}(\mathbf{T})$. In particular, in case \mathbf{G} is connected reductive, $C_{\mathbf{G}}(\mathbf{T}) = \mathbf{T}$ and $W_{\mathbf{G}}(\mathbf{T}) = N_{\mathbf{G}}(\mathbf{T})/\mathbf{T}$, see [GM20, 1.3.1].

Given a (maximal) torus $\mathbf{T} \leq \mathbf{G}$ we write $X(\mathbf{T})$ for the group of *characters* of \mathbf{T} , where a character is an algebraic group homomorphism $\alpha : \mathbf{T} \rightarrow \mathbf{k}^\times$. Note that this definition of characters differs to the one assumed in the previous chapters.

The set of *co-characters* $Y(\mathbf{T})$ on the other hand is the set of algebraic homomorphisms $\varphi : \mathbf{k}^\times \rightarrow \mathbf{T}$. Via a construction using the Lie algebra $\text{Lie}(\mathbf{G})$ of \mathbf{G} and the adjoint representation of \mathbf{G} one can distinguish a set $\Phi(\mathbf{G}) \subset X(\mathbf{T})$ called the *set of roots*, which encodes a lot of information on \mathbf{G} , see [MT11, Section 2.3].

Let \mathbf{G} be a connected reductive group with maximal torus \mathbf{T} and root system $\Phi(\mathbf{G})$. Then given a root $\alpha \in \Phi$, we let $\mathbf{T}_\alpha := (\ker \alpha)^\circ$ and $C_\alpha := C_{\mathbf{G}}(\mathbf{T}_\alpha)$. As $C_{\mathbf{G}}(\mathbf{T}) = \mathbf{T}$, we have $C_{C_\alpha}(\mathbf{T}) = \mathbf{T}$ and given $n_\alpha \in N_{C_\alpha}(\mathbf{T}) \setminus C_{C_\alpha}(\mathbf{T})$, we let s_α be the image of n_α in $W_{\mathbf{G}}(\mathbf{T})$. By [MT11, Proposition 8.20], the Weyl group of a connected reductive group \mathbf{G} is generated by these *reflections* s_α for $\alpha \in \Phi(\mathbf{G})$. In particular, W acts faithfully on the set of roots and co-roots and stabilizes Φ , see [MT11, Proposition 8.4].

Using that the Lie algebra $\text{Lie}(\mathbf{G})$ has a decomposition into eigenspaces, we may associate to each root $\alpha \in \Phi(\mathbf{G})$ a unique one-dimensional connected unipotent subgroup $\mathbf{U}_\alpha \leq \mathbf{G}$ normalized by \mathbf{T} . By [Hum75, 26.3], these groups satisfy the following:

- For $w \in W_{\mathbf{G}}(\mathbf{T})$ with preimage $n \in N_{\mathbf{G}}(\mathbf{T})$, we have $n\mathbf{U}_\alpha n^{-1} = \mathbf{U}_{w\alpha}$;
- $\mathbf{G} = \langle \mathbf{T}, \mathbf{U}_\alpha \mid \alpha \in \Phi \rangle$; and
- $[C_\alpha, C_\alpha] = \langle \mathbf{U}_\alpha, \mathbf{U}_{-\alpha} \rangle$.

Definition 3.2. This one-dimensional connected unipotent subgroup \mathbf{U}_α is called the *root subgroup* of \mathbf{G} with respect to \mathbf{T} .

For semisimple groups, the structure is even more explicit.

Theorem 3.3. [MT11, Theorem 8.21] *Let \mathbf{G} be a semisimple group, \mathbf{T} a maximal torus of \mathbf{G} and Φ a root system of \mathbf{G} . Then the following statements hold:*

- a) $\mathbf{G} = \langle \mathbf{U}_\alpha \mid \alpha \in \Phi \rangle$.
- b) $\mathbf{G} = [\mathbf{G}, \mathbf{G}]$.
- c) \mathbf{G} only has finitely many minimal non-trivial closed connected normal subgroups $\mathbf{G}_1, \dots, \mathbf{G}_r$, where $\mathbf{G} = \mathbf{G}_1 \cdots \mathbf{G}_r$.

Example 3.4. We consider an example to illustrate the definitions introduced above. The group $\mathbf{G} = \text{GL}_n(\mathbf{k})$ of invertible matrices is connected reductive [MT11, Ex. 1.12, 6.17(4)]. The centre $Z(\mathbf{G}) = \{t \cdot E_n \mid t \in \mathbf{k}^\times\} \cong \mathbf{k}^\times$ is a connected, solvable, normal subgroup of \mathbf{G} , hence GL_n is not semisimple. However PGL_n and SL_n are semisimple groups. The group of invertible upper diagonal matrices \mathbf{T}_n is a Borel subgroup of GL_n and the group of invertible diagonal matrices \mathbf{D}_n is a maximal torus. Similarly the groups $\text{SL}_n \cap \mathbf{T}_n$ and $\text{SL}_n \cap \mathbf{D}_n$ are Borel subgroup and maximal torus of SL_n , respectively, see [MT11, Theo. 6.4, Ex. 6.7]. We write $\mathbf{T} := \mathbf{D}_n$ going forward.

The normalizer $N_{\mathrm{GL}_n}(\mathbf{T})$ consists of all monomial matrices with entries in \mathbf{k}^\times , hence

$$W_{\mathrm{GL}_n}(\mathbf{T}) = N_{\mathrm{GL}_n}(\mathbf{T})/\mathbf{T} \cong \mathfrak{S}_n.$$

The set of characters $X(\mathbf{T})$ is parametrized by homomorphisms

$$\chi_i : \mathrm{diag}(x_1, \dots, x_n) \mapsto x_i.$$

Hence, $X(\mathbf{T}) = \{\prod_{i=1}^n \chi_i^{a_i} \mid a_i \in \mathbb{Z}\} \cong \mathbb{Z}^n$. In our case, we can similarly see that given an n -tuple $(a_1, \dots, a_n) \in \mathbb{Z}^n$, the map

$$\varphi_{a_1, \dots, a_n} : \mathbf{k}^\times \rightarrow \mathbf{T}, \quad x \mapsto \mathrm{diag}(x^{a_1}, \dots, x^{a_n})$$

defines a co-character and that $Y(\mathbf{T}) = \{\varphi_{(a_1, \dots, a_n)} \mid a_i \in \mathbb{Z}\} \cong \mathbb{Z}^n$.

Then a root of GL_n is any character $\chi_{i,j} := \chi_i \chi_j^{-1} \in X(\mathbf{T})$ with $i \neq j$ and the set of roots is given by

$$\Phi(\mathbf{G}) = \{\chi_{i,j} \mid 1 \leq i, j \leq n, i \neq j\}.$$

Here the reflection s_α associated to the root $\alpha = \chi_{ij}$ corresponds to the transposition $(i, j) \in \mathfrak{S}_n$. Given $\alpha \in \Phi(\mathbf{G})$, one fixes a morphism $\mathbf{x}_\alpha : (\overline{\mathbb{F}}_p, +) \rightarrow \mathbf{G}$ of algebraic groups, such that if $\alpha = \chi_{i,j}$, $\mathbf{x}_\alpha(t) - \mathrm{id}_n$ is equal to the elementary matrix with t in position (i, j) . Then $\mathbf{U}_\alpha = \mathbf{x}_\alpha(\mathbf{k})$.

We write $\mathbf{n}_\alpha := \mathbf{x}_\alpha(1)\mathbf{x}_{-\alpha(1)}\mathbf{x}_\alpha(1)$, for the monomial matrix. In case $1 \leq i \leq n-1$ we write $\mathbf{n}_i := \mathbf{n}_{i+1,i} \in N_{\mathbf{G}}(\mathbf{T})$.

3.C The classification of semisimple algebraic groups

One can define an *abstract root system* independent of the previous notion of a root system of a connected reductive group as a subset Φ of a finite-dimensional vector space E [MT11, Definition 9.1]. Furthermore, one associates a *base* $\Delta \subseteq \Phi$ of roots which spans E . The elements of Δ are called *simple roots*. Then Δ determines a set of *positive roots* $\Phi^+ \subset \Phi$. One obtains a *Dynkin diagram* of the root system, where the nodes correspond to the simple roots. By [Hum78, 11.1], the basis of a root system can (up to some factor) be recovered given its Dynkin diagram.

We say that two root systems Φ are *isomorphic* if their underlying Dynkin diagrams are isomorphic in the usual sense. A basis Δ is *decomposable* if there exist non-empty subsets Δ_1, Δ_2 such that $\Delta = \Delta_1 \sqcup \Delta_2$ and they are mutually orthogonal. If no such decomposition exists, we call $\Phi \neq \emptyset$ an *indecomposable* root system. Hence, Φ is indecomposable if and only if its associated Dynkin diagram is connected. All connected/indecomposable Dynkin diagrams/root systems have been classified.

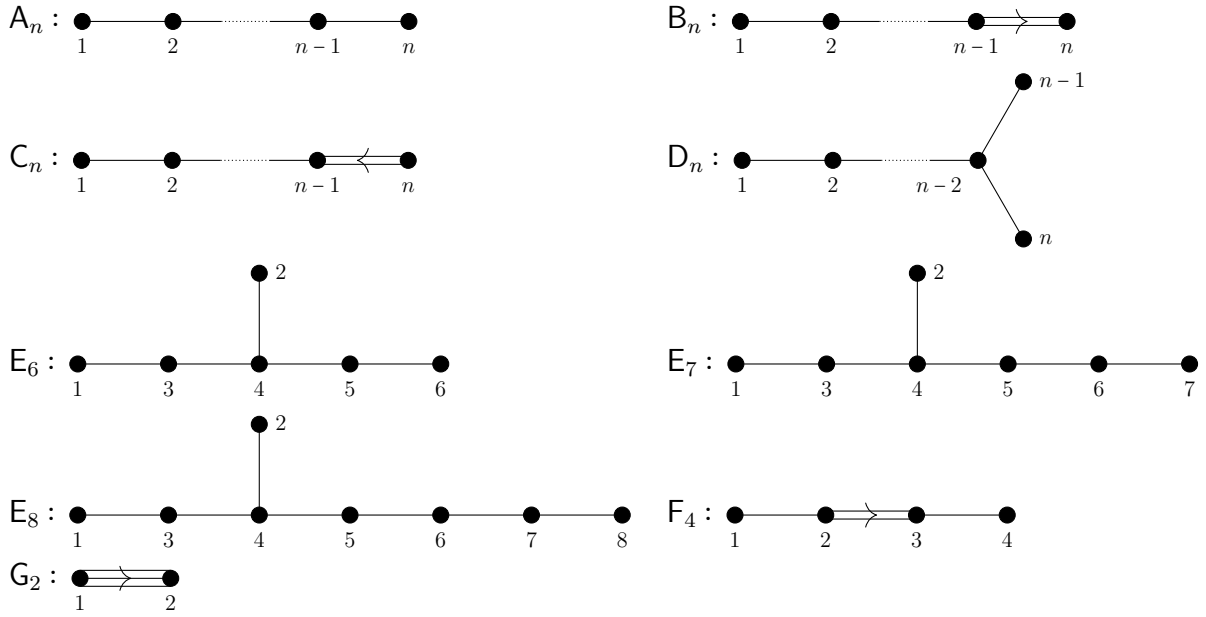
Theorem 3.5. [MT11, Theorem 9.6] *Let Φ be an indecomposable root system in some real vector space $V \cong \mathbb{R}^n$. Then it is isomorphic to one of the following types.*

$$\mathbf{A}_n, (n \geq 1) \quad \mathbf{B}_n, (n \geq 2) \quad \mathbf{C}_n, (n \geq 3) \quad \mathbf{D}_n, (n \geq 4), \quad \mathbf{E}_6, \mathbf{E}_7, \mathbf{E}_8, \mathbf{F}_4, \mathbf{G}_2,$$

with corresponding Dynkin diagrams as in Table 2.1.

Now given a reductive group \mathbf{G} , together with maximal torus \mathbf{T} and root system $\Phi(\mathbf{G}, \mathbf{T})$, we may associate an abstract root system to \mathbf{G} . The set $\Phi(\mathbf{G})$ can be understood as a subset of the real vector space $X(\mathbf{T})_{\mathbb{R}} := X(\mathbf{T}) \otimes_{\mathbb{Z}} \mathbb{R}$. Then by

Table 3.1: Dynkin diagrams of indecomposable root systems



[MT11, Proposition 9.2] we may regard $\Phi(\mathbf{G})$ together with the set of reflections as an abstract root system in $X(\mathbf{T}) \otimes_{\mathbb{Z}} \mathbb{R}$.

In particular, the choice of a Borel subgroup \mathbf{B} containing \mathbf{T} leads in a 1 – 1 correspondence to a choice of a base Δ or to a choice of positive roots, see [Hum75, 27.1].

Note that two semisimple groups may give rise to the same root system, but are non-isomorphic, (e.g., SL_2 and PGL_2 are both of type A_1). Together with the co-characters $Y(\mathbf{T})$ and so called *co-roots* $\Phi^\vee \subseteq Y(\mathbf{T})$, we may give a one-to-one correspondence. Given a connected reductive group \mathbf{G} with maximal torus \mathbf{T} , the quadruple $\Psi = (X(\mathbf{T}), \Phi, Y(\mathbf{T}), \Phi^\vee)$ is the *root datum* of \mathbf{G} .

Theorem 3.6. [Spr10, Theorem 10.1.1, Theorem 9.6.2] *For each root datum Ψ there exists a semisimple linear algebraic group with maximal torus such that the corresponding root datum is isomorphic to Ψ . Furthermore, two semisimple linear algebraic groups over the same field are isomorphic if and only if they have the same root data.*

With regard to the decomposition of semisimple groups into finitely many closed semisimple subgroups, the decomposition $\mathbf{G} = \mathbf{G}_1 \cdots \mathbf{G}_r$ from Theorem 3.3 corresponds to the decomposition of Φ into its indecomposable components. If \mathbf{G} is non-commutative and has no proper closed connected normal subgroups, we say that \mathbf{G} is *almost simple*.

We call the groups of type A_n, B_n, C_n and D_n groups of *classical type* and the groups $E_n, n \in \{6, 7, 8\}, F_4$ and G_2 of *exceptional type*.

Example 3.7. [MT11, Example 9.8] Let $\mathbf{G} = \mathrm{SL}_n$. Then $\mathbf{T} = \mathbf{D}_n \cap \mathrm{SL}_n$ is maximal torus and as in Example 3.4, $\Phi(\mathbf{G}) = \{\chi_{i,j} \mid 1 \leq i, j \leq n, i \neq j\}$. A base of $\Phi(\mathbf{G})$ is given by

$$\Delta := \{\chi_{i,i+1} \mid 1 \leq i \leq n-1\}.$$

Then $\Phi^+ = \{\chi_{i,j} \mid i < j\}$ and the root system of SL_n is of type A_{n-1} .

Let \mathbf{G} be a semisimple algebraic group with maximal torus \mathbf{T} and $X(\mathbf{T})$, Φ as before. We set $\Omega := \text{Hom}(\mathbb{Z}\Phi^\vee, \mathbb{Z})$ and we may regard $X(\mathbf{T}) \subseteq \Omega$ with respect to the inclusion

$$X(\mathbf{T}) \cong \text{Hom}(Y(\mathbf{T}), \mathbb{Z}) \hookrightarrow \text{Hom}(\mathbb{Z}\Phi^\vee, \mathbb{Z}) =: \Omega.$$

Then $\Lambda(\mathbf{G}) := \Omega/X(\mathbf{T})$ is called the *fundamental group* of \mathbf{G} . In case $\Lambda(\mathbf{G}) = 1$, we call \mathbf{G} *simply connected*. In case $\mathbf{X}(\mathbf{T}) = \mathbb{Z}\Phi$, then \mathbf{G} is called of *adjoint type*.

We remind the reader that a surjective homomorphism $\phi : \mathbf{G}_1 \rightarrow \mathbf{G}_2$ of algebraic groups with finite kernel is called an *isogeny*.

Proposition 3.8. [MT11, Proposition 9.15] *Let \mathbf{G} be a semisimple group with root system Φ . Then there exist natural isogenies*

$$\mathbf{G}_{sc} \xrightarrow{\pi_1} \mathbf{G} \xrightarrow{\pi_2} \mathbf{G}_{ad}$$

for a simply connected group \mathbf{G}_{sc} and an adjoint group \mathbf{G}_{ad} , such that each has root system Φ , $Z(\mathbf{G}_{ad}) = 1$ and $\Lambda(\mathbf{G}_{sc}) = \Lambda(\mathbf{G})_{p'}$.

Example 3.9. Let $\mathbf{G} = \text{SL}_n$, with root system $X(\mathbf{T})$ and root system Φ from Example 3.7. The cocharacters are given by

$$Y(\mathbf{T}) = \{\varphi_{(a_1, \dots, a_n)} \mid a_i \in \mathbb{Z}, \sum a_i = 0\}$$

and the co-roots in this case are given by

$$\Phi^\vee = \{\chi_{i,j}^\vee \mid 1 \leq i, j \leq n, i \neq j\},$$

where $\chi_{i,j}^\vee := \varphi_{(b_1, \dots, b_n)}$, with $b_i = -1$, $b_j = 1$, and $b_k = 0$ otherwise. Clearly, $\mathbb{Z}\Phi^\vee = Y(\mathbf{T})$ and SL_n is simply connected. Furthermore, one can show that PGL_n is of adjoint type with root system of type A_{n-1} .

We can furthermore describe the automorphisms of semisimple groups. Let \mathbf{B} be a fixed Borel subgroup of \mathbf{G} . Then Δ be the base of Φ such that $\mathbf{U}_\alpha \leq \mathbf{B}$ for all $\alpha \in \Delta$, i.e., fixing a Borel subgroup is equivalent to fixing a base.

Definition 3.10. Let D be the group of algebraic automorphisms of \mathbf{G} that stabilize the pair $\mathbf{T} \leq \mathbf{B}$. Then any element in $\delta \in D$ induces an automorphism of the roots $\Phi(\mathbf{G})$ that stabilizes the base Δ , hence it induces an automorphism of the Dynkin diagram. We say that δ is a *diagram* or *graph automorphism* of \mathbf{G} . We write $\text{Inn}(\mathbf{G})$ for the subgroup of inner automorphisms.

If \mathbf{G} is semisimple, then the automorphism group is given by $\text{Aut}(\mathbf{G}) = \text{Inn}(\mathbf{G}) \rtimes D$ and $\text{Aut}(\mathbf{G})/\text{Inn}(\mathbf{G})$ is finite, see [Hum75, Theorem 27.4].

Example 3.11. [MT11, Theorem 11.12] The group of graph automorphisms of a Dynkin diagram of type A_{n-1} , for $n \geq 3$ is isomorphic to the cyclic group with two elements. Let $\Delta = \{\alpha_i \mid 1 \leq i \leq n-1\}$ be the base. Here $\alpha_i = \chi_{i,i+1}$ as in Example 3.9. Then the non-trivial graph automorphism sends α_i to α_{n-i} for all $1 \leq i \leq n-1$. This corresponds to the automorphism of SL_n given by transpose-inverse together with conjugation by the element $\begin{pmatrix} 0 & & 1 \\ & \ddots & \\ 1 & & 0 \end{pmatrix} \cdot \text{diag}(-1, 1, -1, \dots)$.

3.D Finite groups of Lie type

Again we let \mathbf{G} be a linear algebraic group over an algebraically closed field $\mathbf{k} = \overline{\mathbb{F}_q}$, with $q = p^m$.

The *standard Frobenius map* $F_{p^m} : \mathrm{GL}_n(\mathbf{k}) \rightarrow \mathrm{GL}_n(\mathbf{k})$ is the endomorphism of varieties that sends each matrix-entry to its p^m -th power $F_{p^m} : (a_{ij})_{1 \leq i, j \leq n} \mapsto (a_{ij}^{p^m})_{1 \leq i, j \leq n}$. An endomorphism $F : \mathbf{G} \rightarrow \mathbf{G}$ is called a *Frobenius map defining an \mathbb{F}_q -structure*, if there exists an embedding $\rho : \mathbf{G} \rightarrow \mathrm{GL}_n(\mathbf{k})$ such that $\rho \circ F = F_q \circ \rho$. An endomorphism $F : \mathbf{G} \rightarrow \mathbf{G}$ is called a *Steinberg map* if some power of F is a Frobenius map. By a theorem of Steinberg, see [MT11, Theorem 21.5], for an almost simple linear algebraic group \mathbf{G} , any endomorphism $\sigma : \mathbf{G} \rightarrow \mathbf{G}$ is either an automorphism of algebraic groups, or the set

$$\mathbf{G}^\sigma := \{g \in \mathbf{G} \mid \sigma(g) = g\}$$

is finite. In the second case, σ is called Steinberg endomorphism. The *Lang-Steinberg theorem* lays the foundation for the theory for groups of Lie type as the theorem ensures that the set of fixed points behaves well.

Theorem 3.12. [MT11, Theorem 21.7] *Let \mathbf{G} be a connected linear algebraic group over $\overline{\mathbb{F}_q}$ and $F : \mathbf{G} \rightarrow \mathbf{G}$ a Steinberg endomorphism. Then the map*

$$\mathcal{L}_F : \mathbf{G} \rightarrow \mathbf{G}, \quad g \mapsto g^{-1}F(g)$$

is surjective. \mathcal{L}_F is called the Lang map.

If \mathbf{G} is a linear algebraic group over \mathbf{k} and $F : \mathbf{G} \rightarrow \mathbf{G}$ is a Steinberg endomorphism, then

$$\mathbf{G}^F := \{g \in \mathbf{G} \mid F(g) = g\}$$

is a finite group and is called a *finite group of Lie type*. In most cases, the Steinberg endomorphism does not commute with taking quotients.

Proposition 3.13. [MT11, Proposition 23.2] *Let \mathbf{G} be a linear algebraic group, $F : \mathbf{G} \rightarrow \mathbf{G}$ a Steinberg endomorphism and \mathbf{H} an F -stable closed connected normal subgroup of \mathbf{G} . Then the map*

$$\mathbf{G}^F / \mathbf{H}^F \rightarrow (\mathbf{G}/\mathbf{H})^F$$

is an isomorphism.

If \mathbf{G} is connected reductive and $F : \mathbf{G} \rightarrow \mathbf{G}$ a Steinberg endomorphism, then there exists an F -stable maximal torus \mathbf{T} contained in an F -stable Borel subgroup \mathbf{B} . All such tori and Borel subgroups $\mathbf{T} \leq \mathbf{B}$ are \mathbf{G}^F -conjugate, see [MT11, Corollary 21.12]. These tori are called *maximally split* with respect to F .

Example 3.14. Let $\mathbf{G} = \mathrm{GL}_n(\mathbf{k})$ and $q = p^m$ some prime power.

- a) Let $F_q : \mathbf{G} \rightarrow \mathbf{G}$ be the standard Frobenius endomorphism. Then the set of upper triangular matrices \mathbf{T}_n is clearly F_q -stable and a Borel subgroup. In particular, it contains the group \mathbf{D}_n of invertible diagonal matrices and \mathbf{D}_n is F_q -stable. This is a maximally split torus of \mathbf{G} and $\mathbf{D}_n^{F_q} \cong (\mathbb{F}_q^\times)^n$ and has order $|\mathbf{D}_n^{F_q}| = (q-1)^n$.

b) Now let $F : \mathbf{G} \rightarrow \mathbf{G}$ be given by $F_q \cdot \sigma$, where σ is the automorphism of \mathbf{G} given by transpose-inverse, see also Example 3.11. Then

$$F : (a_{ij}) \mapsto (a_{ij}^q)^{-T}$$

and $F^2 = F_{q^2}$ so F is a Steinberg endomorphism. In particular, $\mathbf{G}^F \leq \mathbf{G}^{F^2} \cong \mathrm{GL}_n(q^2)$. The group of fixed points \mathbf{G}^F is isomorphic to the *general unitary group* $\mathrm{GU}_n(q)$. We set $\mathrm{SU}_n(q) := \mathrm{SL}_n^F = \mathrm{SL}_n(q^2) \cap \mathrm{GU}_n(q)$ for the *special unitary group*. Note that the group of upper triangular matrices is not F -stable in this case but the group \mathbf{D}_n is. So for $v_0 \in \mathrm{SL}_n$ with

$$v_0 = \begin{pmatrix} 0 & & (-1)^2 \\ & \ddots & \\ (-1)^{n+1} & & 0 \end{pmatrix}, \quad (3.1)$$

we let $F' : \mathbf{G} \rightarrow \mathbf{G}$ be given by

$$F' : (a_{ij}) \mapsto F_q((v_0(a_{ij})v_0^{-1})^{-T}).$$

Now \mathbf{T}_n and \mathbf{D}_n are both F' -stable. Let $t \in \mathbf{D}_n$, $t = \mathrm{diag}(t_1, \dots, t_n)$. Then

$$F'(t) = \mathrm{diag}(t_n^{-q}, t_{n-1}^{-q}, \dots, t_1^{-q}) = \mathrm{diag}(t_1, \dots, t_n),$$

if and only if $t_1 = t_n^{-q}$, $t_2 = t_{n-1}^{-q}, \dots, t_n = t_1^{-q}$. Subsequently we get

$$t_i = t_i^{q^2} \text{ for all } 1 \leq i \leq n \quad \text{and} \quad t_{m+1}^{q+1} = 1 \text{ in case } n = 2m + 1.$$

Notation 3.15. Let $\mathbf{G} = \mathrm{SL}_n(\mathbf{k})$ and $\tilde{\mathbf{G}} := \mathrm{GL}_n(\mathbf{k})$. Then we write γ for the endomorphism of $\tilde{\mathbf{G}}$, defined by

$$\gamma : g \mapsto (v_0 g v_0^{-1})^{-T},$$

where v_0 is as in (3.1). Note that γ defines the graph automorphism of $\tilde{\mathbf{G}}$ by Example 3.11. Furthermore, given the standard Frobenius F_q , we will assume that any Steinberg endomorphism of \mathbf{G} or of $\tilde{\mathbf{G}}$ is of the form F_q or $\gamma \circ F_q$.

Later on, when we are concerned with the Gelfand-Graev characters, we will have to introduce another assumption on our primes.

Definition 3.16. Let \mathbf{G} be a connected reductive simple group over $\overline{\mathbb{F}}_p$ and $F : \mathbf{G} \rightarrow \mathbf{G}$ a Steinberg endomorphism. We say p is a *good prime* for \mathbf{G} if the following conditions are satisfied [Car85, §1.14]:

$$\begin{aligned} A_n &: \text{no further condition} \\ B_n, C_n, D_n &: p \neq 2 \\ G_2, F_4, E_6, E_7 &: p \neq 2, 3 \\ E_8 &: p \neq 2, 3, 5. \end{aligned}$$

In general, we say p is a good prime for \mathbf{G} in case p is good for each simple factor of \mathbf{G} .

Proposition 3.17. [MT11, Corollary 21.8] *Let \mathbf{G} be a connected group with Steinberg endomorphism $F : \mathbf{G} \rightarrow \mathbf{G}$. For any $g \in \mathbf{G}$ we write gF for the endomorphism of \mathbf{G} given by*

$$gF : a \mapsto gF(a)g^{-1}.$$

Then gF is a Steinberg endomorphism and the finite groups \mathbf{G}^{gF} and \mathbf{G}^F are \mathbf{G} -conjugate.

We say that a Steinberg endomorphism $F : \mathbf{G} \rightarrow \mathbf{G}$ of a connected reductive group \mathbf{G} is \mathbb{F}_q -split if there exists an F -stable maximal torus \mathbf{T} of \mathbf{G} such that $F(t) = t^q$ for all $t \in \mathbf{T}$. We call F twisted if F is not split and a product of an \mathbb{F}_q -split endomorphism with an algebraic group automorphism of \mathbf{G} . In the groups of type B_2 , G_2 and F_4 , there furthermore exists the notion of *very twisted* Steinberg endomorphisms.

In particular, all Steinberg endomorphisms of an almost simple simply connected group can be determined up to inner automorphisms by \mathbf{G} , q and a permutation ρ of the positive roots, which induces a Dynkin diagram automorphism, see [MT11, Theorem 22.5]. For groups of type A_{n-1} , the non-trivial graph automorphism is given by γ as in Definition 3.15. This defines a twisted Steinberg endomorphism.

Let \mathbf{G} be an almost simple group of simply connected type and assume that \mathbf{G}^F is perfect and not one of the following exceptions:

$$\mathrm{SL}_2(2), \mathrm{SL}_2(3), \mathrm{SU}_3(2), \mathrm{Sp}_4(2), \mathrm{G}_2(2), {}^2\mathrm{B}_2(2), {}^2\mathrm{G}_2(3), {}^2\mathrm{F}_4(2). \quad (3.2)$$

Then, by [GLS98, 6.1.4], $\mathbf{G}^F/Z(\mathbf{G}^F)$ is simple. In particular, \mathbf{G}^F is a universal covering group of the finite simple group $S := \mathbf{G}^F/Z(\mathbf{G}^F)$ in the sense of [Asc00, 33.1]

Definition 3.18. We will use the notation $E = \langle \gamma, F_p \rangle \leq \mathrm{Aut}(\mathbf{G}^F)$ for the automorphism group generated by the restrictions to \mathbf{G}^F to the field automorphism F_p and the graph automorphism γ from Definition 3.10. Note that in case of type A, the graph automorphism is given by γ from Notation 3.15. Note that in case of D_4 , the group E should contain all graph automorphisms commuting with F . Furthermore note, that in type D_4 , γ has to be replaced by a dihedral group of order 6.

The action of E extends to $\tilde{\mathbf{G}}^F$ in general, see [GM20, Proposition 1.7.5]. In the case of type A, the action of γ and F_p on $\tilde{\mathbf{G}}^F$ are clearly defined and the group $\tilde{\mathbf{G}}^F \rtimes E$ is well-defined according to Definition 2.4.

Note that E can be identified as the group generated by F_p and γ modulo F , i.e., $E \cong \langle F_p, \gamma \rangle / \langle F \rangle$ considered as abstract automorphisms of $\tilde{\mathbf{G}}$. This gives sense to expressions like $N_{\tilde{\mathbf{G}}^F \rtimes E}(\mathbf{S})$, where \mathbf{S} is an F -stable subgroup of $\tilde{\mathbf{G}}$.

Theorem 3.19. [GLS98, 2.5.1] *Let $\mathbf{G} = \mathrm{SL}_n(\mathbf{k})$, $\tilde{\mathbf{G}} = \mathrm{GL}_n(\mathbf{k})$ and $n \geq 3$ such that $\mathbf{G}^F/Z(\mathbf{G}^F)$ is non-abelian simple. The automorphisms of \mathbf{G}^F are generated by $\tilde{\mathbf{G}}^F$ and E . In particular, this implies*

$$\mathrm{Aut}(\mathbf{G}^F) \cong (\tilde{\mathbf{G}}^F \rtimes E) / Z(\tilde{\mathbf{G}}^F).$$

In case $n = 2$, $\mathrm{Aut}(\mathbf{G}^F)$ is generated by the automorphisms induced by $(\tilde{\mathbf{G}}^F \rtimes E)$.

Any finite group of Lie type can once again be described in a combinatorial way with its *complete root datum*: Let \mathbf{G} be a semisimple group and let \mathbf{T} be a F -stable maximal torus. Let $\Phi(\mathbf{G}, \mathbf{T}) \subset X(\mathbf{T})$ be the root system of \mathbf{G} , with set of positive roots Φ^+ with respect to $\mathbf{T} \leq \mathbf{B}$. Then F defines a permutation ρ of Φ^+ and there exists some integer $\delta \geq 1$ such that $F^\delta|_{X(\mathbf{T})} = q^\delta \text{id}_{X(\mathbf{T})}$ and $F = q\phi$ on $X_{\mathbb{R}}$ for some power q of p and some $\phi \in \text{Aut}(X_{\mathbb{R}})$ of order δ that induces ρ^{-1} on Φ^+ , see [MT11, Proposition 22.2]. By the classification of Steinberg endomorphisms ([MT11, Theorem 22.5]), F is uniquely determined up to inner automorphisms of \mathbf{G} , q and the permutation $\rho|_{\Delta}$.

As \mathbf{T} is F -stable, so is the Weyl group W , hence $W\langle F \rangle$ is well-defined. As each left-coset wF , stabilizes \mathbf{T} , it stabilizes the root datum. By Proposition 3.17, the group \mathbf{G}^{wF} is isomorphic to \mathbf{G}^F . As the map $\phi \in \text{Aut}(X_{\mathbb{R}})$ stabilizes Φ and Φ^\vee , the complete root datum $\mathbb{G} = (X, \Phi, Y, \Phi^\vee, W\phi)$, together with q , defines \mathbf{G}^F up to isomorphism.

Example 3.20. [MT11, Example 22.11]

- a) Let $\mathbf{G} = \text{GL}_n(\mathbf{k})$, and F_q the standard Frobenius. Then $\mathbf{T} = \mathbf{D}_n$ is an F_q -stable maximal torus and F_q acts trivially on the Weyl group W . Hence $\phi = \text{id}_{X_{\mathbb{R}}}$ is trivial and the complete root datum is (Γ, W) , where Γ is the root datum of \mathbf{G} .
- b) Let $\mathbf{G} = \text{GL}_n(\mathbf{k})$, and $\gamma F : \mathbf{G} \rightarrow \mathbf{G}$ the Steinberg endomorphism as in Notation 3.15. Then γF acts on $X(\mathbf{T})$ as $-q$, see Example 3.14. The complete root datum of the family of groups $\{\text{GU}_n(q)\}$ is given by $(\Gamma, -\text{id}_{X_{\mathbb{R}}}W)$, where Γ is the root datum of GL_n .

Definition 3.21. We say that $\text{GU}_n(q)$ is *Ennola-dual* to $\text{GL}_n(q)$. We often use the notation $\text{GU}_n(q) = \text{GL}_n(-q)$. Note however, that $\text{GU}_n(q)$ can't be obtained as a subgroup of $\text{GL}_n(q)$.

The order of \mathbf{G}^F is determined by its complete root datum $\mathbb{G} = (X, \Phi, Y, \Phi^\vee, \phi W)$ and the prime power q of p . In particular, one can construct a polynomial

$$|\mathbb{G}| := x^{|\Phi^+|} \prod_{i=1}^{rk(\mathbb{G})} (x^{d_i} - \epsilon_i) \in \mathbb{Z}[x] \quad (3.3)$$

called the *order polynomial* of the complete root datum \mathbb{G} , where $d_i \in \mathbb{Z}^+$ are integers corresponding to the Weyl group and ϵ_i are some roots of unity that correspond to the automorphism F . In particular, $|\mathbb{G}|(q) = |\mathbf{G}^F|$, when F is a Steinberg map. This defines an \mathbb{F}_q -structure on \mathbf{G} .

3.E Sylow ℓ -subgroups and d -tori

As introduced in Definition 1.42, in order to obtain the inductive Galois–McKay conditions we are concerned with groups $N \not\leq G$ such that $N_G(P) \leq N$ for a Sylow ℓ -subgroup P of G , where ℓ is a prime $\neq p$. In particular, the inductive McKay conditions have been verified under the assumption that N is the normalizer of a Sylow d -torus.

If not stated otherwise, \mathbf{G} is a connected reductive group and F a Steinberg endomorphism on \mathbf{G} , $\mathbf{T} \leq \mathbf{B}$ an F -stable torus and Borel subgroup. The fixed point group of the unipotent radical of \mathbf{B} , $\mathbf{U}^F = R_u(\mathbf{B})^F$ is a Sylow p -subgroup of G , see [MT11, Corollary 24.11]. Hence, in defining characteristic, the characters of \mathbf{B}^F have to be studied, see [Ruh21]. So we will assume that ℓ is an odd prime, different from p . Indeed, we will look at other prime divisors of $|G|$ and look at the structure of its Sylow ℓ -subgroups.

Let \mathbf{T} be a maximal F -stable torus of \mathbf{G} . The action of F on the groups of characters and co-characters is given by

$$F(\chi)(t) = \chi(F(t)), \text{ for } \chi \in X(\mathbf{T}), t \in \mathbf{T}, \text{ and}$$

$$F(y)(\lambda) = F(y(\lambda)) \text{ for } y \in Y(\mathbf{T}), \lambda \in \mathbb{G}_m,$$

respectively.

Let \mathbf{T} be a maximally split F -stable torus. Then for $g \in \mathbf{G}$, the torus ${}^g\mathbf{T}$ is F -stable if and only if $g^{-1}F(g) \in N_{\mathbf{G}}(\mathbf{T})$, see [MT11, Theorem 21.11]. So any F -stable conjugate of \mathbf{T} determines an element $w := g^{-1}F(g)\mathbf{T} \in W$, called the q -twist of ${}^g\mathbf{T}$.

Assume that ${}^g\mathbf{T} = {}^{g'}\mathbf{T}$ are F -stable for $g, g' \in \mathbf{G}$. Let w be the element corresponding to $g^{-1}F(g)$ and w' corresponding to $g'^{-1}F(g')$. Then there exists some $x \in W$ such that $w' = x^{-1}wF(x)$ as for $n := g^{-1}g' \in N_{\mathbf{G}}(\mathbf{T})$ we have $g' = gn$ and $g'^{-1}F(g') = n^{-1}g^{-1}F(g)F(n)$. So the element $x = n\mathbf{T} \in W$ fulfils $w' = x^{-1}wF(x)$. In this case we call w and w' F -conjugate.

Proposition 3.22. [MT11, Propositions 25.1, 25.3] *Let \mathbf{G} be a connected reductive group with Steinberg endomorphism F and $\mathbf{T} \leq \mathbf{G}$ an F -stable maximal torus with corresponding Weyl group W . Then the following hold:*

- a) $\{\mathbf{G}^F\text{-classes of } F\text{-stable maximal tori of } \mathbf{G}\} \leftrightarrow \{F\text{-conjugacy classes in } W\}$.
- b) *Let $w := g^{-1}F(g)\mathbf{T} \in W$ and $\mathbf{T}_w = g\mathbf{T}g^{-1}$. Then the \mathbf{T}_w corresponds to the F -class of w via the map of a) and $N_{\mathbf{G}^F}(\mathbf{T}_w)/\mathbf{T}_w^F \cong C_W(w\phi)$.*

Example 3.23. Let $\tilde{\mathbf{G}} = \mathrm{GL}_n$, F_q the standard Frobenius and $\mathbf{T} := \mathbf{D}_n$ the maximally split torus. Let $g \in \tilde{\mathbf{G}}$ such that

$$g^{-1}F(g) = \begin{pmatrix} 0 & \dots & 0 & 1 \\ 1 & & & 0 \\ 0 & \ddots & & \vdots \\ 0 & 0 & 1 & 0 \end{pmatrix} \in \tilde{\mathbf{G}}.$$

Such an element exists by Lang-Steinberg. Let $\mathbf{T}_w = g\mathbf{T}g^{-1}$ be an F_q -stable maximal torus of $\tilde{\mathbf{G}}$. Then F_q acts on \mathbf{T}_w as $F := wF_q$ acts on \mathbf{T} . So for

$$t \in \mathbf{T}, t = \mathrm{diag}(t_1, \dots, t_n) : F(t) = \mathrm{diag}(t_n^q, t_1^q, \dots, t_{n-1}^q).$$

Hence, $\mathbf{T}^{wF_q} = \{\mathrm{diag}(t_1, \dots, t_n) \mid t_2 = t_1^q, t_3 = t_1^{q^2}, \dots, t_n = t_1^{q^{n-1}}, t_1^{q^n} = t_1\} \cong \mathbb{F}_{q^n}^\times$, as w is a preimage of an n -cycle in $W = S_n$. Hence $\mathbf{T}^F \cong \mathbb{F}_{q^n}^\times$ and $|\mathbf{T}^F| = q^n - 1$. Furthermore, by the Lemma above we have $C_W(w) \cong C_n$.

Generally, given a cycle with partition $(a_1, \dots, a_r) \vdash n$, the corresponding torus \mathbf{T} in GL_n satisfies $\mathbf{T}^F \cong \mathbb{F}_{q^{a_1}}^\times \times \dots \times \mathbb{F}_{q^{a_r}}^\times$.

The d -th cyclotomic polynomial over \mathbb{Q} is the irreducible polynomial $\Phi_d \in \mathbb{Z}[x]$ with primitive d -th roots of unity as zeros, given by

$$x^m - 1 = \prod_{d|m, d \geq 1} \Phi_d(x).$$

We can (re)-factorize the order formula (3.3) to obtain:

$$|\mathbf{G}| = x^{|\Phi^+|} \prod_{d \geq 1} \Phi_d(x)^{a(d)} \quad (3.4)$$

for suitable integers $a(d) \geq 0$, see [MT11, 25.1].

Definition 3.24. Let \mathbf{G} be a connected reductive group with Steinberg endomorphism $F : \mathbf{G} \rightarrow \mathbf{G}$. An F -stable torus $\mathbf{S} \leq \mathbf{G}$ is called d -torus of (\mathbf{G}, F) if its generic order equals

$$|\mathbf{S}| = \Phi_d(x)^a$$

for some $a \geq 0$. Here \mathbf{S} is the complete root datum with respect to (\mathbf{S}, F) . In particular, we have $|\mathbf{S}^F| = \Phi_d(q)^a$. The torus \mathbf{S} is called a *Sylow d -torus* of (\mathbf{G}, F) if $|\mathbf{S}| = \Phi_d(x)^{a(d)}$ for the integer $a(d)$ from (3.4).

Example 3.25. For $\mathbf{G} = \mathrm{GL}_n(\mathbf{k})$ and $F_q : \mathbf{G} \rightarrow \mathbf{G}$ the standard Frobenius, the maximally split torus $\mathbf{T} := \mathbf{D}_n$ of diagonal matrices is a Sylow 1-torus with

$$|\mathbf{T}^F| = (q-1)^n \text{ for } |\mathrm{GL}_n(q)| = q^{\binom{n}{2}} \prod_{i=1}^n (q^i - 1).$$

Replacing q by $-q$ as in Example 3.14, one obtains the order of the general unitary group. Note that the generic order of a Sylow d -torus of \mathbf{G} is given by $\Phi_d(x)^{\lfloor \frac{n}{d} \rfloor}$.

The term Sylow torus is justified, as a Sylow-type theorem can be stated for d -tori.

Theorem 3.26. [BM92, Thm. 3.4] *Let \mathbf{G} be connected reductive and $F : \mathbf{G} \rightarrow \mathbf{G}$ a not very twisted Steinberg endomorphism and $d \geq 1$. Then*

- a) *there exists a Sylow d -torus and all Sylow d -tori of (\mathbf{G}, F) are \mathbf{G}^F -conjugate, and*
- b) *any d -torus of \mathbf{G} is contained in some Sylow d -torus.*

We write $W_d := \mathrm{N}_{\mathbf{G}^F}(\mathbf{S}) / \mathrm{C}_{\mathbf{G}^F}(\mathbf{S})$ for the *relative Weyl group*, associated to the Sylow d -torus \mathbf{S} of (\mathbf{G}, F) .

In case \mathbf{S} is a Sylow d -torus of (\mathbf{G}, F) and its centralizer $\mathrm{C}_{\mathbf{G}}(\mathbf{S})$ is abelian, we say that \mathbf{S} is *regular* and that the integer d is a *regular number* of (\mathbf{G}, F) .

Example 3.27. By [Spr74, Section 5], the regular numbers of the group $\mathbf{G} = \mathrm{SL}_n$, with standard Frobenius F , are exactly the integers dividing n or dividing $n-1$. If we assume F to be the twisted Steinberg endomorphism as in Example 3.14, the regular numbers are obtained using the formula

$$\Phi_{2d}(x) = \Phi_d(-x), \quad d \text{ odd,}$$

by replacing q with $-q$. Hence, we obtain the following table for regular numbers of type \mathbf{A}_{n-1} .

Type:	A_{n-1}	${}^2A_{n-1}, 2 \mid n$	${}^2A_{n-1}, 2 \nmid n$
regular numbers d	$d \mid n$ or $d \mid n-1$	$d \mid n$ or $2 \mid d$ and $d \mid 2n-2$	$d \mid n$ or $2 \mid d, d \mid 2n$

We see that d is a regular number for (\mathbf{G}, F) if and only if d divides n or $(n-1)$.

Definition 3.28. Let ℓ be an odd prime, different from the defining prime p . We write $d_\ell(q)$ for the minimal integer such that $\ell \mid q^{d_\ell(q)} - 1$. Then $d_\ell(q)$ is the multiplicative order of q in $(\mathbb{Z}/\ell\mathbb{Z})^\times$.

Each Sylow d -torus of (\mathbf{G}, F) is contained in a maximal torus, with a twist w , such that w has $a(d)$ d -cycles.

Theorem 3.29. [Mal07, Theorem 5.14] *Let \mathbf{G} be quasi-simple with Frobenius endomorphism $F : \mathbf{G} \rightarrow \mathbf{G}$ giving an \mathbb{F}_q structure and $G = \mathbf{G}^F$. Let $\ell \neq 2$ be a prime, $\ell \nmid q$ and S_ℓ a Sylow ℓ -subgroup of G . Let $d := d_\ell(q)$. Then there exists a Sylow d -torus \mathbf{S} of \mathbf{G} such that*

$$N_G(S_\ell) \leq N_G(\mathbf{S}),$$

except if $\ell = 3$ and one of the following holds:

- a) $H = \mathrm{SL}_3(q)$, with $q \equiv 4, 7 \pmod{9}$, or
- b) $H = \mathrm{SU}_3(q)$, with $q \equiv 2, 5 \pmod{9}$, or
- c) $H = \mathrm{G}_2(q)$, with $q = 2, 4, 5, 7 \pmod{9}$.

Example 3.30.

- (i) Assume $\tilde{\mathbf{G}} = \mathrm{GL}_n(\mathbf{k})$, let F be the standard Frobenius and the let integers ℓ and q be as in the previous theorem. Let $d = d_\ell(q)$. We determine the centralizer and relative Weyl group of a Sylow d -torus of (GL_n, F) . In case $d = 1$, we have already determined that the maximal torus of diagonal matrices \mathbf{D}_n is a Sylow 1-torus and

$$C_{\mathrm{GL}_n(q)}(\mathbf{D}_n) = \mathbf{D}_n \cong (\mathrm{GL}_1(q))^n, \text{ see Example 3.25.}$$

The normalizer $N_{\tilde{\mathbf{G}}}(\mathbf{D}_n)$ is given by the group of generalized permutation matrices $\mathbf{D}_n \rtimes \mathfrak{S}_n$. Hence, the relative Weyl group is equal to the Weyl group. Now let $d = n$. In this case the element of $W \cong \mathfrak{S}_n$ corresponding to a Sylow n -torus is any element with cycle type n , see Example 3.23. Hence the Sylow d -torus \mathbf{S} is contained in the cyclic group of order $q^d - 1$ centralized only by itself. Hence by Lemma 3.22

$$C_{\mathrm{GL}_n(q)}(\mathbf{S}) \cong \mathrm{GL}_1(q^d) \text{ and } W_d \cong C_{\mathfrak{S}_n}((1, \dots, n)) = C_n.$$

Given arbitrary d , an element corresponding to a Sylow d -torus is any element with cycle type such that the number of d -cycles is maximal, see [GM20, Ex. 3.5.14]. Hence let $a \cdot d + r = n$ with $d > r \geq 0$. Then the centralizer of \mathbf{S} is isomorphic to the group

$$C_{\mathrm{GL}_n(q)}(\mathbf{S}) \cong \mathrm{GL}_1(q^d)^a \times \mathrm{GL}_r(q) \quad \text{and} \quad W_d \cong C_d \wr \mathfrak{S}_a$$

as all Sylow d -tori are conjugate in $\mathrm{GL}_n(q)$. The corresponding groups in SL_n are obtained by intersection.

- (ii) Now assume that $\tilde{\mathbf{G}} = \mathrm{GL}_n(\mathbf{k})$ and F is the twisted Frobenius from Example 3.14. Then the d -tori, corresponding subgroups and relative Weyl group are obtained through Ennola duality, replacing q with $-q$. In this case one defines the integer d_0 by

$$d_0 = \begin{cases} d & \text{if } d \equiv 0 \pmod{4}, \\ 2d & \text{if } d \equiv \pm 1 \pmod{4}, \\ \frac{d}{2} & \text{if } d \equiv 2 \pmod{4}. \end{cases}$$

In particular, the centralizer and relative Weyl group of a Sylow d -torus of $\mathrm{GU}_n(q) = \mathrm{GL}_n(-q)$ corresponds to the centralizer and relative Weyl group of a Sylow d_0 -torus of $\mathrm{GL}_n(q)$.

- (iii) Furthermore, we see that any Sylow d -torus of (\mathbf{D}_n, F) is also a Sylow d -torus of (\mathbf{G}, F) .

We finish the section with some technical properties of Sylow d -tori, which are of relevance in the later observations. We denote $d_0 = d$ in case $F = F_q$.

Proposition 3.31. *Let $\mathbf{G} = \mathrm{SL}_n(\mathbf{k})$ and $\mathbf{G}_1 = \mathrm{SL}_{n_1}(\mathbf{k})$, $\mathbf{G}_2 = \mathrm{SL}_{n_2}(\mathbf{k})$ subgroups such that $n_1 + n_2 = n$ and $\mathbf{G}_1 \times \mathbf{G}_2 \leq \mathbf{G}$. Let \mathbf{S} be a Sylow d -torus of \mathbf{G}_1 . If $n_2 < d$, then \mathbf{S} is a Sylow d -torus of $\mathbf{G}_1 \times \mathbf{G}_2$. In case $d_0 \mid n_1$ and $n_2 < d_0$, then \mathbf{S} is a Sylow d -torus of \mathbf{G} .*

Proof. This follows immediately from the order formula and the description of Sylow d -tori from the example above. \square

We present one more statement, which will be of importance later on.

Proposition 3.32. *[CS13, Prop. 2.5] Let \mathbf{G} be connected reductive with Steinberg endomorphism F defining an \mathbb{F}_q -structure. Let ℓ be a prime, $\ell \neq p$ and let \mathbf{S} be a Sylow d -torus of (\mathbf{G}, F) with $S_\ell \in \mathrm{Syl}_\ell(\mathbf{G}^F)$ such that $N_{\mathbf{G}^F}(S_\ell) \leq N_{\mathbf{G}^F}(\mathbf{S})$. Then*

$$\mathrm{Aut}(\mathbf{G}^F)_{S_\ell} \leq \mathrm{Aut}(\mathbf{G}^F)_{\mathbf{S}}.$$

4. Conditions for Equivariance

As previously discussed, a proof of the Galois–McKay conjecture is possible by verifying the inductive conditions $(\text{Eq})_{\mathcal{H}_\ell}$ and $(\text{Ext})_{\mathcal{H}_\ell}$ from Definition 1.42 for all non-abelian finite simple groups. In particular, we aim to construct the bijection required in $(\text{Eq})_{\mathcal{H}_\ell}$, where G is the universal covering group of a finite simple group S . Since the condition (Eq) for the McKay conjecture holds, an $\text{Aut}(G)_P$ -equivariant bijection has been constructed using a criterion by Späth in [Spä12, Theorem 2.12].

In the first section of this chapter, we will investigate concretely how the Galois automorphisms of \mathcal{H}_ℓ act on some specific character fields that will appear in the following sections. In particular, we will describe several relationships between the integers ℓ, p, ϵ, q and n , which will be helpful to describe the action of \mathcal{H}_ℓ in various cases.

The second section will be concerned with studying the outer diagonal automorphisms of $G = \text{SL}_n(\epsilon q)$ and how they can be described and compared using two different constructions, one arising from the group $\tilde{G} = \text{GL}_n(\epsilon q)$ and another from the image of the Lang map. Then we will define outer automorphisms of type \mathfrak{t}_ℓ , which will be of importance in the following chapters and play a crucial role in the verification of $(\text{Eq})_{\mathcal{H}_\ell}$. In the last section of this chapter, we will adapt this criterion with respect to the action of \mathcal{H}_ℓ and show how the \mathcal{H}_ℓ -version of the criterion allows the construction of an $\mathcal{H}_\ell \times \text{Aut}(G)_P$ -equivariant bijection in Theorem 4.18.

Following this we continue to refine the conditions to the case where G is a simple group of Lie type over a field of characteristic p , where $p \neq \ell$. The assumptions of Theorem 4.18, then can be adapted and simplified to the three conditions $\text{A}(\infty)_{\mathcal{H}_\ell}$, $\text{A}(d)_{\mathcal{H}_\ell}$ and $\text{B}(d)_{\mathcal{H}_\ell}$. These are the Galois compatible versions of the conditions $\text{A}(\infty)$, $\text{A}(d)$ and $\text{B}(d)$, as introduced in [CS19].

4.A Number-theoretical observations

We remind the reader of the group $\mathcal{H}_\ell \leq \text{Gal}(\mathbb{Q}^{ab}/\mathbb{Q})$ from Definition 1.25, central to the Galois–McKay conjecture. In this section, we collect several notions from number theory that will be used in later chapters to explicitly describe the action of \mathcal{H}_ℓ on certain characters and their fields of values. We may characterize the

group \mathcal{H}_ℓ as the direct product of two subgroups. Let σ_1 be the automorphism that maps each root of unity $\zeta_{\ell'}$ of ℓ' -order to its ℓ -th power while leaving all roots of unity of order a power of ℓ invariant. We further define

$$\mathcal{K} = \{\delta \in \mathcal{H} \mid \delta(\zeta_{\ell'}) = \zeta_{\ell'} \text{ for every } \ell'\text{-root of unity } \zeta_{\ell'} \text{ of } \ell' \text{ - order}\}.$$

Then

$$\mathcal{H}_\ell = \mathcal{K} \times \langle \sigma_1 \rangle. \quad (4.1)$$

Notation 4.1. We fix the following notation for this section. Let p be a prime, $q = p^m$ a prime power, and $\epsilon = \pm 1$. Let ℓ be an odd prime distinct from p , and define $d_0 = d_\ell(\epsilon q)$, as in Definition 3.28, that is, the smallest positive integer d_0 such that

$$\ell \mid (\epsilon q)^{d_0} - 1.$$

Equivalently, d_0 is the order of ϵq in $(\mathbb{Z}/\ell\mathbb{Z})^\times$. In particular, $d_0 \mid \ell - 1$ and thus there exists some positive integer k such that

$$\ell = k \cdot d_0 + 1.$$

Given an arbitrary automorphism $\sigma \in \mathcal{H}_\ell$, we denote by $f(\sigma)$ the integer corresponding to the power of σ_1 appearing in σ , i.e, we write

$$\sigma = \sigma_{\mathcal{K}} \cdot \sigma_1^{f(\sigma)} \text{ for some } \sigma_{\mathcal{K}} \in \mathcal{K}, \text{ see (4.1).}$$

Then σ acts on any root of unity ζ of order ℓ' by $\sigma(\zeta) = \zeta^{\ell' f(\sigma)}$.

The integers d_0 and k encode essential arithmetic information, relating ϵq and ℓ . Thus, the action of \mathcal{H}_ℓ on characters of \mathbf{G}^F and its ℓ -local subgroups can often be described in relation to these integers. In particular, $(2d_0)$ -th roots of unity will be of significant importance in our explicit computations.

Lemma 4.2. *Let ζ_{2d_0} be a primitive $(2d_0)$ -th root of unity and ζ_{d_0} a primitive d_0 -th root of unity. Then:*

- a) $\sigma_1(\zeta_{d_0}) = \zeta_{d_0}$.
- b) $\sigma_1(\zeta_{2d_0}) = (-1)^k \zeta_{2d_0}$.
- c) *The group \mathcal{H}_ℓ acts trivially on the field $\mathbb{Q}(\zeta_{d_0})$. In case $\frac{\ell-1}{d_0}$ is even, it also acts trivially on $\mathbb{Q}(\zeta_{2d_0})$.*

Proof. As we have seen above, we can write $\ell = k \cdot d_0 + 1$, hence

$$\sigma_1(\zeta_{d_0}) = \zeta_{d_0}^\ell = \zeta_{d_0}^{d_0 k} \zeta_{d_0} = \zeta_{d_0}.$$

Similarly for ζ_{2d_0} we have

$$\sigma_1(\zeta_{2d_0}) = \zeta_{2d_0}^\ell = \zeta_{2d_0}^{d_0 k} \zeta_{2d_0} = (-1)^k \zeta_{2d_0}.$$

As \mathcal{K} acts trivially on both ζ_{d_0} and ζ_{2d_0} , we only need to check the action of multiples of σ_1 . As σ_1 only acts non-trivially whenever $k = \frac{\ell-1}{d_0}$ is odd, this shows c). \square

Another important way to capture the connection of our two primes is through the Legendre symbol, which provides information about quadratic residues. We will also introduce the Jacobi symbol, a natural generalization of the Legendre symbol, since in our context we often work with prime powers $q = p^m$.

Definition 4.3. Let p be an odd prime and n an integer. The *Legendre symbol* is the integer given by

$$\left(\frac{n}{p}\right) = \begin{cases} 1, & \text{if } n \text{ is a quadratic residue modulo } p \text{ and } n \not\equiv 0 \pmod{p}, \\ -1, & \text{if } n \text{ is a non-quadratic residue modulo } p, \\ 0 & \text{if } n \equiv 0 \pmod{p}. \end{cases}$$

Let r be any odd positive integer with prime factorization $r = p_1^{a_1} \cdots p_j^{a_j}$. Then the *Jacobi symbol* is the integer given by

$$\left(\frac{n}{r}\right) := \left(\frac{n}{p_1}\right)^{a_1} \cdots \left(\frac{n}{p_j}\right)^{a_j}.$$

Note that one could further generalize this to the Kronecker symbol, for even and negative integers. However we assume all primes to be odd, and by definition $\left(\frac{\ell}{-q}\right) = \left(\frac{\ell}{q}\right)$ for the Kronecker symbol, hence the sign does not appear in our calculations regardless.

Proposition 4.4. Let p and ℓ be two distinct odd primes, let a, b be two integers, and let $q = p^m$ be some prime power. Then the following hold:

- a) $\left(\frac{a}{p}\right) \equiv a^{\frac{p-1}{2}} \pmod{p}$.
- b) $\left(\frac{a}{q}\right)\left(\frac{b}{q}\right) = \left(\frac{ab}{q}\right)$.
- c) $\left(\frac{a}{b}\right)\left(\frac{b}{a}\right) = (-1)^{\frac{a-1}{2}\frac{b-1}{2}}$ with $2 \nmid ab$ and $a, b > 0$.

Proof. See [IR90, Proposition 5.1.2., 5.2.2., 5.3.3.] □

Let $\eta \in \{\pm 1\}$ be the integer such that $p \equiv \eta \pmod{4}$. Another character field that will be of importance later on is $\mathbb{Q}(\sqrt{\eta q})$, hence we need a way to describe the action of \mathcal{H}_ℓ on this square root. Fortunately, this action can be described in arithmetic terms using the *quadratic Gauss sum*, which is defined as

$$g(a, p) := \sum_{i=1}^{p-1} \left(\frac{i}{p}\right) \zeta_p^{a \cdot i}.$$

Proposition 4.5. [IR90, Proposition 6.3.2] Let p be a prime. Then

$$g(1, p)^2 = (-1)^{\frac{p-1}{2}} p$$

and for $p \equiv \eta \pmod{4}$ we have $g(1, p) = \sqrt{\eta p}$.

We observe that the sum $\sum_{i=0}^{p-1} \left(\frac{i}{p}\right)$ equals zero, as there is the same amount of quadratic and non-quadratic residues of p . Hence, for $p \mid a$, the Gauss sum equals zero and in particular $g(a, p) = \left(\frac{a}{p}\right) g(1, p)$. In case $p \nmid a$, we observe

$$\left(\frac{a}{p}\right) g(a, p) = \sum_{i=0}^{p-1} \left(\frac{i \cdot a}{p}\right) \zeta_p^{a \cdot i}$$

and since i as well as $a \cdot i$ run over the full set of possible residues mod p , once again the following equality holds

$$g(a, p) = \left(\frac{a}{p}\right) g(1, p). \quad (4.2)$$

Using this, we obtain a clear description of the action of \mathcal{H}_ℓ on $\mathbb{Q}(\sqrt{\eta p})$.

Lemma 4.6. *Let ℓ and p be two distinct odd primes and $\eta = \pm 1$ such that $p \equiv \eta \pmod{4}$. Then $\mathbb{Q}(\sqrt{\eta p})$ is \mathcal{K} -fixed. Furthermore, $\mathbb{Q}(\sqrt{\eta p})$ is \mathcal{H}_ℓ -fixed if and only if $\left(\frac{\ell}{p}\right) = 1$.*

Proof. Let $\sigma \in \mathcal{H}_\ell = \mathcal{K} \times \langle \sigma_1 \rangle$. If $\sigma = \sigma_1$, then by Proposition 4.4 b)

$$\sigma_1(\sqrt{\eta p}) = \sum_{i=0}^{p-1} \left(\frac{i}{p}\right) \sigma_1(\zeta_p^i) = \sum_{i=0}^{p-1} \left(\frac{\ell}{p}\right)^2 \left(\frac{i}{p}\right) \zeta_p^{\ell \cdot i} = \left(\frac{\ell}{p}\right) \sqrt{\eta p}.$$

Hence, σ_1 acts trivially on $\sqrt{\eta p}$ if and only if $\left(\frac{\ell}{p}\right) = 1$. As \mathcal{K} acts trivially on roots of unity of order ℓ' , it fixes $\sqrt{\eta p}$ and the action of \mathcal{H}_ℓ is completely determined by the Legendre symbol via σ_1 . \square

Note that the relation between $d_\ell(q)$ and $d_\ell(-q)$ has already been observed via Ennola duality and the properties of cyclotomic polynomials (see Definition 3.21), however, we give a purely arithmetic proof here.

Lemma 4.7. *The following relation between $d_\ell(-q)$ and $d_\ell(q)$ holds:*

$$d_\ell(-q) = \begin{cases} 2d_\ell(q) & \text{if } d_\ell(q) \text{ is odd,} \\ d_\ell(q)/2 & \text{if } d_\ell(q) \equiv 2 \pmod{4}, \\ d_\ell(q) & \text{if } d_\ell(q) \equiv 0 \pmod{4}. \end{cases}$$

Furthermore, if q is a square, then $\frac{\ell-1}{d_\ell(q)}$ is even.

Proof. Let $d_\ell(q)$ be odd, then $-q^{d_\ell(q)} = (-q)^{d_\ell(q)} \equiv -1 \pmod{\ell}$, hence $d_\ell(-q) = 2d_\ell(q)$ and is minimal with respect to this property. In case $d_\ell(q) \equiv 2 \pmod{4}$, we may write $d_\ell(q) = 2\delta$, with odd δ and have $q^\delta \equiv -1$ which implies $d_\ell(-q) = \delta$. Finally in case $d_\ell(q)$ is divisible by 4, like before it is clear that $d_\ell(-q) = d_\ell(q)$.

Now assume that $q = p^{2m}$ is a square. Then $q^{d_\ell(q)} \equiv p^{2md_\ell(q)} \equiv 1 \pmod{\ell}$ and $2d_\ell(q) \mid \ell - 1$. Thus $\frac{\ell-1}{d_\ell(q)}$ is even. \square

Finally we can give an explicit connection between the Jacobi symbol $\left(\frac{\ell}{q}\right)$ and the integer d_0 .

Lemma 4.8. *Let ℓ and p be two distinct primes, ℓ odd, $q = p^m$ a prime power of p , $\epsilon \in \{\pm 1\}$ and $d_0 = d_\ell(\epsilon q)$, then the following holds:*

$$\left(\frac{\ell}{q}\right) = -1 \Leftrightarrow \begin{cases} (\ell - 1)_2 = (d_0)_2 \geq 4, \text{ or} \\ (\ell - 1)_2 = (d_0)_2 = 2, q \equiv \epsilon \pmod{4}, \text{ or} \\ (\ell - 1)_2 = 2, (d_0)_2 = 1, q \equiv -\epsilon \pmod{4}. \end{cases}$$

Proof. Assume first that q is a square. In this case $\left(\frac{\ell}{q}\right) = 1$, so we show that none of the above cases can occur. Indeed, by Lemma 4.7, if $\epsilon = 1$, then $(\ell - 1)_2 > (d_0)_2$ and the first two cases can not occur. If q is square, $q \equiv 1 \not\equiv -\epsilon \pmod{4}$, so the statement holds. If $\epsilon = -1$, then by Lemma 4.7, the only case we need to check is $(\ell - 1)_2 = (d_0)_2 = 2$, $q \equiv \epsilon \pmod{4}$. However once again q is square, so $q \equiv 1 \not\equiv \epsilon \pmod{4}$.

We continue with the case that $q = p^m$ is an odd power of p . In particular, we may just assume that $p = q$, as $\left(\frac{\ell}{q}\right) = \left(\frac{\ell}{p}\right)^m = \left(\frac{\ell}{p}\right)$ and $(d_\ell(\epsilon p))_2 = (d_\ell(\epsilon q))_2$ in this case.

According to Proposition 4.4, we can write

$$\left(\frac{\ell}{p}\right) \equiv \left(\frac{p}{\ell}\right) (-1)^{\frac{(\ell-1)(p-1)}{4}} \equiv p^{\frac{\ell-1}{2}} (-1)^{\frac{(\ell-1)(p-1)}{4}} \pmod{\ell}.$$

By definition d_0 is minimal such that $(\epsilon p)^{d_0} \equiv 1 \pmod{\ell}$ and $k \in \mathbb{Z}$ is such that $\ell - 1 = d_0 k$, so we have

$$p^{\frac{\ell-1}{2}} \equiv \epsilon^{\frac{d_0 k}{2}} (\epsilon p)^{\frac{d_0 k}{2}} \equiv \epsilon^{\frac{\ell-1}{2}} (-1)^{\frac{\ell-1}{d_0}} \pmod{\ell}.$$

Hence

$$\left(\frac{\ell}{p}\right) = \epsilon^{\frac{\ell-1}{2}} (-1)^{\frac{\ell-1}{d_0}} (-1)^{\frac{(p-1)(\ell-1)}{4}}. \quad (4.3)$$

It is now straightforward to check that $\left(\frac{\ell}{p}\right) = -1$ holds if and only if one of the three cases above holds. \square

4.B Comparing outer automorphisms

Let $\mathbf{G} = \mathrm{SL}_n(\mathbf{k})$, $\tilde{\mathbf{G}} = \mathrm{GL}_n(\mathbf{k})$ and let $F : \tilde{\mathbf{G}} \rightarrow \tilde{\mathbf{G}}$, $F \in \{F_q, \gamma \circ F_q\}$ be a Steinberg endomorphism defining an \mathbb{F}_q -structure on $\tilde{\mathbf{G}}$ and \mathbf{G} as in Notation 3.15. By Theorem 3.19, the automorphism group of \mathbf{G}^F is generated by the action of $\tilde{\mathbf{G}}^F$ together with the graph and field automorphisms γ and F_p .

Let $\epsilon \in \{\pm 1\}$ such that $F = \gamma^{\frac{1-\epsilon}{2}} \circ F_q$ and write $G := \mathbf{G}^F = \mathrm{SL}_n(\epsilon q)$, $\tilde{G} := \tilde{\mathbf{G}}^F = \mathrm{GL}_n(\epsilon q)$. Let $\mathcal{L} : \mathbf{G} \rightarrow \mathbf{G}$ be the Lang map given by $x \mapsto x^{-1}F(x)$.

As \mathcal{H}_ℓ acts on the characters of G and its subgroups, there often exists some automorphism $\tau \in \mathrm{Aut}(G)$ (stabilizing the subgroup), such that the action of τ and $\sigma \in \mathcal{H}_\ell$ coincide on all irreducible characters. Thus, in this section, we want to study the automorphisms of \mathbf{G}^F induced by the action of $\tilde{\mathbf{G}}^F$ and give two ways to characterize/parametrize them, hoping to relate them to the action of \mathcal{H}_ℓ later on.

We write $\mathrm{Diag}(G)$ for the automorphisms of G induced by \tilde{G} or $(\mathbf{G}/\mathbf{Z}(\mathbf{G}))^F \cong \tilde{G}/\mathbf{Z}(\tilde{\mathbf{G}})^F$. We consider the subgroup of the *inner automorphism* $\mathrm{Inn}(G)$ induced

by conjugation with an element of G . As we are interested in the action of an automorphism on the characters of $\text{Irr}(G)$, it suffices to consider the group of *outer diagonal automorphisms* given by

$$\text{Out}'(G) := \text{Diag}(G)/\text{Inn}(G).$$

In the standard literature, see for example [Bon06, Chapter 6], the outer diagonal automorphisms are parametrized via the group

$$Z(\mathbf{G})_F := Z(\mathbf{G})/[Z(\mathbf{G}), F].$$

Note that by Lang's theorem 3.12 there exists a bijection

$$(\mathbf{G}Z(\tilde{\mathbf{G}}))^F/(\mathbf{G}^F Z(\tilde{\mathbf{G}})^F) \rightarrow Z(\mathbf{G})_F.$$

We write $\text{int}(g) \in \text{Aut}(G)$ for the automorphism of G defined by conjugation with $g \in \tilde{\mathbf{G}}$ and $\text{out}(g)$ for its image in the group $\text{Out}'(G)$. Also note that by Lang's Theorem, for each $z \in Z(\mathbf{G})$, there exists some $g_z \in \mathbf{G}$ such that $g_z^{-1}F(g_z) \in z[Z(\mathbf{G}), F]$.

Proposition 4.9. *For each $z \in Z(\mathbf{G})_F$ fix some $g_z \in \mathbf{G}$ such that $g_z^{-1}F(g_z)[Z(\mathbf{G}), F] = z$. Then the map*

$$Z(\mathbf{G})_F \hookrightarrow \text{Out}'(G), \quad z \mapsto \text{out}(g_z) := \text{int}(g_z) \cdot \text{Inn}(G).$$

is a well-defined injective group homomorphism.

Proof. First let $z \in Z(\mathbf{G})_F$ and let $g, h \in \mathbf{G}$ such that $g^{-1}F(g)$ and $h^{-1}F(h)$ are in $Z(\mathbf{G})$ and both elements represent z . There exists some $x \in Z(\mathbf{G})$ such that $x^{-1}F(x)h^{-1}F(h) = g^{-1}F(g)$. As x lies in the centre, it follows that

$$F(gh^{-1}x^{-1}) = gh^{-1}x^{-1} \in \mathbf{G}^F.$$

Thus $\text{int}(g) = \text{int}(gh^{-1}x^{-1})\text{int}(hx) = \text{int}(gh^{-1}x^{-1})\text{int}(h)$. As $gh^{-1}x^{-1} \in \mathbf{G}^F$, the image under the natural map $\text{Aut}(G) \rightarrow \text{Aut}(G)/\text{Inn}(G)$ coincides for $\text{int}(g)$ and $\text{int}(h)$. Thus the map is well-defined. Let $z_1, z_2 \in Z(\mathbf{G})_F$ with corresponding $g_{z_1}, g_{z_2} \in \mathbf{G}$. Then the element $g_{z_1 z_2}$ can be chosen as $g_{z_2} g_{z_1}$. Hence

$$\text{out}(g_{z_1 z_2}) = \text{int}(g_{z_1 z_2}) \cdot \text{Inn}(G) = \text{int}(g_{z_2} g_{z_1}) \cdot \text{Inn}(G) = \text{int}(g_{z_1})\text{int}(g_{z_2}) \cdot \text{Inn}(G).$$

So the homomorphism property holds. Let $z \in Z(\mathbf{G})_F$ and $g_z \in \mathbf{G}$ as before. Suppose that $\text{out}(g_z)$ is trivial, so there exists some $h \in \mathbf{G}^F$ such that $h^{-1}g_z \in C_{\mathbf{G}}(\mathbf{G}^F) = Z(\mathbf{G})$, see [Bon06, 6.1]. Then for $a = h^{-1}g_z$ we have $g_z^{-1}F(g_z) = (ha)^{-1}F(ha) = a^{-1}F(a) \in [Z(\mathbf{G}), F]$. Thus, $z = 1$ and the map is injective. \square

In our considerations, we aim to describe and compare automorphisms $\text{int}(g)$ given by certain $g \in \tilde{\mathbf{G}}$. In particular, we describe the action first for $g \in \tilde{\mathbf{G}}^F$ and for $g \in \mathbf{G}$ with $\mathcal{L}(g) \in Z(\mathbf{G})$.

We write

$$C_{q-\epsilon} := \{\zeta \in \mathbb{F}_{q^2}^\times \mid \zeta^{q-\epsilon} = 1\} \tag{4.4}$$

for the cyclic group of order $q - \epsilon$ in $\overline{\mathbb{F}}_q^\times$. Let $(n, q - \epsilon)$ denote the greatest common divisor. Then

$$C_{(n, q-\epsilon)} := \{\zeta \in C_{q-\epsilon} \mid \zeta^n = 1\}, \tag{4.5}$$

is the cyclic group of order $(n, q - \epsilon)$.

Proposition 4.10. *The group of outer diagonal automorphisms $\text{Out}'(G)$ is parametrized by $C_{(n,q-\epsilon)}$ via the homomorphism*

$$\tilde{\pi} : \tilde{\mathbf{G}}^F \rightarrow C_{(n,q-\epsilon)}, \quad x \mapsto \det(x)^{\frac{q-\epsilon}{(n,q-\epsilon)}}.$$

Proof. Let $x \in \tilde{\mathbf{G}}^F$ with $\det(x) = \zeta$. Then $\zeta^{q-\epsilon} = 1$, which expresses the fact that $F(\det(x)) = \det(F(x)) = \det(x)$. Thus, we have $\tilde{\pi}(x) = \zeta^{\frac{q-\epsilon}{(n,q-\epsilon)}}$, as well as

$$\left(\zeta^{\frac{q-\epsilon}{(n,q-\epsilon)}}\right)^n = \zeta^{(q-\epsilon)\frac{n}{(n,q-\epsilon)}} = 1, \text{ and}$$

$$\left(\zeta^{\frac{q-\epsilon}{(n,q-\epsilon)}}\right)^{q-\epsilon} = \zeta^{(q-\epsilon)\frac{q-\epsilon}{(n,q-\epsilon)}} = 1,$$

thus, the map is well-defined and clearly surjective. In case $\tilde{\pi}(x) = \zeta^{\frac{q-\epsilon}{(n,q-\epsilon)}} = 1$, there exists some $\zeta_0 \in C_{q-\epsilon}$ such that $\det(x) = \zeta = \zeta_0^n$. In particular, $\zeta_0 \cdot E_n \in Z(\tilde{\mathbf{G}})^F$ and $x \cdot \zeta_0^{-1} \cdot E_n = x_0 \in \mathbf{G}^F$, where E_n is the identity matrix. Hence, $x \in \mathbf{G}^F Z(\tilde{\mathbf{G}})^F$ and $\text{out}(x) = 1$, so x acts as an inner diagonal automorphism on G . \square

We define the group

$$\check{\mathbf{G}} := \mathcal{L}^{-1}(Z(\mathbf{G})) = \{g \in \mathbf{G} \mid g^{-1}F(g) \in Z(\mathbf{G})\} \triangleright \mathbf{G}^F.$$

For $x \in \mathcal{L}^{-1}(Z(\mathbf{G}))$ we identify $\mathcal{L}(x) = z \cdot E_n \in Z(\mathbf{G})$ with the corresponding element $z \in \overline{\mathbb{F}}_q^\times$.

Proposition 4.11. *The group of outer diagonal automorphisms $\text{Out}'(G)$ is parametrized by $C_{(n,q-\epsilon)}$ via the homomorphism*

$$\check{\pi} : \check{\mathbf{G}} \rightarrow C_{(n,q-\epsilon)}, \quad x \mapsto \mathcal{L}(x)^{\frac{-\epsilon n}{(n,q-\epsilon)}}.$$

Proof. Let $\check{x} \in \check{\mathbf{G}}$, and $\mathcal{L}(\check{x}) = z \in Z(\mathbf{G})$ such that $F(\check{x}) = \check{x}z$. Then for $g \in G$ we have

$$F(\check{x}^{-1}g\check{x}) = z^{-1}\check{x}^{-1}g\check{x}z = \check{x}^{-1}g\check{x},$$

and $\text{int}(\check{x})|_G \in \text{Aut}(G)$. As $\mathcal{L}(\check{x}) \in Z(\mathbf{G})$, $z^n = 1$, so $\check{\pi}(\check{x}) \in C_{(n,q-\epsilon)}$ since

$$\left(z^{\frac{-\epsilon n}{(n,q-\epsilon)}}\right)^n = z^{n\frac{-\epsilon n}{(n,q-\epsilon)}} = 1 \text{ and}$$

$$\left(z^{\frac{-\epsilon n}{(n,q-\epsilon)}}\right)^{q-\epsilon} = z^{(-\epsilon n)\frac{q-\epsilon}{(n,q-\epsilon)}} = 1,$$

and the map is well-defined. Since the Lang map is a homomorphism on $\check{\mathbf{G}}$, $\check{\pi}$ is a group homomorphism. Clearly, by definition of $\check{\mathbf{G}}$, \mathcal{L} induces a surjective map $\check{\mathbf{G}} \rightarrow Z(\mathbf{G})$ hence the map $\check{\pi}$ is surjective. In particular, for $\mathcal{L}(\check{x}) = z$ with

$$\check{\pi}(\check{x}) = \left(\mathcal{L}(x)^{-\epsilon}\right)^{\frac{n}{(n,q-\epsilon)}} = \left(z^{-\epsilon}\right)^{\frac{n}{(n,q-\epsilon)}} = 1,$$

there exists a $(q-\epsilon)$ -th root $z_0 \in Z(\mathbf{G})$, such that $z = z_0^{(q-\epsilon)}$. As F is a Steinberg endomorphism, we have $\mathcal{L}(z_0) = z_0^{-1}F(z_0) = z_0^{\epsilon(q-\epsilon)}$. Hence,

$$\mathcal{L}(\check{x})\mathcal{L}(z_0^\epsilon) = \mathcal{L}(\check{x}z_0^\epsilon) = 1$$

and $\check{x}z_0^\epsilon \in \mathbf{G}^F$. Thus, $\check{x} \in \mathbf{G}^F Z(\tilde{\mathbf{G}})$ and $\text{out}(\check{x}) = 1$. \square

We remark that the use of the sign $-\epsilon$ in the above map is not strictly necessary. However this choice gives us the following result, which lets us compare the diagonal automorphism parametrized by \check{G} and \tilde{G} . Note that both \tilde{G} and \check{G} normalize G and each other since they are intermediate subgroups in the inclusion $\mathbf{G}^F \triangleleft \tilde{\mathbf{G}}^F Z(\tilde{\mathbf{G}})$ with abelian quotient.

Lemma 4.12. *Let $\check{\pi}$ and $\tilde{\pi}$ be the two maps parametrizing $\text{Out}'(G)$ from the previous two propositions. Then any two elements $\tilde{x} \in \tilde{G}$ and $\check{x} \in \check{G}$ with $\tilde{\pi}(\tilde{x}) = \check{\pi}(\check{x})$ induce the same outer diagonal automorphism on G . The group $\check{G}\tilde{G}$ acts by conjugation on G and $y \in \check{G}\tilde{G}$ corresponds to the outer diagonal automorphism, parametrized by*

$$\pi : \check{G}\tilde{G} \rightarrow C_{(n, q-\epsilon)}, \quad y \mapsto \mathcal{L}(y)^{\frac{\epsilon n}{(n, q-\epsilon)}} \det(y)^{\frac{q-\epsilon}{(n, q-\epsilon)}}.$$

Proof. Let $\tilde{x} \in \tilde{G}$, with $\det(\tilde{x}) = \zeta \in \mathbb{C}_{q-\epsilon}$. We choose a n -th root ζ' of ζ in $\overline{\mathbb{F}}_q^\times$ and write $z := \zeta' E_n \in Z(\tilde{\mathbf{G}})$. Then \tilde{x} and $\tilde{x}z^{-1}$ have the same action on G and $\tilde{x}z^{-1} \in \mathbf{G}$. Now

$$\mathcal{L}(\tilde{x}z^{-1}) = \tilde{x}^{-1} z F(\tilde{x}) F(z^{-1}) = z^{1-\epsilon q} \in Z(\mathbf{G}),$$

so $\tilde{x}z^{-1} \in \check{G}$ and since $z^n = \zeta \cdot E_n$ we have

$$\check{\pi}(\tilde{x}z^{-1}) = \mathcal{L}(\tilde{x}z^{-1})^{\frac{-\epsilon n}{(n, q-\epsilon)}} = (z^{1-\epsilon q})^{\frac{-\epsilon n}{(n, q-\epsilon)}} = (z^{q-\epsilon})^{\frac{n}{(n, q-\epsilon)}} = \zeta^{\frac{q-\epsilon}{(n, q-\epsilon)}} = \tilde{\pi}(\tilde{x}).$$

As for any $\check{x} \in \check{G}$ with $\check{\pi}(\check{x}) = \check{\pi}(\tilde{x}z^{-1})$, $\check{x}^{-1}\check{x}$ acts as an inner automorphism in G , the first statement follows.

By Propositions 4.10 and 4.11 and the above considerations, $\check{G}\tilde{G}$ acts by conjugation and π is a well-defined homomorphism. \square

Definition 4.13. Let $\ell \neq p$ be two distinct odd primes, $\epsilon = \pm 1$ and $q = p^m$ for some integer m . Let F be a Steinberg endomorphism such that $\mathbf{G}^F = \text{SL}_n(\epsilon q)$. Then for $x \in \check{G}\tilde{G}$ with image

$$\pi(x) = \left(\frac{\ell}{q} \right)^{\frac{n(n-1)}{(n, q-\epsilon)}},$$

we say that x induces the outer automorphism of G of type $\mathfrak{t}_\ell = \mathfrak{t}_\ell(G)$. In the case where $p = 2$, since the Jacobi symbol is not defined in this situation, we say that x induces the outer automorphism of type \mathfrak{t}_ℓ if it induces the trivial outer diagonal automorphism.

In particular, an automorphism of type \mathfrak{t}_ℓ is either an inner automorphism, or the unique outer diagonal automorphism of G of order 2. Furthermore, we may determine the elements of \check{G} and \tilde{G} that induce the automorphism of type \mathfrak{t}_ℓ of G .

Lemma 4.14. *Let $G = \text{SL}_n(\epsilon q)$ and $\ell \neq p$ a prime. We may interpret ℓ as an element in \mathbb{F}_q^\times . Let $\kappa \in \mathbb{F}_{q^2}^\times$, such that $\kappa^{q+1} = \ell$ and $\delta_\ell := \kappa^{q+\epsilon}$. Then any element $\tilde{x} \in \tilde{G}$ with $\det(\tilde{x}) = \delta_\ell^{\frac{n(n-1)}{2}}$ and any $\check{x} \in \check{G}$ with $\mathcal{L}(\check{x}) = \left(\frac{\ell}{q} \right)^{n-1} E_n$ induces the outer automorphism of type \mathfrak{t}_ℓ of G .*

Proof. Using the identity $\left(\frac{\ell}{q}\right) \equiv \ell^{\frac{q-1}{2}}$ in $\mathbb{F}_{q^2}^\times$ from the previous section, we simply calculate

$$\begin{aligned} \pi(\tilde{x}) &= \det(\tilde{x})^{\frac{q-\epsilon}{(n,q-\epsilon)}} = \delta_\ell^{\frac{n(n-1)}{2} \frac{q-\epsilon}{(n,q-\epsilon)}} \\ &= \kappa^{\frac{q^2-1}{2} (n-1) \frac{n}{(n,q-\epsilon)}} \\ &= \ell^{\frac{q-1}{2} (n-1) \frac{n}{(n,q-\epsilon)}} \\ &= \left(\left(\frac{\ell}{q} \right)^{n-1} \right)^{\frac{n}{(n,q-\epsilon)}} \\ &= \mathcal{L}(\check{x})^{\frac{n}{(n,q-\epsilon)}} = \pi(\check{x}). \end{aligned}$$

So by Lemma 4.12 \tilde{x} and \check{x} induce the same outer diagonal action on G . \square

Remark 4.15. Let $p = 2$, $q = p^m$, ℓ an odd prime and let $\kappa^{q+1} = \ell$ and $\delta_\ell = \kappa^{q+\epsilon}$ in $\mathbb{F}_{q^2}^\times$ as above. Let $\tilde{x} \in \tilde{G}$ be an element satisfying $\det(\tilde{x}) = \delta_\ell^{\frac{n(n-1)}{2}}$. Since any odd prime ℓ corresponds to the identity in \mathbb{F}_p^\times , we may take $\kappa = 1$, and hence $\delta_\ell = 1$. Hence, $\det(\tilde{x}) = 1$, which coincides with the definition of an element of type \mathfrak{t}_ℓ above.

4.C Approaching the equivariance statement

To achieve our main goal of verifying the condition $(\text{Eq})_{\mathcal{H}_\ell}$, as described in Definition 1.42, for certain groups of Lie type, we will begin this section by reformulating it. Indeed, we will break it down into several properties that the groups in question and their extension maps need to fulfil. Particularly we will first define an \mathcal{H}_ℓ -equivariant version of the conditions introduced in [CS17, Theorem 2.2] and then rewrite them as the three conditions $\text{A(d)}_{\mathcal{H}_t}$, $\text{B(d)}_{\mathcal{H}_\ell}$ and $\text{A}(\infty)_{\mathcal{H}_t}$, which will be the \mathcal{H}_ℓ -equivariant versions of the conditions found in [CS19, Definition 2.2]. This Galois-compatible version notably grants us some flexibility in selecting the automorphism group, which will be very beneficial in the following chapters. Indeed, moving forward, we will frequently work with groups as in the following definition.

Definition 4.16. Let ℓ be a prime and \mathcal{H}_ℓ be the subgroup of $\mathcal{G} := \text{Gal}(\mathbb{Q}^{ab}/\mathbb{Q})$ as defined in Definition 1.25. That is $\sigma \in \mathcal{H}_\ell$ sends all roots of unity of ℓ' -order to some ℓ -power. Let $n = \ell^a m$, where ℓ does not divide m . Then any n -th root of unity ζ can be written as $\zeta = \omega \delta$ for ω an element of order dividing ℓ^a and δ of order dividing m . Then we have

$$\mathcal{G} \cong \mathcal{K} \times \mathcal{J},$$

where we define the subgroups

$$\mathcal{J} := \{\sigma \in \mathcal{G} \mid \sigma(\delta) = \delta \text{ for all roots of unity } \delta \text{ of } \ell'\text{-order}\}, \text{ and}$$

$$\mathcal{K} := \{\sigma \in \mathcal{G} \mid \sigma(\omega) = \omega \text{ for all roots of unity } \omega \text{ of } \ell\text{-power order}\}.$$

Let $\sigma_1 \in \mathcal{G}$ with $\sigma_1(\delta) = \delta^\ell$ for all roots of unity δ of ℓ' -order. It is obvious that σ_1 generates \mathcal{J} , hence $\mathcal{H}_\ell = \mathcal{K} \times \langle \sigma_1 \rangle$.

Generally we will not consider \mathcal{H}_ℓ directly but rather embed it into a group $B \leq \tilde{G} \times \mathcal{H}_\ell$ acting on their characters. A special version of this group will be given by one element $t \in \tilde{\mathbf{G}}$ with $F(t)t^{-1} \in Z(\mathbf{G})$, acting on G by conjugation and the corresponding group will be labelled as

$$\mathcal{H}_t := \mathcal{K} \times \langle t \cdot \sigma_1 \rangle.$$

We now repeat the original conditions to obtain both inductive conditions for the McKay conjecture. Note that these have been originally stated in [Spä12, Theorem 2.12]. We will however introduce the reader to a slight adaptation of these results, similar to the one in [CS25, Proposition 2.12].

Theorem 4.17. *[CS25, Theorem 2.12]/[Spä12, Proposition 2.12] Let G be a finite group, ℓ a prime such that ℓ divides $|G|$ and $P \in \text{Syl}_\ell(G)$. Let $M < G$ be a proper subgroup of G . Let $\mathbb{G} \subset \text{Irr}(G)$ and $\mathbb{M} \subset \text{Irr}(M)$. Assume the following.*

- (i) *There exists a finite group $G \triangleleft \tilde{G}$ such that \tilde{G}/G is abelian. Furthermore, there exists a finite group E such that $\tilde{G} \rtimes E$ is well-defined, normalizes G , induces all automorphisms of G and $C_{\tilde{G} \rtimes E}(G) = Z(\tilde{G})C_E(G)$*
- (ii) *M is $(\tilde{G} \rtimes E)_P$ -stable, $N_G(P) \leq M$ and both \mathbb{G} and \mathbb{M} are $(\tilde{G} \rtimes E)_M$ -stable.*
- (iii) *Maximal extensibility holds with respect to $G \triangleleft \tilde{G}$ and $M \triangleleft \tilde{M} := N_{\tilde{G}}(M)$ for \mathbb{G} and \mathbb{M} respectively.*
- (iv) *There exists an E -stable \tilde{G} -transversal $\mathbb{G}_0 \subset \mathbb{G}$ and each $\chi \in \mathbb{G}_0$ extends to $(G \rtimes E)_\chi$.*
- (v) *For $\hat{M} := N_{G \rtimes E}(M)$ there exists some \hat{M} -stable \tilde{M} -transversal \mathbb{M}_0 in \mathbb{M} such that each $\psi \in \mathbb{M}_0$ extends to \hat{M}_ψ .*
- (vi) *For $\tilde{\mathbb{G}} := \text{Irr}(\tilde{G} \mid \mathbb{G})$ and $\tilde{\mathbb{M}} := \text{Irr}(\tilde{M} \mid \mathbb{M})$, there exists a $\text{Lin}(\tilde{G}/G) \rtimes \hat{M}$ -equivariant bijection*

$$\tilde{\Omega} : \tilde{\mathbb{G}} \longrightarrow \tilde{\mathbb{M}}$$

such that $\Omega(\tilde{\mathbb{G}} \cap \text{Irr}(\tilde{G} \mid \nu)) = \tilde{\mathbb{M}} \cap \text{Irr}(\tilde{M} \mid \nu)$ for each $\nu \in \text{Irr}(Z(\tilde{G}))$.

Then there exists an $\tilde{M}\hat{M}$ -equivariant bijection $\Omega : \mathbb{G} \longrightarrow \mathbb{M}$ such that for each $\chi \in \mathbb{G}$

$$((\tilde{G} \rtimes E)_\chi, G, \chi) \succeq_c ((\tilde{M}\hat{M})_{\Omega(\chi)}, M, \Omega(\chi)).$$

When one chooses $\mathbb{G} = \text{Irr}_{\ell'}(G)$ and $\mathbb{M} = \text{Irr}_{\ell'}(M)$ as the $\tilde{M}\hat{M}$ -stable sets, this theorem thus implies both conditions (Eq) and (Ext) of Definition 1.31 as $N_{\tilde{G} \rtimes E}(M) = MN_{\tilde{G} \rtimes E}(P)$.

In order to obtain the inductive Galois–McKay conditions (Eq) $_{\mathcal{H}_\ell}$ and (Ext) $_{\mathcal{H}_\ell}$, it is necessary to refine the above statements to include \mathcal{H}_ℓ -equivariance. Indeed, Theorem 3.4 and Corollary 3.5 of [RSST25] provide such a refinement. In the following, we will introduce our own variant, which demands slightly weaker requirements, sufficient to ensure condition (Eq) $_{\mathcal{H}_\ell}$.

Theorem 4.18. *Assume we are in the situation of Theorem 4.17 and that conditions (i)–(iii) hold. Let \mathcal{H} be a subgroup of $\text{Gal}(\mathbb{Q}^{ab}/\mathbb{Q})$, stabilizing \mathbb{G} and \mathbb{M} and assume that the following hold.*

- (iv) *There exists an E -stable \tilde{G} -transversal \mathbb{G}_0 in \mathbb{G} and each $\chi \in \mathbb{G}_0$ extends to $(G \rtimes E)_\chi$.*
- (v) *There exists some \hat{M} -stable \tilde{M} -transversal \mathbb{M}_0 in \mathbb{M} such that each $\psi \in \mathbb{M}_0$ extends to \hat{M}_ψ .*

- (vi) *There exists a group $B \leq \text{Aut}(\tilde{G}E)_{G,GE,\tilde{G},P} \times \mathcal{H}$ whose projection on $\text{Aut}(\tilde{G}E)$ stabilizes M , G , \tilde{G} , \mathbb{G}_0 and \mathbb{M}_0 , and the projection on \mathcal{H} is onto.*
- (vii) *For $\tilde{\mathbb{G}} := \text{Irr}(\tilde{G} | \mathbb{G})$ and $\tilde{\mathbb{M}} := \text{Irr}(\tilde{M} | \mathbb{M})$, there exists a $\text{Lin}(\tilde{G}/G) \times (\tilde{M}B)$ -equivariant bijection*

$$\tilde{\Omega} : \tilde{\mathbb{G}} \longrightarrow \tilde{\mathbb{M}}$$

such that $\Omega(\tilde{\mathbb{G}} \cap \text{Irr}(\tilde{G} | \nu)) = \tilde{\mathbb{M}} \cap \text{Irr}(\tilde{M} | \nu)$ for each $\nu \in \text{Irr}(\mathbb{Z}(\tilde{G}))$.

Then there exists a $\tilde{M}\tilde{M} \times \mathcal{H}$ -equivariant bijection $\Omega : \mathbb{G} \longrightarrow \mathbb{M}$.

Proof. First note that since $\tilde{G} \rtimes E$ induces all the automorphism of G , $\tilde{M}\tilde{M}$ induces all automorphisms of G stabilizing M . Hence, $\tilde{M}\tilde{M}B$ acts as $\tilde{M}\tilde{M} \times \mathcal{H}$ on G .

As \mathbb{G}_0 is an $\tilde{M}B$ -stable \tilde{G} -transversal, by Lemma 2.9 b), for any $\chi \in \mathbb{G}_0$ we have

$$(\tilde{G}E \times \mathcal{H})_\chi = (\tilde{G}EB)_\chi = \tilde{G}_\chi(GEB)_\chi. \quad (4.6)$$

Similarly for each $\psi \in \mathbb{M}_0$ we have

$$(\tilde{M}\tilde{M} \times \mathcal{H})_\psi = (\tilde{M}\tilde{M}B)_\psi = \tilde{M}_\psi(\tilde{M}B)_\psi. \quad (4.7)$$

Our argument follows the construction of the bijection given in [Spä12, Theorem 2.12], adapting it to include the Galois group. Thus, we skip the group-theoretic arguments already present there.

Let $\tilde{\chi} \in \tilde{\mathbb{G}}$. By Clifford theory and condition (iv), there exists exactly one $\chi \in \mathbb{G}_0$ such that

$$\mathbb{G}_0 \cap \text{Irr}(G | \tilde{\chi}) = \{\chi\}.$$

We define $\Omega(\chi)$ as the unique element in $\mathbb{M}_0 \cap \text{Irr}(M | \tilde{\Omega}(\tilde{\chi}))$. As $\tilde{\Omega}$ is $\text{Lin}(\tilde{G}/G) \times (\tilde{M}B)$ -equivariant, this is a well-defined map

$$\Omega : \mathbb{G}_0 \rightarrow \mathbb{M}_0.$$

Furthermore, as \mathbb{G}_0 and \mathbb{M}_0 are $\tilde{M}B$ -stable \tilde{M} -transversals, we may extend Ω to a map $\Omega : \mathbb{G} \rightarrow \mathbb{M}$ by

$$\Omega(\chi^x) = \Omega(\chi)^x \text{ for any } x \in (\tilde{M}\tilde{M}B).$$

It remains to show that this is well-defined, i.e. $(\tilde{M}\tilde{M}B)_\chi = (\tilde{M}\tilde{M}B)_{\Omega(\chi)}$ for all $\chi \in \mathbb{G}$. As $(\tilde{G}EB)_\chi = \tilde{G}_\chi(EB)_\chi$ and \tilde{G}/G is abelian, by Clifford theory we have

$$(EB)_\chi = \{\alpha \in EB \mid \tilde{\chi}^\alpha = \tilde{\chi}\lambda, \text{ for some } \lambda \in \text{Lin}(\tilde{G}/G)\},$$

for some $\tilde{\chi} \in \text{Irr}(\tilde{G} | \chi)$. We abbreviate $\psi := \Omega(\chi) \in \mathbb{M}_0$ and similarly obtain

$$(\tilde{M}B)_\psi = \{\alpha \in \tilde{M}B \mid \tilde{\psi}^\alpha = \tilde{\psi}\lambda, \text{ for some } \lambda \in \text{Lin}(\tilde{M}/M)\}$$

for $\tilde{\psi} = \tilde{\Omega}(\tilde{\chi}) \in \text{Irr}(\tilde{M} | \psi)$. Note that $\tilde{M}/M \cong \tilde{G}/G$.

Furthermore, we have

$$\begin{aligned} \tilde{G}_\chi &= \bigcap_{\{\lambda \in \text{Lin}(\tilde{G}/G) \mid \tilde{\chi}\lambda = \tilde{\chi}\}} \ker(\lambda). \\ \tilde{M}_\psi &= \bigcap_{\{\lambda \in \text{Lin}(\tilde{M}/M) \mid \tilde{\psi}\lambda = \tilde{\psi}\}} \ker(\lambda). \end{aligned}$$

As $\tilde{\Omega}(\chi\lambda) = \tilde{\Omega}(\chi)\lambda_{\tilde{M}}$ we get $G\tilde{M}_\psi = \tilde{G}_\chi$. Furthermore, $GE = G\widehat{M}$ and $GEB = G(\widehat{M}B)$. Thus

$$\begin{aligned} G(\widehat{M}B)_\psi &= G\{\alpha \in \widehat{M}B \mid \tilde{\psi}^\alpha = \tilde{\psi}\lambda, \text{ for some } \lambda \in \text{Lin}(\widehat{M}/M)\} \\ &= G\{\alpha \in \widehat{M}B \mid \tilde{\chi}^\alpha = \tilde{\chi}\lambda, \text{ for some } \lambda \in \text{Lin}(\tilde{G}/G)\} \\ &= G(EB)_\chi. \end{aligned}$$

Together with $G\tilde{M}_\psi = \tilde{G}_\chi$, (4.6) and (4.7) this yields $(\tilde{M}\widehat{M}B)_\chi = (\tilde{M}\widehat{M}B)_\psi$. \square

Corollary 4.19. *Let S be a finite simple group and G its universal covering group. If the conditions of Theorem 4.18 hold for G and the prime ℓ , then $(\text{Eq})_{\mathcal{H}_\ell}$ holds for S and ℓ .*

Proof. We need to show that there exists a $\text{Aut}(G)_Q \times \mathcal{H}_\ell$ -equivariant bijection $\text{Irr}_{\ell'}(G) \rightarrow \text{Irr}_{\ell'}(M)$. Clearly, the sets $\mathbb{G} := \text{Irr}_{\ell'}(G)$ and $\mathbb{M} := \text{Irr}_{\ell'}(M)$ are $(\tilde{G} \rtimes E)_M$ - and \mathcal{H}_ℓ -stable. Furthermore, as by Proposition 3.32, $\text{Aut}(G)_Q \leq \text{Aut}(G)_\mathbf{S}$, the map Ω is an $\text{Aut}(G)_Q$ -equivariant bijection $\text{Irr}_{\ell'}(G) \rightarrow \text{Irr}_{\ell'}(M)$. \square

Remark 4.20. *In addition to condition $(\text{Eq})_{\mathcal{H}_\ell}$, we also obtain the usual Extension condition (Ext) for the McKay conjecture in the theorem above, since all assumptions of Theorem 4.17 are satisfied. However, there exists a strengthened version that yields both inductive conditions simultaneously.*

Theorem 4.21. *[RSST25, Corollary 3.5] Assume the setting and conditions (i)–(vii) of Theorem 4.18. Let Λ_G be an extension map with respect to $G \triangleleft GE$ for \mathbb{G}_0 and Λ_M an extension map with respect to $M \triangleleft \widehat{M}$ for \mathbb{M}_0 . Assume that the following hold.*

(viii) *For all $\chi \in \mathbb{G}_0$ and $\psi \in \mathbb{M}_0$ we have*

$$\text{Irr}(C_E(G) \mid \Lambda_G(\chi)) = \text{Irr}(C_E(G) \mid \Lambda_M(\psi)).$$

(ix) *for each $\chi \in \mathbb{G}_0$, with $\psi \in \mathbb{M}_0$ such that $\tilde{\Omega}(\text{Irr}(\tilde{G} \mid \chi)) = \tilde{\Omega}(\text{Irr}(\tilde{M} \mid \psi))$ and for $\alpha \in (\widehat{M}B)_\psi$ we have*

$$[\Lambda_G(\chi), \alpha] \Big|_{M_\chi}^\wedge = [\Lambda_M(\psi), \alpha].$$

Then the bijection $\Omega : \mathbb{G} \rightarrow \mathbb{M}$ from Theorem 4.18 can be chosen such that it satisfies

$$((\tilde{G} \rtimes E)_{\chi^\mathcal{H}}, G, \chi)_{\mathcal{H}} \geq_c ((\tilde{M}\widehat{M})_{\Omega(\chi)^\mathcal{H}}, M, \Omega(\chi))_{\mathcal{H}} \quad \text{for all } \chi \in \mathbb{G}.$$

In particular, the Theorems above give a possible path towards the verification of the inductive conditions. As we have seen in Section 3.A, except for the alternating groups and the 26 sporadic groups, every finite non-abelian simple group belongs to one of the families of finite groups of Lie type and each such simple group arises as a central quotient of the corresponding simply-connected finite group of Lie type. Hence, G will be \mathbf{G}^F for some pair (\mathbf{G}, F) of a simple algebraic group \mathbf{G} and a Steinberg map F defining an \mathbb{F}_q -structure on \mathbf{G} .

The intermediate subgroup $N_G(P) \leq M < G$ is chosen to be the normalizer of a Sylow d -torus following the approach introduced in [Spä12], where conditions (i)–(iii) were established.

Remark 4.22. Note that by Theorem 3.29, there exists some Sylow ℓ -subgroup Q of G such that its normalizer lies in the normalizer of a Sylow d -torus of (\mathbf{G}, F) for $d = d_\ell(q)$ unless one of the following cases hold.

$$\ell = n = 3 \text{ with } q \equiv 2, 5 \pmod{9}.$$

Furthermore, when $n = 2$, the automorphism group $\text{Aut}(\mathbf{G}^F)$ is not isomorphic to $(\tilde{\mathbf{G}}^F \times E)/Z(\tilde{\mathbf{G}}^F)$, see Theorem 3.19. Therefore, in the following statements, we exclude these exceptional cases. They are however covered in [Joh22] and [Tap25].

Proposition 4.23. [CS17, Lemma 7.1] Let $\mathbf{G} = \text{SL}_n(\overline{\mathbb{F}}_q)$ and $\tilde{\mathbf{G}} = \tilde{\mathbf{G}} = \text{GL}_n(\overline{\mathbb{F}}_q)$. Assume that we are not in any of the exceptions given in Theorem 3.29 and assume $n \geq 3$. Let $F : \tilde{\mathbf{G}} \rightarrow \tilde{\mathbf{G}}$ be a Steinberg endomorphism, such that $G = \mathbf{G}^F = \text{SL}_n(\epsilon q)$ and $\tilde{\mathbf{G}}^F = \text{GL}_n(\epsilon q)$. Let $Q \leq \text{Syl}_\ell(G)$ and let $E = \langle \gamma, F_p \rangle \leq \text{Aut}(\tilde{\mathbf{G}})$ as in Definition 3.18. Let $d := d_\ell(q)$ and \mathbf{S} a Sylow d -torus of (\mathbf{G}, F) , $M = N_G(\mathbf{S})$. Then the group theoretic parts of the conditions (i)-(iii) from Theorem 4.17 hold.

The remaining conditions (iv)- (vii) will be combined and reduced to the following three conditions. These are analogues of the conditions for the McKay conjecture as introduced in [CS19, Definition 2.2]

Definition 4.24. Let p and ℓ be distinct primes and $q = p^m$. Let \mathbf{G} be a connected reductive group defined over $\overline{\mathbb{F}}_p$ and $F : \mathbf{G} \rightarrow \mathbf{G}$ a Steinberg endomorphism such that it gives rise to an \mathbb{F}_q -structure on \mathbf{G} . Let $\tilde{\mathbf{G}}$ be a regular embedding of \mathbf{G} and let $E := E(\mathbf{G}^F)$ be the subgroup of $\text{Aut}(\tilde{\mathbf{G}}^F)$ as in Definition 3.18, acting on $\tilde{\mathbf{G}}^F$ as an extension of the action of E on \mathbf{G}^F . Let $d = d_\ell(q)$ and \mathbf{S} a Sylow d -torus of (\mathbf{G}, F) . We denote

$$M := N_{\mathbf{G}}(\mathbf{S})^F, \tilde{M} := N_{\tilde{\mathbf{G}}}(\mathbf{S})^F, \widehat{M} := N_{\mathbf{G}^E}(\mathbf{S})^F \text{ and } \tilde{L} := C_{\tilde{\mathbf{G}}}(\mathbf{S})^F.$$

Let $t \in \tilde{\mathbf{G}}$, such that $[t, \widehat{M}] \subset Z(\mathbf{G})$, $tZ(\mathbf{G})$ is in an E -stable G -orbit in $\tilde{\mathbf{G}}$ and t stabilizes L and M . Then $\mathcal{H}_t = \mathcal{K} \times \langle \sigma_1 t \rangle$ defines a subgroup of $\text{Aut}(\tilde{\mathbf{G}}^E) \times \mathcal{H}_\ell$ as described in Definition 4.16. With these definitions, we can now reformulate our conditions accordingly.

- A**(∞) $_{\mathcal{H}_t}$: There exists an E -stable $\tilde{\mathbf{G}}^F$ -transversal \mathbb{G}_0 in $\text{Irr}(\mathbf{G}^F)$ such that every $\chi \in \mathbb{G}_0$ extends to $\mathbf{G}^F E_\chi$ and \mathbb{G}_0 is \mathcal{H}_t -stable.
- A**(d) $_{\mathcal{H}_t}$: There exists an \widehat{M} -stable \tilde{M} -transversal \mathbb{M}_0 in $\text{Irr}(M)$, such that every $\chi \in \mathbb{M}_0$ extends to \widehat{M}_χ and \mathbb{M}_0 is \mathcal{H}_t -stable.
- B**(d) $_{\mathcal{H}_\ell}$: There exists a $(\text{Lin}(\widehat{M}/M) \times \widehat{M}\widehat{M}) \times \mathcal{H}_\ell$ -equivariant extension map $\tilde{\Lambda}$ with respect to $\tilde{L} \triangleleft \tilde{M}$.

Remark 4.25. Note that in the above conditions, we have already made a rather explicit choice for the automorphism group $B \leq \text{Aut}(GE) \times \mathcal{G}$. This idea can be generalized to conditions **A**(∞) $_{B_1}$ and **A**(d) $_{B_2}$, where $B_1 \text{Inn}(G) = B_2 \text{Inn}(G)$ and each B_i satisfies sufficient stabilizer properties.

Theorem 4.26. [RSST25, Theorem B] Assume the situation of Condition 4.24. If the group \mathbf{G}^F satisfies the properties **A**(d) $_{\mathcal{H}_t}$ and **B**(d) $_{\mathcal{H}_\ell}$ for all integers $d \geq 1$ for some $t \in \tilde{\mathbf{G}}$, and $\mathbf{G}^F/Z(\mathbf{G}^F)$ is simple, then $(\text{Eq})_{\mathcal{H}_\ell}$ is satisfied for all primes ℓ .

We will call condition $A(\infty)_{\mathcal{H}_t}$ the *global condition* and $A(d)_{\mathcal{H}_t}$ the *local condition*. Indeed, if \mathbf{S}' is another Sylow d -torus of (\mathbf{G}, F) by Theorem 3.26, they must be \mathbf{G}^F -conjugate and the conditions $A(d)$ and $B(d)$ are independent of the choice of \mathbf{S} . In particular, condition $A(d)_{\mathcal{H}_t}$ is equivalent to $A(\infty)_{\mathcal{H}_t}$ for large d as the following Theorem, first stated by Lusztig, gives us maximal extendibility with respect to $\mathbf{G}^F \triangleleft \widetilde{\mathbf{G}}^F$.

Theorem 4.27. [CE04, Theorem 15.11] *Let $\mathbf{G} \subset \widetilde{\mathbf{G}}$ be a regular embedding. Then for any $\chi \in \text{Irr}(\widetilde{\mathbf{G}}^F)$, the restriction $\chi|_{\mathbf{G}^F}$ is multiplicity free.*

We will refrain from giving a complete proof of Theorem 4.26, as this would exceed the scope of this thesis.

Towards the proof of Theorem 4.26. Let $B = \mathcal{H}_t$. As $[t, \widehat{M}] \subseteq Z(\mathbf{G})$, if ℓ does not divide the order of $G := \mathbf{G}^F$, then there is nothing to prove in the inductive conditions. Moreover the statement is already proven for $\ell = 2$ in [RSF25] so we may ignore this case and assume $\ell \geq 3$.

We may ignore all cases from Remark 4.22 as these special cases have been verified to admit the inductive Galois–McKay conditions, see [Joh22; Tap25].

Thus, we assume that $\ell \mid |G|$ and let d be the order of $q \pmod{\ell}$. Let \mathbf{S} be a Sylow d -torus of (\mathbf{G}, F) and define $M := N_{\mathbf{G}}(\mathbf{S})^F$. By Proposition 4.23, conditions (i)–(iii) of Theorem 4.18 hold. Clearly, conditions (iv)–(vi) are contained in $A(\infty)_{\mathcal{H}_t}$ and $A(d)_{\mathcal{H}_t}$ as $\mathbb{G} = \text{Irr}_{\ell'}(G)$ and $\mathbb{M} = \text{Irr}_{\ell'}(M)$ in this case contains all possible $\widetilde{M}\widetilde{M} \times \mathcal{H}$ -stable subsets. Finally the construction of the bijection of condition (iv) is a consequence of $B(d)$ by [CS17, Section 6]. Indeed, the construction is \mathcal{H} -compatible so that (vii) is implied by $B(d)_{\mathcal{H}}$ as can be seen in [RSST25, Section 4.D]. In particular, this construction uses the fact that Jordan decomposition of $\widetilde{\mathbf{G}}$ and d -Harish Chandra theory for $\widetilde{\mathbf{G}}$ are Galois equivariant. See [SV20] and [RSST25, Proposition 3.7]. \square

5. The global condition $A(\infty)_{\mathcal{H}_t}$

In the previous chapter we have established that the verification of the equivariant condition of the inductive Galois–McKay conditions reduces to the verification of the "global condition" $A(\infty)_{\mathcal{H}_t}$ and the "local conditions" $A(d)_{\mathcal{H}_t}$ and $B(d)_{\mathcal{H}_\ell}$. The goal of this chapter is the verification of the global condition for groups of type A.

The result can also be found in [RSST25, Section 5]. There, the group \mathbf{G} is at first assumed to be of type A or C, and for those groups an outer diagonal automorphism is described in terms of principal cocharacters. This automorphism maps u^k into the \mathbf{G} -conjugacy class of u using subsystem subgroups. Earlier results of Taylor are then applied to generalized Gelfand–Graev representations (GGGRs) to describe the stabilizer of a GGGR character in the considered types. Afterwards, the stabilizer of certain irreducible characters of $\mathrm{SL}_n(\epsilon q)$ is determined, which leads to the existence of a transversal \mathbb{T} as required in Definition 4.24.

In contrast, our approach proceeds as follows. We use [TZ04] to determine the outer diagonal automorphism, given explicitly in terms of the value of a determinant. We then describe in detail the action of a diagonal automorphism on GGGRs. Combining that with results from [SFT18] we determine the stabilizer of certain GGGRs in the group of Galois and outer automorphisms. In the end we employ properties of the unipotent support to give an explicit construction of a $\tilde{\mathbf{G}}^F$ -transversal \mathbb{T} , modifying the construction from the original verification of $A(\infty)$ in [CS17].

5.A GGGRs and Galois automorphisms

In this section, we recall the construction and basic properties of generalized Gelfand–Graev representations (GGGRs) for groups of type A. In particular, we investigate how GGGRs are permuted under the action of Galois and outer automorphisms. Since GGGRs are constructed from unipotent elements, a key step in our analysis is to understand how Galois automorphisms act on these elements. This perspective naturally leads to the introduction of a suitable diagonal automorphism t , which is closely related to the global element defining the group \mathcal{H}_t . Such actions have previously been studied by Taylor and Schaeffer Fry in [SFT18].

Let p be prime, $\mathbf{k} = \overline{\mathbb{F}}_p$ and $\mathbf{G} := \mathrm{SL}_n(\mathbf{k})$, $\tilde{\mathbf{G}} := \mathrm{GL}_n(\mathbf{k})$. Let $F: \tilde{\mathbf{G}} \rightarrow \tilde{\mathbf{G}}$ be a Steinberg endomorphism such that $F \in \{F_q, \gamma F_q\}$ as in Definition 3.15. Let $\epsilon \in \{\pm 1\}$ be such that $F = \gamma^{\frac{1-\epsilon}{2}} F_q$. Let E be the group of automorphisms of $\tilde{\mathbf{G}}$, given by $E = \langle \gamma, F_p \rangle$, as in Definition 3.18. We use the notation \mathbf{G}_u for the set of unipotent elements of \mathbf{G} . Note that the construction of GGGR's can be generalized to any connected reductive algebraic group over \mathbf{k} , when p is a good prime for \mathbf{G} as in Definition 3.16, which does not impose a restriction on p in type A.

We write $G := \mathbf{G}^F$, $\tilde{G} := \tilde{\mathbf{G}}^F$ and $G_u := \mathbf{G}_u^F$ for the set of unipotent elements. We denote by $\mathrm{Uni}(\mathbf{G})$ the set of \mathbf{G} -conjugacy classes of unipotent elements of \mathbf{G} . In particular, $\mathrm{Uni}(\mathbf{G}) = \mathrm{Uni}(\tilde{\mathbf{G}})$ as $\tilde{\mathbf{G}} = \mathbf{Z}(\tilde{\mathbf{G}})\mathbf{G}$.

Given the torus \mathbf{T} and the Borel subgroup \mathbf{B} of \mathbf{G} from Example 3.4, we write $\mathbf{U} := R_u(\mathbf{B})$ for the unipotent radical of \mathbf{B} . The root system Φ , together with the set of positive roots and the basis Δ , is then determined by \mathbf{U} and has been studied in Example 3.7. The root subgroups $\mathbf{U}_\alpha \leq \mathbf{U}$ for $\alpha \in \Phi$ are defined as in Definition 3.2.

We fix a unipotent class $\mathcal{C} \in \mathrm{Uni}(\mathbf{G})$ and extend the definition of the weight function of the Dynkin diagram with respect to this unipotent class. We do not give an explicit construction here, but instead use the definition due to Kawanaka [Kaw85, 2.1.1] without further detail.

Definition 5.1. The *weighted Dynkin diagram* associated with \mathcal{C} , is the map $h'_\mathcal{C}: \Delta \rightarrow \{0, 1, 2\}$. We may extend this linearly to the map

$$h_\mathcal{C}: \mathbb{Z}\Delta \rightarrow \mathbb{Z}.$$

For $i \geq 0$ we then set $\mathbf{U}_{i,\mathcal{C}} := \{\mathbf{U}_\alpha \mid \alpha \in \Phi^+, h_\mathcal{C}(\alpha) \geq i\}$.

By Chevalley's commutator formula, see [Hum75, Lemma 32.5], we have

$$\mathbf{U}_{i,\mathcal{C}} \triangleleft \mathbf{P}_{1,\mathcal{C}} =: \mathbf{N}_\mathbf{G}(\mathbf{U}_{1,\mathcal{C}}), \quad \text{for all } i \geq 0$$

and

$$\mathbf{U}_{i,\mathcal{C}}/\mathbf{U}_{i+1,\mathcal{C}} \cong (\overline{\mathbf{k}}^+)^{|h_\mathcal{C}^{-1}(i)|}.$$

Definition 5.2. To each $u \in \mathcal{C} \cap \mathbf{U}_{2,\mathcal{C}}^F$, we may associate a linear character $\psi_u: \mathbf{U}_{2,\mathcal{C}}^F \rightarrow \mathbb{C}^\times$, where one can check the details in [Gec04, Definition 2.1]. Then there exists an irreducible character $\psi'_u \in \mathrm{Irr}(\mathbf{U}_{1,\mathcal{C}}^F)$ such that

$$(\psi_u)^{\mathbf{U}_{1,\mathcal{C}}^F} = |\mathbf{U}_{1,\mathcal{C}}^F : \mathbf{U}_{2,\mathcal{C}}^F|^{\frac{1}{2}} \psi'_u$$

As $|\mathbf{U}_{1,\mathcal{C}}^F : \mathbf{U}_{2,\mathcal{C}}^F|$ is an even power of q , see [Gec04, Definition 2.1], the square root does exist.

Then the associated *generalized Gelfand-Graev character* of \tilde{G} is defined as

$$\tilde{\Gamma}_u := (\psi'_u)^{\tilde{G}}.$$

Accordingly the associated *generalized Gelfand-Graev character* of G is defined as

$$\Gamma_u := (\psi'_u)^G.$$

In particular, $\tilde{\Gamma}_u \in \mathbb{Z}_{\geq 0}\mathrm{Irr}(\tilde{G})$ by this construction.

Note that this is indeed a generalization, as this definition coincides with the Gelfand-Graev character in case u is a regular unipotent element. According to [Kaw85, 2.1.1] the following facts hold

- $\mathcal{C}' := \mathcal{C} \cap \mathbf{U}_{2,\mathcal{C}}$ is dense in $\mathbf{U}_{2,\mathcal{C}}$,
- \mathcal{C}' is a single $\mathbf{P}_{1,\mathcal{C}}$ -orbit and
- $C_{\mathbf{G}}(u) \leq \mathbf{P}_{1,\mathcal{C}}$, for all $u \in \mathcal{C}'$.

The groups $\mathbf{U}_{i,\mathcal{C}}$ and $\mathbf{P}_{1,\mathcal{C}}$ are E -stable, see [Sho97, 2.3.1], [CS17, Section 4.2], and $\mathbf{U}_{1,\mathcal{C}}^F$ and $\mathbf{U}_{2,\mathcal{C}}^F$ are $\mathbf{P}_{1,\mathcal{C}}^F \rtimes E$ -stable.

Proposition 5.3. *Let $u \in \mathcal{C} \cap \mathbf{U}_{2,\mathcal{C}}^F$. Then*

(i) [CS17, Lemma 4.3] *For $g \in \mathbf{P}_{1,\mathcal{C}}^F \rtimes E$*

$$\psi_{u^g} = \psi_u^g \quad \text{and} \quad \psi'_{u^g} = (\psi'_u)^g.$$

(ii) [Gec04, 2.2] *For $g \in \tilde{\mathbf{P}}_{1,\mathcal{C}}^F = \mathbf{N}_{\tilde{\mathbf{G}}}(\mathbf{U}_{1,\mathcal{C}})^F$*

$$\psi_{u^g} = \psi_u^g \quad \text{and} \quad \psi'_{u^g} = (\psi'_u)^g.$$

The second part of the proposition shows that $\tilde{\Gamma}_u$ only depends on the conjugacy class $\mathcal{C} \in \text{Uni}(\tilde{\mathbf{G}})$. Thus, we may write $\tilde{\Gamma}_{\mathcal{C}}$ for the Generalized Gelfand-Graev character of $\tilde{\mathbf{G}}$ associated to u . We also use the abbreviation GGGC in this case.

Note that the class $\mathcal{C} \in \text{Uni}(\tilde{\mathbf{G}})$ may contain several unipotent classes \mathcal{C}_i of $\text{SL}_n(\epsilon q)$. The GGGRs can be used to distinguish the irreducible constituents of the character $\tilde{\chi}|_G$ for $\tilde{\chi} \in \text{Irr}(\tilde{\mathbf{G}})$. Indeed, this is possible due to the next result.

Theorem 5.4. [Kaw85, 3.2.18(iii), 3.2.24(i)] *Let $\tilde{\mathbf{G}} := \text{GL}_n(\mathbf{k})$ and $F: \tilde{\mathbf{G}} \rightarrow \tilde{\mathbf{G}}$ the Steinberg endomorphism of the form $F \in \{F_q, \gamma F_q\}$ as in Notation 3.15. Then for any $\tilde{\chi} \in \text{Irr}(\tilde{\mathbf{G}})$, there exists $\mathcal{C} \in \text{Uni}(\tilde{\mathbf{G}})$ such that*

$$\langle \tilde{\Gamma}_{\mathcal{C}}, \tilde{\chi} \rangle = 1.$$

We may extend this result slightly, obtaining uniqueness regarding the choice of \mathcal{C} .

Proposition 5.5. *For every $\tilde{\chi} \in \text{Irr}(\tilde{\mathbf{G}}^F)$, there exists some conjugacy class $\mathcal{C} = \mathcal{C}(\tilde{\chi}) \in \text{Uni}(\tilde{\mathbf{G}})$ such that*

(i) $\langle \tilde{\chi}, \tilde{\Gamma}_{\mathcal{C}} \rangle = 1$,

(ii) $\langle \tilde{\chi}, \tilde{\Gamma}_{\mathcal{D}} \rangle = 0$, for every other $\mathcal{D} \in \text{Uni}(\tilde{\mathbf{G}})$ with $\mathcal{D} \not\subseteq \bar{\mathcal{C}}$, the Zariski closure of \mathcal{C} .

Additionally \mathcal{C} is uniquely defined by these two properties.

Proof. This result is a combination of two results on GGGR's by Geck [Gec91] and Bonnafé [Bon00] where we use that the root system is of type **A**.

Let $\tilde{\chi} \in \text{Irr}(\tilde{\mathbf{G}})$. By [Gec91, Theorem A], there exists a unipotent class $\mathcal{C} := \mathcal{C}(\tilde{\chi})$ in $\text{Uni}(\tilde{\mathbf{G}})$, such that

$$\langle \tilde{\chi}, \Gamma_{\mathcal{C}} \rangle \neq 0 \quad \text{and} \quad \langle \tilde{\chi}, \Gamma_{\mathcal{D}} \rangle = 0, \text{ when } \mathcal{D} \in \text{Uni}(\tilde{\mathbf{G}}), \mathcal{D} \not\subseteq \bar{\mathcal{C}},$$

that is uniquely defined through these properties. By [Bon00, Theorem 5.2.2], there exists a class $\mathcal{C}' \in \text{Uni}(\tilde{\mathbf{G}})$ such that

$$\langle \tilde{\chi}, \Gamma_{\mathcal{C}'} \rangle = 1 \quad \text{and} \quad \langle \tilde{\chi}, \Gamma_{\mathcal{D}} \rangle = 0, \text{ when } \mathcal{D} \in \text{Uni}(\tilde{\mathbf{G}}), \mathcal{D} \not\subseteq \bar{\mathcal{C}}'.$$

Thus $\mathcal{C}' = \mathcal{C}$ and the statement holds. \square

We now turn to describe the action of \mathcal{H}_ℓ on the GGGR's. Therefore we first describe the ℓ -powers of u , for a unipotent element $u \in G$. Indeed, by work of P.H. Tiep and A.E. Zalesski, each unipotent element $u \in G$ is \tilde{G} -conjugate to its ℓ -power u^ℓ .

We remind the reader of Section 4.A, where we have interpreted the integer ℓ as an element in \mathbb{F}_q^\times . We write κ for the element in $\mathbb{F}_{q^2}^\times$, such that $\kappa^{q+1} = \ell$ and define $\delta_\ell := \kappa^{q+\epsilon}$.

Finally we remind the reader of the definition of an automorphism of G of type \mathfrak{t}_ℓ from Definition 4.13. In particular, an element $t \in \tilde{G}$ is of this type whenever $\det(t) = \delta_\ell^{\frac{n(n-1)}{2}}$ by Lemma 4.14. Now by [TZ04], the element conjugating u and u^ℓ can be explicitly constructed.

Theorem 5.6. [TZ04, Lemma 6.6] *Let $G = \mathrm{SL}_n(\epsilon q)$, $\tilde{G} = \mathrm{GL}_n(\epsilon q)$ and let u be a unipotent element of G . Then there exists an element $t \in \mathrm{GL}_n(\epsilon q)$, such that $tut^{-1} = u^\ell$ and $\det(t) = \delta_\ell^{\frac{n(n-1)}{2}}$, i.e. t induces the outer automorphism of type \mathfrak{t}_ℓ .*

From this, we see that the ℓ -powers of u lie in at most two different conjugacy classes in G , as an outer automorphism of type \mathfrak{t}_ℓ has order at most 2. By work of A. Schaeffer Fry and J. Taylor, we can also determine the action of certain Galois automorphism on the GGGR's.

Proposition 5.7. [SFT18, Prop. 4.10] *Let $\sigma \in \mathrm{Gal}(\mathbb{Q}^{ab}/\mathbb{Q})$ and $j \in \mathbb{Z}/p\mathbb{Z}$ be such that, $\sigma(\zeta) = \zeta^j$ for any root of unity ζ of p -power order. Then for $u \in \mathcal{C}^F \cap \mathbf{U}_{2,\mathcal{C}}^F$ with Γ_u its associated GGGR we have*

$$(\Gamma_u)^\sigma = \Gamma_{u^j}$$

Furthermore, we have some information about the character field of GGGR's.

Proposition 5.8. [SFT18, 10.12] *Let Γ_u be a GGGR of G . Then $\mathbb{Q}(\Gamma_u) \leq \mathbb{Q}(\sqrt{\eta p})$, where $\eta \in \{\pm 1\}$ is such that $p \equiv \eta \pmod{4}$*

In particular, as $\tilde{\Gamma}_u = (\Gamma_u)^{\tilde{G}}$, the character values of $\tilde{\Gamma}_u$ lie in $\mathbb{Q}(\Gamma_u)$. Furthermore, one can check that $\mathbb{Q}(\tilde{\Gamma}_u) = \mathbb{Q}$. Thus, we now have found an element $t \in \tilde{G}$ which has the same action on GGGR's as σ_1 . We have to make sure, that $u^t = u^\ell$ is indeed conjugate to u in $\mathbf{P}_{1,\mathcal{C}}$.

Proposition 5.9. *Let $t \in \tilde{G}$ such that t induces the automorphism of type \mathfrak{t}_ℓ . Then every GGGR Γ_u is \mathcal{H}_t -invariant.*

Proof. Let $u \in \mathcal{C} \cap \mathbf{U}_{2,\mathcal{C}}^F$ be the unipotent element associated to Γ_u . By [Gec04, Remark 2.2] each G -conjugacy class contained in \mathcal{C} has non-trivial intersection with the group $\mathbf{U}_{2,\mathcal{C}}$. Furthermore, any power u^k of u for k prime to p , has the same Jordan normal form as u and thus lies in \mathcal{C} and clearly also in $\mathbf{U}_{2,\mathcal{C}}$. Thus $u, u^k \in \mathcal{C} \cap \mathbf{U}_{2,\mathcal{C}}$ for all $k \in \mathbb{Z} \setminus p\mathbb{Z}$.

Let $\sigma_1 \in \mathcal{H}$ be the element of \mathcal{H}_ℓ , that sends each root of unity of ℓ' -order to its ℓ power and fixes roots of ℓ power order. By Proposition 5.7 we have $(\Gamma_u)^{\sigma_1} = \Gamma_{u^\ell}$ and by Theorem 5.6, there exists some $t \in \tilde{G}$ inducing the outer automorphism of type \mathfrak{t}_ℓ on G with $u^t = u^\ell$.

Since $\mathcal{C} \cap \mathbf{U}_{2,\mathcal{C}}$ is a $\mathbf{P}_{1,\mathcal{C}}$ -orbit, there exists some $x \in \mathbf{P}_{1,\mathcal{C}}$, such that $u^x = u^\ell$. As furthermore $\mathbf{C}_{\mathbf{G}}(u) \leq \mathbf{P}_{1,\mathcal{C}}$ we have $\mathbf{C}_{\tilde{\mathbf{G}}}(u) \leq \mathbf{P}_{1,\mathcal{C}}\mathbf{Z}(\tilde{\mathbf{G}})$.

Now the element xt^{-1} lies in $\mathbf{C}_{\tilde{\mathbf{G}}}(u) \leq \mathbf{Z}(\tilde{\mathbf{G}})\mathbf{P}_{1,\mathcal{C}}$ and thus $t \in \mathbf{Z}(\tilde{\mathbf{G}})\mathbf{P}_{1,\mathcal{C}} \cap \tilde{\mathbf{G}}$. Since $\mathbf{U}_{2,\mathcal{C}} \triangleleft \mathbf{P}_{1,\mathcal{C}} = \mathbf{N}_{\mathbf{G}}(\mathbf{U}_{1,\mathcal{C}})$, t stabilizes $\mathbf{U}_{2,\mathcal{C}} \leq \mathbf{U}_{1,\mathcal{C}}$. Thus, by Proposition 5.3(ii) and by definition of Γ_u , we have

$$(\Gamma_u)^t = ((\psi_u)^G)^t = ((\psi_u)^t)^G = (\psi_{u^t})^G = \Gamma_{u^t}.$$

Hence

$$(\Gamma_u)^\sigma = \Gamma_{u^\ell} = \Gamma_{u^t} = \Gamma_u^t.$$

As the outer automorphism of type \mathfrak{t}_ℓ is of order at most 2, t^{-1} also induces the automorphism of type \mathfrak{t}_ℓ and t^2 defines an inner automorphism. We write $\mathcal{H}_t = \mathcal{K} \times \langle \sigma_1 t^{-1} \rangle$ for the group of automorphism as in Definition 4.16. In particular, the element t is independent of the choice of u .

Hence, all that remains to show is that Γ_u is \mathcal{K} -invariant. Recall that \mathcal{K} is the set of all Galois automorphisms fixing all roots of unity of ℓ' order. By Proposition 5.8, the character field $\mathbb{Q}(\Gamma_u)$ is contained in $\mathbb{Q}(\sqrt{\eta p})$ which is \mathcal{K} -fixed by Lemma 4.6. \square

5.B Proof of condition $A(\infty)_{\mathcal{H}_t}$

From now on we denote by $t_2 \in \text{Out}'(G)$ the outer diagonal automorphism of G with type \mathfrak{t}_ℓ . We choose this notation for easier identification of elements in the later chapters. As condition $A(\infty)$ from [CS17] for the McKay conjecture holds, we are able to extend this result to in order to verify condition $A(\infty)_{\mathcal{H}_t}$, using the previous considerations about the action of Galois automorphisms on the GGGRs. We first recall the following theorem which gives condition $A(\infty)$.

Theorem 5.10. [CS17, Theorem 4.1] *For any $\tilde{\chi} \in \text{Irr}(\tilde{G})$, there exists an irreducible character $\chi \in \text{Irr}(G \mid \tilde{\chi})$, such that $(\tilde{G} \rtimes E)_\chi = \tilde{G}_\chi \rtimes E_\chi$ and χ extends to a character of $G \rtimes E_\chi$.*

In particular, the above theorem is implied by the fact that unipotent classes can be chosen to be compatible with the action of E as the following theorem shows.

Theorem 5.11. [CS17, Theorem 4.6] *Let $\mathcal{C} \in \text{Uni}(\tilde{\mathbf{G}})$. We can choose a unipotent element $u \in \mathcal{C}$ such that Γ_u is E -invariant and extends to $G \rtimes E$.*

Proposition 5.12. *There exists a map*

$$\Pi : \text{Irr}(\tilde{G}) \rightarrow \text{Uni}(\tilde{\mathbf{G}})$$

such that the the following hold:

$$(i) \langle \tilde{\chi}, \tilde{\Gamma}_{\Pi(\tilde{\chi})} \rangle = 1,$$

(ii) Π is constant on $\text{Irr}(\tilde{G}/G) \rtimes E$ -orbits and \mathcal{H}_{t_2} -orbits.

Proof. Let $\tilde{\chi} \in \text{Irr}(\tilde{G})$. By Proposition 5.5, there exists a unique class $\mathcal{C} \in \text{Uni}(\tilde{\mathbf{G}})$ with $\langle \tilde{\chi}, \tilde{\Gamma}_{\mathcal{C}} \rangle = 1$ and such that $\langle \tilde{\chi}, \tilde{\Gamma}_{\mathcal{D}} \rangle = 0$, for every unipotent class $\mathcal{D} \in \text{Uni}(\tilde{\mathbf{G}})$ such that $\mathcal{D} \notin \mathcal{C}$. Set $\Pi(\tilde{\chi}) := \mathcal{C}$ with these properties. According to Lemma 5.11, we can associate to \mathcal{C} a unipotent element $u \in \mathcal{C}^F \cap \mathbf{U}_{2,\mathcal{C}}$, such that the GGGC Γ_u is E -stable and extends to $G \rtimes E$. Furthermore, Γ_u is \mathcal{H}_{t_2} -stable by Proposition 5.9. Since $\tilde{\Gamma}_{\mathcal{C}} = (\Gamma_u)^{\tilde{G}}$, the same properties translate to $\tilde{\Gamma}_{\Pi(\tilde{\chi})}$ and $\tilde{\Gamma}_{\Pi(\tilde{\chi})}$ is stable under multiplication with linear characters in $\text{Lin}(\tilde{G}/G)$.

or $E\mathcal{H}_{t_2}$ Let $\tilde{\chi}' \in \text{Irr}(\tilde{G})$ such that $\tilde{\chi}'$ is either $E\mathcal{H}_{t_2}$ -conjugate to $\tilde{\chi}$, or such that $\tilde{\chi}' = \tilde{\chi} \cdot \lambda$ for some $\lambda \in \text{Lin}(\tilde{G}/G)$. Then

$$\langle \tilde{\chi}', \tilde{\Gamma}_{\mathcal{C}} \rangle = \langle \tilde{\chi}, \tilde{\Gamma}_{\mathcal{C}} \rangle = 1,$$

since $\tilde{\Gamma}_{\mathcal{C}}\lambda = \tilde{\Gamma}_{\mathcal{C}}$ and $\tilde{\Gamma}_{\mathcal{C}}$ is E and \mathcal{H}_{t_2} -stable.

Let $\mathcal{C}' \in \text{Uni}(\tilde{\mathbf{G}})$, such that $\mathcal{C}' \notin \Pi(\tilde{\chi})$. With the same reasoning as above, $\tilde{\Gamma}_{\mathcal{C}'}$ is stable under the action of E , $\text{Lin}(\tilde{G}/G)$ and \mathcal{H}_{t_2} . But since $u' \in \mathcal{C}'^F \cap \mathbf{U}_{2,\mathcal{C}'}$ implies $u' \notin \mathcal{C}$ and there is a unique $\chi'_0 \in \text{Irr}(\tilde{G} \mid \tilde{\chi})$ such that $\langle \chi'_0, \Gamma_{u'} \rangle = 1$, $\tilde{\chi}$ cannot be contained in $\text{Irr}(\tilde{\Gamma}_{\mathcal{C}'})$. Thus $\Pi(\tilde{\chi}') = \Pi(\tilde{\chi})$. \square

We are now able to give an alternative proof of Theorem 5.2 of [RSST25], where condition $A(\infty)$ is strengthened to its \mathcal{H}_{t_2} -equivariant version.

Theorem 5.13. *Let $t_2 \in \tilde{G}$ such that t_2 induces a diagonal automorphism of type \mathfrak{t}_ℓ . Then there exists an E -stable \tilde{G} -transversal \mathbb{T} in $\text{Irr}(G)$, such that \mathbb{T} is \mathcal{H}_{t_2} -stable and each $\chi \in \mathbb{T}$ extends to $G \rtimes E_\chi$.*

Proof. Let \mathcal{C} be any unipotent \tilde{G} -class. Then, according to Lemma 5.11, there exists a unipotent element $u(\mathcal{C}) \in \mathcal{C}^F \cap \mathbf{U}_{2,\mathcal{C}}$, such that $\Gamma_{u(\mathcal{C})}$ is an E -invariant GGGC of G that extends to $G \rtimes E$. We fix such an element $u(\mathcal{C})$ for each class \mathcal{C} .

Given $\chi \in \text{Irr}(G)$ and $\tilde{\chi} \in \text{Irr}(\tilde{G} \mid \chi)$, we let $\mathcal{C} := \Pi(\tilde{\chi})$ be the unipotent $\tilde{\mathbf{G}}$ -conjugacy class associated to $\tilde{\chi}$, given by Proposition 5.12. Note furthermore that by Definition

$$\tilde{\Gamma}_{\Pi(\tilde{\chi})} = (\Gamma_{u(\mathcal{C})})^{\tilde{G}} \quad \text{and} \quad \langle \tilde{\Gamma}_{\Pi(\tilde{\chi})}, \tilde{\chi} \rangle = 1. \quad (5.1)$$

The class $\mathcal{C} = \Pi(\tilde{\chi})$ is independent of the choice of $\tilde{\chi} \in \text{Irr}(\tilde{G} \mid \chi)$, since each GGGC of \tilde{G} is $\text{Lin}(\tilde{G}/G)$ -stable. Equation (5.1) allows us to apply Proposition 2.22 to the above situation with

$$X := G \triangleleft \tilde{G} =: Y, \quad \rho := \Gamma_{u(\mathcal{C})} \quad \text{and} \quad \theta := \tilde{\chi}.$$

Thus, there exists a unique constituent χ_0 of $\tilde{\chi}|_G$ that is also a constituent of $\Gamma_{u(\mathcal{C})}$ and thus χ_0 is \tilde{G} -conjugate to χ . We set

$$\mathbb{T} = \{ \chi \in \text{Irr}(G) \mid \chi \in \text{Irr}(\Gamma_{u(\Pi(\tilde{\chi}))}) \text{ for some } \tilde{\chi} \in \text{Irr}(\tilde{G} \mid \chi) \}.$$

We observe that χ_0 is the constituent that lies in \mathbb{T} . Hence, the set \mathbb{T} is a \tilde{G} -transversal in $\text{Irr}(G)$.

Now let $\delta \in E$, $\chi \in \mathbb{T}$ and $\tilde{\chi} \in \text{Irr}(\tilde{G} \mid \chi)$ such that $\chi \in \text{Irr}(\Gamma_{u(\Pi(\tilde{\chi}))})$. Then $\tilde{\chi}^\delta \in \text{Irr}(\tilde{G} \mid \chi^\delta)$ and by the properties of Π we have $\Pi(\tilde{\chi}^\delta) = \Pi(\tilde{\chi})$. Furthermore $\chi^\delta \in \text{Irr}(\Gamma_{u(\mathcal{C})}^\delta) = \text{Irr}(\Gamma_{u(\mathcal{C})})$ as the character $\Gamma_{u(\mathcal{C})}$ is E -invariant. Since χ is unique with this property we have $\chi^\delta \in \mathbb{T}$.

Similarly for $\sigma \in \mathcal{H}_{t_2}$ and $\chi \in \mathbb{T}$, $\tilde{\chi} \in \text{Irr}(\tilde{G} \mid \chi)$, $\mathcal{C} = \Pi(\tilde{\chi}^\sigma) = \Pi(\tilde{\chi})$, as we have seen that Π is constant on $\text{Lin}(\tilde{G}/G)$ and \mathcal{H}_{t_2} -orbits.

By Proposition 5.9, $\Gamma_{u(\mathcal{C})}$ is \mathcal{H}_{t_2} -invariant, so again $\tilde{\chi}^\sigma \in \text{Irr}(\tilde{G} \mid \chi^\sigma)$, $\chi^\sigma \in \text{Irr}(\Gamma_{u(\mathcal{C})})$ and $\chi^\sigma \in \mathbb{T}$. This finishes the proof as the extension property is provided by Proposition 2.22 (ii). \square

6. Local conditions for type A in the doubly regular case

As we have now verified condition $A(\infty)_{\mathcal{H}_t}$ we continue in this chapter towards the verification of the local conditions $A(d)_{\mathcal{H}_t}$ and $B(d)_{\mathcal{H}_\ell}$ given the integer $d = d_\ell(q)$ for type A, see Definition 4.24.

Our approach begins by focusing on the doubly regular case, where $d_\ell(\epsilon q) \mid n$. In this case, the centralizer of the Sylow d -torus \mathbf{S} is abelian. Consequently, the extension map Λ with respect to $C_G(\mathbf{S}) \triangleleft N_G(\mathbf{S})$ can be constructed in a particularly convenient manner.

We note, however, that this extension map, originally constructed in [CS17], is generally not \mathcal{H}_ℓ -equivariant. The constructions show that, once again, we need to introduce an automorphism $t_1 \in \text{Aut}(G)$ such that \mathcal{H}_{t_1} -equivariance holds. Together with studying the characters of the relative Weyl group W_φ for some $\varphi \in \text{Irr}(C_G(\mathbf{S}))$, this enables us to verify condition $A(d)_{\mathcal{H}_t}$ in this case. This result from the doubly regular case, together with the global condition $A(\infty)_{\mathcal{H}_t}$ is then used to ensure the local condition in the general case, see Chapter 7.

Subsequently, we establish that the choice of t_1 in the local case aligns with the choice of t_2 in the global condition. Finally, as a corollary of the constructions in the verification of Condition $A(d)_{\mathcal{H}_{t_1}}$, we derive condition $B(d)_{\mathcal{H}_\ell}$ in the doubly regular case.

Notation

We use the following notation. Let p be a prime and set $q = p^m$ for some integer $m \geq 1$. Let $\epsilon \in \{\pm 1\}$, and let ℓ be an odd prime different from p . We denote by

$$d = d_\ell(q), \quad d_0 = d_\ell(\epsilon q)$$

the orders of q and ϵq , respectively, in $(\mathbb{Z}/\ell\mathbb{Z})^\times$. Let \mathbf{G} be a group with root system of type A_{n-1} , and let $F' : \mathbf{G} \rightarrow \mathbf{G}$ be the Steinberg endomorphism defined in Definition 3.15, i.e $F' \in \{F_q, \gamma F_q\}$ such that $F' = \gamma^{\frac{1-\epsilon}{2}} F_q$. Note that we use different notation here, as throughout most of this chapter, we will use a twisted version of F' . We

write \mathbf{S} for a Sylow d -torus of (\mathbf{G}, F') as defined in Definition 3.24. Finally, suppose $\mathbf{G} \leq \tilde{\mathbf{G}}$ is a regular embedding such that $\tilde{\mathbf{G}}^{F'} \cong \mathrm{GL}_n(\epsilon q)$. We remind the reader that an integer d is a regular number for (\mathbf{G}, F') , if $C_{\mathbf{G}}(\mathbf{S})$ is a torus, which is equivalent to d_0 dividing n or $(n-1)$, see Example 3.27. We denote by $E(\mathbf{G}^{F'})$ the group of automorphisms of $\mathbf{G}^{F'}$, as defined in Definition 3.18. For any automorphism $t \in \mathrm{Aut}(\mathbf{G}^{F'})$, we write \mathcal{H}_t for the subgroup of $\mathcal{G} \times \mathrm{Aut}(G)$ given by

$$\mathcal{H}_t = \mathcal{K} \times \langle \sigma_1 \cdot t \rangle,$$

as described in Definition 4.16.

Definition 6.1. If (\mathbf{G}, F') is of type A_{n-1} , an integer d is called *doubly regular* for (\mathbf{G}, F') , if $d_0 \mid n$, where d_0 is the integer associated to d as in Proposition 4.7.

We assume that $d = d_\ell(q)$ is doubly regular for the rest of this chapter. The goal of this chapter is the verification of condition $A(d)_{\mathcal{H}_t}$ for the doubly regular case. In particular, the element $t \in \tilde{\mathbf{G}}^{F'}$ induces an outer automorphism of type \mathfrak{t}_ℓ , see Definition 4.13, such that it is consistent with the automorphism of condition $A(\infty)_{\mathcal{H}_t}$.

Theorem 6.2. *Let $\mathbf{G}^{F'} = \mathrm{SL}_n(\epsilon q)$, where $q = p^m$ is a prime power, $\epsilon = \pm 1$, and let ℓ be an odd prime distinct from p and set $d := d_\ell(q)$. Assume that d is doubly regular for (\mathbf{G}, F') . Let \mathbf{S} be a Sylow d -torus of (\mathbf{G}, F') . Then, there exists some $t \in \tilde{\mathbf{G}}$ inducing an automorphism on $\mathbf{G}^{F'}$ of type \mathfrak{t}_ℓ and stabilizing \mathbf{S} , such that the following holds:*

- *there exists a $(\mathbf{G}^{F'} \rtimes E(\mathbf{G}^{F'}))_{\mathbf{S}}$ -stable $N_{\tilde{\mathbf{G}}}(\mathbf{S})^{F'}$ -transversal \mathbb{T} in $\mathrm{Irr}(N_{\mathbf{G}}(\mathbf{S})^{F'})$,*
- *each $\chi \in \mathbb{T}$ extends to $(\mathbf{G}^{F'} \rtimes E(\mathbf{G}^{F'}))_{\mathbf{S}, \chi}$ and*
- *\mathbb{T} is \mathcal{H}_t -stable.*

In particular, $A(d)_{\mathcal{H}_t}$ holds for $\mathbf{G}^{F'}$ in this case.

To construct the $N_{\tilde{\mathbf{G}}}(\mathbf{S})^{F'}$ -transversal in $\mathrm{Irr}(N_{\mathbf{G}}(\mathbf{S})^{F'})$, we follow [CS17, Section 5], where a parametrization of $\mathrm{Irr}(N_{\mathbf{G}}(\mathbf{S})^{F'})$ is established by studying isomorphic groups obtained using a twisted version of the Steinberg endomorphism F' .

In particular, we may first replace the use of F' by the Steinberg endomorphism

$$F : \tilde{\mathbf{G}} \rightarrow \tilde{\mathbf{G}}, \quad F := (\gamma_0)^{\frac{1-\epsilon}{2}} \circ F_q,$$

where γ_0 denotes the automorphism on $\tilde{\mathbf{G}}$ given by the transpose-inverse. Recall that in the global case we used the Steinberg endomorphism $\gamma^{\frac{1-\epsilon}{2}} F_q$, where γ is the graph automorphism as defined in Example 3.11. We remind the reader of the existence of an element $v_0 \in \mathbf{G}$, such that $\mathrm{int}(v_0) \circ \gamma_0 = \gamma$, as established in Notation 3.15. Given some $v \in \mathbf{G}$, we let v^F denote the automorphism $\mathrm{int}(v) \circ F$. We now turn to the choice of the element v . We fix the following notation for this chapter.

Notation 6.3. We write \mathbf{T} and $\tilde{\mathbf{T}}$ for the group of diagonal matrices in \mathbf{G} and $\tilde{\mathbf{G}}$ respectively. Let \tilde{V} be the group of monomial matrices in $\tilde{\mathbf{G}}$ with coefficients in $\{0, 1, -1\}$, and set $V := \tilde{V} \cap \mathbf{G}$. Note that V is generated by the elements

$\{\mathbf{n}_i \mid i = 1, \dots, n-1\}$, where \mathbf{n}_i is the monomial matrix from Example 3.4, constructed via the root system of \mathbf{G} . Since d is a doubly regular number for (\mathbf{G}, F') , the number $a := \frac{n}{d_0}$ is an integer. We define the element $v \in V$ by

$$v := \mathbf{n}_1 \dots \mathbf{n}_{d_0-1} \mathbf{n}_{d_0+1} \dots \mathbf{n}_{ad_0-1} \in N_{\mathbf{G}}(\mathbf{T}).$$

In particular, v has the block-matrix form

$$v = \begin{pmatrix} \boxed{v_{d_0}} & 0 & \dots \\ 0 & \ddots & 0 \\ \vdots & 0 & \boxed{v_{d_0}} \end{pmatrix} \text{ with } v_{d_0} := \begin{pmatrix} 0 & 0 & (-1)^{d_0+1} \\ 1 & 0 & 0 \\ \vdots & \ddots & 0 \\ 0 & 1 & 0 \end{pmatrix}. \quad (6.1)$$

Thus, v corresponds to a product of a disjoint d_0 -cycles in the Weyl group $W \cong \mathfrak{S}_n$ under the natural epimorphism $\rho : N_{\mathbf{G}}(\mathbf{T}) \rightarrow N_{\mathbf{G}}(\mathbf{T})/\mathbf{T}$. Moreover, v is invariant under the action of γ_0 and F_q , so

$$[\gamma_0, v] = [F, v] = 1 \quad \text{and} \quad v \in N_{\mathbf{G}}(\mathbf{T})^{(\gamma_0, F_p)} \quad (6.2)$$

In case p is an odd prime, v satisfies $v^{d_0} = (-E_n)^{d_0+1}$ and

$$\text{ord}(v) = \begin{cases} 2d_0 & \text{if } 2 \mid d_0, \\ d_0 & \text{if } 2 \nmid d_0. \end{cases} \quad (6.3)$$

In case $p = 2$, we have $\text{ord}(v) = d_0$.

From now on, we will work with the Steinberg endomorphism $vF : \tilde{\mathbf{G}} \rightarrow \tilde{\mathbf{G}}$, see Proposition 3.17. We fix a Sylow d -torus \mathbf{S} of (\mathbf{T}, vF) , which is also a Sylow d -torus of (\mathbf{G}, vF) by the considerations in Example 3.30. From now on we set $G := \mathbf{G}^{vF}$ and $\tilde{G} := \tilde{\mathbf{G}}^{vF}$ and write $\mathcal{L}_{vF} : \mathbf{G} \rightarrow \mathbf{G}$ for the corresponding Lang map. We define the groups

$$N := N_{\mathbf{G}}(\mathbf{S})^{vF}, \quad \tilde{N} := N_{\tilde{\mathbf{G}}}(\mathbf{S})^{vF}, \quad L := C_{\mathbf{G}}(\mathbf{S})^{vF}, \quad \tilde{L} := C_{\tilde{\mathbf{G}}}(\mathbf{S})^{vF}.$$

Note that the new choice vF also leads to a new automorphism group. Recall $q = p^m$ and define the group $E := C_{2d_0m} \times C_2$. It acts on the group $\tilde{\mathbf{G}}_{F_p}^{2d_0m}$ by F_p in the first factor and by the transpose-inverse automorphism γ_0 in the second factor.

We see that $\tilde{\mathbf{G}}^{vF} \leq \tilde{\mathbf{G}}_{F_p}^{2d_0m}$ by the previous considerations. Furthermore, note that in $\text{Aut}(G)$, the automorphism of G induced by F_q has order $\text{ord}(vZ(G))$ which divides $2d_0$. Hence, the order of F_p divides $2d_0m$ as an automorphism of G . We write \hat{F} for the element in E inducing the automorphism F on $\mathbf{G}_{F_p}^{2d_0m}$.

6.A Structure of the normalizer and the centralizer in the doubly regular case

In order to construct an extension map, we examine the structure of the Levi subgroup $L = C_{\mathbf{G}}(\mathbf{S})^{vF}$, and observe that most of the relevant information about the characters is contained in a much smaller subgroup, namely the product of cyclic subgroups of order 2, when $p \neq 2$. Let $W_d = N/L$ be the relative Weyl group, and let

$\rho : N_{\mathbf{G}}(\mathbf{T}) \rightarrow N_{\mathbf{G}}(\mathbf{T})/\mathbf{T} \cong \mathfrak{S}_n$ denote the natural epimorphism. By the structure of v from (6.1), we see that $C_{\mathfrak{S}_n}(\rho(v)) \cong C_{d_0} \wr \mathfrak{S}_a$. For quadratic matrices $(A_i)_{i=1}^a \in \mathrm{GL}_{n_i}(q)$ we use the notation $\underline{\mathrm{diag}}(A_1, \dots, A_a)$ for the block-diagonal matrix with A_i on the diagonal, i.e.,

$$\underline{\mathrm{diag}}(A_1, \dots, A_a) := \begin{pmatrix} \boxed{A_1} & 0 & \dots \\ 0 & \ddots & 0 \\ \vdots & 0 & \boxed{A_a} \end{pmatrix} \in \mathrm{GL}_{\sum_{i=1}^a n_i}(q).$$

From now on we assume that p is an odd prime unless specified otherwise. The case for $p = 2$ is considerably simpler and will be explored at the end of section 6.B Recall that we denote by C_i the cyclic group of order i .

Lemma 6.4. *If d is doubly regular for (\mathbf{G}, vF) , then*

$$L = C_{\mathbf{G}}(\mathbf{S})^{vF} \cong (C_{q^{d_0} - \epsilon^{d_0}})^{a-1} \times C_{\frac{q^{d_0} - \epsilon^{d_0}}{q - \epsilon}}.$$

Proof. By Example 3.30, we know that \tilde{L} is the direct product of a copies of $\mathrm{GL}_1((\epsilon q)^{d_0})$, with each group generated by an element in $\mathrm{GL}_{d_0}(\epsilon q)$ of order $(q^{d_0} - \epsilon^{d_0})$. So $\tilde{L} \cong (C_{q^{d_0} - \epsilon^{d_0}})^a$. Since $L = \mathbf{G} \cap \tilde{L} = \mathrm{SL}_n \cap \tilde{L}$, the result follows. \square

Proposition 6.5. *Let $V_d := C_V(v)$, $\tilde{V}_d := C_{\tilde{V}}(v)$, $H_d := \mathbf{T} \cap V_d$ and $\tilde{H}_d := \tilde{V}_d \cap \tilde{\mathbf{T}}$. The following statements hold.*

- (a) $\tilde{H}_d \cong (C_2)^a$ and $H_d \cong (C_2)^{a'}$, with $a' = \begin{cases} a, & \text{if } d_0 \text{ is even,} \\ a - 1, & \text{if } d_0 \text{ is odd.} \end{cases}$
- (b) $W_d = \rho(V_d) = C_{\mathfrak{S}_a}(\rho(v)) \cong C_{d_0} \wr \mathfrak{S}_a$.
- (c) In case d_0 is even we have $V_d \cong C_{2d_0} \wr \mathfrak{S}_a \cong \tilde{V}_d$.
- (d) In case d_0 is odd, we have $V_d \cong \ker(\alpha) \leq C_{2d_0} \wr \mathfrak{S}_a \cong \tilde{V}_d$ for the linear character $\alpha \in \mathrm{Irr}(\tilde{V}_d)$ given by

$$(x_1, x_2, \dots, x_a, \delta) \mapsto \mathrm{sgn}(\delta) \prod \beta(x_i),$$

where β is the character of order 2 on C_{2d_0} .

- (e) $N = V_d L$.

Proof. We follow the proof of Proposition 5.5 in [CS17].

By definition \tilde{H}_d is the group of diagonal matrices $\underline{\mathrm{diag}}(e_1, \dots, e_a)$ with $e_i \in \{\pm E_{d_0}\}$, hence $\tilde{H}_d \cong (C_2)^a$. In case d_0 is even $\det(e_i) = 1$, hence $H_d \cong C_2^a$. In case d_0 is odd, $\det(e_i) = -1$ and $H_d \cong (C_2)^{a-1}$.

As we have

$$N/L = N_{\mathbf{G}}(\mathbf{S})^{vF}/C_{\mathbf{G}}(\mathbf{S})^{vF} = (N_{\mathbf{G}}(\mathbf{S})/C_{\mathbf{G}}(\mathbf{S}))^{vF} \cong C_{\mathfrak{S}_n}(\rho(v)),$$

where the second equality follows from Proposition 3.13 as $C_{\mathbf{G}}(\mathbf{S})$ is connected, part (e) follows from the second isomorphism theorem with $H_d = L \cap V_d$, once $\rho(V_d) =$

$C_{d_0} \wr \mathfrak{S}_a$ is proven.

Assume d_0 is even. By (6.1), v is the matrix of a blocks of $d_0 \times d_0$ matrices v_0 with $v_0^{d_0} = -E_{d_0} \in G$. Hence, an easy computation shows that $V_d = C_V(v)$ is isomorphic to the group $C_{2d_0} \wr \mathfrak{S}_a$, where \mathfrak{S}_a acts on the block matrices by permuting them. In particular, we have

$$\rho(V_d) = V_d/\mathbf{T} \cong V_d/H_d \cong C_{d_0} \wr \mathfrak{S}_a.$$

Now assume that d_0 is odd. Again we see that $\tilde{V}_d = C_{\tilde{V}}(v) \cong (C_2 \times C_{d_0}) \wr \mathfrak{S}_a \cong (C_{2d_0}) \wr \mathfrak{S}_a$. We obtain V_d as the kernel of the determinant map \det , which in this case maps the permutation matrices associated with $\delta \in \mathfrak{S}_a$ to $\text{sgn}(\delta)$, and when restricted to C_{2d_0} , gives the character of order 2. This proves part (d), and since $H_d = \tilde{H}_d \cap \ker(\alpha) \not\cong \tilde{H}_d$, we have

$$\rho(V_d) = V_d/H_d \cong \tilde{V}_d/\tilde{H}_d \cong C_{d_0} \wr \mathfrak{S}_a,$$

which completes the proof of part (b). \square

6.B Construction of an equivariant extension map

We assume the Notation from the previous sections. To describe all irreducible characters of N , we aim to apply Lemma 2.10 to the groups $L \triangleleft N$. For this, we must construct an extension map Λ with respect to $L \triangleleft N$ and analyze the characters of the relative Weyl group N_φ/L for a given $\varphi \in \text{Irr}(L)$. The construction of this map will be carried out in this section.

In particular, the extension map Λ must satisfy certain properties that allow us to study the action of \mathcal{H}_ℓ on $\text{Irr}(N)$ and to build a suitable \tilde{N} -transversal in $\text{Irr}(N)$. Our aim in the next two sections is therefore to construct an extension map Λ with respect to $L \triangleleft N$ such that the following are satisfied. Recall $E = C_{2d_0m} \times C_2$ and its action on G , which is defined after Notation 6.3.

- Λ is $N \rtimes E$ -equivariant
- for each $\varphi \in \text{Irr}(L)$ the character $\Lambda(\varphi)$ has an extension to $\hat{\varphi} \in \text{Irr}((N \rtimes E)_\varphi)$ such that $v\hat{\varphi} \in \ker(\hat{\varphi})$, and
- there exists an automorphism $t \in \text{Aut}(G)$ of type \mathfrak{t}_ℓ such that L and N are t -stable and Λ is \mathcal{H}_t -equivariant.

Together with additional details on the characters of the relative Weyl group, this will allow us to establish a proof of Theorem 6.2, as we will see later.

We begin by constructing an extension map with respect to $H_d \triangleleft V_d$, following Section 5 of [CS17]. Already in this basic construction we observe that the action of \mathcal{H}_ℓ on the irreducible characters of V_d is non-trivial and \mathcal{H}_ℓ -equivariance does not generally hold for the extension map obtained this way. This motivates the introduction of a first iteration of an element $t \in \tilde{G}$ such that the extension map is \mathcal{H}_t -equivariant.

We first explore the case for $a = 1$.

Proposition 6.6. *Assume that $n = d_0$, or equivalently $a = 1$. Then there exists an extension map $\tilde{\Lambda}_0$ with respect to $\tilde{H}_d \triangleleft \tilde{V}_d$, which is \mathcal{H}_{t_0} -equivariant, where $t_0 \in \tilde{\mathbf{G}}$ is defined by*

$$t_0 := \begin{cases} \text{diag}(-1, 1, \dots, 1, -1, 1), & \text{if } (\ell - 1)_2 = (d_0)_2 \\ E_{d_0}, & \text{otherwise.} \end{cases}$$

Proof. Recall that ℓ is an odd prime, hence $2 \mid n = d_0$ if $(\ell - 1)_2 = (d_0)_2$. Hence, t_0 is well-defined. Furthermore, it is clear, that t_0 stabilizes H_d .

By Proposition 6.5 we have $\tilde{H}_d \cong C_2$ and in particular $\tilde{H}_d \cong \langle -E_{d_0} \rangle$. Since $v = v_{d_0}$ from (6.1) computations show that

$$\tilde{V}_d = \langle -E_{d_0}, v \rangle \cong \begin{cases} C_{2d_0}, & \text{if } 2 \mid d_0 \\ C_2 \times C_{d_0}, & \text{if } 2 \nmid d_0 \end{cases}.$$

In particular, as $C_2 \times C_{d_0}$ is cyclic itself for odd d_0 , so we have $\tilde{V}_d \cong C_{2d_0}$ in both cases.

Thus, there exists an extension map Λ_0 with respect to $\tilde{H}_d \triangleleft \tilde{V}_d$. The two characters of $\text{Irr}(C_2)$ are obviously \mathcal{H}_ℓ -invariant and V_d -invariant.

As we have seen in Proposition 4.2, \mathcal{H}_ℓ acts trivially on d_0 -th roots of unity.

Let $\chi \in \text{Irr}(C_{2d_0})$ be a character of \tilde{V}_d of order $2d_0$. Then by Proposition 4.2, $\sigma_1 \in \mathcal{H}_\ell$ changes the sign of the primitive $2d_0$ -th roots of unity whenever $(\ell - 1)_2 = (d_0)_2$, otherwise the action of \mathcal{H}_ℓ is trivial. As ℓ is an odd prime, $\tilde{\Lambda}_0$ is \mathcal{H}_ℓ -equivariant whenever d_0 is odd.

Now assume that $(\ell - 1)_2 = (d_0)_2$. We calculate $v^{t_0} = -v$ which is equal to $v^{d_0}v$ by (6.3). In particular, this also shows that t_0 stabilizes V_d . Calculations show

$$\tilde{\Lambda}_0(\mu_1)^{t_0}(v) = \tilde{\Lambda}_0(\mu_1)(-v) = \tilde{\Lambda}_0(\mu_1)(v)^{\sigma_1},$$

whenever $\mu_1 \in \text{Irr}(\tilde{H}_d)$ and $(\ell - 1)_2 = (d_0)_2$. Thus, $\tilde{\Lambda}_0$ is \mathcal{H}_{t_0} -equivariant. \square

Note that t_0 was chosen such that $v^{t_0} = -v$. We continue with the case where $n = ad_0$.

Lemma 6.7. *Let $n = ad_0$. There exists an extension map $\tilde{\Lambda}_0$ with respect to $\tilde{H}_d \triangleleft \tilde{V}_d$, such that $\tilde{\Lambda}_0$ is $\mathcal{H}_{t'_1}$ -equivariant for $t'_1 \in \tilde{\mathbf{G}}$ with*

$$t'_1 := \underline{\text{diag}}(t_0, \dots, t_0),$$

where $t_0 \in \text{GL}_{d_0}(q)$ is as defined in Proposition 6.6.

Proof. We have $\tilde{H}_d \cong (C_2)^a$ and $\tilde{V}_d \cong C_{2d_0} \wr \mathfrak{S}_a$ by Proposition 6.5.

Any character of $\lambda \in \text{Irr}(\tilde{H}_d)$ is a product of a characters of $\text{Irr}(C_2)$, thus by Proposition 2.17 the inertia group of λ in \tilde{V}_d is given by

$$\tilde{V}_{d,\lambda} \cong C_{2d_0} \wr Y$$

where $Y \leq \mathfrak{S}_a$ is a Young subgroup determined by λ . Let Λ_0 be the extension map with respect to $C_2 \triangleleft C_{2d_0}$ from the previous proposition. Applying this map to each factor, we obtain an extension of λ to some $\tilde{\lambda} \in \text{Irr}((C_{2d_0})^a)$. Note that $\tilde{\lambda}$ then is Y -stable. Since the base group is abelian, we may extend λ on the semidirect product $\tilde{V}_{d,\lambda}$ using Proposition 2.5, to the character that is trivial on Y . While the trivial character would be the canonical choice, we instead use the following construction in anticipation of the next proposition. To each $\lambda \in \text{Irr}(\tilde{H}_d) = \text{Irr}((C_2)^a)$ associate a_+ as the number of trivial factors in the decomposition of λ and a_- as the number of non-trivial factors of λ . Then by Lemma 2.17, the inertia group of λ in \mathfrak{S}_a is given by $Y = \mathfrak{S}_{a_+} \times \mathfrak{S}_{a_-}$ and we define the character $\psi := 1|_{\mathfrak{S}_{a_+}} \times \text{sgn}|_{\mathfrak{S}_{a_-}}$. We define $\tilde{\Lambda}_0(\lambda)$ as the extension of $\tilde{\lambda}$ that restricts to ψ on Y applying Lemma 2.6. By the previous proposition, $\sigma_1 t_0$ fixes $\tilde{\lambda}$ and thus also fixes ψ and $\tilde{\Lambda}_0(\lambda)$. As t'_1 acts trivially on \tilde{H}_d and \tilde{V}_d/\tilde{H}_d the map $\lambda \mapsto \tilde{\Lambda}_0(\lambda)$ has the desired properties. \square

Until now, we have considered the \mathcal{H}_{t_1} -equivariance only for subgroups of \tilde{G} . In the following lemma, we generalize this approach to extension maps with respect to $H_d \triangleleft V_d$. Note that, in our earlier discussions, \mathcal{H}_ℓ -equivariance was always guaranteed for odd d_0 . However, this property no longer holds for the setting in G .

Definition 6.8. If p is odd then we define the element

$$t_1 := \begin{cases} \text{diag}(-1, 1, -1, \dots, -1, 1), & \text{if } (\ell - 1)_2 = (d_0)_2, \\ \frac{\text{diag}(-E_{\frac{a}{2}d_0}, E_{\frac{a}{2}d_0})}{E_{ad_0}}, & \text{if } 2 \nmid d_0, \ell \equiv 3 \pmod{4}, 2 \mid a, \\ E_{ad_0}, & \text{otherwise.} \end{cases}$$

If $p = 2$, we set $t_1 = E_{ad_0}$.

Proposition 6.9. Let p be odd and t_1 the element from above. Then there exists a $(V_d \rtimes \mathcal{H}_{t_1})$ -equivariant extension map Λ_0 with respect to $H_d \triangleleft V_d$.

Proof. If d_0 is even, then by Proposition 6.5, we have $H_d = \tilde{H}_d$ and $V_d = \tilde{V}_d$. Consequently, we can choose the same extension map $\Lambda_0 = \tilde{\Lambda}_0$, preserving the equivariance properties from Lemma 6.7. In particular, in this case $t_1 = t'_1$.

In the case of odd d_0 , we have $H_d \cong (C_2)^{a-1}$ and V_d is the kernel of the character $\alpha \in \text{Irr}(\tilde{V}_d)$ given by Proposition 6.5(d), as

$$\alpha : (x_1, x_2, \dots, x_a, \delta) \mapsto \text{sgn}(\delta) \prod \beta(x_i),$$

where β is the character of order 2 on C_{2d_0} and we identify \tilde{V}_d with $C_{2d_0} \wr \mathfrak{S}_a$. Let $\tilde{\lambda}$ be an irreducible character of $\tilde{H}_d \cong (C_2)^a$ and define $a_+(\tilde{\lambda})$ as the number of trivial factors of $\tilde{\lambda}$ and $a_-(\tilde{\lambda})$ as the number of non-trivial factors of $\tilde{\lambda}$. In particular, $\tilde{\lambda}$ is conjugate in \tilde{V}_d to some character such that the first $a_+(\tilde{\lambda})$ summands are trivial and the other $a_-(\tilde{\lambda})$ are non-trivial. We assume $\tilde{\lambda}$ to be that character.

We define $V_\pm := (C_2 \times C_{d_0}) \wr \mathfrak{S}_{a_\pm}$ and $H_\pm := (C_2)^{a_\pm}$. With these definitions we have

$$\tilde{V}_{d,\tilde{\lambda}} \cong V_+ \times V_-.$$

We rewrite $\tilde{\lambda}$ as a the character given by

$$\tilde{\lambda} = 1_{H_+} \times \alpha|_{H_-}$$

and extend it on $\tilde{V}_{d,\tilde{\lambda}}$ as

$$\tilde{\Lambda}_0(\tilde{\lambda}) := 1_{V_+} \times \alpha|_{V_-}.$$

Note that this is the same extension as chosen in the proof of Lemma 6.7.

We now take an arbitrary character $\lambda \in \text{Irr}(H_d)$. As H_d and \tilde{H}_d are abelian, λ extends to some character $\tilde{\lambda} \in \text{Irr}(\tilde{H}_d)$. As $\alpha|_{H_d} = 1$, the two possible extensions of λ to \tilde{H}_d are given by $\tilde{\lambda}$ and $\tilde{\lambda} \cdot \alpha|_{\tilde{H}_d}$. So for any element $x \in V_{d,\lambda}$ we have $\tilde{\lambda}^x \in \{\tilde{\lambda}, \tilde{\lambda} \cdot \alpha|_{\tilde{H}_d}\}$. In particular, $x \in V_{d,\lambda}$ acts on λ by permuting the factors within $(C_2)^{a-}$ and $(C_2)^{a+}$, or in case $a_-(\tilde{\lambda}) = a_+(\tilde{\lambda})$ the two sets of factors may be interchanged. Thus, for each $\lambda \in \text{Irr}(H_d)$ the map

$$\phi : V_{d,\lambda} \rightarrow \text{Irr}(\tilde{H}_d), \quad x \mapsto \tilde{\lambda}^x \tilde{\lambda}^{-1}$$

is a well-defined homomorphism with $\ker \phi = V_{d,\tilde{\lambda}}$ and non-trivial if and only if some $\tilde{\lambda} \in \text{Irr}(H_d | \lambda)$ satisfies $a_+(\tilde{\lambda}) = a_-(\tilde{\lambda})$. Either way $V_{d,\tilde{\lambda}} \triangleleft V_{d,\lambda}$ with quotient of order

at most 2.

It remains to show that the character $\tilde{\Lambda}_0(\tilde{\lambda})|_{V_{d,\tilde{\lambda}}}$ is $V_{d,\lambda}$ -invariant; then we may choose the cyclic extension to $V_{d,\lambda}$ as our desired extension. Since $x \in \tilde{V}_d$ may only act as a permutation on $\tilde{\lambda}$ by interchanging the factors, we easily see that for all $x \in \tilde{V}_d$ we have

$$\tilde{\Lambda}_0(\tilde{\lambda})^x = \tilde{\Lambda}_0(\tilde{\lambda}^x).$$

Furthermore, multiplication of $\tilde{\lambda}$ with $\alpha|_{H_d}$ just interchanges the groups H_+ and H_- so we have

$$\tilde{\Lambda}_0(\tilde{\lambda} \cdot \alpha|_{H_d}) = \tilde{\Lambda}_0(\tilde{\lambda}) \cdot \alpha|_{\tilde{V}_{d,\tilde{\lambda}}} \text{ for all } \tilde{\lambda} \in \tilde{H}_d.$$

Thus, in case $(V_d)_\lambda \neq (V_d)_{\tilde{\lambda}}$, we have for $x \in (V_d)_\lambda \setminus (V_d)_{\tilde{\lambda}}$

$$\left(\tilde{\Lambda}_0(\tilde{\lambda})|_{V_{d,\tilde{\lambda}}}\right)^x = \tilde{\Lambda}_0(\tilde{\lambda}^x)|_{V_{d,\tilde{\lambda}}} = \left(\tilde{\Lambda}_0(\tilde{\lambda}) \cdot \alpha|_{\tilde{V}_{d,\tilde{\lambda}}}\right)|_{V_{d,\tilde{\lambda}}} = \tilde{\Lambda}_0(\tilde{\lambda})|_{V_{d,\tilde{\lambda}}},$$

since V_d is the kernel of α .

We write Λ_0 for the extension map with respect to $H_d \triangleleft V_d$ obtained by restriction $\tilde{\Lambda}_0$ to V_d . In particular, this means, that for each extension $\tilde{\lambda}$ of λ we have

$$\Lambda_0(\lambda)|_{V_{d,\tilde{\lambda}}} = \tilde{\Lambda}_0(\tilde{\lambda})|_{V_{d,\lambda}}.$$

We now examine the action of \mathcal{H}_ℓ on these extensions.

First we note, that for odd a , $\tilde{V}_{d,\tilde{\lambda}}$ and $V_{d,\lambda}$ agree. So Λ_0 is \mathcal{H}_ℓ -invariant by Lemma 6.7.

In case a is even, there exists a V_d -conjugacy class of characters $\lambda \in \text{Irr}(H_d)$ such that for some $\tilde{\lambda} \in \text{Irr}(\tilde{H}_d | \lambda)$ we have $a_+(\tilde{\lambda}) = a_-(\tilde{\lambda}) = a/2$ and $V_{d,\lambda}/V_{d,\tilde{\lambda}} \cong C_2$. Without loss of generality, select $\lambda \in \text{Irr}(H_d)$ such that $\tilde{\lambda} \in \text{Irr}(\tilde{H}_d | \lambda)$ is of this form and assume that the first $\frac{a}{2}$ factors of $\tilde{\lambda}$ are trivial and the last $\frac{a}{2}$ factors correspond to the non-trivial character $\mu \in \text{Irr}(C_2)$ i.e.,

$$\lambda = \tilde{\lambda}|_{H_d} = (\mu^0 \times \cdots \times \mu^0 \times \mu \times \cdots \times \mu)|_{H_d}.$$

Then the quotient is given by $\langle x, V_{d,\tilde{\lambda}} \rangle / V_{d,\tilde{\lambda}} = V_{d,\lambda} / V_{d,\tilde{\lambda}} \cong C_2$ for some $x \in V_{d,\lambda}$.

We construct an element $x \in V_{d,\lambda}$ to be an element in V_d that swaps the first factor with the $(\frac{ad_0}{2} + 1)$ -st factor, the second with the $(\frac{ad_0}{2} + 2)$ -nd, and so on. Note that each factor of λ corresponds to a character of some $d_0 \times d_0$ -block matrices. We use the notation (i, j) for the permutation matrix in $\text{GL}_n(q)$ that corresponds to the transposition (i, j) in \mathfrak{S}_n . Therefore, we can define x as

$$x := \underline{\text{diag}}(-E_{\frac{a}{2}d_0}, E_{\frac{a}{2}d_0}) \cdot (1, \frac{ad_0}{2} + 1)(2, \frac{ad_0}{2} + 2) \cdots (\frac{ad_0}{2}, ad_0).$$

We see that $\det(x) = 1$ and $[v, x] = [F, x] = 1$, hence $x \in V_d$. Indeed, x acts on $\tilde{\lambda}$ by interchanging the first $\frac{a}{2}$ factors with the last $\frac{a}{2}$ factors. Since $H_d \leq \ker \alpha$ we have

$$\tilde{\lambda}^x = \tilde{\lambda} \cdot \alpha \neq \tilde{\lambda} \text{ and } \lambda^x = \tilde{\lambda}^x|_{H_d} = (\tilde{\lambda} \cdot \alpha)|_{H_d} = \tilde{\lambda}|_{H_d} = \lambda.$$

Thus $x \in V_{d,\lambda} \setminus V_{d,\tilde{\lambda}}$. Furthermore, $x^2 = -E_{ad_0} \in H_d$, so $\text{ord}(x) = 4$ and

$$\Lambda(\lambda)(x^2) = \lambda(x^2) = (-1)^{\frac{a}{2}}.$$

As we have seen in Remark 2.3, the values of $\Lambda_0(\lambda)$ now lie in $\mathbb{Q}(\tilde{\Lambda}(\tilde{\lambda})|_{V_{d,\tilde{\lambda}}}, \zeta_4)$, where ζ_4 is a primitive fourth root of unity. In particular, the character extension in our case is given by

$$\Lambda(\lambda)(x) := \zeta_4^{\frac{a}{2}}$$

Hence, $\Lambda(\lambda)$ takes a primitive fourth root of unity as a value if and only if $a \equiv 2 \pmod{4}$. Let σ_1 be the element of \mathcal{H}_ℓ , that maps ζ_4 to ζ_4^ℓ , as in Definition 4.16. In particular, σ_1 acts trivially on ζ_4 if and only if $\ell \not\equiv 3 \pmod{4}$. So assume that $\ell \equiv 3 \pmod{4}$ and choose the element t_1 as defined above, namely $t_1 = \text{diag}(-E_{\frac{a}{2}d_0}, E_{\frac{a}{2}d_0})$. Indeed t_1 stabilizes H_d and since $v^{t_1} = v$ it also stabilizes V_d . In particular, given any $x_0 \in V_d$ we have $[t_1, x_0] \in H_d$. Specifically, for $x \in V_d$ constructed above, we have:

$$t_1 x t_1^{-1} = -E_{ad_0} x = x^3.$$

Thus

$$\Lambda_0(\lambda)(x)^{t_1} = \Lambda_0(\lambda)(x^2)\Lambda_0(\lambda)(x) = \Lambda_0(\lambda)(xt)^{\sigma_1}.$$

Furthermore, we have

$$\Lambda_0(\lambda)^{t_1}|_{V_{d,\tilde{\lambda}}} = \Lambda_0(\lambda)|_{V_{d,\tilde{\lambda}}} = \Lambda_0(\lambda)^{\sigma_1}|_{V_{d,\tilde{\lambda}}}$$

as $[(V_d)_{\tilde{\lambda}}, t_1] = 1$. This proves $\Lambda_0(\lambda)^{t_1} = \Lambda_0(\lambda)^{\sigma_1}$ as $\Lambda_0(\lambda)$ is linear.

Now assume that λ' is an irreducible character of H_d conjugate to λ via an element $x_0 \in V_d$. Since Λ_0 is V_d -equivariant and $[t, x_0] =: h \in H_d$ we have

$$\Lambda_0(\lambda')^{\sigma_1 t_1} = \Lambda_0(\lambda^{x_0})^{\sigma_1 t_1} = ((\Lambda_0(\lambda)^h)^{\sigma_1 t_1})^{x_0} = \Lambda_0(\lambda)^{x_0} = \Lambda_0(\lambda').$$

Clearly, all characters $\lambda \in \text{Irr}(H_d)$ are \mathcal{H}_{t_1} -invariant, so it remains to show that in the case where d_0 is odd, $\ell \equiv 3 \pmod{4}$ and $2 \mid a$, for any character $\lambda_0 \in \text{Irr}(H_d)$ that is not V_d -conjugate to λ , the character $\Lambda_0(\lambda_0)$ is \mathcal{H}_{t_1} -invariant. Indeed, the element t_1 was chosen such that $t_1 \in \tilde{H}_d$ and hence fixes every extension of $\tilde{\lambda}_0$ to $\tilde{V}_{d,\tilde{\lambda}_0}$. Now, as $V_{d,\lambda_0} = V_{d,\tilde{\lambda}_0}$ we see that $\Lambda_0(\lambda_0)$ is \mathcal{H}_ℓ -invariant. This concludes the statement. \square

Remark 6.10. *In the previous chapter we showed that the global condition $A(\infty)_{\mathcal{H}_t}$ is satisfied for an element $t \in \tilde{G}$ that induces an automorphism of type \mathfrak{t}_ℓ , as defined in Definition 4.13. However, it is not immediately clear from the above statements that t_1 necessarily induces an automorphism of this type. In particular, t_1 is not generally vF -fixed. For example, consider the case $\epsilon = 1$, $\ell = 3$, $p = q = 13$ and $n = 2$. Then we have*

$$d_0 = d_\ell(p) = 1 \text{ and } a = 2.$$

As d_0 is odd and a is even, the previous proposition implies that the extension map Λ_0 with respect to $H_d \triangleleft V_d$ is not \mathcal{H}_ℓ -equivariant and the action of σ_1 on the characters is given by conjugation by some $t_1 \in \tilde{G}$. In this case $[t_1, vF] = 1$, $\det(t_1) = -1$ and t_1 does not induce an inner automorphism of G . On the other hand, the automorphism of type \mathfrak{t}_ℓ is trivial in this situation, since

$$\left(\frac{\ell}{p}\right) = \left(\frac{3}{13}\right) = 1.$$

This discrepancy occurs because the construction above does not depend on the information contained in $q \pmod{4}$. However we will see in Lemma 6.12 that t_1 lies in $\mathbf{G}^{vF}Z(\tilde{G}^{vF})$ (through suitable multiplication by an element in $Z(\tilde{G})$) and this redefined element t_1 indeed induces an automorphism of type \mathfrak{t}_ℓ .

For now, we turn to constructing an extension map with respect to $L \triangleleft N$.

In order to construct such an extension map, we wish to apply our considerations for building extension maps from smaller groups as seen in Lemma 2.6.

In particular, we will extend the characters to their inertia group in $N \rtimes E$. Note that

$$[V_d, E] = 1 \tag{6.4}$$

since $V_d \leq V$ is a subgroup of monomial matrices with entries ± 1 and 0 and all elements are fixed by the standard Frobenius and the transpose-inverse endomorphism γ_0 .

Lemma 6.11. *Assume the Notation 6.3, $p \neq 2$ and let $t_1 \in \tilde{\mathbf{G}}$ as in Definition 6.8. Then there exists an extension map Λ with respect to $L \triangleleft N$ such that the following holds:*

- a) Λ is $N \rtimes E$ -equivariant.
- b) For each $\varphi \in \text{Irr}(L)$ the character $\Lambda(\varphi)$ has an extension $\widehat{\varphi} \in \text{Irr}((N \rtimes E)_\varphi)$ such that $v\widehat{F} \in \ker(\widehat{\varphi})$.
- c) For all $t \in \widetilde{L}$ and $\varphi \in \text{Irr}(L)$, there exists a linear character $\nu_t \in \text{Irr}(N_\varphi)$ such that $\Lambda(\varphi)^t = \Lambda(\varphi)\nu_t$ and $\ker(\nu_t) = N_{\widehat{\varphi}}$, for an extension $\widehat{\varphi}$ of φ to $\langle L, t \rangle$.
- d) Λ is \mathcal{H}_{t_1} -equivariant.

Proof. We begin by constructing an extension map Λ' with respect to $L \triangleleft N \rtimes E / \langle v\widehat{F} \rangle$. We pick some $N \rtimes E$ -transversal in $\text{Irr}(L)$ and let $\varphi \in \text{Irr}(L)$ lie in it. As L is abelian $\lambda_0 := \varphi|_{H_d}$ is irreducible and by Proposition 6.9, there exists an extension $\Lambda_0(\lambda_0) \in \text{Irr}(V_{d,\lambda_0})$. As $[V_d, E] = 1$ we may extend $\Lambda_0(\lambda_0)$ to an irreducible character $\widehat{\lambda}_0 \in \text{Irr}(V_{d,\lambda_0} \rtimes E)$. In particular, as E is abelian and $2d_0$ divides the order of \widehat{F} , we may choose the extension $\widehat{\lambda}_0$ such that $v\widehat{F} \in \ker(\widehat{\lambda}_0)$. Since $[H_d, E] = [V_d, E] = 1$, we have $(V_d \rtimes E)_{\lambda_0} = V_{d,\lambda_0} \rtimes E$.

We apply Lemma 2.6 to this situation with

$$X := L, \quad Y := N \rtimes E, \quad U := V_d \rtimes E, \quad \psi := \varphi \text{ and } \psi_0 := \lambda_0.$$

By Proposition 6.5, the group theoretic assumptions hold, so there exists a unique extension $\widehat{\varphi}$ of φ to $\text{Irr}((N \rtimes E)_\varphi)$ such that

$$\widehat{\varphi}|_{(V_d \rtimes E)_\varphi} = \widehat{\lambda}_0|_{(V_d \rtimes E)_\varphi}.$$

Thus $v\widehat{F} \in \ker(\widehat{\varphi})$.

Let Λ' be the extension map with respect to $L \triangleleft N \rtimes E / \langle v\widehat{F} \rangle$ defined by the above construction through $N \rtimes E$ conjugation. Then we obtain a $N \rtimes E$ -equivariant extension map Λ with respect to $L \triangleleft N$. This concludes the first two statements. Condition c) is implied by Lemma 2.8. Since Λ_0 is \mathcal{H}_{t_1} -equivariant, given $t_1 \in \tilde{\mathbf{G}}$ from Definition 6.8, the map Λ is \mathcal{H}_{t_1} -equivariant taking into account Lemma 2.7 and d) holds. \square

We recall the notation $\left(\frac{\ell}{q}\right)$ for the Jacobi symbol from Definition 4.3 and denote $\check{\mathbf{G}} := \mathcal{L}_{vF}^{-1}(\mathbf{Z}(\mathbf{G}))$. We now show that with our choice of t_1 , this element induces a

diagonal automorphism of type \mathfrak{t}_ℓ on G and thus is consistent with the global case. While the verification is straightforward in principle, it requires checking several distinct cases, thus we give a detailed case-by-case proof.

Lemma 6.12. *The element $t_1 \in \check{\mathbf{G}}$ from Proposition 6.9, induces a diagonal automorphism of \mathbf{G}^{vF} of type \mathfrak{t}_ℓ .*

Proof. In case $p = 2$, the element is trivial, hence the statement holds. Now assume that $p \neq 2$. Recall $d_0 \mid n$. An element $t \in \check{\mathbf{G}}$ induces an automorphism of type \mathfrak{t}_ℓ whenever the map $\pi : \check{\mathbf{G}} \rightarrow C_{(n, q-\epsilon)} := \{x \in \mathbb{F}_{q-\epsilon}^\times \mid x^n = 1\}$, from Lemma 4.12 sends t to $\left(\frac{\ell}{q}\right)^{\frac{n(n-1)}{(n, q-\epsilon)}}$. Recall that a is the integer such that $n = ad_0$ and recall

$$t_1 = \begin{cases} \underline{\text{diag}}(-E_{\frac{a}{2}d_0}, E_{\frac{a}{2}d_0}) & \text{if } 2 \mid a, \ell \equiv 3 \pmod{4}, 2 \nmid d_0 \\ \text{diag}(-1, 1, -1, \dots, 1)^{\frac{\ell-1}{d_0}} & \text{otherwise.} \end{cases}$$

We show in particular, that $\det(t_1) = \pm 1$ and $\mathcal{L}_{vF}(t_1) \in \mathbf{Z}(\mathbf{G})$, so the element t_1 lies in $\check{\mathbf{G}}$.

First assume that a is even, $\ell \equiv 3 \pmod{4}$ and d_0 is odd. Clearly,

$$[t_1, vF] = t_1 t_1^{-1} = E_n \quad \text{and} \quad \det(t_1) = (-1)^{a/2} = (-1)^{n/2}.$$

Thus $t_1 \in \check{\mathbf{G}}$ and

$$\pi(t_1) = (-1)^{\frac{n}{2} \cdot \frac{q-\epsilon}{(n, q-\epsilon)}} = (-1)^{\frac{q-\epsilon}{2} \cdot \frac{n}{(n, q-\epsilon)}}.$$

If the above case does not hold and $(d_0)_2 < (\ell-1)_2$, then $t_1 \in G$ and the automorphism induced by t_1 is an inner automorphism of G . So assume $(d_0)_2 = (\ell-1)_2$. Then

$$[t_1, vF] = -E_n \quad \text{and} \quad \det(t_1) = (-1)^{\frac{n}{2}}.$$

So $t_1 \in \check{\mathbf{G}}$ whenever $4 \mid n = ad_0$. In this case

$$\pi(t_1) = (-1)^{\frac{n}{(n, q-\epsilon)}}.$$

So assume $4 \nmid n = ad_0$. Hence, a is odd and $(\ell-1)_2 = (d_0)_2 = 2$.

Remember that p is an odd prime. We write $C_{q-\epsilon}$ as defined in (4.4) and let $\zeta \in \overline{\mathbb{F}}_q^\times$, be an element of order 4. We write z for the diagonal matrix $\zeta \cdot E_n \in \mathbf{Z}(\check{\mathbf{G}})$. Then zt_1 and t_1 induce the same automorphism by conjugation and

$$[zt_1, vF] = -\zeta^{q-\epsilon} \cdot E_n \quad \text{and} \quad \det(zt_1) = -\zeta^n = 1.$$

In case $q \equiv \epsilon \pmod{4}$, $\zeta^{q-\epsilon} = 1$, so $[zt_1, vF] = -E_n$ and we have $\det([zt_1, vF]) = 1$ which shows $zt_1 \in \check{\mathbf{G}}$. Thus

$$\pi(t_1) = \pi(zt_1) = (-1)^{\frac{n}{(n, q-\epsilon)}}.$$

If $q \equiv -\epsilon \pmod{4}$, then $\zeta^{q-\epsilon} = -1$ so $zt_1 \in \check{\mathbf{G}}$ and has trivial image under π .

We gather the results about the various cases:

$$\pi(t_1) = \begin{cases} (-1)^{\frac{n}{2} \cdot \frac{q-\epsilon}{(n, q-\epsilon)}} & \text{if } \ell \equiv 3 \pmod{4}, 2 \nmid d_0, 2 \mid n, \\ (-1)^{\frac{n}{(n, q-\epsilon)}} & \text{if } (d_0)_2 = (\ell-1)_2, 4 \mid n, \\ (-1)^{\frac{n}{(n, q-\epsilon)}} & \text{if } (d_0)_2 = (\ell-1)_2 = 2, 4 \nmid n, q \equiv \epsilon \pmod{4}, \\ 1 & \text{else.} \end{cases}$$

By Lemma 4.8, the Jacobi symbol satisfies

$$\left(\frac{\ell}{q}\right) = -1 \Leftrightarrow \begin{cases} (\ell-1)_2 = (d_0)_2 \geq 4, \\ (\ell-1)_2 = (d_0)_2 = 2 \text{ and } q \equiv \epsilon \pmod{4}, \\ (\ell-1)_2 = 2, (d_0)_2 = 1 \text{ and } q \equiv -\epsilon \pmod{4}. \end{cases}$$

A comparison between the two expressions may be verified by a straightforward case-by-case check, which we leave to the reader. This leads to

$$\pi(t) = \left(\frac{\ell}{q}\right)^{\frac{n}{(n, q-\epsilon)}},$$

so t_1 induces an automorphism of G of type \mathfrak{t}_ℓ . \square

Proposition 6.13. *The element t_1 as defined above, stabilizes L , N , \tilde{N} and $N_{\mathbf{G}E}(\mathbf{S})$.*

Proof. The torus \mathbf{S} consists of diagonal matrices. As t_1 is a diagonal matrix as well, t_1 commutes with all elements of \mathbf{S} . By definition $L = C_G(\mathbf{S})$ and $N = N_G(\mathbf{S})$. As t_1 induces an automorphism of G and $[t_1, E] = 1$, the statement follows. \square

Remark 6.14. *Let $\varphi \in \text{Irr}(L)$ and let $\hat{\varphi}$ be its extension to $(N \rtimes E)_\varphi$, given by Lemma 6.11. We note that this extension is generally not invariant under the action of \mathcal{H}_{t_1} . In particular, the character φ might be \mathcal{H}_ℓ -fixed, but the choice of extension can yield values that are not \mathcal{H}_ℓ -fixed. Since $[t_1, E] = 1$, this extension can therefore not be \mathcal{H}_{t_1} -invariant either.*

For instance, assume that we have the following situation

$$q = p^m = 3^6, \quad \ell = 5, \quad \epsilon = 1.$$

Then $d_0 = d_\ell(q) = 2$ and by the above considerations the extension map Λ with respect to $L \triangleleft N$ is \mathcal{H}_ℓ -equivariant. Let $v \in V_d$ the element of order $2d_0 = 4$ as defined in (6.1), then $v^2 = -E_n$. Given any $\varphi \in \text{Irr}(L)$ such that $\varphi(v^2) = -1$, we obtain an extension $\hat{\varphi} \in \text{Irr}(N \rtimes E)_\varphi / \langle vF \rangle$, such that $\hat{\varphi}(v) = \zeta_4$, a primitive fourth root of unity and

$$\hat{\varphi}(\hat{F}_p) = \zeta_8,$$

is a primitive 8-th root of unity. Clearly, this character is not \mathcal{H}_ℓ -fixed and hence not \mathcal{H}_{t_1} fixed

As we have restricted our attention to the case where p is an odd prime, we now consider the case $p = 2$. This situation is considerably simpler, since many of the groups constructed above become trivial. In particular, the element t_1 introduced in the previous section is trivial when p is even, which suggests the triviality of the action of σ_1 in this case.

Lemma 6.15. *Let $p = 2$, $q = p^m$, ℓ an odd prime and otherwise assume the Notation 6.3. Let t_1 as in Definition 6.8, equal to the trivial element. Then there exists an extension map Λ with respect to $L \triangleleft N$ such that it satisfies properties a) – c) from Lemma 6.12 and Λ is \mathcal{H}_{t_1} -equivariant.*

Proof. By considerations as in the proof of Proposition 6.5, we have $N = LV_d$ and $H_d = L \cap V_d$ is trivial. As in the construction of Λ in Lemma 6.12, we need to consider an extension map Λ_0 with respect to $H_d \triangleleft V_d$. As in the present setting, however, the group H_d is trivial, so we may therefore choose the trivial extension map Λ_0 with respect to $\{1\} \triangleleft V_d$, which is clearly \mathcal{H}_ℓ -equivariant. We can repeat the arguments of Lemma 6.12, and obtain results a)- c). Since Λ_0 is \mathcal{H}_ℓ -equivariant, so is Λ by Lemma 2.7. \square

6.C Action on the relative Weyl Group

As established in Chapter 2 (see Lemma 2.10), the extension map Λ helps us to parametrize the irreducible characters of N up to N -conjugation by pairs (φ, η) , where $\varphi \in \text{Irr}(L)$ and η is an irreducible character of the relative Weyl group $W_\varphi = N_\varphi/L$, with $W_\varphi \leq W_d = N/L$.

To describe the action of the Galois group \mathcal{H}_ℓ and the group \mathcal{H}_{t_1} on $\text{Irr}(N)$, it is therefore necessary to determine how they act on the characters of W_φ .

Fortunately, in this setting, these groups have a very manageable structure, being wreath products of wreath products. This allows us to extend characters systematically by repeatedly applying Proposition 2.19.

Lemma 6.16. *Let $\varphi \in \text{Irr}(L)$, $\tilde{\varphi} \in \text{Irr}(\tilde{L} \mid \varphi)$, $W_d := N/L$, $W_\varphi = N_\varphi/L$ and $W_{\tilde{\varphi}} = N_{\tilde{\varphi}}/L$. Let η_0 be an irreducible character of $W_{\tilde{\varphi}}$. Let $K := N_{W_d}(W_\varphi) \cap N_{W_d}(W_{\tilde{\varphi}})$. Then there exists a character $\eta \in \text{Irr}(W_\varphi \mid \eta_0)$ such that the following holds:*

- (i) $\{\eta^w \mid w \in K\} \cap \text{Irr}(W_\varphi \mid \eta_0) = \{\eta\}$,
- (ii) η extends to a character in K_η ,
- (iii) η has an extension to a character $\hat{\eta} \in \text{Irr}(K_\eta \times E)$ such that $v\hat{F} \in \ker(\hat{\eta})$ and
- (iv) η is \mathcal{H}_ℓ -invariant.

Remark 6.17. *Note that t_1 from Lemma 6.12 stabilizes $V_d = \rho^{-1}(W_d)$, commutes with W_d , hence (iv) implies that the character η is \mathcal{H}_{t_1} -invariant.*

Proof of Lemma 6.16. The first three conditions follow from [CS17, Lemma 5.12]. We will repeat this construction here in order to check for \mathcal{H}_ℓ -invariance afterwards. Let $\eta_0 \in \text{Irr}(W_{\tilde{\varphi}})$. By Lemma 6.5 we have $W_d \cong C_{d_0} \wr \mathfrak{S}_a$. By Proposition 6.4, $L_{\tilde{\varphi}}$ is isomorphic to a subgroup of

$$\tilde{L} \cong (C_{q^{d_0 - \epsilon d_0}})^a.$$

Thus, the action of W_d on $L_{\tilde{\varphi}}$ is given such that each factor C_{d_0} acts on one of the factors $C_{q^{d_0 - \epsilon d_0}}$ and the elements of \mathfrak{S}_a permute the factors. So (after suitable $W_{\tilde{\varphi}}$ -conjugation) there exists some positive integer $j \in \mathbb{N}$ and

- a set $\{d_i\}_{i=1}^j$ of integers $d_i \mid d_0$,
- a set $\{a_i\}_{i=1}^j$ of integers $a_i \in \mathbb{N}$ with $\sum_{i=1}^j a_i = a$ and
- a partition M_1, \dots, M_j of $\{1, \dots, a\}$, such that $a_i = |M_i|$,

such that $W_{\tilde{\varphi}}$ is of the form

$$\begin{aligned} W_{\tilde{\varphi}} &\cong ((C_{d_1})^{a_1} \times \cdots \times (C_{d_j})^{a_j}) \rtimes (\mathfrak{S}_{M_1} \times \cdots \times \mathfrak{S}_{M_j}) \\ &\cong (C_{d_1} \wr \mathfrak{S}_{M_1}) \times \cdots \times (C_{d_j} \wr \mathfrak{S}_{M_j}), \end{aligned}$$

where \mathfrak{S}_{M_i} is the symmetric group acting on the set M_i . We write $Z_i := (C_{d_i})^{a_i}$ and $Z := Z_1 \times \cdots \times Z_j$ for the base groups and $\hat{Z} := (C_{d_0})^a$. We can identify each of these base groups as a subgroup of $W_{\tilde{\varphi}}$ since $Z_i \cong (C_{d_i}) \wr 1_{\mathfrak{S}_{a_i}}$. We choose some $\nu \in \text{Irr}(Z \mid \eta_0)$, which we write as $\nu = \nu_1 \times \cdots \times \nu_j$ for $\nu_i \in \text{Irr}(Z_i)$. Each ν_i itself is the product of a_i linear characters of C_{d_i} . Note that $(\mathfrak{S}_{M_i})_{\nu}$ is a Young subgroup as in Definition 2.15. Then

$$S_i := (\mathfrak{S}_{M_i})_{\nu}$$

is also a Young subgroup and by Lemma 2.17 we have

$$(C_{d_i} \wr \mathfrak{S}_{M_i})_{\nu} = C_{d_i} \wr S_i$$

We write $S := S_1 \times \cdots \times S_j$ and $(W_{\tilde{\varphi}})_{\nu} = (C_{d_1} \times \cdots \times C_{d_j}) \wr S$. By Proposition 2.19, maximal extendibility holds in wreath products, for $X^n \triangleleft X \wr Y$, hence there exists an extension of ν to $\tilde{\nu} \in \text{Irr}((W_{\tilde{\varphi}})_{\nu})$ and there exists a unique character $\kappa \in \text{Irr}(S)$ with extension $\tilde{\kappa} \in \text{Irr}((W_{\tilde{\varphi}})_{\nu})$ such that $Z \subset \ker(\tilde{\kappa})$ and

$$\eta_0 = \text{Ind}_{(W_{\tilde{\varphi}})_{\nu}}^{W_{\tilde{\varphi}}}(\tilde{\nu} \tilde{\kappa}).$$

For each $d_i \mid d_0$ we choose an extension map $\Pi_i : \text{Irr}(C_{d_i}) \rightarrow \text{Irr}(C_{d_0})$. Let $\hat{\nu}_i \in \text{Irr}((C_{d_0})^{a_i})$ be the extension of ν_i via Π_i and write $\hat{\nu} = \hat{\nu}_1 \times \cdots \times \hat{\nu}_j \in \text{Irr}(\hat{Z})$. The group $\hat{Z}W_{\tilde{\varphi}}$ is isomorphic to $(C_{d_0} \wr \mathfrak{S}_{M_1}) \times \cdots \times (C_{d_0} \wr \mathfrak{S}_{M_j}) \cong C_{d_0} \wr (\mathfrak{S}_{M_1} \times \cdots \times \mathfrak{S}_{M_j})$ and $\hat{Z} \triangleleft \hat{Z}W_{\tilde{\varphi}}$. We write

$$Y := \mathfrak{S}_{M_1} \times \cdots \times \mathfrak{S}_{M_j}$$

for the Young subgroup that defines the wreath product $\hat{Z}W_{\tilde{\varphi}} \cong C_{d_0} \wr Y$. The inertia group of $\hat{\nu}$ in $\hat{Z}W_{\tilde{\varphi}}$ is given by $C_{d_0} \wr S$. So again by Proposition 2.19 there exists an extension ψ of $\hat{\nu}$ to $\text{Irr}(C_{d_0} \wr S)$ such that $S \in \ker(\psi)$ and κ has an extension $\hat{\kappa} \in \text{Irr}(C_{d_0} \wr S)$ such that $\hat{Z} \leq \ker(\hat{\kappa})$. Now the character

$$\hat{\eta}_0 := \text{Ind}_{C_{d_0} \wr S}^{\hat{Z}W_{\tilde{\varphi}}}(\psi \hat{\kappa})$$

is irreducible and we have $\text{Res}_{W_{\tilde{\varphi}}}^{\hat{Z}W_{\tilde{\varphi}}}(\hat{\eta}_0) = \eta_0$.

We define $A := \{a_i \mid 1 \leq i \leq j\} = \{x_1, \dots, x_r\}$ the set of all possible sizes of symmetric groups \mathfrak{S}_{a_j} and $c_i := |\{j \mid |a_j| = x_i\}|$, as the integer that describes how often each possible size occurs. Then we may denote by Y the Young group of the form $\mathfrak{S}_{x_1}^{c_1} \times \cdots \times \mathfrak{S}_{x_r}^{c_r}$. In particular,

$$\hat{Z}W_{\tilde{\varphi}} \cong (C_{d_0} \wr \mathfrak{S}_{x_1})^{c_1} \times \cdots \times (C_{d_0} \wr \mathfrak{S}_{x_r})^{c_r}.$$

For each $1 \leq i \leq r$ we write

$$\hat{\eta}_i \in \text{Irr}((C_{d_0} \wr \mathfrak{S}_{x_i})^{c_i})$$

for the character we obtain from restricting $\hat{\eta}_0$ to the i -th factor, so that $\hat{\eta}_0 = \hat{\eta}_1 \times \cdots \times \hat{\eta}_r$.

Set $\hat{K} := N_{W_d}(\hat{Z}W_{\hat{\varphi}})$. The action of the normalizer of $W_d \cong C_{d_0} \wr \mathfrak{S}_a$ is given such that we may permute the factors $C_{d_0} \wr \mathfrak{S}_{x_i}$ of the same size by the action of \mathfrak{S}_{c_i} . Thus, we have

$$\begin{aligned} \hat{K} &\cong (C_{d_0} \wr \mathfrak{S}_{x_1}) \wr \mathfrak{S}_{c_1} \times \cdots \times (C_{d_0} \wr \mathfrak{S}_{x_r}) \wr \mathfrak{S}_{c_r} \text{ and} \\ \hat{K}_{\hat{\eta}_0} &= ((C_{d_0} \wr \mathfrak{S}_{x_1}) \wr \mathfrak{S}_{c_1})_{\hat{\eta}_1} \times \cdots \times ((C_{d_0} \wr \mathfrak{S}_{x_r}) \wr \mathfrak{S}_{c_r})_{\hat{\eta}_r}. \end{aligned}$$

Once again Lemma 2.20 allows us to extend each factor $\hat{\eta}_i$ to some $\phi_i \in \text{Irr}(((C_d \wr \mathfrak{S}_{x_i}) \wr \mathfrak{S}_{c_i})_{\hat{\eta}_i})$ with $\mathbb{Q}(\hat{\eta}_i) = \mathbb{Q}(\phi_i)$ so we have an extension $\hat{\eta}_0$ to an irreducible character $\phi_1 \times \cdots \times \phi_r =: \phi \in \text{Irr}(\hat{K}_{\eta_0})$. Now $\tilde{\eta}_0 := \text{Res}_{(W_{\varphi})_{\eta_0}}^{\hat{K}_{\eta_0}} \phi$ is an extension of η_0 and we set

$$\eta := \text{Ind}_{(W_{\varphi})_{\eta_0}}^{W_{\varphi}} \tilde{\eta}_0 \quad (6.5)$$

which is well-defined as $K_{\eta_0} \leq \hat{K}_{\eta_0}$. We now check η for \mathcal{H}_{ℓ} -invariance.

Let $\mathbb{Q}(\zeta_{d_0})$ denote the cyclotomic field containing the character fields of all irreducible characters in $\text{Irr}(C_{d_i})$ and $\text{Irr}(C_{d_0})$, and hence of all characters in $\text{Irr}(\hat{Z})$. By Lemma 2.20, the character fields of all irreducible characters of $W_{\hat{\varphi}}$ and of $\text{Irr}((C_{d_0} \wr \mathfrak{S}_{x_i})^{c_i})$ are contained in $\mathbb{Q}(\zeta_{d_0})$. In particular, for each factor $\hat{\eta}_i$ we have $\mathbb{Q}(\hat{\eta}_i) \subseteq \mathbb{Q}(\zeta_{d_0})$. Since $\hat{\eta}_0 = \hat{\eta}_1 \times \cdots \times \hat{\eta}_r$, it follows that

$$\mathbb{Q}(\hat{\eta}_0) \subseteq \mathbb{Q}(\zeta_{d_0}).$$

Applying Lemma 2.20 again to the characters $\phi_i \in \text{Irr}(((C_d \wr \mathfrak{S}_{x_i}) \wr \mathfrak{S}_{c_i})_{\hat{\eta}_i})$, we see that

$$\mathbb{Q}(\phi) \subseteq \mathbb{Q}(\phi_1, \dots, \phi_r) \subseteq \mathbb{Q}(\hat{\eta}_1, \dots, \hat{\eta}_r).$$

Finally, by Proposition 1.23, restriction and induction of characters preserves or shrinks character fields. Hence, for η as defined in (6.5), we have

$$\mathbb{Q}(\eta) \subseteq \mathbb{Q}(\zeta_{d_0}).$$

Since ζ_{d_0} is \mathcal{H}_{ℓ} -invariant by Lemma 4.2, it follows that η is \mathcal{H}_{ℓ} -invariant as claimed in (iv). \square

6.D Construction of the transversal

As established at the beginning of this chapter, our goal is to prove the existence of a $N_{GE}(\mathbf{S})$ -stable \tilde{N} -transversal in $\text{Irr}(N)$ that is \mathcal{H}_{t_1} -stable and has certain extension properties.

Since we have already determined the action of \mathcal{H}_{t_1} on the characters of the relative Weyl group, and established the existence of a suitable extension map Λ with respect to $L \triangleleft N$, we may now begin by constructing an appropriate parametrization of $\text{Irr}(N)$, following the considerations of Lemma 2.10.

Definition 6.18. Applying Lemma 2.10 to the extension map Λ with respect to $L \triangleleft N$, we obtain a surjective map

$$\Pi : \{(\varphi, \eta) \mid \varphi \in \text{Irr}(L), \eta \in \text{Irr}(W_{\varphi})\} \longrightarrow \text{Irr}(N),$$

defined by

$$(\varphi, \eta) \mapsto \text{Ind}_{N_\varphi}^N(\Lambda(\varphi)\eta).$$

By Lemma 6.11(c) together with Lemma 6.15, for an arbitrary $\varphi \in \text{Irr}(L)$ and $t \in \tilde{L}$, there exists a linear character $\nu_t \in \text{Irr}(N_\varphi)$ such that $\Lambda(\varphi)^t = \Lambda(\varphi)\nu_t$ and $\ker(\nu_t) = N_{\tilde{\varphi}}$, for some $\tilde{\varphi} \in \text{Irr}(\langle L, t \rangle \mid \varphi)$. Note that since $t \in L$, $W_d = N/L$ is fixed under the action of t . Hence, $\Pi(\varphi, \eta)^t = \Pi(\varphi, \eta\nu_t)$ for every $\eta \in \text{Irr}(W_\varphi)$. Furthermore, as Λ is $N \rtimes E$ - and \mathcal{H}_t -equivariant by Lemmas 6.11 and 6.15, Π satisfies the following equations for every $n \in N$, $w = \rho(n) \in W_d$, $e \in E$ and $\sigma \in \mathcal{H}_t$:

$$\Pi(\varphi, \eta)^n = \Pi(\varphi^n, \eta^w), \quad \Pi(\varphi, \eta)^e = \Pi(\varphi^e, \eta^e), \quad \Pi(\varphi, \eta)^\sigma = \Pi(\varphi^\sigma, \eta^\sigma), \quad (6.6)$$

for every $\varphi \in \text{Irr}(L)$ and $\eta \in \text{Irr}(W_\varphi)$, see Lemma 2.10.

In particular, given an extension $\tilde{\varphi} \in \text{Irr}(\tilde{L} \mid \varphi)$ and some $\eta_0 \in \text{Irr}(W_{\tilde{\varphi}})$, the set

$$\{\Pi(\varphi, \eta) \mid \eta \in \text{Irr}(W_\varphi \mid \eta_0)\}$$

forms an \tilde{L} -orbit in $\text{Irr}(N)$. Since $\tilde{N} = N\tilde{L}$, this defines an \tilde{N} -orbit in $\text{Irr}(N)$.

Now given any $\chi' \in \text{Irr}(N)$ with

$$\chi' = \Pi(\varphi, \eta'), \quad \tilde{\varphi} \in \text{Irr}(\tilde{L} \mid \varphi) \quad \text{and} \quad \eta_0 \in \text{Irr}(W_{\tilde{\varphi}} \mid \eta'),$$

we may apply Lemma 6.16 and choose $\eta'' \in \text{Irr}(W_\varphi \mid \eta_0)$ as defined therein and set $\chi'' := \Pi(\varphi, \eta'')$. By the above considerations, χ'' and χ' are \tilde{N} -conjugate.

Let \mathbb{T} be the \tilde{N} -transversal in $\text{Irr}(N)$ consisting of $\chi'' \in \text{Irr}(N)$ such that $\chi'' = \Pi(\varphi, \eta'')$ with $\varphi \in \text{Irr}(L)$ and $\eta'' \in \text{Irr}(W_\varphi \mid \eta_0)$ as described above.

Note that we have chosen $E = \langle F_p, \gamma \rangle \leq \text{Aut}(\mathbf{G}^{F_p^{2d_0^m}})$ such that $N_{\mathbf{G}^{vFE}}(\mathbf{S}) = N \rtimes E$. We write \hat{N} for this group going forward.

Let t_1 be the element defined in Proposition 6.9. As t_1 stabilizes \tilde{N} and \hat{N} the product $\tilde{N}\hat{N} \rtimes \mathcal{H}_{t_1}$ is well-defined and $\tilde{N}\hat{N} \rtimes \mathcal{H}_{t_1}$ defines a subgroup of $\text{Aut}(G)_N \times \mathcal{G}$ as a subset of the Galois automorphism and automorphisms of G stabilizing N . Clearly, this group acts on $\text{Irr}(N)$. The following proposition will allow us to obtain an \hat{N} -stable and \mathcal{H}_t -stable \tilde{N} -transversal in $\text{Irr}(N)$.

Proposition 6.19. *Let \mathbb{T} be a \tilde{N} -transversal in $\text{Irr}(N)$ as in Definition 6.18. Then for every $\chi \in \mathbb{T}$ the following holds:*

- a) $(\tilde{N}\hat{N}\mathcal{H}_{t_1})_\chi = \tilde{N}_\chi(\hat{N}\mathcal{H}_{t_1})_\chi$ and $(\tilde{N}\hat{N})_\chi = \tilde{N}_\chi\hat{N}_\chi$.
- b) χ extends to some $\hat{\chi} \in \text{Irr}(N \rtimes E_\chi)$ with $v\hat{F} \in \ker(\hat{\chi})$.

Proof. We follow the proof of Lemma 5.13 in [CS17].

Let $\chi \in \mathbb{T}$, $\varphi \in \text{Irr}(L)$ and $\eta \in \text{Irr}(W_\varphi)$ with $\chi = \Pi(\varphi, \eta)$ and $x \in (\tilde{N}\hat{N}\mathcal{H}_t)_\chi$. As $\tilde{L}N = \tilde{N}$ we may write $x = g \cdot e \cdot \sigma \cdot t$ for some $t \in \tilde{L}$, $g \in N$, $e \in E$ and $\sigma \in \mathcal{H}_{t_1}$. Without loss of generality, we may assume that $\varphi^x = \varphi$ by replacing x with some element in xN .

According to Lemma 6.11(c), given the character $(\varphi^{ge\sigma})^t$ there exists some linear character $\nu \in \text{Irr}(W_{\varphi^{ge\sigma}})$ such that

$$\Pi(\varphi^{ge\sigma}, \eta)^t = \Pi(\varphi^{ge\sigma}, \eta\nu).$$

Furthermore, $W_{\tilde{\varphi}} \leq \ker(\nu)$ for any extension $\tilde{\varphi}' \in \text{Irr}(\langle L, t \rangle \mid \varphi^{ge\sigma})$. Recall the equivariance properties of Λ as seen in Lemma 6.11, i.e., Λ is \tilde{N} - and \mathcal{H}_{t_1} -equivariant. We also use the fact that η is \mathcal{H}_{t_1} -stable and stable under the action of \tilde{L}/L and E . Recall that ρ is the natural epimorphism from $N \rightarrow N/L = W_d$ and write $\rho(g) = w$. The character φ is invariant under conjugation with $t \in \tilde{L}$, as $L \leq \tilde{L}$ and \tilde{L} is abelian. Using the Equations (6.6), this leads to

$$\begin{aligned} \chi^x &= \Pi(\varphi, \eta)^x = \Pi(\varphi, \eta)^{ge\sigma t} \\ &= \Pi(\varphi^{ge\sigma}, \eta^w \nu). \end{aligned}$$

As x is chosen such that $\varphi^x = \varphi$, this shows that $\varphi^{ge\sigma} = \varphi$ and that $x \in N_{\varphi}$. Hence, x also stabilizes $\eta \in \text{Irr}(N_{\varphi}/L)$. Thus, the equality $\chi^x = \chi$ implies

$$\chi^x = \Pi(\varphi^x, \eta^w \nu) = \Pi(\varphi, \eta),$$

and therefore $\eta^w \nu = \eta$. Given

$$\tilde{\varphi} \in \text{Irr}(\tilde{L} \mid \varphi) \quad \text{and} \quad \eta_0 \in \text{Irr}(W_{\tilde{\varphi}} \mid \eta')$$

we observe $\eta^w = \eta\nu^{-1} \in \text{Irr}(W_{\varphi} \mid \eta_0)$. Let $K := N_{W_d}(W_{\varphi}) \cap N_{W_d}(W_{\tilde{\varphi}})$ be the group from Lemma 6.16. If we show that $w \in K$, we can conclude

$$\eta = \eta^w = \eta\nu,$$

as by definition of η we have $\{\eta^z \mid z \in K\} \cap \text{Irr}(W_{\varphi} \mid \eta_0) = \{\eta\}$. This would imply $\chi^t = \Pi(\varphi, \eta\nu) = \chi$ and thus $t \in \tilde{N}$ stabilizes χ . Hence

$$(\tilde{N}\tilde{N}\mathcal{H}_{t_1})_{\chi} = \tilde{N}_{\chi}(\tilde{N}\mathcal{H}_{t_1})_{\chi}. \quad (6.7)$$

So it remains to show

$$w \in N_{W_d}(W_{\varphi}) \cap N_{W_d}(W_{\tilde{\varphi}}).$$

First we observe that $\varphi^x = \varphi$ implies $W_{\varphi}^x = W_{\varphi^x} = W_{\varphi}$. As t, σ_t and e all act trivially on W_{φ} , we obtain $W_{\varphi}^x = W_{\varphi}^w = W_{\varphi}$. Now take $\tilde{\varphi}$ an extension of φ to $\text{Irr}(\tilde{L})$. Then $\tilde{\varphi}^g$ is also an extension of φ , thus there exists some linear character $\mu \in \text{Irr}(\tilde{L})$ with $\tilde{\varphi}^g = \tilde{\varphi}\mu$ and $L \leq \ker(\mu)$. As $[\tilde{N}, \tilde{L}] \leq L$, μ is \tilde{N} -invariant. As $W_d = N/L$, this implies $W_{\tilde{\varphi}} = W_{\tilde{\varphi}\mu}$. Thus

$$W_{\tilde{\varphi}} = W_{\tilde{\varphi}\mu} = W_{\tilde{\varphi}^g} = (W_{\tilde{\varphi}})^w$$

and hence $w \in K$. The equality (6.7) implies also $(\tilde{N}\tilde{N})_{\chi} = \tilde{N}_{\chi}\tilde{N}_{\chi}$, which finishes the proof of the first statement, with $\chi = \Pi(\varphi, \eta)$.

We now verify the extension property for $\chi \in \mathbb{T}$. We set

$$I := (\tilde{N}_{\varphi})_{\Lambda(\varphi)\eta}.$$

Then we have

$$\hat{N}_{\chi} = N\hat{N}_{\Lambda(\varphi)\eta} = (\tilde{N}_{\varphi})_{\Lambda(\varphi)\eta}N = IN.$$

Let $\hat{\varphi} \in \text{Irr}(\widehat{N}_\varphi)$ be the extension of φ from Lemma 6.11 (b), with $v\hat{F} \in \ker(\hat{\varphi})$ and $\hat{\eta} \in \text{Irr}(K_\eta \times E)$ the extension of η from Lemma 6.16 (iii) with $v\hat{F} \in \ker(\hat{\eta})$. Since we have $I/L = (\widehat{N}_\varphi)_{\Lambda(\varphi)\eta}/L \leq K_\eta \times E$, we may extend the character $\text{Infl}_{W_\varphi}^{N_\varphi}(\eta)$ to a character $\hat{\eta}'$ of I with $v\hat{F} \in \ker \hat{\eta}'$. Furthermore let $\hat{\varphi}' = \hat{\varphi}|_I$. Now the character

$$\hat{\chi} := (\hat{\varphi}'\hat{\eta}')^{\widehat{N}_\chi}$$

is the desired extension of χ with $v\hat{F}$ in its kernel. \square

This allows us to obtain a result mirroring Theorem 6.2 for the finite groups with respect to the twisted Steinberg vF . Recall the group E generated by elements acting as γ_0 and F_p on G . As in Section 4.B, we define the group $\check{G} = \mathcal{L}_{vF}^{-1}(Z(\mathbf{G})) \leq \mathbf{G}$.

Proposition 6.20. *Let \mathbf{G} and $\check{\mathbf{G}}$ be as in Theorem 6.2. Assume that d is a doubly regular number for (\mathbf{G}, F) and let v and vF be the elements and E be the group from Notation 6.3, acting on \mathbf{G}^{vF} as F_p and γ_0 . Let \mathbf{S} be a Sylow d -torus of (\mathbf{G}, vF) . Then there exists some $t_1 \in \check{\mathbf{G}}^{vF} \mathcal{L}_{vF}^{-1}(Z(\mathbf{G}))$ such that the following holds:*

- *There exists a $(\mathbf{G}^{vF} \rtimes E)_{\mathbf{S}}$ and \mathcal{H}_{t_1} -stable $N_{\check{\mathbf{G}}}(\mathbf{S})^{vF}$ -transversal \mathbb{M} in $\text{Irr}(N_{\mathbf{G}}(\mathbf{S})^{vF})$.*
- *Each $\chi \in \mathbb{M}$ extends to a character $\hat{\chi}$ of $(\mathbf{G}^{vF} \rtimes E)_{\mathbf{S}, \chi}$ such that $v\hat{F} \in \ker(\hat{\chi})$.*
- *t_1 induces an automorphism of \mathbf{G}^{vF} of type \mathfrak{t}_ℓ and stabilizes $N_{\mathbf{G}^{vF}}(\mathbf{S})$ and $C_{\mathbf{G}^{vF}}(\mathbf{S})$.*

Proof. Let t_1 be the element from Definition 6.8. By the observations of Lemma 6.12, t_1 induces an automorphism of type \mathfrak{t}_ℓ . Furthermore, t_1 stabilizes $N_{\mathbf{G}^{vF}}(\mathbf{S})$ and $C_{\mathbf{G}^{vF}}(\mathbf{S})$ by Proposition 6.13.

Let \mathbb{T} be the transversal from Proposition 6.19. In particular, this is a $N_{\check{\mathbf{G}}}(\mathbf{S})^{vF}$ -transversal such that each $\chi \in \mathbb{T}$ extends to some irreducible character $\hat{\chi}$ of

$$\widehat{N}_\chi = (\mathbf{G}^{vF} \rtimes E)_{\mathbf{S}, \chi}$$

with $v\hat{F} \in \ker(\hat{\chi})$. If we apply Lemma 2.9 a) to the transversal \mathbb{T} in the situation above with

$$X = N_{\check{\mathbf{G}}^{vF}}(\mathbf{S}), \quad Y = (\mathbf{G}^{vF} \rtimes E)_{\mathbf{S}} \mathcal{H}_{t_1} \quad \text{and} \quad U = N_{\mathbf{G}^{vF}}(\mathbf{S}),$$

we obtain a $N_{\check{\mathbf{G}}^{vF}}(\mathbf{S})$ -transversal \mathbb{M} , that is $(\mathbf{G}^{vF} \rtimes E)_{\mathbf{S}} \mathcal{H}_{t_1}$ -stable. Clearly, it is stable with regard to the action of $(\mathbf{G}^{vF} \rtimes E)_{\mathbf{S}}$ and \mathcal{H}_{t_1} . Finally by the construction, each element of \mathbb{M} is of the form $\varphi^{e\sigma}$, for some $\varphi \in \mathbb{T}$ and $e\sigma \in \widehat{N} \mathcal{H}_{t_1}$. By the definition of \mathbb{T} , φ has the desired extension, and so does $\varphi^{e\sigma}$. \square

6.E From the twisted group to the non-twisted

In the previous section we have replaced the groups $N_{\mathbf{G}^{F'}}(\mathbf{S})$ and $C_{\mathbf{G}^{F'}}(\mathbf{S})$ from Theorem 6.2 with isomorphic groups $N_{\mathbf{G}^{vF}}(\mathbf{S})$ and $C_{\mathbf{G}^{vF}}(\mathbf{S})$ for a certain modified endomorphism F and some element $v \in \mathbf{G}$. Furthermore, instead of extending to

the automorphism group $E(\mathbf{G}^{F'})$, generated by the graph and field automorphism of $\mathbf{G}^{F'}$ as in Definition 3.18, we have constructed extensions to subgroups of $G \rtimes E$, where $E = E(\mathbf{G}^{F_p^{2d_0m}})$ is generated by the transpose inverse γ_0 and an element acting as F_p .

As the groups in question are isomorphic, it is clear that the statement of Proposition 6.20 should still hold for the original statement. Indeed, we will see why the condition $v\hat{F}$ to be in the kernel of the extension $\chi \in \text{Irr}(\mathbf{N}_{\mathbf{G}}(\mathbf{S})^{vF})$ to \hat{N}_χ has been added. First we present a technical statement, in order to define a suitable bijection between the two constructions.

Proposition 6.21. *[Spä25, Proposition 4.6] Let \mathbf{H} be a linear algebraic group and let $E(\mathbf{H})$ be an abelian group of abstract automorphisms of \mathbf{H} with $F \in E(\mathbf{H})$ inducing a Frobenius endomorphism on \mathbf{H} and $v \in \mathbf{H}$. Let $g \in \mathbf{H}$ with $v = g^{-1}F(g)$.*

- a) $vF \in \mathbf{H} \rtimes E(\mathbf{H})$ and the map $\iota : \mathbf{H} \rtimes E(\mathbf{H}) \rightarrow \mathbf{H} \rtimes E(\mathbf{H})$ with $x \mapsto x^g$, satisfies $\iota(\mathbf{H}^F) = \mathbf{H}^{vF}$.
- b) If $\tau \in E(\mathbf{H})$, with $\tau(v) = v$, then $\iota(\tau) \in \mathbf{H}^{vF}\tau \leq \mathbf{H}^{vF} \rtimes E(\mathbf{H})$ and $\iota^{-1}(\tau) \in \mathbf{H}^F\tau$.
- c) Let $e \geq 1$ an integer, such that $g \in \mathbf{H}^{F^e}$ and let $E(\mathbf{H}^{F^e}) \leq \text{Aut}(\mathbf{H}^{F^e})$ be the group of automorphisms obtained by restricting $E(\mathbf{H})$ to \mathbf{H}^{F^e} . Then ι induces an automorphism of $\mathbf{H}^{F^e} \rtimes E(\mathbf{H}^{F^e})$.

Proof. We remind the reader of the convention to write vF for the automorphism $\text{int}(v) \circ F$, where v acts on \mathbf{H} by conjugation. We also regard all of the following, as elements in the semidirect product $\mathbf{H} \rtimes E(\mathbf{H})$. Hence, we see that

$$\iota(F) = g^{-1}Fg = g^{-1}F(g)F = vF \in \mathbf{H} \rtimes E(\mathbf{H})$$

and therefore $F(g) = gv$. Given any $x \in \mathbf{H}^F$ we see that

$$vF(\iota(x)) = vF(g^{-1}xg)v^{-1} = g^{-1}F(g)F(g^{-1})F(x)F(g)F(g^{-1})g = g^{-1}xg,$$

and hence $\iota(\mathbf{H}^F) = \mathbf{H}^{vF}$, which gives a).

For b) we consider the element $\iota^{-1}(\tau) = \tau^{g^{-1}} = g\tau g^{-1} = g\tau(g^{-1})\tau$ and calculate $\tau^{-1}\tau^{g^{-1}} = \tau^{-1}(g)g^{-1}\tau^{-1}\tau = \tau^{-1}(g)g^{-1} \in \mathbf{H}$. We compute the element

$$F(\tau^{-1}\tau^{g^{-1}}) = \tau^{-1}(F(g))\tau F(g)^{-1} = \tau^{-1}g v \tau v^{-1} g^{-1} = \tau^{-1}g v \tau(v)^{-1} \tau g^{-1} = \tau^{-1}g \tau g^{-1}.$$

where we have used that τ and F commute with each other, as $E(\mathbf{H})$ is abelian. Thus $\iota^{-1}(\tau) \in \mathbf{H}^F\tau$, and applying a) gives us $\tau \in \mathbf{H}^{vF}\iota(\tau)$.

If e is such an integer, then clearly ι induces an inner automorphism on \mathbf{H}^{F^e} and thus on $\mathbf{H}^{F^e} \rtimes E(\mathbf{H}^{F^e})$ and c) follows. \square

Recall that the Steinberg endomorphism $F' : \mathbf{G} \rightarrow \mathbf{G}$ from Notation 3.15 is defined by $\gamma^{\frac{1-\epsilon}{2}} F_q$, where $\gamma := \text{int}(v_0) \circ \gamma_0$.

Proposition 6.22. *Let \mathbf{G} and $\tilde{\mathbf{G}}$ as before and let $E(\mathbf{G})$ be the subgroup of abstract group automorphisms of \mathbf{G} generated by F_p and γ_0 . There exists some $g \in \mathbf{G}$ such that conjugation with g defines maps*

$$\iota : \mathbf{G}^{vF} \rtimes E(\mathbf{G}) \rightarrow \mathbf{G}^{F'} \rtimes E(\mathbf{G}) \text{ and}$$

$$\iota' : \tilde{\mathbf{G}}^{vF} \rtimes E(\mathbf{G}) \rightarrow \tilde{\mathbf{G}}^{F'} \rtimes E(\mathbf{G}),$$

such that $\iota(vF) = \iota'(vF) = F'$. Furthermore, ι induces an isomorphism

$$\tau : \mathbf{G}^{vF} \rtimes E(\mathbf{G}_{F_p}^{2d_0m}) / \langle vF \rangle \rightarrow \mathbf{G}^{F'} \rtimes E(\mathbf{G}^{F'}),$$

that maps $\tilde{\mathbf{G}}^F$ to $\tilde{\mathbf{G}}^{vF}$.

Proof. First let $\epsilon = 1$. In particular, $F = F'$ in this case. By the Theorem of Lang-Steinberg 3.12 there exists some $g \in \mathbf{G}$ such that $gF(g)^{-1} = v$. We remind the reader that the element F_p in E has order dividing $2d_0m$, with $p = q^m$ and by definition of $F = \gamma_0^{\frac{1-\epsilon}{2}} F_q$, we have $F^{2d_0} = F_q^{2d_0} = F_p^{2d_0m}$. Thus

$$gF^{2d_0}(g^{-1}) = gF(g)F(g^{-1})F^{2d_0}(g^{-1}) = v(F^{2d_0-1}(g^{-1})) = \dots = v^{2d_0} = 1$$

since $[v, F] = 1$ and $\text{ord}(v) \mid 2d_0$, so $g \in \mathbf{G}^{F^{2d_0}} = \mathbf{G}^{F_p^{2d_0m}}$. Finally γ_0 and F_p both fix v , so we may apply Proposition 6.21 to this situation in regard to \mathbf{G} and $\tilde{\mathbf{G}}$ and obtain the maps

$$\begin{aligned} \iota : \mathbf{G}^{vF} \rtimes E(\mathbf{G}) &\rightarrow \mathbf{G}^F \rtimes E(\mathbf{G}), \\ \iota' : \tilde{\mathbf{G}}^{vF} \rtimes E(\mathbf{G}) &\rightarrow \tilde{\mathbf{G}}^{F'} \rtimes E(\mathbf{G}). \end{aligned}$$

Note that the group E as defined in the beginning of this chapter is just $E(\mathbf{G}^{F_p^{2d_0m}})$. Thus, the existence of the epimorphism

$$\varepsilon : \mathbf{G}^F \rtimes E(\mathbf{G}^{F^{2d_0}}) \rightarrow \mathbf{G}^F \rtimes E(\mathbf{G}^F)$$

with $\ker(\varepsilon) = \langle \hat{F} \rangle$ is clear and thereby ε and ι induce the desired isomorphism via Proposition 6.21 c) and

$$\mathbf{G}^F \rtimes E \rightarrow \mathbf{G}^F \rtimes E(\mathbf{G}) / \langle F \rangle \rightarrow \mathbf{G}^{vF} \rtimes E(\mathbf{G}) / \langle vF \rangle.$$

In case $\epsilon = -1$, we have $F' = \text{int}(v_0) \circ F$ by Notation 3.15 and we may apply Proposition 6.21 again, to obtain the desired maps. \square

The image of these isomorphisms with regard to the normalizers of Sylow d -tori together with Proposition 6.20 now finishes the proof of Theorem 6.2.

Proof of Theorem 6.2. By Proposition 6.20, given a Sylow d -torus \mathbf{S} of (\mathbf{G}, vF) there exists some $t_1 \in \tilde{\mathbf{G}}^{\tilde{G}}$ stabilizing \mathbf{S} such that the following holds:

- (i) There exists a \hat{N} and \mathcal{H}_t -stable \tilde{N} -transversal \mathbb{T} in $\text{Irr}(N)$.
- (ii) Each $\chi \in \mathbb{T}$ extends to a character $\hat{\chi}$ of \hat{N}_χ such that $vF \in \ker(\hat{\chi})$.
- (iii) t_1 induces an automorphism of type \mathfrak{t}_ℓ on G .

Let $\iota : \mathbf{G}^{vF} \rtimes E(\mathbf{G}) \rightarrow \mathbf{G}^{F'} \rtimes E(\mathbf{G})$ be the map from the previous proposition. We write

$$N' := \iota(N), \quad \tilde{N}' := \iota(\tilde{N}) \quad \text{and} \quad \mathbf{S}_0 := \iota(\mathbf{S}) = \mathbf{S}^g.$$

By definition \mathbf{S}_0 is a Sylow d -torus of (\mathbf{G}, F) and we have $N' := N_{\mathbf{G}^{F'}}(\mathbf{S}_0)$ and $\tilde{N}' = N_{\tilde{\mathbf{G}}^{F'}}(\mathbf{S}_0)$. We have $\hat{N} = N_{GE}(\mathbf{S})$ and thus $\iota(\hat{N} / \langle vF \rangle) = N_{\mathbf{G}^{F'}E}(\mathbf{S}_0) =: \hat{N}'$. For the element t_1 from Lemma 6.12, we have

$$\mathcal{L}_{F'}(\iota(t_1)) = \mathcal{L}_{vF}(t_1) \in \{\pm E_n\} \quad \text{and} \quad \det(\iota(t_1)) = \det(t_1).$$

and thus the element $t' = \iota(t_1)$ stabilizes \mathbf{S}_0 and induces an automorphism of type $\mathfrak{t}_\ell(\mathbf{G}^{F'})$ of $\mathbf{G}^{F'}$. The action of \mathcal{H}_{t_1} on characters of t_1 -stable subgroups of \mathbf{G}^{vF} translates naturally via ι to the action of $\mathcal{H}_{t'}$. In particular, we may regard $\iota(\mathcal{H}_t)$ as the group $\mathcal{H}_{t'}$. The set $\mathbb{T}' := \iota(\mathbb{T})$ is a \widehat{N}' and $\mathcal{H}_{t'}$ -stable \widetilde{N}' -transversal in $\text{Irr}(N')$. Let $\chi \in \mathbb{T}'$ and $\chi_0 \in \mathbb{T}$, such that $\iota(\chi_0) = \chi$. By the above assumptions χ_0 extends to its stabilizer in $\widehat{N} = N_{\mathbf{G}^{vF}E}(\mathbf{S})$ with vF in its kernel or equivalently to $(\widehat{N}/\langle vF \rangle)_\chi$. Thus, via the isomorphism τ from Proposition 6.22, χ extends to its stabilizer in $\widehat{N}'N_{\mathbf{G}}(\mathbf{S}_0)^{F'}$. \square

Lemma 6.23. *Assume the situation of Theorem 6.2. There exists $t \in \widetilde{G}\check{G}$ stabilizing $\iota(L)$ and $\iota(N)$ and inducing an automorphism of type \mathfrak{t}_ℓ on G , such that there exists an \mathcal{H}_t -equivariant extension map Λ' with respect to $C_{\mathbf{G}}(\mathbf{S}_0)^{F'} \triangleleft N_{\mathbf{G}}(\mathbf{S}_0)^{F'}$*

Proof. Let ι be the map from Proposition 6.22

$$\iota : \mathbf{G}^{vF} \rtimes E(\mathbf{G}) \rightarrow \mathbf{G}^{F'} \rtimes E(\mathbf{G}),$$

such that $\iota(L) = C_{\mathbf{G}}(\mathbf{S}_0)^{F'} =: L'$ and $\iota(N) = N_{\mathbf{G}}(\mathbf{S}_0)^{F'} =: N'$. By Lemma 6.11, there exists some extension map Λ with respect to $L \triangleleft N$, which translates to an extension map Λ' with respect to $L' \triangleleft N'$. By Lemma 6.12, there exists some $t_1 \in \widetilde{\mathbf{G}}^{vF} \mathcal{L}_{vF}^{-1}(Z(\mathbf{G}))$ inducing an automorphism of type \mathfrak{t}_ℓ on \mathbf{G}^{vF} . As in the proof of Theorem 6.2, $t := \iota(t_1)$ is an element in $\widetilde{G}\check{G}$ with the desired properties. As Λ is \mathcal{H}_{t_1} -equivariant, Λ' is \mathcal{H}_t -equivariant, as equivariance of Galois automorphisms clearly translates via isomorphisms. \square

Theorem 6.24. *Let ℓ be an odd prime and $d = d_\ell(q)$ and assume $d \mid n$. Let \mathbf{S}_0 be a Sylow d -torus of (\mathbf{G}, F') . Then condition $B(d)_{\mathcal{H}_\ell}$ holds for $\mathbf{G}^{F'}$ and the prime ℓ .*

Proof. We need to show that there exists a $\text{Lin}(\widetilde{N}/N) \times \mathcal{H}_\ell$ -equivariant extension map $\widetilde{\Lambda}$ with respect to $\widetilde{L} := \widetilde{C}_{\widetilde{G}}(\mathbf{S}_0) \triangleleft N_{\widetilde{G}}(\mathbf{S}_0) =: \widetilde{N}$. We apply Lemma 2.12 with

$$X := \widetilde{L} \triangleleft \widetilde{N} =: Y, U := N$$

We observe that the group theoretic properties are satisfied as $\widetilde{L}N = \widetilde{N}$, $\widetilde{L} \cap N = L$ and $\widetilde{N}/N = \widetilde{L}/L$ is cyclic. Thus maximal extendibility holds with respect to $L \triangleleft \widetilde{L}$ and by Lemma 6.23 there exists a \mathcal{H}_t -equivariant extension map with respect to $L \triangleleft N$. Then Proposition 2.12 shows that there exists an $\text{Lin}(\widetilde{N}/N)$ -equivariant extension map $\widetilde{\Lambda}$ with respect to $\widetilde{L} \triangleleft \widetilde{N}$. As t stabilizes \mathbf{S}_0 , t induces an inner automorphism of \widetilde{N} , and as $\widetilde{\Lambda}$ is \widetilde{N} -equivariant, \mathcal{H}_ℓ -equivariance follows. \square

7. Local conditions in type A

In the previous two chapters, we verified the global condition $A(\infty)_{\mathcal{H}_{t_2}}$ as well as the local conditions $A(d)_{\mathcal{H}_{t_1}}$ and $B(d)_{\mathcal{H}_\ell}$ in the case where the integer d is doubly regular for (\mathbf{G}, F) . Note that in the local condition i.e., Theorem 6.2, that $t_1 \in \tilde{G}\mathcal{L}^{-1}(Z(\mathbf{G}))$, whereas $t_2 \in \tilde{G}$, in Theorem 5.13. Using the considerations in Section 5.A, we compare the outer automorphisms of \mathbf{G}^F induced by t_1 and t_2 .

When studying the structure of the normalizer and centralizer of a Sylow d -torus in the general case, we find that their structure resembles that of block matrices. One block corresponds to the normalizer and centralizer of a Sylow d -torus in the doubly regular case, while the other corresponds to $\mathrm{SL}_m(\epsilon q)$, with $m \leq d$, i.e., the global case. Hence, to finish the local condition in the general case, we may combine the two previous results.

In this chapter, we will first examine the structure of the normalizer and centralizer of some d -torus. We then construct an outer automorphism of \mathbf{G}^F by combining the elements t_1 from the doubly-regular case and t_2 from the global case. Following this, we define a new transversal and verify the remaining inductive condition. Finally, we get conditions $A(d)_{\mathcal{H}_t}$ and $B(d)_{\mathcal{H}_\ell}$ for groups of type A.

7.A Notation

Let p and ℓ be distinct primes with ℓ odd, $\epsilon \in \{\pm 1\}$ and let m be a positive integer such that $q = p^m$ and let $n = n_1 + n_2$ with $n_1, n_2 \geq 0$. Set $\mathbf{G} = \mathrm{SL}_n(\overline{\mathbb{F}}_p)$, $\tilde{\mathbf{G}} = \mathrm{GL}_n(\overline{\mathbb{F}}_p)$. Let γ_0 denote the transpose-inverse automorphism of \mathbf{G} and let F_p be the standard Frobenius endomorphism. Define $F := \gamma_0^{\frac{1-\epsilon}{2}} F_p^m$ such that $\mathbf{G}^F = \mathrm{SL}_n(\epsilon q)$ and $\tilde{\mathbf{G}}^F = \mathrm{GL}_n(\epsilon q)$.

Let $\tilde{\mathbf{T}}$ be the set of diagonal matrices in $\tilde{\mathbf{G}}$ and define $\mathbf{T} = \mathbf{G} \cap \tilde{\mathbf{T}}$.

We write $\tilde{\mathbf{K}} = \tilde{\mathbf{K}}_{n_1, n_2} \cong \mathrm{GL}_{n_1}(\overline{\mathbb{F}}_p) \times \mathrm{GL}_{n_2}(\overline{\mathbb{F}}_p)$ for the subgroup of $\tilde{\mathbf{G}}$ given by the block matrices in $\mathrm{GL}_{n_1}(\overline{\mathbb{F}}_p)$ and $\mathrm{GL}_{n_2}(\overline{\mathbb{F}}_p)$. Let $\tilde{\mathbf{G}}_i$ denote the direct factors of $\tilde{\mathbf{K}}$, so that $\tilde{\mathbf{G}}_i \cong \mathrm{GL}_{n_i}(\overline{\mathbb{F}}_p)$. Let $\mathbf{K} := \tilde{\mathbf{K}} \cap \mathbf{G}$ and $\mathbf{G}_i := \mathbf{K} \cap \tilde{\mathbf{G}}_i$. For the finite groups we similarly define $\tilde{K} := \tilde{\mathbf{K}}^F$ and $K := \mathbf{K}^F$. Note that F stabilizes \mathbf{G}_i and $\tilde{\mathbf{G}}_i$, so $\tilde{G}_i = \tilde{\mathbf{G}}_i^F$ and $G_i := \mathbf{G}_i^F$ are well-defined. We recall the notation $\underline{\mathrm{diag}}(x_1, \dots, x_n)$ for the block-diagonal matrix with x_i 's on the diagonal.

Proposition 7.1. *We have $G_1 \times G_2 \triangleleft K \triangleleft \tilde{K} = \tilde{G}_1 \times \tilde{G}_2$ and $K/(G_1 \times G_2)$ is cyclic.*

Proof. Let $x \in \tilde{K}$, with $x = \underline{\text{diag}}(x_1, x_2)$ for $x_1 \in \tilde{G}_1$ and $x_2 \in \tilde{G}_2$. Then clearly $F(x) = \underline{\text{diag}}(F(x_1), F(x_2))$. Hence, $\tilde{K} = \tilde{K}^F = (\tilde{G}_1 \times \tilde{G}_2)^F = \tilde{G}_1^F \times \tilde{G}_2^F = \tilde{G}_1 \times \tilde{G}_2$. As $\mathbf{K} = \{x \in \tilde{K} \mid \det(x) = 1\}$ is the kernel of $\det : \tilde{G} \rightarrow \tilde{G}$, we have $\mathbf{K}^F \triangleleft \tilde{K}^F$. Similarly $G_1 \times G_2$ is the kernel of the map

$$\det_1 : K \rightarrow K, \quad \det_1(x_1, x_2) = \det(x_1).$$

Hence, both quotients are isomorphic to the cyclic group $\mathbb{F}_{\epsilon q}^\times \cong C_{q-\epsilon}$. \square

Let $d := d_\ell(q)$, $d_0 := d_\ell(\epsilon q)$, and assume $d_0 \mid n_1$ so $n_1 = ad_0$ for some positive integer a . Let \mathbf{S} be a Sylow d -torus of (\mathbf{G}_1, F) . Then \mathbf{S} is a d -torus of (\mathbf{G}, F) , though it is only a Sylow d -torus of (\mathbf{G}, F) if $d_0 > n_2$ by the considerations in Chapter 3.

We set

$$N_1 := N_{G_1}(\mathbf{S}), \quad \tilde{N}_1 := N_{\tilde{G}_1}(\mathbf{S}), \quad N := N_G(\mathbf{S}) \quad \text{and} \quad \tilde{N} := N_{\tilde{G}}(\mathbf{S}).$$

We introduce the notation $E(G_i)$ for the subgroups of $\text{Aut}(\tilde{G}_i)$ generated by F_p , notated as $F_p^{(i)}$ and the automorphism γ_i induced by the action of the transpose inverse γ_0 on \tilde{G} . Then we let $E(G_1) \times E(G_2)$ act on \tilde{K} and the automorphisms of \tilde{K} induced by γ_0 and F_p . They can be identified with $\gamma_1 \times \gamma_2$ and $F_p^{(1)} \times F_p^{(2)}$. This implies

$$(\tilde{G}_1 \times \tilde{G}_2) \rtimes (E(G_1) \times E(G_2)) \geq (\tilde{G}_1 \times \tilde{G}_2) \rtimes E(G).$$

However given $x = \underline{\text{diag}}(x_1, x_2)$, we have $\det(F_p^{(1)}(x_1) \times \text{id}_{G_2}(x_2)) = \det(x_1)^p \det(x_2)$, hence the action of $E(G_1) \times E(G_2)$ does not stabilize K but $G_1 \times G_2$. Accordingly the group

$$(G_1 \times G_2) \rtimes (E(G_1) \times E(G_2))$$

is well-defined and for $\hat{N}_1^\circ := N_{G_1 \rtimes E(G)}(\mathbf{S})$ and $\hat{G}_i := G_i \rtimes E(G_i)$ we have

$$N_{(G_1 \times G_2) \rtimes (E(G_1) \times E(G_2))}(\mathbf{S}) = \hat{N}_1^\circ \times \hat{G}_2 \geq N_{(G_1 \times G_2) \rtimes E(G)}(\mathbf{S}) =: \hat{N}^\circ.$$

We remind the reader of the definition of an automorphism of type \mathfrak{t}_ℓ from Definition 4.13.

Proposition 7.2. *Let $t_1 \in \text{GL}_{n_1}(\epsilon q)$, $t_2 \in \text{GL}_{n_2}(\epsilon q)$, be two elements such that t_1 and t_2 induce the automorphisms of type \mathfrak{t}_ℓ on G_1 and G_2 respectively, then there exists some $t \in \text{GL}_n(\epsilon q)$ that induces an automorphism of type \mathfrak{t}_ℓ on G and such that t acts on G_1 as t_1 and G_2 as t_2 .*

Proof. We remind the reader of the group $C_{(n, q-\epsilon)}$ and $\delta_\ell \in \overline{\mathbb{F}}_q^\times$ as in Section 4.B. By Lemma 4.14 we can assume that $t_i \in \text{GL}_{n_i}(\epsilon q)$ with $\det(t_i) = \delta_\ell^{\frac{n_i(n_i-1)}{2}}$ as this induces an automorphism of type $\mathfrak{t}_{\ell, n_i, q, \epsilon}$. We write $t := \underline{\text{diag}}(t_1, \delta_\ell^{n_1} t_2)$. Then

$$\det(t) = \delta_\ell^{n_1 n_2} \det(t_1) \det(t_2) = \delta_\ell^{\frac{2n_1 n_2 + n_1(n_1-1) + n_2(n_2-1)}{2}} = \delta_\ell^{\frac{n_1 n_2 (n_1 n_2 - 1)}{2}}$$

Hence, t induces an automorphism of type \mathfrak{t}_ℓ and acts as t_1 on G_1 and t_2 on G_2 . \square

Lemma 7.3. *Let $\tilde{N}_1 := N_{\tilde{G}_1}(\mathbf{S})$, $\hat{N}_1 := N_{G_1 \rtimes E(G_1)}(\mathbf{S})$ and $\hat{G}_2 := G_2 \rtimes E(G_2)$.*

- a) $N_1 \times G_2 \triangleleft N := N_G(\mathbf{S})$ with cyclic quotient $N/(N_1 \times G_2)$.
- b) There exists a \widehat{N}_1 -stable \widetilde{N}_1 -transversal \mathbb{T}_1 in $\text{Irr}(N_1)$, such that each $\chi \in \mathbb{T}_1$ extends to $(\widehat{N}_1)_\chi$.
- c) There exists a \widehat{G}_2 -stable \widetilde{G}_2 -transversal \mathbb{T}_2 in $\text{Irr}(G_2)$, such that each $\chi \in \mathbb{T}_2$ extends to $(\widehat{G}_2)_\chi$.
- d) There exists some $t \in \widetilde{G}$ stabilizing G_1, G_2 and K and fixing \mathbf{S} , such that \mathbb{T}_1 and \mathbb{T}_2 are both \mathcal{H}_t -stable and t induces the outer automorphism of type \mathfrak{t}_ℓ on G .

Proof. As $G_1 \times G_2 \triangleleft K$ with cyclic quotient, a) follows.

Since \mathbf{S} is a Sylow d -torus for (\mathbf{G}_1, F) and d is a doubly regular number for (\mathbf{G}_1, F) , because of $d_0 \mid n_1$, we obtain a \widehat{N}_1 -stable \widetilde{N}_1 -transversal \mathbb{T}_1 in $\text{Irr}(N_1)$ from Theorem 6.2. So we get b).

For c) we apply Theorem 5.13 in the case where $n_2 > 1$. In particular, there exists some $E(G_2)$ -stable transversal \mathbb{T}_2 in $\text{Irr}(G_2)$ such that each $\chi \in \mathbb{T}_2$ extends to its stabilizer in $G_2 E(G_2) = \widehat{G}_2$. In case $n_2 = 1$ we have $\mathbf{G}_2 = 1$ and the statement is trivial.

For d) we observe that Theorem 6.2 shows the existence of some $t'_1 \in \widetilde{\mathbf{G}}_1$, fixing \mathbf{S} , such that \mathbb{T}'_1 is $\mathcal{H}_{t'_1}$ -stable and t'_1 induces the outer automorphism of type \mathfrak{t}_ℓ on G_1 . By Lemma 4.14, we may choose some $\widetilde{t}_1 \in \widetilde{\mathbf{G}}_1^F$ such that \widetilde{t}_1 induces the same outer automorphism as t'_1 . Let \widetilde{t}_2 be the element in \widetilde{G}_2 from Theorem 5.13 such that \mathbb{T}_2 is $\mathcal{H}_{\widetilde{t}_2}$ -stable and \widetilde{t}_2 induces the outer automorphism of type \mathfrak{t}_ℓ on G_2 . Then by Proposition 7.2 there exists an element $t \in \widetilde{G}_1 \times \widetilde{G}_2 \leq \widetilde{G}$, that acts as \widetilde{t}_i on G_i and such that t induces the outer automorphism of type \mathfrak{t}_ℓ on G as stated. As \widetilde{t}_2 clearly fixes \mathbf{S} and $t = \underline{\text{diag}}(\widetilde{t}_1, \delta \widetilde{t}_2)$, for some $\delta \in C_{q-\epsilon}$, t fixes \mathbf{S} . \square

Lemma 7.4. *Let $\widehat{N}^\circ := N_{(G_1 \times G_2) \rtimes E(G)}(\mathbf{S})$ and $\mathbb{T}_1, \mathbb{T}_2$ be the transversals defined in Lemma 7.3. Then the following holds:*

- a) $\widehat{N} = N\widehat{N}^\circ$.
- b) Every $\phi \in \mathbb{T}_1 \times \mathbb{T}_2$ satisfies $\widehat{N}_\phi = N_\phi \widehat{N}_\phi^\circ$ and extends to its stabilizer in \widehat{N}° .

Proof. As F and γ_0 stabilize $G_1 \times G_2$ we have $E(G) \leq \text{Aut}(G_1 \times G_2)$. Furthermore, \mathbf{S} is a Sylow d -torus of (\mathbf{G}_1, F) and $E(\mathbf{G})$ stabilizes \mathbf{G}_1 . Hence, as all Sylow d -tori of (\mathbf{G}_1, F) are conjugate, given any $e \in E(G)$, there exists some $h \in G_1$ such that $he \in N_{(G_1 \times G_2)E(G)}(\mathbf{S})$. Now given any $g \cdot e \in N_{GE}(\mathbf{S})$, we have $ge = (gh^{-1})(he) \in N_G(\mathbf{S})N_{(G_1 \times G_2)E(G)}(\mathbf{S})$ hence the first statement follows.

For b), we let $\phi = \phi_1 \times \phi_2 \in \mathbb{T}_1 \times \mathbb{T}_2$. Now by definition of \mathbb{T}_1 , ϕ_1 extends to $N_{G_1 \rtimes E(G_1)}(\mathbf{S})_{\phi_1} =: \widehat{N}_{1, \phi_1}$ and ϕ_2 extends to $(G_2 \rtimes E(G_2))_{\phi_2}$. So ϕ extends to

$$\widehat{N}_{1, \phi_1} \times (G_2 \rtimes E(G_2))_{\phi_2} = (\widehat{N}_1 \times (G_2 \rtimes E(G_2)))_{\phi}.$$

As $\widehat{N}^\circ \leq \widehat{N}_1 \times (G_2 \rtimes E(G_2))$ the statement follows. \square

We note that this statement does not assume \mathbf{S} to be a Sylow d -torus of (\mathbf{G}, F) and describes the stabilizers of characters of normalizers of more general d -tori.

7.B Main results

We are now able to show that the conditions $A(d)_{\mathcal{H}_t}$ and $B(d)_{\mathcal{H}_\ell}$ from Definition 4.24, hold for groups of type A.

Theorem 7.5. *Let $N := N_G(\mathbf{S})$, $\tilde{N} := N_{\tilde{G}}(\mathbf{S})$ and $\hat{N} := N_{G \rtimes E(G)}(\mathbf{S})$. Then there exists some \hat{N} -stable \tilde{N} -transversal \mathbb{T} in $\text{Irr}(N)$ such that \mathbb{T} is \mathcal{H}_t -stable for some $t \in \tilde{G}$ inducing an automorphism of G of type \mathfrak{t}_ℓ . Furthermore, each $\chi \in \mathbb{T}$ extends to its stabilizer in \hat{N} .*

Proof. We use that the groups N , \tilde{N} and \hat{N} are contained in K , \tilde{K} and $K \rtimes E(G)$, respectively. Let \mathbb{T}_1 , \mathbb{T}_2 and $t \in \tilde{G}$ be as in Lemma 7.3. As \mathbb{T}_1 is a \tilde{N}_1 -transversal in $\text{Irr}(N_1)$ and \mathbb{T}_2 a \tilde{G}_2 -transversal in $\text{Irr}(G_2)$, the set $\mathbb{T}_1 \times \mathbb{T}_2$ is a $\tilde{N}_1 \times \tilde{G}_2$ -transversal in $\text{Irr}(N_1 \times G_2)$ as $[\tilde{\mathbf{G}}_1, \tilde{\mathbf{G}}_2] = 1$. Since $\tilde{N}_1 \times \tilde{G}_2 = \tilde{N}$, $\mathbb{T}_1 \times \mathbb{T}_2$ is a \tilde{N} -transversal in $\text{Irr}(N_1 \times G_2)$. We set

$$\mathbb{T} := \text{Irr}(N \mid \mathbb{T}_1 \times \mathbb{T}_2).$$

Given $\chi' \in \text{Irr}(N)$, there exists a $\tilde{N}_1 \times \tilde{G}_2$ -conjugate χ of χ' such that there exists some character χ_0 in the set

$$\text{Irr}(N_1 \times G_2 \mid \chi) \cap (\mathbb{T}_1 \times \mathbb{T}_2) \neq \emptyset$$

as $N_1 \times G_2 \triangleleft N$ and by the properties of $\mathbb{T}_1 \times \mathbb{T}_2$. In particular, $\chi \in \mathbb{T}$ and \mathbb{T} contains a \tilde{N} -transversal of $\text{Irr}(N)$.

Let $\chi, \chi' \in \mathbb{T}$ be two \tilde{N} -conjugate characters in \mathbb{T} . Then there exists some $\chi_0 \in \mathbb{T}_1 \times \mathbb{T}_2$, such that $\chi \in \text{Irr}(N \mid \chi_0)$ and $\chi' \in \text{Irr}(N \mid \chi_0)$. Hence, χ and χ' are \tilde{N}_{χ_0} -conjugate. As the quotients \tilde{N}_1/N_1 and \tilde{G}_2/G_2 are cyclic, maximal extendibility holds with respect to $N_1 \times G_2 \triangleleft \tilde{N}_1 \times \tilde{G}_2 = \tilde{N}$ and by Proposition 2.11, there exist extensions $\tilde{\chi}_0, \tilde{\chi}'_0$ of χ_0 to N_{χ_0} such that

$$(\tilde{\chi}_0)^N = \chi \text{ and } (\tilde{\chi}'_0)^N = \chi'.$$

However, we see that χ_0 also extends to \tilde{N}_{χ_0} . This implies that χ and χ' are \tilde{N}_{χ_0} -invariant. Hence, $\chi = \chi'$ and \mathbb{T} is a \tilde{N} -transversal in $\text{Irr}(N)$.

By Lemma 7.3 both \mathbb{T}_1 and \mathbb{T}_2 are \mathcal{H}_t -stable and t stabilizes \mathbf{S} and N , thus \mathbb{T} is \mathcal{H}_t -stable.

By definition of the transversals, $\mathbb{T}_1 \times \mathbb{T}_2$ is $\hat{N}_1 \times \hat{G}_2$ -stable. The automorphisms of G_i induced by the groups $E(G)$ and $E(G_i)$ coincide for $i = 1, 2$ and thus $\mathbb{T}_1 \times \mathbb{T}_2$ is \hat{N}° -stable, where $\hat{N}^\circ = N_{(G_1 \times G_2) \rtimes E(G)}(\mathbf{S})$. As by Lemma 7.4, $\hat{N} = \hat{N}^\circ N$, the set \mathbb{T} is \hat{N} -stable.

For the extendibility statement, let $\chi \in \mathbb{T}$ and $\chi_0 \in \mathbb{T}_1 \times \mathbb{T}_2$, such that $\chi \in \text{Irr}(N \mid \chi_0)$. Again by Proposition 2.11, there exists some extension $\tilde{\chi}_0$ of χ_0 to N_{χ_0} such that $(\tilde{\chi}_0)^N = \chi$. By Lemma 7.4 b), χ_0 extends to $\hat{N}_{\chi_0}^\circ$. We may apply Lemma 2.6 to the situation

$$X := N_{\chi_0} \triangleleft Y := \hat{N}^\circ N_{\chi_0}, \quad U := \hat{N}^\circ, \quad \psi := \tilde{\chi}_0, \quad \psi_0 := \chi_0.$$

So $\tilde{\chi}_0$ extends to some irreducible character $\hat{\chi}_0$ of $\hat{N}_{\chi_0}^\circ N_{\chi_0} = \hat{N}_{\chi_0}$, see also Lemma 7.3 b). Thus $(\hat{\chi}_0)^{\hat{N}^\circ}$ is irreducible and is the sought after extension of χ . \square

Corollary 7.6. *Let ℓ be an odd prime and (\mathbf{G}, F) be as in Definition 3.15 such that $\mathbf{G}^F = \text{SL}_n(\epsilon q)$. Let $d = d_\ell(q)$ and $t \in \tilde{G}$ from Lemma 7.3 d), that induces the outer automorphism of type \mathfrak{t}_ℓ on G . Then condition $A(d)_{\mathcal{H}_t}$ holds for \mathbf{G}^F and the prime ℓ .*

Proof. In case d is doubly regular for (\mathbf{G}, F) , this follows from Theorem 6.2. Otherwise, let $n_1 := \lfloor \frac{n}{d_0} \rfloor d_0$ for $d_0 = d_\ell(\epsilon q)$ and $n_2 = n - n_1$. Let \mathbf{G}_1 and \mathbf{G}_2 be defined as in the beginning of Section 7.A. Let \mathbf{S} be a Sylow d -torus of (\mathbf{G}_1, F) . By Proposition 3.31, \mathbf{S} is also a Sylow d -torus of (\mathbf{G}, F) . Thus, the conditions are satisfied by Theorem 7.5. \square

Theorem 7.7. *Let $L := C_G(\mathbf{S})$, $\tilde{L} := C_{\tilde{G}}(\mathbf{S})$, $\tilde{N} := N_{\tilde{G}}(\mathbf{S})$, and $\widehat{N} := N_{GE(G)}(\mathbf{S})$. Then there exists an extension map $\tilde{\Lambda}$ with respect to $\tilde{L} \triangleleft \tilde{N}$ such that $\tilde{\Lambda}$ is $\text{Irr}(\tilde{N}/N) \rtimes \widehat{N}$ -equivariant and \mathcal{H}_ℓ -equivariant.*

Proof. Let $T_1 := C_{G_1}(\mathbf{S})$. We wish to apply Lemma 2.12 to the following groups.

$$X := \tilde{L} \triangleleft \tilde{N} =: Y, \quad U := N_1 \times G_2 \text{ and } X_0 = \tilde{L} \cap (N_1 \times G_2) = T_1 \times G_2.$$

We observe that the group theoretic assumptions $T_1 \times G_2 \triangleleft \tilde{L}$, $T_1 \times G_2 \triangleleft N_1 \times G_2$ and $\tilde{L}(N_1 \times G_2) = \tilde{N}$ are satisfied and the quotients $\tilde{L}/(T_1 \times G_2) \cong \tilde{N}/(N_1 \times G_2)$ are abelian.

As maximal extendibility holds with respect to $T_1 \triangleleft C_{\tilde{G}_1}(\mathbf{S})$ and $G_2 \triangleleft \tilde{G}_2$, because of the cyclic quotients, we obtain maximal extendibility with respect to $T_1 \times G_2 \triangleleft \tilde{L} \cong C_{\tilde{G}_1}(\mathbf{S}) \times \tilde{G}_2$. Given the \tilde{G}_2 -transversal \mathbb{T}_2 in $\text{Irr}(G_2)$ of Lemma 7.3 c), the set $\text{Irr}(T_1) \times \mathbb{T}_2$ forms a $N_1 \times G_2$ -stable \tilde{L} -transversal in $\text{Irr}(T_1 \times G_2)$, since $T_1 \leq Z(\tilde{L})$. From the doubly regular case, by Lemma 6.23, there exists an extension map Λ_1 with respect to $T_1 \triangleleft N_1$. We define Λ as the extension map with respect to $T_1 \times G_2 \triangleleft N_1 \times G_2$ via

$$\lambda_1 \times \phi_2 \mapsto \Lambda_1(\lambda_1) \times \phi_2.$$

Thus, all conditions of Lemma 2.12 are satisfied and by a) there exists a \tilde{N} -equivariant extension map $\tilde{\Lambda}$ with respect to $\tilde{L} \triangleleft \tilde{N}$. By Lemma 2.12 b) $\tilde{\Lambda}$ is $\text{Lin}(\tilde{N}/(N_1 \times G_2))$ -equivariant. Clearly, this also implies $\text{Lin}(\tilde{N}/N)$ -equivariance.

The action of \widehat{N} on \tilde{N} stabilizes \tilde{L} and $N_1 \times G_2$. By definition of \mathbb{T}_2 , the transversal $\text{Irr}(T_1) \times \mathbb{T}_2$ is $\widehat{N}_1 \times \tilde{G}_2$ -stable. Once again the automorphisms of G_i induced by the groups $E(G)$ and $E(G_i)$ coincide for $i = 1, 2$ and thus $\text{Irr}(T_1) \times \mathbb{T}_2$ is $\widehat{N}^\circ = N_{(G_1 \times G_2) \rtimes E(G)}(\mathbf{S})$ -stable. As Λ_1 is \widehat{N}_1 -equivariant, Λ is \widehat{N}° -equivariant, so by Lemma 2.12c) $\tilde{\Lambda}$ is \widehat{N}° -equivariant as well. So $\tilde{\Lambda}$ is $\widehat{N}^\circ \tilde{N} = \widehat{N} \tilde{N}$ -equivariant.

Let $t \in \tilde{G}$ be the element from Theorem 6.2, that stabilizes \mathbf{S} . Hence, t lies in $\text{Aut}(Y)_{X,U}$ and $\text{Irr}(T_1) \times \mathbb{T}_2$ is \mathcal{H}_t -stable by definition. Furthermore, Λ is \mathcal{H}_t -equivariant and so is $\tilde{\Lambda}$ by Lemma 2.12 d). As $t \in \tilde{N}$ and $\tilde{\Lambda}$ is \tilde{N} -equivariant already, we obtain \mathcal{H}_ℓ -equivariance of $\tilde{\Lambda}$ which finishes the proof. \square

Corollary 7.8. *Let ℓ be an odd prime and (\mathbf{G}, F) be as in Definition 3.15 such that $\mathbf{G}^F = \text{SL}_n(\epsilon q)$. Let $d = d_\ell(q)$. Then condition $B(d)_{\mathcal{H}_\ell}$ holds for \mathbf{G}^F and the prime ℓ .*

Proof. In case d is doubly regular for (\mathbf{G}, F) , this follows from Theorem 6.24. Again, let $n_1 := \lfloor \frac{n}{d_0} \rfloor d_0$ for $d_0 = d_\ell(\epsilon q)$ and $n_2 = n - n_1$ and let \mathbf{S} be the Sylow d -torus of (\mathbf{G}_1, F) , then by Theorem 6.2, there exists an $\tilde{N} \times \mathcal{H}_\ell$ -equivariant extension map $\tilde{\Lambda}$ with respect to $\tilde{L} \triangleleft \tilde{N}$, which fulfils the properties needed. The existence of an extension map with respect to $N \triangleleft \tilde{N}$ follows from Lemma 2.2. \square

As established in Theorem 4.26, proven in [RSST25, Section 3], if conditions $A(\infty)_{\mathcal{H}_t}$, $A(d)_{\mathcal{H}_t}$ and $B(d)_{\mathcal{H}_\ell}$ hold for some $t \in \tilde{G}$, there exists an $\text{Aut}(G)_N \times \mathcal{H}_\ell$ -equivariant bijection

$$\Omega : \text{Irr}_{\ell'}(G) \xrightarrow{\sim} \text{Irr}_{\ell'}(N).$$

Consequently, the equivariance condition $(\text{Eq})_{\mathcal{H}_\ell}$ of the inductive conditions holds, leading to the following Theorem.

Theorem 7.9. *Let ℓ be an odd prime. The quasi-simple groups $\text{SL}_n(q)$ and $\text{SU}_n(q)$ satisfy the Equivariance condition $(\text{Eq})_{\mathcal{H}_\ell}$ from the inductive Galois–McKay conditions.*

Proof. By Corollary 7.8, condition $B(d)_{\mathcal{H}_\ell}$ holds for $\text{SL}_n(q)$ and $\text{SU}_n(q)$. Furthermore, Theorem 5.10 and Corollary 7.6 ensure that conditions $A(\infty)_{\mathcal{H}_{t_\infty}}$ and $A(d)_{\mathcal{H}_{t_d}}$ hold for elements $t_\infty, t_d \in \tilde{G}$, each inducing an outer automorphism of type \mathfrak{t}_ℓ . Since an outer automorphism of type \mathfrak{t}_ℓ is determined solely by the arithmetic properties of $\text{SL}_n(\epsilon q)$, being either trivial or the unique outer diagonal automorphism of order 2, it follows that $t_\infty t_d^{-1} \in G$. Consequently, both $A(\infty)_{\mathcal{H}_t}$ and $A(d)_{\mathcal{H}_t}$ hold for the choice $t = t_\infty$ or $t = t_d$. \square

Bibliography

- [Asc00] Michael Aschbacher. *Finite group theory*. Second. Cambridge Studies in Advanced Mathematics, 10. Cambridge University Press, Cambridge, 2000, pp. xii+304. ISBN: 0-521-78145-0; 0-521-78675-4. DOI: 10.1017/CB09781139175319.
- [Bon00] Cédric Bonnafé. “Opérateur de torsion dans $SL_n(q)$ et $SU_n(q)$ ”. In: *Bull. Soc. Math. France* 128.3 (2000), pp. 309–345. ISSN: 0037-9484,2102-622X.
- [Bon06] Cédric Bonnafé. “Sur les caractères des groupes réductifs finis à centre non connexe: applications aux groupes spéciaux linéaires et unitaires”. In: *Astérisque* 306 (2006), pp. vi+165. ISSN: 0303-1179,2492-5926.
- [BM92] Michel Broué and Gunter Malle. “Théorèmes de Sylow génériques pour les groupes réductifs sur les corps finis”. In: *Math. Ann.* 292.2 (1992), pp. 241–262. ISSN: 0025-5831,1432-1807. DOI: 10.1007/BF01444619.
- [BN21] Olivier Brunat and Rishi Nath. “The Navarro conjecture for the alternating groups”. In: *Algebra Number Theory* 15.4 (2021), pp. 821–862. ISSN: 1937-0652,1944-7833. DOI: 10.2140/ant.2021.15.821.
- [CE04] Marc Cabanes and Michel Enguehard. *Representation theory of finite reductive groups*. New Mathematical Monographs 1. Cambridge University Press, Cambridge, 2004, pp. xviii+436. ISBN: 0-521-82517-2. DOI: 10.1017/CB09780511542763.
- [CS19] Marc Cabanes and Britta Späth. “Descent equalities and the inductive McKay condition for types B and E”. In: *Adv. in Mathematics* 356 (2019), p. 106820. ISSN: 0001-8708. DOI: <https://doi.org/10.1016/j.aim.2019.106820>.
- [CS13] Marc Cabanes and Britta Späth. “Equivariance and extendibility in finite reductive groups with connected center”. In: *Math. Z.* 275.3-4 (2013), pp. 689–713. ISSN: 0025-5874,1432-1823. DOI: 10.1007/s00209-013-1156-7.
- [CS17] Marc Cabanes and Britta Späth. “Equivariant character correspondences and inductive McKay condition for type A”. In: *J. Reine Angew. Math.* 728 (2017), pp. 153–194. ISSN: 0075-4102,1435-5345. DOI: 10.1515/crelle-2014-0104.
- [CS25] Marc Cabanes and Britta Späth. “The McKay Conjecture on character degrees”. In: *Preprint, to appear in Annals of Mathematics.* (2025).
- [Car85] Roger W. Carter. *Finite groups of Lie type, Conjugacy classes and complex characters*. Pure and Applied Mathematics. John Wiley & Sons, New York, 1985, pp. xii+544. ISBN: 0-471-90554-2.
- [Gec03] Meinolf Geck. *An introduction to algebraic geometry and algebraic groups*. Vol. 10. Oxford Graduate Texts in Mathematics. Oxford University Press, Oxford, 2003, pp. xii+307. ISBN: 0-19-852831-0.
- [Gec91] Meinolf Geck. “On the decomposition numbers of the finite unitary groups in nondefining characteristic”. In: *Math. Z.* 207.1 (1991), pp. 83–89. ISSN: 0025-5874,1432-1823. DOI: 10.1007/BF02571376.

- [Gec04] Meinolf Geck. “On the Schur indices of cuspidal unipotent characters”. In: *Finite groups 2003*. Walter de Gruyter, Berlin, 2004, pp. 87–104. ISBN: 3-11-017447-2.
- [GM20] Meinolf Geck and Gunter Malle. *The character theory of finite groups of Lie type*. Vol. 187. Cambridge Studies in Advanced Mathematics. A guided tour. Cambridge University Press, Cambridge, 2020, pp. ix+394. ISBN: 978-1-108-48962-1.
- [GLS98] Daniel Gorenstein, Richard Lyons, and Ronald Solomon. *The classification of the finite simple groups. Number 3. Part I. Chapter A*. Vol. 40.3. Mathematical Surveys and Monographs. American Mathematical Society, Providence, RI, 1998, pp. xvi+419. ISBN: 0-8218-0391-3. DOI: 10.1090/surv/040.3.
- [Hum95] James E. Humphreys. *Conjugacy classes in semisimple algebraic groups*. Vol. 43. Mathematical Surveys and Monographs. American Mathematical Society, Providence, RI, 1995, pp. xviii+196. ISBN: 0-8218-0333-6. DOI: 10.1090/surv/043.
- [Hum78] James E. Humphreys. *Introduction to Lie algebras and representation theory*. Vol. 9. Graduate Texts in Mathematics. Second printing, revised. Springer-Verlag, New York-Berlin, 1978, pp. xii+171. ISBN: 0-387-90053-5.
- [Hum75] James E. Humphreys. *Linear algebraic groups*. Vol. No. 21. Graduate Texts in Mathematics. Springer-Verlag, New York-Heidelberg, 1975, pp. xiv+247.
- [IR90] Kenneth Ireland and Michael Rosen. *A classical introduction to modern number theory*. Second. Vol. 84. Graduate Texts in Mathematics. Springer-Verlag, New York, 1990, pp. xiv+389. ISBN: 0-387-97329-X. DOI: 10.1007/978-1-4757-2103-4.
- [Isa76] I. Martin Isaacs. *Character theory of finite groups*. Vol. No. 69. Pure and Applied Mathematics. Academic Press [Harcourt Brace Jovanovich, Publishers], New York-London, 1976, pp. xii+303.
- [IMN07] I. Martin Isaacs, Gunter Malle, and Gabriel Navarro. “A reduction theorem for the McKay conjecture”. In: *Invent. Math.* 170.1 (2007), pp. 33–101. ISSN: 0020-9910,1432-1297. DOI: 10.1007/s00222-007-0057-y.
- [JK81] Gordon James and Adalbert Kerber. *The representation theory of the symmetric group*. Vol. 16. Encyclopedia of Mathematics and its Applications. With a foreword by P. M. Cohn, With an introduction by Gilbert de B. Robinson. Addison-Wesley Publishing Co., Reading, MA, 1981, pp. xxviii+510. ISBN: 0-201-13515-9.
- [Joh22] Birte Johansson. “On the inductive McKay-Navarro condition for finite groups of Lie type in their defining characteristic”. In: *J. Algebra* 610 (2022), pp. 223–240. ISSN: 0021-8693,1090-266X. DOI: 10.1016/j.jalgebra.2022.06.033.
- [Kaw85] Noriaki Kawanaka. “Generalized Gelfand-Graev representations and Ennola duality”. In: *Algebraic groups and related topics (Kyoto/Nagoya, 1983)*. Vol. 6. Adv. Stud. Pure Math. North-Holland, Amsterdam, 1985, pp. 175–206. ISBN: 0-444-87711-8. DOI: 10.2969/aspm/00610175.
- [Mal07] Gunter Malle. “Height 0 characters of finite groups of Lie type”. In: *Represent. Theory* 11 (2007), pp. 192–220. ISSN: 1088-4165. DOI: 10.1090/S1088-4165-07-00312-3.
- [MT11] Gunter Malle and Donna Testerman. *Linear Algebraic Groups and Finite Groups of Lie type*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 2011.

- [McK71] John McKay. “A new invariant for simple groups”. In: *Notices Amer. Math. Soc.* Vol. 18, issue no. 128, p. 397 (1971).
- [Nat09] Rishi Nath. “The Navarro conjecture for the alternating groups, $p = 2$ ”. In: *J. Algebra Appl.* 8.6 (2009), pp. 837–844. ISSN: 0219-4988,1793-6829. DOI: 10.1142/S0219498809003667.
- [Nav18] Gabriel Navarro. *Character theory and the McKay conjecture*. Vol. 175. Cambridge University Press, Cambridge, 2018, pp. xviii+234. ISBN: 978-1-108-42844-6. DOI: 10.1017/9781108552790.
- [Nav04] Gabriel Navarro. “The McKay conjecture and Galois automorphisms”. In: *Ann. of Math. (2)* 160.3 (2004), pp. 1129–1140. ISSN: 0003-486X,1939-8980. DOI: 10.4007/annals.2004.160.1129.
- [NSV20] Gabriel Navarro, Britta Späth, and Carolina Vallejo. “A reduction theorem for the Galois-McKay conjecture”. In: *Trans. Amer. Math. Soc.* 373.9 (2020), pp. 6157–6183. ISSN: 0002-9947,1088-6850. DOI: 10.1090/tran/8111.
- [Ruh21] Lucas Ruhstorfer. “The Navarro refinement of the McKay conjecture for finite groups of Lie type in defining characteristic”. In: *J. Algebra* 582 (2021), pp. 177–205. ISSN: 0021-8693,1090-266X. DOI: 10.1016/j.jalgebra.2021.04.025.
- [RSF22] Lucas Ruhstorfer and A. A. Schaeffer Fry. “The inductive McKay-Navarro conditions for the prime 2 and some groups of Lie type”. In: *Proc. Amer. Math. Soc. Ser. B* 9 (2022), pp. 204–220. ISSN: 2330-1511. DOI: 10.1090/bproc/123.
- [RSF25] Lucas Ruhstorfer and A. A. Schaeffer Fry. “The McKay-Navarro conjecture for the prime 2”. In: *Adv. Math.* 477 (2025), Paper No. 110369, 25. ISSN: 0001-8708,1090-2082. DOI: 10.1016/j.aim.2025.110369.
- [RSST25] Lucas Ruhstorfer, A. A. Schaeffer Fry, Britta Späth, and Jay Taylor. “Towards the inductive McKay-Navarro Condition for groups of Lie type”. In: *Preprint* (2025). eprint: arXiv:2506.17123.
- [SFT18] A. A. Schaeffer Fry and Jay Taylor. “On self-normalising Sylow 2-subgroups in type A”. In: *J. Lie Theory* 28.1 (2018), pp. 139–168. ISSN: 0949-5932.
- [Ser77] Jean-Pierre Serre. *Linear Representations of Finite Groups*. Graduate Texts in Mathematics. Springer New York, 1977. ISBN: 0-387-90190-6.
- [Sho97] Toshiaki Shoji. “Shintani descent for special linear groups”. In: 991. Group theory and combinatorial mathematics (Japanese) (Kyoto, 1996). 1997, pp. 68–79. DOI: 10.1006/jabr.1997.7174.
- [Spä25] Britta Späth. “Extensions of characters in type D and the inductive McKay condition, II”. In: *Invent. Math.* 242.1 (2025), pp. 45–122. ISSN: 0020-9910,1432-1297. DOI: 10.1007/s00222-025-01354-9.
- [Spä12] Britta Späth. “Inductive McKay condition in defining characteristic”. In: *Bull. of the Lond. Mathe. Soc.* 44.3 (2012), pp. 426–438. DOI: <https://doi.org/10.1112/blms/bdr100>.
- [Spä10] Britta Späth. “Sylow d -tori of classical groups and the McKay conjecture. II”. In: *J. Algebra* 323.9 (2010), pp. 2494–2509. ISSN: 0021-8693,1090-266X. DOI: 10.1016/j.jalgebra.2010.02.007.
- [Spr74] T. A. Springer. “Regular elements of finite reflection groups”. In: *Invent. Math.* 25 (1974), pp. 159–198. ISSN: 0020-9910,1432-1297. DOI: 10.1007/BF01390173.
- [Spr10] T.A. Springer. *Linear Algebraic Groups*. Modern Birkhäuser Classics. Birkhäuser Boston, 2010. ISBN: 9780817648404.

- [SV20] Bhama Srinivasan and C. Ryan Vinroot. “Galois group action and Jordan decomposition of characters of finite reductive groups with connected center”. In: *Journal of Algebra* **558** (2020). Special Issue in honor of Michel Broué, pp. 708–727. ISSN: 0021-8693. DOI: <https://doi.org/10.1016/j.jalgebra.2019.04.025>.
- [Tap25] Carlos Tapp. “Inductive Feit and Galois-McKay conditions for some small-rank simple groups of Lie Type”. In: *Preprint* (2025). eprint: [arXiv:2507.21650](https://arxiv.org/abs/2507.21650).
- [TZ04] Pham Huu Tiep and Alexandre E. Zalesski. “Unipotent elements of finite groups of Lie type and realization fields of their complex representations”. In: *J. Algebra* 271.1 (2004), pp. 327–390. ISSN: 0021-8693,1090-266X. DOI: 10.1016/S0021-8693(03)00174-1.
- [Wil09] Robert A. Wilson. *The finite simple groups*. Vol. 251. Graduate Texts in Mathematics. Springer-Verlag London, Ltd., London, 2009, pp. xvi+298. ISBN: 978-1-84800-987-5. DOI: 10.1007/978-1-84800-988-2.